Equivalence Between Indirect Controllability and Complete Controllability for Quantum Systems

Domenico D’Alessandro
Department of Mathematics, Iowa State University, Ames IA-5001, Iowa, U.S.A.;
Electronic address: dmdaless@gmail.com.

July 27, 2012

Abstract

We consider a control scheme where a quantum system $S$ is put in contact with an auxiliary quantum system $A$ and the control can affect $A$ only, while $S$ is the system of interest. The system $S$ is then controlled indirectly through the interaction with $A$. Complete controllability of $S + A$ means that every unitary state transformation for the system $S + A$ can be achieved with this scheme. Indirect controllability means that every unitary transformation on the system $S$ can be achieved. We prove in this paper, under appropriate conditions and definitions, that these two notions are equivalent in finite dimension. We use Lie algebraic methods to prove this result.

Keywords: Controllability of quantum systems, Lie algebraic methods, interacting systems.

1 Introduction

In many experimental set-ups, a quantum system $S$, which is the target of control, is put in contact with an auxiliary quantum system $A$ and the control can only directly affect $A$, while $S$ is the system of interest. Therefore $S$ is controlled indirectly via the interaction with $A$. The (indirect) controllability of $S$ with this scheme has been studied in several papers and for various physical examples (see, e.g., [1], [5]). However always conditions have been given so that the full system $S + A$ is completely controllable, i.e., every unitary transformation can be achieved in the Hilbert space associated with the full system. This implies in particular that $S$ is indirectly controllable, i.e., any unitary transformation on the state of $S$ can be obtained. The opposite is in general not true and there are schemes where one can have indirect controllability of the system $S$ without having complete controllability of the full system $S + A$. An example of this was given in [4] (Proposition 5.2) for the case of two coupled qubits $S$ and $A$. Whether or not we can have indirect controllability of $S$ without complete controllability of $S + A$, depends in general on the initial state assumed for $A$. In this paper we shall prove that if the initial state of $A$ is the perfectly mixed state (see definitions below), then complete controllability is also necessary to have indirect controllability. Therefore if we require indirect controllability for an arbitrary state of the system $A$ the two definitions are equivalent. We now describe in mathematical terms the definitions and result of this paper.

The state of a finite dimensional quantum mechanical system is represented by a density matrix, that is, a trace 1, positive semidefinite Hermitian matrix acting as a linear operator on a
Hilbert space associated with the system. The dimension of the system refers to the dimension of this Hilbert space. We shall denote by \( \rho_S \), \( \rho_A \), and \( \rho_{TOT} \), the density matrices for the systems \( S \), \( A \) and \( S+A \), respectively, which have dimensions \( n_S \), \( n_A \) and \( n_S n_A \), respectively. The density matrix \( \rho_S (\rho_A) \) is obtained from \( \rho_{TOT} \) through the operation of partial trace with respect to \( A \) (\( S \)), that is

\[
\rho_S = Tr_A(\rho_{TOT}), \quad \rho_A = Tr_S(\rho_{TOT}).
\]

(1)

The dynamics of the total system is determined by

\[
\rho_{TOT}(t) = X_{TOT}(t)\rho_{TOT}(0)X_{TOT}^\dagger(t),
\]

(2)

where \( X_{TOT} \) is the solution of Schrödinger operator equation

\[
i\dot{X}_{TOT} := H(u)X_{TOT}, \quad X_{TOT}(0) = 1_{n_S n_A}.
\]

(3)

In (3), \( 1_{n_S n_A} \) is the \( n_S n_A \times n_S n_A \) identity and \( H(u) \) is the Hamiltonian operator, an \( n_S n_A \times n_S n_A \) Hermitian matrix which we assume function of a control \( u \).

According to the Lie algebra rank condition \([6]\) applied to quantum control (see, e.g., \([2]\)), the set \( \mathcal{R} \) of possible transformations, \( X_{TOT} \), which can be obtained as solutions of (3) is as follows. Let \( \mathcal{L} \) be the Lie algebra generated by the set

\[
\mathcal{F} := \{ i H(u) \mid u \in U \},
\]

(4)

where \( U \) is the set of possible values for the control \( u \). Denote by \( e^\mathcal{L} \) the associated Lie group. If \( e^\mathcal{L} \) is compact, then \( \mathcal{R} \) is equal to \( e^\mathcal{L} \). If \( e^\mathcal{L} \) is not compact, then \( \mathcal{R} \) is dense in \( e^\mathcal{L} \). In the following, in order not to complicate the exposition, we shall neglect this distinction and always assume \( \mathcal{R} = e^\mathcal{L} \) where sometimes the equality between two topological spaces really means that one space is dense in the other. The Lie algebra \( \mathcal{L} \) is called the dynamical Lie algebra associated with the system \( S + A \). If \( \mathcal{L} \) is the full \( u(n_S n_A) \) or \( su(n_S n_A) \) \([3]\) then the system \( S + A \) is called completely controllable and \( \mathcal{R} \) is \( U(n_S n_A) \) or \( SU(n_S n_A) \), respectively \([\mathcal{L}] \). In this case, every unitary transformation on the initial state \( \rho_{TOT}(0) \) according to (2) is possible.

We shall assume in this paper that systems \( S \) and \( A \) are initially uncorrelated, i.e., the initial state \( \rho_{TOT}(0) \) has the form \( \rho_{TOT}(0) = \rho_S(0) \otimes \rho_A(0) \). The evolution of the target system \( S \) is obtained by combining (2) with (1), i.e.,

\[
\rho_S(t) = Tr_A \left( X_{TOT}(t)\rho_S(0) \otimes \rho_A(0)X_{TOT}^\dagger(t) \right),
\]

(5)

where \( X_{TOT} \) is the solution of (3). Therefore, the set of available states for the system \( S \), starting from \( \rho_S \), and with \( A \) in the initial state \( \rho_A \), is

\[
\mathcal{R}_S := \left\{ Tr_A(X_{TOT} \rho_S \otimes \rho_A X_{TOT}^\dagger)|X_{TOT} \in e^\mathcal{L} \right\}.
\]

(6)

\( ^1 \)In the following, \( 1_v \) denotes the \( v \times v \) identity. We shall omit the index \( v \) when the dimension is obvious from the context.

\( ^2 \)This last statement is a consequence of the fact that for quantum systems \( e^\mathcal{L} \) is always a subgroup of the unitary Lie group \( U(n_S n_A) \) (cf. \([3], [7]\)).

\( ^3 \)The Lie algebras of \( n_S n_A \times n_S n_A \) skew-Hermitian matrices or \( n_{SN} A \times n_{SN} A \) skew-Hermitian matrices with zero trace, respectively.

\( ^4 \)The full Lie group of \( n_{SN} A \times n_{SN} A \) unitary matrices or the full Lie group of \( n_{SN} A \times n_{SN} A \) unitary matrices with determinant equal to one, respectively.
In indirect control schemes, the set of generators of the dynamical Lie algebra $\mathcal{L}$, i.e., $\mathcal{F}$ in (4), is to be taken of the form

$$\mathcal{F} := \{J\} \cup \{\tilde{B}\},$$

(7)

where the set $\tilde{B}$ generates a Lie subalgebra $\mathcal{B}$ of $u(n_{SA})$ of matrices of the form $1_{nS} \otimes B$ with $B$ in $u(n_A)$. This subalgebra describes the control authority we have on the auxiliary system $A$. Transformations in the associated Lie group, $e^B$, are all available and they are of the form $1 \otimes X_A$, with $X_A \in U(n_A)$. Therefore any initial state $\rho_S \otimes \rho_A$ can be transformed as

$$\rho_S \otimes \rho_A \rightarrow (1 \otimes X_A)\rho_S \otimes \rho_A(1 \otimes X_A^\dagger) = \rho_S \otimes (X_A\rho_AX_A^\dagger).$$

(8)

In (7) the (Hamiltonian) matrix $J$ models the autonomous (non-controlled) dynamics of the system $S$, the autonomous dynamics of the system $A$ and the interaction between the system $S$ and the auxiliary system $A$. These three terms, in that order, are the three summands in the definition of $J$

$$J := K \otimes 1 + 1 \otimes L + \sum_{j=1}^{n} iS_j \otimes \sigma_j.$$  

(9)

Here $K$ and $S_j$, $j = 1, \ldots, n$, are in $su(n_S)$, $L$ and $\sigma_j$, $j = 1, \ldots, n$, are in $su(n_A)$ and the $\sigma_j$’s are linearly independent. In the following, we shall assume that $\mathcal{B}$ does not contain any nonzero trace element, so that the dynamical Lie algebra $\mathcal{L}$ is a Lie subalgebra of $su(n_{SA})$. This is done without loss of generality as multiples of the identity only induce a common phase factor in equation (4) which has no effect on the dynamics of $\rho_{TOT}$ in (2).

There are several notions of indirect controllability [4], according to the restrictions we place on the possible initial states for the auxiliary system $A$ and the possible states we require to reach for the system $S$, starting from $\rho_S(0)$ (e.g., unitary equivalent, or general density matrices). We shall adopt, in this paper, the following definitions (cf. [4]).

**Definition 1.1.** The system $S$ is called indirectly controllable given $\rho_A$ (initial state of $A$) if, for every initial density matrix $\rho_S$ and every unitary $X_S \in U(n_S)$, there exists a (reachable) $X_{TOT} \in e^\mathcal{L}$ such that (cf. (5))

$$X_S\rho_SX_S^\dagger = Tr_A(X_{TOT}\rho_S \otimes \rho_A X_{TOT}^\dagger).$$

(10)

**Definition 1.2.** The system $S$ is called strongly indirectly controllable if it is indirectly controllable given $\rho_A$ for every initial state $\rho_A$ of $A$.

In other terms, we are able to steer the state of the system $S$ between any two unitarily equivalent states independently of the state $\rho_A$ of the auxiliary system $A$. The indirect control scheme works just as well as a completely controllable scheme for system $S$. It was proven in [4], for the case where both $S$ and $A$ are qubits, and every unitary is available on the system $A$ (i.e., $\mathcal{B} = su(n_A)$ above), that this property is equivalent to complete controllability of the total system. The goal of this paper is to extend this result to the case where $S$ and $A$ have arbitrary dimensions. In particular, our main result is as follows:

**Theorem 1.** Assume $\mathcal{B} = su(n_A)$. A system $S$ is indirectly controllable given the perfectly mixed state $\rho_A := \frac{1}{n_A}1$ for $A$ if and only if the total system $S + A$ is completely controllable. Therefore it is strongly indirectly controllable if and only if the system $S + A$ is completely controllable.
Indirect controllability can be studied using Lie algebraic methods but the investigation is complicated by the fact that the various controllability notions are not invariant under general (unitary) coordinate transformations in the state space of the system $S + A$. They are invariant only under local transformations, that is, transformations which act on the Hilbert spaces of $S$ and $A$ separately. For instance, if we replace the dynamical Lie algebra $\mathcal{L}$ with $\mathcal{L}' := (T_S \otimes T_A)\mathcal{L}(T_S^\dagger \otimes T_A^\dagger)$, with $T_S \in U(n_S)$ and $T_A \in U(n_A)$, then indirect controllability is not modified as it can be easily seen using the property of the partial trace

$$Tr_A((T_S \otimes T_A)\rho_{TOT}(T_S^\dagger \otimes T_A^\dagger)) = T_STr_A(\rho_{TOT})T_S^\dagger.$$  \hfill (11)

One direction of Theorem 1 follows immediately from the property (11) of the partial trace. In fact, if $S + A$ is completely controllable, $e^L = SU(n_Sn_A)$ in particular contains every matrix of the form $T_A \otimes 1$, with $T_A \in SU(n_S)$, and the claim follows from (11) using $\rho_{TOT} := \rho_S \otimes \rho_A$.

The rest of the paper is devoted to proving the other direction of Theorem 1. In section 2 we give some preliminary technical results after which, the proof is presented in section 3. We give some concluding remarks in section 4.

2 Preliminary Results

**Lemma 2.1.** Consider two matrices $X$ and $Y$ in $su(n)$. Then $X$ and $Y$ are linearly dependent if and only if $[X,Y] = 0$ and, for every $A \in su(n)$,

$$[[A,X], [A,Y]] = 0.$$  \hfill (12)

**Proof.** One direction is straightforward. If $X$ and $Y$ are linearly dependent, then we can write $X = \alpha Y$ (or $Y = \alpha X$), for some $\alpha \in \mathbb{R}$. Then we have $[X,Y] = [\alpha Y,Y] = \alpha [Y,Y] = 0$. Furthermore, for arbitrary $A \in su(n)$, we have

$$[[A,X], [A,Y]] = [[A,\alpha Y], [A,Y]] = \alpha [[A,Y], [A,Y]] = 0.$$  \hfill (13)

To prove the converse implication, we first notice that since $X$ and $Y$ commute, they can be simultaneously diagonalized. By applying the same similarity transformation to all elements in $su(n)$, there is no loss of generality in assuming that $X$ and $Y$ are both diagonal. Moreover this proves the Lemma for $n = 2$, since we can write $X = \alpha \sigma_z$, and $Y = \beta \sigma_z$, for some real numbers $\alpha$ and $\beta$ and $\sigma_z$ denoting the Pauli $z$–matrix which gives $\alpha Y - \beta X = 0$. Therefore, we can assume $n \geq 3$. Let us denote by $A_{jk}$, with $j \neq k$, the matrix in $su(n)$

$$A_{jk} := |j\rangle\langle k| - |k\rangle\langle j|.$$  \hfill (14)

For $j \neq k$, let us also denote by $E_{jk}$ the matrix $E_{jk} := i|j\rangle\langle k| + i|k\rangle\langle j|$. By writing $X := \sum_{l=1}^n ix_l|l\rangle\langle l|$, a straightforward calculation shows that

$$[A_{jk},X] = (x_k - x_j)E_{jk} := X_{kj}E_{jk},$$  \hfill (15)

where we used the definition $X_{kj} := x_k - x_j$. Also, using the definition $Y_{kj} := y_k - y_j$, we have $[A_{jk},Y] = Y_{kj}E_{jk}$. Now, with these notations, fix two indices $a$ and $b$ in $\{1, 2, \ldots, n\}$, with $a \neq b$. **\footnote{$\sigma_z := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$}**
Consider another index $g$ in $\{1, 2, \ldots, n\}$ different from both $a$ and $b$. Consider $A = A_{ab} + A_{ga}$. We have from (12) and (13),

\[ 0 = [[A, X], [A, Y]] = [X_{ba}E_{ab} + X_{ag}E_{ga}, Y_{ba}E_{ab} + Y_{ag}E_{ga}] = (X_{ba}Y_{ag} - X_{ag}Y_{ba})[E_{ab}, E_{ga}], \quad (16) \]

which implies

\[ X_{ba}Y_{ag} = X_{ag}Y_{ba}. \quad (17) \]

By choosing $A = A_{ab} + A_{gb}$, we find analogously

\[ X_{ba}Y_{bg} = X_{bg}Y_{ba}. \quad (18) \]

Summing (17) and (18), we find for any $a$, $b$ and $g$

\[ X_{ba}(y_a + y_b - 2y_g) = Y_{ba}(x_a + x_b - 2x_g). \quad (19) \]

Summing the equations (19) over all $g$ different from $a$ and $b$. We obtain

\[ X_{ba}\left((n - 2)(y_a + y_b) - 2 \sum_{g \neq a, g \neq b} y_g\right) = Y_{ba}\left((n - 2)(x_a + x_b) - 2 \sum_{g \neq a, g \neq b} x_g\right). \quad (20) \]

Using the fact that both $X$ and $Y$ have zero trace we can replace $\sum_{g \neq a, g \neq b} y_g$, with $-(y_a + y_b)$ and $\sum_{g \neq a, g \neq b} x_g$ with $-(x_a + x_b)$ in the above equation. Recalling the definition of $X_{ba}$ and $Y_{ba}$, we have $(x_b - x_a)(y_a + y_b) = (y_b - y_a)(x_a + x_b)$, which gives

\[ x_b y_a = x_a y_b. \quad (21) \]

This equation is valid for any pair $a$ and $b$ in $\{1, 2, \ldots, n\}$. Equation (21) is equivalent to $X$ and $Y$ being linearly dependent. \hfill \Box

**Lemma 2.2.** (Simplicity Lemma) Consider an element $X \in su(n)$ different from zero and the space $\mathcal{V}$ defined as

\[ \mathcal{V} := \bigoplus_{k=0}^{\infty} ad_{su(n)}^k \text{span}\{X\}. \quad (22) \]

Then $\mathcal{V} = su(n)$.

**Proof.** The space $\mathcal{V}$ defined in (22) is an ideal in $su(n)$ and it is nonzero since $X \neq 0$. Since $su(n)$ is simple it has no nontrivial ideals. So it must be $\mathcal{V} = su(n)$. \hfill \Box

---

6 It exists since $n \geq 3$.
7 The equation is obvious for $g = a$ or $g = b$ or $a = b$.
8 In fact, if $X = \alpha Y$ (or $Y = \alpha X$) for some real number $\alpha$, equations (21) are automatically satisfied. Vice versa, assume (21) are verified. If at least one between $X$ and $Y$ is zero, then they are clearly linearly dependent. Assume that they are both nonzero and let $\bar{a}$ be the smallest index $a$ so that at least one between $x_a$ and $y_a$ is different from zero. If $x_a \neq 0$ then from (21) with $a = \bar{a}$ we have that if $y_a = 0$ then $y_b = 0$ for any other $b$ which implies $Y = 0$ which we have excluded. Therefore $y_a$ is also different from zero. For all $b > \bar{a}$, $\frac{y_b}{y_a} = \frac{x_b}{x_a}$.
9 For a general subspace $P$ of $u(n)$, and a Lie subalgebra $\mathcal{L}$ of $u(n)$, the spaces $ad_{\mathcal{L}}^k P$ are defined recursively as $ad_{\mathcal{L}}^0 P := P$, $ad_{\mathcal{L}}^{k+1} P := [\mathcal{L}, ad_{\mathcal{L}}^k P]$.
Lemma 2.3. (Disintegration Lemma) Consider a matrix of the form of $J$ as in (23)

$$J := K \otimes 1 + 1 \otimes L + \sum_{j=1}^{n} iS_j \otimes \sigma_j,$$

(23)

$K$ and $L$, matrices in $su(n_S)$ and $su(n_A)$, respectively, and with $\sigma_j$ linearly independent matrices in $su(n_A)$ and $S_j$ general non-zero matrices in $su(n_S)$. The Lie algebra, $L_1$, generated by $\tilde{B} := \{1 \otimes \sigma | \sigma \in su(n_A)\}$ and $J$, is the same as the Lie algebra, $L_2$, generated by $iS_1 \otimes \sigma_1, \ldots, iS_n \otimes \sigma_n$, $K \otimes 1$ and $\tilde{B} := \{1 \otimes \sigma | \sigma \in su(n_A)\}$.

Proof. The inclusion $L_1 \subseteq L_2$ is obvious since $J$ is a linear combination of $K \otimes 1$ and $iS_1 \otimes \sigma_1, \ldots, iS_n \otimes \sigma_n$ and an element of $\tilde{B}$. For the other inclusion, since $1 \otimes L$ is in $\tilde{B}$, $L_1$ is generated by $\tilde{B}$ and $J' := K \otimes 1 + \sum_{j=1}^{n} iS_j \otimes \sigma_j$.

(24)

Then we show by induction on $n$ that $K \otimes 1$ and $iS_1 \otimes \sigma_1, \ldots, iS_n \otimes \sigma_n$ are in $L_1$. For $n = 0$, this is obvious and for $n = 1$, take the Lie bracket of $J'$ with $1 \otimes T$, for some $T$ in $su(n_A)$ so that $[\sigma_1, T] \neq 0$. Then $iS_1 \otimes [\sigma_1, T]$ is in $L_1$ and, by the simplicity Lemma 2.2, $iS_1 \otimes \sigma_1$ is in $L_1$ so that $K \otimes 1$ is in $L_1$ as well. Assume now $n \geq 2$. There are two cases to be treated separately. In the first case, there exists at least one pair $\{\sigma_j, \sigma_k\}$ in $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that $[\sigma_j, \sigma_k] \neq 0$. In the second case, all the elements, $\sigma_1, \sigma_2, \ldots, \sigma_n$, commute.

Case 1: Assume, without loss of generality, that $\sigma_1$ does not commute with all the remaining $\sigma_2, \ldots, \sigma_n$. Also assume, without loss of generality, that the first $r - 1 > 0$ commutators $[\sigma_2, \sigma_1], \ldots, [\sigma_r, \sigma_1]$ form a linearly independent set while the remaining $n - r$ commutators (if any), $[\sigma_{r+1}, \sigma_1], \ldots, [\sigma_n, \sigma_1]$, can be each written as linear combinations of the first $r - 1$ ones. Therefore, we write

$$[J', 1 \otimes \sigma_1] := \sum_{j=2}^{r} iS_j \otimes [\sigma_j, \sigma_1] + \sum_{j=r+1}^{n} iS_j \otimes [\sigma_j, \sigma_1],$$

(25)

and, for $j = r + 1, \ldots, n$,

$$[\sigma_j, \sigma_1] := \sum_{l=2}^{r} a_j^l [\sigma_l, \sigma_1],$$

(26)

for some coefficients $a_j^l$, $j = r + 1, \ldots, n$, $l = 2, \ldots, r$. Defining, for $j = r + 1, \ldots, n$,

$$X_j := \sigma_j - \sum_{l=2}^{r} a_j^l \sigma_l,$$

(27)

we notice that, from (26), all $X_j$'s commute with $\sigma_1$. Moreover $\{\sigma_1, \sigma_2, \ldots, \sigma_r, X_{r+1}, \ldots, X_n\}$ form a linearly independent set. By replacing $\sigma_j$ with $X_j + \sum_{l=2}^{r} a_j^l \sigma_l$ using (27), we can write $J'$ as

$$J' := K \otimes 1 + iS_1 \otimes \sigma_1 + \sum_{l=2}^{r} i(S_l + \sum_{j=r+1}^{n} a_j^l S_j) \otimes \sigma_l + \sum_{l=r+1}^{n} iS_l \otimes X_l.$$

(28)
Now, if one of the \((S_j + \sum_{j=r+1}^{n} a_j^1 S_j)\)'s, for \(l = 2, 3, \ldots, r\), is zero, the claim follows by induction on \(n\). More precisely, it follows by induction on \(n\), that \(iS_1 \otimes \sigma_1\) belongs to \(L_1\). Applying the inductive assumption to \(J' - iS_1 \otimes \sigma_1\), we obtain that \(K \otimes 1\) and all the matrices \(iS_j \otimes \sigma_j\), \(j = 2, \ldots, n\), also belong to \(L_1\). If all the matrices \((S_j + \sum_{j=r+1}^{n} a_j^1 S_j)\), for \(l = 2, 3, \ldots, r\), are different from zero, with the expression (28) of \(J'\) and using the fact that the \(X_j\)'s commute with \(\sigma_1\), we calculate

\[
[J', 1 \otimes \sigma_1] = \sum_{l=2}^{r} i \left( S_l + \sum_{j=r+1}^{n} a_j^1 S_j \right) \otimes [\sigma_l, \sigma_1],
\]  

and since all \([\sigma_l, \sigma_1], \ l = 2, \ldots, r\), are linearly independent, it follows from induction that all matrices \(i(S_l + \sum_{j=r+1}^{n} a_j^1 S_j) \otimes [\sigma_l, \sigma_1]\), \(l = 2, \ldots, r\), belong to the Lie algebra \(L_1\). Moreover by taking repeated Lie brackets with elements \(1 \otimes \sigma\), with arbitrary \(\sigma \in su(n_A)\), and taking linear combinations, it follows from the simplicity Lemma (22) that every matrix \(i(S_l + \sum_{j=r+1}^{n} a_j^1 S_j) \otimes \sigma_l\), \(l = 2, \ldots, r\), also belongs to \(L_1\). Therefore these matrices can be subtracted from \(J'\) in (28) and the claim follows again by induction on \(n\).

**Case 2:** The proof is similar to the one of Case 1 but with some extra complications due to the fact that all the \(\sigma_j, \ j = 1, \ldots, n\), commute. Again we use induction on \(n\). Given the form of \(J'\) in (24) and the fact that, in particular, \(\sigma_1\) and \(\sigma_2\) are linearly independent, it follows from Lemma (2.1) that there must exist a matrix \(A\) in \(su(n_A)\) such that \([\sigma_2, A], [\sigma_1, A] \neq 0\). By calculating \(J' := [[J', 1 \otimes A], 1 \otimes [\sigma_1, A]]\), we see that \(L_1\) contains the matrix

\[
\tilde{J}' := \sum_{j=2}^{n} iS_j \otimes [[\sigma_j, A], [\sigma_1, A]].
\]  

Let \(m\) be the largest integer (\(\leq n\)) such that all \([[\sigma_j, A], [\sigma_1, A]],\ for \(j = 2, 3, \ldots, m\), are linearly independent. Notice \(m\) is at least 2 because \([[\sigma_2, A], [\sigma_1, A]] \neq 0\). Therefore we can write \(J'\) in (30) as

\[
\tilde{J}' = \sum_{j=2}^{m} iS_j \otimes [[\sigma_j, A], [\sigma_1, A]] + \sum_{j=m+1}^{n} iS_j \otimes [[\sigma_j, A], [\sigma_1, A]],
\]  

with, for every \(j = m + 1, \ldots, n\),

\[
[[\sigma_j, A], [\sigma_1, A]] := \sum_{k=2}^{m} \alpha_j^k [[\sigma_k, A], [\sigma_1, A]],
\]  

for some coefficients \(\alpha_j^k, \ j = m + 1, \ldots, n, \ k = 2, \ldots, m\). Defining, for \(j = m + 1, \ldots, n\),

\[
X_j := \sigma_j - \sum_{k=2}^{m} \alpha_j^k \sigma_k,
\]  

we have that \(\{\sigma_1, \ldots, \sigma_m, X_{m+1}, \ldots, X_n\}\) are linearly independent and, using (32),

\[
[[X_j, A], [\sigma_1, A]] = 0.
\]
With this definition, $J'$ in (24) can be written as

$$J' = K \otimes 1 + iS_1 \otimes \sigma_1 + i \sum_{k=2}^{m} \left( S_k + \sum_{j=m+1}^{n} \alpha_j^k S_j \right) \otimes \sigma_k + \sum_{j=m+1}^{n} iS_j \otimes X_j. \quad (35)$$

If one of the $\left( S_k + \sum_{j=m+1}^{n} \alpha_j^k S_j \right)$’s is zero then the claim follows by induction on $n$. In fact it follows by induction that $iS_1 \otimes \sigma_1$ is in $L_1$ and subtracting this from $J'$ in (24), we can apply the inductive assumption on $n$. If all of these matrices are different from zero, we consider a gain $\tilde{J}'$ in (31) calculated by taking the commutator of $J'$ in (35) with $1 \otimes A$ and then with $1 \otimes [\sigma_1, A]$ and using (34). We have

$$\tilde{J}' := \sum_{k=2}^{m} i \left( S_k + \sum_{j=m+1}^{n} \alpha_j^k S_j \right) \otimes [[\sigma_k, A], [\sigma_1, A]], \quad (36)$$

which, by the inductive assumption, gives that all $i \left( S_k + \sum_{j=m+1}^{n} \alpha_j^k S_j \right) \otimes [\sigma_k, A], [\sigma_1, A], k = 2, \ldots, m$ are in $L_1$. By the simplicity Lemma 2.2 all $i \left( S_k + \sum_{j=m+1}^{n} \alpha_j^k S_j \right) \otimes \sigma_k, k = 2, \ldots, m$, are also in $L_1$. Subtracting them all from (35) and applying again the inductive assumption, we find that $iS_1 \otimes \sigma_1$ is in $L_1$, which subtracted from (24) and applying the inductive assumption once again says that all of the $iS_j \otimes \sigma_j, j = 1, \ldots, n$ as well as $K \otimes 1$ are in $L_1$. This concludes the proof of the Lemma.

3 Proof of Theorem 1

We shall use the following general criterion of indirect controllability which was proved in [4]. Let $\rho_S \otimes \rho_A$ be the initial state of the system $S + A$ and $L$ the dynamical Lie algebra associated with the dynamics of $S + A$. Define the subspace of $u(n_Sn_A)$,

$$V := \bigoplus_{k=0}^{\infty} \text{ad}_L^k \left( \text{span}\{i\rho_S \otimes \rho_A\} \right). \quad (37)$$

Then we have the following theorem [4].

**Theorem 2.** Let $\rho_S \neq \frac{1}{n_S} 1_{n_S}$ and assume that for all $X \in SU(n_S)$ there exists $U \in e^L$ such that

$$\text{Tr}_A(U \rho_S \otimes \rho_A U^\dagger) = X \rho_S X^\dagger. \quad (38)$$

Then

$$\text{Tr}_A(V) = u(n_S). \quad (39)$$

As a corollary, recalling the Definitions 1.1 and 1.2, we have:

**Corollary 3.1.** Assume that the system $S$ is indirectly controllable given $\rho_A$, then the dynamical Lie algebra $L$ is such that, for every $n_S \times n_S$ density matrix of $S$, $\rho_S \neq \frac{1}{n_S} 1_{n_S}$, $V$ in (37) satisfies (39). In particular, if $S$ is strongly indirectly controllable, then (39) is satisfied for every $\rho_A$. 8
We are now ready to prove Theorem 1.

Proof. Let $B_A$ be an orthogonal basis of $\text{su}(n_A)$, $B_A := \{\sigma_1, \ldots, \sigma_{d_A}\}$, where $d_A := n_A^2 - 1$ is the dimension of $\text{su}(n_A)$. Every element of $\mathcal{L}$ can be written as $J$ in (9), and using Lemma 2.3, a basis for $\mathcal{L}$ can be taken of the form

$$\mathcal{B}_\mathcal{L} := \{1 \otimes \sigma_1, \ldots, 1 \otimes \sigma_{d_A}\}, \quad (40)$$

where, for every $j = 1, \ldots, d_A$, $L^j_1, \ldots, L^j_{r_j}$ can be taken orthogonal matrices in $\text{su}(n_S)$. Also $\{D_1, \ldots, D_s\}$ are linearly independent matrices in $\text{su}(n_S)$. To see this in more detail, notice that every element of $\mathcal{L}$ can be written as $J$ in (9), where the $S_j$’s are orthogonal matrices in $\text{su}(n_S)$ and the $\sigma_j$ are orthogonal matrices in $\text{su}(n_A)$ (belonging to a previously chosen basis $\{\sigma_1, \ldots, \sigma_{d_A}\}$). This is true in particular for the elements of a given basis of $\mathcal{L}$. Applying Lemma 2.3, every element in this basis can be broken into single tensor products and all the tensor products so obtained form a spanning set for $\mathcal{L}$.

Select, in this set, a maximum number of linearly independent elements. There will be elements of the form $1 \otimes F_1, \ldots, 1 \otimes F_{d_A}$, with $F_1, \ldots, F_{d_A} \in \text{su}(n_A)$ which can be replaced by the elements as in the first line of (40), as well as the other elements in (40). In summary: It follows from Lemma 2.3 that a basis of $\mathcal{L}$ can be taken made up of tensor product matrices.

Let $d_S := n_S^2 - 1$ be the dimension of $\text{su}(n_S)$. There are three possible cases:

1. $\{D_1, D_2, \ldots, D_s\}$ span $\text{su}(n_S)$, i.e., $s = d_S$.
2. $s = 0$.
3. (intermediate case) $1 \leq s < d_S$.

In the first case, since there is at least one element in the basis of $\mathcal{L}$ of the form $iB \otimes C$ with $B \in \text{su}(n_S)$ and $C \in \text{su}(n_A)$, both different from zero, it follows from the simplicity Lemma 2.2 applied to both the $S$ and the $A$ part of the tensor product the all tensor product matrices of the form $iB \otimes C$ are in $\mathcal{L}$ and therefore $\mathcal{L} = \text{su}(n_S n_A)$ and $S + A$ is completely controllable. To conclude the proof of the theorem, we have to show that, under the indirect controllability (given $\rho_A = \frac{1}{n_A^2}1$) assumption, the other two cases are not possible.

Consider the second case. To see that it is not possible, notice that there are, in the basis of $\mathcal{L}$, at least two matrices $iA \otimes \sigma_1$ and $iB \otimes \sigma_2$ with $A$ and $B$ in $\text{su}(n_S)$ non-commuting and some $\sigma_1$ and $\sigma_2$ matrices in $\text{su}(n_A)$. If this was not the case, we could choose ($\rho_A = \frac{1}{n_A^2}1$ and) $\rho_S$ commuting with all the matrices in the left hand side of the tensor products in the basis.

---

10This is because the interaction term in (9) is assumed different from zero.
of $\mathcal{L}$. With this choice, $\rho_S \otimes \rho_A$ commutes with $\mathcal{L}$ and with all elements in $\mathcal{L}$, and therefore indirect controllability is not verified ($\rho_S$ is a fixed point of the dynamics (5)). From the fact that the matrices $iA \otimes \sigma_1$ and $iB \otimes \sigma_2$ (with $A$ and $B$ non-commuting) are in $\mathcal{L}$, using the simplicity Lemma 2.2, it follows that every matrix of the form $iA \otimes \sigma$ and $iB \otimes \sigma$ with arbitrary $\sigma \in su(n_A)$ also belongs to $\mathcal{L}$. Assume $n_A$ is even and let $\sigma_e$ be the matrix with alternating 1 and $-1$ on the diagonal and zero everywhere else (so that the trace is equal to zero). By calculating $[A \otimes \sigma_e, B \otimes \sigma_e]$, using the formula $[A \otimes C, B \otimes D] = \frac{1}{2} \left( [A, B] \otimes [C, D] + [A \otimes C, B \otimes D] \right)$, we obtain

$$[A \otimes \sigma_e, B \otimes \sigma_e] = [A, B] \otimes 1_{n_A}, \quad (41)$$

which, since $A$ and $B$ do not commute, contradicts our assumption on the basis of $\mathcal{L}$. In the case where $n_A$ is odd, let $\sigma_o^j$, be the diagonal matrix having alternating +1 and $-1$ on the main diagonal, except in the position $j$ which is occupied by 0 (so that $Tr(\sigma_o^j) = 0$) and zeros everywhere else. As before, we calculate

$$\frac{1}{n_A} \sum_{j=1}^{n_A} [A \otimes \sigma_o^j, B \otimes \sigma_o^j] = [A, B] \otimes 1, \quad (42)$$

which also contradicts the assumption on the basis of $\mathcal{L}$.

The third case is also not possible. To see this, choose $\rho_S := \frac{1}{n_S} \mathbf{1}_{n_S} - \alpha iD_1$ with $|\alpha|$ different from zero but small enough so that $\rho_S$ is still positive semi-definite. With $\rho_A = \frac{1}{n_A} \mathbf{1}_{n_A}$, it follows from an inductive argument that $V$ in (37) satisfies

$$V \subseteq \mathcal{L} \oplus \text{span}\{i \mathbf{1}_{n_S} \otimes \mathbf{1}_{n_A}\}. \quad (43)$$

Taking the partial trace of both sides in (43), we have that

$$Tr_A(V) \subseteq \text{span}\{D_1, \ldots, D_s\} \oplus \text{span}\{i \mathbf{1}_{n_S}\}, \quad (44)$$

which since $s < d_S$ contradicts Theorem 2. This concludes the proof of the theorem.

\[\Box\]

4 Concluding Remarks

I have proved that indirect controllability and complete controllability are equivalent notions under appropriate assumptions. This extended the equivalence result proved in [4] (Theorem 4) from the case of two qubits to the case of target system $S$ and accessor system $A$ of arbitrary dimensions. The result in [4] was proven by listing the various possibilities for the dynamical Lie algebra $\mathcal{L}$. This list also showed that, if we choose the initial state of the accessor, $\rho_A$, as a pure state, it is not necessary that $\mathcal{L}$ is the full Lie algebra $su(n_Sn_A)$ in order to have indirect controllability on $S$. In fact a Lie algebra isomorphic to the symplectic Lie algebra $\text{sp}(2)$ [12] is possible and induces arbitrary unitary state transfers for the target system $S$ (Proposition 5.2 in

---

11 $\{A, B\}$ here denotes the anticommutator, $\{A, B\} := AB + BA$.

12 Recall that $\text{sp}(n)$ is the Lie algebra of skew-Hermitian $2n \times 2n$ matrices $A$ satisfying $JA + A^T J = 0$, where $J = \left( \begin{array}{cc} 0 & 1_n \\ -1_n & 0 \end{array} \right)$. 

10
Our result here was proved using the fully mixed, maximum entropy, state for the accessor $A$ (as opposed to a pure state). It is therefore reasonable to expect that, in general, the ‘size’ of the dynamical Lie algebra $\mathcal{L}$ needed in order to have controllability on the state of $S$ will depend on the eigenvalues of $\rho_A$, and this dependence is currently under study.

The two main assumptions of this paper have been 1) that the initial state $\rho_{TOT}$ of the system $S + A$ is a product state, i.e., it is of the form $\rho_S \otimes \rho_A$ and 2) we have full control on the system $A$. The first assumption corresponds to starting an experiment with the two system $S$ and $A$ uncorrelated. From a theory point of view, separating $\rho_S$ and $\rho_A$ in the initial condition, allowed us to separate the role of $S$ and $A$ in the definition of indirect controllability and state it as a property of $S$ only given the set-up for $A$. If the initial state $\rho_{TOT}$ of $S + A$ is not a product state, we can still define indirect controllability by requiring that for every $X \in SU(n_S)$ there exists a $U \in e^\mathcal{L}$ such that $Tr_A(U\rho_{TOT}U^\dagger) = XTr_A(\rho_{TOT})X^\dagger$ for any possible value of $Tr_S(\rho_{TOT})$. However, there are many $\rho_{TOT}$ giving the same value of $Tr_S(\rho_{TOT})$, and one should decide how to restrict in a physical meaningful way the set of such $\rho_{TOT}$’s. In any case, since much of the machinery developed in this paper, and in particular the technical results of section 2, dealt with properties of the Lie algebra $\mathcal{L}$, The results presented here can be used to analyze cases where the initial state of $S + A$ is not a product state. Even Theorem 2 which was proved in [4] can be extended to this case with only notational modifications. The assumption 2) is used in the technical results of section 2 and in particular in the Lemmas 2.2 and 2.3. It allowed us to write the basis of $\mathcal{L}$ in the convenient form (40) from which we could deduce the main result. If this assumption is not verified a basis made up of tensor products might not exist (see, e.g., the examples in section IV-D of [4]). The study of indirect controllability in these cases will probably require further analysis and new tools and it remains an open problem.

Acknowledgement This research was supported by NSF under Grant No. ECCS0824085, and by the ARO MURI grant W911NF-11-1-0268. The author would like to thank Yao Fang who participated in an undergraduate research project on the topic of this paper and provided helpful suggestions.

References

[1] D. Burgarth, S. Bose, C. Bruder, and V. Giovannetti, Local controllability of quantum networks, Physical Review A, 79, 060305(R), (2009).

[2] D. D’Alessandro, Introduction to Quantum Control and Dynamics, CRC-Press, Boca Raton FL, 2007.

[3] D. D’Alessandro, Constructive decomposition of the controllability Lie algebra for quantum systems, IEEE Transactions on Automatic Control June 2010, 1416-1421.

[4] D. D’Alessandro and R. Romano, Indirect controllability of quantum systems: A study of two interacting quantum bits, to appear in IEEE Transactions on Automatic Control, special issue on Quantum Control.
[5] H.C. Fu, H. Dong, X. F. Liu, and C.P. Sun, Indirect control of quantum systems via an accessor: pure coherent control without system excitation, *Journal of Physics A: Mathematical and Theoretical*, **42**, 045303, (2009).

[6] V. Jurdjević and H. Sussmann, Control systems on Lie groups, *Journal of Differential Equations*, **12**, 313-329, (1972).

[7] T. Polack, H. Suchowski and D. Tannor, Uncontrollable quantum systems: A classifications scheme based on Lie subalgebras, *Physical Review A*, vol. 79, p. 053403, 2009.