Impact of massive tau-neutrinos on primordial nucleosynthesis. Exact calculations

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Abstract

The influence of a massive Majorana $\nu_\tau$ on primordial nucleosynthesis is rigorously calculated. The system of three integro-differential kinetic equations is solved numerically for $m_{\nu_\tau}$ in the interval from 0 to 20 MeV. It is found that the usual assumption of kinetic equilibrium is strongly violated and non-equilibrium corrections considerably amplify the effect. Even a very weak restriction from nucleosynthesis, allowing for one extra massless neutrino species, permits to conclude that $m_{\nu_\tau} < 1$ MeV. For a stricter bound, e.g. for $\Delta N_\nu < 0.3$, the limit is $m_{\nu_\tau} < 0.35$ MeV.

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1 Introduction

The mass of the tau-neutrino is very loosely bound by experiments \([1]\):

\[
m_{\nu_\tau} < 18 \text{ MeV}, \tag{1}
\]

whereas it is much stronger restricted by cosmology. In the case that the \(\nu_\tau\) is stable on the scale of the universe age, \(t_U \approx 10 \text{ Gyr}\), its mass obeys the Gerstein-Zel’dovich limit \([2]\), i.e. roughly \(m_{\nu_\tau} < 10 \text{ eV}\). If it is unstable, the cosmological limit is much less restrictive and depends upon the life-time of \(\nu_\tau\) and possible decay channels.

If the \(\nu_\tau\) is unstable on the cosmological time scale, so that the Gerstein-Zel’dovich limit is avoided, but is stable on the nucleosynthesis time scale, i.e. \(\tau_{\nu_\tau} > 200 \text{ sec}\), then considerations of primordial nucleosynthesis lead to a much better bound than \((1)\). The bound on the \(\nu_\tau\)-mass from nucleosynthesis was first found in ref. \([3]\) and in a slightly improved form in ref. \([4]\). The calculations of the second work predict a somewhat larger value of the frozen energy density of \(\nu_\tau\), but in the translation of this result to the effective number of neutrino species, found from the distortion of \(^4\text{He}\) abundance, a numerical error was done which resulted in an overestimation of the number of additional effective neutrino species. Still, even with the correction of this error the results of ref. \([4]\) are stronger than those of the pioneering paper \([3]\). The calculations of both papers were done under the following basic assumptions. It was assumed that the massive \(\nu_\tau\) and the two massless neutrinos, \(\nu_e\) and \(\nu_\mu\), are in complete kinetic equilibrium so that their energy distributions are given by the canonical expressions:

\[
f_j(E) = \frac{1}{\exp(\beta E - \xi_j) + 1}, \tag{2}
\]

where \(\beta = 1/T\) is the inverse temperature and \(\xi_j\) are dimensionless effective chemical potentials. Another two simplifying assumptions were made, namely that the chemical potentials of massless neutrinos are zero, \(\xi_{\nu_e} = \xi_{\nu_\mu} = 0\), and that the distribution function of \(\nu_\tau\) can be approximated by its Boltzmann limit:

\[
f(E) = \exp[(\mu(t) - E)/T(t)] \ll 1, \tag{3}
\]
which is accurate when the temperature is small in comparison with the mass, \( m > T \). Here we use the standard notation for the chemical potential, \( \mu \equiv \xi T \).

It is well known that with these approximations the system of complicated integro-differential kinetic equations (see eq. (6) below) is reduced to a single ordinary differential equation (eq. (7)) for the number density of \( \nu_\tau \) or, equivalently, for the unknown function \( \xi(t) \) (see e.g. books [5, 6]), which easily can be solved numerically. A similar approach was recently used in ref. [7] where the role of a massive \( \nu_\tau \) in the production of all primordial light elements and not only of \(^4\)He, as in refs. [3, 4], was considered.

The assumption of vanishing \( \xi_{\nu_e} \) and \( \xi_{\nu_\mu} \) was relaxed in ref. [8]. If in addition one assumes validity of Boltzmann statistics, then instead of one differential equation for \( \xi_{\nu_e} \) there appear three ordinary coupled equations for three chemical potentials \( \xi_j(t), j = e, \mu, \tau \). If one uses exact Fermi-Dirac statistics, then the unknown functions \( \xi_j(t) \) remain “inside” the collision integrals but the kinetic equations still remain ordinary differential equations in this approximation, and not integro-differential ones for \( f_j(t, p) \) as in the exact case. This approach permits one to take into account an average heating of massless neutrinos by annihilation of massive tau-neutrinos. It naturally results in a smaller neutron-to-proton ratio and in a weaker influence of a possible non-zero \( m_{\nu_\tau} \) on primordial nucleosynthesis. On the other hand, one can easily see that nonzero \( \xi_{\nu_e} \) and \( \xi_{\nu_\mu} \) (in kinetic equilibrium) give rise to a considerably larger frozen number density of \( \nu_\tau \). Correspondingly the net effect on nucleosynthesis is stronger. We do indeed obtain a large increase in \( n_{\nu_\tau} \), however, a \( \chi^2 \)-fit, for different \( \nu_\tau \) masses, of our resulting distribution functions for the massless neutrinos to exact FD-distributions allowing for both an effective temperature (possibly different from the expected neutrino temperature) and a chemical potential gives \( \xi_{\nu_e} \) in the range from \( +10^{-2} \) to \( -2 \cdot 10^{-2} \) with extremum near \( m_{\nu_\tau} = 5MeV \).

It was found in ref. [8] that together with an overall neutrino heating there are considerable distortions at the high energy tail of massless neutrino spectra, created by annihilation of heavy tau-neutrinos into massless (or light) \( \nu \)'s. The spectral
distortion is especially important for the electronic neutrinos because they directly
influence the frozen $n/p$-ratio, through the reactions:

$$\nu_e n \leftrightarrow p e^- \quad \text{and} \quad e^+ n \leftrightarrow p \bar{\nu}_e.$$  \hfill (4)

The calculations of the spectral distortions were made in the Boltzmann limit and the
problem was reduced to the solution of an ordinary differential equation. According
to ref. 9 the deviation from kinetic equilibrium of the electronic neutrinos is quite
essential and has a noticeable influence on nucleosynthesis. These calculations were
refined in 10. The calculations of the present paper are in agreement with the semi-
analytical estimates of the distortion of $\nu_e$ spectrum made in refs. 9, 10, which results
in a larger $n/p$-ratio. The spectral distortion is not just a peak from $\nu_\tau$-annihilation,
since the excessive energy is redistributed by elastic scattering over the whole spectra
of $\nu_e, \mu$. This effect gives rise to an overall neutrino heating or, in other words, to
larger temperatures of the massless neutrinos relative to the photon temperature.
This leads in turn to a later freezing of the reactions 4 and to a lower $n/p$-ratio.
According to our calculations the average heating produces a stronger effect than the
spectral distortion. This agrees with the statement made in ref. 10 (see also ”Note
added” to ref. 8).

In view of this discussion it is interesting to solve the exact system of integro-
differential kinetic equations numerically (with a good precision) and to find exactly,
without any simplifying assumptions, the impact of possibly massive tau-neutrinos on
primordial nucleosynthesis. The first calculations of this kind were done in ref. 11.
In what follows we will correct some matrix elements for scattering of Majorana
neutrinos presented in ref. 11, and solve the kinetic equations numerically, using a
more accurate code. Our results for $n_{\nu_e}$ are in a good agreement with ref. 11. We also
agree with this paper in the calculation of the changes in light element abundances for
low values of $m_{\nu_e}$. In the high mass range our results are noticeably lower. We ascribe
this difference to a lower momentum cut-off made in ref. 11, $y_{\text{max}} \approx (p/T)_{\text{max}} \approx 13$,
in the region being essential for the reactions 4, while we made the calculations with
a considerably larger cut-off, $y_{\max} = 20$.

Our technique repeats that of ref. [12], where similar calculations were done for massless neutrinos, with evident complications in the case of a massive $\nu_\tau$ due to a larger number of equations and different matrix elements. We conclude that the influence of non-equilibrium corrections on primordial nucleosynthesis is considerably larger than what was calculated in the earlier papers under assumptions of kinetic equilibrium and Boltzmann statistics. The dominant effect comes from an increase of the frozen energy density of massive $\nu_\tau$.

2 Qualitative discussion

The complete set of integro-differential kinetic equations has the form:

$$\left(\partial_t - Hp\partial_p\right)f_j(p_j, t) = I_j^{\text{coll}},$$

where the collision integral for two-body reactions $1 + 2 \rightarrow 3 + 4$ is given by the expression:

$$I_1^{\text{coll}} = \frac{1}{2E_1} \sum \int \frac{d^3p_2}{2E_2(2\pi)^3} \frac{d^3p_3}{2E_3(2\pi)^3} \frac{d^3p_4}{2E_4(2\pi)^3} S |A|_{12 \rightarrow 34}^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) F(f_1, f_2, f_3, f_4),$$

where $F = f_3f_4(1 - f_1)(1 - f_2) - f_1f_2(1 - f_3)(1 - f_4)$, $|A|^2$ is the weak interaction amplitude squared summed over spins of all particles, and $S$ is the symmetrization factor which includes $1/2$ from the averaging over the first particle, $1/2!$ for each pair of identical particles in the initial and final states and the factor 2 if there are 2 identical particles in the initial state; the summation is done over all possible sets of leptons 2, 3, and 4.

Numerical solution of such a set of equations is tremendously more difficult than the solution of the well known simple ordinary differential equation for the number density of massive neutrinos, which is valid in the case when the neutrinos are in kinetic equilibrium and obey Boltzmann statistics:

$$\dot{n}_{\nu_\tau} = \langle \sigma_{\text{ann}} v \rangle (n_{\nu_\tau}^2 - n_{\nu_\tau}^{2\text{eq}}).$$

(7)
One would naturally ask if such a complexity is necessary. Eq. (7) gives a very good approximation if the rate of elastic scattering of the massive particles in question is much larger than their annihilation rate. This is usually true for very non-relativistic particles since their annihilation is Boltzmann suppressed while elastic scattering is not. For the case of $\nu_\tau$, both elastic scattering and annihilation are frozen at temperatures which are not much below $m_{\nu_\tau}$ and rather close to each other so the effects of deviations from equilibrium may be significant. When one calculates the frozen number density of massive $\nu_\tau$, a gradual switch-off of elastic scattering results in a distortion of their spectrum and in a faster-than-equilibrium cooling. Since the cross-section of their annihilation is proportional to the energy squared, this faster-than-equilibrium cooling gives rise to an increase in their frozen number and energy densities and results in a larger effect on nucleosynthesis. A simple attempt to take this non-equilibrium cooling into account was made in ref. [4] where it was assumed that at temperatures above a certain value, $T_{eq}$, which is determined by the strength of the elastic scattering, tau-neutrinos have the equilibrium distribution in energy, $f_{\nu_\tau} = \exp[-\sqrt{p^2 + m_{\nu_\tau}^2}/T + \xi]$, while below $T = T_{eq}$ the complete kinetic decoupling was assumed, so that the momentum $p$ was scaled as inverse universe expansion factor $a(t)$. Unfortunately this simple anzats is not very accurate and more elaborate calculations (ref. [13]) show, that the spectrum is distorted in a more complicated way. Rough estimates of paper [13] show that these non-equilibrium corrections give rise to an increase of $n_{\nu_\tau}$ at the level of 10%. As we see below in fig. 1, the calculations of the present paper show an even bigger effect. The frozen number density of heavy $\nu_\tau$ differ from that of ref. [3] by 35 % and from that of ref. [4] by 15 % in the maximum.

Another important effect is a distortion of the equilibrium spectra of massless neutrinos and in particular of electronic neutrinos. An excess of $\nu_e$ at the high energy tail of the spectrum gives rise to a larger frozen number density of neutrons and correspondingly to a larger abundance of $^4$He. There is also an opposite effect related to an overall increase in the number density of electronic neutrinos due to the annihilation
\( \nu_\tau \nu_\tau \rightarrow \nu_e \nu_e \). This results in a later neutron decoupling and to a smaller frozen ratio
\( n/p = \exp(-\Delta m/T_f) \), where \( T_f \) is the freezing temperature of the \((n \leftrightarrow p)\)-reactions and \( \Delta m = 1.293 \text{ MeV} \) is the neutron-proton mass difference. It is difficult to separate rigorously these two effects using the \( \nu_e \) spectrum calculated here, because of an ambiguity in the choice of the unperturbed distribution, in particular because the effective temperature is different in different parts of the spectrum.

It is also important that the ratios of the photon temperature \( T_\gamma \) and the (average) temperatures of the massless neutrinos are different from the canonical value \( T_\gamma/T_\nu = (11/4)^{1/3} \). It can easily be understood that these ratios depend non-monotonically on \( m_{\nu_e} \). They go up with \( m_{\nu_e} \), reach maximum near \( m_{\nu_e} = 5 \text{ MeV} \) and fall down close to the massless value for large \( m_{\nu_e} \).

One sees that there are several essential effects associated with the distortion of kinetic equilibrium which significantly may change the primordial abundances of \(^4\text{He}\) and other light elements. All of them are automatically taken into account in our calculations below.

3 Kinetic equations

We want to solve kinetic equations numerically for the following system of interacting particles: massless \( \nu_e \) and \( \nu_\mu \), massive \( \nu_\tau \), electron-positron pairs, and quanta of electro-magnetic radiation. There are three unknown distribution function of time and momenta, \( f_j(t,p) \), for the three neutrino species. We assume that electron-positrons and photons are in complete equilibrium so that their distributions are given by eq. (2) with vanishing chemical potentials and with temperature \( T(t) \), which is an unknown function of time (of course photons obey Bose statistics). The assumption of equilibrium of the electro-magnetic component of the primeval plasma is very accurate because of a large strength of the electro-magnetic interactions. Thus there are 3 unknown functions of \( t \) and \( p \) and one unknown function of \( t \). For their determination we have three kinetic equations (3). The fourth necessary equation is the covariant
conservation of energy-momentum in the expanding universe:

\[ \dot{\rho} = -3H(\rho + P). \] (8)

Usually one determines the temporal evolution of temperature from the law of entropy conservation. However, the latter is only true in thermal equilibrium, so if one wants to describe deviations from equilibrium consistently, a more general, though more complicated eq. (8) must be used. In this equation \( \rho \) and \( P \) are respectively total energy and pressure densities in the cosmic plasma. They are given by the expressions:

\[ \rho = \frac{\pi^2 T_\gamma^4}{15} + \int \frac{2dq^2}{\pi^2}\sqrt{q^2 + m_e^2} + \sum_{j=\nu_e,\nu_\mu} \int \frac{dq^3 f_j}{\pi^2} + \int \frac{dq^2}{\pi^2}\sqrt{q^2 + m_{\nu_e}^2} f_{\nu_e}, \] (9)

and:

\[ P = \frac{\pi^2 T_\gamma^4}{45} + \int \frac{dq^4}{3\pi^2}\frac{(q^2 + m_e^2)^{-1/2}}{\exp(E/T_\gamma) + 1} + \sum_{j=\nu_e,\nu_\mu} \int \frac{dq^3 f_j}{3\pi^2} + \int \frac{dq^4 f_{\nu_e}}{3\pi^2}\sqrt{q^2 + m_{\nu_e}^2}. \] (10)

The Hubble parameter, \( H = \dot{a}/a \), is related to the total energy density \( \rho \) in the usual way:

\[ H^2 = \frac{8\pi \rho}{3m_{P l}^2}. \] (11)

where \( a(t) \) is the universe expansion factor (scale factor) and \( m_{P l} = 1.22 \cdot 10^{19} \) GeV is the Planck mass.

It is convenient instead of time and momenta to use the following dimensionless variables:

\[ x = ma(t), \quad y_j = p_j a(t), \] (12)

where \( m \) is an arbitrary parameter with dimension of mass, which we took as \( m = 1 \) MeV, and the scale factor \( a(t) \) is normalized so that \( a(t) = 1/T_\nu = 1/T_\gamma \) at high temperatures or at early times. In terms of these variables the kinetic equations (5) can be rewritten as:

\[ H x \partial_x f_j(x, y_1) = I_j^{\text{coll}}. \] (13)

The relevant reactions and the corresponding matrix elements squared are presented in tables 1 and 2 for the cases when the first particle is \( \nu_e \) (\( \nu_\mu \)) and \( \nu_\tau \) respectively.
There are the following differences with ref. [11]: 1) for the reaction $\nu_a\nu_a \rightarrow \nu_a\nu_a$ we took twice larger contribution because of identical particles in the initial state; 2) we have a different expression for the matrix element squared for elastic scattering of massive Majorana $\nu_\tau$.

Our procedure of solving these equations is essentially the same as in our previous paper [12], where non-equilibrium corrections to the spectra were calculated for the case when all neutrinos were massless. The collision integral is reduced from 9 down to 2 dimensions by the method described in ref. [14] with some complications connected with the momentum dependence of the matrix elements (see ref. [12]). The results of the calculations are presented in Appendix A.

### 3.1 The problem of initial conditions

We solve the system of kinetic equations (13) in the “time” interval $x_{in} \leq x \leq x_{out}$. All the collision integrals in the r.h.s. of eq. (13) are suppressed at large $x$ at least by the factor $1/x^2$. We find that at $x \approx 50$ all the variables, we are interested in, reach their asymptotical values so, to be on the safe side, we choose the final time $x_{out} = 100$.

The nucleosynthesis code of ref. [15] requires the final time $x \approx 2000$ for all energy densities and $n \leftrightarrow p$ rates. We applied a separate program, which calculates these quantities, using the energy conservation law and our final values for the distribution functions $f_{\nu_i}$ and the temperature $T_\gamma$.

At early times, $x \ll 1$, the collision rates in eq. (13) are very high due to the factor $1/x^4$ and therefore all non-equilibrium corrections to the distribution functions are suppressed. Correspondingly we could start at some initial ”moment” $x_{in} \ll 1$ with the equilibrium distribution functions:

$$
\begin{align*}
\left. f_{\nu_e}(\nu_e) \right|_{eq} &= \frac{1}{e^{p/T_{\nu_e}(\nu_e)} + 1}, &
\left. f_{\nu_\tau} \right|_{eq} &= \frac{1}{e^{(E_{\nu_\tau} - \mu)/T_{\nu_\tau}} + 1}, &
\left. f_{e^\pm} \right|_{eq} &= \frac{1}{e^{E_e/T_\gamma} + 1},
\end{align*}
$$

where $T_j$ are the temperatures of the particles, $\mu$ is the chemical potential of $\nu_\tau$ and $E_j$ are the energies, $E_j = \sqrt{p^2 + m^2_j}$. 

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At very early times, when $T_\gamma \gg m_{\nu_{\tau}}$, we can neglect the mass of $\nu_{\tau}$ and put $T_{\nu_e} = T_{\nu_\mu} = T_{\nu_\tau} = 1/a = T_\gamma$ and $\mu = 0$. However, in the case of a large $\nu_{\tau}$ mass (e.g. $m_{\nu_{\tau}} = 20 MeV$) we should start at least at $x_{in} = 0.01$, in order to satisfy the condition $T_\gamma \gg m_{\nu_{\tau}}$. Since the collision rates are very high at this time, the time steps in the numerical integration become very small and the program requires much CPU time.\footnote{CPU time, which is proportional to the number of time steps, grows at least as fast as the rates, i.e. as $1/x_{in}^4$; for example, a run with $x_{in} = 0.04$ requires twice as much CPU time as a run with $x_{in} = 0.05$.} Another problem is that a big number of small time steps leads to an accumulation of numerical errors.

Thus it is very difficult to start at $T_\gamma \gg m_{\nu_{\tau}}$, and we need to find reasonable and convenient initial values of $T_j$ and $\mu$. We assume that at $T_\gamma \geq \max(10 MeV, m_{\nu_{\tau}})$ the collision rates are high enough to keep all the neutrinos in kinetic equilibrium with the distributions given by eqs. (14).

A simple self-consistent way to find the evolution of $T_j$ and $\mu$ in the “almost equilibrium” region is to integrate all the kinetic equations over external momenta and to get the system of differential equations for the number densities $n_{\nu_j}$. However, we can express the collision integrals in terms of $n_{\nu_j}$ only in the limit of Boltzmann statistics, which differs from the exact Fermi-Dirac one by approximately 10\%\footnote{\cite{16}}. Therefore in order to keep a high precision in the calculations we need another approximation for the initial conditions. We require that the temperatures of all neutrino species and the photon temperature are equal, $T_{\nu_j} = T_\gamma$, and assume that the initial value of the chemical potential $\xi_{\nu_{\tau}}$ is zero. The unknown value of the temperature can be found from the energy conservation law (i.e. we do not require that $T_\gamma = 1/a$). Thus our initial conditions at $x = x_{in}$ are the following:

$$
 f_{\nu_e(\mu)} = \frac{1}{e^{E_{\nu_e}/T_\gamma} + 1}, \quad f_{\nu_\tau} = \frac{1}{e^{E_{\nu_\tau}/T_\gamma + 1}}, \quad f_{e^\pm} = \frac{1}{e^{E_e/T_\gamma + 1}}, \quad (15)
$$

where $T_\gamma(x_{in}, m_{\nu_{\tau}})$ is found from the equation $\dot{\rho} = -3H(\rho + P)$, evolved from higher temperatures (or smaller $x$), where expressions (15) are very accurate with “preinitial”
$T_\gamma = 1/a$. Note, that for $m_{\nu_\tau} > 1\text{MeV}$ the photon temperature $T_\gamma(x_{in}, m_{\nu_\tau})$ strongly depends on $m_{\nu_\tau}$.

In order to check the validity of our approximation we compared the energy densities and number densities for all neutrino species using distributions given by eqs. (15) with the same quantities found from the solution of the kinetic equations (13) for the case $m_{\nu_\tau} = 20\text{MeV}$ in the time interval $0.04 < x < 0.05$. We found that both the energy densities and number densities differ in these two approaches only by 0.1%.

For such values of neutrino mass where $m_{\nu_\tau} > T_\gamma(x_{in}, m_{\nu_\tau})$ at $x_{in} = 0.1$, non-equilibrium corrections cannot be neglected, so for $m_{\nu_\tau} \geq 6\text{MeV}$ we choose $x_{in} = 0.05$, and for smaller masses, $m_{\nu_\tau} < 6\text{MeV}$, we take $x_{in} = 0.1$.

A careful choice of initial values of $x_{in}$ and $T_\gamma(x_{in}, m_{\nu_\tau})$ is very important for the numerical results. For example, if for $m_{\nu_\tau} = 20\text{MeV}$ we take $x_{in} = 0.1$ and choose to use the “wrong” temperature evolution $T_\gamma = 1/a$, then we find that the function $r_m = m_{\nu_\tau} n_{\nu_\tau}/n_{eq}$ is smaller by about 6% than $r_m$ calculated with $x_{in} = 0.05$. With our procedure using $T_\gamma \neq 1/a$, we are certain that the final results, in particular the abundances of light elements, have reached a plateau as long as we use $x_{in} \leq 0.1$.

### 3.2 Numerical integration of kinetic equations

For the dimensionless momentum $y$ we took 100 point grid equally spaced in the region $0 \leq y \leq 20$. At $y = 0$ the analytical expressions for the collision integrals differ from those at $y \neq 0$ due to 0/0-uncertainty arising from the factor $1/E_1 p_1$ in front of the integrals and similar vanishing factors in D-functions. Because of that we include the point $y = 0$ separately. This permits to compare the collision integrals at $y = 0$ with those in nearby points, $y \ll 1$, in order to check that numerical errors are small in the region of small momentum, $y < 1$. This is especially important for a massive tau-neutrino, because its distribution function rapidly changes in the course of evolution.

Evolution in time is calculated by the simple Euler method for small $x$ ($x < 1$)
and by the Bulirsch-Stoer method (see e.g. ref. [19]) for large $x \ (x > 1)$. We checked that numerical errors in the distribution functions calculated by the Euler method for $x < 1$ are smaller than 0.1%. The Euler method allows us to save a factor of 2 in CPU time compared to the second order Runge-Kutta method and more than a factor of 10 compared to the Bulirsch-Stoer method. For $x > 1$ the situation is opposite. The Euler method requires very small time steps in order to be precise enough, while the more powerful Bulirsch-Stoer method requires fewer time steps in order to reach the same precision.

An important check of our program is a run with a small value of $\nu_\tau$ mass ($m_{\nu_\tau} = 0.1\,\text{MeV}$) which is in a very close agreement with the results of our calculations in the massless case (see ref. [12]).

4 Discussion of results

The results of our calculations are presented in several figures below. The energy density of massive $\nu_\tau$ is characterized by the quantity $r m_{\nu_\tau}$, where $r = n_{\nu_\tau}/n_{\nu_0}$ and $n_{\nu_0}$ is the number density of massless equilibrium neutrinos. This quantity is presented in fig. 1 for asymptotically large $x$. Our results are rather close to the previous calculations of refs. [3, 4, 11] for small masses, while for large masses they are considerably larger than those obtained in refs. [3, 4] and in fine agreement with the calculations of ref. [11]. The energy density of $\nu_\tau$ influences the rate of the universe cooling and affects two quantities which are important for nucleosynthesis: the frozen neutron-to-proton ratio and the time moment corresponding to $T \simeq 0.065 \, \text{MeV}$, when formation of light elements begins. These two quantities determine the neutron number density at the onset of nucleosynthesis and through that the abundances of light elements and in particular of $^4\text{He}$. A larger $r$ results in a larger mass fraction of $^4\text{He}$ both because of a faster cooling rate and because of a more efficient production of non-equilibrium electronic neutrinos [4]. Of course the relative abundance of $\nu_\tau$ is a function of "time" $x$, so in our numerical calculations we used the exact function...
$r(x)$ and not just its asymptotic value. The same is true for the energy density, for
which the exact expression (9) was used.

In fig. 2 the product $T_\gamma a$ is presented as a function of $x$ for different values of $m_{\nu_\tau}$.
In the standard calculations this quantity coincides with the ratio $T_\gamma/T_{\nu_\tau}$, which tends
to 1.40102 for high $x$. For massless $\nu_\tau$ the product $T_\gamma a$ is slightly smaller than this
result because of the energy transfer from the electro-magnetic components of the
primeval plasma to the massless neutrinos. For a sufficiently heavy $\nu_\tau$ the product
$T_\gamma a$ is larger than the canonical value because now the effect of photon heating by
annihilation of massive $\nu_\tau$ is stronger than the energy loss mentioned above. In this
case massless neutrinos are heated more than electrons, because electrons have to
share the given energy with photons. For very heavy $\nu_\tau$’s the final photon temperature
($T_\gamma a$) can be estimated from entropy conservation to be $(43/36)^{1/3}$ times bigger than
in the massless case.

In fig. 3 we plot the relative deviations of the energy densities of $\nu_e$ and $\nu_\mu$ from
the equilibrium value, $\delta \rho/\rho = (\rho - \rho_{eq})/\rho_{eq}$, where $\rho_{eq}(x) = (7/120\pi^2)(1MeV/x)^4$.
The solid and dashed lines in fig. 3 correspond to $\nu_e$ and $\nu_\mu$ respectively. In fig. 3a the
evolution of $\delta \rho/\rho$ as functions of $x$ for several different masses of $\nu_\tau$ is presented. In
fig. 3b the asymptotic values of $\delta \rho/\rho$ as functions of $m_{\nu_\tau}$ are shown. An explanation
of the different and non-monotonic relative behavior of $\delta \rho_{\nu_e}/\rho$ and $\delta \rho_{\nu_\mu}/\rho$ is given
below.

It is convenient to characterize a distortion of neutrino spectrum by the average
neutrino energy, $\langle E \rangle = \rho/n$. In the equilibrium case it is equal to $\langle E_{eq} \rangle = 3.15T$. In
fig. 4 we present the relative deviation $(\langle E \rangle - \langle E_{eq} \rangle)/\langle E_{eq} \rangle$ for electronic and muonic
neutrinos. For small masses of $\nu_\tau$ this quantity is practically mass independent. In
this mass range $\nu_e$ and $\nu_\mu$ are heated by $e^-e^+$-annihilation. Electronic neutrinos are
heated more efficiently because of a larger coupling to electrons/positrons due to
charged current interactions. For higher $m_{\nu_e}$ the situation is opposite. As we have
already mentioned, with a heavy $\nu_\tau$, massless neutrinos ($\nu_e$ and $\nu_\mu$) are ”overheated”,

\[13\]
so that their excessive energy is transferred to the electro-magnetic component of the primeval plasma \((e^-, e^+, \gamma)\). Now a stronger coupling of \(\nu_e\) to electrons results in a more efficient cooling of \(\nu_e\) than \(\nu_\mu\). This explains why muonic neutrinos are heated less than \(\nu_e\) for a low \(m_\nu\), while for a higher \(m_\nu\) the situation is reversed. However, for a very large \(m_\nu\), we return to hotter \(\nu_e\). This is because very heavy tau neutrinos effectively stopped annihilating while \(\nu_e, \nu_\mu\) and \(e^\pm\) were still in equilibrium. At that stage their temperatures were the same and energy was not transferred either way. At smaller temperatures when \(\nu_e\) and \(\nu_\mu\) were already almost decoupled, the “old” process of \(e^+e^-\) annihilation heated up \(\nu_e\) more than \(\nu_\mu\).

The shape of the electronic neutrino spectrum, disturbed by \(\nu_\tau\) annihilation, is presented in figs. 5 and 6. For large masses the relative distortion \(\delta\) is much larger than 1 for high momentum, so we present the distorted function itself and compare it with the equilibrium one (fig. 5). The magnitude and in particular the non-monotonic behavior of the spectral distortions as functions of \(x\) for different masses \((m = 7 \text{ MeV} \text{ and } m = 20 \text{ MeV})\) are in agreement with the semi-analytical calculations of refs. [9, 10]. The effect of the changed neutrino distribution on light element abundance is essentially \(y^2\delta f\), which is plotted as a function of \(y\) for different neutrino masses in fig. 6.

In order to extract implications of a massive \(\nu_\tau\) on light element abundance, we have modified the standard nucleosynthesis code (ref. [15]) in the following way. We calculate various quantities at the correct photon temperature and import the values to the code. These imported quantities are the neutrino energy densities \(\rho_{\nu_i}\), the 6 weak interaction rates for \((n \leftrightarrow p)\)-reactions (this includes reactions [1] and decay and inverse decay), and finally \(d(\ln a^3)/dT_\gamma\) (with the account of neutrinos) which governs the evolution of the photon temperature.

The effect of a nonzero \(m_{\nu_\tau}\) on the abundances of light elements is illustrated in figs. 7 and 8. Comparing our results with the standard calculations of e.g. helium production with a variable number of massless neutrino species, we express the impact
of massive $\nu_\tau$ on nucleosynthesis in terms of the number of equivalent neutrino species
$\Delta N = N_{\text{eff}} - 3$ \cite{20}. The excessive number of neutrino species found from $^4\text{He}$ is
presented in fig. 7, where our results are compared to those of the previous papers \cite{3, 11, 21}. For low masses we
are in a good agreement with ref. \cite{11}, while for high masses our results are noticeably lower. This difference is probably explained by a
lower momentum cut-off made in ref. \cite{11}. The authors calculated spectral functions of
different neutrinos up to $y_{\text{max}} = 12T_\gamma/T_\nu$, while we were able to do that up to $y_{\text{max}} = 20$. When we diminished our cut-off down to the value of ref. \cite{11}, we reproduced
their higher results in the high mass region. A large sensitivity of $^4\text{He}$-production to
the value of $y_{\text{max}}$ is connected to a strong (quadratic) energy dependence of the cross-sections of $n \leftrightarrow p$ reactions. To independently test the effect of the distortion of the high energy neutrinos we calculated the neutron to proton ratio as a function of the
cut-off of the distortion, numerically solving the kinetic equation for the $n/p$-ratio.
The results is, that the $n/p$ ratio is decreasing with an increase of momentum cut-off.
This continues until rather high momenta (around $y = 16$ for $m_{\nu_\tau} = 20$) when the
effect changes sign, so that $n/p$ is increasing when we increase the cut-off in $y$ from
16 to 20 (as predicted by ref. \cite{3} and refined by ref. \cite{14}). The net result is, that the
increase of the cut-off from $y = 12$ to 20 leads to a decrease of $n/p$ in agreement with
the full nucleosynthesis calculation. With our cut-off in momentum, $y_{\text{max}} = 20$, the
error in $\Delta N_\nu$ is approximately an order of magnitude smaller than that at $y_{\text{max}} = 12$,
i.e. $\delta (\Delta N_\nu) < 0.02$.

In fig. 8 the number of effective extra neutrino species obtained from other light
elements: $^2\text{H}, ^3\text{He}$ and $^7\text{Li}$ is presented, using $\eta_{10} = 3.0$. These curves are given for
illustration only because the best bound on $N_\nu$ is obtained from $^4\text{He}$. The results are
in a qualitative agreement with the simplified calculations of ref. \cite{7}, though there are
quantitative differences.

The recent confusion with the data on abundance of primordial deuterium (ref. \cite{17})
makes it difficult to put a stringent bound on $\Delta N$. It seems that it is rather safe to
conclude that $\Delta N < 1$. In this case the consideration of primordial nucleosynthesis excludes the mass of $\nu_\tau$ in the interval $1 - 22$ MeV. Recall that it is valid for the sufficiently long-lived $\nu_\tau$, i.e. for $\tau_{\nu_\tau} > 200$ sec. Together with the direct experimental bound (1) it gives $m_{\nu_\tau} < 1$ MeV. This result is obtained for $\eta_{10} = 3.0$. At lower $\eta_{10} = 1 - 2$ the lower bound is slightly strengthened. Hopefully a resolution of the observational controversies in the light element abundances will permit to shift this limit to even smaller values of $m_{\nu_\tau}$. The result in fig. 7 is exact in the sense, that it is independent of any measured abundance. One can therefore choose ones favorite experiment, extract a $dN$-limit and apply it to fig. 7. In particular, if the limit on $\Delta N_\nu$ would return to the "good old" value, $\Delta N_\nu < 0.3$, one could conclude from fig. 7 that $m_{\nu_\tau} < 0.35$ MeV. If one believes in $\Delta N_\nu < 0.2$ (see e.g. ref. 18) one finds $m_{\nu_\tau} < 0.25$ MeV, and in the case of an over-optimistic limit, $\Delta N_\nu < 0.1$, which was advocated in the literature a few years ago, we obtain $m_{\nu_\tau} < 0.15$ MeV.

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A Kinetic equations

After reducing the collision integral for two-body reactions in eq. (3) from nine to two dimensions the 3 kinetic equations for the distribution functions are solved numerically. For the sake of brevity we introduce some notation: $f_a(p_j) \equiv f_a^{(j)}$, $d_1 = D_1$, $d_2(3, 4) = D_2(3, 4)/E_3E_4$ and $d_3 = D_3/E_1E_2E_3E_4$, with D-functions defined in Appendix A of ref. [12]. The functions $d_1$ and $d_3$ are symmetric in all four arguments.
We introduce one more condensed notation:

\[ F(f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}) = f^{(3)} f^{(4)} (1 - f^{(1)})(1 - f^{(2)}) - f^{(1)} f^{(2)} (1 - f^{(3)})(1 - f^{(4)}), \quad (16) \]

and write the coupled system of kinetic equations as follows:

\[
H x \partial_x f_{\nu e}^{(1)} = \frac{G_F^2}{2\pi^3 p_1} \int dp_2 dp_3 dp_4 \delta(E_1 + E_2 - E_3 - E_4) \\
\{ F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) \\
[6d_1 + 2d_2 (1, 2) + 2d_2 (3, 4) - 2d_2 (1, 4) - 2d_2 (2, 3) + 6d_3] \\
+ [F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) + F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)})] \\
[d_1 - d_2 (1, 4) - d_2 (2, 3) + d_3] \\
- F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) [m_{\nu e}^2 / 2(d_1 + d_2 (1, 2))/E_3 E_4] \\
+ [F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) + F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)})] \\
[2d_1 + d_2 (1, 2) + d_2 (3, 4) - d_2 (1, 4) - d_2 (2, 3) + 2d_3] \\
+ F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) [m_{\nu e}^2 (d_1 - d_2 (1, 3))/E_2 E_4] \\
+ F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) [4(g_L^2 + g_R^2) (d_1 - d_2 (1, 4) - d_2 (2, 3) + d_3) \\
+ 4g_L g_R m_{\nu e}^2 (d_1 + d_2 (1, 2))/E_3 E_4] \\
+ F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) [4(g_L^2 + g_R^2) (2d_1 + d_2 (1, 2) + d_2 (3, 4) \\
- d_2 (1, 4) - d_2 (2, 3) + d_3) - 8g_L g_R m_{\nu e}^2 (d_1 - d_2 (1, 3))/E_2 E_4] \}.
\]

The kinetic equation for \( f_{\nu \mu} \) has the same form with the substitutions \( f_{\nu e} \leftrightarrow f_{\nu \mu} \) and \( g_L \rightarrow \tilde{g}_L = g_L - 1 \).

The equation for \( f_{\nu e} \) reads:

\[
H x \partial_x f_{\nu e}^{(1)} = \frac{G_F^2}{2\pi^3 p_1} \int dp_2 dp_3 dp_4 \delta(E_1 + E_2 - E_3 - E_4) \\
\{ F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) \\
[6d_1 + 2d_2 (1, 2) + 2d_2 (3, 4) - 2d_2 (1, 4) - 2d_2 (2, 3) + 6d_3] \\
+ 6d_1 m_{\nu e}^4 / E_1 E_2 E_3 E_4 + 4m_{\nu e}^2 (2(d_1 - d_2 (1, 4))/E_2 E_3 - (d_1 + d_2 (1, 2))/E_3 E_4) \\
+ [F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)}) + F(f_{\nu e}^{(1)}, f_{\nu e}^{(2)}, f_{\nu e}^{(3)}, f_{\nu e}^{(4)})] \}
\]
\[ d_1 - d_2(1, 4) - d_2(2, 3) + d_3 - \]
\[ -m_\tau^2/2 \left( d_1 + d_2(3, 4)/E_1 E_2 \right) \]
\[ + F(f^{(1)}_{\nu_e}, f^{(2)}_{\nu_e}, f^{(3)}_e, f^{(4)}_e) \left[ 4(g_L^2 + g_R^2) \right] \]
\[ \left( d_1 - d_2(1, 4) - d_2(2, 3) + d_3 - m_{\nu_e}^2/2 \left( d_1 + d_2(3, 4) \right)/E_1 E_2 \right) \]
\[ + 4\tilde{g}_L g_R m_e^2 \left( d_1 + d_2(1, 2) - 2m_{\nu_e}^2 d_1/E_1 E_2 \right) / E_3 E_4 \]
\[ + F(f^{(1)}_{\nu_e}, f^{(2)}_e, f^{(3)}_e, f^{(4)}_e) \left[ 4(g_L^2 + g_R^2) \right] \]
\[ \left( 2d_1 + d_2(1, 2) + d_2(3, 4) - d_2(1, 4) - d_2(2, 3) + 2d_3 + m_{\nu_e}^2 d_1 - d_2(2, 4)/E_1 E_3 \right) \]
\[ - 8\tilde{g}_L g_R m_e^2 \left( \left( d_1 - d_2(1, 3) \right) + 2m_e^2 d_1/E_1 E_3 \right) / E_2 E_4 \].

In these equations the energy \( \delta \)-function can be trivially integrated away and we are left with two-dimensional integral over the energies of incoming particles:

\[ H \partial_x f^{(1)}_{\nu_i} = \frac{G_F^2}{2\pi^3 p_1} \int \int A(F(f), d_k) \theta(E_3 + E_4 - E_1 - m_2) E_2 p_3 dp_3 p_4 dp_4, \] (17)

where \( E_2 = E'_1 + E'_2 - E_1 \), \( p_2 = \sqrt{E_2^2 - m_2^2} \), and \( A(F(f), d_k) \) is the integrand from the equations above. \( \theta(E_3 + E_4 - E_1 - m_2) \) arises from the energy conservation law.
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sec. E.g. for \( \eta_{10} \) near 3 we use: 

\[
Y_{He} = 0.163445 + 0.013133\eta_{10} - 0.0015777\eta_{10}^2 + 0.018999N - 0.0008902N^2.
\]

[21] The authors of ref. [8] used an early version of data for their \( \Delta N \)-graph, so we compared our results to their eqns. (16,17) which are more correct (private communication from K. Kainulainen).
Figure Captions:

**Fig. 1** Relative energy density of a massive tau neutrinos, \( rm = m_{\nu_\tau} n_{\nu_\tau} / n_{\nu_0} \), for asymptotically large \( x \) as a function of \( m_{\nu_\tau} \). The solid line corresponds to our results, while dashed, dashed-dotted and dotted lines correspond respectively to the results of refs. [3, 4, 11].

**Fig. 2** The quantity \( T_{\gamma a} \) as a function of the dimensionless time \( x \) for several values of the tau neutrino mass in the interval \( 0 < m_{\nu_\tau} < 20 MeV \). The lowest and uppermost solid curves correspond respectively to \( m_{\nu_\tau} = 0 \) and to \( m_{\nu_\tau} = 20 MeV \), while the dashed curves correspond to several intermediate masses: 4, 6, 8, 10, 12, 15 MeV.

**Fig. 3** Relative deviation of the energy densities of massless neutrinos from the equilibrium value, \( \delta \rho / \rho = (\rho - \rho_{eq}) / \rho_{eq} \), where \( \rho_{eq}(x) = (7/120\pi^2)(1 MeV/x)^4 \). The solid and dashed lines correspond to \( \nu_e \) and \( \nu_\mu \) respectively. In fig. 3a the evolution of \( \delta \rho / \rho \) with time \( x \) for several different masses of \( \nu_\tau \) is presented. In fig. 3b the asymptotic value of \( \delta \rho / \rho \) as a function of \( m_{\nu_\tau} \) is presented.

**Fig. 4** The relative deviation of the average energy \( \langle (E - E_{eq}) / E_{eq} \rangle \) for electronic and muonic neutrinos as a function of the mass of tau neutrino. The solid line corresponds to electron neutrino, while the dashed one corresponds to muon neutrino.

**Fig. 5** Spectral distribution of electron neutrinos as function of the dimensionless momentum \( y \). Dotted, dashed, and solid lines correspond respectively to equilibrium distributions (initial condition), \( m_{\nu_\tau} = 20 MeV \) and \( m_{\nu_\tau} = 7 MeV \) at asymptotically large time.
**Fig. 6** The change in the electron neutrino spectral distribution times $y^2$ for various masses as a function of momentum $y$. This quantity is related to the change in the neutron to proton ratio, and therefore to the light element abundances.

**Fig. 7** Relative number of equivalent massless neutrino species $\Delta N = N_{\text{eff}} - 3$ as a function of $\nu_\tau$ mass, found from $^4He$. Our result is presented by the solid line. Dashed, dashed-dotted, and dotted lines correspond respectively to the results of refs. [11, 8, 3].

**Fig. 8** The number of equivalent massless neutrino species $\Delta N = N_{\text{eff}} - 3$ as functions of the tau-neutrino mass, calculated from abundances of deuterium (solid), $^7Li$ (long dashed), $^3He$ (dashed) and $^4He$ (dotted).
Figure 1.

Figure 2.
Figure 3a.

Figure 3b.
Figure 4.

Figure 5.
Figure 6.

Figure 7.
Figure 8.
| Process                          | S                  | $2^{-5}G_F^{-2}S |A|^2$                                                                 |
|---------------------------------|--------------------|---------------------------------------------------------------|
| $\nu_e + \nu_e \rightarrow \nu_e + \nu_e$ | $1/4$             | $2 [ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) ]$ |
| $\nu_e + \nu_e \rightarrow \nu_\mu + \nu_\mu$ | $1/4$             | $\frac{1}{2} [ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) ]$ |
| $\nu_e + \nu_e \rightarrow \nu_\tau + \nu_\tau$ | $1/4$             | $\frac{1}{2} [ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - m_{\nu_\tau}^2(p_1 \cdot p_2) ]$ |
| $\nu_e + \nu_\mu \rightarrow \nu_e + \nu_\mu$ | $1/2$             | $(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)$ |
| $\nu_e + \nu_\tau \rightarrow \nu_e + \nu_\tau$ | $1/2$             | $(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_{\nu_\tau}^2(p_1 \cdot p_3)$. |
| $\nu_e + \nu_e \rightarrow e^+ + e^-$ | $1/2$             | $2(g_L^2 + g_R^2) \{ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) \} + 4gLgRm_{\nu_e}^2(p_1 \cdot p_2)$ |
| $\nu_e + e^\pm \rightarrow \nu_e + e^\pm$ | $1/2$             | $2(g_L^2 + g_R^2) \{ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \} - 4gLgRm_{\nu_e}^2(p_1 \cdot p_3)$ |

**Table 1:** Matrix elements for various electron neutrino processes; $g_L = \frac{1}{2} + \sin^2 \theta_W$ and $g_R = \sin^2 \theta_W$. Matrix elements for muon neutrino processes are obtained by the substitutions $\nu_e \rightarrow \nu_\mu$ and $g_L \rightarrow g_L = g_L - 1$. 


| Process | S | $2^{-5}G_F^{-2}S |A|^2$ |
|---------|---|------------------|
| $\nu_\tau + \nu_\tau \rightarrow \nu_\tau + \nu_\tau$ | 1/4 | $2 \left[ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) + 3m_{\nu_\tau}^4 \right] + 2m_{\nu_\tau}^2 \left[ (p_1 \cdot p_3) + (p_1 \cdot p_4) - (p_1 \cdot p_2) \right]$ |
| $\nu_\tau + \nu_\tau \rightarrow \nu_e + \nu_e$ | 1/4 | $\frac{1}{2} \left[ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - m_{\nu_\tau}^2 (p_3 \cdot p_4) \right]$ |
| $\nu_\tau + \nu_\tau \rightarrow \nu_\mu + \nu_\mu$ | 1/4 | $\frac{1}{2} \left[ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - m_{\nu_\tau}^2 (p_3 \cdot p_4) \right]$ |
| $\nu_\tau + \nu_e \rightarrow \nu_\tau + \nu_e$ | 1/2 | $(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_{\nu_\tau}^2 (p_2 \cdot p_4)$ |
| $\nu_\tau + \nu_\mu \rightarrow \nu_\tau + \nu_\mu$ | 1/2 | $(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_{\nu_\tau}^2 (p_2 \cdot p_4)$ |
| $\nu_\tau + \nu_\tau \rightarrow e^+ + e^-$ | 1/2 | $2(\tilde{g}_L^2 + \tilde{g}_R^2) \left[ (p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - m_{\nu_\tau}^2 (p_3 \cdot p_4) \right] + 4\tilde{g}_Lg_Rm_e^2 \left[ (p_1 \cdot p_2) - 2m_{\nu_\tau}^2 \right]$ |
| $\nu_\tau + e^\pm \rightarrow \nu_\tau + e^\pm$ | 1/2 | $2(\tilde{g}_L^2 + \tilde{g}_R^2) \left[ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_{\nu_\tau}^2 (p_2 \cdot p_4) \right] - 4\tilde{g}_Lg_Rm_e^2 \left[ (p_1 \cdot p_3) + 2m_{\nu_\tau}^2 \right]$ |

**Table 2:** Matrix elements for various tau-neutrino processes; $\tilde{g}_L = g_L - 1 = -\frac{1}{2} + \sin^2 \theta_W$ and $g_R = \sin^2 \theta_W$. 

30