Jacobians of singularized spectral curves and completely integrable systems

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Abstract

We state two recent results concerning the linearization of integrable systems on generalized Jacobians. Then we apply this to the (complexified) spherical pendulum.

1 Introduction

Let $M^J$ be the affine vector space of complex matrix polynomials $A(x)$ in a variable $x$, of fixed degree $d$ and dimension $r$

$$A(x) = Jx^d + A_{d-1}x^{d-1} + \cdots + A_0, \quad A_i \in \mathfrak{gl}_r(\mathbb{C})$$

where $J \in \mathfrak{gl}_r(\mathbb{C})$ is a fixed matrix. The matricial polynomial Lax equations

$$\frac{d}{dt} A(x) = \left[ \frac{A^k(a)}{x-a}, A(x) \right], \quad k \in \mathbb{N}, \ a \in \mathbb{C}$$

are well known to be Hamiltonian (with respect to several compatible Poisson structures on $M^J$) and completely integrable. The corresponding Hamiltonian vector fields define a complete set of commuting vector fields on the isospectral manifolds

$$M^J_P = \{ A(x) \in M^J : \det(A(x) - yI_r) = P(x, y) \}. $$

The corresponding complex flows are not complete, but may be completed by adding a suitable divisor $D_{\infty}$ called the Painlevé divisor of the system. The new manifold $\bar{M}^J_P = M^J_P \cup D_{\infty}$ becomes a non-compact complex-analytic commutative group. The most remarkable fact concerning $\bar{M}^J_P$ is that it has a commutative algebraic group structure which is compatible with the structure of an analytic group. This algebraic group is just the generalized Jacobian $J(X')$ of a suitable singularized spectral curve $X'$ (to be defined below). The variety $\bar{M}^J_P$ is non-compact and it admits several algebraic structures. The “right” one is defined by the
symmetry group $G$ of the system \([\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C}; J)]\). Namely, let $G = \mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C}; J)$ be the subgroup of the projective group $\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C})$ formed by matrices which commute with $J$. In the applications $G$ is the symmetry group of the corresponding dynamical system (e.g. a rigid body with an axis of symmetry has a symmetry group $\mathbb{C}^*$ which is the complexified rotation group $S^1$). The group $G$ acts on $M^J$ by conjugation, the action is Poisson, and the reduced Hamiltonian system is completely integrable too. The action of $G$ on $M^J_P$ can be extended on the completed variety $\tilde{M}^J_P$ and it is proper and free. Therefore $\tilde{M}^J_P$ can be considered as the total space of a holomorphic principal fiber bundle $\xi$ with base $\tilde{M}^J_P/G$, structural group $G$, and natural projection map

$$\phi : \tilde{M}^J_P \rightarrow \tilde{M}^J_P/G.$$ 

The fiber bundle $\xi$ is described as follows. When the spectral curve $X$ defined by $\{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\}$ is smooth, then the partially compactified variety $\tilde{M}^J_P$ is smooth and biholomorphic to the generalized Jacobian variety $J(X')$. The curve $X'$ is singular and as a topological space it is just $X$ with its “infinite” points $\infty_1, \infty_2, ..., \infty_r$ identified to a single point $\infty$. Thus $J(X')$ is a non-compact commutative algebraic group and it can be described as an extension of the usual Jacobian $J(X)$ by the algebraic group $G = (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{r-s}$, where $s \leq r$ is the number of distinct eigenvalues of the leading term $J$

$$0 \rightarrow G \rightarrow J(X') \xrightarrow{\phi} J(X) \rightarrow 0 . \tag{2}$$

As analytic spaces $J(X')$ and $J(X)$ are complex tori

$$J(X') = \mathbb{C}^{p_a}/\Lambda', \quad J(X) = \mathbb{C}^{p_g}/\Lambda$$

where $\Lambda', \Lambda$ are lattices of rank $2p_g + s - 1$ and $2p_g$ respectively, $p_g$ is the genus of $X$, and $p_a = p_g + r - 1$ is the arithmetic genus of $X'$. The generalized Jacobian $J(X')$ can be also considered as the total space of a holomorphic principal fiber bundle with base $J(X)$, projection $\phi$, and structural group $G$. The group $G$ is then identified with the symmetry group $\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C}; J)$ of \([\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C})]\), and the manifold $\tilde{M}^J_P/G$ with the usual Jacobian $J(X) = J(X')/G$.

The algebraic description of the reduced invariant manifold $\tilde{M}^J_P/G$ is a well known result proved by A. Beauville \([\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C}; J)]\) and M.R. Adams, J. Harnad, J. Hurtubise \([\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C})]\) (see also M. Adler, P. van Moerbeke \([\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C})]\) and section 8.2 of the survey \([\mathbb{P} \mathbb{G} \mathbb{L}_r(\mathbb{C})]\) by A.G. Reyman and M.A. Semenov-Tian-Shansky).

The above can be illustrated on the following “simple” example. Let $\tau_1, \tau_2 \in \mathbb{C}$ be generic complex numbers. Consider the $\mathbb{Z}$-module of rank three $\Lambda \subset \mathbb{C}^2$

$$\Lambda = \mathbb{Z} \left( \begin{array}{c} \frac{2\pi i}{0} \\ 0 \end{array} \right) \oplus \mathbb{Z} \left( \begin{array}{c} 0 \\ 2\pi i \end{array} \right) \oplus \mathbb{Z} \left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right) . \tag{3}$$

$\mathbb{C}^2/\Lambda$ is a non-compact algebraic group and it can be considered as a (non-trivial) extension of the elliptic curve $\mathbb{C}/\{2\pi i \mathbb{Z} \oplus \tau_1 \mathbb{Z}\}$ by $\mathbb{C}^* \sim \mathbb{C}/2\pi i \mathbb{Z}$

$$0 \rightarrow \mathbb{C}/2\pi i \mathbb{Z} \rightarrow \mathbb{C}^2/\Lambda \xrightarrow{\phi} \mathbb{C}/\{2\pi i \mathbb{Z} \oplus \tau_1 \mathbb{Z}\} \rightarrow 0 , \quad \phi(z_1, z_2) = z_1 . \tag{4}$$

If $d = 2, r = 2$ and $P(x, y)$ is a suitable polynomial, then it may be shown that $\tilde{M}^J_P = \mathbb{C}^2/\Lambda$, the symmetry group is $G = \mathbb{C}/2\pi i \mathbb{Z}$ and acts as

$$G \times \mathbb{C}^2/\Lambda \rightarrow \mathbb{C}^2/\Lambda : g \times \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \rightarrow \left( \begin{array}{c} z_1 \\ z_2 + g \end{array} \right) .$$
The quotient group $\{C^2/\Lambda\}/G$ is the elliptic curve $C/\{2\pi i Z \oplus \tau_1 Z\}$ which is the Jacobian of the spectral curve $X$ with an affine equation $\{P(x, y) = 0\}$. Finally $C^2/\Lambda$ is the generalized Jacobian of the singular spectral curve $X'$ which is obtained from $X$ by a suitable singularization (a procedure opposite to the usual regularization of a singular curve). We note that $C^2/\Lambda$ is also an extension of the elliptic curve $C/\{2\pi i Z \oplus \tau_2 Z\}$ and hence it has a second algebraic structure. The “right” algebraic structure is the one related to the symmetry group $G = C/2\pi i Z$.

Generalized Jacobians as generic invariant manifolds of integrable systems appeared (implicitly) first in the papers of Jacobi [19, 20, 21] which is easily seen from the analytic expressions of the solutions (see also Klein and Sommerfeld [22]). This motivated the further study of the corresponding generalized Jacobi inversion problem (e.g. Clebsch an Gordan [8]). The modern theory of generalized Jacobians, without relation to integrable systems, is due to Rosenlicht [27, 28]. In our exposition we shall follow Serre [29]. In the modern literature on integrable systems generalized Jacobians appeared again in B.A. Dubrovin [10] and E. Previato [25]. For more examples see Fedorov [13, 14, 3] and Faye [11, 12].

In the present note we describe the invariant manifold of the integrable system (1) in the case when the leading term $J$ of the Lax matrix $A(x)$ is either regular (Theorem 2.1), or it is not regular but diagonalizable (Theorem 2.2). The proofs are given in [16, 30]. Then we apply Theorem 2.1 to the complexified spherical pendulum. The description which we obtain of the invariant manifold of the system completes the recent results of M. Audin, F. Beukers and R. Cushman [4, 7, 9].

Acknowledgments I am grateful to R. Cushman for sending me the unpublished text [9] part of which is reproduced in the last section.

2 Singularized spectral curves and their Jacobians

A polynomial

$$P(x, y) = y^r + s_1(x)y^{r-1} + \ldots + s_r(x)$$

is called spectral, provided that the affine curve $\{(x, y) \in C^2 : P(x, y) = 0\}$ is the spectrum of some polynomial $r \times r$ matrix $A(x)$

$$P(x, y) = \det(A(x) - y.I_r) .$$

In this case $\deg(s_i(x)) \leq i.d$, where $d$ is the degree of $A(x)$

$$A(x) = A_dx^d + A_{d-1}x^{d-1} + \ldots + A_0, \quad A_i \in \mathfrak{gl}_r(C) . \quad (5)$$

Consider the weighted projective space $\mathbb{P}^2(d) = C^3\{0\}/C^*$, where the $C^*$-action on $C^3$ is defined by

$$t \cdot (x, y, z) \rightarrow (tx, t^d y, tz), \quad t \in C^*. \quad$$

$\mathbb{P}^2(d)$ is a compact complex surface with one singular point $\{[0, 1, 0]\} = \mathbb{P}^2(d)_{sing}$. The affine curve $\{(x, y) \in C^2 : \det(A(x) - y.I_r) = 0\}$ is naturally embedded in $\mathbb{P}^2(d)$,

$$C^2 \rightarrow \mathbb{P}^2(d) : (x, y) \mapsto [x, y, 1],$$

$$C^2 = \{C^2/\Lambda\}/G \rightarrow \mathbb{P}^2(d).$$
and the condition $\deg(s_i(x)) \leq i.d$ shows that its closure $X$ is contained in the smooth surface $\mathbb{P}^2(d)_{reg} = \mathbb{P}^2(d) \setminus \{(0, 1, 0)\}$. Let $x$ be an affine coordinate on $\mathbb{P}^1$. The surface $\mathbb{P}^2(d)_{reg}$ is identified with the total space of the holomorphic line bundle $O_{\mathbb{P}^1}(d)$ with base $\mathbb{P}^1$ and projection

$$\pi : \mathbb{P}^2(d)_{reg} \to \mathbb{P}^1 : [x, y, z] \to [x, z].$$

The induced projection

$$\pi : X \to \mathbb{P}^1$$

is a ramified covering of degree $r$, and over the affine plane $\mathbb{C}$ it is simply the first projection

$$\pi : \{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\} \to \mathbb{C} : (x, y) \to x.$$

**Definition 1** (spectral curve of $A(x)$) We define the spectral curve $X$ of the matrix polynomial $A(x)$, to be the closure of the affine curve $\{(x, y) \in \mathbb{C}^2 : \det(A(x) - y.I_r) = 0\}$ in the total space of the line bundle $O_{\mathbb{P}^1}(d)$.

From now on we fix the spectral polynomial $P(x, y)$ and suppose that the spectral curve $X$ is smooth and irreducible.

We are going now to singularize the curve $X$. Let $m = \sum_{i=1}^{s} n_i P_i$, $P_i \in X, n_i > 0$, be an effective divisor on $X$. To the pair $(X, m)$ we associate a singular curve $X' = X_{reg} \cup \infty$, where if $S = \cup_{i=1}^{s} P_i$ is the support of $m$, then $X_{reg} = X - S$, and $\infty$ is a single point. The structure sheaf $\mathcal{O}'$ of $X' \sim (X, m)$ is defined in the following way. Let $\mathcal{O}_{X'}$ be the direct image of the structure sheaf $\mathcal{O} = \mathcal{O}_X$ under the canonical projection $X \to X'$. Then

$$\mathcal{O}'_P = \begin{cases} \mathcal{O}_P, & P \in X_{reg} \\ \mathbb{C} + i_\infty, & P = \infty \end{cases}$$

where $i_\infty$ is the ideal of $\mathcal{O}_\infty$ formed by the functions $f$ having a zero at $P_i$ of order at least $n_i$. Thus a regular function $f$ on $X'$ is a regular function $f$ on $X$, and such that for some $c \in \mathbb{C}$ and any $i$ holds $v_{P_i}(f - c) \geq n_i$, where $v_{P_i}(.)$ is the order function. If $p_0$ is the genus of $X$ then the arithmetic genus $p_a$ of the singular curve $X'$ is $p_a = p_0 + \deg(m) - 1$.

**Example** Let $m = P^+ + P^-$ be a divisor on the Riemann surface $X$. Then in a neighborhood of $\infty$ the singularized curve $X'$ is analytically isomorphic either to the germ of analytical curve $xy = 0$ ($P^+ \neq P^-$), or to $y^2 = x^3$ ($P^+ = P^-$).

**Definition 2** (singularized spectral curve of $A(x)$) If $\pi$ is the projection (3) and $\infty = [1, 0] \in \mathbb{P}^1$ the “infinite” divisor, then the effective divisor $m = \pi^*(\infty)$ is called modulus of the spectral curve $X$. We have $\deg(m) = r$ and we denote by $X'$ the singular curve associated to the regular curve $X$ and to the modulus $m$.

The generalized Jacobian of the singular curve $X'$ is the analytic manifold

$$J(X') = H^0(X, \Omega^1(-m))^*/H_1(X_{reg}, \mathbb{Z}) = \mathbb{C}^{p_a}/\Lambda',$$

where $\Lambda'$ is a rank $2p_a + s - 1$ lattice, and $\Omega^1(-m)$ is the sheaf of meromorphic one-forms $\omega$, such that $(\omega) \geq -m$. Similarly, for the usual Jacobian $J(X) \subset J(X')$, we have

$$J(X) = H^0(X, \Omega^1)^*/H_1(X, \mathbb{Z}) = \mathbb{C}^{p_a}/\Lambda,$$
where \( \Lambda \subset \Lambda' \) is a rank 2 lattice. It may be shown that both \( J(X') \) and \( J(X) \) are commutative algebraic groups related as follows. \( J(X') \) is a non-trivial extension of \( J(X) \) by the algebraic group \( G = (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{\deg(m)-s} \)

\[
0 \to G \to J(X') \xrightarrow{\phi} J(X) \to 0
\]

where the map \( \phi \) is induced by the natural homomorphisms

\[
H_1(X_{\text{reg}}, \mathbb{Z}) \to H_1(X, \mathbb{Z}), \quad H^0(X, \Omega^1) \to H^0(X, \Omega^1(-m)) .
\]

This means that the sequence (7) is exact in the usual sense and moreover the algebraic structure of \( G \) (respectively of \( J(X) \)) is induced (respectively quotient) of the algebraic structure of \( J(X') \). Note that \( J(X') \) is non-compact. Indeed, while the topological space of \( J(X) \) is \((S^1)^{2p_g}, \) the one of \( J(X') \) is \((S^1)^{2p_g+s-1} \times \mathbb{R}^{2\deg(m)-s-1}.\)

Let \( M_P \) be the variety of \( r \times r \) polynomial matrices of degree \( d \) (3), which have a fixed spectral polynomial \( P(x, y) \)

\[
M_P = \{ A(x) : \det(A(x) - yI_r) = P(x, y) \}.
\]

and let \( M_P^J = M_P \cap M^J \) be the isospectral manifold formed by matrices of the form (3) with fixed leading term \( A_d = J \)

\[
A(x) = Jx^d + A_{d-1}x^{d-1} + \ldots + A_0, \quad A_i \in \mathfrak{gl}_r(\mathbb{C}).
\]

The stabilizer

\[
\mathbb{P}GL_r(\mathbb{C}; J) = \{ R \in \mathbb{P}GL_r(\mathbb{C}) : RJR^{-1} = J \}
\]

of \( \mathbb{P}GL_r(\mathbb{C}) \) at \( J \in \mathfrak{gl}_r(\mathbb{C}) \) is a commutative algebraic group isomorphic to \((\mathbb{C}^*)^{s-1} \times \mathbb{C}^{\deg(m)-s}.\) It is a well known fact that \( M_P^J \) is a smooth manifold, \( \mathbb{P}GL_r(\mathbb{C}; J) \) acts freely and properly on \( M_P^J \) by conjugation, and the quotient space \( M_P^J/\mathbb{P}GL_r(\mathbb{C}; J) \) is a smooth manifold biholomorphic to \( J(X) - \Theta \) [1, 2, 3].

Consider the holomorphic principal fiber bundle \( \xi \) with total space \( M_P^J, \) structural group \( \mathbb{P}GL_r(\mathbb{C}; J), \) base \( M_P^J/\mathbb{P}GL_r(\mathbb{C}; J), \) and natural projection map \( \varphi : M_P^J \to M_P^J/\mathbb{P}GL_r(\mathbb{C}; J). \) Consider also the associate principal bundle \( \eta \) with base space \( J(X) - \Theta, \) total space \( J(X') - \Theta', \) projection map \( \phi, \) and structural group \( G \) (see [1]).

We are ready to formulate our first result

**Theorem 2.1** ([1], L. Gavrilov) Suppose that the spectral curve \( X \) is smooth. There exists a partial compactification of \( M_P^J \) to a non-compact algebraic manifold \( \bar{M}_P^J = M_P^J \cup D_\infty \) which is isomorphic (as an algebraic manifold) to the generalized Jacobian \( J(X') \) of the singularized spectral curve \( X'. \) The Hamiltonian flows (1) are translation invariant on \( J(X'). \) The action of the symmetry group \( \mathbb{P}GL_r(\mathbb{C}; J) \) on \( M_P^J \) extends to an action on \( \bar{M}_P^J \) which is free, proper and compatible with the algebraic structure of \( \bar{M}_P^J. \) The principal algebraic bundles

\[
0 \to \mathbb{P}GL_r(\mathbb{C}; J) \to \bar{M}_P^J \to \bar{M}_P^J/\mathbb{P}GL_r(\mathbb{C}; J) \to 0
\]

and

\[
0 \to G \to J(X') \xrightarrow{\phi} J(X) \to 0
\]

are isomorphic. The Painlevé divisor \( D_\infty = \bar{M}_P^J/M_P^J \) is the pre-image \( \Phi^{-1}(\Theta) \) of the Riemann theta divisor \( \Theta \subset J(X). \)
A local analysis (at infinity) shows that the smoothness of the compact spectral curve \( X \) implies the regularity of the leading term \( J \) of \( A(x) \) (a matrix is regular if its minimal and characteristic polynomials coincide). We shall suppose now that \( J \) is not regular, and hence \( X \) is never smooth (at infinity). We consider only the simplest case when \( J \) is diagonal. Equivalently, if \( m(\lambda) \) is the minimal polynomial of \( J \), then \( m(\lambda) = \prod_{i=1}^{k}(\lambda - \lambda_i) \), where \( \lambda_i \neq \lambda_j \) for \( i \neq j \), and \( k < r = \dim J \). Geometrically this means that \( X \) has only normal crossing singularities at infinity. Without loss of generality we suppose that \( J \) is diagonal, so \( \{e_i\}_{i=1}^{r} \) are its eigenvectors. If \( J \) is not regular the dimension of the stabilizer \( \mathbb{P}GL_r(\mathbb{C}; J) \) is strictly bigger than \( d - 1 \). This suggests that, in order to compensate the increase of the dimension of the stabilizer of \( J \), additional restrictions should be imposed on the subleading term of \( A \) for \( i > 1 \). If the affine part \( \bar{X} \) of the stabilizer of \( J \) is not smooth (at infinity) we consider only the simplest case when \( \bar{X} \) has only normal crossing singularities at infinity. Without loss of generality we suppose that \( \bar{X} \) is smooth. There exists a partial compactification of \( \bar{X} \) into a non-compact algebraic manifold \( \bar{X} = \bar{X} \cup D_{\infty} \) which is isomorphic (as an algebraic manifold) to the generalized Jacobian \( J(X') \) of the singularized spectral curve \( X' \). The Hamiltonian flows (1) are translation invariant on \( J(X') \). The action of the symmetry group \( \mathbb{P}GL_r(\mathbb{C}; J, K) \) on \( M_{JK}^J \) extends to an action on \( \bar{X} \), which is free, proper and compatible with the algebraic structure of \( M_{JK}^J \). The principal algebraic bundles

\[
0 \to \mathbb{P}GL_r(\mathbb{C}; J, K) \to \bar{X} \to \mathbb{P}GL_r(\mathbb{C}; J, K) / \mathbb{P}GL_r(\mathbb{C}; J, K) \to 0
\]
and
\[ 0 \to G \to J(X') \xrightarrow{\Phi} J(X) \to 0 \]
are isomorphic. The Painlevé divisor \( D_\infty = \overline{\text{bar} M^{J,K}_P \setminus M^{J,K}_P} \) is the pre-image \( \Phi^{-1}(\Theta) \) of the Riemann theta divisor \( \Theta \subset J(X) \).

See [30] for applications.

3 The spherical pendulum

The complexified spherical pendulum is described by the following system of differential equations
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -e_3 + (\langle x, e_3 \rangle - \langle v, v \rangle) x
\end{align*}
(9)
constrained on the (invariant) manifold
\[ M = \{(x, v) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid \langle x, x \rangle = 1, \langle x, v \rangle = 0\} . \]
where \( \langle x, y \rangle = \sum_{i=1}^{3} x_i y_i \) and \( e_1, e_2, e_3 \) is the standard basis of \( \mathbb{C}^3 \). The algebraic manifold \( M \) is the tangent bundle \( TQ \) of the complexified sphere \( Q = \{x : \langle x, x \rangle = 1\} \) and may be also identified to \( T^*Q \) via the bilinear form \( \langle ., . \rangle \) on \( T_xQ \). Therefore \( TQ \) is a simplectic manifold and with respect to this simplectic structure \( \mathfrak{g} \) is a two degrees of freedom completely integrable Hamiltonian system with Hamiltonian function
\[ H(x, v) = \frac{1}{2} \langle v, v \rangle + x_3 . \]
The second integral of motion
\[ K(x, v) = x_1 v_2 - x_2 v_1 \]
is associated to the symmetry group \( G = SO(2, \mathbb{C}) \) (rotations about the axes \( e_3 \)). We denote the corresponding Hamiltonian vector fields by \( X_H, X_K \).

Following Cushman [9], consider the following automorphism of \( M \)
\[ \varphi : M \to M : (x, v) \to (y, u) = (x, x \times v) . \]
(10)
It is straightforward to check that \( \varphi^2(x, v) = (x, -v) \), \( \varphi^{-1}(y, u) = (y, u \times y) \). The push forward under \( \varphi \) of the Hamiltonian vector field \( \mathfrak{h} \) is the Hamiltonian vector field \( X_{\mathcal{H}} \) corresponding to the Hamiltonian
\[ \mathcal{H}(y, u) = (\varphi^{-1})^* H(y, u) = \frac{1}{2} \langle u, u \rangle + \langle y, e_3 \rangle . \]
The integral curves of \( X_{\mathcal{H}} \) satisfy
\begin{align*}
\dot{y} &= \dot{x} = v = u \times y \\
\dot{u} &= \dot{x} \times v + x \times \dot{v} = -x \times e_3 = e_3 \times y.
\end{align*}
(11)
Physically the above Hamiltonian system is obtained by reducing the right $S^1$ symmetry given by rotating the Lagrange top about its symmetry axis at its momentum value 0 and then choosing appropriate time and length scales. For more details see [3, pp. 191–200].

As in [15] we introduce the complex change of variables

\begin{align}
U_1 &= u_3 - i u_2 & U_2 &= y_3 - i y_2 \\
V_1 &= u_1 & V_2 &= y_1 \\
W_1 &= u_3 + i u_2 & W_2 &= y_3 + i y_2.
\end{align}

Consider the matrices

\begin{align}
A(\lambda) &= \begin{pmatrix} V_1 \lambda + V_2 & \lambda^2 + U_1 \lambda + U_2 \\ \lambda^2 + W_1 \lambda + W_2 & -(V_1 \lambda + V_2) \end{pmatrix} \\
B(\lambda) &= \frac{A(\lambda) - A(0)}{\lambda} = \begin{pmatrix} V_1 & \lambda + U_1 \\ \lambda + W_1 & -V_1 \end{pmatrix}.
\end{align}

**Theorem 3.1** ([9, R. Cushman]).

Equation (9) can be written in Lax form as

\begin{equation}
2i \frac{d}{dt} A(\lambda) = [A(\lambda), B(\lambda)] = - \left[ A(\lambda), \frac{A(0)}{\lambda} \right].
\end{equation}

The proof is straightforward. An equivalent Lax pair may be found in the recent preprint of M. Audin (she attributes it to A. Reyman). A Lax pair in the Lie algebra $so(3,1)$ appeared earlier in [26].

A short calculation shows that characteristic polynomial of $A(\lambda)$ is

\begin{align}
\det(\mu I - A(\lambda)) &= \mu^2 + (V_1 \lambda + V_2)^2 + (\lambda^2 + U_1 \lambda + U_2)(\lambda^2 + W_1 \lambda + W_2) \\
&= \mu^2 + \lambda^4 + (U_1 + W_1)\lambda^3 + (U_2 + W_2 + U_1 W_1 + V_1^2)\lambda^2 \\
&\quad + (U_1 W_2 + U_2 W_1 + 2 V_1 V_2)\lambda + (U_2 W_2 + V_2^2).
\end{align}

Using (12) and (10) we find that

\begin{align}
U_1 + W_1 &= 2 \langle u, e_3 \rangle = 2 \langle x \times v, e_3 \rangle = 2k \\
U_2 + W_2 + U_1 W_1 + V_1^2 &= 2 \left( \frac{1}{2} \langle u, u \rangle + \langle y, e_3 \rangle \right) \\
&= 2 \left( \frac{1}{2} \langle v, v \rangle + \langle x, e_3 \rangle \right) = 2h \\
U_1 W_2 + U_2 W_1 + 2 V_1 V_2 &= 2 \langle u, y \rangle = 2 \langle v \times x, x \rangle = 0 \\
U_2 W_2 + V_2^2 &= \langle y, y \rangle = \langle x, x \rangle = 1.
\end{align}

Hence we obtain the spectral curve

\begin{equation}
X_{a,ff} = \{ 0 = \mu^2 + F_c(\lambda) \}
\end{equation}
of the Lax equation (13) where
\[ F_c(\lambda) = \lambda^4 + 2k \lambda^3 + 2h \lambda^2 + 1. \] (18)

Let \( X \) be the compactified and normalized curve \( X_{aff} \). When \( X_{aff} \) is smooth, \( X \) is an elliptic curve, \( X = X_{aff} \cup \infty_1 \cup \infty_2 \). Let \( X' \) be the singularized curve \( X \) with respect to the modulus \( \infty_1 + \infty_2 \) (Definition 2) (the topological space of \( X' \) is pictured in [13, Fig.2] and [4, Fig.2]). As a corollary of Theorem 2.1 we obtain the following

**Theorem 3.2** Suppose that the affine spectral curve \( X_{aff} \) of the spherical pendulum is smooth. Then the invariant manifold \( M_{hk} \) is also smooth. There exists a partial compactification \( \tilde{M}_{hk} = M_{hk} \cup D_\infty \) which is isomorphic (as an algebraic manifold) to the generalized Jacobian \( J(X') \) of the singularized spectral curve \( X' \). The Hamiltonian flows \( X_H \) and \( X_K \) of the spherical pendulum are translation invariant on \( J(X') \). The action of the symmetry group \( G = SO(2, \mathbb{C}) = \mathbb{C}^* \) on \( M_{hk} \) extends to an action on \( \tilde{M}_{hk} \) which is free, proper and compatible with the algebraic structure of \( \tilde{M}_{hk} \). The principal algebraic bundles
\[ 0 \to \mathbb{C}^* \to \tilde{M}_P \to \tilde{M}_{hk}/ \mathbb{C}^* \to 0 \]
and
\[ 0 \to \mathbb{C}^* \to J(X') \to J(X) \to 0 \]
are isomorphic. The Painlevé divisor \( D_\infty \) is isomorphic to \( \mathbb{C}^* \)

**Remarks** The complexified spherical pendulum is a limiting case of the complexified Lagrange top and the algebro-geometric structure of the latter was studied in [13]. The above theorem is the analogue of the main result of [13]. A big part of Theorem 3.2 is proved by Beukers and Cushman [7, 9]. It is shown in [7] that the general invariant manifold
\[ M_{hk} = \{(x, v) \in M : H(x, v) = h, K(x, v) = k\} \]
is the total space of a principal \( \mathbb{C}^* \) bundle with base an elliptic curve. The equations satisfied by the variables \( U_i, V_i, W_i \) as well the Lax pair (13) are obtained by R. Cushman [4].

Our second remark concerns the monodromy of the energy-momentum map
\[ M \to \mathbb{R}^2 : (x, v) \to (H(x, v), K(x, v)) \]
restricted to the set of its regular values. This is a “classical” question and we refer the reader to [3]. As the regular fibers of the energy-momentum map are real parts of the affine part of the generalized Jacobian \( J(X') \), then it suffices to study the monodromy of the flat bundle
\[ H_1(J(X'), \mathbb{Z}) \to (h, k) \]
restricted to the set of real regular \( (h, k) \) (and taking into account the real structure of \( J(X') \). But \( H_1(J(X'), \mathbb{Z}) = H_1(X_{aff}, \mathbb{Z}) \) and hence the last question is equivalent to the study of the familiar Picard-Lefschetz monodromy of the homology bundle associated to the fibration
\[ \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda^4 + 2k \lambda^3 + 2h \lambda^2 + 1 + \mu^2 = 0\} \to (h, k) . \]
In the case of the Lagrange top this was (implicitly) used by O. Vivolo \[31, 32\] and in the case of the spherical pendulum by M. Audin [4].

The non-degeneracy of the frequency map is another remarkable feature of the spherical pendulum. This has been first proved by E. Horozov (as conjecture of J.J. Duistermaat) in relation to applications of KAM theory [18]. Another proof, revealing the hidden algebraic geometry of the problem, can be found in [17].

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