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Аннотация

Подгруппа $A$ группы $G$ называется tcc-подгруппой в $G$, если существует подгруппа $T$ группы $G$ такая, что $G = AT$ и для любого $X \subseteq A$ и $Y \subseteq T$ существует элемент $u \in \langle X, Y \rangle$ такой, что $XY^u \leq G$. Запись $H \leq G$ означает, что $H$ является подгруппой группы $G$. В этой статье мы исследуем группу $G = AB$ при условии, что $A$ и $B$ являются tcc-подгруппами в $G$. Доказано, что такая группа $G$ принадлежит $\mathfrak{F}$, если подгруппы $A$ и $B$ принадлежат $\mathfrak{F}$, где $\mathfrak{F}$ — насыщенная формация такой, что $\mathfrak{U} \subseteq \mathfrak{F}$. Здесь $\mathfrak{U}$ — формация всех сверхразрешимых групп.

Ключевые слова: сверхразрешимая группа, толькоперестановочное произведение, насыщенная формация, tcc-перестановочное произведение, tcc-подгруппа.

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Abstract

A subgroup $A$ of a group $G$ is called tcc-subgroup in $G$, if there is a subgroup $T$ of $G$ such that $G = AT$ and for any $X \leq A$ and $Y \leq T$ there exists an element $u \in \langle X, Y \rangle$ such that $XY^u \leq G$. The notation $H \leq G$ means that $H$ is a subgroup of a group $G$. In this paper we consider a group $G = AB$ such that $A$ and $B$ are tcc-subgroups in $G$. We prove that $G$ belongs to $\mathfrak{F}$, when $A$ and $B$ belong to $\mathfrak{F}$ and $\mathfrak{F}$ is a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. Here $\mathfrak{U}$ is the formation of all supersolvable groups.

Keywords: supersolvable group, totally permutable product, saturated formation, tcc-permutable product, tcc-subgroup.

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1. Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. We use the standard notations and terminology of [1, 2]. The notation $H \leq G$ means that $H$ is a subgroup of a group $G$.

It is well known that the product of two normal nilpotent subgroups of a group $G$ is nilpotent. However, the product of two normal supersolvable subgroups of a group $G$ is not necessarily supersolvable. It seems then natural to consider factorized groups in which certain subgroups of the corresponding factors permute, in order to obtain new criteria of supersolubility. A starting point of this research can be located at M. Asaad and A. Shaalan’s paper [3]. In particular, they proved the supersolubility of a group $G = AB$ such that the subgroups $A$ and $B$ are totally permutable and supersolvable, see [3, Theorem 3.1]. Here the subgroups $A$ and $B$ of a group $G$ are totally permutable if every subgroup of $A$ is permutable with every subgroup of $B$. In [4] Maier showed that this statement is also true for the saturated formations containing the formation $\mathfrak{U}$ of all supersolvable groups. Ballester-Bolinches and Perez-Ramos in [5] extend Maier’s result to non-saturated formations which contain all supersolvable groups. This direction have since been subject of an in-depth study of many authors, see, for example, [6], [7], [8]. The monograph [9, chapters 4–5] contains other detailed information on the structure of groups, which are totally or mutually permutable products of two subgroups.

The following concept was introduced in [8].

Definition . A subgroup $A$ of a group $G$ is called tcc-subgroup in $G$, if it satisfies the following conditions:

1) there is a subgroup $T$ of $G$ such that $G = AT$;
2) for any $X \leq A$ and $Y \leq T$ there exists an element $u \in \langle X, Y \rangle$ such that $XY^u \leq G$.

We say that the subgroup $T$ is a tcc-supplement to $A$ in $G$.

Now, we can state the main result in [10], which is the following:

Theorem 1. ([10, Theorem A]) Let $G = AB$, where $A$ and $B$ are tcc-subgroups in $G$. Let $\mathfrak{F}$ be a saturated formation of soluble groups such that $\mathfrak{U} \subseteq \mathfrak{F}$. Suppose that $A$ and $B$ belong to $\mathfrak{F}$. Then $G$ belongs to $\mathfrak{F}$.

In this article we show that the hypothesis of solubility in Theorem 1 can be removed.

Theorem 2. Let $G = AB$, where $A$ and $B$ are tcc-subgroups in $G$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. Suppose that $A$ and $B$ belong to $\mathfrak{F}$. Then $G$ belongs to $\mathfrak{F}$.
2. Preliminaries

In this section we give some definitions and basic results which are essential in the sequel.

A group whose chief factors have prime orders is called supersoluble. If \( H \trianglelefteq G \) and \( H \neq G \), we write \( H < G \). The notation \( H \trianglelefteq G \) means that \( H \) is a normal subgroup of \( G \). Denote by \( Z(G) \), \( F(G) \) and \( \Phi(G) \) the centre, Fitting and Frattini subgroups of \( G \) respectively, and by \( O_p(G) \) the greatest normal \( p \)-subgroup of \( G \). Denote by \( \pi(G) \) the set of all prime divisors of order of \( G \). The semidirect product of a normal subgroup \( A \) and a subgroup \( B \) is written as follows: \( A \rtimes B \).

The monographs [11], [12] contain the necessary information of the theory of formations. A formation \( \mathfrak{F} \) is said to be saturated if \( G/\Phi(G) \in \mathfrak{F} \) implies \( G \in \mathfrak{F} \). In view of Theorems 3.2 and 4.6 in [12, IV], for any non-empty saturated formation \( \mathfrak{F} \) there exists a formation function \( f \) (that is, any function of the form \( f : \mathbb{P} \to \{ \text{formations} \} \)) such that \( \mathfrak{F} = LF(f) := \{ G \mid G/F_p(G) \in f(p) \} \) for all primes \( p \) dividing \( |G| \). Here \( F_p(G) = O_p'(G) \) is the greatest normal \( p \)-nilpotent subgroup of \( G \) [12, IV, Section 7]. Such a function is called a local definition of \( \mathfrak{F} \). Moreover, in view of Proposition 5.4 in [12, III], every non-empty saturated formation \( \mathfrak{F} \) has a unique local definition \( f \) (called the canonical local definition of \( \mathfrak{F} \)) such that \( f(p) = \mathfrak{N}_p f(p) \subseteq \mathfrak{F} \) for all primes \( p \), where \( \mathfrak{N}_p f(p) = \emptyset \) if \( f(p) = \emptyset \) and \( \mathfrak{N}_p f(p) \) is the class of all groups \( A \) with \( A^{f(p)} \leq O_p(A) \) whenever \( f(p) \neq \emptyset \).

If \( H \) is a subgroup of \( G \), then \( H_G = \bigcap_{x \in G} H^x \) is called the core of \( H \) in \( G \). If a group \( G \) contains a maximal subgroup \( M \) with trivial core, then \( G \) is said to be primitive and \( M \) is its primitivator. A simple check proves the following lemma.

**Lemma 1.** Let \( \mathfrak{F} \) be a saturated formation and \( G \) be a group. Assume that \( G \notin \mathfrak{F} \), but \( G/N \in \mathfrak{F} \) for all non-trivial normal subgroups \( N \) of \( G \). Then \( G \) is a primitive group.

Recall that the product \( G = AB \) is said to be tcc-permutable [7], if for any \( X \leq A \) and \( Y \leq B \) there exists an element \( u \in \langle X, Y \rangle \) such that \( XY^u \leq G \). The subgroups \( A \) and \( B \) in this product are called tcc-permutable.

**Lemma 2.** ([7, Theorem 1, Proposition 1-2]) Let \( G = AB \) be the tcc-permutable product of subgroups \( A \) and \( B \) and \( N \) be a minimal normal subgroup of \( G \). Then the following statements hold:

1. \( \{ A \cap N, B \cap N \} \subseteq \{ 1, N \} \);
2. if \( N \leq A \cap B \) or \( N \cap A = N \cap B = 1 \), then \( |N| = p \), where \( p \) is a prime.

**Lemma 3.** ([13, Theorem 4]) Let \( G = AB \) be the tcc-permutable product of subgroups \( A \) and \( B \). Then \( [A, B] \leq F(G) \).

**Lemma 4.** ([8, Lemma 3.1]) Let \( A \) be a tcc-subgroup in \( G \) and \( Y \) be a tcc-supplement to \( A \) in \( G \). Then the following statements hold:

1. \( A \) is a tcc-subgroup in \( H \) for any subgroup \( H \) of \( G \) such that \( A \leq H \);
2. \( AN/N \) is a tcc-subgroup in \( G/N \) for any \( N \leq G \);
3. for every \( A_1 \leq A \) and \( X \leq Y \) there exists an element \( y \in Y \) such that \( A_1 X^y \leq G \). In particular, \( A_1 M \leq G \) for some maximal subgroup \( M \) of \( Y \) and \( A_1 H \leq G \) for some Hall \( \pi \)-subgroup \( H \) of soluble \( Y \) and any \( \pi \subseteq \pi(G) \);
4. \( A_1 K \leq G \) for every subnormal subgroup \( K \) of \( Y \) and for every \( A_1 \leq A \);
5. if \( T \leq G \) such that \( T \leq A \) and \( T \cap Y = 1 \), then \( T_1 \leq G \) for every \( T_1 \leq A \) such that \( T_1 \leq T \);
6. if \( T \leq G \) such that \( T \cap A = 1 \) and \( T \leq Y \), then \( A_1 \leq N_G(T_1) \) for every \( T_1 \leq T \) and for every \( A_1 \leq A \).

**Lemma 5.** Let \( G \) be a group and \( N \) a unique minimal normal subgroup of \( G \). If \( G \) has a proper tcc-subgroup \( A \) such that \( A \neq 1 \), then \( N \) is abelian.
Proof. Since $A$ is a tcc-subgroup, it follows that $G = AY$, $A$ and $Y$ are tcc-permutable. If $[A, Y] = 1$, then $A \leq C_G(Y)$. It is clear $A$ and $Y$ are normal in $G$. Thus $N \leq A \cap Y$. By Lemma 2, $|N| = p$ and $N$ is abelian. Therefore $[A, Y] \neq 1$ and $N \leq [A, Y] \leq F(G) \neq 1$ by Lemma 3. Hence $N$ is abelian. □

Lemma 6. Let $A \neq 1$ be a proper tcc-subgroup in a primitive group $G$ and $Y$ be a tcc-supplement to $A$ in $G$. Suppose that $N$ is a unique minimal normal subgroup of $G$. If $N \cap A = 1$ and $N \leq Y$, then $A$ is a cyclic group of order dividing $p - 1$.

Proof. Since $N \cap A = 1$ and $N \leq Y$, by Lemma 4(6), $A \leq N_G(K)$ for any $K \leq N$. By Lemma 5, $N$ is an elementary abelian group. We fix an element $a \in A$. If $x \in N$, then $x^a \in \langle x \rangle$, since $A \leq N_G(\langle x \rangle)$ by hypothesis. Hence $x^a = x^{ma_x}$, where $m_x$ is a positive integer and $1 \leq m_x \leq p$. If $y \in N \setminus \{x\}$, then

$$
(xy)^a = (xy)^{ma_y} = x^{ma_y}y^{ma_y}, (xy)^a = x^a y^a = x^{ma_y},
$$

$$
x^{ma_y}y^{ma_y} = x^{ma_y}y^{ma_y}, x^{ma_y} = y^{ma_y} = 1, m_{xy} = m_x = m_y.
$$

Therefore we can assume that $x^a = x^{na}$ for all $x \in N$, where $1 \leq n_a \leq p$ and $n_a$ is a positive integer. Hence we have $A$ induces a power automorphism group on $N$. By the Fundamental Homomorphism Theorem, $A/C_A(N)$ is isomorphic to a subgroup of $P(N)$, where $P(N)$ is the power automorphism group of $N$. Since $N$ is abelian, it follows that $C_G(N) = N$ by [2, Theorem 4.41] and $C_A(N) = 1$. On the other hand, $P(N)$ is a cyclic group of order $p - 1$. Really $P(N)$ is a group of scalar matrices over the field $\mathbb{P}$ consisting of $p$ elements. Hence $P(N)$ is isomorphic to the multiplicative group $\mathbb{P}^*$ of $\mathbb{P}$ and besides, $\mathbb{P}^*$ is a cyclic group of order $p - 1$. Therefore $A$ is a cyclic group of order dividing $p - 1$. □

Lemma 7. Let $\mathcal{F}$ be a formation, $G$ group, $A$ and $B$ subgroups of $G$ such that $A$ and $B$ belong to $\mathcal{F}$. If $[A, B] = 1$, then $AB \in \mathcal{F}$.

Proof. Since

$$
[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle = 1,
$$

it follows that $ab = ba$ for all $a \in A, b \in B$. Let

$$
A \times B = \{ (a, b) \mid a \in A, b \in B \},
$$

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2), \ \forall a_1, a_2 \in A, b_1, b_2 \in B -$$

be the external direct product of groups $A$ and $B$. Since $A \in \mathcal{F}, B \in \mathcal{F}$ and $\mathcal{F}$ is a formation, we have $A \times B \in \mathcal{F}$. Let $\varphi : A \times B \rightarrow AB$ be a function with $\varphi((a, b)) = ab$. It is clear that $\varphi$ is a surjection. Because

$$
\varphi((a_1, b_1)(a_2, b_2)) = \varphi((a_1a_2, b_1b_2)) = a_1a_2b_1b_2 = a_1b_1a_2b_2 = \varphi((a_1, b_1))\varphi((a_2, b_2)),
$$

it follows that $\varphi$ is an epimorphism. The core $\text{Ker} \varphi$ contains all elements $(a, b)$ such that $ab = 1$. In this case $a = b^{-1} \in A \cap B \leq Z(G)$. By the Fundamental Homomorphism Theorem,

$$
A \times B/\text{Ker} \varphi \cong AB.
$$

Since $A \times B \in \mathcal{F}$ and $\mathcal{F}$ is a formation, $A \times B/\text{Ker} \varphi \in \mathcal{F}$. Hence $AB \in \mathcal{F}$. □

Lemma 8. ([14, Lemma 2.16]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ be a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. If $E$ is cyclic, then $G \in \mathcal{F}$.
3. Proof of Theorem 2

Assume that the claim is false and let $G$ be a minimal counterexample. Suppose that $G$ is simple. By Lemma 3, $A$ and $B$ are normal in $G$, a contradiction. Hence let $K$ be an arbitrary non-trivial normal subgroup of $G$. The quotients $AK/K \simeq A/A \cap K$ and $BK/K \simeq B/B \cap K$ are tcc-subgroups in $G/K$ by Lemma 4(2), $AK/K \simeq A/A \cap K \in \mathfrak{F}$ and $BK/K \simeq B/B \cap K \in \mathfrak{F}$, because $\mathfrak{F}$ is a formation. Hence the quotient $G/K = (AK/K)(BK/K) \in \mathfrak{F}$ by induction.

Since $\mathfrak{F}$ is a saturated formation, it follows that $\Phi(G) = 1$, $G$ has a unique minimal normal subgroup $N$ and $G$ is primitive by Lemma 1. By Lemma 5, $N$ is abelian and $F(G) = N = C_G(N) = O_p(G)$, $G = N \rtimes M$, where $|N| = p^n$ and $M$ is a primitivator.

By Lemma 2, $G$ is either $|N| = p$, or $N \leq A$ and $N \cap Y = 1$, or $N \cap A = 1$ and $N \leq Y$, where $Y$ is a tcc-supplement to $A$ in $G$. In the first case, by Lemma 8, $G \in \mathfrak{F}$. Suppose that $N \leq A$ and $N \cap Y = 1$. Since $Y$ is a tcc-subgroup in $G$, it follows that by Lemma 6, $Y$ is a cyclic group of order dividing $p - 1$. Then $Y \leq g(p)$, where $g$ is the canonical local definition of $\mathfrak{F}$. Since $\mathfrak{F} \subseteq \mathfrak{F}$, we have by [12, Proposition IV.3.11], $g(p) \subseteq f(p)$, where $f$ is the canonical local definition of $\mathfrak{F}$. Hence $Y \leq f(p)$.

Let $Q$ be a Sylow $q$-subgroup of $Y$. It is obvious that $Q \leq G_q$ for some Sylow subgroup $G_q$ of $G$. Then we can always choose a primitivator $H$ of $G$ such that $Q \leq H$. Really $G_q = M_q^g$ and $G_q \leq M^g = H$ for some $g \in G$ and some Sylow $q$-subgroup $M_q$ of $M$. It is clear that $H$ is a maximal subgroup of $G$. If $N \leq H$, then $G = NM = NM^q = NH = H$, a contradiction. Hence $NH = G$.

Because $N$ is abelian, then $N \cap H = 1$ and $H$ is a primitivator.

Since $A = A \cap G = A \cap NH = N(A \cap H)$, we have

$$G = AY = N(A \cap H)Y.$$  

Prove that $(A \cap H)Y$ is a primitivator of $G$. Since

$$[A \cap H, Q] \leq [A, Y] = F(G) = N$$

by Lemma 3 and $[A \cap H, Q] \leq H$, it follows that $[A \cap H, Q] \leq H \cap N = 1$. Therefore $A \cap H \leq C_G(Q) = T$. Besides $Y \leq T$. Then

$$T = T \cap G = T \cap N(A \cap H)Y = (A \cap H)Y(N \cap T).$$

It is obvious that $N \cap T$ is normal in $T$ and hence $N \cap T$ is normal in $G = N(A \cap H)Y = NT$, since $N$ is abelian. Thus is either $N \leq T$, or $N \cap T = 1$. In the first case, $T = G$ and $Q \leq Z(G)$, a contradiction. Otherwise, $T = (A \cap H)Y$ and $G = N \rtimes T$. Hence $T = (A \cap H)Y$ is a primitivator of $G$. Thus we can always choose a primitivator $M_1$ of $G$ such that $G = N \rtimes M_1$, $Y \leq M_1$ and $M_1 = (A \cap M_1)Y$.

Because $A \in \mathfrak{F}$, it follows that $A/F_p(A) \in f(p)$. Since $N = C_G(N)$ and $N \leq A$, we have that $N \leq F_p(A) = F(A)$. Let $N_1$ is a minimal normal subgroup of $A$ such that $N_1 \leq N$. Then $F(A) \leq C_A(N_1)$ by [2, Lemma 4.21]. Since $A$ is a tcc-subgroup in $G$, it follows that by Lemma 4(5), $N_1$ is normal in $G$. Hence $N = N_1$ and $C_A(N_1) = C_A(N) = N$. Then $F_p(A) = N$ and $A \cap N_1 \simeq A/N \leq f(p)$.

Since $f(p)$ is a formation, $A \cap M_1 \leq f(p)$, $Y \leq f(p)$ and $[A \cap M_1, Y] = 1$, it follows that $M_1 \leq f(p)$ by Lemma 7. Because $N \in \mathfrak{F}$, we have $G \in \mathfrak{F}f(p) = f(p) \subseteq \mathfrak{F}$. So, we assume that $N \cap A = 1$ and $N \leq Y$. Similarly, we can show that $N \cap B = 1$ and $N \leq X$, where $X$ is a tcc-supplement to $B$ in $G$. By Lemma 6, $A$ and $B$ are cyclic. Hence $G$ is supersoluble and therefore $G \in \mathfrak{F}$. The theorem is proved.
4. Conclusion

Clear that by condition 2 of Definition 1, \( G = AT \) is the tcc-permutable product of the subgroups \( A \) and \( T \). If \( G = AB \) is the tcc-permutable product of subgroups \( A \) and \( B \), then the subgroups \( A \) and \( B \) are tcc-subgroups in \( G \). The converse is false.

**Example 1.** The dihedral group \( G = < a > \rtimes < c > \), \( |a| = 12, |c| = 2 \) ([15], IdGroup=24,6) is the product of tcc-subgroups \( A = < a^3c > \) of order 2 and \( B = < a^{10} > \rtimes < c > \) of order 12. But \( A \) and \( B \) are not tcc-permutable. Indeed, there are the subgroups \( X = A \) and \( Y = < c > \) of \( A \) and \( B \) respectively such that doesn’t exist \( u \in \langle X, Y \rangle = < a^3 > \rtimes < c > \) such that \( XY^u \leq G \).

Hence we have the following result.

**Corollary 1.** 1. Let \( A \) and \( B \) be tcc-subgroups in \( G \) and \( G = AB \). If \( A \) and \( B \) are supersoluble, then \( G \) is supersoluble, ([8, Theorem 4.1])

2. Let \( \mathfrak{F} \) be a saturated formation containing \( \mathfrak{U} \). Let the group \( G = HK \) be the tcc-permutable product of subgroups \( H \) and \( K \). If \( H \in \mathfrak{F} \) and \( K \in \mathfrak{F} \), then \( G \in \mathfrak{F} \), ([13, Theorem 5]).

3. Suppose that \( A \) and \( B \) are supersoluble subgroups of \( G \) and \( G = AB \). Suppose further that \( A \) and \( B \) are totally permutable. Then \( G \) is supersoluble, ([3, Theorem 3.1]).

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