Practical method of vector optimization on the example of optimization of thermal conditions

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Abstract. The practical method of vector optimization is described by the example of optimization of thermal conditions in industrial premises during their design based on the construction of a multi-response regression model. A practical algorithm is proposed for determining the discrete set of Pareto optimal thermal parameters.

1. Introduction to thermal conditions
The calculation of the dissipation power of the heater and the temperature in the production premise is usually calculated according to the following formula [1]:

\[ p = q(t_b - t_n), \]  

where \( p \) is the heater dissipation power (kW), \( t_b \) is the indoor temperature (°C), \( t_n \) is the outdoor temperature (°C), \( q \) is the coefficient describing the thermophysical characteristics of the premise, determined experimentally (kW/°C). The adjacent premises, the temperatures in which are interdependent, and the thermal regime of which obeys formula (1), are considered as production premises. The thermal regime refers to the combination of temperatures in the studied premises.

Such a model for calculating the thermal regime is, firstly, linear, and, secondly, one-dimensional and cannot be used to calculate the thermal regime of the aggregate of production premises. The tasks of designing the thermal conditions of several production premises are multicriteria. In this case, the task of modeling and optimizing thermal conditions is to determine the values of the input independent variables, which are the dissipation power \( d \) of the heating devices, providing the required values of the output dependent variables in the form of a temperature vector inside \( m \) premises:

\[ T = [(t_1 - t_n), (t_2 - t_n), \ldots, (t_d - t_n)]^T, \]  

where \((t_i - t_n)\) is the relative temperature inside the \( i \) premise, \( i = 1, m \).

2. Multiple thermal model
To describe the relationship between the values of independent variables and dependent variables (response), multi-reflection mathematical regression models are used [2]. In this case, the type of multirespone model is determined with accuracy to constant coefficients. Based on the resulting
A multiresponse model can be represented as:

$$ T = F(P, B) + E, \quad (3) $$

where $P = \{p_1, p_2, \ldots, p_d\}^T$ is the power vector of heating appliances, $B = \{b_1, b_2, \ldots, b_r\}^T$ is the vector of coefficients whose values are determined by the results of experiments, $E = \{e_1, e_2, \ldots, e_m\}^T$ is the temperature error observation vector, $T$, $F(P, B)$ is the $m$-dimensional temperature function, depending on the form of the multiresponse model $F$, power dissipation $P$ and coefficients of model $B$.

To model the dependence of temperatures on power dissipation, a linear multiresponse model was constructed with respect to the coefficients, which allows you to include in the model a large number of variables of the real process of designing the thermal regime. In addition, such linear models provide a short response time calculation.

Such a model has the following form:

$$ T = R(P)^T B + E, \quad (4) $$

where $R(P) = \left\{ \frac{\partial f_1(P, B)}{\partial B}, \frac{\partial f_2(P, B)}{\partial B}, \ldots, \frac{\partial f_m(P, B)}{\partial B} \right\}$ is the matrix of derivatives of the model by the coefficients.

The calculation of the coefficient estimates is performed using the following formula:

$$ \hat{B} = \left[ \sum_{i=1}^{n} R(P_i)V_E^{-1}R(P_i)^T \right]^{-1} \sum_{i=1}^{n} R(P_i)V_E^{-1}T_i, \quad (5) $$

where the matrix $V_E$ is the covariance matrix of the observation errors and $n$ is the number of temperature measurements in the premises for a given power dissipation. The covariance matrix of coefficient estimates is

$$ K = \left[ \sum_{i=1}^{n} R(P_i)V_E^{-1}R(P_i)^T \right]^{-1}. $$

As a matrix $V_E$, its estimate is used, which is determined by the results of duplicate experiments for a given vector of independent variables.

Based on the results of observations, model (4) makes it possible to estimate temperatures in production rooms for given dissipation capacities of heating devices and perform the optimization procedure for thermal conditions.

So, the dependence of two output variables $T = \{t_1 - t_{10}, (t_2 - t_{10})\}^T$ on five input variables $P = \{p_1, p_2, \ldots, p_d\}^T$, $d = 5$, was described using a quadratic model, which takes into account the quadratic terms and interactions of the input variables, in the following form:

$$\begin{align*}
t_1 &= b_{00} + \sum_{i=1}^{d} b_{i1}p_i + \sum_{i=1}^{d} \sum_{j=1}^{d} b_{ij}p_ip_j + \sum_{i=1}^{d} b_{i0}p_i, \quad i \neq j \\
t_2 &= b_{20} + \sum_{i=1}^{d} b_{2i}p_i + \sum_{i=1}^{d} \sum_{j=1}^{d} b_{2j}p_ip_j + \sum_{i=1}^{d} b_{20}p_i, \quad i \neq j \quad . (6)
\end{align*}$$

As functions $F(P, B) = \{f_1(P, B), f_2(P, B), \ldots, f_m(P, B)\}^T$ for modeling thermal regimes, quadratic relatively independent variable polynomials with all interactions can be used. In this case, function $R(P)$ in expression (4) takes the following form:
Then, a step-wise regression procedure was applied to this model with a decrease of the number of members. In the step regression procedure, coefficients are found for which the inequality holds
\[
\frac{\hat{b}_i}{\sqrt{k_i}} \leq t_{\frac{\alpha}{2}},
\]
where \(k_i\) is the diagonal matrix \(K\) element for the \(i\)-th coefficient, \(t_{\frac{\alpha}{2}}\) is the quantile of the student distribution at the \(\alpha\) significance level.

Members including coefficients satisfying inequality (8) are excluded from the corresponding functions, and the coefficient estimates are recalculated, and the procedure for reducing the number of model members is repeated \[2\].

3. Admissible areas
There is a so-called admissible area of input variables \(D\) and a corresponding area of output variables that ensure that all the requirements for the thermal regime are met. This area contains a subset of unimprovable or Pareto-optimal output variables and their corresponding input variables \(P\), which cannot be improved simultaneously by all optimized criteria without degrading at least one of these indicators. Among the set of Pareto-optimal variables, the final variant of the thermal regime for all rooms is then selected \[3\].

To solve the problem of finding an admissible area, a method is used that makes it possible to probe the set of parameters with the points of the sequence distributed in the parallelepiped (1).

The study of the parameter space consists of three stages.
1. The first stage is the compilation of test tables.
2. Select \(n\) test points \(P_1, \ldots, P_n\) located in the general set \(G\).
3. At each point, all local criteria \(f_i(P_v), v = 1, m + d\), are calculated. A test table (matrix) \(W\) is compiled for each test point.

The first \(r\) columns of this table represent test points \(P_1, \ldots, P_n\). The next \(m\) columns of this table represent the values of the parameters-criteria of validity \(T_i = F(P_v), i = 1, n\), calculated in accordance with expression (4). The matrix \(W\) has a dimension and is represented in the form:

\[
\begin{pmatrix}
T_1 & T_2 & \ldots & T_m \\
T_1 & T_2 & \ldots & T_m \\
\vdots & \vdots & \ddots & \vdots \\
T_1 & T_2 & \ldots & T_m \\
\end{pmatrix}
\]

This table presents the values of the parameter criteria the parameters-criteria of validity. The matrix \(W\) has dimensions \([n \times (r + m)]\) and is represented in the form:

\[
W = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1v} & f_{11}(P_1) & f_{12}(P_1) & \cdots & f_{1m}(P_1) \\
p_{21} & p_{22} & \cdots & p_{2v} & f_{21}(P_2) & f_{22}(P_2) & \cdots & f_{2m}(P_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nv} & f_{n1}(P_n) & f_{n2}(P_n) & \cdots & f_{nm}(P_n) \\
\end{pmatrix},
\]

where \(n\) is the number of test points, \(m\) is the number of parameters-criteria of validity.
Test points represent a discrete approximation of the area of input parameters, and the values of the parameters-criteria of validity represent a discrete approximation of the area $F(\Pi)$.

The first $r$ columns of this matrix form a discrete admissible area $D$ in the space of input parameters containing $s$ points. The next $m$ columns form the image of the area $D$ in the space of parameters-criteria of validity $F(D)$. At the request of the designer, $k$ parameters can also be included in the optimization process, which are not eligibility criteria and for which technical limitations are not set, but which are desirable to optimize [4].

Usually, optimization (for example, reduction) of one particular criterion leads to an increase in other particular criteria. Points at which the vector optimality criterion $Y$ is not reducible by all particular criteria at the same time are called unimprovable solutions or Pareto optimal [1]. Pareto optimality means that it is impossible to reduce the value of one of the particular criteria without increasing at least one of the others. The point $P^*$ is the best possible solution if there is no such $P$, that

$$
\begin{cases}
f_j(P) \leq f_j(P^*) & i = 1, \ldots, m; \\
f_j(P) < f_j(P^*) & \text{for some } j,
\end{cases}
$$

where $f_j(P)$ is value of a particular optimality criterion. The Pareto-optimal area will be based on the matrix $W$, which determines the allowable set of input parameters. The search for Pareto-optimal points is performed among the rows of the matrix $W$ in accordance with the following conditions. Point $P^* (P \notin D)$ is Pareto optimal if there is no such point. that for all and at least one $n$ exists. An area is called Pareto-optimal if it consists of all Pareto-optimal points.

The construction of the Pareto-optimal area begins after obtaining an admissible set $D$ in the space of input parameters in the form of a matrix $W$. The first $r$ columns of this matrix form a discrete admissible area $D$ in the space of input parameters containing $s$ points. The next $m$ columns form the image of the area $D$ in the space of parameters-criteria of validity.

By checking the conditions for a point to belong to a Pareto-optimal set for all points of the matrix $W$, only Pareto-optimal points remain in this table [5].

Consider a practical method for constructing a Pareto-optimal set of solutions (criteria must be minimized). In figure 1 the Pareto-optimal set in the criterion space is represented by a solid line in the form of the boundary of the region bounded by points $A$ and $B$. Based on the analysis of the Pareto-optimal area, the decision maker (DM) determines the most preferable option $P^0$.

![Figure 1](image-url). Formation of an admissible area of input variables $F(D)$ by crossing the region of limiting temperatures $t_1^*$ and $t_2^*$ and $F(P)$ area.
4. Construction of a discrete Pareto-optimal set on a grid

Using grids to study admissible points in a multi-criterion problem, optimization allows one to rather easily and efficiently recognize and select admissible points from the space of input parameters that belong to an admissible area. However, the admissible area $D$ in problems with continuous parameters and criteria is continuous. Therefore, its approximation usually means the selection of such a finite subset that would approximate any value of each criterion calculated at an arbitrary point from the domain $D$ with a predetermined accuracy [6].

To select test points it is advisable to use distributed in the space of a unit volume with the origin at the origin, the dimension of which is equal to the dimension of the parameter space, sequences (or grids) $Q_1, Q_2 \ldots$ with simple algorithms for calculating the coordinates of their points.

The process of selecting points $P_i$ proceeds as follows. According to the Cartesian coordinates of the next grid point, $Q_i = (q_{i1}, \ldots, q_{id})$, $i = 1, n$, in space we find the Cartesian coordinates of point $P_i = (p_{i1}, \ldots, p_{id})$, where are the maximum and minimum values of the variable, respectively

$$p_{ij} = p_j^* + q_j(p_j^{**} - p_j^*), \quad j = 1, d, \quad i = 1, n,$$

where $p_j^{**}$ and $p_j^*$ are the maximum and minimum values of the variable, respectively.

For example, for one-dimensional parameter space, we generate the following grid, including 5 points in Cartesian space: $Q = \{0;0,25;0,5;0,75;1\}$. Let $p_j^{**} = 20, p_j^* = 5$. Then, based on the expression (8), we obtain the values of the grid points in the parameter space. For example, for one-dimensional parameter space, we generate the following grid, including 5 points in Cartesian space: $Q = \{0;0,25;0,5;0,75;1\}$. Then, based on the expression (8), we obtain the values of the grid points in the parameter space: $\{0;0,25;0,5;0,75;1\}$.

The value $\delta'$ must be binary rational, because otherwise, for example, when rounding, the grid points may be outside the allowable area. The coordinates of the grid points (its nodes) are defined as follows:

$$Q_j = \{p_{1i_j}, p_{2i_j}, \ldots, p_{ni_j}, \ldots, p_{di_j}\}.$$

The coordinates of the grid points (its nodes) are defined as follows:

$$n_v = \left(\frac{p_j^{**} - p_j^*}{2\delta'}\right) + 1.$$

The grid node number depends on the formation order of the grid nodes and can be expressed, for example, as follows:

$$j = i_1 + n_1(i_2 - 1) + n_1n_2(i_3 - 1) + n_1n_2n_3(i_4 - 1) + \ldots,$$

where $i_v = 1, n_v$ is the grid node number.

For illustration, figure 1 shows a grid constructed for three input parameters with a discreteness value. The circles show the grid node numbers. For illustration grid shows built for three parameters $c$ with discrete values $2\delta_1' = \frac{1}{4}; 2\delta_2' = \frac{1}{2}$. In round circles show numbers of nodes, $n_v = \left(\frac{p_j^{**} - p_j^*}{2\delta'}\right) + 1$.

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For illustration figure 2 shows a grid constructed of three input parameters with a discreteness value. The circles show the grid node positions.

![Figure 2](image)

**Figure 2.** Grid size 5×3×3 round circles marked nodes. Point has 10 sets of 45 nodes. For example, node with coordinates (0.25; 0.5; 0.5), which coexists with the index (2,2,2), at a rate of (30), equal \( j = 2 + 5 \times 1 + 5 \times 3 \times 1 = 22 \).

5. **The procedure of vector discrete optimization**

On the first stage, there is an optimization area, and in the widest space, enter the parameter to approximate the sequence, for which you can specify the total vector of the incoming parameter, where, and \( n \) – the number of usable segments.

For all permeable discs with a disconnected diaphragm, \( F(P, B) = [f_1(P, B), f_2(P, B), \ldots, f_{n_{\text{im}}}(P, B)]^T \), where \( i = 1, n \), a \( n \) change the 20-point perimeter of the socket to include 106 nods. It is impossible to explain such a large number of reflections with the help of imitation models, thereby further multiplying the multi regression line of the yielding coefficient model (8). To create a Pareto-optimized solution, suggests the following algorithm.

In terms of specificity, the suboptimal points of the area are more likely to be limited by the number of cut-outs; for partial suboptimal points, create a new cut, and then set new sections for separate partitions and semiconductor areas.

To create a Pareto-optimized solution following algorithm is necessary. On the first stage, there is an optimization area, and in the widest space, enter the parameter to approximate the sequence, for which you can specify the total vector of the incoming parameter, where, and \( n \)-the number of usable segments. For all permeable discs with a disconnected diaphragm, change the 20-point perimeter of the socket to include 106 nodes. It is impossible to explain such a large number of reflections with the help of imitation models, thereby further multiplying the multi-click regression line of the yielding coefficient model (8).

In terms of specificity, the suboptimal points of the area are more likely to be limited by the number of cut-outs; for partial suboptimal points, create a new cut, and then set new sections for separate partitions and semiconductor areas including все suboptimal точки; это значительно step up the rate economic growth множества.

The operations according to p. 1–3 continue until the specified approximation accuracy is reached, i.e. when inequalities will be satisfied,. Here is the grid cell size for the \( i \)-th input variable, is the permissible accuracy of determining the optimal point for this variable.

To construct Pareto-optimal solutions, the following algorithm is proposed.
1. At the first stage of optimization, the search area in the space of input parameters is approximated by a grid, for each node of which the vector of output parameters is calculated, where, and \( n \) is the number of grid nodes. For seven variables with a discretization of the range of variation of variables by 20 points, the grid included more than 106 nodes. It is impossible to calculate such a number of responses even with the help of a simulation model; therefore, for this, we used a multi-response regression linear model with respect to the coefficients (8).

2. For the set of vectors of output parameters over the grid obtained in step 1, by the search method of all points.

3. In the vicinity of each suboptimal point, the search area is limited to several cells of the original mesh; a new grid is constructed for a separate suboptimal point, and then these new grids for individual points are combined and a search area of complex shape of a smaller volume and with a significantly smaller sampling step is obtained, including all suboptimal points; this significantly increases the accuracy of the approximation of the optimal set.

Figure 3 shows the Pareto-optimal discrete set of output parameters. As a result, we obtain a discrete approximation of the Pareto optimal domain of solutions by a set of points, among which we can choose the preferred solution. The search for the final optimal values of the input variables was carried out among the Pareto optimal points by introducing additional restrictions. The resulting Pareto-optimal set of output parameters is shown in figure 3.

**Figure 3.** Discrete optimization Pareto-optimal vector procedure.

6. Conclusion

The proposed practical method for constructing an admissible set of variables is possible to obtain an admissible set in a discrete form, which makes it possible to provide an acceptable probability of an error in its construction. The optimization procedure is based on a multi-response linear with respect to the coefficients of the model, the construction of which is the first stage of multi-criteria optimization. The proposed practical vector optimization method based on multi-feedback models allows vector optimization of thermal conditions for two or more rooms.

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