COMPLEMENTARITY OF KINEMATICS AND GEOMETRY IN GENERAL RELATIVITY THEORY

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Abstract

Relations between kinematics, geometry and law of reference frame motion are considered. We show, that kinematical tensors define geometry up to a space functional arbitrariness when integrability condition for spin tensor is satisfied. Some aspects of geometrization principle and geometrical conventionalism of Poincaré are discussed in a light of the obtained results.

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1 Introduction

According to the equivalence principle which lies at the core of the General Relativity (GR), inertia forces and gravity forces cannot be distinguished from one another if one stays within the boundaries of the theory. In general case semi-Riemannian metrics of space-time describes a gravi-inertial complex which only in the particular case of flat metrics (i.e. when the curvature tensor is zero) can be interpreted as a pure field of inertia and as such can be globally removed by choosing the suitable reference frame. In the case of space-time with a non-zero curvature gravitational and inertial degrees of freedom are mixed and can be go over into one another when the reference frame is changed.

Classical approach to reference frames in the relativity theory [1, 2] means to specify a world line of the referent body (a single observer) or local space-time bundle field of 1-form (monad’s formalism) or a tetrad of 1-forms (tetrad’s formalism) on a manifold with a specified metric \( g \). The kinematic characteristics of the reference frame (the world line curvature for a single observer or forms of acceleration, rotation and strain together with their components for monad’s and tetrad’s formalisms) are calculated according to the standard formulae that explicitly or implicitly include metric of the manifold. In practice the tasks of defining (measuring) of metrics and the kinematic characteristics of the reference frame go together. Metric measurements (for example, by light signals in chronogeometry, [3]) are based on certain hypotheses of a concrete reference frame in which the measuring is taking place, while the kinematic characteristics of that same reference frame are to be defined according to the hypotheses of geometry. Consequently there arises the question of the correlation between kinematics and geometry within the GR, as well as that whether they can be seen as interdependent and maybe even interchangeable. Let us explain the point with a simple example. We’ll consider a world line \( \gamma \) of a test particle with 4-velocity
field $u$. According to the well known theorem of geometry of manifolds \[4\], there can be found a coordinate system, such as

$$u = \frac{\partial}{\partial t} |_{\gamma},$$

and, considering the normalization condition of the vector $u$: $g(u, u) = 1$, the component $g_{tt} = g_{00} = 1$ in the new coordinate system. The question whether the test particle is going to move with or without acceleration, if the metric $g$ is specified, can be then easily solved.

One needs to calculate the first curvature of the $\gamma$ curve according to the formula:

$$a = \nabla_u u,$$

where the covariant derivative $\nabla$ is concordant with the metric $g$. Of course, the result of the calculations doesn’t depend on the coordinate system chosen. Let us now consider a more general problem definition, where we can change the geometry (assuming, for example, it to be unknown and to be defined experimentally), while the coordinate system remains constant (if we assume that it is related to a reference frame of fixed bodies). Let us consider the metric $g'$, which in the coordinate system that straighten the field $u$ (i.e. where (1) is true) takes the form:

$$g' = dt \otimes dt - h_{ik} dx^i \otimes dx^k \quad (i, k = 1, 2, 3).$$

According to the well known theorem of differential geometry (for example, see \[5\]), on the manifold with such metric the line $\gamma$ will be a geodesic. Therefore, its curvature which is equal to 4-acceleration turns to zero. So, through a choice of a suitable geometry such important kinematic characteristic of motion as particle acceleration can be turned to zero (or, if we prefer a more general definition, can be made equal to any 4-vector, specified beforehand and orthogonal to $u$ — see below). Basically this possibility expresses the essence of the geometrization principle of the physical interactions, which is the ground principle in modern theories, considering physical interactions.

These conclusions can be easily applied to the case of 4-velocity field which describes continuous medium, that specifies an extended reference frame. However, in the case of the continuous medium the analysis is complicated by the fact that besides the curvature vector of congruence, such systems have a couple of additional characteristics as well, such as congruence rotation tensor (or spin tensor) $\omega$ and rate of deformation $D$ tensor, which impact the calculations when applied in practice.

Therefore if we reject a priori assumptions related to background geometry and kinematic characteristics of the reference frame, there appears a construction which consists of three ”mobile” parts: geometry (manifold metrics), reference frame (time lines congruence) and kinematics (kinematic tensors). In this case the standard definition of the problem considering determination of kinematic tensors related to decomposition of a covariant derivative of the field of the reference frame into spacetime projections ($\tau$-field, see formula (15) below) is only one of the three fundamental problems shown on the diagram below.

Let us sum up the statements of the problems as shown on the diagram.

1. **Traditional (geometric) statement of the problem.** Kinematic characteristics of the reference frame are being calculated with regard to the specified metric $g$ and specified congruence of the $\tau$-field. This problem has only one solution on provision that the manifold metric is known a priori.

2. **Kinematic statement of the problem.** After this statement, it is the manifold metric that is being calculated on the basis of specified congruence of $\tau$-field and a set of kinematic tensors. As will be shown below, this problem always allows a solution, provided the metric and kinematic tensors satisfy consistency constraints (not overly rigid). Metric in this case has more than one definition.
3. **Mixed (kinematic-geometric) definition of the problem.** Here we reconstruct \( \tau \)-field after specified metric \( g \) and specified kinematic tensors. In order to obtain a solution we need that the metric and kinematic tensors satisfy rather rigid conditions (integrability conditions of kinematic tensors).

All three statements of the problem have a fundamental value and practical applications. The first, geometric statement arises when one calculates the observable effects of the GR, based on the known metrics (for example, with a certain type of symmetry and asymptotic properties). The second (kinematic) statement can be of use when one tries to define the geometry of space-time with the help of experiments that are carried out in a certain reference frame with known properties (see discussion in Conclusion). Lastly, the third (mixed) statement is related to the definition of the motion of the reference frame by the inertia fields "measurements", based on the known geometry of space-time (see discussion in Conclusion).

The aim of the present paper is to investigate two of the unconventional definitions mentioned above, both of which bring to light the interdependence that exist between geometry and kinematics in GR. Our analysis shows that there is a subtle interrelation between geometry, kinematics and practice, and mostly confirms the viewpoint that concerns the conventionality of geometry suggested by A. Poincaré [6]. Physical implications of the results obtained, as well as some of the principal conclusions concerning construction of different physical theories in general will be discussed in Conclusion.

When relaying general statements of fundamental character we will use free of coordinate formulations, mathematical notations of which basically agree with those found in the well-known guidebooks on differential geometry of manifolds [5, 7, 8]. Particularly, we use the following notations and abbreviations:

\( \iota_X \) and \( \varrho_\omega \) — 1-form and vector field dual to vector field \( X \) and 1-form \( \omega \) respectively. For any vector field \( Y \) we have: \( \iota_X (Y) \equiv (X,Y) \), \( \langle \varrho_\omega, Y \rangle \equiv \omega(Y) \), where \( \langle , \rangle \) — Riemannian metric; sometimes we’ll use shortened notation \( \overrightarrow{\omega} \) for \( \varrho_\omega \);

\( T^r_s (\mathcal{M}) \) — fibering of \( r \)-contravariant and \( s \)-covariant tensor fields over \( \mathcal{M} \);

\( \mathcal{T}(\mathcal{M}) = \bigoplus_{r,s} T^r_s(\mathcal{M}) \) — tensor algebra over \( \mathcal{M} \);

For any \( T \in T^0_2(\mathcal{M}) \) we define \( jT \) and \( T_j \) by the formulae:

\[
(jT)(\omega, X) = T(\varrho_\omega, X); \quad (T_j)(X, \omega) = T(X, \varrho_\omega).
\]

Coordinate form: \((jT)^{\alpha\beta}_\gamma = G^{\alpha\gamma}T_{\gamma\beta}, \quad (T_j)^{\alpha\beta}_\gamma = G^{\alpha\gamma}T_{\beta\gamma}\) shows, that \( j \) can be viewed as coordinate free notation of tensor indexes raising. Lowering is defined similarly by means of \( \iota \);

\( \hat{S} \) and \( \hat{A} \) — symmetrization and antisymmetrization operators, acting in spaces \( T^n_0(\mathcal{M}) \) and \( T^n_0\mathcal{M} \) for every \( n \); for example, in case \( T \in T^0_2(\mathcal{M}) \):

\[
(\hat{S}T)(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X)); \quad (\hat{A}T)(X, Y) = \frac{1}{2}(T(X, Y) - T(Y, X));
\]
\( a \lor b \equiv a \otimes b + b \otimes a \) — symmetrized tensor product of vectors or 1-forms;
\( \nabla : T^1_0(\mathcal{M}) \to T^2_0(\mathcal{M}) \) covariant (with respect to some fixed Riemannian metrics \( g \)) derivative;
\( \pi_X : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M}) \) — lowering degree operator, acting on space of external forms of degree \( p \) by the rule:
\[(\pi_X \omega)(Y_1,\ldots,Y_{p-1}) = \omega(X,Y_1,\ldots,Y_{p-1});\]
\( \Delta \equiv \det(g) \) — determinant of the metric tensor matrix;
\( \text{vol} \equiv \sqrt{\left| \Delta \right|} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \) — standard volume form on 4-dimensional manifold,
\[
\text{vol} \equiv \frac{1}{\sqrt{|\Delta|}} \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3
\]
— invariant 4-vector ("volume form in space of 1-forms").
\( * : \Lambda_s(\mathcal{M}) \leftrightarrow \Lambda^{4-s}(\mathcal{M}) \) — covariant operator of dualization isomorphism, mapping fibered of external forms of rank \( s \) in fibered polyvectors of rank \( 4-s \) and vice versa. For example, 3-form \( * X \equiv \pi_X \text{vol} \), and 2-form \( * (X \wedge Y) \equiv (\pi_Y \circ \pi_X) \text{vol} \) and is uniquely defined by its values on the pair of vectors \( Z,W \):
\[* (X \wedge Y)(Z,W) \equiv \text{vol}(X,Y,Z,W);\]
\( \ell(a,b,c,\ldots) \equiv \text{span}(a,b,c,\ldots) \) — span of elements \( a,b,c \) of some linear space.

We use Greek indices \( \alpha,\beta,\gamma,\cdots = 0,1,2,3 \) for the space-time components of geometrical objects and Latin indexes \( i,j,k,\cdots = 1,2,3 \) for a purely space components of geometrical objects.

## 2 Essentials of reference frames theory in GR

The aim of present section is to expose monad’s formalism of general relativity (GR) with some details necessary for future purposes.

### 2.1 Algebra of monad’s formalism

Let \( \mathcal{M} \) be (semi-)Riemannian 4-dimensional manifold with some fixed metric \( g \). The most general way to define 3D submanifolds ("submanifolds of simultaneous events", "embedded" into \( \mathcal{M} \), is to fix some smooth 1-form of time \( \tau \) (\( \tau \)-field, monad field) with the normalization condition:
\[
\tilde{g}(\tau,\tau) = 1,
\]
where \( \tilde{g} \) is standard metric in space of 1-forms, induced by the Riemannian metric \( g \). This form induces decompositions of tangent and cotangent spaces at every point \( p \in \mathcal{M} \):
\[
T^*_p\mathcal{M} = (T^*_p)_{h}\mathcal{M} \oplus \ell_p(j_{\tau}); \quad T^*_p\mathcal{M} = (T^*_p)_{h}\mathcal{M} \oplus \ell_p(\tau),
\]
where space-like tangent and cotangent subspaces are defined by the formulas:
\[
(T^*_p)_{h}\mathcal{M} \equiv \{ v \in T^*_p\mathcal{M} | \tau(v)_p = 0 \}
\]
and
\[
(T^*_p)_{h}\mathcal{M} \equiv \{ \lambda \in T^*_p\mathcal{M} | \lambda(j_{\tau})_p = 0 \}
\]
respectively. Subspaces \( \ell_p(j_{\tau}) \) and \( \ell_p(\tau) \) we’ll call tangent and cotangent time directions at the point \( p \). Let’s note, that the set
\[
T_{h}\mathcal{M} \equiv \bigcup_{p \in \mathcal{M}} (T^*_p)_{h}\mathcal{M}
\]
(or similarly \( T^*_h \mathcal{M} \)) in general does’nt admit local representation \( R \times T(\mathcal{M}_h) \), where \( \mathcal{M}_h \) is some space 3D manifold, since the form \( \tau \) can be anholonomic (nonintegrable). In this situation we’ll refer to \( \mathcal{M}_h \) as anholonomic horizontal manifold \([8]\), such that formally\([4]\) \( T(\mathcal{M}_h) \equiv T_h \mathcal{M} \).

Tensor continuations of \([4]\) give decomposition of a whole tensor algebra \( T(\mathcal{M}) \) on \( \tau - h \) components. Formally, let consider linear operator (affinnor field):

\[
\hat{h} \equiv \hat{I} - \tau \otimes J_\tau \equiv \hat{I} - \hat{\tau},
\]

mapping \( T \mathcal{M} \to T \mathcal{M} \) and \( T^* \mathcal{M} \to T^* \mathcal{M} \). Here \( \hat{I} = \text{id}_{T \mathcal{M}} \) or \( \hat{I} = \text{id}_{T^* \mathcal{M}} \). By the definition it follows, that \( \hat{h}(\hat{h}(X)) = \hat{h}(X) \) and \( \langle \hat{h}(X), Y \rangle = 0 \) for every vector field \( X \) and every vertical \( Y \) (the same is true for 1-forms). So, \( \hat{h} \) is projector: \( T \mathcal{M} \xrightarrow{\hat{h}} T_h \mathcal{M} \) or \( T^* \mathcal{M} \xrightarrow{\hat{h}} T^*_h \mathcal{M} \). Writing \( \hat{I} = \hat{\tau} + \hat{h} \) and taking its \( n \)-th tensor degree, we have:

\[
\hat{I} \otimes^n \equiv \text{id}_{T^n_{\tau - h} (\mathcal{M})} = (\hat{\tau} + \hat{h}) \otimes^n = \sum_\zeta \hat{\pi}_\zeta,
\]

where \( \zeta \) runs all binary sequences of symbols \( \{ \tau, h \} \) of length \( n \), \( \hat{\pi}_\zeta \equiv \) projector on \( \zeta \)-th component of \( T^n_{\tau - h} (\mathcal{M}) \). Acting by initial and final operators of \([6]\) on any tensor field \( T \in T^n_{\tau - h} (\mathcal{M}) \), we have:

\[
T = \sum_\zeta T_\zeta,
\]

where \( \zeta \equiv \hat{\pi}_\zeta(T) \) is \( \zeta \)-th projection of \( T \). In what follows we’ll denote projections by index-like symbols \( \tau \) or \( h \) when it will not lead to ambiguousness. For example, any vector field can be decomposed as follows:

\[
X = X_\tau J_\tau + X_h,
\]

where \( X_\tau \equiv \tau(X), \; X_h \equiv \hat{h}(X) \).

With using \([7]\) it is easy to get decomposition of \( g \):

\[
g = \tau \otimes \tau - \hat{h},
\]

where \( h \) is metric on (anholonomic) manifold \( \mathcal{M}_h \), defined by the rule:

\[
h(X, Y) = -g(\hat{h}(X), \hat{h}(Y))
\]

for any vector fields \( X, Y \). Relation \([10]\) means, that

\[
h(X, Y) = -g(X, Y)
\]

for every horizontal vector fields \( X = X_h \) and \( Y = Y_h \) and

\[
\ker h = \ell(J_\tau).
\]

We define space-like volume forms \( \text{vol}_h \) and \( \text{vol}_h \) by the relations:

\[
\text{vol}_h \equiv \star \hat{\tau}; \quad \text{vol}_h \equiv \star h.
\]

This pair of forms induces operation of space dualization \( \star \), mapping space-like forms and polyvectors with summarized rank 3 in each other.

So, the \( \tau \)-field allows one to make local differentiation of space and time projections of any geometrical objects without more detailed differentiation of space projections onto space components. The latter problem can be solved within the frame of tetrad formalism, which is referred to complete methods of description of reference frame \([1]\).

\[\text{In case of the, so called, complete nonintegrability, }\text{Rashevski-Chow’s theorem }[9][10] \text{ states, that }\mathcal{M}_h = \mathcal{M} \text{ i.e. any two points of }\mathcal{M} \text{ can be joined by a some horizontal curve } \gamma_h.\]
2.2 1+3-analysis on $\mathcal{M}$

By [3] it follows, that $(\nabla_{\tau}\tau)_{\tau} = 0$. Let define space curvature 1-form of $\tau$-congruence:

$$a \equiv \nabla \varphi \tau. \quad (12)$$

From the view point of kinematics this form defines acceleration field of reference frame and characterizes a measure of deflection of integral curves related to the vector field $\tau \tau$ from geodesics (straightest), which corresponds to a free fall. Obviously, the tensor $\mathcal{H} \equiv \nabla \tau - \tau \otimes a$ is space-like. It can be decomposed on symmetric and antisymmetric components: $\mathcal{H} = \mathcal{D} + \omega$, where

$$\mathcal{D} \equiv \hat{S}(\nabla \tau - \tau \otimes a) = \frac{1}{2}(L_{\tau}g - \tau \vee a) \quad (13)$$

— space tensor of external curvature of the manifold $\mathcal{M}$, having kinematical sense of velocity deformations field, related to the reference frame $\tau$, and

$$\omega \equiv \hat{A}(\nabla \tau - \tau \otimes a) = \frac{1}{2}(d\tau - \tau \wedge a) \quad (14)$$

— space rotation tensor of $\tau$-congruence, having kinematical sense of local spin of the reference frame. So, finally we obtain the expression

$$\nabla \tau = \tau \otimes a + \omega + \mathcal{D}. \quad (15)$$

Acting in (15) by $j$ from the right (with using $[\nabla, j] = 0$), we obtain for the vector field $\tau \hat{\tau} = j_{\tau}$:

$$\nabla \tau \hat{\tau} = \tau \otimes \hat{a} + \hat{\omega} + \hat{\mathcal{D}}, \quad (16)$$

where $\hat{\omega} = \omega_{j}$, $\hat{\mathcal{D}} = D_{j}$.

Following to [1], let define operators of time-like and space-like derivatives:

$$\dot{T}_{h} \equiv \frac{d}{d\tau}T_{h} \equiv (L_{\tau}T_{h})_{h}; \quad (3)\nabla T_{h} \equiv (\nabla_{h}T_{h})_{h},$$

where $T_{h}$ is arbitrary space tensor field. On scalar functions by definition we have:

$$\dot{f} = \nabla \hat{\tau}(f); \quad (3)\nabla f = (df)_{h} \equiv d_{h}f.$$

With using (7) the following identity for any vector field $Z$ can be established:

$$\nabla Z = (3)\nabla Z_{h} + Z_{\tau}\hat{\mathcal{H}} + (\dot{Z}_{\tau} - Z_{a})\tau \otimes \hat{\tau} + \tau \otimes (Z_{\tau} \hat{a} + (\dot{Z}_{h} + \hat{\mathcal{H}}(Z_{h}, )) \quad (17)$$

$$+((dZ_{\tau})_{h} - \mathcal{H}( , Z_{h})) \otimes \hat{\tau},$$

where $Z_{a} = a(Z)$. Acting on (17) by $i$ from the right, identifying $i_{Z} \equiv \lambda$ and using the relation

$$i_{Z_{h}} = \frac{d}{d\tau}i_{Z_{h}} - 2\mathcal{D}(Z_{h}, ),$$

we have for 1-forms:

$$\nabla \lambda = (3)\nabla \lambda_{h} + \lambda_{\tau}\mathcal{H} + (\dot{\lambda}_{\tau} - \lambda_{a})\tau \otimes \tau + \tau \otimes (\lambda_{\tau}a + (\dot{\lambda}_{h} - \hat{\mathcal{H}}( , \lambda_{h}))) \quad (18)$$

$$+((d\lambda_{\tau})_{h} - \hat{\mathcal{H}}( , \lambda_{h})) \otimes \tau.$$

Assuming in (18) $\lambda = \tau$, $\lambda_{\tau} = 1$, $\lambda_{h} = \lambda_{0} = 0$ we obtain (15).

Formulas (17)-(18) show, that any 4D tensor expression, including covariant derivatives, can be rewritten in terms of time-like and space-like derivatives. The following useful identities are easy checked:

$$\dot{\tau} = a = L_{\tau}\tau; \quad (3)\nabla \tau \equiv (\nabla_{h}\tau)_{h} = \mathcal{H}; \quad \dot{h} = 2\mathcal{D} = \dot{g}; \quad (3)\nabla h = 0. \quad (19)$$
The latter expression suggests, that operator $^{(3)}\nabla$ should be treated as "covariant" (relatively $h$) derivative on $\mathcal{M}_h$.

Note also, that kinematic tensors $a, \omega$ and $\mathcal{D}$ are generally covariant 4D tensors. This circumstance reveals important difference between reference frames and coordinate systems: the fact that these tensor are nonzero doesn’t depend of choice of coordinate system, while by definition it is drastically dependent on the choice of reference frame.

### 2.3 Monads formalism in special gauge

In spite of independency of kinematical tensors on the choice of coordinate system, its apparent representation can be more or less complicated in the coordinate systems, differently adopted to the $\tau$-field. Concrete realizations of monads formalism in special coordinate systems is called *gauges of monads formalism*. In a difference from a pure mathematical formalism, which development is more compact and convenient when one is based on 1-form $\tau$, using of the vector field $\nabla$ is more preferable in the most of applied problems. Physical (or geometrical) cause of this preference is due to the fact, that 1-form $\tau$ defines local simultaneity space, which we are commonly don’t deal with in our experiments. From the other hand law of motion of a particles, forming some reference frame and vector field $\nabla$ (this is 4-velocity field of the particles) are very often available for experimental observations. Mathematically the difference between fields $\tau$ and $\nabla$ is caused by the theorem stating possibility of straightening of the vector fields and by absence of the similar theorem stating possibility of "straightening" of the 1-forms.

Let consider in more details monads formalism in the coordinate system, which straights $\tau$-field, i.e. in which we have

$$\tau = \frac{\partial}{\partial t}. \tag{20}$$

Such gauge is particular case of the so called chronometric gauge of monads method \[1\], for which

$$\tau = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}. \tag{21}$$

We’ll call this very special gauge \[21\] *strong chronometric gauge* or more briefly *canonical gauge*. In a difference with classical chronometric gauge canonical one assumes, that not only lines of time coincide with coordinate lines $x^i = \text{const}$, ($i = 1, 2, 3$), but also the coordinate $x^0 = t$ is world (i.e. global) time. Note, that the choice of canonical gauges bring no any restrictions on the type of reference frame or on the metric of the manifolds. This choice only assumes fixation of some definite coordinate system on the manifold.

In the canonical gauge the metric $g$ can be written in the form:

$$g = dt \otimes dt + \Omega_i dt \vee dx^i - \sum_{i=1}^3 H_i^2 dx^i \otimes dx^i - \sum_{i=1}^3 \delta_i (dx \vee dx)^i, \tag{22}$$

where

$$\Omega_i \equiv g_{0i}, \quad H_i^2 \equiv -g_{ii}, \quad \delta_1 \equiv -g_{23}; \quad \delta_2 \equiv -g_{13}; \quad \delta_3 \equiv -g_{12}; \quad (dx \vee dx)^1 \equiv dx^2 \vee dx^3, \ldots$$

etc. From the \[22\] we obtain

$$\tau = dt + \Omega_i dx^i, \tag{23}$$

\[2\]In fact, $^{(3)}\nabla$ possesses effective torsion, since direct calculation gives: $\text{Tors}^{(3)}(X_h, Y_h) \equiv ^{(3)}\nabla_{X_h} Y_h - ^{(3)}\nabla_{Y_h} X_h - [X_h, Y_h] = 2\omega(X_h, Y_h)\nabla$. However, with respect to space bracket: $[\cdot, \cdot]_h$ torsion of $^{(3)}\nabla$ is zero.
and for space metric by (9)

\[ h = \sum_{i=1}^{3} (H_i^2 + \Omega_i^2) dx^i \otimes dx^i + \sum_{i=1}^{3} (\delta_i + (\Omega_i \Omega)_i)(dx \wedge dx)^i, \tag{24} \]

where \((\Omega \Omega)_1 = \Omega_2 \Omega_3\), \((\Omega \Omega)_2 = \Omega_1 \Omega_3\), \((\Omega \Omega)_3 = \Omega_1 \Omega_2\).

Finally by (20) and space-like nature of kinematic tensors their components are satisfied to the equations:

\[ a_0 = 0; \quad \omega_{0i} = \omega_{i0} = 0; \quad D_{0i} = D_{0i} = 0. \tag{25} \]

### 2.4 Kinematical tensors and observables

Let us take our attention to the physical sense of kinematical tensors and their relations to observable values. Since results of our discussion will be used in searching for metrics, it is important from the beginning to clear true geometrical nature of the physical values, that are described by the kinematical tensors [11]. Geometrically 4-acceleration is 4-vector. Writing this vector in the form:

\[ \vec{a} = a^a \partial_a \tag{26} \]

and using special properties of canonical gauge, we go to the following kind of space-likeness condition for the vector \(\vec{a}\):

\[ g(\vec{a}, \vec{\tau}) = 0 \Rightarrow a^0 + \Omega_i a^i = 0, \tag{27} \]

that leads to the expression

\[ \vec{a} = -(\Omega_i a^i) \partial_i + a^i \partial_i. \tag{28} \]

 Observable quantities are the so called physical components \(\bar{a}^i\), which are connected with 3D coordinate components \(a^i\) by the well known relations [3].

\[ \bar{a}^i = \sqrt{g_{ii}} a^i = H_i a^i \quad \text{(no summation!)}. \tag{29} \]

So, we obtain the following final expression for the 4-acceleration vector in terms of observable:

\[ \vec{a} = - \left( \sum_{i=1}^{3} \frac{\Omega_i \bar{a}^i}{H_i} \right) \partial_t + \sum_{i=1}^{3} \frac{\bar{a}^i}{H_i} \partial_i. \tag{30} \]

The acceleration 4-covector has the following kind in terms of observable quantities \(\bar{a}^i\) (manipulations with indexes can be yield by the metric (24), taken with opposite sign):

\[ a = - \left( \frac{H_1^2 + \Omega_1^2}{H_1} \bar{a}^1 + \frac{\delta_3 + \Omega_1 \Omega_2}{H_2} \bar{a}^2 + \frac{\delta_2 + \Omega_1 \Omega_3}{H_3} \bar{a}^3 \right) dx^1 + \]

\[ - \left( \frac{\delta^3 + \Omega_1 \Omega_2}{H_1} \bar{a}^1 + \frac{H_2^2 + \Omega_2^2 \bar{a}^2}{H_2} + \frac{\delta_1 + \Omega_2 \Omega_3}{H_3} \bar{a}^3 \right) dx^2 + \]

\[ - \left( \frac{\delta_2 + \Omega_1 \Omega_3}{H_1} \bar{a}^1 + \frac{\delta_1 + \Omega_2 \Omega_3}{H_2} \bar{a}^2 + \frac{H_2^2 + \Omega_3^2 \bar{a}^3}{H_3} \right) dx^3. \tag{31} \]

\[ \text{In fact we use affine (non-orthogonal and non-normalized) coordinate tetrad fields \{\partial_t, \partial_1, \partial_2, \partial_3\}. By the fact physical components contain multipliers } \sqrt{g_{ii}} = ||\partial_i||, \text{ rather than } \sqrt{h_{ii}} = ||(\partial_i)_h||. The difference of descriptions of space values in terms \{\partial_i\} and \{(\partial_i)_h\} has no any influence on final conclusions. Moreover, in what follows we’ll not need complete tetrad reference frame formalism. \]
The angular velocity of continuum media is in its geometrical nature bivector of 3D space, i.e. this is 3D antisymmetric contravariant tensor of the kind

$$\dot{\phi} \equiv \varphi^{ik} \partial_i \wedge \partial_k. \quad (32)$$

In practice the well known isomorphism between vectors and bivectors in 3D space allows to consider angular velocity as the vector $\vec{\omega}$, which direction defines instant axe of rotation and absolute value defines the number of radians, that rotating elements goes per unit of time. Transition from the vector $\vec{\omega}$ to the space spin tensor $\omega$, defined as above, is given by the relation:

$$\omega = \pi \tau (\star \vec{\omega}) = \star \vec{\omega}, \quad (33)$$

or in the components

$$\omega_{\alpha\beta} = \sqrt{\Delta} [\varepsilon_{\gamma\delta\alpha\beta} \tau^\gamma \omega^\delta] = \sqrt{\Delta} [\varepsilon_{0i\alpha\beta} \omega^i], \quad (34)$$

where the special kind of $\tau$ in canonic gauge have been used. Going to the physical components $\bar{\omega}^i$, we finally obtain:

$$\omega_{ik} = \sqrt{\Delta} \sum_{s=1}^{3} \varepsilon_{i k 0 s} \frac{\bar{\omega}^s}{H_s}, \quad (35)$$

Interpretation of rate of strain tensor is based on its contribution to deformation of space metric along the flow of world lines of reference frame (the third formula in (19)). The components of covariant rate of strains tensor are directly connected with the rates of relative deformations of the lengths, measured along coordinate lines, and with the rate of variations of the angles between the coordinate lines \[12\]. In the canonical gauge the following relations between physical components of rate of strains tensor and its coordinate components takes place

$$D_{ik} = \frac{\bar{D}_{ik}}{\sqrt{g^{ii}} \sqrt{g^{kk}}}. \quad (36)$$

3 Kinematical statement of the problem: reconstruction of a metric from the $\tau$-field and kinematical tensors

Let consider some reference frame, defined by the $\tau$-field and let prescribe to the physical components of kinematic tensors, related to this reference frame, some arbitrary values. Without loss of generality reference frame and kinematical tensors can be considered in canonical gauge. Moreover such gauge is the most natural for the observers, attached to material elements of the reference frame. So, we have $\tau$-field, given by the formula (20) and the set of 12 values \{$a^1, a^2, a^3, \bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{D}_{11}, \bar{D}_{12}, \bar{D}_{13}, \bar{D}_{22}, \bar{D}_{23}, \bar{D}_{33}$\} defined by the formulas (31), (33) and (36), every of which contains as unknown values the components of metric $g$ to be defined in future. Expression (16), defining kinematic tensors, one should to consider now as equations for metric. In view of (20) left hand side of expression (16) takes the form:

$$\nabla_\alpha \tau^\beta = \Gamma^\beta_{\alpha 0}, \quad (37)$$

so the expression (16) can be rewritten in the following equivalent form:

$$\frac{\partial g_{\gamma 0}}{\partial x^\alpha} + g_{\sigma \gamma} \frac{\partial g_{\alpha 0}}{\partial t} - \frac{\partial g_{\alpha 0}}{\partial x^\gamma} = 2(\tau_{\alpha \gamma} + \mathcal{H}_{\alpha \gamma}). \quad (38)$$

\[4\]This statement is supported by the consideration that angular velocity will remain bivector in space of any dimensions.
The components of equation (38) with $\alpha = 0$, $\gamma = 0$ and $\alpha = i$, $\gamma = 0$ are satisfied identically by gauge conditions. Nontrivial equations are obtained for the components with $\alpha = 0$, $\gamma = i$

$$\frac{\partial \Omega_i}{\partial t} = a_i$$ \hspace{1cm} (39)

and with $\alpha = i$, $\gamma = k$

$$\frac{\partial g_{ik}}{\partial t} + \frac{\partial \Omega_k}{\partial x^i} - \frac{\partial \Omega_i}{\partial x^k} = 2(\Omega_i a_k + \mathcal{H}_{ik}).$$ \hspace{1cm} (40)

Decomposing symmetric and antisymmetric part in the latter equation, we go to the following pair equations:

$$\frac{\partial g_{ik}}{\partial t} = 2D_{ik} + \Omega_i a_k + \Omega_k a_i,$$ \hspace{1cm} (41)

and

$$\frac{\partial \Omega_k}{\partial x^i} - \frac{\partial \Omega_i}{\partial x^k} = 2\omega_{ik} + \Omega_i a_k - \Omega_k a_i,$$ \hspace{1cm} (42)

that is concordant with coordinateless definitions (13) and (14). The system of equations (39), (41) and (42) will be a subject of further investigations.

### 3.1 Integrability condition

The remarkable circumstance of the subsystem (39), (41), defining time dependency of metric components, is its conditionless integrability. Really, space coordinates, which metric components and kinematic tensors depend on, can be viewed in the system as parameters. So, the system (39), (41) can be treated as ordinary differential equations system of the first order, resolved with respect to the first derivatives on $t$. In view of general theorems of the theory of differential equations solution to the system (39), (41) locally exists and depends on 9 "integration constants" which in fact are "functional constants" i.e. some undefined functions of space coordinates.

Some problems with integrability can arise only for the equation (42), which partially define dependency of the metric on space variables. For the analysis of integrability of the equation let’s introduce the following space 1-forms:

$$\Omega \equiv \Omega_i \ dx^i; \quad a \equiv a_i \ dx^i; \quad \omega = \omega_{ik} \ dx^i \wedge dx^k$$

and let’s define sectional space external differential:

$$\mathcal{d} \equiv d|_{t=\text{const}}.$$ 

Then the subsystem (39), (42) can be rewritten more compactly:

$$\dot{\Omega} = a; \hspace{1cm} (I) \hspace{1cm} \mathcal{d} \Omega = 2\omega + \Omega \wedge a. \hspace{1cm} (II)$$ \hspace{1cm} (43)

Hereafter the dot over letter will denote standard partial derivative on $t$. The equation (43, II) means, that righthand side is exact space form. In view of nilpotency of external differentiation ($\mathcal{d} \circ \mathcal{d} \equiv 0$) the necessary integrability condition for (43, II) (locally this is also sufficient condition!) is closeness condition for the righthand part with respect to $\mathcal{d}$:

$$\mathcal{d}(2\omega + \Omega \wedge a) = 0,$$ \hspace{1cm} (44)

which must be satisfied as a consequences of original equations. Using the well known rules of wedge product and differentiation, commutativity of $\partial_t$ and $\mathcal{d}$ and equations (14), we go to the following chain of equalities for (14):

$$2 \mathcal{d}\omega + \mathcal{d}(\Omega \wedge \dot{\Omega}) = 2 \mathcal{d}\omega + \mathcal{d}\Omega \wedge \dot{\Omega} - \Omega \wedge \mathcal{d}\dot{\Omega} = 2 \mathcal{d}\omega + 2\omega \wedge \dot{\Omega} - \Omega \wedge (\mathcal{d}\Omega).$$
\[2(\mathcal{d}\omega + \omega \wedge \dot{\Omega} - \Omega \wedge \dot{\omega}) = 0.\]  

(45)

In fact the integrability condition (45) allows generally covariant formulation, if one will start from the equation (14). Rewritten it in the form

\[d\tau = 2\omega + \tau \wedge a\]  

(46)

and applying differential to the both sides, we go to the closeness condition of 2-forms in the righthand side of (46) in the following form:

\[d(2\omega + \tau \wedge a) = 0.\]  

(47)

Taking into account the identity

\[\tau = dt + \Omega, \quad d\lambda = (-1)^s \dot{\lambda} dt + \mathcal{d}\lambda,\]  

(48)

where \(s\) — is rank of external form, it is easily to show, that (47) is equivalent to (45).

Lets make some remarks to the obtained results.

1. Integrability condition (47) in the form of vanishing of some 3-form in 4D space-time after applying of dualization operator would be equivalent to vanishing of some 4-vector i.e. four conditions on kinematical tensors. Canonical gauge reveals, that we are dealt with space 3-form in (47) and 3D dualization \(\star\) leads only to one essential condition.

2. Rather simple calculations with \(\star\)-operator, applied to (45), lead to the following form of integrability condition:

\[(3)\text{div} \omega + 2a(\omega) - 2\Omega(D_t \omega) = 0,\]  

(49)

where

\[(3)\text{div} \equiv \mathcal{d} \mathcal{d}; \quad D_t \equiv \mathcal{d}_t \mathcal{d}\]

or in components:

\[
\frac{1}{\sqrt{\Delta}} \partial_i (\sqrt{\Delta} \omega^i) + 2a_i \omega^i - 2\frac{\Omega_i}{\sqrt{\Delta}} \partial_t (\sqrt{\Delta} \omega^i) = 0.\]  

(50)

3. For the formula (14), treated within the context of standard geometrical statement of the problem on kinematical tensors calculation, the condition (47) or equivalent condition (45) are identities and they are satisfied due to definitions of the kinematical tensors. In our kinematical statement of the problem these relations must be interpreted in other manner. In order to avoid additional equations on metric components and chain of integrability conditions, \textit{we should consider relation (14) or its equivalent forms as restrictions on the components of kinematical tensors \(\omega\) and \(a\), or, more exactly, as a restrictions on its physical components}. In other words, in general, kinematical tensors \(\omega\) and \(a\) can’t be given absolutely independently from each other and from geometry, but they must satisfy (49) in special coordinate system. Connection between tensors \(\omega\), \(a\) and the metrics, expressed by (49), should be taken into account by some way, for example, by resolving of (49) with respect to one of the physical components of the tensors.

4. Note, that integrability condition (49) become identity under \(\omega = 0\). From the other hand, this equation is necessary and sufficient condition of integrability of 1-form of time, as it directly follows from (46) and the well known Frobenius theorem. Integrable distribution \(\ker \tau\) defines local foliation of space-time on submanifolds of simultaneous events, parametrized by \(t\). So, the fact of nontrivial relation of kinematics and geometry is in closest relation with non-holonomic aspects of submanifolds of simultaneous events in case of \(\omega \neq 0\).
5. Physically the relation (49) generalizes nonrelativistic 3D hydrodynamical formula \( \text{div} \mathbf{\omega} = 0 \) for the case of relativistic motions on 3D Riemannian non-holonomic submanifolds.

6. Mathematically relation (49) is consequence of commutativity of space partial derivatives. The condition of commutativity of the space partial derivatives with time one can be easily obtained from (41) by differentiating with respect to \( t \) and it does’nt produce any new independent conditions.

7. The components of rate of strain tensor does’nt appear in apparent form in (49), so it can be prescribed arbitrarily.

8. In case of satisfying of (49) the reconstructed metric \( g \) will have considerable functional arbitrariness. It becomes clear after calculation of number of functions and constraints. In case of nonzero spin it will be \( 12 - 1 = 11 \) (total number of components rotation, acceleration and deformation minus one restriction). From the other hand, there are only 9 unknown components of metric, which are to be found (the component \( g_{00} = 1 \) due to chosen gauge), from which only 6 are physically essential in view of possibility of four general coordinate transformation (that may violate canonical gauge). In absence of rotation we have 12 arbitrary functions against 6 physically essential components of a metric.

4 Example 1: space-times with stationar field of acceleration

Let consider the reference frame in canonic gauge, for which \( \mathcal{D} = 0, \omega = 0 \) and \( a \neq 0 \). One of the space coordinate line (for example \( x^1 \)) can always be adopted\(^5\) to the acceleration field so, that \( \mathbf{\bar{a}} = a^1 \partial_1 \). We’ll denote the nonzero physical component of acceleration as \( \bar{\bar{a}} \equiv \bar{\bar{a}}^1 \).

Then with using (31) we go to the following expression for 1-form of acceleration:

\[
a = -H_1^2 \bar{\bar{a}} \partial_1 x^1 - \frac{\delta_3 + \Omega_1 \Omega_2}{H_1} \bar{\bar{a}} \partial_2 x^2 - \frac{\delta_2 + \Omega_1 \Omega_3}{H_1} \bar{\bar{a}} \partial_3 x^3. \tag{51}
\]

Let us go to the new unknown functions:

\[
X_i \equiv \frac{\Omega_i}{H_i}; \quad \Delta_i \equiv \frac{\delta_i}{(\Omega_1 \Omega_2 \Omega_3)^i} \quad (i = 1, 2, 3).
\]

In the variables \( X_i, \Delta_i \) the subsystem of equations (39), (41), defining time dependency of the metric, takes the following form:

\[
\dot{X}_1 = -(1 + X_1^2)^2 \bar{\bar{a}}; \quad \dot{X}_2 = -(1 + X_2^2)(\Delta_3 + X_1 X_2) \bar{\bar{a}}; \quad \dot{X}_3 = -(1 + X_3^2)(\Delta_2 + X_1 X_3) \bar{\bar{a}}; \tag{52}
\]

\[
\dot{\Delta}_1 = [X_2(\Delta_2 - \Delta_1 \Delta_3) + X_3(\Delta_1 - \Delta_1 \Delta_2) - X_1(\Delta_1 X_2^2 + \Delta_1 X_3^2 - 2 X_2 X_3)] \bar{\bar{a}}; \tag{53}
\]

\[
\dot{\Delta}_2 = [(\Delta_2 + X_1 X_3)(X_1 - \Delta_2 X_3) + (1 + X_2^2)(X_3 - \Delta_2 X_1)] \bar{\bar{a}}; \tag{54}
\]

\[
\dot{\Delta}_3 = [(\Delta_3 + X_1 X_2)(X_1 - \Delta_3 X_2) + (1 + X_3^2)(X_2 - \Delta_3 X_1)] \bar{\bar{a}}. \tag{55}
\]

\(^5\)Let us remind, that after canonical gauge fixing we have allowable coordinate transformation of the form:

\[
x'^0 = x^0 + \psi(x^1, x^2, x^3); \quad x'^i = \chi^i(x^1, x^2, x^3),
\]

where \( \psi \) and \( \chi^i (i = 1, 2, 3) \) — arbitrary smooth functions, providing invertibility of the transformations. Using this remained coordinate degrees of freedom, one can transform acceleration field to the form mentioned in the text.
By using new variables we have reduced the system of a nine ordinary differential equations (39), (41) to the system of a six equations (52)-(55). The excluded functions $H_i$ are expressed through the variables $X_i, \Delta_j$ as follows:

\[
\ln H_1 = X_1 (1 + X_1^2) \bar{a}; \quad \ln H_2 = X_2 (\Delta_3 + X_1 X_2) \bar{a}; \quad \ln H_3 = X_3 (\Delta_2 + X_1 X_3) \bar{a};
\]

(56)

Let consider particular case of the system (52)-(55), admitting complete integration: $\Delta_1 = \Delta_2 = \Delta_3 = 0, \bar{a} = 0$. Substituting of the first three conditions into the system (52)-(55) leads to the integrals: $X_2 = X_3 = 0$ and to the following unique equation:

\[
\dot{X}_1 = -(1 + X_1^2) \bar{a},
\]

(57)

which under the imposed stationarity condition can be easily integrated by separation of variables:

\[
-\bar{a}t + F(x^1, x^2, x^3) = \frac{X_1}{2(1 + X_1^2)} + \frac{1}{2} \arctan X_1,
\]

(58)

where $F$ is yet unknown function of space coordinates. Integrating (58), we go to the following expression for Lame coefficients:

\[
H_1 = \frac{\Phi_1(x^1, x^2, x^3)}{\sqrt{1 + X_1^2}}; \quad H_2 = H_2(x^1, x^2, x^3); \quad H_3 = H_3(x^1, x^2, x^3),
\]

(59)

where $\Phi_1$ is unknown function of space coordinates, $H_2, H_3$ are arbitrary functions of space coordinates. For the unique nonzero component $\Omega_1$ we obtain the following expression:

\[
\Omega_1 = H_1 X_1 = \frac{\Phi_1(x^1, x^2, x^3)X_1}{\sqrt{1 + X_1^2}}.
\]

(60)

The equation (42) gives two additional restrictions for dependency of $\Omega_1$ on space coordinates:

\[
\frac{\partial \Omega_1}{\partial x^2} = \frac{\partial \Omega_1}{\partial x^3} = 0.
\]

(61)

These equation will be identically satisfied by the following dependencies:

\[
\bar{a} = \bar{a}(x^1); \quad F = F(x_1); \quad \Phi_1 = \Phi(x^1).
\]

(62)

Without loss of generality the function $\Phi(x^1)$ can be put to 1 by suitable choice of the coordinate $x^1$, which after imposing of all restrictions is defined up to arbitrary diffeomorphism $x^1 = f(x^1)$. Now the obtained metric can be written in the form:

\[
g = dt \otimes dt + \tanh \psi (dt \otimes dx^1 + dx^1 \otimes dt) - (1 - \tanh^2 \psi) dx^1 \otimes dx^1 -
\]

\[
H_2^2(x^1, x^2, x^3) dx^2 \otimes dx^2 - H_3^2(x^1, x^2, x^3) dx^3 \otimes dx^3,
\]

(63)

where the function $\psi$ is defined by the relation:

\[
\frac{1}{2} \tanh \psi + \arctan e^\psi = \bar{a}(x^1) t + F(x^1).
\]

(64)

So, the class of space-times (63) prescribes to the reference frames of the kind $\tau = \partial_t$ zero kinematical tensors $\omega$ and $\mathcal{D}$, while $\bar{a}^1 = \bar{a}(x^1)$, where $\bar{a}(x^1)$ is arbitrary function of $x^1$, defining the field of physical components of acceleration vector.
5 Example 2: space-times with isotropic field of deformations

In present section we are going to derive general kind of space-times, having canonical reference frames with $a = 0$, $\omega = 0$ and

$$D = \sigma h,$$

where $\sigma$ is arbitrary scalar function of all four coordinates of the manifold. The deformations field (55) describes isotropic but non-homogeneous and non-static deformation of the reference frame at every point of the manifold. Assuming $\Omega = 0$, we find, that in the system (39)-(42) the only equations (40) remain non-trivial. They take the following form:

$$\dot{H}_i = -\sigma H_i; \quad \dot{\delta}_i = -2\sigma \delta_i.$$  

These equations can be easily integrated and the solution has the following form:

$$g = dt \otimes dt - e^{-2\int \sigma dt} \left( \sum_{i=1}^{3} H_{0i} dx^i \otimes dx^i - \sum_{i=1}^{3} \delta_{0i} (dx \vee dx)^i \right),$$

where $H_{0i}, \delta_{0i}$ are arbitrary functions of space coordinates. The metric (67) generalizes the well known Friedman-Robertson-Walker cosmological metrics for non-homogeneous case.

6 Example 3: the space-times with stationary rotated reference frames

Finally let us consider one simple class of space-times, admitting the canonical reference frames with stationary rotation. Assuming in the equations (39)-(42) $a = 0$, $D = 0$, $\omega^2 = \omega^3 = 0$, $\omega^1 \neq 0$, $\dot{\omega}^1 = 0$, we go to the following stationarity condition for metric: $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2, x^3)$. Using this condition, one can transform integrability condition (51) to the following simple kind:

$$\partial_1 \left( \sqrt{\Delta} \frac{\bar{\omega}^1}{H_1} \right) = 0,$$

wherefrom

$$\bar{\omega}^1 = \frac{H_1}{\sqrt{\Delta}} \psi(x^2, x^3),$$

where $\psi$ is arbitrary function of the pair of coordinates $x^2, x^3$. Substituting the last expression into the equation (12) and assuming $\Omega_1 = 0$, we get:

$$\Omega_2 = \Omega_2(x^2, x^3); \quad \Omega_3 = \Omega_3(x^2, x^3); \quad \frac{\partial \Omega_3}{\partial x^2} - \frac{\partial \Omega_2}{\partial x^3} = 2\psi(x^2, x^3).$$

Up to the unessential addends, that are proportional to some exact form the solution takes the form:

$$\Omega_2 = -\int \psi \, dx^3; \quad \Omega_3 = \int \psi \, dx^2.$$  

Finally, the class metric, admitting stationary rotated rigid geodesic canonical reference frame, is described by the expression:

$$g = dt \otimes dt - \left( \int \psi(x^2, x^3) \, dx^3 \right) (dt \otimes dx^2 + dx^2 \otimes dt) + \left( \int \psi(x^2, x^3) \, dx^2 \right) (dt \otimes dx^3 + dx^3 \otimes dt)$$

\(^6\)The equation (42) defines $\Omega$ up to an arbitrary exact form, which can be compensated by purely space coordinate transformations.
where $\psi$ and $g_{ik}$ are arbitrary function of the written coordinates.

7 The mixed (kinematically-geometrical) problem

Now let us consider the situation, when the reference frame is unknown, while metric and kinematical tensors are forgiven. In such problem the concordance conditions for the metric and kinematical tensors are much more rigid, then in previous problems. Really, the fact, that kinematic tensors are different projection of covariant derivative of monads field\(^7\) leads to the very complicated system of integrability conditions, following from the basic relation:

$$[\nabla, \nabla] \tau = -\hat{R}(\tau),$$  \hspace{1cm} (71)

where $\hat{R}$ is curvature operator. Moreover, purely algebraic restrictions arise for the kinematical tensors themselves: in view of the fact, that the field $\tau$ must be orthogonal to all three kinematic tensors:

$$a(\vec{\tau}) = 0; \quad \omega(\vec{\tau}) = 0; \quad \mathcal{D}(\vec{\tau}) = 0,$$  \hspace{1cm} (72)

and the fact of non-degeneracy of a metric, the following condition must be satisfied identically:

$$\text{rank}(a, \omega, \mathcal{D})^T \leq 3,$$  \hspace{1cm} (73)

where the braces denote (composed) matrix of the system (72), which one can consider as the system of linear equations with respect to components $\tau^\alpha$. The condition (73) expresses the necessary condition of non-trivial compatibility for this system. The second necessary condition is requirement that the space of solutions $\ker(a, \omega, \mathcal{D})^T$ must include time-like direction. Even these condition are in fact very strong: some “randomly written” system of kinematical tensors under fixed metric will not satisfy these conditions.

In order to analyze integrability conditions, following from the (71), we calculate commutator from the left in (71) using the definition\(^8\) (15) and substitute the result into the (71). After some algebra the result takes the following form:

$$- [\hat{R} + (\tilde{\nabla} \wedge a) \otimes a - (\tilde{\nabla} \wedge \nabla) a](\tau) = 2 \omega \otimes a + \nabla \wedge \mathcal{H}$$  \hspace{1cm} (74)

or in the components

$$- [\hat{R}^{\gamma}_{\sigma\alpha\beta} + (\delta^\gamma_\alpha a_\beta - \delta^\gamma_\beta a_\alpha) a_\sigma + (\delta^\gamma_\beta \nabla_\alpha - \delta^\gamma_\alpha \nabla_\beta) a_\sigma] \tau^\gamma = 2 \omega a_\sigma + \nabla_\alpha \mathcal{H}_{\beta\sigma} - \nabla_\beta \mathcal{H}_{\alpha\sigma}.$$  \hspace{1cm} (75)

The equations (75) can be understood as strongly overdetermined system of linear equations (in general their number will be 24) with respect to the components $\tau^\alpha$. Its non-trivial compatibility implies coincidence of the ranks of matrix and enlarged matrix (Kronecker-Capelli theorem). Differently from the algebraic conditions (73), the conditions on the ranks for (75) will involve derivatives of kinematical tensors. So, kinematical tensors and metric are concordant to each other much more rigidly, then kinematical tensors and $\tau$-field, in spite of the first are commonly defined through the $\tau$-field by relation (15).

\(^7\)Since the metric now is forgiven, one may do not take care on differing of co- and contravariant versions of kinematical tensors. Now it will be more conveniently for us to be dealt with the field of 1-form $\tau$ and covariant representation of kinematical tensors.

\(^8\)It will be more convenient for us to collect the sum $\omega + \mathcal{D}$ into one space-projected tensor $\mathcal{H}$. 

8 Undeformed geodesic reference frames in spherically-symmetric space-times

As example let us consider the problem of finding of canonical reference frames with \( D = 0 \), \( a = 0, \omega \neq 0 \) in space-times of the following kind:

\[
g = e^{2\nu(t,r)} dt \otimes dt - e^{2\lambda(t,r)} dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi). \tag{76}
\]

The metric (76) describes geometry of spherically symmetric space-time in special coordinates (co-moving curvature coordinates). The condition \( a = 0 \) means that vector field \( \vec{\tau} \) is geodesic and satisfies the equation \( \nabla \vec{\tau} \vec{\tau} = 0 \). The condition \( D = 0 \) with the definition (13) and previous condition mean, that the field \( \vec{\tau} \) is Killing’s field, i.e. satisfies Killing’s equations:

\[
L_{\vec{\tau}} g = 0 \tag{77}
\]

and is element of isometry algebra of space-time with the metric (76).

We’ll describe the field \( \vec{\tau} \) by its coordinate components:

\[
\vec{\tau} = \tau^t \partial_t + \tau^r \partial_r + \tau^\theta \partial_\theta + \tau^\varphi \partial_\varphi,
\]

which are restricted by the following normalizing condition:

\[
c^{2\nu} (\tau^t)^2 - e^{2\lambda} (\tau^r)^2 - r^2 (\tau^\theta)^2 - r^2 \sin^2 \theta (\tau^\varphi)^2 = 1. \tag{78}
\]

By technical considerations it will be more conveniently for us to derive integrability conditions directly from the equations for components of kinematical tensors, rather than from the equations (74). In view of linearity of Killing’s equations (77) it is conveniently to begin analysis namely from these equations. In the components the Killing’s equations take the following kind:

\[
\tau^t \dot{\nu} + \tau^r \nu' + \dot{\tau}^t = 0; \quad (1) \quad \tau^t \dot{\lambda} + \tau^r \lambda' + \tau^t \dot{\tau} = 0; \quad (2)
\]

\[
\tau^r + r \tau^\theta \dot{\theta} = 0; \quad (3) \quad \frac{\tau^r}{r} + \cot \theta \tau^\theta + \tau^\varphi \dot{\varphi} = 0; \quad (4)
\]

\[
e^{2\lambda} \dot{\varphi} - e^{2\nu} \varphi' = 0; \quad (5) \quad r^2 \dot{\varphi} - e^{2\nu} \varphi' = 0; \quad (6) \quad r^2 \sin^2 \theta \dot{\varphi} - e^{2\nu} \varphi' = 0; \quad (7)
\]

\[
r^2 \varphi' + e^{2\lambda} \tau^r = 0; \quad (8) \quad r^2 \sin^2 \theta \varphi' + e^{2\lambda} \tau^r = 0; \quad (9) \quad \sin^2 \theta \tau^r + \tau^\varphi = 0. \quad (10)
\]

Exact consecutive analysis of the system we bring out in Appendix. Here we use finite result, expressed by the formulas (108)-(110), (113) for the components \( \tau^a \). In view of the fact that \( \vec{\tau} \)-field must be the field of 4-velocity of reference frame, it should satisfy the normalizing condition (78). Substituting the formulas (108)-(110), (113) into (78) and equating the coefficients under a different combinations of trigonometric functions on the angles \( \theta \) and \( \varphi \) we go to the following simple deduction: the unique time-like unit Killing’s field of space-time with spherical symmetry is the field of the kind \( \tau = \partial_t \) under \( \nu = 0, \lambda = 0, i.e. \) for the case of special static metric. This space-time is direct sum \( R \oplus \mathcal{H}_3 \), where \( \mathcal{H}_3 \) is arbitrary 3-dimensional manifold, possessing SO(3)-symmetry with respect to some point. It is easily to check that the field \( \partial_t \) in this space is geodesic and non-rotating (it describes matter points, that are in a rest with respect to 3-dimensional unmovable spherical coordinate system). So, the unique class of spherically symmetric space-times, admitting rigid geodesic reference frames are those with the metric:

\[
g = dt \otimes dt - e^{2\lambda(r)} dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi). \tag{79}
\]

The said reference frame is automatically non-rotating.
9 Conclusion

9.1 The geometrization principle

Our analysis shows that the three components (metric, reference frame and its kinematic characteristics) lying at the core of all theoretical speculations of the observable phenomena within the scope of the GR are interdependent to a degree. The standard geometric statement of the problems means that the kinematic characteristics of the reference frame can be calculated when the $\tau$-field and metric are known. However, the practical definition of either of these quantities depends, as a rule, on considerable explicit or implicit a priori assumptions concerning both theory behind the experiments being carried out and their interpretation. Any attempt to determine the law of motion of the reference frame or a space-time geometry will be based on the accepted laws of electromagnetic signal propagation, on more or less plausible hypotheses of the laws of distant body motion (stars, galaxies and clusters), global and local properties of space-time etc. Further analysis of these problems discloses even more a priori assumptions, for instance those which allow the laboratory physical laws concerning extremal states and characteristics of physical systems to be extrapolated. The general picture is rather confounded, with observable values only indirectly related to the estimable value. The most popular method used to overcome these difficulties (probably fundamental flaws) that emerged in the course of scientific practice is an attempt to construct a consistent and comprehensive model of the Universe. One of the attractions of the Einstein’s gravity theory (and its modern modifications) is that we can consistently describe, using the same Einstein equations, both interior of the stars and the global evolution of the Universe, the space-time scale of which greatly exceeds the astrophysical one. On the other hand, the law of free-fall
\[ a = \nabla u = 0 \] (80)
maybe with corrections due to the extended structure of test bodies) describes the motion of the particles of cosmic dust, planets, stars and galaxies. This law, slightly generalized (say, in light of Kaluza-Klein theory) but basically the same in form describes the motion of bodies and particles that have electrical and other kinds of charge. On the basis of the geometrization principle which has been introduced to physics as a result of a successful development of the GR ideas, equation \ref{eq:80} is declared to be fundamental and universal. It describes in its ideal form the dynamics of any system by means of choosing convenient metrics $g$ and (or) connection $\nabla$ \cite{13,14}.

In this case geometry in the most general sense becomes a "matrix" of sorts, matrix of the environment that exists in our mind. However, in such situation it loses "objectivity status". Indeed, thanks to the wide array of tools employed by the modern geometry, it appears almost invulnerable to the impact of observable and experimental data. It becomes pretty apparent when we look at the attempts to explain existing cosmological problems (accelerated expansion, non-flat rotation curves of galaxies etc.) in view of nonlinear $f(R)$-theories that are suggested today \cite{15,16}.

9.2 Geometry or kinematics?

This somewhat unexpected status of geometry shown by our study brings to light another aspect that concerns the problem of observables in GR. Any observation is being carried out within the concrete reference frame and well-developed formalisms of reference frames (monad’s, tetrad’s, spinor etc.) enable us to calculate the observable effects with any degree of precision, provided the space-time geometry is known with the same degree of precision. But if the geometry is unknown or known only approximately, or if we accept Poincaré's point of view (see discussion below) and consider geometry as a conditional entity to be chosen...
more or less deliberately, methods of the theory of reference systems lose their ability to predict results.

Let us illustrate the interaction between kinematics and geometry and the nature of problems arising with the attempts to distinguish one from the other in practice by example that refers to theoretical interpretation of test particles motion. Let observable law of the motion of free test particles be determined by relation \( x(t) = \{1, x^i(t)\} \), where \( t \) is world time of the canonical reference frame \( \mathcal{T} = \partial_t \). Let this law of motion be geometrically interpreted within the scope of a certain Riemannian geometry with metric \( g \), such as the dependency \( x(t) \) is the solution of the geodesic equation in \( t \)-parameterization:

\[
\ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma} \dot{x}^\lambda \dot{x}^\nu = \frac{s}{\dot{s}} \ddot{x}^\mu, \quad (81)
\]

where

\[
\dot{s} = \sqrt{g(\dot{x}, \dot{x})}; \quad \ddot{s} = \sqrt{(\dot{x}^\alpha \partial_\alpha g)(\dot{x}, \dot{x}) + 2g(\dot{x}, \dot{\dot{x}})}, \quad (82)
\]

and the point denotes \( t \)-differentiation. It should be noted that geometrization of the law of motion of test bodies, that is fixation of \( g \)-metric at the same time determines all of the kinematic characteristics of the canonical reference frame within which the law has been observed. Let further more detailed experiments with test bodies reveal a new law of motion of the bodies in the same situation:

\[
x'(t) = x(t) + \delta(t), \quad (83)
\]

with a minor correction \( \delta(t) \). Purely geometric approach consists of a suitable modification (correction) to geometry: \( g \rightarrow g' = g + \delta g \), which would account for observable disturbances. Disturbance of \( g \)-metric will determine disturbance \( \delta \) of the law of motion by means of second-order linear differential equation obtained through linearization (81) in the neighborhood of non-disturbed law of motion and metric:

\[
\dddot{x}^\mu - \frac{s}{\dot{s}} \dddot{x}^\mu + 2\Gamma^\mu_{\lambda\sigma} \dot{x}^\lambda \dddot{x}^\nu + (\delta \Gamma^\mu_{\lambda\sigma}) \dot{x}^\lambda \dot{x}^\nu - \frac{s}{\dot{s}} \dddot{x}^\mu = \frac{d}{dt} \left[ \frac{(\delta g \partial_\alpha g + \delta g)(\dot{x}, \dot{x}) + 2g(\dot{x}, \dot{\dot{x}})}{2\dot{s}} \right] = 0, \quad (84)
\]

where

\[
\delta \Gamma^\mu_{\lambda\sigma} = \frac{1}{2} g^{\mu\rho}(\delta g_{\rho\sigma,\lambda} + \delta g_{\rho\lambda,\sigma} - \delta g_{\lambda\sigma,\rho} - \delta g_{\alpha\beta} g^{\alpha\rho} \Gamma^\beta_{\lambda\sigma}).
\]

However, there is another (complementary) method to interpret disturbance. It requires correction of the reference frame characteristics while geometry remains constant. This view of the law of motion disturbances means that the reference frame within which the observations are being carried out is not quite canonical and requires a minor additional transformation of coordinates. At the same time it will compensate for the law of motion disturbance and make the reference system canonical. In order to determine the relation between the reference frame modification \( \delta \mathcal{T} \) and the law of motion \( \delta \) let us select \( \delta \mathcal{T} \), such as the transformation of the reference frame \( \mathcal{T} + \delta \mathcal{T} \) into a canonical one is achieved by coordinate transformation related to the disturbances of the law of motion \( \delta \). The requirement that the \( \tau \)-field normalization is preserved leads in the first order of smallness to orthogonality of disturbances \( \delta \mathcal{T} \) to the basic \( \mathcal{T} \), therefore the disturbance of the reference frame has only the space components \( \delta \tau^i \). If we represent the desired coordinate transformations as:

\[
t' = t, \quad x'^i = x^i + \xi^i(x), \quad (85)
\]

then, using the transformation law for vector components in the first order of smallness, we obtain:

\[
\tau'^0 = 1; \quad \tau'^i = \frac{\partial \xi^i}{\partial t} + \delta \tau^i, \quad (85)
\]

\[
\tau^0 = 1; \quad \tau'^i = \frac{\partial \xi^i}{\partial t} + \delta \tau^i,
\]
consequently, the requirement of canonicity \((\tau^i = 0)\) leads to the relation between the disturbance of the reference frame and transformations that compensate for it:

\[
\xi^i = - \int \delta \tau^i \, dt + f^i(x^1, x^2, x^3).
\]  

(86)

Let us now apply the obtained transformations to the disturbed law of motion of probe particles. In the first order of smallness, after having omitted purely space transformations related to functions \(f_i\) we obtain

\[
(x^i(t) + \delta^i(t))^\prime = x^i(t) + \delta(t) + \xi = x^i(t) + \delta^i(t) - \int \delta \tau^i \, dt.
\]

If we then select \(\delta \tau^i\) such as the equations

\[
\dot{\delta}^i = \delta \tau^i
\]  

are true, we can achieve the desired result. Namely, that the corrections to the law of motion that come across in the course of observations can be eliminated by means of transformation of the reference frame within which these observations are being carried out while the background geometry remains the same.

Both approaches have been used in the course of the science history. The GR and any geometrical gravity theory in general is an example of a purely geometric solution to the problem of the motion of test bodies. On the other hand, the Ptolemaic theory of epicycles and deferents is essentially an example of a purely kinematic solution to the problem of motion, based on the Euclidian geometry of space and time. In light of our present notions concerning the Universe we are forced to take an intermediate point of view on the problem of observable motion, especially as the details are being corrected all the time. In these circumstances both the geometry of ambient space-time and the kinematics of the reference frame related to the Earth are equally important for understanding of the observable motions.

### 9.3 Poincaré’s conventionalism and its limitations

Our analysis makes it clear that in the real situation correlation \([15]\) read as the definition of kinematic values is not sufficient to describe the relation between the reference frame kinematics and geometry. The fundamental equality of the geometric and kinematic approaches discussed above comes most prominently into view if we turn to the general approach to geometry, suggested by A. Poincaré.

"Where do the initial principles of geometry come from? Are they dictated by logic? By founding the non-Euclidean geometries Lobachevsky established that is not the case. Do we discover spaces with the help of the senses? The answer is no once again, because the space such as we can learn about through our senses is totally different from the space of the geometer’s. Can we say that geometry is based on the experience at all? Thorough investigation will show us we can’t. Therefore we may conclude that these principles are in fact conditional assumptions...and if we were transported into another world (which I call non-Euclidean and try to describe), we would form different assumptions. ...Returning now to the question whether the Euclidean geometry can be considered true or not, we must conclude that it makes no sense. It would be the same as asking which system is true — metric one or that of old measures, or which coordinates are more correct, Cartesian or polar ones. No geometry is truer than the other; the question is only which is more convenient for our purposes" (translated in English from [6, p.10]).

This thesis of Poincaré’s concerning “convenience of a specified geometry” has its merits. The principle of geometrization expressed by all-encompassing formula \([80]\) suggests there
is a certain general geometry within the scope of which any motion will be a free-fall. However, if such geometry were founded, the question whether it would become a part of a fundamental theory or remain a temporary half-empiric model depends on its simplicity, and not in the least convenience. Ptolemaic system can be seen as an example of a "working theory" that can hardly be called simple or convenient. Poincaré’s thesis can be applied to the historical relationship between kinematics and geometry too. Indeed, the change of the kinematic paradigm (i.e the switch from the geocentric to the heliocentric model) drastically simplified description of the planet and star motions. It is not a coincidence that only after the discoveries of Copernicus, Brage and Kepler was Newton able to guess from the simple elliptic trajectories of the planets the existence of one of nature’s simple and basic laws, the law of gravity. Further analysis of this law and its consequences led to the geometric paradigm being reconsidered (i.e. the switch from Euclidean geometry to Riemannian one), and as a result all observable disturbances of Kepler’s ellipses can be now easily described through characteristics of Schwarzschild’s geodesic metrics in GR.

Our study helps to establish some of the limitations the conventional approach of Poincaré’s has. If the relations between geometry and kinematics were governed solely by convenience, any more or less objective geometric world picture would be impossible. But even through our over-simplified and schematic approach one can glimpse obstacles standing in the way of such absolute conventionalism. One such obstacle is the existence of correlations (49) or (50) that limits options for independent specification of the metric and kinematic characteristics of the reference frame. The existence of this correlation does not depend on concrete form of metric, i.e. it is metageometric and therefore sets limits to Poincaré’s conventionalism, the conventionalality of the chosen geometry or reference frame with its characteristics notwithstanding. Actually the relations (49)- (50) pose the only purely mathematical obstacle that restricts the conventionalism discussed above. In view of Poincaré’s scientific philosophy it is such relations that can aspire to the role of "nature’s laws", free from convenience’s influence. Of particular interest is the fundamental role which the rotation of the reference frame plays in relations (49)- (50).

9.4 Complementarity of kinematics and geometry

The aspect of relations (49)- (50) mentioned earlier is curiously analogous to the quantum mechanical correlation between coordinate and momentum uncertainty and the quantum mechanical principle of complementarity. The uncertainties relation in quantum mechanics restricts the degree of detalization with which the state of quantum mechanical system can be decribed through coordinates and momentum. Mathematically the uncertainties relation is explained by non-commutativity of dynamical variables operators. Let us now turn back to GR and assume that for some reason we need to explain disturbances of the law of motion $\delta (t)$ in (83) and simultaneously correct the metric (through $\delta g$) and reference frame (through $\delta \tau$). The relations (49)- (50) sets some fundamental limits to our options here. In order to show more clearly their analogy with quantum mechanics let us now represent (49) as follows:

$$\bar{g}(\vec{a}, \vec{\omega}) + \tau(\Xi) = 0,$$

(88)

where $\vec{a}, \vec{\omega}$ are "physical vectors" of acceleration and angular velocity, i.e. they consist of physical components of these vectors, $\bar{g}$ is "physical metric" with components $\bar{g}_{\alpha\beta} \equiv g_{\alpha\beta}/(H_{\alpha}H_{\beta})$, $\Xi$ is 4-vector with components

$$\Xi = (3 \text{div } \vec{\omega}/2, -D_{t} \vec{\omega}).$$

(89)
After we’ve varied relation (88) over all values, we obtain the equation

\[ \delta \bar{g}(\vec{a}, \vec{\omega}) + \bar{g}(\delta \vec{a}, \vec{\omega}) + \bar{g}(\vec{a}, \delta \vec{\omega}) + \delta \tau(\Xi) + \tau(\delta \Xi) = 0. \tag{90} \]

From (90) it follows that there are fundamental restrictions to the detalization of probe bodies motion in GR by means of kinematics or geometry, and these restrictions are expressed through the said integrability condition. Here the kinematic part of the variations \( \delta \bar{g} \) can be seen as a relativistic analogy of the kinematic values (coordinates) in the uncertainties relation, whereas the metric part of variations \( \delta \bar{g} \) can be seen as a relativistic analogy of dynamic values (momenta), because metrics in GR satisfied to the Einstein’s dynamic equations. Variation \( \delta \tau \) of the reference frame does not have a direct quantum mechanical analogy. Note that the geometrical anholonomicity of the space manifold measured by \( \omega \) serves as an analogy to quantum mechanical non-commutativity of the variables. It is possible that anholonomicity of manifolds is even more closely connected to their quantization. Moreover, in this light the analogy between quantum mechanics and gravity can be seen as much more than purely superficial likeness. It gives us both more stronger motivation to use geometric approach to quantization, and the appearance of the new aspects of the gravity quantization problem including geometry, kinematics and reference frames. Investigation of these problems can be material for future studies.

### A Solution to the Killing equations for a general spherically symmetric space-time

For successful separation of variables in the system of equations (1)-(10) in Sec. 8 it is important to find right order of substitutions. With using (3) equations (8) and (9) takes the following kind:

\[ \tau^\theta = e^{2\lambda / r} \tau^\theta_{\theta, \theta}, \quad \tau r = e^{2\lambda / r} \tau^\theta_{\theta, \phi}. \tag{91} \]

Differentiating (10) with respect to \( r \) and substituting (91), we go to the third order equation for \( \tau^\theta \):

\[ \sin^2 \theta \left( \frac{\tau^\theta_{\theta, \varphi}}{\sin^2 \theta} \right) + \tau^\theta_{\theta, \theta, \varphi} = 0. \tag{92} \]

For the new variable \( X = \tau^\theta_{\theta, \varphi} \) one can obtain the following linear equation of the first order:

\[ X_{\theta} - \cot \theta X = 0, \]

which has the solution

\[ X = \tau^\theta_{\theta, \varphi} = \sin \theta \Phi_1(t, r, \varphi), \tag{93} \]

where \( \Phi_1 \) is yet unknown function of the three mentioned variables. Integrating this solution for \( \tau^\theta_{\theta, \varphi} \) with respect to \( \theta \) and with respect to \( \varphi \), we go to the following general expression for \( \tau^\theta \):

\[ \tau^\theta = - \cos \theta \Phi_1(t, r, \varphi) + \Phi_2(t, r, \varphi) + \Phi_3(t, r, \theta), \tag{94} \]

where \( \Phi_i \) are yet unknown function of the written variables\(^\text{10} \). Now taking into account (3), we obtain from (92):

\[ \tau^r = - r(\sin \theta \Phi_1(t, r, \varphi) + \Phi_{3, \theta}(t, r, \theta)). \tag{95} \]

\(^9\)In detailed calculations it is important to remember, that variation \( \delta \) in general does’t commute with spatial partial derivatives \( \partial_i \) in view of relations (85), included in variational procedures.

\(^{10}\)We use economic notations if it is does’t lead to ambiguity. So, the functions \( \Phi_1 \) in (93) and in (92) are different, but they depend on the same variables, are equally arbitrary and play the same role of multipliers in front of the function, depending only on \( \theta \). We may denote the functions in this situation by the same manner.
Differentiating (3) with respect to \( \theta \), expressing \( \tau'_{r \theta} \) and substituting the result into (8), we go to the equation

\[
\tau'_{\theta} = \frac{e^{2\lambda}}{r} \tau_{\theta, \theta, \theta},
\]

which after substitution (94) gives the equation

\[
- \cos \theta \Phi'_1 + \Phi'_2 + \Phi'_3 = \frac{e^{2\lambda}}{r} (\cos \theta \Phi_1 + \Phi_{3, \theta, \theta}).
\]  

(96)

Differentiating it with respect to \( \varphi \) and with respect to \( \theta \), we go to the equation:

\[
\Phi'_{1, \varphi} + \frac{e^{2\lambda}}{r} \Phi_{1, \varphi} = 0,
\]

whose integral has the form

\[
\Phi_1 = \psi_1(t, \varphi) Q + \psi_2(t, r),
\]  

(97)

where

\[
Q \equiv \exp \left\{ - \frac{e^{2\lambda}}{r} dr \right\}; \quad e^{2\lambda} = -r \frac{Q'}{Q}.
\]  

(98)

and \( \psi_2 \) is functional constant of integration. Since its role is to redefine \( \Phi_3 \), one can put \( \psi_2 \) equal to zero. Now in view of equality of the first terms from the left and from the right in (96), we go to the following equations for remainders:

\[
\Phi'_2 + \Phi'_3 = \frac{e^{2\lambda}}{r} \Phi_{3, \theta, \theta}.
\]  

(99)

Differentiating (99) with respect to \( \varphi \) and taking into account independence of \( \Phi_3 \) on \( \varphi \), we go to the equation \( \Phi'_{2, \varphi} = 0 \), wherefrom it follows the expression:

\[
\Phi_2(t, r, \varphi) = \phi_1(t, r) + \phi_2(t, \varphi).
\]  

(100)

Coming back to the equation (99), we obtain another equation

\[
\frac{e^{2\lambda}}{r} \Phi_{3, \theta, \theta} - \Phi'_3 = \phi'_1(t, r).
\]  

(101)

Differentiating now (9) with respect to \( \varphi \), and equation (4) with respect to \( r \) and equating mixed derivatives \( \tau'_{r \varphi} = \tau'_{\varphi, r} \), we go to the following equation:

\[
- \frac{e^{2\lambda}}{r^2 \sin^2 \theta} \tau'_{r \varphi} = (\tau'_{\theta \theta} - \tau^0 \cot \theta)_{,r},
\]

or rewriting with using (94)-(95), (97), (100):

\[
\frac{Q' \psi_{1, \varphi, \varphi}}{\sin \theta} = \frac{Q' \psi_1}{\sin \theta} + \Phi'_{3, \theta} - \cot \theta \Phi_{3} - \cot \theta \phi'_1.
\]  

(102)

Differentiating it with respect to \( \varphi \) and taking into account independency of \( \Phi_3 \) and \( \phi_1 \) on \( \varphi \), we go to the following equation for \( \psi_1 \):

\[
\psi_{1, \varphi, \varphi} + \psi'_{1, \varphi} = 0,
\]

whose general solution reads as follows:

\[
\psi_1 = \psi_0(t) + C_1(t) \sin \varphi + C_2(t) \cos \varphi,
\]

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where \(\psi_0, C_1, C_2\) are yet unknown functional constants of integration. Substituting this solution into (102), we go to the equation:

\[
\frac{\dot{Q}}{\sin \theta} \psi_0 + \Phi'_{3,\theta} - \cot \theta \Phi'_3 - \cot \theta \phi'_1 = 0.
\]

Multiplying both parts of this equation by \(\sin \theta\) and differentiating the result on \(\theta\), we go to the following equation:

\[
\Phi'_{3,\theta,\theta} + \Phi'_3 + \phi'_1 = 0. \tag{103}
\]

Combining (103) with (101), we obtain

\[
\left(\Phi'_3 + \frac{e^{2\lambda}}{r} \Phi_3\right)_{,\theta,\theta} = 0.
\]

General solution to this equation has the following kind:

\[
\Phi_3(t, r, \theta) = Q\chi_1(t, \theta) + \theta \chi_2(t, r) + \chi_3(t, r). \tag{104}
\]

Substituting it back into (103), we go to the expression

\[
Q'\chi_1 + \Phi'_3 + \chi'_3 + \phi'_1 = 0. \tag{105}
\]

Differentiating it twice with respect to \(\theta\) we go to the equation:

\[
\chi_{1,\theta,\theta,\theta} + \chi_{1,\theta,\theta} = 0.
\]

Its general solution has the following kind:

\[
\chi_1 = D_1(t) \sin \theta + D_2(t) \cos \theta + D_3(t) \theta + D_4(t), \tag{106}
\]

where \(D_i(t)\) are yet unknown functional constants of integration. Substituting this solution back into (105) and separating variables, we go to the following system:

\[
Q' \Phi_3 + \Phi'_3 + \phi'_1 = 0; \quad D_4 Q' + \chi'_3 + \phi'_1 = 0.
\]

Integrating it with respect to \(r\) and substituting all into (104), we obtain the following result:

\[
\Phi_3 = Q(D_1(t) \sin \theta + D_2(t) \cos \theta) + a(t) \theta + b(t), \tag{107}
\]

where \(a, b\) are yet unknown functional constants of integration. We have excluded from (107) the term \(-\phi_1\), since it is cancelled with the term \(+\phi_1\) in (104), which is contained in \(\Phi_2\) by (100). So, \(\Phi_2\) becomes the function of only two variables \(t, \varphi\). The equations (101) and (103) are satisfied by (107) identically.

Now let us go to the integrability conditions for \(\tau_\varphi\). Differentiating (4) with respect to \(\varphi\), (9) with respect to \(\varphi\) and equating mixed second derivatives \(\tau_{,\varphi,\varphi} = \tau_{,\varphi,\varphi}\), we go (after some algebra) to the following equation:

\[
\psi_0(t) = D_2(t).
\]

It means that this values can be put to zero, since respective terms in (104), (105) are cancelled. Differentiating (4) with respect to \(\varphi\), (10) with respect to \(\varphi\) and equating mixed second derivatives \(\tau_{,\varphi,\varphi} = \tau_{,\varphi,\varphi}\), we go to the expressions:

\[
a(t) = 0; \quad \Phi_2 = G_1(t) \sin \varphi + G_2(t) \cos \varphi - b(t).
\]

So, in expression for \(\Phi_2\) the value \(-b\) can be omitted, since it is cancelled with \(b\) in (107) for \(\Phi_3\) in all expressions. Now integrating (4), (9), (10) respectively over \(\varphi\), \(r\) and \(\theta\) and
equating results for $\tau^\nu$, one can find general kind of this component. We present general
solution for spatial part of Killing equations in the convenient for future purposes form:

$$\tau^\nu = -rQ[\sin \theta (C_1(t) \sin \varphi + C_2(t) \cos \varphi) + \cos \theta D(t)];$$

(108)

$$\tau^\theta = Q[\sin \theta D(t) - \cos \theta (C_1(t) \sin \varphi + C_2(t) \cos \varphi)] + G_1(t) \sin \varphi + G_2(t) \cos \varphi;$$

(109)

$$\tau^r = -\frac{Q}{\sin \theta}[C_1(t) \cos \varphi - C_2(t) \sin \varphi] + \cot \theta (G_1(t) \cos \varphi - G_2(t) \sin \varphi) + f(t),$$

(110)

where all written functions are yet unknown.

Now let consider the equations (5)-(7) and write their integrability conditions with re-
spect to the function $\tau^t$, combining and equating different mixed second derivatives from
this component with using (108)-(110). The integrability condition for $\tau^t$ with respect to $\theta$
and $\varphi$ leads to the expressions:

$$\dot{G}_i = 0; \ (i = 1, 2) \quad \dot{f} = 0.$$

(111)

The integrability condition for $\tau^t$ with respect to $\theta$ and $r$ adds the expression

$$e^{2\lambda t - 2\nu r} \dot{\tilde{Q}}_i = (r^2 e^{-2\nu r} \tilde{Q}_i)_r,$$

(112)

where $\tilde{Q}_i = C_i(t)Q, \ i = 1, 2, \tilde{Q}_3 = D(t)Q$. The integrability condition for $\tau^t$ with respect
to $r$ and $\varphi$ doesn't add any new expressions. Now integrating the equations (5)-(7) with
respect to spatial variables and comparing the obtained three expressions for $\tau^t$ we go to
the following general expression for $\tau^t$:

$$\tau^t = -r^{-2} e^{-2\nu t} [(\tilde{Q}_1 \sin \varphi + \tilde{Q}_2 \cos \varphi) \sin \theta + \tilde{Q}_3 \cos \theta] + \xi(t),$$

(113)

where $\xi(t)$ is yet unknown function, together with one additional to the (112) condition

$$\tilde{Q}G_i = 0, \ i = 1, 2.$$

(114)

 Easily to check, that condition (112) has the following integral:

$$r^2 e^{-2\nu t} \tilde{Q}_i \tilde{Q}_i = F_i(t), \ (i = 1, 2, 3)$$

(115)

where $F_i$ is arbitrary function of the time.

Substituting now expressions (108)-(110) into equations (1)-(2) of the system of
Killing equations, we go to the following additional restrictions on the metric, which must
be satisfied for its integrability:

$$(r\tilde{Q}_i)' + \lambda r^2 e^{-2\nu} \tilde{Q}_i + \lambda' r \tilde{Q}_i = 0;$$

(116)

$$(e^{-2\nu} \tilde{Q}_i)' + \nu e^{-2\nu} \tilde{Q}_i + \frac{\nu'}{r} \tilde{Q}_i = 0$$

(117)

$i = 1, 2, 3$ and the two more simple conditions:

$$\xi(t) = \xi_0 e^{-\nu t}; \quad \lambda \xi_0 = 0; \quad \xi_0 = \text{const.}$$

(118)

So, in general we have four conditions (112), (116)-(118) on the two metric functions $\nu$
and $\lambda$, wherefrom it follows that integrability of the Killing equations (1)-(10) takes place
for only special cases. Using the integral (115), the equation (117) can be reduced to purely
algebraic kind with respect to $\tilde{Q}_i$, which solutions reads as follows:

$$\tilde{Q}_i = \sqrt{-\nu \dot{F}_i - \dot{F}_i \pm \sqrt{\nu \dot{F}_i + \dot{F}_i}^2 + 4\nu' e^{2\nu} F_i^2 / \nu'^2}$$

(119)
under $\nu' \neq 0$ and

$$\tilde{Q}_i = \pm \frac{F_i e^{\nu'}}{r\sqrt{F_i} + \nu F_i}$$

under $\nu' = 0$.

Further analysis of compatibility of the system (1)-(10) is go beyond the aims of our paper. We present some well known particular isometry fields, satisfying all integrability conditions:

1. the fields of rotation algebra of the group SO(3) ($C_i = 0$, $i = 1, 2, 3$, $\xi_0 = 0$, $G_1 \neq 0$, $f \neq 0$):

$$\tau(1) = \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi; \quad \tau(2) = -\sin \varphi \partial_\theta - \cot \theta \cos \varphi \partial_\varphi; \quad \tau(3) = \partial_\varphi;$$

2. for the metrics with $\dot{\nu} = 0$, $\dot{\lambda} = 0$ ($C_i = 0$, $i = 1, 2, 3$, $\xi_0 \neq 0$, $G_1 = 0$, $f = 0$) (static case):

$$\tau(4) = \partial_t.$$ 

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