BARDEEN–ANOMALY AND WESS–ZUMINO TERM IN THE SUPERSYMMETRIC STANDARD MODEL

S. Ferrara♣∗, A. Masiero♦, M. Porrati♥ and R. Stora♠

♣ Theory Division, CERN, Geneva, Switzerland
♦ Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Italy
♥ Department of Physics, New York University, New York, USA
♠ LAPP, P.O. Box 110 F-74941, Annecy le Vieux Cedex, France

Abstract

We construct the Bardeen anomaly and its related Wess–Zumino term in the supersymmetric standard model. In particular we show that it can be written in terms of a composite linear superfield related to supersymmetrized Chern–Simons forms, in very much the same way as the Green–Schwarz term in four–dimensional string theory. Some physical applications, such as the contribution to the g–2 of gauginos when a heavy top is integrated out, are briefly discussed.
I. Introduction

The experimental fact that the mass of the top quark is much larger than all the other quark and lepton masses has recently led several authors to study the behaviour of the electroweak theory in the regime of infinite top quark mass \([1,2]\). This mathematical limit is not only useful for computational purposes but it is also a rich laboratory where one may investigate the interplay between gauge anomalies and non-decoupling of heavy particles which are originally chiral with respect to a gauge group \(G\) which is then spontaneously broken to an anomaly free subgroup \(H\) by the Higgs mechanism, giving then an \(H\)-invariant mass to the fermions through \((G\text{-invariant})\) Yukawa couplings.

The best evidence of non-decoupling of heavy fermions with large Yukawa couplings (while keeping the Higgs v.e.v. fixed) is the phenomenon of anomaly cancellation via Wess–Zumino terms obtained by integrating out the heavy fermions in loop-diagrams. Since \(H\) is anomaly free only the generators of \(G/H\) are anomalous and then the Wess–Zumino term just cancels these anomalies. In the electroweak theory the structure group is \(G = SU(2)_L \times U(1)_R\) and \(H\) is the electromagnetic \(U(1)\) subgroup of \(G\).

In the present paper we extend this analysis to the supersymmetric version of the standard model where now the entire (chiral) top multiplet is integrated out and we investigate the interplay between gauge anomalies and supersymmetric Wess–Zumino terms.

In the supersymmetric case new phenomena occur with respect to the conventional theory [2]:

i) A gauge anomaly induces a supersymmetry anomaly, because of the Wess–Zumino gauge-fixing condition.

ii) The supersymmetric version of the Wess–Zumino term is not only gauge variant but also violates supersymmetry. Indeed its gauge variation cancels the gauge anomaly while its supersymmetry transformation cancels the supersymmetry anomaly.

iii) The violation of supersymmetry due to supersymmetry anomalies also manifests itself in gauge-invariant processes involving the Higgs particles, the vector bosons and gauginos such as the \(H^\pm W^\mp \gamma\) vertex and the magnetic moment of charged gauginos. It also gives corrections to the Higgs scalar potential.

In the electroweak theory, due to the fact that the non-abelian part of \(G\), based on \(SU(2)\), is anomaly free, and that \(H\), based on \(U(1)_{em}\), is abelian,
the specific form of the Wess–Zumino term which cancels the Bardeen anomaly is fairly simple and in fact can be viewed as a four–dimensional version of the Green–Schwarz mechanism occurring in certain 4D strings\[^3\]. The main difference here is that the antisymmetric tensor $b_{\mu\nu}$ is not a fundamental field but a composite field made out of the Goldstone fields and of the gauge fields.

This observation allows us to derive a fairly simple supersymmetrization of this term, working in $N = 1(4D)$ superspace, without making use of the rather complicated formulae of the non–abelian gauge supersymmetry anomalies derived by different groups some years ago\[^4,5,6\].

II. Bardeen Anomaly Revisited and its Wess–Zumino Term

We start by reminding some properties of the four–dimensional gauge anomaly. Given a gauge group $G$ based on a structure group $G$, for each symmetric invariant polynomial $P$ of degree 3 on Lie $G$, one can construct a corresponding gauge anomaly:

$$G(\alpha, A) = \int_{M_4} P(\alpha d(AdA + \frac{1}{2} A^3)), \quad (1)$$

where $M_4$ is spacetime, $\alpha$ the gauge parameter, and $A$ the gauge field. Here a form language has been used ($AB = A \wedge B$). The corresponding Wess–Zumino action is\[^7\]

$$\Gamma_{WZ}(A, g) = \int_{M_4} \int_0^1 dt G(g^{-1}(t) \frac{d}{dt} g(t), A_{g(t)} = g(t)^{-1} dg(t) + g^{-1}(t) Ag(t)), \quad (2)$$

where $g(0) =$ identity, $g(1) = g$ (e.g. $g(t) = e^{it\xi}$, if $g = e^{i\xi}$, $\xi \in \text{Lie } G$).

To discuss the anomalies and related W–Z action for the standard model we have actually to depart from the consistent chiral anomaly and consider the Bardeen anomaly\[^8\] since we are interested in the electromagnetic gauge invariant W–Z action related to the coset $(SU(2)_L \times U(1)_R / U(1)_{em})$ rather than to a chiral group $\mathcal{G}$.

Here we are using at first a general method, described in ref.\[^7\], to obtain a (generalized) Bardeen anomaly and Bardeen W–Z action on $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is required to be an anomaly free subgroup of $\mathcal{G}$ which means that the invariant polynomial $P$ vanishes when Lie $G$ is restricted to Lie $H$.

The Bardeen anomaly is then

$$G^B(\alpha, A) = G(\alpha, A) + \delta_\alpha \Delta^B(A), \quad (3)$$
where

$$G^B(\alpha, A) = 0$$  \hspace{1cm} (4)$$

for \(\alpha \in \text{Lie } \mathcal{H}\).

Explicit expressions are known for \(G^B, \Delta^B\) as well as for the W–Z action \([4,9]\) which, by virtue of (3), is simply given by

$$\Gamma^B_{WZ}(g, A) = \Gamma^A_{WZ}(g, A) - \Delta(A) + \Delta(A_g)$$

(5)

The point is that \(\Gamma^B_{WZ}\) is \(\mathcal{H}\) gauge invariant and therefore depends only on the coset elements \(\hat{g}\) in \(G/\mathcal{H}\).

Hence

$$\Gamma^B_{WZ}(g, A) = \Gamma^B_{WZ}(\hat{g}, A) \quad .$$

(6)

In the case of the standard model a major simplification occurs because of the particular structure of the \(G = SU(2) \times U(1)\) and \(U(1)_{em}\) groups. In fact \(G\) does not have any non–abelian anomaly and \(\mathcal{H}\) is abelian.

The invariant polynomial is taken to be

$$T_0^3 - T_0 T_a^2 \quad (T_0 \in \text{Lie } U(1), T_a \in \text{Lie } SU(2)), \quad a = 1, 2, 3,$$

(7)

and

$$G(\alpha_0, \vec{\alpha}) = \alpha_0(\text{Tr } F(W)F(W) - F(B)F(B)), \quad \alpha = (\alpha_0, \vec{\alpha}).$$

(8)

The fields \(W\) and \(B\) correspond, respectively, to the \(SU(2)\) and \(U(1)\) components of the gauge connection \(A\). In this case it is easy to get \(\Delta^B\) and we obtain:

$$\Delta^B = BQ(W) + BW_3 dB,$$

(9)

where \(Q(W)\) is the Chern-Simons three-form with the property

$$dQ(W) = \text{Tr } F(W)F(W), \quad (F(W) = dW + W^2).$$

(10)

\(Q(W)\) can be written in terms of a one-parameter family of (interpolating) gauge potentials \(W(t)\) \((0 \leq t \leq 1, W(0) = 0, W(1) = W)\) as follows

$$Q(W) = 2 \int_0^1 dt \text{Tr } (\dot{W}(t)F(t)).$$

(11)
For different choices of $W(t)$ we obtain gauge-equivalent $Q(W)$, i.e. $Q(W) = Q(W') + dP$, where $P$ is a two form. The choice $W(t) = tW$ yields the standard expression

$$Q(W) = Tr(Wdw + \frac{2}{3}W^3),$$

(12)

with gauge transformation

$$\delta_\alpha Q(W) = d(\bar{\alpha}d\bar{W}).$$

We work in the doublet representation of $SU(2)$ so that

$$W_3dB \rightarrow \frac{2}{3}Tr W dB$$

where $B = B\sigma_3/2$.

The $U(1)$ anomaly is:

$$G + \delta_\alpha \Delta^B = -\alpha_0F(B)F(B) - \alpha_0dW_3dB$$

(13)

The $SU(2)$ anomaly is ($\alpha = \alpha_3, \alpha_i, i = 1, 2$):

$$\delta_\alpha \Delta^B = Bd(\bar{\alpha}d\bar{W}) + BD_\alpha W_3dB$$

$$= \alpha_i dW_i dB + \alpha_3 F(B)F(B) - \epsilon_{3ij} \alpha_i W_j B dB$$

(14)

so that

$$G^B(\alpha_i, \alpha_3 - \alpha_0) = (\alpha_3 - \alpha_0)(dB dB + dW_3 dB) +$$

$$+ \alpha_i dW_i dB - \epsilon_{3ij} \alpha_i W_j B dB$$

(15)

The associated $\Gamma^B_{WZ}$ is then given by

$$\Gamma^B_{WZ}(g, A) = \Gamma^B_{WZ}(g, \bar{W}, B) =$$

$$= \int \left[ g_0(Tr F(\bar{W})F(\bar{W}) - F(B)F(B)) - \Delta^B(\bar{W}, B) + \Delta^B(\bar{W}_g, B_{g_0}) \right]$$

(16)

where $(g_0, g_i)$ denotes the group element of $G$ transforming as $g_0 \rightarrow g_0 - \alpha, g \rightarrow \ell g$ under $G$, and $\Delta^B$ is given by eq. (9).

From eq. (16) we get

$$\int \left[ g_0(Tr F(\bar{W})F(\bar{W}) - F(B)F(B)) + \int [(B + dg_0)Q(\bar{W}_g) +$$

$$+ (B + dg_0)W_3^g dB] - \Delta^B(W, B) = \int B[J(g) - J(g = 1)],$$

(17)

where
\[ J(g, A) = Q(\tilde{W}_g) - dg_0 dB + W_3^g dB - dg_0 dW_3^g. \]  

(18)

Hence we recognize that

\[ \int BJ(g, A) = \Gamma_{WZ}(g, A) + \Delta^B(A_g) \]  

(19)

while

\[ -\int BJ(g = 1) = -\Delta^B(A). \]  

(20)

Also \( J(g, A) \) can be made completely \( U(1) \times SU(2) \) invariant by noticing that we may covariantize \( dg_0 \rightarrow dg_0 + B \) by terms which vanish when inserted in eq. (19):

\[ J_{GW}(g, A) = Q(\tilde{W}_g) - (dg_0 + B) dB + W_3^g dB - (dg_0 + B) dW_3^g \]  

(21)

which satisfies

\[ dJ_{GW}(g, A) = Tr(F(\tilde{W}) F(\tilde{W})) - dBdB \]  

(22)

Also the gauge invariance of eq. (18) implies that we may restrict \( g \) to the coset element \( U \)

\[ U = e^{i\xi_i \sigma_i / 2} \]  

(23)

so that

\[ J_{GW}(g, A) = J_{GW}(U, A) = Q(WU) - BdB - 2d(Tr W^U B) \]  

(24)

Therefore we can finally write

\[ \Gamma_{WZ}^B(U, A) = \int B[Q(W) + Q(W_3^U, B) - Q(W) - Q(W_3, B)], \]  

(25)

where \( Q(W_3, B) = W_3 dB - BdW_3 = -d(W_3 B) \). The \( U \)-dependent part reproduces the consistent anomaly, while the other part reproduces \(-\delta \Delta_B^B\).

We may now make some further useful remarks. Let us first consider the object \( J_{GW}^G \) is a three–form dual to the Goldstone–Wilczek current):
\[ J^{GW}(U, A) - J^{GW}(1, A) = J(U, A) \quad (A = W, B). \quad (26) \]

It identically satisfies

\[ dJ(U, A) = 0 \quad (\partial^\mu J_\mu = 0). \quad (27) \]

This implies that \( J(U, A) \) can be locally written as the (exterior) derivative of a two–form \( T \) (antisymmetric tensor \( T_{\mu\nu} \)) as follows

\[ J(U, A) = dT(U, A) \quad (J_\mu(U, A) = \epsilon_{\mu\nu\rho\sigma} \partial_\nu T_{\rho\sigma}(U, A)) \quad (28) \]

and

\[ T(U, A) = T(U, 0) + \Delta T(U, A) \quad (29) \]

By explicit calculation it turns out that

\[ Q(W_U) - Q(W) = -\frac{1}{3}(U^{-1}dU)^3 - d(dU^{-1}W) \quad (30) \]

Hence

\[ Q(W_U) - Q(W) = dT^W(U, W) \quad (31) \]

and

\[ T^W(U, W) = T^W(U, 0) + \Delta T^W(U, W) \quad (32) \]

with

\[ dT^W(U, 0) = -\frac{1}{3} Tr(U^{-1}dU)^3, \quad (33) \]

\[ d\Delta T^W(U, W) = -dTr(dUU^{-1}W). \quad (34) \]

This implies

\[ \Delta T^W(U, W) = -Tr dUU^{-1}W \quad (35) \]

\[ T^W(U, 0) = -Tr \int_0^1 dt U^{-1}(t) \frac{d}{dt} U(t) U^{-1}(t) dU(t) U^{-1}(t) dU(t) \quad (36) \]

\[ (U(t) = e^{it\sigma_i/2}) \]
\[ T^3(U, W) = \text{Tr} dU^{-1} \text{d}W + (W_3 - W_3^U) B \]

Therefore

\[ T(U, A) = T^W + T^3 = T^W(U, 0) - \text{Tr} dU^{-1} \text{d}W + (W_3 - W_3^U) B \]

By virtue of eqs. (15), (16) and (38), the full W–Z term can be written as

\[ \Gamma^{B}_{WZ}(U, A) = \int dBT(U, A) \]  

Note that the SU(2) gauge transformation

\[ \delta(Q(W_U) - Q(W)) = -d \text{Tr} (\alpha \cdot \text{d}W) \]

implies that

\[ \delta T^W(U, W) = -\text{Tr} \alpha \cdot \text{d}W. \]

In an analogous manner, adding the U(1) hypercharge transformation we have

\[ \delta(Q(W_U) - Q(W)) = d(\alpha_0 dW_3^U) - d(\alpha \cdot \text{d}W), \]

which implies:

\[ \delta_\alpha T^W = +\alpha_0 dW_3^U - \alpha \cdot \text{d}W. \]

For \( T^3 \) we have:

\[ \delta T^3 = -\alpha_3 \text{d}B - \alpha_0 dW_3^U + \alpha_0 dW_3 \]

\[ + \alpha_0 \text{d}B + \epsilon_{3ij} \alpha_i W_j B, \]

so that

\[ \delta_{\alpha_0, \alpha}(T^W + T^3) = -\alpha^T dW^T + (\alpha_0 - \alpha_3) dB + \]

\[ + (\alpha_0 - \alpha_3) dW_3 + \epsilon_{3ij} \alpha_i W_j B. \]

We recognize that the transformation laws for \( T \) are those of an antisymmetric tensor in string theory \[3\]. The only difference is that in the standard model \( T \) is a composite field constructed out of the Goldstone matrix \( U \) and the gauge fields \( A(W, B) \).
III. Supersymmetric Gauge Anomalies in the Wess–Zumino Gauge

In a supersymmetric gauge theory in D=4 dimension the supersymmetric form of the Yang–Mills anomaly and the corresponding Wess–Zumino lagrangian were found in refs. [4-6]. It was also found that there is no non–trivial supersymmetric global anomaly \([10]\) in superspace (where the supersymmetry transformation laws are linear, and Lorentz invariance is not broken).

For the physical applications supersymmetry is usually formulated in the Wess–Zumino gauge \([11]\) in which the supersymmetry algebra is modified and the supersymmetry transformations become non–linear \([12]\). In fact:

\[
\delta_{\epsilon'}^{WZ} = \delta_{\epsilon} + \delta_{\Lambda}
\]  

where \(\delta_{\Lambda}\) is a superspace transformation with chiral parameters \([13]\)

\[
\Lambda^a = (0, \frac{i}{\sqrt{2}} \sigma^\mu A_\mu^a \bar{\epsilon}, \bar{\epsilon} \lambda^a).
\]  

Due to this fact it follows that the superspace chiral gauge anomaly induces, in component fields, both an ordinary gauge anomaly \(G(\alpha)\) and a (global) supersymmetry anomaly \(S(\epsilon)\). If we take

\[
\Lambda^a = (\frac{\alpha}{2}, \frac{i}{\sqrt{2}} \sigma^\mu A_\mu^a \bar{\epsilon}, \bar{\epsilon} \lambda^a) = \Lambda^a(\epsilon, \alpha, V_{WZ}),
\]  

and we insert this expression into the superfield formula for the gauge anomaly with the gauge field \(V\) in the Wess–Zumino gauge:

\[
G(\alpha) + S(\epsilon) = \mathcal{A}(\Lambda = \Lambda(\epsilon, \alpha, V_{WZ})),
\]  

eq (49)

eq. (49) gives for \(G(\alpha)\) and \(S(\epsilon)\) the same expressions found for the solutions of the component form of the W–Z consistency conditions \([14]\):

\[
\delta_\alpha G(\alpha') - \delta_{\alpha'} G(\alpha) = G(\alpha \wedge \alpha'),
\]  

\[
\delta_\epsilon G(\alpha) - \delta_{\alpha} S(\epsilon) = 0,
\]  

\[
\delta_\epsilon S(\eta) - \delta_\eta S(\epsilon) = G(\alpha = -2i(\epsilon \sigma^m \bar{\eta} - \eta \sigma^m \bar{\epsilon}) A_m),
\]  

up to \(\delta \mathcal{L}\) where \(\mathcal{L}\) is a local functional of \(V_{WZ}\).
It has been shown in ref. [6] that $S(\epsilon)$ is cohomologically non trivial in the sense that $S(\epsilon) \neq \delta_c \mathcal{L}$, in the Wess–Zumino gauge.

As noted in ref. [15], the non–vanishing of $S(\epsilon)$ implies that, in a gauge theory with anomalous fermion content, supersymmetry is broken by loop effects even if the original classical action is supersymmetric. A typical example is the breaking of the supersymmetric sum rules [16] relating the magnetic transitions of members of charged vector multiplets (containing the W bosons) in the supersymmetric standard model. The feedback of this phenomenon is that the integration over the top supermultiplet produces an effective action with supersymmetry violating terms, due to the supersymmetric Wess–Zumino term previously discussed.

Let us make an excursion into the W–Z terms [17] and their supersymmetric extension.

Because of eq. (49), the W–Z terms of a supersymmetric theory should be given by the usual SUSY W–Z terms, derived in ref. [4], computed in the W–Z gauge.

The supersymmetric W–Z term for the chiral anomaly, $\mathcal{A}(\Lambda, \Lambda^\dagger, V)$, in superspace is obtained through the same rule as the usual W–Z, namely [14]:

\[
\Gamma_{WZ}^S(\xi, \bar{\xi}, V) = - \int_0^1 dt \mathcal{A}(\Sigma^{-1}(t) \dot{\Sigma}(t), (\Sigma(t)^{-1} \dot{\Sigma}(t))^\dagger, \Sigma(t) e^V \Sigma(t)),
\]

where $\dot{\Sigma} \equiv d\Sigma/dt$, and $\Sigma(t) = e^{it\xi}$ is a (one–parameter family) gauge (chiral) group element.

Since $\Gamma_{WZ}^S$ is invariant under (linear) supersymmetry transformations, it exactly follows that:

\[
\delta_\epsilon \Gamma_{WZ}^S = - \mathcal{A}(\Lambda = (0, \frac{i}{\sqrt{2}} \sigma^\mu A^a_\mu \bar{\epsilon}, \bar{\epsilon} \bar{\lambda}^a))
\]

because of eqs. (46) and (47).

Using the expression of the consistent anomaly in eq. (1), one obtains the linearized part of the W–Z action

\[
\Gamma_{WZ} = \text{Str}(T^a T^b T^c) \int \xi^a \partial_\mu (A^b_\nu \partial_\rho A^c_\sigma) \epsilon^{\mu \nu \rho \sigma} dx,
\]

which has an obvious supersymmetric extension

\[
\text{Re} \text{ Str}(T^a T^b T^c) \int dx^4 d^2 \theta i \xi^a \mathcal{F}^b \mathcal{F}^c,
\]
where \( F^a_\alpha = \bar{D}^2 D_\alpha V^a \) is the abelian gauge field strength of the (non-abelian) field \( V^a \).

The component expression of eq. (56) corresponds to the formula of ref. [13]:

\[
\text{Re} \int dx^4 d^2 \theta f^{bc} \mathcal{F}^b \mathcal{F}^c
\]

with

\[
f^{bc} = i \text{Str}(T^a T^b T^c) \xi^a
\]

Eq. (56) reduces to the exact form of the anomaly or of the W–Z term in the abelian case with gauge transformation

\[
\delta \xi = \Lambda
\]

IV. Supersymmetric Bardeen Anomaly and its Wess–Zumino Term

Eq. (5) can be supersymmetrized, provided we know the supersymmetric form of \( \Delta(A) \). Then in superspace we would get

\[
\Gamma_{SB}^{WZ}(\hat{\xi}, V) = \Gamma_{WZ}^{S}(\hat{\xi}, V) - \Delta^S(V) + \Delta^S(V\xi),
\]

where

\[
V\xi = e^{i\hat{\xi}^\dagger} e^V e^{-i\hat{\xi}}
\]

The advantage of eq. (25) is that it involves only Chern–Simons (C–S) forms and therefore it can be supersymmetrized in a simple way using the C–S multiplets \([18]\) \( \Omega(V) \) with the property

\[
D^2 \Omega = \bar{D}^2 \Omega = Tr (F^\alpha (V) F_\alpha (V))
\]

It is possible to give an explicit form for \( \Omega(V) \) in terms of a one-parameter family of interpolating vector superfields \( V(t) \) \((0 \leq t \leq 1, V(0) = 0, V(1) = V)\), which generalizes eq. (14). This expression reduces to the formula given in ref. [18] for the particular choice \( V(t) = tV \).
Let us introduce the quantities

\[
\nabla_\alpha(t) = e^{-V(t)} D_\alpha e^{V(t)} = D_\alpha + \phi_\alpha(t),
\]

\[
\nabla^{\dot{}\alpha}(t) = e^{V(t)} D^{\dot{\alpha}} e^{-V(t)} = D^{\dot{\alpha}} + \bar{\phi}^{\dot{\alpha}}(t),
\]

\[
\mathcal{F}_\alpha(t) = \bar{D}^2 \phi_\alpha(t), \quad \bar{\mathcal{F}}^{\dot{\alpha}}(t) = -D^2 \bar{\phi}^{\dot{\alpha}}(t)
\]

\[
H(t) = e^{-V(t)} \frac{d}{dt} e^{V(t)}, \quad \bar{H}(t) = e^{V(t)} H(t) e^{-V(t)}.
\]

By using the following property of spinorial derivatives

\[
D^\alpha \bar{D}^2 D_\alpha = \bar{D}^{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}},
\]

one can derive the following superspace Bianchi identity

\[
D^\alpha (e^{V(t)} \mathcal{F}_\alpha(t) e^{-V(t)}) = e^{V(t)} \bar{D}^{\dot{\alpha}} (e^{-V(t)} \bar{\mathcal{F}}^{\dot{\alpha}}(t) e^{V(t)}) e^{-V(t)},
\]

which implies

\[
Tr \, H(t) \{ \nabla^\alpha(t), \mathcal{F}_\alpha(t) \} = Tr \, \bar{H}(t) \{ \nabla^{\dot{\alpha}}(t), \bar{\mathcal{F}}^{\dot{\alpha}}(t) \}.
\]

Here we used the property

\[
D^\alpha (e^{V(t)} \mathcal{F}_\alpha(t) e^{-V(t)}) = e^{V(t)} \{ \nabla^\alpha(t), \mathcal{F}_\alpha(t) \} e^{-V(t)}.
\]

By means of eq. (68) and using the same arguments as in ref. [18] one easily finds

\[
\Omega(V) = 2 \int_0^1 dt \, Tr \{ [\nabla^\alpha(t), H(t)] \mathcal{F}_\alpha(t) + [\nabla^{\dot{\alpha}}(t), \bar{H}(t)] \bar{\mathcal{F}}^{\dot{\alpha}}(t) + H(t) \{ \nabla^\alpha(t), \mathcal{F}_\alpha(t) \} \}
\]

Notice that, in analogy with the bosonic case, the choice of the interpolating path \( V(t) \) introduces an arbitrariness in the definition of \( \Omega \). Two different C-S multiplets, \( \Omega_1(V) \) and \( \Omega_2(V) \), corresponding to two different paths \( V_1(t), V_2(t) \), obey, by virtue of eq. (62)

\[
\bar{D}^2 [\Omega_1(V) - \Omega_2(V)] = D^2 [\Omega_1(V) - \Omega_2(V)] = 0.
\]

This equation implies that

\[
\Omega_1(V) = \Omega_2(V) + L(V),
\]
where $L(V)$ is a linear multiplet $^{[11]}$. By dimensional arguments it has the form

$$L(V) = D^\alpha \bar{D}^2 P_\alpha (V) + h.c.,$$

(73)

where $P_\alpha (V)$ is an unconstrained superfield built in terms of $V$ and $D_\alpha V$.

Eqs. (71),(72) express the gauge equivalence of $\Omega_1$ and $\Omega_2$ with respect to superspace gauge transformations.

In order to supersymmetrize the counterterm $\Delta^B$ of eq. (9) we also need the analog of the $Q(W_3, B)$ mixed C-S three-form. It reads

$$\Omega(\hat{W}_3, B) = D^\alpha \hat{W}_3 F_\alpha (B) + \bar{D}^\alpha \hat{W}_3 \bar{F}_\alpha (B) - (\hat{W}_3 \leftrightarrow B).$$

(74)

The superfield $\hat{W}_3$ is defined below, by eqs. (82),(83). Note that $\Omega(\hat{W}_3, B)$ is a linear multiplet since $D^2 \Omega(\hat{W}_3, B) = \bar{D}^2 \Omega(\hat{W}_3, B) = 0$.

The supersymmetrized form of $\Delta^S$ gives rise to the supersymmetric Bardeen anomaly provided that under the infinitesimal $\Lambda_3$ supergauge transformation

$$\delta e^{\sigma_a W_a/2} = -\frac{i}{2} \sigma_3 \Lambda_3^i e^{\sigma_a W_a/2} + \frac{i}{2} e^{\sigma_a W_a/2} \sigma_3 \Lambda_3^i,$$

(75)

$\hat{W}_3$ and $\Omega(W)$ transform as follows

$$\delta_{\Lambda_3} \hat{W}_3 = i(\Lambda_3 - \Lambda_3^i),$$

(66)

$$\delta_{\Lambda_3} \Omega(W) = \frac{i}{2} D^\alpha \bar{D}^2 (\Lambda_3 D_\alpha \hat{W}_3) + h.c..$$

(77)

If this is the case, the supersymmetric Bardeen anomaly takes the form ($i = 1, 2$)

$$\mathcal{A}^B(W, B, \Lambda_0 - \Lambda_3, \Lambda_i) = -\frac{i}{2} \int d^2 \theta (\Lambda_0 - \Lambda_3) F_\alpha (B) F_\alpha (B)$$

$$-\frac{i}{2} \int d^2 \theta (\Lambda_0 - \Lambda_3) F_\alpha (\hat{W}) F_\alpha (B)$$

$$+ h.c. + \delta_{\Lambda_3} \Delta^S.$$  

(78)

This equation is obtained by adding to the $SU(2)$-invariant mixed anomaly

$$i \int d^2 \theta \Lambda_0 [Tr F_\alpha (W) F_\alpha (W) - \frac{1}{2} F_\alpha (B) F_\alpha (B)],$$

(79)

the $SU(2) \times U(1)$ gauge variation of $\Delta^S$, given by

$$\Delta^S = -\int d^4 \theta (\Omega(W) + \frac{1}{2} \Omega(\hat{W}_3, B)).$$

(80)
It is worth commenting on the uniqueness of $A^B$, given by eq. (78), and its related Wess-Zumino term $\Gamma_{WB}^S$, given by eq. (60). $A^B$ is defined only up to an electromagnetic gauge-invariant counterterm $\Delta^I$,
\[
\delta_{e.m.} \Delta^I = 0.
\]

In the bosonic case, the expression given in eq. (15) is unique if CP conservation is assumed\(^[1]\). In the supersymmetric case, due to the non-polynomial structure of the gauge-field transformations, this result does not follow in any obvious way from the non-supersymmetric case.

To obtain eqs. (76) and (77) we first have to construct new gauge field variables $\hat{W}_3, \hat{W}_i$ such that, under $\sigma_3 \Lambda_3$ gauge transformations
\[
\begin{align*}
\delta \hat{W}_3 &= i(\Lambda_3 - \Lambda_3^\dagger), \\
\delta \hat{W}_i &= -\frac{1}{2} \epsilon_{ij} (\Lambda_3 + \Lambda_3^\dagger) \hat{W}_j, \quad (\epsilon_{12} = -\epsilon_{21} = 1).
\end{align*}
\]
As explained in the appendix, this is achieved by defining
\[
\hat{W}_i = \frac{\sinh \xi}{2\xi} W_i,
\]
\[
e^{\hat{W}_3/2}(1 + \hat{W}_i^2)^{1/2} = \cosh \xi + \frac{\sinh \xi}{2\xi} W_3, \quad \xi^2 = \frac{W^2}{4} = \frac{W_a W_a}{4}.
\]

In terms of these variables we have
\[
e^{V} = e^{\sigma_3 W_a/2} = e^{\sigma_3 W_3/2}(1 + \hat{W}_i^2)^{1/2} + \hat{W}_i \sigma_i.
\]

To obtain eq. (77) we first define an $\hat{\Omega}(W)$ through a particular interpolating path $W(t)$, such that, under a $\sigma_3 \Lambda_3$ transformation
\[
e^{W(t)} \to e^{-i\Lambda_3^\dagger(t)\sigma_3/2} e^{W(t)} e^{i\Lambda_3(t)\sigma_3/2},
\]
for some specific choice of the gauge-transformation interpolating path $\Lambda_3(t)$.

If these interpolating paths exist then
\[
\delta_{\Lambda_3} \hat{\Omega}(W) = 2i \int_0^1 dt D^\alpha (Tr \hat{\Lambda} F_\alpha(t)) + h.c..
\]
Here $\hat{\Lambda}(t) \equiv (\sigma_3 d\Lambda_3(t)/dt)/2$. 

14
One easily verifies that eq. (85) holds when the path is chosen as follows

\[(\hat{W}_3(t), \hat{W}_i(t)) = (2t\hat{W}_3, 0), \quad 0 \leq t \leq \frac{1}{2}, \quad (87)\]

\[(\hat{W}_3(t), \hat{W}_i(t)) = (\hat{W}_3, (2t - 1)\hat{W}_i), \quad \frac{1}{2} \leq t \leq 1.\]

and

\[\Lambda_3(t) = 2t\Lambda_3, \quad 0 \leq t \leq \frac{1}{2}, \quad \Lambda_3(t) = \Lambda_3, \quad \frac{1}{2} \leq t \leq 1. \quad (88)\]

Inserting eqs. (87),(88) into eq. (86) we finally obtain eq. (77).

Let us notice that at the linearized level, that is for the terms in \(\Omega\) quadratic in the gauge superfields, the path given in eq. (87) gives the same C-S form of the path \(W(t) = tW\). Thus, eq. (77) reduces to the abelian gauge transformation eq. (21). The same holds for the complete non-abelian transformation in the Wess–Zumino gauge.

We therefore obtain a superfield formula for \(\Gamma^{SB}_{WZ}\) in the supersymmetric standard model as follows

\[
\Gamma^{SB}_{WZ}(U = e^{i\xi_i \sigma_i/2}, W, B) = \\
= \int d^4 \theta d^4 \bar{x}B[\hat{\Omega}(W\xi) - \hat{\Phi}(W) + \frac{1}{2}\Omega(\hat{W}_3^\xi, B) - \frac{1}{2}\Omega(\hat{W}_3, B)], \quad (89)
\]

with

\[W_\xi \rightarrow U^\dagger e^W U. \quad (90)\]

At the linearized level it reduces to

\[\text{Re} \int d^2 \theta (i\xi^a \mathcal{F}_W^a \mathcal{F}_B + i\xi^3 \mathcal{F}_B \mathcal{F}_B) \]

\[(\mathcal{F}_V = \bar{D}D\alpha V \text{ here}) \quad (91)\]

and its component expression gives

\[i\xi^a = (H^a, G^a, \lambda^a, h_1^a, h_2^a)\]

15
\[ \Gamma_{SBWZ} = \int d^4z \frac{1}{2} \left[ H^a(F^a_{\mu\nu}F_{\mu\nu}(B) + i\bar{\lambda}^a\partial \lambda_B + i\bar{\lambda} B \partial \lambda^a \\
-2D^aD^B \\
+G^a(\frac{1}{2} F^a_{\mu\nu}\tilde{F}^B_{\mu\nu} - i\partial_\mu (\bar{\lambda}^a \gamma_5 \gamma_\mu \lambda_B) \\
-i\bar{\chi}^a(D_a \gamma_5 \lambda_B + D_B \gamma_5 \lambda^a + F^a_{\mu\nu}\sigma^{\mu\nu} \lambda_B + \\
+F^B_{\mu\nu}\sigma^{\mu\nu} \lambda^a) + \\
+ih^a_1\bar{\lambda} \lambda B + ih^a_2\bar{\lambda}^a \gamma_5 \lambda^B \right] \\
+ \left[ H^3(\frac{1}{2} F^B_{\mu\nu} F^B_{\mu\nu} + i \lambda B \partial \lambda B - D^B \partial^2 + \\
+G^3(\frac{1}{4} F^B_{\mu\nu}\tilde{F}^B_{\mu\nu} - \frac{i}{2}(\bar{\lambda}^B \gamma_5 \gamma_\mu \lambda^B) \\
- \bar{\chi}^3(D^B \gamma_5 \lambda^B + F^B_{\mu\nu}\sigma^{\mu\nu} \lambda^B) + \\
\frac{i}{2} h^3_1\bar{\lambda}^b \lambda B + \frac{i}{2} h^3_2\bar{\lambda} \gamma_5 \lambda^B \right]. \] (92)

The terms proportional to $G^a$ and $G^3$ in $\Gamma_{SBWZ}$ is the ordinary W–Z term whose gauge variation gives the gauge anomaly, while the terms proportional to $\chi^a$, $\chi^3$ and $h^a$, $h^3$ are responsible for the supersymmetric anomaly due to the anomalous supersymmetry transformation of $\chi$ and $h$. Finally, the term proportional to the pseudogoldstone $H^a$ does not contribute to the anomaly, but is required by supersymmetry.

In the supersymmetric case the multiplet

\[ L^W(U, W) = \hat{\Omega}(W^U) - \hat{\Omega}(W) \] (93)

is a linear multiplet \cite{19} ($\bar{D}\bar{D}L^W = DDL^W = 0$) and so it is the multiplet

\[ L^3(U, W, B) = \Omega(\hat{W}_3^U, B) - \Omega(\hat{W}_3, B). \] (94)

In fact $\Omega(\hat{W}_3, B)$ itself is a linear multiplet given by

\[ \Omega(\hat{W}_3, B) = D^a\bar{D}\bar{D}(\hat{W}_3\hat{D}_a B) + h.c.. \] (95)

Then we conclude that

\[ L(U, W, B) = L^W(U, W) + \frac{1}{2}L^3(U, W, B) \] (96)

is a linear multiplet with $(L(U = 1) = 0)$.

So the full supersymmetric W–Z term can be written as
\[ \int d^4 \theta d^4 x B L(U, W, B), \]  

(97)

and it is precisely analogous to the supersymmetric form of the Green–Schwarz term \cite{3,18} with a composite linear multiplet \( L(U, W, B) \).

Eq. (97) is the supersymmetric version of eq. (25).

A linear multiplet can be locally solved as

\[ L = D^\alpha \bar{D}^2 U_\alpha + \bar{D}_{\dot{\alpha}} D^2 \bar{U}_{\dot{\alpha}}. \]

(98)

(with \( U_\alpha \) defined up to terms \( U_\alpha = i D_\alpha Z(Z = Z^\dagger) \)).

At the linearized level in the \( U \sim 1 + i \xi^i \sigma_i / 2 \)

\[ U_\alpha = \frac{i}{2} \xi^i D_\alpha W^i + \frac{i}{2} \xi^3 D_\alpha W^3. \]

(99)

The most general form of \( U_\alpha(U, W, B) \) can be constructed from the identity

\[ L(U, W, B) = \int_0^1 dt \frac{d}{dt} [\hat{\Omega}(\hat{W} U(t)) + \frac{1}{2} \Omega(W_3 U(t), B)], \]

(100)

with \( U(t) = e^{i \xi(t)} \) (\( \xi(0) = 0, \xi(1) = \xi \)). Indeed, if

\[ \delta_\Lambda (\hat{\Omega}(W) + \frac{1}{2} \Omega(\hat{W}_3, B)) = D^\alpha \bar{D}^2 \Gamma_\alpha(W, B, \Lambda) + h.c., \]

(101)

then we find the following formula for \( U_\alpha \)

\[ U_\alpha(W, B, U) = \int_0^1 dt \Gamma_\alpha(W U(t), B, U^{-1}(t) \frac{d}{dt} U(t)). \]

(102)

By means of eq. (98), integrating by part eq. (97), we obtain the supersymmetric counterpart of eq. (39):

\[ Re \int d^4 x d^4 \theta F^\alpha(B) U_\alpha(U, U^\dagger, V). \]

(103)

At the linearized level, by using eq. (99), the above formula reproduces eq. (91).

V. Conclusions

We would like finally to comment on some physical effects of the supersymmetric Wess–Zumino term as given by eq. (97).

Already at the linearized level (eq. (92)) we see that, beyond the usual terms of the non supersymmetric case, there is a contribution to the \( HW\gamma \) coupling and
to the magnetic moment of the gauginos. Also the auxiliary fields give a correction to the scalar Higgs potential as well as to the gaugino Yukawa couplings. These terms can be obtained, by explicit computation, sending to infinity the top (or bottom) Yukawa couplings in loop diagrams, as already shown for the g–2 in some examples [15,20]. Quite independently, Wess–Zumino terms have broader applications in any context where some fermions contributing to anomalies are replaced by some bosonic fields interacting with gauge fields.

Acknowledgements

S. Ferrara and M. Porrati would like to thank, respectively, the Santa Barbara Institute for Theoretical Physics and the Aspen Institute for Physics, where part of this work was done, for their kind hospitality.

Appendix

In this appendix we describe in some details how to find the hatted variables $\hat{W}_a$, $a = 1, 2, 3$.

The gauge transformation [11]

$$e^V \rightarrow e^{-i\Lambda^\dagger} e^V e^{i\Lambda}$$

implies the following (infinitesimal) gauge transformation for the Lie algebra valued superfield $V$ [21]

$$\delta V = i\mathcal{L}_{V/2}\Delta + \mathcal{L}_{V/2} \coth \mathcal{L}_{V/2}\Sigma,$$

with $\Delta = \Lambda + \Lambda^\dagger$, $\Sigma = i(\Lambda - \Lambda^\dagger)$. For the group $G = SU(2)$ with

$$\Lambda = \frac{1}{2} \Lambda_a \sigma_a, \quad V = \frac{1}{2} W_a \sigma_a,$$

we obtain the component expression

$$\delta W_a = \frac{1}{2} \epsilon_{abc} \Delta_b W_c + \frac{1}{4} f(\xi)(W^2 \delta_{ab} - W_a W_b)\Sigma_b + \Sigma_a,$$

with

$$f(\xi) = \xi^{-2}(\xi \coth \xi - 1), \quad \xi^2 \equiv \frac{W^2}{2}.$$  

Notice that $\delta W^2 = 2 W_a \Sigma_a$.

In the Wess–Zumino gauge

$$\delta W_a = \Sigma_a + \frac{1}{2} \epsilon_{abc} \Delta_b W_c.$$
It is easy to see that no field redefinition of the form

\[ \hat{W}_a = W_a(W), \quad \forall a \]

exists, such that eq. A.6 holds in any supersymmetric gauge. However it is possible to define new \( \hat{W}_a \) such that A.6 is verified for \( \Lambda = \Lambda_3 \sigma_3 / 2 \). Indeed, let us set

\[ \hat{W}_i = g(\xi)W_i, \quad i = 1, 2. \]

From eq. A.6 and A.4, we obtain the following differential equation

\[ \xi f(\xi) = \frac{g'(\xi)}{g(\xi)}, \]

whose solution is

\[ g(\xi) = \lambda \frac{\sinh \xi}{\xi}. \]

We set the arbitrary constant of integration \( \lambda \) equal to 1/2; thus,

\[ \hat{W}_i = \frac{\sinh \xi}{2\xi} W_i. \]

Next, we define \( \hat{W}_3 \) as

\[ \hat{W}_3 = W_3 + F(\hat{W}_2, \xi), \quad \hat{W}_2 \equiv \hat{W}_1^2 + \hat{W}_2^2, \]

and demand

\[ \delta_{\Lambda_3} \hat{W}_3 = \Sigma_3. \]

Note that \( \delta_{\Lambda_3} \hat{W}_2 = 0 \). Eq. A.13 gives a first-order differential equation for \( F \)

\[ \frac{dF}{2\hat{W}_2} = (\sinh^2 \xi - \hat{W}_2)^{-1/2} d\left( \frac{\xi}{\sinh \xi} \right), \]

whose solution is

\[ F = -W_3 + 2 \log(\cosh \xi + \frac{\sinh \xi W_2}{\xi} + \log f(\hat{W}_2^2). \]

In this equation, \( f \) is an arbitrary function, and we used the relation

\[ \sinh^2 \xi - \hat{W}_2^2 = \frac{\sinh^2 \xi W_3^2}{\xi^2 - 4}. \]
The above equations give
\[ e^{\hat{W}_3/2} = (\cosh \xi + \frac{\sinh \xi}{2\xi} W_3) f^{1/2}(\hat{W}^2). \]

By choosing \( f(\hat{W}^2) = (1 + \hat{W}^2)^{-1} \) we arrive at equation (83). This particular form of the function \( f(\hat{W}^2) \) has been chosen so that eq. (84) holds.

References

[1] T. Sterling and M. Veltman, Nucl. Phys. B189 (1981) 557;
   E. D’Hoker and E. Fahri, Nucl. Phys. B248 (1984) 59; Nucl. Phys. B248 (1984) 77;
   J. Preskill, Ann. Phys. (N.Y.) 210 (1991) 323;
   G.L. Lin, H. Steger and Y.P. Yao, Phys. Rev. D44 (1991) 2139.
   F. Feruglio, L. Maiani and A. Masiero, Nucl. Phys. B387 (1992) 523.
   E. D’Hoker, Phys. Rev. Lett. 69 (1992) 1316.

[2] S. Ferrara, A. Masiero and M. Porrati, Phys. Lett. B301 (1993) 358.

[3] M. Green and J. Schwarz, Phys. Lett. 149B (1984) 17.
   M. Dine, N. Seiberg and E. Witten, Nucl. Phys. B289 (1987) 589.

[4] G. Girardi, R. Grimm and R. Stora, Phys. Lett. 156B (1985) 203;
   L. Bonora, P. Pasti and M. Tonin, Phys. Lett. 156B (1985) 341.

[5] E. Guadagnini, K. Konishi and M. Mintchev, Phys. Lett. 157B (1985) 37;
   N.K. Nielsen, Nucl. Phys. B244 (1984) 499.

[6] H. Itoyama, V.P. Nair and H. Ren, Nucl. Phys. B262 (1985) 317.
   E. Guadagnini and M. Mintchev, Nucl. Phys. B269 (1986) 543.

[7] W. A. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421.
   B. Zumino, Wu Yong–Shi and A. Zee, Nucl. Phys. B239 (1984) 477.
   J. Manez, R. Stora and B. Zumino, Comm. Math. Phys. 102 (1985) 157.

[8] W.A. Bardeen, Phys. Rev. 184 (1969) 1848.

[9] E. Witten, Nucl. Phys. B223 (1983) 422,433.
   Chou Kuang–chao, Guo Han–ying, Wu Ke and Song Xing–chang, Phys.
   Lett. 134B (1984) 67.
   H. Kawai and S.-H.H. Tye, Phys. Lett. 140B (1984) 403.

[10] O. Piguet and K. Sibold, Nucl. Phys. B247 (1984) 484.
   J. Dixon, Class. Quant. Grav. 7 (1990) 1511.

[11] J. Wess and B. Zumino, Phys. Lett. B78 (1974) 1; S. Ferrara and B.
    Zumino, Nucl. Phys. B79 (1974) 413.
[12] B. de Wit and D.Z. Freedman, Phys. Rev. **D12** (1975) 2286.
[13] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Nucl. Phys. **B212** (1983) 413.
[14] B. Zumino in “Geometry, Anomaly and Topology”, (World Scientific, Singapore, 1985).
[15] C.L. Bilchak, R. Gastmans and A. Van Proeyen, Nucl. Phys. **B273** (1986) 46.
[16] S. Ferrara and M. Porrati, Phys. Lett. **288B** (1992) 85.
[17] J. Wess and B. Zumino, Phys. Lett. **37B** (1971) 95.
[18] S. Cecotti, S. Ferrara and M. Villasante, Int. J. Mod. Phys. **A2** (1987) 1839.
[19] S. Ferrara, J. Wess and B. Zumino, Phys. Lett. **51B** (1974) 239.
[20] S. Ferrara and A. Masiero, preprint CERN TH-6846/93, to appear in the Proc. of the 26th Workshop, Eloisatron Project, “From Superstrings to Supergravity”, Erice, December 1992.
[21] J. Wess and J. Bagger, “Supersymmetry and Supergravity”, p. 46 (Princeton University Press, Princeton NJ, 1992).