Security flaw of counterfactual quantum cryptography in practical setting

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Abstract. Recently, counterfactual quantum cryptography proposed by T. G. Noh [Phys. Rev. Lett. 103, 230501 (2009)] becomes an interesting direction in quantum cryptography, and has been realized by some researchers (such as Y. Liu et al’s [Phys. Rev. Lett. 109, 030501 (2012)]). However, we find out that it is insecure in practical high lossy channel setting. We analyze the secret key rates in lossy channel under a polarization-splitting-measurement attack. Analysis indicates that the protocol is insecure when the loss rate of the one-way channel exceeds 50%.
1. Introduction

Quantum cryptography allows higher security than classical cryptography as it is based on the laws of physics instead of the difficulty of solving mathematical problems. Quantum key distribution (QKD)\cite{1-3}, which is to provide secure means of distributing secret keys between the sender (Alice) and the receiver (Bob), is often used to represent quantum cryptography as the primary most important part. Now it has been researched and developed in both theoretics and experiments. In theoretic, QKD could offer unconditional security guaranteed by the laws of physics\cite{4}. But due to the limitations of real-life setting\cite{5}, such as the imperfect source, imperfect detector, loss and noise in channel, practical QKD has security loopholes and has suffered some attacks, such as photon number splitting (PNS) attack\cite{6}, Trojan-horse attack\cite{7}, faked state attack\cite{8}. On the other hand, some achievements, such as decoy states method\cite{9}, measurement-device-independent QKD (MDI-QKD) scheme\cite{10} were made to let practical QKD be more secure.

Recently, counterfactual quantum cryptography proposed by Noh\cite{11} has attracted a lot of research, which allows participants to share secret information using counterfactual quantum phenomena. It is believed that the security is based on that quantum particles carrying secret information are seemingly not transmitted through quantum channels. So far, some security proof\cite{12}, improvements\cite{13} and experimental demonstrations\cite{14-18} of counterfactual quantum cryptography have been proposed.

However, we find out the counterfactual quantum cryptography\cite{11} is insecure in practical long distance communication. The secret key rate will be 0 under a polarization-splitting-measurement attack when the loss rate of the one-way channel is no less than 50%. Namely, the eavesdropper (commonly called Eve) can obtain all the secret information. Nevertheless, the cheat is unknowable to Alice and Bob because its effect just likes a reasonable loss in practical channel.

This paper is organized as follows. Sec. II reviews the counterfactual QKD proposed in Ref.\cite{11}. In Sec. III, We analyze the error rate of raw key in lossy channel. A polarization-splitting-measurement attack is given in Sec. IV. In Sec. V, we analyze the secret key rate under the attack. Finally, a short conclusion are provided in Section VI.

2. Counterfactual QKD

Fig.1 is the schematic of counterfactual QKD\cite{11}. For simpleness, we have made some equivalent adjustments on it. Alice triggers the single-photon source $S$, which emits a short optical pulse containing a single photon at a certain time interval. She randomly chooses the photon polarization in $|V\rangle$ representing the bit value 0, or $|H\rangle$ representing the bit value 1. Thereafter, the photon enters a beam splitter $BS_1$ and is split to two wave pulses $s_a$ and $s_b$. Then the system state evolves into one of the following states:

$$|\phi_0\rangle = \sqrt{R}|0\rangle_a|V\rangle_b + \sqrt{T}|V\rangle_a|0\rangle_b,$$

$$|\phi_1\rangle = \sqrt{R}|0\rangle_a|H\rangle_b + \sqrt{T}|H\rangle_a|0\rangle_b.$$ (1a)

(1b)
Figure 1. (color online). The schematic of counterfactual QKD. For simpleness, we have made some equivalent adjustments on the original one. Whole space is divided by dotted line into three sub-spaces, Alice’s site, Bob’s site and public space (i.e., Eve’s space). Alice sends the $i$th single-photon in state $|V\rangle$ or $|H\rangle$, representing bit 0 or 1, to beam splitter $BS_1$. Then the split pulses are transmitted into two paths $a$ which is always in Alice’s site, and $b$ which is in public space toward Bob’s site. Bob randomly uses $|V\rangle$ (representing 0) or $|H\rangle$ (representing 1) PBS to block the pulse in path $b$ when his bit is identical to Alice’s, or let it pass when his bit is differ to Alice’s. When their bits are different, detectors $D_2$ should always click since the interferometry happens in $BS_2$. Else when their bits are same, the detectors $D_1$ and $D_2$ and $D_3$ will click with some probabilities since interaction-free measurement happens. Additional, it is assumed that all of $D_1$, $D_2$ and $D_3$ could detect the state’s polarization $|V\rangle$ or $|H\rangle$. All the $D_2$’s and $D_3$’s clicks and a part of $D_1$’s clicks are used to detect eavesdropping, and the rest of $D_1$’s clicks with correct polarization are used as the raw key.

where subscripts $a$ and $b$ represent the path towards Alice’s site and the path toward Bob’s site, respectively, and $|0\rangle$ denotes the vacuum state in the path $a$ or $b$. $R$ and $T = 1 - R$ are the reflectivity and transmissivity of both $BS_1$ and $BS_2$, respectively.

Bob has two polarizing beam splitter (PBS), $|V\rangle$ PBS (representing the bit value 0) and $|H\rangle$ PBS (representing the bit value 1), where $|V\rangle$ (or $|H\rangle$) PBS means it addresses the state $|V\rangle$ (or $|H\rangle$) towards detector $D_3$, while the state $|H\rangle$ (or $|V\rangle$) is sent towards the beam splitter $BS_2$. He randomly chooses to use the $|V\rangle$ PBS or $|H\rangle$ PBS as his device $PBS_1$.

If Alice and Bob’s bits are different, the pulse on path $b$ will be reflected by Bob and combined again at Alice’s device $BS_2$. The case just likes an interferometry with
a single photon. In the ideal setting, detector $D_2$ will click with certainty. Else if Alice and Bob’s bits are identical, the path $b$ will be blocked by Bob’s $PBS_1$. The case just likes an interaction-free measurement with a single photon. Here the state $|\phi_0\rangle$ will be collapsed to $|0\rangle_a|V\rangle_b$ or $|V\rangle_a|0\rangle_b$. $|\phi_1\rangle$ will be collapsed to $|0\rangle_a|H\rangle_b$ or $|H\rangle_a|0\rangle_b$. In the ideal setting, detector $D_1$, $D_2$ and $D_3$ will click with probability $RT$, $T^2$ and $R$, respectively.

So in the ideal setting, $D_1$ clicks means Alice’s source photon basis and Bob’s $PBS$ basis are identify. Then Alice and Bob have a certain amount of identify bits, some of which could be used to check possible eavesdropping, and the rest of with could be used as raw key bits. And some statistical laws are between $D_2$’s, $D_3$’s clicks and Alice, Bob’s bits, which could be used to check possible eavesdropping and judge error rate. Additional, it is assumed that all the detectors $D_1$, $D_2$ and $D_3$ could detect the state’s polarization $|V\rangle$ and $|H\rangle$, which also could be used to check possible eavesdropping.

Since the raw key bits come from the events of $D_1$ clicks which means that Bob’s measurement result is vacuum state, peoples feel that the participles which carry secret information seemingly have not travelled between Alice and Bob. In fact, its security is based on a type of noncloning principle for orthogonal states[11]: if reduced density matrices of an available subsystem are nonorthogonal and the other subsystem is not allowed access, it is impossible to distinguish two orthogonal quantum states $|\phi_0\rangle$ and $|\phi_1\rangle$ without disturbing them.

3. Users’ error raw key rate depend on lossy channel

Similar to other QKD schemes, the limitations of real-life setting will also bring some troubles to counterfactual QKD. Specially, high lossy channel will be a formidable difficulty to it. In this section, we will analyze the error rate of users’ raw key pair, i.e., the different rate of Alice and Bob’s raw key pair, depend on lossy channel. (Besides the the loss in channel, some other loss also appear in the source and the devices and some noise appear in the source, channel and the devices, but they are not in the paper’s range.)

In the counterfactual QKD, the raw key rate is proportional to the single detector click rate (i.e., the rate of the case in which only one of detectors $D_1$, $D_2$ and $D_3$ clicks), which will be affected by source single photon rate $R_{\text{single}}$ and the loss rate. Symmetrically, we suppose that both the loss rates in channel from Bob to Alice, and that from Alice to Bob are $\eta$, i.e., the single photon will loss with probability $\eta$ in one of the two channels. We recall that (1) Alice’s raw key bits are generated from the source single photons’ bases. (2) Bob’s raw key bits are generated from his $PBS_1$’s basis, i.e., state in which basis would be sent from $PBS_1$ toward $D_3$.

Then we analyze the cases in which the raw key will be generated by Alice and Bob. The analysis will be done on one single photon sent by Alice, which is in state $|V\rangle$ or $|H\rangle$ with probability 1/2 respectively. And we suppose that the channel loss in the channel in public space and Bob’s site, which is denoted as channel $c_{A\rightarrow B\rightarrow A}$ and could
be divided to two parts (the channels from Alice to Bob $b_{A\rightarrow B}$ and from Bob to Alice $b_{B\rightarrow A}$), is independent with state’s polarization $|V\rangle$ and $|H\rangle$, i.e., all the possible wave pulse will loss when channel loss happens. (Note that the channel loss in $c_{A\rightarrow B\rightarrow A}$ is different to the loss happens in Bob’s PBS in which only one polarization is blocked. And also note that channel loss in $c_{A\rightarrow B\rightarrow A}$ does not mean that the photon vanishes in $c_{A\rightarrow B\rightarrow A}$ with certainly since it might go through path $a$ probably.) Theses cases are divided by two elements (i) whether the loss happens or not in the channel in public space and Bob’s site (then we divide the channel $c_{A\rightarrow B\rightarrow A}$ to two parts, the channels from Alice to Bob $b_{A\rightarrow B}$ and from Bob to Alice $b_{B\rightarrow A}$) and (ii) if loss happens, whether it happens in the channels $b_{A\rightarrow B}$ or $b_{B\rightarrow A}$.

**Case I.** Channel loss does not happen either on $b_{A\rightarrow B}$ or $b_{B\rightarrow A}$.

This case just like the single photon has transmitted in a no-lossy channel. Namely, there are not any blocks except the possible block from Bob’s PBS. Case I will generate a raw key bit with probability $P_1 = \frac{RT}{2}$ as the reasons (1) Bob’s PBS blocks the special polarization with probability $\frac{1}{2}$ (2) a raw key bit will be generated with probability $RT$ when Bob’s PBS blocks the special polarization.

As both of the loss rates in the channels $b_{A\rightarrow B}$ and $b_{B\rightarrow A}$ are $\eta$, channel loss will not happen on $b_{A\rightarrow B}$ and $b_{B\rightarrow A}$ with probability $1 - \eta$ respectively. So the Case I, channel loss does not happen either on $b_{A\rightarrow B}$ or $b_{B\rightarrow A}$, will occur with probability $P_I = (1 - \eta)^2$. The raw key rate comes from Case I is

$$R_{raw}^{AB} = P_I \cdot P_1 \cdot R_{single} = (1 - \eta)^2 \cdot \frac{RT}{2} \cdot R_{single}.$$  \hspace{1cm} (2)

Alice and Bob’s raw key are identify in this case.

**Case II.** Channel loss happens in the channel $b_{A\rightarrow B}$, regardless of whether channel loss happens in the channel $b_{B\rightarrow A}$ or not.

When channel loss happened in $b_{A\rightarrow B}$, no wave pulse will pass through $b_{B\rightarrow A}$, so we combine the cases that (II-1) channel loss happens both in the channels $b_{A\rightarrow B}$ and $b_{B\rightarrow A}$ (II-2) channel loss only happens in the channel $b_{A\rightarrow B}$, not in the channel $b_{B\rightarrow A}$ to Case II. Case II will generate an additional raw key bit with probability $P_2 = RT$ as following analysis.

Without loss of generality, we suppose the single photon Alice sent is $|V\rangle$. After $BS_1$, the state could be described as Eq.(1a). When it comes into Bob’s site, the state evolves to $|V\rangle_a|0\rangle_b$ with probability $T$, or $|0\rangle_a|0\rangle_b$ with probability $R$ as the possible pulse wave $|V\rangle_b$ lost in the channel $b_{A\rightarrow B}$. The state $|0\rangle_a|0\rangle_b$ will not lead to any clicks, so no raw key will be generated. But as the state $|V\rangle_a|0\rangle_b$, the photon in path $a$ will fire detector $D_1$ and let Alice generate a raw key bit with probability $R$, fire detector $D_2$ with probability $T$. After Alice announced that $D_1$ clicked, Bob would generate an according raw key bit based on his $PBS$’s basis, i.e., state in which basis is sent toward $D_3$. So Case II will generate an additional raw key bit with probability $T \cdot R$ as following analysis.

Case II will happen with probability $P_{II} = \eta$. Hence, with the loss in channel from
Alice to Bob, additional raw key bits are generated, the totally rate of which is
\[ R_{\text{raw}}^{AB} = P_{II} \cdot P_2 \cdot R_{\text{single}} = \eta \cdot RT \cdot R_{\text{single}}. \]  
(3)

Since Bob has chosen his PBS’s basis randomly, his raw key bit will be identify, and different with Alice’s with equal probability 1/2. So both of the correct and error raw key rates are \( R_{\text{raw}}^{AB} = \frac{1}{2}. \)

**Case III.** Channel loss does not happen in the channel \( b_{A \rightarrow B} \), but happens in the channel \( b_{B \rightarrow A} \). Namely, a complete block is in the channel \( b_{B \rightarrow A} \) except the possible block from Bob’s PBS. Case III will generate a raw key bit with probability \( P_3 = RT \) as following analysis.

We still suppose the single photon Alice sent is \( |V\rangle \). If Bob’s PBS basis is same with Alice’s basis, Bob’s PBS will send possible wave pulse \( |V\rangle_b \) toward \( D_3 \). On one hand, the system state evolves to \( |0\rangle_a |V\rangle_b \) with probability \( R \), which means that the photon went through path \( b \), and it will be destroyed by detector \( D_3 \). So no pulse wave will transmit from Bob to Alice. On the other hand, the system state evolves to \( |V\rangle_a |0\rangle_b \) with probability \( T \), which means that the photon went through path \( a \), then it will fire detectors \( D_1 \) and \( D_2 \) with probabilities \( R \) and \( T \) respectively. Bob will generate a raw key bit which is identify with Alice’s after she announces that \( D_1 \) clicked, whose probability is \( T \cdot R \).

Else if Bob’s PBS basis is different with Alice’s basis, Bob’s PBS would pass possible wave pulse \( |V\rangle_b \), and send it back to Alice. After it lost in the channel from Bob to Alice, the system state evolves to \( |0\rangle_a |0\rangle_b \) (with probability \( R \)) which means that it is destroyed by the lossy channel, or \( |V\rangle_a |0\rangle_b \) (with probability \( T \)) which means that it will fire \( D_1 \) or \( D_2 \) with probabilities \( R \) and \( T \), respectively. Bob will generate a raw key bit which is identify with Alice’s after she announces that \( D_1 \) clicked, whose probability is \( T \cdot R \).

So regardless Bob’s PBS basis is \( |V\rangle \) or \( |H\rangle \), this case will generate a raw key bit with probability \( RT \). But Alice’s and Bob’s bits are same and different with equal probability \( \frac{1}{2} \).

The case will happen with probability \( P_{III} = (1 - \eta) \cdot \eta \) as channel loss does not happen in the channel \( b_{A \rightarrow B} \) with probability \( 1 - \eta \), happens in the channel \( b_{B \rightarrow A} \) with probability \( \eta \). Hence, with the loss in channel from Bob to Alice, additional raw key bits is generated, the totally rate of which is
\[ R_{\text{raw}}^{AB} = P_{III} \cdot P_3 \cdot R_{\text{single}} = (1 - \eta) \cdot \eta \cdot RT \cdot R_{\text{single}}. \]  
(4)

Both of the same and different raw key rates are \( R_{\text{raw}}^{AB} = \frac{1}{2} \).

All in all, the raw key rate is
\[ R_{\text{raw}}^{AB} = R_{\text{raw}}^{AB} + R_{\text{raw}}^{AB} + R_{\text{raw}}^{AB} = \frac{1+2\eta-\eta^2}{2} \cdot TR \cdot R_{\text{single}}. \]  
(5)

The probability of that Alice’s and Bob’s raw keys in a same order are identify is
\[ P_{\text{raw same}} = \frac{R_{\text{raw}}^{AB} + R_{\text{raw}}^{AB} + R_{\text{raw}}^{AB}}{R_{\text{raw}}} = \frac{1}{1+2\eta-\eta^2}. \]  
(6)
the probability of that they are different is

\[
P_{\text{raw, diff}} = \frac{R_{\text{raw,2}}^{AB} + R_{\text{raw,3}}^{AB}}{2} = \frac{2\eta - \eta^2}{1 + 2\eta - \eta^2}.
\]

(7)

Namely, in users’ raw key pair, the error rate is \(P_{\text{raw, diff}}\) which should be correct by some following classical postprocessing such as information reconciliation.

The error in users’ raw key pairs will give a lot of chances to Eve to perform some attacks. But to Eve, the first aim is that her attacks should not be detected by the users. Following polarization-splitting-measurement attack is one of the attacks.

4. Polarization-splitting-measurement attack

Usually, we assume that Eve has unlimited technological, which is only limited by the laws of nature. So Eve could replace the lossy channel by a perfect quantum channel, and use the excess power for her mischievous purposes. In this section, we first give an attack method which can cheat the raw key bits and be concealed by the practical lossy channel with loss rate \(\frac{1}{2}\), then give the special cheat strategies according to special loss rate range for cheating maximal information.

In the attack method, polarization-splitting and measurement will be used to cheat secret information from channel \(b_{A \rightarrow B}\) (shown in fig.2). Eve first replaces the lossy channel \(b_{A \rightarrow B}\) by a perfect quantum channel. She also has two polarizing beam splitters, \(|V\rangle\) PBS representing the bit value 0 and \(|H\rangle\) PBS representing the bit value 1. She randomly chooses the \(|V\rangle\) or \(|H\rangle\) PBS for the \(i\)th order, and inserts it in front of Bob’s site.

If Eve’s \(i\)th bit is identical with Alice’s \(i\)th bit, the detector \(D_4\) will click with probability \(R\), else if her \(i\)th bit is differ to Alice’s \(i\)th bit, the detector \(D_4\) will not click. In other words, the case that \(D_4\) clicks means that Eve’s bit is identical to Alice’s \(i\)th bit, and the case that \(D_4\) does not click means that Eve is uncertain about Alice’s \(i\)th bit now. We recall that Alice and Bob’s raw key pair will product from these uncertain bits corresponding to the case that detector \(D_4\) does not click. So Eve cannot make sure of the raw key bit. However, Eve could easily extract the raw key bit according to what Alice and Bob will announce in the following processing.

Without loss of generality, we consider the case of Eve chooses 0, i.e., she inserts a \(|V\rangle\) PBS. When Alice’s bit is 0, two possible cases are here. (1) When Eve’s detector \(D_4\) clicked, the system state has been collapsed to \(|0\rangle_a |V\rangle_b\) which means Alice’s bit is identical to Eve’s bit 0, and vacuum state will go into Bob’s site. (2) Else when Eve’s detector \(D_4\) did not click, the system has been collapsed to \(|V\rangle_a |0\rangle_b\), and vacuum state still will go into Bob’s site. Altogether, vacuum state (i.e., nothing) always will go into Bob’s site when Eve and Alice’s bits are same, which likes the pulse in path \(b\) has lost completely by the lossy channel.

On the other hand, when Alice’s bit is 1, the pulse in path \(b\) will pass Eve’s PBS\(_2\) completely, so the system state still is \(|\phi_1\rangle = \sqrt{R} |0\rangle_a |H\rangle_b + \sqrt{T} |H\rangle_a |0\rangle_b\) after Eve’s
Figure 2. (color online). The schematic of polarization-splitting-measurement attack on counterfactual QKD. Eve performs attack on channel $b_{A \rightarrow B}$ in front of Bob’s site. Eve replaces the lossy channel $b_{A \rightarrow B}$ by a perfect quantum channel. Then she randomly uses $|V\rangle$ (representing 0) or $|H\rangle$ (representing 1) PBS to block the pulse in path $b_{A \rightarrow B}$ when her bit is identical to Alice’s, or let it pass when her bit is differ to Alice’s. What she does just like a reasonable loss in path $b_{A \rightarrow B}$.

devices. The case is same to that Eve has done nothing, liking the ideal setting. In the point view of Alice and Bob, all the following processes will just like the normal processes. When $D_1$ clicks, the corresponding bit will be chosen as a raw key bit by Alice followed by announcing its order. Then Eve can always make sure that Alice’s raw key bit is 1. In other words, when Eve and Alice’s bits are different, a raw key bit will be produced with probability. And the probability will be revealed with Alice’s announcement. Since the raw key bit is generated from the inverse of Eve’s PBS$_2$’s basis, Eve can not only know the raw key bits, but also decide its value with some probability.

Since Eve chooses bit 0 or 1 randomly, her bit will be same and different with Alice’s bit with probability $1/2$ respectively. The complete loss will happen when their bits are same, and the ideal setting will happen when their bits are different. Totally, the cheat method likes a loss of rate $1/2$ happens in the channel $b_{A \rightarrow B}$. The cheat method could be used on every photon to cheat the secret information when $\eta = \frac{1}{2}$ and will not be detected (the analysis will be given in the following). To other value of $\eta$, more complex strategies should be designed for optimal cheating.

We suppose the amount of Alice sent single photons is $n$. Using the above attack
method, Eve could simulate practical loss channel with loss rate \(0 \leq \eta \leq 1\) and cheat raw key bits with following strategies.

**Cheat strategy (I)** When \(0 \leq \eta < \frac{1}{2}\), Eve performs the attack method on \(2\eta \cdot n\) single photons randomly, and fills the raw key orders which she has not attacked in with random bits.

**Cheat strategy (II)** When \(\frac{1}{2} \leq \eta \leq 1\), Eve performs the attack method on \(2(1-\eta) \cdot n\) single photons randomly, and blocks the remaining \((2\eta - 1) \cdot n\) single photons. After Alice announced in which orders the remaining single photons have fired detector \(D_1\), she fills these raw key orders in with random bits.

Like the loss in practical channel, what Eve did has brought some errors to the protocol (we will analyze the details in next section). For instance, some \(D_1\)'s clicks happened not only when Alice and Bob’s bits were same, but also when they were different as long as Eve blocked the channel. However, since the error rate is same as that brought by practical lossy channel, it will be judged as a legal case by the protocol’s detection process. The basis reason is that, the system state under the above cheat strategies is same to the system state transmitted from a practical channel. We will analyze it as follows.

We suppose the photon Alice sent is \(\ket{V}\). If Eve’s PBS past wave pulse \(\ket{V}\) to Bob’s site, the density matrix of system state is

\[
\rho_{1\text{attack}} = \ket{\phi_0}\bra{\phi_0},
\]

when it comes into Bob’s site. If Eve’s PBS blocked wave pulse \(\ket{V}\), the system state is a mixed state with density matrix

\[
\rho_{2\text{attack}} = R|0\rangle_a|0\rangle_b\bra{0}_a\bra{0}_b + T|V\rangle_a|0\rangle_b\bra{0}_a|V\rangle_a,
\]

when it comes into Bob’s site.

So after the strategy (I), the system state is a mixed state with density matrix

\[
\rho_{I\text{attack}} = (1 - 2\eta) \cdot \rho_{1\text{attack}} + \eta \cdot \rho_{2\text{attack}}
\]

where \(0 \leq \eta < \frac{1}{2}\). After the strategy (II), the system state is a mixed state with density matrix

\[
\rho_{II\text{attack}} = (1 - \eta) \cdot \rho_{I\text{attack}} + [1 - \eta + (2\eta - 1)] \cdot \rho_{2\text{attack}}
\]

where \(\frac{1}{2} \leq \eta \leq 1\).

Now we analyze the system state in practical lossy channel without the attack strategies. If the wave pulse in channel \(b_{A\rightarrow B}\) has not lost, the density matrix of the system state is

\[
\rho_{1\text{loss}} = \ket{\phi_0}\bra{\phi_0},
\]

when it come into Bob’s site. If the wave pulse in channel \(b_{A\rightarrow B}\) has lost, the system state will be a mixed state with density matrix

\[
\rho_{2\text{loss}} = R|0\rangle_a|0\rangle_b\bra{0}_a\bra{0}_b + T|V\rangle_a|0\rangle_b\bra{0}_a|V\rangle_a
\]
when it come into Bob’s site.

Since the loss rate is $\eta$ on the practical lossy channel $b_{A\rightarrow B}$, the general system state is a mixed state with density matrix

$$
\rho^{\text{loss}} = (1 - \eta) \cdot \rho_{1}^{\text{loss}} + \eta \cdot \rho_{2}^{\text{loss}} = (1 - \eta) \cdot \rho_{1}^{\text{attack}} + \eta \cdot \rho_{2}^{\text{attack}},
$$

when it goes into Bob’s site, which is same with $\rho_{i}^{\text{attack}}$ when $0 \leq \eta < \frac{1}{2}$, $\rho_{II}^{\text{attack}}$ when $\frac{1}{2} \leq \eta \leq 1$.

So the states are same either when the protocol suffers a lossy channel or when it is under the cheat strategies. The conclusion is still tenable when the photon Alice sent is $|H\rangle$. Consequently, Alice and Bob could not distinguish between the practical lossy channel and the cheat strategies.

5. Secret key rate under the cheat strategies in lossy channel

In this section, we will analyze the protocol in lossy channel with the secret key rate $R_{QKD}$, a convenient and commonly used quantitate measure of protocol security.

Secret key rate $R_{QKD}$ is the product of the raw key rate $R_{raw}$ and the secret fraction $r_{\infty}$. The secret fraction represents the fraction of secure bits that may be extracted from the raw key. Formally, we have

$$
R_{QKD} = R_{raw} \cdot r_{\infty}.
$$

The expression for the secret fraction extractable using one-way classical postprocessing reads

$$
r_{\infty} = I(A; B) - \min(I_{EA}, I_{EB}),
$$

where $I(A; B)$ is Alice and Bob’s mutual information, $I_{EA} = \max_{Eve} I(A; E)$, $I_{EB} = \max_{Eve} I(B; E)$. Since Alice and Bob’s each raw key pair is randomly in $\{0,1\}$, it should be $H(A) = H(B) = 1$. We also have $P(A = 0, B = 0) = P(A = 1, B = 1) = P_{raw}^{\text{same}} / 2$, $P(A = 0, B = 1) = P(A = 1, B = 0) = P_{raw}^{\text{diff}} / 2$. Combined with Eqs.(6) and (7), it should be that

$$
I(A; B) = H(A) + H(B) - H(A, B) = 1 + 1 + \sum_{A\in\{0,1\}, B\in\{0,1\}} p(A, B) \log p(A, B)
$$

$$
= 2 + 2 \cdot \frac{1}{2(1+2\eta-\eta^2)} \log \frac{1}{2(1+2\eta-\eta^2)} + 2 \cdot \frac{2\eta - \eta^2}{2(1+2\eta-\eta^2)} \log \frac{2\eta - \eta^2}{2(1+2\eta-\eta^2)}.
$$

Then we analyze the secret key rate under the cheat strategies (I) and (II) respectively depend on the loss rate $\eta$ by calculating $\min(I_{EA}, I_{EB})$.

We recall that (1) Alice’s raw key bits are generated from the source single photons’ bases. (2) Bob’s raw key bits are generated from his $PBS_{1}$’s basis, i.e., state in which basis would be sent from $PBS_{1}$ toward $D_{3}$. (3) Eve’s raw key bits are generated from the inverse of her $PBS_{2}$’s basis, i.e., state in which basis would be sent from $PBS_{2}$ toward Bob’s site. Now we analyze the cases in which the raw key will be cheated by
Eve when she cheats in the channel from Alice to Bob. And we still suppose the single photon Alice sent is $|V\rangle$.

5.1. Secret key rate under the cheat strategy (I) in lossy channel

We first analyze the cases in cheat strategy (I), i.e., the strategy with $0 \leq \eta < \frac{1}{2}$. We recall Cheat strategy (I): When $0 \leq \eta < \frac{1}{2}$, Eve performs the attack method on $2\eta \cdot n$ single photons randomly, and fills the raw key orders which she has not attacked in with random bits. We divide the cases with elements (i) whether Eve performs the attack method or not and (ii) if Eve performs the attack method, whether her $PBS$ basis is same with Alice’s basis or not.

Cheated raw key I. The cheated raw key when Eve does not perform the attack method.

For $0 \leq \eta < \frac{1}{2}$, Eve does not perform the attack method on $(1 - 2\eta) \cdot n$ source single photons, in which raw key bits will be generated as the case I and case III (shown in Sec.III). Due to that the loss rate in the channel $b_{A\rightarrow B}$ is $\eta$, case I will happen with probability $(1 - 2\eta) \cdot (1 - \eta)$, case III will happen with probability $(1 - 2\eta) \cdot \eta$. The totally rate of these raw key is

$$R_{raw}^E = ((1 - 2\eta) \cdot (1 - \eta) \cdot P_1 + (1 - 2\eta) \cdot \eta \cdot P_3) \cdot R_{single} \cdot (1 - \frac{\eta - 2\eta^2}{2}) \cdot RT \cdot R_{single}. \quad (18)$$

Eve will guess these raw key bits, so the correct probability is $\frac{1}{2}$. So compared to Alice’s and Bob’s raw keys, both of Eve’s same and different raw key rates in this case are

$$R_{raw}^{EA}_{same} = R_{raw}^{EB}_{same} = R_{raw}^{EA}_{diff} = R_{raw}^{EB}_{diff} = \frac{1}{4} - \eta \cdot 2\eta^2 \cdot RT \cdot R_{single}. \quad (19)$$

Cheated raw key II. The cheated raw key when Eve performs the attack method, and her $PBS$ basis is same with Alice’s basis.

When Eve’s $PBS$ basis is same with Alice’s basis (namely, Eve’s $PBS$ will send wave pulse $|V\rangle$ toward $D_4$), raw key bits will be generated as the case II (shown in Sec.III). It will happen with probability $\eta$. So the totally rate of these raw key is

$$R_{raw}^E = \eta \cdot P_3 \cdot R_{single} = \eta \cdot RT \cdot R_{single}. \quad (20)$$

Since Eve always generates her raw key bit as the inverse of her $PBS_2$’s basis, all her raw key bits are different to Alice’s, and different to Bob’s with probability $\frac{1}{2}$. Compared to Alice’s raw key, Eve’s same and different raw key rates are

$$R_{raw}^{EA}_{same} = 0, \quad R_{raw}^{EA}_{diff} = \eta \cdot RT \cdot R_{single}. \quad (21)$$

respectively. Compared to Bob’s raw key, Eve’s same and different raw key rates are

$$R_{raw}^{EB}_{same} = R_{raw}^{EB}_{diff} = \frac{\eta}{2} \cdot RT \cdot R_{single}. \quad (22)$$

Cheated raw key III. The cheated raw key when Eve performs the attack method, and her $PBS$ basis is different with Alice’s basis.
When Eve’s PBS basis is different with Alice’s basis, Eve’s PBS will send wave pulse $|V\rangle$ toward Bob’s site. It will happen with probability $\frac{1}{2} \cdot 2\eta = \eta$. And raw key bits will be generated as the case I and case III. Due to that the loss rate in channel from Bob to Alice is $\eta$, case I will happen with probability $\eta \cdot (1 - \eta)$, case III will happen with probability $\eta \cdot \eta$. So the totally rate of these raw key is

$$R^E_{\text{raw}} = \eta \cdot (1 - \eta) \cdot P_1 + \eta \cdot \eta \cdot P_3 \cdot R_{\text{single}}$$

Since Eve always generates her raw key bit as the inverse of her PBS’s basis, all her raw key bits are identify with Alice’s. Compared to Alice’s raw key, Eve’s same and different raw key rates are

$$R^E_{\text{raw}}^{\text{same}} = \frac{\eta + \eta^2}{2} \cdot RT \cdot R_{\text{single}}$$

$$R^E_{\text{raw}}^{\text{diff}} = 0$$

Compared to Bob’s raw key, Eve’s same and different raw key rates are

$$R^E_{\text{raw}}^{\text{same}} = \eta \cdot (1 - \eta) \cdot RT \cdot R_{\text{single}}$$

$$R^E_{\text{raw}}^{\text{diff}} = \eta \cdot \eta \cdot RT \cdot R_{\text{single}}$$

All in all, the raw key rate Eve cheated is

$$R^E_{\text{raw}} = R^E_{\text{raw}} + R^E_{\text{raw}} + R^E_{\text{raw}}$$

$$= \frac{1 + 2\eta - \eta^2}{2}RT \cdot R_{\text{single}}$$

which is same as users’ raw key rate. The probabilities of that Eve and Alice’s raw key bits are same and different are

$$P^E_{\text{raw}}^{\text{same}} = \frac{P^E_{\text{raw}}^{\text{same}} + P^E_{\text{raw}}^{\text{same}} + P^E_{\text{raw}}^{\text{same}}}{R^E_{\text{raw}}}$$

$$= \frac{1 + \eta}{2(1 + 2\eta - \eta^2)}$$

$$P^E_{\text{raw}}^{\text{diff}} = \frac{P^E_{\text{raw}}^{\text{diff}} + P^E_{\text{raw}}^{\text{diff}} + P^E_{\text{raw}}^{\text{diff}}}{R^E_{\text{raw}}}$$

$$= \frac{1 + 3\eta - 2\eta^2}{2(1 + 2\eta - \eta^2)}$$

The probabilities of that Eve and Bob’s raw key bits are same and different are

$$P^E_{\text{raw}}^{\text{same}} = \frac{P^E_{\text{raw}}^{\text{same}} + P^E_{\text{raw}}^{\text{same}} + P^E_{\text{raw}}^{\text{same}}}{R^E_{\text{raw}}}$$

$$= \frac{1 + 3\eta - 4\eta^2}{2(1 + 2\eta - \eta^2)}$$

and

$$P^E_{\text{raw}}^{\text{diff}} = \frac{P^E_{\text{raw}}^{\text{diff}} + P^E_{\text{raw}}^{\text{diff}} + P^E_{\text{raw}}^{\text{diff}}}{R^E_{\text{raw}}}$$

$$= \frac{1 + \eta + 2\eta^2}{2(1 + 2\eta - \eta^2)}$$

In fact, Eve’s error rate will not be larger than 50% by using a simple way.[22]

Similar to the calculation of $I(A; B)$, combined with Eqs. (27-30) it should be

$$I(E; A)^i = H(E) + H(A) - H(E, A)$$

$$= 1 + 1 + \sum_{E \in \{0, 1\}, A \in \{0, 1\}} p(E, A) \log p(E, A)$$

$$= 2 + 2 \cdot \frac{1 + \eta}{4(1 + 2\eta - \eta^2)} \log \frac{1 + \eta}{4(1 + 2\eta - \eta^2)}$$

$$+ 2 \cdot \frac{1 + 3\eta - 2\eta^2}{4(1 + 2\eta - \eta^2)} \log \frac{1 + 3\eta - 2\eta^2}{4(1 + 2\eta - \eta^2)}.$$
and
\[
I(E; B)^i = H(E) + H(B) - H(E, B)
\]
\[
= 1 + 1 + \sum_{E \in \{0, 1\}, B \in \{0, 1\}} p(E, B) \log p(E, B)
\]
\[
= 2 + 2 \cdot \frac{\frac{1 + 3\eta - 4\eta^2}{4(1 + 2\eta - \eta^2)}}{\frac{\log \frac{1 + 3\eta - 4\eta^2}{4(1 + 2\eta - \eta^2)}}{4(1 + 2\eta - \eta^2)}}
\]
\[
+ 2 \cdot \frac{\frac{1 + \eta + 2\eta^2}{4(1 + 2\eta - \eta^2)}}{\log \frac{1 + \eta + 2\eta^2}{4(1 + 2\eta - \eta^2)}},
\]

Then the secret fraction is
\[
r_i^\infty = I(A; B) - \min(I_{EA}^i, I_{EB}^i),
\]
where \(0 \leq \eta < \frac{1}{2}\).

For simpleness, we set \(R = T = \frac{1}{2}\). Then secret key rate is
\[
R_{QKD} = R_{raw} \cdot r_i^\infty = \frac{1 + 2\eta - \eta^2}{8} \cdot r_i^\infty \cdot R_{single},
\]
where \(0 \leq \eta < \frac{1}{2}\).

### 5.2. Secret key rate under the cheat strategy (II) in lossy channel

Now we analyze the cases in which the raw key will be cheated by Eve using cheat strategy (II), i.e., the strategy with \(\frac{1}{2} \leq \eta \leq 1\). We recall Cheat strategy (II): When \(\frac{1}{2} \leq \eta \leq 1\), Eve performs the attack method on \(2(1 - \eta) \cdot n\) single photons randomly, and blocks the remaining \((2\eta - 1) \cdot n\) single photons. After Alice announced in which orders the remaining single photons have fired detector \(D_1\), she fills these raw key orders in with random bits.

In the strategy, the attack is performed with probability \(2(1 - \eta)\) replacing the probability \(2\eta\) in cheat strategy (I). So the amount of raw key rate generated by the attack is \(\frac{1 - \eta}{\eta} \cdot (R_{raw2}^E + R_{raw3}^E)\).

In addition, Eve blocks the remaining \((2\eta - 1) \cdot n\) wave pulses in the channel \(b_{A \to B}\) followed by guessing the possible raw key bits. This just likes case II. It will generate raw key bits whose amount is \((2\eta - 1) \cdot P_2 \cdot R_{single}\). And both of the probabilities of them are same and different with Alice (and Bob’s) are \(\frac{1}{2}\).

Hence, the raw key rate is
\[
R_{raw}^E = \left[\frac{1 - \eta}{\eta} \cdot (R_{raw2} + R_{raw3}) + (2\eta - 1) \cdot P_2\right] \cdot R_{single}
\]
\[
= \frac{1 + 2\eta - \eta^2}{2} \cdot R_{raw} \cdot R_{single},
\]
which is same as users’ raw key rate. The probabilities of that Eve’s and Alice’s raw key bits are same and different are
\[
P_{raw, same} = \frac{\frac{1 - \eta}{\eta} \cdot (R_{raw2}^{same} + R_{raw3}^{same}) + (2\eta - 1) \cdot P_2}{R_{raw}^E}
\]
\[
= \frac{2\eta - \eta^2}{1 + 2\eta - \eta^2},
\]
and
\[
P_{raw, diff} = \frac{\frac{1 - \eta}{\eta} \cdot (R_{raw2}^{diff} + R_{raw3}^{diff}) + (2\eta - 1) \cdot P_2}{R_{raw}^E}
\]
\[
= \frac{1}{1 + 2\eta - \eta^2}.
\]
The probabilities of that Eve’s and Bob’s raw key bits are same and different are

\[ P_{raw, same}^{EB} = \frac{1-\eta}{\eta} \left( R_{raw, same}^{EB} + R_{raw, same}^{EB, other} \right) + \frac{(2\eta - 1)}{2} P_2 \]
\[ = 1 - \eta + \eta^2 \] \(1 + 2\eta - \eta^2 = 0\),

and

\[ P_{raw, diff}^{EB} = \frac{1-\eta}{\eta} \left( R_{raw, diff}^{EB} + R_{raw, diff}^{EB, other} \right) + \frac{(2\eta - 1)}{2} P_2 \]
\[ = 3\eta - 2\eta^2 \] \(1 + 2\eta - \eta^2 = 0\).

Then we have

\[ I(E; A)^{ii} = H(E) + H(A) - H(E, A) \]
\[ = 1 + 1 + \sum_{E \in \{0,1\}, A \in \{0,1\}} p(E, A) \log p(E, A) \]
\[ = 2 + 2 \cdot \frac{2\eta - \eta^2}{2(1 + 2\eta - \eta^2)} \log \frac{2\eta - \eta^2}{2(1 + 2\eta - \eta^2)} \]
\[ + 2 \cdot \frac{3\eta - 2\eta^2}{2(1 + 2\eta - \eta^2)} \log \frac{3\eta - 2\eta^2}{2(1 + 2\eta - \eta^2)}, \]

and

\[ I(E; B)^{ii} = H(E) + H(B) - H(E, B) \]
\[ = 1 + 1 + \sum_{E \in \{0,1\}, B \in \{0,1\}} p(E, B) \log p(E, B) \]
\[ = 2 + 2 \cdot \frac{1-\eta + \eta^2}{2(1 + 2\eta - \eta^2)} \log \frac{1-\eta + \eta^2}{2(1 + 2\eta - \eta^2)} \]
\[ + 2 \cdot \frac{3\eta - 2\eta^2}{2(1 + 2\eta - \eta^2)} \log \frac{3\eta - 2\eta^2}{2(1 + 2\eta - \eta^2)}, \]

Then the secret fraction is

\[ r_{\infty}^{ii} = I(A; B) - \min(I_{EA}^{ii}, I_{EB}^{ii}), \]

where \( \frac{1}{2} \leq \eta \leq 1 \).

For simpleness, we set \( R = T = \frac{1}{2} \). Then secret key rate is

\[ R_{QKD} = R_{raw} \cdot r_{\infty}^{ii} \]
\[ = \frac{1+2\eta-\eta^2}{8} \cdot r_{\infty}^{ii} \cdot R_{single}, \]

where \( \frac{1}{2} \leq \eta \leq 1 \).

5.3. Discuss of the secret key rate

Fig.3 shows Alice and Bob’s mutual information \( I(A; B) \), the minimum of Eve’s and Alice’s, Eve’s and Bob’s mutual information \( \min(I(E; A), I(E; B)) \) when Eve uses the cheat strategies (I) and (II), and the secret fraction \( r_{\infty}^{ii} = I(A; B) - \min(I(E; A), I(E; B)) \) compared to the loss rate \( \eta \). It indicates that \( r_{\infty} = 0 \) when \( \frac{1}{2} \leq \eta \leq 1 \) under the cheat strategies.

We explain something about the data. When \( \frac{1}{2} \leq \eta \leq 1 \), \( \min(I(E; A), I(E; B)) = I(E; A) \), which is monotonic. But when \( 0 \leq \eta < \frac{1}{2} \), it will be \( \min(I(E; A), I(E; B)) = I(E; B) \), which is not monotonic. Specially, when \( \eta = \frac{1}{3} \), minimal value \( I(E; B) = 0 \) is here with \( P_{raw, same}^{EB} = P_{raw, diff}^{EB} \). The reason is that information entropy is non-negative. With the increasing of disparity between \( \eta \) and the special value \( \frac{1}{3} \), the disparity between \( P_{raw, same}^{EB} \) and \( P_{raw, diff}^{EB} \) increases, consequently, \( I(E; B) \) increases. (Also see [22])
Fig. 3. (color online). $I(A;B)$ is Alice’s and Bob’s mutual information. $\min(I(E;A), I(E;B))$ is the minimum of Eve’s and Alice’s, Eve’s and Bob’s mutual information. $r_\infty$ is the secret fraction. They are given compared to loss rate $\eta$. The left figure is the whole show of them. In the right figure, the ordinate scale is magnified.

Fig. 4 shows the counterfactual QKD’s raw key rate $R_{\text{raw}}$ and the secret key rate $R_{QKD}$ compared to the loss rate $\eta$. It indicates that $R_{\text{raw}}$ increases with the increasing of $\eta$, $R_{QKD}$ decreases with the increasing of $\eta$. Specially, $R_{QKD}$ will be equal to 0 when $\frac{1}{2} \leq \eta \leq 1$ under the cheat strategies, which means the protocol is insecure.

As QKD applications, they usually need to distribute secret information over long distance, so the high loss rate of channel is inevitable. For instance, let us assume that the transmission line is a fiber-based channel, which is always slightly lossy (about 0.2
Figure 4. (color online). The raw key rate $R_{raw}$ and the secret key rate $R_{QKD}$ compared to loss rate $\eta$. Here the key rate is the key bit rate generated by one single photon, and we set $T = R = 1/2$. The left figure is the whole show of them. In the right figure, the ordinate scale is magnified.

If we want to use the cryptographic system over reasonable distances, say up to 15 km, transmission losses will be as high as 3dB, or about 50%. Then Eve could cheat all the secret information using the cheat strategies proposed without leaving any clues.

6. Conclusion

In conclusion, we pointed out that counterfactual cryptography[11] is insecure in practical high lossy channel. We proposed a polarization-splitting-measurement attack and analyzed the secret key rate in lossy channel. The analysis indicates that the protocol is insecure when the loss rate of the channel from Alice to Bob is up to 50%. Since the attack’s effect just likes a loss channel, it is invisible to the protocol’s participants. Maybe the security flaw could be overcome by using nonorthogonal states as BB84 QKD[1], but the protocol will be more complex and lower efficient.

Acknowledgments

We are very grateful to Professor Horace P. Yuen for encouragement. This work is supported by NSFC (Grant Nos. 61300181, 61272057, 61202434, 61170270, 61100203, 61121061, 61370188, and 61103210), Beijing Natural Science Foundation (Grant No. 4122054), Beijing Higher Education Young Elite Teacher Project, China scholarship council.
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[21] It will be $I_{EB} < I_{EA}$ (or $I_{EB} > I_{EA}$) when $\eta$’s value is in some ranges. It means that in the counterfactual QKD, Bob’s (or Alice’s) raw key should be chosen as a reference raw key followed by classical postprocessing for more secure against Eve’s attack. So the mutual information Eve obtained is $I_{EB}$ (or $I_{EA}$), i.e., $\min(I_{EA}, I_{EB})$.
[22] It will be $R_{EA, same}^{RA} < R_{EA, diff}^{RA}$ ($R_{EB, same}^{RA} < R_{EB, diff}^{RA}$) when $\eta$’s value is in some ranges. In practical attack, Eve should reverse all her cheated bits for decreasing the error rate to less than 50% based on the value of $\eta$. Then the information she obtained is corresponding to the result calculated from information theory. And what she did will not affect the calculation in information theory, so we have not emphasized this in the rest of this paper.