DILOGARITHM AND HIGHER \( \mathcal{L} \)-INVARIANTS FOR \( \text{GL}_3(\mathbb{Q}_p) \)

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Abstract. The primary purpose of this paper is to clarify the relation between previous results in [Schr11], [Bre17] and [BD18]. Let \( E \) be a sufficiently large finite extension of \( \mathbb{Q}_p \) and \( \rho_p \) be a \( p \)-adic semi-stable representation \( \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \text{GL}_3(E) \) such that the Weil–Deligne representation \( \text{WD}(\rho_p) \) associated with it has rank two monodromy operator \( N \) and the Hodge filtration associated with it is non-critical. Then by a computation of extensions of rank one \((\varphi, \Gamma)\)-modules we know that the Hodge filtration of \( \rho_p \) depends on three invariants in \( E \). We construct a family of locally analytic representations \( \Sigma_{\min}(\lambda, L_1, L_2, L_3) \) of \( \text{GL}_3(\mathbb{Q}_p) \) depending on three invariants \( L_1, L_2, L_3 \in E \) with each of the representation containing the locally algebraic representation \( \text{Alg} \otimes \text{Steinberg} \) determined by \( \text{WD}(\rho_p) \) via classical local Langlands correspondence for \( \text{GL}_3(\mathbb{Q}_p) \) and by the Hodge–Tate weights of \( \rho_p \). When \( \rho_p \) comes from an automorphic representation \( \pi \) of \( G(\mathbb{A}_{\mathbb{Q}_p}) \) with a fixed level \( U_p \) prime to \( p \) for a suitable unitary group \( G/\mathbb{Q} \), we show (under some technical assumption) that there is a unique locally analytic representation in the above family that occurs as a subrepresentation of the associated Hecke-isotypic subspace in the completed cohomology with level \( U_p \). We recall that [Bre17] constructed a family of locally analytic representations depending on four invariants (cf. (4) in [Bre17]) and proved that there is a unique representation in the family that embeds into the fixed Hecke-isotypic space above. We prove that if a representation \( \Pi \) in Breuil’s family embeds into a certain Hecke-isotypic subspace of completed cohomology, then it must equally embed into \( \Sigma_{\min}(\lambda, L_1, L_2, L_3) \) for certain choices of \( L_1, L_2, L_3 \in E \) determined explicitly by \( \Pi \). This gives a purely representation theoretic necessary condition for \( \Pi \) to embed into completed cohomology. Moreover, certain natural subquotients of \( \Sigma_{\min}(\lambda, L_1, L_2, L_3) \) give a true complex of locally analytic representations that realizes the derived object \( \Sigma(\lambda, L) \) in (1.14) of [Schr11]. Consequently, the locally analytic representation \( \Sigma_{\min}(\lambda, L_1, L_2, L_3) \) gives a relation between the higher \( \mathcal{L} \)-invariants studied in [Bre17] as well as [BD18] and the \( p \)-adic dilogarithm function which appears in the construction of \( \Sigma(\lambda, L) \) in [Schr11].

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1. Introduction

Let \( p \) be a prime number and \( F \) an imaginary quadratic extension of \( \mathbb{Q} \) such that \( p \) splits in \( F \). We fix a unitary algebraic group \( G \) over \( \mathbb{Q} \) which becomes \( \text{GL}_n \) over \( F \) and such that \( G(\mathbb{R}) \) is compact and \( G \) is split at all places of \( F \) above \( p \). Then to each finite extension \( E \) of \( \mathbb{Q}_p \) and to each prime-to-\( p \) level \( U^p \) in \( G(\mathbb{A}_Q^{\infty, p}) \), one can associate the Banach space of \( p \)-adic automorphic forms \( \tilde{S}(U^p, E) \). One can also associate with \( U^p \) a set of finite places \( D(U^p) \) of \( F \) and a Hecke algebra \( \mathcal{T}(U^p) \) which is the polynomial algebra freely generated by Hecke operators at places of \( F \) lying above \( D(U^p) \). In particular, the commutative algebra \( \mathcal{T}(U^p) \) acts on \( \tilde{S}(U^p, E) \) and commutes with the action of \( G(\mathbb{Q}_p) \cong \text{GL}_n(\mathbb{Q}_p) \) coming from translations on \( G(\mathbb{A}_Q^{\infty}) \).

If \( \rho: \text{Gal}(\overline{F}/F) \to \text{GL}_n(\mathbb{Q}) \) is a continuous irreducible representation, one considers the associated Hecke isotypic subspace \( \tilde{S}(U^p, E)[\mathfrak{m}_\rho] \), which is a continuous admissible representation of \( \text{GL}_n(\mathbb{Q}_p) \) over \( E \), or its locally \( \mathbb{Q}_p \)-analytic vectors \( \tilde{S}(U^p, E)[\mathfrak{m}_\rho]^{an} \), which is an admissible locally \( \mathbb{Q}_p \)-analytic representation of \( \text{GL}_n(\mathbb{Q}_p) \). We fix \( w_p \) a place of \( F \) above \( p \) and it is widely wished that \( \tilde{S}(U^p, E)[\mathfrak{m}_\rho] \) (and its subspace \( \tilde{S}(U^p, E)[\mathfrak{m}_\rho]^{an} \) as well) determines and depends only on \( \rho_p := \rho|_{\text{Gal}(\overline{F}/F_{wp})} \). The case \( n = 2 \) is well-known essentially due to various results in [Col10], [Eme]. The case \( n \geq 3 \) is much more difficult and only some partial results are known. We are particularly interested in the case when the subspace of locally algebraic vectors \( \tilde{S}(U^p, E)[\mathfrak{m}_\rho]^{alg} \subseteq \tilde{S}(U^p, E)[\mathfrak{m}_\rho] \) is non-zero, which implies that \( \rho_p \) is potentially semi-stable. Certain cases when \( n = 3 \) and \( \rho_p \) is semi-stable and non-crystalline have been studied in [Bre17] and [BD18]. We are going to continue their work and obtain some interesting relation between results in [Bre17], [BD18] and previous results in [Schr11] which involve the \( p \)-adic dilogarithm function.

We use the notation \( \lambda \in X(T)_+ \) for a weight \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) (of the diagonal split torus \( T \) of \( \text{GL}_3 \)) which is dominant with respect to the upper-triangular Borel subgroup \( B \) and hence satisfies \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \). Given two locally analytic representations \( V, W \) of \( \text{GL}_3(\mathbb{Q}_p) \), we use the short notation \( V \twoheadrightarrow W \) (resp. the shorten notation \( V \twoheadrightarrow W \)) for a locally analytic representation determined by a non-zero (resp. possibly zero) element in \( \text{Ext}^{\lambda}_{\text{GL}_3(\mathbb{Q}_p)}(V, W) \).

**Theorem 1.1.** ([Proposition 6.29]) For each choice of \( \lambda \in X(T)_+ \) and \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E \), there exists a locally analytic representation \( \Sigma^{\min}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3} \) of \( \text{GL}_3(\mathbb{Q}_p) \) of the form:

\[
\begin{align*}
\begin{array}{c}
\text{St}^{an}_3(\lambda) \\
\rho_1 \mapsto C_{s_1, s_1} & \quad \mathcal{T}(\lambda) \otimes_E v_{P_1}^{\infty} \\
\rho_2 \mapsto C_{s_2, s_2} & \quad \mathcal{T}(\lambda) \otimes_E v_{P_2}^{\infty}
\end{array}
\end{align*}
\]

where \( \text{St}^{an}_3(\lambda) \), \( v_{P_1}^{\infty}(\lambda) \), \( v_{P_2}^{\infty}(\lambda) \), \( \mathcal{T}(\lambda) \) and \( C_{w', w} \) for \( w, w' \in \{s_1, s_2, s_1s_2, s_2s_1\} \) and \( * \in \{\varnothing, 1, 2\} \) are various explicit locally analytic representations defined in Section 2.7. Moreover, different choices of \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E \) give non-isomorphic representations.

We will see in Lemma 6.31 and 6.55 that \( \Sigma^{\min}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3} \) is the minimal locally analytic representation that involves \( p \)-adic dilogarithm, hence explains the notation ‘min’. We also construct
a locally analytic representation \( \Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) of the form

\[
\begin{array}{c}
\text{St}^m_3(\lambda) \\
\downarrow \quad \downarrow \quad \downarrow \\
C_{s_1,s_1} \\
\vdots \\
C_{s_2,s_2} \\
\vdots \\
C_{s_3,s_3} \\
\end{array}
\]

which contains and is uniquely determined by \( \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \).

**Theorem 1.3.** (Theorem 7.5) Assume that \( p \geq 5 \) and \( n = 3 \). Assume moreover that

(i) \( \rho \) is unramified at all finite places of \( F \) above \( D(U^p) \);
(ii) \( \widehat{S}(U^p, E)[m_p]^{alg} \neq 0 \);
(iii) \( \rho_p \) is semi-stable with Hodge–Tate weights \( \{k_1 > k_2 > k_3\} \) such that \( N^2 \neq 0 \);
(iv) \( \rho_p \) is non-critical in the sense of Remark 6.1.4 of [Bre17];
(v) only one automorphic representation contributes to \( \widehat{S}(U^p, E)[m_p]^{alg} \).

Then there exists a unique choice of \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E \) such that \( \widehat{S}(U^p, E)[m_p]^{an} \) contains (copies of) the locally analytic representation

\[
\Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \otimes_E (ur(\alpha) \otimes_E \varepsilon^2) \circ \det
\]

where \( \lambda = (\lambda_1, \lambda_2, \lambda_3) = (k_1 - 2, k_2 - 1, k_3) \) and \( \alpha \in E^\times \) is determined by the Weil–Deligne representation \( WD(\rho_p) \) associated with \( \rho_p \). Moreover, we have

\[
\text{Hom}_{GL_3(Q_p)} \left( \Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \otimes_E (ur(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{an}[m_p] \right) \\
\cong \text{Hom}_{GL_3(Q_p)} \left( \mathcal{T}(\lambda) \otimes_E \text{St}^\infty_3 \otimes_E (ur(\alpha) \otimes_E \varepsilon^2) \circ \det, \widehat{S}(U^p, E)^{an}[m_p] \right).
\]

The assumptions of our Theorem 1.3 are the same as that of Theorem 1.3 of [Bre17]. We do not attempt to obtain any explicit relation between \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E \) and \( \rho_p \), which is similar in flavor to Theorem 1.3 of [Bre17]. On the other hand, Theorem 7.52 of [BD18] does care about the explicit relation between invariants of the locally analytic representation associated with \( \rho_p \), under further technical assumptions such as \( \rho_p \) is ordinary with consecutive Hodge–Tate weights and has an irreducible mod \( p \) reduction but without assuming our condition (v). The improvement of our Theorem 1.3 upon Theorem 1.3 of [Bre17] will be explained in Section 1.2. One can naturally wish that there is a common refinement or generalization of our Theorem 1.3 and Theorem 7.52 of [BD18] by removing as many technical assumptions as possible.

**Remark 1.5.** It is actually possible to construct a locally analytic representation \( \Sigma^{\max}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) of \( GL_3(Q_p) \) containing \( \Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) which is characterized by the fact that it is maximal (for inclusion) among the locally analytic representations \( V \) satisfying the following conditions:

(i) \( \text{soc}_{GL_3(Q_p)}(V) = V^{alg} = \mathcal{T}(\lambda) \otimes_E \text{St}^\infty_3 \);
(ii) each constituent of \( V \) is a subquotient of a locally analytic principal series

where \( V^{alg} \) is the subspace of locally algebraic vectors in \( V \). Moreover, one can use an immediate generalization of the arguments in the proof of Theorem 1.3 (and thus of Theorem 1.1 of [Bre17]) to
show that

\[
(1.6) \quad \text{Hom}_{\text{GL}_3(\mathbb{Q}_p)} \left( \Sigma^\max(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \otimes_E (\text{ur}(\alpha) \otimes E \, \varepsilon^2) \circ \det, \tilde{S}(U^p, E)\text{an}[m_p] \right) \\
\tilde{\sim} \quad \text{Hom}_{\text{GL}_3(\mathbb{Q}_p)} \left( \mathcal{T}(\lambda) \otimes E \, \text{St}_3^\infty \otimes_E (\text{ur}(\alpha) \otimes E \, \varepsilon^2) \circ \det, \tilde{S}(U^p, E)\text{an}[m_p] \right).
\]

We can also show that

\[
\Sigma^\max(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)/\mathcal{T}(\lambda) \otimes E \, \text{St}_3
\]

is independent of the choice of \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E \), which is compatible with the fact that

\[
\Sigma^{\min,*}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)/\mathcal{T}(\lambda) \otimes E \, \text{St}_3
\]

is independent of the choice of \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E \) for each \( * \in \{ \emptyset, + \} \) as mentioned in Remark \ref{rem:independence}. However, the full construction of \( \Sigma^\max(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) is lengthy and technical and thus we decided not to put it in the present article.

1.1. Derived object and dilogarithm. We consider the bounded derived category

\[
\mathcal{D}^b \left( \text{Mod}_{\text{GL}_3(\mathbb{Q}_p), E} \right)
\]

associated with the abelian category \( \text{Mod}_{\text{GL}_3(\mathbb{Q}_p), E} \) of abstract modules over the algebra \( D(\text{GL}_3(\mathbb{Q}_p), E) \) of locally \( \mathbb{Q}_p \)-analytic distributions on \( \text{GL}_3(\mathbb{Q}_p) \). An object

\[
\Sigma(\lambda, \mathcal{L}') \in \mathcal{D}^b \left( \text{Mod}_{\text{GL}_3(\mathbb{Q}_p), E} \right)
\]

(one should not confuse this notation \( \Sigma(\lambda, \mathcal{L}') \) borrowed directly from \cite{Schr11} with our notation \( \Sigma^+(\lambda, \mathcal{L}) \) before Lemma \ref{lemma:key-role}) has been constructed in \cite{Schr11} and plays a key role in Theorem 1.2 of \cite{Schr11}. An interesting feature of \cite{Schr11} is the appearance of the \( p \)-adic dilogarithm function in the construction of \( \Sigma(\lambda, \mathcal{L}') \) in Definition 5.19 of \cite{Schr11}. Roughly, the object \( \Sigma(\lambda, \mathcal{L}') \) was constructed from the choice of an element in \( \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \) together with general formal arguments in triangulated categories (cf. Proposition 3.2 of \cite{Schr11}). In particular, \( \Sigma(\lambda, \mathcal{L}') \) fits into the following distinguished triangle:

\[
F'_\lambda \longrightarrow \Sigma(\lambda, \mathcal{L}') \longrightarrow \Sigma(\lambda, \mathcal{L}, \mathcal{L}')[-1] \overset{+1}{\longrightarrow}
\]

as illustrated in (5.99) of \cite{Schr11}. However, it was not clear in \cite{Schr11} whether there is an explicit complex \( [C_\bullet] \) of locally analytic representations of \( \text{GL}_3(\mathbb{Q}_p) \) such that the object

\[
\mathcal{D}' \in \mathcal{D}^b \left( \text{Mod}_{\text{GL}_3(\mathbb{Q}_p), E} \right)
\]

associated with \( [C'_\bullet] \) satisfies

\[
\mathcal{D}' \cong \Sigma(\lambda, \mathcal{L}') \in \mathcal{D}^b \left( \text{Mod}_{\text{GL}_3(\mathbb{Q}_p), E} \right).
\]

Although our notation are slightly different from \cite{Schr11} in the sense that the notation \( \Sigma(\lambda, \mathcal{L}, \mathcal{L}') \) (resp. the notation \( F_\lambda \)) is replaced with \( \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) (resp. with \( \mathcal{T}(\lambda) \)), we show that

Theorem 1.7. \cite{proposition:6.36, 2.24} and Lemma \ref{lemma:7.15} The complex

\[
(1.8) \quad \left[ \left( \mathcal{T}(\lambda) \otimes_E v_{\mathcal{F}_{3-i}}^\infty \mathcal{T}(\lambda) \right) \longrightarrow \Sigma^+_i(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \right]
\]

realizes the object \( \Sigma(\lambda, \mathcal{L}') \) where \( \mathcal{T}(\lambda) \otimes_E v_{\mathcal{F}_{3-i}}^\infty \mathcal{T}(\lambda) \) is the unique non-split extension of \( \mathcal{T}(\lambda) \otimes_E v_{\mathcal{F}_{3-i}}^\infty \) by \( \mathcal{T}(\lambda) \) thanks to Proposition \ref{proposition:7.3} \( \Sigma^+_i(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) is the locally analytic subrepresentation...
of $\Sigma^{\text{min}}(\lambda, L_1, L_2, L_3)$ of the form

$$
\begin{align*}
\text{St}_3^\text{an}(\lambda) & \longrightarrow v_{P_3}^\text{an}(\lambda) & \longrightarrow \mathcal{T}(\lambda) \otimes_E v_{P_3}^\infty \\
L_1 & \longrightarrow \mathcal{T}(\lambda) & \longrightarrow \mathcal{L}(\lambda) \\
L_2 & \longrightarrow \mathcal{L}(\lambda) & \longrightarrow \mathcal{L}(\lambda)
\end{align*}
$$

and the invariants $L_1, L_2, L_3 \in E$ are determined by the formula

$$
L_1 = -L', \quad L_2 = -L, \quad L_3 = \gamma(L'' - \frac{1}{2} L L')
$$

with the constant $\gamma \in E^\times$ defined in Lemma \ref{lem:gamma}.

**Remark 1.9.** Strictly speaking, the complex \((1.8)\) realizes an object in $D^b(\text{Mod}_{D(\text{GL}_3(Q_p), \lambda)})$ characterized by an element in

$$
\text{Ext}_{\text{GL}_3(Q_p), \lambda}^2(\mathcal{T}(\lambda), \Sigma^{\Sigma^\prime}(\lambda, L_1, L_2))
$$

due to formal arguments from Proposition 3.2 of \cite{Schr1}. However, we can prove that there is a canonical isomorphism

$$
\text{Ext}_{\text{GL}_3(Q_p), \lambda}^2(\mathcal{T}(\lambda), \Sigma(\lambda, L_1, L_2)) \cong \text{Ext}_{\text{GL}_3(Q_p), \lambda}^2(\mathcal{T}(\lambda), \Sigma^{\Sigma^\prime}(\lambda, L_1, L_2))
$$

and hence we can equally say that \((1.8)\) realizes $\Sigma(\lambda, L_1, L_2)$ for a suitable normalization of notation as $\Sigma(\lambda, L)$ has been constructed by choosing a non-zero element in $\text{Ext}_{\text{GL}_3(Q_p), \lambda}^2(\mathcal{T}(\lambda), \Sigma(\lambda, L, L'))$ via Proposition 3.2 of \cite{Schr1}. Note that we have

$$
\Sigma(\lambda, L_1, L_2) \cong \Sigma(\lambda, L_1, L_2)
$$

by \(\cite{Za1}\).

**1.2. Higher $L$-invariants for GL$_3(Q_p)$.** It follows from \(\cite{Za1}\) and \(\cite{Za2}\) that $\Sigma^{\Sigma^\prime}(\lambda, L_1, L_2, L_3)$ can be described more precisely by the following picture:
and therefore contains a unique subrepresentation of the form

\[
\begin{array}{c}
L(\lambda) \otimes E \text{St}_3^\infty \\
\begin{array}{c}
\begin{array}{c}
C_{s_2, s_1, 1} \\
C_{s_2, 1}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_{s_1, s_1, 1} \\
C_{s_1, 1}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_{s_2, s_1, s_2, 1} \\
C_{s_2, s_1, s_2}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_{s_1, s_1, s_2, 1} \\
C_{s_1, s_1, s_2}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_{s_2, s_1, s_2, s_2, 1} \\
C_{s_2, s_1, s_2, s_2}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_{s_1, s_1, s_2, s_2, s_2, 1} \\
C_{s_1, s_1, s_2, s_2, s_2}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

which is denoted by

\[(1.10) \quad \begin{array}{c}
T(\lambda) \otimes E \text{St}_3^\infty \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\Pi^1(k, D) \\
\Pi^2(k, D)
\end{array}
\end{array}
\end{array}
\end{array}
\]

in Theorem 1.1 of [Bre17]. It follows from Theorem 1.2 of [Bre17] that

\[\dim_{E} \text{Ext}^{1}_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left(\Pi^1(k, D), T(\lambda) \otimes E \text{St}_3^\infty\right) = 3\]

for \(i = 1, 2\), and therefore a locally analytic representation of the form \((1.10)\) depends on four invariants. On the other hand, by a computation of extensions of rank one \((\varphi, \Gamma)\)-modules we know that \(\rho_p\) depends on three invariants. As a result, Theorem 1.1 of [Bre17] predicts that not all representations of the form \((1.10)\) can be embedded into \(\hat{S}(U^p, E)^{an}[m_p]\) for a certain pair of \(U^p\) and \(\rho_p\). This is actually the case as we show that

**Theorem 1.11.** [Corollary 7.17] If a locally analytic representation \(\Pi\) of the form \((1.10)\) can be embedded into \(\hat{S}(U^p, E)^{an}[m_p]\) for a certain pair of \(U^p\) and \(\rho_p\), then it can be embedded into

\[\Sigma^{\min, +}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)\]

for a unique choice of \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E\) determined by \(\Pi\).

### 1.3. Sketch of content

Section 2 recalls various well-known facts around locally analytic representations and our notation for a family of specific irreducible subquotients of locally analytic principal series to be used in the rest of the article. We emphasize that our definition of various Ext-groups follows [Bre17] closely and the only difference is that we use the dual notation compared to that of [Bre17]. We also recall the \(p\)-adic dilogarithm function from Section 5.3 of [Schr11] which is part of the main motivation of this article to relate [Schr11] with [Bre17] and [BD18].

Section 3 proves a crucial fact (Proposition 6.14) on the non-existence of locally analytic representations of \(\text{GL}_2(\mathbb{Q}_p)\) of a certain specific form using arguments involving infinitesimal characters of locally analytic representations. We learn such arguments essentially from Y. Ding.

Section 4 is a collection of various computational results necessary for the applications in Section 6. These computations essentially make use of the formula in Section 5.2 and 5.3 of [Bre17].

Section 5 serves as the preparation of Section 6 for the construction of \(\Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)\). It makes full use of the computational results from Section 4 to compute the dimension of various more complicated Ext-groups to be crucially used in various important long exact sequences in Section 6 cf. Lemma 6.4 and Proposition 6.8.

Section 6 finishes the construction of \(\Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)\) as well as \(\Sigma^{\min, +}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)\). Moreover, the construction of \(\Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)\) leads naturally to the construction of an explicit complex as in Theorem 1.7 that realizes the derived object \(\Sigma(\lambda, \mathcal{L})\) constructed in [Schr11].

Section 7 finishes the proof of Theorem 7.5 by directly mimicking arguments from the proof of Theorem 6.2.1 of [Bre17]. In particular, we give a purely representation theoretic criterion for a representation of the form \((1.10)\) to embed into completed cohomology as mentioned in Theorem 1.11.
1.4. Acknowledgement. The author expresses his gratefulness to Christophe Breuil for introducing the problem of relating [Schr11] with [Bre17] and especially for his interest on the role played by the $p$-adic dilogarithm function. The author also benefited a lot from countless discussions with Y. Ding especially for Section 3 of this article. Finally, the author thanks B. Schraen for his beautiful thesis which improved the author’s understanding on the subject.

2. Preliminary

2.1. Locally analytic representations. In this section, we recall the definition of some well-known objects in the theory of locally analytic representations of $p$-adic reductive groups.

We fix a locally $\mathbb{Q}_p$-analytic group $H$ and denote the algebra of locally $\mathbb{Q}_p$-analytic distribution with coefficient $E$ on $H$ by $\mathcal{D}(H, E)$, which is defined as the strong dual of the locally convex $E$-vector space $C_c^\infty(H, E)$ consisting of locally $\mathbb{Q}_p$-analytic functions on $H$. We use the notation $\text{Rep}^\text{la}_{H, E}$ for the abelian category $\text{Mod}_{\text{D}(H, E)}$ of abstract modules over $\mathcal{D}(H, E)$. The $E$-vector space $\text{Ext}^i_{\mathcal{D}(H, E)}(M_1, M_2)$ is well-defined for any two objects $M_1, M_2 \in \text{Mod}_{\mathcal{D}(H, E)}$, and therefore we define

$$\text{Ext}^i_{\mathcal{D}(H, E)}(\Pi_1, \Pi_2) := \text{Ext}^i_{\mathcal{D}(H, E)}(\Pi'_2, \Pi'_1)$$

for any two objects $\Pi_1, \Pi_2 \in \text{Rep}^\text{la}_{H, E}$ where $'$ is the notation for strong dual. We also define the cohomology of an object $M \in \text{Mod}_{\mathcal{D}(H, E)}$ by

$$H^i(H, M) := \text{Ext}^i_{\mathcal{D}(H, E)}(1, M)$$

where $1$ is the strong dual of the trivial representation of $H$. If $H'$ is a closed locally $\mathbb{Q}_p$-analytic normal subgroup of $H$, then $H/H'$ is also a locally $\mathbb{Q}_p$-analytic group. It follows from the fact

$$\mathcal{D}(H, E) \otimes_{\mathcal{D}(H', E)} E \cong \mathcal{D}(H/H', E)$$

(see Section 5.1 of [Bre17] for example) that $H^i(H', M)$ admits a structure of $\mathcal{D}(H/H', E)$-module for each $M \in \text{Mod}_{\mathcal{D}(H, E)}$. We define the $H'$-homology of $\Pi \in \text{Rep}^\text{la}_{H, E}$ as the object (if it exists up to isomorphism) $H_i(H', \Pi) \in \text{Rep}^\text{la}_{H/H', E}$ such that

$$H_i(H', \Pi') \cong H^i(H', \Pi').$$

We emphasize that $H_i(H', \Pi)$ is well defined in the sense above only after we know its existence. We fix a subgroup $Z$ of the center of the group $H$, then the algebra $\mathcal{D}(Z, E)$ consists of locally $\mathbb{Q}_p$-analytic distribution with coefficient $E$ on $Z$ is naturally contained in the center of $\mathcal{D}(H, E)$. For each locally $\mathbb{Q}_p$-analytic $E$-character $\chi$ of $Z$, we can define the abelian subcategory $\text{Mod}_{\mathcal{D}(H, E), \chi}$ consisting of all the objects in $\text{Mod}_{\mathcal{D}(H, E)}$ on which $\mathcal{D}(Z, E)$ acts by $\chi$. Then we consider the functors $\text{Ext}^i_{\mathcal{D}(H, E)}(-, -)$ defined as $\text{Ext}^i_{\text{Mod}_{\mathcal{D}(H, E), \chi}}(-, -)$ which are extensions inside the abelian category $\text{Mod}_{\mathcal{D}(H, E), \chi'}$. Consequently we can define

$$\text{Ext}^i_{\mathcal{D}(H, E), \chi}(\Pi_1, \Pi_2) := \text{Ext}^i_{\mathcal{D}(H, E), \chi}(\Pi'_2, \Pi'_1)$$

for any two objects $\Pi_1, \Pi_2 \in \text{Rep}^\text{la}_{H, E}$ such that $\Pi'_1, \Pi'_2 \in \text{Mod}_{\mathcal{D}(H, E), \chi'}$. In particular, if $Z$ is the center of $H$ and acts on $\Pi \in \text{Rep}^\text{la}_{H, E}$ via the character $\chi$, then $\Pi' \in \text{Mod}_{\mathcal{D}(H, E), \chi'}$, and we usually say that $\Pi$ admits a central character $\chi$.

Assume now $H$ is the set of $\mathbb{Q}_p$-points of a split reductive group over $\mathbb{Q}_p$. We recall the category $\mathcal{O}$ together with its subcategory $\mathcal{O}^\text{alg}_{\text{alb}}$ for each parabolic subgroup $P \subseteq H$ from Section 9.3 of [Hum08] or [OS15]. The construction by Orlik–Strauch in [OS15] gives us a functor

$$\mathcal{F}^P : \mathcal{O}^\text{alg}_{\text{alb}} \times \text{Rep}^\infty_{H, E} \to \text{Rep}^\text{la}_{P, H, E}$$

for each parabolic subgroup $P \subseteq H$ with Levi quotient $L$. We use the notation $\text{Rep}^\text{la}_{H, E}$ for the abelian full subcategory of $\text{Rep}^\text{la}_{H, E}$ generated by the image of $\mathcal{F}^P$ when $P$ varies over all possible parabolic.
subgroups of $H$. Here we say a full subcategory is generated by a family of objects if it is the minimal full subcategory that contains these objects and is stable under extensions.

2.2. Formal properties. In this section, we recall and prove some general formal properties of locally analytic representations of $p$-adic reductive groups.

We fix a split $p$-adic reductive group $H$ and a parabolic subgroup $P$ of $H$. We use the notation $N$ for the unipotent radical of $P$ and fix a Levi subgroup $L$ of $P$.

Lemma 2.1. We have a spectral sequence

$$\text{Ext}^j_{L,*}(H_k(N, \Pi_1), \Pi_2) \Rightarrow \text{Ext}^{j+k}_{H,*}(\Pi_1, \text{Ind}^H_P(\Pi_2)^{an})$$

which implies an isomorphism

$$\text{Hom}_{L,*}(H_0(N, \Pi_1), \Pi_2) \cong \text{Hom}_{H,*}(\Pi_1, \text{Ind}^H_P(\Pi_2)^{an})$$

and a long exact sequence

$$\text{Ext}^1_{L,*}(H_0(N, \Pi_1), \Pi_2) \rightarrow \text{Ext}^1_{H,*}(\Pi_1, \text{Ind}^H_P(\Pi_2)^{an})$$

$$\rightarrow \text{Hom}_{L,*}(H_1(N, \Pi_1), \Pi_2) \rightarrow \text{Ext}^2_{L,*}(H_0(N, \Pi_1), \Pi_2)$$

for each $\Pi_1 \in \text{Rep}^{la}_{H,E}$, $\Pi_2 \in \text{Rep}^{la}_{L,E}$ satisfying the (FIN) condition in Section 6 of [ST05], $* \in \{\emptyset, \chi\}$ where $\chi$ is a locally analytic character of the center of $H$.

Proof. This follows directly from our definition of $\text{Ext}^k$ and $H_k$ in Section 2.1 for $k \geq 0$, the original dual version in (44) and (45) of [Bre17].

We fix a Borel subgroup $B \subseteq H$ together with its opposite Borel subgroup $\overline{B}$. We fix an irreducible object $M \in O^\infty_{alg}$. We choose a parabolic subgroup $P \subseteq H$ such that $P$ is maximal among all the parabolic subgroups $Q \subseteq H$ such that $M \in O^\infty_{alg}$, where $\mathfrak{b}$ is the Lie algebra of the opposite parabolic subgroup $\overline{Q}$ associated with $Q$. We fix a smooth irreducible representation $\pi^\infty$ of $L$ and a smooth character $\delta$ of $H$. We know that [OS15] constructed an irreducible locally analytic representation

$$\mathcal{F}^H_P(M, \pi^\infty)$$

of $H$.

Lemma 2.2. The functor

$$- \otimes_E \delta$$

induces an equivalence of category from $\text{Rep}^{la}_{P,H,E}$ to itself. Moreover, the restriction of $- \otimes_E \delta$ to the subcategory $\text{Rep}^{OS}_{H,E}$ is again an equivalence of category to itself and satisfies

$$(2.3) \quad \mathcal{F}^H_P(M, \pi^\infty) \otimes_E \delta \cong \mathcal{F}^H_P(M, \pi^\infty \otimes_E \delta|_L)$$

for each irreducible object $\mathcal{F}^H_P(M, \pi^\infty) \in \text{Rep}^{OS}_{H,E}$.

Proof. The functor $- \otimes_E \delta$ is clearly an equivalence of category from $\text{Rep}^{la}_{P,H,E}$ to itself with quasi-inverse given by

$$- \otimes_E \delta^{-1}.$$

It is sufficient to prove the formula (2.3) to finish the proof. First of all, we notice by formal reason (equivalence of category) that $\mathcal{F}^H_P(M, \pi^\infty) \otimes_E \delta$ is an irreducible object in $\text{Rep}^{la}_{P,H,E}$ since $\mathcal{F}^H_P(M, \pi^\infty)$ is. We use the notation $\mathfrak{m}$ for the Lie algebra associated with the unipotent radical $\mathfrak{N}$ of the opposite parabolic subgroup $\overline{P}$ of $P$. We define $M_L$ as the (finite dimensional) algebraic representation of $L$ whose dual is isomorphic to $M^\mathfrak{m}$ as a representation of $\mathfrak{t}$ and note that we have a surjection

$$U(\mathfrak{h}) \otimes_{U(\mathfrak{t})} M^\mathfrak{m} \rightarrow M.$$
We observe that $N$ acts trivially on $\delta$, and therefore we have
\[ H_0(N, \mathcal{F}_P^H(M, \pi^\infty) \otimes E \delta) \cong H_0(N, \mathcal{F}_P^H(M, \pi^\infty)) \otimes E \delta|_L \to M_L \otimes E \pi^\infty \otimes E \delta|_L \]
which induces by Lemma 2.1 a non-zero morphism
\[ (2.4) \quad \mathcal{F}_P^H(M, \pi^\infty) \otimes E \delta \to \text{Ind}_{P}^H(M_L \otimes E \pi^\infty \otimes E \delta|_L) \cong \mathcal{F}_P^H(U(h) \otimes U(\mathcal{F}) M^\mathcal{P}, \pi^\infty \otimes E \delta|_L). \]
We finish the proof by the fact that $\mathcal{F}_P^H(M, \pi^\infty) \otimes E \delta$ is irreducible and that
\[ \mathcal{F}_P^H(M, \pi^\infty \otimes E \delta|_L) \cong \text{soc}_H \left( \mathcal{F}_P^H(U(h) \otimes U(\mathcal{F}) M^\mathcal{P}, \pi^\infty \otimes E \delta|_L) \right). \]
due to Corollary 3.3 of [Bre16]. □

We fix a finite length locally analytic representation $V \in \text{Rep}^la_H,E$ equipped with a increasing filtration of subrepresentations \{Fil$_kV\}_{0 \leq k \leq m}$ such that
\[ \text{Fil}_0(V) = 0, \text{Fil}_m(V) = V \text{ and } \text{gr}_{k+1}V := \text{Fil}_{k+1}V/\text{Fil}_kV \neq 0 \text{ for all } 0 \leq k \leq m - 1. \]
Note that the assumption above automatically implies that
\[ \ell(V) \geq m \]
where $\ell(V)$ is the length of $V$.

**Proposition 2.5.** Assume that $W$ is another object of $\text{Rep}^la_H,E$ and $\chi$ is a locally analytic character of the center of $H$.

(i) If $\text{Ext}_{H,\chi}^1(W, \text{gr}_kV) = 0$ for each $1 \leq k \leq m$, then we have
\[ \text{Ext}_{H,\chi}^1(W, V) = 0. \]

(ii) If there exists $1 \leq k_0 \leq m$ such that $\text{Ext}_{H,\chi}^1(W, \text{gr}_kV) = 0$ for each $1 \leq k \neq k_0 \leq m$ and $\dim E\text{Ext}_{H,\chi}^1(W, \text{gr}_{k_0}V) = 1$, then we have
\[ \dim E\text{Ext}_{H,\chi}^1(W, V) \leq 1; \]
if moreover $\text{Ext}_{H,\chi}^2(W, \text{gr}_kV) = 0$ for each $1 \leq k \leq k_0 - 1$ and $\text{Hom}_{H,\chi}(W, \text{gr}_kV) = 0$ for each $k_0 + 1 \leq k \leq m$, then we have
\[ \dim E\text{Ext}_{H,\chi}^1(W, V) = 1. \]

**Proof.** The short exact sequence $\text{Fil}_kV \to \text{Fil}_{k+1}V \to \text{gr}_{k+1}V$ induces a long exact sequence
\[ \text{Ext}_{H,\chi}^1(W, \text{Fil}_kV) \to \text{Ext}_{H,\chi}^1(W, \text{Fil}_{k+1}V) \to \text{Ext}_{H,\chi}^1(W, \text{gr}_{k+1}V) \]
which implies
\[ \dim E\text{Ext}_{H,\chi}^1(W, \text{Fil}_{k+1}V) \leq \dim E\text{Ext}_{H,\chi}^1(W, \text{Fil}_kV) + \dim E\text{Ext}_{H,\chi}^1(W, \text{gr}_{k+1}V). \]
Therefore we finish the proof of part (i) and the first claim of part (ii) by induction on $k$ and the fact that $\text{gr}_1V = \text{Fil}_1V$.

It remains to show the second claim of part (ii). The same method as in the proof of part (i) shows that
\[ (2.6) \quad \text{Ext}_{H,\chi}^1(W, \text{Fil}_{k_0-1}V) = \text{Ext}_{H,\chi}^2(W, \text{Fil}_{k_0-1}V) = 0 \]
and
\[ (2.7) \quad \text{Ext}_{H,\chi}^1(W, V/\text{Fil}_{k_0}V) = \text{Hom}_{H,\chi}(W, V/\text{Fil}_{k_0}V) = 0 \]
The short exact sequence $\text{Fil}_{k_0-1}V \to \text{Fil}_{k_0}V \to \text{gr}_{k_0}V$ induces the long exact sequence
\[ \text{Ext}_{H,\chi}^1(W, \text{Fil}_{k_0-1}V) \to \text{Ext}_{H,\chi}^1(W, \text{Fil}_{k_0}V) \to \text{Ext}_{H,\chi}^1(W, \text{gr}_{k_0}V) \to \text{Ext}_{H,\chi}^2(W, \text{Fil}_{k_0-1}V) \]
which implies that
\[ (2.8) \quad \dim E\text{Ext}_{H,\chi}^1(W, \text{Fil}_{k_0}V) = 1 \]
by [2.7]. The short exact sequence \( \text{Fil}_{k_0} V \rightarrow V \rightarrow V/\text{Fil}_{k_0} V \) induces the long exact sequence

\[
\text{Hom}_{H,X}(W, V/\text{Fil}_{k_0} V) \rightarrow \text{Ext}^1_{H,X}(W, \text{Fil}_{k_0} V) \rightarrow \text{Ext}^1_{H,X}(W, V) \rightarrow \text{Ext}^1_{H,X}(W, V/\text{Fil}_{k_0} V)
\]

which finishes the proof by combining [2.7] and [2.8]. \(\square\)

2.3. Some notation. In this section, we are going to recall some standard notation for the \(p\)-adic reductive groups \(\text{GL}_2(\mathbb{Q}_p)\) and \(\text{GL}_3(\mathbb{Q}_p)\) as well as notation for some locally analytic representations of these groups.

We denote the lower-triangular Borel subgroup (resp. the diagonal maximal split torus) of \(\text{GL}_2(\mathbb{Q}_p)\) by \(B_2\) (resp. by \(T_2\)) and the unipotent radical of \(B_2\) by \(N_{\text{GL}_2}\). We use the notation \(s\) for the non-trivial element in the Weyl group of \(\text{GL}_2\). We fix a weight \(\nu \in \mathcal{X}(T_2)\) of \(\text{GL}_2\) of the following form

\[
\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2
\]

which corresponds to an algebraic character of \(T_2(\mathbb{Q}_p)\)

\[
\delta_{T_2, \nu} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^{\nu_1} b^{\nu_2}.
\]

We denote the upper-triangular Borel subgroup by \(B_2^\circ\). If \(\nu\) is dominant with respect to \(\mathcal{T}_2\), namely if \(\nu_1 \geq \nu_2\), we use the notation \(\mathcal{T}_{\text{GL}_2}(\nu)\) (resp. \(L_{\text{GL}_2}(-\nu)\)) for the irreducible algebraic representation of \(\text{GL}_2(\mathbb{Q}_p)\) with highest weight \(\nu\) (resp. \(-\nu\)) with respect to the positive roots determined by \(\mathcal{T}_2\) (resp. \(B_2^\circ\)). In particular, \(\mathcal{T}_{\text{GL}_2}(\nu)\) and \(L_{\text{GL}_2}(-\nu)\) are the dual of each other. We use the shortening notation

\[
I_{B_2}^{\text{GL}_2}(\chi_{T_2}) := \left(\text{Ind}_{B_2(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)} \chi_{T_2}\right)^{\text{an}}
\]

for any locally analytic character \(\chi_{T_2}\) of \(T_2(\mathbb{Q}_p)\) and set

\[
i_{B_2}^{\text{GL}_2}(\chi_{T_2}) := \left(\text{Ind}_{B_2(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)} \chi_{T_2}\right)^{\infty} \otimes_{E} \mathcal{T}_{\text{GL}_2}^{\text{an}}(\nu)
\]

if \(\chi_{T_2} = \delta_{T_2, \nu} \otimes E \chi_{T_2}^{\infty}\) is locally algebraic where \(\chi_{T_2}^{\infty}\) is a smooth character of \(T_2(\mathbb{Q}_p)\). Then we define the locally analytic Steinberg representation as well as the smooth Steinberg representation for \(\text{GL}_2(\mathbb{Q}_p)\) as follows

\[\text{St}_{2}^{\text{an}}(\nu) := I_{B_2}^{\text{GL}_2}(\delta_{T_2, \nu})/\mathcal{T}_{\text{GL}_2}^{\text{an}}(\nu), \text{ St}_{2}^{\text{inf}} := i_{B_2}^{\text{GL}_2}(1_{T_2})/1_{2}\]

where \(1_2 (\text{resp. } 1_{T_2})\) is the trivial representation of \(\text{GL}_2(\mathbb{Q}_p)\) (resp. of \(T_2(\mathbb{Q}_p)\)).

We denote the lower-triangular Borel subgroup (resp. the diagonal maximal split torus) of \(\text{GL}_3(\mathbb{Q}_p)\) by \(B\) (resp. by \(T\)) and the unipotent radical of \(B\) by \(N\). We fix a weight \(\lambda \in \mathcal{X}(T)\) of \(\text{GL}_3\) of the following form

\[
\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3,
\]

which corresponds to an algebraic character of \(T(\mathbb{Q}_p)\)

\[
\delta_{T, \lambda} := \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mapsto a^{\lambda_1} b^{\lambda_2} c^{\lambda_3}.
\]

We denote the center of \(\text{GL}_3\) by \(Z\) and notice that \(Z(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times\). Hence the restriction of \(\delta_{T, \lambda}\) to \(Z(\mathbb{Q}_p)\) gives an algebraic character of \(Z(\mathbb{Q}_p)\):

\[
\delta_{Z, \lambda} := \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mapsto a^{\lambda_1+\lambda_2+\lambda_3}.
\]

We use the shortening notation

\[
\text{Ext}_*^{i, \lambda}(\cdot, \cdot) := \text{Ext}_*^{i, \delta_{Z, \lambda}}(\cdot, \cdot)
\]

for \(* \in \{ T(\mathbb{Q}_p), L_1(\mathbb{Q}_p), L_2(\mathbb{Q}_p), \text{GL}_3(\mathbb{Q}_p) \}\). In particular, the notation

\[
\text{Ext}_*^{i, 0}(\cdot, \cdot)
\]
Similarly, we use the notation $L(-\lambda)$ for the irreducible algebraic representation of $GL_3(\mathbb{Q}_p)$ with highest weight $\lambda$ (resp. $-\lambda$) with respect to the positive roots determined by $B$ (resp. $B$). In particular, $L(\lambda)$ and $L(-\lambda)$ are dual of each other. We use the notation $P_1 := \begin{pmatrix} 1 & * & 0 \\ * & 1 & 0 \\ * & * & 0 \end{pmatrix}$ and $P_2 := \begin{pmatrix} * & * & 0 \\ * & 1 & 0 \\ 1 & * & * \end{pmatrix}$ for the two standard maximal parabolic subgroups of $GL_3$ with unipotent radical $N_1$ and $N_2$ respectively, and the notation $P_i$ for the opposite parabolic subgroup of $P_i$ for $i = 1, 2$. We set

$$L_i := P_i \cap P_i$$

and set $s_i$ for the simple reflection in the Weyl group of $L_i$ for each $i = 1, 2$. In particular, the Weyl group $W$ of $GL_3$ can be lifted to a subgroup of $GL_3$ with the following elements

$$\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}.$$

We will usually use the short notation $N_i$ (cf. Section IX for its set of $\mathbb{Q}_p$-points $N_i(\mathbb{Q}_p)$ if it does not cause any ambiguity. We use the notation $M(-\lambda)$ for the Verma module in $O^b_{-\lambda}$ with highest weight $-\lambda$ (with respect to $B$) and simple quotient $L(-\lambda)$ for each $\lambda \in X(T)$ (not necessarily dominant). Similarly, we use the notation $M_i(-\lambda)$ for the parabolic Verma module in $O^b_{-\lambda}$ with highest weight $-\lambda$ with respect to $B$ (cf. Section 9.4 of [Hum08]). We define $T_i(\lambda)$ as the irreducible algebraic representation of $L_i(\mathbb{Q}_p)$ with a highest weight $\lambda$ dominant with respect to $B \cap L_i$. For example, if $\lambda \in X(T)_+$, then we know that $\lambda$, $s_i \cdot \lambda$ and $s_is_{i-1} \cdot \lambda$ are dominant with respect to $B \cap L_{3-i}$ for $i = 1, 2$. We use the following notation for various parabolic inductions

$$I^{GL_3}_B(\chi) := \left(\text{Ind}_{B(\mathbb{Q}_p)}^{GL_3(\mathbb{Q}_p)} \chi\right)_{an}, I^{GL_3}_{P_1}(\pi_1) := \left(\text{Ind}_{P_1(\mathbb{Q}_p)}^{GL_3(\mathbb{Q}_p)} \pi_1\right)_{an}$$

if $\chi$ is an arbitrary locally analytic character of $T(\mathbb{Q}_p)$ and $\pi_i$ is an arbitrary locally analytic representation of $L_i(\mathbb{Q}_p)$ for each $i = 1, 2$. Moreover, we use the notation

$$i^{GL_3}_B(\chi) := \left(\text{Ind}_{B(\mathbb{Q}_p)}^{GL_3(\mathbb{Q}_p)} \chi \otimes \pi\right)_{an}, i^{GL_3}_{P_1}(\pi) := \left(\text{Ind}_{P_1(\mathbb{Q}_p)}^{GL_3(\mathbb{Q}_p)} \pi\right)_{an}$$

for $i = 1, 2$ if $\chi = \delta_{T, \lambda} \otimes \pi_1^{\infty}$ and $\pi_1 = \overline{L}_1(\lambda) \otimes \pi_1^{\infty}$ are locally algebraic where $\chi^{\infty}$ (resp. $\pi_1^{\infty}$) is a smooth representation of $T(\mathbb{Q}_p)$ (resp. of $L_i(\mathbb{Q}_p)$). We will also use similar notation for parabolic induction to Levi subgroups such as $i^{GL_3}_{B(i)}$ and $i^{GL_3}_{P_1(i)}$ for $i = 1, 2$. Then we define the locally analytic (generalized) Steinberg representation as well as the smooth (generalized) Steinberg representation for $GL_3(\mathbb{Q}_p)$ by

$$\text{St}^{an}_3(\lambda) := I^{GL_3}_{B(\mathbb{Q}_p)}(\delta_{T, \lambda})/\left(I^{GL_3}_{P_1(\mathbb{Q}_p)}(\overline{L}_1(\lambda)) + I^{GL_3}_{P_2(\mathbb{Q}_p)}(\overline{L}_2(\lambda))\right), \text{St}^{\infty}_3 := i^{GL_3}_{B(\mathbb{Q}_p)}(1)/\left(i^{GL_3}_{P_1(\mathbb{Q}_p)}(1_{L_1}) + i^{GL_3}_{P_2(\mathbb{Q}_p)}(1_{L_2})\right)$$

and

$$v^{an}_{P_1}(\lambda) := i^{GL_3}_{P_1(\mathbb{Q}_p)}(\overline{L}_1(\lambda))/\overline{L}_1(\lambda), v^{\infty}_{P_1} := i^{GL_3}_{P_1(\mathbb{Q}_p)}(1_{L_1})/1_3$$

where $1_3$ (resp. $1_{L_i}$) is the trivial representation of $GL_3(\mathbb{Q}_p)$ (resp. of $L_i(\mathbb{Q}_p)$ for each $i = 1, 2$). We define the following smooth representations of $L_1(\mathbb{Q}_p)$:

$$\pi^1_{1,1} := \text{St}^{\infty}_2 \otimes_E 1$$
$$\pi^1_{1,2} := i^{GL_2}_{B_2}(1 \otimes_E | \cdot |^{-1}) \otimes_E | \cdot |$$
$$\pi^1_{1,3} := (\text{St}^{\infty}_2 \otimes_E (| \cdot | \circ \det_2)) \otimes_E | \cdot |^2$$

and the following smooth representations of $L_2(\mathbb{Q}_p)$:

$$\pi^2_{1,1} := 1 \otimes_E \text{St}^{\infty}_2$$
$$\pi^2_{1,2} := | \cdot |^{-1} \otimes_E i^{GL_2}_{B_2}(1 \otimes_E 1)$$
$$\pi^2_{1,3} := | \cdot |^{-2} \otimes_E (\text{St}^{\infty}_2 \otimes_E (| \cdot | \circ \det_2))$$
Consequently, we can define the following locally analytic representations for $i = 1, 2$:

\begin{align}
C_{1,1}^3 & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 \cdot \lambda), 1_{L_{3-1}}) \\
C_{1,3-1}^3 & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 s_{3-1} \cdot \lambda), 1_{L_{3-1}}) \\
C_{1,1} & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 \cdot \lambda), \pi_{i,1}^\infty) \\
C_{1,3-1} & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 s_{3-1} \cdot \lambda), \pi_{i,1}^\infty) \\
C_{1,1} & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 \cdot \lambda), \pi_{i,2}^\infty) \\
C_{1,3-1} & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 s_{3-1} \cdot \lambda), \pi_{i,2}^\infty) \\
C_{1,1} & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 \cdot \lambda), \pi_{i,3}^\infty) \\
C_{1,3-1} & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 s_{3-1} \cdot \lambda), \pi_{i,3}^\infty)
\end{align}

where

\[ \mathcal{D}_{B_{\lambda}} := | \cdot |^{-1} \det \otimes | \cdot |^{2} \quad \text{and} \quad \mathcal{D}_{B_{\lambda}} := | \cdot |^{-2} \otimes | \cdot | \circ \det. \]

We also define

\begin{align}
C_{1,3-1, w}^w & := \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} (L(-s_1 s_2 s_1 \cdot \lambda), \chi_{\infty}) \\
\end{align}

for each $w \in W$ where

\begin{align}
\lambda_{\infty}^1 & := 1_{T} \\
\lambda_{\infty}^{s_2} & := | \cdot |^{-2} \otimes | \cdot | \circ | \cdot | \\
\lambda_{\infty}^{s_2} & := | \cdot |^{-1} \otimes | \cdot | \\
\lambda_{\infty}^{s_2} & := 1 \otimes | \cdot |^{-1} \otimes | \cdot | \\
\lambda_{\infty}^{s_2} & := | \cdot |^{-1} \otimes | \cdot | \circ | \cdot | \\
\lambda_{\infty}^{s_2} & := | \cdot |^{-2} \otimes | \cdot | \circ | \cdot | \\
\lambda_{\infty}^{s_2} & := | \cdot |^{-2} \otimes | \cdot | \circ | \cdot |.
\end{align}

We notice that the representations considered in (2.4) and (2.10) are all irreducible objects inside $\mathbb{R}^\mathbb{G}_{\mathbb{G}_{\mathbb{L}_3}}(\mathbb{Q}_p).E$ according to the main theorem of [OS15]. We use the notation $\Omega$ for the set whose elements are listed as the following:

\[ \mathcal{L}(\lambda), \mathcal{T}(\lambda), \mathcal{V}_{B_{\lambda}, 2}^\infty, \mathcal{V}_{B_{\lambda}, 3}^\infty, \mathcal{V}_{B_{\lambda}, 2}^\infty, \mathcal{V}_{B_{\lambda}, 3}^\infty, \mathcal{V}_{B_{\lambda}, 2}^\infty, \mathcal{V}_{B_{\lambda}, 3}^\infty \]

Remark 2.11. It is actually possible to show that $\Omega$ is the set of (isomorphism classes of) irreducible objects of the block inside $\mathbb{R}^\mathbb{G}_{\mathbb{G}_{\mathbb{L}_3}}(\mathbb{Q}_p).E$ containing the object $\mathcal{L}(\lambda)$.

Lemma 2.12. The representation $\mathcal{V}_{B_{\lambda}, 2}^\infty(\lambda)$ fits into a non-split extension

\begin{align}
\mathcal{L}(\lambda) \otimes E \mathcal{V}_{B_{\lambda}, 2}^\infty & \rightarrow \mathcal{V}_{B_{\lambda}, 2}^\infty (\lambda) \rightarrow C_{s_1}^{1,3-1, 1}
\end{align}

for $i = 1, 2$. On the other hand, the representation $\mathcal{V}_{B_{\lambda}, 2}^\infty(\lambda)$ has the following form:

\begin{align}
\mathcal{L}(\lambda) \otimes E \mathcal{V}_{B_{\lambda}, 2}^\infty & \rightarrow C_{s_1}^{1,3-1, 1}
\end{align}

Proof. The non-split short exact sequence follows directly from (3.62) of [BD18]. It follows easily from the definition of $\mathcal{V}_{B_{\lambda}, 2}^\infty(\lambda)$ that

\[ J_{H_{\mathbb{G}_{\mathbb{L}_3}}(\mathbb{Q}_p)}(\mathcal{V}_{B_{\lambda}, 2}^\infty(\lambda)) = \{ \mathcal{L}(\lambda) \otimes E \mathcal{V}_{B_{\lambda}, 2}^\infty, C_{s_1}^{1,3-1, 1}, C_{s_2}^{1,3-1, 1}, C_{s_1 s_2}^{1,3-1, 1}, C_{s_1}^{2,3-1, 1}, C_{s_2}^{2,3-1, 1}, C_{s_1 s_2}^{2,3-1, 1}, C_{s_1 s_2}^{2,3-1, 1} \} \]

and each Jordan–Hölder factor occurs with multiplicity one. It follows from Section 5.2 of [Brel7] that

\[ H_0 \left( N_1, \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} \left( L(-s_3 - i s_1 \cdot \lambda), i_{B \otimes \mathbb{L}_3}^L(1_T) \right) \right) = \mathcal{T}(-s_3 - i s_1 \cdot \lambda) \otimes E i_{B \otimes \mathbb{L}_3}^L(1_T) \]

which together with

\[ J_{H_{\mathbb{G}_{\mathbb{L}_3}}(\mathbb{Q}_p)} \left( \mathcal{F}_{B_{\lambda}}^\mathbb{G}_{\mathbb{L}_3} \left( L(-s_3 - i s_1 \cdot \lambda), i_{B \otimes \mathbb{L}_3}^L(1_T) \right) \right) = \{ C_{s_3 - i s_1}^{1,3-1, 1}, C_{s_3 - i s_1}^{2,3-1, 1} \} \]
imply that \( F_{\lambda}^{GL_3}(L(-s_3-i\lambda \cdot \lambda), i_{B/\mathbb{A}L_1}(1_T)) \) fits into a non-split extension

\[
C^1_{s_3-i\lambda, 1} \to F_{\lambda}^{GL_3}(L(-s_3-i\lambda \cdot \lambda), i_{B/\mathbb{A}L_1}(1_T)) \to C^2_{s_3-i\lambda, 1}
\]

for \( i = 1, 2 \). We also observe from Section 5.2 and 5.3 of [Bre17] that

\[
H_2(N_{3-i}, F_{\lambda}^{GL_3}(M_i(-s_3-i\lambda \cdot \lambda), \pi_{i,1}^{\infty}) \neq H_2(N_{3-i}, C^2_{s_3-i\lambda, 1}) \oplus H_2(N_{3-i}, C^2_{s_3-i\lambda, 1})
\]

which together with

\[
JH_{GL_3(Q_p)}(F_{\lambda}^{GL_3}(M_i(-s_3-i\lambda \cdot \lambda), \pi_{i,1}^{\infty})) = \{ C^2_{s_3-i\lambda, 1}, C^2_{s_3-i\lambda, 1} \}
\]

imply that \( F_{\lambda}^{GL_3}(M_i(-s_3-i\lambda \cdot \lambda), \pi_{i,1}^{\infty}) \) fits into a non-split extension

\[
C^2_{s_3-i\lambda, 1} \to F_{\lambda}^{GL_3}(M_i(-s_3-i\lambda \cdot \lambda), \pi_{i,1}^{\infty}) \to C^2_{s_3-i\lambda, 1}
\]

for \( i = 1, 2 \). We notice that both \( F_{\lambda}^{GL_3}(L(-s_3-i\lambda \cdot \lambda), i_{B/\mathbb{A}L_1}(1_T)) \) and \( F_{\lambda}^{GL_3}(M_i(-s_3-i\lambda \cdot \lambda), \pi_{i,1}^{\infty}) \) are subquotients of \( St^{an}_3(\lambda) \) by various properties of the functors \( F_{\lambda}^{GL_3} \) (cf. main theorem of [OS15]) and the definition of \( St^{an}_3(\lambda) \). We finish the proof by combining (2.15) and (2.16) with the results before Remark 3.38 of [BD18].

**Remark 2.17.** It is actually possible to show that all the possibly non-split extensions indicated in (2.15) are non-split, although they are essentially unrelated to the \( p \)-adic dilogarithm function.

### 2.4. \( p \)-adic logarithm and dilogarithm.

In this section, we recall \( p \)-adic logarithm and dilogarithm function as well as their representation theoretic interpretations.

We recall the \( p \)-adic logarithm function \( \log_0 : \mathbb{Q}_p^\times \to \mathbb{Q}_p \) defined by power series on a open subgroup of \( \mathbb{Z}_p^\times \) and then extended to \( \mathbb{Q}_p^\times \) by the condition \( \log_0(p) = 0 \). We also recall the \( p \)-adic valuation function \( \text{val}_p : \mathbb{Q}_p \to \mathbb{Z} \) satisfying \(|\cdot| = p^{-\text{val}_p(\cdot)}\) (and in particular \( \text{val}_p(p) = 1 \)). We notice that

\[ \{ \log_0, \text{val}_p \} \]

forms a basis of the two dimensional \( E \)-vector space

\[ \text{Hom}_{cont}(\mathbb{Q}_p^\times, E). \]

We define \( \log_{\mathcal{L}} := \log_0 - \mathcal{L}\text{val}_p \) for each \( \mathcal{L} \in E \) and consider the following two dimensional locally analytic representation of \( \mathbb{Q}_p^\times \)

\[ V_{\mathcal{L}} : \mathbb{Q}_p^\times \to B_2(E), \ a \mapsto \begin{pmatrix} 1 & \log_{\mathcal{L}}(a) \\ 0 & 1 \end{pmatrix} \]

and therefore

\[
\text{soc}_{\mathbb{Q}_p^\times}(V_{\mathcal{L}}) = \text{cosoc}_{\mathbb{Q}_p^\times}(V_{\mathcal{L}}) = 1
\]

where 1 is the notation for the trivial character of \( \mathbb{Q}_p^\times \). We notice that

\[ \text{Ext}_{\mathbb{Q}_p^\times}^1(1, 1) \cong \text{Hom}_{cont}(\mathbb{Q}_p^\times, E), \]

by a standard fact in (continuous) group cohomology and therefore the set \( \{ V_{\mathcal{L}} \mid \mathcal{L} \in E \} \) exhausts (up to isomorphism) all different two dimensional locally analytic non-smooth \( E \)-representations of \( \mathbb{Q}_p^\times \) satisfying (2.15). We observe that \( V_{\mathcal{L}} \) can be viewed as a representation of \( T_2(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \) by composing with the map

\[
T_2(\mathbb{Q}_p) \to \mathbb{Q}_p^\times : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^{-1}b.
\]

As a result, we can consider the parabolic induction

\[ I_{B_2}^{GL_2}(V_{\mathcal{L}} \otimes E \delta_{T_2,\nu}) \]
which naturally fits into an exact sequence

\[(2.20) \quad I^\text{GL}_2(\delta_{T,\nu}) \hookrightarrow I^\text{GL}_2(V_{L' \otimes E} \delta_{T,\nu}) \twoheadrightarrow I^\text{GL}_2(\delta_{T,\nu}).\]

Then we define \(\Sigma_{\text{GL}_2}(\nu, L')\) as the subrepresentation of \(I^\text{GL}_2(V_{L' \otimes E} \delta_{T,\nu})/\mathcal{L}_{\text{GL}_2}(\nu)\) with cosocle \(\mathcal{L}_{\text{GL}_2}(\nu)\). It follows from (the proof of) Theorem 3.14 of [BD18] that \(\Sigma_{\text{GL}_2}(\nu, L')\) has the form

\[(2.21) \quad \text{St}_2^{an}(\nu) \rightarrow \mathcal{L}_{\text{GL}_2}(\nu)\]

and the set \(\{\Sigma_{\text{GL}_2}(\nu, L') \mid L' \in E\}\) exhausts (up to isomorphism) all different locally analytic \(E\)-representations of \(\text{GL}_2(\mathbb{Q}_p)\) of the form \(2.21\) that do not contain \(\mathcal{L}_{\text{GL}_2}(\nu)\)

as a subrepresentation. We have the embeddings

\[i_i : \text{GL}_2 \hookrightarrow L_i\]

for \(i = 1, 2\) by identifying \(\text{GL}_2\) with a Levi block of \(L_i\), which induce the embeddings

\[\iota_{T,i} : T_2 \hookrightarrow T\]

by restricting \(i_i\) to \(T_2 \subseteq \text{GL}_2\). We use the notation \(v_{\iota,i}(V_{L'})\) for the locally analytic representation of \(T(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)^3\) which is \(V_{L'}\) after restricting to \(T_2\) via \(\iota_{T,i}\) and is trivial after restricting to the other copy of \(\mathbb{Q}_p^\times\). By a direct analogue of \(\Sigma_{\text{GL}_2}(\nu, L')\), we can construct \(\Sigma_{L_i}(\lambda, L')\) as the subrepresentation of \(I_{\text{GL}_2}^\iota_{T,i}(V_{L'} \otimes E \delta_{T,\lambda})/\mathcal{L}_{\iota}(\lambda)\) with cosocle \(\mathcal{L}_{\iota}(\lambda)\). In fact, if we have \(\lambda|_{T_2,\iota_{T,i}} = \nu\), then we obviously know that \(\Sigma_{L_i}(\lambda, L')\mid_{\text{GL}_2,i} \cong \Sigma_{\text{GL}_2}(\nu, L')\) where the notation \(\cdot|_{\iota_{T,i}}\) means the restriction of \(\cdot\) to \(\iota_{T,i}\) via the embedding \(\iota_{T,i}\). We observe that the parabolic induction \(I_{P_i}^\text{GL}_2(\Sigma_{L_i}(\lambda, L'))\) fits into the exact sequence

\[v_{P_i}^{an}(\lambda) \hookrightarrow \text{St}_3^{an}(\lambda) \hookrightarrow I_{P_i}^\text{GL}_2(\Sigma_{L_i}(\lambda, L')) \twoheadrightarrow \mathcal{L}(\lambda) \rightarrow v_{P_i}^{an}(\lambda)\]

According to Proposition 5.6 of [Schr11] for example, we know that

\[
\text{Ext}_1^{\text{GL}_2(\mathbb{Q}_p),\lambda} \left(\mathcal{L}(\lambda), \text{St}_3^{an}(\lambda)\right) = 0
\]

and thus we can define \(\Sigma_i(\lambda, L')\) as the unique quotient of \(I_{P_i}^\text{GL}_2(\Sigma_{L_i}(\lambda, L'))\) that fits into the exact sequence

\[
\text{St}_3^{an}(\lambda) \rightarrow \Sigma_i(\lambda, L') \rightarrow v_{P_i}^{an}(\lambda).
\]

The constructions of \(\Sigma_i(\lambda, L')\) above actually induce canonical isomorphisms

\[(2.22) \quad \text{Hom}_{\text{cont}}(\mathbb{Q}_p^\times, E) \cong \text{Ext}_1^{\text{GL}_2(\mathbb{Q}_p),\lambda}(v_{P_i}^{an}(\lambda), \text{St}_3^{an}(\lambda))
\]

for \(i = 1, 2\). We denote the image of \(\text{log}_0\) (resp. \(\text{val}_p\)) in

\[
\text{Ext}_1^{\text{GL}_2(\mathbb{Q}_p),\lambda}(v_{P_i}^{an}(\lambda), \text{St}_3^{an}(\lambda))
\]

by \(b_{i,\text{log}_0}\) (resp. \(b_{i,\text{val}_p}\)). We use the notation \(1_T\) for the trivial character of \(T(\mathbb{Q}_p)\). We use the same notation \(b_{i,\text{log}_0}\) and \(b_{i,\text{val}_p}\) for the image of \(\text{log}_0\) and \(\text{val}_p\) respectively under the embedding

\[
\text{Ext}_1^{\text{GL}_2(\mathbb{Q}_p),\lambda}(1_T, 1) \hookrightarrow \text{Ext}_1^{\text{GL}_2(\mathbb{Q}_p),\lambda}(v_{P_i}^{an}(\lambda), \text{St}_3^{an}(\lambda))
\]

induced by the maps

\[
T(\mathbb{Q}_p) \twoheadrightarrow T_2(\mathbb{Q}_p) \xrightarrow{\mathcal{L}_i} \mathbb{Q}_p^\times
\]

where \(p_i\) is the section of \(\iota_{T,i}\) which is compatible with the projection \(L_i \rightarrow \text{GL}_2\). Recall the elements

\[
c_i,\text{log}_0, c_i,\text{val}_p \in \text{Ext}_1^{T(\mathbb{Q}_p),0}(1_T, 1_T) \text{ constructed after (5.24) of [Schr11] and observe that}
\]

\[(2.23) \quad \left\{ \begin{array}{l}
c_{1,\text{log}_0} = b_{1,\text{log}_0} + 2b_{2,\text{log}_0}, \quad c_{1,\text{val}_p} = b_{1,\text{val}_p} + 2b_{2,\text{val}_p} \\
c_{2,\text{log}_0} = 2b_{1,\text{log}_0} + b_{2,\text{log}_0}, \quad c_{2,\text{val}_p} = 2b_{1,\text{val}_p} + b_{2,\text{val}_p}
\end{array} \right. \]
We notice that there exists canonical surjections
\[(2.24) \quad \operatorname{Ext}_{GL_3(Q_p)}^1(1_T, 1_T) \to \operatorname{Ext}_{GL_3(Q_p), \lambda}^1(v_{p_1}^a(\lambda), \operatorname{St}_3^{an}(\lambda))\]
with kernel spanned by \(\{c_{1, \log}, c_{1, \val}\}\), according to (5.70) and (5.71) of [Schr11]. Therefore the relation (2.24) reduces via the surjection (2.24) to
\[(2.25) \quad c_{3-i, \log} = -3b_{i, \log}, \quad c_{3-i, \val} = -3b_{i, \val}\]
inside the quotient \(\operatorname{Ext}_{GL_3(Q_p), \lambda}^1(v_{p_1}^a(\lambda), \operatorname{St}_3^{an}(\lambda))\). We define \(\Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)\) as the amalgamated sum of \(\Sigma_1(\lambda, \mathcal{L}_1)\) and \(\Sigma_2(\lambda, \mathcal{L}_2)\) over \(\operatorname{St}_3^{an}(\lambda)\), for each \(\mathcal{L}_1, \mathcal{L}_2 \in E\). Consequently, \(\Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)\) has the following form
\[
\begin{array}{ccc}
\text{St}_3^{an}(\lambda) \\
v_{p_1}^a(\lambda) \\
v_{p_2}^a(\lambda)
\end{array}
\]
and we have
\[(2.26) \quad \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \cong \Sigma(\lambda, \mathcal{L}, \mathcal{L}')\]
if
\[(2.27) \quad \mathcal{L}_1 = -\mathcal{L}', \mathcal{L}_2 = -\mathcal{L} \in E,\]
where \(\Sigma(\lambda, \mathcal{L}, \mathcal{L}')\) is the locally analytic representation defined in Definition 5.12 of [Schr11] using the element
\[
(c_{2, \log} + \mathcal{L}' c_{2, \val}, \ c_{1, \log} + \mathcal{L}_1 c_{1, \val})
\]
in
\[
\operatorname{Ext}_{GL_3(Q_p), \lambda}^1(v_{p_1}^a(\lambda) \oplus v_{p_2}^a(\lambda), \operatorname{St}_3^{an}(\lambda)).
\]

**Remark 2.28.** The appearance of a sign in (2.27) is essentially due to Remark 3.1 of [Ding18], which implies that our invariants \(\mathcal{L}_1\) and \(\mathcal{L}_2\) can be identified with Fontaine–Mazur \(\mathcal{L}\)-invariants of the corresponding Galois representation via local-global compatibility.

We have a canonical morphism by (5.26) of [Schr11]
\[(2.29) \quad \kappa: \operatorname{Ext}_{T(Q_p), 0}^2(1_T, 1_T) \to \operatorname{Ext}_{GL_3(Q_p), \lambda}^2(T(\lambda), \operatorname{St}_3^{an}(\lambda)).\]

Note that we also have
\[
\operatorname{Ext}_{T(Q_p), 0}^2(1_T, 1_T) \cong \wedge^2 \left(\operatorname{Ext}_{T(Q_p), 0}^1(1_T, 1_T)\right)
\]
by (5.24) of [Schr11] and thus the set
\[
\{b_{1, \valp} \wedge b_{2, \valp}, b_{1, \logp} \wedge b_{2, \valp}, b_{1, \valp} \wedge b_{2, \logp}, b_{1, \logp} \wedge b_{2, \logp}, b_{1, \valp} \wedge b_{1, \logp} \wedge b_{2, \valp} \wedge b_{2, \logp}\}
\]
forms a basis of \(\operatorname{Ext}_{T(Q_p), 0}^2(1_T, 1_T)\). It follows from (5.27) of [Schr11] and (2.23) that the set
\[
\{\kappa(b_{1, \valp} \wedge b_{2, \valp}), \kappa(b_{1, \logp} \wedge b_{2, \valp}), \kappa(b_{1, \valp} \wedge b_{2, \logp}), \kappa(b_{1, \logp} \wedge b_{2, \logp})\}
\]
forms a basis of the image of (2.29).

We recall the \(p\)-adic dilogarithm function \(li_2: \mathbb{Q}_p \setminus \{0, 1\} \to \mathbb{Q}_p\) defined by Coleman in [Cole82] and we consider the function
\[
D_{\mathcal{L}}(z) := li_2(z) + \frac{1}{2} \log_{\mathcal{L}}(z) \log_{\mathcal{L}}(1 - z)
\]
as in (5.34) of [Schr11]. We also define
\[
d(z) := \log_{\mathcal{L}}(1 - z) \val_p(z) - \log_{\mathcal{L}}(z) \val_p(1 - z)
\]
as in (5.36) of [Schr11] which is also a locally analytic function over \(\mathbb{Q}_p \setminus \{0, 1\}\) and is independent of the choice of \(\mathcal{L} \in E\). Note by our definition that
\[
D_{\mathcal{L}} - D_0 = \frac{\mathcal{L}}{2}.
\]
It follows from Theorem 7.2 of [Schr11] that \( \{D_0, d\} \) can be identified with a basis of
\[
\Ext_{\GL_2(Q_p),0}^2(1, \text{St}^\text{an}_2) \quad \text{(cf. (5.38) of [Schr11])}
\]
which naturally embeds into \( \Ext_{\GL_2(Q_p),0}^2(1, \text{St}^\text{an}_2) \). Then the map \( \iota_i : \GL_2 \hookrightarrow L_i \)
induces the isomorphisms
\[
(2.30) \quad \Ext_{\GL_2(Q_p),0}^2(1_2, \text{St}^\text{an}_2) \cong \Ext_{L_i(Q_p),0}^2(1_2, \text{St}^\text{an}_2) \cong \Ext_{\GL_2(Q_p),0}^2(1_3, I_{P_i}^{GL_3}(\text{St}^\text{an}_2))
\]
where \( L_i(Q_p) \) acts on \( \text{St}^\text{an}_2 \) via the projection \( p_i \). We abuse the notation for the composition
\[
(2.31) \quad \iota_i : \Ext_{\GL_2(Q_p),0}^2(1_2, \text{St}^\text{an}_2) \cong \Ext_{\GL_2(Q_p),0}^2(1_3, I_{P_i}^{GL_3}(\text{St}^\text{an}_2)) \to \Ext_{\GL_2(Q_p),0}^2(1_3, \text{St}^\text{an}_3)
\]
given by (2.30) and the surjection
\[
I_{P_i}^{GL_3}(\text{St}^\text{an}_2) \to \text{St}^\text{an}_3.
\]
Finally there is canonical isomorphism
\[
\Ext_{\GL_3(Q_p),0}^2(1_3, \text{St}^\text{an}_3) \cong \Ext_{\GL_3(Q_p),\lambda}(\mathbb{T}(\lambda), \text{St}^\text{an}_3(\lambda))
\]
by (5.20) of [Schr11].

**Lemma 2.32.** We have
\[
\dim_E \Ext_{\GL_3(Q_p),\lambda}^2(\mathbb{T}(\lambda), \text{St}^\text{an}_3(\lambda)) = 5
\]
and the set
\[
\{ \kappa(b_{1,\text{val}_p} \wedge b_{2,\text{val}_p}), \kappa(b_{1,\log_6} \wedge b_{2,\text{val}_p}), \kappa(b_{1,\text{val}_p} \wedge b_{2,\log_6}), \kappa(b_{1,\log_6} \wedge b_{2,\log_6}), \iota_i(D_0) \}
\]
forms a basis of \( \Ext_{\GL_3(Q_p),\lambda}^2(\mathbb{T}(\lambda), \text{St}^\text{an}_3(\lambda)) \) for \( i = 1, 2 \).

**Proof.** This follows directly from (5.57) of [Schr11] and (2.23). \( \Box \)

**Lemma 2.33.** There exists \( \gamma \in E^\times \) such that
\[
\iota_1(d) = \iota_2(d) = \gamma \left( \kappa(b_{1,\log_6} \wedge b_{2,\text{val}_p} + b_{1,\text{val}_p} \wedge b_{2,\log_6}) \right).
\]

**Proof.** This follows directly from Lemma 5.8 of [Schr11] and (2.23) if we take
\[
\gamma := -3\alpha
\]
where \( \alpha \in E^\times \) is the constant in the statement of Lemma 5.8 of [Schr11]. \( \Box \)

**Lemma 2.34.** We have
\[
\dim_E \Ext_{\GL_3(Q_p),\lambda}^1(\mathbb{T}(\lambda), \Sigma(\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) = 1 \quad \text{and} \quad \dim_E \Ext_{\GL_3(Q_p),\lambda}^2(\mathbb{T}(\lambda), \Sigma(\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) = 2.
\]
Moreover, the image of
\[
\{ \kappa(b_{1,\text{val}_p} \wedge b_{2,\text{val}_p}), \iota_i(D_0) \}
\]
under
\[
\Ext_{\GL_3(Q_p),\lambda}^2(\mathbb{T}(\lambda), \text{St}^\text{an}_3(\lambda)) \to \Ext_{\GL_3(Q_p),\lambda}^2(\mathbb{T}(\lambda), \Sigma(\lambda, \mathcal{Z}_1, \mathcal{Z}_2))
\]
forms a basis of \( \Ext_{\GL_3(Q_p),\lambda}^2(\mathbb{T}(\lambda), \Sigma(\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) \) for \( i = 1 \) or \( 2 \).

**Proof.** This follows directly from Corollary 5.17 of [Schr11] and (2.23). \( \Box \)

We recall from (5.55) of [Schr11] that
\[
(2.35) \quad c_0 := \alpha^{-1} \iota_i(D_0) - \frac{1}{2} \kappa(c_{1,\log_6} \wedge c_{2,\log_6})
\]
where \( \alpha \) is defined in Lemma 5.8 of [Schr11].
Lemma 2.36. Assume that $\mathcal{L}_3 \in E$ satisfies the equality
\begin{equation}
E(t_1(D_0) + \mathcal{L}_3 \kappa(b_{1,\text{val}_p} \wedge b_{2,\text{val}_p})) = E(c_0 + \mathcal{L}'' \kappa(c_{1,\text{val}} \wedge c_{2,\text{val}})) \subseteq \text{Ext}_{\text{GL}_3(\mathbb{Q}_p),\lambda}^2(\mathcal{T}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)).
\end{equation}
Then we have
\[ \mathcal{L}_3 = \gamma(\mathcal{L}'') - \frac{1}{2} \mathcal{L}_1 \mathcal{L}_2 = \gamma(\mathcal{L}'') - \frac{1}{2} \mathcal{L} \mathcal{L}'. \]

Proof. All the equalities in this lemma are understood to be inside
\[ \text{Ext}_{\text{GL}_3(\mathbb{Q}_p),\lambda}^2(\mathcal{T}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]
without causing ambiguity. It follows from our assumption (2.37) that
\[ t_1(D_0) + \mathcal{L}_3 \kappa(b_{1,\text{val}_p} \wedge b_{2,\text{val}_p}) = \alpha(c_0 + \mathcal{L}'' \kappa(c_{1,\text{val}} \wedge c_{2,\text{val}})) \]
which together with (2.37) imply that
\begin{equation}
\mathcal{L}_3 \kappa(b_{1,\text{val}_p} \wedge b_{2,\text{val}_p}) = \frac{\alpha}{2} \kappa(c_{1,\text{log}} \wedge c_{2,\text{log}}) + \alpha \mathcal{L}'' \kappa(c_{1,\text{val}} \wedge c_{2,\text{val}}).
\end{equation}
We know that
\begin{equation}
\kappa(c_{1,\text{log}} \wedge c_{2,\text{log}}) = \mathcal{L} \mathcal{L}' \kappa(c_{1,\text{val}} \wedge c_{2,\text{val}})
\end{equation}
from the proof of Corollary 5.17 of [Schr11] and that
\begin{equation}
\kappa(c_{1,\text{val}} \wedge c_{2,\text{val}}) = -3\kappa(b_{1,\text{val}_p} \wedge b_{2,\text{val}_p})
\end{equation}
from (2.23). Therefore we finish the proof by combining (2.37c, 2.38) and (2.40) with (2.27) and the equality $\gamma = -3\alpha$ from Lemma 2.35. \hfill $\square$

Remark 2.41. We emphasize that we do not know whether
\[ E_{t_1}(D_0) = E_{t_2}(D_0) \]
in $\text{Ext}_{\text{GL}_3(\mathbb{Q}_p),\lambda}^2(\mathcal{T}(\lambda), \text{St}^\text{an}_3(\lambda))$ or not, which is of independent interest.

3. A Key Result for $\text{GL}_2(\mathbb{Q}_p)$

In this section, we are going to prove Proposition 3.14 which will be a crucial ingredient for the proof of Lemma 5.8 and Proposition 6.8.

We use the following shorten notation
\[ I(\nu) := \mathcal{I}_{\text{GL}_2}^\text{GL}_2(\delta_{T_2,\nu}), \quad \tilde{I}(\nu) := \mathcal{I}_{\text{GL}_2}^\text{GL}_2(\delta_{T_2,\nu} \otimes_E (|\cdot|^{-1} \otimes_E |\cdot|)) \]
for each weight $\nu \in X(T_2)$.

Lemma 3.1. We have
\[ \dim_E \text{Ext}_{\text{GL}_2(\mathbb{Q}_p)}^1(\tilde{I}(s \cdot \nu), \Sigma_{\text{GL}_2}(\nu, \mathcal{L})) = 1. \]

Proof. This is essentially contained in the proof of Theorem 3.14 of [BDIS]. In fact, we know that
\begin{align*}
\text{Ext}_{\text{GL}_2(\mathbb{Q}_p)}^1(\tilde{I}(s \cdot \nu), \Sigma_{\text{GL}_2}(\nu, \mathcal{L})) &= 0, \\
\text{Ext}_{\text{GL}_2(\mathbb{Q}_p)}^2(\tilde{I}(s \cdot \nu), \Sigma_{\text{GL}_2}(\nu, \mathcal{L})) &= 0
\end{align*}
and
\[ \dim_E \text{Ext}_{\text{GL}_2(\mathbb{Q}_p)}^1(\tilde{I}(s \cdot \nu), \Sigma_{\text{GL}_2}(\nu, \mathcal{L})) = 1 \]
which finish the proof by a simple devissage induced by the short exact sequence
\[ (\Sigma_{\text{GL}_2}(\nu, \mathcal{L}) \rightarrow \text{Ext}_{\text{GL}_2(\mathbb{Q}_p)}^1(\tilde{I}(s \cdot \nu), \Sigma_{\text{GL}_2}(\nu, \mathcal{L}))) \]
\hfill $\square$
We fix a split p-adic reductive group $H$ and have a natural embedding
\[ U(\mathfrak{h}) \hookrightarrow D(H, E)_{\{1\}} \hookrightarrow D(H, E) \]
where $D(H, E)_{\{1\}}$ is the closed subalgebra of $D(H, E)$ consisting of distributions supported at the identity element (cf. [Koh07]). The embedding above induces another embedding
\[ Z(U(\mathfrak{h})) \hookrightarrow Z(D(H, E)) \]
(3.2)
by the main result of [Koh07] where $Z(\cdot)$ is the notation for the center of a non-commutative algebra.

We say that $\Pi \in \text{Rep}^{la}_{\text{GL}_2(Q_p), E}$ has an infinitesimal character if $Z(U(\mathfrak{h}))$ acts on $\Pi'$ via a character.

**Lemma 3.3.** If $V,W \in \text{Rep}^{la}_{H,E}$ have both the same central character and the same infinitesimal character as the one for $V$ and $W$.

\[ \text{Hom}_H(V, W) = 0, \]
then any non-split extension of the form $W \longrightarrow V$ has both the same central character and the same infinitesimal character as the one for $V$ and $W$.

**Proof.** This is a direct analogue of Lemma 3.1 in [BD18] and follows essentially from the fact that both $D(Z(H), E)$ and $Z(U(\mathfrak{h}))$ are subalgebras of $Z(D(H, E))$ by [Koh07].

We fix a Borel subgroup $B_H \subseteq H$ as well as its opposite Borel subgroup $B_H^\circ$. We consider the split maximal torus $T_H := B_H \cap B_H^\circ$ and use the notation $N_H \ (\text{resp. } N_H^\circ)$ for the unipotent radical of $B_H \ (\text{resp. of } B_H^\circ)$.

**Lemma 3.4.** If $V \in \text{Rep}^{la}_{H,E}$ has an infinitesimal character, then $U(t_\lambda)^{W_H}$ (as a subalgebra of $U(t_\lambda)$) acts on $J_{B_H}^{\text{la}}(V)$ via a character where $W_H$ is the Weyl group of $H$.

**Proof.** We know by our assumption that $Z(U(\mathfrak{h}))$ acts on $V'$ (and hence on $V$ as well) via a character. We note from (3.2) that $Z(U(\mathfrak{h}))$ commutes with $D(N_H, E) \subseteq D(H, E)$ and thus the action of $Z(U(\mathfrak{h}))$ on $V$ commutes with that of $N_H^\circ$, which implies that $Z(U(\mathfrak{h}))$ acts on $V^{N_H^\circ}$ via a character for each open compact subgroup $N_H^\circ \subseteq N_H$. We use the notation
\[ \theta : Z(U(\mathfrak{h})) \overset{\sim}{\longrightarrow} U(t_\lambda)^{W_H} \]
for the Harish-Chandra isomorphism (cf. Section 1.7 of [Hum08]) and the notation $j_1$ and $j_2$ for the embeddings
\[ j_1 : Z(U(\mathfrak{h})) \hookrightarrow U(\mathfrak{h}) \quad \text{and} \quad j_2 : U(t_\lambda) \hookrightarrow U(\mathfrak{h}). \]

We choose an arbitrary Verma module $M_H(\lambda_H)$ with highest weight $\lambda_H$, namely we have
\[ M_H(\lambda) := U(\mathfrak{h}) \otimes_{U(\overline{\mathfrak{h}})} \lambda_H. \]

We use the notation $M_H(\lambda_H)_{\mu}$ for the subspace of $M_H(\lambda)$ with $t_\lambda$-weight $\mu$ and note that
\[ \dim E M_H(\lambda_H)_{\lambda_H} = 1. \]

We easily observe that
\[ (3.5) \quad Z(U(\mathfrak{h})) \cdot M_H(\lambda_H)_{\lambda_H} = M_H(\lambda_H)_{\lambda_H} \quad \text{and} \quad U(t_\lambda) \cdot M_H(\lambda_H)_{\lambda_H} = M_H(\lambda_H)_{\lambda_H}. \]

It is well-known that the the direct sum decomposition
\[ (3.6) \quad \mathfrak{h} = \mathfrak{n}_H \oplus t_\lambda \oplus \overline{\mathfrak{n}_H} \]
induces a tensor decomposition of $E$-vector space
\[ (3.7) \quad U(\mathfrak{h}) = U(\mathfrak{n}_H) \otimes_E U(t_\lambda) \otimes_E U(\overline{\mathfrak{n}_H}). \]

Hence we can write each element in $U(\mathfrak{h})$ as a polynomial with variables indexed by a standard basis of $\mathfrak{h}$ that is compatible with (3.6). It follows from the definition of $\theta$ as the restriction to $Z(U(\mathfrak{h}))$ of the projection $U(\mathfrak{h}) \twoheadrightarrow U(t_\lambda)$ (coming from (3.7)) that
\[ j_1(z) - j_2 \circ \theta(z) \in U(\mathfrak{h}) \cdot \overline{\mathfrak{n}_H} + \mathfrak{n}_H \cdot U(\mathfrak{h}) \]
It follows from Lemma 3.4 that $U_1$ forms a basis of $\text{Ext}^1_{\mathcal{L}}$. Then the set
\[ \{ \log_0 \circ \epsilon_1, \text{val}_p \circ \epsilon_1, \log_0 \circ \epsilon_2, \text{val}_p \circ \epsilon_2 \} \]
forms a basis of $\text{Ext}^1_{T_2(\mathbb{Q}_p)}(1,1)$. Therefore we deduce by a twisting that the the subspace of

for each $z \in Z(U(\mathfrak{h}))$. If a monomial $f$ in the decomposition of $j_1(z) - j_2 \circ \theta(z)$ belongs to
\[ n_H \cdot U(n_H) \cdot U(\mathfrak{h}), \]
then we have
\[ f \cdot M_H(\lambda_H)_{\lambda_H} \subseteq M_H(\lambda_H)_{\mu} \]
for some $\mu \neq \lambda_H$, which contradicts the fact (3.3). Hence we conclude that
\[ j_1(z) - j_2 \circ \theta(z) \in U(\mathfrak{h}) \cdot \mathfrak{n}_H \]
and in particular
\[ j_1(z) = j_2 \circ \theta(z) \]
on $V^{\mathfrak{n}_H^\circ}$ for each $z \in Z(U(\mathfrak{h}))$. Hence we deduce that $U(\mathfrak{t}_h)^{W_H}$ acts on $V^{\mathfrak{n}_H^\circ}$ via a character. We note by the definition of $J^n_{B_H}(V)$ (cf. [Eme06]) that we have a $T_H^+$-equivariant embedding
\[ J^n_{B_H}(V) \hookrightarrow V^{\mathfrak{n}_H^\circ} \]
where $T_H^+$ is a certain submonoid of $T_H$ containing an open compact subgroup. As a result, (3.8) is also $U(\mathfrak{t}_h)$-equivariant and thus $U(\mathfrak{t}_h)^{W_H}$ acts on $J^n_{B_H}(V)$ via a character which finishes the proof. \qed

We set $H = \text{GL}_2(\mathbb{Q}_p)$, $B_H = B_2$ and $B_H^0 = B_2^0$ in the rest of this section. The idea of the following lemma which is closely related to Lemma 3.20 of [BD18], owes very much to Y.Ding.

**Lemma 3.9.** A locally analytic representation of either the form
\[ (3.10) \quad \mathcal{T}_{\text{GL}_2}(\nu) \otimes_E \text{St}^\infty_2 \quad \text{I}(s \cdot \nu) \quad \mathcal{T}_{\text{GL}_2}(\nu) \otimes_E \text{St}^\infty_2 \]
or the form
\[ (3.11) \quad \mathcal{T}_{\text{GL}_2}(\nu) \quad \bar{\text{I}}(s \cdot \nu) \quad \mathcal{T}_{\text{GL}_2}(\nu) \otimes_E \text{St}^\infty_2 \quad \mathcal{T}_{\text{GL}_2}(\nu) \]
does not have an infinitesimal character.

**Proof.** Assume that a representation $V$ of the form (3.10) has an infinitesimal character. Note that $V$ can be represented by an element in the space $\text{Ext}^1_{\text{GL}_2(\mathbb{Q}_p)}(\mathcal{T}_{\text{GL}_2}(\nu) \otimes_E \text{St}^\infty_2, \Sigma_{\text{GL}_2}(\nu, \mathcal{L}))$ for certain $\mathcal{L} \in E$. We consider the upper-triangular Borel subgroup $B_2$ and the diagonal split torus $T_2$. Then by the proof of Lemma 3.20 of [BD18] we know that the Jacquet functor $J^B_{B_2}$ (cf. [Eme06] for the definition) induces a injection
\[ (3.12) \quad \text{Ext}^1_{\text{GL}_2(\mathbb{Q}_p)}(\mathcal{T}_{\text{GL}_2}(\nu) \otimes_E \text{St}^\infty_2, \Sigma_{\text{GL}_2}(\nu, \mathcal{L})) \]
\[ \hookrightarrow \text{Ext}^1_{T_2(\mathbb{Q}_p)}(\delta_{T_2, \nu} \otimes_E (| \cdot | \otimes E \cdot | \cdot |^{-1}), \delta_{T_2, \nu} \otimes_E (| \cdot | \otimes E \cdot | \cdot |^{-1})). \]
By twisting $\delta_{T_2, \nu} \otimes_E (| \cdot | \otimes E \cdot | \cdot |^{-1})$ we have an isomorphism
\[ (3.13) \quad \text{Ext}^1_{T_2(\mathbb{Q}_p)}(\delta_{T_2, \nu} \otimes_E (| \cdot | \otimes E \cdot | \cdot |^{-1}), \delta_{T_2, \nu} \otimes_E (| \cdot | \otimes E \cdot | \cdot |^{-1})) \cong \text{Ext}^1_{T_2(\mathbb{Q}_p)}(1_{T_2}, 1_{T_2}). \]
It follows from Lemma 3.20 of [BD18] (up to changes on notation) that the image of the composition of (3.13) and (3.12) is a certain two dimensional subspace $\text{Ext}^1_{T_2(\mathbb{Q}_p)}(1,1, \mathcal{L})$ of $\text{Ext}^1_{T_2(\mathbb{Q}_p)}(1,1)$ depending on $\mathcal{L}$. More precisely, if we use the notation $\epsilon_1, \epsilon_2$ for the two charaters
\[ \epsilon_1 : T_2(\mathbb{Q}_p) \to \mathbb{Q}_p^\times, \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \mapsto a \quad \epsilon_2 : T_2(\mathbb{Q}_p) \to \mathbb{Q}_p^\times, \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \mapsto b, \]
then the set
\[ \{ \log_0 \circ \epsilon_1, \text{val}_p \circ \epsilon_1, \log_0 \circ \epsilon_2, \text{val}_p \circ \epsilon_2 \} \]
forms a basis of $\text{Ext}^1_{T_2(\mathbb{Q}_p)}(1,1)$. Therefore we deduce by a twisting that the the subspace of

It follows from Lemma 3.4 that $U(\mathfrak{t}_h)^{W_{\text{GL}_2}}$ acts on $J^n_{B_H}(V)$ via a character where $W_{\text{GL}_2}$ is the notation for the Weyl group of $\text{GL}_2(\mathbb{Q}_p)$. Therefore we deduce by a twisting that the the subspace of

Ext^1_{T_2(Q_p)}(1,1) corresponding to $\mathcal{J}_{T_2}(V)$ is killed by $U(t_2)^{W_{GL_2}}$. We notice that the subspace $M$ of Ext^1_{T_2(Q_p)}(1,1) killed by $U(t_2)^{W_{GL_2}}$ is two dimensional with basis 
\{val_p \circ \epsilon_1, \text{val}_p \circ \epsilon_2\}
and we have
\[M \cap \text{Ext}^1_{T_2(Q_p)}(1,1)_{\mathcal{L}} = E(\text{val}_p \circ \epsilon_1 + \text{val}_p \circ \epsilon_2). \]
However, the representation given by the line $E(\text{val}_p \circ \epsilon_1 + \text{val}_p \circ \epsilon_2)$ has a subrepresentation of the form
\[\overline{L}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow \overline{L}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2\]
which is a contradiction.

The proof of the second statement is a direct analogue as we observe that $\mathcal{J}_{T_2}$ also induces the following embedding
\[\text{Ext}^1_{GL_2(Q_p)} \left( \overline{T}_{GL_2}(\nu), \overline{T}_{GL_2}(\nu) \rightarrow I(s \cdot \nu) \rightarrow \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow \overline{T}_{GL_2}(\nu) \right) \]
\[\hookrightarrow \text{Ext}^1_{T_2(Q_p)}(\nu, \nu, \mathcal{L}) \]

We define $\Sigma^+_2(\nu, \mathcal{L})$ as the unique (up to isomorphism) non-split extension of $\Sigma_{GL_2}(\nu, \mathcal{L})$ by $I(s \cdot \nu)$ given by Lemma 3.1.

**Proposition 3.14.** We have
\[\text{Ext}^1_{GL_2(Q_p)} \left( \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2, \overline{T}_{GL_2}(\nu) , \Sigma^+_2(\nu, \mathcal{L}) \right) = 0.\]

**Proof.** Assume on the contrary that $V$ is a representation given by a certain non-zero element inside
\[\text{Ext}^1_{GL_2(Q_p)} \left( \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2, \overline{T}_{GL_2}(\nu) , \Sigma^+_2(\nu, \mathcal{L}) \right).\]

We deduce that $V$ has both a central character and an infinitesimal character from Lemma 3.3 and the fact
\[\text{Hom}_{GL_2(Q_p)} \left( \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2, \overline{T}_{GL_2}(\nu) , \Sigma^+_2(\nu, \mathcal{L}) \right) = 0.\]

Note that we have
\[\text{Ext}^1_{GL_2(Q_p)}(\overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2, I(s \cdot \nu)) = \text{Ext}^1_{GL_2(Q_p)}(\overline{T}_{GL_2}(\nu), I(s \cdot \nu)) = 0,\]
\[\dim_E \text{Ext}^1_{GL_2(Q_p)}(\overline{T}_{GL_2}(\nu) , \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2) = 1\]
and
\[\dim_E \text{Ext}^1_{GL_2(Q_p)}(\overline{T}_{GL_2}(\nu) , I(s \cdot \nu)) = 1\]
by a combination of Lemma 3.13 of [BD18] with Lemma 2.1 and thus $V$ has a subrepresentation of one of the three following forms
\begin{enumerate}
\item \[\overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2;\]
\item \[\overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow I(s \cdot \nu) \rightarrow \overline{T}_{GL_2}(\nu) \rightarrow \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2;\]
\item \[\overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow I(s \cdot \nu) \rightarrow \overline{T}_{GL_2}(\nu) \rightarrow \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow \overline{T}_{GL_2}(\nu).\]
\end{enumerate}

In the first case, we know from Proposition 4.7 of [Schr11] and the main result of [Or05] that
\[\text{Ext}^1_{GL_2(Q_p),\nu}(\overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2, \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2) = 0\]
and therefore this case is impossible due to the existence of central character for $V$ (and hence for its subrepresentations). In the second case, we deduce from Lemma 3.3 a contradiction as $V$ has an infinitesimal character. In the third case, we thus know that $V$ has a quotient representation of the form
\[\overline{T}_{GL_2}(\nu) \rightarrow \overline{T}_{GL_2}(\nu) \otimes_E \text{St}^\infty_2 \rightarrow \overline{T}_{GL_2}(\nu)\]
which can not have an infinitesimal character due to Lemma 3.9 a contradiction again. Hence we finish the proof.

\[ \text{Remark 3.15. Note that the argument in Proposition 3.14 actually implies that} \]
\[ \Ext^1_{\GL_2(\mathbb{Q}_p)} \bigg( \mathcal{T}_{\GL_2}(\nu) \otimes_E \mathcal{S}_2^\infty, \mathcal{T}_{\GL_2}(\nu), \mathfrak{I}(s \cdot \nu) \bigg) = 0 \]
\[ \text{and we can show by the same method that} \]
\[ \Ext^1_{\GL_2(\mathbb{Q}_p)} \bigg( \mathcal{T}_{\GL_2}(\nu) \mathcal{T}_{\GL_2}(\nu) \otimes_E \mathcal{S}_2^\infty, \mathcal{T}_{\GL_2}(\nu), \mathfrak{I}(s \cdot \nu) \bigg) = 0. \]

4. Computations of Ext I

In this section, we are going to compute various Ext-groups based on known results on group cohomology in Section 5.2 and 5.3 of [Bre17].

\[ \text{Proposition 4.1. The following spaces are one dimensional} \]
\[ \Ext^1_{\GL_3(\mathbb{Q}_p), \lambda} \big( \mathfrak{I}(\lambda), \mathcal{T}(\lambda) \otimes_E \nu^\infty \big) \]
\[ \Ext^1_{\GL_3(\mathbb{Q}_p), \lambda} \big( \mathfrak{I}(\lambda) \otimes \mathcal{E} \mathcal{S}_3^\infty, \mathcal{T}(\lambda) \otimes \nu^\infty \big) \]
\[ \Ext^2_{\GL_3(\mathbb{Q}_p), \lambda} \big( \mathfrak{I}(\lambda) \otimes \mathcal{E} \mathcal{S}_3^\infty, \mathcal{T}(\lambda) \big) \]
\[ \Ext^2_{\GL_3(\mathbb{Q}_p), \lambda} \big( \mathfrak{I}(\lambda) \otimes \nu^\infty \big) \]
\[ \text{for } i = 1, 2. \text{ Moreover, we have} \]
\[ \Ext^k_{\GL_3(\mathbb{Q}_p), \lambda} (V_1, V_2) = 0 \]
\[ \text{in all the other cases where } 1 \leq k \leq 2 \text{ and } V_1, V_2 \in \{ \mathfrak{I}(\lambda), \mathfrak{I}(\lambda) \otimes \nu^\infty, \mathcal{T}(\lambda) \otimes \nu^\infty, \mathfrak{I}(\lambda) \otimes \mathcal{E} \mathcal{S}_3^\infty \}. \]

\[ \text{Proof. This follows from a special case of Proposition 4.7 of [Schr11] together with the main result of [Or03].} \]

\[ \text{Lemma 4.2. We have} \]
\[ \Ext^1_{\GL_3(\mathbb{Q}_p), \lambda} \bigg( \mathcal{T}(\lambda) \otimes_E \nu^\infty, \mathfrak{I}(\lambda) \bigg) = 0 \]
\[ \Ext^2_{\GL_3(\mathbb{Q}_p), \lambda} \bigg( \mathfrak{I}(\lambda) \otimes \mathcal{E} \mathcal{S}_3^\infty, \mathfrak{I}(\lambda) \bigg) = 0 \]
\[ \text{for } i = 1, 2 \text{ and } k = 1, 2. \]

\[ \text{Proof. It is sufficient to prove that} \]
\[ \Ext^1_{\GL_3(\mathbb{Q}_p), \lambda} \bigg( \mathfrak{I}(\lambda) \otimes E \nu^\infty, \mathfrak{I}(\lambda) \bigg) = 0 \]
\[ \text{and} \]
\[ \Ext^2_{\GL_3(\mathbb{Q}_p), \lambda} \bigg( \mathfrak{I}(\lambda) \otimes \mathcal{E} \mathcal{S}_3^\infty, \mathfrak{I}(\lambda) \bigg) = 0 \]
\[ \text{as the other cases are similar. We observe that} \]
\[ \text{is equivalent to the non-existence of a uniserial} \]
\[ \mathfrak{I}(\lambda) \otimes E \mathcal{S}_3^\infty \rightarrow \mathfrak{I}(\lambda) \otimes E \nu^\infty \rightarrow \mathfrak{I}(\lambda) \]
\[ \text{which is again equivalent to the vanishing} \]
\[ \Ext^1_{\GL_3(\mathbb{Q}_p), \lambda} \bigg( \mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes E \mathcal{S}_3^\infty \bigg) = 0 \]
\[ \text{according to the fact} \]
\[ \Ext^1_{\GL_3(\mathbb{Q}_p), \lambda} \bigg( \mathfrak{I}(\lambda), \mathfrak{I}(\lambda) \otimes E \mathcal{S}_3^\infty \bigg) = 0 \]
We also define the (unique up to isomorphism) locally algebraic representation of the form

\[ \left( \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3, \mathcal{L}(\lambda) \otimes_E v_P^\infty \right) \leftarrow \mathcal{F}_{P_1}^{\mathrm{GL}_3}(M_1(-\lambda), \pi_{1,3}^\infty) \rightarrow C_{s_3,s_3-s_1}^2 \]
duces an injection

\[ \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3 \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E v_P^\infty \right) \]

Therefore (4.15) follows from Lemma 2.1 and the facts that

\[ \operatorname{Ext}_{L_i(Q_p),\lambda}^1 \left( H_0(N_i, \mathcal{L}(\lambda)), \mathcal{L}(\lambda) \otimes_E \pi_{i,3}^\infty \right) = \operatorname{Hom}_{L_i(Q_p),\lambda} \left( H_1(N_i, \mathcal{L}(\lambda)), \mathcal{L}(\lambda) \otimes_E \pi_{i,3}^\infty \right) = 0. \]

On the other hand, the short exact sequence

\[ \mathcal{L}(\lambda) \otimes_E v_P^\infty \leftarrow \left( \mathcal{L}(\lambda) \otimes_E v_P^\infty \longrightarrow \mathcal{L}(\lambda) \right) \rightarrow \mathcal{L}(\lambda) \]
duces a long exact sequence

\[ \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E v_P^\infty \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3 \right) \]

and thus we can deduce (4.3) from Proposition 4.1 and (4.4).

We define \( W_0 \) as the unique locally algebraic representation of length three satisfying

\[ \operatorname{soc}_{\mathrm{GL}_3(Q_p)}(W_0) = \mathcal{L}(\lambda) \otimes_E (v_P^\infty \oplus v_P^\infty) \text{ and } \operatorname{cosoc}_{\mathrm{GL}_3(Q_p)}(W_0) = \mathcal{L}(\lambda). \]

We also define the (unique up to isomorphism) locally algebraic representation of the form

\[ W_i := \mathcal{L}(\lambda) \otimes_E v_P^\infty \longrightarrow \mathcal{L}(\lambda) \]

for each \( i = 1, 2 \)

**Lemma 4.6.** We have

\[ \dim_E \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( W_0, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) = 1 \]

and

\[ \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^2 \left( W_0, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) = 0. \]

**Proof.** The short exact sequence

\[ \mathcal{L}(\lambda) \otimes_E v_P^\infty \leftarrow W_0 \rightarrow W_2 \]
duces a long exact sequence

\[ \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( \mathcal{L}(\lambda) \otimes_E v_P^\infty, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^1 \left( W_0, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^2 \left( W_0, \mathcal{L}(\lambda) \otimes_E v_P^\infty, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^2 \left( W_0, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^2 \left( W_2, \mathcal{L}(\lambda) \otimes_E v_P^\infty, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) \rightarrow \operatorname{Ext}_{\mathrm{GL}_3(Q_p),\lambda}^2 \left( W_2, \mathcal{L}(\lambda) \otimes_E \mathrm{St}_3^\infty \right) \]

which finishes the proof by Proposition 4.1, 4.3 and 4.4.

\[ \square \]
We define the following subsets of $\Omega$:
\[
\begin{align*}
\Omega_1 (\mathcal{L}(\lambda)) & := \{ \mathcal{L}(\lambda) \otimes_E v^\infty_{P_1}, \mathcal{L}(\lambda) \otimes_E v^\infty_{P_2}, C_{s_1,1}^1, C_{s_2,1}^1 \} \\
\Omega_1 (\mathcal{L}(\lambda) \otimes_E v^\infty_{P_1}) & := \{ \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E \text{St}^\infty_{3}, C_{s_1,1}, C_{s_2,2}, C_{s_1,s_2}^1 \} \\
\Omega_1 (\mathcal{L}(\lambda) \otimes_E v^\infty_{P_2}) & := \{ \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E \text{St}^\infty_{3}, C_{s_2,1}, C_{s_1,1}, C_{s_1,s_2}^1 \} \\
\Omega_1 (\mathcal{L}(\lambda) \otimes E \text{St}^\infty_{3}) & := \{ \mathcal{L}(\lambda) \otimes E \text{St}^\infty_{3}, C_{s_2,1}, C_{s_1,1}, C_{s_1,s_2}^1 \} \\
\Omega_2 (\mathcal{L}(\lambda)) & := \{ \mathcal{L}(\lambda) \otimes_E \text{St}^\infty_{3}, C_{s_2,1}, C_{s_1,1}, C_{s_1,s_2}^1 \} \\
\Omega_2 (\mathcal{L}(\lambda) \otimes E v^\infty_{P_1}) & := \{ \mathcal{L}(\lambda) \otimes E v^\infty_{P_1}, C_{s_1,1}^1, C_{s_2,2}, C_{s_1,s_2}^1 \} \\
\Omega_2 (\mathcal{L}(\lambda) \otimes E v^\infty_{P_2}) & := \{ \mathcal{L}(\lambda) \otimes E v^\infty_{P_2}, C_{s_2,1}, C_{s_1,1}, C_{s_1,s_2}^1 \} \\
\Omega_2 (\mathcal{L}(\lambda) \otimes E \text{St}^\infty_{3}) & := \{ \mathcal{L}(\lambda), C_{s_1,1,s_2}, C_{s_2,2,s_1}, C_{s_2,2,s_1}^1 \}
\end{align*}
\]

**Proposition 4.7.** We have all explicit formula for
\[
H_k \left( N_i, F^{\GL_3}_{P_j}(M, \pi^\infty_j) \right)
\]
for each smooth admissible representation $\pi^\infty_j$ of $L_j(\mathbb{Q}_p)$, each
\[
M \in \{ L(-\lambda), M_j(-\lambda), L(-s_{3-j} \cdot \lambda), M_j(-s_{3-j} \cdot \lambda), L(-s_{3-j} \cdot s_j \cdot \lambda) \}
\]
and each $0 \leq k \leq 2$, $i, j = 1, 2$.

**Proof.** This follows directly from Section 5.2 and 5.3 of [Bre17]. □

**Lemma 4.8.** For
\[
V_0 \in \{ \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E v^\infty_{P_1}, \mathcal{L}(\lambda) \otimes_E v^\infty_{P_2}, \mathcal{L}(\lambda) \otimes_E \text{St}^\infty_{3} \},
\]
we have
\[
\dim_E \mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (V_0, V) = 1
\]
if $V \in \Omega_1 (V_0)$ and
\[
\mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (V_0, V) = 0
\]
if $V \in \Omega \setminus \Omega_1 (V_0)$.

**Proof.** We only prove the statements for $V_0 = \mathcal{L}(\lambda)$ as other cases are similar. If
\[
V \in \{ \mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E v^\infty_{P_1}, \mathcal{L}(\lambda) \otimes_E v^\infty_{P_2}, \mathcal{L}(\lambda) \otimes_E \text{St}^\infty_{3} \}
\]
then the conclusion follows from Proposition 4.7. If
\[
V = F^{\GL_3}_{P_j}(L(-s_{3-j} \cdot s_j \cdot \lambda), \pi^\infty_j)
\]
for a smooth irreducible representation $\pi^\infty_j$ and $j = 1$ or 2, then it follows from Lemma 2.11 that
\[
\begin{align*}
\mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (H_0(N_j, \mathcal{L}(\lambda)), \mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j) & \hookrightarrow \mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (\mathcal{L}(\lambda), V) \\
& \rightarrow \mathrm{Hom}_{\GL_3(\mathbb{Q}_p), \lambda} (H_1(N_j, \mathcal{L}(\lambda)), \mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j) \\
& \rightarrow \mathrm{Ext}^2_{\GL_3(\mathbb{Q}_p), \lambda} (H_0(N_j, \mathcal{L}(\lambda)), \mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j)
\end{align*}
\]

It follows from Proposition 4.7 and (4.9) that
\[
\begin{align*}
\mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (\mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j) & \hookrightarrow \mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (\mathcal{L}(\lambda), V) \\
& \rightarrow \mathrm{Hom}_{\GL_3(\mathbb{Q}_p), \lambda} (\mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda), \mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j)
\end{align*}
\]

We notice that $Z(L_j(\mathbb{Q}_p))$ acts via different characters on $\mathcal{L}_j(\lambda), \mathcal{L}_j(s_{3-j} \cdot \lambda)$ and $\mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j$, and thus we have the equalities
\[
\begin{align*}
\mathrm{Ext}^1_{\GL_3(\mathbb{Q}_p), \lambda} (\mathcal{L}_j(\lambda), \mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j) & = 0 \\
\mathrm{Hom}_{\GL_3(\mathbb{Q}_p), \lambda} (\mathcal{L}_j(s_{3-j} \cdot \lambda), \mathcal{L}_j(s_{3-j} \cdot s_j \cdot \lambda) \otimes E \pi^\infty_j) & = 0
\end{align*}
\]
which imply that
\[(4.11) \quad \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), \pi_j^\infty) \right) = 0 \]
for each \(\pi_j^\infty\) and \(j = 1, 2\). If
\[ V = \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), \pi_j^\infty) \]
for a smooth irreducible representation \(\pi_j^\infty\) and \(j = 1\) or \(2\), then the short exact sequence
\[ \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), \pi_j^\infty) \rightarrow \mathcal{F}_{P_j}^{GL_3}(M_j(-s_3-j \cdot \lambda), \pi_j^\infty) \rightarrow \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), \pi_j^\infty) \]
duces a long exact sequence
\[ \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), V \right) \rightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_{P_j}^{GL_3}(M_j(-s_3-j \cdot \lambda), \pi_j^\infty) \right) \]
\[ \rightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), \pi_j^\infty) \right) \]
which implies an isomorphism
\[(4.12) \quad \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), V \right) \cong \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_{P_j}^{GL_3}(M_j(-s_3-j \cdot \lambda), \pi_j^\infty) \right) \]
by \[(4.11).\] It follows from Proposition \ref{prop:iso} and Lemma \ref{lem:iso} that
\[(4.13) \quad \text{Ext}^1_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(\lambda), \mathcal{F}_{P_j}^{L_j}(s_3-j \cdot \lambda \otimes_E \pi_j^\infty) \right) \rightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), V \right) \rightarrow \text{Hom}_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(s_3-j \cdot \lambda), \mathcal{F}_{P_j}(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty \right) \rightarrow \text{Ext}^2_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(\lambda), \overline{\mathcal{T}}_j(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty \right). \]
As \(Z(L_j(\mathbb{Q}_p))\) acts via different characters on \(\overline{\mathcal{T}}_j(\lambda)\) and \(\overline{\mathcal{T}}_j(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty\), we have the equalities
\[ \text{Ext}^1_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(\lambda), \overline{\mathcal{T}}_j(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0 \]
\[ \text{Ext}^2_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(\lambda), \overline{\mathcal{T}}_j(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0 \]
which imply that
\[(4.14) \quad \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), V \right) \cong \text{Hom}_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(s_3-j \cdot \lambda), \overline{\mathcal{T}}_j(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty \right). \]
It is then obvious that
\[ \text{Hom}_{L_j(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}_j(s_3-j \cdot \lambda), \overline{\mathcal{T}}_j(s_3-j \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0 \]
for each smooth irreducible \(\pi_j^\infty \neq 1_{L_j}\), and therefore
\[ \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), 1_{L_j}) \right) = 1 \]
and
\[ \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_{P_j}^{GL_3}(L(-s_3-j \cdot \lambda), 1_{L_j}) \right) = 0 \]
for each smooth irreducible \(\pi_j^\infty \neq 1_{L_j}\). Finally, similar methods together with Proposition \ref{prop:iso} also show that
\[ \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{T}}(\lambda), \mathcal{F}_B^{GL_3}(L(-s_1s_2s_1 \cdot \lambda), \chi_w^\infty) \right) = 0 \]
for each \(w \in W\). \[\square\]

We define
\[ \Omega^- := \Omega \setminus \{ \overline{\mathcal{T}}(\lambda), \overline{\mathcal{T}}(\lambda) \otimes_E v_p^\infty, \overline{\mathcal{T}}(\lambda) \otimes_E v_{p_2}^\infty, \overline{\mathcal{T}}(\lambda) \otimes_E St_3^\infty \}. \]
Then we define the following subsets of $\Omega^-$ for $i = 1, 2$:

\[
\begin{align*}
\Omega_i \left( C_{s_i}^1 \right) & := \{ C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^2, C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^2 \}, \\
\Omega_i \left( C_{s_i,s_{3-i}}^2 \right) & := \{ C_{s_i,s_{3-i}}^2, C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^2, C_{s_i,s_{3-i}}^1 \}, \\
\Omega_i \left( C_{s_i,s_{3-i}}^1 \right) & := \{ C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^1 \}, \\
\Omega_i \left( C_{s_i,s_{3-i}}^2 \right) & := \{ C_{s_i,s_{3-i}}^2, C_{s_i,s_{3-i}}^2, C_{s_i,s_{3-i}}^2, C_{s_i,s_{3-i}}^2 \}.
\end{align*}
\]

**Lemma 4.15.** For $V_0 \in \{ C_{s_i}^1, C_{s_i}^2, C_{s_i,s_{3-i}}^1, C_{s_i,s_{3-i}}^2, C_{s_i,s_i} \mid i = 1, 2 \}$, we have

\[
\dim_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(V_0, V) = 1
\]

if $V \in \Omega_i(V_0)$ and

\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(V_0, V) = 0
\]

if $V \in \Omega^- \setminus \Omega_i(V_0)$.

**Proof.** The proof is very similar to that of Lemma 4.15.

**Lemma 4.16.** For $V_0 \in \{ \mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes_E v_{p_1}^\infty, \mathcal{T}(\lambda) \otimes_E v_{p_2}^\infty, \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty \}$, we have

\[
\dim_E \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(V_0, V) = 1
\]

if $V \in \Omega_2(V_0)$ and

\[
\text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(V_0, V) = 0
\]

if $V \in \Omega \setminus \Omega_2(V_0)$.

**Proof.** We only prove the statements for $V_0 = \mathcal{T}(\lambda)$ as other cases are similar. If $V \in \{ \mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes_E v_{p_1}^\infty, \mathcal{T}(\lambda) \otimes_E v_{p_2}^\infty, \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty \}$ then the conclusion follows from Proposition 4.13. We notice that $Z(L_j(\mathbb{Q}_p))$ acts via different characters on $\mathcal{T}_j(\lambda), \mathcal{T}_j(s_{3-j} \cdot \lambda)$ and $\mathcal{T}_j(s_{3-j} s_j \cdot \lambda) \otimes_E \pi_j^\infty$, and thus we have

\[
\begin{align*}
\text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}_j(\lambda), \mathcal{T}_j(s_{3-j} s_j \cdot \lambda) \otimes_E \pi_j^\infty) & = 0, \\
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}_j(s_{3-j} \cdot \lambda), \mathcal{T}_j(s_{3-j} s_j \cdot \lambda) \otimes_E \pi_j^\infty) & = 0, \\
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}_j(s_{3-j} \cdot \lambda), \mathcal{T}_j(s_{3-j} s_j \cdot \lambda) \otimes_E \pi_j^\infty) & = 0.
\end{align*}
\]

On the other hand, we notice that

\[
\text{Hom}_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}_j(s_{3-j} s_j \cdot \lambda), \mathcal{T}_j(s_{3-j} s_j \cdot \lambda) \otimes_E \pi_j^\infty) = 0
\]

for each smooth irreducible $\pi_j^\infty \neq 1_{L_j}$ and

\[
\dim_E \text{Hom}_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}_j(s_{3-j} s_j \cdot \lambda), \mathcal{T}_j(s_{3-j} s_j \cdot \lambda)) = 1.
\]

We combine (4.17), (4.18) and (4.19) with Lemma 2.2 and Proposition 4.17 and deduce that

\[
\text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \mathcal{F}^\text{GL}_3(\mathcal{L}(-s_{3-j} s_j \cdot \lambda), \pi_j^\infty)) = 0
\]

for each smooth irreducible $\pi_j^\infty \neq 1_{L_j}$ and

\[
\dim_E \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \mathcal{F}^\text{GL}_3(\mathcal{L}(-s_{3-j} s_j \cdot \lambda), 1_{L_j})) = 1
\]
which finishes the proof if

\[ V = \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} s_j \cdot \lambda), \pi_j^\infty). \]

Similarly, we have

\begin{equation}
\tag{4.22}
\text{Ext}^2_{L_j(Q_p), \lambda} \left( \mathcal{T}_j(\lambda), \mathcal{T}_j(s_{3-j} \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0,
\end{equation}

\[ \text{Hom}_{L_j(Q_p), \lambda} \left( \mathcal{T}_j(s_{3-j} s_j \cdot \lambda), \mathcal{T}_j(s_{3-j} \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0, \]

\[ \text{Ext}^3_{L_j(Q_p), \lambda} \left( \mathcal{T}_j(\lambda), \mathcal{T}_j(s_{3-j} \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0. \]

On the other hand, we have

\begin{equation}
\tag{4.23}
\text{Ext}^1_{L_j(Q_p), \lambda} \left( \mathcal{T}_j(s_{3-i} \cdot \lambda), \mathcal{T}_j(s_{3-j} \cdot \lambda) \otimes_E \pi_j^\infty \right) = 0
\end{equation}

for each smooth irreducible \( \pi_j^\infty \neq \pi_j^{i,1} \) and

\begin{equation}
\tag{4.24}
\dim_E \text{Ext}^1_{L_j(Q_p), \lambda} \left( \mathcal{T}_j(s_{3-i} \cdot \lambda), \mathcal{T}_j(s_{3-j} \cdot \lambda) \otimes_E \pi_j^{i,1} \right) = 1.
\end{equation}

We combine \( \text{4.22}, \text{4.23} \) and \( \text{4.24} \) with Lemma \( \text{2.1} \) and Proposition \( \text{4.7} \) and deduce that

\begin{equation}
\tag{4.25}
\text{Ext}^2_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_{P_j}^{G_{L_3}}(M_j(-s_{3-j} \cdot \lambda), \pi_j^\infty) \right) = 0
\end{equation}

for each smooth irreducible \( \pi_j^\infty \neq \pi_j^{i,1} \) and

\begin{equation}
\tag{4.26}
\dim_E \text{Ext}^2_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_{P_j}^{G_{L_3}}(M_j(-s_{3-j} \cdot \lambda), \pi_j^{i,1}) \right) = 1.
\end{equation}

The short exact sequence

\[ \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} \cdot \lambda), \pi_j^\infty) \hookrightarrow \mathcal{F}_{P_j}^{G_{L_3}}(M_j(-s_{3-j} \cdot \lambda), \pi_j^\infty) \rightarrow \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} s_j \cdot \lambda), \pi_j^\infty) \]

induces a long exact sequence

\[ \text{Ext}^1_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} s_j \cdot \lambda), \pi_j^\infty) \right) \rightarrow \text{Ext}^2_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} s_j \cdot \lambda), \pi_j^\infty) \right) \]

\rightarrow \text{Ext}^2_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_{G_{L_3}}(M_j(-s_{3-j} \cdot \lambda), \pi_j^\infty) \right) \rightarrow \text{Ext}^2_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} s_j \cdot \lambda), \pi_j^\infty) \right)

which finishes the proof if

\[ V = \mathcal{F}_{P_j}^{G_{L_3}}(L(-s_{3-j} \cdot \lambda), \pi_j^\infty). \]

Finally, similar methods together with Proposition \( \text{4.7} \) also show that

\[ \text{Ext}^2_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{F}_B^{G_{L_3}}(L(-s_1 s_2 s_1 \cdot \lambda), \chi_w) \right) = 0 \]

for each \( w \in W \).

\[ \Box \]

**Lemma 4.27.** We have

\[ \text{Ext}^1_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v_{P_1}^\infty, \mathcal{T}(\lambda), C_{s_1,1}^2 \right) = 0, \]

\[ \text{Ext}^1_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v_{P_1}^\infty, \mathcal{T}(\lambda) \otimes_E S_{s_3}^\infty, C_{s_1,s_3-1}^1 \right) = 0, \]

\[ \text{Ext}^1_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E S_{s_3}^\infty, \mathcal{T}(\lambda) \otimes_E v_{P_1}^\infty, C_{s_1,s_3-1}^2 \right) = 0, \]

\[ \text{Ext}^1_{G_{L_3}(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E S_{s_3}^\infty, \mathcal{T}(\lambda) \otimes_E S_{s_3}^\infty, C_{s_1,s_3-1}^1 \right) = 0. \]

for \( i = 1, 2 \).

**Proof.** We only prove the first vanishing

\begin{equation}
\tag{4.28}
\text{Ext}^1_{G_{L_3}(Q_p), \lambda} \left( W_i, C_{s_1,1}^2 \right) = 0
\end{equation}

as the other cases are similar. The embedding \( C_{s_1,1}^2 \hookrightarrow \mathcal{F}_{P_{3-i}}^{G_{L_3}}(M_{3-i}(-s_i \cdot \lambda), \pi_{3-i,1}^\infty) \)
induces an embedding

\[ (4.29) \quad \Ext^1_{\GL_3(Q_p), \lambda} \left( W_i, C_{s_i,1}^2 \right) \hookrightarrow \Ext^1_{\GL_3(Q_p), \lambda} \left( W_i, \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-s_i \cdot \lambda), \pi_{3-i,1}^\infty) \right). \]

It follows from Proposition 4.7 that

\[ (4.30) \quad \begin{aligned}
H_0(N_{3-i}, W_i) &= \mathcal{T}_{3-i}(\lambda) \otimes_E \left( \mathcal{L}_{B \cap L_{3-i}}(\chi_{s_i,1}^\infty) \otimes \mathcal{O}_{\mathbb{F}_p}^\infty \right), \\
H_1(N_{3-i}, W_i) &= \mathcal{T}_{3-i}(s_i \cdot \lambda) \otimes_E \left( \mathcal{L}_{B \cap L_{3-i}}(\chi_{s_i}^\infty) \otimes \mathcal{O}_{\mathbb{F}_p}^\infty \right).
\end{aligned} \]

We notice that \( \mathcal{Z}(L_{3-i}(Q_p)) \) acts on \( \mathcal{T}_{3-i}(\lambda) \) and \( \mathcal{T}_{3-i}(s_i \cdot \lambda) \) via different characters and that

\[ \Hom_{L_{3-i}(Q_p), \lambda} \left( \mathcal{T}_{3-i}(\lambda) \otimes_E \pi_{3-i,1}^\infty, \mathcal{T}_{3-i}(s_i \cdot \lambda) \otimes_E \pi_{3-i,1}^\infty \right) = 0. \]

Therefore we deduce from (4.30) the equalities

\[ \begin{aligned}
\Ext^1_{L_{3-i}(Q_p), \lambda} \left( H_0(N_{3-i}, W_i), \mathcal{T}_{3-i}(s_i \cdot \lambda) \otimes_E \pi_{3-i,1}^\infty \right) &= 0, \\
\Hom_{L_{3-i}(Q_p), \lambda} \left( H_1(N_{3-i}, W_i), \mathcal{T}_{3-i}(s_i \cdot \lambda) \otimes_E \pi_{3-i,1}^\infty \right) &= 0,
\end{aligned} \]

which imply by Lemma 2.1 that

\[ \Ext^1_{\GL_3(Q_p), \lambda} \left( W_i, \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-s_i \cdot \lambda), \pi_{3-i,1}^\infty) \right) = 0. \]

Hence we finish the proof of (4.25) by the embedding (4.29).

**Lemma 4.31.** We have for \( i = 1, 2 \):

\[ \begin{aligned}
\Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i}^2, C_{s_i,1}^2 \right) &= 0, \\
\Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i}^2, C_{s_i,s_i}^2 \right) &= 0, \\
\Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i,s_i}^2, C_{s_i,1}^2 \right) &= 0, \\
\Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i,s_i}^2, C_{s_i,1}^2 \right) &= 0.
\end{aligned} \]

**Proof.** We only prove that

\[ \Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i}^2, C_{s_i,1}^2 \right) = 0 \]

as the other cases are similar. The surjection

\[ \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-\lambda), \pi_{3-i,2}^\infty) \twoheadrightarrow \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i}^2 \]

and the embedding

\[ C_{s_i,1}^2 \hookrightarrow \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-s_i \cdot \lambda), \pi_{3-i,1}^\infty) \]

induce an embedding

\[ (4.33) \quad \Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E \mathcal{L}_{P_i}^\infty, C_{s_i,s_i}^2, C_{s_i,1}^2 \right) \hookrightarrow \Ext^1_{\GL_3(Q_p), \lambda} \left( \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-\lambda), \pi_{3-i,2}^\infty), \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-s_i \cdot \lambda), \pi_{3-i,1}^\infty) \right). \]

It follows from Proposition 4.7 that

\[ \begin{aligned}
H_0(N_{3-i}, \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-\lambda), \pi_{3-i,2}^\infty)) &= \left( \mathcal{T}_{3-i}(\lambda) \oplus \mathcal{T}_{3-i}(s_i \cdot \lambda) \right) \otimes_E \pi_{3-i,2}^\infty, \\
H_1(N_{3-i}, \mathcal{F}^{\GL_3}_{P_{3-i}}(M_{3-i}(-\lambda), \pi_{3-i,2}^\infty)) &= \left( \mathcal{T}_{3-i}(s_i \cdot \lambda) \oplus \mathcal{T}_{3-i}(s_i,s_i \cdot \lambda) \right) \otimes_E \pi_{3-i,2}^\infty.
\end{aligned} \]
We notice that $Z(L_{3-i}(\mathbb{Q}_p))$ acts on each direct summand of $H_k(N_{3-i}, F_{P_{3-i}}^{GL_3}(M_{3-i}(-\lambda), \pi_{3-i, \lambda}^{\infty}))$ $(k = 0, 1)$ via a different character, and the only direct summand that produces the same character as $\mathcal{T}_{3-i}((s_i \cdot \lambda) \otimes \pi_{3-i,1}^{\infty})$ is $I_{B \cap L_{3-i}}^{L_{3-i}}(\delta_{s_i \cdot \lambda})$. However, we know that

$$\text{cosoc}_{L_{3-i}(\mathbb{Q}_p), \lambda} \left( I_{B \cap L_{3-i}}^{L_{3-i}}(\delta_{s_i \cdot \lambda}) \right) = I_{B \cap L_{3-i}}^{L_{3-i}}(\delta_{s_i \cdot s_i \cdot \lambda})$$

and thus

$$\text{Hom}_{L_{3-i}(\mathbb{Q}_p), \lambda} \left( I_{B \cap L_{3-i}}^{L_{3-i}}(\delta_{s_i \cdot s_i \cdot \lambda}), \mathcal{T}_{3-i}((s_i \cdot \lambda) \otimes \pi_{3-i,1}^{\infty}) \right) = 0.$$

As a result, we deduce the equalities

$$\text{Ext}^1_{L_{3-i}(\mathbb{Q}_p), \lambda} \left( H_0(N_{3-i}, F_{P_{3-i}}^{GL_3}(M_{3-i}(-\lambda), \pi_{3-i,2}^{\infty})), \mathcal{T}_{3-i}((s_i \cdot \lambda) \otimes E \pi_{3-i,1}^{\infty}) \right) = 0$$

and

$$\text{Hom}_{L_{3-i}(\mathbb{Q}_p), \lambda} \left( H_1(N_{3-i}, F_{P_{3-i}}^{GL_3}(M_{3-i}(-\lambda), \pi_{3-i,2}^{\infty})), \mathcal{T}_{3-i}((s_i \cdot \lambda) \otimes E \pi_{3-i,1}^{\infty}) \right) = 0,$$

which imply by Lemma 2.1 that

$$\text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( F_{P_{3-i}}^{GL_3}(M_{3-i}(-\lambda), \pi_{3-i,2}^{\infty}), F_{P_{3-i}}^{GL_3}(M_{3-i}(-s_i \cdot \lambda), \pi_{3-i,1}^{\infty}) \right) = 0.$$

Hence we finish the proof of (4.32) by the embedding (4.33). \qed

**Lemma 4.34.** There exists a unique representation of the form

$$C_{s_i,1} \xrightarrow{C_{s_i,1}^1} \mathcal{T}(\lambda) \otimes_E v_{P_i}^{\infty} \xrightarrow{C_{s_i,1}^2} C_{s_i,1}$$

or of the form

$$C_{s_i,1} \xrightarrow{C_{s_i,1}^1} \mathcal{T}(\lambda) \otimes_E v_{P_{3-i}}^{\infty} \xrightarrow{C_{s_i,1}^2} C_{s_i,1}$$

Proof. We only prove the first statement as the second is similar. It follows from Proposition 4.4.2 of [Bre17] that there exists a unique representation of the form

$$C_{s_i,1} \xrightarrow{C_{s_i,1}^1} \mathcal{T}(\lambda) \otimes_E v_{P_i}^{\infty} \xrightarrow{C_{s_i,1}^2} C_{s_i,1}$$

but it is not proven there whether its quotient

$$C_{s_i,1}^1 \xrightarrow{C_{s_i,1}^1} C_{s_i,1}$$

is split or not. However, If (4.35) is split, then there exists a representation of the form

$$C_{s_i,1}^2 \xrightarrow{C_{s_i,1}^2} \mathcal{T}(\lambda) \otimes_E v_{P_i}^{\infty} \xrightarrow{C_{s_i,1}^2} C_{s_i,1}$$

which contradicts the first vanishing in Lemma 4.31 and thus we finish the proof. \qed

**Remark 4.36.** Our method used in Lemma 4.31 and in Lemma 4.34 is different from the one due to Y. Ding mentioned in part (ii) of Remark 4.4.3 of [Bre17]. It is not difficult to observe that

$$\dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( C_{s_i,1}, C_{s_i,1} \right) = 1$$

(4.37)
and

\[(4.38) \dim_{E} \text{Ext}^{1}_{\text{GL}_{3}(\mathbb{Q}_{p}), \lambda} \left( C_{s_{1}, s_{3}, s_{3} - i}, C_{s_{1}, s_{1}}, C_{s_{3} - i, s_{3} - i} \right) = 1 \]

for \(i = 1, 2\). Similar methods as those used in Proposition 4.4.2 of [Bre17], in Lemma 4.34 and in Lemma 4.31 also imply the existence of a unique representation of the form

\[
\begin{align*}
C_{s_{1}, 1} & \quad \xrightarrow{\mathcal{L}(\lambda)} \quad C_{s_{3} - i, s_{3} - i} \\
C_{s_{3} - i, s_{3} - i} & \quad \xrightarrow{\mathcal{L}(\lambda) \otimes_{E} \mathcal{V}} \quad C_{s_{1}, s_{1}, s_{3}}
\end{align*}
\]

or of the form

\[
\begin{align*}
C_{s_{1}, s_{1}, s_{3} - i} & \quad \xrightarrow{\mathcal{L}(\lambda) \otimes_{E} \mathcal{V}} \quad C_{s_{1}, s_{1}, s_{3}} \\
C_{s_{3} - i, s_{3} - i} & \quad \xrightarrow{\mathcal{L}(\lambda) \otimes_{E} \mathcal{V}} \quad C_{s_{1}, 1}
\end{align*}
\]

5. Computations of Ext II

In this section, we are going to establish several computational results (most notably Lemma 5.8) which have crucial applications in Section 7.

**Lemma 5.1.** We have

\[\dim_{E} \text{Ext}^{1}_{\text{GL}_{3}(\mathbb{Q}_{p}), \lambda} (C_{s_{1}, s_{i}}, \Sigma_{i}(\lambda, \mathcal{Z}_{i})) = 1\]

for \(i = 1, 2\).

**Proof.** We only prove that

\[(5.2) \dim_{E} \text{Ext}^{1}_{\text{GL}_{3}(\mathbb{Q}_{p}), \lambda} (C_{s_{1}, s_{1}}, \Sigma_{1}(\lambda, \mathcal{Z}_{1})) = 1\]

as the other equality is similar. We note that \(\Sigma_{1}(\lambda, \mathcal{Z}_{1})\) admits a subrepresentation of the form

\[
\begin{array}{c}
\mathcal{W} := \mathcal{L}(\lambda) \otimes_{E} \mathcal{V}_{p}^{\infty}_{3} \\
\xrightarrow{C_{s_{2}, s_{1}, 1}} C_{s_{1}, 1} \\
\xrightarrow{\mathcal{L}(\lambda) \otimes_{E} \mathcal{V}_{p}^{\infty}_{3}} C_{s_{3} - i, s_{3} - i}
\end{array}
\]

due to Lemma 3.34, Lemma 3.37 and Remark 3.38 of [BDIS]. Therefore \(\Sigma_{1}(\lambda, \mathcal{Z}_{1})\) admits a filtration such that \(\mathcal{W}\) appears as one term of the filtration and the only reducible graded piece is

\[
\mathcal{V}_{1} := C_{s_{1}, s_{1}, 1} \quad \xrightarrow{\mathcal{L}(\lambda) \otimes_{E} \mathcal{V}_{p}^{\infty}_{3}} \quad C_{s_{2}, s_{1}, 1}
\]

It follows from Lemma 4.4.1 and Proposition 4.2.1 of [Bre17] as well as our Lemma 4.4.7 that

\[(5.3) \text{Ext}^{1}_{\text{GL}_{3}(\mathbb{Q}_{p}), \lambda} (C_{s_{1}, s_{1}}, \mathcal{V}) = 0\]

for all graded pieces \(\mathcal{V}\) such that \(\mathcal{V} \neq \mathcal{V}_{1}\). On the other hand, we have

\[(5.4) \dim_{E} \text{Ext}^{1}_{\text{GL}_{3}(\mathbb{Q}_{p}), \lambda} (C_{s_{1}, s_{1}}, \mathcal{V}_{1}) = 1\]

due to (4.37) and

\[(5.5) \text{Ext}^{2}_{\text{GL}_{3}(\mathbb{Q}_{p}), \lambda} (C_{s_{1}, s_{1}}, \mathcal{L}(\lambda) \otimes_{E} \mathcal{V}_{p}^{\infty}_{3}) = 0\]
by Proposition 4.6.1 of [Bre17]. Hence we finish the proof by combining (5.3), (5.21), (5.5) and part (ii) of Proposition 2.25. \qed

**Lemma 5.6.** We have

\[ \dim E \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_i}^\infty, \Sigma_i^+(\lambda, \mathcal{L}_1) \right) = 3 \]

for \( i = 1, 2 \).

**Proof.** By symmetry, it suffices to prove that

\[ \dim E \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, \Sigma_1^+(\lambda, \mathcal{L}_1) \right) = 3. \]

This follows immediately from Lemma 3.42 of [Bre17] as our \( \Sigma_1^+(\lambda, \mathcal{L}_1) \) can be identified with the locally analytic representation \( \tilde{\Pi}^1(\lambda, \psi) \) defined before (3.76) of [Bre17] up to changes on notation. \( \Box \)

We define \( \Sigma_1^+(\lambda, \mathcal{L}_1) \) (resp. \( \Sigma_2^+(\lambda, \mathcal{L}_2) \)) as the unique non-split extension given by a non-zero element in \( \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( C_{s_1, s_1}, \Sigma_1(\lambda, \mathcal{L}_1) \right) \) (resp. \( \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( C_{s_2, s_2}, \Sigma_2(\lambda, \mathcal{L}_2) \right) \)). Hence we may consider the amalgamate sum of \( \Sigma_1^+(\lambda, \mathcal{L}_1) \) and \( \Sigma_2^+(\lambda, \mathcal{L}_2) \) over \( \text{St}^\text{an}_{3}(\lambda) \) and denote it by \( \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \). In particular, \( \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) has the following form

\[ \text{St}^\text{an}_{3}(\lambda) \rightarrow v_{P_1}^\infty(\lambda) \rightarrow C_{s_1, s_1}. \]

**Lemma 5.7.** We have

\[ \dim E \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 2 \]

for \( i = 1, 2 \).

**Proof.** The short exact sequence

\[ \Sigma_2^+(\lambda, \mathcal{L}_2) \hookrightarrow \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \rightarrow v_{P_1}^\infty(\lambda) \rightarrow C_{s_1, s_1} \]

induces the following long exact sequence

\[ \text{Hom}_{\text{GL}_3(Q_p),\lambda} \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, v_{P_1}^\infty(\lambda) \rightarrow C_{s_1, s_1} \right) \rightarrow \text{Ext}_{\text{GL}_3(Q_p),\lambda} \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, \Sigma_1^+(\lambda, \mathcal{L}_1) \right) \rightarrow \text{Ext}_{\text{GL}_3(Q_p),\lambda} \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, v_{P_1}^\infty(\lambda) \rightarrow C_{s_1, s_1} \right). \]

As a result, we can deduce

\[ \dim E \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 2 \]
from Lemma 5.6 and the facts

\[ \dim E \text{Hom}_{\text{GL}_3(Q_p),\lambda} \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, v_{P_1}^\infty(\lambda) \rightarrow C_{s_1, s_1} \right) = 1 \]
and

\[ \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_1}^\infty, v_{P_1}^\infty(\lambda) \rightarrow C_{s_1, s_1} \right) = 0 \]
by Proposition 4.4.8 and Lemma 5.8. The proof for

\[ \dim E \text{Ext}_{\text{GL}_3(Q_p),\lambda}^1 \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{P_2}^\infty, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 2 \]
is similar. \( \Box \)
Lemma 5.8. We have
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_1, \Sigma^+_i(\lambda, \mathcal{L}_1)) = 0 \]
and in particular
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_1, \Sigma_i(\lambda, \mathcal{L}_1)) = 0 \]
for \( i = 1, 2 \).

Proof. We only need to show the vanishing
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, \Sigma^+_1(\lambda, \mathcal{L}_1)) = 0 \]
as the others are similar or easier. We define \( \nu := \lambda_{T_2 \cdot T_1} \) (which is the restriction of \( \lambda \) from \( T \) to \( T_2 \) via the embedding \( \iota_{T,1} : T_2 \hookrightarrow T \)) and view \( \Sigma^+_{\text{GL}_2}(\nu, \mathcal{L}_1) \) (which is defined before Proposition 3.14) as a locally analytic representation of \( L_1(\mathbb{Q}_p) \) via the projection \( p_1 : L_1 \twoheadrightarrow \text{GL}_2 \) and denote it by \( \Sigma^+_1(\lambda, \mathcal{L}_1) \). We note by definition by of \( \Sigma_1(\lambda, \mathcal{L}_1) \) that we have an isomorphism
\[ \Sigma_1(\lambda, \mathcal{L}_1) \xrightarrow{\sim} I^\text{GL}_3_{P_1}(\Sigma_{L_1}(\lambda, \mathcal{L}_1)) / (\nu_{\mathbb{P}_2}(\lambda) \longrightarrow \mathcal{T}(\lambda)) \].
Therefore we can deduce from the short exact sequence
\[ \Sigma^+_1(\lambda, \mathcal{L}_1) \hookrightarrow \Sigma^+_{\text{GL}_2}(\nu, \mathcal{L}_1) \twoheadrightarrow \tilde{I}(s \cdot \nu) \]
and the fact (up to viewing \( \tilde{I}(s \cdot \nu) \) as a locally analytic representation of \( L_1(\mathbb{Q}_p) \) via the projection \( p_1 \))
\[ C_{s_1, s_1} \cong \text{soc}_{\text{GL}_3(\mathbb{Q}_p)}(I^\text{GL}_3_{P_1}(\tilde{I}(s \cdot \nu))) \]
that we have an injection
\[ \Sigma_1^+(\lambda, \mathcal{L}_1) \hookrightarrow I^\text{GL}_3_{P_1}(\Sigma_{L_1}^+(\lambda, \mathcal{L}_1)) / (\nu_{\mathbb{P}_2}(\lambda) \longrightarrow \mathcal{T}(\lambda)) \]
which induces an injection
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, \Sigma^+_1(\lambda, \mathcal{L}_1)) \hookrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, V) \]
where we use the shorten notation
\[ V := I^\text{GL}_3_{P_1}(\Sigma_{L_1}^+(\lambda, \mathcal{L}_1)) / (\nu_{\mathbb{P}_2}(\lambda) \longrightarrow \mathcal{T}(\lambda)) \].
Note that we have an exact sequence
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, I^\text{GL}_3_{P_1}(\Sigma_{L_1}^+(\lambda, \mathcal{L}_1))) \]
\[ \rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, V) \rightarrow \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, \nu_{\mathbb{P}_2}(\lambda) \longrightarrow \mathcal{T}(\lambda)) \]
It follows from Proposition 4.7 that
\[ H_0(N_1, W_2) = \mathcal{T}_1(\lambda) \otimes_E i^L_{B_1/L_1}(\chi_{s_1}^\infty) \]
\[ H_1(N_1, W_2) = \mathcal{T}_1(s_2 \cdot \lambda) \otimes_E i^L_{B_1/L_1}(\chi_{s_1}^\infty) \]
Therefore we observe that
\[ \text{Hom}_{L_1(\mathbb{Q}_p), \lambda}(H_1(N_1, W_2), \Sigma^+_1(\lambda, \mathcal{L}_1)) = 0 \]
from the action of \( Z(L_1(\mathbb{Q}_p)) \) and
\[ \text{Ext}^1_{L_1(\mathbb{Q}_p), \lambda}(H_0(N_1, W_2), \Sigma^+_1(\lambda, \mathcal{L}_1)) = 0 \]
according to Proposition 3.14 and the natural identification
\[ \text{Ext}^1_{L_1(\mathbb{Q}_p), \lambda}(\lambda, \lambda) \cong \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\lambda, \lambda). \]
As a result, we deduce
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, I^\text{GL}_3_{P_1}(\Sigma_{L_1}^+(\lambda, \mathcal{L}_1))) = 0 \]
from Lemma 2.44. We know that
\[(5.12) \quad \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( W_2, \, v_{P_i}^{an}(\lambda) \longrightarrow \mathcal{L}(\lambda) \right) = 0 \]
due to Proposition 4.11, Lemma 4.16 and a simple devissage, and thus we finish the proof by (5.13). \( \square \)

**Lemma 5.13.** We have
\[(5.14) \quad \dim E \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma^+_i(\lambda, \mathcal{L}_i) \right) = 3 \]
for each \( i = 1, 2, \)
\[(5.15) \quad \dim E \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 2 \]
and
\[(5.16) \quad \dim E \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma^- (\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 1. \]

**Proof.** The equalities (5.15) and (5.16) follow directly from Lemma 2.44 and the fact that
\[(5.17) \quad \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, C_{s_i,s_i} \right) = \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, C_{s_i,s_i} \right) = 0 \]
by Lemma 4.8 and Lemma 4.16 using a long exact sequence induced from the short exact sequence
\[\Sigma_i(\lambda, \mathcal{L}_i) \rightarrow \Sigma^+_i(\lambda, \mathcal{L}_i) \rightarrow C_{s_i,s_i}.\]
Due to a similar argument using (5.17), we only need to show that
\[(5.18) \quad \dim E \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma_i(\lambda, \mathcal{L}_i) \right) = 3 \]
to finish the proof of (5.13). The short exact sequence
\[\text{St}^n_3(\lambda) \rightarrow \Sigma_i(\lambda, \mathcal{L}_i) \rightarrow v_{P_i}^{an}(\lambda)\]
induces a long exact sequence
\[(5.19) \quad \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma_i(\lambda, \mathcal{L}_i) \right) \rightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, v_{P_i}^{an}(\lambda) \right) \]
\[\rightarrow \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \text{St}^n_3(\lambda) \right) \rightarrow \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma_i(\lambda, \mathcal{L}_i) \right) \rightarrow \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, v_{P_i}^{an}(\lambda) \right).\]
We know that
\[(5.20) \quad \dim E \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \text{St}^n_3(\lambda) \right) = 5 \]
by Lemma 4.32. It follows from Proposition 4.11, Lemma 4.16, and a simple devissage that
\[(5.21) \quad \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, v_{P_i}^{an}(\lambda) \right) = 0.\]
Hence it remains to show that
\[(5.22) \quad \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, \Sigma_i(\lambda, \mathcal{L}_i) \right) = 0 \]
to deduce (5.18) from (5.19). The short exact sequence
\[v_{P_i}^{an}(\lambda) \longrightarrow \mathcal{L}(\lambda) \rightarrow I_{P_i}^{\text{GL}_3} (\Sigma_i(\lambda, \mathcal{L}_i) \rightarrow \Sigma_i(\lambda, \mathcal{L}_i) \]
induces
\[\text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, v_{P_i}^{an}(\lambda) \rightarrow \mathcal{L}(\lambda) \right) \]
\[\rightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, I_{P_i}^{\text{GL}_3} (\Sigma_i(\lambda, \mathcal{L}_i)) \rightarrow \Sigma_i(\lambda, \mathcal{L}_i) \right) \]
by the vanishing
\[\text{Ext}^2_{\text{GL}_3(Q_p), \lambda} \left( \mathcal{L}(\lambda), \, v_{P_i}^{an}(\lambda) \rightarrow \mathcal{L}(\lambda) \right) = 0.\]
using Proposition 4.1 and Lemma 6.10. Therefore we only need to show that
\[(5.23) \dim E \text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ v_{P_{i-1}}^\lambda, (\lambda) \to \mathcal{T}(\lambda) \right) = 1 \]
and
\[(5.24) \dim E \text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ I^{GL_3}_{L_i}(\lambda, \mathcal{L}_i) \right) = 1. \]
The equality (5.24) follows from Lemma 2.1 and the facts
\[
\dim E \text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ \Sigma_{L_i}(\lambda, \mathcal{L}_i) \right) = 1, \quad \text{Hom}_{L_i(Q_p), \lambda} \left( H_1(N_i, \mathcal{T}(\lambda)), \ \Sigma_{L_i}(\lambda, \mathcal{L}_i) \right) = 0
\]
where the first equality essentially follows from Lemma 3.14 of [BD15] and the second equality follows from checking the action of \( Z(L_i(Q_p)) \). On the other hand, (5.23) follows from (5.20) and Proposition 1.1 by an easy devissage. Hence we finish the proof.

\textbf{Proposition 5.25.} The short exact sequence
\[
\mathcal{T}(\lambda) \otimes_E v_{P_i}^\infty \to W_i \to \mathcal{T}(\lambda)
\]
induces the following isomorphisms
\[(5.26) \text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v_{P_{i-1}}^\infty, \ \Sigma_{L_i}^+(\lambda, \mathcal{L}_i) \right) \to \text{Ext}^2_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ \Sigma_{L_i}^+(\lambda, \mathcal{L}_i) \right) \]
and
\[(5.27) \text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v_{P_{i-1}}^\infty, \ \Sigma_{L_i}^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) \to \text{Ext}^2_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ \Sigma_{L_i}^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) \]
for \( i = 1, 2 \).

\textbf{Proof.} The vanishing from Lemma 5.5 implies that
\[
\text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v_{P_{i-1}}^\infty, \ \Sigma_{L_i}^+(\lambda, \mathcal{L}_i) \right) \to \text{Ext}^2_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ \Sigma_{L_i}^+(\lambda, \mathcal{L}_i) \right)
\]
is an injection and hence an isomorphism as both spaces have dimension three according to Lemma 5.6 and Lemma 5.13. The proof of (5.26) is similar. We emphasize that both (5.26) and (5.27) can be interpreted as the isomorphism given by the cup product with the one dimensional space
\[
\text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ \mathcal{T}(\lambda) \otimes_E v_{P_{i-1}}^\infty \right).
\]

We define
\[
\Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) := \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) / \mathcal{T}(\lambda) \otimes_E \text{St}_{\infty}^\infty \quad \text{and} \quad \Sigma^\circ_i(\lambda, \mathcal{L}_i) := \Sigma_i(\lambda, \mathcal{L}_i) / \mathcal{T}(\lambda) \otimes_E \text{St}_{\infty}^\infty
\]
for \( i = 1, 2 \).

\textbf{Lemma 5.28.} We have
\[
\dim E \text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 1.
\]

\textbf{Proof.} We define \( \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) as the subrepresentation of \( \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) that fits into the following short exact sequence
\[(5.29) \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \to \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \to \Sigma^1_{s_{2,1}} \oplus C^1_{s_{1,1}} \quad (\text{cf. (2.29) for the definition of } C^1_{s_{2,1}}, C^1_{s_{1,1}}, C^2_{s_{2,1}}, \text{ and } C^2_{s_{1,1}}) \quad \text{and then define } \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \quad \text{as the subrepresentation of } \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \quad \text{that fits into}
\]
\[(5.30) \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \to \Sigma^\circ(\lambda, \mathcal{L}_1, \mathcal{L}_2) \to \left( C^2_{s_{2,1}} \to \mathcal{T}(\lambda) \otimes_E v_{P_i}^\infty \right) \oplus \left( C^2_{s_{2,1}} \to \mathcal{T}(\lambda) \otimes_E v_{P_i}^\infty \right).
\]
It follows from Lemma 5.8 that
\[
\text{Ext}^1_{GL_3(Q_p), \lambda} \left( \mathcal{T}(\lambda), \ V \right) = 0
\]
for each \( V \in \text{JH}_{\text{GL}_3(Q_p)}(\Sigma^{\pm,-}((\lambda, \mathcal{L}_1, \mathcal{L}_2))) \) and therefore

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^{\pm,-}((\lambda, \mathcal{L}_1, \mathcal{L}_2))) = 0
\]

by part (i) of Proposition 2.5. On the other hand, we know from Lemma 4.8 and Lemma 127 that there is no uniserial representation of the form

\[
C_{s,1}^2 \longrightarrow \mathcal{T}(\lambda) \otimes_E v_{p_2}^\infty \longrightarrow \mathcal{L}(\lambda)
\]

which implies that

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^{\pm,-}((\lambda, \mathcal{L}_1, \mathcal{L}_2))) = 0
\]

for \( i = 1, 2 \). Hence we deduce from (5.30), (5.31), (5.32) and Proposition 2.5 that

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^{\pm,-}((\lambda, \mathcal{L}_1, \mathcal{L}_2))) = 0.
\]

Therefore, (5.29) induces an injection

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^3((\lambda, \mathcal{L}_1, \mathcal{L}_2))) \hookrightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), C_{s,1}^1 \oplus C_{s,1}^1).
\]

Assume first that (5.31) is a surjection, then we pick a representation \( W \) represented by a non-zero element in \( \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^3((\lambda, \mathcal{L}_1, \mathcal{L}_2))) \) lying in the preimage of \( \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), C_{s,1}^1) \) under (5.31). We note that there is a short exact sequence

\[
\Sigma^1((\lambda, \mathcal{L}_2)) \hookrightarrow \Sigma^3((\lambda, \mathcal{L}_1, \mathcal{L}_2)) \twoheadrightarrow v_{p_2}^\infty((\lambda)).
\]

We observe that \( \mathcal{T}(\lambda) \) lies above neither \( C_{s,1}^1 \) nor \( \mathcal{T}(\lambda) \otimes_E v_{p_2}^\infty \) inside \( W \) by our definition and (5.32), and thus \( W \) is mapped to zero under the map

\[
f: \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^3((\lambda, \mathcal{L}_1, \mathcal{L}_2))) \rightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), v_{p_2}^\infty((\lambda)))
\]

which means that \( W \) comes from an element in

\[
\ker(f) = \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^1((\lambda, \mathcal{L}_1)))
\]

and in particular

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^1((\lambda, \mathcal{L}_1))) \neq 0.
\]

The short exact sequence

\[
\mathcal{T}(\lambda) \otimes_E v_{p_2}^\infty \rightarrow W_2 \rightarrow \mathcal{T}(\lambda)
\]

induces an injection

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^1((\lambda, \mathcal{L}_1))) \hookrightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \Sigma^1((\lambda, \mathcal{L}_1))).
\]

On the other hand, the short exact sequence

\[
\mathcal{T}(\lambda) \otimes E \text{St}_{3}^\infty \hookrightarrow \Sigma_1((\lambda, \mathcal{L}_1)) \twoheadrightarrow \Sigma^1((\lambda, \mathcal{L}_1))
\]

induces a long exact sequence

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \mathcal{T}(\lambda) \otimes E \text{St}_{3}^\infty) \rightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \Sigma_1((\lambda, \mathcal{L}_1))) \rightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \Sigma^1((\lambda, \mathcal{L}_1)))
\]

which implies

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \Sigma_1((\lambda, \mathcal{L}_1))) \rightarrow \text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \Sigma^1((\lambda, \mathcal{L}_1)))
\]

as we have

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda}(W_2, \mathcal{T}(\lambda) \otimes E \text{St}_{3}^\infty) = \text{Ext}^2_{\text{GL}_3(Q_p), \lambda}(W_2, \mathcal{T}(\lambda) \otimes E \text{St}_{3}^\infty) = 0.
\]
from Lemma 4.2. We combine Lemma 5.8, (5.36) and (5.38) and deduce that

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), \Sigma^0_\iota(\lambda, \mathcal{L}_1) \right) = 0$$

which contradicts (5.39). In all, we have thus shown that

$$\dim_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), \Sigma^0_\iota(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) < \dim_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), C^1_{s_2,1} \oplus C^1_{s_1,1} \right)$$

by combining Lemma 4.8. Finally, the vanishing

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes_E \text{St}^\infty \right) = 0$$

from Proposition 4.1 implies an injection

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) \hookrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right)$$

which finishes the proof by combining Lemma 5.10 and (5.39).

**Lemma 5.40.** We have

$$\dim_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 2.$$

**Proof.** The short exact sequence

$$\Sigma^0_\iota(\lambda, \mathcal{L}_1) \hookrightarrow \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \twoheadrightarrow v^\text{an}_{\mathcal{P}_{s_1-1}}(\lambda)$$

induces a long exact sequence

$$\begin{align*}
\text{Hom}_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, v^\text{an}_{\mathcal{P}_{s_1-1}}(\lambda) \right) & \twoheadrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, \Sigma^0_\iota(\lambda, \mathcal{L}_1) \right) \\
\rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) & \twoheadrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, v^\text{an}_{\mathcal{P}_{s_1-1}}(\lambda) \right).
\end{align*}$$

It is easy to observe that

$$\dim_E \text{Hom}_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, v^\text{an}_{\mathcal{P}_{s_1-1}}(\lambda) \right) = 1$$

and

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, v^\text{an}_{\mathcal{P}_{s_1-1}}(\lambda) \right) = 0$$

from Proposition 4.1 and Lemma 4.8. We can actually observe from Lemma 4.8 that the only $V \in JH_{\text{GL}_3(\mathbb{Q}_p)}(\Sigma^0_\iota(\lambda, \mathcal{L}_1))$ such that

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, V \right) \neq 0$$

is $V = C^2_{s_3-1,1}$ and

$$\dim_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, C^2_{s_3-1,1} \right) = 1.$$

Hence we deduce that

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, \Sigma^0_\iota(\lambda, \mathcal{L}_1) \right) \leq 1$$

and therefore

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E v^\infty_{\mathcal{P}_{s_1-1}}, \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) = 0$$

for $i = 1, 2$. The short exact sequence

$$\mathcal{T}(\lambda) \otimes_E (v^\infty_{\mathcal{P}_1} \oplus v^\infty_{\mathcal{P}_2}) \twoheadrightarrow W_0 \twoheadrightarrow \mathcal{T}(\lambda)$$

induces

$$\begin{align*}
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda), \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) & \twoheadrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( W_0, \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) \\
& \rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \mathcal{T}(\lambda) \otimes_E (v^\infty_{\mathcal{P}_1} \oplus v^\infty_{\mathcal{P}_2}), \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right)
\end{align*}$$
On the other hand, note that by (5.42). Finally, the short exact sequence (5.37) induces which finishes the proof by Lemma 4.8 and thus we have  

\[
\text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma^1(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \to \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) 
\]

which finishes the proof by

\[
\dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 1 \text{ and } \dim_E \text{Ext}^2_{GL_3(Q_p),\lambda}(W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0 
\]

from Lemma 4.6 and by Lemma 5.28 as well as (5.43). □

**Lemma 5.44.** We have the inequality

\[
\dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) \leq 2
\]

for \( i = 1, 2 \).

**Proof.** We know that

\[
\text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) = \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0
\]

for \( i, j = 1, 2 \) from Proposition 4.1 and Lemma 4.8 and thus

\[
\text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0
\]

for \( i, j = 1, 2 \) which together with (5.20) imply that

\[
\dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \leq \dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) = 2 - 1 = 1.
\]

On the other hand, note that

\[
\text{Ext}^1_{GL_3(Q_p),\lambda}(\Sigma(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) = 0
\]

by Lemma 4.8 and thus we have

\[
\dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) \leq \dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) = 1
\]

where the last equality follows again from Lemma 4.8. We finish the proof by combining (5.45) and (5.46) with the inequality

\[
\dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) \leq \dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)) + \dim_E \text{Ext}^1_{GL_3(Q_p),\lambda}(W_0, \Sigma_{i,s}(\lambda, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2)).
\]

□
6. Key exact sequences

Lemma 6.1. We have the inequality
\[
\dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2)) \leq 3.
\]

Proof. The short exact sequence
\[
\Sigma (\lambda, \mathcal{L}_1, \mathcal{L}_2) \hookrightarrow \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2) \twoheadrightarrow C_{s_1, s_1} \oplus C_{s_2, s_2}
\]
induces the exact sequence
\[
(6.2) \quad \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma (\lambda, \mathcal{L}_1, \mathcal{L}_2)) \hookrightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2)) \twoheadrightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, C_{s_1, s_1} \oplus C_{s_2, s_2}).
\]

We know that
\[
\dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, C_{s_1, s_1} \oplus C_{s_2, s_2}) = \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, C_{s_1, s_1}) + \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, C_{s_2, s_2}) = 1 + 1 = 2
\]
by Lemma 4.8 and Lemma 4.16. We also know that
\[
\dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma (\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 2
\]
by Lemma 5.40 and thus we obtain the following inequality:
\[
(6.3) \quad \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2)) \leq \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma (\lambda, \mathcal{L}_1, \mathcal{L}_2)) + \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, C_{s_1, s_1} \oplus C_{s_2, s_2}) = 2 + 2 = 4.
\]
Assume first that
\[
(6.4) \quad \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 4.
\]

The short exact sequence
\[
\Sigma^+_1 (\lambda, \mathcal{L}_1) \hookrightarrow \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2) \twoheadrightarrow \left( v^p_{P_3}(\lambda) \right.
\]
induces a long exact sequence
\[
(6.5) \quad \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+_1 (\lambda, \mathcal{L}_1)) \hookrightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+ (\lambda, \mathcal{L}_1, \mathcal{L}_2)) \twoheadrightarrow \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, v^p_{P_3}(\lambda) \longrightarrow C_{s_2, s_2})
\]
which implies
\[
(6.6) \quad \dim_E \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} (W_0, \Sigma^+_1 (\lambda, \mathcal{L}_1)) \geq 2
\]
by (6.4) and Lemma 5.44. We observe that \(\Sigma^+_1 (\lambda, \mathcal{L}_1)\) admits a filtration whose only reducible graded piece is
\[
C^2_{s_1, 1} \longrightarrow \mathcal{L}(\lambda) \otimes_E v_p^\infty
\]
and thus it follows from Lemma 4.8 and
\[
\text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \mathcal{L}(\lambda) \otimes_E v_p^\infty, C^2_{s_1, 1} \otimes \mathcal{L}(\lambda) \otimes_E v_p^\infty \right) = 0
\]
(coming from Proposition 4.1, Lemma 4.8 together with a simple devisage) that
\[
\text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \mathcal{L}(\lambda) \otimes_E v_p^\infty, V \right) = 0
\]
for all graded pieces of such a filtration except the subrepresentation \(\mathcal{L}(\lambda) \otimes E \text{St}^\infty_3\). Hence we deduce by part (ii) of Proposition 2.3 an isomorphism of one dimensional spaces
\[
(6.7) \quad \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \mathcal{L}(\lambda) \otimes_E v_p^\infty, \mathcal{L}(\lambda) \otimes_E \text{St}^\infty_3 \right) \cong \text{Ext}^1_{GL_3(\mathbb{Q}_p), \lambda} \left( \mathcal{L}(\lambda) \otimes_E v_p^\infty, \Sigma^+_1 (\lambda, \mathcal{L}_1) \right).
\]
Then the short exact sequence
\[
\mathcal{L}(\lambda) \otimes_E v_p^\infty \hookrightarrow W_0 \twoheadrightarrow W_2
\]
induces a long exact sequence

\[ \text{Ext}^1_{GL_3(Q_p), \lambda}(W_2, \Sigma^+(\lambda, \mathcal{L}_1)) \to \text{Ext}^1_{GL_3(Q_p), \lambda}(W_0, \Sigma^+(\lambda, \mathcal{L}_1)) \]
\[ \to \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda) \otimes_E v_{P_1}^\infty, \Sigma^+(\lambda, \mathcal{L}_1)) \]

which together with (6.6) and (6.7) implies that

\[ \dim_E \text{Ext}^1_{GL_3(Q_p), \lambda}(W_2, \Sigma^+(\lambda, \mathcal{L}_1)) \geq 1 \]

which contradicts Lemma 5.8. Hence we finish the proof. \( \square \)

**Proposition 6.8.** We have

\[ \dim_E \text{Ext}^1_{GL_3(Q_p), \lambda}(W_0, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 3. \]

**Proof.** The short exact sequence

\[ \mathcal{T}(\lambda) \otimes_E (v_{P_2}^\infty \oplus v_{P_1}^\infty) \to W_0 \to \mathcal{T}(\lambda) \]

induces a long exact sequence

\[ \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \to \text{Ext}^1_{GL_3(Q_p), \lambda}(W_0, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]
\[ \to \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda) \otimes_E (v_{P_2}^\infty \oplus v_{P_1}^\infty), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]
\[ \to \text{Ext}^2_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]

and thus we have

\[ \dim_E \text{Ext}^1_{GL_3(Q_p), \lambda}(W_0, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]
\[ \geq \dim_E \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) + \dim_E \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda) \otimes_E (v_{P_2}^\infty \oplus v_{P_1}^\infty), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]
\[ \geq \dim_E \text{Ext}^2_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \]
\[ = 1 + 4 - 2 = 3 \]

due to Lemma 5.7 and Lemma 5.13 which finishes the proof by combining with Lemma 6.1. \( \square \)

We define \( \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) as the unique non-split extension of \( \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) by \( \mathcal{T}(\lambda) \) (cf. Lemma 2.34) and then set \( \Sigma^0+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) to be the amalgamate sum of \( \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) and \( \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) over \( \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \). Hence \( \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) has the form

\[ \text{St}^\text{an}(\lambda) \longrightarrow v_{P_2}^\infty(\lambda) \longrightarrow v_{P_1}^\infty(\lambda) \longrightarrow \mathcal{T}(\lambda) \]

and \( \Sigma^0+(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) has the form

\[ \text{St}^\text{an}(\lambda) \longrightarrow v_{P_2}^0(\lambda) \longrightarrow v_{P_1}^0(\lambda) \longrightarrow \mathcal{T}(\lambda) \lor C_{s_1, s_2} \]

It follows from Lemma 2.34, Proposition 4.1, 5.17 and an easy devissage that

\[ \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^0+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0. \]

Then we set

\[ \Sigma^* = \Sigma^*(\lambda, \mathcal{L}_1, \mathcal{L}_2) := \Sigma^*(\lambda, \mathcal{L}_1, \mathcal{L}_2) / \mathcal{T}(\lambda) \otimes_E \text{St}^\infty_3 \]

for \( * = \{+, [0]\} \) and \( \{0, +\} \). It follows from Lemma 5.28, 5.17 and an easy devissage that

\[ \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^0(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = \text{Ext}^1_{GL_3(Q_p), \lambda}(\mathcal{T}(\lambda), \Sigma^0+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0. \]
We emphasize that the isomorphism (6.16) can be naturally interpreted as the cup product map from Proposition 5.25 an isomorphism of two dimensional spaces

Proof. It follows from (5.17) that we only need to show that

$\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^1(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0$ and $\text{dim}_E \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^2(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 2$.

These results follow from combining the long exact sequence

$$
\text{Hom}_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda)) \to \text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\to \text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^1(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \to \text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^1(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\to \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda)),$$

with Lemma 2.34 and the equalities

$$\text{dim}_E \text{Hom}_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda)) = 1$$
$$\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda)) = 0$$
$$\text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda)) = 0$$
due to Proposition 4.1.

Lemma 6.14. We have

$$\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^{1,2}(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0$$

and

$$\text{dim}_E \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^{1,2}(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = \text{dim}_E \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^{1,2}(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \geq 1.$$

Proof. It follows from (5.17) that we only need to show that

$$\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^{1,2}(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 0$$

which follow from combining (6.12), Lemma 6.13 and the long exact sequence

(6.15) $\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda) \otimes E \text{St}^\infty_3) \to \text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\to \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^2(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \to \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^2(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\to \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^2(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\to \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda) \otimes E \text{St}^\infty_3)$

with the equalities

$$\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda) \otimes E \text{St}^\infty_3) = 0$$
$$\text{dim}_E \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \overline{L}(\lambda) \otimes E \text{St}^\infty_3) = 1$$
due to Proposition 4.1.

We use the shortening notation $\mathcal{L} := (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1', \mathcal{L}_2')$ for a tuple of four elements in $E$. We recall from Proposition 5.25 an isomorphism of two dimensional spaces

(6.16) $\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda) \otimes E v_1^\infty, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \sim \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)).$

We emphasize that the isomorphism (6.16) can be naturally interpreted as the cup product map

(6.17) $\text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda) \otimes E v_1^\infty, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \cup \text{Ext}^1_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\to \text{Ext}^2_{GL_3(Q_p),\lambda}(\overline{L}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))$
where $\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p)}(\mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes_E v^\infty_{P_2})$ is one dimensional by Proposition 4.4. We recall from the proof of Lemma 4.3 that there is a canonical isomorphism
\[
\text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p)}(\mathcal{T}(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \cong \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]
which together with Lemma 2.3.3 implies that $\text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))$ admits a basis of the form
\[
\{ \kappa(b_{1,\text{val}_p} \land b_{2,\text{val}_p}), \iota_1(D_0) \},
\]
and therefore the element
\[
\iota_1(D_0) + \mathcal{L}_1 \kappa(b_{1,\text{val}_p} \land b_{2,\text{val}_p})
\]
generates a line in $\text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))$. We define $\Sigma^+_1(\lambda, \mathcal{L}_1, \mathcal{L}_2)$ as the representation represent by the preimage of
\[
\iota_1(D_0) + \mathcal{L}_1 \kappa(b_{1,\text{val}_p} \land b_{2,\text{val}_p})
\]
in $\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda) \otimes_E v^\infty_{P_1}, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))$ via (6.16) for $i = 1, 2$. Then we define $\Sigma^+(\lambda, \mathcal{L})$ as the amalgamate of $\Sigma^+_1(\lambda, \mathcal{L}_1, \mathcal{L}_2)$ and $\Sigma^+_2(\lambda, \mathcal{L}_1, \mathcal{L}_2)$ over $\Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2)$, and therefore $\Sigma^+(\lambda, \mathcal{L})$ has the form
\[
\begin{array}{cccc}
\text{St}_{3}^{	ext{an}}(\lambda) & \overset{v^\text{an}_{P_1}(\lambda)}{\longrightarrow} & C_{s_1, s_1} & \rightarrow \mathcal{T}(\lambda) \otimes_E v^\infty_{P_2} \\
\overset{v^\text{an}_{P_2}(\lambda)}{\longrightarrow} & C_{s_2, s_2} & \rightarrow \mathcal{T}(\lambda) \otimes_E v^\infty_{P_1}
\end{array}
\]
We define $\Sigma^+\beta(\lambda, \mathcal{L})$ as the amalgamate sum of $\Sigma^+(\lambda, \mathcal{L})$ and $\Sigma^+\beta(\lambda, \mathcal{L}_1, \mathcal{L}_2)$ over $\Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2)$, and thus $\Sigma^+\beta(\lambda, \mathcal{L})$ has the form
\[
\begin{array}{cccc}
\text{St}_{3}^{	ext{an}}(\lambda) & \overset{v^\text{an}_{P_1}(\lambda)}{\longrightarrow} & \mathcal{T}(\lambda) \otimes_E v^\infty_{P_2} \\
\overset{v^\text{an}_{P_2}(\lambda)}{\longrightarrow} & \mathcal{T}(\lambda) \otimes_E v^\infty_{P_1}
\end{array}
\]
We also need the quotients
\[
\Sigma^+\beta(\lambda, \mathcal{L}) := \Sigma^+(\lambda, \mathcal{L}) / \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty, \quad \Sigma^+\beta(\lambda, \mathcal{L}_1, \mathcal{L}_2) := \Sigma^+\beta(\lambda, \mathcal{L}) / \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty.
\]

**Lemma 6.18.** We have the inequality
\[
\text{dim}_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+\beta(\lambda, \mathcal{L})) \leq 1.
\]

**Proof.** The short exact sequence
\[
\Sigma^+\beta(\lambda, \mathcal{L}_1, \mathcal{L}_2) \rightarrow \Sigma^+\beta(\lambda, \mathcal{L}) \rightarrow \mathcal{T}(\lambda) \otimes_E (v^\infty_{P_2} \oplus v^\infty_{P_1})
\]
duces an injection
\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma^+\beta(\lambda, \mathcal{L})) \rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes_E (v^\infty_{P_2} \oplus v^\infty_{P_1}))
\]
by Lemma 6.14. Note that we have
\[
\text{dim}_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \mathcal{T}(\lambda) \otimes_E (v^\infty_{P_2} \oplus v^\infty_{P_1})) = 2
\]
by Proposition 4.1. Assume first that (6.19) is a surjection, and thus we can pick a representation $W$ represented by a non-zero element lying in the preimage of $\mathcal{T}(\lambda) \otimes_E v^\infty_{P_2}$ under (6.19). We observe that the very existence of $W$ implies that
\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda}(W_2, \Sigma^+\beta(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \neq 0.
\]
We define
\[
\Sigma^+\beta(\lambda, \mathcal{L}_1) := \Sigma^+\beta(\lambda, \mathcal{L}_1) / \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty
\]
and thus we have an embedding
\[ \Sigma^1_{-i}((\lambda, \mathcal{Z}_1)) \rightarrow \Sigma^1_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) \]
for each \( i = 1, 2 \). We notice that the quotient \( \Sigma^1_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2))/\Sigma^1_{-i}((\lambda, \mathcal{Z}_1)) \) fits into a short exact sequence
\[ \begin{array}{c}
0 \\
\rightarrow \mathcal{N}_i((\lambda, \mathcal{Z}_1)) \\
\rightarrow \Sigma^1_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) \\
\rightarrow \mathcal{N}(\lambda) \\
\rightarrow \Sigma^1_{-i}((\lambda, \mathcal{Z}_1)) \\
\rightarrow 0
\end{array} \]
Hence it remains to show the equality
\[ (6.21) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( W_2, \mathcal{N}_i((\lambda, \mathcal{Z}_1)) \right) = 0 \]
and the equality
\[ (6.22) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} (W_2, C_{s_2, s_2}) = 0 \]
to finish the proof of
\[ (6.23) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( W_2, \Sigma_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2))/\Sigma_{-i}((\lambda, \mathcal{Z}_1)) \right) = 0. \]

The vanishing \( (6.22) \) follows from Lemma \( 4.8 \) and part (i) of Proposition \( 2.5 \). It follows from Proposition \( 4.1 \), Lemma \( 5.8 \) and a simple devissage that
\[ (6.24) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{L}(\lambda) \otimes E v_P^\infty, C_{s_1, 1} \right) = \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{L}(\lambda), C_{s_1, 1} \longrightarrow \overline{L}(\lambda) \right) = 0. \]

Hence if
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( W_2, C_{s_1, 1} \longrightarrow \overline{L}(\lambda) \right) \neq 0 \]
then there exists a uniserial representation of the form
\[ C_{s_1, 1} \longrightarrow \overline{L}(\lambda) \longrightarrow \overline{L}(\lambda) \otimes E v_P^\infty \]
which contradicts \( (6.23) \) and Lemma \( 4.27 \). As a result, we have shown that
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( W_2, C_{s_1, 1} \longrightarrow \overline{L}(\lambda) \right) = 0 \]
which together with Proposition \( 4.1 \) and part (i) of Proposition \( 2.5 \) implies \( (6.21) \) and hence \( (6.23) \) as well concerning \( (6.22) \). Therefore we can combine \( (6.23) \) with Lemma \( 6.8 \) and conclude that
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( W_2, \Sigma_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) \right) = 0 \]
which contradicts \( (6.20) \). Consequently, the injection \( (6.19) \) must be strict and we finish the proof. \( \square \)

According to Lemma \( 6.14 \) the short exact sequence
\[ \Sigma_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) \rightarrow \Sigma_{-i}((\lambda, \mathcal{Z}_2)) \rightarrow \overline{L}(\lambda) \otimes E (v_P^\infty \oplus v_P^\infty) \]
induces a long exact sequence:
\[ (6.25) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{L}(\lambda), \Sigma_{-i}((\lambda, \mathcal{Z}_2)) \right) \rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{L}(\lambda), \overline{L}(\lambda) \otimes E (v_P^\infty \oplus v_P^\infty) \right) \]
\[ \rightarrow \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{L}(\lambda), \Sigma_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2)) \right) \]

**Proposition 6.26.** We have
\[ \dim_E \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} (\overline{L}(\lambda), \Sigma_{-i}((\lambda, \mathcal{Z}_2))) = 1 \]
and the image of \( f \) is not contained in the image of the natural injection
\[ \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda} (\overline{L}(\lambda), \overline{L}(\lambda) \otimes E \text{St}^\infty) \rightarrow \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda} (\overline{L}(\lambda), \Sigma_{-i}((\lambda, \mathcal{Z}_1, \mathcal{Z}_2))) \].
Proof. We use the shorten notation for the two dimensional space
\[ M := \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E (v_P^\infty \oplus v_P^\infty)). \]

We actually have the following commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L})) & \xrightarrow{i} & \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L}_1, \mathcal{L}_2)) \\
h & & \downarrow k \\
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L})) & \xrightarrow{j} & \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\end{array}
\]

where the middle vertical map is just an equality. We know that \( h \) is injective by the vanishing
\[ \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E \text{St}_3^\infty) = 0 \]
and \( k \) has a one dimensional image by (6.15). Both \( i \) and \( j \) are injective due to (6.11) and (6.12). Therefore by a simple diagram chasing we have

\[
\dim_E \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L})) = \dim_E M - \dim_E \text{Im}(g) \geq \dim_E M - \dim_E \text{Im}(k) = 2 - 1 = 1
\]

by Lemma 6.14 and therefore

\[
\dim_E \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L})) = 1
\]

by Lemma 6.18. Moreover, the map \( g \) has a one dimensional image and hence \( k \circ f \) has one dimensional image, meaning that the image of \( f \) has dimension one or two and is not contained in \( \text{Ker}(k) \), which is exactly the image of

\[
\text{Ext}^2_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E \text{St}_3^\infty) \rightarrow \text{Ext}^2_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]

by (6.15). In fact, the restriction of \( f \) to the direct summand \( \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E v_P^\infty) \) is given by the cup product map with a non-zero element in the line of

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda) \otimes_E v_P^\infty, \Sigma^+(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]
given by the preimage of

\[
E \left( t_1(D_0) + \mathcal{L}_1\kappa(b_1, \text{val}_p \wedge b_2, \text{val}_p) \right)
\]
via (6.19) by our definition of \( \Sigma^{\ast, +}(\lambda, \mathcal{L}) \) and it is obvious that \( t_1(D_0) + \mathcal{L}_1\kappa(b_1, \text{val}_p \wedge b_2, \text{val}_p) \) does not lie in the image of (6.28), which is exactly the line \( E\kappa(b_1, \text{val}_p \wedge b_2, \text{val}_p) \).

\[ \square \]

Proposition 6.29. We have

\[
\dim_E \text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L})) = 1
\]

if and only if \( \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 \) for a certain \( \mathcal{L}_3 \in E \).

Proof. It follows from (6.28) that

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L})) = 1
\]

if and only if the image of \( f \) is one dimensional. Then we notice by the interpretation of \( f \) as cup product in Proposition 6.26 that the image of

\[
\text{Ext}^1_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \mathcal{L}(\lambda) \otimes_E v_P^\infty)
\]
under \( f \) is the line of

\[
\text{Ext}^2_{\text{GL}_3(Q_p), \lambda} (\mathcal{L}(\lambda), \Sigma^{\ast, +}(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]
generated by

\[
t_1(D_0) + \mathcal{L}_1\kappa(b_1, \text{val}_p \wedge b_2, \text{val}_p)
\]
for each \( i = 1, 2 \). Therefore the image of \( f \) is one dimensional if and only if the two lines for \( i = 1, 2 \) coincide which means that \( \mathcal{L}'_1 = \mathcal{L}'_2 = \mathcal{L}_3 \) for a certain \( \mathcal{L}_3 \in E \).

We use the notation \( \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) for the representation \( \Sigma^{\geq +}(\lambda, \mathcal{L}) \) when

\[
\mathcal{L} = (L_1, L_2, L_3, L_3).
\]

We define \( \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) as the unique representation (up to isomorphism) given by a non-zero element in \( \text{Ext}^1_{GL_3(V_p)}(L(\lambda), \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)) \) according to Proposition 6.29. Therefore by our definition \( \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) has the following form

\[
(6.30) \quad \text{St}_3^{\min}(\lambda) \longrightarrow \text{Ext}^1_{GL_3(V_p)}(L(\lambda), \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)) = 0.
\]

It follows from Proposition 6.1 and Proposition 6.29 the definition of \( \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) and an easy devissage that

\[
(6.31) \quad \text{dim}_E \text{Ext}^1_{GL_3(V_p)}(W_0, \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2)) = 2.
\]

Moreover, if \( V \) is a locally analytic representation determined by a line

\[
M_V \subseteq \text{Ext}^1_{GL_3(V_p)}(W_0, \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]

satisfying

\[
M_V \neq \text{Ext}^1_{GL_3(V_p)}(W_0, \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2)),
\]

then there exists a unique \( \mathcal{L}_3 \in E \) such that

\[
V \cong \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3).
\]

**Proof.** The short exact sequence

\[
\text{Ext}^1_{GL_3(V_p)}(W_0, V^+) \longrightarrow \text{Ext}^1_{GL_3(V_p)}(V_1^{\text{alg}} \oplus V_2^{\text{alg}}, V^+) \longrightarrow \text{Ext}^2_{GL_3(V_p)}(L(\lambda), V^+)
\]

induce a commutative diagram

\[
(6.34) \quad \begin{array}{ccc}
\text{Ext}^1_{GL_3(V_p)}(W_0, V^+) & \xrightarrow{g_1} & \text{Ext}^1_{GL_3(V_p)}(V_1^{\text{alg}} \oplus V_2^{\text{alg}}, V^+) \\
\downarrow h_1 & & \downarrow h_2 \\
\text{Ext}^1_{GL_3(V_p)}(W_0, V^{\sharp +}) & \xrightarrow{g_2} & \text{Ext}^1_{GL_3(V_p)}(V_1^{\text{alg}} \oplus V_2^{\text{alg}}, V^{\sharp +})
\end{array}
\]

where we use shorten notation \( V_i^{\text{alg}} \) for \( L(\lambda) \otimes_E v_i^{\infty} \), \( V^+ \) for \( \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) and \( V^{\sharp +} \) for \( \Sigma^{\geq +}(\lambda, \mathcal{L}_1, \mathcal{L}_2) \) to save space. We observe that \( g_2 \) is an injection due to Lemma 6.13 and \( h_1 \) is a surjection by the proof of Proposition 6.8 and an easy devissage and finally \( h_2 \) is
an injection. Assume that \( h_2 \) is not surjective, then any representation given by a non-zero element in \( \text{Coker}(h_2) \) admits a quotient of the form
\[
(6.35) \quad C_{A,1}^1 \longrightarrow T(\lambda) \longrightarrow V^\text{alg}_i
\]
for \( i = 1 \) or \( 2 \) due to Lemma 4.38. However, it follows from Lemma 4.27 that there is no uniserial representation of the form \( (6.35) \), which implies that \( h_2 \) is indeed an isomorphism, and hence \( k_2 \) is surjective by a diagram chasing. Therefore we conclude that
\[
\dim E \operatorname{Ext}^1_{GL_3(Q_p),\lambda}(W_0, V^{\ast,+}) = \dim E \operatorname{Ext}^1_{GL_3(Q_p),\lambda}(V^\text{alg}_1 \oplus V^\text{alg}_2, V^{\ast,+}) - \dim E \operatorname{Ext}^2_{GL_3(Q_p),\lambda}(T(\lambda), V^{\ast,=})
\]
\[
= \dim E \operatorname{Ext}^1_{GL_3(Q_p),\lambda}(V^\text{alg}_1 \oplus V^\text{alg}_2, V^{\ast,+}) - \dim E \operatorname{Ext}^2_{GL_3(Q_p),\lambda}(T(\lambda), V^{\ast,=}) = 4 - 2 = 2.
\]
The final claim on the existence of a unique \( \mathcal{L}_3 \) follows from Proposition 6.29; our definition of \( \Sigma^\text{im}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) and the observation that the restriction of \( k_2 \) to the direct summand
\[
\operatorname{Ext}^1_{GL_3(Q_p),\lambda}(V^\text{alg}_i, V^{\ast,+})
\]
duces isomorphisms
\[
\operatorname{Ext}^1_{GL_3(Q_p),\lambda}(V^\text{alg}_i, V^{\ast,+}) \cong \operatorname{Ext}^2_{GL_3(Q_p),\lambda}(T(\lambda), V^{\ast,=})
\]
which can be interpreted as the cup product morphism with the one dimensional space
\[
\operatorname{Ext}^1_{GL_3(Q_p),\lambda}(T(\lambda), V^\text{alg}_i)
\]
for \( i = 1, 2 \).

We define \( \Sigma_i^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) as the subrepresentation of \( \Sigma^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) that fits into the short exact sequence
\[
\Sigma_i^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \hookrightarrow \Sigma^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \twoheadrightarrow T(\lambda) \otimes_E v^{\infty}_P
\]
for each \( i = 1, 2 \). We use the notation \( \mathcal{D}_i(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3') \) for the object in the derived category \( D^b(\text{Mod}_{D(\text{GL}_3(Q_p), \mathcal{E})}) \) associated with the complex
\[
[W_3 \rightarrow \Sigma_i^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3')].
\]

**Proposition 6.36.** The object
\[
\mathcal{D}_i(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3') \in D^b(\text{Mod}_{D(\text{GL}_3(Q_p), \mathcal{E})})
\]
fits into the distinguished triangle
\[
(6.37) \quad T(\lambda)' \longrightarrow \mathcal{D}_i(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3') \longrightarrow \Sigma_i^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2)'[-1] \xrightarrow{+1}
\]
for each \( i = 1, 2 \). Moreover, the element in
\[
(6.38) \quad \operatorname{Ext}^2_{GL_3(Q_p),\lambda}(T(\lambda), \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]
\[
\cong \operatorname{Ext}^2_{GL_3(Q_p),\lambda}(T(\lambda), \Sigma^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2))
\]
\[
\cong \operatorname{Hom}_{D^b(\text{Mod}_{D(\text{GL}_3(Q_p), \mathcal{E})})}(\Sigma^{\ast,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2)'[-2], T(\lambda)')
\]
associated with the distinguished triangle \( (6.37) \) is
\[
(6.39) \quad \iota_1(D_0) + \mathcal{L}_3\kappa(b_{1, \text{val}_p} \land b_{2, \text{val}_p}).
\]
**Proof.** It follows from Proposition 3.2 of \cite{Schr11} that there is a unique (up to isomorphism) object
\[ D(\lambda, L_1, L_2, L_3)' \in D^b(\text{Mod}_{D(GL_3(\mathbb{Q}_p))}) \]
that fits into a distinguished triangle
\[ (6.40) \quad \mathcal{T}(\lambda)' \rightarrow D(\lambda, L_1, L_2, L_3) \rightarrow \Sigma^{\delta,+}(\lambda, L_1, L_2)[{-1}] \rightarrow +1 \]
such that the element in\( \text{Ext}^2_{\text{GL}_3(\mathbb{Q}_p), \lambda}(\mathcal{T}(\lambda), \Sigma(\lambda, L_1, L_2)) \) associated with (6.40) via (6.38) is (6.39).

It follows from TR2 (cf. Section 10.2.1 of \cite{Wei94}) that (6.16) can be reinterpreted as the isomorphism
\[ (6.41) \quad D(\lambda, L_1, L_2, L_3)' \rightarrow \Sigma^{\delta,+}(\lambda, L_1, L_2)'[-1] \rightarrow \mathcal{T}(\lambda)'[1] \rightarrow +1 \]
is another distinguished triangle. The isomorphism (6.16) can be reinterpreted as the isomorphism
\[ (6.42) \quad \text{Hom}_{D^b(\text{Mod}_{D(GL_3(\mathbb{Q}_p))})}(\Sigma^{\delta,+}(\lambda, L_1, L_2)'[-1], (\mathcal{T}(\lambda) \otimes_E v_{p_{\delta,-}}^\infty)'(\lambda)) \rightarrow \text{Hom}_{D^b(\text{Mod}_{D(GL_3(\mathbb{Q}_p))})}(\Sigma^{\delta,+}(\lambda, L_1, L_2)'[-1], \mathcal{T}(\lambda)'[1]) \]
induced by the composition with \( \text{Hom}_{D^b(\text{Mod}_{D(GL_3(\mathbb{Q}_p))})}(\mathcal{T}(\lambda) \otimes_E v_{p_{\delta,-}}^\infty)'(\lambda), \mathcal{T}(\lambda)'[1]) \).

As a result, each morphism
\[ \Sigma^{\delta,+}(\lambda, L_1, L_2)'[-1] \rightarrow \mathcal{T}(\lambda)'[1] \]
uniquely factors through a composition
\[ \Sigma^{\delta,+}(\lambda, L_1, L_2)'[-1] \rightarrow (\mathcal{T}(\lambda) \otimes_E v_{p_{\delta,-}}^\infty)'(\lambda) \rightarrow \mathcal{T}(\lambda)'[1] \]
which induces a commutative diagram with four distinguished triangles
\[ (6.43) \quad \Sigma^{\delta,+}(\lambda, L_1, L_2)'[-1] \]
by TR4. Hence we deduce that
\[ \Sigma^{\delta,+}(\lambda, L_1, L_2)' \rightarrow D(\lambda, L_1, L_2, L_3)' \rightarrow W_{p_{\delta,-}}'[1] \rightarrow +1 \]
or equivalently
\[ W_{3-i}' \rightarrow \Sigma_{i}^{1}((\lambda, L_1, L_2, L_3))' \rightarrow D((\lambda, L_1, L_2, L_3))' \rightarrow \mathbb{1} \]
is a distinguished triangle. On the other hand, it is easy to see that \( D((\lambda, L_1, L_2, L_3))' \) fits into the distinguished triangle
\[ W_{3-i}' \rightarrow \Sigma_{i}^{1}((\lambda, L_1, L_2, L_3))' \rightarrow D((\lambda, L_1, L_2, L_3))' \rightarrow \mathbb{1} \]
and thus we conclude that
\[ \text{Ext}^{0}_{\text{GL}_{3}(\mathbb{Q}_{p})}(L_{3}(\lambda), \Sigma_{i}^{1}(\lambda, L_1, L_2, L_3)) = 0 \]
by the uniqueness in Proposition 3.2 of [Schr11]. Hence we finish the proof. \( \square \)

We define \( \Sigma_{\text{min}^+}(\lambda, L_1, L_2, L_3) \) as the unique subrepresentation of \( \Sigma_{\text{min}}(\lambda, L_1, L_2, L_3) \) of the form
\[
\text{St}_{3}^{\lambda}(\lambda) \rightarrow \text{C}_{s_{1}, s_{1}} - \overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty}
\]
that fits into the short exact sequence
(6.44)
\[ \Sigma_{\text{min}^+}(\lambda, L_1, L_2, L_3) \rightarrow \Sigma_{\text{min}}(\lambda, L_1, L_2, L_3) \rightarrow \overline{T}(\lambda)^{\oplus 2} \]
and \( \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \) as the unique subrepresentation of \( \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \) of the form
\[
\text{St}_{3}^{\lambda}(\lambda) \rightarrow \overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty} \rightarrow \text{C}_{s_{2}, s_{2}}
\]
that fits into the short exact sequence
(6.45)
\[ \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \rightarrow \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \rightarrow (\overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty}) \oplus (\overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty}) \oplus C_{s_{2}, 1}^{1} \oplus C_{s_{1}, 1}^{1} \]
The short exact sequence (6.44) induces a long exact sequence
\[
\text{Hom}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \overline{T}(\lambda)^{\oplus 2}) \rightarrow \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3))
\rightarrow \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3)) \rightarrow \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \overline{T}(\lambda)^{\oplus 2})
\]
which easily implies that
\[ \dim_{E} \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3)) = 2 \]
by Proposition 1.1 and (6.31). On the other hand, we notice that \( \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \) admits a filtration whose only reducible graded piece is
\[ C_{s_{1}, 1}^{1} \rightarrow \overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty} \]
and
\[ \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), V) = 0 \]
for all graded pieces \( V \) of such a filtration by Lemma 1.18 and Lemma 1.27 which implies that
\[ \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3)) = 0. \]
Therefore (6.45) induces an injection of a two dimensional space into a four dimensional space
(6.46) \[ M_{\text{min}^-} := \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3)) \]
\[ \hookrightarrow M^{+} := \text{Ext}_{\text{GL}_{3}(\mathbb{Q}_{p})}(\overline{T}(\lambda), (\overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty}) \oplus (\overline{T}(\lambda) \otimes_{E} v_{P_{2}}^{\infty}) \oplus C_{s_{2}, 1}^{1} \oplus C_{s_{1}, 1}^{1}). \]
It follows from the definition of \( \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \) that we have embeddings
\[ \Sigma(\lambda, L_1, L_2) \hookrightarrow \Sigma^{+}(\lambda, L_1, L_2) \hookrightarrow \Sigma_{\text{min}^-}(\lambda, L_1, L_2, L_3) \]
which allow us to identify
\[ M^- := \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{L}}(\lambda), \left( \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \right) \right) \]
with a line in \( M^{\min} \). We use the number 1, 2, 3, 4 to index the four representations \( \overline{\mathcal{L}}(\lambda) \otimes_E v_{\mathcal{P}_1}, \overline{\mathcal{L}}(\lambda) \otimes_E v_{\mathcal{P}_2}, C_{s,1}^1, C_{s,2}^1 \) respectively, and we use the notation \( M_I \) for each subset \( I \subseteq \{1, 2, 3, 4\} \) to denote the corresponding subspace of \( M^+ \) with dimension the cardinality of \( I \). For example, \( M_{\{1, 2\}} \) denotes the two dimensional subspace
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{L}}(\lambda), \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{\mathcal{P}_1}^\infty \right) \oplus \left( \overline{\mathcal{L}}(\lambda) \otimes_E v_{\mathcal{P}_2}^\infty \right) \right) \]
of \( M^+ \).

**Lemma 6.47.** We have the following characterizations of \( M^{\min} \) inside \( M^+ \):

\[ M^{\min} \ominus M_{\{i,j\}} = 0 \text{ for } \{i, j\} \neq \{3, 4\}, \]
\[ M^{\min} \ominus M_{\{1,3,4\}} = M^{\min} \ominus M_{\{2,3,4\}} = M^{\min} \ominus M_{\{3,4\}} = M^-, \]
and
\[ M^{\min} = (M^{\min} \ominus M_{\{1,2,3\}}) \oplus (M^{\min} \ominus M_{\{1,2,4\}}). \]

**Proof.** As \( C_{s,1}^1 \) and \( C_{s,2}^1 \) are in the cosocle of \( \Sigma(\lambda, \mathcal{L}_1, \mathcal{L}_2) \), it is immediate that
\[ M^- \subseteq M_{\{3,4\}}. \]

It follows from (6.38) that
\[ M^{\min} \nsubseteq M_{\{3,4\}} \]
and thus \( M^{\min} \ominus M_{\{3,4\}} \) is one dimensional which must coincide with \( M^- \). The proof of Lemma 6.1 implies that \( M \nsubseteq M_{\{i,3,4\}} \) for \( i = 1, 2 \) and therefore \( M \ominus M_{\{i,3,4\}} \) is one dimensional, which implies that
\[ M^{\min} \ominus M_{\{3,4\}} = M^- \]
by the inclusion
\[ M^{\min} \ominus M_{\{3,4\}} \subseteq M^{\min} \ominus M_{\{i,3,4\}} \]
for \( i = 1, 2 \). We observe (cf. Lemma 5.8) that
\[ M^- \ominus M_{\{3\}} = M^- \ominus M_{\{4\}} = 0 \]
and thus
\[ M^{\min} \ominus M_{\{i,j\}} = M^- \ominus M_{\{i,j\}} = 0 \]
for each \( \{i, j\} \neq \{3, 4\}, \{1, 2\} \). We define \( \Sigma^{\min,-\tau} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) as the unique subrepresentation of \( \Sigma^{\min,-\tau} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) that fits into the short exact sequence
\[ \Sigma^{\min,-\tau} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \rightarrow \Sigma^{\min,-\tau} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \rightarrow C_{s,1}^1 \oplus C_{s,2}^1 \oplus C_{s,2s,1,1} \]
and then define
\[ \Sigma^{\min,-\tau,\beta} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) := \Sigma^{\min,-\tau} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) / \overline{\mathcal{L}}(\lambda) \otimes_E \text{St}_3^\infty. \]
It is obvious that \( M^{\min} \ominus M_{\{1,2\}} \neq 0 \) if and only if
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{L}}(\lambda), \Sigma^{\min,-\tau,\beta} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \right) \neq 0 \]
which implies that
\[ (6.48) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{L}}(\lambda), \Sigma^{\min,-\tau,\beta} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \right) \neq 0 \]
as
\[ \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p), \lambda} \left( \overline{\mathcal{L}}(\lambda), \overline{\mathcal{L}}(\lambda) \otimes_E \text{St}_3^\infty \right) = 0 \]
due to Proposition 4.1. We notice that we have a direct sum decomposition
\[ \Sigma^{\min,-\tau,\beta} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = V_1 \oplus V_2 \]
where \( V_1 \) is a representation of the form

\[
\begin{align*}
C^2_{s_{1},s_{1},1} & \quad \longrightarrow \quad C^1_{s_{2},s_{1},1} \\
\mathcal{L}(\lambda) \otimes E v^\infty_{\mathbb{P}^3_{1}} & \quad \longrightarrow \quad \mathcal{T}(\lambda) \otimes E v^\infty_{\mathbb{P}^3_{2}}.
\end{align*}
\]

Switching \( V_1 \) and \( V_2 \) if necessary, we can assume by (6.48) that

\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{T}(\lambda), \ V_1 \right) \neq 0.
\]

On the other hand, we have an embedding

\[
V_1 \hookrightarrow \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1) \longrightarrow \mathcal{L}(\lambda) \otimes E v^\infty_{\mathbb{P}^2_{2}}
\]

which induces an embedding

\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ V_1 \right) \hookrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma^{1,\lambda}(\lambda, \mathcal{L}_1) \longrightarrow \mathcal{L}(\lambda) \otimes E v^\infty_{\mathbb{P}^2_{2}} \right)
\]

and in particular

\[
(6.49) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1) \longrightarrow \mathcal{L}(\lambda) \otimes E v^\infty_{\mathbb{P}^2_{2}} \right) \neq 0.
\]

The short exact sequences

\[
\mathcal{L}(\lambda) \otimes E \text{St}^\infty_3 \hookrightarrow \Sigma_1(\lambda, \mathcal{L}_1) \rightarrow \Sigma^1_1(\lambda, \mathcal{L}_1), \quad \mathcal{L}(\lambda) \otimes E \text{St}^\infty_3 \hookrightarrow \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1) \rightarrow \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1)
\]

induce isomorphisms

\[
(6.50) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma_1(\lambda, \mathcal{L}_1) \right) \rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1) \right)
\]

by Lemma 4.2. Hence we deduce that

\[
(6.51) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma_1^{1}(\lambda, \mathcal{L}_1) \right) = \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( W_2, \ \Sigma_1^{1,\lambda}(\lambda, \mathcal{L}_1) \right) = 0
\]

from Lemma 6.5 and (6.50). The surjection \( W_2 \rightarrow \mathcal{L}(\lambda) \) induces an embedding

\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma_1^{1,\lambda}(\lambda, \mathcal{L}_1) \right) \rightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( W_2, \ \Sigma_1^{1,\lambda}(\lambda, \mathcal{L}_1) \right)
\]

which together with (6.51) imply that

\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma_1^{1,\lambda}(\lambda, \mathcal{L}_1) \right) = 0
\]

and hence

\[
(6.52) \quad \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1) \right) = 0
\]

by (6.51) and an easy devissage. It follows from (6.51) and (6.52) that

\[
\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p),\lambda} \left( \mathcal{L}(\lambda), \ \Sigma^{1,\lambda}_1(\lambda, \mathcal{L}_1) \longrightarrow \mathcal{L}(\lambda) \otimes E v^\infty_{\mathbb{P}^2_{2}} \right) = 0
\]

which contradicts (6.49). As a result, we have shown that

\[
M^{\min} \cap M_{(1,2)} = 0.
\]

As \( M^+ \not\subseteq M_{(1,2,i)} \) for \( i = 3, 4 \), we deduce that both \( M^{\min} \cap M_{(1,2,3)} \) and \( M^{\min} \cap M_{(1,2,4)} \) are one dimensional. On the other hand, since we know that

\[
(M^{\min} \cap M_{(1,2,3)}) \cap (M^{\min} \cap M_{(1,2,4)}) = M^{\min} \cap M_{(1,2)} = 0,
\]

we deduce the following direct sum decomposition

\[
M^{\min} = (M^{\min} \cap M_{(1,2,3)}) \oplus (M^{\min} \cap M_{(1,2,4)}).
\]
We use the notation \( T(\lambda)^i \) for copy of \( T(\lambda) \) inside \( T(\lambda)^{\otimes 2} \) corresponding to the one dimensional space \( M^{\text{min}} \cap M_{\{1,2,i+2\}} \) inside \( M^{\text{min}} \), and therefore we have a surjection

\[
\Sigma^{\text{min}}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \twoheadrightarrow \left( C^1_{s_2,1} \rightarrow T(\lambda)^1 \right) \oplus \left( C^1_{s_1,1} \rightarrow T(\lambda)^2 \right).
\]

(6.53)

As a result, the representation \( \Sigma^{\text{min}}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) has the following form:

\[
\begin{align*}
\text{St}_3^n(\lambda) & \twoheadrightarrow C^1_{s_1,s_1} \rightarrow T(\lambda)^1 \oplus C^1_{s_1,s_1} \rightarrow T(\lambda)^2.
\end{align*}
\]

(6.54)

If we clarify the internal structure of \( \text{St}_3^n(\lambda) \), \( v^n_{P_1}(\lambda) \) and \( v^n_{P_2}(\lambda) \) using Lemma 2.12, then \( \Sigma^{\text{min}}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) has the following form:

\[
\begin{align*}
\text{St}_3^3(\lambda) & \twoheadrightarrow v^n_{P_1}(\lambda) \oplus \text{St}_3^2(\lambda) \\
& \twoheadrightarrow v^n_{P_2}(\lambda) \oplus \text{St}_3^1(\lambda).
\end{align*}
\]

Remark 6.56. It is actually possible to show that all the possibly split extensions illustrated in (6.55) are non-split. However, the proof is quite technical and not related to the \( p \)-adic dilogarithm function, and thus we decided not to include the proof here.

We observe that \( \Sigma^{\text{min}}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) admits a unique subrepresentation \( \Sigma^{\text{Ext}^1,-}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \) of the form

\[
\begin{align*}
\text{St}_3^3(\lambda) & \twoheadrightarrow C^1_{s_2,s_1,1} \rightarrow T(\lambda)^1 \oplus C^1_{s_1,s_1} \rightarrow T(\lambda)^2 \\
& \twoheadrightarrow C^1_{s_1,s_1} \rightarrow T(\lambda)^1 \oplus C^1_{s_1,s_1} \rightarrow T(\lambda)^2.
\end{align*}
\]
which can be uniquely extend to a representation $\Sigma^{\text{Ext}^1}(\lambda, L_1, L_2, L_3)$ of the form:

\[
\begin{array}{cccccc}
C^2_{s_1,1} & C^1_{s_2s_1,1} & C^1_{s_2s_1,s_2s_1} & C^2_{s_1s_2s_1} \\
\mathcal{T}(\lambda) \otimes_E v_{P_1}^Q & \mathcal{T}(\lambda) \otimes_E v_{P_2}^Q & \mathcal{T}(\lambda) \otimes_E v_{P_3}^Q & \\
C^2_{s_2s_1} & C^1_{s_2s_1,s_2s_1} & C^2_{s_2s_1,s_2s_1} \\
\end{array}
\]

(6.57) $\mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty$

according to Section 4.4 and 4.6 of [Bre17] together with our Lemma 4.34. Finally, we define $\Sigma^{\text{min,+}}(\lambda, L_1, L_2, L_3)$ as the amalgamate sum of $\Sigma^{\text{min}}(\lambda, L_1, L_2, L_3)$ and $\Sigma^{\text{Ext}^1}(\lambda, L_1, L_2, L_3)$ over $\Sigma^{\text{Ext}^1,-}(\lambda, L_1, L_2, L_3)$.

**Remark 6.58.** It is actually possible to prove (by several technical computations of Ext-groups) that the quotient

\[
\Sigma^{\text{min,+}}(\lambda, L_1, L_2, L_3) / \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty
\]

and the quotient

\[
\Sigma^{\text{min}}(\lambda, L_1, L_2, L_3) / \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty
\]

are independent of the choices of $L_1, L_2, L_3 \in E$.

### 7. Local-Global Compatibility

We are going to borrow most of the notation and assumptions from Section 6 of [Bre17]. We fix embeddings $\iota_\infty : \mathbb{Q} \rightarrow \mathbb{C}$, $\iota_p : \mathbb{Q} \rightarrow \mathbb{Q}_p$, an imaginary quadratic CM extension $F$ of $\mathbb{Q}$ and a unitary group $G/\mathbb{Q}$ attached to the extension $F/\mathbb{Q}$ such that $G \times F \cong \text{GL}_3$ and $G(\mathbb{R})$ is compact. If $\ell$ is a finite place of $\mathbb{Q}$ which splits completely in $F$, we have isomorphisms $\iota_{G,\ell} : G(\mathbb{Q}_\ell) \cong G(F_\ell)$ for each finite place $w$ of $F$ over $\ell$. We assume that $p$ splits completely in $F$, and we fix a finite place $v_0$ of $F$ dividing $p$ and therefore $G(\mathbb{Q}_p) \cong G(F_{v_0}) \cong \text{GL}_3(\mathbb{Q}_p)$.

We fix an open compact subgroup $U^p \subseteq G(A_{\infty,p}^\text{alg})$ of the form $U^p = \prod_{\ell \neq p} U_\ell$ where $U_\ell$ is an open compact subgroup of $G(\mathbb{Q}_\ell)$. For each finite extension $E$ of $\mathbb{Q}_p$ inside $\mathbb{Q}_p$, we consider the following $\mathcal{O}_E$-lattice inside a $p$-adic Banach space:

\[
\hat{S}(U^p, \mathcal{O}_E) := \{ f : G(\mathbb{Q}) \backslash G(\mathbb{A}_\infty^\text{alg}) / U^p \rightarrow \mathcal{O}_E, \ f \text{ continuous} \}
\]

and note that $\hat{S}(U^p, E) := \hat{S}(U^p, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E$. The right translation of $G(\mathbb{Q}_p)$ on $G(\mathbb{Q}_p)$ induces a $p$-adic continuous action of $G(\mathbb{Q}_p)$ on $\hat{S}(U^p, \mathcal{O}_E)$ which makes $\hat{S}(U^p, E)$ an admissible Banach representation of $G(\mathbb{Q}_p)$ in the sense of [ST02]. We use the notation $\hat{S}(U^p, E)_{\text{alg}} \subseteq \hat{S}(U^p, E)_{\text{an}}$ following Section 6 of [Bre17] for the subspaces of locally $\mathbb{Q}_p$-algebraic vectors and locally $\mathbb{Q}_p$-analytic vectors inside $\hat{S}(U^p, E)$ respectively. Moreover, we have the following decomposition:

\[
\hat{S}(U^p, E)_{\text{alg}} \otimes_{\mathcal{O}_E} \mathbb{Q}_p \cong \bigoplus_{\pi} (\pi_{1,\infty})_{U_\ell} \otimes_{\mathcal{O}_E} (\pi_{v_0} \otimes_{\mathcal{O}_E} W_{\ell})
\]

where the direct sum is over the automorphic representations $\pi$ of $G(\mathbb{A}_\infty^\text{alg})$ over $\mathbb{C}$ and $W_{\ell}$ is the $\mathbb{Q}_p$-algebraic representation of $G(\mathbb{Q}_p)$ over $\mathbb{Q}_p$ associated with the algebraic representation $\pi_{1,\infty}$ of $G(\mathbb{R})$ over $\mathbb{C}$ via $\iota_\ell$ and $\iota_{1,\infty}$. In particular, each distinct $\pi$ appears with multiplicity one (cf. the paragraph after (55) of [Bre17] for further references).

We use the notation $D(U^p)$ for the set of finite places $\ell$ of $\mathbb{Q}$ that are different from $p$, split completely in $F$ and such that $U_\ell$ is a maximal open compact subgroup of $G(\mathbb{Q}_\ell)$. Then we consider the commutative polynomial algebra $\mathcal{T}(U^p) := E[T_{w,1}^{(j)}]$ generated by the variables $T_{w,1}^{(j)}$ indexed by $j \in \{1, \cdots, n\}$ and $w$ a finite place of $F$ over a place $\ell$ of $\mathbb{Q}$ such that $\ell \in D(U^p)$. The algebra $\mathcal{T}(U^p)$ acts on $\hat{S}(U^p, E)$, $\hat{S}(U^p, E)_{\text{alg}}$ and $\hat{S}(U^p, E)_{\text{an}}$ via the usual double coset operators. The action of $\mathcal{T}(U^p)$ commutes with that of $G(\mathbb{Q}_p)$. 
We fix now $\alpha \in E^\times$, hence a Deligne–Fontaine module $D$ over $\mathbb{Q}_p = F_{w_0}$ of rank three of the form

$$D = Ee_2 \oplus Ee_1 \oplus Ee_0,$$

with

$$\begin{align*}
\varphi(e_2) &= \alpha e_2 \\
\varphi(e_1) &= p^{-1}\alpha e_1 \\
\varphi(e_0) &= p^{-2}\alpha e_0
\end{align*}$$

and finally a tuple of Hodge–Tate weights $k = (k_1 > k_2 > k_3)$. If $\rho: \text{Gal}(\overline{F}/F) \to \text{GL}_3(\mathbb{Q}_p)$ is non-critical in the sense of (ii) of Remark 6.1.4 of [Bre17], as in (7.3), and gives the Deligne–Fontaine module $D$, then we have

$$\hat{\rho}(\ell) \in \text{Gal}(\mathbb{Q}_p)$$

and finally a tuple of Hodge–Tate weights $k = (k_1 > k_2 > k_3)$. If $\rho: \text{Gal}(\overline{F}/F) \to \text{GL}_3(\mathbb{Q}_p)$ is an absolute irreducible continuous representation which is unramified at each finite place $w$ lying over a finite place $\ell \in D(U^p)$, we can associate to $\rho$ a maximal ideal $m_\rho \subseteq \mathbb{T}(U^p)$ with residual field $E$ by the usual method described in the middle paragraph on Page 58 of [Bre17]. We use the notation $\star_{m_\rho}$ for spaces of localization and $\star[m_\rho]$ for torsion subspaces where $\star \in \{S(U^p), \hat{S}(U^p, E)^{alg}, \hat{S}(U^p, E)^{an}\}$.

We assume that there exists $U^p$ and $\rho$ such that

(i) $\rho$ is absolutely irreducible and unramified at each finite place $w$ of $F$ over a place $\ell$ of $\mathbb{Q}$ satisfying $\ell \in D(U^p)$;

(ii) $\hat{S}(U^p, E)^{alg}[m_\rho] \neq 0$ (hence $\rho$ is automorphic and $\rho_{w_0} := \rho|_{\text{Gal}(\overline{F_{w_0}}/F_{w_0})}$ is potentially semi-stable);

(iii) $\rho_{w_0}$ has Hodge–Tate weights $k$ and gives the Deligne–Fontaine module $D$.

By identifying $\hat{S}(U^p, E)^{alg}$ with a representation of $\text{GL}_3(\mathbb{Q}_p)$ via $\iota_{G,w_0}$, we have the following isomorphism up to normalization from [Ca14] and [Ca22]:

$$\hat{S}(U^p, E)^{alg}[m_\rho] \cong (\overline{\mathbb{L}(\lambda)} \otimes E \text{St}_3^\infty \otimes E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det) \otimes d(U^p, \rho)$$

for all $(U^p, \rho)$ satisfying the conditions (i), (ii) and (iii), where $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (k_1 - 2, k_2 - 1, k_3)$ and $d(U^p, \rho) \geq 1$ is an integer depending only on $U^p$ and $\rho$.

**Theorem 7.5.** We consider $\mathbb{U}^p = \prod_{\ell \neq p} U^p$ and $\rho: \text{Gal}(\overline{F}/F) \to \text{GL}_3(\mathbb{Q}_p)$ such that

(i) $\rho$ is absolutely irreducible and unramified at each finite place $w$ of $F$ lying above $D(U^p)$;

(ii) $\hat{S}(U^p, E)^{alg}[m_\rho] \neq 0$;

(iii) $\rho$ has Hodge–Tate weights $k$ and gives the Deligne–Fontaine module $D$ as in (7.3);

(iv) the filtration on $D$ is non-critical in the sense of (ii) of Remark 6.1.4 of [Bre17];

(v) only one automorphic representation $\pi$ contributes to $\hat{S}(U^p, E)^{alg}[m_\rho]$.

Then there exists a unique choice of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E$ such that:

$$\text{Hom}_{\text{GL}_3(\mathbb{Q}_p)} \left( \sum_{\min}^{+} (\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \otimes E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det, \hat{S}(U^p, E)^{an}[m_\rho] \right)$$

$$\overset{\sim}{\rightarrow} \text{Hom}_{\text{GL}_3(\mathbb{Q}_p)} \left( \overline{\mathbb{L}(\lambda)} \otimes E \text{St}_3^\infty \otimes E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det, \hat{S}(U^p, E)^{an}[m_\rho] \right).$$

We recall several useful results from [Bre17] and [BH18].

**Proposition 7.7.** Suppose that $U^p = \prod_{\ell \neq p} U^p$ is a sufficiently small open compact subgroup of $G(A^0_{\mathbb{Q}_p})$, $\hat{S}(U^p, E)^{an} \to \Pi \to \Pi_1$, a short exact sequence of admissible locally analytic representations of $\text{GL}_3(\mathbb{Q}_p)$, $\chi: T(\mathbb{Q}_p) \to E^\times$ a locally analytic character and $\eta: U(t) \to E$ its derived character, then we have $T(\mathbb{Q}_p)^+ \text{-equivariant short exact sequences of finite dimensional } E\text{-spaces}$

$$\left( \hat{S}(U^p, E)^{an} \right) \left[ t = \eta \right] \hookrightarrow \Pi \left[ t = \eta \right] \to \Pi_1 \left[ t = \eta \right]$$

and

$$\left( \hat{S}(U^p, E)^{an} \right) \left[ t = \eta \right] \hookrightarrow \Pi \left[ t = \eta \right] \to \Pi_1 \left[ t = \eta \right]$$

where $T(\mathbb{Q}_p)^+$ is a submonoid of $T(\mathbb{Q}_p)$ defined by

$$T(\mathbb{Q}_p)^+ := \{ t \in T(\mathbb{Q}_p) \mid t \overline{N}(\mathbb{Z}_p)^t^{-1} \subseteq \overline{\mathbb{N}}(\mathbb{Z}_p) \}.$$

**Proof.** This is Proposition 6.3.3 of [Bre17] and Proposition 4.1 of [BH18].
Proposition 7.8. We fix $U^p$ and $\rho$ as in Theorem 7.3. For a locally analytic character $\chi : T(\mathbb{Q}_p) \to E^\times$, we have
\[
\text{Hom}_{T(\mathbb{Q}_p)^+} \left( \chi \otimes_E (\text{ur}(\alpha) \otimes_E \varepsilon^2) \circ \text{det}, \left( \hat{S}(U^p, E)^{\text{an}}[m_\rho] \right) \right) \neq 0
\]
if and only if $\chi = \delta\chi$.

Proof. This is Proposition 6.3.4 of [Bre17]. □

We recall the notation $i_B^{GL_3}(\chi_\infty)$ for a smooth principal series for each $w \in W$ from Section 2.3. Given three locally analytic representations $V_i$ for $i = 1, 2, 3$ and two surjections $V_1 \twoheadrightarrow V_2$ and $V_3 \twoheadrightarrow V_2$, we use the notation $V_1 \times V_2$ for the representation given by the fiber product of $V_1$ and $V_2$ over $V_2$ with natural surjections $V_1 \times V_2 \twoheadrightarrow V_1$ and $V_1 \times V_2 \twoheadrightarrow V_3$. We also use the shorten notation $V^\text{alg}$ for the maximally locally algebraic subrepresentation of a locally analytic representation $V$. We recall that $U^p$ is sufficiently small if there exists $\ell \neq p$ such that $U_\ell$ has no non-trivial element with finite order.

Proposition 7.9. We fix $U^p$ and $\rho$ as in Theorem 7.3 and assume moreover that $U^p$ is a sufficiently small open compact subgroup of $G(\mathbb{A}_F^\infty \mathbb{Q}_p)$. We also fix a non-split short exact sequence $V_1 \hookrightarrow V_2 \rightarrow V_3$ of finite length representations inside the category $\text{Rep}^{S}_{GL_3(\mathbb{Q}_p), E}$ such that $V_1 \otimes_E (\text{ur}(\alpha) \otimes_E \varepsilon^2) \circ \text{det}$ embeds into $\hat{S}(U^p, E)^{\text{an}}[m_\rho]$. We conclude that:

(i) if $V_3$ is irreducible and not locally algebraic, then we have an embedding

\[
V_2 \otimes_E (\text{ur}(\alpha) \otimes_E \varepsilon^2) \circ \text{det} \hookrightarrow \hat{S}(U^p, E)^{\text{an}}[m_\rho];
\]

(ii) if there is a surjection

\[
\overline{L}(\lambda) \otimes_E i_B^{GL_3}(\chi_\infty) \twoheadrightarrow V_3
\]

for a certain $w \in W$, then there exists a certain quotient $V_1$ of $V_2 \times V_3$ of the form $\overline{L}(\lambda) \otimes_E i_B^{GL_3}(\chi_\infty)$ satisfying

\[
\text{soc}_{GL_3(\mathbb{Q}_p)}(V_1) = V_4^\text{alg} = \overline{L}(\lambda) \otimes_E \text{St}_{3\infty}
\]

such that we have an embedding

\[
V_4 \otimes_E (\text{ur}(\alpha) \otimes_E \varepsilon^2) \circ \text{det} \hookrightarrow \hat{S}(U^p, E)^{\text{an}}[m_\rho].
\]

Proof. This is an immediate generalization (or rather formalization) of Section 6.4 of [Bre17]. More precisely, part (i) (resp. (ii)) generalizes the Étape 1 (resp. the Étape 2) of Section 6.4 of [Bre17]. □

proof of Theorem 7.3. We may assume that $\alpha = 1$ for simplicity of notation thanks to Lemma 2.2. According to the Étape 1 and 2 of Section 6.2 of [Bre17], we may assume without loss of generality that $U^p$ is sufficiently small and it is sufficient to show that there exists a unique choice of $L_1, L_2, L_3 \in E$ such that

\[
\text{Hom}_{GL_3(\mathbb{Q}_p)} \left( \Sigma^{\text{min}^+}(\lambda, L_1, L_2, L_3) \otimes_E (\text{ur}(\alpha) \otimes_E \varepsilon^2) \circ \text{det}, \hat{S}(U^p, E)^{\text{an}}[m_\rho] \right) \neq 0.
\]

We borrow the notation $\Pi^i(k, D)$ from Theorem 6.2.1 of [Bre17]. We observe from (6.56) that $\Sigma^{\text{min}^+}(\lambda, L_1, L_2, L_3)$ contains a unique subrepresentation $\Sigma^{\text{Ext}^1}(\lambda, L_1, L_2, L_3)$ of the form

\[
\overline{L}(\lambda) \otimes_E \text{St}_{3\infty} \xrightarrow{\Pi^1(k, D)} \Pi^2(k, D).
\]

Moreover, $\Sigma^{\text{min}^+}(\lambda, L_1, L_2, L_3)$ is uniquely determined by $\Sigma^{\text{Ext}^1}(\lambda, L_1, L_2, L_3)$ up to isomorphism. It is known by Étape 3 of Section 6.2 of [Bre17] that there is at most one choice of $L_1, L_2, L_3 \in E$ such that

\[
\text{Hom}_{GL_3(\mathbb{Q}_p)} \left( \Sigma^{\text{Ext}^1}(\lambda, L_1, L_2, L_3) \otimes_E (\text{ur}(\alpha) \otimes_E \varepsilon^2) \circ \text{det}, \hat{S}(U^p, E)^{\text{an}}[m_\rho] \right) \neq 0,
\]
and thus there is at most one choice of \( L_1, L_2, L_3 \in E \) such that (7.10) holds. As a result, it remains to show the existence of \( L_1, L_2, L_3 \in E \) that satisfies (7.10). We notice that \( \Sigma_{\min}^{*,+}(\lambda, L_1, L_2, L_3) \) admits an increasing filtration \( \text{Fil}_* \) satisfying the following conditions

(i) the representations \( \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3) \) and \( \Sigma^{*,+}(\lambda, L_1, L_2) \) (cf. their definition after Proposition 6.8 and Proposition 6.29) appear as two consecutive terms of the filtration;

(ii) each graded piece is either locally algebraic or irreducible.

As a result, the only reducible graded pieces of this filtration is the quotient

\[
\Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3)/\Sigma_{\min}^{*,+}(\lambda, L_1, L_2, L_3) \cong W_0.
\]

Then we can prove the existence of \( L_1, L_2, L_3 \in E \) satisfying (7.10) by reducing to the isomorphism

\[
\text{Hom}_{GL_3}(Q_p) \left( \text{Fil}_{k+1} \Sigma_{\max}^{*,}(\lambda, L_1, L_2, L_3) \otimes_E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det, \widetilde{S}(U^p, E)^{an}[m_p] \right) \\
\cong \text{Hom}_{GL_3}(Q_p) \left( \text{Fil}_k \Sigma_{\max}^{*,}(\lambda, L_1, L_2, L_3) \otimes_E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det, \widetilde{S}(U^p, E)^{an}[m_p] \right)
\]

for each \( k \in \mathbb{Z} \). If

\[
\text{Gr}_{k} := \text{Fil}_{k+1} \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3)/\text{Fil}_k \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3)
\]

is not locally algebraic, then (7.12) is true in this case by part (i) of Proposition 7.9. The only locally algebraic graded pieces of the filtration except \( \mathcal{L}(\lambda) \otimes E \text{St}^\infty_3 \) are \( \mathcal{L}(\lambda) \otimes E v^\infty_{p_3} \), \( \mathcal{L}(\lambda) \otimes E v^\infty_{p_2} \) and \( W_0 \). The isomorphism (7.12) when the graded piece \( \text{Gr}_k \) equals \( \mathcal{L}(\lambda) \otimes E v^\infty_{p_3} \) or \( \mathcal{L}(\lambda) \otimes E v^\infty_{p_2} \) has been treated in Étape 2 of Section 6.4 of [Bre17]. As a result, it remains to show that

\[
\text{Hom}_{GL_3}(Q_p) \left( \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3) \otimes_E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det, \widetilde{S}(U^p, E)^{an}[m_p] \right) \\
\cong \text{Hom}_{GL_3}(Q_p) \left( \Sigma^{*,+}(\lambda, L_1, L_2) \otimes_E (\text{ur}(\alpha) \otimes E \varepsilon^2) \circ \det, \widetilde{S}(U^p, E)^{an}[m_p] \right)
\]

(7.13)

to finish the proof of Theorem 7.6. It follows from results in Section 5.3 of [Bre17] (cf. (53) of [Bre17]) that \( \mathcal{I}_B^{\mathcal{L}}(\chi_{s_1s_2s_3}) \) has the form

\[
\begin{array}{c}
\text{St}^\infty_3 \\
\downarrow v^\infty_{p_3} \\
\downarrow v^\infty_{p_2} \\
\downarrow 1_3
\end{array}
\]

and thus there is a surjection

\[
\mathcal{L}(\lambda) \otimes E \mathcal{I}_B^{\mathcal{L}}(\chi_{s_1s_2s_3}) \twoheadrightarrow W_0.
\]

According to part (ii) of Proposition 7.9 we only need to show that any quotient \( V \) of

\[
V^\circ := \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3) \times W_0 \left( \mathcal{L}(\lambda) \otimes E \mathcal{I}_B^{\mathcal{L}}(\chi_{s_1s_2s_3}) \right)
\]

such that

\[
\text{soc}_{GL_3}(Q_p)(V) = V^{\text{alg}} = \mathcal{L}(\lambda) \otimes E \text{St}^\infty_3
\]

must have the form

\[
\Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3)
\]

for certain \( L_3' \in E \). We recall from Proposition 6.29 and our definition of \( \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3) \) afterwards that \( \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3) \) fits into a short exact sequence

\[
\Sigma^{*,+}(\lambda, L_1, L_2) \hookrightarrow \Sigma_{\min}^{*,}(\lambda, L_1, L_2, L_3) \twoheadrightarrow W_0
\]

(7.15)

and thus \( V^\circ \) fits (by definition of fiber product) into a short exact sequence

\[
\Sigma^{*,+}(\lambda, L_1, L_2) \hookrightarrow V^\circ \twoheadrightarrow \mathcal{I}_B^{\mathcal{L}}(\chi_{s_1s_2s_3})
\]

(7.16)

and in particular

\[
\text{soc}_{GL_3}(Q_p)(V^\circ) = (\mathcal{L}(\lambda) \otimes E \text{St}^\infty_3)^{\otimes 2}.
\]
Hence the condition (7.14) implies that $V$ fits into a short exact sequence

$$\mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty \xrightarrow{j} V^\circ \to V$$

and that

$$j(\mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty) \cap \Sigma^{\delta,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2) = 0 \subseteq V^\circ$$

which induces an injection

$$\Sigma^{\delta,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2) \hookrightarrow V.$$

Therefore $V$ fits into a short exact sequence

$$\Sigma^{\delta,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2) \hookrightarrow V \to W_0$$

and thus corresponds to a line $M_V$ inside $\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p)}(W_0, \Sigma^{\delta,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2))$ which is two dimensional by Lemma 6.33. Moreover, the condition (7.14) implies that $M_V$ is different from the line

$$\text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p)}(W_0, \mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty) \hookrightarrow \text{Ext}^1_{\text{GL}_3(\mathbb{Q}_p)}(W_0, \Sigma^{\delta,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2)).$$

Hence it follows from Lemma 6.33 that there exists $\mathcal{L}_3' \in E$ such that

$$V \cong \Sigma^{\min}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3').$$

□

**Corollary 7.17.** If a locally analytic representation $\Pi$ of the form (7.11) is contained in $\hat{S}(U^p, E)^{\text{an}}[m_\rho]$ for a certain $U^p$ and $\rho$ as in Theorem 7.5, then there exists $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E$ uniquely determined by $\Pi$ such that

$$\Pi \hookrightarrow \Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3).$$

**Proof.** We fix $U^p$ and $\rho$ such that the embedding

(7.18) $$\Pi \hookrightarrow \hat{S}(U^p, E)^{\text{an}}[m_\rho]$$

exists. Then (7.18) restricts to an embedding

$$\mathcal{T}(\lambda) \otimes_E \text{St}_3^\infty \hookrightarrow \hat{S}(U^p, E)^{\text{an}}[m_\rho]$$

which extends to an embedding

(7.19) $$\Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \hookrightarrow \hat{S}(U^p, E)^{\text{an}}[m_\rho]$$

for a unique choice of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E$ according to Theorem 7.5. The embedding (7.19) induces by restriction an embedding

$$\Sigma^{\text{Ext}^1}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \hookrightarrow \hat{S}(U^p, E)^{\text{an}}[m_\rho]$$

and therefore we have

$$\Pi \cong \Sigma^{\text{Ext}^1}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$$

by Theorem 6.2.1 of [Bre17]. In particular, we deduce an embedding

$$\Pi \hookrightarrow \Sigma^{\min,+}(\lambda, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$$

for certain invariants $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in E$ determined by $\Pi$. □
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