Strong eigenstate thermalization hypothesis

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We present a generalization of the ETH conjecture. Using this generalization we are able to derive the fact that an arbitrary eigenstate of a general many body system may be used to represent microcanonical ensemble in any many body experiment that involves only local operators and projectors onto eigenstates of the system Hamiltonian. In particular we extend the ETH to include some non-local operators. We present a derivation of this conjecture in the case of a many body model whose Hamiltonian is composed of two parts: an integrable Hamiltonian and a small but finite Gaussian perturbation.

I. INTRODUCTION

Understanding the non-equilibrium long time dynamics of non-integrable many body systems has become a major theoretical and experimental challenge over recent years. This research is driven by the advances in the field of ultracold atoms [1–9]. These advances allow the detailed experimental study of the time evolution (of thermally isolated systems) from given initial states under the influence of well defined Hamiltonians (in particular ones not interacting with any bath). One of the key questions that is under investigation is the validity of the Gibbs ensemble formula for describing equilibrium states of such systems. The ETH hypothesis is one of the cornerstones in deriving the validity of the Gibbs formula for thermodynamic quantities and for local operators. It states that for any non-integrable, non many-body localized Hamiltonian \( H \):

\[
\text{Tr} \{ | \lambda \rangle \langle \lambda | O \} = F_O (E_\lambda)
\]

for most eigenstates \( | \lambda \rangle \), \( H | \lambda \rangle = E_\lambda | \lambda \rangle \). Here \( O \) is any local operator and \( F_O \) is some smooth function that depends only on the energy of the state \( E_\lambda \), \( E_\lambda \), and not on any other of its properties. There currently does not exist a general proof of the ETH conjecture. There are however some derivation for special classes of Hamiltonians. In particular there is a rigorous derivation of the ETH conjecture for integrable Hamiltonians perturbed by small Gaussian perturbations [10]. Furthermore for nuclear models calculations have shown that individual wavefunctions reproduce thermodynamic expectation values [11]. There are various proofs that semiclassical quantum systems, whose classical counterparts are chaotic, satisfy the ETH hypothesis [12–15]. In particular the low density billiards in the semiclassical limit satisfy ETH [16,17]. More generally the ETH conjecture may be derived from Berry’s conjecture which is valid for semiclassical chaotic systems [18]. Furthermore the validity of the ETH hypothesis has been numerically demonstrated for a variety of non-integrable models [19–24].

With the ETH hypothesis it is possible to show that the expectation value with respect to a given state is equal to the one given by the microcanonical ensemble, e.g.

\[
\langle \lambda | O | \lambda \rangle = \frac{1}{N_{E_\lambda, \delta E}} \sum_{| \lambda | - E_\lambda | < \delta E} \langle \alpha | O | \alpha \rangle
\]

Here \( \delta E \) is a small energy width and \( N_{E_\lambda, \delta E} \) is the number of states in the interval \( [E_\lambda - \delta E, E_\lambda + \delta E] \) (this is indeed true since all the entries on the right hand side of Eq. (2) are the same). Furthermore through a saddle point argument it is possible to show that for the purposes of computing local expectation values a single state is equivalent to the canonical ensemble:

\[
\langle \lambda | O | \lambda \rangle = \frac{1}{Z} \sum e^{-\beta E_\alpha} \langle \alpha | O | \alpha \rangle
\]

where \( Z = \sum e^{-\beta E_\alpha} \) and \( \beta \) is the inverse temperature chosen such that \( \frac{1}{Z} \sum e^{-\beta E_\alpha} E_\alpha = E_\lambda \). This is true in the thermodynamic limit since the energies \( E_\alpha \) contribute strongly to the right hand side of Eq. (3) are strongly peaked about one value \( E_\lambda \). Using this it is possible to show that a single eigenstate is equivalent to the canonical ensemble when the system undergoes an arbitrary evolution and is subjected to any set of measurements as long as all these operations are local [23]. However it is of great interest to consider non-local operations. The most important non-local operation is projection onto an eigenstate of the Hamiltonian \( H \): \( \Pi_{\lambda} \) which comes from an accurate measurement of the system energy. To match such measurements it is important to generalize the ETH formalism to include such projectors. A sufficient postulate that would extend ETH to include all such energy projectors is given by:

\[
\text{Tr} \left\{ \prod_{i=1}^{n} O_i | \lambda_i \rangle \langle \lambda_i | \right\} = F_{O_1 \ldots O_n} (\text{Distinct} \{ E_{\lambda_1}, \ldots E_{\lambda_n} \})
\]

The function \( F_{O_1 \ldots O_n} \) depends smoothly on its variables function and the label Distinct means to count only different \( \lambda \)’s, e.g. if \( | \lambda_i \rangle = | \lambda_j \rangle \) then there is only one energy in the set Distinct \( \{ E_{\lambda_1}, \ldots E_{\lambda_n} \} \) corresponding to the elements \( \lambda_i \) and \( \lambda_j \). We note that the case \( n = 1 \) corresponds to the usual ETH hypothesis see Eq. (1). Using Eq. (4) it is possible to prove the ETH hypothesis, Eq. (1), in the case when \( O \) contains non-local parts in

\[\ldots\]
II. DISORDERED HERMITIAN MATRICES

We would like to derive Eq. (4) for a simple model. For simplicity we will prove it in the case when \(|\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n\) (the case when some of the eigenvectors are identical is handled highly similarly). We will consider a model where the Hamiltonian is composed of two part, an integrable part and a Gaussian perturbation, e.g. 

\[ H = \sum_i E_i |i\rangle \langle i| + H_{\text{Gauss}} \]  

(5)

Here it is assumed that the \(E_i\) are dense with average level spacing \(\Delta\) and \(H_{\text{Gauss}}\) has matrix elements \(h_{ij} = \delta_{ij}\) that have gaussian distribution with \(|h_{ij}| \sim \sqrt{\frac{\Delta}{\omega}}\). Let \(c_{ij}\) be the unitary matrix that diagonalizes \(H\), e.g. \(c_{ij}H = D\) for a diagonal matrix \(D\). The the expectation value of the expression in Eq. (4) is given by:

\[ \sum_{i_1, \ldots, i_n} \langle \prod_{\eta=1}^n O_{\eta} | \lambda_{\eta} \rangle \langle \lambda_{\eta} | \prod_{\eta=1}^n c_{i_{\eta}, i_{\eta}+1} \rangle \]  

(6)

where \(Cyc\) means to cyclically identify \(n+1 = 1\). Now the probability distribution of the expression given in Eq. (6) for eigenenergies \(E_{\lambda_1}, \ldots, E_{\lambda_n}\) is given by:

\[ P \left\{ \prod_{\eta=1}^n O_{\eta} | \lambda_{\eta} \rangle \langle \lambda_{\eta} | \right\} \]

\[ \propto \sum_{i_1, \ldots, i_n} \int dc_{ij} \delta(c_{ij} - c) \prod_{i<j} \delta \left( (c^{\dagger}Hc)_{ij} - E_{\lambda_i} \right) \times \prod_{i_1, \ldots, i_n} \delta \left( (c^{\dagger}Hc)_{ii} - E_{\lambda_i} \right) \times \prod_{i} \exp \left( -\frac{1}{\Delta} \sum_{i<j} |H_{ij}|^2 - \frac{1}{\Delta} \sum_i (E_{\lambda_i} - E_i)^2 \right) \]

(7)

The form of expression in the exponential (which is extensive in the system size): 
\[ \prod_{i} \exp \left( -\frac{1}{\Delta} \sum_{i<j} |H_{ij}|^2 - \frac{1}{\Delta} \sum_i (E_{\lambda_i} - E_i)^2 \right) \]  

gives the probability distribution in Eq. (7) a very small spread, with the distribution having width \(\sqrt{\frac{\Delta}{\omega}}\). Therefore for fixed energies \(E_{\lambda_1}, \ldots, E_{\lambda_n}\) the expression in Eq. (7) has uncertainty that scales like \(\sqrt{\frac{\Delta}{\omega}}\) and tends to zero exponentially in the thermodynamic limit. The expression in Eq. (7) depends therefore only on the energies \(E_{\lambda_1}, \ldots, E_{\lambda_n}\), and Eq. (4) is proven.

We would like to note that using a very similar procedure it is possible to prove that for any local operators \(O_1, \ldots, O_n\) and generic Hamiltonians \(H_1, \ldots, H_n\) chosen from the distribution given in Eq. (5) we have that:

\[ \prod_{i=1}^n O_i | \lambda_i \rangle \langle \lambda_i | = F_{O_1 \ldots O_n} (\{E_{\lambda_1}, \ldots, E_{\lambda_n}\}) \]  

(8)

for most states such that \(H_i | \lambda_i \rangle = E_{\lambda_i} | \lambda_i \rangle\).

III. APPLICATIONS

We would like to show that for any system that satisfies Eq. (4) a single eigenstate is completely equivalent to the microcanonical ensemble for the purposes of doing any experiment that involves only local Hamiltonians and projectors \(\Pi_E\) onto eigenstates of the Hamiltonian \(H\). As a warm up we will show that the linear response, e.g. conductivity of an eigenstate is equivalent to that of the canonical ensemble, even though this measurement involves projectors.

A. Linear Response

We would like to show that the linear response of an eigenstate in a model satisfying Eq. (4) has the same linear response as the canonical and microcanonical ensemble. We note that for many operators it is possible to do this directly from the ETH, see Eq. (4).

To do so recall that the linear response function \(D_{\omega_{O_1 O_2}}(\lambda)\) of a system
initialized in an eigenstate \( H |\lambda\rangle = E_\lambda |\lambda\rangle \) at \( t = 0 \) has the following definition. Suppose the system is subjected to the time dependent Hamiltonian \( \hat{H}(t) = H + a e^{-i\omega t} O_1 \) for some infinitesimal constant \( a \) and an arbitrary operator \( O_1 \), then it develops a non-zero expectation value for the operator \( O_2 \) given by \( \langle O_2(t) \rangle = a D^{O_2}_{O_1,O_2}(\lambda) e^{-i\omega t} \) at long times. The linear response coefficient may be calculated, it is given by \(^{[29]}\):

\[
D^{O_2}_{O_1,O_2}(\lambda) = \sum_{\chi} \frac{\langle \lambda | O_2 | \chi \rangle \langle \chi | O_1 | \lambda \rangle}{\omega - (E_\chi - E_\lambda) + i\epsilon} + \sum_{\chi} \frac{\langle \chi | O_1 | \lambda \rangle \langle \lambda | O_2 | \chi \rangle}{-\omega - (E_\chi - E_\lambda) - i\epsilon} \tag{9}
\]

Here \( \epsilon \) is an infinitesimal positive number and \( |\chi\rangle \) is a complete set of states for the Hamiltonian \( H \). From the form of Eq. \(^{[1]}\) we see that based on Eq. \(^{[4]}\) if two states \( |\lambda_1\rangle \) and \( |\lambda_2\rangle \) have the same energy then the linear response coefficients \( D^{O_2}_{O_1,O_2}(\lambda_1) = D^{O_2}_{O_1,O_2}(\lambda_2) \). From this it is straightforward to obtain that:

\[
D^{\rho_{\text{micro}}}_{O_1,O_2}(\lambda) = D^{\rho_{\text{can}}}_{O_1,O_2}(\lambda) \tag{10}
\]

namely, a single eigenstate has the same linear response as the canonical and microcanonical ensembles. We note that in the case when \( O_2 = O_1^\dagger \) the imaginary part of the expression in Eq. \(^{[1]}\) is given by:

\[
\text{Im} \left\{ D^{O_1}_{O_1,O_1}(\lambda) \right\} = -\pi \sum_{\chi} |\langle \chi | O_1 | \lambda \rangle|^2 \delta (\omega - (E_\chi - E_\lambda)) + \pi \sum_{\chi} |\langle \chi | O_1 | \lambda \rangle|^2 \delta (\omega + (E_\chi - E_\lambda)) \tag{11}
\]

This is a highly non-local term that involves a projector \( \Pi_{E_\chi + \omega} \) onto eigenstates of the Hamiltonian \( H \), exactly the scenario our formalism handles. We note that unlike the usual ETH, with Eq. \(^{[4]}\) this result is valid even when the local operators \( O_1, O_2 \) are far apart in real space so that their product is no longer a local operator.

### B. General experiment

We would like to show that a single eigenstate will produce the same measurement outcomes as the microcanonical ensemble for any experiment that involves only local evolution and measurement as well as any projectors \( \Pi_E \) onto eigenstates of the Hamiltonian \( H \). Indeed the most generic experiment \(^{[22]}\) may be described as preparing the system \( S \) in some state and preparing an auxiliary reservoir \( R \) in a well defined initial state \( \rho_R \), then having the system, reservoir ensemble undergo rounds of evolution \( U_i \) and measurement \( M_i \) with the measurement operators being given by \( M_{i,j} \). The probability of some outcome is given by:

\[
\text{Tr} \left\{ M_{i_{n},i_{n}} M_{i_{n+1},i_{n+1}} \ldots M_{i_{1},i_{1}} U_{1} \ldots U_{n} |\lambda\rangle \otimes \rho_R |\lambda\rangle \right\} \tag{12}
\]

By inserting multiple resolutions of identity \( \sum_{\chi} |\chi\rangle \langle \chi| \) between all the operators in Eq. \(^{[12]}\) it is possible to show that the probability of the outcome \( M_{1_{i_{1}},i_{1}} \ldots M_{n_{i_{n}},i_{n}} \) depends only on the energy of the eigenstate \( |\lambda\rangle \): \( E_\lambda \). This proves that the microcanonical ensemble is equivalent to a single eigenstate for any experiment involving only local operations and projectors \( \Pi_E \). In a very similar manner we can prove Eq. \(^{[1]}\) for \( O \) composed of any local operators and projectors onto the eigenstates of \( H \). We note also that unlike the regular ETH our formalism allows for the measurement operators \( M_{k_{i_{k}},i_{k}} \) to be far apart in real space so that their product is non-local. We note that to prove equivalence of a single state to the canonical ensemble, which has some energy spread, we need to further assume that the projectors \( \Pi_{E_i} \) and \( \Pi_{E_j} \) come with an integration over energy so that the total operator depends only on projectors in the combination \( E_i - E_j \) and any local operators. We would like to note that similar results are possible for experiments that involve only local operators and projectors onto eigenstates of generic Hamiltonians that satisfy Eq. \(^{[8]}\) above.

### IV. CONCLUSIONS

In this work we have considered extensions of the ETH hypothesis to include some nonlocal operators, namely projectors onto eigenstates of the Hamiltonian of the system. We extended the ETH hypothesis to include multiple eigenstates thereby proving the usual ETH including the case of projectors and measurements. We supported our results by considering a simple model of an integrable Hamiltonian and a Gaussian perturbation. It is of great interest to extend the ETH to other non-local operators, it is a direction of research the authors are currently pursuing.

**Acknowledgments:** This research was supported by NSF grant DMR 1410583 and Rutgers CMT fellowship.

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