On finite dimensionality of homology of subalgebras of vector fields

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Abstract

We show that the tensor product of modules of tensor fields is a noetherian module as a module over any graded Lie subalgebra of finite codimension in the Lie algebra of polynomial vector fields on \( \mathbb{R}^n \). As a corollary, we prove the conjecture of I.M.Gelfand announced at ICM’1970 at Nice on finite dimensionality of continuous cohomology of graded Lie subalgebras of formal vector fields \( W_n \).

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0 Introduction

This paper answers some questions that remain open from the golden epoch of Gelfand-Fuchs cohomology. We deal with the Lie algebra of formal vector fields \( \mathcal{W}_n \) on \( \mathbb{R}^n \) and its subalgebra of polynomial vector fields \( \mathcal{W}_n^{\text{pol}} \). With each vector bundle \( E \to \mathbb{R}^n \) we assign the \( \mathcal{W}_n^{\text{pol}} \)-module \( T_E \) of polynomial sections of this bundle and the main purpose of this paper is the following

Theorem. (Theorem 2.3.3) For any collection of vector bundles \( E_1, \ldots, E_n \) and any subalgebra \( \mathfrak{g} \subset \mathcal{W}_n^{\text{pol}} \) of finite codimension the tensor product \( T_{E_1} \otimes \ldots \otimes T_{E_r} \) is a noetherian \( \mathfrak{g} \)-module.

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As a corollary, we prove (Theorem 3.2.1) the old-standing Gelfand’s conjecture stated at ICM’70 ([9]), claiming that the homology $H_p(g;\mathbb{R})$ of any graded Lie subalgebra $g$ of finite codimension in $\mathcal{W}_{\text{pol}}$ are finite-dimensional for all $p$.

Suppose that $g \subset \mathcal{W}_{\text{pol}}$ is a graded subalgebra of finite codimension that does not contain vector fields with nonzero constant and linear terms. Then $g$ is pronipotent and finite-dimensionality of the first and the second cohomology of $g$ implies that $g$ is finitely presentable. I.e. there exists a finite set of generators indexing the basis of $g/[g,g]$ and a finite set of relations. However, $g$ is graded $g = \bigoplus_{d=1}^{\infty} g_d$ by degree of the vector field and $\dim g_d \sim d^{n-1}$. Therefore, its universal enveloping algebra $U(g)$ has a finite number of generators and relations but has the intermediate growth of dimensions. In particular, what follows that the sum of all homology for all homological degrees is infinite-dimensional, its generating series is not rational and $U(g)$ is not (left/right)-noetherian and does not admit a finite Gröbner basis. What confirms that noetherianity of modules we discuss is far from being obvious. (We refer the interesting reader to [22] for a detailed description of the theory of the growth of algebras, the Gröbner bases theory, and the examples above, which are the first known examples of the algebras of intermediate growth.)

However, finite-dimensionality of the cohomology of $\mathcal{W}_n$ was the starting point for the big project of computing the appropriate cohomology of the Lie algebras $\text{Vect}(X)$ of smooth vector fields on smooth manifolds $X$ (see e.g. [16]). Note, that $\text{Vect}(X)$ is an infinite-dimensional topological Lie algebra, which means that the Lie bracket is a continuous map. The ordinary Lie cohomology of $\text{Vect}(X)$ is not known and we are far from understanding any answer for this kind of topological Lie algebras, in particular, because of the problems with the exactness of the tensor product operation for topological Lie algebras (see [21]). However, there was a big progress in computations in 70’s and 80’s of the so-called continuous cohomology $H^*_c(-)$ – the cohomology of the subcomplex of continuous cochains of the Chevalley-Eilenberg complex of $\text{Vect}(X)$, consisting of continuous skewsymmetric multilinear maps from $\text{Vect}(X)$. The first known computation of this kind is due to Gelfand and Fuchs [11]. They described the continuous cohomology of the Lie algebra $\mathcal{W}_n$ of formal vector fields on $\mathbb{R}^n$ and showed the applications of this theory to the characteristic classes of foliations. Later on, they showed the finite-dimensionality of $H^*_c(\text{Vect}(X);\mathbb{R})$ for all $X$ and all $k$ in [12, 13]. The conjectural description of $H^*_c(\text{Vect}(X);\mathbb{R})$ was given independently by R.Bott and D.Fuchs in 1972 and first proved by Haefliger in [18], another proof was suggested a bit later by Bott and Segal ([4]). Recently, B.Hennion and M.Kapranov extended the Gelfand-Fuchs computations to the case of algebraic vector fields $\mathcal{T}_X$ on an affine algebraic variety $X$ ([19]). Note that for both computations the smooth vector fields $\text{Vect}(X)$ and the algebraic vector field $\mathcal{T}_X$ the convergence of appropriate spectral sequences is necessary for different computations. We hope that the noetherianity of certain modules is one of the first steps in this direction and was one of the motivating points for this project.

We finish the Introduction with Conjecture 0.0.1 whose particular cases follow from the results of this paper.

**Conjecture 0.0.1.** *The homology of the Lie subalgebra of finite codimension in the Lie algebra of algebraic vector fields on an affine algebraic manifold are finite-dimensional in each homological degree.*

### 0.1 Structure of the paper
The paper is organized as follows. We recall the necessary notations for the Lie algebras of (formal) vector fields $\mathcal{W}_n$, its graded subalgebras $L_d$ and the module of sections $\mathcal{T}_\lambda$ in Section 1.1. We make some general conclusions...
on PBW theory in Section §1.3 that are used later for the proof of the main Theorem 2.3.3. Section §2 is devoted to the detailed proof of the aforementioned Theorem 2.3.3. We suggest three different applications of this Theorem. First, we show that the Hilbert series of dimensions of modules we consider are rational (Section §3.1). Second, we explain the proof of Gelfand’s conjecture and some generalizations in Section §3.2. Third, we show an immediate application of the crucial easily formulated fact for compositions of polynomials; this fact reformulates the finite presentability of $T$-spaces, which happens to be a key step in the proof of Specht theorem (Section §3.3).

Acknowledgement

The main part of this text was written while A.Kh. was visiting Hebrew University in the summer 2022 and A.Kh. would like to thank D.Kazhdan, Hebrew University and ERSC grant 669655 for suggesting perfect working conditions and a stimulating atmosphere. The research of A.K.-B. was supported by RSF grant 22-11-00177.

1 Recollection and initial conclusions

All results of this paper remain to be true for the arbitrary field of characteristics 0. However, for most of the applications, one only needs either $k = \mathbb{R}$ or $k = \mathbb{C}$.

1.1 Lie algebras of vector fields

Denote by $W_n$ be the Lie algebra of formal vector fields on the $n$-dimensional plane and by $W_n^{pol}$ the Lie subalgebra of polynomial vector fields. Let us choose the coordinates $\langle x_1, \ldots, x_n \rangle$ on $n$-dimensional plane. We will write the elements of $W_n$ and $W_n^{pol}$ as sums $\sum_{i=1}^n f_i \partial_i$ where $f_i \in k[[x_1, \ldots, x_n]]$ (resp. $f_i \in k[x_1, \ldots, x_n]$). The assignment $E_{ij} \mapsto x_i \partial_j$ defines an isomorphism of the matrix Lie algebra $\mathfrak{gl}_n$ and the Lie subalgebra of linear vector fields. Let us denote the tautological $n$-dimensional representation $\text{span}(x_1, \ldots, x_n)$ by $V$ then we have the following decomposition of the $\mathfrak{gl}_n$ modules:

$$W_n \simeq \prod_{k=0}^\infty S^kV \otimes V^*; \quad W_n^{pol} \simeq \oplus_{k=0}^\infty S^kV \otimes V^*$$

For each $d \geq 0$ we have the following graded subalgebras

$$L_d(n) := \prod_{k=d}^\infty S^kV \otimes V^*, \quad L^{pol}_d := \oplus_{k\geq d} S^kV \otimes V^*$$

in $W_n$ and in $W_n^{pol}$ respectively. Note that $L_d(n)$ are pronilpotent for $d \geq 1$.

With each irreducible $\mathfrak{gl}_n$-representation $V_\lambda$ we associate the irreducible $L_0(n)$-module $V_\lambda$ with the trivial action of the subalgebra $L_1(n) \subset L_0(n)$. The induced and coinduced modules

$$T_\lambda := \text{Ind}_{L_0(n)}^{W_n} V_\lambda = U(W_n^{pol}) \otimes U(L_0(n)) V_\lambda \simeq \oplus_{k=0}^\infty S^kV^* \otimes V_\lambda, \quad (1.1.1)$$

$$T^{\lambda} := \text{CoInd}_{L_0(n)}^{W_n} V_\lambda = \text{Hom}_{U(L_0(n))}(U(W_n), V_\lambda) \simeq \oplus_{k=0}^\infty S^kV \otimes V_\lambda \quad (1.1.2)$$

of the Lie algebras $W_n^{pol}$ and $W_n$ correspondingly are called the modules of tensor fields.
1.2 \( n = 1 \)

Let us go into some details for the Lie algebra \( W_1 \) of vector fields on the line. This case is much simpler than the general \( n \) since all graded components are one-dimensional. In order to avoid some misprints we will use variable \( z \) as a coordinate on the line. The Lie algebra \( \mathfrak{w}_n^{\text{pol}} \) admits the following monomial basis: 
\[
e_k := z^{k+1} \partial \]

that satisfy the following commutation rules:
\[
\forall k, m \ [e_k, e_m] = (m - k)e_{k+m}.
\]

Each irreducible \( \mathfrak{g}l_1 \)-representation is one-dimensional \( k_\lambda \) where \( \lambda \in k \) is a character. The module \( T_\lambda \) can be symbolically written as \( k[z] \partial^{-\lambda} \) with the following action of \( \mathfrak{w}_n^{\text{pol}} \):
\[
f \partial \cdot g \partial^{-\lambda} := (fg' + \lambda g f') \partial^{-\lambda} \text{ with } f, g \in k[z].
\]

Note that as a vector space \( T_\lambda \) is isomorphic to the polynomial ring \( k[z] \) and the symbol \( \partial^{-\lambda} \) only affects the additional term in the action. We will be interested in the following flat deformation of these modules over the Lie subalgebra \( L_0 \):
\[
T_{\lambda, \mu} := k[x] x^\mu \partial^{-\lambda}, \quad \lambda, \mu \in k
\]

where we have
\[
e_k \cdot z^m (z^\mu \partial^{-\lambda}) := (m + \mu + (k + 1) \lambda) z^{m+k} (z^\mu \partial^{-\lambda}).
\]

Note that \( T_{\lambda, 0} \simeq T_\lambda \), but for \( \mu \neq 0 \) the action of \( e_{-1} \) in \( T_{\lambda, \mu} \) is not well defined. With each pair of collection of numbers \( \lambda = (\lambda_1, \ldots, \lambda_r) \) and \( \bar{\mu} = (\mu_1, \ldots, \mu_r) \) we associate a formal notation \( z^{\bar{\mu}} := \prod_i z_i^{\mu_i} \) and \( \partial^{-\bar{\mu}} := \prod_i \partial^{-\lambda_i} \) and the module
\[
T_{\lambda, \bar{\mu}} := T_{\lambda_1, \mu_1} \otimes \cdots \otimes T_{\lambda_r, \mu_r} \simeq k[z_1, \ldots, z_r] z^{\bar{\mu}} \partial^{-\lambda}.
\]

Remark 1.2.5. The infinite-dimensional Lie algebra \( L_1 \) is graded with one-dimensional graded components: \( \deg(e_k) = k \). The modules \( T_{\lambda, \mu} \) as well as the tensor products \( T_{\lambda, \bar{\mu}} \) are graded
\[
\deg(z^\bar{a} z^{\bar{\mu}} \partial^{-\lambda}) := a_1 + \ldots + a_r
\]
with finite-dimensional graded components.

1.3 Poincaré-Birkhoff-Witt filtration

We recall a few standard facts from Lie theory and derive simple conclusions that are sufficient to show noetherian property for modules over an infinite-dimensional Lie algebra \( \mathfrak{g} \). The representations of a Lie algebra \( \mathfrak{g} \) are in one-to-one correspondence with the left modules of its universal enveloping \( U(\mathfrak{g}) \). The universal enveloping \( U(\mathfrak{g}) \) is the quotient of the free tensor algebra \( T^\otimes \mathfrak{g} \) generated by \( \mathfrak{g} \) subject to relations \( g_1 \otimes g_2 - g_2 \otimes g_1 = [g_1, g_2] \) for all pairs \( g_1, g_2 \in \mathfrak{g} \). In particular, if \( x_1, x_2, \ldots \) constitute a basis of the Lie algebra \( \mathfrak{g} \) with the Lie structure constants \( \epsilon_{ij}^k \):
\[
[x_i, x_j] = \sum_k c_{ij}^k x_k,
\]
then \( U(\mathfrak{g}) \) has the following presentation in generators and relations as associative algebra:

\[
U(\mathfrak{g}) \simeq k\langle x_1, x_2, \ldots | x_i x_j - x_j x_i - \sum c_{ij} x_k \rangle.
\]

Moreover, it is well known that the aforementioned quadratic-linear relations constitute a Gröbner basis with respect to the standard degree-lexicographical ordering of monomials.

Each universal enveloping algebra \( U(\mathfrak{g}) \) admits a PBW-filtration \( \mathcal{F} \) that comes from the standard filtration \( \mathcal{F} \) of the tensor algebra \( T^\otimes \mathfrak{g} \):

\[
k \subset (k \oplus \mathfrak{g}) \subset (k \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g}) \subset \mathcal{F}^n = \oplus_{k=0}^n \mathfrak{g}^\otimes n \subset \ldots
\]

The PBW filtration induces the PBW filtration of any \( \mathfrak{g} \)-module with a chosen set of generators. Indeed, if \( m_1, \ldots, m_k \) are the generators of \( M \), then we have a canonical surjective morphism of \( \mathfrak{g} \)-modules:

\[
U(\mathfrak{g})^k \twoheadrightarrow M.
\]

Thus, the PBW filtration of \( U(\mathfrak{g}) \) induces the PBW-filtration of \( M \).

**Proposition 1.3.1.** Suppose that a finitely generated \( \mathfrak{g} \)-module \( M \) admits a finite set of generators \( \{m_1, \ldots, m_k\} \) and a finite subset \( \{g_1, \ldots, g_r\} \) of \( \mathfrak{g} \) such that each element of \( M \) is a linear combination of the elements of the form

\[
\{g_1^{a_1} g_2^{a_2} \ldots g_r^{a_r} m_s : a_i \geq 0, 1 \leq s \leq k\}.
\]

Then the image of the generators \( m_1, \ldots, m_k \) of the associated graded module \( \text{gr}^{PBW} M \) generates \( M \) as a module over the polynomial subalgebra \( k[g_1, \ldots, g_r] \subset \text{gr}^{PBW} U(\mathfrak{g}) \). In particular, \( M \) is noetherian.

**Proof.** Let us complete the collection \( \{g_1, \ldots, g_k\} \) into a basis \( \{g_i | i \in I\} \) of \( \mathfrak{g} \). We suppose \( I = \{1, \ldots, r\} \cap J \). Thanks to the assumptions in Proposition [1.3.1] for all \( j \in J, s \leq k \) the elements \( g_j m_s \) have presentations as a sum of elements of the form (1.3.2). Hence, there exists a collection of polynomials \( p_{js}^l \in k[g_1, \ldots, g_r] \) such that

\[
g_j m_s = \sum_{l=1}^k \Psi(p_{js}^l)m_t.
\]

Here the map \( \Psi : k[g_1, \ldots, g_r] \to U(\mathfrak{g}) \) is the map that writes each commutative monom in a left-normalized order. That is, \( \Psi \) is defined on a monomial basis in the following form:

\[
\Psi : g_1^{a_1} \ldots g_r^{a_r} \mapsto \underbrace{g_1 \cdot \ldots \cdot g_1}_{a_1} \ldots \underbrace{g_r \cdot \ldots \cdot g_r}_{a_r} \in U(\mathfrak{g}).
\]

The remaining relations of \( M \) do not contain \( g_j \) with \( j \in J \) and are generated by the linear combinations of the elements of the form (1.3.2). Therefore, the defining relations of \( M \) can be given in the following form:

\[
U(\mathfrak{g}) \left\langle m_1, \ldots, m_k \right| g_j m_s = \sum_{t=1}^k \Psi(p_{js}^l)m_t, j \in J, s \leq k \right\rangle
\]

for an appropriate set \( T \). Consequently, the module \( \text{gr}^{PBW} M \) considered as a module over \( k[g_1, \ldots, g_r] \) is the quotient of the finitely generated module generated by the same set of elements \( \{m_1, \ldots, m_k\} \):

\[
\tilde{M} := k[g_1, \ldots, g_r] \left\langle m_1, \ldots, m_k \right| \sum_{s=1}^k q_i^s m_s : q_i^s \in k[g_1, \ldots, g_r], \ t \in T \right\rangle
\]

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Hence, $\mathfrak{g}^{PBW}M$ is a finitely-generated module over a finitely generated ring $\mathbb{k}[g_1, \ldots, g_r]$ and thanks to the Hilbert Basis Theorem $\mathfrak{g}^{PBW}M$ is a noetherian $S(\mathfrak{g})$-module. In particular, this implies that the number $\#T$ of the defining relations is finite and the $U(\mathfrak{g})$-module $M$ is noetherian.

2 Noetherianity of $L_d(n)$-modules $\mathcal{T}_{\lambda_1} \otimes \cdots \otimes \mathcal{T}_{\lambda_r}$

In this section, we will prove that the tensor product of the finite number of copies of modules $\mathcal{T}_\lambda$ for possibly different $\lambda$ are noetherian $L_d(n)$-modules. The proof consists of two steps – first, we discuss in detail $n=1$, second, we reduce general $d$ to $d=1$, and third, we reduce the general $n$ to the case $n=1$.

2.1 $n=1$, $d=1$

In this section we consider the Lie algebra $L_1 := L_1(1)$ and we study the tensor products of a finite number of copies of $L_1$-modules $\mathcal{T}_{\lambda_1, \ldots, \lambda_r}$ which we denote by $\mathcal{T}_{\lambda, \bar{\mu}}$ as in (1.2.3).

We start this section by key technical Lemma 2.1.1 and Lemma 2.1.4 and then show how do they imply the noetherianity of $L_1$-modules $\mathcal{T}_{\lambda, \bar{\mu}}$.

Lemma 2.1.1. For big enough $N \in \mathbb{N}$ the elements

$$\left\{ e_1^{\rho_1} \cdots e_r^{\rho_r} z^\bar{\alpha} z^{\bar{\gamma}+\bar{\mu}} \partial^{-\bar{\lambda}} : a_i < i, \sum_i \rho_i + \sum a_i = r \right\}$$

(2.1.2)

constitute a basis in the degree $r$ component of $\mathcal{T}_{\lambda, \bar{\mu} + \bar{N}}$. Here $\bar{N}$ is a constant tuple $(N, \ldots, N)$.

Proof. Let us consider a particular case $\lambda = \bar{\mu} = 0$. The action of $e_k$ on $\mathcal{T}_{r, \bar{N}}$ coincides with the multiplication by the Newton’s power sum $p_k := \sum_{i=1}^r z_i^k$ multiplied by a nonzero constant $N$. Recall that for the field $\mathbb{k}$ of characteristic zero the ring of symmetric functions $\Lambda_\mathbb{k} := \mathbb{k}[z_1, \ldots, z_r]^{S_n}$ is isomorphic to the polynomial ring generated by Newton’s sums:

$$\Lambda_\mathbb{k} \simeq \mathbb{k}[p_1, \ldots, p_r].$$

Moreover, the polynomial ring $\mathbb{Q}[z_1, \ldots, z_r]$ is the free $\Lambda_\mathbb{Q}$-module with a monomial basis

$$\left\{ z^\alpha := z_1^{\alpha_1} \cdots z_r^{\alpha_r} : a_i < i \right\}.$$

In particular, the elements

$$\left\{ p_1^{\rho_1} \cdots p_r^{\rho_r} z^\alpha : a_i < i, \sum_i \rho_i + \sum a_i = r \right\}$$

constitute a basis of the finite-dimensional vector space $\mathbb{Q}[z_1, \ldots, z_r]^{(r)}$ spanned by homogeneous polynomial of degree $r$ in $r$ variables and the statement of Lemma 2.1.1 is satisfied for $\lambda = \bar{\mu} = 0$.

Let us consider the linear endomorphism $A_r(\bar{\lambda}, \bar{\mu})$ of the degree $r$ component of the module $\mathcal{T}_{\lambda, \bar{\mu}}$ defined in the Newton basis in the following form:

$$p_\rho z^\alpha \partial^{-\bar{\lambda}} \mapsto e_1^{\rho_1} \cdots e_r^{\rho_r} z^\alpha \partial^{-\bar{\lambda}}, \quad a_i < i, \sum_{i=1}^r i \rho_i + \sum_a = r$$

The determinant

$$\Phi_r := \Phi_r(\bar{\lambda}, \bar{\mu}) = \det(A_r(\bar{\lambda}, \bar{\mu})) \in \mathbb{Q}[\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r]$$

(2.1.3)
is a polynomial in variables $\lambda_i$ and $\mu_j$ with rational coefficients. It remains to show that $\Phi_r(\lambda, N + \mu)$ differs from zero for all $N$ sufficiently large. Indeed, for nonzero $\lambda$ and $\mu$ we have the following action of $e_k$ (described in (1.2.2)) on the monomial $z^{N + \mu} \partial^{-\lambda}$

$$e_k \cdot z^{N + \mu} \partial^{-\lambda} = \left( \sum_{i=1}^r (N + \mu_i + (k + 1)\lambda_i)z_i^k \right) z^{N + \mu} \partial^{-\lambda} = (NP_k + q_k(\lambda, \mu))z^{N + \mu} \partial^{-\lambda},$$

where $q_k(\lambda, \mu) \in \mathbb{Q}[\lambda, \mu, z_1, \ldots, z_r]$ is a polynomial in $\lambda$ and $\mu$ with rational coefficients that does not depend on $N$ and whose $z$-degree is equal to $k$. Let us vary $\varphi \rightarrow \bar{\mu} + \bar{\nu}$ with bounded from above integer numbers $\nu_1 + \ldots + \nu_r < r$. Therefore, the action of $e_k$ on elements in $T_{\lambda, \bar{\mu} + \bar{\nu}}$ of degree less then or equal to $r$ is equal to $NP_k$ with lower (with respect to $N$) terms. Therefore, for each partition $\rho = 1^{\rho_1}2^{\rho_2} \ldots r^{\rho_r}$ of length $l(\rho) := \rho_1 + \ldots + \rho_r$ with $\sum i\rho_i = k < r < l(\rho)$ we have

$$e^{\rho_1}_1 \ldots e^{\rho_r}_r \cdot z^{N + \mu} \partial^{-\lambda} = (NP_l + q_l(N, \bar{\lambda}, \bar{\mu}, \bar{\nu}))z^{N + \mu} \partial^{-\lambda},$$

Where $q_l(N, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a polynomial function with rational coefficients, whose degree with respect to $N$ is strictly less than $l(\rho)$ and whose degree with respect to $\bar{\nu}$ is equal to $k$. Thus, $\Phi_r(\lambda, N + \mu)$ is a polynomial in $N$ with leading coefficient 1, what follows that for any given collections of numbers $\lambda, \bar{\mu} \in \mathbb{K}^{r}$ the value of $\Phi_r(\lambda, \bar{\nu} + \bar{\mu})$ differs from zero for all $N$ sufficiently large.

**Lemma 2.1.4.** For all $r \in \mathbb{N}$ and for any given collections $\bar{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{K}^{r}$ and $\bar{\mu} = (\mu_1, \ldots, \mu_r) \in \mathbb{K}^{r}$ there exists a semi-algebraic open domain $U_r \subset \mathbb{Q}^{r}$ containing infinitely many integral points $|\mathbb{N}^r \cap U_r| = \infty$, such that for all points $\nu := (\nu_1, \ldots, \nu_r) \in U_r \cap \mathbb{Q}^{r}$, the elements

$$\left\{ e^{b_1}_1 \ldots e^{b_r}_r \ z_1^{a_1} \ldots z_r^{a_r} \ z^{N + \mu} \partial^{-\lambda} : b_i \in \mathbb{Z}_{\geq 0}, \ 0 \leq a_i < i \right\}$$

constitute a graded basis of $T_{\bar{\lambda}, \bar{\mu} + \bar{\nu}}$.

**Proof.** The proof is by induction on $r$.

For $r = 1$ we know the action of $e_1$ in $T_{\lambda, \mu}$ from (1.2.2). Therefore, whenever $n + \mu + 2\lambda + 1 \neq 0$ we have $e_1 \cdot z^{\mu + \lambda} \partial^{-\lambda} \neq 0$. Thus the domain $U_1$ coincides with the ray $(-\mu - 2\lambda - 1, \infty)$ if $\mu + 2\lambda + 1 \in \mathbb{Z}$ and with the whole line $\mathbb{Q} \subset \mathbb{R}$ otherwise.

The induction step is based on Lemma 2.1.1. Let us fix a pair of $(r + 1)$-tuples $\bar{\lambda} = (\lambda_1, \ldots, \lambda_{r+1})$ and $\bar{\mu} = (\mu_1, \ldots, \mu_{r+1})$. Let $U_{r+1} \subset \mathbb{Q}^{r+1}$ be a desired open domain as asked in Lemma 2.1.4 chosen for the subcollections $\bar{\lambda} := (\lambda_1, \ldots, \lambda_r)$ and $\bar{\mu} := (\mu_1, \ldots, \mu_r)$. Note that the action of $e_k$ can be splitted into the sum of two summands:

$$e_k z^{m}_{r+1} f(z_1, \ldots, z_r) \partial^{-\lambda} = (m + \mu + (k + 1)\lambda) z^{m+k}_{r+1} f(z_1, \ldots, z_r) \partial^{-\lambda} +$$

$$z^{m+\mu+1}_{r+1} \partial^{-\lambda+1} e_k \cdot f(z_1, \ldots, z_r) \partial^{-\lambda}.$$
constitute a graded basis of the $L_1$-module $T_{r+1}^+$ for all $r$-tuples $\tilde{\nu}$ that belongs to $U$. This happens because the components of these elements with the lowest degree with respect to $z_{r+1}$ constitute a graded basis as well.

Consider an algebraic subset $Z_{r+1}^+ \subset Q^r$ consisting of the following elements

$$Z_{r+1}^+ := \left\{ \tilde{\nu} := (\nu_1, \ldots, \nu_r) \in Q^r \Bigg| \Phi_{r+1}(\tilde{\lambda}, \tilde{\nu} + \tilde{\mu}, \mu_{r+1} + y) \neq 0, \right. \left. \text{considered as a polynomial in } y\text{-variable with other arguments substituted by given numbers} \right\}$$

We know that the complement to the set $Z_{r+1}^+$ contains a ray $\{(t, t, \ldots, t) : t \gg 0\}$. Therefore $Z_{r+1}^+$ is a proper algebraic subset of $Q^r$ and $V_r := U_r \cap (Q^r \setminus Z_{r+1})$ is a nonempty open subset of $Q^r \subset R^r$ containing infinitely many integer points. For all points $\tilde{\nu} \in V_r$ we know that the nonzero polynomial $\Psi_{r+1}(\tilde{\lambda}, \tilde{\nu} + \tilde{\mu}, \mu_{r+1} + y)$ has only a finite number of roots and, therefore, there exists an open ray $(c_{r+1}, \infty) \in R$ that does not contain the roots of $\Psi_{r+1}(\tilde{\nu})$. Moreover, the function $\Phi_{r+1} \mapsto c_{r+1}$ is smooth and we claim that the set

$$U_{r+1} := \{ (\tilde{\nu}, \nu_{r+1}) : \tilde{\nu} \in V_r, \nu_{r+1} > c_{r+1} \}$$

is an open subset of $R^{r+1}$ that do satisfy the condition of Lemma 3.1.3 in particular $U_{r+1}$ contains infinitely many integral points. Indeed, for any point $N \in U_{r+1}$ we know that the elements (2.1.2) constitute a graded basis. Moreover, since the determinant $\Phi_{r+1}(\lambda, \tilde{\mu} + N)$ differs from zero we know that the elements of degree $r$ of the following set

$$\left\{ e_{r+1} z^N a^\mu b^\nu \partial^{-\lambda} \bigg| \right. \left. \sum_{i \leq r} a_i < i, m \leq r, \sum_{i < r} a_i + \sum_{i} b_i = r + 1 \right\}$$

are linearly independent. Therefore, the elements of the set

$$\left\{ e_{r+1} z^N a^\mu b^\nu \partial^{-\lambda} : b_1 \in Z_{r+1}, 0 \leq a_i < i \right\}$$

are also linearly independent. Notice, that we choose the subset $U_{r+1}$ in order to have the determinant $\Phi_{r+1}(\lambda, \tilde{\mu} + (N_1, \ldots, N_r, N_{r+1} + 1))$ different from zero. Therefore, the same arguments show that the elements

$$\left\{ e_{r+1} z^N a^\mu b^\nu \partial^{-\lambda} : b_1 \in Z_{r+1}, 0 \leq a_i < i \right\}$$

are also linearly independent. Using the iterative replacement $\mu_{r+1} \mapsto \mu_{r+1} + 1$ we still stay in the area $U_{r+1}$ with the corresponding determinant $\Phi_{r+1}(\lambda, \tilde{\mu} + (N_1, \ldots, N_r, N_{r+1} + k))$ different from zero for all $k \in N$ and we see that the elements of the set

$$\left\{ e_{r+1} z^N a^\mu b^\nu \partial^{-\lambda} : b_1 \in Z_{r+1}, 0 \leq a_i < i \right\}$$

are linearly independent. The number of elements of a given degree $d$ coincides with the number of monomials of degree $d$ in $r + 1$ variables. Thus, the elements of the set (2.1.6) constitute a graded basis of $T_{r+1}^+$. \hfill \Box
Theorem 2.1.7. For all pairs of collections $\lambda := (\lambda_1, \ldots, \lambda_r)$ and $\mu := (\mu_1, \ldots, \mu_r)$ there exists a finite subset $S \subset \mathbb{N}^r$ such that the elements
\[
\{ e_1^{b_1} e_2^{b_2} \cdots e_r^{b_r} \cdot z_1^{a_1} \cdots z_r^{a_r} \prod_{i=1}^r z_i^{\mu_i} \prod_{i=1}^r \partial_i^{\lambda_i} : b_i \geq 0, a_i := (a_1, \ldots, a_r) \in S \}
\]
span the $L_1$-module $T_{\lambda, \mu}$. 

Proof. The proof is by induction on $r$. For $r = 0$ the number of tensor multiples in $T_{\lambda, \mu}$ is empty and the corresponding module is a trivial one-dimensional $L_1$-module and the statement of Theorem is satisfied because the entire module is finite-dimensional.

Suppose we proved Theorem 2.1.7 for all $d < r$. Notice that for any given collection of integer numbers $N := (N_1, \ldots, N_r) \in \mathbb{N}^r$ the elements
\[
\{ f(z_1, \ldots, z_r) z^N \partial^{-\lambda} : f \in \mathbb{k}[z_1, \ldots, z_n] \}
\]
span a submodule $T_{\lambda, \mu+N} \subset T_{\lambda, \mu}$. Moreover, for all $i = 1, \ldots, r$ there is an isomorphism:
\[
T_{\lambda, \mu}/T_{\lambda, (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_r)} \simeq T_{\lambda, (\lambda_1, \ldots, \lambda_r), (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_r)}. 
\]
Therefore, the quotient $T_{\lambda, \mu}/T_{\lambda, \mu+N}$ is an extension of $N_1 + \ldots + N_k$ copies of tensor product of $r-1$-multiples of modules $T_{\lambda, \mu}^{-1}$ for an appropriate collection of pairs $(\alpha, \beta)$. In particular, thanks to the induction hypothesis there exists a finite subset $S_1 \subset \mathbb{N}^r$ such that the latter quotient module is spanned by elements: $\{ e_1^{b_1} \cdots e_r^{b_r} z^s \partial^{-\lambda} | s \in S_1 \}$. Thanks to key Lemma 2.1.4 we know that for all $r$-tuples $\lambda$ and $\mu$ there exists an integral collection $\hat{N} := (N_1, \ldots, N_r) \in U_r$ such that the submodule $T_{\lambda, \mu+N} \subset T_{\lambda, \mu}$ is spanned by the set (2.1.6). Consequently, the finite subset
\[
S := S_1 \cup \{(N_1 + a_1, N_2 + a_2, \ldots, N_r + a_r) | a_i = 0, 1, \ldots, i-1 \}
\]
do satisfy the assumption of Theorem 2.1.7.

Corollary 2.1.8. The $L_1$-module $T_{\lambda, \mu}$ is a noetherian $L_1$-module.

Proof. This is the direct consequence of the PBW theory discussed in Proposition 1.3.1 whose assumptions are satisfied thanks to Theorem 2.1.7.

2.2 Noetherianity for all $d$

For all $d \geq 1$ we have the following embedding of Lie algebras
\[
\iota_d : L_1(1) \rightarrow L_1(d) \text{ defined by } e_k \mapsto \frac{e_k d}{d} \tag{2.2.1}
\]

Proposition 2.2.2. The $L_1$-modules $T_{\lambda, \mu}$ considered as a module over the Lie subalgebra $\iota_d(L_1)$ is isomorphic to the direct sum of $L_1$-modules of the same kind:
\[
T_{\lambda, \mu} = \oplus T_{\lambda, \frac{m+\mu}{d}} + \oplus T_{\lambda, \frac{m+\mu}{d}} - 1. 
\]

Proof. Let us look carefully on the action of $\iota_d(e_k)$ on a particular element of $T_{\lambda, \mu}$:
\[
\iota_d(e_k) \cdot x^{(m+\mu)} \partial^{-\lambda} = \frac{e_k d}{d} \cdot x^{(m+\mu)} \partial^{-\lambda} = m + \mu + \lambda kd x^{kd+m+\mu} \partial^{-\lambda}.
\]
Therefore, the subspace $\mathbb{k}[x^d]x^{\mu+s} \partial^{-\lambda}$ is a $\iota_d(L_1)$-submodule isomorphic to $T_{\lambda, \frac{m+\mu}{d}}$ and the elements $\{x^{s+\mu} \partial^{-\lambda} | s = 0, \ldots, d-1 \}$ generate the $\iota_d(L_1)$-submodules mentioned in Proposition 2.2.2.
Corollary 2.2.3. For all pairs of collections \( \lambda = (\lambda_1, \ldots, \lambda_r) \) and \( \tilde{\mu} := (\mu_1, \ldots, \mu_r) \) there exists a finite set of monomials \( S \subset k[x_1, \ldots, x_r] \) such that the elements
\[
\{ e_d^b e_{2d}^c \cdots e_{rd}^g s^{-\lambda} : b_i \in \mathbb{N}, s \in S \}
\]
span the module \( T_{\lambda, \tilde{\mu}} \).

Proof. Direct inspection of the map \( \iota_d \) and Theorem 2.1.7 \( \square \)

2.3 Noetherianity for arbitrary \( n \) and \( d \)

There are \( n \) copies of pairwise commuting coordinate embeddings \( \alpha_m : L \hookrightarrow L_1(n) : \)
\[
\alpha_m : e_k \mapsto x_m^{k+1} \partial_m \in L_1(n) \subset W_n := \left\{ \sum f_m(x_1, \ldots, x_n) \partial_m \right\}.
\]
For all \( d \geq 1 \) we have the following embedding of Lie algebras
\[
\iota_d : L_1(1) \hookrightarrow L_d(1) \text{ defined by } e_k \mapsto \frac{e_d^k}{d} \tag{2.3.1}
\]
In particular, for all \( d \) and \( n \) there exists the pair of consecutive embeddings of Lie algebras:
\[
L_1^{\oplus n} := L_1^{(x_1)} \oplus \cdots \oplus L_1^{(x_n)} \hookrightarrow L_d^{(x_1)} \oplus \cdots \oplus L_d^{(x_n)} \hookrightarrow L_d(n) \hookrightarrow W_n. \tag{2.3.2}
\]

Theorem 2.3.3. For all \( d \) and \( n \), and all collection of weights \( \lambda_1, \ldots, \lambda_r \in \mathbb{Z}^n \) the tensor product of \( W_n \)-modules \( T_{\lambda_1, \ldots, \lambda_r} := T_{\lambda_1} \otimes \cdots \otimes T_{\lambda_r} \) is a noetherian \( L_1^{\oplus n} \)-module, (with \( L_1^{\oplus n} \hookrightarrow W_n \) considered in (2.3.2)). In particular, \( T_{\lambda_1, \ldots, \lambda_r} \) is a noetherian \( L_d(n) \)-module.

Proof. Let us start to work out first the one tensor multiple \( T_{\lambda} \) for a dominant weight \( \lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \mathbb{Z}^n \). Let \( V_\lambda \) be the irreducible \( GL_n \) module with the highest weight \( \lambda \in \mathbb{Z}^n \). Then from the basics of Lie theory we know that \( V_\lambda \) admits a weight decomposition with respect to the action of the diagonal subalgebra \( \mathfrak{h} \subset \mathfrak{gl}_n \)
\[
V_\lambda \simeq \oplus_{\alpha \leq \lambda} (V_\lambda)_{(\alpha)}, \quad \forall h, v \in V((\alpha)) \; hv = \alpha(h)v.
\]
Let \( \{ v_i^\alpha | \alpha \in \mathbb{Z}^n, i = 1, \ldots, \dim(V_\lambda) \} \) be a basis of \( (V_\lambda)_{(\alpha)} \). For example, if \( \lambda \) is a partition of \( N \) (i.e. \( \lambda_n \geq 0 \)) then the elements of a \( \mathfrak{h} \)-eigen basis are indexed by semi-standard Young tableaux of shape \( \lambda \). Therefore the coinduced module \( T_{\lambda} \) is considered as a module over the Lie subalgebra \( L_1^{(x_1)} \oplus \cdots \oplus L_1^{(x_n)} \) is isomorphic to the following direct sum of tensor product of modules:
\[
T_{\lambda} \simeq \oplus_{\alpha} \oplus_{i=0}^{\dim(V_\lambda)_{(\alpha)}} k[x_1, \ldots, x_n] \prod_{s=1}^{n} \partial_s^{-\alpha_s} \simeq \oplus_{\alpha} \oplus_{i=0}^{\dim(V_\lambda)_{(\alpha)}} (k[x_1] \partial_1^{-\alpha_1} \otimes \cdots \otimes (k[x_n] \partial_n^{-\alpha_n}) \tag{2.3.4}
\]
We know that each tensor multiple in (2.3.4) is a noetherian module and even is noetherian for the action of \( e_1^{x_k} = x_k^2 \partial_k \). Therefore, the full module \( T_{\lambda} \) is isomorphic to a direct sum of noetherian modules over commutative Lie subalgebra \( \langle e_1^{(x_1)}, \ldots, e_1^{(x_n)} \rangle \). Similarly, we know that for all collection of \( \mathfrak{gl}_n \)-weights \( (\lambda^1, \ldots, \lambda^r) \) with \( \lambda^s = (\lambda^1_s \geq \lambda^2_s \geq \cdots \geq \lambda^r_s) \in \mathbb{Z}^n \) the tensor product \( T_{\lambda^1} \otimes \cdots \otimes T_{\lambda^r} \), considered as a module over the Lie subalgebra \( L_1^{(x_1)} \oplus \cdots \oplus L_1^{(x_n)} \) will be a direct sum of tensor products of the form
\[
\otimes_{s=1}^{r} \otimes_{i=1}^{n} T_{\alpha^s_i} \simeq \otimes_{i=1}^{n} T_{(\alpha^1_i, \ldots, \alpha^r_i)}
\]
where \( \alpha^s = (\alpha^s_1, \ldots, \alpha^s_n) \) runs through the weight support of the \( g/V \)-module \( V \). Theorem 2.1.7 and Corollary 2.2.3 predicts that each tensor multiple \( T^{(x_s)}_{(\alpha^s_1, \ldots, \alpha^s_r)} \) admits a finite collection of generators \( S_i \) such that the span of all elements \( \{ (\epsilon^{(x_s)}_{d_1})^{b_{11}} \ldots (\epsilon^{(x_s)}_{d_r})^{b_{rr}} s | b_{ij} \in \mathbb{N}, s \in S_i \} \) coincide with the entire module \( T^{(x_s)}_{(\alpha^s_1, \ldots, \alpha^s_r)} \). Therefore, the elements

\[
\left\{ \left( \prod_{i=1}^n (\epsilon^{(x_s)}_{d_i})^{b_{i1}} \ldots (\epsilon^{(x_s)}_{d_r})^{b_{ri}} \right) \otimes_{i=1}^n s_i \mid b_{ij} \in \mathbb{N}, s_i \in S_i \right\}
\]

span the entire tensor product. Thanks to Proposition 1.3.1 we see that the initial module \( T_{\lambda_1} \otimes \ldots \otimes T_{\lambda_n} \) is a noetherian \( \oplus_{i=1}^n L^{(x_s)}_1 \)-module. \( \square \)

### 3 Applications

#### 3.1 On Hilbert series of \( L_d(n) \)-submodules of \( T_\lambda \)

**Definition 3.1.1.** The abelian category \( \mathcal{O}_d^{pol} \) be the abelian category of finitely generated graded \( L_d(n) \)-modules generated by submodules of finite tensor products of the modules \( T_\lambda \).

**Proposition 3.1.2.** 1. Each module in the category \( \mathcal{O}_d^{pol} \) is noetherian;

2. The category \( \mathcal{O}_d^{pol} \) is closed under taking tensor products.

3. The generating series of dimensions \( F_M(t) := \sum_{k=k_0}^{\infty} \dim M_k t^k \) is a rational function for all modules \( M \in \mathcal{O}_d^{pol} \); Moreover, the function \( f_M(n) := \sum_{k=k_0}^{n} \dim M_k \) is a polynomial in \( n \) for \( n \) sufficiently large, whose leading term is equal to \( c^{nd}_n \) for appropriate integers \( c \) and \( d \).

**Proof.** Item 1 follows from Theorem 2.3.3 item 2 is obvious because the tensor product of modules \( T_{\lambda_1} \otimes \ldots \otimes T_{\lambda_r} \) and \( T_{\nu_1} \otimes \ldots \otimes T_{\nu_s} \) is again a finite tensor product of modules \( T_{\lambda_1} \otimes \ldots \otimes T_{\nu_s} \). For the item 3 we have to look more carefully at Theorem 2.1.7. The PBW theory implies that for each \( T_\lambda \) admits a finite set of generators such that the associated graded will be a finitely generated module over a polynomial ring \( \mathbb{k}[e_1, \ldots, e_k] \) on a finite number of variables. Therefore, each submodule admits a finite set of generators such that the associated graded with PBW filtration will be a finitely generated module over a polynomial ring. We know from the classical commutative algebra \([3] [22]\) that any finitely generated module over a finitely generated commutative algebra admits a finite Gröbner basis and, consequently, its generating series is a rational function. The statement for the polynomiality of \( f_M(n) \) is also a well-known fact observed by I.Bernstein for the purpose of \( D \)-modules theory \([3]\). \( \square \)

#### 3.2 The proof of Gelfand’s conjecture

In this section we prove the conjecture of I.M.Gelfand on finite dimensionality of the (co)homology of the subalgebras \( L_d(n) \) of \( W_n \) stated in \([9]\).

**Theorem 3.2.1.** For any graded Lie subalgebras \( g \subset W_n^{pol} \) of finite codimension and for any graded finitely generated \( L_d(n) \)-module \( M \in \mathcal{O}^{pol} \) the Lie algebra homology of \( g \) with coefficients in \( M \) is finite-dimensional for each homological degree:

\[
\forall p \geqslant 0 \quad \dim H_p(g; M) < \infty.
\]
Proof. Note that $\bigcap_{k=0}^{\infty}L^k_d(n) = 0$ and, therefore, for each $\mathfrak{g} \subset W_n$ of finite codimension there exists $d$ such that $\mathfrak{g} \supset L_d(n)$. The space of $k$-chains of the Chevalley-Eilenberg complex $C^CE_k(\mathfrak{g}; M)$ is isomorphic to the following $\mathfrak{g}$-module:

$$C_p(\mathfrak{g}; M) := \Lambda^p \mathfrak{g} \otimes M \subset \Lambda^p W_n^{pol} \otimes M \subset \otimes^k W_n^{pol} \otimes M$$

In particular, for all $p$ the space of $p$-chains $C_p(\mathfrak{g}; M)$, considered as $L_d(n)$ module belongs to the category $O_{d}^{pol}$ and, in particular, is a noetherian $L_d(n)$-module. The Chevalley-Eilenberg differential is compatible with $\mathfrak{g}$-action and the Lie algebra cohomology has the trivial $\mathfrak{g}$-structure (see e.g. [7]). Therefore, $H_p(\mathfrak{g}; M)$ is a trivial noetherian $L_d(n)$-module, therefore it is finitely-generated trivial module. The number of generators of a trivial module coincides with its dimension, what follows that a trivial finitely-generated module is a finite-dimensional vector space.

Remark 3.2.2. It is worth mentioning that the generating series of homology $F_{H(\mathfrak{g})}(t) := \sum_{k=0}^{\infty} (-t)^k \dim H_k(\mathfrak{g})$ is inverse to the generating series of the universal enveloping $U(\mathfrak{g})$. The latter is not a rational function and, in particular, this implies that the number of $p$ such that $H_p(\mathfrak{g}; \mathbb{R}) \neq 0$ is infinite. So we can check finite dimensionality only for each particular degree, but there is no bound for homological dimension.

Let us now come back to the Lie algebra of formal vector fields $W_n$. This algebra is an infinite dimensional Lie algebra that admits a topology, whose basic open neighborhoods of 0 are generated by Lie subalgebras $L_d(n)$. The correct cohomology theory of topological Lie algebras of these kind computed in the golden epoch of Gelfand-Fuchs cohomology was postulated to be the so-called continuous cohomology (see e.g. [17], [14]). Nowadays, there are several conceptual explanations for why do people have to work out with continuous Chevalley-Eilenberg complex rather than with the ordinary one (see e.g. [1], [21]). It is worth mentioning that the cohomology of the ordinary cohomological Chevalley-Eilenberg complex of topological Lie algebras $W_n$, $L_d(n)$ and Lie algebra of smooth vector fields on a manifold might be infinite-dimensional and we are far from understanding anything about them. We will not present here a detailed definition of continuous cohomology and refer the interesting reader to the aforementioned papers. The rough definition says that the space of continuous $k$-chains $C^k_L(\mathfrak{g}; M)$ consists of continuous (with respect to given topology) skew-symmetric $k$-multilinear maps from $\mathfrak{g} \times \ldots \times \mathfrak{g}$ to $M$. In particular, $C^k_L(L_d(n); \mathbb{R})$ is the complex that is graded linear dual to the homological Chevalley-Eilenberg complex $C_*(L^d(n); \mathbb{R})$. Therefore, the finite-dimensionality of continuous cohomology of subalgebras $L_d(n)$ is equivalent to the homology of $L^d(n)$.

The following Corollary 3.2.3 was stated as a conjecture by I.M. Gelfand on his plenary talk at ICM 1970 (see p.106 of [9]).

Corollary 3.2.3. For all $k$ and $n$ the continuous cohomology $H^k_L(L_d(n); k)$ is finite-dimesional.

Proof. The complex that computes the continuous cohomology of the infinite-dimensional Lie algebra is isomorphic to the graded linear dual complex of $L^d(n)$.

There is not so many results on the cohomology of $L_d(n)$. The homology of $L_1(1)$ were computed first by Goncharova in [14]. Later, another proof was suggested by Gelfand, Feigin and Fuchs in [10]. In [6] Feigin and Fuchs showed how to compute the (co)homology of $L_1(n)$ up to degree $2n - 1$ and, in particular, they found the finite set of generators and relations of $L_1(n)$ for all $n$. 

3By graded linear dual to the graded vector space with finite dimensional graded components we mean the graded vector space with linear dual graded components.
Corollary 3.2.4. The homology of any graded Lie subalgebra of any of the given infinite-dimensional Lie subalgebras of polynomial vector fields are finite-dimensional in each homological degree:

- \( \mathcal{W}^\text{pol}_n \ltimes \mathfrak{g} \otimes k[z_1, \ldots, z_n] \) – subalgebra preserving the fibration with the structure group \( G \) whose Lie algebra is \( \mathfrak{g} \);
- \( \mathcal{W}^\text{pol}_n \ltimes \mathcal{W}^\text{pol}_m \otimes k[z_1, \ldots, z_n] \) – subalgebra preserving the foliation of codimension \( m \).

The finite dimensionality holds for continuous cohomology of any graded subalgebras of the corresponding Lie subalgebras of formal vector fields

\[ \mathcal{W}_n \ltimes \mathfrak{g} \otimes k[[z_1, \ldots, z_n]] \text{ or } \mathcal{W}^\text{pol}_n \ltimes \mathcal{W}_m \otimes k[[z_1, \ldots, z_n]]. \]

Proof. The proof completely repeats the one given for Theorem 3.2.1. For example, the Lie algebra \( \mathcal{W}^\text{pol}_n \ltimes \mathfrak{g} \otimes k[z_1, \ldots, z_n] \) contains Lie subalgebra \( \mathcal{W}^\text{pol}_n \) and the entire Lie algebra is a module that belongs to \( \mathcal{O}_1^{\text{pol}}(\mathfrak{g}) \). Therefore the Chevalley-Eilenberg complex consists of Noetherian \( L_d(n) \)-modules and thus homology are finite-dimensional.

3.3 Around Specht problem

In this section, we consider the problem that happens as an intermediate step in the proof of Specht problem on finite representability of the associative operad. The Specht problem was first proved by A.R. Kemer in [20]. Different other proofs can be found in [15]; Positive characteristic case treated in [2]. In particular, the A.Grishin and A.K.-B. explains that one of the crucial points in the Specht problem is the finite presentability of \( T \)-spaces. The latter is equivalent to the following easily formulated problem.

With each given polynomial \( p(x) \in k[x] \) with \( p(0) = 0 \) we can assign a substitution map in the ring of polynomials on \( n \) variables:

\[ \alpha_p : f(x_1, \ldots, x_n) \mapsto f(p(x_1), \ldots, p(x_n)). \]

The following Corollary 3.3.1 is an important intermediate step in Kemer’s proof of the famous Specht theorem.

Corollary 3.3.1. Suppose that a vector space \( S \subset k[x_1, \ldots, x_n] \) which is closed under substitutions \( \alpha_p \) for all \( p \in k[x] \) then

1. The vector space \( S \) is graded \( S = \oplus_{d=0}^{\infty} S_d \) by the degree of polynomials;
2. The set \( S \) is generated from a finite set \( T \) by substitutions and linear combinations;
3. The generating series \( F_S(t) := \sum_{d=0}^{\infty} \dim S_d t^d \) is a rational function.

Proof. Suppose \( f \in S \) is a nonhomogeneous polynomial with \( f = f_{n_1} + \ldots + f_{n_k} \) its homogeneous components of degrees \( n_1, \ldots, n_k \) correspondingly. The substitution with a linear polynomial \( \lambda x \) gives:

\[ f(\lambda x_1, \ldots, \lambda x_n) = \lambda^{n_1} f_{n_1}(x_1, \ldots, x_n) + \ldots + \lambda^{n_k} f_{n_k}(x_1, \ldots, x_k). \]

Therefore, by choosing substitutions with \( k \) different numbers \( \lambda_1, \ldots, \lambda_k \) and thanks to invertibility of Vandermonde operator \( (\lambda_1^d) \) we conclude that all \( f_{n_i} \) belongs to \( S \).

Let us focus on the infinitesimal substitutions for polynomials \( p = x + \varepsilon p(x) \) with \( \varepsilon \to 0 \) then we conclude that \( S \) is closed under the action of a vector field \( p(x_1) \partial_1 + \ldots + p(x_n) \partial_n \). Therefore, the statement of Corollary 3.3.1 reduces to the statement of noetherianity of \( L_1(1) \)-module \( k[x_1, \ldots, x_n] \). The latter module in our notations is equal to \( T_0 \otimes \ldots \otimes T_0 \) and is noetherian thanks to Theorem 2.3.3. In particular, as outlined in Proposition 3.1.2 the generating series of \( S \) is a rational function.
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