The $S$-weak global dimension of commutative rings

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Abstract

In this paper, we introduce and study the $S$-weak global dimension $S$-w.gl.dim($R$) of a commutative ring $R$ for some multiplicative subset $S$ of $R$. Moreover, commutative rings with $S$-weak global dimension at most 1 are studied. Finally, we investigated the $S$-weak global dimension of factor rings and polynomial rings.

Key Words: $S$-flat modules, $S$-flat dimensions, $S$-weak global dimensions.

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Throughout this article, $R$ always is a commutative ring with unity 1 and $S$ always is a multiplicative subset of $R$, that is, $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S, s_2 \in S$. In 2002, Anderson and Dumitrescu [1] defined $S$-Noetherian rings $R$ for which any ideal of $R$ is $S$-finite. Recall from [1] that an $R$-module $M$ is called $S$-finite provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule $F$ of $M$. An $R$-module $T$ is called uniformly $S$-torsion if $sT = 0$ for some $s \in S$ in [11]. So an $R$-module $M$ is $S$-finite if and only if $M/F$ is uniformly $S$-torsion for some finitely generated submodule $F$ of $M$. The idea derived from uniformly $S$-torsion modules is deserved to be further investigated. In [11], the author of this paper introduced the class of $S$-flat modules $F$ for which the functor $F \otimes_R -$ preserves $S$-exact sequences. The class of $S$-flat modules can be seen as a “uniform” generalization of that of flat modules, since an $R$-module $F$ is $S$-flat if and only if Tor$_1^R(F, M)$ is uniformly $S$-torsion for any $R$-module $M$ (see [11, Theorem 3.2]). The class of $S$-flat modules owns the following $S$-hereditary property: let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an $S$-exact sequence, if $B$ and $C$ are $S$-flat so is $A$ (see [11, Proposition 3.4]). So it is worth to study the $S$-analogue of flat dimensions of $R$-modules and $S$-analogue of weak global dimension of commutative rings.

In this article, we define the $S$-flat dimension $S$-fd$_R(M)$ of an $R$-module $M$ to be the length of the shortest $S$-flat $S$-resolution of $M$. We characterize $S$-flat dimensions of $R$-modules using the uniform torsion property of the “Tor” functors in Proposition [3.2]. Besides, we obtain a new local characterization of flat dimensions.
of \( R \)-modules (see Corollary 3.7). The \( S \)-weak global dimension \( S\text{-w.gl.dim}(R) \) of a commutative ring \( R \) is defined to be the supremum of \( S \)-flat dimensions of all \( R \)-modules. A characterization of \( S \)-weak global dimensions is given in Proposition 3.2. Examples of rings \( R \) for which \( S\text{-w.gl.dim}(R) \neq \text{w.gl.dim}(R_S) \) can be found in Example 3.11. S-von Neumann regular rings are firstly introduced in [11] for which there exists \( s \in S \) such that for any \( a \in R \) there exists \( r \in R \) such that \( sa = ra^2 \). By [11, Theorem 3.11], a ring \( R \) is \( S \)-von Neumann regular if and only if all \( R \)-modules are \( S \)-flat. So \( S \)-von Neumann regular rings are exactly commutative rings with \( S \)-weak global dimension equal to 0 (see Corollary 3.8). We also study commutative rings \( R \) with \( S\text{-w.gl.dim}(R) \leq 1 \). The nontrivial example of commutative rings with \( S\text{-w.gl.dim}(R) \leq 1 \) but infinite weak global dimension is given in Example 3.11.

In the final section, we investigate the \( S \)-weak global dimensions of factor rings and polynomial rings and show that \( S\text{-w.gl.dim}(R[x]) = S\text{-w.gl.dim}(R)+1 \) (see Theorem 4.7).

1. Preliminaries

Recall from [11], an \( R \)-module \( T \) is called a uniformly \( S \)-torsion module provided that there exists an element \( s \in S \) such that \( sT = 0 \). An \( R \)-sequence \( M \overset{f}{\rightarrow} N \overset{g}{\rightarrow} L \) is called \( S \)-exact (at \( N \)) provided that there is an element \( s \in S \) such that \( s \ker(g) \subseteq \text{Im}(f) \) and \( s \text{Im}(f) \subseteq \ker(g) \). We say a long \( R \)-sequence \( \ldots \rightarrow A_{n-1} \overset{f_n}{\rightarrow} A_n \overset{f_{n+1}}{\rightarrow} A_{n+1} \rightarrow \ldots \) is \( S \)-exact, if for any \( n \) there is an element \( s \in S \) such that \( s \ker(f_{n+1}) \subseteq \text{Im}(f_n) \) and \( s \text{Im}(f_n) \subseteq \ker(f_{n+1}) \). An \( S \)-exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is called a short \( S \)-exact sequence. An \( R \)-homomorphism \( f : M \rightarrow N \) is an \( S \)-monomorphism (resp., \( S \)-epimorphism, \( S \)-isomorphism) provided \( 0 \rightarrow M \overset{f}{\rightarrow} N \) (resp., \( M \overset{f}{\rightarrow} N \rightarrow 0 \), \( 0 \rightarrow M \overset{f}{\rightarrow} N \rightarrow 0 \)) is \( S \)-exact. It is easy to verify an \( R \)-homomorphism \( f : M \rightarrow N \) is an \( S \)-monomorphism (resp., \( S \)-epimorphism, \( S \)-isomorphism) if and only if \( \ker(f) \) (resp., \( \text{coker}(f) \), both \( \ker(f) \) and \( \text{coker}(f) \)) is a uniformly \( S \)-torsion module.

**Proposition 1.1.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Suppose there is an \( S \)-isomorphism \( f : M \rightarrow N \) for \( R \)-modules \( M \) and \( N \). Then there is an \( S \)-isomorphism \( g : N \rightarrow M \).

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker}(f) & \rightarrow & M & \overset{f}{\rightarrow} & N & \rightarrow & \text{Coker}(f) & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & \uparrow & & \downarrow & \\
& & \text{Im}(f) & & & & & & &
\end{array}
\]
with \( s \ker(f) = 0 \) and \( sN \subseteq \text{Im}(f) \) for some \( s \in S \). Define \( g_1 : N \to \text{Im}(f) \) where \( g_1(n) = sn \) for any \( n \in N \). Then \( g_1 \) is a well-defined \( R \)-homomorphism since \( sn \in \text{Im}(f) \). Define \( g_2 : \text{Im}(f) \to M \) where \( g_2(f(m)) = sm \). Then \( g_2 \) is well-defined \( R \)-homomorphism. Indeed, if \( f(m) = 0 \), then \( m \in \ker(f) \) and so \( sm = 0 \). Set \( g = g_2 \circ g_1 : N \to M \). We claim that \( g \) is an \( S \)-isomorphism. Indeed, let \( n \) be an element in \( \ker(g) \). Then \( sn = g_1(n) \in \ker(g_2) \). Note that \( s \ker(g_2) = 0 \). Thus \( s^2n = 0 \). So \( s^2 \ker(g) = 0 \). On the other hand, let \( m \in M \). Then \( g(f(m)) = g_2 \circ g_1(f(m)) = g_2(f(sm)) = s^2m \). Thus \( s^2m \in \text{Im}(g) \). So \( s^2M \subseteq \text{Im}(g) \). It follows that \( g \) is an \( S \)-isomorphism. \( \square \)

**Remark 1.2.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( M \) and \( N \) \( R \)-modules. Then the condition “there is an \( R \)-homomorphism \( f : M \to N \) such that \( f_S : M_S \to N_S \) is an isomorphism” does not mean “there is an \( R \)-homomorphism \( g : N \to M \) such that \( g_S : N_S \to M_S \) is an isomorphism”.

Indeed, let \( R = \mathbb{Z} \) be the ring of integers, \( S = R - \{0\} \) and \( \mathbb{Q} \) the quotient field of integers. Then the embedding map \( f : \mathbb{Z} \hookrightarrow \mathbb{Q} \) satisfies \( f_S : \mathbb{Q} \to \mathbb{Q} \) is an isomorphism. However, since \( \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0 \), there does not exist any \( R \)-homomorphism \( g : \mathbb{Q} \to \mathbb{Z} \) such that \( g_S : \mathbb{Q} \to \mathbb{Q} \) is an isomorphism.

Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Suppose \( M \) and \( N \) are \( R \)-modules. We say \( M \) is \( S \)-isomorphic to \( N \) if there exists an \( S \)-isomorphism \( f : M \to N \). A family \( \mathcal{C} \) of \( R \)-modules is said to be closed under \( S \)-isomorphisms if \( M \) is \( S \)-isomorphic to \( N \) and \( M \) is in \( \mathcal{C} \), then \( N \) is also in \( \mathcal{C} \). It follows from Proposition 1.1 that the existence of \( S \)-isomorphisms of two \( R \)-modules is an equivalence relation. Next, we give an \( S \)-analogue of Five Lemma.

**Theorem 1.3.** (\( S \)-analogue of Five Lemma) Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). Consider the following diagram with \( S \)-exact rows:

\[
\begin{array}{cccccc}
A & \xrightarrow{g_1} & B & \xrightarrow{g_2} & C & \xrightarrow{g_3} & D & \xrightarrow{g_4} & E \\
\downarrow{f_A} & & \downarrow{f_B} & & \downarrow{f_C} & & \downarrow{f_D} & & \downarrow{f_E} \\
A' & \xrightarrow{h_1} & B' & \xrightarrow{h_2} & C' & \xrightarrow{h_3} & D' & \xrightarrow{h_4} & E'.
\end{array}
\]

(1) If \( f_B \) and \( f_D \) are \( S \)-monomorphisms and \( f_A \) is an \( S \)-epimorphism, then \( f_C \) is an \( S \)-monomorphism.

(2) If \( f_B \) and \( f_D \) are \( S \)-epimorphisms and \( f_E \) is an \( S \)-monomorphism, then \( f_C \) is an \( S \)-epimorphism.

(3) If \( f_A \) is an \( S \)-epimorphism, \( f_E \) is an \( S \)-monomorphism, and \( f_B \) and \( f_D \) are \( S \)-isomorphisms, then \( f_C \) is an \( S \)-isomorphism.

(4) If \( f_A, f_B, f_D \) and \( f_E \) are all \( S \)-isomorphisms, then \( f_C \) is an \( S \)-isomorphism.
Proof. (1) Let \( x \in \ker(f_C) \). Then \( f_D g_3(x) = h_3 f_C(x) = 0 \). Since \( f_D \) is an \( S \)-monomorphism, \( s_1 \ker(f_D) = 0 \) for some \( s_1 \in S \). So \( s_1 g_3(x) = g_3(s_1 x) = 0 \). Since the top row is \( S \)-exact, there exists \( s_2 \in S \) such that \( s_2 \ker(g_3) \subseteq \im(g_2) \). Thus there exists \( b \in B \) such that \( g_2(b) = s_2 s_1 x \). Hence \( h_2 f_B(b) = f_C g_2(b) = f_C(s_2 s_1 x) = 0 \). Thus there exists \( s_3 \in S \) such that \( s_3 \ker(h_2) \subseteq \im(h_1) \). So there exists \( a' \in A' \) such that \( h_1(a') = s_2 f_B(b) \). Since \( f_A \) is an \( S \)-epimorphism, there exists \( s_4 \in S \) such that \( s_4 A' \subseteq \im(f_A) \). So there there exists \( a \in A \) such that \( s_4 a' = f_A(a) \). Hence \( s_4 s_2 f_B(b) = s_4 h_1(a') = h_1(f_A(a)) = f_B(g_1(a)) \). So \( s_4 s_2 b - g_1(a) \in \ker(f_B) \). Since \( f_B \) is an \( S \)-monomorphism, there exists \( s_5 \in S \) such that \( s_5 \ker(f_B) = 0 \). Thus \( s_5(s_4 s_2 b - g_1(a)) = 0 \). So \( s_5 s_4 s_2 s_2 s_1 x = s_5(g_2(s_4 s_2 b)) = s_5 g_2(g_1(a)) \). Since the top row is \( S \)-exact at \( B \), then there exists \( s_6 \in S \) such that \( s_6 \im(g_1) \subseteq \ker(g_2) \). So \( s_6 s_5 s_4 s_2 s_2 s_1 x = s_5 g_2(s_6 g_1(a)) = 0 \). Consequently, if we set \( s = s_6 s_5 s_4 s_2 s_2 s_1 \), then \( s \ker(f_C) = 0 \). It follows that \( f_C \) is an \( S \)-monomorphism.

(2) Let \( x \in C' \). Since \( f_D \) is an \( S \)-epimorphism, then there exists \( s_1 \in S \) such that \( s_1 D' \subseteq \im(f_D) \). Thus there exists \( d \in D \) such that \( f_D(d) = s_1 h_3(x) \). By the commutativity of the right square, we have \( f_E g_4(d) = h_4 f_D(d) = s_1 h_4(h_3(x)) \). Since the bottom row is \( S \)-exact at \( D' \), there exists \( s_2 \in S \) such that \( s_4 \im(h_3) \subseteq \ker(h_4) \). So \( s_4 f_E(g_4(d)) = s_1 h_4(s_4 h_3(x)) = 0 \). Since \( f_E \) is an \( S \)-monomorphism, there exists \( s_3 \in S \) such that \( s_3 \ker(f_E) = 0 \). Thus \( s_3 s_4 g_4(d) = 0 \). Since the top row is \( S \)-exact at \( D \), there there exists \( s_5 \in S \) such that \( s_5 \ker(g_4) \subseteq \im(g_3) \). So there exists \( c \in C \) such that \( s_5 s_3 s_4 d = g_3(c) \). Hence \( s_5 s_3 s_4 f_D(d) = f_D(g_3(c)) = h_3(f_C(c)) \). Since \( s_5 s_3 s_4 f_D(d) = h_3(s_1 s_5 s_3 s_4 x) \), we have \( f_C(c) - s_1 s_5 s_3 s_4 x \in \ker(h_3) \). Since the bottom row is \( S \)-exact at \( C' \), there exists \( s_6 \in S \) such that \( s_6 \ker(h_3) \subseteq \im(h_2) \). Thus there exists \( b' \in B' \) such that \( s_6(f_C(c) - s_1 s_5 s_3 s_4 x) = h_2(b') \). Since \( f_B \) is an \( S \)-epimorphism, then there exists \( s_7 \in S \) such that \( s_7 b' = \im(f_B) \). So \( s_7 b' = f_B(b) \) for some \( b \in B \). Thus \( f_C(g_2(b)) = h_2(f_B(b)) = s_7 h_2(b') = s_7(s_6(f_C(c) - s_1 s_5 s_3 s_4 x)) \). So \( s_7 s_6 s_1 s_5 s_3 s_4 x = s_7 s_6 f_C(c) - f_C(g_2(b)) = f_C(s_7 s_6 c - g_2(b)) \in \im(f_C) \). Consequently, if we set \( s = s_7 s_6 s_1 s_5 s_3 s_4 \), then \( s C' \subseteq \im(f_C) \). It follows that \( f_C \) is an \( S \)-epimorphism.

It is easy to see (3) follows from (1) and (2), while (4) follows from (3). \(\square\)

Recall from [11, Definition 3.1] that an \( R \)-module \( F \) is called \( S \)-flat provided that for any \( S \)-exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \), the induced sequence \( 0 \rightarrow A \otimes_R F \rightarrow B \otimes_R F \rightarrow C \otimes_R F \rightarrow 0 \) is \( S \)-exact. It is easy to verify that the class of \( S \)-flat modules is closed under \( S \)-isomorphisms by the following result.

**Lemma 1.4.** [11, Theorem 3.2] Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( F \) an \( R \)-module. The following assertions are equivalent:

1. \( F \) is \( S \)-flat;
(2) for any short exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$, the induced sequence $0 \to A \otimes_R F \overset{f \otimes_R F}{\to} B \otimes_R F \overset{g \otimes_R F}{\to} C \otimes_R F \to 0$ is $S$-exact;

(3) $\text{Tor}^R_n(M, F)$ is uniformly $S$-torsion for any $R$-module $M$;

(4) $\text{Tor}^R_n(M, F)$ is uniformly $S$-torsion for any $R$-module $M$ and $n \geq 1$.

The following result says that a short $S$-exact sequence induces a long $S$-exact sequence by the functor “$\text{Tor}$” as the classical case.

**Theorem 1.5.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $N$ an $R$-module. Suppose $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is an $S$-exact sequence of $R$-modules. Then for any $n \geq 1$ there is an $R$-homomorphism $\delta_n : \text{Tor}^R_n(C, N) \to \text{Tor}^R_{n-1}(A, N)$ such that the induced sequence

$$\ldots \to \text{Tor}^R_n(A, N) \to \text{Tor}^R_n(B, N) \to \text{Tor}^R_n(C, N) \overset{\delta_n}{\to} \text{Tor}^R_{n-1}(A, N) \to$$

$$\text{Tor}^R_{n-1}(B, N) \to \ldots \to \text{Tor}^R_1(C, N) \overset{\delta_1}{\to} A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$$

is $S$-exact.

**Proof.** Since the sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is $S$-exact at $B$. There are three exact sequences $0 \to \ker(f) \overset{i_{\ker(f)}}{\to} A \overset{\pi_{\im(f)}}{\to} \im(f) \to 0$, $0 \to \ker(g) \overset{i_{\ker(g)}}{\to} B \overset{\pi_{\im(g)}}{\to} \im(g) \to 0$ and $0 \to \im(g) \overset{i_{\im(g)}}{\to} C \overset{\pi_{\coker(g)}}{\to} \coker(g) \to 0$ with $\ker(f)$ and $\coker(g)$ uniformly $S$-torsion. There also exists $s \in S$ such that $s\ker(g) \subseteq \im(f)$ and $s\im(f) \subseteq \ker(g)$. Denote $T = \ker(f)$ and $T' = \coker(g)$.

Firstly, consider the exact sequence

$$\text{Tor}^R_{n+1}(T', N) \to \text{Tor}^R_n(\im(g), N) \overset{\text{Tor}^R_n(i_{\im(g)}, N)}{\to} \text{Tor}^R_n(C, N) \to \text{Tor}^R_n(T', N).$$

Since $T'$ is uniformly $S$-torsion, $\text{Tor}^R_{n+1}(T', N)$ and $\text{Tor}^R_n(T', N)$ is uniformly $S$-torsion. Thus $\text{Tor}^R_n(i_{\im(g)}, N)$ is an $S$-isomorphism. So there is also an $S$-isomorphism $h^\im_n : \text{Tor}^R_n(C, N) \to \text{Tor}^R_n(\im(g), N)$ by Proposition \[1\]. Consider the exact sequence:

$$\text{Tor}^R_{n-1}(T, N) \to \text{Tor}^R_{n-1}(A, N) \overset{\text{Tor}^R_{n-1}(\pi_{\im(f)}, N)}{\to} \text{Tor}^R_{n-1}(\im(f), N) \to \text{Tor}^R_{n-2}(T, N).$$

Since $T$ is uniformly $S$-torsion, we have $\text{Tor}^R_{n-1}(\pi_{\im(f)}, N)$ is an $S$-isomorphism. So there is also an $S$-isomorphism $h^\im_{n-1} : \text{Tor}^R_{n-1}(\im(f), N) \to \text{Tor}^R_{n-1}(A, N)$ by Proposition \[1\]. We have two exact sequences

$$\text{Tor}^R_{n+1}(T_1, N) \to \text{Tor}^R_n(s\ker(g), N) \overset{\text{Tor}^R_n(i^1_{s\ker(g)}, N)}{\to} \text{Tor}^R_n(\im(f), N) \to \text{Tor}^R_{n+1}(T_1, N)$$

and

$$\text{Tor}^R_{n+1}(T_2, N) \to \text{Tor}^R_n(s\ker(g), N) \overset{\text{Tor}^R_n(i^2_{s\ker(g)}, N)}{\to} \text{Tor}^R_n(\ker(g), N) \to \text{Tor}^R_{n+1}(T_2, N),$$
where \( T_1 = \text{Im}(f)/s\text{Ker}(g) \) and \( T_2 = \text{Im}(f)/s\text{Im}(f) \) is uniformly \( S \)-torsion. So Tor\(_n^R(i_2^s\text{Ker}(g), N) \) and Tor\(_n^R(i_2^s\text{Ker}(g), N) \) are \( S \)-isomorphisms. Thus there is an \( S \)-isomorphisms \( h_s^\text{Ker}(g) : \text{Tor}_n^R(\text{Ker}(g), N) \to \text{Tor}_n^R(s\text{Ker}(g), N) \). Note that there is an exact sequence Tor\(_n^R(B, N) \xrightarrow{\text{Tor}_n^R(\text{im}(g), N)} \text{Tor}_n^R(\text{Im}(g), N) \xrightarrow{\delta_n^{\text{Im}(g)}} \text{Tor}_{n-1}^R(\text{Ker}(g), N) \xrightarrow{\text{Tor}_n^R(\text{im}(g), N)} \text{Tor}_{n-1}^R(B, N) \). Set \( \delta_n = h_n^\text{im}(g) \circ \delta_n^{\text{im}(g)} \circ h_n^s\text{Ker}(g) \circ \text{Tor}_n^R(i_1^s\text{Ker}(g), N) \circ h_n^{\text{im}(f)} : \text{Tor}_n^R(C, N) \to \text{Tor}_{n-1}^R(A, N) \). Since \( h_n^{\text{im}(g)}, \delta_n^{\text{im}(g)}, h_n^s\text{Ker}(g) \) and \( h_n^{\text{im}(f)} \) are \( S \)-isomorphisms, we have the sequence Tor\(_n^R(B, N) \to \text{Tor}_n^R(C, N) \xrightarrow{\delta_n} \text{Tor}_{n-1}^R(A, N) \) and Tor\(_n^R(B, N) \) is \( S \)-exact.

Secondly, consider the exact sequence:

\[
\text{Tor}_{n+1}^R(T, N) \to \text{Tor}_n^R(A, N) \xrightarrow{\text{Tor}_n^R(\text{im}(f), N)} \text{Tor}_n^R(\text{Im}(f), N) \to \text{Tor}_n^R(T, N).
\]

Since \( T \) is uniformly \( S \)-torsion, Tor\(_n^R(\text{im}(f), N) \) is an \( S \)-isomorphism. Consider the exact sequences:

\[
\text{Tor}_{n+1}^R(\text{Im}(g), N) \to \text{Tor}_n^R(\text{Ker}(g), N) \xrightarrow{\text{Tor}_n^R(\text{im}(g), N)} \text{Tor}_n^R(B, N) \to \text{Tor}_n^R(\text{Im}(g), N)
\]

and

\[
\text{Tor}_{n+1}^R(T', N) \to \text{Tor}_n^R(\text{Im}(g), N) \xrightarrow{\text{Tor}_n^R(\text{im}(g), N)} \text{Tor}_n^R(C, N) \to \text{Tor}_n^R(T', N).
\]

Since \( T' \) is uniformly \( S \)-torsion, we have Tor\(_n^R(\text{im}(g), N) \) is an \( S \)-isomorphism. Since Tor\(_n^R(i_1^s\text{Ker}(g), N) \) and Tor\(_n^R(i_2^s\text{Ker}(g), N) \) are \( S \)-isomorphisms as above, Tor\(_n^R(A, N) \to \text{Tor}_n^R(B, N) \to \text{Tor}_n^R(C, N) \) is \( S \)-exact at Tor\(_n^R(B, N) \).

Continue by the above method, we have an \( S \)-exact sequence:

\[
... \to \text{Tor}_n^R(A, N) \to \text{Tor}_n^R(B, N) \to \text{Tor}_n^R(C, N) \xrightarrow{\delta_n} \text{Tor}_{n-1}^R(A, N) \to \\
\text{Tor}_{n-1}^R(B, N) \to ... \to \text{Tor}_1^R(C, N) \xrightarrow{\delta_1} A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.
\]

\[ \square \]

**Corollary 1.6.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( N \) an \( R \)-module. Suppose \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) is an \( S \)-exact sequence of \( R \)-modules where \( B \) is \( S \)-flat. Then Tor\(_{n+1}^R(C, N) \) is \( S \)-isomorphic to Tor\(_n^R(A, N) \) for any \( n \geq 0 \). Consequently, Tor\(_{n+1}^R(C, N) \) is uniformly \( S \)-torsion if and only if Tor\(_n^R(A, N) \) is uniformly \( S \)-torsion for any \( n \geq 0 \).

**Proof.** It follows from Lemma 1.4 and Theorem 1.5. \[ \square \]
2. ON THE $S$-FLAT DIMENSIONS OF MODULES

Let $R$ be a ring. The flat dimension of an $R$-module $M$ is defined as the shortest flat resolution of $M$. We now introduce the notion of $S$-flat dimension of an $R$-module as follows.

**Definition 2.1.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. We write $S$-$fd_R(M) \leq n$ ($S$-$fd$ abbreviates $S$-flat dimension) if there exists an $S$-exact sequence of $R$-modules

$$0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$$

where each $F_i$ is $S$-flat for $i = 0, \ldots, n$. The $S$-exact sequence (\$) is said to be an $S$-flat $S$-resolution of length $n$ of $M$. If such finite $S$-flat $S$-resolution does not exist, then we say $S$-$fd_R(M) = \infty$; otherwise, define $S$-$fd_R(M) = n$ if $n$ is the length of the shortest $S$-flat $S$-resolution of $M$.

Trivially, the $S$-flat dimension of an $R$-module $M$ cannot exceed its flat dimension for any multiplicative subset $S$ of $R$. And if $S$ is composed of units, then $S$-$fd_R(M) = fd_R(M)$. It is also obvious that an $R$-module $M$ is $S$-flat if and only if $S$-$fd_R(M) = 0$.

**Lemma 2.2.** Let $R$ be a ring, $S$ a multiplicative subset of $R$. If $A$ is $S$-isomorphic to $B$, then $S$-$fd_R(A) = S$-$fd_R(B)$.

**Proof.** Let $f : A \to B$ be an $S$-isomorphism. If $\ldots \to F_n \to \ldots \to F_1 \to F_0 \xrightarrow{f_0} A \to 0$ is an $S$-resolution of $A$, then $\ldots \to F_n \to \ldots \to F_1 \to F_0 \xrightarrow{f_0} B \to 0$ is an $S$-resolution of $B$. So $S$-$fd_R(A) \geq S$-$fd_R(B)$. Note that there is an $S$-isomorphism $g : B \to A$ by Proposition 1.1. Similarly we have $S$-$fd_R(B) \geq S$-$fd_R(A)$. \hfill $\Box$

**Proposition 2.3.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. The following statements are equivalent for an $R$-module $M$:

1. $S$-$fd_R(M) \leq n$;
2. $Tor^R_{n+k}(M, N)$ is uniformly $S$-torsion for all $R$-modules $N$ and all $k > 0$;
3. $Tor^R_{n+1}(M, N)$ is uniformly $S$-torsion for all $R$-modules $N$;
4. there exists $s \in S$ such that $s$-$Tor^R_{n+1}(M, R/I) = 0$ for all ideals $I$ of $R$;
5. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an $S$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-flat $R$-modules, then $F_n$ is $S$-flat;
6. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an $S$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules, then $F_n$ is $S$-flat;
7. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-flat $R$-modules, then $F_n$ is $S$-flat;
(8) if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules, then $F_n$ is $S$-flat;

(9) there exists an $S$-exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules and $F_n$ is $S$-flat;

(10) there exists an exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules and $F_n$ is $S$-flat;

(11) there exists an exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_n$ are flat $R$-modules.

Proof. (1) ⇒ (2): We prove (2) by induction on $n$. For the case $n = 0$, we have $M$ is $S$-flat, then (2) holds by [11] Theorem 3.2. If $n > 0$, then there is an $S$-exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where each $F_i$ is $S$-flat for $i = 0, \ldots, n$. Set $K_0 = \text{Ker}(F_0 \to M)$ and $L_0 = \text{Im}(F_1 \to F_0)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to L_0 \to 0$ are $S$-exact. Since $S$-$fd_R(L_0) \leq n - 1$ and $L_0$ is $S$-isomorphic to $K_0$, $S$-$fd_R(K_0) \leq n - 1$ by Lemma 2.2. By induction, $\text{Tor}_R^{n-1+k}(K_0, N)$ is uniformly $S$-torsion for all $S$-torsion $R$-modules $N$ and all $k > 0$. It follows from Corollary 1.6 that $\text{Tor}_R^{n+k}(M, N)$ is uniformly $S$-torsion.

(2) ⇒ (3), (5) ⇒ (6) ⇒ (8) and (5) ⇒ (7) ⇒ (10): Trivial.

(3) ⇒ (4): Let $N = \bigoplus_{I \subseteq R} R/I$. Then there exists an element $s \in S$ such that $s\text{Tor}_R^{n+1}(M, N) = 0$. So $s\bigoplus_{I \subseteq R} \text{Tor}_R^{n+1}(M, R/I) = 0$. It follows that $s\text{Tor}_R^{n+1}(M, R/I) = 0$ for all ideals $I$ of $R$.

(4) ⇒ (3): Suppose $N$ is generated by $\{n_i| i \in \Gamma\}$. Set $N_0 = 0$ and $N_\alpha = \langle n_i| i < \alpha \rangle$ for each $\alpha \leq \Gamma$. Then $N$ have a continuous filtration $\{N_\alpha| \alpha \leq \Gamma\}$ with $N_{\alpha+1}/N_\alpha \cong R/I_{\alpha+1}$ and $I_\alpha = \text{Ann}_R(n_\alpha + N_\alpha \cap Rn_\alpha)$. Since $s\text{Tor}_1^R(M, R/I_\alpha) = 0$ for each $\alpha \leq \Gamma$, it is easy to verify $s\text{Tor}_1^R(M, N_\alpha) = 0$ by transfinite induction on $\alpha$. So $s\text{Tor}_1^R(M, N) = 0$.

(3) ⇒ (5): Let $0 \to F_n \overset{d_n}{\to} F_{n-1} \overset{d_{n-1}}{\to} F_{n-2} \to \ldots \overset{d_2}{\to} F_1 \overset{d_1}{\to} F_0 \overset{d_0}{\to} M \to 0$ be an $S$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat. Then $F_n$ is flat if and only if $\text{Tor}_1^R(F_n, N)$ is uniformly $S$-torsion for all $R$-modules $N$, if and only if $\text{Tor}_2^R(\text{Im}(d_{n-1}), N)$ is uniformly $S$-torsion for all $R$-modules $N$. Following these steps, we can show $F_n$ is flat if and only if $\text{Tor}_R^{n+1}(M, N)$ is uniformly $S$-torsion for all $R$-modules $N$.

(10) ⇒ (11) ⇒ (1) and (10) ⇒ (9) ⇒ (1): Trivial.

(8) ⇒ (10): Let $\ldots \to P_n \to P_{n-1} \overset{d_{n-1}}{\to} P_{n-2} \to \ldots \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0$ be a projective resolution of $M$. Set $F_n = \text{Ker}(d_{n-1})$. Then we have an exact sequence $0 \to F_n \to P_{n-1} \overset{d_{n-1}}{\to} P_{n-2} \to \ldots \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0$. By (8), $F_n$ is flat. So (10) holds. \□
Corollary 2.4. Let $R$ be a ring and $S' \subseteq S$ multiplicative subsets of $R$. Suppose $M$ is an $R$-module, then $S\cdot fd_R(M) \leq S'\cdot fd_R(M)$.

Proof. Suppose $S' \subseteq S$ are multiplicative subsets of $R$. Let $M$ and $N$ be $R$-modules. If $\text{Tor}_{n+1}^R(M,N)$ is uniformly $S'$-torsion, then $\text{Tor}_{n+1}^R(M,N)$ is uniformly $S$-torsion. The result follows by Proposition 2.3.

Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. For any $s \in S$, we denote by $R_s$ the localization of $R$ at $\{s^n | n \in \mathbb{N}\}$ and denote $M_s = M \otimes_R R_s$ as an $R_s$-module.

Corollary 2.5. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. If $S\cdot fd_R(M) \leq n$, then there exists an element $s \in S$ such that $fd_{R_s}(M_s) \leq n$.

Proof. Let $M$ be an $R$-module with $S\cdot fd_R(M) \leq n$. Then there is an element $s \in S$ such that $s\text{Tor}_{n+1}^R(R/I, M) = 0$ for any ideal $I$ of $R$ by Proposition 2.3. Let $I_s$ be an ideal of $R_s$ with $I$ an ideal of $R$. Then $\text{Tor}_{n+1}^R(R_s/I_s, M_s) \cong \text{Tor}_{n+1}^R(R/I, M) \otimes_R R_s = 0$ since $s\text{Tor}_{n+1}^R(R/I, M) = 0$. Hence $fd_{R_s}(M_s) \leq n$.

Corollary 2.6. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Suppose $M$ is an $R$-module, then $S\cdot fd_R(M) \geq fd_{R_s}M_s$. Moreover, if $S$ is composed of finite elements, then $S\cdot fd_R(M) = fd_{R_s}M_s$.

Proof. Let $\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence with each $F_i$ $S$-flat. By localizing at $S$, we can obtain a flat resolution of $M_s$ over $R_s$ as follows:

$$\to (F_n)_S \to \cdots \to (F_1)_S \to (F_0)_S \to (M)_S \to 0.$$ 

So $S\cdot fd_R(M) \geq fd_{R_s}M_s$ by Proposition 2.3. Suppose $S$ is composed of finite elements and $fd_{R_s}M_s = n$. Let $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence, where $F_i$ is flat over $R$ for any $i = 0, \ldots, n-1$. Localizing at $S$, we have $(F_n)_S$ is flat over $R_s$. By [11] Proposition 3.8, $F$ is $S$-flat. So $S\cdot fd_R(M) \leq n$ by Proposition 2.3.

Proposition 2.7. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $0 \to A \to B \to C \to 0$ be an $S$-exact sequence of $R$-modules. Then the following assertions hold.

1. $S\cdot fd_R(C) \leq 1 + \max\{S\cdot fd_R(A), S\cdot fd_R(B)\}$.
2. If $S\cdot fd_R(B) < S\cdot fd_R(C)$, then $S\cdot fd_R(A) = S\cdot fd_R(C) - 1 > S\cdot fd_R(B)$.

Proof. The proof is similar with that of the classical case (see [11] Theorem 3.6.7). So we omit it.
Let \( p \) be a prime ideal of \( R \) and \( M \) an \( R \)-module. We denote by \( p-fd_R(M) (R-p)\)-\( fd_R(M) \) briefly. The next result gives a new local characterization of flat dimension of an \( R \)-module.

**Proposition 2.8.** Let \( R \) be a ring and \( M \) an \( R \)-module. Then

\[
fd_R(M) = \sup \{ p-fd_R(M) | p \in \text{Spec}(R) \} = \sup \{ m-fd_R(M) | m \in \text{Max}(R) \}.
\]

**Proof.** Trivially, \( \sup \{ m-fd_R(M) | m \in \text{Max}(R) \} \leq \sup \{ p-fd_R(M) | p \in \text{Spec}(R) \} \leq fd_R(M) \). Suppose \( \sup \{ m-fd_R(M) | m \in \text{Max}(R) \} = n \). For any \( R \)-module \( N \), there exists an element \( s \in R - m \) such that \( s^m \text{Tor}_n^R(M, N) = 0 \) by Proposition 2.3. Since the ideal generated by all \( s^m \in R - m \) such that \( s^m \text{Tor}_n^R(M, N) = 0 \) for all \( R \)-modules \( N \). So \( fd_R(M) \leq n \). Suppose \( \sup \{ m-fd_R(M) | m \in \text{Max}(R) \} = \infty \). Then for any \( n \geq 0 \), there exists a maximal ideal \( m \) and an element \( s^m \in R - m \) such that \( s^m \text{Tor}_n^R(M, N) \neq 0 \) for some \( R \)-module \( N \). So for any \( n \geq 0 \), we have \( \text{Tor}_n^R(M, N) \neq 0 \) for some \( R \)-module \( N \). Thus \( fd_R(M) = \infty \). So the equalities hold. \( \square \)

### 3. On the \( S \)-weak global dimensions of rings

Recall that the weak global dimension \( w.gl.dim(R) \) of a ring \( R \) is the supremum of \( S \)-flat dimensions of all \( R \)-modules. Now, we introduce the \( S \)-analogue of weak global dimensions of rings \( R \) for a multiplicative subset \( S \) of \( R \).

**Definition 3.1.** The \( S \)-weak global dimension of a ring \( R \) is defined by

\[
S-w.gl.dim(R) = \sup \{ S-fd_R(M) | M \text{ is an } R \text{-module} \}.
\]

Obviously, \( S-w.gl.dim(R) \leq w.gl.dim(R) \) for any multiplicative subset \( S \) of \( R \). And if \( S \) is composed of units, then \( S-w.gl.dim(R) = w.gl.dim(R) \). The next result characterizes the \( S \)-weak global dimension of a ring \( R \).

**Proposition 3.2.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following statements are equivalent for \( R \):

1. \( S-w.gl.dim(R) \leq n \);
2. \( S-fd_R(M) \leq n \) for all \( R \)-modules \( M \);
3. \( \text{Tor}_n^R(M, N) \) is uniformly \( S \)-torsion for all \( R \)-modules \( M, N \) and all \( k > 0 \);
4. \( \text{Tor}_{n+1}^R(M, N) \) is uniformly \( S \)-torsion for all \( R \)-modules \( M, N \);
5. there exists an element \( s \in S \) such that \( s \text{Tor}_{n+1}^R(I, J) \) for any ideals \( I \) and \( J \) of \( R \).

**Proof.** (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (4): Trivial.

(2) \( \Rightarrow \) (3): Follows from Proposition 2.3.
Thus it suffices, by Proposition 2.3, to prove that $\text{Tor}^R_0(F_n, M)$ is uniformly $S$-torsion, where $F_n$ is $S$-flat. Let $N$ be an $R$-module. Thus $S$-fd$_g(N) \leq n$ by (4). It follows from Corollary 1.6 that $\text{Tor}^R_n(N, F_n) \cong \text{Tor}^R_{n+1}(N, M)$ is uniformly $S$-torsion. Thus $F_n$ is $S$-flat.

(4) $\Rightarrow$ (1): Let $M$ be an $R$-module and $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules. To complete the proof, it suffices, by Proposition 2.3, to prove that $F_n$ is $S$-flat. Let $N$ be a $R$-module. Thus $S$-fd$_g(N) \leq n$ by (4). It follows from Corollary 1.6 that $\text{Tor}^R_0(N, F_n) \cong \text{Tor}^R_1(N, M)$ is uniformly $S$-torsion. Thus $F_n$ is $S$-flat.

(4) $\Rightarrow$ (5): Let $M = \bigoplus_{I \subseteq R} R/I$ and $N = \bigoplus_{J \subseteq R} R/J$. Then there exists $s \in S$ such that

$$s\text{Tor}^R_{n+1}(M, N) = s \bigoplus_{I \subseteq R, J \subseteq R} \text{Tor}^R_{n+1}(R/I, R/J) = 0.$$ 

Thus $s\text{Tor}^R_{n+1}(R/I, R/J) = 0$ for any ideals $I, J$ of $R$.

(5) $\Rightarrow$ (4): Suppose $M$ is generated by $\{m_i | i \in \Gamma\}$ and $N$ is generated by $\{n_i | i \in \Lambda\}$. Set $M_0 = 0$ and $M_\alpha = \langle m_i | i < \alpha \rangle$ for each $\alpha \in \Gamma$. Then $M$ has a continuous filtration $\{M_\alpha | \alpha \in \Gamma\}$ with $M_{\alpha+1}/M_\alpha \cong R/I_{\alpha+1}$ and $I_\alpha = \text{Ann}_R(m_\alpha + M_\alpha \cap M_{\alpha+1})$. Similarly, $N$ has a continuous filtration $\{N_\beta | \beta \in \Lambda\}$ with $N_{\beta+1}/N_\beta \cong R/J_{\beta+1}$ and $J_\beta = \text{Ann}_R(n_\beta + N_\beta \cap N_{\beta+1})$. Since $s\text{Tor}^R_{n+1}(R/I_\alpha, R/J_\beta) = 0$ for each $\alpha \in \Gamma$ and $\beta \in \Lambda$, it is easy to verify $s\text{Tor}^R_{n+1}(M, N) = 0$ by transfinite induction on both positions of $M$ and $N$.

The following Corollaries 3.3, 3.4, 3.5 and 3.7 can be deduced by Corollaries 2.5, 2.6, 2.4 and Proposition 2.8.

**Corollary 3.3.** Let $R$ be a ring and $S' \subseteq S$ multiplicative subsets of $R$. Then $S$-w.gl.dim($R$) $\leq S'$-w.gl.dim($R$).

**Corollary 3.4.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. If $S$-w.gl.dim($R$) $\leq n$, then there exists an element $s \in S$ such that w.gl.dim($R_s$) $\leq n$.

**Corollary 3.5.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then $S$-w.gl.dim($R$) $\leq$ w.gl.dim($R_S$). Moreover, if $S$ is composed of finite elements, then $S$-w.gl.dim($R$) = w.gl.dim($R_S$).

The following example shows that the converse of Corollary 3.3 is not true in general.

**Example 3.6.** Let $R = k[x_1, x_2, \ldots, x_{n+1}]$ be a polynomial ring with $n+1$ indeterminates over a field $k$ $(n \geq 0)$. Set $S = k[x_1] - \{0\}$. Then $S$ is a multiplicative subset of $R$ and $R_S = k[x_1][x_2, \ldots, x_{n+1}]$ is a polynomial ring with $n$ indeterminates over the field $k(x_1)$. So w.gl.dim($R_S$) = $n$ by [10] Theorem 3.8.23. Let $s \in S$, we have $R_s = k[x_1][x_2, \ldots, x_{n+1}]$. Since $k[x_1]$ is not a G-domain, $k[x_1][s]$ is not a field (see [7] Theorem 21). Thus w.gl.dim($k[x_1][s]$) = 1. So w.gl.dim($R_s$) = $n+1$ for any $s \in S$ by [10] Theorem 3.8.23] again. Consequently $S$-w.gl.dim($R$) $\geq n + 1$ by Corollary 3.4.
Let \( p \) be a prime ideal of a ring \( R \) and \( p\text{-}w.gl.dim(R) \) denote \((R-p)\text{-}w.gl.dim(R)\) briefly. We have a new local characterization of weak global dimensions of commutative rings.

**Corollary 3.7.** Let \( R \) be a ring. Then

\[
\text{w.gl.dim}(R) = \sup\{p\text{-}w.gl.dim(R)\mid p \in \text{Spec}(R)\} = \sup\{m\text{-}w.gl.dim(R)\mid m \in \text{Max}(R)\}.
\]

The rest of this section mainly consider rings with \( S \)-weak global dimensions at most one. Recall from [11] that a ring \( R \) is called \( S \)-von Neumann regular provided that there exists \( s \in S \) such that for any \( a \in R \) there exists \( r \in R \) such that \( sa = ra^2 \). Thus by [11, Theorem 3.11], the following result holds.

**Corollary 3.8.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following assertions are equivalent:

1. \( R \) is an \( S \)-von Neumann regular ring;
2. for any \( R \)-module \( M \) and \( N \), there exists \( s \in S \) such that \( s\text{Tor}^R_1(M,N) = 0 \);
3. there exists \( s \in S \) such that \( s\text{Tor}^R_1(R/I,R/J) = 0 \) for any ideals \( I \) and \( J \) of \( R \);
4. any \( R \)-module is \( R \)-flat;
5. \( S\text{-}w.gl.dim(R) = 0 \).

Trivially, von Neumann regular rings are \( S \)-von Neumann regular, and if a ring \( R \) is \( S \)-von Neumann regular ring then \( R_S \) is von Neumann regular. It was proved in [11, Proposition 3.17] that if the multiplicative subset \( S \) of \( R \) is composed of non-zero-divisors, then \( R \) is \( S \)-von Neumann regular if and only if \( R \) is von Neumann regular. Examples of \( S \)-von Neumann regular rings that are not von Neumann regular, and a ring \( R \) for which \( R_S \) is von Neumann regular but \( R \) is not \( S \)-von Neumann regular are given in [11].

**Proposition 3.9.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following assertions are equivalent:

1. \( S\text{-}w.gl.dim(R) \leq 1 \);
2. any submodule of \( S \)-flat modules is \( S \)-flat;
3. any submodule of flat modules is \( S \)-flat;
4. \( \text{Tor}^R_2(M,N) \) is uniformly \( S \)-torsion for all \( R \)-modules \( M,N \);
5. there exists an element \( s \in S \) such that \( s\text{Tor}^R_2(R/I,R/J) = 0 \) for any ideals \( I,J \) of \( R \).

**Proof.** The equivalences follow from Proposition 3.2. □
The following lemma can be found in [3, Chapter 1 Exercise 6.3] for integral domains. However it is also true for any commutative rings and we give a proof for completeness.

**Lemma 3.10.** Let $R$ be a ring and $I, J$ ideals of $R$, then $\text{Tor}_2^R(R/I, R/J) \cong \text{Ker}(\phi)$ where $\phi : I \otimes J \to IJ$ is an $R$-homomorphism, where $\phi(a \otimes b) = ab$.

**Proof.** Let $I$ and $J$ be ideals of $R$, then $\text{Tor}_2^R(R/I, R/J) \cong \text{Tor}_1^R(R/I, J)$. Consider the following exact sequence: $0 \to \text{Tor}_1^R(R/I, J) \to I \otimes_R J \xrightarrow{\phi} R \otimes_R J$, where $\phi$ is an $R$-homomorphism such that $\phi(a \otimes b) = ab$. We have $\text{Tor}_2^R(R/I, R/J) \cong \text{Ker}(\phi)$. \qed

Trivially, a ring $R$ with $\text{w.gl.dim}(R) \leq 1$ has $S$-weak global dimension at most one. The following example shows the converse does not hold generally.

**Example 3.11.** Let $A$ be a ring with $\text{w.gl.dim}(A) = 1$, $T = A \times A$ the direct product of $A$. Set $s = (1, 0) \in T$, then $s^2 = s$. Let $R = T[x]/(sx, x^2)$ with $x$ an indeterminate and $S = \{1, s\}$ be a multiplicative subset of $R$. Then $S$-$\text{w.gl.dim}(R) = 1$ but $\text{w.gl.dim}(R) = \infty$.

**Proof.** Note that every element in $R$ can be written as $r = (a, b) + (0, c)x$ where $a, b, c \in A$. Let $f : R \to A$ be a ring homomorphism where $f((a, b) + (0, c)x) = a$. Then $f$ makes $A$ a module retract of $R$. Let $I$ and $J$ be ideals of $R$. Suppose $r_1 = (a_1, b_1) + (0, c_1)x$ and $r_2 = (a_2, b_2) + (0, c_2)x$ are elements in $I$ and $J$ respectively such that $r_1 \otimes r_2 \in \text{Ker}(\phi)$, where $\phi : I \otimes_R J \to IJ$ is the multiplicative homomorphism. Then $r_1r_2 = (a_1a_2, b_1b_2) + (0, b_1c_2 + b_2c_1)x = 0$, so $a_1a_2 = 0$ in $A$. By Lemma [3.10] $a_1 \otimes_A a_2 = 0$ in $f(I) \otimes_A f(J)$ since $\text{w.gl.dim}(A) = 1$. Consequently $s^2r_1 \otimes r_2 = sr_1 \otimes_R sr_2 = (a_1, 0) \otimes_R (a_2, 0) = 0$ in $I \otimes J$. So $s^2\text{Tor}_2^R(R/I, R/J) = 0$ by Lemma [3.10]. It follows that $S$-$\text{w.gl.dim}(R) \leq 1$ by Proposition [3.9]. Since $R_S \cong A$ have weak global dimension 1, $S$-$\text{w.gl.dim}(R) = 1$ by Corollary [3.8] and [11, Corollary 3.14]. Since $R$ is non-reduced coherent ring, then $\text{w.gl.dim}(R) = \infty$ by [4, Corollary 4.2.4]. \qed

4. **S-weak global dimensions of factor rings and polynomial rings**

In this section, we mainly consider the $S$-weak global dimensions of factor rings and polynomial rings. Firstly, we give an inequality of $S$-weak global dimensions for ring homomorphisms. Let $\theta : R \to T$ be a ring homomorphism. Suppose $S$ is a multiplicative subset of $R$, then $\theta(S) = \{\theta(s) | s \in S\}$ is a multiplicative subset of $T$.

**Lemma 4.1.** Let $\theta : R \to T$ be a ring homomorphism, $S$ a multiplicative subset of $R$. Suppose $L$ is a $\theta(S)$-flat $T$-module. Then for any $R$-module $X$ and any $n \geq 0,
\[
\text{Tor}_n^R(X, L) \text{ is } S\text{-isomorphic to } \text{Tor}_n^R(X, T) \otimes_T L. \text{ Consequently, } S\text{-fd}_R(L) \leq S\text{-fd}_R(T).
\]

**Proof.** If \( n = 0 \), then \( X \otimes_R L \cong X \otimes_R (T \otimes_T L) \cong (X \otimes_R T) \otimes_T L \).

If \( n = 1 \), let \( 0 \to A \to P \to X \to 0 \) be an exact sequence of \( R\)-modules where \( P \) is free. Thus we have two exact sequences of \( T\)-module: \( 0 \to \text{Tor}_1^R(X, T) \to A \otimes_R T \to P \otimes_R T \to X \otimes_R T \to 0 \) and \( 0 \to \text{Tor}_1^R(X, L) \to A \otimes_R L \to P \otimes_R L \to X \otimes_R L \to 0 \).

Consider the following commutative diagram with exact sequence:

\[
\begin{array}{cccccc}
0 & \to & 0 & \to & \text{Tor}_1^R(X, L) & \to & A \otimes_R L & \to & P \otimes_R L \\
& & & & \downarrow{h} & & \downarrow{\cong} & & \downarrow{\cong} \\
0 & \to & \text{Ker}(\delta) & \to & \text{Tor}_1^R(X, T) \otimes_T L & \to & (A \otimes_R T) \otimes_T L & \to & (P \otimes_R T) \otimes_T L.
\end{array}
\]

Since \( L \) is a \( \theta(S)\)-flat \( T\)-module, \( \delta \) is a \( \theta(S)\)-monomorphism. By Theorem 1.3, \( h \) is a \( \theta(S)\)-isomorphism over \( T \). So \( h \) is an \( S\)-isomorphism over \( R \) since \( T\)-modules are viewed as \( R\)-modules through \( \theta \). By dimension-shifting, we can obtain that \( \text{Tor}_n^R(X, L) \) is \( S\)-isomorphic to \( \text{Tor}_n^R(X, T) \otimes_T L \) for any \( R\)-module \( X \) and any \( n \geq 0 \).

Thus for any \( R\)-module \( X \), if \( \text{Tor}_n^R(X, T) \) is uniformly \( S\)-torsion, then \( \text{Tor}_n^R(X, L) \) is also uniformly \( S\)-torsion. Consequently, \( S\text{-fd}_R(L) \leq S\text{-fd}_R(T) \). \( \square \)

**Proposition 4.2.** Let \( \theta : R \to T \) be a ring homomorphism, \( S \) a multiplicative subset of \( R \). Suppose \( M \) is an \( T\)-module. Then

\[
S\text{-fd}_R(M) \leq \theta(S)\text{-fd}_T(M) + S\text{-fd}_R(T).
\]

**Proof.** Assume \( \theta(S)\text{-fd}_T(M) = n < \infty \). If \( n = 0 \), then \( M \) is \( \theta(S)\)-flat over \( T \). By Lemma 4.1, \( S\text{-fd}_R(M) \leq n + S\text{-fd}_R(T) \).

Now we assume \( n > 0 \). Let \( 0 \to A \to F \to M \to 0 \) be an exact sequence of \( T\)-modules, where \( F \) is a free \( T\)-module. Then \( \theta(S)\text{-fd}_T(A) = n - 1 \) by Corollary 1.6 and Proposition 2.3. By induction, \( S\text{-fd}_R(A) \leq n - 1 + S\text{-fd}_R(T) \). Note that \( S\text{-fd}_R(T) = S\text{-fd}_R(F) \). By Proposition 2.7, we have

\[
S\text{-fd}_R(M) \leq 1 + \max\{S\text{-fd}_R(F), S\text{-fd}_R(A)\}
\]

\[
\leq 1 + n - 1 + S\text{-fd}_R(T)
\]

\[
= \theta(S)\text{-fd}_T(M) + S\text{-fd}_R(T).
\]

\( \square \)

Let \( R \) be a ring, \( I \) an ideal of \( R \) and \( S \) a multiplicative subset of \( R \). Then \( \pi : R \to R/I \) is a ring epimorphism and \( \pi(S) := \overline{S} = \{s + I \in R/I | s \in S\} \) is naturally a multiplicative subset of \( R/I \).
Proposition 4.3. Let $R$ be a ring, $S$ a multiplicative subset of $R$. Let $a \in R$ be neither a zero-divisor nor a unit. Written $\overline{R} = R/aR$ and $\overline{S} = \{ s + aR \in \overline{R} | s \in S \}$. Then the following assertions hold.

(1) Let $M$ be a nonzero $\overline{R}$-module. If $\overline{S} \cdot \text{fd}_{\overline{R}}(M) < \infty$, then

$$\text{S-fd}_R(M) = \overline{S} \cdot \text{fd}_{\overline{R}}(M) + 1.$$ 

(2) If $\overline{S} \cdot \text{w.gl.dim}(\overline{R}) < \infty$, then

$$\text{S-w.gl.dim}(R) \geq \overline{S} \cdot \text{w.gl.dim}(\overline{R}) + 1.$$ 

Proof. (1) Set $\overline{S} \cdot \text{fd}_{\overline{R}}(M) = n$. Since $a \in R$ be neither a zero-divisor nor a unit, then $\text{S-fd}_R(M) = 1$. By Proposition 1.2, we have $\text{S-fd}_R(M) \leq \overline{S} \cdot \text{fd}_{\overline{R}}(M) + 1 = n + 1$. Since $\overline{S} \cdot \text{fd}_{\overline{R}}(M) = n$, then there is an injective $\overline{R}$-module $C$ such that $\text{Tor}_{\overline{R}}^n(M, C)$ is not uniformly $\overline{S}$-torsion. By [10] Theorem 2.4.22], there is an injective $R$-module $E$ such that $0 \to C \to E \to E \to 0$ is exact. By [10] Proposition 3.8.12(4), $\text{Tor}_{R}^{n+1}(M, E) \cong \text{Tor}_{R}^{n}(M, C)$. Thus $\text{Tor}_{R}^{n+1}(M, E) = \text{Tor}_{R}^{n}(M, C)$ is uniformly $S$-torsion. So $\text{S-fd}_R(M) = \overline{S} \cdot \text{fd}_{\overline{R}}(M) + 1$.

(2) Let $n = \overline{S} \cdot \text{w.gl.dim}(\overline{R})$. Then there is a nonzero $\overline{R}$-module $M$ such that $\overline{S} \cdot \text{fd}_{\overline{R}}(M) = n$. Thus $\text{S-fd}_R(M) = n + 1$ by (1). So $\text{S-w.gl.dim}(R) \geq \overline{S} \cdot \text{w.gl.dim}(\overline{R}) + 1$. □

Let $R$ be a ring and $M$ an $R$-module. $R[x]$ denotes the polynomial ring with one indeterminate, where all coefficients are in $R$. Set $M[x] = M \otimes_R R[x]$, then $M[x]$ can be seen as an $R[x]$-module naturally. It is well-known $\text{w.gl.dim}(R[x]) = \text{w.gl.dim}(R)$ (see [10] Theorem 3.8.23]). In this section, we give a $S$-analogue of this result. Let $S$ be a multiplicative subset of $R$, then $S$ is a multiplicative subset of $R[x]$ naturally.

Lemma 4.4. Let $R$ be a ring, $S$ a multiplicative subset of $R$. Suppose $T$ is an $R$-module and $F$ is an $R[x]$-module. Then the following assertions hold.

(1) $T$ is uniformly $S$-torsion over $R$ if and only if $T[x]$ is a uniformly $S$-torsion $R[x]$-module.

(2) If $F$ is $S$-flat over $R[x]$, then $F$ is $S$-flat over $R$.

Proof. (1) If $sT[x] = 0$ for some $s \in S$, then trivially $sT = 0$. So $T$ is uniformly $S$-torsion over $R$. Suppose $sT = 0$ for some $s \in S$. Then $sT[x] \cong (sT)[x] = 0$. Thus $T[x]$ is a uniformly $S$-torsion $R[x]$-module.

(2) Suppose $F$ is an $S$-flat $R[x]$-module. By [1] Theorem 1.3.11, $\text{Tor}_1^R(F, L)[x] \cong \text{Tor}_1^R(F[x], L[x])$ is uniformly $S$-torsion. Thus there exists an element $s \in S$ such that $s\text{Tor}_1^R(F, L)[x] = 0$. So $s\text{Tor}_1^R(F, L) = 0$. It follows that $F$ is an $S$-flat $R$-module. □
Proposition 4.5. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Then $S\text{-}fd_{R[x]}(M[x]) = S\text{-}fd_{R}(M)$.

Proof. Assume that $S\text{-}fd_{R}(M) \leq n$. Then $\text{Tor}^{R}_{n+1}(M, N)$ is uniformly $S$-torsion for any $R$-module $N$ by Proposition 2.3. Thus for any $R[x]$-module $L$, $\text{Tor}^{R}_{n+1}(M[x], L) \cong \text{Tor}^{R}_{n+1}(M, L)$ is uniformly $S$-torsion for any $R[x]$-module $L$ by [4] Theorem 1.3.11. Consequently, $S\text{-}fd_{R[x]}(M[x]) \leq n$ by Proposition 2.3.

Let $0 \to F_{n} \to \ldots \to F_{1} \to F_{0} \to M[x] \to 0$ be an exact sequence with each $F_{i}$ $S$-flat over $R[x]$ ($1 \leq i \leq n$). Then it is also $S$-flat resolution of $M[x]$ over $R$ by Lemma 4.4. Thus $\text{Tor}^{R}_{n+1}(M[x], N)$ is uniformly $S$-torsion for any $R$-module $N$ by Proposition 2.3. It follows that $s\text{Tor}^{R}_{n+1}(M[x], N) = \bigoplus_{i=1}^{\infty} \text{Tor}^{R}_{n+1}(M, N) = 0$. Thus $\text{Tor}^{R}_{n+1}(M, N)$ is uniformly $S$-torsion. Consequently, $S\text{-}fd_{R}(M) \leq S\text{-}fd_{R[x]}(M[x])$ by Proposition 2.3 again. $\square$

Let $M$ be an $R[x]$-module then $M$ can be naturally viewed as an $R$-module. Define $\psi : M[x] \to M$ by

$$\psi\left(\sum_{i=0}^{n} x^{i} \otimes m_{i}\right) = \sum_{i=0}^{n} x^{i} m_{i}, \quad m_{i} \in M.$$ 

And define $\varphi : M[x] \to M[x]$ by

$$\varphi\left(\sum_{i=0}^{n} x^{i} \otimes m_{i}\right) = \sum_{i=0}^{n} x^{i+1} \otimes m_{i} - \sum_{i=0}^{n} x^{i} \otimes x m_{i}, \quad m_{i} \in M.$$ 

Lemma 4.6. [10] Theorem 3.8.22] Let $R$ be a ring, $S$ a multiplicative subset of $R$. For any $R[x]$-module $M$,

$$0 \to M[x] \xrightarrow{\varphi} M[x] \xrightarrow{\psi} M \to 0$$

is exact.

Theorem 4.7. Let $R$ be a ring, $S$ a multiplicative subset of $R$. Then $S\text{-w.gl.dim}(R[x]) = S\text{-w.gl.dim}(R) + 1$.

Proof. Let $M$ be an $R[x]$-module. Then, by Lemma 4.6 there is an exact sequence over $R[x]$:

$$0 \to M[x] \to M[x] \to M \to 0.$$ 

By Proposition 2.7 and Proposition 4.5

$$S\text{-}fd_{R}(M) \leq S\text{-}fd_{R[x]}(M) \leq 1 + S\text{-}fd_{R[x]}(M[x]) = 1 + S\text{-}fd_{R}(M) \quad (*)$$

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Thus if $S$-w.gl.dim$(R) < \infty$, then $S$-w.gl.dim$(R[x]) < \infty$.

Conversely, if $S$-w.gl.dim$(R[x]) < \infty$, then for any $R$-module $M$, $S$-fd$_R$(M) = $S$-fd$_{R[x]}$(M[x]) < \infty by Proposition 4.5. Therefore we have $S$-w.gl.dim$(R) < \infty$ if and only if $S$-w.gl.dim$(R[x]) < \infty$. Now we assume that both of these are finite. Then $S$-w.gl.dim$(R[x]) \leq S$-w.gl.dim$(R) + 1$ by $(\ast)$. Since $R \cong R[x]/xR[x]$, $S$-w.gl.dim$(R[x]) \geq S$-w.gl.dim$(R) + 1$ by Proposition 4.3. Consequently, we have $S$-w.gl.dim$(R[x]) = S$-w.gl.dim$(R) + 1$. \hspace{1cm} \Box$

**Corollary 4.8.** Let $R$ be a ring, $S$ a multiplicative subset of $R$. Then for any $n \geq 1$ we have

$$S$$-w.gl.dim$(R[x_1, \ldots, x_n]) = S$-w.gl.dim$(R) + n$.

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