Classical formulation of Cosmic Censorship Hypothesis

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Spacetimes admitting appropriate spatial homothetic Killing vectors are called spatially homothetic spacetimes. Such spacetimes conform to the fact that gravity has no length-scale for matter inhomogeneities. The matter density for such spacetimes is (spatially) arbitrary and the matter generating the spacetime admits any equation of state. Spatially homothetic spacetimes necessarily possess energy-momentum fluxes. We first discuss spherically symmetric and axially symmetric examples of such spacetimes that do not form naked singularities for regular initial data. We then consider the most general spatially homothetic spacetime and show that the Cosmic Censorship Hypothesis is equivalent to the statement that gravity has no length-scale for matter properties.

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I. INTRODUCTION

The existence of massive stars, star clusters, the galaxies, the structure hierarchy of the galaxy distribution etc. point to the richness of the phenomenon of gravity, in general. What is strikingly noticeable in this panorama of the universe is that the mass and the involved size grow at each of these steps going from the smaller to the larger objects. This points to the fundamental nature of the mass and length-scale independence of gravity.

Spatial scale-independence of gravity

The phenomenon of gravitation does not provide any length-scale or mass-scale for spatial distributions of matter properties. Newton’s celebrated law of gravitation and its applications in non-relativistic regime of observations instill sufficient confidence in this property for us to consider it as one of the fundamental, observational properties of gravitation. We emphasize that the scale-independence of Newtonian gravity applies only to space and not to time. Moreover, Newton’s law of gravitation does not specify any property of matter that it deals with. It applies irrespective of the form of matter under consideration.

The spatial scale-independence of gravity means that we can construct a gravitating object of any size and of any mass. It can be made from any matter. Further, matter within such an object can be distributed in any desirable manner since gravity does not provide for the spatial distribution of matter within any gravitating object. (It is a separate question as to whether every such object will be stable or not.)

General Relativity is a theory of gravitation. Therefore, if the spatial scale-independence is any basic property of gravity then, General Relativity must admit, in general, a spacetime with matter density as an arbitrary function of each of the three spatial coordinates. We emphasize that such a spacetime metric and all other metric forms that are reducible to it under non-singular coordinate transformations, that is to say, diffeomorphic to it, are the only solutions of the field equations of General Relativity that are consistent with gravity not possessing a length-scale for matter properties.

All other spacetimes that are not diffeomorphic to the aforementioned spacetime then violate the property that gravity has no length-scale for matter properties. Further, we note that spacetimes obtained for matter with some specific equation of state do not conform with the property of gravity that it applies to all forms of matter. In short, not all solutions of the Einstein field equations respect these properties of the phenomenon of gravitation.

The field equations of General Relativity are based on Einstein’s equivalence principle which is, primarily, the principle of equality of the inertial and gravitational masses. The equivalence principle leads to the geometrization of gravity and, hence, from a variational principle, to the field equations of General Relativity. However, General Relativity does not automatically incorporate other basic properties of gravity. This is evident from the fact that we can always construct a spacetime violating the spatial scale-independence and equate its Einstein tensor with the energy-momentum tensor of matter fields to obtain a solution of the field equations. Therefore, we need to separately enforce other basic properties of gravity, such as its spatial scale-independence, on the solutions of the field equations.
In general, a homothetic Killing vector captures [1] the notion of the scale-invariance. A spacetime that conforms to the spatial scale-invariance, to be called a spatially homothetic spacetime, is then required to admit an appropriate spatial homothetic Killing vector $X$ satisfying

$$L_X g_{ab} = 2 \Phi g_{ab}$$  \hspace{1cm} (1)$$

where $\Phi$ is an arbitrary constant. We then expect spatially homothetic spacetimes to possess arbitrary spatial characteristics for matter. This is also the broadest (Lie) sense of the scale-invariance of the spacetime leading not only to the reduction of the Einstein field equations as partial differential equations to ordinary differential equations but leading also to their separation.

Further, in General Relativity, the newtonian notion of scale-invariance or self-similarity of a physical problem [2] can be generalized in different possible ways [1]. The self-similarity of matter fields is that the physical quantities transform according to their respective dimensions. When matter fields exhibit this property of the scale-invariance, a spatially homothetic spacetime admits, in addition to the spatial homothetic Killing vectors, other appropriate homothetic Killing vector(s) and, in this case, we call the spacetime a source self-similar spacetime.

In general, a spatially homothetic spacetime is not a source self-similar spacetime. It must be emphasized that the spatial homothety is the basic property of gravitation and the self-similarity of matter fields is an additional restriction on the spacetime geometry.

Astrophysical considerations

A physically realistic gravitational collapse problem imagines matter, with regular initial data, collapsing under its self-gravity. The resultant compression of matter causes pressure to build-up in it. Further, matter compression generates heat and radiation because of either the onset of thermonuclear fusion reactions or other reasons. The radiation or heat then propagates through the space. The collapsing matter could stabilize to some size when its equation of state is such as to provide pressure support against gravity. If self-gravity dominates, the collapse continues to a spacetime singularity. The issue of Cosmic Censorship Hypothesis [3] relates to whether the singularity is visible to any observer or not, i.e., whether it is naked or not.

Irrespective of what the central object is, matter in the surroundings will accrete onto it. The accreting matter may, initially, be dust in the far away regions. However, it gets compressed as it moves closer to the central object and pressure must build up in it. In many such situations, heat and radiation partly escape the system and partly fall onto the central object together with the accreting matter.

Therefore, any complete spacetime description of the collapse and accretion processes requires us to properly match different spacetimes of various such stages, during which the properties of matter are different from each other, to produce the final spacetime. Note that the final spacetime will have to be a solution of the Einstein field equations. (Note further that the equation of state at extremely high densities is not known.) In any case, the final spacetime description of the gravitational collapse and/or the accretion process must admit a changing equation of state for collapsing/accreting matter. Further, such a spacetime must also admit an energy or heat flux during late collapse or accretion stages. To accomplish this process of matching different such spacetimes is a herculean, if not impossible, task.

Hence, another approach to these problems is essential. We could then demand that a spacetime describing the collapse and/or the process of accretion in its totality admits any equation of state and appropriate energy-momentum fluxes. In other words, the spacetime geometry should be obtainable from considerations that do not involve the equation of state for the matter in the spacetime. Furthermore, these considerations should result in a spacetime admitting energy-momentum fluxes.

We now turn to precisely such considerations in General Relativity that involve the spatially homothetic spacetimes.

II. SPHERICALLY SYMMETRIC SPACETIME

We begin here with a spherically symmetric example of a spatially homothetic spacetime. We impose a spatial homothetic Killing vector

$$X = (0, f(r, t), 0, 0)$$  \hspace{1cm} (2)$$

on a general spherically symmetric metric. This uniquely determines the spherically symmetric metric to that obtained in [3], namely...
\[ ds^2 = -y^2(r) \, dt^2 + \gamma^2 \, (y')^2 \, B^2(t) \, dr^2 + y^2(r) \, Y^2(t) \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right] \] (3)

with \( f(r,t) = y/(\gamma y') \), a prime indicating a derivative with respect to \( r \) and \( \gamma \) being a constant. (We shall always absorb the temporal function in \( g_{tt} \) by suitable redefinition of the time coordinate. The coordinates are co-moving.)

The Einstein tensor for (3) has components

\[
\begin{align*}
G_{tt} &= \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2} + \frac{\dot{Y}^2}{Y} + 2 \frac{\dot{B} \dot{Y}}{BY} \\
G_{rr} &= \gamma^2 B^2 \left( \frac{y'}{y} \right)^2 \left[ -2 \frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y} + \frac{3}{\gamma^2 B^2} - \frac{1}{Y^2} \right] \\
G_{\theta\theta} &= -Y \ddot{Y} - Y^2 \frac{\ddot{B}}{B} - Y \frac{\dot{Y} \dot{B}}{B} + \frac{Y^2}{\gamma^2 B^2} \\
G_{\phi\phi} &= \sin^2 \theta \, G_{\theta\theta} \\
G_{tr} &= 2 \frac{\dot{B} \dot{y}}{\dot{B} y} \\
\end{align*}
\] (4-8)

Notice that the \( t - r \) component of the Einstein tensor is non-vanishing. Hence, matter in the spacetime could be imperfect or anisotropic indicating that its energy-momentum tensor could be

\[
\begin{align*}
^{1}T_{ab} &= (p + \rho) U_{a} U_{b} + p g_{ab} + q_{a} U_{b} + q_{b} U_{a} - 2 \eta \sigma_{ab} \\
^{A}T_{ab} &= \rho U_{a} U_{b} + p_{\parallel} n_{a} n_{b} + p_{\perp} P_{ab} \\
\end{align*}
\] (9-10)

where \( U^{a} \) is the matter 4-velocity, \( q^{a} \) is the heat-flux 4-vector relative to \( U^{a} \), \( \eta \) is the shear-viscosity coefficient, \( \sigma_{ab} \) is the shear tensor, \( n^{a} \) is a unit spacelike 4-vector orthogonal to \( U^{a} \), \( P_{ab} \) is the projection tensor onto the two-plane orthogonal to \( U^{a} \) and \( n^{a} \), \( p_{\parallel} \) denotes pressure parallel to and \( p_{\perp} \) denotes pressure perpendicular to \( n^{a} \). Also, \( p \) is the isotropic pressure and \( \rho \) is the energy density. Note that the shear tensor is trace-free. We will represent by \( \sigma \) the shear-scalar that is given by \( \sqrt{6} \, \sigma \)

Now, the Einstein field equations with imperfect matter yield for (3)
where \( q^a = (0, q, 0, 0) \) is the radial heat-flux vector. The radial function \( y(r) \) is not determined by the field equations and the temporal functions \( B(t) \) and \( Y(t) \) get determined by the properties of matter such as its equation of state.

The spacetime of (3) conforms with the general requirements of a physical collapse then. Consequently, the fate of spherical collapse, i.e., whether the collapse results in a black hole or a naked singularity, can be explored using (3).

The spacetime singularity can result from either \( y(r) = 0 \) for some \( r \) and/or the temporal functions \( B(t), Y(t) \) being zero for some \( t = t_o \) in (3).

**Absence of naked singularities**

The radial null cone equation for (3) is

\[
\frac{dt}{dr} = \pm \gamma \frac{1}{y} \frac{dy}{dr} B(t) \quad (15)
\]

and that it is non-singular for any nowhere-vanishing function \( y(r) \).

Hence, there does not exist an out-going null tangent at the spacetime singularity that results from purely temporal evolution of these spacetimes if, initially, \( y(r) \neq 0 \) for all \( r \). From (14) it requires the density to be spatially non-singular always.

However, it is usual (3) to enforce on self-similar, spherically symmetric spacetimes, the form

\[
\tilde{X}_a = (T, R, 0, 0) \quad (16)
\]

for the homothetic Killing vector. But, in (3) we showed that, for spherically symmetric spacetimes, this form (3) of the homothetic Killing vector is too restrictive and obscures important information about the properties of such spacetimes, for example, the existence of naked singularities. This is understandable since the Killing vector (3) corresponds to the *simultaneous* scale-invariance of the spacetime in \( T \) and \( R \) in the sense of Lie.

We emphasize that, for spherical symmetry, the appropriate form is (3) since it corresponds only to the radial scale-invariance of the spacetime in the sense of Lie. However, (3) is equivalent to (13) under the transformation

\[
R = l(t) \exp \left( \int f^{-1}dr \right) \quad (17)
\]

\[
T = k(t) \exp \left( \int f^{-1}dr \right) \quad (18)
\]

Of course, the transformed metric can always be made diagonal in \( R \) and \( T \) coordinates. The imposition of the homothetic Killing vector (16) on a spherically symmetric spacetime is *over-restrictive* and is not demanded by any basic property of gravitation. It should be noted that spacetimes admitting (14) are included in (3) when the transformations (17) and (18) are non-singular. This simply relates to the coordinate freedom in General Relativity. Naked singularities of spacetimes obtained by enforcing only (14) are then the artefact of the singular transformations (17) and (18) in some appropriate sense.

**Black hole as infinite red-shift surface**

Moreover, a black hole forms (3) in the spacetime of (3) only as an infinite red-shift surface and not as a null hyper-surface. This is easily seen by noticing that \( g_{tt} = -y^2 \) is non-vanishing at all \( r \) for nowhere-vanishing radial function \( y(r) \). No spatially finite null hyper-surface then exists with (3).

That the infinite red-shift surface forms in (3) follows from the vanishing expansion of the radially outgoing null vector

\[
\ell^a \partial_a = \frac{1}{y} \frac{\partial}{\partial t} + \frac{1}{\gamma y'B} \frac{\partial}{\partial r} \quad (19)
\]

of (3). The zero-expansion of (19) yields a condition only on the temporal metric functions as

\[
\frac{\dot{B}}{B} + 2 \frac{\dot{y}}{y} = -\frac{3}{\gamma B} \quad (20)
\]

When this condition is reached during the gravitational collapse, light and, with it, matter trapping occurs. It is only at some “instant” of the co-moving time that the curvature becomes strong enough to trap light and matter. The condition (20) determines this instant of the co-moving time.

The four-velocity of the matter fluid with respect to the co-moving observer is:

\[
U^a = \left( U^t, U^r, 0, 0 \right) \quad (21)
\]

Defining then the radial velocity of matter with respect to the co-moving observer as

\[
V_r = U^r / U^t \quad (22)
\]

we then obtain from the metric (3):

\[
U^a = \frac{1}{y \sqrt{\Delta}} \left( 1, V_r, 0, 0 \right) \quad (23)
\]

\[
\Delta = 1 - \gamma^2 \left( \frac{y'}{y} \right)^2 B^2 V_r^2 \quad (24)
\]

Now, if \( d\tau_{CM} \) is a small time duration for the co-moving observer and if \( d\tau_{RF} \) is the corresponding
time duration for the observer in the rest frame of matter, then we have
\[ d\tau_{CM} = \frac{d\tau_{RF}}{\sqrt{\Delta}} \] (25)
Therefore, the co-moving observer waits for an infinite period of its time to receive a signal from the rest-frame observer when \( \Delta = 0 \). Equation (25) is also the red-shift formula. Clearly, therefore, \( \Delta = 0 \) is the infinite red-shift surface.

Of course, the infinite red-shift surface separates the spacetime of (3) into two regions - one that can communicate to the far away zone and the black hole region that cannot. The inside and outside of the infinite red-shift surface are then causally disconnected regions of the spacetime of (3).

We have then the following possibilities
\[(\Delta > 0) \quad |\gamma (y') B V_r| < y \] (26)
\[(\Delta = 0) \quad |\gamma (y') B V_r| = y \] (27)
\[(\Delta < 0) \quad |\gamma (y') B V_r| > y \] (28)
Matter with an initial density distribution determined by \( y(r) \) begins to collapse under the condition (26) with initial velocity \( V_{r, ini} \) and initial heat flux, determined by \( B(t_o) \). The in-fall velocity of matter and heat flux grow as matter collapse progresses. Matter properties decide whether the collapse becomes unstoppable or not. Then, in any unstoppable collapse, matter reaches the black hole region of (27) and (28) when condition (20) is reached.

### III. AXISYMMETRIC SPACETIME

Encouraged by the example [9, 10, 11] in spherical symmetry, we then considered [12] the implications of such a requirement of homothety for axially symmetric spacetimes.

In axial symmetry we have two spatial variables which can be expected to behave in a homothetic manner, viz. \( r \) and \( z \). In other words, we expect the spacetime to admit arbitrary functions of \( r \) and \( z \) determining the matter characteristics of axially symmetric spacetimes. We then consider the axisymmetric metric
\[ ds^2 = -A^2(t, r, z)dt^2 + C^2(t, r, z)dr^2 + D^2(t, r, z)dz^2 + B^2(t, r, z)d\phi^2 \] (29)
and, guided by our previous considerations [4], impose the existence of two independent homothetic Killing vectors of the form
\[ H_r = (0, f(r), 0, 0) \] (30)
\[ H_z = (0, 0, g(z), 0) \] (31)
on (29). The imposition of (30) and (31) reduces the metric (29) uniquely to
\[ ds^2 = -Z^2(z) y^2(r) dt^2 + \gamma_1^2 Z^2(z) C^2(t) (y')^2 dt^2 + \gamma_2^2 D^2(t) y^2(r) (\tilde{Z})^2 dz^2 + Z^2(z) y^2(r) B^2(t) d\phi^2 \] (32)
where \( \gamma \) s are constants, \( f(r) = y(r)/((\gamma_1 y') \) and \( g(z) = Z(z)/(\gamma_2 \tilde{Z}) \), an overhead prime denotes differentiation with respect to \( r \) and an overhead tilde denotes differentiation with respect to \( z \).
The Einstein tensor for (32) has the following components

\[
G_{tt} = -\frac{1}{\gamma_1^2 D^2} - \frac{1}{\gamma_2^2 C^2} + \frac{\dot{C} \ddot{D}}{CD} + \frac{\dot{B} \ddot{D}}{BD} + \frac{\ddot{B} \dot{C}}{BC} \\
G_{rr} = \gamma_1^2 C^2 \left( \frac{y'}{y} \right) \left[ \frac{\dot{D}}{D} - \frac{\dot{B}}{B} \right] - \frac{B \ddot{D}}{BD} + \frac{3}{\gamma_1^2 C^2} + \frac{1}{\gamma_2^2 D^2} \right) \\
G_{zz} = \gamma_2^2 D^2 \left( \frac{\dot{Z}}{Z} \right) \left[ -\frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right] - \frac{B \ddot{C}}{BC} + \frac{3}{\gamma_2^2 D^2} + \frac{1}{\gamma_1^2 C^2} \right) \\
G_{\phi\phi} = B^2 \left[ -\frac{\dot{D}}{D} - \frac{\dot{C}}{C} - \frac{\dot{D} \ddot{C}}{CD} + \frac{1}{\gamma_1^2 D^2} + \frac{1}{\gamma_2^2 C^2} \right] \\
G_{tr} = 2 \frac{\dot{C} y'}{Cy} \\
G_{tz} = 2 \frac{\dot{D} \dot{Z}}{DZ} \\
G_{rz} = 2 \frac{\dot{Z} y'}{Zy}
\]

where an overhead dot denotes a time derivative. It is clear from the above that the spacetime necessarily possesses energy and momentum fluxes. The matter in the spacetime is imperfect.

This is interesting in its own right. Any mass-particle of an axisymmetric body has a Newtonian gravitational force directed along the line joining it to the origin. This force, which is unbalanced during the collapse, has generally non-vanishing components along \( r \) and \( z \) axes. Hence, a non-static axisymmetric spacetime of (32) will necessarily possess appropriate energy-momentum fluxes!

The coordinates \((t, r, z, \phi)\) are co-moving. The matter 4-velocity, in general, will have all the four components, ie, \( U^a = (U^t, U^r, U^z, U^\phi)\).

In the case that \( U^\phi = 0 \), the spacetime of (32) describes any non-rotating, axisymmetric matter configuration, in particular, a cigar configuration. In the case that \( U^\phi \neq 0 \), the spacetime of metric (32) describes rotating matter configurations. In other words, it represents the “internal” Kerr spacetimes that are also axisymmetric in nature. (It is also clear that non-static “internal” Kerr spacetimes cannot admit any perfect fluid matter since axisymmetry requires the existence of appropriate energy-momentum fluxes in such spacetimes as is evident from the earlier discussion.)

The spacetime (32) has a singularity when either \( C(t) = 0 \) or \( D(t) = 0 \) for some \( t \) or when \( y(r) = 0 \) for some \( r \) and/or \( Z(z) = 0 \) for some \( z \).

Moreover, from (32), the \( r \) and \( z \) null cone equations are

\[
\frac{dt}{dr} = \pm \frac{y'}{y} C(t) \\
\frac{dt}{dz} = \pm \frac{\dot{Z}}{Z} D(t)
\]

and these are non-singular for nowhere-vanishing functions \( y(r) \) and \( Z(z) \). Hence, there does not exist an out-going null tangent at the spacetime singularity when \( y(r) \neq 0 \) and \( Z(z) \neq 0 \). Hence, the singularities of these axisymmetric spacetimes are not naked with these restrictions on the spatial functions.

We note that the nowhere-vanishing of \( y(r) \) and \( Z(z) \) means that the density is initially non-singular. Moreover, it is also clear that the spacetime of (32) will allow an arbitrary density profile in \( r \) and \( z \) since the field equations do not determine these spatial functions. Further, it is also seen that the spacetime of (32) admits any equation of state for the matter in the spacetime and that the properties of matter in the spacetime determine the temporal metric functions. Hence, on the basis of arguments similar in nature to the spherically symmetric case, only a black hole as an infinite red-shift surface forms in the axisymmetric collapse of regular matter distributions in (32).

We also note that the energy-momentum tensor of the imperfect matter in the spacetime can con-
tain contributions from the presence of electromagnetic fields in the matter. (That is why we listed only the Einstein tensor above.) The spacetime of the matter and magnetic fields in the spacetime is all that is determinable from the properties of matter including those of the electromagnetic fields in the spacetime.

IV. MOST GENERAL, SPATIALLY HOMOTHETIC SPACETIME

In general, we then demand that the spacetime admitting no special symmetries, that is no proper Killing vectors, admits three independent homothetic Killing vectors corresponding to the three dimensions for which gravity provides no length-scale for matter inhomogeneities. Such a metric, from the broadest (Lie) sense, admits three functions \( X(x), Y(y), Z(z) \) of three space variables, conveniently called here, \( x, y, z \), each being a function of only one variable.

Based on the above considerations, we then demand that there exist three independent spatial homothetic Killing vectors

\[
H_1 = (0, f(x), 0, 0) \quad (42)
\]
\[
H_2 = (0, 0, g(y), 0) \quad (43)
\]
\[
H_3 = (0, 0, 0, h(z)) \quad (44)
\]

for the general spacetime metric

\[
ds^2 = g_{ab}dx^a dx^b \quad (45)
\]

with \( g_{ab} \) being functions of the coordinates \( (t, x, y, z) \). Then, the spacetime metric is, uniquely, the following

\[
ds^2 = -X^2(x)Y^2(y)Z^2(z)dt^2 + \gamma_1^2 \left( \frac{dX}{dx} \right)^2 Y^2(y)Z^2(z)A^2(t)dx^2 + \gamma_2^2 X^2(x) \left( \frac{dY}{dy} \right)^2 Z^2(z)B^2(t)dy^2 + \gamma_3^2 X^2(x)Y^2(y) \left( \frac{dZ}{dz} \right)^2 C^2(t)dx^2 \quad (46)
\]

This is the most general spacetime compatible with gravity not possessing any length-scale for matter inhomogeneities in its diagonal form.

The coordinates \( (t, x, y, z) \) in which we is separable are co-moving. Hence, the matter 4-velocity is \( U^a = (U^t, U^x, U^y, U^z) \) with all the components non-vanishing in general. Then, using, for example, the software \textsc{sheep}, it is easy to see that the Einstein tensor has appropriate components

\[
G_{tt} = -\frac{1}{\gamma_1^2 A^2} - \frac{1}{\gamma_2^2 B^2} - \frac{1}{\gamma_3^2 C^2} + \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{A} \dot{C}}{AC} + \frac{\dot{B} \dot{C}}{BC} \quad (47)
\]
\[
G_{xx} = \gamma_2^2 A^2 \left( \frac{L_x}{L} \right)^2 \left[ -\frac{B}{A} \frac{\dot{B}}{\dot{C}} - \frac{B \dot{C}}{ABC} + \frac{3}{\gamma_1^2 A^2} + \frac{1}{\gamma_2^2 B^2} + \frac{1}{\gamma_3^2 C^2} \right] \quad (48)
\]
\[
G_{yy} = \gamma_3^2 B^2 \left( \frac{M_y}{M} \right)^2 \left[ -\frac{A}{B} \frac{\dot{A}}{\dot{C}} - \frac{A \dot{C}}{ABC} + \frac{3}{\gamma_1^2 A^2} + \frac{1}{\gamma_2^2 B^2} + \frac{1}{\gamma_3^2 C^2} \right] \quad (49)
\]
\[
G_{zz} = \gamma_3^2 C^2 \left( \frac{N_z}{N} \right)^2 \left[ -\frac{C}{A} \frac{B}{B} - \frac{A B}{AB} + \frac{3}{\gamma_1^2 A^2} + \frac{1}{\gamma_2^2 B^2} + \frac{1}{\gamma_3^2 C^2} \right] \quad (50)
\]
\[
G_{tx} = \frac{2 \dot{A} L_x}{AL} \quad (51)
\]
\[
G_{ty} = \frac{2 \dot{B} M_y}{BM} \quad (52)
\]
corresponding to expected non-vanishing energy-momentum fluxes. It is then easy to see that the field equations do not determine the spatial functions \(X(x), Y(y), Z(z)\). Further, the density is initially non-singular for nowhere-vanishing spatial functions \(X(x), Y(y), Z(z)\). Moreover, the temporal functions \(A(t), B(t), C(t)\) get determined only from the properties of matter generating the spacetime.

From arguments similar to those considered earlier for spherically and axially symmetric spacetimes, it then follows that the spacetime singularity, which results from the vanishing of only the temporal function(s) \(A(t), B(t), C(t)\), is not locally naked for nowhere-vanishing spatial functions \(X(x), Y(y), Z(z)\). Hence, the most general spatially homothetic spacetime, (46), does not result to a naked singularity in the gravitational collapse of matter with initially non-singular properties. A black hole then always results in the gravitational collapse of matter fields with non-singular spatial properties in (46).

### Semi-stable objects

We note that a collapsing object could stabilize, for some co-moving time, by the switching on of some forces opposing gravity. Stable such objects correspond to static spacetimes. Then, by considering temporal functions of the spatially homothetic spacetimes, namely, eqs. (46), (53) and (56), appearing in the energy fluxes to be constants, we could obtain the spacetimes of stabilized objects with corresponding symmetries. However, these are everywhere static spacetimes and, hence, not realistic and not interesting.

When the equation of state of matter in a spatially homothetic, non-static spacetime approximates to that of the corresponding static spacetime during the collapse, we may obtain a semi-stabilized object within these solutions. For such semi-stable objects, we have, in general, non-vanishing heat generation in the matter. Such objects may remain “stable” for a long duration of the co-moving time but may, eventually, collapse due to accretion of matter onto them.

In general, matter collapse may begin as dust but pressure must build up, nucleosynthesis may commence to produce heat and may result in a semi-stabilized object like a star. The star may explode to shed some mass or may collapse under its self-gravity. A black hole as an infinite red-shift surface but not as a null hyper-surface forms in the unstoppable collapse and may accrete matter in its surroundings. The spatially homothetic spacetimes accommodate these features. Their temporal behavior is determined by the properties of matter such as its equation of state.

### Machian nature of (46)

Mach’s principle is the hypothesis of the relativity of inertia. In a machian theory, the inertia of a body gets determined by the presence of all other bodies in the universe.

Mach’s principle states that the inertia of a particle of matter is the result of its interaction with all other particles in the universe. Consequently, there must be energy density of matter “everywhere” in a machian universe. This can be interpreted to mean that we can assemble “masses” to produce another “mass” and that the process of this building up of mass cannot be terminated in space. This is, then, recognized as the principle of the mass-scale and/or spatial scale invariance of the theory of gravity. Therefore, the spatial scale-independence of gravity is (one of) the direct implications of Mach’s hypothesis of the relativity of inertia. Then, we emphasize that the spacetime of (46) is also Machian [13].

## V. DISCUSSION

Gravity does not provide any length-scale for matter properties. This requirement, through spatially homothetic spacetime of (46), is then sufficient to ensure that the spacetime singularities
are not visible to any observers in gravitational collapse of matter with initially non-singular spatial properties. Moreover, a spatially homothetic spacetime admits any equation of state for the matter generating it and, hence, such spacetimes satisfy the general requirements of astrophysical nature needed to be imposed on the gravitational collapse problem.

**Relation with previous results**

We note that some indications already existed in the literature that point to some of the results or conclusions obtained here. For example, a result belonging to this class is that a perfect fluid spacetime cannot admit a non-trivial homothetic Killing vector which is orthogonal to the fluid 4-velocity unless \( p = \rho \). The spacetime of (3) admits a non-trivial, spatial homothetic Killing vector orthogonal to the fluid 4-velocity. Then, from (2) and (12), it follows that the equation of state for the matter when the time derivatives vanish is

\[
p = \frac{1}{y^2} \left( \frac{4}{\gamma^2 B^2} - \frac{2}{Y^2} \right) + \rho \tag{57}\]

where \( B, Y \) and \( \gamma \) are constants. The equation of state for the matter in (3) is (53) when the spacetime is static, in general. It is uniquely \( p = \rho \) since the constants can be chosen appropriately.

Another result is that a non-flat vacuum spacetime can only admit a non-trivial homothetic Killing vector if that vector is neither null nor hyper-surface orthogonal. We interpret this result to mean that a vacuum spacetime can admit spatial homothetic Killing vectors.

There also are the following exceptional situations in which we need not demand the existence of spatial homothetic Killing vectors:

- The first one being the Friedmann-Lemaitre-Robertson-Walker (FLRW) solution. This spacetime corresponds to the homogeneous and isotropic matter distribution. Note that it admits only perfect fluid matter with any equation of state and that the equation of state determines its temporal evolution. This is also the degenerate metric limit of the general spatially homothetic spacetime (46). This spacetime need not admit any spatial homothetic Killing vectors.
- The second one is the case of vacuum spacetimes. The vacuum spacetimes are not required to admit any spatial homothetic Killing vectors. When there is no matter in the spacetime, we do not impose any principle related to matter.

These exceptions arise primarily because the spacetime is either vacuum or has homogeneous and isotropic distribution of matter. In either situation, there is then no necessity for invoking the principle of no-length-scale for matter properties since it is implicitly satisfied by these spacetimes. Moreover, the FLRW spacetime is also the only non-static, perfect-fluid solution that is compatible with gravity not possessing any length-scale for matter inhomogeneities.

Further, the spacetimes admitting homothetic Killing vectors of the form

\[
(T, \bar{x}, \bar{y}, \bar{z}) \tag{58}
\]
or, combinations thereof, are contained with (46) provided the transformations of (42) - (44) leading to (58) are non-singular. (In [7], we provided the example of this type for spherically symmetric spacetimes. This has also been considered earlier in this paper.) It also generally follows that naked singularities can only arise in spacetimes for which these transformations are singular.

Hence, there are no spatially regular matter data which result into naked singularities as end states of gravitational collapse when spatial scale-invariance is respected as our spherical, axisymmetric and general examples of spatially homothetic spacetimes show. Then, all spacetimes reducible to the given spatially homothetic metrics will not result into naked singularities for spatially non-singular, regular data of matter fields.

We must now address the issue of all other solutions of the Einstein field equations apart from the spatially homothetic spacetimes. In this connection, we note that solutions of the field equations obtained for any particular, specific, equation of state need not be reducible, under non-singular coordinate transformations, to spatially homothetic spacetimes for the same equation of state. We emphasize here that such solutions would be seen to violate the spatial scale-invariance of gravity.

However, solutions obtained for specific equation of state could, under restrictions, be reducible to the corresponding spatially homothetic forms, for example, the Vaidya or the Tolman-Bondi spacetimes. But, we must note that such solutions with specific equation of state apply only when the equation of state of the collapsing matter is that of the considered solution. Hence, these are, under applicable restrictions, only a part of the spatially homothetic spacetimes that apply to the entire gravitational collapse problem.

The important point is, however, that the spacetimes that are not reducible to spatially homothetic spacetimes, namely, (3), (12) and (46), violate one of the basic properties - the spatial scale-
invariance of gravity. As a result, even the regular initial data for the matter fields could lead, in such spacetimes, to naked singularities in some cases and to black holes in some others. Therefore, if the spatial scale-invariance is any basic property of gravity then, it is misleading to ask whether the regular initial data results in a naked singularity or a black hole as end state of collapse with complete disregard to this basic property of gravity. Further, if the spatial scale-invariance is any basic property of gravity then, the phenomenon of criticality in gravity must also be reexamined using the spatially homothetic spacetimes.

Importance of spatial scale-independence of gravity

Some further remarks on the relevance of spatial scale-invariance and on solutions violating it.

The field equations of General Relativity were arrived at by demanding only that these reduce to the Newton-Poisson equation in the weak gravity limit. But, the field equations of any theory of gravity should contain the entire weak gravity physics due to the applicability of the laws of weak gravity to any form of matter displaying any physical phenomena. These equations are only the formal equality of the appropriate tensor from the geometry and the energy-momentum tensor of matter. Therefore, the field equations could have been obtained by imposing the requirement that these reduce to the single “equation of the entire weak gravity physics”.

However, there is no “single” equation for the “entire weak gravity physics” since we include different physical effects in an ad-hoc manner in the Newtonian physics.

But, there can be a “single” spacetime containing the entire weak gravity physics. Therefore, we need a principle to identify such a solution of the field equations. In the weak field limit, the spatial scale-invariance is the freedom of specification of matter properties through three independent functions of the three spatial coordinates, in general. To be precise, we can assemble masses to produce another mass, of any desired spatial density distribution as well as of any size. Newtonian law of gravitation permits this even when other physical phenomena are considered together with that of gravitation.

The spatial scale-invariance is then the principle that could help us identify spacetimes containing the entire weak gravity physics. We have seen in this paper that this is indeed the case - the spatial scale invariance identifies as the single such spacetime. It has appropriate energy-momentum fluxes, applicability to any form of matter and, hence, it contains the entire weak gravity physics.

Clearly, the spatial homothety allows us to distinguish between solutions that contain the entire weak gravity physics and those that do not. Spacetimes of the latter kind can only be of two types - those containing a “special” part of the weak gravity physics or “never” any part of the weak gravity physics.

This is seen as follows. Vacuum spacetimes can never contain any part of the weak gravity physics. Newton’s law of gravity and his laws of motion have no meaning for vanishing mass. In the same spirit, solutions for specific equation of state contain “special” part of the weak gravity physics since these apply to only considered type of matter. There may be matter solutions “never” containing any part of the weak gravity physics.

But, why are spacetimes containing “only a part” and “never” any part of the weak gravity physics “physically not-meaningful”? We appeal to observations to answer this question.

Any spatial scale is equivalent with an appropriate mass scale since the fundamental constants of the theory provide only the relation leading to the Schwarzschild radius. Then, the spatial scale-independence of gravity either breaks down at some scale or it holds at all scales. The breakdown of spatial scale invariance at some scale also implies then the break-down of mass scale. This means that we cannot assemble masses to form another mass. This signifies the break-down of the equivalence principle at that scale. Any such breakdown has not been observed to 1 part in 10^{12}.

The equivalence principle and, hence, the spatial scale-invariance are then fundamental to the theory of gravity. The conclusion that the spacetimes obeying the spatial scale-invariance are the only physically meaningful solutions of the field equations is then inescapable. From our results here, the spacetime of is then the only physically meaningful spacetime.

Many puzzling features may result from the use of “physically not-meaningful” spacetimes. As an example, consider the requirement for all if the spacetime that one obtains from the principle of equivalence. This requirement implies constraints on the energy-momentum tensor and, hence, on the forms of matter. But, at the Newtonian level, the law of gravity holds for all forms of matter. This puzzling feature is a result of the use of physically not-meaningful spacetimes that have been used in such considerations. Similar puzzling features will also be obtainable in other theories of gravity if the spatial scale invariance is not respected. In the same spirit, the existence of naked singularities is an artefact of the use of
“physically not-meaningful” spacetimes.

Therefore, the situation with the solutions of the field equations of General Relativity is understandable only if we realize that the field equations are based only on the equivalence principle and do not incorporate the spatial scale-independence of gravity. In factuality, the newtonian law of gravitation gets replaced by the single spacetime of (46) that contains all of the weak gravity physics. But, spatial scale-independence needs to be separately imposed on the field equations to obtain it.

In retrospect, General Relativity replaces a “single” law of weak gravity - Newton’s law - with a multiplicity of “laws of gravity” corresponding to many inequivalent spacetimes that are solutions of the field equations. It is therefore not surprising that it is only one spacetime, that of (46), that alone truly contains Newton’s law, in its entirety, in the weak gravity limit. In order to identify this ‘unique’ spacetime, we need to impose the spatial homothety on the field equations because Newton’s law is based on the spatial homothety while the field equations are more general. Why are spacetimes other than that of (46) to be considered “gravitationally not-meaningful” or “physically not-meaningful”? The reason is then directly related to the fact that some of the results obtained from such spacetimes will ‘contradict’ the corresponding results of the weak field theory. Further, the panorama offered by the universe at small and large spatial scales and also the experiments related to the testing of the equivalence principle do not show the break-down of the spatial scale-invariance of gravity at any scale.

In conclusion, the requirement that General Relativity as a theory of gravitation does not provide any length-scale for matter properties results in (46). This spatially homothetic spacetime does not possess a locally or globally naked singularity for spatially non-singular, regular initial data for matter fields. Hence, Cosmic Censorship [21] is equivalent to the statement that gravity does not provide any length-scale for matter properties. We have, in essence, provided also the proof of this statement here.

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[1] Carr B J and Coley A A (1999) Class. Quantum Grav. 16, R31 - R71
[2] Sedov L I (1967) Similarity and Dimensional Methods in Mechanics (New York: Academic Press)
[3] Penrose R (1969) Rev. Nuovo Cimento 1 252
[4] Wagh S M and Govinder K S (2001) Class. Quantum Grav. submitted. Spherically symmetric, self-similar spacetimes. gr-qc/0112035 Pre-print No. - CIRI/01-swkg01
[5] Wagh S M, Govinder M, Govinder K S, Maharaj S D, Mukitbodh P S and Moodley M (2001) Class. Quantum Grav. 18, 2147 - 2162
[6] See Joshi P S (1993) Global Aspects in Gravitation and Cosmology (Oxford: Clarendon Press) and references therein
[7] Wagh S M and Govinder K S (2002) Phys. Rev. Lett. submitted. Naked singularities in spherically symmetric, self-similar spacetimes. gr-qc/0112064 Pre-print No. - CIRI/01-swkg02
[8] Wagh S M (2002) Phys. Rev. D - Rapid Communications submitted. Spherical Gravitational Collapse and Accretion - Exact General Relativistic Description. astro-ph/012055 Pre-print No. CIRI/02-smw02
[9] Wagh S M, Saraykar R V, Mukitbodh P S and Govinder K S (2001) Class. Quantum Grav. submitted. Title: Conformal Killing vectors in spherically symmetric, inhomogeneous, shear-free, separable metric spacetimes. gr-qc/0112033 Pre-print No. CIRI/01-swrsmpkg02
[10] Wagh S M, Saraykar R V, Mukitbodh P S and Govinder K S (2001) Gen. Rel. Grav. submitted. Title: Spherical gravitational collapse with heat flux and cosmic censorship. gr-qc/0112034 Pre-print No. - CIRI/01-swpmkg
[11] Wagh S M and Govinder K S (2002) Spherically symmetric, self-similar gravitational collapse in preparation. Pre-print No. - CIRI/01-swkg03
[12] Wagh S M and Govinder K S (2002) Phys. Rev. D - Rapid Communications submitted. Title: Axially symmetric, spatially homothetic spacetimes gr-qc/0201018 Pre-print No. - CIRI/01-swkg04
[13] Wagh S M (2002) Phys. Rev. D - Brief Reports submitted. Title: Mach’s Principle and Spatial Scale-Invariance of Gravity gr-qc/0202005 Pre-print No. - CIRI/02-smw03
[14] McIntosh C B G (1975) Gen. Rel. Grav. 7 199
[15] Choptuik, M (1993) Phys. Rev. Lett. 70 9
[16] Einstein A and Grossmann M (1913) Z. Math. Physik 62 225

[17] Pais A (1982) Subtle is the Lord ... The science and the life of Albert Einstein (Oxford: Clarendon Press)

[18] See, for example, articles by Cook A H and by Will C M in (1987) 300 Years of Gravitation (Ed. S W Hawking and W Israel, Cambridge: Cambridge University Press)

[19] Mitra A (2000) Found. of Phys. Letters 13 543

Database: astro-ph/9910408

[20] Leiter D, Mitra A and Robertson S (2002) Database: astro-ph/0111421. Also, private electronic communications.

[21] Penrose R (1998) in Black Holes and Singularities: S. Chandrasekhar Symposium (Ed. R. M. Wald, Yale: Yale University Press) and references therein