On the discrete version of the black hole solution

V.M. Khatsymovsky

Budker Institute of Nuclear Physics
of Siberian Branch Russian Academy of Sciences
Novosibirsk, 630090, Russia
E-mail address: khatsym@gmail.com

Abstract

A Schwarzschild type solution in Regge calculus is considered. Earlier, we considered a mechanism of loose fixing of edge lengths due to the functional integral measure arising from integration over connection in the functional integral for the connection representation of the Regge action. The length scale depends on a free dimensionless parameter that determines the final functional measure. For this parameter and the length scale large in Planck units, the resulting effective action is close to the Regge action.

Earlier, we considered the Regge action in terms of affine connection matrices as functions of the metric inside the 4-simplices and found that this is a difference form of the Hilbert-Einstein action in the leading order over metric variations between the 4-simplices.

Now we take the (continuum) Schwarzschild problem in the form where spherical symmetry is not set a priori and arises just in the solution, take the difference form of the corresponding equations and get the metric (in fact, in the Lemaitre or Painlevé-Gullstrand like frame), which is finite at the origin, just as the Newtonian gravitational potential, obeying the difference Poisson equation with a point source, is cut off at the elementary length and is finite at the source.

1 Introduction

Regge calculus [1] is known as a coordinateless discretization of general relativity (GR). In fact, this is the same GR, but on a certain subclass of the Riemannian manifolds,
piecewise flat ones, able to approximate in a certain sense any Riemannian space-time with arbitrarily high accuracy [2, 3]. Regge calculus was used in classical numerical applications and in constructing quantum models [4, 5]. In quantum gravity, there are applications related to regularization in approaches with functional integrals, both in those that are closer to standard lattice field theory [6], and in the Causal Dynamical Triangulations approach [7]. There is also connection with the spin-foam models [8] and Loop Quantum Gravity [9].

The Schwarzschild and Reissner-Nordström geometries within Regge calculus were considered in Ref [10]. A certain fixed icosahedral decomposition of three-dimensional space into tetrahedra was used there. The task was to provide maximally good approximation to the continuum case. Although effective lattice methods alternative to Regge calculus are also proposed for the numerical study of such systems [11]. Also cosmological models were analysed numerically with the help of Regge calculus [[12] – [15]]. The emergence of cosmological models in the Causal Dynamical Triangulations approach was considered in Ref [16].

A resolution of the Schwarzschild black hole singularity in Loop Quantum Gravity was given in [17, 18]. Although this is not a discrete theory, such a resolution eventually occurs due to the discreteness of spectra of geometrical quantities like area and existence of a minimal quantum of them of the order of the Plank scale.

We would like to analyze a possible black hole solution when, due to specific properties of the discrete path integral measure, the background elementary lengths are loosely fixed dynamically at a certain microscopic scale (defined by the Planck scale and some else parameter characterizing a certain freedom in constructing the functional measure), and we should not pass to the continuum limit.

We discuss dynamically fixing background elementary lengths in [19]. In that paper, we consider the perturbative expansion in Regge gravity and a (loose) fixation of the background configuration for it, in particular, the background length scale, using the functional integral approach. The functional integral measure can be fixed using the canonical Hamiltonian formalism (this implies an intermediate consideration of the simplicial manifold whose edges are arbitrarily strongly shrunk along any given coordinate, which plays the role of a ”continuous time” necessary in such a formalism; the requirement for the full discrete measure is that it should pass to the canonical measure whatever coordinate is taken as a time and such a continuous time limit is performed).
Initially, we consider an extended set of variables with an orthogonal connection, which is an independent variable in addition to the tetrad type variables (the edge vectors or area tensors). Such a connection and curvature in the Regge calculus (finite rotation matrices) were defined by Fröhlich [20].

In the continuum GR, the functional integration over the connection, viewed as an independent variable, is Gaussian and gives a functional integral with the action purely in terms of the tetrad type variables. In the discrete framework, such an integration over the connection gives a certain phase of the result and a certain module of the result.

For the module, it is appropriate to use an expansion over the discrete analogs of the Arnowitt-Deser-Misner [21, 22] lapse-shift functions \((N, N^i)\), for a nontrivial module appears already in the zero order of this expansion. In the continuum GR, \((N, N^i)\) are the tetrad 4-vectors with the temporal world vector index (covariant). These enter the tetrad-connection (Cartan-Weyl) GR action linearly, through the time-like (with spatio-temporal indices) bi-vectors. The discrete \((N, N^i)\) are the 4-vectors of certain ("time-like") edges, and their scales enter the tetrad-connection Regge action still linearly, through the tensors of certain ("time-like") triangles. Each term of the expansion over \((N, N^i)\) (as for zero \((N, N^i)\)) factorizes over "space-like" triangles (analogs of the continuum bi-vectors with spatio-spatial world covariant indices) and is calculable. The resulting module or measure in terms of the purely tetrad type variables has a pronounced maximum at the values of the areas of the space-like triangles defined by the Planck scale and some parameter \((\eta)\) characterizing a certain freedom in constructing the functional measure. We can say that their areas or the lengths of the space-like edges are loosely fixed dynamically. At the same time, the time-like edge vectors \((N, N^i)\) can be manually set (like fixing the gauge in the continuum case) as defining the construction of a simplicial space-time from a time-sequence of simplicial spatial 3D sections of a similar structure.

For the phase, it is appropriate to use the stationary phase expansion, for a nonzero phase appears already in the zero order - this is exactly the Regge action \(S(\ell)\) in terms of the edge lengths \(\ell\) since classically excluding the connection from the tetrad-connection Regge action gives just \(S(\ell)\).

The above said can be resumed by the following expression for the functional integral in terms of the purely length type variables obtained by the functional integration.
over the connection type variables,

\[
\int \exp[iS(\ell)] F(\ell) D\ell, \tag{1}
\]

\(D\ell = \prod_k d\ell_k\) is the collective Lebesgue measure, \(\ell = (l_1, \ldots, l_n)\) is the set of the edge lengths. The Regge action \(S(\ell)\) appears here as the leading term in the stationary phase expansion for the phase, and in \(F(\ell)\) we take the leading term in the expansion over the discrete \((N, N')\) for the module. \(F(\ell)\) turns out to have a maximum at the areas of the space-like triangles being \(a^2/2\), where the length scale

\[a = \sqrt{32G(\eta - 5)/3}.\]  

(2)

Here \(\eta\) is a parameter of the theory defining the 4-simplex volume degree factors \(V^\eta\) in the functional measure analogous to the metric determinant factors \((-g)^{n/2}\) in the continuum measure. (The continuum measure is determined up to such a factor, which leads, eg, to the measures of Misner [23] or DeWitt [24].) This loosely fixes the space-like edge lengths at the scale \(a\). Both these expansions, for the phase and for the module, are consistent and allow to limit ourselves by their leading terms for a sufficiently large \(a\) and thus \(\eta\).

To formulate this in a more quantitative manner, we consider implicitly passing from \(\ell\) to some new collective variable \(u = (u_1, \ldots, u_n)\) such that \(F(\ell)D\ell = Du\) is the Lebesgue measure. Besides, we can choose some stationary collective point \(\ell_0 = (l_{01}, \ldots, l_{0n})\), that is, ensuring the validity of the equations of motion (Regge equations) in it,

\[\frac{\partial S(\ell_0)}{\partial \ell} = 0.\]  

(3)

Then the Taylor expansion of \(S(\ell)\) around \(\ell_0 = \ell(u_0)\), \(u_0 = (u_{01}, \ldots, u_{0n})\), over \(\Delta u = u - u_0\) begins with a second-order term,

\[S(\ell) = \frac{1}{2} \sum_{j,k,l,m} \frac{\partial^2 S(\ell_0)}{\partial l_j \partial l_k} \frac{\partial l_l(u_0)}{\partial u_k} \frac{\partial l_l(u_0)}{\partial u_m} \Delta u_k \Delta u_m + \ldots.\]  

(4)

The equations of motion (3), that is, the requirement of the extremum of the zero order term \(S(\ell_0)\), do not fix all the variables \(l_k\). The latter, in particular, their scale, are loosely fixed by the requirement of an extremum (maximum) of the contribution of the second order term, that is, of the maximum of the determinant of the bilinear form over \(\Delta u\) in (4),

\[F(\ell_0)^2 \det \left\| \frac{\partial^2 S(\ell_0)}{\partial l_i \partial l_k} \right\|^{-1}.\]  

(5)
The matrix $\frac{\partial^2 S(\ell_0)}{\partial l_i \partial l_k}$ has zero order in the scale of edge lengths. Geometrically, the edge length scale can not change rapidly from simplex to simplex, and the matrix $\frac{\partial^2 S(\ell_0)}{\partial l_i \partial l_k}$ is just close to a diagonal one (only those $l_i$ and $l_k$ "interact" in the Regge action $S$ which refer to the same 4-simplex). Therefore, it is expected that the inclusion of the determinant of this matrix in (5) will not lead to an essential change in the extreme point $\ell_0$ of (5) compared to the maximum of only $F(\ell_0)$. This also means some sufficient uniformity of the elementary length scale.

We could also define an elementary length scale, for example, by finding length vacuum expectations. In the above definition of the length scale $\ell_0$, the perturbative interaction with gravitons acts as a probe, which seems quite natural.

In the paper [25], we considered calculating the Regge action in terms of the simplicial metric resembling calculating the Hilbert-Einstein action through intermediate finding the Christoffel symbols. Discrete Christoffel symbols or affine connection matrices were used there, and an exact expression of the Regge action in terms of the piecewise constant metric was considered for the general simplicial structure. A particular periodic simplicial structure with the 4-cube cell divided by the diagonals into 24 4-simplices was considered. For this structure, we found that the Regge action in the considered form, arranged in a series over metric variations between the 4-simplices, is in the leading order a finite-difference form of the Hilbert-Einstein action in terms of the metric variations between the 4-cubes,

$$
\sum_{4\text{-cubes}} K^\mu_{\lambda \rho \nu} \sqrt{g}, \text{ where } K^\lambda_{\mu \nu \rho} = \Delta_\nu M^\lambda_{\rho \mu} - \Delta_\rho M^\lambda_{\nu \mu} + M^\lambda_{\nu \sigma} M^\sigma_{\mu \rho} - M^\lambda_{\rho \sigma} M^\sigma_{\nu \mu},
$$

$$
M^\lambda_{\mu \nu} = \frac{1}{2} g^{\lambda \rho} (\Delta_\nu g_{\mu \rho} + \Delta_\mu g_{\rho \nu} - \Delta_\rho g_{\mu \nu}), \quad \Delta_\lambda = 1 - T_\lambda.
$$

(6)

$K^\lambda_{\mu \nu \rho}$ is the finite-difference form of the Riemannian tensor $R^\lambda_{\mu \nu \rho}$, $T_\lambda$ (respectively, Hermitian conjugate $\overline{T}_\lambda$) is the shift operator along the edge $\lambda$ or the coordinate $x^\lambda$ in the forward (respectively, backward) direction. This can be understandably generalized to coordinate steps other than 1.

Thereby, the analysis of the Regge skeleton equations is reduced to the analysis of a finite-difference Einstein equations (with the possibility of a regular consideration of the further corrections).

The use of a periodic simplicial structure respects the above mentioned uniformity of the elementary length scale and makes the most appropriate that the metric ansatz, substituted into the action (6), be formulated in Cartesian type coordinates without
requiring a priori spherical symmetry. It turns out that such an ansatz should cover the case of the Painlevé-Gullstrand metric in such coordinates, and it is convenient to take the 3+1 ADM form of the metric for it. Before the substitution, we should convert the ansatz into a form in which $N, N^k$, as mentioned, are constant parameters.

In Section 2, the metric ansatz is transformed to that analogous to the Lemaitre metric (as a simplest example with fixed $N = 1, N^k = 0$), and also the region of sufficiency of the leading order over metric variations between the 4-simplices is considered from different viewpoints. In Section 3, the leading order over metric variations is considered. In this order, the finite-difference form of the GR action on the metric ansatz transformed to the Lemaitre type coordinates coincides with this form on the original ansatz in the Painlevé-Gullstrand type coordinates due to invariance. The Einstein equations (in the ADM formalism), their discretization, discretization of the Painlevé-Gullstrand solution that follows, the metric, effective Riemann tensor and Kretschmann scalar at the center are considered.

2 General form of the metric

Now we consider a black hole type solution of the equations of motion (3). In accordance with the above mentioned mechanism for fixing the edge lengths, the space-like edge lengths are loosely fixed dynamically while the discrete lapse-shift vectors are given as parameters. A particular case is (the discrete analog of) the synchronous frame $N = 1, N^i = 0$. For the Schwarzschild black hole, this means using (a discrete analog of) the Lemaitre type metric [26], which we write in the form using a radial type coordinate $r_1$ such that at $r_g = 0$ it be the standard $r$ [27],

$$
\begin{align*}
\quad ds^2 &= -d\tau^2 + \frac{r_1}{r(r_1, \tau)}dr_1^2 + r^2(r_1, \tau)d\Omega^2 \\
&= -d\tau^2 + (dr(r_1, \tau)_{\tau=\text{const}})^2 + r^2(r_1, \tau)d\Omega^2, \quad r^{3/2} = r_1^{3/2} - \frac{2}{3}\sqrt{r_g\tau}.
\end{align*}
(7)
$$

We see that the 3D sections $\tau = \text{const}$ possess the flat metric. We are interested in a finite-difference analogue of the equations which lead to (7). For a periodic simplicial structure and a certain length scale such an analog can be written in Cartesian type coordinates,

$$
y = r_1 n, \quad n^2 \equiv \sum_k n^k n^k = 1, \quad d\Omega^2 = dn^2,$$
\[ ds^2 = -d\tau^2 + \frac{r^2}{r_1^2} dy^2 + \left( \frac{r_1}{r} - \frac{r_2}{r_1^2} \right) \left( \frac{y dy}{r^2} \right)^2. \] (8)

Taking a step back in obtaining (7), we return from \( r_1 \) to the original \( r \) and get the Painlevé-Gullstrand metric \[28, 29],

\[ ds^2 = -d\tau^2 + \left( dr + \sqrt{\frac{r^g}{r}} d\tau \right)^2 + r^2 d\Omega^2. \] (9)

The 3D sections \( \tau = const \) possess the flat metric here too.

According to the above said, we would like to write down the Einstein equations in a finite-difference form and re-solve the problem with this input. Any Regge manifold does not possess the spherical symmetry. The latter is restored when averaging over possible simplicial structures. As for a given structure, the criterium for the solution might be that at large distances, where the metric variations between the 4-simplices are small and the Einstein equations in a finite-difference form are close to their continuum form, the solution of interest should be close to (7).

The criterium for retaining only the leading order over metric variations between the 4-simplices in the above mentioned expansion over such variations is smallness of these variations. When we go to a smaller distances, using the leading order over metric variations between the 4-simplices may become insufficient. Using the suggested required matching between the finite-difference and continuum solutions, we can trace when the metric variations between the 4-simplices can not be small. To this end, we can consider a simplicial structure with 4-cube cells, whose space-like bases are in the (flat) 3D sections \( \tau = const \), time-like edges are geodesic lines \( r_1 = const \), orthogonal to these sections, and their time-like length is \( \Delta \tau \) (the difference between the neighboring sections \( \tau = const \)). In Fig. 1, a triangulation in the neighborhood of \( r = 0 \) in the

![Figure 1](image-url)
Lemaitre type coordinates is shown. Because of the events of ending worldlines at \( r = 0 \), intervals \( \Delta \tau \) between neighboring 3D sections \( \tau = \text{const} \) look non-uniform, although they can be made uniform at the expense of distorting the (hyper)cubic cells in the vicinity of \( r = 0 \).

If a certain line \( r_1 = \text{const} \) corresponds to \( r > 0 \) at \( \tau = \tau_0 \), then it reaches the singularity \( r = 0 \) at \( \tau = \tau_0 + T \). This event means a defect violating the regular coverage of the space-time by the considered orthogonal lattice. The condition that the number of such defects (=1) be negligible compared to the number of the regular vertices along this geodesic \( (T/\Delta \tau) \) is \( T \gg \Delta \tau \). Fixing \( \Delta \tau \) similarly to fixing the lapse function \( N \) contributes to choosing a certain measurement procedure. Having the dynamically fixed elementary length scale \( a \), the choice \( \Delta \tau \ll a \) means an excessive accuracy, \( \Delta \tau \gg a \) lack of accuracy, and the essential deviation of \( \Delta \tau/a \) from unity in both these cases makes the description of the system somewhat singular. Like averaging over possible simplicial structures, averaging over different time-like lengths seems to be appropriate, but now we can take for estimate \( \Delta \tau \simeq a \), which also reflects a symmetry between space and time. Thus, at

\[
T = \frac{2r^{3/2}}{3\sqrt{r_g}} \gg \Delta \tau \simeq a \text{ or } r_g \ll \frac{r^3}{a^2},
\]

(10)

metric variations between the 4-simplices are small. This is a kind of a nonlocal consideration. In a more local approach, since we are interested in a finite-difference form of differential equations, it is sufficient to take \( \tau = 0 \) and a few first 3D sections after \( \tau = 0 \) (enough to form a 4-geometry), that is, \( \tau \simeq a \). The metric from \( \tau = 0 \) to \( \tau \simeq a \) is distorted from being flat, and the measure of this distortion is closeness of \( r/r_1 \) to unity,

\[
\left| \frac{r_1(r, \tau) - r}{r_1(r, \tau)} \right| \ll 1 \text{ at } r_g \ll \frac{r^3}{a^2}.
\]

(11)

The same follows from the typical curvature in the continuum GR \( R \sim r_g/r^3 \) (from the curvature invariants). For the elementary area scale \( a^2 \), this gives a typical value of the angle defect \( \alpha \sim Ra^2 \) and the condition for its smallness,

\[
\alpha \sim a^2 R \sim a^2\frac{r_g}{r^3} \ll 1 \text{ at } r_g \ll \frac{r^3}{a^2}.
\]

(12)

In (6), the metric in the 4-simplices/cubes enters, but these values can be viewed as particular values of some smooth (interpolating) field \( g_{\lambda\mu} \). In the zeroth order over metric variations between the 4-simplices, the Lemaitre metric (7) could be taken as
Having in view the minimal nonzero \( r = a \) of a vertex, we get from the above estimates that the Lemaitre \( g_{\lambda \mu} \) can be prolonged to \( r \geq a \) at

\[
  r_g \ll a, 
\]  

though this is not physically quite an interesting case (there is no horizon as such). Note that \( a \gg 1 \) at \( \eta \gg 1 \) (see (2)), and (13) can be satisfied at \( r_g \gg 1 \).

In non-leading orders, generally speaking, it should be a common synchronous metric. To bring it into the form most adapted to the task at hand, it is convenient to issue from the Painlevé-Gullstrand metric (9), rewritten in Cartesian type coordinates,

\[
ds^2 = -d\tau^2 + \sum_{k=1}^{3} (dx^k + f^k d\tau)^2, \quad f^k = \sqrt{\frac{r_g x^k}{r}}. 
\]  

This can be naturally generalized to the 3+1 ADM form of metric [22],

\[
ds^2 = -(Nd\tau)^2 + g_{kl} (dx^k + f^k d\tau)(dx^l + f^l d\tau). 
\]  

Here \( k, l, \ldots = 1, 2, 3 \); in the leading order over metric variations or in the continuum limit \( f^k \) are given by (14), \( N = 1, g_{kl} = \delta_{kl} \).

We can perform the change of variables in (15) which generalizes the transition from the Painlevé-Gullstrand (9) to Lemaitre (7) metric. Finding this change of variables \((x, \tau) \rightarrow (y, \tau), x^k = x^k(y, \tau)\) amounts to solving the differential equations

\[
\frac{\partial x^k(y, \tau)}{\partial \tau} + f^k(x(y, \tau), \tau) = 0, \quad x^k(y, 0) = y^k \tag{16}
\]

or

\[
x^k(y, \tau) = y^k - \int_0^\tau f^k(x(y, \tau), \tau) d\tau \tag{17}
\]

in the integral form. Then the metric reads

\[
ds^2 = -(Nd\tau)^2 + g_{kl} \frac{\partial x^k}{\partial y^m} \frac{\partial x^l}{\partial y^n} dy^m dy^n. 
\]  

In principle, the task of replacing variables could be complicated by the requirement of reducing \( N \) to unity, that is, obtaining a completely synchronous reference system as a result. But since \( N = 1 \) in the leading order over metric variations, we expect it is bounded on the whole space-time and its possible dependence on the coordinates will not affect convergence properties of the expansion over lapse-shift functions \((N, N^i)\) (here \((N, 0)\)) for the above functional integral measure \( F(\ell) \) in (1). In the specific calculations of the present paper, in the leading order over metric variations, a Lemaitre
type metric is used only intermediately (only the fact of its existence is important), and the specific value of $N$ does not manifest itself.

We can represent the solution of equations (16) as a combination of a certain (expectedly large) part of the transformation $(x, \tau) \to (y, \tau)$, expressed in quadratures, and other less significant transformations.

For that, we separate the radial and angle parts of $x$: $x = r n$ (with respect to the 3-metric $g_{kl} = \delta_{kl}$: $\sum_k n^k n^k = 1$). First, we take the longitudinal part of equations (16),

$$
\frac{\partial r}{\partial \tau} + f \cdot n = 0. \tag{19}
$$

In the case of the Painlevé-Gullstrand metric, solving this via

$$
\int_{r_1}^r \frac{dr}{f \cdot n} + \tau = 0 \tag{20}
$$

introduces the Lemaitre variable $r_1$ as a function of $r$ and $\tau$. Now consider the general case when $f \cdot n$ is a function of $r, n, \tau$. Then (20) defines a function $r = r_1(n, \tau)$, which, as a rule, does not vanish the spatio-temporal components of the metric, but can serve as a substitution, after which it remains to go from $r_1, n$ to some $\tilde{r}_1, \tilde{n}$, close to $r_1, n$, if the metric is close to the Painlevé-Gullstrand one. We write out the differential 1-form included in the metric 2-form (15),

$$
dx + f d\tau = ndr + rdn + f d\tau = n \left( \frac{\partial r}{\partial r_1} \right)_{n,\tau} dr_1 + rdn
+ n \left( \left( \frac{\partial r}{\partial n} \right)_{r_1,\tau} \cdot dn \right) + \left( n \left( \frac{\partial r}{\partial \tau} \right)_{r_1,n} + f \right) d\tau. \tag{21}
$$

The derivatives of $r$ over $r_1, n, \tau$ are found by differentiating (20),

$$
\left( \frac{\partial r}{\partial r_1} \right)_{n,\tau} = (f \cdot n)(r, n, \tau), \quad \left( \frac{\partial r}{\partial n} \right)_{r_1,\tau} = (f \cdot n)(r, n, \tau) \frac{\partial}{\partial n} \int_{r_1}^r \frac{dr}{f \cdot n},

\left( \frac{\partial r}{\partial \tau} \right)_{r_1,n} = (f \cdot n)(r, n, \tau) \left( \frac{\partial}{\partial \tau} \int_{r_1}^r \frac{dr}{f \cdot n} - 1 \right). \tag{22}
$$

Now we pass from $r_1, n$ to some $\tilde{r}_1, \tilde{n}$ and admit that this transformation $r_1 = r_1(\tilde{r}_1, \tilde{n}, \tau), n = n(\tilde{r}_1, \tilde{n}, \tau)$ (weakly) depends on $\tau$ to remove the residual $d\tau$ term in (21). Requiring vanishing this term, we get the equations for this transformation,

$$
\frac{\partial r_1}{\partial \tau} = (f \cdot n)(r_1, n, \tau) \left( \frac{1}{r} f_\perp \cdot \frac{\partial}{\partial n} - \frac{\partial}{\partial \tau} \right) \int_{r}^{r_1} \frac{dr}{f \cdot n}, \quad \frac{\partial n}{\partial \tau} = -\frac{f_\perp}{r},

f_\perp \equiv f - n(f \cdot n), \quad r_1|_\tau = 0 = \tilde{r}_1, \quad n|_\tau = 0 = \tilde{n}. \tag{23}
$$
Then the 1-form (21) reads
\[ n \frac{f \cdot n}{(f \cdot n)(r_1, n, \tau)} dr_1|_\tau + r dn|_\tau + n(f \cdot n)(r, n, \tau) \left( dn|_\tau \cdot \frac{\partial}{\partial n} \int_r^{r_1} \frac{dr}{f \cdot n} \right), \] (24)
where $|_\tau$ means $\tau = \text{const}$,
\[ dr_1|_\tau = \frac{\partial r_1}{\partial \tilde{r}_1} d\tilde{r}_1 + \frac{\partial r_1}{\partial \tilde{n}} d\tilde{n}, \quad dn^k|_\tau = \frac{\partial n^k}{\partial \tilde{r}_1} d\tilde{r}_1 + \frac{\partial n^k}{\partial \tilde{n}} d\tilde{n}. \] (25)
Finally, we pass to Cartesian type coordinates
\[ y = \tilde{r}_1 \tilde{n}, \quad \tilde{r}_1 = \sqrt{y^2}, \quad d\tilde{r}_1 = \tilde{r}_1^{-1} y \cdot dy, \quad d\tilde{n} = \tilde{r}_1^{-1} dy - \tilde{r}_1^{-3} y(y \cdot dy). \] (26)
The metric is
\[ ds^2 = -(N d\tau)^2 + \left( g_{kl} n^k n^l \right) \left( \frac{f \cdot n}{(f \cdot n)(r_1, n, \tau)} \right)^2 \left( \frac{dr_1}{r_1} \right)^2 + (f \cdot n)(r, n, \tau) \left( dn|_\tau \cdot \frac{\partial}{\partial n} \int_r^{r_1} \frac{dr}{f \cdot n} \right)^2 + 2r \frac{(f \cdot n)(r, n, \tau)}{(f \cdot n)(r_1, n, \tau)} \left( g_{kl} n^k dn^l \right) \left( \frac{dr_1}{r_1} \right) \] (27)
Together with (20), (23), (25), (26), this is an exact expression generalizing the Lemaître form for an arbitrary metric (15), a more detailed formula (18) where the leading and non-leading contributions are singled out if the original metric is close to the Painlevé-Gullstrand one. In particular, $f_\perp$ and $\partial(f \cdot n)/\partial n$ are non-leading in this case, therefore, at $\partial f / \partial \tau = 0$, the equations for the transformation (23) give $\partial r_1 / \partial \tau$ being of the second order over deviation from the Painlevé-Gullstrand metric. Also $g_{kl} n^k dn^l = 0$ in $ds^2$ (27) in the leading order and so on. In the discrete framework, this deviation will be characterized by the metric variations from simplex to simplex or finite differences $\Delta$.

3 Field equations in the leading order over metric variations

We can substitute the metric ansatz (18) or, in more detail, (27), into the finite-difference form of the GR action (6). The finite differences approximately obey the
rules for the derivatives, in particular, the chain rule for computing the derivative of
the composition of two functions, for example,
\[
\frac{\Delta g_{kl}(x(y, \tau), \tau)}{\Delta y^m} = \frac{\Delta x^n(y, \tau)}{\Delta y^m} \frac{\Delta g_{kl}(x(y, \tau), \tau)}{\Delta x^n(y, \tau)},
\] (28)
and the product rule. The condition of validity of these properties of the finite differ-
ces of the coordinate change expressions is the same as the condition of smallness of
the metric variations, if we apply this to that point involved in any considered finite
difference which has a smaller \( r \); for example, taking the radial coordinates \( r \) and \( r_1 \)
(which are certain functions of the Cartesian type coordinates \( x^k, y^k \)), we have for the
accuracy of reproducing the typical coordinate derivative by the corresponding finite
difference,
\[
\left| \frac{a}{r_1(r + a, \tau) - r_1(r, \tau)} - \frac{\partial r}{\partial r_1(r + a, \tau)} \right| \left| \frac{\partial r}{\partial r_1(r + a, \tau)} \right|^{-1} \ll 1 \text{ at } r_g \ll \frac{r^3}{a^2},
\] (29)
like (11). It is also taken into account that the minimal nonzero \( r = a \) of a vertex in a
3D section. Here we take into account that it is \( r \) that measures the radial lengths in
the sections \( \tau = \text{const} \), and not \( r_1 \) (since \( r^{-1}r_1\mathrm{d}r_1^2 = \mathrm{d}r^2|_{\tau=\text{const}} \) in \( ds^2 \) (7)), although if
(29) holds, there is no difference, \( \Delta r \) or \( \Delta r_1 \) has the scale \( a \).

If we consider the leading order over metric variations when operating with the
finite differences in the form for the action, we can handle these differences as the
derivatives and, in particular, use the general covariance, reducing this form from the
coordinates \( y^k \) back to \( x^k \) when substituting the metric ansatz (18) into the action.
Thereby we get a finite-difference form of the action with the 3+1 ADM form of metric
(15). Note that such a return to the (finite-difference) action on the metric (15) takes
place only in the leading order over metric variations. In non-leading orders, in the
equations like (28), the corrections of the higher order over \( \Delta \) should be taken into
account in addition to the main terms in which the differences are replaced by the
corresponding derivatives.

Thus, we are interested in (the finite-difference form of) the action, a bulk expression
of which (up to surface terms) takes the form [22, 30],
\[
S = \int \left\{ -g_{kl} \frac{\partial \pi^{kl}}{\partial \tau} + N\sqrt{g} \left[ 3\hat{R} + g^{-1} \left( \frac{1}{2} \pi^k \pi^l_t - \pi^k_t \pi^l_k \right) \right] + 2f_k \pi^{kl} \right\} \mathrm{d}^3x \mathrm{d}\tau, \quad (30)
\]
\[
\pi_{kl} = \sqrt{g} (g_{kl} K_m^m - K_{kl}), \quad K_{kl} = \frac{1}{2N} \left( \dot{f}_{kl} + f_{l|k} - \frac{\partial g_{kl}}{\partial \tau} \right). \quad (31)
\]
Raising and lowering Latin indices is done using the 3D metric $g_{kl}$, the symbol $\mid$ as an index means the covariant derivative, and $^3R$ is the curvature scalar for this metric.

Varying $S$ so that

$$\frac{\delta S}{\delta f}$$

appropriate to take a combination of this equation and form a system for

$$\frac{\delta S}{\delta g}$$

Writing out the equation look at these equations from a slightly different angle. Namely,

$$\frac{\delta S}{\delta g}$$

Equating these expressions to zero gives combinations of the Einstein equations (in empty space), and $\delta S/\delta N = 0$ and $\delta S/\delta f_k = 0$ are the equations for initial conditions in the Hamiltonian formalism, and $\delta S/\delta g_{kl} = 0$ (and analogous one defining $\pi_{kl}$ in terms of $\partial g_{kl}/\partial \tau$ (31)) are the dynamical equations. In the Lagrangian formalism in our case and especially in the stationary problem ($\partial \pi_{kl}/\partial \tau = 0$, $\partial g_{kl}/\partial \tau = 0$), we can look at these equations from a slightly different angle. Namely, $\delta S/\delta g_{kl} = 0$ can be considered as those defining 3D geometry ($^3R_{kl}$ and, therefore, $^3R_{klmn}$) from knowing $N$, $f_k$. $\delta S/\delta f_k = 0$ can be viewed as specifying $f_k$. $\delta S/\delta N = 0$ looks as a condition on $^3R_{kl}$ or $^3R_{klmn}$. Since, however, the 3D curvature is already defined by $\delta S/\delta g_{kl} = 0$, the term $^3R$ can be excluded from $\delta S/\delta N = 0$. This is achieved by forming a combination of $\delta S/\delta N = 0$ and the trace of $\delta S/\delta g_{kl} = 0$. The resulting equation and $\delta S/\delta f_k = 0$ form a system for $N$, $f_k$. To get a more compact dependence on $f_k$, it turns out to be appropriate to take a combination of this equation and $\delta S/\delta f^k = 0$,

$$0 = \frac{N}{\sqrt{g}} \left[ \frac{g_{kl}}{2} \frac{\delta S}{\delta g_{kl}} - f^k \frac{\delta S}{\delta f^k} - N \frac{\delta S}{\delta N} \right] = \frac{1}{2} (f^k f^l f_k^l) - N \frac{\partial g_{kl}}{\partial \tau} \left[ f^l f^m f_n - f^m f^l f_n \right] + \ldots ,$$

so that

$$\frac{1}{2} (f^k f^l f_k^l) - N \frac{\partial g_{kl}}{\partial \tau} \left[ f^l f^m f_n - f^m f^l f_n \right] + \ldots ,$$

Writing out the equation $\delta S/\delta f^k = 0$,

$$0 = \frac{N}{\sqrt{g}} \frac{\delta S}{\delta f^k} = - f^l f_k^l + g_{kl} f^l ,$$
we have

\[ f_{k|l}^l - f_l^{|l} = -3R_{kl}f^l + \left( \ln N \right)^l \left( f_{k|l}^l + f_{l|k}^l - 2g_{kl}f_{m|n}^m \right) \]

\[ + \left( g^{mn}\delta_k^l - g^{lm}\delta_k^n \right) \left[ \frac{\partial g_{lm}}{\partial \tau} \right]_n - \left( \ln N \right)^l_n \frac{\partial g_{lm}}{\partial \tau} \].  

(38)

For \( 3R_{kl} = 0 \), \( N = \text{const} \), \( \partial g_{kl}/\partial \tau = 0 \), these three equations are not independent. These do not determine the longitudinal part of \( f^k \) (which is then defined by substituting into (36)). For a nonzero RHS, a consistency condition should be fulfilled. This should be similar to the current conservation law in electrodynamics (now three-dimensional) \( (f_{k|l}^l - f_{l|k}^l)^{|k} = 0 \) imposed on \( (f_{k|l}^l - f_{l|k}^l)^{|l} = f_{k|l}^l - f_l^{|l} - 3R_{kl}f^l \). Then this condition reads

\[ \left[ \left( \ln N \right)^l \left( 2g_{kl}f_{m|n}^m - f_{k|l}^l - f_{l|k}^l \right) \right]^{|k} = -23R_{kl}f^l + \]

\[ + \left( g^{mn}\delta_k^l - g^{lm}\delta_k^n \right) \left[ \frac{\partial g_{lm}}{\partial \tau} \right]_n \left( \ln N \right)^l_n \frac{\partial g_{lm}}{\partial \tau} \].  

(39)

This can be considered and is written as an equation for \( N \), and actual \( f^k \) for the continuum Painlevé-Gullstrand metric (14) give a non-degenerate (invertible) second order differential operator of the Laplace type acting on \( \ln N \) in the LHS.

Six remaining equations \( \delta S/\delta g_{kl} = 0 \) can be written as

\[ 3R_{kl} - \frac{1}{2}g^{kl}3R = -\frac{1}{2}g^{kl} \left( \frac{1}{2}\pi^m_m\pi^n_n - \pi^m_n\pi^m_n \right) \]

\[ + \frac{2}{g} \left( \frac{1}{2}\pi^m_m\pi^{kl} - \pi^k_m\pi^{ml} \right) + \frac{1}{N\sqrt{g}} \left[ (\pi^{kl}f^m)_m - f^k_m\pi^{ml} - f^l_m\pi^{mk} \right] \]

\[ + N^{-1} \left( N^{kl} - g^{kl}N^{lm}_m \right) - \frac{1}{N\sqrt{g}} \frac{\partial \pi^{kl}}{\partial \tau}. \]

(40)

These take the form of the three-dimensional Einstein equations with a non-trivial RHS. For the Painlevé-Gullstrand metric (14) in the RHS, these give \( 3R_{kl} = 0 \) self-consistently, that is, a flat metric \( g_{kl} \). In particular, the considered \( f^k \) lead to zero bilinear contribution in the RHS (the sum of the terms of the type \( (\partial f)(\partial f) \) and \( (f)(\partial^2 f) \), two first lines of the equation (40)). Simultaneously, these \( f^k \) satisfy (36) and (38) with zero RHS. In (36), these \( f^k \) vanish the bilinear in the LHS of (36) of the type \( \partial^2(ff) \).

Satisfying \( f^k \) more than three equations with trivial \( N, g_{kl} \) seems to be a coincidence, but this degeneracy is removed for the target finite-difference form of the action, and the remaining variables become non-trivial, albeit small. The terms of the type of
\( \partial^2(ff), (\partial f)(\partial f) \) and \((f)(\partial^2 f)\) are related by the product rule of differentiation. If the derivatives \( \partial \) are replaced by the finite differences \( \Delta \), such a rule continue to hold, but only in the leading order over the differences. If \( f^k \) vanish the LHS of (36) and (38), in particular, the bilinear \( \partial^2(ff) \), the bilinears \( (\partial f)(\partial f) \) and \((f)(\partial^2 f)\) in the RHS of (40) will be not zero, but \( O(\Delta) \). This will lead to \( 3R^{kl} \) and metric \( g_{kl} \) differing from \( \delta_{kl} \) by \( O(\Delta) \).

It is important for transition to the discrete form of the equations that the black hole solution be defined without invoking a priori spherical symmetry. We can take \( g_{kl} = \delta_{kl}, N = 1 \), stationarity (independence on \( \tau \)) and a \( \delta \)-function-like nature of the source. So we have the system of (36) and (38),

\[
\begin{align*}
\frac{1}{2} \nabla^2 f^2 - \frac{1}{4} [\nabla \times f] \cdot [\nabla \times f] &= 0, \\
[\nabla \times [\nabla \times f]] &= 0,
\end{align*}
\]

at \( r > 0 \). It follows from (42) that \( f \) is purely longitudinal, and we write instead of (41)

\[
f = \nabla \chi, \quad \nabla^2 ((\nabla \chi)^2) = 0
\]

at \( r > 0 \). This gives

\[
(\nabla \chi)^2 = \frac{r_g}{r}.
\]

The constant \( r_g \) is chosen to reproduce the Painlevé-Gullstrand metric. In the form \( r(\nabla \chi)^2 - r_g = 0 \), this looks as the Hamilton-Jacobi equation for a particle with the action \( \chi \), mass squared \( r_g \) and a three-dimensional positively defined metric \( r\delta_{kl} \) (more precisely, with all the coordinates having the time-like signature). In spherical coordinates, this reads

\[
\left( \frac{\partial \chi}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial \chi}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial \chi}{\partial \phi} \right)^2 = \frac{r_g}{r},
\]

According to the standard approach to the Hamilton-Jacobi equation, we try the additive separation of variables,

\[
\chi(r, \theta, \phi) = R(r) + \Theta(\theta) + \Phi(\phi),
\]

\[
[R'(r)]^2 + \frac{1}{r^2} [\Theta'(\theta)]^2 + \frac{1}{r^2 \sin^2 \theta} [\Phi'(\phi)]^2 = \frac{r_g}{r},
\]

\[
(\Phi')^2 = M^2, \quad (\Theta')^2 + \frac{M^2}{\sin^2 \theta} = L^2, \quad (R')^2 + \frac{L^2}{r^2} = \frac{r_g}{r}.
\]

Here \( L^2, M^2 \) are positive separation constants. It can be seen that \( \Theta \) does not exist for sufficiently small values of \( \theta \), unless \( M = 0 \), and \( R \) does not exist for sufficiently small
values of $r$, unless $L = 0$. Thus, the solution is automatically spherically symmetrical, and we get the Painlevé-Gullstrand metric.

Thus, equations (36), (38) (in particular, the consistency condition (39)) and (40) form a suitable system for the analysis of a metric close to the Painlevé-Gullstrand solution if we require that the spherical symmetry of the latter solution would arise automatically without imposing a priori, but after imposing some other conditions. Of these equations, equation (36) seems to be interesting in that it is a combination of the equations for initial conditions and the dynamical equations (in the Hamiltonian formalism terminology). It is interesting to ask what combination of the components of the Einstein equations is this equation. In the variation of the action $\delta S = \int G_{\lambda\mu} \delta g^{\lambda\mu} d^4x$, we can express $\delta g^{\lambda\mu}$ in terms of $\delta g_{kl}$, $\delta f^k$, $\delta N$ and find

$$\frac{1}{N\sqrt{g}} \frac{\delta S}{\delta \tilde{g}^{kl}} = G_{kl}, \quad \frac{N}{2\sqrt{g}} \frac{\delta S}{\delta f_k} = G_{0k} - G_{kl} f^l,$$

$$\frac{N^2}{2\sqrt{g}} \frac{\delta S}{\delta N} = G_{00} - 2G_{0k} f^k + G_{kl} f^k f^l, \quad (49)$$

and the component (35) includes all the spatio-spatial, spatio-temporal and temporal-temporal components of the Einstein equations, but looks the simplest in terms of the 4D Ricci tensor,

$$\frac{N}{\sqrt{g}} \left[ \frac{g_{kl}}{2} \frac{\delta S}{\delta g_{kl}} - f^k \frac{\delta S}{\delta f_k} - \frac{N}{4} \frac{\delta S}{\delta N} \right] = -\frac{1}{2} N^2 \tilde{g}^{kl} G_{kl} + \frac{3}{2} f^k f^l G_{kl} - f^k G_{0k} - \frac{1}{2} G_{00} = -R_{00} + f^k f^l R_{kl}. \quad (50)$$

Here $\tilde{g}^{kl}$ is the inverse of $g_{kl}$.

A situation in which spherical symmetry is absent is exactly the situation with the Regge equations or, here, finite-difference equations. So we use the above input as a definition of a black hole solution ($g_{kl} = \delta_{kl}$, $N = 1$, independence on $\tau$ and a $\delta$-function-like nature of the source). Now we have a finite-difference form of the equations (36), (38) (in particular, the consistency condition (39)) and (40). Their RHSs differ from zero in the next-to-leading order over the differences $O(\Delta)$. The LHS (including that of the consistency condition (39) for $N$, in particular, in the vicinity of the Painlevé-Gullstrand type metric) can be resolved with respect to $f^k$, $N$ and $3R_{kl}$ (and eventually $g_{kl}$). An iterative process can be carried out. In the leading (zeroth) order over the differences, we have a finite-difference form of equations (41),
(42), leading to

\[ f_k = a^{-1} \Delta_k \chi, \sum_{k=1}^{3} \Delta_k \left( f^2 \right) = \begin{cases} 0 \text{ at } x \neq 0 \\ C \text{ at } x = 0. \end{cases} \]  

(51)

Here \( \Delta_k h(x^k) \equiv h(x^k) - h(x^k - a) \), and the simplest Hermitian form of the difference Laplacian is written out (in the leading order over the differences \( \Delta \), such forms are the same); \( C \) is a constant chosen so that \( f \) at large distances would lead to the Painlevé-Gullstrand metric for a given \( r_g \). With the help of passing to the momentum representation, we get

\[ f^2(x) = C \int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3 \sum_k 4 \sin^2(p_k a/2)} \exp(ip x) \]  

(52)

At large \( r \), small \( p \) are significant, \( \sum_k 4 \sin^2(p_k a/2) \approx a^2 p^2 \), and to reproduce the Painlevé-Gullstrand metric we must have \( C = 4\pi a^2 r_g \). There also \( f_k \approx \partial_k \chi \), and as discussed above, \( \chi \) is a function of only \( r \), and the required metric is restored.

The metric function \( f^2 \), as well as the other metric functions and any value depending on them, could be averaged over the orientation of the Regge manifold with respect to the center \( r = 0 \) and any given point \( x \). For \( f^2 \) (52), this is equivalent to averaging over the orientation of \( x \) under the integral sign in (52),

\[ \langle f^2 \rangle = \int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3 \sum_k 4 \sin^2(p_k a/2)} \exp(ip x) \frac{d^2n}{4\pi} \]  

(53)

where \( n = x/r \). This is the simplest analogue of averaging over the simplicial structures.

It is also interesting to see this solution in the center \( r = 0 \). At \( x = 0 \), the formula (52) reduces to some table integral [31],

\[ f^2(0) = \frac{8 r_g}{\pi a} (18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}) K^2[(2 - \sqrt{2})/(\sqrt{3} - \sqrt{2})] \approx 1.05 \frac{\pi r_g}{a}, \]  

(54)

\( K(k) \) is the complete elliptic integral of the first kind. It is also easy to see that

\[ f^2(\pm a, 0, 0) = f^2(0, \pm a, 0) = f^2(0, 0, \pm a) = f^2(0) \]  

(55)

To find \( \chi \), we should solve the equation

\[ \sum_{k=1}^{3} (\Delta_k \chi)^2 = \int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3 \sum_k 4 \sin^2(p_k a/2)} \exp(ip x). \]  

(56)
Looking at (55), we can assume the validity of the continuum equation for $\chi$ with an accuracy of better than 20% at the vertices with $r \geq a$ and determine $\chi(0)$, knowing $f^2(0)$,

$$\chi(x) = 2 \sqrt{r_g} \text{ at } r \geq a, \quad 3a^{-2}[(\chi(0) - \chi(-a,0,0))^2 = f^2(0) \approx 1.05 \pi \frac{r_g}{a},$$

$$\chi(0) \approx \left(2 - \sqrt{\frac{1.05 \pi}{3}}\right) \sqrt{r_g} a \approx 0.95 \sqrt{r_g} a. \quad (57)$$

Now checking for $f^2(a,0,0)$ with this $\chi(x)$ (since the equation for $\chi$ is not symmetrical w.r.t. the change $a \rightarrow -a$ due to the finite differences),

$$a^2 f^2(a,0,0) = [\chi(a,0,0) - \chi(0)]^2 + [\chi(a,0,0) - \chi(-a,0)]^2 + [\chi(a,0,0) - \chi(a,-a,0)]^2 \approx \frac{1.05 \pi}{3} r_g a + 8(1 - 2^{1/4})^2 r_g a \approx 1.39 r_g a. \quad (58)$$

This is at variance with the exact solution (55) within the same 20% (but in the opposite direction compared to the continuum formula (44)), which seems to be satisfactory.

This allows us to make a crude estimate of the effective (shown by the discretized Riemann tensor (6)) curvature value at the center. The Riemann tensor components in the continuum GR at $N = 1$, $g_{kl} = \delta_{kl}$ have the form

$$R_{klmn} = \frac{1}{4}[(f_{k,m} + f_{m,k})(f_{l,n} + f_{n,l}) - (f_{k,n} + f_{n,k})(f_{l,m} + f_{m,l})], \quad (59)$$

$$R_{0klm} = \frac{1}{2}(f_{m,l} - f_{l,m}) + f_n R_{nklm}, \quad (60)$$

$$R_{0k0l} = -\frac{1}{2} f^2_{,kl} + \frac{1}{4} (f_{m,k} - f_{k,m})(f_{m,l} - f_{l,m}) + f_m f_n R_{mknl}, \quad (61)$$

or, on the solution $f_k = \partial_k \chi$,

$$R_{klmn} = \chi_{,km} \chi_{,ln} - \chi_{,kn} \chi_{,lm}, \quad R_{0k0l} = f_n R_{nk0l}, \quad R_{0k0l} = -\frac{1}{2} f^2_{,kl} \quad (62)$$

The transition from the derivatives $\partial$ to the finite differences $\Delta$ is not unique, but this non-uniqueness is at the level of higher orders in $\Delta$; their inclusion makes sense if we also take them into account when passing from the original Regge action to the finite-difference action (6). Here we should provide a consistence with the considered field equations, in which we use the simplest Hermitian form of the difference Laplacian (51). The field equations are combinations of the components of the Ricci tensor and, therefore, of the Riemann tensor. To get such a Laplacian, the second derivative in
$R_{0klm}$ (61) should be analogously substituted by such an operator, $\partial_i \partial_k \to a^{-2}(\Delta_i \Delta_k + \Delta_i \Delta_k)/2$.

We can go directly to the piecewise-difference form of equations (62) (after the substitution $f_k = a^{-1}\Delta_k \chi$, (51), in (59-61)). The normal order of magnitude of the Riemann tensor components at $r_g a^{-1} \ll 1$ is $r_g a^{-3}$; $R_{0klm}$ and the second term in $R_{0klm}$ (62) are of a smaller order and, in particular, the Kretschmann scalar in this order is a sum of only positive terms,

$$R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} = (R_{klmn})^2 + 4(R_{0klm})^2. \quad (63)$$

Here $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is taken to raise the indices in the leading order over $r_g a^{-1}$. To estimate this at the center, we need to find a few typical finite differences there,

$$\Delta_1 \Delta_1 \chi(0) = \chi(0) - 2\chi(-a, 0, 0) + \chi(-2a, 0, 0)$$
$$\approx [(2 - \sqrt{0.35\pi}) + 2\sqrt{2} - 4]\sqrt{r_g a} \approx -0.22\sqrt{r_g a},$$
$$\Delta_2 \Delta_1 \chi(0) = \chi(0) - \chi(-a, 0, 0) - \chi(0, -a, 0) + \chi(-a, -a, 0)$$
$$\approx [(2 - \sqrt{0.35\pi}) + 2 \cdot 2^{1/4} - 4]\sqrt{r_g a} \approx -0.67\sqrt{r_g a},$$

$$\Delta_1 \Delta_1 f^2(0) = 2f^2(0) - f^2(-a, 0, 0) - f^2(a, 0, 0) = \frac{4}{3} r_g \approx 4.19 \frac{r_g}{a},$$

$$\Delta_2 \Delta_1 + \Delta_1 \Delta_2) f^2(0) = 2f^2(0) + f^2(-a, a, 0) + f^2(a, -a, 0)$$
$$- f^2(-a, 0, 0) - f^2(0, a, 0) - f^2(a, 0, 0) - f^2(0, -a, 0)$$
$$\approx 2 \cdot 1.05 \pi r_g \frac{a}{\sqrt{2}} + 2 \pi \frac{r_g}{a} - 4 \left(1.05 - \frac{2}{3}\right) \pi \frac{r_g}{a} \approx 3.19 \frac{r_g}{a}. \quad (64)$$

This yields typical components of the effective Riemann tensor of interest,

$$R_{1212} \approx -0.40 \frac{r_g}{a^3}, \quad R_{1213} \approx -0.30 \frac{r_g}{a^3}, \quad R_{0101} \approx 2.09 \frac{r_g}{a^3}, \quad R_{0102} \approx 0.80 \frac{r_g}{a^3}, \quad (65)$$

and the required Kretschmann scalar,

$$R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho}(0) = 12(R_{1212})^2 + 24(R_{1213})^2 + 12(R_{0101})^2 + 24(R_{0102})^2 \approx 71.9 \frac{r_g^2}{a^6}. \quad (66)$$

This is to be compared with that one at large distances (that is, the continuum value),

$$R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} = 12 \frac{r_g^2}{a^6}. \quad (67)$$

(66) looks like (67), cut off at $r$ slightly less than $a$ ($r = 0.74 a$).

The Schwarzschild metric follows approximately at large distances, $r^3 \gg a^2 r_g$, when $f \approx x(r_g/r^3)^{1/2}$, by the standard additive redefinition of $\tau$ by a certain function of $r$.

If $r_g r^{-3} a^2 \sim 1$, then the defect angles are also $\sim 1$, and probably we need to directly solve the Regge skeleton equations instead, and the above consideration reformulated in terms of edge lengths can give some initial approximation for this.
4 Conclusion

Remarkable is that the Schwarzschild geometry can be obtained from general Einstein equations without requiring a priori spherical symmetry. This allows to formulate the discrete version of the Schwarzschild problem and get its solution which tends at sufficiently large distances to the Painlevé-Gullstrand metric, which is described by pure shift ADM functions. The equations obtained from the 3 + 1 ADM formalism are convenient for this, this time not in the Hamiltonian form, but in the Lagrangian form. It is appropriate to solve iteratively these equations rewritten in a certain form (the black hole solution follows just in the zeroth order). Namely, the dynamical equations (for $\partial g_{kl}/\partial \tau$) turn into 3D Einstein equations for $g_{kl}$ with certain RHSs; the equations for initial conditions, Hamiltonian and diffeomorphism constraints (of which the Hamiltonian constraint being combined in a certain way with the dynamical equations and the diffeomorphism constraints) turn into the equations for the lapse-shift functions (here $N, f^k$). This scheme also seems convenient for analyzing any small perturbations of the Schwarzschild geometry in the Painlevé-Gullstrand coordinates, and not just its discretization.

Passing to the simplest periodic Regge lattice consists in rewriting the GR action in the finite-difference form in the leading order over metric variations from simplex to simplex. A complication is that fixing the edge lengths implies loose dynamical fixation of the space-like lengths while the discrete lapse-shifts should be fixed manually, like gauge conditions. Therefore, the metric (in the 3+1 ADM form) should be transformed to the synchronous frame (like the Lemaitre frame in the present context) and only then substituted into the finite-difference form of the action. However, in the particular case of the leading order over metric variations or over finite differences, the action in the transformed coordinates reduces to the action in the original coordinates with the 3+1 ADM metric by invariance, and we return to the above ADM equations, now in the discrete form and with the dynamically fixed elementary length scale, and get the discrete Painlevé-Gullstrand metric with the resolved singularity.

Turning to a comparison with the result of the resolution of the Schwarzschild black hole singularity in Loop Quantum Gravity [17, 18], we note that the quantum of area or area gap responsible for this resolution is proportional to the Barbero-Immirzi parameter $\gamma$. In our case, we have a discrete version of the Barbero-Immirzi parameter
\( \gamma \), and the factor in the functional measure \( F \) at \( \gamma \ll 1 \) has a local maximum at the elementary area scale \( a^2 \) proportional to \( \gamma \) (and to Planck scale, of course). However, there is another maximum, which at the value of another parameter \( \eta \gg 1 \) arises at \( a^2 \) proportional to \( \eta \) (2), and the maximum at \( a^2 \sim \gamma \) is negligibly smaller [19]. Then the resolution of the singularity in our case occurs on a certain scale \( a \) larger than the Planck scale, at least formally.

**Acknowledgments**

The present work was supported by the Ministry of Education and Science of the Russian Federation.

**References**

[1] T. Regge, General relativity theory without coordinates, *Nuovo Cimento* **19**, 558 (1961).

[2] G. Feinberg, R. Friedberg, T. D. Lee, and M. C. Ren, Lattice gravity near the continuum limit, *Nucl. Phys. B* **245**, 343 (1984).

[3] J. Cheeger, W. Müller, and R. Shrader, On the curvature of the piecewise flat spaces, *Commun. Math. Phys.* **92**, 405 (1984).

[4] R. M. Williams and P. A. Tuckey, Regge calculus: a brief review and bibliography, *Class. Quantum Grav.* **9**, 1409 (1992).

[5] T. Regge and R. M. Williams, Discrete structures in gravity, *Journ. Math. Phys.* **41**, 3964 (2000); (Preprint arXiv:gr-qc/0012035).

[6] H. W. Hamber, Quantum Gravity on the Lattice, *Gen. Rel. Grav.* **41**, 817 (2009); (Preprint arXiv:0901.0964[gr-qc]).

[7] J. Ambjorn, A. Goerlich, J. Jurkiewicz, and R. Loll, Nonperturbative Quantum Gravity, *Physics Reports* **519**, 127 (2012); (Preprint arXiv:1203.3591[hep-th]).

[8] A. Perez, The spin-foam approach to quantum gravity, *Living Rev. Relativity* **16** (2013), DOI: 10.12942/lrr-2013-3; (Preprint arXiv:1205.2019[gr-qc]).
[9] M. Dupuis, J. P. Ryan, and S. Speziale, Discrete gravity models and Loop Quantum Gravity: a short review, *SIGMA* **8**, 052 (2012); ([Preprint arXiv:1204.5394[gr-qc]]).

[10] C.-Y. Wong, Application of Regge calculus to the Schwarzschild and Reissner-Nordstrom geometries, *Journ. Math. Phys.* **12**, 70 (1971).

[11] L. C. Brewin, Einstein-Bianchi system for smooth lattice general relativity. I. The Schwarzschild spacetime, *Phys. Rev. D* **85**, 124045 (2012); ([Preprint arXiv:1101.3171[gr-qc]])

[12] P. A. Collins and R. M. Williams, Dynamics of the Friedmann universe using Regge calculus, *Phys. Rev. D* **7**, 965 (1973).

[13] A. P. Gentle, A cosmological solution of Regge calculus, *Class. Quantum Grav.* **30**, 085004 (2013); ([Preprint arXiv:1208.1502[gr-qc]])

[14] L C Brewin, A numerical study of the Regge calculus and Smooth Lattice methods on a Kasner cosmology, *Classical and Quantum Gravity* **32**, 195008 (2015); ([Preprint arXiv:1505.00067[gr-qc]])

[15] R. G. Liu and R. M. Williams, Regge calculus models of closed lattice universes, *Phys. Rev. D* **93**, 023502 (2016); ([Preprint arXiv:1502.03000[gr-qc]])

[16] L. Glaser and R. Loll, CDT and cosmology, *Comptes Rendus Physique* **18**, 265 (2017); [Preprint arXiv:1703.08160[gr-qc]]

[17] A. Ashtekar, J. Olmedo and P. Singh, Quantum Transfiguration of Kruskal Black Holes, *Phys. Rev. Lett.* **121**, 241301 (2018); [Preprint arXiv:1806.00648[gr-qc]]

[18] A. Ashtekar, J. Olmedo and P. Singh, Quantum extension of the Kruskal spacetime, *Phys. Rev. D* **98**, 126003 (2018); [Preprint arXiv:1806.02406[gr-qc]]

[19] V.M.Khatymovsky. On the non-perturbative graviton propagator. - *Int. J. Mod. Phys. A*, Vol. 33, No. 36, pp. 1850220, 2018; arXiv:1804.11212.

[20] J. Fröhlich, Regge calculus and discretized gravitational functional integrals, in *Nonperturbative Quantum Field Theory: Mathematical Aspects and Applications, Selected Papers* (World Scientific, Singapore, 1992), p. 523, IHES preprint 1981 (unpublished).
[21] R. Arnowitt, S. Deser, and C. W. Misner, Canonical variables for general relativity, *Phys. Rev.* **117**, 1595 (1960).

[22] R. Arnowitt, S. Deser, and C. W. Misner, The Dynamics of General Relativity, in *Gravitation: an introduction to current research*, Louis Witten ed. (Wiley, 1962), chapter 7, p. 227; (*Preprint* arXiv:gr-qc/0405109).

[23] C. W. Misner, Feynman quantization of general relativity, *Rev. Mod. Phys.* **29**, 497 (1957).

[24] B. S. DeWitt, Quantization of fields with infinite-dimensional invariance groups. III. Generalized Schwinger-Feynman theory, *Journ. Math. Phys.* **3**, 1073 (1962).

[25] V.M.Khatsymovsky. On the discrete Christoffel symbols. - *Int. J. Mod. Phys. A*, Vol. 34, No. 30, pp. 1950186, 2019; arXiv:1906.11805.

[26] G. Lemaitre, *Ann. Soc. Sci. Bruxelles* A **53**, 51 (1933).

[27] K. P. Stanyukovich, On the question of the Schwarzschild metric in a synchronous reference frame, *Reports of the USSR Acad. Sci.* 187, 75 (1969).

[28] P. Painlevé, *C. R. Acad. Sci. (Paris)* 173(October 24), 677 (1921).

[29] A. Gullstrand, *Arkiv. Mat. Astron. Fys.* 16(8), 1 (1922).

[30] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman and Company, San Francisco, 1973).

[31] W. Magnus und F. Oberhettinger, Formeln und Satze für die speziellen Funktionen der mathematischen Physik, Springer Verlag Berlin - Göttingen - Heidelberg, 1948.