Homogenization of biomechanical models of plant tissues with randomly distributed cells

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Abstract
In this paper homogenization of a mathematical model for biomechanics of a plant tissue with randomly distributed cells is considered. Mechanical properties of a plant tissue are modelled by a strongly coupled system of reaction-diffusion-convection equations for chemical processes in plant cells and cell walls, the equations of poroelasticity for elastic deformations of plant cell walls and middle lamella, and the Stokes equations for fluid flow inside the cells. The nonlinear coupling between the mechanics and chemistry is given by the dependence of elastic properties of plant tissue on densities of chemical substances as well as by the dependence of chemical reactions on mechanical stresses present in a tissue. Using techniques of stochastic homogenization we derive rigorously macroscopic model for plant tissue biomechanics with random distribution of cells. Strong stochastic two-scale convergence is shown to pass to the limit in the non-linear reaction terms. Appropriate meaning of the boundary terms is introduced to define the macroscopic equations with flux boundary conditions and transmission conditions on the microscopic scale.

Keywords: stochastic homogenization, poroelasticity, Stokes system, biomechanics of plant tissues, stochastic two-scale convergence

(Some figures may appear in colour only in the online journal)

1. Introduction

Formation of plant tissues and organs is a result of the coordinated expansion of hundreds of thousands of cells, different in size, shape, and composition. Plant organs are composed of...
several types of tissues, e.g. epidermis, cortex, endodermis, vascular tissue [55]. While the turgor pressure, the main force for cell expansion, acts isotropically, the anisotropic deformation and growth of plant cells and tissues rely on the mechanics of cell walls, surrounding plant cells, and the microstructure of cell walls and tissues. Plant tissues have complex hierarchical microstructures given by the size and arrangement of cells, connected by cross-linked pectin network of middle lamella, on one scale, and by the heterogeneous structure of cell walls on the other scale [39]. In some tissues, such as wood or cork, the geometric arrangement of cells is very regular and can be regarded as periodic [33], however many plant tissues exhibit random variations in their microstructure [28, 47, 48]. Plant cell walls mainly consist of cellulose microfibrils, pectin, hemicellulose, macromolecules, and water. The orientation of microfibrils, their length, high tensile strength and interaction with wall matrix macromolecules determine the wall stiffness. For irreversible deformation, the deposition of new wall materials and the loosening of the cell wall through the breaking of the load-bearing cross-links between microfibrils, pectin and hemicellulose by enzymes activity are required [62]. It is supposed that calcium-pectin cross-linking chemistry strongly influences elastic properties of plant cell walls [70]. Pectin is produced in Golgi apparatus inside the cells and is deposited to a cell wall in a methyl-esterified form, where it can be de-methylesterified by the enzyme pectin methylesterase (PME), which removes methyl groups by breaking ester bonds. The de-methyl-esterified pectin is able to form calcium-pectin cross-links, and so stiffen the cell wall and reduce its expansion, see e.g. [69], whereas mechanical stresses can break calcium-pectin cross-links and hence increase the extensibility of plant cell walls and middle lamella.

Considering the complex structure of plant tissues and organs, for a better understand and improvement of plant growth and development, it is important to model and analyse how microscopic structure and interactions between chemical processes and mechanical properties of individual cell walls and cells contribute to the properties of the plant tissues and organs [8, 39]. Different approaches were applied to analyse the interplay between micro- and macro-mechanics and transport processes in plant tissues [7]. Many results can be found for multiscale modelling and analysis of the periodic microstructure of wood [26, 46, 60]. Multiscale modelling and analysis of the impact of the microscopic structure of plant cell walls, especially orientation and distribution of microfibrils, on mechanical properties of cell walls were conducted in [58]. A vertex-element model and hybrid vertex-midline model for plant tissue deformation and growth, coupled with the cell-scale transport of plant hormone, were considered in [30, 31]. The impact of microfibrils on the mechanical properties of cell walls was accounted for by introducing an anisotropic viscous stress which depends on a pair of microfibril directions. A simple constitutive model at the cell scale which characterises cell walls via yield and extensibility parameters together with an appropriate averaging over a cross-section were used to derive the analogous tissue-level model describing elongation and bending of a plant root [27]. A mesh-free particle method was proposed in [43] to simulate the mechanics of both individual plant cells and cell aggregates in response to external stresses and to study how plant tissue mechanics is related to the micromechanics of cells. The interior of the cell is regarded as liquid phase and simulated using the smoothed particle hydrodynamics (SPH) method, where in the domain corresponding to the viscoelastic material of cell walls the particles are connected by pairwise interactions holding them together. A multiscale method for the simulation of large viscoelastic deformations of a plant tissue presented in [32] combines particle method on the microscopic level with standard finite elements methods on the macroscopic scale. The effect of non-periodic microstructure on effective (homogenized) elastic properties of two-dimensional cellular materials (honeycombs) was studied in [66] by considering non-periodic arrangement of cell walls in random Voronoi honeycombs and applying finite element analysis. The finite-edge centroidal Voronoi tessellation (FECVT) was introduced in
To generate a realistic model of a non-periodic tissue microstructure and, combined with finite elements analysis, was used to determine the effective elastic properties of plant tissues, especially plant petioles and stems [29]. Smoothed particles hydrodynamics (SPH) framework was used in [52] to model plant tissue growth. The framework identifies the SPH particle with individual cells in a tissue, but the tissue growth is performed at the macroscopic level using SPH approximations and plant tissue is represented as an anisotropic poro-elastic material. A coarse-grained multiscale numerical model is proposed in [68] to predict macroscale deformations of food-plant tissues (e.g. apple tissues) during drying.

In [57] we derived and analysed a mathematical model for plant tissue biomechanics, which describes the interactions between calcium–pectin dynamics and deformations of a plant tissue. The microscopic model, at the length scale of plant cells, comprises a strongly coupled system of the Stokes equations modelling water flow inside plant cells, the equations of poro-elasticity defining elastic deformations of plant cell walls and middle lamella, and reaction-diffusion-convection equations describing the dynamics of the methyl-esterified pectin, de-methyl-esterified pectin, calcium ions, and calcium–pectin cross-links. The interplay between the mechanics and the chemistry comes in by assuming that the elastic properties of cell walls and middle lamella depend on the density of the calcium–pectin cross-links and the stress within cell walls and middle lamella can break the cross-links. Assuming periodic distribution of cells in a plant tissue in [57] we derived rigorously macroscopic model for plant tissue biomechanics. The two-way coupling between chemical processes and mechanics is the main novelty of the model, which also induces some non-standard elements in the analysis of the model and in the rigorous derivation of macroscopic equations. In this paper we generalise the results obtain in [57] by considering random distribution of cells in a plant tissue, observed experimentally in many plant tissues and organs [28, 48]. The derivation of macroscopic equations from a continuum description of the microscopic processes on the cell level using stochastic homogenization techniques results into a continuum macroscopic two-scale model containing the information on the microscopic interactions. Our microscopic model incorporates microscopic properties of plant cell walls, essential for plant tissue mechanics. The macroscopic model takes into account the microscopic structure of a plant tissue via effective (macroscopic) elasticity and permeability tensors and includes the interplay between the fluid in cell inside and poroelastic nature of cell walls and middle lamella. The effect of the microstructure and heterogeneity of the processes is also reflected in the equations for calcium–pectin chemistry via effective (macroscopic) diffusion coefficients, reaction terms and advective velocity. In the relation to particle and vertex-elements methods, continuum modelling approach proposed here may be beneficial when consider large size plant tissues and organs.

To analyse macroscopic mechanical properties of plant tissues with a random distribution of cells, we derive rigorously a macroscopic model for plant biomechanics using techniques of stochastic homogenization. The stochastic two-scale convergence [74] is applied to obtain the macroscopic equations. The main mathematical difficulties in the derivation of the macroscopic equations arise from the strong coupling between the equations of poro-elasticity and the system of reaction-diffusion-convection equations, as well as due to transmission conditions between the free fluid and poro-elastic material. The strong stochastic two-scale convergence for the displacement gradient and flow velocity is proven to pass to the limit in the nonlinear reactions terms. Extension arguments and formulations of surface integrals as volume integrals are used to pass to the stochastic two-scale limit in the equations with non-homogeneous Neumann boundary conditions and transmission conditions. To pass to the limit in the flux boundary conditions defined on the surfaces of the microstructure, Palm measure and the proven here trace inequality for $H^1$-function in the probability space, see lemma 8.1, are used.
Some of the first results on the stochastic homogenization of linear second-order elliptic equations were obtained in [41, 56, 72]. The homogenization of quasi-linear elliptic and parabolic equations with stochastic coefficients and convex integral operators was considered in [9, 21, 24, 25]. Subadditive ergodic theory and the method of viscosity solutions were applied to homogenize Hamilton-Jacobi, viscous Hamilton–Jacobi equations, and fully nonlinear elliptic and parabolic equations in stationary ergodic media [4, 20, 40, 44, 45] (see also references therein). The stochastic two-scale convergence introduced in [74] has been extended to Riemannian manifolds and has been applied to analyze heat transfer through composite and polycrystalline materials with nonlinear conductivities [36, 37]. The two-scale convergence in the mean [15] has been applied to derive macroscopic equations for single- and two-phase fluid flows in randomly fissured media [13, 71].

The poro-elastic equations, modelling interactions between fluid flow and elastic deformations of a porous medium, has been first obtained by Biot using a phenomenological approach [10–12] and subsequently derived by applying formal asymptotic expansion [5, 18, 42, 61, 64] or the two-scale convergence method [22, 34, 38, 49, 50, 54]. Along many results for poroelastic equations, only few studies of interactions between a free fluid and a deformable porous medium can be found. In [65] nonlinear semigroup method was used for mathematical analysis of a system of poroelastic equations coupled with the Stokes equations for free fluid flow. A rigorous derivation of interface conditions between a poroelastic medium and an elastic body was considered in [51]. Numerical methods for coupled system of poroelastic and Navier–Stokes equations were studied in [6, 19].

One of the approaches commonly used in numerical homogenization to approximate the effective coefficients of a microscopic problem describing some processes in a random medium is the so-called periodization [14]. The key idea of this method is to choose a large enough sample of the random medium, to extend it periodically, and to take the effective coefficients of the obtained periodic problem as an approximation of the effective coefficients of the original random problem. Recent years an essential progress was achieved in this approach, see the work [35], and references therein. Justification of this method for the model studied in the present paper is an interesting problem. Mixed multiscale finite element method [1] or stochastic variational multiscale method [2] can also be used for numerical simulation of multiscale stochastic problems.

The paper is organised as follows. In section 2 we formulate the microscopic model for plant tissue biomechanics. The main results of the paper are summarised in section 3. The a priori estimates and convergence results are given in sections 4 and 5. In section 6 we derive macroscopic equations for the coupled poro-elastic and Stokes problem. The strong stochastic two-scale convergence for displacement gradient and flow velocity is proven in section 7. The macroscopic equations for the system of reaction-diffusion-convection equations are derived in section 8.

2. Microscopic model

We consider a probability space \((\Omega, \mathcal{F}, \mathcal{P})\) with probability measure \(\mathcal{P}\). We define a 3-dimensional dynamical system \(\mathcal{T}_x : \Omega \to \Omega\), i.e. a family \(\{\mathcal{T}_x : x \in \mathbb{R}^3\}\) of invertible maps, such that for each \(x \in \mathbb{R}^3\), \(\mathcal{T}_x\) is measurable and satisfy the following conditions:

(a) \(\mathcal{T}_0\) is the identity map on \(\Omega\), and for all \(x_1, x_2 \in \mathbb{R}^3\) the semigroup property holds:

\[
\mathcal{T}_{x_1 + x_2} = \mathcal{T}_{x_1} \mathcal{T}_{x_2}.
\]
(b) $\mathcal{P}$ is an invariant measure for $T_x$, i.e. for each $x \in \mathbb{R}^3$ and $F \in \mathcal{F}$ we have that
\[ \mathcal{P}(T_x^{-1}F) = \mathcal{P}(F). \]
(c) For each $F \in \mathcal{F}$, the set $\{(x, \omega) \in \mathbb{R}^3 \times \Omega : T_x \omega \in F\}$ is a $\mathcal{L} \times \mathcal{F}$-measurable subset of $\mathbb{R}^3 \times \Omega$, where $\mathcal{L}$ denotes the Lebesgue $\sigma$-algebra on $\mathbb{R}^3$.

We consider a fixed measurable set $\Omega_f$ such that $\mathcal{P}(\Omega_f) > 0$ and $\mathcal{P}(\Omega \setminus \Omega_f) > 0$ and denote $\Omega_\varepsilon = \Omega \setminus \Omega_f$. We also consider $\Omega_T \subset \Omega$, with $\mathcal{P}(\Omega_T) > 0$ and $\mathcal{P}(\Omega_T \cap \Omega_f) > 0$, for $f = e, f$.

For $\mathcal{P}$-a.a. $\omega \in \Omega$ we define the following random subdomains in $\mathbb{R}^3$$
G_f(\omega) = \{x \in \mathbb{R}^3 : T_{x,f} \omega \in \Omega_f\}, \quad \text{for } j = e, f, \quad G_T(\omega) = \{x \in \mathbb{R}^3 : T_{x,\varepsilon} \omega \in \Omega_T\},$
and surfaces
\[\Gamma(\omega) = \partial G_f(\omega), \quad \tilde{\Gamma}(\omega) = \Gamma(\omega) \cap G_T(\omega).\]

We shall consider the following assumptions on $G_f(\omega)$, $\Gamma(\omega)$, and $\tilde{\Gamma}(\omega)$:

**Assumption 1.**

(a) $G_f(\omega)$ consists of countable number of disjoined Lipschitz domains for $\mathcal{P}$-a.a. $\omega \in \Omega$ with a uniform Lipschitz constant.

(b) The distance between two connected components of $G_f(\omega)$ is uniformly bounded from above and below.

(c) The diameter of connected components of $G_f(\omega)$ is uniformly bounded from above and below by some positive constants.

(d) The surface $\tilde{\Gamma}(\omega) \subset \Gamma(\omega)$ is open on $\Gamma(\omega)$ and Lipschitz continuous.

Consider a bounded $C^{1,\alpha}$-domain $G \subset \mathbb{R}^3$, with $\alpha > 0$, representing a part of a plant tissue. In a plant tissue individual cells, consisting of cell inside and cell walls, are connected by the pectin network of middle lamella. Then the microscopic structure of a plant tissue with a random distribution of cells is defined as
\[ G_f = \{x \in \mathbb{R}^3 : T_{x,f} \omega \in \Omega_f\} \cap G, \quad G_T = \{x \in \mathbb{R}^3 : T_{x,\varepsilon} \omega \in \Omega_T\} \cap G, \quad G_c = G \setminus G_f, \]
\[ \Gamma_c = \partial G_f, \quad \tilde{\Gamma}_c = \Gamma_c \cap G_T, \]
$\mathcal{P}$-a.s., where $G_f$ represent the subdomains occupied by cell walls and middle lamella, $G_f$ denotes the cell inside, and $\Gamma_c$ defines a part of cell membrane which is impermeable to calcium ions.

Assumption 1.2 states that the thickness of cell walls and middle lamella is uniformly bounded from above and below and assumption 1.3 postulates that the diameter of cells is bounded from above and below.

Due to assumed random distribution of cells in a plant tissue, the permeability and elastic properties of plant cell walls and middle lamella are characterised by the corresponding random variables. For this we define statistically homogeneous random fields $E_i(x, \omega, \xi) = E_i(T_{x,\varepsilon} \omega, \xi)$ and $K_p(x, \omega) = K_p(T_{x,\varepsilon} \omega)$, where $E_i(\cdot, \xi)$ and $K_p(\cdot)$ are given measurable functions from $\Omega$ to $\mathbb{R}^3$ and $\mathbb{R}^{3 \times 3}$, respectively, for $\xi \in \mathbb{R}$ representing the dependence of the elastic properties on the calcium–pectin cross-links density. It is observed experimentally that the load bearing calcium–pectin cross-links reduce cell wall expansion, see e.g. [70], and hence we assume that elastic properties of cell walls and middle lamella depend on the density of calcium–pectin cross-links.
Then for each \( \omega \in \Omega \) and \( \xi \in \mathbb{R} \) the microscopic elasticity tensor \( \mathbf{E}_i^\varepsilon \) and permeability tensor \( \mathbf{K}_e^\varepsilon \) are defined as

\[
\mathbf{E}_i^\varepsilon(x, \varepsilon, \omega, \xi) = \mathbf{E}_i(x/\varepsilon, \omega, \xi), \quad \mathbf{K}_e^\varepsilon(x) = \mathbf{K}_e(x/\varepsilon, \omega).
\]

In the mathematical model for biomechanics of a plant tissue we consider concentration of calcium \( c_e^\varepsilon \) and \( c_f^\varepsilon \) in cell walls and middle lamella \( G_e^\varepsilon \) and in cell cytoplasm \( G_f^\varepsilon \) (cell inside), respectively. In addition, in the domain of cell walls and middle lamella \( G_e^\varepsilon \) densities of methylsterified and de-methylsterified pectins \( n_e^\varepsilon \) and \( n_f^\varepsilon \) and of calcium–pectin cross-links \( n_c^\varepsilon \) are considered. We shall use the notation \( b_e^\varepsilon = (n_e^\varepsilon, n_f^\varepsilon, n_c^\varepsilon) \) and \( D_b(b_e^\varepsilon, x) = \text{diag}(D_{n_e}(n_e^\varepsilon), D_{n_f}(n_f^\varepsilon), D_{n_c}(n_c^\varepsilon)) \) denotes the diagonal matrix of diffusion coefficients for \( n_e^\varepsilon, n_f^\varepsilon, \) and \( n_c^\varepsilon \) respectively. We assume that the inflow of new calcium is facilitated only on parts of the cell membrane \( \Gamma^c \). Here we consider a passive flow of calcium between cell wall and cell inside. The regulatory mechanism for calcium inflow by mechanical properties of cell walls will be considered in further studies. For elastic deformations of plant cell walls and middle lamella we consider homogenized equations of poro-elasticity reflecting the microscopic structure of cell walls composed of elastic cellulose microfibrils and cell wall matrix permeable for the fluid flow. The differences in the elastic properties of cell walls and middle lamella are reflected in the elasticity tensor \( \mathbf{E}_i^\varepsilon \), which depends on the microscopic variable \( x/\varepsilon \). Here we consider diffusion coefficients depending on calcium–pectin cross-links density. The analysis in the case of diffusion coefficients depending additionally on microscopic and macroscopic variables will follow the same steps.

We shall use the notations \( G_f^\varepsilon(T) = (0, T) \times G, (\partial G) \varepsilon = (0, T) \times \partial G, G_{f,T}^\varepsilon = (0, T) \times G_f^\varepsilon \) for \( j = e, f, \Gamma_f^\varepsilon = (0, T) \times \Gamma^c, \) and \( \Gamma_f^\varepsilon = (0, T) \times \hat{\Gamma}^c \). By \( \Pi \cdot w \) we define the tangential projection of a vector \( w \), i.e. \( \Pi_w = w - (w \cdot n)n \), where \( n \) is a normal vector and \( \tau \) indicates the tangential subspace to the boundary.

For \( \mathcal{P} \cdot \text{a.a.} \) realizations \( \omega \in \Omega \) the microscopic model for the concentration of calcium and densities of pectins and calcium–pectin cross-links reads

\[
\begin{align*}
\partial_t b_e^\varepsilon & = \text{div}(D_b(b_e^\varepsilon)\nabla b_e^\varepsilon) + g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)), & \text{in } G_e^\varepsilon, \\
\partial_t c_e^\varepsilon & = \text{div}(D_c(c_e^\varepsilon)\nabla c_e^\varepsilon) + g_c(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)), & \text{in } G_e^\varepsilon, \\
\partial_t c_f^\varepsilon & = \text{div}(D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon)c_f^\varepsilon) + g_f(c_f^\varepsilon), & \text{in } G_f^\varepsilon, \\
D_b(b_e^\varepsilon)\nabla b_e^\varepsilon \cdot n & = \varepsilon \mathbf{R}(b_e^\varepsilon) & \text{on } \Gamma_f^\varepsilon, \\
c_f^\varepsilon & = c_e^\varepsilon & \text{on } \Gamma_f^\varepsilon \setminus \hat{\Gamma}_f^c, \\
D_b(b_e^\varepsilon)\nabla c_f^\varepsilon \cdot n & = (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon)c_f^\varepsilon) \cdot n & \text{on } \Gamma_f^\varepsilon \setminus \hat{\Gamma}_f^c, \\
D_b(b_e^\varepsilon)\nabla c_f^\varepsilon \cdot n & = 0, & \text{on } \hat{\Gamma}_f^c.
\end{align*}
\]

Here \( u_f^\varepsilon \) stands for the displacement from the equilibrium position in poroelastic material of cell wall and middle lamella, \( \mathbf{e}(u_f^\varepsilon) = (\mathbf{e}(u_f^\varepsilon))_{ij} \) for its symmetrized gradient, with \( \mathbf{e}(u_f^\varepsilon)_{ij} = (\partial_i u_f^\varepsilon_j + \partial_j u_f^\varepsilon_i)/2 \), and \( \partial_t u_f^\varepsilon \) denotes the fluid velocity in the cell inside. The pressures in the poroelastic and fluid domains are denoted by \( p_e^\varepsilon \) and \( p_f^\varepsilon \), respectively. The function \( \mathcal{G} \) defines the velocity field in the convection term in cell inside and is a Lipschitz continuous bounded function of the intracellular flow velocity \( \partial_t u_f^\varepsilon \). The condition that \( \mathcal{G} \) is bounded is natural from the biological and physical point of view, because the flow velocity in plant tissues is bounded. This condition is also essential for the derivation of \textit{a priori} estimates.
The water flow inside the cells and elastic deformations of plant cell walls and middle lamella are modelled by a coupled system of poro-elastic and Stokes equations

\[
\begin{align*}
\rho_e \partial_t^2 u_e^j - \text{div}(E^e(b_{e,3}^j)(e(u_e^j))) + \nabla p_e^j &= 0 & \text{in } G_e^T, \\
\rho_f \partial_t^2 p_f^j - \text{div}(K_p^e \nabla p_e^j - \partial_t u_e^j) &= 0 & \text{in } G_e^T, \\
\rho_f \partial_t^2 u_f^j - \varepsilon^2 \mu \text{div}(e(\partial_t u_f^j)) + \nabla p_f^j &= 0 & \text{in } G_f^T, \\
\text{div} \partial_t u_f^j &= 0 & \text{in } G_f^T, \\
(E^e(b_{e,3}^j)(e(u_e^j)) - p_f^j)n &= (\varepsilon^2 \mu e(\partial_t u_f^j) - p_f^j)n & \text{on } \Gamma_e^T, \\
\Pi_e \partial_t u_e^j = \Pi_f \partial_t u_f^j, & n \cdot (\varepsilon^2 \mu e(\partial_t u_f^j) - p_f^j)n = -p_e^j & \text{on } \Gamma_f^T, \\
(-K_p^e \nabla p_e^j + \partial_t u_e^j) \cdot n = \partial_t u_f^j \cdot n & \text{on } \Gamma_f^T, \\
u_e^j(0, x) = u_{e,0}^j(x), & \partial_t u_e^j(0, x) = u_{e,0}^j(x), & p_e^j(0, x) = p_{e,0}^j(x) & \text{in } G_e^0, \\
\partial_t u_f^j(0, x) = u_{f,0}^j(x) & \text{in } G_f^0,
\end{align*}
\]
A2 Elasticity tensor $\tilde{E}(\omega, \xi) = (\tilde{E}_{ijkl}(\omega, \xi))_{1 \leq i, j, k, l \leq 3}$ satisfies $\tilde{E}_{ijkl} = \tilde{E}_{klij} = \tilde{E}_{jilk}$ and $\alpha_1 |A|^2 \leq \tilde{E}(\omega, \xi)A \cdot A \leq \alpha_2 |A|^2$ for all symmetric $A \in \mathbb{R}^{3 \times 3}$, $\xi \in \mathbb{R}^3$, $\mathcal{P}$-a.a. $\omega \in \Omega$, and $0 < \alpha_1 \leq \alpha_2 < \infty$.

$\tilde{E}(\omega, \xi) = E_{3}(\omega, K(\xi))$, where $E_{1} \in C(\Omega; C_{b}^{0}(\mathbb{R}))$ and $K(\xi) = \int_{0}^{\beta} \kappa(t - \tau) \kappa(t, x) dt$, with a smooth function $\kappa: \mathbb{R}_{+} \to \mathbb{R}_{+}$ such that $\kappa(0) = 0$.

A3 $\tilde{K}_{p} \in L^{\infty}(\Omega)$ and $\tilde{K}_{p}(\omega) \eta \cdot \eta \geq k_{1} |\eta|^{2}$ for $\eta \in \mathbb{R}^{3}$, $\mathcal{P}$-a.a. $\omega \in \Omega$, and $k_{1} > 0$.

A4 The convection function $G$ is a Lipschitz continuous function on $\mathbb{R}^{3}$ such that $|G(r)| \leq \rho$, for some $\rho > 0$.

A5 For functions $g_{b}, g_{c}, g_{f}, R, F_{b},$ and $F_{c}$ we assume

$$g_{b} \in C(\mathbb{R} \times \mathbb{R}^{3} \times \mathbb{R}^{6}; \mathbb{R}), \quad g_{c} \in C(\mathbb{R} \times \mathbb{R}^{3} \times \mathbb{R}^{6}), \quad F_{b}, R \in C(\mathbb{R}^{3}; \mathbb{R}^{3}),$$

and $F_{c}$ and $g_{f}$ are Lipschitz continuous. Moreover, the following estimates hold:

$$|g_{b}(s, r, A)| \leq C_{1}(1 + |s| + |r|) + C_{2}|r| |A|, \quad |F_{b}(r)| + |R(r)| \leq C(1 + |r|),$$

$$|g_{c}(s, r, A)| \leq C_{1}(1 + |s| + |r|) + C_{2}(|s| + |r|)|A|, \quad |F_{c}(s)| + |g_{f}(s)| \leq C(1 + |s|),$$

where $s \in \mathbb{R}_{+}, r \in \mathbb{R}^{3}_{+}$, and $A$ is a symmetric $3 \times 3$ matrix. Here and in what follows we identify the space of symmetric $3 \times 3$ matrices with $\mathbb{R}^{9}$.

It is also assumed that for any symmetric $3 \times 3$ matrix $A$ we have that $g_{b}(s, r, A),$ $F_{b}(r), R(r)$ are non-negative for $r_{j} = 0, s \geq 0$, and $r_{j} \geq 0$, with $i = 1, 2, 3$ and $j \neq i$, and $g_{c}(s, r, A), g_{f}(s)$, and $F_{c}(s)$ are non-negative for $s = 0$ and $r_{j} \geq 0$, with $j = 1, 2, 3$.

We assume also that $F_{b}$ and $R$ are locally Lipschitz continuous, and

$$|g_{b}(s_{1}, r_{1}, A_{1}) - g_{b}(s_{2}, r_{2}, A_{2})| \leq C_{1}(|r_{1}| + |r_{2}|)|s_{1} - s_{2}| + C_{2}(|s_{1}| + |s_{2}| + |A_{1}| + |A_{2}|)|r_{1} - r_{2}|$$

$$+ C_{3}(|r_{1}| + |r_{2}|)|A_{1} - A_{2}|,$$

$$|g_{c}(s_{1}, r_{1}, A_{1}) - g_{c}(s_{2}, r_{2}, A_{2})| \leq C_{1}(|r_{1}| + |r_{2}| + |A_{1}| + |A_{2}|)|s_{1} - s_{2}|$$

$$+ C_{2}(|s_{1}| + |s_{2}| + |A_{1}| + |A_{2}|)|r_{1} - r_{2}|$$

$$+ C_{3}(|r_{1}| + |r_{2}| + |s_{1}| + |s_{2}|)|A_{1} - A_{2}|,$$

for $s_{1}, s_{2} \in (-\mu, +\infty)$, $r_{1}, r_{2} \in (-\mu, +\infty)^{3}$, for some $\mu > 0$, and $A_{1}, A_{2}$ are symmetric $3 \times 3$ matrices.

A6 $h_{0} \in L^{\infty}(\Gamma)$, $c_{0} \in L^{\infty}(\Gamma)$, and $h_{0} > 0, c_{0} > 0$ a.e. in $G$, where $j = 1, 2, 3$, $u_{0}^{(e)} \in H^{1}(\Gamma)^{3}, u_{0}^{(f)} \in H^{2}(\Gamma)^{3}$ and div $u_{0}^{(f)} = 0$ in $G_{f}$ for $\mathcal{P}$-a.a. realization $\omega \in \Omega$, $u_{0}^{(e)} \in H^{1}(\Gamma)^{3}, p_{0}^{(e)} \in H^{1}(\Gamma)$, are defined as solutions of

$$\text{div}(\mathbf{E}(b_{0,e})\mathbf{e}(u_{0}^{(e)})) = f_{a} \quad \text{in } G_{c},$$

$$\Pi_{r}(\mathbf{E}(b_{0,e})\mathbf{e}(u_{0}^{(e)})) n = \varepsilon_{2} \mu \Pi_{r}(\mathbf{e}(u_{0}^{(f)})) n \quad \text{on } \Gamma^{c},$$

$$n \cdot \mathbf{E}(b_{0,e})\mathbf{e}(u_{0}^{(e)}) n = 0 \quad \text{on } \Gamma^{c}, \quad u_{0}^{(e)} = 0 \text{ on } \partial G,$$

$$\text{div}(K_{p}^{(e)}\nabla p_{0}^{(e)}) = f_{p} \quad \text{in } G, \quad p_{0}^{(e)} = 0 \text{ on } \partial G.$$

$\mathcal{P}$-a.s., for given $f_{a} \in L^{2}(G)^{3}$ and $f_{p} \in L^{2}(G),

$$F_{p} \in H^{1}(0, T; L^{2}(\partial G)), \quad F_{a} \in H^{2}(0, T; L^{2}(\partial G)^{3}).$$

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Remark 2.1. Under our assumptions on \( \tilde{u}_{e0} \) and \( \tilde{p}_{e0} \) by the standard stochastic homogenization arguments we obtain

\[
\tilde{u}_{e0} \to u_{e0}, \quad \tilde{p}_{e0} \to p_{e0} \quad \text{strongly in } L^2(G),
\]

\[
\mathbf{e}(\tilde{u}_{e0}) \to \mathbf{e}(u_{e0}) + U_{e}^{\text{sym}} \quad \text{strongly stochastically two \(-\) scale, } U_{e}^{\text{sym}} \in L^2(G; L^2_{\text{pot}}(\Omega))^3,
\]

for \( \mathcal{P} \)-a.a. \( \omega \in \Omega \), where \( \tilde{u}_{e0} \) is an extension of \( u_{e0} \), and \( u_{e0} \in H^1(G) \) and \( p_{e0} \in H^1(G) \) are solutions of the corresponding macroscopic (homogenized) equations.

Here the subscript sym is used to emphasize that the corresponding matrix is symmetric.

Notice that in the equation for calcium \( c^f \) inside plant cells we consider a bounded function of the water velocity \( u_f \). This technical assumption is biologically justified, since only bounded velocities are possible inside plant cells.

By \( \langle \cdot, \cdot \rangle_{H^1(G);L^1} \) we shall denote the duality product between \( L^1(0,T; H^1(G))' \) and \( L^2(0,T; H^1(G)) \), and

\[
\langle \phi, \psi \rangle_{G_f} = \int_0^T \int_G \phi \psi \, dx \, dt \quad \text{for } \phi \in L^q(0,T; L^p(G)) \text{ and } \psi \in L^{q'}(0,T; L^{p'}(G)),
\]

and

\[
\langle \phi, \psi \rangle_{G_f;\Omega} = \int_0^T \int_{G_f\Omega} \phi \psi \, d\mathcal{P}(\omega) \, dx \, dt \quad \text{for } \phi \in L^q(0,T; L^p(G \times \Omega)) \text{ and } \psi \in L^{q'}(0,T; L^{p'}(G \times \Omega)),
\]

where \( 1 < p, q < +\infty, 1/q + 1/q' = 1 \) and \( 1/p + 1/p' = 1 \).

Definition 2.2. Weak solution of \((2)-(4)\) are functions

\[
u_{e}^f \in [L^2(0,T; H^1(G_e^f))] \cap H^2(0,T; L^2(G_e^f))^3,
\]

\[
u_{f}^e \in [L^2(0,T; H^1(G_f^e))] \cap H^2(0,T; L^2(G_f^e))^3,
\]

\[
u_{f}^j \in [L^2(0,T; H^1(G_f^j))] \cap H^2(0,T; L^2(G_f^j))^3,
\]

\[
\text{div } \nu_{e}^f = 0 \text{ in } G_f^f \text{, } \Pi e \nu_{e}^f = \Pi e \nu_{e}^f \text{ on } \Gamma_f^e \text{,}
\]

\[
\nu_{f}^j \in L^2((0,T) \times G_f^j)
\]

for \( \mathcal{P} \)-a.a. \( \omega \in \Omega \), that satisfy the integral relation

\[
\langle p_e \partial_{\nu}^e \nu_{e}^f, \phi \rangle_{G_f^e} + \langle \mathbf{E}(b_{e}^f) \mathbf{e}(\nu_{e}^f), \mathbf{e}(\phi) \rangle_{G_f^e} + \langle \nabla p_{e}^f, \phi \rangle_{G_f^e} + \langle \mathbf{e}(\partial_{\nu}^e \nu_{f}^e) \cdot \mathbf{n}, \psi \rangle_{\Gamma_f^e} - \langle \partial_{\nu}^e \nu_{f}^e \cdot \mathbf{n}, \psi \rangle_{\Gamma_f^e} - \langle \nu_{f}^e, \eta \rangle_{\Gamma_f^e}
\]

\[
\langle \nu_{f}^j, \partial_{\nu}^j \nu_{f}^j, \eta \rangle_{G_f^j} + \mu \varepsilon^2 \langle \mathbf{e}(\partial_{\nu}^j \nu_{f}^j), \mathbf{e}(\eta) \rangle_{G_f^j} = \langle F_n, \phi \rangle_{\partial G_f^j} + \langle F_p, \psi \rangle_{\partial G_f^j}
\]

for all \( \phi \in L^2(0,T; H^1(G_e^f))^3, \psi \in L^2(0,T; H^1(G_f^e))^3, \eta \in L^2(0,T; H^1(G_f^j))^3 \) such that \( \Pi_{\nu} \phi = \Pi_{\nu} \eta \text{ on } \Gamma_e \text{ and } \text{div } \eta = 0 \text{ in } G_f^j \), and functions

\[
b_{e}^f \in [L^2(0,T; H^1(G_e^f)) \cap L^\infty(0,T; L^2(G_e^f)))^3,
\]

\[
c^e = c_{e}^c \chi_{G_e^c} + c_{e}^f \chi_{G_e^f} \in L^2(0,T; H^1(G_\Gamma^c)) \cap L^\infty(0,T; L^2(G))
\]
that satisfy the integral relations
\begin{equation}
\langle \partial_t b^*_e, \varphi_1 \rangle_{H^1(G^*_e; \mathbb{H})} + \langle D_b(b^*_e \nabla b^*_e; \nabla \varphi_1) \rangle_{G^*_e} = \langle g_b(c^*_e, b^*_e, \mathbf{e}(u^*_e)), \varphi_1 \rangle_{\mathbb{H}}
\end{equation}

(6)

and
\begin{equation}
\langle \partial_t c^*_e, \varphi_2 \rangle_{H^1(G^*_e; \mathbb{H})} + \langle D_c(b^*_e \nabla c^*_e; \nabla \varphi_2) \rangle_{G^*_e} = \langle g_c(c^*_e, b^*_e, \mathbf{e}(u^*_e)), \varphi_2 \rangle_{\mathbb{H}}
\end{equation}

(7)

for all \( \varphi_1 \in L^2(0, T; H^1(G^*_e)) \) and \( \varphi_2 \in L^2(0, T; H^1(G^*_e)) \), and for \( \mathcal{P} \)-a.a. \( \omega \in \Omega \). Moreover the initial conditions are satisfied in \( L^2 \)-sense, i.e. \( u^*_e(t) \rightarrow u^*_0 \), \( \partial_t u^*_e(t) \rightarrow u^*_0 \), \( p^*_e(0) \rightarrow p^*_0 \), \( b^*_e(t) \rightarrow b_0 \), \( c^*_e(t) \rightarrow c_0 \) in \( L^2(G^*_e) \) as \( t \rightarrow 0 \), \( \partial_t u^*_e(t) \rightarrow u^*_0 \), \( c^*_e(t) \rightarrow c_0 \) in \( L^2(G^*_e) \) as \( t \rightarrow 0 \), \( \mathcal{P} \)-almost sure.

Examples of random geometries

- Let \( Q \) be a smooth domain, \( Q \subset (0, 1)^3 \), and assume that \( \gamma = \text{dist}(Q, \partial(Q, 1)^3) > 0 \). Let \( \xi \) be i.i.d. random vectors in \( \mathbb{R}^3 \) such that \( |\xi| \leq \gamma/4 \), and \( \eta_j, j \in \mathbb{Z}^3 \), be random variables with values in the interval \([1/2, 1] \). Letting \( Q_j = j + \xi_j + \eta_j Q \) we define

\[ G_j(\omega) = \bigcup_{j \in \mathbb{Z}^3} Q_j(\omega). \]

- Let \( \mathcal{P} \) be a stationary ergodic point process in \( \mathbb{R}^3 \) such that

* almost surely for any two points \( x_j \) and \( x_k \) from \( \mathcal{P}(\omega) \) the inequality \( |x_j - x_k| \geq c > 0 \) holds with a deterministic constant \( c \);

* There exists \( r > 0 \) such that the intersection of the process with any ball of radius \( r \) is a.s. non-empty. We then set \( Q_j = \{ x \in \mathbb{R}^3 : \text{dist}(x, x_j) < \frac{1}{2} \text{dist}(x, \mathcal{P}(\omega \backslash x_j)) \} \) and define

\[ G_j(\omega) = \bigcup_{j \in \mathbb{Z}^3} Q_j(\omega). \]

- The last example admits the following modifications: for the same stationary point process \( \mathcal{P} \) we consider the Voronoi tessellation

\[ Q_j(\omega) = \{ x \in \mathbb{R}^3 : \text{dist}(x, x_j) < \text{dist}(x, \mathcal{P}(\omega \backslash x_j)) \}. \]

Then \( \bigcup_j Q_j = \mathbb{R}^3 \) and, under the assumptions on \( \mathcal{P} \), the diameters of the polyhedrons \( Q_j \) are uniformly bounded and their boundaries are uniformly Lipschitz continuous.

Given \( \delta > 0 \) we then set

\[ G_\delta(\omega) = \{ x \in \mathbb{R}^3 : \text{dist}(x, \bigcup_j \partial Q_j) < \delta \}. \]

Notice that in this case the volume fraction of \( G_\delta \) is of order \( \delta \), if \( \delta \) is sufficiently small. This allows to model cell structures with relatively small volume fraction of cell walls and middle lamella.
3. Main results

The main result of the paper is the derivation of the macroscopic equations for the microscopic problem (2)–(4) using methods of stochastic homogenization.

First we shall introduce the following notations. Denote by \( \partial_L \) the generator of a strongly continuous group of unitary operators in \( L^2(\Omega) \) associated with \( \tilde{T}_\omega \) along \( e_j \)-direction, i.e.

\[
\partial_L u(\omega) = \lim_{\delta \to 0} \frac{u(T_{\omega, \delta}) - u(\omega)}{\delta}.
\]

The domains of \( \partial_L \), with \( j = 1, 2, 3 \), are dense in \( L^2(\Omega) \). We denote \( \nabla_v u = (\partial_L^1 u, \partial_L^2 u, \partial_L^3 u)^T \) and \( H^1_v(\Omega) = \{ v : v, \nabla_v v \in L^2(\Omega) \} \). By \( C_T(\Omega) \) we denote the space of functions with continuous realizations and \( C^0_T(\Omega) \) defines the set of functions from \( C_T(\Omega) \) such that \( (\partial_L^j u) \in C_T(\Omega) \), for \( j = 1, 2, 3 \).

First we introduce the spaces of potential and solenoidal vector fields:

\[
L^2_{\text{pot}}(\Omega) = \{ \nabla_v u(\omega) : u \in C^0_T(\Omega) \} \quad \text{and} \quad L^2_{\text{sol}}(\Omega) = (L^2_{\text{pot}}(\Omega))^\perp,
\]

where the closure in the definition of \( L^2_{\text{pot}}(\Omega) \) is with respect to the \( L^2(\Omega) \)-norm, see [73]. To introduce correctors we also need the space of functions whose realizations are discontinuous along the surface \( \Gamma(\omega) \). We define

\[
L^2_{\text{pot}, \Gamma}(\Omega) = \{ \nabla_v u(T_{\omega, \Gamma})|_{x=0} , \ u(T_{\omega, \Gamma}) \in H^1_\text{loc}(\mathbb{R}^3, \Gamma(\omega)) \cap C^1(\mathbb{R}^3, \tilde{\Gamma}(\omega)) \}
\]

with the norm

\[
\| u \|^2 = \int_{\Omega, \Gamma} \int_{[0,1]^3, \tilde{\Gamma}(\omega)} |\nabla_v u(T_{\omega, \Gamma})|^2 dx d\mathcal{P},
\]

and

\[
L^2_{\text{sol}, \Gamma}(\Omega) = (L^2_{\text{pot}, \Gamma}(\Omega))^\perp.
\]

We also denote

\[
C_{T, \Gamma}(\Omega) = \{ u : u(T_{\omega, \Gamma}) \in C(\mathbb{R}^3, \tilde{\Gamma}(\omega)) \}.
\]

We start with the definition of effective coefficients for macroscopic poro-elastic equations, which are obtained by deriving the macroscopic equations for the microscopic problem (3) and (4). The macroscopic elasticity tensor \( E^\text{hom} = (E^\text{hom}_{ijkl}) \) and permeability tensor \( K^\text{hom} = (K^\text{hom}_{ijkl}) \), along with \( K_\omega = (K_{\omega,ijkl}) \), are defined by

\[
E^\text{hom}_{ijkl}(h_{e,3}) = \int_\Omega \left[ \tilde{E}_{ijkl}(\omega, h_{e,3}) + (\tilde{E}(\omega, h_{e,3}) W^k_{e,\text{sym}})_{ijkl} \right] \chi_{\Omega_3} d\mathcal{P}(\omega),
\]

\[
K^\text{hom}_{ijkl} = \int_\Omega \left[ \tilde{K}_{ijkl}(\omega) + (\tilde{K}(\omega) W^j_{e})_{ijkl} \right] \chi_{\Omega_3} d\mathcal{P}(\omega),
\]

\[
K_{\omega,ijkl} = \int_\Omega \left[ \delta_{ij} - (\tilde{K}(\omega) W^j_{e})_{ijkl} \right] \chi_{\Omega_3} d\mathcal{P}(\omega),
\]

where \( \chi_{\Omega_3} \) stands for the characteristic function of the set \( \Omega_3 \), \( W^k_{e,\text{sym}} \) denotes the symmetric part of the matrix \( W^k_{e} \), and \( W^{ijkl}_{e} \in L^\infty(\mathbb{R}^3, L^2_{\text{pot}}(\Omega)^3) \) together with \( W^k_{e} \) and \( W^k_{s} \in L^2_{\text{pot}}(\Omega) \) are solutions.
of cell problems
\[
\begin{align*}
\int_\Omega (w_{i,3} e_{ij} + b_{ij}) \Phi \chi_{\Omega_t} d\mathcal{P}(\omega) &= 0 \quad \text{for all } \Phi \in L^2_{\text{pol}}(\Omega)^3, \text{ a.a. } (t,x) \in G_T, \\
\int_\Omega \tilde{K}_f(\omega) W^k + e_k) \zeta \chi_{\Omega_t} d\mathcal{P}(\omega) &= 0 \quad \text{for all } \zeta \in L^2_{\text{pol}}(\Omega), \\
\int_\Omega \tilde{K}_p(\omega) W_k - e_k) \zeta \chi_{\Omega_t} d\mathcal{P}(\omega) &= 0 \quad \text{for all } \zeta \in L^2_{\text{pol}}(\Omega),
\end{align*}
\]
for \( k, l = 1, 2, 3 \), with \( b_{ij} = \frac{1}{2}(e_k \otimes e_l + e_l \otimes e_k) \)
and \( \{e_j\}_{j=1}^3 \) is the canonical basis of \( \mathbb{R}^3 \).

We also define \( Q(\partial u_f) \) as
\[
Q(\partial u_f) = \int_\Omega \partial u_f \chi_{\Omega_t} d\mathcal{P}(\omega) - \int_\Omega \tilde{K}_f(\omega) Q_f(\omega, \partial u_f) \chi_{\Omega_t} d\mathcal{P}(\omega),
\]
where \( Q_f(\cdot, \partial u_f) \in L^2_{\text{pol}}(\Omega) \) is a solution of the problem
\[
\int_\Omega \left( \tilde{K}_f(\omega) Q_f(\omega, \partial u_f) + \partial u_f \chi_{\Omega_t} \right) \zeta d\mathcal{P}(\omega) = 0 \quad \text{for all } \zeta \in L^2_{\text{pol}}(\Omega).
\]

Then the macroscopic equations for the microscopic problem (3) and (4) are formulated as follows.

**Theorem 3.1.** A sequence of solutions \( \{u_\varepsilon, p_\varepsilon, \partial u_\varepsilon, p_f\} \) of the microscopic problem (3) and (4) converges to a solution \( u \in H^2(0, T; L^2(G)) \cap L^2(0, T; H^1(G)) \), \( p_e \in L^2(0, T; H^1(G)) \cap H^1(0, T; L^2(G)) \), \( \partial u_f \in L^2(G_T; H^1(\Omega)) \cap H^1(0, T; L^2(G \times \Omega)) \), \( p_f \in L^2(G_T \times \Omega) \) of the macroscopic equations
\[
\begin{align*}
\partial_t \rho_e + \nabla \cdot \left( \mathbf{E}^\text{hom}(b_{e,3}) \mathbf{e}(u_\varepsilon) \right) \rho_e &= F_e \quad \text{in } G_T, \\
\partial_t \rho_e \partial u_e - \nabla \cdot (K^\text{hom} \nabla p_e - K_e \partial u_e - Q(\partial u_f)) &= 0 \quad \text{in } G_T
\end{align*}
\]
with boundary and initial conditions
\[
\begin{align*}
\mathbf{E}^\text{hom}(b_{e,3}) \mathbf{e}(u_\varepsilon) \cdot n &= F_e \quad \text{on } (0, T) \times \partial G, \\
(K^\text{hom} \nabla p_e - K_e \partial u_e - Q(\partial u_f)) \cdot n &= F_p \quad \text{on } (0, T) \times \partial G, \\
u_\varepsilon(0) &= u_0, \quad \partial u_\varepsilon(0) = u^1_0, \quad p_e(0) = p_0 \quad \text{in } G,
\end{align*}
\]
and the equations for the flow velocity
\[
\begin{align*}
\int_\Omega \left[ \rho_f \partial^2 u_f \varphi + \mu \mathbf{e}(\partial u_f) \mathbf{e}(\varphi) + \nabla p_e \varphi \right] \chi_{\Omega_t} d\mathcal{P}(\omega) - \int_\Omega P^1 \chi_{\Omega_t} \varphi d\mathcal{P}(\omega) &= 0, \\
\text{div}_\omega \partial u_f &= 0 \quad \text{in } G_T \times \Omega, \\
\partial u_f(0) &= u^{j_0}_f \quad \text{in } G \times \Omega,
\end{align*}
\]
\( \Pi_\varepsilon \partial u_f(t, x, \tilde{\omega}) = \Pi_\varepsilon \partial u_\varepsilon(t, x) \) for \( (t, x) \in G_T \) and \( \tilde{\omega} \in \Gamma(\omega), \ \mathcal{P} - \text{a.s. in } \Omega, \)
and

$$\begin{align*}
P_i^f(t, x, \omega) &= \sum_{k=1}^3 \partial_{q_k} p_i(t, x) W_k^f(\omega) + \partial_{q_k} u_i^f(t, x) W_k^f(\omega) + Q_f(\omega, \partial_t u_f),
\end{align*}$$

for all $\varphi \in L^2(G_T; H^1(\Omega))^3$, with $\text{div}_w \varphi = 0$ in $G_T \times \Omega$, and $\Pi_\varphi(t, x, \tau; \omega) = 0$ for $(t, x) \in G_T$, $\tau \in \Gamma(\omega)$ and $\mathcal{P}$-a.s. in $\Omega$.

Here $e_\omega(\psi) = \left(\frac{1}{2} \partial_{e_k} \psi + \partial_{\psi} \psi\right)_{j=1,2,3}$ denotes a symmetric gradient for $\psi \in H^1(\Omega)^3$, $\varphi = \int_\Omega \chi_{\partial_\Omega} d\mathcal{P}(\omega)$, and $\text{div}_w \psi = \partial_1^e \psi_1 + \partial_2^e \psi_2 + \partial_3^e \psi_3$.

**Remark.** Notice that the equations for correctors (9) and (11), as well as problem (14) for $\partial_t u_f$ are formulated in the weak form as integral identities. This is due to the fact that the equations are define on $\Omega_\varepsilon \subset \Omega$ and $\Omega_{\bar{\varepsilon}} \subset \Omega$, respectively, and have strong formulation only for $\mathcal{P}$-a.a. realizations $\omega \in \Omega$.

The homogenized coefficients in reaction-diffusion-convection equations that will be obtained by deriving macroscopic equations for microscopic problem (2) and (4), are defined as

$$\begin{align*}
D_{i,\text{eff}}^j(b_{c,3}) &= \int_\Omega \left[ D_i^j(b_{c,3}) + (D_0(b_{c,3}) w_0^j) \right] \chi_{\Omega_\varepsilon} d\mathcal{P}(\omega),

D_{i,\text{eff}}^j(b_{c,3}) &= \int_\Omega \left[ D_i^j(b_{c,3}, \omega) + (D_0(b_{c,3}, \omega) w_i^j) \right] d\mathcal{P}(\omega),
\end{align*}$$

where $D(b_{c,3}, \omega) = D_\varepsilon(b_{c,3}) \chi_{\Omega_\varepsilon}(\omega) + D_f \chi_{\Omega_f}(\omega)$ for $\omega \in \Omega$, with $w_0^j \in L^2_{\text{pot}}(\Omega)$ and $w_i^j \in L^2_{\text{pot},1}(\Omega)$ are solutions of the cell problems

$$\begin{align*}
\int_\Omega D_\varepsilon(b_{c,3})(w_0^j + e_j) \zeta \chi_{\Omega_\varepsilon} d\mathcal{P}(\omega) = 0 \quad \text{for all } \zeta \in L^2_{\text{pot}}(\Omega),
\end{align*}$$

and

$$\begin{align*}
\int_\Omega D(\omega, b_{c,3})(w_i^j + e_j) \eta \chi \mathcal{P}(\omega) = 0 \quad \text{for all } \eta \in L^2_{\text{pot},1}(\Omega).
\end{align*}$$

The effective velocity is defined by

$$u_{\text{eff}}(t, x) = \int_\Omega \left( G(\partial_t u_f) - D_f Z(t, x, \omega) \right) \chi_{\Omega_f} d\mathcal{P}(\omega),$$

where $Z \in L^\infty_1(G_T; L^2_{\text{pot}}(\Omega))$ satisfies

$$\begin{align*}
\int_\Omega (D_f Z - G(\partial_t u_f)) \zeta \chi_{\Omega_f} d\mathcal{P}(\omega) = 0 \quad \text{for all } \zeta \in L^2_{\text{pot}}(\Omega), \text{ for a.a. } (t, x) \in G_T.
\end{align*}$$
Theorem 3.2. A sequence of solutions of microscopic problem (2) and (4) converges to a solution $b_{\varepsilon}, c \in L^2(0, T; H^1(\Omega))$, with $\partial_p b_{\varepsilon}, \partial_t c \in L^2(0, T; (H^1(\Omega)))$, of the macroscopic equations

$$\begin{align*}
\partial_t b_{\varepsilon} - \mathrm{div}(D_{\mathrm{eff}}(b_{\varepsilon}) \nabla b_{\varepsilon}) &= \int_\Omega g_b(c, b_{\varepsilon}, U(b_{\varepsilon}, \omega)) \chi_\Omega \, d\mathcal{P}(\omega) + \int_\Omega R(b_{\varepsilon}) \, d\mu(\omega) \quad \text{in } G_T, \\
\partial_t c - \mathrm{div}(D_{\mathrm{eff}}(b_{\varepsilon}) \nabla c - u_{\mathrm{eff}}(c)) &= \partial_f \mathcal{G}(c) + \int_\Omega g_c(c, b_{\varepsilon}, U(b_{\varepsilon}, \omega)) \chi_\Omega \, d\mathcal{P}(\omega) \quad \text{in } G_T,
\end{align*}$$

$$D_{\mathrm{eff}}(b_{\varepsilon}) \nabla b_{\varepsilon} \cdot n = F_b(b_{\varepsilon}) \quad \text{on } (\partial G)_T,$$

$$D_{\mathrm{eff}}(b_{\varepsilon}) \nabla c - u_{\mathrm{eff}}(c) \cdot n = F_c(c) \quad \text{on } (\partial G)_T,$$

$$b_{\varepsilon}(0, x) = b_{\varepsilon,0}(x), \quad c(0, x) = c_0(x) \quad \text{in } G,$$

where $\partial_j = \int_\Omega \chi_\Omega(j) \, d\mathcal{P}(\omega)$, for $j = e, f$, and

$$\|U(b_{\varepsilon}, \omega)\| = \{U_{ij}(b_{\varepsilon}, \omega)\}_{i,j=1}^{3} = \left\{b_{ij} + W_{\text{sym},ij}^e\right\}_{i,j=1}^{3},$$

with $W_{ij}^e$ being solutions of cell problems (9) and $b_{ij} = (b_{ij})_{i,j=1}^{3}$, where $b_{ij} = e_k \otimes e_l$.

Here $\mu$ is the Palm measure of the random measure $\mu_{\omega}$ of surfaces $\Gamma(\omega)$, see e.g. [23] for the definition of Palm measure.

4. A priori estimates

Considering assumptions on $G_j$, with $j = e, f$, in the same way as in the periodic case [57], for $P$-a.a. realizations $\omega \in \Omega$, we obtain the existence, uniqueness and a priori estimates, uniform in $\varepsilon$, for solutions of microscopic problem (2)-(4).

Lemma 4.1. Under assumption 2 there exists a unique weak solution of microscopic problem (2)-(4).

Proof. Sketch. For each realization $\omega$ the proof of the existence and uniqueness results follows the same steps as the proof of theorem 7 in [57].

Lemma 4.2. Under assumptions 2 solutions of microscopic problem (2)-(4) satisfy a priori estimates for elastic displacement $u_{\varepsilon}$, pressure $p_{\varepsilon}$, and fluid flow velocity $\partial_t u_{\varepsilon}$

$$\begin{align*}
\|u_{\varepsilon}\|_{L^2(0, T; H^1(G_\omega))} + \|\partial_t u_{\varepsilon}\|_{L^2(0, T; H^1(G_\omega))} + \|\partial_t^2 u_{\varepsilon}\|_{L^2(G_\omega)} &\leq C, \\
\|P_{\varepsilon}\|_{L^\infty(0, T; L^2(G_\omega))} + \|\partial_t P_{\varepsilon}\|_{L^2(G_\omega)} &\leq C, \\
\|\partial_t u_{\varepsilon}\|_{L^2(0, T; L^2(G_\omega))} + \|\partial_t^2 u_{\varepsilon}\|_{L^2(G_\omega)} + \varepsilon \|\nabla \partial_t u_{\varepsilon}\|_{L^2(G_\omega)} + \|P_{\varepsilon}\|_{L^2(G_\omega)} &\leq C,
\end{align*}$$

and for the concentration of calcium $c_j$ and $c^c_j$ and densities of pectins and calcium–pectin cross-links $b^c_j$ we obtain

$$\begin{align*}
\|b^c_j\|_{L^2(0, T; H^1(G_\omega))} + \|b^c_j\|_{L^\infty(0, T; L^\infty(G_\omega))} + \varepsilon^{1/2} \|b^c_j\|_{L^2(\Gamma_j)} &\leq C, \\
\|c^c_j\|_{L^2(0, T; L^2(G_\omega))} + \|c^c_j\|_{L^\infty(0, T; L^\infty(G_\omega))} &\leq C, \quad j = e, f,
\end{align*}$$
and
\[
\|\theta_j b^\varepsilon_j - b^\varepsilon_j\|_{L^2(0,T;G_j)} + \|\theta_j c^\varepsilon_j - c^\varepsilon_j\|_{L^2(0,T;G_j)} \leq Ch^{1/4}, \quad j = e, f,
\]
for \(\tilde{T} \in (0, T - h]\) and for \(\mathcal{P}\)-a.a. \(\omega \in \Omega\), where the constant \(C\) is independent of \(\varepsilon\) and \(\theta_j v(t,x) = v(t+h,x)\) for \((t,x) \in (0, T-h] \times G_j\), with \(j = e, f\).

**Proof.** For \(\mathcal{P}\)-a.a. realizations \(\omega \in \Omega\) the proof of the \textit{a priori} estimates follows the same lines as in \([57, \text{lemma 6}].\) We shall denote \(c^\varepsilon_j(t,x) = c^\varepsilon_j(t,x)\chi_G + c^\varepsilon_j(t,x)\chi_{G_j}\).

Using the assumptions on the random microscopic structure of \(G^\varepsilon_j\) and \(G_j\), we obtain the following extension results for functions defined on \(G^\varepsilon_j\) and on a subdomain \(\tilde{G}_{ij} \subset G\), which will be specified below.

**Lemma 4.3.**

(a) There exist extensions \(\tilde{b}^\varepsilon\) and \(\tilde{c}^\varepsilon\) of \(b^\varepsilon\) and \(c^\varepsilon\) respectively, from \(L^2(0,T;H^1(G^\varepsilon_j))\) to \(L^2(0,T;H^1(G))\) such that
\[
\|\tilde{b}^\varepsilon\|_{L^2(G)} \leq C \|b^\varepsilon\|_{L^2(G^\varepsilon_j)}, \quad \|\nabla \tilde{b}^\varepsilon\|_{L^2(G)} \leq C \|\nabla b^\varepsilon\|_{L^2(G^\varepsilon_j)},
\]
\[
\|\tilde{c}^\varepsilon\|_{L^2(G)} \leq C \|c^\varepsilon\|_{L^2(G^\varepsilon_j)}, \quad \|\nabla \tilde{c}^\varepsilon\|_{L^2(G)} \leq C \|\nabla c^\varepsilon\|_{L^2(G^\varepsilon_j)}.
\]

(b) There exists an extension \(\tilde{c}^\varepsilon\) of \(c^\varepsilon\) from \(L^2(0,T;H^1(\tilde{G}_{ij}))\) to \(L^2(0,T;H^1(G))\) such that
\[
\|\tilde{c}^\varepsilon\|_{L^2(G)} \leq C \left(\|c^\varepsilon\|_{L^2(G^\varepsilon_j \cap \tilde{G}_{ij})} + \|c^\varepsilon\|_{L^2(G^\varepsilon_j \cap \tilde{G}_{ij})}\right),
\]
\[
\|\nabla \tilde{c}^\varepsilon\|_{L^2(G)} \leq C \left(\|\nabla c^\varepsilon\|_{L^2(G^\varepsilon_j \cap \tilde{G}_{ij})} + \|\nabla c^\varepsilon\|_{L^2(G^\varepsilon_j \cap \tilde{G}_{ij})}\right).
\]

Here \(\tilde{G}_{ij} = G \setminus \tilde{G}_{i}\), with \(\tilde{G}_{i} = \tilde{G}_{i,\varepsilon}(\omega) \cap \tilde{G}_j\), where \(\tilde{G}_{i,\varepsilon}(\omega)\) is a \(\varepsilon\)-\(\sigma\)-neighbourhood of \(\tilde{G}_i\) for \(\mathcal{P}\)-a.a. realizations \(\omega \in \Omega\) and \(0 < \sigma < d_{\text{dim}}/4\), with \(d_{\text{min}}\) being the minimal distance between connected components of \(G_j(\omega)\).

**Proof.** The uniform boundedness of the diameter of cell walls and cell interiors, independent on realizations \(\omega \in \Omega\), implies the existence of the corresponding extension operators, see \([3]\].

Extensions for \(u^\varepsilon\) and \(p^\varepsilon\) are defined in the similar way as for \(b^\varepsilon\).

**Lemma 4.4.** For extensions of \(b^\varepsilon, c^\varepsilon, u^\varepsilon, p^\varepsilon\) from \(G^\varepsilon_{ij}\) to \(G_T\) and \(c^\varepsilon\) from \(\tilde{G}_{ij}\) to \(G_T\) (denoted again by \(b^\varepsilon, c^\varepsilon, u^\varepsilon, p^\varepsilon, c^\varepsilon\)) we have the following estimates
\[
\|u^\varepsilon\|_{W^{1,\infty}(0,T;H^1(G))} + \|\partial_t u^\varepsilon\|_{L^2(G_T)} + \|p^\varepsilon\|_{W^{1,\infty}(0,T,H^1(G))} + \|\partial_t p^\varepsilon\|_{L^2(G_T)} \leq C,
\]
\[
\|b^\varepsilon\|_{L^2(0,T;H^1(G))} + \|c^\varepsilon\|_{L^2(0,T;H^1(G))} \leq C,
\]
\[
\|\theta_h b^\varepsilon - b^\varepsilon\|_{L^2(0,T;G)} + \|\theta_h c^\varepsilon - c^\varepsilon\|_{L^2(0,T;G)} \leq Ch^{1/4},
\]

(27)
where the constant $C$ is independent of $\varepsilon$. An extension of $\partial_t u_j^\varepsilon$ from $G^\varepsilon_{j,T}$ to $G_1$, constructed below and denoted again by $\partial_t u_j^\varepsilon$ satisfies the following estimates

$$
\|\partial_t u_j^\varepsilon\|_{L^\infty(0,T;L^2(G_0))} + \|\partial_t^2 u_j^\varepsilon\|_{L^2(G_T)} + \varepsilon \|\nabla \partial_t u_j^\varepsilon\|_{L^2(G_T)} \leq C,
$$

(28)

where the constant $C$ is independent of $\varepsilon$. Also we have that

$$
\|ar{p}_j\|_{L^3(G_T)} + \|p_j\|_{L^3(0,T;G_T)} \leq C,
$$

(29)

and the constant $C$ does not depend on $\varepsilon$.

**Proof.** The estimates for $b_j^\varepsilon$, $c_j^\varepsilon$, $c^\varepsilon$, $u_j^\varepsilon$, and $p_j^\varepsilon$ follow directly from estimates (21)--(23), lemma 4.3, and the linearity of extension considered in lemma 4.3.

Using geometrical assumptions on $G_j(\omega)$, for $\mathcal{P}$-a.a. $\omega \in \Omega$, we can extend $\partial_t u_j^\varepsilon$ from $G^\varepsilon_{j,T}$ to $G_1$ in the following way. For each connected component $G_j^\varepsilon(\omega)$ of $G_j(\omega)$, where $j \in \mathbb{N}$, we can consider a $\sigma$-neighbourhood $G^\varepsilon_{j,\sigma}(\omega)$ of $G_j(\omega)$, where $\sigma = d_{\min}/4$ and $d_{\min}$ is the minimal distance between $G_j(\omega)$ for $j \in \mathbb{N}$. Then since $\partial_t u_j^\varepsilon \in L^2(0,T;H^1(G^\varepsilon_{j,\sigma}))$, i.e. $\partial_t u_j^\varepsilon \in L^2(0,T;H^{1/2}(G^\varepsilon_{j,\sigma}))$, there exists $\partial_t \bar{u}_j^\varepsilon \in L^2(0,T;H^1(G^\varepsilon_{j,\sigma}))$ satisfying the problem

$$
\begin{align*}
\text{div}_j \partial_t \bar{u}_j^\varepsilon &= 0 & \text{in } G^\varepsilon_{j,\sigma}(\omega), \\
\partial_t \bar{u}_j^\varepsilon &= \partial_t u_j^\varepsilon(t,\cdot,\varepsilon) & \text{on } \Gamma_j(\omega), \\
\partial_t \bar{u}_j^\varepsilon &= 0 & \text{on } \partial G^\varepsilon_{j,\sigma}(\omega)
\end{align*}
$$

(30)

for $\mathcal{P}$-a.a. realizations $\omega \in \Omega$ and $j \in \mathbb{N}$, see e.g. [67, theorem 2.4, lemma 2.4]. Each $\partial_t \bar{u}_j^\varepsilon$ we extend by zero to $G_j(\omega) \setminus G^\varepsilon_{j,\sigma}$. Considering a scaling $y = x/\varepsilon$ in $\partial_t \bar{u}_j^\varepsilon$ and collecting all $\partial_t \bar{u}_j^\varepsilon$ for $j \in \mathbb{N}$ we obtain an extension $\partial_t \bar{\pi}_j$ of $\partial_t u_j^\varepsilon$ from $G^\varepsilon_{j,T}$ to $G$ such that $\partial_t \bar{\pi}_j \in L^2(0,T;H^1(G))$ and

$$
\begin{align*}
\text{div} \partial_t \bar{\pi}_j &= 0 & \text{in } G, \\
\|\partial_t u_j^\varepsilon\|_{L^3(G_T)} + \varepsilon \|\nabla \partial_t u_j^\varepsilon\|_{L^2(G_T)} &\leq C,
\end{align*}
$$

(31)

where the constant $C$ is independent of $\varepsilon$.

Similar to the periodic case to show the a priori estimates for $p_j^\varepsilon$, we consider the first and third equations in (3) and use the a priori estimates for $u_j^\varepsilon$, $p_j^\varepsilon$, and $\partial_t u_j^\varepsilon$ to obtain

$$
\begin{align*}
\langle p_j^\varepsilon, \text{div } \phi \rangle_{G^\varepsilon_{j,T}} + \langle p_j^\varepsilon, \text{div } \phi \rangle_{G_{j,T}} &= \langle \varepsilon^2 \mu e(\partial_t u_j^\varepsilon), e(\phi) \rangle_{G^\varepsilon_{j,T}} + \langle \rho_j \partial_t^2 u_j^\varepsilon, \phi \rangle_{G^\varepsilon_{j,T}} \\
&\quad + \langle \rho_j \partial_t^2 u_j^\varepsilon, \phi \rangle_{G_{j,T}} + \langle \rho_j \partial_t^2 u_j^\varepsilon, \phi \rangle_{G_{j,T}} \\
&\quad + \langle \rho_j n - F_\varepsilon, \phi \rangle_{\partial \Omega_\varepsilon} \\
&\leq C \|\phi\|_{L^2(0,T;H^1(G))},
\end{align*}
$$

(32)

with $\phi \in L^2(0,T;H^1(G))^3$. Here we used the extension of $p_j^\varepsilon$ from $G^\varepsilon_{j,T}$ to $G$, see lemma 4.3, and the trace estimate $\|p_j^\varepsilon\|_{L^2(0,T;\partial G)} \leq C_1 \|p_j^\varepsilon\|_{L^2(0,T;H^1(G))} \leq C_2 \|p_j^\varepsilon\|_{L^2(0,T;H^1(G))}$.
For any \( q \in L^2(G_T) \) there exists \( \phi \in L^2(0, T; H^1(G)) \) satisfying

\[
\text{div } \phi = q \quad \text{in } G, \quad \phi \cdot n = \frac{1}{|\partial G|} \int_{\partial G} q(t, x) \, dx \quad \text{on } \partial G
\]

and \( \|\phi\|_{L^2(0,T;H^1(G))} \leq C \|q\|_{L^2(G_T)} \). Thus using (29), the definition of the \( L^2 \)-norm, and the a priori estimates for \( p' \) we obtain

\[
\|\tilde{p}'\|_{L^2(G_T)} \leq C \quad \text{and} \quad \|p'\|_{L^2((0,T)\times G_T^c)} \leq C,
\]

where the constant \( C \) is independent of \( \varepsilon \).

\[\square\]

5. Convergence results

From a priori estimates derived in lemma 4.4 we obtain corresponding strong and stochastic two-scale convergences for a subsequence of solutions of microscopic problem (2)–(4). First we recall the definition of the stochastic two-scale convergence introduced in [74].

**Definition 5.1.** Let \( G \) be a domain in \( \mathbb{R}^3 \), \( \mathcal{T}_\varepsilon \) be an ergodic dynamical system, and \( \tilde{\omega} \) be a ‘typical realization’. Then, we say that a sequence \( \{v^\varepsilon\} \subset L^2(0, T; L^2(G)) \) converges stochastically two-scale to \( v \in L^2(G_T; L^2(\Omega; dP)) \) if

\[
\lim_{\varepsilon \to 0} \sup \int_0^T \int_G |v^\varepsilon(t, x)|^2 \, dx \, dt < \infty \tag{33}
\]

and

\[
\lim_{\varepsilon \to 0} \int_0^T \int_G v^\varepsilon(t, x) \varphi(t, x) \psi(\mathcal{T}_\varepsilon \omega) \, dx \, dt \tag{34}
\]

\[
= \int_0^T \int_G \int_\Omega v(t, x, \omega) \varphi(t, x) \psi(\omega) \, dP(\omega) \, dx \, dt
\]

for all \( \varphi \in C^\infty_c((0, T) \times G) \) and \( \psi \in L^2(\Omega) \).

As a ‘typical realization’ we denote such realization \( \omega \in \Omega \) that Birkhoff’s theorem is satisfied for \( \mathcal{T}_\varepsilon \omega \), i.e.

\[
\lim_{\varepsilon \to \infty} \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}} g(\mathcal{T}_\varepsilon \omega) \, dx = \int_\Omega g(\omega) \, dP(\omega)
\]

\( \mathcal{P} \)-a.s. for all bounded Borel sets \( \mathcal{A} \), \( |\mathcal{A}| > 0 \), and all \( g(\omega) \in C(\Omega) \). Let us note that realizations are typical \( \mathcal{P} \)-a.s., see e.g. [74].

Using compactness properties of stochastic two-scale convergence, see [74], we obtain the following result.

**Lemma 5.2.** There exist functions \( u_\varepsilon \in H^1(0, T; H^1(G)) \cap H^2(0, T; L^2(G)) \), \( p_\varepsilon \in L^2(0, T; H^1(G)) \cap H^1(0, T; L^2(G)) \), \( U_\varepsilon^1, \partial_\varepsilon U_\varepsilon^1 \in L^2(G_T; L^2_{\text{pot}}(\Omega))^3 \), \( P_\varepsilon^1 \in L^2(G_T; L^2_{\text{pot}}(\Omega)) \), and
\[ \partial_t u^\varepsilon \chi_{\Omega_f}, \nabla \partial_t u^\varepsilon \chi_{\Omega_f}, \partial^2 \varepsilon u^\varepsilon \chi_{\Omega_f}, p_f \chi_{\Omega_f} \in L^2(G_T \times \Omega), \text{ such that, up to a subsequence,} \]
\[ u^\varepsilon \to u \quad \text{strongly in } H^1(0, T; L^2(G)), \]
\[ p^\varepsilon \to p \quad \text{strongly in } L^2(0, T) \times G, \]
\[ \partial^2 u^\varepsilon \to \partial^2 u, \quad \partial_t p^\varepsilon \to \partial_t p \quad \text{stochastically two-scale,} \]
\[ \nabla u^\varepsilon \to \nabla u + U_1^\varepsilon \quad \text{stochastically two-scale,} \]
\[ \nabla p^\varepsilon \to \nabla p + p_1^\varepsilon \quad \text{stochastically two-scale,} \]

and for fluid velocity and pressure we have
\[ \chi_{G_f} \partial_t u_f^\varepsilon \to \chi_{\Omega_f} \partial_t u_f \quad \text{stochastically two-scale,} \]
\[ \varepsilon \chi_{G_f} \nabla \partial_t u_f^\varepsilon \to \chi_{\Omega_f} \nabla \partial_t u_f \quad \text{stochastically two-scale,} \]
\[ \chi_{G_f} p_f^\varepsilon \to \chi_{\Omega_f} p_f \quad \text{stochastically two-scale.} \]

**Proof.** The estimates (27), the compactness of the embedding of \( H^1(0, T; L^2(G)) \cap L^2(0, T; H^1(G)) \) in \( L^2(G_T) \), and the compactness theorem for stochastic two-scale convergence, see e.g. [74], yield the convergence results in (35).

For the extension of \( u_f^\varepsilon \) from \( G_f \) to \( G \) we have the stochastic two-scale convergence of
\[ \partial_t u_f^\varepsilon \to \partial_t u_f \quad \text{and} \quad \varepsilon \nabla \partial_t u_f^\varepsilon \to \varepsilon \nabla \partial_t u_f, \quad \text{with} \partial_t u_f, \nabla \partial_t u_f \in L^2(G_T \times \Omega), \]

Additionally we have that
\[ U_1^\varepsilon \chi_{\Omega_f}, p_1^\varepsilon \chi_{\Omega_f}, \partial_t u_f \chi_{\Omega_f}, \text{ and } \varepsilon \nabla \partial_t u_f \chi_{\Omega_f} \text{ do not depend on the extension of } u_f^\varepsilon, \quad \text{from } G_f \text{ to } G \quad \text{and of } \partial_t u_f^\varepsilon \text{ from } G_f \text{ to } G. \]

The estimate and definition of \( p^\varepsilon \) in (29) and (32) ensure the stochastic two-scale convergence of \( \chi_{G_f} p_f^\varepsilon \).

In the following lemma, we shall use the same notation for \( b^\varepsilon, c^\varepsilon \) and their extensions from \( G_f \) to \( G \), whereas the extension for \( c^\varepsilon \) from \( G^c \) to \( G \) will be denoted by \( \bar{c}^\varepsilon \).

**Lemma 5.3.** There exist functions \( b^\varepsilon, c \in L^2(0, T; H^1(G)), b^\varepsilon \in L^\infty(0, T; L^\infty(G)), c \in L^\infty(0, T; L^1(G)), \) and correctors \( B_1^\varepsilon \in L^2(G_T; L^2_{pot}(\Omega)) \) and \( C^1 \in L^2(G_T; L^2_{pot}(\Omega)) \), such that, up to a subsequence,
\[ b^\varepsilon \to b, \quad c^\varepsilon \to c \quad \text{strongly in } L^2(G_T), \]
\[ \nabla b^\varepsilon \to \nabla b + B_1^\varepsilon \quad \text{stochastically two-scale,} \]
\[ \nabla c^\varepsilon \to \nabla c + C^1 \quad \text{stochastically two-scale,} \quad \text{as } \varepsilon \to 0. \]

**Proof.** The estimates in (27), together with compactness results for stochastic two-scale convergence, see e.g. [74], ensure that for every “typical” realization \( \tilde{\omega} \in \Omega \) there exist \( b^\varepsilon, c^\varepsilon, c \in L^2(0, T; H^1(G)) \) and \( B_1^\varepsilon, C^1 \in L^2(G_T; L^2_{pot}(\Omega)) \), such that \( \nabla b^\varepsilon \to \nabla b + B_1^\varepsilon, \nabla c^\varepsilon \to \nabla c + C^1, \) and \( \nabla \bar{c} \to \nabla c + C^1 \) stochastically two-scale. Estimates (22) and (27) and the compactness of the embedding of \( H^1(G) \) in \( L^2(G_T) \), together with the Kolmogorov compactness theorem, see e.g. [16, 53], yield the strong convergence \( b^\varepsilon \to b, \quad c^\varepsilon \to c, \) and \( \bar{c} \to c \) in \( L^2(G_T) \) for \( P \)-a.a. realizations \( \tilde{\omega} \in \Omega \). Since \( G^c_{T, T} \cap \tilde{G}^c_{T, T} \neq \emptyset, \) \( c^\varepsilon(t, x) = \bar{c}(t, x) \) for \( \text{a.a. } (t, x) \in G^c_{T, T} \cap \tilde{G}^c_{T, T} \), and \( c^\varepsilon \) and \( c \) are independent of \( \omega \in \Omega \), we obtain that \( c^\varepsilon(t, x) = \bar{c}(t, x) \) for \( \text{a.a. } (t, x) \in G_T \) and \( \bar{c} \)-a.s. in \( \Omega \).

From the estimates for \( c^\varepsilon = c^\varepsilon \chi_{G_f} + c^\varepsilon \chi_{G_f} \) in (22) we obtain that there exists \( C^1 \in L^2(G_T; L^2_{pot}(\Omega)) \) such that \( \nabla c^\varepsilon \to \nabla c + C^1 \) stochastically two-scale.
6. Derivation of macroscopic equations for flow velocity and elastic deformations.

To show the convergence of boundary terms we shall prove the relation between convergence with respect to $\mathcal{P}$ in $G$ and Palm measure $\mu$ on the oscillating surfaces $\Gamma^e$.

**Definition 6.1.** [23] The Palm measure of the random stationary measure $\mu_{\omega}$ is the measure $\mu$ on $(\Omega, \mathcal{F})$ defined as

$$\mu(F) = \int_\Omega \int_{\mathbb{R}^3} \chi_{[0,1]}(x) \chi_F(T_\omega \omega) d\mu_{\omega}(x) dP(\omega) \quad \text{for } F \in \mathcal{F}.$$  

**Lemma 6.2.** For $u \in H^1(\Omega, \mathcal{P})$ we have that $u \in L^2(\Omega, \mu)$, where $\mu$ is the Palm measure of the random stationary measure $\mu_{\omega}$ of surfaces $\Gamma(\omega)$ for realizations $\omega \in \Omega$, and the embedding is continuous.

**Proof.** Consider $u \in H^1(\Omega, \mathcal{P})$ and a random stationary measure $\mu_{\omega}$ given by $d\mu_{\omega}(x) = 1_{\Gamma(\omega)} d\sigma(x)$, where $d\sigma(x)$ is the standard surface measure. By $\mu$ we denote the Palm measure of the random stationary measure $\mu_{\omega}$. Let $Q_\rho$ be the ball in $\mathbb{R}^3$ of radius $\rho$ centered at the origin. Since $u \in H^1(\Omega, \mathcal{P})$, then a.s. $u(T(\omega)) \in H^1_{\text{loc}}(\mathbb{R}^3)$. Under our assumptions by the trace theorem there exist $\delta > 0$ and $C > 0$ such that

$$\int_{\Gamma(\omega) \cap Q_\rho} |u(T(\omega))|^2 d\sigma(x) \leq C \int_{Q_{\rho + \delta}} |u(T(\omega))|^2 dx + C \int_{Q_{\rho + \delta}} |\nabla u(T(\omega))|^2 dx \quad (38)$$

$\mathcal{P}$-a.s. in $\Omega$. We divide the left- and the right-hand sides of this relation by $\rho^3$ and pass to the limit, as $\rho \to \infty$. By the Birkhoff theorem we obtain

$$\int_{\Gamma(\omega)} |u(\omega)|^2 d\mu \leq C \left[ \int_{\Omega} |u(\omega)|^2 d\mathcal{P} + \int_{\Omega} |\nabla \omega u(\omega)|^2 d\mathcal{P} \right].$$

This yields the desired statement. \qed

**Proof of Theorem 3.1.** To derive macroscopic equations for the system of poro-elastic and Stokes equations, first we consider as test functions in (5) the following functions

- $\phi(t, x) = \varepsilon \phi_1(t, x) \phi_2(T(\omega) \omega)$, $\phi_1 \in C^1(G_T)$, $\phi_2 \in C^1(\Omega)^3$
- $\psi(t, x) = \varepsilon \psi_1(t, x) \psi_2(T(\omega) \omega)$, $\psi_1 \in C^1(G_T)$, $\psi_2 \in C^1(\Omega)^3$, $\eta_1 \in C^0(G_T)$, $\eta_2 \in C^0(\Omega)^3$, and $\phi_1(t, x) \Pi \phi_2(T_\omega \omega) = \eta_1(t, x) \Pi \eta_2(T_\omega \omega)$ for $(t, x) \in G_T$, $\tilde{x} \in \Gamma(\omega)$, and $\mathcal{P}$-a.a. realizations $\omega \in \Omega$. To apply stochastic two-scale convergence of $u_\varepsilon$, $p_\varepsilon$, and $\partial_t u_\varepsilon$, we rewrite the boundary integrals over $\Gamma^e$ in the weak formulation (5) as volume integrals

$$\langle \rho \partial^2_{tt} u_\varepsilon, \phi \chi_{G_T} \rangle_{G_T} + \langle E(\varepsilon^2 b_j) \epsilon(\varepsilon) \chi_{G_T} \rangle_{G_T} + \langle \eta \chi_{G_T} \rangle_{G_T} + \langle \rho \partial_t p_\varepsilon, \psi \chi_{G_T} \rangle_{G_T}$$

$$\quad + \langle K_p \nabla p_\varepsilon - \partial_t u_\varepsilon, \nabla \psi \chi_{G_T} \rangle_{G_T} - \langle \partial_t u_\varepsilon, \nabla \psi \chi_{G_T} \rangle_{G_T} + \langle \nabla p_\varepsilon, \eta \chi_{G_T} \rangle_{G_T} + \langle p_\varepsilon, \delta u_\varepsilon \rangle_{G_T} - \langle \mu \varepsilon^2 (\epsilon(\varepsilon) u_\varepsilon), \epsilon(\varepsilon) \chi_{G_T} \rangle_{G_T} - \langle \eta, \delta u_\varepsilon \rangle_{G_T}$$

$$= \langle F_g, \phi \rangle_{\partial G_T} + \langle F_p, \psi \rangle_{\partial G_T}. \quad (39)$$

Here we have used the relation $\delta u_\varepsilon = 0$ in $G_T$ and the fact that $\chi_{G_T}(x, \omega) = \chi_{\Omega_j(T_j \omega)}$ $\mathcal{P}$-a.s. in $\Omega$, where $j = e, f$. Using the convergence results in lemma 5.2 and passing
to the limit $\varepsilon \to 0$ we obtain
\[
\langle \bar{E}(\omega, b_{\varepsilon, \lambda}) \rangle \psi(u_{\varepsilon}) + U_{\text{sym}}^1 \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega + \langle \bar{K}_P(\omega) \rangle \psi(u_{\varepsilon}) + P_{\varepsilon}^1 \rangle \psi(u_{\varepsilon}) + P_{\varepsilon}^1 \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \Omega + \langle \bar{P}_F(\omega) \rangle \psi(u_{\varepsilon}) + P_{\varepsilon}^1 \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \Omega = 0.
\]
Letting first $\psi_1 \equiv 0$ and $\eta_1 \equiv 0$ and then $\phi_1 \equiv 0$ and $\eta_1 \equiv 0$ we obtain the equations for the correctors $U_{\text{sym}}^1$ and $P_{\varepsilon}^1$, i.e.
\[
\langle \bar{E}(\omega, b_{\varepsilon, \lambda}) \rangle \psi(u_{\varepsilon}) + U_{\text{sym}}^1 \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega = 0,
\]
and
\[
\langle \bar{K}_P(\omega) \rangle \psi(u_{\varepsilon}) + P_{\varepsilon}^1 \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega = 0.
\]
From (40) considering $\phi_1 \equiv 0$ and $\psi_1 \equiv 0$ also yields
\[
p_f \chi \chi_f = p_f \chi \chi_f \quad \text{in} \quad G_T \times \Omega.
\]
Next, choosing in (5) test functions of the form \((\phi(t, x), \psi(t, x), \eta(t, x, x/\varepsilon))\), where
\begin{itemize}
  \item $\phi \in C^\infty(G_T)^1$ and $\psi \in C^\infty(G_T)$,
  \item $\eta(t, x, x/\varepsilon) = \eta_1(t, x) \eta_2(T_{\varepsilon}(t))$, where $\eta_1 \in C^\infty(G_T)$, $\eta_2 \in C^\infty(\Omega)^1$, with $\text{div}_{\varepsilon} \eta_2 = 0$ for $\mathcal{P}$-a.a. $\omega \in \Omega$, and $\eta_1(t, x) \Pi_{\varepsilon} \eta_2(T_{\varepsilon}(t)) = \Pi \phi(t, x)$ for $(t, x) \in G_T$, $\varepsilon \in \Gamma(\omega)$, and $\mathcal{P}$-a.s. in $\Omega$,
\end{itemize}
we obtain
\[
\langle p_\varepsilon \partial_\varepsilon^2 u_{\varepsilon} \rangle \chi \rangle G_{T} \rangle \Omega + \langle \bar{E}(b_{\varepsilon}) \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega + \langle \bar{K}_P(\omega) \rangle \psi(u_{\varepsilon}) + P_{\varepsilon}^1 \rangle \psi(u_{\varepsilon}) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega = 0.
\]
Letting $\varepsilon \to 0$ and using the stochastic two-scale and strong convergences of $u_{\varepsilon}$ and $p_{\varepsilon}$, the strong convergence of $b_{\varepsilon}$, and the stochastic two-scale convergence of $\partial_\varepsilon u_f$ we obtain
\[
\langle p_f \partial_\varepsilon^2 u_f \rangle \chi \rangle G_{T} \rangle \Omega + \langle \bar{E}(b_f, b_{\varepsilon}) \rangle \psi(u_f) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega + \langle \bar{K}_P(\omega) \rangle \psi(u_f) \rangle G_{T} \rangle \chi \rangle G_{T} \rangle \Omega = 0.
\]
Here we used the fact that $\chi \rangle G_{T} \rangle p_f = \chi \rangle G_{T} \rangle p_f \chi \rangle G_{T} \rangle \Omega$. Since $P_{\varepsilon}^1 \in L^2(G_T; L^2_{\text{sym}}(\Omega))$ and $\eta_1 \in C(G_T)$, $\eta_2 \in L^2_{\text{sym}}(\Omega)$ we obtain that
\[
\langle P_{\varepsilon}^1 \rangle \eta_1 \rangle G_{T} \rangle \Omega = 0.
\]
The stochastic two-scale convergence of $\partial_t u^\varepsilon_j$ and the fact that $\partial_t u^\varepsilon_j$ is divergence-free in $G_T$ (we identify here $\partial_t u^\varepsilon_j$ with its extension constructed in lemma 4.4) imply

$$0 = \lim_{\varepsilon \to 0} (\text{div } \partial_t u^\varepsilon_j, \varepsilon_1(t, x, \cdot \varepsilon))_{G_T} = \lim_{\varepsilon \to 0} (\partial_t u^\varepsilon_j, \varepsilon \nabla \sigma + \nabla \cdot \eta)_{G_T} = -\langle \partial_t u^\varepsilon_j, \nabla \cdot \eta \rangle_{G_T \times \Omega} = (\text{div } \partial_t u^\varepsilon_j, \eta)_{G_T \times \Omega}.$$

Thus $\text{div}_\omega \partial_t u^\varepsilon_j = 0$ a.e. in $G_T$ and $\mathcal{P}$-a.s. in $\Omega$.

Choosing $\phi \equiv 0$ and $\psi \equiv 0$, and taking $\eta = \eta_1 \eta_2$, where $\eta_1 \in C^1_0(G_T)$ and $\eta_2 \in C^1(\Omega)^3$, with $\text{div}_\omega \eta_2 = 0$ and $\Pi_1 \eta_2(T, \omega) = 0$ on $\Gamma(\omega)$ $\mathcal{P}$-a.s. in $\Omega$, we conclude that $\partial_t u^\varepsilon_j$ is a solution to problem (14). Taking $\eta = \eta_1 \eta_2$ with $\eta_2 = \text{const}$ and $\eta_1 \in C^1(0, T; C^1(\Omega)^3)$ as a test function in (14) yields

$$\langle \rho_1 t, \eta_1 \chi_\Omega, \rangle_{G_T, \Omega} = (\rho_1 \partial_t^2 u^\varepsilon_j + \nabla p_e, \eta_1 \chi_\Omega)_{G_T, \Omega}.$$  (45)

Next we have to determine the boundary conditions for tangential components of $\partial_t u^\varepsilon_j$ on $\Gamma(\omega)$ for $\mathcal{P}$-a.a. $\omega \in \Omega$. From a priori estimates for $\partial_t u^\varepsilon_j$ and $\partial_t u^\varepsilon_j$ we have that

$$\epsilon \|\partial_t u^\varepsilon_j\|_{L^2(\Gamma_T)}^2 \leq C_1 \left( \|\partial_t u^\varepsilon_j\|_{L^2(G_T)}^2 + \epsilon^2 \|\nabla \partial_t u^\varepsilon_j\|_{L^2(G_T)}^2 \right) \leq C_2,$$

$$\epsilon \|\partial_t u^\varepsilon_j\|_{L^2(\Gamma_T)}^2 \leq C_3 \left( \|\partial_t u^\varepsilon_j\|_{L^2(G_T)}^2 + \epsilon^2 \|\nabla \partial_t u^\varepsilon_j\|_{L^2(G_T)}^2 \right) \leq C_4,$$

where the constants $C_j$, with $j = 1, 2, 3, 4$, are independent of $\epsilon$. Thus using lemmas 6.2 and 8.1 and the fact that $\partial_t u^\varepsilon_j \in L^2(G_T; H^1(\Omega))$ and $\partial_t u^\varepsilon_j \in L^2(0, T; H^1(\Omega))$ we obtain

$$\int_{G_T} \int_{\Omega_T} \Pi_1 \partial_t u^\varepsilon_j(t, x, \omega) \psi_1(t, x) \psi_2(\omega) d\mu dt = \lim_{\epsilon \to 0} \int_{\Omega_T} \Pi_1 \partial_t u^\varepsilon_j(t, x) \psi_1(t, x) \psi_2(T, \omega) d\sigma d\tau dt \leq C_3 \epsilon \int_{\Omega_T} \Pi_1 \partial_t u^\varepsilon_j(t, x) \psi_1(t, x) \psi_2(\omega) d\mu d\tau dt$$

for $\psi_1 \in C^1_0(G_T)$, $\psi_2 \in C^1(\Omega)$ and typical realizations $\tilde{\omega} \in \Omega$. Thus for each typical realization $\tilde{\omega} \in \Omega$ we have

$$\Pi_1 \partial_t u^\varepsilon_j = \Pi_1 \partial_t u^\varepsilon_j \text{ on } G_T \times \Gamma(\tilde{\omega}).$$

Considering first $\phi \in C^\infty_0(G_T)^3$, $\psi \in C^\infty_0(G_T)$, and then $\phi \in C^\infty(G_T)^3$, $\psi \in C^\infty(G_T)$, and using equality (45) together with

$$U^\varepsilon = \sum_{k,l=1}^3 \epsilon(u_k(t, x), \eta_\kappa) W^k_{\varepsilon t}(t, x, \omega),$$  (46)

where $W^k_{\varepsilon t}$ are solutions of the first equations in (9), yield the macroscopic equations for $u_\varepsilon$:

$$\partial_\varepsilon \rho_\varepsilon \partial_t^2 u_\varepsilon - \text{div} (E_{\text{hom}}(b_{\varepsilon 3}) \mathbf{e}(u_\varepsilon)) + \nabla p_e + \int_{\Omega_T} \rho_\varepsilon \partial_t^2 u_\varepsilon \chi_\Omega d\mathcal{P}(\omega) = 0 \quad \text{in } G_T,$$

$$E_{\text{hom}}(b_{\varepsilon 3}) \mathbf{e}(u_\varepsilon) n = F_u \quad \text{on } (\partial G_T),$$  (47)

where $E_{\text{hom}}$ is defined by (8), as well as the equation
\[ \partial_t p^j \partial_t p^j - \text{div} \left( \int_{\Omega} \left[ \left( \bar{K}_p(\omega)(\nabla p^j + P^j) - \partial_t u^j \right) \chi_{\Omega} - \partial_t u^j \chi_{\Omega} \right] \text{d} \mathcal{P}(\omega) \right) = 0 \quad \text{in } G_T, \]

\[ \left( \int_{\Omega} \left[ \left( \bar{K}_p(\omega)(\nabla p^j + P^j) - \partial_t u^j \right) \chi_{\Omega} - \partial_t u^j \chi_{\Omega} \right] \text{d} \mathcal{P}(\omega) \right) \cdot n = F_p \quad \text{on } (\partial G)_T, \]

(48)

together with problem (42) for \( P^j \). The structure of the problem (42) suggests that \( P^j \) should be of the form

\[ P^j(t, x, \omega) = \sum_{k=1}^{3} \frac{\partial p^j}{\partial x_k}(t, x) W^j_k(\omega) + \sum_{k=1}^{3} \partial_t u^j_k(t, x) W^j_k(\omega) + Q_j(\omega, \partial_t u^j), \]

(49)

where \( W^j_k \) and \( W^j_k \) are solutions of cell problems (9), and \( Q_j \) is a solution of problem (11). Substituting the right-hand side of (49) for \( P^j \) in (48) we obtain the macroscopic equations for \( p^j \) in (12), where \( K_p^{\text{hom}} \) and \( K_u \) are defined in (8).

7. Strong stochastic two-scale convergence of \( \mathbf{e}(u^j), \nabla p^j, \) and \( \partial_t u^j \).

Due to the presence of nonlinear functions depending on \( \mathbf{e}(u^j) \) and \( \partial_t u^j \) in equations for \( b^j, c^j, \) and \( e^j, \) in order to derive the macroscopic equations for \( b, c \) and \( e \) we have to show that \( \mathbf{e}(u^j) \) and \( \partial_t u^j \) converge stochastically two-scale strongly.

**Lemma 7.1.** For a subsequences of \( \{u^j\}, \{p^j\} \) and \( \{\partial_t u^j\} \) as in lemma 5.2 (denoted again by \( \{u^j\}, \{p^j\}, \) and \( \{\partial_t u^j\} \)) we have

\[ \chi_{G^j} \mathbf{e}(u^j) \to \chi_{\Omega} (\mathbf{e}(u) + U_{e, \text{sym}}) \quad \text{strongly stochastic two-scale,} \]

\[ \chi_{G^j} \nabla p^j \to \chi_{\Omega} (\nabla p + P^j) \quad \text{strongly stochastic two-scale,} \]

\[ \chi_{G^j} \partial_t u^j \to \chi_{\Omega} \partial_t u \quad \text{strongly stochastic two-scale.} \]

**Proof.** Similar to the periodic case [57], to show the strong stochastic two-scale convergence of \( \mathbf{e}(u^j), p^j, \) and \( \partial_t u^j \) we prove the convergence of the energy related to the equations for \( u^j, \)

\[ E^j(u^j, p^j, \partial_t u^j) = \frac{1}{2} \rho_p \| \partial_t u^j(\cdot)\|_G^2 - \rho_p \langle g(\cdot) \partial_t u^j, \partial_t u^j \rangle_G, \]

\[ + \frac{1}{2} \langle \mathbf{E}(b_{e,3}^j)\mathbf{e}(u^j)(\cdot), \mathbf{e}(u^j)(\cdot) \rangle_G, \]

\[ - \frac{1}{2} \langle (2g(\cdot) \mathbf{e}(\cdot))\mathbf{E}(b_{e,3}^j) + g^2 \partial_t \mathbf{E}(b_{e,3}^j) \rangle \mathbf{e}(u^j), \mathbf{e}(u^j) \rangle_G, \]

\[ + \frac{1}{2} \rho_p \| p^j(\cdot) \|_G^2 - \rho_p \langle g(\cdot) p^j(\cdot), p^j(\cdot) \rangle_G, \]

\[ + \frac{1}{2} \rho_f \| \partial_t u^j(\cdot) \|_G^2 - \rho_f \langle g(\cdot) \partial_t u^j(\cdot), \partial_t u^j(\cdot) \rangle_G, \]

\[ + \mu \| \mathbf{e}(\cdot) \mathbf{e}(\partial_t u^j(\cdot)) \|_G^2 \]

(51)
for \( s \in (0, T) \) and \( P \)-a.a. \( \omega \in \Omega \). Considering \( \partial u_\varepsilon^s g^2, p_\varepsilon^s g^2 \), and \( \partial u_\varepsilon^s g^2 \) as test functions in (5) yields the equality

\[
\mathcal{E}^\varepsilon(u_\varepsilon, p_\varepsilon, \partial u_\varepsilon) = \frac{1}{2} \rho_\varepsilon \| \partial u_\varepsilon(0) \|_{L^2(\mathcal{G})}^2 + \frac{1}{2} \langle \mathbf{E}'(b_\varepsilon^s) \mathbf{e}(u_\varepsilon(0)), \mathbf{e}(u_\varepsilon(0)) \rangle_{\mathcal{G}} + \frac{1}{2} \rho_\gamma \| \partial u_\gamma(0) \|_{L^2(\mathcal{G})}^2 + \frac{1}{2} \langle F_u, \partial u_\varepsilon^s g^2 \rangle_{\partial \mathcal{G}\Omega} + \langle F_p, p_\varepsilon^s g^2 \rangle_{\partial \mathcal{G}\Omega}.
\]

Due to assumptions on \( \mathbf{E} \) and \( \partial \mathbf{E} \) there exists such \( \gamma > 0 \) that

\[
\left( 2 \gamma \mathbf{E}_1(\omega, K(\eta)) - \partial \mathbf{E}_1(\omega, K(\eta)) \right) A \cdot A \geq 0 \quad \text{for all symmetric matrices } A \text{ and } \eta \in \mathbb{R},
\]

and \( P \) – a.a. \( \omega \in \Omega \).

The weak stochastic two-scale convergence of \( (\mathbf{E}'(b_\varepsilon^s))^1/2 \mathbf{e}(u_\varepsilon), (\gamma \mathbf{E}'(b_\varepsilon^s) - \partial \mathbf{E}') (b_\varepsilon^s)^1/2 \mathbf{e}(u_\varepsilon), \) and \( (K_\rho^s)^1/2 \nabla p_\varepsilon, \) as \( \varepsilon \to 0 \), and the lower-semicontinuity of the norm ensure

\[
\frac{\rho_\varepsilon}{2} \| \partial u_\varepsilon e(s)g(s) \chi_\Omega \|_{L^2(G \times \Omega)}^2 + \gamma \rho_\varepsilon \| \partial u_\varepsilon g \chi_\Omega \|_{L^2(G \times \Omega)}^2 + \rho_\gamma \| \partial u_\gamma g \chi_\Omega \|_{L^2(G \times \Omega)}^2 + \langle F_u, \partial u_\varepsilon^s g^2 \rangle_{\partial \mathcal{G}\Omega} + \langle F_p, p_\varepsilon^s g^2 \rangle_{\partial \mathcal{G}\Omega}.
\]

Here we also used the strong convergence of \( b_\varepsilon^s \) and the stochastic two-scale convergence of \( \nabla p_\varepsilon, \mathbf{e}(u_\varepsilon), \partial u_\varepsilon, \) and \( \partial \mathbf{e}(u_\varepsilon) \). Considering the limit equations for \( u_\varepsilon, U_1^e, p_\varepsilon, P_1^e, \) and \( \partial \mathbf{u}_\varepsilon \) and taking \( (\partial u_\varepsilon^s g^2, p_\varepsilon^s g^2, \partial \mathbf{u}_\varepsilon g^2) \) as a test function yield
Thus we obtain that

\[
\frac{\rho_c}{2} \| \partial_t u_e(s) \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 - \frac{\rho_p}{2} \| \partial_t u_e(0) \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 + \gamma_p \rho_c \| \partial_t u_e \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 \\
+ \left( \mathbf{E}(\omega, b_{c,3})(e(u_e) + U_{e,\text{sym}}^1) \right) \cdot (\partial_t u_e) \varphi(s) \chi_{\Omega_e} + (\nabla \rho_e + P^1_{e, \partial u_e \varphi(s) \chi_{\Omega_e}}) \\
+ \frac{\rho_p}{2} \| p_e(s) \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 + \frac{\rho_p}{2} \| p_e(0) \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 + \gamma \rho_p \| p_e \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 \\
+ \left( \mathbf{K}_{p}(\omega)(\nabla \rho_e + P^1_{e}) - \partial u_e \right) \varphi(s) \chi_{\Omega_e} + (\nabla \rho_e, \partial u_e \varphi(s) \chi_{\Omega_e}) \\
+ \frac{\rho_f}{2} \| \partial_t u_e(s) \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 + \frac{\rho_f}{2} \| \partial_t u_e(0) \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 + \gamma \rho_f \| \partial_t u_e \varphi(s) \chi_{\Omega_e} \|_{L^2(G \times \Omega)}^2 \\
+ \mu \left( e_{c}(\partial_t u_f), e_{c}(\partial_t u_f) \varphi(s) \chi_{\Omega_e} \right) + (\nabla \rho_f, \partial_t u_f \varphi(s) \chi_{\Omega_e}) - (P^1_{e} \varphi(s) \chi_{\Omega_e}, \partial_t u_f \varphi(s) \chi_{\Omega_e}) \\
= (F_{ue}, \partial u_e \varphi(s) \chi_{\Omega_e})_{\partial \Omega_{\Omega_e}} + (F_{pe}, \varphi(s) \chi_{\Omega_e})_{\partial \Omega_{\Omega_e}} \\
\tag{54}
\]

for \( s \in (0, T) \). Taking \( P^1_e \) as a test function in the equation for \( P^1_e \) yields

\[
\langle P^1_e, \partial_t u_e \varphi(s) \chi_{\Omega_e} \rangle_{G, \Omega_e} = (\mathbf{K}_{p}(\omega)(\nabla \rho_e + P^1_{e}) - \partial u_e, P^1_{e} \varphi(s) \chi_{\Omega_e} \rangle_{G, \Omega_e}. \\
\tag{55}
\]

Since \( P^1_e \in L^2(GT; L^2_{\text{rad}}(\Omega)) \) and \( \partial u_f \in L^2(GT; L^2_{\text{rad}}(\Omega)) \) we obtain

\[
\langle P^1_e, \partial_t u_e \varphi(s) \chi_{\Omega_e} \rangle_{G, \Omega_e} = 0 \quad \text{and} \quad \langle P^1_e, \partial_t u_f \varphi(s) \chi_{\Omega_e} \rangle_{G, \Omega_e} = - \langle P^1_e, \partial_t u_f \varphi(s) \chi_{\Omega_e} \rangle_{G, \Omega_e}. \\
\]

Considering equation (41) for the corrector \( U^1_e \) and taking \( \partial_t U^1_e \varphi(s) \) as a test function imply

\[
\left( \mathbf{E}(\omega, b_{c,3})(e(u_e) + U_{e,\text{sym}}^1), e(\partial_t u_e) \varphi(s) \chi_{\Omega_e} \right)_{G, \Omega_e} \\
\quad - \frac{1}{2} \left( \mathbf{E}(\omega, b_{c,3})(e(u_e(s)) + U_{e,\text{sym}}^1(s)) \varphi(s) \chi_{\Omega_e}, e(u_e(s)) + U_{e,\text{sym}}^1(s) \right)_{G, \Omega_e} \\
\quad - \frac{1}{2} \left( \mathbf{E}(\omega, b_{c,3})(e(u_e(0)) + U_{e,\text{sym}}^1(0)) \chi_{\Omega_e}, e(u_e(0)) + U_{e,\text{sym}}^1 \right)_{G, \Omega_e} \\
\quad + \frac{1}{2} \left( 2 \gamma \mathbf{E}(\omega, b_{c,3}) - \partial_t \mathbf{E}(\omega, b_{c,3}) \right) \varphi(s) (e(u_e) + U_{e,\text{sym}}^1) \chi_{\Omega_e}, e(u_e) + U_{e,\text{sym}}^1 \right)_{G, \Omega_e}. \\
\tag{56}
\]

Thus we obtain that

\[
\mathcal{E}(u_e, p_e, \partial u_f) \leq \liminf_{\varepsilon \to 0} \mathcal{E}(u^\varepsilon, p^\varepsilon, \partial u^\varepsilon_f) \leq \limsup_{\varepsilon \to 0} \mathcal{E}(u^\varepsilon, p^\varepsilon, \partial u^\varepsilon_f) = \mathcal{E}(u_e, p_e, \partial u_f),
\]

and, hence the strong stochastic two-scale convergence stated in lemma. \( \square \)

8. Derivation of macroscopic equations for \( b_{c} \) and \( c_{c} \).

Using strong stochastic two-scale convergence of \( e(u^\varepsilon) \) and \( \partial u^\varepsilon_f \) we derive macroscopic equations for concentrations of pectins \( b_{c} \) and calcium \( c_{c} \). First we shall prove convergence of sequences defined on the boundaries of the random microstructure \( \Gamma^\varepsilon \).
Lemma 8.1. Consider the random measure \( \mu_\omega \) denoting the surface measure of \( \Gamma(\omega) \) and define \( d\mu_\omega(x) = \varepsilon^d d\mu_\omega(x/\varepsilon) \).

(a) If \( \|b^\varepsilon\|_{L^p(G_T')} + \|\nabla b^\varepsilon\|_{L^p(G_T')} \leq C \) and \( b^\varepsilon \to b \) stochastic two-scale, \( b \in L^p(0, T; W^{1,p}(G)) \), with \( p \in (1, \infty) \), then for any \( \phi, \psi \in C_0^\infty(0, T; C_0^\infty(\mathbb{R}^3)) \) and any \( \psi \in C(\Omega) \) we have

\[
\lim_{\varepsilon \to 0} \int_{G_T'} b^\varepsilon(t, x) \phi(t, x) \psi(T_{\varepsilon, T}(\omega))d\mu_\omega(x)dt = \int_{G_T} b(t, x) \phi(t, x) \psi(x)d\mu(x)dxdt
\]

and

\[
\int_{G_T} \int |b|^p d\mu(x) dx dt \leq C \int_{G_T} \int |b|^p d\mu(x) dx dt.
\]

(b) If \( \|b^\varepsilon\|_{L^p(G_T')} + \varepsilon \|\nabla b^\varepsilon\|_{L^p(G_T')} \leq C \) and \( b^\varepsilon \to b \) stochastic two-scale, \( b \in L^p(G_T; W^{1,p}(\Omega, d\mathcal{P})) \), with \( p \in (1, \infty) \), then convergence (57) holds, and

\[
\int_{G_T} \int |b|^p d\mu(x) dx dt \leq C.
\]

Proof. For \( \mathcal{P} \)-a.a. realizations \( \omega \in \Omega \), using the assumptions on the geometry of \( G_\varepsilon' \) and the trace inequality in each \( G_\varepsilon' = G_{T'}^{G_\varepsilon}(\omega) \setminus G_{T'}^{G_\varepsilon}(\omega) \), see proof of lemma 4.4 for the definition of \( G_{T'}^{G_\varepsilon}(\omega) \), applying the scaling \( x/\varepsilon \) and summing up over \( j \) we obtain

\[
\int_{G_T} |\nabla b^\varepsilon|^p d\mu_\omega(x) dx dt = \varepsilon \int_{G_{T'}^G} |\nabla b|^p d\sigma dx dt \leq C_1 \int_{G_{T'}^G} |\nabla b|^p dx dt + C_2 \varepsilon^p \int_{G_{T'}^G} |\nabla b|^p dx dt \leq C.
\]

Moreover, in the case (a) the limit function \( b \) does not depend on \( \omega \); its trace on \( \Gamma_{\varepsilon, T}^\varepsilon \) is well defined, and

\[
\varepsilon^p \int_{G_{T'}^G} |\nabla b^\varepsilon - \nabla b|^p dx dt \xrightarrow{\varepsilon \to 0} 0.
\]

Choosing \( \phi(x, t) = \phi_1(t)\phi_2(x) \) we conclude that \( \breve{b}^\varepsilon(x) = \int_0^T b^\varepsilon(x, t) \phi_1(t) dt \) converges in \( L^p(G) \) strongly to \( \breve{b}(x) = \int_0^T b(x, t) \phi_1(t) dt \), and

\[
\int_{G} |\breve{b}^\varepsilon - \breve{b}|^p d\mu_\omega(x) \leq C_1 \int_{G_{T'}^G} |\breve{b}^\varepsilon - \breve{b}|^p dx + C_2 \varepsilon^p \int_{G_{T'}^G} |\nabla \breve{b}^\varepsilon - \nabla \breve{b}|^p dx \xrightarrow{\varepsilon \to 0} 0.
\]

This yields (57).

In the case (b), for \( b \in L^p(G_T; W^{1,p}(\Omega, d\mathcal{P})) \) using the same arguments as in the proof of lemma 6.2 one can show that \( b \in L^p(G_T; L^p(\Omega, \mu)) \). This yields (59).

To justify (57) we regularize measures \( \mu_\omega \) as follows. Let \( k = k(x) \) be a non-negative symmetric \( C_0^\infty(\mathbb{R}^d) \) function such that \( \int_{\mathbb{R}^d} k(x) dx = 1 \), where here \( d = 3 \). We set

\[
d\mu_{\omega, \delta}(x) = \rho^\delta(T_{\varepsilon, T}\omega) dx \quad \text{with} \quad \rho^\delta(\omega) = \delta^{-d} \int_{\mathbb{R}^d} \left( \frac{\chi}{\delta} \right) d\mu_\omega(y).
\]
It is easy to check that a.s. for any test functions \( \phi \in C^\infty(0, T; C_0^\infty(\mathbb{R}^d)) \) and \( \psi \in C(\Omega) \) we have

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{T} \left| \int_{G_T} b'(t, x) \phi(t, x)\psi(T_{\varepsilon}x) d\mu_{\varepsilon, \delta}(x) dt - \int_{G_T} b'(t, x) \phi(t, x)\psi(T_{\varepsilon}x) d\mu_{\varepsilon, 0}(x) dt \right| = 0.
\]

The Palm measure of \( d\mu_{\varepsilon, \delta}(x) = d\mu_{\varepsilon}(\omega) = \rho^\varepsilon(\omega) d\mathcal{P} \). Since for each \( \delta > 0 \) the measure \( \mu_{\varepsilon} \) is absolutely continuous with respect to \( d\mathcal{P} \) and the density \( \rho^\varepsilon \) is bounded, the two-scale limit of \( b' \) with respect to \( d\mu_{\varepsilon, \delta} \) is \( b \) that is

\[
\lim_{\varepsilon \to 0} \int_{G_T} b'(t, x) \phi(t, x)\psi(T_{\varepsilon}x) d\mu_{\varepsilon, \delta}(x) dt = \int_{G_T} b(t, x, \omega)\phi(t, x)\psi(\omega) d\mu_{\varepsilon}(\omega) dx dt.
\]

By the trace theorem a.s.

\[
\lim_{\varepsilon \to 0} \sup_{T} ||b'||_{L^p(G_T, d\mu_{\varepsilon})} \leq C.
\]

Therefore, for a subsequence \( b' \) stochastically two-scale converge in \( L^p(G_T, d\mu_{\varepsilon}) \) to some function \( B \in L^p(G_T; L^p(\Omega, d\mu)) \). As was proved in [74], the measures \( d\mu_{\varepsilon} \) converge weakly to the measure \( d\mu \). Using one more time the same arguments as in the proof of lemma 6.2 we obtain

\[
\lim_{\delta \to 0} \int_{G_T} b(x, t)\phi(t, x)\psi(\omega) d\mu_{\varepsilon, \delta}(\omega) dx dt = \int_{G_T} b(x, t)\phi(t, x)\psi(\omega) d\mu(\omega) dx dt.
\]

Passing to the limit \( \delta \to 0 \) and combining the above relations, we conclude that

\[
\int_{G_T} b(x, t)\phi(t, x)\psi(\omega) d\mu(\omega) dx dt = \int_{G_T} B(x, t)\phi(t, x)\psi(\omega) d\mu(\omega) dx dt.
\]

In view of arbitrariness of \( \phi \) and \( \psi \) this implies that \( B = b \) in \( L^p(G_T; L^p(\Omega, d\mu)) \).

Proof of Theorem 3.2. We can rewrite the microscopic equation for \( b' \) as

\[
\begin{align*}
- \langle b'_\varepsilon \chi_{G_T}, \partial_t \phi_1 \rangle_{G_T} + \langle D_b(b'_\varepsilon) \nabla \phi_1 \chi_{G_T} \rangle_{G_T} + \langle b_0 \chi_{\Gamma_T}, \phi_1(0) \rangle_{G_T} \\
+ \langle g_b(c'_\varepsilon, \varepsilon \chi_{G_T}), \phi_1 \chi_{G_T} \rangle_{G_T} + \varepsilon \langle [R(b'_\varepsilon), \phi_1]_{\Gamma_T} + [F_b(b'_\varepsilon), \phi_1]_{\partial \Gamma_T} \rangle
\end{align*}
\]

for \( \phi_1 = \phi_1(t, x) + \varepsilon \phi_2(t, x)\phi_3(T_{\varepsilon}x) \), where \( \phi_1 \in C^\infty(\Gamma_T) \), with \( \phi_1(T, x) = 0 \) for \( x \in \overline{G} \), \( \phi_2 \in C_0^\infty(\Gamma_T) \), and \( \phi_3 \in C_1^T(\Omega) \).

From the \textit{a priori} estimates for \( b' \) and assumptions on \( R \) we have that

\[
\varepsilon \int_0^T \int_{\Gamma_T} |R(b'_\varepsilon)|^2 d\sigma dt \leq C,
\]

where the constant \( C \) is independent of \( \varepsilon \). Thus considering the stochastic two-scale convergence we obtain that there exists \( \tilde{R} \in L^2(G_T \times \Omega, dx \times d\mu(\omega)) \)

\[
\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_T} R(b'_\varepsilon) \phi_1 d\sigma dt = \lim_{\varepsilon \to 0} \int_{G} R(b'_\varepsilon) \phi_1 d\mu_{\varepsilon, \delta}(x) dt = \int_0^T \int_{G_T} \tilde{R} \phi_1 d\mu(\omega) dx dt,
\]

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where \(\mu_\omega\) is the random measure of \(\Gamma(\omega)\). Using the assumptions on the geometry and on the function \(R\) together with the strong convergence of \(b_\varepsilon^r\) in \(L^2(G_T)\) we have that
\[
\varepsilon \int_0^T \int_{G_T} |R(b_\varepsilon^r) - R(b_\varepsilon)|^2 \, d\sigma^r \, dt \leq C \int_{G_T} \left[ |b_\varepsilon^r - b_\varepsilon|^2 + \varepsilon^2 |\nabla(b_\varepsilon^r - b_\varepsilon)|^2 \right] \, dx \, dt \to 0 \text{ as } \varepsilon \to 0.
\]

Then using the strong convergence of \(b_\varepsilon\), the continuity of \(R\) and the convergence result in lemma 8.1 we obtain that \(R = R(b_\varepsilon)\) \(\mathcal{P}\text{-a.s.}\) in \(G_T \times \Omega\).

Taking the stochastic two-scale limit and using the strong convergence of \(b_\varepsilon^r\) and \(c_\varepsilon^r\) and the strong stochastic two-scale convergence of \(e(u_\varepsilon^r)\), shown in lemma 7.1, we obtain
\[
- \langle \partial_t b_\varepsilon, \partial_t \phi_1 \rangle_{G_T} + \langle D_\varepsilon(b_\varepsilon) \nabla b_\varepsilon + B_1^\varepsilon \nabla \phi_1 + \phi_2 \nabla \omega \phi_3 \rangle_{G_T \Omega} = \langle \partial_t b_\varepsilon, \phi_1(0) \rangle_{G_T} + \langle g_b(c, b_\varepsilon, e(u_\varepsilon^r)) + U_1^1|_{\text{sym}} \nabla \phi_1 \rangle_{G_T \Omega} + \int_0^T \int_{G_T} R(b_\varepsilon) \phi_1 \, d\mu(\omega) \, dx \, dt + \langle F_b(b_\varepsilon), \phi_1 \rangle_{(0)G_T}.
\]

(63)

To show the convergence of \(g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r))\) we considered an approximation of \(U^1_{\varepsilon, \text{sym}} \in L^2(G_T \times \Omega)\) by \(U_\varepsilon \in C(G_T; \mathcal{C}_T(\Omega))\), such that \(U_\varepsilon \to U^1_{\varepsilon, \text{sym}} \text{ in } L^2(G_T \times \Omega)\) as \(\delta \to 0\). For \(\mathcal{P}\text{-a.s.}\), \(\omega \in \Omega\) we define \(U_\varepsilon(t, x) = U_\varepsilon(t, x, T_{\varepsilon/\omega})\). Using the strong stochastic two-scale convergence of \(U_\varepsilon\) to \(U_3\) we obtain
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \|U_\varepsilon\|_{L^2(G_T)} = \lim_{\delta \to 0} \|U_\varepsilon\|_{L^2(G_T \times \Omega)} = \|U^1_{\varepsilon, \text{sym}}\|_{L^2(G_T \times \Omega)}.
\]

(64)

Then for \(\phi_2 \in C_0([0, T) \times G)\) and \(\phi_3 \in C_T(\Omega)\) we can write
\[
\langle g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r))\rangle_{(t, x) \mathcal{T}_\varepsilon(\omega) \Gamma G_T} = \langle g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r)) - g_b(c, b_\varepsilon, e(u_\varepsilon^r)) + U_\varepsilon^3\rangle_{(t, x) \mathcal{T}_\varepsilon(\omega) \Gamma G_T} + \langle g_b(c, b_\varepsilon, e(u_\varepsilon^r)) + U_\varepsilon^3\rangle_{(t, x) \mathcal{T}_\varepsilon(\omega) \Gamma G_T}.
\]

Assumptions on \(g_b\), together with (64), the strong convergence of \(U_\varepsilon\) to \(U^1_{\varepsilon, \text{sym}}\), and the strong stochastic two-scale convergence of \(e(u_\varepsilon^r)\), imply
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \|g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r)) - g_b(c, b_\varepsilon, e(u_\varepsilon^r))\|_{L^2(G_T \times \Omega)} = \|g_b(c, b_\varepsilon, e(u_\varepsilon^r))\|_{L^2(G_T \times \Omega)}.
\]

and
\[
\|g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r)) - g_b(c, b_\varepsilon, e(u_\varepsilon^r))\|_{L^2(G_T \times \Omega)} \leq C \left( \|c_\varepsilon^r - c\|_{L^1(\Omega)} + \|b_\varepsilon^r - b_\varepsilon\|_{L^1(G_T \times \Omega)} + \|e(u_\varepsilon^r) - e(u_\varepsilon^r)\|_{L^1(G_T \times \Omega)} \right)
\]

\[
\leq C(\varepsilon + C \left( \|e(u_\varepsilon^r)\|_{L^2(\Omega)} + \|e(u_\varepsilon^r)\|_{L^2(G_T \times \Omega)} + 2 \langle e(u_\varepsilon^r), \chi_{G_T} \rangle_{G_T} + \|U_\varepsilon^3\|_{L^2(G_T \times \Omega)} \right) \rightarrow 0
\]

as \(\varepsilon \to 0\) and \(\delta \to 0\). Assumptions on \(g\) and \(a \text{ priori}\) estimates for \(b_\varepsilon^r, c_\varepsilon^r\), and \(e(u_\varepsilon^r)\) ensure that
\[
\|g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r))\|_{L^2(G_T \times \Omega)} \leq C,
\]

where the constant \(C\) is independent of \(\varepsilon\). Thus
\[
g_b(c_\varepsilon^r, b_\varepsilon^r, e(u_\varepsilon^r)) \rightarrow g_b(c, b_\varepsilon, e(u_\varepsilon^r) + U^1_{\varepsilon, \text{sym}}) \text{ stochastically two-scale.}
\]
Considering $\phi_1 = 0$ and using the linearity of the resulted equation we obtain

$$B^i_j(t, x, \omega) = \sum_{j=1}^{3} \partial_j b_i(t, x) w_j^i(\omega)$$

and the unit cell problem (17) for $w_j^i$. Choosing $\phi_2 = 0$ yields macroscopic equations for $b_i$.

Taking $\varphi_2(t, x) = \psi_1(t, x) + \varepsilon \psi_2(t, x) \psi_3(T_{\varepsilon}, \omega)$ with $\psi_1 \in C^\infty(\overline{G_T})$, $\psi_1(T, x) = 0$ for $x \in \overline{\mathcal{G}}$, $\psi_2 \in C^1_0(G_T)$, and $\psi_3 \in C^1_{T, \Gamma}(\Omega)$ as a test function in (7) we obtain

$$- (e \varphi \chi_{G_T} + \partial \varphi)(\varphi \chi_{G_T}) + (Dv \partial c \partial c_e + \partial \varphi \chi_{G_T}) + (g_e, \varphi \varepsilon_e(\varepsilon_e, \varepsilon_e, \varepsilon_e)) = (g_e, \varphi \varepsilon_e(\varepsilon_e, \varepsilon_e, \varepsilon_e))$$

In the same way as for $g_e$, using the strong stochastic two-scale convergence of $e(\varepsilon_e)\chi_{G_T}$ and $\partial \varphi \chi_{G_T}$, the strong convergence of $b^*_e$ and $c^*_e$, and assumptions on $g_e$ and $\mathcal{G}$, we obtain

$$\chi_{G_T} g_e(\varepsilon_e, \varepsilon_e, \varepsilon_e) = \chi_{G_T} g_e(\varepsilon_e, \varepsilon_e, \varepsilon_e) + U_{\varepsilon_{sym}}^1 \quad \text{stochastically two-scale},$$

$$\chi_{G_T} g_e(\varepsilon_e, \varepsilon_e, \varepsilon_e) = \chi_{G_T} g_e(\varepsilon_e, \varepsilon_e, \varepsilon_e) \quad \text{stochastically two-scale}.$$

Thus applying the stochastic two-scale and the strong convergences of $b^*_e$ and $c^*_e$ together with strong stochastic two-scale convergence of $e(\varepsilon_e)\chi_{G_T}$ and $\partial \varphi \chi_{G_T}$, yields

$$- (e, \partial \psi_1)(\varphi \chi_{G_T}) + (Dv \partial c \partial c_e + \partial \varphi)(\varphi \chi_{G_T}) + (g_e, \varphi \varepsilon_e(\varepsilon_e, \varepsilon_e, \varepsilon_e))$$

Considering $\psi_1 = 0$ yields

$$\langle Dv \partial c \partial c_e + \partial \varphi \rangle = 0.$$

From here we obtain that

$$C^i(t, x, \omega) = \sum_{j=1}^{3} \partial_j c(t, x) w_j^i(\omega) + c(t, x) Z(t, x, \omega) \chi_{\Omega_T},$$

where $w^j \in L^2_{pot}(\Omega)$, with $j = 1, 2, 3$, and $Z \in L^\infty(G_T; L^2_{pot}(\Omega))$ are solutions of the cell problems (18) and (19). Then considering $\psi_2 = 0$ and first $\psi_1 \in C^1_0(G_T)$ and then $\psi_1 \in C^3_0(0, T; C^1(\overline{G_T}))$, and using the expression (46) for the corrector $U_1^j$ we obtain the macroscopic equation and the boundary conditions for $c$ in (20). The equations for $b_i$ and $c$ and the fact that $b_i, c \in L^2(0, T; H^1(G))$ imply that $\partial_t b_i, \partial_t c \in L^2(0, T; H^1(G))$. Thus $b_i, c \in C(0, T; L^2(G))$ and using equations (63) and (65) we obtain that $b_i$ and $c$ satisfy initial conditions. □
9. Well-posedness of the macroscopic problem

In the same way as in the case of periodic microstructure [57], using fixed point iteration we show existence of a unique solution of the limit problem.

**Lemma 9.1.** There exists a unique weak solution of the limit problem (12)–(14), (20).

**Proof.** First we show estimates for two iterations \((u^{l-1}_e, \partial_t p^{l-1}_e, \partial_t u^{l-1}_e), (b^{l-1}_e, c^{l-1})\) and \((u^l_e, \partial_t p^l_e, \partial_t u^l_e), (b^l_e, c^l)\) for limit problem (12)–(14), (20).

We begin with the equations for fluid flow velocity \(\partial_t \tilde{u}_e\) and for elastic displacement \(u_e\).

Taking \(\partial_t \tilde{u}_e - \partial_t u_e\) as a test function in the equation for the difference \(\partial_t \tilde{u}_e\) and \(\partial_t u_e\) as a test function in the equations for the difference \(\tilde{u}_e\) we obtain

\[
\rho_e \|\partial_t \tilde{u}_e(s)\|^2_{L^2(G)} + \left\langle \mathbf{E}^{\text{hom}}(b^{l-1}_e) \mathbf{e}(\tilde{u}_e(s)), \mathbf{e}(\tilde{u}_e(s)) \right\rangle_G - \left\langle \partial_t \mathbf{E}^{\text{hom}}(b^{l-1}_e) \mathbf{e}(\tilde{u}_e), \mathbf{e}(\tilde{u}_e) \right\rangle_G + 2\left\langle \mathbf{E}^{\text{hom}}(b^{l-1}_e) - \mathbf{E}^{\text{hom}}(b^{l-1}_e) \mathbf{e}(u_e(s)), \mathbf{e}(\tilde{u}_e(s)) \right\rangle_G - 2\left\langle \partial_t \mathbf{E}^{\text{hom}}(b^{l-1}_e) - \mathbf{E}^{\text{hom}}(b^{l-1}_e) \mathbf{e}(u_e), \mathbf{e}(\tilde{u}_e) \right\rangle_G,
\]

where \(\tilde{u}_e = u^l_e - u^{l-1}_e\), \(\tilde{u}_e = p^l_e - p^{l-1}_e\), \(\tilde{u}_e = u^l_e - u^{l-1}_e\), and \(\tilde{u}_e = p^l_e - p^{l-1}_e\). The equation (12) for \(p^l_e\) and \(p^{l-1}_e\) yields

\[
\rho_p \|\tilde{p}_e(s)\|^2_{L^2(G)} + 2\left\langle \mathbf{K}_p \nabla \tilde{p}_e, \nabla \tilde{p}_e \right\rangle_G = 2\left\langle \mathbf{K}_e \partial_t \tilde{u}_e + Q(\partial_t u^{l-1}_e), \nabla \tilde{p}_e \right\rangle_G + \rho_p \|\tilde{p}_e(0)\|^2_{L^2(G)}.
\]

Due to the assumptions in A1 on \(\mathbf{E}\), the definition of the macroscopic elasticity tensor \(\mathbf{E}^{\text{hom}}\) and the properties of a solution \(W^l_e\), with \(k, l = 1, 2, 3\), of the corresponding cell problems in (9), we have

\[
\|\mathbf{E}^{\text{hom}}(b^l_e) - \mathbf{E}^{\text{hom}}(b^{l-1}_e)\|_{L^\infty(G_e)} + \|\partial_t \mathbf{E}^{\text{hom}}(b^l_e) - \mathbf{E}^{\text{hom}}(b^{l-1}_e)\|_{L^\infty(G_e)} \leq C\|\tilde{b}_e\|_{L^\infty(0,T;L^\infty(G_e))}
\]

for \(s \in (0,T]\), where \(\tilde{b}_e = b^l_e - b^{l-1}_e\). The expression (15) for \(P^{l,j}_e\) and \(P^{l,j-1}_e\) and the estimates for the \(H^1\)-norm of the solutions of the cell problems (9) and (11) yield

\[
\|\tilde{P}^{l,j}_e\|_{L^2(G_e)} \leq C \left( \|\nabla \tilde{p}_e\|_{L^2(G_e)} + \|\partial_t \tilde{u}_e\|_{L^2(G_e)} + \|\partial_0 \tilde{u}_e\|_{L^2(G_e)} \right).
\]

Adding (67) and (68), and applying the Hölder and Gronwall inequalities yield

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for all $s \in (0, T]$ and the constant $C$ does not depend on $s$ and solutions of the macroscopic problem.

In the same way as in the case of periodic microstructure [57] we obtain the following estimates for $\tilde{b}^j_e$ and $\tilde{c}^j_e$:

$$
\| \tilde{b}^j_e \|_{L^\infty(0,s; L^\infty(G))} + \| \tilde{c}^j_e \|_{L^\infty(0,s; L^2(G))} \leq C_1 \left[ \| \mathbf{e}( \tilde{u}^j_{0e} ) \|_{L^1(0,s; L^2(G))} + \| \partial_t \tilde{u}^j_{0f} \chi_{\Omega_f} \|_{L^2(G_s \times \Omega_f)} \right],
$$

(70)

for $s \in (0, T]$ and any $0 < \sigma < 1/9$, the constant $C$ being independent of $s$ and solutions of the problem, and

$$
\| b^j_e \|_{L^\infty(0,T; L^\infty(G))} + \| c^j_e \|_{L^\infty(0,T; L^\infty(G))} + \| b^j_{0e} \|_{L^\infty(0,T; L^\infty(G))} + \| c^j_{0e} \|_{L^\infty(0,T; L^\infty(K))} \leq C.
$$

Then combining (69) and (70) and applying a fixed point argument we obtain existence of a unique solution of the coupled macroscopic problem (12)–(14), (20).

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