Towards a classification of bifurcations in Vlasov equations

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We propose a classification of bifurcations of Vlasov equations, based on the strength of the resonance between the unstable mode and the continuous spectrum on the imaginary axis. We then identify and characterize a new type of generic bifurcation where this resonance is weak, but the unstable mode couples with the Casimirs of the system to form a size 3 Jordan block. We derive a three-dimensional reduced noncanonical Hamiltonian system describing this bifurcation. Comparison of the reduced dynamics with direct numerical simulations on a test case gives excellent agreement. We finally discuss the relevance of this bifurcation to specific physical situations.

A wide variety of physical systems are governed over certain time scales by mean-field forces rather than collisions between their constituents. The appropriate kinetic description is then a Vlasov, or Vlasov-like, equation. These equations possess both regular features (such as an infinite number of conserved quantities) and chaotic ones (such as the development of infinitely fine structures in phase space) which make both the understanding of their qualitative behavior and their numerical simulation famously difficult problems, relevant in various physical fields. In particular, Vlasov-like equations have an uncountable number of stationary states, and linear and nonlinear stability studies of these states have led, among other important physical and mathematical concepts, to the discovery of Landau damping close to stable stationary states. We are concerned in this Letter with the question: What happens close to weakly unstable stationary states? This amounts to a study of local bifurcations of Vlasov equations. The rationale is that these bifurcations i) should be universal, i.e. be relevant for all types of Vlasov equations, and ii) could provide basic building blocks to describe the qualitative behavior of these equations.

Vlasov equations are Hamiltonian systems, and while bifurcations in Hamiltonian systems are well known and classified, the specificities of Vlasov equations bring difficulties. First, their Hamiltonian structure is noncanonical, and highly degenerate, which is the origin of the infinite number of conserved quantities, called Casimir invariants. Second, the linearized Vlasov evolution typically features continuous spectrum on the imaginary axis, and a growing unstable mode can create resonances with part of this spectrum, triggering complex dynamics.

The study of Vlasov bifurcations is an old topic. One of the most common Vlasov bifurcations describes the destabilization of a homogeneous stationary state, and is relevant in plasma physics (bump on tail, or two beams instabilities) and fluid dynamics (shear flow instability); the nonlinear development of the instability gave rise to a debate starting in the 60s and concluded in the 90s, when it became clear that it was governed by resonances. This instability is characterized by strong nonlinear effects, at the origin of ”trapping scaling” and an infinite dimensional ”normal form” for the near threshold dynamics called the Single Wave Model (SWM). It is clear that however important this SWM example may be, it is just one case among many other possible bifurcations, which are much less studied.

First, a kind of ”very strong” resonance has already been identified in the literature. The critical eigenvector associated with the instability is in this case singular and entails stronger nonlinear effects altering the trapping scaling characteristic of the SWM bifurcation. While this bifurcation was only studied for two species plasmas, it can likely be found in other contexts as well.

Second, some instabilities do not give rise to resonances, or only to weak ones. The weak resonance happens for homogeneous stationary states with some special velocity distributions, and, more importantly, this is a generic situation for nonhomogeneous stationary states. At first sight, it seems that such nonresonant bifurcations can be studied through standard center manifold reduction and hence would fall into the class of normal finite dimensional canonical Hamiltonian bifurcations. While it is true in some cases, we highlight in this Letter a new type of generic bifurcation for weakly resonant non-oscillatory instabilities: a neutral mode associated with a Casimir invariant combines with a stable and an unstable modes, thereby forms a three-dimensional Jordan block, and controls the bifurcation.

Summarizing the above discussion, Table sketches a
TABLE I. Classification sketch for bifurcations in Vlasov-like systems. VSIR (SR,WR) represents (very) strong (weak) resonance which occur without (no-C) or with (with-C) coupling with the Casimir modes at the linear level. SWM represents the single wave model. SE (RE) represents singular (regular) eigenvectors. The "Scaling" column precise the perturbation amplitude at which non linear effects kick in; $\lambda$ is the instability rate. The fourth line is highlighted, as the main subject of this work.

| Resonance | Casimirs | Reduction | E.Vec. | Scaling Ref. | |
|-----------|----------|-----------|--------|--------------|---|
| VSIR      | no-C     | SE        | $\lambda^{5/2}$ | [18–20] | |
| SR        | no-C     | SWM       | $\lambda^2$     | [15–17] | |
| WR        | no-C     | Finite dim. | RE $\lambda^{1/2}$ | [15] Sec. IVA | |
| WR        | with-C   | Finite dim. | RE $\lambda^2$ | This work | |

classification of bifurcation in Vlasov-like equations. It is the first product of this Letter.

In the following, we turn to the main contribution of this work: we will first show that the scenario involving the weak resonance and coupling with the Casimirs is generic, and study it at linear and nonlinear levels, until we obtain a reduced three-dimensional Hamiltonian which plays the role of a normal form for this bifurcation. We then provide an illustration in a spatially one-dimensional model, where all computations can be performed explicitly and the reduced dynamics can be quantitatively compared with direct numerical simulations of the Vlasov equation.

**Dimension reduction** - The starting point is the following noncanonical Hamiltonian system:

$$\dot{y} = J(y)\nabla H(y)$$

(1)

where $J$ is a degenerate Poisson operator, depending on the state $y$, and $H$ is the Hamiltonian. For the Vlasov equation, one should think of the gradient $\nabla$ as a functional derivative. Our setting is as follows: we consider a family of stationary state $\{y_\mu\}$, where $\mu = 0$ is the critical point changing the stability and $\{y_\mu\}$ are close to $y_0$ for $\mu$ small. We reduce Eq. (1) by projecting it on a lower dimensional space governing the slow dynamics, which is extracted from the linearized equation. We assume that the imaginary part of the unstable eigenvalue is zero (steady state bifurcation), otherwise a coupling with Casimir invariants, which at the linear level correspond to zero modes, is impossible. We also assume no resonance, or weak resonance, with the continuous spectrum on the imaginary axis: hence one expects that a reduction to an effective finite dimensional dynamics is possible close to $\mu = 0$. We will use the matrix formalism for simplicity.

We assume that $y_0$, the stationary point of interest, is not singular for $J$. i.e. $J$ has a constant rank in $V(y_0)$, a neighborhood of $y_0$. The Weinstein’s splitting theorem [23] implies that, up to a local coordinate change, the Poisson operator can be written as

$$J = \begin{pmatrix} J_0 & O_{m,2n} \\ O_{2n,m} & O_{m,m} \end{pmatrix}, \quad J_0 = \begin{pmatrix} O_{n,n} & I_n \\ -I_n & O_{n,n} \end{pmatrix}$$

(2)

in $V(y_0)$ including a part of $\{y_\mu\}$. Here, $O_{k,l}$ is the zero matrix of size $k \times l$ and $I_n$ is the unit matrix of size $n$. The appropriate change of variable can be built order by order [23], where the procedure is called "heatification". In practice, we will only need the lowest order. The degenerate part, $O_{m,m}$ in $J$, corresponds to the Casimir invariants $z$ with the notation $y = (x,z) (x \in \mathbb{R}^{2n}, z \in \mathbb{R}^m)$; it permits $\nabla H(y_0) \neq 0$ even at the stationary point $y_0$. Nevertheless, we may assume $\nabla H(y_0) = 0$ by adding to $H$ a linear combination of the Casimir invariants. The linearized equation around $y_\mu$ is, therefore, $\dot{\eta}_\mu = J S_\mu \eta_\mu$, where $\eta_\mu = y - y_\mu$ and $S_\mu$ is the Hessian matrix of $H$ at $y_\mu$.

Our first result is to show that the linearized operator at $y_0$ generically has a three-dimensional Jordan block with generalized eigenvalue 0. The following proof by the matrix formalism is justified by the weak resonance condition (see Supplemental Material (SM) [24] for an example in an explicitly infinite setting). Omitting the subscript 0, we write the linearized equation at $y_\mu$ as

$$\dot{\psi} = J S \psi =: L \psi, \quad S = \begin{pmatrix} S_{xx} & S_{xz} \\ S_{zx} & S_{zz} \end{pmatrix}.$$  

(3)

where $S_{xx} \in \mathbb{R}^{2n \times 2n}$ and $S_{zz} \in \mathbb{R}^{m \times m}$. Clearly, rank($L$) $\leq 2n$. Furthermore, rank($S_{xx}$) $< 2n$ because $y_0$ is a critical stationary point at which the stability changes. The generic case gives rank($S_{xx}$) $= 2n - 1$ together with rank($L$) $= 2n$. Denoting the inner product on $\mathbb{R}^{2n}$ (resp. $\mathbb{R}^{2n+m}$) by $\langle \cdot, \cdot \rangle_n$ (resp. $\langle \cdot, \cdot \rangle_{n,m}$) and $\psi = (\xi, \zeta)$ (\(\xi \in \mathbb{R}^{2n}, \zeta \in \mathbb{R}^m\)), we make two remarks: i) The kernel of $L^\dagger$, adjoint operator of $L$, is Ker$L^\dagger = \{0_n\} \times \mathbb{R}^m$, where $0_n$ is the origin of $\mathbb{R}^{2n}$; hence the equation $L \psi = v$ has a solution if and only if $\langle \zeta, v \rangle_{n,m} = 0$ for any $\zeta \in \text{Ker}L^\dagger$, i.e. if the last $m$ coordinates of $v$ vanish. ii) The equation $J_0 S_{xx} \xi = w$ has a solution if and only if $\langle J_0 \xi_0, w \rangle_n = 0$, where $\xi_0$ is a vector spanning Ker($S_{xx}$).

The critical eigenvector is $\psi_0 = (\xi_0, 0)$. By i) the equation $L \psi_1 = \psi_0$ has a solution. This equation writes for $\psi_1 = (\xi_1, \xi_1)$:

$$J_0 S_{xx} \xi_1 + J_0 S_{xz} \xi_1 = \xi_0.$$  

(4)

Since $\langle J_0 \xi_0, \xi_0 \rangle_n = 0$, by ii) the above equation has a solution with $\xi_1 = 0$, and we have found the first generalized eigenvector $\psi_1 = (\xi_1, 0)$. Again by i) $L \psi_2 = \psi_1$ has a solution because the last $m$ coordinates of $\psi_1$ vanish. However, in general $\langle J_0 \xi_0, \xi_1 \rangle_n \neq 0$, hence by ii) it is impossible to find $\psi_2$ with a vanishing second component: $\psi_2 = (\xi_2, \xi_2 \neq 0)$. It is now clear, by i) again,
that $L\psi = \psi_2$ has no solution, and there are only two
generalized eigenvectors, forming a size 3 Jordan block.
Furthermore, $\psi_2$ has a zero non component $\zeta_2$ along the
direction of the Casimir, this is the specificity of this
bifurcation. If the Hamiltonian does not induce any cou-
pling with the Casimir modes at the linear level, i.e.
$S_{xz} = 0$, the assumption rank($L$) = $2n$ breaks down,
and rank($L$) = $2n - 1$ instead. Generically $L$ then has a
size 2 Jordan block, without coupling with the Casimir
modes.

We now study the bifurcation at the nonlinear level
by projecting the infinite dimensional dynamics onto
$E_0 = \text{Span}(\psi_0, \psi_1, \psi_2)$. We build an invariant three-
dimensional manifold whose tangent space at $y = y_0$
is $E_0$, through a local Taylor expansion together with
the expansion on the bifurcation parameter $\mu$. Here,
the nonlinearity of the Vlasov equation is quadratic,
hence at leading order it will be enough to approxi-
mate this manifold by $E_0$. We take an initial condi-
tion $y(t = 0) = y_0 + \sum_i A_i(0)\psi_i$; up to quadratic or-
der in the $A_i$’s, the evolution is $y(t) = y_0 + \sum_i A_i(t)\psi_i$,
and we aim to determine the $A_i(t)$. The strategy is:
$\text{i) restrict the Poisson structure to the subspace } E_0$;
$\text{ii) expand and truncate the Hamiltonian up to cubic or-
der in the } A_i$’s. After appropriate changes of variables
$(A_i) \rightarrow u = (u_0, u_1, u_2)$, we are left with the reduced
dynamics $\dot{u} = J_{\text{red}}\nabla H(u, \mu)$ with

$$J_{\text{red}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H(u, \mu) = H_2(u, \mu) + H_3(u),$$

where $H_k$ are homogeneous polynomials of degree $k$ in
its $u$ variables. We keep only the leading terms in $\mu$, of
order $\mu u^2$.

Normal form of the bifurcation- Our last task is to
provide a normal form for the reduced Hamiltonian, i.e.
make it as simple as possible through changes of vari-
bles. To recover the size 3 Jordan block at $\mu = 0$ from
$J_{\text{red}}\nabla H_2$, we set the quadratic Hamiltonian $H_2$
as

$$H_2(u, \mu) = (u_1^2 - \mu u_0^2)/2 - u_0 u_2.$$  

The parameter $\mu$ controls the stability of the origin. The
idea to derive the normal form is to transform the coor-
dinates to $U = (Q, P, Z)$, defined as $u = U + T(U)$, in
order to simplify $H(U + T(U)) =: \tilde{H}(U)$. We assume that
$T(U)$ is a homogeneous polynomial of order 2 and thus
has $6 \times 3 = 18$ parameters. Imposing to keep the standard
form [21] for $J_{\text{red}}$ reduces the number of free parameters
to 10. The cubic term of $\tilde{H}(U)$ is $H_3(U) + T \cdot \nabla H_2(U)$.
All terms in $H_3(U)$ can be eliminated by appropriate
choices of the 10 free parameters left in $T$, except the $Q^3$
and the $ZQ^2$ ones, see SM [22]. Moreover, the coefficient
of the remaining $Q^3$ term can be scaled to 1, and, since
$Z$ is conserved by the dynamics, the $ZQ^2$ term can be
absorbed in a redefinition of $\mu$.

Consequently, around the critical point, the normal
form of the reduced Hamiltonian is

$$H_{\text{red}} = P^2/2 + \Phi(Q, Z), \quad \Phi = -\mu Q^2/2 - QZ + Q^3$$

up to the cubic order. This Hamiltonian provides a kind
of three-dimensional “fish”-shape bifurcation [1], with an
important observation: the value of the Casimir invari-
ant, $Z$, controls the bifurcation; the details are shown
on Fig. [1]. Note that if $Z = 0$ the oscillation amplitude
scales as $\lambda^2 \sim \mu$.

FIG. 1. Sketch of the phase space portraits according to the
reduced Hamiltonian. Since $Z$ is a conserved quantity de-
dpending on the initial perturbation, it can be thought of as an-
other parameter controlling the bifurcation. For $\mu^2 + 12Z > 0$,
$\partial \Phi / \partial Q = 0$ has two real solutions corresponding to one sta-
ble and one unstable stationary states. Depending on the ini-
tial condition, trajectories in the $(Q, P)$ plane can be trapped
around the stable state, or unbounded, eventually leaving the
perturbative regime. For $\mu^2 + 12Z < 0$, there is no stationary
state, and all trajectories are unbounded. At variance with the
finite dimensional cases, when $(Q, P, Z) = (0, 0, 0)$ is a sta-
ble stationary state (i.e. $\mu < 0$), there are not necessar-
ily purely imaginary eigenvalues close to 0, and the reduced
Hamiltonian may not be meaningful.

We conclude that under the hypotheses: i) steady state
bifurcation and ii) weak resonance with the continuous
spectrum, Fig. [1] describes a new type of generic bifur-
cation expected in Vlasov-like systems. We turn now to
explicit computations in Vlasov equations in a two-
dimensional phase space to demonstrate that this bifur-
cation indeed occurs and accurately describes the dynam-
ical behavior of the system in the vicinity of the bifurca-
tion point.

Explicit example- We consider a periodic domain $T = [0, 2\pi]$ in
the space variable $q$, hence the phase space is
$(q, p) \in T \times R$, and the Vlasov equation reads for the

density $F(q,p,t)$

$$\frac{\partial F}{\partial t} + \frac{\partial H[F]}{\partial p} \frac{\partial F}{\partial q} - \frac{\partial H[F]}{\partial q} \frac{\partial F}{\partial p} = 0,$$  \hspace{1cm} (8)

where $H[F] = p^2/2 + V[F] + \text{Cst}$ is the one-body Hamiltonian with

$$V[F](q,t) = \int v(q - q')F(q',p',t)dq'dp'$$  \hspace{1cm} (9)

and $v(q)$ is the two-body potential. To simplify explicit computations and numerical simulations, we will use $v(q) = -\cos q$, i.e., the so-called Hamiltonian mean-field (HMF) model [23][27].

We take the family of “Fermi-Dirac” stationary states

$$F_\mu(J) = \mathcal{N}^{-1} \frac{1}{1 + e^{\beta[H(J) - (\mu - \kappa)]}},$$  \hspace{1cm} (10)

where $\mathcal{N}^{-1}$ is the normalization factor, $\mu$ controls the bifurcation, and $\kappa(\beta)$ is chosen so that the critical point is $\mu = 0$. The rotational symmetry of the HMF model permits to write the stationary potential as $V[F_\mu] = -M_\mu \cos q$ without loss of generality, where $M_\mu = M[F_\mu] = \int \cos qF_\mu(J)dqdp \neq 0$ is the stationary magnetization. The action variable $J$ is defined for the pendulum Hamiltonian $H_\mu = H[F_\mu] = p^2/2 + M_\mu(1 - \cos q)$. The family (10) undergoes a bifurcation [21]: $F_{\mu < 0}$ is stable, and $F_{\mu > 0}$ is unstable. The trajectories for $H_\mu$ are oscillating around $(q,p) = (0,0)$ or rotating along the torus $T$, their frequency begin $\Omega_\mu(J) = dH_\mu/dJ$. The definitions of $J$ and $\Omega_\mu$ depend on $\mu$ through $M_\mu$; however, this dependence does not enter in the equations up to the second order, hence we may use $\mu = 0$ for the definitions of these quantities. As pointed out in [21], trajectories near the zero frequency correspond to trajectories close to the separatrix, hence their density is small. The small density ensures that the weak resonance condition is satisfied.

As described in [24], we start from the Vlasov dynamics [8] in a neighborhood of the critical stationary state $F_0$, and reduce it to Hamiltonian [7], with explicit expressions of the changes of variables, coefficients and initial conditions involved: hence we can directly and quantitatively compare the predictions of Fig. [1] with numerical simulations of [8]. This comparison is presented on Fig. [2]. We have used the initial condition

$$F(t = 0) = F_\mu + \varepsilon \cos q e^{-\beta Tr^2},$$  \hspace{1cm} (11)

with a parameter $\varepsilon$ controlling the size and amplitude of the initial perturbation. The Vlasov simulations are performed using the algorithm of [28], and we use the analytic solution of the reduced dynamics in terms of Weierstrass $\wp$–function [24]. We give four remarks. i) The agreement is good, both in terms of frequency and amplitude, over fairly long time scales. ii) There is a small damping (and frequency shift) acting on the direct numerical simulation (DNS), an effect that we attribute to the numerical dissipation well known in Vlasov simulations [23][30]. We confirmed that hypothesis by varying grid sizes (see SM [24]). However, one cannot exclude the possibility of a weak Landau damping like effect not described by the reduced Hamiltonian. iii) Changing the initial perturbation amplitude has an important effect on the dynamics: this is a signature of the importance of the $z$ coordinate, representing the coupling with the Casimirs. iv) The divergence in the reduced dynamics may be suppressed by the quartic order Hamiltonian [24].

To summarize, we have first identified and described on general theoretical grounds a new type of bifurcation for Vlasov systems, and then proved that it indeed occurs on a simple system. We discuss now its possible relevance in more realistic physical systems; we need to find situations where the basic conditions of the weak resonance and possible coupling with Casimir modes are satisfied. Instabilities of Bernstein-Greene-Kruskal modes in plasmas provide a vast class of natural candidates. The simplest cases, based on the 1D Vlasov-Poisson equation, are similar to the HMF example studied above, and we know that bifurcations do occur (see for instance [31]): we expect some of these bifurcations to be described by the theory put forward in this paper. Radial orbit instability is well known in astrophysics (see for instance [32]), and believed to play a role in determining the structure of galaxies. It occurs in self-gravitating systems, when the amount of particles (usually stars) with small angular momentum increases. The nonlinear analysis in [33]...
suggests similarities with the phenomenology of Fig. 1; in particular, the instability is non-oscillating, and, depending on the initial perturbation, the saturated state may be close to the reference stationary state, or far away. Still in astrophysics, gravitational loss cone instability (see for instance [54]) could also present a similar phenomenology, however we are not aware of a nonlinear analysis of this situation. Pressure waves in bubbly fluids have also been described, somewhat unexpectedly, by a collisionless Hamiltonian kinetic equation, that is a kind of Vlasov equation [25, 30]. An instability due to the coherent oscillations of the bubbles may appear; since a single bubble behaves as a nonlinear oscillator, and its frequency never vanishes, we expect that nonresonant Vlasov bifurcations can also be observed in this case if the instability is non-oscillatory. We conclude that many physical systems from very different fields can be expected to follow the phenomenology in Fig. 1 specific studies and simulations in each case are now needed to confirm or infirm these predictions, and assess their physical importance.

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