Triaxial stellar systems following the $r^{1/n}$ luminosity law: an analytical mass–density expression, gravitational torques and the bulge/disc interplay

I. Trujillo,1* A. Asensio Ramos,1 J. A. Rubiño-Martín,1 Alister W. Graham,1,2 J. A. L. Aguerri,3 J. Cepa1 and C. M. Gutiérrez1

1Instituto de Astrofísica de Canarias, E-38205 La Laguna, Tenerife, Spain
2Department of Astronomy, University of Florida, Gainesville, Florida, USA
3Astronomisches Institut der Universität Basel, Venusstrasse 7, CH-4102 Binningen, Switzerland

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ABSTRACT
We have investigated the structural and dynamical properties of triaxial stellar systems whose surface brightness profiles follow the $r^{1/n}$ luminosity law – extending the analysis by Ciotti, who explored the properties of spherical $r^{1/n}$ systems. A new analytical expression that accurately reproduces the spatial (i.e., deprojected) luminosity density profiles (error less than 0.1 per cent) is presented for detailed modelling of the Sérsic family of luminosity profiles. We evaluate both the symmetric and the non-axisymmetric components of the gravitational potential and force, and compute the torques as a function of position. For a given triaxiality, stellar systems with smaller values of $n$ have a greater non-axisymmetric gravitational field component. We also explore the strength of the non-axisymmetric forces produced by bulges with differing $n$ and triaxiality on systems having a range of bulge-to-disc ratios. The increasing disc-to-bulge ratio with increasing galaxy type (decreasing $n$) is found to greatly reduce the amplitude of the non-axisymmetric terms, and therefore reduce the possibility that triaxial bulges in late-type systems may be the mechanism or perturbation for non-symmetric structures in the disc.

Using seeing-convolved $r^{1/n}$-bulge plus exponential-disc fits to the K-band data from a sample of 80 nearby disc galaxies, we probe the relations between galaxy type, Sérsic index $n$ and the bulge-to-disc luminosity ratio. These relations are shown to be primarily a consequence of the relation between $n$ and the total bulge luminosity. In the K band, the trend of decreasing bulge-to-disc luminosity ratio along the spiral Hubble sequence is predominantly, though not entirely, a consequence of the change in the total bulge luminosity; the trend between the total disc luminosity and Hubble type is much weaker.

Key words: stellar dynamics – galaxies: elliptical and lenticular, cD – galaxies: kinematics and dynamics – galaxies: photometry – galaxies: spiral – galaxies: structure.

1 INTRODUCTION
As the quality of photometric data has improved over the years (largely due to the use of CCDs), the applicability of a fitting-function which can account for variations in the curvature of a light profile has been demonstrated for elliptical galaxies (Capaccioli 1987, 1989; Davies et al. 1988; Caon, Capaccioli & D’Onofrio 1993; Young & Currie 1994; Graham et al. 1996), and for the bulges of spiral galaxies (Andredakis, Peletier & Balcells 1995; Moriondo, Giovannardi & Hunt 1998; Seigar & James 1998; Khosroshahi, Wadadekar & Kembhavi 2000; Graham 2001; Möllenhoff & Heidt 2001; Prieto et al. 2001). These systems are not universally described with either an exponential profile or an $r^{1/n}$ law (de Vaucouleurs 1948, 1959), but rather a continuous range of light profile shapes exist which are well described by the Sérsic (1968) $r^{1/n}$ model.

In ellipticals, the shape parameter $n$ from the Sérsic model is strongly correlated with the other global properties derived independently of the $r^{1/n}$ model, such as total luminosity and effective radius (Caon et al. 1993; Young & Currie 1994, 1995;
Triaxial stellar systems following the $r^{1/n}$ luminosity law

Jerjen & Bingelli 1997; Trujillo, Graham & Caon 2001b), central velocity dispersion (Graham, Trujillo & Caon 2001a), and also central supermassive black hole mass (Graham et al. 2001b). Additionally, the spiral Hubble type has been shown to correlate with the bulge index $n$ such that early-type spiral galaxy bulges have larger values of $n$ than late-type spiral galaxy bulges (Andredakis et al. 1995; Graham 2001). This correlation arises from the fact that the index $n$ is well correlated with the bulge-to-disc luminosity ratio (B/D; see, e.g., Simien & de Vaucouleurs 1986), and this is one of the parameters used to establish morphological type (Sandage 1961).

Given the abundance of observational work and papers now using the Sérsic model, it seems timely that a theoretical study is performed on realistic, analytical models following the $r^{1/n}$ law. Structural and dynamical properties of isotropic, spherical galaxies following $r^{1/n}$ models have already been studied in detail in the insightful paper of Ciotti (1991). However, most elliptical galaxies and bulges of spiral galaxies are known to be non-spherical objects. Typically, the mass models which have been used for the study of triaxial galaxies have followed analytical expressions which were selected to reproduce the properties of the de Vaucouleurs $r^{1/4}$ profile (e.g. Jaffe 1983; Hernquist 1990; Dehnen 1993), or more recently the modified Hubble law (Chakraborty & Thakur 2000). For that reason, previous studies based on these kinds of analytical models, though certainly useful, are unable to probe the full range of properties which are now observed in real galaxies. Consequently, it is of importance to know how much room for improvement exists in the study of triaxial objects following the $r^{1/n}$ family of profiles.

Due to the fact that the observed $r^{1/n}$ luminosity profiles cannot be deprojected to yield analytical expression for the spatial density, the $r^{1/n}$ law has been considered less useful for studies of detailed modelling. An analytical representation (approximation) for the mass density profiles which accurately reproduces the observed $r^{1/n}$ luminosity profiles would be of great interest for simulations of real galaxies. We have therefore derived just such an analytical expression for the mass density profiles of the Sérsic family of models. Our approximation surpasses the accuracy of both the Dehnen models for representing the specific $r^{1/4}$ profile and also their extension to the double-power-law models of Zhao (1997).

In this paper we present a detailed study of how the physical properties of triaxial stellar systems change as a function of the index $n$. An accurate analytical expression for modelling the spatial density is presented in Section 2. In Section 3 we explore the axisymmetric and the non-axisymmetric components of the potential, forces and torques associated with a Sérsic light distribution. Finally, by using literature available K-band observations of a sample of 80 spiral galaxies, the physical basis of the $n-T$ (or $n-B/D$) relation is investigated in Section 4.

2 THE $r^{1/n}$ MODEL

The projected, elliptically symmetric Sérsic $r^{1/n}$ intensity distribution $I(r)$ can be written in terms of the projected, elliptical radial coordinate $\xi$ (see details in Trujillo et al. 2001a) such that

$$ I(\xi) = I(0) e^{-\beta \xi (f(\xi)/I^{1/n})}, $$

where $I(0)$ is the central intensity, and $r_e$ is the effective radius of the projected major axis. Curves of constant $\xi$ on the plane of the sky are the isophotes. The quantity $b_n$ is a function of the shape parameter $n$, and is chosen so that the effective radius encloses half of the total luminosity. The exact value is derived from $\Gamma(2n) = 2\gamma(n, b_n)$, where $\Gamma(a)$ and $\gamma(a, x)$ are the gamma function and the incomplete gamma function respectively (Abramowitz & Stegun 1964, p. 260). The index $n$ increases monotonically with the central luminosity concentration of the surface brightness distribution (Trujillo et al. 2001b).

The total projected luminosity $L$ associated with this model is given by

$$ L = I(0) r_e^2 (1 - e) 2\pi n \Gamma(2n), $$

where $e$ is the ellipticity of the isophotes. For a homologous triaxial ellipsoid, the spatial (deprojected) luminosity density profile $\nu(\xi)$ can be obtained by an Abel integral equation (Stark 1977):

$$ \nu(\xi) = \frac{f^{1/2}}{\pi} \int_0^{\xi} \frac{d}{d\xi} I(\xi) \left(\xi^2 - \xi'^2\right)^{-1/2} d\xi', $$

where $f^{1/2}$ is a constant that depends on the three-dimensional spatial orientation of the object (Varela, Muñoz-Tuñón & Simonneau 1996; Simonneau, Varela & Muñoz-Tuñón 1998), and $\xi$ parametrizes the ellipsoids of constant volume brightness. $f^{1/2}$ equals 1 when the proper axis frame of the object has the same orientation as the observer axis frame (i.e., when the Euler angles between the two frames equal zero).

2.1 Mass density profiles

Assume a triaxial object whose mass is stratified over ellipsoids with axis ratios $a:b:c$ ($a \geq b > c$) and the $x$- ($z$-) is the long (short) axis (see Fig. 1). The symmetry of the problem motivates us to work with ellipsoidal coordinates where

$$ x = \xi \sin \psi \cos \theta $$
$$ y = \alpha \xi \sin \psi \sin \theta $$
$$ z = \beta \xi \cos \psi, $$

Figure 1. A surface of constant density for the triaxial ellipsoid described in equations (5) and (6).
and where \( \alpha = \beta a \) and \( \beta = c/a \). The mass models considered in this study are the triaxial generalizations of the spherical models discussed in detail by Ciotti (1991). The mathematical singularities present in equation (3) were considered and solved by Simonneau & Prada (1999, equation 16). Substituting equation (1) into equation (3), letting \( \xi = \zeta \cosh \varphi \), and multiplying by the mass-to-light ratio \( Y = M/L \), we obtain a similar expression to the one found by these authors:

\[
\rho(\xi) = \frac{f^{1/2}(0)}{\pi n r_0^{1/n}} \int_0^\infty e^{-b_0(\zeta \cosh \varphi)^{1/n-1} d\varphi},
\]

with

\[
x^2 + \left( \frac{\zeta}{\alpha} \right)^2 + \left( \frac{\zeta}{\beta} \right)^2 = \xi^2.
\]

The dimensionless mass density profiles \( \rho(\xi) = r_0^2/M \rho(\xi) \), where \( M \) is the total mass, are shown for different values of \( n \) in Fig. 2(a). It should be noted that the inner density profile decreases more slowly with increasing radius for systems having lower values of \( n \).

The mass density profiles of the \( r^{1/n} \) family (equation 5) can be extremely well approximated by the analytical expression

\[
\rho_{\text{app}}(\xi) = f^{1/2}(0) b_0 (\zeta h_2)^n h^{p(1/n-1) K_N(b_0 h_2^{1/n})},
\]

where \( h = g r_0 \), \( C(h) = h_1(\log h)^2 + h_2 \log h + h_3 \), and \( K_N \) is the \( N \)-th order modified Bessel function of the third kind (Abramowitz & Stegun 1964, p. 374). In Appendix A we show the values of the parameters \( (r, \pi, h_1, h_2, h_3) \) as functions of the index \( n \). This approximation contains two exact cases, \( n = 0.5 \) and \( n = 1 \), and provides relative errors less than 0.1 per cent for the rest of the cases (Fig. 2b) in the radial range \( 10^{-3} \leq g r_0 \leq 10^3 \). This approximation surpasses (by a factor of \( 10^2 - 10^3 \)) the expression presented in Lima Neto, Gerbal & Márquez (1999).

3 NON-AXISYMMETRIC PERTURBATIONS DUE TO A TRIAXIAL \( r^{1/n} \) STRUCTURE

For three different triaxiality mass distributions: (a) spherical \( (\alpha = \beta = 1) \); (b) moderately triaxial \( (\alpha = 0.75, \beta = 0.5) \), and (c) highly triaxial \( (\alpha = 0.5, \beta = 0.25) \), we have explored, in detail, the non-axisymmetric gravitational field over the \( z = 0 \) plane (i.e., the disc plane when studying spiral galaxies).

Figure 2. (a) The dimensionless mass density profiles (see Section 2.1) for the values of \( n = 0.5, 1, 2, 4 \) and 10. (b) The relative errors between the analytical approximation proposed in equation (7) and the exact solution are shown for the previous values of \( n \).

Figure 3. The parameters \( G(r) \), \( N(r) \) and \( \Pi(\theta) \) are shown for different values of \( n \) and triaxiality: (a) First Row: \( G(r) \), \( N(r) \) and \( \Pi(\theta) \) for a moderately triaxial object and different \( n \); (b) Second Row: \( G(r) \), \( N(r) \) and \( \Pi(\theta) \) for a highly triaxial structure; (c) Third Row: \( G(r) \), \( N(r) \) and \( \Pi(\theta) \) for three moderately triaxial objects with \( n = 1 \) and the same axis ratio along the \( y \)- and \( z \)-axes.

3.1 Non-spherical component of the gravitational potential in the plane \( z = 0 \)

We evaluate this quantity by calculating

\[
G(r) = \frac{\Phi_2(r)}{\Phi_0(r)},
\]

where \( \Phi_2(r) \) and \( \Phi_0(r) \) are the \( m = 2 \) and \( m = 0 \) components of the gravitational potential, such that the \( m \)-th order term \( \Phi_m(r) \) is evaluated from the gravitational potential on the \( z = 0 \) plane \( \Phi(r, \theta) \) by using the Fourier decomposition (see, e.g., Combes & Sanders 1981). Gravitational potential and gravitational force expressions are shown in Appendix B.

The profiles of \( G(r) \) for different triaxialities and values of \( n \) are shown in Fig. 3. As expected, as the triaxiality increases the non-spherical nature of the gravitational field increases. Also, we highlight the fact that for a given triaxiality, smaller values of \( n \) (i.e., less concentrated mass distribution) give greater non-spherical gravitational fields. The maximum non-axisymmetric behaviour of the potential is obtained at radial distances less than \( 2r_c \). This radial distance is also a function of the index \( n \), decreasing as \( n \) increases, and remains quite independent of the triaxiality of the object. For a moderately triaxial object with \( n = 1 \), the non-axisymmetric component of the potential can vary some 6 per cent between \( r = 0 \) and \( r = 2r_c \), and varies some 15 per cent for our highly triaxial model.

For an \( n = 1 \) model, and starting from our moderately triaxial case \( (\alpha = 0.75, \beta = 0.50) \), we increased the value of \( \beta \) to 0.75. The results are shown in Fig. 3(c) and reveal that \( G(r) \) varied only...
3.2 Non-spherical component of radial gravitational forces in the \( z = 0 \) plane

The non-spherical component of the radial gravitational forces in the \( z = 0 \) can be estimated by

\[
N(r) = \frac{\partial F_2(r)/\partial r}{\partial \Phi_0(r)/\partial r}. \tag{9}
\]

In Fig. 3 the \( N(r) \) profiles (equation 9) are evaluated for the same cases as was the \( G(r) \) profiles. A remarkable point is that \( N(r) \) reaches its maximum value in the radial range \( 2r_c < r < 4r_c \). For a spiral bulge structure, this means that the most important non-axisymmetric effects take place in a zone which is dominated by the disc. As with the \( G(r) \) parameter, the strongest distortions occur as the triaxiality increases and the index \( n \) decreases. The mechanism which controls this distortion is basically determined by the mass distribution in the \( z = 0 \) plane (Fig. 3c).

It is noted that the relative (i.e., percentage change) non-axisymmetric effects on the radial forces are larger than the relative distortion on the potential. As an example, for a moderately triaxial structure with \( n = 1 \) the non-axisymmetric component of the radial forces can reach 8 per cent.

3.3 Torques on \( z = 0 \) plane

The torques provoked by the triaxial structures along the angular coordinate are evaluated around the circle of radius \( r_{\text{max}} \) where the maximum non-axisymmetric distortion of the radial forces is produced [i.e., at the peak of the \( N(r) \) profile]. Given the gravitational potential \( \Phi(r, \theta) \) in the \( z = 0 \) plane, we have at the radius \( r_{\text{max}} \)

\[
\Pi(\theta) = \frac{F_T(r_{\text{max}})}{F_G(r_{\text{max}})}, \tag{10}
\]

where \( F_T(r_{\text{max}}) = [\partial \Phi(r_{\text{max}}, \theta)/\partial \theta]r_{\text{max}} \) represents the amplitude of the tangential force along the angular coordinate at radius \( r_{\text{max}} \), and \( F_G(r_{\text{max}}) = (\partial \Phi(r_{\text{max}}, \theta)/\partial r) \) is the radial force at this radius. Due to the symmetry of the ellipsoid, the values of \( \Pi(\theta) \) need to be plotted for only one quadrant in the \( z = 0 \) plane; we use \( 0^\circ < \theta < 90^\circ \) (Fig. 3). Depending on the quadrant, \( \Pi(\theta) \) is either negative or positive, because the sign of the tangential force changes from quadrant to quadrant. The maximum torque around a circle of radius \( r_{\text{max}} \) depends on the triaxiality of the object. As the triaxiality increases the maximum torque tends to be closer to the major axis – as one would expect. The position of this peak is quite independent of the value of \( n \).

The absolute value of the torque for any given triaxiality increases as \( n \) decreases. For our highly triaxial bulge, \( \Pi(\theta) \) ranges from 0.17 (\( n = 10 \)) to 0.24 (\( n = 1 \)), which would be considered a ‘bar strength’ class of 2 in the classification scheme of Buta & Block (2001). In the case of our moderately triaxial object, the maximum absolute value of \( \Pi(\theta) \) ranges between 0.06 and 0.09. These values correspond to a ‘bar strength’ class of 1. A detailed study separating the torque contribution from both bars and bulges would of course be of interest, and it is our intention to add a range of bar potentials to our models in the future.

As with the previous parameters, for the range of triaxialities investigated and a given \( n \), varying the mass distribution along the \( z \)-axis (i.e., varying the triaxiality parameter \( \beta \)) results in only a slight change to \( \Pi(\theta) \) (see Fig. 3c). For a spherical distribution all above parameters are 0.

4 LINKING THEORY AND OBSERVATIONS: THE CONNECTION BETWEEN \( n \) AND THE \( B/D \) LUMINOSITY RATIO

In the previous section we have seen how the non-axisymmetric effects (in the \( z = 0 \) plane) from a triaxial bulge increase as \( n \) decreases. Taken with the correlation between \( n \) and galaxy type (Andredakis et al. 1995) shown in Fig. 4, this invokes the natural question: How, if at all, are the structural properties of the bulges (i.e., \( n \)) related (that is, cause and effect) with the non-axisymmetric components (i.e., arms) observed in the disc? The results obtained in the previous sections were evaluated without any mention of the relative mass of the bulge and disc. It turns out that the axisymmetric mass distribution of the disc causes a strong softening of the non-axisymmetric perturbation caused by the non-sphericity of the bulge. The degree of ‘smoothing’ is an increasing function of the \( B/D \) ratio.

Fig. 5 shows the \( N(r) \) profile for a moderately triaxial bulge with \( n = 1 \) and \( B/D = 0.1 \) and 0.01, and for a bulge with \( n = 4 \) and

![Figure 4](attachment:figure4.png)

**Figure 4.** The best-fitting bulge index \( n \) is plotted against the \( B/D \) luminosity ratio (left) and versus the galaxy morphological type (right). The galaxies come from the samples of de Jong (1996) (filled circles) and Andredakis et al. (1995) (open diamonds). See text for details.

![Figure 5](attachment:figure5.png)

**Figure 5.** The \( N(r) \) profiles for a moderately triaxial bulge with \( n = 1 \) and 4, and \( B/D = 1, 0.1 \) and 0.01 are shown.
4.1 Why does the n-Type (or n-B/D) relation exist?

To explore the connection between bulges and discs in spiral galaxies (see, e.g., Fuchs 2000), we have used the data from two independent samples of galaxies observed in the K band. The K band provides a good tracer of the mass due to the near absence of dust extinction and the reduced biasing effect of a few per cent (in mass) of young stars. We used the data from Andredakis et al. (1995) and the structural analysis of the de Jong (1996) data performed by Graham (2001). Both studies were done by fitting a seeing-convolved Sérsic law to the spiral galaxy bulges. In both samples we have removed those objects which contained a clear bar structure, leaving a total of 28 objects from Andredakis et al. (1995) and 52 objects from Graham (2001). The relations present in Fig. 4 between n and the B/D luminosity ratio, and n versus the morphological type T have Spearman rank-order correlation coefficients of \( r_s = 0.77 \) and \( -0.73 \), respectively, for the combined sample.

Andredakis et al. (1995) suggest ‘although other possibilities cannot be excluded, the most straightforward explanation for this trend is that the presence of the disc affects the density distribution of the bulge in such a way as to make the bulge profile steeper in the outer parts. One mechanism to produce such an effect might be that a stronger disc truncates the bulge, forcing its profile to become exponential’. Following this line of thought, via collisionless N-body simulations, Andredakis (1998) studied the adiabatic growth of the disc on to an existing \( r^{1/4} \) spheroid. He found that the disc potential modifies the bulge surface brightness profile, lowering the index n. This decrease was larger with more massive and more compact discs. This mechanism, however, saturated at around \( n = 2 \), and exponential bulges could not be produced.

We believe that this line of reasoning is not the most appropriate explanation for the relation between n and B/D. First, we find that the index n is not only well correlated with the luminous B/D ratio, but is equally well correlated (\( r_s = -0.75 \)) with bulge luminosity \( M_K(\text{bulge}) \) (Fig. 6a). Additionally, the correlation between n and disc luminosity \( M_K(\text{disc}) \) is relatively poor (\( r_s = -0.53 \)). Secondly, \( M_K(\text{bulge}) \) is more strongly correlated (\( r_s = -0.86 \)) with the B/D ratio than \( M_K(\text{disc}) \) and the B/D ratio (\( r_s = -0.50 \)).

Although there is a range of bulge-to-disc size ratios, a median value for \( h/r_c \) in the K-band is 5 (Graham & Prieto 1999).

4.2 \( M_K(\text{bulge}) \) versus B/D for classifying morphological types

Due to the strong correlation between the B/D luminosity ratio and \( M_K(\text{bulge}) \), it might be of interest to ask which one of these quantities is preferred to establish the morphological type T of a galaxy.\(^3\) Working from B-band images (which are good for observing the young star population, and consequently the spiral arm structure, which is one of the basic criteria to the Hubble galaxy classification), Simien & de Vaucouleurs (1986) fitted \( r^{1/4} \) profiles and exponential discs to a sample of 64 spiral galaxies and 34 S0-type galaxies. They presented a good correlation between the bulge-to-disc luminosity and T, but not between \( M_{B_\text{r}}(\text{bulge}) \) and T. Consequently, their B-band observations suggested that the B/D

\(^3\) We refer here to the morphological type established on the basis of B-band observations. Infrared images have shown that the appearance of galaxies can be substantially different (Block et al. 1999).
We were able to repeat this analysis using $B$-band data from de Jong (1996) (excluding barred galaxies), which gave $r_s = -0.64$ between the $B/D$ luminosity ratio and $T$, but only $r_s = 0.54$ between $M_B(bulge)$ and $T$. Also, $K$-band data do not support the idea that the $B/D$ luminosity ratio can be preferred over $M_K(bulge)$ as an indicator to establish galaxy morphological type ($T$). Both parameters present equally good correlations with galaxy type $T$.

### References

Abramowitz M., Stegun I., 1964, Handbook of Mathematical Functions. Dover, New York

Aguerri J. A. L., Balcells M., Peletier R., 2001, A&A, 367, 428

Andreadakis Y. C., 1998, MNRAS, 295, 72

Andreadakis Y. C., Peletier R., Balcells M., 1995, MNRAS, 275, 874

Binney J., Tremaine S., 1987, Galactic Dynamics. Princeton Univ. Press, Princeton

Block D. L., Puerari I., Frogel J. A., Eskridge P. B., Stockton A., Burkhard F., 1999, ApJS, 269, 5

Bureau R., Block D. L., 2001, ApJ, 550, 243

Caon N., Capaccioli M., D’Onofrio M., 1993, MNRAS, 265, 1013

Capaccioli M., 1987, in de Zeeuw T., ed., Proc IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies. Reidel, Dordrecht, p. 47

Capaccioli M., 1989, in Corwin H. G., Bottinelli L., eds, The World of Galaxies. Springer-Verlag, Berlin, p. 208

Chakraborty D. K., Thakur P., 2000, MNRAS, 318, 1273

Chandrasekhar S., 1969, Ellipsoidal Figures of Equilibrium. Dover, New York

Ciotti L., 1991, A&A, 249, 99

Combes F., Sanders R. H., 1981, A&A, 96, 164

Davies J., Philips S., Caswion M., Disney M., Kibblewhite E., 1988, MNRAS, 232, 239

Dekenn W., 1993, MNRAS, 265, 250

dejong R. S., 1996, A&P, 118, 557

de Vaucouleurs G., 1948, Ann. Astrophys., 11, 247

de Vaucouleurs G., 1959, Hand. Phys., 53, 311

Fuchs B., 2000, in Combes F., Combes M., eds, ASP Conf. Ser. Vol. 197, Galaxy Dynamics: from the Early Universe to the Present. Astron. Soc. Pac., San Francisco, p. 53

Graham A. W., 2001, AJ, 121, 820

Graham A. W., Prieto M., 1999, ApJ, 524, L23

Graham A. W., Lauer T., Colles M. M., Postman M., 1996, ApJ, 465, 534

Graham A. W., Trujillo I., Caon N., 2000, AJ, 122, 1707

Graham A. W., Erwin P., Caon N., Trujillo I., 2001b, ApJ, 563, L11

Hernquist L., 1990, ApJ, 356, 359

Jaffe W., 1983, MNRAS, 202, 995

Jerjen E., Bingelli B., 1997, in Arnaboldi M., da Costa G. S., Saha P., eds, ASP Conf. Ser. Vol. 116, The Nature of Elliptical Galaxies. Astron. Soc. Pac., San Francisco, p. 239

Khosroshahi H. G., Wadadekar Y., Kembhavi A., 2000, ApJ, 533, 162

Lima Neto G. B., Gerbal D., Marquez I., 1999, MNRAS, 309, 481

Mazure A., Capelato H. V., 2002a, A&A, 383, 384

Möllenhoff C., Heidt J., 2001, A&A, 368, 16

Moriondo G., Giovanardi C., Hunt L. K., 1998, A&AS, 130, 81

Prieto M., Aguerri J. A. L., Varela A. M., Muñoz-Tunon C., 2001, A&A, 367, 405

Sandage A., 1961, The Hubble Atlas of Galaxies. Carnegie Institution, Washington

Seigar M., James P., 1998, MNRAS, 299, 672

Sérsic J., 1968, Atlas de Galaxias Australes. Córdoba. Obs. Astronómico Simien F., de Vaucouleurs G., 1986, ApJ, 302, 564

Simonneau E., Prada F., 1999, preprint (astro-ph/9906151)

Simonneau E., Varela A., Muñoz-Tuñon C., 1998, II Nuovo Cimento, 113B, 927

Stark A., 1977, ApJ, 213, 368

Trujillo I., Aguerri J. A. L., Cepa J., Gutiérrez C. M., 2001a, MNRAS, 321, 269
The gravitational potential at position $\Phi(x, y, z)$ may be written as

$$\Phi(x) = \pi Gabc \int_0^m \frac{[\psi(a) - \psi(\zeta)] d\tau}{\sqrt{(\tau + a^2)(\tau + b^2)(\tau + c^2)}}$$

(B1)

(Chandrasekhar 1969, p. 52, theorem 12), with

$$\psi(\zeta) = \frac{2}{a^2} \int_0^z \xi^2 d\xi ,$$

(B2)

It follows from equation (B1) that the potential at an internal point is a result of two contributions: that due to the ellipsoidal interior to the point $x$ considered, and that due to the homogeneous shell exterior to $x$:

$$\Phi(x) = 4\pi Ga\beta \frac{1}{\sqrt{1 - \beta^2}}$$

$$\times \left[ F \left( \text{arcsin}\sqrt{1 - \beta^2} \right) - \int_0^\zeta \zeta^2 d\zeta \right] + \int_0^\zeta \xi^2 d\xi$$

$$+ \int_0^\zeta \xi^2 \left[ F \left( \text{arcsin}\sqrt{1 - \beta^2} \right) - \int_0^\zeta \zeta^2 d\zeta \right]$$

$$\times \frac{1}{\sqrt{(\zeta^2 + \lambda)\left(1 - \beta^2\right)^2}} F \left( \text{arcsin}\sqrt{1 - \beta^2} \right).$$

(B3)

with $F(p, q)$ the elliptic integral of the first kind (Abramowitz & Stegun 1964, p. 589), and with the restriction

$$\frac{x^2}{\zeta^2 + \lambda} + \frac{y^2}{(\zeta^2 \alpha^2 + \lambda)} + \frac{z^2}{(\zeta^2 \beta^2 + \lambda)} = 1 .$$

(B4)

The strength of the central potential decreases as the triaxiality of the object increases:

$$\frac{\Phi(0)}{\Phi_{sph}(0)} = \frac{\int^1 \alpha \beta}{\sqrt{1 - \beta^2}} F \left( \text{arcsin}\sqrt{1 - \beta^2} \right).$$

(B5)

As reported in Ciotti (1991), the models with low $n$ have an inner ($r < r_s$) potential which is much flatter than models with high $n$. As the triaxiality increases there is no important change to the gradient of the gravitational potential along the semimajor axis; the main effect is to shift the gravitational potential profile inwards from the spherical case, resulting in a lower potential at intermediate radii.

### APPENDIX B: GRAVITATIONAL POTENTIAL AND FORCES OF A TRIAXIAL HOMOLOGOUS STRUCTURE

#### B1 Gravitational potential

The gravitational potential at position $x = (x, y, z)$ may be written as

$$\Phi(x) = \pi Gabc \int_0^m \frac{[\psi(a) - \psi(\zeta)] d\tau}{\sqrt{(\tau + a^2)(\tau + b^2)(\tau + c^2)}}$$

(B1)

(Chandrasekhar 1969, p. 52, theorem 12), with

$$\psi(\zeta) = \frac{2}{a^2} \int_0^z \xi^2 d\xi ,$$

(B2)

It follows from equation (B1) that the potential at an internal point is a result of two contributions: that due to the ellipsoidal interior to the point $x$ considered, and that due to the homogeneous shell exterior to $x$:

$$\Phi(x) = 4\pi Ga\beta \frac{1}{\sqrt{1 - \beta^2}}$$

$$\times \left[ F \left( \text{arcsin}\sqrt{1 - \beta^2} \right) - \int_0^\zeta \zeta^2 d\zeta \right] + \int_0^\zeta \xi^2 d\xi$$

$$+ \int_0^\zeta \xi^2 \left[ F \left( \text{arcsin}\sqrt{1 - \beta^2} \right) - \int_0^\zeta \zeta^2 d\zeta \right]$$

$$\times \frac{1}{\sqrt{(\zeta^2 + \lambda)\left(1 - \beta^2\right)^2}} F \left( \text{arcsin}\sqrt{1 - \beta^2} \right).$$

(B3)

with $F(p, q)$ the elliptic integral of the first kind (Abramowitz & Stegun 1964, p. 589), and with the restriction

$$\frac{x^2}{\zeta^2 + \lambda} + \frac{y^2}{(\zeta^2 \alpha^2 + \lambda)} + \frac{z^2}{(\zeta^2 \beta^2 + \lambda)} = 1 .$$

(B4)

The strength of the central potential decreases as the triaxiality of the object increases:

$$\frac{\Phi(0)}{\Phi_{sph}(0)} = \frac{\int^1 \alpha \beta}{\sqrt{1 - \beta^2}} F \left( \text{arcsin}\sqrt{1 - \beta^2} \right).$$

(B5)

As reported in Ciotti (1991), the models with low $n$ have an inner ($r < r_s$) potential which is much flatter than models with high $n$. As the triaxiality increases there is no important change to the gradient of the gravitational potential along the semimajor axis; the main effect is to shift the gravitational potential profile inwards from the spherical case, resulting in a lower potential at intermediate radii.

#### B2 Gravitational forces

The gravitational forces for a triaxial structure are given by the expression

$$- \frac{\partial \Phi(x)}{\partial x_i} = 4\pi Ga\beta \int_0^\zeta \xi^2 d\xi$$

$$\times \left[ F \left( \text{arcsin}\sqrt{1 - \beta^2} \right) - \int_0^\zeta \zeta^2 d\zeta \right]$$

$$+ \int_0^\zeta \xi^2 \left[ F \left( \text{arcsin}\sqrt{1 - \beta^2} \right) - \int_0^\zeta \zeta^2 d\zeta \right]$$

$$\times \frac{1}{\sqrt{(\zeta^2 + \lambda)\left(1 - \beta^2\right)^2}} F \left( \text{arcsin}\sqrt{1 - \beta^2} \right).$$

(B7)

The restrictions given in equations (6) and (B4) also apply here.