A $q$-Difference Equation and Fourier Series Expansions of $q$-Lidstone Polynomials

Maryam Al-Towailb

Department of Computer Science and Engineering, King Saud University, Riyadh 11451, Saudi Arabia; mtowaileb@ksu.edu.sa

**Abstract:** In this paper, we present the $q$-Lidstone polynomials which are $q$-Bernoulli polynomials generated by the third Jackson $q$-Bessel function, based on the Green’s function of a certain $q$-difference equation. Also, we provide the $q$-Fourier series expansions of these polynomials and derive some results related to them.

**Keywords:** $q$-difference equation; $q$-Lidstone polynomials; Green’s function; $q$-Fourier series

**MSC:** 05A30; 39A13; 41A58; 30E20; 35J08; 42A16

1. Introduction

In 1929, Lidstone [1] introduced a generalization of Taylor’s series that approximates an entire function $f(z)$ of exponential type less than $\pi$ in a neighborhood of two points instead of one:

$$f(z) = \sum_{n=0}^{\infty} A_n(z) f^{2n}(1) + \sum_{n=0}^{\infty} A_n(1-z) f^{2n}(0),$$

where the set $\{A_n(z]\}_n$ called Lidstone polynomials. In [2], Whittaker proved that

$$A_n(z) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}(\frac{z+1}{2}),$$

where $B_n(x)$ is the Bernoulli polynomial of degree $n$, which may be defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Recently, Ismail and Mansour [3] introduced a $q$ analog of the Lidstone expansion theorem where they expand a class of entire functions of $q$-exponential growth in terms of Jackson $q$-derivatives of even degree at 0 and 1. See also [4–6] for some results and applications to the $q$-Lidstone theorem.

In [7], the authors constructed another formula of $q$-Lidstone expansion theorem by using the symmetric $q$-difference operator $\delta_q$ (see Section 2), that is

$$f(z) = \sum_{n=0}^{\infty} \left[ \delta_q^{2n} f(1) z^n A_n(z) - \delta_q^{2n} f(0) z^n B_n(z) \right],$$

(1)

where $A_n(z)$ and $B_n(z)$ are the $q$-Lidstone polynomials defined by the generating functions

$$\frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)} = \sum_{n=0}^{\infty} \hat{A}_n(z) w^{2n},$$

$$\frac{\exp_q(zw)\exp_q(-w) - \exp_q(-zw)\exp_q(w)}{\exp_q(w) - \exp_q(-w)} = \sum_{n=0}^{\infty} \hat{B}_n(z) w^{2n}. $$

(2)
Moreover, it turns out that
\[
\tilde{B}_n(z) = \frac{2^{2n+1}}{[2n+1]_q!} \tilde{B}_{2n+1}(z/2; q), \tag{3}
\]
where \(\tilde{B}_n(z; q)\) are \(q\)-Bernoulli polynomials generated by
\[
\frac{\exp_q(zw) \exp_q(\frac{-zw}{2})}{\exp_q(\frac{w}{2}) - \exp_q(\frac{-w}{2})} = \sum_{n=0}^\infty \frac{B_n(z; q)}{[n]_q!} w^n, \tag{4}
\]
and the function \(\exp_q(\cdot)\) is the \(q\)-exponential function which has the series representation
\[
\exp_q(z) = \sum_{n=0}^\infty \frac{a(n+1)}{[n]_q!} z^n, \quad z \in \mathbb{C}. \tag{5}
\]

In this paper, we assume that \(q\) is a positive number less than one and the set \(A_q^*\) is defined by
\[
A_q^* := \{q^n : n \in \mathbb{N}_0\} \cup \{0\},
\]
where \(\mathbb{N}_0 := \{0, 1, 2, \ldots\}\). We present the \(q\)-Lidstone polynomials \(\tilde{A}_n(z)\) and \(\tilde{B}_n(z)\) based on the Green’s function of a \(q\)-boundary value problem
\[
\begin{cases}
\frac{\partial^n f(z)}{\partial z^n} = \phi(z), \\
\frac{\partial^k f(0)}{\partial z^k} = a_k, \quad \frac{\partial^k f(1)}{\partial z^k} = b_k \quad (k = 0, 1, \ldots, n - 1),
\end{cases} \tag{6}
\]
where \(f\) and \(\phi\) are assumed to be continuous functions on \(A_q^*\). Also, we introduce the \(q\)-Fourier series expansions of these functions and derive some results related to them. For other recent contributions on this area, one may refer to [8–10].

This article is organized as follows: In the next section, we present some background on \(q\)-analysis which we need in our investigations. In Section 3, we establish the existence of a solution for the system (51). In Section 4, we introduce the \(q\)-Fourier series expansions of some functions. As an application, in Section 5, we define \(q\)-Lidstone polynomials based on the Green’s function of the system (51), and we provide the \(q\)-Fourier series expansions of these polynomials. Moreover, relying on the obtained \(q\)-Fourier series, we derive a close approximation to \(\tilde{A}_n(z)\) and \(\tilde{B}_n(z)\) for large \(n\).

2. Preliminaries

Recall that the \(q\)-derivative \(D_q\) of the function \(f\) is defined by
\[
D_q f(z) := \frac{f(z) - f(qz)}{z - qz}, \quad \text{for } z \neq 0, \tag{7}
\]
and the \(q\)-derivative at zero is defined to be \(f'(0)\) if it exists, see [11]. The \(q\)-shifted fractional \((a; q)_n\) of \(a \in \mathbb{C}\) is defined by
\[
(a; q)_0 := 1 \quad \text{and} \quad (a; q)_n := \prod_{j=0}^n (1 - aq^j), \quad \text{for } n \in \mathbb{N},
\]
and the \(q\)-number factorial \([n]_q!\) is defined for \(q \neq 1\) by
\[
[n]_q! = \prod_{j=0}^n [j]_q, \quad [j]_q = \frac{1 - q^j}{1 - q}. \]
Jackson [12] introduced the following integral, as a right inverse of the $q$-derivative (7), by
\[
\int_a^b f(t) \, dq_t := \int_0^b f(t) \, dq_t - \int_0^a f(t) \, dq_t \quad (a, b \in \mathbb{C}),
\]
where
\[
\int_0^z f(t) \, dq_t := (1 - q) \sum_{n=0}^{\infty} zq^n f(zq^n),
\]
provided that the series converges at $z = a$ and $z = b$. We can interchange the order of double $q$-integral by
\[
\int_0^z \int_0^v f(t) \, dq_t \, dq_v = \int_0^z \int_q^v f(t) \, dq_v \, dq_t = \int_0^z (z - qt) f(t) \, dq_t.
\] (8)

The symmetric $q$-difference operator $\delta_q$ which is acting on a function $f$ defined by
\[
\frac{\delta_q f(z)}{\delta_q z} := \frac{f(qz \cdot z) - f(qz^{-1} \cdot z)}{z(q^2 - q^{-2})}, \quad \text{for } z \neq 0.
\] (9)
(see [11,13]). From (7) and (9), it follows
\[
\frac{\delta_q f(z)}{\delta_q z} := D_q f\left( \frac{z}{\sqrt{q}} \right).
\]

Therefore, we have
\[
\int_0^a \frac{\delta_q f(z)}{\delta_q z} \, dq_z = q \left[ f(q^{-1} \cdot 2) - f(0) \right].
\] (10)

A function $f$ defined on $A_q^*$ is called $q$-regular at zero if it satisfies
\[
\lim_{n \to \infty} f(xq^n) = f(0), \quad \text{for all } x \in A_q^*.
\]
The $q$-integration by parts rule on $A_q^*$ (see [13]) is
\[
\int_0^a g(q^{-1} \cdot t) \frac{\delta_q f(t)}{\delta_q t} \, dq_t = q \left( f(q^{-1} \cdot t) \right) \bigg|_0^a - \int_0^a f(q^{-1} \cdot t) \frac{\delta_q g(t)}{\delta_q t} \, dq_t,
\] (11)
where $f$ and $g$ are complex valued $q$-regular functions at zero.

We will use a $q$-exponential function $exp_q(.)$ defined in (5) and the $q$-linear sine and cosine, $S_q(z)$ and $C_q(z)$, which defined by
\[
S_q(z) := \frac{exp_q(iz) - exp_q(-iz)}{2i} = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(1 - 1)} q^{2n+1}}{[2n+1]_q} z^{2n+1},
\] (12)
\[
C_q(z) := \frac{exp_q(iz) + exp_q(-iz)}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(-1 - 1)} q^{2n}}{[2n]_q} z^{2n}.
\]

They can be written in terms of the third Jackson $q$-Bessel function $J_v^{(3)}(z; q)$ [14,15] as follows
\[
S_q(z) := q^{1/8} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} z^{1/2} J^{(3)}_v (z^{-1/4} q; q^2),
\] (13)
\[
C_q(z) := q^{-3/8} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} z^{1/2} J^{(3)}_v (z^{-3/4} q; q^2).
\]
These functions satisfy
\[
\frac{\delta_q C_q(wz)}{\delta_q z} = -w S_q(wz), \quad \frac{\delta_q S_q(wz)}{\delta_q z} = w C_q(wz),
\]
(14)
see [11,13]. We denote to the derivative of \( S_q(z) \) by \( S'_q(z) \) and we assume that \( \{w_k : k \in \mathbb{N} \text{ with } w_1 < w_2 < w_3 < \ldots \} \) is the set of positive zeroes of \( S_q(z) \).

3. Existence Solutions of \( q \)-Differential System

In this section, we construct the solution of the \( q \)-differential system (51). Let \( C^2_q(A_q^*) \) denote the space of all continues functions with continuous \( q \)-derivatives up to order \( n-1 \) on \( A_q^* \) with values in \( \mathbb{R} \).

**Lemma 1.** Let \( f, \phi \in C^2_q(A_q^*) \). Then, the solution of the \( q \)-differential equation
\[
\frac{\delta^2_q f(z)}{\delta q^2 z} - \sqrt{q} \phi(qz) = 0,
\]
(15)
subject to the boundary conditions \( f(0) = f(1) = 0 \) is equivalent to the basic Fredholm \( q \)-integral equation
\[
f(z) = \int_0^1 \tilde{G}(z,qt) \phi(qt) \, dq t,
\]
(16)
where \( \tilde{G}(z,t) \) is the Green’s function defined on \( A_q^* \) by
\[
\tilde{G}(z,t) := \begin{cases} \sqrt{q}(t-1), & z < t; \\ \sqrt{q}(z-1), & t < z. \end{cases}
\]
(17)

**Proof.** The \( q \)-differential Equation (15) can be written as
\[
D^2_q f(z) - \sqrt{q} \phi(qz) = 0 \quad (z \in A_q^*). \]
(18)
By taking double \( q \)-integral for (18) and using (8), we obtain
\[
f(z) = c_0 + c_1 z + \sqrt{q} \int_0^z (z - qt) \phi(qt) \, dq t,
\]
(19)
where \( c_0 \) and \( c_1 \) are arbitrary constant. Using the boundary conditions, we get \( c_0 = 0 \) and
\[
c_1 = -\sqrt{q} \int_0^1 (1 - qt) \phi(qt) \, dq t.
\]
Substituting in (19), we have
\[
f(z) = -\sqrt{q} \int_0^1 (1 - qt) \phi(qt) \, dq t + \sqrt{q} \int_0^z (z - qt) \phi(qt) \, dq t,
\]
and then we obtain (16). \( \Box \)

**Remark 1.** By induction on \( n \), one can verify that if \( f, \phi \in C^{2n}_q(A_q^*) \), then the function
\[
f(z) = \int_0^1 \tilde{G}_n(z,qt) \phi(qt) \, dq t,
\]
(20)
is the solution of the q-boundary value problem

\[
\begin{cases}
\frac{d^n f(z)}{dz^n} = \phi(z), \\
\frac{d^k f(t)}{dt^k} = 0 & (k = 0, 1, \ldots, n - 1),
\end{cases}
\]

(21)

where \( \tilde{G}_1(z, t) \) is the Green’s function defined as in (17) and

\[
\tilde{G}_n(z, qt) = \int_0^1 \tilde{G}(z, qw) \tilde{G}_{n-1}(qw, qt) \, dqw
\]

(22)

Theorem 1. If \( f(z) \) and \( \phi(z) \) are functions of class \( \mathcal{C}^{2n}_{q^n}(A_q^n) \), then any solution of the system

\[
\begin{cases}
\frac{d^n f(z)}{dz^n} = \phi(z), \\
\frac{d^k f(t)}{dt^k} = a_k, \quad \frac{d^k f(1)}{dt^k} = b_k & (k = 0, 1, \ldots, n - 1)
\end{cases}
\]

(23)

is given by

\[
f(z) = a_0(z - 1) + \sum_{k=1}^{n-1} a_k \int_0^1 (qt - 1) \tilde{G}_k(z, qt) \, dq t + b_0 z
\]

(24)

\[+ \sum_{k=1}^{n-1} b_k \int_0^1 (qt) \tilde{G}_k(z, qt) \, dq t + \int_0^1 \tilde{G}_n(z, qt) \phi(qt) \, dq t,
\]

where the functions \( \tilde{G}_n(z, qt) \) \( (n \in \mathbb{N}) \) defined as in (17) and (22).

Proof. From (17), (22) and Equation (23) we get

\[
R_n(z) = \int_0^1 \tilde{G}_n(z, qt) \phi(qt) \, dq t
\]

(25)

\[
= \int_0^1 \tilde{G}_{n-1}(z, qw) \int_0^1 \tilde{G}(qw, qt) \frac{\partial^{2n} f(qt)}{\partial q^{2n}} \, dq t \, dq w
\]

\[
= \int_0^1 \tilde{G}_{n-1}(z, qw) \left[ \sqrt{q}(qw - 1) \int_0^{qw} (qt) \frac{\partial^{2n} f(qt)}{\partial q^{2n}} \, dq t \right]
\]

\[+ q \sqrt{qw} \int_0^{qw} (qt - 1) \frac{\partial^{2n} f(qt)}{\partial q^{2n}} \, dq t \, dq w.
\]

Using the rule (11), after some simplifications, we obtain

\[
R_n(z) = \frac{\partial^{2n-2} f(0)}{\partial q^{2n-2}} \int_0^1 (qw - 1) \tilde{G}_{n-1}(z, qw) \, dq w
\]

(26)

\[+ \frac{\partial^{2n-2} f(1)}{\partial q^{2n-2}} \int_0^1 (qw) \tilde{G}_{n-1}(z, qw) \, dq w + \int_0^1 \tilde{G}_{n-1}(z, qw) \frac{\partial^{2n-2} f(qw)}{\partial q^{2n-2}} \, dq w.
\]
Repeating the \( q \)-integration by parts on the last \( q \)-integral of Equation (26) \((n - 1)\) times, we get

\[
R_n(z) = \sum_{k=1}^{n-1} \frac{\delta^2 q f(0)}{\delta^2 q z^2k} \int_0^1 (qw - 1) \tilde{G}_k(z, qt) dq w - \frac{\delta^2 q f(1)}{\delta^2 q z^2k} \int_0^1 (qw) \tilde{G}_k(z, qw) dq w + \frac{\delta^2 f(qw)}{\delta^2 q z^2} dq w.
\]

(27)

Computing the last integral of (27), we get

\[
\int_0^1 \tilde{G}(z, qw) \frac{\delta^2 f(qw)}{\delta^2 q z^2} dq w = \sqrt{q}(1 - z) \int_0^z (-qw) \frac{\delta^2 f(qw)}{\delta^2 q z^2} dq w - \sqrt{q}z \int_1^1 (1 - qw) \frac{\delta^2 f(qw)}{\delta^2 q z^2} dq w
\]

\[
= a_0(1 - z) + b_0 z - f(z).
\]

Now, by substituting (28) in (27), we obtain the required result. \( \square \)

4. Certain \( q \)-Fourier Expansions

In this section, we consider the \( q \)-trigonometric functions \( C_q(z) \) and \( S_q(z) \) which are defined in (12). Our aim is to obtain the \( q \)-Fourier expansions of certain \( q \)-integral transforms involving the Green’s functions \( \tilde{G}_q(z, qt) \) defined in Section 3.

Recall that the \( q \)-Fourier series expansion for \( f(x) = 1 \) and \( g(x) = x \) are given [13,16] by

\[
1 = 2 \sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)} S_q(qw_k x), \quad x \in A_q^*,
\]

\[
x = - \frac{1}{q} \sum_{k=1}^{\infty} \frac{2}{w_k S_q'(w_k)} S_q(qw_k x),
\]

(29)

where \( \{ w_k : k \in \mathbb{N} \} \) is the set of positive zeroes of \( S_q(z) \).

**Lemma 2.** Let \( z \in A_q^* \). Then

\[
\int_0^1 \tilde{G}(z, qt) S_q(qw_k t) dq t = - \frac{1}{w_k^2} S_q(w_k z).
\]

**Proof.** From (17), we get

\[
\int_0^1 \tilde{G}(z, qt) S_q(qw_k t) dq t = \sqrt{q}(1 - z) \int_0^z (-qt) S_q(qw_k t) dq t - \sqrt{q}z \int_1^1 (1 - qt) S_q(qw_k t) dq t
\]

(30)

Using \( q \)-integration by parts (11), we obtain

\[
\int_0^z (-qt) S_q(qw_k t) dq t = \frac{z}{\sqrt{q} w_k} C_q(w_k z) - \frac{1}{\sqrt{q} w_k^2} S_q(w_k z),
\]

(31)

\[
\int_1^1 (1 - qt) S_q(qw_k t) dq t = \frac{(1 - z)}{\sqrt{q} w_k} C_q(w_k z) + \frac{1}{\sqrt{q} w_k^2} S_q(w_k z).
\]

(32)

Substituting from (31) and (32) into (30), we have the required result. \( \square \)
Lemma 3. For $z \in \mathbb{A}_q^*$, the following q-Fourier series expansion holds:

$$\int_0^1 \tilde{G}(z, qt) \, dq \, dt = - \sum_{k=1}^{\infty} \frac{L_k}{w_k^2} S_q(w_k z),$$

where

$$L_k := \frac{2 - 2C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}.$$

Proof. According to (29), we have

$$1 = 2 \sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)} S_q(qw_k t), \quad t \in \mathbb{A}_q^*.$$

Multiplying (34) by $\tilde{G}(z, qt)$, and integrating with respect to $t$ from zero to unity, we get

$$\int_0^1 \tilde{G}(z, qt) \, dq \, dt =$$

$$2 \sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)} \int_0^1 \tilde{G}(z, qt) S_q(w_k q t) \, dq \, dt.$$

By setting $L_k := \frac{2 - 2C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}$ and using Lemma 2, we obtain the result.

\[\square\]

Theorem 2. For $z \in \mathbb{A}_q^*$, the following q-Fourier series expansion holds:

$$\int_0^1 \tilde{G}_n(z, qt) \, dq \, dt = (-1)^n \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2n}} S_q(w_k z).$$

Proof. We prove the result by mathematical induction with respect to $n$. We first observe that for $n = 1$, the Formula (36) reduces to the formula in Lemma 3; that is, Equation (36) is true for $n = 1$.

Next, assume that (36) is true for some $n \geq 2$. Then

$$\int_0^1 \tilde{G}_{n+1}(z, qt) \, dq \, dt = \int_0^1 \int_0^1 \tilde{G}(z, qy) \tilde{G}_n(qy, qt) \, dq \, dy \, dt$$

$$= \int_0^1 \tilde{G}(z, qy) \left[ \int_0^1 \tilde{G}_n(qy, qt) \, dq \right] \, dy$$

$$= (-1)^n \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2n}} \int_0^1 \tilde{G}(z, qy) S_q(w_q q y) \, dq \, dy$$

$$= (-1)^n \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2n+1}} \left[ -1 \frac{1}{w_k} S_q(w_k z) \right]$$

$$= (-1)^{n+1} \sum_{k=1}^{\infty} \frac{L_k}{w_k^{2(n+1)}} S_q(w_k z).$$

\[\square\]

Lemma 4. For $z \in \mathbb{A}_q^*$, the following q-Fourier series expansion holds:

$$\int_0^1 (qt) \tilde{G}(z, qt) \, dq \, dt = 2 \sum_{k=1}^{\infty} \frac{1}{w_k^3 S_q'(w_k)} S_q(w_k z), \quad k \in \mathbb{N}. \quad (37)$$
Proof. Consider the function \( g(t) = t \). From (29), we have

\[
t = -\frac{1}{q} \sum_{k=1}^{\infty} \frac{2}{w_k S_q'(w_k)} S_q(qw_k t), \quad 0 < t < 1.
\]  

(38)

Hence, the proof can be performed by using (38) similar to the proof of Lemma 3.

Theorem 3. For \( z \in A_q^* \), the following \( q \)-Fourier series expansion holds:

\[
\int_0^1 (qt) \tilde{G}_n(z, qt) \, dt = (-1)^n \sum_{k=1}^{\infty} \frac{2}{w_k^{2n+1} S_q'(w_k)} S_q(w_k z).
\]

(39)

Proof. The proof can be performed by induction similar to the proof of Theorem 2. So, we omit it.

5. Fourier Series Expansions of the \( q \)-Lidstone Polynomials

The Fourier expansion of special polynomials has been studied by some mathematicians; see [17–20]. In this section, we consider the \( q \)-Lidstone polynomials \( \tilde{A}_n(z) \) and \( \tilde{B}_n(z) \) defined in (2). We define these polynomials by using the Green’s functions \( \tilde{G}_n(z, qt) \) defined in (17) and (22) and then, we introduce the \( q \)-Fourier Series Expansions for them.

We begin with the following result from [7]:

Lemma 5. For \( n \in \mathbb{N} \), the \( q \)-polynomials \( \tilde{A}_n(z) \) and \( \tilde{B}_n(z) \) satisfy the \( q \)-difference equations

\[
\frac{\delta^2 \tilde{A}_n(z)}{\delta q^2} = \tilde{A}_{n-1}(z) \quad \text{and} \quad \frac{\delta^2 \tilde{B}_n(z)}{\delta q^2} = \tilde{B}_{n-1}(z),
\]

with the boundary conditions \( \tilde{A}_0(0) = \tilde{A}_n(1) = 0 = \tilde{B}_n(0) = \tilde{B}_n(1) = 0. \) Moreover,

\[
\tilde{A}_0(z) = z, \quad \tilde{B}_0(z) = z - 1.
\]

We have the following:

Proposition 1. The \( q \)-Lidstone polynomials \( \tilde{A}_n \) and \( \tilde{B}_n \) can be expressed as \( \tilde{A}_0(z) = z, \tilde{B}_0(z) = z - 1 \), and for \( n \in \mathbb{N} \)

\[
\tilde{A}_n(z) = q \int_0^1 t \tilde{G}_n(z, qt) \, dt, \quad \tilde{B}_n(z) = \int_0^1 (qt - 1) \tilde{G}_n(z, qt) \, dt,
\]

where

\[
\tilde{G}(z, t) := \tilde{G}_1(z, t) = \begin{cases} \sqrt{q} (t - 1), & 0 \leq t < 1; \\ \sqrt{q} (z - 1), & 0 \leq z < 1. \end{cases}
\]

(42)

\[
\tilde{G}_n(z, qt) = \int_0^1 \tilde{G}(z, qw) \tilde{G}_{n-1}(qw, qt) \, dqw \quad (n = 2, 3, \ldots).
\]

Proof. We use the induction on \( n \). By Lemma 5, we have

\[
\begin{align*}
\frac{\delta^2 \tilde{A}_n(z)}{\delta q^2} &= \tilde{A}_{n-1}(z) \quad (n \in \mathbb{N}), \\
\tilde{A}_0(z) &= \tilde{A}_1(z) = 1.
\end{align*}
\]

(43)
So, if \( n = 1 \) we get the \( q \)-boundary value problem
\[
\begin{aligned}
\delta_q^2 \tilde{A}_1(z) &= z \quad (z \in A_q^*), \\
\tilde{A}_1(0) &= \tilde{A}_1(1) = 0.
\end{aligned}
\]
(44)

According to Lemma 1, we have the result.

Next, assume that (40) is true for \( n \geq 1 \). According to Remark (1), the solution \( \tilde{A}_{n+1}(z) \) of the \( q \)-boundary value problem
\[
\begin{aligned}
\delta_q^2 \tilde{A}_{n+1}(z) &= \tilde{A}_n(z), \\
\tilde{A}_{n+1}(0) &= \tilde{A}_{n+1}(1) = 0,
\end{aligned}
\]
(45)
is given by
\[
\tilde{A}_{n+1}(z) = \int_0^1 \tilde{G}(z, qw) \tilde{A}_n(qw) \, dqw = \int_0^1 \tilde{G}(z, qw) \left[ \int_0^1 qt \tilde{G}_n(qw, qt) \, dqw \right] \, dqt = \int_0^1 \tilde{G}(z, qt) \, dqt.
\]

Similarly, we can prove Equation (41). Finally, by induction on \( n \ (n \geq 2) \) again, it is easy to see that
\[
\tilde{G}_n(z, qt) = \int_0^1 \tilde{G}_{n-1}(z, qw) \tilde{G}(qw, qt) \, dqw.
\]

\( \square \)

The following result offers the explicit representation of the interpolating \( q \)-Lidstone polynomials and the associated error function \( R_n(z) \).

**Theorem 4.** Let \( 0 < q < 1 \) and \( f \in C_q^2(A_q^*) \). Then
\[
f(z) = \sum_{k=0}^{n-1} \left[ \frac{\delta_q^k f(1)}{\delta_q z^{2k}} \tilde{A}_k(z) + \frac{\delta_q^2 f(0)}{\delta_q z^{2k}} \tilde{B}_k(z) \right] + R_n(z),
\]
(46)

where
\[
R_n(z) = \int_0^1 \tilde{G}_n(z, qt) \frac{\delta_q^n f(qz)}{\delta_q z^{2n}} \, dqt.
\]

**Proof.** The proof follows immediately from Theorem 1 and Proposition 1, if we replace \( a_k, b_k \) and \( \phi(z) \) in Equation (24) by their values in terms of \( f(z) \) as given by the system (23). \( \square \)
Proposition 2. For \( z \in A_q^* \) and \( n \in \mathbb{N} \), the Fourier series for \( q \)-Lidstone polynomials \( \tilde{A}_n(z) \) and \( \tilde{B}_n(z) \) are given by

\[
\tilde{A}_n(z) = (-1)^n \sum_{k=1}^{\infty} \frac{2}{w_k^{n+1} S'_q(w_k)} S_q(w_k z),
\]

(47)

\[
\tilde{B}_n(z) = (-1)^n \sum_{k=1}^{\infty} \frac{2}{w_k^{n+1} S'_q(w_k) C_q(q^{1/2} w_k)} S_q(w_k z),
\]

(48)

where \( \{w_k : k \in \mathbb{N} \text{ with } w_1 < w_2 < w_3 < \ldots \} \) is the set of positive zeroes of \( S_q(z) \).

Proof. By using Equation (40) and Theorem 3 we get (47). Similarly, Equation (48) follows immediately from (41), (36) and (37).

We end this section by determining the asymptotic behavior of \( \tilde{A}_n(z) \) and \( \tilde{B}_n(z) \) for large \( n \).

Proposition 3. Let \( z \in A_q^* \). Then, there exist some constants \( K_q \) and \( L_q \) such that

\[
\left| (-1)^n \tilde{A}_n(z) - \frac{2 S_q(w_1 z)}{w_1^{n+1} S'_q(w_1)} \right| < \frac{K_q}{w_1^n},
\]

(49)

\[
\left| (-1)^n \tilde{B}_n(z) - \frac{2 S_q(w_1 z)}{w_1^{n+1} S'_q(w_1) C_q(q^{1/2} w_1)} \right| < \frac{L_q}{w_1^n},
\]

(50)

where \( w_1 \) is the smallest positive zero of \( S_q(z) \).

Proof. From Equation (47), we get

\[
\left| (-1)^n \tilde{A}_n(z) - \frac{2}{w_1^{n+1} S'_q(w_1)} S_q(w_1 z) \right| = \left| \sum_{k=2}^{\infty} \frac{2}{w_k^{n+1} S'_q(w_k)} S_q(w_k z) \right|
\]

Since the function \( S_q(\cdot) \) is bounded on \( A_q^* \), there exists a constant \( M > 0 \) such that

\[
\left| \sum_{k=2}^{\infty} \frac{2}{w_k^{n+1} S'_q(w_k)} S_q(w_k z) \right| < \frac{M}{w_2^{n+1} S'_q(w_2)} 
\]

\[
\left[ 1 + \left( \frac{w_3}{w_2} \right)^{2n+1} S'_q(w_2) S'_q(w_3) + \left( \frac{w_4}{w_3} \right)^{2n+1} S'_q(w_2) S'_q(w_3) + \ldots \right].
\]

Note that \( w_1 < w_2 < \ldots \), this implies the series in brackets tends to unity when \( n \to \infty \). Set \( K_q = \frac{M}{w_2^{n+1} S'_q(w_2)} \), we get (49). Inequality (50) can be proved in the same manner by using Equation (48).

6. Conclusions and Future Work

In this paper, we have introduced some definitions of the \( q \)-Lidstone polynomials which are \( q \)-Bernoulli polynomials generated by the third Jackson \( q \)-Bessel function, based on the Green’s function of the \( q \)-difference equation

\[
\begin{cases}
\phi^q_n f(z) = \phi(z), \\
\phi^q_n f(0) = a_k, \frac{\phi^q_k f(1)}{\phi^{q-1}_k} = b_k (k = 0, 1, \ldots, n - 1).
\end{cases}
\]

(51)

New results are obtained; particularly the \( q \)-Fourier series expansions of these functions.
Another study to give a characterization of those functions on the plane given by absolutely convergent of $q$-Lidstone series expansion (1), using the results in Section 5, is in progress.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The author is grateful to the referees for their valuable comments and remarks, which have improved the manuscript in its present form. Also, the author thanks Zeinab S. Mansour for many discussions and her interest in this work.

**Conflicts of Interest:** The author declares that she has no conflict of interest.

**References**

1. Lidstone, G. Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types. Proc. Edinb. Math. Soc. 1930, 2, 16–19. [CrossRef]

2. Whittaker, J. On Lidstone's series and two-point expansions of analytic functions. Proc. Lond. Math. Soc. 1934, 2, 451–469. [CrossRef]

3. Ismail, M.; Mansour, Z. $q$-analogs of Lidstone expansion theorem, two point Taylor expansion theorem, and Bernoulli polynomials. Anal. Appl. 2018, 17, 853–895. [CrossRef]

4. AL-Towailb, M. A generalization of the $q$-Lidstone series. J. AIMS Math. 2022, 7, 9339–9352. [CrossRef]

5. Mansour, Z.; AL-Towailb, M. $q$-Lidstone polynomials and existence results for $q$-boundary value problems. J. Bound Value Probl. 2017, 2017, 178. [CrossRef]

6. Mansour, Z.; AL-Towailb, M. The Complementary $q$-Lidstone Interpolating Polynomials and Applications. Math. Comput. Appl. 2020, 25, 34. [CrossRef]

7. Mansour, Z.; AL-Towailb, M. The $q$-Lidstone Series Involving $q$-Bernoulli and $q$-Euler Polynomials Generated by the Third Jackson $q$-Bessel Function. 2021, submitted to publication. Available online: https://arxiv.org/abs/2202.03340.pdf (accessed on 5 April 2022).

8. Andrei, L.; Caus, V.-A. Starlikeness of New General Differential Operators Associated with $q$-Bessel Functions. Symmetry 2021, 13, 2310. [CrossRef]

9. Samei, M.E.; Ghaffari, R.; Yao, S.-W.; Kaabar, M.K.A.; Martinez, F.; Inc, M. Existence of Solutions for a Singular Fractional $q$-Differential Equations under Riemann Liouville Integral Boundary Condition. Symmetry 2021, 13, 1235. [CrossRef]

10. Shokri, A. The symmetric P-stable hybrid Obrechkoff methods for the numerical solution of second order IVPs, TWMS. J. Pure Appl. Math. 2014, 5, 28–35.

11. Gasper, G.; Rahman, M. Basic Hypergeometric Series; Cambridge University Press: Cambridge, UK, 2004.

12. Jackson, F. On $q$-definite integrals. Quart. J. Pure Appl. Math. 1910, 41, 193–203.

13. Cardoso, J. Basic Fourier series: convergence on and outside the $q$-linear grid. J. Fourier Anal. Appl. 2011, 17, 96–114. [CrossRef]

14. Ismail, M. The zeros of basic Bessel functions, the functions $J_n + ax$ and associated orthogonal polynomials. J. Math. Anal. Appl. 1982, 86, 11–19. [CrossRef]

15. Swarttouw, R. The Hahn-Exton $q$-Bessel Function, Ph.D. Thesis, Technische Universiteit Delft, Delft, The Netherlands, 1992.

16. Bustoz, J.; Cardoso, J. Basic analog of Fourier series on a $q$-linear grid. J. Approx. Theory 2001, 112, 154–157. [CrossRef]

17. Araci, S.; Acikgoz, M. Construction of Fourier expansion of Apostol Frobenius Euler polynomials and its applications. Adv. Differ. Equ. 2018, 2017, 67. [CrossRef]

18. Bayad, A. Fourier expansions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Math. Comput. 2011, 80, 2219–2221. [CrossRef]

19. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. Fourier series of sums of products of Genocchi functions and their applications. J. Nonlinear Sci. Appl. 2017, 10, 1683–1694. [CrossRef]

20. Luo, Q.M. Fourier expansions and integral representations for the Apostol Bernoulli and Apostol Euler polynomials. Math. Comput. 2009, 78, 2193–2208. [CrossRef]