Asymptotically Flat Space -Times and its Hidden Recesses :
An Enigma from GR

Ezra T. Newman
Feb.19, 2016

Abstract
We begin by emphasizing that we are dealing with standard Ein-
stein or Einstein-Maxwell theory - absolutely no new physics has been
inserted. The fresh item is that the well-known asymptotically flat so-
lutions of the Einstein-Maxwell theory are transformed to a new coor-
dinate system with surprising and (seemingly) inexplicable results. We
begin with the standard description of (Null) Asymptotically Flat Space-
Times described in conventional Bondi-coordinates. After transforming
the variables (mainly the asymptotic Weyl tensor components) to a very
special set of NU (Newman-Unti) coordinates, we find a series of relations
totally mimicking standard Newtonian classical mechanics and Maxwell
theory. The surprising and troubling aspect of these relations is that
the associated motion and radiation does not take place in physical space-
time. Instead these relations takes place in an unusual inherited complex
four-dimensional manifold referred to as H-Space that has no immediate
relationship with space-time. In fact these relations appear in two such
spaces, H-Space and its dual space $\overline{H}$.

1 Introduction

We begin by emphasizing that the material described here is based only on stan-
dard Maxwell theory and classical general relativity. The matter and charged
sources are totally conventional - and absolutely no new physical ideas have
been introduced. The enigma lies in the fact that when a particular very
special coordinate transformation is applied to any asymptotically flat (in the
null sense, $[2][3][4]$) solution of the Einstein-Maxwell equations in the neigh-
borhood of future null infinity (Scri),$[10][11]$, that had been given initially in
Bondi-Coordinates, certain very strange, seemingly inexplicable, relations arise.
First we obtain standard classical mechanic and classical Maxwell theory rela-
tionships having nothing to do, in any obvious sense, with GR. Among many
more relations, described in detail later, we have, for example, $\vec{P}=M \vec{v}$ or
$\vec{L}=\vec{\pi} \times \vec{P}$ or the rate of quadrupole radiation of angular momentum via a
Maxwell field. There is even a special case of $\vec{F} = M \vec{a}$. More startling is the fact that these relations do not relate to activities in physical space-time - but rather to activities in an unusual inherited four-complex dimensional space, (a parameter space) referred to as $H$-space - and in fact, the activities occur in both $H$-space and/or in its dual $\overline{H}$-space\[12\]. The $H$-space ($\overline{H}$-space) arises naturally via the solutions of a differential equation associated with the coordinate transformation. We find these results to be, at a minimum, surprising - and (so-far) inexplicable and/or (maybe?) meaningless. They are also disturbing - why should GR produce such strange relationships - without an underlying reason.

In Sec. II, we briefly review some of the well known results of asymptotically flat space-times \[5\][10][11][1][14] while in Sec. III the special transformation from Bondi to the NU coordinates[10] will be given\[1\] - essentially as a review. The transformation is applied mainly to the Weyl tensor components (in spin-coefficient form). The results of the transformation, i.e., the new relations, are described in Sec. IV and followed by a discussion of these relations in Sec.V and VI.

The details of the transformation from Bondi to the special NU coordinates\[10\] are both long and complicated and have already been reported in detail in a recent Living Review article. Rather than duplicate these details we will, when needed, simply quote from the Living Review.\[1\]

The basic argument, though appearing to be rather complicated, has, for its genesis, a simple analogue in the case of electro & magneto-statics; the determination of center of charge motion and the complex center of charge motion.

Consider a charge distribution, an origin and the associated electric dipole $\vec{D}_E$. A shift of origin by $\vec{R}$, so that $\vec{r}^* = \vec{r} - \vec{R}$, leads to the dipole transformation, $\vec{D}^*_E = \vec{D}_E - q\vec{R}$. Setting $\vec{D}^*_E = 0$, defines the center of charge by $\vec{R} = \vec{D}_E / q$. Formally this can be generalized to the complex center of charge, by including the magnetic dipole, $\vec{D}_M$, via the complex dipole moment, $\vec{D}_C = \vec{D}_E + i\vec{D}_M$. Assume that it transforms under the complex translation $\vec{r}^* = \vec{r} - \vec{R}_C$, as $\vec{D}^*_C = \vec{D}_C - q\vec{R}_C$. Setting $\vec{D}^*_C = 0$, we have, by definition,

$$\vec{R}_C = \frac{\vec{D}_C}{q},$$

the position of the complex center of charge.

This idea will be generalized to the asymptotically flat Einstein-Maxwell fields where we will have, in addition to the complex electro-magnetic dipole, a generalization, by definition, to a complex mass dipole (essentially, mass dipole + i angular momentum).

From asymptotic information we will search for both the complex center of mass and complex center of charge.

The major issue will be what replaces the role of $\vec{R}_C$. The enigma then centers on the meaning of that replacement.
2 Asymptotically Flat Space-Times

As there is a great deal of existing literature on the asymptotic behavior of the Einstein-Maxwell equations we will simply take what is needed from this literature. Our two main sources are Newman-Penrose, (in Scholarpedia) and Adamo-Newman (in Living Reviews). In all the discussions we make heavy use of the NP formalism.

The study of asymptotically flat space-times was born from the early brilliant work of H. Bondi\[2\] \[3\] where a one-parameter family of null (i.e., characteristic) surfaces, $\mathcal{C}_u$ labeled by $u$, was introduced as a (space-time) coordinate. Each of these surfaces is generated by a two-parameter family of null geodesics, $\mathcal{G}$, each labeled by sphere coordinates ($\theta, \phi$) or equivalently (used by us) by complex stereographic coordinates ($\zeta, \overline{\zeta}$), where $\zeta = e^{i\phi} \cot(\theta/2)$. The ‘length’ along the geodesics is given by the affine parameter, $r$.

The limit, $r \rightarrow \infty$, is given as the null boundary of the compactified space-time and referred to as Scri. It is coordinatized by ($u, \zeta, \overline{\zeta}$). The full set of coordinates, ($u, \zeta, \overline{\zeta}, r$), called Bondi coordinates (Bondi himself did not use an affine parameter, but what is referred to as a “luminosity distance” parameter), is not unique; there is a large class of such coordinates. (We point out shortly that there is a generalization of the Bondi coordinates, often referred to as NU coordinates\[10\], a subset of which will shortly play a major role.) With the choice of (any) one Bondi set, there is a natural choice of (Bondi) null tetrad system, ($l^a, n^a, m^a, \overline{m}^a$). The vector $l^a$ is taken as the tangent vector to the null geodesics of $u$, $n^a$ is tangent to the null generators of Scri, while the ($m^a, \overline{m}^a$) are space-like tangent to Scri. They are normalized by all products vanishing except $l^a n_a - 1 = m^a \overline{m}_a + 1 = 0$. Their remaining freedom is greatly limited by having the tetrad parallel propagated along the null geodesics of $u$. When these restrictions are translated to the tetrad, metric and the derivative operators become:

\[
D = l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r} \quad (2)
\]
\[
\nabla = n^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A} \quad x^A = (x^3, x^4) = (\zeta, \overline{\zeta}) \quad (3)
\]
\[
\delta = m^a \frac{\partial}{\partial x^a} = \omega \frac{\partial}{\partial r} + \xi^A \frac{\partial}{\partial x^A} \quad (4)
\]
\[
\overline{\delta} = \overline{m}^a \frac{\partial}{\partial x^a} = \overline{\omega} \frac{\partial}{\partial r} + \overline{\xi}^A \frac{\partial}{\partial x^A} \quad (5)
\]

The metric takes the form

\[
g^{ab} = \begin{bmatrix}
0 & 1 & 0 \\
1 & g^{22} & g^{2A} \\
0 & g^{2A} & g^{AB}
\end{bmatrix} \quad (6)
\]
with
\[
\begin{align*}
g^{22} &= 2(U - \omega\overline{\omega}), \\
g^{2A} &= X^A - (\overline{\xi}^A + \omega\overline{\xi}^A), \\
g^{AB} &= - (\xi^A\xi^B + \overline{\xi}^A\overline{\xi}^B),
\end{align*}
\]

Our major interest and concern will center on the Weyl and Maxwell tensors; their asymptotic behavior, physical meaning, evolution and transformation properties. We use the five complex self-dual NP components of the Weyl tensor and three complex Maxwell components:\(^5\)

\[
\begin{align*}
\Psi_0 &= -C_{abcd}t^a m^b t^c m^d = -C_{1313}, \\
\Psi_1 &= -C_{abcd}t^a n^b l^c m^d = -C_{1213}, \\
\Psi_2 &= -C_{abcd}t^a m^b \overline{m}^c n^d = -C_{1342}, \\
\Psi_3 &= -C_{abcd}n^a \overline{m}^b m^c n^d = -C_{1242}, \\
\Psi_4 &= -C_{abcd}n^a m^b \overline{m}^c n^d = -C_{2442},
\end{align*}
\]

\[
\begin{align*}
\phi_0 &= F_{ab} t^a n^b, \\
\phi_1 &= \frac{1}{2} F_{ab} (t^a n^b + m^a \overline{m}^b), \\
\phi_2 &= F_{ab} t^a \overline{m}^b.
\end{align*}
\]

By integrating the radial asymptotic Bianchi identities and Maxwell equations, we have what is known as the ‘peeling’ theorem:\(^5\):

\[
\begin{align*}
\Psi_0 &= \Psi_0^0 r^{-5} + O(r^{-6}), \\
\Psi_1 &= \Psi_1^0 r^{-4} + O(r^{-5}), \\
\Psi_2 &= \Psi_2^0 r^{-3} + O(r^{-4}), \\
\Psi_3 &= \Psi_3^0 r^{-2} + O(r^{-3}), \\
\Psi_4 &= \Psi_4^0 r^{-1} + O(r^{-2}),
\end{align*}
\]

\[
\begin{align*}
\phi_0 &= \phi_0^0 r^{-3} + O(r^{-4}), \\
\phi_1 &= \phi_1^0 r^{-2} + O(r^{-3}), \\
\phi_2 &= \phi_2^0 r^{-1} + O(r^{-2}),
\end{align*}
\]

with
\[
\begin{align*}
\Psi_0^n &= \Psi_0^n (u, \zeta, \overline{\zeta}), \\
\phi_0^n &= \phi_0^n (u, \zeta, \overline{\zeta}).
\end{align*}
\]
The remaining (non-radial) Bianchi Identities and Maxwell equations yield the evolution equations:

\[
\begin{align*}
\dot{\Psi}_2^0 &= -\phi_3^0 + \sigma^0 \Psi_3^0 + k \phi_3^0 \phi_2^0, \\
\dot{\Psi}_1^0 &= -\phi_2^0 + 2\sigma^0 \Psi_3^0 + 2k \phi_1^0 \phi_2^0, \\
\dot{\Psi}_0^0 &= -\Psi_1^0 + 3\sigma^0 \Psi_2^0 + 3k \phi_0^0 \phi_2^0, \\
k &= 2Gc^{-1},
\end{align*}
\]

(13) (14) (15) (16)

These 5 equations, \(13-18\), after the coordinate transformation to the special NU coordinates, contain our mechanical equations of motion.

The quantity \(\sigma^0(u, \zeta, \bar{\zeta})\) is often called the asymptotic shear, being the leading term in the shear of the geodesic congruence, \(l^i\); i.e.,

\[\sigma = r^{-2}\sigma^0(u, \zeta, \bar{\zeta}) + O(r^{-4})\]

while the first \(u\)-derivative of \(\sigma^0\) is referred to as the Bondi news function. We consider \(\sigma^0(u, \zeta, \bar{\zeta})\) as a free function. It, as such, plays a significant role in what later follows. From the spin-coefficient equations one finds that

\[
\begin{align*}
\dot{\phi}_1^0 &= -\phi_2^0, \\
\dot{\phi}_0^0 &= -\phi_1^0 + \sigma^0 \phi_2^0.
\end{align*}
\]

(17) (18)

Defining the \textit{mass aspect}, \(\Psi\), (real from field equations) by

\[\Psi = \Psi = \Psi_2^0 + \phi^2 \phi^0 + \sigma^0 (\phi^0)\]

(19)

Bondi defines the asymptotic mass, \(M_B\), and 3-momentum, \(P_B^i\) as the \(l = 0\) \& \(l = 1\) harmonic coefficients of \(\Psi\). Specifically,

**Definition 1** Identification of Physical Quantities:

\[
\begin{align*}
\Psi_0^0 &= \phi^2 \phi^0 + \sigma^0 (\phi^0) \\
\Psi_0^0 &= -\phi_2^0, \\
\Psi_0^0 &= -\phi_1^0 + \sigma^0 \phi_2^0.
\end{align*}
\]

(20) (21) (22)

By rewriting Eq.(13), replacing the \(\Psi_2^0\) by \(\Psi\) via Eq.(19), we have

\[\Psi = (\sigma^0) (\phi^0) + k \phi_2^0 \phi_2^0,\]
and one immediately has the Bondi mass/energy loss theorem:

\[ \dot{M} = -\frac{c^2}{2\sqrt{2}G} \int \left( (\sigma^0)^* (\sigma^0) + k \phi_2^0 \phi_2^0 \right) d^2S, \]  

the integral taken over the unit 2-sphere. This relationship is at the basis of all the contemporary work on the detection of gravitational radiation.

**Definition 2** Though there has been no universal agreement, we adopt the definition of the complex mass dipole moment, \( D^{i}_{\text{(complex)}} = D^{i}_{\text{(mass)}} + i c^{-1} J^i \), as the \( l = 1 \) harmonic component of \( \Psi^0_1 \),

\[ \Psi^0_1 = -6\sqrt{2}Gc^{-2}(D^{i}_{\text{(mass)}} + i c^{-1} J^i)Y^1_1 + \ldots. \]  

\( D^i \) the mass dipole and \( J^i \), the total angular momentum, as seen at null infinity.

**Definition 3** Our physical identification for the complex E&M dipole, (electric and magnetic dipoles, \( D^i_{\text{Elec}} + i D^i_{\text{Mag}} \)) as the \( l = 1 \) harmonic component of \( \phi^0_0 \) is standard

\[ \phi^0_0 = 2(D^i_{\text{Elec}} + i D^i_{\text{Mag}})Y^1_1. \]  

Later we will connect these three physical identifications with the complex center of mass and the complex center of charge. For the general situation these two complex centers, are not yet defined and are independent.

However, here, for simplicity, we will assume that they coincide. This is not necessary but is a restriction.

# 3 Transformation To NU coordinates

The Bondi coordinates (with their associated freedom, the BMS group) are not the only useful coordinates to be used in the neighborhood of Scri. Often the generalization, to what is often referred to as NU coordinates, is called for. The basic idea of NU coordinates is to modify the use of the uniform stacking of the constant \( u \)-slices of Scri and instead allow for both the deformation of individual slices and the arbitrary stacking of the new slices. Analytically the coordinate transformation between Bondi, \( (u, \zeta, \zeta^\star) \), and NU, \( (\tau, \zeta, \bar{\zeta}) \) coordinates is

\[ u = G(\tau, \zeta, \bar{\zeta}) \]

with \( G \) an arbitrary analytic function of \( (\tau, \zeta, \bar{\zeta}) \). Often we will allow the coordinates to be complex but close to the real - which means that \( \bar{\zeta} \) can take values close to the complex conjugate value of \( \zeta \). This will be denoted by \( \bar{\zeta} \Rightarrow \bar{\zeta} \). The NU coordinates can be thought of as being analogous to a coordinate system attached or associated with an arbitrary world line. That analogy will shortly be taken a step further.

The arbitrariness in the choice of \( G \) is greatly restricted by the following argument. Null Geodesic Congruences (NGC) coming from the interior of the
space-time and intersecting with Scri are usually classified by the values of the optical parameters near Scri, their divergence, shear and twist. We now ask for slicings of Scri, (i.e., choices of \(G(\tau, \zeta, \overline{\zeta})\)) so that the NGCs normal to the slicing have vanishing asymptotic shear. The basic result from the extensive literature ([7], [1]) is that \(G\) must satisfy the so-called good-cut equation, 

\[
\delta^2 G = \sigma^0(G, \zeta, \overline{\zeta}). \tag{26}
\]

With regularity conditions, the solution space for this equation is a four-complex dimensional parameter space, \(z^a\), referred to as H-Space [8][9]. Solutions, with appropriate choice of H-space coordinates, can be written as 

\[
\begin{align*}
 u &= Z(z^a, \zeta, \overline{\zeta}) = z^a l^a_\#(\zeta, \overline{\zeta}) + \Sigma_{l \geq 2} Z_{lm}(z^a) Y_{lm}(\zeta, \overline{\zeta}) \quad \tag{27} \\
l^a_\#(\zeta, \overline{\zeta}) &= \left(\frac{\sqrt{2}}{2} Y_{00}, \frac{1}{2} Y_{1m}\right) = \frac{\sqrt{2}}{2(1 + \zeta \overline{\zeta})}(1 + \zeta \overline{\zeta}, \zeta + \overline{\zeta}, i \zeta - i \overline{\zeta}, -1 + \zeta \overline{\zeta}) \quad \tag{28}
\end{align*}
\]

H-Space plays several roles. First of all, it possesses a natural complex metric that automatically is Ricci flat and anti-self dual [8]. Penrose uses the H-Space for his non-linear graviton construction. In the metric context there are strong suggestions that H-space and its properties are closely related to the ideas of the present work. The details, however, are not yet clear. We use H-Space without reference to its metric.

We choose an arbitrary (for the time being) one-complex dimensional curve, \(z^0 = \xi^a(\tau)\), and insert it into Eq.(27) so that we get a one parameter family of cuts 

\[
 u = Z(\xi^a(\tau), \zeta, \overline{\zeta}) = G(\tau, \zeta, \overline{\zeta}) \tag{29}
\]

depending on choice of the world line. The solutions (with an appropriate choice of H-Space coordinates) has the form 

\[
 u = G(\tau, \zeta, \overline{\zeta}) = \xi^0(\tau) l^\#(\zeta, \overline{\zeta}) + \Sigma_{l \geq 2} Z_{lm}(\xi^a(\tau)) Y_{lm}(\zeta, \overline{\zeta}).
\]

This world-line in H-space will become our complex center of mass and center of charge, the analogue of the electro-magnetic, Eq.(1). The inverse function is written as 

\[
\tau = T(u, \zeta, \overline{\zeta}). \tag{30}
\]

Note that though \(G(\tau, \zeta, \overline{\zeta})\) is in general complex one can construct from \(G\), a one parameter family of real cuts by taking the real part of \(G\).

### 3.1 The Transformation

Our task is to transform the Weyl tensor physical identification, i.e., the complex mass (or gravitational) dipole, Eq. (24), from the Bondi coordinates (\(u = \text{constant slicings}\)) to the NU coordinates (\(\tau = \text{constant slicings}\)). This is a very tedious and long task - done by approximations and Clebsch-Gordon expansions - that is
only outlined here. When this complex mass (or gravitational) dipole is expressed on the \( \tau \) slices, we set it (and, by assumption, the complex E&M dipole) to zero. This determines the world-line, \( z^a = \xi^a(\tau) \), which, by definition, is our complex center of mass world-line (and, by our assumption, our complex center of charge).

**Remark 1** At this point there is no reason to take this definition of a "complex world-line" seriously or for it to have any physical significance. The "world-line" seems to have no relationship with space-time points. It is defined on the rather mysterious four-complex dimensional \( H \)-space. (The Penrose non-linear graviton.) Nevertheless, in the following section we will see that in fact it does have a very close relationship with things of obvious great physical significance.

It is the fact of the existence of these relationships that is our enigma - why do they exist?

The transformation of the complex mass dipole from \( u \)-coordinates to \( \tau \)-coordinates is, as we said, long and painful. It begins with the null rotation of the Bondi tetrad [based on the (Bondi) null vector, \( l^a \), normal to the Bondi slices] to the tetrad based on the null vector \( l^*a \), (normal to the \( \tau \)-slices);

\[
\begin{align*}
l^*a &= l^a + b m^a + \overline{b} n^a + \overline{\overline{b}} m^a \\
m^*a &= m^a + b n^a \\
n^*a &= n^a
\end{align*}
\]

where

\[
b = - r^{-1} L + O(r^{-2}),
\]

and \( L \), the angle field at Scri between \( l^a \) and \( l^a \),

\[
L(u, \zeta, \overline{\zeta}) = \partial G(\tau, \zeta, \overline{\zeta}) |_{\tau = T(u, \zeta, \overline{\zeta})}.
\]

Our procedure for finding the complex center of mass now centers on Eq.(33).

We search for and set to zero the \( l = 1 \) spherical harmonic coefficient of \( \Psi^0_0 \) on a constant \( \tau \) slice. The right-side of Eq.(33), a function of \( u \), is first converted to a function of \( \tau \) via Eq.(29). All the variables on the right side are then
expanded in spherical harmonics and simplified by Clebsch-Gordan expansions. The isolation of the \( l = 1 \) harmonics is still difficult and approximations - to second order in the variables - are needed.

The result from these operations on Eq. (33) leads to an expression for the \( l = 1 \) coefficients of Bondi \( \Psi_0^i \), namely the following:

\[
\Psi_0^i = -\frac{6\sqrt{2}G}{c^2} M_B \xi^i + i \frac{6\sqrt{2}G}{c^3} P^k \xi^i \epsilon_{kji} - \frac{576G}{5c^5} P^k \xi^i + i \frac{6912\sqrt{2}}{5} \xi^i \epsilon_{kji} \\
- \frac{2\sqrt{2}G}{c^6} q^2 \xi^i \xi^{ij} \epsilon_{kji} - \frac{48G}{5c^6} q^2 \xi^i \xi^{ij} \epsilon_{kji} - \frac{4G}{5c^7} q^2 \xi^i Q_C^{ij} \epsilon_{kji} - \frac{16\sqrt{2}G}{5c^7} q \xi^i Q_C^{ij} \epsilon_{kji}.
\]

where

\[
\xi^i = \frac{\sqrt{2}G}{24c^4} (Q_{mass}^{ij} + iQ_{spin}^{ij})
\]

are the time-derivatives of the gravitational quadrupoles and \( Q_C^{ij} \) the electric and magnetic quadrupoles.

Treating the quadrupoles and higher powers of \( c^{-1} \) as small, writing

\[
\xi^i = \xi^i_R + i \xi^i_I,
\]

and remembering our physical identifications, Eq. (24), we are left with

\[
D^{(\text{mass})}_i = M_B \xi^i_R - c^{-1} P^k \xi^i \epsilon_{kji} + \ldots, \quad (38)
\]

\[
J^i = c M_B \xi^i_I + P^k \xi^i_R \epsilon_{kji} + \ldots. \quad (39)
\]

In other words we have the conventional definition of mass dipole \((M \vec{T})\) augmented with an unusual term, \( \vec{D} \times \vec{S} \), discussed later, i.e.,

\[
\vec{D}^{(\text{mass})} = M_B \vec{T} + c^{-2} M_B \vec{P} \times \vec{S}. \quad (40)
\]

In addition we have an expression for angular momentum; the intrinsic spin \( \vec{S} \) (same as for the Kerr metric) plus the orbital angular momentum

\[
\vec{J} = \vec{S} + \vec{P} \times \vec{S} = c M_B \vec{\xi}^i_R + \vec{P} \times \vec{S}. \quad (41)
\]

We find this to be a rather startling result; standard classical mechanical relationships where ordinary space-time has been replaced (not by any assumption but automatically) by the mysterious H-space. It’s our first set of enigmas. We return to them later in connection with the relativistic angular momentum tensor and the \( \vec{D} \times \vec{S} \) term.

Further classical mechanical relations, even more startling, are given in the next section.
4 More Results

The relations for the mass dipole and the angular momentum, given by Eqs. (38) & (39) are now substituted into the evolutionary Bianchi Identities Eqs.(14), (15) and Maxwell Eq.(17) and (18)

\[
\dot{\Psi}_0^2 = -\partial \Psi_3^0 + \sigma_0^0 \Psi_3^0 + k \phi_2^0 \phi_2^0,
\]

(42)

\[
\dot{\Psi}_1^2 = -\partial \Psi_2^0 + 2 \sigma_0^0 \Psi_2^0 + 2 k \phi_1^0 \phi_2^0,
\]

(43)

\[
\dot{\phi}_0^1 = -\partial \phi_0^1 + \sigma_0^0 \phi_0^1.
\]

(44)

\[
\dot{\phi}_0^0 = -\partial \phi_0^0 + \sigma_0^0 \phi_2^0.
\]

(45)

From the real part of Eq.(43) we get an expression for the Bondi momentum in terms of the time derivative of the mass dipole;

\[
P^i = M_B \xi_R^i - \frac{2q^2}{3c^3} \xi_R^i + H.O.
\]

(46)

\[
H.O. = \text{higher order terms}
\]

i.e., we obtain, for the momentum, the kinematic \( \vec{M} \) term and a term familiar from electrodynamics, the radiation reaction contribution to the linear momentum. From the imaginary part of the Bianchi Identity we have the momentum loss equation;

\[
J^i = \frac{2q^2}{3c^3} \xi_R^i + \frac{2q^2}{3c^3} (\xi_R^k \xi_R^l + \xi_I^k \xi_I^l) \epsilon_{kli} + \text{Mass&E&M quadrupole terms}. \tag{47}
\]

To our knowledge this might be a new expression for angular momentum loss. We have not found anything like this in the literature.

Going to the next Bianchi identity, Eq(12), we get relations for both the \( l = 0 \) and the \( l = 1 \) harmonic terms. First we have the (Bondi) mass loss expression but now augmented by the electromagnetic loses.

\[
M_B = -\frac{G}{5c^7} (Q_{mass}^{km} Q_{mass}^{km} + Q_{spin}^{km} Q_{spin}^{km}) - \frac{4q^2}{3c^5} (\xi_R^i \xi_R^i + \xi_I^i \xi_I^i)
\]

(48)

where the first term is the standard Bondi quadrupole mass loss (including now the spin quadrupole contribution to the loss), the second term and third terms are the standard E & M dipole and quadrupole energy loss.

From the \( l = 1 \) terms we get the momentum loss

\[
P^i = F_{\text{recoil}}^i \tag{49}
\]
where $F_{\text{recoil}}$ is composed of many non-linear radiation terms involving the time derivatives of the gravitational quadrupole and the E&M dipole and quadrupole moments whose details are not now relevant for us. We however can substitute Eq. (46) into Eq. (49) which leads Newton’s second law;

$$M_B\xi''_R = F^i = M'_B\xi''_R + \frac{2q^2}{3c^3}\xi''_R + F_{\text{recoil}}.$$  

The first force term, on the right, is the standard rocket mass loss expression while the second is the "famous" radiation reaction force of classical Maxwell theory.

There are two further items worth mentioning.

1. From our earlier results,
   i. $\xi^i_R$ = center of mass position
   ii. $S^i = Mc\xi^i = \text{Spin-Angular momentum}$
   iii. $D^i_M = q\xi^i = \text{Magnetic dipole Moment}$

   We have, from comparing the classical gyromagnetic ratio $\gamma = \frac{q}{2Mc}$, with ours

$$\gamma = \frac{D^i_M}{L_{\text{spin,ang,mom}}} = \frac{q\xi^i}{M_B\xi^i} = \frac{q}{M_Bc}, \quad (51)$$

we find the Dirac value of the $g$-factor, i.e., $g=2$.

2. In classical relativistic mechanics one defines the angular momentum tensor, $M^{ab}$,

$$M^{ab} = L^{ab} + S^{ab}$$

$$L^{ab} = 2MX^{[a}V_{b]}$$

$$S^{ab} = -\eta^{abcd}S^c_{\ast}V_d, \quad S^c_{\ast}V^c = 0$$

so that

$$M^{ij} = L^{ij} + S^{ij} \quad (52)$$

$$= M(X^iV^j - V^iX^j) - \epsilon^{ijk}(S^k_{\ast}V_0 - V_kS^0_{\ast})$$

Using our results, $S^i = cS^0^i = Mc\xi^i$, $S_0 = 0$, $V_0 \sim 1$, $V_k \sim 0$ and multiplying by $\epsilon_{ijk}$, we have agreement with our Eq. (39).

Then from

$$M^{0i} = L^{0i} + S^{0i} \quad (53)$$

$$= 2MX^{[0}V^i] - \eta^{0ijk}S^\ast_jV_k$$

$$= M_B\xi^i_R - \epsilon_{ijk}c^{-1}\xi^j_Pk,$$

we have agreement with our Eq. (38), i.e., with our $\overrightarrow{P_X}\vec{S}$ term.

The relativistic angular momentum tensor (unrelated to physical space-time) is sitting in our Weyl tensor.
It is the collection of results, Eqs. (38), (39), (46), (47), (48) and (50), mimicking or imitating classical mechanics, that constitute our enigma. Though they appear to be space-time equations of motion there is no space-time that is associated with the equations. They (as we said earlier) take place on the strange H-space. Is this just a giant coincidence? That is difficult to believe. What possible meaning can one give to them - and to the H-space? What is the physical meaning of $\xi^i(\tau) = \xi^i_0(\tau) + i\xi^i_1(\tau)$.

As a final comment we point out that starting with the same asymptotically flat space time but now looking for the conjugate slicing of its Scri, i.e., looking for the slicing that is associated with complex conjugate shear-free null surfaces yields at alternate (conjugate) NU coordinate system and a construction of the conjugate classical mechanical relations. We obtain instead of H-space, the conjugate $\overline{H}$-space. In other words there is a dual description of everything that was described here.

5 An Example

If we consider the very special (radiationless) case where the Bondi asymptotic shear, vanishes, i.e., $\sigma^0(u, \zeta, \bar{\zeta}) = 0$, the Good Cut Equation simplifies to

$$\delta^2 G = 0,$$

with regular solutions being

$$u = Z(z^a, \zeta, \bar{\zeta}) = z^a l^a_\#(\zeta, \bar{\zeta}),$$

$$l^a_\#(\zeta, \bar{\zeta}) = \left(\frac{\sqrt{2}}{2}Y_{00}, \frac{1}{2}Y_{1m}\right) = \frac{\sqrt{2}}{2(1 + \zeta\bar{\zeta})}(1 + \zeta\bar{\zeta}, \zeta + \bar{\zeta}, i\zeta - i\bar{\zeta}, -1 + \zeta\bar{\zeta})$$

The four dimensional solutions space, $z^a$, i.e., the H-space, becomes in this case 'complex Minkowski space", with metric $ds^2 = \eta_{ab} dz^a dz^b$ [1]. Choosing a one-complex dimensional curve, $z^a = \xi^a(\tau)$, and inserting it into Eq. (55), we get a one parameter family of cuts

$$u = \xi^a(\tau) l^a_\#(\zeta, \bar{\zeta}),$$

that mimic the cuts of Scri of flat space.

All our equations of motion can now be thought of as taking place in Minkowski space.

Our kinematic equations, [40] and [41]

$$\overline{D}_{(mass)} = M_B \overline{\nabla} + e^{-2} M_B^{-1} \overline{\nabla} \times \overline{\mathbf{S}}.$$  

$$\overline{J} = \overline{\mathbf{S}} + \overline{\nabla} \times \overline{\mathbf{P}} = c M_B \overline{\xi}_I + \overline{\nabla} \times \overline{\mathbf{P}}.$$  

are now 3-D flat space relations.
6 Discussion & Comments

We want in this section to reiterate and emphasize the strangeness of the results presented.

a. In the search of null surfaces that resemble, at infinity, ordinary Minkowski light-cones - i.e., shear-free null and diverging (as $r^{-1}$) surfaces - we find a four complex parameter family of such surfaces, H-space. This family DOES NOT CONSTITUTE THE POINTS OF SPACE-TIME. Nevertheless we find that much of classical mechanics seems to live on H-space. If the starting space-time had been flat, the H-space would have been complex Minkowski space and its real part could be identified with the physical space-time. One might think of H-space as an observation or 'Mirage' space - though this does not appear to clarify its physical meaning.

b. If we are dealing with asymptotic Einstein-Maxwell, the H-space equations of motion contain the classical Radiation Reaction term without the standard use of mass renormalization - Just the "Asymptotic" Bondi Mass.

c. If (and we do not know if it is true) the Einstein-Maxwell Solutions are stable - then something in the equations suppresses radiation reaction run away behavior? It is easily seen that the equations of motion of a charged particle contain not only the radiation reaction force but other non-standard terms. What is the effect of these terms?

d. One can see the details in some special cases - e.g., all asymptotic Static and Stationary metrics and the Robinson-Trautman class [1].

e. For no obvious reason, one obtains the Dirac value of the gyromagnetic ratio. This is related to the assumption made at the start that the complex center of mass coincides with the complex electric dipole moment.

f. It definitely appears as if there are further structures to be explored. They are related to the question of the real and imaginary parts of the shear-free NU cuts of Scri.

7 Questions

a. We feel that these results might be profound but we are stuck how to further interpret or further use them. On Tuesdays, Thursdays, Saturdays and Sundays we are rather certain that they are profound but unfortunately on Mondays, Wednesdays and Fridays we wonder if there is anything of substance?

b. We have the "motion" of center of mass of the entire system. Can one do two body Equations of Motion? How? It is not clear but there seems to be a real possibility to do this by treating the space between two distant separated masses as approximately flat and behaving similarly to Scri.

c. How do these results enter into the issues of Quantum Gravity - or even ordinary Quantum Mechanics? Perhaps on the profound side?

d. Our complex world-line sits in H-Space whose metric satisfies the COMPLEX Einstein Eqs. What does this mean? Is there any further significance of this?
e. H-Space plays a fundamental role in Penrose’s Non-Linear Graviton construction? [12] So What? Is this just a fact with no further significance?

f. Is our present enigma at all related to the so-called holographic principle[15]?

g. If these H-Space equations of motion were to contain potentially new phenomena, do we take that seriously and look for experimental confirmation?

References

[1] Adamo, T.M., Newman, E.T, Kozameh, C., Living Rev. Relativity, 15, (2012), 1, http://www.livingreviews.org/lrr-2012-1 (Update of lrr-2009-6)

[2] Bondi, H., van der Burg, M.G.J. and Metzner, A.W.K., Proc.Roy.Soc.Lond.A, 269, p21 (1962),

[3] Sachs, R.K., Proc.Roy.Soc.Lond. A270, pp. 103-126 (1963)

[4] Penrose, R., Rindler, W., Spinors and Space-Time Vol 2, Cambridge Univ Press, (1986) Cambridge UK

[5] Newman, E.T. and Penrose, R., J. Math. Phys., 3, 566–578, (1962).

[6] Newman, E.T. & Penrose, R. (2009). "Spin-coefficient formalism," Scholarpedia, 4(6): 7445.

[7] Kozameh, C. & Newman, Ezra T. Class. Quantum Grav. 22 (2005) 4659–4665.

[8] Hansen, R. O., Newman,E. T., Penrose, R., and Tod, K. P., Proc. R. Soc. Lond. A 1978 363, 445-468

[9] Newman, E.T., General Relativity and Gravitation · Volume 7, Issue 1 , pp 107-111;

[10] Newman, E. T , Uniti, T., J. Math. Phys. 3, 891, (1962).

[11] Newman, E. T, Tod, K.P., Asymptotically Flat Space-Times, in GENERAL RELATIVITY AND GRAVITATION, VOL 2 edited by A. Held, Plenum Publishing, NY, 1980

[12] Penrose, R., Gen.Rel.Grav. 7 (1976) 171-176.

[13] Kozameh, C and Newman, E.T , Class. Quantum Grav. 22 (2005.) 4659–4665.

[14] Adano,T., Newman, E.T., Phys.Rev.D.83.044023

[15] Adano,T., Newman, E.T., Class. Quantum Grav. 26 (2009) 155003 (9pp).
Acknowledgement 2

We thank Timothy Adamo for both a careful critical reading of the manuscript, for hours of enlightening discussions and collaboration on an earlier manuscript where many of the present ideas were developed.