$S$-matrix unitarity ($S^\dagger S = 1$) in $R^2_{\mu\nu}$ gravity

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We show that in the quadratic curvature theory of gravity, or simply $R^2_{\mu\nu}$ gravity, unitarity bound is violated but $S$-matrix unitarity ($SS^\dagger = 1$) is satisfied. This theory is renormalizable, and hence the failure of unitarity bound is a counter example of Llewellyn Smith’s conjecture on the relation between unitarity and renormalizability. We have recently proposed a new conjecture that $S$-matrix unitarity gives the same conditions as renormalizability. We verify that $S$-matrix unitarity holds in the matter-graviton scattering at tree level in the $R^2_{\mu\nu}$ gravity, demonstrating our new conjecture.

Subject Index E05

1. Introduction

Renormalizability and unitarity are the two key conditions when we wish to quantize a gravitational theory. Llewellyn Smith gave a conjecture on the relation between the two conditions in quantum field theories (QFT) \cite{1-5}.

Einstein gravity is classically a beautiful theory, but it is a non-renormalizable theory. Related to this point, the unitarity bound ($E^n$, $n \leq 0$ as $E \to \infty$) on $2 \to 2$ scattering amplitudes of graviton \cite{8, 9} is violated.

Quadratic curvature gravity, or simply $R^2_{\mu\nu}$ gravity, is a renormalizable theory \cite{10}, but it contains negative norm states of massive graviton and hence the unitarity is violated. We have pointed out that the unitarity bound and the $S$-matrix unitarity ($SS^\dagger = 1$) are two separate notions \cite{11}. The former often fails in theories containing negative norm states, but the latter provides useful conditions on the UV behavior of a wide class of theories.

In this letter we evaluate the amplitudes of matter-graviton two-body scattering in $R^2_{\mu\nu}$ gravity, and show that they obey the $S$-matrix unitarity ($SS^\dagger = 1$) after taking account of negative norm states (massive graviton) in the sum over intermediate normalized states $\Phi$.

\footnote{An interesting connection is derived \cite{6, 7} between perturbative unitarity constrains on S-matrix and finiteness (rather than renormalizability) of physical quantities in some class of Higgs field theories.}
in the optical theorem,
\[ 2\text{Im} T(\Psi \rightarrow \Psi) = \Sigma_\Phi \varepsilon_\Phi |T(\Psi \rightarrow \Phi)|^2, \]
where \( \varepsilon_\Phi = +1(-1) \) (using an adequate normalization) if the norm of \( \Phi \) is positive (negative). 
S-matrix unitarity is a generalization of the unitarity bound applicable to scattering in theories containing negative norm states too. We formulate the conjecture that the S-matrix unitarity gives the same conditions as renormalizability in gravity theories, as well as QFT, thus extending Llewelyn Smith’s conjecture regarding the unitarity bound [1–5]. This letter presents a positive result of our conjecture by computing the matter-graviton two-body scattering at tree-level in the quadratic curvature gravity, which is the lowest order in \( \kappa \).

2. \( R^2_{\mu\nu} \) gravity coupled with a scalar field \( \phi \) We begin by writing the renormalizable action \( S = S_{\text{gravity}} + S_{\text{matter}} \) of \( R^2_{\mu\nu} \) gravity coupled with a matter scalar field \( \phi \) [12],
\[
S_{\text{gravity}} = \int d^4x \sqrt{-g} \left( \Lambda + \frac{1}{\kappa^2} R + \alpha R^2 + \beta R^2_{\mu\nu} \right),
\]
\[
S_{\text{matter}} = \int d^4x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 + \xi \phi^2 R \right). \tag{3}
\]
We consider the tree-level amplitude of the matter-graviton scattering in the flat spacetime, where the assumption of \( \Lambda = 0 \) is required and the \( \phi^4 \) term does not contribute to matter-graviton scattering. Graviton field \( h_{\mu\nu}(= g_{\mu\nu} - \eta_{\mu\nu}) \) contains massless field \( H_{\mu\nu} \) and massive one \( I_{\mu\nu} \),
\[
h_{\mu\nu} = H_{\mu\nu} + I_{\mu\nu}. \tag{4}
\]
\( H_{\mu\nu} \) is composed of a positive-norm massless spin-2 field with 2 degrees of freedom (DOF) denoted by \( H^{(\sigma)} \) with \( (\sigma) = (2, e), (2, o) \), while the \( I_{\mu\nu} \) is composed of a negative-norm massive spin-2 field with 5 DOF denoted by \( I^{(\tau)} \) with \( (\tau) \) being \( (2, e), (2, o), (1, e), (1, o), (0) \), and a positive-norm scalar (spin-0) graviton \( I^{(S)} \). The precise meaning of the suffix \( (\sigma) \) and \( (\tau) \) will be explained later. The fields \( H_{\mu\nu} \) and \( I_{\mu\nu} \) are expanded in terms of spin-polarization bases \( \epsilon_{\mu\nu}^{(\sigma)}, \epsilon_{\mu\nu}^{(\tau)} \) and \( \theta_{\mu\nu}/\sqrt{3} \). In momentum space, they are written as
\[
H_{\mu\nu}(p) = \sum_{\sigma} H^{(\sigma)}(p) \epsilon_{\mu\nu}^{(\sigma)}(p), \tag{5}
\]
\[
I_{\mu\nu}(p) = \sum_{\tau} I^{(\tau)}(p) \epsilon_{\mu\nu}^{(\tau)}(p) + \frac{1}{\sqrt{3}} I^{(S)}(p) \theta_{\mu\nu}. \tag{6}
\]
where summations of \( \sigma \) and \( \tau \) are done with respect to 2 massless and 5 massive spin-2 degrees of freedom respectively. Concrete forms of \( \epsilon_{\mu\nu}^{(\sigma)}(p), \epsilon_{\mu\nu}^{(\tau)}(p) \) and \( \theta_{\mu\nu} \) will be given later.

We give Feynman rules. \( H_{\mu\nu} \) and \( I_{\mu\nu} \) may be denoted by \( h_{\mu\nu} \) collectively in Feynman diagrams, since graviton field is expressed in terms of \( h_{\mu\nu} \) in the action. Propagators and the vertices \( h_{\mu\nu} h_{\alpha\beta} h_{\gamma\lambda}, h_{\mu\nu} \phi, h_{\mu\nu} h_{\alpha\beta} \phi \phi \) are shown in Fig. 1 and 2. These are the minimum requirements for computing \( h_{\mu\nu} \phi \) scattering at tree level. Feynman rules are obtained by expanding the action (2) and (3) in powers of \( h_{\mu\nu} \), as obtained in [13]. It is useful to define the transverse part \( \tilde{h}_{\mu\nu} \) of graviton \( h_{\mu\nu} \), obeying \( \partial^\mu \tilde{h}_{\mu\nu} = 0 \). Note that \( \tilde{h}_{\mu\nu} \) includes all on-shell states, that is, massless graviton \( H_{\mu\nu} \) and massive graviton \( I_{\mu\nu} \). We can write Feynman rules with \( h_{\mu\nu} \) replaced by \( \tilde{h}_{\mu\nu} \). This is because, in tree-level approximation which we will
take, the calculation requires only degrees of freedom appearing as physical onshell modes, even for off-shell propagators. The propagators for $\phi$ and $\tilde{h}_{\mu \nu}$ are then given by

$$G_\phi = -\frac{i}{p^2 + m^2},$$

$$G_{\alpha \beta; \mu \nu} = \frac{2i}{\beta} \frac{1}{p^4 + m_I^2 p^2} P^{(2)}_{\alpha \beta; \mu \nu} + \frac{i}{2(3\alpha + \beta)} \frac{1}{p^4 + m_S^2 p^2} P^{(0)}_{\alpha \beta; \mu \nu},$$

$$m_I^2 = -\left(\beta \kappa^2\right)^{-1}, \quad m_S^2 = \left(2\kappa^2 (3\alpha + \beta)\right)^{-1},$$

where $P^{(2)}_{\alpha \beta; \mu \nu}$ and $P^{(0)}_{\alpha \beta; \mu \nu}$ are the projections to the transverse-traceless and the transverse-trace part [14]. The first term in Eq.(8) shows the propagation of spin-2 degree of freedom and may be written as

$$\frac{1}{\beta} \frac{i}{p^4 + m_I^2 p^2} = -i\kappa^2 \left(\frac{1}{p^2} - \frac{1}{p^2 + m_I^2}\right),$$

where $m_I$ is the massive graviton mass. The minus sign in the second term corresponds to $I_{\mu \nu}$ being a negative norm field. The second term in the right hand side of Eq.(8) is also decomposed in the similar way. The mass of the scalar graviton is $m_S$. The massless pole is gauge mode, and thus it does not appear as a physical onshell degree of freedom. The vertex functions are lengthy, and thus we do not show the explicit form here. They will be shown in the forthcoming paper [13]. The difference between $H_{\mu \nu}$ and $I_{\mu \nu}$ appears on the external lines, that is, according to the initial and final gravitons being massless spin-2, massive spin-2 or scalar graviton, we operate the corresponding basis $e^{(c)}_{\mu \nu}(p)$, $e^{(r)}_{\mu \nu}(p)$ or $\theta_{\mu \nu}(p)/\sqrt{3}$, respectively, on the external graviton legs.

3. Scattering amplitudes

The purpose of this letter is to study the optical theorem (1) for $h_{\mu \nu}$-$\phi$ scattering in the lowest order of perturbation in $\kappa$. The invariant amplitude $A$ is defined by

$$\langle \Phi | T | \Psi \rangle = \delta^4(p_\Psi - p_\Phi) A(\Psi \to \Phi).$$

For $2 \to 2$ scattering $A$ has the mass dimension $[A] = 0$. 
We are concerned about the interplay between the massless and massive gravitons appearing as the intermediate states $\Phi$ in the optical theorem (1). Hence we consider $h_{\mu\nu}$-\phi scattering involving both massless (positive norm) and massive (negative norm) gravitons. The simplest such $2 \to 2$ processes are:

$$
H^{(\sigma)} + \phi \to H^{(\sigma')} + \phi, \quad H^{(\sigma)} + \phi \to I^{(r')} + \phi, \quad I^{(r)} + \phi \to I^{(r')} + \phi, \quad I^{(r)} + \phi \to I^{(s)} + \phi.
$$

We fix the scattering kinematics by

$$
h(k_1, e_{1,\mu\nu}(k_1)) + \phi(k_2) \to h(k_3, e_{3,\alpha\beta}(k_3)) + \phi(k_4),
$$

where $h$ indicates all modes of graviton $H^{(\sigma)}$, $I^{(r)}$ and $I^{(s)}$. $e_{i,\mu\nu}(k_i)$ stands for the corresponding bases $e^{(\sigma)}_{\mu\nu}(k_i)$, $e^{(r)}_{\mu\nu}(k_i)$ and $\theta_{\mu\nu}(k_i)/\sqrt{3}$ ($i = 1, 3$). We take the center of mass (CoM) frame, and set

$$
k_{1,\mu} = \left(\sqrt{k^2 + m_1^2}, k, 0, 0\right), \quad k_{3,\mu} = \left(\sqrt{q^2 + m_3^2}, q \cos \theta, q \sin \theta, 0\right),
$$

$$
k_{2,\mu} = \left(\sqrt{k^2 + m_2^2}, -k, 0, 0\right), \quad k_{4,\mu} = \left(\sqrt{q^2 + m_2^2}, -q \cos \theta, -q \sin \theta, 0\right),
$$

where $m_1^2, m_2^2 = 0$, $m_1^2$ or $m_2^2$ for massless, massive spin-2 or massive scalar graviton, respectively.

To define bases $e^{(\sigma)}_{\mu\nu}$, $e^{(r)}_{\mu\nu}$ and $\theta_{\mu\nu}/\sqrt{3}$, we introduce longitudinal vector $l_{i,\mu}$ and transvers vectors $t_{i,\mu}$ and $u_{\mu}$ ($i = 1, 3$),

$$
l_{1,\mu} = m_1^{-1} \left( k, \sqrt{k^2 + m_1^2}, 0, 0 \right), \quad t_{1,\mu} = (0, 0, 1, 0), \quad u_{\mu} = (0, 0, 0, 1),
$$

$$
l_{3,\mu} = m_3^{-1} \left( q, \sqrt{q^2 + m_3^2} \cos \theta, \sqrt{q^2 + m_3^2} \sin \theta, 0 \right), \quad t_{3,\mu} = (0, -\sin \theta, \cos \theta, 0),
$$

where $t_{i,\mu}$ ($i = 1, 3$) is tangent to the spatial scattering plane but $u$ is normal to. The bases for graviton are expressed with these vectors,

$$
e^{(0)}_{i,\mu\nu} = \frac{2}{\sqrt{6}} l_{i,\mu} l_{i,\nu} - \frac{1}{\sqrt{6}} t_{i,\mu} t_{i,\nu} - \frac{1}{\sqrt{6}} u_{\mu} u_{\nu}, \quad e^{(1,e)}_{i,\mu\nu} = \frac{1}{\sqrt{2}} (l_{i,\mu} t_{i,\nu} + t_{i,\mu} l_{i,\nu}),
$$

$$
e^{(1,o)}_{i,\mu\nu} = \frac{1}{\sqrt{2}} (l_{i,\mu} u_{\nu} + u_{\mu} l_{i,\nu}), \quad e^{(2,e)}_{i,\mu\nu} = \frac{1}{\sqrt{2}} (t_{i,\mu} t_{i,\nu} - u_{\mu} u_{\nu}),
$$

$$
e^{(2,o)}_{i,\mu\nu} = \frac{1}{\sqrt{2}} (t_{i,\mu} u_{\nu} + u_{\mu} t_{i,\nu}), \quad e^{(s)}_{i,\mu\nu} = \frac{1}{\sqrt{3}} (l_{i,\mu} l_{i,\nu} + t_{i,\mu} t_{i,\nu} + u_{\mu} u_{\nu}) = \frac{1}{\sqrt{3}} \theta_{i,\mu\nu}.
$$

Here, the numbers in the index show the helicity. For massless graviton basis $e^{(\sigma)}_{\mu\nu}$, these bases except helicity-2 are ill-defined, since the inverse of mass appears in $l_{i,\mu}$. However, it does not matter, because massless graviton has only helicity-2 degrees of freedom. Indices $e$ and $o$ mean even and odd; in the bases with $e$ (or) index, all terms have even (odd) number of $u$. The scattering amplitude from even to odd (or vice versa) vanishes.

We compute the amplitudes of (12) at tree level. Four types of graphs contribute to the scattering, contact term, $s$-channel and $u$-channel exchanges of $\phi$ propagator, $t$-channel exchange of $h_{\mu\nu}$ propagator (Fig.3), which are denoted by $A_c$, $A_s$, $A_u$, $A_t$, respectively. $A_s$ and $A_u$ are related by the crossing symmetry. Because the difference among the amplitudes of (12) caused only by the on-shell bases operated to the external lines 1 and 3, these amplitudes
are related to each other. Especially, the three amplitudes for helicity-2 mode on the first line of (12) and two amplitudes for helicity-2 mode of the first two on the second line of (12) are related, respectively, by the replacement \( m_1^2 \leftrightarrow m_H^2 = 0 \).

The computation requires many pages and is relegated to the forthcoming paper [13]. Here we only give the amplitudes in the high energy (HE) limit \( E \to \infty \) (which is equivalent to \( k \to \infty \)), where the CoM energy \( E = k_{10} + k_{20} = 2k + (m_1^2 + m^2)/(2k) \). The scattering amplitude \( A \) is the sum of the four terms, \( A = A_c + A_s + A_u + A_t \).

\[
A \left( h^{(2,o)} + \phi \to h^{(2,o)} + \phi \right) = A \left( h^{(2,e)} + \phi \to h^{(2,e)} + \phi \right) = -\kappa^2 k_2 \frac{1 + \cos \theta}{1 - \cos \theta} + O \left( k^0 \right),
\]

\[
A \left( h^{(2,o)} + \phi \to I^{(1,o)} + \phi \right) = -A \left( h^{(2,e)} + \phi \to I^{(1,e)} + \phi \right) = \kappa^2 \frac{m_1 k \sin \theta}{2 \left( 1 - \cos \theta \right)} + O \left( k^{-1} \right),
\]

\[
A \left( I^{(1,o)} + \phi \to I^{(1,o)} + \phi \right) = A \left( I^{(1,e)} + \phi \to I^{(1,e)} + \phi \right)
= -\frac{\kappa^2 m_1^2 \theta^2}{8} + \frac{1 + \cos \theta}{\left( 1 - \cos \theta \right)^2} + O \left( k^{-2} \right),
\]

\[
A \left( h^{(2,e)} + \phi \to I^{(0,e)} + \phi \right) = O \left( k^0 \right),
\]

\[
A \left( I^{(1,e)} + \phi \to I^{(0,e)} + \phi \right) = O \left( k^{-1} \right),
\]

\[
A \left( I^{(0,e)} + \phi \to I^{(0,e)} + \phi \right) = -\frac{\kappa^2 m_1^2 \left( 1 + \cos \theta \right)}{2 \left( 1 - \cos \theta \right)} - \frac{\kappa^2}{8} \left( m_1^2 + 2m^2 \right) + O \left( k^{-2} \right),
\]

\[
A \left( h^{(2,e)} + \phi \to I^{(S)} + \phi \right) = -\frac{\sqrt{6} \kappa^2}{48} \left( m_1^2 + 2m^2 \right) \left( 3 + \cos^2 \theta \right) - \sqrt{6} \kappa^2 \theta^2 + O \left( k^{-2} \right),
\]

\[
A \left( I^{(1,e)} + \phi \to I^{(S)} + \phi \right) = O \left( k^{-1} \right),
\]

\[
A \left( I^{(0,e)} + \phi \to I^{(S)} + \phi \right) = O \left( k^0 \right),
\]

\[
A \left( I^{(S)} + \phi \to I^{(S)} + \phi \right) = \frac{\kappa^2}{24 m_1^2} \left( 4 m_1^4 - 8 \theta^2 + 9 m^4 \right)
- \frac{1}{72 \kappa^2 \beta} \left( 1 - \frac{12 (1 - \cos \theta)}{(1 + \cos \theta)^2} \right) + 8 \kappa^2 \theta^2 + O \left( k^{-2} \right),
\]
and the others are zero, where $h$ is $H$ or $I$.

The HE behavior of these amplitudes may also be written as

$$A(h(a,b) + \phi \rightarrow h(a',b') + \phi) \sim \beta(a,b), (a' b') E^{\alpha_{a'b'}}; \tag{27}$$

$$A(h(a,b) + \phi \rightarrow I(S) + \phi) \sim \beta_{a,b} E^{\alpha_{aS}}; \tag{28}$$

$$A(I(S) + \phi \rightarrow I(S) + \phi) \sim \beta E^{\alpha_{SS}}; \tag{29}$$

where $a, a' = 2, 1, 0$ and $b, b' = o, e$. Equations (17)-(26) imply

$$\alpha_{22} = 2, \quad \alpha_{21} = 1, \quad \alpha_{20} = 0, \quad \alpha_{11} = 0, \quad \alpha_{10} \leq -1, \quad \alpha_{00} = 0, \quad \alpha_{2S} = 0, \quad \alpha_{1S} \leq -1, \quad \alpha_{0S} \leq 0, \quad \alpha_{SS} = 0. \tag{30}$$

The HE behavior of the elastic $2 \rightarrow 2$ amplitudes for the massless graviton-matter scattering is

$$A \left( H(2,e) + \phi \rightarrow H(2,e) + \phi \right) \sim E^2. \tag{31}$$

It is the same as those in Einstein gravity [8, 9], and the unitarity bound, $A \sim E^n$, $n \leq 0$, is apparently violated. The immediate question is whether the optical theorem (1) is still obeyed or not. We will study this question using the the amplitudes obtained above.

4. S-matrix unitarity  Finally we will show the mechanism how the condition $\langle \Psi | S^\dagger S | \Psi \rangle = 1$ is met due to the cancellation of the two contributions, one from positive norm graviton $H_{\mu\nu}$ and the other from negative norm graviton $I_{\mu\nu}$ in the intermediate sum in the optical theorem (1). We apply Eq.(1) to the elastic scattering $h(\sigma) + \phi \rightarrow h(\sigma) + \phi$, i.e. $\Psi = h(\sigma) + \phi$ as depicted in Fig 4. We take the two-particle approximation of the intermediate states $\Phi$, which is the lowest order in the perturbation in $\kappa$, hence $\Phi = \{ H_{\mu\nu} + \phi \}$ and $\{ I_{\mu\nu} + \phi \}$ in Eq.(1).

We demonstrate Eq.(1) by $H(2,e) + \phi \rightarrow H(2,e) + \phi$. In the present two-particle approximation for $\Phi$, noting $\epsilon_{H(\tau)+\phi} = +1$, $\epsilon_{I(\tau)+\phi} = -1$ and $\epsilon_{I(S)+\phi} = +1$, we have\footnote{To be accurate, the sum over the intermediate states on the right side includes the integration with respect to three-dimensional momenta, but it is not shown explicitly. Furthermore, the momentum-dependent normalization factor is required. In $2 \rightarrow 2$ scattering in four-dimensional spacetime, such dependences accidentally cancel to each other. Generic cases are discussed in Ref. [5]. Moreover, we use the fact that the amplitudes involving both even and odd modes vanish.}

$$\text{Im} A \left( H(2,e) + \phi \rightarrow H(2,e) + \phi \right) = \left| A \left( H(2,e) + \phi \rightarrow H(2,e) + \phi \right) \right|^2$$

$$- \sum_{\tau} \left| A \left( H(2,e) + \phi \rightarrow I(\tau) + \phi \right) \right|^2 + \left| A \left( H(2,e) + \phi \rightarrow I(S) + \phi \right) \right|^2. \tag{32}$$
For Eq. (32) to be satisfied in HE limit, the $k$ of Eq. (33) cannot be weaker than that of the right hand side. We evaluate the both sides using the tree amplitudes obtained in sec. 3. The left hand side of Eq. (33) is bounded as

$$|A \left( H^{(2,e)} + \phi \rightarrow H^{(2,e)} + \phi \right) | \geq |A \left( H^{(2,e)} + \phi \rightarrow H^{(2,e)} + \phi \right) |^2 - \sum \tau \left| A \left( H^{(2,e)} + \phi \rightarrow I^{(\tau)} + \phi \right) \right|^2 + \left| A \left( H^{(2,e)} + \phi \rightarrow I^{(S)} + \phi \right) \right|^2. \quad (33)$$

For Eq. (32) to be satisfied in HE limit, the $k$ dependence of the left hand side of inequality of Eq. (33) cannot be weaker than that of the right hand side. We evaluate the both sides using the tree amplitudes obtained in sec. 3. The left hand side of Eq. (33) is bounded as

$$|A \left( H^{(2,e)} + \phi \rightarrow H^{(2,e)} + \phi \right) | = k^2 k^2 \frac{1 + \cos \theta}{1 - \cos \theta} + O \left( k^0 \right). \quad (34)$$

We explicitly write down the right side of inequality (33),

$$|A \left( H^{(2,e)} + \phi \rightarrow H^{(2,e)} + \phi \right) |^2 - \left| A \left( H^{(2,e)} + \phi \rightarrow I^{(2,e)} + \phi \right) \right|^2 - \left| A \left( H^{(2,e)} + \phi \rightarrow I^{(1,e)} + \phi \right) \right|^2 - \left| A \left( H^{(2,e)} + \phi \rightarrow I^{(0,e)} + \phi \right) \right|^2 + \left| A \left( H^{(2,e)} + \phi \rightarrow I^{(S)} + \phi \right) \right|^2. \quad (35)$$

Since Eq. (17) shows that the first term has $k^4$ dependence in the leading order, inequality of Eq. (33) seems to be violated in the first glance. However, if we look at the first two terms of the right hand side of Eq. (35), their leading order $k^4$ dependences cancel to each other, and there remains the next leading $k^2$ term. Since the other terms are $O(k^2)$, Eq. (35) is indeed $O(k^2)$. Hence, the $k$ dependence is the same in the both sides of inequality of Eq. (33), thus satisfying the necessary condition for the $S$-matrix unitarity.

For other elastic $2 \rightarrow 2$ amplitudes, Eqs. (17) through (26) show us that the similar inequality is satisfied due to the same cancelation between $H^{(2,b)}$ and $I^{(2,b)}$ ($b = o, e$) appearing in the sum of $\sigma$ and $\tau$, namely $\Phi$ in the intermediate sum of Eq. (1). Hence $S$-matrix unitarity holds in all matter-graviton scattering.

5. Discussion We suggested in Ref. [11] that $S$-matrix unitarity can be a guideline to renormalizability, and we have shown in this letter that it is true in gravitational theory. If a kinetic term in the action is degenerate, such as that in gauge theory, because the power counting theorem does not work, the proof of renormalizability is hard. Renormalizability of the quadratic gravity was proved by Stelle [10] in BRST method, where additional degrees of freedom, BRST ghosts and gauge modes, to on-shell physical states need to be introduced. On the other hand, the discussion of $S$-matrix unitarity done here is at the tree level, and hence it does not involve unphysical modes. Hence, the discussion of renormalizability often involves unphysical degrees of freedom, but that of $S$-matrix unitarity does not. The result is an evidence of our conjecture that the conditions for $S$-matrix unitarity and renormalizability are identical. Although our conjecture is not proved completely yet, it is useful to study renormalizability of various theories; the on-shell amplitude at tree level would show renormalizability in full order.
Two-graviton scattering in the quadratic gravity is expected to satisfy $S$-matrix unitarity too. We leave the project to demonstrate it in future work.

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