A GEOMETRIC APPROACH TO THE CONCEPT OF
EXTENSIVITY IN THERMODYNAMICS

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Abstract. This paper presents a rigorous treatment of the concept of exten-
sivity in equilibrium thermodynamics from a geometric point of view. This
is achieved by endowing the manifold of equilibrium states of a system with
a smooth atlas that is compatible with the pseudogroup of transformations
on a vector space that preserve the radial vector field. The resulting geo-
metric structure allows for accurate definitions of extensive differential forms
and scaling, and the well-known relationship between both is reproduced. This
structure is represented by a global vector field that is locally written as a
radial one. The submanifolds that are transversal to it are embedded, and
locally defined by functions with extensive differential. These submanifolds
are a geometric generalization of the space of states of a closed system in
equilibrium.

1. Introduction

The concept of extensivity plays a central role in equilibrium thermodynamics.
Remarkably, it lacks a precise geometric formulation, in spite of the increasing
interest in the application of differential geometry to this branch of physics.

Of particular importance in the geometric approaches to thermodynamics are
extensive functions and extensive differential forms. The definition of extensive
function that is commonly used in equilibrium thermodynamics (and its geometric
approaches) relies on a rather vague concept from the mathematical point of view:
that of extensive variables. This paper seeks to define them rigorously, in a geo-
metric setting, so that the resulting notion of extensivity coincides locally with the
usual one.

Throughout this paper, $M$ denotes the manifold of equilibrium states of a ther-
modynamic system, which is assumed both finite-dimensional and smooth. All
tensor fields on $M$ (including functions) are considered smooth.

It is well known that a thermodynamic system in equilibrium is completely de-
scribed by a fundamental equation [1], which is simply the coordinate expression
of a smooth (global) function $\Phi$ defined on $M$, called thermodynamic potential. It
has the particular feature of being an extensive function: a degree-1 homogeneous
function of the extensive variables of the system. This attribute turns out to be
important in Geometrothermodynamics and in Ruppeiner Geometry, the two main
geometric approaches to thermodynamics nowadays. In the former, the extensivity
of thermodynamic potentials has been suggested to classify thermodynamic sys-
tems, in order to determine uniquely metrics suitable to describe them [2]. On
the other hand, using extensive potentials in the Ruppeiner Geometry of black hole
thermodynamics —where the customary potentials are quasihomogeneous [3]— sen-
sibly affects the results this formalism yields [4].
Extensive differential forms are also an important subject in the geometric approaches to thermodynamics. As has been pointed out \[5\], if \( \vartheta \) is an extensive differential form representing infinitesimal heat within a given thermodynamic system, then the Euler vector field \( \rho \) of the system is a symmetry thereof and \( 1/\vartheta(\rho) \) is an integrating factor for the former (see also Ref. \[6\]). Notice that the notion of Euler vector field is coordinate dependent, resting on the aforementioned idea of extensive variables. As expected, it turns out that providing a geometric definition of extensive variables amounts to describing \( \rho \) in a coordinate-free fashion, as we shall show. The fact that \( \vartheta \) is proportional to the derivative of entropy, an extensive function (which is unique up to a constant scale factor), follows from a more general feature that share the manifolds that are transversal to \( \rho \): they are locally defined by functions whose differential is extensive.

We have organized this paper as follows. In Section 2, we state the main definitions of this paper, viz., that of extensive variables, extensive functions, and extensive differential forms. Besides, we show that our definition of extensivity agrees with the common one that relies on scaling. The Euler equation also holds in this case. As we mentioned before, endowing \( M \) with a suitable structure to describe extensive variables is equivalent to defining a global vector field \( \rho \) having locally the form of an Euler vector field. Hence, the question of existence of an extensive structure on \( M \) may be translated to the analysis of the singularities of vector fields. We briefly describe this relationship in Section 3. The extensivity and uniqueness of entropy in thermodynamic systems is the consequence of the transversality between the distribution defined by the heat 1-form and \( \rho \), as we show in Section 4, where we deal with submanifolds that are transversal to the latter. Finally, Section 5 is devoted to concluding remarks.

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2. Definitions and basic results

In what follows, \( n \) denotes the dimension of \( M \). The set of vector fields defined on an open subset \( U \) of \( M \) is represented by \( \mathfrak{X}(U) \), whereas \( \Omega^k(U) \) stands for the set of differential \( k \)-forms defined thereon. The set of real-valued functions on \( U \) is written as \( C^\infty(U) \). We use Einstein’s summation convention over repeated indices.

We have mentioned above that the usual concept of extensive functions in thermodynamics relies on the well-known notion of homogeneous function\[1\]. Recall that if \( V \) is an \( n \)-dimensional real vector space and \( U \) is an open subset of \( V \), we say \( f : U \to \mathbb{R} \) is a degree-1 homogeneous function if
\[
(1) \quad f(\lambda v) = \lambda f(v),
\]
for all \( v \in U \) and \( \lambda \in [0, \infty) \) for which \( \lambda v \in U \). Immediate examples of degree-1 homogeneous functions are real-valued linear functions on \( V \).

A remarkable feature of Eq. (1) is that it does not involve any particular set of coordinates. If we wished to make a similar definition on \( M \), we would require a group action of \([0, \infty)\) on \( M \) whose definition is coordinate independent. Physically, this amounts to describing the scaling of systems without referring to

\[1\] These functions are particular instances of the broader class of quasi-homogeneous complex-valued functions \[8\]. The use of the latter in thermodynamics is not discussed in this paper.
extensive variables. We will circumvent this task and follow an alternative route to extensivity, pointed out by a well-known theorem by Euler, which establishes that smooth degree-1 homogeneous functions may be written in terms of their derivative along the radial vector field $R$ on $V$. Before focusing to this result, we remind the reader that, since the tangent bundle of $V$ may be canonically identified with $V \times V$, $R$ can be written in a coordinate-free fashion as $R = \text{id} \times \text{id}$, where $\text{id}$ represents the identity mapping. As usual, we denote by $d$ the exterior derivative of k-forms.

**Theorem 1.** Let $U \subset V$ be open and $f \in C^\infty(U)$. Then $f$ is a degree-1 homogeneous function if and only if
\[
\frac{d(f(R))}{dt} = f.
\]

**Proof.** Let $\gamma$ denote the integral curve of $R$ starting at $v$ (this is, $\gamma : \mathbb{R} \to V$ is given by $\gamma(t) = e^tv$, for all $t \in \mathbb{R}$). If $U$ is an open subset of $V$ and $f \in C^\infty(U)$ is a degree-1 homogeneous function, then $f \circ \gamma(t) = e^t f(v)$ for all $t$ lying in some open interval around $t = 0$. Therefore, $d(f(R_v)) = \frac{d(f \circ \gamma)}{dt}|_{t=0} = f(v)$. Since $v$ is an arbitrary element of $V$, we obtain Eq. (2).

Conversely, if $U$ is an open subset of $V$ and $f \in C^\infty(U)$ satisfies $d(f(R)) = f$, then $\frac{d(f \circ \gamma)}{dt} = f \circ \gamma(t)$ for all $t \in \mathbb{R}$ such that $\gamma(t) \in U$. Integrating the last equation yields $f \circ \gamma(t) = e^tf \circ \gamma(0)$. This means that $f(e^tv) = e^tf(v)$, for any $v \in V$ and $t \in \mathbb{R}$ satisfying $e^tv \in U$, whence $f$ is a degree-1 homogeneous function. \hfill \square

For the sake of definiteness, in what follows we shall consider $V$ simply as $\mathbb{R}^n$. Eq. (2) suggests that pushing $R$ forward to $M$ consistently through local parameterizations might help to define extensive functions thereon. By consistently, we mean both that the resulting vector field is globally defined and that it does not depend on the parameterization. To be more precise, if $(U_1, \phi_1)$ and $(U_2, \phi_2)$ are two overlapping charts belonging to the smooth atlas of $M$, then for any $x \in U_1 \cap U_2$,
\[
\phi_1^{-1}(\phi_1(x)) \left( R_{\phi_1(x)} \right) = \phi_2^{-1} \phi_2(x) \left( R_{\phi_2(x)} \right)
\]

must hold (we denote by $F_{p}$, the derivative of a mapping $F$ on a point $p$). Equivalently, both the transition function $\psi_{12} = \phi_2 \circ \phi_1^{-1}$ and its inverse $\psi_{21}$ must leave $R$ invariant, i.e., $\psi_{12}$ has to satisfy $\psi_{12} \circ (R_{p}) = R_{\psi_{12}(p)}$, for any $p \in \phi_1(U_1 \cap U_2)$, and $\psi_{21} \circ (R_{q}) = R_{\psi_{21}(q)}$, for any $q \in \phi_2(U_1 \cap U_2)$. These last conditions motivate the following.

**Definition 1.** We refer to diffeomorphisms $F$ defined on an open subset of $\mathbb{R}^n$ satisfying
\[
F_{*p}(R_p) = R_{F(p)}
\]
as degree-1 homogeneous diffeomorphisms. The set of all such diffeomorphisms will be denoted by $H$.

Observe that Eq. (4) can be regarded as the analogue of Eq. (2), provided that $R_{F(p)}$ may be identified with $F(p)$. In fact, the similarity goes beyond a mere analogy, as the elements of $F$ behave like degree-1 homogeneous functions under scaling.

**Proposition 1.** A diffeomorphism $F$ defined on an open set $U$ of the Euclidean space belongs to $H$ if and only if
\[
F(\lambda p) = \lambda F(p),
\]
for all $p \in U$ and all $\lambda \in [0, \infty]$ for which $\lambda p \in U$.

Proof. Let $U$ be an open subset of $\mathbb{R}^n$ and $(u^1, \ldots, u^n)$ denote the cartesian coordinates thereon. By defining $F^i := u^i \circ F$ for each $i \in \{1, \ldots, n\}$, it can readily be seen that $F \in H$ if and only if every $F^i$ satisfies Eq. (2), whence the result follows. \hfill \Box

Stemming from the result above, linear operators on $\mathbb{R}^n$ are straightforward examples of degree-1 homogeneous diffeomorphisms.

The aim of introducing degree-1 homogeneous diffeomorphisms is to state Eq. (3) in the language of atlases compatible with pseudogroups of transformations. This can be achieved owing to the fact below.

**Proposition 2.** The set $H$ is a group.

Proof. Since the identity mapping $\text{id}$ belongs to $H$, the latter is nonempty.

Suppose that $F \in H$. Then, for any $p \in \mathbb{R}^n$, $F_*\!\!_{\!\!F^{-1}(p)}(R_{F^{-1}(p)}) = R_p$. Besides, $R_p = F_*\!\!_{\!\!F^{-1}(p)} \circ F^{-1}_*\!\!_{\!\!R_p}(R_p)$. The last two expressions imply that $F^{-1}_*\!\!_{\!\!R_p}(R_p) = R_{F^{-1}(p)}$, whence $F^{-1} \in H$.

Finally, given $F_1, F_2 \in H$ and $p \in \mathbb{R}^n$, we have that $(F_1 \circ F_2)_*\!\!_{\!\!R_p}(R_p) = F_1\!\!_{\!\!F_2(p)}(F_2\!\!_{\!\!R_p}(R_p)) = F_1\!\!_{\!\!F_2(p)}(R_{F_2(p)}) = R_{F_1 \circ F_2(p)}$. Thus, $F_1 \circ F_2 \in H$, which completes the proof. \hfill \Box

We denote by $\mathcal{H}$ the pseudogroup of transformations on $\mathbb{R}^n$ formed by restrictions of elements of $H$ to open subsets of $\mathbb{R}^n$.

Demanding that $R$ is pushed forward to $M$ consistently by local parameterizations means that the corresponding transition functions must belong to $\mathcal{H}$. In more sophisticated terms, we need to furnish $M$ with an atlas compatible with $\mathcal{H}$. We may readily see that the vector field $\rho$ defined on $M$ as

$$\rho_p := \phi^{-1}_*\!\!_{\!\!\phi(p)}(R_{\phi(p)}),$$

for each $p \in M$, is both well and globally defined, provided that $\phi$ corresponds to a chart whose domain contains $p$ and that belongs to the aforementioned atlas.

Using the vector field above and inspired by Theorem 1, we can define extensive functions on $M$. In what follows, we shall assume that the latter is furnished with an atlas compatible with $\mathcal{H}$, which will be denoted by $\mathcal{A}_\mathcal{H}$.

**Definition 2.** Let $U$ be an open subset of $M$. We say that $f \in C^\infty(U)$ is an extensive function if $df(\rho) = df$.

A straightforward example of extensive functions are the coordinate functions that correspond to charts belonging to $\mathcal{A}_\mathcal{H}$. This follows upon observing that if $f$ is an extensive function defined on a neighborhood of a point $p \in M$ and $\phi$ is a coordinate transformation corresponding to an element of $\mathcal{A}_\mathcal{H}$ around $p$, then

$$df_p(\rho_p) = d(f \circ \phi^{-1})_{\phi(p)}(R_{\phi(p)}).$$

The equation above has two important, straightforward consequences. We express the first one in the next proposition.

**Theorem 2.** Let $U$ be an open subset of $M$. A function $f \in C^\infty(U)$ is extensive if and only if for any chart $(W, \phi) \in \mathcal{A}_\mathcal{H}$ with $W \subset U$, $f \circ \phi^{-1}$ is a degree-1 homogeneous function on $\mathbb{R}^n$. 

It is worth observing that, if \((W', \psi) \in \mathcal{A}_H\) is any other chart whose domain overlaps with the above-mentioned \(W\), then \(f \circ \psi^{-1}\) is also a degree-1 homogeneous function. This follows from writing \(f \circ \psi^{-1}\) as \(f \circ \phi^{-1} \circ \phi \circ \psi^{-1}\) and applying the chain rule.

The other significant by-product of Eq. (7) is that \(\rho|_U = x^i \partial_i\), provided that \((U, (x^1, \ldots, x^n)) \in \mathcal{A}_H\). This means that an extensive function \(f\) whose domain overlaps with \(U\) is locally written as \(x^i \partial_i f\), which prompts the following definition.

**Definition 3.** An extensive variable on \(M\) is a coordinate function of a chart belonging to \(\mathcal{A}_H\). The latter shall be referred to as an extensive structure on \(M\). The charts belonging to the extensive structure of \(M\) are called extensive charts. The pair \((M, \mathcal{A}_H)\) is called extensive manifold.

Notice that the contents of Theorem 2 and Definition 3 together may be rephrased in the standard terms of equilibrium thermodynamics: extensive functions are degree-1 homogeneous functions of any extensive variables of the system.

**Remark 1.** We write any extensive variables and not the extensive variables, because these are defined up to an extensive function. In other words, any non-zero extensive function is itself an extensive variable.

We have mentioned that not only functions, but also extensive differential forms are an important concept in thermodynamics. The latter may be readily defined using \(d\) to extend Definition 2.

**Definition 4.** Let \(U\) be an open subset of \(M\), and \(k \in \mathbb{N}\). A differential \(k\)-form \(\omega \in \Omega^k(U)\) is extensive if \(\mathcal{L}_\rho \omega = \omega\).

Two straightforward instances of extensive differential forms are those representing infinitesimal heat and infinitesimal work in thermodynamics. The heat form is of particular importance, due both to its geometric properties and the fact that it is sufficient to determine uniquely a thermodynamic system, as we shall explain in the sequel.

Before concluding this section, we consider worth mentioning that extensive structures may be portrayed differently, depending on what feature of extensivity is considered to be the most important. For instance, instead of demanding that the transition functions on \(M\) leave the radial vector field invariant, we could have required that they leave the homogeneity of functions invariant, i.e., that they map (via the pull-back of functions) degree-1 homogeneous functions to degree-1 homogeneous functions. The next result establishes that the geometric structure that corresponds to this requirement is actually an extensive one.

**Proposition 3.** A diffeomorphism \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is degree-1 homogenous if and only if for any degree-1 homogeneous function \(f\) defined on an open subset of \(\mathbb{R}^n\), \(F^* f\) is a degree-1 homogeneous function.

**Proof.** Observe that if \(f\) is a real-valued function defined on an open subset of \(\mathbb{R}^n\) and \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a diffeomorphism, then for each \(p \in \mathbb{R}^n\), \(d(F^* f)_p(R_p) = df_{F(p)} \circ F^*_p(R_p)\).

Let \(f\) be a degree-1 homogeneous function, and suppose that \(F \in H\). Then \(d(F^* f)_p(R_p) = df_{F(p)}(R_{F(p)}) = f(F(p)) = F^* f(p)\). Therefore, \(F^* f\) is a degree-1 homogeneous function.
Conversely, if $F$ pulls back any degree-1 homogeneous function to a degree-1 homogeneous function, then for any such $f$ defined on an open subset of $\mathbb{R}^n$ we have that $df_{F(p)}(R_p) = d(F^*f)_p(R_p) = F^*f(p) = f(F(p)) = df_{F(p)}(R_{F(p)})$. Hence, the derivative of $f$ at $F(p)$ annihilates $F_{*p}(R_p) - R_{F(p)}$. Since the canonical projections $\pi^i : \mathbb{R}^n \to \mathbb{R}$, with $i \in \{1, \ldots, n\}$, are degree-1 homogeneous functions, we have that $F_{*p}(R_p) - R_{F(p)} \in \bigcap_{i=1}^n \ker d\pi^i = \{0\}$. This implies that $F \in H$. \hfill $\square$

So far, we have not made any reference to the relationship between scaling and extensivity that is established by Eq. (1). As expected, defining an extensive structure on $M$ provides a means to define scaling of equilibrium states geometrically. Indeed, let $\phi_t$ denote the (local) flow of $\rho$. For each $p \in M$, the integral curve of $\rho$ starting at $p$, $\gamma(t) := \phi_t(p)$, is defined on an open neighborhood of $t = 0$, which may be written as $]-\varepsilon_p, \varepsilon_p[$ for some $\varepsilon_p > 0$ that varies pointwise. We let $\lambda \in ]e^{-\varepsilon_p}, e^{\varepsilon_p}[\text{ and define } \lambda p := \phi_{\log \lambda}(p)$. This operation is not exactly an action of the positive real numbers on $M$. However, it satisfies the familiar properties of uniform scaling on the Euclidean space, and reproduces on $M$ the well-known relationship between scaling and extensivity.

**Proposition 4.** Let $U$ be an open subset of $M$ and $\omega \in \Omega^k(U)$, with $k \in \mathbb{N} \cup \{0\}$. Then $\omega$ is extensive if and only if

$$\phi_t^* \omega = e^t \omega,$$

for every value of $t$ for which Eq. (8) makes sense.

**Proof.** We first show that differential forms satisfying Eq. (8) are extensive. Suppose that $\omega \in \Omega^k(U)$ is such a differential form, defined on an open subset $U$ of $M$. Then, there is an open interval $I$ containing $t = 0$ such that $(\phi_t^* \omega - \omega)/t = \omega(e^t - 1)/t$, for every $t \in I \setminus \{0\}$. This approaches $\omega$ as $t \to 0$, i.e., $\mathcal{L}_\rho \omega = \omega$.

Conversely, let $\omega \in \Omega^k(U)$. Recall that

$$\frac{d}{dt}(\phi_t^* \omega) = \phi_t^* (\mathcal{L}_\rho \omega).$$

Suppose now that $\omega$ is extensive. Then, for any (time-independent) $X_1, \ldots, X_k \in \mathcal{X}(M)$, Eq. (9) is written as

$$\frac{d}{dt}[(\phi_t^* \omega)(X_1 \wedge \cdots \wedge X_k)] = (\phi_t^* \omega)(X_1 \wedge \cdots \wedge X_k).$$

We have thus that $(\phi_t^* \omega)(X_1 \wedge \cdots \wedge X_k) = e^t \omega(X_1 \wedge \cdots \wedge X_k)$, for values of $t$ around $t = 0$ for which $\phi_t^* \omega$ is defined. Since the vector fields $X_1, \ldots, X_k$ are arbitrary, Eq. (8) follows. \hfill $\square$

When $k = 0$, Eq. (8) reads precisely $f(\lambda p) = \lambda f(p)$, for all $\lambda \in ]e^{-\varepsilon_p}, e^{\varepsilon_p}[\text{ and } p \in M$ (cf. Eq. (1)).

The notion of uniform scaling of states may also be taken as starting point to define an extensive structure on $M$. Namely, for any $x \in M$ we may intuitively define $\lambda x$ as $\phi^{-1}(\lambda \phi(x))$, where $(U, \phi)$ is a smooth chart whose domain contains $x$. Demanding that this definition be coordinate-independent amounts to requiring the transition functions on $M$ satisfy Eq. (5) for all points on the Euclidean space and all values of $\lambda$ for which the latter makes sense. Hence, an extensive structure on $M$ may be regarded as a smooth atlas whose charts preserve locally dilations on $\mathbb{R}^n$. 

\hfill $\square$
We have seen so far that any extensive manifold is endowed with a global vector field that has locally the form of a radial vector field. In the next section, we will show that actually such vectors embody extensive structures. Therefore, studying the conditions under which a manifold accepts an extensive structure can be translated to questions regarding the existence of the aforementioned vector fields.

3. Existence of extensive structures

This section is devoted to the following question: what kinds of manifolds may be endowed with an extensive structure? We provide a partial answer by means of identifying extensive structures with global vector fields.

As is established in Eq. (6), any extensive structure defines a global vector field that is locally written like a radial vector field. It is natural to ask whether any vector field that is locally written like a radial one defines an extensive structure. The answer is in the affirmative, as we now show.

**Proposition 5.** If $M$ is endowed with a vector field $X$ and a smooth subatlas comprising charts where $X$ has the form of a radial vector field, then $M$ is furnished with an extensive structure.

*Proof.* The result above follows upon observing that $X \in \mathcal{X}(M)$ has the form of a radial vector field, if and only if for all $p \in M$, there exists a smooth chart $(U, \phi)$ with $p \in U$, that satisfies $X_q = \phi^{-1} \ast_{\phi(q)} (R_{\phi(q)})$, for all $q \in U$. If $(U', \psi)$ is another smooth chart whose domain contains $p$ and such that $X_q = \psi^{-1} \ast_{\psi(q)} (R_{\psi(q)})$, for all $q \in U'$, then the corresponding transition function belongs to $\mathcal{H}$. Indeed, for all $q \in U \cap U'$, $\phi^{-1} \ast_{\phi(q)} (R_{\phi(q)}) = \psi^{-1} \ast_{\psi(q)} (R_{\psi(q)})$. Upon applying $\psi \circ \phi^{-1}$ to both hand sides of the last equation, we obtain that $(\psi \circ \phi^{-1}) \ast_{\phi(p)} (R_{\phi(p)}) = R_{\psi(p)}$, whence $\psi \circ \phi^{-1} \in \mathcal{H}$.

As a consequence, the set of all smooth charts on which $X$ is written as a radial vector field forms an atlas compatible with $\mathcal{H}$. This atlas is contained in a maximal atlas compatible with $\mathcal{H}$, which yields the desired result.

We have thus proven that extensive structures are equivalent to global vector fields that are locally written as radial vector fields, which we call *locally-radial vector fields*. Hence, any manifold admitting a locally-radial vector field admits an extensive structure.

A particular instance of locally-radial vector field is a non-vanishing one, as we now show.

**Proposition 6.** Let $X \in \mathcal{X}(M)$. If $p \in M$ is such that $X_p \neq 0$, then $X\{|U\} = \phi^{-1} \ast_{\phi(p)} (R_{\phi(p)})$, for some smooth chart $(U, \phi)$ around $p$.

*Proof.* Let $p \in M$ be such that $X_p \neq 0$. Then, there exists a chart $(U', (y^1, \ldots, y^n))$ around $p$ such that $X\{|U\} = \partial_1$. We wish to show that there exist $n$ independent smooth functions $x^1, \ldots, x^n \in C^\infty(U)$, with $U \subset U'$, satisfying $dx^i(X) = x^i$. Because of the form that $X$ has on $U'$, the last expression is equivalent to $\partial x^i/\partial y^1 = x^i$, whose general solution is given by $x^i = e^y^i G^i$, for all $i \in \{1, \ldots, n\}$. In the last expression, $G^1, \ldots, G^n \in C^\infty(U)$, for some open set $U$ contained in $U'$. Furthermore, each function $G^i$ satisfies $\partial G^i/\partial y^1 = 0$. A choice of $n$ independent functions yields a coordinate chart $(U, \phi)$ around $p$, with $\phi = (x^1, \ldots, x^n)$. This coordinate
mapping satisfies $X_p = \phi^{-1} \ast_{\phi(p)} (R_{\phi(p)})$. The result then follows from Proposition 5. □

As we mentioned before, a direct consequence of Proposition 6 is the following.

**Corollary 1.** If $M$ admits a non-vanishing vector field then $M$ admits an extensive structure.

It is evident though that being endowed with a non-vanishing vector field is not a necessary condition for a manifold to possess an extensive structure. A straightforward illustration of this claim is the Euclidean space: its radial vector field vanishes at the origin. Yet more, the vector field $\rho$ on an extensive manifold $(M, A_H)$ may contain countably many singularities. This is because $p$ is a singularity of $\rho$ (i.e., $\rho_p = 0$) if and only if an extensive coordinate chart (and hence, all of them containing $p$) is centered at $p$. Since coordinate mappings are diffeomorphisms, each extensive domain may contain only one singularity of $\rho$, which implies that the set of singularities of $\rho$ is discrete. Because $M$ is second-countable, $\rho$ has countably many singularities.

According to the previous paragraph, we might think that a global vector field on $M$ with a discrete set of singularities yields an extensive structure on $M$ (if that were the case, any manifold would admit an extensive structure). Nonetheless, this turns out to be false, as we make evident in the next example.

**Example 1.** Consider the Euclidean plane $\mathbb{R}^2$ with its canonical linear and smooth structures. We define $X \in \mathfrak{X}(\mathbb{R}^2)$ as $X = -x \partial_y + y \partial_x$, where $(x, y)$ are the cartesian coordinates on the plane and the symbols $\partial_x$ and $\partial_y$ denote the vector fields of the holonomic frame thereby induced.

Let $o$ denote the point $(0, 0) \in \mathbb{R}^2$. According to Proposition 6, there must exist a coordinate chart $(U, (w, z))$ around each $p \neq o$, such that $X|_U = w \partial_w + z \partial_z$. Indeed, if $\theta$ and $r$ denote the polar coordinates on the plane, then $w = e^\theta$ and $z = re^\theta$ are extensive coordinate functions of the extensive structure that $X$ defines on $\mathbb{R}^2 \setminus \{o\}$.

Nevertheless, we claim that it is impossible to construct similar functions around $o$. This is the case because the integral curves of $X$ are circles, whereas those of a radial vector field are lines.

The example above provides some information about the local structure of the flow of a vector field around a singularity, if this vector is to define an extensive structure on a manifold. As we have observed, $X$ may have countably many singularities, but these must be of a particular kind. Studying the conditions over a vector field so that it is locally radial around a singularity is a path to answering the question of existence of extensive structures, and shall be the topic of future work.

We return to the main subject of this paper in the next section, where we deal in general terms with an important geometric property of the heat 1-form in the context of extensive structures.

### 4. Submanifolds transversal to the extensive structure

We begin this section by pointing out a well-known class of manifolds that are extensive. Recall that a manifold is affine if it is endowed with an atlas compatible with the pseudogroup of affine transformations on the Euclidean space $\mathbb{S}$. If
the coordinate transformations on this manifold are further restricted to be linear mappings, the resulting geometric structure thereon is called radiant. The latter turns out to be relevant in the context of thermodynamics, since it provides an appropriate setting for a rigorous description of Ruppeiner Geometry. Because every linear transformation is a degree-1 homogeneous diffeomorphism, radiant manifolds are examples of extensive manifolds. Obviously, every radiant manifold is endowed with a locally-radial vector field $\rho$. The immersed submanifolds that are transversal to $\rho$ are locally defined by extensive functions (and are therefore embedded). In this section, we show that the same holds for extensive manifolds in general.

Before proving the above-mentioned result, let us briefly discuss its importance in the context of equilibrium thermodynamics.

As we have said, the manifold of states of a thermodynamic system in equilibrium is a smooth $n$-dimensional extensive manifold. Different thermodynamic systems may share a common space of states (including its extensive structure), as is the case of hydrostatic systems, for instance. The difference between one system and another (e.g., an ideal gas and a van der Waals gas) lies in the so-called fundamental equation. This is a coordinate expression for a thermodynamic potential $\Phi \in C^\infty(M)$.

Two basic thermodynamic potentials (from which any other can be derived via a Legendre transform) are the internal energy of the system and its entropy. The knowledge of any of these two determines uniquely a thermodynamic system. This is true because the entropy of a system is the only (up to a constant scale factor) extensive function whose derivative is proportional to $\vartheta$.

In order to prove that $\vartheta$ determines the fundamental equation of a thermodynamic system, we must rely on two particular properties of this 1-form: it is integrable and transversal to $\rho$ (by transversal, we mean that the integral manifolds of the distribution defined by $\ker \vartheta$ are transversal to the integral curves of $\rho$, which in less sophisticated terms means that $\vartheta_p(\rho_p) \neq 0$ at every nonsingular point $p$ of $\rho$).

As we have said before, the two features mentioned above yield an integrating factor for the heat form, viz., $1/\vartheta(\rho)$. With the aid of the latter, the desired extensive function whose differential is proportional to $\vartheta$ may be readily obtained. However, the fact that $\vartheta$ is proportional to the differential of an extensive function follows from the more general result that we mentioned at the beginning of this section. In order to establish it, we recall that if $\iota : N \hookrightarrow M$ is a smooth embedded submanifold of $M$, a smooth function $f$ defined on an open subset $U$ of $M$ is a local defining function for $N$ if $U \cap \iota(N)$ is a regular level set of $f$, this is, if $U \cap \iota(N) = f^{-1}(c)$, for some regular value $c$ of $f$. We say that $N$ is locally defined by extensive functions if $N$ admits an extensive function as a local defining function in a neighborhood of each of its points.

**Theorem 3.** A submanifold of $M$ containing no singular points of $\rho$ is transversal to $\rho$ if and only if it is locally defined by nonvanishing extensive functions.

**Proof.** We prove first that if $U$ is an open subset of $M$ and $f \in C^\infty(U)$ is extensive, then $f^{-1}(c)$ is transversal to $\rho$, provided that $c \neq 0$ is a regular value of $f$. In
order to do this, it suffices to show that the regular level sets of extensive variables are transversal to $\rho$, since an extensive function is an extensive variable in a neighborhood of any $p \in U$ such that $d\rho_p \neq 0$. Thus, we let $(U, (x^1, \ldots, x^n))$ be an extensive chart and $c$ be a regular value of $x^1$. If we denote by $i$ the inclusion of $(x^1)^{-1}(c)$ into $M$, then $d(i^*x^1) = 0$. This means that for any $p \in (x^1)^{-1}(c)$ and $v \in T_p(x^1)^{-1}(c)$, $\iota(v) = a^1\partial_{x^1(p)} + \cdots + a^n\partial_{x^n(p)}$. The only common element of $\iota_*(T_p(x^1)^{-1}(c))$ and the span of $\rho(t)$ is zero. Indeed, if $\iota(v) = \alpha\rho(t)$, for some $\alpha \in \mathbb{R}$, $0 = d(i^*x^1)_p(v) = d\rho(p)(\rho(t)) = a^1x^1(p) = ac$, which implies that $\alpha = 0$.

Hence, the tangent space to $M$ at $p$, $T_pM$, may be written as the direct sum of $\iota_*(T_p(x^1)^{-1}(c))$ and the span of $\rho(t)$, for any $p \in (x^1)^{-1}(c)$, and thus $(x^1)^{-1}(c)$ is transversal to $\rho$.

In consequence, the regular level hypersurfaces (submanifolds of codimension 1) of extensive functions are transversal to $\rho$. Hence, if $i : N \rightarrow M$ is a manifold locally defined by nonvanishing extensive functions, it is transversal to $\rho$.

Conversely, suppose that $N$ is transversal to $\rho$ and let $p \in N$. Since $\rho(t) \neq 0$, there exists a chart $(U, (y^1, \ldots, y^n))$ around $i(p)$ such that $\rho(t) = \partial_1$, where $i$ denotes the inclusion of $N$ into $M$. The transversality of $N$ to $\rho$ implies that $d(i^*y^1)_p = 0$. Hence, $(U, (y^1, \ldots, y^n))$ is a slice chart for $N$ around $p$, and because the existence of such a chart is guaranteed for any point of $N$, it follows that it is embedded.

We define $f \in C^\infty(U)$ as $f := e^{y^1}$, which is both nonvanishing and extensive: $df(\rho) = dy^1(\rho) = f$. Furthermore, $d(i^*f)_p = 0$, meaning that $i^*f$ is constant, i.e., $U \cap i(N) = f^{-1}(c)$, for some nonzero $c \in \mathbb{R}$. The latter is a regular value of $f$, since $dy^1|_p \neq 0$, as follows from $dy^1|_p(\rho(t)) = 1$. We have thus proven that $N$ is locally defined by nonvanishing extensive functions, as desired.

The existence of entropy in the context of thermodynamics is a corollary of the result above. Indeed, if $f \in C^\infty(U)$ is a local defining function for $i : N \rightarrow M$, we have that for any $p \in i^{-1}(U)$, $\ker df = T_pN$. When $N$ is an integral manifold of the distribution defined by an integrable 1-form $\vartheta$, then $\ker \vartheta_p = \ker df_p$, which implies that $\vartheta_p \propto df_p$. It follows that $\vartheta|_U \propto df$. In particular, if $N$ is transversal to $\rho$, $f$ may be chosen extensive. This argument proves that $\vartheta$ is locally proportional to the differential of an extensive function, which is unique up to a constant scale factor, as we now show.

**Proposition 7.** The local defining functions of manifolds transversal to $\rho$ are unique modulo scale.

**Proof.** Let $i : N \rightarrow M$ be an immersed submanifold that is transversal $\rho$. Let $f \in C^\infty(U)$ be a local defining function for $i$ around a point $p \in N$, and suppose that $\Phi \in C^\infty(U)$ is another local defining function for $i$ around $p$. Then, $df \propto d\Phi$, which means that $df \wedge d\Phi = 0$. Thus, $\Phi$ may be written as a function of $f$, this is, there exists a real-valued function $\hat{\Phi}$ defined on an open interval $I$ containing the image of $f$ such that $\Phi = \hat{\Phi} \circ f$. Since $\Phi$ is extensive, we have that $\hat{\Phi}$ must satisfy

$$
\hat{\Phi}(t) = tf'(t),
$$

for every $t$ lying in the image of $f$, where $\hat{\Phi}'$ denotes the derivative of $\hat{\Phi}$. The solution to Eq. (11) in an open interval containing the image of $f$ is $\hat{\Phi}(t) = kt$, with $t \in I$. This means that $\Phi = kf$, as we wished to prove. \(\square\)
The function that defines locally the integral manifolds of $\vartheta$ is precisely the (local) entropy of the system. Hence, regarding $\vartheta$ as a basic geometric thermodynamic object justifies distinguishing entropy from all other thermodynamic potentials, as done, for instance, in Ruppeiner Geometry [10].

We conclude this section by noting that the three conditions imposed upon the heat 1-form of a thermodynamic system—integrability, extensivity, and transversality—are independent between each other.

Observe that forms that are transversal to $\rho$ are not necessarily extensive. Given an extensive 1-form $\alpha$ that is transversal to $\rho$, a straightforward example of a 1-form that is transversal to $\rho$ but is not extensive is $f\alpha$, where $f$ is an extensive function. If the manifold in question is 2-dimensional, then $f\alpha$ is integrable, whence transversality and integrability do not guarantee extensivity.

It is also true that extensivity is not a sufficient condition for transversality, as we illustrate below.

**Example 2.** Consider the set of points $(x, y)$ in the Euclidean plane with both $x > 0$ and $y > 0$, which we denote by $\mathbb{R}^2_+$, furnished with the smooth and extensive structures that the Euclidean plane induces thereon.

The global 1-form $\alpha := (1 + y/x)dx - (1 + x/y)dy$ is extensive, yet $\alpha(\rho) = 0$, whence it is not transversal to $\rho$.

Notice that the 1-form $\alpha$ of the example above is integrable. This shows that not even integrable extensive forms are necessarily transversal to the extensive structure of a manifold. In brief words, extensivity and integrability do not imply transversality.

Likewise, 1-forms transversal to $\rho$ are not necessarily integrable, as we now show.

**Example 3.** We denote by $\mathbb{R}^3_+$ the set of all points $(x, y, z)$ in the Euclidean space with $x > 0$, $y > 0$ and $z > 0$.

We define the global 1-form $\beta := dz + x^2/ydy$. It can readily be seen that $\beta(\rho) \neq 0$. Furthermore, $\alpha \wedge d\alpha \neq 0$, which means that $\alpha$ is not integrable.

The example above also illustrates that not every extensive differential form is integrable, whence extensivity and transversality do not imply integrability.

5. **Concluding remarks**

In several instances, differential geometry seems to be a powerful tool for the study of equilibrium thermodynamics. The aim of this paper was to contribute in establishing firmly the foundations of these geometric approaches.

We point out that the concept of extensive structures presented herein allows for a geometric depiction of the natural variables of a thermodynamic system: they arise by restricting the structure group of the atlas on $M$ to $\text{GL}(n)$. This means that the choice of a preferred thermodynamic potential (and its natural variables) to describe a thermodynamic system amounts to defining a flat affine connection on $M$ which, together with $\rho$, makes $M$ a radiant manifold [10]. The relationship between the natural variables of different potentials is thus a relationship between flat affine connections, which has not been treated geometrically.

The fact that global locally-radial vector fields and extensive structures are equivalent might restrict the topology of a manifold of states a priori. For instance, if a two-dimensional manifold of states has a nowhere-vanishing $\rho$, then it cannot be
a sphere. The physical consequences that the topological properties of $M$ might yield has not received any attention so far.

In equilibrium thermodynamics, closed systems are of particular theoretical importance. Under the approach we have presented in this paper, the latter are readily generalized by the submanifolds of $M$ that are transversal to $\rho$. The geometric features of the foliation defined by these transversal submanifolds might also carry physical information, especially in the context of Ruppeiner Geometry.

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