Quantum vs Classical Mechanics: 
role of elementary excitations

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Abstract

Simple theorems relating a quantum mechanical system to the corresponding classical one at equilibrium and connecting the quantum eigenvalues to the frequencies of normal modes oscillations are presented. Corresponding to each quantum eigenfunction, a ‘classical eigenfunction’ is associated. Those belonging to ‘elementary excitations’ play an important role.

1 Introduction

In this paper we will address a general and a most fundamental issue in multi-particle quantum mechanics; the correspondence/contrast between quantum and classical mechanics. Usually such correspondence/contrast is discussed in the “quasi-classical” or “quasi-macroscopic” regime of quantum mechanics in which the expectation values are good representations of the classical variables. These are exemplified in Ehrenfest’s well-known theorem or in the WKB method. Here we ask a rather different question. Suppose a multi-particle quantum mechanical system has a unique ground state and discrete energy spectrum. We naturally expect that the corresponding classical potential has a well-defined minimum,
which gives an equilibrium point. Near the equilibrium, the system is reduced to a collection of harmonic oscillators as many as the degrees of freedom.

We will ask and answer in simple terms the following universal question in multi-particle quantum mechanics:

How can we relate the knowledge of the eigenfunctions and eigenvalues of a multi-particle quantum mechanical system to the properties of the corresponding classical system, in particular, at equilibrium?

In fact we show that for each quantum eigenfunction a corresponding “classical eigenfunction” is defined as the $\hbar \to 0$ limit, see (16). These classical eigenfunctions satisfy simple eigenvalue equations (17). Among them, there are “elementary excitations”, as many as the degrees of freedom, corresponding to each normal mode of the small oscillations at equilibrium (20). They are the generators of the classical eigenfunctions. The quantum eigenfunctions are the $\hbar$-deformations of these classical eigenfunctions. We show that the main part of the generic quantum eigenvalues, which is proportional to Planck’s constant $\hbar$, is given by a linear combination of the (angular) frequencies of small oscillations with integer coefficients (22).

Another motivation of the present research is to provide an analytical proof for the recent results on the quantum vs classical integrability in Calogero-Moser (C-M) systems by Corrigan-Sasaki [1]. C-M systems [2] are classical and quantum integrable multi-particle dynamics based on root systems and the quantum eigenvalues are expressed in terms of roots and weights. In other words, they have ‘integer’ energy spectra. It was shown by direct numerical calculation [1] that most of the classical data, for example, the (angular) frequencies of small oscillations at equilibrium are also ‘integers’. The Propositions 1–3 in section 2 give a simple analytic proof for these interesting observations. Thanks to the integrability (exact solvability), classical and quantum eigenfunctions for C-M systems based on any root system can be constructed explicitly. They will be shown in a subsequent paper [3] in some details, in particular, those for the elementary excitations. These provide excellent explicit examples of the main results of this paper.

The Planck’s constant $\hbar$ is always written explicitly in this article. This paper is organised as follows. In section 2, the formulation of multi-particle quantum mechanics in terms of the prepotential is introduced and the basic results on the quantum and classical eigenfunctions are derived in an elementary way. Section 3 is devoted for summary and comments.
2 Multi-particle Quantum Mechanics

We will discuss a multi-particle quantum mechanical system and its relationship with the corresponding classical ($\hbar \to 0$) dynamics. The dynamical variables are the coordinates \( \{q_j\mid j = 1, \ldots, r\} \) and their canonically conjugate momenta \( \{p_j\mid j = 1, \ldots, r\} \), subject to the Heisenberg commutation relations or the Poisson bracket relations. We will adopt the standard vector notation in \( \mathbb{R}^r \):

\[
q = (q_1, \ldots, q_r), \quad p = (p_1, \ldots, p_r), \quad q^2 \equiv \sum_{j=1}^r q_j^2, \quad p^2 \equiv \sum_{j=1}^r p_j^2, \ldots,
\]

in which \( r \) is the number of particles. In quantum theory, the momentum operator \( p_j \) acts as a differential operator:

\[
p_j = -i\hbar \frac{\partial}{\partial q_j}, \quad j = 1, \ldots, r.
\]

Throughout this paper we discuss the standard Hamiltonian system

\[
H = \frac{1}{2} p^2 + V(q), \quad (1)
\]

in which we have assumed for simplicity that all the particles have the same mass, which is rescaled to unity. Let us start with mild assumptions that the system has a unique and square integrable ground state \( \psi_0 \):

\[
H \psi_0 = 0, \quad \int |\psi_0|^2 \, dq < \infty, \quad E_0 = 0, \quad (2)
\]

and that it has a finite (or an infinite) number of discrete eigenvalues:

\[
H \psi_n = E_n \psi_n, \quad E_n = \mathcal{E}_n \hbar + \mathcal{O}(\hbar^2). \quad (3)
\]

Here we adopt the convention that the ground state energy is vanishing \( E_0 = 0 \), by adjusting the constant part of the potential \( V \), see below.

Since the above time-independent Schrödinger equation is real for a self-adjoint Hamiltonian and that the ground state has no node we express the ground state eigenfunction as

\[
\psi_0(q) = e^{\frac{i}{\hbar} W(q)}, \quad (4)
\]

in which a real function \( W = W(q) \) is called a prepotential \[4]. By simple differentiation of \( W \), we obtain

\[
p_j \psi_0 = -i \frac{\partial W}{\partial q_j} \psi_0, \quad p^2 \psi_0 = - \sum_{j=1}^r \left[ (\frac{\partial W}{\partial q_j})^2 + \hbar \frac{\partial^2 W}{\partial q_j^2} \right] \psi_0, \quad (5)
\]
which results in
\[
\left\{ \frac{1}{2}p^2 + \frac{1}{2}\sum_{j=1}^{r} \left[ \left( \frac{\partial W}{\partial q_j} \right)^2 + \hbar \frac{\partial^2 W}{\partial q_j^2} \right] \right\} \psi_0 = 0. \tag{6}
\]

In other words, we can express the Hamiltonian and the potential in terms of the prepotential \(^1\)
\[
H(W) = \frac{1}{2}p^2 + V(q), \quad V(q) = \frac{1}{2}\sum_{j=1}^{r} \left[ \left( \frac{\partial W}{\partial q_j} \right)^2 + \hbar \frac{\partial^2 W}{\partial q_j^2} \right]. \tag{7}
\]

By removing the obvious \(\hbar\)-dependent terms, let us define a classical potential \(V_C(q)\):
\[
V_C(q) = \frac{1}{2}\sum_{j=1}^{r} \left( \frac{\partial W}{\partial q_j} \right)^2. \tag{8}
\]

Equivalently one could introduce the classical Hamiltonian \(H_C\) as an ‘average’ of the original Hamiltonian \(H(W)\) with the one whose ground state is the inverse of the original ground state\(^2\) \(H(-W)\):
\[
H_C = \frac{1}{2}(H(W) + H(-W)) = \frac{1}{2}p^2 + V_C(q). \tag{9}
\]

Conversely, (7) is a Riccati equation determining the prepotential \(W\) for a given potential \(V\) (or \(V_C\)). Needless to say, it does not matter if the prepotential can be expressed in terms of elementary functions or not.

### 2.1 Equilibrium position and frequencies of small oscillations

Now let us consider the equilibrium point of the classical potential \(V_C\) \(^5\). The classical Hamiltonian \(^9\) has a stationary solution at the classical equilibrium point, \(p = 0, q = \bar{q}\).

There could be, in general, many stationary points of the classical potential \(V_C\), among which we will focus on the ‘maximum’ point \(\bar{q}\) of the ground state wavefunction \(\psi_0\) \(^11\):
\[
\frac{\partial W(\bar{q})}{\partial q_j} = 0, \quad \Rightarrow \quad \frac{\partial V_C(\bar{q})}{\partial q_j} = \sum_{k=1}^{r} \frac{\partial^2 W(\bar{q})}{\partial q_j \partial q_k} \frac{\partial W(\bar{q})}{\partial q_k} = 0, \quad j = 1, \ldots, r. \tag{10}
\]

By expanding the classical potential \(V_C\) around \(\bar{q}\), we obtain
\[
V_C(q) = \frac{1}{2}\sum_{j,k=1}^{r} \frac{\partial^2 V_C(\bar{q})}{\partial q_j \partial q_k} (q - \bar{q})_j (q - \bar{q})_k + \mathcal{O}((q - \bar{q})^3)
\]
\[
= \frac{1}{2}\sum_{j,k,l=1}^{r} \frac{\partial^2 W(\bar{q})}{\partial q_j \partial q_l} \frac{\partial^2 W(\bar{q})}{\partial q_l \partial q_k} (q - \bar{q})_j (q - \bar{q})_k + \mathcal{O}((q - \bar{q})^3), \tag{11}
\]

\(^1\) Similar formulas can be found within the context of supersymmetric quantum mechanics \(^6\). Here we stress that supersymmetry is not necessary.

\(^2\) This is the main ingredient of the well-known Darboux transformation \(^7\).
since \( V_C(\bar{q}) = 0 \). Thus the eigen (angular) frequencies (frequency squared) of small oscillations near the classical equilibrium are given as the eigenvalues of the Hessian matrix \( \tilde{W} (\tilde{V}_C) \):

\[
\tilde{W} = \text{Matrix} \left[ \frac{\partial^2 W(\bar{q})}{\partial q_j \partial q_k} \right], \quad \tilde{V}_C = \text{Matrix} \left[ \frac{\partial^2 V_C(\bar{q})}{\partial q_j \partial q_k} \right] = \tilde{W}^2.
\] (12)

### 2.2 Quantum & Classical Eigenfunctions

Let us express the discrete eigenfunctions in product forms

\[
\psi_n(q) = \phi_n(q) \psi_0(q), \quad n = 0, 1, \ldots, \quad \phi_0 \equiv 1,
\] (13)

in which \( \phi_n \) obeys a simplified equation with the similarity transformed Hamiltonian \( \tilde{H} \):

\[
\tilde{H} \phi_n = E_n \phi_n, \quad (14)
\]

\[
\tilde{H} = e^{-\frac{1}{\hbar}W} He^{\frac{1}{\hbar}W} = -\frac{\hbar^2}{2} \sum_{j=1}^{r} \frac{\partial^2}{\partial q_j^2} - \hbar \sum_{j=1}^{r} \frac{\partial W}{\partial q_j} \frac{\partial}{\partial q_j}. \quad (15)
\]

Here we adjust the normalisation of the eigenfunctions \( \{\phi_n\} \) so that the corresponding “classical” eigenfunctions \( \{\varphi_n\} \) are finite (non-vanishing) in the limit \( \hbar \to 0 \):

\[
\lim_{\hbar \to 0} \phi_n(q) = \varphi_n(q), \quad n = 1, 2, \ldots.
\] (16)

By taking the classical limit \( (\hbar \to 0) \) of \( \{14\} \) and considering \( \{3\}, \{15\} \), we arrive at an ‘eigenvalue equation’ for the “classical” wavefunctions

\[
- \sum_{j=1}^{r} \frac{\partial W}{\partial q_j} \frac{\partial \varphi_n}{\partial q_j} = \mathcal{E}_n \varphi_n, \quad n = 1, 2, \ldots.
\] (17)

Conversely one could define the classical eigenfunctions as solutions of the above eigenvalue equation. In this case the classical eigenfunctions must satisfy certain regularity conditions. Then the quantum eigenfunction \( \phi_n \) could be considered as an \( \hbar \)-deformation of the classical eigenfunction \( \varphi_n \). For the Calogero and Sutherland systems to be discussed in a subsequent paper \( \{3\} \), there is a one-to-one correspondence between the classical and quantum eigenfunctions. For generic multi-particle quantum mechanical systems, the situation is less clear.
2.3 Main Results

The classical eigenfunctions have the following remarkable properties:

**Proposition 1**  
The product of two classical eigenfunctions \((\varphi_n, E_n)\) and \((\varphi_m, E_m)\) is again a classical eigenfunction with the eigenvalue \(E_n + E_m\),

\[
-\sum_{j=1}^{r} \frac{\partial W}{\partial q_j} \frac{\partial (\varphi_n \varphi_m)}{\partial q_j} = (E_n + E_m)\varphi_n \varphi_m.
\]  

**Proposition 2**  
The classical eigenfunctions vanish at the equilibrium \(\bar{q}\)

\[
\varphi_n(\bar{q}) = 0, \quad n = 1, 2, \ldots.
\]  

**Proposition 3**  
The derivatives of a classical eigenfunction at the equilibrium \(\bar{q}\) form an eigenvector of the Hessian matrix \(\tilde{W}\), iff

\[
-\tilde{W} \cdot \nabla \varphi_n|_{\bar{q}} = \mathcal{E}_n \nabla \varphi_n|_{\bar{q}}, \quad n = 1, 2, \ldots,
\]

or

\[
-\sum_{j=1}^{r} \frac{\partial^2 W(\bar{q})}{\partial q_k \partial q_j} \frac{\partial \varphi_n(\bar{q})}{\partial q_j} = \mathcal{E}_n \frac{\partial \varphi_n(\bar{q})}{\partial q_k}, \quad n = 1, 2, \ldots.
\]

Obviously the Hessian matrix \(\tilde{W}\) has at most \(r\) different eigenvalues and eigenvectors. The classical eigenfunctions \(\{(\varphi_j, \mathcal{E}_j)\}, \quad j = 1, \ldots, r\) for which \(\nabla \varphi_j|_{\bar{q}} \neq 0\) will be called “elementary excitations”. At equilibrium, each corresponds to the normal coordinate of the small oscillations with the eigen (angular) frequency \(\mathcal{E}_j\). That is, the ‘main part’ \(\mathcal{E}_n\) \((i.e, \mathcal{O}(\hbar)\) part) of the quantum energy eigenvalue \(E_n\) is given by the classic eigenfrequencies of the normal mode oscillations at the classical equilibrium. The elementary excitations are the generators of the classical eigenfunctions. In other words, any classical eigenfunction can be expressed as

\[
\varphi_{1}^{n_1} \cdots \varphi_{r}^{n_r}, \quad \mathcal{E} = n_1 \mathcal{E}_1 + \cdots + n_r \mathcal{E}_r, \quad n_j \in \mathbb{Z}_+,
\]

or a linear combination thereof with the same eigenvalue \(\mathcal{E}\). The above type of classical eigenfunctions are obviously non-elementary and they have zero gradient at equilibrium, for example, \(\nabla (\varphi_j \varphi_k)|_{\bar{q}} = 0\).
3 Summary and comments

We have shown that for any multi-particle quantum mechanical system,

the main part i.e. the $\mathcal{O}(\hbar)$ part, of the quantum energy eigenvalue is determined
solely by the corresponding classical data, i.e. the eigenfrequencies of the normal
mode oscillations at the classical equilibrium.

This is a very powerful result, since for most multi-particle systems the quantum eigenvalues
are hard to evaluate, whereas the eigenfrequencies of the normal mode oscillations at
classical equilibrium are easily calculated. The Calogero-Moser (C-M) systems based on
any root system [2] provide ideal explicit examples in which the above Propositions 1–3
are thoroughly verified in Corrigan-Sasaki paper [1]. Thanks to the exact solvability, all
the quantum eigenvalues of the C-M systems are known [4, 5] and they are compared with
the eigenfrequencies of the normal mode oscillations at classical equilibrium evaluated in
[1]. The classical and quantum eigenfunctions for the elementary excitations are reported
in some detail in Loris-Sasaki paper [3]. It should be mentioned that Perelomov’s recent
work [6] asserts essentially our Proposition 3 for the special cases of the quantum-classical
eigenvalue correspondence of the Sutherland systems.

Let us present a few elementary examples of one degree of freedom quantum mechanics
to illustrate the prepotential method and the main results explicitly.

**Harmonic oscillator** The ground state wavefunction, the prepotential, the quantum
and classical potential, etc are ($\omega > 0$):

\[
\psi_0 = e^{-\omega q^2/2\hbar}, \quad W = -\omega q^2/2, \quad V(q) = \omega^2 q^2/2 - \omega \hbar/2, \quad V_C(q) = \omega^2 q^2/2. \tag{23}
\]

The quantum eigenvalues and eigenfunctions are:

\[
E_n = n\hbar\omega, \quad \mathcal{E}_n = n\omega, \quad \phi_n(q) = H_n(\sqrt{\omega/\hbar} q), \tag{24}
\]

in which $H_n$ is the Hermite polynomial. The classical equilibrium point is the origin $\bar{q} = 0$
and the classical eigenfunction and the Hessian $\tilde{W}$ are:

\[
\varphi_n(q) = \lim_{\hbar \to 0} \hbar^{n/2} H_n(\sqrt{\omega/\hbar} q) = \omega^{n/2} q^n, \quad -\tilde{W} = \omega. \tag{25}
\]

It is trivial to check (17) and the Propositions 1–3.

**‘Soliton’ potential** Let us consider a simple Pöschl-Teller [8] potential

\[
\psi_0 = 1/(\cosh q)^{g/\hbar}, \quad W = -g \log \cosh q.
\]
\[ V(q) = -\frac{g(g + \hbar)}{2 \cosh^2 q} + \frac{g^2}{2}, \quad V_C(q) = -\frac{g^2}{2 \cosh^2 q} + \frac{g^2}{2}. \]  \hspace{1cm} (26)

The quantum eigenvalues and eigenfunctions are:

\[ E_n = gn - n^2 \hbar^2 / 2, \quad \mathcal{E}_n = gn, \quad \phi_n(q) = (\cosh q)^n P_n^{(\alpha, \alpha)}(\tanh q), \quad \alpha \equiv g/\hbar - n > 0, \] \hspace{1cm} (27)

in which \( P_n^{(\alpha, \beta)}(x) \) is the Jacobi polynomial of degree \( n \). The classical equilibrium point is the origin \( \bar{q} = 0 \) and the classical eigenfunction and the Hessian \( \widetilde{W} \) (12) are:

\[ \varphi_n(q) = n! \lim_{\hbar \to 0} \hbar^n \phi_n(q) = g^n (\sinh q)^n, \quad -\widetilde{W} = g. \] \hspace{1cm} (28)

It is easy to verify (17) and the Propositions 1–3. For integer \( g/\hbar \), \( V(q) \) (26) (without the constant term) is the reflectionless potential corresponding to a KdV soliton. In both examples, \( \varphi_1 \) is the elementary excitation.

Throughout this Letter we have assumed that the prepotential \( W \) is independent of the Planck’s constant \( \hbar \), for simplicity of the presentation. The main content of this Letter is valid even if \( W \) depends on \( \hbar \), so long as \( \lim_{\hbar \to 0} W = W_0 \) is well-defined. A celebrated example that \( \lim_{\hbar \to 0} W \) diverges is the hydrogen atom, for which the classical equilibrium does not exist. In this case the quantum-classical correspondence does not make sense and the present formulation does not apply.

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