CORRECTION OF METRICS

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Metric triple \((X, \rho, \mu)\) is a space with structures of metric \(\rho\) and Borel probabilistic measure \(\mu\). The studying of them was initiated by M. L. Gromov in [2]. Namely, it was proved that metric triples are uniquely parametrized by measures on the set of distance matrices \((\rho(x_i, x_j))_{i,j}\), generated by random and independent choice of points \(x_1, x_2, \ldots \in X\). A. M. Vershik gave another formulation of this fact and proposed its new proof, based on the Ergodic theorem, and initiated the studying of variable metrics on the measurable space. In particular, in [3, 4] he suggested to use dynamics of metrics for studying the entropy and other invariants of dynamical systems. It requires the preliminary studying of the set of matrices and “almost metrics” on the standard measurable space. The present note is devoted to correction of almost semimetrics to real metrics on the set of full measure. We are going to use the obtained theorems in further work on application of metrics dynamics to ergodic theory.

In this article we will consider only Lebesgue spaces with continuous measure.

Definition 1. Let \((X, \mu)\) be Lebesgue space with probabilistic measure, \(\rho(x, y)\) be measurable non-negative function on \((X \times X, \mu \times \mu)\) such that \(\rho(x, y) = \rho(y, x)\) and \(\rho(x, z) \leq \rho(x, y) + \rho(y, z)\) for almost all \(x, y, z\) in \(X\). We say that \(\rho\) is almost-metric on \(X\).

We say that almost-metric \(\rho\) is essentially separable, if for any \(\varepsilon > 0\) one may cover \(X\) by a countable set of measurable subsets of essential diameter less then \(\varepsilon\) (essential diameter of the set \(A \subset X\) is defined as the essential supremum of the function \(\rho(x, y)\), restricted to \(A \times A\)).

Two almost-metrics which coincide on the set of full \((\mu \times \mu)\)-measure as functions of 2 variables, are called equivalent. Sometimes we also say that one of two equivalent metrics is a correction of another (usually corrected metric is in a sense better then the initial one).

Metric (or semimetric) on the Lebesgue space is called admissible, if the corresponding (semi-)metric space is separable on the set of full measure.

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We say that a semimetric $\rho$ on the space with Borel probabilistic measure has a finite $\varepsilon$-entropy $H_\varepsilon(\rho) < \infty$, if there exist a finite number of balls of radius $\varepsilon$, which cover the set of measure at least $1 - \varepsilon$.

Closed ball of radius $r$ centered in $x$ is denoted by $B(x, r)$.

The following lemma lists several equivalent definitions of admissible metric.

Lemma 1. Let $(X, \rho)$ be a semimetric space and $\mu$ be Borel measure on it. Then the following statements are equivalent:
1) For any $\varepsilon > 0$ the semimetric $\rho$ has finite $\varepsilon$-entropy.
2) There exists a set of full measure $\mu$ such that restriction of $\rho$ on this set is separable.
3) For $\mu$-almost all $x$ all balls (in semimetric $\rho$) centered in $x$ have positive measure.

Proof. Let’s prove implication from 1) to 3). Consider the sets $T_n = \{ x \in X : \mu(B(x, \frac{1}{n})) = 0 \}$. It suffices to prove that $\mu(T_n) = 0$ for all $n$. A priori we do not even know that $T_n$ is measurable, but we do not need it. Choose any $\varepsilon \in (0, \frac{1}{10n})$. Condition 1) says that $X$ may be partitioned onto disjoint sets $X_0, X_1, \ldots, X_k$ such that $\mu(X_0) < \varepsilon$ and diam($X_j$) $< 2\varepsilon$ for all $j \in \{1, \ldots, k\}$. Without loss of generality sets $X_j$ have positive measure (if $\mu(X_j) = 0$, remove it and replace $X_0$ to $X_0 \cup X_j$). Then for any point $x \notin X_0$ the ball $B(x, 2\varepsilon)$ contains one of the sets $X_j$, hence has strictly positive measure. It follows that $T_n \subset X_0$. So we have proved that for arbitrarily small $\varepsilon > 0$ the set $T_n$ is contained in a set of measure less then $\varepsilon$. It follows that $T_n$ is measurable and have zero measure. The union of $T_n$’s taken by all positive integers $n$ has zero measure aswell. This proves 3).

Let’ prove that 3) implies 2). Consider the set $Y$ of points such that all balls centered in those points have positive measure. Then $Y$ is a set of full measure and it suffice to prove that for any $\varepsilon > 0$ it has a countable $\varepsilon$-net. Assume that for some $\varepsilon > 0$ it does not have a countable $\varepsilon$-net, then by Zorn’s lemma $Y$ has a more then countable subset with mutual distances at least $\varepsilon$. But then the balls of radius $\varepsilon/3$ centered in those points have positive measures and are mutually disjoint, so the sum of their measures does not exceed 1. But any summable family is at most countable. A contradiction.

Let’ finally prove that 2) implies 1). Let a countable set $\{x_n\}_{n=1}^\infty$ be dense in a set $X'$ of full measure. For fixed $\varepsilon > 0$ consider the balls $B_n = B(x_n, \varepsilon)$. The union of them has full measure. But then for some finite $N$ the finite union $\bigcup_{n=1}^N B_n$ has measure more then $1 - \varepsilon$, as desired. \qed

Theorem 1. 1) Let $(X, \mu)$ be a Lebesgue space, $\rho$ be almost metric on $X$. Then $\rho$ may be corrected to everywhere finite semimetric on $X$. 2) Let $(X, \rho)$ be a semimetric space and $\mu$ be Borel measure on it. Then the following statements are equivalent: 1) For any $\varepsilon > 0$ the semimetric $\rho$ has finite $\varepsilon$-entropy. 2) There exists a set of full measure $\mu$ such that restriction of $\rho$ on this set is separable. 3) For $\mu$-almost all $x$ all balls (in semimetric $\rho$) centered in $x$ have positive measure.
2) If \( \rho \) was essentially separable, then corrected semimetric may be chosen so that \((X, \rho)\) is separable. In other words, the corrected semimetric may be chosen admissible.

Proof. 1) Note that for almost all \( x \) the function \( \rho(x, \cdot) \) is measurable and inequality \( \rho(x, y) + \rho(x, z) \geq \rho(y, z) \) holds for almost all pairs \( (y, z) \). Fix such point \( x_0 \). At first, we replace the measure \( \mu \) to some equivalent measure so that the function \( f(t) = \rho(x_0, t) \) becomes summable. For example, we may take \( A_n = f^{-1}([n - 1, n)) \) and put \( \hat{\mu}(B) = c \sum_{n=1}^{\infty} 2^{-n} \mu(B \cap A_n) \) for any measurable \( B \) and the constant \( c \) is chosen so that \( \hat{\mu}(X) = 1 \). Note that now the function \( \rho(y, z) \) becomes summable as a function of two variables by triangle inequality.

Now we identify Lebesgue space and the unit circle \( S = \mathbb{R}/\mathbb{Z} \) equipped by the Lebesgue probabilistic measure. Define the new metric \( \tilde{\rho} \) (possibly infinite somewhere) by equality

\[
\tilde{\rho}(x, y) = \lim_{T \to +0} T^{-2} \int_0^T \int_0^T \rho(x + t, y + s) dt ds \quad (1)
\]

Note that by Lebesgue theorem on differentiation of integral the limit exists and equals \( \rho(x, y) \) almost surely (i.e. for almost all pairs \( (x, y) \)). This function \( \tilde{\rho} \) is obviously symmetric. Let’s prove triangle inequality for it. Note that for almost all \( (s, t, \tau) \in [0, T]^3 \) one has

\[
\rho(y + s, z + t) \leq \rho(y + s, x + \tau) + \rho(x + \tau, z + t).
\]

Let’s integrate it by \([0, T]^3\), divide by \( T^3 \) and take an upper limit for \( T \to +0 \). Using inequality \( \lim \sup (F + G) \leq \lim \sup F + \lim \sup G \) we get a triangle inequality for \( \tilde{\rho} \). If \( \tilde{\rho} \) does not vanish in some points on diagonal, replace those values to 0. Note that the function \( \tilde{\rho} \) is finite almost everywhere (since equivalent function \( \rho \) is almost everywhere finite). Hence one may choose a point \( x_0 \) so that \( \tilde{\rho}(x_0, x) < \infty \) for almost all \( x \in S \), i.e. for all \( x \) in the set \( S_1 \) of full measure. Now change the semimetric \( \tilde{\rho} \) outside \( S_1 \times S_1 \) by the rules \( \tilde{\rho}(x, y) := \tilde{\rho}(x_0, y) \) for \( x \notin S_1, y \in S_1 \), \( \tilde{\rho}(x, y) := \tilde{\rho}(x, x_0) \) for \( y \notin S_1, x \in S_1 \) and \( \tilde{\rho}(x, y) := 0 \) for \( x, y \notin S_1 \). This semimetric is almost everywhere finite.

2) Using the statement of p. 1), we may suppose that the semimetric \( \rho \) is defined on whole \( X \) and satisfies triangle inequality everywhere. Let’s prove that there exists a set \( Y \) of full measure so that restriction of \( \rho \) on \( Y \) is a separable semimetric space. The correction outside \( Y \) is done as described above in the end of proof of 1), and separability is saved after such correction. It suffices for any fixed positive integer \( n \) to find a countable union of balls of radius \( 1/n \) which has a full measure, then define \( Y \) as the intersection of such sets. Let’s cover \( X \) by a countable number of measurable sets of diameter at most \( 1/n \). Consider one such set \( A \). For almost all points \( x \in A \) the distances from \( x \) to almost all points of \( A \) do not exceed \( 1/n \). Hence \( A \mod 0 \) is covered by a ball of radius \( 1/n \), and we are done. \( \square \)
Usually when one says about the space with metric and measure, she considers the metric structure as “main”, and restricts measure to satisfy some properties in terms of metric (Borel, regular measures). We follow A. M. Vershik’s approach, considering the metric as measurable function. Note that if \((X, \rho)\) is separable semimetric space, and \(\mu\) is a Borel measure on \(X\), then \(\rho\) as a function on \(X \times X\) is measurable w.r.t. the measure \(\mu \times \mu\) (since it is continuous and hence Borel measurable).

The following theorem shows that in separable case those two conditions (“measure is Borel” and “metric is measurable”) are equivalent.

**Theorem 2.** Let \((X, \mu)\) be the Lebesgue space, \(\rho\) be admissible semimetric on \(X\). Then the measure \(\mu\) is Borel w.r.t. topology of metric \(\rho\).

**Proof.** For any rational \(r > 0\) the set \(\{(x, y) : \rho(x, y) < r\}\) is measurable, hence almost all its sections (balls of radius \(r\), hereafter balls mean open balls). Hence for almost all points \(x \in X\) all balls with rational radius center in \(x\) are measurable. In this case all balls centered in \(x\) are measurable. Denote by \(X_1\) the set of such points, let \(X_2\) be the complement of \(X_1\) (\(\mu(X_2) = 0\)). Consider a countable dense subset \(X' \subset X_1\). Let’s prove that any ball is measurable, from separability it follows that measure is Borel. Consider the ball \(B = B(x_0, r_0)\) centered in \(x_0\) with radius \(r_0\). For any point \(x \in X' \cap B\) consider the ball \(B(x, r_0 - \rho(x, x_0))\). It lies in \(B\) and is measurable. Let’s prove that the union of such balls contains \(X_1 \cap B\). Then \(B\) contains their union \(U\) and is contained in \(U \cup X_2\), hence \(B\) is measurable and \(B = U \mod 0\). Take any point \(x \in X_1 \cap B\). Let’s find a point \(y \in X'\) such that \(\rho(x, y) < (r_0 - \rho(x_0, x))/2\). Then by triangle inequality \(\rho(x_0, y) < (r_0 + \rho(x_0, x))/2\), so \(y \in B\) and moreover the ball \(B(y, r_0 - \rho(x_0, y))\) contains a point \(x\), as we wish. \(\square\)

If the metric is not separable, the conclusion of this theorem may fail. Indeed, let \(A\) be a non-measurable subset of \(X\), \(x_0 \in X \setminus A\) be a point, define a metric

\[
\rho(x, y) = \begin{cases} 
0, & \text{if } x = y; \\
1, & \text{if } x = x_0, y \in A \text{ or } y = x_0, x \in A; \\
2, & \text{otherwise.}
\end{cases}
\]

It equals 2 almost everywhere and is so measurable, but the ball \(B(x_0, 3/2)\) is not measurable.

One may ask an opposite question. Let be given some (not separable) semimetric \(\rho\) on the space \(X\). Consider its Borel sigma-algebra \(\mathcal{B}\). Is it true that \(\rho\), as a function on \(X \times X\), is measurable w.r.t. sigma-algebra \(\mathcal{B} \times \mathcal{B}\)? We do not know the answer.
Now let’s prove that two equivalent admissible metrics coincide on the set of full measure. Of course, this statement fails without admissibility condition (for example, for metrics with distances values 1 and 2.)

**Theorem 3.** Let two admissible metrics \( \rho_1, \rho_2 \) coincide almost everywhere on \( X \times X \) w.r.t. measure \( \mu \times \mu \). Then there exists a subset \( X' \subset X \) of full measure such that \( \rho_1 \) and \( \rho_2 \) coincide on \( X' \times X' \).

**Proof.** Note that for almost all \( x \in X \), we have \( \rho_1(x, y) = \rho_2(x, y) \) for almost all \( y \in X \). Removing the set of zero measure from \( X \), we may suppose that it holds for all \( x \). Now for any \( x \in X \) and any positive \( r \) the balls \( \{ y \in X : \rho_1(x, y) < r \} \) and \( \{ y \in X : \rho_2(x, y) < r \} \) have equal measure. Using \( \square \) we may find a subset \( X' \) of full measure such that for any \( x \in X' \) any ball \( \{ y \in X : \rho_1(x, y) < r \} \) has a positive measure. Then the same holds for \( \rho_2 \). Let’s prove that \( \rho_1 \) and \( \rho_2 \) coincide on \( X' \times X' \) Take \( x_1, x_2 \in X' \). Take any \( r > 0 \) and note that for almost all \( y \) such that \( \rho_1(x_1, y) < r \) one has \( \rho_1(x_1, y) = \rho_2(x_1, y) \) and \( \rho_1(x_2, y) = \rho_2(x_2, y) \). Since the set of such \( y \)'s (it is a ball) has positive measure, we may find at least one point \( y \) in it. Then

\[
\rho_2(x_1, x_2) \leq \rho_2(x_1, y) + \rho_2(y, x_2) = \rho_1(x_1, y) + \rho_1(y, x_2) \leq 2r + \rho_1(x_1, x_2).
\]

Since it holds for any \( r > 0 \), we conclude that \( \rho_2(x_1, x_2) \leq \rho_1(x_1, x_2) \). Opposite inequality is analogous. So, \( \rho_1 \) and \( \rho_2 \) coincide on \( X' \times X' \) as desired. \( \square \)

The natural question to ask is the following: which structures on measure spaces (except semimetric space structure) admit correction theorems like Theorem 3? May one correct almost-group to the group, almost-vector space to vector-space and so on? We do not know non-trivial examples, in which the answer is negative (the trivial examples include, say, the observation that almost-metric may be corrected only to semimetric, not to metric). We present another positive result

**Theorem 4.** Let semimetric \( \rho \) be defined on a Lebesgue space \( X \), and let it satisfy ultrametric inequality \( \rho(x, z) \leq \max(\rho(x, y), \rho(y, z)) \) for almost all triples \( (x, y, z) \in X^3 \). Then there exists an ultrametric on \( X \), which coincides with \( \rho \) on almost all pairs and satisfies ultrametric inequality for all triples.

**Proof.** For any \( n \) consider the correction of almost-metric \( \rho^n \) given by the formula

\[
(\rho_n(x, y))^n = \lim_{T \to +0} \sup_{T} T^{-2} \int_0^T \int_0^T \rho^n(x + t, y + s) \, dt \, ds
\]

By power means inequalities the sequence \( \rho_n(x, y) \) increases by \( n \) for fixed \( x, y \), so it has (finite or infinite) limit \( \tilde{\rho}(x, y) \). By Lebesgue’s integrals differentiation theorem this limit is almost everywhere finite
and equal to $\rho(x, y)$. The function $\tilde{\rho}^n$ is a semimetric for all $n$ (this is so for $\rho_m$ instead $\tilde{\rho}$ with $m \geq n$, and one may pass to limit in the corresponding triangle inequality). The infinite values may be avoided on the same way as in the proof of Theorem 1 for usual metrics. □

We also formulate the following general statement, which includes Theorems 1, 4 as partial cases.

**Conjecture 1.** Given positive integers $k \leq n$ and measurable real-valued function $f(x_1, x_2, \ldots, x_k)$. For almost all $y_1, \ldots, y_n$ the vector $\{f(y_{i_1}, \ldots, y_{i_k})\}_{1 \leq i_j \leq n}$ of dimension $n^k$ belongs to the given closed subset in the space of dimension $n^k$. Then there exists a function $\tilde{f}$, equivalent to $f$, for which this condition holds for all $y_1, \ldots, y_n$.

The questions of this paper arised in the program in ergodic theory, initiated by A. M. Vershik. We are also grateful to him for support and numerous helpful discussions.

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Zatitskiy P. B., Petrov F. V. Correction of metrics.

We prove that a symmetric nonnegative function of two variables on a Lebesgue space that satisfies the triangle inequality for almost all triples of points is equivalent to some semimetric. Some other properties of metric triples (spaces with structures of a measure space and a metric space) are discussed.