Magnetization plateaus of spin-1/2 system on a 5/7 skewed ladder

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Magnetization plateaus are some of the most striking manifestations of frustration in low-dimensional spin systems. We present numerical studies of magnetization plateaus in the fascinating spin-1/2 skewed ladder system obtained by alternately fusing five- and seven-membered rings. This system exhibits three significant plateaus at \( m = 1/4, 1/2 \) and \( 3/4 \), consistent with Oshikawa-Yamanaka-Affleck condition. Our numerical as well as perturbative analysis show that the ground state can be approximated by three weakly coupled singlet dimers and two free spins, in the absence of a magnetic field. With increasing applied magnetic field, the dimers progressively become triplets with large energy gaps to excited states, giving rise to stable magnetization plateaus. Finite temperature studies show that \( m = 1/4 \) and \( 1/2 \) plateaus are robust and survive thermal fluctuations while \( m = 3/4 \) plateau shrinks rapidly due to thermal noise. The cusps at the ends of a plateau follow the algebraic square root dependence on \( B \).

I. INTRODUCTION

The study of quantum phase transition in frustrated low dimensional magnets has been an active area of research in the last few decades. The simplest model of frustrated magnet is the one dimensional (1D) \( J_1 - J_2 \) model where each spin is interacting with its nearest and next nearest neighbors with exchange parameters of strength \( J_1 \) and \( J_2 \) respectively\(^1-21\). This system can be mapped on to a zigzag ladder if odd and even sites of the chain are arranged on the two legs of the ladder as shown in Fig. 1(a). The frustration in these magnetic systems can give rise to many exotic phases in the ground state (gs) like the gapless spin liquid\(^2,3,7,9,10\), dimer\(^1-11\) and spiral phases\(^3,5,7,8,10,12\), in the absence of an applied magnetic field \( B \). These systems become a veritable zoo of phases in the presence of a finite magnetic field \( B \)\(^18,19\); for example a system with exchange interactions \( J_1 \) ferromagnetic and \( J_2 \) antiferromagnetic shows multi-magnon condensation\(^14,15,20-22\), vector chiral phase\(^3,14,15,22\) and magnetization plateaus\(^18,23,24\). In addition to a magnetization plateau phase, frustrated systems can also show kinks and jumps in magnetization\(^18\).

Theoretical understanding of the magnetization pro-

cess in quantum spin chains and ladders has attracted considerable attention in recent past\(^18,23-33\). The magnetization plateau indicates the existence of energy gaps between two consecutive magnetic spin sectors in the thermodynamic limit and one such example is the integer spin Heisenberg antiferromagnetic (HAF) chain\(^34-36\). The energy gap between the gs \( (S = 0) \) and next magnetic excited state \( (S = 1) \) is finite for an integer spin model with periodic boundary condition, which is the well known Haldane gap\(^34-36\). Therefore, there is a \( M = 0 \) plateau in the gs and a finite field \( B \) of strength \( E_{\text{gs}}/g\mu_B \) \( (E_g \) is the Haldane gap, \( g \) is the gyromagnetic ratio and \( \mu_B \) is the Bohr magneton) is required to spin polarize the system. In fact the Heisenberg spin-1 chain with single ion anisotropy also shows a plateau at \( m = M/M_{\text{max}} = 1/2 \) where \( M \) is the magnetization at the plateau and \( M_{\text{max}} \) is the saturation magnetization\(^37\). The 1D \( J_1 - J_2 \) model with exchange interactions \( J_1 \) ferromagnetic and \( J_2 \) antiferromagnetic as well as both \( J_1 \) and \( J_2 \) antiferromagnetic shows a plateau phase at \( m = 1/3 \) for \( |J_2/J_1| > 0.6\)\(^18,23,24\). Magnetization plateau at \( M = 0 \) has also been predicted for the ordinary two legged ladder which has finite spin gap in the gs\(^25,26\). In the 2D system, magnetization plateaus in Kagomé lattice are predicted to be at \( m = 1/9, 1/3, 5/9 \) and \( 7/9\)\(^38-43\), whereas for a triangular lattice, magnetization plateau shows up at \( m = 1/3\)\(^44\).

Experimentally, the magnetization plateau at \( m = 1/3 \) is found in frustrated spin chains like Cu\(_3\)(CO\(_3\))\(_2\) (OH)\(_2\)\(^45-47\) and spin-1/2 trimer compound Cu\(_3\)(P\(_2\)O\(_6\)) (OH)\(_2\)\(^48\). Other compounds showing 1/3 plateau are Ca\(_2\)Co\(_2\)O\(_6\)\(^49-51\), Sr\(_3\)Co\(_2\)O\(_6\)\(^52\), Sr\(_3\)HoCrO\(_6\)\(^53\), SrCo\(_2\)O\(_{11}\)\(^54\) and Co\(_2\)O\(_6\)\(^55-57\). Frustrated ladder compound NH\(_4\)CuCl\(_3\)\(^58\) shows two plateaus at \( m = 1/4 \) and \( 3/4 \).

Oshikawa, Yamanaka, and Affleck (OYA)\(^25\) established the necessary condition for the occurrence of plateaus in a 1D spin-S system by generalizing the Lieb-Schultz-Mattis (LSM) theorem\(^36,59\). The OYA condition for observing

FIG. 1. (a) The regular zigzag chain and (b) the 5/7 skewed ladder lattice. A unit cell consisting of eight spins is shown in the box with dotted edges.
a plateau at m is given by \( S_p(1 - m) \in \mathbb{Z} \) where S is the spin of a site, p is the number of lattice sites per unit cell and \( \mathbb{Z} \) represents the set of positive integers. The condition is further generalized to n-leg ladder\(^{26,60} \) which is given as \( n S_p (1 - m) \in \mathbb{Z} \). Haldane chain is a special case of this condition where \( n = p = 1 \) and for integer S a plateau could occur at \( m = 0 \).

In this paper we are interested in the skewed ladder system which is a variant of a zigzag ladder with periodically missing rung bonds\(^{61} \). The 5/7 system which corresponds to fused azulenes shows high spin gs for large \( J_1/J_2 \) in the thermodynamic limit. Thomas et al. showed that fused azulene systems have high spin gs even for finite system sizes\(^{62} \). The behavior of these systems in the presence of a static axial magnetic field had not been studied so far. Here, we analyze the behavior of this skewed ladder in the presence of an external magnetic field.

The skewed ladders can be of various types\(^{61} \), but in this paper we study only the 5/7 skewed ladder shown in Fig. 1(b). In this system there are eight spins and ten bonds per unit cell. There are two bonds exchanged with exchange interaction \( J_1 \) and eight bonds with \( J_2 \). This system shows a high spin gs in large \( J_1 (> 2.35/J_2) \) limit and one-quarter of spins in each unit cell are connected through the effective ferromagnetic interaction while the remaining six spins form three singlet dimers\(^{61} \). The OYA condition predicts possibility of plateaus at \( m = 0, 1/4, 1/2 \) and \( 3/4 \) for this system. In this paper we show that there are indeed three plateaus at \( m = 1/4, 1/2 \) and \( 3/4 \) in the 5/7 skewed ladder system: in the regular zigzag chain a lone plateau at \( m = 1/3 \) is observed although OYA conditions predict two more plateaus at \( m = 0 \) and \( 2/3,18,23,24 \). These plateaus in the 5/7 ladder are formed because of the strong dimer formations in the system. The existence of four plateaus in a ladder is not found in the literature. We also explore the cusp at the ends of plateaus.

This paper is divided into four sections. In section II we discuss the model Hamiltonian and numerical method. The results are presented in section III which has three subsections. The discussion of results are presented in section IV.

II. MODEL AND METHOD

In Fig. 1(b) we show schematically a 5/7 ladder. All exchange interactions between the spins are antiferromagnetic in nature. The sites are numbered such that odd numbered sites are on the bottom leg and even numbered sites are on the top leg. In this scheme the rung bonds are the nearest neighbor exchanges \( J_1 \) and the bonds on the legs are the next nearest neighbor exchanges. The nearest neighbor exchange \( J_1 \) is taken in units of \( J_2 \) which defines the energy scale. In the presence of an axial magnetic field \( B \), the model Hamiltonian can be written as

\[
H_{5/7} = J_1 \sum_{i=0}^{n} \left( \vec{S}_{8i+1} \cdot \vec{S}_{8i+2} + \vec{S}_{8i+4} \cdot \vec{S}_{8i+5} \right) + \sum_{i=0}^{n} \sum_{k=1}^{8} \vec{S}_{8i+k} \cdot \vec{S}_{8i+k+2} - B \sum_{i=0}^{n} \sum_{k=1}^{8} S_{8i+k}^z (1)
\]

where \( i \) labels the unit cell and \( k \) the spin in the unit cell as shown in Fig. 1(b). The first and second terms denote the rung exchange interaction \( J_1 \) and the interaction along the legs \( J_2 \), and the third term of the Hamiltonian gives interaction of the spins with an axial field \( B \) in units of \( J_2/g\mu_B \).

The Hamiltonian in Eq. 1 is many-body in nature, therefore we need to deal with a large number of degrees of freedom. We use density matrix renormalization group (DMRG) method to handle the large degrees of freedom\(^{63-65} \). The DMRG method is based on systematic truncation of irrelevant degrees of freedom. The dimension of effective density matrix \( m \) (which is also the number of block states and not to be confused with fractional magnetization) chosen, varies up to 400 and the truncation error of the density matrix is less than \( 10^{-11} \). We also carry out five to six finite DMRG sweeps for satisfactory convergence of the eigenstate. The growth sequence of the system is the same as that in our earlier work\(^{61} \).

The lowest eigen states in all \( M_S \) sectors, from zero to \( N/2 \) are calculated to find the total spin \( S_{gs} \) of the gs from the condition \( E_0(S_{gs} - 1) = E_0(S_{gs}) < E_0(S_{gs} + 1) \). An exact diagonalization (ED) technique is used to calculate gs properties of small systems.

III. RESULTS AND DISCUSSIONS

The 5/7 ladder system which has eight spins per unit cell, and in large \( J_1 \) limit, the ground state can be represented as six spins forming three singlet dimers and weak ferromagnetic interaction between the remaining two spins on the same legs in the two adjacent rings. Therefore, in the thermodynamic limit, for \( J_1/J_2 > 2.35 \), the magnetization in gs is \( m = 1/4 \). In the strong \( J_1 (> 2.35) \) limit, the gs has \( m = 1/4 \) for \( B = 0 \), even for small systems. In this paper we focus on the gs properties in the presence of an axial magnetic field and we show that there are three plateau phases at \( m = 1/4, 1/2 \) and \( 3/4 \) as predicted by the generalized Lieb-Schultz-Mattis theorem\(^{25,36,59} \). To understand the plateau phases we analyze the gs energies, spin densities and spin-spin correlations in presence of the applied magnetic field, \( B \). The Hamiltonian in Eq. 1 conserves \( M_S \), therefore field dependent gs energies are obtained simply by adding the Zeeman term to the zero field energies,

\[
E(M_S, B) = E_0(M_S, B = 0) - BM_S,
\]

where \( M_S \) is the z-component of the total spin. The change in magnetization from \( M_S \) to \( M_S' \) occurs at the
crossing of the $E(M_S, B)$ and $E(M'_S, B)$ lines.

The gs of this model was studied in the absence of magnetic field in Ref. 61 and it shows a high spin ferromagnetic gs even at $J_1 = 1$ and a re-entrant antiferromagnetic phase for $1.75 < J_1 < 2.18$. The gs has $S = n$ (the number of unit cells) for $J_1 > 2.35$. In this phase the spins at sites 3, 7, 11, . . . have effective ferromagnetic interactions. The spin density $\rho_i$ at site $i$ and axial spin correlation $C(r)$ between spins at sites $i$ and $i + r$ are defined as

$$
\rho_i = \langle gs | S^z_i | gs \rangle
$$

$$
C(r) = \langle gs | S^z_i S^z_{i+r} | gs \rangle.
$$

The results are presented in the following four subsections. First the $m - B$ curves and their corresponding energies are presented in IIIA. To understand all three plateau phases spin density $\rho_i$ at site ‘i’ and spin-spin correlation function $C(r)$ (Eqn. 3) are studied for different $M_S$ values in IIIB. From these we can infer the gs spin configurations at different magnetization plateaus. Here we also study the cusp singularities and thermal stability of plateaus. The analytical perturbation theory to support this is discussed in IIIC.

**A. Plateau phases in 5/7 ladder**

We plot the $m - B$ curve for different strength of the rung exchange $J_1$ in Fig. 3 for $N = 498$ spins (62 unit cells), and note that the $m=1/4$ and $3/4$ plateaus appear for $J_1 \geq 0.4$, whereas, the other plateau at $m = 1/2$ appears for $J_1 \geq 0.9$. Let us define $B^L_i$ and $B^U_i$ as the lower and upper critical value of the magnetic field for the $i^{th}$ plateau where $i = 1, 2, 3$ for the three plateaus at $m = 1/4, 1/2$ and $3/4$ respectively. We note that onset field of the first plateau at $m = 1/4$ decreases as $J_1$ increases while the width of the plateau $w_1 = (B^U_1 - B^L_1)$ increases with increasing $J_1$. The width of the $1/4$ plateau is strongly dependent on the term linear in $J_2$ in the large $J_1$ limit as shown analytically in subsection IIIC. We notice that a finite field is required to reach $m = 1/4$ for small $J_1$; in this limit of $J_1$, the rung dimers are weak and a finite field is required to align spins 3 and 7 in the field direction. In the $J_1 > 2.35$ limit, all the rung singlets are strong and effective ferromagnetic interaction develops between spin 3 and 7 as shown in Fig. 2(a) and the field required to attain the $1/4$ plateau decreases to zero. Along both legs most bonds are weak except the generic 6-8 bond in each unit cell. Therefore, the next plateau occurs when the 6-8 bond breaks and the system enters another locked phase with $m = 1/2$ (Fig. 2(b)). In this case, most of the magnetic contribution comes from the ferromagnetically aligned spins at sites 3, 6, 7 and 8 in the unit cell (Fig. 2(b)). On further ramping the field $B$, the singlet involving sites 1, 2, 4 and 5 flips to yield a triplet and the system locks into the $m = 3/4$ plateau phase (Fig. 2(c)). At fields greater than $B_{sat}$, the saturation field, all singlet bonds in every unit cells are broken and the system goes to a completely polarized state.

To estimate the width of plateaus we plot the magnetic field, $B$, required to achieve successively higher $M_S$ states for a given $J_1$. Here, we define magnetic field $B_M$ as the magnetic field required to close the gap between the $M_S = M$ and $M_S = M + 1$ state i.e.,

$$
B_M = \frac{E_0(M + 1) - E_0(M)}{g\mu_B}.
$$

In Fig. 4(a) we notice that the $B_M$ curves collapse into four bands for large $J_1$: the first band corresponds to $m = 1/4$, the second band corresponds to the $m = 1/2$, the third band corresponds to $3/4$ and the fourth to $m = 1$. This is a consequence of the fact that for large

![FIG. 2. Schematic representation of gs in terms of singlet dimers and aligned free spins at (a) $m = 1/4$ plateau, (b) $m = 1/2$ plateau and (c) $m = 3/4$ plateau, where broken lines imply a triplet delocalized over four sites.](image)

![FIG. 3. (a) $m - B$ curve for 5/7 skewed ladder for $J_1 = 0.5, 1.0, 1.5$ and 3.0 for $N = 498$ sites. (b) The finite size effect of the $m - B$ curve with $J_1 = 1.0$ for five system sizes.](image)
$J_1$, the ground state has a spin $p/8$ and excited states with finite gap in the thermodynamic limit have spins $p/4, 3p/8$ and $p/2$ where $p$ is the number of spins in a unit cell. There are three plateaus of significance corresponding to $m = 1/4, 1/2$ and $3/4$. The width of the $1/4$ plateau at large $J_1$ corresponds to the magnetic field difference between the $m = 1/4$ and $m = 1/2$ band. This remains independent of $J_1$ at large $J_1$. The magnetic field $B_M$ for $m = 1/2$ is independent of $J_1$ while that of $B_M$ with $m = 3/4$ increases linearly with $J_1$ at large $J_1$. Therefore in this limit, the plateau width for $m = 1/2$ increases linearly with $J_1$. The width of the $m = 3/4$ plateau is independent of $J_1$ as the $B_M$ for $m = 3/4$ and $m = 1$ both increase linearly with $J_1$ and their difference is independent of $J_1$, for large $J_1$. This can be seen in Fig. 4(b) where the plateau width for magnetization calculation is plotted as a function of $J_1$.

Cusp singularity indicates the minima of dispersion relation of magnons in a frustrated magnet at an incommensurate value of the wave vector. The magnetization should vary as $m(B) - m(B_c) \propto (B - B_c)^{1/2}$ where $B_c$ is the magnetic field at the cusp. In fact for 5/7 ladder, there are at least four cusp singularities (at $m = 1/4, 1/2$ and $3/4$) in the $m - B$ curve for $J_1 = 1$ case with $N = 498$ spins (62 unit cells) as shown in Fig. 5. The behavior of the magnetization curves near the plateau are fitted with $m \propto (B - B_c)^{1/2}$. The $B_c^{L}$ for this system for $J_1 = 1$ are 0.759 (1.192), 1.490 (1.687), and 2.126 for $m = 1/4, 1/2$ and $3/4$ respectively. These plateaus have cusp singularities and can be fitted to a square root function of the magnetic field. Fig. 5 shows the cusp singularities at various plateaus.

For sufficiently large $J_1$, the plateaus are wide and the energy gap near each plateau is large. Any practical application of these plateaus are feasible only if they are stable to thermal fluctuations. The finite temperature

![Figure 4](https://example.com/figure4.png)

**FIG. 4.** (a) The magnetic field $B_M$ required to close the energy gap between successive lowest energy $M_S$ states vs the strength of nearest neighbor exchange $J_1$. $B_M$ is in units of $J_2/g\mu_B$. The $M$ values increase from $M = 1$ for the bottom curve to $M = 49$ for the upppermost curve. At large $J_1$, these curves collapse into four groups with $m = M/M_{\text{max}} = 1/4$, $1/2$, $3/4$ and 1. (b) The width of the plateaus $w_1$, $w_2$ and $w_3$ for $m = 1/4$, $1/2$ and $3/4$ plotted as a function of $J_1$ for a system with $N = 498$ spins. The solid lines are the plateau widths calculated from the perturbation theory in the large $J_1$ limit (see III C).

![Figure 5](https://example.com/figure5.png)

**FIG. 5.** Cusps near magnetization plateaus for 5/7 ladder at $J_1 = J_2 = 1.0$ for $N = 498$ spins (62 unit cells) (a) $m = 1/4$ plateau at $B_c^{L}$, (b) $m = 1/4$ plateau at $B_c^{L}$, (c) $m = 1/2$ plateau at $B_c^{L}$ and (d) $3/4$ plateau at $B_c^{L}$. The cusps near the plateaus obey $\propto (B - B_c)^{1/2}$ and the numerical $m$ vs $B$ dependence at the cusps are shown in the figure.

![Figure 6](https://example.com/figure6.png)

**FIG. 6.** $m - B$ curve at four temperatures $T/J_2 = 0.02, 0.05, 0.10$ and $0.20$ for $J_1 = 1.60$ and $N = 98$ spins.
behavior of the magnetization plateau of the system is shown in Fig. 6 (a) with $J_1 = 1.6$ for $N = 98$ spins. We have used hybrid ED-DMRG method with average density matrix taken over many states in each $M_S$ sector. We have used 300 energy eigenvalues from each $M_S$ sector for thermal averaging of the magnetization. Since we are focusing on the low temperature properties of the system, retaining 300 low lying states in each $M_S$ sector should be accurate as higher excited states are practically inaccessible at low temperatures. As shown in Fig. 6 (a), the plateau at $m = 1/4$ and $1/2$ are robust at finite temperatures, while the plateau at $3/4$ survives only up to $T/J_2 = 0.05$. The derivative $\chi = \frac{dm}{dB}$ vs $B$ is shown in Fig. 6 (b). In the plateau regime the susceptibility $\chi$ vanishes at $T = 0$ whereas, it is finite when the plateaus are perturbed by the thermal fluctuations. We notice that $3/4$ plateau is very much susceptible to the thermal fluctuation as $\chi$ is finite even at low temperatures, however $m = 1/4$ and $1/2$ plateaus survive thermal fluctuations.

B. Spin density and correlation function in the plateau phase

To identify the spins that are aligned along the applied field in a given plateau we study the spin densities and spin-spin correlation functions with 24 spins under periodic boundary condition (PBC). For a system of 24 spins, $M_S$ varies from 0 to 12. We obtain spin densities at all the sites for the lowest eigenstate in each positive $M_S$ sector. Based on the calculated spin densities we find that sites $(1, 5), (2, 4), (3, 7)$ and $(6, 8)$ have very nearly same spin densities in all the $M_S$ sectors of the system. We show in Fig. 7 the variation of spin densities with $m = M_S/12$ for $J_1 = 1.8$ and 5 corresponding to below and above the critical rung interaction $J_{1c} = 2.35$.

The spin densities at site numbers 3 and 7 increase quickly with $m$ and it attains value $\rho_3 \simeq \rho_7 \sim 0.5$ at $m = 1/4$ for both $J_1 = 1.8$ and 5. In fact for large value of $J_1 > 2.35$, $\rho_3$ and $\rho_7$ are both 0.5 without any field as $m = 1/4$ is the gs. The spin densities at 6 and 8 ($\rho_6$ and $\rho_8$) increases linearly with $m$ between $m = 1/4$ and $1/2$, and these go to 0.5 for $M_S = 6$. At the magnetic field at which this state becomes the gs, the 6-8 singlet bond breaks and becomes a triplet bond. The spins at the remaining sites, namely 1, 2, 4, and 5 form a singlet and have very low spin densities. This singlet state transitions to a triplet for the $m = 3/4$ plateau and the gs has $M_S = 9$. In this state the singlet formed by the spins 1, 2, 4 and 5 is canted to a triplet while the spins at other sites are ferromagnetically aligned. Hence, the spin densities $\rho_1, \rho_2, \rho_4$ and $\rho_5$ are almost equal beyond $m = 0.75$.

To understand the effective interactions between the neighboring spins we study the bond energies or magnetic bond order $b_{ij} = -\langle \psi_{gs} | \vec{S}_i \cdot \vec{S}_j - \frac{1}{4} | \psi_{gs} \rangle$ where the sites $i$ and $j$ are connected either by a $J_1$ or a $J_2$ interaction. There are ten such bonds in a unit cell and we show only dominant bond-orders in a unit cell in Fig. 8.

Only the bonds $(1, 2), (4, 5), (6, 8)$ and $(2, 4)$ are significant and others are small and increase with $m$. We find $b_{12} = b_{45}$ and $b_{13} = b_{35}$ for all $M_S$ values. For degenerate states we obtained $b_{ij}$ by diagonalizing the bond order matrix in the basis of degenerate states. In Fig. 8, we show the variation of $b_{ij}$ with $m$. In the large $J_1$ case, we note that for $m = 1/4$, the $(6, 8), (1, 2)$ and $(4, 5)$ bonds are singlets. The $(1, 3), (3, 5)$ and $(2, 4)$ bonds are weak and have a large triplet component. The (3,
7) bond (not shown in the figure) always remain triplet. When \( m \) reach the value of 0.5, the (6, 8) singlet breaks and a triplet is obtained. This is true even for small \( J_1 \) (Fig. 8a). As the \( m \) value increases further, (1, 2) and (4, 5) bonds resonate between a singlet and a triplet, but the (2, 4) bond becomes a singlet. This can be represented as a state in which resonance involving the valence bond pairing \([1, 2](4, 5) \leftrightarrow [2, 4](1, 5) \leftrightarrow (1, 2)[4, 5]\) where square brackets imply singlet pairing and round brackets imply triplet pairing of spins at sites indicated inside the brackets. All the singlet bonds are broken when \( m = 1 \) as is to be expected.

C. Perturbation calculations for plateau phases

The numerical studies indicate that we can develop an understanding of the plateau phases from a perturbative approach in the large \( J_1 \) limit. Hence, we treat the \( J_2 \) terms in the spin Hamiltonian as a perturbation over the \( J_1 \) terms. Within a unit cell the \( J_1 \) term provides interactions between spins \( S_1 \) and \( S_2 \), \( S_4 \) and \( S_6 \) while the \( J_2 \) term operates between spin pairs \( S_2, S_4; S_4, S_6; S_6, S_8 \), \( S_1, S_3; S_3, S_5; S_5, S_7 \). Since the spins \( S_6 \) and \( S_8 \) experience only \( J_2 \) interaction, we include this interaction also in the unperturbed Hamiltonian. Thus for a perturbation calculation, the zeroth order Hamiltonian over one unit cell under periodic boundary condition is given by

\[
H_0 = J_1 \left( \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_4 \cdot \vec{S}_6 \right) + J_2 \vec{S}_6 \cdot \vec{S}_8 - B \left( S_3^z + S_7^z \right).
\]

(5)

Other terms of the Hamiltonian involve \( J_2 \) interaction and can be treated as perturbation \( H_1 \) given by

\[
H_1 = J_2 \sum_{k=1}^{8} \vec{S}_k \cdot \vec{S}_{k+2} - J_2 \vec{S}_6 \cdot \vec{S}_8 - B \left( S_3^z + S_7^z \right),
\]

(6)

where cyclic boundary condition is implied in the summation. The ground state of the unperturbed system in terms of the spin couplings can be written as

\[
\langle \psi_0 \rangle = \begin{pmatrix} \uparrow_1 & 4 \uparrow_5 \uparrow_6 \uparrow_7 \uparrow_8 \end{pmatrix}
\]

(7)

In Eq. 7, the convention we follow is \( \langle \uparrow|\downarrow \rangle = \frac{1}{\sqrt{2}} \{ |\uparrow_1\downarrow_3 \rangle - |\downarrow_1\uparrow_3 \rangle \} \). The zeroth order energy of the system can be obtained by operating the zeroth order Hamiltonian \( H_0 \) on the state \( \langle \psi_0 \rangle \) in Eq. 7 and is given by

\[
\mathcal{E}_0 = -\frac{3}{2} J_1 - \frac{3}{4} J_2 - B,
\]

(8)

The first order contribution from \( H_1 \) vanishes as the matrix elements \( \langle \psi_0 | H_1 | \psi_0 \rangle = 0 \). The second order correction to the energy is given by

\[
E^{(2)} = \sum_{\text{ex}} \frac{\langle \psi_{\text{ex}} | H_1 | \psi_0 \rangle^2}{\mathcal{E}_{\text{ex}}^{(0)} - \mathcal{E}_{gs}^{(0)}}.
\]

(9)

We can obtain the excited states that connect to the ground states \( |\psi_0 \rangle \) by operating each term in \( H_1 \) on \( | \psi_0 \rangle \); the resulting state will be an excited state of \( H_0 \), whose unperturbed energy is computed by acting on this resulting \( | \psi_{\text{ex}} \rangle \) by \( H_0 \). For example, consider the first exchange interaction term in \( H_1 \) which is \( J_2 (\vec{S}_1 \cdot \vec{S}_3) \). When this operates on \( | \psi_0 \rangle \), we get the state

\[
J_2 (\vec{S}_1 \cdot \vec{S}_3) | \psi_0 \rangle = -\frac{J_2}{2\sqrt{2}} \psi_{\text{ex}}
\]

\[
= -\frac{1}{2\sqrt{2}} | \uparrow_1 \uparrow_2 \downarrow_3 \uparrow_4 \uparrow_5 \uparrow_6 \uparrow_7 \uparrow_8 \rangle.
\]

(10)

The unperturbed energy of the \( | \psi_{\text{ex}} \rangle \), \( \mathcal{E}_{\text{ex}}^{(0)} \) is given by \(-\frac{J_2}{2} - \frac{J_2}{4} - B \). Similarly, we can calculate the matrix elements of other exchange operators occurring in \( H_1 \), in the basis of the eigenstates of \( H_0 \). This gives the second order corrected energy of the ground state of \( H_0 \) to be

\[
\mathcal{E}_{gs}^{(2)} = -\frac{3J_1}{2} + \frac{3J_2}{4} - \frac{9J_2^2}{16J_1} - \frac{J_2^2}{4(J_1 + J_2)} - B.
\]

(11)

Exact diagonalization of the skewed ladder Hamiltonian of 24 sites, for \( J_1 = 5.0 \) and \( J_2 = 1.0 \) gives a per site energy of \(-1.067 \) as against the perturbation theory prediction of \(-1.05 \) corresponding to an error of \( \sim 1.6\% \). Similarly the exact and second order corrected ground state energy per site from perturbation theory, for \( J_1 = 2.5 \) and \( J_2 = 1.0 \) are \(-0.63 \) and \(-0.60 \) respectively. Thus the error in perturbation theory is less than 5.0% in this case as well.

Our numerical results show that the next plateau occurs at \( m = 1/2 \), corresponding to the breaking of (6, 8) singlet bond. The eigenstate of the unperturbed Hamiltonian in this case is \( \begin{pmatrix} \uparrow_1 \uparrow_2 \uparrow_3 \uparrow_4 \uparrow_5 \uparrow_6 \uparrow_7 \uparrow_8 \end{pmatrix} \). The energy of the state correct to second order in perturbation is given by

\[
\mathcal{E}_{m=1/2}^{(2)} = -\frac{3J_1}{2} + \frac{J_2}{4} - \frac{13J_2^2}{16J_1} - 2B.
\]

(12)

The error in energy per site of perturbation theory compared with exact results for 24 sites in the absence of an applied magnetic field is \(< 0.5\% \) for \( J_1 = 5.0 \) and \( < 0.6\% \) for \( J_1 = 2.5 \). From our numerical studies the ground state with \( m = 3/4 \) is obtained by creating a triplet superposition of the two states formed by nearest neighbor singlets and triplets from spins at sites 1, 2, 4 and 5 while all other spins have \( m_S = \pm 1/2 \) in the unit cell. Thus
we consider the $|\psi_{m=3/4}^{(0)}\rangle$ as given by
\begin{equation}
|\psi_{m=3/4}^{(0)}\rangle = \frac{1}{\sqrt{2}} \left[ |1\downarrow 2\uparrow 3\uparrow 4\downarrow 5\uparrow 6\uparrow 7\uparrow 8\rangle + |1\uparrow 2\downarrow 3\uparrow 4\downarrow 5\uparrow 6\uparrow 7\uparrow 8\rangle \right],
\end{equation}
where $|i\rightarrow j\rangle$ corresponds to $|\uparrow\downarrow i\rangle$, which is an $M_S=1$ triplet. Using this as the unperturbed state we can obtain the ground state energy of the unit cell in the $m=3/4$ state as
\begin{equation}
\epsilon_{m=3/4}^{(2)} = -\frac{J_1}{2} + \frac{3J_2}{4} - \frac{3J_2^2}{8J_1} - 3B.
\end{equation}
This again has an error of $\sim 1\%$ for $J_1 = 5.0$ and $< 5\%$ for $J_1 = 2.5$ when compared with exact diagonalization results for 24 site skewed ladder under periodic boundary condition in zero external field. In the fully polarized case, which corresponds to
\begin{equation}
|\psi_{m=1}^{(0)}\rangle = |1\uparrow 2\uparrow 3\uparrow 4\uparrow 5\uparrow 6\uparrow 7\uparrow 8\rangle,
\end{equation}
the exact energy is trivially given by
\begin{equation}
\epsilon_{m=1} = \frac{J_1}{2} + 2J_2 - 4B.
\end{equation}

From the perturbation calculation, we obtain the critical fields for the onset of the plateaus at $m = 1/2, 3/4$ and 1 which also correspond to the end of $m = 1/4, 1/2$ and $3/4$ plateaus. The critical fields $B_{c1}$, $B_{c2}$ and $B_{c3}$ are given by
\begin{align}
B_{c1} &= J_2 - \frac{J_2^2}{4J_1} + \frac{J_2^2}{4(J_1 + J_2)}, \\
B_{c2} &= J_1 + \frac{1}{2}J_2 + \frac{7J_2^2}{16J_1}, \\
B_{c3} &= J_1 + \frac{5}{4}J_2 + \frac{3J_2^2}{8J_1}.
\end{align}
These critical fields as well as the plateau widths are in good agreement with the numerical results, as seen in Fig. 4 (b), for $J_1 > 1.5$.

\section{Summary}

In this paper we have studied magnetic properties of an antiferromagnetically interacting spin-$1/2$ system arranged on a $5/7$ skewed ladder lattice (Fig. 1(b)). This system shows high spin gs with $m = 1/4$ even in the absence of a magnetic field $B$, for $J_1 > 2.35^{18}$; $J_1$ is the rung interaction while the interaction between the nearest neighbors on the leg $J_2$ is set to 1. This ladder system is also interesting and unique as it exhibits many plateaus unlike a single plateau at $m = 1/3$ found in a $J_1 - J_2$ model$^{18,23,24}$. We have obtained $m - B$ curve, and find that there are three magnetization plateaus as a function of $B$. These three plateaus are at $m = 1/4, 1/2$ and $3/4$, consistent with the OYA condition. In the gs at $B = 0$, each unit cell has three singlet dimers and two ferromagnetically arranged free spins (Fig. 2(a)). We find that each plateau formation corresponds to successive breaking of a singlet dimer (Fig. 2). We have analytically obtained the widths of these plateaus as a function of $J_1$ from a simple perturbation theory.

The plateaus at $m = 1/4$ and $3/4$ appear even for small $J_1 (> 0.4)$, but the $m = 1/2$ plateau appears only for $J_1 > 0.8$. The width $w_i$ of the $i^{th}$ plateau at $m_i$ represents the magnitude of energy gap in the system for that particular magnetization. We notice that $w_i$ always increases with $J_1$, however, for $m = 1/4$ and $1/2$ it weakly depends on $J_1$; at $m = 3/4$ the plateau width shows almost linear variation with $J_1$. This is consistent with the perturbation theory results. As usual this system also shows cusps in $m - B$ curve at the beginning and end of plateaus and follows the square root dependence, $m \propto (B - B_c)^{1/2}$. The stability of the magnetization plateaus in presence of thermal fluctuation is an important factor for its observation. We notice that the plateaus at $m = 1/4$ and $1/2$ are robust against small thermal fluctuation, while the plateau at $3/4$ survives only up to $T/J_2 = 0.05$. The skewed ladder system can be mapped to a molecular system corresponding to fused five- and seven-membered carbon rings$^{62}$. Such a system corresponds to fused azulene lattice$^{62}$, and may be engineered at the grain boundary of a graphene sheet.

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1. C. K. Majumdar and D. K. Ghosh, J. Math. Phys. \textbf{10}, 1388 (1969); J. Math. Phys. \textbf{10}, 1399 (1969).
2. T. Hamada, J.-i. Kane, S.-i. Nakagawa, and Y. Natsume, J. Phys. Soc. Jpn. \textbf{57}, 1891 (1988).
3. A. V. Chubukov, Phys. Rev. B \textbf{44}, 4693 (1991).
4. R. Chitra, S. Pati, H. R. Krishnamurthy, D. Sen, and S. Ramasesha, Phys. Rev. B \textbf{52}, 6581 (1995).

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M. Kumar, S. Ramasesha, and Z. G. Soos, Phys. Rev. B 10, 335230 (2008).
J. Sirker, Phys. Rev. B 81, 014419 (2010).
M. Kumar, A. Parvej, and Z. G. Soos, J. Phys.: Condens. Matter 20, 316001 (2015).
Z. G. Soos, A. Parvej, and M. Kumar, J. Phys.: Condens. Matter 28, 175603 (2016).
M. Kumar, S. Ramasesha, and Z. G. Soos, Phys. Rev. B 81, 064413 (2010).
M. Kumar and Z. G. Soos, Phys. Rev. B 85, 144415 (2012).
T. Vekua, A. Honecker, H.-J. Mikeska, and F. Heidrich-Meisner, Phys. Rev. B 76, 174420 (2007).
T. Hikihara, L. Kecke, T. Momoi, and A. Furusaki, Phys. Rev. B 78, 144404 (2008).
J. Sudan, A. Lüschter, and A. M. Läuchli, Phys. Rev. B 80, 140402(R) (2009).
D. V. Dmitriev and V. Y. Krivnov, Phys. Rev. B 77, 024401 (2008).
F. Heidrich-Meisner, A. Honecker, and T. Vekua, Phys. Rev. B 74, 020403(R) (2006).
F. Heidrich-Meisner, I. A. Sergienko, A. E. Feiguin, and E. R. Dagotto, Phys. Rev. B 75, 064413 (2007).
F. Heidrich-Meisner, I. P. McCulloch, and A. K. Kolezhuk, Phys. Rev. B 80, 144417 (2009).
A. Parvej and M. Kumar, Phys. Rev. B 96, 054413 (2017).
L. Kecke, T. Momoi, and A. Furusaki, Phys. Rev. B 76, 060407(R) (2007).
A. Parvej and M. Kumar, J. Magn. Magn. Mater. 401, 96 (2016).
K. Okunishi and T. Tonegawa, J. Phys. Soc. Jpn. 72, 479 (2003).
K. Okunishi and T. Tonegawa, Phys. Rev. B 68, 224422 (2003).
M. Oshikawa, M. Yamanaka, and I. Affleck, Phys. Rev. Lett. 78, 1984 (1997).
D. C. Cabra, A. Honecker, and P. Pujol, Phys. Rev. Lett. 79, 5126 (1997).
K. Totsuka, Phys. Rev. B 57, 3454 (1998).
T. Sakai and M. Takahashi, Phys. Rev. B 57, R3201 (1998).
T. Sakai and S. Yamamoto, Phys. Rev. B 60, 4053 (1999).
A. Honecker, F. Mila, and M. Troyer, Eur. Phys. J. B 15, 227 (2000).
K. Okamoto, N. Okazaki, and T. Sakai, J. Phys. Soc. Jpn. 70, 636 (2001).
K. Hida and I. Affleck, J. Phys. Soc. Jpn. 74, 1849 (2005).
J. Alicea and M. P. A. Fisher, Phys. Rev. B 75, 144411 (2007).
F. D. M. Haldane, Phys. Lett. 93A, 464 (1983).
F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).
I. Affleck and E. H. Lieb, Lett. Math. Phys. 12, 57 (1986).
H. Nakano and M. Takahashi, J. Phys. Soc. Jpn. 67, 1126 (1998).
K. Morita, T. Sugimoto, S. Sota, and T. Tohyama, Phys. Rev. B 97, 014412 (2018).
J. Schlenenberg, A. Honecker, J. Schnack, J. Richter, and H.-J. Schmidt, Phys. Rev. Lett. 88, 167207 (2002).
H.-J. Schmidt, J. Richter, and R. Moessner, J. Phys. A: Math. Gen. 39, 10673 (2006).
N. Nishimoto, N. Shibata, and C. Hotta, Nat. Commun. 4, 2287 (2013).
S. Capponi, O. Derzhko, A. Honecker, A. M. Läuchli, and J. Richter, Phys. Rev. B 88, 144416 (2013).
S. Pal, A. Mukherjee, and S. Lal, “Non-perturbative approach to quantum liquid ground states on geometrically frustrated heisenberg antiferromagnets,” (2018), arXiv:1810.03935 [cond-mat.str-el].
D. J. J. Farnell, R. Zinke, J. Schulenburg, and J. Richter, J. Phys.: Condens. Matter 21, 406002 (2009).
H. Kikuchi, Y. Fujii, M. Chiba, S. Mitsudo, T. Idehara, T. Tonegawa, K. Okamoto, T. Sakai, T. Kuwai, and H. Ohta, Phys. Rev. Lett. 94, 227201 (2005).
H. Kikuchi, Y. Fujii, M. Chiba, S. Mitsudo, T. Idehara, T. Tonegawa, K. Okamoto, T. Sakai, T. Kuwai, and H. Ohta, Phys. Rev. Lett. 97, 089702 (2006).
B. Gu and G. Su, Phys. Rev. Lett. 97, 089701 (2006).
M. Hase, M. Kohno, H. Kitaazawa, N. Tsujii, O. Suzuki, K. Ozawa, G. Kido, M. Imai, and X. Hu, Phys. Rev. B 73, 104419 (2006).
Y. Zhao, S.-S. Gong, W. Li, and G. Su, Applied Physics Letters 96, 162503 (2010).
A. Maigman, V. Hardy, S. Hbert, M. Drillon, M. R. Lees, O. Petrenko, D. M. K. Paul, and D. Khomskii, J. Mater. Chem. 14, 1231 (2004).
V. Hardy, D. Flahaut, M. R. Lees, and O. A. Petrenko, Phys. Rev. B 70, 214439 (2004).
X. X. Wang, J. J. Li, Y. G. Shi, Y. Tsujimoto, Y. F. Guo, S. B. Zhang, Y. Matsushita, M. Tanaka, Y. Katsuya, K. Kobayashi, K. Yamaura, and E. Takayama-Muromachi, Phys. Rev. B 83, 100410(R) (2011).
V. Hardy, C. Martin, G. Martinet, and G. André, Phys. Rev. B 74, 064413 (2006).
S. Ishiwata, D. Wang, T. Saito, and M. Takano, Chemistry of Materials 17, 2789 (2005).
X. Yao, J. Phys. Chem. A 116, 2278 (2012).
M. Lerner, J. Alaria, D. Stoefferl, S. Colis, and A. Dinia, J. Phys. Chem. C 115, 17190 (2011).
Z. He, J.-I. Yamaura, Y. Ueda, and W. Cheng, J. Am. Chem. Soc. 131, 7554 (2009).
W. Shiramura, K.-i. Takatsu, B. Kurniawan, H. Tanaka, H. Uekusa, Y. Ohashi, K. Takizawa, H. Mitamura, and T. Goto, J. Phys. Soc. Jpn. 67, 1548 (1998).
E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. 16, 407 (1961).
D. C. Cabra, A. Honecker, and P. Pujol, Phys. Rev. B 58, 6241 (1998).
G. Giri, D. Dey, M. Kumar, S. Ramasesha, and Z. G. Soos, Phys. Rev. B 95, 224408 (2017).
S. Thomas, S. Ramasesha, K. Hallberg, and D. Garcia, Phys. Rev. B 86, 180403(R) (2012).
S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
S. R. White, Phys. Rev. B 48, 10345 (1993).
M. Kumar, Z. G. Soos, D. Sen, and S. Ramasesha, Phys. Rev. B 81, 104406 (2010).
K. Okunishi, Y. Hieida, and Y. Akutsu, Phys. Rev. B 60, R6953 (1999).
K. Okunishi, Progress of Theoretical Physics Supplement 145, 119 (2002).
S. K. Saha, D. Dey, M. Kumar, and Z. G. Soos, Phys. Rev. B 99, 195144 (2019).