A SHARP BOUND
FOR THE STEIN-WAINGER OSCILLATORY INTEGRAL

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Abstract. Let \( P_d \) denote the space of all real polynomials of degree at most \( d \). It is an old result of Stein and Wainger that

\[
\sup_{P \in P_d} \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \leq C_d
\]

for some constant \( C_d \) depending only on \( d \). On the other hand, Carbery, Wainger and Wright claim that the true order of magnitude of the above principal value integral is \( \log d \). We prove that

\[
\sup_{P \in P_d} \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \sim \log d.
\]

1. Introduction

Let \( P_d \) be the vector space of all real polynomials of degree at most \( d \) in \( \mathbb{R} \). For \( P \in P_d \) we consider the principal value integral

\[
I(P) = \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right|.
\]

We wish to estimate the quantity \( I(P) \) by a constant \( C(d) \) depending only on the degree of the polynomial \( d \). This amounts to estimating the integral

\[
I(\epsilon,R)(P) = \left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right|
\]

by some constant \( C(d) \) independent of \( \epsilon, R \) and \( P \).

This problem is quite old and in fact was answered some thirty years ago by Stein and Wainger in [4] and [6]. They showed that the quantity \( I(P) \) is bounded by a constant \( C_d \) depending only on \( d \). Their proof is very simple and uses a combination of induction and van der Corput’s lemma. Let us recall the latter since we’ll also be using it in what follows.

Proposition 1.1 (van der Corput). Let \( \phi : [a, b] \to \mathbb{R} \) be a \( C^k \) function and suppose that \( |\phi^{(k)}(t)| \geq 1 \) for some \( k \geq 1 \) and all \( t \in [a, b] \). If \( k = 1 \) suppose in addition...
that $\phi'$ is monotonic. Then, for every $\lambda \in \mathbb{R}$,
\[
\left| \int_a^b e^{i\lambda \phi(x)} \, dx \right| \leq \frac{Ck}{|\lambda|^\frac{1}{k}},
\]
where $C$ is an absolute constant independent of $a, b, k$ and $\phi$.

For a proof of this very well-known result with $Ck$ replaced by $Ck$, see, for example, [3]. A proof that the constant $Ck$ can be taken to be linear in $k$ can be found in [1].

On the other hand, Carbery, Wainger and Wright have conjectured in [2] that the true order of magnitude of the principal value integral is $\log d$. The main result of this paper is the proof of this conjecture. This is the content of:

**Theorem 1.2.** There exist two absolute positive constants $c_1$ and $c_2$ such that
\[
c_1 \log d \leq \sup_{P \in \mathcal{P}_d} \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c_2 \log d.
\]

**Remark 1.3.** Suppose that $K$ is a $-n$ homogeneous function on $\mathbb{R}^n$, odd and integrable on the unit sphere. Then, by the one-dimensional result, we trivially get that there is an absolute positive constant $c$, such that
\[
\left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) \, dx \right| \leq c \|K\|_{L^1(S^{n-1})} \log d,
\]
for every polynomial $P$ on $\mathbb{R}^n$, of degree at most $d$.

**Notation.** We will use the letter $c$ to denote an absolute positive constant, which might change even in the same line of text. Also, the notation $A \sim B$ means that there exist absolute positive constants $c_1$ and $c_2$ such that $c_1B \leq A \leq c_2B$.

### 2. The Lower Bound in the Theorem

In this section we will construct a real polynomial $P$ of degree at most $d$ such that the inequality
\[
I(P) = \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \geq c \log d
\]
holds. The general plan of the construction is as follows. We will first construct a function $f$ (which will not be a polynomial) such that $I(f) \geq c \log n$. We will then construct a polynomial $P$ of degree $d = 2n^2 - 1$ that approximates the function $f$ in a way that $|I(f) - I(P)|$ is small (small means $o(\log n)$ here). Since $\log n \sim \log d$, this will yield our result.

**Lemma 2.1.** For $n$ a large positive integer, let $f(t)$ be the continuous function that is equal to 1 for $\frac{1}{n} \leq t \leq 1 - \frac{1}{n}$, equal to -1 for $-1 + \frac{1}{n} \leq t \leq -\frac{1}{n}$, equal to 0 for $|t| \geq 1$ and linear in each interval $[-1, -1 + \frac{1}{n}]$, $[-\frac{1}{n}, \frac{1}{n}]$ and $[1 - \frac{1}{n}, 1]$. Then,
\[
I(f) = \left| \text{p.v.} \int_{\mathbb{R}} e^{if(t)} \frac{dt}{t} \right| \geq c \log n.
\]
Proof. The proof is more or less straightforward.

\[ I(f) = 2 \left| \int_0^1 \sin \left( \frac{f(t)}{t} \right) dt \right| \]

\[ \geq 2 \left| \int_{\frac{1}{n}}^{1 - \frac{1}{n}} \sin \left( \frac{f(t)}{t} \right) dt \right| - 2 \left| \int_0^{\frac{1}{n}} \sin \left( \frac{f(t)}{t} \right) dt \right| - 2 \left| \int_{1 - \frac{1}{n}}^1 \sin \left( \frac{f(t)}{t} \right) dt \right| \]

\[ \geq 2 \sin 1 \log(n - 1) - 2 \int_0^{\frac{1}{n}} \frac{f(t)}{t} dt - 2 \int_{1 - \frac{1}{n}}^1 \frac{f(t)}{t} dt \]

\[ = 2 \sin 1 \log(n - 1) - 2 - 2n \log \frac{n}{n - 1} + 2 \]

\[ \geq 2 \sin 1 \log(n - 1) - 4 \geq c \log n. \quad \square \]

We now want to construct a polynomial which approximates the function \( f \). We will do so by convolving the function \( f \) with a “polynomial approximation to the identity”. To be more specific, for \( k \in \mathbb{N} \) and \( x \in \mathbb{R} \) define the function

\[ \phi_k(x) = c_k \left( 1 - \frac{x^2}{4} \right)^{k^2}, \]

where the constant \( c_k \) is defined by means of the normalization

\[ \int_{-2}^{2} \phi_k(x) dx = 1. \]

Observe that

\[ 1 = c_k \int_{-2}^{2} \left( 1 - \frac{x^2}{4} \right)^{k^2} dx = 4c_k \int_0^1 (1 - x^2)^{k^2} dx = 2c_k B \left( \frac{1}{2}, k^2 + 1 \right), \]

where \( B(\cdot, \cdot) \) is the beta function. Using standard estimates for the beta function we see that \( c_k \sim k \).

Define, next, the functions \( P_k \) in \( \mathbb{R} \) as

\[ P_k(t) = \int_{-1}^{1} f(x) \phi_k(t - x) dx, \]

where \( f \) is the function of Lemma 2.1. It is clear that the functions \( P_k \) are polynomials of degree at most \( 2k^2 \). The following lemma deals with some technical issues concerning the polynomials \( P_k \).

Lemma 2.2. Let \( P_k \) be defined as in (2.5) above.

(i) \( P_k \) is an odd polynomial of degree \( 2k^2 - 1 \) with leading coefficient

\[ a_k = (-1)^{k^2 + 1} \frac{2c_k k^2}{4^{k^2}} \left( 1 - \frac{1}{n} \right). \]

That is,

\[ P_k(t) = a_k t^{2k^2 - 1} + \cdots. \]

(ii) As a consequence of (i) we have for all \( t \),

\[ |P_k^{(2k^2 - 1)}(t)| \geq c(2k^2 - 1)! \frac{k^3}{4^{k^2}}. \]
(iii) For \( t \in [-1, 1] \) we have
\[
P_k(t) = \int_0^2 (f(t + x) + f(t - x))\phi_k(x)\,dx.
\]

Proof. (i) Using (2.5) we have
\[
P_k(-t) = \int_{-1}^1 f(x)\phi_k(-t - x)\,dx = \int_{-1}^1 f(x)\phi_k(t + x)\,dx = \int_{-1}^1 f(-x)\phi_k(t - x)\,dx = -P_k(t).
\]

Next, from (2.5) we have that
\[
P_k(t) = c_k \int_{-1}^1 f(x) \sum_{m=0}^{k^2} \binom{k^2}{m} \left(-\frac{(t - x)^2}{4}\right)^m \,dx
\]
\[
= c_k \sum_{m=0}^{k^2} \binom{k^2}{m} \frac{(-1)^m}{4^m} \int_{-1}^1 f(x)(t - x)^{2m} \,dx
\]
\[
= c_k \frac{(-1)^{k^2}}{4^{k^2}} \int_{-1}^1 f(x)(x - t)^{2k^2} \,dx
\]
\[
+ c_k \sum_{m=0}^{k^2-1} \binom{k^2}{m} \frac{(-1)^m}{4^m} \int_{-1}^1 f(x)(t - x)^{2m} \,dx.
\]

It is now easy to see that the two highest-order terms come from the first summand in the above formula. Therefore,
\[
P_k(t) = c_k \frac{(-1)^{3k^2}}{4^{k^2}} \int_{-1}^1 f(x)dx \cdot t^{2k^2} - c_k \frac{(-1)^{k^2}2k^2}{4^{k^2}} \int_{-1}^1 f(x)dx \cdot t^{2k^2-1} + \ldots
\]
\[
= (-1)^{k^2+1} \frac{2ck^2}{4^{k^2}} \left(1 - \frac{1}{n}\right) t^{2k^2-1} + \ldots.
\]

(ii) We just use the result of (i) and the fact that \( c_k \sim k \).

(iii) Fix a \( t \in [-1, 1] \). Then,
\[
\int_{-2}^2 f(t - x)\phi_k(x)\,dx = \int_{\mathbb{R}} f(t - x)\phi_k(x)\chi_{[-2,2]}(x)\,dx
\]
\[
= \int_{-1}^1 f(x)\phi_k(t - x)\chi_{[-2,2]}(t - x)\,dx
\]
\[
= \int_{-1}^1 f(x)\phi_k(t - x)\,dx
\]
\[
= P_k(t).
\]

However, since \( \phi_k \) is even,
\[
P_k(t) = \int_{-2}^2 f(t - x)\phi_k(x)\,dx = \int_{0}^2 (f(t + x) + f(t - x))\phi_k(x)\,dx.
\]

We are now ready to prove the lower bound for \( I(P) \).
Proposition 2.3. Let $P_n$ be the polynomial defined in (2.5) where $n$ is the large positive integer used to define the function $f$ in Lemma 2.1. Then $P_n$ is a polynomial of degree $d = 2n^2 - 1$ and

$$I(P_n) = \left| \text{p.v.} \int_{\mathbb{R}} e^{itP_n(t)} \frac{dt}{t} \right| \geq c \log d.$$ 

Proof. Since $P_n$ is odd,

$$I(P_n) = 2 \left| \int_0^{+\infty} \sin \frac{P_n(t)}{t} \frac{dt}{t} \right|,$$

and it suffices to show that for all $R \geq 1$,

$$\left| \int_0^{R} \sin \frac{P_n(t)}{t} \frac{dt}{t} \right| \geq c \log d \sim c \log n.$$ 

By part (ii) of Lemma 2.2 and a standard application of Proposition 1.1 (van der Corput) we see that

$$\left| \int_{1}^{R} \sin \frac{P_n(t)}{t} \frac{dt}{t} \right| \leq c$$

for all $R \geq 1$. As a result, the proof will be complete if we show that

$$I_1(P_n) = \left| \int_0^{1} \sin \frac{P_n(t)}{t} \frac{dt}{t} \right| \geq c \log n.$$ 

Using Lemma 2.1 and the triangle inequality we get

$$I_1(P_n) \geq c \log n - |I_1(P_n) - I(f)|$$

and, in order to show (2.7), it suffices to show that

$$|I_1(P_n) - I(f)| = o(\log n).$$

We have that

$$|I_1(P_n) - I(f)| = \left| \int_0^{1} \frac{P_n(t) - f(t)}{t} dt \right| \leq \int_0^{1} \frac{|P_n(t) - f(t)|}{t} dt.$$

Using part (iii) of Lemma 2.2 and (2.4), we get

$$|P_n(t) - f(t)| \leq \int_0^{1} |f(t + x) + f(t - x) - 2f(t)| \phi_n(x) dx$$

for $0 \leq t \leq 1$. Hence

$$|I_1(P_n) - I(f)| \leq \int_0^{1} \int_0^{1} \frac{|f(t + x) + f(t - x) - 2f(t)|}{t} dt \phi_n(x) dx.$$ 

Now, the desired result, condition (2.9), is the content of the following lemma. □

Lemma 2.4. Let $A(x, t) = |f(t + x) + f(t - x) - 2f(t)|$. Then,

$$\int_0^{2} \int_0^{1} \frac{A(x, t)}{t} dt \phi_n(x) dx = o(\log n).$$
Proof. Firstly, it is not difficult to establish that

\begin{align}
A(x, t) & \leq 4 \min(nx, nt, 1), \\
A(x, t) & = 0, \quad \text{when } \frac{1}{n} \leq t - x \leq t + x \leq 1 - \frac{1}{n}.
\end{align}

Indeed,

\[ A(x, t) \leq |f(t + x) - f(t)| + |f(t - x) - f(t)| \leq nx + nx \leq 2nx. \]

On the other hand,

\[ A(x, t) = |f(t + x) - f(t) + f(t - x) - f(-x) - 2f(t)| \leq |f(t + x) - f(t)| + |f(t - x) - f(-x)| + 2|f(t)| \leq nt + nt + 2nt = 4nt. \]

Inequality (2.10) now follows by the fact that \(|f|\) is bounded by 1 and (2.11) is trivial to prove.

We split the integral \(\int_0^2 \int_0^1 \cdots dt \, dx\) into seven integrals:

\[ \int_0^2 \int_0^1 \cdots dt \, dx + \int_0^\frac{1}{2} \int_0^x \cdots dt \, dx + \int_0^\frac{1}{2} \int_0^1 \cdots dt \, dx + \int_0^\frac{1}{2} \int_0^{\frac{x+\frac{1}{n}}{2}} \cdots dt \, dx + \int_0^{\frac{x}{nx}} \int_0^1 \cdots dt \, dx + \int_0^{\frac{x}{2}} \int_0^{\frac{x+\frac{1}{n}}{2}} \cdots dx + \int_0^{\frac{x}{2}} \int_0^{\frac{x+\frac{1}{n}}{2}} \cdots dt \, dx. \]

We estimate each of the seven integrals separately:

\[ \int_0^2 \int_0^1 \frac{A(x, t)}{t} \, dt \, \phi_n(x) \, dx \leq 4 \log 2 \int_0^2 \phi_n(x) \, dx = 2 \log 2, \]

\[ \int_0^\frac{1}{n} \int_0^x \frac{A(x, t)}{t} \, dt \, \phi_n(x) \, dx \leq \int_0^\frac{1}{n} \int_0^x \frac{4nt}{t} \, dt \, \phi_n(x) \, dx = \int_0^\frac{1}{n} 4nx \phi_n(x) \, dx \leq 2, \]

\[ \int_0^\frac{1}{n} \int_0^\frac{1}{n} \frac{A(x, t)}{t} \, dt \, \phi_n(x) \, dx \leq \int_0^\frac{1}{n} \int_0^\frac{1}{n} \frac{4nt}{t} \, dt \, \phi_n(x) \, dx = \int_0^\frac{1}{n} \frac{4n\phi_n(x)}{dx} \, dx \leq 2. \int_0^\frac{1}{n} \int_x^{x+\frac{1}{n}} \frac{A(x, t)}{t} \, dt \, \phi_n(x) \, dx \]

\[ \leq \int_0^\frac{1}{n} \int_x^{x+\frac{1}{n}} \frac{4nx}{t} \, dt \, \phi_n(x) \, dx = \int_0^\frac{1}{n} 4nx \log \left(1 + \frac{1}{nx}\right) \phi_n(x) \, dx \leq 2. \]

For \(\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \frac{A(x, t)}{t} \, dt \, \phi_n(x) \, dx\) we have \(\frac{1}{n} \leq t - x \leq t + x \leq 1 - \frac{1}{n}\) and, by (2.11), \(A(x, t) = 0\). Hence

\[ \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \frac{A(x, t)}{t} \, dt \, \phi_n(x) \, dx = 0. \]
Next
\[
\int_{\frac{1}{n}}^{\frac{1}{n} + \frac{\xi}{n}} \int_{\frac{1}{n}}^{\frac{1}{n} + \frac{\xi}{n}} \frac{A(x,t)}{t} dt \phi_n(x) dx \leq \int_{\frac{1}{n}}^{\frac{1}{n} + \frac{\xi}{n}} \int_{\frac{1}{n}}^{\frac{1}{n} + \frac{\xi}{n}} \frac{4}{t} dt \phi_n(x) dx \\
\leq 4 \int_{\frac{1}{n}}^{1} \log(n x + 1) \phi_n(x) dx.
\]

Now, fix some \( \alpha \in (0, 1) \). Write
\[
\int_{\frac{1}{n}}^{1} \log(n x + 1) \phi_n(x) dx = \int_{\frac{1}{n}}^{1} \cdots dx + \int_{\frac{1}{n}}^{1} \cdots dx \\
\leq \frac{\log(n^{1-\alpha} + 1)}{2} + c n \log(n + 1) \int_{\frac{1}{n}}^{1} \left(1 - \frac{x^2}{4}\right)^2 dx \\
\leq \frac{\log(n^{1-\alpha} + 1)}{2} + c n \log(n + 1) e^{-\frac{1}{4} n^{2(1-\alpha)}}.
\]

Therefore,
\[
\limsup_{n \to \infty} \frac{\int_{\frac{1}{n}}^{1} \log(n x + 1) \phi_n(x) dx}{\log n} \leq \frac{1 - \alpha}{2}
\]
and, since \( \alpha \) is arbitrary in \((0, 1)\),
\[
\int_{\frac{1}{n}}^{\frac{1}{n} + \frac{\xi}{n}} \int_{\frac{1}{n}}^{\frac{1}{n} + \frac{\xi}{n}} \frac{A(x,t)}{t} dt \phi_n(x) dx = o(\log n).
\]

Finally,
\[
\int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \frac{A(x,t)}{t} dt \phi_n(x) dx \leq \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \frac{4}{t} dt \phi_n(x) dx \\
\leq 4 \log \frac{n}{2} c n \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \left(1 - \frac{x^2}{4}\right)^2 dx \\
\leq cn \log ne^{-\frac{1}{4} n^2} = o(1).
\]

3. The upper bound in the theorem

We set
\[
K_d = \sup_{P \in \mathcal{P}_d, \epsilon, R} \left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right|.
\]

We take any polynomial \( P \), of degree at most \( d \), which we can assume has no constant term, that is, \( P(0) = 0 \). We set \( k = \left\lfloor \frac{d}{2} \right\rfloor \) and we write
\[
P(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k + a_{k+1} t^{k+1} + \cdots + a_d t^d = Q(t) + R(t),
\]
where \( Q(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k \) and \( R(t) = a_{k+1} t^{k+1} + \cdots + a_d t^d \). Let \( |a_l| = \max_{k+1 \leq j \leq d} |a_j| \) for some \( k + 1 \leq l \leq d \). By a change of variables in the
integral in (3.1) we can assume that \(|a_i| = 1\) and thus that \(|a_j| \leq 1\) for every \(k + 1 \leq j \leq d\). Now split the integral in (3.1) into two parts as follows:

\[
\int_{|t| \leq R} e^{iP(t)\frac{dt}{t}} \leq \left| \int_{|t| \leq 1} e^{iP(t)\frac{dt}{t}} \right| + \left| \int_{1 \leq |t| \leq R} e^{iP(t)\frac{dt}{t}} \right| = I_1 + I_2.
\]

For \(I_1\) we have that

\[
I_1 \leq \left| \int_{|t| \leq 1} \left[ e^{iP(t)} - e^{iQ(t)} \right] \frac{dt}{t} \right| + \left| \int_{|t| \leq 1} e^{iQ(t)\frac{dt}{t}} \right| 
\leq \int_{0 \leq |t| \leq 1} \frac{|R(t)|}{t} dt + K_{\frac{\epsilon}{2}} 
\leq 2 \sum_{j=k+1}^{d} \frac{|a_j|}{j} + K_{\frac{\epsilon}{2}} 
\leq \sum_{j=k+1}^{d} \frac{1}{j} + K_{\frac{\epsilon}{2}} \leq c + K_{\frac{\epsilon}{2}}.
\]

For the second integral in (3.2) we have that

\[
I_2 \leq \left| \int_{1 \leq t \leq R} e^{iP(t)\frac{dt}{t}} \right| + \left| \int_{-R \leq t \leq -1} e^{iP(t)\frac{dt}{t}} \right| = I_2^+ + I_2^-.
\]

For some \(\alpha > 0\) to be defined later split \(I_2^+\) into two parts as follows:

\[
I_2^+ \leq \left| \int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} e^{iP(t)\frac{dt}{t}} \right| + \left| \int_{\{t \in [1, R] : |P'(t)| > \alpha\}} e^{iP(t)\frac{dt}{t}} \right|.
\]

Since \(\{t \in [1, R] : |P'(t)| > \alpha\}\) consists of at most \(O(d)\) intervals where \(P'\) is monotonic, using Proposition 1.1 we get the bound

\[
\left| \int_{\{t \in [1, R] : |P'(t)| > \alpha\}} e^{iP(t)\frac{dt}{t}} \right| \leq c \frac{d}{\alpha}.
\]

For the logarithmic measure of the set \(\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}\), observe that

\[
\int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} \frac{dt}{t} \leq \sum_{m=0}^{\infty} \int_{\{t \in [2^m, 2^{m+1}] : |P'(t)| \leq \alpha\}} \frac{dt}{t} 
= \sum_{m=0}^{\infty} \int_{2^m t \in [2^m, 2^{m+1}] : |P'(2^m t)| \leq \alpha} \frac{dt}{t} 
= \sum_{m=0}^{\infty} \int_{\{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}} \frac{dt}{t}
= \sum_{m=0}^{\infty} \int_{\{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}} \frac{dt}{t}.
\]

We have thus shown that

\[
\int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} \frac{dt}{t} \leq \sum_{m=0}^{\infty} |\{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}|.
\]

(3.3)
In order to finish the proof we need a suitable estimate for the sublevel set of a polynomial. This is the content of the following lemma.

**Lemma 3.1 (Vinogradov).** Let \( h(t) = b_0 + b_1 t + \cdots + b_n t^n \) be a real polynomial of degree \( n \). Then,

\[
|\{ t \in [1, 2] : |h(t)| \leq \alpha \}| \leq c \left( \frac{\alpha}{\max_{0 \leq k \leq n} |b_k|} \right)^{\frac{1}{n}}.
\]

This lemma is due to Vinogradov [5]. We postpone the proof of Lemma 3.1 until after the end of the proof of the upper bound.

Consider the polynomial \( P'(2^m t) \) with coefficients \( ja_j 2^{m(j-1)} \), \( 1 \leq j \leq d \). Clearly, \( \max_{1 \leq j \leq d} |ja_j 2^{m(j-1)}| \geq |a_j 2^{m(l-1)}| \geq ([\frac{d}{2}] + 1) 2^{m([\frac{d}{2}])} \). Using Lemma 3.1 and (3.3), we get

\[
\int_{\{ t \in [1, +\infty) : |P'(t)| \leq \alpha \}} \frac{dt}{t} \leq c \alpha^{\frac{1}{m-1}} \sum_{m=0}^{\infty} \left( \frac{1}{([\frac{d}{2}] + 1) 2^{m([\frac{d}{2}])}} \right)^{\frac{1}{m}} \leq c \alpha^{\frac{1}{m-1}}.
\]

Obviously, a similar estimate holds for \( I_2' \). Summing the estimates we get

\[
\left| \int_{|t| \leq R} e^{iP(t)} \frac{dt}{t} \right| \leq c + c \frac{d\alpha^{\frac{1}{m-1}}}{\alpha} + K_{[\frac{d}{2}]},
\]

Optimizing in \( \alpha \) we get that

\[
(3.4) \quad \left| \int_{|t| \leq R} e^{iP(t)} \frac{dt}{t} \right| \leq c + K_{[\frac{d}{2}]},
\]

and hence

\[
K_d \leq c + K_{[\frac{d}{2}]}. \]

In particular we have

\[
K_{2^n} \leq c + K_{2^{n-1}}.
\]

Using induction on \( n \) we get that \( K_{2^n} \leq c n \). It is now trivial to show the inequality for general \( d \). Indeed, if \( 2^{n-1} < d \leq 2^n \), then \( K_d \leq K_{2^n} \leq c n \leq c \log d \).

For the sake of completeness we give the proof of Lemma 3.1.

**Proof of Lemma 3.1.** The set \( E_\alpha = \{ t \in [1, 2] : |h(t)| \leq \alpha \} \) is a union of intervals. We slide them together to form a single interval \( I \) of length \( |E_\alpha| \) and pick \( n + 1 \) equally spaced points in \( I \). If we slide the intervals back to their original position, we end up with \( n + 1 \) points \( x_0, x_1, x_2, \ldots, x_n \in E_\alpha \) that satisfy

\[
(3.5) \quad |x_j - x_k| \geq \frac{|E_\alpha| |j-k|}{n}.
\]

The Lagrange polynomial that interpolates the values \( h(x_0), h(x_1), \ldots, h(x_n) \) coincides with \( h(x) \):

\[
h(x) = \sum_{j=0}^{n} h(x_j) \frac{(x-x_0)(x-x_1) \cdots (x-x_{j-1})(x-x_{j+1}) \cdots (x-x_n)}{(x_j-x_0)(x_j-x_1) \cdots (x_j-x_{j-1})(x_j-x_{j+1}) \cdots (x_j-x_n)}.
\]

Therefore we get for the coefficients of \( h \) that

\[
b_k = \sum_{j=0}^{n} h(x_j) \frac{(-1)^{n-k} \sigma_{n-k}(x_0, \ldots, x_j, \ldots, x_n)}{(x_j-x_0)(x_j-x_1) \cdots (x_j-x_{j-1})(x_j-x_{j+1}) \cdots (x_j-x_n)}.
\]
for \( k = 0, 1, \ldots, n \). In the above formula, \( \sigma_{n-k}(x_0, \ldots, \hat{x}_j, \ldots, x_n) \) is the \((n-k)\)-th elementary symmetric function of \( x_0, \ldots, \hat{x}_j, \ldots, x_n \) where \( x_j \) is omitted. Using the estimate \( \sigma_{n-k}(x_0, \ldots, \hat{x}_j, \ldots, x_n) \leq \binom{n}{n-k}2^{n-k} \) together with (3.5) we get that, for every \( k = 0, 1, \ldots, n \),

\[
|b_k| \leq \binom{n}{n-k}2^{n-k} \frac{n^n}{|E_\alpha|^n} \sum_{j=0}^n \frac{1}{j!(n-j)!} \leq \binom{n}{n-k} \frac{8^n}{\sqrt{n}} \frac{n^n}{|E_\alpha|^n} \frac{\alpha}{\sqrt{n}}.
\]

where we used the estimate \( \binom{n}{n-k} \leq \left( \frac{n}{2} \right)^n \leq c \frac{2^n}{\sqrt{n}} \). Hence

\[
\max_{0 \leq k \leq n} |b_k| \leq c \frac{8^n}{\sqrt{n}} \frac{n^n}{|E_\alpha|^n} \frac{\alpha}{\sqrt{n}},
\]

and solving with respect to \(|E_\alpha|\) we get

\[
|E_\alpha| \leq c \left( \frac{\alpha}{\max_{0 \leq k \leq n} |b_k|} \right)^{\frac{1}{n}}.
\]

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