Learning in Restless Bandits under Exogenous Global Markov Process

Tomer Gafni, Michal Yemini, Kobi Cohen

Abstract—We consider an extension to the restless multi-armed bandit (RMAB) problem with unknown arm dynamics, where an unknown exogenous global Markov process governs the rewards distribution of each arm. Under each global state, the rewards process of each arm evolves according to an unknown Markovian rule, which is non-identical among different arms. At each time, a player chooses an arm out of $N$ arms to play, and receives a random reward from a finite set of reward states. The arms are restless, that is, their local state evolves regardless of the player’s actions. Motivated by recent studies on related RMAB settings, the regret is defined as the reward loss with respect to a player that knows the dynamics of the problem, and plays at each time $t$ the arm that maximizes the expected immediate value. The objective is to develop an arm-selection policy that minimizes the regret. To that end, we develop the Learning under Exogenous Markov Process (LEMP) algorithm. We analyze LEMP theoretically and establish a finite-sample bound on the regret. We show that LEMP achieves a logarithmic regret order with time. We further analyze LEMP numerically and present simulation results that support the theoretical findings and demonstrate that LEMP significantly outperforms alternative algorithms.

I. INTRODUCTION

The multi-armed bandit (MAB) problem is a popular model for sequential decision making with unknown information: A player chooses actions repeatedly among $N$ different arms. After each action it receives a random reward having an unknown probability distribution that depends on the chosen arm. The objective is to maximize the expected total reward over a finite horizon of $T$ steps. Restless multi-armed bandit (RMAB) problems are generalizations of the MAB problem. Differing from the classic MAB, where the states of passive arms remain frozen, in the RMAB setting, the state of each arm (active or passive) can change. In this paper we consider an extension to the RMAB problem, in which we assume that an exogenous (global) Markov process governs the distribution of the restless arms, and thus the reward depends on both the state of the global process, and the local state of the chosen (active) arm.

RMAB problems have attracted much attention for their wide application in diverse areas such as manufacturing systems [2], economic systems [3], biomedical engineering [4], wireless communication systems [5], [6] and communication network [7], [8]. A particularly relevant application captured by the extended RMAB model considered in this paper is the Dynamic Spectrum Access (DSA) paradigm [9], [10], where primary users (licensed) occupy the spectrum occasionally, and a secondary user is allowed to transmit over a single channel when the channel is free. This model is captured as our exogenous (global) process. The statistical model for the arms establishes the relationship between a physical channel and its finite-state Markov model for a packet transmission system. We adopt the view of previous studies, forming a finite-state Markov channel model to reflect the fading channel effect [11].

A. Performance Measure of RMAB under Exogenous Global Markov Process

Computing the optimal policy for RMABs is P-SPACE hard even when the Markovian model is known [12], therefore, alternative tractable policies and objective functions have been proposed. Nevertheless, always playing the arm with the highest expected reward is optimal in the classic MAB under i.i.d. or rested Markovian rewards, up to an additional constant term [13]. Thus, a commonly used approach in classic RMAB (i.e., without exogenous process) with unknown dynamics settings to measure the algorithm performance in a tractable way defines the regret as the reward loss of the algorithm with respect to a genie that always plays the arm with the highest expected reward, also known as weak regret [14]–[16].

However, in our setting, due to the exogenous process, each global state is associated with different "best" arm (i.e., the arm with the highest expected reward). To accommodate the effect of the exogenous global Markovian state, we extend the definition of regret, and measure the performance of the algorithm by the reward loss of the algorithm with respect to a genie that plays in each time step the arm with the highest expected reward given the global state. Furthermore, since the next global state is unknown before choosing the arm for the next time step, we adopt a myopic performance measure, as considered also in [17], [18]. That is, the objective in this paper is to select the arm that has the highest immediate expected
value at each time slot under unknown arm dynamics. The expected value, and thus also the arm selection, depend on both the transition probabilities of the global exogenous Markov process and the mean reward of the arms, which depends on the global state. Consequently, we define the regret as the reward loss of an algorithm with respect to a genie that knows the transition probabilities of the global process and the expected rewards of the local arms. Thus, we note that the regret is not defined with respect to the best arm on average (that would result in a weak regret), but with respect to a strategy tracking the best arm at each step, which is stronger. This notion of regret was also considered in Section 8 of [19] and in [20], for the non-stochastic bandit problem.

B. Main Results

Due to the restless nature of both active and passive arms, learning the Markovian reward statistics requires that arms will be played in a consecutive manner for a period of time (i.e., phase) [14]–[16]. Thus we divide the time horizon into two phases, an exploration phase and exploitation phase. The goal of the exploration phase is to identify the best arm for each global state before entering the exploitation phase.

Upper Confidence Bound (UCB)-based policies, that are used to identifying the best arm, require parameter tuning depending on the unobserved hardness of the task [21]–[23]. The hardness parameter is a characteristic of the hardness of the problem, in the sense that it determines the order of magnitude of the sample complexity required to find the best arm with a required probability. In the classic MAB formulation, the hardness of the task is characterized by $H_i = \frac{1}{(\mu^i - \mu^*)^2}$, where $\mu^*$, $\mu^i$ are the means of the best arm and arm $i$, respectively. However, since the hardness parameter is unknown, existing algorithms use an upper bound on $\max_i H_i$ (e.g., [15]), which increases the order of magnitude of exploration phases, and consequently the regret. Considering the above, we summarize our main results and contributions.

1) An Extended Model for RMAB: RMAB problems have been investigated under various models of observation distributions in past and recent years. The extended model considered in this paper is capable of capturing more complex scenarios and requires an adaptation of the regret measure as discussed above. Handling this extension in the RMAB setting leads to different algorithm design and analysis as compared to existing methods.

2) Algorithm Development: We develop a novel algorithm, dubbed Learning under Exogenous Markov Process (LEMP), that estimates online the appropriate hardness parameter from past observations (Sec. III-A). Based on these past observations, the LEMP algorithm generates adaptive sizes of exploration phases, designed to explore each arm in each global state with the appropriate number of samples. Thus, LEMP avoids over-sampling bad arms, and at the same time identifies the best arms with sufficient high probability. To ensure the consistency of the restless arms’ mean estimation, LEMP performs regenerative sampling cycles (Sec. III-B). In the exploitation phases, LEMP dynamically chooses the best estimated arm, based on the evaluation of the global state (Sec. III-C). The rules that decide when to enter each phase are adaptive in the sense that they are updated dynamically and controlled by the current sample means and the estimated global transition probabilities in a closed-loop manner (Sec. III-D). Interestingly, the size of the exploitation phases is deterministic and the size of the exploration phases is random.

3) Performance Analysis: We provide a rigorous theoretical analysis of LEMP algorithm. Specifically, we establish a finite sample upper bound on the expected regret, and show that its order is logarithmic with time. We also characterize the appropriate hardness parameter for our model (the $D_i$ parameter defined in [3]), and we demonstrate that estimating the hardness parameter indeed results in a scaled regret proportional to the hardness of the problem. The result in Theorem 1 also clarifies the impact of different system parameters (rewards, mean hitting times of the states, eigenvalues of the transition probability matrices, etc.) on the regret. We provide numerical simulations that support the theoretical results presented in this paper.

C. Related Work

The extended RMAB model considered here is a generalization of the classic MAB problem [13], [24]–[27]. RMAB problems have been extensively studied under both the non-Bayesian [14]–[16], [17], [28]–[32], and Bayesian [33]–[36], [10], [37]–[41] settings. Under the non-Bayesian setting, special cases of Markovian dynamics have been studied in [14], [17]. There are a number of studies that focused on special classes of RMABs. In particular, the optimality of the myopic policy was shown under positively correlated two-state Markovian arms [37]–[39], [42] under the model where a player receives a unit reward for each arm that was observed in a good state. In [40], [43], the indexability of a special classes of RMAB has been established. In [29], the traditional restless bandit is extended by relaxing the restriction of a risk-neutral target function, and a general risk measure is introduced to construct a performance criterion for each arm. Our work is also related to models of partially observed Markov decision process (POMDP) [44], with the goal of balancing between increasing the immediate reward and the benefits of improving the learning accuracy of the unknown states. In [7], the offloading policy design in a large-scale asynchronous MEC system with random task arrivals, distinct workloads, and diverse deadlines is formulated as an RMAB problem, and the authors in [45] considered a tracking problem with independent objects and uses an approximated Gittins index approach for finding policies.
II. System Model and Problem Formulation

We consider a set of $N$ arms, indexed by $\{1, \ldots, N\} \triangleq \mathcal{N}$, and a global system state process $\{s_t\}_{t=1,2,\ldots}$, which is governed by a finite space, irreducible, and aperiodic discrete time Markov chain $S$ with unknown transition matrix $P_S$. We denote the transition probability between states $s$ and $\hat{s}$ in $S$ by $p_{s\hat{s}}$, and we denote by $\pi_s$ the stationary distribution of states $s \in S$. For each global state $s \in S$, the $i^{th}$ arm is modeled as a finite space, irreducible, and aperiodic discrete time Markov chain $\mathcal{X}_i^s$ with unknown transition matrix $P_{\mathcal{X}_i^s}$. We denote the transition probability between states $x$ and $y$ in $\mathcal{X}_i^s$ by $p_{x,y}^i$. We assume that $\mathcal{X}_i^s \cap \mathcal{X}_{\hat{s}}^i = \emptyset$ for all $i, s, \hat{s}$ (i.e., we can recover the global state in each time slot). We also define the stationary distribution of state $x$ in arm $i$ at global state $s$ to be $\pi_s^i(x)$. An illustration for the model is given in Fig. 1.

At each time $t$, the player chooses one arm to play. When played, each arm offers a certain positive reward that defines the current state of the arm, $x_{s,t}$. The player receives the reward of the chosen arm, and infers the current global state $s_t$. Then, the global state transitions to a new state, which is unknown to the player before choosing the next arm to play. We assume that the arms are mutually independent and restless, i.e., the local states of the arms continue to evolve regardless of the player’s actions according to the unknown Markovian rule $P_{\mathcal{X}_i}$. The unknown stationary reward mean of arm $i$ at global state $s$, $\mu^i_s$, is given by:

$$\mu^i_s = \sum_{x \in \mathcal{X}_i^s} x \pi_s^i(x).$$

We further define the expected value of arm $i$ in global state $s$ to be

$$V^i_s \triangleq \sum_{\hat{s} \in S} p_{s\hat{s}} \mu^i_{\hat{s}}. \quad (1)$$

Let $\sigma$ be a permutation of $\{1, \ldots, N\}$ such that

$$V^{\sigma(1)}_s \geq V^{\sigma(2)}_s \geq \cdots \geq V^{\sigma(N)}_s.$$ 

Let $V^{i*}_s(t)$ denote the value of arm $i$ at time $t$, let $i^*_s$ be the arm with the highest expected value at time $t$, i.e., $i^*_s \triangleq \arg\max_i V^{i*}_s(t)$, and let $\phi(t) \in \{1, 2, \ldots, N\}$ be a selection rule indicating which arm is chosen to be played at time $t$, which is a mapping from the observed history of the process to $\mathcal{N}$. Denote

$$V_{\phi}(n) = \{i : V^i_{s,n}(n) < V^{\sigma_1(1)}_{s,n}(n)\}.$$

The expected regret of policy $\phi$ is defined as:

$$\mathbb{E}_\phi[r(t)] = \mathbb{E}_\phi \left[ \sum_{n=1}^t \sum_{i \in V_{\phi}(n)} (x^i_{s,n}(n) - x^i_{s,n}(n)) \mathbb{I}_{\{\phi(n) = i\}} \right], \quad (2)$$

where hereafter $\mathbb{I}_{\{\cdot\}}$ denotes an indicator of an event $\cdot$.

The objective is to find a policy that minimizes the growth rate of the regret with time (this notion of regret is similar to the “regret against arbitrary strategies” introduced in Section 8 of [19] and in [20] for the non-stochastic bandit problem). We note that, in this paper, the regret is not defined with respect to the best arm on average (e.g., as in RMAB models in [14–16]), but with respect to the best arm at each step according to the instantaneous global state, which is a stronger regret.

III. The Learning Under Exogenous Markov Process (LEMP) Algorithm

The LEMP algorithm divides the time horizon into two types of phases, namely exploration and exploitation. In order to ensure sufficient small regret in exploitation phases (i.e., to reduce the probability for choosing sub-optimal arms in exploitation), our strategy estimates the required exploration rate of each arm, and updates the arm selection dynamically with time, controlled by the random sample means and transition probability estimates in a closed loop manner.

A. Design principles of LEMP

For sufficient small regret during exploitation phases, we should take a sufficiently large number of samples in exploration phases. From (1) we observe that we should estimate accurately two terms: the mean reward of each arm $i$ in each global state $s$, $\mu^i_s$, and the transition probabilities of the global Markovian $S$, $p_{s\hat{s}}$.

In the analysis, we show that in each global state $s$, we must explore a suboptimal arm $i$ with a local exploration rate of at least $D_s^i \log(t)$ times for being able to distinguishing it from $i^*_s \triangleq \arg\max_i V^i_s$ (i.e., the arm that maximizes the expected value in state $s$) with a sufficiently high accuracy, where

$$D_s^i \triangleq \frac{4L}{(V^i_s - V^i_{s^*})^2}, \quad (3)$$

where $V^i_s \triangleq \max_i V^i_s$, and $L$ is a constant that depends on the system parameters, defined in (12). The $D_s^i$ parameter is a type of hardness parameter [21], appropriate for the setting considered in this paper, in the sense that it determines the order of magnitude of the sample size required to find the best arm in each global state with a required probability.
Algorithm 1 LEMP Algorithm

initialize: $t = 0$, $N_s = 0$, $n_I = 0$, $n^i_O = 1$, $T^0_i = 0$, $s^i_* = 0, \forall i = 1, \ldots, N$

for $i=1:N$ do
  play arm $i$; observe global state $s$ (denote the previous global state by $s$) and local state as $x$ and set $\gamma^i(n^i_O) = x$
  $t := t + 1; T^1_i := T^0_i + 1; n^i_O := n^i_O + 1; r^i_* := r^i_* + r^i_s$, $N_s = N_s + 1$;

while (1) do
  for $i = 1 : N$ do
    set $\hat{\mu}_i^0(t) \hat{p}_{ss}(t), \hat{D}_s^i(t)$ according to (4),(5),(6), respectively;
    while condition (10) holds for some arm $i$ (or condition (11) holds) do
      play arm $i$ (or arm $i_M$); observe global state $s$ and local state as $x$;
      while $x \neq \gamma^i(n^i_O - 1)$ (SB1) do
        $t := t + 1; N_s = N_s + 1$; play arm $i$; observe global state $s$ and local state as $x$;
        $t := t + 1; T^1_i := T^0_i + 1; r^i_* := r^i_* + r^i_s$, $N_s = N_s + 1$;
      for $n = 1 : 4^{n^i_O - 1}$ (SB2) do
        play arm $i$; observe global state $s$ and local state as $x$;
        $t := t + 1; T^2_i := T^1_i + 1; r^i_* := r^i_* + r^i_s$, $N_s = N_s + 1$;
        $n^i_O := n^i_O + 1$; set $\hat{\mu}_i^0(t) \hat{p}_{ss}(t)$ according to (4),(5),(6), respectively; $\gamma^i(n^i_O) = x$
      set $V^0_i(t)$ as according to (7)
      set $s^* = \text{arg max}_s \hat{V}_s^i(t) \& s$;
      for $n = 1 : 2^{4^{n^i_O - 1} - 1}$ do
        play arm $i$; observe new global state $s$ and local state as $x$;
        $t := t + 1; N_s = N_s + 1$;
        $n_I = n_I + 1$;
  end while

end for

We point out that in order to derive $\hat{D}_s^i$, we should know the system parameters $\{p_{ss}\}$, $\{\mu_s^i\}$. Since the reward means and the transition probabilities are unknown, we estimate $\hat{D}_s^i$ by replacing $\mu_s^i, p_{ss}$ by their estimators:

$$\hat{\mu}_s^i(t) = \frac{1}{T^1_i(t)} \sum_{n=1}^{T^1_i(t)} x^i_s(t^i_s(n)),$$

(4)

$$\hat{p}_{ss} = \frac{N_{ss}(t)}{N_s(t)}.$$

(5)

where $t^i_s(n)$ is the time index of the $n^{th}$ play on arm $i$ in global state $s$ in sub-block SB2 only (SB2 is detailed in Sec. III-B). $T^1_i(t)$ is the number of samples from arm $i$ in global state $s$ in sub-block SB2 up to time $t$, $N_s(t)$ is the number of occurrences of the state $s$ until time $t$, and $N_{ss}(t)$ is the number of transitions from $s$ to $s$ up to time $t$. We also define: $\Delta_s^i \triangleq (\hat{V}_s^i(t) - \hat{V}_s^i(t))^2$, $\Delta_s \triangleq \min \Delta_s^i$ and $\Delta_s \triangleq \min_s \Delta_s$.

Denote the estimator of $\hat{D}_s^i$ by:

$$\hat{D}_s^i(t) \triangleq \frac{4L}{\max \{\Delta_s, (\hat{V}_s^i(t) - \hat{V}_s^i(t))^2 - \epsilon\}},$$

(6)

where:

$$\hat{V}_s^i(t) \triangleq \sum_{s \in S} \hat{p}_{ss}(t) \hat{\mu}_s^i(t),$$

(7)

$\hat{V}_s^i(t)$ max, $\hat{V}_s^i(t)$, and $\epsilon > 0$ is a fixed tuning parameter.

Using $\{\hat{D}_s^i(t)\}$, which are updated dynamically overall time and controlled by the corresponding estimators, we can design an adaptive arm selection for sampling arm $i$ at state $s$ that will converge to its exploration rate required for efficient learning, as time increases. Whether we succeed to obtain a logarithmic regret order depends on how fast $\hat{D}_s^i(t)$ converges to a value which is no smaller than $\hat{D}_s$ (so that we take at least $\hat{D}_s$ samples from bad arms in most of the times).

B. Description of the exploration phases:

Due to the restlessness nature of both active and passive arms, learning the Markovian reward statistics requires that arms will be played in a consecutive manner for a period of time (i.e., phase). Therefore, the exploration phases are divided into sub-blocks SB1 and SB2. Consider time $t$ (and we remove the time index $t$ for convenience). We define $n^i_O(t)$ as the number of exploration phases in which arm $i$ was played up to time $t$. Let $\gamma^i(n^i_O - 1)$ be the last reward state observed at the $(n^i_O - 1)^{th}$ exploration phase for arm $i$. As illustrated in Fig. 2, once the player starts the $(n^i_O)^{th}$ exploration phase, it first plays a random period of time, also known as a random hitting time, until observing state $\gamma^i(n^i_O - 1)$.
This random period of time is referred to as SB1. Then, the player plays arm \( i \) until it observes \( 4^\lambda \) samples. This period of time is referred to as SB2. The player stores the \( (4^\lambda)_t \)th state \( \gamma^t(n^t) \) observed at the current \( (n^O)_t \)th exploration phase, and so on. We define the set of time indices during SB2 sub-blocks by \( V_t \). This procedure ensures that each interval in \( V_t \) starts from the last state that was observed in the previous interval. Therefore, cascading these intervals forms a sample path which is equivalent to a sample path generated by continuously sampling the Markov chain.

C. Description of the exploitation phases:

Let \( n^t(t) \) be the number of exploitation phases up to time \( t \). The player plays the exploitation phase for a deterministic period of time with length \( 2 \cdot 4^\lambda \cdot (t) - 1 \) according to the following rule: at each time slot the player computes the expected value, \( V^\lambda(t) \), of each arm given the observed global state when entering the \( (n^t) \)th exploitation phase, and plays the arm that maximizes the expected value.

D. Phase Selection Conditions:

At the beginning of each phase, the player needs to decide whether to enter an exploration phase for one of the \( N \) arms, or whether to enter an exploitation phase. We recall that the purpose of the exploration phases is to estimate both the expected rewards of the arms, and the transition probabilities of the global process. We therefore define:

\[
I_L \triangleq \frac{\lambda_{\min}}{3072((x_{\max} + 2)^2 \cdot |X_{\max}| \cdot \pi_{\max} \cdot |S| \cdot (V^*_{\max} + 2)^2)} \\
I_G \triangleq \frac{1}{128((x_{\max} + 2) \cdot |S| \cdot (V^*_{\max} + 2)^2)}
\]

The decision to explore or exploit will be made due to the next two conditions: first, if there exists an arm \( i \) and a global state \( s \) such that the following condition holds:

\[
T^i_s (t) \leq \max \left\{ \frac{D^i_s(t)}{I_L}, \frac{2}{c^T \cdot I_G} \right\} \cdot \log t,
\]

then the player enters an exploration phase for arm \( i \). Second, if there exists a global state \( s \in S \) where

\[
N_s(t) \leq \frac{2}{c^T \cdot I_G} \cdot \log t,
\]

then the player enters an exploration phase for arm \( i \), where \( i_M \triangleq \arg \min \{ \min_s D^i_s(t) \} \). Otherwise, the player enters an exploitation phase.

IV. REGRET ANALYSIS

In the following theorem we establish a finite-sample bound on the expected regret as the function of time, resulting in a logarithmic regret order.

**Theorem 1.** Assume that LEMP algorithm is implemented and the assumptions on the system model described in Section 2 hold, and an upper bound on \( \Delta \) in known. Let \( \lambda_{\max} \) be the second largest eigenvalue of \( P_{X^t} \), and let \( M_{s, s, \max}^i \) be the mean hitting time of state \( y \) starting at initial state \( x \) in global state \( s \). Define \( x_{\max} \triangleq \max_{x \in S, i \in K} |X^i_s| \cdot |X^i_s| \cdot \lambda_{\max} \triangleq \max_{x \in S,i \in K} \frac{1}{x_{\max}} \lambda_{\max} \triangleq \frac{1}{x_{\max} \cdot |X^i_s| \cdot \lambda_{\max}} \triangleq \frac{M_{s, s, \max}^i}{\max_s M_{s, s, \max}^i} \triangleq \max_{x \in X^i_s} \pi_{\max} \cdot |S| \cdot (V^*_{\max} + 2)^2 \cdot \log \left( \frac{1}{1 - \lambda_{\max}} \right) + O(1),

Then, the regret at time \( t \) is upper bounded by:

\[
\mathbb{E}[r(t)] \leq x_{\max} \cdot \left[ \sum_{i=1}^{N} \frac{1}{3} (4A_i \cdot \log(t) + 1) \right] + N|S| \left[ \log_4 4(A_i \cdot \log(t) + 1) \right] + O(1),
\]

where

\[
A_i \triangleq \max \left\{ \frac{2}{c^T I_L}, \frac{2}{c^T I_G}, \frac{2}{c^T \cdot D_{s,\max}^i} \right\}, \quad \text{if } \forall s : i \in K_s \text{,}
\]

\[
D_{s,\max}^i \triangleq \frac{4L}{(V^*_s - V^i_s)^2 - 2\epsilon}, \quad \text{and } K_s \text{ is defined as the set of all indices } i \in \{2, \ldots, N \} \text{ in global state } s \text{ that satisfy: }
\]

\[
(V^*_s - V^i_s)^2 - 2\epsilon > \Delta.
\]

Before proceeding to prove Theorem 1, we define the following auxiliary notation.
\textbf{Definition 1.} Let $T_1$ be the smallest integer, such that for all $t \geq T_1$ the following holds: $\tilde{D}_{i}^s(t) \leq \tilde{D}_{i}^s(t)$ for all $i \in \mathcal{N}$, $s \in \mathcal{S}$, and also $\tilde{D}_{i}^s(t) \leq \tilde{D}_{s, \max}^s$ for all $i \in \mathcal{K}_s$, $s \in \mathcal{S}$.

The term $T_1$ captures the random time by which the exploration rates for all arms are sufficiently close to the desired exploration rates needed for achieving the desired logarithmic regret bound (as shown later).

\textit{Proof of Theorem 7.} The layout of the proof is as follows, first we show in Lemma 1 that the expectation of the random time $T_1$ is bounded independently of $t$. Then, based on Lemma 2 we show in Lemma 3 and Lemma 4 that a logarithmic regret is obtained for all $t > T_1$, which yields the desired expected regret.

In the next Lemma we show that the expected value of $T_1$ is bounded under the LEMP algorithm.

\textbf{Lemma 1.} Assume that the LEMP algorithm is implemented as described in Section III. Then, $\mathbb{E}[T_1] < \infty$ is bounded independent of $t$.

The proof of the Lemma 1 is given in Appendix A.

The second step of the proof is to show that a logarithmic regret is obtained for all $t > T_1$, which yields the desired expected regret.

\textbf{Lemma 2.} Let $\tilde{T}_i(t) \triangleq \sum_{n=1}^{t} \mathbb{I}_{\{\phi(n) = i \neq i^*\}}$ denote the number of times arm $i$ was played when it was not the best arm during the $t$ first rounds. Then the expected regret is upper bounded by:

$$\mathbb{E}[r(t)] \leq x_{\max} \mathbb{E}[T_1] + x_{\max} \sum_{i=1}^{N} \mathbb{E}[\tilde{T}_i(t)].$$

We present the proof of this lemma in Appendix B.

From (15) we observe that it is sufficient to upper-bound the expected number of times an arm $i$ is played when this arm is sub-optimal. We will bound (15) for the exploration and exploitation phases separately. Specifically, let $T_{i}^s(t)$, and $\tilde{T}_i(t)$, denote the time spent on sub-optimal arm $i$ in exploration and exploitation phases, respectively, by time $t$. Thus,

$$\tilde{T}_i(t) = T_{i}^s(t) + \tilde{T}_i(t).$$

The following two lemmas show that both $\mathbb{E}[T_{i}^s(t)]$ and $\mathbb{E}[\tilde{T}_i(t)]$ have a logarithmic order with time.

\textbf{Lemma 3.} The time spent by time $t$ in exploration phases for sub-optimal arm $i$ is bounded by:

$$\mathbb{E}[T_{i}^s(t)] \leq \sum_{i=1}^{N} \left[ \frac{1}{2} \left[ 4(3A_i \cdot \log(t) + 1) - 1 \right] + M_{i, \max} \cdot \log_4(3A_i \cdot \log(t) + 1) \right].$$

\textbf{Lemma 4.} The time spent by time $t$ in exploitation phases for sub-optimal arm $i$ is bounded by:

$$\mathbb{E}[\tilde{T}_i(t)] \leq 6|\mathcal{S}| \cdot \left( \frac{|\mathcal{S}| \cdot X_{\max}^s}{\pi_{\min}^s} + 2|\mathcal{S}| \right) \cdot \max_{s} \pi_{s,i} \cdot \log_4 \left( \frac{3}{2} t + 1 \right).$$

The proofs of Lemma 3 and Lemma 4 are given in Appendix C and Appendix D, respectively.

Finally, we show at Appendix E, that combining the above four lemmas concludes the proof of Theorem 1.

\section{Simulation Results}

In this section we evaluate the regret of the LEMP algorithm numerically in four different scenarios.

We compare the LEMP algorithm to an extended version of the DSEE algorithm \cite{13}, which is an efficient and widely used algorithm in the RMAB settings, and to a strategy that chooses in the exploitation phases the best arm on average (i.e., competing against weak regret as in \cite{14, 16}). The DSEE algorithm uses deterministic sequencing of exploration and exploitation phases, however, it does not estimate the hardness parameter, and explores each arm $\Delta \cdot \log(t)$ times, which results in oversampling bad arms to achieve the desired logarithmic regret. We averaged 1000 Monte-Carlo experiments for generating the simulation results.

In Fig. 3, we simulated a scenario with $|\mathcal{S}| = 2$, $N = 3$, $|A_i^s| = 2 \forall s, i$. Here, the global state models the presence of the primary user that uses the entire bandwidth by a Gilbert-Eliot model \cite{53}, that comprises a Markov chain with two binary states, where global state $s = 1$ denotes a transmitting primary user and $s = 0$ denotes a vacant channel, i.e., inactive primary user. To limit the interference to the primary user, a secondary user may choose to transmit over one of three possible channels (i.e., $N = 3$), where the channels are modeled by a Finite-State Markovian Channel (FSMC), which is a tractable model widely used to capture the time-varying behavior of a radio communication channel (e.g., a Rayleigh fading channels \cite{11}). The transition probabilities of the global chain are $p_{00} = 0.4$, $p_{10} = 0.75$. In global state 1, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: $p_{11} = [0.5, 0.6, 0.7]$, $p_{21} = [0.5, 0.4, 0.3]$, and the rewards for all arms at states 1, 2, respectively, are $r_1 = [4, 5, 8, 1]$, $r_2 = [6, 8, 2, 2]$. In global state 2, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: $p_{11} = [0.55, 0.65, 0.75]$, $p_{21} = [0.45, 0.35, 0.25]$, and the rewards for all arms at states 1, 2, respectively, are $r_1 = [10, 9, 2, 5]$, $r_2 = [14, 11, 3]$. It can be seen that LEMP significantly outperforms the extended DSEE algorithm. Fig. 3 also shows the superior of LEMP against a strategy that chooses the best arm on average, demonstrating the gain in tracking the best arm at each step according to the global process evolution.

Next, we increased the number of arms to 6, and chose $|\mathcal{S}| = 2$, $|A_i^s| = 2 \forall s, i$. The transition probabilities
of the global process is as in the previous scenario. In global state 1, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.5, 0.6, 0.7, 0.7, 0.6, 0.5], p_{21} = [0.5, 0.4, 0.3, 0.3, 0.4, 0.5] \), and the rewards for all arms at states 1, 2, respectively, are \( r_1 = [4, 5, 8, 1, 1, 1, 0.6, 1.2], r_2 = [6, 8, 2, 1.9, 0.9, 2.2] \). In global state 2, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.55, 0.65, 0.75, 0.75, 0.65, 0.55], p_{21} = [0.45, 0.35, 0.25, 0.25, 0.35, 0.45] \), and the rewards for all arms at states 1, 2, respectively, are \( r_1 = [10, 9, 2.5, 3, 2.56, 2.7], r_2 = [14, 11, 3, 2.8, 3.1, 3.3] \). Increasing the number of arms is expected to decrease the performance under the extended DSEE algorithm, since more arms are sampled by the worst exploration rate. Indeed, it can be seen in Fig. 3 that the gap in the regret between LEMP and the extended DSEE algorithm is increased compared to the previous simulation. This is due to the fact that in the proposed LEMP algorithm, each arm is played according to its unique exploration rate (as a result of the online estimation of the hardness parameter), thus adding “bad” arms (i.e., arms with high exploration rate) does not significantly affect the LEMP performances. LEMP outperforms the strategy that chooses the best arm on average also in this scenario.

Next, we increased the number of global states to 3, and chose \( N = 3, |X| = 2 \) \( s, i \). The transition probabilities of the global chain are \( p_{00} = 0.85, p_{01} = 0.1, p_{02} = 0.05, p_{10} = 0.08, p_{11} = 0.85, p_{12} = 0.07, p_{20} = 0.06, p_{21} = 0.09, p_{22} = 0.85 \). In global state 1, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.5, 0.6, 0.7], p_{21} = [0.5, 0.4, 0.3] \), and the rewards for all arms at states 1, 2 are \( r_1 = [4, 1, 1.2], r_2 = [6, 3, 1.8] \), respectively. In global state 2, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.55, 0.65, 0.75], p_{21} = [0.45, 0.35, 0.25] \), and the rewards for all arms at states 1, 2 are \( r_1 = [7, 11, 8.5], r_2 = [7, 11, 8.5] \), respectively. In global state 3, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.52, 0.62, 0.72], p_{21} = [0.48, 0.38, 0.28] \), and the rewards for all arms at states 1, 2 are \( r_1 = [9.9, 9.5, 14], r_2 = [10.3, 11.5, 16] \), respectively. In this scenario, for each global state \( s \), there is a different best arm that is significantly better then the other two arms. Thus, playing the best arm on average in each time slot results in poor performances compared to LEMP, that tracks the best arm at each step, as can be seen in Fig. 5 LEMP outperforms the extended DSEE algorithm also in this scenario.

Finally, we simulated a scenario with \( |S| = 2, N = 3, |X| = 2 \) \( s, i \), and decreased the difference between the highest and the second highest values in global state 1, compared to this difference in the first simulation (i.e., in Fig. 3). The transition probabilities of the global chain are \( p_{00} = 0.4, p_{10} = 0.75 \). In global state 1, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.5, 0.6, 0.7], p_{21} = [0.5, 0.4, 0.3] \), and the rewards for all arms at states 1, 2, respectively, are \( r_1 = [4, 5, 8, 1], r_2 = [6, 9, 2, 2] \). In global state 2, the local transition probabilities for all arms to transition from 1 to 1 and from 2 to 1, respectively, are: \( p_{11} = [0.55, 0.65, 0.75], p_{21} = [0.45, 0.35, 0.25] \), and the rewards for all arms at states 1, 2, respectively, are \( r_1 = [10, 9, 2.5], r_2 = [14, 11, 3] \). Decreasing the difference between the highest and the second highest values in global state 1 results in a high exploration rate used to distinguish between the two best arms in this state. As discussed in Section III LEMP explores only these two arms using the high exploration rate, while the extended version of the DSEE algorithm explores all the arms with the high exploration rate. Indeed, as can be seen in Fig. 6, this effect results in a high regret under the extended DSEE algorithm as compared to LEMP, which also outperforms the strategy that chooses the best arm on average.

VI. CONCLUSION

We developed a novel Learning under Exogenous Markov Process (LEMP) algorithm for an extended ver-
sion of the RMAB problem, where an exogenous Markov global process governs the distribution of the arms. Inspired by recent developments of sequencing methods of exploration and exploitation phases, LEMP estimates the hardness parameter of the problem which controls the size of exploration phases. During the exploitation phases, LEMP switches arms dynamically according to the global process evolution. Simulation results support the theoretical analysis, and shows superior performances of LEMP against competitive strategies.

\[ \sum_{s \in S} \sum_{i \in \mathcal{K}_s} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \mathbb{P}(\hat{D}_s^i(j) < \mathcal{D}_s^i \text{ or } \hat{D}_s^i(j) > \mathcal{D}_s^{i,\text{max}}) \leq |\mathcal{S}|NC \left[ \sum_{j=1}^{\infty} j^{-(2+\delta)} + \sum_{n=2}^{\infty} \int_{n-1}^{\infty} j^{-(2+\delta)} \, dj \right] \]

Note that if we show that

\[ \mathbb{P}(\hat{D}_s^i(j) < \mathcal{D}_s^i \text{ or } \hat{D}_s^i(j) > \mathcal{D}_s^{i,\text{max}}) \leq C \cdot j^{-(2+\delta)}, \]

for some constants \( C > 0, \delta > 0 \) for all \( i \in \mathcal{K}_s, s \in \mathcal{S} \) for all \( j \geq n \), then we get:

\[ \sum_{s \in \mathcal{S}} \sum_{i \in \mathcal{K}_s} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \mathbb{P}(\hat{D}_s^i(j) < \mathcal{D}_s^i \text{ or } \hat{D}_s^i(j) > \mathcal{D}_s^{i,\text{max}}) \leq |\mathcal{S}|NC \left[ \sum_{j=1}^{\infty} j^{-(2+\delta)} + \sum_{n=2}^{\infty} \int_{n-1}^{\infty} j^{-(2+\delta)} \, dj \right] \]

\[ = |\mathcal{S}|NC \left[ \sum_{j=1}^{\infty} j^{-(2+\delta)} + \frac{1}{1+\delta} \sum_{n=2}^{\infty} (n-1)^{-(1+\delta)} \right] \]

which is bounded independent of \( t \). Similarly, showing that \( \mathbb{P}(\hat{D}_s^i(j) < \mathcal{D}_s^i) \leq C \cdot j^{-(2+\delta)} \) for some constants \( C, \delta > 0 \) for all \( i \in \mathcal{K}_s, s \in \mathcal{S} \) for all \( j \geq n \) completes the statement.

**Step 1: Simplifying (16):** First,

\[ \mathbb{P}(\hat{D}_s^i(t) < \mathcal{D}_s^i \text{ or } \hat{D}_s^i(t) > \mathcal{D}_s^{i,\text{max}}) \]

\[ = \mathbb{P}\left( \left( \frac{4L}{\max(\Delta, (V_s^* - V_s^i)^2 - \epsilon)} \right) < \frac{4L}{(V_s^* - V_s^i)^2 - 2\epsilon} \right) \]

\[ = \mathbb{P}\left( \left( (V_s^* - V_s^i)^2 - \epsilon > (V_s^* - V_s^i)^2 \right) \cap \left( \hat{V}_s^i(t) - \hat{V}_s^i(t) < \Delta \right) \right) \]

\[ \cup \left( \left( (V_s^* - V_s^i)^2 - \epsilon < (V_s^* - V_s^i)^2 - 2\epsilon \right) \cap \left( \hat{V}_s^i(t) - \hat{V}_s^i(t) < \Delta \right) \right) \]

\[ \cup \left( \left( (V_s^* - V_s^i)^2 - \epsilon < (V_s^* - V_s^i)^2 - 2\epsilon \right) \cap \left( \hat{V}_s^i(t) - \hat{V}_s^i(t) < \Delta \right) \right) \]

\[ \cup \left( \left( (V_s^* - V_s^i)^2 - \epsilon < (V_s^* - V_s^i)^2 - 2\epsilon \right) \cap \left( \hat{V}_s^i(t) - \hat{V}_s^i(t) < \Delta \right) \right) \]

**Proof of Lemma 1:** First note that \( \mathbb{E}[T_1] \) can be written as follows:

\[ \mathbb{E}[T_1] = \sum_{n=1}^{\infty} n \cdot \mathbb{P}(T_1 = n) = \sum_{n=1}^{\infty} \mathbb{P}(T_1 \geq n) \]

\[ \leq \sum_{s \in \mathcal{S}} \sum_{i \in \mathcal{K}_s} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \mathbb{P}(\hat{D}_s^i(j) < \mathcal{D}_s^i \text{ or } \hat{D}_s^i(j) > \mathcal{D}_s^{i,\text{max}}) \]

\[ + \sum_{s \in \mathcal{S}} \sum_{i \in \mathcal{K}_s} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \mathbb{P}(\hat{D}_s^i(j) < \mathcal{D}_s^i) \cdot \mathbb{P}(\hat{D}_s^i(j) > \mathcal{D}_s^{i,\text{max}}) \]
The probability for the second event on the RHS is zero, and the forth event lies inside the measure of the third event due to the fact that \( i \in K_s \). Hence,

\[
P(\tilde{D}_s^i(t) < D_s^i \text{ or } \tilde{D}_s^i(t) > D_s^i,_{\text{max}}) \leq P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))^2 - (V_s^* - V_i^*)^2 | \tilde{V}_i^*(t) > \epsilon) \]

\[
\leq P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))^2 - (V_s^* - V_i^*)^2 > \epsilon) = P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))^2 - (\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(V_s^* - V_i^*) + (\tilde{V}_s^*(t) - \tilde{V}_i^*(t))V_s^* - V_s^*)^2 | \tilde{V}_i^*(t) > \epsilon) \]

\[
= P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2 | \tilde{V}_i^*(t) > \epsilon) \]

\[
+ (V_s^* - V_i^*)|\tilde{V}_s^*(t) - \tilde{V}_i^*(t) - (V_s^* - V_i^*) | > \epsilon) \]

\[
\leq P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2 | \tilde{V}_i^*(t) > \epsilon) + P((V_s^* - V_i^*)|\tilde{V}_s^*(t) - \tilde{V}_i^*(t) - (V_s^* - V_i^*) | > \epsilon) \]

\[
\leq \frac{\epsilon}{2} \cdot P(\tilde{V}_s^*(t) - \tilde{V}_i^*(t) - (V_s^* - V_i^*) | > \epsilon) > \frac{\epsilon}{2}. \tag{17} \]

We continue by bounding the first term on the RHS of (17). For every \( R > 0 \), we have:

\[
P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2) > \frac{\epsilon}{2} \]

\[
\leq P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2) > 1 \]

\[
+ P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2 | > \frac{\epsilon}{2(R+1)})) \]

\[
+ P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) + 1 > R) \]

\[
\leq 2P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2 | > \frac{\epsilon}{2(R+1)}) \]

\[
+ P(V_s^* + 1 > R). \tag{18} \]

We choose \( R = V_s^* + 1 \). Then, the second term is equal to 0. We proceed with the first term:

\[
2 \cdot P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2 | > \frac{\epsilon}{2(V_s^* + 2)}) \]

\[
\leq 2P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) > \frac{\epsilon}{4(V_s^* + 2)}) \]

\[
+ 2P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) > \frac{\epsilon}{4(V_s^* + 2)}) \]

\[
\tag{19} \]

We next bound the second term on the RHS of (17). For every \( R' > 0 \), we have:

\[
P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2) > \frac{\epsilon}{2} \]

\[
\leq P(V_s^* > R) \]

\[
+ P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) > \frac{\epsilon}{2(R'^{2+1})}) \]

\[
\tag{20} \]

We now choose \( R' = R = V_s^* + 1 \), so the first term is equal to 0. We continue with the second term:

\[
P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t))(\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) - (V_s^* - V_i^*)^2) > \frac{\epsilon}{2(R'^{2+1})} \]

\[
P((\tilde{V}_s^*(t) - \tilde{V}_i^*(t)) > \frac{\epsilon}{4(V_s^* + 2)}) \]

\[
\tag{21} \]

By combining (18) and (19) we get:

\[
P(\tilde{D}_s^i(t) < \tilde{D}_s^i \text{ or } \tilde{D}_s^i(t) > D_s^i,_{\text{max}}) \leq 6 \cdot \max \left\{ \frac{\epsilon}{4(V_s^* + 2)} \right\} \]

\[
\tag{20} \]

Step 2: Bounding (20): We first bound the second term in (20) (the first term is bounded similarly).

\[
P((\tilde{V}_s^*(t) - V_s^*) > \frac{\epsilon}{4(V_s^* + 2)}) \]

\[
= P\left( \sum_{s' \in S} \tilde{p}_{ss'}(t)\tilde{\mu}_{s'}(t) - \sum_{s' \in S} p_{ss'}\mu_{s'} > \frac{\epsilon}{4(V_s^* + 2)} \right) \]

\[
\leq P\left( \sum_{s' \in S} (\tilde{p}_{ss'}(t)\tilde{\mu}_{s'}(t) - p_{ss'}\mu_{s'}) > \frac{\epsilon}{4(V_s^* + 2)} \right) \]

Following similar steps as we did to obtain (20) from (17), we get:

\[
P((\tilde{V}_s^*(t) - V_s^*) > \frac{\epsilon}{4(V_s^* + 2)}) \]

\[
\leq 2P((\tilde{p}_{ss'}(t)\tilde{\mu}_{s'}(t) - p_{ss'}\mu_{s'}) > \frac{\epsilon}{16(V_s^* + 2)(x_{\text{max}} + 2)|S|}) \]

\[
+ P((\tilde{\mu}_{s'}(t) - \mu_{s'}) > \frac{\epsilon}{16(V_s^* + 2)(x_{\text{max}} + 2)|S|}) \]

\[
\tag{22} \]

To complete the statement, we need to bound (21) and (22). To bound (22), we will use Lezaud’s result [54].

**Lemma 5.** [54] Consider a finite-state, irreducible Markov chain \( \{X_t\}_{t \geq 1} \) with state space \( S \), matrix of transition probabilities \( P \), an initial distribution \( q \), and stationary distribution \( \pi \). Let \( N_q = \|\left( \frac{q_s}{\pi_s}, x \in S \right)\|_2 \). Let \( \tilde{P} = PP' \) be the multiplicative symmetrization of \( P \) where \( P' \) is the adjoint of \( P \) on \( l_2(\pi) \). Let \( \epsilon = 1 - \lambda_2 \), where \( \lambda_2 \) is the second largest eigenvalue of the matrix \( P' \). \( \epsilon \) will be referred to as the eigenvalue gap of \( P' \). Let \( f : S \rightarrow R \) be such that \( \sum_{y \in S} \pi_y f(y) = 0, \|f\|_2 \leq 1 \) and \( 0 \leq \|f\|_2^2 \leq 1 \) if \( P' \) is irreducible. Then, for any positive integer \( n \) and all \( 0 < \lambda \leq 1 \), we have:

\[
Pr\left( \frac{1}{n} \sum_{t=1}^{n} f(X_t) \geq \lambda \right) \leq N_q \exp\left(-\frac{n\lambda^2 \epsilon}{12}\right). \]

Consider an initial distribution \( q^i_s \) for the \( i \)th arm in global state \( s \). We have:

\[
\|\left( \frac{q^i_s(x)}{\pi^i_s(x)}, x \in X^i_s \right)\|_2 \leq \sum_{x \in X^i_s} \|\frac{q^i_s(x)}{\pi^i_s(x)}\|_2 \leq \frac{1}{\pi_{\text{min}}}. \]
Before applying Lezaud’s bound, we pay attention for the following: (i) The sample means \( \{ \hat{\mu}_i(t) \} \) are calculated only from measurements in the set \( \mathcal{V}_i \). As discussed in Section III-B, these measurements are equivalent to a sample path generated by continuously sampling the Markov chain. Hence, we can apply Lezaud’s bound to upper bound (22). (ii) By the construction of the algorithm, (10) ensures that once exploitation phases are executed (which are deterministic), the event \( T^i_\delta(t) \geq \frac{(2+\delta) \log(t)}{c^2 t^3} \) for \( \delta > 0 \) arbitrarily small surely occurs during exploration phases, the randomness of SB1 (say for arm \( r \neq i \)) affects \( T^i_\delta(t) \) since SB1 can be very long (with small probability) and then \( T^i_\delta(t) \geq \frac{(2+\delta) \log(t)}{c^2 t^3} \) might not hold until the end of the phase once the algorithm corrects the exploration gap by condition (10). Therefore, we define \( E_i(t) \) as the event when all SB1 phases that have been executed by time \( t \) are smaller than \( \delta \cdot t \). When event \( E_i(t) \) occurs we have \( T^i_\delta(t) \geq \frac{(2+\delta) \log(t)}{c^2 t^3} \) (for all \( t > D \), for a sufficiently large finite deterministic value \( D \)). Then, for all \( i \) and \( s' \), we have:

\[
\Pr(\hat{\mu}^i_s(t) - \mu^i_s \geq \frac{\epsilon}{16(V^*_s + 2)(x_{\max} + 2)|S|}, E_i(t) \text{ occurs})
\] 

\[+ \Pr( E_i(t) \text{ does not occur})
\] 

(23)

We next bound (22) by bounding (24) and (24):

We define \( O^i_{x[s]}(t) \) as the number of occurrences of local state \( x \) on arm \( i \) in global state \( s \) up to time \( t \), and we first look at:

\[
\Pr(\mu^i_s(t) - \hat{\mu}^i_s > \frac{\epsilon}{16(V^*_s + 2)(x_{\max} + 2)|S|}, E_i(t) \text{ occurs})
\]

\[
= \Pr \left( \sum_{x \in \mathcal{X}_s^i} x \cdot O^i_{x[s]}(t) - T^i_\delta(t) \sum_{x \in \mathcal{X}_s^i} x \cdot \pi^i_s(x) > \frac{\epsilon}{16(V^*_s + 2)(x_{\max} + 2)|S|} \right)
\]

\[
\leq \sum_{x \in \mathcal{X}_s^i} \Pr \left( \sum_{n=1}^{T^i_\delta(t)} x^i_s(x) - T^i_\delta(t) \pi^i_s(x) > \frac{\epsilon}{16(V^*_s + 2)(x_{\max} + 2)|S|} \pi^i_s(x) \right), E_i(t) \) 
\]

\[
\leq |\mathcal{X}_s^i| N^i_{\epsilon \pi_{\max}^i}(t)
\]

1We point out that a precise statement requires to set \( (2 + 2\delta) \) in (10) and the statement holds for all \( t > D \), where \( D \) is a finite deterministic value. However, since \( \delta > 0 \) is arbitrarily small and is not a design parameter, we do not present it explicitly when describing the algorithm to simplify the presentation.
Next, we upper bound \( D_s^t \). Note that for all \( t > T_1 \), we have:
\[
D_s^t \leq \bar{D}_s^i (t) \leq D_{s,max}^i
\]  
(28)
for all \( s \in S, i \in K_s \), and we have the LHS of the inequality for \( i \notin K_s \). For convenience, we will develop \( (27) \) between \( n = 1 \) and \( t \) with \( (28) \) (and the LHS for \( i \notin K_s \)) holds for all \( 1 \leq n \leq t \), which upper bounds \( (27) \) between \( n = T_1 + 1 \) and \( t \):
\[
\mathbb{E}_\phi \left[ \sum_{n=T_1+1}^{t} \sum_{i \in V_s(n)} (x_{s,n}^i (n) - x_{s,n}^i (n)) \mathbb{I}_{\phi(n) = i} \right] \
\leq \mathbb{E}_\phi \left[ \sum_{n=1}^{t} \sum_{i \in V_s(n)} (x_{s,n}^i (n) - x_{s,n}^i (n)) \mathbb{I}_{\phi(n) = i} \right].
\]  
(29)
Finally, note that:
\[
\mathbb{E}_\phi \left[ \sum_{n=1}^{t} \sum_{i \in V_s(n)} (x_{s,n}^i (n) - x_{s,n}^i (n)) \mathbb{I}_{\phi(n) = i} \right] \
\leq x_{max} \sum_{i=1}^{N} \mathbb{E}_\phi[\bar{T}^i(t)].
\]  
(30)

**APPENDIX C**

**PROOF OF LEMMA 2**

**Proof of Lemma 2** We first upper bound the number of exploration phases \( n_{i,0}^t (t) \) for each arm (say \( i \)) by time \( t \). If the player has started the \( n_{i,0}^t \)th exploration phase, we have by (10) and the fact that \( n_{i,0}^t \) is observed again (i.e., SB1 sub-block), and another \( 4^{n_{i,0}^t} \) time slots.

Thus, the time spent by time \( t \) in exploration phases for arm \( i \) is bounded by:
\[
\mathbb{E}[\hat{T}_{s,i}^t (t)] \leq \sum_{n=0}^{n_{i,0}^t (t) - 1} \left( 4^n + M_{max}^i \right) = \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + M_{max}^i \right) \leq \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + M_{max}^i + n_{i,0}^t (t) \right) \leq \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + 1 + M_{max}^i \right) \leq \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + 1 + M_{max}^i \right) \leq \\
\frac{1}{3} \left[ 4(3A_i \cdot \log(t) + 1) \right] + M_{max}^i \cdot \log(3A_i \cdot \log(t) + 1).
\]

**APPENDIX D**

**PROOF OF LEMMA 3**

**Proof of Lemma 3** We first upper bound the number of exploration phases \( n_{i,0}^t (t) \) for each arm (say \( i \)) by time \( t \). If the player has started the \( n_{i,0}^t \)th exploration phase, we have by (10) and the fact that \( n_{i,0}^t \) is observed again (i.e., SB1 sub-block), and another \( 4^{n_{i,0}^t} \) time slots.

Thus, the time spent by time \( t \) in exploration phases for arm \( i \) is bounded by:
\[
\mathbb{E}[\hat{T}_{s,i}^t (t)] \leq \sum_{n=0}^{n_{i,0}^t (t) - 1} \left( 4^n + M_{max}^i \right) = \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + M_{max}^i \right) \leq \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + M_{max}^i + n_{i,0}^t (t) \right) \leq \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + 1 + M_{max}^i \right) \leq \\
\frac{1}{3} \left( 4^{n_{i,0}^t (t) - 1} + 1 + M_{max}^i \right) \leq \\
\frac{1}{3} \left[ 4(3A_i \cdot \log(t) + 1) \right] + M_{max}^i \cdot \log(3A_i \cdot \log(t) + 1).
\]
Using (36), we can bound (32), and therefore:

\[ \mathbb{P}(V_s^i < V_s^i + C_{s,t,n}^i + C_{s,t,n}) \]
\[ = \mathbb{P}(V_s^i - V_s^i < \frac{L \log t_n}{T_i^s(t_n)} + \sqrt{\frac{L \log t_n}{T_i^s(t_n)}}) \]
\[ \leq \mathbb{P}(V_s^i - V_s^i < 2 \frac{L \log t_n}{\min(T_i^s(t_n), T_j^i(t_n))}) \]
\[ = \mathbb{P}(V_s^i - V_s^i < 2 \frac{4L \log t_n}{\min(T_i^s(t_n), T_j^i(t_n))}) \]
\[ = \mathbb{P}(\min\{T_i^s(t_n), T_j^i(t_n)\} < \Delta_i \cdot \log t_n) = 0. \]

Therefore, we can rewrite (33) as follows:

\[ \mathbb{P}(i,s,n) \leq \mathbb{P}(V_s^i(t_n) \geq V_s^i(t_n)) \]
\[ \leq \mathbb{P}(V_s^i(t_n) \leq V_s^i - C_{s,t,n}) \]
\[ + \mathbb{P}(V_s^i(t_n) \geq V_s^i + C_{s,t,n}). \] (34)

Next, we bound both terms on the RHS of (34). Using similar steps as we used for bounding the second term in (20), we get:

\[ \mathbb{P}(V_s^i(t_n) - V_s^i \geq C_{s,t,n}) \]
\[ \leq 2|S| \mathbb{P}(|\hat{p}_{s,s'}(t_n) - p_{s,s'}| \geq \frac{1}{4|S||X_{\max}|^2} \cdot C_{s,t,n}) \]
\[ + |S| \mathbb{P}(|\hat{p}_{s,s}(t_n) - \mu_{s,s}'| \geq \frac{1}{4|S||X_{\max}|^2} \cdot C_{s,t,n}). \] (35)

The second term in (35) is bounded similarly as in (22):

\[ |S| \mathbb{P}(\hat{p}_{s,s}(t_n) - \mu_{s,s}' \geq C_{s,t,n}) \]
\[ \leq \frac{|S||X_{\max}|}{\pi_{\min}} \cdot \exp(-T_i^s(t_n) \frac{L \log t_n}{T_i^s(t_n)} \frac{1}{16(\sigma_{s,s}^2 + 2)s|S|^2 \pi_{\min}^2}) \]
\[ \leq \frac{|S||X_{\max}|}{\pi_{\min}} \cdot t_n^{-1}, \]

where the last inequality is due to (12). The first term in (35) is bounded similarly as in (21):

\[ 2|S| \mathbb{P}(|\hat{p}_{s,s}(t_n) - p_{s,s'}| \geq \frac{1}{4|S||X_{\max}|^2} \cdot C_{s,t,n}) \]
\[ \leq 2|S| \exp\left(-2N_s(t_n) \cdot \frac{L \log t_n}{16(\sigma_{s,s}^2 + 2)s|S|^2 \pi_{\min}^2} \cdot N_s(t_n)\right) \]
\[ \leq 2|S|t_n^{-1}, \]

where the last inequality is due to (12), and the fact that \( \forall i : N_s(t) > T_i^s(t) \). The first term in (34) is bounded similarly, and therefore:

\[ \mathbb{P}(i,s,n) \leq 2|S| \left(\frac{|S||X_{\max}|}{\pi_{\min}} + 2|S|\right \cdot t_n^{-1}. \] (36)

Using (36), we can bound (32), and therefore:

\[ \mathbb{E}[T_i^s(t_n)] \leq 6|S| \left(\frac{|S||X_{\max}|}{\pi_{\min}} + 2|S|\right \cdot t_n \max_{s,s'} |\log_4 (\frac{3}{2}t + 1)|. \] (37)

**APPENDIX E**

**INCORPORATING THE REGRET EVENTS TO PROVE THEOREM 1**

We conclude the proof of Theorem 1 by incorporating the regret events discussed above. i.e., regret that is caused by imprecise estimation of the exploration rate, regret that is caused by playing bad arms in exploration phases, and regret that is caused by playing bad arms in exploitation phases.

Lemmas 1 and 2 result in the additional constant term \( O(1) \) in the regret bound (15), which is independent of \( t \).

From Lemmas 3 and 2, the regret caused by playing bad arms in exploitation phases by time \( t \) is bounded by:

\[ x_{\max}^* \cdot \max_{i,s} \left(\frac{|S||X_{\max}|}{\pi_{\min}} + 2|S|\right \cdot t \max_{s,s'} |\log_4 (\frac{3}{2}t + 1)|, \]

which coincides with the first and second terms on the RHS of (13).

From Lemma 4 and Lemma 2, the regret caused by playing sub-optimal arms in exploitation phases by time \( t \) is bounded by:

\[ x_{\max} \cdot |S| \geq 1 \left(\frac{|S||X_{\max}|}{\pi_{\min}} + 2|S|\right \cdot t \max_{s,s'} |\log_4 (\frac{3}{2}t + 1)|, \]

which coincides with the third term on the RHS of (13), and thus Theorem 1 follows.

**REFERENCES**

[1] T. Gafni, M. Yemini, and K. Cohen, “Restless multi-armed bandits under exogenous global markov process,” to appear in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), May 2022.

[2] J. Sun, Y. Zhao, N. Zhang, X. Chen, Q. Hu, and J. Song, “A dynamic distributed energy storage control strategy for providing primary frequency regulation using multi-armed bandits method,” IET Generation, Transmission & Distribution, 2021.

[3] S. L. Scott, “Multi-armed bandit experiments in the online service economy,” Applied Stochastic Models in Business and Industry, vol. 31, no. 1, pp. 37–45, 2015.

[4] C. Buluca, Personalizing treatments via contextual multi-armed bandits by identifying relevance. PhD thesis, Bilkent University, 2019.

[5] Y.-P. Hsu, E. Modiano, and L. Duan, “Scheduling algorithms for minimizing age of information in wireless broadcast networks with random arrivals,” IEEE Transactions on Mobile Computing, vol. 19, no. 12, pp. 2903–2915, 2019.

[6] T. Gafni, N. Slivkins, K. Cohen, Y. C. Eldar, and H. V. Poor, “Federated learning: A signal processing perspective,” to appear in the IEEE Signal Processing Magazine, arXiv preprint arXiv:2108.17175, 2021.

[7] Y. Xu, P. Cheng, Z. Chen, M. Ding, Y. Li, and B. Vucetic, “Task offloading for large-scale asynchronous mobile edge computing: An index policy approach,” IEEE Trans. Signal Process., vol. 69, pp. 401–416, 2021.

[8] O. Amar and K. Cohen, “Online learning for shortest path and backpressure routing in wireless networks,” in IEEE International Symposium on Information Theory (ISIT), pp. 2702–2707, 2021.

[9] Q. Zhao and B. M. Sadler, “A survey of dynamic spectrum access,” IEEE signal processing magazine, vol. 24, no. 3, pp. 79–89, 2007.

[10] K. Wang and L. Chen, “On optimality of myopic policy for restless multi-armed bandit problem: An axiomatic approach,” IEEE Trans. Signal Process., vol. 60, no. 1, pp. 300–309, 2011.

[11] H. S. Wang and N. Moayeri, “Finite-state markov channel-a useful model for radio communication channels,” IEEE Trans. Veh. Technol., vol. 44, no. 1, pp. 163–171, 1995.

[12] C. H. Papadimitriou and J. N. Tsitsiklis, “The complexity of optimal queueing network control,” in Proceedings of IEEE 9th Annual Conference on Structure in Complexity Theory, pp. 318–322, 1994.
[13] V. Anantharam, P. Varaiya, and J. Walrand, “Asymptotically efficient allocation rules for the multiarmed bandit problem with multiple plays-part ii: Markovian dynamics,” IEEE Trans. Autom. Control, vol. 32, no. 11, pp. 977–982, 1987.

[14] C. Tekin and M. Liu, “Online learning of rested and restless bandits,” IEEE Trans. Inf. Theory, vol. 58, no. 8, pp. 5588–5611, 2012.

[15] H. Liu, K. Liu, and Q. Zhao, “Learning in a changing world: Restless multi-armed bandit with unknown dynamics,” IEEE Trans. Autom. Control, vol. 66, no. 10, pp. 5029–5036, 2020.

[16] A. Slivkins and E. Upfal, “Adapting to a changing environment: The restless multiarmed bandit problem,” European Journal of Operational Research, vol. 290, no. 2, pp. 622–639, 2021.

[17] P. Karthik and R. Sundaresan, “Learning to detect an odd restless markov arm,” in 2021 IEEE International Symposium on Information Theory (ISIT), pp. 1457–1462, 2021.

[18] T. Gafni and K. Cohen, “A distributed stable strategy learning algorithm for restless markovian dynamic spectrum access,” in 2019 57th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 347–351, 2019.

[19] T. Gafni and K. Cohen, “Distributed learning over markovian fading channels for stable spectrum access,” arXiv preprint [arXiv:2101.11292] 2021.

[20] P. Whittle, “Restless bandits: Activity allocation in a changing world,” J. Appl. Probab., vol. 25, no. A, pp. 287–298, 1988.

[21] R. R. Weber and G. Weiss, “On an index policy for restless bandits,” J. Appl. Probab., vol. 27, no. 3, pp. 637–648, 1990.

[22] N. Ehsan and M. Liu, “On the optimality of an index policy for bandwidth allocation with delayed state observation and differentiated services,” in IEEE INFOCOM 2004, vol. 3, pp. 1974–1983, 2004.

[23] K. Cohen, Q. Zhao, and A. Scaglione, “Restless multi-armed bandits under time-varying activation constraints for dynamic spectrum access,” in 2014 48th Asilomar Conference on Signals, Systems and Computers, pp. 1575–1578, 2014.

[24] Q. Zhao, B. Krishnamachari, and K. Liu, “On myopic sensing for multi-channel opportunistic access: structure, optimality, and performance,” IEEE Trans. Wireless Commun., vol. 7, no. 12, pp. 5431–5440, 2008.

[25] S. H. A. Ahmad, M. Liu, T. Javid, Q. Zhao, and B. Krishnamachari, “Optimality of myopic sensing in multichannel opportunistic access,” IEEE Trans. Inf. Theory, vol. 55, no. 9, pp. 4040–4050, 2009.

[26] S. H. A. Ahmad and M. Liu, “Multi-channel opportunistic access: A case of rested bandits with multiple plays,” in 2009 47th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1361–1368, 2009.

[27] K. Liu and Q. Zhao, “Indexability of restless bandit problems and optimality of whittle index for dynamic multichannel access,” IEEE Trans. Inf. Theory, vol. 56, no. 11, pp. 5547–5567, 2010.

[28] K. Wang, L. Chen, and Q. Liu, “On optimality of myopic policy for opportunistic access with nonidentical channels and imperfect sensing,” IEEE Trans. Veh. Technol., vol. 63, no. 5, pp. 2478–2483, 2013.

[29] Q. Zhao and B. Krishnamachari, “Structure and optimality of myopic sensing for opportunistic spectrum access,” in 2007 IEEE International Conference on Communications, pp. 6476–6481, 2007.

[30] K. Liu, R. Weber, and Q. Zhao, “Indexability and whittle index for restless bandit problems involving reset processes,” in 2011 50th IEEE Conference on Decision and Control and European Control Conference, pp. 7690–7696, 2011.

[31] V. Krishnamurthy and B. Wahlberg, “Partially observed markov decision process multiarmed bandits—structural results,” Mathematics of Operations Research, vol. 34, no. 2, pp. 287–302, 2009.

[32] V. Krishnamurthy and R. J. Evans, “Hidden markov model multichain bandits: a methodology for beam scheduling in multtarget tracking,” IEEE Trans. Signal Process., vol. 49, no. 12, pp. 2893–2908, 2001.

[33] C. Hartland, S. Gelly, N. Baskiotis, O. Teytaud, and M. Sebag, “Multi-armed bandit, dynamic environments and meta-bands”, in nIPS-2006 Workshop, Online Trading Between Exploration and Exploitation, Whistler, Canada, 2006.

[34] J. Y. Yu and S. Mannor, “Piecewise-stationary bandit problems with side observations,” in Proceedings of the 26th annual international conference on machine learning, pp. 1177–1184, 2009.

[35] A. Slivkins and E. Upfal, “Adapting to a changing environment: the brownian restless bandits,” in COLT, pp. 343–354, 2008.

[36] Z. Wang, R. Zhou, and C. Shen, “Regional multi-armed bandits with partial informativeness,” IEEE Trans. Signal Process., vol. 66, no. 21, pp. 5705–5717, 2018.

[37] S. Baltaoglu, L. Tong, and Q. Zhao, “Online learning and optimization of markov jump linear models”, in 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 2289–2293, 2016.

[38] S. Baltaoglu, L. Tong, and Q. Zhao, “Online learning and optimization of markov jump affine models,” arXiv preprint [arXiv:1605.02213] 2016.

[39] M. Yemini, A. Leishem, and A. Somekh-Baruch, “The restless hidden markov bandit with linear rewards and side information,” IEEE Trans. Signal Process., vol. 69, pp. 1108–1123, 2021.

[40] K. Liu and Q. Zhao, “Link throughput of multi-channel opportunistic access with limited sensing,” in 2008 IEEE International Conference on Acoustics, Speech and Signal Processing, pp. 2977–3000, IEEE, 2008.

[41] P. Lezaud, “Chernoff-type bound for finite markov chains,” Annals of Applied Probability, pp. 849–867, 1998.