ON NEW GENERAL INTEGRAL INEQUALITIES FOR QUASI-CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are quasi-convex. Some applications to special means of real numbers are also given.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality

\begin{equation}
 f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\end{equation}

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [1, 3, 4, 6, 7, 9], the results of the generalization, improvement and extension of the famous integral inequality (1.1).

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$
 f(tx + (1 - t)y) \leq \sup \{f(x), f(y)\},
$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [2]).

The following inequality is well known in the literature as Simpson’s inequality. Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $\|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$
 \left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} \right] + 2f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b - a)^2.
$$

In recent years many authors have studied error estimations for Simpson’s inequality; for refinements, counterparts, generalizations and new Simpson’s type inequalities, see [2, 5, 10, 11, 12].
In [7], Ion introduced two inequalities of the right hand side of Hadamard’s type for quasi-convex functions, as follow:

**Theorem 1.** Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable function on $(a, b)$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds true

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \sup \{|f'(a)|, |f'(b)|\}.
$$

**Theorem 2.** Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable function on $(a, b)$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds true

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left( \sup \left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p}{p-1}}.
$$

In [8], Alomari et al. established some new upper bound for the right-hand side of Hadamard’s inequality for quasi-convex mappings, which is better than the inequality had done in [7]. The authors obtained the following results:

**Theorem 3.** Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^o$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^{q/(p-1)}$ is a quasi-convex on $[a, b]$, for $p > 1$, then the following inequality holds:

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4(p+1)^{1/p}} \left[ \left( \sup \left\{ |f'(a+b/2)|^{\frac{1}{p-1}}, |f'(b)|^{\frac{1}{p-1}} \right\} \right)^{\frac{p}{p-1}} + \left( \sup \left\{ |f'(a)|^{\frac{1}{p-1}}, |f'(a)|^{\frac{1}{p-1}} \right\} \right)^{\frac{p}{p-1}} \right].
$$

**Theorem 4.** Let $f : I^o \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$. If $|f|^q$ is a quasi-convex on $[a, b]$, for $q \geq 1$, then the following inequality holds:

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{8} \left[ \left( \sup \left\{ |f'(a+b/2)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} + \left( \sup \left\{ |f'(a+b/2)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \right].
$$

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson’s inequality for functions whose derivatives in absolute value at certain power are quasi-convex, we need the following lemma given by Iscan in [8]:

**Lemma 1.** Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^o$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. Then the following equality
holds:

\[
(1.6) \quad \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx
\]

\[
= (b-a) \left[ \int_0^{1-\alpha} (t - \alpha \lambda) f'(tb + (1-t)a) \, dt + \int_{1-\alpha}^1 (t - 1 + \lambda (1-\alpha)) f'(tb + (1-t)a) \, dt \right].
\]

2. Main results

**Theorem 5.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) such that \( f' \in L[a,b] \), where \( a, b \in I^0 \) with \( a < b \) and \( \alpha, \lambda \in [0,1] \). If \( |f'|^q \) is quasi-convex on \( [a,b] \), \( q \geq 1 \), then the following inequality holds:

\[
(2.1) \leq \left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

where

\[
\gamma_1 = (1 - \alpha) \left[ \alpha \lambda - \frac{(1-\alpha)}{2} \right], \quad \gamma_2 = (\alpha \lambda)^2 - \gamma_1,
\]

\[
v_1 = \frac{1 - (1-\alpha)^2}{2} - \alpha [1 - \lambda (1-\alpha)],
\]

\[
v_2 = \frac{1 + (1-\alpha)^2}{2} - (\lambda + 1) (1-\alpha) [1 - \lambda (1-\alpha)],
\]

and

\[
A = \sup \{ |f'(a)|^q, |f'(b)|^q \}.
\]

**Proof.** Suppose that \( q \geq 1 \). From Lemma 1 and using the well known power mean inequality, we have

\[
\left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq (b-a) \left[ \int_0^{1-\alpha} |t - \alpha \lambda| |f'(tb + (1-t)a)| \, dt + \int_{1-\alpha}^1 |t - 1 + \lambda (1-\alpha)| |f'(tb + (1-t)a)| \, dt \right]
\]

\[
\leq (b-a) \left\{ \left( \int_0^{1-\alpha} |t - \alpha \lambda| \, dt \right)^{1-\alpha} \left( \int_0^1 |t - \lambda (1-\alpha)| |f'(tb + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}}.
\]
The inequality (2.1) we get the following trapezoid inequality
\[ 4\dot{t} \leq \frac{1}{4} \left( \int_{1-\alpha}^{1} |t - 1 + \lambda (1 - \alpha)| dt \right)^{1 - \frac{1}{2}} \left( \int_{1-\alpha}^{1} |t - 1 + \lambda (1 - \alpha)| |f'(tb + (1 - t)a)|^q dt \right)^{\frac{1}{q}}. \]

Since \(|f'|^q\) is quasi-convex on \([a, b]\), we know that for \(t \in [0, 1]\)
\[ |f'(tb + (1 - t)a)|^q \leq \sup \{|f'(a)|^q, |f'(b)|^q\}, \]
hence, by simple computation
\[ \int_{0}^{1-\alpha} |t - \alpha \lambda| dt = \left\{ \begin{array}{ll} \gamma_2, & \alpha \lambda \leq 1 - \alpha, \\ \gamma_1, & \alpha \lambda \geq 1 - \alpha, \end{array} \right. \]
\[ \int_{1-\alpha}^{1} |t - 1 + \lambda (1 - \alpha)| dt = \left\{ \begin{array}{ll} \nu_1, & 1 - \lambda (1 - \alpha) \leq 1 - \alpha, \\ \nu_2, & 1 - \lambda (1 - \alpha) \geq 1 - \alpha, \end{array} \right. \]
Thus, using (2.3) and (2.4) in (2.2), we obtain the inequality (2.1). This completes the proof. \(\square\)

**Corollary 1.** Under the assumptions of Theorem 5 with \(q = 1\), the inequality (2.1) reduced to the following inequality
\[ \lambda (af(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \left\{ \begin{array}{ll} \{ (b-a)(\gamma_2 + \nu_2) \sup \{|f'(a)|, |f'(b)|\} & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha), \\ \{ (b-a)(\gamma_2 + \nu_1) \sup \{|f'(a)|, |f'(b)|\} & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha, \end{array} \right. \]
where \(\gamma_1, \gamma_2, \nu_1\) and \(\nu_2\) are defined as in Theorem 5.

**Corollary 2.** Under the assumptions of Theorem 6 with \(\alpha = \frac{1}{2}\) and \(\lambda = \frac{1}{3}\), from the inequality (2.7) we get the following Simpson type inequality
\[ \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq (b-a) \left( \frac{5}{36} \right) \sup \{|f'(a)|^q, |f'(b)|^q\}. \]

**Corollary 3.** Under the assumptions of Theorem 6 with \(\alpha = \frac{1}{2}\) and \(\lambda = 0\), from the inequality (2.7) we get the following midpoint inequality
\[ \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{b-a}{4} \sup \{|f'(a)|^q, |f'(b)|^q\}. \]

**Corollary 4.** Under the assumptions of Theorem 6 with \(\alpha = \frac{1}{2}\) and \(\lambda = 1\), from the inequality (2.7) we get the following trapezoid inequality
\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{b-a}{4} \sup \{|f'(a)|^q, |f'(b)|^q\}. \]
which is the same of the inequality (1.2) for $q = 1$.

Using Lemma 11 we shall give another result for convex functions as follows.

**Theorem 6.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^*$ such that $f' \in L[a,b]$, where $a, b \in I^*$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q > 1$, then the following inequality holds:

\[
\left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a)
\]

\[
\times \left( \frac{1}{p+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} \left\{ \begin{array}{ll}
(1 - \alpha)^{\frac{1}{q}} \varepsilon_1^\frac{1}{q} + \alpha^n \varepsilon_2^\frac{1}{q}, & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
(1 - \alpha)^{\frac{1}{q}} \varepsilon_3^\frac{1}{q} + \alpha^n \varepsilon_4^\frac{1}{q}, & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
(1 - \alpha)^{\frac{1}{q}} \varepsilon_5^\frac{1}{q} + \alpha^n \varepsilon_6^\frac{1}{q}, & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{array} \right.
\]

where

\[
A = \sup \{ |f'(a)|^q, |f'(b)|^q \},
\]

\[
\varepsilon_1 = (\alpha \lambda)^{p+1} + (1 - \alpha - \alpha \lambda)^{p+1}, \quad \varepsilon_2 = (\alpha \lambda)^{p+1} - (\alpha \lambda - 1 + \alpha)^{p+1},
\]

\[
\varepsilon_3 = [\lambda (1 - \alpha)]^{p+1} + [\alpha - \lambda (1 - \alpha)]^{p+1}, \quad \varepsilon_4 = [\lambda (1 - \alpha)]^{p+1} - [\lambda (1 - \alpha) - \alpha]^{p+1},
\]

and $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** From Lemma 5 and by Hölder’s integral inequality, we have

\[
\left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq (b-a) \left[ \int_0^{1-\alpha} |t - \alpha \lambda| |f'(tb + (1-t)a)| dt + \int_{1-\alpha}^1 |t - 1 + \lambda (1 - \alpha)| |f'(tb + (1-t)a)| dt \right]
\]

\[
\leq (b-a) \left\{ \left( \int_0^{1-\alpha} |t - \alpha \lambda|^p dt \right)^\frac{1}{p} \left( \int_0^{1-\alpha} |f'(tb + (1-t)a)|^q dt \right)^\frac{1}{q} + \left( \int_{1-\alpha}^1 |t - 1 + \lambda (1 - \alpha)|^p dt \right)^\frac{1}{p} \left( \int_{1-\alpha}^1 |f'(tb + (1-t)a)|^q dt \right)^\frac{1}{q} \right\}.
\]

Since $|f'|^q$ is quasi-convex on $[a, b]$, for $\alpha \in [0, 1]$, we get

\[
\int_0^{1-\alpha} |f'(tb + (1-t)a)|^q dt = (1-\alpha) \sup \{ |f'(a)|^q, |f'(b)|^q \}
\]

\[
\int_{1-\alpha}^1 |f'(tb + (1-t)a)|^q dt = (\alpha - 1) \sup \{ |f'(a)|^q, |f'(b)|^q \}
\]
Similarly, for \( \alpha \in [0,1] \), we have

\[
(2.8) \quad \int_{1-\alpha}^{1} |f'(tb + (1-t)a)|^q \, dt = \alpha \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\}.
\]

By simple computation

\[
(2.9) \quad \int_{0}^{1-\alpha} |t - \alpha\lambda|^p \, dt = \begin{cases} \frac{(\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}}{(\alpha\lambda)^{p+1} - (\alpha\lambda - 1 + \alpha\lambda)^{p+1}}, & \alpha \lambda \leq 1 - \alpha \vspace{1mm} \\ \frac{(\alpha\lambda)^{p+1} - (\alpha\lambda - 1 + \alpha\lambda)^{p+1}}{\alpha \lambda \geq 1 - \alpha} \end{cases},
\]

and

\[
(2.10) \quad \int_{1-\alpha}^{1} |t - \lambda(1-\alpha)|^p \, dt = \begin{cases} \frac{|\alpha(1-\alpha)|^{p+1} + |\alpha - \lambda(1-\alpha)|^{p+1}}{|\alpha(1-\alpha)|^{p+1} - |\alpha - \lambda(1-\alpha) - \alpha|^{p+1}}, & 1 - \alpha \leq 1 - \lambda(1 - \alpha) \vspace{1mm} \\ \frac{|\alpha(1-\alpha)|^{p+1} - |\alpha - \lambda(1-\alpha) - \alpha|^{p+1}}{1 - \alpha \geq 1 - \lambda(1 - \alpha)} \end{cases},
\]

thus, using (2.7)-(2.10) in (2.6), we obtain the inequality (2.5). This completes the proof. \( \square \)

Corollary 5. Under the assumptions of Theorem 6 with \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), from the inequality (2.8) we get the following Simpson type inequality

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{2} \left( 1 + \frac{2^{p+1}}{3(p+1)} \right)^{\frac{1}{2}} \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{2}}.
\]

Corollary 6. Under the assumptions of Theorem 6 with \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), from the inequality (2.8) we get the following midpoint inequality

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{p}}.
\]

Corollary 7. Let the assumptions of Theorem 6 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), from the inequality (2.8) we get the following trapezoid inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{p}}.
\]

Theorem 7. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \) such that \( f' \in L[a,b] \), where \( a, b \in I^o \) with \( a < b \) and \( \alpha, \lambda \in [0,1] \). If \( |f|^q \) is quasi-convex on \( [a,b] \), \( q > 1 \), then the following inequality holds:

\[
(2.11) \quad \lambda (\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq (b-a)
\]
\[
\begin{align*}
\left\{ \left( 1 - \alpha \right)^{\frac{1}{p}} B^{\frac{1}{p}} + \alpha \frac{1}{p} C^{\frac{1}{q}} \right\}^{\frac{1}{p+1}}, \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
\left( 1 - \alpha \right)^{\frac{1}{q}} B^{\frac{1}{q}} + \alpha \frac{1}{q} C^{\frac{1}{p}} \right\}^{\frac{1}{p+1}}, \quad \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
\left( 1 - \alpha \right)^{\frac{1}{p}} B^{\frac{1}{p}} + \alpha \frac{1}{p} C^{\frac{1}{q}} \right\}^{\frac{1}{p+1}}, \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{align*}
\]

where
\[ B = \sup \left\{ |f'(a)|^q, |f'(aa + (1 - \alpha) b)|^q \right\}, \]
\[ C = \sup \left\{ |f'(b)|^q, |f'(aa + (1 - \alpha) b)|^q \right\}, \]
and \(\frac{1}{p} + \frac{1}{q} = 1\), and \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\) numbers are defined as in Theorem 6.

**Proof.** From Lemma 1 and by Hölder’s integral inequality, we have the inequality (2.6). Since \(|f'|^q\) is convex on \([a, b]\), for all \(t \in [0, 1]\) and \(\alpha \in [0, 1]\) we get
\[ |f'\left( ta + (1 - t) \left( aa + (1 - \alpha) b \right) \right)|^q \leq B = \sup \left\{ |f'(a)|^q, |f'(aa + (1 - \alpha) b)|^q \right\} \]
then
\[ (2.12) \]
\[ \int_0^1 |f'\left( ta + (1 - t) \left( aa + (1 - \alpha) b \right) \right)|^q dt \leq \left( \frac{1}{(1 - \alpha)(b-a)} \right)^{(1-\alpha)k+\alpha a} \int_a^b |f'(x)|^q dx \leq B. \]

By the inequality (2.12), we get
\[ (2.13) \]
\[ \int_0^{1-\alpha} |f'\left( tb + (1 - t)a \right)|^q dt = \left( 1 - \alpha \right)^\left( \frac{1}{(1 - \alpha)(b-a)} \right)^{(1-\alpha)k+\alpha a} \int_a^b |f'(x)|^q dx \]
\[ \leq \left( 1 - \alpha \right) B. \]

The inequality (2.13) also holds for \(\alpha = 1\) too. Since \(|f'|^q\) is convex on \([a, b]\), for all \(t \in [0, 1]\) and \(\alpha \in (0, 1]\) we have
\[ |f'\left( tb + (1 - t) \left( aa + (1 - \alpha) b \right) \right)|^q \leq C = \sup \left\{ |f'(b)|^q, |f'(aa + (1 - \alpha) b)|^q \right\} \]
then
\[ (2.14) \]
\[ \int_0^1 |f'\left( tb + (1 - t) \left( aa + (1 - \alpha) b \right) \right)|^q dt = \left( \frac{1}{\alpha(b-a)} \right)^{(1-\alpha)b+\alpha a} \int_0^b |f'(x)|^q dx \leq B. \]

By the inequality (2.14), we get
\[ (2.15) \]
\[ \int_0^{1-\alpha} |f'\left( tb + (1 - t)a \right)|^q dt = \alpha \left[ \left( \frac{1}{\alpha(b-a)} \right)^{(1-\alpha)b+\alpha a} \int_0^b |f'(x)|^q dx \right] \]
\[ \leq \alpha C. \]

The inequality (2.15) also holds for \(\alpha = 0\) too. Thus, using (2.4), (2.11), (2.13) and (2.15) in (2.6), we obtain the inequality (2.11). This completes the proof. \(\square\)
Corollary 8. Under the assumptions of Theorem \[7\] with \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), from the inequality \( (2.11) \) we get the following Simpson type inequality

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \\
\leq \frac{b-a}{12} \left( \frac{1+2p+1}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left( \sup \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{q}, \left| f'(a) \right|^{q} \right\} \right)^{\frac{1}{q}} \right. \\
+ \left( \sup \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{q}, \left| f'(b) \right|^{q} \right\} \right)^{\frac{1}{q}} \right\}.
\]

Corollary 9. Under the assumptions of Theorem \[7\] with \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), from the inequality \( (2.11) \) we get the following trapezoid inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \\
\leq \frac{b-a}{4(p+1)^{1/p}} \left\{ \left( \sup \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{q}, \left| f'(a) \right|^{q} \right\} \right)^{\frac{1}{q}} \right. \\
+ \left( \sup \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{q}, \left| f'(b) \right|^{q} \right\} \right)^{\frac{1}{q}} \right\},
\]

which is the same of the inequality \( (1.4) \).

Corollary 10. Under the assumptions of Theorem \[7\] with \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), from the inequality \( (2.11) \) we get the following midpoint inequality

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \\
\leq \frac{b-a}{4(p+1)^{1/p}} \left\{ \left( \sup \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{q}, \left| f'(a) \right|^{q} \right\} \right)^{\frac{1}{q}} \right. \\
+ \left( \sup \left\{ \left| f'\left( \frac{a+b}{2} \right) \right|^{q}, \left| f'(b) \right|^{q} \right\} \right)^{\frac{1}{q}} \right\},
\]

which is the better than the inequality in \[1\] Corollary 8.

3. Some applications for special means

Let us recall the following special means of arbitrary real numbers \( a, b \) with \( a \neq b \) and \( \alpha \in [0,1] \):

1. The weighted arithmetic mean
   \[ A_{\alpha}(a, b) := \alpha a + (1-\alpha)b, \ a, b \in \mathbb{R}. \]

2. The unweighted arithmetic mean
   \[ A(a, b) := \frac{a+b}{2}, \ a, b \in \mathbb{R}. \]

3. The weighted harmonic mean
   \[ H_{\alpha}(a, b) := \left( \frac{\alpha}{a} + \frac{1-\alpha}{b} \right)^{-1}, \ a, b \in \mathbb{R} \setminus \{0\}. \]
Proposition 1. Let \( a, b, \gamma \in \mathbb{R} \setminus \{0\} \).

Proof. The assertion follows from Theorem 5, for \( \gamma \).

Proposition 2. Let \( a, b \in \mathbb{R} \) with \( a < b \), and \( n \in \mathbb{N} \), \( n \geq 2 \). Then, for \( \alpha, \lambda \in [0, 1] \) and \( q \geq 1 \), we have the following inequality:

\[
|\lambda A_n (a^n, b^n) + (1 - \lambda) A^n_n (a, b) - L^n_n (a, b)|
\leq \begin{cases} 
  n(b - a)(\gamma_2 + \nu_2)E^{\frac{1}{q}} & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
  n(b - a)(\gamma_2 + \nu_1)E^{\frac{1}{q}} & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
  n(b - a)(\gamma_1 + \nu_2)E^{\frac{1}{q}} & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{cases}
\]

where

\[
E = \sup \left\{ |a^{(n-1)q}|, |b^{(n-1)q}| \right\},
\]

\( \gamma_1, \gamma_2, \nu_1 \) and \( \nu_2 \) are defined as in Theorem 6.

Proof. The assertion follows from Theorem 5 for \( f(x) = x^n, x \in \mathbb{R} \).

Proposition 2. Let \( a, b \in \mathbb{R} \) with \( a < b \), and \( n \in \mathbb{N} \), \( n \geq 2 \). Then, for \( \alpha, \lambda \in [0, 1] \) and \( q > 1 \), we have the following inequality:

\[
|\lambda A_n (a^n, b^n) + (1 - \lambda) A^n_n (a, b) - L^n_n (a, b)| \leq (b - a) \left( \frac{1}{p + 1} \right)^{\frac{1}{q}} n
\]

\[
\times \begin{cases} 
  (1 - \alpha)\frac{1}{q} F \epsilon_1^{\frac{1}{q}} + \alpha \frac{1}{q} G \epsilon_3^{\frac{1}{q}} & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
  (1 - \alpha)\frac{1}{q} F \epsilon_2^{\frac{1}{q}} + \alpha \frac{1}{q} G \epsilon_4^{\frac{1}{q}} & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
  (1 - \alpha)\frac{1}{q} F \epsilon_3^{\frac{1}{q}} + \alpha \frac{1}{q} G \epsilon_3^{\frac{1}{q}} & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{cases}
\]

where

\[
F = \sup \left\{ |a^{(n-1)q}|, |A_n (a, b)|^{(n-1)q} \right\},
\]

\[
G = \sup \left\{ |b^{(n-1)q}|, |A_n (a, b)|^{(n-1)q} \right\},
\]

\[
\frac{1}{p} + \frac{1}{q} = 1, \text{ and } \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \text{ numbers are defined as in Theorem 7}
\]

Proof. The assertion follows from Theorem 7 for \( f(x) = x^n, x \in \mathbb{R} \).
Proposition 4. Let $a, b \in \mathbb{R}$ with $a < b$, $0 \notin [a, b]$. Then, for $\alpha, \lambda \in [0,1]$ and $q \geq 1$, we have the following inequality:

$$\left| \lambda H_{\alpha}^{-1}(a, b) + (1 - \lambda) A_{\alpha}^{-1}(a, b) - L^{-1}(a, b) \right| \leq \begin{cases} 
(b - a) (\gamma_2 + \nu_2) \frac{K}{\lambda} & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
(b - a) (\gamma_1 + \nu_1) \frac{K}{\lambda} & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
(b - a) (\gamma_1 + \nu_2) \frac{K}{\lambda} & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha) 
\end{cases}$$

where

$$K = \sup \left\{ a^{-2q}, b^{-2q} \right\},$$

$\gamma_1, \gamma_2, \nu_1$, and $\nu_2$ are defined as in Theorem [4].

Proof. The assertion follows from Theorem [5], for $f(x) = \frac{1}{x}$, $x \in (0, \infty)$.

Proposition 4. Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0,1]$ and $q > 1$, we have the following inequality:

$$\left| \lambda H_{\alpha}^{-1}(a, b) + (1 - \lambda) A_{\alpha}^{-1}(a, b) - L^{-1}(a, b) \right| \leq (b - a) \left( \frac{1}{p+1} \right) \frac{1}{\lambda}$$

$$\times \left\{ (1 - \alpha) \frac{1}{p} M \frac{1}{\lambda} \epsilon_1 + \alpha \frac{1}{q} N \frac{1}{\lambda} \epsilon_3 \right\}, \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)$$

$$\times \left\{ (1 - \alpha) \frac{1}{p} M \frac{1}{\lambda} \epsilon_2 + \alpha \frac{1}{q} N \frac{1}{\lambda} \epsilon_4 \right\}, \quad \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha$$

$$\times \left\{ (1 - \alpha) \frac{1}{p} M \frac{1}{\lambda} \epsilon_2 + \alpha \frac{1}{q} N \frac{1}{\lambda} \epsilon_4 \right\}, \quad 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)$$

where

$$M = \sup \left\{ a^{-2q}, A_{\alpha} (a, b)^{-2q} \right\},$$

$$N = \sup \left\{ b^{-2q}, A_{\alpha} (a, b)^{-2q} \right\},$$

$$\frac{1}{p} + \frac{1}{q} = 1, \text{ and } \epsilon_1, \epsilon_2, \epsilon_3, \text{ and } \epsilon_4 \text{ are defined as in Theorem [7]}.$$

Proof. The assertion follows from Theorem [7] for $f(x) = \frac{1}{x}$, $x \in (0, \infty)$.

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