HERMITIAN vs. ANTI-HERMITIAN 1-MATRIX MODELS AND THEIR HIERARCHIES

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ABSTRACT

Building on a recent work of Č. Crnković, M. Douglas and G. Moore, a study of multi-critical multi-cut one-matrix models and their associated \( sl(2, \mathbb{C}) \) integrable hierarchies, is further pursued. The double scaling limits of hermitian matrix models with different scaling ansätze, lead, to the KdV hierarchy, to the modified KdV hierarchy and part of the non-linear Schrödinger hierarchy. Instead, the anti-hermitian matrix model, in the two-arc sector, results in the Zakharov-Shabat hierarchy, which contains both KdV and mKdV as reductions. For all the hierarchies, it is found that the Virasoro constraints act on the associated tau-functions. Whereas it is known that the ZS and KdV models lead to the Virasoro constraints of an \( sl(2, \mathbb{C}) \) vacuum, we find that the mKdV model leads to the Virasoro constraints of a highest weight state with arbitrary conformal dimension.

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Introduction

Most of the interesting properties of the matrix model formulation of two dimensional gravity were originally extracted for the special case of lagrangians with even potentials [1]. In the hermitian 1-matrix model it was later shown that the introduction of odd terms in the potential gives rise to a doubling of the critical degrees of freedom and a doubling of the critical equations. In the 1-arc sector one gets two decoupled Painlevé I equations, for the first critical point; the underlying integrable structure being two decoupled Korteweg-de Vries (KdV) hierarchies [2,3]. The situation, however, turns out to be different in the 2-arc sector of the theory, which, for an even potential, has a Painlevé II equation as the lowest multi-critical point; the underlying integrable structure being the modified Korteweg-de Vries (mKdV) hierarchy [4,5]. Explicit calculations show that the introduction of odd terms do not lead to decoupled equations and to the doubling of the mKdV system [6].

This paper originated from our attempt to explore the integrable structures associated with the 2-arc sector of the hermitian 1-matrix model, with a generic potential. Indeed, one of the most interesting features of matrix models is the fact that the known 2d quantum gravity models (both pure and coupled to minimal conformal matter) are described by an integrable hierarchy, supplemented with an additional condition known as the ‘string equation’ [7]. The common belief is that the (anti-) hermitian n-matrix model should correspond to a hierarchy associated to the Lie algebra $\text{sl}(n+1, \mathbb{C})$, in the sense that, choosing different scaling ansätze for the double scaling limit of the matrix model, one gets a field theory, where the free energy (or a function related to it), satisfies the string equation of such a hierarchy. Obviously, the case $n = 1$ is the simplest and also virtually the only one where examples can be worked out explicitly. More general one-matrix models have also been considered in ref. [8], which deals with the case for complex matrices.

We start by deriving in an explicit way the higher multi-critical points of the hermitian 1-matrix model in the 2-arc sector with generic polynomials. We find that the resulting string equations are associated with only the ‘even’ subset of flows of the nonlinear Schrödinger (NLS) hierarchy. This, indeed, seems to be the hierarchy behind the 2-arc sector of the hermitian 1-matrix model, as one can check by computing explicitly the Lax operator.

Interestingly, we notice that if the odd terms in the potential were purely imaginary, i.e. if the starting matrix model were anti-hermitian instead of hermitian, the multi-critical points in the 2-arc sector reproduce all the string equations associated to the ZS hierarchy. Indeed the Lax operator one gets in this case is that of the Zakharov-Shabat (ZS) hierarchy\(^1\). The reason why we do not find the odd string equations of the NLS in our derivation of multi-critical points of the hermitian matrix model is that those equations are complex. Instead all the flows of the ZS hierarchy are real as we will see in section 2.5.

While the NLS hierarchy is known to contain, as reduction, the mKdV hierarchy, the

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\(^1\) The connexion of the ZS hierarchy with the (anti-) hermitian matrix model in the 2-arc sector was first noticed by the authors of ref. [9].
ZS hierarchy contains both the mKdV and KdV hierarchies, [10], (which can equivalently be obtained directly from the matrix model by restricting the ansatz made for the double scaling limit). In addition, we find that the string equations of the NLS hierarchy reduce to those of the mKdV hierarchy, and the string equations of ZS to those of the KdV and mKdV hierarchies. From this one deduces that solutions of the KdV and mKdV theories correspond also to solutions of the NLS and ZS theories, in a particular subspace spanned by the even flows of the two hierarchies. The results are conveniently illustrated in diagram 1, which shows how the critical points of the various hierarchies relate both in the 1-arc and 2-arc sectors.

We then go on to discuss the tau-function formalism of the ZS hierarchy. In this case we find that the partition function of the theory is equal to the tau-function of the hierarchy, rather than to its square, as happens instead in the 1-arc KdV model. As it was first shown in ref. [9], the partition function of the matrix model leading to the ZS hierarchy, satisfies the Virasoro constraints of an untwisted boson with an, a priori, arbitrary value for the zero-mode or ‘momentum’. This implies that the Virasoro constraints act on the tau-function, to mirror the situation for the 1-arc KdV model [11]. However, in the ZS case, the tau-function carries an additional quantum number due to the zero-mode of the untwisted field. This additional quantum number seems to play the rôle of
a non-perturbative parameter which labels different sectors of the theory and arguments connected with the tau-function formalism suggest that it takes discrete values.

An interesting side issue concerns the existence of Virasoro constraints for the models described by the mKdV hierarchy; these include both the 2-arc (anti-) hermitian model, with even potential, and the unitary matrix models [12,13]. We find that there are, indeed, Virasoro constraints for the mKdV model; however, the situation is more complicated than in the KdV case. In the KdV case, the Virasoro constraints act on the tau-function: that is the square root of the partition function. For the mKdV model there are two tau-functions; the partition function being the product. We find that the string equation and the mKdV hierarchy imply a set of Virasoro constraints for each tau-function separately, however, in contrast to the KdV case, only the $L_n$ for $n \geq 0$ appear, and the eigenvalue of $L_0$ is an, \emph{a priori}, undetermined integration constant. In other words, the tau-functions satisfy the Virasoro constraints of a highest weight state of the conformal algebra.

Under the reduction from ZS to KdV, the Virasoro constraints are transformed into those of a twisted boson acting on the square root of the partition function (as expected [11]). More interestingly, the reduction to KdV of the $t_2 = \text{constant}, t_0 = x$, ZS scaling theory gives rise to the topological point of the KdV hierarchy! In other words, from the anti-hermitian 1-matrix model with a potential of the fourth order it is possible, after the double scaling limit, to get Topological Gravity [14].

For the reduction to mKdV we get a series of constraints acting on the mKdV partition function which are not Virasoro-like; this is to be expected because they act on the product of the mKdV tau-functions and not on each of them separately. We also show that the ‘conformal dimension’ of the mKdV partition function is fixed under the reduction. Finally in both reductions the value of the above-mentioned discrete parameter is fixed.

Another important issue which arises from this analysis, is the possible existence of new continuum theories described by the ‘odd’ critical points of the ZS model. The first such model was discussed in [9], and it seems to display a topological nature, similar to the topological point in the 1-arc KdV model [14]. Nevertheless, the theory is sufficiently different from the ‘normal’ topological theory, to make it unclear as to what continuum theory it describes. However, like its 1-arc cousin, this topological theory cannot be obtained from the matrix model. The possible higher ‘odd’ multi-critical points have yet to be explored. We will not address the mathematical technicalities required to prove that solutions of these new scaling points exist; however, since the mathematical apparatus exists [15] we hope these questions will be tackled elsewhere.

Finally we notice that models with anti-hermitian matrices seem to arise in relation with topological gravity (the Kontsevich model [16]) and with the Penner model [17]. Indeed, whereas hermitian matrix models naturally emerge from the study of quantum gravity as a theory of random surfaces [1], anti-hermitian matrices seem to arise when one tries to make connexions between matrix models and moduli spaces of Riemann surfaces.

1. \emph{(Anti-)} Hermitian 1-Matrix Models and Orthogonal Poly-
nomials

In this section we will consider the double scaling limit of a general (anti-) hermitian 1-matrix model. Most of what follows has been already extensively discussed in the literature, so we will recall only the most important results, add some new ones and develop a few explicit examples to fix our notation.

1.1 Double Scaling Limit and String Equations of Hermitian Matrix Models

Let $M$ be a hermitian $N \times N$ matrix and consider

$$ Z_N = \int dM e^{-\beta \text{tr}V(M)} \quad (1.1) $$

where $V(\lambda) = g_1 \lambda + \frac{g_2}{2} \lambda^2 + \frac{g_3}{3} \lambda^3 + \ldots$ and $\lambda$ denotes a (real) eigenvalue of $M$.

As it is well known, one can introduce orthonormal polynomials $P_n(\lambda)$ such that

$$ \int d\lambda e^{-\beta V(\lambda)} P_n(\lambda) P_m(\lambda) = \delta_{n,m} \quad (1.2) $$

with

$$ \lambda P_n(\lambda) = \sqrt{R_{n+1}} P_{n+1}(\lambda) + S_n P_n(\lambda) + \sqrt{R_n} P_{n-1}(\lambda) \quad , \quad (1.3) $$

then

$$ Z_N = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{i=1}^{N-1} R_i^{N-i} \quad (1.4) $$

where $R_n = h_n/h_{n-1}$.

We will consider, for the moment, the case with real potentials. Notice that this implies that both $R_n$ and $S_n$ are real. Moreover, if $V(\lambda) = V(-\lambda)$ then $S_n = 0$.

The most important equation is the so-called ‘string equation’, which can be written as:

$$ n \beta = g_2 R_n + g_3 (R_n S_n + R_n S_{n-1}) + \cdots \quad (1.5) $$

$$ 0 = g_1 + g_2 S_n + g_3 (S_n^2 + R_n + R_{n+1}) + \cdots . $$

In the double scaling limit one assumes that $\beta/N \to 1$, $n/\beta = 1 - x/N^\alpha$ and

$$ R_n = 1 + (-1)^n \frac{f(x)}{N^{\gamma_1}} + \frac{r(x)}{N^{\gamma_2}} + \frac{j(x)}{N^{\gamma_3}} + \frac{z(x)}{N^{\gamma_4}} + \cdots \quad (1.6) $$

$$ S_n = b + (-1)^n \frac{g(x)}{N^{\delta_1}} + \frac{s(x)}{N^{\delta_2}} + \frac{p(x)}{N^{\delta_3}} + \frac{v(x)}{N^{\delta_4}} + \cdots $$

where $b$ is an arbitrary constant.
For an even potential, \( V(\lambda) = V(-\lambda) \), and \( V(\lambda) = \frac{g_2}{2} \lambda^2 + \cdots + \frac{g_{2k}}{2k} \lambda^{2k} \), one sets in the 1-arc sector

\[
f(x) = 0 = g(x), \quad \alpha = \frac{2k}{2k + 1}, \quad \gamma_i = \delta_i = \frac{i}{2k + 1} \tag{1.7}
\]

and in the 2-arc sector

\[
\alpha = \frac{2k - 2}{2k - 1}, \quad \gamma_i = \delta_i = \frac{i}{2k - 1}, \tag{1.8}
\]

and one needs to introduce scaling functions up to the order \( 2k - 2 \).

Let us introduce \( a \) such that

\[
a = N^{\frac{1}{2k+1}} \text{ in the 1-arc sector} \tag{1.9}
\]

\[
a = N^{\frac{1}{2k-1}} \text{ in the 2-arc sector},
\]

then eqs. (1.5) become

\[
1 - \frac{x}{a^\epsilon} = F_0(g_i) + \frac{f(x)}{a} F_1(g_i) + \frac{1}{a^2} F_2(f, g, r, s, f', g'; g_i) + \cdots \tag{1.10}
\]

\[
0 = G_0(g_i) + \frac{g(x)}{a} G_1(g_i) + \frac{1}{a^2} G_2(f, g, r, s, f', g'; g_i) + \cdots
\]

where \( \epsilon = 2k \) in the 1-arc sector, \( \epsilon = 2k - 2 \) in the 2-arc sector and \( \partial \) means \( \partial \frac{\partial}{\partial x} \). The functions \( F_i \) and \( G_i \) can be explicitly obtained from eqs. (1.5) and depend on the coupling constants and on the scaling functions.

It is now easy to solve order by order in \( 1/a \) the string equation (1.10), for example the zeroth order in \( 1/a \) (\( F_0 = 0 = G_0 \)) fixes the value of \( g_1 \) and \( g_2 \):\(^3\)

\[
g_2 = 1 - 2bg_3 - (3 + 3b^2)g_4 - (12b + 4b^3)g_5 - (10 + 30b^2 + 5b^4)g_6 + \cdots
\]

\[
g_1 = -bg_2 - (2 + b^2)g_3 - (6b + b^3)g_4 - (6 + 12b^2 + b^4)g_5 - (10 + 30b^2 + 5b^4)g_6 + \cdots
\] \tag{1.11}

Notice that the \( g_i \)'s depend on the free parameter \( b \).

At order \( 1/a \) one finds two possible solutions: either \( f(x) = 0 \) and \( g(x) = 0 \) or one fixes the value of \( g_4 \). Choosing the solution \( f(x) = 0 = g(x) \) one gets the 1-arc sector string equations; fixing \( g_4 \) instead, one gets the 2-arc string equations. In the case of an even potential of order 4 (\( k = 2 \)) these are the well-known Painlevé I and Painlevé II string equations.

Moreover, from eq. (1.4), in the double scaling limit, we find the ‘specific heat’ (up to non-universal terms)

\[
F''' = \frac{1}{2} f^2(x) - r(x) \tag{1.12}
\]

\(^2\) This is not the ‘\( a \)’ defined by \( Na^{2+1/m} = 1 \) usually introduced in the literature.

\(^3\) Most of the computations of this section have been done with the help of Mathematica™.
where $Z = \exp(-F)$ and $\langle PP \rangle = \partial_x^2 \log Z = -F''$.

Let us consider now a real potential $V(\lambda)$ with both even and odd couplings. In
the 1-arc sector ($f(x) = 0 = g(x)$) one gets the well known ‘doubling’ phenomenon. For
example, from

$$V(\lambda) = (\frac{1}{3}b^3 - 2b)\lambda + \frac{1}{2}(2 - b^2)\lambda^2 + \frac{1}{3}b\lambda^3 - \frac{1}{12}\lambda^4$$

one has the string equations

$$x = \frac{1}{3}\chi'' + \chi^2, \quad x = \frac{1}{3}\overline{\chi}'' + \overline{\chi}^2$$

where $\chi(x) = r(x) - s(x)$ and $\overline{\chi}(x) = r(x) + s(x)$. Notice that these equations do not depend
on the parameter $b$ and that the $g_{2i+1}$ are proportional to $b$. In the same way, in the 1-arc
sector one obtains all the string equations of the KdV hierarchy, which are described in the
following chapter. The ‘specific heat’ for these models is given by $F'' = -r(x) = -\frac{1}{2}(\chi + \overline{\chi})$.

The 2-arc sector with a general real potential is more interesting, indeed many different
scaling solutions and string equations appear. For example, with

$$V(\lambda) = (2b - b^3)\lambda + \frac{1}{4}(3b^2 - 2)\lambda^2 - b\lambda^3 + \frac{1}{4}\lambda^4$$

one gets

$$f'' - \frac{1}{4}f(g^2 + f^2) + \frac{1}{2}fx = 0$$

$$g'' - \frac{1}{4}g(g^2 + f^2) + \frac{1}{2}gx = 0$$

and from

$$V(\lambda) = (b + b^3 - \frac{1}{2}b^5)\lambda + \frac{1}{2}(-1 - 3b^2 + \frac{5}{2}b^4)\lambda^2 + \frac{1}{3}(3b - 5b^3)\lambda^3 +$$

$$\frac{1}{4}(-1 + 5b^2)\lambda^4 - \frac{1}{2}b\lambda^5 + \frac{1}{12}\lambda^6$$

one gets

$$0 = xf - \frac{3}{8}f^3(g^2 + f^2) - \frac{3}{8}fg^2(f^2 + g^2) + \frac{5}{2}f(f')^2 + 3fg'g' +$$

$$- \frac{1}{2}f(g')^2 + \frac{5}{2}f^2f'' + \frac{3}{2}g^2f'' + fg'' - f^{(4)}$$

$$0 = xg - \frac{3}{8}g^3(g^2 + f^2) - \frac{3}{8}gf^2(f^2 + g^2) + \frac{5}{2}g(g')^2 + 3fg'g' +$$

$$- \frac{1}{2}g(f')^2 + \frac{5}{2}g^2g'' + \frac{3}{2}f^2g'' + fgg'' - g^{(4)}.$$ (1.18)

Again these equations do not depend on the parameter $b$; the $g_{2i+1}$ are proportional to $b$ and putting $g = 0$ these equations will turn out to be the first two string equations of
the mKdV hierarchy [5]. When $g$ is not put to zero, these two string equations will be seen,
in the next chapter, to be the second and fourth string equation of the Non-Linear-
Schrödinger (NLS) hierarchy. It is not possible to get the first and third string equation of
the NLS hierarchy from the hermitian 1-matrix model. It seems that the NLS hierarchy
does not have a full ‘matrix model’ realization, in the sense that we do not get the multi-
critical points corresponding to the ‘odd’ flows; a fact that will be explained in section
§2.5.

For all these models the ‘specific heat’ turns out to be

$$F'' = \frac{1}{4} \left[ f^2(x) + g^2(x) \right]$$

since $r(x) = \frac{1}{2}(f^2(x) - g^2(x))$. 

6
1.2 Anti-Hermitian Matrix Models and Their String Equations

Let us first consider, in more detail, the NLS string equations (1.16) and (1.18). Sending \( g \to ig \) one obtains string equations that are nothing but the string equations associated to the Zakharov-Shabat (ZS) hierarchy, as will be apparent from the next chapter. But this substitution is incompatible with our ansatz, eq. (1.6), for the double scaling limit. Indeed, sending \( g \to ig \) implies that \( S_n \) becomes complex. But if the potential is real, \( V^*(\lambda) = V(\lambda) \), it is easy to show that \( S_n \) must be real. On the other hand, if \( V^*(\lambda) = V(-\lambda) \) then \( S_n \) is pure imaginary. (This can be easily shown using the orthogonality of the polynomials \( P_n(\lambda) \) \( (P_n(\lambda) = \sqrt{h_n} P_n(\lambda)) \), the fact that \( P_n(\lambda) = \lambda^n + O(\lambda^{n-1}) \) and the reality of the partition function (see (1.4)).) Therefore, we are led to consider the potential \( V(\lambda) = ig_1 \lambda + \frac{1}{2} g_2 \lambda^2 + \frac{1}{3} i g_3 \lambda^3 + \frac{1}{4} g_4 \lambda^4 + \cdots \) with \( \lambda, g_k \in \mathbb{R} \) and some of the \( g_{2i+1} \) different from zero. The double scaling limit is now done in exactly the same way as in the previous section except for the ansatz for \( S_n \) (eq. (1.6)) which is instead pure imaginary, i.e.

\[
S_n = ib + (-1)^n \frac{ig(x)}{a} + \frac{is(x)}{a^2} + \frac{ip(x)}{a^3} + \frac{iv(x)}{a^4} + \cdots
\]  

(1.20)

with \( b, g(x), s(x), \ldots \) all real.

Notice that we can get rid of the ‘i’ in the potential introducing anti-hermitian matrices \( \tilde{M} = iM \) with pure imaginary eigenvalues \( \tilde{\lambda} = i\lambda \). Again the \( R_n \) are real and in the same way it is easy to prove that the \( S_n \) are pure imaginary.

For notational simplicity in the rest of this section we will continue to use hermitian matrices with complex potentials instead of using directly anti-hermitian matrices.

In the 1-arc sector one gets again the double KdV string equation, now however in the variables \( \chi = r - is \) and \( \overline{\chi} = r + is = \lambda^* \).

In the 2-arc sector we found the family of scaling solutions corresponding to the ZS hierarchy. For example, from

\[
V(\lambda) = i(2b + b^3) \lambda + \frac{1}{2}(-2 - 3b^2) \lambda^2 - ib \lambda^3 + \frac{1}{4} \lambda^4
\]  

(1.21)

with \( \epsilon = 2 \) one gets

\[
0 = \frac{1}{2} xf + f'' + \frac{1}{2} f(g^2 - f^2)
\]

\[
0 = \frac{1}{2} xg + g'' + \frac{1}{2} g(g^2 - f^2).
\]  

(1.22)

From

\[
V(\lambda) = i \left[ -1 + b + b^2 - b^3 + \frac{1}{2} b^4 - \frac{1}{2} b^5 \right] \lambda +
\frac{1}{2} \left[ -1 - 2b + 3b^2 - 2b^3 + \frac{5}{2} b^4 \right] \lambda^2 +
\frac{1}{3} i \left[ -1 + 3b - 3b^2 + 5b^3 \right] \lambda^3 + \frac{1}{4} \left[ -1 + 2b - 5b^2 \right] \lambda^4 +
\frac{1}{5} i \left[ \frac{1}{2} - \frac{5}{2} b \right] \lambda^5 + \frac{1}{12} \lambda^6
\]  

(1.23)

and \( \epsilon = 3 \) one has

\[
0 = xf + g'' + \frac{3}{2} g'(g^2 - f^2)
\]

\[
0 = xg + f'' + \frac{3}{2} f'(g^2 - f^2).
\]  

(1.24)
Finally, from
\[ V(\lambda) = i \left[ b - b^3 - \frac{1}{2} b^5 \right] \lambda + \frac{1}{2} \left[ -1 + 3b^2 + \frac{5}{2} b^4 \right] \lambda^2 + \frac{1}{3} i \left[ 3b + 5b^3 \right] \lambda^3 + \frac{1}{4} \left[ -1 - 5b^2 \right] \lambda^4 - \frac{1}{2} i b \lambda^5 + \frac{1}{12} \lambda^6 \] (1.25)
with \( \epsilon = 4 \) one has
\[
0 = x f + \frac{3}{8} f^3 \left( g^2 - f^2 \right) - \frac{3}{8} f g^2 \left( g^2 - f^2 \right) + \frac{5}{2} f \left( f' \right)^2 - 3g f' g' + \frac{1}{2} f \left( f'' \right) + \frac{5}{2} f^2 f'' - \frac{3}{2} g^2 f'' - f g g'' - f(4) \\
0 = x g - \frac{3}{8} g^3 \left( g^2 - f^2 \right) + \frac{3}{8} g f^2 \left( g^2 - f^2 \right) - \frac{5}{2} g \left( g' \right)^2 + 3f f' g' + \frac{1}{2} g \left( f' \right)^2 - \frac{5}{2} g^2 g'' + \frac{3}{8} f^2 g'' + f g f'' - g(4) .
\] (1.26)

These are the first three string equations of the ZS hierarchy (excluding the topological-like point [9] which cannot be obtained from the matrix model). More precisely, they correspond to the points \( t_\epsilon = \) constant, \( t_0 = x \) and \( t_i = 0 \) for \( i \neq \{0, \epsilon\} \) in the ZS hierarchy (see next section).

A few comments are in order. As usual the string equations do not depend on \( b \). The \( g_{2i+1} \) are proportional to \( b \) except for the case of eq. (1.23). Indeed, it is not possible to get the string equation eq. (1.24) from a real potential \( (V^*(\lambda) = V(\lambda)) \). Moreover, setting \( g = 0 \) one obtains the first two string equations of the mKdV hierarchy (eq. (1.24) vanishes identically) and setting \( \frac{1}{2} \left( f + g \right) = -1 \) in eqs. (1.22) and (1.26) one gets the first two string equations (topological point included) of the KdV hierarchy in the variable \( \psi = \frac{1}{2} (f - g) \).

For all these models it turns out that
\[ F'' = -\frac{1}{4} \left[ g^2(x) - f^2(x) \right] \] (1.27)

since \( r(x) = \frac{1}{4} \left[ f^2(x) + g^2(x) \right] \).

### 1.3 Lax Operators from Matrix Models

The basic idea, due to Douglas [7], is that it is possible to make the double scaling limit not only on the string equations but also directly on \( \lambda \) and \( \frac{d}{d\lambda} \) seen as operators acting on \( P_n(\lambda) \). This operator is then related to the Lax operator of the integrable hierarchy which underlies the behaviour of the continuum theory associated to the particular sequence of scaling ansätze chosen. Indeed, under a double scaling limit from eq. (1.3) it is easy to see that \( \lambda \) becomes a differential operator \( (\hat{\lambda}) \) of order 2 in \( x \). The obvious relation
\[ \left[ \frac{\partial}{\partial \lambda}, \lambda \right] = 1 \] (1.28)
after the double scaling limit becomes the string equation [7]. Eq. (1.3) under the double scaling limit becomes
\[ \hat{\lambda} \Psi = \Lambda \Psi \] (1.29)
where $\Psi$, which can be a vector, is related to the rescaled polynomials and $\Lambda$ is a differential operator of degree 2 in $x$. Analogously, there exists an equation of the form

$$\frac{\hat{\partial}}{\partial \lambda} \Psi = M \Psi \quad (1.30)$$

These two equations can be rewritten as

$$L \Psi \overset{\text{def}}{=} \left( \hat{\lambda} - \Lambda \right) \Psi = 0, \quad \left( \frac{\hat{\partial}}{\partial \lambda} - M \right) \Psi = 0 \quad (1.31)$$

and then the string equation becomes the compatibility condition for these differential equations [15], i.e.

$$\left[ L, \left( \frac{\hat{\partial}}{\partial \lambda} - M \right) \right] = 0 \quad (1.32)$$

It is easy to get the explicit expression of $L$ from eq. (1.29). $L$ is then the Lax operator of the corresponding hierarchy. Using the techniques of Zakharov-Shabat and Drinfeld-Sokolov [18,19], one can also explicitly compute $M$ [7,4,9].

We now explicitly compute $L$ from eq. (1.3). Consider first the case of a hermitian matrix with a real potential. Let $\Pi(x, \lambda)$ denote $P_n(\lambda)$ after the double scaling limit\(^4\), thus eq. (1.3) becomes

$$\lambda P_n(\lambda) \sim (2 + b)\Pi(x) + \frac{1}{a^2} [r(x)\Pi(x) + s(x)\Pi(x) + \Pi''(x)] + \cdots \quad (1.33)$$

where for notational simplicity we have dropped the dependence on $\lambda$ in $\Pi$. Setting $\lambda - (b + 2) \to \hat{\lambda}/a^2$ one has

$$L \Pi(x, \lambda) = 0 \quad (1.34)$$

where

$$L = \partial_x^2 + (r(x) + s(x)) - \hat{\lambda} \quad (1.35)$$

This is the Lax operator of the KdV hierarchy.

Notice that setting $\Pi(x, \lambda) \sim (-1)^n P_n(\lambda)$ one gets $L = \partial_x^2 + (r(x) - s(x)) + \hat{\lambda}$. Thus there are two KdV Lax operators associated to the hermitian 1-matrix model in the 1-arc sector, this is nothing but the doubling phenomenon [2].

In a similarly way, for the anti-hermitian 1-matrix model in the 1-arc sector one gets the previous two Lax operators with $s(x) \to is(x)$.

Consider now the case of the 2-arc sector with a hermitian matrix and a real potential. In order to get a non trivial Lax operator in the double scaling limit, following [4], we need to introduce two scaling functions depending on parity, $\Pi(\lambda, x) \sim (-1)^m P_{2m}(\lambda)$ and $\Omega(\lambda, x - \frac{1}{a}) \sim (-1)^m P_{2m+1}(\lambda)$ where $x = x_{2m}$. Thus for $n$ even eq. (1.3) becomes

$$\lambda (-1)^{n/2} P_n(\lambda) \sim b\Pi(x) + \frac{1}{a} [g(x)\Pi(x) - f(x)\Omega(x) - 2\Omega'(x)] + \cdots \quad (1.36)$$

\(^4\) Actually, as explained in refs. [7,4,15], under the double scaling limit $\Pi(x, \lambda) \sim e^{-NV(\lambda)/2} P_n(\lambda)$ has a smooth behaviour and should be considered for this computation.
and for $n$ odd one has
\[
\lambda (-1)^{(n-1)/2} \mathcal{P}_n(\lambda) \sim b\Omega(x) + \frac{1}{a} \left[-g(x)\Omega(x) - f(x)\Pi(x) + 2\Pi'(x)\right] + \cdots .
\] (1.37)

Thus, setting $\lambda - b \to \hat{\lambda}/a$ one has
\[
L \Psi = 0
\] (1.38)

where
\[
L = \frac{\hat{\lambda}}{2} \frac{g(x)}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{f(x)}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] (1.39)

and $\Psi = \begin{pmatrix} \Pi \\ \Omega \end{pmatrix}$. Now, after having conjugated $L$ and rotated $\Psi$, we get eq. (1.38) with
\[
L = i\sigma_3 \hat{\lambda} + \sigma_2 g - \sigma_1 f + \partial_x
\] (1.40)

and $[\sigma_i, \sigma_j] = i\epsilon_{ijk} \sigma_k$. As we expected, this is the Lax operator of the NLS hierarchy.

In the case of an anti-hermitian matrix with pure imaginary eigenvalues $\tilde{\lambda}$ one can do the same computation with $\mathcal{P}_{2m+1}$ and $S_n$ pure imaginary (see eq. (1.20)). Letting $\tilde{\lambda} - ib \to i\tilde{\lambda}/a$, one finally gets
\[
L = \partial_x + i\sigma_2 g - \sigma_1 f - \sigma_3 \tilde{\lambda} ,
\] (1.41)

and this is the Lax operator of the ZS hierarchy.

Notice that the mKdV string equations can be obtained both from the NLS (hermitian matrix, real potential) and ZS (anti-hermitian matrix, complex potential) hierarchy. This is obvious since setting $S_n = 0$ ($b = 0 = g = \ldots$) hermitian and anti-hermitian matrix models coincide.

2. Hierarchies and String Equations

In this section we will discuss the hierarchies of integrable equations which lie behind the non-perturbative structure of the one matrix model. The relevant hierarchies are well-known in the mathematical physics literature, and are the Zakharov-Shabat (ZS), for the Lie algebra $sl(2, \mathbb{C})$, the Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV) and non-linear Schrödinger (NLS) hierarchies [10].

Interestingly, all these hierarchies are intimately related to the Lie algebra $sl(2, \mathbb{C})$. This is apparent in the matrix Lax formalism [18] and also the ‘Hirota’ or ‘tau-function’ formalism [20]. It is also significant that the complex-ZS hierarchy is the ‘master’ hierarchy for $sl(2, \mathbb{C})$, since the ZS, NLS, KdV and mKdV hierarchies can all be obtained by appropriate reductions. This will be explained in more detail below. We shall also discuss how the hierarchies are related to the matrix models.
2.1 The \( sl(2, \mathbb{C}) \) Hierarchies

In section \( \S 1.3 \) we showed how the matrix models are connected with the integrable hierarchies presented in the Lax formalism. In fact, the most economical way of explicitly introducing the hierarchies is via their recursion relations. These can be extracted from the Lax formalism, see for example ref. \[18\]. In what follows we often write \( x \) for the flow \( t_0 \), of the hierarchy under discussion. For the KdV equation the hierarchy can be presented, in the following way

\[
\frac{\partial u}{\partial t_k} = \partial_x R_{k+1} \quad k \geq 0, \tag{2.1}
\]

where \( R_k \) is a polynomial in \( u \) and its \( x \)-derivatives (the \( R_k \)'s are known as the Gel’fand-Diki polynomials \[21\], and are not to be confused with the \( R_k \) used in the previous section). The integrability of the hierarchy is summed up in the recursion relation

\[
\partial_x R_{k+1} = \left( \frac{1}{2} \partial_x^3 + 2u \partial_x + u' \right) R_k. \tag{2.2}
\]

Specifying \( R_0 = 1 \) along with the recursion relation completely determines the hierarchy. To make contact with the notation used in the previous section, for example for the hermitian matrix model in the 1-arc sector with even potential, one should set \( u = r \).

For the complex-ZS hierarchy a similar structure of recursion relations is found. For two independent complex variables \( \psi \) and \( \bar{\psi} \), if

\[
\frac{\partial \psi}{\partial t_k} = \frac{1}{2}(F_{k+1} - G_{k+1}), \quad \frac{\partial \bar{\psi}}{\partial t_k} = \frac{1}{2}(F_{k+1} + G_{k+1}), \tag{2.3}
\]

with \( k \geq -1 \), then

\[
F_{k+1} = G_k' + (\bar{\psi} - \psi)H_k \]
\[
G_{k+1} = F_k' + (\bar{\psi} + \psi)H_k \]
\[
H_k' = \bar{\psi}(G_k - F_k) - \psi(G_k + F_k). \tag{2.4}
\]

Specifying \( F_0 = \bar{\psi} - \psi \) and \( G_0 = \bar{\psi} + \psi \) along with the recursion relations then determines the whole hierarchy. Notice that the flow \( t_{-1} \) is particularly simple

\[
\frac{\partial \psi}{\partial t_{-1}} = -\psi, \quad \frac{\partial \bar{\psi}}{\partial t_{-1}} = \bar{\psi}, \tag{2.5}
\]

implying the following dependence on \( t_{-1} \): \( \psi \sim e^{-t_{-1}} \) and \( \bar{\psi} \sim e^{t_{-1}} \). To make contact with the notation of the previous section \( \psi = \frac{1}{2}(f - g) \) and \( \bar{\psi} = \frac{1}{2}(f + g) \), which also agrees with the conventions of ref. \[9\] when \( t_k \to t_{k+1} \). The NLS hierarchy is recovered from the complex-ZS hierarchy by choosing \( \psi \) to be the complex conjugate of \( \psi \) (up to a term involving \( t_{-1} \)), a choice which is easily seen to be consistent with the recursion relations.
of the hierarchy. The ZS hierarchy itself, simply corresponds to the complex-ZS hierarchy with $\psi$ and $\bar{\psi}$ both real.

The mKdV hierarchy is best introduced through its relation to the KdV hierarchy via the *Miura Map*. This map takes a solution of the mKdV $\nu$ hierarchy into a solution of the KdV $u$ hierarchy as
\[ u = -\nu' - \nu^2. \]  
(2.6)

The pull-back of the KdV flow $t_k^{\text{KdV}}$ under the Miura Map is then the mKdV flow $t_k^{\text{mKdV}}$. We now make this more explicit. Defining $D = -\partial_x - 2\nu$ and $D^* = \partial_x - 2\nu$, the recursion relation (2.2) becomes $\partial_x R_{k+1} = D(-\frac{1}{2}\partial_x)D^* R_k$ [22]. Since $\partial_x u = D\partial_x \nu$, we have
\[ \frac{\partial u}{\partial t_k^{\text{KdV}}} = D \frac{\partial \nu}{\partial t_k^{\text{mKdV}}} = D \left(-\frac{1}{2}\partial_x\right) D^* R_k, \]  
(2.7)
and so the mKdV flows are
\[ \frac{\partial \nu}{\partial t_k} = -\frac{1}{2}\partial_x D^* R_k \quad k \geq 0, \]  
(2.8)
where the polynomial $R_k = R_k[-\nu' - \nu^2]$ is expressed in terms of $\nu$ and its $x$-derivatives by substituting $u$ for $\nu$ via the Miura Map. To make contact with the notation of the previous section $\nu = f/2$.

2.2 THE ZS, NLS, KDV AND MKDV HIERARCHIES AS REDUCTIONS OF THE COMPLEX-ZS HIERARCHY

In this section we explain how the various hierarchies that we have introduced can all be obtained as reductions of the complex-ZS hierarchy. The situation is conveniently summarized by diagram 2 which shows how the various hierarchies are related by reduction.

\[
\begin{array}{ccc}
\text{NLS} & \rightarrow & \text{mKdV} \\
\text{complex-ZS} & \rightarrow & \text{KdV} \\
\text{ZS} & \rightarrow & \text{mKdV} \\
\end{array}
\]

Diagram 2

We have already noted how the complex-ZS hierarchy is reduced to the ZS and NLS hierarchies by taking two different ‘real slices’. For the former one takes $\psi$ and $\bar{\psi}$ to be real, whilst for the latter one takes the complex conjugate of $\psi$ to be $\psi^* = e^{-2t-1} \bar{\psi}$.
The KdV hierarchy is obtained from the ZS hierarchy by setting

\[ \psi = -e^{t-1}, \]  

(2.9)

the KdV variable being given by \( u = -\psi \psi \). One can readily prove that \( F_{2k+1} + G_{2k+1} = 0 \), when evaluated at \( \psi = -e^{t-1} \), whereas \( F_{2k} + G_{2k} \neq 0 \). This means that only the even flows preserve the condition (2.9), and so only the even flows reduce to flows of the KdV hierarchy. One finds

\[ \frac{\partial(-\psi \psi)}{\partial t_{2k}} \bigg|_\ast = \frac{1}{2} H_{2k+1}^\prime \bigg|_\ast, \]  

(2.10)

where \( \ast \) means 'evaluate at \( \psi = -e^{t-1} \). If one now compares the recursion relations of the ZS hierarchy (2.4), evaluated at \( \psi = -e^{t-1} \), to those of the KdV hierarchy (2.2), then one deduces that

\[ H_{2k+1}[\psi, \psi = -e^{t-1}] = 2^{k+1} R_{k+1}[-\psi \psi]. \]  

(2.11)

This means that the relation between the flow variables is \( \lambda_{2k}^{ZS} = 2^{-k} \lambda_{2k}^{KdV} \). It is straightforward to see that the KdV hierarchy cannot be obtained from the NLS hierarchy by a similar reduction.

The mKdV hierarchy is obtained from both the NLS and ZS hierarchies by setting

\[ \psi = e^{2t-1} \psi, \]  

(2.12)

the mKdV variable being given by \( \nu^2 = \psi \psi \). One finds that \( G_{2k+1} = F_{2k} = H_{2k} = 0 \), when evaluated at (2.12). So the situation is similar to that for the KdV reduction, in that only the even flows preserve the reduction. By pursuing a similar analysis of the recursion relations, one discovers that the flow variables are related via \( \lambda_{2k}^{ZS} = 2^{-k} \lambda_{2k}^{mKdV} \) since \( F_{2k+1} = -2^k \partial_x D^\ast R_k[-\nu' - \nu^2] \), when evaluated at (2.12).

2.3 The String Equations and Their Reductions

The hierarchies that we have discussed above admit many types of solution. However, in applications to matrix models and two-dimensional field theories, very particular solutions are required. In addition to boundary conditions, these are specified by adjoining to the hierarchy an extra condition called the string equation [7]. The string equation must be consistent with the flows of the hierarchy, in the sense that it must be preserved by the flows of the hierarchy. It turns out that the string equation admits certain scaling, or multi-critical solutions. It is these solutions which are found in the matrix model, after the double scaling limit. They are obtained by restricting to the subspace \( t_k = \text{constant}, t_0 = x \) and \( t_j = 0 \) otherwise, for some \( k \). We call the resulting reduced equation the \( k^{th} \) string equation. The string equations can be found in general following the analysis of §1.3.

The string equation associated to the KdV hierarchy was found originally in [7]. In our conventions it takes the form

\[ \sum_{k=1}^{\infty} (2k+1) t_k \frac{\partial u}{\partial t_{k-1}} = -1, \]  

(2.13)
which may be integrated using (2.1) to give

\[ \sum_{k=0}^{\infty} (2k + 1)t_k R_k = 0. \] (2.14)

The \( k^{th} \) multi-critical point corresponding to \( t_k = \) constant, \( t_0 = x \), and \( t_j = 0 \) otherwise, is described by the string equation

\[ (2k + 1)t_k R_k[u] = -x. \] (2.15)

For the ZS hierarchy, the string equation was found in [9]

\[ \sum_{k=0}^{\infty} (k + 1)t_k \frac{\partial \psi}{\partial t_{k-1}} = 0, \quad \sum_{k=0}^{\infty} (k + 1)t_k \frac{\partial \bar{\psi}}{\partial t_{k-1}} = 0. \] (2.16)

The \( k^{th} \) multi-critical point for which \( t_k = \) constant, \( t_0 = x \) and otherwise \( t_j = 0 \), is described by the string equation

\[ (k + 1)t_k (F_k - G_k) = 2x\psi, \quad (k + 1)t_k (F_k + G_k) = 2x\bar{\psi}. \] (2.17)

The above equations also apply to the NLS hierarchy by taking \( \psi^* = e^{-2t-1}\bar{\psi} \).

The string equation of the mKdV hierarchy has been obtained in [4], however, one can obtain it in a simple way given the string equation of the KdV hierarchy by using the Miura map. The idea is to pull back the string equation of the KdV hierarchy via the Miura map \( u = -\nu' - \nu^2 \); the result is then guaranteed to be consistent with the flows of the mKdV hierarchy, because of the Hamiltonian property of the Miura map. We act on (2.14) with the operator \( \frac{1}{2} \partial_x D^* \) and use eq. (2.8) to obtain

\[ \sum_{k=1}^{\infty} (2k + 1)t_k \frac{\partial \nu}{\partial t_k} = -x\nu' - \nu. \] (2.18)

This can be rewritten as

\[ \sum_{k=0}^{\infty} (2k + 1)t_k \frac{\partial \nu}{\partial t_k} + \nu = 0. \] (2.19)

Using eq. (2.8) again, we can integrate with respect to \( x \) (discarding, an integration constant) obtaining the mKdV string equation

\[ \sum_{k=0}^{\infty} (2k + 1)t_k D^* R_k = 0. \] (2.20)

where \( R_k = R_k[-\nu' - \nu^2] \). The \( k^{th} \) multi-critical point for which \( t_k = \) constant, \( t_0 = x \) and otherwise \( t_j = 0 \), is described by the string equation

\[ (2k + 1)t_k D^* R_k[-\nu' - \nu^2] = 2x\nu. \] (2.21)
We now show that the string equation of the ZS hierarchy consistently reduces to the string equations of the KdV and mKdV hierarchies, respectively, for the reductions we discussed in §2.2. For the reduction to KdV we evaluate (2.16) at \( \psi = -e^{t-1}, t_{2k+1} = 0 \) and use the relations obtained in §2.2 to get

\[
\sum_{k=0}^{\infty} (k + \frac{1}{2}) t_k^{KdV} 2^{-k} \left( \frac{1}{2} \partial_x^2 + \psi \right) H_{2k-1} = 0
\]

\[
\sum_{k=0}^{\infty} (k + \frac{1}{2}) t_k^{KdV} 2^{-k} H_{2k-1} = 0.
\]

These equations are obviously equivalent to the KdV string equations (2.13) since \( H_{2k-1} = 2^k R_k \) at \( \psi = -e^{t-1}, t_{2k+1} = 0 \).

Analogously, in the case of the mKdV reduction with \( \psi = e^{2t-1} \psi, t_{2k+1} = 0 \), one of the equations (2.16) is trivially solved and the other equation becomes

\[
\sum_{k=0}^{\infty} (k + \frac{1}{2}) t_k^{mKdV} 2^{-k} G_{2k} = 0
\]

But \( G_{2k} = -2^k D^* R_k [-\nu' - \nu^2] \) at \( \psi = e^{2t-1} \psi, t_{2k+1} = 0 \), which gives the mKdV string equation (2.20).

The significance of the fact that the string equation of the ZS model reduces to that of the KdV and mKdV models, is that solutions of the KdV and mKdV string equations are, by pulling back, solutions of the ZS string equation (on the subspace \( t_{2k+1} = 0 \) \( \forall k \)). This will lead us to conclude that the model described by the ZS hierarchy includes the models described by the KdV and mKdV hierarchies. We further discuss these facts in §3.4 and §3.5.

2.4 The Tau-Function Formalism

There is an alternative formalism for constructing integrable hierarchies which was originally developed as a direct solution technique of the non-linear equations of the hierarchy. The central objects of this approach are the tau-functions, which satisfy a hierarchy of non-linear ‘Hirota bilinear equations’, see [20] for example. For the hierarchies that we are considering, the Hirota hierarchies are intimately related to the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) (= \( A_1 \)). In fact, they use the two vertex operator representations of the basic representations of the affine algebra \( A_1^{(1)} \). This works in the following way. The basic representations of \( A_1^{(1)} \) are carried by the Fock space of a scalar field, either twisted or untwisted. The Hirota equations for the tau-function are equivalent to the condition that the tau-function lies in the orbit of the highest weight state of the group associated to the affine algebra. The untwisted construction underlies the ZS and NLS hierarchy, whilst the twisted construction underlies the KdV and mKdV hierarchies [20].
For the joint KdV and mKdV system (related by the Miura Map) there are two tau-functions \( \tau_0 \) and \( \tau_1 \) arising from the two basic representations of the Kac-Moody algebra \( A_1^{(1)} \). The relationships between these and the functions \( u \) and \( \nu \) are

\[
\begin{align*}
  u &= 2\partial_x^2 \log \tau_0, \\
  \nu &= \partial_x \log \left( \frac{\tau_1}{\tau_0} \right).
\end{align*}
\]  

One of the equations of the hierarchy is

\[
\tau_0'' \tau_1 - 2\tau_0' \tau_1' + \tau_0 \tau_1'' = 0,
\]

from which one can extract the Miura Map \( u = -\nu' - \nu^2 \) and the relation

\[
\nu^2 = -\partial_x^2 \log (\tau_0 \tau_1).
\]

The Hirota hierarchy which leads to the complex-ZS hierarchies has an infinite set of tau-functions. This is because the relevant vertex operator construction, in this case, involves an untwisted scalar field, which has a zero-mode. In order that the operator product expansions of the vertex operators are local, the zero-mode must be quantized, taking values in the weight lattice of the finite Lie algebra; \( A_1 \) in this case, whose weight lattice is simply isomorphic to \( \mathbb{Z} \). We will label elements of the weight lattice with half-integers, so that the sub-lattice generated by the root consists of the integers. With this labelling, the integers and half-integers correspond to the two distinct basic representations of the affine algebra \( A_1^{(1)} \). To make a connexion with the complex-ZS hierarchy one chooses a fixed element of the weight lattice, that is a half-integer \( n \). Then

\[
\psi = \tilde{\tau}_{n+1} / \tilde{\tau}_n, \quad \bar{\psi} = \tilde{\tau}_{n-1} / \tilde{\tau}_n,
\]

where we have used a tilde in order to avoid confusion with the mKdV/KdV tau-functions. In addition, the equations of the hierarchy imply

\[
\psi \bar{\psi} = -\partial_x^2 \log \tilde{\tau}_n.
\]

The hierarchies for different choices of \( n \) are isomorphic.

2.5 From Hierarchies to Matrix Models

Given a hierarchy and its string equation, to obtain the field theory describing a matrix model after a double-scaling limit, one must first identify the partition function of the field theory with some variable in the hierarchy.

If \( x \) is the cosmological constant then the ‘specific heat’ is \( F'' = -\partial_x^2 \log \mathcal{Z} \). Clearly \( -F'' \) has a well defined scaling dimension, from a hierarchical point of view. So, in principle, one can construct all the terms of the correct dimension from the hierarchy, and then see which can be integrated twice with respect to \( x \). However, even this would not
determine the normalization of the ‘specific heat’. In the absence of any additional physical requirements, one has to appeal to the matrix model. It transpires that the partition function of each particular model is related to the tau-function of the hierarchy in a simple way:

\[ Z_{ZS} = \tilde{\tau}_n, \quad Z_{KdV} = \tau_0^2, \quad Z_{mKdV} = \tau_0 \tau_1, \quad (2.29) \]

where the tau-functions for the hierarchies where introduced in section §2.4.

With \( x \) interpreted as the cosmological constant, the ‘specific heat’ is nothing but the correlation function

\[ \langle PP \rangle = -\partial_x^2 \log Z, \]

where the operator which couples to the cosmological constant is conventionally denoted \( P \) and called the puncture operator. Finally, the insertion of an operator in a correlation function is given by the corresponding flow in the hierarchy, for example

\[ \langle O_i PP \rangle = \partial_i \langle PP \rangle \quad (2.30) \]

where \( O_0 = P \) and the right-hand side is computed using the equation of the hierarchy. Moreover, with the identifications (2.29)

\[ \langle PP \rangle_{ZS} = -\psi \bar{\psi} = \langle PP \rangle_{NLS}, \quad \langle PP \rangle_{KdV} = u, \quad \langle PP \rangle_{mKdV} = -\nu^2. \quad (2.31) \]

Let us consider first the hierarchies which should correspond to the anti-hermitian matrix models and their correlation functions constructed using the prescription above. The anti-hermitian 1-matrix model in the 1-arc sector should correspond to the sum of two KdV hierarchies in the variables \( \chi = r - is, \quad \bar{\chi} = r + is = \chi^* \). Although these variables are complex \((r \text{ and } s \text{ are real functions of } x)\), it turns out that the correlation functions and the string equations are real. Indeed, the ‘specific heat’ is given by the sum of the ‘specific heats’,

\[ \langle PP \rangle = \frac{1}{2} (\chi + \bar{\chi}) = r \quad (2.32) \]

and the correlation functions are given by

\[ \langle O_k PP \rangle = \frac{1}{2} \partial_x (R_{k+1}(\chi) + R_{k+1}(\bar{\chi})) \quad. \quad (2.33) \]

Using the recursion relations of the KdV hierarchy it is easy to show that \( R_k(\bar{\chi}) = R_k(\chi)^* \) and hence the correlation functions are real. Moreover, the sum and the difference of the string equations (2.14) for \( \chi \) and \( \bar{\chi} \) give exactly eqs. (1.5) for this model.

The anti-hermitian 1-matrix model in the 2-arc sector corresponds to the ZS hierarchy. First of all, notice that for the ZS and NLS models one has

\[ \langle O_k P \rangle = \frac{1}{2} H_{k+1} \quad (2.34) \]

implying, for example, \( \langle PP \rangle = \frac{1}{2} H_1 = -\psi \bar{\psi} \). For the ZS hierarchy both \( \psi \) and \( \bar{\psi} \) are real, and thus the whole hierarchy and all the physical quantities are real. The string equations (2.16) correspond to the sum and difference of the two equations (1.5) where the second equation is multiplied by \( i \).

We now turn our attention to the hermitian 1-matrix model. In the 1-arc sector we have a double KdV hierarchy where everything is expressed in terms of real functions. In
the 2-arc sector, instead, we found only some of the string equations of the NLS hierarchy. Indeed, for the NLS hierarchy, one can easily show that $F^{2k}$, $H^{2k}$ and $G^{2k+1}$ are purely imaginary, whereas $F^{2k+1}$, $H^{2k+1}$ and $G^{2k}$ are real. Thus, although the ‘specific heat’ is real, many correlation functions are complex or pure imaginary. This obviously forbids an interpretation of the full NLS hierarchy as a field theory obtained after a double scaling limit of a hermitian 1-matrix model in the 2-arc sector. Taking the sum and the difference of the string equations (2.16) and using the recursion relations of the hierarchy one gets

$$\sum_{k=0}^{\infty} (k+1) t_k G_k = 0, \quad \sum_{k=0}^{\infty} (k+1) t_k F_k = 0. \quad (2.35)$$

The string equations (1.5) should correspond to the multi-critical points $t_0 = x, t_n = \text{constant}$ and $t_k = 0$ otherwise. Thus, if $n$ is even the first equation is real and the second is pure imaginary and they correspond to eq. (1.5) where the second equation is multiplied by $i$. These are the string equations we found in the first chapter. For $n$ odd instead, both equations are complex and so they could not arise from the direct study of the matrix model.

3. Virasoro Constraints

In [11] it was shown, for the models described by the KdV hierarchy, that the string equation, along with the hierarchy equations, could be reformulated as an infinite number of Virasoro-like constraints on the square-root of the partition function of the model. These constraints have a natural interpretation in terms of the Schwinger-Dyson equations for the loops of the matrix model. For the KdV model the square-root of the partition function is the tau-function of the hierarchy. The fact that the Virasoro constraints act on the tau-function of the hierarchy seems to be a universal feature of all the models, as will become apparent. First we briefly review the case for the KdV hierarchy. Using (2.29), (2.24) and the string equation (2.13), and integrating twice we deduce

$$\left( \sum_{k=1}^{\infty} (k+\frac{1}{2}) t_k \frac{\partial}{\partial t_{k-1}} + \frac{1}{8} t_k^2 \right) \tau_0 = 0. \quad (3.1)$$

Following [11], we use the recursion relations of the hierarchy (2.2) and the relation between $\tau_0$ and $u$ in (2.24), which together imply

$$\frac{\partial^2}{\partial x \partial t_{k+1}} \log \tau_0 = \left( \frac{1}{2} \partial^3_x + 2u \partial_x + u' \right) \frac{\partial}{\partial t_k} \log \tau_0, \quad (3.2)$$

in order to express the $t_k$ derivative in terms of the $t_{k+1}$ derivative. With this relation, one finds that $L_k \tau_0 = 0$ implies $L_{k+1} \tau_0 = 0$, where the $L_k$ are written below. All of the constants of integration encountered are set to zero, on the grounds that they would otherwise introduce spurious scales into the theory, except for the constant in the $L_0$ constraint.
which is dimensionless and fixed by the requirement that the algebra of constraints closes. The end result is that the tau-function satisfies an infinite number of constraints of the form

\[ L_n \tau_0 = 0, \quad n \geq -1, \quad (3.3) \]

where the \( L_n \)'s are the Virasoro generators of a \( \mathbb{Z}_2 \)-twisted scalar field

\[
L_{-1} = \sum_{m=1}^{\infty} (m + \frac{1}{2}) t_m \frac{\partial}{\partial t_{m-1}} + \frac{1}{8} t_0^2 \\
L_0 = \sum_{m=0}^{\infty} (m + \frac{1}{2}) t_m \frac{\partial}{\partial t_m} + \frac{1}{16} \\
L_n = \sum_{m=0}^{\infty} (m + \frac{1}{2}) t_m \frac{\partial}{\partial t_{m+n}} + \frac{1}{2} \sum_{m=1}^{n} \frac{\partial^2}{\partial t_{m-1} \partial t_{n-m}}.
\]

### 3.1 Virasoro Constraints for the MKDV Hierarchy

Although the matrix model which leads to the mKdV hierarchy has been discussed in the literature [4], the analogue of the Virasoro constraints do not seem to have been determined before (although ref. [12] does discuss Virasoro constraints before taking the double scaling limit). In this section we find the constraints using the mKdV string equation (2.20) and the recursion relations for the hierarchy.

The mKdV string equation (2.20) is obtained from integrating (2.19). From (2.19) one easily deduces

\[
\sum_{k=0}^{\infty} (2k+1) t_k \frac{\partial \nu^2}{\partial t_k} + 2 \nu^2 = 0.
\]

(3.5)

Recall that the partition function of the mKdV model is equal to the product of the tau-functions \( \tau_0 \) and \( \tau_1 \). We now express \( \nu \) in (2.19) in terms of the tau-functions \( \tau_0 \) and \( \tau_1 \), using (2.24), and \( \nu^2 \) in (3.5) using (2.26). The resulting two equations can be decoupled to arrive at

\[
\partial_x^2 \left[ \sum_{k=0}^{\infty} (2k+1) t_k \frac{\partial}{\partial t_k} \log \tau_j \right] = 0, \quad j = 0, 1.
\]

(3.6)

Integrating twice, and eliminating dimensionful integration constants, the two resulting equations may be written simply as

\[ L_0 \tau_j = \mu_j \tau_j, \quad j = 0, 1, \quad (3.7) \]

where \( L_0 \) is identical to the Virasoro constraint of the KdV model, eq. (3.4), and \( \mu_0 \) and \( \mu_1 \) are two, \( \textit{a priori} \) undetermined, dimensionless integration constants. They are not, however, independent as we now show. By substituting the expression for \( \nu \) in terms of the ratio \( \tau_1/\tau_0 \), in the equation for the flows (2.8), we deduce

\[
\frac{\partial}{\partial t_k} \log \left( \frac{\tau_1}{\tau_0} \right) = -\frac{1}{2} D^* R_k.
\]

(3.8)
This can now be substituted directly in (2.20) to yield
\[
\sum_{k \geq 0} (2k + 1) t_k \left( \frac{1}{\tau_1 \partial \tau_1} - \frac{1}{\tau_0 \partial \tau_0} \right) = 0. \tag{3.9}
\]

The above, along with eq. (3.7) implies that \( \mu_0 = \mu_1 = \mu \).

Before we discuss the possible meaning of the constant \( \mu \), we first present a simple argument for determining the higher Virasoro constraints. We already know from the construction of the KdV constraints, that \( L_k \tau_0 = 0 \) implies \( L_{k+1} \tau_0 = 0 \). It is also straightforward to verify that if \( L_0 \tau_0 = \mu \tau_0 \) then \( L_1 \tau_0 = 0 \), regardless of the value of \( \mu \). Therefore we deduce that \( \tau_0 \) satisfies the infinite set of constraints
\[
L_n \tau_0 = \mu \tau_0 \delta_{n,0}, \quad n \geq 0. \tag{3.10}
\]
To find the constraints satisfied by \( \tau_1 \), we notice that \( \tau_1 \) satisfies exactly the same recursion relation as \( \tau_0 \), that is (3.2), except that \( u = 2\partial^2 x \log \tau_0 \) is replaced with \( \tilde{u} = 2\partial^2 x \log \tau_1 \). Therefore, the same arguments that were applied to determine the constraints on \( \tau_0 \) will be applicable to \( \tau_1 \), hence we deduce
\[
L_n \tau_j = \mu \tau_j \delta_{n,0}, \quad n \geq 0, \tag{3.11}
\]
for \( j = 0 \) and 1. So the mKdV partition function is the product of two factors which separately satisfy a set of Virasoro constraints, however, in contrast to the KdV case there is no \( L_{-1} \) constraint and the \( L_0 \) constraint includes an undetermined constant. Notice that the requirement that the constraints form a closed algebra under commutation, does not in any way constrain the value of the constant. It is important to realize that \( \tau_0 \) and \( \tau_1 \) are not independent, in fact they satisfy a whole hierarchy of equations for which (2.25) is but the first. So although, at first sight, the mKdV Virasoro constraints look less restrictive than the KdV constraints, one must bear in mind the additional equations which tie \( \tau_0 \) and \( \tau_1 \) together.

The appearance of a parameter, which is not determined from the matrix model, seems, at first sight, to be surprising. However, it is not totally unexpected, indeed, such an occurrence is found at the first critical point of the mKdV model. At this point, the square root of the specific heat, \( \nu \), satisfies the Painlevé II equation \[4\]
\[
\nu'' - 2\nu^3 + x\nu = 0. \tag{3.12}
\]
This equation is known to admit a one-parameter family of solutions [23]. The actual solution which describes the matrix model, requires a scaling behaviour \( \nu \sim z^\xi \), as \( x \to \infty \). Ref. [24] discusses how, for one particular value of the parameter, such a physical solution does exists and is unique. It would be natural to suggest that the parameter is related to \( \mu \), the eigenvalue of the \( L_0 \) constraint. Indeed, (3.12) admits the trivial solution \( \nu = 0 \), which corresponds to the situation when \( \mu = \frac{1}{16} \), for which the Virasoro constraints have the solution \( \tau = 1 \) (i.e. \( \tau \) being the vacuum of the twisted Fock space). Notice that, for the KdV model, the Virasoro constraints are those of an \( \mathfrak{sl}(2,C) \) vacuum, whereas, for the mKdV model, the Virasoro constraints are those of a highest weight state of conformal dimension \( \mu \).
3.2 Virasoro Constraints for the ZS Hierarchy

The analogous constraints for the ZS hierarchy and string equation were found in [9]. Here, we briefly repeat their derivation which leads to Virasoro type constraints for an untwisted boson.

The string equations are (see eq. (2.16))

\[ \sum_{k \geq 0} (k+1)t_k F_k = 0 , \quad \sum_{k \geq 0} (k+1)t_k G_k = 0 . \tag{3.13} \]

Consider first the objects

\[ I_j = \sum_{k \geq 0} (k+1)t_k H'_{k+j} = \sum_{k \geq 0} (k+1)t_k (g G_{k+j} - f F_{k+j}) . \tag{3.14} \]

They can be reduced, using the hierarchy, to sums involving only \( F_k, G_k \) and their derivatives, which are related to the string equations.

From \( I_0 \), integrating twice over \( x \) and introducing an arbitrary integration constant\(^5\) \( \alpha \), we get

\[ \sum_{k \geq 1} (k+1)t_k \langle O_{k-1} \rangle + \frac{\alpha t_0}{2} = 0 \tag{3.15} \]

which will lead to the \( L_{-1} \) constraint.

From \( I_1 \) we get

\[ \sum_{k \geq 0} (k+1)t_k \langle O_k \rangle + \beta = 0 , \tag{3.16} \]

where we have picked up a new dimensionless integration constant, \( \beta \). This leads to the \( L_0 \) constraint.

Using a similar procedure, from \( I_2 \) we get

\[ \sum_{k \geq 0} (k+1)t_k \langle O_{k+1} \rangle + \alpha \langle P \rangle = 0 \tag{3.17} \]

which leads to the \( L_1 \) constraint.

Finally, with a few more steps from \( I_3 \) we obtain

\[ \sum_{k \geq 0} (k+1)t_k \langle O_{k+2} \rangle + \langle P \rangle^2 + \langle PP \rangle + \alpha \langle O_1 \rangle = 0 . \tag{3.18} \]

Whilst the previous equations involve only first order derivatives of the partition function, the equation coming from \( I_3 \) has second order terms, which fix the function on which the Virasoro constraints act. In fact, we can write

\[ \langle P \rangle^2 + \langle PP \rangle = (F')^2 - F'' = \frac{1}{\mathcal{Z}} \partial_x^2 \mathcal{Z} \tag{3.19} \]

\(^{5}\) We will discard all integration constants which would have non-trivial dimension.
and, therefore, the Virasoro constraints act on the partition function. Since for this model the partition function is equal to the tau-function, we find that the Virasoro constraints act on the tau-function, mirroring the situation for the KdV model.

By consistency, the commutator of two constraints should be a new constraint on the partition function. Therefore, using \( [L_n, L_1] \equiv (n - 1) L_{n+1} \) with \( n \geq 2 \), we get an infinite set of constraints acting on the partition function. These constraints are the Virasoro constraints

\[
L_n \tau_{\mathcal{ZS}} = 0, \quad n \geq -1 \tag{3.20}
\]

where

\[
\begin{align*}
L_{-1} &= \sum_{k \geq 1} (k + 1) t_k \frac{\partial}{\partial t_{k-1}} + \frac{\alpha t_0}{2}, \\
L_0 &= \sum_{k \geq 0} (k + 1) t_k \frac{\partial}{\partial t_k} + \frac{\alpha^2}{4}, \\
L_1 &= \sum_{k \geq 0} (k + 1) t_k \frac{\partial}{\partial t_{k+1}} + \alpha \frac{\partial}{\partial t_0}, \\
L_n &= \sum_{k \geq 0} (k + 1) t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n-2} \frac{\partial^2}{\partial t_k \partial t_{n-k-2}} + \alpha \frac{\partial}{\partial t_{n-1}}
\end{align*}
\tag{3.21}
\]

and \( \beta = \alpha^2/4 \) has been fixed by the relation \([L_1, L_{-1}] = 2(L_0 + \alpha^2/4 - \beta) = 2L_0\).

3.3 Connection with the Tau-Function Formalism

It was noticed in ref. [9] that the Virasoro constraints (3.21) are those of an untwisted scalar field. In the convention for which

\[
\varphi(z) = q - ip \log z + i \sum_{n \in \mathbb{Z} \neq 0} a_n \frac{z^{-n}}{n}, \tag{3.22}
\]

and \([a_m, a_n] = n \delta_{n+m,0}\) and \([q, p] = i\), we have for \( k \geq 0 \)

\[
t_k = \frac{\sqrt{2}}{k+1} a_{-k-1}, \quad \frac{\partial}{\partial t_k} = \frac{1}{\sqrt{2}} a_{k+1}. \tag{3.23}
\]

The zero-mode \( p \) is related to the integration constant \( \alpha \) via \( p = \alpha/\sqrt{2} \). The conjugate variable to \( p \) does not appear in the Virasoro operators.

The fact that the Virasoro constraints are those of an untwisted scalar field is very natural from the point of view of the tau-function approach, which we explained in section §2.4 based on ref. [20]. Recall that the zero mode of the scalar field of the construction of ref. [20] must have the quantized values \( m/\sqrt{2} \), for \( m \in \mathbb{Z} \), in order that the vertex operators have local expansions. The tau-function can be projected onto eigenspaces of the zero-mode, these are precisely the \( \tilde{\tau}_n \) which were introduced in §2.4, with \( \sqrt{2} n \) being the eigenvalue of the zero-mode.
From eq. (2.29) the partition function of the ZS model is equal to \( \tilde{\tau}_n \), for some fixed half-integer \( n \). It is very natural to identify the scalar field of the Virasoro constraints with the scalar field of the Hirota equations of [20]. We do not yet have a direct proof of this, however, below we present some arguments which support this view. Given this identification, one is led to a relation between the parameter \( n \) and the integration constant of the Virasoro constraints \( \alpha \):

\[
p = \frac{\alpha}{\sqrt{2}} = -\sqrt{2n}, \quad n \in \frac{1}{2}\mathbb{Z} .
\] 

(3.24)

The possibility that \( \alpha \) is quantized seems to be consistent with the results that we obtain in §3.4 and §3.5, for the KdV and mKdV reductions which require \( \alpha = 0 \) and \( \alpha = -1 \), respectively. To substantiate this identification we now show that if \( \tilde{\tau}_n \) satisfies the Virasoro constraints eq. (3.21) with \( \alpha \), then \( \tau_{n \pm 1} \) satisfy the same constraints but with \( \alpha \to \alpha \mp 2 \). We prove this fact following a similar demonstration as the previous paragraph. Let us consider the objects

\[
Y_j = \sum_{k \geq -1} (k + 1) t_k \frac{\partial}{\partial t_{k+j-1}} \left( \frac{\tilde{\tau}_{n \pm 1}}{\tilde{\tau}_n} \right) .
\] 

(3.25)

and use the string equations, the hierarchy of Hirota equations for the \( \tilde{\tau}_n \)'s, and the fact that \( \tilde{\tau}_n \) satisfies Virasoro constraints with \( \alpha \), \( L_m \tilde{\tau}_n \equiv L_m(\alpha)\tilde{\tau}_n = 0, \ m \geq -1 \). Considering \( Y_0, Y_1 \) and \( Y_2 \), it is easy to obtain

\[
L_{-1}(\alpha \mp 2)\tilde{\tau}_{n \pm 1} = L_0(\alpha \mp 2)\tilde{\tau}_{n \pm 1} = L_1(\alpha \mp 2)\tilde{\tau}_{n \pm 1} = 0 .
\] 

(3.26)

Again, the \( L_2 \) constraint is more tricky. Considering \( Y_3 \), it is easy to show that

\[
L_2(\alpha \mp 2)\tilde{\tau}_{n \pm 1} = (\alpha \mp 2) \left\{ \tilde{\tau}_n \left( \frac{\partial}{\partial t_1} + \frac{\partial^2}{\partial t_0} \right) \left( \frac{\tilde{\tau}_{n \pm 1}}{\tilde{\tau}_n} \right) \mp 2\tilde{\tau}_{n \pm 1}\partial^2_x \log \tilde{\tau}_n \right\} \] 

(3.27)

The right hand side of this equation vanishes because of one of the Hirota equations satisfied by the tau-functions. In particular, (see ref. [20] pg. 232 (III)\(_{1; n,n+1}\)),

\[
L_2(\alpha \mp 2)\tilde{\tau}_{n \pm 1} = (\alpha \mp 2) \frac{1}{\tilde{\tau}_n} (D_1^2 \mp D_2)\tilde{\tau}_{n \pm 1} \cdot \tilde{\tau}_n = 0 .
\] 

(3.28)

The \( D_i \)'s are operators of the Hirota calculus which are defined in ref. [20] for example. Therefore, we get

\[
L_m(\alpha)\tilde{\tau}_n = L_m(\alpha \mp 2)\tilde{\tau}_{n \pm 1} = 0 , \quad m \geq -1 .
\] 

(3.29)

If we now consider \( p = \alpha/\sqrt{2} \) as the zero-mode, ‘momentum’ operator of the scalar field, with \( p\tilde{\tau}_n = -\sqrt{2n}\tilde{\tau}_n \), in accordance with (3.24), then its conjugate variable or ‘position’ operator is \( q = -it_{-1}/\sqrt{2} \). This is deduced from equation (2.5) and (3.29).

\[\text{6 However, the ‘topological’ point described in ref. [9] does not seem to require any particular value of } \alpha.\]
The above result (3.29) also implies that the whole Hirota hierarchy admits a ‘master’ string equation. It is most suggestively written in terms of the full tau-function $\tilde{\tau}$, for which the $\tilde{\tau}_n$ are the projections onto eigenspaces of the zero-mode. The master string equation is the $m = -1$ version of the the following Virasoro constraints

$$L_m \tilde{\tau} = 0, \quad m \geq -1,$$

where the $L_m$ are the Virasoro generators of the bosonic field. So the final set of constraints are exactly analogous to those for the 1-arc KdV case, the difference being that there one has a twisted scalar field, whereas here we have an untwisted scalar field.

The appearance of the parameter $\alpha$ is rather mysterious, since it was not manifest in the matrix model. It seems to label different sectors in the theory which are not connected by the flows. It is clearly desirable to have a better understanding of its origin and meaning.

3.4 KdV Reduction of Virasoro Constraints

We have already shown how the the ZS hierarchy can be reduced to the KdV hierarchy, and how the string equations respect the reduction. On general grounds, one would anticipate that this would extend to all the Virasoro constraints, and we now prove this. The reduction involves taking $\psi = -e^{t-1}$ and $t_{2k+1} \to 0 \forall k$. Then $u = -\psi \tilde{\psi}$ satisfies the KdV hierarchy with

$$t_{2k}^{\text{ZS}} \equiv 2^{-k} t_k^{\text{KdV}}, \quad \tau^{\text{ZS}} = (\tau^{\text{KdV}})^2$$

and

$$\frac{\partial u}{\partial t_{2k}^{\text{KdV}}} = \partial_x R_{k+1}.$$

Notice that the second equation of (3.30) implies that under the reduction, i.e. on the subspace $t_{2k+1} = 0 \forall k$ with $\psi = -e^{t-1}$, $Z_{\text{ZS}} \to Z_{\text{KdV}}$, as it should. Under such reduction, $H_{2k} = -H_{2k-1}^{\prime}$:

$$F_{2k} + G_{2k} = (F_{2k-1} + G_{2k-1})' + (g + f)H_{2k-1} = (g - f)H_{2k-1}$$

$$0 = F_{2k+1} + G_{2k+1} = (F_{2k} + G_{2k})' + (g + f)H_{2k}$$

$$\Rightarrow ((g + f)H_{2k-1})' + (g + f)H_{2k} = 0$$

but $(g + f)' = 2\psi' = 0$, and the result follows.

We now show that the ZS Virasoro constraints on $\tau^{\text{ZS}}(= \tilde{\tau}_n)$ reduce to Virasoro constraints on $\tau^{\text{KdV}}(= \tau_0)$ for a precise value of the zero-mode $\alpha$ (or $n$). Let us first consider the equation corresponding to $I_0$, under the reduction

$$\sum_{k \geq 1} (2k + 1)t_{2k}H_{2k} + \alpha = - \sum_{k \geq 1} (2k + 1)t_{2k}H_{2k-1}^{\prime} + \alpha = 0$$

Using the relations with the correlation functions, and integrating twice over $t_0 = x$, we get

$$\sum_{k \geq 1} (2k + 1)t_{2k} \langle O_{2k-2} \rangle - \frac{\alpha t_0^2}{4} = 0$$

24
which will produce the reduced $L_{-1}$ constraint. The reduction of $I_1$ is direct, and leads to

$$\sum_{k \geq 0} (2k + 1)t_{2k} \langle O_{2k} \rangle + \alpha^2/4 = 0 \quad (3.35)$$

which produces the $L_0$ constraint. This constraint is also obtained from $I_2$, the $L_1$ constraint in the ZS hierarchy. The corresponding equation is

$$\sum_{k \geq 0} (k + 1)t_k H'_{k+2} + \alpha H'_1 + 2H_2 = 0$$

$$\Rightarrow \sum_{k \geq 0} (2k + 1)t_{2k} H''_{2k+1} + (2 - \alpha) H'_1 = 0 \quad (3.36)$$

Using the relations with the correlation functions, and integrating three times over $t_0 = x$, we get

$$\partial^3_x \left[ \sum_{k \geq 0} (2k + 1)t_{2k} \langle O_{2k} \rangle + (1 + \alpha) F \right] = 0 \quad (3.37)$$

which, by consistency with eq. (3.35), requires $\alpha = -1$. Therefore, the KdV reduction is only consistent for this value of the, $a$ priori, arbitrary parameter $\alpha$.

Let us now consider the equation corresponding to $I_3$, which, again, will to fix the function on which the constraints act:

$$\sum_{k \geq 0} (2k + 1)t_{2k} H'_{2k+3} + 4 \langle PPP \rangle \langle P \rangle + 4 \langle PP \rangle^2 + 2H_3 + 2(1 - \alpha) \langle PPPP \rangle = 0 \quad (3.38)$$

In the usual way, we get

$$\sum_{k \geq 0} (2k + 1)t_{2k} \langle O_{2k+2} \rangle + \langle P \rangle^2 + (1 - \alpha) \langle PP \rangle = 0 \quad (3.39)$$

This equation will lead to the $L_1$ constraint, and, again, the second order terms fix the functional on which the Virasoro constraint act. In this case $\alpha = -1$, and

$$\langle P \rangle^2 + (1 - \alpha) \langle PP \rangle = \langle P \rangle^2 + 2 \langle PP \rangle = \frac{4}{\sqrt{Z_{KdV}}} \partial_x^2 \sqrt{Z_{KdV}} \equiv \frac{4}{\tau_{KdV}} \partial_x^2 \tau_{KdV} \quad (3.40)$$

Notice that the required value of $\alpha$ is consistent with the quantization proposed in (3.24). Therefore, the reduced Virasoro constraints act on the square root of the partition function, which is the tau-function of the KdV hierarchy, in agreement with [11].

In terms of the variables $t_k \equiv t_k^{KdV} = 2^k \langle \tilde{Z}_k \rangle$, the reduced Virasoro constraints are $\tilde{L}_n \tau_{KdV} = 0$, with $n \geq -1$, where the operators $\tilde{L}_n$ are those of eq. (3.4).
3.5 MKDV Reduction

In the case of the mKdV reduction we have already shown in section §2.2 that $G_{2k+1}$, $F_{2k}$ and $H_{2k}$ vanish. It is straightforward to verify that under the reduction, i.e. on the subspace $t_{2k+1} = 0 \forall k$ with $\psi = e^{2t-1}\psi$, $Z_{ZS} \rightarrow Z_{mKdV}$. Using these results one can apply the reduction directly on the constraints.

From $I_0$ ($L_1$ constraint), we get

$$
\sum_{k \geq 1} (2k + 1)t_{2k}H_{2k} + \alpha = 0 \quad \Rightarrow \quad \alpha = 0 .
$$

(3.41)

Therefore, the mKdV reduction requires $\alpha = 0$, which is clearly consistent with the quantization of $\alpha$ proposed in (3.24). From $I_1$, we get

$$
\sum_{k \geq 0} (2k + 1)t_{2k}H'_{2k+1} + 2H_1 = 0 .
$$

(3.42)

But,

$$
H'_{2k+1} = 2\langle PP\sigma_{2k} \rangle \equiv -2\frac{\partial F''}{\partial t_{2k}}
$$

(3.43)

and we get the equation

$$
\sum_{k \geq 0} (2k + 1)t_{2k}\frac{\partial F''}{\partial t_{2k}} + 2F'' = 0
$$

(3.44)

which has the form of an $L_0$ constraint. This equation can also be rewritten as

$$
\sum_{k \geq 0} (2k + 1)t_{2k}\frac{\partial \nu}{\partial t_{2k}} + \nu = 0
$$

(3.45)

which, after integration, becomes the string equation of the mKdV hierarchy, eq. (2.20).

If we reduce the $L_0$ constraint of the ZS hierarchy itself, we find that the partition of the mKdV model satisfies

$$
\sum_{k \geq 0} (2k + 1)t_k^{mKdV} \frac{\partial}{\partial t_k^{mKdV}} Z_{mKdV} = 0 .
$$

(3.46)

Notice that $\mu$, the parameter of the mKdV Virasoro constraints of §3.1, is determined by the reduction to be $\frac{1}{16}$.

In a similar way, from $I_{2k}$ we get equations which are identically zero and do not give rise to any constraint. Instead, from $I_3$ ($L_2$ constraint) we get

$$
\left(\sum_{k \geq 0} (2k + 1)t_k^{mKdV} \frac{\partial}{\partial t_k^{mKdV}} + \frac{1}{2} \frac{\partial^2}{\partial x^2}\right) Z_{mKdV} = 0
$$

(3.47)
where \( t^\text{mKdV}_k = 2^k t^\text{ZS}_k \). We can write the above constraint in the following way. Firstly, we express the partition function in terms of the tau-functions \( Z^\text{mKdV} = \tau_0 \tau_1 \). Then we use the relation (2.25) to write
\[
\partial_x^2 (\tau_0 \tau_1) = 2\tau_0'' \tau_1 + 2\tau_0 \tau_1'',
\]
from which we deduce that (3.47) may be rewritten as
\[
(L_1 \tau_0) \tau_1 + \tau_0 (L_1 \tau_1) = 0,
\]
which is clearly a consequence of the \( L_1 \) constraints for \( \tau_0 \) and \( \tau_1 \) that we found for the mKdV model in §3.1.

One could carry on this process of reducing the higher Virasoro constraints. The resulting constraints would act directly on the partition function of the mKdV model; and hence would not be Virasoro constraints. Nevertheless, we expect that the constraints on the partition function should be expressible in terms of Virasoro constraints acting on each tau-function separately, as we found for \( \tau_0 \) and \( \tau_1 \) that we found for the mKdV model in §3.1.

4. Discussion and Open Problems

In this paper we have attempted to analyse all the possible double scaling limits of the hermitian and anti-hermitian 1-matrix models. As it is clear from the fact that eq. (1.3), after the double scaling limit, gives rise to a differential operator of degree two in \( x \), the hermitian and anti-hermitian 1-matrix models are related to the \( sl(2, \mathbb{C}) \) hierarchies. Hermitian and anti-hermitian matrix models have many common properties: in the 1-arc sector they both give rise to a KdV hierarchy for an even potential and to a doubled KdV hierarchy for a general potential; in the 2-arc sector, with even potential, they both give rise to the mKdV hierarchy. Instead they differ in the 2-arc sector with a general potential where the hermitian models give rise to only half of the critical points associated to the NLS hierarchy whereas the anti-hermitian model gives all the critical points associated to the ZS hierarchy (except for the ‘Topological’ one).

For the hermitian models the multi-critical points obtained, although described by a NLS hierarchy, actually have solutions which are described by a mKdV hierarchy. We do not know whether these solutions are the only ones. Furthermore, these critical points correspond to purely even potentials; in this sense the NLS structure is irrelevant, and one is really dealing with a mKdV structure. Instead the ZS hierarchy admits also a reduction to KdV, in other words in the 2-arc sector of the anti-hermitian matrix model we found a new series of multi-critical points described by the KdV hierarchy, besides the one already known from the 1-arc sector. This set of KdV multi-critical points are not in any simple way connected with those in the 1-arc sector, since, for example, the topological critical
point describing topological gravity [14], which cannot be obtained from the 1-arc sector, is obtained from the 2-arc sector with anti-hermitian matrices with a fourth order potential.

The situation in the 2-arc sector with anti-hermitian matrices and a general potential seems to be the richest, being described by the ZS hierarchy. The ‘even’ multi-critical points admit solutions described by KdV and mKdV hierarchies, which require two particular values of the parameter $\alpha$. Clearly, it would desirable to understand the role of the parameter $\alpha$, from the point of view of the matrix model, and also to know whether the KdV and mKdV solutions exhaust the possible solutions for the ‘even’ multi-critical points. For instance, are there other solutions for different values of $\alpha$, and do solutions exist only for the discrete values suggested by the tau-function formalism? An interesting open question regards the nature of the ‘odd’ multi-critical points of the ZS string equation. It is now well-known that solutions for the KdV and mKdV systems describing multi-critical behaviour exist [1,12,24]; we do not have any arguments to show that solutions can be found for the ‘odd’ scaling points of the ZS hierarchy, except for the first (or ‘topological’) point, corresponding to $t_1 \neq 0$, which was investigated in [9]. This point cannot be obtained from the matrix model and, as described in ref. [9], could give rise to a new kind of ‘topological’ theory. Anyway, since the mathematical apparatus exists for tackling the issue of the existence of solutions to these non-linear differential equations [15], we hope these questions will be addressed and solved elsewhere.

One of the results of this paper is the realization of the rather universal nature of the Virasoro constraints and the fact that they act on the tau-functions of the appropriate hierarchy, and not, necessarily, directly on the partition function. For the KdV and ZS hierarchies, the Virasoro constraints are

$$L_n \tau = 0 \quad n \geq -1,$$

where in both cases $\tau$ is an element of the basic representation of the Kac-Moody algebra $A^{(1)}_1$, for the twisted and untwisted constructions, respectively. In the mKdV case the partition function is the product of two tau-functions, which arise from the two basic representations of $A^{(1)}_1$, which both satisfy the Virasoro constraints of a highest weight vector:

$$L_n \tau_j = \mu \tau_j \delta_{n,0},$$

for $j = 0$ and 1.

Near the completion of this work our attention was drawn to refs. [25,26] which discuss similar Virasoro constraints to those above.

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