ON THE JUMPING PHENOMENON OF $\dim_{\mathbb{C}} H^q(X_t, E_t)$

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Abstract. Let $X$ be a compact complex manifold and $E$ be a holomorphic vector bundle on $X$. Given a deformation $(X, E)$ of the pair $(X, E)$ over a small polydisk $B$ centered at the origin, we study the jumping phenomenon of the cohomology groups $\dim_{\mathbb{C}} H^q(X_t, E_t)$ near $t = 0$. We show that there are precisely two cohomological obstructions to the stability of $\dim_{\mathbb{C}} H^q(X_t, E_t)$, which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye [7, 8]. Finally, we give an application concerning the jumping phenomenon of the dimension of the cohomology group $H^1(X_t, \text{End}(T_{X_t}))$.

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1. Introduction

Let $X$ be a compact complex manifold and $\pi : \mathcal{X} \rightarrow B$ be a small deformation of $X = \pi^{-1}(0)$ over a small polydisk $B$ centered at the origin in some complex vector space. Suppose that $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$ which is flat over $B$. Then the sheaf $\mathcal{F}$ can be viewed as a deformation of the sheaf $\mathcal{F}|_X$ on $X$. It is known by Grauert’s direct image theorem that the dimension $\dim_{\mathbb{C}} H^q(X_t, \mathcal{F}_t)$ is an upper semi-continuous function in $t$. Moreover, we have the following characterization for when the dimension $\dim_{\mathbb{C}} H^q(X_t, \mathcal{F}_t)$ is locally constant, also due to Grauert.

Theorem 1.1 (Grauert [2]). Let $\pi : \mathcal{X} \rightarrow B$ be a flat proper holomorphic map between complex analytic space $\mathcal{X}, B$ with $B$ being reduced and connected. Suppose that $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$ that is flat over $B$. Let $k(t) := \mathcal{O}_{B,t}/m_t$ be the residue field at $t \in B$ and $\mathcal{F}_t$ be the pullback of $\mathcal{F}$ to $X_t$. Then the following are equivalent:

(a) The function $t \mapsto \dim_{\mathbb{C}} H^q(X_t, \mathcal{F}_t)$
is locally constant in $t \in B$.
(b) The sheaf $R^q\pi_*\mathcal{F}$ is locally free and the natural map

$$R^q\pi_*\mathcal{F} \otimes k(t) \to H^q(X_t, \mathcal{F}_t)$$

is an isomorphism.

Nevertheless, condition (b) in the above theorem is not easy to check in general even when $\mathcal{F}$ is locally free. In [7, 8], X. Ye studied the jumping phenomenon of the dimensions $\dim C H^q(X, \bullet)$ under small deformations of $X$, where $\bullet = \Omega_X^p, T_X$. He found two obstructions $o_{n,n-1}^q$, $o_{n,n-1}^{q-1}$ and proved that the dimension of $H^q(X, \bullet)$ does not jump if and only if $o_{n,n-1}^q = 0$, $o_{m,m-1}^{q-1} = 0$ for all $n, m \geq 1$.

In this note, we generalize Ye’s results to a much more general setting, namely, when $X$ is a compact complex manifold and $E$ is an arbitrary holomorphic vector bundle on $X$. Let $(\mathcal{X}, \mathcal{E})$ be a small deformation of $(X, E)$ over a polydisk $B$ centered at the origin. We assume that $\mathcal{E}$ is flat over $B$ via the proper holomorphic submersion $\pi : \mathcal{X} \to B$. Let $\mathcal{E}_t := \mathcal{E}|_{X_t}$. We are interested in characterizing when the dimension $\dim C H^q(\mathcal{X}_t, \mathcal{E}_t)$ stays constant near $t = 0$.

Following [7, 8], we formulate the jumping phenomenon of $\dim C H^q(\mathcal{X}_t, \mathcal{E}_t)$ as an extension problem, namely, whether we can extend a nonzero element in $H^q(X, E)$ to one in a nearby fiber $H^q(\mathcal{X}_t, \mathcal{E}_t)$. In general, such extensions may not exist, and there are obstructions to this extension problem. In [7, 8], Ye gave explicit formulae for these obstructions in the cases when $\mathcal{E} = \Omega_{X/B}^p$ and $\mathcal{E} = T_{X/B}$. We will see that his formulae can be generalized to the above general setting.

While Ye [7, 8] employed an algebraic approach to study the extension problem by applying a version of Grauert’s direct image theorem which states that $R^q\pi_*\mathcal{F}$ is a quotient of two locally free sheaves of finite ranks over $B$, here we adopt a differential-geometric approach, following [11].

We will formulate the problem directly as extending $E$-valued differential forms over $B$, which means that, in contrary to [7, 8], we are going to work with sheaves of infinite rank. A key step is to get an explicit description of $R^q\pi_*\mathcal{E}$ by using an acyclic resolution $(\mathcal{D}^*, \mathcal{D}^*)$ of the sheaf $\mathcal{E}$ constructed from the differential operators $\mathcal{D}^*$ studied in [11] (see Section 3). The operators $\mathcal{D}^*$ capture the holomorphic structures of the deformed pairs $\{X_t, \mathcal{E}_t\}_{t \in B}$ (see [11] and also Section 2). Then more or less the same strategy as in Ye’s proofs will work. An advantage of this geometric approach is that the computation of the obstructions becomes much simpler and more transparent, as compared to the Čech calculations in [7, 8]. Our main result is as follows (see Section 4 in particular, Theorem 4.11 and Equations (1) & (2) for the details):

**Theorem 1.2.** Let $\{(A(t), \varphi(t))\}_{t \in B}$ be the family of Maurer-Cartan elements associated to the small deformation $(\mathcal{X}, \mathcal{E})$ of $(X, E)$. We define the $n$-th order obstruction maps $O^i_{n,n-1} : H^i((\pi_*\mathcal{D}^*)_0 \otimes \mathcal{O}_{B,0}/m_0) \to H^{i+1}((\pi_*\mathcal{D}^*)_0 \otimes \mathcal{O}_{B,0}/m_0)$, where $i = q, q - 1$, by

$$O^i_{n,n-1} ([\alpha_{n-1}]) = \left[ t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \nabla + A^{n-j}) \alpha^j_{n-1} \right].$$

Then the function $t \mapsto \dim C H^q(\mathcal{X}_t, \mathcal{E}_t)$ is locally constant if and only if $O^q_{m,m-1} \equiv 0$ and $O^{q-1}_{n,n-1} \equiv 0$ for all $m, n \geq 1$. 

We apply this theorem to study the jumping phenomenon of the dimension \( \dim \mathbb{C} H^1(X_t, \text{End}(T_{X_t})) \). The main result of Section 5 is the following

**Theorem 1.3.** (=Theorem 5.3) Suppose \( X \) is a Calabi-Yau manifold such that the deformation of the pair \((X, T_X)\) is unobstructed, then \( \dim \mathbb{C} H^1(X_t, \text{End}(T_{X_t})) \) does not jump at \( t = 0 \) for any deformation of \( X \).

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2. Deformations of Pairs

In this section, we briefly review the deformation theory of a pair \((X, E)\), where \( X \) is a compact complex manifold and \( E \) is a holomorphic vector bundle on \( X \), following the exposition in [1] (cf. [4]), and recall several useful facts.

**Definition 2.1.** Let \( B \) be a small polydisk in some complex vector space containing the origin. A deformation of \((X, E)\) consists of a surjective proper submersion \( \pi : X \to B \) between complex manifolds \( X \) and \( B \), together with a holomorphic vector bundle \( E \) on \( X \), such that \( \pi^{-1}(0) = X \) and \( E|_{\pi^{-1}(0)} = E \).

Given such a deformation of \((X, E)\), we put \( X_t := \pi^{-1}(t) \) and \( E_t := E|_{X_t} \). Since we have assumed that \( B \) is a polydisk, it is contractible and thus we can choose a diffeomorphism \( F : X \to X \times B \) and a bundle isomorphism \( F' : E \to E \times B \) covering \( F \) such that \( F, F' \) are holomorphic with respective to \( t \). There are two complex structures on \( X \times B \). One comes from pushing forward the complex structure on \( X \) and the other comes from the product structure on \( X \times B \). We denote these complex structures by \( J \) and \( J_0 \), respectively. Let \( \varphi(t) \in \Omega^{0,1}(T_X) \) be the family of Maurer-Cartan elements which corresponds to the family \( \mathcal{X} \to B \). In [1], we considered a holomorphic family of differential operators \( \bar{D}_t^q : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E) \) defined locally by

\[
\bar{D}_t \left( \sum_j \alpha_j \otimes e_j(t) \right) := \sum_j (\bar{\partial} + \varphi(t), \bar{\partial}) \alpha_j \otimes e_j(t),
\]

where \( \{e_j(t)\} \) are the push-forward of local holomorphic frames of \( \mathcal{E}_t \) by \( F' \).

By choosing a Hermitian metric on \( E \) and the Chern connection \( \nabla \), one can express \( \bar{D}_t \) as

\[
\bar{D}_t^q = \bar{\partial}_E + \varphi(t), \bar{\partial} + A(t),
\]

for some \( A(t) \in \Omega^{0,1}(\text{End}(E)) \). Then \( (A(t), \varphi(t)) \in \Omega^{0,1}(A(E)) \) is the family of Maurer-Cartan elements which corresponds to the deformation \((\mathcal{X}, \mathcal{E})\), namely, we have

\[
\bar{\partial}_{A(E)}(A(t), \varphi(t)) + \frac{1}{2} \left( (A(t), \varphi(t)), (A(t), \varphi(t)) \right) = 0
\]

for \( t \in B \); here \( A(E) \) is the Atiyah extension of \( E \). This family of operators satisfies the integrability condition \( \bar{D}_t^q \bar{D}_t^{q-1} = 0 \) (which is equivalent to the Maurer-Cartan equation).
Another important feature of the operator $\bar{D}_t$, which will be useful later, is that its cohomology computes the Dolbeault cohomology of $(X_t, \mathcal{E}_t)$.

**Proposition 2.2** ([1], Proposition 3.13). For each fixed $t \in B$, we have

$$H^q(X_t, \mathcal{E}_t) \cong H^q((\pi_*\mathcal{D}^\bullet_t)_{\mathcal{E}} \otimes k(t)) \cong H^q(\Omega^{0, \bullet}(E), \bar{D}_t),$$

for any $q \geq 0$.

3. An acyclic resolution for $\mathcal{E}$

From now on, for the purpose of simplifying computations and formulae, we will assume that the base $B$ of the deformation is complex 1-dimensional. We abuse the notation $X$ for the complex manifold $(X \times B, \mathcal{J})$ and $\mathcal{E}$ for the vector bundle $E \times B$ with holomorphic structure induced from $\mathcal{E}$ via pushing forward by $F'$.

In this section, we will construct an acyclic resolution of the sheaf $\mathcal{E}$ in order to get an explicit description of the direct image sheaf $R^q\pi_*\mathcal{E}$.

Define an operator $\bar{\partial}_{E,B} : \Omega^{0,q}_J(\mathcal{E}) \to \Omega^{0,q+1}_J(\mathcal{E})$ by

$$\bar{\partial}_{E,B} \left( \sum_j s_j e_j(t) \right) := \sum_j \partial_B s_j \otimes e_j(t).$$

Here $\Omega^{0,\bullet}_J$ is the space of all smooth $(0, \bullet)$-form on $X \times B$ with respect to the product complex structure $J_0$. If we choose a different frame $f_k(t)$, then $e_j(t) = \sum_k g_j^k(t)f_k(t)$ for some local smooth function $g_j^k$ on $X \times B$ which is holomorphic in $t$. Hence $\bar{\partial}_{E,B}$ is well-defined. Since the kernel of $\bar{D}_t : \Omega^0(E) \to \Omega^{0,1}(E)$ is precisely holomorphic sections of $\mathcal{E}_t$, the sheaf $\mathcal{E}$ can be identified with the sheaf of $\mathcal{O}_X$-module:

$$\mathcal{U} \mapsto \{ s \in \Gamma_{\text{smooth}}(\mathcal{U}, \mathcal{E}) : \bar{D}s = \bar{\partial}_{E,B}s = 0 \}.$$ 

Now, we define another sheaf of $\mathcal{O}_X$-modules $\mathcal{D}^q$. Let $\pi_X : X \to X$ be the projection onto $X$ (not necessary holomorphic). The sheaf $\mathcal{U} \mapsto \Omega^{0,k}_J(\mathcal{U}, \mathcal{E})$ has a typical direct summand given by

$$\tilde{\mathcal{D}}^{q,p} = \pi^*\Omega^{0,p}_B \otimes \pi_X^*\Omega^{0,q} \otimes \mathcal{E}, \quad p + q = k,$$

which also carries an $\mathcal{O}_X$-module structure by multiplication of $\mathcal{J}$-holomorphic functions. The operator $\partial_{E,B}$ acts on $\tilde{\mathcal{D}}^{q,p}$. We define the sheaf of $\mathcal{O}_X$-modules $\mathcal{D}^q$ by

$$\mathcal{D}^q : \mathcal{U} \mapsto \{ s \in \Gamma_{\text{smooth}}(\mathcal{U}, \mathcal{E}) : \partial_{E,B}s = 0 \}.$$ 

The pushforward of $\mathcal{D}^q$ by $\pi : X \to B$ carries a natural $\mathcal{O}_B$-module structure by multiplication. Since $\bar{D}_t$ varies holomorphically in the variable $t$, it induces a sheaf map $\tilde{\mathcal{D}}^q : \mathcal{D}^q \to \mathcal{D}^{q+1}$ for each $q \geq 0$. Clearly, $\mathcal{D}^\bullet \subset \tilde{\mathcal{D}}^{\bullet,0}$ as $\mathcal{O}_X$-submodules. We obtain a complex $(\tilde{\mathcal{D}}^{q, \bullet}, \partial_{E,B})$ for each $q$.

**Lemma 3.1.** For each $p, q \geq 0$ the sheaf $\pi_*\tilde{\mathcal{D}}^{q, p}$ is fine and the complex $(\pi_*\tilde{\mathcal{D}}^{q, \bullet}, \pi_*\partial_{E,B})$ has no higher cohomology sheaves, that is,

$$H^p(\pi_*\tilde{\mathcal{D}}^{q, \bullet}) = 0$$

for all $p \geq 1$. 

Proof. Fineness is clear, for we can apply a partition of unity to conclude that $\tilde{D}^q,p$ has no higher direct images.

To prove that $(\pi_*,\tilde{D}^\bullet_{E,i},\pi_*,\tilde{\partial}^\bullet_{E,i})$ has no higher cohomology, we recall that $\mathcal{H}^p(\pi_*\tilde{D}^\bullet_{E,i})$ is the sheafification of

$$W \mapsto H^p(\Gamma(\pi^{-1}(W),\tilde{D}^\bullet_{E,i})).$$

It suffices to prove that $H^p(\Gamma(\pi^{-1}(W),\tilde{D}^\bullet_{E,i})) = 0$ for any polydisk $W \subset B$ and all $p \geq 1$. Let $\alpha \in \Gamma(\pi^{-1}(W),\tilde{D}^\bullet_{E,i})$ and $\{U_i\}$ be a locally finite open covering of $X \subset \pi^{-1}(W)$ by coordinates charts. Let $\alpha_i$ be the restriction of $\alpha$ on $U_i \times W$. Write

$$\alpha_i = \sum_{I,J} \alpha_{I,J}(z,\bar{z},t) d\bar{z}^I \otimes dz^J =: \sum_I \alpha_{I,i} \otimes dz^I.$$ 

Then $\bar{\partial}_{E,i} \alpha_i = 0$ simply means for each $I$,

$$0 = \bar{\partial}_{E,i} \left( \sum_j \alpha_{I,J,j} d\bar{z}^J \right) = \bar{\partial}_B \alpha_{I,i}.$$ 

Hence, for fixed $z$, we can apply the Dolbeault lemma on $W$ to conclude that

$$\alpha_{I,i} = \bar{\partial}_B \beta_{I,i},$$

for some $\beta_{I,i} \in \Omega^0_B(W)$. Since $\alpha_{I,i}$ varies smoothly in $z$ and $\bar{z}$, we see from the proof of the Dolbeault-Grothendieck lemma that $\beta_{I,i}$ can be chosen to be smooth in $z$ as well. Let $\{\psi_i\}$ be a partition of unity on $X$ subordinate to the covering $\{U_i\}$. Define

$$\beta := \sum_{I,i} \psi_I \beta_{I,i} \otimes dz^I.$$ 

Then $\beta \in \Gamma(\pi^{-1}(W),\tilde{D}^\bullet_B)$ and

$$\bar{\partial}_{E,B} \beta = \sum_{I,i} \psi_I (\bar{\partial}_B \beta_{I,i}) \otimes dz^I = \sum_{I,i} \psi_I \alpha_{I,i} \otimes dz^I = \left( \sum_i \psi_i \right) \alpha = \alpha.$$ 

We have the first equality simply because $\{\psi_i\}$ are all independent of $t$ and $\bar{t}$, and the second last equality follows from the fact that $\alpha$ is a global section on $\pi^{-1}(W)$.

\[\square\]

Lemma 3.2. For each $q \geq 0$, the sheaf $D^q$ is acyclic with respective to the left-exact functor $\pi_*$. 

Proof. Lemma 3.1 shows that $(\tilde{D}^q\bullet_{E,i},\tilde{\partial}^q\bullet_{E,i})$ is a fine resolution of $D^q$ and so $R^q\pi_*D^q \cong \mathcal{H}^q(\pi_*\tilde{D}^q\bullet_{E,i}) = 0$ for all $q \geq 1$. \[\square\]

Proposition 3.3. The complex of sheaves $(D^\bullet_{E,i},\tilde{\partial}^\bullet_{E,i})$ is an acyclic resolution of $\mathcal{E}$ with respective to the left-exact functor $\pi_*$. In particular, we have

$$R^q\pi_*\mathcal{E} \cong \mathcal{H}^q(\pi_*D^\bullet_{E,i})$$

as $\mathcal{O}_B$-modules.

Proof. By Lemma 3.2, $R^p\pi_*D^q = 0$ for all $p \geq 1$. It remains to prove that it defines a resolution of $\mathcal{E}$. We need to show that for any point $(p, t) \in X = X \times B$, the sequence of stalks

$$0 \to \mathcal{E}_{(p,t)} \to D^0_{(p,t)} \to D^1_{(p,t)} \to \cdots$$
is exact. The exactness of
\[ 0 \to \mathcal{E}_{(p,t)} \to \mathcal{D}_0 \to \mathcal{D}_{(p,t)}^1 \]
follows from the fact that \( \bar{D}_0 \) and \( \bar{\Delta}_\mathcal{E} \) share the same kernel. For the remaining exactness, we will focus on the case \( t = 0 \); same argument works for general \( t \in B \).

Recall that \( \bar{D}_t \) is locally defined by
\[ \bar{D}_t \left( \sum_j \alpha_j \otimes e_j(t) \right) := \sum_j (\bar{\partial} + \varphi(t) \cdot \partial) \alpha_j \otimes e_j(t), \]
so it suffices to prove the exactness for the case \( \mathcal{E} = \mathcal{O}_X \).

We would like to first work over \( \mathbb{C}[[t]] \) instead of \( \mathbb{C}\{t\} \). Let \( U \subset X \) be a polydisk and denote \( \Omega^{0,*}(U) \{t\} := \Omega^{0,*}(U) \otimes \mathbb{C}\{t\} \) and \( \Omega^{0,*}(U) [[t]] := \Omega^{0,*}(U) \otimes \mathbb{C}[[t]] = \Omega^{0,*}(U) \{t\} \otimes _{\mathbb{C}\{t\}} \mathbb{C}[[t]] \). The Maurer-Cartan element \( \varphi(t) \) is gauge equivalent to 0 on \( U \). Hence
\[ \bar{\partial} + \varphi(t) \cdot \partial = e^{v(t)} \bar{\partial} e^{-v(t)}, \]
for some \( v(t) \in \Omega^{0}(T_U) [[t]] \) and \( e^{v(t)} \) acts on \( \Omega^{0,q}(U) [[t]] \) by
\[ e^{v(t)} \alpha(t) = \sum_{n=0}^{\infty} \frac{(v(t) \cdot \partial)^n}{n!} \alpha(t). \]

Hence we can apply the Dolbeault-Grothendieck lemma with analytic parameter (the \( t \)-variable) to conclude that \( (\Omega^{0,*}(U) [[t]], \bar{D}^*_t) \) is an exact complex.

Now, as \( \mathbb{C}[[t]] \) is a flat-\( \mathbb{C}\{t\} \) module (because \( \mathbb{C}[[t]] \) is torsion free and \( \mathbb{C}\{t\} \) is a PID), we have
\[ H^q(\Omega^{0,*}(U) [[t]]) = H^q(\Omega^{0,*}(U) \{t\} \otimes \mathbb{C}[[t]]) \cong H^q(\Omega^{0,*}(U) \{t\} \otimes \mathbb{C}[[t]]). \]

But we have shown that \( H^q(\Omega^{0,*}(U) \{t\} \otimes \mathbb{C}[[t]]) = 0 \). Therefore, \( H^q(\Omega^{0,*}(U) \{t\} \otimes \mathbb{C}[[t]]) = 0 \). If we can show that \( H^q(\Omega^{0,*}(U) \{t\}) \) is torsion free, we see that \( H^q(\Omega^{0,*}(U) \{t\}) \) vanishes. Assuming this, we conclude that every \( \bar{D}_t \)-closed \( (0,q) \)-form valued power series on \( U \) is locally exact.

Now, for any \( \bar{D}_{(p,0)} \)-closed element \( \alpha \in \mathcal{D}^q_{(p,0)} \), we can represent it by a \( \bar{D}_t \)-closed element \( \alpha(t) \in \Omega^{0,q}(U) \{t\} \), for some polydisk \( U \subset X \). The vanishing of \( H^q(\Omega^{0,*}(U) \{t\}) \) shows that \( \alpha(t) = \bar{D}_t \beta(t) \) for some \( \beta(t) \in \Omega^{0,q-1}(U) \{t\} \). This \( \beta(t) \) defines an element \( \beta \in \mathcal{D}^{q-1}_{(p,0)} \) such that \( \bar{D}^{q-1} \beta = \alpha \). This proves the exactness of the complex \( (\mathcal{D}^*, \bar{D}^* \cdot) \).

To complete the proof of the proposition, we need to prove that \( H^q(\Omega^{0,*}(U) \{t\}) \) is a torsion free \( \mathbb{C}\{t\} \)-module for \( q > 1 \). That is, if \([\alpha(t)] \in H^q(\Omega^{0,*}(U) \{t\})\) is a nonzero element, then \( f(t) \cdot [\alpha(t)] \) is nonzero for all \( f(t) \in \mathbb{C}\{t\} - \{0\} \). Since \( f(0) \) is invertible if \( f(0) \neq 0 \), we may assume \( f(t) \in (t^N) \) for some \( N \geq 1 \). We may assume \( f(0) = 0 \) is chosen such that \( f(t) = t^N g(t) \) with \( g(0) \neq 0 \). Again, we can invert \( g(t) \), so we can further assume \( f(t) = t^N \). Then the vanishing of \( f(t) \cdot [\alpha(t)] = [f(t) \cdot \alpha(t)] \)
means
\[ t^N \alpha(t) = \bar{D}_t \beta(t) = (\bar{\partial} + \varphi(t) \cdot \partial) \beta(t), \]
for some \( \beta(t) \in \Omega^{0,q-1}(U) \{t\} \). Since both \( \alpha(t) \) and \( \beta(t) \) is holomorphic in \( t \), the equation shows that \( \beta(t) \) is in fact \( \bar{D}_t \)-closed up to order \( N - 1 \).

We first prove the following
Lemma 3.4. For any $\bar{\partial}$-closed $\beta \in \Omega^{0,q-1}(U)$, $q > 1$, there exists $\beta(t) \in \Omega^{0,q-1}(U)\{t\}$ such that

$$\beta(0) = \beta \text{ and } \bar{D}_t \beta(t) = 0.$$ 

Proof of Lemma 3.4. Since $\beta$ is $\bar{\partial}$-closed on the polydisk $U$, it must be $\bar{\partial}$-exact. Write $\beta = \bar{\partial} \alpha$ for some $\alpha \in \Omega^{0,q-2}(U)$. Define

$$\beta(t) := \beta + \varphi(t) \bar{\partial} \alpha \in \Omega^{0,q-1}(U)\{t\}.$$ 

Then $\beta(0) = \beta$. Since $\bar{D}_t^2 = 0$, we have $\bar{D}_t \beta(t) = 0$. \hfill $\square$

With this lemma in hand, we see that $\alpha(t)$ is $\bar{D}_t$-exact and this proves that $H^q(\Omega^{0,n}(U)\{t\})$ is torsion free.

Since $\beta_0$ is $\bar{\partial}$-closed, we can choose $\beta_1(t) \in \Omega^{0,q-1}(U)\{t\}$ such that

$$\beta_1(0) = \beta_0 \text{ and } \bar{D}_t \beta_1(t) = 0.$$ 

Then we have

$$t^{N-1} \alpha(t) = \bar{D}_t \left( \frac{\beta(t) - \beta_1(t)}{t} \right) = \bar{D}_t \gamma_1(t).$$

If $N = 1$, we are done. Otherwise, we see that $\gamma_1(0)$ is $\bar{\partial}$-closed. Hence we can find $\beta_2(t)$ such that

$$\beta_2(0) = \gamma_1(0) \text{ and } \bar{D}_t \beta_2(t) = 0.$$ 

Hence

$$t^{N-2} \alpha(t) = \bar{D}_t \left( \frac{\gamma_1(t) - \beta_2(t)}{t} \right).$$

Repeating this process, we will arrive at the conclusion that

$$\alpha(t) = \bar{D}_t \gamma_N(t),$$

for some $\gamma_N(t) \in \Omega^{0,q-1}(U)\{t\}$. This completes the proof of the proposition. \hfill $\square$

4. Obstructions

In this section, we will find out explicitly the obstruction maps for extending a given element of $H^q(X, E)$. In [7, 8], the author used the Grauert direct image theorem to obtain a complex of locally free $O_B$-modules of finite ranks to compute the obstruction maps, while we use the infinite-dimensional complex of $O_B$-modules $(\pi_* D^\bullet, \bar{D}^\bullet)$. We are going to see that more or less the same strategy of proofs in [7, 8] will work in our infinite-dimensional setting as well. We give most details of the proofs in order to be more self-contained.

Recall that Proposition 1.3 gives an isomorphism of $O_B$-modules:

$$R^q \pi_* \mathcal{E} \cong \mathcal{H}^q(\pi_* D^\bullet).$$

Together with Proposition 2.2 we see that it is equivalent to work with the sheaf $\mathcal{H}^q(\pi_* D^\bullet)$ and the cohomology group $H^q((\pi_* D^\bullet)_0 \otimes k(0))$. Tensoring the stalk $(\pi_* D^\bullet)_0$ with $O_{B,0}/m_0^{n+1}$ over $O_{B,0}$, we obtain a complex

$$((\pi_* D^\bullet)_0 \otimes O_{B,0}/m_0^{n+1}, \bar{D}^\bullet_n).$$

Given $\alpha \in \ker(\bar{\partial}^\bullet_E)$, and suppose that we have a local extension $\alpha_{n-1} \in \Gamma(U, \pi_* D^n)$ of $\alpha$ such that

$$j_0^{n-1}(\bar{D}^\bullet \alpha_{n-1})(t) = 0,$$
we define the obstruction map $O_{n_n-1}^q : H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n) \rightarrow H^{q+1}((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n)$ by
\begin{equation}
O_{n_n-1}^q(j_0^{-1}(\alpha_{n-1})(t)) := [t^{n-1} \cdot (j_0^q(\bar{D}^q\alpha_{n-1})(t)/t^n)]
\end{equation}

**Remark 4.1.** The $(n-1)$-st jet can be viewed as an element in $(\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n$. The map $O_{n_n-1}^q$ factors through a map
\begin{equation}
O_n^q : H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n) \rightarrow H^{q+1}((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n),
\end{equation}
given by
\begin{equation}
O_n^q(j_0^{-1}(\alpha_{n-1})(t)) := [j_0^q(\bar{D}^q\alpha_{n-1})(t)/t^n].
\end{equation}
This is well-defined because the cohomology class of $j_0^q(\bar{D}^q\alpha_{n-1})(t)/t^n$ only depends on the cohomology class of the $(n-1)$-st jet $j_0^{-1}(\alpha_{n-1})(t)$.

For later use, we define
\begin{equation}
O_n^q(i_0^{-1}(\alpha_{n-1})(t)) := [t^i \cdot (j_0^q(\bar{D}^q\alpha_{n-1})(t)/t^n)],
\end{equation}
for $i \geq 0$ and $n \geq 1$.

The following proposition characterizes when an extension exists up to order $n \geq 1$.

**Proposition 4.2.** For a fixed $n \geq 1$, the following are equivalent:

1. For any local section $\alpha_{n-1}$ around $t = 0$ such that $j_0^{-1}(\bar{D}^q\alpha_{n-1})(t) = 0$, there exists a local section $\alpha_n$ around $t = 0$ such that $j_0^q(\alpha_n - \alpha_{n-1}) = 0$ and $j_0^q(\bar{D}^q\alpha_n)(t) = 0.$
2. For any $c_{n-1} \in H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n)$, there exists $c_n \in H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^{n+1})$ such that $c_n|_{t=0} = c_{n-1}|_{t=0} \in H^q((\pi_*, D^*)_0 \otimes k(0)).$
3. For any local section $\alpha_{n-1}$ around $t = 0$ such that $j_0^{-1}(\bar{D}^q\alpha_{n-1})(t) = 0$, $O_{n_n-1}^q[j_0^{-1}(\alpha_{n-1})(t)] = 0.$

**Proof.** We shall prove that (1) $\iff$ (2) $\iff$ (1).

For (1) $\Rightarrow$ (2): Let $c_{n-1} \in H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^n)$ and $\alpha_{n-1}$ be a local section around $t = 0$ such that $j_0^{-1}(\bar{D}^q\alpha_{n-1})(t) \in \ker(\bar{D}^q_{n-1})$ represents the class $c_{n-1}$. Then $j_0^{-1}(\bar{D}^q\alpha_{n-1})(t) = 0.$ By assumption, we can extend $\alpha_{n-1}$ to a local section $\alpha_n$ around $t = 0$ such that $j_0^q(\alpha_n - \alpha_{n-1}) = 0$ and $j_0^q(\bar{D}^q\alpha_n)(t) = 0.$ Then $\bar{D}^q(j_0^q(\alpha_n)(t)) = 0 \in (\pi_*, D^{q+1})_0 \otimes O_{B,0}/m_0^{n+1}.$ Set $c_n := [j_0^q(\alpha_n)(t)] \in H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^{n+1}).$ Since $j_0^q(\alpha_n - \alpha_{n-1}) = 0$, we have $c_{n-1}|_{t=0} = j_0^q(\alpha_n)(t) = c_{n-1}|_{t=0} = 0.$

For (2) $\Rightarrow$ (1): Let $\alpha_{n-1}$ be such that $j_0^{-1}(\bar{D}^q\alpha_{n-1})(t) = 0.$ Extend $c_{n-1} := [j_0^{-1}(\alpha_{n-1})(t)]$ to a class $c_n \in H^q((\pi_*, D^*)_0 \otimes O_{B,0}/m_0^{n+1}).$ Let $\alpha_n$ be local section around $t = 0$ such that $j_0^q(\alpha_n)(t)$ represents the class $c_n.$ Then $j_0^q(\bar{D}^q\alpha_n)(t) = 0.$ Since $c_{n-1}|_{t=0} = c_{n-1}|_{t=0} = 0,$ we have
\begin{equation}
j_0^q(\alpha_n - \alpha_{n-1}) = \bar{D}^q_{n-1}\gamma,
\end{equation}
for some $\gamma \in (\pi_*, D^{q-1})_0 \otimes k(0).$ Choose any representative $\gamma'$ of $\gamma$ and define
\begin{equation}
\alpha_n' := \alpha_n - \bar{D}^q_{n-1}\gamma'.
\end{equation}
Then $j_0^q(\bar{D}^q\alpha_n')(t) = j_0^q(\bar{D}^q\alpha_n)(t) = 0$ and $j_0^q(\alpha_n' - \alpha_{n-1}) = 0.$

For (1) $\Rightarrow$ (3): Let $\gamma := \alpha_n - \alpha_n.$ Then
\begin{equation}
\bar{D}^q_n \gamma = j_0^q(\bar{D}^q(\alpha_n - \alpha_{n-1}))(t) = t^n \cdot (j_0^q(\bar{D}^q\alpha_n)(t)/t^n),
\end{equation}
for some $\gamma \in (\pi_*, D^{q-1})_0 \otimes k(0).$ Choose any representative $\gamma'$ of $\gamma$ and define
\begin{equation}
\alpha_n' := \alpha_n - \bar{D}^q_{n-1}\gamma'.
\end{equation}
Then $j_0^q(\bar{D}^q\alpha_n')(t) = j_0^q(\bar{D}^q\alpha_n)(t) = 0$ and $j_0^q(\alpha_n' - \alpha_{n-1}) = 0.$

For (1) $\Rightarrow$ (3): Let $\gamma := \alpha_n - \alpha_n.$ Then
\begin{equation}
\bar{D}^q_n \gamma = j_0^q(\bar{D}^q(\alpha_n - \alpha_{n-1}))(t) = t^n \cdot (j_0^q(\bar{D}^q\alpha_n)(t)/t^n),
\end{equation}
for some $\gamma \in (\pi_*, D^{q-1})_0 \otimes k(0).$ Choose any representative $\gamma'$ of $\gamma$ and define
\begin{equation}
\alpha_n' := \alpha_n - \bar{D}^q_{n-1}\gamma'.
\end{equation}
Then $j_0^q(\bar{D}^q\alpha_n')(t) = j_0^q(\bar{D}^q\alpha_n)(t) = 0$ and $j_0^q(\alpha_n' - \alpha_{n-1}) = 0.$

For (1) $\Rightarrow$ (3): Let $\gamma := \alpha_n - \alpha_n.$ Then
\begin{equation}
\bar{D}^q_n \gamma = j_0^q(\bar{D}^q(\alpha_n - \alpha_{n-1}))(t) = t^n \cdot (j_0^q(\bar{D}^q\alpha_n)(t)/t^n),
\end{equation}
for some $\gamma \in (\pi_*, D^{q-1})_0 \otimes k(0).$ Choose any representative $\gamma'$ of $\gamma$ and define
\begin{equation}
\alpha_n' := \alpha_n - \bar{D}^q_{n-1}\gamma'.
\end{equation}
Then $j_0^q(\bar{D}^q\alpha_n')(t) = j_0^q(\bar{D}^q\alpha_n)(t) = 0$ and $j_0^q(\alpha_n' - \alpha_{n-1}) = 0.$
Remark 4.3. The radius of convergence of each extension $j$ fixing $\alpha$ as an element $\alpha \in H$. Hence choose a basis, for instance, consisting of harmonic forms with respective to a certain $X$ be nonzero. Let $\beta$ be a local section around $t = 0$ representing the germ $\beta$ and set $\alpha_n := \alpha_{n-1} - t\beta$. Then $j_0^n(\bar{D}^q\alpha_n)(t) = j_0^n(\bar{D}^q\alpha_{n-1})(t) - t\cdot j_0^{n-1}(\bar{D}^q\beta')(t) = t^n(\bar{j}_0^n(\bar{D}^q\alpha_{n-1})(t)/t^n) - t\cdot \bar{D}_n^q\beta = 0$. Hence $\alpha_n$ defines an $n$-th order extension of $\alpha$. □

Therefore, if $O_{n,n-1}^q \equiv 0$ for all $n \geq 1$, then by (1) above we obtain a formal element $\alpha(t)$ such that $D_t\alpha(t) = 0$. In Appendix A we will show that after a gauge fixing, $\alpha(t)$ is analytic in a neighborhood around $0 \in B$.

Remark 4.3. The radius of convergence of each extension $\alpha(t)$ may be different as $\alpha = \alpha(0)$ varies. However, since $H^q(X,E)$ is finite dimensional, we can simply choose a basis, for instance, consisting of harmonic forms with respective to a certain hermitian metric. Then we obtain a minimum radius of convergence, uniform in all $[\alpha] \in H^q(X,E)$.

Next we shall demonstrate that there is another obstruction for an extension to be nonzero.

Proposition 4.4. A non-exact element $\beta \in \ker(\bar{D}_E^q)$ admits a local extension $\beta(t) \in \Gamma(U,\pi_*D^q)$ such that $\beta(t)$ is exact for $t \neq 0$ if and only if there exist $n \geq 1$ and $[\bar{j}_0^{n-1}(\alpha_{n-1})] \in H^{n-1}(\pi_*D^q) \otimes \mathcal{O}_{B,0}/m_0^n$ such that

$$O_n^{q-1}[\bar{j}_0^{n-1}(\alpha_{n-1})] = [\beta].$$

Proof. Suppose that $O_n^{q-1}[\bar{j}_0^{n-1}(\alpha_{n-1})] = [\beta]$. Then

$$\beta = \bar{j}_0^n(\bar{D}^{q-1}\alpha_{n-1})(t)/t^n + \bar{D}_E^{q-1}\gamma$$

for some $\gamma \in \Omega^{0,q-1}(E)$. Define $\beta(t)$ by

$$\beta(t) := \bar{D}^{q-1}(\alpha_{n-1}(t)/t^n) + \bar{D}_E^{q-1}\gamma(t), \quad t \neq 0,$$

where $\gamma(t)$ is any extension of $\gamma$. Clearly $\beta(t)$ can be extended through the origin by setting $\beta(0) = \beta$. Then $\beta(t)$ is a $\bar{D}^{q-1}$-exact class and equals $\beta$ at $t = 0$. Hence $\beta(t)$ serves as an extension of $\beta$ which is $\bar{D}^{q-1}$-exact for $t \neq 0$.

Conversely, if $\beta(t)$ is an extension of $\beta$ such that

$$\beta(t) = \bar{D}^{q-1}\gamma(t)$$

for $t \neq 0$. Then $\gamma(t)$ can be chosen to be meromorphic in $t$ with pole order $n \geq 1$ at $t = 0$. Let $\alpha_{n-1}(t) := t^n\gamma(t)$. Then $\alpha_{n-1}(t)$ is holomorphic in $t$ and

$$O_n^{q-1}[\bar{j}_0^{n-1}(\alpha_{n-1})] = [\bar{j}_0^n(\bar{D}^{q-1}(t^n\gamma(t)))/t^n] = [\bar{j}_0^n(t^n\beta(t))/t^n] = [\beta].$$

This completes the proof. □
Proposition 4.5. Let \([j_0^{n-1}(\alpha_{n-1})(t)] \in H^{q-1}((\pi_*D^\bullet)_0 \otimes O_{B,0}/m_0^q)\) such that 
\(O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0\). Then there exist \(n' \leq n\) and \([j_0^{n'-1}(\alpha_{n'-1})(t)] \in H^{q-1}((\pi_*D^\bullet)_0 \otimes O_{B,0}/m_0^{q'})\) such that 
\(O^{q-1}_{n',n'-1}[j_0^{n'-1}(\alpha_{n'-1})(t)] = O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0\).

Proof. If \(O^{q-1}_{n,n-1}[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0\), we can simply take \(n' = n\) and \(\alpha_{n'-1} = \alpha_{n-1}\). Otherwise, there exists \(\alpha_1'\) such that 
\(\bar{D}_n^{q-1}[\alpha_1'] = O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)]\).

Then we have 
\(O^{q-1}_{n-1,n-2}[\alpha_1'] = O^{q-1}_{n,n-2}[j_0^{n-1}(\alpha_{n-1})(t)]\).

Since \(O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0\), we finally arrive at some \(n'\) such that 
\(O^{q-1}_{n',n'-1}[j_0^{n'-1}(\alpha_{n'-1})(t)] = O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0\).

□

These two propositions together prove the following

Corollary 4.6. Every local extension of every non-exact element \(\beta \in \ker(\partial^n_E)\) is non-exact if and only if \(O^{q-1}_{n,n-1} \equiv 0\) for all \(n \geq 1\).

Proof. For a fixed non-exact \(\beta \in \ker(\partial^n_E)\), if any extension of \(\beta\) is non-exact, then \([\beta] \notin \text{Im}(O^{q-1}_{n,n-1})\) for all \(n \geq 1\). Hence \(O^{q-1}_{n,n-1} \equiv 0\).

Conversely, if there is an extension of \(\beta\) such that it is exact for \(t \neq 0\), then there exist \(n \geq 1\) and \([j_0^{n-1}(\alpha_{n-1})(t)] \in H^{q-1}((\pi_*D^\bullet)_0 \otimes O_{B,0}/m_0^q)\) such that 
\(O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)] = [\beta] \neq 0\).

But we can also choose \(n' \leq n\) and \([j_0^{n'-1}(\alpha_{n'-1})(t)] \in H^{q-1}((\pi_*D^\bullet)_0 \otimes O_{B,0}/m_0^{n'})\) such that 
\(O^{q-1}_{n',n'-1}[j_0^{n'-1}(\alpha_{n'-1})(t)] = O^{q-1}_n[j_0^{n-1}(\alpha_{n-1})(t)] \neq 0\).

This proves the corollary. □

Lemma 4.7. For each \(q \geq 0\), \(\pi_*D^q\) is a flat \(O_B\)-module.

Proof. This follows from the fact that \((\pi_*D^q)_t\) is torsion free and \(O_{B,t} \cong C\{x - t\}\) is a PID for every \(t \in B\). □

We will need the following fact in homological algebra, whose proof can be found, e.g. in [3].

Proposition 4.8. Let \(A\) be a Noetherian ring and \(C^\bullet\) be a finite cochain complex of flat \(A\)-modules whose cohomology \(H^i(C^\bullet)\) is finitely generated for all \(i\). Then there exists a cochain complex of finitely generated flat \(A\)-modules \(K^\bullet\) and a cochain map \(C^\bullet \to K^\bullet\), which is a quasi-isomorphism. Moreover, for any \(A\)-module \(M\), the natural map \(C^\bullet \otimes M \to K^\bullet \otimes M\) is a quasi-isomorphism. Furthermore, if the dimension 
\(\dim_{k(p)} H^q(K^\bullet \otimes k(p))\)

is locally constant in \(p \in \text{Spec}(A)\), then for \(i = q, q - 1\), the \(\delta\)-functors \(T^i(M) := H^i(K^\bullet \otimes M)\) commute with base change.
We apply this proposition to the case \( A = \mathcal{O}_{B,0} \), \( C^* = (\pi_* \mathcal{D}^*)_0 \) to prove the following

**Proposition 4.9.** If \( \dim_{k(t)} H^q((\pi_* \mathcal{D}^*)_t \otimes k(t)) \) is locally constant around 0 \( \in B \), then the canonical map

\[
H^q((\pi_* \mathcal{D}^*)_0 \otimes k(0)) \to H^q((\pi_* \mathcal{D}^*)_0 \otimes k(0))
\]

is an isomorphism.

**Proof.** Since \( (\pi_* \mathcal{D}^*)_0 \) is a flat \( \mathcal{O}_{B,0} \)-module, using Proposition 4.8, we obtain a complex of finitely generated flat \( \mathcal{O}_{B,0} \)-modules \( K^* \) such that

\[
H^*(\pi_* \mathcal{D}^*)_0 \otimes M \cong H^*(K^* \otimes M)
\]

for any \( \mathcal{O}_{B,0} \)-module \( M \). We claim that the dimension

\[
\dim_{k(p)} H^q(K^* \otimes k(p))
\]

is locally constant in \( p \in \text{Spec}(\mathcal{O}_{B,0}) \).

First of all, since \( \dim_{k(t)} H^q((\pi_* \mathcal{D}^*)_t \otimes k(t)) \) is locally constant, by Theorem 3.3 and Proposition 3.3, the sheaf \( H^q(\pi_* \mathcal{D}^*) \cong R^q \pi_* \mathcal{E} \) is a locally free \( \mathcal{O}_B \)-module. Hence

\[
H^q((\pi_* \mathcal{D}^*)_0 \otimes k(0)) \cong (R^q \pi_* \mathcal{E})_0 \otimes k(0) \cong H^q(X, E) \cong H^q((\pi_* \mathcal{D}^*)_0 \otimes k(0)).
\]

In particular

\[
\dim_{k(0)} H^q(K^* \otimes k(0)) = \dim_{k(0)} H^q((\pi_* \mathcal{D}^*)_0 \otimes k(0)) = \dim_{k(0)} H^q((\pi_* \mathcal{D}^*)_0 \otimes k(0)) = \dim_{k(0)} H^q(K^* \otimes k(0)).
\]

Note that \( \text{Spec}(\mathcal{O}_{B,0}) = \text{Spec}(\mathbb{C}\{x\}) = \{(0), (x)\} \). Let \( Q := (\mathcal{O}_{B,0})_{(0)} \) be the localization of \( \mathcal{O}_{B,0} \) at the ideal \( (0) \), which is the field of quotients of \( \mathcal{O}_{B,0} \). We obtain

\[
H^q(K^* \otimes k((0))) = H^q(K^* \otimes Q) \cong H^q(K^*) \otimes Q,
\]

since localization is flat. On the other hand, as \( (\mathcal{O}_{B,0})_{(x)} \cong \mathcal{O}_{B,0} \), we have

\[
H^q(K^* \otimes k((x))) \cong H^q(K^* \otimes \mathcal{O}_{B,0}/m_0) = H^q(K^* \otimes k(0)),
\]

and so

\[
\dim_{k((x))} H^q(K^* \otimes k((x))) = \dim_{k(0)} H^q(K^* \otimes k(0)) = \dim_{k(0)} H^q(K^*) \otimes k(0).
\]

As \( H^q(K^*) \cong H^q((\pi_* \mathcal{D}^*)_0) \) is a free \( \mathcal{O}_{B,0} \)-module and \( \mathcal{O}_{B,0} \) is a local integral domain, we have

\[
\dim_{Q} H^q(K^*) \otimes Q = \dim_{k(0)} H^q(K^*) \otimes k(0).
\]

In summary, we conclude that

\[
\dim_{Q} H^q(K^* \otimes Q) = \dim_{k((x))} H^q(K^* \otimes k((x))),
\]

which means that \( \dim_{k(p)} H^q(K^* \otimes k(p)) \) is constant in \( p \in \text{Spec}(\mathcal{O}_{B,0}) \). Hence \( T^q \) commutes with base change. The required isomorphism now follows from taking \( M = k(0) \) in Proposition 4.8. \( \square \)

**Remark 4.10.** By replacing 0 \( \in B \) by nearby \( t \in B \), we note that the isomorphism holds in a neighborhood of 0.

We are now ready to prove our main result.
Theorem 4.11. \( \dim_{k(t)} H^q(X_t, \mathcal{E}_t) \) is locally constant if and only if \( O_{m,m-1}^q \equiv 0 \) and \( O_{n,n-1}^{q-1} \equiv 0 \) for all \( m,n \geq 1 \).

Proof. If \( \dim_{k(t)} H^q(X_t, \mathcal{E}_t) = \dim_{k(t)} H^q((\pi_\ast \mathcal{D})^\ast_t \otimes k(t)) \) is locally constant, then Proposition 4.2 shows that the natural map

\[
H^q((\pi_\ast \mathcal{D})^\ast_0) \otimes k(0) \to H^q((\pi_\ast \mathcal{D})^\ast_0) \otimes k(0)
\]

is an isomorphism.

Now, let \( c \in H^q((\pi_\ast \mathcal{D})^\ast_0) \otimes k(0) \). We can extend it to a nonzero local holomorphic section of \( H^q((\pi_\ast \mathcal{D})^\ast) \) since \( H^q((\pi_\ast \mathcal{D})^\ast) \) is locally free. Denote this extension by \( \tilde{c} \). Consider the germ of this section \( \tilde{c}_0 \in H^q((\pi_\ast \mathcal{D})^\ast_0) \). Choose a representative \( \tilde{\alpha}_0 \in (\pi_\ast \mathcal{D})^\ast_0 \) in this cohomology class. For each \( m \geq 1 \), maps \( \tilde{\alpha}_0 \) to \( (\pi_\ast \mathcal{D})^\ast_0 \otimes \mathcal{O}_{B,0/m_0^{m+1}} \) via the quotient map \( p_m \). Then the class \( [p_m(\tilde{\alpha}_0)] \in H^q((\pi_\ast \mathcal{D})^\ast_0 \otimes \mathcal{O}_{B,0/m_0^{m+1}}) \) is an \( m \)-th order extension of \( c \). Hence \( O_{m,m-1}^q \equiv 0 \) by Proposition 4.2. Since \( m \) is arbitrary, \( O_{m,m-1}^q \equiv 0 \) for all \( m \geq 1 \).

For the obstruction map \( O_{n,n-1}^{q-1} \), if \( O_{n,n-1}^{q-1} \equiv 0 \) for some \( n \geq 1 \) and \( [\alpha_{n-1}](t)] \neq 0 \) for \( n \geq 1 \), then we can find some nonzero \( \beta \in H^q((\pi_\ast \mathcal{D})^\ast_0 \otimes k(0)) \) and a local holomorphic extension \( \tilde{\beta} \) of \( \beta \) such that it is exact only when \( t \neq 0 \). But \( H^q((\pi_\ast \mathcal{D})^\ast) \) is locally free, any extension is locally nonzero by continuity. Therefore, \( O_{n,n-1}^{q-1} \equiv 0 \) for all \( n \geq 1 \) by Proposition 4.6.

Conversely, if both obstruction maps vanish, then for each \( [\alpha] \in H^q((\pi_\ast \mathcal{D})^\ast_0) \otimes k(0) \), we obtain \( \alpha(t) \in \Gamma(U, \pi_\ast \mathcal{D}^\ast) \) such that \( \tilde{D}_t \alpha(t) = 0 \) in some neighborhood \( U \subset B \) containing 0 and \( [\alpha(0)] = [\alpha] \). Moreover, \( \alpha(t) \) is non-exact since \( O_{n,n-1}^{q-1} \equiv 0 \) for all \( n \geq 1 \). Hence for fixed \( t \in U \) we obtain an injective linear map

\[
H^q((\pi_\ast \mathcal{D})^\ast_0) \otimes k(0) \to H^q((\pi_\ast \mathcal{D})^\ast_t) \otimes k(t), \quad [\alpha] \mapsto [\alpha(t)].
\]

Therefore,

\[
\dim_{k(t)} H^q((\pi_\ast \mathcal{D})^\ast_0) \otimes k(0) \leq \dim_{k(t)} H^q((\pi_\ast \mathcal{D})^\ast_t) \otimes k(t).
\]

By upper semi-continuity, \( \dim_{k(t)} H^q((\pi_\ast \mathcal{D})^\ast_t) \otimes k(t) = \dim_{k(t)} H^q((X_t, \mathcal{E}_t)) \) is locally constant.

Recall that by choosing a Hermitian metric on \( E \) and using the associated Chern connection, we can write

\[
\tilde{D}_t = \tilde{\partial} + \varphi(t) \cdot \nabla + A(t),
\]

where \( \{(A(t), \varphi(t))\}_{t \in B} \) is the family of Maurer-Cartan elements which controls the deformations of \( (X, E) \). Hence the \( n \)-th order obstruction maps \( O_{n,n-1}^i : H^i((\pi_\ast \mathcal{D})^\ast_0) \otimes \mathcal{O}_{B,0/m_0^n} \to H^{i+1}((\pi_\ast \mathcal{D})^\ast_0) \otimes \mathcal{O}_{B,0/m_0^n} \), for \( i = q, q-1 \), defined in (1) can be rewritten as

\[
O_{n,n-1}^i ([\alpha_{n-1}]) = \left[ t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \cdot \nabla + A^{n-j}) \alpha_{n-1}^j \right].
\]

as claimed in Theorem 4.7.

Example 4.12. We first consider the case when \( E = T_X \), the holomorphic tangent bundle of \( X \). We deform the pair \( (X_t, T_X) \) to \( (X_t, T_X) \), where \( T_X \) is the holomorphic tangent bundle to \( X_t \) (note that \( T_X \) may have other deformations which
are not isomorphic to the holomorphic tangent bundle on \(X_i\). In this case, the \(\text{End}(T_X)\)-part of the Maurer-Cartan element \(\langle A(t), \varphi(t) \rangle\) is given by

\[
A(t) = -T(\varphi(t), \bullet) - \nabla_{\bullet} \varphi(t),
\]

where \(T: \Omega^{0,\bullet}(T_X) \times \Omega^{0,\bullet}(T_X) \to \Omega^{0,\bullet}(T_X)\) is the graded torsion on \(T_X\) defined by

\[
T(\varphi, \psi) := \varphi \nabla \psi - (-1)^{|\varphi|} |\psi| \varphi \nabla - [\varphi, \psi].
\]

So we have

\[
\bar{D}_t^\bullet = \bar{\partial}_{T_X}^\bullet + [\varphi(t), -].
\]

For \(\alpha_{n-1} \in \Omega^{0,q}(T_X) \otimes \mathcal{O}_{B,0}\) such that \(\bar{D}_t^\bullet \alpha_{n-1} = 0 \mod t^{n-1}\), we have

\[
t^{n-1}(j_0^*(\bar{D}_t^\bullet \alpha_{n-1})/t^n) = t^{n-1} \left( \partial_{T_X} \alpha_{n-1} + \sum_{j=0}^{n-1} [\varphi^{n-j}, \alpha_n]\right).
\]

As a class in \(H^{q+1}(\Omega^{0,\bullet}(T_X) \otimes \mathcal{O}_{B,0}/m_{B,0}^n, \bar{D}_t^\bullet)\), it is equal to

\[
\left[ t^{n-1} \sum_{j=0}^{n-1} [\varphi^{n-j}, \alpha_n]\right].
\]

Hence the obstruction is given by

\[
O^q_{n,n-1}[j_0^{n-1}(\alpha_{n-1})(t)] = \left[ t^{n-1} \sum_{j=0}^{n-1} [\varphi^{n-j}, \alpha_n]\right].
\]

**Example 4.13.** For the case \(E = T_X^\bullet\), we have

\[
\bar{D}_t^\bullet = \bar{\partial}_{T_X^\bullet}^\bullet + [\varphi(t), -]^\bullet,
\]

where \([\varphi(t), -]^\bullet : \Omega^{0,q}(T_X^\bullet) \to \Omega^{0,q+1}(T_X^\bullet)\) is given by

\[
[\varphi(t), \eta]^\bullet(v) := [\varphi(t), \eta(v)] - (-1)^q \eta([\varphi(t), v]) = \varphi(t) \cdot \partial(\eta(v)) - (-1)^q \eta([\varphi(t), v])
\]

for \(v \in \Omega^{0}(T_X)\). Since

\[
\varphi(t) \cdot \partial(\eta(v)) - (-1)^q \eta([\varphi(t), v]) = (\varphi(t) \cdot \partial \eta)(v) + v \cdot \partial(\varphi(t) \cdot \eta),
\]

the obstruction is given by

\[
O^q_{n,n-1}[j_0^{n-1}(\alpha_{n-1})(t)] = \left[ t^{n-1} \sum_{j=0}^{n-1} (\varphi^{n-j} \cdot \partial \alpha_n + \partial(\varphi^{n-j} \cdot \alpha_n))\right].
\]

For \(E = \wedge^q T_X^\bullet\), we have

\[
\bar{D}_t(\alpha_1 \wedge \cdots \wedge \alpha_p) = \sum_{j=1}^{p-1} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge \bar{D}_t \alpha_j \wedge \cdots \wedge \alpha_p
\]

\[
= \bar{\partial}(\alpha_1 \wedge \cdots \wedge \alpha_p) + \sum_{j=1}^{p} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge [\varphi(t), \alpha_j] \wedge \cdots \wedge \alpha_p,
\]

where \(\alpha_j \in \Omega^{0}(T_X^\bullet)\). Then

\[
\sum_{j=1}^{p} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge [\varphi(t), \partial \alpha_j] \wedge \cdots \wedge \alpha_p + \sum_{j=1}^{p} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge \partial(\varphi(t) \cdot \alpha_j) \wedge \cdots \wedge \alpha_p = [\varphi(t), \partial(\alpha_1 \wedge \cdots \wedge \alpha_p)] + \partial(\varphi(t) \cdot (\alpha_1 \wedge \cdots \wedge \alpha_p)).
\]
Hence the obstruction map is given by

\[
O^q_{n,n-1}[\tilde{j}^{n-1}(\alpha_{n-1})(t)] = \left[ \partial^n - \sum_{j=0}^{n-1} (\varphi^n - j \partial \alpha^j_{n-1} + \partial(\varphi^n - j \partial \alpha^j_{n-1})) \right].
\]

These two examples recover the obstruction formulae in [7].

5. Application to Jumping Phenomenon of \( \dim_C H^1(X_t, \text{End}(T_{X_t})) \)

Physicists are interested in knowing whether the dimension \( \dim_C H^1(X_t, \text{End}(T_{X_t})) \) is locally constant under small deformations \( X \). In this section, we prove the constancy of this dimension in the case when \( X \) is a Calabi-Yau manifold that satisfies some unobstructedness assumption.

In stead of proving directly that \( \dim_C H^1(X_t, \text{End}(T_{X_t})) \) does not jump at \( t = 0 \) for any deformation of \( X \), we first prove that in some nice cases, \( \dim_C H^1(X, A(E)) \) does not jump at \( t = 0 \) for any deformation of \((X,E)\). To do this, we choose a harmonic basis \( \{(A_i, \varphi_i)\}_{i=1}^m \) for \( H^1(X, A(E)) \). In [7], we proved that the obstruction map \( \text{Ob}_{(X,E)} : H^1(X, A(E)) \rightarrow H^2(X, A(E)) \) of the deformation theory of \((X,E)\) is given by

\[
\text{Ob}_{(X,E)} : \sum_{i=1}^m t_i(A_i, \varphi_i) \mapsto H[(A(t), \varphi(t)), (A(t), \varphi(t))],
\]

where \((A(t), \varphi(t))\) satisfies

\[
(A(t), \varphi(t)) = \sum_{i=1}^m t_i(A_i, \varphi_i) - \frac{1}{2} \hat{\partial}^2_{A(E)} G[(A(t), \varphi(t)), (A(t), \varphi(t))].
\]

Moreover, \((A(t), \varphi(t))\) satisfies the Maurer-Cartan equation if and only if \( \text{Ob}_{(X,E)} = 0 \).

Suppose now \( \text{Ob}_{(X,E)} = 0 \). Then we have

\[
\hat{\partial}_{A(E)}(A(t), \varphi(t)) + \frac{1}{2} [(A(t), \varphi(t)), (A(t), \varphi(t))] = 0.
\]

Differentiate \((A(t), \varphi(t))\) with respective to \( t_i \) and set \( t = 0 \), we get

\[
\frac{\partial}{\partial t_i} |_{t=0}(A(t), \varphi(t)) = (A_i, \varphi_i).
\]

Hence, for each \( i = 1, \ldots, m \), if we define \((B(t), \psi(t))_i\) to be

\[
(B(t), \psi(t))_i = \frac{\partial}{\partial t_i}(A(t), \varphi(t)),
\]

then \((B(t), \psi(t))_i\) satisfies

\[
\hat{\partial}_{A(E)}(B(t), \psi(t))_i + [(A(t), \varphi(t)), (B(t), \psi(t))_i] = 0
\]

and \( \{(B_0, \psi_0)_i\}_{i=1}^m \) forms a basis for \( H^1(X, A(E)) \).

Note that the differential operator \( \hat{D}_{A(E)} \) defined by

\[
\hat{D}_{A(E)} := \hat{\partial}_{A(E)} + [(A(t), \varphi(t)), -]
\]

satisfies \( \hat{D}_{A(E)}^2 = 0 \) and the Leibniz rule

\[
\hat{D}_{A(E)}(fs) = (\hat{\partial} + \varphi(t), \partial)f \otimes s + f \hat{D}_{A(E)}(s).
\]
It follows that $\mathcal{D}_{A(E_t)}$ defines a deformation $\{(X_t, A(E_t))\}_{t \in \text{Def}(X,E)}$ of the pair $(X, A(E))$. In fact, $A(E_t)$ is the Atiyah extension of the deformed bundle $E_t$ on $X_t$.

**Lemma 5.1.** Suppose $\text{Ob}(X,E) = 0$. Then for any $[(B, \psi)] \in H^1(X, A(E))$, there exists $(B(t), \psi(t))$ such that $\mathcal{D}_{A(E_t)}(B(t), \psi(t)) = 0$ and $[(B_0, \psi_0)] = [(B, \psi)]$. Hence all element in $H^1(X, A(E))$ admits an extension to $H^1(X_t, A(E_t))$ for any deformation of $(X, E)$. In particular $G_{n,n-1}^1 = 0$ for all $n \geq 1$.

**Proof.** Since this is true for the harmonic basis $\{(B_0, \psi_0)_i\}_{i=1}^m$, it is true for any element in $H^1(X, A(E))$. 

**Lemma 5.2.** Let $X$ be a compact complex manifold and $E \to X$ be a holomorphic vector bundle. Suppose the deformation of the pair $(X, E)$ is always unobstructed and $\dim_{\mathbb{C}} H^0(X_t, A(E_t))$ does not jump at $t = 0$ along any deformations of $(X, E)$. Then $\dim_{\mathbb{C}} H^1(X_t, A(E_t))$ does not jump at $t = 0$ along any deformation of $(X, E)$.

**Proof.** Since $\text{Ob}(X,E) = 0$, Lemma 5.1 allows us to extend any element in $H^1(X, A(E))$ to $H^1(X_t, A(E_t))$. Since

$$H^0(X_t, A(E_t)) = \ker(\mathcal{D}_{A(E_t)} : \Omega^0(A(E)) \to \Omega^{0,1}_t(A(E_t)))$$

$$= \ker(\mathcal{D}_{A(E_t)} : \Omega^0(A(E)) \to \Omega^{0,1}_0(A(E)))$$

the assumption that $\dim_{\mathbb{C}} H^0(X_t, A(E_t))$ does not jump at $t = 0$ implies $H^1(X_t, A(E_t))$ does not jump at $t = 0$ implies $O_{n,n-1}^0 \equiv 0$ for all $n \geq 1$. Now apply Theorem 4.11.

We are now going to prove that under certain assumptions, $\dim_{\mathbb{C}} H^1(X_t, \text{End}(TX_t))$ does not jump at $t = 0$ along any deformation of $X_t$.

First, when $E = TX$, we have a canonical lift $L : H^1(X, TX) \to H^1(X, A(TX))$, defined by

$$L : \varphi \mapsto (-\nabla \varphi - T(\varphi, \bullet), \varphi),$$

where $T : \Omega^{0,p}(TX) \otimes \Omega^{0,q}(TX) \to \Omega^{0,p+q}(TX)$ is the graded torsion, defined by

$$T(\varphi, \psi) = \varphi \wedge \nabla \psi - (-1)^{pq} \psi \wedge \nabla \varphi - [\varphi, \psi].$$

Moreover, if $\text{Ob}_X = 0$, then we have a Maurer-Cartan element $\varphi(t) \in \Omega^{0,1}(TX)$ and we obtain a deformation of $(X, TX)$ by

$$\mathcal{D}_t = \mathcal{D}_{TX} + \varphi(t) \wedge \nabla \varphi(t) - T(\varphi(t), \bullet) = \mathcal{D}_{TX} + [\varphi(t), \bullet].$$

In fact, the deformation induced by this operator is isomorphic to the family $\{(X_t, TX_t)\}_{t \in \text{Def}(X,X)}$, where $TX_t$ is the holomorphic tangent bundle of $X_t$. Therefore, $L$ induces a natural embedding

$$\text{Def}(X) \subset \text{Def}(X, TX).$$

By a Calabi-Yau $n$-fold we mean an $n$-dimensional compact Kähler manifold $X$ with trivial canonical line bundle $K_X$ and $H^{0,p}(X) = 0$ for all $p \neq 0, n$.

**Theorem 5.3.** Suppose $X$ is a Calabi-Yau manifold such that the deformation of the pair $(X, TX)$ is unobstructed, then $\dim_{\mathbb{C}} H^1(X_t, \text{End}(TX_t))$ does not jump at $t = 0$ for any deformation of $X$. 

Proof. Let $E = T_X$. Since the pair $(X, E)$ admits unobstructed deformation, Lemma 5.1 allows us to extend any element in $H^1(X, A(E))$ to element in $H^1(X_t, A(E_t))$, where $t \in \text{Def}(X, E)$. Consider the Atiyah exact sequence of $E_t$ (NOT the tangent bundle of $X_t$ in general) over $X_t$:

$$0 \rightarrow \text{End}(E_t) \rightarrow A(E_t) \rightarrow T_{X_t} \rightarrow 0,$$

which gives rise to the injective map $\iota_t^*: H^0(X_t, \text{End}(E_t)) \rightarrow H^0(X_t, A(E_t))$. Since $X$ has a Calabi-Yau manifold is a stable bundle, we have $H^0(X, \text{End}_0(T_X)) = 0$ and so

$$H^0(X, A(E)) \cong H^0(X, \text{End}(T_X)) \cong H^0(X, \mathcal{O}_X) = \mathbb{C}.$$ 

Since the identity map $\text{id}_{E_t}$ is always a non-zero holomorphic section of $H^0(X_t, \text{End}(E_t))$ and $\iota_t^*: H^0(X_t, \text{End}(E_t)) \rightarrow H^0(X_t, A(E_t))$ is injective, we get

$$1 \leq \dim_{\mathbb{C}} H^0(X_t, A(E_t)) \leq \dim_{\mathbb{C}} H^0(X, A(E)) = 1,$$

for $|t|$ small. By Lemma 5.2, we conclude that $\dim_{\mathbb{C}} H^1(X_t, A(E_t))$ does not jump at $t = 0$ along any deformation of $(X, T_X)$. In particular, along the deformation of $X$ itself.

For $t \in \text{Def}(X) \subset \text{Def}(X, T_X)$, we have a family of canonical lifts $L_t : H^1(X_t, T_{X_t}) \rightarrow H^1(X_t, A(E_t))$, since $A(E_t)$ is the Atiyah extension of $T_{X_t}$ for $t \in \text{Def}(X)$. So the map $\pi_t^* : H^1(X_t, A(E_t)) \rightarrow H^1(X_t, T_{X_t})$ is surjective and we obtain the following exact sequence

$$0 \rightarrow H^1(X_t, \text{End}(T_{X_t})) \rightarrow H^1(X_t, A(E_t)) \rightarrow H^1(X_t, T_{X_t}) \rightarrow 0.$$

Since $\dim_{\mathbb{C}} H^1(X_t, T_{X_t}) = \dim_{\mathbb{C}} H^{n-1,1}(X_t)$ does not jump at $t = 0$ for $t \in \text{Def}(X)$ with $|t|$ small, we see that $\dim_{\mathbb{C}} H^1(X_t, \text{End}(T_{X_t}))$ does not jump at $t = 0$ for any deformation of $X$. □

Appendix A. Convergence

Consider an element $\alpha \in \ker(\bar{\partial}^r)$. Suppose that the obstruction maps $O^n_{n, n-1}$ vanish for all $n \geq 1$. Then we obtain a formal extension $\alpha(t)$ of $\alpha$, that is, as a formal power series in $\Omega^0, 0^r(E)$,

$$\bar{\partial}^r\alpha(t) = 0.$$ 

In this appendix, we show that one can always choose an extension $\alpha(t)$ with nonzero radius of convergence. To achieve this, we shall work on the Kuranishi family of $(X, E)$ [6], following the approach of the book [5].

We choose a hermitian metric for $E$ and consider the equation

$$\alpha(t) + \bar{\partial}_E^r G_E(\varphi(t), \bar{\partial}^r + A(t))\alpha(t) = 0, \quad \alpha(0) = \alpha \in \ker(\bar{\partial}_E^r),$$

with $\alpha(t)$ holomorphic in the variable $t$. Then $\alpha(t)$ can be solved by the recursive relations:

$$\alpha^n + \sum_{i=0}^{n-1} \bar{\partial}_E^r G(\varphi_{n-i}\bar{\partial}^r + A_{n-i})\alpha^i = 0, \quad n \geq 1.$$ 

We shall prove that $\alpha(t) := \sum_{n=0}^{\infty} \alpha^n t^n$ converges uniformly in the Hölder norm $\| \cdot \|_{k+\alpha}$. First of all, let us recall the obvious estimates

$$\| [(A, \varphi), (B, \psi)] \|_{k+\alpha} \leq C_{k, \alpha} \| (A, \varphi) \|_{k+\alpha+1} \| (B, \psi) \|_{k+\alpha+1},$$

$$\| (\varphi \bar{\partial}^r + A)\delta \|_{k+\alpha} \leq C'_{k, \alpha} \| (A, \varphi) \|_{k+\alpha+1} \| \delta \|_{k+\alpha+1}$$

and

$$\| \bar{\partial}_E^r G(\varphi_{n-i}\bar{\partial}^r + A_{n-i})\alpha^i \|_{k+\alpha} \leq C_{k, \alpha} \| \alpha^i \|_{k+\alpha} \leq C_{k, \alpha} \sum_{j=0}^{n-i} \| \alpha^j \|_{k+\alpha} \leq C_{k, \alpha} \| \alpha \|_{k+\alpha}^n.$$
for any \((A, \varphi), (B, \psi) \in \Omega^\bullet(E)\) and \(\delta \in \Omega^0\bullet(E)\), where \(C_{k,\alpha}, C'_{k,\alpha}\) are positive constants which depend only on \(k, \alpha\). We may assume that \(C_{k,\alpha}\) is larger so that
\[
\|[(A, \varphi), (B, \psi)]\|_{k+\alpha} \leq C_{k,\alpha} \|A, \varphi\|_{k+\alpha} \|B, \psi\|_{k+\alpha},
\]
\[
\|[(\varphi, \nabla + A)\delta]_{k+\alpha} \leq C_{k,\alpha} \|(A, \varphi)\|_{k+\alpha} \|\delta\|_{k+\alpha}
\]
for any \((A, \varphi), (B, \psi) \in \Omega^\bullet(A(E))\) and \(\delta \in \Omega^0\bullet(E)\). Next, we have the estimates
\[
\|\partial_*^n G_E \delta\|_{k+\alpha} \leq C_{k,\alpha} \|\delta\|_{k-1+\alpha},
\]
\[
\|\partial^*_\lambda(G(A, E)(A, \varphi))\|_{k+\alpha} \leq C'_{k,\alpha} \|(A, \varphi)\|_{k-1+\alpha}
\]
for all \((A, \varphi) \in \Omega^0\bullet(A(E))\) and \(\delta \in \Omega^0\bullet(E)\), where \(G(A, E), G_E\) are Green’s operators corresponding to \(A(E), E\), respectively, and \(C_{k,\alpha}, C'_{k,\alpha}\) are positive constants depending only on \(k, \alpha\). Again we assume that \(C_{k,\alpha}\) is larger.

**Proposition A.1.** For \(|t|\) small, \(a(t) = \sum_{n=0}^{\infty} \alpha^n t^n\) converges in the norm \(\|\cdot\|_{k+\alpha}\) and \(a(t)\) is a smooth solution.

**Proof.** The proof is rather standard, and we follow the book \cite{5} very closely.

First we observe that \(\delta(t) := t \cdot a(t)\) also satisfies the equation
\[
\delta(t) + \partial^*_\lambda G_E ((\varphi(t), \nabla + A(t))\delta(t)) = 0.
\]
Denote \(\delta_n(t) = \delta(t) \mod t^{n+1}\) (similar meaning for \(A^n(t)\) and \(\varphi^n(t)\)). Let
\[
B(t) := \frac{\beta}{16\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n^2 t^n} := \sum_{n=1}^{\infty} B^n t^n,
\]
where \(\beta, \gamma\) are positive constants which are to be chosen. We want to choose \(\beta, \gamma\) such that \(\|\delta^n\|_{k+\alpha} \leq B^n\) for all \(n \geq 1\) (this condition will be denoted by \(\|\delta_n(t)\|_{k+\alpha} \ll B(t)\)). This is of course possible for \(n = 1\). Hence we assume that this is possible up to order \(n - 1\), for some \(n > 1\).

For any \((A, \varphi)\) and \(\delta\), we have
\[
\|\partial^*_\lambda G_E ((\varphi, \nabla + A)\delta)\|_{k+\alpha} \leq C_{k,\alpha} \|\varphi, \nabla + A\|_{k-1+\alpha} \leq C_{k,\alpha} C_{k,\alpha} \|(A, \varphi)\|_{k+\alpha} \|\delta\|_{k+\alpha},
\]
so the induction hypothesis gives
\[
\|\delta_n(t)\|_{k+\alpha} \leq C_{k,\alpha} C_{k,\alpha} \|(A^n(t), \varphi^n(t))\|_{k+\alpha} \|\delta_{n-1}(t)\|_{k+\alpha} \ll C_{k,\alpha} C_{k,\alpha} \|(A^n(t), \varphi^n(t))\|_{k+\alpha} B(t).
\]
It follows from Proposition 2.4, p.162 in \cite{5} that, when \(\beta, \gamma\) are chosen such that
\[
C_{k,\alpha} C_{k,\alpha} \frac{\beta}{\gamma} < 1 \quad \text{and} \quad \|(A^1(t), \varphi^1(t))\|_{k+\alpha} \ll B(t),
\]
we have \(\|(A^n(t), \varphi^n(t))\|_{k+\alpha} \ll B(t)\) for any \(n \geq 1\). Hence
\[
\|\delta_n(t)\|_{k+\alpha} \ll C_{k,\alpha} C_{k,\alpha} (B(t))^2.
\]
It can also be proved (see Lemma 3.6, p. 50 in \cite{5}) that
\[
(B(t))^2 \ll \frac{\beta}{\gamma} B(t).
\]
Therefore, for the above choices of \(\beta, \gamma\), we have
\[
\|\delta_n(t)\|_{k+\alpha} \ll B(t).
\]
Since \(B(t)\) converges on \(|t| < \gamma^{-1}\), we see that \(\delta(t)\), and hence \(a(t)\), also converges there.
Finally, \( \alpha(t) \) satisfies

\[
\left( \frac{\partial^2}{\partial t \partial t} + \Delta_E + \bar{\partial}_E (\varphi(t) \nabla + A(t)) \right) \alpha(t) = 0.
\]

Since the operator

\[
\frac{\partial^2}{\partial t \partial t} + \Delta_E + \bar{\partial}_E (\varphi(t) \nabla + A(t))
\]

is elliptic for \(|t|\) small, regularity guarantees smoothness of \( \alpha(t) \).

Next we have the following

**Proposition A.2.** The \( \alpha(t) \) defined above satisfies

\[
D_t \alpha(t) = (\bar{\partial}_E + \varphi(t) \nabla + A(t)) \alpha(t) = 0 \mod t^n
\]

if and only if \( \mathbb{H}((\varphi(t) \nabla + A(t)) \alpha(t)) = 0 \mod t^n \).

**Proof.** If \( D_t \alpha(t) = 0 \mod t^n \), then it is clear that \( \mathbb{H}((\varphi(t) \nabla + A(t)) \alpha(t)) = 0 \mod t^n \) since \( \mathbb{H} \bar{\partial}_E = 0 \).

Conversely, suppose that \( \mathbb{H}((\varphi(t) \nabla + A(t)) \alpha(t)) = 0 \mod t^n \). Let \( \psi(t) := D_t \alpha(t) \). Since \( \alpha(t) \) satisfies

\[
\alpha(t) + \bar{\partial}_E G_E (\varphi(t) \nabla + A(t)) \alpha(t) = 0,
\]

applying \( \bar{\partial}_E \) gives

\[
\bar{\partial}_E \alpha(t) = -\bar{\partial}_E \bar{\partial}_E G_E (\varphi(t) \nabla + A(t)) \alpha(t).
\]

Then

\[
\psi(t) = -\bar{\partial}_E \bar{\partial}_E G_E (\varphi(t) \nabla + A(t)) \alpha(t) + (\varphi(t) \nabla + A(t)) \alpha(t).
\]

Since \( (\varphi(t) \nabla + A(t)) \alpha(t) \) mod \( t^n \) has no harmonic part, we have

\[
\psi(t) = \bar{\partial}_E \bar{\partial}_E G_E (\varphi(t) \nabla + A(t)) \alpha(t) - (\varphi(t) \nabla + A(t)) \bar{\partial}_E \alpha(t)
\]

\[
= \bar{\partial}_E G_E \left[ -\frac{1}{2} (\varphi(t), \varphi(t)) \cdot \alpha(t) - (\varphi(t) \nabla + A(t)) \psi(t) - (\varphi(t) \nabla + A(t)) \alpha(t) \right]
\]

\[
= -\bar{\partial}_E G_E ((\varphi(t) \nabla + A(t)) \psi(t)) \mod t^n,
\]

where the Lie bracket acts by

\[
[(\varphi(t), \varphi(t)), (A(t), \varphi(t))] \cdot \alpha(t) := (2\varphi(t) \nabla + [A(t), A(t)] + [\varphi(t), \varphi(t)] \nabla) \alpha(t).
\]

Since the leading order term of \( (\varphi(t) \nabla + A(t)) \) is at least 1, the leading order of \( \bar{\partial}_E G_E ((\varphi(t) \nabla + A(t)) \psi(t)) \) is of order at least 2. Hence \( \psi(t) \) has no first order term. Inductively, we conclude \( \phi(t) = 0 \mod t^n \).

Finally, we claim that the harmonic part of \( (\varphi(t) \nabla + A(t)) \alpha(t) \) vanishes under the assumption that \( O^d_{n,n-1} \equiv 0 \) for all \( n \geq 1 \).

**Proposition A.3.** The obstructions \( O^d_{n,n-1} \equiv 0 \) for all \( n \geq 1 \) if and only if for any \( \alpha(t) \) satisfying

\[
\alpha(t) + \bar{\partial}_E G_E (\varphi(t) \nabla + A(t)) \alpha(t) = 0
\]

and \( \bar{\partial}_E \alpha(0) = 0 \), we have \( D_t \alpha(t) = 0 \).
Proof: If $\mathbb{H}((\varphi(t)\nabla + A_1(t))\alpha(t)) = 0$ for any $\alpha = \alpha(0) \in \ker(\bar{\partial}_E^n)$, then $\alpha(t)$ is an extension of $\alpha$. Hence $O_{n,n-1}^0 \equiv 0$ for all $n \geq 1$.

For the converse direction, we proceed by induction on $n$. For $n = 1$, we have

$$j_0^1(\bar{D}_t\alpha_0)(t) = \bar{D}_0(j_0^0(\beta)(t)) = j_0^0(\bar{D}_t\beta)(t)$$

for some local section $\beta = \sum_{n=0}^{\infty} \beta_n t^n$, i.e.

$$(\varphi_1\nabla + A_1)\alpha_0 = \bar{\partial}_E\beta_0.$$  

Hence $\mathbb{H}((\varphi(t)\nabla + A(t))\alpha(t)) = 0$ mod $t$. Assume $\mathbb{H}((\varphi(t)\nabla + A(t))\alpha(t)) = 0$ mod $t^{n-1}$. Then $\alpha(t)$ is an $(n-1)$-th order extension of $\alpha_0$. By assumption, we have $O_{n,n-1}[j_0^{n-1}(\alpha(t))] \equiv 0$. Therefore,

$$t^{n-1} \sum_{j=0}^{n-1} (\varphi_{n-j}\nabla + A_{n-j})\alpha^j = \bar{D}_{n-1}(j_0^{n-1}(\beta)(t)) = j_0^{n-1}(\bar{D}_t\beta)(t).$$

Hence

$$\sum_{j=0}^{n-1} (\varphi_{n-j}\nabla + A_{n-j})\alpha^j = \bar{\partial}_E\beta_0^{n-1} + \sum_{j=0}^{n-2} (\varphi_{n-1-j}\nabla + A_{n-1-j})\beta^j$$

and

$$\bar{\partial}_E\beta^k + \sum_{j=0}^{k-1} (\varphi_{k-j}\nabla + A_{k-j})\beta^j = 0$$

for $k \leq n - 2$.

The last $(n-2)$ equations simply means that $\beta$ defines an extension of $\beta_0$ of order $n - 2$. By assumption, we have $O_{n-1,n-2}[j_0^{n-2}(\beta)(t)] = 0$, and so

$$\sum_{j=0}^{n-2} (\varphi_{n-1-j}\nabla + A_{n-1-j})\beta^j = \bar{\partial}_E\gamma^{n-2} + \sum_{j=0}^{n-3} (\varphi_{n-2-j}\nabla + A_{n-2-j})\gamma^j$$

for some $\gamma = \sum_{n=0}^{\infty} \gamma_n t^n$. Keep repeating the previous argument, this reduces to the $n = 1$ case and so

$$\sum_{j=0}^{n-1} (\varphi_{n-j}\nabla + A_{n-j})\alpha^j$$

is $\bar{\partial}_E$-exact and therefore, has no harmonic part. This completes the induction argument. \qed

REFERENCES

1. K. Chan and Y.-H. Suen, A differential-geometric approach to deformations of pairs $(X,E)$, Complex Manifolds (2016), to appear. arXiv:1406.6758

2. H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Inst. Hautes Études Sci. Publ. Math. (1960), no. 5, 64. MR 0121814 (22 #12544)

3. R. Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

4. L. Huang, On joint moduli spaces, Math. Ann. 302 (1995), no. 1, 61–79. MR 1329447 (96d:32021)

5. K. Kodaira and J. Morrow, Complex manifolds, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the 1971 edition with errata. MR 2214741 (2006j:32001)

6. M. Kuranishi, New proof for the existence of locally complete families of complex structures, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, pp. 142–154. MR 0176496 (31 #768)
7. X. Ye, *The jumping phenomenon of Hodge numbers*, Pacific J. Math. **235** (2008), no. 2, 379–398. MR 2386229 (2009a:32016)
8. ______, *The jumping phenomenon of the dimensions of cohomology groups of tangent sheaf*, Acta Math. Sci. Ser. B Engl. Ed. **30** (2010), no. 5, 1746–1758. MR 2778644 (2011m:32015)

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