ON THE BEHAVIOR OF STRINGY MOTIVES UNDER GALOIS QUASI-ÉTALE COVERS

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Abstract. We investigate the behavior of stringy motives under Galois quasi-étale covers. We prove them to descend under such covers in a sense defined via their Poincaré realizations. Further, we show such descent to be strict in the presence of ramification. As a corollary, we reduce the problem regarding the finiteness of the étale fundamental group of KLT singularities to a DCC property for their stringy motives. We verify such DCC property for surfaces in arbitrary characteristic. As an application, we give a characteristic-free proof for the finiteness of the étale fundamental group of log terminal surface singularities, which was unknown in equal characteristics 2 and 3 and in mixed characteristics.

1. Introduction

Kawamata log terminal singularities (KLT for short) are arguably the most important class of singularities in algebraic geometry. For instance, they are the gold-standard for mild log canonical singularities and are the singularities required to run the Minimal Model Program. Hence, there has been a great effort in the last decades to understand how mild these singularities actually are. In general, much is known over fields of characteristic zero but the situation over positive characteristic fields—let alone mixed characteristics—is rather thorny. A typical example of this is their rationality. We know KLT singularities are rational in characteristic zero thanks to vanishing theorems such as Kodaira vanishing and their generalizations. However, they are not rational in general over positive characteristic fields and it is yet to be determined when exactly they are rational. Another but related problem has to do with the purity of the branch locus over KLT singularities, or; to be more precise, with the finiteness of their local étale fundamental groups. In this note, we study this problem by means of stringy motives; which is an invariant of KLT singularities amalgamating their log discrepancies with (generalized) Euler characteristics. Let us recall next what the problem inspiring this work is.

Let \((X, \Delta)\) be a log pair over an algebraically closed field \(k\) of characteristic \(p \geq 0\) and \(K_X\) be a canonical divisor on \(X\). That is, \(X\) is a normal \(k\)-variety and \(\Delta\) is a \(\mathbb{Q}\)-divisor on \(X\) with coefficients in \([0, 1]\) such that \(K_{(X, \Delta)} := K_X + \Delta\) is a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor. We may also refer to \(\Delta\) as a boundary on \(X\). Let \(r_{(X, \Delta)}\) denote the index of \((X, \Delta)\), i.e., the minimal positive integer \(r\) such that \(rK_{(X, \Delta)}\) is Cartier. A morphism \(g: (X', \Delta') \to (X, \Delta)\) between log pairs is said to be a Galois quasi-étale log-cover if \(g\) is a finite dominant morphism such that \(K_{X'} = g^*K_X\), \(\Delta' = g^*\Delta\), and the extension of function fields \(\mathcal{H}(X')/\mathcal{H}(X)\) is Galois. In such case, \(g\) is crepant and \(r_{(X', \Delta')}\) divides \(r_{(X, \Delta)}\).

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This work is concerned with the following fundamental problem.

**Question 1.1.** Assume that $(X, \Delta)$ is a Kawamata log terminal log pair and consider a tower of Galois quasi-étale log-covers

$$(1.1.1) \quad (X, \Delta) = (X_0, \Delta_0) \leftarrow f_0 (X_1, \Delta_1) \leftarrow f_1 (X_2, \Delta_2) \leftarrow f_2 \cdots$$

Does there exist $N \in \mathbb{N}$ such that $f_i$ is étale for all $i \geq N$?

**Remark 1.2.** In Question 1.1, the compositions $g_n := f_n \circ \cdots \circ f_0$ are the ones being required to be Galois quasi-étale log-covers (not the $f_i$’s), which then implies that the maps $f_i$ are Galois quasi-étale. Further, the Galois hypothesis is essential; see [GKP16, Proposition 11.4]. Additionally, if $(X, \Delta)$ is KLT then so is $(X_i, \Delta_i)$ for all $i$; see [Kol13, Corollary 2.43].

**Question 1.1** has attracted great attention in recent years; see [GKP16, Xu14, BCRG+19]. It is well-known that Question 1.1 is intimately related to the problem of determining the finiteness of the (regional) local étale fundamental group of KLT singularities; see [St17]. This “twin” problem has been considered in the works [Xu14, CRST18, CSK20, Bra21, BFMS22, CLM+22], cf. [Kaw84, Corollary 1.9] which settles this for surfaces in characteristic zero. Despite of these efforts, Question 1.1 remains open in positive characteristics. Nonetheless, we know its answer to be affirmative for $F$-regular pairs (in full generality) as well as in dimensions $\leq 3$ but characteristics $\geq 7$; see [BCRG+19, CSK20]. In dimension 2, we may add $p = 5$ to the known cases by putting together [Kaw99, Art77]; see Remark 5.5.

It is worth noting that, in dimensions $\leq 3$, Question 1.1 can be answered in the tame case due to recent advances in the minimal model program; see [HW19b, Proposition 5.2], cf. [XZ19, Theorem 3.4]. Note also that the local étale fundamental group of an $F$-regular singularity as well as the one of a KLT singularity in dimension 3 and characteristic $\geq 7$ is not only finite but tame as shown in [CRST13, CSK20]. On the other hand, even in dimension two, there are rational double points (hence KLT singularities) having no non-trivial tame quasi-étale cover, equivalently, whose local étale fundamental group has no non-trivial tame quotient [Art77, p.15]. In general, the wild aspects in low characteristics make these problems much more difficult. For instance, Kawamata’s method using index-1 covers does not apply to characteristics $\leq 3$ in dimension 2 as log terminal singularities are not preserved by index-1 covers. Thus, a new approach is needed in this case which we provide here.

In this work, we propose a novel strategy to attack Question 1.1 via *stringy motives*; at least in dimensions $\leq 3$ where resolution of singularities and $(W\mathcal{O})$-rationality of KLT singularities are available. Loosely speaking, stringy motives are a hybrid between log discrepancies and Euler characteristics and are well-suited to the study of wild quotients.

We sketch our approach next. Following [Yas14, §6], we consider the stringy motive

$$M_{st}(X, \Delta) := \int_{j_{\infty}X} \mathbb{L}^{F(X, \Delta)} d\mu_X \in \hat{M}_{k, r(X, \Delta)}$$

of a log pair $(X, \Delta)$ which, by definition, is convergent (i.e., exists) if and only if $(X, \Delta)$ is *stringily KLT*. Such pairs form a subclass of KLT log pairs. Conversely, a KLT log pair is stringily KLT if it has a log resolution. Hence, these two notions agree in characteristic zero and in dimensions $\leq 3$ in positive characteristics.

Then, our goal is to examine the behavior of stringy motives under Galois quasi-étale log-covers $g: (X', \Delta') \rightarrow (X, \Delta)$. Let $G := \text{Gal}(\mathcal{H}(X')/\mathcal{H}(X))$ be the corresponding Galois
group. To do so, we are going to require a notion of ordering among stringy motives, for which we use the Poincaré realization, which is a ring homomorphism:

\[ P: \mathcal{M}_{k,r} \to \mathbb{Z}[T^{-1/r}] \]

where \( r := r(X, \Delta_X) \) and \( \mathbb{Z}[T^{-1/r}] \) is the ring of integral Laurent series on \( T^{-1/r} \). Thus, to a stringy motive we may associate its Poincaré realization \( P_{st}(X, \Delta_X) := P(M_{st}(X, \Delta_X); T) \in \mathbb{Z}[T^{-1/r}] \). For instance, in the surface case, any stringy motive we consider here will belong to the subring \( \mathbb{Z}[\mathbb{L}^{-1/r}] \subset \mathcal{M}_{k,r} \) on which the Poincaré realization map is given by \( \mathbb{L} \mapsto T^2 \) (where \( \mathbb{L} := \{ \mathbb{A}^1 \} \)).

We obtain an ordering on \( \mathbb{Z}[T^{-1/r}] \) as follows. For \( 0 \neq f \in \mathbb{Z}[T^{-1/r}] \), \( f > 0 \) if and only if \( f = \sum_{i-n}^\infty a_iT^{-i/r} \) we have \( a_{-n} > 0 \). This is none other than the lexicographic ordering. We lift this ordering to the ring \( \mathcal{M}_{k,r} \) via the Poincaré realization. For instance, \( M_{st}(X, \Delta_X) > 0 \) means \( P_{st}(X, \Delta_X) > 0 \). In general, we shall say that an element of \( \mathcal{M}_{k,r} \) (e.g. motivic measures and integrals) is positive if so is its Poincaré realization.

Since \( (Y, \Delta_Y) \) admits a log \( G \)-action (with log quotient \( (X, \Delta_X) \)) so does \( M_{st}(Y, \Delta_Y) \). Then, we may form the quotient \( M_{st}(Y, \Delta_Y)/G \); see Section 2.2. Our first observation is the equality:

\[ M_{st}(Y, \Delta_Y)/G = \int_{g_{\infty}(J_{\infty}Y)} L_{F_X, \Delta_X} d\mu_X, \]

where \( g_{\infty}: J_{\infty}Y \to J_{\infty}X \) is the induced morphism on the spaces of arcs; see Corollary 2.10. Consequently, \( M_{st}(X, \Delta_X) \) and \( M_{st}(Y, \Delta_Y)/G \) can be compared as follows:

\[ M_{st}(X, \Delta_X) = \int_{J_{\infty}X} L_{F_X, \Delta_X} d\mu_X = \int_{g_{\infty}(J_{\infty}Y)} L_{F_X, \Delta_X} d\mu_X + \int_{J_{\infty}X \setminus g_{\infty}(J_{\infty}Y)} L_{F_X, \Delta_X} d\mu_X \]

\[ = M_{st}(Y, \Delta_Y)/G + \int_{J_{\infty}X \setminus g_{\infty}(J_{\infty}Y)} L_{F_X, \Delta_X} d\mu_X. \]

Our main result is then the following.

**Main Theorem** [Theorem 3.12]. With notation as above, assume that \( (X, \Delta_X) \) is KLT pair of dimension \( \leq 3 \) and that \( g \) is not étale. Further, assume \( G \) to be a \( p \)-group if \( d = 3 \) and \( 0 < p \leq 5 \). Then, the motivic integral \( \int_{J_{\infty}X \setminus g_{\infty}(J_{\infty}Y)} L_{F_X, \Delta_X} d\mu_X \) is positive and so:

\[ M_{st}(X, \Delta_X) > M_{st}(Y, \Delta_Y)/G. \]

In particular, if there were a non-affirmative answer to Question 1.1 in any of the following cases:

(a) \( \dim X = 2 \) and \( p \geq 0 \),
(b) \( \dim X = 3 \) and \( p \notin \{2, 3, 5\} \),
(c) \( \dim X = 3, p \in \{2, 3, 5\} \), and \( \text{Gal}(X_i/X) \) is a \( p \)-group for all \( i \geq 0 \),

then, the corresponding tower \((1.1.1)\) would yield a strictly descending chain:

\[ M_{st}(X_0, \Delta_0)/G_0 > M_{st}(X_1, \Delta_1)/G_1 > M_{st}(X_2, \Delta_2)/G_2 > \cdots \]

where \( G_i := \text{Gal} \left( K(X_i)/(X_0) \right) \). Hence, we reach a contradiction if we can prove a descending chain condition (DCC) for stringy motives in those cases. We are able to achieve this in case (a); see Proposition 4.6, but leave it open in the remaining two cases. Noteworthy, a DCC property for another version of stringy invariants was discussed by Takahashi; see [Tak11].
As an application, we give a characteristic-free proof for the finiteness of the local étale fundamental group of KLT surface singularities; see Theorem 5.3. Of course, if the (purely wild) DCC condition holds for threefolds, the same would hold for KLT threefold singularities.

In proving our main theorem, the main technical step is showing that \( J_\infty X \setminus g_\infty(J_\infty Y) \) has positive measure if \( g \) is not étale. To this end, we have improved upon results of Kato and Kerz–Schmidt; see [Kat18, Lemma 3.5], [KS10, Lemma 2.4]. Our argument was inspired by those of Nakamura–Shibata in [NS22]. We let \( D \) denote the formal disk \( \text{Spec} \, k[[t]] \) and \( \delta, \eta \in D \) denote its closed and generic points; respectively.

**Theorem A** (Corollary 3.7). Let \( G \) be a finite group and \( H \subset G \) be a subgroup. Let \( g: Y \to X \) be a \( G \)-cover between normal varieties so that \( Y \) is smooth and there is \( y \in Y(\bar{k}) \) being fixed by \( H \). Let \( E \to D \) be \( G \)-cover such that \( H \) preserves some connected component \( E_0 \) of \( E \). Let \( \mathcal{N} \subset J_\infty X \) be the subset of arcs \( \gamma: D \to X \) such that: \( \delta \mapsto g(y) \), \( g \) is étale over \( \gamma(\eta) \), and the pullback of \( \gamma \) along \( g \) induces \( E \to D \). Then, \( \mathcal{N} \) has positive measure.

We have discussed so far only the equal characteristic case, where we have a suitable theory of motivic integration. Motivic integration makes the proof of our Main Theorem rather formal. Unfortunately, such a theory is not available yet in mixed characteristics. Nevertheless, at least for surfaces, we may define the required stringy motives directly via minimal resolutions and we may rely on the strong factorization theorem. In Section 6, we prove our Main Theorem in this case as well without relying on motivic integration. See Theorem 6.3. This let us prove the finiteness of the étale fundamental group of log terminal surface singularities in all equal/mixed characteristics.

**Theorem B** (Corollary 6.4). Let \( (R, m, k) \) be a log terminal 2-dimensional complete local ring with algebraically closed residue field \( k \). Then, \( \pi_1^{\text{ét}}(\text{Spec} \, R \setminus \{m\}) \) is finite.

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**Convention 1.3.** We fix an algebraically closed field \( k \) of characteristic \( p \geq 0 \). Except for Section 6, every relative notion/object is defined over \( k \) unless otherwise explicitly stated. Also, \( 0 \in \mathbb{N} \).

2. Stringy Motives

For the reader’s convenience, we recall how to define the stringy motive \( M_{\text{st}}(X, \Delta) \) of a KLT log pair \( (X, \Delta) \) as well as its main properties. We follow [Yas19], where the reader can find full details.
2.1. Motivic integration. We briefly review the theory of motivic integration over Deligne–Mumford stacks and in the equivariant setting. Motivic measures and integrals take values in a version of the Grothendieck ring of varieties. In this paper, we choose the one $\hat{M}_{k,r}$ used in [Yas19]. The class of a variety $X$ in this ring is denoted by $\{X\}$ (square brackets $[\cdot]$ are reserved to express quotient stacks). More generally, we can define the class $\{C\}$ of a constructible subset $C$ of a variety via its partition into locally closed subsets: if $C = \bigsqcup_{i=1}^n C_i$ with $C_i$ locally closed, then $\{C\} := \sum_{i=1}^n \{C_i\}$. As usual, $\mathbb{L} := \mathbb{A}^1$. The subscript $r \in \mathbb{N}$ of $\hat{M}_{k,r}$ means that the ring is adjoined with the fractional power $\mathbb{L}^{1/r}$ of $\mathbb{L}$. We often take a sufficiently divisible $r$ so that every rational number showing up sits in $\frac{1}{r}\mathbb{Z}$.

We work with the Poincaré realization map $\hat{M}_{k,r} \rightarrow \mathbb{Z}[T^{-1/r}]$, which sends $\{X\}$ to the Poincaré polynomial $P(X) = P(X; T)$ of $X$; see [Nic11] §8. For example, $P(\mathbb{L}) = T^2$. We define an order $>\!\!> \text{ on } \mathbb{Z}[T^{-1/r}]$ by comparing coefficients lexicographically. That is, for two distinct elements $f, g \in \mathbb{Z}[T^{-1/r}]$, $f > g$ (resp. $f < g$) if and only if the leading coefficient (i.e., the coefficient of the highest degree term) of $f - g$ is positive (resp. negative). As usual, the symbol $\leq$ means either $<$ or $=$ (and likewise for $\geq$). Note that Poincaré polynomials $P(X)$ of varieties are positive (i.e., $>\!\!> 0$ with respect to this order) unless $X = \emptyset$. In fact, after applying $P$, all the motivic measures and integrals that we consider become convergent countable sums of the form $\sum_i P(X_i) T^{e_i}$ with $e_i \in \frac{1}{r}\mathbb{Z}$, which are $\geq 0$. In general, we shall say that $\alpha \in \hat{M}_{k,r}$ is positive (i.e., $>\!\!> 0$) if $P(\alpha) > 0$.

We denote the formal disk by $\mathbb{D} := \text{Spec } \mathbb{K}[[t]]$. We let $\delta, \eta \in \mathbb{D}$ denote its closed and generic points, respectively. The punctured formal disk is $\mathbb{D}^* := \text{Spec } \mathbb{K}[t] = \mathbb{D} \setminus \{\delta\}$. An arc of a variety $X$ is a morphism $\mathbb{D} \rightarrow X$. Given $n \in \mathbb{N}$, an $n$-jet of $X$ is a morphism $\text{Spec } \mathbb{K}[[t]]/(t^{n+1}) \rightarrow X$. The $n$-jet scheme $J_nX$ of $X$ is the moduli scheme of $n$-jets of $X$, which is a separated scheme of finite type. For $n' \geq n$, there is a natural map $J_nX \rightarrow J_{n'}X$. The arc space $J_\infty X$ is the moduli space of arcs of $X$ and is identified with the projective limit $\lim_{n \rightarrow \infty} J_nX$. For each $n \in \mathbb{N}$, we have a truncation map $\pi_n : J_\infty X \rightarrow J_n X$. A subset $C \subset J_\infty X$ is said to be a stable if there is an $n > 0$ such that: $\pi_n(C) \subset J_n X$ is constructible, $C = \pi_n^{-1}\{\pi_n(C)\}$, and $\pi_{m+1}(C) \rightarrow \pi_m(C)$ is a piecewise trivial $\hat{M}_{k,\dim X}$-bundle for all $m \geq n$. The arc space is equipped with the motivic measure $\mu_X$; for a stable subset $C \subset J_\infty X$, we write

\begin{equation}
\mu_X(C) := \{\pi_n(C)\} \mathbb{L}^{-n \dim X} \in \hat{M}_{k,r}, \quad n > 0.
\end{equation}

Thus, we may say that a measurable subset $C \subset J_\infty X$ has positive measure if $P(\mu_X(C)) > 0$.

If $X$ is endowed with an action of a finite group $G$ and if $C$ is a $G$-invariant stable subset, the same formula as in (2.0.1) gives an element of the $G$-equivariant version $\hat{M}_{k,r}^G$ of $\hat{M}_{k,r}$; we denote this by $\mu_X(C)^G$ too. The ring $G - \hat{M}_{k,r}$ is constructed from the Grothendieck ring of $G$-varieties (varieties given with a $G$-action) by the same procedure as the construction of $\hat{M}_{k,r}$ from the Grothendieck ring of varieties. Via the forgetting map $G - \hat{M}_{k,r} \rightarrow \hat{M}_{k,r}$, the equivariant version of $\mu_X(C)$ maps to its non-equivariant version. However, there is a quotient map $G - \hat{M}_{k,r}^G \rightarrow \hat{M}_{k,r}^G$, which extends $\{X\} \mapsto \{X/G\}$ (e.g. for $X$ quasi-projective). The image of $\alpha \in G - \hat{M}_{k,r}^G$ under this map is denoted by $\alpha/G$. This let us define $\mu_X(C)/G$.

Let $G$ be a finite abstract group. A $G$-cover of $\mathbb{D}$ is a morphism $E \rightarrow \mathbb{D}$ together with a $G$-action on $E$ such that: $E$ is regular, $E \rightarrow \mathbb{D}$ is flat of rank $\#G$, and $E^* := \mathbb{D}^* \times_{\mathbb{D}} E \rightarrow \mathbb{D}^*$ is a $G$-torsor. Equivalently, a $G$-cover $E \rightarrow \mathbb{D}$ is the same as the normalization of $\mathbb{D}$ along a
$G$-torsor $E^* \to \mathbb{D}^*$. A twisted formal disk is the quotient stack $\mathcal{E} = [E/G]$ associated to a $G$-cover $E \to \mathbb{D}$, which is equipped with a morphism $\mathcal{E} \to \mathbb{D}$. Note that isomorphism classes of twisted formal disks are in one-to-one correspondence with isomorphism classes of Galois extensions of $\mathcal{E}[t]$ and that these are parametrized by an infinite dimensional space. See [Yas14, §2.1], [Yas19, §5].

Let $\mathcal{X}$ be a separated, irreducible, and reduced Deligne–Mumford stack of finite type. A twisted arc of $\mathcal{X}$ is a representable morphism $\mathcal{E} \to \mathcal{X}$ from a twisted formal disk $\mathcal{E}$. The moduli stack of twisted arcs of $\mathcal{X}$ is denoted by $\mathcal{J}_\infty \mathcal{X}$. For a twisted formal disk $\mathcal{E}$, we denote by $\mathcal{J}_\mathcal{E} \mathcal{X}$ the locus of twisted arcs $\mathcal{E} \to \mathcal{X}$, so $\mathcal{J}_\infty \mathcal{X} = \bigsqcup_{\mathcal{E}} \mathcal{J}_\mathcal{E} \mathcal{X}$. An untwisted arc of $\mathcal{X}$ is a (necessarily representable) morphism $\mathbb{D} \to \mathcal{X}$. Since $\mathbb{D}$ is a special case of twisted formal disks, untwisted arcs are twisted arcs. The moduli stack of untwisted arcs is denoted by $\mathcal{J}_\infty \mathcal{X}$; which is the substack $\mathcal{J}_{\infty, \mathcal{E}} \mathcal{X}$ of $\mathcal{J}_\infty \mathcal{X}$.

To define the motivic measure $\mu_\mathcal{X}$ on $\mathcal{J}_\infty \mathcal{X}$, we need first to define the class $\{ \mathcal{Y} \} \in \mathcal{M}_{k, \mathcal{X}}$ of a Deligne–Mumford stack $\mathcal{Y}$ of finite type. If $\mathcal{Y}$ is an algebraic space, it admits a partition $\mathcal{Y} = \bigsqcup_{i=1}^n \mathcal{Y}_i$ into schemes of finite type and we define $\{ \mathcal{Y} \} := \sum_{i=1}^n \{ \mathcal{Y}_i \}$. For a Deligne–Mumford stack $\mathcal{Y}$, we define $\{ \mathcal{Y} \}$ to be the class of its coarse moduli space. We can also define the class $\{ C \}$ of a constructible subset $C \subset \mathcal{Y}$ via its partition into locally closed subsets. If $C \subset \mathcal{J}_\infty \mathcal{X}$ is a stable subset and if $\mathcal{J}_n \mathcal{X}$ is a suitably defined stack of twisted $n$-jets with the truncation map $\pi_n : \mathcal{J}_\infty \mathcal{X} \to \mathcal{J}_n \mathcal{X}$, we define

$$\mu_\mathcal{X}(C) := \{ \pi_n(C) \}_L^{-n \dim \mathcal{X}}, \quad n \gg 0,$$

by mimicking (2.0.1).

Remark 2.1. The above definition of $\mu_\mathcal{X}$ is implicit in [Yas19], which deals with schemes/stacks over $\text{Spf} \mathcal{E}[[t]]$. To pass to this setting, we just base change from $\mathcal{E}$ to $\text{Spf} \mathcal{E}[[t]]$. Then, $\mu_\mathcal{X}$ was defined to be the motivic measure of the untwisted arc space $J_\infty(Utg_{\mathcal{X}})^{\text{pur}}$ of the untwisting stack $Utg_{\mathcal{X}}$. In turn, the motivic measure of $J_\infty(Utg_{\mathcal{X}})^{\text{pur}}$ was defined in terms of truncation maps $J_\infty(Utg_{\mathcal{X}})^{\text{pur}} \to J_n(Utg_{\mathcal{X}})^{\text{pur}}$. However, since $\mathcal{J}_n \mathcal{X} = J_n(Utg_{\mathcal{X}})^{\text{pur}}$ by definition, we may define the motivic measure on $\mathcal{J}_\infty \mathcal{X}$ in the usual way in terms of truncation maps $\mathcal{J}_\infty \mathcal{X} \to \mathcal{J}_n \mathcal{X}$ eventually.

When $\mathcal{X}$ is the quotient stack $[V/G]$ associated to an action of a finite group $G$ on a variety $V$, we then have $J_\infty \mathcal{X} = [(J_\infty V)/G]$. To describe twisted arcs of $\mathcal{X}$ in terms of the $G$-action on $V$, we need the notion of $G$-arcs. For a $G$-cover $E \to \mathbb{D}$, an $E$-twisted $G$-arc of $V$ is a $G$-equivariant morphism $E \to V$. Two $G$-arcs $E \to V$ and $E' \to V$ are isomorphic if there is a $V$-morphism $E \to E'$ that is an isomorphism as $G$-covers of $\mathbb{D}$. We denote the space of $E$-twisted $G$-arcs of $V$ by

$$J_{\infty, E}^G V := \{ (G\text{-equivariant } E \to V)/\text{Aut}(E)^{\text{opp}} \},$$

where $\text{Aut}(E)^{\text{opp}}$ is the opposite group of the automorphism group of $E$ as a $G$-cover of $\mathbb{D}$ and isomorphic to the centralizer $C_G(H)$ of the stabilizer $H$ of a connected component of $E$. In particular, for the trivial $G$-cover $E^{\text{triv}} = \mathbb{D} \times G \to \mathbb{D}$, we have $J_{\infty, E}^{G\text{triv}} V = (J_\infty V)/G$. Further, we have identifications $J_{\infty, E} \mathcal{X} = J_{\infty, E}^G V$, where $\mathcal{E} = [E/G]$. Thus,

$$J_{\infty} \mathcal{X} = \bigsqcup_{\mathcal{E}} J_{\infty, \mathcal{E}} \mathcal{X} = \bigsqcup_{\mathcal{E}} J_{\infty, E}^G V.$$
left action of $\text{Aut}(E)^{\text{opp}} \simeq C_G(H)$ on $J_\infty^E V$. This action is identical to the one induced by restricting the $G$-action on $V$ to $C_G(H)$.

The measure $\mu_X$ restricted to $J_\infty X = [(J_\infty V)/G]$ and the measure $\mu_V$ are related as follows (with $X = [V/G]$ as above). For a measurable subset $C \subset J_\infty X$, its preimage $\tilde{C} \subset J_\infty V$ is a $G$-invariant measurable subset and the following equality holds

$$\mu_X(C) = \mu_V(\tilde{C})/G.$$  

Let $\mathcal{Y}$ be another Deligne–Mumford stack satisfying the same conditions as $X$ and let $f: \mathcal{Y} \to X$ be a (not necessarily representable) morphism. For a twisted arc $\gamma: \mathcal{E} \to \mathcal{Y}$, the composition $f \circ \gamma: \mathcal{E} \to X$ is not generally a twisted arc as it may not be representable. Nevertheless, it factors uniquely as

$$f \circ \gamma: \mathcal{E} \to \mathcal{E}' \xrightarrow{\gamma'} X,$$

where $\mathcal{E} \to \mathcal{E}'$ is a $\mathbb{D}$-morphism of twisted formal disks and $\gamma'$ is a twisted arc of $X$. If $X$ is an algebraic space, then $\mathcal{E}' = \mathbb{D}$ and $\gamma'$ is induced from the universality of the coarse moduli space $\mathcal{E} \to \mathbb{D}$. Sending $\gamma$ to $\gamma'$ defines a map $f_\infty: J_\infty \mathcal{Y} \to J_\infty X$. When $f$ is proper and birational, $f_\infty$ is almost bijective (meaning bijective outside subsets of motivic measure zero). On the other hand, the restriction of $f_\infty$ to untwisted arcs $f_\infty|_{J_\infty \mathcal{Y}}: J_\infty \mathcal{Y} \to J_\infty X$ is not necessarily almost bijective, which is the main reason for introducing twisted arcs. When $\mathcal{Y}$ is the quotient stack $[V/G]$ and $X$ is the corresponding quotient scheme $V/G$, the map

$$J_\infty \mathcal{Y} = \bigsqcup E J_\infty^E V \to J_\infty (V/G) = J_\infty X$$

sends a $G$-arc $E \to V$ to the associated morphism of quotient schemes $\mathbb{D} = E/G \to V/G$.

**Theorem 2.3** (The change of variables formula; [Yas19, Theorem 16.1]). Let $f: \mathcal{Y} \to X$ be a morphism of separated, irreducible, and reduced Deligne–Mumford stack of finite type over $k$. Let $h: f_\infty(A) \subset J_\infty X \to \frac{1}{2}\mathbb{Z} \cup \{\infty\}$ be a measurable function where $A \subset J_\infty \mathcal{Y}$ is a measurable subset over which $f_\infty$ is almost geometrically injective. Then,

$$\int_{f_\infty(A)} L^{h+sx} d\mu_X = \int_A L^{h \circ f_\infty - j_f + sy} d\mu_\mathcal{Y},$$

where $j_f$ is the Jacobian order function associated to $f$ and $s_\mathcal{X}$ and $s_\mathcal{Y}$ are the shift functions of $\mathcal{X}$ and $\mathcal{Y}$; respectively.

Recall that the shift function $s_\mathcal{X}$ is constantly zero on the space $J_\infty \mathcal{X}$ of untwisted arcs.

### 2.2. Stringy motives of log pairs.

Let $X$ be a $d$-dimensional normal variety. If $x \in X$ is a closed point, $(J_\infty X)_x \subset J_\infty X$ denotes the preimage of $x$ along the projection $J_\infty X \to X$. Let $\Delta$ be a boundary on $X$ so that $(X, \Delta_X)$ has index $r$. Let $J_{X,\Delta} \subset \mathcal{O}_X$ be defined by:

$$J_{X,\Delta} \mathcal{O}_X (r(K_X + \Delta)) = \text{Im} \left( (\Omega^d_{X/k})^{\otimes r} \to \mathcal{O}_X (r(K_X + \Delta)) \right).$$

In other words, $J_{X,\Delta} \subset \mathcal{O}_X$ is the image of the pairing map:

$$(\Omega^d_{X/k})^{\otimes r} \otimes \mathcal{O}_X (-r(K_X + \Delta)) \to \mathcal{O}_X.$$
Further, we consider the associated order function \( \text{ord} \mathcal{J}_{X,\Delta} : J_{\infty}X \to \mathbb{N} \cup \{\infty\} \) (namely, \( \gamma \mapsto \text{length} \kappa[[t]]/\gamma^{-1}\mathcal{J}_{X,\Delta} \)) and set for notation ease
\[
F = F_{(X,\Delta)} = \frac{1}{r} \text{ord} \mathcal{J}_{X,\Delta} : J_{\infty}X \to \frac{1}{r} \cdot \mathbb{N} \cup \{\infty\}.
\]
Note that \( F^{-1}(\infty) = J_{\infty}Z \), where \( Z = V(\mathcal{J}_{X,\Delta}) \), and that this is a subset of measure zero with respect to the measure \( \mu_X \) on \( J_{\infty}X \).

**Definition 2.4** (Stringy motive of a log pair). Suppose that \((X,\Delta)\) is a KLT log pair that admits a log resolution. The *stringy motive of \((X,\Delta)\)* is defined as
\[
M_{st}(X,\Delta) := \int_{J_{\infty}X} \mathcal{L}^{F_{(X,\Delta)}} d\mu_X \in \hat{\mathcal{M}}^r_{\kappa,x}.
\]
If \( x \in X \) is a closed point, the stringy motive of \((X,\Delta)\) at \( x \) is defined as:
\[
M_{st}(X,\Delta)_x := \int_{(J_{\infty}X)_x} \mathcal{L}^{F_{(X,\Delta)}} d\mu_X \in \hat{\mathcal{M}}^r_{\kappa,x}.
\]
If \( X \) is log terminal, we define \( M_{st}(X) \) and \( M_{st}(X)_x \) as the corresponding stringy motives associated to the log pair \((X,0)\).

**Remark 2.5.** Since \((X,\Delta)\) has a log resolution, the KLT condition implies that the above integral converges. In fact, let \( \phi : \tilde{X} \to (X,\Delta) \) be a log resolution and write:
\[
K_{\tilde{X}} \sim_Q \phi^*(K_X + \Delta) + \sum_{i \in I} b_i(E,X,\Delta) \cdot E_i,
\]
where the \( E_i \) are prime divisors on \( \tilde{X} \). For notation ease, we set
\[
K_{\tilde{X}/(X,\Delta)} := \sum_{i \in I} b_i(E,X,\Delta) \cdot E_i =: \sum_{i \in I} b_i \cdot E_i.
\]
If \( r \) is the index of \((X,\Delta)\), then \( b_i = \frac{1}{r} \cdot \mathbb{Z} \). Moreover, \((X,\Delta)\) being KLT means that the log discrepancies \( a_i := 1 + b_i \) are positive. The change of variables formula and the explicit computation of stringy motives for simple normal crossing divisors yields:
\[
M_{st}(X,\Delta) = M_{st}(\tilde{X},-K_{\tilde{X}/(X,\Delta)}) = \sum_{J \subseteq I} \{E^\circ_j\} \prod_{j \in J} \frac{L-1}{\sum_{a_j} -1}, \quad \text{where } E^\circ_j := \bigcap_{j \in J} E_j \bigcup_{j \notin J} E_j.
\]
Likewise, \( M_{st}(X,\Delta)_x \) is computed by the same formula by writing \( \{E^\circ_j \cap \phi^{-1}(x)\} \) in place of \( \{E^\circ_j\} \). Also, one readily sees that \( M_{st}(X,\Delta)_x \) depends only on the formal completion (or the henselization) of \( X \) at \( x \) and so we may regard it as an invariant of \( \mathcal{O}_{X,x} \) (or \( \mathcal{O}^h_{X,x} \)).

**Example 2.6** (Surfaces and minimal resolutions). With notation as above, suppose that \((x,\Delta)\) is a log terminal surface singularity (with Gorenstein index \( r \in \mathbb{N} \)) and \( \phi : \tilde{X} \to (x,\Delta) \) is a minimal log resolution, so that \( a_i \in (0,1) \) (see [Kol13 Claim 2.26.4, p. 56]). In this case, \((x,\Delta)\) is further rational and so the dual graph \( \Gamma \) associated to \( \phi : \tilde{X} \to (x,\Delta) \) is a tree of \( \mathbb{P}^1 \)'s, which means that \( E_i \cong \mathbb{P}^1 \) for all \( i \in I \). Let us recall that the set of vertices of \( \Gamma \) is \( I \) and there is an edge connecting two different vertices \( i,j \in I \) if and only if \( E_i \cap E_j \neq \emptyset \). Let \( H \) be the set of edges of \( \Gamma \). We denote an element of \( H \) by \([i,j]\) where \( i,j \in I \) are the vertices
that edge connects (of course, \([i, j] = [j, i]\)). Further, denote by \(m_i \geq 1\) the number of edges sticking out of a vertex \(i \in I\). In this way, we may compute \(M_{st}(X)_x\) as

\[
M_{st}(X)_x = \sum_{i \in I} \left\{ \frac{E_i}{\bigcup_{j \neq i} E_j} \right\} \frac{L - 1}{a_i - 1} + \sum_{\{i, j\} \subseteq I} \left\{ \frac{E_i \cap E_j}{(L - 1)^2} \right\} \frac{(L - 1)^2}{(L_{a_i - 1})(L_{a_j - 1})} = \sum_{i \in I} (L + 1 - m_i) \frac{L - 1}{a_i - 1} + \sum_{\{i, j\} \in H} \frac{(L - 1)^2}{(L_{a_i - 1})(L_{a_j - 1})} \in \mathbb{Z}[(L - 1)/r] \subset \hat{\mathcal{M}}_r^\tau.
\]

Its Poincaré realization then is

\[
P_{st}(X)_x = \sum_{i \in I} (T^2 + 1 - m_i) \frac{T^2 - 1}{T^{2a_i} - 1} + \sum_{h \in H} \frac{(T^2 - 1)^2}{(T^{2a_i} - 1)(T^{2a_j} - 1)} \in \mathbb{Z}[(T - 1)/r],
\]

which is a formal Laurent series in \(T^{2/r}\) with integral coefficients.

**Example 2.7** (Cones). Let \(V\) be a smooth Fano variety and write \(K_V + aL \sim_{\mathbb{Q}} 0\) for some ample Cartier divisor \(L\) on \(V\) and some \(0 < a \in \mathbb{Q}\). Then, the affine cone \(X = \text{Spec} \bigoplus_{i \in \mathbb{N}} H^0(V, iL)\) is log terminal; let \(x \in X\) denote its vertex. See [Kol13, Lemma 3.1]. The blowup \(Y := \text{Bl}_x X \to X\) is a resolution with exceptional divisor \(E \cong V\). Hence,

\[
M_{st}(X) = (L - 1)\{V\} + \{V\} \frac{L - 1}{a_i - 1} \text{ whereas } M_{st}(X)_x = \{V\} \frac{L - 1}{a_i - 1}.
\]

Observe that \(\{X\} = (L - 1)\{V\} + 1\). For instance, if \(V\) were a smooth hypersurface of degree \(d\) inside \(\mathbb{P}^n\) (with \(L\) being the hyperplane section), then \(a = n + 1 - d\). For example, if \(V \subset \mathbb{P}^d\) is the rational normal curve, then \(\{V\} = L + 1\) and \(a = 2/d\) so that \(M_{st}(X)_x = (L^2 - 1)/(L^{2/d} - 1)\).

The formula above formula can be generalized to the case in which \(V\) is a log terminal Fano variety as follows:

\[
M_{st}(X)_x = M_{st}(V) \frac{L - 1}{a + 1 - 1}.
\]

To show this, let us take a log resolution \(\hat{V} \to V\). Since \(Y\) is a line bundle over \(V\), we have the corresponding log resolution \(f : \hat{Y} \to Y\). Let us write \(K_{\hat{V}/V} = \sum_{i \in I} b_i E_i\) and \(K_{\hat{Y}/Y} = \sum_{i \in I} b_i F_i\), where the \(E_i\) are prime divisors on \(\hat{V}\) and the \(F_i\) are their corresponding prime divisors on \(\hat{Y}\); respectively. Then

\[
M_{st}(V) = \sum_{J \subseteq I} \{E_J\} \prod_{j \in J} \frac{L - 1}{a_{b_j - 1}}.
\]

To compute \(M_{st}(X)_x\), observe that

\[
K_{\hat{Y}/X} = K_{\hat{Y}/Y} + f^* K_{Y/X} = \sum_{i \in I} b_i F_i + aG,
\]

where \(G\) denotes the strict transform of the exceptional divisor of \(Y \to X\), which is isomorphic to \(V\). Setting \(F_0 := G\) and \(b_0 := a\), we may write \(K_{\hat{Y}/X} = \sum_{i \in I \cup \{0\}} b_i F_i\). Thus, letting \(g\)
denote the map $\tilde{Y} \rightarrow Y \rightarrow X$, we have:
\[
M_{st}(X)_x = \sum_{J \subset I \cup \{0\}} \left\{ F^o_J \cap g^{-1}(x) \right\} \prod_{j \in J} \frac{L - 1}{L_{b_j + 1} - 1} \\
= \sum_{J \subset I} \left\{ F^o_J \cap g^{-1}(x) \right\} \prod_{j \in J} \frac{L - 1}{L_{b_j + 1} - 1} + \sum_{0 \in J \subset I \cup \{0\}} \left\{ F^o_J \cap g^{-1}(x) \right\} \prod_{j \in J} \frac{L - 1}{L_{b_j + 1} - 1}.
\]
Since $g^{-1}(x) = F_0$, we have that $F^o_J \cap g^{-1}(x) = \emptyset$ for every $J \subset I$. Therefore,
\[
M_{st}(X)_x = \sum_{0 \in J \subset I \cup \{0\}} \left\{ F^o_J \cap g^{-1}(x) \right\} \prod_{j \in J} \frac{L - 1}{L_{b_j + 1} - 1} \\
= \left( \sum_{J' \subset I} \left\{ F^o_{J'} \right\} \prod_{j \in J' \cup \{0\}} \frac{L - 1}{L_{b_j + 1} - 1} \right) \frac{L - 1}{L_{b_0 + 1} - 1} \\
= M_{st}(V) \frac{L - 1}{L_{b_0 + 1} - 1};
\]
as required. This concludes our example about stringy motives of cone singularities.

If a finite group $G$ acts on $X$ preserving the boundary $\Delta$, there exists a canonical lift of $M_{st}(X, \Delta)$ to $\mathcal{M}'_{k,r}$, which we denote by $\tilde{M}_{st}(X, \Delta)$ by abuse of notation. The same lifting principle holds for $M_{st}(X, \Delta)_x$ if $G$ fixes $x$. This let us define $M_{st}(-)/G$ where “−” denotes any of the data we had considered above. Once again, the invariant $M_{st}(X, \Delta)_x/G$ depends only on the completion/henselization of $X$ at $x$ (and the $G$-action on it).

We may also define $M_{st}(-)/G$ in terms of motivic integration over Deligne–Mumford stacks. This interpretation will be useful when a change of variables formula is required. Let $\mathcal{X}$ be the quotient stack $[X/G]$. Then, $M_{st}(X/G) = M_{st}(\mathcal{X})$ (use [Yas19, Theorem 1.2]) whereas $M_{st}(X)/G$ is equal to the untwisted stringy motive $M_{st}^{\text{utd}}(\mathcal{X})$ [Yas19, Definition 13.2], which is by definition a motivic integral over the space $J_{\infty} \mathcal{X}$ of untwisted arcs of $\mathcal{X}$. That is:
\[
M_{st}(X) / G = \left( \int_{J_{\infty} X} \mathbb{L}^F_X \mu_X \right) / G = \int_{J_{\infty} \mathcal{X}} \mathbb{L}^F_{\mathcal{X}} \mu_{\mathcal{X}} =: M_{st}^{\text{utd}}(\mathcal{X}),
\]
where $F_{\mathcal{X}}$ is the function corresponding to the $G$-invariant function $F_X$ by identifying $J_{\infty}X$ with $(J_{\infty}X)/G$. Analogous descriptions apply to $M_{st}(X, \Delta)/G$ and $M_{st}(X, \Delta)_x/G$, e.g.
\[
M_{st}(X, \Delta)_x / G = M_{st}^{\text{utd}}(\mathcal{X}, \Delta)_{\bar{x}} := \int_{(J_{\infty} \mathcal{X})_{\bar{x}}} \mathbb{L}^F_{\mathcal{X}, \Delta} \mu_{\mathcal{X}}
\]
where $\Delta$ and $\bar{x}$ are, respectively, the divisor and point of $\mathcal{X}$ induced by $\Delta$ and $x$. Further, observe that the stringy motive $M_{st}(\mathcal{X}, \Delta)_{\bar{x}}$ (without the adjective “untwisted”) is instead a motivic integral over the whole space $J_{\infty} \mathcal{X}$ of twisted arcs of $\mathcal{X}$:
\[
M_{st}(\mathcal{X}, \Delta)_{\bar{x}} = \int_{(J_{\infty} \mathcal{X})_{\bar{x}}} \mathbb{L}^F_{\mathcal{X}, \Delta + s_{\mathcal{X}}} \mu_{\mathcal{X}} = M_{st}(X/G, \Delta/G)_{\bar{x}},
\]
where the function $F_{\mathcal{X}, \Delta}$ (as defined above) extends to $J_{\infty} \mathcal{X}$, $s_{\mathcal{X}}$ is the shift function, $\Delta/G$ and $x/G$ are respectively the divisor and the point on $X/G$ induced from $\Delta$ and $x$, and the
Moreover, if the inequality is strict if \( J \). Let us consider a commutative diagram
\[
\begin{array}{ccc}
(X_1, \Delta_1) & \xleftarrow{f} & (X_2, \Delta_2) \\
g_1 \downarrow & & \downarrow g_2 \\
(X_0, \Delta_0) & & \\
\end{array}
\]
of crepant finite covers among log pairs. Moreover, suppose that \( g_i \) is generically Galois with Galois group \( G_i \). Let us define \( \mathcal{X}_i := [X_i/G_i] \) and let \( \Delta_i \) be the boundary divisor on \( \mathcal{X}_i \) induced by \( \Delta_i \). Let \( x_0 \) be a \( k \)-point of \( X_0 \) and let \( \bar{x}_i \) be its unique lifts to \( \mathcal{X}_i \).

**Proposition 2.9** ([Yas19, Theorem 16.2]). Work in Setup 2.8. Then, for all \( i \in \{1, 2\} \):
\[
M_{st}(X_0, \Delta_0) = M_{st}(\mathcal{X}_i, \bar{\Delta}_i) \quad \text{and likewise} \quad M_{st}(X_0, \Delta_0)_{x_0} = M_{st}(\mathcal{X}_i, \bar{\Delta}_i)_{\bar{x}_i}.
\]

**Corollary 2.10.** Work in Setup 2.8. Then,
\[
M_{st}(X_1, \Delta_1)/G_1 = M_{st}^{utd}(\mathcal{X}_1, \bar{\Delta}_1) \geq M_{st}^{utd}(\mathcal{X}_2, \bar{\Delta}_2) = M_{st}(X_2, \Delta_2)/G_2
\]
Moreover, if \( J_{\infty}X_1 \setminus f_{\infty}(J_{\infty}X_2) \) has positive measure then the inequality is strict.

Let \( x_i \in X_i \) be the preimage of \( x_0 \) (which we are assuming to be a point and so that \( G_i \) fixes \( x_i \)). Then,
\[
M_{st}(X_1, \Delta_1)_{x_1}/G_1 = M_{st}^{utd}(\mathcal{X}_1, \bar{\Delta}_1)_{\bar{x}_1} \geq M_{st}^{utd}(\mathcal{X}_2, \bar{\Delta}_2)_{\bar{x}_2} = M_{st}(X_2, \Delta_2)_{x_2}/G_2.
\]
Moreover, the inequality is strict if \( J_{\infty}X_1 \setminus f_{\infty}((J_{\infty}X_2)_{x_2}) \) has positive measure.

**Proof.** From the change of variables formula,
\[
M_{st}^{utd}(\mathcal{X}_1, \bar{\Delta}_1) - M_{st}^{utd}(\mathcal{X}_2, \bar{\Delta}_2) = \int_{J_{\infty}\mathcal{X}_1 \setminus f_{\infty}(J_{\infty}\mathcal{X}_2)} \mathcal{L}^{F_{x_1, \Delta_1} + s_{x_1}} d\mu_{\mathcal{X}_1} \geq 0.
\]
The inequality is strict if \( J_{\infty}\mathcal{X}_1 \setminus f_{\infty}(J_{\infty}\mathcal{X}_2) \) has positive measure. By (2.2.1) this is the case if \( J_{\infty}X_1 \setminus f_{\infty}(J_{\infty}X_2) \) has positive measure. The local statements are shown likewise. \( \square \)

### 2.3. Explicit description for log terminal surface singularities.
In this section, we compute \( M_{st}(X)_x/G \) explicitly for a log terminal surface singularity \( x \in X \) with a given action by a finite group \( G \) fixing \( x \). Our results will be analogous to those in Example 2.6 but we will need to use \( G \)-equivariant minimal log resolutions instead.

Recall that if \( (X, \Delta) \) is a KLT surface pair then \( X \) is necessarily \( \mathbb{Q} \)-factorial and so log terminal; see [Tan18, Corollary 4.11], [KM98, Corollary 2.35]. Thus, since we are ultimately interested in the topology around \( x \in X \), there is no harm in assuming \( \Delta = 0 \). This will turn out being a considerable simplification on the possible shapes of the dual graph associated to the minimal resolution (i.e., in the terminology of [Kol13, §2.2], we shall avoid dealing with
extended dual graphs). Further, we may shrink $X$ so that $X \setminus \{x\}$ is nonsingular. In such case:

\[
M_{st}(X)/G = \left\{ (X \setminus \{x\})/G \right\} + M_{st}(X)/G.
\]

More generally, if $X \setminus \{x_1, \ldots, x_k\}$ is regular and $G$ acts on $\{x_1, \ldots, x_k\}$ then

\[
M_{st}(X) = \left\{ X \setminus \{x_1, \ldots, x_k\} \right\} + \sum_{i=1}^{k} M_{st}(X)_{x_i}.
\]

and so

\[
M_{st}(X)/G = \left\{ (X \setminus \{x_1, \ldots, x_k\})/G \right\} + \sum_{j=1}^{l} M_{st}(X)_{x_j}/\text{Stab}(x_j),
\]

where the $x_1, \ldots, x_k$ are representatives for the orbits of the action of $G$ on $\{x_1, \ldots, x_k\}$; in particular $\{x_1, \ldots, x_k\} = \bigsqcup_{j=1}^{l} Gx_j$.

Let us commence by making a general remark on how a $G$-equivariant log resolution can be used to compute $M_{st}(X)/G$. Let us take a $G$-equivariant log resolution $\phi: \tilde{X} \to (x, X)$ and use the notation of [Remark 2.5](#Remark2.5) (for $I$, $E_i$, $a_i$, and so on). Following [Bat99, Definition 5.1](#Bat99), we introduce the following notion:

**Definition 2.11.** We say that $\phi$ is $G$-normal if, for every node $w \in E$, the stabilizer \(\text{Stab}(w)\) preserves each of the two irreducible components $E_i$ including $w$. In terms of the dual graph $\Gamma$ of $E$, this means that if an element $g \in G$ fixes some edge $e$ then it fixes the two vertices adjacent to $e$ (instead of swapping them).

Let us suppose that $\phi$ is $G$-normal. Then, for all $\iota \in I/G$, the prime divisors $\{E_i\}_{\iota \in \iota}$ are mutually disjoint. Here, we think of $\iota$ as an orbit of $G$ acting on $I$. Thus, we may define

\[
E_i := \bigsqcup_{\iota \in \iota} E_i \text{ and } E^\circ_i := E_i \setminus \bigsqcup_{\kappa \in (I/G) \setminus \iota} E_\kappa.
\]

Observe that, for all $\iota \in I/G$, the prime divisors $\{E_i\}_{\iota \in \iota}$ have the same log discrepancy; which we denote by $a_\iota$ (i.e., $a_\iota = a_i$ for all $i \in \iota$). Our general remark is the following:

**Lemma 2.12.** With notation and hypotheses as above, the following formula holds

\[
M_{st}(X)/G = \sum_{\iota \in I/G} \{E^\circ_i\}/G \frac{\mathbb{L} - 1}{\mathbb{L}^{a_\iota} - 1} + \sum_{\{\iota, \kappa\} \subset I/G} \{(E_i \cap E_\kappa)\}/G \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_\iota} - 1)(\mathbb{L}^{a_\kappa} - 1)},
\]

where $\{\iota, \kappa\}$ runs over the subsets of $I/G$ with two elements.

**Proof.** This is basically the $g = 1$ part of the formula defining the orbifold $E$-function [Bat99, Definition 6.3](#Bat99). Before taking the quotient by $G$, we have

\[
\begin{align*}
M_{st}(X)_{x} &= \int_{(X, x)} \mathbb{L}^{-\text{ord}_xK_X} \, d\mu_{\tilde{x}} \\
&= \sum_{\iota \in I} \{E^\circ_i\}(\mathbb{L} - 1) \sum_{n \geq 0} \mathbb{L}^{-a_\iota n} + \sum_{\{\iota, \kappa\} \subset I} \{(E_i \cap E_\kappa)\}(\mathbb{L} - 1)^2 \sum_{n, m \geq 0} \mathbb{L}^{-n_{\iota} - m_{\kappa}}.
\end{align*}
\]
The first equality follows from the change of variables formula. The term \( \{E_i\}(L - 1)L^{-a_i n} \) is the contribution of arcs on \( \tilde{X} \) meeting \( E_i \) with order \( n \) but not meeting any exceptional prime divisor; \( \{E_i\}(L - 1)L^{-a_i n} \) is the measure of the set of those arcs and, since \( b_i = a_i - 1 \) is the multiplicity of \( E_i \) in \( K_{\tilde{X}/X} \), then \( L^{-b_i n} \) is the value of the function \(-\text{ord } K_{\tilde{X}/X} \) there. Likewise, the term \( \{E_i \cap E_j\}(L - 1)^2L^{-na_i - mb_j} \) is the contribution of arcs that meet \( E_i \) and \( E_j \) with orders \( n \) and \( m \) respectively: \( \{E_i \cap E_j\}(L - 1)^2L^{-2n} \) is the measure of the set of those arcs and \( L^{-nb_i - mb_j} \) is the value of the function \(-\text{ord } K_{\tilde{X}/X} \) there.

Let \( \iota \in I/G \) and consider the set \( C_n \) (resp. \( C_{\geq n} \)) of arcs on \( \tilde{X} \) that meet \( E_i \) for some \( i \in \iota \) with order \( n \) (resp. \( \geq n \)) but do not meet any exceptional prime divisor (with index) not belonging to \( \iota \). These are cylinders of level \( n \) and their images \( \bar{C}_n \) and \( \bar{C}_{\geq n} \) in the \( n \)-th jet scheme \( J_n \tilde{X} \) are; respectively, a \( G_m \)-bundle and an \( A_1 \)-bundle over \( E^\circ \iota \). Therefore,

\[
\{\bar{C}_{\geq n}/G\} = \{E^\circ \iota/G\}\mathbb{L} \quad \text{and} \quad \{(\bar{C}_{\geq n} - \bar{C}_n)/G\} = \{E^\circ \iota/G\}
\]

by the relation imposed in the definition of our complete Grothendieck ring of varieties [Yas19, Definitions 9.5 and 9.6]. Hence,

\[
\{\bar{C}_n/G\} = \{E^\circ \iota/G\}(L - 1).
\]

This explains the first summation on the right-hand side in the formula of the lemma. The second summation is explained likewise. \( \square \)

### 2.3.1. On the \( G \)-normality of the minimal resolution of a surface.

Lemma 2.12 raises the question of whether a \( G \)-normal log resolution exists. We discuss next the \( G \)-normality of the minimal resolution of \( X \), say \( \psi: X' \rightarrow (x, X) \); see [Kol13, Theorem 2.25]. By [Kol13, p. 123] (also see [Ke92, §3]), the exceptional set of \( \psi \) is a simple normal crossing divisor whose dual graph \( \Gamma' \) is either a straight line (Figure 1) or has three straight branches sticking out of a single vertex (Figure 2). In the latter case, we refer to the single vertex as the node.

![figure 1](image1.png)

**Figure 1.** Straight dual graph.

![figure 2](image2.png)

**Figure 2.** Three branches dual graph.

We study the \( G \)-normality of \( \psi \) by checking three distinct cases separately:

**The case where \( \Gamma' \) is a straight line and has an odd number of vertices:** In this case, \( \Gamma' \) has only two automorphisms; namely, the identity and the involution switching the two ends. The involution does not fix any edge. Thus, the minimal resolution \( \psi \) is \( G \)-normal (for all \( G \)) and we choose it as our \( G \)-normal log resolution \( \phi: \tilde{X} \rightarrow X \).

**The case where \( \Gamma' \) is a straight line and has an even number of vertices:** As in the previous case, \( \Gamma' \) has only two automorphisms. This time, however, the involution does fix the middle edge and switches the two vertices adjacent to it. Hence, \( \psi \) may not be \( G \)-normal. To fix it,
we take the blowup $\tilde{X} \to X'$ at the point corresponding to the middle edge. The dual graph associated to the resulting log resolution $\phi: \tilde{X} \to X$ is a straight line with an odd number of vertices. As before, $\phi$ is $G$-normal and we choose it as our $G$-normal log resolution.

The case where $\Gamma'$ has three branches: Any automorphism of $\Gamma'$ fixes the node. If it fixes some edge $e$, then it fixes the branch including $e$. Since one end of the branch; the node, is fixed, the automorphism fixes all the edges and the vertices in the branch. In particular, it fixes the two vertices adjacent to $e$. This shows that the minimal resolution $\psi: X' \to X$ is $G$-normal. We choose $\psi$ to be our log resolution $\phi: \tilde{X} \to X$.

Summing up, we have constructed a $G$-normal log resolution $\phi: \tilde{X} \to X$ in each of the above three cases, which is the minimal resolution possibly followed by the blowup at a $G$-fixed point. Furthermore, it is independent of the group action and is minimal among $G$-normal log resolutions.

Definition 2.13. We refer to $\phi: \tilde{X} \to (x, X)$ as the modified minimal resolution of $x \in X$.

According to [Kol13] Claim 2.26.4, p. 56], an exceptional prime divisor of the minimal resolution $\psi: X' \to X$ has log discrepancy $\leq 1$. In the case where $\tilde{X}$ is a one-point blowup of $X'$, then the exceptional divisor of this blowup has log discrepancy $\leq 2$ over $X$. Thus:

Lemma 2.14. All but one exceptional prime divisors of the modified minimal resolution have log discrepancy $\leq 1$. Moreover, the possible exception has log discrepancy $\leq 2$.

With the above in place, we can now specialize the computation of $M_{st}(X)_{x}/G$ in [Lemma 2.12] to the case where $\phi$ is the modified minimal resolution. Let $\Gamma$ be the dual graph associated to the modified minimal resolution $\phi: \tilde{X} \to X$. Note that $\Gamma$ also has one of the forms of either [Figure 1] or [Figure 2] and the same is true for the quotient $\Gamma/G$. We may think of a vertex $i \in I/G$ as a set of orbits of vertices of $I$. Note that two vertices $i, j \in I$ correspond to disjoint exceptional prime divisors (precisely because the modified minimal resolution is $G$-normal). Further, the log discrepancies are constant across $G$-orbits, i.e., $a_i = a_j$ if $i, j \in I$. This let us define the quotient log discrepancy $a_i$ of $i \in I/G$. An edge $[i, \kappa] \in H/G$ of $\Gamma/G$ corresponds to at least one vertex $i \in I$ being connected to a vertex in $j \in \kappa$ (by an edge $[i, j] \in H$). Note that the set of edges $\{(i, j) \in H \mid i \in I, j \in \kappa\}$ corresponds (bijectively) to the intersection points of $E_i := \bigcup_{i \in I} E_i$ and $E_\kappa := \bigcup_{j \in \kappa} E_j$, which get all identified under the action of $G$. Thus, an edge $[i, \kappa]$ of $\Gamma/G$ corresponds to $E_i \cap E_\kappa \neq \emptyset$ and $\{(E_i \cap E_\kappa)/G\} = \delta_{i, \kappa} \in \hat{\mathcal{M}}_{k,r}$ (where $\delta_{i, \kappa}$ is Kronecker’s delta) as there is only one edge connecting two vertices in $\Gamma/G$. Further, we see that $\{(E_i \cup_{\kappa \neq i} E_\kappa)/G\} \in \hat{\mathcal{M}}_{k,r}$ equals $L + 1 - m_i$ where $m_i$ is the number of edges of $\Gamma/G$ sticking out of $i$—this uses Lüroth’s theorem to see that $(E_i \cup_{\kappa \neq i} E_\kappa)/G$ are rational curves.

In conclusion:

\[
M_{st}(X)_{x}/G = \sum_{i \in I/G} (L + 1 - m_i) \frac{L - 1}{L^{a_i} - 1} + \sum_{[i, \kappa] \in H/G} \frac{(L - 1)^2}{(L^{a_i} - 1)(L^{a_\kappa} - 1)} \in \mathbb{Z}[L^{-1}] \subset \hat{\mathcal{M}}_{k,r}.
\]

Example 2.15. Rational double points (also known as Du Val singularities) are exactly the canonical surface singularities or, equivalently, the rational Gorenstein surface singularities. These can be further characterized as those surface singularities whose minimal resolution is crepant. As such, the dual graphs associated to their minimal resolution are the simply laced Dynkin diagrams. See [Kol13] Example 3.26 for further details. Following [Art77] §3, it is
standard to denote the formal germ at a rational double point by $X^r_n$ where $X \in \{A, D, E\}$ and $n, r$ vary in between certain intervals of nonnegative integers. However, if $p \geq 7$, the index $r$ is superfluous and in that case we only have the so-called standard forms of the rational double points. Else; $p = 2, 3, 5$, we obtain an interesting zoo of non-standard forms; [Art77, Ibid.] for details. In general, we let $X_n = X^0_n$ denote the standard forms, which can be realized as linear quotients $\hat{\mathbb{A}}^2 / G$ by a finite subgroup $G \subset \text{SL}_2(k)$. As far as stringy motives are concerned, what matters is that the dual graph of the minimal resolution of $X^r_n$ is the simply laced Dynkin diagram $X_n$ with vertices of weight 2. In particular, the vertices correspond to rational smooth curves intersecting transversely at exactly one point on the minimal resolution. Let us denote by $M_{st}(X^r_n)$ the stringy invariant $M_{st}(X)_x$ associated to a rational double point $x \in X$. Since the minimal resolution $\tilde{X} \to (x, X)$ is crepant (i.e., $a_i = 1$), $M_{st}(X^r_n)$ equals the motive associated to the exceptional set of curves (which is a tree of $n$ $\mathbb{P}^1$’s intersecting as prescribed by the diagram $X_n$). An easy calculation then shows:

$$M_{st}(X^r_n) = nL + 1.$$ 

Further, using (2.10.1) we see that:

$$M_{st}(\mathbb{A}^2 / G) = \{\mathbb{A}^2 / G \setminus \{0\}\} + M_{st}(X_n) = \{\mathbb{A}^2 / G\} - 1 + nL + 1 = L^2 + nL$$

for the standard forms. Here, we used $\{\mathbb{A}^2 / G\} = L^2$, which follows from the relations imposed in the definition of $\mathcal{M}_{k,r}$.

It is well-known that there are many Galois quasi-étales among rational double points; see [CRMP+21, p. 15]. For instance, if $p \neq 3$, there is a degree-3 Galois quasi-étaient cover $D_4 \to E_6$. Let us consider the action of $\mathbb{Z}/3$ on $D_4$. Recall that the dual graph of $D_4$ is the one in Figure 3. Then, one readily sees that the action of $\mathbb{Z}/3$ on this graph is given by fixing the node and permuting the other three vertices cyclically. The quotient graph is then displayed in Figure 4. One vertex represents a copy of $\mathbb{P}^1$ (which is the quotient of the disjoint union of three $\mathbb{P}^1$’s under the trivial cyclic action) whereas the other vertex represents the quotient of $\mathbb{P}^1$ under the action of $\mathbb{Z}/3$ (where the points 0, 1, $\infty$ are being permuted cyclically). This lets us conclude that

$$M_{st}(D_4)/(\mathbb{Z}/3) = L + \left\{ (\mathbb{P}^1 \setminus \{0, 1, \infty\}) / (\mathbb{Z}/3) \right\} + 1 = 2L + 1.$$

Similarly, if $p \neq 2$, there is an action of $\mathbb{Z}/2$ on $E_6$ whose quasi-étales quotient is $E_7$. To compute $M_{st}(E_6)/(\mathbb{Z}/2)$, we note that the induced action of $\mathbb{Z}/2$ on the dual graph of $E_6$—being it depicted in Figure 5—is given by reflection across the symmetry axis. This implies that the quotient graph is the one in Figure 6.
As before, we then have:

$$M_{st}(E_6)/(\mathbb{Z}/2) = \mathbb{L} + \left\{ (\mathbb{P}^1 \setminus \{0, 1, \infty\}) / (\mathbb{Z}/2) \right\} + \mathbb{L} - 1 + \mathbb{L} + 3 = 4\mathbb{L} + 1.$$  

Finally, we consider the example $A_{2n-5} \xrightarrow{\mathbb{Z}/2} D_n$ where $p \neq 2$ and $n \geq 4$. The dual graph is as in Figure 1 with $2n - 5$ vertices, which is an odd number. The only automorphism of such graph is the one switching the two branches (with $n - 3$ vertices each) sticking out of the middle point which remains fixed under the action. Thus, the quotient graph is another straight graph but with $n - 2$ vertices. By a similar computation to the ones above, we get:

$$M_{st}(A_{2n-5})/(\mathbb{Z}/2) = (n - 2)\mathbb{L} + 1.$$  

There is also the trivial example $A_0 \xrightarrow{G} X_n$ whenever $(p, \#G) = 1$. Since $A_0$ is the smooth germ, $M_{st}(A_0) = 1$ and so $M_{st}(A_0)/G = 1$ for all $G$.

Observe that we have built a tower (for $p \neq 2, 3$):

$$E_7 \xleftarrow{\mathbb{Z}/2} E_6 \xleftarrow{\mathbb{Z}/3} D_4 \xleftarrow{\mathbb{Z}/2} A_3 \xleftarrow{\mathbb{Z}/4} A_0,$$

where the displayed arrows are all Galois quasi-étale covers whose Galois group is the one displayed on top of them. Note that the composite arrows $E_7 \xleftarrow{S_3} D_4$ and the ones $X_n \xleftarrow{A_0}$ (i.e., those with $A_0$ as the source) are all Galois quasi-étale covers as well (with Galois group displayed on top). Here, $S_n$ denotes the $n$-th symmetric group. Nonetheless, the composite arrow $E_6 \xleftarrow{A_3}$ is not Galois although it is quasi-étale.\footnote{Indeed, if it were Galois its Galois group would have to be cyclic as $S_3$ does not have $\mathbb{Z}/2$ as a normal subgroup. In that case, $\mathbb{Z}/3$ would have to act non-trivially on $A_3$ which is impossible as it cannot act non-trivially on its dual graph.}

In particular,

$$E_7 \xleftarrow{} E_6 \xleftarrow{} D_4 \xleftarrow{} A_0$$

is a tower of Galois quasi-étale covers inducing (by the computations we had done above) the following sequence on stringy motives

$$M_{st}(E_7)/1 = 7\mathbb{L} + 1 > M_{st}(E_6)/(\mathbb{Z}/2) = 4\mathbb{L} + 1 > M_{st}(D_4)/S_3 = 2\mathbb{L} + 1 > M_{st}(A_0)/BO = 1,$$

where $BO$ denotes the binary octahedral group.

**Example 2.16.** Let $X_d$ be the (affine) cone over the rational normal curve $C \subset \mathbb{P}^d$ and $0 \in X_d$ be its vertex. As we had seen in **Example 2.7**, $M_{st}(X_d)_0 = (\mathbb{L}^2 - 1)/([\mathbb{L}^2/d] - 1)$ and its minimal resolution consists of blowing up $0 \in X_d$ so that its dual graph is just one vertex. On the other hand, we may think of $X_d$ as a quotient singularity $\mathbb{A}^2/\mu_d$ as it is the spectrum of the $d$-th Veronese subring of $k[x, y]$. In fact, $\mathbb{A}^2 \rightarrow X_d$ is a connected $\mu_d$-torsor over $X_d \setminus \{0\}$.
Thus, if \( d = nm \), we may think of \( X_d \) as a quotient \( X_n / \mu_m \). Assume now \( p \nmid m \). By the triviality of the dual graphs in this example, \( M_{st}(X_n)_0 / \mu_m = M_{st}(X_n)_0 \). Furthermore,
\[
M_{st}(X_d)_0 = \frac{L^2 - 1}{L^{2/d} - 1} = (L^2 - 1) \sum_{i=1}^{\infty} L^{\frac{-2i}{d}} = (L^2 - 1) \sum_{(i,m)=1}^{\infty} L^{\frac{-2im}{d}} = M_{st}(X_n)_0 + (L^2 - 1) \sum_{(i,m)=1} \text{L}^{\frac{-2i}{d}} > M_{st}(X_n)_0.
\]

3. Strict Descent in the Presence of Ramification

In this section, we establish the core result of this work. That is, we explain why stringy motives get smaller across ramified Galois quasi-étale covers. The main step is to show that there is an abundance of non-liftable arcs.

3.1. Abundance of non-liftable arcs. Let us consider the following setup.

Setup 3.1. Let \((X, \Delta_X)\) be a \( d \)-dimensional KLT log pair. Let \( g: (Y, \Delta_Y) \rightarrow (X, \Delta_X) \) be a Galois quasi-étale log-cover with Galois group \( G \). As customary, \( g_\infty: J_\infty Y \rightarrow J_\infty X \) is the induced map by functoriality. Further, let \( \phi: (\bar{X}, \Delta_{\bar{X}}) \rightarrow (X, \Delta) \) be a log resolution; where we mean that \( \phi \) is crepant and \( \phi_* \Delta_{\bar{X}} = \Delta_X \). Let \( \psi: (\bar{Y}, \Delta_{\bar{Y}}) \rightarrow (Y, \Delta_Y) \) be the normalization of \( \phi \) along \( g \) (i.e., with respect to \( K(Y) \)) and \( \tilde{g}: (\bar{Y}, \Delta_{\bar{Y}}) \rightarrow (\bar{X}, \Delta_{\bar{X}}) \) be the induced morphism. Schematically:

\[
(\bar{X}, \Delta_{\bar{X}}) \xleftarrow{\tilde{g}} (\bar{Y}, \Delta_{\bar{Y}}) \xrightarrow{\psi} (Y, \Delta_Y)
\]

Note that \( \Delta_{\bar{Y}} \) is defined as the \( \mathbb{Q} \)-divisor on \( \bar{Y} \) that makes both \( \psi \) and so \( \tilde{g} \) crepant and \( \psi_* \Delta_{\bar{Y}} = \Delta_Y \). Also, \( \psi \) is a proper birational morphism whereas \( \tilde{g} \) is a \( G \)-cover; i.e., a finite cover whose extension of function fields is Galois with Galois group \( G \).

Remark 3.2 (On the nature of étaleness for \( G \)-covers). It is well-known to experts that a finite dominant morphism \( g: Y \rightarrow X \) between smooth varieties is étale if and only if it is unramified. In fact, for this to work, we only need \( X \) and \( Y \) to be; respectively, regular and Cohen–Macaulay. Indeed, one uses [Eis95, Corollary 18.17] to conclude that finiteness implies flatness in that case. In particular, if \( g \) is further a \( G \)-cover, then \( g \) is étale if and only if for all \( \mathbb{k} \)-points \( x \in X \) the set-theoretic fiber \( Y_x(\mathbb{k}) \) is a \( G \)-torsor under the induced action. Nevertheleseed, the same principle works with no hypotheses whatsoever on the singularities of \( Y \) and \( X \) except for normality. Indeed:

Claim 3.3. With notation as above, suppose that \( g \) is a \( G \)-cover between normal varieties. If the set-theoretic fiber \( Y_x(\mathbb{k}) \) is a \( G \)-torsor for all \( x \in X(\mathbb{k}) \) then \( g \) is étale.

Proof of claim. We need to prove that \( g_* \mathcal{O}_Y \) is locally free and that \( g \) is unramified, which can be checked at every point \( x \in X(\mathbb{k}) \). Set \( \mathcal{O}_{Y, x} := (g_* \mathcal{O}_Y)_x \). Observe that the generic rank of \( \mathcal{O}_{Y, x} / \mathcal{O}_{X, x} \) is \( [K(Y) / K(X)] = \# G \). By [Har77, II, Lemma 8.9], the local freeness claim would follow if we prove this to be its residual rank as well. To do so, let us pullback \( \mathcal{O}_{Y, x} / \mathcal{O}_{X, x} \) to the completion of \( \mathcal{O}_{X, x} \) to get \( \hat{\mathcal{O}}_{X, x} \rightarrow \hat{\mathcal{O}}_{Y, x} = \bigprod_{y \in Y_x(\mathbb{k})} \hat{\mathcal{O}}_{Y, y} \), where \( \# Y_x(\mathbb{k}) = \# G \).
hypothesis. Now, the generic rank of $\hat{O}_{Y,x}/\hat{O}_{X,x}$ is still $\# G$ as ranks are invariant under completion. Therefore, $\hat{O}_{Y,y}/\hat{O}_{X,x}$ must have generic rank 1. Since these rings are assumed normal, we have $\hat{O}_{Y,y} \simeq \hat{O}_{X,x}$; as required. Along the way, we proved that $\hat{O}_{Y,y}/\hat{O}_{X,x}$ and so $\hat{O}_{Y,y}/\hat{O}_{X,x}$ are unramified. □

We use this freely in what follows. This finishes our remarks.

In proving that there are abundant non-liftable arcs of $X$ across $g: Y \to X$, the following two lemmas are crucial.

**Lemma 3.4** (Reduction to the log smooth case). Work in Setup 3.1 and assume $d \leq 3$. In case $d = 3$ and $0 < p \leq 5$, assume that $G$ is a $p$-group. If $\tilde{g}$ is étale then so is $g$. Equivalently, if $g$ is not étale then neither is $\tilde{g}$.

**Elementary proof of Lemma 3.4 for surfaces.** We present first a proof in the surfaces case and discuss its extension to threefolds in Section 3.2 below. We may assume that $X$ is affine with a $k$-rational isolated singularity $x \in X$; so $Y$ is affine too. Since $X$ has rational singularities, $\text{Exc} \phi$ is a tree of $\mathbb{P}^1$’s (see [EV10, Theorem 4.1] for a characterization of when this is the case). In particular, $\text{Exc} \phi$ is algebraically simply connected which implies that the restriction of $\tilde{g}$ to $\text{Exc} \phi$ must consist of the disjoint union of $\# G$ copies of $\text{Exc} \phi$. However, such disjoint union is the exceptional locus of $\psi$. This means that over $x$ there must lie $\# G$ many points in $Y$. That is, $\# G = \# Y_x(k)$ and so $g$ is étale. □

The following lemma and corollary should be thought of as an strengthening of results of Kato and Kerz–Schmidt; see [Kat18, Lemma 3.5] which is based on [KS10, Lemma 2.4]. Our proofs of these results were inspired by arguments in the recent paper of Nakamura–Shibata [NS22]. In the proof of this lemma, we use the untwisting technique, which we briefly recall next. For details, we refer the reader to [Yas16, §4] or to [Yas19, §6] for the stacky formulation.

Consider a $G$-cover $g: Y \to X$ between normal varieties and let $g_{\mathbb{D}}: Y_{\mathbb{D}} \to X_{\mathbb{D}}$ be its base change to $\mathbb{D}$. By an arc of $X_{\mathbb{D}}$, we mean a section $\mathbb{D} \to X_{\mathbb{D}}$. We denote by $J_{x,\mathbb{D}}$ the space of arcs of $X_{\mathbb{D}}$. We have the obvious identification $J_{x,\mathbb{D}} = J_{x,\mathbb{D}}$. To each $G$-cover $E \to \mathbb{D}$, we can associate a separated and flat $\mathbb{D}$-scheme $Y^{[E]}_{\mathbb{D}}$ of finite type called the untwisting scheme. It comes with a natural $\mathbb{D}$-morphism $g^{[E]}_{\mathbb{D}}: Y^{[E]}_{\mathbb{D}} \to X_{\mathbb{D}}$. The morphisms at the generic fiber

$g^{[E]}_{\mathbb{D},\eta}: Y^{[E]}_{\mathbb{D},\eta} \to X_{\mathbb{D},\eta}$ and $g_{\mathbb{D},\eta}: Y^{[E]}_{\mathbb{D},\eta} \to X_{\mathbb{D},\eta}$

are twisted forms of each other. That is, letting $\bar{\eta}$ denote the geometric generic point of $\mathbb{D}$, there exists an isomorphism $Y^{[E]}_{\mathbb{D},\bar{\eta}} \to Y_{\mathbb{D},\bar{\eta}}$ over $X_{\mathbb{D},\eta}$. It follows that these two morphisms have the same branch locus in $X_{\mathbb{D},\eta}$. It also follows that if $Y/k$ is smooth then $Y^{[E]}_{\mathbb{D},\eta}$ is smooth over $\mathbb{D}^\ast$. The key property of the untwisting scheme is that there is a one-to-one correspondence

$$J^{[E]}_{\mathbb{D}} = Y^{[E]}_{\mathbb{D}} \cong J_{x,\mathbb{D}} / \text{Aut}(E)$$

indeed, let $M$ be a finitely generated module over a local excellent normal domain $(R, m)$ of generic rank $r$. Write a short exact sequence $0 \to R^{\text{nr}} \to M \to T \to 0$ where $T$ is torsion; such a sequence is obtained by choosing a basis of $M_{(0)}$ over $R_{(0)}$ consisting of elements in the image of $M \to M_{(0)}$. Base changing the sequence by the flat extension $R'/R$ and noting that $T \otimes_R R'$ is torsion gives the desired result. Note that $R'$ is necessarily a domain. □

Alternatively, one may use that $\text{Exc} \phi$ is a tree of $\mathbb{P}^1$’s to show that $\tilde{g}$ induces a local isomorphism over the corresponding dual graph, which is contractible and so simply connected.
which is compatible with the maps to $J_\infty X_D$.

Let $(J^\infty E Y_D)^\sharp \subset J^\infty E Y_D$ be the locus of those $E$-twisted $G$-arcs that map some (and hence every) generic point into the étale locus of $g_D: Y_D \to X_D$ and let $(J^\infty Y_D^{|E|})^\sharp \subset J^\infty Y_D^{|E|}$ be the locus of arcs mapping $\eta$ into the étale locus of $g_D^{|E|}: Y_D^{|E|} \to X_D$. Note that both of these loci have complement of measure zero. The above correspondence restricts to:

$$
(J^\infty E Y_D)^\sharp \xleftarrow{1:1} (J^\infty Y_D^{|E|})^\sharp / \aut (E)
$$

and maps from these sets to $J_\infty X_D$ are injective.

**Remark 3.5.** By [Yas19] §6.3 and Remark 14.5, the untwisting scheme $Y_D^{|E|}$ is the irreducible component (with the reduced structure) of the $G$-fixed locus of the Hom scheme $\Hom_D^G(E, Y_D)$ (i.e., the Hom scheme parametrizing $G$-equivariant morphisms $E \to Y_D$) that dominates $D$. In the stacky language, this corresponds to the irreducible component of the Hom stack $\Hom_D^G(E, [Y_D/G])$ of representable morphisms. The generic fiber of the latter Hom stack, which is defined over $D$, is

$$\Hom_D^G(D^*, [Y_D/G]_\eta) = [Y_D/G]_\eta.$$

See [Yas14] Remark 6.23. Thus, in the stacky setting, untwisting leaves the generic fiber unchanged.

**Lemma 3.6.** Let $g: Y \to X$ be a $G$-cover between normal varieties. Suppose that $Y$ is smooth and that $J^\infty E Y_D$ is nonempty. Then, the subset

$$g_D,\infty \left( (J^\infty E Y_D)^\sharp \right) = g_D^{|E|,\infty} \left( (J^\infty Y_D^{|E|})^\sharp \right) \subset J_\infty X_D = J_\infty X$$

is a measurable subset of positive measure.

**Proof.** Since the generic fiber $Y_{D,\eta}$ is smooth, there exists a Néron smoothing $h: W \to Y_D^{|E|}$; see [BLR90] §3.1. We have the following properties: $W$ is smooth over $D$, the induced morphism $W_\eta \to Y_D^{|E|}$ of generic fibers is an isomorphism, and $h_\infty: J_\infty W \to J_\infty Y_D^{|E|}$ is bijective. Since $J_\infty Y_D^{|E|}$ is non-empty, then so is $J_\infty W$. Since $W$ is smooth, the total space $J_\infty W$ is a cylinder of measure $\{ W \times_D E \} \neq 0$. Let $j_h$ be the Jacobian order function of $h$. This function is bounded above as $h$ is an isomorphism on generic fibers. Thus, $J_\infty W = \bigsqcup_{n=0}^\infty j_{h^{-1}}(i)$ for all $n \gg 0$. For some $i$, $\mu_W(j_{h^{-1}}(i)) > 0$. By the change of variables formula [Seb04 Theorem 8.0.5], it follows that

$$\mu_{Y_D^{|E|}}(h_\infty(j_{h^{-1}}(i))) = \mathbb{L}^{-i} \mu_W(j_{h^{-1}}(i)) > 0$$

for any such $i$. This shows that

$$\mu_{Y_D^{|E|}}((J^\infty Y_D^{|E|})^\sharp) = \mu_{Y_D^{|E|}}(J^\infty Y_D^{|E|}) > 0.$$

To deduce the desired positivity from this, we apply a similar argument as above to $g_D^{|E|}: Y_D^{|E|} \to X_D$. However, we need a result on motivic integration in the equivariant situation. In characteristic zero, such a theory was developed in [DL02]. We will use the stacky formulation developed in [Yas19] as an available theory having the necessary generality.
Set $\mathcal{Y} := [Y_D^{[E]}/\text{Aut}(E)]$. The morphism $g_D^{[E]} : Y_D^{[E]} \to X_D$ factors through $\mathcal{Y}$ and hence the map $g_{D,\infty}$ factors as

$$J_\infty Y_D^{[E]} \to J_\infty Y_D^{[E]} / \text{Aut}(E) = J_\infty \mathcal{Y} \to J_\infty X_D.$$ 

The right map is injective outside a subset of measure zero. Further, we have

$$\mu_{\mathcal{Y}}(J_\infty \mathcal{Y}) = \mu_{Y_D^{[E]}}(J_\infty Y_D^{[E]}) / \text{Aut}(E) > 0.$$ 

Let $j$ be the Jacobian order function of $\mathcal{Y} \to X_D$. We have the partition

$$J_\infty \mathcal{Y} = \bigsqcup_{i \in \mathbb{N} \setminus \{\infty\}} j^{-1}(i)$$

by countably many measurable subsets. Since $j^{-1}(\infty)$ has measure zero, there exists $i \neq \infty$ such that $j^{-1}(i)$ has positive measure. For this $i$, from the change of variables ([Yas19 Theorem 10.26 or 11.13]), we have

$$\mu_{X_D}(J_D^{[E]}(j^{-1}(i))) = \mathbb{L}^{-i}\mu_{\mathcal{Y}}(j^{-1}(i)) > 0.$$ 

The subset of the statement has the same measure as $g_{D,\infty}(J_\infty \mathcal{Y})$, which contains $J_D^{[E]}(j^{-1}(i))$. This proves the lemma.

**Corollary 3.7.** Let $G$ be a finite group and $H \subset G$ be a subgroup. Let $g : Y \to X$ be a $G$-cover so that $Y$ is smooth and there is $y \in Y(E)$ being fixed by $H$. Let $E \to D$ be $G$-cover such that $H$ preserves some connected component $E_0$ of $E$. Let $\mathcal{N} \subset J_\infty X$ be the subset of arcs $\gamma : D \to X$ so that: $\delta \mapsto g(y)$, $g$ is étale over $\gamma(\eta)$, and the pullback of $\gamma$ along $g$ induces the given $G$-cover $E \to D$. Then, $\mathcal{N}$ has positive measure.

**Proof.** First of all, note that $\mathcal{N}$ is none other than $g_{E,\infty}(J_\infty^{G}(D_Y^{G}))$. Thus, in order to apply Lemma 3.6, it suffices to show that $J_\infty^{G,E} Y \neq \emptyset$; which we do next. The $D$-scheme $Y_D$ has the $H$-invariant section

$$D \xrightarrow{\cong} \{y\} \times D \subset Y_D.$$ 

The composition $E_0 \to D \to Y_D$ is clearly an $H$-equivariant $D$-morphism. There exists a unique extension of this morphism to a $G$-equivariant morphism $E \to Y_D$, i.e., an $E$-twisted $G$-arc (a corresponding arc $D \to Y_D^{[E]}$, which is unique up to the Aut($E$)-action, corresponds to the “trivial arc” by Nakamura–Shibata; see Remark 3.8 below.)

**Remark 3.8.** Nakamura–Shibata work over $\mathbb{A}^1$, following the setting in [DL02]. In [NS22 Claim 5.2], they use a result of Hacon–McKernan [HM07] about rational chain connectedness, which is based on a result by Graber–Harris–Starr [CHS03], to show that the “trivial arc” deforms. We work instead over the formal disk $D$ and use the Néron smoothening to deform the “trivial arc.”

**Proposition 3.9** (Abundance of non-liftable arcs). Work in [Setup 3.1] with $d \leq 3$. If $d = 3$ and $0 < p \leq 5$, further assume that $G$ is a $p$-group. If $g$ is not étale then there is an arc $\gamma : D \to X$ that does not lift to $Y$ and so $g_\infty : J_\infty Y \to J_\infty X$ is not surjective. Moreover, $J_\infty X \setminus g_\infty(J_\infty Y)$ has positive measure. Similarly, if $G$ fixes $y \in Y(E)$ and $x = g(y)$, then $(J_\infty X)_x \setminus g_\infty((J_\infty Y)_y)$ has positive measure.
Proof. Since $g$ is not étale, neither is $\tilde{g}: \tilde{Y} \to \tilde{X}$ by Lemma 3.4. The map $\phi_\infty: J_\infty \tilde{X} \to J_\infty X$ is almost bijective and the subsets $g_\infty(J_\infty Y)$ and $\tilde{g}_\infty(J_\infty Y)$ correspond to each other through this almost bijection (outside subsets of measure zero). Moreover, the map sends a subset of positive measure of $J_\infty \tilde{X}$ to such of $J_\infty X$. Therefore, it suffices to show the proposition for $\tilde{g}$ in place of $g$. Now, there exists a point $y \in \tilde{Y}(\kappa)$ fixed by a nontrivial cyclic subgroup $1 \neq H \subset G$ of prime order. In particular, there exists a connected $H$-cover $\mathbb{E}_0 \to \mathbb{D}$. Indeed, if the order of $H$ is a prime $\ell \neq p$, we can then take the cover $\Spec \kappa[t^{1/\ell}] \to \Spec \kappa[t]$. If the order is $p$, we can take, for example, the Artin–Schreier extension $\mathbb{E}[t]/(u^p - u - t^{-1})$ of $\kappa_t$ and the corresponding cover of $\mathbb{D}$. We can extend $\mathbb{E}_0 \to \mathbb{D}$ to a $G$-cover $\mathbb{E} \to \mathbb{D}$. We now just apply Corollary 3.7 to the above $y$ and $\mathbb{E}$ to obtain the result. □

Remark 3.10. In proving Proposition 3.9 we did not use Corollary 3.7 in its full power. The authors believe a sharper analysis may give a stronger version of Proposition 3.9 which may help in showing the DCC property by providing a stronger descent. See Question 4.7 below.

3.2. On the proof of Lemma 3.4. We come back now to the proof of Lemma 3.4. We concentrate now on the case of threefolds. The substantial difference is that in the 2-dimensional case we can rely on rationality whereas in the 3-dimensional case only on $W\Theta$-rationality, which is much less elementary than rationality. Recall that KLT threefold singularities are $W\Theta$-rational; see [HW19b, GNT19]. However, these are rational in characteristics $p \geq 7$; see [ABL22, HW19a]. It is then far from obvious how what we wrote above in the proof of Lemma 3.4 can be generalized to threefolds and beyond. The key idea was to exploit the topological simplicity of $\text{Exc} \phi$ granted by the rationality of $X$. Due to the low dimensions involved, the theorem of formal functions can be utilized to prove that $H^0(\tilde{X}_x, \mathcal{O}_{\tilde{X}_x}) = \kappa$, $H^1(\tilde{X}_x, \mathcal{O}_{\tilde{X}_x}) = 0$ from the vanishing $\mathcal{O}_X \cong R\phi_* \mathcal{O}_{\tilde{X}}$ (where “$\cong$” means a quasi-isomorphism in the appropriate derived category); see [Art69]. In particular, $\chi(\tilde{X}_x, \mathcal{O}_{\tilde{X}_x}) = 1$. This can be further exploited to prove that $\text{Exc} \phi = (\tilde{X}_x)_{\text{red}}$ is a tree of $\mathbb{P}^1$’s; see [Lip69, EV10], which is what we had used above. We start by reproving Lemma 3.4 only knowing that $\chi(\tilde{X}_x, \mathcal{O}_{\tilde{X}_x}) = 1$.

Proof of Lemma 3.4. Surface case. Since $\tilde{g}$ is étale, $\psi$ is a resolution of singularities. In particular, $\mathcal{O}_Y \cong R\psi_* \mathcal{O}_Y$ as $Y$ also has rational singularities. Now, let us consider the fiber of (3.1.1) at a point $x \in X(\kappa)$:

\[
\begin{array}{ccc}
\tilde{X}_x & \xleftarrow{\tilde{g}_x} & \tilde{Y}_x \\
\phi_x & \downarrow & \psi_x \\
x & \leftarrow & Y_x
\end{array}
\]

By the Hirzebruch–Riemann–Roch theorem, $\chi(\tilde{Y}_x, \mathcal{O}_{\tilde{Y}_x}) = \#G \cdot \chi(\tilde{X}_x, \mathcal{O}_{\tilde{X}_x}) = \#G$ as $\tilde{g}_x$ is finite étale of degree $\#G$; see [Ful84, Example 18.3.9]. That is the result of computing the Euler characteristic of $\tilde{Y}_x$ along $\tilde{Y}_x \to \tilde{X}_x \to x$. If we compute it along the other leg $\tilde{Y}_x \to Y_x \to x$, we obtain that $\chi(\tilde{Y}_x, \mathcal{O}_{\tilde{Y}_x}) = \dim_k \mathcal{O}_{\tilde{Y}_x}$, which uses the vanishing $\mathcal{O}_Y \cong R\psi_* \mathcal{O}_Y$. In conclusion, $\#G = \dim_k \mathcal{O}_{\tilde{Y}_x}$ and so $g_x, \mathcal{O}_Y$ is locally free; say by [Har77, II, Exercise 5.8.(c)]. Since $g$ is further quasi-étale, we conclude that $g$ is étale by the purity of the branch locus for faithfully flat finite covers [AK70, VI, Theorem 6.8]. □
Next, we remark that the above proof can be carried out without passing to the fibers by means of the Grothendieck–Riemann–Roch theorem (GRR). This has the salient feature of letting us extend our proof to higher dimensions yet still assuming rationality.

Proof of Lemma 3.4: General rational case. Here, we may assume that \( d \) is arbitrary but that \( X \) and \( Y \) have rational singularities (e.g. KLT surface singularities or KLT threefold singularities in characteristic \( p \neq 2, 3, 5 \) \cite{ABL22, HW19a}). As before, we consider the proper morphism \( \gamma : \tilde{Y} \to X \) where \( \phi \circ \tilde{g} = \gamma = g \circ \psi \). We use the rationality of \( X \) to say that \( \phi \mathcal{O}_X = \mathcal{O}_X \). Similarly, since \( \tilde{g} \) is étale and so \( \psi \) is a resolution, we may say \( \psi_! \mathcal{O}_{\tilde{X}} = \mathcal{O}_Y \). Now, being \( g \) and \( \tilde{g} \) affine morphisms, we have \( g_! \mathcal{O}_Y = g_* \mathcal{O}_Y \) and likewise for \( \tilde{g} \).

The point now is that we may compute \( \text{ch}(\gamma! \mathcal{O}_Y) \) in two different ways. First:

\[
\text{ch}(\gamma! \mathcal{O}_Y) = \text{ch}(\phi_* \tilde{g}_* \mathcal{O}_Y) = \phi_* \left( \text{ch}(\tilde{g}_* \mathcal{O}_Y) \cdot \text{td}(\mathcal{T}_\phi) \right) = \phi_* (\tilde{g}_* (\text{td}(\mathcal{T}_{\tilde{g}})) \cdot \text{td}(\mathcal{T}_\phi)),
\]

where we used GRR in the last two equalities. Next, we use that \( \tilde{g} \) is a finite étale map of degree \( \# G \) to say \( \tilde{g}_* (\text{td}(\mathcal{T}_{\tilde{g}})) = \# G \) as \( \mathcal{T}_{\tilde{g}} = 0 \). This let us conclude that:

\[
\text{ch}(\gamma! \mathcal{O}_Y) = \# G \cdot \phi_* (\text{td}(\mathcal{T}_\phi)) = \# G \cdot \text{ch}(\phi_! \mathcal{O}_{\tilde{X}}) = \# G \cdot \text{ch}(\mathcal{O}_X) = \# G,
\]

where the second equality uses GRR. On the other hand, we may compute \( \text{ch}(\gamma! \mathcal{O}_Y) \) as follows:

\[
\text{ch}(\gamma! \mathcal{O}_Y) = \text{ch}(g_* \psi_* \mathcal{O}_{\tilde{Y}}) = \text{ch}(g_* \mathcal{O}_Y).
\]

In this manner, we conclude:

\[
\text{ch}(g_* \mathcal{O}_Y) = \# G.
\]

Next, we recall that Chern characters commute with pullbacks. Then,

\[
\# G = \text{ch} \left( g_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \kappa(x) \right) = \dim_{\kappa(x)} \left( g_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \kappa(x) \right)
\]

for all \( x \in X \). Using \cite{Har77, II, Exercise 5.8.(c)} gives that \( g_* \mathcal{O}_Y \) is locally free, i.e., \( g \) is faithfully flat. We conclude as before that \( g \) is étale by purity \cite{AK70, VI, Theorem 6.8}. \( \Box \)

As mentioned in Section 1 in studying Question 1.1 in dimensions \( \leq 3 \), the most important case is the purely wild one (i.e., when all the Galois groups are \( p \)-groups) as it is the one left unsolved by the minimal model program. Under such hypothesis, it turns out that one can give simpler proofs for Lemma 3.4. Let us start with the rational case.

Proof of Lemma 3.4: Rational and purely wild case. Say \( \# G = p^e \) and both \( X \) and \( Y \) have rational singularities of any dimension. There is short exact sequence \( 0 \to \mathbb{Z}/p \to G \to H \to 1 \) where \( \# H = p^{e-1} \). Thus, by induction on \( e \), we may assume \( G = \mathbb{Z}/p \). Consider the following diagram induced from the Artin–Schreier theory:

\[
\begin{array}{ccc}
H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^1_{\acute{e}t}(\tilde{X}, \mathbb{Z}/p) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \\
\downarrow & & \downarrow \quad \downarrow \\
H^0(X, \mathcal{O}_X) & \longrightarrow & H^1_{\acute{e}t}(X, \mathbb{Z}/p) \longrightarrow H^1(X, \mathcal{O}_X)
\end{array}
\]

Since our problem is local, we may assume that \( X \) is affine and so \( H^1(X, \mathcal{O}_X) = 0 \). Since \( X \) has rational singularities, then \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \). Therefore, the cover \( \tilde{g} : \tilde{Y} \to \tilde{X} \); which can be regarded as an element of the upper middle group \( H^1_{\acute{e}t}(\tilde{X}, \mathbb{Z}/p) \), is realized as the normalized pullback of a \( \mathbb{Z}/p \)-torsor over \( X \). Namely, the original cover \( g : Y \to X \) is étale. \( \Box \)
There is the notion of $W\mathcal{O}$-rationality, which is weaker than rationality. There are KLT singularities which are not rational but $W\mathcal{O}$-rational. Therefore, it is natural to look for generalization of above arguments using $W\mathcal{O}$-cohomology. We invite the reader to consult [PZ21, §3] for an excellent account on Witt vector and $p$-adic étale cohomology as well as $W\mathcal{O}$-rationality. Also see [GNT19, CR12, BBE07, CL04, LS07]. This approach works well at least in the case where the Galois group is a $p$-group, as in the proof below.

**Proof of Lemma 3.4:** $W\mathcal{O}$-rational and purely wild case. Suppose that $\# G$ is a $p$-group, both $X$ and $Y$ have $W\mathcal{O}$-rational rational singularities (e.g. KLT singularities in dimension $\leq 3$ [HW19b, GNT19]), and their dimension $d$ is arbitrary. In particular, $\phi$ is $W\mathcal{O}$-rational: $W\mathcal{O}_{X,Q} \rightarrow \phi_* W\mathcal{O}_{X,Q}$ and $R^i \phi_* W\mathcal{O}_{X,Q} = 0$ for all $i > 0$. According to [PZ21, Lemma 3.19], $\phi$ is further $\mathbb{Q}_p$-rational which means that $\phi_* \mathbb{Q}_p = \mathbb{Q}_p$ and $R^i \phi_* \mathbb{Q}_p = 0$ for all $i > 0$.

Now, let $x \in X(\overline{k})$ be arbitrary and consider the fiber diagram (3.10.1), where $\tilde{X}_x$ is a proper connected $\overline{k}$-scheme. Using proper base change for $p$-adic étale cohomology [PZ21, Proposition 3.16] (c.f. [SGA77, Exposé VI, 2.2.3 B]), we conclude that $H^0_{\et}(\tilde{X}_x, \mathbb{Q}_p) = \mathbb{Q}_p$ and $H^i_{\et}(\tilde{X}_x, \mathbb{Q}_p) = 0$ if $i > 0$. In particular, the $p$-adic Euler–Poincaré characteristic of the fiber is $\chi_{\et}(\tilde{X}_x, \mathbb{Q}_p) = 1$. Now, since $\tilde{g}_x$ is a $G$-torsor with $G$ a $p$-group, we may use Crew’s formula; see [Cre84, CL04, Lemma 8.5], to get that $\chi_{\et}(\tilde{Y}_x, \mathbb{Q}_p) = \# G \cdot \chi_{\et}(\tilde{X}_x, \mathbb{Q}_p) = \# G$.

Next, since $\tilde{g}$ is étale, $\psi$ is a resolution and so it is $\mathbb{Q}_p$-rational. Hence, by proper base change for $p$-adic étale cohomology once again, we get that $\chi_{\et}(\tilde{Y}_x, \mathbb{Q}_p) = H^0(\tilde{Y}_x, \mathbb{Q}_p) = \# Y_x(\overline{k})$.

Putting these two observations together, $\# G = \# Y_x(\overline{k})$ for all $x \in X(\overline{k})$. We finish by using Claim 3.3.

We may wonder whether $W\mathcal{O}$-rationality and/or rationality are necessary for the statement of Lemma 3.4 (and of Proposition 3.9). The following example shows this to be the case.

**Example 3.11 (cf. [NS22, Remark 5.3]).** Let $E$ be an ordinary elliptic curve and $O \neq P \in E$ be an $p$-torsion point. The automorphism of $E$ sending a point $Q$ to $Q + P$ defines an action of $\mathbb{Z}/p\mathbb{Z}$ on $E$. This action lifts to the one on the ample invertible sheaf $\mathcal{O}_E(P_0 + \cdots + P_{p-1})$ with $P_i$ the $p$-torsion points. We also have natural actions on the affine cone $C$ and the standard resolution $\tilde{C}$ over $E$ associated to this invertible sheaf; the latter is a line bundle over $E$. The action on $C$ has the unique fixed point; namely the vertex of the cone, whereas the action on $\tilde{C}$ is free. The Galois cover $C \rightarrow C/G$ with $G = \mathbb{Z}/p\mathbb{Z}$ ramified while the Galois cover $\tilde{C} \rightarrow \tilde{C}/G$ is étale. This shows that every arc on $C/G$ lifts to $C$ unless it maps $\eta \in \mathbb{D}$ to the branch point.

3.3. Main result. We are now ready to establish our main result:

**Theorem 3.12 (Strict descent).** Work in Setup 2.8. Suppose that $(X_0, \Delta_0)$ is KLT of dimension $d \leq 3$. Assume that $g_1$ and $g_2$ are quasi-étale. In the case $d = 3$ and $0 < p \leq 5$, further assume that $G_1$ and $G_2$ are $p$-groups. If $f$ is not étale, then:

$$M_{st}(X_1, \Delta_1)/G_1 > M_{st}(X_2, \Delta_2)/G_2.$$  

Similarly, if $x_i \in X_i$ is the preimage of $x_0$ then

$$M_{st}(X_1, \Delta_1)_{x_1}/G_1 > M_{st}(X_2, \Delta_2)_{x_2}/G_2.$$  

**Proof.** Put together Corollary 2.10 and Proposition 3.9. 

□
4. DCC for Stringy Motives of Log Terminal Surface Singularities

In this section, we discuss the Descending Chain Condition (DCC) for stringy motives of log terminal surface singularities. We work in the setup of Section 2.3. We commence by using the formula (2.14.1) to see that, as a Laurent power series in $L^{-1/r}$, the invariant $M_{st}(X)_{x}/G$ can be expressed as:

$$
\sum_{i \in I/G} (\mathbb{L} + m_i)(\mathbb{L} - 1)(\mathbb{L}^{-a_i} + \mathbb{L}^{-2a_i} + \ldots)
+ \sum_{[\iota, \kappa] \in H/G} (\mathbb{L} - 1)(\mathbb{L}^{-a_i} + \mathbb{L}^{-2a_i} + \ldots)(\mathbb{L}^{-a_\kappa} + \mathbb{L}^{-2a_\kappa} + \ldots),
$$

which has degree $2 - \min\{a_i\}$, where we set $\deg \mathbb{L}^{-1/r} = 1/r$. Modulo terms of degrees $< 1$, the above series becomes

$$(4.0.1) \sum_{i \in I/G} \mathbb{L}^2(\mathbb{L}^{-a_i} + \mathbb{L}^{-2a_i} + \ldots) + \sum_{[\iota, \kappa] \in H/G} \mathbb{L}^2(\mathbb{L}^{-a_\iota} + \mathbb{L}^{-2a_\iota} + \ldots)(\mathbb{L}^{-a_\kappa} + \mathbb{L}^{-2a_\kappa} + \ldots).$$

This shows that $M_{st}(X)_{x}/G$ has nonnegative coefficients for the terms $\mathbb{L}^b$ with $1 \leq b < 2$, which motivates the following definition.

**Definition 4.1.** With notation as above, we define $N(x, X, G) \in \mathbb{N}[\mathbb{L}^{-1/r}]$ to be $M_{st}(X)_{x}/G$ with all the terms of degree $< 1$ removed. We define $C(x, X, G) \in \mathbb{N}$ to be the sum of the coefficients of $N(x, X, G)$.

**Remark 4.2.** Clearly, if $M_{st}(X)_{x}/G \geq M_{st}(X')_{x'}/G'$ then $N(x, X, G) \geq N(x', X', G')$.

Note that the expression (4.0.1) shows that each vertex of $\Gamma/G$ with log discrepancy $\leq 1$ contributes at least 1 to $C(x, X, G)$. From Lemma 2.14 we then obtain:

**Lemma 4.3.** The graph $\Gamma/G$ has at most $C(x, X, G)$ non-special vertices (i.e., the vertices coming from the minimal resolution) and at most $C(x, X, G) + 1$ vertices.

**Definition 4.4.** Let $r$ be a positive integer. We define $\mathcal{A}_r^2 \subset \mathbb{Z}[\mathbb{L}^{-1/r}]$ to be the subset of elements $M_{st}(X)_{x}/G$ such that $X$ is a log terminal surface with an action of a finite group $G$ and so that $rK_X$ is Cartier and the $G$-action fixes a point $x \in X$.

**Lemma 4.5.** Fix a positive integer $r$ and a polynomial $N \in \mathbb{N}[\mathbb{L}^{-1/r}]$. Then, there exist only finitely many elements $\alpha \in \mathcal{A}_r^2$ such that $\alpha = M_{st}(X)_{x}/G$ and $N(x, X, G) = N$.

**Proof.** The invariant $M_{st}(X)/G$ is determined by the data of the dual graph $\Gamma/G$ associated to the modified minimal resolution of $X$ and the log discrepancies $a_i$ assigned to their vertices. Since $N(x, X, G)$ determines $C(x, X, G)$, the number of vertices of $\Gamma/G$ is bounded by Lemma 4.3. Thus, there are only finitely many possibilities for the graph $\Gamma/G$. For each possibility, there are only finitely many ways to assign numbers $a_i$ to vertices as these numbers must belong to the finite set $\frac{1}{r}\mathbb{Z} \cap (0, 2]$. The result then follows.

**Proposition 4.6** (DCC for surface stringy motives). Fix a positive integer $r$. Then, $\mathcal{A}_r^2$ satisfies DCC: every descending chain

$$M_{st}(X_0)_{x_0}/G_0 \geq M_{st}(X_1)_{x_1}/G_1 \geq M_{st}(X_2)_{x_2}/G_2 \geq \cdots$$

of elements in $\mathcal{A}_r^2$ eventually stabilizes.
Proof. According to Remark 4.2, from the chain in the statement, we obtain the following decreasing chain

\[ N(x_0, X_0, G_0) \geq N(x_1, X_1, G_1) \geq N(x_2, X_2, G_2) \geq \cdots \]

of polynomials in \( \mathbb{N}[\mathbb{L}^{1/r}] \). Therefore, it must stabilize. By Lemma 4.5 the chain of the statement stabilizes as well. \( \square \)

**Question 4.7** (DCC for threefold stringy motives). Fix a positive integer \( r \). Let \( \mathcal{A}_r^3 \subset \mathcal{M}_{k,r}^l \) be the subset of stringy motives \( M_{st}(X, \Delta)_x/G \) where \( (X, \Delta) \) is a KLT 3-dimensional log pair so that \( r(K_X + \Delta) \) is Cartier and \( G \) acts on \( X \) fixing the boundary \( \Delta \) as well as \( x \in X \). Does \( \mathcal{A}_r^3 \) satisfy the DCC? If \( 0 < p \leq 5 \), does DCC hold when we consider \( p \)-groups only? See Remark 3.10.

5. Applications to Étale Fundamental Groups

We finish by establishing some consequences of the above results. The main consequence is that [Question 1.1] has an affirmative answer if \( (X, \Delta) \) is a KLT surface. Indeed:

**Theorem 5.1.** Let

\[ X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots \]

be a tower of Galois quasi-étale covers of log terminal surfaces over an algebraically closed field. Then, \( f_i \) is étale for all \( i \gg 0 \).

**Proof.** Suppose for the sake of contradiction that \( f_i \) is not étale for infinitely many \( i \). Let \( G_i := \text{Gal}(X_i/X) \). Since each \( f_i \) is étale over the regular locus of \( X_0 \), which has only finitely many singular points, there exists a singular point \( x_0 \in X_0 \) such that for infinitely many \( i \), \( f_i \) is not étale over \( x_0 \). Shrinking \( X_0 \) around this point, we may suppose that \( X_0 \) has the unique singular point \( x_0 \). Note that, if \( r \) is the Gorenstein index of \( X \), then \( r \) is a multiple of the Gorenstein index of every \( X_i \). By Theorem 3.12 we get a decreasing sequence

\[ M_{st}(X_0)/G_0 \geq M_{st}(X_1)/G_1 \geq M_{st}(X_2)/G_2 \geq \cdots \]

such that infinitely many inequalities are strict. For each \( i \), let us choose a point \( x_i \in X_i \) lying over \( x_0 \). Then,

\[ M_{st}(X_i)/G_i = \{X_0 \setminus \{x_0\}\} + M_{st}(X_i)_{x_i}/\text{Stab}(x_i) \]

by using (2.10.2). Hence, by subtracting \( \{X_0 \setminus \{x_0\}\} \) from the above sequence, we obtain a non-terminating decreasing sequence

\[ M_{st}(X_0)_{x_0}/\text{Stab}(x_0) \geq M_{st}(X_1)_{x_1}/\text{Stab}(x_1) \geq M_{st}(X_2)_{x_2}/\text{Stab}(x_2) \geq \cdots \]

of elements in the set \( \mathcal{A}_r^2 \subset \mathbb{Z}[\mathbb{L}^{-1/r}] \) (which was defined in Section 4). This contradicts Proposition 4.6. \( \square \)

**Remark 5.2.** A little more effort enables us to prove the above theorem by the DCC of the sequence \( M_{st}(X_i)/\text{Stab}(x_i) \) without shrinking \( X_0 \) (but still reducing to the DCC of the “local” stringy motives \( M_{st}(X_i)_{x_i}/\text{Stab}(x_i) \)). This approach would be a more faithful practice of the strategy explained in Section 1.

A similar (but simpler) argument shows the following; see Corollary 6.4 and its proof.

**Theorem 5.3.** The local étale fundamental group of a log terminal surface germ is finite.

As an immediate consequence, we obtain:
Corollary 5.4. Big opens of weak del Pezzo surfaces have finite étale fundamental group.

Proof. See [CSK20, Theorem 2.6] and its proof. □

Remark 5.5. To the best of the author’s knowledge, the results of this section were unknown at least in characteristics 2 and 3. In characteristic $p > 5$, one may use that log terminal surfaces are strongly $F$-regular and use [CRST18, BCRG+19]. In characteristic $p = 5$, one may use that the canonical cover of log terminal surface singularities is a rational point; see [Kaw99, Ari04], and then use Artin’s explicit description of them in [Art77] to conclude. In characteristics $p = 2, 3$, we know the canonical cover trick fails by [Kaw99] hence a new, different approach was needed. Our methods addressed these cases with the salient feature of providing an unified approach which is conceptual and independent of the characteristic of the groundfield. For instance, it gives a conceptual proof for the finiteness of the étale local fundamental group of rational double points in the low characteristics, which Artin had accomplished only on a case-by-case classification analysis [Art77].

6. Mixed Characteristic Surface Case

In this section, we aim to establish Theorem 5.3 in the mixed characteristic case as well. We may use the exact same proof as long as we establish Theorem 3.12 in mixed characteristics. The first problem we face with this is that there is no suitable theory of motivic integration in mixed characteristic; see [Yas21, Problem 10.1]. Therefore, we cannot see the strict decent of stringy motives formally as the non-liftability of arcs as we did in Section 3. However, the surface case is simple enough for this descent to be seen by direct analysis, which is what we do in this section.

Let $(R, m, k, K)$ be a 2-dimensional complete normal integral domain with algebraically closed residue field $k$ and field of fractions $K$. Suppose that it has mixed characteristic $(\text{char } K = 0, \text{char } k = p > 0)$. In that case, $R$ admits a canonical module which is a reflexive module of rank 1 and unique up to isomorphism. For instance, we may use that $R$ is a finite extension over $A = \Lambda[t]$ where $(\Lambda, (p), k)$ is a complete DVR (see [The21, Tag 032D]) to set $\omega_R := \omega_{R/A} := \text{Hom}_A(R, A)$. We let $K_X$ denote a corresponding canonical divisor on $X := \text{Spec } R$. The choice of $K_X$ is unique up to linear equivalence and so the canonical class $K_X \in \text{Cl } X$ is well-defined. Further, we write $x := \text{Spec } k$ and identify it with the closed point of $X$.

We will assume that $X$ is $Q$-Gorenstein, i.e. $K_X \in \text{Cl } X$ is torsion, and denote by $r \geq 1$ the index of $K_X$ in $\text{Cl } X$. In particular, we may define log discrepancies with respect to a minimal resolution and define $(x, X)$ as log terminal if these are positive. In doing this, we are setting our base to be $A = \Lambda[t]$; with notation as in the previous paragraph.

Remark 6.1. Sometimes one considers a relative canonical module $\omega_{R/B}$ with respect to a different base $B$. For example, one may consider a scheme $Y$ of finite type over a complete DVR $B$ with an algebraically closed reside field and suppose that $R$ is the complete local ring $\mathcal{O}_{Y,y}$ at a closed point $y$. Then, the relative canonical module is defined as $\omega_{R/B} = \omega_{Y/B} \otimes_{\mathcal{O}_{Y,y}}$. However, whether we use $\omega_R$ or $\omega_{R/B}$ does not make any change to the following argument; for we get the same discrepancies in either way. See “Comments” on pages 7 and 8 of [Kol13].

In what follows, all canonical divisors are defined with respect to $A$ as above. However, as in [Kol13], we will omit the base in our notation for canonical divisors.
Suppose that \((x, X)\) is a log terminal singularity and let \(\phi: \tilde{X} \to (x, X)\) be the minimal resolution. Then, we define \(M_{st}(X)_x\) by the explicit formula in Example 2.6. Namely,

\[
M_{st}(X)_x := \sum_{i \in I} (\mathbb{L} + 1 - m_i) \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1} + \sum_{[i,j] \in H} \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_i} - 1)(\mathbb{L}^{a_j} - 1)} \in \mathbb{Z}[\mathbb{L}^{-1/r}] \subset \hat{\mathcal{M}}_{\mathbb{K}, r},
\]

where we use the exact same notation as in Example 2.6. Recall that \(a_i \in \frac{1}{r}\mathbb{Z} \cap (0, 1]\) as \((x, X)\) is log terminal with Gorenstein index \(r\).

Likewise, if \(G\) is a finite (discrete) group acting on \(X\) and fixing \(x\), we may further define the quotient stringy invariant \(M_{st}(X)_x/G\) using the modified minimal log resolution as explained in Section 2.3. That is, we take \((2.14.1)\) as the definition of \(M_{st}(X)_x/G\) in mixed characteristics:

\[
M_{st}(X)_x/G := \sum_{i \in I/G} (\mathbb{L} + 1 - m_i) \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1} + \sum_{[i,j] \in H/G} \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_i} - 1)(\mathbb{L}^{a_j} - 1)} \in \mathbb{Z}[\mathbb{L}^{-1/r}] \subset \hat{\mathcal{M}}_{\mathbb{K}, r}.
\]

For the sake of concreteness, we have used minimal resolutions and minimal \(G\)-normal resolutions to define the stringy motives above. However, using strong factorization, we may take any log resolution as the following lemma makes precise.

**Lemma 6.2.** With notation as above, let \(\psi: Y \to (x, X)\) be a log resolution and \(\Gamma\) be the corresponding dual graph with set of vertices \(I\) and set of edges \(H\). Further, assume that \(E_i\) has log discrepancy \(a_i\) for all \(i \in I\). Then,

\[
M_{st}(X)_x = \sum_{i \in I} (\mathbb{L} + 1 - m_i) \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1} + \sum_{[i,j] \in H} \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_i} - 1)(\mathbb{L}^{a_j} - 1)},
\]

where \(m_i\) is the number of edges of \(\Gamma\) sticking out of \(i\). Suppose further that \(\psi\) is a \(G\)-normal log resolution, then

\[
M_{st}(X)_x/G = \sum_{i \in I/G} (\mathbb{L} + 1 - m_i) \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1} + \sum_{[i,j] \in H/G} \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_i} - 1)(\mathbb{L}^{a_j} - 1)},
\]

where \(m_i\) is the number of edges of \(\Gamma/G\) sticking out of \(i\) and \(a_i = a_i + a_j\) for all \(i \in I\).

**Proof.** Any two log resolutions are related by a sequence of blowups at a point (strong factorization). Thus, it suffices to consider \(\theta: Y' \to Y\) the blowup of \(Y\) at an edge of \(\Gamma\) and prove that our formula remains invariant when computed using the log resolution \(Y' \to Y \to (x, X)\). Note that the dual graph of \(Y'\), say \(\Gamma'\), is obtained from \(\Gamma\) by replacing the edge being blown up, say \([i, j]\), by \([i, k] \cup [k, j]\) where \(k\) is the new vertex of \(\Gamma'\) corresponding to the exceptional divisor of \(Y' \to Y\), whose log discrepancy over \(X\) is \(a_k = a_i + a_j\). All other discrepancies remain unchanged. In particular, it suffices to prove that

\[
(\mathbb{L} - 1) \frac{\mathbb{L} - 1}{\mathbb{L}^{a_k} - 1} + \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_i} - 1)(\mathbb{L}^{a_k} - 1)} + \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_k} - 1)(\mathbb{L}^{a_j} - 1)} = \frac{(\mathbb{L} - 1)^2}{(\mathbb{L}^{a_i} - 1)(\mathbb{L}^{a_j} - 1)},
\]

which is a straightforward consequence of the equality \(a_k = a_i + a_j\). \(\square\)

We aim to prove the following.
Theorem 6.3. Let \((x, X)\) be a log terminal surface singularity as above and \(G\) be a nontrivial finite group acting on \(X\) and fixing \(x\). Let \(\bar{x}\) be the image of \(x\) on the quotient \(X/G\). Assume that the quotient morphism \(f: (x, X) \to (\bar{x}, X/G)\) is étale away from \(\bar{x}\). Then,

\[ M_{\text{st}}(X/G)_{\bar{x}} > M_{\text{st}}(X)_x/G, \]

with respect to the lexicographic order in \(\mathbb{Z}[\mathbb{L}^{-1/r}]\).

As a corollary, we obtain:

Corollary 6.4. Let \((x, X)\) be a log terminal surface singularity as above. Then, the étale fundamental group \(\pi_{\text{\acute{e}t}}(X \setminus \{x\})\) is finite.

Proof. The same arguments in Section 4 shows that the set \(\mathcal{A}^2_{k,r} \subset \mathcal{M}_{k,r}\) of stringy motives \(M_{\text{st}}(X)_x/G\) such that \((X, x)\) is a log terminal singularity with \(rK_X = 0 \in \text{Cl} X\) and admitting an action of a finite group \(G\) fixing \(x = \text{Spec} k\) satisfies a descending chain condition (DCC).

If \(\pi_{\text{\acute{e}t}}(X \setminus \{x\})\) were not finite, then there would be an infinite chain of finite Galois local covers \((x, X) \leftarrow (x_1, X_1) \leftarrow (x_2, X_2) \leftarrow \cdots\) such that \(X_{i+1} \setminus \{x_{i+1}\} \to X_i \setminus \{x_i\}\) is étale but \(X_{i+1} \to X_i\) is not étale. In particular, the Galois group \(1 \neq \text{Gal}(X_i/X)\) acts on \(X_i\) fixing \(x_i\) and the quotient is \((x, X)\). Further, every \((x_i, X_i)\) is a log terminal surface singularity with \(x_i = \text{Spec} k\) and such that \(rK_{X_i} = 0 \in \text{Cl} X_i\). Then, by applying Theorem 6.3 we obtain an strictly ascending chain

\[ M_{\text{st}}(X)_x > M_{\text{st}}(X_1)_{x_1}/G_1 > M_{\text{st}}(X_2)_{x_2}/G_2 > \cdots \]

violating the DCC on \(\mathcal{A}^2_{k,r}\). \(\square\)

The rest of this section is dedicated to the proof of Theorem 6.3. Since we cannot use motivic integration to compare \(M_{\text{st}}(X/G)_{\bar{x}}\) and \(M_{\text{st}}(X)_x/G\), we have to compare the explicit formulas defining them. However, the main difficulty in making this comparison is that the \(G\)-quotient of a \(G\)-normal log resolution \(\phi: \tilde{X} \to (x, X)\) is not a log resolution of \(X/G\). Our strategy to bypass this issue is to compare stringy motives along some constructible subsets and then take limits. To this end, we shall need some auxiliary stringy motives along closed subsets. First, we explain the setup in which discrepancies can be compared directly.

6.1. Comparing discrepancies. Consider the following commutative square between normal integral schemes

\[
\begin{array}{c}
\tilde{W} \\
\phi \downarrow \\
(w, W) \leftarrow (x, X)
\end{array}
\]

\[
\begin{array}{c}
\tilde{X} \\
\psi \downarrow \\
(x, X)
\end{array}
\]

where \((w, W)\) is a log terminal surface singularity, \(f\) is a quasi-étale cover of surface singularity germs, and the vertical arrows are proper birational morphisms. When \(f\) is a Galois cover, we also assume that the Galois action on \(X\) lifts to \(\tilde{X}\). Let \(F \subset \tilde{X}\) be a prime divisor over \(X\) and let \(E \subset \tilde{W}\) be its image. Suppose that these divisors have (non-log) discrepancies \(a_F\) and \(a_E\) over \(X\) and \(W\); respectively.

Proposition 6.5. With notation as above, suppose that either

(a) the free \(\hat{\mathcal{O}}_{W,\eta_E}\)-module \(\hat{\mathcal{O}}_{\tilde{X},\eta_F}\) has prime-to-\(p\) rank, or
(b) \(f\) is a Galois cover.
Then, $a_F \geq a_E$. Moreover, if $\tilde{f}$ is ramified along $F$, then $a_F > a_E$.

Proof. Concerning divisors appearing in this proof, we are mainly interested in coefficients of $F$ and $E$ so that we will omit other prime divisors by writing “…”.

We can write

$$K_{\tilde{X}} = \psi^* K_X + a_F F + \cdots$$

and

$$K_{\tilde{W}} = \phi^* K_W + a_E E + \cdots.$$  

Let $e$ be the ramification index at $F$. Namely, the uniformizer of $\hat{O}_{\tilde{X,F}}$ has order $e$ with respect to the normalized valuation of $\hat{O}_{\tilde{X,F}}$. Let $\delta$ denote the different at $F$, which is defined to be the length of the $\hat{O}_{\tilde{X,F}}$-module $\hat{\Omega}_{\tilde{X}/\hat{W},F}$. Since $f$ is quasi-étale, pulling-back the second equality gives

$$K_{\tilde{X}} = \tilde{f}^* K_{\tilde{W}} + \delta F + \cdots = \psi^* K_X + (ea_E + \delta)F + \cdots.$$  

Comparing the coefficients of $F$, we get

$$a_F = ea_E + \delta.$$  

Proof of case (a). In this tame case, it is well-known that $\delta = e - 1$. It follows that

$$a_F + 1 = e(a_E + 1) \geq a_E + 1.$$  

Moreover, if $\tilde{f}$ is ramified along $F$, then $e > 1$ and hence $a_F + 1 > a_E + 1$. The assertion follows.  

Proof of case (b). From Hyodo’s formula [Hyo87, (1-4)], we have

$$\delta = e - 1 + d_L(M/L),$$  

where $M$ and $L$ are the fraction fields of $\hat{O}_{\tilde{X,\eta_F}}$ and $\hat{O}_{\tilde{W,\eta_E}}$, and $d_L(M/L) \in \mathbb{N}$ is the depth of ramification of the extension $M/L$. We have that $d_L(M/L) = 0$ if and only if $\tilde{f}$ is tamely ramified along $F$. Let $G = \text{Gal}(L/K)$ be the Galois group.

The case $G \cong \mathbb{Z}/p$: Let $k_F$ and $k_E$ be the residue fields at $F$ and $E$ respectively. This case is further divided into three subcases according to ramification type: unramified ($e = 1$ and $k_F/k_E$ is separable), wildly ramified ($e = p$ and $[k_F : k_E] = 1$), and fiercely ramified ($e = 1$ and $k_F/k_E$ is inseparable). The unramified case is contained in the tamely ramified case (a). In the other two cases, from [XZ14, Section 2.1], $\delta \geq p - 1$. In the wildly ramified case,

$$a_F = pa_E + \delta \geq p(a_E + 1) - 1,$$

and

$$a_F + 1 > a_E + 1,$$

as $a_E + 1 > 0$ from the log terminal condition. In the fiercely ramified case,

$$a_F - a_E = \delta = d_L(M/L) > 0.$$  

Thus we get the desired assertion in every subcase.

The case $G$ is a p-group: Suppose that $G$ has order $p^b$ with $b > 0$. Let $C \subset G$ be a central subgroup of order $p$. It is well-known that every $p$-group admits such a subgroup. Suppose that the assertion holds when the Galois group is a $p$-group of order $\leq p^{b-1}$. Let $F'$ be the image of $F$ in $\tilde{X}/C$. From the degree $p$ case and the induction hypothesis, we get

$$a_E \leq a_F' \leq a_F.$$  

If $\tilde{X} \to \tilde{W}$ is ramified along $F$, then either $\tilde{X} \to \tilde{X}/C$ is ramified along $F$ or $\tilde{X}/C \to \tilde{W}$ is ramified along $F'$. Thus, at least one of the above intermediate inequalities is strict, and hence $a_E < a_F$, if $\tilde{X} \to \tilde{W}$ is ramified along $F$. We have proved the assertion in the case of $p$-groups.

The general Galois case: Let $H \subset G$ be the stabilizer of $F$ and $S \subset H$ be a Sylow $p$-subgroup. Let $F'$ be the image of $F$ in $\tilde{X}/S$. From the $p$-group case and the tame (non-Galois) case, we get

$$a_E \leq a'_F \leq a_F.$$

In the ramified case, we get the strict inequality $a_E < a_F$ as in the last case. We have completed the proof of (b). $\square$

This concludes the proof of Proposition 6.5. $\square$

6.2. Tweak by constructible subsets. In this subsection, we define a version of stringy motives, denoted by $M_{\text{st}}(X)_{x,C}/G$, in the following situation. We consider a $G$-equivariant proper birational morphism $\phi: \tilde{X} \to (x, X)$ and a $G$-invariant constructible subset $C \subset \phi^{-1}(x)$ such that $(\tilde{X}, \phi^{-1}(x))$ is a $G$-normal SNC pair in a neighborhood of $C$. We write $\phi^{-1}(x) = \bigcup_{i \in I} E_i$ as usual, where $E_i$ are prime divisors. We define:

$$M_{\text{st}}(X)_{x,C}/G := \sum_{i \in I} \left\{ C \cap \left( E_i \setminus \bigcup_{j \neq i} E_j \right) \right\} G \frac{L - 1}{1}$$

$$+ \sum_{\{i,j\} \subset I} \left\{ (C \cap E_i \cap E_j)/G \right\} \frac{(L - 1)^2}{(L^{a_i} - 1)(L^{a_j} - 1)}$$

(6.5.1)

Note that curves $E_i$ as well as their quotients by finite group actions are all rational. Thus, their classes in our Grothendieck ring are of the form $L^n$ with $n$ an integer. Therefore, the above invariant belongs to $\mathbb{Z}[L^{-1/r}]$ with $r$ the Gorenstein index of $X$ or any multiple of it. It is worth noting that if $\phi$ is a $G$-normal log resolution, then

$$M_{\text{st}}(X)_{x}/G = M_{\text{st}}(X)_{x,\phi^{-1}(x)}/G.$$ 

As a special case, we define $M_{\text{st}}(X)_{x,C} := M_{\text{st}}(X)_{x,C}/\{e\}$ where $\{e\}$ is the trivial group. Thus, we may readily generalize Lemma 6.2 as follows:

Lemma 6.6. Consider the following commutative diagram of $G$-equivariant proper birational morphisms:

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\psi} & \tilde{X}' \\
\phi \downarrow & & \phi' \\
(x, X) & & (x, X')
\end{array}$$

Let $C \subset \phi^{-1}(x)$ be a $G$-invariant constructible subset. Suppose that $(\tilde{X}, \phi^{-1}(x))$ is a $G$-normal SNC pair in a neighborhood of $C$ and $(\tilde{X}', \phi'^{-1}(x))$ is a $G$-normal SNC pair in a neighborhood of $\psi^{-1}(C)$. Then, the following equality holds:

$$M_{\text{st}}(X)_{x,C}/G = M_{\text{st}}(X)_{x,\psi^{-1}(C)}/G.$$ 

The following lemma is a direct consequence of the definition:
Lemma 6.7. Let $\phi: \tilde{X} \to (x, G)$ be a $G$-equivariant proper birational morphism and let $C_\lambda \subset \phi^{-1}(x)$ be pairwise disjoint $G$-invariant constructible subsets. Suppose that $(\tilde{X}, \phi^{-1}(x))$ is a $G$-normal SNC pair in a neighborhood of $\bigcup C_\lambda$. Then

$$M_{st}(X)_{x, \bigcup C_\lambda}/G = \sum \lambda M_{st}(X)_{x, C_\lambda}/G.$$  

6.3. Approximation by constructible subsets. The following construction will let us approximate $M_{st}(X/G)_{x}$ and $M_{st}(X)_{x}/G$ by stringy motives of the form $M_{st}(X/G)_{x, \overline{\lambda}}$ and $M_{st}(X)_{x, C}/G$ where the inequality $M_{st}(X/G)_{x, \overline{\lambda}} \geq M_{st}(X)_{x, C}/G$ holds. We first construct the following diagram in a way explained below:

$$\begin{array}{cccc}
X = X_0 & \cong & \cdots & \cong X_i \\
| & | & | & | \\
X/G = X_0/G & \cong & \cdots & \cong X_i/G \\
\end{array}$$

Suppose that we have constructed up to $X_i$ and $X_i/G$. We then take a log resolution $Y_i/G \to X_i/G$ and define $Y_i$ to be the normalization of the fiber product $X_i \times_{X_i/G} Y_i/G$ so that $Y_i/G$ is the quotient of $Y_i$ by the induced $G$-action as the notation suggests. Then, we take a $G$-normal log resolution $X_{i+1} \to Y_i$ and define $X_{i+1}/G$ to be its quotient. We suppose that the log resolutions $Y_i/G \to X_i/G$ and $X_{i+1} \to Y_i$ are isomorphisms over the SNC loci of the targets paired with the exceptional loci of $X_i/G \to X/G$ and $Y_i \to X$; respectively. Repeating this procedure produces a diagram as above.

Let $E_i \subset X_i$ be the exceptional locus of $X_i \to X$ and let $E_i \subset X_i/G$ be its image. Let $\tilde{E}_i \subset \tilde{E}_i$ be the largest open subset along which the pair $(X_i/G, E_i)$ is SNC. Let $C_i$ be the preimage of $\tilde{C}_i$ in $X_i$, which is an open subset of $E_i$. From the construction, $X_{i+1} \to X_i$ is an isomorphism over $C_i$.

Lemma 6.8. With the above notation, the following statements hold:

(a) $M_{st}(X/G)_{x, \tilde{C}_i} \geq M_{st}(X)_{x, C_i}/G$

(b) $M_{st}(X/G)_{x} = \lim_{i \to \infty} M_{st}(X(G))_{x, \tilde{C}_i}$

(c) $M_{st}(X)_{x}/G = \lim_{i \to \infty} M_{st}(X)_{x, C_i}/G$

(d) $M_{st}(X/G)_{x} \geq M_{st}(X)_{x}$

(e) If $X_i \to X_i/G$ is ramified along a prime divisor $E \subset X_i$ and $C_i$ contains its generic point, then $M_{st}(X(G))_{x, \tilde{C}_i} > M_{st}(X)_{x, C_i}/G$.

Proof. The statement (a) follows from the defining formula (6.5.1) of a stringy motive and the comparison of discrepancies in Proposition 6.5.

For (b), note that the preimage of $\tilde{C}_i$ in $X_{i+1}/G$ is contained in $\tilde{C}_{i+1}$. This shows that

$$M_{st}(X(G))_{x, \tilde{C}_{i+1}} \geq M_{st}(X(G))_{x, \tilde{C}_i}.$$  

Let $D \subset Y_i/G$ be the preimage of $\tilde{E}_i \setminus \tilde{C}_{i+1}$, which has pure dimension 1. We have

$$M_{st}(X(G))_{x} - M_{st}(X(G))_{x, \tilde{C}_i} = M_{st}(X(G))_{x, D}.$$  

Let $D^\circ$ be the open dense subset of $D$ obtained by removing the image of $\tilde{E}_{i+1} \setminus \tilde{C}_{i+1}$ from $D$. Since $\tilde{C}_i$ and $D^\circ$ are disjoint and the preimage of their union in $X_{i+1}/G$ is contained in $\tilde{C}_{i+1}$, we have

$$M_{st}(X(G))_{x, \tilde{C}_{i+1}} \geq M_{st}(X(G))_{x, \tilde{C}_i} + M_{st}(X(G))_{x, D^\circ}.$$  

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Since the preimage of $D \setminus D^o$ in $X_{i+1}/G$ contains $E_{i+1} \setminus C_{i+1}$, we have
\[
M_{st}(X/G)_{\bar{x}} - M_{st}(X/G)_{\bar{x},C_{i+1}} = M_{st}(X/G)_{\bar{x},E_{i+1}\setminus C_{i+1}} \leq M_{st}(X/G)_{\bar{x},D\setminus D^o}.
\]
To show (b), it suffices to show
\[
(6.8.1) \quad \dim (M_{st}(X/G)_{\bar{x}} - M_{st}(X/G)_{\bar{x},\bar{C}_i}) < \dim (M_{st}(X/G)_{\bar{x}} - M_{st}(X/G)_{\bar{x},\bar{C}_i}).
\]
Note that for every $y \in D \setminus D^o$,
\[
(6.8.2) \quad \dim M_{st}(X/G)_{\bar{x},y} < \dim M_{st}(X/G)_{\bar{x},D}.
\]
If $y$ is a smooth point of $E_{i+1}$ and if $a$ denotes the log discrepancy at the prime divisor containing $y$, then for some integer $n$, we have
\[
\dim M_{st}(X/G)_{\bar{x},y} = \dim \frac{L - 1}{L^a - 1} < \dim (L + n) \frac{L - 1}{L^a - 1} \leq \dim M_{st}(X/G)_{\bar{x},D}.
\]
If $y$ is a node of $E_{i+1}$ and if $a$ and $b$ are the log discrepancies at the two prime divisors containing $y$, then for some integer $n$, we have
\[
\dim M_{st}(X/G)_{\bar{x},y} = \dim \frac{(L - 1)^2}{(L^a - 1)(L^b - 1)} < \dim (L + n) \frac{L - 1}{L^a - 1} \leq \dim M_{st}(X/G)_{\bar{x},D}.
\]
Thus, in either case, $(6.8.2)$ is valid, and $(6.8.1)$ follows as desired.

Point (c) can be shown as in (b). The statement (d) is a direct consequence of the previous ones. The last statement (e) follows from the defining formula of $M_{st}(X)_{x,C}/G$ and the strict inequality of discrepancies proved in Proposition 6.5.

\[\square\]

6.4. Proof of Theorem 6.3. In this subsection, we prove Theorem 6.3. Let $(x, X) \to (\bar{x}, X/G)$ as in the theorem. We then construct diagram $(6.7.1)$ and follow the notation there.

**Lemma 6.9.** There exists a prime divisor $F \subset \bar{E}_1 \subset Y_1/G$ along which $Y_1 \to Y_1/G$ is ramified.

**Proof.** This is basically Lemma 3.4 for surfaces in mixed characteristic. The same proof as "Elementary proof of Lemma 3.4 for surfaces" shows the lemma.

We fix a divisor $F$ as in the last lemma. Let $a > 0$ be its log discrepancy. For every $i > 0$, we have $M_{st}(X/G)_{\bar{x},\bar{C}_i} - M_{st}(X)_{x,C_i}/G$ has positive leading coefficient and has dimension
\[
\dim (M_{st}(X/G)_{\bar{x},\bar{C}_i} - M_{st}(X)_{x,C_i}/G) \geq \dim (L + n) \frac{L - 1}{L^a - 1} = 2 - a.
\]
From Lemma 6.8 if we take sufficiently large $i$, then
\[
\dim (M_{st}(X/G)_{\bar{x},\bar{C}_i} - M_{st}(X)_{x,C_i}/G) > \dim (M_{st}(X/G)_{\bar{x}} - M_{st}(X/G)_{\bar{x},\bar{C}_i}),
\]
\[
\dim (M_{st}(X/G)_{\bar{x},\bar{C}_i} - M_{st}(X)_{x,C_i}/G) > \dim (M_{st}(X)_{x}/G - M_{st}(X)_{x,C_i}/G).
\]

These inequalities together with
\[
M_{st}(X/G)_{\bar{x}} - M_{st}(X)_{x}/G = (M_{st}(X/G)_{\bar{x},\bar{C}_i} - M_{st}(X)_{x,C_i}/G)
+ (M_{st}(X/G)_{\bar{x}} - M_{st}(X/G)_{\bar{x},\bar{C}_i})
+ (M_{st}(X)_{x}/G - M_{st}(X)_{x,C_i}/G)
\]
show that $M_{st}(X/G)_{\bar{x}} - M_{st}(X)_{x}/G$ and $M_{st}(X/G)_{\bar{x},\bar{C}_i} - M_{st}(X)_{x,C_i}/G$ have the same leading term, which has positive coefficient. Thus, $M_{st}(X/G)_{\bar{x}} > M_{st}(X)_{x}/G$; as desired.
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