Surjectivity of quotient maps for algebraic \((\mathbb{C}, +)\)-actions and polynomial maps with contractible fibres

Philippe Bonnet

Departamento de Algebra,
Geometria y Topologia
Universidad de Valladolid
47005 Valladolid, Spain.
e-mail: pbonnet@agt.uva.es

Abstract

In this paper, we establish two results concerning algebraic \((\mathbb{C}, +)\)-actions on \(\mathbb{C}^n\). First of all, let \(\varphi\) be an algebraic \((\mathbb{C}, +)\)-action on \(\mathbb{C}^3\). By a result of Miyanishi, its ring of invariants is isomorphic to \(\mathbb{C}[t_1, t_2]\). If \(f_1, f_2\) generate this ring, the quotient map of \(\varphi\) is the map \(F : \mathbb{C}^3 \to \mathbb{C}^2, x \mapsto (f_1(x), f_2(x))\). By using some topological arguments, we prove that \(F\) is always surjective. Secondly we are interested in dominant polynomial maps \(F : \mathbb{C}^n \to \mathbb{C}^{n-1}\) whose connected components of their generic fibres are contractible. For such maps, we prove the existence of an algebraic \((\mathbb{C}, +)\)-action \(\varphi\) on \(\mathbb{C}^n\) for which \(F\) is invariant. Moreover we give some conditions so that \(F^*(\mathbb{C}[t_1, \ldots, t_{n-1}])\) is the ring of invariants of \(\varphi\).

1 Introduction

In this paper, we are going to study some properties of algebraic \((\mathbb{C}, +)\)-actions on \(\mathbb{C}^n\). An algebraic \((\mathbb{C}, +)\)-action on \(\mathbb{C}^n\) is a regular map \(\varphi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n\) such that \(\varphi(u; \varphi(v; x)) = \varphi(u + v; x)\) for all \(u, v, x\). It is well-known that \(\varphi\) is obtained by integrating a locally nilpotent derivation \(\partial\) on \(\mathbb{C}[x_1, \ldots, x_n]\), that is a derivation \(\partial\) such that, for any polynomial \(R\), there exists an integer \(k > 0\) such that \(\partial^k(R) = 0\). A polynomial \(R\) is invariant if \(R \circ \varphi = R\), or equivalently if \(\partial(R) = 0\). These polynomials form a ring called the ring of invariants of \(\varphi\), and denoted by \(\mathbb{C}[x_1, \ldots, x_n]^{\varphi}\). We say that \(\varphi\) satisfies condition \((H)\) if its ring of invariants is isomorphic to a polynomial ring in \((n - 1)\) variables. In this case, \(\varphi\) is provided with a quotient map \(F\) defined as follows: If \(f_1, \ldots, f_{n-1}\) denote a system of
generators of \( \mathbb{C}[x_1, \ldots, x_n]^\varphi \), then \( F \) is the map:

\[
F : \mathbb{C}^n \to \mathbb{C}^{n-1}, \quad x \mapsto (f_1(x), \ldots, f_{n-1}(x))
\]

Note that for \( n > 3 \), the assumption (H) need not be satisfied ([Wi]). Conversely a dominant polynomial map \( F = (f_1, \ldots, f_{n-1}) \) is the quotient map of a \((\mathbb{C}, +)\)-action on \( \mathbb{C}^n \) if there exists an algebraic \((\mathbb{C}, +)\)-action \( \varphi \) on \( \mathbb{C}^n \) such that:

\[
\mathbb{C}[f_1, \ldots, f_{n-1}] = \mathbb{C}[x_1, \ldots, x_n]^\varphi
\]

First of all, we establish a property concerning algebraic \((\mathbb{C}, +)\)-actions on \( \mathbb{C}^3 \). According to a result of Miyanishi (see [Miy]), such an action always satisfies condition (H), and is therefore provided with a quotient map \( F : \mathbb{C}^3 \to \mathbb{C}^2 \). In [Kr], Kraft conjectures that every fixed-point free \((\mathbb{C}, +)\)-action \( \varphi \) on \( \mathbb{C}^3 \) is trivial, which means that it is conjugate via an automorphism of \( \mathbb{C}^3 \) to the action:

\[
\varphi^0(t; x_1, x_2, x_3) = (x_1 + t, x_2, x_3)
\]

This is known if its quotient space is separated. More precisely, a fixed-point free \((\mathbb{C}, +)\)-action \( \varphi \) on \( \mathbb{C}^3 \) is trivial if and only if:

- \( F \) is non-singular,
- \( F \) is surjective,
- Every fibre of \( F \) is connected.

Daigle proved in [Da] that \( F \) is non-singular in codimension 1, i.e. its singular set has codimension \( \geq 2 \). Moreover he derived a jacobian formula for the locally nilpotent derivation generating \( \varphi \). His formula implies in particular that \( F \) is non-singular if \( \varphi \) is fixed-point free. In an attempt to understand the behaviour of \((\mathbb{C}, +)\)-actions on \( \mathbb{C}^3 \), we are going to study the second condition on \( F \) given above. More precisely:

**Theorem 1** Let \( \varphi \) be any algebraic \((\mathbb{C}, +)\)-action on \( \mathbb{C}^3 \). Then its quotient map \( F \) is surjective.

Consequently a fixed-point free \((\mathbb{C}, +)\)-action \( \varphi \) on \( \mathbb{C}^3 \) is trivial if and only if every fibre of \( F \) is connected.

The proof uses both algebraic and topological methods. First we check that the complement of the image of \( F \) is at most finite. We assume that this complement is not empty, and denote by \( x \) one of its points. Let \( K \) be an homological 3-sphere, i.e. a singular 3-cycle whose class generates the group \( H_3(\mathbb{C}^2 - \{x\}) \). We construct a singular 3-cycle \( \Sigma \) in \( \mathbb{C}^3 \), such that \( F \) maps its homological class \([\Sigma]\) in \( \mathbb{C}^3 \) \( p \) times on \([K]\), where \( p \) is a positive integer. In other words, \( F \) behaves from an homological viewpoint as a \( p \)-sheeted covering from \( \Sigma \) to \( K \). Since \( \mathbb{C}^3 \) is contractible, the class of \( \Sigma \) in \( H_3(\mathbb{C}^3) \) is zero. So the class \([K]\)
in $H_3(C^2 - \{x\})$ is zero, hence a contradiction. Thus the main step in the proof is the construction of the singular 3-chain $\Sigma$. We proceed to this construction in sections 2-3-4.

Secondly, we are going to characterize from a topological viewpoint the morphisms $F : C^n \to C^{n-1}$ which are quotient maps of a $(C, +)$-action on $C^n$, and more generally which are invariant with respect to such an action. Let $\varphi$ be an algebraic $(C, +)$-action in $C^n$ distinct from the identity, i.e. $\varphi(t; x) \neq x$. If $F : C^n \to C^{n-1}$ is a dominant invariant morphism, then its generic fibres are finite union of orbits of $\varphi$, hence their connected components are contractible. Moreover if $F$ is a quotient map for $\varphi$, then its generic fibres are connected contractible, since they are one and only one orbit of $\varphi$. Conversely polynomial maps with contractible generic fibres correspond to $(C, +)$-actions. More precisely:

**Theorem 2** Let $F : C^n \to C^{n-1}$ be a dominant polynomial map. Assume that the connected components of its generic fibres are contractible. Then there exists an algebraic $(C, +)$-action $\varphi$, distinct from the identity, for which $F$ is invariant. If moreover $F$ is non-singular in codimension 1 and its generic fibres are connected, then $F$ is the quotient map of a $(\mathbb{C}, +)$-action on $\mathbb{C}^n$.

Let $f_1, ..., f_{n-1}$ be the coordinate functions of $F$. The main idea is to introduce the following derivation:

$$\partial : C[x_1, ..., x_n] \to C[x_1, ..., x_n], \quad R \mapsto J(R, f_1, ..., f_{n-1})$$

where $J$ denotes the jacobian of $n$ functions in $n$ complex variables, and to prove that $\partial$ is locally nilpotent. Therefore its integration leads to an algebraic $(\mathbb{C}, +)$-action $\varphi$ on $\mathbb{C}^n$ for which $F$ is invariant, because $\partial(f_i) = 0$ for any $i$. If the generic fibres are connected and $F$ is non-singular in codimension 1, there remains to check that $\mathbb{C}[f_1, ..., f_{n-1}] = \mathbb{C}[x_1, ..., x_n]^\varphi$, and this can be done by using Zariski’s Main Theorem. For more details, see section 5.

As a consequence, theorem [2] can be rewritten in an entirely topological way, as follows: If $F$ is a polynomial map that is non-singular in codimension 1 and whose generic fibres are connected contractible, then $F$ is surjective.

We end up this paper with two examples of polynomials maps which are not surjective, and we will explain why in light of the arguments given in the proof of theorem [2]. Moreover we will see with the first example that there does not exist any torical analogue of theorem [2]. More precisely there exists a dominant map whose generic fibres are isomorphic to $\mathbb{C}^*$ and that is not invariant with respect to any $\mathbb{C}^*$-action. The second one is an example of a non-surjective quotient map $F : \mathbb{C}^4 \to \mathbb{C}^3$. Both examples appear in section 6.

## 2 Some preliminary results

We begin with some standard results concerning algebraic $(\mathbb{C}, +)$-actions on $\mathbb{C}^3$. Recall that a $(\mathbb{C}, +)$-action $\varphi$ on $\mathbb{C}^3$ induces a degree function $\text{deg}$ on $\mathbb{C}[x_1, x_2, x_3]$, defined by:

$$\text{deg}(R) = \text{deg}_t(R \circ \varphi(t; x))$$
Therefore $R$ is invariant with respect to $\varphi$ if and only if $\deg(R) = 0$ or $R = 0$. The existence of this degree implies in particular that the ring of invariants is factorially closed (see [Da]). This means that if a polynomial is invariant, then all its irreducible factors are invariant. Let $\Gamma$ be the complement in $\mathbb{C}^2$ of $F(\mathbb{C}^3)$. For any polynomial $R$ in $\mathbb{C}[x_1, \ldots, x_n]$, we set by convention:

$$V(R) = \{x \in \mathbb{C}^n, R(x) = 0\}, \quad D(R) = \{x \in \mathbb{C}^n, R(x) \neq 0\}$$

**Lemma 3** The set $\Gamma$ is at most finite.

*Proof:* Since $f_1, f_2$ are algebraically independent, $F$ is a dominant map. Moreover $\Gamma$ is a constructible set of codimension $\geq 1$. Let us prove by absurd that $\Gamma$ has codimension $\geq 2$. Suppose that $\Gamma$ contains a Zariski open set $U$ of an irreducible curve in $\mathbb{C}^2$. We may assume that:

$$U = D(Q) \cap V(P)$$

where $P$ is irreducible and $Q$ is not divisible by $P$. Then $D(Q(F)) \cap V(P(F)) = \emptyset$, and $V(P(F)) \subset V(Q(F))$. By Hilbert’s Nullstellensatz, there exists an integer $n$ and a polynomial $R$ such that:

$$Q(F)^n = P(F)R$$

Since $\mathbb{C}[f_1, f_2]$ is factorially closed, $R$ is of the form $S(F)$, where $S$ is a polynomial. Therefore

$$Q^n = PS$$

and $P$ divides $Q$, hence a contradiction. ■

The following lemma is standard and asserts the existence of a rational slice for any $(\mathbb{C}, +)$-action on $\mathbb{C}^3$ (see [Da], [De], [D-F]).

**Lemma 4** Let $F$ be the quotient map of $\varphi$. Then there exists a hypersurface $V(f)$ in $\mathbb{C}^3$, and a principal open set $D(P)$ in $\mathbb{C}^2$ such that $F : V(f) \cap D(P(F)) \to D(P)$ is an isomorphism.

*Proof:* Let $\partial$ be the locally nilpotent derivation generating $\varphi$. Since $\partial \neq 0$, there exists a polynomial $f$ such that $\partial(f) \neq 0$ and $\partial^2(f) = 0$. Since $\mathbb{C}[f_1, f_2]$ is the kernel of $\partial$, there exists a polynomial $P$ such that $\partial(f) = P(F)$. By induction on the degree, we easily check that every polynomial $R$ can be written as $P(F)^nR = T(f, f_1, f_2)$, where $T$ is an element of $\mathbb{C}[x_1, x_2, x_3]$. This yields the equality:

$$\mathbb{C}[x_1, x_2, x_3]_{P(F)} = \mathbb{C}[f, f_1, f_2]_{P(F)}$$

So the map $G = (f, f_1, f_2)$ defines an isomorphism from $D(P(F))$ to $\mathbb{C} \times D(P)$. Moreover $G$ maps $V(f) \cap D(P(F))$ on $\{0\} \times D(P)$, and its restriction is equal to $F$ via the identification $\{0\} \times D(P) \simeq D(P)$. Therefore $F : V(f) \cap D(P(F)) \to D(P)$ is an isomorphism. ■
3 Construction of coverings

Let us denote by $Q$ an irreducible polynomial in $\mathbb{C}[t_1, t_2]$, and by $F$ a polynomial map from $\mathbb{C}^3$ to $\mathbb{C}^2$ that is non-singular in codimension 1, that is whose singular set has codimension $\geq 2$ in $\mathbb{C}^3$. In this section, we will show how to construct some coverings over a neighborhood of a compact set contained in $V(Q)$. More precisely:

**Proposition 5** Let $F : \mathbb{C}^3 \to \mathbb{C}^2$ be a polynomial map that is nonsingular in codimension 1. Let $Q$ be an irreducible polynomial in $\mathbb{C}[t_1, t_2]$. Then there exists a Zariski open set $U$ of $V(Q)$ satisfying the following property: For any compact set $K$ contained in $U$, there exist an analytic subvariety $X_K$ in $\mathbb{C}^3$ and an open set $U_K$ in $\mathbb{C}^2$, containing $K$, such that $F : X_K \to U_K$ is a finite unramified covering.

This result applies in particular for any quotient map of an algebraic $(\mathbb{C}, +)$-action on $\mathbb{C}^3$, since Daigle proved in [Da] that any such map is non-singular in codimension 1.

**Lemma 6** There exists a plane $H$ in $\mathbb{C}^3$ and a point $x$ in $H \cap V(Q(F))$ such that $F : H \to \mathbb{C}^2$ is non-singular at $x$.

**Proof:** Since $F$ is nonsingular in codimension 1, there exists a point $x$ in the hypersurface $V(Q(F))$ such that $dF(x)$ has rank 2. In particular the wedge product $df_1 \wedge df_2(x)$ is non-zero. So there exists a linear form $l$ on $\mathbb{C}^3$ such that $dl \wedge df_1 \wedge df_2(x) \neq 0$. Let us set:

$$H = V(l - l(x))$$

By construction $F : H \to \mathbb{C}^2$ is non-singular at $x$, and $x$ belongs to $H \cap V(Q(F))$.

**Lemma 7** Let $x$ and $H$ be a point and a plane in $\mathbb{C}^3$ satisfying the conditions of the previous lemma. Then there exists an irreducible curve $C$ in $H$, passing through $x$ such that the map $F : C \to V(Q)$ is dominant.

**Proof:** Denote by $F_H$ the restriction map $F : H \to \mathbb{C}^2$. By assumption $F_H$ is smooth at the point $x$. So $F_H$ is dominant, $F^{-1}(V(Q))$ cannot contain $H$ and $F^{-1}(V(Q)) \cap H$ is a union of irreducible curves. Since $Q(F(x)) = 0$, there exists an irreducible curve passing through $x$ and contained in $F^{-1}(V(Q)) \cap H$. Let us fix such a curve and denote it by $C$. Let us show by absurd that the restriction $F : C \to V(Q)$ is dominant. Assume it is not. Then $F$ maps $C$ to a point, and $F : C \to V(Q)$ is everywhere singular. For any smooth point $x'$ of $C$, the differential $dF(x')$ must vanish on $T_{x'}C$. So $dF(x')$ must have rank $< 2$, and the smooth part of $C$ is contained in the singular set $Sing(F_H)$. Since this set is closed, $C$ is contained in $Sing(F_H)$. But this is impossible because $x$ belongs to $C$ and is not a singular point of $F_H$.
Lemma 8 Let $x$ be a point, $H$ be a plane and $C$ be an irreducible curve satisfying the conditions of the previous lemmas. Then there exists a Zariski open set $U$ of $V(Q)$ such that:

- $F : F^{-1}(U) \cap C \to U$ is proper for the metric topology,
- $U$ does not contain any critical value of $F : H \to \mathbb{C}^2$.

Proof: The singular set of $F_H$ is closed in $H$ and does not contain $x$. Since $x$ belongs to $C$ and $C$ is irreducible, the intersection $C \cap \text{Sing}(F_H)$ is at most finite. Let $U'$ be a Zariski open set of $V(Q)$ such that $U'$ does not meet the finite set $F(C \cap \text{Sing}(F_H))$. By assumption, $F : F^{-1}(U') \cap C \to U'$ is a dominant map of irreducible curves. So this is a quasi-finite morphism, and there exists a Zariski open set $U$ in $U'$ such that $F : F^{-1}(U) \cap C \to U$ is finite, hence proper for the metric topology.

Proof of proposition 5 Let $F : \mathbb{C}^3 \to \mathbb{C}^2$ be a polynomial map that is nonsingular in codimension 1. Let $Q$ be an irreducible polynomial in $\mathbb{C}[t_1, t_2]$. Let $H$ and $C$ be the plane in $\mathbb{C}^3$ and the irreducible curve found in the previous lemmas. Let $U$ be the Zariski open set of $V(Q)$ satisfying the conditions of lemma. Let $K$ be a compact set contained in $U$. Since $F : F^{-1}(U) \cap C \to U$ is a proper map, $L = F^{-1}(K) \cap C$ is compact. Since $F^{-1}(U)$ does not meet the singular set of $F_H$, there exists a relatively compact open set $U_1$ of $H$ that contains $L$ and does not meet $\text{Sing}(F_H)$. Then the restriction map:

$$F : U_1 \to F(U_1)$$

is proper because its source is compact. Moreover the set $F(U_1)$ contains $K$, and is open because $F : U_1 \to \mathbb{C}^2$ is non-singular. By the localisation lemma (see [Ch], p.29), there exist two open sets $X_K$ of $H$ and $U_K$ of $\mathbb{C}^2$, containing $L$ and $K$ respectively, such that $X_K$ is contained in $U_1$ and the map $F : X_K \to U_K$ is proper for the metric topology. By construction $X_K$ is an analytic subvariety of $\mathbb{C}^3$, and the map $F : X_K \to U_K$ is proper and non-singular. Since its fibres are compact analytic sets, they are finite (see [Ch]). Therefore $F : X_K \to U_K$ is a finite unramified covering.

4 Proof of the first theorem

From now on, we assume that the quotient map $F$ is not surjective, or in other words that $\Gamma \neq \emptyset$. Up to a translation, we may suppose that $\Gamma$ contains the origin in $\mathbb{C}^2$. In what follows, we will always consider singular homology with integer coefficients.

Since $F$ is a continuous map from $\mathbb{C}^3$ to $\mathbb{C}^2 - \{0\}$, it induces a morphism $F_*$ from the space of singular 3-chains in $\mathbb{C}^3$ to the space of singular 3-chains in $\mathbb{C}^2 - \{0\}$. If $\Delta^3$ denotes
the standard 3-simplex, recall that a singular 3-chain \( K \) in a topological space \( X \) is a formal sum:

\[
K = \sum n_\alpha \Delta_\alpha
\]

where each \( n_\alpha \) is an integer and each \( \Delta_\alpha \) is a continuous map from \( \Delta^3 \) to \( X \). \( \Delta_\alpha \) is called a singular 3-simplex and its image is denoted by \( \Delta_\alpha(\Delta^3) \). In this section, we are going to construct two singular 3-chains \( \Sigma \) in \( C^3 \) and \( K \) in \( C^2 - \{0\} \) such that:

- The boundaries \( \partial \Sigma \) and \( \partial K \) are equal to zero, and there exists an integer \( p > 0 \) such that \( F_*(\Sigma) = pK \),

- The class of \( K \) in \( H_3(C^2 - \{0\}) \simeq \mathbb{Z} \) is a generator of this group.

Assume this is done for the moment. Since \( \Sigma \) and \( K \) have no boundaries, they define homological classes \( [\Sigma] \) and \( [K] \) in \( H_3(C^3) \) and in \( H_3(C^2 - \{0\}) \) respectively. Moreover if \( F_* \) denotes the morphism induced by \( F \) on singular homology, we get \( F_*(\Sigma) = pK \).

Since \( C^3 \) is contractible, we have \( [\Sigma] = 0 \) and \( p[K] = 0 \), hence contradicting the fact that \( [K] \) is a generator of \( H_3(C^2 - \{0\}) \simeq \mathbb{Z} \).

So in order to complete the proof of theorem 1, there only remains to construct these singular chains. We proceed to their construction in the following subsections.

### 4.1 Construction of \( K \)

Let \( P \) be the polynomial appearing in lemma 4 and let \( P_1, \ldots, P_s \) be its irreducible factors in \( \mathbb{C}[t_1, t_2] \). For each factor \( P_i \), we denote by \( U_i \) a Zariski open set of \( V(P_i) \) satisfying the conditions of proposition 5. Let \( S \) be a 3-sphere in \( C^2 \) centered at the origin in \( C^2 \). Since the sets \( \bigcup_i U_i, \Gamma \) and \( U_i \cap U_j \) for \( i \neq j \) are at most finite, we can choose its radius small enough so that:

- \( S \) does not meet the sets \( V(P) - \bigcup_i U_i, \Gamma \) and \( U_i \cap U_j \) for \( i \neq j \),

As every open set \( U_i \) is connected for the metric topology, and \( U_i \cap U_j \cap S = \emptyset \) if \( i \neq j \), every connected component of \( V(P) \cap S \) is contained in one and only one open set \( U_i \). Let us denote by \( \gamma_1, \ldots, \gamma_r \) the connected components of \( V(P) \cap S \). Since every \( \gamma_i \) is contained in an open set \( U_j \), there exists an open set \( U_{\gamma_i} \) containing \( \gamma_i \) and satisfying the conditions of proposition 5. Thus \( S \) is covered by the open sets \( D(P) \) and \( U_{\gamma_i}, 1 \leq i \leq r \).

**Lemma 9** There exists a singular 3-cycle \( K = \sum n_\alpha \Delta_\alpha \) in \( C^2 - \{0\} \) satisfying the following conditions:

- Every image \( \Delta_\alpha(\Delta^3) \) is contained either in \( S \cap D(P) \) or in one of the sets \( S \cap U_{\gamma_i} \),

- Every image \( \Delta_\alpha(\Delta^3) \) cannot meet two different sets \( \gamma_i \) and \( \gamma_j \),

- If \( \Delta_\alpha(\Delta^3) \) meets \( \gamma_i \) and \( \Delta_\beta(\Delta^3) \) meets \( \gamma_j \) with \( i \neq j \), then \( \Delta_\alpha(\Delta^3) \cap \Delta_\beta(\Delta^3) = \emptyset \),
• The homological class of $K$ is a generator of $H_3(\mathbb{C}^2 - \{0\})$.

Proof: Since the sphere $S$ is a deformation retract of $\mathbb{C}^2 - \{0\}$, the inclusion map $i : S \hookrightarrow \mathbb{C}^2 - \{0\}$ induces an isomorphism:

$$i_* : H_3(S) \to H_3(\mathbb{C}^2 - \{0\})$$

So we can find a singular 3-cycle $K'$ generating $H_3(\mathbb{C}^2 - \{0\})$ of the form:

$$K' = \sum n'_a \Delta'_a$$

where the image of every $\Delta'_a$ lies in $S$. Let $d$ be the distance function defined by the canonical Hermitian metric on $\mathbb{C}^2$. We provide $S$ with the metric topology induced by the embedding $i : S \hookrightarrow \mathbb{C}^2$. Since $S$ is compact and covered by the open sets $D(P), U_{\gamma_1}, \ldots, U_{\gamma_r}$, there exists an $\epsilon$ such that any ball of radius $\leq \epsilon$ in $S$ is contained in one of these open sets. For any compact sets $\gamma, \gamma'$ in $\mathbb{C}^2$, we denote by $\text{dist}(\gamma, \gamma')$ the distance between these two sets. Up to choosing a smaller $\epsilon$, we may even assume that:

$$\epsilon \leq \frac{\text{dist}(\gamma_i, \gamma_j)}{3}$$

whenever $i \neq j$. By performing enough barycentric subdivisions of every simplex $\Delta'_a$ in $K'$, we can get a new 3-cycle $K$, homologous to $K'$, such that:

$$K = \sum n_a \Delta_a$$

where every image $\Delta_a(\Delta^3)$ is contained in $S$ and has diameter $\leq \epsilon$. By construction, the homological class of $K$ is a generator of $H_3(\mathbb{C}^2 - \{0\})$. Moreover since every set $\Delta_a(\Delta^3)$ has diameter $\leq \epsilon$, it is contained in a ball of radius $\epsilon$. Hence $\Delta_a(\Delta^3)$ is contained in one of the open sets $S \cap D(P), S \cap U_{\gamma_1}, \ldots, S \cap U_{\gamma_r}$ in $S$. The other two conditions are as easy to check.

Let $K$ be the singular 3-chain of the previous lemma. By the second condition of this lemma, we can perform the following partition:

• $\{\Delta_{i,j}\}_j$ is the set of 3-simplices of $K$ meeting $\gamma_i$,

• $\{\Delta_k\}_k$ is the set of 3-simplices of $K$ meeting none of the $\gamma_i$.

That enables us to rewrite this singular 3-cycle in the following way:

$$K = \sum_{i,j} n_{i,j} \Delta_{i,j} + \sum_k n_k \Delta_k$$

Note that by the third condition of the lemma, the images of $\Delta_{(i,j)}$ and $\Delta_{(i',j')}$ intersect only if $i = i'$. 

8
4.2 Construction of $\Sigma$

In this subsection, we construct the singular 3-chain $\Sigma$ by lifting the 3-simplices of $K$ in a suitable way. Let $X_{\gamma_i}$ be the analytic variety given by proposition 5, and let $p_i$ be the degree of the unramified covering:

$$F : X_{\gamma_i} \to U_{\gamma_i}$$

Since the image of every $\Delta_{(i,j)}$ is contained in $U_{\gamma_i}$, we can lift it in $p_i$ different ways. More precisely there exist $p_i$ different maps $\Delta_{(i,j)}^l : \Delta^3 \to X_{\gamma_i}$ making the following diagram commute:

$$\begin{array}{ccc}
X_{\gamma_i} & \xrightarrow{F} & U_{\gamma_i} \\
\Delta^3 & \downarrow & \Delta^3 \to U_{\gamma_i} \\
\end{array}$$

where the arrow at the bottom stands for the map $\Delta_{(i,j)}$. Let us denote by $\Delta_k^1$ the lifting of the 3-simplex $\Delta_k$ via the isomorphism:

$$F : V(f) \cap D(P(F)) \to D(P)$$

More precisely, if $G$ is the restriction of $F$ to $V(f) \cap D(P(F))$, then $\Delta_k^1$ is the map $G^{-1} \circ \Delta_k$. This yields the other following commutative diagram:

$$\begin{array}{ccc}
V(f) \cap D(P(F)) & \xrightarrow{F} & D(P) \\
\Delta^3 & \downarrow & \Delta^3 \to D(P) \\
\end{array}$$

where the arrow at the bottom stands for the map $\Delta_k$. Now, and this is the key-point of the construction, we are going to modify these simplices so that the boundary of $\bigcup_{i,j,l} \Delta_{(i,j)}^l$ coincides with the boundary of $\bigcup_k \Delta_k^1$. That will enable us to get a singular 3-chain $\Sigma$ with no boundary. In order to do so, we will use the fact that the generic fiber of $F$ is contractible, in the following way. Let $V$ be the complement in $S$ of $\bigcup_k \Delta_k(\Delta^3)$. By construction $V$ is an open neighborhood of the union $\bigcup_i \gamma_i$ in $S$. There exists a continuous function $g$ on $S$ such that:

- $g$ is equal to 1 outside $V$,
- $g$ vanishes in a neighborhood of each $\gamma_i$.

If $\varphi$ is the algebraic ($\mathbb{C}$, $+$)-action of the beginning, we define the map $L$ on $F^{-1}(S)$ by the formula:

$$L(x) = \varphi(t(x); x), \quad t(x) = \frac{-f(x)g(F(x))}{P(F(x))}$$
outside $\cup_i F^{-1}(\gamma_i)$, and $L$ is the identity on $\cup_i F^{-1}(\gamma_i)$. Note that $L$ is continuous since $g$ vanishes on a neighborhood of each $\gamma_i$ in $S$. Moreover $F \circ L = F$ on $S$. The new 3-simplices $\mathcal{D}_{(i,j)}^l$ and $\mathcal{D}_k^1$ are given by the formulas:

$$\mathcal{D}_{(i,j)}^l = L \circ \Delta_{(i,j)}^l$$ and $$\mathcal{D}_k^1 = L \circ \Delta_k^1$$

By construction we get:

$$F \circ \mathcal{D}_{(i,j)}^l = F \circ \Delta_{(i,j)}^l = \Delta_{(i,j)}$$ and $$F \circ \mathcal{D}_k^1 = F \circ \Delta_k^1 = \Delta_k$$

We set $p = \prod_i p_i$ and define the singular 3-chain $\Sigma$ by the sum:

$$\Sigma = \sum_{i,j,l} p_i n_{(i,j)} \mathcal{D}_{(i,j)}^l + p \sum_k n_k \mathcal{D}_k^1$$

**4.3 Properties of these singular 3-chains**

We are going to derive the properties announced at the beginning of this section, and then conclude the proof of theorem 1. Recall that a face of a 3-simplex $\Delta$ is the restriction of $\Delta$ to one of the faces of $\Delta^3$. By extension a face of a 3-chain is a face of one of the 3-simplices of its decomposition. Let $\Sigma'$ be the singular 3-chain:

$$\Sigma' = \sum_{i,j,l} p_i n_{(i,j)} \Delta_{(i,j)}^l + p \sum_k n_k \Delta_k^1$$

For commodity we introduce the following sets:

- $E$ is the set of faces $\delta$ of $\Sigma$ such that $F(\delta)$ belongs to the boundary of both a $\Delta_{(i,j)}$ and a $\Delta_k$,

- $E'$ is the set of faces $\delta'$ of $\Sigma'$ such that $F(\delta')$ belongs to the boundary of both a $\Delta_{(i,j)}$ and a $\Delta_k$.

**Proposition 10** If $F_*$ is the morphism induced by $F$ on the space of singular 3-chains, then $F_*(\Sigma) = pK$.

**Proof:** By construction we have the following relations:

$$F_*(\mathcal{D}_{(i,j)}^l) = \Delta_{(i,j)}$$ and $$F_*(\mathcal{D}_k^1) = \Delta_k$$

Since every 3-simplex $\Delta_{(i,j)}$ has been lifted $p_i$ times, and every 3-simplex $\Delta_k$ has been lifted once, we obtain:

$$F_*(\Sigma) = \sum_{i,j} p_i n_{(i,j)} \left( \sum_l F_*(\mathcal{D}_{(i,j)}^l) \right) + p \sum_k n_k F_*(\mathcal{D}_k^1)$$

$$= \sum_{i,j} p_i n_{(i,j)} \Delta_{(i,j)} + p \sum_k n_k \Delta_k$$

$$= p \sum_{i,j} n_{(i,j)} \Delta_{(i,j)} + p \sum_k n_k \Delta_k$$

$$= pK$$
Lemma 11 Let $\delta_1, \delta_2$ be any 2-faces of $\Sigma$ such that $\delta = F(\delta_1) = F(\delta_2)$ belongs to the boundary of a $\Delta_k$. Then $\delta_1 = \delta_2$.

Proof: Let us show that $\delta_1$ is equal to the map $G^{-1} \circ \delta$ (see the previous subsection). If $\delta_1$ belongs to the boundary of a $D^k_1$, then $\delta_1 = G^{-1} \circ \delta$ by construction. If $\delta_1$ belongs to the boundary of a $D^{l(i,j)}_k$, then $\delta_1$ is of the form $L \circ \delta'_1$, where $\delta'_1$ is a face of $\Delta^l_{i,j}$. Since $F(\delta'_1) = F(\delta_1) = \delta$ belongs to the boundary of a $\Delta_k$, the function $g$ is equal to 1 on the image of $F(\delta'_1)$. So $g \circ F \circ \delta'_1 = 1$ and we have the following equality for any point $x$ in the image of $\delta'_1$:

$$L(x) = \varphi(-f(x)/P(F(x)); x)$$

By using the exponential map, we get:

$$f \circ \varphi(t; x) = f(x) + P(F(x)) t$$

After substitution, that implies:

$$f(L(x)) = 0$$

Therefore $f \circ \delta_1 = f \circ L \circ \delta'_1 = 0$ and the image of $\delta_1$ lies in the set $V(f)$. Since $\delta$ is a face of a $\Delta_k$, and $\Delta_k$ does not meet the hypersurface $V(P)$, the function $P \circ \delta$ never vanishes. Since $F(\delta_1) = \delta$, the function $P \circ F \circ \delta_1$ never vanishes and the image of $\delta_1$ lies in the intersection $V(f) \cap D(P(F))$. Since $F \circ \delta_1 = \delta$, and since the map $G$ defined as the restriction:

$$F : V(f) \cap D(P(F)) \to D(P)$$

is an isomorphism, $\delta_1$ is equal to $G^{-1} \circ \delta$.

Lemma 12 For any $i$, the boundary of $\sum_{j,l} n(i,j) D^l_{i,j}$ is a linear combination of elements of $E$.

Proof: Assume first that the boundary of $\sum_{j,l} n(i,j) \Delta^l_{i,j}$ can be written as:

$$\partial \left( \sum_{j,l} n(i,j) \Delta^l_{i,j} \right) = \sum_{\delta' \in E'} n_{\delta'} \delta'$$

Let $L_*$ be the morphism induced by $L$. Since $L \circ \Delta^l_{i,j} = D^l_{i,j}$ and $L \circ \Delta^l_k = D^l_k$, $L_*$ maps elements of $E'$ to elements of $E$. Since the boundary operator commutes with $L_*$, that implies:

$$\partial \left( \sum_{j,l} n(i,j) D^l_{i,j} \right) = \sum_{\delta \in E} \left( \sum_{L_*(\delta') = \delta} n_{\delta'} \right) \delta$$
So there only remains to show that the boundary of $\sum_{j,l} n_{(i,j)} \Delta_{i,j}^l i,j$ is a linear combination of elements of $E'$. Let us prove that any face $\delta'$ not belonging to $E'$ cannot appear with a non-zero coefficient into that boundary.

Let $\delta'$ be a face of a $\Delta_{i,j}^l$, that does not belong to $E'$. Then $\delta = F(\delta')$ is a face of $\Delta_{i,j}$. Moreover $\delta$ is not a face of any $\Delta_k$. By lemma 9 the only simplices of $\Sigma'$ that may have $\delta$ as a face are of the form $\Delta_{i,j}$. We write them as $\Delta_{i,j_1}, \ldots, \Delta_{i,j_r}$. We now use the lifting property of the covering:

\[ F: X_{\gamma_i} \to U_i \]

For any $\alpha$, there exists a unique lifting $\Delta_{i,j_{\alpha}}^l$ of $\Delta_{i,j_{\alpha}}$ such that $\delta'$ is one of its faces. So $\Delta_{i,j_1}, \ldots, \Delta_{i,j_r}$ are the only simplices of $\Sigma'$ and of $\sum_{j,l} n_{(i,j)} \Delta_{i,j}^l$, having $\delta'$ as a face. Let $\epsilon_{\alpha}$ be the coefficient of $\delta$ in the boundary of $\Delta_{i,j_{\alpha}}$. Since the singular 3-chain $K$ has no boundary, we have:

\[ \sum_{\alpha} n_{(i,j_{\alpha})} \epsilon_{\alpha} = 0 \]

But $\epsilon_{\alpha}$ is also the coefficient of $\delta'$ in the boundary of $\Delta_{i,j_{\alpha}}^l$. Therefore, the coefficient of $\delta'$ in the boundary of $\sum_{j,l} n_{(i,j)} \Delta_{i,j}^l$ is equal to the number given above, hence zero, and the result follows.

\[ \blacksquare \]

**Lemma 13** The boundary of $\sum_k n_k D_k^l$ is a linear combination of elements of $E$.

**Proof:** The proof is entirely similar to the proof of the previous lemma. The only difference is the use of the isomorphism $F: V(f) \cap D(P(F)) \to D(P)$ in place of the covering $F: X_{\gamma_i} \to U_i$ for the definition of the $\Delta_k$.

\[ \blacksquare \]

**Proposition 14** The singular 3-chain $\Sigma$ has no boundary.

**Proof:** By applying lemmas 12 and 13 to the definition of $\Sigma$, we see that the boundary of $\Sigma$ can be written as:

\[ \partial \Sigma = \sum_{\delta' \in E} n_{\delta'} \delta' \]

Since the boundary operator commutes with the morphism $F$, induced by $F$, we get by lemma 10:

\[ \sum_{\delta \in F(E)} \left( \sum_{F_*(\delta') = \delta} n_{\delta'} \right) \delta = p\partial(K) = 0 \]

Thus all the sums $\sum_{F_*(\delta') = \delta} n_{\delta'}$ are equal to zero. By lemma 11 for any face $\delta$ in $F(E)$, there exists a unique face $\delta'$ in $E$ such that $F(\delta') = \delta$. This implies the equality $n_{\delta'} = 0$ for any $\delta'$, and the result follows.

\[ \blacksquare \]
5 Morphisms with contractible generic fibres

In this section, we pass on to polynomial maps with contractible fibres, and we are going to prove theorem \[\Box\] We begin with the following lemma, which corresponds to the first assertion of this theorem.

Lemma 15 Let \( F \) be a dominant polynomial map from \( \mathbb{C}^n \) to \( \mathbb{C}^{n-1} \). Assume that the connected components of its generic fibres are contractible. Then there exists an algebraic \((\mathbb{C},+)\)-action \( \varphi \), distinct from the identity, for which \( F \) is invariant. In particular, the generic fibres of \( F \) are finite unions of orbits of \( \varphi \).

Proof: Write \( F = (f_1, \ldots, f_{n-1}) \), and let \( \partial \) be the derivation on \( \mathbb{C}[x_1, \ldots, x_n] \) defined for any \( R \) by:
\[
\partial(R) = J(R, f_1, \ldots, f_{n-1})
\]
where \( J \) denotes the jacobian of \( n \) functions in \( n \) variables. If we show that \( \partial \) is a locally nilpotent derivation, then it will generate an algebraic \((\mathbb{C},+)\)-action \( \varphi \) on \( \mathbb{C}^n \) for which each \( f_i \) is invariant, because \( \partial(f_i) = 0 \) for any \( i \). Moreover \( \varphi \) will be distinct from the identity because \( \partial \neq 0 \). So let us prove by absurd that \( \partial \) is locally nilpotent.

Assume there exists a polynomial \( R \) such that \( \partial^k(R) \neq 0 \) for any \( k \). By assumption on \( F \), there exists a Zariski open set \( U \) in \( \mathbb{C}^{n-1} \) such that:
- For any \( y \) in \( U \), \( y \) is not a critical value of \( F \) and \( F^{-1}(y) \) is not empty,
- For any \( y \) in \( U \), the connected components of \( F^{-1}(y) \) are contractible.

For any \( k \), let us set \( U_k = F^{-1}(U) \cap D(\partial^k(R)) \). Then \( U_k \) is a non-empty Zariski open set which is dense in \( \mathbb{C}^n \) for the metric topology. By Baire’s property of complete topological spaces, we get:
\[
\bigcap_{k \geq 0} U_k \neq \emptyset
\]
Let \( x \) be a point of this intersection, and let \( C \) be the connected component of \( F^{-1}(F(x)) \) containing \( x \). Since \( \partial(f_i) = 0 \) for any \( i \), \( \partial \) corresponds to a vector field that is tangent to \( F^{-1}(F(x)) \), hence to \( C \). Therefore it induces a derivation \( \Delta \) on the ring \( \mathbb{C}[C] \) of regular functions on \( C \). Since \( C \) is a smooth contractible algebraic curve, it is isomorphic to \( \mathbb{C} \), and \( \Delta \) appears as a derivation on \( \mathbb{C}[t] \). Write it as:
\[
\Delta = P(t) \frac{\partial}{\partial t}
\]
By construction the singularities of \( \partial \), considered as a vector field, are the singular points of \( F \). Since \( F(x) \) is not a critical value of \( F \), \( \partial \) has no singularities along \( C \). Thus the polynomial \( P(t) \) must never vanish, hence it is a constant. Therefore \( \Delta \) is locally nilpotent on \( \mathbb{C}[t] \), and there exists an order \( k \) such that:
\[
\Delta^k(R) = \partial^k(R)|_C = 0
\]
In particular \( \partial^k(R)(x) = 0 \), hence contradicting the construction of \( x \).
Proof of theorem 2. There only remains to show the second assertion of this theorem. Let $F$ be a dominant polynomial map that is non-singular in codimension 1, and whose generic fibres are connected contractible. By the previous lemma, there exists a $(\mathbb{C},+)$-action $\varphi$ distinct from the identity and for which $F$ is invariant. Let us check that:

$$\mathbb{C}[x_1, \ldots, x_n]^\varphi = \mathbb{C}[F]$$

Let $R$ be an invariant polynomial. Since the generic fibres of $F$ are smooth and connected, each of them is exactly one and only one orbit of $\varphi$. Let $U$ be a Zariski open set in $\mathbb{C}^{n-1}$ such that, for any $y$ in $U$, $F^{-1}(y)$ is an orbit of $\varphi$. Since $R$ is invariant, it is constant along any such fibre. Consider the following correspondence:

$$\alpha : U \longrightarrow \mathbb{C}, \quad y \longmapsto \text{"value of } R \text{ along } F^{-1}(y)\text{"}$$

Its graph corresponds to the image of $F^{-1}(U)$ by the map $(f_1, \ldots, f_{n-1}, R)$. This is a constructible set whose Zariski closure is irreducible. Thus $\alpha$ defines a rational correspondence in the sense of Zariski. By Zariski’s Main Theorem (see [Mum]), $\alpha$ is rational and $R$ can be written as $R = \alpha(F)$. Let us prove by absurd that $\alpha$ is a polynomial.

Assume that $\alpha = A/B$, where $A$ and $B$ have no common factors and $B$ is not constant. Since $R = A(F)/B(F)$ is a polynomial, $A(F)$ and $B(F)$ have a common irreducible factor $h$. So $F$ maps the hypersurface $V(h)$ into the set $V(A) \cap V(B)$. For any smooth point $x$ of $V(h)$, we get:

$$\operatorname{rank}(d\{F_{|V(h)}\}(x)) = \operatorname{rank}(dF(x)|_{T_xV(h)}) \leq \dim V(A) \cap V(B) \leq (n-3)$$

Since $T_xV(h)$ is an hyperplane of $\mathbb{C}^n$, this yields:

$$\operatorname{rank}(dF(x)) \leq (n-2)$$

By upper semi-continuity, $dF(x)$ has rank $\leq (n-2)$ for any point $x$ of $V(h)$. Therefore the singular set of $F$ contains the hypersurface $V(h)$, hence contradicting the fact that $F$ is nonsingular in codimension 1.

6 Two examples

Finally we are giving two examples of polynomial maps between affine spaces which are not surjective, and we are trying to understand why. Moreover we are going to see with the first example that there is no torical analogue of theorem 2, namely that a polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$, whose generic fibres are isomorphic to $\mathbb{C}^*$, need not be the quotient map of an algebraic $\mathbb{C}^*$-action on $\mathbb{C}^n$. 

■
6.1 First example

The construction of the singular 3-chains $K$ and $\Sigma$, which is the main argument of the proof of theorem, is made possible because first $F$ is non-singular in codimension 1 and second, the generic fibre of $F$ is contractible. This is clear by theorem. These are the reasons why we can lift the singular 3-chain $K$, and then adjust the boundaries of the singular 3-simplices forming $\Sigma$ so as to assure that $\Sigma$ has no boundary. Consider the following map:

$$F: \mathbb{C}^3 \to \mathbb{C}^2, \quad (x, y, z) \to (1 + xz, y + z + xyz)$$

Then $F$ is not surjective because its image is the set $\mathbb{C}^2 \setminus \{0\}$. Moreover its singular set is the line $\{x = 0, z = 0\}$ in $\mathbb{C}^3$, and so $F$ is non-singular in codimension 1. The obstruction to surjectivity lies in the fact that the generic fibre of $F$ is isomorphic to $\mathbb{C}^\ast$, hence it is not contractible.

As in theorem, we might expect that $F$ is invariant with respect to an algebraic $\mathbb{C}^\ast$-action, that is a regular map $\psi: \mathbb{C}^\ast \times \mathbb{C}^3 \to \mathbb{C}^3$ such that $\psi(t; \psi(s; p)) = \psi(ts; p)$ for any $t, s, p$. We are going to see that this is not the case. Assume that $F$ is invariant with respect to such an action $\psi$, whose parameter in $\mathbb{C}^\ast$ is denoted by $t$. Then the polynomial $xz$ is invariant with respect to $\psi$, and there exists an integer $r$ such that:

$$x \circ \psi = t^r x, \quad z \circ \psi = t^r z$$

Since $y + z + xyz$ is invariant, this yields the equality:

$$(y \circ \psi - y)(1 + xz) = z(1 - t^{-r})$$

and this is impossible since $(1 + xz)$ cannot divide $z(1 - t^{-r})$ in $\mathbb{C}[t, 1/t, x, y, z]$.

6.2 Second example

Considering the class of algebraic $(\mathbb{C}, +)$-actions on $\mathbb{C}^n$ satisfying condition $(H)$ (see the introduction), we may ask if theorem extends with no restriction to higher dimension, that is for $n > 3$. More precisely, if $\varphi$ is an algebraic $(\mathbb{C}, +)$-action on $\mathbb{C}^n$ satisfying condition $(H)$, is its quotient map always surjective?

The answer is no. Let us denote by $x, y, u, v$ the coordinates in $\mathbb{C}^4$, let $t$ be a parameter in $\mathbb{C}$ and consider the $(\mathbb{C}, +)$-action on $\mathbb{C}^4$ defined as follows:

$$\varphi(t; x, y, u, v) = (x, y, u - ty, v + tx)$$

It is easy to check that its ring of invariants is generated by the polynomials $x, y, xu + yv$. So $\varphi$ satisfies condition $(H)$, and its quotient map is given by:

$$F: \mathbb{C}^4 \to \mathbb{C}^3, \quad (x, y, u, v) \mapsto (x, y, xu + yv)$$

The map $F$ is not surjective, since its image is the set:

$$F(\mathbb{C}^4) = \mathbb{C}^3 \setminus \{(x_1, x_2, x_3), x_1 = x_2 = 0, x_3 \neq 0\}$$
We may wonder why this map is not surjective, since surjectivity is automatically satisfied for quotient maps if \( n = 3 \). In fact, given a singular 3-cycle \( K \) in \( F(\mathbb{C}^4) \), we can reproduce in exactly the same way our previous construction, and find a singular 3-cycle \( \Sigma \) in \( \mathbb{C}^4 \) and an integer \( p > 0 \) such that:

\[
F_*(\Sigma) = pK
\]

But this will not lead us to any contradiction as in the proof of theorem \( \Pi \) because the group \( H_3(F(\mathbb{C}^4)) \) is reduced to zero. Indeed the set \( F(\mathbb{C}^4) \) is contractible, since it retracts by deformation to the origin via the following map:

\[
R : [0, 1] \times F(\mathbb{C}^4) \longrightarrow F(\mathbb{C}^4), \quad (t, x_1, x_2, x_3) \longmapsto (tx_1, tx_2, tx_3)
\]

**References**

[Ch] E.M.Chirka *Complex analytic sets*, Kluwer Academic Publishers.

[Da] D.Daigle *On some properties of locally nilpotent derivations*, Journal of Pure and Applied Algebra 114 (1997) 221-230.

[De] J.K.Deveney *\( G_a \)-actions on \( \mathbb{C}^3 \) and \( \mathbb{C}^7 \)*, Communications in Algebra, 22(15), 6295-6302, 1994.

[D-F] J.K.Deveney, D.R.Finston *On locally trivial \( G_a \)-actions*, Transformation Groups, Vol.2, 2, 1997, pp. 137-145.

[Kr] H.Kraft *Challenging problems on affine n-space*, Séminaire Bourbaki, 802, 1994-95.

[Miy] M.Miyanishi *Normal affine subalgebras of a polynomial ring*, in: Algebraic and Topological Theories-to the memory of Dr.Takehito Miyata, Kinokuniya, Tokyo (1985) 37-51.

[Mum] D.Mumford *Algebraic geometry I: Complex projective varieties*, Springer Verlag.

[Wi] J.Winkelmann *On free holomorphic \( \mathbb{C} \)-actions on \( \mathbb{C}^n \) and homogeneous Stein manifolds*, Math. Ann. 286, 593-612.