TG–EQUIVARIANCE OF CONNECTIONS AND GAUGE TRANSFORMATIONS

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December 6, 1994

Abstract

We consider the notion of a connection on a principal bundle $P(M, G)$ from the point of view of the action of the tangential group $TG$. This leads to a new definition of a connection and allows to interpret gauge transformations as effect of the $TG$ action.

1 Introduction

The central role the gauge potential plays in the theories of fundamental interactions of nature always justifies new investigations concerning its nature. If we take into account gauge transformations it becomes quite apparent that gauge potential is something special compared to any other particle field. It is not a field which transforms itself like a tensor field. It is something more complicated than a tensor field with more involved and deeper lying properties.

A gauge potential is carrier of a geometric structure in the sense of the metric structure in the general relativity, but in a more general sense. Because of its complexities, its deepness and its importance, there exist several formulations of it [1, 2]. In this paper, we would like to present a new property of the gauge potential: the $TG$–equivariance of a connection on a principal bundle $P(M, G)$ with $M$ the space-time (basis manifold) and $G$ the structure group. The $G$–equivariance property of a connection is of course well known. Here we mean, as we shall explain, the equivariance respectively to the bigger group, the tangential group $TG$, of the structure group $G$. It is also important to note that the $G$–equivariance property is shared by the usual tensor fields, considered as vector-valued functions on a principal bundle $P(M, G)$. But only the connection has in addition the $TG$–equivariance property.
The motivation for this paper came from considerations within the $G$–theory framework and applications of it in investigating the structure of the space of reducible connections for Yang-Mills theories [3], anomalies [4, 5] and recently within a non commutative geometrical approach to the standard model [6]. The $G$–theory point of view emphasizes particularly the action of a group and all what follows from it on a relevant space of physical objects.

Our starting point is as already mentioned a principal bundle $P(M, G)$ with a connection $\omega$ on it and the structure group $G$ acting by definition freely on $P$ from the right. Going now to the tangential space $TP$ and $TG$ of $P$ and $G$ respectively, it is important to realize that $TG$ may be considered as a Lie group and that it acts freely on $TP$. The multiplication law $TG \times TG \to TG$ and the action $TP \times TG \to TP$, are to be defined below. This leads to the fact that the tangential space $TP$ is also a principal bundle with basis manifold $TM$ and structure group $TG$. We now consider the space $TP/G$ as a fiber bundle (non-vector bundle) with typical fibre the Lie algebra $G$ of $G$. We show that $TP/G$ is isomorphic to the associated fiber bundle of $TP \times TG \mathcal{G}$ where the action of $TG$ on $\mathcal{G}$ is deduced from the multiplication law in $TG$. We give a new definition of a connection on $P$: a connection on $P(M, G)$ is a linear section $\tilde{H}$ on the fiber bundle $TP \times TG \mathcal{G}$.

$$\tilde{H} : TM \to TP \times TG \mathcal{G}. \quad (1)$$

It can be shown that $\omega$ (from the usual definition of a connection) and $\tilde{H}$, as above, are equivalent. The section $\tilde{H}$ may also be considered as a $TG$–equivariant and linear function $H$ on $TP$ with values on $\mathcal{G}$.

$$\tilde{H} : TP \to \mathcal{G}, \quad X_p \mapsto \tilde{H}(X_p) \quad \text{with} \quad \tilde{H}(X_p \cdot \xi_g) = \xi_g^{-1} \cdot \tilde{H}(X_p), \quad (2)$$

and $X_p \in TP$, $\xi_g \in TG$. $X_p \cdot \xi_g$ represents the action of $TG$ on $TP$ from the right and $\xi_g \cdot \eta$ with $\eta \in \mathcal{G}$ the action of $TG$ on $\mathcal{G}$ (from the left). It follows immediately that this property of $TG$–equivariance of $\tilde{H}$ is also adapted by the $\omega$. As we shall see, the following equation is valid:

$$\omega(X_p) = -\tilde{H}(X_p). \quad (3)$$

The last topic we would like to discuss, are the gauge tranformations on $\omega$ in the light of the above development. It is well known that a gauge transformation $\Psi$ can be described by the equivariant function $U$ on $P$ with values on $G$. So we have $\Psi(p) = pU(p)$ with

$$U : P \to G$$

and

$$U(pg) = g^{-1}U(p)g. \quad (4)$$
This makes the $G$–action point of view transparent and induces the gauge transformation of the connection $\omega$ given by:

$$(\Psi^*\omega)(X_p) = \text{Ad}_{U(p)^{-1}}\omega(X_p) + U^{-1}dU(X_p).$$

This transformation may also be expressed by the equation

$$(\Psi^*\omega)(X_p) = -(TU X_p)^{-1} \cdot (-\omega(X_p)).$$

As we see, gauge transformations expressed pointwise are nothing else but the $T G$–action on the values of connection. The non–linear term in the transformation of a connection may now be explained by the $T G$–action on $\mathcal{G}$. $T G$ is a group of affine type which acts non linearly on $\mathcal{G}$.

In what follows we discuss in the next section the multiplication rule in $T G$, the action of $T G$ on $TP$ and the space $TP$ as a principal bundle over $TM$. In section three we present the new definition of a connection and focus on its $T G$–equivariance property. In section four we consider gauge transformations on a connection from the point of view of the $T G$–action and we finally give a summary in section five.

In the physical literature we found no explicit treatment of the $T G$–equivariance aspect of connections. The same is valid, to our surprise, for the recent mathematical literature we examined. The only place we found some, for us very inspiring, but short and compact indications on $T G$–equivariance, is the very early paper [7]. Parts of section two and three are implicitly or explicitly contained in [7] but without connection to the gauge transformations which were the main motivation for the present paper. For all these reasons we tried to be quite detailed and explicit in our presentations.

## 2 $TP$ as $TG$ principal bundle

The action of the Lie–group $G$ in the principal bundle $P(M, G)$ is a part of its definition. Therefore we expect the $G$–action to be also relevant for further structures defined on $P$. This is the case i.e. for the connection $\omega$ on $P$ and for the vector valued functions on $P$ which correspond to particle fields in the space time $M$.

In this paper we would like to investigate the action of the group $T G$ which of course contains $G$ and in addition its Lie algebra $\mathcal{G} = \text{Lie}(G)$. For this purpose we have first to determine the multiplication law on $T G$ and then the action of $T G$ on $TP$. For the reasons explained in the introduction, we shall proceed in a quite detailed way. We start with the multiplication law in $T G$, although the space $T G$ is isomorphic to the Cartesian product of $\mathcal{G}$ and $G$, the relevant multiplication law is not the so expected trivial one, but it makes $T G$ a semi–direct product of $\mathcal{G}$ and $G$:

$$(\xi_g, \eta_h) \rightarrow \xi_g \cdot \eta_h := TR_h \xi_g + TL_g \eta_h.$$
The notation $R$ and $L$ is used to the right and left action of $G$ on $G$. The inverse of $\xi_g$ is given by

$$\xi_g^{-1} = -TL_{g^{-1}}TR_{g^{-1}}\xi_g.$$ \hspace{1cm} (8)

If we denote with $e$ and $0_e$ the neutral elements in $G$ and $TG$ we obtain by explicit calculation $\xi_g \cdot (\xi_g)^{-1} = 0_e$. We have of course the injections:

$$G \rightarrow TG \quad \text{and} \quad G \rightarrow TG \quad g \mapsto 0_g \quad \xi \mapsto \xi_e.$$ \hspace{1cm} (9)

The verification of associativity is slightly involved and it is therefore an advantage to use the isomorphism which contains the semi–direct product splitting of $TG$ given by

$$\alpha : TG \rightarrow G \times G,$$

$$\xi_g \mapsto (\xi, 0_g) := (TR_{g^{-1}}\xi_g, g).$$ \hspace{1cm} (10)

Denoting as usual $\text{Ad}_G = \text{Aut}(G)$ with

$$\text{Ad}_g\xi = TL_gTR_{g^{-1}}\xi = g\xi 0_{g^{-1}},$$ \hspace{1cm} (11)

we have for the multiplication rule the multiplication

$$\xi_g \cdot \eta_h \mapsto (\xi, g)(\eta, h) = (\xi + \text{Ad}_g\eta, gh).$$ \hspace{1cm} (12)

At this point it should be noticed that of course all the above structures are induced naturally by the $T$–functor and the canonical isomorphism given in eq. (10).

In what follows we need several times the action of $TG$ on $G$. It may be derived from the multiplication law above and the action of $TG$ on the $G$–orbit space $TG/G$ given by

$$TG/G = \{\xi_gG = TR_h\xi_g|h \in G\}$$ \hspace{1cm} (13)

which we also denote as $G = TG/G$ by misuse of notation. The rule of addition in $TG/G$ may be defined by

$$\xi_gG + \eta_gG := (\xi_g + \eta_g)G.$$ \hspace{1cm} (14)

The $TG$–action on $G = TG/G$ is given by

$$TG \times G \rightarrow G,$$$$

(\eta_h, \xi) \mapsto \eta_h \cdot \xi = \text{Ad}_h\xi + TR_{h^{-1}}\eta_h.$$ \hspace{1cm} (15)

\footnote{The restriction to the subgroup $G$ of $TG$ leads as expected to the adjoined action of $G$ on $G$:}

$$G \times G \rightarrow G,$$$$

(g, \eta) \mapsto TR_{g^{-1}}(0_g \cdot \eta) = TR_{g^{-1}}TL_g\eta = \text{Ad}_g\eta.$$
The action of \( TG \) on \( TP \) is given canonically from the lifting of the \( G \)-action on \( P \) to the tangential space (\( T \)-functor). So we have in an obvious notation from

\[
R : P \times G \to P,
(p, g) \mapsto pg = R_g p = R_p g,
\]

the \( TG \)-action on \( TP \):

\[
TP \times TG \to TP,
(X_p, \xi_g) \mapsto X_p \cdot \xi_g := TR_g X_p + TR_p \xi_g.
\]

In order to facilitate the reading, we give some further notations and some examples. For the Maurer–Cartan form \( C \) on \( G \) given by

\[
C(\xi_g) = TL_{g^{-1}} \xi_g.
\]

we often use the expression \( \xi(g) = C(\xi_g) \in \mathcal{G} \). We denote a fundamental vector field on \( P \) with \( \xi \) where we have \( \xi \in \mathcal{G} \) and \( \xi_p = TR_p \xi \).

We now give some useful examples of the \( TG \) action on \( TP \).

\[
\begin{align*}
X_p \cdot 0 &= X_p, \\
X_p \cdot \xi &= X_p + \hat{\xi}_p, \\
X_p \cdot 0_g &= TR_g X_p, \\
0_p \cdot \xi &= TR_p \xi =: \hat{\xi}_p, \\
0_p \cdot \xi_g &= TR_{pg} C(\xi_g) =: TR_{pg} \xi(g).
\end{align*}
\]

After this preparation, everything is fixed and as a first result of the \( TG \)-action we may formulate the following proposition:

*The space \( TP \) is a principal bundle \( TP(TM, TG) \) which structure group \( TG \) basis manifold \( TM \) and canonical projection \( T\pi : TP \to TM \). \( \pi \) is the canonical projection in \( P(M, G) \), \( \pi : P \to M \).*

This can be seen from the fact that

i) \( TG \) acts freely on \( TP \).

ii) \( T\pi \) commutes with the group action \( TG \).

For i) we have to show that with \( X_p \cdot \xi_g = X_p \),

\[
\xi_g = 0_c
\]

must be valid.

From

\[
X_p \cdot \xi_g = TR_g X_p + TR_p \xi_g = X_p
\]
we must first have \( g = e \) so that using eq. (19), we obtain
\[
X_p \cdot \xi_e = X_p + \hat{\xi}_p = X_p \quad \text{and} \quad \hat{\xi}_p = 0_p
\]
which shows that
\[
\xi = 0 \quad \text{and} \quad \xi_g = 0_e.
\]
For ii) we have to show that
\[
T_\pi X_p = T_\pi (X_p \cdot \xi_g)
\]
is valid. Using eq. (19) and
\[
X_p \cdot \xi_g = X_p \cdot 0_g + 0_p \cdot \xi_g
\]
we obtain
\[
X_p \cdot \xi_g = TR_g X_p + TR_g \xi(g).
\]
\( TR_p \xi(g) \) is vertical and
\[
T_\pi TR_g X_p = T_\pi X_p,
\]
so that ii) is indeed valid.

3 The \( TG \)–equivariance of connection

Having in mind the \( G \)–equivariance property of connections and the \( TG \)–action on \( TP \), as discussed in the previous section, we study the space \( TP/G \) and ask ourselves what its connection with the connections on \( P(M, G) \) is. This leads us to a new definition of a connection which is of course, as we shall show, equivalent to the usual one. The original definition of connections is closely related to the \( G \)–action aspect whereas the definition in this section will reveal the \( TG \)–action aspect and bring to light the most characteristic property of connections: the \( TG \)–equivariance.

For that purpose we shall first discuss the \( G \) orbit space
\[
TP/G = \{ X_p G = \{ TR_g X_p | g \in G \} | p \in P \}
\]
and we shall show that it is isomorph to the \( TP \) associated fibre bundle with typical fiber the space \( \mathcal{G} \):
\[
TP/G \simeq TP \times_{TG} \mathcal{G}.
\]
The \( TG \)–action on \( \mathcal{G} \) is given in eq. (15). This action is non-linear, therefore we do not expect \( TP \times_{TG} \mathcal{G} \) to be a vector bundle although it may be represented as \( TP \times_{TG} \mathcal{G} = TM \times \mathcal{G} \).
The isomorphism in eq. (29) follows from the fact that \( TP \) is a principal bundle and that \( G \) is a subgroup of its structure group \( TG \):

\[
TP/G \rightarrow TP \times_{TG} (TG/G),
\]

\[
[X_p \cdot \xi_g]_G \rightarrow [X_p, \xi_g]_{TG}.
\]

If we take \( \xi_g = 0_e \), this may be seen more perspicuously:

\[
X_p G \rightarrow [X_p, 0_e]_{TG}.
\]

In the fibre bundle \( TP/G \), we have to define an addition: for \( \pi(p) = \pi(q) \) and \( p = qg \), we have:

\[
X_p G + Y_q G := X_p G + (TR_q Y_q)G = (X_p + TR_q Y_q)G.
\]

In what follows we shall give three new definitions of a connection. These are closely related to the horizontal lifting of the (original) definition of connection: A connection on a principal bundle \( P(M, G) \) is a linear section in the associated fibre bundle \( TP \times_{TG} G \) (\( G = \text{Lie}(G) \)).

The action of \( TG \) on \( G \) is given in eq. (15). If we denote by \( \tilde{H} \) this section, we have:

\[
\tilde{H} : TM \rightarrow TP/G,
\]

\[
V_x \mapsto \tilde{H}(V_x).
\]

The existence of such a section leads to an equivalent but more abstract definition:

A connection on a principal bundle \( P(M, G) \) is a reduction of the structure group \( TG \) of the principal bundle \( TP(TM, TG) \) to \( G \).

The most transparent definition we obtain if we consider the above section \( \tilde{H} \) in \( TP/G \) as a \( G \) valued linear function \( \bar{H} \) on \( TP \):

A connection on a principal bundle \( P(M, G) \) is a linear \( TG \)-equivariant function on \( TP \) with values in \( G = \text{Lie}(G) \). \( \bar{H} \) is given by

\[
\bar{H} : TP \rightarrow G,
\]

\[
X_p \mapsto \bar{H}(X_p) \quad \text{with}
\]

\[
\bar{H}(X_p \cdot \xi_g) = (\xi_g)^{-1} \cdot \bar{H}(X_p).
\]

At this point it is interesting to note that the function \( \bar{H} \) is, as we shall show, directly related to the original definition of a connection \( \omega \) (as a linear function on \( TP \)) whereas the section \( \tilde{H} \) is directly related to the horizontal lift \( H^\omega \) which stems from the \( \omega \). The relation between \( \tilde{H} \) and \( \bar{H} \) is given by:

\[
\tilde{H}(T\pi X_p) = [X_p, \bar{H}(X_p)]_{TG}.
\]
The linearity of $\tilde{H}$ is related as expected to the linearity of $\tilde{H}$:

$$[X_p, \xi]_{TG} + [Y_p, \eta]_{TG} = [X_p + \xi_p, 0]_{TG} + [Y_p + \eta_p, 0]_{TG} ,$$

$$= [X_p \cdot \xi, 0]_{TG} + [Y_p \cdot \eta, 0]_{TG} ,$$

$$= (X_p + \xi_p)G + (Y_p + \eta_p)G ,$$

$$= [X_p + \xi_p + Y_p + \eta_p, 0]_{TG} ,$$

$$= [X_p + Y_p, \xi + \eta]_{TG} .$$

(36)

where we have used the isomorphism of eq. (31) and the definition of the eq. (32).

This corresponds to

$$[X_p, \tilde{H}(X_p)]_{TG} + [Y_p, \tilde{H}(Y_p)]_{TG} = [X_p + Y_p, \tilde{H}(X_p) + \tilde{H}(Y_p)]_{TG} .$$

(37)

So we obtain from the linearity of $\tilde{H}$

$$\tilde{H}(T\pi X_p) + \tilde{H}(T\pi Y_p) = \tilde{H}(T\pi (X_p + Y_p))$$

(38)

the linearity of $\tilde{H}$:

$$[X_p, \tilde{H}(X_p)]_{TG} + [Y_p, \tilde{H}(X_p)]_{TG} = [X_p + Y_p, \tilde{H}(X_p) + \tilde{H}(Y_p)]_{TG}$$

$$\tilde{H}(X_p + Y_p) = \tilde{H}(X_p) + \tilde{H}(Y_p) .$$

(39)

For the relation between the horizontal lift $H_\omega$ and $\tilde{H}$, taking $\pi(p) = x$ and $T\pi X_p = V_x$, we have:

$$H_\omega : T_x M \rightarrow T_p P ,$$

$$V_x \mapsto H_\omega(V_x) .$$

(40)

Using the projection:

$$\mu : TP \rightarrow TP/G ,$$

$$X_p \mapsto X_p G ,$$

(41)

we have

$$\tilde{H}_x = \mu \circ H_\omega .$$

(42)

The relation of $\tilde{H}$ to connection $\omega$ is given by

$$\omega(X_p) = -\tilde{H}(X_p) .$$

(43)

All we have to show is that the so-defined $\omega = \omega(\tilde{H})$ fulfills the original axioms of a connection:
i) \( \omega \) is \( G \)-equivariant.

ii) the restriction of \( \omega \) to the fundamental vector fields on \( P \) corresponds, after the identification of the fundamental vector fields with the Lie algebra \( \mathcal{G} \), to the identity.

The condition i) is clearly valid since the \( \omega \) given by \( \bar{H} \) is \( TG \)-equivariant, \( G \) is a subgroup of \( TG \) and acts on \( \mathcal{G} \) like \( \text{Ad}_G \).

The condition ii) follows from the fact that for \( \hat{\xi}_p = TR_p \xi \) with \( \xi \in \mathcal{G} \)

\[
\omega(\hat{\xi}_p) = -\bar{H}(\hat{\xi}_p) = -\bar{H}(0_p \cdot \xi_e),
\]

\[
= -\xi^{-1} \cdot \bar{H}(0_p),
\]

\[
= -\xi^{-1} \cdot 0,
\]

\[
= -(-\xi) = \xi
\]

where we have used the \( TG \)-equivariance of \( \bar{H} \) (eq. (34)) and the linearity of \( \bar{H} \) (eq. (39)) to evaluate \( \bar{H}(0_p) \).

The above considerations showed the equivalence of all the notions of connection represented by \( \omega, H^\omega, \tilde{H}, \bar{H} \) in every combination and particularly in \( (\omega, H^\omega) \) with \( (\bar{H}, \tilde{H}) \).

At this point we give, for those who feel uneasy with the use of the fibre bundle \( (TP/G \to TM) = TP \times TG \mathcal{G} \) in the definition of a connection in eq. (33), a further definition in terms of vector bundles. For that purpose we observe that \( TP/G \to M \) is a vector bundle with typical fiber \( IR^m \times \mathcal{G} \) and \( m = \text{dim}(M) \). So, denoting this vector bundle by \( E = (TP/G \to M) \), we have \( E = M \tilde{\times} (IR^m \times \mathcal{G}) \).

This leads to the definition of a connection as a special vector bundle homomorphism between \( TM \) and \( E \):

\[ A \text{ connection on a principal bundle } P(M, G) \text{ is a vector bundle homomorphism} \]

\[ \hat{H} : TM \to E = (TP/G \to M) \quad (45) \]

with the restriction \( \nu \circ \hat{H} = id_{TM} \) and \( \nu = T\pi \circ \mu^{-1} \) the projection \( \nu : TP/G \to TM \). In the case of a trivial \( P(M, G) \), this definition corresponds to the graph of a gauge potential \( A \):

\[ A : TM \to \mathcal{G} \quad (46) \]

as it is used in particle physics phenomenology.

4 \( TG \)-equivariance and gauge transformations

In this section we deal with the implications for gauge transformations from the \( TG \)-action point of view. In this way we obtain a new formulation for
the well known expression for gauge transformations of connections in terms of $T G$–elements instead of $G$–elements.

We consider a gauge transformation as an automorphism on $P(M, G)$ which may be expressed equivalently by a function on $P$ with values on $G$ \[8\]:

\[
\Psi : P \rightarrow P, \quad p \mapsto \Psi(p) = pU(p) \tag{47}
\]

and

\[
U : P \rightarrow G, \quad p \mapsto U(p) \tag{48}
\]

with

\[
U(pg) = g^{-1}U(p)g. \tag{49}
\]

The automorphism $\Psi$ induces in the space of connections the transformation $\Psi^*$ and we have for the connection $\omega$ pointwise the expression

\[
(\Psi^*\omega)(X_p) = \text{Ad}_{U(p)^{-1}}\omega(X_p) + (U^*C)(X_p) \tag{50}
\]

with $\xi_p \in TP$ and where $C$ is the Maurer–Cartan form on $G$:

\[
C : TG \rightarrow \mathcal{G}, \quad \xi_g \mapsto C(\xi_g) = TL_g^{-1}\xi_g. \tag{51}
\]

We would like now to reexpress the right hand side of eq. \((49)\) in terms of the $TG$–action on $TP$ and $\mathcal{G}$. For the action of $TG$ on $\mathcal{G}$ we obtain from

\[
\xi_g \cdot \eta = \text{Ad}_g\eta + TR_{g^{-1}}\xi_g, \tag{52}
\]

given in eq. \((53)\), the $(\xi_g)^{-1}$ action on $\eta$:

\[
(\xi_g)^{-1} \cdot \eta = \text{Ad}_{g^{-1}}\eta - TL_{g^{-1}}\xi_g \tag{53}
\]

or

\[
(\xi_g)^{-1} \cdot \eta = \text{Ad}_{g^{-1}}\eta - C(\xi_g). \tag{54}
\]

For the $\Psi^*\omega$, expressed in $\tilde{H}$, we have pointwise:

\[
(\Psi^*\omega)(X_p) = \omega(T\Psi X_p) = -\tilde{H}(T\Psi X_p). \tag{55}
\]

Using the equivariance property of $\tilde{H}$ (eq. \((56)\)) and eq. \((57)\), we obtain:

\[
\tilde{H}(T\Psi X_p) = \tilde{H}(X_p \cdot TU X_p) = (TU X_p)^{-1} \cdot \tilde{H}(X_p) \tag{58}
\]
and
\[(TXP)^{-1} \cdot \tilde{H}(X_p) = \text{Ad}_{U^{-1}} \tilde{H}(X_p) - C(TUX_p). \tag{55}\]

Eqs. (53 - 55) lead to
\[\Psi^*(\omega)(X_p) = -(TXP)^{-1} \cdot (-\omega(X_p)) \tag{56}\]
or equivalently to
\[\Psi^*(\omega)(X_p) = \text{Ad}_{U^{-1}} \omega(X_p) + U^* C(X_p). \tag{57}\]

On the right hand side of eq. (56), the minus in front of \(\omega(X_p)\) cannot be eliminated because the \(TG\)–action is not linear. Notice also that we obtained in this way eq. (49) in a very straightforward manner.

5 Summary

The aim of this paper was to set some new lights on several aspects of the notion of a connection on a principal bundle \(P(M, G)\) and especially on its gauge transformations. We expect this to be very useful in physical applications.

We proceeded essentially by exploring the properties of the \(T\)–functor, considering the action of the tangential group \(TG\) on \(\mathcal{G} = \text{Lie}(G)\) and \(TP\). It was perceived that \(TP\) is also a principal bundle \(TP(TM, TG)\) and that the fibre bundle \(TP/G \to TM\) associated with it \((TP \times_{TG} \mathcal{G})\) is useful for a new definition of a connection as a linear section on it.

This definition revealed the most characteristic property of a connection, the \(TG\)–equivariance:
\[\omega(X_p \cdot \xi_g) = -\xi^{-1} \cdot (-\omega(X_p)) \tag{58}\]

where we have used the symbol \(\cdot\) for the \(TG\)–action on \(TP\) and \(\mathcal{G}\).

The \(TG\)–equivariance property of the connection also gives the most plausible explanation for the specific form the expression for gauge transformations takes. In particular, the nonlinearity in those transformations stems from the nonlinear action of the affine group \(TG\).

Acknowledgements

We would like to thank S. Klaus and M. Kreck for very helpful discussions and careful reading of the manuscript. We also thank R. Coquereaux for an interesting discussion, and R. Buchert and F. Scheck for useful comments.
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