Unitary Matrix Models with a topological term and
discrete time Toda equation

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June 14, 2021

Abstract

We study the full unitary matrix models. Introducing a new term $l \log U$, $l$ plays the role of the discrete time. On the other hand, the full unitary matrix model contains a topological term. In the continuous limit it gives rise to a phase transition at $\theta = \pi$. The ground state is characterize by the discrete time $l$. The discrete time $l$ plays like the instanton number.
1 Introduction

Models of the symmetric unitary matrix model are solved exactly in the double scaling limit, using orthogonal polynomials on a circle. The partition function is the form $\int dU \exp\{-\frac{N}{\lambda} \text{tr} V(U)\}$, where $U$ is an $N \times N$ unitary matrix and $\text{tr} V(U)$ is some well defined function of $U$. When $V(U)$ is the self adjoint we call the model symmetric. The simplest case is given by $V(U) = U + U^\dagger$. This unitary models has been studied in connection with the large-$N$ approximation to QCD in two dimensions. The full non-reduced unitary model was first discussed in \cite{4}. The simplest potential is divided into the symmetric and anti-symmetric part. The symmetric part is the usual Wilson action. The anti-symmetric part becomes the topological term. In the continuous limit we can reconstruct the topological information encoded in the theta term. We introduce a new term $l \log U$. The full unitary matrix model can be embedded in the two-dimensional Toda lattice hierarchy. Adding the new term to the full unitary matrix model, $l$ becomes the discrete time and the partition function satisfies the discrete time Toda equation. This equation is sometimes called as “Hirota equation”. Discrete time integrable system is currently under extensive study. The partition function for the Hermite matrix model with the new term also satisfy the discrete time Toda equation. In the full unitary matrix model the discrete time $l$ couples to the theta term and characterizes the ground state. The partition function for the $SU(N)$ case is the sum of the partition function for the $U(N)$ case about all the discrete time $l$. Thus the theta term would have been zero for $SU(N)$ case.

This letter is organized as follows. In the section 2 we introduce a new term to the full unitary matrix model. We show that the partition function of our model satisfy the discrete time Toda equation. In the section 3 we see the role of the discrete time $l$ in the Painlevé equation. Coupling the Toda equation and the string equation, we obtain the Painlevé V with $\delta = 0$. This becomes the Painlevé III. In the section 4 we consider the physical meaning of our model.
We derive the continuum limit of the action which will turn out to be the heat kernel with the theta term. We interpret the continuum partition function in terms of the underlying topology and show the presence of a phase transition of at $\theta = \pi$. The ground state is indicated by the discrete time $l$. The latter half of this section we calculate the partition function for $SU(N)$ and study an effect of the theta term. The last section is devoted to concluding remarks.

2 The partition function on the lattice

We consider the partition function of the full unitary matrix model with a discrete time. We calculate explicitly the determinant form of the partition function.

We consider the unitary matrix model

\[ Z_{N,l}^U = \int dU \exp\left(-\frac{N}{g^2} V(U)\right), \]  

(2.1)

where $V(U)$ is a potential

\[ V(U) = t_1 U + t_{-1}U^\dagger - l \log U. \]  

(2.2)

$U$ is the gauge group $U(N)$. Here we consider the case

\[ \frac{t_1}{t_{-1}} = O(1). \]  

(2.3)

We parameterize $t_1$ and $t_{-1}$ by $\epsilon$:

\[ t_1 = -e^\epsilon, \quad t_{-1} = -e^{-\epsilon}. \]  

(2.4)

The measure $dU$ may be written as

\[ dU = \prod_{m}^{N} \frac{d\alpha_m}{2\pi} \Delta(\alpha)\bar{\Delta}(\alpha). \]  

(2.5)

Here the eigenvalues of $U$ are $\{\exp(i\alpha_1), \exp(i\alpha_2), \cdots, \exp(i\alpha_N)\}$ and $\Delta\bar{\Delta}$ is the Jacobian for the change of variables,

\[ \Delta(\alpha) = \det_{j,k} e^{i\alpha_j(N-k)}, \]

\[ \bar{\Delta}(\alpha) = \det_{j,k} e^{-i\alpha_j(N-k)}. \]  

(2.6)
In terms of these variables the potential \( V(U) \) is
\[
V(\alpha) = -\sum_{j=1}^{N} (\cosh \epsilon \cos \alpha_j + i \sinh \epsilon \sin \alpha_j + il\alpha_j). \tag{2.7}
\]

Putting this all together and introducing
\[
f(\alpha) = \cosh \epsilon \cos \alpha + i \sinh \epsilon \sin \alpha + il\alpha, \tag{2.8}
\]
we obtain
\[
Z_{U}^{N,l}(N/g^2) = \text{const.} \prod_{m=1}^{N} \int_{-\pi}^{\pi} \frac{d\alpha_m}{2\pi} e^{f(\alpha_m)}|\Delta(\alpha)|^2,
\]
\[
= \text{const.} \prod_{m=1}^{N} \int_{-\pi}^{\pi} \frac{d\alpha_m}{2\pi} e^{f(\alpha_m)} \left[ \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{j} \frac{N}{g^2} e^{i\alpha_m(N-\sigma_j)} \right]
\]
\[
\times \left[ \sum_{\eta \in S_N} (-1)^{\eta} \prod_{j} \frac{N}{g^2} e^{i\alpha_m(N-\eta_j)} \right],
\]
\[
= \text{const.} \sum_{\mu \in S_N} (-1)^{\mu} \sum_{\sigma \in S_N} \prod_{m=1}^{N} \int_{-\pi}^{\pi} \frac{d\alpha_m}{2\pi} \exp[f(\alpha_m) - i\alpha_m(\sigma(m) - \mu\sigma(m))],
\]
\[
= \text{const.} N! \det_{jk} \left[ \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{f(\alpha)-i\alpha(k-j)} \right], \tag{2.9}
\]
where the summations are on all the permutations of the \( N \) elements. We can calculate the \((j, k)\) element of the determinant as
\[
\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \exp\left[\frac{N}{g^2} \frac{\cosh \epsilon \cos \alpha + i \sinh \epsilon \sin \alpha}{-i\alpha} \right] = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \exp\left[\frac{N}{g^2} \cos(\alpha - i\epsilon) - i\alpha \right] = e^{n\alpha} I_m(N/g^2), \tag{2.10}
\]
where \( m = l - j + k \). Here \( I_m \) is the modified Bessel function of order \( m \).

Substituting this into (2.9) we can obtain the partition function
\[
Z_{N,l}^{U} = \text{const.} N! \det_{jk} e^{(l-j+k)} I_{l-j+k}(N/g^2). \tag{2.11}
\]

Here we introduce symbols \( D \), \( D_{a,b} \) and \( D_{b,c} \), which are the determinants of the \((N+1) \times (N+1)\), \( N \times N \) and \((N-1) \times (N-1)\) matrices respectively with
the respective definitions. We define \( D = Z_{N+1,l}^U \). \( D_b \) is same as \( D \) except that the \( a \)-th row and \( b \)-th column are removed from it. \( D_{b,a}^o \) is same as \( D \) except that the \( a \)-th and \( c \)-th rows, and \( b \)-th and \( d \)-th columns are removed from it. Then the Jacobi formula

\[
D_1^1 D_{N+1}^{N+1} - D_1^{N+1} \cdot D_{N+1}^1 = D \cdot D_{1,N+1}^{1,N+1},
\]

becomes the recurrent relations

\[
(Z_{N,l})^2 - Z_{N,l+1} \cdot Z_{N,l-1} = Z_{N+1,l} \cdot Z_{N-1,l}
\]

(2.13)

This is the discrete time Toda equation. \( l \) is the discrete time variable. In the continuous limit of \( l \), (2.13) becomes the Toda molecule equation. (2.13) is the same relations which is obtained from the symmetric model. [9]

3 Unitary Matrix Model and Painlevé III

It is well known that the partition function \( Z_N^U \) of the unitary matrix model can be presented as a product of norms of the biorthogonal polynomial system. Namely, let us introduce a scalar product of the form

\[
<A, B> = \oint d\mu(z) \exp\{-V(z)\} A(z) B(z^{-1}),
\]

where

\[
V(z) = \sum_{m>0} (t_m z^m + t_{-m} z^{-m}) - l \log z.
\]

(3.2)

Let us define the system of the polynomials biorthogonal with respect to this scalar product

\[
<\Phi_{n,l}, \Phi_{k,l}^* = h_{k,l} \delta_{nk}.
\]

(3.3)

Then, the partition function \( Z_{N,l}^U \) is equal to the product of \( h \)'s:

\[
Z_{N,l}^U = \prod_{k=0}^{N-1} h_{k,l}, \quad \tau_0 = 1.
\]

(3.4)
The polynomials are normalized as follows (we should stress that superscript \('*\) does not mean the complex conjugation):

\[
\Phi_{n,l} = z^n + \cdots + S_{n-1,l}, \quad \Phi^*_{n,l} = z^n + \cdots + S^*_{n-1,l}, \quad S_{-1,l} = S^*_{-1,l} = 1. \tag{3.5}
\]

Now it is easy to show that these polynomials satisfy the following recurrent relations,

\[
\Phi_{n+1,l}(z) = z\Phi_{n,l}(z) + S_{n,l}z^n\Phi^*_{n,l}(z^{-1}),
\]

\[
\Phi^*_{n+1,l}(z^{-1}) = z^{-1}\Phi^*_{n,l}(z^{-1}) + S^*_{n,l}z^{-n}\Phi_{n,l}(z), \tag{3.6}
\]

and

\[
\frac{h_{n+1,l}}{h_{n,l}} = 1 - S_{n,l}S^*_{n,l}. \tag{3.7}
\]

Note that \(h_{n,l}, S_{n,l}, S^*_{n,l}, \Phi_{n,l}(z)\) and \(\Phi^*_{n,l}(z)\) depend parametrically on \(t_1, t_2, \cdots\) and \(t_{-1}, t_{-2}, \cdots\), but for convenience of notation we suppress this dependence.

Hereafter we call \(t_1, t_2, \cdots\) and \(t_{-1}, t_{-2}, \cdots\), time variables.

Using (3.3) and integration by parts, we can obtain next relations:

\[
\oint \frac{d\mu(z)}{2\pi iz} V'(z)\Phi_{n+1,l}(z)\Phi^*_{n,l}(z^{-1}) = (n+1)(h_{n+1,l} - h_{n,l}), \tag{3.8}
\]

and

\[
\oint \frac{d\mu(z)}{2\pi iz} z^2 V'(z)\Phi^*_{n+1,l}(z^{-1})\Phi_{n,l}(z) = (n+1)(h_{n+1,l} - h_{n,l}). \tag{3.9}
\]

(3.8) and (3.9) are string equations of the full unitary matrix model.

If \(t_1\) and \(t_{-1}\) are free variables while \(t_2 = t_3 = \cdots = 0\) and \(t_{-2} = t_{-3} = \cdots = 0\), (3.8) and (3.9) become

\[
(n+1)S_{n,l}S^*_{n,l} = t_{-1}(S_{n,l}S^*_{n+1,l} + S^*_{n,l}S_{n-1,l})(1 - S_{n,l}S^*_{n,l}) - l(1 - S_{n,l}S^*_{n,l}), \tag{3.10}
\]

\[
(n+1)S_{n,l}S^*_{n,l} = t_1(S^*_{n,l}S_{n+1,l} + S_{n,l}S^*_{n-1,l})(1 - S_{n,l}S^*_{n,l}) + l(1 - S_{n,l}S^*_{n,l}). \tag{3.11}
\]

Next we introduce a useful relation. Using (3.3) and integration by parts, we can show

\[
\oint \frac{d\mu(z)}{2\pi iz} zV'(z)\Phi_{n,l}(z)\Phi^*_{n,l}(z^{-1}) = 0. \tag{3.12}
\]
This corresponds to the Virasoro constraint:

$$L_0 = \sum_{k=-\infty}^{\infty} k t_k \frac{\partial}{\partial t_n}. \tag{3.13}$$

This relation constrains a symmetry like the complex conjugate between $t_k$ and $t_{-k}$. If we set that $t_1$ and $t_{-1}$ are free variables while $t_2 = t_3 = \cdots = 0$ and $t_{-2} = t_{-3} = \cdots = 0$, from (3.12) we get

$$t_1 S_{n,l} S_{n-1,l}^* = t_{-1} S_{n,l}^* S_{n-1,l} - l. \tag{3.14}$$

Using (3.14), (3.10) and (3.11) can be written

$$\begin{align*}
(n + 1) S_{n,l} &= (t_{-1} S_{n+1,l} + t_1 S_{n-1,l})(1 - S_{n,l} S_{n,l}^*), \\
(n + 1) S_{n,l}^* &= (t_1 S_{n+1,l}^* + t_{-1} S_{n-1,l}^*)(1 - S_{n,l} S_{n,l}^*).
\end{align*} \tag{3.15}$$

Using the orthogonal conditions, it is also possible to obtain the equations which describe the time dependence of $\Phi_{n,l}(z)$ and $\Phi_{n,l}^*(z)$. Namely, differentiating (3.3) with respect to times $t_1$ and $t_{-1}$ gives the following evolution equations:

$$\begin{align*}
\frac{\partial \Phi_{n,l}(z)}{\partial t_1} &= -S_{n,l} \frac{h_{n+1,l}}{S_{n-1,l} h_{n-1,l}} (\Phi_{n,l}(z) - z \Phi_{n-1,l}), \\
\frac{\partial \Phi_{n,l}(z)}{\partial t_{-1}} &= h_{n,l} \frac{\Phi_{n-1,l}(z)}{h_{n-1,l}}, \\
\frac{\partial \Phi_{n,l}^*(z^{-1})}{\partial t_1} &= S_{n,l}^* \frac{h_{n,l}}{S_{n-1,l}^* h_{n-1,l}} (\Phi_{n,l}^*(z^{-1}) - z^{-1} \Phi_{n-1,l}^*), \\
\frac{\partial \Phi_{n,l}^*(z^{-1})}{\partial t_{-1}} &= -S_{n,l}^* \frac{h_{n+1,l}}{S_{n-1,l}^* h_{n-1,l}} (\Phi_{n,l}^*(z^{-1}) - z^{-1} \Phi_{n-1,l}^*), \tag{3.17-3.20}
\end{align*}$$

The compatibility condition gives the following nonlinear evolution equations:

$$\begin{align*}
\frac{\partial S_{n,l}}{\partial t_1} &= -S_{n,l} \frac{h_{n+1,l}}{h_{n,l}}, \\
\frac{\partial S_{n,l}}{\partial t_{-1}} &= -S_{n-1,l} \frac{h_{n+1,l}}{h_{n,l}}, \\
\frac{\partial S_{n,l}^*}{\partial t_1} &= S_{n+1,l} \frac{h_{n+1,l}}{h_{n,l}}, \\
\frac{\partial S_{n,l}^*}{\partial t_{-1}} &= -S_{n-1,l}^* \frac{h_{n+1,l}}{h_{n,l}}, \\
\frac{\partial h_{n,l}}{\partial t_1} &= S_{n,l} S_{n-1,l} h_{n,l}, \\
\frac{\partial h_{n,l}}{\partial t_{-1}} &= S_{n,l} S_{n-1,l} h_{n,l}. \tag{3.21-3.23}
\end{align*}$$
Here we define \( a_{n,l}, b_{n,l} \) and \( b^*_{n,l} \):

\[
a_{n,l} \equiv 1 - S_{n,l}S_{n,l}^* = \frac{h_{n+1,l}}{h_{n,l}},
\]

(3.24)

\[
b_{n,l} \equiv S_{n,l}S_{n-1,l}^*.
\]

(3.25)

\[
b^*_{n,l} \equiv S_{n,l}^*S_{n-1,l}.
\]

(3.26)

Notice that since the definitions \( a_{n,l}, b_{n,l} \) and \( b^*_{n,l} \) satisfy the following identity:

\[
b_{n,l}b^*_{n,l} = (1 - a_{n,l})(1 - a_{n-1,l}).
\]

(3.27)

It can be shown using (3.14) that

\[
t_1b_{n,l} = t_{-1}b^*_{n,l} - l.
\]

(3.28)

In terms of \( a_{n,l}, b_{n,l} \) and \( b^*_{n,l} \), (3.21) and (3.22) become the two-dimensional Toda equations:

\[
\frac{\partial a_{n,l}}{\partial t_1} = a_{n,l}(b_{n+1,l} - b_{n,l}), \quad \frac{\partial b_{n,l}}{\partial t_1} = a_{n,l} - a_{n-1,l},
\]

(3.29)

and

\[
\frac{\partial a_{n,l}}{\partial t_{-1}} = a_{n,l}(b^*_{n+1,l} - b^*_{n,l}), \quad \frac{\partial b^*_{n,l}}{\partial t_1} = a_{n,l} - a_{n-1,l}.
\]

(3.30)

Using \( a_{n,l}, b_{n,l} \) and \( b^*_{n,l} \), we rewrite (3.10) and (3.11)

\[
\frac{n + 1}{t_1} \frac{1 - a_{n,l}}{a_{n,l}} = b_{n+1,l} + b_{n,l},
\]

(3.31)

and

\[
\frac{n + 1}{t_{-1}} \frac{1 - a_{n,l}}{a_{n,l}} = b^*_{n+1,l} + b^*_{n,l}.
\]

(3.32)

From (3.29) and (3.31) we eliminate \( b_{n+1,l} \),

\[
2b_{n,l} = \frac{1}{a_{n,l}} \left[ \frac{n + 1}{t_1} (1 - a_{n,l}) - \frac{\partial a_{n,l}}{\partial t_1} \right].
\]

(3.33)

In the same way, from (3.30) and (3.32) we eliminate \( b^*_{n+1,l} \),

\[
2b^*_{n,l} = \frac{1}{a_{n,l}} \left[ \frac{n + 1}{t_{-1}} (1 - a_{n,l}) - \frac{\partial a_{n,l}}{\partial t_{-1}} \right].
\]

(3.34)
Using (3.27) and (3.28), (3.29) and (3.30) can be written
\[
\frac{\partial b_{n,l}}{\partial t} - \frac{1}{t-1} = \left( a_{n,l} - \frac{1}{t-1} \right) + 1 \quad \text{(3.35)}
\]
\[
\frac{\partial b_{n,l}^*}{\partial t_{1}} = \left( a_{n,l} - \frac{1}{t_{1}} \right) + 1 \quad \text{(3.36)}
\]

Using (3.33) and (3.35) to eliminate \( b_{n,l} \) we obtain a second order ODE for \( a_{n,l} \)
\[
\frac{\partial^2 a_{n,l}}{\partial t \partial t_{1}} = \frac{n+1}{t-1 a_{n,l}} \frac{\partial a_{n,l}}{\partial t_{1}} - \frac{2a_{n,l}(a_{n,l} - 1)}{2t_{1} t_{1}} \frac{\partial a_{n,l}}{\partial t_{1}} - \frac{(n+1)^2 a_{n,l} - 1}{2t_{1} t_{1} a_{n,l} - 1} \quad \text{(3.37)}
\]

In the same way, we eliminate \( b_{n,l}^* \) using (3.34) and (3.35) and obtain an ODE for \( a_{n,l} \)
\[
\frac{\partial^2 a_{n,l}}{\partial t \partial t_{1}} = \frac{n+1}{t_{1} a_{n,l}} \frac{\partial a_{n,l}}{\partial t_{1}} - \frac{2a_{n,l}(a_{n,l} - 1)}{2t_{1} t_{1}} \frac{\partial a_{n,l}}{\partial t_{1}} - \frac{(n+1)^2 a_{n,l} - 1}{2t_{1} t_{1} a_{n,l} - 1} \quad \text{(3.38)}
\]

Using (3.28), (3.29) and (3.30) we can obtain
\[
\frac{t_{1} \partial a_{n,l}}{\partial t_{1}} = \frac{1}{t-1} \frac{\partial a_{n,l}}{\partial t_{1}} \quad \text{(3.39)}
\]

So \( a_{n,l} \) are functions of the radial coordinate
\[
x = t_{1} t_{-1} \quad \text{(3.40)}
\]

only. Then from (3.37) and (3.38) we can obtain
\[
\frac{\partial^2 a_{n,l}}{\partial x^2} = \frac{1}{2} \left( \frac{1}{a_{n,l} - 1} + \frac{1}{a_{n,l}} \right)(\frac{\partial a_{n,l}}{\partial x})^2 - \frac{1}{x} \frac{\partial a_{n,l}}{\partial x}
\]
\[
- \frac{2}{x} a_{n,l}(a_{n,l} - 1) + \frac{(n+1)^2 a_{n,l} - 1}{2x^2} \frac{a_{n,l}}{a_{n,l} - 1} - \frac{l^2}{2x^2} a_{n,l} - 1 \quad \text{(3.41)}
\]

(3.41) is an expression of the fifth Painlevé equation (PV) with \( \delta_V = 0 \). It is interesting that (3.41) has a symmetry \( a_{n,l} \leftrightarrow a_{n,l} - 1 \) and \( (n+1)^2 \leftrightarrow -l^2 \). To see this, we define \( c_{n,l} \):
\[
c_{n,l} = 1 - a_{n,l} \quad \text{(3.42)}
\]
Using $c_{n,l}$, through the transformation $x \rightarrow -x$ we rewrite (3.41)

\[
\frac{\partial^2 c_{n,l}}{\partial x^2} = \frac{1}{2} \left( \frac{1}{c_{n,l} - 1} + \frac{1}{c_{n,l}} \right) \left( \frac{\partial c_{n,l}}{\partial x} \right)^2 - \frac{1}{x} \frac{\partial c_{n,l}}{\partial x} + \frac{2}{x} c_{n,l} (c_{n,l} - 1) + \frac{l^2}{2x^2} \frac{c_{n,l} - 1}{c_{n,l}} - \frac{(n + 1)^2}{2x^2} c_{n,l} - 1. \tag{3.43}
\]

In (3.43) the role of $l^2$ and $(n + 1)^2$ exchange.

This equation can be transformed into the third Painlevé equation (P III). (see [10], [5]) (3.41) is the same equation obtained from the O(4) nonlinear $\sigma$-model [11].

4 Topology on the lattice

We divide the potential into the symmetric and the anti-symmetric part.

\[ V(U) = t_1^+ s_w + t_1^- s_\theta, \tag{4.1} \]

where

\[ t_1^+ = t_1 + t_{-1}, \quad t_1^- = \frac{t_1 - t_{-1}}{2}. \tag{4.2} \]

$s_w$ is the symmetric part, the usual Wilson action

\[ s_w = \frac{1}{2} (\text{tr} U + \text{tr} U^\dagger). \tag{4.3} \]

Here we choose

\[ s_\theta = \frac{1}{2} (\text{tr} U - \text{tr} U^\dagger). \tag{4.4} \]

for the theta term, the anti-symmetric part. In the naive continuum limit, the sum of the theta term gives the first Chern number of the bundle on which the gauge field lives. We parametrise the one plaquette action as

\[ s(U) = - \cosh \epsilon s_w(U) - \sinh \epsilon s_\theta(U) - l \log U, \tag{4.5} \]

where $\epsilon$ is a real parameter that determines the relative weight of the two terms, introducing in the section 2.
We calculated the partition function in (2.11). We consider the continuum limit of the partition function to leading order in $g \to 0$. As $g \to 0$ we want the coupling of the theta term to approach its continuum value $\theta/2\pi$. Accordingly $\epsilon$ has to be scaled in this limit as

$$\epsilon = \frac{\theta^2}{2\pi N}.$$ (4.6)

The asymptotic representation of the modified Bessel functions are

$$I_n(z) \to \frac{e^z}{\sqrt{2\pi z}} e^{-\frac{z^2}{2}} \text{ as } z \to \infty.$$ (4.7)

Substituting (4.6) for $\epsilon$ and (4.7) into (2.11), we obtain the small $g$

$$Z_U(g, \theta) \sim \det_{j,k} \exp\left[-\frac{g^2}{2N} \{(l - j + k)^2 - \frac{\theta}{\pi} (l - j + k)\}\right]$$

$$= \det_{j,k} \exp\left[-\frac{g^2}{2N} \{(l - j + k - \frac{\theta}{2\pi})^2 - \frac{1}{4}(\frac{\theta}{\pi})^2\}\right].$$ (4.8)

Notice that $\theta$ couples to the discrete time $l$. For our purpose we are interested in the coupling of $\theta$ and $l$. Then in the first step we drop the independent factors.

The action is given by

$$s(\theta, l) = -\log(\det_{j,k} \exp[-\frac{g^2}{2N} \{(l - j + k - \frac{\theta}{2\pi})^2 - \frac{1}{4}(\frac{\theta}{\pi})^2\}]).$$ (4.9)

If $\theta$ is varied, the ground state might change at some points due to the level crossing which causes phase transition.

For $\theta = 0$ using next relation

$$\det_{j,k} \exp -\alpha (m - j + k)^2 = e^{-\alpha m^2 N} \det_{j,k} \exp -\alpha (-j + k)^2,$$ (4.10)

the action is

$$s(0, l) = -\log(\det_{j,k} \exp[-\frac{g^2}{2N} (l-j+k)]) \geq -\log(\det_{j,k} \exp[-\frac{g^2}{2N} (-j+k)]) = s(0.0).$$ (4.11)

It is clear that at $\theta = 0$ the ground state is $l = 0.$
For any $\theta$ using (4.10) we can obtain that
\[
 s(\theta, l) = -g^2 \frac{\theta}{8} \pi^2 - \log \det_{jk} \exp[-g^2 \frac{\theta}{2N} (l - j + k - \frac{\theta}{2\pi})^2]
\]
\[
= -g^2 \frac{\theta}{8} \pi^2 + g^2 \frac{\theta}{2} (l - \frac{\theta}{2\pi})^2 - \log \det_{jk} \exp[-g^2 \frac{\theta}{2N} (-j + k)^2]
\]
(4.12)
The first term is a constant shift of the free energy. The second term shows that by increasing $\theta$ starting from zero we do not encounter any phase transition up to $\theta = \pi$. At this point however there is a phase transition. If $\theta$ is above $\pi$, $l = 1$ becomes the ground state. Due to the periodicity in $\theta$ the same type of phase transition occurs whenever $\theta$ becomes an odd integer times $\pi$. (Fig. 1.)

From (4.12) the free energy density around the phase transition is given by
\[
f(\theta) = 0, \quad -\pi \geq \theta \geq \pi,
\]
\[
f(\theta) = g^2 \frac{\theta}{2} (1 - \frac{\pi}{\theta}), \quad -3\pi \geq \theta \geq -\pi, \quad 3\pi \geq \theta \geq \pi,
\]
\[
f(\theta) = g^2 (2 - \frac{\pi}{\theta}), \quad -5\pi \geq \theta \geq -3\pi, \quad 5\pi \geq \theta \geq 3\pi,
\]
(4.13)
We have to note that (4.13) gives only the part of the energy without the constant term.

Next we consider the $SU(N)$ case. In this case there is an additional $\sum_{k=1}^{N} \alpha_k = 0$ constraint on the eigenvalues that must be enforced by a delta function in the measure. If the delta function is written as
\[
\delta(\sum_{k=1}^{N} \alpha_k) = \sum_{m=-\infty}^{\infty} \exp[i m \sum_{k=1}^{N} \alpha_k],
\]
(4.14)
The calculation (2.9) goes through without any modification giving the final result,
\[
Z_{SU}^{N,l} = \text{const.} N! \sum_{m=-\infty}^{\infty} \det_{jk} e^{(l+m-j+k)I_{l+m-j+k}(N/g^2)}
\]
\[
= \text{const.} N! \sum_{m=-\infty}^{\infty} \det_{jk} e^{(m-j+k)I_{m-j+k}(N/g^2)}.
\]
(4.15)
In the last step the discrete time $l$ disappears for the summing about $m$. Using
for small $g$, we obtain

$$Z_l^{SU}(g, \theta) = \sum_{m=-\infty}^{\infty} \text{const.} N! \sum_{m=-\infty}^{\infty} \det_{jk} e^{\epsilon (m-j+k)} I_{l+m-j+k} (N/g^2)$$

$$\sim \sum_{m=-\infty}^{\infty} \det_{jk} \exp \left[-\frac{g^2}{2N} \left( (m-j+k-\frac{\theta}{2\pi})^2 - \frac{1}{4} (\frac{\theta}{\pi})^2 \right) \right]$$

$$= \sum_{m=-\infty}^{\infty} \exp \left[-\frac{g^2}{2} (m-\frac{\theta}{2\pi})^2 \right] \det_{jk} \exp \left[-\frac{g^2}{2N} \left( (j+k)^2 - \frac{1}{4} (\frac{\theta}{\pi})^2 \right) \right]$$

In the $g \to 0$ limit this sum can be explicitly evaluated by the approximating it with a Gaussian integral. Then we can obtain

$$Z_l^{SU}(g, \theta) \sim \sqrt{\frac{2\pi}{g}} \exp \frac{g^2}{8} \left( \frac{\theta}{\pi} \right)^2 \det_{jk} \exp \left[-\frac{g^2}{2N} \left( (j-k)^2 - \frac{1}{4} (\frac{\theta}{\pi})^2 \right) \right]. \quad (4.17)$$

Then in the $SU(N)$ case $\theta$ has influence on a constant shift of the free energy only. We can rewritten (4.15) as

$$Z_{N,l}^{SU} = \text{const.} N! \sum_{l=-\infty}^{\infty} \det_{jk} e^{\epsilon (l-j+k)} I_{l-j+k} (N/g^2)$$

$$= \sum_{l=-\infty}^{\infty} Z_{N,l}^{U} \quad (4.18)$$

The partition function for the $SU(N)$ case is the sum of the partition function for the $U(N)$ case about all the discrete time $l$. Thus the theta term would have been zero for $SU(N)$ case.

### 5 Concluding remarks

In this letter we introduce a new term to the full unitary matrix model. This term is $l \log U$. We show that the partition function of this model satisfy the discrete time Toda equation and $l$ is the discrete time. Furthermore we consider the relation the partition function of this model and Painlevé III. We can see the symmetry between $n$ and $l$.

We derive the continuum limit of this model in the presence of a topological term. The theta term remained to be coupled to the dynamics in the continuum,
giving rise to phase transition at \( \theta_c = \pm \pi, \pm 3\pi, \pm 5\pi, \cdots \). The ground state is indicated by the discrete time \( l \). The discrete time plays like the instanton number. The partition function for the \( SU(N) \) case is the sum of the partition function for the \( U(N) \) case about all the discrete time \( l \). Then for the \( SU(N) \) case the theta term has influence on a constant shift of the free energy only.

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Figure 1: The free energy density around the phase transition.