THE $p$-HARMONIC BOUNDARY FOR METRIC MEASURE SPACES

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Abstract. Let $p$ be a real number greater than one and let $X$ be a locally compact, noncompact metric measure space that satisfies certain conditions. The $p$-harmonic boundary of $X$ is constructed by using the spectrum of the $p$-Royden algebra on $X$. We use this boundary to solve a Dirichlet type problem at infinity.

1. Introduction

Throughout this paper $p$ will always denote a real number greater than one. Let $\Omega$ be a domain in the complex plane $\mathbb{C}$. The extended boundary of $\Omega$ is the usual boundary if $\Omega$ is bounded and is the usual boundary along with the point at infinity if it is unbounded. The domain $\Omega$ is known as a Dirichlet domain if the Dirichlet problem is solvable on $\Omega$. That is, if $f$ is a continuous real-valued function on the extended boundary of $\Omega$, then there exists a function $h$ that is harmonic in the interior of $\Omega$ and is equal to $f$ on the extended boundary of $\Omega$. It is well known that the unit disk in $\mathbb{C}$ is a Dirichlet domain. In fact, any simply connected domain in $\mathbb{C}$ is a Dirichlet domain.

More recently, Dirichlet type problems have been investigated in the more general setting of a metric measure space $X$. A good introduction to this topic is [1, Chapter 10]. With some additional assumptions on $X$, the following theorem is proved in [1, Theorem 10.24]

Theorem 1.1. Let $\Omega$ be a bounded domain in $X$ and assume that the Sobolev capacity of $X \setminus \Omega$ is greater than zero. If $f$ is a continuous function on $\partial \Omega$, then there exists a unique bounded $p$-harmonic function $h$ in $\Omega$ such that

$$\lim_{\Omega \ni y \to x} h(y) = f(x)$$

for quasieverywhere $x \in \partial \Omega$.

A natural question to ask is what can we say about the Dirichlet problem if $\Omega$ is unbounded? Shanmugalingam showed in [11, Proposition 5.3] that with some assumptions on the metric space $X$, and if the measure on $X \setminus \Omega$ is positive, then for every function $f \in N^{1,p}(X)$ (Newtonian space) there is a solution to the $p$-Dirichlet problem on $\Omega$ with boundary data $f$. In [3], Hansevi proved a similar result for the more general obstacle problem. However, the papers [3, 11] do not take into account the behavior of $f$ at infinity, or apply in the case $\Omega = X$. The papers [3, 11]

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also illustrate that the main issue we come up against is that there is no natural boundary for \( X \) that corresponds to the extended boundary of \( C \). What we need is a compactification for \( X \) that will allow us to define a suitable boundary for \( X \). The authors of [4] use the Gromov boundary to study the Dirichlet problem at infinity for Gromov hyperbolic metric measure spaces. Another possible compactification for \( X \) is the \( p \)-Royden compactification. From the \( p \)-Royden compactification, the \( p \)-harmonic boundary can be defined. The purpose of this paper is to construct the \( p \)-harmonic boundary for a metric measure space and prove a Dirichlet type problem at infinity for \( X \) using this boundary.

In [8] Royden introduced the harmonic boundary of a Riemann surface \( R \), also see Chapter III of [9]. It was also shown in [8] that a bounded harmonic function with finite Dirichlet integral on an open Riemann surface \( R \) can be determined by its behavior on the harmonic boundary of \( R \). The results for Riemann surfaces were extended to noncompact orientable Riemannian manifolds in [2]. In [5] the concept of the harmonic boundary was generalized to the \( p \)-harmonic boundary. The harmonic boundary corresponds to the case \( p = 2 \). In [5] Lee proved the following Dirichlet type result: Suppose \( M \) is a complete Riemannian manifold of bounded geometry and that the \( p \)-harmonic boundary of \( M \)-consists of \( n \) points, where \( n \) is finite. Then every bounded \( p \)-harmonic function on \( M \) with finite \( p \)-Dirichlet integral is determined uniquely by its value on the \( p \)-harmonic boundary. Consequently, there is a nonconstant bounded \( p \)-harmonic function with finite \( p \)-Dirichlet integral on \( M \) if and only if the \( p \)-harmonic boundary of \( M \) consists of more than one point. In [7] many of the results in [5, 6] were shown to be true in the setting of graphs with bounded degree.

In 1975 Yau [13] proved that on a complete Riemannian manifold of non-negative Ricci curvature, every positive harmonic function is constant. Since then a number of papers have appeared studying various Liouville type problems, not only in the linear setting of harmonic functions, but also in the nonlinear setting of \( p \)-harmonic functions. See the introduction of [4] for an excellent review of work done related to these issues. One reason for studying Dirichlet type problems at infinity is to determine when a subclass of \( p \)-harmonic functions on a space are constant or not. This of course, tells us if the space has a Liouville type property or not.

In this paper \( X \) will always be a locally compact, noncompact, complete metric measure space with metric \( d \) and positive complete Borel measure \( \mu \). Furthermore, we will also assume that \( \mu \) is doubling and that if \( B \) is a nonempty open ball in \( X \), then \( 0 < \mu(B) < \infty \). It will be assumed throughout that \( X \) contains at least two points. The main result of this study is the following theorem. All unexplained notation will be defined in later parts of the paper.

**Theorem 1.2.** Let \( X \) be a metric measure space. Suppose \( X \) satisfies a \((1, p)\)-Poincaré inequality. Let \( f \) be a continuous real-valued function on the \( p \)-harmonic boundary of \( X \). Then there exists a real-valued, \( p \)-harmonic function \( h \) on \( X \), such that \( h \) has finite \( p \)-Dirichlet integral and \( f = h \) on the \( p \)-harmonic boundary.

In Section 2 we define many of the terms that will be used throughout the paper. We also define an algebra of functions on \( X \), the \( p \)-Royden algebra, which is critical for defining the \( p \)-harmonic boundary. We end Section 2 by giving the definition for a function to be \( p \)-harmonic on \( X \). Section 3 is devoted to the construction of the \( p \)-harmonic boundary for \( X \). We also give a characterization of when the
p-harmonic boundary is empty. We conclude the paper by proving Theorem 1.2 in Section 4.

2. Preliminaries

In this section we will define some terms and notation that will be used throughout the paper. We also define the p-Royden algebra for X. This algebra is crucial for defining the p-harmonic boundary of X. We end this section by giving the definition of a p-harmonic function on X.

If \( x \in X \) and \( r \) is a positive real number, then \( B_r(x) \) will denote the metric ball of radius \( r \) centered at \( x \). We will write a.e. to indicate that a given property holds almost everywhere with respect to the measure \( \mu \). A connected open set is a domain.

A measurable real-valued function \( f \) on \( X \) is \( p \)-integrable if \( \int_X |f|^p d\mu < \infty \). Let \( L^p(X) \) be the set of extended real-valued measurable functions on \( X \) that are \( p \)-integrable. Observe that in \( L^p(X) \) we do not identify those functions that differ only on a set of measure zero. We will write \( L^p_{\text{loc}}(X) \) to indicate the set obtained from \( L^p(X) \) by identifying functions that agree a.e..

The main ingredient used in the construction of the p-harmonic boundary for \( X \) is the p-Royden algebra of functions on \( X \). For Riemannian manifolds and graphs the p-Royden algebra consists of all bounded continuous functions \( f \) such that \( |\nabla f| \), where \( \nabla f \) denotes the gradient of \( f \), is \( p \)-integrable. This poses a problem for us because there is no differentiable structure on a general metric measure space.

Over the past couple of decades the theory of the \( p \)-weak upper gradient of a function \( f \) on a metric measure space has been developed as a generalization of \( |\nabla f| \). In particular, \( p \)-weak upper gradients have been very useful in the development of Newtonian spaces, which are Sobolev type spaces on metric measure spaces. See [1, Chapter 1] and the references therein for information concerning Newtonian spaces.

Let \( \mathcal{P} \) be the family of all locally rectifiable curves in \( X \). For \( \Gamma \subset \mathcal{P} \) let \( Q(\Gamma) \) be the set of all Borel functions \( \rho : X \to [0, \infty] \) that satisfy \( \int_\gamma \rho ds \geq 1 \) for every \( \gamma \in \Gamma \).

The \( p \)-modulus of \( \Gamma \) is defined to be

\[
\text{Mod}_p(\Gamma) = \inf_{\rho \in Q(\Gamma)} \int_X \rho^p d\mu.
\]

We shall say that a property of curves holds for \( p \)-almost every curve if the family of curves for which the property fails has zero \( p \)-modulus.

Suppose \( u : X \to \mathbb{R} \) is a Borel function. A Borel function \( g : X \to [0, \infty] \) is defined to be an upper gradient of \( u \) if

\[
|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma g ds
\]

for every rectifiable curve \( \gamma : [a, b] \to X \). Note that \( g = \infty \) is an upper gradient for \( u \). We shall say that \( g \) is a \( p \)-weak upper gradient of \( u \) if the above inequality
holds on $p$-almost every curve $\gamma$ in $\mathcal{P}$. It is worth mentioning that if $g$ is a $p$-weak upper gradient for $u$ and $v$ is a function on $X$ that satisfies $u = v$ a.e., then it is not necessarily true that $g$ is a $p$-weak upper gradient of $v$. However, what is true is that if $g'$ is a nonnegative function on $X$ for which $g' = g$ a.e., then $g'$ is also a $p$-weak upper gradient for $u$, [1 Corollary 1.44]. The following proposition is a direct consequence of the fact $\text{Mod}_p(\bigcup_{j=1}^{\infty} \Gamma_j) \leq \sum_{j=1}^{\infty} \text{Mod}_p(\Gamma_j)$, which is [1 Lemma 1.34(b)].

**Proposition 2.1.** Let $u, v$ be extended real-valued functions on $X$ and let $a$ and $b$ be real numbers. Suppose that $g$ and $h$ are $p$-weak upper gradients for $u$ and $v$ respectively. Then $|au + bv|$ is a $p$-weak upper gradient for $pu + ph$.

The following lemma will be needed in the sequel.

**Lemma 2.2.** Assume that $(u_n)$ is a uniformly bounded sequence of continuous functions on $X$ and that $g_n \in L^p(X)$ is a $p$-weak upper gradient of $u_n$ for all $n \in \mathbb{N}$. Suppose that $u_n \rightarrow u$ pointwise and suppose that $g_n \rightarrow g$ in $L^p(X)$ and that $g$ is nonnegative. Then $g$ is a $p$-weak upper gradient for $u$.

**Proof.** By Fuglede’s lemma [1 Lemma 2.1] $\int_{\gamma} g_n ds \rightarrow \int_{\gamma} g ds \in \mathbb{R}$ for all $\gamma \in \Gamma(X) \setminus \Gamma$, where $\text{Mod}_p(\Gamma) = 0$. Also, [1 Lemma 1.40] says that $p$-a.e. curve $\gamma$ is such that $g_n$ is an upper gradient of $u_n$ along $\gamma$, and neither $\gamma$ nor any of its subcurves belong to $\Gamma$. Consider such a curve $\gamma$: $[0, l_\gamma] \rightarrow X$, where $l_\gamma$ is the length of $\gamma$. Now

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \leq \lim_{n \rightarrow \infty} |u_n(\gamma(l_\gamma)) - u_n(\gamma(0))| \leq \limsup_{n \rightarrow \infty} \int_{\gamma} g_n ds = \int_{\gamma} g ds.$$ 

Thus $g$ is a $p$-weak upper gradient of $u$ and $g \in L^p(X)$. \hfill $\square$

A much more general version of this lemma was proved in [1 Proposition 2.3].

We shall say that $g$ is a minimal $p$-weak upper gradient for $u$ if $g_1 \geq g$ a.e. for any $p$-weak upper gradient $g_1$ of $u$. The following theorem is [1 Theorem 2.5].

**Theorem 2.3.** Let $1 < p \in \mathbb{R}$. If $u \in L^p_\text{loc}(X)$ and $u$ has a $p$-weak upper gradient in $L^p(X)$, then there exists a minimal $p$-weak upper gradient $g$ of $u$. Moreover, $g$ is unique up to sets of measure zero.

A minimal $p$-weak upper gradient of a function $u$ turns out to be an adequate replacement for the norm of the gradient of $u$. With this in mind we will write $|\nabla u|$ to indicate the minimal $p$-weak upper gradient of $u$.

There is no guarantee that if $|\nabla u| = 0$, then $u$ is constant. In order to rectify this situation we need to assume that $X$ satisfies a $(1, p)$-Poincaré inequality, which we now describe. Let $q \geq 1$. We shall say that $X$ satisfies a $(q, p)$-Poincaré inequality if there exists constants $C > 0$ and $\sigma \geq 1$ such that

$$\left(\int_{B_r(x)} |u - u_{B_r(x)}|^q d\mu\right)^{1/q} \leq r^C \left(\int_{B_{r/\sigma}(x)} g^p d\mu\right)^{1/p}$$

for all metric balls $B_r(x), u \in L^p_\text{loc}(X)$ and $g$ a $p$-weak upper gradient of $u$, where

$$u_{B_r(x)} = \int_{B_r(x)} u d\mu = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} u d\mu.$$
For the rest of this paper we will assume that $X$ satisfies the $(1, p)$-Poincaré inequality. By [1] Proposition 4.2 $X$ is connected. Thus the $(1, p)$-Poincaré inequality assumption on $X$ implies that $g = 0$ is a $p$-weak upper gradient of $u$ if and only if $u$ is constant a.e..

Define $BD^p(X)$ to be the set of bounded continuous functions on $X$ with minimal $p$-weak upper gradient in $L^p(X)$. An immediate consequence of Proposition 2.1 is that $BD^p(X)$ is a vector space. It is clear that $BD^p(X)$ is closed under the operations of pointwise addition of functions and scalar multiplication. Furthermore, $BD^p(X)$ is closed under pointwise multiplication. To see this let $u, v \in BD^p(X)$. Then

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_{\gamma} |\nabla u|ds$$

for all nonconstant rectifiable curves $\gamma: [a, b] \to X$ not in $\Gamma_u$, where $\text{Mod}_p(\Gamma_u) = 0$. Also, there exists a set $\Gamma_v$ of rectifiable curves such that $\text{Mod}_p(\Gamma_v) = 0$ and

$$|v(\gamma(b)) - v(\gamma(a))| \leq \int_{\gamma} |\nabla v|ds$$

for all nonconstant rectifiable curves not in $\Gamma_v$. Now [1] Lemma 1.34(b) says that $\text{Mod}_p(\Gamma_u \cup \Gamma_v) = 0$. Let $\gamma: [a, b] \to X$ be a nonconstant rectifiable curve that does not belong to $\Gamma_u \cup \Gamma_v$. Then

$$|(uv)(\gamma(b)) - (uv)(\gamma(a))| \leq \|u\|_{\infty}|v(\gamma(b)) - v(\gamma(a))| + \|v\|_{\infty}|u(\gamma(b)) - u(\gamma(a))|$$

$$\leq \int_{\gamma} (\|u\|_{\infty}|\nabla v| + \|v\|_{\infty}|\nabla u|)ds.$$

Thus, $\|u\|_{\infty}|\nabla v| + \|v\|_{\infty}|\nabla u|$ is a $p$-weak upper gradient for $uv$ and it is also in $L^p(X)$. Hence, $uv \in BD^p(X)$. The algebra $BD^p(X)$ is known as the $p$-Royden algebra of $X$.

Let $(u_n)$ be a sequence of functions and $u$ a function on $X$. We shall say that $(u_n) \to u$ in the $CD^p$-topology if

$$\limsup_K |u_n - u| \to 0$$

for all compact subsets $K$ of $X$

and

$$\int_X |\nabla (u_n - u)|^p d\mu \to 0.$$

If the sequence $(u_n)$ is uniformly bounded, in addition to the above conditions, then we will say that $(u_n) \to u$ in the $BD^p$-topology.

**Theorem 2.4.** Let $1 < p \in \mathbb{R}$, with respect to the $BD^p$-topology, $BD^p(X)$ is complete.

**Proof.** Let $(u_n)$ be a uniformly bounded sequence in $BD^p(X)$ and suppose $(u_n) \to u$ in the $BD$-topology. Since $X$ is locally compact and $(u_n) \to u$ uniformly on compact sets, $u$ is a bounded continuous function on $X$. We will now finish the proof of the theorem by constructing a $p$-weak upper gradient in $L^p(X)$ for $u$. By taking a subsequence, if necessary, of the sequence $|\nabla (u_n - u)|$, we may and do assume $\|\nabla (u_n - u)\|_p < 2^{-n}$. It now follows from Proposition 2.1 that

$$|\nabla (u_{n+1} - u_n)| \leq |\nabla (u_{n+1} - u)| + |\nabla (u - u_n)|.$$

Thus

$$\|\nabla (u_{n+1} - u_n)\|_p \leq \|\nabla (u_{n+1} - u)\|_p + \|\nabla (u_n - u)\|_p < 3 \cdot 2^{-(n+1)}.$$
Since \( f_n = f_1 + (f_2 - f_1) + \cdots + (f_n - f_{n-1}) \) for \( n \geq 2 \), Proposition 2.1 tells us that
\[
g_n = |∇u_1| + \sum_{i=1}^{n-1} |∇(u_{i+1} - u_i)|
\]
is a \( p \)-weak upper gradient in \( L^p(X) \) for \( f_n \). Let \( m > n \). We now have
\[
\|g_m - g_n\|_p < 3 \cdot 2^{-n}.
\]
Thus \((g_n)\) is a Cauchy sequence in \( L^p(X) \). Denote the limit of \((g_n)\) in \( L^p(X) \) by \( g \).
Let \( E = \{ x \mid g(x) < 0 \} \). Since \( g_n \geq 0 \) for all \( n, \mu(E) = 0 \). By redefining \( g \) on \( E \) we may assume that \( g \geq 0 \) on \( X \). Since \((u_n) \rightarrow u\) pointwise, Proposition 2.2 says that \( g \) is a \( p \)-weak upper gradient of \( u \) in \( L^p(X) \). Therefore, \( u \in BD^p(X) \) and the proof of the theorem is complete. \( \square \)

Before we move on to the definition of a \( p \)-harmonic function we need to define the Sobolev \( p \)-capacity of a set in \( X \). Let \( N^{1,p}(X) \) be the set of functions \( u \in L^p(X) \) that have a \( p \)-weak upper gradient in \( L^p(X) \). A seminorm can be defined on \( N^{1,p}(X) \) via
\[
\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \int_X |∇u|^p \, d\mu \right)^{1/p}.
\]
The space \( N^{1,p}(X) \) is known as a Newtonian space, and was originally studied by Shanmugalingan in [10]. Newtonian spaces were developed in order to establish a Sobolev space type theory on metric measure spaces.

Now suppose \( E \subset X \). The Sobolev \( p \)-capacity of \( E \) is the number
\[
C_p(E) = \inf \|u\|_{N^{1,p}(X)},
\]
where the infimum is taken over all \( u \in N^{1,p}(X) \) for which \( u \geq 1 \) on \( E \). We shall say that a property \( P \) holds quasieverywhere (q.e.) if the set of points on which \( P \) fails has Sobolev \( p \)-capacity zero. One nice fact about Sobolev \( p \)-capacity is that two functions which agree q.e. on \( X \) have the same set of \( p \)-weak upper gradients. \( \square \) Corollary 1.49]. Compare this to the fact we mentioned earlier that two functions that agree a.e. do not necessarily have the same set of \( p \)-weak upper gradients.

Define \( N^{1,p}_{loc}(X) \) to be the set consisting of all real-valued functions \( f \) on \( X \) with \( f \in L^p_{loc}(X) \) and \( |∇f| \in L^p(X) \). Identify functions on \( N^{1,p}_{loc}(X) \) that agree q.e.. For an open set \( U \) in \( X \) let \( C_0(U) \) denote the set of functions that equal zero q.e. on \( X \setminus U \). We shall say that a function \( h \in N^{1,p}_{loc}(X) \) is a \( p \)-minimizer in \( X \) if
\[
\int_{Ω} |∇h|^p \, d\mu \leq \int_{Ω} |∇v|^p \, d\mu
\]
holds for every open \( Ω \subset X \) and every \( v \in N^{1,p}_{loc}(X) \) for which \( h - v \in C_0(Ω) \). The function \( h \) is said to be \( p \)-harmonic on \( X \) if it is a continuous \( p \)-minimizer on \( X \). We will write \( HBD^p(X) \) to indicate the \( p \)-harmonic functions contained in \( BD^p(X) \).

3. The \( p \)-Harmonic Boundary

In this section we define the \( p \)-Royden and \( p \)-harmonic boundaries of \( X \) by constructing an appropriate compactification of \( X \). The algebra \( BD^p(X) \) is crucial for this construction. We finish the section by characterizing the metric measure spaces whose \( p \)-harmonic boundary is the empty set. We begin with

Lemma 3.1. The space \( BD^p(X) \) separates points from closed sets in \( X \).
Proof: Let $A$ be a closed set in $X$ and let $x \in X \setminus A$. Pick $\epsilon > 0$ such that $B_{2\epsilon}(x) \cap A = \emptyset$. Define $u: X \to \mathbb{R}$ by

$$u(y) = \begin{cases} d(x, y), & y \in B_{\epsilon}(x) \\ \epsilon, & y \notin B_{\epsilon}(x) \end{cases}.$$ 

Since $u(x) = 0 \notin \overline{u(A)}$, $u$ separates $x$ from $A$. The proof of the lemma will be complete once we show $u \in \text{BD}^p(X)$, which we now do. Define $g: X \to [0, \infty]$ by

$$g(y) = \begin{cases} 1 & y \in B_{2\epsilon}(x) \\ 0 & y \notin B_{2\epsilon}(x) \end{cases}.$$ 

Clearly $g \in \mathcal{L}^p(X)$. Let $\gamma: [a, b] \to X$ be a rectifiable curve and suppose that $\gamma(a)$ and $\gamma(b)$ are elements of $B_{\epsilon}(x)$. It follows from the triangle inequality that $d(\gamma(b), x) - d(\gamma(a), x) \leq d(\gamma(a), \gamma(b))$. Hence

$$|d(\gamma(b), x) - d(\gamma(a), x)| = |u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma g ds.$$ 

Similar calculations show that this inequality is also true in the cases $\gamma(b) \in B_{\epsilon}(x), \gamma(a) \notin B_{\epsilon}(x)$ and both $\gamma(b), \gamma(a) \notin B_{\epsilon}(x)$. Therefore, $g$ is an upper gradient, and hence a $p$-weak upper gradient of $u$. Thus, $u \in \text{BD}^p(X)$ and the proof of the lemma is now complete. \hfill \square

For $u \in \text{BD}^p(X)$ let $I_u$ be a closed bounded interval in $\mathbb{R}$ that contains the image of $u$. Let $Y$ denote the product space

$$Y := \prod_{u \in \text{BD}^p(X)} I_u$$

with the Tychonoff topology. The space $Y$ can be thought of as the set of real-valued functions with domain $\text{BD}^p(X)$. Furthermore, $Y$ is a compact Hausdorff space, and a sequence $(x_n)$ converges to $x$ in $Y$ if $(x_n(f))$ converges to $x(f)$ for all $f \in \text{BD}^p(X)$. The evaluation map $e: X \to Y$ is given by

$$e(x)f = f(x).$$

We saw in Lemma 3.4 that $\text{BD}^p(X)$ separates points from closed sets in $X$, so [12] Theorem 8.16] tells us that $e$ is actually an embedding of $X$ into $Y$. We identify $X$ with $e(X)$. Let $\overline{X} = e(X)$, where the closure is taken in $Y$. Thus, $X$ is an open dense subset of the compact set $\overline{X}$. Also, every function in $\text{BD}^p(X)$ can be extended to a continuous function on $\overline{X}$. Denote by $C(\overline{X})$ the set of continuous functions on $\overline{X}$ with the uniform norm. By the Stone-Wierstrass theorem, $\text{BD}^p(X)$ is dense in $C(\overline{X})$.

Set $R_p(X) = \overline{X} \setminus X$. The compact Hausdorff space $R_p(X)$ is known as the $p$-Royden boundary of $X$. We will write $\text{BD}^p_\epsilon(X)$ to indicate the set of functions in $\text{BD}^p(X)$ that have compact support. Denote by $\overline{\text{BD}^p_\epsilon(X)}_{\text{BD}^p}$ the closure of $\text{BD}^p_\epsilon(X)$ with respect to the $\text{BD}^p$-topology. It follows from Theorem 2.3 that $\overline{\text{BD}^p_\epsilon(X)}_{\text{BD}^p}$ is contained in $\text{BD}^p(X)$. The $p$-harmonic boundary of $X$ is the following subset of $R_p(X)$:

$$\Delta_p(X) := \{ x \in R_p(X) \mid x(u) = 0 \text{ for all } u \in \overline{\text{BD}^p_\epsilon(X)}_{\text{BD}^p} \}.$$ 

Sometimes it will be the case $\Delta_p(X) = \emptyset$. The following theorem will be useful in determining when this happens.
Theorem 3.2. Let $F$ be a closed subset of $\overline{X}$ such that $F \cap \Delta_p(X) = \emptyset$. Then there exists $u \in BD^p_c(X)_{BD^p}$ such that $u = 1$ on $F$ and $0 \leq u \leq 1$.

Proof. Let $x \in F$. Since $x \notin \Delta_p(X)$ there exists $u_x \in BD^p_c(X)_{BD^p}$ such that $u_x(x) \neq 0$. Replacing $u_x$ by $-u_x$ if need be, we assume that $u_x(x) > 0$. Pick a neighborhood $U_x$ of $x$ for which $u_x > 0$ on $U_x$. Using $\max\{0, u_x\}$ instead of $u_x$ if necessary, we also assume that $u_x \geq 0$ on $X$. Since $F$ is compact, there exists $x_1, \ldots, x_k$ that satisfy $F \subseteq \bigcup_{i=1}^k U_{x_i}$. Let

$$g = \sum_{i=1}^k u_{x_i}.$$ 

Then $g \in BD^p_c(X)_{BD^p}$. Set $c = \inf\{g(x) \mid x \in F\}$. So $c > 0$. Now let

$$u = \min\{1, c^{-1}g\}.$$ 

Since $|\nabla(c^{-1}g)| \in L^p(X)$ is a $p$-weak upper gradient of $u$, we have that $u \in BD^p(X)$. In fact, $u \in BD^p_c(X)_{BD^p}$. Indeed, let $(g_n)$ be a sequence in $BD^p_c(X)$ that converges to $g$ in the $BD^p$-topology. Define $u_n \in BD^p_c(X)$ by

$$u_n = \min\{1, c^{-1}g_n\}.$$ 

Then $|\nabla(u - u_n)| \leq |\nabla(c^{-1}g - c^{-1}g_n)|$ since $|\nabla(c^{-1}g - c^{-1}g_n)|$ is a $p$-weak upper gradient of $u - u_n$. Thus

$$\int_X |\nabla(u - u_n)|^pd\mu \leq \int_X |\nabla(g - g_n)|^pd\mu.$$ 

Hence, $\int_X |\nabla(u - u_n)|^pd\mu \to 0$ as $n \to \infty$. Because $(g_n) \to g$ uniformly on compact sets, it follows that $(u_n) \to u$ uniformly on compact sets. Therefore, $u \in BD^p_c(X)_{BD^p}$ and $0 \leq u \leq 1$ on $X$.

Let $1_X$ denote the function that equals one for all $x \in X$. Since $|\nabla(1_X)| = 0, 1_X \in BD^p(X)$. We shall say that $X$ is $p$-parabolic if $1_X \in BD^p_c(X)_{BD^p}$. If $X$ is not $p$-parabolic, then it is said to be $p$-hyperbolic. A consequence of the above theorem is the following characterization for $\Delta_p(X)$.

Corollary 3.3. Let $X$ be a metric measure space. Then $X$ is $p$-parabolic if and only if $\Delta_p(X) = \emptyset$.

Proof. If $\Delta_p(X) = \emptyset$, then by the above theorem $1_X \in BD^p_c(X)_{BD^p}$ and $X$ is $p$-parabolic.

Conversely, suppose $X$ is $p$-parabolic and assume that $\Delta_p(X) \neq \emptyset$. Let $x \in \Delta_p(X)$. Then there exists a sequence $(x_n)$ in $X$ such that $(x_n(f)) \to x(f)$ for each $f \in BD^p(X)$. Since $x_n(1_X) = 1_X(x_n) = 1$ for all $n, x(1_X) = 1$. However, this contradicts our hypothesis that $1_X \in BD^p_c(X)_{BD^p}$. Hence, $\Delta_p(X) = \emptyset$.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We start by giving a crucial lemma that is a slightly modified version of [11] Theorem 10.24. We then use the lemma in the proof of a proposition that is Theorem 1.2 for the special case $f \in BD^p(X)$. With this result in hand, we use an approximation argument to prove our main result.

Recall that $X$ represents a metric measure space that is locally compact, non-compact, complete and satisfies the $(1, p)$-Poincaré inequality.
Lemma 4.1. Let $\Omega$ be a relatively compact domain of $X$. Suppose $f \in BD^p(X)$. Then there exists a unique $p$-harmonic function $h$ in $\Omega$, such that $h = f$ q.e. on $\partial \Omega$ and $|\nabla h| \in L^p(X)$.

Proof. Let $f \in BD^p(X)$. By [1] Theorem 10.24 there exists a unique bounded $p$-harmonic function $h$ in $\Omega$ such that

$$\lim_{\Omega \ni y \rightarrow x} h(y) = f(x) \text{ for } x \in \partial \Omega \setminus E,$$

where $E \subseteq \partial \Omega$ and has Sobolev $p$-capacity zero, so $h = f$ q.e. on $\partial \Omega$. Extend $h$ to all of $X$ by setting $h = f$ on $X \setminus \Omega$. It now follows that $|\nabla f|$ is a $p$-weak upper gradient for $h$ on $X$ because $\int_\Omega |\nabla h|d\mu \leq \int_\Omega |\nabla f|d\mu$. Consequently, $|\nabla h| \in L^p(X)$. \hfill \Box

Proposition 4.2. Let $f \in BD^p(X)$. Then there is a function $h \in BD^p(X)$ such that $h$ is $p$-harmonic on $X$ and $h = f$ on $\Delta_p(X)$.

Proof. Let $f \in BD^p(X)$. Since $X$ is connected ($(1, p)$-Poincare implies connected), second countable and locally compact, there exists an exhaustion $(\Omega_k)$ of $X$ by relatively compact domains. By Lemma 4.1 there exists for each $k \in \mathbb{N}$, an unique function $h_k$ such that $h_k$ is $p$-harmonic in $\Omega_k$ and $h_k = f$ q.e. on $\Omega \setminus \Omega_k$. Combining the strong maximum principe, [1] Theorem 8.13], with $f \in BD^p(X)$ we obtained a constant $M$ that is a uniform bound for the sequence $(h_k)$. Let $j \in \mathbb{N}$. It follows from [1] Theorem 8.15 that

$$|h_n(x) - h_n(y)| \leq C2Md(x, y)^\alpha,$$

where $x, y \in \Omega_j$ and $n > j$. Furthermore, the constant $C$ does not depend on $n$, and $0 < \alpha < 1$. Thus the family $(h_n)_{n \geq j}$ is equicontinuous on $\Omega_j$. For $j = 1$, the Ascoli-Arzela theorem yields a subsequence $(h_{1,k})$ of $(h_k)$ such that $(h_{1,k})$ converges uniformly on $\Omega_1$ to a continuous function $v_1$. Now there exists a subsequence $(h_{2,k})$ of $(h_{1,k})$ such that $(h_{2,k})$ converges uniformly on $\Omega_2$ to a continuous function $v_2$. Continue inductively in this manner for each $j$. The diagonal construction produces a subsequence $(h_{i,k})$ of $(h_k)$ such that $(h_{i,k})$ converges to a sequence $h$ on $X$. The sequence $(h_{i,k})$ converges locally uniformly on $X$ because $(h_{j,k})$ converges uniformly on each $\Omega_j$ and $(h_{i,j})$ is a subsequence of $(h_{j,k})$ for each $j$. By [1] Theorem 9.36, $h$ is $p$-harmonic on $X$.

Now, $h$ is bounded because the sequence $(h_k)$ is uniformly bounded. We saw in the proof of Lemma 4.1 that on $X$, $|\nabla f|$ is a $p$-weak upper gradient of $h_k$ for each $k \in \mathbb{N}$. It now follows from Proposition 2.4 of [1] that $|\nabla f|$ is a $p$-weak upper gradient of $h$. Thus, $|\nabla h| \in L^p(X)$ and $h \in BD^p(X)$.

We now show that $h = f$ on $\Delta_p(X)$. Let $x \in \Delta_p(X)$ and let $(x_n)$ be a sequence in $X$ such that $(x_n) \rightarrow x$. Since $h_k - f \in BD^p(X)_{RBD}$ for all $k$,

$$\lim_{n \rightarrow \infty} (h_k - f)(x_n) = (h_k - f)(x) = 0.$$

Now,

$$|(h - f)(x)| \leq \max \{|h - h_j, k)(x)| + |h_j - f)(x)| = \max \{|h - h_j, k)(x)|,$$

which implies $h(x) = f(x)$. Hence, $h = f$ on $\Delta_p(X)$. \hfill \Box

We can now prove Theorem 1.2. For convenience we restate the result.
Theorem 4.3. Let $f$ be a continuous real-valued function on $\Delta_p(X)$. Then there exists a $p$-harmonic function $h$ such that $h = f$ on $\Delta_p(X)$.

Proof. Let $f$ be a continuous function on $\Delta_p(X)$. By Tietze’s extension theorem there exists a continuous extension of $f$, which we shall also denote by $f$, to all of $\bar{X}$. Let $(f_n)$ be a sequence in $\text{BD}^p(X)$ such that $(f_n) \to f$ in $C(\bar{X})$. So for $\epsilon > 0$ there exists $N$ for which $\sup_X |f_n - f_m| < \epsilon$ for $n, m > N$. Let $(\Omega_i)$ be an exhaustion of $X$ by relatively compact sets. For each $n$, Proposition [12] shows that there exists $h_n \in H\text{BD}^p(X)$ that is $p$-harmonic on $X$ and $h_n = f_n$ on $\Delta_p(X)$. Furthermore, we also saw in the proof of Proposition [12] that for each $n$, there exists a sequence $(h_{i,n})$ such that $(h_{i,n}) \to h_n$ locally uniformly, where $h_{i,n}$ is $p$-harmonic on $\Omega_i$ and $h_{i,n} = f_n$ on $X \setminus \Omega_i$. Thus $\sup_{\partial \Omega_i} |h_{i,n} - h_{i,m}| < \epsilon$ for all $i \in \mathbb{N}$. Combining $h_{i,m} - \epsilon < h_{i,n} < h_{i,m} + \epsilon$ with the comparison principle, [1] Theorem 9.39] yields $\sup_{\Omega_i} |h_{i,n} - h_{i,m}| < \epsilon$ for each $i \in \mathbb{N}$. Since $(h_{i,n}) \to h_n$ and $(h_{i,m}) \to h_m$ it follows that $\sup_{\Omega} |h_n - h_m| < 3\epsilon$ for any relatively compact subset $\Omega$ of $X$. Consequently $\sup_X |h_n - h_m| \leq 3\epsilon$.

Thus the Cauchy sequence $(h_n)$ converges locally uniformly to a continuous function $h$ on $X$. Also, [1] Theorem 9.36] tells us that $h$ is $p$-harmonic on $X$.

We will now finish the proof of the theorem by showing $h = f$ on $\Delta_p(X)$. Let $x \in \Delta_p(X)$ and let $(x_n)$ be a sequence in $X$ for which $(x_n) \to x$. Choose $\epsilon > 0$. Then there exists an $N$ such that if $n > N$,

$$\sup_X |f_n - f| < \epsilon/3 \text{ and } \sup_X |h_n - h| < \epsilon/3.$$ 

We also have for sufficiently large $j$

$$|h_n(x_j) - f_n(x)| < \epsilon/3,$$

because $f_n = h_n$ on $\Delta_p(X)$ and both $f_n$ and $h_n$ are continuous on $X$. It now follows that

$$|h(x_j) - f(x)| < \epsilon.$$

Thus $\lim_{j \to \infty} h(x_j) = f(x)$. Now define $h(x) = f(x)$. Hence, $h = f$ on $\Delta_p(X)$. \qed

References

[1] Anders Björn and Jana Björn. Nonlinear potential theory on metric spaces, volume 17 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2011.
[2] Moses Glasner and Richard Katz. The Royden boundary of a Riemannian manifold. Illinois J. Math., 14:488–495, 1970.
[3] Daniel Hansevi. The obstacle and Dirichlet problems associated with $p$-harmonic functions in unbounded sets in $\mathbb{R}^n$ and metric spaces. arXiv: 1311.5955, 2013.
[4] Ilkka Holopainen, Urs Lang, and Aleksi Väähäkangas. Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces. Math. Ann., 339(1):101–134, 2007.
[5] Yong Hah Lee. Rough isometry and energy finite solutions of elliptic equations on Riemannian manifolds. Math. Ann., 318(1):181–204, 2000.
[6] Yong Hah Lee. Rough isometry and $p$-harmonic boundaries of complete Riemannian manifolds. *Potential Anal.*, 23(1):83–97, 2005.

[7] Michael J. Puls. Graphs of bounded degree and the $p$-harmonic boundary. *Pacific J. Math.*, 248(2):429–452, 2010.

[8] H. L. Royden. On the ideal boundary of a Riemann surface. In *Contributions to the theory of Riemann surfaces*, Annals of Mathematics Studies, no. 30, pages 107–109. Princeton University Press, Princeton, N. J., 1953.

[9] L. Sario and M. Nakai. *Classification theory of Riemann surfaces*. Die Grundlehren der mathematischen Wissenschaften, Band 164. Springer-Verlag, New York, 1970.

[10] Nageswari Shanmugalingam. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana*, 16(2):243–279, 2000.

[11] Nageswari Shanmugalingam. Some convergence results for $p$-harmonic functions on metric measure spaces. *Proc. London Math. Soc. (3)*, 87(1):226–246, 2003.

[12] Stephen Willard. *General topology*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

[13] Shing Tung Yau. Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.*, 28:201–228, 1975.

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