Curie–Weiss magnet—a simple model of phase transition

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Abstract
The Curie–Weiss model is an exactly solvable model of ferromagnetism that allows one to study thermodynamic functions in detail, in particular their properties near the critical temperature. In this model every magnetic moment interacts with every other magnetic moment. Because of its simplicity and the correctness of at least some of its predictions, the Curie–Weiss model occupies an important place in the statistical mechanics literature and its application to information theory. It is frequently presented as an introduction to the Ising model or to spin-glass models, and usually only general features of the Curie–Weiss model are presented. In this paper, we discuss the properties of this model in a rather detailed way. We present the exact, approximate and numerical results for this particular model. The exact expression for the limiting magnetic field is derived.

1. Introduction

A phase transition is the transformation of a thermodynamic system from one phase or state to another. During a phase transition of a given medium certain properties change, often discontinuously, as a result of some external conditions, such as temperature, pressure or the magnetic field. For example, a liquid may become gas upon heating to its boiling point, resulting in an abrupt change in volume.

In the case of a ferromagnet, one should predict the dependence of such quantities as the free and internal energy, entropy, heat capacity and magnetic susceptibility on temperature \( T \) and magnetic induction \( B \). Generally, this is a very complicated task. Ma [1] stressed the distinction between the direct approach to the problem of phase transitions and the approach exploiting symmetries of the problem. Here we shall illustrate the former approach. This means calculations of the physical properties of interest in terms of parameters given in a particular model, i.e. solving a model. The calculations may be done analytically or numerically; exactly or approximately.
One of the simplest classical systems exhibiting phase transition was introduced by Pierre Curie and then by Pierre Weiss in their development of simplified theory of ferromagnetism [2, 3]. Recently, it was named the mean field theory. Curie and Weiss considered a set of magnetic moments interacting with their nearest neighbours. They replaced the actual interactions experienced by each magnetic moment with the mean interaction given by the mean magnetization. With growing number of nearest neighbours the mean field theory becomes a better approximation. One can enlarge the number of nearest neighbours by considering magnetic moments in spaces of higher dimensions (cf [4]). Kac considered a model where every magnetic moment interacts with every other magnetic moment, and called it the Curie–Weiss (CW) model [2]. Gould and Tobochnik called this model the fully connected Ising model [5].

Although the CW model leads to the same results as the mean field theory for the behaviour of the system in the vicinity of the critical temperature, generally the thermodynamic functions calculated in the frame of these models are different.

We shall focus on the CW model of a magnet. The CW model is an exactly solvable model of ferromagnetism that allows one to study in detail the behaviour of thermodynamic functions. Since not all predictions of this model agree with experiments, other models must be considered. However, because of its simplicity and because of the correctness of at least some of its predictions, the classical CW model occupies a central place in statistical mechanics literature. It is frequently presented as an introduction to the Ising model or to spin-glass models [3, 6–12]. However, in these references only general features of the CW model are discussed. This is why, here, we shall discuss the properties of this model in a rather detailed way. We present the exact, approximate and numerical results for this particular model.

We hope that this paper will be of use to readers such as graduate and postgraduate students as well as beginner research workers. We expect also that teachers might find our paper interesting enough to incorporate it in their courses and thereby introduce students to an example that is richer than the mean field theory.

2. The CW model

Let us call the set of integers from 1 to $N$ a lattice, and each element $i$ a site. We assign a variable $s_i$ (the Ising spin) to each site. The Ising spin is characterized by the binary value: $+1$ if the microscopic magnetic moment is pointing up or $-1$ if it is pointing down. Particles with Ising spins interact via the Hamiltonian

$$H_{\text{int}} = -J \sum_{1 \leq i < j \leq N} s_i s_j.$$  \hspace{1cm} (1)

The constant $J$ is positive. The interaction energy of all pairs of spins in the CW magnet is the same and their interaction depends on $N$. The normalization by $1/N$ makes $H_{\text{int}}$ a quantity of the order $N$, i.e. an extensive quantity. The underlying assumption of an infinite-range interaction is clearly unphysical. The Hamiltonian (1) does not depend on the dimensions of the space which the CW magnet occupies.

The magnetic moment of a particle is proportional to the spin $\mu_i = \mu s_i$, where $\mu$ is the magnetic moment. In an applied magnetic field with the magnetic induction vector $\mathbf{B}$, particles with magnetic moments being parallel or antiparallel to $\mathbf{B}$ acquire the energy

$$H_f = -\mu B \sum_{i=1}^{N} s_i.$$  \hspace{1cm} (2)
The complete Hamiltonian consists of two terms

$$H = -\frac{J}{N} \sum_{1 \leq i < j \leq N} s_is_j - \mu B \sum_{i=1}^{N} s_i. \quad (3)$$

The Hamiltonian (3) does not change if we reverse the signs of all spins $s_i \rightarrow -s_i \ (i = 1, 2, \ldots, N)$ and the direction of the induction vector $B \rightarrow -B$. 

$$H(s_1, \ldots, s_n; B) = H(-s_1, \ldots, -s_n; -B). \quad (4)$$

Denote a particular configuration $(s_1, s_2, \ldots, s_N)$ by $\{s\}$. To each configuration $\{s\}$ there corresponds an energy $E(\{s\}) = H(\{s\})$.

The nature of the phase transitions of magnetic systems is well understood. At temperature 0 K, magnetic systems, in particular the CW magnet, are in a lowest energy state with all spins being parallel. Thus, their magnetization $M$ is finite and our magnet is ferromagnetic. As temperature is increased from zero the thermal noise randomizes the spins. A fraction of them becomes antiparallel. This disorder grows with rising temperature and a diminishing fraction of them points in the initial direction. At temperature $T_c$—the critical temperature, and beyond, magnetization vanishes and the material becomes paramagnetic. For $T$ above $T_c$, there must be macroscopically large regions in which a net fraction of spins become aligned. However, their magnetization mutually compensates—they cannot make a finite fraction of all regions agree. For $T$ just below $T_c$ the compensation is not complete and a small, but finite, fraction points in the same direction.

For left-hand vicinity of $T_c$ thermodynamic functions depend on the dimensionless parameter $t = (T - T_c)/T_c$ and consist of terms that are regular and singular in $t$. The singular terms depend on powers of $|t|^{-1}$. These powers are called the critical indices (or critical exponents) and are defined for $B = 0$ and $t \rightarrow 0$. The specific heat $c$ per particle and the magnetization and the magnetic susceptibility $\chi_T$ for the paramagnetic phase are characterized respectively by critical indices $\alpha$, $\beta$ and $\gamma$

$$c \sim |t|^{-\alpha}, \quad m \sim |t|^{-\beta}, \quad \chi_T \sim |t|^{-\gamma}. \quad (5)$$

As we shall show, these critical indices characterizing the critical behaviour of both phases are the same. We shall calculate these sets of critical indices, as well as the dependence of the internal energy and entropy on $|t|$, for ferromagnetic and paramagnetic phases of the CW magnet.

In the presence of a magnetic field, the dependence of magnetization on the magnetic field at $|t| = 0$ is characterized by the critical index $\delta$

$$m \sim B^{1/\delta}. \quad (6)$$

In the following, in place of temperature we will use $\theta = k_BT$, $k_B$ being the Boltzmann constant.

3. Calculation of free energy

Since $s_i^2 = 1$ the Hamiltonian (1) can be written as

$$H_{\text{int}} = -\frac{J}{2N} \left( \sum_{i=1}^{N} s_i \right)^2 + \frac{J}{2}. \quad (7)$$

The partition function is defined as usual as [13]

$$Z_N = \sum_{\{s\}} e^{-E(\{s\})/\theta}. \quad (8)$$
The summation is performed over all $2^N$ configurations $\{s\}$. Note that a different method for calculating spin configurations can be used [5].

Introduce two dimensionless quantities $K = J/\theta$ and $h = \mu B/\theta$. Then, the partition function (8) can be written as

$$Z_N(\theta) = \sum_{\{s\}} \exp \left[ \frac{K}{2N} \left( \sum_{i=1}^{N} s_i \right)^2 - \frac{K}{2} \sum_{i=1}^{N} s_i + h \sum_{i=1}^{N} s_i \right]$$

$$= e^{-\frac{\theta}{2N}} \sum_{\{s\}} \exp \left[ \left( \frac{K}{2N} \sum_{i=1}^{N} s_i \right)^2 + h \sum_{i=1}^{N} s_i \right]. \quad (9)$$

Thermodynamic functions can be obtained via the Helmholtz free energy $F_N(\theta, B)$ [13]

$$F_N(\theta, B) = -\theta \ln Z_N(\theta, B). \quad (10)$$

The derivative of $F_N(\theta, B)$ (10) with respect to $B$ gives

$$\frac{\partial F_N(\theta, B)}{\partial B} = -\mu \left( \sum_{i=1}^{N} s_i \right) = -\mu \sum_{i=1}^{N} \langle s_i \rangle, \quad (11)$$

where $\langle s_i \rangle$ means the mean value of $s_i$ calculated with the canonical distribution function [13].

The minus derivative of free energy (11) defines the magnetization $M$ of a magnet [13], thus

$$M = \mu \sum_{i=1}^{N} \langle s_i \rangle. \quad (12)$$

The second derivative of $F_N(\theta, B)$ with respect to $B$ is positive [14]

$$\frac{\partial^2 F_N(\theta, B)}{\partial B^2} = \frac{\mu^2}{\theta} \left\{ \left( \sum_{i=1}^{N} \langle s_i \rangle \right)^2 - \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \langle s_i s_j \rangle \right) \right\} > 0. \quad (13)$$

Since $[\partial^2 F(\theta, B)/\partial B^2]_0$ defines the magnetic susceptibility $\chi_T$ (cf. equation (35)), we conclude that the magnetic susceptibility is positive. Therefore, the change of induction $\delta B$ increases the magnet energy by $\delta B \cdot \chi \cdot \delta B > 0$. The CW magnet fulfills the thermodynamic (macroscopic) condition of stability [12].

A simple evaluation of the partition function (9) is precluded only by the square of magnetization in the exponential. One can get rid of this square using the Gaussian linearization of the form

$$\exp \frac{\theta}{2} \int_{-\infty}^{+\infty} \, d\xi \, e^{-\xi^2/2 + \sqrt{\theta} \xi}. \quad (14)$$

In the present case $a = \sqrt{K/(2N)} \sum_{i=1}^{N} s_i$. Now, the partition function factors with respect to individual summations over the state $s_i$:

$$Z_N = \left( \frac{e^{-\frac{\theta}{2N}}}{\sqrt{2\pi}} \right)^{2^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \, d\xi \, e^{-\xi^2/2 + \sqrt{\frac{KN}{2\pi}}} \sum_{s_1=-1}^{+1} \cdots \sum_{s_N=-1}^{+1} \int_{-\infty}^{+\infty} \, dh \, e^{-\xi^2/2} e^{\sqrt{KN} s_1 \xi + h s_1} e^{\sqrt{KN} s_2 \xi + h s_2} \cdots e^{\sqrt{KN} s_N \xi + h s_N}$$

$$= 2^N \frac{e^{-\frac{\theta}{2N}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \, d\xi \, e^{-\xi^2/2} \left( \cosh(\sqrt{K/2\pi} \xi + h) \right)^N. \quad (15)$$

Performing the change of variable $\sqrt{K/2\pi} \xi = Ky$ we get

$$Z_N = 2^N \left( \frac{KN}{2\pi} \right)^{1/2} \frac{e^{-\frac{\theta}{2N}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \, dy \left[ \Phi_K(y) \right]^N, \quad (16)$$
where
\[ \Phi_{K,h}(y) = e^{-Ky^2/2} \cosh(Ky + h). \] (17)

In addition to \( y \), the function \( \Phi_{K,h}(y) \) depends on two dimensionless parameters \( K \) and \( h \), i.e. on \( \theta \) and \( B \).

The free energy per particle is proportional to \( (\ln Z_N)/N \). Since we are interested in analysing the system in the large size limit

\[ \lim_{N \to \infty} \ln \frac{Z_N}{N} = \lim_{N \to \infty} \ln (Z_N^{1/N}) = \ln \left( \lim_{N \to \infty} Z_N^{1/N} \right). \]

Using \( Z_N \) (16) we obtain free energy per particle \( f(\theta, B) \)

\[ \frac{f(\theta, B)}{\theta} = \lim_{N \to \infty} \frac{1}{N} \ln \left( e^{-\frac{K}{2}KN/2\pi} \right) + \ln 2 + \ln \left( \int_{-\infty}^{+\infty} dy[\Phi_{K,h}(y)]^N \right)^{1/N}. \] (18)

In order to obtain the explicit form of the function \( f(\theta, B) \) we use the Laplace theorem [15].

**Theorem.** Let functions \( \varphi(y) \) and \( \psi(y) \) be continuous and positive in a range \( c \leq x \leq d \), then

\[ \lim_{n \to \infty} \left\{ \int_c^d \varphi(x) \psi(x) dx \right\}^{1/n} = \max_{c \leq x \leq d} \varphi(x). \] (19)

For \( \varphi(y) = 1 \) and \( \psi(y) = \Phi_{K,h} \) in the limit \( N \to \infty \) this theorem yields

\[ \frac{f(\theta, B)}{\theta} = \ln \max_{-\infty \leq y \leq \infty} \Phi_{K,h}(y) + \ln 2. \] (20)

Let us introduce a function of \( y \) related to free energy

\[ f_{\theta,h}(y) = -\theta[\ln 2 + \ln \Phi_{\theta,h}(y)]. \] (21)

According to equation (20), to find the dependence of free energy on thermodynamic variables \( \theta \) and \( B \) one should find extreme points of \( f_{\theta,h}(y) \). For these points \( (df_{\theta,h}(y)/dy)_{\theta,h} = 0 \).

Calculating the derivative of \( \Phi_{K,h}(y) \) (17) we find

\[ \left( \frac{\partial f}{\partial y} \right)_{\theta,h} = -\frac{\theta K}{\Phi_{K,h}(y)} \left[ \tanh(Ky + h) - y \right] \Phi_{K,h}(y) = -\theta \left[ \tanh(Ky + h) - y \right]. \]

Thus, the variable \( y \) obeys the equation

\[ y = \tanh(Ky + h). \] (22)

For various values of \( \theta(K) \) and \( B(h) \) solutions of this equation provide the function \( y = y(\theta, B) \) of state variables. Therefore, the function \( \Phi(\theta, B) = \max_{-\infty \leq y \leq \infty} \Phi_{K,h}(y) \) is a composite function of \( \theta \) and \( B \), namely \( \Phi_{K,h}(y(\theta, B)) \) and also depends implicitly on these two state variables. Now equation (20) can be rewritten in the form

\[ f(\theta, B) = -\theta \ln 2 - \theta \ln \Phi(\theta, B), \]

or

\[ f(\theta, B) = -\theta \ln 2 - \theta \ln \left[ e^{-K/2(\theta,B)/2} \cosh(Ky(\theta, B) + h) \right]. \] (23)

Using two familiar identities [16] \( d \tanh x/dx = \cosh^{-2} x \) and

\[ \cosh^2 x = (1 - \tanh^2 x)^{-1}, \] (24)
we calculate the second partial derivative of \( f \) with respect to \( y \)
\[
\left( \frac{\partial^2 f}{\partial y^2} \right)_{\theta, h, y = y(\theta, B)} = -\theta K [1 - y^2(\theta, B)] - 1].
\] (25)
In the appendix we show that there exist solutions of equation (22) for which this derivative is positive, hence for them free energy is minimal.

For the reversed magnetic field the solution of equation (22) is
\[
-y = \tanh[K(-y - h)].
\] (26)
Therefore, free energy \( f(\theta, B) \) (20) is an even function of \( B \) (and \( h \))
\[
-f(\theta, -B) = \ln 2 + \ln[e^{-K(-y(\theta, B))} \cosh[K(-y(\theta, B)) - h]]
\]
\[
= -f(\theta, B).
\] (27)
These properties of free energy and solutions of equation (22) are the reason why in the following we will always have in mind the positive value of \( B \).

Calculating derivatives \( (\partial y/\partial h)_{\theta} \) and \( (\partial^2 f/\partial y^2)_{K, h} \) one may show that the product of \( (\partial y/\partial h)_{\theta} \) and \( (\partial^2 f/\partial y^2)_{K, h} \) is positive because
\[
\left( \frac{\partial y}{\partial h} \right)_{\theta} \left( \frac{\partial^2 f_{K, h}(y)}{\partial y^2} \right)_{K, h} = \frac{\theta K}{\cosh(Ky + h)}.
\] (28)
Introduce here \( K_c = 1 \)—the critical value of the parameter \( K \). From the definition of \( K \) it is seen that the critical value of \( \theta \) is \( \theta_c = J \).

4. Free energy of a CW magnet in the absence of a magnetic field

Suppose that \( B = 0 \) (\( h = 0 \)). If we plot \( g(y) = y \) and \( q(y) = \tanh(Ky) \) as functions of \( y \), the points of intersection determine the solutions of equation (22). Referring to figure 1 (left panel), we have to make a distinction between the cases. If \( \theta > \theta_c \) (\( \theta > J \)) the slope of the function \( q(y) \) at the origin \( K = J/\theta = \theta_c/\theta < 1 \) is smaller than the slope of linear function \( g(y) = y \), which is 1, thus these graphs intersect only at the origin. It is easy to check that for this solution the second derivative of free energy (20) is positive (cf appendix). Therefore, the extreme is indeed a minimum (cf figure 2).
5. Magnetization and magnetic susceptibility of a CW magnet

Consider the CW magnet when the magnetic field $B$ is brought back. Magnetization per particle, $m$, is a partial derivative of free energy $f(\theta, B)$ after $B$

$$m(\theta, h) = - \left[ \frac{\partial f(\theta, h)}{\partial B} \right]_\theta = \mu \left[ \frac{\partial \ln \Phi(\theta, h)}{\partial h} \right]_\theta$$

$$= \mu \frac{\partial \ln \Phi_{K,h}(y)}{\partial h} \Big|_{y=y(\theta, B)} + \mu \frac{\partial \ln \Phi_{K,h}(y)}{\partial y} \bigg|_{y=y(\theta, B)} \frac{\partial y(\theta, B)}{\partial h}. \quad (30)$$

Since $\Phi_{K,h}(y)$ attains an extremum at $y(\theta, B)$, the second term on the right-hand side of (30) vanishes. The contribution of the first term yields the equation of state

$$m = \mu \tanh \left( \frac{K}{\mu} m + h \right), \quad (31)$$
which has a closed analytic form. According to equation (22), the study of $y$ is equivalent to the study of magnetization $m$.

Consider the solutions of equation (22) for $B > 0$. When $\theta < \theta_c$, the plots of functions $g(y) = y$ and $Q(y) = \tanh(Ky + h)$ intersect at three non-symmetric and nonzero points (cf the right panel of figure 1). For $h \neq 0$, equation (22) has two solutions if $|h| \leq h_t$, where $h_t$ is some limiting value of $h$. This problem is discussed in sections 6 and 8.

Only for a positive value of $y = y(\theta, B)$ does free energy $f$ attain the global minimum. To one of the negative values there corresponds a local minimum, to the remainder a maximum (cf figure 2). The negative values of $y(\theta, B)$ (as well as $m(\theta, B)$) do not correspond to stable states and should be omitted. One should notice that negative values of $y$ for positive $h$ are not compatible with the symmetry (26) of the state equation. When $\theta \geq \theta_c$, the graphs of $g(y)$ and $Q(y)$ intersect at one point $y(\theta, B) > 0$. For this value of $y(\theta, B)$ free energy has global minimum (cf figure 2).

Consider equation (22) for small values of $|t|$ and $h = 0$. We can expand $\tanh(Ky)$ into Taylor’s series. Using equations (29) we obtain

$$y \simeq \sqrt{3} \frac{(K - 1)^{1/2}}{K^{3/2}} \sim |t|^{1/2}. \quad (32)$$

Hence, in agreement with figure 3, we conclude that

$$m \sim |t|^{1/2}. \quad (33)$$

When positive $B \to 0$ then $y(\theta, h) \to y(\theta)$ of figure 3. Thus, we note the existence of spontaneous magnetization $m(\theta) = \mu y(\theta)$

$$m(\theta, h = 0^+) = \begin{cases} 0 & \theta_c \leq \theta, \\ m_0 & \theta_c > \theta. \end{cases} \quad (34)$$

Even after turning off the magnetic field, below critical temperature the system remains magnetized, depending on the sign of $B$ before its removal. The dependence of $y$ for $B = 0$ on $\theta$ is singular. It is seen in figure 3 that at the point $\theta = \theta_c$ the tangents of two branches of the curve are different. We note that the point $\theta = \theta_c$ is the boundary between the region of existence and nonexistence of magnetization, i.e. it is a critical point. We conclude that magnetization $m$ is the order parameter.

In the presence of even an arbitrarily weak ($0 \leq h \ll 1$), the magnetic field magnetization $m$ does not vanish below and above the critical point (cf figure 3 the right panel). The external
magnetic field lowers the symmetry of the paramagnetic phase. From the point of view of magnetization the difference between paramagnetic and ferromagnetic phases vanishes and the critical point ceases to exist.

Consider the magnetic susceptibility
\[
\chi_T(\theta, B) = \left[ \frac{\partial^2 m(\theta, B)}{\partial B^2} \right]_{\partial \theta} .
\]
According to the definition (35) and equation (30) the susceptibility is related to the second derivative of free energy with respect to induction
\[
\chi_T(\theta, B) = -\left[ \frac{\partial^2 f(\theta, B)}{\partial B^2} \right]_{\partial \theta}. \tag{36}
\]
Differentiating both sides of equation (31) with respect to \(B\) and solving the obtained equation, we obtain the general expression for the susceptibility
\[
\chi_T(\theta, B) = \mu^2 K \left[ 1 - y^2(\theta, B) \right]. \tag{37}
\]
Using equation (25) the susceptibility can be written as
\[
\chi_T(\theta, B) = \mu^2 K \left( \frac{\partial^2 f_K(y)}{\partial y^2} \right)_{y=y(\theta, B)} . \tag{38}
\]
Since for a stable state the second derivative \((\partial^2 f_K,y)/\partial y^2)_{y=y(\theta, B)}\) is nonnegative and \(0 \leq |y| \leq 1\), the susceptibility is nonnegative too. Notice that we succeed in linking together the macroscopic (13) and microscopic (38) stability conditions.

For the paramagnetic phase \((\theta \geq \theta_c (t \geq 0)) y = 0, and in the vicinity of the critical temperature we obtain
\[
\chi_T \sim \frac{\mu^2}{\theta_c(1+t)} \frac{(1+t)}{t} \sim \frac{\mu^2}{\theta_c} t^{-1}. \tag{39}
\]
This means that the critical index \(\gamma = 1\).

The function \(\text{artanh} y\) obeys the equation
\[
\text{artanh} y = Ky + h. \tag{40}
\]
We shall study solutions of equation (40) in the vicinity of the critical temperature. For small \(y\) one can expand \(\text{artanh} y\) into Taylor’s series [16]. For \(\theta < \theta_c (t < 0)\) equation (40) reduces to
\[
y^3 - 3 \frac{|t|}{1 - |t|} y - 3h = 0. \tag{41}
\]
In the ferromagnetic phase and in the absence of the magnetic field, the order parameter does not vanish
\[
y^2 = \frac{3|t|}{1 - |t|} (t < 0). \tag{42}
\]
When \(B = 0\) from equation (42) it follows that for a ferromagnetic phase in the vicinity of critical temperature \(m \sim \sqrt{3|t|}\). Thus the critical index of magnetization is \(\beta = 1/2\).

Consider the susceptibility (38) in a ferromagnetic phase in the vicinity of the critical temperature. Using the expressions (29) and (42) for the ferromagnetic phase we obtain
\[
\lim_{B \to 0} \chi_T(\theta, B) \sim \frac{\mu^2}{\theta_c} |t|^{-1}, \tag{43}
\]
and according to the definition (5) for both phases the critical index of magnetization is $\gamma = 1$. We shall note that in the case of a ferromagnetic phase one should use the relation (41). We conclude that for both phases the magnetic susceptibility of a CW magnet is divergent at the critical temperature. This singular behaviour is shown in figure 4.

At the critical point $t = 0$, hence from equation (41) it follows that $y = \sqrt{3}h^{1/3}$. Therefore, according the definition (6), the critical index for the critical isotherm $\delta = 3$.

Continuous phase transitions occur when a new state of reduced symmetry develops continuously from the disordered (high temperature) phase. The ordered phase of a CW magnet has lower symmetry than the symmetry (4) of the Hamiltonian, thus the symmetry is spontaneously broken. There exist two equivalent symmetry related states of a CW magnet with magnetization $m_i + m$ and $m_i - m$ respectively, with equal free energies. These states are macroscopically different, so thermal fluctuations will not bring them into contact in the thermodynamic limit. To describe the ordered state we introduced magnetization—the macroscopic order parameter that describes the character and strength of the broken symmetry.

6. Approximate theory—the analysis of roots of the cubic equation for magnetization

Now we shall study the roots of the cubic equation (41) for the ferromagnetic phase. Since free energy is an even function of $B$ we assume that $h > 0$. We shall study the so-called incomplete cubic equation

$$y^3 + 3(-p)y + 2q = 0,$$

where $p = |t|/(1 - |t|)$, $q = -3h/2$.

It is worthwhile recalling that the assumptions proposed by Landau for an incompressible magnet result in free energy depending only on even powers of magnetization [4, 12, 17]

$$f(m, T) = f_0(T) + \alpha(T)m^2 + \frac{1}{2}\beta(T)m^4.$$ 

This form of free energy also leads to a cubic equation for magnetization.

Introduce a characteristic value of the parameter $h$

$$h_t = \frac{2}{3}|t|^3 \left(|t| \ll 1\right).$$  

(45)
The function $\Lambda_{xy}(y) = y^3 - 3py$ has two extreme points at $y = \pm |t|^{1/2}$. For these values of $y$ the parameter $h$ is equal to $\pm h_i$. Such a method of finding the limiting value of a magnetic field was used by Landau and Lifshits [19].

The roots of equation (44) depend on the sign of the discriminant [16] $D = (q^2 - p^3)$

$$D = \left( \frac{3}{2} h \right)^2 - \left( \frac{|t|}{1 - |t|} \right)^3 = \left( \frac{3}{2} \right)^2 (h^2 - h_i^2).$$

(46)

If $D < 0$ inequalities $-h_i < h < h_i$ hold. If $D > 0$, then $h > h_i$ or $h < -h_i$.

If $D < 0$ all tree roots are real

$$y^{(1)}_1(t, h) = u_<(t, h) + v_< (t, h),$$
$$y^{(1)}_2(t, h) = \varepsilon_2 u_<(t, h) + \varepsilon_1 v_< (t, h),$$
$$y^{(1)}_3(t, h) = \varepsilon_1 u_<(t, h) + \varepsilon_2 v_< (t, h),$$

where

$$u_<(t, h) = \sqrt[3]{-q + i \sqrt{|D|}}, \quad v_< (t, h) = [u_<(t, h)]^*,$$
$$\varepsilon_1 = (-1 + i \sqrt{3})/2, \quad \varepsilon_2 = \varepsilon_1^*.$$ 

(47)

Using relations (48) we can show that

$$y^{(1)}_1(t, h) = 2 \text{Re} u_<(t, h),$$
$$(y^{(1)}_1(t, h))^* = y^{(1)}_1(t, h)(\sigma = 2, 3).$$

If $D > 0$ (i.e. $h^2 > h_i^2$) only one root is real

$$y_>(t, h) = u_>(t, h) + v_>(t, h),$$

where

$$u_>(t, h) = \sqrt[3]{\frac{3h}{2} + \sqrt{D}}, \quad v_>(t, h) = \sqrt[3]{\frac{3h}{2} - \sqrt{D}}.$$ 

(50)

The remaining two roots are complex.

If we combine equations (45)–(50) we obtain the functions $u_<(t, u), v_<(t, u)$ and $u_>(t, u), v_>(t, u)$ in the useful form

$$u_<(t, h) = (3/2)^{1/3} \sqrt[3]{h + \sqrt{h^2 - h_i^2}}, \quad v_<(t, h) = (3/2)^{1/3} \sqrt[3]{h - \sqrt{h^2 - h_i^2}},$$
$$u_>(t, h) = (3/2)^{1/3} \sqrt[3]{h + i \sqrt{h_i^2 - h_i^2}}, \quad v_>(t, h) = (3/2)^{1/3} \sqrt[3]{h - i \sqrt{h_i^2 - h_i^2}}.$$ 

(51)

Consider functions $u_<(t, h)$ and $v_<(t, h)$ for negative $h$

$$u_>(t, -h) = \sqrt[-1]{3} \sqrt[3]{h - \sqrt{h^2 - h_i^2}} = \sqrt[-1]{3} v_>(t, h),$$
$$v_>(t, -h) = \sqrt[-1]{3} \sqrt[3]{h + \sqrt{h^2 - h_i^2}} = \sqrt[-1]{3} u_>(t, h).$$

Similar relations hold also for $u_<$ and $v_<$. Among the three roots of unity $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3 = -1$ only the latter root yields a proper symmetry relation (26)

$$y^{(1)}_j(t, -h) = -y^{(1)}_j(t, h) (\sigma = >, <; \; j = 1, 2, 3).$$ 

(52)

In figure 5 we have plotted the dependence of roots (47) and (50) on the magnetic field $h$. For positive values of $y^{(1)}_1$ and $-h_i < h < 0$, as well as for negative values of $y^{(1)}_3$ and $0 < h < h_i$, the signs of $h$ and these two roots do not agree. The symmetry (26) of the
equation of state is broken. On the line BB’ (corresponding to the root \(y^{(2)}_\alpha\)) the derivative \((\partial y(\theta, h)/\partial h)_h < 0\) is negative. From equation (28) it follows that \((\partial^2 f_{\theta, h}(y)/\partial y^2)_{\theta, h}\) is negative too. In this interval of values of \(h\), the free energy has a maxima, therefore the CW magnet is not in a stable state. We conclude that for the root \(y^{(2)}_\alpha(\theta, h)\) free energy attains maximal values.

For \(0 < h < h_t\) the root \(y^{(1)}_\alpha\) is positive. The remaining two roots are negative and \(|y^{(3)}_\alpha| < y^{(1)}_\alpha\). Further \(y^{(2)}_\alpha \to y^{(3)}_\alpha\) when \(h \to h^- \equiv h_t - \varepsilon;\) \((0 < \varepsilon \ll 1)\) and

\[
\lim_{h \to h^-} \left( \frac{\partial y^{(2)}_\alpha(\theta, h)}{\partial h} \right)_{\theta, h} = -\infty.
\]

From equation (28) it follows that

\[
\lim_{h \to h^-} \left( \frac{\partial^2 f_{\theta, h}(y)}{\partial y^2} \right)_{\theta, h} = 0.
\]
This means that for \( h = \pm h_t \) one should consider the third derivative of free energy with respect to \( y \).

In the interval \(-h_t < h < h_t\) the derivative of the root \( y_{(3)} \) is positive, therefore the derivative \((\partial^2 f_{h_t}(y)/\partial y^2)_{y, h_t(y_{(3)})}\) is also positive. This means that in this interval free energy has a (local) minimum. Free energy exhibits the global minimum for the root \( y_{(1)} \) for \( h_t > h > 0 \). This behaviour of free energy and of the solutions of the cubic equation is shown in figure 5.

For \( h > h_t \) and \( h < -h_t \) free energy has the global maximum for the real root \( y_{(t, h)} \) (cf figure 5).

Note that as a result of the approximation yielding equation (41) the values of \( |y| \) may exceed the limiting value 1. This means that we shall restrict ourselves to small values of \( |t| \) and \( h_t \).

We shall point out that value \( h_t \) (45) of parameter \( h \) defines such a value \( B_t \) of magnetic induction \( B \), for which the value of induced magnetization \( m_{ind} \sim \chi T \cdot B_t \) \((B_t = \theta/h_t/\mu)\) with \( h_t \) given by equation (39) is in accordance with equation (42).

If \( h \ll h_t \) \((|t| \neq 0)\) the magnetic field \( B \) is weak and does not influence the thermodynamic quantities characterizing the system. If \( h \gg h_t \) the field \( B \) is strong. If \( t = 0 \) \((T = T_c)\) all magnetic fields are strong. As we have shown, if \( t \sim 0 \) and the field is strong, \( m \sim h^{1/3} \).

The parameter \( h_t \) (45) divides the positive \( h \)-semiaxis into two parts. For \( 0 < h \leq h_t \) there exist three roots of equation (44) and to one of them there corresponds a global minimum of the free energy. For \( h > h_t \) there exists one root corresponding to a minimum of free energy.

7. Properties of the internal energy, entropy and specific heat of the CW magnet

To find the internal energy \( U \) we shall use the familiar thermodynamic identity \([13]\)

\[
U = -\theta^2 \frac{\partial}{\partial \theta} \left( \frac{F}{\theta} \right) = N\theta^2 \frac{\partial}{\partial \theta} \left( -\frac{f}{\theta} \right) B.
\]

For the internal energy per spin this formula gives

\[
u = -\theta^2 \frac{\partial}{\partial \theta} \left( -Ky^2/2 + \ln \cosh(Ky + h) \right).
\]

As a result of simple calculations we obtain

\[
u(\theta, B) = -\frac{J}{2}y^2(\theta, B) - \mu By. \tag{53}\]

The first term of this equation is the interaction energy per spin, whereas the second term is the energy of a spin in the magnetic field. When \( \theta = 0 \), \( y = 1 \), hence \( u = -J/2 - B \mu \). When \( B = 0 \), in the paramagnetic phase \( y = 0 \) and \( u = 0 \). When \( B = 0 \) in the ferromagnetic phase is in the vicinity of \( \theta_c \) one has \( K \sim (1 - |t|)^{-1} \) and

\[
y \approx \pm \sqrt{3} |t|/(1 - |t|), \tag{54}\]

thus, \( u \sim |t|^1 \). As we can see from figure 6, in the absence of the magnetic field the behaviour of internal energy is singular at \( K = 1 \). The curve representing the function \( u(\theta, h = 0) \) consists of two branches. At \( \theta = \theta_c \), their derivatives are different. When the magnetic field is turned on this singularity is washed out (figure 6, right panel).

Consider entropy

\[
S = -Nk_B \left( \frac{\partial f(\theta, B)}{\partial \theta} \right) B, \tag{55}\]

By calculating the derivative we obtain for the entropy per particle the familiar thermodynamic identity [13]

\[ s(\theta, B) = \frac{k_B}{\theta} [\theta f(\theta, B) + u(\theta, B)]. \]  

(56)

For low temperatures

\[ f(\theta, B) \sim u(\theta, B). \]

Therefore, even when the magnetic field is turned on, at \( \theta = 0 \) entropy vanishes. The spins are completely ordered and entropy acquires the lowest value. Since in the absence of the magnetic field the internal energy is singular at \( K_c \), entropy is also singular at the critical point. When \( B = 0 \) in the paramagnetic phase \( (\theta \geq \theta_c) \) \( \gamma = 0 \) and the internal energy vanishes. From the definition (20) it follows that \(-k_B f(\theta \geq \theta_c, B = 0)/\theta = k_B \ln 2 \), and \( s = k_B \ln 2 \approx 0.7 \times k_B \). Spins in the paramagnetic phase are completely disordered and entropy reaches its greatest value. The dependence of entropy on temperature is shown in figure 7.

If \( B = 0 \) in the paramagnetic phase \( \gamma = 0 \), hence \( u(\theta > \theta_c, h = 0) = 0 \). In the ferromagnetic phase and in the vicinity of \( \theta_c \) according to equation (32) \( u \sim |t|^3 \). For \( \theta = \theta_c \) the internal energy vanishes \( u = 0 \). This means that the internal energy is a continuous function of temperature \( u(\theta^-) = u(\theta^+) = 0 \), with \( \theta^\pm = \theta_c \pm \epsilon \) and \( 0 < \epsilon \ll 1 \).
Entropy $s$ is also a continuous function of temperature. To show this property in the case of the ferromagnetic phase and the vicinity of $\theta_c$ we use equations (23) and (32) and Taylor’s series for $\ln\tanh(Ky)$. We get

$$s \approx kB \left( \ln 2 - \frac{3|t|}{2} \right).$$

We see that, as one may expect, the entropy of the ferromagnetic phase is smaller than the entropy of the paramagnetic phase and for $|t| = 0$ attains its maximal value.

The heat capacity behaviour in the vicinity of the critical temperature is more complex. To calculate the heat capacity per particle one can use one of two thermodynamic relations, namely \[ c(\theta, B) = kB \left( \frac{\partial u(\theta, B)}{\partial \theta} \right)_B, \tag{57} \]
or

$$c(\theta, B) = \theta \left( \frac{\partial s(\theta, B)}{\partial \theta} \right)_B. \tag{58}$$

It is easy to show that these identities yield the same result. We shall consider equation (57).

From equation (53) it follows that heat capacity $c(\theta, B)$ depends on derivative \( \left( \frac{\partial y}{\partial \theta} \right)_B \)

$$c = -k_B \theta(Ky + h) \left( \frac{\partial y}{\partial \theta} \right)_B.$$

Differentiating both sides of equation (22) with respect to $y$, solving the obtained equation for $\left( \frac{\partial y}{\partial \theta} \right)_B$ and applying the identity (24), we obtain an analytic expression for this derivative

$$\left( \frac{\partial y}{\partial \theta} \right)_B = \frac{1}{\theta (K + Ky^2 - \theta)}.$$

With the help of the above relation we obtain the final form of the expression for heat capacity per particle

$$c = \frac{k_B}{2} \frac{(1 - y^2)(Ky + h)^2}{(1 - K) + Ky^2}. \tag{59}$$

When $B = 0$ in the paramagnetic phase ($\theta \geq \theta_c$) magnetization $m$ vanishes, i.e. $y = 0$. Hence, $c = 0$. Since $\lim_{\theta \to 0} y = 1$ in this limit $c$ also vanishes.

Consider heat capacity in the ferromagnetic phase in the vicinity of the critical temperature. Using expressions (29) and (54) we obtain

$$c \approx \frac{3}{2} (1 - |t|). \tag{60}$$

Heat capacity is discontinuous at $\theta_c$

$$c(\theta_c^+, B = 0) - c(\theta_c^-, B = 0) = 3kB/2. \tag{61}$$

The zero field heat capacity is singular at the critical temperature, whereas in the magnetic field it exhibits a peak at the transition point (cf figure 8).

8. The magnetic limiting field

Until now we have described the phase transition using the approximate expression for $\text{artanh} y$.

Now we shall stop using this approximation, which means that we shall rely on numerical calculations. Such an approach will shed light on the approximation used in section 6. To the best of our knowledge, ours is the first systematic study of the limiting magnetic field of the CW model.
Figure 8. Dependence of heat capacity per particle on temperature. Left panel: $h = 0$. Right panel: $h > 0$.

Figure 9. Dependence of $y(h)$ on the magnetic field $h$. The dashed line represents the dependence of the roots of equation (44) on the magnetic field. The full line is a plot of the solution to equation (62).

Let us rewrite equation (40) in the form
\[ \text{artanh} \, y - Ky = h. \] (62)

The function $\Lambda(y) = (\text{artanh} \, y - Ky)$ has two extreme points at $y = \pm |t|^{1/2}$. For these values of $y$ the limiting value of the parameter $h$ is equal to
\[ h_\nu = \pm \left[ -\text{artanh} (\sqrt{|t|}) + \frac{\sqrt{|t|}}{1 - |t|} \right] \quad (0 \leq T \leq T_c). \]
Figure 10. Plot of the dependence of the critical value of the magnetic field $h_t$ on the parameter $t$.

The full line shows the result of numerical calculation, the dashed line represents the dependence resulting from the cubic equation (44).

For $|t| \ll 1$ with the accuracy to terms proportional to $|t|^{3/2}$ the limiting value of $h$ is equal to $h_t^{(app)}$ (45) (cf figure 10).

For a given value of $|t|$ we numerically solve equation (62) for various values of $h$. We plot $y = y_t(h)$ in figure 9. The obtained plot resembles the plot obtained for the roots of equation (44). However, the critical values of the parameter $h_t$ are smaller than $h_t^{(app)}$ given by equation (45), which we shall call $h_t^{(app)}$. Besides, unlike for the approximate theory, the values of $y_{num}$ are confined in the interval $[-1,1]$.

In figure 10 we compare values of $h_t^{(app)}$ and $h_t$. It is seen that for small values of $|t|$ both plots differ a little. With growing $|t|$ the difference is more pronounced.

Appendix

We shall show that there exist solutions $y(\theta, B)$ of equation (40) for which the second derivative of function $f_{K,h}(y)$ (equation (21)) with respect to $y$ calculated at $y = y(\theta, B)$

$$
\left( \frac{\partial^2 f_{\theta,h}(y)}{\partial y^2} \right)_{\theta,h |_{y = y(\theta, B)}} = -\theta K[1 - y^2(\theta, B)] - 1 \tag{A.1}
$$

is positive.

Consider temperatures higher than $\theta_c$.

(i) For $B = 0$ in the paramagnetic phase $K \leq 1$ and magnetization vanishes. In this case $y = 0$, therefore, the second derivative (A.1) $[-\theta K(K - 1)] > 0$ is positive.

(ii) If $B > 0$ magnetization is positive, hence $1 > y > 0$. The double inequality $0 < K[1 - y^2(\theta, B)] < 1$ holds. Therefore, $K[1 - y^2(\theta, B)] - 1 < 0$. Hence, the second derivative (A.1) is positive. The same arguments are valid for the negative value of induction ($B < 0$ and $y < 0$).

In the case of $\theta < \theta_c$ the parameter $K$ is greater than unity. As we know (cf section 4), if $B = 0$, equation (22) has three solutions, namely $y = 0$ and $y = \pm y(\theta) \equiv \pm y_0$.

(i) For $y = 0$ the derivative (A.1) is negative

$$
\left( \frac{\partial^2 f_{\theta,h}(y)}{\partial y^2} \right)_{\theta,h |_{y = 0}} = \theta K(1 - K) < 0. \tag{A.2}
$$
This means that in the ferromagnetic phase the solution \( y = 0 \) corresponds to a maximum of free energy.

(ii) In the case of the two remaining solutions the second derivative reads

\[
\left( \frac{\partial^2 f_{K,h}(y)}{\partial y^2} \right)_{\theta,h} \bigg|_{y = \pm y_0} = -\theta K \left[ \arctanh \frac{y_0}{y_0} (1 - y_0^2) - 1 \right].
\]

We shall express the parameter \( K \) by \( y_0 \). Using equation (40) we can write

\[
K = \frac{\text{artanh} \ y_0}{y_0}.
\]

With the help of this relation we find

\[
\left( \frac{\partial^2 f_{K,h}(y)}{\partial y^2} \right)_{\theta,h} \bigg|_{y = \pm y_0} = -\theta K \left[ \frac{\text{artanh} \ y_0}{y_0} (1 - y_0^2) - 1 \right].
\]

(A.3)

For small \( y_0 \) we can use the approximate expression \( \text{artanh} \ y_0 \approx y_0 + y_0^3/3 \). This yields the inequality

\[
(1 - y_0^2) \text{artanh} \ y_0 \approx (1 + y_0^2/3) (1 - y_0^2) < (1 - y_0^2) < 1,
\]

and the second derivative of free energy is positive. In the ferromagnetic phase, in the vicinity of critical temperature, for solutions \( \pm y_0 \) free energy reaches a minimum.

The function

\[
\psi(y_0) = \frac{\text{artanh} \ y_0}{y_0} (1 - y_0^2)
\]

monotonically decreases in the interval \((0, 1]\) and \( 0 \leq \psi(y_0) < 1 \).
Using the logarithmic representation \[ \arctanh y_0 = \frac{1}{2} \ln \left( \frac{y_0 + 1}{y_0 - 1} \right) \] and the limiting value \[ \lim_{y_0 \to 1^{-}} \frac{\arctanh y_0}{y_0} = 0, \]

hence, for low temperatures the right-hand side of equation (A.4) is positive. The plot of function (A.4) is shown in figure A1. We conclude that in the ferromagnetic phase minima of free energy occur for \( y = \pm y_0 \).

(iii) Assume that \( B > 0 \) \((h > 0)\) and \( \theta < \theta_c \) \((K > 1)\). In this case, according to section 5, equation (22) has three solutions \( y_1^{(c)}(t, h) > 0, \ y_2^{(c)}(t, h) < 0, \ y_3^{(c)}(t, h) < 0 \). Proceeding as before we express \( K \) by \( y_1^{(c)} \) and \( h \)

\[ K = \frac{\arctanh y_1^{(c)} - h}{y_1^{(c)}}. \]  

Now, equation (A.1) takes the form

\[
\left. \left( \frac{\partial^2 f_{K,h}(y)}{\partial y^2} \right) \right|_{\theta,B|y=y_1^{(c)}} = -\theta K \left\{ \frac{\arctanh y_1^{(c)} - h}{y_1^{(c)}} \left[ 1 - \left( y_1^{(c)} \right)^2 \right] \right\} - 1 \]

Since \( 0 < y_1^{(c)} \leq 1 \), the inequality \( 1 - \left( y_1^{(c)} \right)^2 < 1 \) holds. Thus, the additional term of equation (A.6) is non-positive

\[ -\frac{h}{y_1^{(c)}} \left[ 1 - \left( y_1^{(c)} \right)^2 \right] \leq 0, \]

and, even in this case, the second derivative of free energy is positive.

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