Amplitude and phase mode in a Bose Einstein condensate

Aranya B Bhattacherjee
Department of Physics, ARSD College, University of Delhi (South Campus), New Delhi-110021, India

We show that starting from the Heisenberg equations of motion for Bose annihilation and creation operators and using an appropriate transformation, we can split the Bogoliubov mode into a free particle mode and the amplitude mode. We show this for both the free Bose gas as well as the Bose gas in an optical lattice.

PACS numbers: 03.75.Kk,03.75.Lm

I. INTRODUCTION

The dynamical behaviour of Bose-Einstein condensate (BEC) such as collective modes is one of the important source of information about the physical characteristics of the condensate. More ever, the spectrum of elementary excitations of the condensate is utilized to derive the thermodynamic properties. The properties of elementary excitations may be investigated by considering small deviations of the state of the BEC from equilibrium and finding periodic solutions of the time-dependent Gross-Pitaevskii equation. The resulting excitation spectrum is called the Bogoliubov spectrum investigated by considering small deviations of the state of the BEC from equilibrium and finding periodic solutions of the time-dependent Gross-Pitaevskii equation.

II. THE UNIFORM BOSE GAS

Let us consider a uniform gas of interacting bosons contained in a box of volume $V$. Within the Bogoliubov approach (equivalent to including terms which are no more than quadratic in $a_q$ (annihilation operator for a Bose particle with momentum $q$) and $a_q^\dagger$ (creation operator for a Bose particle with momentum $q$) ) the Hamiltonian is written as

$$H = \frac{N^2 U}{2V} + \sum_{q(q\neq 0)} [(\epsilon_q^0 + n_0 U)(a_q^\dagger a_q + a_{-q}^\dagger a_{-q}) + n_0 U(a_q^\dagger a_{-q}^\dagger + a_q a_{-q})],$$

(1)

where $N$ is the expectation value of $\hat{N} = \sum_q a_q^\dagger a_q$. In Eqn.(1), we have also taken into account that the total number of particles is fixed. The chemical potential is $\mu = n_0 U$. Note that in Eqn.(1), the summation is to be taken only over one half of momentum space, since the terms corresponding to $q$ and $-q$ must be counted only once. We now write down the Heisenberg equation of motion for $a_q$ and $a_{-q}$ from Eqn.(1). This yields

$$\frac{da_q}{dt} = -i\epsilon_q a_q - i\epsilon_{-q} a_{-q}^\dagger,$$

(2)
\[ \frac{da_{-q}}{dt} = -i\epsilon_q a_{-q} - i\epsilon_1 a_q^\dagger, \]  \hspace{1cm} (3)

where \( \epsilon_q = (\epsilon_q^o + n_0 U)/\hbar \) and \( \epsilon_1 = n_0 U/\hbar \). We now make a transformation \( a_q \rightarrow \tilde{a}_q e^{i\omega t} \) and \( a_{-q} \rightarrow \tilde{a}_{-q} e^{-i\omega t} \). This yields from Eqns.(2) and (3), the usual Bogoliubov mode

\[ \omega_{\text{Bog}} = \sqrt{\tilde{\epsilon}_q^o (\tilde{\epsilon}_q^o + 2n_0 \tilde{U})}, \]  \hspace{1cm} (4)

where \( \tilde{\epsilon}_q^o = \frac{\hbar a^2}{2m} \) and \( \tilde{U} = \frac{U}{\hbar} \). On the other hand a transformation of the type \( a_q \rightarrow \tilde{a}_q e^{-i\omega t} \) and \( a_{-q} \rightarrow \tilde{a}_{-q} e^{i\omega t} \) yields two modes

\[ \omega_+ = \tilde{\epsilon}_q^o + 2n_0 \tilde{U}, \]  \hspace{1cm} (5)

\[ \omega_- = \tilde{\epsilon}_q^o. \]  \hspace{1cm} (6)

The \( \omega_- \) branch is the free particle mode while \( \omega_+ \) branch is the amplitude or the gapped branch with the property that \( \omega_+ \rightarrow 2\tilde{\mu} \) as \( q \rightarrow 0 \), \( \tilde{\mu} = n_0 \tilde{U} \). For a Bose gas in an optical lattice it was shown that the gap is exactly \( 2\tilde{\mu} \). The above result is independent of the fact whether \( n_0 \tilde{U} << 1 \) or \( n_0 \tilde{U} >> 1 \).

The above result is independent of the fact whether \( n_0 \tilde{U} << 1 \) or \( n_0 \tilde{U} >> 1 \).

Figure 1: Figure depicting the amplitude mode (solid think line), free particle mode (thin solid line) and the Bogoliubov mode (dashed lined) as a function of \( qd \) for the uniform Bose gas (left plot) and the Bose gas in an optical lattice (right plot). The mode frequencies are dimensionless with respect to \( n_0 \tilde{U} \) for the uniform gas and with respect to \( \tilde{n} \tilde{U} \) for the Bose-Hubbard model.

III. BOSE GAS IN AN OPTICAL LATTICE

We now consider a Bose gas confined in an optical lattice. We ignore the parabolic confining potential and write down the corresponding Bose-Hubbard Hamiltonian in the spirit of the Bogoliubov approximation as in the previous section

\[
H_{\text{BH}} = \frac{N^2 U}{2I} + \sum_{j,q(q\neq0)} \left[ \epsilon_q + 2\tilde{n}U \right] (a_{j,q}^\dagger a_{j,q} + a_{j,-q}^\dagger a_{j,-q})
- t \sum_{j,q(q\neq0)} \left[ (a_{j,q}^\dagger a_{j+1,q}^\dagger + a_{j,q}^\dagger a_{j-1,q}^\dagger) + (a_{j,-q}^\dagger a_{j+1,-q}^\dagger + a_{j,-q}^\dagger a_{j-1,-q}^\dagger) \right]
+ \tilde{n}U \sum_{j,q(q\neq0)} (a_{j,q}^\dagger a_{j,-q}^\dagger + a_{j,q} a_{j,-q}).
\]  \hspace{1cm} (7)
Here $\epsilon_{j,q}$ are the onsite energies, $t$ is the hopping parameter and $I$ is the total number of sites. The Heisenberg equation of motion for $a_{j,q}$ and $\tilde{a}_{j,-q}$ is

$$\frac{d a_{j,q}}{d t} = -i \tilde{\epsilon}_o a_{j,q} - i \epsilon_1 a_{j,-q} + it(a_{j+1,q} + a_{j-1,q}), \tag{8}$$

$$\frac{d a_{j,-q}}{d t} = -i \tilde{\epsilon}_o a_{j,-q} - i \epsilon_1 a_{j,q} + it(a_{j+1,-q} + a_{j-1,-q}), \tag{9}$$

where $\tilde{\epsilon}_o = \epsilon_q + 2\hbar U$ and $\epsilon_1 = \hbar U$. We have assumed $\epsilon_{j,q} = \epsilon_q$. Again a transformation of the type $a_{j,q} \rightarrow \tilde{a}_{j,q} e^{iqj\sigma} e^{i\omega t} e^{-i\mu t}$ and $a_{j,-q} \rightarrow \tilde{a}_{j,-q} e^{-iqj\sigma} e^{-i\omega t} e^{-i\mu t}$ ($\mu = \hbar U - 2t$ and $d$ is the lattice spacing) yields the Bogoliubov mode for the Bose-Hubbard Hamiltonian

$$\omega^B_{\text{BH}} = \sqrt{\left(\tilde{\epsilon}_q + 4t \sin^2 qd/2\right)^2 \left(\tilde{\epsilon}_q + 2\hbar U + 4t \sin^2 qd/2\right)}. \tag{10}$$

Here $\omega^B_{\text{BH}} \rightarrow 0$ as $q \rightarrow 0$. Similarly the transformations $a_{j,q} \rightarrow \tilde{a}_{j,q} e^{iqj\sigma} e^{-i\omega t} e^{i\mu t}$ and $a_{j,-q} \rightarrow \tilde{a}_{j,-q} e^{-iqj\sigma} e^{-i\omega t} e^{-i\mu t}$ splits the Bogoliubov mode into two modes

$$\omega^B_+ = \tilde{\epsilon}_p + 2\hbar U + 4t \sin^2 qd/2, \tag{11}$$

$$\omega^B_- = \tilde{\epsilon}_p + 4t \sin^2 qd/2, \tag{12}$$

The amplitude mode $\omega^B_+ \rightarrow 2\hbar U$ as $q \rightarrow 0$. Note that the above theory makes use of the “breaking of symmetry” in frequency space to separate the amplitude mode from the phase mode. Take for example the uniform Bose gas. The terms $(a^\dagger_j a_q + a^\dagger_q a_{-q})$ and $(a_j^\dagger a_{-q} + a_{q} a_{-q})$ evolve under the transformation $a_q \rightarrow \tilde{a}_q e^{i\omega t}$ and $a_{-q} \rightarrow \tilde{a}_{-q} e^{-i\omega t}$ as $(\tilde{a}^\dagger_q \tilde{a}_q + \tilde{a}^\dagger_{-q} \tilde{a}_{-q})$ and $(\tilde{a}^\dagger_q \tilde{a}_{-q} + \tilde{a}_{q} \tilde{a}_{-q})$. On the other hand, the transformations $a_q \rightarrow \tilde{a}_q e^{-i\omega t}$ and $a_{-q} \rightarrow \tilde{a}_{-q} e^{i\omega t}$ yield $(\tilde{a}^\dagger_q \tilde{a}_q + \tilde{a}^\dagger_{-q} \tilde{a}_{-q})$ and $(\tilde{a}^\dagger_q \tilde{a}_{-q} - \tilde{a}^\dagger_{-q} \tilde{a}_q e^{-2i\omega t})$, indicating a breaking of symmetry due to a two-photon process. Such asymmetry can be created by exposing the BEC to two counter-propagating Bragg pulses such that the atoms undergo two-photon process and the amplitude mode can be detected only at higher Bragg frequency [7].

IV. ACKNOWLEDGEMENTS

The author thanks Axel Pelster for some stimulating discussions on amplitude mode.

[1] C. J. Pethick and H. Smith, Bose-Einstein Condensation in Dilute Gases, Cambridge University Press, 2002.
[2] S.D. Huber et al., Phys. Rev. B, 75, 085106, (2007).
[3] S. D. Huber et al., Phys. Rev. Letts., 100, 050404, (2008).
[4] M.A. Cazalilla et al., New J. Phys., 10, 158, (2006).
[5] T.D. Grass et al., [arXiv:1003.4497] (2010).
[6] A. F. Ho, M. A. Cazalilla, and T. Giamarchi, Phys. Rev. Lett. 92, 130405 (2004); K. Sengupta and N. Dupuis, Phys. Rev. A 71, 033629 (2005); P. Pippan, H. G. Evertz, and M. Hohenadler, Phys. Rev. A 80, 033612 (2009); C. Menotti and N. Trivedi, Phys. Rev. B 77, 235120 (2008); Y. Ohashi, M. Kitaura, and H. Matsumoto, Phys. Rev. A 73, 033617 (2006).
[7] U. Bissbort et al., Phys. Rev. Lett., 106, 205303, (2011).