Adeles on \( n \)-dimensional schemes and categories \( C_n \).

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Abstract

We consider categories \( C_n \) which are very close to the iterated functor \( \lim \), which was introduced by A.A. Beilinson in [2]. We prove that an adelic space on \( n \)-dimensional Noetherian scheme is an object of \( C_n \).

1 Introduction

In this note we want to introduce by induction some class of infinite-dimensional vector spaces and morphisms between them. These spaces depend on integer \( n \), we call such spaces as \( C_n \)-spaces.

\( C_0 \)-spaces are finite dimensional spaces. A first basic example of \( C_1 \)-space is the field of Laurent series \( k((t)) \). If \( \mathcal{O}(n) = t^n k[[t]] \), then we have a filtration

\[ \ldots \mathcal{O}(m) \subset \mathcal{O}(m - 1) \subset \mathcal{O}(m - 2) \ldots \]

and every factor space \( \mathcal{O}(m - k)/\mathcal{O}(m) \) is a finite dimensional vector space over \( k \).

We can consider the morphisms between two such spaces as continuous linear maps. We remark that these morphisms can be described only in terms of filtration \( \mathcal{O}_n \), without considering topology on \( k((t)) \). From this point of view, the space of adeles on an algebraic curve has a structure of \( C_1 \)-space, which is filtered by partially ordered set of coherent sheaves on the curve.

We construct an iterated version of \( C_1 \)-spaces, which we call a \( C_n \)-space. But in our construction we do not consider the structure of completion. So, we consider the filtered vector spaces. For example, the discrete valuation field is also a \( C_1 \)-space, the space of rational adeles from [14] is also a \( C_1 \)-space.

The constructions of similar categories were introduced also in [2], [7], [6]. Our construction of \( C_n \) is very close to the iterated functor \( \lim \), which was introduced by A.A. Beilinson in appendix to [2]. The main difference is that we consider non-completed version of \( \lim \), i.e., filtered spaces, but with morphisms which come from \( \lim \). From this point of view the categories \( C_n \) are rather closed to the dir-inv modules which were considered by A. Yekutieli in [15] for \( n = 1 \).

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The main result of this note is the theorem [1] where we prove that the space of Parshin-Beilinson adeles on an \( n \)-dimensional Noetherian scheme \( V \) over \( k \) is a \( C_n \)-space, which is filtered by partially ordered set of coherent sheaves on the scheme. We calculate also the endomorphism algebra of \( n \)-dimensional local field.

Adeles on algebraic surfaces were introduced by A.N.Parshin in [13]. A.A. Beilinson generalized it to arbitrary Noetherian schemes in [1]. Adeles on higher-dimensional schemes were applied to a lot of problems of algebraic geometry, see [4].

A.N. Parshin has pointed out to me that the categories \( C_n \) can be useful for constructing of harmonic analysis on higher-dimensional schemes and higher-dimensional adeles and local fields, see [12].

In this note we consider the categories \( C_n \) and the schemes over a field \( k \). But all the constructions, for example, can be moved to arithmetical schemes, where \( C_0 \) are finite abelian groups, \( C_1 \) are filtered abelian groups with finite abelian group factors and so on.

2 Categories \( C_n \)

2.1 Constructions

2.1.1 Objects in \( C_n \)

Definition 1 We say that \((I, F, V)\) is a filtered \( k \)-vector space, if

1. \( V \) is a vector space over the field \( k \),
2. \( I \) is a partially ordered set, such that for any \( i, j \in I \) there are \( k, l \in I \) with \( k \leq i \leq l \) and \( k \leq j \leq l \),
3. \( F \) is a function from \( I \) to the set of \( k \)-vector subspaces of \( V \) such that if \( i \leq j \) are any from \( I \), then \( F(i) \subset F(j) \),
4. \( \bigcap_{i \in I} F(i) = 0 \) and \( \bigcup_{i \in I} F(i) = V \).

Definition 2 We say that a filtered vector space \((I_1, F_1, V)\) dominates another filtered vector space \((I_2, F_2, V)\) when there is a preserving order function \( \phi : I_2 \to I_1 \) such that

1. for any \( i \in I_2 \) we have \( F_1(\phi(i)) = F_2(i) \)
2. for any \( j \in I_1 \) there are \( i_1, i_2 \in I_2 \) such that \( \phi(i_1) \leq j \leq \phi(i_2) \).

Now we define by induction the category of \( C_n \)-spaces and morphisms between them.

Definition 3 1. The category \( C_0 \) is the category of finite-dimensional vector spaces over \( k \) with morphisms coming from \( k \)-linear maps between vector spaces.
2. The triple from \( C_0 \)

\[
0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0
\]

is admissible when it is an exact triple of vector spaces.
Now we define the objects of the category $C_n$ by induction. We suppose that we have already defined the objects of the category $C_{n-1}$ and the notion of admissible triple in $C_{n-1}$.

**Definition 4**

1. Objects of the category $C_n$, i.e. $\text{Ob}(C_n)$, are filtered $k$-vector spaces $(I, F, V)$ with the following additional structures
   (a) for any $i \leq j \in I$ on the $k$-vector space $F(j)/F(i)$ it is given a structure $E_{i,j} \in \text{Ob}(C_{n-1})$,
   (b) for any $i \leq j \leq k \in I$

   $$0 \to E_{i,j} \to E_{i,k} \to E_{j,k} \to 0$$

   is an admissible triple from $C_{n-1}$.

2. Let $E_1 = (I_1, F_1, V_1), E_2 = (I_2, F_2, V_2)$ and $E_3 = (I_3, F_3, V_3)$ be from $\text{Ob}(C_n)$. Then we say that

   $$0 \to E_1 \to E_2 \to E_3 \to 0$$

   is an admissible triple from $C_n$ when the following conditions are satisfied
   (a)

   $$0 \to V_1 \to V_2 \to V_3 \to 0$$

   is an exact triple of $k$-vector spaces
   (b) the filtration $(I_1, F_1, V_1)$ dominates the filtration $(I_2, F'_1, V_1)$, where $F'_1(i) = F_2(i) \cap V_1$ for any $i \in I_2$,   (c) the filtration $(I_3, F_3, V_3)$ dominates the filtration $(I_2, F'_3, V_3)$, where $F'_3(i) = F_2(i)/F_2(i) \cap V_1$,   (d) for any $i \leq j \in I_2$

   $$0 \to \frac{F'_1(j)}{F'_1(i)} \to \frac{F_2(j)}{F_2(i)} \to \frac{F'_3(j)}{F'_3(i)} \to 0$$

   is an admissible triple from $C_{n-1}$. (By definition of $\text{Ob}(C_n)$, on every vector space from triple (1) it is given the structure of $\text{Ob}(C_{n-1})$).

### 2.1.2 Morphisms in $C_n$

By induction, we define now the morphisms in the category $C_n$. We suppose that we have already defined the morphisms in $C_{n-1}$.

**Definition 5** Let $E_1 = (I_1, F_1, V_1)$ and $E_2 = (I_2, F_2, V_2)$ be from $\text{Ob}(C_n)$. Then $\text{Mor}_{C_n}(E_1, E_2)$ consists of elements $A \in \text{Hom}_k(V_1, V_2)$ such that the following conditions hold.

1. for any $i \in I_1$ there is an $j \in I_2$ such that $A(F_1(i)) \subset F_2(j)$,
2. for any $j \in I_2$ there is an $i \in I_1$ such that $A(F_1(i)) \subset F_2(j)$,
3. for any $i_1 \leq i_2 \in I_1$ and $j_1 \leq j_2 \in I_2$ such that $A(F_1(i_1)) \subset F_2(j_1)$ and $A(F_1(i_2)) \subset F_2(j_2)$ we have that the induced $k$-linear map

$$\tilde{A} : \frac{F_1(i_2)}{F_1(i_1)} \rightarrow \frac{F_2(j_2)}{F_2(j_1)}$$

is an element from

$$\text{Mor}_{C_n-1}(\frac{F_1(i_2)}{F_1(i_1)}, \frac{F_2(j_2)}{F_2(j_1)})$$

Now we want to prove that the compositions of so defined morphisms in $C_n$ will be again a morphism. We need the following definition.

**Definition 6** Let $E_1, E_2$ be from $\text{Ob}(C_n)$.

1. A $k$-linear map $C : E_1 \rightarrow E_2$ is an admissible $C_n$-monomorphism when it is the part of an admissible triple from $C_n$

$$0 \rightarrow E_1 \xrightarrow{C} E_2 \rightarrow E_3 \rightarrow 0$$

2. A $k$-linear map $D : E_1 \rightarrow E_2$ is an admissible $C_n$-epimorphism when it is the part of an admissible triple

$$0 \rightarrow E_3 \rightarrow E_1 \xrightarrow{D} E_2 \rightarrow 0$$

We have the following proposition.

**Proposition 1** Let $E_1 = (I_1, F_1, V_1)$, $E_2 = (I_2, F_2, V_2)$, $E'_1$, $E'_2$ be from $\text{Ob}(C_n)$ and $A$ is a $k$-linear map form $\text{Hom}_k(V_1, V_2)$.

1. If $B : E_3 \rightarrow E_1$ is an admissible $C_n$-epimorphism then $A \in \text{Mor}_{C_n}(E_1, E_2)$ if and only if $A \circ B \in \text{Mor}_{C_n}(E_3, E_2)$.

2. If $B : E_2 \rightarrow E_3$ is an admissible $C_n$-monomorphism, then $A \in \text{Mor}_{C_n}(E_1, E_2)$ if and only if $B \circ A \in \text{Mor}_{C_n}(E_1, E_3)$.

3. If the filtered vector space $E_1$ dominates the filtered vector space $E'_1$ and the filtered vector space $E_2$ dominates the filtered vector space $E'_2$, then $A \in \text{Mor}_{C_n}(E_1, E_2)$ if and only if $A \in \text{Mor}_{C_n}(E'_1, E'_2)$.

4. $\text{Mor}_{C_n}(E_1, E_2)$ is a $k$-linear subspace of $\text{Hom}_k(V_1, V_2)$.

5. If $E_3$ is an object of $C_n$, then

$$\text{Mor}_{C_n}(E_2, E_3) \circ \text{Mor}_{C_n}(E_1, E_2) \subset \text{Mor}_{C_n}(E_1, E_3)$$

**Proof**. The first two statements follow by induction on $n$.

The third statement follows from the first and the second statement.

The other statements follow by induction on $n$ using the previous statements.

We give the proof of the fifth statement. Let $A \in \text{Mor}_{C_n}(E_1, E_2)$ and $B \in \text{Mor}_{C_n}(E_2, E_3)$. We have to prove that $B \circ A \in \text{Mor}_{C_n}(E_1, E_3)$. We have to check for $B \circ A$ the conditions of definition Let $E_3 = (I_3, F_3, V_3)$. 

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For any $i_1 \in I_1$ there is $i'_2 \in I_2$ such that $A(F_1(i_1)) \subset F_2(i'_2)$. For $i'_2 \in I_2$ there is $i''_3 \in I_3$ such that $B(F_2(i'_2)) \subset F_3(i''_3)$. Therefore $B \circ A(F_1(i_1)) \subset F_3(i''_3)$.

Analogously for any $j_3 \in I_3$ we find $j'_2 \in I_2$ such that $B(F_2(j'_2)) \subset F(j_3)$. For $j'_2 \in I_2$ we find $j''_1 \in I_1$ such that $A(F_1(j''_1)) \subset F_2(j'_2)$. Then $B \circ A(F_1(j''_1)) \subset F_3(j_3)$.

Now let $i_1 \geq j_1 \in I_1$ and $i_3 \geq j_3 \in I_3$ such that

$$B \circ A(F_1(j_1)) \subset F_3(j_3) \quad \text{and} \quad B \circ A(F_1(i_1)) \subset F_3(i_3).$$

Now we fix any $i''_3 \in I_3$ such that $i''_3 \geq i'_3$ and $i''_3 \geq i_3$. We fix any $j''_1 \in I_1$ such that $j''_1 \leq j_1$ and $j''_1 \leq j'_1$. Then, by items 1 and 2 of this proposition, the map induced by $B \circ A$ belongs to $\text{Mor}_{C_{n-1}}(F_1(i_1)/F_2(j_1), F_3(i''_3))$. But the last induced map is the composition of the induced map by $A$ in $\text{Mor}_{C_{n-1}}(F_1(i_1)/F_2(j_1), F_3(i''_3))$. By induction, the composition of morphisms from $C_{n-1}$ is a morphism from $C_{n}$. The proposition is proved.

### 2.2 Examples

#### 2.2.1 Linearly locally compact spaces.

**Definition 7** A topological $k$-vector space $V$ (over a discrete field $k$) is linearly compact (see [8] ch.2 §6, [9] ch.III, §2, ex.15-21) when the following conditions hold

1. $V$ is complete and Hausdorff,
2. $V$ has a base of neighbourhoods of 0 consisting of vector subspaces,
3. each open subspace of $V$ has finite codimension.

**Definition 8** A topological $k$-vector space $W$ (over a discrete field $k$) is linearly locally compact (see [9]) when it has a basis of neighborhoods of 0 formed by linearly compact open subspaces.

Any topological linearly locally compact space is a $C_1$-space, where filtration is given by linearly compact open subspaces.

Any field of discrete valuation is $C_1$-space. And the completion functor gives us the linearly locally compact vector space. Moreover, if $E = (I,F,V)$ is a $C_1$-space, we take

$$\Phi_1(E) = \lim_{i \to I} \lim_{j \leq i} F(i)/F(j).$$

The space $\Phi_1(E)$ is a linearly locally compact space. And all the linearly locally compact spaces can be obtained in a such way.

Moreover, we define by induction on $n$ the functor of completion $\Phi_n$ from $C_n$-spaces to $C_n$-spaces. We put

$$\Phi_n(E) = \lim_{i \to I} \lim_{j \leq i} \Phi_{n-1}(F(i)/F(j))$$

where $E = (I,F,V)$ is a $C_n$-space. From properties of $C_n$ we obtain that this functor $\Phi_n$ is well-defined.
2.2.2 Contragredient space

We will define by induction on \( n \) a contravariant functor of contragredient space \( D_n \) from \( C_n \)-spaces to \( C_n \)-spaces. Let \( E = (I, F, V) \) be a \( C_n \)-space.

If \( n = 0 \), then \( D_0(V) = V^* \), where \( V^* \) is the dual space to \( V \).

If \( n \geq 1 \), then we put
\[
D_n(V) = \lim_{\rightarrow} \lim_{\leftarrow} D_{n-1}(F(i)/F(j)).
\]

Then \( D_n(E) = (I^0, F^0, D_n(V)) \) is a \( C_n \)-space, where \( I^0 \) is a partially ordered set, which has the same set as \( I \), but with the inverse order then \( I \), and
\[
F^0(j) = \lim_{\rightarrow} D_{n-1}(F(i)/F(j)).
\]

It is easy to see by induction on \( n \) that \( D_n(V) \subset V^* \), the functor \( D_n \) maps admissible triples to admissible triples, and \( D_n(D_n(E)) = \Phi_n(E) \).

2.2.3 Adelic space.

The next example of \( C_n \)-space is the space of adeles on a Noetherian \( n \)-dimensional scheme \( V \) (see \([1],[5],[10]\)).

Let \( P(V) \) be the set of points of the scheme \( V \). Consider \( \eta, \nu \in P(V) \). Define \( \eta \geq \nu \) if \( \nu \in \overline{\{\eta\}} \). \( \geq \) is a half ordering on \( P(X) \). Let \( S(V) \) be the simplicial set induced by \( (P(X), \geq) \), i.e.
\[
S(V)_m = \{(\nu_0, \ldots, \nu_m) \mid \nu_i \in P(V) ; \nu_i \geq \nu_{i+1}\}
\]
is the set of \( m \)-simplices of \( S(V) \) with the usual boundary and degeneracy maps.

In \([1],[5]\) for any \( K \subset S(V)_m \) it was constructed the space \( \mathbb{A}(K, \mathcal{F}) \) for any quasicoherent sheaf \( \mathcal{F} \) on \( V \) such that
\[
\mathbb{A}(K, \mathcal{F}) \subset \prod_{\delta \in K} \mathbb{A}(\delta, \mathcal{F})
\]

**Theorem 1** Let \( V \) be a Noetherian \( n \)-dimensional scheme over \( k \), \( K \subset S(V)_n \), and \( \mathcal{F} \) be a coherent sheaf on \( V \). Then the adelic space \( \mathbb{A}(K, \mathcal{F}) \) has a structure of \( C_n \)-space.

**Proof**. We denote
\[
K_0 = \{\eta \in S(V)_0 \mid (\eta > \eta_1 \ldots > \eta_n) \in K \text{ for some } \eta_i \in P(V)\}.
\]

For \( \eta \in K_0 \) we denote
\[
\eta K = \{(\eta_1 > \ldots > \eta_n) \in S(V)_{n-1} \mid (\eta > \eta_1 \ldots > \eta_n) \in K\}.
\]

We have
\[
\mathbb{A}(K, \mathcal{F}) = \prod_{\eta \in K_0} \mathbb{A}(\eta K, \mathcal{F}_\eta).
\]
If \((I_1,F_1,V_1)\) and \((I_2,F_2,V_2)\) are \(C_n\)-spaces, then \((I_1 \times I_2,F_1 \times F_2,V_1 \times V_2)\) is a \(C_n\)-space as well. Moreover, any finite product of \(C_n\)-spaces is a \(C_n\)-space in the same way. The set \(K_0\) is finite, therefore it is enough to define a \(C_n\)-structure on \(\mathcal{A}(\eta K,F_{\eta})\) for every \(\eta \in K_0\).

For \(\eta \in K_0\) we define a partially ordered set

\[
I_\eta(\mathcal{F}) = \{G \subset F_\eta \mid G \text{ is a coherent sheaf on } \eta, \ G_\eta = F_\eta\},
\]

which is ordered by inclusions of sheaves.

The functor \(\mathcal{A}(\eta K, )\) is an exact functor. Therefore for any \(G \in I_\eta(\mathcal{F})\) we have an embedding \(\mathcal{A}(\eta K, G) \longrightarrow \mathcal{A}(\eta K,F_{\eta})\).

If \(G_1 \subset G_2\) are from \(I_\eta(\mathcal{F})\), then from the exactness of the functor \(\mathcal{A}(\eta K, )\) we have

\[
\mathcal{A}(\eta K,G_2)/\mathcal{A}(\eta K,G_1) = \mathcal{A}(\eta K,G_2/G_1).
\]

We have \((G_2/G_1)_\eta = 0\). Therefore \(G_2/G_1\) is a coherent sheaf on some subscheme \(Y\) of dimension \(n - 1\). Therefore \(\mathcal{A}(\eta K,G_2/G_1) = \mathcal{A}(S(Y) \cap \eta K,G_2/G_1)\). We apply induction on dimension of scheme to the \(n - 1\)-dimensional scheme \(Y\). Therefore we have that \(\mathcal{A}(S(Y) \cap \eta K,G_2/G_1)\) is a \(C_{n-1}\)-space.

We check now that this \(C_n\)-structure is well defined. It is enough to prove only conditions 2b and 2c of definition 4. They follow from the following three statements.

The first statement claims that every exact triple of quasicoherent sheaves on \(V\) is a direct limit of exact triples of coherent sheaves on \(V\), see [5 lemma 1.2.2].

The second statement claims that on any irreducible Noetherian scheme \(X\) for any exact triple of coherent sheaves

\[
0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0
\]

and any coherent subsheaves \(G_1 \subset F_1\) and \(G_3 \subset F_3\) such that \((F_1/G_1)_\eta = 0\) and \((F_3/G_3)_\eta = 0\), where \(\eta\) is the general point of \(V\), there exists a coherent subsheaf \(G_2 \subset F_2\) such that \((G_2 \cap F_1)_\eta \subset G_1\), \((G_2/(G_2 \cap F_1))_\eta \subset G_3\) and \((F_2/G_2)_\eta = 0\). It is enough to construct \(F_2\) locally, where this sheaf exists by the Artin-Rees lemma.

The third statement claims that for any two quasicoherent subsheaves \(F_1, F_2\) of a quasicoherent sheaf \(F_3\) on \(V\) we have

\[
\mathcal{A}(K,F_1 \cap F_2) = \mathcal{A}(K,F_1) \cap \mathcal{A}(K,F_2),
\]

\[
\mathcal{A}(K,F_1 + F_2) = \mathcal{A}(K,F_1) + \mathcal{A}(K,F_2),
\]

where the intersection and sum is taken inside of \(\mathcal{A}(K,F_3)\). It follows from exactness of the functor \(\mathcal{A}(K, )\) and the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{A}(K,F_1 + F_2) & = & \mathcal{A}(K,F_2) \\
\mathcal{A}(K,F_1) & = & \mathcal{A}(K,F_3) \\
\end{array}
\]

which gives that the vertical arrows are isomorphisms.

The theorem is proved.
Remark 1 For the smooth surface $V$ a $C_2$-structure on $\mathbb{A}(K, \mathcal{F})$ can be defined by filtration of Cartier divisors. (We use that the Cartier divisors coincide with the Weil divisors for the smooth varieties).

Remark 2 The structure of $C_n$ space can be defined on introduced in [5] §5.2 spaces of rational adeles $a(K, \mathcal{F})$ as well. Then the functor of completion $\Phi_n$ (see section 2.2.1) applied to the space of rational adeles gives the space $\mathbb{A}(K, \mathcal{F})$.

2.3 The endomorphism algebra of $n$-dimensional local field

We suppose now that $\delta = (\eta_0 > \ldots > \eta_n) \in S(V)_n$ on the $n$-dimensional scheme $V$ has the following property: $\eta_n$ is a smooth point on every scheme $\eta_i$.

Then $(\mathcal{O}_V)_\delta = \mathbb{A}(\delta, \mathcal{O}_V) = k'(t_n)\ldots(t_1)$ is an $n$-dimensional local field, where $k' = k(\eta_n)$. We demand that the local parameters $t_i \in (\mathcal{O}_V)_{\eta_i}$ for any $i$.

Then we define the filtration on $k'(t_n)\ldots(t_1)$ by $E_{l} = t_1^l k'(t_n)\ldots(t_2)[t_1]$ for $l \in \mathbb{Z}$. On each factor $E_{l_1}/E_{l_2}$ of this filtration we define the new filtration given in $E_{l_1}/E_{l_2}$ by images of $t_m^2 t_1^l k'(t_n)\ldots(t_3)[t_2, t_1]$ for $m \in \mathbb{Z}$ and so on.

We obtained the structure of $C_n$-space on $(\mathcal{O}_V)_\delta$. And the structure of $C_n$-space constructed on $(\mathcal{O}_V)_\delta$ in theorem I dominates the constructed now structure of $C_n$-space.

Now let $K = k((t_n))\ldots(t_1))$. We define the $k$-algebra

$$\text{End}_K = \text{Mor}_{C_n}(K, K)$$

Let $\bar{K} = k((t_n))\ldots(t_2))$. Then $K = \bar{K}(t_1))$. For any element $A \in \text{End}_K$ we consider the matrix $\{A_{ij}\}_{i,j \in \mathbb{Z}}$ if $A_{ij} \in \text{End}_k(\bar{K})$ given by

$$A(x t_n^i) = \sum_j A_{ij}(x) t_n^j \quad \text{with} \quad x \in \bar{K}.$$ 

Proposition 2 An endomorphism $A \in \text{End}_k(\bar{K})$ belongs to $\text{End}_K$ if and only if the following conditions are satisfied.

1. There is a nondecreasing function $a : \mathbb{Z} \rightarrow \mathbb{Z}$, $a(i) \rightarrow \infty$ when $i \rightarrow \infty$ such that for $j < a(i)$ all elements $A_{ij} = 0$.

2. Any element $A_{ij}$ belongs to $\text{End}_{\bar{K}}$.

Proof follows directly from the definition of morphisms between $C_n$-spaces and the definition of $C_n$-structure on $K$. The proposition is proved.

We consider on $K$ the topology of $n$-dimensional local field (see [I]). Let $\text{End}^c_{k}(K)$ be the algebra of continuous $k$-linear endomorphisms of $K$.

Proposition 3 1. $\text{End}_K \subset \text{End}^c_{k}(K)$

2. If $n = 1$, then $\text{End}_K = \text{End}^c_{k}(K)$.

3. If $n > 1$, then $\text{End}^c_{k}(K)$ bigger then $\text{End}_K$. 

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Proof. We denote by $O_K = \bar{K}[[t_1]]$. The base of neighbourhoods of 0 in $\bar{K}((t_1))$ consists of the following $k$-vector subspaces:

$$\sum_i U_i t_i + O_K t_1^m,$$

where $U_i$ are open $k$-vector subspaces from $\bar{K}$ and $m$ is an integer.

Then statement 2 is the same as condition 2 of definition 5. Statement 1 follows by induction on $n$ from the definition 5, for $n = 2$ see [9, lemma 2]. Now we give an example for statement 3.

Let $n = 2$. Then we consider a $k$-linear map $\phi: K \rightarrow K$ defined as following. For $U = k[[t_2]]((t_1)) + k((t_2))[[t_1]]$ we put $\phi(U) = 0$ and the induced map $\phi : K/U \rightarrow K$ we put on monomials $t_2 t_1^m$ by the rule: if $m = -1$ then $\phi(t_2 t_1^m) = t_2 t_1^{m+1}$, for other monomials we put $\phi = 0$. Then $\phi$ is continuous, but $\phi$ is not from $End_K$. The proposition is proved.

**Proposition 4**  
1. For any $m$ we have an embedding $End_K^{\oplus m} \hookrightarrow End_K$.

2. For any $m$ we have an embedding $gl(m, K) \hookrightarrow End_K$.

Proof We have $K = \bar{K}((t_n))$. Let $e_i$, $1 \leq i \leq m$ be the standard basis of $K^{\oplus m}$, i.e. $K^m = \oplus Ke_m$. We consider a $\bar{K}$-isomorphism $\phi$ of $\bar{K}$-vector spaces

$$\phi : K^{\oplus m} \rightarrow K \quad \phi(t_n^{i} e_i) = t_n^{m+1} 1 \leq i \leq m.$$ 

This isomorphism induces an isomorphism

$$Mor_{C_n}(K^{\oplus m}, K^{\oplus m}) \hookrightarrow End_K.$$ 

Now the proposition follows from the natural embeddings:

$$End_K^{\oplus m} \hookrightarrow Mor_{C_n}(K^{\oplus m}, K^{\oplus m}) \quad \text{and} \quad gl(m, K) \hookrightarrow Mor_{C_n}(K^{\oplus m}, K^{\oplus m}).$$

**Remark 3** From proposition 4 we have embeddings of

$$K^* \hookrightarrow End_K$$

and toroidal Lie algebras

$$gl(m, k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]) \hookrightarrow End_K.$$ 

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References

[1] Beilinson A.A., Residues and Adeles, Funct. Anal. Pril., 14 (1980), no. 1, 44-45; English transl. in Func. Anal. Appl., 14, no. 1 (1980), 34-35.

[2] Beilinson A.A., How to glue perverse sheaves, in K-theory, Arithmetic, Geometry, Lecture Notes in Mathematics 1289.

[3] Bourbaki N., Algebre commutative, Hermann, 1961.

[4] Fimmel T., Parshin A.N., An introduction to the higher adelic theory, preprint 1999.

[5] Huber A., On the Parshin-Beilinson Adeles for Schemes, Abh. Math. Sem. Univ. Hamburg, 61 (1991), 249-273.

[6] Kapranov M.M., Semi-infinite symmetric powers, e-print math.QA/0107089.

[7] Kato K., Existence theorem for higher local class field theory, Invitation to higher local fields (Münster, 1999), 165–195 (electronic), Geom. Topol. Monogr., 3, Geom. Topol. Publ., Coventry, 2000, see also math.AG/0012150

[8] Lefschetz S., Algebraic topology, AMS Colloquium Publications 27, Amer. Math. Soc. New York, (1942).

[9] Osipov D.V., Central extensions and reciprocity laws on algebraic surfaces, (Russian) Mat. Sb. 196:10, 111-136; translation in Sb. Math. 196:10, 1503-1527, see also math.NT/0501155

[10] Osipov D.V., n -dimensional local fields and adeles on n -dimensional schemes, e-print math.AG/0508205

[11] Parshin A. N., Local class field theory, Proc. Steklov Inst. Math. 165 (1985), 157-185.

[12] Parshin A. N., Higher-dimensional local fields and L-functions, Invitation to higher local fields (Münster, 1999), 199–213 (electronic), Geom. Topol. Monogr., 3, Geom. Topol. Publ., Coventry, 2000, see also math.AG/0012151

[13] Parshin A.N., On the arithmetic of two-dimensional schemes I, Repartitions and residues, Math. USSR Izv. Vol.10 (1976) No.4, 695-729 (Engl.)

[14] Serre J.-P., Groupes algebriques et corps de classes, Hermann, 1959.

[15] Yekutieli A., Continuous and twisted $L_{\infty}$ morphisms, math.QA/0502137, to appear in Journal of Pure and Applied Algebra.

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