The Weierstrass $\wp$-function of the hexagonal lattice

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Abstract

We present some properties of the Weierstrass $\wp$-function associated to the hexagonal (or triangular) lattice. In particular, with the help of an old theorem of I.N. Baker [2] on the characterization of meromorphic solutions of the equation $X^3 + Y^3 = 1$ we determine the zeros of the function $\wp'(z) \pm \sqrt{3}$.

Keywords. Weierstrass $\wp$-function; Weierstrass $\sigma$-function; Weierstrass $\zeta$-function; hexagonal or triangular lattice; Eisenstein integers; Dixon elliptic functions; uniformization; meromorphic solutions.

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1 Foreword

Among the planar lattices two are singled out for their special symmetries: The square lattice and the hexagonal or triangular lattice. In this short article we discuss certain properties of the Weierstrass $\wp$-function whose period lattice is hexagonal.

2 Preliminaries

The general facts about elliptic functions used in this section can be found in the books [1], [3], [4], [6], [7], and [8].
Let \( \wp(z) \) be the Weierstrass \( \wp \)-function with \( g_2 = 0 \) and \( g_3 = 1 \), namely the \( \wp \)-function satisfying

\[
\wp'(z)^2 = 4\wp(z)^3 - 1 = 4[\wp(z) - e_1][\wp(z) - e_2][\wp(z) - e_3]
\]

(with \( \lim_{z \to 0} z^2 \wp(z) = 1 \)), where

\[
e_1 = \frac{1}{4^{1/3}}, \quad e_2 = \frac{1}{4^{1/3}} e^{-2\pi i/3} \quad \text{and} \quad e_3 = \frac{1}{4^{1/3}} e^{2\pi i/3}.
\]

If we set

\[
\wp_k(z) := k^2 \wp(kz), \quad k \in \mathbb{C} \setminus \{0\},
\]

then \( \wp_k(z) \) is the Weierstrass \( \wp \)-function satisfying

\[
\wp_k'(z)^2 = 4\wp_k(z)^3 - k^6.
\]

By choosing \( k = e^{\pi i/3} \) we see that \( \wp(z) \) possesses the symmetry

\[
\wp\left(e^{\pi i/3}z\right) = e^{-2\pi i/3} \wp(z), \quad \wp'\left(e^{\pi i/3}z\right) = -\wp'(z),
\]

in particular, \( \wp(z) \) is even and \( \wp'(z) \) is odd.

Now, in view of (1), one period of \( \wp(z) \) is

\[
\varpi = \int_{\sqrt[3]{-1/4}}^{\infty} \frac{dx}{\sqrt{x^3 - 1/4}} = \frac{2^{1/3}}{3} \int_0^1 \xi^{-5/6}(1 - \xi)^{-1/2}d\xi \]

\[
= \frac{2^{1/3} \Gamma(1/6) \sqrt{\pi}}{3 \Gamma(2/3)} = \frac{1}{2\pi} \Gamma(1/3)^3
\]

(to obtain the last equation we used Legendre’s duplication formula for the Gamma function [1], [4], together with the fact that \( \Gamma(1/3) \Gamma(2/3) = 2\pi/\sqrt{3} \)). Then, with the help of (6) we can see that

the quantity \( e^{\pi i/3} \varpi \) is also a period of \( \wp(z) \)

and that \( \varpi \) and \( e^{\pi i/3} \varpi \) are, actually, primitive periods of \( \wp(z) \). In fact, in view of (1), (2), (6), and (5), we have

\[
\wp\left(\frac{\varpi}{2}\right) = \frac{1}{4^{1/3}} \quad \text{and} \quad \wp\left(e^{\pi i/3} \varpi \right) = e^{-2\pi i/3} \wp\left(\frac{\varpi}{2}\right) = \frac{1}{4^{1/3}} e^{-2\pi i/3}.
\]

It follows that the period lattice of \( \wp(z) \) is the hexagonal lattice

\[
\mathbb{T} := \{m \varpi + n e^{\pi i/3} \varpi : m, n \in \mathbb{Z}\},
\]

where \( \varpi \) is given by (6). In other words

\[
\mathbb{T} = \varpi \mathbb{Z}[e^{\pi i/3}] = \varpi \mathbb{Z}[e^{2\pi i/3}]
\]
(the second equality follows from the fact that \( e^{2\pi i/3} = e^{\pi i/3} - 1 \), where \( \mathbb{Z}[e^{2\pi i/3}] \) is the ring of Eisenstein integers. Notice that \( T \) possesses the rotational symmetry

\[
e^{\pi i/3} T = \left\{ e^{\pi i/3} \omega : \omega \in T \right\} = T
\]

(11) and (since, e.g., \( e^{-\pi i/3} T = T \)) the symmetry with respect to complex conjugation, i.e.

\[
T^* := \{ \bar{\omega} : \omega \in T \} = T
\]

(12) (as usual, \( \bar{\omega} \) denotes the complex conjugate of \( \omega \)). Finally, the fundamental cell of \( T \)

\[
K = \{ s\varpi + te^{\pi i/3} \varpi : 0 \leq s, t < 1 \}
\]

(13).

From now on, unless otherwise stated, \( \wp(z) \) will denote the specific Weierstrass \( \wp \)-function whose period lattice is \( T \).

We continue by considering the Weierstrass \( \sigma \)-function and the Weierstrass \( \zeta \)-function associated to \( T \), defined as

\[
\sigma(z) = \sigma(z; T) := z \prod_{\omega \in T'} \left( 1 - \frac{z}{\omega} \right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}, \quad \text{where } T' := \bar{T} \setminus \{0\},
\]

(14) and

\[
\zeta(z) = \zeta(z; T) := \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega \in T'} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).
\]

(15) For typographical convenience the \( \sigma \)- and the \( \zeta \)-functions associated to the specific lattice \( T \) of (9) will be denoted by \( \sigma(z) \) and \( \zeta(z) \) respectively, instead of \( \sigma(z; T) \) and \( \zeta(z; T) \).

The function \( \sigma(z) \) is entire of order 2 and it is obvious from (14) that its zeros are simple and located at the points of \( T \). In particular, at \( z = 0 \) we have

\[
\sigma(0) = 0 \quad \text{and} \quad \sigma'(0) = 1.
\]

(16) Furthermore, from the symmetries (11) and (12) of \( T \) it follows that

\[
\sigma \left( e^{\pi i/3} z \right) = e^{\pi i/3} \sigma(z) \quad \text{and} \quad \overline{\sigma(z)} = \sigma(\bar{z}),
\]

(17) in particular, \( \sigma(z) \) is odd and if \( x \) denotes a real variable, then \( \sigma(x) \) is real.

The function \( \zeta(z) \) is meromorphic with simple poles located at the points of \( T \), and from (15) and (17) we obtain that it satisfies the symmetry relations

\[
\zeta \left( e^{\pi i/3} z \right) = e^{-\pi i/3} \zeta(z) \quad \text{and} \quad \overline{\zeta(z)} = \zeta(\bar{z}),
\]

(18) in particular, \( \zeta(z) \), too, is odd. Furthermore, in each of the intervals

\[
I_n := (n\varpi, n\varpi + \varpi), \quad n \in \mathbb{Z},
\]

(19) \( \zeta(x) \) is decreasing with \( \zeta(n\varpi) = +\infty \) and \( \zeta((n + 1)\varpi) = -\infty \).
Recall that, having \( \sigma(z) \) and \( \zeta(z) \), the function \( \wp(z) \) can be constructed as

\[
\wp(z) = -\zeta'(z) = \frac{\sigma'(z)^2 - \sigma''(z)\sigma(z)}{\sigma(z)^2} = \frac{1}{z^2} + \sum_{\omega \in \mathbb{T}} \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]
\]

(20)

and from (20) it follows immediately that

\[
\wp'(z) = -\sum_{\omega \in \mathbb{T}} \frac{2}{(z - \omega)^3}.
\]

(21)

We remind the reader that the differential equation (1) is obtained by noticing that its left- and right-hand sides are elliptic functions with matched poles. As for the symmetry relations (5), let us notice that they also follow directly from the first equality of (18).

The function \( \wp'(z) \) has three zeros in \( K \) and the fact that it is odd and \( \mathbb{T} \)-periodic implies that these zeros are \( \varpi/2, e^{\pi i/3} \varpi/2 \), and \( (1 + e^{\pi i/3}) \varpi/2 = \sqrt{3}e^{\pi i/6} \varpi/2 \).

If we differentiate both sides of (1) and eliminate \( \wp'(z) \), we obtain the equation

\[
\wp''(z) = 6\wp(z)^2,
\]

(22)

thus, in the interval, say, \( (0, \varpi) \) the function \( \wp(z) \) is convex, with \( \wp(0^+) = \wp(\varpi^-) = +\infty \), while \( \wp'(0^+) = -\infty \) and \( \wp'(\varpi^-) = +\infty \). And since \( \wp'(\varpi/2) = 0 \), the minimum value of \( \wp(z) \) is \( \wp(\varpi/2) = 4^{-1/3} \).

In connection to the periods \( \varpi \) and \( e^{\pi i/3} \varpi \) of \( \wp(z) \), it is expedient to introduce the quantities

\[
\eta_1 := \zeta(z + \varpi) - \zeta(z) \quad \text{and} \quad \eta_2 := \zeta(z + e^{\pi i/3} \varpi) - \zeta(z).
\]

(23)

From the fact that the derivative \( \zeta'(z) \) is elliptic with periods \( \varpi \) and \( e^{\pi i/3} \varpi \) we infer that \( \eta_1 \) and \( \eta_2 \) are independent of \( z \). By setting \( z = -\varpi/2 \) in the first equation and \( z = -e^{\pi i/3} \varpi/2 \) in the second equation of (23) we obtain, in view of (18),

\[
\eta_1 = 2\zeta \left( \frac{\varpi}{2} \right) \quad \text{and} \quad \eta_2 = 2\zeta \left( \frac{e^{\pi i/3} \varpi}{2} \right) = 2e^{-\pi i/3} \zeta \left( \frac{\varpi}{2} \right) = e^{-\pi i/3}\eta_1.
\]

(24)

Now, a well-known property of \( \wp(z) \) is that \( \eta_1 \) and \( \eta_2 \) satisfy Legendre’s relation

\[
\eta_1 e^{\pi i/3} \varpi - \eta_2 \varpi = 2\pi i
\]

(25)

(which follows by integrating \( \zeta(z) \) along the perimeter of the parallelogram with vertices \( a, a + \varpi, a + e^{\pi i/3} \varpi, \) and \( a + \varpi + e^{\pi i/3} \varpi \), where \( a \) is any complex number in the interior of \( K \)).

By combining (24) and (25) we obtain

\[
2\varpi \zeta \left( \frac{\varpi}{2} \right) = \eta_1 \varpi = \frac{2\pi}{\sqrt{3}}.
\]

(26)
thus, in view of (6),
\[
\eta := \eta_1 = 2 \zeta \left( \frac{\varpi}{2} \right) = \frac{2 \pi}{\sqrt{3}} \frac{1}{\varpi} = \frac{4 \pi^2}{\sqrt{3}} \frac{1}{\Gamma(1/3)^3}
\] (27)

(from now on, for typographical convenience we will write \( \eta \) instead of \( \eta_1 \)).

Incidentally, by using (26) in (15) we get a little bonus:
\[
\frac{2 \pi}{\sqrt{3}} = 2 \varpi \zeta \left( \frac{\varpi}{2} \right) = 4 + \sum_{\kappa \in \mathbb{Z}[e^{\pi i/3}] \setminus \{0\}} \frac{1}{(1 - 2\kappa)\kappa^2}
\] (28)

(3) The zeros of \( \wp(z) \)

The following sets will be useful in the sequel:
\[ T_1 := \text{the equilateral triangle with vertices } 0, \varpi, e^{\pi i/3} \varpi; \] (31)
\[ T_2 := \text{the equilateral triangle with vertices } \varpi, e^{\pi i/3} \varpi, \sqrt{3} e^{\pi i/6} \varpi \] (32)

(notice that \( \sqrt{3} e^{\pi i/6} \varpi = \varpi + e^{\pi i/3} \varpi \));
\[ K' := \{ s \varpi + t e^{2\pi i/3} \varpi : 0 \leq s, t < 1 \} \] (33)

the cell formed by the primitive periods \( \varpi \) and \( e^{2\pi i/3} \varpi \). Notice that the union of \( T_1 \) and \( T_2 \) is the fundamental cell \( K \) of (13), while the intersection of \( K' \) with \( K \) is the equilateral triangle \( T_1 \).
The following proposition is a special case of a result regarding the location of the zeros of the general Weierstrass \( \wp \)-function, which can be found in [3]. We include a proof here for the sake of completeness.

**Proposition 1.** The zeros of \( \wp(z) \) are

\[
\pm r \omega + \omega, \quad \omega \in \mathbb{T}, \quad \text{where} \quad r := \frac{\sqrt{3}}{3} e^{\pi i/6}.
\]

Furthermore, they are all simple.

**Proof.** The zeros of \( \wp(z) \) are simple, since if \( \wp(z^*) = 0 \), then (1) implies that \( \wp'(z^*) \neq 0 \). We, also, know that \( \wp(z) \) has exactly two zeros in each period cell. Furthermore, as we have seen, \( \wp(x) > 0 \) for \( x \in (0, \omega) \). Therefore, by (5) we get that \( \wp(x e^{\pi i/3}) \neq 0 \) and consequently \( \wp(z) \neq 0 \) for \( z \in \partial K \cup \partial K' \). Thus, the two zeros of \( \wp(z) \) in \( K \) lie in the union of the interiors of the two equilateral triangles \( T_1 \) and \( T_2 \) of (31) and (32).

Suppose \( z^* \) is a zero of \( \wp(z) \) in the interior of \( T_1 \). Then, since \( \wp(z) \) is even and \( T \)-periodic, \( 0 = \wp(-z^*) = \wp(\omega + e^{\pi i/3} \omega - z^*) \). Thus, \( \omega + e^{\pi i/3} \omega - z^* \) is a zero of \( \wp(z) \) lying in the interior of \( T_2 \) (likewise, if \( z^* \) is a zero in the interior of \( T_2 \), then \( \omega + e^{\pi i/3} \omega - z^* \) is a zero of the interior of \( T_1 \)). It follows that \( \wp(z) \) has one zero, say \( z_1 \) in the interior of \( T_1 \) and one zero, say \( z_2 \) in the interior of \( T_2 \), and these are the only zeros of \( \wp(z) \) in \( K \).

Now by (5) we get that \( e^{\pi i/3} z_1, \ell = 0, 1, \ldots, 5 \), are zeros of \( \wp(z) \). These zeros lie at the vertices of a regular hexagon centered at \( 0 \). Hence, the \( T \)-periodicity of \( \wp(z) \) implies that for every point \( \omega \in \mathbb{T} \) we have an associated set of zeros \( \mathcal{Z}(\omega) := \{ \omega + e^{\pi i/3} z_1 : \ell = 0, 1, \ldots, 5 \} \) lying at the vertices of a regular hexagon centered at \( \omega \).

From the above observations it follows that

\[
\mathcal{Z}(0) \cap T_1 = \mathcal{Z}(\omega) \cap T_1 = \mathcal{Z}(e^{\pi i/3} \omega) \cap T_1 = \{ z_1 \}
\]

and, consequently, \( z_1 \) is equidistant from the vertices 0, \( \omega \), and \( e^{\pi i/3} \omega \) of \( T_1 \). Therefore, \( z_1 \) is located at the center of \( T_1 \) and, likewise, \( z_2 \) is located at the center of \( T_2 \). In other words, the zeros of \( \wp(z) \) in \( K \) are

\[
\frac{\sqrt{3}}{3} e^{\pi i/6} \omega \quad \text{and} \quad \frac{2\sqrt{3}}{3} e^{\pi i/6} \omega.
\]

and the proof is completed by noticing that

\[
\frac{2\sqrt{3}}{3} e^{\pi i/6} \omega = -\frac{\sqrt{3}}{3} e^{\pi i/6} \omega \quad (\text{mod } \mathbb{T})
\]

(due to the fact that \( \sqrt{3} e^{\pi i/6} \omega = (1 + e^{\pi i/3}) \omega \in \mathbb{T} \)).

The knowledge of its zeros and its poles yields an expression of \( \wp(z) \) in terms of the \( \sigma \)-function, which is quite different from the one of [20] [1], [7]

\[
\wp(z) = -\frac{1}{\sigma(r \omega)^2} \frac{\sigma(z - r \omega) \sigma(z + r \omega)}{\sigma(z)^2}.
\]
In the special case studied in this article, where the period lattice is the hexagonal lattice $T$, there is, yet, one more expression of $\wp(z)$ in terms of $\sigma(z)$.

**Proposition 2.** Let $\wp(z)$ be the $T$-periodic Weierstrass $\wp$-function, where $T$ is the hexagonal lattice of (39), and $\sigma(z)$ the associated $\sigma$-function. Then,

$$\wp(z) = \frac{\sigma(z; rT)}{\sigma(z)^3} = \frac{r\sigma(r^{-1}z)}{\sigma(z)^3},$$

(39)

where $r$ is given by (34).

**Proof.** First, let us observe that

$$T \cup (T + r\varpi) \cup (T - r\varpi) = rT,$$

(40)

from which it follows that the functions

$$\sigma(z)\sigma(z - r\varpi)\sigma(z + r\varpi) \quad \text{and} \quad \sigma(z; rT) = r\sigma(r^{-1}z)$$

(41)

have the same zeros (including multiplicities). Both functions in (41) are odd and entire of order 2, while their derivatives at $z = 0$ are $-\sigma(r\varpi)^2$ and 1 respectively. Therefore, there is a constant $a \in \mathbb{C}$ such that

$$\sigma(z)\sigma(z - r\varpi)\sigma(z + r\varpi) = -r\sigma(r\varpi)^2 e^{az^2} \sigma(r^{-1}z)$$

(42)

and, consequently, (38) implies

$$\wp(z) = \frac{r\sigma(r^{-1}z)}{\sigma(z)^3} e^{az^2}$$

(43)

(actually, the symmetries $rT = re^{-\pi i/3}T = rT = rT^*$ imply that $a \in \mathbb{R}$). Since $\wp(z + \varpi) = \wp(z)$, formula (39) yields

$$\frac{\sigma(r^{-1}z + r^{-1}\varpi)}{\sigma(z + \varpi)^3} e^{2az\varpi + a\varpi^2} = \frac{\sigma(r^{-1}z)}{\sigma(z)^3}.$$

(44)

Noticing that

$$r^{-1} = \sqrt{3}e^{\pi i/6} = e^{-\pi i/3} + 1$$

(45)

we have, in view of (29) and (20),

$$\sigma(r^{-1}z + r^{-1}\varpi) = \sigma\left(r^{-1}z + e^{-\pi i/3}\varpi + \varpi\right)$$

$$= -e^{\pi i/3} \exp\left(-\eta z + \frac{2\pi}{\sqrt{3}} e^{-\pi i/3}\right) \sigma\left(r^{-1}z + e^{-\pi i/3}\varpi\right).$$

(46)

Now, in view of (17) and (30),

$$\sigma\left(r^{-1}z + e^{-\pi i/3}\varpi\right) = \sigma\left(r^{-1}z + e^{-\pi i/3}\varpi\right)$$

$$= -e^{\pi i/3} \exp\left(-\eta z + \frac{2\pi}{\sqrt{3}} e^{-\pi i/3}\right) \sigma\left(r^{-1}z\right)$$

$$= -e^{\pi i/3} \exp\left(e^{\pi i/3} r^{-1}\eta z\right) \sigma\left(r^{-1}z\right).$$

(47)
Substituting (47) in (46) gives

\[
\sigma \left( r^{-1}z + r^{-1}z' \right) = e^{2\pi/\sqrt{3}} \exp \left( \left( 1 + e^{\pi i/3} \right) r^{-1}\eta z + \frac{2\pi}{\sqrt{3}} e^{-\pi i/3} \right) \sigma \left( r^{-1}z \right)
\]

\[
= e^{2\pi/\sqrt{3}} \exp \left( 3\eta z + \frac{\pi}{\sqrt{3}} + i\pi \right) \sigma \left( r^{-1}z \right)
\]

\[
= -e^{3\pi/\sqrt{3}} e^{3\eta z} \sigma \left( r^{-1}z \right). \tag{48}
\]

Also, again by (29)

\[
\sigma (z + \omega)^3 = -e^{3\pi/\sqrt{3}} e^{3\eta z} \sigma (z)^3 \tag{49}
\]

and, finally, by substituting (48) and (49) in (44) we get that \(a = 0\).

Formula (39) is reminiscent of the well-known equation \([4]\)

\[
\wp' (z) = -\frac{\sigma (2z)}{\sigma (z)^3}, \tag{50}
\]

which is valid for any lattice, not just \(T\).

4 The zeros of \(\wp' (z) \pm \sqrt{3}\)

Regarding the Weierstrass \(\wp\)-function associated to any period lattice, a consequence of the addition theorem is that if \(n\) is an integer, then \(\wp(nz)\) can be expressed as a rational function of \(\wp(z)\). Thus, for any rational number \(q\), the quantity \(\wp(qz)\) is an algebraic function of \(\wp(z)\) and the same is true for the derivatives \(\wp^{(k)}(qz)\). For instance,

\[
\wp \left( \frac{z}{2} \right) = \wp(z) + \sqrt{[\wp(z) - e_1] [\wp(z) - e_2]}
\]

\[
+ \sqrt{[\wp(z) - e_2] [\wp(z) - e_3]} + \sqrt{[\wp(z) - e_3] [\wp(z) - e_1]} \tag{51}
\]

In particular, if \(\omega\) is a period of \(\wp(z)\) and \(g_2, g_3\) are algebraic numbers, then for any \(q \in \mathbb{Q}\) and any \(k = 0, 1, 2, \ldots\) the number \(\wp^{(k)}(qz)\) is algebraic or \(\infty\).

In this section we will determine the values of \(\wp(z)\) and \(\wp'(z)\) for certain arguments \(z\) without following the above general approach.

We start with a little detour. It is well known that the algebraic curve

\[
X^3 + Y^3 = 1 \tag{52}
\]

can be “uniformized” by elliptic functions (this is a consequence of the fact that the genus of the curve is 1). In fact, by using (41) and the evenness of \(\wp(z)\) it is easy to check that

\[
X = f(z) := \frac{\wp'(z) + \sqrt{3}}{2\sqrt{3} \wp(z)} \quad \text{and} \quad Y = f(-z) \tag{53}
\]

is a uniformization (i.e. global parametrization) of (52).
Incidentally, the function $f(z)$ above is related to the Dixon elliptic functions $\wp$, which, too, uniformize the curve $\mathcal{C}$.

Formula (52) can be viewed as a meromorphic (in $\mathbb{C}$) solution $(X, Y)$ of the Diophantine-type equation (52). And, as it turns out, this solution is unique in the following sense.

**Theorem** (I.N. Baker [2, 1966]). Every meromorphic solution of (52) is of the form

$$X = f(h(z)), \quad Y = \rho f(-h(z)), \quad \rho^3 = 1,$$

where $f(z)$ is as in (53) and $h(z)$ is an entire function.

Here is a brief sketch of Baker’s neat and instructive argument: The function $f(z)$ of (53) is elliptic and has exactly three simple poles in each period cell; hence it takes every value three times, counting multiplicities. Since $[1 - f(z)]^{1/3} = pf(-z)$ is meromorphic it follows that $f(z) - e^{2\pi ki/3}$ has a triple zero for $k = 0, 1, 2$. Thus, in any fixed period cell, for each such $k$ there is a unique $z_k$ for which $f(z_k) = e^{2\pi ki/3}$. Now $f'(z)$ has exactly three double poles in that period cell, hence it takes every value there exactly six times. In particular, $z_k, k = 0, 1, 2$, are double zeros of $f'(z)$ and, consequently, these are the only zeros of $f'(z)$ in that period cell. Since the singularities of the (multi-valued) inverse function $f^{-1}(w)$ arise at the values $w = f(z)$ for which $f'(z) = 0$, it follows that the singularities of $f^{-1}(w)$ lie over $w = 1, e^{2\pi i/3}, e^{-2\pi i/3}$. From the above observations it is not hard to show that if $X(z), Y(z)$ is any meromorphic solution of (52), then the function element $h(z) = f^{-1}(X(z))$ can be analytically continued indefinitely along any curve $\gamma$ in the complex plane, even if $\gamma$ passes through a point $z_s$ for which $X(z_s) = w_s$, since the fact that $[1 - X(z)]^{1/3}$ is meromorphic implies that $X(z) = w_s + \phi(z)^3$, where $\phi(z)$ is analytic near $z_s$ and $\phi(z) = 0$. Therefore, by the monodromy theorem $h(z)$ is an entire function.

Baker’s theorem has a remarkable implication regarding the values of $\psi'(z)$.

**Proposition 3.** The zeros of $\psi'(z) + \sqrt{3}$ are

$$\frac{\omega}{3} + \omega, \quad e^{\pi i/3} \frac{2\omega}{3} + \omega, \quad e^{2\pi i/3} \frac{\omega}{3} + \omega, \quad \omega \in \mathbb{T},$$

(clearly, they are all simple).

**Proof.** As we have seen, $\psi'(x)$ is increasing in $(0, \omega)$, with $\psi'(0+) = -\infty$ and $\psi'(\omega/2) = 0$. Thus, there is a unique $x_+ \in (0, \omega)$ such that $\psi'(x_+) = -\sqrt{3}$, and we must have $x_+ < \omega/2$. From the fact that $\psi'(z)$ is odd and $\omega$-periodic it follows that if $x_- := \omega - x_+$, then $\psi'(x_-) = \sqrt{3}$.

Each of the functions $\psi'(z) + \sqrt{3}$ and $-\psi'(z) + \sqrt{3}$ has exactly three zeros in $\mathcal{K}$ (counting multiplicities). Therefore, the second equation in (55) implies that the zeros of $\psi'(z) + \sqrt{3}$ are

$$\omega + x_+, \quad \omega + e^{\pi i/3} x_-, \quad \omega + e^{2\pi i/3} x_+, \quad \omega \in \mathbb{T},$$

and it remains to show that $x_+ = \omega/3$. 

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From the fact that \( X \) and \( Y \) of (53) satisfy (52) we obtain that
\[
\left[ \psi'(z) + \sqrt{3} \right]^3 + \left[ \psi'(-z) + \sqrt{3} \right]^3 = \left[ 2\sqrt{3} \psi(z) \right]^3, \\
\]
which, by dividing by \( \left[ \psi'(z) + \sqrt{3} \right]^3 \) implies (since \( \psi'(z) \) is odd)
\[
\left[ \frac{2\sqrt{3} \psi(z)}{\psi'(z) + \sqrt{3}} \right]^3 + \left[ \frac{\psi'(z) - \sqrt{3}}{\psi'(z) + \sqrt{3}} \right]^3 = 1. \\
\]
(57)

Since (57) yields another solution of (52), Baker’s theorem implies that there is an entire function \( h(z) \) such that
\[
\rho^2 2\sqrt{3} \psi(z) + \psi'(z) + \sqrt{3} = \psi'(h(z)) + \sqrt{3} \\
\]
(58)
The quantity in the left-hand side of (58) is a \( T \)-periodic elliptic function with exactly three simple poles in \( K \). Therefore, its period lattice must be \( T \) (i.e. it cannot be larger than \( T \)). Thus, the same must be true for the quantity in the right-hand side of (58). Consequently,
\[
h(z) = \alpha z + b, \quad \text{where } |\alpha| = 1. \\
\]
(59)

Furthermore, (58) also implies that the functions \( \psi'(h(z)) + \sqrt{3} \) and \( \sigma(z)\psi'(z) \) have the same zeros (including multiplicities). Thus, in view of (39), the set of zeros of \( \psi'(h(z)) + \sqrt{3} \) is the hexagonal lattice
\[
e^{\pi i/6} \sqrt{3} T \\
\]
(60)
(and, of course, all these zeros are simple).

Since the map \( z \mapsto \alpha z + b \) preserves (Euclidean) distances in the complex plane, the matching of the zeros of \( \psi'(h(z)) + \sqrt{3} \) and \( \sigma(z)\psi'(z) \) implies that the distance between the zeros \( x_+ \) and \( e^{\pi i/3}x_- \) of \( \psi'(h(z)) + \sqrt{3} \) must be equal to the distance of neighboring zeros of \( \sigma(z)\psi'(z) \), namely \( \varpi/\sqrt{3} \) (in view of (60)).

Thus, since \( x_- = \varpi - x_+ \),
\[
x_+ - e^{\pi i/3}(\varpi - x_+) = \frac{\varpi}{\sqrt{3}}, \quad \text{i.e.} \quad \frac{(3x_+ - \varpi)^2}{4} + \frac{3(\varpi - x_+)^2}{4} = \frac{\varpi^2}{3}, \\
\]
which yields \( x_+ = \varpi/3 \) or \( x_+ = 2\varpi/3 \). Therefore, \( x_+ = \varpi/3 \) since, as we have seen, \( x_+ < \varpi/2 \).

An immediate consequence of Proposition 3 is that the zeros of \( \psi'(z) - \sqrt{3} \) are
\[
\frac{2\varpi}{3} + \omega, \quad e^{\pi i/3}\frac{\varpi}{3} + \omega, \quad e^{2\pi i/3}\frac{2\varpi}{3} + \omega, \quad \omega \in T, \\
\]
(61)
(and, of course, they are all simple).
Finally, let us notice that the equation (1) implies
\[ \wp'(z)^2 - 3 = 4 [\wp(z)^3 - 1] \]  
from which we get that (since \( \wp(x) \) is real for \( 0 < x < \wp \))
\[ \wp\left( \pm \frac{\wp}{3} \right) = 1 \]  
or, equivalently,
\[ \int_{1}^{\infty} \frac{dx}{\sqrt{4x^3 - 1}} = \int_{\wp\left( \frac{\wp}{3} \right)}^{\infty} \frac{dx}{\sqrt{4x^3 - 1}} = \frac{\wp}{3} = \frac{1}{6\pi} \Gamma(1/3)^3, \]
where the last equality is obtained from (6).

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