Weak Harnack inequalities for eigenvalues and constant rank theorems

Gábor Székelyhidi\(^a\) and Ben Weinkove\(^b\)

\(^a\)Department of Mathematics, University of Notre Dame, Notre Dame, IN, USA; \(^b\)Department of Mathematics, Northwestern University, Evanston, IL, USA

**ABSTRACT**

We consider convex solutions of nonlinear elliptic equations which satisfy the structure condition of Bian and Guan. We prove a weak Harnack inequality for the eigenvalues of the Hessian of these solutions. This can be viewed as a quantitative version of the constant rank theorem.

**1. Introduction**

Constant rank theorems in PDE have a long history, starting with work of Caffarelli and Friedman [1], Yau (see [2]) and then developed further by Korevaar and Lewis [3], Caffarelli et al. [4], Bian and Guan [5, 6] and others [7–13]. These results assert that a convex solution \(u\) of a certain class of elliptic or parabolic equations has Hessian \(D^2 u\) of constant rank.

Constant rank theorems, also known as the “microscopic convexity principle”, have been used to establish “macroscopic” convexity properties of solutions to PDEs on convex domains, now a vast area of research (see [1, 14–34] and the references therein).

One method for establishing a constant rank theorem is to compute with an expression involving the elementary symmetric polynomials \(\sigma_k\) of the eigenvalues

\[ \lambda_1 \leq \cdots \leq \lambda_n \]

of the Hessian \(D^2 u\). Bian and Guan [5] considered solutions of nonlinear elliptic equations

\[ F(D^2 u, Du, u, x) = 0, \]

under a convexity condition for \(F\) (see (1.4) below) and proved a constant rank theorem using a differential inequality for the quantity \(\sigma_{\ell+1} + \frac{\sigma_{\ell+2}}{\sigma_{\ell+1}}\). The authors [13] gave a new proof of the Bian and Guan result using the simple linear expression

\[ \ell \lambda_\ell + 2\lambda_{\ell-1} + \cdots + \lambda_1, \]

and a method of induction. This approach exploited the concavity of the sums of the lowest eigenvalues.
In this paper we will also assume the Bian and Guan structure conditions. Building on the method of [13] and making use again of the expression (1.1) we directly prove a weak Harnack inequality for each of the eigenvalues $\lambda_i$. This states that the $L^q$ norm for some $q > 0$ is bounded above by the infimum. We view this Harnack inequality as a quantitative version of the constant rank theorem of Bian and Guan, which follows as an immediate consequence.

Another difference between the current paper and [13] is that we compute differential inequalities directly, holding almost everywhere, for sums of the eigenvalues $\sum \lambda_i$. In particular we avoid here the device of approximation by polynomials.

We now state our results precisely. Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and let $F$ be a real-valued function

$$ F = F(A, p, u, x) \in C^2(\text{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R} \times B), $$

where $\text{Sym}_n(\mathbb{R})$ is the vector space of symmetric $n \times n$ matrices with real entries. We assume that $F$ satisfies the condition of 00-Guan [5] that for each $p \in \mathbb{R}^n$,

$$(A, u, x) \in \text{Sym}_n^+(\mathbb{R}) \times \mathbb{R} \times B \mapsto F(A^{-1}, p, u, x)$$ is locally convex, (1.2)

where $\text{Sym}_n^+(\mathbb{R})$ is the subset of $\text{Sym}_n(\mathbb{R})$ that are strictly positive definite. Suppose that $u \in C^3(B)$ is a convex solution of

$$ F(D^2u, Du, u, x) = 0, $$

subject to the ellipticity condition that for all $\xi \in \mathbb{R}^n$,

$$ \Lambda^{-1}|\xi|^2 \leq F^{ij}(D^2u, Du, u, x)\xi^i \xi^j \leq \Lambda|\xi|^2, \quad \text{on } B, $$

for a positive constant $\Lambda > 0$, where $F^{ij}$ is the derivative of $F$ with respect to the $(i, j)$th entry $A_{ij}$ of $A$. Our main result is as follows.

**Theorem 1.1.** Let $u$ be as above and let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $D^2u$. Then there exist positive constants $C_0, q$ depending only on $n$, $\Lambda$, $\|u\|_{C^1(B)}$ and $\|F\|_{C^2}$ such that for each $\ell = 1, \ldots, n$,

$$ \|\lambda_{\ell}\|_{L^q(B_{1/2})} \leq C_0 \inf_{B_{1/2}} \lambda_{\ell}, $$

where $B_{1/2} = B_{1/2}(0)$ is the ball in $\mathbb{R}^n$ centered at 0 of radius 1/2.

This implies in particular the constant rank theorem of Bian and Guan [5]:

**Corollary 1.1.** The Hessian $D^2u$ has constant rank in $B$.

Indeed, applying Theorem 1.1 on appropriately scaled balls, the sets

$$ \{x \in B|\text{rank}(D^2u(x)) \leq k\}, \quad k = 0, 1, 2, \ldots, n, $$

are open in $B$. On the other hand the sets $\{x \in B|\text{rank}(D^2u(x)) \geq k\}$ are open in $B$ by continuity of the eigenvalues of $D^2u$. A consequence is that the sets

$$ \{x \in B|\text{rank}(D^2u(x)) = k\} $$

are open and closed in $B$, giving the corollary.

We now give an outline of the paper. In Section 2 we recall some definitions and known results about semi-concave functions and in particular we provide a proof of the
semi-concavity of the sum of the first $k$ eigenvalues of $D^2u$ for $u$ in $C^4$. We also give a version of the weak Harnack inequality for subsolutions of elliptic equations.

In Section 3, under the assumption that $u$ is in $C^4$, we prove that the key differential inequality

$$F^{ab} Q_{ab} \leq CQ + \sum_{i=1}^{k} b^{i,j}(\lambda_i)_j, \quad \text{for } Q = Q_k = \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1,$$

(1.5)

holds almost everywhere, where $C$ and $b^{i,j}$ are bounded. This improves on the analogous result in [13] where the inequality is proved for approximating polynomials. We note that the method of Section 3 includes proofs of a first variation formula for $\lambda_i$ (Lemma 3.2) and a second variation inequality (Lemma 3.3), which hold almost everywhere.

In Section 4 we complete the proof of Theorem 1.1. We cannot directly apply the weak Harnack inequality to $Q$ satisfying (1.5) because its right hand side includes derivatives of $\lambda_1, \ldots, \lambda_k$. We get around this difficulty by considering a new quantity

$$R = \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{1/2}, \quad \text{for } \varepsilon > 0.$$ 

By exploiting the concavity of the square root function, $R$ is shown to satisfy the differential inequality

$$F^{ab} R_{ab} \leq C R,$$

almost everywhere. We apply the weak Harnack inequality to $R$ and then let $\varepsilon \to 0$ to obtain Theorem 1.1.

2. Preliminaries

In this section we collect some elementary and well-known results which we will need in the sequel.

2.1. Semi-concave functions

Let $U$ be a bounded convex subset of $\mathbb{R}^n$. A real-valued function $f$ on $U$ is semi-concave if there exists a constant $M$ such that $g = f - M|x|^2$ is concave. We call $M$ the semi-concavity constant for $f$ on $U$. Observe that every function in $C^2(U)$ is automatically semi-concave.

Equivalently, a continuous function $f$ is semi-concave if for some $M'$

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) \leq M' |x - y|^2, \quad \text{for all } x, y \in U.$$  

(2.1)

It is a classical result that a concave function $f$ is Lipschitz continuous and hence differentiable almost everywhere (we write its derivative as $Df(x)$ if it exists at $x$). Moreover, by a theorem of Alexandrov, the second derivative of $f$ exists almost everywhere in the sense that there is a second order Taylor expansion at almost every $x$ (see for example [35, Theorem 2.6.4]). This holds too then for semi-concave functions $f$ on
More explicitly, at almost every \( x \in U \), the derivative \( Df(x) = (f_1(x), \ldots, f_n(x)) \) exists and there is a symmetric matrix which we write as \( D^2f(x) = (f_{ij}(x)) \) such that

\[
f(y) = f(x) + f_1(x)(y - x)_1 + \frac{1}{2} f_{ij}(x)(y - x)_i(y - x)_j + o(|y - x|^2), \quad \text{as } |y - x| \to 0,
\]

where as usual we are summing repeated indices from 1 to \( n \).

The following proposition is well-known (see for example [36]), but we include a brief proof for the reader’s convenience.

**Proposition 2.1.** Let \( u \in C^4(\bar{U}) \), for a bounded convex set \( U \subset \mathbb{R}^n \). Denote by \( \lambda_1(x) \leq \cdots \leq \lambda_n(x) \) the eigenvalues of the Hessian \( D^2u(x) \). Then for each \( k = 1, \ldots, n \) the map \( U \to \mathbb{R} \) given by

\[
x \mapsto \lambda_1(x) + \cdots + \lambda_k(x)
\]

is semi-concave.

**Proof.** We claim the following: the map \( \sigma : \text{Sym}_n(\mathbb{R}) \to \mathbb{R} \) given by

\[
\sigma(A) = \lambda_1(A) + \cdots + \lambda_k(A)
\]

is increasing and concave on \( \text{Sym}_n(\mathbb{R}) \), and is Lipschitz continuous with Lipschitz constant depending only on \( n \) and \( k \). To see the claim, note that given fixed unit vectors \( V_1, \ldots, V_k \in \mathbb{R}^n \), the function

\[
A \mapsto \sum_{i=1}^k A_{ij} V_i V_j^T,
\]

is linear, increasing and has bounded Lipschitz constant depending only on \( n \) and \( k \). Here we are writing \( V_x = (V_x^1, \ldots, V_x^n) \) and \( A = (A_{ij})_{i,j=1}^n \). But we can define

\[
\sigma(A) = \inf \left\{ \sum_{i=1}^k A_{ij} V_i V_j^T | V_1, \ldots, V_k \text{ are orthonormal} \right\}.
\]

The map \( \sigma \) is clearly increasing, and it is concave since the infimum of concave functions is concave. Moreover, it is an elementary fact that for any normed vector space \((X, || \cdot ||)\), if \( f_s : X \to \mathbb{R} \), for \( s \in S \), is a family of functions which are uniformly Lipschitz continuous:

\[
|f_s(x) - f_s(y)| \leq C||x - y||, \quad x, y \in X
\]

then \( f := \min_{s \in S} f_s \), assuming the minimum is attained at each point and is finite, is also Lipschitz continuous with the same constant \( C \). The claim follows.

For \( x, y \in \bar{U} \), using concavity of \( \sigma \),

\[
\frac{\sigma(D^2u(x)) + \sigma(D^2u(y))}{2} - \sigma \left( D^2u \left( \frac{x + y}{2} \right) \right) \\
\leq \sigma \left( \frac{D^2u(x) + D^2u(y)}{2} \right) - \sigma \left( D^2u \left( \frac{x + y}{2} \right) \right).
\]

But then since \( u \) is in \( C^4(\bar{U}) \) we have that \( D_{ij}u \) is semi-concave for every \( i, j \) and hence
\[
\frac{D_{ij}u(x) + D_{ij}u(y)}{2} - D_{ij}u \left( \frac{x + y}{2} \right) \leq M|x - y|^2.
\]

Then, increasing \(M\) if necessary, we have the following inequality of symmetric matrices:

\[
\frac{D^2u(x) + D^2u(y)}{2} - D^2u \left( \frac{x + y}{2} \right) \leq M|x - y|^2 \text{Id}.
\]

Since \(\sigma\) is increasing and Lipschitz continuous,

\[
\frac{\sigma(D^2u(x)) + \sigma(D^2u(y))}{2} - \sigma \left( D^2u \left( \frac{x + y}{2} \right) \right)
\]

\[
\leq \sigma \left( D^2u \left( \frac{x + y}{2} \right) + M|x - y|^2 \text{Id} \right) - \sigma \left( D^2u \left( \frac{x + y}{2} \right) \right)
\]

\[
\leq C|x - y|^2,
\]

completing the proof of the proposition.

We end the subsection with an elementary proposition.

**Proposition 2.2.** Let \(U\) be a bounded convex set in \(\mathbb{R}^n\) and \(V\) an open interval in \(\mathbb{R}\). Suppose that \(f : U \to V\) is semi-concave and \(h : V \to \mathbb{R}\) is increasing, Lipschitz continuous and concave. Then \(h \circ f : U \to \mathbb{R}\) is semi-concave.

**Proof.** We have for \(x, y \in U\),

\[
h(f(x)) + h(f(y)) \leq h \left( f \left( \frac{x + y}{2} \right) \right)
\]

\[
\leq h \left( f \left( \frac{x + y}{2} \right) \right) - h \left( f \left( \frac{x + y}{2} \right) \right) \quad (h \text{ concave})
\]

\[
\leq h \left( f \left( \frac{x + y}{2} \right) + M|x - y|^2 \right) - h \left( f \left( \frac{x + y}{2} \right) \right) \quad (f \text{ semi-concave, } h \text{ increasing})
\]

\[
\leq C|x - y|^2, \quad (h \text{ Lipschitz})
\]

as required.

**2.2. The weak Harnack inequality**

We state a weak Harnack inequality for semi-concave functions. It follows by approximation from the classical weak Harnack inequality for functions in \(W^{2,n}\) (see [37, Theorem 9.22]). Similar statements, with slightly different hypotheses, can be found in [38] and [13].

**Proposition 2.3.** Consider the operator \(Lv = a^{ij}D_{ij}v + b^iD_i v + cv\) with bounded coefficients on the unit ball \(B \subset \mathbb{R}^n\) and with \(a^{ij}\) satisfying the uniform ellipticity condition \(\Lambda^{-1} |\xi|^2 \leq a^{ij}\xi_i \xi_j \leq \Lambda |\xi|^2\) for all \(\xi \in \mathbb{R}^n\) for \(\Lambda > 0\). Let \(v\) be a semi-concave nonnegative function on \(B\) satisfying \(Lv \leq f\) almost everywhere in \(B\) for \(f \in L^n(B)\). Then on the half size ball \(B'\),
for positive constants $C$ and $q$ depending only on $n, \Lambda$, bounds for $b^i$ and $c$ and the radius of the ball $B$.

**Proof.** We include a brief proof for the sake of completeness. Let $\tilde{B}$ be a ball such that $B' \subset \subset \tilde{B} \subset \subset B$. Since $v$ is semi-concave, it is Lipschitz continuous and twice differentiable almost everywhere. For $\varepsilon > 0$ let $v_\varepsilon$ be a standard mollification of $v$. Then $v_\varepsilon \to v$ uniformly and $D^k v_\varepsilon \to D^k v$ for $k = 1, 2$ almost everywhere on $\tilde{B}$ as $\varepsilon \to 0$ (the second assertion is a direct consequence of the expansion (2.2)). Let $\delta > 0$ be given. Then by Egorov’s Theorem there exists a set $K_\delta \subset \tilde{B}$ such that $|\tilde{B} \setminus K_\delta| \leq \delta$ and $D^k v_\varepsilon \to D^k v$ uniformly on $K_\delta$ for $k = 1, 2$. Then we may choose $\varepsilon > 0$ sufficiently small so that

$$Lv_\varepsilon = Lv + L(v_\varepsilon - v) \leq f + \delta,$$

almost everywhere on $K_\delta$.

On the other hand, since $v$ is semi-concave we have an upper bound for $D^2 v_\varepsilon$ and hence on $\tilde{B} \setminus K_\delta$

$$Lv_\varepsilon \leq M,$$

for a constant $M \geq 1$ depending on the $C^0$ norm, semi-concavity constant and Lipschitz bound of $v$ as well as bounds on the coefficients of $L$. It follows that almost everywhere on $\tilde{B}$ we have

$$Lv_\varepsilon \leq f + g,$$

for a function $g \in L^\infty(\tilde{B})$ with $\|g\|_{L^\infty(\tilde{B})} \leq CM\delta^{1/n}$. Then we apply [37, Theorem 9.22] to the smooth nonnegative function $v_\varepsilon$ to obtain

$$\left(\int_{B'} v^q_\varepsilon\right)^{1/q} \leq C\inf_{B'} v_\varepsilon + C\|f + g\|_{L^n(\tilde{B})} \leq C\left(\inf_{B'} v_\varepsilon + \|f\|_{L^n(B)} + M\delta^{1/n}\right),$$

for uniform positive constants $C, q$. Then let $\delta \to 0$, so that in addition $\varepsilon \to 0$ and we obtain (2.3).

We emphasize that the constants $C, q$ in the above proposition are independent of the semi-concavity constant of $v$.

### 3. A Differential inequality

As in the introduction, let $u$ solve the equation (1.3) subject to the conditions (1.2) and (1.4). In this section, we make the following:

**Additional assumption:** $u \in C^4(B)$.

Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $D^2 u$. By Proposition 2.1, the map $x \mapsto \lambda_1 + \cdots + \lambda_k$ is semi-concave on compact convex subsets of $B$. It follows that $\lambda_1, \ldots, \lambda_n$ are twice differentiable almost everywhere on $B$. The goal of this section is the following differential inequality.
Lemma 3.1. For $1 \leq k \leq n$, define
\[ Q := Q_k := \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1. \]

Then
\[ F^{ab}Q_{ab} \leq CQ + \sum_{i=1}^{k} b^{i,j}(\lambda_i)^2, \quad \text{almost everywhere on } B, \tag{3.1} \]
where $C$ is a uniform constant and the $b^{i,j}$ are uniformly bounded functions on $B$.

In the above, “uniform” means that the constants depend only on $n$, $\Lambda$, $||u||_{C^2(B)}$ and $||F||_{C^2}$. In particular, the constants do not depend on a $C^4$ bound for $u$.

To establish the lemma, we prove two rather general results about the first and second derivatives of the eigenvalues $\lambda_i$, which will hold almost everywhere. In the case when the eigenvalues $\lambda_i$ are all distinct, there are well-known formulæ for their first and second derivatives (see for example [39]). To deal with eigenvalues with multiplicity we adapt an approach of Brendle-Choi-Daskalopoulos [40, Lemma 5] where similar statements to Lemmas 3.2 and 3.3 below are proved for $\lambda_1$. Crucially, these lemmas only hold at a point where the eigenvalues are twice differentiable.

Fix an $x_0$ at which the $\lambda_i$ are twice differentiable, and choose coordinates at $x_0$ such that $D^2u$ is diagonal with entries $u_{ii} = \lambda_i$. Moreover, we suppose that $N$ of the $\lambda_i$’s are distinct at $x_0$, and define $1 \leq \mu_1 < \cdots < \mu_N = n$ by
\[ \lambda_1 = \cdots = \lambda_{\mu_1} < \lambda_{1+\mu_1} = \cdots = \lambda_{2+\mu_1} < \lambda_{1+\mu_2} = \cdots = \lambda_{2+\mu_2} = \cdots = \lambda_{1+\mu_N} = \lambda_N. \]

We also define $\mu_0 = 0$ so that the multiplicities of the eigenvalues of $D^2u$ at $x_0$ are $\mu_1 - \mu_0, \mu_2 - \mu_1, \ldots, \mu_N - \mu_{N-1}$.

Lemma 3.2. For each $j = 1, 2, \ldots, N$ we have at $x_0$
\[ u_{k\ell i} = (\lambda_{1+\mu_{j-1}})^i \delta_{k\ell}, \quad \text{for } 1 + \mu_{j-1} \leq k, \ell \leq \mu_j. \tag{3.2} \]

for $i = 1, 2, \ldots, n$.

Proof. We prove this by induction on $j$. We first prove the case $j = 1$, which states that
\[ u_{k\ell i} = (\lambda_1)^i \delta_{k\ell}, \quad \text{for } 1 \leq k, \ell \leq \mu_1, \quad 1 \leq i \leq n. \tag{3.3} \]

Let $V = (V^1, \ldots, V^n)$ be a constant unit vector field defined in a neighborhood of $x_0$. Then by definition of $\lambda_1$, the function $h$ defined by
\[ h := u_{k\ell} V^k V^\ell - \lambda_1, \]
has $h(x) \geq 0$ for $x$ near $x_0$. Choose $V$ with $V^k(x_0) = 0$ for $k > \mu_1$ so that we have $h(x_0) = 0$ and $h$ has a local minimum at $x_0$. Moreover, $h$ is twice differentiable at $x_0$.

Then at $x_0$,
\[ 0 = h_i = \sum_{k, \ell \leq \mu_1} u_{k\ell i} V^k V^\ell - \sum_{k, \ell \leq \mu_1} (\lambda_1)^i \delta_{k\ell} V^k V^\ell, \]
using the fact that $V$ is a unit vector. Then (3.3) follows since we are free to choose $V^k(x_0)$ for $k \leq \mu_1$.

For the inductive step, assume (3.2) holds for $1 \leq j \leq p$. Let $V_1, \ldots, V_{\mu_p}$ be the constant unit vector fields in the $\partial/\partial x_1, \ldots, \partial/\partial x_{\mu_p}$ directions. That is, writing $V_x$ in
Consider the quantity
\[ h = \sum_{x=1}^{1+p} u_{k\ell} V^k_x V^\ell_x - \sum_{j=1}^p (\mu_j - \mu_{j-1}) \dot{\lambda}_{1+\mu_{j-1}} - \dot{\lambda}_{1+\mu_p} \]
\[ = \sum_{x=1}^{1+p} u_{k\ell} V^k_x V^\ell_x - \sum_{j=1}^p \sum_{1+\mu_{j-1} \leq x \leq \mu_j} \dot{\lambda}_{1+\mu_{j-1}} \delta_{k\ell} V^k_x V^\ell_x - \dot{\lambda}_{1+\mu_p} \]
\[ = \sum_{j=1}^p \sum_{1+\mu_{j-1} \leq x \leq \mu_j} (u_{k\ell} - \dot{\lambda}_{1+\mu_{j-1}} \delta_{k\ell}) V^k_x V^\ell_x + u_{k\ell} W^k W^\ell - \dot{\lambda}_{1+\mu_p}. \]

Now note that \( h(x_0) = 0 \) and \( h(x) \geq 0 \) for \( x \) near \( x_0 \). Since \( h \) achieves a minimum at \( x_0 \) we have
\[ 0 = h_1(x_0) = \sum_{j=1}^p \sum_{1+\mu_{j-1} \leq x \leq \mu_j} (u_{k\ell} - (\dot{\lambda}_{1+\mu_{j-1}}) \delta_{k\ell}) V^k_x V^\ell_x + u_{k\ell} W^k W^\ell - (\dot{\lambda}_{1+\mu_p})_i \]
\[ = (u_{k\ell} - \delta_{k\ell}(\dot{\lambda}_{1+\mu_p})_i) W^k W^\ell, \]
where for the last line we used the inductive hypothesis and the fact that \( W \) is a unit vector. Since \( W \) is an arbitrary unit vector in the span of \( \partial/\partial x_{1+\mu_p}, \ldots, \partial/\partial x_{\mu_p+1}, \) we have proved (3.2) for \( j = p + 1 \) and the result follows by induction.

As above, pick coordinates at \( x_0 \) such that \( D^2 u \) is diagonal with entries \( u_{ii} = \dot{\lambda}_i \). Fix \( m \) between 1 and \( n \). Define \( \rho \in \{m, m+1, \ldots, n\} \) to be the largest integer such that \( \dot{\lambda}_\rho = \dot{\lambda}_m \) at \( x_0 \), so that
\[ 0 \leq \dot{\lambda}_1 \leq \cdots \leq \dot{\lambda}_m = \dot{\lambda}_{m+1} = \cdots = \dot{\lambda}_\rho < \dot{\lambda}_{\rho+1} \leq \cdots \leq \dot{\lambda}_n. \]
Then we have the following lemma on the second derivatives of the \( \dot{\lambda}_x \). We emphasize that this only holds at the point \( x_0 \) where the \( \dot{\lambda}_i \) are twice differentiable.

**Lemma 3.3.** As symmetric \( n \times n \) matrices we have at \( x_0 \)
\[ \sum_{x=1}^m (\dot{\lambda}_x)_{ab} \leq \sum_{x=1}^m u_{x\alpha \beta} + 2 \sum_{x=1}^m \sum_{q > p} u_{q\alpha} u_{q\beta} \dot{\lambda}_x - \dot{\lambda}_q. \quad (3.4) \]

**Proof.** Let \( V_1, \ldots, V_m \) be smoothly mutually orthogonal unit vector fields defined in a neighborhood of \( x_0 \) with \( V_x(x_0) \) the unit vector in the \( \partial/\partial x_x \) direction. In particular, writing \( V_x = (V^1_x, \ldots, V^n_x) \) we have \( V^1_x = \delta_{q1} \) at \( x_0 \).

We consider the quantity
\[ h(x) = \sum_{x=1}^m u_{k\ell} V^k_x V^\ell_x - \sum_{x=1}^m \dot{\lambda}_x, \]
which has \( h(x_0) = 0 \) and \( h(x) \geq 0 \) for \( x \) near \( x_0 \). In particular \( h \) achieves a minimum at \( x_0 \), and moreover, \( h \) is twice differentiable at \( x_0 \).
We prescribe the first and second derivatives of the $V_x$ at $x_0$ as follows. For $1 \leq x \leq m$, and $1 \leq a \leq n$,

$$\partial_a V_x^q = \begin{cases} 
0, & q \leq \rho \\
\frac{u_{zqa}}{\lambda_x - \lambda_q}, & q > \rho.
\end{cases}$$

For $1 \leq x, \beta \leq m$ and $1 \leq a, b \leq n$,

$$\partial_a \partial_b V_x^q = -\sum_{q>\rho} \frac{u_{zqa}u_{zqb}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)} - \sum_{q>\rho} \frac{u_{zqb}u_{zqa}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)}$$

noting that when $x = \beta$ we have

$$\partial_a \partial_b V_x^q = -\sum_{q>\rho} \frac{u_{zqa}u_{zqb}}{(\lambda_x - \lambda_q)^2}.$$

We take $\partial_a \partial_b V_x^q = 0$ with $q > m$.

We first check these definitions are consistent with the $V_x$ being orthonormal vectors. Compute at $x_0$, for $x, \beta = 1, \ldots, m$,

$$\partial_a \left( \sum_q V_x^q V_\beta^q \right) = \sum_q (\partial_a V_x^q) V_\beta^q + \sum_q V_x^q \partial_a V_\beta^q = 0,$$

since $\partial_a V_x^q$ and $\partial_b V_\beta^q$ vanish when $q \leq m \leq \rho$. And

$$\partial_a \partial_b \left( \sum_q V_x^q V_\beta^q \right) = \sum_{q>\rho} \frac{u_{zqa}u_{zqb}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)} + \sum_{q>\rho} \frac{u_{zqb}u_{zqa}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)}$$

$$- \sum_{q>\rho} \frac{u_{zqa}u_{zqb}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)} - \sum_{q>\rho} \frac{u_{zqb}u_{zqa}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)} = 0,$$

as required.

Since $h$ has a minimum at $x_0$ we have the inequality of matrices:

$$0 \leq h_{ab} = \sum_{x=1}^m \left\{ u_{zxa} - (\lambda_2)_{ab} + 2u_{zka}(\partial_b V_x^k)(\partial_\nu V_x^\nu) + 2u_{zkb}(\partial_a V_x^k)(\partial_\nu V_x^\nu) 
+ 2u_{kla}(\partial_b V_x^k)(\partial_\nu V_x^\nu) + 2u_{kla}(\partial_a V_x^k)(\partial_\nu V_x^\nu) \right\}$$

$$= \sum_{x=1}^m \left\{ u_{zxa} - (\lambda_2)_{ab} + 4 \sum_{q>\rho} \frac{u_{zxa}u_{zqb}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)} + 2 \sum_{q>\rho} \frac{u_{zxa}u_{zqb}}{(\lambda_x - \lambda_q)} - 2 \lambda_2 \sum_{q>\rho} \frac{u_{zxa}u_{zqb}}{(\lambda_x - \lambda_q)^2} \right\}$$

$$= \sum_{x=1}^m \left\{ u_{zxa} - (\lambda_2)_{ab} + 2 \sum_{q>\rho} \frac{u_{zxa}u_{zqb}}{(\lambda_x - \lambda_q)(\lambda_\beta - \lambda_q)} \right\},$$

giving (3.4).
We can now complete the proof of Lemma 3.1, making crucial use of the convexity condition. Following [5], we observe that the convexity condition (1.2) can be written as: for every symmetric matrix \((X_{ab}) \in \text{Sym}_n(\mathbb{R})\), vector \((Z_a) \in \mathbb{R}^n\) and \(Y \in \mathbb{R}\),

\[
0 \leq F^{ab,rs} X_{ab} X_{rs} + 2 F^{ar} A^{br} X_{ab} X_{rs} + F^{x_r, s_r} Z_a Z_b - 2 F^{ab, u} X_{ab} Y - 2 F^{ab, x_r} X_{ab} Z_r + 2 F^{u, x_s} Y Z_a + F^{u, u} Y^2,
\]

(3.5)

where we are evaluating the derivatives of \(F\) at \((A, p, u, x)\) for a positive definite matrix \(A\). We are writing \(A^q\) for the \((i, j)\)th entry of the inverse matrix of \(A\). We remark that we do not require the condition (1.2) to hold for the entire space \(\text{Sym}_n^+(\mathbb{R}) \times \mathbb{R} \times B\), but only for a certain subset which depends on the solution \(u\).

**Proof of Lemma 3.1.** Observe that

\[
Q = \sum_{m=1}^{k} \sum_{x=1}^{m} \lambda_x.
\]

We will compute at a point \(x_0\) at which the \(\lambda_x\) are twice differentiable and \(D^2 u\) is diagonal with entries \(u_{ij} = \lambda_i\). From Lemma 3.3 we have

\[
F^{ab} Q_{ab} = \sum_{m=1}^{k} \sum_{x=1}^{m} F^{ab}(\lambda_x)_{ab} \\
\leq \sum_{m=1}^{k} \sum_{x=1}^{m} F^{ab} u_{xx} + 2 \sum_{m=1}^{k} \sum_{q=1}^{m} \sum_{q > \rho_m} F^{ab} u_{xq} u_{q2b} \lambda_x - \lambda_q,
\]

where we are writing \(\rho_m\) for the largest integer \(\rho_m \in \{m, m + 1, \ldots, n\}\) with the property that \(\lambda_{\rho_m} = \lambda_m\) at \(x_0\). Differentiating the equation

\[
F(D^2 u, Du, u, x) = 0,
\]

twice in the \(x_x\) direction gives

\[
0 = F^{ab} u_{ab2x} + F^{ps} u_{a2x} + F^{d} u_{xx} \\
+ F^{ab, rs} u_{abx} u_{rsx} + F^{ps, p} u_{a2x} u_{bx} + F^{u, u} u_{xx} + F^{x_x, x_x} \\
+ 2 F^{ab, p} u_{abx} u_{rx} + 2 F^{ab, u} u_{abx} u_{x} + 2 F^{ab, x_s} u_{abx} \\
+ 2 F^{ps, u} u_{ax} u_{ux} + 2 F^{ps, u} u_{ax} u_{ux} + 2 F^{u, x_x} u_{ax}.
\]

(3.6)

Hence,

\[
F^{ab} Q_{ab} \leq \sum_{m=1}^{k} \sum_{x=1}^{m} \left\{ 2 \sum_{q > \rho_m} F^{ab} u_{q2a} u_{q2b} \lambda_x - \lambda_q - \sum_{a, b, r, s > \rho_k} F^{ab, rs} u_{abx} u_{rx} - F^{u, u} u_{xx} - F^{x_x, x_x} \right\} \\
- 2 \sum_{a, b > \rho_k} F^{ab, u} u_{abx} u_{ax} - 2 \sum_{a, b > \rho_k} F^{ab, x_s} u_{abx} - 2 F^{u, x_x} u_{ax} \\
+ CQ + \sum_{i=1}^{k} b^{i,j}(\lambda_i)_{j} + \sum_{1 \leq x < \beta \leq \rho_k} c^{i, x, \beta} u_{x \beta j},
\]

for uniformly bounded \(b^{i,j}, c^{i, x, \beta}\). Here we are using Lemma 3.2 which implies that if \(1 \leq x \leq \rho_k\) then at \(x_0\),
for some $1 \leq i \leq k$ with $\lambda_i = \lambda_x$. But

$$
\sum_{1 \leq x < \beta \leq \rho_k} c_{x, \beta}^i u_{\beta x} \leq C \sum_{1 \leq x < \beta \leq \rho_k} (\lambda_x - \lambda_{\beta}) + 2 \sum_{1 \leq x < \beta \leq \rho_k, \lambda_x \neq \lambda_{\beta}} F_{\alpha \beta} \frac{U_{x \alpha \beta} U_{\beta x}}{\lambda_x - \lambda_{\beta}}
$$

\leq CQ + 2 \sum_{x=1}^{k-1} \sum_{\rho_x < q \leq \rho_k} \frac{F_{\alpha \beta} U_{q \alpha \beta} U_{q x}}{\lambda_q - \lambda_x}

\leq CQ + 2 \sum_{m=1}^{k-1} \sum_{\rho_x < q \leq \rho_k} \sum_{\rho_m < q \leq \rho_k} \frac{F_{\alpha \beta} U_{q \alpha \beta} U_{q x}}{\lambda_q - \lambda_x},

(3.7)

where for the first line we note that if $1 \leq \alpha < \beta \leq \rho_k$ with $\lambda_x = \lambda_{\beta}$ then by Lemma 3.2 we have $u_{\alpha \beta x} = 0$.

We now apply the convexity assumption (3.5), taking for each fixed $\alpha$,

$$
X_{\alpha \beta} = \begin{cases} -u_{\alpha \beta x} & \text{if } a, b > \rho_k \\ 0 & \text{otherwise} \end{cases}, \quad Z_{\alpha} = \begin{cases} 1 & \text{if } a = \alpha \\ 0 & \text{otherwise} \end{cases}, \quad Y = u_x.

(3.8)

This gives for each $\alpha = 1, \ldots, m$,

$$
0 \leq \sum_{a, b > \rho_k} F_{\alpha \beta} u_{\alpha \beta x} u_{\beta x} + 2 \sum_{a, b > \rho_k} F_{\alpha \beta} U_{q \alpha \beta} U_{q x} + \frac{F_{\alpha \beta} U_{q \alpha \beta} U_{q x}}{\lambda_q - \lambda_x} + 2 F_{\alpha \beta} u_{\alpha x} + 2 F_{\alpha \beta} u_{\alpha x} + 2 F_{\alpha \beta} u_{\alpha x} + 2 F_{\alpha \beta} u_{\alpha x} + F_{\alpha \beta} u_{\alpha x} + F_{\alpha \beta} u_{\alpha x}.

(3.9)

Observe that

$$
2 \sum_{m=1}^{k-1} \sum_{x=1}^{m} \sum_{a, b > \rho_k} \frac{F_{\alpha \beta} U_{q \alpha \beta} U_{q x}}{\lambda_q} \leq 2 \sum_{m=1}^{k-1} \sum_{x=1}^{m} \sum_{q > \rho_k} \frac{F_{\alpha \beta} U_{q \alpha \beta} U_{q x}}{\lambda_q - \lambda_x}.

(3.10)

Combining all of the above gives

$$
F_{\alpha \beta} Q_{\alpha \beta} \leq CQ + \sum_{i=1}^{k} b^{i, j}(\lambda_i),

(3.11)

as required.

\Box

4. Proof of Theorem 1.1

We will first assume that $u \in C^4(B)$ and then remove this assumption by approximation. The quantity $Q_k = \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1$ is semi-concave and from Lemma 3.1 we have almost everywhere

$$
F_{\alpha \beta}(Q_k)_{\alpha \beta} \leq CQ_k + \sum_{i=1}^{k} b^{i, j}(\lambda_i),

$$

for $b^{i, j}$ bounded. For $\varepsilon > 0$ and a fixed $\ell \in \{1, 2, \ldots, n\}$, consider
\[ R = \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{1/2}. \]

Then by Proposition 2.2, since the map \( x \mapsto (x + \varepsilon)^{1/2} \) is increasing, Lipschitz continuous and concave for \( x \geq 0 \), we see that \( R \) is semiconcave (but note that its constant of semi-concavity will depend on \( \varepsilon \)). \( R \) is twice differentiable almost everywhere. At such a point, we compute

\[
F^{ab} R_{ab} = \frac{1}{2} \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} F^{ab}(Q_k)_{ab} \leq \frac{1}{2} \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} \left( C Q_k + \sum_{i=1}^{k} b^{ij}(\lambda_i) \right) - c \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-3/2} |DQ_k|^2
\]

for uniform constants \( C, c > 0 \) (the constant \( C \) may differ from line to line). We claim that

\[
\sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} \sum_{i=1}^{k} |D\lambda_i| \leq C \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} |DQ_k|,
\]

for a universal constant \( C \). We do this by induction on \( \ell \). The case \( \ell = 1 \) is trivial. Assume it holds for \( \ell - 1 \). We need to show that

\[
(Q_{\ell} + \varepsilon)^{-1/2} \sum_{i=1}^{\ell} |D\lambda_i| \leq C \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} |DQ_k|.
\]

But using \( Q_{\ell-1} \leq Q_\ell \), the inductive hypothesis and the definition of \( Q_\ell \) we have

\[
(Q_{\ell} + \varepsilon)^{-1/2} \sum_{i=1}^{\ell} |D\lambda_i| \leq (Q_{\ell-1} + \varepsilon)^{-1/2} \sum_{i=1}^{\ell-1} |D\lambda_i| + (Q_{\ell} + \varepsilon)^{-1/2} |D\lambda_\ell|
\]

\[
\leq C \sum_{k=1}^{\ell-1} (Q_k + \varepsilon)^{-1/2} |DQ_k| + (Q_{\ell} + \varepsilon)^{-1/2} \left( |DQ_\ell| + c \sum_{i=1}^{\ell-1} |D\lambda_i| \right)
\]

\[
\leq C \sum_{k=1}^{\ell-1} (Q_k + \varepsilon)^{-1/2} |DQ_k| + (Q_{\ell} + \varepsilon)^{-1/2} |DQ_\ell|
\]

\[
+ (Q_{\ell-1} + \varepsilon)^{-1/2} C \sum_{i=1}^{\ell-1} |D\lambda_i|
\]

\[
\leq C \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} |DQ_k|.
\]

This completes the proof of (4.1).
It follows that, for constants \(C, c > 0\) independent of \(\varepsilon\),
\[
F_{ab} R_{ab} \leq CR + C \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-1/2} |DQ_k| - c \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{-3/2} |DQ_k|^2
\]
\[
\leq CR,
\]
using the bound
\[
(Q_k + \varepsilon)^{-1/2} |DQ_k| \leq \frac{\delta}{2} (Q_k + \varepsilon)^{-3/2} |DQ_k|^2 + \frac{1}{2\delta} (Q_k + \varepsilon)^{1/2},
\]
which holds for any \(\delta > 0\).

Since \(R\) is semi-concave, the weak Harnack inequality (Proposition 2.3 above) implies that for a uniform \(q > 0\) and \(C\),
\[
||R||_{L^q(B_{1/2})} \leq C \inf_{B_{1/2}} R. \tag{4.2}
\]
In particular, the constant \(C\) is independent of \(\varepsilon\). Hence we can let \(\varepsilon \to 0\) to obtain the same estimate for \(\sum_{k=1}^{\ell} Q_k^{1/2}\). Write
\[
S = \left( \sum_{k=1}^{\ell} Q_k^{1/2} \right)^2.
\]
Now observe that for a \(C\) depending only on \(n\) we have
\[
\frac{1}{C} \lambda^{-} \leq S \leq C \lambda^{+}.
\]
We have
\[
||S||_{L^{q/2}(B_{1/2})}^{1/2} \leq C \inf_{B_{1/2}} S^{1/2},
\]
and squaring both sides gives
\[
||S||_{L^q(B_{1/2})} \leq C \inf_{B_{1/2}} S
\]
and hence
\[
||\lambda^{-}\|_{L^{q/2}(B_{1/2})} \leq C \inf_{B_{1/2}} \lambda^{+}.
\]
This completes the proof of the theorem in the case that \(u \in C^4(B)\).

For \(u \in C^3(B)\) as in the statement of the theorem, the elliptic equation satisfied by \(u\) implies that \(u \in W^{4,p}_{loc}(B)\) for all \(p\). Fix \(p > n\) and a ball \(\tilde{B}\) with \(B_{1/2} \subset \subset \tilde{B} \subset \subset B\). Then we can find a sequence of smooth convex functions \(u^{(s)}\) in a neighborhood of \(\tilde{B}\) such that \(u^{(s)} \to u\) in \(W^{4,p}(B)\) as \(s \to \infty\). This also implies that \(u^{(s)} \to u\) in \(C^3(\tilde{B})\). We wish to apply Lemma 3.1 to each \(u^{(s)}\). Although \(u^{(s)}\) does not solve \(F(D^2 u, Du, u, x) = 0\) we see from (3.6) and \(F \in C^2\) that \(\bar{u} := u^{(s)}\) satisfies
\[
0 = \tilde{F}^{ab} \tilde{u}_{ab}x + \tilde{F}^{pa} \tilde{u}_{ax} + \tilde{F}^{u} \tilde{u}_{x}x
+ \tilde{F}^{ab,rs} \tilde{u}_{ab} \tilde{u}_{ry} + \tilde{F}^{pa,p} \tilde{u}_{ax} \tilde{u}_{by} + \tilde{F}^{u,u} \tilde{u}_{x}^2 + \tilde{F}^{x,x} \tilde{u}^2
+ 2\tilde{F}^{ab,p} \tilde{u}_{ab} \tilde{u}_{ax} + 2\tilde{F}^{ab,u} \tilde{u}_{ax} + 2\tilde{F}^{ab,x} \tilde{u}_{ax}
+ 2\tilde{F}^{pa,u} \tilde{u}_{ax} + 2\tilde{F}^{pa,x} \tilde{u}_{ax} + 2\tilde{F}^{u,u} \tilde{u}_{x} - f^{(s)},
\]

where \(f^{(s)} \to 0\) in \(L^p(\tilde{B})\) as \(s \to \infty\). Here we are using \(\tilde{F}\) to indicate that we are evaluating the derivatives of \(F\) at \(\tilde{u}\). Carrying out the rest of the proof of Lemma 3.1 with this extra term \(f^{(s)}\) gives almost everywhere on \(\tilde{B}\),

\[
\tilde{F}^{ab} \tilde{Q}_{ab} \leq CQ + \sum_{i=1}^{k} b_{i,j} (\tilde{\lambda}_{i})_j + f^{(s)},
\]

modifying \(f^{(s)}\) if necessary, and evaluating \(\tilde{\lambda}_i, \tilde{Q}\) etc. with respect to \(\tilde{u}\). Then \(\tilde{R}\) satisfies almost everywhere on \(\tilde{B}\),

\[
\tilde{F}^{ab} \tilde{R}_{ab} \leq C\tilde{R} + \frac{1}{2} e^{-1/2} f^{(s)}.
\]

Applying Proposition 2.3 we obtain

\[
||\tilde{R}||_{L^q(B_{1/2})} \leq C \inf_{B_{1/2}} \tilde{R} + C e^{-1/2} ||f^{(s)}||_{L^p(\tilde{B})}
\]

and letting \(s \to \infty\) gives

\[
||R||_{L^q(B_{1/2})} \leq C \inf_{B_{1/2}} R,
\]

for a constant \(C\) independent of \(\varepsilon\). The rest of the proof goes through as above, and this completes the proof of Theorem 1.1.

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