EQUIVARIANT-CONSTRUCTIBLE KOSZUL DUALITY
FOR DUAL TORIC VARIETIES

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Abstract. For affine toric varieties $X$ and $X^\vee$ defined by dual cones, we define an equivalence of categories between mixed versions of the equivariant derived category $D^b_T(X)$ and the derived category of sheaves on $X^\vee$ which are locally constant with unipotent monodromy on each orbit. This equivalence satisfies the Koszul duality formalism of Beilinson, Ginzburg, and Soergel.

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1. Introduction

1.1. Let $T$ and $T^\vee$ be dual complex tori. The ring $\mathcal{O}(T)$ of regular functions on $T$ is canonically isomorphic to the group ring $\mathbb{C}[\pi_1(T^\vee)]$ of the fundamental group $\pi_1(T^\vee)$. Thus the category of quasi-coherent sheaves on $T$ is identified with the category of local systems on $T^\vee$. Let $\mathfrak{t}$ be the Lie algebra of $T$. The (evenly graded) algebra $A$ of polynomial functions on $\mathfrak{t}$ is canonically isomorphic to the equivariant cohomology
The exponential map
\[ \exp : \mathfrak{t} \to T \]
identifies \( A \)-modules supported at the origin with quasi-coherent sheaves on \( T \) supported at the identity and hence with unipotent local systems on \( T^\vee \).

Notice that under this correspondence the logarithm of the monodromy operator becomes the multiplication by the first Chern class of a line bundle. This phenomenon also occurs in mirror symmetry, where the logarithm of the monodromy of the Gauss-Manin local system around certain loops in the moduli space of complex structures of a Calabi-Yau manifold is identified with multiplication by a Chern class on the cohomology of the mirror Calabi-Yau. This is worked out in more generality for the case of complete intersections in toric varieties in [H].

1.2. In this paper we extend the above idea to affine toric varieties. Let \( X \) be an affine \( T \)-toric variety whose fan consists of a cone \( \sigma \) and all its faces, and let \( X^\vee \) be the toric variety whose fan consists of the dual cone \( \sigma^\vee \) and all its faces. The \( T \)-orbits of \( X \) are indexed by the faces \( \tau \) of the cone \( \sigma \). The \( T \)-orbit \( O_\tau \subset X \) is identified with a quotient \( T/T_\tau \) by a subtorus \( T_\tau \). The dual variety has a corresponding orbit \( O_{\tau^\perp} \) which is isomorphic as a \( T^\vee \)-space to \( T^\vee/T_{\tau^\perp} \), where \( T_{\tau^\perp} \) is the “perpendicular” subtorus whose Lie algebra is the annihilator of \( t_\tau = \text{Lie} T_\tau \).

The ring of regular functions on \( T_\tau \) is canonically isomorphic to the group ring of \( \pi_1(T^\vee/T_{\tau^\perp}) \). Let \( A_\tau \) be the algebra of polynomial functions on \( t_\tau \); it is canonically isomorphic to the equivariant cohomology \( H^*_T(T/T_\tau) \). In the same way as before, the exponential map identifies \( A_\tau \)-modules supported at the origin with unipotent local systems on \( O_{\tau^\perp} \).

We use this idea (together with a more combinatorial duality, see [72]) to relate \( T \)-equivariant sheaves on \( X \) to complexes of sheaves with unipotent monodromy on \( X^\vee \). We define an equivalence of triangulated categories \( K \), which fits into the following diagram of categories and functors.

\[
\begin{array}{ccc}
D^b(A\text{-mod}) & \xrightarrow{K} & D^b(LC_f(X^\vee)) \\
\downarrow F_r & & \downarrow F_{cf} \\
D^b_T(X) & & D^b(LC_{cf}(X^\vee))
\end{array}
\]

The categories on the bottom are topological categories of sheaves on the dual toric varieties \( X \) and \( X^\vee \). The categories above them are
“mixed” versions of these categories, where objects have been given an (extra) grading, and the vertical functors forget the grading. On the left hand side we have $T$-equivariant sheaves on $X$, while on the right we have complexes of orbit-constructible sheaves on $X^\vee$ with unipotent monodromy.

Let us describe these categories in more detail. $D^b_T(X)$ is the (bounded, constructible) $T$-equivariant derived category of sheaves on $X$ defined in [BeL]. By results of [L] it is equivalent to a full subcategory of the category of differential graded modules (DG-modules) over a sheaf $A = A_{[\sigma]}$ of rings on the finite poset $[\sigma]$ of faces of $\sigma$. Sections of this sheaf on a face $\tau \prec \sigma$ are complex valued polynomial functions on $\tau$, graded so linear functions have degree 2.

Our mixed version of this category is $D^b(A\text{-mod}_f)$, the derived category of finitely generated graded $A$-modules; it has two gradings – the module grading, and the grading in the complex. The forgetful functor $F_T$ combines these two gradings into the single grading on DG-modules. There is a “twist” automorphism of $D^b(A\text{-mod}_f)$, denoted $\langle 1 \rangle$, which shifts both the complex and module gradings so that $F_T\langle 1 \rangle = F_T$.

The right side of (1.2.1) involves sheaves on $X^\vee$. $LC_{cf}(X^\vee)$ denotes the category of sheaves of $C$-vector spaces on $X^\vee$ which are

1. “locally constant” – the restriction to each $T$-orbit $O \subset X^\vee$ is a local system,
2. “unipotent” – for any orbit $O$ and any $\gamma \in \pi_1(O)$, the action of $\gamma - 1$ on the stalk at a point $p \in O$ is locally nilpotent.
3. “cofinite” – the $\pi_1(O)$ invariants of the stalk at $p$ is finite dimensional, for all orbits $O$ and $p \in O$.

$D^b(LC_{cf}(X^\vee))$ is then the derived category of this abelian category. Note that although objects of $D^b(LC_{cf}(X^\vee))$ are locally constant on orbits, they are not constructible in the usual sense, since the stalks need not be finite-dimensional. As we will see, though, conditions (1)–(3) imply that these objects are still well-behaved. In particular, the full subcategory of objects all of whose stalk cohomology groups are finite-dimensional is equivalent to a full subcategory of the usual constructible derived category. See Proposition 5.6.1.

To define the mixed version $LC_F(X^\vee)$ of $LC_{cf}(X^\vee)$, we use a self-map $F : X^\vee \to X^\vee$ which is a lift of the Frobenius map to characteristic zero defined for toric varieties. An object in $LC_F(X^\vee)$ is an object $S \in LC_{cf}(X^\vee)$ together with an isomorphism $\theta : F^{-1}F \to F$ whose eigenvalues on the stalks at points of $(X^\vee)^F$ are powers of $2^{1/2}$. We again have a “shift of grading” functor $\langle 1 \rangle$, which multiplies $\theta$ by $2^{1/2}$. It is clear that $F_{cf}\langle 1 \rangle = F_{cf}$.
1.3. \(t\)-structures. All four categories in (1.2.1) come equipped with natural perverse \(t\)-structures, whose abelian cores we denote by \(P_T(X), P(A), P_{cf}(X^\vee), \) and \(P_F(X^\vee)\). These \(t\)-structures are particularly nice, in that each triangulated category is equivalent to the bounded derived category of its core: \(D^b_T(X) \cong D^b(P_T(X)), \) etc. The forgetful functors \(F_T, F_{cf}\) are \(t\)-exact, so they restrict to functors \(P(A) \to P_T(X)\) and \(P_{cf}(X^\vee) \to P_{cf}(X^\vee)\).

The twist functors \(\langle 1 \rangle\) are also \(t\)-exact, and so they give automorphisms of \(P(A)\) and \(P_{cf}(X^\vee)\). This induces bijections between isomorphism classes of “ungraded” simple objects and “graded” simples up to twists:

\[
\text{Irr}(P_T(X)) \leftrightarrow \text{Irr}(P(A))/\mathbb{Z},
\]
\[
\text{Irr}(P_{cf}(X^\vee)) \leftrightarrow \text{Irr}(P_F(X^\vee))/\mathbb{Z}.
\]

The set \(\text{Irr}(P_T(X))\) consists of all equivariant IC-sheaves supported on the closures of \(T\)-orbits. For each IC-sheaf \(IC_T(O_\tau)\) we will fix a certain lift to \(P(A)\), which we denote \(L_\tau;\) it is a complex of \(A\)-modules with nonzero cohomology in a single degree. Up to a grading shift, it is the combinatorial equivariant intersection cohomology sheaf studied in [BBFK, BrL, Ka].

1.4. The abelian category \(P(A)\) has enough projective objects. Let \(P \in P(A)\) be a projective cover of \(\oplus_{r \in [a]} \mathcal{L}^r\), and let \(R\) be the opposite ring to the graded ring

\[
\text{end}(P) := \oplus_{i \geq 0} \text{Hom}_{P(A)}(P, P(i)).
\]

In the usual way we see that \(P(A)\) is equivalent to \(R\text{-mod}_{\text{f}}\), the category of finitely generated \(R\)-modules. Furthermore, \(P_T(X)\) is equivalent to \(R\text{-Mod}_{\text{f}}\), the category of finitely generated ungraded \(R\)-modules. It follows that we have equivalences \(D^b(A\text{-mod}_{\text{f}}) \cong D^b(R\text{-mod}_{\text{f}}), D^b_T(X) \cong D^b(R\text{-Mod}_{\text{f}})\). With respect to these equivalences, \(F_T\) is the functor of forgetting the grading.

A similar story holds on the right-hand side of (1.2.1). The simple objects in \(P_{cf}(X^\vee)\) are the intersection cohomology complexes \(IC^*_*(\overline{O_\alpha})\) for \(O_\alpha \subset X^\vee\) a \(T\)-orbit. We will single out a distinguished lift \(L_\alpha^* \in P_F(X^\vee)\) of \(IC^*_*(\overline{O_\alpha})\).

There are enough injectives in \(P_F(X^\vee)\); let \(I\) denote the injective hull of \(\oplus L_\alpha^*\), and put \(R^\vee = \text{end}(I)^{\text{opp}}\). The functor \(\oplus_{i \geq 0} \text{Hom}_{P_F(X^\vee)}(-, I(-i))^*\) gives an equivalence between \(P_F(X^\vee)\) and \(R^\vee\text{-mod}_{\text{cf}}\), the category of “co-finite” graded \(R^\vee\)-modules (see [2.1]). Similarly \(P_{cf}(X^\vee)\) is equivalent to the category \(R^\vee\text{-Mod}_{\text{cf}}\) of ungraded co-finite modules. With
respect to these equivalences, $F_{cf}$ is the functor of forgetting the grading.

The full subcategory $P_{u,fl}(X^\vee)$ of $P_{cf}(X^\vee)$ consisting of objects of finite length is the full subcategory of the usual topological category of orbit-constructible perverse sheaves consisting of objects with unipotent monodromy. Further, the category $P_{\mathcal{F},fl}(X^\vee)$ of finite length objects in $P_{\mathcal{F}}(X^\vee)$ is a mixed version of $P_{u,fl}(X^\vee)$. These categories are equivalent to finite-dimensional ungraded and graded $R^\vee$-modules, respectively.

1.5. The functor $K$ which relates the two sides of (1.2.1) is not $t$-exact, but it does have an interesting relationship with the $t$-structures: it is a Koszul equivalence. Roughly this means that $K$ takes simples of weight 0 in $P(A)$ to indecomposable injectives in $P_{\mathcal{F}}(X^\vee)$ and indecomposable projectives in $P(A)$ to simples in $P_{\mathcal{F}}(X^\vee)$. It also implies that $R$ and $R^\vee$ are Koszul graded rings, and are naturally Koszul dual to each other, in the sense of [BGS]. We explain this in more detail in §2.2 below.

1.6. A similar Koszul functor was constructed for the varieties $X$ and $X^\vee$ in [B]. In that construction the source and target categories were combinatorially defined triangulated categories $D_\Phi(X)$ and $D_{\Phi^\vee}(X^\vee)$, which model mixed sheaves on $X$ and $X^\vee$ with “conditions at infinity” described by auxiliary data $\Phi$, $\Phi^\vee$. This auxiliary choice (essentially the choice of a toric normal slice to each stratum) is somewhat artificial, and as a result it is not clear how to relate these categories directly to a topological category, although the corresponding abelian category of perverse objects in $D_\Phi(X)$ is a mixed version of a category $P_\Phi(X)$ of perverse sheaves on $X$.

The construction in this paper removes these defects, and it is our hope that this more canonical approach will in turn inspire a more intrinsic point of view on this phenomenon, in which the duality functor is defined directly, perhaps in terms of filtered $D$-modules.

1.7. Ideas for future work. We expect that our work can be extended in several directions.

a) We hope that a particular instance of our constructible-equivariant correspondence can be considered as a “limit case” of the mirror symmetry between dual families of Calabi-Yau hypersurfaces in dual toric varieties.

b) We believe that the same constructible-equivariant component should be present in the Koszul duality on flag manifolds constructed in [BGS]. This should be in agreement with Soergel’s conjectures [S].
One should be able to extend to toric varieties the full correspondence \((\text{local systems on } T) \leftrightarrow (\text{quasi-coherent sheaves on } T^\vee)\). In our work we restricted ourselves to unipotent local systems and quasi-coherent sheaves supported at the identity. The language of configuration schemes \([L2]\) may be appropriate in this problem.

1.8. We briefly describe the structure of the paper. Section 2 contains some basic background from homological algebra, including definitions on graded modules, discussion of Koszul equivalences, and mixed categories and gradings. Section 3 introduces the formalism of sheaves on fans considered as finite partially ordered sets, and defines three sheaves of rings on fans which are important later.

In Section 4 we consider sheaves on toric varieties which are locally constant on orbits. We prove that the derived category of these sheaves is the same as the category of complexes of sheaves with locally constant cohomology; this means that locally constant sheaves have enough flexibility for our homological calculations. We also show that the category \(LC(X)\) of locally constant sheaves is equivalent to comodules over a sheaf of rings \(B\) defined in section 3.

In section 5 we define our “mixed” version \(LC_F(X)\) of locally constant sheaves, and show that they are equivalent to graded comodules over \(B\). We define a perverse \(t\)-structure on \(D^b(LC_F(X))\) and prove some basic properties of perverse objects, including the local purity of simple objects. This purity allows us to define a mixed structure on the category of perverse objects, which is the first step to proving that they are equivalent to modules over some graded algebra.

Section 6 turns to the equivariant side of our picture. We first describe a topological realization functor from complexes of \(\mathcal{A}\)-modules on a fan \(\Sigma\) to equivariant complexes on the corresponding toric variety \(X\); this was originally defined in \([L]\). We next study the homological algebra of \(\mathcal{A}\)-modules; the main result is that \(D^b(\mathcal{A}\text{-mod}_f)\) is equivalent to the homotopy category of complexes of “pure” \(\mathcal{A}\)-modules, which are direct sums of shifts of the combinatorial intersection cohomology sheaves \(\mathcal{L}^\tau\) studied in \([BBFK, BrL]\).

In Section 7 we finally define our toric Koszul functor \(K\) and prove that it has the asserted properties. In fact the existence of enough projectives in \(P(\mathcal{A})\) and enough injectives in \(P(LC_F(X^\vee))\) is deduced by applying \(K\) and \(K^{-1}\) to the appropriate simple perverse objects, and all the assertions of \(\S 1.4\) are proved here.

We banish a few technical proofs to Section 8.
1.9. **Acknowledgments.** The first author would like to thank David Cox for suggesting that the results of [B] should be reformulated equivariantly. The second author would like to thank V. Golyshev for stimulating discussions.

2. **Ideas from homological algebra**

2.1. **Conventions on graded rings and modules.** Fix a field $k$, and let $R = \oplus_{n \geq 0} R_n$ be a positively graded $k$-algebra whose zeroth graded piece $R_0$ is isomorphic to $k^{\oplus s}$ for some $s \geq 1$. Suppose that all graded pieces $R_n$ are finite-dimensional (this holds if $R$ is either left or right Noetherian, for instance).

Let $R$-mod, $R$-Mod denote the abelian categories of graded (resp. ungraded) left $R$-modules. Let $R$-mod$_f$, $R$-Mod$_f$ be their respective full subcategories of finitely generated modules; they are abelian subcategories if and only if $R$ is left Noetherian.

The shift of grading functors $\langle j \rangle$, $j \in \mathbb{Z}$ act on $R$-mod by $(M(\langle j \rangle))_n = M_{n+j}$ (note that this is the opposite convention to [BGS]). They preserve the subcategory $R$-mod$_f$.

Given $M$ in $R$-mod, define the “graded dual” $M^*$ of $M$ by $(M^*)_n = \text{Hom}_k(M_{-n}, k)$. Then $M \mapsto M^*$ is a functor $R$-mod $\to (R^{opp})$-mod$^{opp}$. Put $R^\circ := (R^{opp})^*$; it is an injective object of $R$-mod.

We will also need “cofinite” graded $R$-modules, the dual notion to finitely generated modules. If $n \geq 0$, put $R^\rangle_n = \oplus_{k>n} R_k$.

**Proposition 2.1.1.** Let $M \in R$-mod or $R$-Mod. The following are equivalent:

1. $\dim_k \{m \in M \mid R^\rangle_0 \cdot m = 0 \} < \infty$, and every $m \in M$ is annihilated by some $R^\rangle_n$.
2. $M^*$ is contained in a finite direct sum of shifted copies of $R^\circ$.

Such modules are called $R^\rangle$-cofinite, or simply “cofinite”. Let $R$-mod$_{cf}$, $R$-Mod$_{cf}$ denote the category of cofinite graded and ungraded $R$-modules, respectively. The graded dual gives an equivalence of categories $R$-mod$_f \to (R^{opp})$-mod$^{opp}_{cf}$. It follows that $R$-mod$_f$ is an abelian subcategory of $R$-mod if and only if $R$ is right Noetherian.

2.2. **Koszul functors and Koszul duality.** We present the ideas of Koszul duality on derived categories at a level of generality appropriate for our purposes. For a more general discussion, see [BGS].

Fix a field $k$. Let $R$ and $R^\nu$ be algebras of the type considered in the previous section.
Definition 2.2.1. A covariant functor
\[ K : D^b(R\text{-mod}_f) \to D^b(R^\vee\text{-mod}_{cf}) \]
is a Koszul equivalence if the following are satisfied:
(1) \( K \) is a triangulated equivalence of categories. In particular \( K(M[1]) = (KM)[1] \) for all \( M \in D^b(R\text{-mod}_f) \).
(2) For all \( M \in D^b(R\text{-mod}_f) \), we have \( K(M\langle 1 \rangle) = (KM)\langle -1 \rangle[1] \).
(3) \( KR_0 \cong (R^\vee)^\otimes \).
(4) \( KR \cong (R_0^\vee)^\otimes \).

Since \( R_0, R_0^\vee \) are semisimple, \( R \) is projective and \( (R^\vee)^\otimes \) is injective, the conditions (3) and (4) can be replaced by the following.

(3') \( K \) sends simple objects of grading degree 0 to injective hulls of simples of degree 0.
(4') \( K \) sends projective covers of simples of grading degree 0 to simples of degree 0.

Here we use the standard embeddings of \( R\text{-mod}_f, R^\vee\text{-mod}_{cf} \) into their derived categories as complexes with cohomology only in degree 0.

Theorem 2.2.2. If a Koszul functor K exists as in Definition 2.2.1 then
(a) \( R, R^\vee \) are Koszul graded rings, i.e.
\[ \text{Ext}^i_R(R_0, R_0\langle j \rangle) = 0 \text{ for } i \neq -j, \]
and similarly for \( R^\vee \).
(b) \( R^\vee \) is the Koszul dual ring to \( R \), i.e. we have an isomorphism of rings
\[ R^\vee \cong \oplus_{i \geq 0} \text{Ext}^i_R(R_0, R_0\langle -i \rangle). \]

The proof is immediate from the definition 2.2.1.

The conclusion (a) implies in particular that \( R \) is a quadratic algebra, with generators in degree 1 and relations in degree 2. (b) implies that \( R \) and \( R^\vee \) are quadratic dual rings: \( R_0 = R_0^\vee \) canonically, \( R_1 \) and \( R_1^\vee \) are dual \( R_0 \)-modules, and the relations for \( R \) and \( R^\vee \) are orthogonal. See [BGS] for more precise statements and a proof.

Remark. The original example of a Koszul equivalence was defined by Bernstein, Gelfand, and Gelfand [BGG] for \( R \) a polynomial ring and \( R^\vee \) the dual exterior algebra. This example is related to a duality between equivariant and ordinary cohomology, and underlies the “local”, one-orbit case of our toric Koszul duality.
In [BGS] it is shown that under mild finiteness conditions (e.g. if \( \dim_k R^\vee < \infty \), so \( R^\vee \text{-mod}_{\text{cf}} = R^\vee \text{-mod}_f \)), then a Koszul dual pair of rings \((R, R^\vee)\) gives rise to a Koszul equivalence \(D^b(R\text{-mod}_f) \to D^b(R^\vee\text{-mod}_f)\). We are taking the opposite point of view and considering the functor \(K\) as the primary object.

2.3. Mixed categories. We will need the notion of a “mixed” abelian category. This generalizes the category of finitely generated graded modules over a finitely generated positively graded ring. We mostly follow [BGS], but we do not wish to assume our algebras are finite-dimensional, so our abelian categories are not assumed to be Artinian.

Fix a field \(k\). Consider triples \((M, W_\bullet, \langle 1 \rangle)\), where

- \(M\) is an abelian \(k\)-category,
- \(\langle 1 \rangle\) is an automorphism of \(M\), and
- For each \(M \in M\), \(\{W_j M\}_{j \in \mathbb{Z}}\) is a functorial increasing filtration of \(M\).

We call such a triple a mixed category if the following are satisfied:

1. The filtration \(W\) is strictly compatible with morphisms, so \(\text{Gr}_j W = W_j / W_{j-1}\) is an exact functor.
2. For any \(M \in M\), we have \(W_j (M \langle 1 \rangle) = W_{j-1} (M) \langle 1 \rangle\).
3. If \(\text{Gr}_j^W M = 0\) for \(j \neq w\) (we call such an object pure of weight \(w\)), then \(M\) is a finite direct sum of simple objects.
4. There are only finitely many isomorphism classes of simples of weight 0.

Define automorphisms \(\langle n \rangle\), \(n \in \mathbb{Z}\) of \(M\) by taking powers: \(\langle n \rangle = \langle 1 \rangle^n\).

We say an object \(M\) of \(M\) has weights \(\leq j\) (resp. has weights \(\geq j\)) if \(W_j M = M\) (resp. \(W_{j-1} M = 0\)). If both hold, we say \(M\) is pure of weight \(j\); such an object is semisimple of finite length by (3).

Given a mixed category \((M, W_\bullet, \langle 1 \rangle)\), and objects \(X, Y \in M\), define the graded hom and graded ext by

\[
\text{hom}(X, Y)_n = \text{Hom}_M(X, Y \langle n \rangle)
\]
\[
\text{ext}^i(X, Y)_n = \text{Ext}^i_M(X, Y \langle n \rangle).
\]

The graded vector space \(\text{end}(X) := \text{hom}(X, X)\) naturally has the structure of a graded ring.

Let \(L \in M\) be the direct sum of one object from each isomorphism class of weight 0 simples. We call a projective object \(M \in M\) a mixed projective generator (resp. a mixed injective generator) if

1. \(M\) is projective (resp. injective),
2. \(M/W_{-1} M \cong L\) (resp. \(W_0 M \cong L\), and
(3) for any $X \in \mathcal{M}$ there exist $r_k \in \mathbb{Z}$ and a surjection $\bigoplus_{k=1}^{n} M\langle r_k \rangle \to X$ (resp. an injection $X \to \bigoplus_{k=1}^{n} M\langle r_k \rangle$).

If $P$ is a mixed projective generator, then it is is a projective cover of $L$, and the graded endomorphism ring $\text{end}(P)$ is positively graded. If in addition the endomorphisms of simple objects in $\mathcal{M}$ are reduced to scalars, then $\text{end}(P)_0$ is isomorphic to $k^r$, where $r$ is the number of isomorphism classes of weight 0 simples in $\mathcal{M}$. Similar statements hold for mixed injective generators.

The main examples of mixed categories are categories of graded modules over graded rings. Given a positively graded ring $R$ with $R_0$ semisimple, we have a mixed category $(\mathcal{M}, W^*, \langle 1 \rangle)$ where $\mathcal{M}$ consists of graded $R$-modules $M$ with $\dim_k M_j < \infty$ for all $j$, $\langle 1 \rangle$ is the degree shift as defined previously, and $W_j M = \bigoplus_{i \geq -j} M_i$. If $R$ is left (resp. right) Noetherian, then this restricts to a mixed structure on $R\text{-mod}_{df}$ (resp. $R\text{-mod}_{cf}$).

We want sufficient conditions for a mixed category to be of the form $R\text{-mod}_{df}$ or $R\text{-mod}_{cf}$.

**Proposition 2.3.1.** Let $(\mathcal{M}, W^*, \langle 1 \rangle)$ be a mixed category.

(a) If $\mathcal{M}$ has a mixed projective generator $P$ and $\text{end}(P)$ is Noetherian, then $\text{hom}(P, -)$ defines an equivalence of categories $\mathcal{M} \to R\text{-mod}_{df}$, where $R = \text{end}(P)^{\text{opp}}$.

(b) If $\mathcal{M}$ has a mixed injective generator $I$ and $\text{end}(I)$ is Noetherian, then $\text{hom}(-, I)^*$ defines an equivalence of categories $\mathcal{M} \to R\text{-mod}_{cf}$, where $R = \text{end}(I)^{\text{opp}}$.

In either case the mixed structure on $\mathcal{M}$ agrees with the one on graded modules.

2.3.1. **Gradings on abelian categories.** Let $\mathcal{C}$ be an abelian category. By a pre-grading on $\mathcal{C}$ we mean a collection $(\mathcal{M}, W^*, \langle 1 \rangle, v, \epsilon)$, where $(\mathcal{M}, W^*, \langle 1 \rangle)$ is a mixed category, $v : \mathcal{M} \to \mathcal{C}$ is an exact functor, and $\epsilon$ is a natural isomorphism $v \to v \circ \langle 1 \rangle$, satisfying: (1) $v$ sends simples to simples and (2) for any $X, Y$ in $\mathcal{M}$, the map

$$\text{hom}_\mathcal{M}(X, Y) \to \text{Hom}_\mathcal{C}(vX, vY)$$

induced by $v, \epsilon$ is bijective.

**Proposition 2.3.2.** Let $(\mathcal{M}, W^*, \langle 1 \rangle, v, \epsilon)$ be a pre-grading on $\mathcal{C}$.

(a') If part (a) of Proposition 2.3.1 holds, and in addition $vP$ is a projective generator of $\mathcal{C}$, then $\mathcal{C}$ is equivalent to $R^{\text{opp}}\text{-Mod}_{df}$.

(b') If part (b) of Proposition 2.3.1 holds, and in addition $vI$ is an injective generator of $\mathcal{C}$, then $\mathcal{C}$ is equivalent to $R^{\text{opp}}\text{-Mod}_{cf}$.

In either case $v$ is the functor of forgetting the grading.
In either situation \((a')\) or \((b')\) it follows that for any \(X, Y\) in \(M\) and \(i \geq 0\), the induced map

\[
\text{ext}^i_M(X, Y) \to \text{Ext}^i_C(X, Y)
\]

is bijective. Thus our pre-grading is what in [BGS] was termed a “grading” on \(C\).

### 2.4. Triangulated gradings.

Let \(D\) be a triangulated category. A triangulated grading on \(D\) is defined to be a tuple \((D_m, \langle 1 \rangle, v, \epsilon)\), where \(D_m\) is a triangulated category, \(\langle 1 \rangle\) is a triangulated automorphism of \(D_m\), \(v: D_m \to D\) is a triangulated functor, and \(\epsilon: v \to v \circ \langle 1 \rangle\) is a natural isomorphism, subject to the condition that the induced map

\[
\text{hom}_{D_m}(X, Y) \to \text{Hom}_D(vX, vY)
\]

is an isomorphism for any \(X, Y \in D_m\), where \(\text{hom}_{D_m}\) is defined as in the previous section.

If we have a grading on an abelian category as in the previous section, we get a triangulated grading by letting \(D_m = D^b(M)\), \(D = D^b(C)\). One can also go in the other direction, starting from a triangulated grading as defined above, and endowing \(D_m\) and \(D\) with \(t\)-structures for which \(\langle 1 \rangle\) and \(v\) are \(t\)-exact. Letting \(M\) and \(C\) be the abelian cores of \(D_m\) and \(D\), respectively, we get functors \(\langle 1 \rangle: M \to M\) and \(v: M \to C\) as above. Endowing \(M\) with a suitable mixed structure, we get a pre-grading. If Proposition 2.3.2 applies, it is a grading. In §7 we use this approach to prove the fact stated in the introduction that the functors \(F_T\) and \(F_{cf}\) are gradings on the appropriate perverse abelian categories.

### 3. Sheaves of rings associated to toric varieties

#### 3.1. Ringed quivers.

Let \(\Gamma\) be a finite partially ordered set which we consider as a category: for any \(\alpha, \beta \in \Gamma\) the set of morphisms \(\text{Hom}(\beta, \alpha)\) contains a single element if \(\beta \geq \alpha\) and is empty otherwise. A (covariant) functor from \(\Gamma\) to the category of rings is called a sheaf of rings on \(\Gamma\). Let \(A = A_\Gamma\) be such a sheaf of rings, i.e. \(A\) is a collection of rings \(\{A_\alpha\}_{\alpha \in \Gamma}\) with ring homomorphisms \(\phi_{\beta \alpha}: A_\beta \to A_\alpha\), if \(\beta \geq \alpha\), satisfying \(\phi_{\gamma \beta} \phi_{\beta \alpha} = \phi_{\gamma \alpha}\) if \(\gamma \geq \beta \geq \alpha\). We call the pair \((\Gamma, A_\Gamma)\) a ringed quiver.

Assume that for every \(\alpha \in \Gamma\) there is given a \(A_\alpha\)-module \(M_\alpha\) with a morphism \(\psi_{\beta \alpha}: M_\beta \to M_\alpha\) of \(A_\beta\) modules (for \(\beta \geq \alpha\), such that \(\psi_{\gamma \beta} \psi_{\beta \alpha} = \psi_{\gamma \alpha}\) if \(\gamma \geq \beta \geq \alpha\). This data will be called an \(A\)-module. If each \(M_\alpha\) is finitely generated over \(A_\alpha\), we call the resulting \(A\)-module locally finitely generated. \(A\)-modules (resp. locally finitely generated
\( \mathcal{A} \)-modules) form an abelian category which we denote \( \mathcal{A} \text{-Mod} \) (resp. \( \mathcal{A} \text{-Mod}_f \)).

Notice that \( \Gamma \) can be viewed as a topological space where \( \beta \) is in the closure of \( \alpha \) iff \( \beta \geq \alpha \). So the subsets \( [\beta] = \{ \alpha \mid \alpha \leq \beta \} \) are the irreducible open subsets in \( \Gamma \). Then \( \mathcal{A} \) induces a sheaf of rings on this topological space, so that the corresponding category of sheaves of modules is equivalent to \( \mathcal{A} \text{-Mod} \).

We call \((\Gamma, \mathcal{A})\) a graded ringed quiver if rings \( \mathcal{A}_\alpha \) are graded and \( \phi_{\beta\alpha} \) are morphisms of graded rings. In this case let \( \mathcal{A} \text{-mod} \) (resp. \( \mathcal{A} \text{-mod}_f \)) denote the abelian category of graded \( \mathcal{A} \)-modules (resp. locally finitely generated graded \( \mathcal{A} \)-modules) with morphisms of degree zero.

**Remark.** The category \( \mathcal{A} \text{-mod} \) can also be described as graded modules over the quiver algebra \( R = R_{\Gamma, \mathcal{A}} \) generated by idempotents \( e_\alpha \), \( \alpha \in \Gamma \) in degree 0, maps \( \psi_{\beta\alpha} \), \( \beta \geq \alpha \) in degree 1, together with all the elements of the rings \( \mathcal{A}_\alpha \), \( \alpha \in \Gamma \), and satisfying obvious relations (for instance, for \( a \in \mathcal{A}_\gamma \), \( \psi_{\beta\alpha}a = \phi_{\beta\alpha}(a) \) if \( \gamma = \alpha \), and is zero otherwise). \( \mathcal{A} \text{-mod}_f \) is then the category of finitely generated \( R \)-modules.

### 3.1.1. Co-sheaves of modules on a ringed quiver

Given a ringed quiver \((\Gamma, \mathcal{A}_\Gamma)\), by a co-sheaf of \( \mathcal{A}_\Gamma \)-modules we mean the following data: for every \( \alpha \in \Gamma \) there is given a \( \mathcal{A}_\alpha \)-module \( M_\alpha \) with a morphism \( \phi_{\beta\alpha}: M_\alpha \to M_\beta \) of \( \mathcal{A}_\beta \) modules if \( \beta \geq \alpha \), so that \( \phi_{\gamma\beta}\phi_{\alpha\beta} = \phi_{\alpha\gamma} \) for \( \gamma \geq \beta \geq \alpha \). We call co-sheaves of \( \mathcal{A} \)-modules co-\( \mathcal{A} \)-modules and denote this abelian category by co-\( \mathcal{A} \text{-Mod} \) (resp. co-\( \mathcal{A} \text{-mod} \) in the graded case).

### 3.2. DG ringed quiver

A sheaf of DG algebras \( \mathcal{C} = \mathcal{C}_\Gamma \) on \( \Gamma \) is defined in the same way as a sheaf of rings, except the stalks \( \mathcal{C}_\alpha \) are DG algebras and morphisms \( \phi_{\beta\alpha}: \mathcal{C}_\beta \to \mathcal{C}_\alpha \) are homomorphisms of DG algebras. Similarly, a DG \( \mathcal{C} \)-module \( \mathcal{N} \) is a collection \( \{ N_\alpha, \psi_{\beta\alpha} \}_{\alpha \leq \beta} \), where \( N_\alpha \) is a DG module over \( \mathcal{C}_\alpha \) and \( \psi_{\beta\alpha}: \mathcal{N}_\beta \to \mathcal{N}_\alpha \) is a homomorphism of DG modules over \( \mathcal{C}_\beta \). Denote by \( \mathcal{C} \text{-DG-Mod} \) the abelian category of DG \( \mathcal{C} \)-modules. One can define a natural triangulated category \( D(\text{DG-} \mathcal{C}) \) which is called the derived category of DG \( \mathcal{C} \)-modules (see [L]).

As mentioned before it is sometimes convenient to consider \( \Gamma \) as a topological space. Then \( \mathcal{C} \) induces a sheaf of DG algebras on this space and DG \( \mathcal{C} \)-modules become sheaves of DG modules over that sheaf of DG algebras.

**Definition 3.2.1.** Let \( \mathcal{M} \) be a sheaf (or a co-sheaf, or a DG-module) on a quiver \( \Gamma \).

a) If \( \Phi \subset \Gamma \) is a locally closed subset (i.e. the difference of two open sets), denote by \( \mathcal{M}_\Phi \) the extension by zero to \( \Gamma \) of the restriction \( \mathcal{M}|_\Phi \).
Thus \((M_\Phi)_\alpha = M_\alpha\) if \(\alpha \in \Phi\), and 0 otherwise. The restriction map \((M_\Phi)_\alpha \to (M_\Phi)_\beta\) is the one from \(M\) if \(\alpha, \beta \in \Phi\), and is zero otherwise.

b) In case \(M\) is graded and \(k \in \mathbb{Z}\) denote by \(M\{k\}\) the same object shifted “down” by \(k\), i.e. \(M\{k\}_i = M_{k+i}\).

Example 3.2.2. Let \((\Gamma, A)\) be a graded ringed quiver. Assume that the algebras \(A_\alpha\) are evenly graded. We may consider \(A\) as a sheaf of DG algebras with zero differential. There is a natural “forgetful” exact functor between the corresponding derived categories

\[ \nu: D(A\text{-mod}) \to D(DG\text{-}A). \]

Namely, an object of \(D(A\text{-mod})\) is a complex \(M\) of graded \(A\)-modules (thus it has a double grading), whereas an object of \(D(DG\text{-}A)\) is a single DG \(A\)-module. We put \(\nu(M) = \bigoplus_i M_i\{\ell - i\}\) as \(A\)-modules, with the obvious induced differential. It is easy to see that \(\nu\) is a triangulated grading (§2.4).

3.3. Generalities on tori and toric varieties. Fix a complex torus \(T \simeq (\mathbb{C}^*)^n\) with Lie algebra \(t\). The lattice \(N = N_T = \text{Hom}(\mathbb{C}^*, T)\) of co-characters embeds naturally into \(t\). Namely, given a group homomorphism \(\phi: \mathbb{C}^* \to T\), the corresponding point in \(t\) is \(d\phi\ast(1)\), where \(d\phi\ast: \mathbb{C} \to t\) is the induced map of Lie algebras. Then \(t \cong N_C := N \otimes \mathbb{Z} \mathbb{C}\).

Note that \(N\) is naturally isomorphic to the fundamental group \(\pi_1(T)\): given an element \(n \in N\) the corresponding map \(f: [0, 1] \to T\) is defined by the formula

\[ f(t) = e^{2\pi i t n}. \]

Clearly this correspondence is functorial with respect to homomorphisms of tori.

There is also a natural lattice of characters \(M = M_T \subset t^*\) defined similarly. The abelian groups \(M\) and \(N\) are dual to each other: \(M = \text{Hom}(N, \mathbb{Z})\). The dual torus \(T\) is the torus for which \(M_{T^\vee} = N_T\) and \(N_{T^\vee} = M_T\); it is isomorphic to \(T\), but not canonically.

3.4. Toric varieties and fans. Let \(X\) be a normal \(T\)-toric variety. The \(T\)-orbits \(\{O_\alpha\}\) in \(X\) are indexed by the cones \(\alpha\) in a finite polyhedral fan \(\Sigma\) in the vector space \(N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}\) which is rational with respect to the lattice \(N\). If \(\alpha \in \Sigma\), then \(\text{Span}_\mathbb{C}(\alpha) \subset t\) is the Lie algebra of the stabilizer \(T_\alpha\) of the corresponding orbit \(O_\alpha\) (since \(T\) is abelian, the stabilizer can be taken at any point).

We put the natural inclusion order on \(\Sigma\), where \(\alpha \leq \beta\) if and only if \(\alpha\) is a face of \(\beta\). Then \(\alpha \leq \beta\) if and only if \(O_\beta \subset \overline{O_\alpha}\). Thus open unions of orbits correspond to subfans of \(\Sigma\).
More generally we will want to consider locally closed unions of orbits in $X$. Such a subvariety $Y$ corresponds to a locally closed subset $\Lambda \subset \Sigma$, which satisfies all the fan properties, except that it need not be closed under taking faces. Instead if $\alpha \leq \beta$ are cones in $\Lambda$, then $\Lambda$ must contain all faces of $\beta$ which contain $\alpha$. We call such subsets quasifans.

3.5. Let $X$ be a $T$-toric variety. For each orbit $O_\alpha$ in $X$ denote by $$\text{St}(O_\alpha) = \bigcup_{\alpha \leq \beta} O_\beta$$ its star in $X$. Consider the stabilizer $T_\alpha \subset T$ of the orbit $O_\alpha$. Since $X$ is normal, the group $T_\alpha$ is connected, and hence is a torus. There exists a non-canonical homeomorphism $$\text{St}(O_\alpha) \simeq X' \times (T/T_\alpha) \simeq X' \times O_\alpha,$$ where $X'$ is an affine $T_\alpha$-toric variety with a single fixed point.

The action of $T_\alpha$ on $\text{St}(O_\alpha)$ defines a canonical projection $$p_\alpha : \text{St}(O_\alpha) \to O_\alpha,$$ which is compatible with the product decomposition above.

If $O_\beta \subset \text{St}(O_\alpha)$ we denote by $p_{\beta \alpha}$ the restriction of $p_\alpha$ to $O_\beta$. The collection of projections $\{p_{\beta \alpha}\}$ is compatible in the sense that $\phi_{\beta \alpha}p_{\gamma \beta} = p_{\gamma \alpha}$ wherever all maps are defined.

**Lemma 3.5.1.** Each point in $O_\alpha$ has a fundamental system of distinguished contractible neighborhoods $U \subset X$, such that $U \cap \text{St}(O_\alpha) \subset p_{\alpha}^{-1}(U \cap O_\alpha)$ and for each orbit $O_\beta \subset \text{St}(O_\alpha)$ the inclusion $U \cap O_\beta \hookrightarrow p_{\alpha}^{-1}(U \cap O_\alpha)$ is a homotopy equivalence.

3.6. **The ringed quiver** $(\Sigma, \mathcal{A}_\Sigma)$. Let $\Sigma$ be a finite polyhedral fan, rational or not, with the inclusion partial order on faces. There is a natural graded ringed quiver $\mathcal{A} = \mathcal{A}_\Sigma$ on $\Sigma$: for $\tau \in \Sigma$ the stalk $\mathcal{A}_\tau$ is the graded ring of complex-valued polynomial functions on the span of $\tau$. The structure homomorphisms $\phi$ are the restrictions of functions. We consider linear functions as having degree 2, so that $\mathcal{A}$ is evenly graded.

Note that the ringed quiver $\mathcal{A}$ makes sense even for non-rational fans (which do not correspond to toric varieties).

**Remark.** Notice that the topological space associated to the partially ordered set $\Sigma$ is homeomorphic to the quotient space $\overline{X} = X/T$. If $Y \subset X$ is $T$-invariant and locally closed, the space of sections of $\mathcal{A}$ on $\overline{Y}$ is canonically identified with the equivariant cohomology $H^*_T(Y; \mathbb{C})$. This is why this sheaf is useful for studying the equivariant topology of
 Later in §6 we will use the categories $\mathcal{A}_X$-$\text{mod}_\beta$, $\text{DG-}\mathcal{A}_X$, and their derived categories to model equivariant sheaves and complexes on $X$.

3.7. **The ringed quiver** $(\Sigma^0, \mathcal{B}_\Sigma)$. Now consider the partially ordered set $\Sigma^0$ which is $\Sigma$ with the opposite ordering. One may think about $\Sigma^0$ as the partially ordered set of orbits of $X$, where $O_\alpha \leq O_\beta$ iff $O_\alpha \subseteq \overline{O_\beta}$. There is a natural sheaf of rings $\mathcal{B}_X = \mathcal{B}$ on $\Sigma^0$: for an orbit $O_\alpha$ take $\mathcal{B}_\alpha$ to be the group ring $\mathbb{C}[\pi_1(\overline{O_\alpha})]$. If $O_\alpha \leq O_\beta$ the canonical projections $p_{\alpha\beta}: O_\beta \to O_\alpha$ induce homomorphisms $\mathcal{B}_\beta \to \mathcal{B}_\alpha$. Thus we obtain a ringed quiver $(\Sigma^0, \mathcal{B})$.

As with the ringed quiver $\mathcal{A}$, $\mathcal{B}$ can be described entirely in terms of the fan $\Sigma$, without reference to the toric variety. For any orbit $O_\alpha$, there is a canonical identification $\pi_1(\overline{O_\alpha}) = \mathcal{N}_\alpha$, where $\mathcal{N}_\alpha$ is the lattice $N/(N \cap \text{Span}(\alpha))$. If $\alpha$ is a face of $\beta$, so $\beta \leq \alpha$ in $\Sigma^0$, the homomorphism $\pi_1(\overline{O_\alpha}) \to \pi_1(\overline{O_\beta})$ comes from the natural map $\mathcal{N}_\alpha \to \mathcal{N}_\beta$.

3.8. **The ringed quiver** $(\Sigma^0, \mathcal{T}_\Sigma)$. We can define another ringed quiver on $\Sigma^0$ as follows. For $\alpha \in \Sigma$, let $\mathcal{T}_\alpha = \text{Sym}(\mathcal{N}_{\alpha,\mathbb{C}})$. We consider it as a graded polynomial algebra, where elements of $\mathcal{N}_{\alpha,\mathbb{C}}$ have degree 2. If $\alpha$ is a face of $\beta$, the homomorphism $\mathcal{T}_\alpha \to \mathcal{T}_\beta$ comes from the natural map $\mathcal{N}_\alpha \to \mathcal{N}_\beta$.

Notice that $\mathcal{N}_{\alpha,\mathbb{C}}$ is canonically isomorphic to the Lie algebra of $T/T_\alpha$, which can be canonically identified with $O_\alpha$. Thus our graded ringed quiver can be described more geometrically: if $O_\alpha \leq O_\beta$, the canonical projections $p_{\alpha\beta}: O_\beta \to O_\alpha$ induce morphisms of tori $T/T_\beta \to T/T_\alpha$, hence they induce homomorphisms of graded polynomial algebras $\mathcal{T}_\beta \to \mathcal{T}_\alpha$.

3.9. **Dual affine toric varieties.** Let $X$ be an affine $T$-toric variety with a single fixed point. The corresponding fan $\Sigma = \Sigma_X$ consists of a single full-dimensional cone $\sigma = \sigma_X \subset N_{T,\mathbb{R}}$ together with its faces. We have the dual cone $\sigma^\vee$ in the dual vector space $M_{T,\mathbb{R}} = (N_{T,\mathbb{R}})^*$, defined by

$$\sigma^\vee = \{y \in M_{T,\mathbb{R}} \mid \langle x, y \rangle \geq 0 \text{ for all } x \in \sigma\}.$$ 

Let $T^\vee$ be the dual torus to $T$; then $N_{T^\vee,\mathbb{R}} = (N_{T,\mathbb{R}})^*$ canonically.

**Definition 3.9.1.** The dual toric variety $X^\vee$ to $X$ is the affine $T^\vee$-toric variety defined by the fan $\Sigma^\vee$ consisting of $\sigma^\vee$ and all its faces, with respect to the lattice $M_T = N_{T^\vee}$. In other words, we have $\sigma_X^\vee = \sigma_X$. 

There is an order-reversing isomorphism $\alpha \mapsto \alpha^\perp$ between $\Sigma$ and $\Sigma^\vee$, defined by $\alpha^\perp = \sigma^\vee \cap \text{Span}(\alpha)^\perp$. In particular we have $\text{Span}(\alpha^\perp) = \text{Span}(\alpha)^\perp$. This map gives an identification $\Sigma^\vee = \Sigma^0$ of partially ordered sets.
With respect to this identification, the ringed quivers \((\Sigma, A_\Sigma)\) and \(((\Sigma')^0, T_{\Sigma'})\) are identical. This will be important for the definition of our equivariant-constructible duality in \cite{37}.

4. Locally constant sheaves on toric varieties

4.1. Some lemmas about sheaves on toric varieties. For a topological space \(Y\) denote by \(\text{Sh}(Y)\) the abelian category of sheaves of complex vector spaces on \(Y\).

Let \(X\) be a normal toric variety. We consider \(X\) as a topological space in the classical topology. Let \(Z\) be a \(T\)-invariant subspace of \(X\). Denote by \(\text{LC}(Z) \subset \text{Sh}(Z)\) the full subcategory of sheaves which are locally constant on each orbit.

Consider the full subcategory \(D^b_{\text{LC}}(\text{Sh}(Z))\) of the bounded derived category \(D^b(\text{Sh}(Z))\), consisting of complexes with cohomologies in \(\text{LC}(Z)\).

Fix an orbit \(O_\alpha \subset X\) and choose a locally closed \(T\)-invariant subset \(W \subset \text{St}(O_\alpha)\), which contains \(O_\alpha\). Denote by \(i: O_\alpha \hookrightarrow W\) the corresponding closed embedding. Let \(j: U \hookrightarrow W\) be the complementary open embedding of \(U = W - O_\alpha\). Denote by \(q: W \rightarrow O_\alpha\), \(p: U \rightarrow O_\alpha\) the restrictions of the projection \(p_\alpha\) to \(W\) and \(U\) respectively.

**Lemma 4.1.1.** In the above notation the functors \(Rq_*\) and \(i^*\) from \(D^b_{\text{LC}}(W)\) to \(D^b_{\text{LC}}(O_\alpha)\) are naturally isomorphic. In particular, the functors \(Rp_*\) and \(i^*j_*\) from \(D^b_{\text{LC}}(U)\) to \(D^b_{\text{LC}}(O_\alpha)\) are naturally isomorphic. Hence, the functors \(p_*\) and \(i^*j_*\) from \(\text{LC}(U)\) to \(\text{LC}(O_\alpha)\) are naturally isomorphic.

**Proof.** Using distinguished neighborhoods of points in \(O_\alpha\) (Lemma 3.5.1), we see that there exists a natural morphism of functors \(Rq_* \rightarrow i^*\). Let us show that it is an isomorphism.

The category \(D^b_{\text{LC}}(W)\) is the triangulated envelope of objects \(Rj_\beta^*L\), where \(j_\beta: O_\beta \hookrightarrow W\) is the embedding of an orbit \(O_\beta\) and \(L\) is an object in \(\text{LC}(O_\beta)\). So we may assume that \(U = O_\beta\) and it suffices to show that \(i^*Rj_\alpha^*L = Rp_*L\) (the case \(\alpha = \beta\) is clear).

Choose a distinguished neighborhood \(V \subset X\) of a point in \(O_\alpha\). Then the complex \(\Gamma(V \cap O_\alpha, i^*Rj_\alpha^*L)\) is quasi-isomorphic to the complex \(\text{R}\Gamma(V \cap O_\beta, L)\). But the inclusion \(V \cap O_\beta \subset p^{-1}(V \cap O_\alpha)\) is a homotopy equivalence, hence it induces a quasi-isomorphism \(\text{R}\Gamma(p^{-1}(V \cap O_\alpha), L) \simeq \text{R}\Gamma(V \cap O_\beta, L)\). This proves that \(i^*Rj_\alpha^*L = Rp_*L\).

The last statement now follows by taking \(H^0\). \(\square\)

4.2. Equivalence of derived categories.

**Theorem 4.2.1.** The natural functor \(D^b(\text{LC}(X)) \rightarrow D^b_{\text{LC}}(\text{Sh}(X))\) is an equivalence.
Proof. Let $i = i_{O}: O \hookrightarrow X$ be the (locally closed) embedding of an orbit $O$. For $F \in LC(O)$ we may consider two different (derived) direct images of $F$ under the embedding $i$: one in the category $D_{LC}^{b}(Sh(X))$, denoted as usual by $R_{i*}F$, and the other in the category $D^{b}(LC(X))$, which we denote by $R_{LCi*}F$. It is clear that the category $D_{LC}^{b}(Sh(X))$ (resp. $D^{b}(LC(X))$) is the triangulated envelope of the objects $R_{i*}F$ (resp. $R_{LCi*}F$) for various orbits $O$ and locally constant sheaves $F$ on them. So it suffices to prove the following two claims.

Claim 1. The complexes $R_{i*}F$ and $R_{LCi*}F$ are quasi-isomorphic.

Claim 2. Let $i$ and $F$ be as above, $j: O' \hookrightarrow X$ be the embedding of an orbit and $G \in LC(O')$. Then

$$\text{Ext}^{*}_{D^{b}(LC(O))}(R_{i*}F,G) = \text{Ext}^{*}_{D_{LC}^{b}(Sh(X))}(R_{LCi*}F,G).$$

Let us prove the second claim first. Using the adjunction we need to prove that

$$\text{Ext}^{*}_{D^{b}(LC(O))}(i^{*}R_{j*}F,G) = \text{Ext}^{*}_{D_{LC}^{b}(Sh(X))}(i^{*}R_{LCi*}F,G).$$

By deissage this is a consequence of the following lemma.

**Lemma 4.2.2.** Let $Y$ be a $K(\pi,1)$-space, $LC(Y)$ – the category of locally constant sheaves on $Y$. Then for any $A,B \in LC(Y)$

$$\text{Ext}^{*}_{LC(Y)}(A,B) = \text{Ext}^{*}_{Sh(Y)}(A,B).$$

**Proof.** Let $f: \tilde{Y} \to Y$ be the universal covering map. Then the functor $f^{*}$ establishes an equivalence of abelian categories

$$f^{*}: \text{Sh}(Y) \to \text{Sh}_{\pi}(\tilde{Y}),$$

where $\text{Sh}_{\pi}(\tilde{Y})$ is the category of $\pi$-equivariant sheaves on $\tilde{Y}$. Clearly $f^{*}$ preserves locally constant sheaves.

**Remark.** It is well known that the category $\text{Sh}_{\pi,LC}(\tilde{Y})$ of locally constant ($\equiv$constant) $\pi$-equivariant sheaves on $\tilde{Y}$ is equivalent to the category of $\pi$-modules. The equivalence is provided by the functor of global sections $\Gamma$.

Put $\tilde{A} = f^{*}A$, $\tilde{B} = f^{*}B$. We will prove that $\text{Ext}^{*}_{\text{Sh}_{\pi,LC}(\tilde{Y})}(\tilde{A},\tilde{B}) = \text{Ext}^{*}_{\text{Sh}_{\pi}(\tilde{Y})}(\tilde{A},\tilde{B})$. Choose an injective resolution

$$\tilde{B} \to I^{0} \to I^{1} \to \ldots$$

in the category $\text{Sh}_{\pi,LC}(\tilde{Y})$. It suffices to prove that $\text{Ext}^{k}_{\text{Sh}_{\pi}(\tilde{Y})}(\tilde{A},I^{t}) = 0$ for any $t$ and $k > 0$. Put $I = I^{t}$ and choose a resolution $0 \to I \to \ldots$
$J^0 \to J^1 \to \ldots$, where $J$'s are injective objects in $\text{Sh}_\pi(\tilde{Y})$. So

$$\text{Ext}^k_{\text{Sh}_\pi(\tilde{Y})}(\tilde{A}, I) = H^k(\text{Hom}^\bullet_{\text{Sh}_\pi(\tilde{Y})}(\tilde{A}, J^s)).$$

Notice that $I$, as a sheaf, is constant and each $J^s$, as a sheaf, is injective \cite{Gr}. Hence the complex of global sections

$$0 \to \Gamma(I) \to \Gamma(J^0) \to \Gamma(J^1) \to \ldots$$

is exact ($\tilde{Y}$ is contractible). Since $\tilde{A}$, as a sheaf, is also constant, the complex

$$0 \to \text{Hom}_{\text{Sh}_\pi(\tilde{Y})}(\tilde{A}, I) \to \text{Hom}_{\text{Sh}_\pi(\tilde{Y})}(\tilde{A}, J^0) \to \ldots$$

is isomorphic to the complex

$$0 \to \text{Hom}_\pi(\Gamma(\tilde{A}), \Gamma(I)) \to \text{Hom}_\pi(\Gamma(\tilde{A}), \Gamma(J^0)) \to \ldots$$

The $\pi$-module $\Gamma(I)$ is injective and so are the $\pi$-modules $\Gamma(J^s)$ for all $s \geq 0$. Hence the last complex is exact. This proves the lemma and Claim 2. \hfill \square

Let us prove Claim 1. Choose a locally constant sheaf $I$ on $O$ which is injective in the category $L\text{C}(O)$. It suffices to prove that the complex of sheaves $Ri_* I$ is acyclic except in degree zero. Choose an orbit $O_\alpha \subset \overline{O}$ and let $p: O \to O_\alpha$ be the canonical projection. Fix a distinguished neighborhood $U \subset X$ of a point in $O_\alpha$ (Remark 5.5.1). By Lemma 4.1.1 above the complex $\Gamma(U \cap O_\alpha, R_i I)$ is quasi-isomorphic to the complex

$$R\Gamma(p^{-1}(U \cap O_\alpha), I) = \mathbf{R}\text{Hom}^\bullet(C_{p^{-1}(U \cap O_\alpha)}, I|_{p^{-1}(U \cap O_\alpha)}).$$

Since the space $p^{-1}(U \cap O_\alpha)$ is $K(\pi, 1)$, and the restriction of the local system $I$ to $p^{-1}(U \cap O_\alpha)$ remains injective, it follows from the above lemma that the complex $\mathbf{R}\text{Hom}^\bullet(C_{p^{-1}(U \cap O_\alpha)}, I|_{p^{-1}(U \cap O_\alpha)})$ is acyclic in positive degrees. This proves Claim 1 and the theorem. \hfill \square

**Remark.** The key property of toric varieties which is used in the proof of the above theorem is that the star of an orbit is homotopy equivalent to the orbit itself. For example, the analogue of the above theorem does not hold for $\mathbb{P}^1$ which is stratified by two cells: $C$ and a point.

The category $L\text{C}(X)$ has enough injectives: injective objects are sums of objects of the form $i_* I$, where $i: O \hookrightarrow X$ is an embedding of an orbit and $i \in L\text{C}(O)$ is an injective local system. Furthermore, $L\text{C}(X)$ has finite cohomological dimension, so objects in $D^b(L\text{C}(X))$ can be represented by bounded complexes of injectives.

Thus if $j: Y \hookrightarrow X$ is an embedding of a locally closed $T$-invariant subspace, we can take derived functors of $j_*$ and $\Gamma_Y$ (sections with
support in $Y$), giving functors $R_{LCj_*}: D^b(LC(Y)) \to D^b(LC(X))$ and $R_{LC\Gamma_Y}: D^b(LC(X)) \to D^b(LC(X)).$ Define $j^!_{LC} = j^* R_{LC\Gamma_Y}$.

On the other hand, the usual derived functors restrict to functors $R_j: D^b_{LC}(Y) \to D^b_{LC}(X)$ and $R\Gamma_Y: D^b_{LC}(X) \to D^b_{LC}(X)$, and we have $j^! = j^* R\Gamma_Y$. The following corollary to the proof of Theorem 4.2.1 will be used later when we discuss the intersection cohomology sheaves.

**Corollary 4.2.3.** The functors $R_{LCj_*}$ and $R_j$, are isomorphic under the equivalence of Theorem 4.2.1 as are $j^!_{LC}$ and $j^!$.

**Proof.** Since $D^b(LC(X))$ is generated by injective objects of $LC(X)$, for the first claim it will be enough to consider $i_* I$, where $i: O \hookrightarrow Y$ is the inclusion of an orbit and $I$ is injective in $LC(O)$. Then $R_{LCj_*}(i_* I) = j^! i_* I = (j \circ i)_* I \simeq R_{LC}(j \circ i)_* I$, since $i_* I$ is injective. On the other hand, we have

$$R_j(i_* I) = R_j R_{LC} i_* I \simeq R_j R i_* I \simeq R (j \circ i)_* I,$$

using Claim 1 from the proof of Theorem 4.2.1. Applying Claim 1 once more gives $R_{LCj_*}(i_* I) \simeq R_j(i_* I)$.

For the second part, let $i: O \hookrightarrow X$ be the inclusion of an orbit, and let $I \in LC(O)$ be injective; we will show that $j^! (i_* I)$ and $j^!_{LC}(i_* I)$ are quasi-isomorphic. The inclusion $j$ can be factored as the composition of an open embedding and a closed embedding. The required isomorphism is obvious when $Y$ is open, so we can assume that $Y$ is closed. If $Y$ contains $\overline{O}$, then $j^! i_* I = j^* i_* I = j^!_{LC} i_* I$. Otherwise, we have $j^!_{LC}(i_* I) = j^! \Gamma_Y (i_* I) = 0$, since $i_* I$ is injective and all nonzero sections of $i_* I$ must contain points of $O$ in their support. On the other hand, $i_* I \simeq R i_* I$, so $j^! (i_* I) = 0$ as well. This completes the proof. \[\square\]

### 4.3. Quiver description of the category $LC(X)$

Recall the ringed quiver $(\Sigma^\circ, \mathcal{B})$ associated with the toric variety $X$. We are going to define a functor

$$\eta: LC(X) \to \text{co-B-Mod}.$$

For this we need to recall how to glue sheaves on topological spaces. Surely this construction is well known, but we do not know a reference.

Let $Y$ be a topological space, $i: Z \hookrightarrow Y$ the embedding of a closed subset and $j: U = Y - Z \hookrightarrow Y$ the complementary open embedding. Consider the abelian category $\text{Sh}(Y, Z)$ consisting of triples $(G, H, \xi)$, where $G \in \text{Sh}(Z)$, $H \in \text{Sh}(U)$ and $\xi$ is a morphism of sheaves in $\text{Sh}(Z)$ $\xi: G \to i^* j_* H$. We have a natural functor $\tau: \text{Sh}(Y) \to \text{Sh}(Y, Z)$ which associates to a sheaf $F \in \text{Sh}(Y)$ its restrictions $i^* F \in \text{Sh}(Z)$,
Lemma 4.3.1. The functor $\tau$ is an equivalence.

Proof. Let us define the inverse functor $\eta: \text{Sh}(Y, Z) \to \text{Sh}(Y)$. Given $(G, H, \xi) \in \text{Sh}(Y, Z)$ define a presheaf $\overline{F}$ on $Y$ as follows. For an open subset $V \subset Y$ put $\overline{F}(V) = H(V \cap Z)$ if $V \subset U$. Otherwise set $\overline{F}(V) = \{(g, h) \in G(V \cap Z) \times H(V \cap U) \mid \xi(g) = h'\}$, where $h'$ is the image of $h$ in $i^*j_*H(V \cap Z)$. Then let $\eta(G, H, \xi) \in \text{Sh}(Y)$ be the sheafification of $\overline{F}$. □

Let $F \in \text{LC}(X)$. Denote the stalk of $F$ at the distinguished point in an orbit $O_{\alpha}$ by $F_{\alpha}$. Then $F_{\alpha}$ is a $B_{\alpha}$-module. Given two orbits $O_{\alpha} \subset O_{\beta}$ consider the canonical projection $p = p_{\beta\alpha}: O_{\beta} \to O_{\alpha}$. By Lemma 6.7 and Lemma 6.1 the restriction of the sheaf $F$ to the union of the two orbits defines a morphism of sheaves $F|_{O_{\alpha}} \to p_*(F|_{O_{\beta}})$, or equivalently a morphism of sheaves $p^{-1}(F|_{O_{\alpha}}) \to (F|_{O_{\beta}})$.

Such a morphism is equivalent to a homomorphism of $B_{\beta}$-modules $F_{\alpha} \to F_{\beta}$. So the sheaf $F$ defines a co-$B$-module. This is our functor

$$\eta: \text{LC}(X) \to \text{co-}B\text{-Mod}.$$

Theorem 4.3.2. The functor $\eta$ is an equivalence.

For example, in case the toric variety $X$ is the affine line with two orbits, $\mathbb{C}^*$ and the origin, an object in $\text{LC}(X)$ is the same as a vector space $P$, a $\mathbb{C}[\pi_1(\mathbb{C}^*)]$-module $Q$ and a linear map $P \to Q^{\pi_1(\mathbb{C}^*)}$.

Proof. We will prove the theorem by induction on the number of orbits in $X$. For one orbit the statement of the theorem is a well known equivalence between the category of locally constant sheaves and that of $\pi_1$-modules.

Now we proceed with the induction step. Pick an orbit $O_{\alpha} \subset X$ of smallest dimension. We may assume that $X = \text{St}(O_{\alpha})$. Indeed, otherwise $X$ may be covered by open $T$-invariant subsets $V$, which are strictly smaller than $X$. By induction, the theorem is true for each $V$ and so we obtain the equivalence for $X$ by gluing the corresponding equivalences $\text{LC}(V) \simeq \text{co-}B_V\text{-Mod}$.

Put $U = X - O_{\alpha}$ and let $j: U \hookrightarrow X$ and $i: O_{\alpha} \hookrightarrow X$ be the open and closed embeddings respectively. By Lemma 6.7 a sheaf $F \in \text{LC}(X)$ is
the same as a triple \((G, H, \xi)\), where \(G \in \text{LC}(O_\alpha)\), \(H \in \text{LC}(U)\) and \(\xi: G \to i^* j_* H\). By Lemma 6.1 \(i^* j_* H = p_{\alpha*} H\). Thus by adjunction the morphism \(\xi\) is the same as a morphism \(\iota: p_{\alpha}^{-1} G \to H\). Let \(G_\alpha\) be the \(\mathcal{B}_\alpha\)-module corresponding to \(G\). It is easy to see that the sheaf \(p_{\alpha}^{-1} G\) considered as a co-\(\mathcal{B}_\alpha\)-module is the constant one equal to \(G_\alpha\). Thus the triple \((G, H, \iota)\) is the same as a co-\(\mathcal{B}_\alpha\)-module. This proves the theorem. \(\square\)

4.4. Unipotent sheaves.

**Definition 4.4.1.** A sheaf \(F \in \text{LC}(X)\) is called **unipotent** if for each orbit \(O_\alpha\) and \(x \in \pi_1(O_\alpha)\) the action of the operator \(x - 1\) on the stalk \(F_\alpha\) of \(F\) at a point of \(O_\alpha\) is locally nilpotent. It is called **co-finite** if in addition the space of invariants \(F_\alpha^{\pi_1(O_\alpha)}\) is finite-dimensional for all \(\alpha\). Let \(\text{LC}_u(X)\) and \(\text{LC}_{cf}(X)\) be the full subcategories of \(\text{LC}(X)\) consisting of unipotent (resp. co-finite) sheaves.

The next result describes the corresponding subcategories of co-\(\mathcal{T}\)-\text{Mod} under the equivalence \(\eta\). Let co-\(\mathcal{T}\)-\text{Mod}_n be the full subcategory of co-\(\mathcal{T}\)-modules \(\mathcal{M}\) which are “supported at the origin”, i.e. for which every \(m \in \mathcal{M}_\alpha\) is annihilated by some power of the homogeneous maximal ideal \(m_\alpha \subset \mathcal{T}_\alpha\). Let co-\(\mathcal{T}\)-\text{Mod}_{cf} be the further full subcategory of modules \(\mathcal{M}\) for which each \(\mathcal{M}_\alpha\) is a cofinite \(\mathcal{T}_\alpha\)-module (§2.1). In other words, in addition to being supported at the origin, for each \(\alpha\) the space \(\{m \in \mathcal{M}_\alpha | m_\alpha \cdot m = 0\}\) should be finite dimensional.

**Theorem 4.4.2.** The functor \(\eta\) restricts to give equivalences of full abelian subcategories

\[
\text{LC}_u(X) \simeq \text{co-}\mathcal{T}\text{-Mod}_n, \quad \text{and}
\]

\[
\text{LC}_{cf}(X) \simeq \text{co-}\mathcal{T}\text{-Mod}_{cf}.
\]

**Proof.** Take a sheaf \(F \in \text{LC}_u(X)\), and let \(\mathcal{M} = \eta(F) \in \text{co-}\mathcal{B}\text{-Mod}\) be the corresponding co-\(\mathcal{B}\)-module. Since the action of any \(x \in \pi_1(O_\alpha) \cong N_\alpha\) on \(\mathcal{M}_\alpha\) is unipotent, the action of the power series \(\frac{1}{2\pi i} \ln x\) is well-defined. We can extend this uniquely to a map \(v \mapsto \frac{1}{2\pi i} \ln v\) from \(N_\alpha \otimes \mathbb{C}\) to \(\text{End}(\mathcal{M}_\alpha)\). Any two of these operators commute, since \(\pi_1(O_\alpha)\) is abelian.

This gives \(\mathcal{M}_\alpha\) the structure of a \(\text{Sym}(N_\alpha \otimes \mathbb{C}) = \mathcal{T}_\alpha\)-module, and in fact makes \(\mathcal{M}\) into a co-\(\mathcal{T}\)-module. The resulting co-\(\mathcal{T}\)-module is clearly supported at the origin. Conversely, given a co-\(\mathcal{T}\)-module \(\mathcal{M}\) supported at the origin, we can exponentiate the action of elements of \(N_\alpha\) to get an action of \(\pi_1(O_\alpha)\) on \(\mathcal{M}_\alpha\). These actions combine to give the structure of a co-\(\mathcal{B}\)-module on \(\mathcal{M}\), and then applying \(\eta^{-1}\) gives the required object in \(\text{LC}_u(X)\).
The second equivalence follows immediately, since the maximal ideal $m_\alpha \subset \mathcal{T}_\alpha$ is generated by $\frac{1}{2\pi i} \ln x, x \in \pi_1(O_\alpha)$. \qed

5. Mixed locally constant sheaves

5.1. pre-$\mathcal{F}$-sheaves. For toric varieties the Frobenius endomorphism has a natural lift to characteristic zero – see [W]. We will use it to define a mixed version of the category $LC(X)$.

Consider the group homomorphism $\phi: T \to T, a \mapsto a^2$. For any toric variety $X$ the homomorphism $\phi$ extends uniquely to a morphism $\mathcal{F} = \mathcal{F}_X: X \to X$. Namely, recall that each orbit $O_\alpha$ is identified with the quotient torus $T/T_\alpha$; then the map $\mathcal{F}: O_\alpha \to O_\alpha$ is again squaring.

The maps $\phi$ and $\mathcal{F}$ have degree $2^n$.

Definition 5.1.1. A pre-$\mathcal{F}$-sheaf is a pair $(F, \theta)$, where $F \in LC(X)$ and $\theta$ is an isomorphism

$$\theta: \mathcal{F}^{-1}F \to F.$$ 

Let us describe the inverse image functor $\mathcal{F}^{-1}: LC(X) \to LC(X)$ in terms of co-$\mathcal{B}$-modules. The map $\mathcal{F}$ induces the endomorphism of the sheaf $\mathcal{B}$, where each element $x \in \pi_1(O_\alpha)$ maps to $x^2$. Denote this endomorphism $\psi: \mathcal{B} \to \mathcal{B}$. Fix $F \in LC(X)$ and let $\mathcal{M}$ be the corresponding co-$\mathcal{B}$-module. Then the sheaf $\mathcal{F}^{-1}F$ corresponds to the co-$\mathcal{B}$-module $\psi_*\mathcal{M}$, i.e. it is obtained from $\mathcal{M}$ by restriction of scalars via $\psi$. So the isomorphism $\theta: \psi_*\mathcal{M} \to \mathcal{M}$ corresponding to the isomorphism $\theta: \mathcal{F}^{-1}F \to F$ amounts to a compatible system of linear maps $\theta_\alpha: \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ such that for $x \in \pi_1(O_\alpha), m \in \mathcal{M}_\alpha$

$$\theta_\alpha(x^2m) = x\theta_\alpha(m).$$

5.2. $\mathcal{F}$-sheaves and graded co-finite co-$\mathcal{T}$-modules.

Definition 5.2.1. A pre-$\mathcal{F}$-sheaf $(F, \theta)$ is called an $\mathcal{F}$-sheaf if $F$ is co-finite (unipotent) and for each $\alpha$ the endomorphism $\theta_\alpha: F_\alpha \to F_\alpha$ is diagonalizable with eigenvalues $2^{n/2}$, $n \in \mathbb{Z}$. We will refer to $\mathcal{F}$-sheaves on a single orbit as “$\mathcal{F}$-local systems”. Denote by $LC_\mathcal{F}(X)$ the category of $\mathcal{F}$-sheaves on $X$.

For any $n \in \mathbb{Z}$ define an automorphism $\langle n \rangle$ of $LC_\mathcal{F}(X)$ by $(F, \theta) \mapsto (F, 2^{n/2}\theta)$.

Let $(F, \theta)$ be an $\mathcal{F}$-sheaf, and take $x \in \pi_1(O_\alpha)$. The relation

$$\theta \cdot x^2 = x \cdot \theta$$

in $\text{End}(F_\alpha)$ is equivalent to

$$\theta \cdot 2\left(\frac{1}{2\pi i} \ln x\right) = \left(\frac{1}{2\pi i} \ln x\right) \cdot \theta.$$
Thus $F$ considered as the co-$\mathcal{T}$-module $\mathcal{M}$ via Theorem 4.4.2 is graded:

$$\mathcal{M}_k = \{a \in \mathcal{M} \mid \theta(a) = 2^{-k/2}a\},$$

and the operators $\frac{1}{2\pi i} \ln x$ map $\mathcal{M}_k$ to $\mathcal{M}_{k+2}$.

**Definition 5.2.2.** Let co-$\mathcal{T}$-mod denote the category of graded co-$\mathcal{T}$-modules which are supported at the origin, and let co-$\mathcal{T}$-Mod$_{cf}$ denote the full subcategory of objects $\mathcal{M}$ for which $\mathcal{M}_\alpha$ is a co-finite graded $\mathcal{T}_\alpha$-module for all $\alpha$.

The following result is an immediate consequence of Theorem 4.4.2 and the above discussion.

**Theorem 5.2.3.** There is a natural equivalence of abelian categories

$$\text{LC}_\mathcal{F}(X) \simeq \text{co-$\mathcal{T}$-mod}_{cf},$$

and hence an equivalence

$$D^b(\text{LC}_\mathcal{F}(X)) \simeq D^b(\text{co-$\mathcal{T}$-mod}_{cf}).$$

Under these equivalences the twist operator $\langle n \rangle$ goes to the grading shift $\{n\}$.  

Note that these isomorphisms and the isomorphisms of Theorem 4.4.2 are compatible with the forgetful functors $\text{LC}_\mathcal{F}(X) \to \text{LC}_{cf}(X)$ and co-$\mathcal{T}$-mod$_{cf} \to$ co-$\mathcal{T}$-Mod$_{cf}$. This means that $D^b(\text{LC}_\mathcal{F}(X)) \to D^b(\text{LC}_{cf}(X))$ is a triangulated grading in the sense of §2.4.

### 5.3. Simple and injective mixed sheaves.

Since the category co-$\mathcal{T}$-mod$_{cf}$ has enough injectives, so does $\text{LC}_\mathcal{F}(X)$. It will be helpful to have a concrete description of simple and injective objects in this category.

First consider the case of a single $T$-orbit $O = O_\alpha$. We have an equivalence $\text{LC}_\mathcal{F}(O_\alpha) \simeq \text{co-$\mathcal{T}_\alpha$-mod}_{cf}$. Up to degree shifts there is one simple object of co-$\mathcal{T}_\alpha$-mod$_{cf}$, namely $(\mathcal{T}_\alpha/\mathfrak{m}_\alpha \mathcal{T}_\alpha)^*$. The corresponding object in $\text{LC}_\mathcal{F}(O_\alpha)$ is the constant local system $\mathbb{C}_{O_\alpha}$, with $\mathcal{F}$-structure given by $\theta_\alpha = 1$. We will denote this $\mathcal{F}$-local system by $\mathbb{C}_\alpha$.

The injective hull of $(\mathcal{T}_\alpha/\mathfrak{m}_\alpha \mathcal{T}_\alpha)^*$ is $\mathcal{T}_\alpha^*$; let $\Theta_\alpha$ denote the corresponding injective object in $\text{LC}_\mathcal{F}(O_\alpha)$. It has the following topological description. Let $q_\alpha : \tilde{O}_\alpha \to O_\alpha$ be the universal cover of $O_\alpha$. Then $\Theta_\alpha$ is the largest subsheaf of the local system $q_\alpha^* \mathbb{C}_{\tilde{O}_\alpha}$ on which all the monodromy operators $x \in \pi_1(O_\alpha)$ act (locally) unipotently.

Let $b \in O_\alpha$ be the distinguished point. Since there is a canonical identification $\pi_1(O_\alpha) \cong N_\alpha$, we can identify the stalk $(q_\alpha^* \mathbb{C}_{\tilde{O}_\alpha})_b$ with the space of functions $N_\alpha \to \mathbb{C}$, at the price of choosing a point $\tilde{b} \in \tilde{O}_\alpha$.
The action of $x \in \pi_1(O_\alpha)$ on this stalk is identified with the pushforward by the translation $\tau_x : n \mapsto n + x$ of the lattice $N_\alpha$.

The stalk $(\Theta_\alpha)_b$ is thus the space of all functions $N_\alpha \to \mathbb{C}$ which are annihilated by some power of $x - 1$ for every $x \in N_\alpha$. This is the space of polynomial functions $N_\alpha \to \mathbb{C}$. The logarithm of $x$ acts on these functions as the differential operator $\partial_x$.

We can now make $\Theta_\alpha$ into an $\mathcal{F}$-sheaf by letting the operator $\theta_\alpha$ be the pullback by $N_\alpha \to N_\alpha$, $x \mapsto 2x$. The corresponding grading is just our usual even grading on polynomial functions. The resulting $\mathcal{F}$-sheaf is the injective hull of $\mathbb{C}_\alpha$.

If $X$ has more than one orbit, then up to grading shift the injective objects of $\mathcal{T}$-mod$_{\mathcal{F}}$ are the sheaves $\mathcal{T}_\alpha$ for $\alpha \in \Sigma_X$, where the closure is taken in the fan topology, so $\overline{\alpha} = \{ \beta \in \Sigma_X \mid \alpha \leq \beta \}$.

The corresponding injective objects in $LC_\mathcal{F}(X)$ are (up to twists $\langle n \rangle$) the sheaves $j_\alpha \Theta_\alpha$, $\alpha \in \Sigma_X$ where $j_\alpha : O_\alpha \to X$ is the inclusion. $j_\alpha \Theta_\alpha$ is the injective hull of the extension by zero $j_\alpha \mathbb{C}_\alpha$ of $\mathbb{C}_\alpha$.

Note that the forgetful functor $F_{\text{cf}} : LC_\mathcal{F}(X) \to LC_{\text{cf}}(X)$ preserves injectivity, as does the inclusion $LC_{\text{cf}}(X) \subset LC(X)$. In particular, this implies that $D^b(LC_{\text{cf}}(X))$ is a full subcategory of $D^b(LC(X))$.

5.4. Extension and restriction functors. Let $j : Y \hookrightarrow X$ be the inclusion of a $T$-invariant locally closed subset of $X$. Since $LC_\mathcal{F}(X)$, $LC_\mathcal{F}(Y)$ have enough injectives, we can take derived functors of the left exact functors $j_*$ and $j^* \Gamma_Y$ to get functors $Rj_* : D^b(LC_\mathcal{F}(Y)) \to D^b(LC_\mathcal{F}(X))$ and $j^! : D^b(LC_\mathcal{F}(X)) \to D^b(LC_\mathcal{F}(Y))$. The restriction and extension by zero functors $j_!$ and $j^*$ are already exact, so they do not need to be derived.

In the same way we get derived functors between $D^b(LC(X))$ and $D^b(LC(Y))$. We will denote them by the same symbols $Rj_*, j_!, j^*, j^!$; context will make clear which functor is meant. These functors correspond to the ones on $\mathcal{F}$-sheaves: $Rj_* F_{\text{cf}} = F_{\text{cf}} Rj_*$, etc. Furthermore, by Corollary 5.1.3, these functors agree with the usual topological versions.

5.5. Perverse $t$-structure. These functors satisfy the usual adjunctions and distinguished triangles which allow one to define perverse $t$-structures; see [GM] or [B].

To do this, define $c(\alpha) = \dim N - \dim \alpha = \dim_\mathbb{C} O_\alpha$ for any $\alpha \in \Sigma$, and define full subcategories of $D = D^b(LC_\mathcal{F}(X))$ by

$$D^\leq_{\mathcal{F}}(X) = \{ F^* \in D \mid H^i(j_\alpha^* F^*) = 0 \text{ for } i > -c(\alpha) \},$$

$$D^\geq_{\mathcal{F}}(X) = \{ F^* \in D \mid H^i(j_\alpha^! F^*) = 0 \text{ for } i < -c(\alpha) \}.$$
The core \( P_\mathcal{F}(X) = D^{\leq 0}_\mathcal{F}(X) \cap D^{\geq 0}_\mathcal{F}(X) \) is an abelian category whose objects will be called perverse \( \mathcal{F} \)-sheaves.

The same formulas define a perverse \( t \)-structure \((D^{\leq 0}_\mathcal{F}(X), D^{\geq 0}_\mathcal{F}(X))\) on \( D^b(LC_{\mathcal{F}}(X)) \). The resulting core of perverse objects will be denoted \( P_{\mathcal{F}}(X) \). The forgetful functor \( F_{\mathcal{F}}: D^b(LC_{\mathcal{F}}(X)) \to D^b(LC_{\mathcal{F}}(X)) \) is \( t \)-exact, so it restricts to an exact functor \( P_\mathcal{F}(X) \to P_{\mathcal{F}}(X) \).

Simple objects in \( P_\mathcal{F}(X) \) and \( P_{\mathcal{F}}(X) \) are obtained as usual by applying the Deligne-Goresky-MacPherson middle extension \( j_{\alpha!} \) to a simple local system on an orbit \( O_\alpha \), shifted so as to be perverse. In particular,

\[
L^\bullet_\alpha := j_{\alpha!}^* \mathcal{C}_\alpha [c(\alpha)] \langle -c(\alpha) \rangle
\]

is simple in \( P_\mathcal{F}(X) \), and all simple objects are isomorphic to \( L^\bullet_\alpha \langle n \rangle \) for some \( \alpha \in \Sigma \), \( n \in \mathbb{Z} \) (we add the twist by \(-c(\alpha)\) so \( L^\bullet_\alpha \) will have weight 0 in the mixed structure we define below). Applying the forgetful functor \( F_{\mathcal{F}} \) to \( L^\bullet_\alpha \) gives the usual intersection cohomology sheaf \( IC^\bullet(\mathcal{O}_\alpha; \mathbb{C}) \); these give all the simple objects of \( P_{\mathcal{F}}(X) \).

Note that unlike the usual category of constructible perverse sheaves, \( P_\mathcal{F}(X) \) is not Artinian, since even if \( X \) has only one stratum, objects like \( \Theta_\alpha \) have infinite length. However, Hom are finite-dimensional in \( D^b(LC_{\mathcal{F}}(X)) \), and hence in \( P_\mathcal{F}(X) \).

**Proposition 5.5.1.** Let \( i: O \hookrightarrow X \) be the inclusion of an orbit. Then the functor \( R_i^*: D^b(LC_\square(O)) \to D^b(LC_\square(X)) \) is \( t \)-exact, \( \square = \mathcal{F}, cf. \).

**Proof.** Suppose that \( O = O_\alpha \). We have \( R_i^*(D^{\leq 0}_\mathcal{F}(O)) \subset D^{\leq 0}_\mathcal{F}(X) \) automatically, since \( j_\beta^* R_i^* = 0 \) for any \( \beta \neq \alpha \). Suppose that \( S^\bullet \in D^{\leq 0}(LC_{\mathcal{F}}(O)) \), and let \( M^\bullet \in D^b(T_\alpha\text{-mod}_{j^\beta}) \) be the corresponding complex of \( T_\alpha \)-modules; we have \( H^d(M^\bullet) = 0 \) for \( d > -c(\alpha) \). Using the description of injective \( \mathcal{F} \)-sheaves from Section 5.3, we see that the complex in \( D^b(T_\beta\text{-mod}_{j^\beta}) \) corresponding to \( j^\beta_\ast R_i^* S^\bullet \) is \( M^\bullet_\beta = R \text{hom}_{T_\beta}(T_\beta, M^\bullet) \). The functor \( R \text{hom} \) can be defined by deriving either the first or the second variable, so the fact that \( H^d(M^\bullet_\beta) = 0 \) for \( d > -c(\beta) \) follows from the fact that \( T_\beta \) has a resolution of length \( c(\alpha) - c(\beta) \) by free \( T_\alpha \)-modules. \( \square \)

Thus for any \( \alpha \in \Sigma \) we can define an object in \( P_\mathcal{F}(X) \) by

\[
\nabla_\alpha^\bullet := R j_{\alpha!*} \Theta_\alpha [c(\alpha)] \langle -c(\alpha) \rangle.
\]

Note that since \( \Theta_\alpha \) is an injective \( \mathcal{F} \)-local system, taking \( j_{\alpha!*} \) instead of \( R j_{\alpha!*} \) defines the same object. Under the isomorphism of Theorem 5.2.3, \( \nabla_\alpha^\bullet \) corresponds to \( T_\alpha^\bullet [c(\alpha)] \langle -c(\alpha) \rangle \).

This object will be important in the proof of the main properties of our Koszul duality functor in Section 7 below.
5.6. **Constructible \( \mathcal{F} \)-sheaves.** We can also consider the full subcategories \( D^b_{\text{ct}}(\text{LC}_T(X)) \subset D^b(\text{LC}_T(X)) \) and \( D^b_{\text{ct}}(\text{LC}_c(X)) \subset D^b(\text{LC}_c(X)) \) consisting of complexes \( S^\bullet \) whose cohomology sheaves have finite dimensional stalks on each orbit \( O_\alpha \). We call such objects “constructible”. Note that by Theorem 4.2.1, \( D^b_{\text{ct}}(\text{LC}_c(X)) \) is equivalent to a full subcategory of the usual constructible derived category of \( D^b_c(X) \): namely the category of objects whose cohomology sheaves are orbit-constructible (and have finite-dimensional stalks), with unipotent monodromy on each orbit.

The \( t \)-structures we defined in the previous section restrict to \( t \)-structures on these subcategories, giving abelian cores \( P_{\mathcal{F},c}(X) \subset P_{\mathcal{F}}(X) \) and \( P_{c,\mathcal{F}}(X) \subset P_{\mathcal{F}}(X) \).

**Proposition 5.6.1.** \( P_{\mathcal{F},c}(X) \) (resp. \( P_{c,\mathcal{F}}(X) \)) is the full subcategory of objects in \( P_{\mathcal{F}}(X) \) (resp. \( P_{\mathcal{F}}(X) \)) consisting of all objects of finite length. In particular, \( P_{c,\mathcal{F}}(X) \) is equivalent to the full subcategory of the category of constructible perverse sheaves on \( X \) consisting of objects all of whose simple constituents are of the form \( \text{IC}^\bullet(O_\alpha; \mathbb{C}) \), \( \alpha \in \Sigma \).

**Remark.** In [B] a triangulated category \( D(\Sigma) \) was defined for any fan \( \Sigma \) to model mixed \( T \)-constructible complexes on the toric variety \( X_\Sigma \) (in the case \( \Sigma \) is rational). It can be shown that \( D^b(\text{LC}_T(X)) \) is equivalent to \( D(\Sigma) \); under this equivalence the \( t \)-structure here is the same as the \( t \)-structure in [B].

5.7. **Mixed structure and pure \( \mathcal{F} \)-sheaves.** In the categories of mixed \( l \)-adic sheaves or mixed Hodge modules, simple perverse objects are pure. We need the following analog of this fact in our combinatorial setting. We call an object \( S^\bullet \in D^b(\text{LC}_T(X)) \) pure of weight 0 if for any orbit \( O_\beta \) and any \( i \in \mathbb{Z} \), the \( \mathcal{F} \)-local systems \( H^i(j_{\beta}^* S^\bullet) \) are direct sums of finitely many copies of \( \mathbb{C}_{\beta}(i) \). More generally we say \( \mathcal{F}^\bullet \) is pure of weight \( k \) if \( S^\bullet[-k] \) is pure of weight 0.

**Proposition 5.7.1.** If \( S^\bullet_1, S^\bullet_2 \in P_{\mathcal{F},c}(X) \) are pure of weights \( r_1 \) and \( r_2 \), respectively, then \( \text{Hom}_{D^b(\text{LC}_T(X))}(S^\bullet_1, S^\bullet_2[k]) = 0 \) unless \( r_2 = r_1 - k \). In particular, \( \text{Ext}^1_{P_{\mathcal{F}}(X)}(S^\bullet_1, S^\bullet_2) = 0 \) unless \( r_2 = r_1 - 1 \).

**Proof.** There is a spectral sequence with \( E_1 \) term

\[
E_1^{p,q} = \bigoplus_{\text{dim}_\alpha = p} \text{Hom}_{D^b(\text{LC}_T(O_\alpha))}(j_{\alpha}^* S^\bullet_1, j_{\alpha}^! S^\bullet_2),
\]

which converges to \( \text{Hom}_{D^b(\text{LC}_T(X))}(S^\bullet_1, S^\bullet_2[p+q]) \). Theorem 5.7.2 implies that \( E_1^{p,q} = 0 \) unless \( p + q = r_1 - r_2 \), which implies the result. \( \square \)

**Theorem 5.7.2.** The simple perverse sheaf \( L^\bullet_\alpha \) is pure of weight 0.
The proof will be given in §8.

Remark. Purity of IC sheaves enters our main argument twice, once via Theorem 5.7.2 and once in the next section, where equivariant IC sheaves are used. In that setting the purity follows from a proof of Karu [Ka], which makes sense even for non-rational fans. In fact Theorem 5.7.2 can also be stated and proved for non-rational fans, without reference to a toric variety. Although the category $LC_{T}(X)$ doesn’t make sense, co-$T$-mod$_{co}$ still does, and one can define a functor from the combinatorial equivariant sheaves ($A$-modules) considered in the next section to $D^{b}(co\text{-}T$-mod$_{co}$), which sends the equivariant IC sheaves to the $L_{\alpha}$’s. The required purity can then be deduced from Karu’s result.

We now define a mixed structure on $P_{F}(X)$. We have already defined the twist functor $\langle 1 \rangle$. What remains is to construct the filtration $W_{\bullet}$.

**Theorem 5.7.3.** There exists a unique functorial increasing filtration $W_{\bullet}$ on objects of $P_{F}(X)$ satisfying the following:

(a) For any $S^{\bullet} \in P_{F}(X)$ there exists $n \in \mathbb{Z}$ depending on $S^{\bullet}$ so that $W_{n}S^{\bullet} = 0$.

(b) For all $i$ and all $S^{\bullet} \in P_{F}(X)$, $\text{Gr}_{W}^{i}S^{\bullet} = W_{i}S^{\bullet}/W_{i-1}S^{\bullet}$ is isomorphic to a finite direct sum of objects $L_{\alpha}^{\bullet}(i)$ (thus $\text{Gr}_{W}^{i}S^{\bullet}$ is pure of weight $i$), and

(c) $(P_{F}(X), W_{\bullet}, \langle 1 \rangle)$ is a mixed category (2.3).

For finite length objects, i.e. objects in $P_{F,c}(X)$, this follows in a standard way from Theorem 5.7.2 and Proposition 5.7.1. We give the complete proof in §8.

**Corollary 5.7.4.** For any $\alpha \in \Sigma$, we have $W_{0}\nabla_{\alpha}^{\bullet} \cong L_{\alpha}^{\bullet}$. In particular, $\nabla_{\alpha}^{\bullet}$ has weights $\geq 0$.

**Proof.** If $m$ is the minimum weight in $\nabla_{\alpha}^{\bullet}$, then $W_{m}\nabla_{\alpha}^{\bullet}$ is a semisimple subobject of $\nabla_{\alpha}^{\bullet}$. But by adjunction $\text{Hom}(L_{\beta}^{\bullet}(k), \nabla_{\alpha}^{\bullet})$ is one-dimensional if $\alpha = \beta$ and $k = 0$, and vanishes otherwise. \qed

6. **Equivariant sheaves**

6.1. Let us very briefly recall the notion of the bounded, constructible equivariant derived category $D^{b}_{T}(X)$ [BeL] (note that in [BeL] this category was denoted $D^{b}_{T,c}(X)$). Let $E$ be a constructible space with a free $T$-action, and put $X_{T} = (X \times E)/T$. Then $E/T = BT$ is the classifying space for $T$ and $X_{T} \to BT$ is a locally trivial fibration with the fiber $X$. Similarly, a $T$-invariant subspace $U \subset X$ induces the corresponding subspace $U_{T} \subset X_{T}$. The triangulated category $D^{b}_{T}(X)$ can be
canonically identified as a full triangulated subcategory of the bounded derived categories of sheaves on $X_T$. For example, it can be defined as the triangulated envelope of the collection of all sheaves $\{C_{U_T}\}$ ($C_{U_T}$ is the extension by zero to $X_T$ of the constant sheaf $C$ on $U_T$), where $U \subset X$ is a star of an orbit. The following theorem is one of the main results in [L].

**Theorem 6.1.1.** There exists a natural equivalence of triangulated categories $$\epsilon: D^b_T(X) \to D_f(DG^-A_X).$$

The equivalence of categories in the above theorem is of “local nature” and actually comes from a continuous map of topological spaces. Namely there exists a natural map $$\mu: X_T \to X/T,$$

and the functor $\epsilon$ is essentially the derived direct image functor $R\mu_*$. In particular, if $U \subset X$ is the star of an orbit in $X$ and $\tau \in \Sigma$ is the cone corresponding to that orbit, then $\epsilon(C_{U_T}) = A_{[\tau]}$. Also $\epsilon$ takes the constant sheaf $C_{X_T}$ on $X_T$ to the sheaf $A$. (In [L] the sheaf $A$ is denoted by $H$).

This allows us to define the functor $F_T: D^b(A) \to D^b_T(X)$ described in the introduction (§1.2). It is obtained by composing the functor of Example 3.2.2 with $\epsilon^{-1}$.

6.2. **Combinatorial equivariant complexes.** Let $\Sigma$ be a fan in the vector space $V$, and let $A = A_\Sigma$ be the sheaf of conewise polynomial functions introduced in §3.6. For the remainder of this section all our $A$-modules will be assumed to be locally finite, so to simplify notation we put $D^b(A\text{-mod}) = D^b(A)$.

If $\Delta \subset \Sigma$ is a subfan or more generally a difference of subfans, the space of sections of a sheaf $M$ on $\Delta$ will be denoted $M(\Delta)$. If $\Lambda \subset \Delta$ is another subfan, we put $M(\Delta, \Lambda) = \ker(M(\Delta) \to M(\Lambda))$.

If $\sigma \in \Sigma$, recall that $[\sigma]$ is the fan of all faces of $\sigma$. It follows that $M([\sigma])$ is isomorphic to the stalk $M_\sigma$. Define $\partial\sigma = [\sigma] \setminus \{\sigma\}$. To simplify notation, we write $M(\sigma, \partial\sigma) = M([\sigma], \partial\sigma)$.

Note that a map of $A$-modules $M \to N$ is determined by the collection of induced maps $M(\sigma) \to N(\sigma)$ over all cones $\sigma \in \Sigma$. A sequence $E \to M \to N$ is exact if and only if $E(\sigma) \to M(\sigma) \to N(\sigma)$ is exact for all $\sigma \in \Sigma$.

**Definition 6.2.1.** Let $M$ be an $A$-module. If the stalk $M(\sigma)$ is a free $A_\sigma$-module for every $\sigma \in \Sigma$, we say that $M$ is locally free. If the restriction $M(\sigma) \to M(\partial\sigma)$ is surjective for every $\sigma \in \Sigma$, we say that
\( M \) is \emph{flabby}. If both conditions hold, we say \( M \) is \emph{combinatorially pure} ("pure" for short).

Note that flabbiness of \( M \) even implies that \( M(\Delta) \to M(\Lambda) \) is surjective for any subfans \( \Lambda \subset \Delta \subset \Sigma \).

Let \( \text{Pure}(A) \) denote the full subcategory of \( A\text{-mod} \) consisting of all pure sheaves, and let \( K^b(\text{Pure}(A)) \) be the category of bounded complexes of pure sheaves, with morphisms taken up to chain homotopy. Our main theorem is the following.

**Theorem 6.2.2.** The natural functor \( K^b(\text{Pure}(A)) \to D^b(A) \) is an equivalence of categories.

This sort of theorem is familiar when instead of pure objects we have complexes of projective or injective objects in an abelian category. The idea of the theorem is that a pure sheaf \( M \) is half injective and half projective: the locally free condition says that the stalk \( M(\sigma) \) is a projective \( A_\sigma \)-module, and flabbiness means that \( M \) is injective as a sheaf of vector spaces.

### 6.3. Flabby sheaves.

Let us make more precise in what sense flabby sheaves are injective. We first need the following definition.

**Definition 6.3.1.** An injective map \( M \to N \) is called \emph{strongly injective} if the inclusion of stalks \( M(\sigma) \to N(\sigma) \) splits for every \( \sigma \in \Sigma \).

**Proposition 6.3.2.** Suppose that \( I \) is a flabby \( A \)-module. If \( \eta: M \to N \) is a strongly injective map, and \( N \) is locally free, then the induced homomorphism \( \text{Hom}_A(N,I) \to \text{Hom}_A(M,I) \) is surjective.

**Proof.** Take a map \( \phi: M \to I \). We define a lift \( \psi: N \to I \) of \( \phi \) inductively on an increasing sequence of subfans. Defining \( \psi \) on the zero cone is trivial. So suppose \( \Delta \subset \Sigma \) is a subfan with more than one cone, that \( \tau \in \Delta \) is a maximal cone, and that \( \psi|_{\Delta(\tau)} \) has been defined already.

Since \( \eta \) is strongly injective, we can choose a splitting of \( A_\tau \)-modules \( \mathcal{N}(\tau) = \eta(M(\tau)) \oplus M \); since \( \mathcal{N}(\tau) \) is a free \( A_\tau \)-module, so are \( M(\tau) \) and \( M \). Define the restriction of \( \psi \) to \( \eta(M(\tau)) \) to be \( \phi \eta^{-1} \). To define \( \psi \) on \( M \), we need a map \( M \to \mathcal{I}(\tau) \) making the square

\[
\begin{array}{ccc}
M & \longrightarrow & \mathcal{I}(\tau) \\
\downarrow & & \downarrow \\
\mathcal{N}(\partial \tau) & \longrightarrow & \mathcal{I}(\partial \tau)
\end{array}
\]

commute. The right-hand vertical map is surjective, since \( \mathcal{I} \) is flabby, and since \( M \) is free, the required map exists. \( \square \)
Proposition 6.3.3. If \( \phi : M^\bullet \to N^\bullet \) is a quasi-isomorphism of pure complexes, then it has a homotopy inverse.

Lemma 6.3.4. If \( Z^\bullet \) and \( M^\bullet \) are bounded complexes of pure sheaves, and \( Z^\bullet \) is acyclic, then any map \( Z^\bullet \to M^\bullet \) is chain-homotopic to zero.

Proof. Proposition 6.3.2 allows us to copy the standard argument used when the objects \( Z_i \) are injective, provided that we know that each coker \( d_Z^i \to Z^{i+2} \) is strongly injective. In other words, we need to show that coker\( (d_{Z^i}(\sigma)) \to Z^{i+2}(\sigma) \) is a split injection for all \( \sigma \in \Sigma \). This follows from the fact that \( Z^i(\sigma) \) is an acyclic complex of free \( A_\sigma \)-modules. □

Proof of Proposition 6.3.3. Let \( Z^\bullet \) be the mapping cone of \( \phi \). Applying the lemma to the connecting map \( Z^\bullet \to M^\bullet[1] \) gives a chain homotopy whose components are maps \( h_i : Z_i \to M_i[1] = M_i \). But \( Z_i = N_i \oplus M_{i+1} \), so the first component of \( h_i \) gives a map \( \psi_i : N_i \to M_i \). The resulting map of complexes \( \psi : N^\bullet \to M^\bullet \) is a homotopy inverse of \( \phi \). □

6.4. Locally free sheaves. Locally free sheaves act like projective objects in a very similar way. We do not need the following result, but we include it to illustrate the parallels with the situation for flabby sheaves.

Definition 6.4.1. A surjective morphism \( M \to N \) between \( A_\Sigma \)-modules is called strongly surjective if the induced homomorphism \( M(\sigma, \partial \sigma) \to N(\sigma, \partial \sigma) \) is surjective for every \( \sigma \in \Sigma \).

Proposition 6.4.2. Let \( P \) be a locally free sheaf. If \( M \to N \) is strongly surjective and \( M \) is flabby, then the homomorphism \( \text{Hom}_A(P, M) \to \text{Hom}_A(P, N) \) is surjective.

The proof is left to the reader.

6.5. There are enough pure sheaves. To finish the proof of Theorem 6.2.2 we need to show that there are “enough” pure sheaves to represent any complex in \( D^b(A) \). This follows from a two-step resolution process, using the following result.

Proposition 6.5.1. Take any object \( M \in A\text{-mod}_f \).

(a) There exists a locally free \( A \)-module \( P \) and a strong surjection \( P \to M \). If \( M \) is flabby, then \( P \) can be chosen to be pure.

(b) There exists a flabby \( A \)-module \( I \) and a strong injection \( M \to I \). If \( M \) is locally free, then \( I \) can be chosen to be pure.
(c) If $\mathcal{M}$ is zero on the subfan $\partial \sigma$ for some $\sigma \in \Sigma$, then the maps in (a) and (b) can be chosen to be isomorphisms on all of $[\sigma]$.

Assuming this result, we can now prove Theorem 6.2.2. Consider any complex $\mathcal{M}^\bullet \in D^b(A)$. Part (a) of the proposition allows us to find a complex $\mathcal{P}^\bullet$ of locally free sheaves and a quasi-isomorphism $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{M}^\bullet$. Note that (c) implies that $\mathcal{P}^\bullet$ can be chosen to be a bounded complex.

Part (b) then implies that there is a complex $\mathcal{I}^\bullet$ of pure sheaves and a quasi-isomorphism $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet$. Note that here it is crucial that (b) gives strong injections: to construct $\mathcal{I}^j$ we apply (b) to $\mathcal{M} = \text{coker}(\mathcal{P}^j \to \mathcal{P}^j \oplus \mathcal{I}^j)$, which is locally free since $\mathcal{P}^j$ and $\mathcal{I}^j$ are and $\mathcal{P}^j \to \mathcal{I}^j$ is a strong injection. Using (c) again, we see that $\mathcal{I}^\bullet$ can be chosen to be a bounded complex.

Thus the functor $K^b(\text{Pure}(A)) \to D^b(A)$ is essentially surjective. The same two-step resolution process also shows it is fully faithful, by the following well-known result. Let $\mathcal{C}$ be an abelian category, $K(\mathcal{C})$ the homotopy category of complexes in $\mathcal{C}$.

Lemma 6.5.2. Let $K_1 \subset K_2$ be full triangulated subcategories of $K(\mathcal{C})$, and let $D_1$, $D_2$ be the corresponding derived categories. If either of the following conditions holds, then the functor $D_1 \to D_2$ is fully faithful.

1. For any quasi-isomorphism $X^\bullet \to Y^\bullet$ in $K_2$, with $X^\bullet$ in $K_1$, there exists a quasi-isomorphism $A^\bullet \to X^\bullet$ with $A^\bullet \in K_1$.
2. For any quasi-isomorphism $X^\bullet \to Y^\bullet$ in $K_2$, with $Y^\bullet$ in $K_1$, there exists a quasi-isomorphism $Y^\bullet \to B^\bullet$ with $B^\bullet \in K_1$.

Corollary 6.5.3. Suppose $\mathcal{M}$, $\mathcal{P}$ are $A$-modules, $\mathcal{M}$ is locally free, and $\mathcal{P}$ is pure. Then $\text{Hom}_{D^b(A)}(\mathcal{M}, \mathcal{P}[i]) = 0$ unless $i = 0$.

Proof. We can replace $\mathcal{M}$ by a quasi-isomorphic complex $\mathcal{I}^\bullet$ of pure sheaves, with $\mathcal{I}^j = 0$ for $j < 0$ and $\text{coker}(\partial^j) \to \mathcal{I}^{j+1}$ strongly injective for all $j \geq 0$. Now apply Proposition 6.3.2.

Proof of Proposition 6.5.1. To prove (a), we construct the object $\mathcal{P}$ and the map $\phi: \mathcal{P} \to \mathcal{M}$ simultaneously, by induction on subfans. For the base case when $\Delta = \{o\}$, we set $\mathcal{I}(o) = \mathcal{M}(o)$ and let $\phi|_o$ be the identity map.

Now suppose $\Delta$ is a fan with at least two cones, $\tau$ is a maximal cone, and the restrictions of $\mathcal{P}$ and $\phi$ to $\Delta \setminus \{\tau\}$ have already been defined. To define them on all of $\Delta$ it is enough to define them on $[\tau]$, since the resulting sheaves and maps can be glued.

This amounts to choosing a free $A_\tau$-module $\mathcal{P}(\tau)$ and homomorphisms $\partial_\tau: \mathcal{P}(\tau) \to \mathcal{P}(\partial \tau)$ and $\phi_\tau: \mathcal{P}(\tau) \to \mathcal{M}(\tau)$ so that $\phi_\tau$ and the
induced map \( \ker \partial_P \to \ker \partial_M \) are surjective and the square

\[
\begin{array}{c}
P(\tau) \xrightarrow{\phi_\tau} M(\tau) \\
\partial_P \downarrow \quad \downarrow \partial_M \\
P(\partial \tau) \xrightarrow{\phi|_{\partial \tau}} M(\partial \tau)
\end{array}
\]

commutes. To do this, find free \( A_\tau \)-modules \( M_1, M_2 \) and maps \( p_1 : M_1 \to P(\partial \tau) \) and \( p_2 : M_2 \to M(\tau) \) so that \( \text{Im} p_1 = (\phi|_{\partial \tau})^{-1}(\text{Im} \partial_M) \) and \( \text{Im} p_2 = \ker(\partial_M) \). Then let \( P(\sigma) = M_1 \oplus M_2 \), and \( \partial_P = p_1 \oplus 0 \). To define \( \phi_\tau \), let \( \phi_\tau|_{M_2} = p_2 \), and for each \( a \) in a basis of \( M_1 \), define \( \phi_\tau(a) \) to satisfy \( (\phi|_{\partial \tau})(p_1(a)) = \partial_M \phi_\tau(a) \).

To prove (b), we again proceed by induction. The base case is again trivial, and we are reduced to the problem of extending the sheaf \( I \) and morphism \( \phi : M \to I \) from \( \partial \tau \) to \( [\tau] \) as before. This, in turn, amounts to finding an \( A_\tau \)-module \( I(\tau) \), a surjective restriction homomorphism \( \partial_I : I(\tau) \to I(\partial \tau) \), and a split injection \( \phi_\tau : M(\tau) \to I(\tau) \), such that the square

\[
\begin{array}{c}
M(\tau) \xrightarrow{\phi_\tau} I(\tau) \\
\partial_M \downarrow \quad \downarrow \partial_I \\
M(\partial \tau) \xrightarrow{\phi|_{\partial \tau}} I(\partial \tau)
\end{array}
\]

commutes. This can be done by letting \( I(\tau) = I(\partial \tau) \oplus M(\tau) \), and letting \( \phi_\tau = (0, \text{id}_{M}) \) and \( \partial_I = \text{id}_{I(\partial \tau)} \oplus (\phi|_{\partial \tau} \circ \partial_M) \).

For the second statement of (b), we make a different choice at the inductive step: take a free \( A_\tau \)-module \( M \) and a surjective homomorphism \( p : M \to I(\partial \tau) \). We then define \( I(\tau) = M \oplus M(\tau) \) and \( \partial_I = p \oplus 0 \). Since \( M(\tau) \) is free by assumption, \( I(\tau) \) is free as well. The required map \( \phi_\tau \) now exists because \( \partial_I \) is surjective and \( M(\tau) \) is free.

Checking that these constructions satisfy (c) is easy. \( \square \)

6.6. **Indecomposable pure sheaves.** The notion of pure \( A \)-module was first used in [BrL, BBFK] to model direct sums of (shifted) intersection cohomology sheaves. The indecomposable pure objects are models of single intersection cohomology sheaves. We recall here their basic properties.

For a cone \( \sigma \in \Sigma \), let \( c(\sigma) \) denote the codimension of \( \sigma \) in the ambient vector space. For \( n \in \mathbb{Z} \), recall that \( \{n\} : D^b(A) \to D^b(A) \) is the functor which shifts the degree down by \( n \).
Theorem 6.6.1. For every $\sigma \in \Sigma$, there is an indecomposable pure $A$-module $L^{\sigma}$, unique up to a scalar isomorphism, for which (1) $L^{\sigma}(\tau) = 0$ unless $\sigma \prec \tau$, and (2) $L^{\sigma}(\sigma) = A_{\sigma}\{c(\sigma)\}$. These objects satisfy the following:

1. Every pure sheaf is isomorphic to a finite direct sum $\oplus L^{\sigma_{i}}\{n_{i}\}$ with $\sigma_{i} \in \Sigma$ and $n_{i} \in \mathbb{Z}$.
2. For all $\tau \in \Sigma \setminus \{\sigma\}$, $L^{\sigma}(\tau)$ is generated in degrees $< -c(\tau)$.
3. For all $\tau \in \Sigma$, $L^{\sigma}(\tau, \partial\tau)$ is a free $A_{\tau}$-module; it is generated in degrees $> -c(\tau)$, unless $\sigma = \tau$.

Remark. We put the generator of $L^{\sigma}(\sigma)$ in degree $-c(\sigma)$ (rather than degree 0 as in [BrL, BBFK]) so that the resulting object will be perverse, i.e., in the core of the $t$-structure which we define in the next section.

A proof of (1) can be found in [BrL, BBFK], while (2) and (3) follow from work of Karu [Ka].

Next we look more carefully at homomorphisms between pure sheaves. Because of Theorem 6.6.1, it is enough to look at the objects $L^{\sigma}\{n\}$.

Theorem 6.6.2. Take $\sigma, \tau \in \Sigma$, $n \in \mathbb{Z}$. Let $H = \text{Hom}_{A}(L^{\sigma}, L^{\tau}\{n\})$.

(a) If $n < 0$, then $H = 0$.
(b) If $n = 0$, then $H = 0$ unless $\sigma = \tau$, in which case $\dim_{R} H = 1$, with a basis given by the identity map.
(c) If $n = 1$ and $\sigma \prec \tau$, then restricting to $\tau$ gives an isomorphism $H \cong \text{Hom}_{A_{\tau}}(L^{\sigma}(\tau), L^{\tau}(\tau, \partial\tau)) = L^{\sigma}(\tau, c(\tau)_{c(\tau)_{-1}}$,
while if $\tau \prec \sigma$, restricting to $\sigma$ gives an isomorphism $H \cong \text{Hom}_{A_{\sigma}}(L^{\sigma}(\sigma), L^{\sigma}(\sigma, \partial\sigma)) = L^{\sigma}(\sigma, \partial\sigma)_{c(\sigma)+1}$.

If $\tau$ and $\sigma$ are not comparable, then $\text{Hom}_{A}(L^{\sigma}, L^{\tau}\{1\}) = 0$.

The proof is by induction, using Theorem 6.6.1 and the following lemma.

Lemma 6.6.3. Suppose $\mathcal{M}, \mathcal{N}$ are $A$-modules, $\mathcal{M}$ is locally free, and $\mathcal{N}$ is flabby. If $\sigma \in \Sigma$ is a maximal cone, then there is a short exact sequence

$$0 \to \text{Hom}_{A_{\sigma}}(\mathcal{M}(\sigma), \mathcal{N}(\sigma, \partial\sigma)) \to \text{Hom}_{A}(\mathcal{M}, \mathcal{N}) \to \text{Hom}_{A|_{\Sigma \setminus \{\sigma\}}}(\mathcal{M}|_{\Sigma \setminus \{\sigma\}}, \mathcal{N}|_{\Sigma \setminus \{\sigma\}}) \to 0.$$
Corollary 6.6.4. If \( M \to N \) is a morphism between two pure sheaves, each of which is a direct sum of various \( L^\sigma \) (without degree shifts), then the kernel and cokernel are both pure.

We will also need the following technical lemma.

Lemma 6.6.5. Suppose that \( M \) is a pure \( A \)-module. Then the graded endomorphism ring

\[
R = \text{End}_{A\text{-Mod}}(M) = \oplus_{n \in \mathbb{Z}} \text{Hom}_{A\text{-mod}}(M, M\{n\})
\]

is Noetherian.

Proof. There is a homomorphism from \( A_0 \) (polynomial functions on \( N \otimes \mathbb{C} \)) to \( R \) given by pointwise multiplication. The ring \( R \) is contained in \( \oplus_{\tau \in \Sigma} \text{End}_{A_0\text{-Mod}}(M(\tau)) \), which is a finitely generated \( A_0 \)-module. \( \square \)

6.7. Perverse \( t \)-structure. We define a \( t \)-structure on the triangulated category \( D^b(A) \), analogous to the usual one on the equivariant derived category \( D^b_T(X) \) whose core consists of equivariant perverse sheaves.

Let \( K_{\geq 0} \) (respectively \( K_{\leq 0} \)) be the full subcategory of \( K^b(\text{Pure}(A)) \) consisting of complexes which are quasi-isomorphic to a complex \( M^* \), where \( M^i = \bigoplus k L^\sigma_k \{n_k\} \), \( \sigma_k \in \Sigma \), \( n_k \leq i \) (respectively \( n_k \geq i \)).

Theorem 6.7.1. This defines a \( t \)-structure on \( K^b(\text{Pure}(A)) \). The heart \( P(\Sigma) = K_{\geq 0} \cap K_{\leq 0} \) is equivalent to the full subcategory of \( P(\Sigma) \) consisting of bounded complexes \( P^* \) so that for any \( i \) we have

(*)

\[
P^i \cong \bigoplus_{k=1}^l L^\sigma_k \{i\}
\]

with \( \sigma_1, \ldots, \sigma_l \in \Sigma \).

Since \( D^b(A) \) and \( K^b(\text{Pure}(A)) \) are equivalent categories, this defines a \( t \)-structure \( (D_{\leq 0}, D_{\geq 0}) \) on \( D^b(A) \) as well.

Remark. Note that all chain homotopies between complexes satisfying (*) automatically vanish, by Theorem 6.6.2. Thus if objects in \( P(\Sigma) \) are represented by such complexes, morphisms are just morphisms of complexes.

The resulting category of mixed equivariant perverse sheaves is similar to a construction of Vybornov \[\overline{V}\].

Proof. There are four things to check to show that \( (K_{\geq 0}, K_{\leq 0}) \) forms a \( t \)-structure. It is clear that \( K_{\geq 0} \subset K_{\geq 0}[1] \) and \( K_{\leq 0}[1] \subset K_{\leq 0} \). If
$\mathcal{M}^\bullet \in K^{\leq 0}$ and $\mathcal{N}^\bullet \in K^{\geq 1} = K^{\geq 0}[-1]$, we have $\text{Hom}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) = 0$, by Theorem 6.6.2(a). Given a distinguished triangle

$$E^\bullet \to \mathcal{M}^\bullet \to \mathcal{N}^\bullet \to [1],$$

where $E^\bullet$ and $\mathcal{N}^\bullet$ are both in $K^{\leq 0}$ (resp. $K^{\geq 0}$), then $\mathcal{M}^\bullet \in K^{\leq 0}$ (resp. $\mathcal{M}^\bullet \in K^{\geq 0}$), since the triangle comes from a short exact sequence $0 \to E^\bullet \to \mathcal{M}^\bullet \to \mathcal{N}^\bullet \to 0$, with $\widetilde{\mathcal{M}}^\bullet \cong \mathcal{M}^\bullet$.

Finally, we need to show that for any $M^\bullet \in K^{\leq 0}$ (resp. $M^\bullet \in K^{\geq 0}$), there exists a triangle $E^\bullet \to M^\bullet \to N^\bullet \to [1]$, where $E^\bullet$ and $N^\bullet$ are both in $K^{\leq 0}$ (resp. $K^{\geq 0}$), since the triangle comes from a short exact sequence $0 \to E^\bullet \to \widetilde{\mathcal{M}}^\bullet \to N^\bullet \to 0$, with $\widetilde{\mathcal{M}}^\bullet \cong M^\bullet$.

To do this, we write each $M^i$ as a direct sum $\bigoplus_{j < 0} M^i_j$, where $M^i_j$ is isomorphic to a sum of various $L^\sigma_j$. Then we can write the differential $d^i: M^i \to M^{i+1}$ as a sum $\sum_{j \in \mathbb{Z}, k \geq 0} d^i_{jk}$, where $d^i_{jk}: M^i_j \to M^{i+1}_{j+k}$.

We then let

$$E^i = \ker d^i_{i,0} \oplus \bigoplus_{i-j < 0} M^i_j,$$

which is pure by Corollary 6.6.4. It is a subcomplex of $M^\bullet$, and it clearly lies in $K^{\leq 0}$. Moreover, it is compatible with the decomposition $M^i = \bigoplus_j M^i_j$, so if we let $N^\bullet = M^\bullet/E^\bullet$, we have a decomposition $N^i = \bigoplus N^i_j$ compatible with the quotient map. Let $\bar{d}$ be the differential of $N^\bullet$; it can be decomposed $\bar{d} = \bigoplus \bar{d}_{jk}$ as before.

We then let $N^i = \widetilde{N}^i_i \oplus \text{Im}(\bar{d}_{i,0})$ defines a subcomplex $N^\bullet$ of $N^\bullet$ which is quasi-isomorphic to 0. Since $N^\bullet/N^\bullet$ is clearly in $K^{\geq 1}$, so is $N^\bullet$.

For the second statement, suppose $M^\bullet$ is in $K^{\leq 0} \cap K^{\geq 0}$. Then $\widetilde{M}^i = (\ker d^i_{i,0})/(\text{Im} d^{i-1}_{i-1,0})$ gives a quasi-isomorphic complex which satisfies (*) \hfill $\square$

6.8. $t$-exactness. In this section, we prove

**Theorem 6.8.1.** The functor $F_T: D^b(A) \to D^b_T(X_{\Sigma})$ defined in \[6.7\] is $t$-exact.

Here $D^b(A)$ has the $t$-structure just defined, and $D^b_T(X)$ has the perverse $t$-structure from [BeL].

In terms of the presentation we use of $D^b_T(X)$ as a full subcategory of $D^b(X_T)$, we define this second $t$-structure in terms of the usual perverse $t$-structure $(D^{\leq 0}(X), D^{\geq 0}(X))$ on $D^b(X_T)$. Since $X_T$ is a fiber bundle
over $BT$ with fiber $X$, we get an embedding of $X$ into $X_T$ by choosing a basepoint in $BT$. This gives rise to a “forgetful functor”

$$For: D^b_T(X) \to D^b(X)$$

which is simply restriction to $X$. Then our perverse $t$-structure is $(For^{-1} D^{\leq 0}(X), For^{-1} D^{\geq 0}(X))$.

For a face $\sigma \in \Sigma$ let $j_\sigma$ be the inclusion of $O_\sigma \hookrightarrow X$. Let $F_{T,\Sigma}: D^b(A) \to D^b_T(X)$ and $F_{T,\sigma}: D^b(A_\sigma\text{-mod}) \to D^b_T(O_\sigma)$ be the realization functors. We have restriction functors

$$j_\sigma^*, j_\sigma^! : D^b_T(X) \to D^b_T(O_\sigma)$$

which are simply restriction and corestriction to $O_{\sigma,T}$ (note that although the space $X_T$ is not locally compact, the corestriction can be defined using the derived “restriction with supports” functor $\mathsf{R}\Gamma_{O_\sigma,T}$; all the usual adjunction and base change properties still apply).

Theorem 6.8.1 follows from Theorem 6.6.1 and the following result, which describes the stalk and costalk functors in terms of $A$-modules.

The proof will be given in §8.

**Theorem 6.8.2.** If $M^\bullet \in K^b(\text{Pure}(A))$, there are natural isomorphisms in $D^b_T(O_\sigma)$:

(a) $j_\sigma^* F_{T,\Sigma} M^\bullet \simeq F_{T,\sigma}(M^\bullet(\sigma))$,

(b) $j_\sigma^! F_{T,\Sigma} M^\bullet \simeq F_{T,\sigma}(M^\bullet(\sigma, \partial \sigma))$.

Let $i_x$ be the inclusion of a point $x$ into $O_{\sigma,T}$. Then there are natural isomorphisms in $D^b(C\text{-mod}) = D^b(pt)$

(c) $i_x^* j_\sigma^* F_{T,\Sigma} M^\bullet \simeq C \otimes_{A_\sigma} \nu M^\bullet(\sigma)$,

(d) $i_x^* j_\sigma^! F_{T,\Sigma} M^\bullet \simeq C \otimes_{A_\sigma} \nu M^\bullet(\sigma, \partial \sigma)$.

Here $\nu$ is the functor $D(A_\sigma\text{-mod}) \to D(DG\text{-}A_\sigma)$ of Example 3.2.2.

We can make a similar statement for general complexes $M^\bullet \in D^b(A)$ if we replace the functors on the right hand sides of (a), (b), (c), and (d) with the appropriate derived functors: replace $\otimes_{A_\sigma}$ by $\otimes_{A_\sigma}$ and $-(\sigma, \partial \sigma) = \Gamma_\sigma(-)|_{\{\sigma\}}$ by $R\Gamma_\sigma(-)|_{\{\sigma\}}$.

6.9. **Mixed structure.** For any $n \in \mathbb{Z}$, define an automorphism of $D^b(A)$ by $\langle n \rangle = [n]\{-n\}$. It is obviously $t$-exact, so it induces an automorphism of $P(\Sigma)$. Define a functorial filtration on objects of $P(\Sigma)$ by $W_j^p = \oplus_{i \geq -j} P^i$, assuming $P^\bullet$ is a complex satisfying condition $(\ast)$ of Theorem 6.7.1. It is easy to see this defines a mixed structure on $P(\Sigma)$. We will show in the next section that the functor $F_T: P(\Sigma) \to P_T(X_\Sigma)$ is a grading on $P_T(X_\Sigma)$ in the sense of §2.3.1.
7. The toric Koszul functor

7.1. Let $X = X_\sigma$ be a normal affine $\mathcal{T}$-toric variety with a single fixed point defined by a cone $\sigma \subset N_\mathcal{T} \otimes \mathbb{R}$, with $\dim \sigma = \text{rank } N_\mathcal{T} = n$. Let $X^\vee$ be the dual $\mathcal{T}^\vee$-toric variety defined by the dual cone $\sigma^\vee \subset N_{\mathcal{T}^\vee} \otimes \mathbb{R}$. Put $\Sigma = \Sigma_X = [\sigma], \Sigma^\vee = \Sigma_{X^\vee} = [\sigma^\vee], \mathcal{A} = \mathcal{A}_\Sigma$, and $\mathcal{T} = \mathcal{T}_{\Sigma^\vee}$. Recall the identification of ringed quivers

\[(\Sigma, \mathcal{A}) = ((\Sigma^\vee)^\circ, \mathcal{T})\]

from §3.9.

In this section we define our Koszul equivalence $K : D^b(\mathcal{A}\text{-mod}_f) \to D^b(\text{LC}_F(X^\vee))$.

It will be a composition of three equivalences:

\[D^b(\mathcal{A}\text{-mod}_f) \overset{\kappa}{\to} D^b(\text{co-}\mathcal{A}\text{-mod}_{cf}) \to D^b(\text{LC}_F(X^\vee)) \overset{(-n)}{\to} D^b(\text{LC}_F(X^\vee));\]

The last functor $(-n)$ is the twist defined in §5.2. The middle functor is the equivalence of Theorem 5.2.3 combined with (7.1.1). The functor $\kappa$ is a combinatorial form of Koszul duality which makes sense for any fan, rational or not. We define it in the next section.

Our main results can be summarized as follows.

**Theorem 7.1.2.** $K$ is a Koszul equivalence in the sense of Definition 2.2.1. Here we use the $t$-structures and mixed structure on $D^b(\mathcal{A}\text{-mod}_f)$ and $D^b(\text{LC}_F(X^\vee))$ defined in §5.2.2 and §5.9, and the ring $R$, resp. $R^\vee$, is the opposed ring of the graded endomorphism ring of a mixed projective generator of $P(\mathcal{A}_\Sigma)$ (resp. a mixed injective generator of $P_{\mathcal{T}}(X^\vee)$).

7.2. Combinatorial Koszul functor. Fix a fan $\Sigma$ (rational or not) in $\mathbb{R}^n$ with the corresponding “structure sheaf” $\mathcal{A} = \mathcal{A}_\Sigma$. We define the functor $\kappa : D^b(\mathcal{A}\text{-mod}_f) \to D^b(\text{co-}\mathcal{A}\text{-mod}_{cf})$ as follows.

**Definition 7.2.1.** Fix an abelian category $\mathcal{C}$.

a) A $\Sigma$-diagram in $\mathcal{C}$ is a collection $\{\mathcal{M}_\tau\}_{\tau \in \Sigma}$ of objects of $\mathcal{C}$ together with with morphisms $p_{\tau \xi} : \mathcal{M}_\tau \to \mathcal{M}_\xi$ for $\tau \geq \xi$, satisfying $p_{\rho \tau} p_{\tau \xi} = p_{\tau \xi}$ whenever $\tau \geq \rho \geq \xi$.

b) Fix an orientation of each cone in $\Sigma$. Then every $\Sigma$-diagram $\mathcal{M} = \{\mathcal{M}_\tau\}$ gives rise to the corresponding cellular complex in $\mathcal{C}$:

\[C^\bullet(\mathcal{M}) = \bigoplus_{\dim(\tau) = n} \mathcal{M}_\tau \to \bigoplus_{\dim(\xi) = n-1} \mathcal{M}_\xi \to \ldots\]

where the terms $\mathcal{M}_\rho$ appear in degree $-\dim \rho$, and the differential is the sum of the maps $p_{\tau \xi}$ with $\pm$ sign depending on whether the orientations of $\tau$ and $\xi$ agree or not.
Lemma 7.2.2. Let $M = \{M_\tau\}$ be a constant $\Sigma$-diagram supported between cones $\eta$ and $\xi$ in $\Sigma$. That is $M_\tau = M$ for a fixed $M$ if $\eta \geq \tau \geq \xi$, and $M_\tau = 0$ otherwise; for $\eta \geq \tau_1 \geq \tau_2 \geq \xi$ the maps $p_{\tau_1 \tau_2}$ are the identity. If $\eta \neq \xi$, then the cellular complex $C^\bullet(M)$ is acyclic.

Proof. The complex $C^\bullet(M)$ is isomorphic to an augmented cellular chain complex of a closed ball of dimension $\dim(\eta) - \dim(\xi) - 1$. □

Recall that the sheaves $A^*_\{\tau\}$ are injective objects of co-$A$-Mod for every $\tau \in \Sigma$. Consider the $\Sigma$-diagram $K = \{K_\tau\}$ in co-$A$-Mod, where $K_\tau = A^*_{\{\tau\}}$ and the maps $p_{\tau\xi}$ are the projections. This diagram $K$ defines a covariant functor $\kappa: D^b(\text{A-mod}_f) \to D^b(\text{co-}A\text{-mod}_{cf})$ in the following way. If $N \in \text{A-mod}_f$ is locally free, the collection $K \otimes N = \{K_\tau \otimes A^*_{\{\tau\}} N\}$ is a $\Sigma$-diagram of co-$A$-modules. Thus its cellular complex $C^\bullet(K \otimes N)$ is a complex of co-$A$-modules. By Theorem 6.2.2, every object in $D^b(\text{A-mod}_f)$ is quasi-isomorphic to a complex of locally free (in fact, pure) $A$-modules, so we obtain a derived functor

$$\kappa(\cdot) = C^\bullet(K \otimes \cdot): D^b(\text{A-mod}_f) \to D^b(\text{co-}A\text{-mod}_{cf}).$$

We call it the combinatorial Koszul functor.

Lemma 7.2.3. For any cone $\tau \in \Sigma$, there are isomorphisms $\kappa(A_{\{\tau\}}) \cong A^*_{\{\tau\}}[\dim \tau]$ and $\kappa(A_{\{\tau\}}) \cong A^*_{\{\tau\}}[\dim \tau]$.

The first isomorphism is obvious; the second follows from Lemma 7.2.2.

Proposition 7.2.4. The functor $\kappa$ is an equivalence of triangulated categories.

Proof. The category $D^b(\text{A-mod}_f)$ is the triangulated envelope of either all objects of the form $A_{\{\tau\}}[k]$ or all objects of the form $A_\{\tau\}[k]$, in either case taken over all $\tau \in \Sigma$ and $k \in \mathbb{Z}$. Similarly, $D^b(\text{co-}A_{cf}\text{-mod})$ is the triangulated envelope of all objects of the form $A^*_{\{\tau\}}[k]$ or all objects of the form $A^*_{\{\tau\}}[k]$.

So it suffices to show that for any $k, l, \tau, \xi$ the functor $\kappa$ induces an isomorphism

$$\kappa: \text{Hom}_{A_{\{\tau\}}}(A^*_{\{\tau\}}[k], A_{\{\xi\}}[l]) \to \text{Hom}_{\text{co-}A_{cf}\text{-mod}}(A^*_{\{\tau\}}[k], A^*_{\{\xi\}}[l]).$$

Both sides are equal to the $l - k$ graded part of $A_{\tau}$ if $\tau = \xi$ and vanish otherwise. □
Thus $K$ satisfies property (1) of the definition of a Koszul equivalence (Definition 2.2.1). Property (2) follows immediately from the definition of the twist functors $\langle n \rangle$ in the categories $D^b(A)$ and $D^b(LC_F(X^\vee))$. Showing that $K$ sends simples to injectives and indecomposable projectives to simples (properties (3) and (4)) will take up the remainder of the paper.

**Remark.** We can think of $\kappa$ as convolution with kernel $K$. This is more enlightening if one considers $K$ as a sheaf on $\Sigma \times (\Sigma^\vee)^\circ$ by the natural identification $\Sigma = (\Sigma^\vee)^\circ$. The support of $K$ is then the “combinatorial conormal variety”

$$\Lambda = \{(\tau, \alpha) \in \Sigma \times (\Sigma^\vee)^\circ \mid \tau^\perp \leq \alpha\}.$$ 

If $p_1: \Lambda \to \Sigma$, $p_2: \Lambda \to (\Sigma^\vee)^\circ$ are the projections, then $K = p_2^{-1}T^*$, using the identification (7.1.1). Note that $K$ has a natural action of $p_1^{-1}A$ which commutes with the action of $p_2^{-1}T$. In fact, $\Lambda$ is the largest subset of $\Sigma \times (\Sigma^\vee)^\circ$ for which this is true.

7.3. **Proof of Theorem 7.1.2, part I:** $K$ (simple) is injective. Let us examine what the functor $K$ does to indecomposable pure sheaves. For a face $\alpha \in [\sigma^\vee]$, define $I^\bullet_\alpha = K(L\alpha^\perp)$. It is perverse, as follows from the following more general statement. Recall the objects $\nabla^\bullet_\alpha \in P_F(X^\vee)$ from §5.5.

**Proposition 7.3.1.** If $M^\bullet \in D^b(A\text{-mod}_f)$ is given by placing a locally free $A$-module in degree 0, then $K(M^\bullet)$ is perverse, and has a filtration whose graded pieces are objects $\nabla^\bullet_\alpha \langle k \rangle$, $\alpha \in \Sigma^\vee$, $k \in \mathbb{Z}$.

**Proof.** A locally free $A$-module has a filtration whose subquotients are sheaves $A_{\{\tau\}}\{k\}$, $\tau \in \Sigma$, $k \in \mathbb{Z}$. By Lemma 7.2.3, we have $K(A_{\{\tau\}}) \cong \nabla^\bullet_{\tau^\perp}(\dim \tau - n)$. The result follows. \hfill \Box

**Theorem 7.3.2.** $I^\bullet_\alpha$ is an injective object in $P_F(X)$, and its image $F_{cf}(I^\bullet_\alpha)$ is injective in $P_{cf}(X)$.

With respect to the mixed structure defined in §5.7.2, we have $W_0I^\bullet_\alpha \cong L^\bullet_\alpha$.

**Proof.** To show that $I^\bullet_\alpha$ is injective, we will show that the following statement holds for any $S^\bullet \in P_F(X^\vee)$:

\[ (\ast) \quad \text{Hom}_{D^b(LC_F(X^\vee))}(S^\bullet, I^\bullet_\alpha[k]) = 0 \text{ for all } k > 0. \]

First note that if

\[ 0 \to S^\bullet_1 \to S^\bullet_2 \to S^\bullet_3 \to 0 \]

is a short exact sequence in $P_F(X^\vee)$ and $(\ast)$ holds for $S^\bullet_1$ and $S^\bullet_3$ or for $S^\bullet_2$ and $S^\bullet_3$, then it also holds for all three objects.
Note that (*) holds for $S^\bullet = \nabla^\bullet_{\beta}(k)$ for any $\beta \in [\sigma^\vee]$ and $k \in \mathbb{Z}$, by applying $K$ to Corollary 5.5.3. Thus (*) holds for any object of the form $S^\bullet = \mathbf{R}j_{\beta*}E_\beta[\dim O_\beta]$, where $E_\beta$ is any object in $LC_F(O_\beta)$, since $\mathbf{R}j_{\beta*}$ is $t$-exact (Proposition 5.5.1), and $E_\beta$ can be resolved by a finite complex of injective $F$-local systems, i.e., by direct sums of copies of objects $\Theta_\beta(k), k \in \mathbb{Z}$.

Now we prove (*) for general $S^\bullet$, by induction on the number of orbits in the support. If $O_\beta$ is an open orbit contained in $\text{Supp } S^\bullet$, then $S^\bullet|_{O_\beta}$ is a $F$-local system placed in degree $-\dim O_\beta$. Consider the adjunction map $\phi: S^\bullet \to \mathbf{R}j_{\beta*}(S^\bullet|_{O_\beta})$, and note that (*) holds for the target of $\phi$ by the previous paragraph. If $\text{Supp } S^\bullet$ consists of the unique closed orbit $O_{\sigma^\vee}$, so $\beta = \sigma^\vee$, then $\phi$ is an isomorphism, and we are done. Otherwise note that (*) holds by induction for the kernel and cokernel of $\phi$, since they have strictly smaller support, and thus it holds for $S^\bullet$.

Thus $I^\bullet_\alpha$ is injective. The same argument shows that $F_{cf}(I^\bullet_\alpha)$ is injective in $P_{cj}(X)$.

Apply Proposition 7.3.1 to obtain a filtration

$$M^\bullet_0 \subset \cdots \subset M^\bullet_i = I^\bullet_\alpha$$

with $M^\bullet_0 = \nabla^\bullet_{\alpha}$, and where the $M^\bullet_i/M^\bullet_{i-1}$ for $i > 0$ are sums of objects $\nabla^\bullet_{\beta}(k)$ with $\beta \in [\alpha] \setminus \{\alpha\}$ and $k \in \mathbb{Z}$. In fact, using Lemma 7.2.3 and property (2) of Theorem 6.6.1, we see that only twists $k > 0$ can occur. The remaining statements of the theorem follow using Corollary 5.7.4.

Let $I^\bullet = \oplus_{\alpha \in [\sigma^\vee]} I^\bullet_\alpha$, so $I^\bullet = K(L)$, where $L = \oplus_{\tau \in [\sigma]} L^\tau$.

**Proposition 7.3.3.** $I^\bullet$ is a mixed injective generator (8.2.3) of $P_E(X^\vee)$; $F_{cf}(I^\bullet)$ is an injective generator of $P_{cj}(X^\vee)$.

**Proof.** Let $Inj$ be the category of all finite direct sums of objects of the form $I^\bullet_\alpha(k), \alpha \in [\sigma^\vee], k \in \mathbb{Z}$. We need to show that any object of $P_E(X^\vee)$ embeds into an object of $Inj$.

We showed in the previous proof that $\nabla^\bullet_{\alpha}(k) = j_{\alpha*}\Theta_\alpha[\dim O_\alpha](k)$ embeds into $I^\bullet_\alpha(k)$. Let $E_\alpha$ be a $F$-local system on $O_\alpha$. It embeds into a finite direct sum of injective $F$-local systems $\Theta_\alpha(n)$, so $j_{\alpha*}E_\alpha[\dim O_\alpha]$ embeds into a finite sum of $I^\bullet_\alpha(k)$.

We prove that an arbitrary $S^\bullet \in P_E(X^\vee)$ embeds into an object of $Inj$ by induction on the number of orbits in $\text{Supp } S^\bullet$. The case when $\text{Supp } S^\bullet$ is a single orbit follows from the previous paragraph. Otherwise, take an open orbit $O_\beta$ in $\text{Supp } S^\bullet$, and let $\phi: S^\bullet \to \mathbf{R}j_{\beta*}(S^\bullet|_{O_\beta})$ be the adjunction map. We have seen that the target of $\phi$ embeds
into an object $I_1^• \in \text{Inj}$, so the image of $\phi$ does as well. Since $\ker \phi$ has support strictly smaller than $S^•$, by induction it embeds into an object $I_2^• \in \text{Inj}$. Since $I_2^•$ is injective, this embedding extends to a map $S^• \to I_2^•$. Thus we get an embedding of $S^•$ into $I_1^• \oplus I_2^• \in \text{Inj}$.

The argument for $P_{cf}(X^\vee)$ is essentially the same. \qed

Now define

$$R = \text{end}(I^•)^{opp} \cong \oplus_{n \in \mathbb{Z}} \text{Hom}_{A\text{-mod}}(L, L\{n\})^{opp}. $$

By Lemma 6.6.5, $R$ is (left and right) Noetherian. Then applying Propositions 2.3.1 and 2.3.2, we conclude that $P_{cf}(X^\vee)$ is equivalent to $R\text{-mod}_{cf}$, $P_{cf}(X^\vee)$ is equivalent to $R\text{-Mod}_{cf}$, and $F_{cf}$ is the functor of forgetting the grading.

**Corollary 7.3.4.** There are equivalences of triangulated categories:

$$D^b(LC_{\mathcal{F}}(X^\vee)) \simeq D^b(P_{\mathcal{F}}(X^\vee))$$

and

$$D^b(LC_{\mathcal{F}_{cf}}(X^\vee)) \simeq D^b(P_{\mathcal{F}_{cf}}(X^\vee)).$$

The argument for the two equivalences is the same, so we concentrate on the first one. Both $D^b(LC_{\mathcal{F}}(X^\vee))$ and $D^b(P_{\mathcal{F}}(X^\vee))$ are generated by the injectives in $P_{\mathcal{F}}(X^\vee)$; note that any complex has a bounded injective resolution, since any object of $D^b(A)$ can be represented by a bounded complex of pure sheaves. We thus need to show that for any injectives $I_1^•, I_2^• \in P_{\mathcal{F}}(X^\vee)$ and any $d \in \mathbb{Z}$ there is an isomorphism

$$\text{Ext}^d_{P_{\mathcal{F}}(X^\vee)}(I_1^•, I_2^•) \sim \text{Hom}_{D^b(LC_{\mathcal{F}}(X^\vee))}(I_1^•, I_2^•[d]).$$

Both sides are automatically isomorphic for $d \leq 0$, while for $d > 0$ the left side vanishes by the injectivity of $I_2^•$. The vanishing of the right side is just (*) from the proof of Theorem 7.3.2.

**7.4. Proof of Theorem 7.1.2, part II: $K^{-1}(\text{simple})$ is projective.**

For any $\tau \in [\sigma]$, let $P_\tau^• = K^{-1}(L_{\tau^\perp})$. Analogously to Theorem 7.3.2, we have

**Theorem 7.4.1.** $P_\tau^•$ lies in the core of the perverse $t$-structure on $D^b(A)$ defined in §6.4. In the abelian category $P(A)$, it is the projective cover of $L^\tau$.

**Proof.** Let $L^• = L_{\tau^\perp}$. Consider an injective resolution of $L^•$ (as remarked before, it can be chosen to be bounded):

$$L^• \sim (J_0^• \to J_1^• \to \cdots \to J_k^•).$$

Taking $K^{-1}$ gives a complex $\mathcal{M}_0 \to \mathcal{M}_1 \to \cdots \to \mathcal{M}_k$ of pure $A$-modules which represents the object $P_\tau^•$.

This complex will be perverse if $\mathcal{M}_j[-j]$ is perverse for $j = 0, \ldots, k$, or in other words, if each $J_l^•$ is a direct sum of objects $I_\alpha^•(l), \alpha \in \mathbb{Z}.$
The existence such a resolution follows from Proposition 5.7.1 and Corollary 7.3.4.

Since objects of $P(A)$ have finite length, to show that $P^\tau$ is the projective cover of $L^\tau$ it will be enough to show that for any $\rho \in \Sigma$, $k, l \in \mathbb{Z}$ we have $\text{Hom}_{D^b(A)}(P^\tau, L^\rho[k])$ is one-dimensional if $k = l = 0$ and $\rho = \tau$, and vanishes otherwise. By applying $K$, this follows from Theorem 7.3.2. □

Define $P^\tau := \bigoplus_{\tau \in [\sigma]} P^\tau$, and let $R^\vee = \text{end}_{P(A)}(P^\bullet)^{op}$.

Corollary 7.4.2. $P^\bullet$ is a graded projective generator of $P(A)$; $F_T(P^\bullet)$ is a projective generator of $P_T(X)$.

There are equivalences of abelian categories $P(A) \simeq R^\vee$-mod and $P_T(X) \simeq R^\vee$-Mod; with respect to these equivalences $F_T$ is the functor of forgetting the grading.

Corollary 7.4.3. There are equivalences of triangulated categories: $D^b(A) \simeq D^b(P(A))$ and $D^b_T(X) \simeq D^b(P_T(X))$.

The proofs are the same as in the previous section; note that since objects of $P(A)$ have finite length the ring $R^\vee$ is automatically Noetherian.

8. Some proofs

8.1. Proof of Theorem 5.7.2. The functor $[k](-k)$ on $D^b(LC_{\mathcal{F}}(X))$ preserves the property of being pure of weight 0, so we can instead prove that

$$S^\bullet := L_\alpha^\bullet[-c(\alpha)]\langle c(\alpha) \rangle = j_{\alpha!}^* \mathcal{C}_\alpha$$

is pure of weight 0. The support of $S^\bullet$ is $O_\alpha$, which is itself a toric variety (for a smaller torus). Thus we can assume that $\alpha = 0$ is the zero cone and the support of $S^\bullet$ is all of $X$.

Let $S^\bullet = F_{j_f}S^\bullet$; it is an intersection cohomology sheaf shifted so that the restriction to the open orbit $O_o$ is a local system in degree 0. The $\mathcal{F}$-structure on $S^\bullet$ defines an isomorphism $\theta: \mathcal{F}^{-1}S^\bullet \to S^\bullet$. It induces an action on the stalk of the cohomology sheaves $H^i(j_{\beta!}^*S^\bullet)$ and $H^i(j_{\beta!}^*S^\bullet)$; we need to show this action is multiplication by $2i/2$.

Note that if $O_\beta$ has positive dimension, there is an $\mathcal{F}$-stable normal slice to $O_\beta$ at a point of $(O_\beta)^{\mathcal{F}}$ which is itself an affine toric variety. By restricting to this slice we can restrict to the case when $O_\beta = \{b\}$ is a single point.

Note that $\mathcal{F}^{-1}S^\bullet \simeq S^\bullet$ (see [BM]), and all automorphisms of $S^\bullet$ are multiplication by scalars. Therefore $\theta$ is uniquely determined by its
action on the stalk at an $F$-fixed point of the open orbit $O_o$, where it acts as the identity.

Let $\pi: \tilde{X} \to X$ be a toric resolution of singularities, and let $\tilde{F}$ be our geometric Frobenius map on $\tilde{X}$; we have $\tilde{F}\pi = \pi F$.

Since $\tilde{F}^*C_{\tilde{X}} \cong C_{\tilde{X}}$, we can put a $\tilde{F}$-structure on the constant sheaf $C_{\tilde{X}}$ by letting $\tilde{\theta}: \tilde{F}^*C_{\tilde{X}} \to C_{\tilde{X}}$ act as the identity on the stalk at a point of $(O_o)^{\tilde{F}}$. By adjunction $\tilde{\theta}$ induces a map $C_{\tilde{X}} \to R\tilde{F}_*C_{\tilde{X}}$, and applying $R\pi_*$ and adjunction again gives $\theta': F^*R\pi_*C_{\tilde{X}} \to R\pi_*C_{\tilde{X}}$.

By the decomposition theorem [BBD], $S^*$ is a direct summand of $R\pi_*C_{\tilde{X}}$ and of $F^*R\pi_*C_{\tilde{X}}$. Composing $\theta'$ with the inclusion and projection gives a map $S^* \to S^*$; it is easy to see that it agrees with $\theta$ on the open orbit, so it must equal $\theta$ on all of $X$.

The cohomology groups of $j_\beta^*R\pi_*C_{\tilde{X}}$ and $j_\beta^*R\pi_*C_{\tilde{X}}$ are $H^*(\pi^{-1}(b))$ and $H^*(\tilde{X}, \tilde{X} \setminus \pi^{-1}(b))$, respectively, and the action of $\theta'$ is the action of the pullback $\tilde{V}^*$. Thus we have reduced the proof of the theorem to showing that this action is multiplication by $2^{i/2}$ on the cohomology in degree $i$.

Here is one way to see this: $\tilde{X}$ has a completion to a smooth complete toric variety $Y$. There is a homomorphism $\mathbb{C}^* \to T$ so that the induced action of $\mathbb{C}^*$ on $X$ is “attractive”: $\lim_{t \to 0} t \cdot x = b$ for all $x \in X$, and the induced action on $Y$ has isolated fixed points. Then by Bialynicki-Birula $\pi^{-1}(b)$ has a decomposition into $\bigcup x C_x$ into affine cells, so $H^*(\pi^{-1}(b)) \cong \oplus_r H^r_t(C_x)$. The cells are $T$- and $F$-invariant, and $\tilde{F}$ acts on each $k$-dimensional cell as the map $C^k \to C^k$, $(x_1, \ldots, x_k) \mapsto (x_1^2, \ldots, x_k^2)$. The result for $H^*(\pi^{-1}(b))$ follows immediately.

For $H^*(\tilde{X}, \tilde{X} \setminus \pi^{-1}(b))$, we use the Bialynicki-Birula cells for the opposite character $\mathbb{C}^* \to T$. Then $\tilde{X}$ is an open union of these cells which deformation retracts onto $\pi^{-1}(b)$ by our action. Therefore we have $H^*(\tilde{X}, \tilde{X} \setminus \pi^{-1}(b)) \cong H^*_t(\tilde{X})$, and can use the argument of the previous paragraph.

8.2. Proof of Theorem 57.53 We begin by defining the filtration $W_\cdot S^*$ when $S^* \in P_{F, c}(X)$, i.e. when $S^*$ has finite length. We proceed by induction on the length of $S^*$. If $S^*$ has length 1, it is simple, say of weight $m$, and we can let $W_k S^* = 0$ if $k < m$, $W_k S^* = S^*$ for $k \geq m$.

Otherwise suppose the filtration has already been defined for objects of smaller length. Find a simple subobject $L^*$ of $S^*$, and suppose it is pure of weight $m$. Let $\phi: S^* \to C^* = S^*/L^*$ be the corresponding quotient map. By induction we can assume we have already defined our filtration on $C^*$. 

For any $k < m$ consider the exact sequence

$$0 \to L^\bullet \to \phi^{-1}W_kC^\bullet \to W_kC^\bullet \to 0.$$ 

Since the simple constituents of $W_kC^\bullet$ are all pure of weights $< m$, Proposition 5.7.1 implies

$$\text{Hom}(L^\bullet, W_kC^\bullet) = \text{Ext}^1(L^\bullet, W_kC^\bullet) = 0,$$

and so the exact sequence splits canonically. We then define $W$ to be the image of $W_kC^\bullet \to \phi^{-1}W_kC^\bullet \to S^\bullet$, where the first map is the splitting map. For $k \geq m$ we let $W_kS^\bullet = \phi^{-1}(W_kC^\bullet)$. Then $\text{Gr}^W S^\bullet \cong \text{Gr}^W C^\bullet \oplus L^\bullet$, while $\text{Gr}^W S^\bullet \cong \text{Gr}^W C^\bullet$ if $k \neq m$, so $\text{Gr}^W S^\bullet$ is pure of weight $k$ for all $k \in \mathbb{Z}$, since the same was true for $C^\bullet$ by induction.

Next we extend this filtration to arbitrary objects of $P_\mathcal{F}(X)$, which may not have finite length. In order to do this, we need to show that for any object $S^\bullet \in P_\mathcal{F}(X)$ there is a lower bound on the weights of the simple constituents of $S^\bullet$. To see this, use induction on the number of orbits in the support of $S^\bullet$. If the support is a single orbit $O_\alpha$, the result follows from the equivalence $P_\mathcal{F}(O_\alpha) \simeq L_\mathcal{F}(O_\alpha) \simeq \text{co-\mathcal{T}}_\alpha$-mod.\text{cf}.

Otherwise, let $O_\alpha$ be an open orbit in the support of $S^\bullet$. Let $\phi$ denote the natural adjunction morphism $S^\bullet \to Rj_{\alpha*}(S^\bullet|_{O_\alpha})$, and consider the short exact sequence

$$0 \to \ker \phi \to S^\bullet \to \text{Im} \phi \to 0.$$ 

The support of $\ker \phi$ is strictly smaller, so its weights are bounded below by the inductive hypothesis. Thus it will suffice to show the weights of $\text{Im} \phi$ are bounded below as well. But by the preceding paragraph the weights in $S^\bullet|_{O_\alpha}$ are bounded below, say by $w$. Since $Rj_{\alpha*}$ is a $t$-exact functor, this implies that the weights of $Rj_{\alpha*}(S^\bullet|_{O_\alpha})$, and hence of $\text{Im} \phi$, are bounded below by $w + w'$, where $w'$ is a lower bound for the weights of $Rj_{\alpha*}C_\alpha[\dim O_\alpha]$. This lower bound exists because $Rj_{\alpha*}C_\alpha[\dim O_\alpha]$ has finite length by Proposition 5.6.1 (in fact, $w' = 0$ works).

The existence of the filtration $W^\bullet$ follows immediately: if $S^\bullet \in P_\mathcal{F}(X)$ and $k \in \mathbb{Z}$, the collection $W_kS^\bullet$ forms a directed system over all finite-length subobjects $\hat{S}^\bullet$ contained in $S^\bullet$. It vanishes identically for $k \ll 0$, so we can proceed by induction on $k$: assume that $W_{k-1}S^\bullet$ has been defined in such a way that $S^\bullet_k = S^\bullet/W_{k-1}S^\bullet$ has only simple constituents of weights $\geq k$. The family $\{W_kS^\bullet_k\}$ must be eventually constant, since $\text{Hom}(J^\alpha(k), S^\bullet_k)$ is finite-dimensional for all $\alpha$. Thus $\{W_kS^\bullet\}$ also stabilizes, so the limit is a finite-length subobject.

The properties of a mixed category are easy to verify.
8.3. Proof of Theorem 6.8.1. Let us briefly recall how the functor $\epsilon: D^b_T(X) \to D^b(A_X:\text{mod}_T)$ of [L] is defined. Since $X$ is affine, we can choose a $T$-equivariant embedding $X \hookrightarrow \mathbb{P}^n$, where the action of $T$ on $\mathbb{P}^n$ is linear. We can choose a representative for the classifying space $BT$ so that $\mathbb{P}^n_T$ is an infinite dimensional manifold in the sense of [Bel] – essentially this means it is a limit of finite-dimensional manifolds by closed embeddings. $\mathbb{P}^n_T$ has a “de Rham complex” $\Omega^{\bullet}_{\mathbb{P}^n_T}$ which is a resolution of the constant sheaf $\mathbb{R}\mathbb{P}^n_T$ by soft sheaves. It is also naturally a supercommutative sheaf of DG-algebras. We then let $\Omega^{\bullet}_{X_T} = \Omega^{\bullet}_{\mathbb{P}^n_T}{|_{X_T}}$.

Let $\pi: X_T \to X/T$ be the map sending $O_T$ to $O/T$ for any $T$-orbit $O$. Given $S^\bullet \in D^b_T(X)$, the complex $M^\bullet = \pi_*(\Omega^{\bullet}_{X_T} \otimes S^\bullet)$ is naturally a DG-module over the DG-sheaf $\tilde{A} := \pi_*(\Omega^{\bullet}_{X_T})$. Here $\otimes$ is tensoring over $\mathbb{R}$. Since all $\mathbb{R}$-sheaves are flat, $\Omega^{\bullet}_{X_T} \otimes S^\bullet$ is quasi-isomorphic to $S^\bullet$.

The DG-sheaf $\tilde{A}$ is formal, i.e. it is quasi-isomorphic to its cohomology $\tilde{H}(\tilde{A})$. Under the natural identification of $X/T$ with the fan $\Sigma$ defining $X$, $\tilde{H}(\tilde{A})$ is canonically isomorphic to our sheaf of rings $A_\Sigma$. This gives an equivalence of categories

$$D(\text{DG}-\tilde{A}) \simeq D(\text{DG}-A)$$

which commutes with restriction and corestriction. The functor $\epsilon$ is the composition of this equivalence with $\pi_*(\Omega^{\bullet}_{X_T} \otimes \cdot )$.

Let us prove (a). Since the functor $F_{T,\Sigma}$ is defined locally, we can assume that $\Sigma = [\sigma]$, so $O_\sigma$ is the unique closed orbit in $X$. Let $S^\bullet = F_{T,\Sigma}M^\bullet$. Let $j: O_\sigma \to X_T$, $j: \{\sigma\} \to \Sigma$ denote the inclusions, and let $\pi_\sigma$ be the constant map $O_\sigma \to \{\sigma\}$, so $\pi \circ j = j \circ \pi_\sigma$. We will show that there are quasi-isomorphisms

$$j^* \pi_*(\Omega^{\bullet}_{X_T} \otimes S^\bullet) \simeq \pi_\sigma j^*(\Omega^{\bullet}_{X_T} \otimes S^\bullet) \simeq \pi_\sigma(\Omega^{\bullet}_{O_\sigma} \otimes j^*S^\bullet).$$

This will imply our result – the equivalence (8.3.1) commutes with taking stalks, so the left hand side is $j^*\nu M^\bullet = \nu(M(\sigma))$, while the right hand side is $\epsilon(j^*F_{T,\Sigma}M^\bullet)$. Applying $\epsilon^{-1}$ gives (a).

The second isomorphism is standard; see [KS] Proposition 2.3.5, for instance. For the first isomorphism, note that since the smallest open subset of the fan $[\sigma]$ containing $\sigma$ is $[\sigma]$ itself, the functor $j^*$ is naturally isomorphic to $\hat{p}_*$, where $\hat{p}: [\sigma] \to \{\sigma\}$ is the constant map. Therefore it will be enough to construct a quasi-isomorphism

$$\hat{p}_* \pi_*(\Omega^{\bullet}_{X_T} \otimes S^\bullet) \simeq \pi_\sigma p_* \pi_*(\Omega^{\bullet}_{X_T} \otimes S^\bullet) \simeq \pi_\sigma j^*(\Omega^{\bullet}_{X_T} \otimes S^\bullet),$$

where

$$p = p_{\sigma, T}: X_T \to O_{\sigma, T}.$$
is the map induced by the projection map $p_{\sigma}$ defined in §3.5.

Since $p \circ j$ is the identity on $O_{\sigma,T}$, adjunction gives a natural transformation $p_{\sigma} \to p_{\sigma} j_{\ast} j^{\ast} = j^{\ast}$. We will show that applying it to $\tilde{S}^{\ast} = \Omega_{X_{T}}^{\ast} \otimes S^{\ast}$ gives a quasi-isomorphism. Without loss of generality we can assume that $S^{\ast} = R i_{\tau}^{\ast} R O_{\tau,T}$, where $\tau \in [\sigma]$ and $i_{\tau} : O_{\tau,T} \to X_{T}$ is the inclusion. Note that $\tilde{S}^{\ast}$ is a complex of soft sheaves, so applying $p_{\ast}$ to is the same as applying $R p_{\ast}$. The stalk cohomology of $R p_{\ast} \tilde{S}^{\ast}$ and $j_{\ast} \tilde{S}^{\ast}$ at a point $x \in O_{\sigma,T}$ are both isomorphic to the cohomology of the torus $p^{-1}(x) \cap O_{\tau,T}$, which implies the claim.

For (b), consider the chain of maps

$$R \pi_{\sigma}^{\ast} (j_{\ast}^{\ast} \Omega_{X_{T}}^{\ast} \otimes j_{\ast}^{\ast} S^{\ast}) \to R \pi_{\sigma}^{\ast} j_{\ast}^{\ast} (\Omega_{X_{T}}^{\ast} \otimes S^{\ast}) \sim j_{\ast}^{\ast} R \pi_{\ast} (\Omega_{X_{T}}^{\ast} \otimes S^{\ast}).$$

For the first map see [KS, Proposition 3.1.11]; the second map is the usual base change. To check this is an isomorphism it is enough to consider the case $S^{\ast} = R i_{\tau}^{\ast} R O_{\tau,T}$ as before. If $\tau = \sigma$ the isomorphism is clear, while if $\tau \neq \sigma$ both sides vanish. (b) then follows.

The isomorphisms (c) and (d) follow from these statements using results of [BeL]. In the case of a single orbit $O_{\sigma}$, the equivalence $\epsilon : D_{b}^{f}(O_{\sigma}) \to D_{f}(DG\cdot A_{\sigma})$ can be factored as an equivalence

$$D_{b}^{f}(O_{\sigma}) \sim D_{b}^{f}(T/T_{\sigma})(pt) \quad (8.3.2)$$

[BeL Theorem 2.6.2] (here $T_{\sigma}$ is the stabilizer of any point of $O_{\sigma}$) followed by an equivalence $D_{b}^{f}(T/T_{\sigma})(pt) \sim D_{f}(DG\cdot A_{\sigma})$ [BeL Theorem 12.7.2(ii)]. The pullback functor $i_{\ast}^{\sigma}$ is $Q_{f}^{\ast}$, where $f : \{y\} \to O_{\sigma}$ is the inclusion of a point $\{y\}$, which is a $\phi$-map for the homomorphism $\phi : \{1\} \to T$ (for the definition and properties of $Q_{f}^{\ast}$, see [BeL, §3.6]). The equivalence (8.3.2) is $Q_{g}^{\ast}$, where $g : O_{\sigma} \to pt$ is the quotient map and $pt$ carries a $T/T_{\sigma}$-action. Thus $i_{\ast}^{\sigma} = Q_{f}^{\ast} Q_{g}^{\ast} = Q_{gf}^{\ast}$. Applying [BeL Theorem 12.7.2(iii)] completes the proof.

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