Research Article

Lyapunov-Type Inequalities for Second-Order Boundary Value Problems with a Parameter

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In this paper, we will establish some new Lyapunov-type inequalities for a class of second-order boundary value problems with a parameter. The inequalities generalize some early results in the literature.

1. Introduction

Up until now, integral inequalities have attracted the attention of many researchers, due to its wide applications in the research of qualitative and quantitative properties such as global existence, boundedness, and stability of differential and integral equations (see [1–26] and the references therein). Among these inequalities, one important kind is the Lyapunov-type inequality, which was originally presented by Lyapunov in [27] as follows.

If \( u(t) \) is a solution of

\[
\frac{d}{dt} \left[ u(t) \right] + q(t) u(t) = 0,
\]

satisfying \( u(a) = u(b) = 0 \) \((a < b)\) and \( u(t) \neq 0 \) for \( t \in (a, b) \), then

\[
\int_a^b |q(t)| dt > \frac{4}{b-a},
\]

and afterwards by Wintner [28] as

\[
\int_a^b q^+(t) dt > \frac{4}{b-a},
\]

where \( q^+(t) := \max\{q(t), 0\} \).

Following Lyapunov’s landmark work, there have been plenty of references focused on the Lyapunov-type inequality and its generalizations which are widely used in various problems such as asymptotic theory, disconjugacy, and eigenvalue problems of differential equations and difference equations (see [29–41] and the references therein).

For example, in 2003, Yang [29] obtained the following result for the second-order half-linear equation:

\[
\begin{align*}
\left( r(t) |u'(t)|^{p-1} u'(t) \right)' + q(t)|u(t)|^{p-1} u(t) = 0, \\
u(a) = u(b) = 0, & u(t) \neq 0, t \in (a, b),
\end{align*}
\]

where \( q, r \in C([a, b], \mathbb{R}) \) such that \( r(t) > 0 \) for \( t \in [a, b] \), and \( p > 0 \).

Theorem 1 (see [29]). Assume boundary value problem (4) has a solution \( u(t) \); then, the following inequality holds:

\[
\int_a^b q_+(t) dt \geq \frac{2^{p+1}}{\left( \int_a^b r^{-1/p} (t) dt \right)^{p}},
\]

where \( q_+(t) := \max\{q(t), 0\} \).

In 2012, Tiryaki et al. [34] established an inequality for boundary value problem of the form

\[
\begin{align*}
\left( r(t) |u'(t)|^{\alpha-2} u'(t) \right)' + q(t)|u(t)|^{\alpha-2}, & u(t) = 0, \\
u(a) = u(b) = 0, & u(t) \neq 0, t \in (a, b),
\end{align*}
\]

where \( \alpha > 1 \) and \( \alpha_* = (\alpha/\alpha - 1) \). Their result is as follows.
Theorem 2 (see [34]). Assume boundary value problem (6) has a solution $u(t)$; then, the following inequality holds:

$$\int_{a}^{b} q_{+}(t) \, dt \geq 1,$$

(7)

where $h_{1}(t) = \int_{a}^{t} r^{-a}(s) \, ds$, $h_{2}(t) = \int_{a}^{b} r^{-a}(s) \, ds$, and $q_{+}(t) = \max\{q(t), 0\}$.

In 2015, Agarwal et al. [36] established a Lyapunov-type inequality for the second-order forced boundary value problem of the form:

$$\begin{cases}
\left(r(t)|u'(t)|^{\beta-1}u'(t)\right)' + q(t)|u(t)|^{\gamma-1} - u(t) = f(t), \\
u(a) = u(b) = 0, \quad u(t) \neq 0, \quad t \in (a, b),
\end{cases}$$

(8)

in the subhalf-linear ($0 < \gamma < \beta$) and the super-half-linear ($0 < \beta < \gamma < 2\beta$) cases, where $r(t)$ and $q(t)$ are integrable on $[a, b]$ with $r(t) > 0$ on $[a, b]$. Their result is as follows.

Theorem 3 (see [36]). Suppose that $a, b, a < b$, are consecutive zeros of a nontrivial solution of the first part of equation (8), then the inequality

$$2\Gamma_{\beta}^{\gamma} \int_{a}^{b} q_{+}(t) \, dt + \int_{a}^{b} |f(t)| \, dt > 2^{\beta+1} \sqrt{\Gamma_{\beta}^{\gamma}} \int_{a}^{b} r^{-\beta}(t) \, dt$$

(9)

holds, where $\gamma \in (0, 2\beta)$ and $\Gamma_{\beta}^{\gamma} = (2\beta - \gamma)\gamma^{\beta-2\beta}(\beta-\gamma) > 0$.

where $p, q, r, h \in C([a, b], \mathbb{R})$ such that $p(t) > 0$, $q(t) > 0$, $r(t) > 0$ for $t \in [a, b]$, $0 < a < \gamma < \beta$, and $\lambda \geq 0$ is a real parameter, and

$$\begin{align*}
(p(t)|u'|^{\alpha-2}u')' + \lambda(q(t)|u'|^{\beta-2}u')' + \lambda r(t)|u'|^{\gamma-2}u' + h(t)|u|^{\alpha-2}u &= 0, \quad t \in [a, b], \\
u(a) = u(b) = 0, \quad u(t) \neq 0, \quad t \in (a, b),
\end{align*}$$

(14)

with the boundary condition (13), where $p, q, r, h \in C([a, b], \mathbb{R})$ such that $p(t) > 0$, $q(t) > 0$, $r(t) > 0$ for $t \in [a, b]$, $1 < a < \gamma < \beta$, and $\lambda \geq 0$ is a real parameter. Our results extend and compliment the results of [29, 36].

2. Main Results

Lemma 1. If $u$ is differential on $[a, b]$ satisfying $u(a) = u(b) = 0$ and $u(t) \neq 0$ for $t \in (a, b)$, then

$$\sup_{a \leq t \leq b} |u(t)| \leq \frac{1}{2} \int_{a}^{b} |u'(t)| \, dt.$$  

(15)

Proof. Since $u$ is differential on $[a, b]$ satisfying $u(a) = u(b) = 0$, then we have

$$u(t) = \frac{1}{2} \int_{a}^{t} u'(s) \, ds - \frac{1}{2} \int_{t}^{b} u'(s) \, ds, \quad t \in [a, b].$$

(16)

So,

$$|u(t)| \leq \frac{1}{2} \int_{a}^{t} |u'(s)| \, ds + \frac{1}{2} \int_{t}^{b} |u'(s)| \, ds = \frac{1}{2} \int_{a}^{b} |u'(s)| \, ds,$

(17)

t \in [a, b].

Therefore, (15) holds.

Lemma 2 (see [13]). Let $A > 0$, $B > 0$, and $1 < a < \gamma < \beta$ be given. Then, for each $x \geq 0$,

$$Ax^\gamma - Bx^\beta \leq \frac{A(\beta - \gamma)}{\beta - \alpha} \left(\frac{B - a}{\beta - \alpha}\right)^{\gamma-\alpha}(\gamma-\beta)x^\alpha,$$

(18)

holds.
Theorem 5. Assume $u$ is a solution of equation (12) satisfying the boundary conditions (13). Then,

$$
\frac{\lambda}{2^\alpha} \left( \frac{\beta - \alpha}{\beta - \alpha} \right)^{\frac{(\gamma - \alpha)}{(\beta - \alpha)}} \frac{\beta - \gamma}{\beta - \alpha} \int_a^b [r^{\beta - \alpha} (t) q^\alpha (t)]^{1/(\beta - \gamma)} dt
+ \frac{1}{2^\alpha} \int_a^b h^+ (t) dt \geq \left( \int_a^b p^{1/(1-a)} (t) dt \right)^{1-a},
$$

where $h^+ (t) := \max \{h(t), 0\}$.

Proof. Multiplying (12) by $u(t)$ and integrating over $[a, b]$, yields:

$$
\int_a^b \left( \frac{p(t)[u'(t)]^{\alpha-2}}{u(t)} + q(t)u(t) \right) dt - \lambda \int_a^b r(t)[u(t)]^\beta dt
+ \lambda \int_a^b r(t)[u(t)]^\beta dt + \int_a^b h(t)[u(t)]^\alpha dt = 0.
$$

Using integration by parts to the first integral on the left-hand side of (20) and from (13), we have

$$
- \int_a^b \left( p(t)[u'(t)]^{\alpha-2} \frac{u'(t)}{u(t)} \right) dt - \lambda \int_a^b q(t)[u(t)]^{\alpha-2} dt + \int_a^b r(t)[u(t)]^\beta dt
+ \int_a^b h(t)[u(t)]^\alpha dt = 0.
$$

Then, we obtain

$$
\int_a^b \left( \frac{p(t)[u'(t)]^{\alpha-2}}{u(t)} + q(t)u(t) \right) dt - \lambda \int_a^b r(t)[u(t)]^\beta dt
+ \int_a^b h^+ (t) dt \geq 0,
$$

i.e.,

$$
\int_a^b \left( p(t)[u'(t)]^{\alpha-2} \frac{u'(t)}{u(t)} \right) dt \leq \lambda \int_a^b r(t)[u(t)]^\beta dt
+ \int_a^b h^+ (t) dt.
$$

By using Hölder’s inequality with indices $(1/\tau) + (1/\rho) = 1$,

$$
\int_a^b f(t)g(t)dt \leq \left( \int_a^b f(t)^\tau dt \right)^{1/\tau} \left( \int_a^b g(t)^\rho dt \right)^{1/\rho},
$$

with $f(t) = p^{1/\alpha} (t)[u'(t)]^\alpha$, $g(t) = p^{-1/\alpha} (t)$, $\tau = \alpha$, and $\rho = \alpha/\alpha - 1$, we obtain that

$$
\int_a^b [u'(t)]^\alpha dt \leq \left( \int_a^b p(t)[u'(t)]^\beta dt \right)^{1/\alpha} \left( \int_a^b p^{1/(1-a)} (t) dt \right)^{(\alpha-1)/a},
$$

i.e.,

$$
\frac{\left( \int_a^b [u'(t)]^\alpha dt \right)^a}{\left( \int_a^b p^{1/(1-a)} (t) dt \right)^{a-1}} \leq \int_a^b p(t)[u'(t)]^\alpha dt.
$$

On the contrary, from Lemma 1, we obtain

$$
\int_a^b h^+ (t) dt \leq \left( \sup_{u \in \mathbb{E}} |u(t)| \right)^\alpha \int_a^b h^+ (t) dt
$$

Then, from (23), (26), and (27), we obtain

$$
\int_a^b h^+ (t) dt \leq \left( \sup_{u \in \mathbb{E}} |u(t)| \right)^\alpha \int_a^b h^+ (t) dt.
$$

\[ \text{(28)} \]
For the first integral on the right-hand side of (28), inequality (18) in Lemma 2 with \( A = r(t), B = q(t), \) and \( x = |u(t)| \geq 0 \) for \( t \in [a,b] \) implies that
\[
\frac{\int_{a}^{b} |u'(t)| \, dt}{\int_{a}^{b} p^{1/(1-a)}(t) \, dt} \geq \lambda \int_{a}^{b} \left( \frac{r^{\beta-a}(t)q^{-\gamma}(t)}{\beta - \alpha} \right)^{1/(\beta-\gamma)} \left( \frac{\gamma - \alpha}{\beta - \alpha} \right)^{\frac{\beta - \gamma}{\beta - \alpha}} |u(t)|^a. \tag{29}
\]
From Lemma 1, we have
\[
\left( \sup_{t \in [a,b]} |u(t)| \right)^a \leq \frac{1}{2^n} \left( \int_{a}^{b} |u'(t)| \, dt \right)^a. \tag{31}
\]
By (28) and (29), we obtain
\[
\left( \int_{a}^{b} |u'(t)| \, dt \right)^a \left( \int_{a}^{b} p^{1/(1-a)}(t) \, dt \right)^{a-1} \geq \frac{\lambda}{2^n} \int_{a}^{b} \left( \frac{r^{\beta-a}(t)q^{-\gamma}(t)}{\beta - \alpha} \right)^{1/(\beta-\gamma)} \left( \frac{\gamma - \alpha}{\beta - \alpha} \right)^{\frac{\beta - \gamma}{\beta - \alpha}} |u(t)|^a \left( \int_{a}^{b} |u'(t)| \, dt \right)^a. \tag{32}
\]
In view of (30) and (31), we obtain that
\[
(p(t)u') - q(t)|u|^{\beta-2}u + r(t)|u|^{-\gamma}u + h(t)u = 0, \quad t \in [a,b], \tag{34}
\]
satisfying boundary condition (13). Then,
\[
\frac{1}{4} \left( \frac{\gamma - 2}{\beta - 2} \right)^{\frac{\beta - \gamma}{\beta - 2}} \int_{a}^{b} \left[ r^{\beta-2}(t)q^{-\gamma}(t) \right]^{1/(\beta-\gamma)} dt + \frac{1}{2} \int_{a}^{b} h^+(t) \, dt \geq \left( \int_{a}^{b} p^{-1}(t) \, dt \right)^{-1}, \tag{35}
\]
which also leads to (19). The proof is complete.

If we take \( \alpha = 2 \) and \( \lambda = 1 \) in inequality (19), we obtain the following result.

**Corollary 1.** Assume \( u \) is a solution of equation

\[
\qquad
\text{By (28) and (29), we obtain}
\]

where \( h^+(t) := \max(h(t), 0) \).

**Theorem 6.** Assume \( u \) is a solution of equation (14) satisfying boundary condition (13). Then,
\[
\lambda \left( \frac{1}{2} \right)^{(\beta - \alpha)/(\beta - \gamma)} \left( \frac{\gamma - \alpha}{\beta - \alpha} \right)^{(\gamma - \alpha)/(\beta - \gamma)} \frac{\beta - \gamma}{\beta - \alpha} \\
\cdot \left( \int_a^b q^{\gamma/(1-\beta)} (t) \, dt \right)^{(1-\beta)/(\gamma-\alpha)} \left( \int_a^b r(t) \, dt \right)^{(\beta - \alpha)/(\beta - \gamma)} + \frac{1}{2^\alpha} \int_a^b h^+(t) \, dt
\]  

(36)

where \( h^+(t) := \max[h(t), 0] \).

Proof. Multiplying (14) by \( u(t) \) and integrating over \([a, b]\) yields

\[
\int_a^b p(t) |u'(t)|^{\alpha - 2} u'(t) \, u(t) \, dt + \nu \int_a^b q(t) |u'(t)|^{\beta - 2} u'(t) \, u(t) \, dt + \nu \int_a^b r(t) |u(t)|^\rho \, dt + \int_a^b h(t) |u(t)|^\alpha \, dt = 0.
\]  

(37)

Using integration by parts to the first and second integrals on the left-hand side of (37) and from (13), we have

\[
-\int_a^b p(t) |u'(t)|^{\alpha} \, dt - \nu \int_a^b q(t) |u'(t)|^{\beta} \, dt + \nu \int_a^b r(t) |u(t)|^\rho \, dt + \int_a^b h(t) |u(t)|^\alpha \, dt = 0,
\]  

(38)

i.e.,

\[
\int_a^b p(t) |u'(t)|^{\alpha} \, dt
\]

\[
= \nu \left[ \int_a^b r(t) |u(t)|^\rho \, dt - \int_a^b q(t) |u'(t)|^{\beta} \, dt \right] + \int_a^b h(t) |u(t)|^\alpha \, dt
\]

(39)

By using Hölder’s inequality (24) with \( f(t) = q^{1/\beta} (t)|u'(t)| \), \( g(t) = q^{1/(\beta - 1)}(t) \), \( \tau = \beta \), and \( \rho = (\beta/\beta - 1) \), we obtain that

\[
\int_a^b |u'(t)|^{\beta} \, dt \leq \left( \int_a^b q(t) |u'(t)|^{\beta} \, dt \right)^{1/\beta} \left( \int_a^b q^{(\beta - 1)/(\beta - \beta)}(t) \, dt \right)^{(\beta - 1)/(\beta - \beta)}.
\]  

(40)

From (26), (39), (41), and Lemma 1, we obtain
\[
\frac{\left( \int_a^b |u'(t)| dt \right)^a}{\left( \int_a^b p^{1/(1-a)}(t) dt \right)^{a-1}} \leq \int_a^b p(t)|u'(t)|^a dt
\]

From (27), (42), and (43), we have
\[
\lambda \left( \frac{1}{2} \right)^{\gamma/(\beta-a)/(\gamma-\beta)} \left( \frac{\gamma-a}{\beta-a} \right)^{(\gamma-a)/(\beta-\gamma)} \frac{\beta-y}{\beta-a} \\
\left( \int_a^b q^{1/(1-\beta)}(t)dt \right)^{(1-\beta)/(\gamma-\alpha)/(\gamma-\beta)} \left( \int_a^b r(t)dt \right)^{(\beta-a)/(\beta-\gamma)} \\
+ \frac{1}{2^a} \int_a^b h^+(t)dt \geq \left( \int_a^b p^{1/(1-\alpha)}(t)dt \right)^{1-\alpha} \\
\]

(45)

which also leads to (36). The proof is complete.

\[\square\]

**Remark 1.** We note that when \( \lambda = 0, \alpha = p + 1, \) and (5) can be obtained from Theorems 5 and 6, respectively.

If we take \( \alpha = 2 \) and \( \lambda = 1 \) in inequality (36), we obtain the following result.

**Corollary 2.** Assume \( u \) is a solution of equation

\[ (p(t)u')' + (q(t)|u|^{p-2}u) + r(t)|u|^\gamma u + h(t)|u|^\beta u = 0, \quad t \in [a,b], \]

satisfying boundary condition (13). Then,

\[ (1) \quad y/(\beta-2)/(\gamma-\beta) \frac{\beta-y}{\beta-2} \]

\[ \left( \int_a^b q^{1/(1-\beta)}(t)dt \right)^{(1-\beta)/(\gamma-\beta)} \\
\left( \int_a^b r(t)dt \right)^{(\beta-2)/(\gamma-\beta)} + \frac{1}{2^a} \int_a^b h^+(t)dt \geq \left( \int_a^b p^{-1}(t)dt \right)^{-1}, \]

(47)

where \( h^+(t) := \max[h(t), 0] \).

**Theorem 7.** Assume \( u \) is a solution of equation (14) satisfying boundary condition (13). Then,

\[ \lambda \left( \frac{1}{2} \right)^{\gamma/(\beta-a)/(\gamma-\beta)} \left( \frac{b-a}{\beta-a} \right)^{\gamma-a} \left( \frac{\gamma-a}{\beta-\gamma} \right) \beta - y \]

\[ \left( \frac{\beta-a}{\beta-\gamma} \right) \frac{\beta-y}{\beta-a} \\
\left( \int_a^b r(t)dt \right)^{(\beta-a)/(\beta-\gamma)} + \frac{1}{2^a} \int_a^b h^+(t)dt \geq p(b-a)^{1-\alpha}, \]

(48)

where \( p := \min[p(t): t \in [a,b]], \quad q := \min[q(t): t \in [a,b]], \)

and \( h^+(t) := \max[h(t), 0] \).

**Proof.** From the proof of Theorem 6, we have (39) holds. By (39) and Lemma 1, we obtain

\[
p \int_a^b |u'(t)|^\alpha dt = \min_{a \leq t \leq b} p(t) \cdot \int_a^b |u'(t)|^\alpha dt \\
\leq \int_a^b p(t)|u'(t)|^\alpha dt \\
= \lambda \left[ \int_a^b r(t)|u(t)|^\gamma dt - \int_a^b q(t)|u'(t)|^\beta dt \right] + \int_a^b h^+(t)|u(t)|^\alpha dt \\
\leq \lambda \left[ \int_a^b r(t)|u(t)|^\gamma dt - \min_{a \leq t \leq b} q(t) \int_a^b |u'(t)|^\beta dt \right] + \int_a^b h^+(t)|u(t)|^\alpha dt \\
\leq \lambda \left[ \left( \sup_{a \leq t \leq b} |u(t)| \right)^\gamma \int_a^b r(t)dt - q \int_a^b |u'(t)|^\beta dt \right] + \int_a^b h^+(t)|u(t)|^\alpha dt \\
\leq \lambda \left[ \left( \frac{1}{2} \right)^\gamma \int_a^b |u'(t)|dt \right]^{\gamma/(\beta-2)/(\gamma-\beta)} \left( \frac{\beta-a}{\beta-\gamma} \right) \beta - y \]

\[ + \left( \frac{\beta-a}{\beta-\gamma} \right) \frac{\beta-y}{\beta-a} \left( \frac{\beta-a}{\beta-\gamma} \right) \beta - y \\
\left( \int_a^b r(t)dt \right)^{(\beta-a)/(\beta-\gamma)} + \frac{1}{2^a} \int_a^b h^+(t)dt \geq \left( \int_a^b p^{-1}(t)dt \right)^{-1}, \]

(49)

where \( p := \min[p(t): t \in [a,b]], \quad q := \min[q(t): t \in [a,b]], \)

and \( h^+(t) := \max[h(t), 0] \).
By using Hölder’s inequality (24) with \( f(t) = |u'(t)|, g(t) = 1, \tau = \alpha, \beta \) and \( \rho = (\alpha/\alpha - 1), (\beta/\beta - 1) \), respectively, we obtain that

\[
\int_a^b |u'(t)|dt \leq \left( \int_a^b |u'(t)|^\alpha dt \right)^{1/\alpha} (b-a)^{(\alpha-1)/\alpha},
\]
\[
\int_a^b |u'(t)|dt \leq \left( \int_a^b |u'(t)|^\beta dt \right)^{1/\beta} (b-a)^{(\beta-1)/\beta}.
\]

(50)

Therefore,

\[
(b-a)^{1-\alpha} \left( \int_a^b |u'(t)|dt \right)^\alpha = \int_a^b |u'(t)|^\alpha dt,
\]
\[
(b-a)^{1-\beta} \left( \int_a^b |u'(t)|dt \right)^\beta \leq \int_a^b |u'(t)|^\beta dt.
\]

(51) (52)

From (49), (51), and (52), we have

\[
p(b-a)^{1-\alpha} \left( \int_a^b |u'(t)|dt \right)^\alpha \\
\leq \lambda \left( \frac{1}{2} \right)^{\gamma(\beta-a)/\beta-\gamma} \left( \int_a^b r(t)dt \right)^{\beta-a}/\beta-\gamma} \left( \frac{b-a}{q(\beta-\alpha)} \right)^{(\gamma-a)/\beta-\gamma} \right)^{\beta-\gamma-(\beta-a)/\beta-\gamma} \right)
\]
\[
\frac{(b-a)^{\beta-1}}{q(\beta-\alpha)} \left( \frac{b-a}{q(\beta-\alpha)} \right)^{(\gamma-a)/\beta-\gamma} \right)^{\beta-\gamma-(\beta-a)/\beta-\gamma} \right)
\]
\[
\int_a^b h^+(t)dt + \frac{1}{2\beta-a} \int_a^b \left( \int_a^b |u'(t)|dt \right)^\alpha.
\]

(53) (54)

For the right-hand side of (53), inequality (18) in Lemma 2 with \( A = (1/2)^{\gamma(\beta-a)/\beta-\gamma} \int_a^b r(t)dt, B = q(b-a)^{1-\beta} \), and \( x = \int_a^b |u'(t)|dt > 0 \) implies that

\[
\int_a^b |u'(t)|dt = 1_\alpha \left( \int_a^b |u'(t)|dt \right)^a.
\]

From (53) and (54), we have

\[
p(b-a)^{1-\alpha} \left( \int_a^b |u'(t)|dt \right)^\alpha \\
\leq \lambda \left( \frac{1}{2} \right)^{\gamma(\beta-a)/\beta-\gamma} \left( \int_a^b r(t)dt \right)^{\beta-a}/\beta-\gamma} \left( \frac{b-a}{q(\beta-\alpha)} \right)^{(\gamma-a)/\beta-\gamma} \right)^{\beta-\gamma-(\beta-a)/\beta-\gamma} \right)
\]
\[
\frac{(b-a)^{\beta-1}}{q(\beta-\alpha)} \left( \frac{b-a}{q(\beta-\alpha)} \right)^{(\gamma-a)/\beta-\gamma} \right)^{\beta-\gamma-(\beta-a)/\beta-\gamma} \right)
\]
\[
\int_a^b h^+(t)dt + \frac{1}{2\beta-a} \int_a^b \left( \int_a^b |u'(t)|dt \right)^\alpha.
\]

(55)

Thus, dividing both sides of (55) by \( \left( \int_a^b |u'(t)|dt \right)^a \), we obtain

\[
\lambda \left( \frac{1}{2} \right)^{\gamma(\beta-a)/\beta-\gamma} \left( \int_a^b r(t)dt \right)^{\beta-a}/\beta-\gamma} \left( \frac{b-a}{q(\beta-\alpha)} \right)^{(\gamma-a)/\beta-\gamma} \right)^{\beta-\gamma-(\beta-a)/\beta-\gamma} \right)
\]
\[
\frac{(b-a)^{\beta-1}}{q(\beta-\alpha)} \left( \frac{b-a}{q(\beta-\alpha)} \right)^{(\gamma-a)/\beta-\gamma} \right)^{\beta-\gamma-(\beta-a)/\beta-\gamma} \right)
\]
\[
\int_a^b h^+(t)dt + \frac{1}{2\beta-a} \int_a^b \left( \int_a^b |u'(t)|dt \right)^\alpha.
\]

(56)

which also leads to (48). The proof is complete.

\[\square\]

Remark 2. We note that when \( \lambda = 0, \alpha = 2, \) and \( p(t) \equiv 1, \) classical result (3) can be obtained from Theorems 5–7, respectively.
\[
\left( \frac{1}{2} \right)^{\left( \beta-2 \right)\left( \beta-\gamma \right)} \left( \frac{(b-a)^{\beta-1}(\gamma-2)}{q(\beta-2)} \right) \left( \frac{\gamma-2}{\beta-\gamma} \right) \frac{\beta-\gamma}{\beta-2} \cdot \left( \int_{a}^{b} r(t) \, dt \right)^{\left( \beta-2 \right)\left( \beta-\gamma \right)} + \frac{1}{4} \int_{a}^{b} h(t) \, dt \geq \frac{p}{b-a},
\]

where \( p := \min\{p(t) : t \in [a, b]\}, \ q := \min\{q(t) : t \in [a, b]\}, \) and \( h^{+}(t) := \max\{h(t), 0\} \).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this article.

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**References**

[1] H. D. Liu, “On some nonlinear retarded Volterra-Fredholm type integral inequalities on time scales and their applications,” *Journal of Inequalities and Applications*, vol. 2018, p. 211, 2018.

[2] D. L. Zhao, S. L. Yuan, and H. D. Liu, “Stochastic Dynamics of the delayed chemostat with Lévy noises,” *International Journal of Biomathematics*, vol. 12, no. 5, Article ID 1950056, 2019.

[3] H. Liu, F. Meng, and P. Liu, “Oscillation and asymptotic analysis on a new generalized Emden-Fowler equation,” *Applied Mathematics and Computation*, vol. 219, no. 5, pp. 2739–2748, 2012.

[4] H. D. Liu and C. Q. Ma, “Oscillation criteria for second-order neutral delay dynamic equations with nonlinearities given by Riemann-Stieltjes integrals,” *Abstract and Applied Analysis*, vol. 2013, Article ID 530457, 9 pages, 2013.

[5] E. Tunç and H. D. Liu, “Oscillatory behavior for second-order damped differential equation with nonlinearities including Riemann-Stieltjes integrals,” *Electronic Journal of Differential Equations*, vol. 2018, no. 54, 2018.

[6] H. D. Liu and F. W. Meng, “Existence of positive periodic solutions for a predator-prey system of Holling type IV function response with mutual interference and impulsive effects,” *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 138984, 12 pages, 2015.

[7] D. Zhao and H. Liu, “Coexistence in a two species chemostat model with Markov switchings,” *Applied Mathematics Letters*, vol. 94, pp. 266–271, 2019.

[8] H. D. Liu and F. W. Meng, “Some new generalized Volterra-Fredholm type discrete fractional sum inequalities and their applications,” *Journal of Inequalities and Applications*, vol. 2016, no. 1, p. 213, 2016.

[9] H. D. Liu and C. C. Yin, “Some generalized Volterra-Fredholm type dynamical integral inequalities in two independent variables on time scale pairs,” *Advances in Difference Equations*, vol. 2020, no. 1, p. 31, 2020.

[10] J. Gu and F. Meng, “Some new nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales,” *Applied Mathematics and Computation*, vol. 245, pp. 235–242, 2014.

[11] H. D. Liu and F. W. Meng, “Nonlinear retarded integral inequalities on time scales and their applications,” *Journal of Mathematical Inequalities*, vol. 12, no. 1, pp. 219–234, 2018.

[12] B. Zhang, J. S. Zhuang, H. D. Liu, J. D. Cao, and Y. H. Xia, “Master-slave synchronization of a class of fractional-order Takagi-Sugeno fuzzy neural networks,” *Advances in Difference Equations*, vol. 2018, p. 473, 2018.

[13] H. D. Liu, C. Y. Li, and F. C. Shen, “A class of new nonlinear dynamic integral inequalities containing integration on infinite interval on time scales,” *Advances in Difference Equations*, vol. 2019, p. 311, 2019.

[14] H. D. Liu and C. Q. Ma, “Oscillation criteria of even order delay dynamic equations with nonlinearities given by Riemann-Stieltjes integrals,” *Abstract and Applied Analysis*, vol. 2014, Article ID 395381, 8 pages, 2014.

[15] D. Zhao, “Study on the threshold of a stochastic SIR epidemic model and its extensions,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 38, pp. 172–177, 2016.

[16] H. D. Liu, “A class of retarded Volterra-Fredholm type integral inequalities on time scales and their applications,” *Journal of Inequalities and Applications*, vol. 2017, p. 293, 2017.

[17] D. L. Zhao, S. L. Yuan, and H. D. Liu, “Random periodic solution for a stochastic SIS epidemic model with constant population size,” *Advances in Difference Equations*, vol. 2018, Article ID 64, 2018.

[18] H. D. Liu and F. W. Meng, “Interval oscillation criteria for second-order nonlinear forced differential equations involving variable exponent,” *Advances in Difference Equations*, vol. 2016, Article ID 291, 2016.

[19] D. Chen, K. I. Kou, and Y. H. Xia, “Linear quaternion-valued dynamic equations on time scales,” *Journal of Applied Analysis and Computation*, vol. 8, pp. 172–201, 2018.

[20] H. D. Liu, “Some new integral inequalities with mixed nonlinearities for discontinuous functions,” *Advances in Difference Equations*, vol. 2018, Article ID 22, 2018.

[21] F. Meng and J. Shao, “Some new Volterra-Fredholm type dynamic integral inequalities on time scales,” *Applied Mathematics and Computation*, vol. 223, pp. 444–451, 2013.

[22] H. D. Liu, “Some new half-linear integral inequalities on time scales and applications,” *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 395381, 8 pages, 2014.

[23] Y. Xia, L. Chen, K. I. Kou, and D. O’Regan, “Holder regularity of Grobman-Hartman Theorem for dynamic equations on measure chains,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 41, no. 3, pp. 1153–1180, 2018.

[24] H. D. Liu and F. W. Meng, “Some new nonlinear integral inequalities with weakly singular kernel and their applications to FDEs,” *Journal of Inequalities and Applications*, vol. 2015, p. 209, 2015.

[25] H. D. Liu and P. C. Liu, “Oscillation criteria for some new generalized Emden-Fowler dynamic equations on time scales,” *Abstract and Applied Analysis*, vol. 2013, Article ID 962590, 16 pages, 2013.

[26] H. D. Liu, “Half-linear Volterra-Fredholm type integral inequalities on time scales and their applications,” *Journal of Applied Analysis and Computation*, vol. 10, no. 1, pp. 234–248, 2020.
[27] A. M. Lyapunov, “Probleme général de la stabilité du mouvement (French Translation of a Russian paper dated 1893),” *Annales de la Faculté des Sciences de Toulouse*, vol. 2, pp. 27–247, 1907.

[28] A. Wintner, “On the non-existence of conjugate points,” *American Journal of Mathematics*, vol. 73, no. 2, pp. 368–380, 1951.

[29] X. J. Yang, “On inequalities of lyapunov type,” *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 293–300, 2003.

[30] J. P. Pinasco, “Lower bounds for eigenvalues of the one-dimensional $p$-Laplacian,” *Abstract and Applied Analysis*, vol. 2004, no. 2, pp. 147–153, 2004.

[31] C.-F. Lee, C.-C. Yeh, C.-H. Hong, and R. P. Agarwal, “Lyapunov and wirtinger inequalities,” *Applied Mathematics Letters*, vol. 17, no. 7, pp. 847–853, 2004.

[32] R. A. Mashiyev, G. Alisoy, and S. Ogras, “Lyapunov, Opial and Beesack inequalities for one-dimensional $p(t)$-Laplacian equations,” *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3459–3467, 2010.

[33] Y. Wang, “Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions,” *Applied Mathematics Letters*, vol. 25, no. 12, pp. 2375–2380, 2012.

[34] A. Tiryaki, D. Cakmak, and M. F. Aktas, “Lyapunov-type inequalities for a certain class of nonlinear systems,” *Computers & Mathematics with Applications*, vol. 64, no. 1, pp. 1804–1811, 2012.

[35] H. D. Liu, “Lyapunov-type inequalities for certain higher-order half-linear differential equations,” *Journal of Mathematical Inequalities*, vol. 13, no. 4, pp. 1159–1170, 2019.

[36] R. P. Agarwal and A. Özbecker, “Disconjugacy via Lyapunov and Vallée-Poussin type inequalities for forced differential equations,” *Applied Mathematics and Computation*, vol. 265, pp. 456–468, 2015.

[37] H. D. Liu, “Lyapunov-type inequalities for certain higher-order difference equations with mixed non-linearities,” *Advances in Difference Equations*, vol. 2018, Article ID 229, 2018.

[38] X.-H. Tang and M. Zhang, “Lyapunov inequalities and stability for linear Hamiltonian systems,” *Journal of Differential Equations*, vol. 252, no. 1, pp. 358–381, 2012.

[39] Q. M. Zhang and X. H. Tang, “Lyapunov-type inequalities for the Quasilinear difference systems,” *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 860598, 16 pages, 2012.

[40] H. D. Liu, “An improvement of the Lyapunov inequality for certain higher order differential equations,” *Journal of Inequalities and Applications*, vol. 2018, Article ID 215, 2018.

[41] M. Jleli and B. Samet, “On Lyapunov-type inequalities for $(p,q)$-laplacian systems,” *Journal of Inequalities and Applications*, vol. 2017, Article ID 100, 2017.