RIGIDITY OF FLAG MANIFOLDS

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Abstract. Let $N \subset \text{GL}(n, \mathbb{R})$ be the group of upper triangular matrices with 1s on the diagonal, equipped with the standard Carnot group structure. We show that quasiconformal homeomorphisms, and more generally Sobolev mappings with nondegenerate Pansu differential, are rigid when $n \geq 4$; this settles the Regularity Conjecture for such groups. This result is deduced from a rigidity theorem for the manifold of complete flags in $\mathbb{R}^n$. Similar results also hold in the complex and quaternion cases.

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1. Introduction

This is part of a series of papers [KMX20, KMX21b, KMX, KMX21a] on geometric mapping theory in Carnot groups, in which we establish

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regularity, rigidity, and partial rigidity results for bilipschitz, quasiconformal, or more generally, Sobolev mappings, between Carnot groups. Our focus in this paper is on the Iwasawa group for $\text{SL}(n, \mathbb{F})$ and the associated flag manifold, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

To state our main theorem we first briefly recall a few facts; see Section 2 for details and references. For simplicity we will stick to the case $\mathbb{F} = \mathbb{R}$ in this introduction.

Fix $n \geq 3$. Let $N \subset \text{GL}(n, \mathbb{R})$ denote the subgroup of upper triangular matrices with 1s on the diagonal, with Lie algebra $\mathfrak{n} \subset \mathfrak{gl}(n, \mathbb{R})$, and Carnot group structure given by the grading $\mathfrak{n} = \bigoplus_i V_i$, where $V_i = \{ A \in \mathfrak{n} \mid A_{jk} = 0 \text{ if } k \neq j + i \}$ corresponds to the $i$th superdiagonal. Let $\mathcal{F}$ be the manifold of complete flags in $\mathbb{R}^n$, i.e. the collection of nested families of linear subspaces

$$\{0\} \subsetneq W_1 \subsetneq \ldots \subsetneq W_{n-1} \subsetneq \mathbb{R}^n.$$

We consider a standard subbundle $W$ of the tangent bundle $T\mathcal{F}$ which is invariant under both the natural action $\text{GL}(n, \mathbb{R}) \curvearrowright \mathcal{F}$ and the diffeomorphism $\psi : \mathcal{F} \to \mathcal{F}$ induced by orthogonal complement:

$$\psi(W_1, \ldots, W_{n-1}) = (W_1^\perp, \ldots, W_{n-1}^\perp).$$

There is a dense open subset $\hat{N} \subset \mathcal{F}$ and a diffeomorphism $\alpha : N \to \hat{N}$ carrying the horizontal bundle $V_1 \subset TN$ to $W \big|_N$, i.e. $(\hat{N}, W \big|_{\hat{N}})$ is contact diffeomorphic to $(N, V_1)$.

We equip $(\mathcal{F}, W)$ with a Carnot-Carathéodory distance $d_{CC}$, and let $\nu$ denote the homogenous dimension of $(\mathcal{F}, W)$.

Our main result is a rigidity theorem for Sobolev mappings satisfying a nondegeneracy condition:

**Theorem 1.1.** Suppose $n \geq 4$. Let $U \subset \mathcal{F}$ be a connected open subset, and $f : U \to \mathcal{F}$ be a $W^{1,p}_{\text{loc}}$-mapping for $p > \nu$, such that the Pansu differential is an isomorphism almost everywhere. Then $f$ is the restriction of a diffeomorphism $\mathcal{F} \to \mathcal{F}$ of the form $\psi^\epsilon \circ g$ where $g \in \text{GL}(n, \mathbb{R})$, $\epsilon \in \{0, 1\}$.

In the theorem and below we use the convention that $\psi^0 = \text{id}$ and $\psi^1 = \psi$. Note that the rigidity assertion is false when $n = 3$, because $(\mathcal{F}, W)$ is locally equivalent to the Heisenberg group, and hence has an infinite dimensional group of contact diffeomorphisms, which are all Sobolev mappings with nondegenerate Pansu differential. Quasiconformal homeomorphisms are $W^{1,p}_{\text{loc}}$-mappings for some $p > \nu$ \cite{HK98}, and therefore we obtain:
Corollary 1.2. When $n \geq 4$, then any quasiconformal homeomorphism $F \supset U \to U' \subset F$ between connected open subsets is the restriction of a diffeomorphism of the form $\psi^\varepsilon \circ g$ for some $g \in \text{GL}(n, \mathbb{R})$, $\varepsilon \in \{0, 1\}$.

Since $(N, V_1)$ is contact diffeomorphic to $(\hat{N}, W|_{\hat{N}})$, the above results give a classification of Sobolev mappings $N \supset U \to N$ with nondegenerate Pansu differential; this applies in particular quasiconformal homeomorphisms and quasiregular maps by [KMX Section 5]. Corollary 1.2 was previously known for real analytic diffeomorphisms [Tan70, Yam93], $C^2$ diffeomorphisms [CDMKR02, CDMKR05], and Euclidean bilipschitz homeomorphisms [Le].

We also obtain more refined rigidity for global quasiconformal homeomorphisms of $N$:

Theorem 1.3. Any quasiconformal homeomorphism $N \to N$ is affine, i.e. the composition of a graded automorphism and a left translation.

Historical notes. We give only the briefest indication of some of the context, and the references provided are only a sampling of the literature. See [KMX20, KMX21b] for more extensive discussion and references.

For a diffeomorphism $F \supset U \to U' \subset F$, the quasiconformality condition reduces (locally) to the condition of being contact, i.e. preserving the subbundle $W \subset T F$. The study of contact diffeomorphisms has a long history in differential geometry and exterior differential systems going back to Cartan [Car04, SS65, Tan70, Yam93, CDMKR05, OW11b]. The particular case considered in this paper – the flag manifold $F$ equipped with the subbundle $W$ – was treated in the papers [CDMKR02, CDMKR05] along with other parabolic geometries.

Quasiconformal homeomorphisms in $\mathbb{R}^n$, $n \geq 3$, have been heavily studied since the 1960s. In the Carnot group setting they first appeared in Mostow’s rigidity theorem [Mos73], and have been investigated intensely since roughly 1990, due to influential contributions of Gromov, Pansu, and applications to geometric group theory. For almost 30 years, these historical threads have merged with other developments in analysis on metric spaces [HK98, Che99] and PDE [Cap99, CC06, IM93]. The rigidity (or regularity) problem for quasiconformal mappings of Carnot groups crystallized over time, first appearing implicitly in the seminal paper [Pan89], then in [Hei95, Remark 6.12], [Kor96] and finally taking shape with Ottazzi-Warhurst’s notion of a rigid Carnot group [OW11b, p.2], [OW11a], explicitly upgraded to
a conjecture in [KMX20]. Corollary 1.2 settles the Regularity Conjecture for the Carnot group $N$. Rigidity of quasiconformal mappings is an instance of rigidity for (solutions to) weak differential inclusions, and more generally partial differential relations, about which there is a vast literature, see for instance [Res89, IM93, Vod07b, Nas54, Tar79, Mur81, Gro86, Sch93, DM99, Mül99, MS03, DLS09, DLS17, Ise18, BV19].

**Discussion of the proof.** For every $1 \leq j \leq n - 1$, let $\pi_j : \mathcal{F} \to G(j, n)$ be the $\text{GL}(n, \mathbb{R})$-equivariant fibration given by $\pi_j(W_1, \ldots, W_{n-1}) = W_j$; here $G(j, n)$ is the Grassmannian of $j$-planes in $\mathbb{R}^n$.

The first step of the proof of Theorem 1.1, which is implemented in Sections 2 and 3, is to show that the mapping $f$ preserves the fibers of the fibration $\pi_j$ for all $1 \leq j \leq n - 1$ (possibly after first post-composing $f$ with the orthogonal complement mapping $\psi : \mathcal{F} \to \mathcal{F}$). The heart of the argument is to exclude a certain type of oscillatory behavior in the Pansu differential, and this is achieved by using the Pullback Theorem from [KMX20].

The second step of the proof (Section 4) is to show that a continuous map $U \to \mathcal{F}$ which preserves the fibrations $\pi_j$, and satisfies a certain nondegeneracy condition, must agree with the diffeomorphism $\mathcal{F} \to \mathcal{F}$ induced by some element $g \in \text{GL}(n, \mathbb{R})$. Our approach to this was inspired by incidence geometry – the Fundamental Theorem of Projective Geometry and its generalization by Tits – although the final form is rather different. See Section 4.1 for a sketch of this argument in the $n = 3$ case.

The approaches taken in [Tan70, Yam93, CDMKR02, CDMKR05, Lel] are all based on classifying the contact vector fields, and then using an “integration” argument to obtain rigidity for contact mappings [Kob95, Theorem 3.1], [Pal57]. To carry out the latter step one has to know that the pushforward of a smooth contact vector field under a contact mapping is still a contact vector field (although possibly of low regularity, a priori). This pushforward assertion is obvious for $C^2$ diffeomorphisms and was shown to hold for Euclidean bilipschitz homeomorphisms in [Lel]; this appears to be the minimal regularity needed to implement such an argument.

**Organization of the paper.** The proof of Theorem 1.1 is carried out in Sections 3-5. The statement and proofs of the analogous results in the complex and quaternionic cases are in Section 6. Rigidity for global quasiconformal homeomorphisms (Theorem 1.3) is proven in Section 7.
2. Preliminaries

2.1. The Iwasawa \( N \) group, and the manifold of complete flags.

Most of this subsection is standard material from Lie theory. We have tried to give a more self-contained treatment accessible to readers less familiar with Lie theory; see for instance [CDMKR05] for another approach.

**Notation.** Let \( n \geq 2 \) be an integer. We will use the following notation:

- \( P^+, P^- \subset \text{GL}(n, \mathbb{R}) \) will denote the subgroups of upper and lower triangular matrices, respectively.
- \( N \subset P^+ \) denotes the subgroup of upper triangular matrices with 1s on the diagonal.
- \( G := \text{PGL}(n, \mathbb{R}) = \text{GL}(n, \mathbb{R}) / \{ \lambda \text{id} | \lambda \neq 0 \} \) is projective linear group.
- We let \( P := P^- / \{ \lambda \text{id} | \lambda \neq 0 \} \subset \text{GL}(n, \mathbb{R}) / \{ \lambda \text{id} | \lambda \neq 0 \} = G \) be the image of \( P^- \subset \text{GL}(n, \mathbb{R}) \) under quotient map \( \text{GL}(n, \mathbb{R}) \to \text{PGL}(n, \mathbb{R}) \).
- Since \( N \cap P^- = \{ e \} \) the quotient map \( \text{GL}(n, \mathbb{R}) \to G \) restricts to an embedding \( N \hookrightarrow G \), and we identify \( N \) with its image in \( G \).
- \( n, p^\pm, p, \text{gl}(n, \mathbb{R}), \) and \( g \) denote the Lie algebras of \( N, P^\pm, P, \text{GL}(n, \mathbb{R}), \) and \( G \), respectively.
- \( X_{ij} \in \text{gl}(n, \mathbb{R}) \) is the matrix with a 1 in the \( ij \)-entry and 0s elsewhere, so

\[
X_{i_1,j_1}X_{i_2,j_2} = \delta_{i_2,j_1}X_{i_1,j_2} \quad [X_{i_1,j_1}, X_{i_2,j_2}] = \delta_{i_2,j_1}X_{i_1,j_2} - \delta_{i_1,j_2}X_{i_2,j_1}.
\]

The Carnot group structure on the upper triangular matrices \( N \) is defined by the grading \( n = \bigoplus_i V_i \) where

\[
V_i = \{ A \in \text{gl}(n, \mathbb{R}) | A_{jk} = 0 \text{ if } k - j \neq i \}.
\]

For \( r \neq 0 \) the Carnot dilation \( \delta_r : N \to N \) is given by conjugation with diagonal matrix \( \text{diag}(r^{-1}, \ldots, r^{-n}) \).

**The flag manifold.** The **flag manifold** \( F \) is the set of (complete) flags in \( \mathbb{R}^n \), i.e. the collection of nested families of linear subspaces of \( \mathbb{R}^n \)

\[
W_1 \subset \ldots \subset W_{n-1}
\]

where \( W_j \) has dimension \( j \); this inherits a smooth structure as a smooth submanifold of the product of Grassmannians \( G(1, n) \times \ldots \times G(n-1, n) \).

**Lemma 2.2 (Properties of \( F \)).**
(1) The action of \( GL(n, \mathbb{R}) \curvearrowright G(j, n) \) induces transitive actions
\[
GL(n, \mathbb{R}) \curvearrowright \mathcal{F} \quad \text{and} \quad G \curvearrowright \mathcal{F}.
\]

(2) Letting \( W_j^+ := \text{span}(e_1, \ldots, e_j) \), \( W_j^- := \text{span}(e_n, \ldots, e_{n-j+1}) \),
the stabilizer of the flag \( (W_j^+_{1 \leq j \leq n-1}) \in \mathcal{F} \) in \( GL(n, \mathbb{R}) \) is \( P^\pm \).

(3) The stabilizer of \( (W_j^-) \) in \( G \) is \( P \). Henceforth we identify \( \mathcal{F} \)
with the homogeneous space (coset space) \( G/P \) using the orbit map \( g \mapsto g \cdot (W_j^-) \).

(4) The differential of the fibration \( G \to G/P \simeq \mathcal{F} \) at \( e \in G \) yields
an isomorphism \( \mathfrak{g}/\mathfrak{p} \simeq T_p(G/P) \simeq T_{(W_j^-)}\mathcal{F} \).

(5) The fibration map \( G \to G/P \) is \( P \)-equivariant w.r.t. the action
of \( P \) on \( G \) by conjugacy and on \( G/P \) by left translation.

(6) The differential at \( e \in G \) induces an isomorphism of \( P \xrightarrow{\text{Ad}} \mathfrak{g}/\mathfrak{p} \)
to the (isotropy) representation \( P \curvearrowright T_p(G/P) \); here \( G \xrightarrow{\text{Ad}} \mathfrak{g} \)
is the Adjoint representation and \( \text{Ad} \big|_P \) is the restriction to \( P \).

(7) The composition \( \mathfrak{n} \hookrightarrow \mathfrak{g} \to \mathfrak{g}/\mathfrak{p} \) is an isomorphism.

(8) The image \( \hat{\mathcal{V}}_1 \) of \( V_1 \subset \mathfrak{n} \) under \( \mathfrak{n} \to \mathfrak{g}/\mathfrak{p} \simeq T_p(G/P) \)
is a \( P \)-invariant subspace of \( T_p(G/P) \). This defines a \( G \)-invariant subbundle \( \mathcal{H} \subset T(G/P) \simeq T\mathcal{F} \).

(9) The orbit map \( \alpha : N \to \mathcal{F} \) given by \( \alpha(g) = g \cdot (W_j^-) \) is an
\( N \)-equivariant embedding of \( N \) onto an open subset of \( \mathcal{F} \), which
we denote by \( \tilde{N} \subset \mathcal{F} \). The image \( \tilde{N} \) may be characterized as
the collection of flags \( (W_j) \) such that \( W_j \cap W_{n-j}^- = \{0\} \) for every
\( 1 \leq j \leq n-1 \).

(10) The embedding \( \alpha \) is contact with respect to the subbundles \( V_1 \subset TN \) and \( \mathcal{H} \subset T(G/P) \).

(11) For every \( r \in (0, \infty) \) we have \( \alpha \circ \delta_r = \hat{\delta}_r \circ \alpha \), where \( \hat{\delta}_r : \mathcal{F} \to \mathcal{F} \)
is given by \( \hat{\delta}_r(x) = g_r \cdot x \) with \( g_r = \text{diag}(r^{-1}, \ldots, r^{-n}) \).

Proof. \( \Box \) \( (1)-(7) \) follow readily from linear algebra or basic theory of manifolds.

\( \Box \). To see this, it suffices to show that the image of \( V_1 \subset \mathfrak{n} \) under \( \mathfrak{n} \hookrightarrow \mathfrak{gl}(n, \mathbb{R})/\mathfrak{p}^- \) is invariant under \( \text{Ad} \big|_{\mathfrak{p}^-} \). Since \( \mathfrak{p}^- \) is connected, it therefore suffices to see that \( [\mathfrak{p}^-, V_1] \subset V_1 + \mathfrak{p}^- \); this follows readily by applying \( (2.1) \).

\( \Box \). The orbit map \( \alpha : N \to \mathcal{F} \) is smooth and \( N \)-equivariant. Since
the differential of \( \alpha \) at \( e \in N \) is an isomorphism \( D\alpha(e) : \mathfrak{n} \to \mathfrak{g}/\mathfrak{p} \),
the map \( \alpha \) is an \( N \)-equivariant immersion onto an open subset. But
\( N \cap \mathfrak{p}^- = \{e\} \) so it is also injective, and hence \( \alpha \) is an embedding.
Let \( Z := \{(W_j) \in \mathcal{F} \mid W_j \cap W_{n-j}^+ = \{0\}\} \) for all \( 1 \leq j \leq n-1 \). If \((W_j)\) belongs to the orbit \( \hat{N} = N \cdot (W_j^-) \) then \((W_j) \in Z\). Now suppose \((W_j) \in Z\). We prove by induction on \( k \) that the \( N \)-orbit of \((W_j)\) contains a flag \((W_{kj})\) such that \( W_{kj} = W_{-j}^- \) for all \( j \leq k \). Since \( W_1 \cap W_{n-1}^+ = \{0\} \) we may choose \( v = (v_1, \ldots, v_n) \in W_1 \) with \( v_n = 1 \). If \( n \in N \) is the block matrix
\[
\begin{bmatrix}
I & -b \\
0 & 1
\end{bmatrix}
\]
with \( b = [v_1, \ldots, v_{n-1}]^t \) we have \( n \cdot v \in W_1^- \), so letting \((W_j) := n \cdot (W_j)\) we have \( W_1^t = W_1^- \), establishing the \( k = 1 \) case. The inductive step follows similarly, by working in the quotient \( \mathbb{R}^n/W^1_{k-1} \).

(11). Both \( V_1 \) and \( H \) are \( N \)-invariant subbundles, and \( \alpha \) is \( N \)-equivariant, so the contact property follows from the fact that \( H(P) \subset T_P(G/P) \) is the image of \( V_1 \subset \mathfrak{n} \) under \( D\alpha(e) \).

(11). Since \( g_r \in P^- \) we have
\[
\alpha(\delta_r(g)) = \delta_r(g) \cdot P = g_r g_r^{-1} \cdot P = g_r g \cdot P = \hat{\delta}_r(\alpha(g)).
\]

\[\square\]

**Automorphisms.** We collect some properties of automorphisms of \( G \) and graded automorphisms of \( N \).

**Lemma 2.3 (Properties of Aut(\( G \))).**

1. **Transpose inverse** \( T \mapsto (T^t)^{-1} \) descends to a Lie group automorphism of \( G \). The group \( \text{Aut}(G) \) of Lie group automorphisms of \( G \) is generated by inner automorphisms and transpose inverse.

2. Let \( \tau : \text{GL}(n, \mathbb{R}) \to \text{GL}(n, \mathbb{R}) \) be the automorphism given by
\[
\tau(T) = \Pi (T^t)^{-1} \Pi^{-1}
\]
where \( \Pi \in \text{GL}(n, \mathbb{R}) \) is the permutation matrix with \( \Pi e_i = e_{n-i+1} \). Then the induced automorphism \( \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}) \) – which we also denote by \( \tau \) by abuse of notation – is given by
\[
(\tau(A))_{i,j} = -A_{n-j+1,n-i+1}.
\]
Then \( \tau \) preserves \( N \), and induces a graded automorphism \( \mathfrak{n} \to \mathfrak{n} \).

3. The automorphism \( \tau : \text{GL}(n, \mathbb{R}) \to \text{GL}(n, \mathbb{R}) \) descends to an automorphism \( G \to G \), which we also denote by \( \tau \) (by further abuse of notation).
Proof.

(1). This follows from Theorem \[6.1\] since the only automorphism of \(\mathbb{R}\) is the identity map. Therefore there exists \(\Phi_1 \in \text{Aut}(G)\) in the group generated by inner automorphisms and transpose-inverse, such that \(D\Phi_1 = D\Phi\). Hence \(\Phi_2 := \Phi_1^{-1} \circ \Phi \in \text{Aut}(G)\) is the identity on the connected component of \(e \in G\). By considering centralizers of elements \(g \in G\) of order 2, it is not hard to see that \(\Phi_2 = \text{id}\), and hence \(\Phi = \Phi_1\).

(2) and (3) are straightforward. \(\Box\)

We now consider the group \(\text{Aut}_{gr}(N)\) of graded automorphisms of \(N\).

Lemma 2.5 (Properties of \(\text{Aut}_{gr}(N)\)). Assume \(n \geq 4\).

(1) If \(\Phi \in \text{Aut}(G)\) and \(\Phi(N) = N\), then \(\Phi\) induces a graded automorphism of \(N\) if and only if \(\Phi\big|_{N} = \tau^\varepsilon \circ I_g\big|_{N}\) for some \(\varepsilon \in \{0, 1\}\), \(g = \text{diag}(\lambda_1, \ldots, \lambda_n) \in G\).

(2) Every graded automorphism of \(N\) arises as in (1).

(3) For \(1 \leq j \leq n-1\) let \(\mathfrak{k}_j \subset \mathfrak{n}\) be the Lie subalgebra generated by \(\{X_{i, i+1}\}_{i \neq n-j}\), and \(K_j \subset N\) be the Lie subgroup with Lie algebra \(\mathfrak{k}_j\). A graded automorphism \(N \rightarrow N\) is induced by conjugation by some \(g = \text{diag}(\lambda_1, \ldots, \lambda_n)\) if and only if it preserves the subgroups \(K_j\) for \(1 \leq j \leq n-1\).

Proof. \(\square\). Suppose \(\Phi = \tau^\varepsilon \circ I_g\) for some \(g = \text{diag}(\lambda_1, \ldots, \lambda_n)\). Note that \(I_g\) induces a graded automorphism of \(N\) since

\[I_g \circ \delta_r = I_g \circ I_{gr} = I_{agr} = I_{g\cdot g} = I_g \circ I_g = \delta_r \circ I_g.\]

Combining with Lemma 2.3(2) we get that \(\Phi\) induces a graded automorphism of \(N\), proving the “if” direction.

Now suppose \(\Phi \in \text{Aut}(G)\) preserves \(N\), and induces a graded automorphism \(N \rightarrow N\).

By Lemma 2.3(1) we have \(\Phi = \tau^\varepsilon \circ I_g\) for some \(\varepsilon \in \{0, 1\}\), \(g \in G\). By Lemma 2.3(2), after postcomposing with \(\tau\) if necessary, we may assume without loss of generality that \(\Phi = I_g\) is an inner automorphism. The condition \(\Phi(N) = N\) is equivalent to saying that \(g\) belongs to the normalizer of \(N\) in \(G\). This is just the image of \(P^+\) under the quotient map \(\text{GL}(n, \mathbb{R}) \rightarrow G\). (To see this, note that if \(\hat{g} \in \text{GL}(n, \mathbb{R})\) is a lift of \(g\), then \(\hat{g}\) normalizes \(N \subset \text{GL}(n, \mathbb{R})\). But \(\text{span}(e_1)\) is the unique fixed point of the action \(N \acts G(1, n)\), so \(\hat{g}\) fixes \(\text{span}(e_1)\); passing to the quotient \(\mathbb{R}^n / \text{span}(e_1)\) and arguing by induction, one gets that \(\hat{g} \in \text{GL}(n, \mathbb{R})\)).
Furthermore, after multiplying \( g \) by a diagonal matrix, we may assume without loss of generality that \( g \in N \).

Now \( I_g : N \to N \) is a graded automorphism, so for all \( r > 0 \) we have \( I_g \circ \delta_r = \delta_r \circ I_g \), i.e.

\[
I_g \circ I_g = I_g \\
\Rightarrow I_g g_{g^{-1}} = \text{id}_N \\
\Rightarrow g_{g^{-1}} g_{g^{-1}} \in \text{Center}(N) \\
\Rightarrow g_{g^{-1}} - X \in \text{Center}(n)
\]

where \( g = \exp X \) for \( X \in n \); here we are using the fact that the exponential \( \exp : n \to N \) is a diffeomorphism. Calculating in \( \mathfrak{gl}(n, \mathbb{R}) \) and letting \( X = \sum_{j > i} a_{ij} X_{ij} \) we have

\[
g_{g^{-1}} - X = \sum_{j > i} a_{ij} (r^{j-i} - 1)X_{ij} \in \text{Center}(n)
\]

Therefore \( X \in \text{Center}(n) \), and \( g = \exp X \in \text{Center}(N) \), and \( I_g = \text{id} \).

(2). Let \( \phi : n \to n \) be a graded automorphism of \( n \). We will use the fact that for any \( x \in V_1 \) and any \( j \geq 1 \), we have \( \phi \circ (\text{ad } x |_{V_j}) = (\text{ad } \phi x |_{V_j}) \circ \phi \), and in particular \( \text{rank}(\text{ad } x |_{V_j}) = \text{rank}(\text{ad } \phi x |_{V_j}) \).

First suppose \( n = 4 \).

Set

\[
X_0 = X_{23}, \ X_1 = X_{12}, \ X_2 = X_{34}, \\
Y_1 = X_{13}, \ Y_2 = -X_{24}, \ Z = X_{14}
\]

Then the following are the only nontrivial bracket relations between the basis elements:

\[
[X_0, X_1] = -Y_1, \ [X_0, X_2] = -Y_2, \\
[X_1, Y_2] = -Z = [X_2, Y_1]
\]

Let \( W_0 := \text{span}(X_0) \), and \( W := \text{span}(X_1, X_2) \). If \( x \in V_1 \) then \( \text{rank}(\text{ad } x |_{V_1}) \leq 1 \) iff \( x \in W \), and \( \text{rank}(\text{ad } x |_{V_2}) \) = 0 iff \( x \in W_0 \). It follows that \( \phi(W) = W \) and \( \phi(W_0) = W_0 \), and so

\[
\phi X_0 = a_0 X_0, \ \phi X_1 = a_{11} X_1 + a_{21} X_2, \ \phi X_2 = a_{12} X_1 + a_{22} X_2
\]
for some $0 \neq a_0 \in \mathbb{R}$, $(a_{ij}) \in \text{GL}(2, \mathbb{R})$. Hence
\[
\begin{align*}
\phi Y_1 &= [\phi X_1, \phi X_0] = a_0 a_{11} Y_1 + a_0 a_{21} Y_2 \\
\phi Y_2 &= [\phi X_2, \phi X_0] = a_0 a_{12} Y_1 + a_0 a_{22} Y_2 \\
0 &= [\phi X_1, \phi Y_1] = -2a_0 a_{11} a_{21} Z \\
0 &= [\phi X_2, \phi Y_2] = -2a_0 a_{12} a_{22} Z.
\end{align*}
\]

(2.8)

(2.9)

Case 1. $a_{11} \neq 0$. By (2.8) we have $a_{21} = 0$, and since $\det(a_{ij}) \neq 0$ it follows that $a_{22} \neq 0$, and so $a_{12} = 0$ by (2.9). Thus $\phi|_{V_i}$ is diagonal in the basis $X_0, X_1, X_2$. Hence it is induced by an inner automorphism $I_g : G \to G$, where $g$ is diagonal.

Case 2. $a_{11} = 0$. Since $\det(a_{ij}) \neq 0$ it follows that $a_{12}$ and $a_{21}$ are both nonzero. Then (2.9) gives $a_{22} = 0$. Therefore modulo postcomposition with the automorphism $\tau$ we get that $\phi|_{V_i}$ is diagonal, and so (2) holds.

Now suppose $n \geq 5$.

First observe that
\[
\{ x \in V_1 | \text{rank}(ad x|_{V_1}) = 1 \} = (\mathbb{R}X_{1,2} \cup \mathbb{R}X_{n-1,n}) \setminus \{0\}.
\]

So $\phi$ either preserves or switches the two lines $\mathbb{R}X_{1,2}$, $\mathbb{R}X_{n-1,n}$, and therefore after postcomposing with $\tau$ if necessary, we may assume without loss of generality that $\phi$ preserves them. Then we observe that
\[
\{ x \in V_1 | \text{rank}(ad x|_{V_{n-2}}) = 0 \} = \text{span}\{X_{2,3}, \ldots, X_{n-2,n-1}\}.
\]

So $\phi(\mathfrak{h}_0) = \mathfrak{h}_0$, where $\mathfrak{h}_0$ is the Lie subalgebra of $\mathfrak{n}$ generated by $X_{2,3}, \ldots, X_{n-2,n-1}$. Note $\mathfrak{h}_0$ is isomorphic to $\mathfrak{n}_{n-2}$, i.e. the strictly upper triangular matrices in $\mathfrak{gl}(n-2, \mathbb{R})$.

Suppose $n = 5$. In this case $\mathfrak{h}_0 \cap V_1 = \text{span}\{X_{2,3}, X_{3,4}\}$. We have
\[
\{ x \in \mathfrak{h}_0 \cap V_1 | [x, X_{1,2}] = 0 \} = \mathbb{R}X_{3,4}
\]
and
\[
\{ x \in \mathfrak{h}_0 \cap V_1 | [x, X_{4,5}] = 0 \} = \mathbb{R}X_{2,3}.
\]

Because $\phi$ preserves the two lines $\mathbb{R}X_{1,2}$, $\mathbb{R}X_{1,5}$ then it also preserves the two lines $\mathbb{R}X_{2,3}$, $\mathbb{R}X_{3,4}$. Hence $\phi$ is diagonal in the basis $\{X_{i,i+1}\}_{1 \leq i \leq 4}$, and therefore (2) holds.

Now we assume $n \geq 6$ and that (2) holds for $\mathfrak{n}_k$ with $k \leq n-2$. We already observed $\phi(\mathfrak{h}_0) = \mathfrak{h}_0$, and have reduced to the case that the two lines $\mathbb{R}X_{1,2}, \mathbb{R}X_{n-1,n}$ are preserved by $\phi$. Since $\mathfrak{h}_0$ is isomorphic to $\mathfrak{n}_{n-2}$, the induction hypothesis implies that either $\phi(\mathbb{R}X_{i,i+1}) = \mathbb{R}X_{i,i+1}$ for all $i = 2, \ldots, n-2$ or $\phi(\mathbb{R}X_{i,i+1}) = \mathbb{R}X_{n-i,n+1-i}$ for all $i = 2, \ldots, n-2$. 


Since \([X_{1,2}, X_{2,3}] \neq 0\) and \([X_{1,2}, X_{n-2,n-1}] = 0\), and \(\phi\) is an automorphism of Lie algebra, we see that \(\phi(RX_{i,i+1}) = RX_{i,i+1}\) for all \(i = 1, \cdots, n-1\), and so (2) holds.

(3). Let \(\Phi\) be a graded automorphism, so by (2) we have \(\Phi = \tau^\varepsilon \circ I_g\) for \(\varepsilon \in \{0, 1\}\) and some \(g\) diagonal. If \(\varepsilon = 0\) then \(D\Phi(X_{i,i+1}) \in R X_{i,i+1}\), so \(\Phi\) preserves \(K_j\) for all \(j\). On the other hand, by (2.4) we have \(\tau(K_j) = K_{n-j}\), so if \(\Phi(K_j) = K_j\) for all \(1 \leq j \leq n-1\) then \(\varepsilon = 0\).

\(\square\)

We now consider the action of \(\text{Aut}(G)\) on the flag manifold.

**Lemma 2.10.**

(1) There is a 1-1 correspondence between cosets of \(P\) and their stabilizers in \(G\) with respect to the action \(G \curvearrowright G/P\): if \(g_1, g_2 \in G\) then

\[
\text{Stab}(g_1 P) = \text{Stab}(g_2 P) \iff g_1 P g_1^{-1} = g_2 P g_2^{-1} \iff g_1 P = g_2 P.
\]

(2) Every \(\Phi \in \text{Aut}(G)\) permutes the conjugates of \(P\); by (1) we thereby obtain an action of \(\text{Aut}(G) \curvearrowright G/P\) defined by

\[
\text{Stab}(\Phi \cdot g P) = \Phi(\text{Stab}(g P)).
\]

More explicitly, if \(\Phi(P) = h P h^{-1}\) then

\[
\Phi \cdot g P = \Phi(g) h P;
\]

in particular \(\rho(\Phi)\) defines a smooth diffeomorphism.

(3) For every \(\Phi \in \text{Aut}(G)\) the map

\[
\rho(\Phi) : G/P \rightarrow G/P
\]

is \(G\)-equivariant from \(G \curvearrowleft G/P\) to \(G \curvearrowleft G/P\), where \(\ell(\bar{g})(g P) := \bar{g} g P\) and \(\ell_{\Phi}(\bar{g})(g P) := \Phi(\bar{g}) g P\).

(4) The action \(\text{Aut}(G) \curvearrowleft G/P\) preserves the horizontal subbundle \(\mathcal{H} \subset T(G/P)\).

(5) If \(\Phi_0 \in \text{Aut}(G)\) is transpose-inverse, then \(\Phi_0 \cdot (W_j) = (W_j^+)\) for every \((W_j) \in \mathcal{F}\), i.e. \(\rho(\Phi_0) = \psi\).

**Proof.** (1). Note that the normalizer of \(P\) in \(G\) is \(P\) itself. Therefore

\[
\text{Stab}(g_1 P) = \text{Stab}(g_2 P) \iff g_1 P g_1^{-1} = g_2 P g_2^{-1} \iff (g_2^{-1} g_1) P (g_2^{-1} g_1)^{-1} = P \iff g_2^{-1} g_1 \in P \iff g_1 P = g_2 P.
\]
(2). Note that $\tau(P) = P$ so $\tau$ permutes the conjugates of $P$; by Lemma 2.3(1) it follows that every $\Phi \in \text{Aut}(G)$ permutes the conjugates of $P$. Since $\text{Stab}(gP) = gPg^{-1}$, if $\Phi(P) = hPh^{-1}$, then for every $g \in G$

$$\text{Stab}(\Phi \cdot gp) = \Phi(\text{Stab}(gP)) = \Phi(gPg^{-1})$$

$$= \Phi(g)hPh^{-1}(\Phi(g))^{-1} = \text{Stab}(\Phi(g)hP)$$

so $\Phi \cdot gP = \Phi(g)hP$.

(3). For $g, \bar{g} \in G$, by (2) we have

$$\Phi \cdot (\ell(\bar{g}) \cdot gP) = \Phi \cdot (\bar{g}gP) = \Phi(\bar{g})hP = \ell(\Phi(\bar{g})) \cdot (\Phi(g)hP) = \ell(\Phi(g)) \cdot (\Phi \cdot gP)$$

as claimed.

(4). If $\Phi = I_{\bar{g}}$ is an inner automorphism, then $I_{\bar{g}} \cdot gP = \bar{g}gP$ by (2.12); hence $I_{\bar{g}} \cdot gP$ preserves $H$ by Lemma 2.2(9). In the case $\Phi = \tau$, we have $\tau(P) = P$, so by (2.12) we have $\tau \cdot gP = \tau(g)P$, and since $\tau(X_{i,j+1}) = X_{n-i,n-i+1}$ we have $\tau(V_1) = V_1$. It follows that the differential of $\tau : G/P \to G/P$ at $P$ preserves $H(P)$. Using (3) and the fact that the actions $\ell$ and $\ell_\tau$ preserve $H$, we conclude that $\tau \cdot$ preserves $H$. By Lemma 2.3 the group $\text{Aut}(G)$ is generated by $\tau$ and the inner automorphisms, so (4) follows.

(5). Let $\Pi \in \text{GL}(n, \mathbb{R})$ be the permutation matrix with $\Pi e_i = e_{n-i}$, so $\Phi_0(P) = \Pi P \Pi^{-1}$. If $(W_j) = g \cdot (W_j^-)$, then using (2.12) we have

$$\Phi_0 \cdot (W_j) = \Phi_0 \cdot (g \cdot (W_j^-)) = (\Phi_0(g)\Pi) \cdot (W_j^-)$$

$$=((g^{-1})^t \Pi W_j^-) = ((g^{-1})^t W_j^+).$$

Since $(g^{-1})^t W_j^+$ is orthogonal to $gW_{n-j}$ for every $1 \leq j \leq n - 1$, assertion (5) follows.

\[\square\]

**Fibrations between flag manifolds.** We now consider partial flags and the associated flag manifolds.

If $\Sigma \subset \{1, \ldots, n-1\}$, we let $\mathcal{F}_{\Sigma}$ be the collection of partial flags $(W_j)_{j \in \Sigma}$ where $W_j \subset \mathbb{R}^n$ has dimension $j$. Then, as in the case when $\Sigma = \{1, \ldots, n-1\}$, the set $\mathcal{F}_{\Sigma}$ is a smooth submanifold of the product $\prod_{j \in \Sigma} G(j,n)$ of Grassmannians, and the actions $G \curvearrowright G(j,n)$ yield a transitive smooth action $G \curvearrowright \mathcal{F}_{\Sigma}$. Taking $(W_j^-)_{j \in \Sigma}$ to be the basepoint, and letting $P_{\Sigma} = \text{Stab}((W_j^-)_{j \in \Sigma}) \subset G$ be its stabilizer, we obtain a $G$-equivariant diffeomorphism $G/P_{\Sigma} \to \mathcal{F}_{\Sigma}$; we identify $\mathcal{F}_{\Sigma}$ with $G/P_{\Sigma}$.
Lemma 2.13.

(1) If \( \Sigma_1 \subset \Sigma_2 \subset \{1, \ldots, n-1\} \) then we obtain a smooth \( G \)-equivariant fibration \( \pi_{\Sigma_1, \Sigma_2} : \mathcal{F}_{\Sigma_2} \to \mathcal{F}_{\Sigma_1} \) by “forgetting subspaces”, i.e. \( \pi_{\Sigma_1, \Sigma_2}((W_j)_{j \in \Sigma_2}) := (W_j)_{j \in \Sigma_1} \).

(2) The fiber of \( \pi_{\Sigma_1, \Sigma_2} \) passing through \( (W_j^-) \) is the \( P_{\Sigma_1} \) orbit \( P_{\Sigma_1} \cdot (W_j^-) \), which is diffeomorphic to \( P_{\Sigma_1}/P_{\Sigma_2} \). To simplify notation, for \( j \in \{1, \ldots, n-1\} \) we let \( \mathcal{F}_j := \mathcal{F}_{\{j\}} = G(j, n) \) and

\[ \pi_j := \pi_{\{j\}, \{1, \ldots, n-1\}} : \mathcal{F} = \mathcal{F}_{\{1, \ldots, n-1\}} \to \mathcal{F}_j, \]

so the fiber of \( \pi_j \) passing through the basepoint \( P \) is \( P_j P \).

(3) The intersection of the fiber \( \pi_j^{-1}(W_j^-) \) with \( \hat{N} = N \cdot (W_j^-) \) is \( K_j \cdot (W_j^-) \), where \( K_j \subset N \) is as in Lemma 2.5(3). Hence the orbit map \( \alpha : N \to \hat{N} \) carries the coset foliation of \( K_j \) to the foliation defined by \( \pi_j \).

Proof. (1) and (2) are a special case of a general fact: if \( K \) is a Lie group and \( H_1 \subset H_2 \subset K \) are closed subgroups, the quotient map \( K/H_1 \to K/H_2 \) is a \( K \)-equivariant fibration; fiber passing through \( H_1 \) is \( H_2/H_1 \subset K/H_1 \) and it is diffeomorphic to \( H_2/H_1 \).

(4). The subbundle of \( T(G/P) \) tangent to the fibers of the fibration \( \pi_j : G/P \to G/P_j \) is \( G \)-invariant, and its value at the basepoint \( P \in G/P \) is the subspace \( p_j/p \). The subbundle of \( TN \) tangent to the coset foliation of \( K_j \) maps under the orbit map \( \alpha : N \to \hat{N} \) to an \( N \)-invariant subbundle of \( T\hat{N} \) whose value at \( P \) is the subspace \( (\mathfrak{k}_j + p)/p \). But \( \mathfrak{k}_j + p = p_{j,n} \), so the two subbundles are the same. It follows that \( \pi_j^{-1}(W_j^-) \cap \hat{N} \) is the union of a closed set of \( K_j \)-orbits in \( \hat{N} \); however, \( \pi_j^{-1}(W_j^-) \cap \hat{N} \) is invariant under the action of \( \delta_r \), so it can contain only one \( K_j \)-orbit. \( \square \)

Dynamics of Carnot dilations. The following result is a special case of [Tit72, Lemma 3.9]; we give a self-contained proof for the reader’s convenience.

Lemma 2.14. Pick \( 1 \leq j \leq n-1 \).

(1) Let \( K_1 \subset \mathcal{F}_1 \) be a compact subset disjoint from \( Z_1 := \{W_1 \in \mathcal{F}_1 \mid W_1 \subset W_{n-j}^+\} \), and \( U_1 \subset \mathcal{F}_1 \) be an open subset containing \( Z_2 := \{W_1 \in \mathcal{F}_1 \mid W_1 \subset W_j^-\} \). Then there exists \( \bar{r}_1 = \bar{r}_1(K_1, U_1) > 0 \) such that for every \( r \leq \bar{r}_1 \) we have \( \hat{\delta}_r(K_1) \subset U_1 \).
(2) Let $K \subset \mathcal{F}_j$ be a compact subset such that $W_j \cap W^\pm_{n-j} = \{0\}$ for every $W_j \in K$, and $U \subset F_j$ be an open subset containing $W_j^-$. Then there exists $\tilde{r} = \tilde{r}(K, U) > 0$ such that for every $r \leq \tilde{r}$ we have $\hat{\delta}_r(K) \subset U$.

Proof. (1). Let $\pi_{W^+_{n-j}}$ and $\pi_{W^-_{n-j}}$ be the projections onto the summands of the decomposition $\mathbb{R}^n = W^+_{n-j} \oplus W^-_{n-j}$. Define $\hat{\beta} : \mathbb{R}^n \setminus W^+_{n-j} \to [0, \infty)$ by $\hat{\beta}(v) := \|\pi_{W^+_{n-j}} v\| / \|\pi_{W^-_{n-j}} v\|$. Note that by homogeneity $\hat{\beta}$ descends to $\beta : \mathcal{F}_1 \setminus Z_1 \to [0, \infty)$.

Since $K_1$ is compact and $U_1$ is open, we may choose $C_2 \leq C_1 < \infty$ such that $\beta(K_1) \subset [0, C_1]$ and $\beta^{-1}([0, C_2]) \subset U$. If $v \in \mathbb{R}^n \setminus W^+_{n-j}$ and $r \leq 1$ then

$$
\|\pi_{W^+_{n-j}}(\hat{\delta}_r v)\| = \left\| \sum_{i=1}^{n-j} r^{-i} v_i \right\| \leq r^{-(n-j)} \|\pi_{W^+_{n-j}} v\|,
$$

$$
\|\pi_{W^-_{n-j}}(\hat{\delta}_r v)\| = \left\| \sum_{i=n-j+1}^n r^{-i} v_i \right\| \geq r^{-(n-j+1)} \|\pi_{W^-_{n-j}} v\|.
$$

Hence for $W_1 \in \mathcal{F}_1 \setminus Z_1$ and $r \leq 1$ we have $\beta(\hat{\delta}_r W_1) \leq r \beta(W_1)$. Taking $\bar{r}_1 := C_2 / C_1$, if $r \leq \bar{r}_1$ then

$$
\hat{\delta}_r(K_1) \subset \hat{\delta}_r(\beta^{-1}([0, C_1])) \subset \beta^{-1}([0, C_2]) \subset U_1.
$$

(2). Note that

$$
K_1 := \{ W_1 \in \mathcal{F}_1 \mid \exists W_j \in K \text{ s.t. } W_1 \subset W_j \}
$$

is a compact subset of $\mathcal{F}_1 \setminus Z_1$, and there is an open subset $U_1 \subset F_1$ containing $Z_2$ such that if $W_j \in F_j$ and $\{ W_1 \in \mathcal{F}_1 \mid W_1 \subset W_j \} \subset U_1$, then $W_j \subset U$. Let $r = \bar{r}_1(K_1, U_1)$ be as in (1), and choose $W_j \in K$, $r \leq \bar{r}$. Then by the choice of $r$, for every $W'_1 \in \mathcal{F}_1$ with $W'_1 \subset W_j$ we have $\hat{\delta}_r(W'_1) \in U_1$. Therefore $\{ W_1 \in \mathcal{F}_1 \mid W_1 \subset \hat{\delta}_r(W_j) \} \subset U_1$, and hence $\hat{\delta}_r(W_j) \in U$, by our assumption on $U_1$. $\square$
2.2. Sobolev mappings and the Pullback Theorem. We give a very brief discussion here, and refer the reader to [KMX20, Section 2] and the references therein for more details.

Let $H$, $H'$ be Carnot groups with gradings $\mathfrak{h} = \oplus_j V_j$, $\mathfrak{h}' = \oplus_j V'_j$, Carnot dilations denoted by $\delta_r$, and Carnot-Carathéodory distance functions $d_{CC}$, $d'_{CC}$. Let $\nu = \sum_j j \dim V_j$ be the homogeneous dimension of $H$.

Choose $p > \nu$. If $U \subset H$ is open and $f : U \to H'$ is a continuous map then $f$ is a $W^{1,p}_{\text{loc}}$-mapping if there exists $g \in L^p_{\text{loc}}(U)$ such that for every 1-Lipschitz function $\psi : H' \to \mathbb{R}$ and every unit length $X \in V_1$, the distribution derivative $X(\psi \circ f)$ satisfies $|X(\psi \circ f)| \leq g$ almost everywhere. If $f : H \supset U \to U' \subset H'$ is a quasiconformal homeomorphism then $f$ is a $W^{1,p}_{\text{loc}}$-mapping for some $p > \nu$.

If $f : U \to H'$ is a $W^{1,p}_{\text{loc}}$-mapping then for a.e. $x \in U$ the map $f$ has a Pansu differential, i.e. letting $f_x := \ell_{f(x)^{-1}} \circ f \circ \ell_x$, $f_{x,r} := \delta^{-1}_r \circ f_x \circ \delta_r$ there is a graded group homomorphism $D_pf(x) : H \to H'$ such that

$$f_{x,r} \overset{C^0_{\text{loc}}}{\to} D_pf(x)$$

as $r \to 0$. A Lie group homomorphism $\Phi : H \to H'$ is graded if $\delta_r \circ \Phi = \Phi \circ \delta_r$ for all $r > 0$. We will often use $D_pf(x)$ to denote the induced graded homomorphism of Lie algebras $\mathfrak{h} \to \mathfrak{h}'$.

We let $\{X_j\}_{1 \leq j \leq \dim H}$ be a graded basis for $\mathfrak{h}$, and $\{\theta_j\}_{1 \leq j \leq \dim H}$ be the dual basis.

The weight of a subset $I \subset \{1, \ldots, \dim H\}$ is given by

$$\text{wt } I := -\sum_{i \in I} \deg i$$

where $\deg : \{1, \ldots, \dim H\} \to \{1, \ldots, s\}$ is defined by $\deg i = j$ iff $X_i \in V_j$. For a non-zero left-invariant form $\alpha = \sum_i a_i \theta_i$ we define $\text{wt}(\alpha) = \max\{\text{wt } I : a_I \neq 0\}$ and set $\text{wt}(0) := -\infty$; here $\theta_I$ denotes the wedge product $\Lambda_{i \in I} \theta_i$.

We use primes for the objects associated with $H'$.

Fix $p > \nu$, and let $f : U \to H'$ be a $W^{1,p}_{\text{loc}}$-mapping for some open subset $U \subset H$. If $\omega \in \Omega^k(H')$ is a differential $k$-form with continuous coefficients, the Pansu pullback of $\omega$ is the $k$-form with measurable coefficients $f^*_p \omega$ given by

$$f^*_p \omega(x) := (D_pf(x))^* \omega(f(x)),$$

where $D_pf(x) : \mathfrak{h} \to \mathfrak{h}'$ is the Pansu differential of $f$ at $x \in U$.

We will use the following special case of [KMX20, Theorem 4.2]:
Theorem 2.15 (Pullback Theorem (special case)). Suppose $\varphi \in C^\infty_c(U)$ and that $\alpha$ and $\beta$ are closed left invariant forms which satisfy

\begin{equation}
\deg \alpha + \deg \beta = N - 1 \quad \text{and} \quad \text{wt}(\alpha) + \text{wt}(\beta) \leq -\nu + 1.
\end{equation}

Then

\begin{equation}
\int_U f^*_P(\alpha) \wedge d(\varphi \beta) = 0.
\end{equation}

We now consider Sobolev mappings on the flag manifold. Let $f : F \supset U \to F$ be a continuous map. Then:

- $f$ is a $W^{1,p}_{\text{loc}}$-mapping if for every $x \in U$ there is an open neighborhood $V$ of $x$ and group elements $g, g' \in G = \text{PGL}(n, \mathbb{R})$ such that $V \subset g \cdot \hat{N}$, $f(V) \subset g' \cdot \hat{N}$, and the composition

\begin{equation}
(\rho(g) \circ \alpha)^{-1}(V) \xrightarrow{\rho(g) \circ \alpha} V \xrightarrow{f} g' \cdot \hat{N} \xrightarrow{(\rho(g') \circ \alpha)^{-1}} N
\end{equation}

is a $W^{1,p}_{\text{loc}}$-mapping.

- The map $f$ is Pansu differentiable at $x \in U$ if for some $x \in U, V \subset U$, $g, g' \in G$ as above, the composition (2.18) is Pansu differentiable at $(\rho(g) \circ \alpha)^{-1}(x)$. By the chain rule for Pansu differentials, Pansu differentiability is independent of the choice of $g, g'$, and the resulting Pansu differential $n \to n$ is well-defined up to pre/post composition with graded automorphisms. In particular, the property being an isomorphism is well-defined.

Equivalently, one may work directly with the flag manifold as an equiregular subriemannian manifold, and use the notion of Pansu differential in that setting (see [Gro96, Section 1.4], [Vod07a], and [KMX20, Appendix A]).

3. Sobolev mappings and the (virtual) preservation of coset foliations

The main result in this section is the following:

Lemma 3.1. Let $n \geq 4$. Let $U \subset N$ be a connected open subset, and for some $p > \nu$ let $f : N \supset U \to N$ be a $W^{1,p}_{\text{loc}}$-mapping whose Pansu differential is an isomorphism almost everywhere. Then after possibly composing with $\tau$, if necessary, for a.e. $x \in U$ the Pansu differential $D_P f(x)$ preserves the subspace $\mathbb{R}X_{i,i+1} \subset V_1$ for every $1 \leq i \leq n - 1$. 

Corollary 3.2. Let \( n \geq 4 \) and \( f : U \to N \) be as in Lemma 3.1 and \( K_j \) be as in Lemma 2.5(3). Then after possibly composing with \( \tau \), if necessary, for every \( 1 \leq j \leq n-1 \), \( f \) locally preserves the coset foliation of \( K_j \), i.e. for every \( x \in U \) there is an \( r > 0 \) such that for every \( g \in N \) the image of \( gK_j \cap B(x,r) \) under \( f \) is contained in a single coset of \( K_j \).

The corollary follows from Lemma 3.1 and [KMX21b, Lemma 2.30]. Note that Lemma 3.1 fails when \( n = 3 \) (when \( N \) is a copy of the Heisenberg group), even for automorphisms. Examples from [KMSX] show that Lemma 3.1 can fail even for bilipschitz mappings whose Pansu differential preserves the splitting \( V_1 = \mathbb{R}X_{1,2} \oplus \mathbb{R}X_{2,3} \) for a.e. \( x \).

Proof of Lemma 3.1. We begin with the \( n = 4 \) case of the lemma.

As in the proof of Lemma 2.7, we let
\[
X_0 = X_{23}, \quad X_1 = X_{12}, \quad X_2 = X_{34},
Y_1 = X_{13}, \quad Y_2 = -X_{24}, \quad Z = X_{14}.
\]
Let \( \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \) be the basis of left invariant 1-forms that are dual to \( X_0, X_1, X_2, Y_1, Y_2, Z \). Then using (2.7) we have
\[
\begin{align*}
d\alpha_0 &= d\alpha_1 = d\alpha_2 = 0; \\
d\beta_1 &= \alpha_0 \wedge \alpha_1, \quad d\beta_2 = \alpha_0 \wedge \alpha_2; \\
d\gamma &= \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1.
\end{align*}
\]

Let \( \omega := \alpha_1 \wedge \beta_1 \wedge \gamma \). By Lemma 2.5(2) on graded automorphisms,
\[
f^*_p \omega = u_1 \alpha_1 \wedge \beta_1 \wedge \gamma + u_2 \alpha_2 \wedge \beta_2 \wedge \gamma
\]
where \( u_1, u_2 \) are measurable and with \( S_i := \{ u_i \neq 0 \} \) the union \( S_1 \cup S_2 \) has full measure and \( S_1 \cap S_2 = \emptyset \).

Let \( \eta = \alpha_2 \wedge \beta_2 \) and pick \( \varphi \in C_c^\infty(U) \). Then \( \omega \) and \( \eta \) are closed left invariant forms and their degrees and weights satisfy the assumption of the pullback theorem (Theorem 2.15), and so
\[
\int_U f^*_p \omega \wedge d(\varphi \eta) = 0.
\]
As \( d(\varphi \eta) = d\varphi \wedge \eta \) we have \( f^*_p \omega \wedge d(\varphi \eta) = \pm u_1(X_0 \varphi) d\text{vol} \) and therefore \( \int_U u_1(X_0 \varphi) d\text{vol} = 0 \). Since \( \varphi \) was arbitrary, it follows that \( X_0 u_1 = 0 \) in the sense of distributions. Similarly by picking \( \eta = \alpha_0 \wedge \beta_2 \) we obtain \( X_2 u_1 = 0 \). It follows that \( Y_2 u_1 = [X_2, X_0] u_1 = X_2 X_0 u_1 - X_0 X_2 u_1 = 0 \). Let \( H_i \ (i = 1, 2) \) be the Lie subgroup of \( N \) whose Lie algebra is generated by \( X_0 \) and \( X_i \). Then we see that for almost every left coset \( gH_2 \), \( u_1 \) is locally constant almost everywhere. Consequently \( X_0 \chi_{S_i} = X_2 \chi_{S_i} = Y_2 \chi_{S_i} = 0 \), where \( \chi \) denotes the characteristic
function of a subset $S \subset U$. Similarly by using $\eta = \alpha_1 \wedge \beta_1$, $\alpha_0 \wedge \beta_1$, we obtain $X_0 u_2 = X_1 u_2 = Y_1 u_2 = 0$ and $X_0 \chi_{S_2} = X_1 \chi_{S_2} = Y_1 \chi_{S_2} = 0$. As $\chi_{S_1} + \chi_{S_2} = 1$, we infer that $X_i \chi_{S_j} = 0$ for all $0 \leq i \leq 2$ and $j = 1, 2$. Since $X_0, X_1, X_2$ generates $n$, this gives $X \chi_{S_j} = 0$ for every $X \in n$, and hence $\chi_{S_j}$ is locally constant a.e. By the connectedness of $U$, $\chi_{S_j}$ is constant a.e. As $\chi_{S_1}$ takes on only two possible values ($0$ or $1$), we have $\chi_{S_1} = 0$ a.e. or $\chi_{S_1} = 1$ a.e. If $\chi_{S_1} = 1$ a.e., then by the definition of $S_1$ we have $Df_x(\mathbb{R}X_1) = \mathbb{R}X_1$ for all $0 \leq i \leq 2$ and a.e. $x \in U$. If $\chi_{S_1} = 0$ a.e., then $\tau \circ Df_x(\mathbb{R}X_1) = \mathbb{R}X_1$ for a.e. $x \in U$.

Now assume $n \geq 5$.

Let $\{\theta_{i,j} \mid 1 \leq i < j \leq n\}$ be the basis of left invariant 1-forms on $N$ that are dual to the basis $\{X_{i,j} \mid 1 \leq i < j \leq n\}$ of $n$. Then we have

$$d\theta_{i,j} = \begin{cases} 0 & \text{if } j = i + 1 \\ -\sum_{k=1}^{j-1} \theta_{i,k} \wedge \theta_{k,j} & \text{if } j > i + 1 \end{cases}$$

(3.3)

Let $\omega_+ = \theta_{1,2} \wedge \theta_{2,3} \wedge \cdots \wedge \theta_{1,n}$ and $\omega_- = \theta_{n-1,n} \wedge \theta_{n-2,n} \wedge \cdots \wedge \theta_{1,n}$. Note that $\omega_+$ is closed since by (3.3) we have $d\theta_{1,2} = 0$, and $d\theta_{1,j} = -\sum_{k=2}^{j-1} \theta_{1,k} \wedge \theta_{k,j}$ for $j \geq 3$ and so $\theta_{1,2} \wedge \cdots \wedge \theta_{1,j-1} \wedge d\theta_{1,j} = 0$. Similarly $\omega_-$ is closed. By Lemma 2.5 we have

$$f^*_p(\omega_+) = u_+ \omega_+ + u_- \omega_-$$

for some measurable functions $u_+, u_-$. For $2 \leq k \leq n - 1$, let

$$\eta_{k-} = \bigwedge_{2 \leq i < j \leq n, (i,j) \neq (k,k+1)} \theta_{i,j},$$

and for $1 \leq k \leq n - 2$, let

$$\eta_{k+} = \bigwedge_{1 \leq i < j \leq n-1, (i,j) \neq (k,k+1)} \theta_{i,j}.$$  

We show that $\eta_{k-}$ is closed. First we have $d\theta_{l,l+1} = 0$. For $m > l + 1$, $d\theta_{l,m} = -\sum_{s=l+1}^{m-1} \theta_{l,s} \wedge \theta_{s,m}$ and at least one of the two terms $\theta_{l,s}, \theta_{s,m}$ is already present in

$$\bigwedge_{2 \leq i < j \leq n, (i,j) \neq (k,k+1), (l,m)} \theta_{i,j};$$

it follows that $d\theta_{l,m} \wedge \bigwedge_{2 \leq i < j \leq n, (i,j) \neq (k,k+1), (l,m)} \theta_{i,j} = 0$. Similarly $\eta_{k+}$ is closed. We next show that the degree and weight conditions in the pullback theorem (Theorem 2.15) are satisfied: for $\omega_+$ and $\eta = \eta_{k-}$ or $\eta_{k+}$ we have $\deg(\omega_+) + \deg(\eta) = N - 1$ and $\wt(\omega_+) + \wt(\eta) = -\nu + 1$. First these conditions are satisfied by $\omega_+$ and $\eta$ as $\omega_+ \wedge \eta_- = \pm \bigwedge_{(i,j) \neq (k,k+1)} \theta_{ij}$ and $\theta_{k,k+1}$ has degree one and weight $-1$. These
conditions are also satisfied by \( \omega_+ \) and \( \eta_+ \) as \( \tau(\eta_-) = \pm \eta_+ \) and so \( \eta_- \), \( \eta_+ \) have the same degree and weight. Hence by Theorem 2.15, for any \( \varphi \in C_c^\infty(U) \), we have \( \int_U f^*_p(\omega_+) \wedge d(\varphi \eta) = 0 \). As \( d(\varphi \eta) = d\varphi \wedge \eta \), we get \( f^*_p(\omega_+) \wedge d(\varphi \eta_k) = \pm u_+ X_{k,k+1} \varphi \text{Vol} \) (for \( 2 \leq k \leq n-1 \)) and \( f^*_p(\omega_+) \wedge d(\varphi \eta_k) = \pm u_- X_{k,k+1} \varphi \text{Vol} \) (for \( 1 \leq k \leq n-2 \)). It follows that distributionally we have \( X_{k,k+1} u_+ = 0 \) for all \( 2 \leq k \leq n-1 \) and \( X_{k,k+1} u_- = 0 \) for all \( 1 \leq k \leq n-2 \).

Let \( S_+ \subset U \) be the subset where the Pansu differential of \( f \) fixes all the directions \( \mathbb{R}X_{i,i+1} \). Similarly let \( S_- \subset U \) be the set of points \( x \in U \) such that \( \tau \circ Df_x \) fixes all the directions \( \mathbb{R}X_{i,i+1} \). Again by Lemma 2.5 (2), \( S_+ \cup S_- \) has full measure in \( U \) and \( S_+ \cap S_- \) is null. Notice that \( S_+ \) agrees with \( \{ x \in U \mid u_+(x) \neq 0 \} \) up to a set of measure 0 and \( S_- \) agrees with \( \{ x \in U \mid u_-(x) \neq 0 \} \) up to a set of measure 0. Hence \( X_{k,k+1} \chi_{S_+} = 0 \) for all \( 2 \leq k \leq n-1 \) and \( X_{k,k+1} \chi_{S_-} = 0 \) for all \( 1 \leq k \leq n-2 \). Since \( \chi_{S_+} + \chi_{S_-} = 1 \) we have \( X_{k,k+1} \chi_{S_+} = 0 \) for all \( 1 \leq k \leq n-2 \). So \( X_{k,k+1} \chi_{S_+} = 0 \) for all \( 1 \leq k \leq n-1 \). Since the \( X_{k,k+1}, 1 \leq k \leq n-1 \) generates the Lie algebra \( \mathfrak{n} \), we see that \( X \chi_{S_+} = 0 \) for any \( X \in \mathfrak{n} \) and we conclude as in the \( n = 4 \) case.

\( \square \)

4. Rigidity for fibration preserving maps

In the section we show that mappings of \( \mathcal{F} \) which respect the fibrations \( \pi_j : \mathcal{F} \to \mathcal{F}_j \) for all \( 1 \leq j \leq n-1 \) are locally projective.

**Definition 4.1.** A mapping \( f : \mathcal{F} \supset U \to \mathcal{F} \) is **fibration-preserving** if for every \( 1 \leq j \leq n-1 \), and every \( W_j \in \mathcal{F}_j \), the image of \( U \cap \pi_j^{-1}(W_j) \) under \( f \) is contained in \( \pi_j^{-1}(W'_j) \) for some \( W'_j \in \mathcal{F}_j \). Equivalently, letting \( U_j := \pi_j(U) \), there exist mappings \( f_j : U_j \to \mathcal{F}_j \) such that \( \pi_j \circ f = f_j \circ \pi_j \). A mapping \( f : \mathcal{F} \supset U \to \mathcal{F} \) is **locally fibration preserving** if it is fibration preserving near any point in \( U \).

**Proposition 4.2.** Suppose \( n \geq 3 \) and \( f : U \to \mathcal{F} \) is a continuous, locally fiber preserving mapping, where \( U \subset \mathcal{F} \) is a connected open subset. If for some \( x \in U \) the Pansu differential \( D_P f(x) \) exists and is an isomorphism (see Subsection 2.2) then \( f \) agrees with \( g : \mathcal{F} \to \mathcal{F} \) for some \( g \in \text{PGL}(n, \mathbb{R}) \).
4.1. **Proof of Proposition 4.2 in the $n = 3$ case.** In this subsection we fix $n = 3$, and without further mention $\mathcal{F}$ and $\mathcal{F}_i$, and $\pi_i : \mathcal{F} \to \mathcal{F}_i$ will be the objects associated with $\mathbb{R}^3$.

**Definition 4.3.** A projective frame is an indexed collection $\{W_i\}_{0 \leq i \leq 3} \subset \mathcal{F}_1$ where any three elements span $\mathbb{R}^3$. The standard projective frame $\{\hat{W}_i\}_{0 \leq i \leq 3}$ is given by $\hat{W}_1 = \text{span}(e_i)$ for $1 \leq i \leq 3$, and $\hat{W}_0 = \text{span}(e_1 + e_2 + e_3)$.

Note that for any projective frame $\{W_i\}_{0 \leq i \leq 3} \subset \mathcal{F}_1$ there exists a unique $g \in \text{PGL}(3, \mathbb{R})$ such that $g(\hat{W}_i) = \hat{W}_i$ for $0 \leq i \leq 3$, i.e. $\text{PGL}(3, \mathbb{R})$ acts freely transitively on the collection of projective frames.

The proof of Proposition 4.2 is inspired by the following elementary classical result:

**Theorem 4.4** (Fundamental Theorem of Projective Geometry). Every fibration-preserving bijection $f : \mathcal{F} \to \mathcal{F}$ agrees with some $g \in \text{PGL}(3, \mathbb{R})$.

Note that this is an equivalent reformulation of the classical result using the flag manifold rather than the (traditional) projective plane.

**Proof of Theorem 4.4.** Since $f$ is a bijection, so are the maps $f_i : \mathcal{F}_i \to \mathcal{F}_i$ for $i \in \{1, 2\}$. It follows that $f_1$ maps $\pi_1(\pi_2^{-1}(W_2))$ bijectively to $\pi_1(\pi_2^{-1}(f_2(W_2)))$ for every plane $W_2 \in \mathcal{F}_2$. Therefore three elements of $\mathcal{F}_1$ lie in a plane if and only if their images under $f_1$ lie in a plane. Hence $\{f_1(W_i)\}_{0 \leq i \leq 3}$ is a projective frame, so after composing $f$ with some element of $\text{PGL}(3, \mathbb{R})$, we may assume that $f_1(\hat{W}_1) = \hat{W}_i$ for $0 \leq i \leq 3$. It follows that $f_1$ also fixes $\hat{W}_1 := \hat{W}_1^3 \cap \hat{W}_1^2$.

We identify $\mathbb{R}^2$ with the affine plane $\{W_1 \in \mathcal{F}_1 \mid W_1 \not\subset \hat{W}_1^2 = \text{span}(e_1, e_2)\}$ by $(x_1, x_2) \leftrightarrow \text{span}(x_1, x_2, 1)$. We may define a bijection $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi(x_1, x_2, 1) = f_1(\text{span}(x_1, x_2, 1))$. Using the fact that $f_1(\hat{W}_1) = \hat{W}_1$ for $0 \leq i \leq 4$, one sees that:

(a) $\phi$ fixes $(0, 0), (1, 1)$.
(b) $\phi$ maps lines bijectively to lines.
(c) For every $v \in \{e_1, e_2, e_1 + e_2\}$ and every line $L$ parallel to $v$, the image $f(L)$ is a line parallel to $v$.

Using a geometric construction (see Figure 1) it follows that the restriction of $\phi$ to $\mathbb{R} \times \{0\} \simeq \mathbb{R}$ is a field isomorphism. Since $\text{id}_\mathbb{R}$ is the only automorphism of $\mathbb{R}$, it follows that $\phi$ fixes $\mathbb{R} \times \{0\}$. Hence it also fixes $\{0\} \times \mathbb{R}$, then all of $\mathbb{R}^2$; and $f_1$ fixes $\mathcal{F}_1$, and $f$ fixes $\mathcal{F}$. \qed
The $n = 3$ case of Proposition 4.2 may be viewed as a localized version of Theorem 4.4 for mappings which are continuous, but not necessarily bijective.

Before proceeding, we first give a rough idea how the argument goes. The first step is to show that local projectivity of $f$ propagates: if $f$ locally agrees with some $g \in \text{PGL}(3, \mathbb{R})$ near some point $x \in U$, then $f$ will locally agree with $g$ near points in the fibers of $\pi_1, \pi_2$ passing through $x$ (see Lemma 4.5). This readily implies that the subset of $U$ where $f$ agrees with $g$ is a connected component of $U$. Thus we are reduced to finding a single point near which $f$ is locally projective. To that end, we consider $(W_1, W_2) \in U$ a point of differentiability where differential is an isomorphism. Using the definition of differentiability, we argue that the map $f$ is nondegenerate near $(W_1, W_2)$, in the sense that the map $f_1$ induced on $\mathcal{F}_1$ carries a projective frame localized near $W_1$ to a projective frame (see below the claim in the proof of the $n = 3$ case of Proposition 4.2). Then by pre/post-composing with suitable elements of $\text{PGL}(3, \mathbb{R})$, and working in $\mathcal{F}_1$, we are able to reduce to a map $\phi : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ which satisfies a localized version of the conditions (a)-(c) appearing in the proof of Theorem 4.4; this is shown.
Lemma 4.5 (Propagation of projectivity). Suppose $U \subset \mathcal{F}$ is open and $f : U \rightarrow \mathcal{F}$ is continuous and fibration preserving.

(1) If $f$ agrees with $g \in \text{PGL}(3, \mathbb{R})$ near $(\hat{W}_1, \hat{W}_2) \in U$, then $f$ agrees with $g$ near the fiber $\pi_i^{-1}(\hat{W}_i) \cap U$, for $i \in \{1, 2\}$.

(2) For every $g \in \text{PGL}(3, \mathbb{R})$, the set where $f$ locally agrees with $g$ is a connected component of $U$.

Proof. (1). We prove the assertion for $i = 2$; the case when $i = 1$ is similar.

After postcomposing with an element of $\text{PGL}(3, \mathbb{R})$, may assume without loss of generality that $f \equiv \text{id}$ on an open set $V \subset U$ with $(\hat{W}_1, \hat{W}_2) \in V$.

Pick $(W_1, W_2) \in U \cap \pi_2^{-1}(\hat{W}_2)$, i.e. $W_2 = \hat{W}_2$. Choose $(W_1, W_2') \in U \setminus \{(W_1, W_2)\}$ such that $\pi_2^{-1}(W_2') \cap V \neq \emptyset$.

Now suppose $(W_1', W_2') \to (W_1, W_2)$. We may choose $\{W_2^{(j)}\} \subset \mathcal{F}_2$ such that $W_1' \subset W_2^{(j)}$ and $W_2^{(j)} \to W_2'$ as $j \to \infty$. Since $V$ is open, after dropping finitely many terms, we may assume that $\pi_2^{-1}(W_2^{(j)}) \cap V \neq \emptyset$ and $\pi_2^{-1}(W_2^{(j)}) \cap V \neq \emptyset$. Since $f$ is fibration-preserving and $f \equiv \text{id}$ on $V$, it follows that $f(\pi_2^{-1}(W_2^{(j)}) \cap U) \subset \pi_2^{-1}(W_2^{(j)})$ and $f(\pi_2^{-1}(W_2^{(j)}) \cap U) \subset \pi_2^{-1}(W_2^{(j)})$. The map $f$ is fibration preserving, so $f(W_1', W_2'), f(W_1', W_2^{(j)}) \in \pi_2^{-1}(f_1(W_1'))$. Therefore $\pi_2^{-1}(f_1(W_1'))$ intersects both $\pi_2^{-1}(W_2^{(j)})$ and $\pi_2^{-1}(W_2^{(j)})$ nontrivially, forcing $f_1(W_1') = W_2' \cap W_2^{(j)} = W_1'$. Thus $f(W_1', W_2') = (W_1', W_2')$. Since the sequence $\{(W_1', W_2')\}$ was arbitrary, this proves (1).

(2). Define an equivalence relation on $U$ where $x, x' \in U$ are equivalent if there is a path $\gamma : [0, 1] \to U$ from $x$ to $x'$ which is piecewise contained in a fiber of one of the fibrations $\pi_i : \mathcal{F} \to \mathcal{F}_i$. We claim that the equivalence classes are open subsets of $U$. To see this, pick $x = g \cdot (W_1', W_2') \in U$, and note that the vector fields $X_{1,2}, X_{2,3}$ on $N$ are bracket generating and tangent to the cosets of $K_1$ and $K_2$, respectively; and hence their pushforwards $X_{1,2}', X_{2,3}'$ under the composition $N \overset{\pi_1}{\to} \hat{N} \overset{\pi_2}{\to} g \cdot \hat{N}$ are bracket generating vector fields on $g \cdot \hat{N}$ which are tangent to the fibers of $\pi_1$ and $\pi_2$ respectively, by Lemma 2.13(4). By [Mon02, pp. 50-52] the equivalence class of $x$ contains a neighborhood of $x$. It follows that the equivalence classes are connected components of
U. Since the set where \( f \) locally agrees with \( g \) is a union of equivalences classes by (1), assertion (2) follows.

\[
\Box
\]

**Lemma 4.6.** Suppose \( U \subset \mathbb{R}^2 \) is a connected open subset, and
\[
\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \supset U \to \mathbb{R}^2
\]
is a continuous map such for every \( v \in \{e_1, e_2, e_1 + e_2\} \) and every line \( L \) parallel to \( v \), the image of \( L \cap U \) is contained in a line parallel to \( v \). Then \( \phi \) is of the form
\[
(4.7) \quad \phi(x, y) = (mx + b_1, my + b_2)
\]
for some \( m, b_1, b_2 \in \mathbb{R} \).

**Proof.** First assume that \( \phi \) is smooth. Applying the hypothesis with \( v \in \{e_1, e_2\} \) implies that \( \partial_2 \phi_1 = \partial_1 \phi_2 \equiv 0 \), so \( \phi_i \) depends only on \( x_i \).

Applying the hypothesis when \( v = e_1 + e_2 \), we have
\[
0 \equiv (\partial_1 + \partial_2)(\phi_2 - \phi_1) = \partial_2 \phi_2 - \partial_1 \phi_1.
\]

But since \( \partial_i \phi_i \) depends only on \( x_i \) this forces \( \partial_1 \phi_1 = \partial_2 \phi_2 = \text{const} \) and so (4.7) holds. Since the conditions are preserved by taking linear combinations and precomposing with translations, the general case follows by mollification.

Alternatively, one may argue as follows. Without loss of generality, one may assume that \( \phi(0, 0) = (0, 0) \). By geometric contruction, \( \phi(x + x', 0) = \phi(x, 0) + \phi(x', 0) \) when \( x, x' \in (-r, r) \) for \( r \) small. Hence \( \phi(x_1, 0) = mx_1 \) for some \( m \in \mathbb{R} \), for \( x_1 \in (-r, r) \). Invoking the hypotheses again, we get \( \phi(x_1, x_2) = (mx_1, mx_2) \). Thus the lemma holds locally, i.e. for every \( (x_1, x_2) \in U \) there is an open set \( V_{x_1,x_2} \) containing \( (x_1, x_2) \) and \( m, b_1, b_2 \) depending on \( (x_1, x_2) \) such that (4.7) holds in \( V_{x_1,x_2} \); since \( U \) is connected and \( m, b_1, b_2 \) are locally constant, it follows that they are independent of \( x_1, x_2 \), and so (4.7) holds.

\[
\Box
\]

**Definition 4.8.** An indexed tuple \( \{W^i_1\}_{0 \leq i \leq 4} \subset \mathcal{F}_1 \) is an **augmented projective frame** if \( \{W^i_1\}_{0 \leq i \leq 3} \) is a projective frame and
\[
W^4_1 = \text{span}(W^3_1, W^0_1) \cap \text{span}(W^1_1, W^2_1).
\]

The **standard augmented projective frame** \( \{\hat{W}^i_1\}_{0 \leq i \leq 4} \) is given by
\[
(4.9) \quad \hat{W}^i_1 = \begin{cases} 
  e_1 + e_2 + e_3, & i = 0 \\
  e_i, & 1 \leq i \leq 3 \\
  e_1 + e_2, & i = 4
\end{cases}
\]
Given a subset $\Sigma \subset F$, we obtain (possibly empty) subsets of $F$ and $F$:

\begin{equation}
\begin{aligned}
F(\Sigma) &= \{ \text{span}(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in \Sigma, \sigma_1 \neq \sigma_2 \}, \\
F(F) &= \{ (W_1, W_2) \mid W_1 \in \Sigma, W_2 \in F(\Sigma) \}.
\end{aligned}
\end{equation}

**Lemma 4.11.** There is an augmented projective frame $\{\hat{W}_i\}_{0 \leq i \leq 4} \subset F_1$ such that $F(\{\hat{W}_i\}_{0 \leq i \leq 4}) \subset \hat{N}$ and

$$
(\hat{W}_3, \text{span}(\hat{W}_1^3, \hat{W}_2^3)) = (W_1^-, W_2^-).
$$

**Proof.** Since $\hat{N}$ is an open dense subset of $F$ by Lemma 2.2, we may choose $g \in G$ such that

$$
g \cdot F(\{\hat{W}_i\}_{0 \leq i \leq 4}) = F(\{g \cdot \hat{W}_i\}_{0 \leq i \leq 4})
$$

lies in $\hat{N}$. Then we may let $\tilde{W}_i := (ng) \cdot \hat{W}_i$ for $0 \leq i \leq 4$, where $n \in N$ satisfies $n \cdot (g \cdot (\hat{W}_3, \text{span}(\hat{W}_1^3, \hat{W}_2^3))) = (W_1^-, W_2^-)$. \[\square\]

**Proof of Proposition 4.2 in the $n = 3$ case.** By Lemma 4.5(2) and the connectedness of $U$, it suffices to show that $f$ locally agrees with some element of $\text{PGL}(3, \mathbb{R})$ near $x$. Therefore after shrinking $U$ we may assume that $f$ is fibration-preserving, not just locally fibration-preserving.

After pre/postcomposing with elements of $\text{PGL}(3, \mathbb{R})$, we assume without loss of generality that $x = (W_1^-, W_2^-) \in U$, $f(x) = x$, and $D_pf(x)$ is defined and an isomorphism.

Let $\{\hat{W}_i\}_{0 \leq i \leq 4} \subset \hat{N}$ be the augmented projective frame from Lemma 4.11. For $r > 0$ let $\{W_i\}_{0 \leq i \leq 4}$ be the image of $\{\hat{W}_i\}_{0 \leq i \leq 4}$ under $\hat{\delta}_r$; we may assume $r$ is small enough that $F(\{W_i\}_{0 \leq i \leq 4}) \subset U$.

**Claim.** If $r$ is sufficiently small, then $\{f_1(W_i)\}_{0 \leq i \leq 4}$ is a projective frame.

**Proof.** Since $f$ is Pansu differentiable at $x$ and $D_pf(x)$ is an isomorphism

\begin{equation}
\hat{\delta}_r^{-1} \circ f \circ \hat{\delta}_r \to D_pf(x)
\end{equation}

uniformly on compact sets as $r \to 0$, where $D_pf : \hat{N} \to \hat{N}$ is a graded automorphism; here we identify $N$ with $\hat{N}$. Note that $D_pf(x) : \hat{N} \to \hat{N}$ is fibration-preserving because it is a limit of fibration-preserving maps. By Lemma 2.5(3) there exist $\lambda_1, \lambda_2, \lambda_3 \neq 0$ such that $\hat{\Phi} = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \in \text{PGL}(3, \mathbb{R})$ agrees with $D_pf(x)$ on $\hat{N}$. Therefore $(\hat{\Phi})_1$ maps $\{\hat{W}_1\}_{0 \leq i \leq 4}$ to a projective frame. It follows that for $r$ small
both \( \{ \hat{\delta}_{r-1} \circ f \circ \hat{\delta}_r \}_1(\hat{W}^i_1) \} \) and \( \{ f_1(W^i_1) = (f \circ \hat{\delta}_r)_1(\hat{W}^i_1) \} \) \( 0 \leq i \leq 3 \) are projective frames.

Let \( \{ \hat{W}^i_1 \} \) be the standard augmented projective frame, and let \( \hat{W}^{ij}_2 := \text{span}(\hat{W}^i_1, \hat{W}^j_1) \) for \( 0 \leq i \neq j \leq 4 \). Since \( \{ \hat{W}^i_1 \} \) and \( \{ f_1(W^i_1) \} \) are both projective frames, there are elements \( g_1, g_2 \in \text{PGL}(3, \mathbb{R}) \) such that \( g_1(\hat{W}^i_1) = W^i_1 \) and \( g_2(f_1(W^i_1)) = \hat{W}^i_1 \) for \( 0 \leq i \leq 3 \).

We let \( \hat{U} := g_1^{-1}(U) \subset \mathcal{F} \) and \( \hat{f} := g_2 \circ f \circ g_1 : \hat{U} \to \mathcal{F} \). Then \( \mathcal{F}(\{ \hat{W}^i_1 \}) \subset \hat{U} \), the map \( \hat{f} \) is fibration-preserving, and \( (\hat{f})_1(\hat{W}^i_1) = \hat{W}^i_1 \) for all \( 0 \leq i \leq 3 \). Since two distinct lines lie in a unique plane, and two distinct planes intersect in a line, the fact that \( \hat{f} \) is fibration-preserving implies that:

1. \( \hat{f}_2(\hat{W}^{ij}_2) = \hat{W}^{ij}_2 \) for \( 0 \leq i \neq j \leq 3 \).
2. \( \hat{f}_1(\hat{W}^i_1) = (\hat{f})_1(\hat{W}^{0i}_2 \cap \hat{W}^{ij}_2) \).
3. \( \hat{f}_2(\hat{W}^{ij}_2) = \hat{W}^{ij}_2 \) for \( 0 \leq i \neq j \leq 4 \).
4. \( \hat{f}(\hat{W}^1_1, \hat{W}^{ij}_2) = (\hat{W}^1_1, \hat{W}^{ij}_2) \) for \( 0 \leq i \neq j \leq 4 \).

Since \( \hat{f}_1([e_3]) = [e_3] \), for small \( r > 0 \) we may define \( \phi : \mathbb{R}^2 \to B(0, r) \to \mathbb{R}^2 \) by \( \text{span}(\phi(x_1, x_2), 1) = \hat{f}_1(\text{span}(x_1, x_2, 1)) \). The map \( \hat{f} \) is fibration-preserving and it fixes the standard augmented projective frame, so the hypotheses of Lemma 4.6 hold for \( \phi \). Applying Lemma 4.6 to \( \phi \), we get that for some \( m \in \mathbb{R} \) we have \( \hat{f}_1([x_1, x_2, 1]) = [mx_1, mx_2, 1] \) for \( x_1, x_2 \in \mathbb{R} \) small.

Suppose \( m = 0 \). Then \( \hat{f} \), and hence also \( f \), takes values in a single fiber of \( \pi_1 \) near \( x = (W^1_1, W^2_1) \). It follows that the Pansu differential \( D_P f(x) = \lim_{r \to 0} \delta_{r-1} \circ f \circ \delta_r \) takes values in a single fiber of \( \pi_1 \); this contradicts the nondegeneracy of the Pansu differential. Hence \( m \neq 0 \).

Let \( \hat{g} := \text{diag}(m, m, 1) \in \text{PGL}(3, \mathbb{R}) \). Then \( \hat{g}_1 : \mathcal{F}_1 \to \mathcal{F}_1 \) agrees with \( \hat{f}_1 \) near \( [e_3] \). Since \( \hat{g} \) and \( \hat{f} \) are both fibration-preserving, they agree near \( x = (W^1_1, W^2_1) \).

\[ \square \]

4.2. Proof of Proposition 4.2, general case. The \( n \geq 4 \) case is similar to the \( n = 3 \) case. The replacement for Lemma 4.6 is:

**Lemma 4.13.** Suppose \( V \subset \mathbb{R}^n \) is a connected open subset, and \( \phi = (\phi_1, \ldots, \phi_n) : V \to \mathbb{R}^n \)
is a continuous map such for every \( v \in \{ e_1, \ldots, e_n, e_1 + \ldots + e_n \} \) and every line \( L \) parallel to \( v \), the image \( \phi(L \cap V) \) is contained in a line parallel to \( v \). Then \( \phi \) is of the form

\[
\phi(x_1, \ldots, x_n) = (mx_1 + b_1, \ldots, mx_n + b_n)
\]

for some \( m, b_1, \ldots, b_n \in \mathbb{R} \).

We omit the proof as it is similar to the proof of Lemma 4.6.

The \( n \geq 4 \) version of Lemma 4.5 is:

**Lemma 4.15.** Suppose \( U \subset \mathcal{F} \) is open and \( f : U \to \mathcal{F} \) is fibration preserving.

1. Suppose \( f \) agrees with \( g \in \text{PGL}(n, \mathbb{R}) \) near \( (\bar{W}_1, \ldots, \bar{W}_{n-1}) \in U \). For \( i \in \{1, \ldots, n-1\} \), let \( V_i \) be the connected component of \( \pi_i^{-1}(\bar{W}_i) \cap U \) containing \( (\bar{W}_1, \ldots, \bar{W}_{n-1}) \). Then \( f \) agrees with \( g \) near \( V_i \).

2. For every \( g \in \text{PGL}(n, \mathbb{R}) \), the set where \( f \) locally agrees with \( g \) is a connected component of \( U \).

**Proof.** We prove the lemma by induction on the dimension \( n \geq 3 \). Lemma 4.5 covers the case \( n = 3 \), so we may assume inductively that the lemma holds for dimensions strictly smaller than \( n \).

(1). We may assume without loss of generality that \( g = \text{id} \). Suppose \( f \equiv \text{id} \) on an open subset \( V \subset U \) containing \( (\bar{W}_1, \ldots, \bar{W}_{n-1}) \). Arguing by contradiction, suppose \( i \in \{1, \ldots, n-1\} \), and for some \( (\bar{W}_1, \ldots, \bar{W}_{n-1}) \in V_i \), there is a sequence \( \{(W_1^j, \ldots, W_{n-1}^j)\} \subset U \) which converges to \( (\bar{W}_1, \ldots, \bar{W}_{n-1}) \) as \( j \to \infty \), but

\[
f(W_1^j, \ldots, W_{n-1}^j) \neq (W_1^j, \ldots, W_{n-1}^j)
\]

for all \( j \). After passing to a subsequence, we may assume that for all \( j \) the connected component of \( \pi_i^{-1}(W_1^j) \cap U \) containing \( (W_1^j, \ldots, W_{n-1}^j) \) intersects \( V \). Since \( f \) is fibration preserving and \( f \equiv \text{id} \) on \( V \), it follows that \( f \) maps \( \pi_i^{-1}(W_1^j) \cap U \) into \( \pi_i^{-1}(W_i^j) \). Identifying \( \pi_i^{-1}(W_i^j) \) with the flag manifold in \( \mathbb{R}^{n-1} \), the restriction \( f \) to \( \pi_i^{-1}(W_i^j) \) induces a fibration-preserving mapping; by the induction assumption, since \( f \) fixes \( V \) it will also fix \( (W_1^j, \ldots, W_{n-1}^j) \). This is a contradiction. Hence (1) holds.

(2). Note that each element of the basis \( X_{1,2}, \ldots, X_{n-1,n} \) for \( V_1 \subset \mathfrak{n} \) is tangent to one of the subgroups \( K_j \) for \( 1 \leq j \leq n-1 \). Hence we may use Lemma 2.13(4) and argue as in Lemma 4.5(2).

□
Definition 4.16. An indexed tuple \( \{W_i\}_{0 \leq i \leq n} \subset \mathcal{F}_1 \) is a projective frame if any subset of \( n \) elements spans \( \mathbb{R}^n \). The standard projective frame \( \{W_i\}_{0 \leq i \leq n} \) is given by \( \hat{W}_i = e_i \) for \( 1 \leq i \leq n \) and \( \hat{W}_0 = e_1 + \ldots + e_n \). An indexed tuple \( \{W_i\}_{0 \leq i \leq n+1} \subset \mathcal{F}_1 \) is an augmented projective frame if \( \{W_i\}_{0 \leq i \leq n} \) is a projective frame and \( W_1^{n+1} = \text{span}(W_n, W_1^n) \cap \text{span}(W_1^1, \ldots, W_1^{n-1}) \). The standard augmented projective frame is \( \{\hat{W}_i\}_{0 \leq i \leq n+1} \) with \( \hat{W}_1^{n+1} = e_1 + \ldots + e_n \).

Given a subset \( \Sigma \subset \mathcal{F}_1 \), we obtain (possibly empty) subsets of \( \mathcal{F}_j \) and \( \mathcal{F} \):

\[
\mathcal{F}_j(\Sigma) := \{\text{span}(\Sigma') \mid \Sigma' \subset \Sigma, \ |\Sigma'| = j, \ \dim \text{span}(\Sigma') = j\}.
\]

\[
\mathcal{F}(\Sigma) := \{(W_1, \ldots, W_{n-1}) \in \mathcal{F} \mid W_j \in \mathcal{F}_j(\Sigma)\}.
\]

Lemma 4.17. There is an augmented projective frame \( \{\hat{W}_i\}_{0 \leq i \leq n+1} \subset \mathcal{F}_1 \) such that:

- \( \mathcal{F}(\{\hat{W}_i\}_{0 \leq i \leq n+1}) \) is contained in \( \hat{\mathcal{N}} \).
- \( \text{span}(W_1^n, \ldots, W_1^{n-j+1}) = \text{span}(e_n, \ldots, e_{n-j+1}) \) for all \( 1 \leq j \leq n-1 \).

Proof. This follows as in the proof of Lemma 4.11.

Proof of Proposition 4.2. \( n \geq 4 \) case. The proof parallels the \( n = 3 \) case closely, so we will be brief.

It suffices to show that \( f \) locally agrees with some element of \( \text{PGL}(n, \mathbb{R}) \) near \( x \). Also, we may assume without loss of generality that \( f \) is fibration-preserving, \( x = (W_1^1, \ldots, W_{n-1}^1) \in U, \ f(x) = x \), and that \( D_p f(x) \) is well-defined and an isomorphism.

For \( r > 0 \) let \( \hat{W}_i := \hat{\delta}_r(\hat{W}_i) \) for \( 0 \leq i \leq n+1 \); we take \( r \) small enough that \( \mathcal{F}(\{\hat{W}_i\}_{0 \leq i \leq n+1}) \subset U \).

Claim. For \( r \) small \( \{f_1(\hat{W}_i)\}_{0 \leq i \leq n} \subset \mathcal{F}_1 \) is a projective frame.

We omit the proof, as it is similar to claim in the proof of the \( n = 3 \) case.

Let \( g_1, g_2 \in \text{PGL}(n, \mathbb{R}) \) be such that \( g_1(\hat{W}_i) = W_i \), \( g_2(f_1(\hat{W}_i)) = \hat{W}_i \) for \( 0 \leq i \leq n \). We now define \( \hat{U} = g_1^{-1}(U) \) and \( \hat{f} := g_2 \circ f \circ g_1 : \hat{U} \rightarrow \mathcal{F} \).

Arguing as in the \( n = 3 \) case, one obtains that \( \hat{f} \) is fibration-preserving, \( \mathcal{F}(\{\hat{W}_i\}_{0 \leq i \leq n+1}) \subset \hat{U} \), \( f_j \) fixes \( \mathcal{F}_j(\{\hat{W}_i\}_{0 \leq i \leq n+1}) \) elementwise and \( f \) fixes \( \mathcal{F}(\{\hat{W}_i\}_{0 \leq i \leq n+1}) \) elementwise.

For \( r > 0 \) small we define \( \phi : \mathbb{R}^n \supset B(0, r) \rightarrow \mathbb{R}^n \) by

\[
\text{span}(\phi(x_1, \ldots, n), 1) = f_1(\text{span}(x_1, \ldots, x_n, 1)).
\]
Applying Lemma 4.13, for some \( m \in \mathbb{R} \) we get
\[
\phi(x_1, \ldots, x_n) = (mx_1, \ldots, mx_n).
\]
As in the \( m = 3 \) case we see that \( m \neq 0 \), and that \( f_1 \) agrees with \( g := \text{diag}(m, \ldots, m, 1) \) near \( e_n \). This implies that \( f \) agrees with \( g \) near \( x = (W_1^-, \ldots, W_{n-1}^-) \).

5. The proof of Theorem 1.1

For every \( x \in U \) choose a connected open set \( U_x \subset U \) containing \( x \) and group elements \( \mathcal{N} \) such that \( \mathcal{N} \subset \mathcal{N} \). Let \( f_x := \text{diag}(m, \ldots, m, 1) \) near \( e_n \). This implies that \( f \) agrees with \( g \) near \( x = (W_1^-, \ldots, W_{n-1}^-) \).

6. The complex and quaternionic cases

The arguments from the previous sections are also valid in the complex and quaternion cases, with some straightforward modifications. In this section we indicate what modifications are needed in these cases. The necessity for these modifications are due to the presence of non-trivial automorphisms of \( \mathbb{C} \) and \( \mathbb{H} \) and the non-commutativity of the quaternions.

We first recall some facts about quaternions. Given any quaternion \( x = x_0 + x_1i + x_2j + x_3k \in \mathbb{H} \), the conjugation of \( x \) is \( \bar{x} = x_0 - x_1i - x_2j - x_3k \). It is easy to check that \( \bar{xy} = \bar{y}\bar{x} \) for any \( x, y \in \mathbb{H} \).
Let \( \lambda, \mu \) be unit quaternions satisfying \( \lambda^2 = \mu^2 = (\lambda \mu)^2 = -1 \). Set \( \nu = \lambda \mu \). Then we have \( \mu = \nu \lambda \) and \( \lambda = \mu \nu \). Define a map \( h = h_{\lambda, \mu, \nu} : \mathbb{H} \to \mathbb{H} \) by

\[
h(a_0 + ia_1 + ja_2 + ka_3) = a_0 + \lambda a_1 + \mu a_2 + \nu a_3.
\]

Then it is easy to check that \( h \) is an automorphism of \( \mathbb{H} \): it is a real linear isomorphism and \( h(xy) = h(x)h(y) \) for any \( x, y \in \mathbb{H} \). Conversely, for any automorphism \( h : \mathbb{H} \to \mathbb{H} \), if we set \( \lambda := h(i) \), \( \mu := h(j) \), \( \nu := h(k) \), then \( \lambda^2 = \mu^2 = \nu^2 = -1 \), \( \nu = \lambda \mu \) and \( h = h_{\lambda, \mu, \nu} \). By the Skolem-Noether theorem, every automorphism of \( \mathbb{H} \) is inner.

6.1. **Changes needed for Section 2.** Let \( F = \mathbb{C}, \mathbb{H} \). Let \( GL(n, F) \) be the group of invertible elements in the ring \( M_n(F) \) of \( n \times n \) matrices with entries in \( F \). The objects \( P^+_F, P^-_F \) and \( N_F \) are defined as before with \( \mathbb{R} \) replaced by \( F \). The group \( G_F \) is defined as before in the complex case with \( \mathbb{R} \) replaced with \( \mathbb{C} \):

\[
G_{\mathbb{C}} = GL(n, \mathbb{C})/\{\lambda \text{id} | 0 \neq \lambda \in \mathbb{C}\}.
\]

In the quaternion case it is defined by

\[
G_{\mathbb{H}} = GL(n, \mathbb{H})/\{\lambda \text{id} | 0 \neq \lambda \in \mathbb{R}\}.
\]

Note that \( \{aI | a \in \mathbb{H}\setminus\{0\}\} \) is not normal in \( GL(n, \mathbb{H}) \) and so we cannot quotient out by this subgroup. Similarly, \( P_{\mathbb{C}} = P^-_{\mathbb{C}}/\{\lambda \text{id} | 0 \neq \lambda \in \mathbb{C}\} \) and \( P_{\mathbb{H}} = P^-_{\mathbb{H}}/\{\lambda \text{id} | 0 \neq \lambda \in \mathbb{R}\} \).

**The flag manifold.** We view \( F^n \) as a right \( F \) module. The **flag manifold** \( F_F \) is the set of (complete) flags in \( F^n \), i.e. the collection of nested families of submodules of \( F^n \)

\[
W_1 \subset \ldots \subset W_{n-1}
\]

where \( W_j \) has dimension (rank) \( j \). Matrix multiplication yields an action \( GL(n, F) \curvearrowright F^n \) by \( F \)-module automorphisms in the usual way, which induces actions \( GL(n, F) \curvearrowright F_{k,F} \), where \( F_{k,F} \) is the Grassmanian of submodules of dimension (rank) \( k \).

Lemma 2.2 holds without changes.
Automorphisms. The map \( A \mapsto (A^*)^{-1} \) is a Lie group automorphism of \( \text{GL}(n, F) \), where \( A^* \) denotes the conjugate transpose of \( A \). So the map \( \tau : \text{GL}(n, F) \to \text{GL}(n, F) \) given by \( \tau(A) = \Pi(A^*)^{-1}\Pi^{-1} \) is also an automorphism and induces an automorphism (still denoted by \( \tau \)) of \( \text{gl}(n, F) \) which is given by 

\[
(\tau(A))_{ij} = -\overline{A_{n-j+1,n-i+1}}.
\]

For any automorphism \( h \) of \( F \), the automorphism \( \text{GL}(n, F) \to \text{GL}(n, F) \), \( (a_{ij}) \mapsto (h(a_{ij})) \) of \( \text{GL}(n, F) \) induces an automorphism of \( G \), which we denote by \( \hat{h} \).

Theorem 6.1. ([Die51, Theorems 1 and 2]) Every automorphism of \( G_F \) is induced by an automorphism of \( \text{GL}(n, F) \). The group \( \text{Aut}(G_F) \) is generated by \( \tau \), maps of the form \( \hat{h} \) (with \( h \in \text{Aut}(F) \)) and the inner automorphisms.

Notice that the automorphisms \( h \) associated with Lie group automorphisms are continuous. We recall that there are only two continuous automorphism of \( \mathbb{C} \): the identity map and the complex conjugation. On the other hand, by the Skolem-Noether theorem, every automorphism \( h : \mathbb{H} \to \mathbb{H} \) is inner. It follows that the automorphism \( \hat{h} \) of \( G_{\mathbb{H}} \) is also inner: if \( h = I_A \) for some \( a \in \mathbb{H} \), then \( \hat{h} = I_g \) with \( g = \text{diag}(a,\ldots,a) \).

The following is the counterpart of Lemma 2.5 in the quaternion and complex cases.

Lemma 6.2. Let \( F = \mathbb{C}, \mathbb{H} \). Let \( n \geq 3 \) if \( F = \mathbb{H} \) and \( n \geq 4 \) if \( F = \mathbb{C} \).

(1) If \( \Phi \in \text{Aut}(G_F) \) and \( \Phi(N_F) = N_F \), then \( \Phi \) induces a graded automorphism of \( N_F \) if and only if \( \Phi|_{N_F} = \tau^\varepsilon \circ \hat{h} \circ I_g|_{N_F} \) for some \( \varepsilon \in \{0,1\} \), some continuous automorphism \( h \) of \( F \) and \( g = \text{diag}(\lambda_1,\ldots,\lambda_n) \in G_F \).

(2) Every graded automorphism of \( N_F \) arises as in (1).

(3) For \( 1 \leq j \leq n-1 \) let \( \mathfrak{k}_j \subset n_F \) be the Lie subalgebra generated by \( \{aX_{i,i+1}|a \in F, i \neq n-j\} \), and \( K_j \subset N_F \) be the Lie subgroup with Lie algebra \( \mathfrak{k}_j \). A graded automorphism \( N_F \to N_F \) is induced by conjugation by some \( g = \text{diag}(\lambda_1,\ldots,\lambda_n) \) if and only if it preserves the subgroups \( K_j \) for \( 1 \leq j \leq n-1 \).

We remark that Lemma 6.2 (2) implies that every graded automorphism of \( n_{n,\mathbb{C}} \) is either complex linear or complex antilinear.

Proof of Lemma 6.2 in the quaternion case. Let \( n \geq 3 \). The proof of (1) and (3) are the same (by using Theorem 6.1) as that of (1) and (3) in Lemma 2.5. Here we prove (2).
The proof is by induction on \( n \). We first consider the case \( n = 3 \). Denote \( X = X_{12}, Y = X_{23} \) and \( Z = X_{13} \). We have \([aX, bY] = abZ\) for \( a,b \in \mathbb{H} \). Note that the Lie bracket is not linear over \( \mathbb{H} \). Let \( \mathbb{H}X = \{aX \mid a \in \mathbb{H}\} \) be the subspace spanned by \( X \). It has dimension 4 over \( \mathbb{R} \). Similarly we have \( \mathbb{H}Y \) and \( \mathbb{H}Z \). The grading \( n_{3,\mathbb{H}} = V_1 \oplus V_2 \) is given by \( V_1 = \mathbb{H}X \oplus \mathbb{H}Y \) and \( V_2 = \mathbb{H}Z \).

Let \( A : n_{3,\mathbb{H}} \to n_{3,\mathbb{H}} \) be a graded automorphism. We claim that \( A \) satisfies either \( \mathcal{A}(\mathbb{H}X) = \mathbb{H}X, \mathcal{A}(\mathbb{H}Y) = \mathbb{H}Y \) or \( \mathcal{A}(\mathbb{H}X) = \mathbb{H}Y, \mathcal{A}(\mathbb{H}Y) = \mathbb{H}X \). There are \( a,b \in \mathbb{H} \) such that \( A(X) = aX + bY \). To prove the claim it suffices to show that \( a = 0 \) or \( b = 0 \). Suppose \( a, b \neq 0 \) we shall get a contradiction. There are \( c, d \in \mathbb{H} \) such that \( (ad - cb)X \) (being careful about the order in \( cb \)), which yields \( a^{-1}c = db^{-1} \). Set \( \lambda = a^{-1}c \). We have \( c = a\lambda \) and \( d = \lambda b \) and so \( A(iX) = a\lambda X + \lambda bY \).

Similarly there are \( \mu, \nu \in \mathbb{H} \) such that \( A(jX) = a\mu X + \mu bY \) and \( A(kX) = a\nu X + \nu bY \). By further considering the brackets between \( A(iX), A(jX), A(kX) \) we get \( a\lambda \mu b = a\mu \lambda b, a\nu \lambda b = a\lambda \nu b \) and \( a\mu \nu b = a\nu \mu b \). It follows that \( \lambda, \mu \) and \( \nu \) commute with each other. Recall the fact that two quaternions commute with each other if and only if their imaginary parts are real multiples of each other. Hence there are real numbers \( r_i, t_i \ (i = 1, 2, 3) \) and a purely imaginary quaternion \( h \) such that \( \lambda = r_1 + t_1 h, \mu = r_2 + t_2 h, \nu = r_3 + t_3 h \). We then have \( A(iX) = r_1(aX + bY) + t_1(ahX + hbY) \) and so \( A(iX) \) lies in the 2-dimensional real vector subspace spanned by \( aX + bY \) and \( ahX + hbY \).

Similarly \( A(jX) \) and \( A(kX) \) also lie in this subspace, contradicting the fact that \( A \) is an isomorphism. This finishes the proof of the claim.

After possibly composing \( A \) with \( \tau \) we may assume that \( \mathcal{A}(\mathbb{H}X) = \mathbb{H}X \) and \( \mathcal{A}(\mathbb{H}Y) = \mathbb{H}Y \). There are \( a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3 \in \mathbb{H} \) such that \( A(X) = a_0X, A(iX) = a_1X, A(jX) = a_2X, A(kX) = a_3X \) and \( A(Y) = b_0Y, A(iY) = b_1Y, A(jY) = b_2Y, A(kY) = b_3Y \). By applying \( A \) to \([X, Y] = [iX, Y] = [jX, kY] = [kX, iY] = [kX, jY] = [X, jY] = [iX, kY] = [iX, jY] = [jX, iY] \) we obtain

\[
\begin{align*}
    a_0b_1 &= a_1b_0 = a_2b_3 = -a_3b_2, \\
    a_0b_2 &= a_2b_0 = a_3b_1 = -a_1b_3, \\
    a_0b_3 &= a_3b_0 = a_1b_2 = -a_2b_1.
\end{align*}
\]

From these we get

\[
\begin{align*}
    a_0^{-1}a_1 &= b_1b_0^{-1} = -b_2b_3^{-1} = b_3b_2^{-1}, \\
    a_0^{-1}a_2 &= b_1b_3^{-1} = b_2b_0^{-1} = -b_3b_1^{-1}, \\
    a_0^{-1}a_3 &= -b_1b_2^{-1} = b_2b_1^{-1} = b_3b_0^{-1}.
\end{align*}
\]
Set $\lambda = a_0^{-1}a_1$, $\mu = a_0^{-1}a_2$ and $\nu = a_0^{-1}a_3$. From $\lambda = -b_2b_3^{-1}b_3b_2^{-1}$, we get $\lambda^2 = -1$. Similarly from $\mu = b_1b_3^{-1} - b_3b_1^{-1}$, we get $\mu^2 = -1$.

Finally $\nu = b_2b_0^{-1}b_3b_2^{-1}b_2b_0^{-1} = \lambda\mu$. Also notice $a_1 = a_0\lambda$, $a_2 = a_0\mu$, $a_3 = a_0\nu$ and $b_1 = \lambda b_0$, $b_2 = \mu b_0$, $b_3 = \nu b_0$. So the automorphism $A$ is given by $A(X) = a_0X$, $A(iX) = a_0\lambda X$, $A(jX) = a_0\mu X$, $A(kX) = a_0\nu X$ and $A(Y) = b_0Y$, $A(iY) = \lambda b_0Y$, $A(jY) = \mu b_0Y$, $A(kY) = \nu b_0Y$. Now it is easy to check that $A = \text{Ad}_g|_{\mathfrak{n}_3} \circ \hat{h}_{\lambda,\mu,\nu}$, where $g = \text{diag}(a_0, 1, b_0^{-1})$.

Now assume $n \geq 4$. By an argument similar to the real case (using rank and the induction hypothesis), we get (after possibly composing with $\tau$) $A(\mathbb{H}X_{i,i+1}) = \mathbb{H}X_{i,i+1}$ for each $1 \leq i \leq n - 1$. Then there are nonzero quaternions $a_1, \cdots, a_{n-1}$ such that $A(X_{i,i+1}) = a_iX_{i,i+1}$.

Set $b_n = 1$ and $b_i = (a_i \cdots a_{n-1})^{-1}$ for $1 \leq i \leq n - 1$, and let $g = \text{diag}(b_1, \cdots, b_n)$. By composing $A$ with $\text{Ad}_g|_{\mathfrak{n}_3}$ we may assume that $A(X_{i,i+1}) = X_{i,i+1}$ for each $1 \leq i \leq n - 1$. Now for each $1 \leq i \leq n - 2$, $\mathbb{H}X_{i,i+1}$ and $\mathbb{H}X_{i+1,i+2}$ generate a Lie subalgebra isomorphic to $\mathfrak{n}_3$. By the previous paragraph we see that there are $\lambda_i, \mu_i$ satisfying $\lambda_i^2 = \mu_i^2 = (\lambda_i\mu_i)^2 = -1$ such that $A((a_0 + a_1i + a_2j + a_3k)X) = (a_0 + a_1\lambda_i + a_2\mu_i + a_3\nu_i)X$ for $X = X_{i,i+1}, X_{i+1,i+2}, X_{i,i+2}$, where $\nu_i = \lambda_i\mu_i$ and $a_0, a_1, a_2, a_3 \in \mathbb{R}$. By considering the Lie subalgebra generated by $X_{i+1,i+2}$ and $X_{i+2,i+3}$ and comparing the values for $A((a_0 + a_1i + a_2j + a_3k)X_{i+1,i+2})$ we get $\lambda_i = \lambda_{i+1}$, $\mu_i = \mu_{i+1}$ and $\nu_i = \nu_{i+1}$. It follows that $A = \hat{h}_{\lambda,\mu,\nu}$ with $\lambda = \lambda_1$, $\mu = \mu_1$ and $\nu = \lambda\mu$.

Proof of Lemma 6.2 in the complex case. Let $n \geq 4$. The proofs of (1) and (3) are the same as in the real case by using Theorem 6.1. As remarked above, the automorphism $\hat{h}$ is either the identity map or the complex conjugation.

(2) The proof is by induction on $n$. We first consider the case $n = 4$. Let $A : \mathfrak{n}_{4,\mathbb{C}} \to \mathfrak{n}_{4,\mathbb{C}}$ be a graded automorphism. We observe that $\mathfrak{n}_{n,\mathbb{C}}$ is the complexification of $\mathfrak{n}_n$. An easy calculation shows $\text{rank}(\text{ad } x) = \dim(\text{ad } x(\mathfrak{n}_{4,\mathbb{C}})) \geq 4$ for any nonzero element $x$ in the first layer of $\mathfrak{n}_{4,\mathbb{C}}$. Clearly, $\text{rank}(\text{ad } x) \leq 6$. So the condition

$$\max\{\text{rank}(\text{ad } x) | x \in V_1\} < 2 \min\{\text{rank}(\text{ad } x) | 0 \neq x \in V_1\}$$

in [KMX20] Lemma 4.7 is satisfied and we conclude that every graded automorphism of $\mathfrak{n}_{4,\mathbb{C}}$ is either complex linear or complex antilinear. So after possibly composing $A$ with the complex conjugation we may assume $A$ is complex linear. The rest of the argument in the case $n = 4$ is the same as in the real case.
Now let \( n \geq 5 \) and assume the statement holds for all integers less than \( n \). Let \( A : \mathfrak{n}_{n, \mathbb{C}} \to \mathfrak{n}_{n, \mathbb{C}} \) be a graded automorphism. Arguing as in the real case using rank, we see that after possibly composing \( A \) with \( \tau \) we have \( A(CX_{i,i+1}) = CX_{i,i+1} \) for each \( i \). So \( A(\mathfrak{n}_+) = \mathfrak{n}_+ \) and \( A(\mathfrak{n}_-) = \mathfrak{n}_- \), where \( \mathfrak{n}_+ \) is the Lie sub-algebra of \( \mathfrak{n}_{n, \mathbb{C}} \) generated by \( \{X_{i,i+1}, 1 \leq i \leq n-2\} \) and \( \mathfrak{n}_- \) is the Lie sub-algebra of \( \mathfrak{n}_{n, \mathbb{C}} \) generated by \( \{X_{i,i+1}, 2 \leq i \leq n-1\} \). Since \( \mathfrak{n}_+ \) and \( \mathfrak{n}_- \) are isomorphic to \( \mathfrak{n}_{n-1, \mathbb{C}} \), the induction hypothesis implies that each of \( A|_{\mathfrak{n}_+}, A|_{\mathfrak{n}_-} \) is either complex linear or complex antilinear. Hence after possibly composing \( A \) with the complex conjugation we may assume \( A \) is complex linear. As \( A \) also satisfies \( A(CX_{i,i+1}) = CX_{i,i+1} \) for all \( i \), we see that \( A = \text{Ad}_g \) for some \( g = \text{diag}(\lambda_1, \cdots, \lambda_n) \in G_{\mathbb{C}} \). This finishes the proof of (2). □

The “Hermitian product” on \( \mathbb{H}^n \) is defined by \( < z, w > = \sum_i \bar{z}_i w_i \) for \( z = (z_i), w = (w_i) \in \mathbb{H}^n \). Then one can check by direct calculation that \( < z, w > = < w, z > \) and \( < A^* z, w > = < z, Aw > \) for \( z, w \in \mathbb{H}^n \) and \( A \in M_n(\mathbb{H}) \). For any \( \mathbb{H} \)-linear subspace (submodule) \( W \) of \( \mathbb{H}^n \), the “orthogonal complement” \( W^\perp \) is defined by \( W^\perp = \{ z \in \mathbb{H}^n | < z, w > = 0 \ \forall w \in W \} \).

Lemma 2.10 holds for \( F = \mathbb{C}, \mathbb{H} \) where in (5) the automorphism \( \Phi_0 \) is induced by the map \( A \mapsto (A^*)^{-1} \). The proof of Lemma 2.10 (5) goes through in the quaternion case since

\[
< (g^{-1})^* W_j^+, g W_{n-j}^- > = < W_j^+, g^{-1} g W_{n-j}^- > = 0.
\]

Of course, the proof is also valid in the complex case if we use the standard Hermitian product in \( \mathbb{C}^n \).

Lemmas 2.13 and 2.14 hold in the complex and quaternion cases without change.

6.2. Changes needed for Section 3. Lemma 3.1 and Corollary 3.2 hold in the complex case for \( n \geq 4 \) and in the quaternion case for \( n \geq 3 \). Note that the analog of Lemma 3.1 in the \( F = \mathbb{C} \) case fails when \( n = 3 \), as in the real \( n = 3 \) case. We indicate the changes needed in the quaternion case below. We skip the complex case since it is similar. Alternatively the complex case also follows from Corollary 8.2 of [KMX20] as by Lemma 6.2 (2) every graded automorphism of \( \mathfrak{n}_{n, \mathbb{C}} \) is either complex linear or complex antilinear.

Lemma 6.3. Let \( n \geq 3 \). Let \( U \subset N_\mathbb{H} \) be a connected open subset, and for some \( p > \nu \) let \( f : N_\mathbb{H} \supset U \to N_\mathbb{H} \) be a \( W^{1,p}_{\text{loc}} \)-mapping whose Pansu...
differential is an isomorphism almost everywhere. Then after possibly composing with \( \tau \), if necessary, for a.e. \( x \in U \) the Pansu differential \( D_p f(x) \) preserves the subspace \( \mathbb{H} X_{i+1} \subset V_1 \) for every \( 1 \leq i \leq n - 1 \).

Here we indicate the differential forms used in the calculations, the rest of the argument being the same. For \( 1 \leq s < t \leq n \), denote \( Y_{st} = iX_{st}, Z_{st} = jX_{st}, W_{st} = kX_{st} \). Then \( \{ X_{st}, Y_{st}, Z_{st}, W_{st} \mid 1 \leq s < t \leq n \} \) form a basis of left invariant vector fields on \( N \). The only nontrivial bracket relations between the basis elements are given by (for \( 1 \leq s_1 < s_2 < s_3 \leq n \)):

\[
-[X_{s_1 s_2}, X_{s_2 s_3}] = [Y_{s_1 s_2}, Y_{s_2 s_3}] = [Z_{s_1 s_2}, Z_{s_2 s_3}] = [W_{s_1 s_2}, W_{s_2 s_3}] = -X_{s_1 s_3}
\]

\[
[X_{s_1 s_2}, Y_{s_2 s_3}] = [Y_{s_1 s_2}, X_{s_2 s_3}] = [Z_{s_1 s_2}, W_{s_2 s_3}] = -[W_{s_1 s_2}, Z_{s_2 s_3}] = X_{s_1 s_3}
\]

\[
[X_{s_1 s_2}, Z_{s_2 s_3}] = [Z_{s_1 s_2}, X_{s_2 s_3}] = [W_{s_1 s_2}, Y_{s_2 s_3}] = -[Y_{s_1 s_2}, W_{s_2 s_3}] = Z_{s_1 s_3}
\]

\[
[X_{s_1 s_2}, W_{s_2 s_3}] = [W_{s_1 s_2}, X_{s_2 s_3}] = [Y_{s_1 s_2}, Z_{s_2 s_3}] = -[Z_{s_1 s_2}, Y_{s_2 s_3}] = W_{s_1 s_3}
\]

Let \( \alpha_{st}, \beta_{st}, \gamma_{st}, \eta_{st} \) be the dual basis of left invariant 1-forms. We have

\[
d\alpha_{s_1 s_3} = \sum_{s_1 < s_2 < s_3} (-\alpha_{s_1 s_2} \wedge \alpha_{s_2 s_3} + \beta_{s_1 s_2} \wedge \beta_{s_2 s_3} + \gamma_{s_1 s_2} \wedge \gamma_{s_2 s_3} + \eta_{s_1 s_2} \wedge \eta_{s_2 s_3})
\]

\[
d\beta_{s_1 s_3} = \sum_{s_1 < s_2 < s_3} (-\alpha_{s_1 s_2} \wedge \beta_{s_2 s_3} - \beta_{s_1 s_2} \wedge \alpha_{s_2 s_3} - \gamma_{s_1 s_2} \wedge \eta_{s_2 s_3} + \eta_{s_1 s_2} \wedge \gamma_{s_2 s_3})
\]

\[
d\gamma_{s_1 s_3} = \sum_{s_1 < s_2 < s_3} (-\alpha_{s_1 s_2} \wedge \gamma_{s_2 s_3} - \gamma_{s_1 s_2} \wedge \alpha_{s_2 s_3} - \eta_{s_1 s_2} \wedge \beta_{s_2 s_3} + \beta_{s_1 s_2} \wedge \eta_{s_2 s_3})
\]

\[
d\eta_{s_1 s_3} = \sum_{s_1 < s_2 < s_3} (-\alpha_{s_1 s_2} \wedge \eta_{s_2 s_3} - \eta_{s_1 s_2} \wedge \alpha_{s_2 s_3} - \beta_{s_1 s_2} \wedge \gamma_{s_2 s_3} + \gamma_{s_1 s_2} \wedge \beta_{s_2 s_3}.
\]

We pull back the following closed left invariant forms

\[
\omega_+ = \bigwedge_{2 \leq s \leq n} (\alpha_{1s} \wedge \beta_{1s} \wedge \gamma_{1s} \wedge \eta_{1s})
\]

\[
\omega_- = \bigwedge_{1 \leq s \leq n-1} (\alpha_{sn} \wedge \beta_{sn} \wedge \gamma_{sn} \wedge \eta_{sn}).
\]

By Lemma 6.2 on graded automorphisms, the pull-back has the form

\[
f_p^* \omega_+ = u_+ \omega_+ + u_- \omega_-
\]

as before. Let

\[
\eta_- = \bigwedge_{2 \leq s \leq t \leq n} (\alpha_{st} \wedge \beta_{st} \wedge \gamma_{st} \wedge \eta_{st})
\]

\[
\eta_+ = \bigwedge_{1 \leq s \leq t \leq n-1} (\alpha_{st} \wedge \beta_{st} \wedge \gamma_{st} \wedge \eta_{st}).
\]

We apply the pull-back theorem to \( f_p^* \omega_+ \) and \( \eta = i_X \eta_- + i_X \eta_+ \), where \( X \in \{ X_{s(s+1)}, Y_{s(s+1)}, Z_{s(s+1)}, W_{s(s+1)} \} \) and \( i_X \) denotes the interior product with respect to \( X \). As before, this yields \( Xu_+ = 0 \) for \( X \in \)
\{X_{s(s+1)}, Y_{s(s+1)}, Z_{s(s+1)}, W_{s(s+1)}\}$ with $2 \leq s \leq n - 1$ and $Xu_\_ = 0$ for $X \in \{X_{s(s+1)}, Y_{s(s+1)}, Z_{s(s+1)}, W_{s(s+1)}\}$ with $1 \leq s \leq n - 2$. The rest of the argument is the same as in the real case.

6.3. Changes needed for Section 4. In the quaternion case, we note that the action of $PGL(n, \mathbb{H})$ on the projective frames is still transitive, but is no longer free. The reason is that $aI_n$ defines a nontrivial element in $PGL(n, \mathbb{H})$ for $a \in \mathbb{H} \setminus \mathbb{R}$, but fixes the standard projective frame.

Below is a version of Lemma 4.6 for the quaternion case. A similar statement holds for the complex case. A line in $H$ is a continuous map such that for every $q \in \{i, j, k\}$, lines parallel to $e_1 + qe_2$ are mapped into lines (not necessarily parallel to $e_1 + qe_2$). Then $\phi$ is of the form

$$\phi(x, y) = (ah(x) + b_1, ah(y) + b_2)$$

where $a, b_1, b_2 \in \mathbb{H}$ and $h : \mathbb{H} \to \mathbb{H}$ is either an automorphism of $\mathbb{H}$ or the zero map.

**Proof.** The argument of Lemma 4.6 yields that $\phi$ is of the form $\phi(x, y) = (m(x) + b_1, m(y) + b_2)$, where $b_1, b_2 \in \mathbb{H}$ and $m : \mathbb{H} \to \mathbb{H}$ is a real linear map. We shall show that either $m$ is the zero map or there is some automorphism $h$ of $\mathbb{H}$ and some $a \in \mathbb{H}$ such that $m(x) = ah(x)$. We may assume $b_1 = b_2 = 0$ after possibly composing $\phi$ with a translation. Then the assumption implies that the line $(1, i)\mathbb{H}$ is mapped by $\phi$ into a line $(a_1, a_2)\mathbb{H}$ (at least one of $a_1, a_2$ is nonzero) through the origin. There are $t_1, t_2 \in \mathbb{H}$ such that $(m(1), m(i)) = \phi(1, i) = (a_1t_1, a_2t_1)$ and $(m(i), m(-1)) = \phi(i, -1) = (a_1t_2, a_2t_2)$. By comparing the components we get $m(i) = a_2t_1 = a_1t_2$, $m(1) = a_1t_1 = -a_2t_2$. Suppose $m(1) = 0$. As at least one of $a_1, a_2$ is nonzero we have $t_1 = 0$ or $t_2 = 0$, which implies $m(i) = 0$. Similarly $m(j) = m(k) = 0$. In this case $m$ is the zero map.

Now we assume $m(1) \neq 0$. Then there is some $\lambda \in \mathbb{H}$ such that $\phi(i, -1) = \phi(1, i)\lambda$. By comparing the two components of both sides we get $m(i) = m(1)\lambda$ and $-m(1) = m(-1) = m(i)\lambda$, which yields $\lambda^2 = -1$. Similarly by considering the lines $(1, j)\mathbb{H}$ and $(1, k)\mathbb{H}$ we see that
there are \( \mu \) and \( \nu \) satisfying \( \mu^2 = \nu^2 = -1 \) such that \( m(j) = m(1)\mu, \quad -m(1) = m(-1) = m(j)\mu, \quad m(k) = m(1)\nu \) and \( -m(1) = m(-1) = m(k)\nu \).

Since \( (j, k) = (1, i) j \in (1, i) \mathbb{H} \), there is some \( c \in \mathbb{H} \) such that \( \phi(j, k) = \phi(1, i)c \). This gives us \( m(j) = m(1)c \) and \( m(k) = m(i)c \). As we also have \( m(\phi) = m(1)\lambda, \quad m(j) = m(1)\mu \) and \( m(k) = m(1)\nu \), we conclude \( c = \mu \) and \( \nu = \lambda \mu \). The three numbers \( \lambda, \mu, \nu \) satisfy \( \lambda^2 = \mu^2 = \nu^2 = -1 \) and \( \nu = \lambda \mu \). As \( m \) is \( \mathbb{R} \)-linear, for any \( x = x_0 + ix_1 + jx_2 + kx_3 \) \( (x_i \in \mathbb{R}) \) we get \( m(x) = m(1)h_{\lambda, \mu, \nu}(x) \).

\[ \square \]

The arguments in Section 4 show that we may assume the map \( \phi \) sends lines parallel to \( v \in \{ e_1, e_2, e_1 + e_2 \} \) into lines parallel to \( v \). We claim that for any \( q \in \{ i, j, k \} \), lines parallel to \( e_1 + ge_2 \) are mapped by \( \phi \) into a family of parallel lines (not necessarily parallel to \( e_1 + ge_2 \)). To see this, we notice that for a suitable diagonal matrix \( g = \text{diag}(a_1, a_2, 1) \in GL(3, \mathbb{H}) \), \( g \circ \hat{f}_1 \) satisfy \( g \circ \hat{f}_1(\text{span}(e_1 + ge_2 + e_3)) = \text{span}(e_1 + ge_2 + e_3) \) and \( g \circ \hat{f}_1(W^j_1) = W^j_1 \) for \( i = 1, 2, 3 \). Since \( \mathbb{H}^3 \) is a right \( \mathbb{H} \) module, here for any \( (x_1, x_2, x_3) \in \mathbb{H}^3 \), \( \text{span}(x_1, x_2, x_3) = \{ (x_1, x_2, x_3)|x| \in \mathbb{H} \} \). Then it follows that \( g \circ \phi \) sends lines parallel to \( e_1 + ge_2 \) to lines parallel to \( e_1 + ge_2 \), where \( \hat{g} : \mathbb{H}^2 \to \mathbb{H}^2 \) is the linear map given by the diagonal matrix \( \text{diag}(a_1, a_2) \). Consequently \( \phi \) sends lines parallel to \( e_1 + ge_2 \) into a family of parallel lines.

**Proof of Counterpart of Lemma 4.2 in the case \( n = 3 \), \( F = \mathbb{H} \).** Only the last paragraph and the third last paragraph of the proof of Lemma 4.2 need some changes. As before we have \( f_1([e_3]) = [e_3] \). Hence for small \( r > 0 \) we may define \( \phi : \mathbb{H}^2 \supset B(0, r) \to \mathbb{H}^2 \) by \( \text{span}(\phi(x_1, x_2), 1) = \hat{f}_1(\text{span}(x_1, x_2, 1)) \). The fact \( \hat{f}_1([e_3]) = [e_3] \) implies \( \phi(0, 0) = (0, 0) \).

By Lemma 6.4 the map \( \phi : \mathbb{H}^2 \supset B(0, r) \to \mathbb{H}^2 \) in this case has the form \( \phi(x_1, x_2) = (ah(x_1) + b_1, ah(x_2) + b_2) \) where \( a, b_1, b_2 \in \mathbb{H} \) and \( h : \mathbb{H} \to \mathbb{H} \) is either an automorphism of \( \mathbb{H} \) or the zero map. As \( \phi(0, 0) = (0, 0) \), we have \( b_1 = b_2 = 0 \) and so \( \phi(x_1, x_2) = (ah(x_1), ah(x_2)) \). The argument for \( m \neq 0 \) applies here and shows that \( h \) is an automorphism. Since any automorphism of \( \mathbb{H} \) is inner, \( h(x) = bxb^{-1} \) for some \( 0 \neq b \in \mathbb{H} \) and so \( \phi(x_1, x_2) = (abx_1b^{-1}, abx_2b^{-1}) \). Let \( \hat{g} := \text{diag}(ab, ab, b) \in \text{PGL}(3, \mathbb{H}) \). Then \( \hat{g}_1 : \mathcal{F}_1 \to \mathcal{F}_1 \) agrees with \( \hat{f}_1 \) near \( [e_3] \). Since \( \hat{g} \) and \( \hat{f} \) are both fibration-preserving, they agree near \( x = (W^1_1, W^2_1) \). \[ \square \]
6.4. Changes needed for Section 5. The counterpart of Theorem 1.1 for the complex and quaternion cases is:

**Theorem 6.6.** Let $U \subset F$ be a connected open subset, and $f: U \to F$ be a $W^{1,p}_{\text{loc}}$-mapping for $p > \nu$, such that the Pansu differential is an isomorphism almost everywhere.

1. If $F = \mathbb{C}$ and $n \geq 4$, then $f$ is the restriction of a diffeomorphism $\mathcal{F} \to \mathcal{F}$ of the form $\psi^{\epsilon_1} \circ C^{\epsilon_2} \circ g$ where $g \in \text{PGL}(n, \mathbb{C})$, $\epsilon_i \in \{0, 1\}$ and $C$ is complex conjugation.

2. If $F = \mathbb{H}$ and $n \geq 3$, then $f$ is the restriction of a diffeomorphism $\mathcal{F} \to \mathcal{F}$ of the form $\psi^{\epsilon} \circ g$ where $g \in \text{PGL}(n, \mathbb{H})$, $\epsilon \in \{0, 1\}$.

The proof of Theorem 6.6 is the same as that of Theorem 1.1 except in the complex case where we may need to compose $f$ with the complex conjugation if necessary.

7. Global quasiconformal homeomorphisms

In this section we identify all global nondegenerate Sobolev maps $N \to N$. These are exactly the graded affine maps of $N$. This result is an immediate consequence of Theorem 1.1.

An affine map of a Lie group $G$ is a map of the form $L_g \circ \phi$, where $\phi$ is an automorphism of $G$ and $L_g$ is left translation by $g \in G$. A graded affine map of a Carnot group $N$ is an affine map where the automorphism is a graded automorphism of $N$.

The following result applies to global quasiconformal homeomorphisms since quasiconformal maps are nondegenerate Sobolev maps.

**Theorem 7.1.** Let $N$ be the Iwasawa group of $\text{GL}(n, F)$, with $n \geq 4$ for $F = \mathbb{R}, \mathbb{C}$ and $n \geq 3$ for $F = \mathbb{H}$. Suppose $f: N \to \hat{N}$ is a $W^{1,p}_{\text{loc}}$-mapping for $p > \nu$, such that the Pansu differential is an isomorphism almost everywhere. Then $f$ is a graded affine map of $N$.

**Proof.** Let $f$ be as above. We shall show that there is a graded affine map $f_0$ such that $f^{-1}_0 \circ f$ is the identity map. After replacing $f$ with $L_{f(0)}^{-1} \circ f$ we may assume $f(0) = 0$. We identify $N$ with $\hat{N}$ and view $f$ as a map $\mathcal{F} \supset \hat{N} \to \hat{N} \subset \mathcal{F}$.

We first consider the case $F = \mathbb{R}$. By Theorem 1.1, $f$ is the restriction to $\hat{N}$ of a map of the form $\psi^{\epsilon} \circ g$ where $g \in \text{GL}(n, \mathbb{R})$, $\epsilon \in \{0, 1\}$. Recall that the automorphism $\tau = I_\Pi \circ \Phi_0$ of $\text{GL}(n, \mathbb{R})$ induces a graded automorphism (again denoted by $\tau$) of $N = \hat{N}$ and acts on $\mathcal{F}$ as $\Pi \circ \psi$. 


By replacing $f$ with $\tau \circ f$ if necessary (when $\epsilon = 1$), we may assume $\epsilon = 0$ and so $f = g|_{\hat{N}}$. This implies $g \in P^-$ as the stabilizer of 0 is $P^-$. So $g = (g_{ij})$ is a lower triangular matrix. Now we can further assume that the entries on the diagonal of $g$ are 1, after replacing $g$ with $D^{-1}g$, where $D = \text{diag}(g_{11}, \cdots, g_{nn})$. So now $g$ is a lower triangular matrix with 1s on the diagonal and such that $g(\hat{N}) = \hat{N}$. We next show that $g = I_n$.

Suppose $g \neq I_n$. We shall find a flag $F \in \hat{N}$ such that $g(F) \notin \hat{N}$, contradicting $g(\hat{N}) = \hat{N}$. Let $1 \leq k \leq n - 1$ be the integer such that $g_{ij} = 0$ for all $i > j > k$ and $g_{jk} \neq 0$ for some $j > k$. Let $j_0 > k$ be such that $g_{j_0k} \neq 0$ and $g_{jk} = 0$ for all $j > j_0$. Denote $v_{j_0} = e_k - \sum_{j=k+1}^{j_0} g_{jk}e_j$. Let $F = \{W_j\}$ be the flag defined by $W_j = W_j^-$ for $1 \leq j \leq n-j_0$ and $n-k < j \leq n$, $W_{n-j_0+1} = \text{span}\{e_n, \cdots, e_{j_0+1}, v_{j_0}\}$ and $W_j = \text{span}\{e_n, \cdots, e_{j_0+1}, v_{j_0}, e_{j_0-1}, \cdots, e_{n-j+1}\}$ for $n-j_0+2 \leq j \leq n-k$.

It is straightforward to check that $F \in \hat{N}$. However, $g(v_{j_0}) = e_k$ and $g(W_{n-j_0+1}) \cap W_{j_0-1}^+$ contains the nontrivial element $e_k$ and so $g(F) \notin \hat{N}$.

The proof in the complex and quaternion cases are the same except we use Theorem 6.6 and in the complex case we may need to compose $f$ with the complex conjugation.

\[\square\]

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