Some remarks on contractive and existence sets

Maciej Ciesielski¹ · Grzegorz Lewicki²

Received: 29 January 2021 / Accepted: 16 September 2022 / Published online: 30 September 2022
© The Author(s) 2022

Abstract
Let $X$ be a real or complex Banach space and let $F \subset X$ be a non-empty set. $F$ is called an existence set of best coapproximation (existence set for brevity), if for any $x \in X$, $R_F(x) \neq \emptyset$, where

$$R_F(x) = \{d \in F : \|d - c\| \leq \|x - c\| \text{ for any } c \in F\}.$$

It is clear that any existence set is a contractive subset of $X$. The aim of this paper is to present some conditions on $F$ and $X$ under which the notions of existence set and contractive set are equivalent.

Keywords  Banach spaces · Reflexivity · Strict convexity · Contractive and existence sets · One complemented spaces

Mathematics Subject Classification 47B37 · 46E30 · 47H09

1 Introduction

Let $X$ be a real or complex Banach space and let $F \subset X$ be a non-empty set. A continuous mapping $P : X \to F$ is called a projection onto $F$, whenever $P|_F = Id$. 

Communicated by Gerald Teschl.

Maciej Ciesielski
maciej.ciesielski@put.poznan.pl

Grzegorz Lewicki
grzegorz.lewicki@im.uj.edu.pl

1 Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965 Poznań, Poland

2 Department of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland
that is $P^2 = P$. Setting

$$\text{Min}(F) = \{ z \in X : \text{ for every } c \in F, x \in X, \text{ if } \|z - c\| \geq \|x - c\| \text{ then } x = z \},$$

we say that $F \subset X$ is optimal if $\text{Min}(F) = F$. Observe that for any $F \subset X$, $F \subset \text{Min}(F)$. This notion has been introduced by Beauzamy and Maurey in [5], where basic properties concerning optimal sets can be found.

A set $F \subset X$ is called an existence set of best coapproximation (existence set for brevity), if for any $x \in X$, $R_F(x) \neq \emptyset$, where

$$R_F(x) = \{ d \in F : \|d - c\| \leq \|x - c\| \text{ for any } c \in F \}.$$

Notice that any contractive set is an existence set. Indeed, if $P : X \to F$ is a contractive projection, then $Px \in R_F(x)$ for any $x \in X$. Also it is clear that any existence set is an optimal set. The converse, in general, is not true. However, the following result is satisfied.

**Theorem 1** [5, Prop. 2] If $X$ is one-complemented in $X^{**}$ and strictly convex, then any optimal subset of $X$ is an existence set in $X$, which, in particular, holds true for strictly convex spaces $X$, such that $X = Z^*$ for some Banach space $Z$.

Existence and optimal sets have been studied by many authors from different points of view, mainly in the context of approximation theory (see e.g. [2, 3, 9–15, 17, 18, 20–31]).

Recall that a closed subspace $F$ of a Banach space $X$ is called one-complemented if there exists a linear projection of norm one from $X$ onto $F$. It is obvious that any one-complemented subspace is an existence set. The converse, in general, is not true. By a deep result of Lindenstrauss, [21] there exists a Banach space $X$ and $F$ a linear subspace of $X$, $\text{codim}(F) = 2$, such that:

(a) $F$ is one-complemented in any containing it hyperplane $Y$ of $X$;
(b) $F$ is not one-complemented in $X$.

This gives you immediately an example of a subspace being an existence set which is not one-complemented. However, in [5] (see also [20, p. 121]), the following result has been proven.

**Theorem 2** (see [5, Prop. 5]) Let $V$ be a linear subspace of a smooth, reflexive and strictly convex Banach space. If $V$ is an optimal set then $V$ is one-complemented in $X$. If $X$ is a smooth Banach space, then any subspace of $X$ which is an existence set is one-complemented. Moreover, in both cases a norm-one projection from $X$ onto $V$ is uniquely determined.

The aim of this paper is to present some conditions on a Banach space $X$ and a convex and closed set $F \subset X$ under which the notions of existence set and contractive set are equivalent. In other words, we will study the problem of existence of a non-expansive selection $P : X \to F$ such that $Px \in R_F(x)$ for any $x \in X$. In Sect. 1 some preliminary results will be demonstrated. The main results of the paper will
be presented in Sect. 2 (see Theorems 6 and 7). In Sect. 2 we assume that \( F \) has a nonempty interior in \( X \). In Sect. 3 we consider the general case. All the results will be demonstrated for real Banach spaces. However, in Sect. 4, we show how to apply the results obtained in the real case to the case of complex Banach spaces (see Lemmas 8, 9 and Theorem 16). Now we present some notions which will be used in this paper. In the sequel by \( S(X) \) we denote the unit sphere in a Banach space \( X \) and by \( S(X^*) \) the unit sphere in its dual space. A functional \( f \in S(X^*) \) is called a supporting functional for \( x \in X \), if \( f(x) = \|x\| \). Analogously, a point \( x \in S(X) \) is called a norming point for \( f \in X^* \) if \( f(x) = \|f\| \). A point \( x \in X \) is called a smooth point if it has exactly one supporting functional. A Banach space \( X \) is called smooth if any \( x \in S(X) \) is a smooth point. By \( ext(X) \) we denote the set of all extreme points of \( S(X) \). A Banach space \( X \) is called strictly convex if \( ext(X) = S(X) \). If \( F \) is a linear subspace of a Banach space \( X \), by \( \mathcal{P}(X, F) \) we will denote the set of all linear, continuous projections from \( X \) onto \( F \). If \( X \) is a Banach space and \( Z \subset X \) is an affine subspace of \( X \), for \( F \subset Z \) by \( int_Z(F) \) (\( int(F) \) if \( Z = X \)), we denote the interior of \( F \) with respect to \( Z \). If \( X \) is a normed space and \( C \subset X \), by \( \sigma(C) \) we denote the boundary of \( C \) with respect to the norm topology in \( X \) and by \( \text{Span}(C) \) the linear space generated by \( C \). If \( T : X \to X \) is a mapping, by \( \text{Fix}(T) \) we denote the set of all fixed points of \( T \).

To the end of this paper, unless otherwise stated, all Banach spaces are real. Also we will need

**Definition 1** It is said that a Banach space \( X \) has (CFPP) if and only if for any nonempty, convex, closed and bounded set \( F \subset X \) and any nonexpansive mapping \( T : X \to X \) such that \( T(F) \subset F \), \( T \) has a fixed point in \( F \).

## 2 Section

We start with a lemma presenting basic properties of existence sets.

**Lemma 1** Let \( X \) be a Banach space and let \( F \subset X \) be an existence set. Then for any \( d \in X \), \( F + d \) is an existence set. Also, for any \( t \in \mathbb{R} \), \( tF \) is an existence set. If \( F \) is a linear subspace, then \( F \) is an existence set if and only if for any \( x \in X \setminus F \) there exists \( P_x \in \mathcal{P}(F_x, F) \), \( \|P_x\| = 1 \), where \( F_x = \text{span}[x] \oplus F \).

**Proof** Fix \( d \in X \) and \( x \in X \). Let \( p \in R_F(x - d) \). Let \( c \in F + d \), \( c = c_1 + d \), wher \( c_1 \in F \). Then

\[
\|c - (p + d)\| = \|c_1 - p\| \leq \|x - d - c_1\| = \|x - c\|,
\]

which shows that \( p + d \in R_{F+d}(x) \). Now fix \( t \in \mathbb{R} \). If \( t = 0 \), then \( F = \{0\} \) is obviously an existence set. If \( t \neq 0 \), fix \( p \in R_F(x/t) \). Let \( c \in tF \), \( c = tc_1 \). Observe that

\[
\|p - c\| \leq \|p - x/t\|
\]
and consequently
\[ |t| \|p - c_1\| \leq |t| \|p - x/t\|, \]

which implies that \( \|tp - c\| \leq \|tp - x\| \). Hence \( tp \in R_tF(x) \). Now assume that \( F \) is a linear subspace and fix \( x \in X \setminus F \). Then any \( y \in F_x \) can be represented in the unique way as \( y = tx + v_o \), where \( v_o \in F \). Fix \( p \in R_F(x) \). We show that \( tp + v_o \in R_F(tx + v_o) \). This is obvious if \( t = 0 \). So assume that \( t \neq 0 \). Fix \( v \in F \). Since \( \frac{v - v_o}{t} \in F \),
\[ \| p - \frac{v - v_o}{t} \| \leq \| x - \frac{v - v_o}{t} \| \]

and consequently
\[ |t| \|p - \frac{v - v_o}{t}\| \leq |t| \|x - \frac{v - v_o}{t}\|, \]

which implies that \( \|tp + v_o - v\| \leq \|tx + v_o - v\| = \|y - v\| \). Notice that a linear mapping \( P_x : F_x \to F \) defined by \( P_x(tx + v_o) = tp + v_o \) belongs to \( P(F_x, F) \). By the above reasoning, for any \( y \in F_x \), \( \|P_x y\| = \|P_x y - 0\| \leq \|y\| \), which shows that \( \|P_x\| = 1 \). Conversely, if there exists \( P_x \in P(F_x, F) \), \( \|P_x\| = 1 \), then for any \( v \in F \), and \( y \in F_x \),
\[ \|P_x y - v\| = \|P_x(y - v)\| \leq \|y - v\| \]

which shows our claim. \( \square \)

**Lemma 2** (compare with [7, Lemma 1]) Let \( X \) be a Banach space and let \( C \subset X \) be a locally weakly compact and convex set. For \( F \subset C \), \( F \neq \emptyset \), define
\[ N(F) = \{ f : C \to C : \|y - f(x)\| \leq \|y - x\| \text{ for any } y \in F, x \in C \}. \]

Then \( N(F) \) is compact in the topology of pointwise-weak convergence.

**Proof** Fix \( x_o \in F \). For \( x \in X \), set \( C_x = \{ y \in C : \|y - x_o\| \leq \|x - x_o\| \} \). Notice that for \( f \in N(F) \)
\[ \|f(x) - x_o\| \leq \|x - x_o\|. \]

Hence \( N(F) \subset P = \prod_{x \in C} C_x \). Since \( C \) is convex and locally weakly compact, by the Mazur Theorem, \( C_x \), as a bounded, convex and closed subset of \( C \), is weakly compact. By the Tychonoff Theorem, \( P \) is a compact set in the topology of pointwise-weak convergence, which we denote by \( \tau \). Hence to show that \( N(F) \) is compact, we need to demonstrate that \( N(F) \) is a \( \tau \) closed subset of \( P \). Let \( \{f_\gamma\} \subset N(F) \) be a net
τ-converging to \( f \in P \). Hence for any \( y \in F \), \( f_\gamma(y) \to f(y) \) in the weak topology. Since \( f_\gamma(y) = y \), for any \( \gamma \), \( f(y) = y \). Moreover, for \( y \in F \), \( x \in C \),

\[
\|y - f(x)\| \leq \liminf_{\gamma} \|y - f_\gamma(x)\| \leq \|y - x\|.
\]

Hence \( f \in N(F) \), which completes the proof. \( \square \)

**Lemma 3** (compare with [7, Lemma 2]) Let \( F, C, X \) and \( N(F) \) be as in Lemma 2. Then there exists \( r \in N(F) \) such that for any \( f \in N(F), x \in C \) and \( y \in F \)

\[
\|y - f(r(x))\| = \|y - r(x)\|.
\]

**Proof** First we define a partial ordering in \( N(F) \). For any \( f, g \in N(F) \) it is said that \( f < g \) if and only if for any \( x \in C \) and \( y \in F \)

\[
\|f(x) - y\| \leq \|g(x) - y\| \text{ and } \|f(x_0) - y_0\| < \|g(x_0) - y_0\|
\]

for some \( y_0 \in F, x_0 \in C \). It is said that \( f \leq g \) if and only if \( f < g \) or \( f = g \). It is easy to see that \((N(F), \leq)\) is a partially ordered set. Observe that \( N(F) \neq \emptyset \), since \( \text{id}_C \in N(F) \). Now we show that there exists a minimal element in \((N(F), \leq)\). First notice that for any \( f \in N(F), \) the set \( A_f = \{ g \in N(F) : g \leq f \} \) is \( \tau \)-closed in \( N(F) \), which follows easily from definition of our partial ordering. Since by Lemma 2, \( N(F) \) is a \( \tau \)-compact set, \( A_f \) is a \( \tau \)-compact set too. Now let \( G \subseteq N(F) \) be a chain. We show that there exists a smallest element in \( G \). Notice that if \( f_1, \ldots, f_n \in G \) then without loss of generality we can assume that \( f_n \leq f_{n-1} \leq \cdots \leq f_1 \). Hence

\[
\bigcap_{j=1}^{n} A_{f_j} = A_{f_n} \neq \emptyset.
\]

Since the sets \( A_f \) are closed and \( N(F) \) is compact, \( \bigcap_{f \in G} A_f \neq \emptyset \). Hence any \( g \in \bigcap_{f \in G} A_f \) is the smallest element in \( G \). By the Kuratowski-Zorn lemma, there exist \( r \in N(F) \) a minimal element in \((N(F), \leq)\). Let \( f \in N(F) \). Hence for any \( x \in C \) and \( y \in F \),

\[
\|y - (f \circ r)(x)\| \leq \|y - r(x)\| \leq \|y - x\|,
\]

which shows that \( f \circ r \leq r \). If for some \( x_0 \in C \) and \( y_0 \in F \), \( \|y_0 - (f \circ r)(x_0)\| < \|y_0 - r(x_0)\| \), then \( f \circ r \neq r \) and \( f \circ r \leq r \), a contradiction with the minimality of \( r \). \( \square \)

**Theorem 3** (compare with [7, Theorem 1]) Let \( F, C, X \) and \( N(F) \) be as in Lemma 2. Assume that for any \( x \in C \) there exists \( h \in N(F) \) such that \( h(x) \in F \). Then there exists \( g \in N(F) \) such that for any \( x \in C \) \( g(x) \in F \). In particular, \( r(x) \in R_F(x) \) for any \( x \in C \), where \( r \) is a maximal element from Lemma 3. If \( C = X \), then \( F \) is an existence set in \( X \).
Proof Let $r \in N(F)$ be as in Lemma 3. We show that for any $x \in X$, $r(x) \in F$. Notice that, since $r(x) \in C$, there exists $h \in N(F)$ such that $h(r(x)) \in F$. Let $y = h(r(x))$. Observe that by Lemma 3,

$$0 = \|h(r(x)) - y\| = \|y - r(x)\|.$$

Hence $r(x) = y \in F$, as required. □

Theorem 4 Let $X$ be a reflexive space. For $f \in X^* \setminus \{0\}$ define

$$G = \{x \in X : f(x) \leq 0\}.$$

If $G$ is an existence set then $F = \ker(f)$ is one-complemented in $X$.

Proof Since $G$ is an existence set, for any $x \in S(X)$, we can select $P_0 x \in R_G(x)$. Define a mapping $P : X \to G$ by $P0 = 0$, $Px = \|x\| P_0(x/\|x\|)$ for $x \neq 0$. Notice that $Px \in R_G(x) \subset G$ for any $x \in X$. This is obvious if $x = 0$. If $x \neq 0$, for any $d \in G$,

$$\|P_0(x/\|x\|) - d/\|x\|\| \leq \|(x - d)/\|x\|\|$$

and consequently

$$\|Px - d\| = \|\|x\| P_0(x/\|x\|) - d\| \leq \|x - d\|.$$

In particular, for any $d \in F = \ker(f)$, $\|Px - d\| \leq \|x - d\|$ and $P|_F = id_F$. This shows that $P \in N(F)$ (in our case $C = X$). Now we show that for any $x \in X$ there exists $Q \in N(F)$ such that $Qx \in F$. Fix $x \in X$. If $f(x) = 0$, then $1d_F(x) \in F$. Now assume that $f(x) > 0$. If $f(Px) = 0$, then $Px \in F$. If $f(Px) < 0$, then there exists $0 < \alpha < 1$ such that $f(\alpha x + (1 - \alpha)Px) = 0$. Since $N(F)$ is convex, $Q = \alpha Id_X + (1 - \alpha)P \in N(F)$ and $Qx \in F$. Now assume that $f(x) < 0$ and set $G_1 = \{x \in X : f(x) \geq 0\}$. By Lemma 1 for any $x \in X$, $P_1 x = -P(-x) \in R_{G_1}(x) \subset G_1$. Since $F \subset G_1$, for any $d \in F$, $\|P_1 x - d\| \leq \|x - d\|$ and $P_1|_F = id_F$. This shows that $P_1 \in N(F)$. If $f(P_1 x) = 0$, then $P_1 x \in F$. If $f(P_1 x) > 0$, then there exists $0 < \alpha < 1$ such that $f(\alpha x + (1 - \alpha)P_1 x) = 0$. Since $N(F)$ is convex, $Q_1 = \alpha Id_X + (1 - \alpha)P_1 \in N(F)$ and $Q_1 x \in F$. By Theorem 3 applied to $C = X$, there exists $Q : X \to \ker(f)$ such that $Q_{\ker(f)} = id_{\ker(f)}$ and $\|Qx - d\| \leq \|x - d\|$ for any $x \in X$ and $d \in \ker(f)$. Hence $\ker(f)$ is an existence set. By Lemma 1, $\ker(f)$ is one-complemented in $X$. □

Theorem 5 Let $X$ be a Banach space and let $f \in X^* \setminus \{0\}$. If $\ker(f)$ is one-complemented in $X$, then for any $d \in \mathbb{R}$ the set $G_d = \{x \in X : f(x) \leq d\}$ is a contractive subset of $X$.

Proof By Lemma 1, we can assume that $d = 0$. Fix $P \in \mathcal{P}(X, \ker(f))$, $\|P\| = 1$. Define $Q : X \to G_o$ by $Qx = x$ if $f(x) \leq 0$ and $Qx = Px$ if $f(x) > 0$. We show
that for any $x, z \in X$ $\|Qx - Qz\| \leq \|x - z\|$. If $f(x) \leq 0$ and $f(z) \leq 0$, then

$$\|Qx - Qz\| = \|x - z\|.$$  

If $f(x) \geq 0$ and $f(z) \geq 0$, then

$$\|Qx - Qz\| = \|Px - Pz\| \leq \|x - z\|.$$  

Now assume that $f(x) > 0$ and $f(z) < 0$. Let $y \in X$ be so chosen that $Py = 0$ and $f(y) = 1$. Then, it is easy to see that for any $w \in X$, $Pw = w - f(w)y$. Hence if $w = \alpha y + v$ where $v \in \ker(f)$ and $\alpha \in \mathbb{R}$, then

$$\|Pw\| = \|\alpha Py + Pv\| = \|v\| \leq \|w\| = \|\alpha y + v\|.$$  

This means that for any $v \in \ker(f)$ and $t \geq 0$ the function $g_v(t) = \|ty + v\|$ satisfies

$$g_v(t) \geq g_v(0) = \|v\|.$$  

Since $g_v$ is a convex function, $g_v$ is increasing in $[0, +\infty)$. Since $f(x) > 0$, if $x = \alpha y + v$, then $f(x) = \alpha > 0$. Analogously, since $f(z) < 0$, if $z = \beta y + w$, then $f(z) = \beta < 0$. Notice that, since the function $g_{v-w}$ is increasing in $[0, +\infty)$,

$$\|Qx - Qz\| = \|Px - z\| = \|v - (\beta y + w)\| = \|\beta y + (v - w)\|$$  

$$= g_{v-w}(\beta) \leq g_{v-w}(\beta + \alpha) =$$  

$$= \|(-\beta + \alpha)y + v - w\| = \|\alpha y + v - (\beta y + w)\| = \|x - z\|,$$

as required. Since $Q|_{G_0} = id_{G_0}$, $G_0$ is a contractive set. The proof is complete.  

The next lemma can be treated as a generalization of the Minkowski functional for non-symmetric, closed convex and bounded sets.

**Lemma 4** Let $X$ be a Banach space and let $C \subset X$ be a closed, convex and bounded set such that $0 \in \text{int}(C)$. Define for $x \in X$,

$$f(x) = \inf\{t > 0 : x/t \in C\}.$$  

Then $f$ is a convex and continuous function.

**Proof** First we show that for any $x_1, x_2 \in X$, $f(x_1 + x_2) \leq f(x_1) + f(x_2)$. Fix $x_1, x_2 \in X$ and $\varepsilon > 0$. Then there exists $t_1 > 0$ and $t_2 > 0$ satisfying $t_1 < f(x_1) + \varepsilon$ and $t_2 < f(x_2) + \varepsilon$ such that $\frac{x_1}{t_1}, \frac{x_2}{t_2} \in C$ for $i = 1, 2$. Notice that

$$\frac{x_1 + x_2}{t_1 + t_2} = \frac{t_1}{t_1 + t_2} \frac{x_1}{t_1} + \frac{t_2}{t_1 + t_2} \frac{x_2}{t_2}.$$  


Since $C$ is convex, by definition of $t_1$ and $t_2$, 
\[ \frac{x_1 + x_2}{t_1 + t_2} \in C. \]
This shows that
\[ f(x_1 + x_2) \leq f(x_1) + f(x_2) + 2\varepsilon \]
and consequently $f$ is a subadditive function. Moreover, it is easy to see that $f(ax) = af(x)$ for any $a \geq 0$ and $x \in X$. Hence for any $a, b \geq 0$, $a + b = 1$, and $x, y \in X$
\[ f(ax + by) \leq f(ax) + f(by) \leq af(x) + bf(y), \]
which shows that $f$ is a convex function. Now we prove that $f$ is continuous. Let 
\[ \|x_n - x\| \to 0. \]
Fix $\varepsilon > 0$ and $t > 0$ such that $f(x) < t < f(x) + \varepsilon$. Since $0 \in \text{int}(C)$, there exists $r > 0$ such that $B_0(x/t, r) \subset C$. Hence for $n \geq n_0$, $x_n/t \in B_0(x/t, r) \subset C$. Hence for any $n \geq n_0$, $f(x_n) \leq t \leq f(x) + \varepsilon$ and consequently,
\[ \limsup_n f(x_n) \leq f(x). \]

Now, assume on the contrary that
\[ 0 \leq d = \liminf_n f(x_n) < f(x). \]
Fix $d < t < f(x)$. Hence $f(x) > 0$, so $x \neq 0$. Then there exists a subsequence $x_{n_k}$ such that $f(x_{n_k}) < t$ and consequently $x_{n_k}/t \in C$. Since $C$ is closed, $x/t \in C$. Hence $f(x) \leq t$, which is a contradiction. Finally we get
\[ \limsup_n f(x_n) \leq f(x) \leq \liminf_n f(x_n) \]
which shows that $f(x) = \lim_n f(x_n)$, as required. \(\square\)

**Lemma 5** Let $X$ be a Banach space and let $C \subset X$ be a closed, convex and bounded set such that $0 \in \text{int}(C)$. Denote by $\sigma(C)$ the boundary of $C$. Let \( \{x_n\} \subset \sigma(C) \) be a dense subset of $\sigma(C)$ such that for any $n \in \mathbb{N}$ there exists exactly one supporting functional $f_n$ for $x_n$, with $\|f_n\| = 1$. Assume that $f_n(x_n) = d_n$ and $f_n(v) \leq d_n$ for any $v \in C$. Put
\[ D_n = \{x \in X : f_n(x) \leq d_n\}. \]
Then $C = \bigcap_{n \in \mathbb{N}} D_n$.

**Proof** By definition of $f_n$ and $D_n$, $C \subset \bigcap_{n \in \mathbb{N}} D_n$. Now assume on the contrary that there exists $y_o \in \bigcap_{n \in \mathbb{N}} D_n \setminus C$. Since $0 \in \text{int}(C)$, there exists $t \in (0, 1)$ such that $ty_o \in \sigma(C)$. Fix $\{y_n\} \subset \{x_n\}$ such that
\[ \|y_n - ty_o\| \to 0. \]
This is possible, since \( \{x_n\} \) is a dense subset of \( \sigma(C) \). Let \( d_n \) and \( f_n \) be so chosen that \( f_n(y_n) = d_n \) and \( V \subseteq D_n = \{x \in X : f_n(x) \leq d_n\} \). Now we show that \( \{d_n\} \) is bounded. Assume on the contrary that this is not true. Without loss of generality, passing to a subsequence, if necessary, we can assume that \( d_n \to +\infty \). Let \( R > 0 \) be so chosen that \( C \subseteq B(0, R) \). Since \( f_n(y_n) = d_n \), and \( y_n \in C \), for \( n \geq n_o \),

\[
\sup\{f_n(z) : z \in B(0, R)\} \geq f_n(y_n) \geq 2R.
\]

Hence

\[
\sup\{f_n(z) : z \in B(0, 1)\} \geq 2
\]

and consequently, \( \|f_n\| > 2 \) for \( n \geq n_o \) which is a contradiction. Now we show that \( e_o = \lim \inf_n d_n > 0 \). Assume on the contrary that \( \lim \inf_n d_n = 0 \). Without loss of generality, passing to a subsequence, if necessary, we can assume that \( d_n \to 0 \). Fix \( r > 0 \) such that \( B(0, r) \subseteq \text{int}(C) \). Fix \( n_o \geq 2 \) such that \( \frac{r}{n_o} < 1 \). Fix \( k_o \) such that \( 0 \leq d_k < \frac{r}{n_o} \) for \( k \geq k_o \). Fix \( x \in B(0, r) \). If \( f_k(x) \geq 0 \), then

\[
0 \leq f_k(x) \leq d_k < \frac{r}{n_o}.
\]

If \( f_k(x) < 0 \), then \( f_k(-x) \geq 0 \) and \( f_k(-x) \leq d_k < \frac{r}{n_o} \). Hence

\[
\sup\{|f_k(x)| : x \in B(0, r)\} \leq \frac{r}{n_o}
\]

and consequently \( \|f_k\| \leq \frac{1}{n_o} < 1 \) which is a contradiction. Without loss of generality, passing to a subsequence if necessary, by (1), we can assume that \( \|y_n - ty_o\| \to 0 \) and \( d_n \to e_o > 0 \). Notice that

\[
f_n(ty_o) = f_n(y_n) - f_n(y_n - ty_o) = d_n - f_n(y_n - ty_o) \geq d_n - \|y_n - ty_o\|.
\]

Hence

\[
\lim \inf_n f_n(y_o) = \lim \inf_n f_n(ty_o) \geq \lim_n d_n = \frac{e_o}{t}.
\]

Since \( t \in (0, 1) \) and \( e_o > 0 \), \( f_n(y_o) > d_n \) for \( n \geq n_o \). Hence \( y_o \notin \bigcap_{n \in \mathbb{N}} D_n \), which is a contradiction. The proof is complete. \( \square \)

**Lemma 6** Let \( X \) be a reflexive Banach space. Let \( \{F_t\}_{t \in T} \) be a family of convex existence sets directed by \( \subseteq \). Then \( F = \text{cl}(\bigcup_{t \in T} F_t) \) is also an existence set, where the closure is taken with respect to the norm topology in \( X \).

**Proof** Since \( \{F_t\}_{t \in T} \) is directed by \( \subseteq \), for any \( t_1, t_2 \in T \) there exists \( t_3 \in T \) such that \( F_{t_1} \cup F_{t_2} \subseteq F_{t_3} \). Define a partial ordering \( \leq \) in \( T \) by \( t_1 \leq t_2 \) provided \( F_{t_1} \subseteq F_{t_2} \). Fix \( F_{t_o} = F_o \in \{F_t\}_{t \in T} \). Since \( \{F_t\}_{t \in T} \) is directed by \( \subseteq \), \( F = \text{cl}(\bigcup_{t \geq t_o} F_t) \). Since \( X \) is
Lemma 7

Let $X$ be a reflexive Banach space. Let

By the axiom of choice, for any $t \geq t_0$ we can select $P_t \in N(F_t)$ (see Lemma 2) such that $P_t(x) \in RF_t(x)$ for any $x \in X$. Observe that for any $t \geq t_0$, $N(F_t) \subset N(F_o)$, since $F_o \subset F_t$. Thus $\{P_t\}_{t \geq t_0} \subset N(F_o)$. Set for any $t \geq t_0, A_t = \tau - cl((P_s : s \geq t))$, where $\tau - cl$ denotes the closure in the topology $\tau$ of of pointwise weak convergence, cf. Lemma 2. Since by Lemma 2, the set $N(F_o)$ is $\tau$-compact, so is the set $A_t$ for any $t \geq t_0$. Since $\{F_t\}_{t \in T}$ is directed by $\supset$, for any $n \in \mathbb{N}$ and $t_1, ...t_n \geq t_0$ there exists $u \in T$ such that

$$\bigcap_{i=1}^{n} A_{t_i} \supset A_u \neq \emptyset.$$  

By $\tau$-compactness of the sets $A_t$ for all $t \geq t_0$, $\bigcap_{t \geq t_0} A_t \neq \emptyset$. Fix $P \in \bigcap_{t \geq t_0} A_t$. We show that for any $x \in X$, $Px \in RF(x)$. Fix $x \in X$ and $d \in \bigcup_{t \geq t_0} F_t$. Then there exists $s \geq t_o$ such that $d \in F_s$ and consequently $d \in F_u$ for $u \geq s$. Since $P \in A_s$,

$$\|Px - d\| \leq \liminf_{u \geq s} \|P_u x - d\| \leq \|x - d\|.$$  

If $d \in F$, then there exists a sequence $\{d_n\} \subset \bigcup_{t \geq t_0} F_t$ such that $\|d_n - d\| \to 0$. By the previous reasoning,

$$\|Px - d\| = \lim_{n} \|Px - d_n\| \leq \lim_{n} \|x - d_n\| = \|x - d\|.$$  

as required. \qed

Corollary 1

Let $X$ be a reflexive Banach space and let $F$ be a convex existence set. Define for any $v \in F$ and $t > 0$, $F_t = (1 - t)v + tF$ and let $C_v = cl(\bigcup_{t \geq 0} F_t)$. Then $F$ is an existence set.

Proof

By Lemma 1, $F_t$ is an existence set for any $t > 0$. Since $F_t \subset F_s$ for $t \leq s$, by Lemma 6 $C_v$ is an existence set too. \qed

Lemma 7

Let $X$ be a reflexive Banach space. Let $\{F_t\}_{t \in T}$ be a family of convex existence sets directed by $\supset$. If $F = \bigcap_{t \in T} F_t \neq \emptyset$, then $F$ is also an existence set.

Proof

Since $\{F_t\}_{t \in T}$ is directed by $\supset$, for any $t_1, t_2 \in T$ there exists $t_3 \in T$ such that $F_{t_1} \cap F_{t_2} \supset F_{t_3}$. Define a partial ordering $\leq$ in $T$ by $t_1 \leq t_2$ provided $F_{t_1} \supset F_{t_2}$. Since $X$ is reflexive, it is locally weakly compact and we will consider the sets $N(F_t)$ defined in Lemma 2 with the set $C = X$. By the axiom of choice, for any $t \in T$ we can select $P_t \in N(F_t)$ such that $P_t(x) \in RF_t(x)$ for any $x \in X$. Observe that for any $t \in T, N(F_t) \subset N(F)$, since $F \subset F_t$. Thus $\{P_t\}_{t \in T} \subset N(F)$. Set for any $t \geq t_0$, $A_t = \tau - cl((P_s : s \geq t))$. Since $\{F_t\}_{t \in T}$ is directed by $\supset$, for any $n \in \mathbb{N}$ and $t_1, ..., t_n \geq t_0$ there exists $u \in T$ such that

$$\bigcap_{i=1}^{n} A_{t_i} \supset A_u \neq \emptyset.$$  

\textcopyright{} Springer
By \( \tau \)–compactness of the set \( N(F) \), \( \bigcap_{t \in T} A_t \neq \emptyset \). Fix \( P \in \bigcap_{t \in T} A_t \). We show that for any \( x \in X \), \( Px \in RF(x) \). Fix \( x \in X \) and \( d \in F \). Hence \( d \in F_s \) for any \( s \in T \) and consequently

\[
\| Px - d \| \leq \liminf_{s} \| P_s x - d \| \leq \| x - d \|.
\]

To finish the proof we need to show that \( Px \in F \) for any \( x \in X \). Assume on the contrary that there exists \( x_o \in X \) such that \( Px_o \notin F \). Then there exists \( t_o \in T \) such that \( Px_o \notin F_{t_o} \). Hence \( Px_o \notin F_t \) for any \( t \geq t_o \). Since \( P_t x_o \in F_t \subset F_{t_o} \) for \( t \geq t_o \), by the Mazur Theorem \( Px_o \in F_{t_o} \), which is a contradiction. \( \square \)

**3 Section**

The next two theorems are the main results of this paper.

**Theorem 6** Let \( X \) be a reflexive, separable Banach space. Let \( F \subset X \) be a nonempty, closed, bounded and convex existence set with nonempty interior in \( X \). Then \( F \) is an intersection of a countable family of contractive half-spaces.

**Proof** By Lemma 1 we can assume that \( 0 \in \text{int}(F) \). Define a function \( f : X \to \mathbb{R} \) by

\[
f(x) = \inf\{t > 0 : x/t \in F\}.
\]

By Lemma 4, \( f \) is a convex and continuous function. By the Mazur Theorem, \( f \) is Gateaux differentiable on a countable, dense subset of \( X \). Since \( f(tx) = tf(x) \) for any \( x \in X \) and \( t > 0 \), there exists a dense countable subset \( Z = \{z_n\} \) of \( \sigma(C) \) such that \( f \) is Gateaux differentiable at any \( z \in Z \), where \( \sigma(C) \) denotes the boundary of \( C \). Let for \( n \in \mathbb{N} \) \( D_n = cl(\bigcup_{t=0} \mathbb{F}_{t,n}) \), where \( \mathbb{F}_{t,n} = \{(1 - t)z_n + tF\} \). By Corollary 1, \( D_n \) is an existence set for any \( n \in \mathbb{N} \). Since \( f \) is Gateaux differentiable at \( z_n \), there exists \( f_n \in S(X^*) \) and \( d_n \in \mathbb{R} \) such that \( D_n = \{x \in X : f_n(x) \leq d_n\} \) and \( f_n(x_n) = d_n \). (Since \( 0 \in \text{int}(C), d_n > 0 \).) By Theorem 4 and Lemma 1, \( \ker(f_n) \) is one-complemented in \( X \). By Theorem 5, \( D_n \) is a contractive half-space. By Lemma 5, \( F = \bigcap_{n=1}^{\infty} D_n \), as required. \( \square \)

Now we will show a sufficient condition under which intersections of countable families of contractive sets are contractive.

**Theorem 7** Let \( (X, \| \cdot \|) \) be a reflexive Banach space. Let \( \| \cdot \|_n \) be a sequence of strictly convex norms on \( X \) such that there exists a sequence \( \{s_n\} \) of nonnegative numbers, \( s_n \to 0 \) satisfying for any \( n \in \mathbb{N} \) and \( x \in X \)

\[
(1 - s_n)\|x\|_n \leq \|x\| \leq (1 + s_n)\|x\|_n
\]

Assume that \( \{F_k\} \) is a countable family of convex sets such that for any \( k, n \in \mathbb{N} \) \( F_k \) is a contractive set with respect to \( \| \cdot \|_n \). Assume that \( F = \bigcap_{k \in \mathbb{N}} F_k \neq \emptyset \). Then \( F \) is a contractive subset of \( X \) with respect to \( \| \cdot \| \).
Proof By Lemma 1, we can assume that $0 \in F$. Fix $n \in \mathbb{N}$. Let for $k \in \mathbb{N}$ $P_{k,n} : X \to F_k$ be a contractive mapping with respect to $\| \cdot \|_n$. Fix a sequence of positive numbers $\{c_k\}$ such that $\sum_{k=1}^{\infty} c_k = 1$. Define $P_n : X \rightarrow X$ by

$$P_n x = \sum_{k=1}^{\infty} c_k P_{k,n} x.$$ 

(Since $\sum_{k=1}^{\infty} c_k = 1$, $P_n$ is well-defined.) Now we show that $Fix(P_n) = F$. By definition of $P_{k,n}$, $F \subset Fix(P_n)$. Now assume that $x \in Fix(P_n) \setminus \{0\}$. Since $0 \in F$, for any $k \in \mathbb{N}$ $\| P_{k,n}(x) \|_n \leq \| x \|_n$. Fix $f_n \in S(X^*)$ (with respect to $\| \cdot \|_n$) such that $f_n(x) = \| x \|_n$. Notice that

$$\| x \|_n = f_n \left( \sum_{k=1}^{\infty} c_k P_{k,n} x \right) = \sum_{k=1}^{\infty} c_k f_n(P_{k,n} x) \leq \sum_{k=1}^{\infty} c_k \| P_{k,n} x \|_n \leq \| x \|_n.$$ 

Hence for any $k \in \mathbb{N}$ $\| P_{k,n} x \|_n = \| x \|_n$ and $f_n(P_{k,n} x) = \| x \|_n$. Hence for any $k \in \mathbb{N}$ $\| P_{k,n} x \|_n = \| x \|_n$ is a norming point for $f_n$. Since $(X, \| \cdot \|_n)$ is strictly convex, $f_n$ has exactly one norming point with respect to $\| \cdot \|_n$. Hence for any $k \in \mathbb{N}$, $P_{k,n} x = x$, which shows that $x \in F$. By [7, Theorem 2 and Lemma 3], $F = Fix(P_n)$ is a compact subset of $(X, \| \cdot \|_n)$. Let $Q_n : X \rightarrow F$ be a contractive mapping with respect to $\| \cdot \|_n$. Define for any $M \geq 1$

$$N_M(F) = \{ f : X \rightarrow X : \| y - f(x) \| \leq M \| y - x \| \text{ for any } y \in F, x \in X \}.$$ 

Reasoning as in Lemma 2, we can show that $N_M(F)$ is a compact set with respect to the topology of weak pointwise convergence. Put

$$M = \sup \left\{ \frac{1 + s_n}{1 - s_n} : n \in \mathbb{N} \right\}.$$ 

Notice that for any $x \in X$ and $y \in F$,

$$\| Q_n x - y \| \leq (1 + s_n) \| Q_n x - y \|_n \leq (1 + s_n) \| x - y \|_n \leq \frac{1 + s_n}{1 - s_n} \| x - y \| \leq M \| x - y \|.$$ 

Hence $Q_n \in N_M(F)$ for any $n \in \mathbb{N}$. Reasoning as in Lemma 3, we can show that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$, where $A_n = cl(\{Q_k : k \geq n\})$. Fix $Q \in \bigcap_{n \in \mathbb{N}} A_n$. We show that $Q$ is a contractive mapping from $X$ onto $F$ with respect to $\| \cdot \|$. Notice that for any $x, y \in X$

$$\| Q_n x - Q_n y \|_n \leq \| x - y \|_n \leq \frac{\| x - y \|}{1 - s_n}.$$
Since \( s_n \to 0 \),
\[
\liminf_n \frac{\|Q_n x - Q_n y\|}{1 - s_n} \leq \|x - y\|.
\]

Since \( Q \in \bigcap_{n \in \mathbb{N}} A_n \), this shows that
\[
\|Qx - Qy\| \leq \liminf_n \|Q_n x - Q_n y\| \leq \liminf_n \frac{\|Q_n x - Q_n y\|}{1 - s_n} \leq \|x - y\|,
\]
as required. Moreover, since for any \( n \in \mathbb{N} \) \( Q_n x \in F \) and \( F \) is convex, by the Mazur
Theorem, \( Qx \in F \). Since for any \( n \in \mathbb{N} \) and \( x \in F \), \( Q_n x = x \), \( Qx = x \). The proof
is complete. ⊓⊔

**Remark 1** Observe that, in general, the countable intersection of contractive sets
need not to be even an existence set. In [17, Example 2.10], it was shown that
\( F = \text{ker}(f_1) \cap \text{ker}(f_2) \subset l^{(4)}_4 \) is not an existence set. Here \( f_1 = (1, 0, 0, 0) \) and
\( f_2 = (1/2, 1/6, 1/6, 1/6) \). (We understand that \( f_i(x) = \sum_{j=1}^{4} x_j f_j \), for \( x \in l^{(4)}_4 \) and
\( i = 1, 2 \).) Observe that \( F = \bigcap_{n=1}^{\infty} F_n \), where
\[
F_n = \{ x \in l^{(4)}_4 : |f_i(x)| \leq 1/n, i = 1, 2 \}.
\]
By Lemma 7 and Theorem 1 for any \( n \in \mathbb{N} \) \( F_n \) is not an existence set. Observe that
for any \( n \in \mathbb{N} \) \( F_n = \bigcup_{k=1}^{\infty} F_{n,k} \), where
\[
F_{n,k} = \{ x \in F_n : |x_i| \leq k \text{ for } k = 1, ..., 4 \}.
\]
By Lemma 6 and Theorem 1, for any \( k, n \in \mathbb{N} \), \( F_{n,k} \) is not an existence set either.

An immediate consequence of Theorems 6 and 7 is

**Theorem 8** Let \((X, \| \cdot \|)\) be a reflexive, separable Banach space and let \( F \) be a
nonempty, closed, bounded and convex set with nonempty interior in \( X \). Assume that
\( F = \bigcap_{k \in \mathbb{N}} F_k \), where \( F_k \) are half-spaces determined in Theorem 6. Assume furthermore that for any \( k \in \mathbb{N} \), \( X \) and \( F_k \) satisfy the assumptions of Theorem 7 (in particular, for any \( k \in \mathbb{N} \), \( F_k \) is a contractive set with respect to any \( \| \cdot \|_n \) defined in the statement of Theorem 7). Then the following conditions are equivalent:

a. \( F \) is an existence set;

b. \( F \) is a contractive set;

c. \( F \) is an intersection of a countable number of contractive half-spaces.

**Proof** Assume that \( F \) is an existence set. By Theorem 6, \( F \) is is an intersection of
a countable number of contractive half-spaces \( F_k \). By our assumptions on \( \{F_k\} \) and
Theorem 7, \( F \) is a contractive set. Since each contractive set is an existence set, the
proof is complete. ⊓⊔

Now we present two applications of Theorems 8 and 6.
Theorem 9 Let $X = l_1^{(n)}$ and let $F \subset X$ be a convex, bounded set with nonempty interior in $X$. Then the following conditions are equivalent:

a. $F$ is an existence set;
b. $F$ is a contractive set;c. $F$ is an intersection of a countable number of contractive half-spaces.

Proof Assume that $F$ is a convex existence set in $X$. By Theorem 6, $F$ is an intersection of a countable family $\{F_k\}$ of half-spaces. By Theorem 4, each $F_k$ is determined by $f_k \in S(X^*)$ such that $\ker(f_k)$ is one-complemented in $X$. By [4, Th. 3, p. 220] each $f_k$ has at most two coordinates different from zero. By [6] and [1, Lemma 3.2, p. 58] (see also [17]) for each $k$, $\ker(f_k)$ is also one-complemented in $l_p^{(n)}$ for $p \geq 1$. Fix a sequence $p_l > 1$, $p_l \to 1$. By Theorem 5, for any $k$, $lF_k$ is a contractive subset of $l_p^{(n)}$. Applying Theorem 8 to $\|\cdot\|_l = \|\cdot\|_{p_l}$ and $\{F_k\}$ we get the result.

Theorem 10 Let $X = l_\infty^{(n)}$ and let $F \subset X$ be a convex set. Assume that $F = \bigcap_{k=1}^{\infty} F_k$, where for each $k$ $F_k$ is a half space determined by $f_k \in S(X^*)$ having at most two coordinates different from zero. Then the following conditions are equivalent:

a. $F$ is an existence set;b. $F$ is a contractive set;

Proof Fix a sequence $p_l \to +\infty$. By [1, 6], and Theorem 5, for any $k$, $lF_k$ is a contractive subset of $l_p^{(n)}$. Reasoning as in Theorem 9 we get the result.

Now we present a sufficient condition (in which strict convexity is not assumed), under which the intersection of a countable family of contractive half-spaces is a contractive set.

Theorem 11 Let $X$ be a reflexive Banach space satisfying (CFPP) (see Definition 1). Assume that $F = \bigcap_{k=1}^{\infty} F_k$, where for each $k$ $F_k$ is a contractive half-space determined by $f_k \in S(X^*)$. (By Theorem 4, for each $k$ $\ker(f_k)$ is one complemented in $X$.) Let $P_k$ be a norm one projection from $X$ onto $\ker(f_k)$. (By [4], for any $x \in X$ $P_k x = x - f_k(x)y_k$ where $y_k \in X$ satisfies $f_k(y_k) = 1$.) Assume furthermore that there exists a sequence $\{d_k\}$ of nonnegative numbers such that $F_k = \{x \in X : f_k(x) \leq d_k\}$. If $0 \notin \text{cl}(\text{conv}(\{y_k\}_{k\in\mathbb{N}}))$, then $F$ is a contractive set.

Proof We follow the idea included in [7, Lemma 3]. Fix a sequence $a_n$ of positive numbers such that $\sum_{n=1}^{\infty} a_n = 1$. By the proof of Theorem 5 and Lemma 1 the mapping $Q_n : X \to F_n$ defined by $Q_n x = x - f_n(x - z_n)y_n$ if $f_n(x) > d_n$ and $Q_n x = x$ in the opposite case, is a contractive projection from $X$ onto $F_n$. (Here for each $n \in \mathbb{N}$ $z_n$ is chosen so that $f_n(z_n) = d_n$.) Define $Q : X \to X$ by

$$Q x = \sum_{n=1}^{\infty} a_n Q_n x.$$ 

It is easy to see that $Q$ is a nonexpansive mapping. We show that $\text{Fix}(Q) = F$, where $\text{Fix}(Q)$ denotes the set of all fixed points of $Q$. It is easy to see that $F \subset \text{Fix}(Q)$. 

\[ Springer\]
Assume on the contrary that there exists $x \in \text{Fix}(Q) \setminus F$. Since $F = \bigcap_{k=1}^{\infty} F_k$, the set

$$Z = \{ n \in \mathbb{N} : Q_k(x) \neq x \} \neq \emptyset.$$  

By definition of $Q$

$$x = \sum_{k \notin Z} a_k x + \sum_{k \in Z} a_k Q_k x.$$  

(We assume that $\sum_{k \notin Z} = 0$ if $Z = \mathbb{N}$.) By definition of $Q_k$,

$$0 = \sum_{k \in Z} a_k f_k(x - z_k) y_k.$$  

Since $a_k > 0$ and $f_k(x - z_k) > 0$ for $k \in Z$, $0 \in \text{cl}(\text{conv}([y_k]_{k \in \mathbb{N}}))$, which is a contradiction. □

**Remark 2** In [7, Theorem 4], sufficient conditions for a Banach space $X$ satisfying (CFPP) are presented. In particular, any finite-dimensional Banach space satisfies (CFPP).

**Example 1** Let $X = l_{\infty}^{(n)}$. Let for $k \in \mathbb{N}$, $f^k \in S(X^*)$, $f^k = (f^k_1, \ldots, f^k_n)$ be so chosen that

$$|f^k_j| \geq \sum_{i \neq j} |f^k_i|$$  

for $j \in \{1, \ldots, n\}$ depending on $k$. By [4], for any $k$, $\text{ker}(f^k)$ is one-complemented in $X$. Fix a sequence of nonnegative numbers $\{d_k\}$. By Theorem 5 the half spaces $F_k = \{ x \in X : f^k(x) \leq d_k \}$ are contractive subsets of $X$. Assume that $F = \bigcap_{k=1}^{\infty} F_k$, is a nonempty set. Let for any $j \in \{1, \ldots, n\}$, $Z_j = \{ k \in \mathbb{N} : |f^k_j| \geq \sum_{i \neq j} |f^k_i| \}$. Assume that for some $j \in \{1, \ldots, n\}$ such that $Z_j \neq \emptyset$, $f^k_j > 0$ for any $k \in Z$. By [4, Th. 1, p. 217] and the proof of Theorem 11, for each $k \in \mathbb{N}$, $Q_kx = x - f^k (x - z_k) y_k$, where $y_k = (0, (f^k_j/j), \ldots, 0)$ where $j \in \{1, \ldots, n\}$ satisfies (3). Hence by our assumption on $Z_j$, $0 \notin \text{cl}(\text{conv}([y_k]_{k \in \mathbb{N}}))$. By Theorem 11, $F$ is a contractive subset of $X$. Observe that if for some $k$ $f^k$ has more than two coordinates different from zero, this result cannot be deduced from Theorem 10.

**Remark 3** The assumption that $F$ is a bounded set in Theorem 8 can be weakened. If we assume that $F = \text{cl}(\bigcup_{t \in T} F_t)$ where $\{F_t\}_{t \in T}$ is a directed by $\subset$ family of bounded and convex existence sets such that $\text{int}(F_{t_0}) \neq \emptyset$ for some $t_0 \in T$, by Lemma 6 the conditions (a) and (b) from Theorem 8 are equivalent. If $T = \mathbb{N}$ and $\text{int}(F) \neq \emptyset$, then by the Baire Property $\text{int}(F_{t_0}) \neq \emptyset$ for some $t_0 \in \mathbb{N}$.

Now we prove the last result of this section concerning strictly convex spaces. Observe that by Remark 1, this result is true only for strictly convex spaces.

Springer
Theorem 12 Let \((X, \| \cdot \|)\) be a strictly convex, reflexive and separable Banach space and let \(F \subset X\) be a bounded set with nonempty interior (compare with Remark 3.) Then the following conditions are equivalent

a. \(F\) is an existence set;
b. \(F\) is a contractive set;c. \(F\) is an intersection of a countable number of contractive half-spaces.d. \(F\) is an optimal set.

Proof If \(F\) is an existence set, then, by definition, \(F\) is an optimal set. Since \(X\) is strictly convex, if \(F\) is an optimal set, by \([5, \text{Prop. 3, p. 110}]\), \(F\) is a convex set. By Theorem 1, \(F\) is an existence set. Hence by Theorem 8, the proof is complete. \(\square\)

Remark 4 There exists Banach spaces such that there is no bounded, convex sets with nonempty interior being existence sets. For example, if we take \(X = L_p[0, 1]\) for \(1 < p < +\infty\), then there is no \(F \subset X\), \(F\) convex with \(\text{int}(F) \neq \emptyset\) being an existence set. Indeed, if such an \(F\) exists, then by Theorem 6 \(F\) would be an intersection of a countable family of contractive half-spaces \(\{D_n\}\). By Theorem 4, each \(D_n\) is determined by \(f_n \in S(X^*)\) with \(\ker(f_n)\) being one-complemented. But by \([5]\) (see also \([13, 19, 23, 30]\)), no hyperplane in \(X\) is one-complemented.

4 Section

Now, we prove some results on existence sets \(F\) without assumptions that they have nonempty interior in the whole space \(X\). To the end of this section, if is not otherwise stated, we assume that \(\text{int}_Z(F) \neq \emptyset\) where \(Z = \text{cl}(\text{Span}(F))\) and \(\text{int}_Z(F)\) means the interior with respect to \(Z\).

Remark 5 It may happen that \(\text{int}_Z(F) = \emptyset\). For example, if we take \(X = L_p[0, 1]\) for \(1 \leq p < +\infty\), and \(F = \{f \in X : f \geq 0\}\) then it is easy to see that \(\text{Span}(F) = X\) and \(\text{int}_X(F) = \emptyset\). Observe that \(F\) is a contractive subset of \(X\). Indeed, it is easy to see that the mapping \(Pf = f \chi(D_f,+)\) is a contractive projection onto \(F\). Here \(\chi(D_f,+)\) denote the characteristic function of the set \(D_{f,+} = \{t \in [0, 1] : f(t) \geq 0\}\). If \(X\) is finite-dimensional, then \(\text{int}_Z(F) \neq \emptyset\) for any \(F \subset X, F \neq \emptyset\).

We start with

Theorem 13 Let \(X\) be a Banach space and let \(F \subset X\) be a convex and bounded (compare with Remark 3) existence set. Assume that \(Z = \text{cl}(\text{Span}(F))\) satisfies the assumption of Theorem 8. If \(Z\) is a contractive subset of \(X\) then \(F\) is a contractive subset of \(X\).

Proof Since \(F\) is an existence set in \(X\), \(F\) is an existence set in \(Z\). By Theorem 8 applied to \(Z\) there exists a contractive projection \(Q : Z \to F\). Let \(P : X \to Z\) be a contractive projection. The \(Q \circ P\) is a contractive projection from \(X\) onto \(F\). The proof is complete. \(\square\)

Now we present examples of Banach spaces \(X\) in which any subspace that is an existence set is a contractive set. In \([20]\), the following result was shown.
Theorem 14 Let $X$ be a Banach space and let $Z \subset X$, be a linear subspace, which is an existence set. Put

$$G_Z = \{ z \in Z \setminus \{0\} : \text{there exists exactly one } f \in S(X^*) : f(z) = \|z\| \}. \quad (4)$$

Assume that the norm closure of $G_Z$ in $X$ is equal to $Z$. Then there exists exactly one projection $P \in \mathcal{P}(X, Z)$ such that $\|P\| = 1$, which means that $Z$ is a contractive subset of $X$.

Remark 6 In particular, if $X$ is a smooth space, the norm closure of $G_Z$ is equal to $Z$ for any subspace $Z$. Applying Theorem 14 it was shown in [20], that in $c_0, l_1$ and some Musielak-Orlicz sequence spaces any subspace $Z$ which is an existence set is one-complemented. Also it was shown in [18] that any subspace $Z$ of the Lorentz sequence space $l_{1,w}$ which is an existence set, is contractive.

Corollary 2 Let $X$ be a reflexive Banach space. Let $F \subset X$ be a bounded and convex existence set such that $\dim(Z = \text{span}(F)) < \infty$. If $X$ is strictly convex and smooth then $F$ is contractive. If $X$ is finite-dimensional, strictly convex and smooth Banach space then any bounded optimal set is contractive.

Proof If $\dim(Z = \text{span}(F)) < \infty$, then $\text{int}_Z(F) \neq \emptyset$. By Theorem 13 and Theorem 14 $F$ is contractive set. If $X$ is strictly convex and reflexive and $F$ is a bounded optimal set then by [5], $F$ is a convex existence set. If $X$ is finite-dimensional, then $\text{int}_Z(F) \neq \emptyset$ for any $F \subset X$. The proof is complete. □

The following result can be applied to sets satisfying $\text{int}_Z(F) = \emptyset$. The proof of it is the same as in [17].

Theorem 15 Let $X$ be a reflexive smooth Banach space and let $F \subset X$ be a convex existence set such that $F = \overline{\bigcup_{t \in T} F_t}$, where $\{F_t\}_{t \in T}$ is a family of convex and compact existence sets ordered by $\subset$. Then $F$ is a contractive set. If $F$ is a convex, compact existence set then the assumption of reflexivity can be omitted.

Proof Let $x \in X$ and fix $t \in T$. By [17, Th. 3.3] there exists $P_t x \in RF_t(x)$ such that for any $d \in F_t$ and $s \geq 0$

$$\|P_t x - d\| \leq \|sx + (1 - s)P_t x - d\|.$$ 

Since $X$ is smooth, by [8, Lemma 1], $P_t x$ is uniquely determined. By [8, Th. 1] $F_t$ is a contractive set. (Up to now the reflexivity is not needed.) By [7, Lemma 4], $F$ is a contractive set. □

Remark 7 In general, we do not know under which conditions on a Banach space $X$ any existence and convex set can be represented as $F = \overline{\bigcup_{t \in T} F_t}$, where $\{F_t\}_{t \in T}$ is a family of convex and compact existence sets ordered by $\subset$. Such a result has been proven [17, Lemma 3.7] for reflexive Köthe sequence spaces. For $l_p$-spaces, $1 \leq p < \infty$, such a result has been demonstrated in [9].
Remark 8  Notice that in general a contractive set does not need to be convex. Let, for example $X = l^{(2)}_{\infty}$. Define

$$F = \{(x, y) \in X : |y| \leq |x|\}.$$  

Then after elementary, but tedious calculations, one can see that the mapping $P : X \to F$ defined by

$$P(x, y) = \begin{cases} (x, \text{sgn}(y)x) & \text{for } x \geq 0, |y| > |x| \\ (x, -\text{sgn}(y)x) & \text{for } x < 0, |y| > |x| \\ (x, y) & \text{for } |y| \leq |x| \end{cases}$$

is a contractive projection from $X$ onto $F$. It would be interesting to obtain a relation between existence and contractive sets in nonconvex case.

5 The complex case

Notice that the following well-known lemma permit us to adopt the results proved for real Banach spaces in the case of complex Banach spaces. Since we cannot find a proper reference for it, we present a proof of it for a convenience of the reader.

Lemma 8  Let $X$ be a complex Banach space with a Hamel basis $H = \{h_t\}_{t \in T}$. Let $X_R$ be a real linear space spanned by $\{h_t, ih_s\}_{s, t \in T}$. Then $H_R = \{h_t, ih_s\}_{s, t \in T}$ is a Hamel basis of $X_R$ over $\mathbb{R}$. Let us equip $X_R$ with a norm induced from $X$, i.e.,

$$\left\| \sum_{j=1}^{n} a_j h_{t_j} + \sum_{j=1}^{m} b_j i h_{t_j} \right\| = \left\| \sum_{j=1}^{n} (a_j + ib_j) h_{t_j} \right\|$$

Then the mapping $I_Z : X \to X_R$ defined by

$$I_Z \left( \sum_{j=1}^{n} a_j h_{t_j} \right) = \sum_{j=1}^{n} \text{re}(a_j) h_{t_j} + \sum_{j=1}^{n} \text{im}(a_j) i h_{t_j}$$

is a linear surjective isometry over $\mathbb{R}$ which means that $\|I_Z(x)\| = \|x\|$, $I_Z(x + y) = I_Z(x) + I_Z(y)$, $I_Z(ax) = aI_Z(x)$ for $a \in \mathbb{R}$, and $x, y \in X$. In particular, $X_R$ is a real Banach space.

Proof  Since $H$ is a Hamel basis over $\mathbb{C}$, the set $\{h_t, ih_s\}_{s, t \in T}$ is linearly independent over $\mathbb{R}$. The fact that $I_Z$ is a linear isometry over $\mathbb{R}$ follows immediately from the definition of the norm in $X_R$. To prove surjectivity, fix $x \in X_R$. Then $x = \sum_{j=1}^{n} a_j y_{t_j} + \sum_{k=1}^{m} b_k y_{s_k}$ with $a_j \neq 0$ and $b_k \neq 0$. Let $S = \{t_1, ..., t_n, s_1, ..., s_m\}$. Let $z = \sum_{j \in S} c_j y_{j}$, where $c_j = a_j + ib_j$. (We put $a_j = 0$ if $j \notin \{s_1, ..., s_m\}$ and $b_j = 0$ if $j \notin \{t_1, ..., t_n\}$). It is clear that $I_Z(x) = x$, which completes the proof.  \[\square\]
We also need

**Lemma 9** Let $X$ be a complex Banach space. Then $X$ is reflexive if and only if $X_R$ is reflexive, $X$ is strictly convex if and only if $X_R$ is strictly convex, $X$ is smooth if and only if $X_R$ is smooth. Moreover $X$ satisfies (CFPP) if and only if $X_R$ satisfies (CFPP).

**Proof** The proofs of all above properties follow from Lemma 8. Before presenting them, fix $f \in X^*$. Then $f = re(f) + i(im(f))$. Put $g = re(f)$. Since $f(ix) = if(x)$ for any $x \in X$, we easily get that $f(x) = g(x) - ig(ix)$. Moreover, $\|f\| = \|g\|$. Indeed, it is immediate that $\|g\| \leq \|f\|$. To prove a converse, fix $x \in X$, $\|x\| = 1$. Then $f(e^{it}x) = g(e^{it}x)$ for some $t \in [0, 2\pi]$. Hence

$$|f(x)| = |f(e^{it}x)| = |g(e^{it}x)| \leq \|g\||e^{it}x| = \|g\|$$

which shows our claim. Moreover, by the above reasoning if $g \in (X_R)^*$ then the mapping $f : X \to \mathbb{C}$ given by $f(x) = g(x) - ig(ix)$ belongs to $X^*$ and $\|f\| = \|g\|$. Hence, in particular, any $f \in S(X^*)$ attains its norm at some $x \in S(X)$ if and only if any $g \in (X_R)^*$ attains its norm in some $z \in S(X_R)$. Consequently, by the James Theorem $X$ is reflexive if and only if $X_R$ is reflexive. Analogously, for any $x \in S(X)$ there exists exactly one $f \in S(X^*)$ satisfying $f(x) = \|x\| = 1$ if and only if for any $z \in S(X_R)$ there exists exactly one $g \in S((X_R)^*)$ satisfying $g(z) = \|z\| = 1$. Hence $X$ is smooth if and only if $X_R$ is smooth. The equivalence of strict convexity for $X$ and $X_R$ and the equivalence of (CFPP) for $X$ and $X_R$ follows immediately from Lemma 8, so we omit the proofs. \[\square\]

**Theorem 16** Theorems 6, 7, 8 (By a half-space determined by $f \in S(X^*)$ we understand the set \{ $x \in X : re(f)(x) \leq d$ \}). Theorems 9, 10, 11, 12, 13, 14 and 15 hold true for any complex Banach space $X$.

**Proof** Let $X$ be a complex Banach space and let $F \subset X$. Then by Lemma 8, $F$ is a convex set if and only if $Iz(F)$ is a convex set, $F$ is an existence set in $X$ if and only if $Iz(F)$ is a and existence set in $X_R$ and $F$ is a contractive subset of $X$ if and only if $Iz(F)$ is a contractive subset of $X_R$. By Lemma 9, we can adopt the proofs of the above mentionned theorems given in the real case to the complex case. \[\square\]

**Acknowledgements** The first author (Maciej Ciesielski) is supported by Poznań University of Technology, Poland, Grant No. 0213/SIGR/2154.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.
References

1. Baronti, M., Papini, P.L.: Norm-one projections onto subspaces of $l_p$. Ann. Mat. Pura Appl. 4(152), 53–61 (1988)
2. Beauzamy, B.: Projections contractantes dans les espaces de Banach. Bull. Sci. Math. 102, 43–47 (1978)
3. Beauzamy, B., Enflo, P.: Théorèmes de point fixe et d’approximation. Ark. Mat. 23(1), 19–34 (1985)
4. Blatter, J., Cheney, E.W.: Minimal projections onto hyperplanes in sequence spaces. Ann. Mat. Pura ed Appl. 101, 215–227 (1974)
5. Beauzamy, B., Maurey, B.: Points minimaux et ensembles optimaux dans les espaces de Banach. J. Funct. Anal. 24, 107–139 (1977)
6. Bohnenblust, F.: Subspaces of $l_{p,q}$ spaces. Am. J. Math. 63, 64–72 (1941)
7. Bruck, R.E., Jr.: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. 179, 251–262 (1973)
8. Bruck, R.E., Jr.: Nonexpansive projections on subsets of Banach spaces. Pac. J. Math. 47(2), 341–355 (1973)
9. Davis, V., Enflo, P.: Contractive projections on $l_p$-spaces. Lond. Math. Soc. Lect. Notes Ser. 137, 151–161 (1989)
10. de Figueiredo, D.G., Karlovitz, L.A.: On the extensions of contractions of normed spaces. In: Nonlinear Functional Analysis, Proceedings of Symposia in Pure Mathematics, vol. 18,1, pp. 95–104. Providence: Amer. Math. Soc (1970)
11. Enflo, P.: Contractive projections onto subsets of $L^1(0, 1)$. Lond. Math. Soc. Lect. Notes Ser. 137, 162–184 (1989)
12. Enflo, P.: Contractive projections onto subsets of $L^p$-spaces. In: Lecture Notes in Pure and Applied Mathematics, Function Spaces, vol. 136, pp. 79–94. New York, Basel, Marcel Dekker Inc., (1992)
13. Franchetti, C.: The norm of minimal projection onto hyperplanes in $L^p[0, 1]$ and the radial constant. Boll. Un. Mat. Ital. B 7(4), 803–821 (1990)
14. Gruber, P.M.: Fixpunktmengen von Kontraktionen in endlichdimensionalen normierten Räumen. Geom. Dedicata 4, 179–198 (1975)
15. Hetzelt, L.: On suns and cosuns in finite dimensional normed real vector spaces. Math. Hungar. 45, 53–68 (1985)
16. Jamison, J., Kamińska, A., Lewicki, G.: One-complemented subspaces of Musielak–Orlicz sequence spaces. J. Approx. Theory 130, 1–37 (2004)
17. Kamińska, A., Lewicki, G.: Contractive and optimal sets in modular spaces. Math. Nachr. 268, 74–95 (2004)
18. Kamińska, A., Lewicki, G.: Extreme and smooth points in Lorentz and Marcinkiewicz Spaces with applications to contractive projections. Rocky Mt. J. Math. 39(5), 1533–1572 (2009)
19. Lewicki, G., Skrzypek, L.: Minimal projections onto hyperplanes in $l_{p,n}^p$. J. Approx. Theory 202, 42–63 (2016)
20. Lewicki, G., Trombetta, G.: Optimal and one-complemented subspaces. Monatsh. Math. 153, 115–132 (2008)
21. Lindenstrauss, J.: On projections with norm 1: an example. Proc. Am. Math. Soc. 15, 403–406 (1964)
22. Papini, P.L., Singer, I.: Best co-approximation in normed linear spaces. Monatsh. Math. 88, 27–44 (1979)
23. Randrianantoanina, B.: Norm one projections in Banach spaces. Taiwan. J. Math. 5, 35–95 (2001)
24. Rao, T.S.S.R.K.: On ideals and generalized centers of finite sets in Banach spaces. J. Math. Anal. Appl. 398(2), 866–888 (2013)
25. Rao, T.S.S.R.K.: On intersections of ranges of projections of norm one in Banach spaces. Proc. Am. Math. Soc. 141(10), 3579–3586 (2013)
26. Rao, T.S.S.R.K.: Existence sets of coapproximation and projections of norm one. Monatsh. Math. 176(4), 607–614 (2015)
27. Rao, T.S.S.R.K.: Coproximinality for quotient spaces. Zeitschrift für Analysis und Ihre Anwendungen 36(2), 151–157 (2017)
28. Rao, T.S.S.R.K.: Into isometries of Banach spaces. Recent Trends in Operator Theory and Applications, Book Series: Contemporary Mathematics, vol. 137, pp. 135–144 (2019)
29. Reich, S.: Product formulas, nonlinear semigroups and accretive operators. J. Funct. Anal. 36, 147–168 (1980)
30. Rolewicz, S.: On projections on subspaces of codimension one. Studia Math. 96(1), 17–19 (1990)
31. Westphal, U.: Cosums in $l^p(n)$. J. Approx. Theory 54, 287–305 (1988)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.