The imprint of large-scale flows on turbulence

A. Alexakis, P.D. Mininni and A. Pouquet
NCAR, P.O. Box 3000, Boulder, Colorado 80307-3000, U.S.A.
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We investigate the locality of interactions in hydrodynamic turbulence using data from a direct numerical simulation on a grid of 1024³ points; the flow is forced with the Taylor-Green vortex. An inertial range for the energy is obtained in which the flux is constant and the spectrum follows an approximate Kolmogorov law. Nonlinear triadic interactions are dominated by their non-local components, involving widely separated scales. The resulting nonlinear transfer itself is local at each scale but the step in the energy cascade is independent of that scale and directly related to the integral scale of the flow. Interactions with large scales represent 20% of the total energy flux. Possible explanations for the deviation from self-similar models, the link between these findings and intermittency, and their consequences for modeling of turbulent flows are briefly discussed.

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Flows in nature are often in a turbulent state driven by large scale forcing (e.g. novae explosions in the interstellar medium) or by instabilities (e.g. convection in the sun). Such flows involve a huge number of coupled modes leading to great complexity both in their temporal dynamics and in the physical structures that emerge. Many scales are excited, for example from the planetary scale to the kilometer for convective clouds in the atmosphere, and much smaller scales when considering microprocesses such as droplet formation. The question then arises concerning the nature of the interactions between such scales: are they predominantly local, involving only eddies of similar size, or are they as well non-local? It is usually assumed that the dominant mode of interaction is the former, and this hypothesis is classically viewed as underlying the Kolmogorov phenomenology that leads to the prediction of a $E(k) \sim k^{-5/3}$ energy spectrum; such a spectrum has been observed in a variety of contexts although there may be small corrections to this power-law due to the presence in the small scales of strong localized structures, such as vortex filaments.

Several studies have been devoted to assess the degree of locality of nonlinear interactions, either through modeling of turbulent flows, as is the case with rapid distortion theory (RDT) or Large Eddy Simulations (LES), or through the analysis of direct numerical simulations (DNS) of the Navier-Stokes equations (see e.g. [4, 8, 5]), and more recently through rigorous bounds. The spatial resolution in the numerical investigations was moderate, without a clearly defined inertial range and the differentiation between local and non-local interactions was somewhat limited. Thus, a renewed analysis at substantially higher Reynolds numbers in the absence of any modeling is in order; we address this issue by analyzing data stemming from a newly performed DNS on a grid of 1024³ points using periodic boundary conditions.

The governing Navier-Stokes equation for an incompressible velocity field $\mathbf{v}$, with $P$ the pressure, $\mathbf{F}$ a forcing term and $\nu = 3 \times 10^{-4}$ the viscosity, reads:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F} \quad (1)$$

together with $\nabla \cdot \mathbf{v} = 0$. Specifically, we consider the swirling flow resulting from the Taylor-Green vortex [7]:

$$\mathbf{F}_{TG}(k_0) = 2F \begin{bmatrix} \sin(k_0 x) \cos(k_0 y) \cos(k_0 z) \\ -\cos(k_0 x) \sin(k_0 y) \cos(k_0 z) \\ 0 \end{bmatrix}, \quad (2)$$

with $k_0 = 2$. This forcing generates cells that have locally differential rotation and helicity, although its net helicity is zero. The resulting flow models the fluid between two counter-rotating cylinders, and has been used widely to study turbulence, including studies in the context of the generation of magnetic fields through dynamo instability. The Reynolds number based on the integral scale $L \equiv 2\pi \int E(k)k^{-1}dk/E \approx 1.2$ (where $E$ is the total energy), is $R_\infty \equiv U L / \nu \approx 4000$, where $U$ is the r.m.s velocity. The Reynolds number based on the Taylor scale $\lambda \equiv 2\pi (E/\int k^2E(k)dk)^{1/2} \approx 0.24$, is $R_\lambda \approx 800$.

The code uses a dealiased pseudo-spectral method, with maximum wavenumber $k_{max} = 341$ and $k_{max}\eta = 1.15$, where $2\pi\eta = 2\pi (\nu^3/\epsilon)^{1/4}$ is the dissipation scale and $\epsilon$ is the energy injection rate; the flow is sufficiently resolved since $1/\eta$ is within the boundaries of the wavenumbers handled explicitly in the computation.

Details of the flow dynamics will be reported elsewhere; suffice it to say that the flow reproduces classical features of isotropic turbulence: the energy spectrum is well-developed (see Fig. 1) with a constant energy flux for $k \in [5, 20]$ and maximally helical vortex tubes are found, as predicted in [10] and shown in [11, 12]. Finally, the anomalous exponents of longitudinal structure functions are in excellent agreement with previous studies up to order $p = 8$ (see Table I), including analysis without using the extended self-similarity (ESS) hypothesis.

To investigate the interactions between different scales we split the velocity field into spherical shells in Fourier
space of unit width, i.e. \( \mathbf{v} = \sum_K \mathbf{v}_K \) where \( \mathbf{v}_K \) is the filtered velocity field with \( K \leq |k| < K + 1 \) (from now on called shell \( K \)). Usually, octave bands are used to define the shells (i.e. shells of width \( \Delta K \) are used, where \( \Delta K \) is a constant often set to 2). This choice is based on the usual hypothesis that interactions are mostly local and self-similar in Fourier space, i.e. the nonlinear term in eq. (1) couples triads of modes \((k,p,q)\) in Fourier space with \( k \sim p \sim q \). Since we want to verify these hypotheses, we choose to use a linear step for the shells. This election does not imply any loss of generality, and if interactions are indeed local our results should be compatible with results using octave bands.

From equation (1), the rate of energy transfer \( T_3(K,P,Q) \) (a third-order correlator) from energy in shell \( Q \) to energy in shell \( K \) due to the interaction with the velocity field in shell \( P \) is defined as usual [12, 16] as:

\[
T_3(K,P,Q) = -\int \mathbf{v}_K \cdot (\mathbf{v}_P \cdot \nabla)\mathbf{v}_Q d\mathbf{x}^3. \quad (3)
\]

If we sum over the middle wave number \( P \) we obtain the total energy transfer \( T_2(K,Q) \) from shell \( Q \) to shell \( K \):

\[
T_2(K,Q) = \sum_P T_3(K,P,Q) = -\int \mathbf{v}_K \cdot (\mathbf{v} \cdot \nabla)\mathbf{v}_Q d\mathbf{x}^3. \quad (4)
\]

Positive transfer implies that energy is transferred from shell \( Q \) to \( K \), and negative from \( K \) to \( Q \); thus, both \( T_3 \) and \( T_2 \) are antisymmetric in their \((K,Q)\) arguments. \( T_2(K,Q) \) gives information on the shell-to-shell energy transfer between \( K \) and \( Q \), but not about the locality or non-locality of the triadic interactions themselves. The energy flux plotted in Fig. 1 is reobtained from these transfer functions as \( \Pi(k) = -\sum_{K=0}^{k=0} \tau_1(K) = -\sum_{K=0}^{k=0} \sum_{Q} T_2(K,Q) \). Note that the transfer terms defined in Eqs. (3,4) are integrated over all volume in real space. Since in periodic boundary conditions there is no net flux of energy through the walls, this is enough to ensure that contributions due to advection (which do not lead to cascade of energy to smaller scales) cancel out (see e.g. [6]).

Figure 2 shows the energy transfer \( T_2(K,Q) \) plotted as a function of \( K - Q \) for 70 different values of \( Q \) varying from 10 to 80. For each value of \( Q \), the \( x \) axis shows the different \( K \) shells giving or receiving energy from that shell \( Q \). All curves collapse to a single one: the energy in shell \( K \) is received locally from shells with wavenumber \( K - \Delta K \) and deposited mostly in the vicinity of \( K + \Delta K \), with \( \Delta K \sim k_0 \) for all values in the inertial range. In other words, the integral scale of the flow, related to the forcing scale \( k_0^{-1} \), plays a determinant role in the process of energy transfer. As a result, the transfer is not self-similar, and the integral length scale is remembered even deep inside the constant-flux inertial range.

This break down of self-similarity indicates that dominant triadic interactions can be non-local. To examine further this point, we need to investigate individual triadic interactions between Fourier shells by considering the tensorial transfer \( T_3(K,P,Q) \). We will study three values of \( Q \), \((Q = 10, 20, \text{and } 40)\); for each case, \( P \) will run from 1 to 80, and \( K \) from \( Q - 12 \) to \( Q + 12 \).

In figure 3 we show contour levels of the transfer \( T_3(K,P,Q) \) for \( Q = 40 \). This figure represents energy going from a shell \( Q \) to a shell \( K \) through interactions with modes in the shell \( P \). As in Fig. 2 positive transfer means the shell \( K \) receives energy from the shell \( Q \), while negative transfer implies the shell \( K \) gives energy to \( Q \). The strongest interactions occur with \( P \sim k_0 \), and therefore the large scale flow is involved in most of the \( T_2 \) transfer of energy from small scales to smaller scales.

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**TABLE I**: Order \( p \) and anomalous exponents \( \zeta_p \) computed on two snapshots of the velocity field using the interval of scales with constant energy flux; the anomalous exponents \( \zeta_{p,ESS} \) are computed using the ESS hypothesis.

| \( p \) | \( \zeta_p \) | \( \zeta_{p,ESS} \) |
|-------|---------|----------------|
| 1     | 0.366   | 0.364          |
| 2     | 0.704   | 0.699          |
| 3     | 1.005   | 1              |
| 4     | 1.271   | 1.264          |
| 5     | 1.502   | 1.495          |
| 6     | 1.703   | 1.695          |
| 7     | 1.878   | 1.869          |
| 8     | 2.029   | 2.020          |

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**FIG. 1**: Compensated energy spectrum and (inset) absolute value of the energy flux \( \Pi(k) \) in the stationary regime.

**FIG. 2**: Normalized energy transfer from the shell \( Q \) to the shell \( K \) with \( Q \in [10, 80] \). The width of the lobes is independent of \( K \) and all the peaks are at \( K - Q \sim k_0 \).
Note that the individual triadic interactions with $P \sim k_0$ and $K \sim Q \pm k_0$ are two orders of magnitude larger than local triadic interactions.

When $T_3(K, P, Q)$ in Fig. 4 is summed over all values of $P$, the transfer function $T_2(K, Q)$ is recovered. This allows us to define the transfer rate due to interactions with the large scale flow, and due to local interactions, summing $P$ over different ranges. Indeed, to further illustrate the dominance of the large scale flow in the involved interactions, we compare in Fig. 4 the total transfer function $T_2(K, Q)$ with the transfer due to the large scale flow $T_3(K, P = 3, Q)$, and with the transfer due to local interactions in octave bands $T_2^{\text{Loc}}(K, Q) = \sum_{P=Q/2}^{2Q} T_3(K, P, Q)$. The figure indicates that the transfer due to the local interactions $(Q/2 < P < 2Q)$ is smaller than the transfer due to the integral length scale velocity field, and this behavior appears to be stronger as the value of $Q$ is increased. The remaining transfer comes from interactions with $P$-shells with wavenumbers between 1 and $Q/2$ (excluding $P = 3$), which are also non-local in nature. Therefore, as $K$ and $Q$ get larger (as we go further down in the inertial range), the dominant triads $(K, P, Q)$ become more and more elongated, corresponding to more nonlocal interactions. As a result, detailed interactions between triads of modes are nonlocal, while the transfer of energy $T_2(K, Q)$ takes place between neighboring shells: local energy transfer occurs through non-local interactions. These results support previous claims at smaller resolution $[3, 4, 5]$ that a significant role in the cascade of energy in the inertial range is played by the large scale components of the velocity field.

However, when computing the energy flux through a shell $k$, i.e. integrating $T_2(K, Q)$ over all values of $Q$, and $K$ from 0 to $k$, these non-local interactions give $\sim 20\%$ of the total flux, since many more local triads contribute in the global summation. We note that this fraction (20%) is independent of $k$, provided that $k$ is large enough and in the inertial range.

We are left therefore with two puzzles. First, why is the large scale flow more effective (at the level of individual triadic interactions) in “destroying” small size eddies than similar size eddies, when phenomenological arguments in the Kolmogorov spirit suggest otherwise? And secondly, why is the energy spectrum so close to $k^{-5/3}$ in the constant flux region, when just advection by the large scale flow would suggest a shallower spectrum $\sim k^{-1}$? (see e.g. 2). In what follows, we give a brief review of possible answers as well as a simple model that shows how a $k^{-5/3}$ energy spectrum can be obtained by advection and stretching of the small scales just by the large scale flow.

A possible answer to explain the strong non-local triadic interactions is that the Reynolds number in the present simulation is not high enough to observe dominance of local triads, and the decrease in amplitude of the small scale fields due to viscosity makes this interactions (when compared to the large scale flow) smaller.

Another possible answer would be that the wavenumber bands defining the local interactions (i.e. the range of values in $P$ used to define $T_2^{\text{Loc}}$), that were arbitrarily taken here to have a width of $2^n$, could be as wide as $10^n$ as some authors suggest 3. If this is the case, a DNS with an inertial range that spans at least three orders of magnitude in wavenumbers would be required to actually observe strong local interactions.

However, neither of these answers address the second question concerning why a Kolmogorov energy spectrum is observed at moderate values of the Reynolds number. If we look at phenomenological scaling arguments, we see that there is one major assumption that may not be satisfied. Current models assume that the energy is distributed in a hierarchy of vortices of size $L, L/\alpha, L/\alpha^2, \ldots$ (with $\alpha > 1$), with no specific geometry. However, experiments as well as numerical simulations have shown that enstrophy is distributed in vortex tubes, where two distinct length scales can be identified: one is the width of the tube $l$ that is typically small and varies, and one is its length $L$, typically of the order of the integral scale.

FIG. 3: Contour levels of the transfer function $T_3(K, P, Q)$ for $Q = 40$. Solid lines correspond to positive transfer, and dotted lines to negative transfer.
It is not clear therefore when two such structures interact, which length scale is responsible for determining the time scale of the cascade.

From the analysis presented here, a simple model for turbulent flows consistent with several features observed in simulations and experiments can emerge (see below). First, recall that Ref. [12] found that helical vortex tubes capture 99% of the energy, give a $k^{-5/3}$ spectrum, and are responsible for the strong wings in the PDF of velocity gradients. Furthermore, it was shown in [2] that, when decomposing the velocity field in a large scale component $U$ and a small scale one $u$, artificially dropping local interactions in a simulation (an operation akin to RDT) gives enhanced intermittency (in the sense that a stronger departure from linear scaling of anomalous exponents is observed), while when non-local interactions are dropped the intermittency of the flow decreases [17].

The data analyzed in the present paper implies that, at low order of correlators, i.e. when considering the energy flux, the interactions are mostly local. But when going to third-order individual triadic interactions (such as with $T_3$), the non-local components are dominant and involve the integral scale. We note that this is consistent with the fact that departures from a linear scaling by anomalous exponents with the order of structure function is stronger as the order is increased, since it involves more non-local interactions linked to the geometrical structure of vortex tubes. This leads to a model of small-scale interactions involving three small scales that are substantially weakened and gaussian, thus in agreement with the findings in [2] that such $uu$-like terms weaken intermittency as well when included in the full dynamics.

As a result, if we take into account the vortex tube structure of a turbulent flow, the picture of the classical Richardson cascade may change: a possible model to explain the aforementioned results is to take the time scale of the cascade as given by the geometric average of the length scales involved, based on the cubic root of the volume of the vortex tube. If this is the case, the energy dissipation rate of vortex tubes with velocity $u$ due to the large scale flow $U_L$ is given by $\epsilon \sim u^2 \cdot U_L / (l^2 L)^{1/3}$. This implies that, for constant flux, $u_l \sim t^{1/3} \sqrt{\epsilon L^{1/3} / U_L}$; this scaling recovers the Kolmogorov spectrum, although in a different spirit [18]. Note that the spirit of this derivation is close to multifractal models used to explain intermittency corrections [2].

Finally, we would like to point out that the success of models involving as an essential agent of nonlinear transfer the distortion of turbulent eddies by a large-scale flow – as in RDT and its variants [2] or as in the alpha model [10] where the flow is interacting with a smooth velocity field (see also [21]) – may be in part explained by the present findings that confirm that nonlinear triadic interactions are mostly nonlocal and involve the integral scale. Similar results have already been obtained for flows coupled to a magnetic field [10], where the weakening of nonlinear interactions may occur in different fashions, e.g. Alfvenization or force-free fields, and where the second order transfer $T_2$ between velocity and magnetic field in the induction equation is itself non-local.

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