Strassen’s invariance principle for random walk in random environment

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Abstract

In this paper, we consider random walk in random environment on \( \mathbb{Z}^d \) \((d \geq 1)\) and prove the Strassen’s strong invariance principle for this model, via martingale argument and the theory of fractional coboundaries of Derriennic and Lin [4], under some conditions which require the variance of the quenched mean has a subdiffusive bound. The results partially fill the gaps between law of large numbers and central limit theorems.

Keywords: random walk in random environment, Strassen’s invariance principle, the law of iterated logarithm, fractional coboundaries.

2000 MR Subject Classification : 60B10, 60F15.

1 Introduction

Random motions in random media gather a variety of probability models often originated from physical science, such as solid physics, biophysics and so on. Random walk in a random environment is one of the basic models. Various interesting problems arise when we consider the different possible limit theorems, such as the 0−1 law, law of large numbers, central limit theorems, large deviations and so on. See also the lecture notes given by Sznitman [16], Molchanov [8] and Zeitouni [17] for a survey. The main object of this work is to prove the invariance principle for the law of iterated logarithm for a class of random walks in certain random environments. In this model, an environment is a collection of transition probabilities \( \omega = (\pi_{x,y})_{x,y \in \mathbb{Z}^d} \in \varphi^{\mathbb{Z}^d} \), where \( \varphi = \{(p_z)_{z \in \mathbb{Z}^d}, p_z \geq 0, \sum p_z = 1\} \) a family of distributions on \( \mathbb{Z}^d \). We denote the space of all such transition probabilities by \( \Omega \). The space \( \Omega \) is endowed with the canonical product \( \sigma \)-algebra \( \mathfrak{F} \). On the space of

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environments \((\Omega, \mathcal{F})\), we are given a \(T\)-invariant probability \(P\) with \((\Omega, \mathcal{F}, (T_z)_{z \in \mathbb{Z}^d}, P)\) ergodic, where \((T_z, z \in \mathbb{Z}^d)\) denote the canonical shift on \(\Omega\), i.e. \(\pi_{x,y}(T_z(\omega)) = \pi_{x+y, z}(\omega)\). The environments are called independent identical distribution (in short i.i.d.), if the family of random probabilities vectors \(\{\pi_{x,y}(\omega)\}_{x \in \mathbb{Z}^d}\) are i.i.d.

We now turn to describe the walk related to the environment. First, an environment \(\omega\) is chosen from the distribution \(P\), and kept fixed throughout the time evolution. Then the random walk in environment \(\omega\), is a canonical Makov chain \(X := (X_n, n \geq 0)\) on \((\mathbb{Z}^d)^N\), with state space \(\mathbb{Z}^d\) and law \(P^\omega_z\), under which

\[
P^\omega_z(X_0 = z) = 1, \quad P^\omega_z(X_{n+1} = y|X_n = x) = \pi_{x,y}(\omega).
\]

The law \(P^\omega_z\) is called the quenched law. Then we can also define a measure in the sense of averaging the environments,

\[
P_z = \int P^\omega_z dP.
\]

The law \(P_z\) is called the annealed law. Obviously, under \(P_z\), \((X_n, n \geq 0)\) is not a Markov chain in general. We also use \(E, E_z, E^\omega_z\) for the expectations operators with respect to \(P\), \(P_z\), and \(P^\omega_z\) respectively.

Our goal in this paper, is to consider the invariance principle for the law of iterated logarithm for the random walk in environment, under some assumptions introduced in Section 2. The results of iterated logarithm types for Sinai’s random walk in random environment can be found in Hu and Shi [6].

It is well known that the law of iterated logarithm (in short LIL) is closely related to the central limit theorems (in short CLT) in some sense. By the technique of split chains and regeneration, Chen [2] systematically studied the CLT and LIL for ergodic Markov chain under the frame of Harris recurrent. Bhattacharya [1] gave the functional CLT and LIL for Markov processes. And Kifer [7] obtained the CLT and LIL for Markov chain in random environment via certain mixing assumptions and via the martingale approach.

Note that, Rassoul-Agha and Seppäläinen [14] mainly rely on the invariance principle for vector-valued martingale, so it is possible to obtain the invariance principle for LIL for random walk in random environment under suitable conditions, only if we can develop the corresponding theory for vector-valued martingale. In the case of real-valued martingale, the Skorokhod representation plays an important role, for example, Hall and Heyde [5]. However, we encounter the essential difficulties, when considering the vector-valued martingale, since Monrad and Philipp [9] proved that it is impossible to embed a general \(\mathbb{R}^d\)-valued martingale in an \(\mathbb{R}^d\)-valued Gaussian process.

In the present paper, we will use essentially the strategy of Maxwell and Woodrooife [12] and the method developed by Morrow and Philipp [14] and Zhang [18]. Moreover, we identify the lim sup in LIL just the square root of the trace of the diffusion matrix corresponding to functional CLT. And this partially fills the gaps between law of large numbers and central limit theorems.
2 Some preliminaries and main results

In this section, we will give some assumptions and state our main results. Let us start with the construction of the auxiliary Markov chain.

For any \( \omega \in \Omega \), let \( \bar{\omega} := (\bar{\omega}(n) = T_X \omega, n \geq 0) \), then \( \bar{\omega} \) is a Markov chain on \( \Omega \) with transition operator

\[
\Pi f(\omega) = \sum_{x \in \mathbb{Z}^d} \pi_{0,x}(\omega) f(T_x \omega),
\]

(2.1)

where \( f \) is a bounded measurable function defined on \( \Omega \), and with the one step transition kernel,

\[
q(\omega, A) = P_0^\omega(T_X \omega \in A), \quad A \in \mathcal{F}.
\]

(2.2)

In this paper, we always assume that there exists a probability measure \( P_\infty \) on the measurable space \((\Omega, \mathcal{F})\) that is invariant for the transition \( \Pi \) and ergodic for the Markov process with generator \( \Pi - I \). Then, the operator \( \Pi \) can be extended to a contraction on \( L^p(P_\infty) \), for every \( p \in [1, \infty] \). When the initial distribution is \( P_\infty \), we will denote this Markov process by \( \hat{P}_0^\infty \). Let \( \hat{P}_0^\infty := \int P_0^\omega dP_\infty \), and \( \mathbb{E}_\infty, E_0^\infty \) the corresponding expectation operators. Note that \( \hat{P}_0^\infty \) is the probability measure induced by \( P_0^\infty \) and \( (T_X \omega) \) onto \( \Omega^N \).

With these notations, the measure

\[
\nu^\infty(d\omega_0, d\omega_1) := q(\omega_0, d\omega_1)P_\infty(d\omega_0)
\]

(2.3)

describes the law of \( (\omega, T_X \omega) \) under \( P_0^\infty \).

Next, we consider the asymptotic Poisson’s equation. For any \( \epsilon > 0 \), let \( h_\epsilon \) be the solution to the equation

\[
(1 + \epsilon)h - \Pi h = g,
\]

(2.4)

where \( g \) is a function defined on \( \Omega \) such that \( \int g dP_\infty = 0 \) and \( \int g^2 dP_\infty < \infty \). In fact,

\[
h_\epsilon = \sum_{k=1}^{\infty} (1 + \epsilon)^{-k} \Pi^{k-1} g \in L^2(P_\infty)
\]

(2.5)

is the solution of the equation (2.4). We also define

\[
S_n(g) := \sum_{k=0}^{n-1} g(T_X \omega)
\]

(2.6)

and

\[
H_\epsilon(\omega_0, \omega_1) := h_\epsilon(\omega_1) - \Pi h_\epsilon(\omega_0).
\]

(2.7)
Then we have
\[
S_n(g) := \sum_{k=0}^{n-1} g(T_X^k \omega)
\]
\[
= \sum_{k=0}^{n-1} \{(1 + \epsilon)h_\epsilon(T_X^k \omega) - \Pi h_\epsilon(T_X^k \omega)\}
\]
\[
= M^\epsilon_n + R^\epsilon_n + \epsilon S_n(h_\epsilon),
\]
(2.8)

where \(M^\epsilon_n = \sum_{k=0}^{n-1} H_\epsilon(T_X^k \omega, T_X^{k+1} \omega)\), \(R^\epsilon_n = h_\epsilon(\omega) - h_\epsilon(T_X^n \omega)\).

In order to discuss the Poisson’s equation ultiorily, we need introduce some assumptions.

**Assumptions**

(A1) There exists a constant \(M < \infty\) such that
\[
\mathbb{P}_\infty(\pi_{x,y}(\omega) = 0, \text{ when } |x - y| > M) = 1,
\]
(2.9)

where \(|\cdot|\) denotes the Euclidean distance.

(A2) There exists an \(\alpha < 1/2\) such that
\[
\sqrt{\mathbb{E}_\infty(E_0^n(X_n) - n\nu)^2} = O(n^\alpha).
\]
(2.10)

**Remarks 2.1.** The assumption (A1) implies that the particle has finite jump at each transition. For instance, setting \(M = 1\), we get the nearest neighbor random walk in random environment. The assumption (A2) shows that the variance of the quenched mean has a subdiffusive bound under the invariant and ergodic measure \(\mathbb{P}_\infty\).

Define the drift for the random walk in random environment as follows
\[
D(\omega) = E_0^\omega X_1 = \sum_{z \in \mathbb{Z}^d} z\pi_{0,z}(\omega).
\]
(2.11)

Notice that the assumption (A1) yields \(D \in L^2(\mathbb{P}_\infty)\). Denote \(\nu = \mathbb{E}_\infty D\) the drift under the annealed law \(P_0^\infty\). If set \(g = D - \nu\), then
\[
X_n - n\nu = X_n - \sum_{k=0}^{n-1} D(T_X^k \omega) + M^\epsilon_n + R^\epsilon_n + \epsilon S_n(h_\epsilon)
\]
\[
= W_n + M^\epsilon_n + R^\epsilon_n + \epsilon S_n(h_\epsilon).
\]
(2.12)

where \(W_n = X_n - \sum_{k=0}^{n-1} D(T_X^k \omega)\) is a martingale under \(P_0^\omega\) with respect to the filtration \(\{G_n = \sigma(X_0, X_1, \cdots, X_n), n \geq 0\}\) for \(\mathbb{P}_\infty\)-a.s. \(\omega\).

Under the assumptions (A1) and (A2), Rassoul-Agha and Seppäläinen [14] obtained the invariance principle for random walks in random environments. We summarize their results in the following theorems.
Theorem RS Let $d \geq 1$ and assume that (A1) and (A2) are satisfied.

1. The limit $H = \lim_{\varepsilon \to 0} H_\varepsilon$ exists in $L^2(\nu^\infty)$.

2. Denote $M_n = \sum_{k=0}^{n-1} H(T_{X_k}, T_{X_{k+1}})$, then $X_n - n\nu = W_n + M_n + R_n$. $E_0^\infty(|R_n|^2) = O(n^{2\alpha})$, and for $P_\infty$-almost surely $\omega$, $(M_n, n \geq 1)$ is a $P_0^\omega$-square integrable martingale relative to the filtration $(\mathcal{G}_n, n \geq 0)$.

3. For $P_\infty - a.s. \omega$, $n^{-1/2}(X_{1n} - [n\nu])$ converges in distribution to the Brown motion with diffusion matrix $\Sigma$, under $P_\omega$. Furthermore, $n^{-1/2} \max_{k \leq n}|E_\omega^0 X_k - kv|$ converges to zero, $P_\infty - a.s. \omega$, and then the same invariance principle holds also for $n^{-1/2}(X_{[n]} - E_0^\omega X_{[n]})$.

Where $\nu = \nu_\omega D$ is the drift under the annealed law $P_0^\omega$, and $\Sigma = E_0^\infty[(X_1 - D(\omega) + H(\omega, T_{X_1}(\omega))(X_1 - D(\omega) + H(\omega, T_{X_1}(\omega))^T]$ is the diffusion matrix ($A^T$ denotes the transpose of matrix or vector $A$).

Remarks 2.2. Furthermore, we know that $H \in L^q(\nu^\infty)$ for some $q \in (2,5/2)$ since the environment is finite (See Theorem 1 in [13]). For the more detailed discussions on the above theorem, please see Rassoul-Agha and Seppäläinen [14].

In order to obtain the invariance principle for the law of iterated logarithm for random walks in random environments, we need the additional assumption,

(A3) For any $\omega \in \Omega$, there exist an integer $l \geq 1$, $0 < \lambda \leq 1$ and a measure $\mu$ on $(\Omega, \mathcal{F})$ such that

$$\sum_{x_1, x_2, \ldots, x_l \in E, |x| \leq M, 1 \leq i \leq l} \pi_{0x_1}(\omega) \cdots \pi_{0x_l}(T_{x_1+\ldots+x_{l-1}}(\omega))1_A(T_{x_1+\ldots+x_l}(\omega)) \geq \lambda \mu(A), \ A \in \mathcal{F}. \quad (2.13)$$

Remarks 2.3. This assumption is a technical condition, since we need the auxiliary Markov chain constructed above has the space $\Omega$ as its a small set. It is the further task to explain and remove this assumption.

For introduce our main results, we firstly give some notations. Let $C([0,1],\mathbb{R}^d)$ be the Banach space of continuous maps from $[0,1]$ to $\mathbb{R}^d$, endowed with the supremum norm $\| \cdot \|$, using the Euclidean norm in $\mathbb{R}^d$. Denote $K$ the set of absolutely continuous maps $f \in C([0,1],\mathbb{R}^d)$, such that

$$f(0) = 0, \quad \int_0^1 |\dot{f}(t)|^2 \, dt \leq 1, \quad (2.14)$$

where $\dot{f}$ denotes the derivative of $f$ determined almost everywhere with respect to Lebesgue measure. Obviously, $K$ is relatively compact and closed.

Let $d \geq 1, X = (X_n, n \geq 0)$ be a random walk in random environment. Define for $t \in [0,1]$,

$$\xi_n(t) = (2v_n^2 \log \log v_n)^{-1/2}(X_k - kv + (X_{k+1} - X_k) - \nu)(v_{k+1}^2 - v_k^2)^{-1}(v_n^2 - v_k^2))$$

for $v_k^2 \leq r v_{k+1}^2 \leq v_{k+1}^2$, $k = 0, 1, 2, \ldots, n - 1$, where $v_n^2$ denotes the trace of the matrix given in Section [3]. In order to avoid difficulties in specification, we adopt the convention that $\log \log x = 1$, if $0 < x \leq e^e$. Then, $\xi_n$ is a random element with values in $C([0,1],\mathbb{R}^d)$. 5
After these preparations, we are now in a position to state our main results.

**Theorem 1.** Under the Assumptions (A1), (A2) and (A3), for $P_0^\infty - a.s.$, the sequence of functions $(\xi_n(\cdot), n \geq 1)$ is relatively compact in the space $C([0, 1], \mathbb{R}^d)$, and the set of its limit points as $n \to \infty$, coincides with $K$.

**Theorem 2.** If assumptions (A1), (A2) and (A3) are satisfied, then

$$
\limsup |X_n - nv|/ \sqrt{2n \log \log n} < +\infty, \quad P_0^\infty - a.s. \quad (2.15)
$$

Furthermore, we have

$$
\limsup |X_n - nv|/ \sqrt{2n \log \log n} = \sqrt{\text{tr}(\Omega)}, \quad P_0^\infty - a.s. \quad (2.16)
$$

where $\text{tr}(\cdot)$ denotes the trace operator of a matrix.

**Remarks 2.4.** It is clear that the statement, almost surely relatively compact of sequence $((X_n - nv)/ \sqrt{2n \log \log n}, n \geq 1)$ in $\mathbb{R}^d$, is equivalent to (2.15).

**Remarks 2.5.** In generally, Theorem 1 is called Strassen’s strong invariance principle or functional LIL. Moreover, Theorem 1 and Theorem 2 also hold, when the normalized center is random, $E_0^\omega X_n$, by Theorem RS.

### 3 The proof of main results

In this section, we will prove our main results, Theorem 1 and Theorem 2, mentioned in Section 2, via the martingale approach and the theory of fractional coboundaries.

Denote $Z_n = W_n + M_n$ and let $w_n$, $m_n$ and $z_n$ the martingale difference corresponding to $W_n$, $M_n$ and $Z_n$ respectively. It is easy to see the following facts,

- $(\bullet) (w_n, n \geq 1)$ is uniformly bounded martingale difference sequence under $P_0^\omega$.
- $(\bullet) (m_n, n \geq 1)$ is stationary and ergodic sequence under $P_0^\infty$.

Define for each $n$ the conditional covariance matrix

$$
A_n := \sum_{k=1}^n E^\omega_0(\xi_k z^\prime_k | \mathcal{G}_{k-1}), \quad (3.1)
$$

and set $v_n^2 = \text{tr}(A_n)$. Note that, $v_n^2 = \sum_{k=1}^n E^\omega_0(|z_k|^2 | \mathcal{G}_{k-1})$. We know by the Markov property,

$$
A_n = \sum_{k=1}^n E^\omega_0(T_{x_k-1} \omega (z_k z^\prime_k))
= \sum_{k=1}^n E^\omega_0(k-1) (z_1 z^\prime_1), \quad (3.2)
$$
where \( \bar{\omega} = (\bar{\omega}(n) = T_{X_n}(\bar{\omega}, n \geq 0) \) a stationary and ergodic Markov chain under \( P_0^\infty \) with initial distribution \( \mathbb{P}_\omega \), since the discussions in Section \( \text{II} \). Hence, from the Birkhoff and Khinchin’s ergodic theory, we know that,

\[
\lim_{n} n^{-1} A_n = \mathbb{D}, \quad P_0^\infty - a.s. \quad (3.3)
\]

And we also have,

\[
\lim_{n} n^{-1} v_n^2 = \text{tr}(\mathbb{D}), \quad P_0^\infty - a.s. \quad (3.4)
\]

For any \( d \times d \) matrix \( A \), define the matrix norm,

\[
\|A\|_m := \sup_{u \in \mathbb{R}^d, |u|=1} |Au|.
\]

(3.5)

### 3.1 Proof of Theorem 1

We will prove separately, in a succession of steps.

**Step I:** Denote \( B(\cdot) \) the Brownian Motion in \( \mathbb{R}^d \) with mean 0 and diffusion matrix \( \Sigma \). Define

\[
B_n(t) = (2n \log \log n)^{-1/2} B(nt)
\]

for \( t \in [0, 1] \) and \( n \geq 3 \). Then we have, by the Theorem 1 of Strassen \( [15] \) the sequence \( \{B_n(\cdot), n \geq 3\} \) is almost surely relatively compact in \( C([0, 1], \mathbb{R}^d) \) and the set of its limit points coincides with \( \sqrt{\text{tr}(\Sigma)} K \).

**Step II:** In this step, we mainly consider the almost sure approximation of the martingale \( Z_n \) introduced in the above section by a suitable \( \mathbb{R}^d \)-valued Brown motion. Since Theorem 1.2 of Zhang \( [18] \), under suitable conditions, i.e., the following (B1) and (B2):

\[
(B1) \quad \sum_{n \geq 1} E_0^\infty(|z_n|^2 1_{|z_n|^2 \geq f(n)})/f(n)) < \infty,
\]

\[
(B2) \quad \|A_n - n \mathbb{D}\|_m = o(f(n)), \quad P_0^\infty - a.s.
\]

we have,

\[
\left| \sum_{n \geq 1} z_n 1_{|z_n| \leq t} - B(t) \right| = O(t^{1/2}(f(t)/t)^{1/50d}), \quad d < \infty, \quad P_0^\infty - a.s. \quad (3.7)
\]

where, \( f(x) \) is non-decreasing and tends to \( \infty \), along the positive axis, \( f(x)(\log x)^\varphi /x \) is non-increasing for some \( \varphi > 50d \), and \( f(x)/x^\delta \) is non-decreasing for some \( 0 < \delta < 1 \). Please notice the difference, conditions (B1) and (B2) here, with the equations (1.12) and (1.13) in Zhang \( [18] \). Hence, the main object turns to check the conditions (B1) and (B2).

Firstly, we consider the condition (B1). If set \( y_n := z_n/ \sqrt{f(n)} \), then \( (y_n, n \geq 1) \) is also a martingale difference sequence under \( P_0^\infty \). Then the above problem turns to be

\[
\sum_{n \geq 1} E_0^\infty(|y_n|^2 1_{|y_n| \geq 1}) < \infty.
\]

(3.8)
It is enough to show that for some $\kappa > 0$,

$$E_0^\infty(|y_n|^2 \textbf{1}_{|y_n| \geq 1}) = O(n^{-(1+\kappa)}). \quad (3.9)$$

Note that $z_n = w_n + m_n$, then

$$|y_n|^2 \textbf{1}_{|y_n| \geq 1} = |w_n + m_n|^2 \textbf{1}_{|y_n| \geq 1} / f(n) \leq (M^2 + 2M|m_n| + |m_n|^2) \textbf{1}_{|y_n| \geq 1} / f(n). \quad (3.10)$$

Hence, we need to deal with three terms,

$$I_n := \frac{1}{f(n)} ||w_n + m_n||^2 \textbf{1}_{|y_n| \geq 1},$$

$$\Pi_n := |m_n| \frac{1}{f(n)} \textbf{1}_{|y_n| \geq 1},$$

$$\PiI_n := |m_n|^2 \frac{1}{f(n)} \textbf{1}_{|y_n| \geq 1}.$$  

Since $(m_n, n \geq 1)$ is stationary under $P_0^\infty$, the key estimation naturally is the probability of the event $\{|y_n| \geq 1\}$. Now put $f(x) = x^\gamma$ for some $\gamma \in (2/q, 1)$, hence there exists a constant $\kappa \in (0, \gamma q/2 - 1)$ such that $\gamma q/2 \geq 1 + \kappa$. If we assume that

$$P_0^\infty(|y_n| \geq 1) = O((n)^{-(q(1+\kappa-\gamma)/(q-2))}. \quad (3.11)$$

Then

$$E_0^\infty(\frac{1}{f(n)} |y_n|^2 \textbf{1}_{|y_n| \geq 1}) \leq n^{-\gamma} E_0^\infty(|m_n|^2 \textbf{1}_{|m_n| \geq 1})^{1/q} P_0^\infty(|y_n| \geq 1)^{1-1/q}$$

$$= n^{-\gamma} E_0^\infty(|m_n|^q)^{1/q} P_0^\infty(|y_n| \geq 1)^{1-1/q}$$

$$= O(n^{-(q-1)(1+\kappa-\gamma)/(q-2)})$$

$$= o(n^{-(1+\kappa)}/f(n)). \quad (3.13)$$

Next we consider $(\Pi_n)$ and $(\PiI_n)$.

$(\Pi_n)$

$$E_0^\infty(|m_n| \textbf{1}_{|y_n| \geq 1}) \leq n^{-\gamma} E_0^\infty(|m_n|^q)^{1/q} P_0^\infty(|y_n| \geq 1)^{1-1/q}$$

$$= O(n^{-(q-1)(1+\kappa-\gamma)/(q-2)})$$

$(\PiI_n)$

$$E_0^\infty(|m_n|^2 \textbf{1}_{|y_n| \geq 1}) \leq n^{-\gamma} E_0^\infty(|m_n|^q)^{2/q} P_0^\infty(|y_n| \geq 1)^{1-2/q}$$

$$= O(n^{-(1+\kappa)}). \quad (3.14)$$
Finally we analyze the equation (3.11). It is sufficiently to consider the following two parts,

\[ P_0^\infty(|m_n| \geq 2^{-1}n^{\gamma/2}) \quad \text{and} \quad P_0^\infty(|w_n| \geq 2^{-1}n^{\gamma/2}). \] (3.15)

By Markov inequality and the facts (♣) and (♠), we get

\[ P_0^\infty(|m_n| \geq 2^{-1}n^{\gamma/2}) \leq (2^qE_0^\infty|m_1|^q)n^{-\gamma q/2}, \] (3.16)

\[ P_0^\infty(|w_n| \geq 2^{-1}n^{\gamma/2}) \leq (2^{M})^q n^{-\gamma q/2}. \] (3.17)

If setting \( \kappa := \gamma q/2 - 1 > 0 \), we have

\[ P_0^\infty(|y_n| \geq 1) = O(n^{-\gamma q/2}) = O((n)^{-q(1+\kappa-\gamma)/(q-2)}). \] (3.18)

Then this completes the discussion of condition (B1).

As for condition (B2), we need to estimate the rate of convergence of

\[ \|A_n - n\mathfrak{V}\|_m. \]

That is to say, we want to have the following order estimations,

\[ \left\| \frac{A_n}{n} - \mathfrak{V} \right\|_m = o(n^{\gamma-1}), \quad P_0^\infty - a.s. \] (3.19)

or

\[ \left\| \frac{\gamma^2}{n} - \text{tr}(\mathfrak{V}) \right\|_m = o(n^{\gamma-1}), \quad P_0^\infty - a.s. \] (3.20)

Denote \( \phi(\omega) := E_0^\infty(|z_1|^2) \), then the above problem turns to be the problem of ergodic convergence rate for additive functionals of stationary and ergodic Markov chain, i.e., the rate of

\[ \frac{1}{n} \sum_{k=0}^{n-1} \phi(\tilde{\omega}(k)) \longrightarrow \int \phi(\omega)d\mathfrak{P}_\infty = \text{tr}(\mathfrak{V}), \quad P_0^\infty - a.s. \]

The above problem can be rewritten as follows,

"Given ergodic Markov Chain \((Y_k, k \geq 1)\) and function \(f\) with \(f \in L^{q/2}\) and \(\int f d\pi = 0\) (where \(\pi\) is invariant distribution) under what condition do we have

\[ n^{-\gamma} \sum_{k=1}^{n} f(Y_k) \longrightarrow 0 \quad \text{almost surely}?" \]

To answer this problem, we need Chen’s theorem [3].

**Theorem Chen** Let \( \{Y_n\}_{n \geq 0} \) be an ergodic Markov chain with state space \((H, \mathcal{H})\), \(f\) a measurable function from \(H\) to some separable Banach space \(B\), and let \(1 \leq p < 2\). Then the following two statements (1) and (2) are equivalent:
(1) For some (all) small set $C$,
\[
\int_C \pi(dx)E_x \max_{n \leq \tau_C} \left\| \sum_{0}^{n-1} f(Y_k) \right\|^p < \infty, \tag{3.21}
\]
and
\[
n^{-1/p} \sum_{0}^{n-1} f(Y_k) \longrightarrow 0 \quad \text{in probability.} \tag{3.22}
\]

(2) The Marcinkiewicz-Zygmund’s law of the large numbers holds, i.e.,
\[
\lim_{n \to \infty} n^{-1/p} \sum_{0}^{n-1} f(Y_k) = 0, \quad \text{a.s.} \tag{3.23}
\]

Since the remarks followed this theorem in Chen [2], we know that the equation (3.21) and the condition
\[
\int f(x)\pi(dx) = 0,
\]
imply the equation (3.22) when $B = \mathbb{R}$. Hence we only need to find the suitable conditions to describe (3.21).

Define
\[
S := \{ \text{all small sets} \},
\]
\[
S_{\varphi} := \{ A \in S : \int_C \pi(dx)E_x \max_{n \leq \tau_A} \left\| \sum_{1}^{n} f(Y_k) \right\| \},
\]
where $\psi(x) = x^{1/\gamma}$.

Notice the dichotomy results obtained in two of Chen’s works [2] and [3], it is easy to show the following statement by applying Chen’s idea,
\[
S_{\varphi} = \emptyset \quad \text{or} \quad S.
\]

By the assumption (A3), for all $\omega \in \Omega$,
\[
q^{(l)}(\omega, A) \geq \lambda u(A), \quad A \in \mathfrak{F},
\]
we have the space $\Omega$ is a small set. Hence, the equation (3.21) turns to be,
\[
E^\infty_0 |f(\omega)|^{1/\gamma} < \infty. \tag{3.24}
\]

However
\[
f(\omega) = E^\infty_0 |z_1|^2 - E^\infty_0 |z_1|^2, \quad f \in L^{q/2},
\]
\[
(2 < q < 5/2, \quad 1/2 < 4/5 < 2/q < \gamma < 1),
\]
these yield the above equation (3.24). Hence we complete the discussion on the condition (B2).

**Step III:** For \( t \in [0, 1] \), we define the \( \mathbb{R}^d \)-valued functions, 

\[
\eta_n(t) := \eta(tv_n^2), \\
\bar{\eta}_n(t) := \sum_{k=1}^{n} z_k 1_{[v_k^2 \leq n^2]} + \sum_{k=0}^{n-1} z_{k+1} (v_{k+1}^2 - v_k^2)^{-1} (tv_n^2 - v_k^2) 1_{[v_k^2 \leq n^2 \leq v_{k+1}^2]},
\]

where \( v_0^2 = 0 \) and \( \eta(t) = \sum_{n \geq 1} z_n 1_{[v_n^2 \leq t]} \).

Then, since the **Step II**, we need to show

\[
\sup_{t \in [0,1]} |\bar{\eta}_n(t) - \eta_n(t)| = o((2v_n^2 \log \log v_n^2)^{1/2}). \tag{3.25}
\]

In fact,

\[
\sup_{t \in [0,1]} |\bar{\eta}_n(t) - \eta_n(t)|
\]

\[
= \max_{0 \leq k \leq n-1} \sup_{v_k^2 \leq n^2 \leq v_{k+1}^2} |z_{k+1} (v_{k+1}^2 - v_k^2)^{-1} (tv_n^2 - v_k^2)|
\]

\[
= \max_{0 \leq k \leq n-1} |z_{k+1}|. \tag{3.26}
\]

Hence, we only need prove the below estimation

\[
\max_{0 \leq k \leq n-1} |z_{k+1}| = o((2v_n^2 \log \log v_n^2)^{1/2}). \tag{3.27}
\]

If we notice that

\[
z_k = w_k + m_k, \tag{3.28}
\]

and the fact (▲), then the problem turns to be

\[
\max_{1 \leq k \leq n} |m_k| = o((2v_n^2 \log \log v_n^2)^{1/2}), \quad P_0^\omega \text{ a.s.} \tag{3.29}
\]

It is easy to see, for any \( \epsilon > 0 \),

\[
P_0^\omega (\max_{1 \leq k \leq n} |m_k| \geq \epsilon (2n \log \log n)^{1/2})
\]

\[
\leq E_0^\omega (\max_{1 \leq k \leq n} |m_k|^q) / \epsilon^q (2n \log \log n)^{q/2}. \tag{3.30}
\]

Next, let us give the estimation of \( E_0^\omega (\max_{1 \leq k \leq n} |m_k|^q) \). The following important inequality is a moment inequality from Môricz [1]\).

**Lemma M** Let \( p > 0 \) and \( \beta > 1 \) be two positive real numbers and \( Z_i \) be a sequence of random variables. Assume that there are nonnegative constants \( a_j \) satisfying

\[
E|\sum_{j=1}^{i} Z_j|^p \leq (\sum_{j=1}^{i} a_j)\beta, \tag{3.31}
\]
for $1 \leq i \leq n$. Then
\[ E(\max_{1 \leq i \leq n} |\sum_{j=1}^{i} Z_j|^p) \leq C_{p, \beta} (\sum_{i=1}^{n} a_i)^\beta, \] (3.32)
for some positive constant $C_{p, \beta}$ depending only on $p$ and $\beta$.

**Lemma 3.1.** For any enough large $n$, there exists a positive constant $C$ such that
\[ E^\infty_0 (\max_{1 \leq i \leq n} |m_i|^q) \leq CE^\infty_0 |m_1|^q. \] (3.33)

**Proof.** Let $E^\infty_0 |m_1|^q = a^2(q)$. Since the fact (∗), for any $k \geq 1$, we have the following relation,
\[ E^\infty_0 |m_k|^q \leq (\sum_{i=1}^{k} a_i)^2, \] (3.34)
where $a_1 = a(q)$ and $a_i = 0$ for $2 \leq i \leq k$. Hence, by Lemma M, there exists a constant $C > 0$, such that
\[ E^\infty_0 (\max_{1 \leq i \leq n} |m_i|^q) \leq C(\sum_{i=1}^{n} a_i)^2 = CE^\infty_0 (|m_1|^q). \] (3.35)

This completes the proof of the lemma. \qed

Lemma 4.1 together with equation (3.30) immediately yields,
\[ P^\infty_n (\max_{1 \leq i \leq n} |m_i| \geq \epsilon) = \epsilon^2(2n \log \log n)^{1/2} \leq O((n \log \log n)^{-q/2}). \] (3.36)

Hence, the above estimation (3.29) is obtained, by Borel-Cantelli’s lemma and equation (3.4).

**Step IV:** We want to give the order of
\[ \sup_{t \in [0,1]} |\tilde{\eta}_n(t) - B(tv_n^2)|. \] (3.37)

Firstly, we rewrite it as follows,
\[ \sup_{t \in [0,1]} |\tilde{\eta}_n(t) - B(tv_n^2)| \]
\[ = \sup_{t \in [0,1]} |\tilde{\eta}_n(t) - \eta_n(t) + \eta_n(t) - B(tv_n^2)| \]
\[ \leq \sup_{t \in [0,1]} |\tilde{\eta}_n(t) - \eta_n(t)| + \sup_{t \in [0,1]} |\eta_n(t) - B(tv_n^2)|. \] (3.38)

From above **Step III**, we know
\[ \sup_{t \in [0,1]} |\tilde{\eta}_n(t) - \eta_n(t)| = o((2v_n^2 \log \log v_n^2)^{1/2}). \]
However, by the equation (3.7) in Step II, we have the following estimation,

\[
\sup_{t \in [0,1]} |\eta_n(t) - B(t v_n^2)| = O((t v_n^2)^{1/2}(f(t v_n^2)/tv_n^2)^{1/50d}) = o((2v_n^2 \log \log v_n^2)^{1/2}).
\] (3.39)

This gives the following order estimation,

\[
\sup_{t \in [0,1]} |\tilde{\eta}_n(t) - B(t v_n^2)| = o((2v_n^2 \log \log v_n^2)^{1/2}), \quad P_0^\infty - a.s.
\] (3.40)

Step V: Notice that,

\[
\sup_{t \in [0,1]} |\tilde{\eta}_n(t) - (2v_n^2 \log \log v_n^2)^{1/2} \xi_n(t)| = \max_{0 \leq k \leq n-1} \sup_{v_n^2 k \leq v_n^2 k + 
\leq 3 \max_{1 \leq k \leq n} |R_k|.
\] (3.41)

Hence, all things will boil down, if we show the following estimation,

\[
\max_{1 \leq k \leq n} |R_k| = o((2v_n^2 \log \log v_n^2)^{1/2}), \quad P_0^\infty - a.s.
\] (3.42)

Define

\[
\varphi(\omega_0, \omega_1) := g(\omega_0) - H(\omega_0, \omega_1),
\] (3.43)

and by a simple calculation,

\[
R_n = \sum_{k=0}^{n-1} [g(T_{X_k} \omega) - H(T_{X_k} \omega, T_{X_{k+1}} \omega)] = \sum_{k=0}^{n-1} \varphi(T_{X_k} \omega, T_{X_{k+1}} \omega).
\] (3.44)

For a sequence \(\hat{\omega} = (\omega^{(i)})_{i \in \mathbb{N}} \in \Omega^\mathbb{N}\), define

\[
\Phi(\hat{\omega}) = \varphi(\omega^{(0)}, \omega^{(1)}) \quad \text{and} \quad \hat{R} = \sum_{k=0}^{n-1} \Phi \circ \theta^k,
\]

where \(\theta\) is the shift map on the sequence space \(\Omega^\mathbb{N}\) and is also a contraction on the space \(L^2(\hat{P}_0^\infty)\). Then \(\Phi \in L^2(\hat{P}_0^\infty)\) and the process \((\hat{R})_{n \geq 1}\) has the same distribution under \(\hat{P}_0^\infty\) as the process \((R)_{n \geq 1}\) has under \(P_0^\infty\).
The part (2) of Theorem RS tells $E_0^\infty(|R_0|^2) = O(n^{2\alpha})$, then there exists a constant $1/2 < c_0 < 1 - \alpha$, such that

$$\sup_n \|n^{c_0-1} \sum_{k=0}^{n-1} \Phi \circ \theta^k\| < \infty. \quad (3.45)$$

By the Theorem 2.17 of Derrienne and Lin [4] and $\alpha < 1/2$, we have $\Phi \in (I - \theta)^\eta L^2(\hat{P}_0^\infty)$, where $\eta \in (1/2, 1 - \alpha)$. Using again Derrienne and Lin’s Theorem 3.2 of [4], we get

$$|\hat{R}_n| = o((2n \log \log n)^{1/2}), \quad \hat{P}_0^\infty - a.s.. \quad (3.46)$$

Hence, $|R_n| = o((2n \log \log n)^{1/2}), \ P_0^\infty - a.s..$ Moreover, applying an elementary property of real convergent sequences, we immediately get

$$\max_{1 \leq k \leq n} |R_k| = o((2n \log \log n)^{1/2}), \quad P_0^\infty - a.s.. \quad (3.47)$$

Together with the equation (3.4), we prove the above equation (3.42).

### 3.2 Proof of Theorem 2

Here, we take along the lines of the proof of Theorem 4.8 in Hall and Heyde [5]. For any $\mathbb{R}^d$-valued function $f$, denote $f = (f_1, f_2, \ldots, f_d)'$. By the definition of $K$, we have, for any $f \in K$,

$$|f(t)|^2 = \sum_{i=1}^{d} (\int_{0}^{t} \dot{f}_i(s) ds)^2 \leq \sum_{i=1}^{d} (\int_{0}^{t} \dot{f}_i(s)^2 ds) \int_{0}^{t} 1 ds \leq t,$$

where the first inequality by the Cauchy-Schwartz’s inequality. So, $|f(t)| \leq \sqrt{t}$. It follows that $\sup_{t \in [0,1]} |f(t)| \leq 1$. Hence, by Theorem 1,

$$\limsup_{t \in [0,1]} |\xi_n(t)| \leq 1, \quad P_0^\infty - a.s. \quad (3.48)$$

and, setting $t = 1$, with the equation (3.4) again,

$$\limsup_{n \in [0,1]} |X_n - nv|/ \sqrt{2n \log \log n} \leq \sqrt{\text{tr}(\Sigma)}, \quad P_0^\infty - a.s. \quad (3.49)$$

On the other hand, we can put $f(t) = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i$, $t \in [0,1]$. Then, $f \in K$ and so for $P_0^\infty - a.s. \omega^*$, there exists a sequence $n_k = n_k(\omega^*)$, such that

$$\xi_{n_k}(\cdot)(\omega^*) \longrightarrow f(\cdot). \quad (3.50)$$

Particularly, $f(1) = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i$, $|\xi_{n_k}(1)(\omega^*)| \longrightarrow |f(1)|$. That is to say,

$$|X_{n_k} - n_k v|/ \sqrt{2n_k \log \log n_k} \longrightarrow \sqrt{\text{tr}(\Sigma)}, \quad P_0^\infty - a.s. \quad (3.51)$$

This completes the proof of Theorem 2.
Acknowledgements

The first author wishes to thank Prof. F.Q. Gao, Dr. Zh.H. Du, and Dr. P. Lv for their kindly help, especially Professor X. Chen for his valuable insight to the Step II in the proof of Theorem 1. The second author wishes to thank Prof. L.M. Wu of Université Blaise Pascal and Wuhan University, for his helpful discussions and suggestions during writing this paper.

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