REPRESENTATION FORMULA AND BI-LIPSCHITZ CONTINUITY OF SOLUTIONS TO INHOMOGENEOUS BIHARMONIC DIRICHLET PROBLEMS IN THE UNIT DISK

PEIJIN LI AND SAMINATHAN PONNUSAMY

Abstract. The aim of this paper is twofold. First, we establish the representation formula and the uniqueness of the solutions to a class of inhomogeneous biharmonic Dirichlet problems, and then prove the bi-Lipschitz continuity of the solutions.

1. Introduction and statement of the main results

Let $\mathbb{C}$ denote the complex plane. For $a \in \mathbb{C}$, let $D(a, r) = \{ z : |z - a| < r \}$, where $r > 0$, and $\mathbb{D}_r = \mathbb{D}(0, r)$. In particular, let $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial \mathbb{D}$, the boundary of $\mathbb{D}$. For domains $D$ and $\Omega$ be domains in $\mathbb{C}$, a function $p : D \to \Omega$ is said to be $L_1$-Lipschitz (resp. $L_2$-co-Lipschitz) if for all $z, w \in D$,

$$|p(z) - p(w)| \leq L_1 |z - w| \quad \text{(resp. } |p(z) - p(w)| \geq L_2 |z - w|)$$

for some positive constants $L_1$ and $L_2$. We say that $p$ is bi-Lipschitz if it is both Lipschitz and co-Lipschitz.

The main aim of this paper is to discuss the representation formula, the uniqueness and the bi-Lipschitz continuity of the solutions to the following inhomogeneous biharmonic Dirichlet problem (briefly, IBDP in the following):

$$\begin{cases}
\Delta^2 \Phi = g & \text{in } \mathbb{D}, \\
\Phi = f & \text{on } \mathbb{T}, \\
\partial_n \Phi = h & \text{on } \mathbb{T},
\end{cases}$$

(1.1)

where $\Delta$ denotes the Laplacian given by

$$\Delta = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

$\partial_n$ denotes the differentiation in the inward normal direction and the boundary data $f$ and $h \in \mathcal{D}'(\mathbb{T})$, the space of distributions in $\mathbb{T}$.

Note that a solution to the biharmonic equation $\Delta^2 \Phi = 0$ is called a biharmonic function. See Almansi [6], Vekua [30] and [1, 2, 3, 13, 14] for properties of biharmonic functions.

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The present article is motivated from the related studies in [7, 18, 23, 26, 28]. In [26], Olofsson considered the representation formula of the solutions to the following homogeneous biharmonic Dirichlet problem (briefly, HBDP in the following):

\[
\begin{aligned}
\Delta^2 u &= 0 \quad \text{in } \mathbb{D}, \\
u &= f \quad \text{on } \mathbb{T}, \\
\partial_n u &= h \quad \text{on } \mathbb{T}.
\end{aligned}
\]

In particular, Olofsson proved the following (see [26, Theorems 3.1 and 3.2]).

**Theorem A.** (i) Suppose \(u\) satisfies the growth conditions:

\[
|u(z)| \leq C(1 - |z|)^{-N} \quad \text{and} \quad |\Delta u(z)| \leq C(1 - |z|)^{-N},
\]

where \(C\) and \(N\) are positive constants. If \(u\) satisfies HBDP (1.2), then \(u\) admits the representation

\[
u(z) = F_0[f](z) + H_0[h](z),
\]

where

\[
F_0[f](z) = \frac{1}{2\pi} \int_0^{2\pi} F_0(ze^{-i\theta}) f(e^{i\theta}) d\theta \quad \text{and} \quad H_0[h](z) = \frac{1}{2\pi} \int_0^{2\pi} H_0(ze^{-i\theta}) h(e^{i\theta}) d\theta.
\]

Here the kernels \(H_0(z)\) and \(F_0(z)\) are given by

\[
H_0(z) = \frac{1}{2} \frac{(1 - |z|^2)^2}{|1 - z|^2} \quad \text{and} \quad F_0(z) = H_0(z) + \frac{1}{2} \frac{(1 - |z|^2)^3}{|1 - z|^4}.
\]

(ii) If \(u\) is defined by (1.3), then \(u\) satisfies HBDP (1.2).

In fact, the function \(F_0\) is a certain biharmonic Poisson kernel introduced by Abkar and Hedenmalm [5]. Moreover, in [9, 10] the authors solved a certain Dirichlet boundary value problem for the polyharmonic equation in \(\mathbb{D}\). In [28], Pavlović proved that the quasiconformality of harmonic homeomorphisms between \(\mathbb{D}\) can be characterized in terms of their bi-Lipschitz continuity ([28, Theorem 1.2]). In [7], Arsenović et al. showed that the Lipschitz continuity of \(\phi: \mathbb{S}^{n-1} \to \mathbb{R}^n\) implies the Lipschitz continuity of its harmonic extension \(P[\phi]: \mathbb{B}^n \to \mathbb{R}^n\) provided that \(P[\phi]\) is a \(K\)-quasiregular mapping ([7, Theorem 1]), where \(\mathbb{B}^n\) (resp. \(\mathbb{S}^{n-1}\)) denotes the unit ball (resp. the boundary of \(\mathbb{B}^n\)) in \(\mathbb{R}^n\) and \(P\) stands for the usual Poisson kernel with respect to \(\Delta\). The assumption “\(P[\phi]\) being \(K\)-quasiregular” in [7, Theorem 1] is necessary as [7, Example 1] demonstrates. Meanwhile, by assuming that \(P[\phi]: \mathbb{B}^n \to \mathbb{B}^n\) is a \(K\)-quasiconformal harmonic mapping with \(P[\phi](0) = 0\) and \(\phi \in C^{1,\alpha}\), Kalaj [18, Theorem 2.1] also proved the Lipschitz continuity of \(P[\phi]\).

In particular, in [23], Kalaj and Pavlović discussed the bi-Lipschitz continuity of quasiconformal self-mappings in \(\mathbb{D}\) satisfying Poisson’s equation \(\Delta u = \psi\) ([23, Theorem 1.2]). See [5, 7, 11, 12, 17, 18, 19, 20, 21, 22, 24, 25, 28, 29] and the references therein for detailed discussions on this topic.
In order to state our results, we need the representation formula of the biharmonic Green function in $\mathbb{D}$. The biharmonic Green function $G$ is the solution to the boundary value problem:

$$
\begin{cases}
\Delta^2 z(G(z, \zeta)) = \delta_\zeta(z) & \text{for } z \in \mathbb{D}, \\
G(z, \zeta) = 0 & \text{for } z \in \mathbb{T}, \\
\frac{\partial G}{\partial n_z}(z, \zeta) = 0 & \text{for } z \in \mathbb{T}
\end{cases}
$$

(1.5)

for each $\zeta \in \mathbb{D}$, where $\delta_\zeta(z)$ denotes the Dirac distribution concentrated at the point $\zeta \in \mathbb{D}$ and $\partial/\partial n_z$ stands for the inward normal derivative with respect to the variable $z \in \mathbb{D}$. In $\mathbb{D}$, the biharmonic Green function $G$ is given by (cf. [6, 9, 10]):

$$
G(z, \zeta) = \frac{|z - \zeta|^2 \log |\zeta - \bar{z}|}{|z - \bar{\zeta}|^2} - (1 - |z|^2)(1 - |\zeta|^2).
$$

(1.6)

For convenience, we let

\[
G[g](z) = \int_{\mathbb{D}} G(z, \zeta)g(\zeta) \, dA(\zeta),
\]

where $dA(\zeta) = (1/\pi) \, dx \, dy$ denotes the normalized area measure in $\mathbb{D}$.

Our first objective of this paper is to establish a representation formula and uniqueness of the solutions to IBDP (1.1), which is as follows.

**Theorem 1.1.** (1) Suppose that $u$ satisfies HBDP (1.2) and that $w \in C^\infty(\overline{\mathbb{D}})$ satisfies the following IBDP:

$$
\begin{cases}
\Delta^2 w = g & \text{in } \mathbb{D}, \\
w = 0 & \text{on } \mathbb{T}, \\
\partial_n w = 0 & \text{on } \mathbb{T}.
\end{cases}
$$

(1.7)

Then $\Phi = u + w$ is the only solution to IBDP (1.1), where $u(z) = F_0[f](z) + H_0[h](z)$ and $w(z) = G[g](z)$.

(2) If $g \in C(\overline{\mathbb{D}})$, then $\Phi$ solves IBDP (1.1), where

$$
\Phi(z) = F_0[f](z) + H_0[h](z) - G[g](z).
$$

(1.8)

The second objective is to discuss the bi-Lipschitz continuity of solutions to IBDP (1.1), which is formulated in the following form.

**Theorem 1.2.** Suppose that $\Phi$ has the representation formula (1.8), $h \in C(\mathbb{T})$, $g \in C(\overline{\mathbb{D}})$ and that $f$ satisfies the Lipschitz condition:

$$
|f(e^{i\theta}) - f(e^{i\varphi})| \leq L|e^{i\theta} - e^{i\varphi}|
$$

where $L$ is a constant. Then for $z_1, z_2 \in \mathbb{D}$,

$$
P\left(\frac{Q}{P^2} - 2\right)|z_1 - z_2| \leq |\Phi(z_1) - \Phi(z_2)| \leq P|z_1 - z_2|,
$$

(1.9)

where

\[
P = \frac{220}{3}L + 4\|h\|_{\infty, \mathbb{T}} + \frac{23}{3}\|g\|_{\infty},
\]

(1.10)
$$\|h\|_{\infty, T} = \sup\{|h(z)| : z \in T\}, \quad \|g\|_{\infty} = \sup\{|g(z)| : z \in \mathbb{D}\},$$

$$A = \left| \frac{1}{4\pi} \int_0^{2\pi} e^{-i\theta} (3f(e^{i\theta}) + h(e^{i\theta})) d\theta - \int_{\mathbb{D}} \bar{\zeta} (\log |\zeta|^2 + 1 - |\zeta|^2) g(\zeta) dA(\zeta) \right|^2,$$

$$B = \left| \frac{1}{4\pi} \int_0^{2\pi} e^{i\theta} (3f(e^{i\theta}) + h(e^{i\theta})) d\theta - \int_{\mathbb{D}} \zeta (\log |\zeta|^2 + 1 - |\zeta|^2) g(\zeta) dA(\zeta) \right|^2,$$

and $Q = A - B$.

It is worth pointing out that if $Q > 2P^2$, then the condition (1.9) shows that $\Phi$ is bi-Lipschitz, otherwise, $\Phi$ is Lipschitz. We would like to point out that this Lipschitz extension property is indeed interesting and does not hold true for the classical Poisson kernel

$$P(z) = \frac{1 - |z|^2}{|1 - z|^2}.$$ 

The rest of this article is organized as follows. In Section 2, some necessary notations will be introduced and several useful lemmas will be proved. In Section 3, Theorem 1.1 will be proved with the aid of Theorem A. Section 4 will be devoted to the proof of Theorem 1.2.

2. Preliminaries

In order to prove Theorems 1.1 and 1.2 we need some preparation.

2.1. Matrix norm. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$ 

We will consider the matrix norm

$$\|M\| = \sup\{|Mz| : z \in \mathbb{C}, \ |z| = 1\}$$

and the matrix function

$$l(M) = \inf\{|Mz| : z \in \mathbb{C}, \ |z| = 1\}.$$ 

Let $D$ and $\Omega$ be plane domains in $\mathbb{C}$. With $p(z) = u(z) + iv(z), \ z = x + iy$, and $p : D \to \Omega$, we can express the Jacobian matrix $\nabla p$ of $p$ and Jacobian (determinant) $J(p)$ as

$$\nabla p = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \quad \text{and} \quad J(p) = u_x v_y - v_x u_y.$$ 

Obviously

$$\|\nabla p(z)\| = \sup\{|\nabla p(z)\zeta| : |\zeta| = 1\} = |p_z(z)| + |p_{\bar{z}}(z)|$$

and

$$l(\nabla p(z)) = \inf\{|\nabla p(z)\zeta| : |\zeta| = 1\} = ||p_z(z)|| - |p_{\bar{z}}(z)||,$$

where

$$p_z = \frac{\partial p}{\partial z} = \frac{1}{2} \left( \frac{\partial p}{\partial x} - i \frac{\partial p}{\partial y} \right) \quad \text{and} \quad p_{\bar{z}} = \frac{\partial p}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y} \right).$$
We denote by $C^\infty_0(\mathbb{D})$ the space of functions which are infinitely differentiable and have compact support in $\mathbb{D}$. And $L^1_{\text{loc}}(\mathbb{D})$ denotes the space of locally integrable functions in $\mathbb{D}$.

Functions $\psi \in L^1_{\text{loc}}(\mathbb{D})$ with distributions in $\mathbb{D}$ having the action

$$\langle \psi, \varphi \rangle = \int_{\mathbb{D}} \psi \varphi \, dA, \; \varphi \in C^\infty_0(\mathbb{D}).$$

Explicitly, we mean that the distribution $\psi_z$ have the action

$$\langle \psi_z, \varphi \rangle = -\int_{\mathbb{D}} \psi \varphi_z \, dA, \; \varphi \in C^\infty_0(\mathbb{D}),$$

and similarly for the distribution $\psi^{\bar{z}}$ (cf. [8]).

2.2. Auxiliary results. The following result is easy to derive: For $\beta > 0$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{2\beta}} = \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \beta)}{n!\Gamma(\beta)} \right)^2 |z|^{2n}, \; z \in \mathbb{D},$$

where $\Gamma$ denotes the Gamma function. Indeed, for $z \in \mathbb{D}$, $|\zeta| = 1$, and $\beta > 0$, one has

$$\frac{1}{1 - z\zeta^\beta} = \sum_{n=0}^{\infty} a_n z^n \zeta^n, \quad a_n = \frac{\Gamma(n + \beta)}{n!\Gamma(\beta)},$$

and thus, by Parseval’s theorem, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{2\beta}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|(1 - ze^{i\theta})^\beta|^2} = \sum_{n=0}^{\infty} |a_n|^2 |z|^{2n}$$

as required.

In particular, for $\beta = 1$, $\beta = 2$ and $\beta = 3$, we obtain

(2.2) \[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ee^{-i\theta}|^2} = \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1 - |z|^2}, \]

(2.3) \[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - e^{-i\theta}|^4} = \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n} = \frac{1 + |z|^2}{(1 - |z|^2)^3} \]

and

(2.4) \[ \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - e^{-i\theta}|^6} \, d\theta = \sum_{n=0}^{\infty} \frac{(n + 1)^2(n + 2)^2}{4} |z|^{2n} \]

Note that the above identities also hold when $e$ is replaced by $z$. Thus, by the Hölder inequality, (2.2) and (2.3), we easily have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^3} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - ze^{-i\theta}|^2} \, d\theta \right)^\frac{1}{2} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - ze^{-i\theta}|^4} \, d\theta \right)^\frac{1}{2}$$

(2.5) \[ = \sqrt{1 + |z|^2} \]

and this will be also used in the proof of Lemma 4.1.
2.3. Useful lemmas.

**Lemma 2.1.** Let $F_0$ be given by (1.4). Then for $z \in \mathbb{D}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} F_0(ze^{-i\theta}) \, d\theta = 1.$$  

**Proof.** By (2.2) and (2.3), we obtain that

$$\frac{1}{2\pi} \int_0^{2\pi} F_0(ze^{-i\theta}) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |z|^2)^2}{2|1 - \overline{z}e^{i\theta}|^2} \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |z|^2)^3}{2|1 - \overline{z}e^{i\theta}|^4} \, d\theta$$

and the desired conclusion follows. \qed

**Lemma 2.2.** Let $H_0$ and $F_0$ be given by (1.4), and $G$ be defined by (1.6).

(a) For any $\theta \in [0, 2\pi]$,

$$\frac{\partial H_0(ze^{-i\theta})}{\partial z} = \frac{(1 - |z|^2)[e^{-i\theta}(1 - |z|^2) - 2\overline{z}(1 - ze^{-i\theta})]}{2(1 - \overline{z}e^{i\theta})(1 - ze^{-i\theta})^2}$$

and

$$\frac{\partial H_0(ze^{-i\theta})}{\partial \overline{z}} = \left( \frac{\partial H_0(ze^{-i\theta})}{\partial z} \right);$$

(b) For any $\theta \in [0, 2\pi]$,

$$\frac{\partial F_0(ze^{-i\theta})}{\partial z} = \frac{(1 - |z|^2)[e^{-i\theta}(1 - |z|^2) - 2\overline{z}(1 - ze^{-i\theta})]}{2(1 - \overline{z}e^{i\theta})(1 - ze^{-i\theta})^2} + \frac{(1 - |z|^2)^2[2e^{-i\theta}(1 - |z|^2) - 3\overline{z}(1 - ze^{-i\theta})]}{2(1 - \overline{z}e^{i\theta})^2(1 - ze^{-i\theta})^3}$$

and

$$\frac{\partial F_0(ze^{-i\theta})}{\partial \overline{z}} = \left( \frac{\partial F_0(ze^{-i\theta})}{\partial z} \right);$$

(c) For any fixed $\zeta \in \mathbb{D}$,

$$G_z(z, \zeta) = (\overline{z} - \overline{\zeta}) \log \left| \frac{z - \zeta}{1 - \overline{\zeta}z} \right|^2 + \frac{(\overline{z} - \overline{\zeta})(1 - |\zeta|^2)}{1 - \overline{\zeta}z} - \overline{\zeta}(1 - |\zeta|^2)$$

and

$$G_{\overline{z}}(z, \zeta) = G_z(z, \zeta).$$

**Lemma 2.3.** Suppose that $f, h \in C(\mathbb{T})$. Then

$$\frac{\partial F_0[f](z)}{\partial z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial z} f(e^{i\theta}) \, d\theta,$$

$$\frac{\partial F_0[f](z)}{\partial \overline{z}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial \overline{z}} f(e^{i\theta}) \, d\theta.$$
\[
\frac{\partial H_0[h](z)}{\partial z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial H_0(ze^{-i\theta})}{\partial z} h(e^{i\theta}) \, d\theta
\]

and
\[
\frac{\partial H_0[h](z)}{\partial \bar{z}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial H_0(ze^{-i\theta})}{\partial \bar{z}} h(e^{i\theta}) \, d\theta.
\]

**Proof.** By Lemma 2.2, we see that the functions
\[
F_0(ze^{-i\theta}) f(e^{i\theta}), \quad \frac{\partial F_0(ze^{-i\theta})}{\partial z} f(e^{i\theta}) \quad \text{and} \quad \frac{\partial F_0(ze^{-i\theta})}{\partial \bar{z}} f(e^{i\theta}),
\]
are all continuous on \( \mathbb{D}_r \times [0, 2\pi] \), where \( r \in [0, 1) \).

Let \( z = \rho e^{i\varphi} \in \mathbb{D}_r \). Then we have
\[
(2.7) \quad \frac{\partial F_0(ze^{-i\theta})}{\partial \rho} = \frac{1}{\rho} \left( \frac{\partial F_0(ze^{-i\theta})}{\partial z} z + \frac{\partial F_0(ze^{-i\theta})}{\partial \bar{z}} \bar{z} \right)
\]
and
\[
(2.8) \quad \frac{\partial F_0(ze^{-i\theta})}{\partial \varphi} = i \left( \frac{\partial F_0(ze^{-i\theta})}{\partial z} z - \frac{\partial F_0(ze^{-i\theta})}{\partial \bar{z}} \bar{z} \right)
\]
so that
\[
(2.9) \quad \frac{\partial F_0(ze^{-i\theta})}{\partial z} = e^{-i\varphi} \left( \frac{\partial F_0(ze^{-i\theta})}{\partial \rho} - \frac{i}{\rho} \frac{\partial F_0(ze^{-i\theta})}{\partial \varphi} \right).
\]

It follows from (2.7) and (2.8) that both
\[
\frac{\partial F_0(ze^{-i\theta})}{\partial \rho} f(e^{i\theta}) \quad \text{and} \quad \frac{\partial F_0(ze^{-i\theta})}{\partial \varphi} f(e^{i\theta})
\]
are continuous in \( \mathbb{D}_r \times [0, 2\pi] \). Hence
\[
\int_0^\rho \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial \rho} f(e^{i\theta}) \, d\theta \, d\rho = \int_0^{2\pi} f(0) \int_0^\rho \frac{\partial F_0(ze^{-i\theta})}{\partial \rho} f(e^{i\theta}) \, d\rho \, d\theta
\]
\[
= \int_0^{2\pi} \left( F_0(ze^{-i\theta}) - F_0(0) \right) f(e^{i\theta}) \, d\theta.
\]

By differentiating with respect to \( \rho \), we get
\[
(2.10) \quad \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial \rho} f(e^{i\theta}) \, d\theta = \frac{\partial}{\partial \rho} \int_0^{2\pi} F_0(ze^{-i\theta}) f(e^{i\theta}) \, d\theta.
\]

On the other hand,
\[
\int_0^\varphi \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial \varphi} f(e^{i\theta}) \, d\theta \, d\varphi = \int_0^{2\pi} \int_0^\varphi \frac{\partial F_0(ze^{-i\theta})}{\partial \varphi} f(e^{i\theta}) \, d\varphi \, d\theta
\]
\[
= \int_0^{2\pi} \left( F_0(ze^{-i\theta}) - F_0(\rho e^{-i\theta}) \right) f(e^{i\theta}) \, d\theta.
\]

By differentiating with respect to \( \varphi \), we get
\[
(2.11) \quad \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial \varphi} f(e^{i\theta}) \, d\theta = \frac{\partial}{\partial \varphi} \int_0^{2\pi} F_0(ze^{-i\theta}) f(e^{i\theta}) \, d\theta.
\]
It follows from (2.9), (2.10), and (2.11) that (2.6) holds. The remaining results of the lemma follows similarly.

Lemma 2.4. Suppose that $F$ is defined on $D \times D$ and satisfies the following conditions:

(a) $\frac{\partial F(z, \zeta)}{\partial z}$ and $\frac{\partial F(z, \zeta)}{\partial \zeta}$ exist;
(b) For all $z \in D$, $\int_D |F(z, \zeta)| dA(\zeta) < \infty$;
(c) For all $z \in D_r$ $(r \in (0, 1))$

\begin{equation}
\int_D \left| \frac{\partial F(z, \zeta)}{\partial z} \right| dA(\zeta) < \infty \text{ and } \int_D \left| \frac{\partial F(z, \zeta)}{\partial \zeta} \right| dA(\zeta) < \infty.
\end{equation}

Then for every $z \in D$, we have

\begin{equation}
\frac{\partial}{\partial z} \int_D F(z, \zeta)dA(\zeta) = \int_D \frac{\partial F(z, \zeta)}{\partial z} dA(\zeta)
\end{equation}

and

\begin{equation}
\frac{\partial}{\partial \zeta} \int_D F(z, \zeta)dA(\zeta) = \int_D \frac{\partial F(z, \zeta)}{\partial \zeta} dA(\zeta).
\end{equation}

Proof. Obviously, we only need to prove the equality (2.13) since the proof of (2.14) is similar. Let $z = re^{i\phi} \in D$. Then, (2.7) and (2.8) continue to hold if $F_0(ze^{-i\theta})$ is replaced by $F(z, \zeta)$. Thus, (2.12) implies

$$
\int_0^\rho \int_D \left| \frac{\partial F(z, \zeta)}{\partial \rho} \right| dA(\zeta) d\rho < \infty \text{ and } \int_0^\rho \int_D \left| \frac{\partial F(z, \zeta)}{\partial \phi} \right| dA(\zeta) d\phi < \infty.
$$

Now Fubini’s theorem guarantees that

$$
\int_0^\rho \int_D \frac{\partial F(z, \zeta)}{\partial \rho} dA(\zeta) d\rho = \int_D \int_0^\rho \frac{\partial F(z, \zeta)}{\partial \rho} d\rho dA(\zeta) = \int_D (F(z, \zeta) - F(0, \zeta)) dA(\zeta).
$$

By differentiating with respect to $\rho$, we get

$$
\int_D \frac{\partial F(z, \zeta)}{\partial \rho} dA(\zeta) = \frac{\partial}{\partial \rho} \int_D F(z, \zeta) dA(\zeta).
$$

Similarly, we can obtain that

$$
\int_D \frac{\partial F(z, \zeta)}{\partial \phi} dA(\zeta) = \frac{\partial}{\partial \phi} \int_D F(z, \zeta) dA(\zeta).
$$

Since (2.9) continues to hold with $F(z, \zeta)$ in place $F_0(ze^{-i\theta})$, (2.13) obviously holds, and thus, the proof of the lemma is complete.

Recall that for $p \geq 1$ and $a > -1$, we have

\begin{equation}
\int_0^1 t^a \left( \log \frac{1}{t} \right)^{p-1} dt = \frac{\Gamma(p)}{(1 + a)^p}
\end{equation}
and the change of variable $t = r^2$ gives

\[
\int_0^1 r^{2a + 1} \left( \log \frac{1}{r^2} \right)^{p-1} dr = \frac{\Gamma(p)}{2(1+a)^p}.
\]

**Lemma 2.5.** For $g \in C(\mathbb{D})$, we have the following:

1. \[ \int_{\mathbb{D}} |G(z, \zeta)g(\zeta)|\,dA(\zeta) \leq \frac{3}{4} \|g\|_{\infty}; \]
2. (a) \[ \frac{\partial G[g](z)}{\partial z} = \int_{\mathbb{D}} \frac{\partial G(z, \zeta)}{\partial z} g(\zeta)\,dA(\zeta), \text{ and} \]
   (b) \[ \left| \frac{\partial G[g](z)}{\partial z} \right| \leq \int_{\mathbb{D}} |G_z(z, \zeta)g(\zeta)|\,dA(\zeta) \leq \frac{23}{6} \|g\|_{\infty}; \]
3. (a) \[ \frac{\partial G[g](z)}{\partial \zeta} = \int_{\mathbb{D}} \frac{\partial G(z, \zeta)}{\partial \zeta} g(\zeta)\,dA(\zeta), \text{ and} \]
   (b) \[ \left| \frac{\partial G[g](z)}{\partial \zeta} \right| \leq \int_{\mathbb{D}} |G_\zeta(z, \zeta)g(\zeta)|\,dA(\zeta) \leq \frac{23}{6} \|g\|_{\infty}; \]

**Proof.** It follows from Lemma 2.2(c) that for $z \in \mathbb{D}$,

\[ \int_{\mathbb{D}} |G_z(z, \zeta)|\,dA(\zeta) \leq J_1 + J_2 + J_3, \]

where

\[ J_1 = \int_{\mathbb{D}} |z - \zeta| \log \left| \frac{1 - \overline{\zeta} z}{z - \zeta} \right|^2 dA(\zeta), \quad J_2 = \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)|z - \zeta|}{1 - \overline{\zeta} z} dA(\zeta) \]

and

\[ J_3 = \int_{\mathbb{D}} |z|(1 - |\zeta|^2)\,dA(\zeta). \]

Next, we estimate $J_1$, $J_2$ and $J_3$, respectively. In order to estimate $J_1$, we let

\[ \zeta \mapsto \eta = \phi(\zeta) = \frac{z - \zeta}{1 - \overline{\zeta} z} = re^{i\theta} \]

so that $\phi = \phi^{-1}$,

\[ \zeta = \frac{z - \eta}{1 - \eta \overline{z}}, \quad z - \zeta = \frac{\eta(1 - |\zeta|^2)}{1 - \eta \overline{z}}, \quad \phi'(\zeta) = -\frac{1 - |z|^2}{(1 - \overline{\zeta} z)^2}, \]

and thus,

\[ dA(\zeta) = |(\phi^{-1})'(\eta)|^2 dA(\eta) = \frac{(1 - |\zeta|^2)^2}{|1 - \eta \overline{z}|^4} dA(\eta). \]

Consequently, switching to polar coordinates yields

\[ J_1 = \int_{\mathbb{D}} \frac{|\eta|(1 - |\zeta|^2)^3}{|1 - \eta \overline{z}|^5} \log \frac{1}{|\eta|^2} dA(\eta) = \frac{(1 - |\zeta|^2)^3}{\pi} \int_0^{2\pi} \int_0^1 \frac{r^2}{|1 - \overline{\zeta} re^{i\theta}|^5} \log \frac{1}{r^2} d\theta dr. \]
By the Hölder inequality, (2.3) and (2.4), we get

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^3} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - zre^{i\theta}|^2} d\theta \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - zre^{i\theta}|^6} d\theta \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{n=0}^{\infty} (n+1)^2 |z|^{2n} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{(n+1)^2(n+2)^2}{4} |z|^{2n} \right)^{\frac{1}{2}}
\]

\[
\leq \sum_{n=0}^{\infty} \frac{(n+1)^2(n+2)^2}{4} |z|^{2n}
\]

so that

\[
J_1 \leq 2(1 - |z|^2)^3 \sum_{n=0}^{\infty} \frac{(n+1)^2(n+2)^2}{4} |z|^{2n} \int_0^1 r^{2n+2} \log \frac{1}{r^2} dr
\]

\[
= 2(1 - |z|^2)^3 \sum_{n=0}^{\infty} \frac{(n+1)^2(n+2)^2}{2(2n+3)^2} |z|^{2n} \quad \text{(by (2.16))}
\]

\[
\leq 2(1 - |z|^2)^3 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{8} |z|^{2n} = \frac{1}{2}
\]

because \(4(n+1)(n+2) \leq (2n+3)^2\) and \(\sum_{n=0}^{\infty} (n+1)(n+2)r^n = 2/(1-r)^3\). By the triangle inequality, we get

\[
J_2 \leq \int_D \frac{(1 - |\zeta|^2)(1 + |\zeta|)}{1 - |\zeta|} dA(\zeta) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (1 + \rho^2) \rho d\rho d\varphi = \frac{17}{6},
\]

where \(\zeta = \rho e^{i\varphi}\). Finally, we obtain

\[
J_3 \leq \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (1 - \rho^2) \rho d\rho d\varphi = \frac{1}{2}
\]

The bounds on \(J_1, J_2\) and \(J_3\) give \(\int_D |G_z(z, \zeta)| dA(\zeta) \leq 23/6\) and thus,

\[
\int_D |G_z(z, \zeta)g(\zeta)| dA(\zeta) \leq \frac{23}{6} \|g\|_{\infty} \quad \text{and} \quad \int_D |G_z(z, \zeta)g(\zeta)| dA(\zeta) \leq \frac{23}{6} \|g\|_{\infty},
\]

where the second inequality above is a consequence of Lemma 2.2. Moreover, by the similar reasoning as above, we deduce that

\[
\int_D |G(z, \zeta)| dA(\zeta) \leq I + \int_D (1 - |\zeta|^2) dA(\zeta) = I + \frac{1}{2},
\]
where

\[ I = \int_D |z - \zeta|^2 \log \left| \frac{1 - \bar{\zeta}z}{z - \zeta} \right|^2 dA(\zeta) \]

\[ = \int_D \frac{\eta^2 (1 - |z|^2)^4}{|1 - \eta \bar{\zeta}|^6} \log \frac{1}{|\eta|^2} dA(\eta) \]

\[ = \frac{(1 - |z|^2)^4}{\pi} \int_0^1 \int_0^{2\pi} \frac{r^3}{|1 - \bar{\eta}re^{i\theta}|^6} \log \frac{1}{r^2} d\theta dr \]

\[ = 2(1 - |z|^2)^4 \sum_{n=0}^{\infty} \frac{(n+1)^2(n+2)^2}{4}|z|^{2n} \int_0^1 r^{2n+3} \log \frac{1}{r^2} dr \]

\[ = (1 - |z|^2)^4 \sum_{n=0}^{\infty} \frac{(n+1)^2}{4}|z|^{2n} \quad \text{(by (2.16))} \]

\[ = (1 - |z|^2)^4 \left( \frac{1 + |z|^2}{4(1 - |z|^2)^3} \right) = 1 - \frac{|z|^4}{4} \]

and thus, we have

\[ \int_D |G(z, \zeta)| dA(\zeta) \leq I + \frac{1}{2} \leq \frac{3}{4} \]

Hence the proof of the lemma is complete, since the rest of it follows from Lemma 2.4.

\[ \square \]

3. Solutions to IBDP (1.1)

First, let us recall a useful result from [26].

**Theorem B.** ([26, Theorem 2.1]) Suppose \( u \) satisfies the conditions:

\[ \Delta^2 u = 0 \quad \text{and} \quad \lim_{r \to 1} \frac{u(re^{i\theta})}{1 - r} = 0 \quad \text{in} \; \mathcal{D}'(\mathbb{T}), \]

where \( re^{i\theta} \in \mathbb{D} \). Then \( u = 0 \) in \( \mathbb{D} \).

Our next result concerns the representation formula and the uniqueness of the solutions to the IBDP (1.7).

**Lemma 3.1.** The function \( G(z, \zeta) \) given by (1.6) satisfies the following:

(a) \( \int_D |G(z, \zeta)| dA(z) < \infty \) and \( \int_D |G(z, \zeta)| dA(z) < \infty \);

(b) For fixed \( \zeta \in \mathbb{D} \),

\[ G_{\zeta \bar{z}}(z, \zeta) = H_2(z, \zeta) = \log \left| \frac{1 - \bar{\zeta}z}{z - \zeta} \right|^2 - \frac{(1 - |\zeta|^2)(1 - |z|^2)|\zeta|^2}{|1 - \bar{\zeta}z|^2} \]

in the sense of distributions in \( \mathbb{D} \);

(c) \( \int_D |G_{\zeta \bar{z}}(z, \zeta)| dA(\zeta) < \infty \) and \( \int_D |G_{\zeta \bar{z}}(z, \zeta)| dA(z) < \infty \).
Proof. By (1.6), we find that
\[ \int_{D} |G(z, \zeta)| \, dA(z) \leq J_4 + \int_{D} (1 - |z|^2) \, dA(z) \leq J_4 + \frac{1}{2}, \]
where
\[ J_4 = \int_{D} |z - \zeta|^2 \log \left| \frac{1 - \zeta z}{z - \zeta} \right|^2 \, dA(z). \]
In order to estimate \( J_4 \), we let
\[ \psi = \psi^{-1}, \]
so that \( \psi = \psi^{-1} \),
\[ z = \frac{\zeta - \tau}{1 - \zeta \tau} \quad \text{and} \quad dA(z) = \frac{(1 - |\zeta|^2)^2}{|1 - \tau \zeta|^4} \, dA(\tau), \]
and consequently, as in the proof of Lemma 2.5, we have
\[ J_4 = \int_{D} |z - \zeta|^2 \log \left| \frac{1 - \zeta z}{z - \zeta} \right|^2 \, dA(\tau) = \frac{1 - |\zeta|^4}{4}. \]
Moreover, by the similar reasoning as above, we find that
\[ \int_{D} |G_z(z, \zeta)| \, dA(z) \leq \int_{D} |z - \zeta| \log \left| \frac{1 - \zeta z}{z - \zeta} \right|^2 \, dA(z) + 2(1 - |\zeta|^2) \int_{D} dA(z) \]
\[ \leq \frac{1}{2} + 2(1 - |\zeta|^2) \leq \frac{5}{2}. \]
Thus the assertion (a) in the lemma is true. Therefore, \( G_z(z, \zeta) \in L^1(D) \) so that its derivative has the action
\[ \langle G_z(\zeta, \zeta), \varphi(z) \rangle = - \int_{D} G_z(z, \zeta) \varphi(z) \, dA(z), \quad \varphi \in C_{0}^\infty(D). \]
By Lebesgue’s dominated convergence theorem we get
\[ \int_{D} G_z(z, \zeta) \varphi(z) \, dA(z) = \lim_{\varepsilon \to 0} \int_{D \setminus D(\zeta, \varepsilon)} G_z(z, \zeta) \varphi(z) \, dA(z). \]
For any small \( \varepsilon > 0 \), let \( D_\varepsilon = D(\zeta, \varepsilon) \). Partial integration gives
\[ \int_{D \setminus D_\varepsilon} G_z(z, \zeta) \varphi(z) \, dA(z) = \int_{\partial D_\varepsilon} G_z(z, \zeta) \varphi(z) v(z) \, ds(z) - \int_{D \setminus D_\varepsilon} G_z(\zeta, \zeta) \varphi(z) \, dA(z), \]
where \( v \) is the unit outward normal of \( D \setminus D_\varepsilon \), that is, the inward unit normal of \( D_\varepsilon \) and \( ds \) denotes the normalized arc length measure. It follows from Lemma 2.2 that
\[ |G_z(z, \zeta)| \leq (1 + |\zeta|) \log \left| \frac{1 - \zeta z}{z - \zeta} \right|^2 + 2(1 - |\zeta|^2) \leq 2 \left( 1 - \log \frac{z - \zeta}{1 - \zeta z} \right). \]
Then
\[
\left| \int_{\partial D_\varepsilon} G_z(z, \zeta) \varphi(z) v(z) \, ds(z) \right| \leq C_\varphi \int_{\partial D_\varepsilon} |G_z(z, \zeta)| \, ds(z) \\
\leq 2C_\varphi (1 - \log \varepsilon^2) \varepsilon \to 0
\]
as \varepsilon \to 0, where \(C_\varphi\) is a constant depending only on \(\sup \varphi\). A straightforward computation shows that for \(z \neq \zeta\) we have
\[
G_{z\bar{z}}(z, \zeta) = H_2(z, \zeta) = \log \left| \frac{1 - \zeta \bar{z}}{z - \zeta} \right|^2 - \frac{(1 - |\zeta|^2)(1 - |z|^2|\zeta|^2)}{|1 - z\bar{\zeta}|^2}
\]
so that
\[
(3.1) \quad |G_{z\bar{z}}(z, \zeta)| \leq \log \left| \frac{1 - \zeta \bar{z}}{z - \zeta} \right|^2 + 2(1 + |\zeta|) + 1 - |\zeta|^2 \leq 5 \left( 1 - \log \left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right|^2 \right).
\]
Therefore
\[
\langle G_{z\bar{z}}(z, \zeta), \varphi(z) \rangle = \lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} G_{z\bar{z}}(z, \zeta) \varphi(z) \, dA(z) = \int_D H_2(z, \zeta) \varphi(z) \, dA(z) = \langle H_2(z, \zeta), \varphi(z) \rangle.
\]
The assertion (b) in the lemma is held. To prove the assertion (c), it follows from the assertion (b) and the first inequality in (3.1) that
\[
\int_D |G_{z\bar{z}}(z, \zeta)| \, dA(\zeta) \leq \int_D \log \left| \frac{1 - \zeta \bar{z}}{z - \zeta} \right|^2 \, dA(\zeta) + C.
\]
We make a convention that in the course of the proof, the value of constants may change from one occurrence to the next, but we always use the same letter \(C\) to denote them.

It follows directly from Green representation formula that
\[
\int_D \log \left| \frac{1 - \zeta \bar{z}}{z - \zeta} \right|^2 \, dA(\zeta) = 1 - |z|^2 \leq 1,
\]
which implies that \(\int_D |G_{z\bar{z}}(z, \zeta)| \, dA(\zeta) < \infty\). By the similar reasoning as above, one obtains that \(\int_D |G_{z\bar{z}}(z, \zeta)| \, dA(z) < \infty\). The proof of the lemma is complete. \(\square\)

**Lemma 3.2.** The function \(G(z, \zeta)\) satisfies:

(a) For fixed \(\zeta \in \mathbb{D}\),
\[
G_{z\bar{z}}(z, \zeta) = H_3(z, \zeta) = -\frac{1 - |\zeta|^2}{(z - \zeta)(1 - \zeta \bar{z})} - \frac{\zeta(1 - |\zeta|^2)}{(1 - z\bar{\zeta})^2}
\]
in the sense of distributions in \(\mathbb{D}\);

(b) \(\int_D |G_{z\bar{z}}(z, \zeta)| \, dA(\zeta) < \infty\) and for fixed \(\zeta \in \mathbb{D}\), \(\int_D |G_{z\bar{z}}(z, \zeta)| \, dA(z) < \infty\).
Proof. It follows from Lemma 3.1(c) that for fixed \( \zeta \in \mathbb{D} \), \( G_{z\bar{z}}(z, \zeta) \in L^1(\mathbb{D}) \). Then
\[
\langle G_{z\bar{z}}(z, \zeta), \varphi(z) \rangle = -\int_\mathbb{D} G_{z\bar{z}}(z, \zeta) \varphi(z) \, dA(z), \quad \varphi \in C_0^\infty(\mathbb{D}).
\]
By Lebesgue’s dominated convergence theorem we get
\[
\int_\mathbb{D} G_{z\bar{z}}(z, \zeta) \varphi(z) \, dA(z) = \lim_{\varepsilon \to 0} \int_{\mathbb{D} \setminus D_\varepsilon} G_{z\bar{z}}(z, \zeta) \varphi(z) \, dA(z).
\]
Partial integration gives
\[
\int_{\mathbb{D} \setminus D_\varepsilon} G_{z\bar{z}}(z, \zeta) \varphi(z) \, dA(z) = -\int_{\partial D_\varepsilon} G_{z\bar{z}}(z, \zeta) \varphi(z) v(z) \, ds(z) - \int_{\mathbb{D} \setminus D_\varepsilon} G_{z\bar{z}z}(z, \zeta) \varphi(z) \, dA(z).
\]
By Lemma 3.1 and the second inequality in (3.1), we get
\[
\left| \int_{\partial D_\varepsilon} G_{z\bar{z}}(z, \zeta) \varphi(z) v(z) \, ds(z) \right| \leq C \varphi \int_{\partial D_\varepsilon} \left| G_{z\bar{z}}(z, \zeta) \right| \, ds(z) \leq 5C \varphi (1 - \log \varepsilon^2) \varepsilon \to 0
\]
as \( \varepsilon \to 0 \). Moreover, for \( z \neq \zeta \) we have
\[
G_{z\bar{z}}(z, \zeta) = H_3(z, \zeta) = -\frac{1 - |\zeta|^2}{(z - \zeta)(1 - \zeta \bar{z})} - \frac{\zeta(1 - |\zeta|^2)}{(1 - \zeta \bar{z})^2}
\]
and thus,
\[
\langle G_{z\bar{z}}(z, \zeta), \varphi(z) \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{D} \setminus D_\varepsilon} G_{z\bar{z}}(z, \zeta) \varphi(z) \, dA(z)
\]
\[
= \int_\mathbb{D} H_3(z, \zeta) \varphi(z) \, dA(z)
\]
\[
= \langle H_3(z, \zeta), \varphi(z) \rangle.
\]
The assertion (a) in the lemma holds.
By Lemma 3.2(a),
\[
\int_\mathbb{D} |G_{z\bar{z}}(z, \zeta)| \, dA(\zeta) \leq \int_\mathbb{D} \frac{1 - |\zeta|^2}{|z - \zeta| \cdot |1 - \zeta \bar{z}|} \, dA(\zeta) + \int_\mathbb{D} \frac{1 - |\zeta|^2}{|1 - \zeta \bar{z}|^2} \, dA(\zeta).
\]
Moreover, as before, the transformation
\[
\zeta \mapsto \eta = \phi(\zeta) = \frac{z - \zeta}{1 - \zeta \bar{z}} = re^{i\theta}
\]
gives after some computation that
\[
\int_D \frac{1 - |\zeta|^2}{|z - \zeta| \cdot |1 - \zeta z|} dA(\zeta) = \int_D \frac{(1 - |\zeta|^2)(1 - |\eta|^2)}{|1 - \eta^2|^4} dA(\eta)
\]
\[
= \frac{(1 - |\zeta|^2)}{\pi} \int_0^1 \int_0^{2\pi} \frac{1 - r^2}{|1 - zr e^{i\theta}|^4} d\theta \, dr
\]
\[
= 2(1 - |\zeta|^2) \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n} \int_0^1 r^{2n}(1 - r^2) \, dr \quad \text{(by (2.3))}
\]
\[
= 4(1 - |\zeta|^2) \sum_{n=0}^{\infty} \frac{(n + 1)^2}{(2n + 1)(2n + 3)} |z|^{2n}
\]
\[
\leq \frac{4(1 - |\zeta|^2)}{3} \sum_{n=0}^{\infty} |z|^{2n} = \frac{4}{3}
\]
and similarly, it is easy to see that
\[
\int_D \frac{1 - |\zeta|^2}{|1 - \zeta z|^2} dA(\zeta) = \int_D \frac{(1 - |\zeta|^2)(1 - |\eta|^2)}{|1 - \eta z|^4} dA(\eta) \leq 1,
\]
Thus, we conclude that
\[
\int_D |G_{\zeta\zeta}(z, \zeta)| \, dA(\zeta) < \infty.
\]
On the other hand, since \(|G_{\zeta\zeta}| \leq 2(1 + |\zeta|)/(1 - |\zeta|)\), we have
\[
\int_D |G_{\zeta\zeta}(z, \zeta)| \, dA(z) \leq 2 \frac{1 + |\zeta|}{1 - |\zeta|} < \infty
\]
for each fixed \(\zeta \in \mathbb{D}\). The proof of the lemma is complete. \(\Box\)

**Lemma 3.3.** (a) If \(w \in C^\infty(\mathbb{D})\) is a solution to (1.7), then

\[
(3.2) \quad w(z) = G[g](z) = \int_D G(z, \zeta) g(\zeta) \, dA(\zeta),
\]

where \(G(z, \zeta)\) is defined by (1.6).

(b) If \(g \in C(\overline{\mathbb{D}})\) and \(w\) is defined by (3.2), then \(w\) is the only solution to (1.7).

**Proof.** Part (a) is a consequence of applying Green’s formula twice (cf. [4] or [16]).
To prove the second part, we assume that \(w\) admits the representation formula (3.2). First, we prove that \(w\) satisfies the first equation in IBDP (1.7).

**Claim 3.1.** \(\Delta_z^2 \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) = g(z)\).

Obviously, it follows from (1.5) that
\[
\int_D \Delta_z^2 (G(z, \zeta)) g(\zeta) \, dA(\zeta) = g(z)
\]
and thus, it suffices to show that
\[ \Delta_z^2 \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) = \int_D \Delta_z^2 (G(z, \zeta)) g(\zeta) \, dA(\zeta) \]
in the sense of distributions in \( \mathbb{D} \).

To prove this equality, we divide the proof into four steps.

**Step 3.1.** By Lemma 2.5, we obtain that
\[ \frac{\partial}{\partial z} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) = \int_D G_z(z, \zeta) g(\zeta) \, dA(\zeta). \]

**Step 3.2.** We prove that \( \frac{\partial^2}{\partial z^2} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) = \int_D G_{zz}(z, \zeta) g(\zeta) \, dA(\zeta) \) in the sense of distributions in \( \mathbb{D} \).

To prove the second step, we recall from Lemma 2.5 that
\[ \langle \frac{\partial^2}{\partial z^2} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right), \varphi(z) \rangle = \langle \int_D G(z, \zeta) g(\zeta) \, dA(\zeta), \varphi(z) \rangle \]
\[ = \int_D \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \varphi(z) \right) dA(z), \]
where \( \varphi \in C_0^\infty(\mathbb{D}) \). By Lemmas 2.5 and 3.1, we know that
\[ \int_D \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \varphi(z) \right) dA(z) = \int_D \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \varphi(z) \right) dA(z) \]
\[ = \int_D \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) dA(z) \]
\[ = \int_D \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) dA(z) \]
\[ = \int_D \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) dA(z) \]
\[ = \int_D \left( \int_D G_{zz}(z, \zeta) g(\zeta) \, dA(\zeta) \right) dA(z) \]
\[ = \langle \int_D G_{zz}(z, \zeta) g(\zeta) \, dA(\zeta), \varphi(z) \rangle. \]

Hence
\[ \langle \frac{\partial^2}{\partial z^2} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right), \varphi(z) \rangle = \langle \int_D G_{zz}(z, \zeta) g(\zeta) \, dA(\zeta), \varphi(z) \rangle \]
as required. Hence, the proof of Step 3.2 is finished.

**Step 3.3.** We prove that \( \frac{\partial^3}{\partial z^2 \partial \zeta} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right) = \int_D G_{zz\zeta}(z, \zeta) g(\zeta) \, dA(\zeta) \) in the sense of distributions in \( \mathbb{D} \).
By the similar reasoning as in the proof of Step 3.2, it follows from Lemmas 2.5, 3.1 and 3.2 and Step 3.2, we see that
\[
\left\langle \frac{\partial^3}{\partial z \partial \bar{z}^2} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right), \varphi(z) \right\rangle = \left\langle \int_D G_z \bar{z}^2(z, \zeta) g(\zeta) \, dA(\zeta), \varphi(z) \right\rangle,
\]
where \( \varphi \in C^\infty_0(D) \). Hence, the proof of Step 3.3 is finished.

**Step 3.4.** We prove that
\[
\left\langle \frac{\partial^4}{\partial z \partial \bar{z}^3} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right), \varphi(z) \right\rangle = \left\langle \int_D G_z \bar{z}^3(z, \zeta) g(\zeta) \, dA(\zeta), \varphi(z) \right\rangle
\]
in the sense of distributions in \( D \).

It follows from \( \Delta^2 z(G(z, \zeta)) = \delta_\zeta(z) \) that
\[
\left| \int_D G_{z \bar{z} \bar{z}}(z, \zeta) g(\zeta) \, dA(\zeta) \right| \leq \|g\|_\infty.
\]

By the similar reasoning as in the proof of Step 3.2, it follows from Lemmas 2.5, 3.1 and 3.2, Steps 3.2 and 3.3, that
\[
\left\langle \frac{\partial^4}{\partial z \partial \bar{z}^3} \left( \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) \right), \varphi(z) \right\rangle = \left\langle \int_D G_{z \bar{z}^3}(z, \zeta) g(\zeta) \, dA(\zeta), \varphi(z) \right\rangle,
\]
where \( \varphi \in C^\infty_0(D) \). This completes the proof of Step 3.4 and this confirms Claim 3.1.

Next, we check that \( w|_T = 0 \), i.e., the middle equation in IBDP (1.7) holds.

**Claim 3.2.** \( \lim_{z \to e^{i\tau}} w(z) = 0 \).

Again, it follows from (1.5) that
\[
\int_D \lim_{z \to e^{i\tau}} G(z, \zeta) g(\zeta) \, dA(\zeta) = 0
\]
and thus, to prove the claim, it suffices to show that
\[
\lim_{z \to e^{i\tau}} \int_D G(z, \zeta) g(\zeta) \, dA(\zeta) = \int_D \lim_{z \to e^{i\tau}} G(z, \zeta) g(\zeta) \, dA(\zeta).
\]

To show this, we use the Vitali theorem (cf. [15, Theorem 26] or [23, p. 4050]) which asserts that if \( X \) is a measurable space with finite measure \( \mu \) and that \( p_n : X \to \mathbb{C} \) is a sequence of functions such that
\[
\lim_{n \to \infty} p_n(x) = p(x) \text{ a.e. and } \sup_n \int_X |p_n|^q \, d\mu < \infty \text{ for some } q > 1,
\]
then
\[
\lim_{n \to \infty} \int_X p_n \, d\mu = \int_X p \, d\mu.
\]

By the Vitali theorem and (1.5), we only need to demonstrate that
\[
\sup_{z \in D} \int_D |G(z, \zeta) g(\zeta)|^2 \, dA(\zeta) < \infty.
\]

Applying the Minkowski inequality, we conclude that
\[
\left( \int_D |G(z, \zeta)|^2 \, dA(\zeta) \right)^{\frac{1}{2}} \leq I_1 + I_2,
\]
where
\[ I_1 = \left( \int_D |z - \zeta|^4 \log^2 \left| \frac{1 - \zeta z}{z - \zeta} \right|^2 dA(\zeta) \right)^{\frac{1}{2}} \]
and
\[ I_2 = \left( \int_D (1 - |z|^2)^2 (1 - |\zeta|^2)^2 dA(\zeta) \right)^{\frac{1}{2}} \leq C. \]
Again, the transformation
\[ \zeta \mapsto \eta = \phi(\zeta) = \frac{z - \zeta}{1 - \zeta z} = re^{i\theta} \]
gives after some computation that
\[ I_2^2 = \int_D \frac{\eta^4 (1 - |\eta|^2)^6}{|1 - \eta z|^8} \log^2 \frac{1}{|\eta|^2} dA(\eta) \]
\[ = 2(1 - |z|^2)^6 \sum_{n=0}^{\infty} \frac{(n + 1)^2(n + 2)^2(n + 3)^2}{36} |z|^{2n} \int_0^1 r^{2n+5} \log^2 \frac{1}{r^2} dr \]
\[ = (1 - |z|^2)^6 \sum_{n=0}^{\infty} \frac{(n + 1)^2(n + 2)^2}{18(n + 3)} |z|^{2n} \quad \text{(by (2.16))} \]
\[ \leq (1 - |z|^2)^6 \sum_{n=0}^{\infty} \frac{(n + 1)^2(n + 2)^2}{18} |z|^{2n} \]
\[ = \frac{1}{18} (1 - |z|^2)^2(2 + 4|z|^2) \leq \frac{1}{3}. \]

Then
\[ \sup_{z \in D} \int_D |G(z, \zeta)g(\zeta)|^2 dA(\zeta) < \infty, \]
since \( g \in C(\overline{D}) \), which is what we want.

To finish the proof, we have to show that \( w \) satisfies the third equation in (1.7).

**Claim 3.3.** \( \lim_{z \to e^{it}} \partial_{n(z)}w(z) = 0. \)

Once again, it following from (1.5) that
\[ \lim_{z \to e^{it}} \int_D |G(z, \zeta)g(\zeta)|^2 dA(\zeta) = 0. \]

To prove this claim, the similar reasoning stated as in the beginning of the proof of Claim 3.2 shows that it suffices to check the boundedness of the integral
\[ (3.3) \quad \int_D \left| \partial_{n(z)}G(z, \zeta)g(\zeta) \right|^2 dA(\zeta). \]

By letting \( z = re^{i\theta} \), we write \( \partial_{n(z)}G(z, \zeta) = G_z(z, \zeta)e^{i\theta} + G_{\overline{\zeta}}(z, \zeta)e^{-i\theta} \). Since \( G_{\overline{\zeta}}(z, \zeta) = \overline{G_z(z, \zeta)} \), to prove the boundedness of (3.3), we only need to show that
\[ \sup_{z \in D} \int_D |G(z, \zeta)g(\zeta)|^2 dA(\zeta) < \infty. \]
By the expression for $G_z$ from Lemma 2.2(c) and the Minkowski inequality, we have

$$\left( \int_D |G_z(z, \zeta)|^2 dA(\zeta) \right)^{\frac{1}{2}} \leq J + C, \quad J^2 = \int_D |z - \zeta|^2 \log^2 \frac{1 - \overline{z} \zeta}{|z - \zeta|^2} dA(\zeta).$$

Now, as before, by letting $\eta = \frac{z - \zeta}{1 - \overline{\zeta} z} = re^{i\varphi}$, we find that

$$J^2 = \int_D |\eta|^2 (1 - |z|^2)^4 \log^2 \frac{1}{|\eta|^2} dA(\eta) = 2(1 - |z|^2)^4 \sum_{n=0}^{\infty} \frac{(n+1)^2(n+2)^2}{n+2} |z|^{2n} \int_0^1 r^{2n+3} \log^2 \frac{1}{r^2} dr \leq \frac{(1 - |z|^2)^4}{2} \cdot \frac{1}{(1 - |z|^2)^2} \leq \frac{1}{2}.$$ 

This observation implies that

$$\sup_{z \in \mathbb{D}} \int_D |G_z(z, \zeta)g(\zeta)|^2 dA(\zeta) < \infty,$$

since $g \in C(\overline{\mathbb{D}})$, which is what we need.

Finally, to complete the proof of the lemma, it remains to verify the uniqueness of $w$. If $w_1$ is also a solution to (1.7), then

$$\begin{cases} \Delta^2 (w - w_1) = 0 & \text{in } \mathbb{D}, \\ w - w_1 = 0 & \text{on } \mathbb{T}, \\ \partial_n(w - w_1) = 0 & \text{on } \mathbb{T}, \end{cases}$$

which by Theorem B gives that $w_1 = w$. The proof of Lemma 3.3 is complete. □

**Proof of Theorem 1.1.** It follows from Theorems A and B together with Lemma 3.3 that Theorem 1.1 holds. □

4. Bi-Lipschitz continuity of solutions to IBDP (1.1)

First, we prove the Lipschitz continuity of $f$ implies the Lipschitz continuity of $u$.

**Lemma 4.1.** Let $u$ be defined by (1.3). Assume that $f$ satisfies the Lipschitz condition

$$|f(e^{i\theta}) - f(e^{i\varphi})| \leq L|e^{i\theta} - e^{i\varphi}|$$

and that $h \in C(\mathbb{T})$. Then for $z_1, z_2 \in \mathbb{D}$,

$$|u(z_1) - u(z_2)| \leq \sup_{z \in \mathbb{D}} |\nabla u(z)| \cdot |z_1 - z_2| \leq 4 \left( \frac{55}{3} L + \|h\|_{\infty, \mathbb{T}} \right) |z_1 - z_2|.$$ 

That is, $u$ is Lipschitz continuous in $\mathbb{D}$. □
Proof. To prove this lemma, obviously, it suffices to show the boundedness of $\nabla u$ in $\mathbb{D}$ since for $z_1, z_2 \in \mathbb{D}$,

$$|u(z_2) - u(z_1)| = \left| \int_{[z_1, z_2]} (u_z(z) \, dz + u_\pi(z) \, d\pi) \right| \leq \sup_{z \in \mathbb{D}} \|\nabla u(z)\| \cdot |z_1 - z_2|,$$

(4.1) where $[z_1, z_2]$ stands for the segment in $\mathbb{D}$ with end points $z_1$ and $z_2$. It follows from (2.1) that we only need to demonstrate the boundedness of $u_z$ and $u_\pi$. We first prove the boundedness of $u_z$, which is formulated in the following claim.

**Claim 4.1.** For $z \in \mathbb{D}$,

$$\left| \frac{\partial u}{\partial z}(z) \right| \leq \frac{110}{3} L + 2 \|h\|_{\infty, T}.$$  

(4.2) For any $z \in \mathbb{D}$ and $\varphi \in [0, 2\pi]$, it follows from (1.3) and Lemma 2.1 that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} F_0(z e^{-i\theta}) (f(e^{i\theta}) - f(e^{i\varphi})) \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} H_0(z e^{-i\theta}) h(e^{i\theta}) \, d\theta + f(e^{i\varphi}),$$

and then Lemma 2.3 implies that

$$\frac{\partial u}{\partial z}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial F_0(z e^{-i\theta})}{\partial z} (f(e^{i\theta}) - f(e^{i\varphi})) \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial H_0(z e^{-i\theta})}{\partial z} h(e^{i\theta}) \, d\theta.$$

To finish the proof of (4.2), we let $z = re^{i\varphi}$, where $r \in [0, 1)$. Then we deduce from Lemma 2.2 together with $1 - |z|^2 \leq |1 - z e^{-i\theta}||(1 + |z|)$ that

$$\left| \frac{\partial F_0(z e^{-i\theta})}{\partial z} \right| \leq \frac{(1 - r^2)(1 + 3r)}{2|1 - z e^{-i\theta}|^2} + \frac{(1 - r^2)^2(2 + 5r)}{2|1 - z e^{-i\theta}|^4}$$

and

$$\left| \frac{\partial H_0(z e^{-i\theta})}{\partial z} \right| \leq \frac{(1 - r^2)(1 + 3r)}{2|1 - z e^{-i\theta}|^2},$$

from which we conclude that

$$\left| \frac{\partial u}{\partial z}(z) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial F_0(z e^{-i\theta})}{\partial z} \right| |f(e^{i\theta}) - f(e^{i\varphi})| \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial H_0(z e^{-i\theta})}{\partial z} \right| |h(e^{i\theta})| \, d\theta \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)(1 + 3r)}{2|1 - z e^{-i\theta}|^2} |f(e^{i\theta}) - f(e^{i\varphi})| \, d\theta,$$

$$I_2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)^2(2 + 5r)}{2|1 - z e^{-i\theta}|^4} |f(e^{i\theta}) - f(e^{i\varphi})| \, d\theta$$

and

$$I_3 = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)(1 + 3r)}{2|1 - z e^{-i\theta}|^2} |h(e^{i\theta})| \, d\theta.$$
Obviously, (2.2), together with the estimate $|f(e^{i\theta}) - f(e^{i\varphi})| \leq 2L$, gives
\[ I_1 \leq 4L \text{ and } I_3 \leq 2\|h\|_{\infty,T}. \]

In order to estimate $I_2$, we split $\theta \in [0, 2\pi]$ into two subsets
\[ E_1 = \{ \theta : |e^{i\theta} - e^{i\varphi}| \leq 1 - r \} \text{ and } E_2 = \{ \theta : |e^{i\theta} - e^{i\varphi}| > 1 - r \}, \]
where $z = re^{i\varphi} \in \mathbb{D}$. Then
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{|e^{i\theta} - e^{i\varphi}|}{|1 - ze^{-i\theta}|^4} d\theta = \frac{1}{2\pi} \int_{E_1} \frac{|e^{i\theta} - e^{i\varphi}|}{|1 - ze^{-i\theta}|^4} d\theta + \frac{1}{2\pi} \int_{E_2} \frac{|e^{i\theta} - e^{i\varphi}|}{|1 - ze^{-i\theta}|^4} d\theta. \]

First, we find that
\[ \frac{1}{2\pi} \int_{E_1} \frac{|e^{i\theta} - e^{i\varphi}|}{|1 - ze^{-i\theta}|^4} d\theta \leq \frac{1 - r}{2\pi(1 - r)^4} \int_{E_1} d\theta \leq \frac{2 \arcsin \frac{1-r}{2}}{\pi(1-r)^3}. \]

Secondly, since $|e^{i\theta} - z| \geq 1 - r$ and
\[ |e^{i\theta} - e^{i\varphi}| \leq |e^{i\theta} - z| + |e^{i\varphi} - z| = |e^{i\theta} - z| + 1 - r, \]
we deduce that $|e^{i\theta} - e^{i\varphi}| \leq 2|e^{i\theta} - z|$, and thus,
\[ \frac{1}{2\pi} \int_{E_2} \frac{|e^{i\theta} - e^{i\varphi}|}{|1 - ze^{-i\theta}|^4} d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{2d\theta}{|1 - ze^{-i\theta}|^3} \leq \frac{2\sqrt{1 + r^2}}{(1 - r)^2}, \]
where the second inequality is a consequence of (2.5). Hence
\[ I_2 \leq L(1 + r)^2(2 + 5r) \left( \frac{\arcsin \frac{1-r}{2}}{\pi(1-r)} + \frac{\sqrt{1 + r^2}}{(1 + r)^2} \right) \leq \frac{98}{3} L. \]

Now, using the estimates for $I_1$, $I_2$ and $I_3$, we conclude that
\[ \left| \frac{\partial u}{\partial z}(z) \right| \leq \frac{110}{3} L + 2\|h\|_{\infty,T}, \]
as required.

Now, we are ready to finish the proof of the lemma based on Claim 4.1. It follows from Lemma 2.3 that
\[ u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial F_0(ze^{-i\theta})}{\partial \overline{z}} (f(e^{i\theta}) - f(e^{i\varphi})) d\theta + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial H_0(ze^{-i\theta})}{\partial \overline{z}} h(e^{i\theta}) d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial F_0(ze^{-i\theta})}{\partial z} \right) (f(e^{i\theta}) - f(e^{i\varphi})) d\theta \]
\[ + \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial H_0(ze^{-i\theta})}{\partial z} \right) h(e^{i\theta}) d\theta \]
and thus, Claim 4.1 leads to
\[ \left| \frac{\partial u}{\partial \overline{z}}(z) \right| \leq \frac{110}{3} L + 2\|h\|_{\infty,T}. \]

Finally, we conclude the proof of the lemma from (4.1). \[ \Box \]

Now we concern the Lipschitz continuity of $w$. 

...
Lemma 4.2. Assume that \( w = G[g] \) and \( g \in C(\mathbb{D}) \). Then for \( z_1, z_2 \in \mathbb{D} \),
\[
|w(z_1) - w(z_2)| \leq \sup_{z \in \mathbb{D}} \|\nabla w(z)\| \cdot |z_1 - z_2| \leq \frac{23}{3} \|g\|_\infty \cdot |z_1 - z_2|,
\]
which implies the Lipschitz continuity of \( w \) in \( \mathbb{D} \).

Proof. It follows from Lemma 2.5 that
\[
|w_z(z)| = \left| \int_{\mathbb{D}} G_z(z, \zeta) g(\zeta) \, dA(\zeta) \right| \leq \frac{23}{6} \|g\|_\infty.
\]
and
\[
|w_{\zeta}(z)| = \left| \int_{\mathbb{D}} G_{\zeta}(z, \zeta) g(\zeta) \, dA(\zeta) \right| \leq \frac{23}{6} \|g\|_\infty.
\]
We conclude from (2.1) that \( \|\nabla w(z)\| \leq (23/3) \|g\|_\infty \), which completes the proof. □

Proof of Theorem 1.2. By Lemmas 4.1 and 4.2, we obtain that
\[
\|\nabla \Phi(z)\| \leq \|\nabla u(z)\| + \|\nabla w(z)\| \leq P,
\]
where \( P \) is given by (1.10). Hence it follows from (4.1) that for \( z_1, z_2 \in \mathbb{D} \),
\[
|\Phi(z_1) - \Phi(z_2)| \leq \sup_{z \in \mathbb{D}} \|\nabla \Phi(z)\| \cdot |z_1 - z_2| \leq P|z_1 - z_2|.
\]
Moreover, since \( A = |\Phi_z(0)|^2 \) and \( B = |\Phi_{\zeta}(0)|^2 \), we have
\[
Q = |\Phi_z(0)|^2 - |\Phi_{\zeta}(0)|^2 = \|\nabla \Phi(0)\| \cdot l(\nabla \Phi(0)) \leq Pl(\nabla \Phi(0)),
\]
which gives that \( l(\nabla \Phi(0)) \geq Q/P \). Finally, for any \( z_1, z_2 \in \mathbb{D} \), we have
\[
|\Phi(z_1) - \Phi(z_2)| \geq \left| \int_{[z_1,z_2]} (\Phi_z(z) \, dz + \Phi_{\zeta}(z) \, d\zeta) \right| - \left| \int_{[z_1,z_2]} (\Phi_z(0) \, dz + \Phi_{\zeta}(0) \, d\zeta) \right| - \left| \int_{[z_1,z_2]} (\Phi(z) - \Phi_z(0) \, dz + \Phi_{\zeta}(z) - \Phi_{\zeta}(0) \, d\zeta) \right|
\]
\[
\geq \left( \frac{Q}{P} - 2P \right) |z_1 - z_2|.
\]
Hence the proof of this theorem is complete. □

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P. Li, Department of Mathematics, Hunan First Normal University, Changsha, Hunan 410205, People’s Republic of China

E-mail address: wokeyi99@163.com

S. Ponnusamy, Stat-Math Unit, Indian Statistical Institute (ISI), Chennai Centre, 110, Nelson Manickam Road, Aminjikarai, Chennai, 600 029, India.

E-mail address: samy@iitm.ac.in