Moonshine paths for 3A and 6A nodes of the extended $E_8$-diagram

Robert L. Griess Jr.
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109 USA
rlg@umich.edu

Ching Hung Lam
Institute of Mathematics
Academia Sinica
Taipei 10617, Taiwan
chlam@math.sinica.edu.tw

Abstract

We continue the program, begun in [13], to make a moonshine path between a node of the extended $E_8$-diagram and the Monster. Our theory is a concrete model expressing some of the mysterious connections identified by John McKay, George Glauberman and Simon Norton. In this article, we treat the 3A and 6A-nodes. We determine the orbits of triples $(x, y, z)$ in the Monster where $z \in 2B$, $x, y \in 2A \cap C(z)$ and $xy \in 3A \cup 6A$. Such $x, y$ correspond to a rootless $EE_8$-pair in the Leech lattice. For the 3A and 6A cases, we shall say something about the “half Weyl groups”, which are proposed in the Glauberman-Norton theory. Most work in this article is with lattices, due to their connection with dihedral subgroups of the Monster. These lattices are $M+N$, where $M, N$ is the relevant pair of $EE_8$-sublattices, and their annihilators in the Leech lattice. The isometry groups of these four lattices are analyzed.
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1 Introduction

Moonshine path theory [13] is intended to understand the discoveries of McKay [20] and Glauberman-Norton [7] which connect the extended $E_8$-diagram and the Monster, denoted $M$, and place these relationships in a broader mathematical context. The paths involve series of small steps, each using established mathematical theories. The introduction of [13] has a detailed discussion of context, which involves lattices, vertex operator algebras (VOAs), Lie theory and finite groups. In [13] and [14], we treated the cases of the 3A-node and the 5A-node in detail. The present article treats the cases of the 3A-node and the 6A-node.
Let us first review the background and the main ideas in [13]. It is well known (cf. [1]) that $2A$-involutions of the Monster simple group $\mathbb{M}$ satisfy a 6-transposition property, that is, given a pair of $2A$-involutions $(x, y)$ in $\mathbb{M}$, the product $xy$ has order $\leq 6$. John McKay [20] noticed a remarkable correspondence with the extended $E_8$-diagram $\tilde{E}_8$ as follows.

There are 9 conjugacy classes of such pairs $(x, y)$, and the orders of the 8 products $|xy|$, for $x \neq y$, are the coefficients of the highest root in the $E_8$-root system. The 9 nodes are labeled with 9 conjugacy classes of $\mathbb{M}$ containing the $xy$.

In 2001, George Glauberman and Simon Norton [7] enriched this theory by adding details about the centralizers in the Monster of such pairs of involutions and relations involving the associated modular forms. Let $(x, y)$ be such a pair and let $n(x, y)$ be its associated node. Let $n'(x, y)$ be the subgraph of $\tilde{E}_8$ which is supported at the set of nodes complementary to $\{n(x, y)\}$. If $z$ is a certain $2B$ involution which commutes with $\langle x, y \rangle$, Glauberman and Norton give a lot of detail about $C(x, y, z)$. In particular, they proposed that $C(x, y, z)$ has a “new” relation to the extended $E_8$-diagram, namely that $C(x, y, z)/O_2(C(x, y, z))$ looks roughly like “half” of the Weyl group corresponding to the subdiagram $n'(x, y)$. This article shows that this new relation is not valid for the $6A$-case. See our main theorems. The relations for $3C$ and $5A$-cases are valid [13, 14].

1.1 About the proof

The main idea of [13] is to transfer a problem in group theory to a study of certain subVOAs of the Moonshine VOA $\mathbb{V}$ and some lattices of the Leech lattice. Thus, our the articles on moonshine paths involve a mixture of techniques, finite group theory, internal analysis of lattices spanned by rootless $EE_8$-pairs and analysis of sublattices of the Leech lattice. The bijection between $2A$ involutions of $\mathbb{M}$ and conformal vectors of central charge $1/2$ (abbreviated as cvcc$\frac{1}{2}$) in the Moonshine VOA $\mathbb{V}$ is foundational. See the
theory of Miyamoto involutions [21].

The first observation is that the dihedral group $\langle x, y \rangle$ is uniquely determined by the subVOA generated by the associated cvcc $\frac{1}{2} e'$ and $f'$ [1, 19]. We noticed that the subdiagram $n'(x, y)$ defines an automorphism $r = r(x, y)$ of exponential type in $Aut(V_{Es})$ and one can construct a pair of conformal vectors $e$ and $f$ of central charge $1/2$ in a lattice VOA $V_{EE\tilde{s}}$ by using $r$. We also explained in [13] that $r(x, y)$ is conjugate in $Aut(V_{E8})$ to an automorphism $\hat{h}(x, y)$ in a torus normalizer in $Aut(V_{E8}) \cong E_8(\mathbb{C})$. This approach led us to consider a pair of $EE_8$-sublattices $M$ and $N$ in $E_8 \perp E_8$. We showed that the pair $(M, N)$ can be isometrically embedded into the Leech lattice $\Lambda$ and that the subVOA $\langle e, f \rangle$ of $V_{EE\tilde{s}}$ generated by $e$ and $f$ can be embedded into the VOA $V_\Lambda \subset V^\perp$. Many properties of the dihedral group $\langle \tau_e, \tau_f \rangle$ generated by the Miyamoto involutions can be studied by examining embeddings of the pair $(M, N)$ in $\Lambda$. In particular, the centralizer $C(\tau_e, \tau_f, z)$ has a factor subgroup which looks like the common stabilizer of $M$ and $N$ in $O(\Lambda)/\{\pm 1\}$ (see Corollary 3.5 and 4.5).

1.2 Statements of main results.

**Main Theorem 1** (Proposition 3.3 and Theorem 3.25). Let $x, y, z \in M$ such that $x, y \in 2A$, $xy \in 3A$, and $z \in 2B \cap C_M(x, y)$. Then the triple $(x, y, z)$ is unique up to conjugation by $M$. Moreover, there exist an $EE_8$ pair $(M, N)$ in $\Lambda$ such that $M + N \cong DIH_6(14)$ and the centralizer $C_M(x, y, z)$ has a homomorphism onto $C_M(\tau_e, \tau_f) / O_2(C_M(\Lambda))$, which is isomorphic to “half” of the Weyl group of type $A_2 + E_6$.

**Main Theorem 2** (Theorem 4.2 and 4.4). Consider triples $(x, y, z)$ so that $x, y, z \in M$, $x, y \in 2A$, $xy \in 6A$, and $z \in 2B \cap C_M(x, y)$. There are two orbits on the set of such triples $(x, y, z)$ under conjugacy by $M$.

** Orbit 6A.1:** $(xy)^3 \in O_2(C_M(z))$, and the triple $(x, y, z)$ is conjugate to $(\tau_{\alpha M}, \tau_{\varphi_G(\alpha N)} z)$, where $M$ and $N$ are $EE_8$-sublattices of $\Lambda$, $M + N = DIH_6(14)$ and $\alpha \in M \cap N(4)$.

** Orbit 6A.2:** $(xy)^3 \not\in O_2(C_M(z))$, and the triple $(x, y, z)$ is conjugate to $(\tau_{\alpha M}, \tau_{\varphi_N} z)$, where $M$ and $N$ are $EE_8$-sublattices of $\Lambda$ and $M + N = DIH_{12}(16)$.

In all cases, the centralizer of $(x, y, z)$ is determined.
Main Theorem 3 (Theorem 4.8). Suppose that the triple \((x, y, z)\) is in the orbit 6A.1. Then \(C_M(x, y, z)\) has a homomorphism onto

\[
C_{\text{Stab}_O(\Lambda)}(t_M, t_N)/O_2(C_{\text{Stab}_O(\Lambda)}(t_M, t_N)) \cong PSU(4,2).2.
\]

The kernel \(K\) is a 2-group of order \(2^{11}\) and we have an exact sequence

\[
1 \to \langle z \rangle \to K \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z}\} \to 1.
\]

Main Theorem 4 (Theorem 4.15). Suppose that the triple \((x, y, z)\) is in the orbit 6A.2. Then the natural map \(\theta\) of \(C_M(x, y, z)\) to \(C_M(z)/O_2(C_M(z))\) has image isomorphic to

\[
C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle \cong (3 \times 2.(\text{Alt}_4 \times \text{Alt}_4).2).2.
\]

The kernel \(\tilde{K}\) of \(\theta\) is a group of order \(2^9\) and the sequence

\[
1 \to \langle z \rangle \to \tilde{K} \to \{\varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z}\} \to 1
\]

is exact. (For a description of the group \(C_{O(\Lambda)}(t_M, t_N)\) as an index 2 subgroup of \(2.(\text{Dih}_6 \times O^+(4,3))\), see (4.15) and (B.9).

We also show that \(O(\Lambda)\) has one orbit on ordered pairs \((M, N)\) of \(EE_8\)-sublattices so that \(M + N\) has type \(\text{DIH}_{6}(14)\) (3.19) and one orbit on ordered pairs \((M, N)\) of \(EE_8\)-sublattices so that \(M + N\) has type \(\text{DIH}_{12}(16)\) (4.11). There are analogous transitivity results in [13], [14].

A striking feature of the Glauberman-Norton theory is that the stabilizer of a triple \((x, y, z)\) (modulo \(O_2\)) seemed to be roughly “half” the Weyl group of the corresponding node of the extended \(E_8\)-diagram. Our results so far confirm this for several nodes, [13, 14] but this is not the case for the 6A-node, for either of the two orbits, 6A.1 or 6A.2.

The Weyl group associated to removal of the 6A-node has shape

\[
\text{Weyl}(A_5) \times \text{Weyl}(A_2) \times \text{Weyl}(A_1) \cong \text{Sym}_5 \times \text{Sym}_3 \times \text{Sym}_2.
\]

The quotients \(C_M(x, y, z)/O_2(C_M(x, y, z))\) are described in Main Theorems 3 and 4. Neither can be interpreted as half the above Weyl group.

Our moonshine path theories for the 3C, 5A, 3A and 6A cases have different degrees of confirmation of the Glauberman-Norton observations. Aspects of the 3C path [13] are especially nice.
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Notation and Terminology

| Notation | Explanation | Examples in text |
|----------|-------------|-----------------|
| $2A, 2B, 3A, \ldots$ | conjugacy classes of the Monster: the first number denotes the order of the elements and the second letter is arranged in descending order of the size of the centralizers | Introduction |
| $A_1, \cdots, E_8$ | root lattice for root system $\Phi_{A_1, \ldots, \Phi_{E_8}}$ | (2.3) |
| $AA_1, \cdots, EE_8$ | lattice isometric to $\sqrt{2}$ times the lattice $A_1, \cdots, E_8$ | (2.3) |
| $A \circ B$ | central product of groups $A$ and $B$ | (3.27) |
| $cvcc_{\frac{1}{2}}$ | conformal vector of central charge $1/2$ | (3.3) |
| $D(L)$ | discriminant group of integral lattice $L$: $D(L) = L^* / L$ | (3.6), (4.9) |
| $DIH_n(k)$ | the sum of a $EE_8$ pair $(M, N)$ such that the corresponding SSD involutions generate a dihedral of order $n$ and $M + N$ has rank $k$ | (3.6) |
| $E$ | a particular $EE_8$-sublattice | (4.12) |
| $EE_8$-involution | SSD involution whose negated space is isometric to $EE_8$ | (3.6) |
| $e_M$ | a $cvcc_{\frac{1}{2}}$ associated to an $EE_8$ sublattice $M$ | (2.4) |
| $G$ | complex isometry group of the Coxeter-Todd lattice $K_{12}$ | (3.8) |
| $g_1$ | an order 3 isometry in $O_3(O(K_{12}))$ | (3.6) |
| $H$ | the subgroup generated by complex reflections defined by norm 4 vectors of $K_{12}$ | (3.10) |
### Notation

| Notation | Explanation | Examples in text |
|----------|-------------|------------------|
| $J$      | a sublattice of $\Lambda$ isometric to $K_{12}$ | (3.6) |
| $\mathcal{H}$ | the hexacode | (3.11) |
| $K$      | a sublattice of $\Lambda$ isometric to $K_{12}$ | (3.6) |
| $K_{12}$ | the Coxeter-Todd lattice of rank 12 | (3.7) |
| $\Lambda$ | the Leech lattice, rank 24 | |
| $M$      | Monster sporadic group | Introduction |
| $\mu$    | the natural surjection from $C_M(z) \to Aut(V^+_\Lambda) \cong 2^{24}.Co_1$ | (2.8) |
| $O(X)$   | the isometry group of the quadratic space $X$ | (3.6), (3.13) |
| $O_p(G)$ | largest normal $p$-subgroup of a finite group $G$ ($p$ is a prime) | Introduction |
| $O^\varepsilon(2n,q)$ | orthogonal group of dimension $2n$ over $\mathbb{F}_q$ of type $\varepsilon = \pm$ | (4.14) |
| $P_N$    | the orthogonal projection from a lattice $L$ to $\mathbb{Q} \otimes N^*$ | (2.14) |
| RSSD     | relatively semi self dual | (2.2) |
| SSD      | semi self dual | (2.2) |
| $t_M$    | SSD involution associated to $M$ | (3.6) |
| $\varphi(u \otimes e^\beta)$ | a cvcc $\frac{1}{2}$ | (2.7) |
| $\tau_e$ | the Miyamoto involution defined by a cvcc $\frac{1}{2} e$ | (2.5), (2.10) |
| $\xi$    | the natural surjection from $Aut(V^+_\Lambda) \to O(\Lambda)/\langle \pm 1 \rangle$ | (2.8) |
| $\omega^\varepsilon(\alpha)$ | a cvcc $\frac{1}{2}$ associated to a norm 4 vector | (2.9) |
| $\omega$ | an automorphism of $V^\varepsilon$ such that $z|_{V^\varepsilon_{\Lambda}^+} = 1$ and $z|_{V^\varepsilon_{\Lambda}^-} = -1$ | (2.8) |

## 2 Preliminary

We review terminology about rational lattices and involutions. For background, see [11].

**Definition 2.1.** Let $X$ be a subset of Euclidean space. Define $t_X$ to be the orthogonal transformation which is $-1$ on $X$ and is $1$ on $X^\perp$. 
Definition 2.2. A sublattice $M$ of an integral lattice $L$ is RSSD (relatively semiselfdual) if and only if $2L \leq M + \text{ann}_L(M)$. This implies that $t_M$ maps $L$ to $L$ and is equivalent to this property when $M$ is a direct summand.

The property that $2M^* \leq M$ is called SSD (semiselfdual). It implies the RSSD property, but the RSSD property is often more useful.

Notation 2.3. We use $XX_n$ to denote the lattice which is isometric to $\sqrt{2}$ times the root lattice of type $X_n$. For example, $EE_8$ is the $\sqrt{2}$ times of the root lattice $E_8$.

2.1 Conformal vectors and the Monster

Next we shall recall some facts about conformal vectors of central charge $1/2$ in the lattice type VOA $V^+_\Lambda$ and the Moonshine VOA $V^\natural$.

We use the standard notation for the lattice vertex operator algebra

$$V_L = M(1) \otimes \mathbb{C}\{L\}$$

associated with a positive definite even lattice $L$ [5]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} L$ is an abelian Lie algebra and we extend the bilinear form to $\mathfrak{h}$ by $\mathbb{C}$-linearity. Also, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$ is the corresponding affine algebra and $\mathbb{C}k$ is the 1-dimensional center of $\hat{\mathfrak{h}}$. The subspace $M(1) = \mathbb{C}[\alpha_i(n) | 1 \leq i \leq d, n < 0]$ for a basis $\{\alpha_1, \ldots, \alpha_d\}$ of $\mathfrak{h}$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\mathfrak{h}$-module such that $\alpha(n) \cdot 1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n$ nonnegative, and $k$ acts as the scalar 1. Also, $\mathbb{C}\{L\} = \text{span}\{e^\beta | \beta \in L\}$ is the twisted group algebra of the additive group $L$ such that $e^\beta e^\alpha = (-1)^{\langle\alpha, \beta\rangle} e^\alpha e^\beta$ for any $\alpha, \beta \in L$. The vacuum vector $1$ of $V_L$ is $1 \otimes e^0$ and the Virasoro element $\omega$ is $\frac{1}{2} \sum_{i=1}^d \beta_i (-1)^2 \cdot 1$ where $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$. For the explicit definition of the corresponding vertex operators, we shall refer to [5] for details.

Notation 2.4. Let $M \cong EE_8$. Define

$$e_M = \frac{1}{16} \omega_M + \frac{1}{32} \sum_{\alpha \in M(4)} e^\alpha,$$

where $\omega_M$ is the Virasoro element of $V_M$ and $M(4) = \{\alpha \in M | \langle\alpha, \alpha\rangle = 4\}$. It is shown in [6] that $e_M$ is a simple conformal vector of central charge $1/2$.

For $x \in M^*$, define a $\mathbb{Z}$-linear map

$$\langle x, \cdot \rangle : M \rightarrow \mathbb{Z}_2$$

$$y \mapsto \langle x, y \rangle \mod 2.$$
Clearly the map
\[ \varphi : M^* \longrightarrow \text{Hom}_\mathbb{Z}(M, \mathbb{Z}_2) \]
\[ x \mapsto \langle x, \cdot \rangle \]
is a group homomorphism and \( \ker \varphi = 2M^* = M \). For any \( x \in M^* = \frac{1}{2}M \), \( \langle x, \cdot \rangle \) induces an automorphism \( \varphi_x \) of \( V_M \) given by
\[ \varphi_x(u \otimes e^\alpha) = (-1)^{\langle x, \alpha \rangle} u \otimes e^\alpha \quad \text{for } u \in M(1) \text{ and } \alpha \in M. \tag{4} \]
Note that \( \varphi_x(e_M) \) is also a simple conformal vectors of central charge 1/2.

**Remark 2.5.** Given a simple conformal vector \( e \) of central charge 1/2, one can define an involutive automorphism \( \tau_e \in \text{Aut}(V) \), called Miyamoto involution. If \( V = V^\natural \) is the Moonshine VOA, then \( \tau_e \) defines a 2A involution in the Monster [21].

**Notation 2.6.** Let \( \Lambda \) be the Leech lattice and \( V_\Lambda \) the lattice VOA associated with \( \Lambda \). Let \( \theta \) be a lift of the \(-1\)-isometry of \( \Lambda \) to \( \text{Aut}(V_\Lambda) \). We use \( V_\Lambda^T \) to denote the unique \( \theta \)-twisted module of \( V_\Lambda \) and \( V_\Lambda^+, V_\Lambda^{T,+} \) to denote the fixed point subspaces of \( \theta \) in \( V_\Lambda \) and \( V_\Lambda^T \), respectively. [5].

**Notation 2.7.** Let \( \Lambda \) be the Leech lattice. For each \( \alpha \in L \), we define \( \varphi_\alpha \in \text{Aut}(V_\Lambda) \) by
\[ \varphi_\alpha(u \otimes e^\beta) = (-1)^{\langle \alpha, \beta \rangle} u \otimes e^\beta \quad \text{for } u \in M(1) \text{ and } \beta \in \Lambda. \]
Note that \( \varphi_\alpha = \varphi_\alpha' \) if and only if \( \alpha - \alpha' \in 2\Lambda \). In addition, \( \varphi_\alpha \) commutes with \( \theta \) and thus \( \varphi_\alpha \) also defines an automorphism on \( V_\Lambda^+ \).

**Notation 2.8.** Let \( \theta, V_\Lambda^+, \text{ and } V_\Lambda^{T,+} \) be defined as in (2.6). Then the Moonshine VOA \( V^\natural \) [5] is constructed as a \( \mathbb{Z}_2 \)-orbifold of the Leech lattice VOA \( V_\Lambda \), i.e.,
\[ V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,+}. \]
Let \( z \) be an involution of \( V^\natural \) acting as 1 and \(-1\) on \( V_\Lambda^+ \text{ and } V_\Lambda^{T,+} \) respectively. Then \( z \) defines an involution on \( V^\natural \), which is in conjugacy class 2B [5, 21]. Recall from [18] that
\[ C_{\text{Aut}(V^\natural)}(z)/\langle z \rangle \cong C_{\text{Aut}(V_\Lambda)}(\theta)/\langle \theta \rangle \cong \text{Aut}(V_\Lambda^+). \]
and the sequences

\[ 1 \to \langle z \rangle \to C_{\text{Aut}(V^\Lambda)}(z) \xrightarrow{\mu} \text{Aut}(V_\Lambda^+) \to 1 \]

and

\[ 1 \to \text{Hom}(\Lambda, \mathbb{Z}_2) \to \text{Aut}(V_\Lambda^+) \xrightarrow{\xi} O(\Lambda)/\langle \pm 1 \rangle \to 1. \]

are exact.

The following results can also be found in [18].

**Theorem 2.9** (Theorem 5.18 of [18]). Let \( \alpha \in \Lambda(4) \) and let

\[ e := \omega^+(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot 1 + \frac{1}{4}(e^\alpha + e^{-\alpha}). \]

Then there is an exact sequence

\[ 1 \to K \to C_{\text{Aut}(V^\Lambda)}(z, \tau_e)/\langle z \rangle \to \text{Stab}_{O(\Lambda)}(Z\alpha) \cong C_{O_2} \to 1, \]

where \( K \cong \{ \varphi_\beta \in \text{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \alpha, \beta \rangle \in 2\mathbb{Z} \} \).

**Remark 2.10.** Recall from [18] that for any \( \alpha \in \Lambda(4) \) and

\[ e = \omega^\pm(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot 1 \pm \frac{1}{4}(e^\alpha + e^{-\alpha}), \]

the Miyamoto involution \( \tau_e \) acts on \( V_\Lambda^+ \) as \( \varphi_\alpha \) defined in (2.7).

**Lemma 2.11.** Let \( M \) be a sublattice of \( \Lambda \) isomorphic to \( EE_8 \). Then the sequence

\[ 1 \to X \to \text{Stab}_{C_{\text{Aut}(V^\Lambda)}}(V_\Lambda^+)/(z) \xrightarrow{\xi} C_{O(\Lambda)}(t_M)/\langle \pm 1 \rangle \to 1 \]

is exact, where \( X = \{ \varphi_\alpha \in \text{Hom}(\Lambda, \mathbb{Z}_2) \mid \langle \alpha, M \rangle \in 2\mathbb{Z} \} \).

**Theorem 2.12.** Let \( M \) be a sublattice of \( \Lambda \) isomorphic to \( EE_8 \). Let \( e = \varphi_x(e_M) \) be defined as in Notation (2.4). Then the centralizer \( C_{\text{Aut}(V^\Lambda)}(\tau_e, z) \) stabilizes the subVOA \( V_M^+ \) and has the structure \( 2^{2+8+16} \cdot \Omega^+(8, 2) \). Moreover, the map

\[ \xi \circ \mu : C_{\text{Aut}(V^\Lambda)}(\tau_e, z) \to C_{O(\Lambda)}(t_M)/\langle \pm 1 \rangle \]

is surjective and \( C_{\text{Aut}(V^\Lambda)}(\tau_e, z) \) acts on \( V_M^+ \) as \( \Omega^+(8, 2) \), which is the quotient of the commutator subgroup of the Weyl group of \( E_8 \) by its center.
The following result can be found in [12, Appendix F] (see also [13, 14]).

**Theorem 2.13.** Except for $\text{DIH}_4(15)$, every $\text{EE}_8$ pair $(M, N)$ in Table 1 of [12] can be embedded into $\Lambda$.

**Definition 2.14.** Let $L$ be an integral lattice and $N$ a sublattice. We denote the orthogonal projection of $L$ to $\mathbb{Q} \otimes N^*$ by $P_N$.

## 3 3A-triples

In this section, we consider a triple of elements $x, y, z \in M$ such that $x, y \in 2A$, $xy \in 3A$ and $z \in 2B \cap C_M(x, y)$. We shall show that there is only one orbit of such triples under the action of the Monster and determine their centralizer in $M$.

**Lemma 3.1.** Let $x, y, z \in M$ be such that $x, y \in 2A$, $xy \in 3A$ and $z \in 2B \cap C_M(x, y)$. Then, there exists a pair of $\text{EE}_8$-sublattices $(M, N)$ of $\Lambda$ and $a \in M^*$, $b \in N^*$ such that $M + N \sim \text{DIH}_6(14)$ and the triple $(x, y, z)$ is conjugate to $(\tau_\phi a(e_M), \tau_\phi b(e_N), z)$ in $M$.

**Proof.** Up to conjugation, we may assume $z$ acts as 1 on $V^+_\Lambda$ and as $-1$ on $V^T \Lambda + V_\Lambda$. By the 1-1 correspondence between cvcc $\frac{1}{2}$ in $V^+_\Lambda$ and 2A involutions of $M$ [15, 21], we have $x = \tau_e$ and $y = \tau_f$ for some cvcc $\frac{1}{2} e, f \in V^_\Lambda$. Since $z$ centralizes $x$ and $y$, we have $\tau_{ze} = z\tau_ez^{-1} = \tau_e$ and $\tau_{zf} = z\tau_fz^{-1} = \tau_f$. Hence $e$ and $f$ are fixed by $z$ by the 1-1 correspondence.

Since $\tau_e \tau_f$ has order 3, both $e$ and $f$ must be of $\text{EE}_8$-type. That means there exists $\text{EE}_8$-sublattices $M$ and $N$ and $a \in M^*$ and $b \in N^*$ such that $e = \varphi_a(e_M)$ and $f = \varphi_b(e_N)$ (see (2.4)).

Recall that there are two types of cvcc $\frac{1}{2}$ in $V^+_\Lambda$. If $e = \omega^+(\alpha)$ is of $\text{AA}_1$-type, then $\tau_e = \varphi_\alpha$ on $V^+_\Lambda$ and $\tau_e \in O_2(\text{Aut}(V^+_\Lambda)) = \{\varphi_\alpha \mid \alpha \in \Lambda\} = \text{Hom}(\Lambda, \mathbb{Z}_2) \cong 2^{24}$ (see [18],(2.7) and (2.10)). Hence $\tau_{f} \tau_e \tau_f = \varphi_\beta$ for some $\beta \in \Lambda$ and $(\tau_e \tau_f \tau_e \tau_f)^2 = (\varphi_\alpha \varphi_\beta)^2 = 1$. Therefore, $\tau_e \tau_f$ is of order 1, 2 or 4.

Since $xy \in 3A$, $\xi \circ \mu(xy) = t_M t_N$ has order 3 and $\langle e, f \rangle = \frac{13}{24}$. Hence $M + N \cong \text{DIH}_6(14)$ [12]. □

The following lemma can be obtained easily by direct calculation (see [19], for example ).

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Lemma 3.2. Let \((M, N)\) be an \(EE_8\)-pair such that \(M + N \cong DIH_6(14)\). Then

\[
\langle e_M, \varphi_B e_N \rangle = \begin{cases}
13/210 & \text{if } \langle b, M \cap N \rangle \in 2\mathbb{Z}, \\
5/210 & \text{otherwise}.
\end{cases}
\]

By Lemma 3.2, we can refine the statement of Lemma 3.1 as follows.

Proposition 3.3. Let \(x, y, z \in M\) be such that \(x, y \in 2A, xy \in 3A\) and \(z \in 2B \cap C_M(x, y)\). Then, the triple \((x, y, z)\) is conjugate to \((\tau_{e_M}, \tau_{e_N}, z)\) in \(M\), where \((M, N)\) is an \(EE_8\)-pair in \(\Lambda\) such that \(M + N \cong DIH_6(14)\).

Proof. Since \((\det(M), \det(\Lambda)) = 1\) and \(M\) is a direct summand of \(\Lambda\), there is \(\beta \in \Lambda\) such that \(P_M(\beta) \in a + 2M\), where \(P_M\) is the natural projection from \(\Lambda\) to \(M^*\). Then \(\varphi_B(\varphi_a(e_M)) = e_M\). Hence, we may assume \(x = e_M\) and \(y = \varphi_B(e_N)\). Since \(\langle x, y \rangle = \langle e_M, \varphi_B(e_N) = \frac{13}{210}\), we have \(\langle b, M \cap N \rangle \in 2\mathbb{Z}\) by Lemma (3.2) and \(P_M(\langle b, M \cap N \rangle) \in 2(M \cap N)^*\).

Recall that \(N^* = \frac{1}{2}N\). Thus, \(b = \frac{1}{2}\eta\) for some \(\eta \in N\). We may assume without loss that \(\eta \in \text{ann}_N(M \cap N) \mod 2N\). Let \(h = t_N t_M\). Then \(h\) is of order 3 and \(h(N) = M\). Define \(\alpha = -h(\eta) \in M\). Then \(\alpha = \frac{1}{2}(\eta + (h^2 - h)\eta)\) and we have \(P_N(\alpha) = b\). Thus,

\[
\varphi_B(y) = e_N \quad \text{and} \quad \varphi_B(x) = \varphi_B(e_M) = e_M
\]

as desired. \(\square\)

The next theorem is important to our study, which translates a problem about \(V^2\) to the study of the Leech lattice.

Proposition 3.4. Let \((M, N)\) be an \(EE_8\)-pair in \(\Lambda\) such that \(M + N \cong DIH_6(14)\). Then

\[
\xi \circ \mu : C_{\text{Aut}(V^2)}(\tau_{e_M}, \tau_{e_N}, z) \to C_{\text{O}(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle
\]

is surjective.

Proof. Let \(s \in C_{\text{O}(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle\). Then by Lemma (2.11), there is \(g \in \text{Aut}(V^2)\) such that \(g\) stabilizes both \(V^+_M\) and \(V^+_N\) and \(\xi \circ \mu(g) = s\).

By the classification of \(\text{cvcc}(\frac{1}{2})\) in \(V^+_M\) \([8, 18]\), \(g(e_M) = \varphi_a(e_M)\) for some \(a \in M^*\). By the same argument as in Proposition 3.3, we may also assume \(g(e_M) = e_M\).
Since \( g \) also stabilizes \( V_N^+ \), we have \( g(e_N) = \varphi_b(e_N) \) for some \( b \in N^* \). Moreover,
\[
\langle e_M, \varphi_b e_N \rangle = \langle g(e_M), g(e_N) \rangle = \langle e_M, e_N \rangle = \frac{13}{2^{10}}
\]
Thus, \( \langle b, M \cap N \rangle \in 2\mathbb{Z} \) by Lemma (3.2). Then by the same argument as in Proposition 3.3, there is an \( \alpha \in M \) such that \( P_N(\alpha) = b \). Hence
\[
\varphi_\alpha(g(e_N)) = e_N \quad \text{and} \quad \varphi_\alpha(g(e_M)) = \varphi_\alpha(e_M) = e_M.
\]
Therefore, \( g' = \tau_\omega^{\tau(\alpha)}g \in C_{\text{Aut}(V)}(\tau e_M, \tau e_N, z) \) and \( \xi \circ \mu(g') = s \). \( \square \)

**Corollary 3.5.** Let \((M, N)\) be an \( EE_8 \)-pair in \( \Lambda \) such that \( M + N \cong DIH_6(14) \). The centralizer \( C_{\text{Aut}(V)}(\tau e_M, \tau e_N, z) \) contains a subquotient isomorphic to the common stabilizer of \( M \) and \( N \) in \( O(\Lambda)/\{\pm 1\} \).

**Proof.** Since \( O(\Lambda)(t_M, t_N) \) is the common stabilizer of \( M \) and \( N \) in \( O(\Lambda) \), we have the conclusion by (3.4). \( \square \)

### 3.1 \( DIH_6(14) \)

Next we shall compute the group \( C_{O(\Lambda)}(t_M, t_N) \). First we recall some notations and facts about \( DIH_6(14) \) from [12].

**Notation 3.6.**

1. Let \( M \) and \( N \) be \( EE_8 \)-sublattices of the Leech lattice \( \Lambda \) such that \( Q := M + N \) is isometric to \( DIH_6(14) \) as obtained in [12]. We have \( F := M \cap N \cong AA_2 \).

   Let \( t_M \) and \( t_N \) be the SSD involutions associated to \( M \) and \( N \), respectively. Then the subgroup \( D := \langle t_M, t_N \rangle \) generated by \( t_M \) and \( t_N \) is a dihedral group \( Dih_6 \).

2. Let \( J := \text{ann}_Q(F) \). Then \( J \cong K_{12} \) is isometric to the Coxeter-Todd lattice and \( Q \) contains a sublattice isometric to \( F \perp J \).

3. \( D(Q) = 2^2 \times 3^5 \).

4. Set \( g := t_M t_N \). Then \( g \) has order 3 and it acts on \( \Lambda \) with trace 6. Let \( K := \text{Fix}_\Lambda(g) \) be the fixed point sublattice of \( g \) in \( \Lambda \). Then \( \text{ann}_\Lambda(K) = J \) (see (2)). Moreover, \( K \cong J \cong K_{12} \).

5. Let \( g_1 \) be an isometry of order 3 in \( O(\Lambda) \) such that \( g_1 \) acts fixed point free on \( K \) but acts trivially on \( J \). In this case, \( g \) and \( g_1 \) generate an elementary abelian group of shape \( 3^2 \) and \( gg_1 \) has trace \(-12 \) on \( \Lambda \).
Next we recall some basic properties of the Coxeter-Todd lattice $K_{12}$. It is well-known (cf. [3, 4]) that $K_{12}$ can also be viewed as a rank 6 complex lattice over the ring of Eisenstein integers $\mathbb{Z}[\omega]$ as follows:

**Notation 3.7.** Let $\omega := (-1 + \sqrt{-3})/2$ be a primitive cubic root of unity and let $\mathcal{E} := \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. Then $\mathcal{E}/2\mathcal{E} \cong \mathbb{F}_4$. Let $\sigma : \mathcal{E} \to \mathcal{E}/2\mathcal{E}$ be the natural quotient map and $\mathcal{H}$ the hexacode over $\mathcal{F}_4$. Then the Coxeter-Todd lattice can be defined as the sublattice

$$K_{12} = \{(x_1, \ldots, x_6) \in \mathcal{E}^6 \mid (\sigma(x_1), \ldots, \sigma(x_6)) \in \mathcal{H}\}.$$

The norm of a vector $v$ in $K_{12}$ is defined by $\langle v, v \rangle$, where $\langle , \rangle$ is the standard Hermitian inner product on $\mathbb{C}^6$.

By direct calculation, it is easy to show that $K_{12}$ has 756 vectors of norm 4, 4032 vectors of norm 6 and 20412 vectors of norm 8 [4].

**Notation 3.8.** We denote the complex isometry group of $K_{12}$ by $G$. In other word, $G$ is the set of all complex linear automorphisms of $\mathbb{C}^6$ that preserve the norm and stabilize $K_{12}$.

**Remark 3.9.** Let $\rho$ be a root of unity in $\mathbb{Z}[\omega]$, i.e., $\rho = \pm 1, \pm \omega, \pm \omega^2$, and let $\lambda_\rho$ be the linear map defined by $v \to \rho \cdot v$. Then $\lambda_\rho$ defines a complex isometry on $K_{12}$ and clearly, it is contained in the center of $G$.

The following result can be found in [4].

**Theorem 3.10.** Let $H$ be the subgroup generated by complex reflections defined by the minimal (norm 4) vectors of $K_{12}$. Note that there are $126 (= 756/6)$ such reflections. Then

1. $H$ acts transitively on the sets of vectors of norms 4, 6 and 8, respectively.
2. $H = G$.
3. The order of $G$ is $2^9 \cdot 3^7 \cdot 5 \cdot 7$.
4. The center of $G$ is given by $\{\lambda_\rho \mid \rho = \pm 1, \pm \omega, \pm \omega^2\}$ and has order 6.

**Remark 3.11.** 1. For each of the 756 minimal vectors $v \in K_{12}$, the sublattice $\mathbb{Z}[\omega]v \cong 2\mathbb{Z}[\omega]$ has exactly 6 minimal vectors. Moreover, $\mathbb{Z}[\omega]v \cong 2\mathbb{Z}[\omega]$ is isometric to $AA_2$ as an integral lattice. Note also that $\mathbb{Z}[\omega]v \cong 2\mathbb{Z}[\omega] \cong AA_2$. 

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is invariant under the action of $\lambda_\omega$. Therefore, we have exactly 126 $\lambda_\omega$-invariant $AA_2$ sublattices in $K_{12}$ and a complex reflection on a minimal vector corresponds to a RSSD-involution associated to a $\lambda_\omega$-invariant $AA_2$-sublattice of $K_{12}$.

2. Let $\nu$ be the anti-automorphism defined by coordinatewise complex conjugation and let $\phi$ be the linear transformation on $\mathbb{C}^6$ defined by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{\omega}
\end{pmatrix}.
\]

Then the anti-automorphism $\nu \circ \phi$ defines an isometry of the real lattice $K_{12}$. Note that $\nu \circ \phi$ preserves the hexacode $H$ and is an involution (see Proposition 4.5 of [9]). Adjoining this anti-automorphism to $G$ will give the real isometry group of $K_{12}$ [3, Section 4.9].

The next theorem follows from Theorem 3.10 and Remark 3.11.

**Lemma 3.12.** Let $g$ be an order 3 element in $O_3(O(K_{12}))$ and let $H$ be the subgroup generated by RSSD involutions associated to $g$-invariant $AA_2$ sublattices of $K_{12}$. Then $H$ is an index 2 subgroup of $O(K_{12})$ and its center has order 6. Moreover, $H$ acts transitively on the sets of vectors of norms 4, 6 and 8, respectively.

We also note that the discriminant group $\mathcal{D}(K_{12}) = K_{12}^*/K_{12} \cong 3^6$. It forms a non-singular quadratic space of minus type with respect to the standard bilinear $\frac{1}{3}\mathbb{Z}/\mathbb{Z}(\cong F_3)$ form [3, 4]. The isometry group $O(K_{12})$ acts on $\mathcal{D}(K_{12})$ as the orthogonal group $O^-(6,3) = \Omega^-(6,3).2^2 = 2.P\Omega^-(6,3).2^2$ and the kernel of the action is a subgroup of order 3. Since $P\Omega^-(6,3) \cong PSU(4,3)$, we have the following result.

**Lemma 3.13** (Section 4.9 of [3] and [4]). Let $K_{12}$ be the Coxeter-Todd lattice of rank 12. Then

1. $O(K_{12})$ has the order $2^{10}.3^7.5.7$ and the shape $(6.PSU(4,3).2).2$. 2. The complex isometry group $G$ is an index 2 subgroup of $O(K_{12})$ and has the shape $6.PSU(4,3).2$. 

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Lemma 3.14. Let $J \cong K_{12}$ and $g$ be defined as in Notation 3.6. Let $H$ be the subgroup of $O(J)$ generated by RSSD involutions associated to $g$-invariant $AA_2$ sublattices in $J$. Then $H$ acts transitively on the set of all $g$-invariant $AA_2$ sublattices in $J$.

Proof. Let $A$ and $B$ be two $g$-invariant $AA_2$ sublattices. Let $\alpha$ be a norm 4 vector of $A$. Then there exists $h \in H$ such that $h\alpha \in B$ by Lemma (3.12). Then $hA = \text{span}_\mathbb{Z}\{h\alpha, h\alpha\} = B$ since $B$ is $g$-invariant and $g \in Z(H)$. □

Lemma 3.15. The centralizer $C_{O(\Lambda)}(g)$ is transitive on the set of all norm 4 vectors in $J$.

Proof. We shall use the notion of hexacode balance to denote the codewords of the Golay code and the vectors in the Leech lattice \([9, 3]\). Namely, we shall arrange the index set $\Omega = \{1, 2, \ldots, 24\}$ into a $4 \times 6$ array such that the six columns form a sextet.

Since there is a unique conjugacy class of order 3 element with trace 6 on $\Lambda$ \([2]\), we may assume

$$g = \begin{bmatrix} 
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow 
\end{bmatrix}$$

and

$$J = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
z_1 & z_2 & z_3 & z_4 & z_5 & z_6 
\end{bmatrix} \in \Lambda \mid x_i + y_i + z_i = 0 \text{ for all } i = 1, \ldots, 6 \right\}.$$

There are two types of norm 4 elements in $J$:

**Type I**

$$\pm \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}$$

where the $\pm 4$'s are supported on a tetrad and on the second, third or fourth rows. There are 36 vectors of this type;

**Type II**

$$\pm \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}$$

Each weight 4 codeword of the Hexacode $\mathcal{H}$ will give $2^4$ vectors of this kind and there are in total $720(=2^4 \times 45)$ norm 4 vectors of this type.
Recall that the subgroup of $M_{24}$ that stabilizes the standard sextet and fixes the first row is given by automorphism group of the Hexacode $\operatorname{Aut}(\mathcal{H}) \cong 3.S_6$ \cite{3, 9}. Let $U$ be the subgroup generated by all $\varepsilon_{\mathcal{O}}$, where $\mathcal{O}$ is a union of any two tetrads of the standard sextet. Then $U \cong \mathbb{Z}_2^5$ and we obtain a subgroup $U.\operatorname{Aut}(\mathcal{H}) \cong 2^5.(3.S_6)$, which commutes with $g$ and is transitive on each of these two types of norm 4 vectors.

Now it remains to show there is an element in $C_{O(\Lambda)}(g)$ which mixes these two types of vectors.

Let $\mathcal{T} = \{T_1, \ldots, T_6\}$ be a sextet given as below:

\[
\begin{array}{cccccc}
\ast & \circ & \bullet & \bullet & \bullet & \bullet \\
\circ & \ast & \times & \times & \times & \times \\
\circ & \ast & \circ & \circ & \circ & \circ \\
\circ & \ast & & & & \\
\end{array}
\]

We take $T_1$ to be the tetrad marked by $\ast$.

Let $\xi_T$ be the linear map defined by

\[
\begin{align*}
v_i & \mapsto v_i - \frac{1}{2}v_{T_1} & \text{if } i \in T_1, \\
v_i & \mapsto \frac{1}{2}v_T - v_i & \text{if } i \in T, T \in \mathcal{T}, T \neq T_1,
\end{align*}
\]

where $\{\pm v_i \mid i \in \Omega = \{1, \ldots, 24\}\}$ is a standard frame of norm 8 vectors in $\Lambda$ and $v_S = \sum_{i \in S} v_i$ for any $S \subset \Omega$. Then $\xi_T$ is an isometry of $\Lambda$ (cf. \cite[p. 288]{3} and \cite[p. 97]{9}).

It is easy to see that $\xi_T$ commutes with $g$ and that

\[
\xi_T \left( \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) = \left( \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right)
\]

Hence $C_{O(\Lambda)}(g)$ is transitive on the set of all norm 4 vectors in $J$. \hfill \Box

**Lemma 3.16** (cf. \cite{3, 9}). *The centralizer $C_{O(\Lambda)}(g)$ is transitive on the set $A = \{A \subset J \mid A \cong AA_2 \text{ and is } g\text{-invariant}\}$.*

**Proof.** Let $A, B \in \mathcal{A}$ and let $\alpha \in A$. Then by Lemma (3.15), there exists $h \in C_{O(\Lambda)}(g)$ such that $h\alpha \in B$. Since $h$ commutes with $g$ and $B$ is $g$-invariant, we have

\[
hA = \text{span}_\mathbb{Z}\{h\alpha, h\gamma\} = \text{span}_\mathbb{Z}\{h\alpha, gh\alpha\} = B
\]

as desired. \hfill \Box
Lemma 3.17. Let \( A \) be a \( g \)-invariant \( AA_2 \)-sublattice in \( J \). There exists an \( EE_8 \)-sublattice \( E \) of \( \Lambda \) such that \( E \cap J = A \). In this case, \( E \cap K \cong EE_6 \) (3.6).

Proof. Clearly, there exists an \( EE_8 \)-sublattice \( X \) such that \( X \cap J \cong AA_2 \) (cf. Case 1: \( DIH_6(14) \) of Appendix A). By Lemma (3.16), \( C_{O(\Lambda)}(g) \) is transitive on \( A := \{ A \subset J \mid A \cong AA_2 \text{ and is } g \text{-invariant} \} \). Thus, there is an \( h \in C_{O(\Lambda)}(g) \) such that \( h(X \cap J) = A \). Now take \( E = hX \). Then \( E \cap J = hX \cap J = h(X \cap J) = A \) since \( hJ = J \). □

Remark 3.18. Let \( E^1 \) and \( E^2 \) be \( EE_8 \) sublattices of \( \Lambda \) such that \( E^1 \cap J = E^2 \cap J \cong A \). Then \( E^1 + E^2 \cong DIH_6(14) \) (cf. [12]). Moreover, \( t_{E^1}t_{E^2} \) acts trivially on \( J \) and hence \( t_{E^1}t_{E^2} \in \langle g_1 \rangle \). Therefore, \( E^2 = g^i_1E^1 \) for some \( i = 1, 2 \).

Theorem 3.19. Let \( M, N \) be \( EE_8 \)-sublattices of \( \Lambda \) such that \( M + N \cong DIH_6(14) \). Then the pair \((M,N)\) is unique, up to the action of \( O(\Lambda) \).

Proof. Let \((M,N)\) be such a pair. Then \( A = M \cap N \cong AA_2 \) and \( g = t_Mt_N \) has order 3 and trace 6 on \( \Lambda \). Hence \( K = \text{Fix}_A(g) \cong K_{12} \) and \( J = \text{ann}_A(K) \cong K_{12} \). Since such a \( g \) is unique up to conjugacy, \( K \) and \( J \) are uniquely determined, up to the action \( O(\Lambda) \).

Now note that \( A \subset K \) since \( A \) is the common \((-1)\)-eigenlattice of \( t_M \) and \( t_N \). By Lemma (3.16), \((A,J)\) is unique, up to the action of \( O(\Lambda) \). Now by Lemma (3.17) and Remark (3.18), the pair \((M,N)\) is unique up to the action of \( O(\Lambda) \). □

Lemma 3.20 ([9]). Let \( g \in O(\Lambda) \) be an element of order 3 and trace 6. Then \( C_{O(\Lambda)}(g) \) has the shape \((2 \times 3^2).PSU(4,3).2\).

Proof. First we note that \( C_{O(\Lambda)}(g) \) stabilizes both \( K = \text{Fix}_A(g) \) and \( J = \text{ann}_A(K) \) and hence \( C_{O(\Lambda)}(g) \) acts on \( J \) and induces a group homomorphism \( \varphi : C_{O(\Lambda)}(g) \to C_{O(J)}(g) \cong H \).

Now let \( A \in \mathcal{A} \). Then there is an \( EE_8 \)-sublattice \( E \) of \( \Lambda \) such that \( E \cap J = A \). In this case, the SSD involution \( t_E \) acts as \( t_A \) on \( J \). Since the RSSD involutions \( t_A, A \in \mathcal{A}, \) generate \( H \), we have \( \text{Im}\varphi = H \).

Claim: \( \text{ker}\varphi = \langle g_1 \rangle \).

Proof. Let \( \sigma \in O(\Lambda) \) such that \( \phi(\sigma) = \text{id}_J \), i.e., \( \sigma \) fixes \( J \) pointwise. Thus \( \sigma \) acts trivially on \( J^*/J \). Moreover, \( \sigma \) acts on \( K \) and must act trivially on \( K^*/K \) since \( \sigma \) preserves the gluing map from \( K \perp J \) to \( \Lambda \). By the
discussion before Lemma (3.13), \( \sigma \in \langle g_1 \rangle \). Therefore, \( C_{O(\Lambda)}(g)/\langle g_1 \rangle \cong H \cong 6.\text{PSU}(4,3).2 \) and \( C_{O(\Lambda)}(g) \) has the shape \((2 \times 3^2).\text{PSU}(4,3).2\). □

**Remark 3.21.** We shall note that for any \( \text{EE}_8 \)-sublattice \( E \) of \( \Lambda \) such that \( E \cap J \in A \), the SSD involution \( t_E \) inverts \( g_1 \) and so \( t_E \) and \( g_1 \) generate a subgroup isomorphic to \( \text{Sym}_3 \).

**Lemma 3.22.** Let \( g \) and \( J \) be defined as in Notation (3.6). Then the centralizer \( C_{O(\Lambda)}(g) \) is transitive on the set
\[
\mathcal{B} = \{ A \perp B \subset J \mid A, B \text{ are } g\text{-invariant } \text{AA}_2 \text{ sublattices of } J \}.
\]

**Proof.** Let \( H \) be the subgroup generated by RSSD involutions associated to \( g\)-invariant \( \text{AA}_2 \) sublattices as defined in Lemma (3.12). Then \( H \) is an index 2 subgroup of \( O(J) \). By Lemma (3.20), the image of \( C_{O(\Lambda)}(g) \) in \( O(J) \) is \( H \cong 6.\text{PSU}(4,3).2 \).

Recall that \( J/(g - 1)J \cong J^*/J \cong 3^6 \) is a non-singular quadratic space of \((-\))-type. The group \( O(J) \) acts on \( J/(g - 1)J \) as the full orthogonal group \( O^{-}(6,3) \cong 2.\text{PSU}(4,3).2 \) while \( C_{O(\Lambda)}(g) \) acts on \( J/(g - 1)J \) as \( 2.\text{PSU}(4,3).2 \).

In \( J/(g - 1)J \), the image of a \( g\)-invariant \( \text{AA}_2 \) sublattice is a non-singular 1-space and the image of \( g\)-invariant \( \text{AA}_2 \perp \text{AA}_2 \) is a definite 2-space. By Witt Theorem, \( O^{-}(6,3) \) is transitive on definite 2-spaces. Therefore, \( O(J) \) is transitive on \( \mathcal{B} \).

Recall from Notation (3.7) that
\[
(2\mathbb{Z}[\omega])^6 \subset J \subset \mathbb{Z}[\omega]^6.
\]
Let \( A \perp B \) be the sum of the first and fourth copies of \( 2\mathbb{Z}[\omega] \cong \text{AA}_2 \). Then the anti-automorphism \( \nu \circ \phi \) defined in Remark (3.11) gives an isometry of the real lattice \( K_{12} \) and by definition, it stabilizes \( A \perp B \). Hence the index \([\text{Stab}_{O(J)}(A \perp B) : \text{Stab}_{H}(A \perp B)] = 2\) and \( C_{O(\Lambda)}(g) \) is transitive on \( \mathcal{B} \). □

**Theorem 3.23.** Let \( M \) and \( N \) be defined as in Notation (3.6) and let \( D \) be the dihedral group generated by \( t_M \) and \( t_N \). Then \( C_{O(\Lambda)}(D) \cong (3 \times 2 \times \text{PSU}(4,2)).2 \).

**Proof.** Set \( G = C_{O(\Lambda)}(D) \). Since \( g \in D \), \( G \) stabilizes \( K = \text{Fix}_{\Lambda}(g) \) and \( J = \text{ann}_{\Lambda}(K) \). In addition, \( G \) centralizes \( t_M \) and hence it stabilizes the \((−1)\)-eigenlattice of \( t_M \), which is \( M \). Therefore, \( G \) acts on \( F = M \cap K \cong \text{AA}_2 \) and \( M \cap J \cong \text{EE}_6 \).
Let \( \alpha \in M \cap J \cong EE_6 \) be a norm 4 vector and \( A = \text{span}_\mathbb{Z}\{\alpha, g\alpha\} \). Let \( E \) be an \( EE_8 \) sublattice such that \( E \cap J = A \). Then \( t_E \) acts as \( t_{(\alpha)} \) on \( M \cap J \), which is a reflection at \( \alpha \). Since \( M \cap E \neq 0 \) and \( t_E \) commutes with \( t_M \), we have \( M \cap E \cong AA_1^2 \) or \( DD_4 \). Thus, \( E \cap F = E \cap M \cap K \cong AA_1 \) and \( t_E \) acts as a reflection at a root, also. Moreover, the \(-1\) map clearly acts on both \( M \cap J \) and \( F \). Thus, \( G \) acts on \( M \cap J \) as the full isometry group \( O(EE_6) = 2 \times \text{Weyl}(E_6) \cong 2 \times \text{PSU}(4,2) \) and acts as \( O(AA_2) \cong 2.S_3 \) on \( F = M \cap K \). Hence \( G = C_{O(\Lambda)}(D) \) has the shape \((3 \times 2 \times \text{PSU}(4,2)).2\). \( \square \)

**Corollary 3.24.** Let \( D \) be the dihedral group generated by \( t_M \) and \( t_N \). Then \( C_{O(\Lambda)}(D)/O_2(C_{O(\Lambda)}(D)) \cong (3 \times \text{PSU}(4,2)).2 \), which is isomorphic to “half” of the Weyl group of type \( A_2 + E_6 \).

The above results may be lifted to a statement about the Monster as follows.

**Theorem 3.25.** Let \( (x, y, z) \) be a triple of elements in the Monster such that \( x, y \in 2A \), \( xy \in 3A \), \( z \in 2B \cap C_M(x, y) \). Then \( (x, y, z) \) is unique up to the conjugation of \( M \). Moreover, \( C_M(x, y, z) \) has a homomorphism onto \( C_{O(\Lambda)}(D)/O_2(C_{O(\Lambda)}(D)) \), which is isomorphic to “half” of the Weyl group of type \( A_2 + E_6 \).

**Proof.** That the triple is unique up to conjugation follows from Proposition 3.3 and 3.19. The last statement follows from Corollary 3.24, Proposition (3.4) and Corollary (3.5). \( \square \)

### 3.2 Isometry groups of \( Q \cong DIH_6(14) \) and \( \text{ann}_\Lambda(Q) \)

Let \( Q \cong DIH_6(14) \) and \( R = \text{ann}_\Lambda(Q) \). We shall determine the isometry group of \( Q \) and \( R \) in this subsection.

**Notation 3.26.** Let \( M \) and \( N \) be \( EE_8 \) sublattices of \( \Lambda \) such that \( Q = M + N \cong DIH_6(14) \) and let \( R = \text{ann}_\Lambda(Q) \).

Let \( g = t_M t_N \in O(\Lambda) \). Set \( K := \text{Fix}_\Lambda(g) \), \( J := \text{ann}_\Lambda(\text{Fix}_\Lambda(g)) \) and \( F := M \cap N \). Then \( K \cong J \cong K_{12} \), \( F \cong AA_2 \) and \( F \) is orthogonal to \( J \). In addition, we denote the order 3 fixed point free isometry in \( O_3(O(K)) \) by \( g_1 \).

We shall use the embedding of \( Q \) and \( R \) in \( \Lambda \) as discussed in Appendix A. Then \( R \) can be obtained by gluing \( AA_2^5 \) with the glue code

\[
C = \text{span}_\mathbb{F}_4\{(1,1,1,1,0), (1,0,\omega,\omega^2,1)\}.
\]
Note that $C$ has 15 codewords of weight 4 and $|R(4)| = 15 \times 2^6 + 5 \cdot 6 = 270$. Moreover, $\text{ann}_R(A) \cong A_2 \otimes D_4$ for an $AA_2$-sublattice $A$ in $AA_2^5$.

**Lemma 3.27.** $O(A_2 \otimes D_4) \cong O(A_2) \circ O(D_4)$, which has order $2^8 \cdot 3^3$.

**Proof.** It is clear that $O(A_2 \otimes D_4)$ has a subgroup isomorphic to $O(A_2) \circ O(D_4)$. Since the minimal vectors of $A_2 \otimes D_4$ have the form $\alpha \otimes \beta$, where $\alpha$ and $\beta$ are roots of $A_2$ and $D_4$, respectively [12], we have $O(A_2 \otimes D_4) \cong O(A_2) \circ O(D_4) \, \Box$

**Lemma 3.28.** The order of $O(R)$ is $2^8 \cdot 3^5 \cdot 5$.

**Proof.** Since $g_1$ is fixed point free on $R$, $R$ has $45 (= 270/6)$ distinct $g_1$-invariant $AA_2$ sublattices.

Now let $A$ be an $AA_2$-sublattice in $AA_2^5$. Then the stabilizer $\text{Stab}_{O(R)}(A)$ of $A$ acts on the sublattice $R' := \text{ann}_R(A) \cong A_2 \otimes D_4$. Let $\varphi : D(A) \to D(R')$ be the gluing map. Then

$$\text{Stab}_{O(R)}(A) = \{ (h, h') \in O(A) \times O(R') \mid \varphi \circ h = h' \circ \varphi \}.$$ 

Let $p_{R'} : \text{Stab}_{O(R)}(A) \to O(R') \cong O(A_2) \circ O(D_4)$ be the restriction map. Then $\ker(p_{R'}) = \langle t_A \rangle \cong \mathbb{Z}_2$ and $\text{Im}(p_{R'})$ is an index 2 subgroup of $O(R')$. Note that the field automorphism $\omega \to \omega^2$ defines an isometry on $R'$ but it does not lift to $O(R)$.

Therefore, $|\text{Stab}_{O(R)}(A)| = (|O(R')| \cdot 2)/2 = 2^8 \cdot 3^3$ and hence $|O(R)| = |\text{Stab}_{O(R)}(A)| \cdot (45) = 2^8 \cdot 3^5 \cdot 5 \, \Box$

**Proposition 3.29.** The isometry group $O(R)$ has the shape $6.PSU(4,2).2$.

**Proof.** First, we note that $R = \text{ann}_{K_{12}}(A)$ for an $AA_2$ sublattice $A$ of $K_{12}$. Hence, $\text{Stab}_{O(K_{12})}(A) = C_{O(K_{12})}(t_A)$ acts on $R$ with the kernel $\langle t_A \rangle$.

Since $O(K_{12}) \cong 6.PSU(4,3).2^2$, by [2, Page 52,53],

$$C_{O(K_{12})}(t_A) \cong 6.PSU(4,2).2 \times 2.$$ 

Hence $C_{O(K_{12})}(t_A)/\langle t_A \rangle \cong 6.PSU(4,2).2$ acts faithfully on $R$.

Since $|6.PSU(4,2).2| = 2^8 \cdot 3^5 \cdot 5$, $O(R)$ has the shape $6.PSU(4,2).2$ by Lemma (3.28). \, \Box$

**Theorem 3.30.** Let $Q \cong DIH_6(14)$ and $J$ be defined as in (3.26). Let $\varphi : O(Q) \to O(J)$ be the restriction map. Then $\text{Im}(\varphi) \cong O(A_2) \circ O(E_6)$ and $\ker(\varphi) \cong \text{Sym}_3$. Therefore, $O(Q)$ has the shape $\text{Sym}_3.(O(A_2) \circ O(E_6))$. 21
Proof. Let $A$ and $J$ be defined as in (3.26). Then $O(Q)$ stabilizes both $A$ and $J$. Hence, by Lemma (A.7), $O(Q)$ stabilizes the set $\{M \cap J, gM \cap J, g^2M \cap J\}$ and the sublattice $M \cap J + gM \cap J \cong A_2 \otimes E_6$. Recall that $M \cap J \cong EE_6$.

By Theorem (3.23), $O(Q)$ acts as the full isometry group $O(E_6)$ on the sublattices $M \cap J, gM \cap J$ and $g^2M \cap J$, respectively. On the other hand, $\langle t_M, t_{gM} \rangle \cong \text{Dih}_6$ acts as permutations on the set $\{M \cap J, gM \cap J, g^2M \cap J\}$. Thus, we have $\text{Im}(\varphi) \cong O(A_2 \otimes E_6) = O(A_2) \circ O(E_6)$.

Clearly, $\text{Ker}(\varphi)$ can be viewed as a subgroup of $O(A) \cong \langle t_A \rangle.\text{Weyl}(A_2)$. Since $[Q : A \perp J] = 3$, $A$ is not an RSSD in $Q$ and thus $t_A$ is not an isometry of $Q$. Hence we have $\text{Ker}(\varphi) < \text{Weyl}(A_2)$.

Let $p_J : Q \rightarrow \mathcal{D}(J)$ and $p_A : Q \rightarrow \mathcal{D}(A) \cong 2^2 \cdot 3$ be the natural maps. Let $\xi : p_A(Q) \rightarrow p_J(Q)$ be the gluing map from $A \perp J$ to $Q$. Since $O(Q)$ stabilizes $A \perp J$, we have

$$O(Q) = \{y \in O(A) \times O(J) \mid y \text{ preserves the gluing map, i.e., } y\xi = \xi y\}.$$ 

Thus $y \in \text{Ker}(\varphi)$ if and only if $y \in O(A) \times O(J)$ acts trivially on $J$ and $y\xi = \xi y$. Therefore, we have $\text{Ker}(\varphi) = \{y \in O(A) \mid \xi = y\xi\}$.

By the discussion in Case 1 of Appendix A, we know that $p_A(Q) = 2A^*/A$ is a subgroup of order 3. Since $\text{Weyl}(A_2)$ fixes all cosets in $2AA_2^*/AA_2$, we have $\text{Ker}(\varphi) \cong \text{Weyl}(A_2) \cong \text{Sym}_3$. □

Remark 3.31. We note that $\mathbb{Z}\alpha$ is an RSSD sublattice in $Q$ for any norm 4 vector $\alpha \in A$. The reason is as follows:

Let $\alpha \in A$ with $(\alpha, \alpha) = 4$. We want to show that $2Q \leq \text{ann}_Q(\alpha) + \mathbb{Z}\alpha$. That is equivalent to $(\alpha, Q) \leq 2\mathbb{Z}$. First we notice that the index $[Q : J + A] = 3$ since $\mathcal{D}(J) \cong 3^6$, $\mathcal{D}(A) \cong 2^2 \times 3$ and $\det(Q) = 2^23^5$. Moreover, we have $(\alpha, J + A) \leq 2\mathbb{Z}$ since $A \cong AA_2$ is doubly even. Also, there is an integer $r$ so that $(\alpha, Q) = r\mathbb{Z}$. Since $3Q \leq J + A$, $3r\mathbb{Z} \leq 2\mathbb{Z}$, whence $r$ is even. We conclude that $(\alpha, Q) \leq 2\mathbb{Z}$.

4 6A-triples

In this section, we consider a triple $(x, y, z)$ in $\mathbb{M}$ such that $x, y \in 2A$, $xy \in 6A$ and $z \in 2B \cap C_\mathbb{M}(x, y)$. We shall study the orbits of such triples under the action of $\mathbb{M}$.

Notation 4.1. Let $\mathcal{S}_A = \{(x, y, z) \mid x, y \in 2A, xy \in 6A, z \in 2B \cap C_\mathbb{M}(x, y)\}$. Note that the Monster $\mathbb{M}$ acts on $\mathcal{S}_A$ by conjugation.
Take $(x, y, z) \in S_4$. Then $z \in 2B$ and we may again assume $z$ acts as $1$ on $V_\Lambda^+$ and as $-1$ on $V_\Lambda^{T,+}$ by conjugation. Moreover, $x = \tau_e$ and $y = \tau_f$ for some $cc_{\frac{1}{2}} e$ and $f$ in $V_\Lambda^+$, by the Miyamoto bijection [15, 21]. By our assumption, $xy \in 6A$ and thus $(xy)^3 = 2A$. There are two cases:

1. $(xy)^3 \in O_2(C_M(z)) \cong 2^{1+24}$;
2. $(xy)^3 \notin O_2(C_M(z))$.

**Case 1 (6A.1):** $(xy)^3 \in O_2(C_M(z)) \cong 2^{1+24}$.

In this case, $\xi \circ \mu(xy)$ has order $3$ in $O(\Lambda)/\{\pm 1\}$. Thus, by the same arguments as in (3.1) and (3.3), we may assume $e = e_M$ and $f = \varphi_b(e_N)$ for some $EE_8$-pair $(M, N)$ and $b \in N^*$ such that $M + N \cong DIH_6(14)$. Since $xy \in 6A$, we have $(e, f) = \frac{5}{2} \alpha$ and hence $(b, M \cap N) \notin 2Z$ by (3.2). Then $b = \frac{1}{2}\alpha \mod 2N$ for some $\alpha \in M \cap N(4)$.

**Theorem 4.2.** Let $(x, y, z) \in S_4$. Suppose $(xy)^3 \in O_2(C_M(z))$. Then $(x, y, z)$ is conjugate to $(\tau_{e_M}, \tau_{\varphi_a/2(e_N)}, z)$ for some $EE_8$-pair $(M, N)$ such that $M + N = DIH_6(14)$ and $\alpha \in M \cap N(4)$.

**Proposition 4.3.** Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong DIH_6(14)$. Let $\alpha \in N(4)$ such that $\{\frac{1}{2}\alpha, M \cap N\} \notin 2Z$. Then

$$\xi \circ \mu : C_{Aut(V^2)}(\tau_{e_M}, \tau_{\varphi_a/2(e_N)}, z) \to C_{Stab_{O(\Lambda)}(Z\alpha)}(t_M, t_N)/\{\pm 1\}$$

is surjective.

**Proof.** By Lemma (3.2), $(e_M, \varphi_a/2(e_N)) = \frac{5}{2} \alpha$ and $\tau_{e_M}, \tau_{\varphi_a/2(e_N)}$ generate a dihedral group of order $12$ in $Aut(V^2)$. In this case, $(\tau_{e_M} \tau_{\varphi_a/2(e_N)})^3 = \tau_{\omega^+(\alpha)}$ or $\tau_{\omega^-(\alpha)} [19]$. Thus, the subgroup generated by $\tau_{e_M}, \tau_{\varphi_a/2(e_N)}, z$ is the same as the group generated by $\tau_{e_M}, \tau_{e_N}, \tau_{\omega^+(\alpha)}$ and $z$. Recall that $\tau_{\omega^+(\alpha)} z = \tau_{\omega^-(\alpha)} [18]$. The result now follows by Lemma (2.9) and Proposition (3.4). □

**Case 2 (6A.2):** $(xy)^3 \notin O_2(C_M(z))$.

Then $\xi \circ \mu(xy)$ has order $6$ in $O(\Lambda)/\{\pm 1\}$. By the analysis in [12], $e = \varphi_a(e_M)$ and $f = \varphi_b(e_N)$ for some $EE_8$-pair $(M, N)$ in $\Lambda$ such that $M + N \cong DIH_{12}(16)$ and $a \in M^*$, $b \in N^*$

As in Proposition 3.3, we may also assume $e = e_M$, up to conjugation. Since $M \cap N = 0$ and $DIH_{12}(16)$ is a direct summand of $\Lambda$, there is a $\beta \in \Lambda$ such that $P_N(\beta) = b \mod 2N$ and $(P_M(\beta), M) \in 2Z$ by (D.2). Then, $\varphi_a(f) = \varphi_a(\varphi_b(e_N)) = e_N$ and $\varphi_b(e) = \varphi_b(e_M) = e_M$.

Thus, up to conjugation, $e = e_M$ and $f = e_N$ for some $EE_8$-pair $(M, N)$ such that $M + N \cong DIH_{12}(16)$.
Theorem 4.4. Let \((x, y, z) \in \mathcal{S}4\). Suppose \((xy)^3 \notin O_2(C_M(z))\). Then \((x, y, z)\) is conjugate to \((\tau e_M, \tau e_N, z)\) for some EE_8-pair \((M, N)\) such that \(M + N \cong DIH_{12}(16)\).

Proposition 4.5. Let \((M, N)\) be an EE_8-pair in \(\Lambda\) such that \(M + N \cong DIH_{12}(16)\). Then

\[
\xi \circ \mu : C_{Aut(V^2)}(\tau e_M, \tau e_N, z) \to C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle
\]

is surjective.

Proof. Let \(s \in C_{O(\Lambda)}(t_M, t_N)/\langle \pm 1 \rangle\). Then by Lemma (2.11), there is \(g \in Aut(V^2)\) such that \(g\) stabilizes both \(V_M^+\) and \(V_N^+\) and \(\xi \circ \mu(g) = s\).

As in Proposition 3.4, we may assume \(g(e_M) = e_M\) and \(g(e_N) = \varphi_b(e_N)\) for some \(b \in N^*\). Since \(M \cap N = 0\) and \(DIH_{12}(16)\) is a direct summand of \(\Lambda\), there is a \(\beta \in \Lambda\) such that \(P_N(\beta) = b\) mod \(2N\) and \((P_M(\beta), M) \in 2\mathbb{Z}\) by (D.2). Then, \(\varphi_\beta(g(e_N)) = \varphi_\beta(\varphi_b(e_N)) = e_N\) and \(\varphi_\beta(g(e_M)) = \varphi_\beta(e_M) = e_M\).

Therefore, \(g' = \tau_{\omega + (\beta)}g \in C_{Aut(V^2)}(\tau e_M, \tau e_N, z)\) and \(\xi \circ \mu(g') = s\). \(\square\)

4.1 Case 6A.1 and \(M + N \cong DIH_6(14)\)

First we consider the case 6A.1. In this case, \(M + N \cong DIH_6(14)\) and \(f = e_M\) and \(f = \varphi_{\tau_2}(e_N)\) for some \(\alpha \in (M \cap N)(4)\).

Proposition 4.6. Let \(M\) and \(N\) be defined as in Notation (3.6) and let \(\alpha \in M \cap N\) be a norm 4 vector. Then \(C_{Stab_{O(\Lambda)}(Z\alpha)}(t_M, t_N) \cong 2 \times PSU(4,2).2\).

Proof. First, we note that \(t_M, t_N\) stabilize \(Z\alpha\) since \(\alpha \in M \cap N\). Moreover, the group

\[
C_{Stab_{O(\Lambda)}(Z\alpha)}(t_M, t_N) = \{h \in C_{O(\Lambda)}(t_M, t_N) \mid h(Z\alpha) = Z\alpha\}.
\]

By Theorem (3.23), \(C_{O(\Lambda)}(t_M, t_N)\) has the shape \(2.(3 \times PSU(4,2)).2\) which acts as \(2.S_3\) on \(M \cap N\) and acts as the isometry group of \(ann_M(M \cap N) \cong EE_6\). Thus, the subgroup that fixes \(Z\alpha\) has the shape \(2 \times PSU(4,2).2\). \(\square\)

By Lemma (2.11) and Proposition (4.3), we have the corollary.

Corollary 4.7. Let \(M, N\) and \(\alpha\) be defined as in Proposition (4.6). Let \(z\) be the automorphism of \(V^2\) such that \(z|_{V_M^+} = 1\) and \(z|_{V_M^{T, +}} = -1\) as defined as
in (2.8). Then there is a homomorphism that maps \( C_{Aut(V)}(\tau_{e_M}; \tau_{\varphi_{Z}(e_N)}, z) \) onto 
\[
C_{Stab_{O}(\Lambda)}(t_M, t_N)/O_2(C_{Stab_{O}(\Lambda)}(t_M, t_N)) \cong PSU(4, 2).2.
\]
The kernel \( K \) is a 2-group of order \( 2^{11} \) and we have an exact sequence 
\[
1 \to \langle z \rangle \to K \to \{ \varphi_{\beta} \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z} \} \to 1.
\]

**Proof.** To compute the kernel, note that the natural map \( \Lambda \to Hom(M + N, \mathbb{Z}_2) \) is onto since \( M + N \) is a direct summand and \( det(\Lambda) = 1. \square \)

By Theorem 4.2 and Corollary 4.7, we have our main theorem as follows.

**Theorem 4.8.** Let \( (x, y, z) \) be a triple of elements in the Monster such that \( x, y \) in \( 2A \), \( xy \) in \( 6A \), \( z \) in \( 2B \cap C_M(x, y) \) and \( (xy)^3 \in O_2(C_M(z)) \). Then such triples form one orbit under the conjugation action of \( M \) and \( C_M(x, y, z) \) has a homomorphism onto 
\[
C_{Stab_{O}(\Lambda)}(t_M, t_N)/O_2(C_{Stab_{O}(\Lambda)}(t_M, t_N)) \cong PSU(4, 2).2.
\]
The kernel \( K \) is a 2-group of order \( 2^{11} \) and we have an exact sequence 
\[
1 \to \langle z \rangle \to K \to \{ \varphi_{\beta} \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z} \} \to 1.
\]

### 4.2 Case 6A.2 and \( M + N \cong DIH_{12}(16) \)

Next we consider the case 6A.2. In this case, \( M + N \cong DIH_{12}(16), e = e_M \) and \( f = e_N. \)

**Notation 4.9** ([12]). 1. Let \( M \) and \( N \) be \( EE_8 \) sublattice of \( \Lambda \) such that \( Q' := M + N \) is isometric to \( DIH_{12}(16) \) (Table 1 of [12]). Then \( \text{ann}_M(N) \cong \text{ann}_N(M) \cong AA_2 \). We also denote \( R' := \text{ann}_A(Q') \).

2. Let \( F' := \text{ann}_M(N) \perp \text{ann}_N(M) \) and \( J := \text{ann}_{Q'}(F') \). Then \( J \cong K_{12} \) is isometric to the Coxeter-Todd lattice and \( Q' \) contains a sublattice isometric to \( F' \perp J \).

3. \( D(Q') \cong D(R') \cong 6^4 \).

By explicit calculation in the Leech lattice (see Appendix A), one can show that \( R' \) contains a sublattice isometric to \( AA_2^{14} \) and \( R' \) is isometric to 
\[
\text{span}_\mathbb{Z}\{ AA_2^{14}, \frac{1}{2}(\beta_1, \beta_1, \beta_1, \beta_1), \frac{1}{2}(\beta_2, \beta_2, \beta_2, \beta_2) \},
\]
25
where $\beta_1 = \sqrt{2}\alpha_1$, $\beta_2 = \sqrt{2}\alpha_2$ and $\{\alpha_1, \alpha_2\}$ is a set of fundamental roots for $A_2$. In fact, $R' \cong A_2 \otimes D_4$ (see (A.1)).

**Notation 4.10** ([12]). Let $t_M$ and $t_N$ be the SSD involutions associated to $M$ and $N$. Then the group $\Delta := \langle t_M, t_N \rangle \cong \text{Dih}_{12}$. Set $h := t_M t_N$, $g := h^2$ and $u = h^3$. Then $h$ has order 6, $g$ has order 3 and $u$ has order 2. The traces of $h$, $g$ and $u$ on $\Lambda$ are 2, 6 and 8, respectively. Note also that $\langle u \rangle = Z(\Delta)$ and $g = uh^{-1}$.

**Proposition 4.11.** Let $(M, N)$ be an $EE_8$-pair in $\Lambda$ such that $M + N \cong \text{Dih}_{12}(16)$. Then the pair $(M, N)$ is unique up to the action of $O(\Lambda)$.

**Proof.** Let $(M, N)$ be such a pair. Then $g = (t_M t_N)^2$ has order 3 and trace 6 on $\Lambda$. Let $K := \text{Fix}_\Lambda(g) \cong K_{12}$ and $J := \text{ann}_\Lambda(K) \cong K_{12}$. Since such a $g$ is unique up to conjugacy, $K$ and $J$ are uniquely determined, up to the action $O(\Lambda)$.

Let $g'$ be the fixed point free order 3 element in $O_3(O(K))$. Then, $F_M := M \cap K$ and $F_N := N \cap K$ are both $g'$-invariant $AA_2$-sublattices in $K$ and $F_M$ is orthogonal to $F_N$. By Lemma (3.22), $F_M \perp F_N \perp J$ is unique up to the action of $O(\Lambda)$. Since $M + N$ is a direct summand of $\Lambda$, we have $\text{ann}_\Lambda(\text{ann}_\Lambda(F_M \perp F_N \perp J)) = M + N \cong \text{Dih}_{12}(16)$.

By Lemma (3.17), there exists $EE_8$ sublattices $E^1$ and $E^2$ such that $E^1 \cap K = F_M$ and $E^2 \cap K = F_N$. Then $E^1 + E^2 \cong \text{Dih}_{12}(16)$ (cf. [12]) and thus $E^1 + E^2 = M + N$. Therefore, $E^1 = g^i M$ and $E^2 = g^j N$ for some $i, j = 0, 1, 2$ (see (A.6)) and $(E^1, E^2)$ is conjugate to $(M, N)$ by the action of the dihedral group $\langle t_M, t_N \rangle$. Hence $(M, N)$ is unique up to the action of $O(\Lambda)$. □

Next we recall few facts about the lattice $\text{Dih}_{12}(16)$ from [12].

**Lemma 4.12** ([12]). Let $h$, $g$ and $u$ be defined as in Notation (4.10). Then

1. $N \cap g^{-1} M \cong DD_4$ and $N + g^{-1} M \cong \text{Dih}_4(12)$.
2. $u = t_N t_{g^{-1} M}$ is an SSD involution associated to an $EE_8$ sublattice $E$ of $N + g^{-1} M$

**Lemma 4.13** (Proposition 6.44 of [12]). Let $P = g^{-1} M \cap J$ and let $P^-(t_N)$ be the $(-1)$-eigenspace of $t_N$ in $P$. Then $P^-(t_N) = N \cap g^{-1} M \cong DD_4$ and $N \cap g^{-1} M < J$. 

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Lemma 4.14. Let $\Delta = \langle t_M, t_N \rangle$ be defined as in Notation (4.10). The centralizer $C_{O(\Lambda)}(\Delta)$ has the shape

$$(3 \times 2 \times 2. (\text{Alt}_4 \times \text{Alt}_4). 2). 2,$$

which is an index 2 subgroup of $2.(\text{Dih}_6 \times O(D_4)).$

Proof. Set $G_1 := C_{O(\Lambda)}(\Delta)$.

Let $L = M + gM$. Then $L \cong \text{Dih}_6(14)$ and the group $\langle t_M, t_gM \rangle \cong \text{Dih}_6$ is a subgroup of $\Delta$. By the analysis of $\text{Dih}_6(14)$, the centralizer $C_1 := C_{O(\Lambda)}(\langle t_M, t_gM \rangle) \cong (3 \times 2 \times \text{PSU}(4, 2)). 2$ (5) and it stabilizes $g^i M \cap J$ for each $i = 0, 1, 2$ and acts as $O(EE_6) (\cong 2 \times \text{PSU}(4, 2). 2)$ on each $g^i M \cap J$, where $J = \text{ann}_\Lambda(K)$ and $K = \text{Fix}_\Lambda(g)$.

Note that $G_1$ also centralizes $t_N$. Thus $G_1$ stabilizes $N$ and $N \cap g^{-1} M \cong \text{DD}_4$. By Lemma 4.13, $N \cap g^{-1} M < J \cap g^{-1} M \cong EE_6$. Therefore, $G_1$ acts as the stabilizer of $N \cap g^{-1} M \cong \text{DD}_4$ on $g^{-1} M \cap J$, which is isomorphic to $2.O(\text{DD}_4) \cong 2.O(\text{D}_4)$. Note that $O(\text{D}_4) \cong \text{Weyl}(F_4) \cong O^+(4, 3)$ and has the shape $2.(\text{Alt}_4 \times \text{Alt}_4).2^2$ (see (B.3) and Appendix B of [14]).

Since $u$ commutes with $D = \langle t_M, t_gM \rangle$, we have

$$G_1 = C_{O(\Lambda)}(D_4) = C_{O(\Lambda)}(\langle u, t_M, t_gM \rangle) = C_{C_1}(u).$$

Next we study the action of $u$ on $(g^{-1} M) \cap J$. First we note that $u = t_E$ acts as $-1$ on $g^{-1} M \cap K \cong A_2$ and $E \cap g^{-1} M \cong \text{DD}_4$. Thus $E \cap g^{-1} M \cap J \cong 2A_2$. Notice that $E \cap g^{-1} M \cap J$ is the $(-1)$-eigenvector of $u$ in $g^{-1} M \cap J$ and $N \cap g^{-1} M$ is the fixed point sublattice of $u$ in $g^{-1} M \cap J$. Hence $u$ acts as $-t_{N \cap g^{-1} M}$ on $(g^{-1} M) \cap J$. Thus we have

$$C_{C_1}(u) / \langle -1, g_1 \rangle = C_{C_1}(t_{N \cap g^{-1} M}) / \langle -1, g_1 \rangle$$

since $-1$ is in the center of $C_1$. Recall that $g_1$ is an order 3 isometry of $\Lambda$ as defined in (5) of (3.6), which acts fixed point free on $K$ and trivially on $J$. Note also that $C_1 / \langle -1, g_1 \rangle \cong \text{PSU}(4, 2). 2 \cong \text{Weyl}(E_6)$ by (5). Since $t_{N \cap g^{-1} M}$ has trace $-2$ on $g^{-1} M \cap J$, by the character table of $\text{PSU}(4, 2)$ [2, Page 26], we know that $C_{C_1}(u) / \langle -1, g_1 \rangle$ has the order $2 \times 576$ and has the shape $2.(\text{Alt}_4 \times \text{Alt}_4).2^2$.

Thus, $C_{O(\Lambda)}(\Delta)$ has the order $12 \cdot 576 = 2^9 \cdot 3^3$ and has the shape

$$(3 \times 2 \times 2. (\text{Alt}_4 \times \text{Alt}_4). 2). 2$$

as desired. □
By Lemma (2.11), Theorem 4.4 and Proposition 4.5, we deduce the main theorem of this section.

**Theorem 4.15.** Let \((x, y, z)\) be a triple of elements in the Monster such that \(x, y \in 2A\), \(xy \in 6A\), \(z \in 2B \cap C_{24}(x, y)\) and \((xy)^3 \notin O_2(C_{24}(z))\). Then such triples form one orbit under the conjugation action of \(\mathbb{M}\) and \(C_{24}(x, y, z)\) has a homomorphism onto

\[C_{O(\Lambda)}(\Delta)/\langle \pm 1 \rangle \cong (3 \times 2.(\text{Alt}_4 \times \text{Alt}_4).2).2.\]

The kernel \(\bar{K}\) is a group of order \(2^9\) and the sequence

\[1 \to \langle z \rangle \to \bar{K} \to \{ \varphi_\beta \mid \beta \in \Lambda \text{ and } \langle \beta, M + N \rangle \in 2\mathbb{Z} \} \to 1\]

is exact.

**Proof.** The kernel can be computed by the same argument as in Corollary (4.7). \(\square\)

### 4.3 Isometry groups of \(Q' \cong DIH_{12}(16)\) and \(\text{ann}_\Lambda(Q')\)

**Notation 4.16.** Let \(Q' = DIH_{12}(16)\) and \(R' = \text{ann}_\Lambda(Q')\). Let \(M\) and \(N\) be \(EE_8\)-sublattices of \(Q'\) such that \(Q' = M + N\). As in Notation (4.10), we set \(h = t_M t_N\), \(g = h^2\) and \(u = h^3\). Then \(\langle u \rangle\) is the center of \(\langle t_M, t_N \rangle\). Let \(E\) be the \((-1)\)-eigenlattice of \(u\) in \(Q'\). Then \(E\) is also an \(EE_8\)-sublattice of \(Q'\) and \(E \cap M \cong E \cap N \cong DD_4\) (see [12]).

Set \(F_M := \text{ann}_M(N)\) and \(F_N := \text{ann}_N(M)\). Then by [12], we have \(F_M \cong F_N \cong AA_2\) and \(J := \text{ann}_{Q'}(F_M \perp F_N) \cong K_{12}\). In addition, \(E \cap J \cong AA_2 \perp AA_2\).

**Lemma 4.17.** The isometry group \(O(R') \cong O(A_2) \circ O(D_4)\) and has the order \(2^8 \cdot 3^3\).

**Proof.** Since \(R' \cong A_2 \otimes D_4\) (A.1), we have \(O(R') \cong O(A_2 \otimes D_4) \cong O(A_2) \circ O(D_4)\) by (3.27). \(\square\)

**Proposition 4.18.** Let \(M, N\) be defined as in (4.16). Then the image of \(\langle t_M, t_N \rangle\) in \(O(Q')\) is normal in \(O(Q')\).

**Proof.** It follows from the classification of \(EE_8\)-sublattices in \(Q'\) (Lemma (A.6)). \(\square\)
Proposition 4.19. Let \( y \in O(Q') \). Then \( y \) normalizes the subgroup \( \langle g \rangle \) and thus it stabilizes the sublattice \( J = \text{ann}_Q(\text{Fix}_{Q'}(g)) \).

Proof. Since \( y \) normalizes \( \langle t_M, t_N \rangle \) and \( \langle g \rangle \) is the unique subgroup of order 3 in \( \langle t_M, t_N \rangle \cong \text{Dih}_{12} \), \( y \) also normalizes \( \langle g \rangle \). \( \square \)

Remark 4.20. By the discussion in Case 2 of Appendix A, it is easy to see that \( R'' := \text{ann}_{Q'}(E) = \text{ann}_J(E \cap J) \cong A_2 \otimes D_4 \cong R' \).

Theorem 4.21. Let \( Q' \cong \text{DIH}_{12}(16) \) and \( E \) be defined as in Notation (4.16). Let \( R'' = \text{ann}_{Q'}(E) \) and let \( \varphi' : O(Q') \to O(R'') \) be the restriction map. Then \( \text{Im}(\varphi') = O(R'') \cong O(A_2) \circ O(D_4) \) and \( \text{Ker}(\varphi') \cong 2.O^+(4,2) \). Therefore, \( O(Q') \) has the shape \( 2. O^+(4,2). (O(A_2) \circ O(D_4)) \).

Proof. First we note that \( O(Q') \) stabilizes the \( E E_4 \) sublattice \( E \) and hence it also stabilizes \( R'' = \text{ann}_{Q'}(E) \cong A_2 \otimes D_4 \). Let \( p_E : Q' \to \text{D}(E) \) and \( p_{R''} : Q' \to \text{D}(R'') \) be the natural maps and let \( \xi' : p_E(Q') \to p_{R''}(Q') \) be the gluing map from \( E \perp R'' \to Q' \). Then

\[
O(Q') = \{ y \in O(E) \times O(R'') \mid y\xi' = \xi'y \}.
\]

By Lemma (4.14), we know that

\[
R'' \cap N = R'' \cap g^{-1}M = N \cap g^{-1}M \cong D D_4
\]

and \( O(Q') \) acts as the full isometry group \( O(D_4) \) on \( R'' \cap N \). Similarly, we also have \( R'' \cap g^iN = R'' \cap g^{i-1}M \cong D D_4 \) and \( O(Q') \) acts as \( O(D_4) \) on each of \( R'' \cap g^iN \) for \( i = 0, 1, 2 \). Moreover, the dihedral group \( \langle t_M, t_N \rangle \) acts on \( R'' \) as \( \text{Dih}_6 \) with the kernel \( \langle t_E \rangle \). More precisely, \( \langle t_N, t_M \rangle \) acts as permutations on the set \( \{ R'' \cap M, R'' \cap gM, R'' \cap g^2M \} = \{ R'' \cap N, R'' \cap gN, R'' \cap g^2N \} \). Hence, \( O(Q') \) acts on \( R'' \) as the full isometry group \( O(R'') \cong O(A_2) \circ O(D_4) \) and we have

\[
\text{Im}(\varphi') = O(R'') \cong O(A_2) \circ O(D_4).
\]

In this case, the kernel of \( \varphi' \) is given by

\[
\text{Ker}(\varphi') = \{ y \in O(E) \mid y \text{ fixes } p_E(Q') \text{ pointwise } \}.
\]

By our discussion in Case 2 of Appendix A, \( [Q' : E \perp R''] = 2^4 \) and \( p_E(Q') \) forms a 4-dimensional non-degenerate quadratic spaces of \((+)-\)type; in fact, it is isometric to the 2-part of \( \text{D}(AA_2 \perp AA_2) \) and is a direct sum of two nonsingular 2-dimensional quadratic spaces of \((-)-\)type. Recall that \( O(E) \cong
2. $O^+(8, 2)$ and hence the subgroup $\text{Ker}(\varphi')$ that fixes $p'(Q')$ pointwise is, by Witt’s theorem, isomorphic to $2. O^+(4, 2)$. This group $\text{Ker}(\varphi')$ is actually isomorphic to a direct product $2 \times O^+(4, 2)$ (for if $Y$ denotes the normal subgroup of $\text{Ker}(\varphi')$ generated by reflections on $R$, $Y \cong \text{Dih}_6 \times \text{Dih}_6$. Thus $\text{Ker}(\varphi')/Y$ has order 4. Now use (C.3)). □

On the lattice $\text{ann}_{Q'}(J) \cong AA_2 \perp AA_2$, $t_M$ negates one of the $AA_2$ summands and fixes the other, while $t_N$ behaves analogously, negating the summand which $t_M$ fixes. It is clear from the analysis of $\text{Ker}(\varphi')$ that there exists an element of $\text{Ker}(\varphi')$ which interchanges the two $AA_2$ summands. Therefore, since $\langle t_M, t_N \rangle$ is normal in $O(Q')$, $O(Q')$ induces by conjugation the full automorphism group of $\langle t_M, t_N \rangle$ ($\text{Aut}(\text{Dih}_12) \cong \text{Dih}_12$).

**Corollary 4.22.** $O(Q')$ leaves invariant the sublattice $E \perp R''$ and the restriction maps to $E, R'$ give an embedding of $O(Q')$ in $(2, [\text{Dih}_6 \times O^+(4, 2)]) \times (\text{Dih}_6 \times O^+(4, 3))$.

**Remark 4.23.** We have $O(D_4) \cong \text{Weyl}(F_4) \cong O^+(4, 3)$.

The group $O(Q')$ acts on $\mathcal{D}(Q') \cong 6^4 \cong 2^4 \times 3^4$ as the full isometry group $O^+(4, 2) \times O^+(4, 3)$ and the kernel of the action is given by $\langle t_M, t_N \rangle \cong \text{Dih}_{12}$.

## A Embeddings of $\text{DIH}_6(14)$ and $\text{DIH}_{12}(16)$ into the Leech lattice

Let $L \cong AA_2^{12}$ be the orthogonal sum of 12 copies of $AA_2$ and let $\mathcal{H}$ be the hexacode over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ with the generating matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
1 & \omega & 0 & \omega^2 & 0 & \omega^2
\end{pmatrix}
$$

Let $\mathcal{D}$ be the ternary code generated by

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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Then $\mathcal{D}$ is isomorphic to a sum of 3 ternary tetra-codes.

We can construct the Leech lattice by using $L$ and $\mathcal{D}$ and $\mathcal{H} \perp \mathcal{H}$ as glue codes [17].

**Case 1:** $DIH_6(14)$.

Let $X$ be the sum of the first 5 copies of $AA_2$. Then by the above construction,

$$Q := DIH_6(14) \cong \text{ann}_\Lambda(X).$$

Moreover,

$$R := \text{ann}_\Lambda(Q) = \text{ann}_\Lambda(S),$$

where $S$ is the sum of the last 7 copies of $AA_2$.

Note that $[R : AA_2^5] = 2^4$. In fact, one can glue $AA_2^5$ to $R$ by using the glue code generated by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & \omega & \omega^2 & 0 & \omega^2 \end{pmatrix}$$

$$\Lambda$$

$$\begin{array}{c} \nearrow \\ EE_8 \nearrow J' \quad J' \quad J \searrow \quad C' \quad C \searrow \quad AA_2^{12} \end{array}$$

**Case 2:** $DIH_{12}(16)$.

Let $Q' := DIH_{12}(16)$ and $R' := \text{ann}_\Lambda(Q')$. Then $\mathcal{D}(Q') \cong \mathcal{D}(R') \cong 6^4$.

Let $X'$ be the sum of the 1st, 2nd, 3rd and 4th copies of $AA_2$. Then

$$DIH_{12}(16) \cong \text{ann}_\Lambda(X').$$

By explicit calculation in the Leech lattice, one can show that $R'$ contains a sublattice isometric to $AA_2^{14}$ and $R'$ is isometric to

$$\text{span}_\mathbb{Z}\{AA_2^{14}, \frac{1}{2}(\beta_1, \beta_1, \beta_1, \beta_1), \frac{1}{2}(\beta_2, \beta_2, \beta_2, \beta_2)\},$$

where $\beta_1 = \sqrt{2}\alpha_1$, $\beta_2 = \sqrt{2}\alpha_2$ and $\{\alpha_1, \alpha_2\}$ is a set of fundamental roots for $A_2$. Note that $\mathcal{D}(AA_2) = 2 \cdot 6$.

By the analysis in [12], $Q'$ contains a sublattice

$$AA_2 \perp AA_2 \perp J,$$

where $J$ is isometric to the Coxeter-Todd lattice.
Remark A.1. The lattice $R'$ is isometric to $A_2 \otimes D_4$.

Proof. Recall that $R'$ is isometric to

$$\text{span}_\mathbb{Z}\{AA_2^{14}, \frac{1}{2}(\beta_1, \beta_1, \beta_1), \frac{1}{2}(\beta_2, \beta_2, \beta_2)\},$$

where $\beta_1 = \sqrt{2}\alpha_1, \beta_2 = \sqrt{2}\alpha_2$ and $\{\alpha_1, \alpha_2\}$ is a set of fundamental roots for $A_2$. Then the sublattices

$$D^1 = \text{span}\{ (\beta_1, 0, 0, 0), (0, \beta_1, 0, 0), (0, 0, \beta_1, 0), \frac{1}{2}(\beta_1, \beta_1, \beta_1, \beta_1)\} \text{ and }$$

$$D^2 = \text{span}\{ (\beta_2, 0, 0, 0), (0, \beta_2, 0, 0), (0, 0, \beta_2, 0), \frac{1}{2}(\beta_2, \beta_2, \beta_2, \beta_2)\}$$

are both isometric to $DD_4$ and $D^2 = g(D^1)$, where $g = (\sigma, \sigma, \sigma, \sigma)$ and $\sigma \in O(AA_2)$ is given by $\beta_1 \rightarrow \beta_2 \rightarrow -(\beta_1 + \beta_2) \rightarrow \beta_1$. Then $D^1 + D^2 \cong A_2 \otimes D_4$ by Lemma 3.1 [12]. Hence $R' \cong A_2 \otimes D_4$ since they have the same determinant. $\Box$

In this appendix, we shall determine all $EE_8$-sublattices in $Q' \cong DIH_{12}(16)$ and $Q \cong DIH_4(14)$.

Lemma A.2. Let $J$ be isometric to the rank 12 Coxeter-Todd lattice $K_{12}$. Then $J$ contains no sublattices isometric to $EE_8$.

Proof. Suppose $Y \cong EE_8$ is a sublattice of $J$. Then by Lemma A.3 of [12], the 3-rank of $\mathcal{D}(Y)$ is at least $6 + 8 - 12 = 2$. It is a contradiction since $\mathcal{D}(Y) \cong 2^8$. Note that the 3-rank of $\mathcal{D}(J) = 6$, $\text{rank}(J) = 12$ and $\text{rank}(E) = 8$. $\Box$

Lemma A.3. Let $A$ be a rank 2 even lattice with $\mathcal{D}(A) \cong 2^2 \cdot 3$ and minimal norm at least 4. Then $A \cong AA_2$.

Proof. Let $\left(\begin{array}{cc} 2a & c \\ c & 2b \end{array}\right)$ be the Gram matrix of $A$. Then $4ab - c^2 = 12$ or $ab - (c/2)^2 = 3$. Hence, $c/2$ is an integer. Therefore, $A' = \frac{1}{\sqrt{2}}A$ is integral and $\text{det}(A') = 3$.

Let $H(n, d) = (\frac{4}{3})^{(n-1)/2}d^{1/n}$ be the Hermite function (cf. [11, 16]). Then $H(2, 3) = 2.000$. Since $A$ has minimal norm at least 4, the minimal norm of $A'$ is at least 2. Thus $A'$ has a norm 2 vector $u$. Then $ann_{A'}(u)$ has determinant $\frac{1}{2}\text{det}(A') = 3/2$ or $2\text{det}(A') = 6$. Since $A'$ is integral, $ann_{A'}(u)$ has determinant 6. Let $v$ be a basis of $ann_{A'}(u)$. Then $w := \frac{1}{2}(u + v) \in A'$. Moreover, $w$ has norm $\frac{1}{4}(2 + 6) = 2$ and $\langle u, -w \rangle = -1$. Thus $A' \cong A_2$ and $A \cong AA_2$. $\Box$
Notation A.4. Let $Q' \cong DIH_{12}(16)$. Let $M$ and $N$ be $EE_8$ sublattices of $Q'$ such that $Q' = M + N$. In this case, the $EE_8$-involutions $t_M$, $t_N$ generate a dihedral group of order 12. As in Notation (4.10), we set $h = t_M t_N$, $g = h^2$ and $u = h^3$. Then $\langle u \rangle$ is the center of $\langle t_M, t_N \rangle$. Let $E$ be the $(-1)$-eigenlattice of $u$ in $Q'$. Then $E$ is also an $EE_8$-sublattice of $Q'$ and $E \cap M \cong E \cap N \cong DD_4$ (see [12]).

Set $F_M := \text{ann}_M(N)$ and $F_N := \text{ann}_N(M)$. Then by [12], we have $F_M \cong F_N \cong AA_2$ and $J := \text{ann}_{Q'}(F_M \perp F_N) \cong K_{12}$. In addition, $E \cap J \cong AA_2 \perp AA_2$.

Lemma A.5. Let $Y$ be an $EE_8$ sublattice of $Q'$. Then either $Y = E$ or $E \cap Y \cong DD_4$.

Proof. Suppose $Y \neq E$. Then by the classification of $EE_8$-pairs [12], we have $E \cap Y \cong 0$, $AA_1$, $AA_2^2$, $AA_2$ or $DD_4$.

Case 1: $E \cap Y = 0$. In this case, $E + Y$ is a full rank sublattice of $Q'$. By the classification of $EE_8$-pairs and $D(Q') = 6^4$, the only possible case is $E + Y = Q'$. Then we have $F_E := \text{ann}_E(Y) \cong AA_2$, $F_Y := \text{ann}_Y(E) \cong AA_2$ and $\text{ann}_E(F_E) \cong EE_6$. Note also that $F_Y \subset \text{ann}_{Q'}(E) \cong R'$ and $\text{ann}_{R'}(AA_2) \cong AA_2^3$. Thus, we obtain a full sublattice of type $EE_6 \perp AA_2^3$ in $\text{ann}_{Q'}(F_E \perp F_Y) \cong K_{12}$. It is impossible since the 3-rank of $D(EE_6 \perp AA_2^3)$ is 4 and $D(K_{12}) = 3^6$.

Case 2: $E \cap Y \cong AA_1$. Then $E + Y \cong DIH_8(15)$. In this case, the $EE_8$ involutions $t_E$, $t_Y$ generate a dihedral group of order 8. Let $z = (t_E t_Y)^2$ and let $Z$ be the $(-1)$-eigenlattice of $z$ in $Q'$. Let $F_E$ and $F_Y$ be the fixed point sublattices of $z$ in $E$ and $Y$, respectively. By the analysis in [12], we know that $F_E \cap F_Y \cong AA_1$, $F_E \cong F_Y \cong DD_4$ and $Z \cong EE_8$. Then $\text{ann}_{E+Y}(F_E \perp Z) \supset \text{ann}_{F_Y}(F_E \cap F_Y) \cong AA_1^3$. However, $\text{ann}_{Q'}(F_E \perp Z) \cong \sqrt{6}DD_4$ has no norm 4 vectors. It is a contradiction.

Case 3: $E \cap Y \cong AA_2^3$. Then $R' \supset \text{ann}_Y(E) \cong DD_6$ but $R' \cong A_2 \otimes D_4$ doesn’t have such a sublattice.

Recall that the norm 4 vectors of $R'$ have the form $\alpha \otimes \beta$, where $\alpha, \beta$ are roots of $A_2$ and $D_4$, respectively. Suppose $R'$ contains a sublattice isometric to $DD_6$. Since $DD_6 \supset AA_1^3$, there exist roots $\alpha_1, \ldots, \alpha_6 \in A_2$ and $\beta_1, \ldots, \beta_6 \in D_4$ such that

$$\langle \alpha_i \otimes \beta_i, \alpha_j \otimes \beta_j \rangle = \langle \alpha_i, \alpha_j \rangle \cdot \langle \beta_i, \beta_j \rangle = 4 \delta_{i,j},$$

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for any \(i, j = 1, \ldots, 6\). Since \(\langle \alpha_i, \alpha_j \rangle \neq 0\) for any roots \(\alpha_i, \alpha_j \in A_2\), we must have \(\langle \beta_i, \beta_j \rangle = 2\delta_{i,j}\) for \(i, j = 1, \ldots, 6\). It is impossible because \(D_4\) has rank 4.

**Case 4:** \(E \cap Y \cong AA_2\). Then \(E + Y \cong DIH_6(14)\) and \(\text{ann}_{E+Y}(E \cap Y) \cong K_{12}\).

By the analysis in [12], we have

\[
\text{ann}_E(E \cap Y) \cong EE_6 \quad \text{and} \quad (E + Y) \cap R' = \text{ann}_{E+Y}(E) \cong \sqrt{6}E^*_6. \tag{6}
\]

Let \(A = \text{ann}_{Q'}(E + Y)\). Then \(\mathcal{D}(A) \cong 2^2 \cdot 3\). Then \(A \cong AA_2\) by Lemma A.3. Then \(\text{ann}_{R}(A) \cong AA_2^3\). However, \(\text{ann}_{R}(A) = (E + Y) \cap R' \cong \sqrt{6}E^*_6\) by (6). It is a contradiction.

Therefore, the only possible case is \(E \cap Y \cong DD_4\). Note that such a case occurs since \(E \cap M \cong E \cap N \cong DD_4\) (see [12]). \(\square\)

**Proposition A.6.** We use the same notation as in Notation (4.16). Let \(Y\) be an \(EE_8\)-sublattice in \(Q'\). Then \(Y = E, g^i(M)\) or \(g^i(N)\), where \(g = (t_M t_N)^2\) and \(i = 0, 1, 2\).

**Proof.** Let \(Y\) be an \(EE_8\) sublattice in \(Q'\) and \(E \neq Y\). Then \(E \cap Y \cong DD_4\) and \(E + Y \cong DIH_4(12)\). Moreover, we have \(Y \cap R' = \text{ann}_Y(Y \cap E) \cong DD_4\). Since there are only 3 \(DD_4\)-sublattices in \(R'\), we must have \(Y \cap R' = g^i(M) \cap R'\) for some \(i = 0, 1, 2\). It now follows \(E + Y = E + g^i(M)\) and we have the desired conclusion since \(DIH_4(12)\) has only 3 \(EE_8\) sublattices in it. \(\square\)

**Proposition A.7.** Let \(Q \cong DIH_6(14)\). Then \(Q\) contains exactly three \(EE_8\)-sublattices.

**Proof.** Let \(M\) and \(g\) be defined as in Notation A.4. Then \(M + gM \cong DIH_6(14)\). Thus we can view \(Q \cong DIH_6(14)\) as a sublattice of \(Q'\). The result now follows from Proposition (A.6). \(\square\)

**B** The containment \(C_{O(\Lambda)}(\langle t_M, t_N \rangle)\) in \(C_{O(\Lambda)}(t_E)\)

The group \(C_{O(\Lambda)}(t_E)\) has the form \(2.BRW^+(2^4) \cong (2 \times 2^{1+8})\Omega^+(8, 2)\). The groups \(BRW^+(2^d)\) may be analyzed by the methods of [11]. We give some discussion of \(C_{O(\Lambda)}(t_E)\) and \(C_{O(\Lambda)}(\langle t_M, t_N \rangle)\) from the BRW-viewpoint.

**Lemma B.1.** Let \(u\) and \(g\) be defined as in (4.16). The element \(g \in C_{O(\Lambda)}(u)\) acts on the Frattini quotient of \(O_2(C_{O(\Lambda)}(u))\) with 4-dimensional fixed points. In the notation of [9], \(g\) is in the class of \(\theta_2\).
Proof. An element of order 3 in the class of $\theta_i$ acts on $\text{Fix}(u)$ with trace $-8, 4, -2, 1$ for $i = 1, 2, 3, 4$, respectively.

By (4.10), the sum of the traces in $\langle g, u \rangle$ is 48. By the orthogonality relations, $\text{rank}((\text{Fix}(g) \cap \Lambda^+(u))) = 8$ whence $\text{rank}((\text{Fix}(g) \cap \Lambda^-(u))) = 4$. This implies that the trace of $g$ on $\Lambda^+(u)$ is $8 - \frac{1}{2}(16 - 8) = 4$, whence $i = 2$, as claimed. □

Lemma B.2. Let $t \in \{t_M, t_N\}$. Then $t$ acts with trace 8 on $\Lambda^+(u)$ and trace 0 on $\Lambda^-(u)$.

Proof. Define $U := O_2(C_{O(\Lambda)}(u))$. The trace of $t$ on $\Lambda^+(u)$ is 0 if and only if $t$ is conjugate in $C_{O(\Lambda)}(u)$ to $tu$ and otherwise the trace has the form $\pm 2^d$ where $2d$ is the dimension of the fixed points of $t$ in its action on $U/U'$.

We claim that $t$ does not have trace 0. Supposing otherwise, we see that $t$ has trace 8 on $\Lambda^-(u)$, which means that $t \in U$. This is impossible since $t$ inverts $g$. The claim follows.

Suppose $t = t_M$. The rank of $M \cap E$ is 4 by (4.12). Therefore, $\Lambda^+(u) \cap M$ has rank 4. This leads to trace 12 - 4 = 8 on $\Lambda^+(u)$. Trace 0 on $\Lambda^-(u) = E$ follows. □

Lemma B.3. $\text{Weyl}(F_4) \cong O(D_4) \cong O^+(4,3)$.

Proof. The group $\text{Weyl}(F_4)$ acts faithfully as isometries on the $D_4$-lattice, which is spanned by the long roots of $F_4$. This lattice has determinant 4, whence its reduction modulo 3 is nonsingular. The $F_3$-valued form is split since the sublattice of type $A_4^1$ maps onto $D_4/3D_4$ (take an orthogonal set of roots $p, q, r, s$ and note that $p + q + r, q - r + s$ generates a direct summand of $\mathbb{Z}p + \mathbb{Z}q + \mathbb{Z}r + \mathbb{Z}s$ and that their image in $D_4/3D_4$ is totally singular). We therefore get a homomorphism $\text{Weyl}(F_4) \rightarrow O^+(4,3)$. Both groups have order $2^73^2$ and this map is monic because any normal subgroup of $\text{Weyl}(F_4)$ contains the central involution which acts as $-1$ on $D_4/3D_4$. □

Notation B.4. Let $G := BRW^+(2^4)$, $U := O_2(G) \cong 2^{1+8}_+$. $\tau \in G$ an involution of trace 8 on the natural module (dimension 16) and $\theta \in G$ an element of order 3 such that $[U, \theta]/U'$ has rank 4. Define $G_1 := C_G([U, \theta])$, $G_2 := C_G(G_1)$. Then $G_1 \cong G_2 \cong BRW^+(2^2)$.

Lemma B.5. $[U, \tau] \leq [U, \theta]$.

Proof. Since the trace of $\tau$ is 8, $[U, \tau]/U'$ has rank 2. Since $\tau$ inverts $\theta$ by conjugation and leaves invariant $[U, \theta]$, the containment follows. □
Corollary B.6. $\tau \in G_2$.

**Proof.** Note that the direct product decomposition $[U, \theta]/U' \times C_U(\theta)/U'$ is orthogonal in the sense of the natural quadratic form on $U/U'$. The stabilizer of one summand stabilizes both and that stabilizer is $G_1G_2$, a central product. So $\tau \in G_1G_2$ and since $\tau$ stabilizes both summands and has commutator rank 2 on the quadratic space, $\tau \in UG_2$. If $\tau \notin G_2$, there exists $x \in (U \cap G_1) \cap G_2 \setminus U'$ so that $\tau \in xG_2$. There exists $y \in U \cap G_1$ so that $[x, y]$ generates $U'$. Then $[y, \tau]$ generates $U'$. This means that $\tau$ has trace 0 on the natural module, a contradiction. □

The next two results apply to $\theta, \tau \in G_2 \cong \text{Weyl}(F_4)$ acting on the $D_4$-lattice.

Lemma B.7. In $O(D_4)$, if $P \cong 3^2$ is a Sylow 3-group and $H$ is its normalizer, $H = Z(O_2(O(D_4))) \times H_1 \times H_2$, for a unique pair of dihedral subgroups $H_1, H_2$ generated by reflections. Furthermore, an element of order 3 in an $H_i$ has trivial fixed points on the Frattini quotient of $O_2(O(D_4))$. Elements of order 3 in $P \setminus (H_1 \cup H_2)$ have rank 2 fixed points on the $D_4$-lattice.

**Proof.** We observe that $\text{Weyl}(F_4)$ contains a natural $\text{Weyl}(A_2) \times \text{Weyl}(A_2)$ generated by reflections (at roots of different lengths). Since $P$, a Sylow 3-group of this (also a Sylow 3-group of $O(D_4)$), acts without fixed points on the Frattini factor of $O_2(O(D_4))$, $H \cong Z(O_2(O(D_4))) \times \text{Dih}_6 \times \text{Dih}_6$.

The subgroup of this generated by roots is the group $\text{Weyl}(A_2) \times \text{Weyl}(A_2)$ mentioned above.

For the second, let $x$ be an element of order 3 in $H_1$. Since $x$ has trace 1 on the lattice, $x$ has trivial fixed points on the Frattini quotient of $O_2(O(D_4))$. The remaining four elements of order 3 in $P$ have rank 2 fixed points on the Frattini factor and 0 fixed point sublattice.

Under conjugacy by $H$, the elements of order 3 are partitioned into orbits $\{x, x^{-1}\}, \{y, y^{-1}\}$ and the remaining set of four elements of order 3 (the ones which have nontrivial fixed points on the Frattini quotient of $O_2(O(D_4))$). The final statement follows. □

**Corollary B.8.** We use the notation of (B.7). Let $\{i, j\} = \{1, 2\}$. If $\theta \in H_i$, then $C_{O(D_4)}(\theta) = Z(O_2(O(D_4))) \times \langle \theta \rangle \times H_j$. Also, $\tau \in H_i$ and $C_{O(D_4)}(\langle \theta, \tau \rangle) = Z(O_2(O(D_4)))H_j$.

**Proof.** Since $\theta$ centralizes just $Z(O_2(O(D_4)))$ on $O_2(O(D_4))$, the form of $C(\theta)$ follows from (B.7). An element of $H$ which inverts $P$ has trace 0 on
the lattice since it induces outer automorphisms on each quaternion group in $O_2(O(D_4))$. Therefore, $\tau \in Z(O_2(O(D_4)))r$, where $r \in H_i$ is a reflection. Since $r$ has trace 2 and since the central involution acts as $-1$ on the lattice, $\tau$ has trace $\pm 2$. Since $\tau$ has trace 8 on the rank 16 representation (B.4) and this module is a tensor product for the central product $G_1G_2$, it follows that in the action of $G_2$ on $D_4$, $\tau$ has trace 2 and $\tau$ is in fact a reflection. □

Notation B.9. Let $t_M, t_N, t_E, h, g$ be as in Section (on $\text{DIH}_{12}(16)$). Then $u := t_E$ is the central involution of $\langle t_M, t_N \rangle$.

Let $\psi$ be a homomorphism of $C_{O(\Lambda)}(u)$ onto $\text{BRW}^+(2^4)$. Denote $\tau := \psi(t_M), \theta := \psi(g)$. Note that (B.1), (B.2) imply that Notation (B.4) applies to $\tau, \theta$.

The centralizer in $G := \text{BRW}^+(2^4)$ of $\langle \tau, \theta \rangle$ has been discussed in (B.8). Note that $\text{Ker}(\psi) = \langle -u \rangle$ and that $\psi$ maps $C_{O(\Lambda)}(\langle t_M, t_N \rangle)$ to a subgroup of $C_G(\langle \tau, \theta \rangle) \cong O^+(4,3) \times \text{Dih}_6$. According to (4.14), $C_{O(\Lambda)}(\langle t_M, t_N \rangle)$ has order $2^53^3$, the same as $O^+(4,3) \times \text{Dih}_6$. This means that $\psi(C_{O(\Lambda)}(\langle t_M, t_N \rangle))$ has index 2 in $C_G(\langle \tau, \theta \rangle) \cong O^+(4,3) \times \text{Dih}_6$.

C Trivial action on lattices mod 2

Lemma C.1. Suppose that the involution $t$ acts on the abelian group $L$ which has no 2-torsion. Assume that $t$ is trivial on $L/2L$. Then $L$ is the direct sum of eigenlattices for $t$.

Proof. There exists an endomorphism $a$ of $L$ so that $t = 1 - 2a$. Then $1 = t^2 = (1 - 2a)^2 = 1 - 4a + 4a^2 = 1 + 4(a^2 - a)$ and absence of 2-torsion imply that $0 = a^2 - a = a(a - 1)$.

For $x \in L$, $x = ax + (1-a)x$, whence $L$ is the sum of subgroups $\text{Ker}(1-a)$ and $\text{Ker}(a)$.

Let $x \in \text{Ker}(a)$. Then $tx = (1 - 2a)x = x$. If $y \in \text{Ker}(1-a)$, $ty = (1 - 2a)y = (-1 + 2(1-a))y = -y$. Therefore $L$ is the sum of the 1-eigenlattice and $(-1)$-eigenlattice of $t$. Their intersection is 0 since $L$ is free of 2-torsion. □

Corollary C.2. Suppose that the involution $t$ is an isometry of the orthogonally indecomposable rational lattice $L$ such that $t$ acts trivially on $L/2L$. Then $t = 1$ or $t = -1$. 37
**Proof.** By the Lemma, \( L \) is a direct sum of eigenlattices for \( t \). Since \( t \) is an isometry of \( L \), this is an orthogonal direct sum, whence indecomposability of \( L \) implies that one of the summands is 0. \( \Box \)

**Lemma C.3.** Let \( L \) be a finite rank positive definite orthogonally indecomposable lattice. Suppose that \( u \in O(L) \) has the property that \( u^2 \) acts trivially on \( L/2L \) and \( (u - 1)(L/2L) \) has dimension less than \( \frac{1}{2} \text{rank}(L) \). Then \( u \) is an involution.

**Proof.** Since \( L \) is an orthogonally indecomposable lattice, \( u^2 = \pm 1 \) by (C.2). Suppose that \( u^2 = -1 \). Then \( L \) is a torsion free module for \( \mathbb{Z}[u] \cong \mathbb{Z}[\sqrt{-1}] \), a PID. Therefore, \( L \) is a free module and \( L/(u - 1)L \) has \( \mathbb{F}_2 \)-dimension exactly \( \frac{1}{2} \text{rank}(L) \), a contradiction. \( \Box \)

### D  Facts about discriminant groups

**Lemma D.1** (Lemma A.9 of [14]). Let \( X, Y \) be sublattices of the lattice \( L \) where \( Y \) is a direct summand of \( L \), \( L = X + Y \) and \( (\det(Y), \det(L)) = 1 \). Then the natural map of \( X \) to the discriminant group \( \mathcal{D}(Y) = Y^*/Y \) is onto.

**Lemma D.2.** Let \( X \) and \( Y \) be sublattices of the unimodular lattice \( L \) such that \( X \cap Y = 0 \) and \( X + Y \) is a direct summand of \( L \). Then for any \( y \in Y^* \), there exists \( \beta \in L \) such that (1) \( P_Y(\beta) = y \mod 2Y \) and (2) \( \langle P_X(\beta), X \rangle \leq 2\mathbb{Z} \).

**Proof.** Since \( X \cap Y = 0 \), Conditions (1) and (2) define uniquely a \( \mathbb{Z} \)-linear map \( \phi : X + Y \to \mathbb{Z}_2 \) such that \( \phi(\alpha) = \langle y, \alpha \rangle \mod 2 \) for \( \alpha \in Y \) and \( \phi(\alpha) = 0 \) for \( \alpha \in X \). Moreover, \( X + Y \) is a direct summand of \( L \) and thus the natural map from \( L \) to \( (X + Y)^* \) is a surjection. Hence there is a \( \beta \in L \) such that \( \phi(\alpha) = \langle \beta, \alpha \rangle \mod 2 \) for all \( \alpha \in X + Y \) and \( \beta \) satisfies Conditions (1) and (2). \( \Box \)

### References

[1] J.H. Conway, A simple construction for the Fischer-Griess Monster group. *Invent. Math.* 79 (1985), 513–540.

[2] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, ATLAS of finite groups. Clarendon Press, Oxford, 1985.
[3] J. H. Conway and N. J. A. Sloane, Sphere packings, lattices and groups, Third edition. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 290. Springer-Verlag, New York, 1999.

[4] J. H. Conway and N. J. A. Sloane, The Coxeter-Todd lattice, the Mitchell group, and related sphere packings, Math. Proc. Camb. Phil. Soc. (1983), 93, 421-440.

[5] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York, 1988.

[6] C. Dong, H. Li, G. Mason and S.P. Norton, Associative subalgebras of Griess algebra and related topics. Proc. of the Conference on the Monster and Lie algebra at the Ohio State University, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin - New York, 1998.

[7] G. Glauberman and S. P. Norton, On McKay’s connection between the affine $E_8$ diagram and the Monster, Proceedings on Moonshine and related topics (Montreal, QC, 1999), 37–42, CRM Proc. Lecture Notes, 30, Amer. Math. Soc., Providence, RI, 2001.

[8] R. L. Griess, Jr., A vertex operator algebra related to $E_8$ with automorphism group $O^+(10,2)$. The Monster and Lie algebras (Columbus, OH, 1996), 43–58, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.

[9] R. L. Griess, Jr., Twelve Sporadic Groups, Springer Monographs in Mathematics, 1998, Springer-Verlag.

[10] R. L. Griess, Jr., Positive definite lattices of rank at most 8, Journal of Number Theory, 103 (2003), 77-84.

[11] R. L. Griess, Jr., “An introduction to groups and lattices: finite groups and positive definite rational lattices”; to be published simultaneously by Higher Education Press (in China) and by the International Press (Boston) for the rest of the world.
[12] R. L. Griess, Jr. and C. H. Lam, Dihedral groups and pairs of $EE_8$ lattices, Pure and applied mathematics quarterly, 7(2011), no. 3, 621-743.

[13] R. L. Griess, Jr. and C. H. Lam, A moonshine path from $E_8$ to the Monster, J. Pure and Applied Algebra, 215(2011), 927-948.

[14] R. L. Griess, Jr. and C. H. Lam, A moonshine path for and associated lattices of ranks 8 and 16, J. Algebra, 331(2011), 338-361.

[15] G. Höhn, The group of symmetries of the shorter moonshine module. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 80 (2010), 275–283, arXiv:math/0210076.

[16] M. Kneser, Theorie der Kristalgitter, Math. Ann. 127, 105-106 (1954).

[17] M. Kitazume, C. H. Lam and H. Yamada, 3-state Potts model, Moonshine vertex operator algebra, and 3A-elements of the Monster group. Internat. Math. Res. Notices 23 (2003), 1269–1303.

[18] C. H. Lam and H. Shimakura, Ising vectors in the vertex operator algebra $V^+_\Lambda$ associated with the Leech lattice $\Lambda$, Int. Math. Res. Not. (2007), Vol. 2007: article ID rnm132, 21 pages.

[19] C.H. Lam, H. Yamada and H. Yamauchi, McKay’s observation and vertex operator algebras generated by two conformal vectors of central charge 1/2. Internat. Math. Res. Papers 3 (2005), 117–181.

[20] J. McKay, Graphs, singularities, and finite groups. Proc. Symp. Pure Math., Vol. 37, Amer. Math. Soc., Providence, RI, 1980, pp. 183–186.

[21] M. Miyamoto, A new construction of the Moonshine vertex operator algebra over the real number field, Ann. of Math. 159 (2004), 535–596.