A refinement of weak order intervals into distributive lattices

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Abstract. In this paper we consider arbitrary intervals in the left weak order of the symmetric group. We show that the Lehmer codes of permutations in an interval forms a distributive lattice under the product order. Furthermore, the rank-generating function of this distributive lattice matches that of the weak order interval. We construct a poset such that the order ideals of the poset, ordered by inclusion, is isomorphic to the poset of Lehmer codes of permutations in the interval. We show that there are at least $\left(\left\lfloor \frac{n}{2} \right\rfloor\right)!$ permutations in $S_n$ that form a rank-symmetric interval in the weak order.

1 Introduction and preliminaries

1.1 Introduction

Our results concern intervals in the weak order of the symmetric group $S_n$. Intervals in this fundamental order can arise in unexpected contexts. For example, Björner and Wachs [3, Theorem 6.8] showed that the set of linear extensions of a regularly labeled two-dimensional poset forms an interval in the weak order. The Bell classes defined by Rey in [5] are also weak order intervals [3, Theorem 4.1].

Stembridge [7, Theorem 2.2] showed that the interval $\Lambda_w = [\text{id}, w]$ in the weak order is a distributive lattice if and only if $w$ is a fully commutative element. Recall that the Lehmer code of a permutation $w \in S_n$ is an $n$-tuple
that encodes information about the inversions of $w$. Our main theorem, Theorem 3.3, states that the set of Lehmer codes for permutations in $\Lambda_w$, ordered by the product order on $\mathbb{N}^n$, is a distributive lattice. Furthermore, the rank-generating function of $\Lambda_w$ matches that of the corresponding distributive lattice.

![Diagram](image)

**Figure 1.** The left weak order interval $\Lambda_{32514}$ on the left is not a distributive lattice due to the subinterval $[12435, 32415]$. Restricted to the Lehmer codes of permutations in $\Lambda_{32514}$, the product order on $\mathbb{N}^5$ refines the left weak order. By Theorem 3.3, this refinement results in a distributive lattice.

The fundamental theorem of finite distributive lattices states that any finite distributive lattice is isomorphic to the set $J(P)$ of down-closed subsets of a finite poset $P$, ordered by inclusion. In light of Theorem 3.3 we construct a finite poset $M_w$ associated to the set of Lehmer codes of permutations in $\Lambda_w$. In section 4, we give a chain decomposition of $M_w$ in which the chains are determined by the Lehmer code. The relations between the chains are determined by an extension to the Lehmer code that we introduce in section 2. The construction of $M_w$ and its properties are summarized by Theorem 4.12.

Our current work is partially motivated by questions given at the end of [8] regarding the rank-generating function of $\Lambda_w$. One question asks which $w \in S_n$ are such that the interval $\Lambda_w$ is rank-symmetric. In Proposition 5.2, we show that there are at least $\left(\left\lfloor \frac{n}{2} \right\rfloor \right)!$ such permutations in $S_n$.  

2
1.2 Preliminaries

We use the conventions that \(\mathbb{N} = \{0, 1, 2, \ldots\}\) and \([n] = \{1, \ldots, n\}\). To specify permutations, we use 1-line notation. That is, we say \(w = w_1 w_2 \cdots w_n\) to specify the permutation satisfying \(w(i) = w_i\) for all \(i \in [n]\).

For any poset \((P, \leq)\), we say that \(P\) is ranked if there is a function \(\rho : P \to \mathbb{N}\) satisfying \(\rho(x) = 0\) for minimal elements \(x \in P\) and \(\rho(y) = \rho(x) + 1\) whenever \(y\) covers \(x\). Whenever \(P\) is ranked and finite, the rank-generating function for \(P\) is defined by

\[
F(P, q) = \sum_{x \in P} q^{\rho(x)}.
\]

For any poset \((P, \leq)\), a down-closed subset \(I \subseteq P\) is called an order ideal. That is, a subset \(I \subseteq P\) is an order ideal if \(y \in I\) whenever \(x \in I\) and \(y \leq x\). We denote the weak order interval \([\text{id}, w]\) by \(\Lambda_w\).

The standard convention for what is called “the weak order on \(S_n\)” is the notion of right weak order described in [2, Chapter 3]. Our constructions are based on the left weak order described below.

Fix \(n \in \mathbb{N}\) throughout the sequel.

**Definition 1.1.** Let \(w \in S_n\) and set

\[
\text{Inv}(w) = \{(i, j) \in [n] \times [n] : i < j \text{ and } w(i) > w(j)\}.
\]

The set \(\text{Inv}(w)\) is called the inversion set of \(w\) and each pair \((i, j) \in \text{Inv}(w)\) is called an inversion of \(w\). Regarding \(w \in S_n\) as a permutation in \(S_{n+1}\) satisfying \(w(n+1) = n+1\), set

\[
\overline{\text{Inv}}(w) = \{(i, j) \in [n] \times [n+1] : i \leq j \text{ and } w(i) \leq w(j)\}.
\]

We call \(\overline{\text{Inv}}(w)\) the set of non-inversions of \(w\) and each pair \((i, j) \in \overline{\text{Inv}}(w)\) is called a non-inversion of \(w\).

The choice to include pairs of the form \((i, i)\) or \((i, n+1)\) in the definition of non-inversion simplifies later characterizations and proofs. Note that \(\overline{\text{Inv}}(w)\) is the complement of \(\text{Inv}(w)\) relative to a set of ordered pairs \((i, j)\) satisfying \(i \leq j\). In particular, when \((i, j) \in \overline{\text{Inv}}(w)\), we have \(i \leq j\).
The length $\ell(w)$ of $w$ is defined by $\ell(w) = |\text{Inv}(w)|$. The left weak order $(S_n, \leq_L)$ is defined by the covering relations

$$v \prec_L w \text{ if and only if } w = s_i v \text{ and } \ell(w) = \ell(v) + 1,$$

where $s_i = (i \ i + 1)$ is an adjacent transposition in $S_n$. It is known that $(S_n, \leq_L)$ is a ranked poset, where length is the rank function.

The right weak order $(S_n, \leq_R)$ has a similar definition where the condition $w = s_i v$ is replaced by $w = vs_i$. Thus $u \leq_R w$ if and only if $u^{-1} \leq_L w^{-1}$.

The results of our paper can be translated to the right weak order by using the fact that

$$(\Lambda_w, \leq_R) \cong (\Lambda_{w^{-1}}, \leq_L).$$

Also, the dual of [2, Proposition 3.1.6] states that $[\text{id}, wv^{-1}] \cong [v, w]$ for intervals the left weak order. Thus our results for principal order ideals can be translated to arbitrary intervals in the left weak order.

For this paper, the following characterization of left weak order will be more convenient to use than the definition.

**Lemma 1.2.** Let $v, w \in S_n$. Then $v \leq_L w$ if and only if $\text{Inv}(v) \subseteq \text{Inv}(w)$. Consequently, we have $v \leq_L w$ if and only if $\overline{\text{Inv}(w)} \subseteq \overline{\text{Inv}(v)}$.

**Proof.** This is a dual version of [3, Proposition 3.1].

For each $i \in [n]$, let $c_i(w)$ be the number of inversions of $w$ that have first coordinate equal to $i$. The finite sequence

$$c(w) = (c_1(w), \ldots, c_n(w))$$

is called the Lehmer code for $w$. We view $c$ as a function

$$c : S_n \to \prod_{i=1}^{n} [0, n - i]$$

mapping each $w \in S_n$ to an $n$-tuple that satisfies the bound $0 \leq c_i(w) \leq n-i$. It is known that $c$ is a bijection and that

$$\sum_{i=1}^{n} c_i(w) = \ell(w).$$

Whenever we need $c_{n+1}(w)$ to be defined, we make the reasonable convention that $c_{n+1}(w) = 0$. 

4
2 Extended codes and the weak order

We define an extension of the standard Lehmer code. This extended code is used to characterize weak order in terms of codes and is central to the construction given in Section 4.

**Definition 2.1.** Let $w \in S_n$. For $i \in [n]$ and $j \in [n+1]$, define $c_{i,j}(w)$ as the number of inversions $(i, k) \in \text{Inv}(w)$ satisfying $k < j$. This defines a matrix of values that we call the extended Lehmer code for $w$.

Given the extended Lehmer code for $w$, we can recover the original Lehmer code.

**Lemma 2.2.** Let $w \in S_n$. Then $c_i(w) = c_{i,n+1}(w)$ for all $i \in [n]$.

**Proof.** The number of inversions $(i, k) \in \text{Inv}(w)$ satisfying $k < n + 1$ is precisely the number of inversions of the form $(i, k)$.

**Example 2.3.** Let $w = 31524$. The extended Lehmer code of $w$ (in matrix form) is

\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and the Lehmer code of $w$ is $(2, 0, 2, 0, 0)$.

**Lemma 2.4.** Let $v, w \in S_n$ and suppose $v \leq_L w$. Then, for all $i \in [n]$ and $j \in [n+1]$, we have

(a) $c_{i,j}(v) \leq c_{i,j}(w)$;

(b) $c_i(v) \leq c_i(w)$.

**Proof.** Suppose $v \leq_L w$. By Lemma 1.2, we have $(i, k) \in \text{Inv}(w)$ whenever $(i, k) \in \text{Inv}(v)$. The first statement follows from Definition 2.1, which, by Lemma 2.2, proves the second statement.

**Remark 2.5.** There exist $v, w \in S_n$ satisfying the inequality $c_i(v) \leq c_i(w)$ for all $i \in [n]$, but $v \not\leq_L w$. Thus the code inequality given in part (b) of Lemma 2.4 is not enough to characterize the left weak order. Proposition 2.8...
gives an inequality characterization of the left weak order using the extended Lehmer code.

Whether a pair is an inversion or a non-inversion can be detected using the extended Lehmer code. The hypothesis $i \leq j$ below guarantees that either $(i, j) \in \text{Inv}(w)$ or $(i, j) \in \overline{\text{Inv}}(w)$.

**Lemma 2.6.** Let $w \in S_n$, let $i \in [n]$, and let $j, k \in [n+1]$. Suppose $i \leq j \leq k$. Then the following are equivalent:

(a) $(i, j) \in \overline{\text{Inv}}(w)$;
(b) $c_{i,k}(w) \leq c_{i,j}(w) + c_{j,k}(w)$;
(c) $c_i(w) \leq c_j(w) + c_{i,j}(w)$.

**Proof.** By Lemma 2.2, we have $c_i(w) = c_{i,n+1}(w)$ and $c_j(w) = c_{j,n+1}(w)$. Thus the specialization $k = n + 1$ proves that (b) implies (c). Define the following subsets of $\text{Inv}(w)$:

- $A = \{(i, l) \in \text{Inv}(w) : l < k\}$;
- $B = \{(i, l) \in \text{Inv}(w) : l < j\}$;
- $C = \{(i, l) \in \text{Inv}(w) : l = j\}$;
- $D = \{(i, l) \in \text{Inv}(w) : j < l < k\}$.

It is clear that $A = B \cup C \cup D$ and that the union is pairwise disjoint. By Definition 2.1, we have $|A| = c_{i,k}(w)$ and $|B| = c_{i,j}(w)$. Therefore

$$c_{i,k}(w) = c_{i,j}(w) + |C| + |D|.$$  

The remaining implications are proven by comparing $|C| + |D|$ to $c_{j,k}(w)$.

Suppose $(i, j) \in \overline{\text{Inv}}(w)$ so that $w(i) \leq w(j)$ and $|C| = 0$. If $(i, l) \in D$, then $l < k$ and $(j, l) \in \text{Inv}(w)$ since $w(j) \geq w(i) > w(l)$. Thus

$$|C| + |D| = |D| \leq c_{j,k}(w).$$

Therefore (a) implies (b).

Suppose $(i, j) \in \text{Inv}(w)$ so that $w(i) > w(j)$. If $(j, l) \in \text{Inv}(w)$ and $j < l < k$, then $(i, l) \in D$ since $w(i) > w(j) > w(l)$. Thus $|D| \geq c_{j,k}(w)$. Since $(i, j) \in \text{Inv}(w)$, we have $|C| = 1$, which implies $c_{i,k}(w) > c_{i,j}(w) + c_{j,k}(w)$. Specializing to $k = n + 1$ gives the contrapositive of (c) implies (a). \qed
The following lemma, which we frequently use in the sequel, is a simple consequence of transitivity on the usual order of $\mathbb{N}$.

**Lemma 2.7.** Let $w \in S_n$ and let $i, j, k \in [n + 1]$. Suppose $i \leq j \leq k$. Then

(a) If $(i, j) \in \text{Inv}(w)$ and $(j, k) \in \text{Inv}(w)$, then $(i, k) \in \text{Inv}(w)$;
(b) If $(i, j) \in \overline{\text{Inv}}(w)$ and $(j, k) \in \overline{\text{Inv}}(w)$, then $(i, k) \in \overline{\text{Inv}}(w)$.

Proof. Each statement follows from Definition 1.1 and transitivity.

The numerical characterization of the weak order given in Proposition 2.8 below plays a central role in the theorems we obtain. For any pair $(i, j)$, we call the difference $j - i$ the **height** of $(i, j)$.

**Proposition 2.8.** Let $v, w \in S_n$. The following statements are equivalent:

(a) The inequality $v \leq_L w$ holds in the left weak order.
(b) For all $(i, j) \in \overline{\text{Inv}}(w)$, we have

$$c_i(v) \leq c_j(v) + c_{i,j}(w).$$

Proof. Suppose $v \leq_L w$. Suppose $(i, j) \in \overline{\text{Inv}}(w)$. By Lemma 2.4, we have

$$c_{i,j}(v) \leq c_{i,j}(w),$$

and by Lemma 1.2, we have $(i, j) \in \overline{\text{Inv}}(v)$. By Lemma 2.6, this implies

$$c_i(v) \leq c_j(v) + c_{i,j}(v).$$

It follows that $c_i(v) \leq c_j(v) + c_{i,j}(w)$.

For the converse, suppose towards a contradiction that $\text{Inv}(v) \not\subseteq \text{Inv}(w)$. Choose a pair $(i, k)$ of minimal height $k - i$ satisfying the following property:

$$(i, k) \in \text{Inv}(v) \text{ and } (i, k) \in \overline{\text{Inv}}(w).$$

(P)

By Lemma 2.6, we have

$$c_i(v) - c_k(v) > c_{i,k}(v).$$
By hypothesis, we have
\[ c_{i,k}(w) \geq c_i(v) - c_k(v). \]
Therefore \( c_{i,k}(w) > c_{i,k}(v) \). By Definition 2.11 this implies the existence of \( j < k \) such that \((i, j) \in \text{Inv}(w)\) and \((i, j) \in \text{Inv}(v)\). By Lemma 2.7 parts (c) and (d), we have \((j, k) \in \text{Inv}(w)\) and \((j, k) \in \text{Inv}(v)\). Since \( k - j < k - i \), this contradicts the minimality of the height of \((i, k)\) with respect to property (P).

Remark 2.9. Since \( c_{n+1}(v) = 0 \) and \( c_{i,n+1}(w) = c_i(w) \), one of the requirements for \( v \leq_L w \) is that \( c_i(v) \leq c_i(w) \) for each \( i \in [n] \).

3 The distributive lattice \((c(\Lambda_w), \leq_S)\)

We mix partial order and lattice theoretic language in the usual way. When we say that “\((P, \leq)\) is a lattice”, we mean that the join and meet operations are given by least upper bound and greatest lower bound, respectively.

By [1, Section 1.6], the product space \( \mathbb{N}^n \) is a distributive lattice, as is any sublattice of \( \mathbb{N}^n \). We denote the partial order relation on the product space \( \mathbb{N}^n \) by \( \leq_S \). Thus we use the symbol “\( \leq \)" for the usual order on \( \mathbb{N} \), the symbol “\( \leq_S \)" for the product order on the product space \( \mathbb{N}^n \), and the symbol “\( \leq_L \)" for the left weak order on \( S_n \). The product order on \( \mathbb{N}^n \) is given by
\[(x_1, \ldots, x_n) \leq_S (y_1, \ldots, y_n) \text{ if and only if } x_i \leq y_i \text{ for all } i \in [n].\]
The meet and join on \( \mathbb{N}^n \) are given by
\[
(x_1, \ldots, x_n) \lor (y_1, \ldots, y_n) = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}) \text{ and } \\
(x_1, \ldots, x_n) \land (y_1, \ldots, y_n) = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).
\]
For an arbitrary \( w \in S_n \), consider the subposet \((c(\Lambda_w), \leq_S)\) of \( \mathbb{N}^n \). This is the set of Lehmer codes for all \( v \in S_n \) satisfying \( v \leq_L w \) ordered by the product order \( \leq_S \). By Lemma 2.4 we know that \( v \leq_L w \) implies \( c(v) \leq_S c(w) \). The converse is false in general. Therefore the set \( c(\Lambda_w) \) contains as many elements as \( \Lambda_w \), but there are more pairs of permutations related by \( \leq_S \) than by \( \leq_L \). We use Proposition 2.8 to show that the subset \( c(\Lambda_w) \) of \( \mathbb{N}^n \) is a sublattice of \((\mathbb{N}^n, \leq_S)\).
Lemma 3.1. Let $w \in S_n$. The set $c(\Lambda_w)$ of Lehmer codes for the order ideal $\Lambda_w$ is closed under the join and meet of $N^n$.

Proof. Let $x, y \in c(\Lambda_w)$, let $x = (x_1, \ldots, x_n)$, and let $y = (y_1, \ldots, y_n)$. For some $u_1, u_2 \in S_n$ such that $u_1, u_2 \leq_L w$, we have $x = c(u_1)$ and $y = c(u_2)$. Let $v \in S_n$ be the permutation satisfying $c(v) = x \wedge y$. Now suppose $(i, j) \in \Inv(w)$.

Suppose without loss of generality that $\min\{x_j, y_j\} = x_j$. We have

$$x_i \leq x_j + c_{i,j}(w),$$

by Proposition 2.8 applied to $u_1$. Since $\min\{x_i, y_i\} \leq x_i$, we have

$$\min\{x_i, y_i\} \leq \min\{x_j, y_j\} + c_{i,j}(w).$$

Since $\min\{x_i, y_i\} = c_i(v)$ and $\min\{x_j, y_j\} = c_j(v)$, it follows that

$$c_i(v) \leq c_j(v) + c_{i,j}(w).$$

Proposition 2.8 implies that $v \leq_L w$. It follows that $x \wedge y \in c(\Lambda_w)$.

Lemma 3.2. Every finite distributive lattice is ranked.

Proof. See [6, Theorem 3.4.1] and [6, Proposition 3.4.4].

Theorem 3.3. Let $w \in S_n$. The subposet $c(\Lambda_w)$ of $N^n$ is a distributive lattice. Furthermore, we have $F(\Lambda_w, q) = F(c(\Lambda_w), q)$.

Proof. Lemma 3.1 implies $c(\Lambda_w)$ is a sublattice of $N^n$. Every sublattice of a distributive lattice is itself distributive, so $c(\Lambda_w)$ is a distributive lattice. By Lemma 3.2, there is a rank function $\rho$ for $c(\Lambda_w)$.

Let $v \leq_L w$. Let $\id = v_0 <_L \cdots <_L v_k = v$ be a maximal chain in the weak order interval $[\id, v]$. Since $v_{i-1} <_L v_i$, we have $c(v_{i-1}) \leq_S c(v_i)$ by Lemma 2.3. Since $v_i$ covers $v_{i-1}$ in the weak order, we have

$$\sum_{k=1}^n c_k(v_i) = \ell(v_i) = \ell(v_{i-1}) + 1 = \sum_{k=1}^n c_k(v_{i-1}) + 1.$$ 

This implies that $c(v_i)$ covers $c(v_{i-1})$ in the product order. It follows that $\rho(c(v_i)) = \rho(c(v_{i-1})) + 1$ for $i \in [k]$. Since $\rho(c(\id)) = \ell(\id) = 0$, we have $\rho(c(v)) = \ell(v)$ for all $v \in c(\Lambda_w)$. 

9
4 A description of the base poset for $c(\Lambda_w)$

For this section, fix $w \in S_n$.

4.1 Identifying the base poset $M_w$

For any finite poset $P$, we denote the set of order ideals of $P$ by $J(P)$. The set of order ideals of a poset, ordered by inclusion, is a distributive lattice. Conversely, the fundamental theorem of finite distributive lattices states that every finite distributive lattice $L$ is isomorphic to $J(P)$ for some finite poset $P$. We call $P$ the base poset for the distributive lattice $L$.

Recall that a join-irreducible $z \in L$ is a nonzero lattice element that cannot be written as $x \lor y$, where $x$ and $y$ are nonzero lattice elements. It is known that the base poset $P$ of a distributive lattice $L$ is isomorphic to the set of join-irreducibles for $L$. See [6, Theorem 3.4.1] and [6, Proposition 3.4.2] for details.

In this section, we construct the base poset $M_w$ for $c(\Lambda_w)$ by identifying its join-irreducibles.

We denote the $j$-th coordinate of $x \in \mathbb{N}^n$ by $\pi_j(x)$.

**Definition 4.1.** If $i \in [n]$ and $x \in [c_i(w)]$, define $m_{i,x}(w)$ coordinate-wise by

$$
\pi_j(m_{i,x}(w)) = \begin{cases} 
0 & \text{if } j < i; \\
0 & \text{if } (i, j) \in \text{Inv}(w); \\
\max\{0, x - c_{i,j}(w)\} & \text{if } (i, j) \in \overline{\text{Inv}}(w).
\end{cases}
$$

Note that the coordinates of $m_{i,x}(w)$ are as small as possible while satisfying the constraints of Proposition 2.8.

**Lemma 4.2.** Suppose $i \in [n]$ and $x \in [c_i(w)]$. Then $m_{i,x}(w) \in c(\Lambda_w)$.

**Proof.** Let $v \in S_n$ be the permutation such that $c(v) = m_{i,x}(w)$. We use Proposition 2.8 to show that $v \leq_L w$.

Suppose $(j, k) \in \overline{\text{Inv}}(w)$. There are two cases: either $c_j(v) = 0$ or $c_j(v) > 0$. 

10
Suppose that $c_j(v) = 0$. Then $c_j(v) \leq c_k(v) + c_{j,k}(w)$.

Suppose instead that $c_j(v) > 0$. By Definition 4.1, we have $(i, j) \in \text{Inv}(w)$ and $c_j(v) = c_i(v) - c_{i,j}(w)$. By Lemma 2.7 part (b), we have $(i, k) \in \text{Inv}(w)$. By Lemma 2.6, we have

$$c_i,k(w) - c_{i,j}(w) \leq c_{j,k}(w),$$

and by Definition 4.1, we have

$$c_i(v) - c_{i,k}(w) \leq \max\{0, c_i(w) - c_{i,k}(w)\} = c_k(v).$$

Adding the inequalities gives

$$c_i(v) - c_{i,j}(w) \leq c_k(v) + c_{j,k}(w).$$

Since $c_j(v) = c_i(v) - c_{i,j}(w)$, it follows that $c_j(v) \leq c_k(v) + c_{j,k}(w)$. By Proposition 2.8, we have $v \leq_L w$. □

**Lemma 4.3.** Suppose $i \in [n]$ and $x \in [c_i(w)]$. Then $m_{i,x}(w)$ is the unique minimal element of $c(\Lambda_w)$ with $i$-th coordinate equal to $x$.

**Proof.** Suppose $y \in c(\Lambda_w)$ satisfies $\pi_i(y) = x$. Suppose $(i, j) \in \text{Inv}(w)$. By Proposition 2.8, we have $\pi_j(y) \geq x - c_{i,j}(w)$. Since $\pi_j(y) \geq 0$, we have $\pi_j(y) \geq \max\{0, x - c_{i,j}(w)\}$. Therefore, by Definition 4.1, each coordinate of $y$ is at least as large as the corresponding coordinate of $m_{i,x}(w)$.

Uniqueness follows from the finiteness of $c(\Lambda_w)$ and the fact that the meet of all elements with $i$-th coordinate equal to $x$ is an element whose $i$-th coordinate is $x$. □

**Lemma 4.4.** Suppose $m_{i,x}(w) = m_{j,y}(w)$ for some $i, j \in [n]$, $x \in [c_i(w)]$, and $y \in [c_j(w)]$. Then $i = j$ and $x = y$.

**Proof.** Let $v$ be the permutation whose Lehmer code is $m_{i,x}(w)$. Since $x > 0$, there is a permutation $u \in \Lambda_w$ such that $u$ is covered by $v$ in the left weak order. The codes of $u$ and $v$ differ in only one coordinate.

Suppose $i \neq j$. Then the $i$-th coordinate or the $j$-th coordinate of $c(u)$ is the same as $c(v)$. This either contradicts that $c(v)$ has the property of being the minimal element of $c(\Lambda_w)$ with $i$-th coordinate equal to $x$ or that it is the minimal element with $j$-th coordinate equal to $y$. Thus $i = j$. Definition 4.1 then implies that $x = y$. □
Lemma 4.5. Let \( x = (x_1, \ldots, x_n) \) and suppose \( x \in c(\Lambda_w) \). Then
\[
x = \bigvee m_{i,x_i}(w),
\]
where the join is over all \( i \in [n] \) such that \( x_i > 0 \).

Proof. By Lemma 4.3, we have \( m_{i,x_i}(w) \leq_S x \) for all \( i \in [n] \) such that \( x_i > 0 \). Therefore,
\[
\bigvee_{i:x_i>0} m_{i,x_i}(w) \leq_S x.
\]

Since the \( i \)-th coordinate of \( x \) is \( x_i \), the \( i \)-th coordinate of \( x \) is 0 or the same as the \( i \)-th coordinate of \( m_{i,x_i}(w) \). Therefore,
\[
x \leq_S \bigvee_{i:x_i>0} m_{i,x_i}(w).
\]

Proposition 4.6. The set
\[
M_w = \{ m_{i,x}(w) : i \in [n] \text{ and } x \in [c_i(w)] \}
\]
is the set of join-irreducibles for \( c(\Lambda_w) \).

Proof. Suppose \( x \lor y = m_{i,x}(w) \). Then either \( x \) or \( y \) has \( i \)-th coordinate equal to \( x \). Suppose without loss of generality that \( x \) has \( i \)-th coordinate equal to \( x \). By Lemma 4.3, we have \( m_{i,x}(w) \leq_S x \). Since \( m_{i,x}(w) \) is the join of \( x \) and another element, we also have \( x \leq_S m_{i,x}(w) \). Therefore \( m_{i,x}(w) \) is a join-irreducible of \( c(\Lambda_w) \).

For the converse, suppose \( x \) is a join-irreducible of \( c(\Lambda_w) \). By Lemma 4.5
\[
x = \bigvee_{i:x_i>0} m_{i,x_i}(w).
\]
Since \( x \) is a join-irreducible, we have \( x = m_{i,x_i}(w) \) for some \( i \in [n] \).
4.2 A chain decomposition for $M_w$

We can describe the set $M_w$ defined in Proposition 4.6 more explicitly. There is a partition of $M_w$ into chains.

**Definition 4.7.** Let

$$C_i(w) = \{m_{i,x}(w) \in M_w : 1 \leq x \leq c_i(w)\},$$

where $C_i(w)$ is possibly empty. We call the sets $C_1(w), \ldots, C_n(w)$ the *chain decomposition* of $M_w$.

The terminology is justified by the following lemma.

**Lemma 4.8.** Let $C_1(w), \ldots, C_n(w)$ be the chain decomposition of $M_w$. Then each $C_i(w)$ is a chain of $M_w$. Furthermore, we have

$$M_w = C_1(w) \cup \cdots \cup C_n(w),$$

where the union is pairwise disjoint.

**Proof.** By Definition 4.1, we have $m_{i,x}(w) \leq m_{i,y}(w)$ whenever $x \leq y$. By Lemma 4.4, the chains are pairwise disjoint as sets.

**Lemma 4.9.** Suppose $i < j$ and suppose $m_{i,x}(w), m_{j,y}(w)$ are defined. Then

$$m_{i,x}(w) \not\leq_S m_{j,y}(w).$$

**Proof.** By Definition 4.1, the $i$-th coordinate of $m_{i,x}(w)$ is $x > 0$. Since $i < j$ by hypothesis, the $i$-th coordinate of $m_{j,y}(w)$ is 0. Therefore, we have $m_{i,x}(w) \not\leq_S m_{j,y}(w)$.

**Lemma 4.10.** Suppose $(i, j) \in \text{Inv}(w)$. Then, every element of $C_i(w)$ is incomparable with every element of $C_j(w)$.

**Proof.** Let $m_{i,x}(w) \in C_i(w)$ and let $m_{j,y}(w) \in C_j(w)$. If $(i, j) \in \text{Inv}(w)$, then by Definition 4.1, the $j$-th coordinate of $m_{i,x}(w)$ is 0 and the $j$-th coordinate of $m_{j,y}(w)$ is $y > 0$. Therefore, we have $m_{j,y}(w) \not\leq_S m_{i,x}(w)$.

By Lemma 4.9 we have $m_{i,x}(w) \not\leq_S m_{j,y}(w)$. Thus, the chains $C_i(w)$ and $C_j(w)$ are pairwise incomparable. 

13
Lemma 4.11. Suppose \((i, j) \in \text{Inv}(w)\), \(x \in [c_i(w)]\), and \(y \in [c_j(w)]\). Then we have \(m_{j,y}(w) \leq_S m_{i,x}(w)\) if and only if \(y \leq x - c_{i,j}(w)\).

Proof. If \(m_{j,y}(w) \leq_S m_{i,x}(w)\), then by Definition 4.1, we have
\[
y \leq \max\{0, x - c_{i,j}(w)\}.
\]
Since \(y > 0\), we have \(y \leq x - c_{i,j}(w)\).

Conversely, suppose that \(y \leq x - c_{i,j}(w)\). Then \(y \leq \pi_j(m_{i,x}(w))\), which implies \(m_{j,y}(w) \leq m_{i,x}(w)\) by Lemma 4.3. \(\square\)

The theorem below summarizes important properties of \(M_w\). There are no relations between chains \(C_i(w)\) and \(C_j(w)\) when \((i, j) \in \text{Inv}(w)\). Otherwise, if \((i, j) \in \text{Inv}(w)\), then the relations are determined by the extended Lehmer code entry \(c_{i,j}(w)\).

Theorem 4.12. Let \(w \in S_n\) and let
\[
M_w = \{m_{i,x}(w) : i \in [n] \text{ and } x \in [c_i(w)]\} \quad \text{and} \quad C_i(w) = \{m_{i,x}(w) : x \in [c_i(w)]\}.
\]
Then

(a) The set of join-irreducibles for \(c(\Lambda_w)\) is \(M_w\).

(b) As distributive lattices, we have \((J(M_w), \subseteq) \cong (c(\Lambda_w), \leq_S)\).

(c) If \(i < j\) and \(m_{i,x}(w)\), \(m_{j,y}(w)\) are defined, then \(m_{i,x}(w) \not\leq_S m_{j,y}(w)\).

(d) If \((i, j) \in \text{Inv}(w)\) then every element of \(C_i(w)\) is incomparable with every element of \(C_j(w)\).

(e) If \((i, j) \in \text{Inv}(w)\), \(x \in [c_i(w)]\), and \(y \in [c_j(w)]\), then
\[
m_{j,y}(w) \leq_S m_{i,x}(w) \iff y \leq x - c_{i,j}(w).
\]

Proof. Part (a) is given by Proposition 4.6. Part (b) can be proved by using \(\square\) Proposition 3.4.2.

Part (c) is given by Lemma 4.9, part (d) is given by Lemma 4.10, and Part (e) is given by Lemma 4.11. \(\square\)
Example 4.13. Let \( w = 41528637 \). Then \( c(w) = (3,0,2,0,3,1,0,0) \). To construct \( M_w \) we first form the chains \( C_i(w) \) whenever \( c_i(w) > 0 \). Then we add the inter-chain relations using the last part of Theorem 4.12. To refine the disjoint union of the chains, we need the following values of \( c_{i,j}(w) \):

\[
\begin{align*}
  c_{1,3}(w) &= 1, \\
  c_{1,5}(w) &= 2, \\
  c_{1,6}(w) &= 2, \\
  c_{3,5}(w) &= 1, \\
  c_{3,6}(w) &= 1.
\end{align*}
\]

As \( (5, 6) \in \text{Inv}(w) \), the associated chains are pairwise incomparable.

Figure 2. We construct the poset \( M_w \) in two steps. We begin with the chain decomposition of Definition 4.7. Then we use Theorem 4.12 part (e) to add relations between the chains.

5 Rank-symmetry of \( \Lambda_w \)

Given a polynomial \( f \), we denote the polynomial whose coefficients are in the reverse order as \( f \) by \( f^R \). More precisely, we can define \( f^R \) by

\[
f^R(q) = q^{\deg(f)} f(1/q).
\]

A polynomial is symmetric if the coefficients, when read left-to-right, are the same as when read right-to-left. So, a polynomial is symmetric if and only if \( f = f^R \).

A ranked poset \( P \) is rank-symmetric if its rank-generating function \( F(P, q) \) is symmetric. By [8, Corollary 3.11], if a permutation \( w \) is separable, then the interval \( \Lambda_w \) is rank-symmetric. We give another class of rank-symmetric weak order intervals.

A poset is self-dual whenever \( P \cong P^* \). If a ranked poset \( P \) is self-dual, it is rank-symmetric, but the converse is false. The following proposition is not
a characterization of rank-symmetry, but it provides a large class of weak order intervals that are rank-symmetric.

**Proposition 5.1.** Let \( w \in W \). If \( M_w \) is self-dual, then the weak order interval \((\Lambda_w, \leq_L)\) is rank-symmetric.

**Proof.** By Theorem 3.3 and Theorem 4.12 part(a), we have

\[
F(J(M_w), q) = F(\Lambda_w, q).
\]

The result then follows from the fact that \( J(P^*) \cong J(P) \) for any poset \( P \).

There is a standard embedding of \( S_m \times S_n \) into \( S_{n+m} \). If \( v = v_1 \cdots v_m \in S_m \) and \( w = w_1 \cdots w_n \in S_n \), then

\[
v \times w = v_1 \cdots v_m(w_1 + m)(w_2 + m) \cdots (w_n + m)
\]
defines the embedding via \((v, w) \mapsto v \times w\). In \( S_{n+m} \), each \( u \leq_L v \times w \) can be decomposed as \( v' \times w' \), where \( v' \leq_L v \) and \( w' \leq_L w \). Therefore, we have

\[
F(\Lambda_{v \times w}, q) = F(\Lambda_v, q) F(\Lambda_w, q)
\]

By [2] Proposition 3.1.2], an alternative characterization of left weak order is given by

\[
u \leq_L w \iff \ell(u) + \ell(wu^{-1}) = \ell(w).
\]

Using this characterization, it is straightforward to show that

\[
u \leq_L w \iff uw^{-1} \leq_L w^{-1} \iff \ell(uw^{-1}) = \ell(w) - \ell(u).
\]

It follows that \( F(\Lambda_{w^{-1}}, q) = F(\Lambda_w, q) \).

**Proposition 5.2.** For any \( w \in S_n \), the permutation \( w \times w^{-1} \in S_{2n} \) has a rank-symmetric rank-generating function. It follows that there are at least \( \left( \begin{array}{c} n \end{array} \right)! \) permutations with rank-symmetric rank-generating functions.

**Proof.** The rank-generating function of \( \Lambda_{w \times w^{-1}} \) in the left weak order is given by

\[
F(\Lambda_{w \times w^{-1}}, q) = F(\Lambda_w, q) F(\Lambda_{w^{-1}}, q) = F(\Lambda_w, q) F(\Lambda_w, q).
\]

Since \((f \cdot f^R)^R = f \cdot f^R\) for any polynomial \( f \), it follows that \( F(\Lambda_{w \times w^{-1}}, q) \) is symmetric. \( \square \)
6 Counterexamples

We give two orders similar to the weak order on $S_n$ whose intervals do not necessarily have the same rank-generating function as a distributive lattice.

The strong Bruhat order $(S_n, \leq_B)$ is defined similarly to the weak order. The condition $w = s_i v$ where $s_i$ is an adjacent transposition is replaced by the condition $w = t v$ where $t$ is any transposition. Under the strong Bruhat order, the permutation $w = 3412$ has rank-generating function given by

$$F((\Lambda_{3412}, \leq_B), q) = 1 + 3q + 5q^2 + 4q^3 + q^4.$$ 

If there exists a distributive lattice $L$ such that $F(L, q) = F((\Lambda_{3412}, \leq_B), q)$, then the dual $L^*$ is a distributive lattice with rank-generating function

$$F(L^*, q) = 1 + 4q + 5q^2 + 3q^3 + q^4.$$ 

By the fundamental theorem of finite distributive lattices, there is a finite poset $P$ such that $L^* \cong J(P)$. Such a poset $P$ would have 4 minimal elements, which means that there would be at least $\binom{4}{2} = 6$ two-element ideals. Thus no such distributive lattice $L$ exists.

The Coxeter group of type $D_4$ has distinguished generating set

$$S = \{s_1, s_2, s_3, s_4\}$$

subject to the relations

$$s_i^2 = 1 \text{ for all } i \in \{1, 2, 3, 4\};$$

$$(s_i s_j)^2 = 1 \text{ for all } i, j \in \{1, 3, 4\};$$

$$(s_2 s_i)^3 = 1 \text{ for all } i \in \{1, 3, 4\}.$$ 

Let $w = s_2 s_1 s_3 s_4 s_2 s_4 s_3 s_1 s_2$. This element of $D_4$ appeared in [4] as an example of an element with a non-contractible inversion triple. The rank-generating function for the interval $(\Lambda_w, \leq_L)$ is given by

$$F(\Lambda_w, q) = 1 + q + 3q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + 3q^7 + q^8 + q^9.$$ 

Suppose there is a distributive lattice with rank-generating function $F(\Lambda_w, q)$. Then there is a poset $P$ such that $F(J(P), q) = F(\Lambda_w, q)$. Such a poset $P$
has a unique minimal and maximal element. By deleting the maximal and minimal elements, this implies the existence of $P'$ such that

$$F(J(P'), q) = 1 + 3q + 3q^2 + 4q^3 + 4q^4 + 3q^5 + 3q^6 + q^7.$$  

One can exhaustively check that there is no seven element poset $P'$ with three minimal elements and three maximal elements such that $J(P')$ has the above rank-generating function. Therefore, there is no distributive lattice with the same rank-generating function as $\Lambda_w$.

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References

[1] G. Birkhoff. *Lattice Theory*, volume 25. Amer. Math. Soc., 3. edition, 1967.

[2] A. Björner and F. Brenti. *Combinatorics of Coxeter Groups*. Springer, New York, NY, 2005.

[3] A. Björner and M. Wachs. Permutations statistics and linear extensions of posets. *J. Combinatorial Theory, Ser. A*, 58:85 – 114, 1991.

[4] R. M. Green and J. Losonczy. Freely braided elements in Coxeter groups. *Ann. Comb.*, 6:337 – 348, 2002.

[5] M. Rey. Algebraic constructions on set partitions. *Formal Power Series and Algebraic Combinatorics*, 2007.

[6] R.P. Stanley. *Enumerative Combinatorics, Volume I*. Wadsworth and Brooks/Cole, Belmont, CA, 1986.

[7] J.R. Stembridge. On the fully commutative elements of Coxeter groups. *J. Algebraic Combin.*, 5:353 – 385, 1996.

[8] F. Wei. The Weak Bruhat Order and Separable Permutations. arXiv:1001.1080v1 [math.CO], 2010.