$\mathcal{N} = 1$ Supersymmetric $SU(2)^r$ Moose Theories∗

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Abstract
We study the quantum moduli spaces and dynamical superpotentials of four dimensional $SU(2)^r$ linear and ring moose theories with $\mathcal{N} = 1$ supersymmetry and link chiral superfields in the fundamental representation. Nontrivial quantum moduli spaces and dynamical superpotentials are produced. When the moduli space is perturbed by a generic tree level superpotential, the vacuum space becomes discrete. The ring moose is in the Coulomb phase and we find two singular submanifolds with a nontrivial modulus that is a function of all the independent gauge invariants needed to parameterize the quantum moduli space. The massive theory near these singularities confines. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose is produced.

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1 Introduction

There are good motivations to study four dimensional moose \[\text{[1]}\] (or quiver \[\text{[2]}\]) theories. On one hand, a class of these theories has been shown to give a description of extra dimensions \[\text{[3, 4]}\]. Consequently, extra dimensions can be naturally incorporated within a familiar setting of four dimensional gauge theories. What is amusing in this “deconstruction” of extra dimensions is that the extra dimensions could be generated with few number of nodes and links and the “size” between the nodes gives a UV completion of the four dimensional gauge theory. Furthermore, deconstruction has provided a framework for model building and investigating various issues such as electroweak symmetry breaking and accelerated grand unification \[\text{[5]}\]. On the other hand, from a different direction, the supersymmetric versions of similar moose diagrams appear in type IIA string theory with D6 branes wrapped on \(S^3\) of Calabi-Yau threefold of \(T^*S^3\) and also in type IIB string theory with D3, D5 and D7 branes wrapped over various cycles of Calabi-Yau threefold. See \[\text{[6]}\] for a recent discussion on this.

Moose diagrams contain nodes and links. Each node represents a gauge group and each link represents a matter field that transforms as some nontrivial representation of the gauge groups directly linked to it and as singlet under the rest. The original motivation for moose diagrams was to give a succinct graphical representation for encoding the transformations of fermions under various gauge (and global) symmetries. The transformation of a moose diagram into a description of extra dimensions occurs when the link fields develop vacuum expectation value (VEV) and “hop” between the nodes. It has been well know for sometime that four dimensional supersymmetric gauge theories have classical moduli space of vacua. If quantum fluctuations do not give rise to a non-vanishing dynamical superpotential, the quantum theory will have a quantum moduli space of vacua \[\text{[7]}\]. In fact, supersymmetric gauge theories have larger moduli spaces of vacua than non-supersymmetric theories. Therefore, supersymmetric moose theories could furnish a richer framework for model building based on the idea of deconstruction.

In this note we are interested in \(\mathcal{N}=1\) supersymmetric \(SU(2)^r\) linear and ring moose theories where the gauge group at each node is \(SU(2)\) and the links are chiral superfields that transform as fundamentals under the nearest gauge groups and as singlets under the rest. The linear moose will look like \(A_r\) Dynkin diagram with additional link fields at the ends. We will call a chiral superfield \(Q_i\) that links two nodes \(SU(2)_i\) and \(SU(2)_{i+1}\) internal and \(Q_i\) transforms as \((\Box, \Box)\) under \(SU(2)_i \times SU(2)_{i+1}\). We will call the superfields \(Q_0\) and \(Q_r\) at the ends of a linear moose external. The external link \(Q_0\) transforms as \(\Box\) under \(SU(2)_1\) and \(Q_r\) transforms as \(\Box\) under \(SU(2)_r\). The internal links are doublets carrying two \(SU(2)\) colors indices while the external links are each two doublets with \(SU(2)\) subflavor symmetry and they carry one color and one subflavor indices. The ring moose will look like affine \(\tilde{A}_r\) Dynkin diagram with all links carrying two color indices. Both the linear and the ring moose theories are asymptotically free and anomaly free.

We will obtain nontrivial quantum moduli space constraints and dynamical superpotentials starting from simple pure gauge theories of disconnected nodes by exploiting simple and efficient integrating in \[\text{[8, 9]}\] and out procedures. We will find that the linear moose with both external links present has a quantum moduli space of vacua. Explicit parameterization of the vacua in terms of the gauge invariant objects constructed out of the chiral superfields will be found. We will also
study the vacuum structure of this theory for a specific case of a moose with two nodes when perturbed by a tree level superpotential that includes a non-quadratic gauge singlet. We will find that this leads to a discrete vacuum space. The linear mooses without one or both external links have non-vanishing dynamical superpotentials and we will explicitly compute these superpotentials. A generic point in the moduli space of the ring moose has an unbroken $U(1)$ gauge symmetry and the ring moose is in the Coulomb phase. We will find two singular submanifolds with modulus that is a nontrivial function of all the independent gauge invariant objects needed to parameterize the moduli space of the ring moose. The massive theory near these singularities leads to confinement. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose will follow from our computation.

The Seiberg-Witten elliptic curve of the ring moose was computed in [11] using a different method where it was started with the curve for a ring with two nodes given in [10] and various asymptotic limits and symmetry arguments were used to obtain the curve for a ring moose with three nodes. The result was then generalized to the curve for a ring with arbitrary number of nodes. Here we will directly and explicitly compute the singularities of the quantum moduli space and the corresponding Seiberg-Witten elliptic curve for a ring moose with arbitrary number of nodes. The curve we obtain agrees with [10] for a ring moose with two nodes and with [11] for a ring moose with three nodes. We believe that the curve given in [11] is incorrect for ring mooses with four or more nodes.

2 Integrating in

In this section we will briefly summarize the integrating in procedure of [8, 9] in the context of the moose theories we will be studying. Consider $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $G = \prod_{i=1}^{r} SU(2)_i$ and matter chiral superfield $Q_i$ transforming as $\square$ under the gauge groups that are directly linked to it. The parameters we need to describe the dynamical superpotential of this theory are gauge singlet fields $X_j$ constructed out of $Q_i$ and the nonperturbative dynamical scale $\Lambda_i$ of each $SU(2)_i$. Let us denote this superpotential by $W_u(X_j, \Lambda_i)$. Now suppose we give mass $m_k$ to one of the chiral superfields $Q_k$. For energies below $m_k$, we integrate out $Q_k$. This can be achieved by integrating out all gauge singlets that contain $Q_k$. All gauge invariant objects that contain $Q_k$ will then be absent in the lower energy theory. If we denote those gauge singlets in $X_j$ that do not contain $Q_k$ by $Y_j$ and those that contain $Q_k$ by $Z_j$, then the dynamical superpotential of the lower energy theory can be written as $W_d(Y_j, \Lambda_{id})$, where $\Lambda_{id}$ is the nonperturbative dynamical scale of $SU(2)_i$ in the lower energy theory. The integrating in procedure takes us from $W_d$ to $W_u$.

First suppose we know $W_u(X_j, \Lambda_i)$. In order to compute $W_d(Y_j, \Lambda_{id})$, first we add the tree level superpotential

$$W_{\text{tree}} = \sum_j g_j Z_j$$  \hspace{1cm} (2.1)$$

to $W_u(X_j, \Lambda_i)$, where $g_j$ are coupling constants. One of the terms in $W_{\text{tree}}$ is $m_k M_k$, where $M_k = \det(Q_k)$ is a quadratic gauge singlet. This term gives mass $m_k$ to $Q_k$. In general, $Z_j$ also consists
of gauge singlets that are not quadratic in $Q_k$. Integrating out $Q_k$ in

$$W = W_u + W_{\text{tree}}$$

(2.2)

gives

$$W = W_d + W_{\text{tree,d}} + W_\Delta,$$

(2.3)

where

$$W_{\text{tree,d}} = W_{\text{tree}}|_{\langle Q_k \rangle}.$$  

(2.4)

We will see in Section (3) that $W_\Delta = 0$ in all the mooses we will be studying.

Suppose we know $W_d(X_j, \Lambda_{id})$ instead. Integrating in $Q_k$ is equivalent to making a Legendre transformation from $W_d(Y_j, \Lambda_{id})$ to $W_u(X_j, \Lambda_i)$. Matching the high energy and low energy scales $\Lambda_i$ and $\Lambda_{id}$ at $m_k$, $\Lambda_{id}$ can be expressed in terms of $\Lambda_i$ and $m_k$. Let us write $\Lambda_{id} = \Lambda_i(m_k)$. The higher energy dynamical superpotential $W_u(X_j, \Lambda_i)$ is then obtained by integrating out $g_j$ (which consists of $m_k$) in

$$W = W_d(\text{with } \Lambda_{id} \rightarrow \Lambda_i(m_k)) + W_{\text{tree,d}} - W_{\text{tree}}.$$  

(2.5)

3 Linear moose

In this section we study the quantum moduli space of a linear moose of $\mathcal{N} = 1$ supersymmetric $SU(2)^r$ gauge theory. An internal chiral superfield $Q_i$ links the $i^{th}$ and $(i+1)^{th}$ nodes. The internal link $Q_i$ transforms as $(\Box, \Box)$ under $SU(2)_i \times SU(2)_{i+1}$ and as singlet under all the other gauge groups. One of the external links $Q_0$ transforms as $\Box$ under $SU(2)_1$ and the second external link $Q_r$ transforms as $\Box$ under $SU(2)_r$. Each external link is two doublets with $SU(2)$ subflavor symmetry. We will compute the quantum moduli space of this theory starting from pure disconnected gauge groups and integrating in all the link fields. Gaugino condensation in the pure gauge theory gives a nonperturbative superpotential,

$$W = \sum_{i=1}^{r} 2\epsilon_i \Lambda_0^3,$$

(3.1)

where each $\epsilon_i = \pm 1$ labels the two vacua due to the breaking of the $Z_4$ $R$ symmetry to $Z_2$ and $\Lambda_0$ is the nonperturbative dynamical scale of $SU(2)_i$. We will no more use “d” and “u” subscripts in $W$ as it should be obvious in all cases. Our notation for the dynamical scales is $\Lambda_0$ for the scale of $SU(2)_i$ with no matter linked, $\Lambda_{id}$ when there is one link, and $\Lambda_i$ when there are two links attached. The scale $\Lambda_i$ is related to $\Lambda_0$ by threshold matching of the gauge coupling running at the masses $m_{i-1}$ and $m_i$ of $Q_{i-1}$ and $Q_i$ respectively,

$$\Lambda_0^6 = \Lambda_i^4 m_{i-1} m_i.$$  

(3.2)

1Quantum moduli space constraint relations for a linear moose with two and more nodes were first shown to us by Howard Georgi. Many results in this section overlap with results in [14].
In order to appreciate the power and simplicity of the integrating in procedure in producing quantum moduli space constraints and exact dynamical superpotentials, we will start with building up the chains shown in Figures 1(b) - 1(e). Later, we will directly and more formally compute the moduli space constraint for a general case of a linear moose with arbitrary number of nodes.

Figure 1: Linear moose with (a) One node and no link, (b) One node and one link, (c) One node and two links, (d) Two nodes and one internal and one external links, and (e) Two nodes and three links. The external links each have one color and one subflavor indices and each internal link has two color indices.

First let us integrate in \((Q_0)_{f \beta_0}\), where we have explicitly put the subflavor \(f = 1, 2\) and color \(\beta_0 = 1, 2\) indices of \(Q_0\), and build Figure 1(b) starting from Figure 1(a). We use indices \(\alpha_i\), \(\beta_i\) for color and indices \(f, g\) for subflavor. The integrating in procedure in this case is simple. There is only one gauge and flavor invariant mass term given by \(W_{\text{tree}} = m_0 M_0\), where \(M_0 = \frac{1}{2} (Q_0)_{f \beta_0} (Q_0)_{f' \beta_0'} \epsilon^{f' \beta_0'} \epsilon^{\beta_0} = \det (Q_0)\) and \(m_0\) is a constant. Threshold matching at energy \(m_0\) gives \(\Lambda^3_{01} = (m_0 \Lambda^5_{ld})^{1/2}\). To integrate in \(Q_0\), first we replace \(\Lambda^3_{01} \rightarrow (m_0 \Lambda^5_{ld})^{1/2}\) and subtract \(m_0 M_0\) in (3.1) for \(r = 1\),

\[
W = 2 \epsilon_1 (m_0 \Lambda^5_{ld})^{1/2} - m_0 M_0. \tag{3.3}
\]

We then minimize (3.3) with \(m_0\) to obtain

\[
W = \frac{\Lambda^5_{ld}}{M_0}. \tag{3.4}
\]

This is exactly the Affleck-Dine-Seiberg [12] superpotential of \(SU(2)\) with one flavor coming from a single instanton in the completely broken \(SU(2)_1\). We can go back to the pure gauge theory by integrating out \(M_0\) in

\[
W = \frac{\Lambda^5_{ld}}{M_0} + m_0 M_0 \tag{3.5}
\]

which gives the original superpotential (3.1) for \(r = 1\).
Next let us add a second external link $Q_1$ and build the moose diagram with one node and two links shown in Figure 1(c). In addition to $M_0$, there are five more gauge invariants given by $M_1 = \frac{1}{2}(Q_1)_{\alpha i}(Q_1)_{\beta j}e^{\alpha_j\beta_i}\epsilon^{ij} = \det(Q_1)$ and a $2 \times 2$ matrix $T$ with components $(T)_{fg} = (Q_0)_{f\beta}(Q_1)_{\alpha i}e^{\beta_\alpha i}\epsilon^j$. Now $T$ is a non-quadratic gauge singlet and we need to compute $W_{\text{tree},d}$. We can get to Figure 1(c) either from Figure 1(b) or directly from Figure 1(a). First let us go from Figure 1(b). The new gauge singlets are $M_1$ and $T$. In this case, $W_{\text{tree},d}$ is easily computed by integrating out $Q_1$ in $W_{\text{tree}} = \text{tr}(cT) + m_1 M_1$, where $c$ is a constant $2 \times 2$ matrix, which gives $W_{\text{tree},d} = -\det(c) M_0/m_1$. Let us now show that symmetries and asymptotic limits give $W_\Delta = 0$ in (2.3) for this example. $W_\Delta$ can only be a function of $M_0$, $\Lambda_{1d}$, $m_1$ and $c$. Moreover, there is a $U(1)_{Q_0} \times U(1)_{Q_1} \times U(1)_{R_1}$ symmetry and $M_0$, $\Lambda_{1d}$, $m_1$ and $c$ have the following $(Q_0, Q_1, R_1)$ charges: $M_0 : (2, 0, 0)$, $\Lambda_{1d}^5 : (2, 0, 2)$, $m_1 : (0, -2, 2)$ and $c : (-1, -1, 2)$. Since $W_\Delta$ must have charges $(0, 0, 2)$, the most general $W_\Delta$ is given by

\[ W_\Delta = -\frac{\det(c) M_0}{m_1} f(t), \quad \text{where} \quad t \equiv \frac{m_1 \Lambda_{1d}^5}{M_0^2 \det(c)} \quad (3.6) \]

and the argument $t$ has charge $(0, 0, 0)$. Since any dependence of $W_\Delta$ on $\Lambda_{1d}$ can only come from instantons in the completely broken $SU(2)_1$, we can expand $f(t)$ as $f(t) = \sum_{n=1}^{\infty} a_n t^n$. The case $n = 0$ would have simply reproduced $W_{\text{tree},d}$. On the other hand, $W_\Delta$ should obey the limits

\[ \lim_{\Lambda_{1d} \to 0} W_\Delta = 0, \quad \lim_{m_1 \to \infty} W_\Delta = 0. \quad (3.7) \]

That is because when $\Lambda_{1d} \to 0$, the quantum superpotential reduces to the classical superpotential with only $W_{\text{tree},d}$. Furthermore, in the limit $m_1 \to \infty$, $Q_1$ should completely decouple from the low energy superpotential except for its effect on the scale of the lower energy theory. It follows from (3.6), (3.7) and the above expansion of $f(t)$ that $W_\Delta = 0$. Similar arguments can be used to show that $W_\Delta = 0$ for all the moose diagrams we consider in this note, and we will not talk about $W_\Delta$ any further. We then minimize

\[ W = \frac{\Lambda_{1d}^5}{M_0} - \frac{\det(c) M_0}{m_1} - m_1 M_1 - \text{tr}(cT). \quad (3.8) \]

with $m_1$ and $c$ to obtain $W = 0$ and a moduli space of vacua with constraint

\[ \det T - M_0 M_1 + \Lambda_{1d}^4 = 0. \quad (3.9) \]

Now let us go directly from Figure 1(a) to 1(c). Which and how many chiral superfields do we need to integrate out of tr$(c T)$ to compute $W_{\text{tree},d}$ in this case? We will need to integrate out only one and either one of $Q_0$ or $Q_1$ will do the job. We can look at this by thinking in terms of building a linear moose chain that has both external links. Such a linear moose has one non-quadratic $2 \times 2$ matrix gauge singlet. This non-quadratic gauge singlet disappears if any one of the link fields is removed. If we think in terms of building the whole moose chain by putting in a link at a time, the need for $W_{\text{tree},d}$ arises only when we put in the last link where the non-quadratic gauge singlet comes in. For the current example, let us first choose to integrate out $Q_1$. This is done by minimizing the tree level superpotential tr$(c T) + m_1 M_1$ with $Q_1$ which gives $W_{\text{tree},d} = -\det(c) M_0/m_1$. In
fact, once we have added the $W_{\text{tree},d}$ we obtain in this way to the superpotential as in (2.5), we can integrate in all the independent gauge invariants at the same time. The superpotential we need for integrating in the two flavors is then

$$W = 2\epsilon_1 \left( m_0 m_1 \Lambda^4_1 \right)^{1/2} - \frac{\det (c) M_1}{m_1} - m_0 M_0 - m_1 M_1 - \text{tr} (c T).$$

(3.10)

Minimizing this with $m_0$, $m_1$ and $c$ gives $W = 0$ and (3.9). If we had chosen to compute $W_{\text{tree},d}$ by integrating out $Q_0$ in $\text{tr} (c T) + m_0 M_0$ instead, we would have obtained $W_{\text{tree},d} = -\det (c) M_1/m_0$ and (3.10) would become

$$W = 2\epsilon_1 \left( m_0 m_1 \Lambda^4_1 \right)^{1/2} - \frac{\det (c) M_1}{m_0} - m_0 M_0 - m_1 M_1 - \text{tr} (c T).$$

(3.11)

The result we obtain by minimizing (3.11) with $m_0$, $m_1$ and $c$ is again $W = 0$ and exactly (3.9). The lesson is that it does not matter which one chiral superfield we integrate out in computing $W_{\text{tree},d}$. However, we can integrate in the independent gauge singlet matter fields all at one time. We will do the same when we consider a general linear moose with arbitrary number of nodes later in this section and we will give an explicit proof that the final result does not depend on which particular chiral superfield we integrate out in computing $W_{\text{tree},d}$.

For Figure 1(d), we have $Q_0 \sim (\square, 1)$ and $Q_1 \sim (\square, \square)$ under the $SU(2)_1 \times SU(2)_2$ gauge symmetry. The gauge singlets are $M_0$, $M_1$ and $\det (Q_0 Q_1)$. As we will explain later when we discuss a general linear moose with arbitrary number of nodes, the superpotential can be completely expressed in terms of the gauge singlets $M_0$ and $M_1$. The superpotential is then obtained by minimizing

$$W = 2\epsilon_1 \left( m_0 m_1 \Lambda^4_1 \right)^{1/2} + 2\epsilon_2 \left( m_1 \Lambda_5^{(2)} \right)^{1/2} - m_0 M_0 - m_1 M_1$$

(3.12)

with $m_0$ and $m_1$ which gives

$$W = \frac{\Lambda_5^{(2)} M_0}{M_0 M_1 - \Lambda^4_1}.$$

(3.13)

Using the constraint we obtained in (3.9) for the moduli space of $SU(2)_1$ with two flavors, this can be rewritten as $W = \Lambda_5^{(2)} M_0 /\det (Q_0 Q_1)$. Note that (3.13) contains a single instanton contribution from $SU(2)_2$ and an infinite series of multi-instanton contributions from $SU(2)_1$ as it can be seen by making a Taylor expansion of $1/(1 - \Lambda^4_1/(M_0 M_1))$ in powers of $\Lambda^4_1/(M_0 M_1)$.

Next let us consider Figure 1(e). The gauge singlets are $M_0$, $M_1$, $M_2$, $\det (Q_0 Q_1)$, $\det (Q_1 Q_2)$ and the $2 \times 2$ matrix $T = Q_0 Q_1 Q_2$. As we will explain later, the moduli space constraint we are looking for can be parameterized by $M_0$, $M_1$, $M_2$, and $T$. Now $T$ is non-quadratic and we can compute $W_{\text{tree},d}$ by minimizing $\text{tr} (c T) + m_2 M_2$ with $Q_2$ which gives $W_{\text{tree},d} = -\det (c) (M_0 M_1 - \Lambda^4_1)/m_2$.

Integrating out $m_0$, $m_1$, $m_2$ and $c$ in

$$W = 2\epsilon_1 \left( m_0 m_1 \Lambda^4_1 \right)^{1/2} + 2\epsilon_2 \left( m_1 m_2 \Lambda^4_2 \right)^{1/2} - \frac{\det (c)}{m_2} (M_0 M_1 - \Lambda^4_1) - m_0 M_0 - m_1 M_1 - m_2 M_2 - \text{tr} (c T)$$

(3.14)

gives $W = 0$ and a quantum moduli space constrained by

$$\det T - M_0 M_1 M_2 + \Lambda^4_1 M_2 + \Lambda^4_2 M_0 = 0.$$

(3.15)
Finally, let us consider the general case of a linear moose with $r$ nodes and $r + 1$ links shown in Figure 2. There are $r + 1$ link chiral superfields each with four complex degrees of freedom. We can construct a total of $\frac{1}{2}(r^2 + 3r + 8)$ gauge singlets given by determinants of products of one to $r$ consecutive link superfields, and the product of all the chiral superfields:

$$\det(Q_i),$$

$$\det(Q_i Q_{i+1}), \ldots, \det(Q_0 Q_1 \cdots Q_{r-1}), \det(Q_1 Q_2 \cdots Q_r),$$

and

$$Q_0 Q_1 \cdots Q_r.$$  

(3.16)

(3.17)

(3.18)

For a generic linear moose, the gauge symmetry is completely broken and $3r$ of the complex degrees of freedom become massive or are eaten by the super Higgs mechanism. Consequently, there are only $4(r + 1) - 3r = r + 4$ massless complex degrees of freedom left. Because we have $\frac{1}{2}(r^2 + 3r + 8)$ gauge singlets, there must be $\frac{1}{2}(r^2 + 3r + 8) - (r + 4) = r(r + 1)/2$ constraints. We claim that a constraint involving the determinants of only subsegments of the moose chain given in (3.17) are not modified by the extra links and nodes. We can see that as follows: Consider the determinant of a subsegment $\det(Q_i Q_{i+1} \cdots Q_j)$. The color indices from the gauge groups $SU(2)_i$ and $SU(2)_j$ are not contracted with the colors of $SU(2)_{i-1}$ and $SU(2)_{j+1}$ respectively. Consequently, these adjoining gauge groups behave like global subflavor symmetries. This amounts to saying that as far as $\det(Q_i Q_{i+1} \cdots Q_j)$ is concerned, the moose chain is cut off at the $(i - 1)^{th}$ and $(j + 1)^{th}$ nodes. Therefore, finding a constraint for $\det(Q_i Q_{i+1} \cdots Q_j)$ is not an independent problem. Thus all the $r(r + 1)/2$ moduli space constraints can be easily deduced from the one constraint which can be parameterized by the $r + 5$ independent set of gauge singlets:

$$M_i \equiv \frac{1}{2}(Q_i)_{\alpha_i \beta_i} (Q_i)_{\alpha'_i \beta'_i} \epsilon^{\alpha_i \alpha'_i} \epsilon^{\beta_i \beta'_i} = \det(Q_i)$$

(3.19)

and

$$T_{fg} \equiv \frac{1}{2}(Q_0)_{f \beta_0} (Q_1)_{\alpha_1 \beta_1} (Q_2)_{\alpha_2 \beta_2} \cdots (Q_r)_{\alpha_r g} \epsilon^{\beta_0 \alpha_1} \epsilon^{\beta_1 \alpha_2} \cdots \epsilon^{\beta_{r-1} \alpha_r}.$$  

(3.20)

where $\alpha_i$, $\beta_i$ are color indices and $f$, $g$ are subflavor indices. For $M_0$ and $M_r$ one of the indices in $Q_0$ and $Q_r$ is for subflavor.

Now, as we have discussed in detail earlier in this section when we considered the $r = 1$ linear moose with two links, we can compute $W_{\text{tree,d}}$ by integrating out $Q_k$ in the gauge and flavor invariant tree level superpotential

$$\text{tr}(cT) + m_k M_k,$$

(3.21)
where $c$ is a constant $2 \times 2$ matrix and $k$ can take any one value from 0 to $r$. This gives

$$W_{\text{tree},d} = -\frac{\det(c)}{m_k} \det(Q_0 Q_1 \ldots Q_{k-1}) \det(Q_{k+1} Q_{k+2} \ldots Q_r).$$  

(3.22)

We will see that the final result on the moduli space constraint does not depend on $k$. Note that (3.22) contains two determinants which can be completely expressed in terms of $M$’s and $\Lambda$’s. For simplicity of notation, we introduce a more general way of representing consecutive products of link chiral superfields and define

$$T_{(i,j)} \equiv Q_i Q_{i+1} \ldots Q_j.$$  

(3.23)

Note that $T_{(i,j)}$ is a $2 \times 2$ matrix with hidden indices. The tree level superpotential that contains all the independent gauge invariants is

$$W_{\text{tree}} = \text{tr}(c T_{(0,r)}) + \sum_{i=0}^{r} m_i M_i.$$  

(3.24)

The superpotential we need for integrating in all the matter superfields starting from a pure gauge theory of disconnected nodes is then obtained by using (3.1), (3.2), (3.22) and (3.24) in (2.5),

$$W = 2 \sum_{i=1}^{r} \epsilon_i (\Lambda_i^4 m_{i-1} m_i)^{\frac{1}{2}} - \frac{\det(c)}{m_k} \det T_{(0,k-1)} \det T_{(k+1,r)} - \text{tr}(c T_{(0,r)}) - \sum_{i=0}^{r} m_i M_i.$$  

(3.25)

Integrating out $m_i$ and $c$ in (3.25) gives

$$\epsilon_1 \left( \frac{\Lambda_1^4 m_0}{m_0} \right)^{\frac{1}{2}} - M_0 = 0,$$

$$\epsilon_1 \left( \frac{\Lambda_1^4 m_0}{m_1} \right)^{\frac{1}{2}} + \epsilon_2 \left( \frac{\Lambda_2^4 m_1}{m_1} \right)^{\frac{1}{2}} - M_1 = 0,$$

$$\epsilon_{k-1} \left( \frac{\Lambda_{k-1}^4 m_{k-2}}{m_{k-1}} \right)^{\frac{1}{2}} + \epsilon_k \left( \frac{\Lambda_k^4 m_k}{m_{k-1}} \right)^{\frac{1}{2}} - M_{k-1} = 0,$$

$$\epsilon_{k} \left( \frac{\Lambda_k^4 m_{k-1}}{m_k} \right)^{\frac{1}{2}} + \epsilon_{k+1} \left( \frac{\Lambda_{k+1}^4 m_{k+1}}{m_k} \right)^{\frac{1}{2}} - \frac{\det(c)}{m_k^2} \det T_{(0,k-1)} \det T_{(k+1,r)} - M_k = 0,$$

$$\epsilon_{k+1} \left( \frac{\Lambda_{k+1}^4 m_{k+1}}{m_{k+1}} \right)^{\frac{1}{2}} + \epsilon_{k+2} \left( \frac{\Lambda_{k+2}^4 m_{k+2}}{m_{k+1}} \right)^{\frac{1}{2}} - M_{k+1} = 0,$$

$$\epsilon_{r} \left( \frac{\Lambda_r^4 m_{r-1}}{m_r} \right)^{\frac{1}{2}} - M_r = 0,$$
\[ T_{(0,r)} + \frac{\det(c)}{m_k} c^{-1} \det T_{(0,k-1)} \det T_{(k+1,r)} = 0. \quad (3.26) \]

Recursively solving for \( m_i \) and \( c \), and putting into (3.25) gives \( W = 0 \) and a quantum moduli space constrained by

\[ \det T_{(0,r)} - \frac{\det T_{(0,k-1)} \det T_{(k+1,r)}}{\Omega_{(0,k-1)} \Omega_{(k+1,r)}} \Omega_{(0,r)} = 0, \quad (3.27) \]

where we have introduced functions \( \Omega_{(i,j)} \) to simplify our notation. The \( \Omega \) functions are defined by

\[
\Omega_{(i,j)} \equiv \prod_{q=i}^{j} M_q - \sum_{p=i+1}^{j} \left( \Lambda_p^4 \prod_{q \neq p-1,p} M_q \right) + \sum_{p=i+1}^{j-2} \sum_{l=0}^{j-p-2} \left( \Lambda_{p+l+2}^4 \prod_{q \neq p-1,p,p+l+1,p+l+2} M_q \right)
- \cdots + (-1)^{(j-i)/2} \prod_{p=1}^{(j-i+1)/2} \Lambda_{i+2p-1}^4,
\]

if \( j - i \) is odd, and

\[
\Omega_{(i,j)} \equiv \prod_{q=i}^{j} M_q - \sum_{p=i+1}^{j} \left( \Lambda_p^4 \prod_{q \neq p-1,p} M_q \right) + \sum_{p=i+1}^{j-2} \sum_{l=0}^{j-p-2} \left( \Lambda_{p+l+2}^4 \prod_{q \neq p-1,p,p+l+1,p+l+2} M_q \right)
- \cdots + (-1)^{(j-i)/2} \sum_{q=0}^{(j-i)/2} \left( M_{i+2q} \prod_{p=0}^{q-1} \Lambda_{i+2q-2p-1}^4 \prod_{l=1}^{(j-i)/2-2q} \Lambda_{i+2q+2l}^4 \right), \quad (3.29)
\]

if \( j - i \) is even. We take \( j > i \) unless explicitly stated. When \( i = j \), we have \( \Omega_{(i,i)} = \det M_i \). The first few \( \Omega \) functions are given in Appendix A and some important recursion relations are given in Appendix B.

Thus the quantum moduli space is constrained by the recursion relations given by (3.27). Note that \( k \) in (3.27) is arbitrary and could take any value from 0 to \( r \). As we have argued earlier in this section, a similar relation as (3.27) should hold for a subset of the linear chain, and we write a more general form of the moduli space constraints as

\[ \det T_{(i,j)} = \frac{\det T_{(i,k-1)} \det T_{(k+1,j)}}{\Omega_{(i,k-1)} \Omega_{(k+1,j)}} \Omega_{(i,j)} = 0. \quad (3.30) \]

Now we can easily prove that the result (3.30) is independent of \( k \), since we can repeatedly use the same recursion relations to simplify the fractional factor in the second term, and (3.30) gives

\[ \det T_{(i,j)} - \Omega_{(i,j)} = 0. \quad (3.31) \]

Note that (3.31) gives \( r(r+1)/2 \) constraints that completely remove all the redundancy in the set of gauge singlets.
4 Integrating out link fields

In this section we will see that the quantum moduli space constraints of the linear moose we found in Section 3 give correct and known dynamical superpotentials when we integrate out some link chiral superfields. We will consider only the cases of \( r = 1 \) and \( r = 2 \), since we can compare the results with established dynamical superpotentials in these cases. The low energy superpotentials we will obtain after integrating out the link fields are correct and consistent with the results in Section 3 and [8]. We will integrate out the external links in a linear moose with arbitrary number of nodes in Section 6.

First let us consider the \( r = 1 \) linear moose shown in Figure 1(c). This theory has a quantum moduli space of vacua given by (3.9). The superpotential can be written as

\[
W = A(\det T_{(0,1)} - M_0M_1 + \Lambda_1^4),
\]

where \( A \) is a Lagrange multiplier. Integrating out \( A \) in (4.1) simply gives the constraint (3.9). We integrate out \( Q_1 \) by minimizing

\[
W = A(\det T_{(0,1)} - M_0M_1 + \Lambda_1^4) + m_1M_1
\]

with \( M_1, T_{(0,1)} \) and \( A \) to obtain

\[
-A M_0 + m_1 = 0, \quad AT_{(0,1)}^{-1} \det T_{(0,1)} = 0, \quad \det T_{(0,1)} - M_0M_1 + \Lambda_1^4 = 0
\]

with solution

\[
M_1 = \frac{\Lambda_1^4}{M_0}, \quad \det T_{(0,1)} = 0, \quad A = \frac{m_1}{M_0}.
\]

Putting (4.3) in (4.2) gives exactly the superpotential of a single node with one link given in (3.4).

If we choose to integrate out both \( Q_0 \) and \( Q_1 \) at the same time, we minimize

\[
W = A(\det T_{(0,1)} - M_0M_1 + \Lambda_1^4) + M_0M_0 + m_1M_1
\]

with \( M_0, M_1, T_{(0,1)} \) and \( A \) to obtain the same equations as in (4.3) and

\[
-A M_1 + m_0 = 0.
\]

There are two sets of solutions given by

\[
M_0 = \epsilon \left( \frac{m_1\Lambda_1^4}{m_0} \right)^{1/2}, \quad M_1 = \epsilon \left( \frac{m_0\Lambda_1^4}{m_1} \right)^{1/2}, \quad \det T_{(0,1)} = 0, \quad A = \epsilon \left( \frac{m_0m_1}{\Lambda_1^4} \right)^{1/2},
\]

where \( \epsilon = \pm 1 \). Putting (4.7) in (4.5) gives exactly (3.1) for \( r = 1 \).

Next let us consider the \( r = 2 \) linear moose with external links shown in Figure 1(e). First let us integrate out \( Q_2 \). This is done by minimizing

\[
W = A(\det T_{(0,2)} - M_0M_1M_2 + \Lambda_1^4M_2 + \Lambda_2^4M_0) + m_2M_2
\]
with $M_2$, $T_{(0,2)}$, and $A$, which gives exactly (3.13). Note also that the superpotential (3.13) vanishes if we set $\Lambda_{2d} \equiv 0$; and the theory with one node of $SU(2)_1$ linked to $Q_0$ and $Q_1$ has a quantum moduli space as expected and seen in (3.9). On the other hand, if we set $\Lambda_1 \equiv 0$ in (3.13), we obtain $W = \Lambda_{2d}^5/M_1$ which is exactly the superpotential of $SU(2)_2$ with a single flavor. We can further integrate out $Q_0$ and obtain a moose diagram with two nodes and an internal link. This is done by minimizing

$$
\frac{\Lambda_{2d}^5M_0}{M_0M_1 - \Lambda_1} + m_0M_0
$$

(4.9)

with $M_0$ which gives

$$
W = \frac{\Lambda_{id}^5}{M_1} + \frac{\Lambda_{2d}^5}{M_1} \pm 2\left(\frac{\Lambda_{id}^5\Lambda_{2d}^5}{M_1}\right)^{1/2},
$$

(4.10)

where $\Lambda_{id}^5 = \Lambda_1^4m_0$. Note that in this case the original $SU(2)_1 \times SU(2)_2$ gauge symmetry is broken by $M_1$ into a diagonal $SU(2)_D$. The first term in (4.10) comes from a single instanton contribution in the completely broken $SU(2)_1$, the second term also comes from a single instanton in the completely broken $SU(2)_2$, and the last term comes from gaugino condensation in the unbroken $SU(2)_D$. By threshold matching the gauge couplings at energy $M_1^{1/2}$, $\Lambda_D^6/M_1^3 = e^{-8\pi^2/g_D^2} = e^{-8\pi^2(g_1^{-2} + g_2^{-2})} = \Lambda_{id}^5\Lambda_{2d}^5/M_1^5$, where we used $g_D^{-2} = g_1^{-2} + g_2^{-2}$ for the gauge coupling constants, we see that the scale of the low energy $SU(2)_D$ is $\Lambda_D = [(\Lambda_{id}^5\Lambda_{2d}^5)^{1/2}/M_1]^{1/3}$.

We can integrate out $Q_1$ instead of $Q_2$ by minimizing

$$
W = A(\det T_{(0,2)} - M_0M_1M_2 + \Lambda_1^4M_2 + \Lambda_2^4M_0) + m_1M_1
$$

(4.11)

with $M_1$, $T_{(0,2)}$, and $A$, which gives

$$
W = \frac{\Lambda_{id}^5}{M_0} + \frac{\Lambda_{2d}^5}{M_2},
$$

(4.12)

where $\Lambda_{id} = (\Lambda_1^4m_1)^{1/5}$ are the scales for the low energy theory. (4.12) is exactly the superpotential for two disconnected gauge groups with a single flavor attached to each. We can further integrate out the two remaining fields $M_0$ and $M_2$ by adding $m_0M_0 + m_2M_2$ to (4.12) and minimizing with $M_0$ and $M_2$. This gives

$$
W = 2\epsilon_1\Lambda_{01}^3 + 2\epsilon_2\Lambda_{02}^3,
$$

(4.13)

where $\Lambda_{01} = (\Lambda_{id}^5m_0)^{1/6}$ and $\Lambda_{02} = (\Lambda_{2d}^5m_2)^{1/6}$ are the scales of the pure $SU(2)_1$ and $SU(2)_2$ gauge theories respectively.

We can, if we wish, integrate out all the matter fields at the same time by minimizing the superpotential

$$
W = A(\det T_{(0,2)} - M_0M_1M_2 + \Lambda_1^4M_2 + \Lambda_2^4M_0) + m_0M_0 + m_1M_1 + m_2M_2
$$

(4.14)

with $M_0$, $M_1$, $M_2$, $T_{(0,2)}$, and $A$. We obtain four sets of solutions

$$
M_0 = \epsilon_1\left(\frac{\Lambda_1^4m_1}{m_0}\right)^{1/6}, \quad M_2 = \epsilon_2\left(\frac{\Lambda_2^4m_2}{m_2}\right)^{1/6},
$$

(4.14)
\[ M_1 = \epsilon_1 \left( \frac{\Lambda_1^4 m_0}{m_1} \right)^{\frac{1}{2}} + \epsilon_2 \left( \frac{\Lambda_2^4 m_2}{m_1} \right)^{\frac{1}{2}}, \quad \det T_{(0,2)} = 0, \quad A = \epsilon_1 \epsilon_2 \left( \frac{m_0 m_2}{\Lambda_1^4 \Lambda_2^4} \right)^{\frac{1}{2}}. \] (4.15)

Putting (4.15) in (4.14) gives exactly (4.13) with \( \Lambda_{01} = (\Lambda_1^4 m_0 m_1)^{1/6} \) and \( \Lambda_{02} = (\Lambda_2^4 m_1 m_2)^{1/6} \). Thus we have consistently reproduced the pure low energy dynamical superpotential.

5 Tree level perturbations

In this section we will study the vacuum structure of the \( r = 2 \) linear moose shown in Figure 1(e) when perturbed by the tree level superpotential

\[ W_{\text{tree}} = m_0 M_0 + m_1 M_1 + m_2 M_2 + \text{tr} \left( c T_{(0,2)} \right), \] (5.1)

which includes a non-zero coupling to the non-quadratic gauge singlet \( T_{(0,2)} \). The lesson we will learn is that the inclusion of the non-quadratic gauge singlet term in \( W_{\text{tree}} \) leads to discrete vacua and also the math becomes complicated. We will explicitly compute the discrete vacua.

Semi-classically, there are two vacuum states. One is at the origin,

\[ M_0 = M_1 = M_2 = T_{(0,2)} = 0, \] (5.2)

where the original \( SU(2)_1 \times SU(2)_2 \) gauge symmetry is preserved. The second vacuum is at

\[ M_0 = \frac{m_1 m_2}{\det (c)}, \quad M_1 = \frac{m_0 m_2}{\det (c)}, \quad M_2 = \frac{m_0 m_1}{\det (c)}, \quad T_{(0,2)} = \frac{m_0 m_1 m_2}{\det (c)} c^{-1}. \] (5.3)

where the gauge symmetry is completely broken.

In the quantum theory, the vacuum structure is much richer. The vacuum expectation values in the quantum theory perturbed by (5.1) are obtained by minimizing the superpotential

\[ W = A (\det T_{(0,2)} - M_0 M_1 M_2 + \Lambda_1^4 M_2 + \Lambda_2^4 M_0) + m_0 M_0 + m_1 M_1 + m_2 M_2 + \text{tr} \left( c T_{(0,2)} \right) \] (5.4)

with \( M_0, M_1, M_2, T_{(0,2)} \) and \( A \). The solution is given in Appendix C. All we need for our discussion here is that the expectation values of \( M_0, M_1, M_2, T_{(0,2)} \) and \( A \) can be parameterized by \( x \) such that

\[
\begin{align*}
\Lambda_1^8 \Lambda_2^8 m_1 x^5 & - 2 \Lambda_1^4 \Lambda_2^4 m_0 m_1 m_2 x^3 - \Lambda_1^4 \Lambda_2^4 \det (c) x^2 \\
& - (\Lambda_1^4 m_0 \det (c)^2 + \Lambda_2^4 m_2 \det (c)^2 - m_0 m_1 m_2^2) x - m_0 m_2 \det (c) = 0.
\end{align*}
\] (5.5)

Note how messy the solution given in Appendix C is even for the case of only two nodes. The complication comes because of the presence of \( \text{tr} \left( c T_{(0,2)} \right) \) in \( W_{\text{tree}} \). There are in general five solutions to \( x \) which give five sets of expectation values with non-zero \( \det (T_{(0,2)}) \) and the vacuum space becomes discrete in the perturbed theory. Let us simplify and interpret the expectation values in some limits of the coupling constants. If the mass \( m_1 \) is set to zero, there are only two solutions to (5.5) given by

\[ x = \left( -\frac{m_0}{\Lambda_2^4}, -\frac{m_2}{\Lambda_1^4} \right). \] (5.6)
which give, using Appendix C two vacua at

\[ M_0 = \left(-\frac{\det(c)\Lambda_1^4}{m_0^2}, 0\right), \quad M_1 = \left(\frac{m_0m_2 - m_0^2\Lambda_1^4/\Lambda_2^4}{\det(c)}, \frac{m_0m_2 - m_0^2\Lambda_2^4/\Lambda_1^4}{\det(c)}\right), \]

\[ M_2 = (0, -\frac{\det(c)\Lambda_1^4}{m_2^2}), \quad T_{(0,2)} = \left(\frac{\det(c)\Lambda_2^4}{m_0c^{-1}}, \frac{\det(c)\Lambda_1^4}{m_2c^{-1}}\right). \]  

(5.7)

Note that for large \( m_0 \) and \( m_2 \) in (5.7), the expectation values of \( M_0, M_2 \) and \( T_{(0,2)} \) vanish and the links \( Q_0 \) and \( Q_2 \) are missing in the low energy theory. In this case, we have only the internal link \( Q_1 \). The \( SU(2)_1 \times SU(2)_2 \) gauge symmetry is then broken by \( M_1 \) down to a diagonal \( SU(2)_D \). Gaugino condensation in \( SU(2)_D \) breaks the \( Z_4 \) \( R \) symmetry to \( Z_2 \).

If we set \( m_0 = m_2 = 0 \) in (5.4), then

\[ x = (0, 0, \frac{\det c^2}{m_1\Lambda_1^4\Lambda_2^4})^{1/2}, \quad e^{i\pi/3}(\frac{\det c^2}{m_1\Lambda_1^4\Lambda_2^4})^{1/2}, \quad e^{i2\pi/3}(\frac{\det c^2}{m_1\Lambda_1^4\Lambda_2^4})^{1/2} \]  

and there are four distinct vacua. In this case, \( SU(2)_1 \times SU(2)_2 \) gauge symmetry is completely broken. For large \( m_1 \), we have \( M_1 = 0 \) and there are two disconnected \( SU(2) \)'s with a single link attached to each in the low energy theory.

If we set \( \det(c) = 0 \) in (5.5), we obtain

\[ x = (0, -(\frac{m_0m_2}{\Lambda_1^4\Lambda_2^4})^{1/2}, -(\frac{m_0m_2}{\Lambda_1^4\Lambda_2^4})^{1/2}, (\frac{m_0m_2}{\Lambda_1^4\Lambda_2^4})^{1/2}, (\frac{m_0m_2}{\Lambda_1^4\Lambda_2^4})^{1/2}). \]  

(5.9)

In this case, all the links are absent in the low energy theory and the \( Z_4 \times Z_4 \) \( R \) symmetry is broken down to \( Z_2 \times Z_2 \) by gaugino condensation in the two nodes. The vacua at \( x = \pm(m_0m_2/(\Lambda_1^4\Lambda_2^4))^{1/2} \) are near the origin and they correspond to the semi-classical vacuum at the origin. The vacuum state at \( x = 0 \) is far out in moduli space and it corresponds to the second semi-classical vacuum.

6 Ring moose

Now we can construct the quantum moduli space of the ring moose shown in Figure 4 starting from the linear moose shown in Figure 2. First we list \( r^2 + 1 \) gauge singlets in the ring moose given by \( M_i \) defined in (3.10), where \( 0 \leq i \leq r - 1 \) now,

\[ U_{(0,r-1)} \equiv \frac{1}{2}(Q_0)_{\alpha_0\beta_0}(Q_1)_{\alpha_1\beta_1}(Q_2)_{\alpha_2\beta_2}\cdots(Q_{r-1})_{\alpha_{r-1}\beta_{r-1}}e^{\beta_{r-1}\alpha_0}e^{\beta_1\alpha_2}\cdots e^{\beta_{r-1}\alpha_0}, \]  

(6.1)

and

\[ \det(Q_iQ_{i+1}), \quad \det(Q_iQ_{i+1}Q_{i+2}), \quad \cdots, \quad \det(Q_iQ_{i+1}\cdots Q_{i-1}). \]  

(6.2)

We identify \( i \sim i + r \). We have already found the constraints that relate the determinants listed in (6.2) to \( M_i \) and \( \Lambda_j \) in Section 3. Therefore, we only need to find the one constraint that relates \( U_{(0,r-1)} \) to \( M_i \) and \( \Lambda_j \). In fact, one can check that there are only \( r + 1 \) independent gauge invariant as follows: The link chiral superfields have a total of \( 4r \) complex components. On the other hand,
there are $3r$ D-flatness conditions and only $3r - 1$ of these conditions are independent because of the unbroken $U(1)$ gauge symmetry. Thus there must be $4r - (3r - 1) = r + 1$ independent complex degrees of freedom which we can choose as $U_{(0, r - 1)}$ and $M_i$.

We will start with the moduli space of the linear moose with external links found in Section 3. We will then integrate out the external links and construct the superpotential for the moose with only internal links shown in Figure 3(b). Finally, a link field that transforms as $(\Box, \Box)$ under $SU(2)_r \times SU(2)_1$ will be integrated in to build the ring moose shown in Figure 4. Since we can not compute the superpotential with only one external link, let us first integrate out $Q_r$ and obtain the superpotential for Figure 3(a). This is done by integrating out $M_r, T_{(0, r)}$ and $A$ in

$$
W = A \left( \det T_{(0, r)} - \Omega_{(0, r)} \right) + m_r M_r. 
$$

(6.3)

As shown in Appendix D, the resulting superpotential is

$$
W = \frac{\Lambda^5_{rd} \Omega_{(0, r-2)}}{\Omega_{(0, r-1)}}, 
$$

(6.4)

where $\Lambda^5_{rd} = \Lambda^4_{r} m_r$.

Next we integrate out $Q_0$ by adding $m_0 M_0$ to (6.4) and minimizing with $M_0$ which, as shown in Appendix E, gives the superpotential for Figure 3(b),

$$
W = \frac{\Lambda^5_{id} \Omega_{(2, r-1)}}{\Omega_{(1, r-1)}} + \frac{\Lambda^5_{rd} \Omega_{(1, r-2)}}{\Omega_{(1, r-1)}} \pm 2 \frac{(\Lambda^5_{id} \Lambda^5_{rd} \prod_{i=2}^{r-1} \Lambda^4_i)^{1/2}}{\Omega_{(1, r-1)}}, 
$$

(6.5)

where $\Lambda^5_{id} = \Lambda^4_{r} m_0$. This superpotential can be interpreted as follows: For the moose chain shown in Figure 3(b), the original $SU(2)^r$ gauge symmetry is completely broken and there is a new unbroken diagonal $SU(2)_D$. The first and second terms come from a single instanton in the broken $SU(2)_1$ and a single instanton in the broken $SU(2)_r$ respectively and infinite series of multi-instantons from the broken $SU(2)_2$ to $SU(2)_{r-2}$. This can be seen by using the explicit form of the $\Omega$ functions and making an expansion of $\Omega^{-1}_{(1, r-1)}$ in powers of the scales of $SU(2)_2$ to $SU(2)_{r-2}$. The last term comes from gaugino condensation in the unbroken diagonal $SU(2)_D$. In fact, we can read off from (6.5) that the scale of the diagonal $SU(2)_D$ is

$$
\Lambda_D = \left( \frac{(\Lambda^5_{id} \Lambda^5_{rd} \prod_{i=2}^{r-1} \Lambda^4_i)^{1/2}}{\Omega_{(1, r-1)}} \right)^{1/4}. 
$$

(6.6)
Finally, we can construct the quantum moduli space of the ring moose shown in Figure 4. The tree level superpotential that contains the new gauge invariant fields is

$$W_{\text{tree}} = b U_{(0,r-1)} + m_0 M_0.$$  \hfill (6.7)

where $b$ and $m_0$ are constants. Because $U_{(0,r-1)}$ is a non-quadratic singlet, minimizing $b U_{(0,r-1)} + m_0 M_0$ with $Q_0$ gives

$$W_{\text{tree},d} = -\frac{b^2}{4m_0} \Omega_{(1,r-1)}.$$  \hfill (6.8)

The superpotential we need to integrate in $Q_0$ is then obtained by setting $\Lambda^5_{ld} \rightarrow m_0 \Lambda^4_{1}$, $\Lambda^5_{rd} \rightarrow m_0 \Lambda^4_{r}$ in (6.5), adding (6.8) and subtracting (6.7),

$$W = \frac{m_0 \Lambda^4_{1} \Omega_{(2,r-1)}}{\Omega_{(1,r-1)}} + \frac{m_0 \Lambda^4_{2} \Omega_{(1,r-2)}}{\Omega_{(1,r-1)}} \pm 2 \frac{m_0 (\prod_{i=1}^{r} \Lambda^4_{i})^{1/2}}{\Omega_{(1,r-1)}} \pm \frac{b^2}{4m_0} \Omega_{(1,r-1)} - m_0 M_0 - b U_{(0,r-1)}.$$  \hfill (6.9)

Minimizing (6.9) with $m_0$ and $b$ gives, see Appendix F, $W = 0$ and

$$U^2_{(0,r-1)} + \Lambda^4_{1} \Omega_{(2,r-1)} + \Lambda^4_{2} \Omega_{(1,r-2)} - M_0 \Omega_{(1,r-1)} \pm 2 (\prod_{i=1}^{r} \Lambda^4_{i})^{1/2} = 0.$$  \hfill (6.10)

This is symmetric in all links and scales.

Before we interpret (6.10), let us first recall that according to the Seiberg-Witten hypothesis, the quantum moduli space of an $SU(2)$ gauge theory with $\mathcal{N} = 2$ supersymmetry coincides with the moduli space of the elliptic curve $y^2 = (x^2 - u)^2 - \Lambda^4$, where $u$ is a gauge invariant coordinate and $\Lambda$ is the dynamical scale of the theory. The singularities of this curve are given by the zeros of the discriminant $\Delta_\Lambda = (u^2 - \Lambda^4)^2 (2\Lambda)^8$. This occurs at $u = \pm \Lambda^2$ and $u = \infty$. The first two singularities at $u = \pm \Lambda^2$ are in the strong coupling region, and there is a massless monopole at one and a massless dyon at the other of these singularities. The singularity at $u = \infty$ is in the semi-classical region.
Now let us rewrite (6.10) as

\[ u_r = \pm \Lambda^2_{(1,r)}, \]  

where

\[ u_r \equiv U^2_{(0,r-1)} + \Lambda^4_{1}\Omega_{(2,r-1)} + \Lambda^4_{r}\Omega_{(1,r-2)} - M_0\Omega_{(1,r-1)} \quad \text{and} \quad \Lambda^2_{(1,r)} \equiv 2\left( \prod_{i=1}^r \Lambda^4_i \right)^{1/2}. \]  

Note that the modulus \( u_r \) contains all the independent gauge invariants we needed to parameterize the moduli space of the ring. What (6.11) is telling us is that the function \( u_r \) is locked at \( \pm \Lambda^2_{(1,r)} \). In other words, (6.11) gives two \( r \)-complex dimensional singular submanifolds in the \( r + 1 \)-complex dimensional moduli space spanned by all the independent gauge invariants. The vacua are fixed to the singularities because of the tree level deformation of the theory. That is why the integrating in procedure is relevant to the Seiberg-Witten curve. Giving large VEVs to the link fields breaks the original \( SU(2)^r \) gauge symmetry into a diagonal \( SU(2)_D \) with matter in the adjoint representation. The two singularities given by (6.11) on the \( u_r \) plane can be nothing but the two singularities in the strong coupling region of the \( SU(2)_D \) gauge theory with \( \mathcal{N} = 2 \) supersymmetry.

The monodromies around these singularities on the \( u_r \) plane must be the same as in Seiberg-Witten and the charge at the singularity \( u_r = +\Lambda^2_{(1,r)} \) is that of a monopole and the charge at \( u_r = -\Lambda^2_{(1,r)} \) is that of a dyon. A generic point in the moduli space of the ring moose has unbroken \( U(1) \) gauge symmetry and the ring moose is in the Coulomb phase. Having obtained these singularities and because the \( U(1) \) coupling coefficient is holomorphic, we have determined the elliptic curve that parameterizes the Coulomb phase of the ring moose.

Thus the quantum moduli space of the ring moose can be parameterized by the elliptic curve

\[ y^2 = \left( x^2 - \left( U^2_{(0,r-1)} + \Lambda^4_{1}\Omega_{(2,r-1)} + \Lambda^4_{r}\Omega_{(1,r-2)} - M_0\Omega_{(1,r-1)} \right) \right)^2 - 4\left( \prod_{i=1}^r \Lambda^4_i \right). \]  

The first few \( u \) functions are listed in Appendix [C].

Seiberg-Witten curves for the ring moose were computed in [11] using a different method. A method used in [10] to obtain the curve for the \( r = 2 \) ring was continued in [11] to compute the curve for \( r = 3 \). The idea was as follows: Because giving large VEVs to the link fields breaks the \( SU(2)^r \) gauge symmetry into a diagonal \( SU(2)_D \) with matter in the adjoint representation, the theory in effect becomes that of a single \( SU(2) \) with \( \mathcal{N} = 2 \) supersymmetry. The curve for \( r = 3 \) was obtained by taking various asymptotic limits of the gauge singlet fields and the nonperturbative scales, comparing with the \( \mathcal{N} = 2 \) \( SU(2) \) curve and imposing symmetries. The result for \( r = 3 \) was then generalized to the curve for a ring moose with arbitrary \( r \). Our results agree with [10] for \( r = 2 \) and with [11] for \( r = 3 \). However, we do not agree with the curves in [11] for \( r \geq 4 \). Only few terms in \( u_r \) were obtained in [11], which would give incorrect singular submanifolds in moduli space. We are not suggesting that the method used in [11] is incorrect. Furthermore, work on applications to deconstruction [15] - [18] should not be affected by the missing terms as they

\[ ^2 \text{I like to thank the referee for suggesting the last two statements.} \]
did not rely on the parameterization of the modulus in terms of the independent gauge invariant coordinates. Here we have obtained the quantum moduli space directly by integrating in all the independent link fields starting from a pure gauge theory of disconnected nodes and building the ring moose via the linear moose. This is done for a ring with arbitrary number of nodes without any need of imposing symmetries in the nodes or links and without taking asymptotic limits; and the result is automatically symmetric in all nodes and links.

7 Monopole condensates

We will now look at the effective field theory near the singularities of the quantum moduli space of the ring moose. It is believed that the singularities correspond to massless particles [13] and there is a massless monopole at \( u_r = +\Lambda^2_{(1,r)} \) and a massless dyon at \( u_r = -\Lambda^2_{(1,r)} \). As these massless states move out of the singularities, they get masses of order \( u_r \mp \Lambda^2_{(1,r)} \). The leading order effective superpotential near the singularities can thus be written as

\[
W \sim \left( U_{(0,r-1)}^2 + \Lambda^4_{1}(2,r-1) + \Lambda^4_{r}(1,r-2) - M_0\Omega_{(1,r-1)} - \Lambda^2_{(1,r)} \right) E_mE_m \\
+ \left( U_{(0,r-1)}^2 + \Lambda^4_{1}(2,r-1) + \Lambda^4_{r}(1,r-2) - M_0\Omega_{(1,r-1)} + \Lambda^2_{(1,r)} \right) \tilde{E}_dE_d + \sum_{i=0}^{r-1} m_i M_i, \tag{7.1}
\]

where \( E_m \) and \( E_d \) are chiral superfields of the monopole and dyon states respectively. The last (mass) term in (7.1) is added to lift up the flat directions and give nonzero VEV to the condensates. Note that although the singularities look the same as in Seiberg-Witten on the \( u_r \) plane, they are \( r \)-dimensional submanifolds with a very nontrivial modulus given by \( \tilde{\Omega}(1,r) \). The equations of motion are obtained by minimizing (7.1) with \( M_i, U_{(0,r-1)}, E_m, \tilde{E}_m, E_d \) and \( \tilde{E}_d \) which, using the properties of the \( \Omega \) functions given in Appendix [B] gives two sets of equations. One for the first singularity at \( u_r = +\Lambda^2_{(1,r)} \) with \( E_d = 0 \) and

\[
-\Omega_{(0,i-1)}\Omega_{(i+1,r-1)} \tilde{E}_m E_m + m_i = 0, \\
U_{(0,r-1)} = 0, \\
U_{(0,r-1)}^2 + \Lambda^4_{1}(2,r-1) + \Lambda^4_{r}(1,r-2) - M_0\Omega_{(1,r-1)} - \Lambda^2_{(1,r)} = 0. \tag{7.2}
\]

The second set of equations at the second singularity give \( E_m = 0 \) and (7.2) with \( \Lambda^2_{(1,r)} \rightarrow -\Lambda^2_{(1,r)} \) and \( m \rightarrow d \).

Let us explicitly solve (7.2) for \( r = 2 \) and \( r = 3 \). When \( r = 2 \), (7.2) with the \( \Omega \) functions given in Appendix [A] give

\[
-M_1 \tilde{E}_m E_m + m_0 = 0, \\
-M_0 \tilde{E}_m E_m + m_1 = 0, \\
\Lambda^4_{1} + \Lambda^4_{2} - M_0 M_1 - 2\Lambda^2_1 \Lambda^2_2 = 0. \tag{7.3}
\]
The solutions are
\[ M_0 = \epsilon \left( \frac{m_1}{m_0} \right)^{1/2} (\Lambda_1^2 - \Lambda_2^2), \quad M_1 = \epsilon \left( \frac{m_0}{m_1} \right)^{1/2} (\Lambda_1^2 - \Lambda_2^2) \] (7.4)
and the expectation values of the monopole condensate
\[ \tilde{E}_m E_m = \epsilon \left( \frac{m_0 m_1}{\Lambda_1^2 - \Lambda_2^2} \right)^{1/2}, \] (7.5)
where \( \epsilon = \pm 1 \). Therefore, the monopole gets confined and a singular submanifold corresponds to the confining branch of the moduli space. Note that the above are two solutions at the first singularity. The second set of equations give two more solutions at the second singularity with \( \Lambda_2^2 \rightarrow -\Lambda_2^2 \) in (7.5). Thus there are a total of four vacuum states which match the four phases from gaugino condensation in the low energy pure gauge theory of two disconnected nodes.

Next consider \( r = 3 \). The equations of motion at the first singularity in this case are
\[
\begin{align*}
(-M_1 M_2 + \Lambda_2^4) \tilde{E}_m E_m + m_0 &= 0, \\
(-M_0 M_2 + \Lambda_3^4) \tilde{E}_m E_m + m_1 &= 0, \\
(-M_0 M_1 + \Lambda_1^4) \tilde{E}_m E_m + m_2 &= 0, \\
-M_0 M_1 M_2 + \Lambda_1^4 M_2 + \Lambda_2^4 M_0 + \Lambda_3^4 M_1 - 2\Lambda_1^2 \Lambda_2^2 \Lambda_3^2 &= 0. \quad (7.6)
\end{align*}
\]
The same equations of motion as (7.6) were obtained in \[11\] for \( r = 3 \). Denoting \( \tilde{E}_m E_m \) by \( x \), the expectation values of the condensate are given by the solutions of
\[
\begin{align*}
(2\Lambda_1^8 \Lambda_2^4 \Lambda_3^4 m_0 m_1 + 2\Lambda_1^4 \Lambda_2^4 \Lambda_3^8 m_0 m_2 + 2\Lambda_1^4 \Lambda_2^8 \Lambda_3^4 m_1 m_2 - \Lambda_1^8 \Lambda_2^8 m_1^2 - \Lambda_1^8 \Lambda_3^8 m_2^2 - \Lambda_2^8 \Lambda_3^8 m_1^2) x^4 \\
+ \Lambda_1^4 \Lambda_2^4 \Lambda_3^4 m_0 m_1 m_2 x^3 + 2(\Lambda_1^4 \Lambda_2^4 m_0 m_2^2 + \Lambda_1^4 \Lambda_3^4 m_0 m_1 m_2 + \Lambda_2^4 \Lambda_3^4 m_0 m_1 m_2) x^2 \\
- m_0^2 m_1^2 m_2^2 &= 0. \quad (7.7)
\end{align*}
\]
Now (7.7) is a fourth order polynomial equation with four solutions. Exactly the same equation holds at the second singularity, since (7.7) contains even powers of \( \Lambda_i^2 \). Thus there are a total of eight vacuum states in the massive theory, four at each singularity. This exactly matches the eight phases of the low energy theory with all the matter fields integrated out, where the \( Z_4 \times Z_4 \times Z_4 \) \( R \) symmetry is broken down to \( Z_2 \times Z_2 \times Z_2 \) by gaugino condensation. Note that \( \tilde{E}_m E_m \) in (7.4) is a Lagrange multiplier in the language of Sections 4 - 6. In fact, if we set \( \Lambda_3 \equiv 0 \) in (7.7), we obtain
\[ x = \left(-\frac{m_0 m_2}{\Lambda_1^4 \Lambda_2^4}\right)^{\frac{1}{2}}, \left(-\frac{m_0 m_2}{\Lambda_1^4 \Lambda_3^4}\right)^{\frac{1}{2}}, \left(-\frac{m_0 m_2}{\Lambda_2^4 \Lambda_3^4}\right)^{\frac{1}{2}}, \left(-\frac{m_0 m_2}{\Lambda_1^4 \Lambda_2^4}\right)^{\frac{1}{2}} \] (7.8)
which exactly matches the non-zero solutions of the linear mode with two nodes given in (7.9). The extra solution at \( x = 0 \) in (7.9) is far out in moduli space.
8 Summary

We started with simple pure gauge theories of disconnected nodes and produced very nontrivial quantum moduli space constraints and dynamical superpotentials for $\mathcal{N} = 1$ $SU(2)_r$ linear and ring moose theories by integrating in and out matter link chiral super fields. We showed that we could consistently add and remove link fields by exploiting simple and efficient integrating in and out procedures. The linear moose with both external links present has quantum moduli space of vacua. We have explicitly computed the constraints on the vacua. We have also shown that when the moduli space is perturbed by a generic tree level superpotential, the vacuum space becomes discrete. The linear mooses without one or both external links have non-vanishing low energy dynamical superpotentials. We have explicitly computed these superpotentials. For the ring moose, we found two singular submanifolds with a nontrivial modulus that is a function of all the gauge singlets we needed to parameterize the quantum moduli space. The massive theory near the singularities led to confinement. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose followed from our computation naturally.

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A The $\Omega$ functions

Here we list the first few $\Omega$ functions defined in (3.28) and (3.29).

$$\Omega_{(i,i+1)} = M_i M_{i+1} - \Lambda^4_{i+1}$$  \hfill (A.1)

$$\Omega_{(i,i+2)} = M_i M_{i+1} M_{i+2} - \Lambda^4_{i+1} M_{i+2} - \Lambda^4_{i+2} M_i$$  \hfill (A.2)

$$\Omega_{(i,i+3)} = M_i M_{i+1} M_{i+2} M_{i+3} - \Lambda^4_{i+1} M_{i+2} M_{i+3} - \Lambda^4_{i+2} M_i M_{i+3}$$

$$- \Lambda^4_{i+3} M_i M_{i+1} + \Lambda^4_{i+1} \Lambda^4_{i+3}$$  \hfill (A.3)

$$\Omega_{(i,i+4)} = M_i M_{i+1} M_{i+2} M_{i+3} M_{i+4} - \Lambda^4_{i+1} M_{i+2} M_{i+3} M_{i+4} - \Lambda^4_{i+2} M_i M_{i+3} M_{i+4}$$

$$- \Lambda^4_{i+3} M_i M_{i+1} M_{i+4} - \Lambda^4_{i+4} M_i M_{i+1} M_{i+2}$$

$$+ \Lambda^4_{i+1} \Lambda^4_{i+3} M_{i+4} + \Lambda^4_{i+1} \Lambda^4_{i+4} M_{i+2} + \Lambda^4_{i+2} \Lambda^4_{i+4} M_i$$  \hfill (A.4)

$$\Omega_{(i,i+5)} = M_i M_{i+1} M_{i+2} M_{i+3} M_{i+4} M_{i+5} - \Lambda^4_{i+1} M_{i+2} M_{i+3} M_{i+4} M_{i+5}$$

$$- \Lambda^4_{i+2} M_i M_{i+3} M_{i+4} M_{i+5} - \Lambda^4_{i+3} M_i M_{i+1} M_{i+4} M_{i+5}$$
Some properties of the $\Omega$ functions

Here we write important recursion relations we used in our computations that involve the $\Omega$ functions. We take $j > i$ in all cases unless explicitly stated.

\begin{align}
\Omega(i,i) &= M_i \quad \text{(B.1)} \\
\Omega(i,j) &= \Omega(i,j-1)M_j - \Lambda_i^4\Omega(i,j-2) \quad \text{(B.2)} \\
\Omega(i,j) &= M_i\Omega(i+1,j) - \Lambda_i^4\Omega(i+2,j) \quad \text{(B.3)} \\
\frac{\partial}{\partial M_k}\Omega(i,j) &= \Omega(i,k-1)\Omega(k+1,j) \quad \text{(B.4)} \\
\end{align}

\begin{align}
\Omega(i,j-2)\Omega(i+1,j-1) - \Omega(i,j-1)\Omega(i+1,j-2) &= \prod_{k=i+1}^{j-1} \Lambda_i^4 \
\text{(B.5)}
\end{align}

(B.4) for $k = i$ and $k = j$ gives

\begin{align}
\frac{\partial}{\partial M_i}\Omega(i,j) &= \Omega(i+1,j) \quad \text{(B.6)} \\
\frac{\partial}{\partial M_j}\Omega(i,j) &= \Omega(i,j-1) \quad \text{(B.7)}
\end{align}

Discrete vacua

Here we will give the discrete vacuum states obtained by minimizing (5.4) with $M_0$, $M_1$, $M_2$, $T$ and $A$. The equations of motion are

\begin{align}
A(-M_1M_2 + \Lambda_2^4) + m_0 &= 0, \\
-AM_0M_2 + m_1 &= 0, \\
A(-M_0M_1 + \Lambda_1^4) + m_2 &= 0 \\
det T_{(0,2)} - M_0M_1M_2 + \Lambda_1^4M_2 + \Lambda_2^4M_0 &= 0, \\
AT_{(0,2)}^{-1} \det T_{(0,2)} + c &= 0. \quad \text{(C.1)}
\end{align}
The solution is

\[ M_0 = \frac{1}{m_0^2 \det(c)[\Lambda_1^4 m_0 - \Lambda_2^4 m_2]} \left( m_2 \det(c)^2 \Lambda_2^8 + m_0^2 m_1 m_2 [\Lambda_1^2 m_0 - \Lambda_2^2 m_2] \right. \\
+ \Lambda_1^4 \Lambda_2^4 [\det(c)^2 \Lambda_2^4 - m_0^2 m_1 m_2] x + \left. \Lambda_1^4 \Lambda_2^4 m_0 m_1 [-\Lambda_1^4 m_0 + 2 \Lambda_2^4 m_2] x^2 \right. \\
+ \left. \Lambda_1^8 \Lambda_2^8 m_0 m_1 x^3 - \Lambda_1^8 \Lambda_2^8 m_1 x^4 \right), \]  

\[ \text{(C.2)} \]

\[ M_1 = \frac{1}{\det(c)} \left( m_0 m_2 - \Lambda_1^4 \Lambda_2^4 x^2 \right), \]  

\[ \text{(C.3)} \]

\[ M_2 = M_0(m_0 \leftrightarrow m_2, \Lambda_1^4 \leftrightarrow \Lambda_2^4), \]  

\[ \text{(C.4)} \]

\[ T = - \frac{c^{-1} x}{\det(c)^2 m_0^2 m_2^2} \left[ m_1^4 m_1^2 m_2^4 - 2 m_0^2 m_1 m_2^2 \det(c)^2 [\Lambda_1^4 m_0 + \Lambda_2^4 m_2] \right. \\
+ \left. \det(c)^2 [\Lambda_1^4 m_0^2 + \Lambda_2^4 m_2^2 + \Lambda_1^4 \Lambda_2^4 m_0 m_2] \right. \\
+ \left. \det(c)^2 [\Lambda_1^4 \Lambda_2^4 (\det(c)^2 [\Lambda_1^4 m_0 + \Lambda_2^4 m_2] - 3 m_0^2 m_1 m_2^2)] x \right. \\
+ \left. 2 \Lambda_1^4 \Lambda_2^4 m_0 m_1 m_2 (\det(c)^2 [\Lambda_1^4 m_0 + \Lambda_2^4 m_2] - m_0^2 m_1 m_2^2) x^2 \right. \\
+ \left. \det(c)^2 [\Lambda_1^4 \Lambda_2^4 m_0 m_1 m_2 x^3 \right. \\
- \left. \Lambda_1^8 \Lambda_2^8 m_1 (\det(c)^2 [\Lambda_1^4 m_0 + \Lambda_2^4 m_2] - m_0^2 m_1 m_2^2) x^4 \right]. \]  

\[ \text{(C.5)} \]

\[ A = x, \]  

\[ \text{(C.6)} \]

where

\[ \Lambda_1^8 \Lambda_2^8 m_1 x^5 - 2 \Lambda_1^4 \Lambda_2^4 m_0 m_1 m_2 x^3 - \Lambda_1^4 \Lambda_2^4 \det(c)^2 x^2 \\
- \Lambda_1^4 m_0 \det(c)^2 + \Lambda_2^4 m_2 \det(c)^2 - m_0^2 m_1 m_2 x - m_0 m_2 \det(c)^2 = 0. \]  

\[ \text{(C.7)} \]

D Derivation of (6.4)

We want to minimize \[ \text{(D.3)}, \]

\[ A \left( \det T_{(0,r)} - \Omega_{(0,r)} \right) + m_r M_r \]  

\[ \text{(D.1)} \]

with \( M_r, T_{(0,r)} \) and \( A \). This gives, using the properties given in Appendix B

\[ - A \Omega_{(0,r-1)} + m_r = 0, \]  

\[ \text{(D.2)} \]

\[ A T_{(0,r)}^{-1} \det T_{(0,r)} = 0, \]  

\[ \text{(D.3)} \]

\[ \det T_{(0,r)} - \Omega_{(0,r)} = 0. \]  

\[ \text{(D.4)} \]

\[ \text{(D.3)} \] gives

\[ \det T_{(0,r)} = 0. \]  

\[ \text{(D.5)} \]
Using (D.4) and (D.5) with the identity given in (B.5) in (D.4) and solving for $M_r$, we get

$$M_r = \frac{\Lambda^4_1 \Omega(0, r-2)}{\Omega(0, r-1)}. \quad (D.6)$$

Finally, (D.6) and (D.4) in (D.1) gives

$$W = \frac{\Lambda^5_r \Omega(0, r-2)}{\Omega(0, r-1)}. \quad (D.7)$$

### E Derivation of (6.5)

Minimizing

$$W = \frac{\Lambda^5_r \Omega(0, r-2)}{\Omega(0, r-1)} + m_0 M_0 \quad (E.1)$$

with $M_0$ and using the identity (B.6) gives

$$\Lambda^5_{rd} \Omega(1, r-2) \Omega(0, r-1) - \Lambda^5_{rd} \Omega(0, r-2) \Omega(1, r-1) + m_0 \Omega^2(0, r-1) = 0. \quad (E.2)$$

Using the identity (B.3) in (E.2) gives a quadratic equation for $M_0$,

$$\Lambda^5_{rd} \Omega(1, r-2) (\Omega(1, r-1) M_0 - \Lambda^4_1 \Omega(2, r-1) - \Lambda^5_{rd} \Omega(1, r-1) M_0) - \Lambda^4_1 \Omega(2, r-2) \Omega(1, r-1) + m_0 (\Omega(1, r-1) M_0 - \Lambda^4_1 \Omega(2, r-1))^2 = 0 \quad (E.3)$$

with solution

$$M_0 = \frac{\Lambda^5_{id} \Omega(2, r-1)}{m_0 \Omega(1, r-1)} \pm 2 \left[ \frac{\Lambda^5_{id} \Lambda^5_{rd} \Omega(1, r-2) \Omega(2, r-1) - \Omega(1, r-1) \Omega(2, r-2)}{m_0 \Omega(1, r-1)} \right]^{1/2}. \quad (E.4)$$

Using (E.4) and the identity (B.3) in (E.1) gives

$$W = \frac{\Lambda^5_{id} \Omega(2, r-1)}{\Omega(1, r-1)} + \frac{\Lambda^5_{rd} \Omega(1, r-2)}{\Omega(1, r-1)} \pm 2 \left( \frac{\Lambda^5_{id} \Lambda^5_{rd} \prod_{i=2}^{r-1} \Lambda^4_i}{\Omega(1, r-1)} \right)^{1/2}. \quad (E.5)$$

### F Derivation of (6.10)

From (6.9)

$$W = \frac{m_0 \Lambda^4_1 \Omega(0, r-1)}{\Omega(1, r-1)} + \frac{m_0 \Lambda^4_0 \Omega(1, r-2)}{\Omega(1, r-1)} \pm 2 \frac{m_0 (\prod_{i=1}^{r-1} \Lambda^4_i)^{1/2}}{\Omega(1, r-1)}$$

$$- \frac{b^2}{4m_0} \Omega(1, r-1) - m_0 M_0 - b U(0, r-1). \quad (F.1)$$
Minimizing \( F.1 \) with \( m_0 \) and \( b \) gives \( W = 0 \) and

\[
\frac{\Lambda_4^4 \Omega_{(2, r-1)}}{\Omega_{(1, r-1)}} + \frac{\Lambda_r^4 \Omega_{(1, r-2)}}{\Omega_{(1, r-1)}} + \frac{2 (\prod_{i=1}^{r} \Lambda_i^4)^{1/2}}{\Omega_{(1, r-1)}} + \frac{b^2}{4m_0^2} \Omega_{(1, r-1)} - M_0 = 0,
\]

\( \text{(F.2)} \)

Putting \( \text{(F.3)} \) in \( \text{(F.2)} \) gives

\[
u_r \pm \Lambda_{(1, r)}^2 = 0,
\]

\( \text{(F.4)} \)

where

\[
u_r = U_{(0, r-1)}^2 + \Lambda_1^4 \Omega_{(2, r-1)} + \Lambda_r^4 \Omega_{(1, r-2)} - M_0 \Omega_{(1, r-1)}
\]

\( \text{(F.5)} \)

and

\[
\Lambda_{(1, r)}^2 = 2 (\prod_{i=1}^{r} \Lambda_i^4)^{1/2}.
\]

\( \text{(F.6)} \)

**G The \( u \) functions**

Using the definition \( \text{(6.12)} \) or \( \text{(F.3)} \) for \( u_r \) and the \( \Omega \) functions given in Appendix A, the first few \( u \) functions are

\[
\begin{align*}
u_2 &= U_{(0, 1)}^2 + \Lambda_1^4 + \Lambda_2^4 - M_0 M_1, \quad \text{(G.1)} \\
u_3 &= U_{(0, 2)}^2 + \Lambda_1^4 M_2 + \Lambda_2^4 M_0 + \Lambda_3^4 M_1 - M_0 M_1 M_2, \quad \text{(G.2)} \\
u_4 &= U_{(0, 3)}^2 + \Lambda_1^4 M_2 M_3 + \Lambda_2^4 M_0 M_3 + \Lambda_3^4 M_0 M_1 + \Lambda_4^4 M_1 M_2 \\
&
- \Lambda_1^4 \Lambda_3^4 - \Lambda_2^4 \Lambda_4^4 - M_0 M_1 M_2 M_3, \quad \text{(G.3)} \\
u_5 &= U_{(0, 4)}^2 + \Lambda_1^4 M_2 M_3 M_4 + \Lambda_2^4 M_0 M_3 M_4 + \Lambda_3^4 M_0 M_1 M_4 + \Lambda_4^4 M_0 M_1 M_2 \\
&+ \Lambda_2^4 M_1 M_2 M_3 - \Lambda_1^4 \Lambda_3^4 M_4 - \Lambda_1^4 \Lambda_4^4 M_2 - \Lambda_2^4 \Lambda_5^4 M_3 - \Lambda_3^4 \Lambda_5^4 M_1 \\
&- \Lambda_4^4 \Lambda_6^4 M_0 - M_0 M_1 M_2 M_3 M_4, \quad \text{(G.4)} \\
u_6 &= U_{(0, 5)}^2 + \Lambda_1^4 M_2 M_3 M_4 M_5 + \Lambda_2^4 M_0 M_3 M_4 M_5 + \Lambda_3^4 M_0 M_1 M_4 M_5 \\
&+ \Lambda_4^4 M_0 M_1 M_2 M_5 + \Lambda_5^4 M_0 M_1 M_2 M_3 + \Lambda_6^4 M_0 M_1 M_2 M_4 \\
&- \Lambda_1^4 \Lambda_5^4 M_5 - \Lambda_1^4 \Lambda_6^4 M_5 - \Lambda_1^4 \Lambda_7^4 M_5 - \Lambda_2^4 \Lambda_6^4 M_0 M_5 \\
&- \Lambda_2^4 \Lambda_7^4 M_0 M_3 - \Lambda_2^4 \Lambda_7^4 M_3 M_4 - \Lambda_3^4 \Lambda_6^4 M_0 M_1 - \Lambda_3^4 \Lambda_6^4 M_1 M_4 \\
&- \Lambda_4^4 \Lambda_6^4 M_1 M_2 + \Lambda_1^4 \Lambda_3^4 \Lambda_4^4 + \Lambda_2^4 \Lambda_4^4 \Lambda_6^4 - M_0 M_1 M_2 M_3 M_4 M_5. \quad \text{(G.5)}
\end{align*}
\]
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