SOME IDENTITIES FOR THE QUANTUM MEASURE 
AND ITS GENERALIZATIONS

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After a brief review of classical probability theory (measure theory), we present an observation (due to Sorkin) concerning an aspect of probability in quantum mechanics. Following Sorkin, we introduce a generalized measure theory based on a hierarchy of "sum-rules." The first sum-rule yields classical probability theory, and the second yields a generalized probability theory that includes quantum mechanics as a special case. We present some algebraic relations involving these sum-rules. This may be useful for the study of the higher-order sum-rules and possible generalizations of quantum mechanics. We conclude with some open questions and suggestions for further work.

1. Introduction

One could take the point of view that what is at the essence of quantum mechanics is the failure of "the classical additivity of probabilities," as demonstrated by the famous two-slit experiment.

Consider the set \( C \) of all electron worldlines (histories) that leave the emitter at a given instant and arrive at a particular detector on the other side of a double-slit screen at a later instant. Suppose we block off only the second slit. Let \( A \) be the subset of those worldlines which pass through the first slit. Similarly, we block off only the first slit and let \( B \) be the subset of those worldlines which pass through the second slit. Ignoring the possibility of the electron winding around so that it passes through both slits, we have \( C = A \sqcup B \), where \( \sqcup \) denotes disjoint-union. This suggests \( 4 (= 2^2) \) experimental combinations of 2 disjoint alternatives: "both slits open," "only slit-A open," "only slit-B open," and "no slits open."

Classical measure theory (probability theory) assigns to each measurable set \( X \) of histories a non-negative number \( P(X) \). So, we can ask about the validity of the "sum rule"

\[
P(A \sqcup B) \overset{?}= P(A) + P(B),
\]

which we write as

\[
P(A) + P(B) - P(A \sqcup B) \overset{?}= 0.
\]

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Implicitly, we assume that $P(\emptyset) = 0$.

Of course, the probability function $P$ in classical physics assigns to a set $A$ a positive additive quantity, $a_1$, which we call the “classical amplitude.” So,

$$P(A) = a_1$$

$$P(B) = b_1$$

$$P(A \sqcup B) = a_1 + b_1.$$ 

We verify that the sum-rule is satisfied

$$P(A) + P(B) - P(A \sqcup B) = (a_1) + (b_1) - (a_1 + b_1) = 0.$$ 

However, the probability function $P_2$ in quantum physics assigns to a set $A$ a sum of the square-norms of additive quantities, $a_2$. (Refer to for details.) We call these additive quantities the “quantum amplitudes.” So,

$$P_2(A) = \sum_i a_{i,2}^* a_{i,2}$$

$$P_2(B) = \sum_i b_{i,2}^* b_{i,2}$$

$$P_2(A \sqcup B) = \sum_i (a_{i,2} + b_{i,2})^* (a_{i,2} + b_{i,2})$$

The corresponding sum-rule, however, fails

$$P_2(A) + P_2(B) - P_2(A \sqcup B)$$

$$= \sum_i a_{i,2}^* a_{i,2} + \sum_i b_{i,2}^* b_{i,2} - \sum_i (a_{i,2} + b_{i,2})^* (a_{i,2} + b_{i,2})$$

$$= - \sum_i (a_{i,2}^* b_{i,2} + b_{i,2}^* a_{i,2})$$

$$\neq 0.$$ 

This is the failure of the additivity of probabilities in quantum mechanics.

Let us define the “interference term”:

$$I_2(A, B) \equiv P(A) + P(B) - P(A \sqcup B),$$

which measures the failure of the additivity of probabilities. Then, we can say that classical probabilities have the property that “$I_2 = 0$”, and that quantum probabilities have the property that “$I_2 \neq 0$.”

\*This expression for $I_2$ is minus the definition given in for details. In general, our expressions for $I_{\text{even}}$ differ in sign. Our convention is chosen for mathematical convenience.
Following Sorkin, consider next the not-so-famous three-slit experiment, with 8 \((= 2^3)\) experimental combinations of 3 disjoint alternatives. Consider the following function

\[ I_3(A, B, C) \equiv P(A) + P(B) + P(C) - P(A \sqcup B) - P(A \sqcup C) - P(B \sqcup C) + P(A \sqcup B \sqcup C). \]

With classical probabilities, this evaluates to zero.

\[
I_3(A, B, C) = (a_1) + (b_1) + (c_1) - (a_1 + b_1) - (a_1 + c_1) - (b_1 + c_1) + (a_1 + b_1 + c_1) = 0
\]

This result can be expected from a simple application of classical measure theory, as we show in the next section.

With quantum probabilities, this, surprisingly, also evaluates to zero!

\[
I_3(A, B, C) = \sum_i a_{i,2}^* a_{i,2} + \sum_i b_{i,2}^* b_{i,2} + \sum_i c_{i,2}^* c_{i,2} - \sum_i (a_{i,2} + b_{i,2})^* (a_{i,2} + b_{i,2}) - \sum_i (a_{i,2} + c_{i,2})^* (a_{i,2} + c_{i,2}) - \sum_i (b_{i,2} + c_{i,2})^* (b_{i,2} + c_{i,2}) + \sum_i (a_{i,2} + b_{i,2} + c_{i,2})^* (a_{i,2} + b_{i,2} + c_{i,2}) = 0
\]

We can say that, for disjoint alternatives, classical probabilities have the property that “\(I_2 = 0\) and \(I_3 = 0\),” and quantum probabilities have the property that “\(I_2 \neq 0\) and \(I_3 = 0\).”

This was Sorkin’s observation concerning an aspect of probability in quantum mechanics. It seems to say that quantum probabilities reveal themselves as a rather mild generalization of classical probabilities in the sense that the probability sum-rules are only slightly different: additivity is lost for two disjoint alternatives but not three. Such a generalization allows one to work directly with the quantum probabilities (quantum measures) instead of indirectly with the quantum amplitudes (wavefunctions defined over a spacelike hypersurface). This generalized viewpoint is probably needed to formulate, say, a quantum theory of causal sets or any other quantum theory of gravity which does not naturally have hypersurfaces.

2. Classical Measure Theory

In this section, we review classical measure theory (á la Kolmogorov). A measure space \((S, M, | \cdot |)\) consists of a set \(S\), a collection \(M\) of certain subsets of \(S\) (called
“measurable sets”), and a function (called the “measure”) \(| \cdot | : M \to R^+\), where \(R^+ = [0, \infty)\), such that

1. \(\emptyset \in M\), and \(|\emptyset| = 0\)  
   “the empty set is measurable and has measure zero”

2. \(S \in M\), and \(|S| = 1\)  
   “the universal set is measurable and has measure one”

3. if \(A \in M\), then \(A^c \in M\)  
   “the complement of a measurable set is also measurable”

4. for \(A_1, A_2, \ldots \in M\), we have \(\cup A_i \in M\)  
   “the union of a countable collection of measurable sets is also measurable”

5. for mutually-disjoint \(S_1, S_2, \ldots \in M\), we have \(|\biguplus S_i| = \sum |S_i|\)  
   “the disjoint-union of a countable collection of mutually-disjoint measurable sets is also measurable, and its measure is the sum of the individual measures”

Axiom \(5\) permits a “frequency” or “area” interpretation for classical measure theory. In particular, this condition directly yields

\[
|A \cup B| = |A| + |B|, \quad (1)
\]

(which is “\(I_2 = 0\)” and

\[
|A \cup B \cup C| = |A| + |B| + |C|. \quad (2)
\]

By reapplying this axiom and using the associativity and commutativity of \(\cup\), we obtain for the case of three mutually disjoint sets:

\[
\begin{align*}
|A \cup (B \cup C)| &= |A| + |B \cup C| \\
|(A \cup B) \cup C| &= |C| + |A \cup B| \\
|B \cup (C \cup A)| &= |B| + |C \cup A| \\
-2|A \cup B \cup C| &= -2|A| -2|B| -2|C|
\end{align*}
\]

(which is “\(I_3 = 0\)”).

For later comparison, let us define the “generalized interference term” for any collection of mutually-disjoint subsets \(S_1, S_2, \cdots \in M\):

\[
I_n(S_1, S_2, S_3, \cdots, S_n) \equiv \sum_i |S_i| - \sum_{\text{distinct } i,j} |S_i \cup S_j| + \sum_{\text{distinct } i,j,k} |S_i \cup S_j \cup S_k| \mp \cdots \\
-(-1)^{n-1} \sum_i |S_1 \cup S_2 \cup \cdots \cup (S_i) \cup \cdots \cup S_n| \\
-(-1)^n|S_1 \cup S_2 \cup \cdots \cup S_n|, \quad (3)
\]
where each $I_n$ is a real-valued symmetric set-function on $n$ mutually-disjoint measurable sets. Clearly, the vanishing of the $n^{th}$ generalized interference term encodes the $n^{th}$-order sum-rule.

Then, axiom 3 (for finite sums) may be re-expressed by

3. For all $n \geq 2$, and for any collection of mutually-disjoint subsets $S_1, S_2, \cdots, S_n \in M$, we have $I_n(S_1, S_2, \cdots, S_n) = 0$, which yields

$$|S_1 \cup S_2 \cup \cdots \cup S_n| \equiv (-1)^n \left( \sum_i |S_i| - \sum_{\text{distinct } i,j} |S_i \cup S_j| + \sum_{\text{distinct } i,j,k} |S_i \cup S_j \cup S_k| \mp \cdots - (-1)^{n-1} \sum_i |S_1 \cup S_2 \cup \cdots \cup (S_i) \cup \cdots \cup S_n| \right).$$

This will be generalized in the next section.

Let us conclude this section with the following interesting fact. Let $A$ and $B$ be disjoint subsets, and let $a$ and $b$ denote their amplitudes, respectively. Since amplitudes are additive, the amplitude of, say, their disjoint union $A \cup B$ is $a + b$. Let $r$ be a non-negative integer and let $P_r$ denote the probability function which assigns to any disjoint union of subsets $A \cup B \cup \cdots \cup M$ the $r^{th}$-power of its amplitude $(a + b + \ldots + m)^r$. Then, it will be shown that if $r < n$, then $I_n = 0$.

In order to see this, let us express $I_n$ in terms of $P_r$:

$$I_n(S_1, S_2, S_3, \cdots, S_n) \equiv \sum_i (s_i)^r - \sum_{\text{distinct } i,j} (s_i + s_j)^r + \sum_{\text{distinct } i,j,k} (s_i + s_j + s_k)^r \mp \cdots - (-1)^{n-1} \sum_i (s_1 + s_2 + \ldots + (s_i) + \cdots + s_n)^r$$

$$-(-1)^n(s_1 + s_2 + \ldots + s_n)^r. \quad (4)$$

Note that every term has degree $r$. The strategy is to consider any term of the form $(s_i)^{r_1}(s_j)^{r_2} \cdots (s_k)^{r_3}$, where $r_1, r_2, \ldots, r_k$ are positive integers whose sum is $r$, and show that its coefficient in $I_n$ vanishes for $r < n$.

Consider such a term $(s_i)^{r_1}(s_j)^{r_2} \cdots (s_k)^{r_3}$, which involves $\ell$ of $n$ possible (atomic) amplitudes. Observe that this term occurs with coefficient $r!/(r_1!r_2!\cdots r_\ell)!$ in the $r^{th}$-power of any multinomial which contains $(s_i + s_j + \ldots + s_\ell)$. From all of the multinomials with exactly $m$ terms, raised to the $r^{th}$ power, i.e., $(s_i + s_j + \ldots + s_\ell)^r$, the term $(s_i)^{r_1}(s_j)^{r_2} \cdots (s_k)^{r_3}$ appears \( \binom{n}{\ell} \cdot \binom{\ell}{m} \cdot (n - \ell)!/(m - \ell)!(n - m)! \) times. So,
the sum of the coefficients of the term \((s_i)^r(s_j)^r \cdots (s_l)^r\) in \(I_n\) is
\[
\left( - \sum_{m=1}^{n} (-1)^m \binom{n - \ell}{m - \ell} \right) \frac{r!}{r_1!r_2! \cdots r_\ell!}.
\]

Note that
\[
\sum_{m=1}^{n} (-1)^m \binom{n - \ell}{m - \ell} = \sum_{m=\ell}^{n} (-1)^m \binom{n - \ell}{m - \ell} = \sum_{p=0}^{n-\ell} (-1)^{p+\ell} \binom{n - \ell}{p}.
\]

Since \(0 \leq r_i + r_j + \cdots + r_\ell = r\), with each of \(r_i, r_j, \ldots, r_\ell > 0\), we have that \(\ell\) ranges from 0 to \(r\). Now, suppose \(r < n\). Thus, we have \(\ell < n\), and the sum and, therefore, \(I_n\) is zero.

For the cases \(r \geq n\), the only surviving terms are those for which \(\ell = n\). So, for \(r = n\), we have \(I_n = -(-1)^n n! s_1 s_2 \cdots s_n\). For \(r > n\), we have \(I_n = -(-1)^n \sum_{r_1=r_2=\cdots=r_n} (s_1)^r s_2 s_3 \cdots (s_n)^r\), where the sum is over all positive-integer partitions of \(r\) into \(n\) parts.

3. Generalized Measure Theory

Following Sorkin, we make a replacement of axiom \(\mathcal{B}\).

\(\mathcal{B}'\). There exists an \(n \geq 2\), such that, for any collection of mutually-disjoint subsets \(S_1, S_2, \cdots, S_n \in M\), we have \(I_n(S_1, S_2, \cdots, S_n) = 0\) but \(I_{n-1}(S_1, S_2, \cdots, (S_j), \cdots, S_n) \neq 0\), in general.

This encodes the requirement that the Generalized Measure satisfies the \(n^{th}\)-order sum-rule but not the \((n-1)^{st}\) sum-rule.

A concomitant of this new axiom is the following lemma:

Lemma 1. \(\mathcal{B}\)

\[I_n(A, B, C, \cdots, N) = I_{n-1}(A, C, \cdots, N) + I_{n-1}(B, C, \cdots, N) - I_{n-1}(A \sqcup B, C, \cdots, N).\]

It is easy to verify this for \(n = 2\) and \(n = 3\).

\(\mathcal{B}\) If \(\ell = 0\), then the sum of the \(r_i\) is empty and, so, \(r = 0\).

\(\mathcal{B}'\) Using \((1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^i\) with \(k > 0\) and \(x = -1\), we find that the sum is equal to 0.

\(\mathcal{B}''\) This expression is minus the corresponding expression given in \(\mathcal{B}'\). This provides consistency with the sign-convention we chose earlier.
For $n = 2$, which is at the level of classical measure theory,

$$I_2(A, B) \equiv I_1(A) + I_1(B) - I_1(A \sqcup B)$$

$$\equiv |A| + |B| - |A \sqcup B|,$$

where we used the definition of $I_1$ in the last step.

For $n = 3$, which is at the level of “Quantum Measure theory,”

$$I_3(A, B, C) \equiv I_2(A, C) + I_2(B, C) - I_2(A \sqcup B, C)$$

$$\equiv (|A| + |C| - |A \sqcup C|) + (|B| + |C| - |B \sqcup C|)$$

$$- (|A \sqcup B| + |C| - |(A \sqcup B) \sqcup C|)$$

$$\equiv |A| + |B| + |C| - |A \sqcup B| - |B \sqcup C| - |C \sqcup A|$$

$$+ |A \sqcup B \sqcup C|,$$

where we used the definition of $I_2$ in the second step.

We will give a full proof for all $n$ later in Section 5. For now, let us see what the lemma implies.

First, for any collection of disjoint subsets,

if $I_{n-1} = 0$, then $I_n = 0$,

and therefore $I_{n+1} = 0, I_{n+2} = 0$, etc. That is, the vanishing of $I_n$ for some $n$ implies that all higher functions vanish. So, one can characterize classical probability as an “$I_2 = 0$” Generalized Measure theory, and quantum probability as an “$I_3 = 0$” Generalized Measure theory.

Secondly,

if $I_n = 0$, then $I_{n-1}$ is additive in its arguments.

For the “$I_3 = 0$” theories, this means that $I_2$ is a bi-additive function. This suggests the quadratic relation between amplitudes and probabilities. In fact, Sorkin used this relation to show that every Quantum Measure comes from an extension of the $I_2$ function applied to general (non-disjoint) arguments.

Another concomitant of axiom is the following lemma:

**Lemma 2.**

$$|A \sqcup B \sqcup C \sqcup \cdots \sqcup N| =$$

$$\sum_i I_1(S_i) - \sum_{\text{distinct } i,j} I_2(S_i, S_j) + \sum_{\text{distinct } i,j,k} I_3(S_i, S_j, S_k) + \cdots$$

$$- (-1)^{n-1} \sum_i I_{n-1}(A, B, \cdots, S_i, \cdots, N)$$

$$- (-1)^n I_n(A, B, C, \cdots, N).$$
This expresses the Generalized Measure of a disjoint union of a finite collection of subsets in terms of the generalized interferences among all of these subsets. This expression resembles the definition of the generalized interference term (see Eq. (3)). Indeed, we will show that there is a kind of duality relation between the two, and this will be used in Section 5 to give a complete proof of this lemma.

4. An Algebraic Formulation

4.1. The Ring $\mathcal{ZP}(S)$

Consider a set $S$ (of histories) and the set $\mathcal{P}(S)$ of all of its subsets [i.e., the power set of $S$]. In practice, one would only use a smaller collection $M$ of measurable subsets which is at least closed under disjoint union and contains the empty set.

We wish to define the set $\mathcal{ZP}(S)$ of finite “formal linear combinations” of the elements of $\mathcal{P}(S)$ with integer coefficients [i.e., the free module on $\mathcal{P}(S)$ over the integers $\mathbb{Z}$]. A typical element of $\mathcal{ZP}(S)$ is of the form $\sum n_i S_i$, where $S_i \in \mathcal{P}(S)$ and $n_i \in \mathbb{Z}$, of which only a finite number are nonzero. We denote the additive-identity (“zero”) by “0”. For clarity, we write “$A$” for the element “1$A$” and “$-A$” for its additive inverse (“$-1)A$”.

We endow this set with a multiplication rule $\cdot : \mathcal{ZP}(S) \times \mathcal{ZP}(S) \to \mathcal{ZP}(S)$ by

$$\left( \sum_i a_i S_i \right) \cdot \left( \sum_j b_j S_j \right) := \left( \sum_k p_k S_k \right)$$

where

$$p_k = \sum_{S_i \sqcup S_j = S_k} a_i b_j$$

with the understanding that we take $p_k = 0$ if the summation is empty. This rather complicated definition is a generalization of the product of two monomials $A \cdot B = (A \sqcup B)$.

By definition, the empty set $\emptyset \in \mathcal{P}(S)$ is disjoint with every element of $\mathcal{P}(S)$. So, it makes sense to form $A \sqcup \emptyset$, which evaluates to $A$. Thus, it follows that the multiplicative-identity (“unit”) is “\emptyset”:

$$A \cdot \emptyset = \emptyset \cdot A = A.$$  

This should not be confused with the fact that “products with zero are zero”:

$$A \cdot 0 = 0 \cdot A = 0.$$

This multiplication rule is obviously commutative. In the appendix, we prove the associativity and distributivity over addition, which shows that $(\mathcal{ZP}(S), +, \cdot)$ is a ring.
4.2. The Circle-Product

We now define a multiplication rule on $\mathcal{P}(S)$, called the “circle-product” or “circle composition operator.” For any pair of subsets $A, B \in \mathcal{P}(S)$,

$$(1A) \circ (1B) := (1A) + (1B) + (-1)(A \cdot B)$$

or, simply,

$$A \circ B := A + B - A \cdot B.$$ 

Note that

$$0 \circ B = 0 + B - 0 \cdot B = 0 + B - 0 = B$$

But

$$\emptyset \circ B = \emptyset + B - \emptyset \cdot B = \emptyset + B - \emptyset = \emptyset.$$ 

So, viewing the circle-product multiplicatively, the “circle-identity” coincides with the additive-identity 0, and the “circle-zero” coincides with the multiplicative-identity $\emptyset$.

Clearly, the circle-product is commutative. Associativity of the circle-product, however, arises in a nontrivial way from that of the multiplication rule.

First, observe that the circle-product generally does not distribute over addition. In fact, this is a consequence of the distributivity of multiplication over addition.

$$(A + B) \cdot C = A \cdot C + B \cdot C$$

$$(A + B) + C - (A + B) \circ C = A + C - A \circ C + B + C - B \circ C$$

$$- (A + B) \circ C = -A \circ C - B \circ C + C$$

$$(A + B) \circ C \neq A \circ C + B \circ C.$$ 

Instead, the circle-product is said to be “quasi-distributive” over addition:

$$(A + B) \circ C = A \circ C + B \circ C - C.$$ 

This implies, for example, that

$$(nA) \circ C = n(A \circ C) - (n - 1)C$$

$$(nA) \circ C \neq n(A \circ C).$$ 

Such an operator is used to define the Jacobson radical of a ring. See reference 7.

Note that if we had defined the circle-operator with the opposite sign-convention, $A \circ B = A \cdot B - A - B$, then we would have had $0 \circ B = 0 \cdot B - 0 - B = -B$ and $\emptyset \circ B = \emptyset \cdot B - \emptyset - B = -\emptyset$. 


However, the circle-product does distribute over “affine sums,” i.e., sums whose coefficients add up to 1.

\[
\left( \sum n_i S_i \right) \circ X = \sum n_i \left( S_i \circ X \right) \\
\left( \sum n_i S_i \right) + X - \left( \sum n_i S_i \right) \cdot X = \sum n_i \left( S_i + X - S_i \cdot X \right) \\
\sum (n_i S_i) + X - \sum (n_i S_i \cdot X) = \sum \left( \sum n_i \right) X - \sum (n_i S_i \cdot X) \\
\sum (n_i S_i) + X = \left( \sum n_i \right) X \\
1 \geq \left( \sum n_i \right).
\]

Such a condition could be characterized as “affine distributivity.”

In particular, this implies that

\[
(A + B - A \cdot B) \circ C = A \circ C + B \circ C - (A \cdot B) \circ C
\]

or, using the definition of the circle-product,

\[
(A \circ B) \circ C = A \circ C + B \circ C - (A \cdot B) \circ C.
\]

This equation will be used to prove lemma 1.

Now, it can be easily shown that the associativity of the circle-product arises from that of the multiplication rule.

\[
(A \circ B) \circ C \geq A \circ (B \circ C) \\
(A + B - A \cdot B) \circ C \geq A \circ (B + C - B \cdot C) \\
A \circ C + B \circ C - (A \cdot B) \circ C \geq A \circ B + A \circ C - A \circ (B \cdot C) \\
(B + C - B \cdot C) \\
-(A \cdot B + C - (A \cdot B) \cdot C) \geq (A + B - A \cdot B) \\
-(A + B \cdot C - A \cdot (B \cdot C)) \\
(A \cdot B) \cdot C \geq A \cdot (B \cdot C).
\]

\(^9\)Note further that if we had defined the circle-operator with the opposite sign-convention, \(A \circ B = A \cdot B - A - B\), associativity would have failed.

\[
(A \circ B) \circ C \geq A \circ (B \circ C) \\
(A \cdot B - A - B) \circ C \geq A \circ (B \cdot C - B - C) \\
(A \cdot B - A - B) \cdot C - (A \cdot B - A - B) - C \geq A \cdot (B \cdot C - B - C) - A - (B \cdot C - B - C) \\
A \cdot B \cdot C - A \cdot C - B \cdot C \\
-A \cdot B + A + B - C \geq A \cdot B \cdot C - A \cdot B - A \cdot C \\
-A - B \cdot C + B + C \\
A - C \neq -A + C
\]
We now derive an algebraic identity which underlies the generalized interference term.

**Lemma 3.** For mutually-disjoint subsets $S_1, S_2, \ldots, S_n \in \mathcal{P}(S)$, the circle-product can be expressed as

\[
S_1 \circ S_2 \circ \ldots \circ S_n = \sum_{i=1}^{n} S_i - \sum_{1 \leq i < j}^{n} S_i \cdot S_j + \sum_{1 \leq i < j < k}^{n} S_i \cdot S_j \cdot S_k \mp \ldots
\]

\[
-(-1)^{n-1} \sum_{i=1}^{n} S_i \cdot S_2 \cdot \ldots \cdot \text{omit} S_i \cdot \ldots \cdot S_n
\]

\[
-(-1)^n S_1 \cdot S_2 \cdot \ldots \cdot S_n.
\]

(7)

**Proof.**

Consider a pair of disjoint subsets $A, B \in \mathcal{P}(S)$. For $n = 2$, the lemma is true by definition.

\[
A \circ B = A + B - A \cdot B.
\]

By forming 1 minus the left-hand side, observe that

\[
1 - A \circ B = 1 - (A + B - A \cdot B)
\]

\[
= (1 - A) \cdot (1 - B).
\]

In fact, for a set of $n$ mutually disjoint subsets $A, B, C, \ldots, N \in \mathcal{P}(S)$, we have

\[
1 - A \circ B \circ C \circ \ldots \circ N = 1 - A \circ (B \circ C \circ \ldots \circ N)
\]

\[
= (1 - A) \cdot (1 - (B \circ C \circ \ldots \circ N))
\]

\[
= (1 - A) \cdot [(1 - B) \cdot (1 - (C \circ \ldots \circ N))]
\]

\[
= (1 - A) \cdot [(1 - B) \cdot (1 - C) \cdot \ldots \cdot (1 - N)]
\]

\[
= (1 - A) \cdot (1 - B) \cdot (1 - C) \cdot \ldots \cdot (1 - N)
\]

(8)

By expanding out the right-hand side of Eq. (8), we obtain 1 minus the right-hand side of Eq. (7).

\[\square\]

**4.3. Duality**

We note the following duality between the multiplication rule and the circle-product. With simple algebra, the definition of the circle-product can be reversed to read

\[A \cdot B := A + B - A \circ B.\]

Formally, it appears that one can swap the roles of “·” and “⊙” in a valid equation and obtain another valid equation. Let us make this more precise.
For any \( A \in \mathcal{ZP}(\mathcal{S}) \), define its dual to be \( A' := \emptyset - A \). Note that \( 0' = \emptyset \). Clearly, we have \((A')' = A\).

Consider \( C = A \circ B \).

\[
C = A \circ B \\
= (\emptyset - A') \circ (\emptyset - B') \\
= (\emptyset - A') + (\emptyset - B') - (\emptyset - A') \cdot (\emptyset - B') \\
= 2\emptyset - A' - B' - \emptyset^2 + \emptyset \cdot B' + A' \cdot \emptyset - A' \cdot B' \\
= 2\emptyset - A' - B' - \emptyset + B' + A' - A' \cdot B' \\
\emptyset - C' = \emptyset - A' \cdot B'.
\]

So, we find that

\[
(A \circ B)' = A' \cdot B'.
\]

A similar calculation verifies that

\[
(A \cdot B)' = A' \circ B'.
\]

Now consider \( C = A + B \), then

\[
C = A + B \\
= (\emptyset - A') + (\emptyset - B') \\
\emptyset - C' = 2\emptyset - A' - B'.
\]

So,

\[
(A + B)' = A' + B' - \emptyset.
\]

Thus, for general sums, the duality-operation does not distribute over addition. However, the duality-operation does distribute over affine sums.

\[
\left( \sum n_iS_i \right)' \overset{?}{=} \sum n_i (S_i') \\
\emptyset - \sum n_iS_i \overset{?}{=} \sum n_i (\emptyset - S_i) \\
\overset{?}{=} \sum n_i\emptyset - \sum n_iS_i \\
\emptyset \overset{?}{=} \left( \sum n_i \right) \emptyset \\
1 \overset{?}{=} \left( \sum n_i \right).
\]
Without proof, we state the following duality theorem.

**Duality Theorem**

Let \( S_1 \) and \( S_2 \) be expressions formed from 0, \( \emptyset \) and indeterminates \( A, B, C, \) etc., using “·”, “◦” and affine linear combination. Let \( S_1' \) and \( S_2' \) be their dual expressions obtained by swapping all occurrences of “◦” with “·” and of \( \emptyset \) with 0. If \( S_1 = S_2 \) is an identity in \( ZP(S) \), then \( S_1' = S_2' \) is also an identity in \( ZP(S) \).

These results will be used to prove lemma 2.

4.4. **The Extended Generalized Measure**

Consider a linear map, which we will call the “Extended Generalized Measure,” \( \mu : ZP(S) \to \mathbb{R} \), where \( \mathbb{R} \) denotes the real numbers.

Consider a pair of disjoint subsets \( A, B \in P(S) \). Applying this map to the circle-product of \( 1A, 1B \in ZP(S) \), we have:

\[
A \circ B = A + B - A \cdot B \\
= A + B - (A \cup B) \\
\mu(A \circ B) = \mu(A) + \mu(B) - \mu(A \cup B). \tag{9}
\]

In order to make the connection with the Generalized Measure \(| \cdot |\) as defined by Sorkin\(^1\), let us impose the following conditions on \( \mu \). We require that \( \mu(\emptyset) = 0 \) and that \( \mu(1X) \geq 0 \) for all \( X \in P(S) \). In other words, we require that \( \mu \) be a Generalized Measure extended to \( ZP(S) \) by linearity.

Now, let us notice the following. For mutually-disjoint \( S_1, S_2, \cdots, S_n \in P(S) \), we have

\[
I_n(S_1, S_2, \cdots, S_i, \cdots, S_n) = \mu(\bigcap_{i=1}^n S_i).
\]

So, Eq. (9) can be written

\[
I_2(A, B) = I_1(A) + I_1(B) - I_1(A \cup B),
\]

which agrees (up to an overall sign) with the definition given by Sorkin\(^1\). Similarly, the corresponding higher-order expressions clearly agree for all \( n \).

We are now prepared to give proofs of lemma 1 and lemma 2.
5. Proofs of the lemmas

Lemma 1.

\[ I_n(A, B, C, \cdots, N) = I_{n-1}(A, C, \cdots, N) + I_{n-1}(B, C, \cdots, N) - I_{n-1}(A \sqcup B, C, \cdots, N). \]

Proof. First, consider a collection of three mutually disjoint subsets \( A, B, X \in \mathcal{P}(S) \). Forming the triple circle-product in \( \mathbb{Z} \mathcal{P}(S) \),

\[ (A \circ B) \circ X = (A + B - A \cdot B) \circ X, \]

and using the associativity and affine-distributivity of the circle-product (see Eq. (6)), we have

\[ (A \circ B) \circ X = A \circ X + B \circ X - (A \cdot B) \circ X, \]

\[ A \circ B \circ X = A \circ X + B \circ X - (A \sqcup B) \circ X, \]

where we used the definition of multiplication on the right-hand side. Applying the linear map \( \mu \) and making the identifications defined in the last section, we find

\[ \mu(A \circ B \circ X) = \mu(A \circ X) + \mu(B \circ X) - \mu((A \sqcup B) \circ X) \]

\[ I_3(A, B, X) = I_2(A, X) + I_2(B, X) - I_2(A \sqcup B, X). \]

By taking \( 1X \in \mathbb{Z} \mathcal{P}(S) \) to be of the form

\[ X = \bigodot_{i=1}^{n-2} X_i = C \circ D \circ \cdots \circ N, \]

where \( A, B, C, D, \cdots, N \in \mathcal{P}(S) \) are mutually disjoint, we complete the proof for all \( n > 1 \):

\[ A \circ B \circ X = A \circ X + B \circ X - (A \cdot B) \circ X \quad (10) \]

\[ \mu(A \circ B \circ X) = \mu(A \circ X) + \mu(B \circ X) - \mu((A \sqcup B) \circ X) \]

\[ \mu(A \circ B \circ (\bigodot_{i=1}^{n-2} X_i)) = \mu(A \circ (\bigodot_{i=1}^{n-2} X_i)) + \mu(B \circ (\bigodot_{i=1}^{n-2} X_i)) - \mu((A \sqcup B) \circ (\bigodot_{i=1}^{n-2} X_i)) \]

\[ I_n(A, B, C, \cdots, N) \triangleq I_{n-1}(A, C, \cdots, N) + I_{n-1}(B, C, \cdots, N) - I_{n-1}(A \sqcup B, C, \cdots, N). \]
Lemma 2.

\[ |S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n| = \sum_i I_1(S_i) - \sum_{\text{distinct } i,j} I_2(S_i, S_j) + \sum_{\text{distinct } i,j,k} I_3(S_i, S_j, S_k) - \ldots \]

\[-(-1)^{n-1} \sum_i I_{n-1}(S_1, S_2, \cdots, S_n) \]

\[-(-1)^n I_n(S_1, S_2, \cdots, S_n). \]

Proof. Consider lemma 3. For mutually-disjoint subsets \( S_1, S_2, \ldots, S_n \in \mathcal{P}(S) \), the circle-product can be expressed as

\[ S_1 \circ S_2 \circ \ldots \circ S_n = \sum_{i=1}^n S_i - \sum_{1 \leq i < j}^n S_i \cdot S_j + \sum_{1 \leq i < j < k}^n S_i \cdot S_j \cdot S_k - \ldots \]

\[-(-1)^{n-1} \sum_{i=1}^n S_1 \cdot S_2 \cdot \ldots \cdot S_i \circ \ldots \circ S_n \]

\[-(-1)^n S_1 \cdot S_2 \cdot \ldots \cdot S_n. \]

So, applying the duality-operation \( h \) of Section 4 to lemma 3, we obtain

\[ S_1 \cdot S_2 \cdot \ldots \cdot S_n = \sum_{i=1}^n S_i - \sum_{1 \leq i < j}^n S_i \circ S_j + \sum_{1 \leq i < j < k}^n S_i \circ S_j \circ S_k - \ldots \]

\[-(-1)^{n-1} \sum_{i=1}^n S_1 \circ S_2 \circ \ldots \circ S_i \circ \ldots \circ S_n \]

\[-(-1)^n S_1 \circ S_2 \circ \ldots \circ S_n. \]

Writing “\( 1(S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n) \)” for \( S_1 \cdot S_2 \cdot \ldots \cdot S_n \) and then applying the linear map \( \mu \), we obtain the statement of lemma 2.

6. Open Questions

We have found an interesting algebraic structure underlying the Quantum Measure and its generalizations. What is its physical interpretation for both classical and quantum physics?

Note that each term of the form \( S_1 \cdot S_2 \cdot \ldots \cdot S_m \) has coefficient equal to 1. The sum of the coefficients of these terms is \(- \sum_{i=1}^n (-1)^i \binom{n}{i}\). Using \( (1 + x)^n = \sum_{i=0}^n (-1)^i \binom{n}{i} x^n \) with \( n > 0 \) and \( x = -1 \), we find that the sum of the coefficients is equal to 1.
By themselves, the sum rules for the “$I_3 = 0$” case do not uniquely yield the standard quantum theory. What additional axioms are needed to select the standard quantum theory from all possible “$I_3 = 0$” theories?

In this paper, we were mainly concerned with the special case of mutually disjoint subsets, which is sufficient to prove these identities involving the Quantum Measure. Sorkin\cite{Sorkin1} showed how one could extend the definition of $I_2$ to general (non-disjoint) arguments, which he used to show that every Quantum Measure [which satisfies the ”$I_3 = 0$” sum rule] can arise in this way. Can a similar extension to general arguments be carried out for the higher-order interference functions $I_n$?

In particular, we showed that a probability function $P_r$ that assigns to any disjoint union of subsets the $r^{th}$-power of its amplitude satisfies the “$I_n = 0$” sum rule if $r < n$. Is the converse true? Or are there other functional relationships between the probabilities and the amplitudes that satisfy these sum rules?

Sorkin\cite{Sorkin1} has proposed a “null” three-slit experiment to test the validity of standard quantum mechanics. A non-null result would indicate that a more general dynamics was at work. In light of this possibility, can an “$I_4 = 0$” generalization of quantum mechanics be formulated?

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Appendix A

In this section, we prove the associative and distributive properties of the multiplication rule $\cdot : Z\mathcal{P}(S) \times Z\mathcal{P}(S) \to Z\mathcal{P}(S)$. To simplify the notation used in this proof, we will write

$$\left( \sum_i a_i S_i \right) \cdot \left( \sum_j b_j S_j \right) := \sum_k [ab]_k S_k$$

(A.1)

where $a_i, b_j \in Z$ and $S_i \in \mathcal{P}(S)$, with

$$[ab]_k = \sum_{S_i \cup S_j = S_k} a_i b_j,$$

with the understanding that we take $[ab]_k = 0$ if the summation is empty. We will also use the summation convention by writing $a_i S^i$ for $\sum_i a_i S_i$.

First, we show distributivity over addition.

$$(a_i S^i) \cdot ((b_j + c_j) S^j) \equiv (a_i S^i \cdot b_j S^j) + (a_i S^i \cdot c_j S^j)$$
\[ (a_i S^i) \cdot (b_j S^j) \cdot (c_k S^k) \equiv (a_i S^i) \cdot ((b_j S^j) \cdot (c_k S^k)) \]
\[ [ab]_m s^m \cdot (c_k S^k) \equiv (a_i S^i) \cdot ([bc]_n s^n) \]
\[ [[[ab]_p]_q s^p]_r \equiv [a[bc]_q]_r s^p \]
\[ \sum_{m,k}^{m,k} a_i b_j c_k \equiv \sum_{i,j,k}^{i,j,k} a_i b_j c_k \]
\[ \sum_{m,k}^{m,k} a_i b_j c_k \equiv \sum_{i,j,k}^{i,j,k} a_i b_j c_k \]

where we have used the associativity of ordinary multiplication in \( Z \).

Finally, we show associativity.

This completes the proof that \( (Z \mathcal{P}(S), +, \cdot) \) is a ring.

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