The Optimal Decay Rate of Strong Solution for the Compressible Nematic Liquid Crystal Equations with Large Initial Data

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Abstract

This paper is devoted to establishing the optimal decay rate of the global large solution to compressible nematic liquid crystal equations when the initial perturbation is large and belongs to $L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$. More precisely, we show that the first and second order spatial derivatives of large solution $(\rho - 1, u, \nabla d)(t)$ converges to zero at the $L^2$ rate $(1 + t)^{-\frac{5}{4}}$ and $L^2$ rate $(1 + t)^{-\frac{7}{4}}$ respectively, which are optimal in the sense that they coincide with the decay rates of solution to the heat equation. Thus, we establish optimal decay rate for the second order derivative of global large solution studied in [12, 18] since the compressible nematic liquid crystal flow becomes the compressible Navier-Stokes equations when the director is a constant vector. It is worth noticing that there is no decay loss for the highest-order spatial derivative of solution although the associated initial perturbation is large. Moreover, we also establish the lower bound of decay rates of $(\rho - 1, u, \nabla d)(t)$ itself and its spatial derivative, which coincide with the upper one. Therefore, the decay rates of global large solution $\nabla^2(\rho - 1, u, \nabla d)(t)$ ($k = 0, 1, 2$) are actually optimal.

Keywords: Compressible nematic liquid crystal equations; optimal decay rate; large initial data.

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1 Introduction

In this paper, we are concerned with the upper and lower bounds of decay rates for a class of global large solution to the three dimensional compressible nematic liquid crystal equations:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P &= \text{div} T - \nabla d \cdot \Delta d, \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d,
\end{align*}
\]

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. The unknown functions $\rho, u = (u_1, u_2, u_3)$ and $P$ represent the density, velocity and pressure respectively. $d(t, x) \in S^2$, the unit sphere in $\mathbb{R}^3$, represents the macroscopic average of the nematic liquid crystal orientation field. The pressure $P$ is given by a smooth function $P = P(\rho) = \rho^\gamma$ with the adiabatic exponent $\gamma \geq 1$. And $T$ is the stress tensor given by $T = \mu(\nabla u + \nabla^T u) + \lambda(\text{div} u)I_{3 \times 3}$ with $I_{3 \times 3}$ the identity matrix. The constants $\mu$ and $\lambda$ are the viscosity coefficients, which satisfy the following conditions: $\mu > 0$, $2\mu + 3\lambda \geq 0$. To complete system (1.1), the initial data is given by

$$(\rho, u, d)(t, x)|_{t=0} = (\rho_0(x), u_0(x), d_0(x)).$$

As the space variable tends to infinity, there follows

$$\lim_{|x| \to \infty} (\rho - 1, u, \nabla d)(t, x) = (0, 0, 0),$$

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where \( \mathbf{d} \) is a unit constant vector in \( S^2 \). The systems \((1.1)\) are a coupling between the compressible Navier-Stokes equations and a heat flow, which is a macroscopic continuum description of the development for the liquid crystal of nematic type. In the sequence, we will describe some mathematical results related to the Navier-Stokes and nematic liquid crystal equations.

(I) Some results for the incompressible nematic liquid crystal equations. The hydrodynamic theory of incompressible liquid crystals was first derived by Ericksen and Leslie in the 1960s (see \([8, 30]\)). It simplified to the incompressible nematic liquid crystal equations, which has been successfully studied. For examples, when density is a constant, Lin et al. \([32]\) obtained the global existence of the weak solutions in any smooth bounded domain in two dimensions. Gong et al. \([15]\) obtained a strong global solution if the initial orientation field vector lying in a two-dimensional plane. The uniqueness of Leray-Hopf type global weak solution was proved by Lin and Wang \([40]\). Later, Li, Titi and Xin \([35]\) extended their results to the general Ericksen-Leslie system. For more related results in \( \mathbb{R}^2 \), one can refer to \([21, 22, 23, 33, 34, 55]\). In the case of \( \mathbb{R}^3 \), Wang \([50]\) showed a global well-posedness theory under the condition that \( \|u_0\|_{BMO} + \|\mathbf{d}_0\|_{BMO} \leq \epsilon_0 \) for some \( \epsilon_0 > 0 \). Hineman and Wang \([19]\) obtained the local well-posedness in the condition that initial data with small \( L^3_{uloc}(\mathbb{R}^3) \)-norm. Lin and Wang \([41]\) obtained the global existence of a weak solution in the case that the initial director field on the unit upper hemisphere. Recently, Gong et al. \([14]\) constructed infinitely many weak solutions for suitable initial and boundary data. For the results of density-dependent incompressible nematic liquid crystal system, one can refer to \([11, 17, 53]\) and references therein.

(II) Some results for the compressible nematic liquid crystal equations. Let us introduce some related mathematical results. In one-dimensional space, Ding et al. \([4]\) obtained both global existence and uniqueness of classical solution of \((1.1)\) with Hölder continuous initial data and non-negative initial density. This result was generalized to the case of fluid with vacuum in \([9]\). In dimension three, Jiang et al. \([27]\) obtained the global existence of weak solution with large initial energy and without any smallness condition on the initial density and velocity in a bounded domain \( \Omega \subset \mathbb{R}^3 \), \( (N = 2, 3) \). The local-in-time well-posedness of strong solution and some blow-up criterions of breakdown of strong solution were studied in \([23, 26]\). The local-in-time strong solutions in \( \mathbb{R}^3 \) which under stricter regularity assumptions turn out to be classical was obtained in \([44]\). The global existence of classical solution with smooth initial data which are of small energy but possibly large oscillations in \( \mathbb{R}^3 \) were established in \([36]\). As a byproduct, they also studied the large-time behavior of the solution. Recently, Gao et al. \([11]\) obtained the global well-posedness of classical solution under the condition of small perturbation of constant equilibrium state in the \( H^N(\mathbb{R}^3)(N \geq 3)\)-framework. Furthermore, the optimal decay rate of \( k - th(k \leq N - 1) \) order spatial derivative of solution was obtained in \([11]\) if the initial perturbation data belongs to \( L^1 \) additionally. For more results on the compressible nematic liquid crystal flows \((1.1)\), we refer to \([24, 55, 12]\) and references therein.

(III) Some decay results for the compressible Navier-Stokes equations. The compressible nematic liquid crystal flow \((1.1)\) becomes the compressible Navier-Stokes equations(CNS) when the director is a constant vector. There are many interesting works about the long-time behavior of solution to CNS. First of all, Matsumura and Nishida \([40]\) obtained the decay rate of global small classical solution converging to some constant equilibrium state in the three-dimensional whole space. Later, Ponce \([47]\) established the optimal \( L^p(p \geq 2) \) decay rate for small initial perturbation in \( H^l \cap L^1 \) with \( l \geq 3 \). With the help of the study of Green function, the optimal \( L^p \) \((1 \leq p \leq \infty) \) decay rate in \( \mathbb{R}^n \), \( n \geq 2 \), were obtained by Hoff, Zumbrun \([20]\) and Liu, Wang \([43]\) when the small initial perturbation bounded in \( H^s \cap L^1 \) with the integer \( s \geq \lfloor n/2 \rfloor + 3 \). For the compressible Navier-Stokes system with an external potential force, the authors obtained the optimal decay rate in \([1, 19]\). For more results about decay problem for the Navier-Stokes equations, one can refer to \([32, 48, 52]\). If the initial perturbation belongs to some negative Sobolev space \( H^{-s} \) rather than some Lebesgue space \( L^p \), Guo and Wang \([17]\) built the time decay rate for the solution of CNS by using a general energy method. For the case of compressible fluid, there are many results about lower bound of decay rate for the solution itself of the compressible Navier-Stokes equations \([28, 32]\), compressible viscoelastic flows \([23]\), and compressible Navier-Stokes-Poisson equations \([31, 56]\). Later, Gao et al. \([9]\) studied the lower bound of decay rate for the higher order spatial derivative of solution to the compressible Navier-Stokes and Hall-MHD equations in three-dimensional whole space. Recently, Wang and Wen \([51]\) established the optimal time-decay rate for strong solution of the full compressible Navier-Stokes equations with reaction diffusion when the initial perturbation is small in \( H^2 \). Moreover, they developed a new estimate to avoid the decay loss for the highest-order spatial derivatives of the solution.

However, most of above decay results for the compressible nematic liquid crystal equations and the compressible
Navier-Stokes equations are established under the condition that the initial data is a small perturbation of constant equilibrium state. Recently, He, Huang and Wang [18] proved global stability of large solution to the compressible Navier-Stokes equations. Specifically, under the assumption that \( \sup_{t \in \mathbb{R}^+} \| \rho(t, \cdot) \|_{C^\alpha} \leq M \) for some \( 0 < \alpha < 1 \), they established upper decay rate
\[
\| (\rho - 1)(t) \|_{H^1} + \| u(t) \|_{H^1} \leq C(1 + t)^{-\frac{3}{2}(\frac{5}{2} - 1)}.
\]  
(1.2)

Here the initial perturbation \( (\rho_0 - 1, u_0) \in L^p(\mathbb{R}^3) \cap H^2(\mathbb{R}^3) \) with \( p \in [1, 2) \). The decay result (1.2) indicates that the first order spatial derivative of solution converges to zero at the \( L^2 \)-rate \( (1 + t)^{-\frac{3}{2}(\frac{5}{2} - 1)} \), which seems not optimal. Meanwhile, the decay rate (1.2) does not establish the decay rate for the second order spatial derivative of solution. Thus, for the the global solution studied in [18], our recent article [12] established the following decay estimate
\[
\| \nabla (\rho - 1)(t) \|_{H^1} + \| \nabla u(t) \|_{H^1} \leq C(1 + t)^{-\frac{3}{2}(\frac{5}{2} - 1) - \frac{3}{2}}.
\]  
(1.3)

Compared with (1.2), our result not only established optimal decay rate for the solution’s first order spatial derivative, but also proved the second order spatial derivative of global solution will converge to zero. However, (1.3) shows that the second order spatial derivative of solution converges to zero at the \( L^2 \)-rate \( (1 + t)^{-\frac{3}{2}(\frac{5}{2} - 1) - \frac{3}{2}} \), which still seems not optimal. Recently, Chen et al. [1] generalized the results in [18] to compressible nematic liquid crystal equations (1.1) and shown that \( (\rho - 1, u, \nabla d) \) converges to zero at the \( H^1 \)-rate \( (1 + t)^{-\frac{5}{2}} \) when the initial perturbation bounded in \( L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3) \). It is worth nothing that the optimal decay rate of first order spatial derivative of solution converging to zero in \( L^2 \)-norm can be proved to be \( (1 + t)^{-\frac{5}{2}} \) just taking the method in our article [12]. However, the decay rate of second order spatial derivative of solution for the compressible Navier-Stokes and nematic liquid crystal equations obtained in [12] is still not optimal. The essential reason is that there are not enough dissipative estimates for the density to control the energy since the compressible Navier-Stokes equations are the hyperbolic-parabolic system.

The purpose of this paper is to establish the optimal decay rate for the large solution of compressible nematic liquid crystal equations (1.1) when the initial perturbation is bounded in \( L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3) \). Our first target is to establish the optimal decay rate of large solution of (1.1) for its first and second order derivatives converging to zero. Here the decay rate of solution is called optimal just in the sense that it coincides with the rate of heat equation. The second purpose is to establish the lower bound of decay rates for the global strong solution itself and its first and second spatial derivatives for the compressible nematic liquid crystal equations (1.1). These lower bound of decay rates will coincide with the upper one. Therefore, these decay rates obtained in this article are actually optimal. As a byproduct, we also obtain the optimal decay rate for the second order spatial derivative of large solution studied in [18] for the compressible Navier-Stokes in three dimensional whole space.

Before stating the main results of this paper, we would like to introduce some notation which will be used throughout this paper.

**Notation:** In this paper, we use \( H^s(\mathbb{R}^3) \) to denote the usual Sobolev space with norm \( \| \cdot \|_{H^s} \) and \( L^p(\mathbb{R}^3) \) to denote the usual \( L^p \) space with norm \( \| \cdot \|_{L^p} \). \( \hat{f} = \mathcal{F}(f) \) represents the usual Fourier transform of the function \( f \). For the sake of simplicity, we write \( \int f \, dx := \int_{\mathbb{R}^3} f \, dx \) and \( \| (A, B) \|_X := \| A \|_X + \| B \|_X \). The constant \( C \) denotes the generic positive constant independent of time, and may change from line to line. Let \( \Lambda^s \) be the pseudodifferential operator defined by \( \Lambda^s = \mathcal{F}^{-1}(\xi^{s} \hat{f}) \), for \( s \in \mathbb{R} \). We note that \( \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \) and for a multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), \( \partial^\alpha_x = \partial^\alpha_{x_1} \partial^\alpha_{x_2} \partial^\alpha_{x_3} \).

First of all, we recall the following results obtained in [1], which will be used in this paper frequently.

**Theorem 1.1.** Let \( \mu > \frac{1}{2} \lambda \), and \( (\rho, u, d) \) be a global and smooth solution of (1.1) with \( 0 \leq \rho \leq M \), and initial data \( (\rho_0, u_0, d_0) \) verifying that \( \rho_0 \geq c > 0 \) and the admissible condition
\[
\begin{cases}
  u_{i}|_{t=0} = -u_0 \cdot \nabla u_0 + \frac{1}{\rho_0} (\text{div} \, T_0 - \nabla d_0 \cdot \Delta d_0 - \nabla \rho_0), \\
  d_{i}|_{t=0} = \Delta d_0 + |\nabla d_0|^2 d_0 - u_0 \cdot \nabla d_0,
\end{cases}
\]
and \( \sup_{t \in \mathbb{R}^+} \| \nabla d(t, \cdot) \|_{L^\infty} \leq \| \nabla d_0 \|_{L^\infty} + \sup_{t \in \mathbb{R}^+} \| \rho(t, \cdot) \|_{C^\alpha} \leq M \) for some \( 0 < \alpha < 1 \). Then if \( (\rho_0 - 1, u_0, \nabla d_0) \in L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3) \), then there exists a constant \( \rho = \rho(c, M, \mathcal{M}) > 0 \) such that for all \( t \geq 0 \), we have
\[
\rho(t, x) \geq \rho.
\]  
(1.4)
We have the uniform-in-time bounds for the regularity of the solutions, assuming that \( \rho = \rho - 1, n = d - 1 \).

\[
\begin{align*}
\| \varphi \|_{L^\infty_t(H^2)}^2 + \| u \|_{L^\infty_t(H^2)}^2 + \| \nabla n \|_{L^\infty_t(H^2)}^2 + \int_0^\infty \left( \| \nabla \varphi(\tau) \|_{H^1}^2 + \| \nabla u(\tau) \|_{H^2}^2 + \| \nabla^2 n(\tau) \|_{H^2}^2 \right) d\tau \\
\leq C(\rho, M, \| (\varphi_0, u_0, \nabla n_0) \|_{L^1_t(H^2)}, \| d_0 \|_{L^2}).
\end{align*}
\]

Moreover, we have the decay estimate for the solution

\[
\| \varphi(t) \|_{H^1} + \| u(t) \|_{H^1} + \| \nabla n(t) \|_{H^1} \leq C(\rho, M, \| \varphi_0 \|_{L^1_t(H^2)}, \| (u_0, \nabla n_0) \|_{L^1_t(H^2)}, \| n_0 \|_{L^2})(1 + t)^{-\frac{3}{4}-\frac{k}{4}}.
\]

Our first result can be stated as follows:

**Theorem 1.2.** Suppose all the conditions in Theorem 1.1 hold on, and let \((\rho, u, d)\) be the global solution of \((\rho, u, d)\). Then, it holds on for \(k = 0, 1, 2\)

\[
\| \nabla^k(\rho - 1)(t) \|_{L^2} + \| \nabla^k u(t) \|_{L^2} + \| \nabla^{k+1} d(t) \|_{L^2} \leq C(1 + t)^{-\frac{3}{4}-\frac{k}{4}},
\]

when \(t \geq T_2\). Here \(C\) is a constant independent of time, and \(T_2\) is a large constant given in Lemma 2.5.

**Remark 1.1.** Compared with decay rate \((1.6)\), our decay result \((1.7)\) not only implies that the second order spatial derivative of solution \((\rho, u, \nabla d)\) converges to zero, but also shows that the decay rates for the first and second order spatial derivatives of solution are optimal in the sense that they coincide with the decay rates of solution to the heat equation. Specially, as the vector \(d(t, x)\) is a constant vector field, our result also implies the optimal decay rate of the second order spatial derivative of some class large solution \((\rho, u)\) (see \([12, 18]\)) of compressible Navier-Stokes equation is \((1 + t)^{-\frac{5}{4}}\).

**Remark 1.2.** By the Sobolev interpolation inequality, it is shown that the solution \((\rho, u, \nabla d)\) converges to the constant equilibrium state \((1, 0, 0)\) at the \(L^\infty\)-rate \((1 + t)^{-\frac{3}{2}}\).

The second result can be stated as follows:

**Theorem 1.3.** Suppose all the assumption of Theorem 1.1 hold on. Denote \(m_0 := \rho_0 u_0, w_0 := \Delta n_0\), assume that the Fourier transform \(F(\rho_0, m_0, w_0) = (\hat{\varphi}_0, \hat{m}_0, \hat{w}_0)\), satisfies \(\hat{\varphi}_0 \geq c_0, \hat{m}_0 = 0, |\hat{w}_0| \geq c_0, 0 < \| \xi \| \ll 1\) with \(c_0 > 0\) a constant. Then the global solution \((\varphi, u, n)\) obtained in Theorem 1.1 has the decay rates for large time \(t\)

\[
\begin{align*}
c_1(1 + t)^{-\frac{3}{4}-\frac{k}{4}} \leq \| \nabla^k u(t) \|_{L^2} \leq c_2(1 + t)^{-\frac{3}{4}-\frac{k}{4}}, & \text{ for } k = 0, 1, 2 \\
c_3(1 + t)^{-\frac{3}{4}-\frac{k}{4}} \leq \| \nabla^k \varphi(t) \|_{L^2} \leq c_4(1 + t)^{-\frac{3}{4}-\frac{k}{4}}, & \text{ for } k = 0, 1, 2
\end{align*}
\]

Here \(c_i, (i = 1, 2, 3, 4)\) are constants independent of time.

**Remark 1.3.** Theorems 1.2 and 1.3 shows that the lower bound of decay rates for \(\nabla^k(\rho - 1, u, \nabla d)(t)\) \((k = 0, 1, 2)\) coincide with the upper one, which means these decay rates are actually optimal.

Next, we would like to introduce the main idea for the proof of Theorem 1.2. First, we shall establish the second order spatial derivative of solution for system (1.1) with large initial data. Since we have known that \(\| (\varphi, u, \nabla n) \|_{H^1} \leq C(1 + t)^{-\frac{3}{4}}, t \geq 0\). Then these quantities can be small enough after a long time. Therefore, as the strategy mentioned in [48] for flow with small initial data, we established the second order energy estimate:

\[
\frac{d}{dt} \| \nabla^2(\varphi, u, \nabla n)(t) \|_{L^2}^2 + \mu \| \nabla^3 u \|_{L^2}^2 + (\mu + \lambda) \| \nabla^2 \text{div} u \|_{L^2}^2 + \| \nabla^4 n \|_{L^2}^2 \\
\leq Q_1(t)(\| \nabla^2 \varphi \|_{L^2}^2 + \| \nabla^2 u \|_{H^1}^2 + \| \nabla^3 n \|_{L^2}^2),
\]

where \(Q_1(t)\) consists of some difficult terms, such as \(\| (\varphi, n) \|_{L^\infty}, \| \nabla(\varphi, u, \nabla n) \|_{L^3}\). According to Sobolev interpolation inequality, these terms should be controlled by the product of \(\| (\varphi, u, \nabla n)(t) \|_{L^2}\) and \(\| \nabla^2(\varphi, u, \nabla n) \|_{L^2}\). Therefore,
where $Q_1(t)$ is a small quantity after a long time according to the decay result (1.6) and uniform result (1.5). In order to get the dissipative estimate for $\nabla^2 \varrho$, we establish the following estimate

$$
\frac{d}{dt} \int \nabla u \cdot \nabla^2 \varrho dx + C \| \nabla^2 \varrho \|_{L^2}^2 \lesssim \| \nabla^2 u(t) \|_{H^1}^2 + Q_2(t)(\| \nabla^2 \varrho \|_{L^2}^2 + \| \nabla^2 u \|_{H^1}^2 + \| \nabla^3 n \|_{H^1}^2),
$$

where $Q_2(t)$ is similar as $Q_1(t)$. The combination of (1.8) and (1.9) implies that

$$
\frac{d}{dt} X_h(t) + \frac{1}{2} (\mu \| \nabla^3 u \|_{L^2}^2 + (\mu + \lambda) \| \nabla^2 \text{div } u \|_{L^2}^2 + \| \nabla^4 n \|_{L^2}^2) + C \delta \| \nabla^2 \varrho \|_{L^2}^2
\leq C_2 \delta (\| \nabla^2 u \|_{L^2}^2 + \| \nabla^3 n \|_{L^2}^2), \quad \text{for } t \geq T_1,
$$

(1.10)

where

$$
X_h(t) := \frac{1}{2} (\| \nabla^2 \varrho \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 + \| \nabla^3 n \|_{L^2}^2 + 2 \delta \| \nabla u \cdot \nabla^2 \varrho \|_{L^2}^2),
$$

\(\delta\) is a small constant and \(T_1\) is a large constant. In the following, we focus our attention on how to obtain the optimal decay rate of the second order spatial derivative of large solution for system (1.1). Actually, the presence of the term $\int \nabla u \cdot \nabla^2 \varrho dx$ is the reason that the decay rate of the second order derivative of solution, i.e. $\| \nabla^2 (\varrho, u, \nabla n) \|_{L^2}$ is not optimal. Indeed, $X_h(t) \sim \| (\nabla \varrho, \nabla u) \|_{H^1}^2 + \| \nabla^3 n \|_{L^2}^2$ since the smallness of $\delta$. Thus the second order and the first order spatial derivatives of the solution have the same time decay rate. In order to avoid this obstacle, by using a method as in [51], we need to establish a new estimate for $\int \nabla u \cdot \nabla^2 \varrho dx$, where $\varrho^L$ stand for the low-medium-frequency part of $\varrho$ (see the definition in Appendix B). Then removing this term from (1.10), using the good properties of the low-frequency and high-frequency decomposition, we deduce that

$$
\frac{d}{dt} (X_h(t) - \delta \int \nabla u \cdot \nabla^2 \varrho^L dx) + C_4 (X_h(t) - \delta \int \nabla u \cdot \nabla^2 \varrho^L dx) \leq C \| \nabla^2 (\varrho^L, u^L, \nabla n^L) \|_{L^2}, \quad t \geq T_1.
$$

By noting that

$$
X_h(t) - \delta \int \nabla u \cdot \nabla^2 \varrho^L dx = \frac{1}{2} \| \nabla^2 (\varrho, u, \nabla n) \|_{L^2}^2 + \delta \int \nabla u \cdot \nabla^2 \varrho^L dx \sim \| \nabla^2 (\varrho, u, \nabla n) \|_{L^2}^2,
$$

it is easy to obtain that

$$
\| \nabla^2 (\varrho, u, \nabla n)(t) \|_{L^2}^2 \leq C e^{-C t} \| \nabla^2 (\varrho, u, \nabla n)(T_1) \|_{L^2}^2 + C \int T_1 e^{-C(t-T_1)} \| \nabla^2 (\varrho^L, u^L, \nabla n^L)(\tau) \|_{L^2}^2 d\tau.
$$

The Duhamel’s principle allow us get fast enough decay rate of $\| \nabla^2 (\varrho^L, u^L, \nabla n^L)(\tau) \|_{L^2}^2$, which make it possible to prove the optimal decay rate of $\| \nabla^2 (\varrho, u, \nabla n) \|_{L^2}^2$. The rest of this paper is organized as follows. In Section 2, we will give the proof of Theorem 1.2. In Section 3, we establish the lower bound decay of strong solution to (1.1) with large initial data, which completes the proof of Theorem 1.3. In Appendix A we give the decay estimates of the low-medium-frequency part for the linearized system. In Appendix B we give the definition of the frequency decomposition and some known inequalities.

2 Proof of Theorem 1.2

In this section, we will give the proof for the Theorem 1.2. The analysis proceeds in several steps, which we will give now in detail below.

2.1. Energy estimate

Since the solution itself and its first order spatial derivatives admit the same $L^2$–rate $(1 + t)^{-\frac{1}{4}}$, these quantities can be small enough essentially if the time is large. Thus, we will take the strategy of the frame of small initial data(cf. [10]) to establish the energy estimate. Denoting $\varrho := \rho - 1$, $n := d - \varrho$, we rewrite (1.1) in the perturbation form as follows

$$
\begin{cases}
\partial_t \varrho + \text{div } u = S_1,
\partial_t u - \mu \Delta u - (\mu + \lambda) \text{div } u + P'(1) \nabla \varrho = S_2,
\partial_t n - \Delta n = S_3,
\end{cases}
$$

(2.1)
where the nonlinear terms $S_1$, $S_2$ and $S_3$ are defined by
\[
\begin{align*}
S_1 & := -\varrho \text{div} u - u \cdot \nabla \varrho, \\
S_2 & := -u \cdot \nabla u - h(\varrho)(\mu \Delta u + (\mu + \lambda) \nabla \text{div} u) - f(\varrho)\nabla \varrho - g(\varrho)\nabla d \cdot \Delta d, \\
S_3 & := -u \cdot \nabla n + |\nabla n|^2(n + d),
\end{align*}
\]
where
\[
\begin{align*}
h(\varrho) & := \frac{\varrho}{\varrho + 1}, & f(\varrho) & := \frac{P'(\varrho + 1)}{\varrho + 1} - \frac{P'(1)}{1}, & g(\varrho) & := \frac{1}{\varrho + 1}.
\end{align*}
\]

First, we give the first order spatial derivative estimate as follows.

**Lemma 2.1.** Under the assumptions of Theorem 1.1, the global solution $(\varrho, u, n)$ of Cauchy problem (2.1) has the estimate
\[
\frac{1}{2} \frac{d}{dt} \int (|\nabla \varrho|^2 + |\nabla u|^2 + |\nabla^2 n|^2) dx + \int (\mu |\nabla^2 u|^2 + (\mu + \lambda) |\nabla \text{div} u|^2 + |\nabla^3 n|^2) dx \\
\leq C(\|\varrho\|_{H^1} + \|u\|_{H^1} + \|g\|_{L^2}^2 + \|\nabla n\|_{H^1}^2 + \|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2 + \|\nabla \text{div} u\|_{L^2}^2).
\]

**Proof.** Applying differential operator $\nabla$ to (2.1), then multiplying the resulting identities by $\nabla \varrho$, $\nabla u$ and $\nabla^2 n$ respectively and integrating over $\mathbb{R}^3$, it is easy to obtain
\[
\frac{1}{2} \frac{d}{dt} \int (|\nabla \varrho|^2 + |\nabla u|^2 + |\nabla^2 n|^2) dx + \int (\mu |\nabla^2 u|^2 + (\mu + \lambda) |\nabla \text{div} u|^2 + |\nabla^3 n|^2) dx \\
= \int \nabla S_1 \cdot \nabla \varrho dx + \int \nabla S_2 \cdot \nabla u dx + \int \nabla^2 S_3 \cdot \nabla^2 n dx. \tag{2.2}
\]

Integrating by part and applying the Hölder inequality yield
\[
| \int \nabla S_1 \cdot \nabla \varrho dx | \leq \| S_1 \|_{L^2} \| \nabla^2 \varrho \|_{L^2}. \tag{2.3}
\]

Using the Hölder and Sobolev inequalities, we show that
\[
\| S_1 \|_{L^2} \leq \| \varrho \|_{L^3} \| \text{div} u \|_{L^6} + \| u \|_{L^3} \| \nabla \varrho \|_{L^6} \leq C(\| \varrho \|_{H^1} + \| u \|_{H^1}) (\| \nabla^2 u \|_{L^2} + \| \nabla^2 \varrho \|_{L^2}). \tag{2.4}
\]

This and inequality (2.3) give
\[
| \int \nabla S_1 \cdot \nabla \varrho dx | \leq C(\| \varrho \|_{H^1} + \| u \|_{H^1}) (\| \nabla^2 u \|_{L^2}^2 + \| \nabla^2 \varrho \|_{L^2}^2). \tag{2.5}
\]

Applying the Hölder and Sobolev inequalities, we get
\[
\| u \cdot \nabla u \|_{L^2} \leq \| u \|_{L^3} \| \nabla u \|_{L^6} \leq C \| u \|_{H^1} \| \nabla^2 u \|_{L^2}. \tag{2.6}
\]

Using the lower bound of density (1.3), Sobolev inequality and uniform estimate (1.6), we have
\[
\| h(\varrho)(\mu \Delta u + (\mu + \lambda) \nabla \text{div} u) \|_{L^2} \leq C \| \varrho \|_{L^6} \| \nabla^2 u \|_{L^2} \leq C \| \varrho \|_{L^2}^2 \| \nabla^2 \varrho \|_{L^2}^2 \| \nabla u \|_{L^2} \leq C \| \varrho \|_{L^2}^2 \| \nabla^2 u \|_{L^2}^2. \tag{2.7}
\]

Using the Taylor expression, Hölder and Sobolev inequalities, we get
\[
\| h(\varrho) \nabla \varrho \|_{L^2} \leq C \| \varrho \|_{L^3} \| \nabla \varrho \|_{L^6} \leq C \| \varrho \|_{H^1} \| \nabla^2 \varrho \|_{L^2}. \tag{2.8}
\]

By the lower bound of density (1.3) and the Sobolev inequality, we obtain that
\[
\| g(\varrho) \nabla n \Delta n \|_{L^2} \leq C \| \nabla n \|_{L^3} \| \Delta n \|_{L^6} \leq C \| \nabla n \|_{H^1} \| \nabla^3 n \|_{L^2}. \tag{2.9}
\]

The combination of (2.4), (2.6), (2.8) and (2.9) gives
\[
| \int \nabla S_2 \cdot \nabla u dx | \leq \| S_2 \|_{L^2} \| \nabla^2 u \|_{L^2} \leq (\| u \|_{H^1} + \| \varrho \|_{L^2}^2 + \| \varrho \|_{H^1} + \| \nabla n \|_{H^1}) (\| \nabla^2 u \|_{L^2}^2 + \| \nabla^2 \varrho \|_{L^2}^2 + \| \nabla^3 n \|_{L^2}^2). \tag{2.10}
\]
The routine calculation yields directly
\[ \nabla S_3 = -\nabla u \cdot \nabla n - u \cdot \nabla^2 n + 2|\nabla n| \cdot \nabla^2 n \cdot (n + d) + |\nabla n|^2 \nabla n. \]

It is easy to see that
\[
\|\nabla u \cdot \nabla n\|_{L^2} + \|u \cdot \nabla^2 n\|_{L^2} \leq \|\nabla n\|_{L^3}\|\nabla u\|_{L^6} + \|u\|_{L^3}\|\nabla^2 n\|_{L^6} \leq C(\|\nabla n\|_{L^1}\|\nabla^2 u\|_{L^2} + \|u\|_{H^1}\|\nabla^3 n\|_{L^2}). \tag{2.11}
\]

Using the uniform bound \((1.5)\), we have
\[
\|2|\nabla n| \cdot \nabla^2 n \cdot (n + d)\|_{L^2} \leq C(\|n\|_{L^\infty} + d)\|\nabla n\|_{L^3}\|\nabla^2 n\|_{L^6} \leq C(\|\nabla n\|_{L^2}^{\frac{1}{2}}\|\nabla^2 n\|_{L^2}^{\frac{1}{2}} + d)\|\nabla n\|_{H^1}\|\nabla^3 n\|_{L^2} \leq C\|\nabla n\|_{H^1}\|\nabla^3 n\|_{L^2}. \tag{2.12}
\]

The Sobolev inequality gives
\[
\|\nabla n\|_{L^2} \leq 2\|\nabla n\|_{L^3}\|\nabla^2 n\|_{L^2} \leq C(\|\nabla n\|_{L^2}^2\|\nabla^2 n\|_{L^6} \leq C\|\nabla n\|_{H^1}^2\|\nabla^3 n\|_{L^2}. \tag{2.13}
\]

The combination of \((2.11)\), \((2.12)\) and \((2.13)\) implies
\[
\int \nabla^2 S_3 \cdot \nabla^2 n dx \leq C(\|\nabla n\|_{H^1} + \|u\|_{H^1} + \|\nabla n\|_{H^1}^2)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2). \tag{2.14}
\]

Therefore, combine the estimates \((2.9)\), \((2.10)\), \((2.11)\) and \((2.14)\), we completes the proof of this lemma.

Next, we establish the energy estimate for the second order spatial derivative of solution \((\varrho, u, n)\) for the Cauchy problem \((2.1)\), which can help us achieve the decay rate for them.

**Lemma 2.2.** Under the assumptions of Theorem 1.7, the global solution \((\varrho, u, n)\) of Cauchy problem \((2.1)\) has the estimate
\[
\frac{1}{2} \frac{d}{dt} \int (|\nabla^2 \varrho|^2 + |\nabla^2 u|^2 + |\nabla^3 n|^2) dx + \int (\mu|\nabla^3 u|^2 + (\mu + \lambda)|\nabla^2 \text{div } u|^2 + |\nabla^4 n|^2) dx \leq C(\|\varrho, u, \nabla n\|_{L^2}^2 + \|u, \nabla n\|_{H^1}^2 + \|\nabla^2 u\|_{H^1}^2 + \|\nabla^3 n\|_{H^1}^2). \tag{2.15}
\]

**Proof.** Applying \(\nabla^2\) to \((2.1)\), \((2.1)\) respectively and \(\nabla^3\) to \((2.1)\), then multiplying the resulting identities by \(\nabla^2 \varrho\), \(\nabla^2 u\) and \(\nabla^3 n\) respectively and integrating over \(\mathbb{R}^3\), it is easy to obtain
\[
\frac{1}{2} \frac{d}{dt} \int (|\nabla^2 \varrho|^2 + |\nabla^2 u|^2 + |\nabla^3 n|^2) dx + \int (\mu|\nabla^3 u|^2 + (\mu + \lambda)|\nabla^2 \text{div } u|^2 + |\nabla^4 n|^2) dx \leq \int \nabla^2 S_1 \cdot \nabla^2 \varrho dx + \int \nabla^2 S_2 \cdot \nabla^2 u dx + \int \nabla^3 S_3 \cdot \nabla^3 n dx. \tag{2.16}
\]

Recall that \(S_1 := -\varrho \text{div } u - u \cdot \nabla \varrho\), the direct computation gives
\[ \nabla^2 (\varrho \text{div } u) = \varrho \nabla^2 \text{div } u + 2\nabla \varrho \cdot \nabla \text{div } u + \nabla^2 \varrho \cdot \text{div } u, \]
then it holds
\[
\|\nabla^2 (\varrho \text{div } u)\|_{L^2} \leq \|\varrho\|_{L^\infty} \|\nabla^2 \text{div } u\|_{L^2} + \|\varrho\|_{L^3} \|\nabla \text{div } u\|_{L^6} \|\nabla^2 \varrho\|_{L^2} + \|\text{div } u\|_{L^\infty} \|\nabla^2 \varrho\|_{L^2} \leq C(\|\varrho\|_{L^\infty} + \|\varrho\|_{L^3})(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 \varrho\|_{L^2}^2) + C\|\text{div } u\|_{L^\infty} \|\nabla^2 \varrho\|_{L^2}^2. \tag{2.17}
\]

By routine checking, one may check that
\[ \nabla^2 (u \cdot \nabla \varrho) = u \cdot \nabla (\nabla^2 \varrho) + 2\nabla u \cdot \nabla^2 \varrho + \nabla^2 u \cdot \nabla \varrho. \]

The integration by part yields directly
\[
\int u \cdot \nabla (\nabla^2 \varrho) \cdot \nabla^2 \varrho dx = \frac{1}{2} \int u \cdot \nabla (|\nabla^2 \varrho|^2) dx = -\frac{1}{2} \int \text{div } u |\nabla^2 \varrho|^2 dx,
\]
and hence, we have

\[
\left| \int \nabla^2 (u \cdot \nabla \varphi) \cdot \nabla^2 \varphi dx \right| \\
\leq \| \text{div } u \|_{L^\infty} \| \nabla^2 \varphi \|_{L^2}^2 + (\| \nabla u \|_{L^\infty} \| \nabla^2 \varphi \|_{L^2} + \| \nabla \varphi \|_{L^3} \| \nabla^2 u \|_{L^6}) \| \nabla^2 \varphi \|_{L^2}
\]

(2.18)

The combination of (2.17) and (2.18) gives

\[
\left| \int \nabla^2 S_1 \cdot \nabla^2 \varphi dx \right| \leq C(\| \nabla \varphi \|_{L^3} + \| \varphi \|_{L^\infty}) (\| \nabla^2 \varphi \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2) + C\| \nabla u \|_{L^\infty} \| \nabla^2 \varphi \|_{L^2}^2.
\]

(2.19)

By virtue of the Sobolev inequality and the uniform estimate (1.5), it follows that

\[
\| \varphi \|_{L^\infty} + \| \nabla \varphi \|_{L^3} + \| \nabla u \|_{L^3} \leq C(\| \varphi \|_{L^2}^2 \| \nabla^2 \varphi \|_{L^2}^2 + \| \varphi \|_{L^2} \| \nabla^2 \varphi \|_{L^2}^2 + \| \varphi \|_{L^2} \| \nabla^2 u \|_{L^2}^2)
\]

\[
\leq C(\| \varphi \|_{L^2}^2 + \| \varphi \|_{L^2}^2),
\]

(2.20)

and

\[
\| \nabla u \|_{L^\infty} \| \nabla^2 \varphi \|_{L^2}^2 \leq C\| \nabla u \|_{L^2}^2 \| \nabla^3 u \|_{L^2} \| \nabla^2 \varphi \|_{L^2}^2
\]

\[
\leq C\| \nabla u \|_{L^2}^2 (\| \nabla^3 u \|_{L^2}^2 + \| \nabla^2 \varphi \|_{L^2}^2).
\]

(2.21)

Substituting (2.20) and (2.21) into (2.17), we get

\[
\left| \int \nabla^2 S_1 \cdot \nabla^2 \varphi dx \right| \leq C(\| \varphi \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) (\| \nabla^2 \varphi \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2).
\]

(2.22)

Next, we shall estimate the second term on the right hand side of (2.19). Integrating by part, we get

\[
\int \nabla^2 S_1 \cdot \nabla^3 u dx = - \int \nabla S_2 \cdot \nabla^4 u dx.
\]

(2.23)

Straightforward calculation shows that

\[
\nabla S_2 = - \nabla u \cdot \nabla u - u \cdot \nabla (\nabla u) - \varphi \left( \frac{1+\varphi}{1+\varphi} \right) [\mu \nabla \Delta u + (\mu + \lambda) \nabla^2 \text{div } u]
\]

\[
- \left[ \frac{P'(1 + \varphi)}{1+\varphi} \right] \nabla^2 \varphi - \frac{\varphi}{(1+\varphi)^2} [\mu \Delta u + (\mu + \lambda) \nabla \text{div } u]
\]

\[
- \frac{P''(1 + \varphi)(1 + \varphi) - P'(1 + \varphi)}{(1+\varphi)^2} \nabla \varphi \nabla \varphi - \frac{1}{\varphi + 1} \nabla (\nabla \varphi) \Delta n
\]

\[
- \frac{1}{\varphi + 1} \nabla \varphi \cdot \nabla \Delta n + \frac{\varphi}{(1+\varphi)^2} \nabla \varphi \cdot \Delta n.
\]

(2.24)

Observe that

\[P'(1 + \varphi) = \gamma (1 + \varphi)^{\gamma - 1}, \quad P''(1 + \varphi) = \gamma (1 + \varphi)^{\gamma - 2},\]

and hence, it holds on

\[P''(1 + \varphi)(1 + \varphi) - P'(1 + \varphi) = \gamma (1 + \varphi)^{\gamma - 2} \quad \frac{P'(1 + \varphi)}{1+\varphi} = \gamma (1 + \varphi)^{\gamma - 2}.
\]

(2.25)

The combination of (2.24) and (2.25) yields directly

\[
\| \nabla S_2 \|_{L^2} \leq C(\| \nabla u \|_{L^3} \| \nabla u \|_{L^6} + \| \nabla u \|_{L^3} \| \nabla^2 u \|_{L^6} + \| \nabla \varphi \|_{L^\infty} \| \nabla \varphi \|_{L^3} \| \nabla^2 \varphi \|_{L^2})
\]

\[
+ C(\| \varphi \|_{L^\infty} \| \nabla^2 \varphi \|_{L^2} + \| \nabla \varphi \|_{L^3} \| \nabla^2 u \|_{L^6} + \| \nabla \varphi \|_{L^3} \| \nabla^2 \varphi \|_{L^6})
\]

\[
+ C(\| \nabla^2 u \|_{L^3} \| \nabla^2 n \|_{L^6} + \| \nabla n \|_{L^3} \| \nabla^3 n \|_{L^6} + \| \nabla n \|_{L^\infty} \| \nabla \varphi \|_{L^3} \| \nabla^2 \varphi \|_{L^6})
\]

\[
\leq C(\| \nabla u \|_{L^3} + \| \nabla \varphi \|_{L^3} + \| \nabla \varphi \|_{L^3} + \| \nabla \varphi \|_{H^1} + \| \nabla^2 \varphi \|_{L^2})
\]

\[
+ (\| \nabla^2 n \|_{L^3} + \| \nabla n \|_{L^\infty} \| \nabla \varphi \|_{L^3} + \| \nabla n \|_{H^1}) \| \nabla^3 n \|_{H^2}.
\]

(2.26)
By virtue of the Sobolev inequality and the uniform estimate (1.5), it follows that
\[
   \| \nabla^2 u \|_{L^2}^2 + \| \nabla u \|_{L^\infty}^2 \| \nabla \phi \|_{L^3}^2 \leq C(\| \nabla n \|_{L^2}^{\frac{1}{2}} \| \nabla^2 n \|_{L^2}^{\frac{3}{2}} + \| \nabla n \|_{H^2}^2 \| \nabla^2 \phi \|_{L^2}^{\frac{3}{2}}) 
   \leq C(\| \nabla n \|_{L^2}^{\frac{1}{2}} + \| \phi \|_{L^2}^{\frac{3}{2}}). 
\] (2.27)

Substituting (2.21) and (2.27) into (2.20), we obtain that
\[
   \| \nabla S_2 \|_{L^2} \leq C(\| u \|_{L^2}^{\frac{1}{2}} + \| u \|_{H^1} + \| \phi \|_{L^2}^{\frac{1}{2}} + \| \nabla n \|_{L^2}^{\frac{1}{2}} + \| \nabla n \|_{H^1}^{\frac{3}{2}})(\| \nabla^2 u \|_{H^1} + \| \nabla^2 \phi \|_{L^2} + \| \nabla^3 n \|_{H^1}). 
\] (2.28)

This together with (2.23) implies that
\[
   \left| \int \nabla^2 S_2 \cdot \nabla^2 u dx \right| \leq C(\| u \|_{L^2}^{\frac{1}{2}} + \| u \|_{H^1} + \| \phi \|_{L^2}^{\frac{1}{2}} + \| \nabla n \|_{L^2}^{\frac{1}{2}} + \| \nabla n \|_{H^1}^{\frac{3}{2}})(\| \nabla^2 u \|_{H^1}^2 + \| \nabla^2 \phi \|_{L^2}^2 + \| \nabla^3 n \|_{H^1}^2). 
\] (2.29)

Finally, we focus on the last term on the right hand side of (2.10). Regularly computation shows that
\[
   \nabla^2 S_3 = -\nabla^2 u \cdot \nabla n - 2\nabla u \cdot \nabla^2 n - u \cdot \nabla^3 n + 2|\nabla^2 n|^2(n + d) + 2|\nabla n| |\nabla^3 n|(n + d) + 3|\nabla^2 n|^2 |\nabla n|^2. 
\]

Using the Hölder and Sobolev inequalities, we have
\[
   \| \nabla^2 S_3 \|_{L^2} \leq \| \nabla n \|_{L^\infty} \| \nabla^3 u \|_{L^5} + \| \nabla u \|_{L^3} \| \nabla^2 n \|_{L^6} + \| u \|_{L^\infty} \| \nabla^3 n \|_{L^2} 
   + \| u \|_{L^\infty} \| \nabla^3 n \|_{L^2} \| \nabla^2 n \|_{L^6} 
   \leq C(\| \nabla n \|_{L^\infty} + \| u \|_{L^3} + \| u \|_{L^\infty})(\| \nabla^3 n \|_{L^2} + \| \nabla^3 u \|_{L^2}) 
   + C(\| u \|_{L^\infty} + \| u \|_{L^3})(\| \nabla^2 n \|_{L^3} \| \nabla^3 n \|_{L^2} + \| \nabla n \|_{L^\infty} \| \nabla^4 n \|_{L^2}) 
   + C \| \nabla n \|_{L^\infty} \| \nabla n \|_{L^3} \| \nabla^3 n \|_{L^2}. 
\] (2.30)

By the Sobolev inequality and the uniform estimate (1.5), it is easy to check that
\[
   \| n \|_{L^\infty} + \| \nabla n \|_{L^3} \leq C(\| \nabla n \|_{L^2}^{\frac{1}{2}} \| \nabla^2 n \|_{L^2}^{\frac{3}{2}}) \leq C(\| \nabla^2 n \|_{L^2}^{\frac{1}{2}}), 
\] (2.31)

and
\[
   \| \nabla^2 n \|_{L^2} \leq C\| \nabla^2 n \|_{L^2}^{\frac{1}{2}} \| \nabla^3 n \|_{L^2}^{\frac{1}{2}} \leq C\| \nabla^2 n \|_{L^2}^{\frac{1}{2}}. 
\] (2.32)

Substituting (2.21), (2.31) and (2.32) into (2.30), we get
\[
   \| \nabla^2 S_3 \|_{L^2} \leq C(\| \nabla^2 n \|_{L^2}^{\frac{1}{2}} + \| u \|_{L^2}^{\frac{1}{2}} + \| \nabla^2 n \|_{L^2})(\| \nabla^3 n \|_{H^1} + \| \nabla^3 u \|_{L^2}), 
\]

which implies that
\[
   \left| \int \nabla^3 S_3 \cdot \nabla^3 n dx \right| \leq \| \nabla^2 S_3 \|_{L^2} \| \nabla^4 n \|_{L^2} \leq C(\| \nabla^2 n \|_{L^2}^{\frac{1}{2}} + \| u \|_{L^2}^{\frac{1}{2}} + \| \nabla^2 n \|_{L^2})(\| \nabla^3 n \|_{H^1}^2 + \| \nabla^3 u \|_{L^2}^2). 
\] (2.33)

Then the combination of (2.22), (2.29) and (2.33) completes the proof.

In order to close the energy estimate, it is necessary to establish the dissipation estimate for $\nabla^2 \phi$.

**Lemma 2.3.** Under the assumptions of Theorem 1.1, the global solution $(\phi, u, n)$ of Cauchy problem (2.1) has the estimate
\[
   \frac{d}{dt} \int \nabla u \cdot \nabla \phi dx + \frac{3P'(1)}{4} \int |\nabla^2 \phi|^2 dx \leq C(\| \nabla u \|_{H^1}^2 + \| (\phi, u, \nabla n) \|_{H^1}^2 + \| (\phi, u, \nabla n) \|_{H^1}^2)(\| \nabla^2 \phi \|_{L^2}^2 + \| \nabla^2 u \|_{H^1}^2 + \| \nabla^3 n \|_{H^1}^2). 
\] (2.34)

**Proof.** Applying $\nabla$ operator to (2.1) and multiplying the resulting by $\nabla^2 \phi$, and integrating over $\mathbb{R}^3$, we have
\[
   \int \partial_t (\nabla u) \cdot \nabla^2 \phi dx - \int (\mu \Delta \nabla u + (\mu + \lambda) \nabla^2 \div u) \cdot \nabla^2 \phi dx + P'(1) \int |\nabla^2 \phi|^2 dx = \int \nabla S_2 \cdot \nabla^2 \phi dx.
\]
Using the first equation in (2.1), it holds on
\[
\int \partial_t (\nabla u) \cdot \nabla^2 \varphi dx = \frac{d}{dt} \int \nabla u \cdot \nabla^2 \varphi dx - \int \nabla u \cdot \nabla^2 \varphi dx
\]
\[
= \frac{d}{dt} \int \nabla u \cdot \nabla^2 \varphi dx + \int \nabla \div u \cdot \varphi_t dx
\]
\[
= \frac{d}{dt} \int \nabla u \cdot \nabla^2 \varphi dx + \int \nabla \div u \cdot (\nabla S_1 - \nabla \div u) dx.
\]
Then integrating by part, and using the Hölder and Cauchy inequalities, we obtain that
\[
\frac{d}{dt} \int \nabla u \cdot \nabla^2 \varphi dx + P'(1) \int \nabla^2 \varphi^2 dx
\]
\[
= \int (\mu \Delta \nabla u + (\mu + \lambda) \nabla^2 \varphi u) \cdot \nabla^2 \varphi dx + \int \nabla S_2 \cdot \nabla^2 \varphi dx + \int \nabla \div u \cdot (\nabla \div u - \nabla S_1) dx
\]
\[
\leq \frac{1}{4} P'(1) \|\nabla^2 \varphi\|_{L^2}^2 + C \|\nabla^2 u\|_{H^1}^2 + C(\|\nabla S_2\|_{L^2}^2 + \|S_1\|_{L^2}^2),
\]
which, together with (2.4) and (2.28) completes the proof of this lemma.

Combining the estimates obtained in Lemmas 2.2 and 2.3 we derive the following energy estimate.

**Lemma 2.4.** Under the assumptions of Theorem 1.1, we define
\[
X_h(t) := \frac{1}{2} \left( \|\nabla^2 \varphi\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2 + 2 \delta \int \nabla u \cdot \nabla^2 \varphi dx \right).
\]
Then there exists a large enough time $T_1 > 0$, such that
\[
\frac{d}{dt} X_h(t) + \frac{1}{2} \int (\mu |\nabla^3 u|^2 + (\mu + \lambda) |\nabla^2 \div u|^2 + |\nabla^4 n|^2) dx + \frac{P'(1) \delta}{2} \|\nabla^2 \varphi\|_{L^2}^2 \leq \delta C_2 \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2)
\]
for all $t \geq T_1$, where $C_2$ is a constant independent of time and $\delta$ is a small constant.

**Proof.** Multiplying $\delta$ to (2.34) and summing with (2.15), choosing $\delta$ small enough and using the uniform estimate (1.6), we get
\[
\frac{d}{dt} X_h(t) + \frac{3}{4} \left( \mu |\nabla^3 u|^2 + (\mu + \lambda) |\nabla^2 \div u|^2 + |\nabla^4 n|^2 \right) dx + \frac{3P'(1) \delta}{4} \|\nabla^2 \varphi\|_{L^2}^2
\]
\[
\leq C \left( \|\varphi, u, \nabla u, \nabla n\|_{L^{\frac{5}{2}}} + \|(\varphi, u, \nabla u)\|_{H^1} + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2 \right)
\]
\[
+ C \left( \|\varphi, u, \nabla u, \nabla n\|_{L^{\frac{5}{2}}} + \|(u, \nabla n)\|_{H^1} + \|\nabla^2 n\|_{L^2}^2 \right) \|\nabla^3 n\|_{L^2}^2
\]
\[
+ C \delta \|\nabla^2 u\|_{L^2}^2 + C_1 \delta \|\nabla^3 n\|_{L^2}^2.
\]
According to the decay result (1.6), one may conclude that
\[
\|\varphi, u, \nabla u, \nabla n\|_{L^{\frac{5}{2}}} + \|(u, \nabla n)\|_{H^1} + \|\nabla^2 n\|_{L^2}^2 \leq C(1 + t)^{-\frac{\mu}{2}}.
\]
Thus, there exists a large time $T_1 > 0$ such that
\[
\|\varphi, u, \nabla u, \nabla n\|_{L^{\frac{5}{2}}} + \|(u, \nabla n)\|_{H^1} + \|\nabla^2 n\|_{L^2}^2 \leq \frac{1}{4} \min \{\mu, 3, P'(1) \delta, 4C_1 \delta\}
\]
holds on for all $t \geq T_1$. Therefore, we obtain that
\[
\frac{d}{dt} X_h(t) + \frac{1}{2} \int (\mu |\nabla^3 u|^2 + (\mu + \lambda) |\nabla^2 \div u|^2 + |\nabla^4 n|^2) dx + \frac{P'(1) \delta}{2} \|\nabla^2 \varphi\|_{L^2}^2 \leq \delta C_2 \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2),
\]
which completes the proof of this lemma.
2.2. Cancellation of a low-medium-frequency part

In this subsection, based on the second order energy estimate (2.35), we get $L^2_t L^2_x$ norm estimate on $\nabla ^2 (g, u, \nabla n)$ by removing the low-medium-frequency part of the term $\int \nabla u \cdot \nabla ^2 g dt$.

Lemma 2.5. It holds that

$$\|\nabla ^2 (g, u, \nabla n)(t)\|_2^2 \leq C e^{-C_4 t} \|\nabla ^2 (g, u, \nabla n)(T_1)\|_2^2 + C \int _{T_1} ^{t} e^{-C_4 (t-\tau)} \|\nabla ^2 (g, u, \nabla n)(\tau)\|_2^2 d\tau,$$

where the positive constant $C_4$ is independent of time, $T_1$ is the large time given in Lemma 2.4.

Proof. Applying $\nabla$ on the second equation of (1.1), then taking $L^2$ inner product with $\nabla ^2 g$ (see (2.4)), we obtain that

$$\int \nabla u_t \cdot \nabla ^2 g dt = \int (\mu \Delta u + (\mu + \lambda) \nabla ^2 \text{div } u) \cdot \nabla ^2 g dt - P'(1) \int \nabla ^2 g \cdot \nabla ^2 g dt + \int \nabla S_2 \cdot \nabla ^2 g. \tag{2.36}$$

Integrating by part and using (2.1), we get

$$\int \nabla u_t \cdot \nabla ^2 g dt = \frac{d}{dt} \int \nabla u \cdot \nabla ^2 g = \int \nabla u \cdot \nabla ^2 g dt - \int \nabla u_t \cdot \nabla ^2 g dt \tag{2.37} \int \nabla \text{div } u \cdot \nabla g dt.$$ 

Therefore, substituting (2.37) into (2.36), and using the Hölder and Cauchy inequalities, we get

$$-\frac{d}{dt} \int \nabla u \cdot \nabla ^2 g dt \leq \frac{\mu}{2} \|\nabla ^3 u\|_2^2 + \frac{\mu + \lambda}{2} \|\nabla ^2 \text{div } u\|_2^2 + \frac{P'(1)}{4} \|\nabla ^2 g\|_2^2 + \frac{1}{2} \|\nabla S_1\|_2^2 + \frac{1}{2} \|\nabla S_2\|_2^2,$$

By routine calculation, it is easy to see that

$$\|\nabla S_1\|_2 \leq \|\nabla g\|_L^2 \|\nabla \text{div } u\|_L^6 + \|g\|_L^\infty \|\nabla \text{div } u\|_L^2 + \|u\|_L^\infty \|\nabla ^2 g\|_L^2 + \|\nabla g\|_L^1 \|\nabla u\|_L^6 \leq C (\|\nabla g\|_L^2 + \|g\|_L^\infty + \|u\|_L^\infty) (\|\nabla ^2 u\|_L^2 + \|\nabla ^2 g\|_L^2) \tag{2.38} \leq C (\|\nabla g\|_L^2 + \|u\|_L^2) (\|\nabla ^2 u\|_L^2 + \|\nabla ^2 g\|_L^2),$$

where we used (2.20) in the last inequality. According to the frequency decomposition (2.4), and using Lemma 3.1, we have

$$\|\nabla S_1\|_L^2 \leq \|\nabla S_1\|_L^2 + \|\nabla S_1\|_L^2 \leq C \|\nabla S_1\|_L^2,$$

this together with (2.28) and (2.38) gives rise to

$$\|\nabla S_1\|_2^2 + \|\nabla S_2\|_2^2 \leq C (\|\nabla g\|_L^2 + \|u\|_L^2) (\|\nabla ^2 u\|_H^2 + \|\nabla ^2 g\|_L^2 + \|\nabla ^3 u\|_H^2).$$

Then we have

$$-\frac{d}{dt} \int \nabla u \cdot \nabla ^2 g dt \leq \frac{\mu}{2} \|\nabla ^3 u\|_2^2 + \frac{\mu + \lambda}{2} \|\nabla ^2 \text{div } u\|_2^2 + \frac{P'(1)}{4} \|\nabla ^2 g\|_2^2 + \|\nabla \text{div } u\|_L^6 + \|\nabla ^2 g\|_L^2 \tag{2.39} \leq C (\|\nabla g\|_L^2 + \|u\|_L^2) (\|\nabla ^2 u\|_L^2 + \|\nabla ^2 g\|_L^2 + \|\nabla ^3 u\|_H^2).$$

The Optimal Decay Rate of Strong Solution for the CNLC Equations with Large Initial Data
Using Lemma B.1, we have
\begin{equation}
\delta \leq C_2(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2) + \frac{\mu}{2}\|\nabla^3 u\|_{L^2}^2 + \frac{\mu + \lambda}{2}\|\nabla^2 \text{div } u\|_{L^2}^2
+ \frac{P'(1)}{4}\|\nabla^2 \theta\|_{L^2}^2 + \delta\|\nabla \text{div } u\|_{L^2}^2 + \frac{1}{2}\delta\|\nabla^2 \text{div } u\|_{L^2}^2 + C\delta\|\nabla^2 \theta\|_{L^2}^2
+ \delta C_3(\|\nabla^2 u\|_{H^1}^2 + \|\nabla^2 \theta\|_{L^2}^2 + \|\nabla^3 n\|_{H^1}^2).
\end{equation}

Using Lemma B.1 we have
\begin{equation}
\frac{\mu}{2}\|\nabla^3 u\|_{L^2}^2 + \frac{1}{2}\|\nabla^4 n\|_{L^2}^2 \geq \frac{\mu}{4}R_0^2\|\nabla^2 u\|_{L^2}^2 + \frac{\mu}{4}\|\nabla^3 u\|_{L^2}^2 + \frac{\mu + \lambda}{4}\|\nabla^2 \text{div } u\|_{L^2}^2
\end{equation}

Substituting (2.41) into (2.40), and adding \(\frac{\mu}{4}R_0^2\|\nabla^2 u\|_{L^2}^2 + \frac{1}{4}R_0^2\|\nabla^3 n\|_{L^2}^2\) on both side of the resulting inequality, we get
\begin{equation}
\frac{d}{dt}(X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta^L dx) + \frac{\mu}{8}R_0^2\|\nabla^2 u\|_{L^2}^2 + \frac{\mu}{8}\|\nabla^3 u\|_{L^2}^2 + \frac{\mu + \lambda}{8}\|\nabla^2 \text{div } u\|_{L^2}^2
+ \frac{P'(1)}{8}\|\nabla^2 \theta\|_{L^2}^2
\leq \delta C_2(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2) + \frac{\mu}{2}\|\nabla^3 u\|_{L^2}^2 + \frac{\mu + \lambda}{2}\|\nabla^2 \text{div } u\|_{L^2}^2
\end{equation}

Choosing \(\delta \leq \min\{\frac{\mu}{4}R_0^2, \frac{\mu}{8}R_0^2, \frac{\mu + \lambda}{8}\}\), \(R_0^2 \geq \max\{\frac{6\mu}{4}, \frac{\mu}{4}, \frac{6\mu + \lambda}{4}\}\), \(\delta C_3\), then we have
\begin{equation}
\frac{d}{dt}(X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta^L dx) + \frac{\mu}{16}R_0^2\|\nabla^2 u\|_{L^2}^2 + \frac{\mu}{16}\|\nabla^3 u\|_{L^2}^2 + \frac{\mu + \lambda}{16}\|\nabla^2 \text{div } u\|_{L^2}^2
+ \frac{P'(1)}{16}\|\nabla^2 \theta\|_{L^2}^2
\leq C\|\nabla^2 (\theta^L, u^L, \nabla^{H})\|_{L^2}^2.
\end{equation}

According to the decomposition (4.4), we have
\begin{equation}
X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta^L dx = \frac{1}{2}(\|\nabla^2 \theta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2) + \delta \int \nabla u \cdot \nabla^2 \theta dx.
\end{equation}

Integrating by part, and using Lemma B.1 we get
\begin{equation}
\delta \int \nabla u \cdot \nabla^2 \theta dx = -\delta \int \text{div } u \cdot \nabla \theta dx \leq \delta \|\nabla \theta\|_{L^2}^2 + \delta \|\text{div } u\|_{L^2}^2 \leq \delta \|\nabla \theta\|_{L^2}^2 + \delta \|\text{div } u\|_{L^2}^2,
\end{equation}

which implies that
\begin{equation}
X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta dx \sim \|\nabla (\theta, u, \nabla n)\|_{L^2}^2,
\end{equation}

where we have used the fact that \(0 < \delta \leq \frac{\mu}{8}\). Thanks to (2.42) and (2.43), there exists a constant \(C_4\) such that
\begin{equation}
\frac{d}{dt}(X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta dx) + C_4(X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta dx) \leq C\|\nabla^2 (\theta^L, u^L, \nabla^{H})\|_{L^2}^2.
\end{equation}

Multiplying (2.44) by \(e^{C_4t}\) and integrating with respect to time over \([T_1, t]\), we get
\begin{equation}
X_h(t) - \delta \int \nabla u \cdot \nabla^2 \theta dx \leq e^{-C_4(t-T_1)}(X_h(T_1) - \delta \int \nabla u(T_1) \cdot \nabla^2 \theta(T_1) dx) + C \int_{T_1}^{t} e^{-C_4(t-\tau)}\|\nabla^2 (\theta^L, u^L, \nabla^{H})\|_{L^2}^2 d\tau.
\end{equation}

Using the equivalent equation (2.44) again, we complete the proof of this lemma. \(\square\)
2.3. Decay estimates of the low-medium-frequency part

In this subsection, based on the classical semigroup method and the $L^2$-norm decay estimate for spectral analysis on the linearized system, we obtain the estimate of the low-medium-frequency part of the solution to the Cauchy problem (2.1). In order to get the decay estimate of $\|\nabla n\|_{H^2}$, we applying $\nabla$ operator to the third equation of system (2.1), then (2.1) becomes

$$
\begin{aligned}
\partial_t \rho + \text{div} u &= S_1, \\
\partial_t u - \mu \Delta u - (\mu + \lambda)\nabla \text{div} u + P'(1)\nabla \rho &= S_2, \\
\partial_t \nabla n - \Delta \nabla n &= \nabla S_3,
\end{aligned}
$$

(2.45)

Denote

$$
\mathbb{U}(t) := (\rho(t), u(t), \nabla n(t))^T,
$$

and the differential operator $\mathcal{G}$:

$$
\mathcal{G} = \begin{pmatrix}
0 & \text{div} & 0 \\
-P'(1)\nabla & -\mu \Delta - (\mu + \lambda)\nabla \text{div} & 0 \\
0 & 0 & -\Delta
\end{pmatrix}
$$

Then we can rewrite the system (2.45) as

$$
\begin{aligned}
\partial_t \mathbb{U} + \mathcal{G} \mathbb{U} &= S(\mathbb{U}), \\
\mathbb{U}|_{t=0} &= \mathbb{U}(0),
\end{aligned}
$$

(2.46)

where

$$
S(\mathbb{U}) := (S_1, S_2, \nabla S_3)^T, \quad \mathbb{U}(0) := (\rho_0, u_0, \nabla n_0).
$$

(2.47)

Moreover, we define

$$
\mathbb{U}(t) := (\bar{\rho}(t), \bar{u}(t), \nabla \bar{n}(t))^T,
$$

then we have the following corresponding linearized problem

$$
\begin{aligned}
\partial_t \mathbb{U} + \mathcal{G} \mathbb{U} &= 0, \\
\mathbb{U}|_{t=0} &= \mathbb{U}(0),
\end{aligned}
$$

(2.48)

Taking the Fourier transform on (2.48) with respect to space variable and solving the ODE, we get

$$
\mathbb{U}(t) = \mathcal{G}(t) \mathbb{U}(0),
$$

where $\mathcal{G}(t) = e^{-t\mathcal{G}} (t \geq 0)$ is the semigroup generated by the operator $\mathcal{G}$ and $\mathcal{G}(t)f := \mathcal{F}^{-1}(e^{-t\mathcal{G}_\xi} \hat{f}(\xi))$ with

$$
\mathcal{G}_\xi = \begin{pmatrix}
0 & i\xi^T & 0 \\
0 & \mu |\xi|^2 \delta_{ij} + (\mu + \lambda)\xi_i\xi_j & 0 \\
0 & 0 & |\xi|^2
\end{pmatrix}
$$

Next, according to the decay estimate of solution to the linearized system (2.48) in frequency regimes (see Appendix $\text{A}$), we give the following estimate of the low-medium-frequency part of the solution.

**Lemma 2.6.** Assume $1 \leq p \leq 2$, for any integer $k \geq 0$, there holds

$$
\|\nabla^k (\mathcal{G}(t) \mathbb{U}(0))\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} + \frac{1}{p} - \frac{1}{2}}\|\mathbb{U}(0)\|_{L^p}.
$$

**Proof.** Set $b := \Lambda^{-1} \text{div} u$ be the “compressible part”, $\mathcal{P}u := \Lambda^{-1} \text{curl} u$ be the “incompressible part”, where $\Lambda^2 = \Delta$. 

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then \( u = -\Lambda^{-1} \nabla b - \Lambda^{-1} \text{div} P u \). Using the Plancherel theorem, (A.12) and (A.14), we obtain that

\[
\| \partial_x^\alpha (\hat{\omega}^L, \hat{b}^L, \nabla \hat{\omega}^L) (t) \|_{L^2}^2 = \| (i \xi)^\alpha (\hat{\omega}^L, \hat{b}^L, \nabla \hat{\omega}^L) \|_{L^2}^2
\]

\[
= \int |(i \xi)^\alpha (\hat{\omega}^L, \hat{b}^L, \nabla \hat{\omega}^L)(t, \xi)|^2 d\xi
\]

\[
\leq C \int_{|\xi| \leq R_0} |\xi|^{2\alpha} (\hat{\omega}, \hat{b}, \nabla \hat{\omega})(t, \xi)|^2 d\xi
\]

\[
\leq C \int_{|\xi| \leq \rho_0} |\xi|^{2\alpha} e^{-C_\alpha |\xi|^2 t} (\hat{\omega}, \hat{b}, \nabla \hat{\omega})(0, \xi)|^2 d\xi
\]

\[
+ \int_{\rho_0 < |\xi| \leq R_0} |\xi|^{2\alpha} e^{-\alpha t} (\hat{\omega}, \hat{b}, \nabla \hat{\omega})(0, \xi)|^2 d\xi.
\]

Using the Hölder, Hausdorff-Young inequalities, we have

\[
\| \partial_x^\alpha (\hat{\omega}^L, \hat{b}^L, \nabla \hat{\omega}^L)(t) \|_{L^2} \leq C \| (\hat{\omega}, \hat{b}, \nabla \hat{\omega})(0) \|_{L^2} \| 1 + t \|_{L^p}^{\frac{1}{p} - \frac{1}{2} - \frac{\alpha}{2}} \leq C \| (\hat{\omega}, \hat{b}, \nabla \hat{\omega})(0) \|_{L^p} \| 1 + t \|_{L^p}^{\frac{1}{p} - \frac{1}{2} - \frac{\alpha}{2}},
\]

(2.49)

where \( 1 \leq p \leq 2 \leq q \leq +\infty \). Similarly, according to (A.15), we have

\[
\| \partial_x^\alpha (\hat{\omega}^L)(t) \|_{L^2} \leq C \left( \int_{|\xi| \leq R_0} |\xi|^{2\alpha} \| \hat{\omega}(t, \xi) \|_{L^2}^2 d\xi \right)^{\frac{1}{2}} \leq C \left( \int_{|\xi| \leq R_0} e^{-2\mu |\xi|^2 t} \| \hat{\omega}(0, \xi) \|_{L^2}^2 d\xi \right)^{\frac{1}{2}}
\]

(2.50)

The combination of (2.49) and (2.50) completes the proof of this lemma. \( \square \)

Next, we establish the decay estimates of the solution to the nonlinear problem (2.46)-(2.47). According to Duhamel principle, we rewrite the solution of system (2.46) as follows

\[
U(t) = G(t)U(0) + \int_0^t G(t-\tau)S(U)(\tau) d\tau.
\]

Then we get the following estimates on the low-medium-frequency part of the solution to the nonlinear problem (2.46)-(2.47).

**Lemma 2.7.** For any integer \( k \geq 0 \), it holds true

\[
\| \nabla^k U(t) \|_{L^2} \leq C_0 (1 + t)^{-\frac{7}{2} - \frac{k}{2}} \| U(0) \|_{L^1} + C_0 \int_0^t \int_0^\tau (1 + t - \tau)^{-\frac{7}{2} - \frac{k}{2}} \| S(U)(\tau) \|_{L^1} d\tau d\tau
\]

\[
+ C_0 \int_0^t (1 + t - \tau)^{-\frac{7}{2}} \| S(U)(\tau) \|_{L^2} d\tau,
\]

where the positive constant \( C_0 \) independent of time.

2.4. Decay rate for the nonlinear system

In this subsection, we will establish the time decay rate of the solution to the original nonlinear problem (2.45).

**Lemma 2.8.** Under the assumptions of Theorem 1.1, there exists a positive constant \( T_2 \), such that the global solution \( (\rho, u, n) \) of Cauchy problem (2.41) has the estimate

\[
\| \nabla^k (\rho, u, \nabla n)(t) \|_{L^2} \leq C (1 + t)^{-\frac{3}{2} - \frac{k}{2}}, \quad k = 0, 1, 2,
\]

(2.51)

for all \( t \geq T_2 \). Here \( C \) is a positive constant independent of time.

**Proof.** Let us denote

\[
M(t) := \sup_{0 \leq \tau \leq t} \sum_{i=0}^2 (1 + \tau)^{\frac{1}{2} + \frac{i}{2}} \| \nabla^i (\rho, u, \nabla n)(\tau) \|_{L^2},
\]

(2.52)
and hence, we have for $0 \leq l \leq 2$

$$\|\nabla^l (\varrho, u, \nabla n)(\tau)\|_{L^2} \leq C(1 + \tau)^{-\frac{4}{5} - \frac{l}{2}} M(t), \quad 0 \leq \tau \leq t. \quad (2.53)$$

According to the definition of $S(\mathbb{U})$, we get

$$\|S(\mathbb{U})(\tau)\|_{L^1} \leq \|S_1(\tau)\|_{L^1} + \|S_2(\tau)\|_{L^1} + \|\nabla S_3(\tau)\|_{L^1}. \quad (2.54)$$

By routine checking, it is easy to see

$$\|S_1\|_{L^1} \leq \|\varrho(u)\|_{L^2} \|\nabla (\varrho, u)\|_{L^2}. \quad (2.55)$$

By the Hölder and Sobolev inequalities, we have

$$\|S_2\|_{L^1} \leq \|u\|_{L^6} \|\nabla u\|_{L^6} + \|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^2 n\|_{L^2} \leq C\|\varrho(u, \nabla n)\|_{L^6} \|\nabla (\varrho, u, \nabla n)\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^3 u\|_{L^2}, \quad (2.56)$$

and

$$\|\nabla S_3\|_{L^1} \leq \|\nabla u\|_{L^6} \|\nabla n\|_{L^6} + \|u\|_{L^6} \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^6} \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^6} \|\nabla^2 n\|_{L^2} \leq C\|\varrho(u, \nabla n)\|_{L^6} \|\nabla (\varrho, u, \nabla n)\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^3 n\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^2 n\|_{L^2}, \quad (2.57)$$

where we used interpolation inequalities in the last inequality as follows

$$\|\n\|_{L^\infty} + \|\nabla n\|_{L^3} \leq \|\nabla n\|_{L^2} \|\nabla^2 n\|_{L^2}. \quad (2.58)$$

Adding $(2.55)$, $(2.56)$ and $(2.57)$ into $(2.54)$, and using the decay estimate $(1.6)$, we get

$$\|S(\mathbb{U})(\tau)\|_{L^1} \leq C(1 + \tau)^{-\frac{4}{5} - \frac{l}{2}} + \|\nabla u\|_{L^6} \|\nabla^3 u\|_{L^2}. \quad (2.58)$$

Next, we need estimate $\|S(\mathbb{U})(\tau)\|_{L^2}$. According to the definition of $S(\mathbb{U})$, we get

$$\|S(\mathbb{U})(\tau)\|_{L^2} \leq \|S_1(\tau)\|_{L^2} + \|S_2(\tau)\|_{L^2} + \|\nabla S_3(\tau)\|_{L^2}. \quad (2.59)$$

Using the decay estimate $(1.6)$ and $(2.58)$, we get

$$\|S_1(\tau)\|_{L^2} \leq C\|\varrho(u)\|_{H^s} \|\nabla^2 (\varrho, u)\|_{L^2} \leq (1 + \tau)^{-\frac{4s}{5}} M(t). \quad (2.59)$$

Using the Hölder, Sobolev inequalities, the uniform estimate $(1.5)$ and decay estimate $(1.6)$, we have

$$\|S_2(\tau)\|_{L^2} \leq \|u\|_{L^6} \|\nabla u\|_{L^6} + \|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla^2 n\|_{L^2} \leq C((1 + \tau)^{-\frac{4}{5} + (1 + \tau)^{-\frac{s}{2}}}) \|\nabla (\varrho, u, \nabla n)\|_{L^2} \leq C(1 + \tau)^{-\frac{4s}{5}} M(t). \quad (2.60)$$

Similarly, we have

$$\|\nabla S_3(\tau)\|_{L^2} \leq \|\nabla n\|_{L^3} \|\nabla u\|_{L^6} + \|\nabla u\|_{L^6} \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^3} \|\nabla^2 n\|_{L^2} \leq C((1 + \tau)^{-\frac{4}{5}} + (1 + \tau)^{-\frac{s}{2}}) \|\nabla (\varrho, u, \nabla n)\|_{L^2} \leq C(1 + \tau)^{-\frac{4s}{5}} M(t). \quad (2.61)$$

By the Sobolev inequality, the uniform estimate $(1.5)$, and $(2.58)$, we get

$$\|\n\|_{L^\infty} \leq C\|\nabla n\|_{L^2} \|\nabla^2 n\|_{L^2} \leq C.$$

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and 
\[ \| \nabla n \|_{L^\infty} \leq C \| \nabla^2 n \|_{L^2}^{\frac{1}{2}} \| \nabla^3 n \|_{L^2}^{\frac{1}{2}} \leq (1 + \tau)^{-\frac{3}{4}} M(t) (1 + \tau)^{-\frac{7}{8}} M(t) \leq C(1 + \tau)^{-\frac{10}{8}} M(t). \]
This together with the decay estimate (1.0) yields
\[ \| \nabla S_3(\tau) \|_{L^2} \leq C(\| u, \nabla n \|_{H^1} \| \nabla^2 u, \nabla n \|_{L^2} + C(1 + \tau)^{-\frac{4}{5}} M(t) \| \nabla n \|_{H^1} \| \nabla^3 n \|_{L^2} \]
\[ \leq C(1 + \tau)^{-\frac{4}{5}} M(t) + C(1 + \tau)^{-3} M(t) \]
\[ \leq C(1 + \tau)^{-\frac{10}{8}} M(t). \]
Collecting the above estimates, we obtain that
\[ \| S(U)(\tau) \|_{L^2} \leq C(1 + \tau)^{-\frac{4}{5}} M(t). \] (2.59)
By Lemma 2.7, 2.68 and 2.69, we have for 0 \leq k \leq 2
\[ \| \nabla^k U^L(t) \|_{L^2} \leq C_5 (1 + t)^{-\frac{k}{2} - \frac{1}{4}} \| U(0) \|_{L^1} + C_5 \int_0^t (1 + t - \tau)^{-\frac{k}{2}} (1 + \tau)^{-\frac{k}{2} - \frac{1}{4}} M(t) d\tau \]
\[ + C_5 \int_0^{\frac{t}{2}} (1 + \tau)^{-\frac{k}{2}} + \| \partial_t \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^3 u \|_{L^2} (1 + t - \tau)^{-\frac{3}{4} - \frac{k}{8}} d\tau. \] (2.60)
Direct calculation gives rise to
\[ \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{k}{2}} (1 + \tau)^{-\frac{k}{2} - \frac{1}{4}} M(t) d\tau \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}} M(t). \] (2.61)
and
\[ \int_0^{\frac{t}{2}} (1 + \tau)^{-\frac{k}{2}} (1 + t - \tau)^{-\frac{k}{2} - \frac{1}{4}} d\tau \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}}. \] (2.62)
Using the Young inequality, the decay estimate (1.0) and the uniform estimate (1.1), we get
\[ \int_0^{\frac{t}{2}} (1 + \tau)^{-\frac{k}{2} - \frac{1}{4}} \| \partial_t \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^3 u \|_{L^2} d\tau \]
\[ \leq C \int_0^{\frac{t}{2}} (1 + \tau)^{-\frac{k}{2} - \frac{1}{4}} (\| \partial_t \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2) d\tau \]
\[ \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}} \int_0^{\frac{t}{2}} (1 + \tau)^{-\frac{k}{2} - \frac{1}{4}} \| \nabla^3 u(\tau) \|_{L^2}^2 d\tau \]
\[ \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}}. \] (2.63)
Substituting the estimates (2.61), (2.62) and (2.63) into (2.60), we get
\[ \| \nabla^k U^L(t) \|_{L^2} \leq C (\| U(0) \|_{L^1} + C + (1 + t)^{-\frac{k}{2}} M(t))(1 + t)^{-\frac{k}{2} - \frac{1}{4}} \]
which together with Lemma 2.5 yields
\[ \| \nabla^2 U(t) \|_{L^2}^2 \leq C e^{-C_4 t} \| \nabla^2 U(T_1) \|_{L^2}^2 + C (\| U(0) \|_{L^1}^2 + 1) \int_{T_1}^t e^{-C_4 (t-\tau)} (1 + \tau)^{-\frac{5}{8}} d\tau \]
\[ + C \int_{T_1}^t e^{-C_4 (t-\tau)} (1 + \tau)^{-\frac{7}{8}} M^2(\tau)(1 + \tau)^{-\frac{7}{8}} d\tau \]
\[ \leq C e^{-C_4 t} \| \nabla^2 U(T_1) \|_{L^2}^2 + C (\| U(0) \|_{L^1}^2 + 1)(1 + t)^{-\frac{7}{8}} + C M^2(t)(1 + t)^{-\frac{7}{8}} \]
\[ \leq C e^{-C_4 t} \| \nabla^2 U(T_1) \|_{L^2}^2 + C (\| U(0) \|_{L^1}^2 + 1 + (1 + t)^{-\frac{7}{8}} M^2(t))(1 + t)^{-\frac{7}{8}} \]
for \( t \geq T_1 \). Therefore, using the decomposition (1.4) and Lemma (5.1) for 0 \leq k \leq 2, we obtain that
\[ \| \nabla^k U(t) \|_{L^2}^2 \leq C \| \nabla^k U^L(t) \|_{L^2}^2 + C \| \nabla^k U^H(t) \|_{L^2}^2 \leq C \| \nabla^k U^L(t) \|_{L^2}^2 + C \| \nabla^2 U(t) \|_{L^2}^2. \] (2.66)
Putting (2.63) and (2.65) into (2.66), for all \( t \geq T_1 \), we obtain that
\[
\|\nabla^k U(t)\|_{L^2} \leq C\left(\|U(0)\|_{L^1}^2 + 1 + (1 + t)^{-\frac{3}{4}} M^2(t)\right)\left(1 + t\right)^{-\frac{k}{2}} + C e^{-C_4 t} \|\nabla^2 U(T)\|_{L^2}^2 + C \left(\|U(0)\|_{L^1}^2 + 1 + (1 + t)^{-\frac{3}{4}} M^2(t)\right)\left(1 + t\right)^{-\frac{2}{3}}.
\]  (2.67)

Recalling the definition of \( M(t) \), from (2.67), we know there exists a positive constant \( C_6 \) such that
\[
M^2(t) \leq C_6\left(\|U(0)\|_{L^1}^2 + 1 + (1 + t)^{-\frac{3}{4}} M^2(t) + \|\nabla^2 U(T)\|_{L^2}^2\right).
\]

Choosing \( T_2 \), such that for all \( t \geq T_2 \), there holds
\[
C_6(1 + t)^{-\frac{2}{3}} \leq \frac{1}{2}.
\]

Then we have
\[
M^2(t) \leq 2C_6\left(\|U(0)\|_{L^1}^2 + 1 + \|\nabla^2 U(T)\|_{L^2}^2\right), \quad t \geq T_2,
\]

which together with the uniform estimate (1.5) implies
\[
M(t) \leq C, \quad \text{for all } t \in [T_2, +\infty).
\]
By the definition of \( M(t) \) in (2.52), we complete the proof of this lemma. \( \Box \)

**Proof of Theorem 1.2:** With the decay estimates stated in Lemma 2.8, we can complete the proof of the Theorem 1.2.

### 3 Proof of Theorem 1.3

In this section, we will establish the lower bound of decay rate for the global solution of Cauchy problem (2.45).

**Lemma 3.1.** Denote \( u_0 := \Delta n_0 \), and assume that the Fourier transform \( \mathcal{F}(u_0) = \hat{\omega}_0 \) satisfies \( |\hat{\omega}_0| \geq c_0, \ 0 < |\xi| \ll 1 \), with \( c_0 > 0 \) a constant. Then the solution \( n(t,x) \) obtained in Theorem 1.2 has the decay rate for all \( t \geq T_3 \)
\[
\|\nabla^{k+1} n(t)\|_{L^2} \geq C(1 + t)^{-\frac{k+2}{3}}, \quad k = 0, 1, 2,
\]
where \( T_3 \) is defined below.

**Proof.** Step 1: Consider the linearized equation corresponding to the third equation of (2.45):
\[
\partial_t \nabla \tilde{n}(t, x) - \Delta (\nabla \tilde{n}) = 0, \quad \nabla \tilde{n}(t, x)|_{t=0} = \nabla n_0(x), \quad x \in \mathbb{R}^3.
\]

Using the semigroup method, it is easy to check that
\[
\int |\nabla \tilde{n}|^2 dx = \int |\hat{\omega}_0|^2 e^{-2|\xi|^2 t} d\xi \geq c_0^2 \int_{0<|\xi|<1} e^{-2|\xi|^2} d\xi \geq C(1 + t)^{-\frac{3}{2}}.
\]  (3.1)

Similarly, we have
\[
\int |\nabla^2 \tilde{n}|^2 dx = \int |\hat{\omega}_0|^2 |\xi|^2 e^{-2|\xi|^2 t} d\xi \geq C(1 + t)^{-\frac{7}{2}},
\]  (3.2)

and
\[
\int |\nabla^3 \tilde{n}|^2 dx = \int |\hat{\omega}_0|^4 |\xi|^4 e^{-2|\xi|^2 t} d\xi \geq C(1 + t)^{-\frac{7}{2}}.
\]  (3.3)

Step 2: Define \( \nabla n_\delta(t,x) := \nabla n(t,x) - \nabla \tilde{n}(t,x) \), then \( \nabla n_\delta(t,x) \) satisfies
\[
\partial_t \nabla n_\delta - \Delta (\nabla n_\delta) = -\nabla (u \cdot \nabla n) + \nabla (|\nabla n|^{2}(n + d)) := \nabla S_3, \quad \nabla n_\delta(t,x)|_{t=0} = 0.
\]  (3.4)

Using Duhamel principle, for \( k = 0, 1 \), we obtain that
\[
\|\nabla n_\delta\|_{L^2} \leq C \int_0^t (1 + t - \tau)^{-\frac{7}{2}} \left(\|u \cdot \nabla n\|_{L^1} + \|\nabla n\|_{L^1} + ||\nabla n\|_{L^1} + ||\nabla S_3\|_{L^2}\right) d\tau.
\]  (3.5)
By routine calculation, it is easy to see
\[
    \| u \cdot \nabla n \|_{L^1} + \| (n + \mathcal{L}) n \|_{L^1} \leq \| u \|_{L^2} \| \nabla n \|_{L^2} + \| \nabla n \|_{L^2} \| \nabla n \|_{L^3} \| n \|_{L^6} + \| \nabla n \|_{L^2} \| \nabla n \|_{L^2} \leq C(\| u \|_{L^2} \| \nabla n \|_{L^2} + \| \nabla n \|_{L^2} \| \nabla n \|_{L^2} + \| \nabla n \|_{L^2}^2),
\]  
(3.6)

and
\[
    \| \nabla S_3 \|_{L^2} \leq \| \nabla u \|_{L^2} \| \nabla n \|_{L^\infty} + \| \nabla^2 n \|_{L^2} \| u \|_{L^\infty} + 2 \| \nabla^2 n \|_{L^2} \| \nabla n \|_{L^3} \| n \|_{L^6} + \| \nabla n \|_{L^2} \| \nabla n \|_{L^5} + \| \nabla^2 n \|_{L^3} \| \nabla n \|_{L^6} \leq \| \nabla u \|_{L^2} \| \nabla^2 n \|_{L^2} \| \nabla n \|_{L^2} \| \nabla^3 n \|_{L^2} + \| \nabla^2 n \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \leq C \| \nabla u \|_{L^2} \| \nabla^2 n \|_{L^2} \| \nabla n \|_{L^2} \| \nabla^3 n \|_{L^2},
\]
(3.7)

where we used the Sobolev inequalities
\[
    \| \nabla n \|_{L^\infty} + \| \nabla^2 n \|_{L^3} \leq \| \nabla^2 n \|_{L^2} \| \nabla^3 n \|_{L^2}, \quad \| u \|_{L^\infty} \leq \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2}, \quad \| \nabla n \|_{L^3} \leq \| \nabla n \|_{L^2} \| \nabla^2 n \|_{L^2}.
\]
(3.8)

Adding (3.6), (3.7) into (3.5), and using the decay estimate (1.6), we obtain that
\[
    \| \nabla n_\delta(t) \|_{L^2} \leq C \int_0^t (1 + t - \tau)^{-\frac{5}{2}} \left((1 + \tau)^{\frac{3}{2}} + (1 + \tau)^{-\frac{5}{2}}\right) \| \nabla^2(u, \nabla n) \|_{L^2}^2 d\tau \leq C(1 + t)^{-\frac{5}{2}} + C \int_0^t (1 + t - \tau)^{-\frac{5}{2}} \left((1 + \tau)^{\frac{3}{2}} + (1 + \tau)^{-\frac{5}{2}}\right) \| \nabla^2(u, \nabla n) \|_{L^2}^2 d\tau.
\]

Using the H"older inequality, and the uniform estimate (1.5), we have
\[
    \int_0^t (1 + t - \tau)^{-\frac{5}{2}} \left((1 + \tau)^{\frac{3}{2}} + (1 + \tau)^{-\frac{5}{2}}\right) \| \nabla^2(u, \nabla n) \|_{L^2}^2 d\tau \leq \left( \int_0^t (1 + t - \tau)^{-\frac{5}{2}} (1 + \tau)^{\frac{3}{2}} d\tau \right)^{\frac{1}{2}} \left( \int_0^t \| \nabla^2(u, \nabla n) \|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \leq C(1 + t)^{-\frac{5}{4}}.
\]

Applying \( \nabla \) to (3.5), then multiplying the resulting equation by \( \nabla^2 n_\delta \) and integrating over \( \mathbb{R}^3 \), we get
\[
    \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 n_\delta|^2 dx + \int_{\mathbb{R}^3} |\nabla^3 n_\delta|^2 dx = \int_{\mathbb{R}^3} \nabla^2 S_3 \cdot \nabla^2 n_\delta dx \leq \frac{1}{2} \| \nabla S_3 \|_{L^2}^2 + \frac{1}{2} \| \nabla^3 n_\delta \|_{L^2}^2.
\]

(3.10)

According to (3.7) and the decay estimate (2.5.1), for all \( t \geq T_2 \), there holds
\[
    \| \nabla S_3(t) \|_{L^2} \leq C(1 + t)^{-\frac{11}{4}},
\]

which together with (3.10) gives rise to
\[
    \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 n_\delta|^2 dx + \int_{\mathbb{R}^3} |\nabla^3 n_\delta|^2 dx \leq C(1 + t)^{-\frac{11}{4}} \quad \text{for} \quad t \geq T_2.
\]

(3.11)

Define \( S_0 := \{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \left( \frac{R}{1 + t} \right)^{\frac{1}{2}} \} \), then we can split the phase space \( \mathbb{R}^3 \) into two time-dependent regions. Here \( R \) is a constant defined below. By straightforward calculation, we get
\[
    \int_{\mathbb{R}^3 \setminus S_0} |\xi|^4 |\nabla n_\delta|^2 d\xi \geq \frac{R}{1 + t} \int_{\mathbb{R}^3} |\xi|^2 |\nabla n_\delta|^2 d\xi - \frac{R}{1 + t} \int_{S_0} |\xi|^2 |\nabla n_\delta|^2 d\xi \geq \frac{R}{1 + t} \int_{\mathbb{R}^3} |\xi|^2 |\nabla n_\delta|^2 d\xi - \frac{R^2}{(1 + t)^2} \int_{S_0} |\nabla n_\delta|^2 d\xi.
\]

(3.12)
Adding (3.12) into (3.11), and using Plancherel equality, then for \( t \geq T_2 \), there holds
\[
\frac{d}{dt} \int |\nabla^2 n_{\delta}|^2 dx + \frac{R}{1 + t} \|\nabla^2 n_{\delta}\|_{L^2}^2 \leq \frac{R^2}{(1 + t)^2} \|\nabla n_{\delta}\|_{L^2}^2 + C(1 + t)^{-\frac{15}{4}} \leq CR^2(1 + t)^{-\frac{15}{4}},
\]
where we used \((3.10)\) in the last step. Choosing \( R = \frac{15}{8} \), and multiplying the resulting inequality by \((1 + t)^{\frac{15}{4}}\), it follows that
\[
\frac{d}{dt} \left[ (1 + t)^{\frac{15}{8}} \|\nabla^2 n_{\delta}(t)\|_{L^2}^2 \right] \leq C(1 + t)^{\frac{6}{7}}, \quad t \geq T_2.
\]
The integration over \([T_2, t]\) and the uniform bound \((3.3)\) give rise to
\[
\|\nabla^2 n_{\delta}(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{15}{4}}, \quad t \geq T_2.
\] (3.13)
This together with \((3.2)\), we have
\[
\|\nabla^2 n\|_{L^2} \geq \|\nabla^2 n\|_{L^2} - \|\nabla^2 n_{\delta}\|_{L^2} \geq C(1 + t)^{-\frac{15}{4}} - C(1 + t)^{-\frac{15}{4}} \geq C(1 + t)^{-\frac{15}{4}},
\] (3.14)
for large time \( t \).

Next, we shall estimate the lower bound decay rate of \( \nabla^3 n \). Applying \( \nabla^2 \) to \((3.3)\), then multiplying the resulting equation by \( \nabla^3 n_{\delta} \) and integrating over \( \mathbb{R}^3 \), we obtain that
\[
\frac{1}{2} \int |\nabla^3 n_{\delta}|^2 dx + \int |\nabla^4 n_{\delta}|^2 dx = \int \nabla^3 S_{\delta} \cdot \nabla^3 n_{\delta} dx \leq \|\nabla^2 S_{\delta}\|_{L^2} \|\nabla^4 n_{\delta}\|_{L^2}.
\] (3.15)
Recalling the definition of \( S_{\delta} \), routine calculation gives
\[
\|\nabla^2 S_{\delta}\|_{L^2} \leq \|\nabla u\|_{L^2} \|\nabla n\|_{L^2} + \|\nabla u\|_{L^1} \|\nabla u\|_{L^2} + \|\nabla^2 n\|_{L^2} + \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^2} \|\nabla^3 n\|_{L^2}
\]
\[
\leq C \|\nabla(u, \nabla n)\|_{L^2} \|\nabla^2 u, \nabla n\|_{L^2} \|\nabla^2 u, \nabla n\|_{L^2} + \|\nabla n\|_{L^2} \|\nabla^2 n\|_{L^2} \|\nabla^3 n\|_{L^2}^2,
\]
where we used \((3.8)\) and the interpolation inequality \( \|n\|_{L^2} \leq \|n\|_{L^2}^\frac{5}{2} \|\nabla^2 n\|_{L^2}^\frac{5}{2} \). According to the decay estimate \((2.51)\), for all \( t \geq T_2 \), there holds
\[
\|\nabla^2 S_{\delta}(t)\|_{L^2} \leq C(1 + t)^{-\frac{15}{4}},
\]
which together with \((3.15)\) yields directly
\[
\frac{d}{dt} \int |\nabla^3 n_{\delta}|^2 dx + \int |\nabla^4 n_{\delta}|^2 dx \leq C(1 + t)^{-\frac{15}{4}}, \quad t \geq T_2.
\]
Similarly, we have
\[
\|\nabla^4 n_{\delta}\|_{L^2}^2 \geq \frac{R}{1 + t} \|\nabla^3 n_{\delta}\|_{L^2}^2 - \frac{R^2}{(1 + t)^2} \|\nabla^2 n_{\delta}\|_{L^2}^2.
\]
Therefore, we have
\[
\frac{d}{dt} \int |\nabla^3 n_{\delta}|^2 dx + \frac{R}{1 + t} \|\nabla^3 n_{\delta}\|_{L^2}^2 \leq \frac{R^2}{(1 + t)^2} \|\nabla^2 n_{\delta}\|_{L^2}^2 + C(1 + t)^{-\frac{15}{4}} \leq CR^2(1 + t)^{-\frac{15}{4}},
\] (3.16)
where we used \((3.13)\) in the last step. Multiplying \((1 + t)^{\frac{15}{4}}\) on the both side of \((3.16)\), and choosing \( R = \frac{15}{8} \), we get
\[
\frac{d}{dt} \left[ (1 + t)^{\frac{15}{4}} \|\nabla^3 n_{\delta}(t)\|_{L^2}^2 \right] \leq C(1 + t)^{-\frac{15}{4}}, \quad t \geq T_2.
\]
Integrating from \([T_2, t]\) and using the uniform estimate \((3.3)\), we get
\[
\|\nabla^3 n_{\delta}(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{15}{4}}, \quad t \geq T_2.
\]
Then we have
\[
\|\nabla^3 n(t)\|_{L^2} \geq \|\nabla^3 n(t)\|_{L^2} - \|\nabla^3 n_{\delta}(t)\|_{L^2} \geq C(1 + t)^{-\frac{15}{4}} - C(1 + t)^{-\frac{15}{4}} \geq C(1 + t)^{-\frac{15}{4}},
\] (3.17)
for large time. The combination of \((3.11)\), \((3.14)\) and \((3.17)\) completes the proof of this lemma. \qed
We are now concerned with the lower bound of decay rate for \((\varrho, u)\). Let us define \(m := \rho u\), rewrite (1.1) and (1.2) in the perturbation form

\[
\begin{cases}
\partial_t \varrho + \text{div} m = 0, \\
\partial_t m - \mu \Delta m - (\mu + \lambda) \text{div} m + P'(1) \varrho = F,
\end{cases}
\]

where

\[
F := -\mu \Delta(\varrho m) - (\mu + \lambda) \text{div}(\varrho m) - \text{div}(1 + \varrho) u \otimes u
- \nabla (P(1 + \varrho) - P(1) - P'(1) \varrho) - \text{div}(\nabla d \otimes \nabla d) - \frac{1}{2} \nabla (|\nabla d|^2),
\]

here \(\nabla d \otimes \nabla d = (\langle \partial_d, \partial_d d \rangle)_{1 \leq i, j \leq 3}\). It is easy to check

\[
\nabla d \cdot \Delta d = \text{div}(\nabla d \otimes \nabla d) - \frac{1}{2} \nabla (|\nabla d|^2).
\]

In order to obtain the lower decay estimate, we need to consider the linearized system

\[
\begin{cases}
\partial_t \bar{\varrho} + \text{div} \bar{m} = 0, \\
\partial_t \bar{m} - \mu \Delta \bar{m} - (\mu + \lambda) \text{div} \bar{m} + P'(1) \bar{\varrho} = 0,
\end{cases}
\tag{3.18}
\]

with the initial data

\[\langle \bar{\varrho}, \bar{m} \rangle_{t=0} = (\varrho_0, m_0)\].

Next, we introduce the following lemma, which can be found in [32].

**Lemma 3.2.** If \(|\hat{\varrho}_0| \geq c_0, \hat{m}_0 = 0, 0 \leq |\xi| \ll 1\), with \(c_0 > 0\) a constant, then the solution \((\bar{\varrho}, \bar{m})\) of the linearized system (3.18) has the following estimate

\[
\min\{\|\nabla^k \bar{\varrho}\|_{L^2}, \|\nabla^k \bar{m}\|_{L^2}\} \geq C(1 + t)^{-\frac{5}{4} - \frac{k}{2}}, \quad k = 0, 1, 2
\]

for large time \(t\).

Finally, we establish the lower bound of decay rate for the global solution of Cauchy problem (2.1).

**Lemma 3.3.** If \(|\hat{\varrho}_0| \geq c_0, \hat{m}_0 = 0, 0 \leq |\xi| \ll 1\), with \(c_0 > 0\) a constant, then the solution \((\varrho, u)\) of the system (2.1) has the following estimate

\[
\min\{\|\nabla^k \varrho(t)\|_{L^2}, \|\nabla^k u(t)\|_{L^2}\} \geq C(1 + t)^{-\frac{5}{4} - \frac{k}{2}}, \quad k = 0, 1, 2
\]

for large time \(t\).

**Proof.** Define \(\varrho_\delta := \varrho - \bar{\varrho}, m_\delta := m - \bar{m}\), then \((\varrho_\delta, m_\delta)\) satisfies

\[
\begin{cases}
\partial_t \varrho_\delta + \text{div} m_\delta = 0, \\
\partial_t m_\delta - \mu \Delta m_\delta - (\mu + \lambda) \text{div} m_\delta + P'(1) \varrho_\delta = F,
\end{cases}
\]

with the initial data

\[\langle \varrho_\delta, m_\delta \rangle_{t=0} = (0, 0)\].

First, we investigate the time decay rate for \(L^2\)-norm of the low-frequency part \(\nabla^k (\varrho_\delta, m_\delta), k = 0, 1\). By Duhamel principle, we have

\[
\|\nabla^k (\varrho_\delta, m_\delta)\|_{L^2} \leq C \int_0^t (1 + t - \tau)^{-\frac{5}{4} - \frac{k}{2}} \left[\|(1 + \varrho) u \otimes u\|_{L^1} + \|P(1 + \varrho) - P(1) - P'(1) \varrho\|_{L^1}ight.
+ \|\nabla d \otimes \nabla d\|_{L^1} + \|\nabla |d|^2\|_{L^1}) d\tau + C \int_0^t (1 + t)^{-\frac{5}{4} - \frac{k}{2}} \|\varrho u\|_{L^1} d\tau.
\tag{3.19}
\]
It is easy to check that
\[
\|(1 + \varrho)u \otimes u\|_{L^1} + \|P(1 + \varrho) - P(1) - P'(1)\varrho\|_{L^1} + \|\nabla d \otimes \nabla d\|_{L^1} + \|\nabla d\|_{L^1} + \|\varrho u\|_{L^1} \\
\leq C(\|\varrho\|_{L^\infty} + \|u\|^2_{L^2} + \|\varrho\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\varrho\|_{L^2} + \|u\|_{L^2}) \\
\leq C(\|u\|^2_{L^2} + \|\varrho\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\varrho\|_{L^2} + \|u\|_{L^2}),
\]
this together with (3.19) gives for \(k = 0, 1\)
\[
\left\|\nabla^k (\varrho^l, m^l)\right\|_{L^2} \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{k}{2}} (1 + \tau)^{-\frac{5}{4} - \frac{k}{2}} d\tau + C \int_0^t (1 + t - \tau)^{-\frac{7}{4} - \frac{k}{2}} (1 + \tau)^{-\frac{7}{4} - \frac{k}{2}} d\tau \\
\leq C(1 + t)^{-\frac{7}{4} - \frac{k}{2}}.
\]

Recalling the fact that
\[
\varrho^l = \hat{\varrho}^l + \varrho^l_0, \quad m^l = \hat{m}^l + m^l_0,
\]
and hence we have for \(k = 0, 1\)
\[
\left\|\nabla^k (\varrho(t))\right\|_{L^2} \geq C(\left\|\nabla^k \varrho^l(t)\right\|_{L^2} - \left\|\nabla^k \varrho^l_0(t)\right\|_{L^2}) \geq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}} - C(1 + t)^{-\frac{3}{4} - \frac{k}{2}} \geq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}},
\]
for large time. Using (3.14), it is easy to see that
\[
\|u(t)\|_{L^2} = \left\|\frac{m}{\rho}(t)\right\|_{L^2} \geq \rho(m(t))_{L^2} \geq C\|m^l(t)\|_{L^2} \geq C(1 + t)^{-\frac{3}{4}}.
\]

Because \(u = m - \frac{am}{n}\), we have
\[
\left\|\nabla u^l(t)\right\|_{L^2} \geq \left\|\nabla m^l(t)\right\|_{L^2} \geq \left\|\nabla (\frac{am}{n})^l(t)\right\|_{L^2}
\]
According to Bernstein inequality (B.3) and Hausdorff-Young inequality (B.5), we have
\[
\left\|\nabla (\frac{am}{n})^l\right\|_{L^2} \leq C\left(\left\|\nabla (\frac{am}{n})^l\right\|_{L^2} = C\|F^{-1}(\chi_0(\xi)(\frac{am}{n}))\|_{L^2} = C\|\chi_0(\xi)(\frac{am}{n})\|_{L^2} \leq C\|\chi_0(\xi)\|_{L^\infty}\left\|\frac{am}{n}\right\|_{L^\infty} \\
\leq C\left\|\frac{am}{n}\right\|_{L^1} \leq C\|\varrho\|_{L^2} \|m\|_{L^2} \leq C\|\varrho\|_{L^2} \|u\|_{L^2} \leq C(1 + t)^{-\frac{5}{4}},
\]
this together with (3.22) yields
\[
\left\|\nabla u(t)\right\|_{L^2} \geq \left\|\nabla u^l(t)\right\|_{L^2} \geq C(1 + t)^{-\frac{3}{4}} - C(1 + t)^{-\frac{3}{4}} \geq C(1 + t)^{-\frac{3}{4}}
\]
for large \(t\). The combination of (3.20), (3.21) and (3.23) gives that
\[
\min\{\left\|\nabla^k \varrho(t)\right\|_{L^2}, \|\nabla^k u(t)\|_{L^2}\} \geq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}}, \quad k = 0, 1
\]
for large \(t\). Next, we apply interpolation inequality to obtain the lower bound of decay rate for the second order spatial derivative of large solution (cf. [5.4]). The decay estimate (3.24) together with interpolation inequality:
\[
\left\|\nabla f\right\|_{L^2} \leq \left\|f\right\|_{L^2} \left\|\nabla^2 f\right\|_{L^2}
\]
gives rise to
\[
\min\{\left\|\nabla^2 \varrho(t)\right\|_{L^2}, \|\nabla^2 u(t)\|_{L^2}\} \geq C(1 + t)^{-\frac{3}{4}}
\]
for large time \(t\). The combination of (3.24) and (3.25) completes the proof of this lemma. \(\square\)

**Proof of Theorem 1.3.** With the combination of Lemmas 3.1 and 3.3 we can complete the proof of Theorem 1.3.

**Appendices**

**A** Estimates on the linearized equations

We adopt the following notations
\[
\Lambda := (-\Delta)^{-\frac{1}{2}}, \quad b := \Lambda^{-1} \text{div} u.
\]
Then \( u = -\Lambda^{-1} \nabla b - \Lambda^{-1} \text{div}(\Lambda^{-1} \text{curl} u) \), we get from (2.1) that
\[
\begin{cases}
\partial_t \phi + \Lambda b = S_1, \\
\partial_t b - (2\mu + \lambda) \Delta b - P'(1) \Lambda \phi = \Lambda^{-1} \text{div} S_2, \\
\partial_t n - \Delta n = S_3,
\end{cases}
\] (A.1)

While \( P u = \Lambda^{-1} \text{curl} u \) satisfies
\[
\begin{cases}
\partial_t Pu - \mu \Delta Pu = PS_2, \\
P u(t, x)|_{t=0} = P u_0(x).
\end{cases}
\] (A.2)

In fact, to derive the estimate of \( u \), we only need to estimate \( b \) and \( P u \). Taking the Fourier transform of the first and second equations in (A.1), applying \( \nabla \) operator to the third equation in (A.1) then taking the Fourier transform, then we get
\[
\begin{cases}
\partial_t \hat{\phi} + |\xi| \hat{b} = \hat{S}_1, \\
\partial_t \hat{b} + (2\mu + \lambda)|\xi|^2 \hat{b} - P'(1)|\xi| \hat{\phi} = \Lambda^{-1} \text{div} S_2, \\
\partial_t \hat{n} + |\xi|^2 \hat{n} = \nabla \hat{S}_3.
\end{cases}
\] (A.3)

In other words, it is
\[
\frac{d}{dt} \hat{U} + G(|\xi|) \hat{U} = (\hat{S}_1, \Lambda^{-1} \text{div} S_2, \nabla \hat{S}_3)^T,
\]
where
\[
\hat{U} = (\hat{\phi}, \hat{b}, \hat{n})^T,
\]
and
\[
G(|\xi|) = \begin{pmatrix}
0 & |\xi| & 0 \\
-2P'(1)|\xi| & (2\mu + \lambda)|\xi|^2 & 0 \\
0 & 0 & |\xi|^2
\end{pmatrix}
\] (A.4)

Thus the linearized system of (A.3) could be rewritten as:
\[
\begin{cases}
\partial_t \hat{\phi} + |\xi| \hat{b} = 0, \\
\partial_t \hat{b} + (2\mu + \lambda)|\xi|^2 \hat{b} - P'(1)|\xi| \hat{\phi} = 0, \\
\partial_t \hat{n} + |\xi|^2 \hat{n} = 0.
\end{cases}
\] (A.5)

Next, we show the estimates of the low-frequency part for the solution of the linearized system (A.3).

### A.1. Low-frequency analysis

From (A.3), we easily obtain
\[
\frac{1}{2} \frac{d}{dt} (P'(1)|\hat{\phi}|^2 + |\hat{b}|^2 + |\nabla \hat{n}|^2) + (2\mu + \lambda)|\xi|^2 |\hat{b}|^2 + |\xi|^2 |\nabla \hat{n}|^2 = 0.
\] (A.6)

Multiplying (A.5)₁ by \( \hat{b} \) and (A.5)₂ by \( \hat{\phi} \), respectively, and summing up the resulting equations, we get
\[
\frac{d}{dt} \text{Re}(\hat{\phi} \hat{\phi}) - P'(1)|\xi||\hat{\phi}|^2 + |\xi||\hat{b}|^2 = -(2\mu + \lambda)|\xi|^2 \text{Re}(\hat{\phi} \hat{b}).
\] (A.7)

Adding (A.6) with \( -\delta_1 |\xi| \times (A.7) \) gives rise to
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & (P'(1)|\hat{\phi}|^2 + |\hat{b}|^2 + |\nabla \hat{n}|^2 - 2\delta_1 |\xi| \text{Re}(\hat{\phi} \hat{b})) + (2\mu + \lambda)|\xi|^2 |\hat{b}|^2 \\
& + |\xi|^2 |\nabla \hat{n}|^2 + \delta_1 P'(1)|\xi|^2 |\hat{\phi}|^2 - \delta_1 |\xi|^2 |\hat{b}|^2 \\
& = \delta_1 (2\mu + \lambda)|\xi|^3 \text{Re}(\hat{b} \hat{b}) \\
& \leq \delta_1 \frac{2P'(1)}{2P'(1)} (2\mu + \lambda)^2 |\xi|^4 |\hat{b}|^2 + \delta_1 \frac{P'(1)}{2} |\xi|^2 |\hat{b}|^2.
\end{align*}
\] (A.8)
we choose the constant $\delta_1$ is a small fixed constant which satisfies

$$\delta_1 \leq \min\left\{ \frac{1}{2}, \frac{2\mu + \lambda}{4}, \sqrt{\frac{P'(1)}{2}} \right\}.$$ 

Then (A.8) implies that

$$\frac{1}{2} \frac{d}{dt} \left( P'(1)|\hat{g}|^2 + |\hat{b}|^2 + \sqrt{n}|\xi|^2 - 2\delta_1 |\xi|\text{Re}(\hat{\beta}) \right) + \frac{3}{2} \frac{2\mu + \lambda}{4} |\xi|^2 |\hat{b}|^2$$

$$+ |\xi|^2 \sqrt{n}|\xi|^2 + \frac{\delta_1 P'(1)}{2} |\xi|^2 |\hat{g}|^2$$

$$\leq \frac{\delta_1}{2P'(1)}(2\mu + \lambda)^2 |\xi|^4 |\hat{g}|^2.$$ 

Recalling that $r_0$ is a small constant in (B.2), then for $|\xi| \leq r_0 \leq \min\left\{ \frac{1}{2}, \sqrt{\frac{P'(1)}{2\mu + \lambda}}, \sqrt{\frac{P'(1)}{2}} \right\}$, it follows from (A.9) that

$$\frac{1}{2} \frac{d}{dt} \left( P'(1)|\hat{g}|^2 + |\hat{b}|^2 + \sqrt{n}|\xi|^2 - 2\delta_1 |\xi|\text{Re}(\hat{\beta}) \right) + \frac{2\mu + \lambda}{2} |\xi|^2 |\hat{b}|^2$$

$$+ |\xi|^2 \sqrt{n}|\xi|^2 + \frac{\delta_1 P'(1)}{2} |\xi|^2 |\hat{g}|^2 \leq 0.$$ 

We denote

$$\mathcal{L}_1(t, \xi) := \frac{1}{2} \left( P'(1)|\hat{g}|^2 + |\hat{b}|^2 + \sqrt{n}|\xi|^2 - 2\delta_1 |\xi|\text{Re}(\hat{\beta}) \right).$$

Since $\delta_1 r_0 \leq \min\left\{ \frac{1}{2}, \frac{P'(1)}{2} \right\}$, then we get

$$\mathcal{L}_1(t, \xi) \sim |\hat{g}|^2 + |\hat{b}|^2 + \sqrt{n}|\xi|^2.$$ 

Then there exists a positive constant $C_5$ such that for any $|\xi| \leq r_0$, there holds

$$C_5 |\xi|^2 \mathcal{L}_1(t, \xi) \leq \frac{2\mu + \lambda}{2} |\xi|^2 |\hat{b}|^2 + |\xi|^2 \sqrt{n}|\xi|^2 + \frac{\delta_1 P'(1)}{2} |\xi|^2 |\hat{g}|^2.$$ 

The combination of (A.10) and (A.11) yields

$$\mathcal{L}_1(t, \xi) \leq C e^{-C_5|\xi|^2} \mathcal{L}_1(0, \xi), \quad \text{for} \quad |\xi| \leq r_0.$$ 

Next, we establish the estimates of the medium-frequency part for the solution of the linearized system (A.5).

### A.2. Medium-frequency analysis

First, we shall check the characteristic polynomial of the matrix $G$ defined in (A.4),

$$|G(|\xi|) - \beta I| = |(\xi|^2 - \beta)(\beta^2 - (2\mu + \lambda)|\xi|^2 \beta + P'(1)|\xi|^2)|$$

$$:= -\beta^3 + a_1(|\xi|)\beta^2 - a_2(|\xi|)\beta + a_3(|\xi|),$$

where

$$a_1(|\xi|) = (2\mu + \lambda)|\xi|^2 + |\xi|^2,$$

$$a_2(|\xi|) = P'(1)|\xi|^2 + (2\mu + \lambda)|\xi|^4,$$

$$a_3(|\xi|) = P'(1)|\xi|^4.$$

According to Liouville-Clipart criterion \[3\], the roots of the function $|G(|\xi|) - \beta I|$ have positive real part if and only if the following inequalities hold

$$a_1(|\xi|) > 0, \quad \text{and} \quad a_1(|\xi|)a_2(|\xi|) - a_3(|\xi|) > 0.$$ 

It is easy to check that

$$a_1(|\xi|) = (2\mu + \lambda)|\xi|^2 + |\xi|^2 > 0,$$

$$a_1(|\xi|)a_2(|\xi|) - a_3(|\xi|) = (2\mu + \lambda)^2|\xi|^4 + (2\mu + \lambda)^2|\xi|^6 + (2\mu + \lambda)|\xi|^6 > 0.$$ 

Thus, following the discussion in Section 3.3 of \[2\], we claim that this fact implies the following lemma.
Lemma A.1. For any given constants $r$ and $R$ with $0 < r < R$, there exists a positive constant $\kappa$ (depending only on $r$, $R$, $\mu$ and $\lambda$), such that

$$|e^{-tG(|\xi|)}| \leq Ce^{-\kappa t}, \quad \text{for all } \ r \leq |\xi| \leq R \ \text{and} \ t \in \mathbb{R}^+. \quad (A.13)$$

For the system (A.5), the inequality (A.13) yields

$$|(\hat{\phi}, \hat{b}, \hat{\nabla}n)(t, \xi)| = |e^{-tG(|\xi|)}(\hat{\phi}, \hat{b}, \hat{\nabla}n)(0, \xi)|$$

$$\leq Ce^{-\kappa t}|(\hat{\phi}, \hat{b}, \hat{\nabla}n)|, \quad \text{for all } |\xi| \in [r, R],$$

where $r$ and $R$ are any given positive constants.

Finally, we give the estimates of the linearized system (A.2).

A.3. Estimates on $\hat{P}u(t, \xi)$

Now we give the estimate of $\hat{P}u$. The linearized equations of (A.2) in Fourier variables has the following form

$$\partial_t \hat{P}u + \mu |\xi|^2 \hat{P}u = 0.$$

Direct calculations gives that for all $|\xi| \geq 0$

$$|\hat{P}u(t, \xi)|^2 \leq Ce^{-\mu |\xi|^2} |\hat{P}u(0, \xi)|^2. \quad (A.15)$$

B. The frequency decomposition

Based on the Fourier transform, we can define a frequency decomposition $(f^l(x), f^m(x), f^h(x))$ for a function $f(x) \in L^2(\mathbb{R}^3)$ as follows

$$f^l(x) = \chi_0(D_x)f(x), \quad f^m(x) = (I - \chi_0(D_x) - \chi_1(D_x))f(x), \quad f^h(x) = \chi_1(D_x)f(x), \quad (B.1)$$

where $\chi_0(D_x)$ and $\chi_1(D_x)$, $D_x = \frac{1}{\sqrt{\mu + \lambda}} \nabla = \frac{1}{\sqrt{\mu}}(\partial_x, \partial_y, \partial_z)$, are the pseudo-differential operators with symbols $\chi_0(\xi)$ and $\chi_1(\xi)$, respectively. Here, $\chi_0(\xi)$ and $\chi_1(\xi)$ are two smooth cut-off functions satisfying $0 \leq \chi_0(\xi), \chi_1(\xi) \leq 1,(\xi \in \mathbb{R}^3)$ and

$$\chi_0(\xi) = \begin{cases} 1, & |\xi| < \frac{r_0}{2}, \\ 0, & |\xi| > r_0, \end{cases} \quad \chi_1(\xi) = \begin{cases} 0, & |\xi| < R_0, \\ 1, & |\xi| > R_0 + 1, \end{cases}$$

for some fixed constants $r_0$ and $R_0$ satisfying

$$0 < r_0 \leq \min\left\{ \frac{1}{2} \sqrt{\frac{P'(1)}{2\mu + \lambda}}, \frac{\sqrt{2P'(1)}}{2} \right\}, \quad (B.2)$$

and

$$R_0 \geq \max\{ \sqrt{\frac{6C_2}{\mu}}, \sqrt{\frac{3}{\mu}}, \sqrt{\frac{6C_3}{\mu}}, \sqrt{4C_2}, \sqrt{4C_3} \}. \quad (B.3)$$

Therefore, we have

$$f(x) = f^l(x) + f^m(x) + f^h(x) := f^L(x) + f^H(x) := f^l(x) + f^H(x),$$

where we denote

$$f^L(x) = f^l(x) + f^m(x) \quad \text{and} \quad f^H(x) = f^m(x) + f^h(x). \quad (B.4)$$

According to the definition (B.1) and using the Plancherel theorem, we have the following inequalities.
Lemma B.1. For $f(x) \in H^m(\mathbb{R}^3)$ and any given integers $k$, $k_0$ and $k_1$ with $k_0 \leq k \leq k_1 \leq m$, it holds that

$$\|\nabla^k f\|_{L^2} \leq r_0^{k-k_0} \|\nabla^{k_0} f\|_{L^2}, \quad \|\nabla^k f\|_{L^2} \leq \|\nabla^{k_1} f\|_{L^2},$$

$$\|\nabla^k f^h\|_{L^2} \leq \frac{1}{R_0^k} \|\nabla^{k_1} f^h\|_{L^2}, \quad \|\nabla^k f^h\|_{L^2} \leq \|\nabla^{k_1} f\|_{L^2},$$

and

$$r_0^k \|f^m\|_{L^2} \leq \|\nabla^k f^m\|_{L^2} \leq R_0^k \|f^m\|_{L^2}.$$

The following Hausdorff-Young inequality is useful in this paper. The proof can be found in [16] (see Proposition 2.2.16).

Lemma B.2. When $f \in L^p(\mathbb{R}^3)$, $1 \leq p \leq 2$, then $\hat{f} \in L^{p'}(\mathbb{R}^n)$, and there holds

$$\|\hat{f}\|_{L^{p'}} \leq C \|f\|_{L^p}, \quad (B.5)$$

where $1/p + 1/p' = 1$.

At last, we introduce Bernstein inequality as follows

Lemma B.3. Let $k$ be in $\mathbb{N}$. Let $(R_1, R_2)$ satisfy $0 < R_1 < R_2$. There exists a constant $C$ depending only on $R_1$, $R_2$, $k$, such that for all $1 \leq a \leq b \leq \infty$ and $u \in L^a$, we have

$$\text{Supp} \hat{u} \subset B(0, R_1 \eta) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \eta^{k+N(1/b-1)} \|u\|_{L^a}.$$

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