On the distance $\alpha$-spectral radius of a connected graph

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Abstract
For a connected graph $G$ and $\alpha \in [0, 1)$, the distance $\alpha$-spectral radius of $G$ is the spectral radius of the matrix $D_\alpha(G)$ defined as $D_\alpha(G) = \alpha T(G) + (1 - \alpha)D(G)$, where $T(G)$ is a diagonal matrix of vertex transmissions of $G$ and $D(G)$ is the distance matrix of $G$. We give bounds for the distance $\alpha$-spectral radius, especially for graphs that are not transmission regular, propose some graft transformations that decrease or increase the distance $\alpha$-spectral radius, and determine the unique graphs with minimum and maximum distance $\alpha$-spectral radius among some classes of graphs.

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1 Introduction
We consider simple and undirected graphs. Let $G$ be a connected graph of order $n$ with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, the distance between $u$ and $v$ in $G$, denoted by $d_G(u,v)$ or simply $d_{uv}$, is the length of a shortest path from $u$ to $v$ in $G$. The distance matrix of $G$ is the $n \times n$ matrix $D(G) = (d_G(u,v))_{u,v \in V(G)}$. For $u \in V(G)$, the transmission of $u$ in $G$, denoted by $T_G(u)$, is defined as the sum of distances from $u$ to all other vertices of $G$, i.e., $T_G(u) = \sum_{v \in V(G)} d_G(u,v)$. The transmission matrix $T(G)$ of $G$ is the diagonal matrix of transmissions of $G$. Then $Q(G) = T(G) + D(G)$ is the distance signless Laplacian matrix of $G$.

Throughout this paper we assume that $\alpha \in [0, 1)$. We consider the convex combinations $D_\alpha(G)$ of $T(G)$ and $D(G)$, defined as

$$D_\alpha(G) = \alpha T(G) + (1 - \alpha)D(G).$$

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Obviously, $D_0(G) = D(G)$ and $2D_{1/2}(G) = Q(G)$. We call the eigenvalues of $D_\alpha(G)$ the distance $\alpha$-eigenvalues of $G$. As $D_\alpha(G)$ is a symmetric matrix, the distance $\alpha$-eigenvalues of $G$ are all real, which are denoted by $\mu_\alpha^{(1)}(G), \ldots, \mu_\alpha^{(n)}(G)$, arranged in nonincreasing order, where $n = |V(G)|$. The largest distance $\alpha$-eigenvalue $\mu_\alpha^{(1)}(G)$ of $G$ is called the distance $\alpha$-spectral radius of $G$, written as $\mu_\alpha(G)$. Obviously, $\mu_\alpha^{(1)}(G), \ldots, \mu_\alpha^{(n)}(G)$ are the distance eigenvalues of $G$, and $2\mu_\alpha^{(1)}(G), \ldots, 2\mu_\alpha^{(n)}(G)$ are the distance signless Laplacian eigenvalues of $G$. Particularly, $\mu_0(G)$ is just the distance spectral radius and $2\mu_{1/2}(G)$ is just the distance signless Laplacian spectral radius of $G$. The distance eigenvalues and especially the distance spectral radius have been extensively studied, see the recent survey [1] and references therein. The distance signless Laplacian eigenvalues and especially the distance signless Laplacian spectral radius have also received much attention, see, e.g., [2, 3, 4, 7, 11, 12, 19].

In this paper, we give sharp bounds for the distance $\alpha$-spectral radius, and particularly an upper bound for the distance $\alpha$-spectral radius of connected graphs that are not transmission regular, and propose some types of graft transformations that decrease or increase the distance $\alpha$-spectral radius. We also determine the unique graphs with minimum distance $\alpha$-spectral radius among trees and unicyclic graphs, respectively, as well as the unique graphs (trees) with maximum and second maximum distance $\alpha$-spectral radii, and the unique graph with maximum distance $\alpha$-spectral radius among connected graphs with given clique number, and among odd-cycle unicyclic graphs, respectively.

2 Preliminaries

Let $G$ be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$. A column vector $x = (x_{v_1}, \ldots, x_{v_n})^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex $v_i$ to $x_{v_i}$, i.e., $x(v_i) = x_{v_i}$ for $i = 1, \ldots, n$. Then

$$x^T D_\alpha(G)x = \alpha \sum_{u \in V(G)} T_G(u) x_u^2 + 2 \sum_{\{u,v\} \subseteq V(G)} (1 - \alpha) d_G(u,v) x_u x_v,$$

or equivalently,

$$x^T D_\alpha(G)x = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) \left(\alpha(x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v\right).$$

Since $D_\alpha(G)$ is a nonnegative irreducible matrix, by the Perron-Frobenius theorem, $\mu_\alpha(G)$ is simple and there is a unique positive unit eigenvector corresponding to $\mu_\alpha(G)$, which is called the distance $\alpha$-Perron vector of $G$. If $x$ is the distance $\alpha$-Perron vector of $G$, then for each $u \in V(G)$,

$$\mu_\alpha(G)x_u = \alpha T_G(u)x_u + (1 - \alpha) \sum_{v \in V(G)} d_G(u,v) x_v,$$

or equivalently,

$$\mu_\alpha(G)x_u = \sum_{v \in V(G)} d_G(u,v)(\alpha x_u + (1 - \alpha)x_v),$$
which is called the \( \alpha \)-eigenequation of \( G \) at \( u \). For a unit column vector \( x \in \mathbb{R}^n \) with at least one nonnegative entry, by Rayleigh’s principle, we have \( \mu_\alpha(G) \geq x^\top D_\alpha(G)x \) with equality if and only if \( x \) is the distance \( \alpha \)-Perron vector of \( G \).

**Lemma 2.1.** Let \( G \) be a connected graph with \( \eta \) being an automorphism of \( G \), and \( x \) a distance \( \alpha \)-Perron vector of \( G \). Then for \( u, v \in V(G) \), \( \eta(u) = v \) implies that \( x_u = x_v \).

*Proof.* Let \( P = (p_{uv})_{u,v \in V(G)} \) be the permutation matrix such that \( p_{uv} = 1 \) if and only if \( \eta(u) = v \) for \( u,v \in V(G) \). We have \( D_\alpha(G) = P^\top D_\alpha(G)P \) and \( Px \) is a positive unit vector. Thus \( \mu_\alpha(G) = x^\top D_\alpha(G)x = (Px)^\top D_\alpha(G)(Px) \), implying that \( Px \) is also a distance \( \alpha \)-Perron vector of \( G \). Thus \( Px = x \), and the result follows. \( \square \)

Let \( G \) be a graph. For \( v \in V(G) \), let \( N_G(v) \) be the set of neighbors of \( v \) in \( G \), and \( d_G(v) \) be the degree of \( v \) in \( G \). Let \( G - v \) be the subgraph of \( G \) obtained by deleting \( v \) and all edges containing \( v \). For \( S \subseteq V(G) \), let \( G[S] \) be the subgraph induced by \( S \). For a subset \( E' \subseteq E(G) \), \( G - E' \) denotes the graph obtained from \( G \) by deleting all the edges in \( E' \), and in particular, we write \( G - xy \) instead of \( G - \{xy\} \) if \( E_1 = \{xy\} \). Let \( \overline{G} \) be the complement of \( G \). For a subset \( E' \subseteq E(\overline{G}) \), denote \( G + E' \) the graph obtained from \( G \) by adding all edges in \( E' \), and in particular, we write \( G + xy \) instead of \( G + \{xy\} \) if \( E' = \{xy\} \).

For a nonnegative square matrix \( A \), the Perron-Frobenius theorem implies that \( A \) has an eigenvalue that is equal the maximum modulus of all its eigen-values; this eigenvalue is called the spectral radius of \( A \), denoted by \( \rho(A) \). Obviously, \( \mu_\alpha(G) = \rho(D_\alpha(G)) \) for a connected graph \( G \).

Restating Corollary 2.1 in [14] p. 38, we have

**Lemma 2.2.** [14] Let \( A \) and \( B \) be square nonnegative matrices. If \( A \) is irreducible, \( A \geq B \), and \( A \neq B \), then \( \rho(A) > \rho(B) \).

By Lemma 2.2, we have

**Lemma 2.3.** Let \( G \) be a connected graph with \( u, v \in V(G) \). If \( u \) and \( v \) are not adjacent, then \( \mu_\alpha(G + uv) < \mu_\alpha(G) \).

The transmission of a connected graph \( G \), denoted by \( \sigma(G) \), is the sum of distance between all unordered pairs of vertices in \( G \). Clearly, \( \sigma(G) = \frac{1}{2} \sum_{v \in V(G)} T_G(v) \). A graph is said to be transmission regular if \( T_G(v) \) is a constant for each \( v \in V(G) \).

**Lemma 2.4.** Let \( G \) be a connected graph of order \( n \). Then

\[
\mu_\alpha(G) \geq \frac{2\sigma(G)}{n}
\]

with equality if and only if \( G \) is transmission regular.
Proof. Let $x = \frac{1}{\sqrt{n}}(1,1,\ldots,1)^\top$. Obviously, $xx^\top = 1$. Then

$$\mu_\alpha(G) \geq x^\top D_\alpha(G)x = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) \left(\alpha \left(x_u^2 + x_v^2\right) + 2(1-\alpha)x_u x_v\right) = \frac{2\sigma(G)}{n}.$$  

Equality holds if and only if $x$ is the distance $\alpha$-Perron vector of $G$, equivalently,

$$\mu_\alpha(G)x_u = \alpha T_G(u)x_u + (1-\alpha) \sum_{v \in V(G)} d_G(u,v)x_v = T_G(u)x_u \text{ for } u \in V(G),$$  

i.e., $T_G(u) = \mu_\alpha(G)$ for $u \in V(G)$.

Let $J_{s \times t}$ be the $s \times t$ matrix of all 1’s, $0_{s \times t}$ the $s \times t$ matrix of all 0’s, and $I_s$ the identity matrix of order $s$.

Let $K_n$, $P_n$, and $S_n$ be the complete graph, the path, and the star on $n$ vertices, respectively.

3 Bounds for the distance $\alpha$-spectral radius

In this section, we give some sharp bounds for the distance $\alpha$-spectral radius, some of which may serve as a gentle warm-up exercise.

Note that $D_\alpha(K_n) = \alpha(n-1)I_n + (1-\alpha)(J_{n \times n} - I_n)$, and thus $\mu_\alpha(K_n) = n-1$. By Lemma 2.3, we have

**Theorem 3.1.** Let $G$ be a connected graph of order $n$. Then

$$\mu_\alpha(G) \geq n - 1$$

with equality if and only if $G \cong K_n$.

If $(d_1,\ldots,d_n)$ is the nonincreasing degree sequence of a graph $G$ of order at least 2, then $d_1$ (resp. $d_2$) is the maximum (resp. second maximum) degree, $d_n$ (resp. $d_{n-1}$) is the minimum (resp. second minimum) degree of $G$. The diameter of $G$ is the maximum distance between all vertex pairs of $G$.

We use the techniques from [24].

**Theorem 3.2.** Let $G$ be a connected graph of order $n \geq 2$ with maximum degree $\Delta$ and second maximum degree $\Delta'$. Then

$$\mu_\alpha(G) \geq \frac{1}{2} \left(\alpha(4n - 4 - \Delta - \Delta') + \sqrt{\alpha^2(4n - 4 - \Delta - \Delta')^2 - 4(2\alpha - 1)(2n - 2 - \Delta)(2n - 2 - \Delta')}\right)$$

with equality if and only if $G$ is regular with diameter at most 2.

**Proof.** Let $x$ be the distance $\alpha$-Perron vector of $G$. Let

$$x_u = \min\{x_w : w \in V(G)\} \text{ and } x_v = \min\{x_w : w \in V(G) \setminus \{u\}\}.$$
From the $\alpha$-eigenequations of $G$ at $u$ and $v$, we have
\[
(\mu_{\alpha}(G) - \alpha T_G(u))x_u = (1 - \alpha) \sum_{w \in V(G)} d_G(u, w)x_w \\
\geq (1 - \alpha) \sum_{w \in V(G)} d_G(u, w)x_v \\
= (1 - \alpha)T_G(u)x_v
\]
and
\[
(\mu_{\alpha}(G) - \alpha T_G(v))x_v = (1 - \alpha) \sum_{w \in V(G)} d_G(v, w)x_w \\
\geq (1 - \alpha) \sum_{w \in V(G)} d_G(v, w)x_u \\
= (1 - \alpha)T_G(v)x_u.
\]
Thus
\[
(\mu_{\alpha}(G) - \alpha T_G(u))(\mu_{\alpha}(G) - \alpha T_G(v)) \geq (1 - \alpha)^2T_G(u)T_G(v),
\]
i.e.,
\[
\mu_{\alpha}^2(G) - \alpha(T_G(u) + T_G(v))\mu_{\alpha}(G) + (2\alpha - 1)T_G(u)T_G(v) \geq 0.
\]
Note that $\mu_{\alpha}(G) > \alpha T_G(u)$, $\mu_{\alpha}(G) > \alpha T_G(v)$, and thus $\mu_{\alpha}(G) > \frac{\alpha(T_G(u)+T_G(v))}{2}$.

Thus
\[
\mu_{\alpha}(G) \geq f(T_G(u), T_G(v))
\]
with
\[
f(s, t) = \frac{\alpha(s + t) + \sqrt{\alpha^2(s + t)^2 - 4(2\alpha - 1)st}}{2}.
\]

It is easily seen that $T_G(u) \geq d_G(u) + 2 \cdot (n - 1 - d_G(u)) = 2n - 2 - d_G(u)$. Similarly, $T_G(v) \geq 2n - 2 - d_G(v)$. Assume that $d_G(u) \geq d_G(v)$. Then
\[
T_G(u) \geq 2n - 2 - \Delta \text{ and } T_G(v) \geq 2n - 2 - \Delta'.
\]
Obviously, $f(s, t)$ is strictly increasing for $s, t \geq 1$. Thus
\[
\mu_{\alpha}(G) \geq f(2n - 2 - \Delta, 2n - 2 - \Delta')
\]
as desired.

Suppose that the lower bound for $\mu_{\alpha}(G)$ is attained. Then all entries of $x$ are equal to $x_u$ or $x_v$, and hence are the same. Therefore all transmissions are equal, and the diameter $d$ is at most 2. If $d = 1$, then $G$ is complete. If $d = 2$, then $\mu_{\alpha}(G) = T_G(w) = 2n - 2 - d_w$ for $w \in V(G)$, and thus $G$ is regular.

Conversely, if $G$ is regular with diameter at most 2, then $T_G(w) = 2n - 2 - d_G(w)$ for $w \in V(G)$, and thus the lower bound for $\mu_{\alpha}(G)$ is attained. \qed
**Theorem 3.3.** Let $G$ be a connected graph of order $n \geq 2$ with minimum degree $\delta$ and second minimum degree $\delta'$. Let $d$ be the diameter of $G$. Let $S = dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)$ and $S' = dn - \frac{d(d-1)}{2} - 1 - \delta'(d-1)$. Then

$$
\mu_\alpha(G) \leq \frac{1}{2} \left( \alpha(2dn - 2 - (d-1)(d + \delta + \delta')) \right. \\
+ \sqrt{\alpha^2(2dn - 2 - (d-1)(d + \delta + \delta'))^2 - 4(2\alpha - 1)SS'} 
$$

with equality if and only if $G$ is regular with $d \leq 2$.

**Proof.** Let $x$ be the distance $\alpha$-Perron vector of $G$. Let

$$x_u = \max\{x_w : w \in V(G)\} \text{ and } x_v = \max\{x_w : w \in V(G) \setminus \{u\}\}.$$

From the $\alpha$-eigenequations of $G$ at $u$ and $v$, we have

$$(\mu_\alpha(G) - \alpha T_G(u))x_u = (1 - \alpha) \sum_{w \in V(G)} d_G(u, w)x_w$$

$$\leq (1 - \alpha) \sum_{w \in V(G)} d_G(u, w)x_v$$

$$= (1 - \alpha)T_G(u)x_v$$

and

$$(\mu_\alpha(G) - \alpha T_G(v))x_v = (1 - \alpha) \sum_{w \in V(G)} d_G(v, w)x_w$$

$$\leq (1 - \alpha) \sum_{w \in V(G)} d_G(v, w)x_u$$

$$= (1 - \alpha)T_G(v)x_u.$$

Thus we have

$$\mu_\alpha^2(G) - \alpha(T_G(u) + T_G(v))\mu_\alpha(G) + (2\alpha - 1)T_G(u)T_G(v) \leq 0.$$ 

Thus

$$\mu_\alpha(G) \leq f(T_G(u), T_G(v))$$

with

$$f(s, t) = \frac{s + t}{2} \alpha(s + t) + \sqrt{\alpha^2(s + t)^2 - 4(2\alpha - 1)st}.$$ 

Assume that $d_G(u) \leq d_G(v)$. Note that

$$T_G(u) \leq d_G(u) + \sum_{i=2}^{d-1} i + d \left( n - 1 - d_G(u) - \sum_{i=2}^{d-1} 1 \right)$$

$$= dn - \frac{d(d-1)}{2} - 1 - d_G(u)(d-1)$$

$$\leq dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)$$

...
and similarly,
\[ T_G(v) \leq dn - \frac{d(d-1)}{2} - 1 - \delta'(d-1). \]
Since \( f(s,t) \) is strictly increasing for \( s, t \geq 1 \), we have
\[ \mu_a(G) \leq f \left( dn - \frac{d(d-1)}{2} - 1 - \delta(d-1), dn - \frac{d(d-1)}{2} - 1 - \delta'(d-1) \right), \]
as desired.

Suppose the upper bound for \( \mu_a(G) \) is attained. Then all entries of \( x \) are equal, and thus all transmissions are equal. If \( d \geq 3 \), then from the the above argument, for every vertex \( w \), there is exactly one vertex \( w' \) with \( d_G(w, w') = 2 \), and thus \( d = 3 \), and for a vertex \( z \) of eccentricity 2,
\[ d_G(z) + (n - 1 - d_G(z)) \cdot 2 = T_G(z) = 3n - \frac{3 \times (3-1)}{2} - 1 - (3-1)\delta, \]
implying that \( \delta \geq n - 2 \). Obviously, \( G \not\cong P_4 \). For a diametrical path \( P = v_0v_1v_2v_3, v_0 \) and \( v_3 \) should be adjacent to all vertices outside \( P \), implying that \( d = 2 \), a contradiction. Therefore \( G \) is regular with \( d \leq 2 \).

Conversely, if \( G \) is regular with \( d \leq 2 \), then \( T_G(w) = 2n - 2 - d_G(w) \) for \( w \in V(G) \), and thus the upper bound for \( \mu_a(G) \) is attained. \( \square \)

For an \( n \times n \) nonnegative matrix \( A = (a_{ij}) \), let \( r_i \) be the \( i \)-th row sum of \( A \), i.e., \( r_i = \sum_{j=1}^{n} a_{ij} \) for \( i = 1, \ldots, n \), and let \( r_{\min} \) and \( r_{\max} \) be the minimum and maximum row sums of \( A \), respectively.

**Lemma 3.1.** Let \( A = (a_{ij}) \) be an \( n \times n \) nonnegative matrix with row sums \( r_1, \ldots, r_n \). Let \( S = \{1, \ldots, n\} \), \( r_{\min} = r_p \), \( r_{\max} = r_q \) for some \( p \) and \( q \) with \( 1 \leq p, q \leq n \), \( \ell = \max \{r_i - a_{ip} : i \in S \setminus \{p\}\} \), \( m = \min \{r_i - a_{iq} : i \in S \setminus \{q\}\} \), \( s = \max \{a_{ip} : i \in S \setminus \{p\}\} \) and \( t = \min \{a_{iq} : i \in S \setminus \{q\}\} \). Then
\[
\begin{align*}
& a_{qq} + m + \sqrt{(m - a_{qq})^2 + 4t(r_{\max} - a_{qq})} \\
& \leq \rho(A) \\
& \leq a_{pp} + \ell + \sqrt{\ell^2 - a_{pp}^2} + 4s(r_{\min} - a_{pp}) \end{align*}
\]
Moreover, the first equality holds if \( r_i - a_{iq} = m \) and \( a_{iq} = t \) for all \( i \in S \setminus \{q\} \), and the second equality holds if \( r_i - a_{ip} = \ell \) and \( a_{ip} = s \) for all \( i \in S \setminus \{p\} \).

A connected graph \( G \) on \( n \) vertices is distinguished vertex deleted regular (DVDR) if there is a vertex \( v \) of degree \( n - 1 \) such that \( G - v \) is regular.

**Lemma 3.2.** Let \( G \) be a non-complete connected graph of order \( n \). Then \( G \) is a DVDR graph if and only if each vertex of \( G \) except one vertex \( v \) of degree \( n - 1 \) has the same transmission.

For a connected graph \( G \), let \( T_{\min}(G) \) and \( T_{\max}(G) \) be the minimum and maximum transmissions of \( G \), respectively. As in \( \text{(3)} \), we have the following bounds. For completeness, we include a proof here.
Lemma 3.1 to $D$ by replacing some non-diagonal entries of each row with row sum greater than $m_2$ in $D$. Let $D(1)$ be the matrix obtained from $D_\alpha(G)$ by replacing all $(w, v)$-entries by $1 - \alpha$ for $w \in V(G) \setminus \{v\}$, and replacing the submatrix $M$ by $M'$. Obviously, $D_\alpha(G)$ and $D(1)$ are nonnegative and irreducible, $D_\alpha(G) \geq D(1)$. By Lemma 2.2, $\mu_\alpha(G) \geq \rho(D(1))$ with equality if and only if $D_\alpha(G) = D(1)$. By applying Lemma 3.1 to $D(1)$, we obtain the lower bound for $\mu_\alpha(G)$. Suppose that this lower bound is attained. Then $D_\alpha(G) = D(1)$. As all $(w, v)$-entries are equal to $1 - \alpha$ for $w \in V(G) \setminus \{v\}$, implying that $d_G(v) = n - 1$. As $T_G(v) = T^\text{max}(G)$, $G$ is a complete graph. Conversely, if $G$ is a complete graph, then it is obvious that the lower bound for $\mu_\alpha(G)$ is attained.

Let $C$ be the submatrix of $D_\alpha(G)$ obtained by deleting the row and column corresponding to vertex $u$. Let $C'$ be the matrix obtained from $C$ by adding positive numbers to non-diagonal entries of each row with row sum less than $m_1$ in $C$ such that each row sum in $C'$ is $m_1$. Let $D(2)$ be the matrix obtained from $D_\alpha(G)$ by replacing all $(w, u)$-entries by $(1 - \alpha)e(u)$ for $w \in V(G) \setminus \{u\}$, and replacing the submatrix $C$ by $C'$. Obviously, $D_\alpha(G)$ and $D(2)$ are nonnegative and irreducible, $D(2) \geq D_\alpha(G)$. By Lemma 2.2, $\mu_\alpha(G) \leq \rho(D(2))$ with equality if and only if $D_\alpha(G) = D(2)$. By applying Lemma 3.1 to $D(2)$, we obtain the upper bound for $\mu_\alpha(G)$. Suppose that this upper bound is attained. By Lemma 2.2 then $D_\alpha(G) = D(2)$. As all $(w, u)$-entries are equal to $(1 - \alpha)e(u)$ for $w \in V(G) \setminus \{u\}$, implying that $e(u) = 1$, i.e., $d_G(u) = n - 1$. Note that $T_G(w) = m_1 + 1 - \alpha$ for all $w \in V(G) \setminus \{u\}$ and $T^\text{min}(G) = T_G(u) = n - 1$. If $m_1 + 1 - \alpha = n - 1$, then $G$ is a complete graph, which is a DVDR graph. Otherwise, $m_1 + 1 - \alpha > n - 1$, and thus by Lemma 3.2 $G$ is a DVDR graph.

Conversely, if $G$ is a DVDR graph, then $G$ is either complete or non-complete, and by Lemma 3.2 when $G$ is non-complete and the above argument, it is obvious that the upper bound for $\mu_\alpha(G)$ is attained.

We mention that more bounds for $\mu_\alpha(G)$ may be derived from some known bounds for nonnegative matrices, see, e.g., [4].
Let $G$ be a connected graph on $n$ vertices. As $\mu_{\alpha}(G) \leq T_{\text{max}}(G)$ with equality if and only if $G$ is transmission regular. Recently, Liu et al. [13] showed that

$$T_{\text{max}}(G) - \mu_0(G) > \frac{nT_{\text{max}}(G) - 2\sigma(G)}{(nT_{\text{max}}(G) - 2\sigma(G) + 1)n}$$

and

$$T_{\text{max}}(G) - \mu_1(G) > \frac{nT_{\text{max}}(G) - 2\sigma(G)}{(2(nT_{\text{max}}(G) - 2\sigma(G)) + 1)n}.$$ 

**Theorem 3.5.** Let $G$ be a connected non-transmission-regular graph of order $n$. Then

$$T_{\text{max}}(G) - \mu_{\alpha}(G) > \frac{(1 - \alpha)nT_{\text{max}}(G)(nT_{\text{max}}(G) - 2\sigma(G))}{(1 - \alpha)n^2T_{\text{max}}(G) + 4\sigma(G)(nT_{\text{max}}(G) - 2\sigma(G))}.$$ 

**Proof.** Let $x$ be the $\alpha$-Perron vector of $G$. Denote by $x_u = \max\{x_w : w \in V(G)\}$ and $x_v = \min\{x_w : w \in V(G)\}$. Since $G$ is not transmission regular, we have $x_u > x_v$, and thus:

$$\mu_{\alpha}(G) = x^\top D_{\alpha}(G)x = \alpha \sum_{w \in V(G)} T_G(w)x_w^2 + 2(1 - \alpha) \sum_{\{w, z\} \subseteq V(G)} d_{wz}x_wx_z < 2\alpha\sigma(G)x_u^2 + 2(1 - \alpha)\sigma(G)x_u^2,$$

implying that $x_u^2 > \frac{\mu_{\alpha}(G)}{2\sigma(G)}$. Note that

$$T_{\text{max}}(G) - \mu_{\alpha}(G) = T_{\text{max}}(G) - \alpha \sum_{w \in V(G)} T_G(w)x_w^2 - 2(1 - \alpha) \sum_{\{w, z\} \subseteq V(G)} d_{wz}x_wx_z$$

$$= \sum_{w \in V(G)} (T_{\text{max}}(G) - T_G(w))x_w^2 + (1 - \alpha) \sum_{\{w, z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2$$

$$\geq \sum_{w \in V(G)} (T_{\text{max}}(G) - T_G(w))x_v^2 + (1 - \alpha) \sum_{\{w, z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2$$

$$= (nT_{\text{max}}(G) - 2\sigma(G))x_v^2 + (1 - \alpha) \sum_{\{w, z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2.$$ 

We need to estimate $\sum_{\{w, z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2$. Obviously,

$$\sum_{\{w, z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2 \geq N_1 + N_2,$$

where $N_1 = \sum_{w \in V(G) \setminus V(P)} \sum_{z \in V(P)} d_{wz}(x_w - x_z)^2$ and $N_2 = \sum_{\{w, z\} \subseteq V(P)} d_{wz}(x_w - x_z)^2$. Let $P = w_0, w_1, \ldots, w_\ell$ be the shortest path connecting $u$ and $v$, where $w_0 = u$, $w_\ell = v$, and $\ell \geq 1$. For $w \in V(G) \setminus V(P)$, by Cauchy-Schwarz inequality, we have

$$d_{wu}(x_w - x_u)^2 + d_{wv}(x_w - x_v)^2 \geq (x_w - x_u)^2 + (x_w - x_v)^2 \geq \frac{1}{2}(x_u - x_v)^2,$$
and thus
\[ N_1 \geq \sum_{w \in V(G) \setminus V(P)} (d_{wu}(x_w - x_u)^2 + d_{wv}(x_w - x_v)^2) \]
\[ \geq \sum_{w \in V(G) \setminus V(P)} \frac{1}{2}(x_u - x_v)^2 \]
\[ = \frac{n - \ell - 1}{2}(x_u - x_v)^2. \]

For \(1 \leq i \leq \ell - 1\), by Cauchy-Schwarz inequality, we have
\[ d_{w_0w_i}(x_{w_0} - x_{w_i})^2 + d_{w_iw_{\ell}}(x_{w_i} - x_{w_{\ell}})^2 \]
\[ \geq \min\{i, \ell - i\} \left( (x_{w_0} - x_{w_i})^2 + (x_{w_i} - x_{w_{\ell}})^2 \right) \]
\[ \geq \min\{i, \ell - i\} \cdot \frac{1}{2}(x_{w_0} - x_{w_{\ell}})^2 \]
\[ = \frac{1}{2} \min\{i, \ell - i\}(x_u - x_v)^2, \]

and thus
\[ N_2 \geq d_{uv}(x_u - x_v)^2 + \sum_{i=1}^{\ell-1} \left( d_{w_0w_i}(x_{w_0} - x_{w_i})^2 + d_{w_iw_{\ell}}(x_{w_i} - x_{w_{\ell}})^2 \right) \]
\[ \geq \ell(x_u - x_v)^2 + \sum_{i=1}^{\ell-1} \frac{1}{2} \min\{i, \ell - i\}(x_u - x_v)^2 \]
\[ = \left( \ell + \frac{1}{2} \sum_{i=1}^{\ell-1} \min\{i, \ell - i\} \right)(x_u - x_v)^2 \]
\[ = \begin{cases} \frac{\ell^2 + 8\ell}{8}(x_u - x_v)^2 & \text{if } \ell \text{ is even}, \\ \frac{\ell^2 + 8\ell - 1}{8}(x_u - x_v)^2 & \text{if } \ell \text{ is even}. \end{cases} \]

**Case 1.** \(u\) and \(v\) are adjacent, i.e., \(\ell = 1\).

In this case, we have
\[ \sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2 \geq N_1 + N_2 \]
\[ \geq \frac{n - 1 - 1}{2}(x_u - x_v)^2 + (x_u - x_v)^2 \]
\[ = \frac{n}{2}(x_u - x_v)^2. \]

Thus
\[ T_{\max}(G) - \mu_\alpha(G) \geq (nT_{\max}(G) - 2\sigma(G))x_v^2 + (1 - \alpha) \sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2 \]
\[ \geq (nT_{\max}(G) - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{n}{2}(x_u - x_v)^2. \]
Viewed as a function of $x_v$, $(nT_{\max}(G) - 2\sigma(G))x_v^2 + (1 - \alpha)^\frac{1}{2}(x_u - x_v)^2$ achieves its minimum value \( \frac{(1 - \alpha)(nT_{\max}(G) - 2\sigma(G))}{(1 - \alpha)n + 2(nT_{\max}(G) - 2\sigma(G))}x_u^2 \). Recall that $x_u^2 > \frac{\mu_\alpha(G)}{2\sigma(G)}$. Then we have

\[
T_{\max}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)n(nT_{\max}(G) - 2\sigma(G))}{(1 - \alpha)n + 2(nT_{\max}(G) - 2\sigma(G))T_{\max}(G)} - \frac{\mu_\alpha(G)}{2\sigma(G)} \frac{2\sigma(G)((1 - \alpha)n + 2(nT_{\max}(G) - 2\sigma(G)))}{2}\]

which implies that

\[
T_{\max}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)nT_{\max}(G)(nT_{\max}(G) - 2\sigma(G))}{(1 - \alpha)n^2T_{\max}(G) + 4\sigma(G)(nT_{\max}(G) - 2\sigma(G))}. \]

**Case 2.** $u$ and $v$ are not adjacent, i.e., $\ell \geq 2$.

Suppose first that $\ell$ is even. Then

\[
\sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2 \geq N_1 + N_2
\]

\[
\geq \frac{n - \ell - 1}{2}(x_u - x_v)^2 + \frac{\ell^2 + 8\ell}{8}(x_u - x_v)^2
\]

\[
= \frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2.
\]

Thus

\[
T_{\max}(G) - \mu_\alpha(G) \geq (nT_{\max}(G) - 2\sigma(G))x_v^2 + (1 - \alpha)\sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2
\]

\[
\geq (nT_{\max}(G) - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2.
\]

Viewed as a function of $x_v$, $(nT_{\max}(G) - 2\sigma(G))x_v^2 + (1 - \alpha)^\frac{1}{2}(x_u - x_v)^2$ achieves its minimum value \( \frac{(1 - \alpha)(nT_{\max}(G) - 2\sigma(G))}{(1 - \alpha)(\ell^2 + 4\ell + 4n - 4 + 8(nT_{\max}(G) - 2\sigma(G)))}x_u^2 \). As $x_u^2 > \frac{\mu_\alpha(G)}{2\sigma(G)}$, we have

\[
T_{\max}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)(nT_{\max}(G) - 2\sigma(G))(\ell^2 + 4\ell + 4n - 4)}{(1 - \alpha)(\ell^2 + 4\ell + 4n - 4)nT_{\max}(G) + 16\sigma(G)(nT_{\max}(G) - 2\sigma(G))} \cdot \frac{\mu_\alpha(G)}{2\sigma(G)},
\]

i.e.,

\[
T_{\max}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)(nT_{\max}(G) - 2\sigma(G))(\ell^2 + 4\ell + 4n - 4)T_{\max}(G)}{(1 - \alpha)(\ell^2 + 4\ell + 4n - 4)nT_{\max}(G) + 16\sigma(G)(nT_{\max}(G) - 2\sigma(G))}.
\]

As a function of $\ell$, the expression in the right hand side in the above inequality is strictly increasing for $\ell \geq 2$. Thus we have

\[
T_{\max}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)(nT_{\max}(G) - 2\sigma(G))(n + 2)T_{\max}(G)}{(1 - \alpha)(n + 2)nT_{\max}(G) + 4\sigma(G)(nT_{\max}(G) - 2\sigma(G))}\]
As a function of $\ell$ is strictly increasing for $\ell$ and thus, as early, we have

$$T_{\text{max}}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)nT_{\text{max}}(G)(nT_{\text{max}}(G) - 2\sigma(G))}{(1 - \alpha)n^2T_{\text{max}}(G) + 4\sigma(G)(nT_{\text{max}}(G) - 2\sigma(G))}.$$ 

Now suppose that $\ell$ is odd.

Then

$$\sum_{\{w,z\} \subseteq V(G)} d_{wz}(x_w - x_z)^2 \geq N_1 + N_2 \geq \frac{n - \ell - 1}{2}(x_u - x_v)^2 + \frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2 = \frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2.$$ 

Thus, as early, we have

$$T_{\text{max}}(G) - \mu_\alpha(G) \geq (nT_{\text{max}}(G) - 2\sigma(G))x_v^2 + (1 - \alpha)\frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2 \geq \frac{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5)(nT_{\text{max}}(G) - 2\sigma(G))}{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5) + 8(nT_{\text{max}}(G) - 2\sigma(G))} x_v^2 \geq \frac{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5)(nT_{\text{max}}(G) - 2\sigma(G))}{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5) + 8(nT_{\text{max}}(G) - 2\sigma(G))} \frac{\mu_\alpha(G)}{2\sigma(G)},$$

implying that

$$T_{\text{max}}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)(nT_{\text{max}}(G) - 2\sigma(G))(\ell^2 + 4\ell + 4n - 5)T_{\text{max}}(G)}{(1 - \alpha)(\ell^2 + 4\ell + 4n - 5)nT_{\text{max}}(G) + 16\sigma(G)(nT_{\text{max}}(G) - 2\sigma(G))}.$$ 

As a function of $\ell$, the expression in the right hand side in the above inequality is strictly increasing for $\ell \geq 3$. Thus we have

$$T_{\text{max}}(G) - \mu_\alpha(G) > \frac{(1 - \alpha)(nT_{\text{max}}(G) - 2\sigma(G))(4 + n)T_{\text{max}}(G)}{(1 - \alpha)(4 + n)nT_{\text{max}}(G) + 4\sigma(G)(nT_{\text{max}}(G) - 2\sigma(G))} > \frac{(1 - \alpha)nT_{\text{max}}(G)(nT_{\text{max}}(G) - 2\sigma(G))}{(1 - \alpha)n^2T_{\text{max}}(G) + 4\sigma(G)(nT_{\text{max}}(G) - 2\sigma(G))}.$$ 

The result follows by combining Cases 1 and 2. \hfill \square

If $\alpha = 0, \frac{1}{2}$, then the bound for $T_{\text{max}}(G) - \rho_0(G)$ in Theorem 4.1 reduces to

$$T_{\text{max}}(G) - \mu_0(G) > \frac{(nT_{\text{max}}(G) - 2\sigma(G))nT_{\text{max}}(G)}{n^2T_{\text{max}}(G) + 4(nT_{\text{max}}(G) - 2\sigma(G))\sigma(G)}$$

and

$$T_{\text{max}}(G) - \mu_{\frac{1}{2}}(G) > \frac{(nT_{\text{max}}(G) - 2\sigma(G))nT_{\text{max}}(G)}{n^2T_{\text{max}}(G) + 8(nT_{\text{max}}(G) - 2\sigma(G))\sigma(G)}.$$
4 Effect of graft transformations on distance $\alpha$-spectral radius

In this section, we study the effect of some graft transformations on distance $\alpha$-spectral radius.

A path $u_0 \ldots u_r$ (with $r \geq 1$) in a graph $G$ is called a pendant path (of length $r$) at $u_0$ if $d_G(u_0) \geq 3$, the degrees of $u_1, \ldots, u_{r-1}$ (if any exists) are all equal to 2 in $G$, and $d_G(u_r) = 1$. A pendant path of length 1 at $u_0$ is called a pendant edge at $u_0$.

A vertex of a graph is a pendant vertex if its degree is 1. The neighbor of the pendant vertex in a graph is called a quasi-pendant vertex. A non-pendant edge in a graph is an edge such that both end vertices are not pendant vertices. A cut edge of a connected graph is an edge whose removal yields a disconnected graph.

If $P$ is a pendant path of $G$ at $u$ with length $r \geq 1$, then we say $G$ is obtained from $H$ by attaching a pendant path $P$ of length $r$ at $u$ with $H = G[V(G) \setminus (V(P) \setminus \{u\})]$. If the pendant path of length 1 is attached to a vertex $u$ of $H$, then we also say that a pendant vertex is attached to $u$.

**Theorem 4.1.** Let $G$ be a connected graph and $uv$ a non-pendant cut edge of $G$. Let $G_{uv}$ be the graph obtained from $G$ by identifying vertices $u$ and $v$ to vertex $v$ and attaching a pendant vertex $u$ to $v$. If at least one of $\{u, v\}$ is a quasi-pendant vertex in $G$, then $\mu_\alpha(G) > \mu_\alpha(G_{uv})$.

*Proof.* Assume that $v$ is a quasi-pendant vertex in $G$. Let $v'$ be a pendant neighbor of $v$, and let $G_1$ and $G_2$ be the components of $G - uv$ containing $u$ and $v$, respectively, see Fig. 1.

Let $x$ be the distance $\alpha$-Perron vector of $G_{uv}$. By Lemma 2.1, $x_u = x_{v'}$.

As we pass from $G$ to $G_{uv}$, the distance between a vertex in $V(G_1) \setminus \{u\}$ and a vertex in $V(G_2)$ is decreased by 1, the distance between a vertex $V(G_1) \setminus \{u\}$ and $u$ is increased by 1, and the distances between all other vertex pairs remain unchanged. Thus

\[
\mu_\alpha(G) - \mu_\alpha(G_{uv}) \\
\geq x^\top (D_\alpha(G) - D_\alpha(G_{uv})) x \\
= \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V(G_2)} \left( \alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_w x_z \right)
\]
and
\[ \bigcup_{i \in [18, 12]} \]

Produced subgraphs or remain unchanged. Thus at most \( d \) a vertex in \( V \) 

\[ \text{Theorem 4.2.} \]

\[ \text{Proof.} \]

\[ \text{Let} \]

\[ G \]

\[ \text{be a connected graph with} \]

\[ k \]

\[ \text{edge–disjoint nontrivial induced subgraphs} \]

\[ G_1, \ldots, G_k \]

\[ \text{such that} \]

\[ V(G_i) \cap V(G_j) = \{u\} \]

\[ \text{for} \]

\[ 1 \leq i < j \leq k \]

\[ \text{and} \]

\[ \bigcup_{i=1}^k V(G_i) = V(G), \]

\[ \text{where} \]

\[ k \geq 3. \]

\[ \text{Let} \]

\[ K \]

\[ \text{be a nonempty subset of} \]

\[ \{3, \ldots, k\} \]

\[ \text{and let} \]

\[ N_K = \bigcup_{i \in K} N_{G_i}(u). \]

\[ \text{For} \]

\[ v' \in V(G_1) \setminus \{u\} \]

\[ \text{and} \]

\[ v'' \in V(G_2) \setminus \{u\}, \]

\[ \text{let} \]

\[ G' = G \setminus \{uw : w \in N_K\} \]

\[ \text{and} \]

\[ G'' = G \setminus \{uw : w \in N_K\} \]

\[ \{v''w : w \in N_K\}. \]

\[ \text{Then} \]

\[ \mu_\alpha(G) < \mu_\alpha(G') \]

\[ \text{or} \]

\[ \mu_\alpha(G) < \mu_\alpha(G''). \]

\[ \text{Proof.} \]

\[ \text{Let} \]

\[ x \]

\[ \text{be the distance} \]

\[ \alpha \text{-Perron vector of} \]

\[ G. \]

\[ \text{Let} \]

\[ V_K = (\bigcup_{i \in K} V(G_i)) \setminus \{u\}. \]

\[ \text{Let} \]

\[ \Gamma = \sum_{w \in V(G_2) \setminus \{u\}} \sum_{z \in V_K} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1 - \alpha)x_wx_z \right) \]

\[ - \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1 - \alpha)x_wx_z \right). \]

\[ \text{As we pass from} \]

\[ G \]

\[ \text{to} \]

\[ G' \]

\[ \text{the distance between a vertex in} \]

\[ V(G_2) \]

\[ \text{and a vertex in} \]

\[ V_K \]

\[ \text{is increased by} \]

\[ d_G(u, v'), \]

\[ \text{the distance between a vertex} \]

\[ w \]

\[ \text{in} \]

\[ V(G_1) \setminus \{u\} \]

\[ \text{and a vertex in} \]

\[ V_K \]

\[ \text{is decreased by} \]

\[ d_G(w, u) - d_G(w, v'), \]

\[ \text{which is} \]

\[ \text{at most} \]

\[ d_G(u, v'), \]

\[ \text{and the distances between all other vertex pairs are increased} \]

\[ \text{or remain unchanged. Thus} \]

\[ \mu_\alpha(G') - \mu_\alpha(G) \]

\[ \geq x^T(D_\alpha(G') - D_\alpha(G))x \]

\[ \geq \sum_{w \in V(G_2)} \sum_{z \in V_K} (d_G(u, v') \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1 - \alpha)x_wx_z \right)) \]

\[ - \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} (d_G(u, v') \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1 - \alpha)x_wx_z \right)) \]

\[ \text{Therefore} \]

\[ \mu_\alpha(G) - \mu_\alpha(G_{uv}) > 0, \text{ i.e.,} \]

\[ \mu_\alpha(G) > \mu_\alpha(G_{uv}). \]

\[ \square \]

The previous theorem has been established for \( \alpha = 0, \frac{1}{2} \) in \[18, 12\].
\[ d_G(u, v') \left( \Gamma + \sum_{z \in V} (\alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z) \right) \]
\[ > d_G(u, v') \Gamma. \]

If \( \Gamma \geq 0 \), then \( \mu_\alpha(G') - \mu_\alpha(G) > d_G(u, v') \Gamma \geq 0 \), implying that \( \mu_\alpha(G) < \mu_\alpha(G') \).

Suppose that \( \Gamma < 0 \). As we pass from \( G \) to \( G'' \), the distance between a vertex in \( V(G_1) \) and a vertex in \( V_K \) is increased by \( d_G(u, v'') \), the distance between a vertex \( w \) in \( V(G_2) \setminus \{u\} \) and a vertex in \( V_K \) is decreased by \( d_G(w, u) - d_G(w, v') \), which is at most \( d_G(u, v'') \), and the distances between all other vertex pairs are increased or remain unchanged. Thus

\[
\begin{align*}
\mu_\alpha(G'') - \mu_\alpha(G) & \geq x^T (D_\alpha(G'') - D_\alpha(G))x \\
& \geq \sum_{w \in V(G_1)} \sum_{z \in V_K} (d_G(u, v'') (\alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z)) \\
& \quad - \sum_{w \in V(G_2) \setminus \{u\}} \sum_{z \in V_K} (d_G(u, v'') (\alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_wx_z)) \\
& = d_G(u, v'') \left( -\Gamma + \sum_{z \in V_K} (\alpha (x_u^2 + x_z^2) + 2(1 - \alpha)x_ux_z) \right) \\
& > d_G(u, v'')(\Gamma) \\
& > 0,
\end{align*}
\]

implying that \( \mu_\alpha(G'') - \mu_\alpha(G) > 0 \), i.e., \( \mu_\alpha(G) < \mu_\alpha(G'') \). \( \square \)

Weak versions of previous theorem for \( \alpha = 0 \) have been given in [21, 20] and a weak version for \( \alpha = \frac{1}{2} \) may be found in [12].

For positive integer \( p \) and a graph \( G \) with \( u \in V(G) \), let \( G(u; p) \) be the graph obtained from \( G \) by attaching a pendant path of length \( p \) at \( u \). Let \( G(u; 0) = G \), and in this case a pendant path of length 0 is understood the trivial path consisting of a single vertex \( u \).

For nonnegative integers \( p, q \) and a graph \( G \), let \( G_u(p, q) \) or simply \( G_{p, q} \) be the graph \( H(u; q) \) with \( H = G(u; p) \).

The following corollary has been given for \( \alpha = 0 \) in [17, 20] and \( \alpha = \frac{1}{2} \) in [11, 12].

**Corollary 4.1.** Let \( H \) be a nontrivial connected graph with \( u \in V(H) \). If \( p \geq q \geq 1 \), then \( \mu_\alpha(H_u(p, q)) < \mu_\alpha(H_u(p + 1, q - 1)) \).

**Proof.** Let \( G = H_u(p, q) \). Let \( P = uu_1 \ldots u_p \) and \( Q = uv_1 \ldots v_q \) be two pendant paths of lengths \( p \) and \( q \), respectively in \( G \). Using the notations in Theorem 4.2 with \( k = 3 \), \( G_1 = P \), \( G_2 = Q \), \( G_3 = H \), \( v' = u_{p-q+1} \) and \( v'' = v_1 \), we have \( G' \cong G'' \cong H_u(p + 1, q - 1) \), and thus by Theorem 4.2 we have \( \mu_\alpha(H_u(p, q)) < \mu_\alpha(H_u(p + 1, q - 1)) \). \( \square \)

**Theorem 4.3.** Let \( G \) be a connected graph with three edge-disjoint induced subgraphs \( G_1, G_2 \) and \( G_3 \) such that \( V(G_1) \cap V(G_3) = \{u\} \), \( V(G_2) \cap V(G_3) = \{v\} \),
\( \cup_{i=1}^{3} V(G_i) = V(G) \), and \( G_1-u, G_2-v, \) and \( G_3-u-v \) are all nontrivial. Suppose that \( uv \in E(G_3) \). For \( u' \in N_{G_1}(u) \) and \( v' \in N_{G_2}(v) \), let

\[
G' = H + \{ u'w : w \in N_{G_3-u}(u) \} + \{ uw : w \in N_{G_3-u-v}(v) \}
\]

and

\[
G'' = H + \{ vw : w \in N_{G_3-u}(u) \} + \{ v'w : w \in N_{G_3-u-v}(v) \},
\]

where \( H = G - \{ uw : w \in N_{G_3-u}(u) \} - \{ vw : w \in N_{G_3-u-v}(v) \} \). Then \( \mu_\alpha(G) < \mu_\alpha(G') \) or \( \mu_\alpha(G) < \mu_\alpha(G'') \).

**Proof.** Let \( x \) be the distance \( \alpha \)-Perron vector of \( G \). Let

\[
\Gamma = \sum_{w \in V(G_2)} \sum_{z \in V(G_3) \setminus \{u,v\}} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1-\alpha)x_wx_z \right) - \sum_{w \in V(G_1)} \sum_{z \in V(G_3) \setminus \{u,v\}} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1-\alpha)x_wx_z \right).
\]

As we pass from \( G \) to \( G' \), the distance between a vertex in \( V(G_2) \) and a vertex in \( V(G_3) \setminus \{u,v\} \) is increased by 1, the distance between a vertex in \( V(G_1) \) and a vertex in \( V(G_3) \setminus \{u,v\} \) may be increased, unchanged, or decreased by 1, and the distances between any other vertex pairs remain unchanged. Thus

\[
\mu_\alpha(G') - \mu_\alpha(G) \geq x^\top (D_\alpha(G') - D_\alpha(G))x \\
\geq \sum_{w \in V(G_2)} \sum_{z \in V(G_3) \setminus \{u,v\}} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1-\alpha)x_wx_z \right) - \sum_{w \in V(G_1)} \sum_{z \in V(G_3) \setminus \{u,v\}} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1-\alpha)x_wx_z \right) \\
= \Gamma.
\]

If \( \Gamma \geq 0 \), then \( \mu_\alpha(G') - \mu_\alpha(G) \geq 0 \), i.e., \( \mu_\alpha(G) \leq \mu_\alpha(G') \). If \( \mu_\alpha(G) = \mu_\alpha(G') \), then \( \mu_\alpha(G') = x^\top D_\alpha(G')x \), implying that \( x \) is the distance \( \alpha \)-Perron vector of \( G' \). By the \( \alpha \)-eigenequations of \( G \) and \( G' \) at \( v \), we have

\[
0 = \mu_\alpha(G')x_v - \mu_\alpha(G)x_v \\
= \sum_{w \in V(G_3) \setminus \{u,v\}} (d_{G'}(v,w) - d_G(v,w))(\alpha x_v + (1-\alpha)x_w) \\
= \sum_{w \in V(G_3) \setminus \{u,v\}} (\alpha x_v + (1-\alpha)x_w) \\
> 0,
\]

a contradiction. Thus, if \( \Gamma \geq 0 \), then \( \mu_\alpha(G) < \mu_\alpha(G') \).

Suppose that \( \Gamma < 0 \). As earlier, we have

\[
\mu_\alpha(G'') - \mu_\alpha(G) \geq x^\top (D_\alpha(G'') - D_\alpha(G))x \\
\geq \sum_{w \in V(G_1)} \sum_{z \in V(G_3) \setminus \{u,v\}} \left( \alpha \left( x_w^2 + x_z^2 \right) + 2(1-\alpha)x_wx_z \right)
\]
Corollary 4.2.

\[ \sum_{w \in V(G_2)} \sum_{z \in V(G_3) \setminus \{u,v\}} (\alpha (x_w^2 + x_z^2) + 2(1 - \alpha)x_w x_z) \]
\[ = -\Gamma \]
\[ > 0, \]
and thus \( \mu_\alpha(G) < \mu_\alpha(G'') \). \qed

A weak version of previous theorem for \( \alpha = \frac{1}{2} \) has been established in [23].

For nonnegative integers \( p, q \) and a graph \( G \) with \( u, v \in V(G) \), let \( G_{u,v}(p, q) \) be the graph \( H(v; q) \) with \( H = G(u; p) \).

Similar versions for the following corollary have been given for \( \alpha = 0, \frac{1}{2} \) in [11].

**Corollary 4.2.** Let \( H \) be a connected graph of order at least 3 with \( uv \in E(H) \). Suppose that \( \eta(u) = v \) for some automorphism \( \eta \) of \( G \). For \( p \geq q \geq 1 \), we have \( \mu_\alpha(H_{u,v}(p, q)) < \mu_\alpha(H_{u,v}(p+1, q-1)) \).

**Proof.** Let \( G = H_{u,v}(p, q) \). Let \( P = uu_1 \ldots u_p \) and \( Q = vv_1 \ldots v_q \) be two pendant paths of lengths \( p \) and \( q \) in \( G \) at \( u \) and \( v \), respectively. Using the notations of Theorem 4.3 with \( G_1 = P, G_2 = Q, G_3 = H, u' = u_1 \) and \( v' = v_1 \), we have \( G' \cong H_{u,v}(p-1, q+1) \) and \( G'' \cong H_{u,v}(p+1, q-1) \), and thus by Theorem 4.3 we have

\[ \mu_\alpha(H_{u,v}(p, q)) < \max\{\mu_\alpha(H_{u,v}(p-1, q+1)), \mu_\alpha(H_{u,v}(p+1, q-1))\}. \] (1)

If \( p = q \) (\( p = q + 1 \), respectively), then \( H_{u,v}(p-1, q+1) \cong H_{u,v}(p+1, q-1) \) (\( H_{u,v}(p, q) \cong H_{u,v}(p-1, q+1) \), respectively) as \( \eta(u) = v \) for some automorphism \( \eta \) of \( G \), and thus from (1), we have \( \mu_\alpha(G) < \mu_\alpha(H_{u,v}(p+1, q-1)) \). Suppose that \( p \geq q + 2 \) and \( \mu_\alpha(G) < \mu_\alpha(H_{u,v}(p-1, q+1)) \).

If \( p \not\equiv q \) (mod 2), then by using (1) repeatedly, we have

\[ \mu_\alpha(G) \leq \mu_\alpha(H_{u,v}(\frac{p+q+3}{2}, \frac{p+q-3}{2})) \]
\[ < \mu_\alpha(H_{u,v}(\frac{p+q+1}{2}, \frac{p+q-1}{2})) \]
\[ < \mu_\alpha(H_{u,v}(\frac{p+q+3}{2}, \frac{p+q-3}{2})) , \]
which is impossible. If \( p \equiv q \) (mod 2), then by using (1) repeatedly, we have

\[ \mu_\alpha(G) \leq \mu_\alpha(H_{u,v}(\frac{p+q}{2} + 1, \frac{p+q}{2} - 1)) \]
\[ < \mu_\alpha(H_{u,v}(\frac{p+q}{2}, \frac{p+q}{2})) \]
\[ < \mu_\alpha(H_{u,v}(\frac{p+q}{2} - 1, \frac{p+q}{2} + 1)) , \]
which is also impossible. Therefore \( \mu_\alpha(H_{u,v}(p, q)) < \mu_\alpha(H_{u,v}(p+1, q-1)) \). \qed
5 Graphs with small distance $\alpha$-spectral radius

In this section, we will determine the graphs with minimum distance $\alpha$-spectral radius among trees and unicyclic graphs.

**Theorem 5.1.** Let $G$ be a tree of order $n \geq 4$. Then $\mu_\alpha(G) \geq \mu_\alpha(S_n)$ with equality if and only if $G \cong S_n$.

**Proof.** Let $G$ be a tree of order $n$ with minimum distance $\alpha$-spectral radius. Let $d$ be the diameter of $G$. Obviously, $d \geq 2$. Suppose that $d \geq 3$. Let $v_0v_1 \ldots v_d$ be a diametral path of $G$. Note that $v_1$ is a quasi-pendant vertex in $G$. By Theorem 4.1, $\mu_\alpha(Gv_1v_2) < \mu_\alpha(G)$, a contradiction. Thus $d = 2$, i.e., $G \cong S_n$.

In Theorem 5.1, the case $\alpha = 0$ has been known in [17] and the case $\alpha = \frac{1}{2}$ has been known in [12, 19].

For $n - 1 \geq 3$ and $1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor$, let $D_{n,a}$ be the tree obtained from vertex-disjoint $S_{a+1}$ with center $u$ and $S_{n-a-1}$ with center $v$ by adding an edge $uv$. Let $T$ be a tree of order $n$ with minimum distance $\alpha$-spectral radius, where $T \not\cong S_n$. Let $d$ be the diameter of $T$. Obviously, $d \geq 3$. Suppose that $d \geq 4$. Let $v_0v_1 \ldots v_d$ be a diametral path of $T$. Note that $v_1$ is a quasi-pendant vertex in $T$ and $T_{v_1v_2} \not\cong S_n$. By Theorem 4.1, $\mu_\alpha(T_{v_1v_2}) < \mu_\alpha(T)$, a contradiction. Thus $d = 3$, implying that $T \cong D_{n,a}$ for some $a$ with $1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor$.

**Lemma 5.1.** [19] Let $G$ be a unicyclic graph of order $n \geq 6$ different from $S_n^+$, where $S_n^+$ is the graph obtained from $S_n$ by adding an edge between two vertices of degree one. Then

$$
\sigma(G) \geq n^2 - n - 4 > \sigma(S_n^+) = n^2 - 2n.
$$

**Theorem 5.2.** Let $G$ be a unicyclic graph of order $n \geq 8$. Then $\mu_\alpha(G) \geq \mu_\alpha(S_n^+)$ with equality if and only if $G \cong S_n^+$.

**Proof.** Suppose that $G \not\cong S_n^+$. We only need to show that $\mu_\alpha(G) > \mu_\alpha(S_n^+)$. By Lemmas 2.4 and 5.1 we have

$$
\mu_\alpha(G) \geq \frac{2\sigma(G)}{n} \geq \frac{2(n^2 - n - 4)}{n}.
$$

As $\mu_\alpha(G)$ is bound above by the maximum row sum of $D_\alpha(G)$, and it is attained if and only if all row sums of $D_\alpha(G)$ are equal [14, p. 24, Theorem 1.1]. Thus

$$
\mu_\alpha(S_n^+) = T_{\text{max}}(S_n^+) = 2n - 3.
$$

Since $n \geq 8$, we have

$$
\mu_\alpha(G) \geq \frac{2(n^2 - n - 4)}{n} \geq 2n - 3 > \mu_\alpha(S_n^+),
$$

as desired. □

The result in Theorem 5.2 for $\alpha = 0, \frac{1}{2}$ has been known in [22, 19].
6 Graphs with large distance $\alpha$-spectral radius

In this section, we will determine the graphs with maximum distance $\alpha$-spectral radius among some classes of graphs. For examples, we determine the unique connected graphs of order $n \geq 4$ with maximum and second maximum distance $\alpha$-spectral radius, respectively in Theorem 6.2 and the unique graph with maximum distance $\alpha$-spectral radius among connected graphs with fixed clique number in Theorem 6.3.

For $2 \leq \Delta \leq n-1$, let $B_{n,\Delta}$ be a tree obtained by attaching $\Delta - 1$ pendant vertices to a terminal vertex of the path $P_{n-\Delta+1}$. In particular, $B_{n,2} = P_n$ and $B_{n,n-1} = S_n$. The following theorem for $\alpha = 0, \frac{1}{2}$ was given in [17, 12] for trees.

**Theorem 6.1.** Let $G$ be a connected graph of order $n \geq 5$ with maximum degree $\Delta$, where $2 \leq \Delta \leq n-1$. Then $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta})$ with equality if and only if $G \cong B_{n,\Delta}$.

**Proof.** Let $G$ be a graph with maximum distance $\alpha$-spectral radius among connected graphs of order $n$ with maximum degree $\Delta$. Obviously, $G$ has a spanning tree $T$ with maximum degree $\Delta$. By Lemma 2.3, $\mu_\alpha(G) \leq \mu_\alpha(T)$ with equality if and only if $G \cong T$. Thus $G$ is a tree.

It is trivial if $\Delta = 2, n-1$. Suppose that $3 \leq \Delta \leq n-2$. We only need to show that $G \cong B_{n,\Delta}$.

Let $u \in V(G)$ with $d_G(u) = \Delta$. Suppose that there exists a vertex different from $u$ with degree at least 3. Then we may choose such a vertex $w$ of degree at least 3 such that $d_G(u,w)$ is as large as possible. Obviously, there are two pendant paths, say $P$ and $Q$, at $w$ of lengths at least 1. Let $p$ and $q$ be the lengths of $P$ and $Q$, respectively. Assume that $p \geq q$. Let $H = G[V(G) \setminus ((V(P) \cup V(Q)) \setminus \{w\})]$. Then $G \cong H_w(p,q)$. Obviously, $G' = H_w(p+1,q-1)$ is a tree of order $n$ with maximum degree $\Delta$. By Corollary 4.1, $\mu_\alpha(G) < \mu_\alpha(G')$, a contradiction. Then $u$ is the unique vertex of $G$ with degree at least 3, and thus $G$ consists of $\Delta$ pendant paths, say $Q_1, \ldots, Q_\Delta$ at $u$. If two of them, say $Q_i$ and $Q_j$ with $i \neq j$ are of lengths at least 2, then $G \cong H'_u(r,s)$, where $H' = G[V(G) \setminus ((V(Q_i) \cup V(Q_j)) \setminus \{u\})]$, and $r$ and $s$ are the lengths of $Q_i$ and $Q_j$, respectively. Assume that $r \geq s$. Obviously, $G'' = H'_u(r+1,s-1)$ is a tree of order $n$ with maximum degree $\Delta$. By Corollary 4.1, $\mu_\alpha(G) < \mu_\alpha(G'')$, also a contradiction. Thus there is exactly one pendant path at $u$ of length at least 2, implying that $G \cong B_{n,\Delta}$. \hfill $\square$

If $G$ is a connected graph of order 1 or 2, then $G \cong P_n$. If $G$ is a connected graph of order 3, then $G \cong P_3, K_3$, and by Lemma 2.3, $\mu_\alpha(K_3) < \mu_\alpha(P_3)$.

Ruzieh and Powers [16] showed that $P_n$ is the unique connected graph of order $n$ with maximum distance 0-spectral radius, and it was proved in [18] that $B_{n,3}$ is the unique tree of order $n$ different from $P_n$ with maximum distance 0-spectral radius. For $\alpha = \frac{1}{2}$, the following theorem was given in [12].

**Theorem 6.2.** Let $G$ be a connected graph of order $n \geq 4$, where $G \not\cong P_n$. Then $\mu_\alpha(G) \leq \mu_\alpha(B_{n,3}) < \mu_\alpha(P_n)$ with equality if and only if $G \cong B_{n,3}$. 

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Proof. First suppose that \( G \) is a tree. If \( n = 4 \), then the result follows from Theorem \ref{thm:4.1}. Suppose that \( n \geq 5 \). Let \( \Delta \) be the maximum degree of \( G \). Since \( G \not\cong P_2 \), we have \( \Delta \geq 3 \). By Theorem \ref{thm:6.1} \( \mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta}) \) with equality if and only if \( G \cong B_{n,\Delta} \). By Corollary \ref{cor:4.1} \( \mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta}) \leq \mu_\alpha(B_{n,3}) < \mu_\alpha(P_n) \) with equalities if and only if \( \Delta = 3 \) and \( G \cong B_{n,\Delta} \), i.e., \( G \cong B_{n,3} \).

Now suppose that \( G \) is not a tree. Then \( G \) contains at least one cycle. If there is a spanning tree \( T \) with \( T \not\cong P_n \), then by Lemma \ref{lem:2.3} and the above argument, we have \( \mu_\alpha(G) < \mu_\alpha(T) \leq \mu_\alpha(B_{n,3}) \). If any spanning tree of \( G \) is a path, then \( G \) is a cycle \( C_n \). Now we only need to show that \( \mu_\alpha(C_n) < \mu_\alpha(B_{n,3}) \).

Let \( C_n = u_1u_2 \ldots u_nu_1 \) and \( T' = C_n - u_1u_2 - u_2u_3 + u_2u_n \). Obviously, \( T' \cong B_{n,3} \). Let \( x \) be the distance \( \alpha \)-Perron vector of \( C_n \). By Lemma \ref{lem:2.3} we have \( x_{u_1} = \cdots = x_{u_n} \). As we pass from \( C_n \) to \( T' \), the distance between \( u_2 \) and \( u_1 \) is increased by 1, the distance between \( u_2 \) and \( u_i \) with \( 3 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil \) is increased by \( n - 2i + 3 \), the distance between \( u_2 \) and \( u_i \) with \( \left\lceil \frac{n+1}{2} \right\rceil \) \( \leq i \leq n \) is decreased by 1, and the distances between all other vertex pairs are increased or remain unchanged. Thus

\[
\begin{align*}
\mu_\alpha(T') - \mu_\alpha(C_n) &= x^\top(D_\alpha(T') - D_\alpha(G))x \\
&\geq \alpha \left( x_{u_2}^2 + x_{u_1}^2 \right) + 2(1 - \alpha)x_{u_2}x_{u_1} - \sum_{i = \left\lceil \frac{n+1}{2} \right\rceil + 2}^n \left( \alpha \left( x_{u_2}^2 + x_{u_i}^2 \right) + 2(1 - \alpha)x_{u_2}x_{u_i} \right) \\
&\quad + \sum_{i = 3}^{\left\lceil \frac{n+1}{2} \right\rceil} (n - 2i + 3) \left( \alpha \left( x_{u_2}^2 + x_{u_i}^2 \right) + 2(1 - \alpha)x_{u_2}x_{u_i} \right) \\
&= 2x_{u_1}^2 \left( 1 - \left( n - \left\lceil \frac{n+1}{2} \right\rceil \right) - 1 \right) + \sum_{i = 3}^{\left\lceil \frac{n+1}{2} \right\rceil} (n - 2i + 3) \\
&= 2x_{u_1}^2 \left( 1 + \left( n - 1 - \left\lceil \frac{n+1}{2} \right\rceil \right) \left( \left\lceil \frac{n+1}{2} \right\rceil - 2 \right) \right) \\
&\geq 2x_{u_1}^2 > 0,
\end{align*}
\]

and therefore \( \mu_\alpha(C_n) < \mu_\alpha(B_{n,3}) \), as desired. \( \square \)

A clique of \( G \) is a subset of vertices whose induced subgraph is a complete graph, and the clique number of \( G \) is the maximum number of vertices in a clique of \( G \). For \( 2 \leq \omega \leq n \), let \( K_{\infty,\omega} \) be the graph obtained from a complete graph \( K_\omega \) and a path \( P_{n-\omega} \) by adding an edge between a vertex of \( K_\omega \) and a terminal vertex of \( P_{n-\omega} \) if \( \omega < n \) and let \( K_{\infty,\omega} = K_n \) if \( \omega = n \). In particular, \( K_{\infty,2} \cong P_n \) for \( n \geq 2 \). The following result for \( \alpha = 0, \frac{1}{2} \) was given in \cite{15, 11}.

**Theorem 6.3.** Let \( G \) be a connected graph of order \( n \geq 2 \) with clique number \( \omega \geq 2 \). Then \( \mu_\alpha(G) \leq \mu_\alpha(K_{\infty,\omega}) \) with equality if and only if \( G \cong K_{\infty,\omega} \).

Proof. It is trivial if \( \omega = n \) and it follows from Theorem \ref{thm:6.2} if \( \omega = 2 \). Suppose that \( 3 \leq \omega \leq n - 1 \). Let \( G \) be a graph with maximum distance \( \alpha \)-spectral
radius among connected graphs of order $n$ with clique number $\omega$. We only need
to show that $G \cong Ki_{n,\omega}$.

Let $S = \{v_1, \ldots, v_{\omega}\}$ be a clique of $G$. By Lemma 6.3 $G - E(G[S])$ is a
tree. Let $T_i$ be the component of $G - E(G[S])$ containing $v_i$, where $1 \leq i \leq \omega$.
For $1 \leq i \leq \omega$, by Corollary 4.1 if $T_i$ is nontrivial, then $T_i$ is a pendant path at
$v_i$. Note that any two distinct vertices in $G[S]$ are adjacent. By Corollary 4.2 there is only one nontrivial $T_i$, and thus $G \cong Ki_{n,\omega}$. \hfill $\square$

Recall that $Ki_{n,3}$ is the unique unicyclic graph of order $n \geq 3$ with maximum
distance 0-spectral radius [22], and the unique odd-cycle unicyclic graph of order
$n \geq 3$ with maximum distance 1-spectral radius [12].

**Theorem 6.4.** Let $G$ be a unicyclic odd-cycle graph of order $n \geq 3$. Then
$\mu_\alpha(G) \leq \mu(Ki_{n,3})$ with equality if and only if $G \cong Ki_{n,3}$.

**Proof.** If $n = 3, 4$, the result is trivial. Suppose that $n \geq 5$. Let $G$ be a graph
with maximum distance $\alpha$-spectral radius among unicyclic odd-cycle graphs of
order $n$. We only need to show that $G \cong Ki_{n,3}$.

Let $C = v_1 \ldots v_{2k+1}v_1$ be the unique cycle of $G$, where $k \geq 1$. Let $T_i$ be
the component of $G - E(C)$ containing $v_i$ for $1 \leq i \leq 2k + 1$. Let $U_1 = V(T_{2k}) \cup V(T_{2k+1}), U_2 = \cup_{k+1 \leq i \leq 2k-1} V(T_i)$ and $U_3 = \cup_{1 \leq i \leq k-1} V(T_i)$. Let $x$ be the distance $\alpha$-Perron vector of $G$. Let

$$\Gamma = \sum_{u \in U_1} \sum_{v \in U_3} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v) - \sum_{u \in U_1} \sum_{v \in U_2} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v).$$

Suppose that $k \geq 2$. Let $G' = G - v_1v_{2k+1} + v_{2k+1}v_{2k-1}$. Note that the length
of $C$ is odd. As we pass from $G$ to $G'$, the distance between a vertex in $S_1$ and
a vertex in $S_3$ is increased by at least 1, the distance between $S_2$ and $V(T_{2k+1})$
is decreased by 1, and the distance between all other vertex pairs are increased or
remain unchanged. Thus

$$\mu_\alpha(G') - \mu_\alpha(G) \geq x^\top (D_\alpha(G') - D_\alpha(G))x$$

$$\geq \sum_{u \in U_1} \sum_{v \in U_3} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v) - \sum_{u \in V(T_{2k+1})} \sum_{v \in U_2} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v)$$

$$> \sum_{u \in U_1} \sum_{v \in U_3} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v) - \sum_{u \in U_1} \sum_{v \in U_2} (\alpha (x_u^2 + x_v^2) + 2(1 - \alpha)x_u x_v).$$

If $\Gamma \geq 0$, then $\mu_\alpha(G') > \mu_\alpha(G)$, a contradiction. Thus $\Gamma < 0$. Let $G'' = G - v_kv_{2k-1} + v_{2k}v_1$. As we pass from $G$ to $G''$, the distance between a vertex in $S_1$ and a vertex in $U_2$ is increased by at least 1, the distance between $U_3$ and
$V(T_{2k})$ is decreased by 1, and the distance between all other vertex pairs are
increased or remain unchanged. As above, we have

$$\mu_\alpha(G'') - \mu_\alpha(G) \geq x^\top (D_\alpha(G'') - D_\alpha(G))x$$

$$> 0,$$
for the distance $\alpha$

study the distance $\alpha$

Some spectral properties of $G$.

Lemma 7.1. Let $A$, $B$ be $n \times n$ Hermitian matrices. Then

\[
\lambda_j(A + B) \leq \lambda_i(A) + \lambda_{j-i+1}(B) \quad \text{for} \quad 1 \leq i \leq j \leq n,
\]

and

\[
\lambda_j(A + B) \geq \lambda_i(A) + \lambda_{j-i+n}(B) \quad \text{for} \quad 1 \leq j \leq i \leq n.
\]

As in the recent work of Atik and Panigrahi [3], we have

Theorem 7.1. Let $G$ be a connected graph and $\lambda$ be any eigenvalue of $D_\alpha(G)$ other than the distance $\alpha$-spectral radius. Then

\[
2\alpha T_{\text{min}}(G) - T_{\text{max}}(G) + (1 - \alpha)(n - 2) \leq \lambda \leq T_{\text{max}}(G) - (1 - \alpha)n.
\]

Proof. Let $D_\alpha(G) = A + B$, where $A = (\alpha T_{\text{min}}(G) - (1 - \alpha))I_n + (1 - \alpha)J_{n \times n}$. Then $B$ is a nonnegative symmetric matrix with maximum row sum $T_{\text{max}}(G) - \alpha T_{\text{min}}(G) - (1 - \alpha)(n - 1)$. Thus $|\lambda_n(B)| \leq \lambda_1(B) \leq T_{\text{max}}(G) - \alpha T_{\text{min}}(G) - (1 - \alpha)(n - 1)$. Thus $\mu_\alpha(G') > \mu_\alpha(G)$, also a contradiction. It follows that $k = 1$, i.e., the unique cycle of $G$ is of length 3.

Obviously, $T_i$ is a tree for $1 \leq i \leq 3$. For $1 \leq i \leq 3$, by Corollary 4.1 if $T_i$ is nontrivial, then it is a path with a terminal vertex $v_i$. Then by Corollary 4.2, only one $T_i$ is nontrivial. Thus $G \cong K_{i_n, 3}$. \qed

7 Remarks

Some spectral properties of $D_\alpha(G)$ have been established in [5]. In this paper, we study the distance $\alpha$-spectral radius of a connected graph. We consider bounds for the distance $\alpha$-spectral radius, local transformations to change the distance $\alpha$-spectral radius, and the characterizations for graphs with minimum and/or maximum distance $\alpha$-spectral radius in some classes of connected graphs. Lots of results in the literature are generalized and/or improved.

Besides the distance $\alpha$-spectral radius, we may concern other eigenvalues of $D_\alpha(G)$ for a connected graph $G$. We give examples.

For an $n \times n$ Hermitian matrix $A$, let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues, arranged in a non-increasing order.

Lemma 7.1. [6] Let $A$, $B$ be $n \times n$ Hermitian matrices. Then

\[
\lambda_j(A + B) \leq \lambda_i(A) + \lambda_{j-i+1}(B) \quad \text{for} \quad 1 \leq i \leq j \leq n,
\]

and

\[
\lambda_j(A + B) \geq \lambda_i(A) + \lambda_{j-i+n}(B) \quad \text{for} \quad 1 \leq j \leq i \leq n.
\]
For matrix $A$, we have $\lambda_1(A) = \alpha T_{\min}(G) + (1 - \alpha)(n - 1)$ and $\lambda_j(A) = \alpha T_{\min}(G) - 1 + \alpha$ for $j = 2, \ldots, n$. For $j = 2, \ldots, n$, we have by Lemma 7.1 that

$$
\lambda_j(D_\alpha(G)) \leq \lambda_1(B) + \lambda_j(A) \\
\leq T_{\max}(G) - \alpha T_{\min}(G) - (1 - \alpha)(n - 1) + \alpha T_{\min}(G) - 1 + \alpha \\
= T_{\max}(G) - (1 - \alpha)n.
$$

Similarly, for $j = 2, \ldots, n$,

$$
\lambda_j(D_\alpha(G)) \geq \lambda_n(B) + \lambda_j(A) \\
\geq -T_{\max}(G) + \alpha T_{\min}(G) + (1 - \alpha)(n - 1) + \alpha T_{\min}(G) - 1 + \alpha \\
= 2\alpha T_{\min}(G) - T_{\max}(G) + (1 - \alpha)(n - 2).
$$

This completes the proof.

Let $G$ be a connected graph and $\lambda$ be any eigenvalue of $D_\alpha(G)$ other than the distance $\alpha$-spectral radius. By previous theorem, we have

$$
|\lambda| \leq T_{\max}(G) - (1 - \alpha)(n - 2).
$$

The distance $\alpha$-energy of a connected graph $G$ of order $n$ is defined as

$$
E_\alpha(G) = \sum_{i=1}^{n} \left| \mu_{\alpha}^{(i)}(G) - \frac{2\alpha \sigma(G)}{n} \right|.
$$

Obviously, $E_0(G)$ is the distance energy of $G$ [10, 24], while

$$
E_{1/2}(G) = \frac{1}{2} \sum_{i=1}^{n} \left| 2\mu_{1/2}^{(i)}(G) - \frac{2\sigma(G)}{n} \right|
$$

is half of the distance signless Laplacian energy of $G$ [8]. Thus, it is possible to study the distance energy and the distance signless Laplacian energy in a unified way.

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