A path-dependent stochastic Gronwall inequality and strong convergence rate for stochastic functional differential equations

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Friday 3rd June, 2022

Abstract

We derive a stochastic Gronwall lemma with suprema over the paths in the upper bound of the assumed affine-linear growth assumption. This allows applications to Itô processes with coefficients which depend on earlier time points such as stochastic delay equations or Euler-type approximations of stochastic differential equations. We apply our stochastic Gronwall lemma with path-suprema to stochastic functional differential equations and prove a strong convergence rate for coefficient functions which depend on path-suprema.

1 Introduction

There are numerous applications of the classical (deterministic) Gronwall lemma. Scheutzow [25] derived a powerful stochastic version of the Gronwall lemma with an $L^p$-estimate with $p \in (0, 1)$. Makasu [20] extended this to the case $p = 1$. Hudde et al. [13] extended this to the case $p \in (1, \infty)$. For related stochastic Gronwall lemmas see, e.g., [29, 32, 16, 21, 31, 9, 10].

Recently, Mehri and Scheutzow [24] relaxed the affine-linear growth assumption and allowed running path-suprema in the upper bound. More precisely, Mehri and Scheutzow [24, Theorem 2.2] in particularly prove that if $\alpha : [0, \infty) \to [0, \infty)$ is measurable, if $X, H : [0, \infty) \times \Omega \to [0, \infty)$ are adapted stochastic processes on a filtered probability space $(\Omega, F, \mathbb{P}, (F_t)_{t \in [0, \infty)})$ with continuous sample paths and if

$$X_t \leq \int_0^t \alpha_s \sup_{r \in [0, s]} X_r \, ds + M_s + H_s,$$  \hspace{1cm} (1)

Key words and phrases: stochastic Gronwall lemma, functional stochastic differential equations, path-dependent stochastic differential equations, stochastic delay equations

AMS 2010 subject classification: 60E15, 65C30, 34K50
then it holds for all \( p \in (0, 1) \), \( t \in [0, \infty) \) that

\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X_s|^p \right] \leq \frac{1}{(1-p)p^{1/p}} \mathbb{E} \left[ \sup_{s \in [0,t]} |H_s|^p \right] \exp \left( \frac{1}{(1-p)p} \int_0^t \alpha_s \, ds \right),
\]

(2)

The main goal of this article is to complement this path-dependent stochastic Gronwall inequality with a result in the case \( p \in (1, \infty) \); see Theorem 2.1 below for the precise formulation.

The second goal of this article is to demonstrate an application of our stochastic Gronwall lemma. Stochastic functional differential equations (SFDEs, which are also denoted as stochastic delay equations or path-dependent SDEs in the literature) appear in a wide range of applications; see, e.g., [3, 4, 5, 8, 27, 28]. Some SFDEs can be transformed to classical stochastic differential equations by the linear chain trick; see, e.g., [26]. In general, however, solutions are typically not known explicitly and the linear chain trick does not work. We will prove that Euler-type approximations of SFDEs converge with strong rate 0.5— if the drift coefficient is one-sided global Lipschitz continuous, the diffusion coefficient is globally Lipschitz continuous and both coefficients grow at most linearly with respect to the supremum-norm on path space. We emphasize that our path-dependent stochastic Gronwall lemma allows us to consider the global monotonicity condition (3) jointly on \( \mu \) and \( \sigma \) with path-supremum on the right-hand side. This was not possible before. The following Theorem 1.1 illustrates our main result in this direction.

**Theorem 1.1.** Let \( d \in \mathbb{N} \), \( T, \tau, c \in [0, \infty) \), \( p \in [2, \infty) \), \( \mu \in C([0, T] \times \Omega ; \mathbb{R}^d), \mathbb{R}^d) \), \( \sigma \in C([-\tau, 0] \times \mathbb{R}^d) \), assume for all \( t \in [0, T] \), \( s \in [0, t] \), \( x, y \in C([-\tau, T], \mathbb{R}^d) \) that \( \| \mu(t, x) \| + \| \sigma(t, x) \| \leq c \sup_{s \in [-\tau, t]} \| a + \| x(s) \|_2^2 \| \), that \( \mu(t, \cdot), \sigma(t, \cdot) \) only depend on the interval \( [-\tau, t] \), that

\[
2 \langle x(t) - y(t), \mu(t, x) - \mu(t, y) \rangle + p \| \sigma(t, x) - \sigma(t, y) \| \leq c \sup_{s \in [-\tau, t]} \| x(s) - y(s) \|_2^2,
\]

and that

\[
\max \left\{ \| \mu(s, x) - \mu(t, x) \|_2^2, \| \sigma(s, x) - \sigma(t, x) \|_2^2 \right\} \leq c \left[ |t - s| + \sup_{u, v \in [0, t]} \| x(u) - x(v) \|_2^2 \right],
\]

(4)

let \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})\) be a filtered probability space which satisfies the usual conditions, let \( W : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) be a standard \( \mathbb{F} \)-Wiener process, let \( \zeta = (\xi_t(\omega))_{\omega \in \Omega, t \in [-\tau, 0]} : \Omega \rightarrow C([-\tau, 0], \mathbb{R}^d) \) be \( \mathbb{F}_0 \)-measurable, assume that \( \zeta \) and \( W \) are independent, let \( X : [-\tau, T] \times \Omega \rightarrow \mathbb{R}^d \) be adapted, have continuous sample paths, and satisfy for each \( r \in [-\tau, 0] \), \( t \in (0, T] \) a.s. that

\[
X_r = \xi_r \quad \text{and} \quad X_t = \xi_0 + \int_0^t \mu(s, X) \, ds + \int_0^t \sigma(s, X) \, dW_s,
\]

(5)

and for every \( n \in \mathbb{N} \) let \( Y^n : [-\tau, T] \times \Omega \rightarrow \mathbb{R}^d \) satisfy assume for all \( r \in [-\tau, 0] \), \( k \in [0, n - 1] \cap \mathbb{Z}, t \in \left( \frac{k}{n}, \frac{k + 1}{n} \right] \) that \( Y^n_r = \xi_r, Y^n_t = (k + 1 - \frac{kT}{n})Y^n_k + \frac{kT}{n} - k\) and

\[
Y^n_0 = \xi_0, \quad \text{and} \quad Y^n_{k+1} = Y^n_k + \mu(\frac{kT}{n}, Y^n) \frac{T}{n} + \sigma(\frac{kT}{n}, Y^n)(W_{(k+1)\frac{T}{n}} - W_{k\frac{T}{n}}).
\]

(6)

Then for every \( q \in [1, p) \) there exists \( C \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \) it holds that

\[
\left( \mathbb{E} \left[ \sup_{k \in [1, n] \cap \mathbb{N}} \| Y^n_k - X^n_k \|_2^q \right] \right)^{\frac{1}{q}} \leq Cn^{\frac{1}{2} - \frac{1}{q}}.
\]

(7)
Theorem 1.1 follows immediately from Theorem 3.1. Next we discuss the assumptions of Theorem 1.1. The coefficients of the SFDE (5) are assumed to depend only on the path up to the current time point to ensure that the solution process is progressively measurable. The condition (4) is satisfied for example for the running-path-supremum \([0, T] \times C([-\tau, T], \mathbb{R}^d) \ni (t, x) \mapsto \sup_{s \in [-\tau, t]} x_s\). Moreover, the coefficients are assumed to satisfy the global monotonicity condition (3). They do not need to be globally Lipschitz continuous. However, for convenience we assume that \(\mu\) and \(\sigma\) grow at most linearly. Otherwise the Euler-Maruyama approximations typically diverge in the strong sense; see [14, 15].

We selectively mention results from an incomplete selection of approximation results in the huge literature on SFDEs is \([17, 6, 22, 23, 2, 12, 1, 29, 24, 30, 19, 11, 19]\). A closely related result is Wu and Mao [30, Theorem 5.1] which establishes \(L^2\)-rate 0.5— if the coefficients functions are globally Lipschitz continuous with respect to the path-supremum. If the diffusion coefficient is globally Lipschitz continuous, then our stochastic Gronwall lemma is not needed and one can apply the Burkholder-Davis-Gundy inequality to the diffusion part. Another closely related result is Mehri and Scheutzow [24, Theorem 3.2] which proves that Euler approximations converge in probability if a local version of the monotonicity condition (3) holds.

2 A path-dependent stochastic Gronwall inequality

The following Theorem 2.1 is the main result of this article and establishes a path-dependent stochastic Gronwall inequality. The function \(V\) is typically chosen as and the squared norm \(V = ([0, T] \times O \ni (t, x) \mapsto \|x\|_H^2 \in [0, \infty)\) and then the Lyapunov-type condition (9) becomes a one-sided linear growth condition.

**Theorem 2.1.** Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) and \((U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)\) be separable \(\mathbb{R}\)-Hilbert spaces, let \(O \subseteq H\) be an open set, let \(T \in (0, \infty), p \in [1, \infty), V \in C^{1,2}([0, T] \times O, [0, \infty))\), let \(\alpha, \lambda; [0, T] \to [0, \infty)\) be measurable, let \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}\) be a filtered probability space which satisfies the usual conditions, let \((W_t)_{t \in [0, T]}\) be an \(\text{Id}_U\)-cylindrical \((\mathcal{F}_t)_{t \in [0, T]}\)-Wiener process, let \(X; [0, T] \times \Omega \to O, \beta, \gamma; [0, T] \times \Omega \to [0, \infty)\) be adapted, let \(a; [0, T] \times \Omega \to H\) be measurable, let \(b; [0, T] \times \Omega \to \text{HS}(U, H)\) be progressively measurable, assume that \(X\) has continuous sample paths, assume that for all \(t \in [0, T]\) it holds a.s. that \(\int_0^T (\|a_s\|_H + \|b_s\|^2_{\text{HS}(U, H)})\, ds < \infty\) and

\[
X_t = X_0 + \int_0^t a_s\, ds + \int_0^t b_s\, dW_s, \quad (8)
\]

and assume that a.s. it holds for all \(s \in [0, T]\) that

\[
\left(\frac{\partial}{\partial s} V\right)(s, X_s) + \left(\frac{\partial}{\partial x} V\right)(s, X_s) a_s + \frac{1}{2} \text{trace}\left(b_s b_s^*(\text{Hess}_x V)(s, X_s)\right)
+ \frac{p-1}{2} \frac{\|b_s^*(\nabla_x V)(s, X_s)\|_U^2}{V(s, X_s)} \leq \alpha_s \left[\sup_{r \in [0, s]} V(r, X_r)\right] + \beta_s \lambda_s + \gamma_s. \quad (9)
\]

Then for all \(q \in (0, p)\) it holds that

\[
\mathbb{E} \left[ \sup_{r \in [0, T]} V(r, X_r)^q \right] \leq \mathbb{E} \left[ (V(0, X_0))^p + \int_0^T (\beta_s^p \lambda_s + \gamma_s^p)\, ds \right]^{\frac{q}{p}} \frac{\exp\left(\int_0^T p \alpha_s + (p-1)(\lambda_s + 1)\, ds\right)}{\left(\frac{p}{2}\right)^{\frac{q}{2}} \left(1 - \frac{q}{p}\right)^{\frac{q}{2}}}.
\quad (10)
\]
Proof of Theorem 2.1. W.l.o.g. we assume that the right-hand side of (10) is finite, otherwise the assertion is trivial. First, (9), the fact that \( V, \alpha, \beta, \gamma, \lambda \geq 0 \), and the fact that \( \forall A, B \in (0, \infty): A^{1-\frac{1}{p}}B^{\frac{1}{p}} \leq (1 - \frac{1}{p})A + \frac{1}{p}B \) yield that a.s. it holds for all \( s \in [0, T], \varepsilon \in (0, 1) \) that

\[
p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial s} V \right)(s, X_s) + p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V \right)(s, X_s)a_s
\]

\[
+ \frac{1}{2}p(p - 1)(\varepsilon + V(s, X_s))^{p-2} ||b_s^\alpha(\nabla_x V)(s, X_s)||^2
\]

\[
= p(\varepsilon + V(s, X_s))^{p-1} \left[ \left( \frac{\partial}{\partial s} V \right)(s, X_s) + \left( \frac{\partial}{\partial x} V \right)(s, X_s)a_s
\right]
\]

\[
+ \frac{1}{2} \text{trace} (b_s b_s^\alpha(\nabla_x V)(s, X_s)) + \frac{p - 1}{2} \left[ \frac{b_s^\alpha(\nabla_x V)(s, X_s)}{\varepsilon + V(s, X_s)} \right]
\]

\[
\leq p \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right)^{p-1} \left[ \alpha_s \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right) + \beta_s \lambda_s + \gamma_s \right]
\]

\[
= p\alpha_s \left( \varepsilon + \sup_{r \in [0,s]} V(s, X_s) \right)^{p} + \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right)^{p} \left[ (\beta_s^p)^\frac{1}{p}\lambda_s + (\gamma_s^p)^\frac{1}{p} \right]
\]

\[
\leq p\alpha_s \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right)^{p} + \left( 1 - \frac{1}{p} \right) \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right)^{p} + \frac{1}{p} \beta_s^p \lambda_s
\]

\[
+ \frac{1}{p} \left( 1 - \frac{1}{p} \right) \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right)^{p} + \frac{1}{p} \gamma_s^p
\]

\[
= \left( p\alpha_s + (p - 1)(\lambda_s + 1) \right) \varepsilon + \sup_{r \in [0,s]} V(r, X_r) + \beta_s^p \lambda_s + \gamma_s^p. \tag{11}
\]

This, Itô’s formula, the fact that \( V \in C^{1,2}([0, T] \times O, [0, \infty)) \), and (8) show that for all \( \varepsilon \in (0, 1), t \in [0, T] \) it holds a.s. that

\[
(\varepsilon + V(t, X_t))^p = (\varepsilon + V(0, X_0))^p + \int_0^t p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V \right)(s, X_s)b_s dW_s
\]

\[
+ \int_0^t p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial s} V \right)(s, X_s) + p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V \right)(s, X_s)a_s
\]

\[
+ \frac{1}{2}p(p - 1)(\varepsilon + V(s, X_s))^{p-2} ||b_s^\alpha(\nabla_x V)(s, X_s)||^2 \right) ds
\]

\[
\leq (\varepsilon + V(0, X_0))^p + \int_0^t p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V \right)(s, X_s)b_s dW_s
\]

\[
+ \int_0^t \left[ p\alpha_s + (p - 1)(\lambda_s + 1) \right] \left( \varepsilon + \sup_{r \in [0,s]} V(r, X_r) \right)^{p} + \beta_s^p \lambda_s + \gamma_s^p \right] ds.
\]

This, [24, Theorem 2.2] (applied for every \( \varepsilon \in (0, 1), q \in (0, p) \) with \( X \sim \left( \varepsilon + V(\min\{t, T\}, X_{\min\{t,T\}}))^{\frac{p}{p-q}} \right)_{t \in [0, \infty)}, \ A \sim \left( J_{0}^{\min\{t,T\}} (p\alpha_s + (p - 1)(\lambda_s + 1)) ds \right)_{t \in [0, \infty)}, \ M \sim
\]

\[
4
\]
\[ (\int_0^T p(\varepsilon + V(s, X_s))^{p-1} \frac{\partial}{\partial x} V(s, X_s)b_s \, dW_s)_{t \in [0, \infty)}, \quad H \sim (\varepsilon + V(0, X_0))^p + \int_0^T (\beta^p \lambda_s + \gamma^p_s) \, ds \] 

in the notation of [24, Theorem 2.2]), the measurability and regularity assumptions of \( X \) and \( V \), the fact that \( p \geq 1 \), and nonnegativity of \( \alpha, \lambda, \varepsilon \in (0, 1) \) show for all \( q \in (0, p) \) that

\[
\mathbb{E} \left[ \left( \varepsilon + \sup_{r \in [0, T]} V(r, X_r) \right)^q \right] 
\leq \mathbb{E} \left[ \left( (\varepsilon + V(0, X_0))^p + \int_0^T \beta^p \lambda_s + \gamma^p_s \, ds \right)^{\frac{q}{p}} \right] \exp \left( \frac{\beta^T \left( p \sigma_{s+\eta} + (p-1)(\lambda_{s+\eta} + 1) \right) \mu_{s+\eta}}{(\frac{q}{p})^{\frac{q}{p}}(1 - \frac{q}{p})} \right).
\]

This, the dominated convergence theorem, and finiteness of the right-hand side of (10) complete the proof of Theorem 2.1.

\[ \square \]

### 3 Strong convergence rate for SFDEs

The following Theorem 3.1 is our main result on strong convergence rates for SFDEs. We note that if \( \sigma \) is globally Lipschitz continuous, then we may choose \( p \) arbitrarily large and then we obtain rate 0.5 –. We also note that all upper bounds are explicit and thus allow us to control dependencies, e.g., on the dimension to see which high-dimensional SFDEs can be approximated without curse of dimensionality.

**Theorem 3.1.** Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) and \((U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)\) be separable \( \mathbb{R} \)-Hilbert spaces, let \( T \in (0, \infty), \tau \in [0, \infty), c, a \in [1, \infty), \varepsilon \in (0, 1), \beta \in [0, \infty), p \in [2, \infty), \mu \in C([0, T] \times C([-\tau, T], H), H), \sigma \in C([0, T] \times C([-\tau, T], H), HS(U, H)), \) let \( n \in \mathbb{N}, t_0, t_1, \dotsc, t_n \in [0, T] \) satisfy for all \( t \in [0, T] \) that \( 0 = t_0 < t_1 < \dotsc < t_n = T, \) assume for all \( t \in [0, T], t \in [0, t], x_1, x_0 \in C([-\tau, T], H) \) that

\[ \forall s \in [-\tau, t]: x_1(s) = x_0(s) \implies [(\mu(t, x_1) = \mu(t, x_0)) \text{ and } (\sigma(t, x_1) = \sigma(t, x_0))], \]

\[ \|\mu(t, x_1)\|_H \leq c \sup_{s \in [-\tau, t]} \left[ a + \|x_1(s)\|_H^2 \right]^{\frac{1}{2}}, \quad \|\sigma(t, x_1)\|_{HS(U, H)}^2 \leq c \sup_{s \in [-\tau, t]} \left[ a + \|x_1(s)\|_H^2 \right]. \]

\[ 2 \langle \mu(t, x_1) - \mu(t, x_0), x_1(t) - x_0(t) \rangle_H + (p - 1)(1 + \varepsilon) \|\sigma(t, x_1) - \sigma(t, x_0)\|_{HS(U, H)}^2 \]

\[ \leq c \sup_{s \in [-\tau, T]} \|x_1(s) - x_0(s)\|_H^2, \]

\[ \max \left\{ \|\mu(t, x_1) - \mu(s, x_1)\|_H^2, \|\sigma(t, x_1) - \sigma(s, x_1)\|_H^2 \right\} \]

\[ \leq c \left[ |t - s| + \sup_{u, v \in [0, T]: |u - v| \leq |t - s|} \|x_1(u) - x_1(v)\|_H^2 \right] \left[ \sup_{s \in [-\tau, t]} \left( a + \|x_1(s)\|_H^2 \right) \right]. \]

let \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})\) be a filtered probability space which satisfies the usual conditions, let \( W = (W_s)_{s \in [0, T]}: [0, T] \times \Omega \to U \) be an \( \mathcal{F}_t \)-cylindrical \((\mathcal{F}_t)_{t \in [0, T]}\)-Wiener process, let \( \xi = (\xi_t(\omega))_{\omega \in \Omega, t \in [-\tau, 0]}: \Omega \to C([-\tau, 0], H) \) be \( \mathcal{F}_0 \)-measurable, assume that \( \xi \) and \( W \) are independent,
let $X^1, \mathcal{X}^1, X^0: [-\tau, T] \times \Omega \rightarrow H$ have continuous sample paths, assume that $(X^0_s)_{s \in [0,T]}$ is adapted, assume that for all $r \in [-\tau, 0], t \in (0, T]$ it holds a.s. that

$$X^0_r = \xi_r \quad \text{and} \quad X^0_t = \xi_0 + \int_0^t \mu(s, X^0) \, ds + \int_0^t \sigma(s, X^0) \, dW_s,$$  \hspace{1cm} (18)$$

and assume for all $r \in [-\tau, 0], k \in [0, n - 1] \cap \mathbb{Z}, t \in (t_k, t_{k+1}]$ that

$$X^1_t = \frac{t_{k+1} - t}{t_{k+1} - t_k} X^1_{t_k} + \frac{t - t_k}{t_{k+1} - t_k} X^1_{t_{k+1}},$$  \hspace{1cm} (19)$$

and assume for all $r \in [-\tau, 0]$ and assume for all $r \in [0, T]$ that

$$X^1_r = \xi_r, \quad \text{and} \quad X^1_t = X^1_{t_k} + \mu(t_k, \mathcal{X}^1)(t - t_k) + \sigma(t_k, \mathcal{X}^1)(W_t - W_{t_k}).$$  \hspace{1cm} (20)$$

Then

i) for all $i \in \{0, 1\}, q \in [2, \infty)$ it holds that

$$\mathbb{E} \left[ \left( \sup_{s \in [-\tau, T]} \left[ a + \|X^i_s \|_H^2 \right] \right)^{\frac{q}{2}} \right] \leq 7e^{38T^2 q^2} \mathbb{E} \left[ \left( a + \sup_{r \in [-\tau, 0]} \|\xi_r \|_H^2 \right)^{\frac{q}{2}} \right],$$  \hspace{1cm} (21)$$

ii) for all $u \in [0, T], \overline{u} \in [u, T], i \in \{0, 1\}, q \in [2, \infty)$ it holds that

$$\left\| \sup_{\Delta, \overline{\Delta} \in [u, \overline{u}]} \|X^i_\Delta - X^i_\overline{\Delta} \|_H \right\|_{L^q(\mathbb{P}; \mathbb{R})} \leq 7c q e^{39Tq} \left( \left( a + \sup_{r \in [-\tau, 0]} \|\xi_r \|_H^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} |\overline{u} - u|^\frac{1}{q},$$  \hspace{1cm} (22)$$

and

iii) for all $q \in [1, p)$ it holds that

$$\mathbb{E} \left[ \left( \sup_{s \in [0, T]} \|X^1_s - X^0_s \|_H^2 \right)^{\frac{q}{2}} \right] \leq \exp \left( \frac{r_{p+1}(\frac{e}{\varepsilon}(\frac{e}{\varepsilon} - 1))}{\frac{p+1}{2} r_{\frac{e}{\varepsilon}}(1-\frac{1}{2})} \right) T^{\frac{q}{2}} \mathbb{E} \left[ \left( a + \sup_{r \in [-\tau, 0]} \|\xi_r \|_H^2 \right)^{q \max(3, 1)} \right] \cdot \left\{ c + \varepsilon + \frac{e-p+e-p}{e} \right\} 202300c^2 p^2 e^{230Tq \max\{\beta^2, 1\}} |\delta_1| \left( |T| / |\delta_1| \right)^{1/4}.\right\}^{\frac{q}{2}}$$  \hspace{1cm} (23)$$

**Proof of Theorem 3.1.** By the fact that $\xi$ is $\mathbb{F}_0$-measurable and by conditioning on $\mathbb{F}_0$ it suffices to assume that $\xi$ is deterministic. Throughout the rest of this proof let $\mathcal{X}^0: [0, T] \times \Omega \rightarrow H$ satisfy that $\mathcal{X}^0 = X^0$ and let $\delta_0, \delta_1: [0, T] \rightarrow \mathbb{R}$ satisfy for all $t \in [0, T]$ that

$$\delta_0(t) = t, \quad |\delta_0| = 0, \quad \delta_1(t) = \mathbb{1}_{[0]}(t) + \sum_{i=1}^n \mathbb{1}_{[t_{i-1}, t_i]}(t), \quad \text{and} \quad |\delta_1| = \sup_{i \in [1, n] \cap \mathbb{Z}} |t_i - t_{i-1}|.$$  \hspace{1cm} (24)$$

First, (20) shows for all $k \in [0, n - 1] \cap \mathbb{Z}, t \in (t_k, t_{k+1}]$ that $\delta_1(t) = t_k$ and hence a.s. it holds that

$$X^1_t = X^1_{t_k} + \mu(t_k, \mathcal{X}^1)(t - t_k) + \sigma(t_k, \mathcal{X}^1)(W_t - W_{t_k})$$

$$= X^1_{t_k} + \int_{t_k}^t \mu(s, \mathcal{X}^1) \, ds + \int_{t_k}^t \sigma(s, \mathcal{X}^1) \, dW_s.$$  \hspace{1cm} (25)$$
This, (20), induction, (18), and (24) show for all \( i \in \{0,1\}, r \in [-\tau,0], t \in [0,T] \) that a.s. it holds that

\[
X_i^t = \xi_r \quad \text{and} \quad X_i^t = \xi_0 + \int_0^t \mu(\delta_i(s), \mathcal{X}^i) \, ds + \int_0^t \sigma(\delta_i(s), \mathcal{X}^i) \, dW_s.
\]  

(26)

This, a standard property of affine linear interpolations, (19), and the fact that \( \mathcal{X}^0 = X^0 \) show for all \( i \in \{0,1\}, t \in [0,T] \) that a.s. it holds that

\[
\sup_{s \in [-\tau,t]} \|X_s^i\|_H \leq \sup_{s \in [-\tau,t]} \|X_s\|_H.
\]

(27)

In addition, note for all \( \mathbf{b} \in \text{HS}(U,H), x \in H \) that

\[
\text{trace}(bb^*) = \|b\|_{\text{HS}(U,H)}^2 \quad \text{and} \quad \|b^*x\|_U^2 \leq \|b^*\|_{\text{HS}(U,U)}^2 \|x\|_H^2 \leq \|b\|_{\text{HS}(U,H)}^2 \|x\|_H^2.
\]

(28)

This, the Cauchy–Schwarz inequality, (15), (27), and the fact that \( \forall i \in \{0,1\}, r \in [-\tau,0] : X_i^t = \xi_r \) (see (26)) show that for all \( i \in \{0,1\}, t \in [0,T], q \in [1,\infty) \) it holds a.s. that

\[
\langle \mu(\delta_i(t), \mathcal{X}^i), 2X_i^t \rangle_H + \frac{1}{2} \text{trace} \left( ((\sigma \sigma^*)(\delta_i(t), \mathcal{X}^i)2\text{Id}_H \right) + \frac{1}{4} \frac{1}{a} \|X_i^t\|_H^2 \leq \frac{1}{a} \|X_i^t\|_H^2 + \|\sigma(\delta_i(t), \mathcal{X}^i)\|_{\text{HS}(U,H)}^2 + 2(q-1) \|\sigma(\delta_i(t), \mathcal{X}^i)\|_{\text{HS}(U,H)}^2
\]

(29)

\[
\leq c \sup_{s \in [-\tau,t]} \left[ a + \|X_s^i\|_H^2 \right] + (2q-1) c \sup_{s \in [-\tau,t]} \left[ a + \|X_s^i\|_H^2 \right] \leq 3qc \sup_{s \in [-\tau,t]} \left[ a + \|X_s^i\|_H^2 \right] + 3qc \sup_{s \in [-\tau,t]} \|\xi_r\|_H^2.
\]

This, Theorem 2.1 (applied for every \( q \in [1,\infty) \), \( i \in \{0,1\} \) with \( p \subset q \), \( O \subset H \), \( V \subset ([0,T] \times H \ni (t,x) \mapsto (a + \|x\|_H^2 \in [0,\infty)) \), \( \alpha \subset ([0,T] \ni t \mapsto 3qc \in [0,\infty)) \), \( \lambda \subset ([0,T] \ni t \mapsto 3qc \in [0,\infty)) \), \( \beta \subset ([0,T] \times \Omega \ni (t,\omega) \mapsto \sup_{s \in [-\tau,t]} \|\xi_s^i\|_H^2 \in [0,\infty]) \), \( \gamma \subset ([0,T] \ni T \sup_{s \in [-\tau,t]} \|\xi_s\|_H^2 \in [0,\infty]) \), \( q \subset 0.5q \) in the notation of Theorem 2.1), the fact that \( \xi \) is deterministic, the fact that \( q \cdot 3qc + (q-1)(3qc+1) = 3q^2c + 3q^2c - 3qc + q - 1 \leq 6q^2c \), the fact that \( \frac{0.5q}{q} \left( 1 - \frac{0.5q}{q} \right)^{q/4} = 0.5^{2.5} \), the fact that \( \frac{0.5q}{q} \left( 1 - \frac{0.5q}{q} \right)^{q/4} = 0.5^{2.5} \), the fact that \( q_{-1}^{-2.5} \leq 6 \), and the fact that

\[
\sqrt{1 + 3Tq^2c} \leq e^{1.5Tq^2c} \text{ show for all } q \in [1,\infty), \, i \in \{0,1\} \text{ that}
\]

\[
\mathbb{E} \left[ \left( \sup_{s \in [0,T]} \left[ a + \|X_s^i\|_H^2 \right] \right)^{q/2} \right] \leq \exp \left( \frac{Q_0 \left( 3qc + (q-1)(3qc+1) \right)}{\frac{0.5q}{q} \left( 1 - \frac{0.5q}{q} \right)^{q/4} + \frac{0.5q}{q} \left( 1 - \frac{0.5q}{q} \right)^{q/4}} \right) \mathbb{E} \left[ \left( a + \|X_s^i\|_H^2 \right)^q + 3Tq^2 \sup_{r \in [-\tau,t]} \|\xi_r\|_H^2 \right] \left( \sup_{r \in [-\tau,t]} \|\xi_r\|_H^2 \right)^{q/4}
\]

(30)

\[
\leq 6e^{6Tq^2c} \sqrt{1 + 3Tq^2} \left( a + \sup_{r \in [-\tau,t]} \|\xi_r\|_H^2 \right)^{q/4} \leq 6e^{38Tq^2c} \left( a + \sup_{r \in [-\tau,t]} \|\xi_r\|_H^2 \right)^{q/4}.
\]
This shows for all \( q \in [1, \infty), i \in \{0, 1\} \) that
\[
\mathbb{E}\left[ \left( \sup_{s \in [-\tau, T]} \left[ a + \|X_s^i\|_H^2 \right] \right)^{\frac{q}{2}} \right] \leq 7e^{38Tq^2c} \left( a + \sup_{r \in [-\tau, 0]} \|\xi_r\|_H^2 \right)^{\frac{q}{2}}. \tag{31}
\]

This and the fact that \( \xi \) is deterministic show (i).

Next, (26), the triangle inequality, the Burkholder-Davis-Gundy inequality (see, e.g., [7, Lemma 7.7]), (15), (27), the fact that \( c \geq 1 \), the fact that \( \forall q \in [2, \infty): 1 + \sqrt{0.5q(q-1)} \leq \sqrt{2(1 + 0.5q(q-1))} = \sqrt{q^2 - q + 2} \leq q \), the fact that \( \forall s, t \in [0, T]: |t - s| \leq \sqrt{T}|t - s| \leq e^{0.5T} \sqrt{|t - s|} \), and (31) show for all \( \underline{u} \in [0, T], \bar{u} \in [\underline{u}, T], i \in \{0, 1\}, q \in [2, \infty) \) that
\[
\left\| \sup_{\underline{u} \leq s \leq \bar{u}} \left| X_s^i - X_{\bar{u}}^i \right| \right\|_{L^q(\mathbb{P}; \mathbb{R})} 
= \left\| \sup_{\underline{u} \leq s \leq \bar{u}} \left\| \int_{\underline{u}}^{\bar{u}} \mu(\delta_i(r), \mathcal{X}^i) \, dr + \int_{\underline{u}}^{s} \sigma(\delta_i(r), \mathcal{X}^i) \, dW_r \right\| \right\|_{L^q(\mathbb{P}; \mathbb{R})} 
\leq \int_{\underline{u}}^{\bar{u}} \left\| \mu(\delta_i(r), \mathcal{X}^i) \right\|_{L^q(\mathbb{P}; \mathbb{R})} \, dr + \sqrt{\frac{q(q-1)}{2}} \left\| \int_{\underline{u}}^{\bar{u}} \left\| \sigma(\delta_i(r), \mathcal{X}^i) \right\|_{H^s(U, H)} \right\|_{L^q(\mathbb{P}; \mathbb{R})} \, dr 
\leq \int_{\underline{u}}^{\bar{u}} c \left\| \sup_{s \in [-\tau, T]} \left[ a + \|X_s^i\|_H^2 \right] \right\|_{L^q(\mathbb{P}; \mathbb{R})} \, dr 
+ \sqrt{\frac{q(q-1)}{2}} \left[ \int_{\underline{u}}^{\bar{u}} c \left\| \sup_{s \in [-\tau, T]} \left[ a + \|X_s^i\|_H^2 \right] \right\|_{L^q(\mathbb{P}; \mathbb{R})} \, dr \right]^{\frac{q}{2}} \tag{32}
\leq c \left( 1 + \sqrt{\frac{q(q-1)}{2}} \right) \max \left\{ |\bar{u} - \underline{u}|, |\bar{u} - \underline{u}|^{\frac{1}{2}} \right\} \left( \mathbb{E} \left[ \left( \sup_{s \in [-\tau, T]} \left[ a + \|X_s^i\|_H^2 \right] \right)^{\frac{q}{2}} \right] \right)^{\frac{1}{q}} \n\leq c q e^{0.5T} |\bar{u} - \underline{u}|^{\frac{1}{2}} \mathbb{E}^{38Tq^2c} \left( a + \sup_{r \in [-\tau, 0]} \|\xi_r\|_H^2 \right)^{\frac{q}{2}} \n\leq 7c q e^{38Tq^2c} \left( a + \sup_{r \in [-\tau, 0]} \|\xi_r\|_H^2 \right)^{\frac{q}{2}} |\bar{u} - \underline{u}|^{\frac{1}{2}}. \]

This and the fact that \( \xi \) is deterministic show (ii).

For the next step for every \( i \in \{0, 1\}, t \in [0, T] \) let \( \mathcal{X}^{i,t}: [0, T] \times \Omega \to H \) have continuous sample paths and satisfy for all \( s \in [0, T] \) that a.s. it holds that
\[
\mathcal{X}^{1,t}_s = \mathbb{1}_{[-\tau, \delta_1(t)]}(s) \mathcal{X}^{1}_s + \mathbb{1}_{(\delta_1(t), t)}(s) \left[ \frac{t - s}{t - \delta_1(t)} \mathcal{X}^{1}_s + \frac{s - \delta_1(t)}{t - \delta_1(t)} X^1_t \right] + \mathbb{1}_{[t, T]}(s) X^1_t, \tag{33}
\]
\[
\mathcal{X}^{0,t} = X^0. \]

Then (27) yields for all \( t \in [0, T], i \in \{0, 1\} \) that a.s. it holds that
\[
\sup_{s \in [-\tau, t]} \|\mathcal{X}_s^{i,t}\|_H \leq \sup_{s \in [-\tau, t]} \|X_s^i\|_H. \tag{34}
\]
Furthermore, (33) and (19) show that for all $k, \ell \in [0, n-1] \cap \mathbb{Z}, \mathfrak{s}, \mathfrak{t}, \tau \in [0, T]$ with $\tau \in [t_k, t_{k+1}], \mathfrak{s} \in [t_\ell, t_{\ell+1}], \mathfrak{s} \leq \mathfrak{t} \leq \mathfrak{t}$, $\max\{t_\ell, t_k\} < t$ it holds a.s. that

$$\mathcal{X}_{\mathfrak{s}}^{1,t} = \frac{\mathfrak{s} - t_k}{\min\{t, t_{k+1}\} - t_k} X_{\min\{t, t_{k+1}\}}^{1} + \frac{\min\{t, t_{k+1}\} - \mathfrak{s}}{\min\{t, t_{k+1}\} - t_k} X_{t_k}^{1}$$

(35)

and

$$\mathcal{X}_{\mathfrak{t}}^{1,t} = \frac{\mathfrak{t} - t_\ell}{\min\{t, t_{\ell+1}\} - t_\ell} X_{\min\{t, t_{\ell+1}\}}^{1} + \frac{\min\{t, t_{\ell+1}\} - \mathfrak{t}}{\min\{t, t_{\ell+1}\} - t_\ell} X_{t_\ell}^{1}.$$  

(36)

This and the triangle inequality show that for all $k, \ell \in [0, n-1] \cap \mathbb{Z}, \mathfrak{s} \in [t_k, t_{k+1}], \mathfrak{s} \in [t_\ell, t_{\ell+1}], \mathfrak{u}, \mathfrak{v}, t \in [0, T]$ with $\mathfrak{u} \leq \mathfrak{s} \leq \mathfrak{t} \leq \mathfrak{v} \leq t$ it holds a.s. that

$$\|\mathcal{X}_{\mathfrak{s}}^{1,t} - \mathcal{X}_{\mathfrak{t}}^{1,t}\|_H = \left\| \frac{\mathfrak{s} - t_k}{\min\{t, t_{k+1}\} - t_k} (X_{\min\{t, t_{k+1}\}}^{1} - X_{\min\{t, t_{k+1}\}}^{1}) \right\| + \left\| \frac{\min\{t, t_{k+1}\} - \mathfrak{s}}{\min\{t, t_{k+1}\} - t_k} (X_{t_k}^{1} - X_{\min\{t, t_{k+1}\}}^{1}) + \frac{\min\{t, t_{k+1}\} - \mathfrak{t}}{\min\{t, t_{\ell+1}\} - t_\ell} (X_{t_\ell}^{1} - X_{\min\{t, t_{\ell+1}\}}^{1}) \right\|_H$$

(37)

$$\leq 2 \sup_{r_1, r_2 \in [\delta_1(t), \min\{\mathfrak{s} + \delta_1, \mathfrak{t}\}]} \left\| X_{r_1}^{1} - X_{r_2}^{1} \right\|_H.$$  

This, (32), and the triangle inequality show that for all $t, \mathfrak{u}, \mathfrak{v} \in [0, T], q \in [2, \infty)$ with $\mathfrak{u} \leq \mathfrak{v} \leq \delta_1(t)$ it holds that

$$\left\| \mathcal{X}_{\mathfrak{u}}^{1,t} - \mathcal{X}_{\mathfrak{v}}^{1,t} \right\|_{L^q(\mathcal{P}; \mathbb{R})} \leq 14 c q e^{39 T^q} \left( a + \sup_{r \in [-\tau, 0]} \left\| \xi_r \right\|_H^{2} \right)^{\frac{1}{2}} \left[ \mathfrak{v} - \mathfrak{u} + 2 |\delta_1| \right]^{\frac{1}{2}}.$$  

(38)

This, (32), (33), and the fact that $\forall A, B \in [0, \infty): \sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$ show that for all $t, \mathfrak{u}, \mathfrak{v} \in [0, T], q \in [2, \infty), i \in \{0, 1\}$ with $\mathfrak{u} \leq \mathfrak{v} \leq t$ it holds that

$$\left\| \sup_{\mathfrak{s}, \mathfrak{t} \in [\mathfrak{u}, \mathfrak{v}]} \left\| \mathcal{X}_{\mathfrak{s}}^{i,t} - \mathcal{X}_{\mathfrak{t}}^{i,t} \right\|_{H} \right\|_{L^q(\mathcal{P}; \mathbb{R})} \leq 14 c q e^{39 T^q} \left( a + \sup_{r \in [-\tau, 0]} \left\| \xi_r \right\|_H^{2} \right)^{\frac{1}{2}} \left[ \mathfrak{v} - \mathfrak{u} + 2 |\delta_1| \right]^{\frac{1}{2}}.$$  

(39)

This, the triangle inequality, and the fact that $\forall A, B \in [0, \infty), q \in [1, \infty): (A + B)^q \leq 2^{q-1} (A^q + B^q)$ show for all $i \in \{0, 1\}$ that

$$\left( E \left[ \sup_{u, v \in [-\tau, \tau]} \left\| \mathcal{X}_{u}^{i,t} - \mathcal{X}_{v}^{i,t} \right\|_H^{q} \right] \right)^{\frac{1}{q}} \leq 2^{q-1} E \left[ \sum_{k=0}^{\lceil T/|\delta_1| \rceil} \sup_{u, v \in [k |\delta_1|, \min\{k+1 |\delta_1|, T\}] \left\| \mathcal{X}_{u}^{i,t} - \mathcal{X}_{v}^{i,t} \right\|_H^{q} \right]^{\frac{1}{q}}$$

(40)

$$\leq 2 \frac{T}{|\delta_1|} \frac{1}{q} 14 c q e^{39 T^q} \left( a + \sup_{r \in [-\tau, 0]} \left\| \xi_r \right\|_H^{2} \right)^{\frac{1}{2}} 3 |\delta_1|^{\frac{1}{2}} \frac{T}{|\delta_1|}$$

$$= 84 c q e^{39 T^q} \left( a + \sup_{r \in [-\tau, 0]} \left\| \xi_r \right\|_H^{2} \right)^{\frac{1}{2}} 3 |\delta_1|^{\frac{1}{2}} \frac{T}{|\delta_1|}.$$
Furthermore, (19), (33), and the triangle inequality show for all $k \in \mathbb{Z} \cap [0, n - 1]$, $s \in (t_k, t_{k+1}]$, $t \in [s, T]$ that a.s. it holds that

$$
X_s^{1,t} = \frac{\min\{t, t_{k+1}\} - s}{\min\{t, t_{k+1}\} - t_k} X_{t_k}^1 + \frac{s - t_k}{\min\{t, t_{k+1}\} - t_k} X_{t_{min(t,k+1)}}^1
$$

and hence

$$
\|X_s^{1,t} - X_s^0\|_H
= \left\| \frac{\min\{t, t_{k+1}\} - s}{\min\{t, t_{k+1}\} - t_k} (X_{t_k}^1 - X_s^0) + \frac{s - t_k}{\min\{t, t_{k+1}\} - t_k} (X_{t_{min(t,k+1)}}^1 - X_s^0) \right\|_H
$$

(41)

$$
\quad
+ \frac{\min\{t, t_{k+1}\} - s}{\min\{t, t_{k+1}\} - t_k} (X_{t_k}^0 - X_s^0) + \frac{s - t_k}{\min\{t, t_{k+1}\} - t_k} (X_{t_{min(t,k+1)}}^0 - X_s^0)
$$

(42)

$$
\leq \left[ \sup_{r \in [-\tau, t]} \|X_r^1 - X_r^0\|_H \right] + \left[ \sup_{s \in [-\tau, t]} \|X_s^0 - X_s^0\|_H \right].
$$

For the next step let $\Gamma: [0, T] \times \Omega \to \mathbb{R}$ satisfy that for all $t \in [0, T]$ it holds a.s. that

$$
\Gamma_t = (c + \varepsilon) \left[ \sup_{s \in [-\tau, t]} \|X_s^0 - X_s^0\|_H \right]
$$

(43)

$$
+ \frac{\beta + \varepsilon + p}{\varepsilon} c \left[ |\delta_1| + \sup_{u,v \in [-\tau, t], [a-b] \subseteq [\delta_1]} \|X_u^1 - X_v^1\|_H \right] \left[ \sup_{s \in [-\tau, t]} (a + \|X_s^1\|^2)^\beta \right].
$$

Then Hölder’s inequality, the triangle inequality, (40), and (31) show for all $t \in [0, T]$ that

$$
\|\Gamma_t\|_{L^2(\mathbb{P}; \mathbb{R})} \leq (c + \varepsilon) \left[ \sup_{s \in [-\tau, t]} \|X_s^0 - X_s^0\|_H \right] \left[ \sup_{s \in [-\tau, t]} (a + \|X_s^1\|^2)^\beta \right]
$$

(44)
In addition, (28) and the fact that \( p \in [2, \infty) \) show for all \( a, x \in H, b \in \text{HS}(U, H) \) that
\[
2 \langle a, x \rangle_H + \frac{1}{2} \text{trace} (bb^*2I_d_H) + \frac{0.5p-1}{2} \frac{\|2b^*x\|_U^2}{\|x\|_H^2} \leq 2 \langle a, x \rangle_H + (p-1)\|b\|_{\text{HS}(U,H)}^2. \tag{45}
\]
This, (14), (33), the fact that \( \forall A, B \in H : 2 \langle A, B \rangle_H \leq \frac{1}{\epsilon} \|A\|_H^2 + \epsilon \|B\|_H^2 \), and the fact that \( \forall A, B \in H : \|A + B\|_H \leq (1 + \epsilon)\|A\|_H + (1 + \frac{1}{\epsilon})\|B\|_H \) show for all \( t \in [0, T] \) that a.s. it holds that
\[
\begin{align*}
2 \langle a, x \rangle_H &+ \frac{1}{2} \text{trace} (bb^*2I_d_H) + \frac{0.5p-1}{2} \frac{\|2b^*x\|_U^2}{\|x\|_H^2} \\
&\leq 2 \langle \mu(\delta(t), X^t) - \mu(t, X^0), X^t \rangle_H + (p-1) \|\sigma(\delta(t), X^t) - \sigma(t, X^0)\|_{\text{HS}(U,H)}^2 \\
&= 2 \langle \mu(\delta(t), X^{\alpha,t}) - \mu(t, X^{\alpha,0}), X^t \rangle_H \\
&\quad + (p-1) \|\sigma(\delta(t), X^{\alpha,t}) - \sigma(t, X^{\alpha,0})\|_{\text{HS}(U,H)}^2 \\
&= 2 \langle \mu(\delta(t), X^{\alpha,t}) - \mu(t, X^{\alpha,0}), X^t \rangle_H + 2 \langle \mu(\delta(t), X^{\alpha,t}) - \mu(t, X^{\alpha,t}), X^t \rangle_H \\
&\quad + (p-1) \|\sigma(\delta(t), X^{\alpha,t}) - \sigma(t, X^{\alpha,0})\|_{\text{HS}(U,H)}^2 \\
&\leq 2 \langle \mu(\delta(t), X^{\alpha,t}) - \mu(t, X^{\alpha,0}), X^t \rangle_H \\
&\quad + \frac{1}{\epsilon} \|\mu(\delta(t), X^{\alpha,t}) - \mu(t, X^{\alpha,t})\|_H^2 + \epsilon \|X^t - X_0\|_H^2 \\
&\quad + (p-1)(1 + \epsilon) \|\sigma(\delta(t), X^{\alpha,t}) - \sigma(t, X^{\alpha,0})\|_{\text{HS}(U,H)}^2 \\
&\quad + (p-1)(1 + \frac{1}{\epsilon}) \|\sigma(\delta(t), X^{\alpha,t}) - \sigma(t, X^{\alpha,t})\|_{\text{HS}(U,H)}^2
\end{align*}
\]
This, (16), (17), the fact that \( \frac{1}{\epsilon} + (p-1)(1 + \frac{1}{\epsilon}) = \frac{e^{p-\epsilon}+p}{\epsilon}, \) (42), (34), and (43) show for all \( t \in [0, T] \) that a.s. it holds that
\[
\begin{align*}
2 \langle a, x \rangle_H &+ \frac{1}{2} \text{trace} (bb^*2I_d_H) + \frac{0.5p-1}{2} \frac{\|2b^*x\|_U^2}{\|x\|_H^2} \\
&\leq (c + \epsilon) \sup_{s \in [-\tau, t]} \|X^{\alpha,s}_t - X^{\alpha,0}_t\|_H \\
&\quad + \frac{e^{p-\epsilon}+p}{\epsilon} \left[ \langle \delta \rangle + \sup_{u,v \in [-\tau,t] : |u-v| \leq |\delta|} \|X^{\alpha,u}_t - X^{\alpha,v}_t\|_H^2 \right] \\
&\quad + \frac{e^{p-\epsilon}+p}{\epsilon} \left[ \langle \delta \rangle + \sup_{u,v \in [-\tau,t] : |u-v| \leq |\delta|} \|X^{\alpha,u}_t - X^{\alpha,v}_t\|_H^2 \right] \\
&\quad + \frac{e^{p-\epsilon}+p}{\epsilon} \langle \delta \rangle + \sup_{r \in [-\tau,t]} \langle X^0_r - X^0_r \rangle_H^2 \\
&\quad + (c + \epsilon) \left[ \langle \delta \rangle + \sup_{r \in [-\tau,t]} \langle X^0_r - X^0_r \rangle_H^2 \right] \\
&\quad + (c + \epsilon) \left[ \langle \delta \rangle + \sup_{r \in [-\tau,t]} \langle X^0_r - X^0_r \rangle_H^2 \right] + \Gamma_t.
\end{align*}
\]
This, (26), (18), Theorem 2.1 (applied for every \( q \in [1, p] \) with \( p \leq 0.5p, O \subseteq H, V \subseteq ([0, T] \times H \ni (t,x) \mapsto \|x\|_H \in [0, \infty)) \), \( \alpha \subseteq ([0, T] \ni t \mapsto c + \epsilon \in [0, \infty)) \), \( \lambda \subseteq ([0, T] \ni t \mapsto 0 \in [0, \infty)) \), \( X \subseteq X^1 - X^0, a \subseteq \langle \mu(\delta(s), X^1) - \mu(s, X^0) \rangle_{s \in [0,T]} \), \( b \subseteq \langle \sigma(\delta(s), X^1) - \sigma(s, X^0) \rangle_{s \in [0,T]} \), \( \beta \subseteq ([0, T] \times \Omega \ni (t,x) \mapsto 0 \in [0, \infty)) \), \( \gamma \subseteq \Gamma_t \), \( q \leq 0.5q \) in the notation of Theorem 2.1, the
fact that \( c \geq 1 \), Jensen’s inequality, Tonelli’s theorem, and (44) show for all \( q \in [1, p) \) that

\[
\mathbb{E} \left[ \left( \sup_{s \in [0, T]} \| X_s^1 - X_s^0 \|_H^2 \right)^{\frac{q}{2}} \right] \leq \exp \left( \frac{\int_0^T \left( 0.5(p-c\varepsilon) + 0.5(p-1) \right) ds}{\left( \frac{q}{p} \right)^{\frac{q}{2}+1} (1 - \frac{q}{p})} \right) \mathbb{E} \left[ \left( \int_0^T \Gamma_s^{0.5p} ds \right)^{\frac{0.5q}{p+q}} \right]
\]

\[
\leq \exp \left( \frac{T(p+c\varepsilon)}{\frac{q}{p} \left( 1 - \frac{q}{p} \right)} \right) \left( \int_0^T \mathbb{E} [\Gamma_s^{0.5p}] ds \right)^{\frac{0.5q}{p+q}} \leq \exp \left( \frac{T(p+c\varepsilon)}{\frac{q}{p} \left( 1 - \frac{q}{p} \right)} \right) \mathbb{E} \left[ \left( \int_0^T \Gamma_s^{0.5p} ds \right)^{\frac{0.5q}{p+q}} \right] \sup_{s \in [0, T]} \| \Gamma_s \|_{L^{\frac{q}{p}}(P; \mathbb{R})}^{\frac{q}{p}}
\]

\[
\leq \frac{T(p+c\varepsilon)}{\left( \frac{q}{p} \right)^{\frac{q}{2}+1} (1 - \frac{q}{p})} T^{-\frac{q}{2}} \left\{ c + \varepsilon + \frac{\varepsilon^{p-c\varepsilon}}{c^2} \right\} 202300c^2 \varepsilon^{230Tpc \max \{\beta^2, 1\}}
\]

\[
\left[ a + \sup_{\tau \in [-\tau, 0]} || \xi_{\tau} ||_H^2 \right]^{\max \{\beta, 1\}} | \delta_1 | | T | | \delta_1 | | \frac{1}{p} \right\}^{\frac{q}{2}}
\]

The proof of Theorem 3.1 is thus completed. \( \Box \)

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