Polaron bubble stabilised by medium-induced three-body interactions

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Important note: This manuscript contains some errors that will be addressed in a future version. For a proper treatment of the mixed-bubble phenomenon (partial miscibility) described in this manuscript, please refer to the more recent work: arXiv/2008.05870.

Abstract
Mixing two kinds of particles that repel each other usually results in either a homogeneous mixture when the repulsion is weak, or a complete phase separation of the two kinds when their repulsion is too strong. It is shown however that there is an intermediate regime where the two kinds can coexist in their ground state as a bubble immersed in a gas of one kind. Such a situation is obtained by adding heavy repulsive impurities into a Bose-Einstein condensate. Above a certain strength of the mutual repulsion, a stable bubble of impurities and bosons can be formed, resulting from the equilibrium between the interactions induced by the bosons inside the bubble and the outside pressure from the surrounding bosons. At some particular strength, the effective interactions between the impurities consist of only three-body interactions. Finally, above a critical strength, the bosons are ejected from the bubble and the impurities collapse into a pure bubble of impurities. This phenomenon could be observed with an imbalanced mixture of ultracold atoms of different masses. Moreover, it appears possible to reach a regime where the impurities form a dense bubble of strongly-interacting particles.

1 Introduction
Mixtures of particles in the quantum regime have been a topic of research in various fields of physics, starting from experiments on liquid helium mixtures [1, 2]. With the development of ultracold atom experiments, it has been possible to realise mixtures of atoms at low temperature and study their properties with full control over their parameters such as density and interactions. Originally, the experiments focused on the more stable mixtures of bosonic atoms with repulsive interactions [3–6]. These experiments exhibited the phenomenon of phase separation for large enough interspecies repulsion, as anticipated by theoretical works based on the mean-field approximation [7–12]. More recently, there has been an interest in mixtures of bosonic atoms with attractive interactions [13–15], originally thought to be unstable, after it was discovered that they can form self-bound liquid droplets [16]. Even the properties of a single or a few particles mixed with another kind of particles constitute a challenging problem for theory. Such impurities immersed in a medium become quasi-particles known as polarons, and have been the subject of recent experimental investigations both in fermionic [17–23] and bosonic [24–30] ultracold atomic media, as well as many related theoretical works [31–73]. One compelling aspect of these polarons is their effective interactions mediated by the medium [74, 75].

So far, theoretical works have focused on the pairwise interactions induced by the medium. In this work, it is shown that the medium may also induce three-body interactions, which can have a crucial effect on the macroscopic properties of the impurities. In the case of bosonic impurities immersed in a Bose-Einstein condensate, a bubble of polarons can be formed and stabilised by these three-body interactions. This paper first gives an exact calculation of the induced two-body and three-body interactions between impurities induced by the surrounding condensate, in the specific limit of infinitely heavy impurities and perturbative interactions. In a second step, a simple mean-field theory is used to characterise the resulting state formed by the impurities. Finally, a possible implementation with ultracold atoms is discussed.

2 Mediated interactions
Let us consider a system of $N_f$ particles of mass $M$, referred to as impurities, immersed in a homogeneous gas of $N_B$ bosonic particles of mass $m$, referred to as bosons. This system is described in the second-quantisation formalism by the following Hamiltonian,

\[
H = \sum_k \epsilon_k b_k^\dagger b_k + \frac{1}{2V} \sum_{k,k',p} U_B(p) b_{k'}^\dagger b_{-p} b_{k+p}^\dagger b_{k'} + \frac{1}{2V} \sum_{k,k',p} U_I(p) c_{k'}^\dagger c_{-p} c_{k+p}^\dagger c_{k'} + \frac{1}{V} \sum_{k,k',p} U(p) b_{k'}^\dagger c_{-p}^\dagger b_{k+p} c_k b_k
\]

(1)

where $V$ is the system’s volume, $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ and $b_k$ are the kinetic energy and annihilation operator for
a boson with momentum $k$, and $\varepsilon_k = \frac{n^2 k^2}{2M}$ and $c_k$ are the kinetic energy and annihilation operator for an impurity with momentum $k$. The potential $U_B$ and $U_I$ describe the interaction between two bosons, and between two impurities, respectively, while the potential $U$ describes the interactions between a boson and an impurity. The potential $U_B$ is assumed to be repulsive to guarantee the stability of the medium of condensed bosons. Moreover, all potentials $U_B$, $U_I$, and $U$ are assumed to be in the perturbative regime satisfying the Born approximation [76], i.e. their respective scattering lengths $a_B$, $a_I$, and $a$ can be expanded in a convergent perturbative series of the form $a = a_1 + a_2 + a_3 + \ldots$ that is dominated by the first-order term $a_1 = \frac{2}{\pi \hbar v_{\text{F}}} U(0)$, where $\hbar$ is the reduced mass of the considered particles. For interactions $U(k)$ that become negligible for $k \gtrsim \Lambda$, where $\Lambda^{-1}$ corresponds to the range of the interaction (for instance, the nanometre range for neutral atoms), then $a_2 \approx -\frac{1}{\hbar^2 v_{\text{F}}^2} \Lambda$ and the Born approximation requires that the scattering length $a_1$ be much smaller than the range $\Lambda^{-1}$.

In this regime, the bosons can be treated with the Bogoliubov approach [12,77], which consists in applying the following substitution,

$$b_0 \equiv \sqrt{N_0}$$
$$b_k \equiv u_k \beta_k - v_k \beta_k^\dagger \quad \text{for } k \neq 0$$

where $N_0$ represents the macroscopic number of bosons occupying the condensate mode $k = 0$, and $\beta_k$ is the annihilation operator for bosonic quasi-particles (Bogoliubov quasi-particles), corresponding to elementary excitations of the bosonic system. The coefficients $u_k$ and $v_k$ are chosen to diagonalize the quadratic form in $\beta$ and $\beta^\dagger$ appearing in the first line of Eq. (1) when the substitution is applied. The Hamiltonian then reads $H = H^{(F)} + H^{(NF)}$ with

$$H^{(F)} = E_0 + \sum_k E_k \beta_k^\dagger \beta_k + H'$$
$$+ \sum_k (\varepsilon_k + n_B U(0)) c_k^\dagger c_k + \sum_k C_k (\beta_k^\dagger \beta_k + \beta_k \beta_k^\dagger) n_k$$
$$+ \frac{1}{2V} \sum_{k,k',p} U_I(p) c_{k-p}^\dagger c_{k'+p}^\dagger c_k c_{k'}$$

$$H^{(NF)} = \frac{1}{V} \sum_{p \neq 0 \neq k \neq 0} \sum_{\mp} \left(A_{k,p} \beta_k^\dagger \beta_p + B_{k,p} \beta_k \beta_p^\dagger + C_{k,p} \beta_{k-p}^\dagger \beta_{k+p}^\dagger + \frac{1}{2} \left(\varepsilon_k + n_0 U_B(0)\right) n_k\right)$$

where $E_0 \approx \frac{1}{2} U_B(0) V n_0^2$ and $E_k = \sqrt{\varepsilon_k + 2 n_0 U_B(0)}$ are the Bogoliubov ground-state and excitation energies, and $n_0 = N_0/V$ and $n_B = N_B/V$ are the condensate and total densities of bosons. In the perturbative limit of small $U_B$, the term $H'$, which contains orders higher than quadratic in $\beta$ and $\beta^\dagger$, may be neglected. Likewise, the condensate is almost pure, and $n_0$ may be approximated by $n_B$. In Eqs. (4-5), the notation $u_k \equiv \sum_p c_{p-k}^\dagger c_p = n_k^\dagger$ has been used, and the coefficients $A_{k,p}$, $B_{k,p}$, and $C_{k}$ are given by the following expressions,

$$A_{k,p} \equiv U(p) (v_{k+p} |v_k + v_{k+p}| v_k)$$
$$B_{k,p} \equiv U(p) (u_{k+p} |v_k)$$
$$C_k \equiv \frac{1}{V} \sqrt{N_0} U(k) (u_k - v_k)$$

where the Bogoliubov amplitudes $u_k$ and $v_k$ are given by $u_k^2 = \frac{1}{2} (\varepsilon_k + n_0 U_B(0)) + 1$ and $v_k^2 = \frac{1}{2} (\varepsilon_k + n_0 U_B(0)) - 1$.

The Hamiltonian $H^{(F)}$ of Eq. (4) corresponds to the Fröhlich polaron model [78,79], in which impurities can move in the bosonic medium and either create or absorb excitations of the medium through the term proportional to $C_k$. It is known [79] that this model can be solved exactly in the limit of static impurities, i.e. large mass $M$. Indeed, the following substitution,

$$\beta_k \equiv \beta_k - \frac{C_k}{E_k} n_k$$

formally turns $H^{(F)}$ into

$$H^{(F)} = E_0 + \sum_k E_k \phi_k^\dagger \phi_k + \sum_k (\varepsilon_k + E_p^{(F)} \phi_k^\dagger \phi_k + V(p) \phi_k^\dagger \phi_k)$$

$$+ \frac{1}{2V} \sum_{k,k',p} U_I(p) \phi_{k-p}^\dagger \phi_{k'+p}^\dagger \phi_k \phi_{k'}$$

with

$$E_p^{(F)} = n_B U(0) - n_0 \frac{1}{V} \sum_p \frac{U(p)^2 \varepsilon_p}{E_p^2}$$

$$V(p) = -2 n_0 \frac{U(p)^2 \varepsilon_p}{E_p^2}$$

Equation (10) formally represents the Hamiltonian of a system of bosonic quasi-particles (hereafter referred to as “Fröhlich quasi-particles”) with annihilation operator $\phi_k$ and another system of quasi-particles (polarons) with annihilation operator $c_k$. One can check that $\phi_k$ satisfies the canonical commutation relations $[\phi_k, \phi_q] = 0$ and $[\phi_k, \phi_q^\dagger] = \delta_{k,q}$. However, the two systems are independent only in the limit of infinite mass $M$. In this limit, the ground state of the system consists of the vacuum of Fröhlich quasi-particles and the ground state of a system of heavy polarons with self-energy $E_p^{(F)}$ and effective two-body interaction potential $V(p)$.

The Hamiltonian $H^{(NF)}$ is proportional to the boson-impurity interaction $U$, which is assumed to be weak. One may therefore treat this term as a perturbation to the exact ground state of the Fröhlich Hamiltonian $H^{(F)}$ obtained in the limit of infinite mass. Upon making the substitution of Eq. (9), $H^{(NF)}$ becomes in the vacuum of Fröhlich quasi-particles

$$H^{(NF)} = \frac{1}{V} \sum_{p \neq 0 \neq k \neq 0} \sum_{\mp} \frac{D_{k,p} C_k C_{k+p} \beta_{k-p} \beta_{k+p}^\dagger n_k n_p}{E_k E_{k+p}}$$

(13)
with \( D_{k,p} = A_{k,p} - B_{k,p} - B_{-k,-p,p} = U(p)(u_k - v_k)(u_{k+p} - v_{k+p}) \). Using the commutation relations for \( c_k \) and \( c_{k'}^\dagger \), one arrives at the perturbed Hamiltonian:

\[
(H) = E_0 + \sum_k (\varepsilon_k + E_p) c_k^\dagger c_k
\]
\[
+ \frac{1}{2V} \sum_{k,k',p} (U_{1}(p) + V(p)) c_{k+p}^\dagger c_{k'-p} c_k^\dagger c_k
\]
\[
+ \frac{1}{6V^2} \sum_{p,q} W(q,p) \sum_{k,k',k''} c_{k+p+q}^\dagger c_{k'-q} c_{k''}^\dagger c_{k''-p} c_k^\dagger c_k
\]
(14)

where \( E_P = E_{P}^{(F)} + E_{P}^{(NF)} \) and \( V = V^{(F)} + V^{(NF)} \) with

\[
E_{P}^{(NF)} = \frac{n_0}{V^2} \sum_{p,k} U(p)U(k)U(s)f_k f_s,
\]
(15)

\[
V^{(NF)}(p) = 2n_0 \frac{U(p)}{V} \sum_q U(q)U(s)(2f_p + f_q),
\]
(16)

\[
W(q,p) = 2n_0 U(p)U(q)U(s) (f_q f_s + f_s f_p + f_p f_q),
\]
(17)

and \( s \equiv p + q \) and \( f_k = \epsilon_k / E_k^2 \).

The expressions for \( E_P, V, \) and \( W \) are exact in the limit of a perturbative interaction \( U \). The perturbative nature of these expressions is apparent as they can be easily represented as perturbative diagrams up to third order in \( U \) - see Fig. 1. Interestingly, the non-Fröhlich part of the Hamiltonian not only modifies the self-energy and induced two-body force between impurities, but also gives rise to a three-body interaction \( W \) between impurities. This three-body interaction may be interpreted as the leading-order three-body process in which an impurity creates a Bogoliubov excitation out of the condensate, which scatters onto a second impurity and is annihilated back to the condensate by a third impurity, as shown in Fig. 1.

Let us consider an interaction \( U(k) \) that becomes negligible for \( k \gtrsim \Lambda \). Then the range \( \Lambda^{-1} \) is much smaller than the coherence length of the bosons \( \xi = (8\pi n_0 a_B)^{-1/2} \), and at distances larger than \( \Lambda^{-1} \), one can write \( E_P, V, \) and \( W \) in real space as follows:

\[
E_P = \frac{2\pi \hbar^2}{m} n_B \left( a_1 + a_2 + a_3 + \frac{a_1^2 + 2a_1a_2 + a_3^2}{\xi^2} \right)
\]
(18)

\[
\tilde{V}(r) = \frac{4\pi \hbar^2}{m} n_B \left( -\frac{a_1^2 + 2a_1a_2}{\xi^2} f(r) + \frac{a_3^2}{\xi} f(r)^2 \right)
\]
(19)

\[
\tilde{W}(x, y) = \frac{4\pi \hbar^2}{m} n_B a_1^3 \left( f(z)f(y) + f(x)f(y) + f(z)f(x) \right)
\]
(20)

where \( \tilde{\xi} = \xi / \sqrt{2}, z = |x - y|, \) and \( f(r) = \frac{f(r)}{r/\sqrt{\xi}} \).

The above expressions are of course not applicable to non-Born interactions such as contact interactions \( (\Lambda \rightarrow \infty, \text{ for which } a_1, a_2, a_3 \rightarrow 0) \). One can nevertheless easily generalise these expressions to the case of non-Born interactions, by completing the first terms of the Born expansion of \( a \) (i.e. performing the summation of ladder diagrams in the language of perturbation theory). One arrives at:

\[
E_P = \frac{2\pi \hbar^2}{m} n_B a \left( 1 + a/\tilde{\xi} + (a/\tilde{\xi})^2 \right)
\]
(21)

\[
\tilde{V}(r) = \frac{4\pi \hbar^2}{m} n_B \left( -a/\tilde{\xi} f(r) + (a/\tilde{\xi})^2 f(r)^2 \right)
\]
(22)

\[
\tilde{W}(x, y) = \frac{4\pi \hbar^2}{m} n_B a^3 \left( f(z)f(y) + f(x)f(y) + f(z)f(x) \right)
\]
(23)

It can be checked that Eq. (21) is indeed consistent with the result \( E_P = \frac{2\pi \hbar^2}{m} n_B a (1 - \sqrt{2a/\xi})^{-1} \) obtained by the coherent ansatz of Ref. [47] and to second order in \( a \) with the diagrammatic result of Ref. [39]. In the rest of this paper, we will restrict our consideration to the low-density limit \( \Lambda^{-1}, a \ll \tilde{\xi}, \) which gives identical results for both Born and non-Born interactions. In this situation, the term proportional to \( f(r)^2 \) in the two-body potential of Eqs. (19) and (22) becomes negligible, and one retrieves the well-known Yukawa attraction mediated between two impurities by a bosonic medium [12, 57, 58, 74, 75].

3 Mean-field theory

An important observation is that the three-body potential, being of third order in \( a \), can be either attractive or repulsive depending of the sign of \( a \), whereas the two-body potential, being of second order, is always attractive irrespective of the sign of \( a \). In the case of a repulsive interaction \( a > 0 \), there is therefore a competition between the induced two-body attraction and three-body repulsion. In this situation, it is tempting to think to that this competition could stabilise the impurities into a liquid state of a certain density [80]. However, this is possible only if the impurities remain mixed with the bosonic medium. On the contrary, the repulsion between the impurities and the bosons tends
to separate them, reducing the strength of the mediated interactions.

To understand this interplay, let us consider the case of bosonic impurities. So far we have obtained an essentially exact form of the Hamiltonian in a state corresponding to the vacuum of Fröhlich quasiparticles. The wave function for the impurities remains to be specified to obtain a variational ansatz for the Hamiltonian. Let us choose the Hartree-Fock ansatz in which all the impurities occupy the $k = 0$ mode. In this state, the Hamiltonian of Eq. (14) gives the following mean-field energy:

$$E = \frac{1}{2} g_B n_B^2 + g_n n_B N_I + \frac{g_1 + g_2}{2} N_I^2 + \frac{g_3}{6V^2} N_I^3 \tag{24}$$

where $g_B = U_B(0)$, $g = U(0)$, $g_1 = U_I(0)$, $g_2 = V(0) = -g^2/g_B$ and $g_3 = W(0,0) = 3g^2/(2g_B n_B)$ represent the low-energy coupling constants of the system. In contrast, the usual mean-field theory, on which previous theoretical works [7–12] are based, is obtained by taking a Hartree-Fock ansatz for the original Hamiltonian (1), resulting in Eq. (24) with $g_2 = g_3 = 0$. As we shall see, the mean-field energy of Eq. (24) can give a lower variational upper bound of the exact ground-state energy.

It should be noted that within the Hartree-Fock ansatz, the coupling constants $g_2$ and $g_3$ are obtained in the Born approximation. However, the induced two- and three-body interactions $V$ and $W$ may or may not be perturbative, depending on the density of the bosons $n_B$ and the mass $M$ of the impurities. The Born expansion of $g_2$ and $g_3$ reads:

$$g_2 = V(0) - \frac{1}{V} \sum_p V(p)^2 + \ldots$$

$$= \frac{\pi \hbar^2 a_B^2}{m a_B} + \frac{\pi \hbar^2 M a_B^4}{2m M a_B^4} + \ldots \tag{25}$$

$$g_3 = W(0,0) - \frac{1}{V^2} \sum_{k,p} \frac{k^2}{m} (k + p/2)^2 + \frac{\hbar^2 a_B^4}{M m \lambda^2} + \ldots \tag{26}$$

For the two-body and three-body potentials to sat-
isfy the Born approximation, it is thus required that
\[ \alpha_2 = \frac{M}{m^2} \sqrt{\frac{\pi a_n}{a_B}} \ll 1 \quad \text{and} \quad \alpha_3 = \frac{4}{3} \frac{M}{m} n_B a_0^3 \ll 1. \]  
(27)

In addition to these requirements, the validity of the above mean-field energy is limited by the two-body and three-body diluteness conditions,
\[ \lambda_2 \equiv n_I (L_2)^3 \ll 1 \quad \text{and} \quad \lambda_3 \equiv n_I (L_3)^3 \ll 1 \]  
(28)

where \( n_I = N_I / V \) is the density of impurities, \( L_2 = (g_I + g_2)/(4\pi \hbar^2/M) \) is the effective two-body interaction length, and \( L_3 = [g_3/(4\pi \hbar^2/M)]^{1/4} \) is the three-body interaction length. In any case, however, the variational nature of the mean-field energy Eq. (24) ensures that it remains an upper bound of the exact ground-state energy.

The mean-field energy Eq. (24) has been written assuming that the whole system is homogeneous. This discards the possibility of the impurities forming a separate phase. We thus need to make a more general ansatz, in which the \( N_I \) impurities occupy a volume \( V' \) within the total volume \( V \). Among the total number of bosons \( N_B \), a certain number \( N_B' \) may permeate in-between the impurities inside the volume \( V' \), forming a density \( n_B' = N_B' / V' \). Let us define the reduced energy \( \tilde{E} = (E - \frac{2}{3} g_B V V_B')/(g_B N_I) \) as the energy difference between the total energy \( E \) and the energy of the pure homogeneous Bose gas of density \( n_B \), normalised by the interaction energy \( g_B N_I \).

In the limit of large volume \( V \), the reduced energy of the completely segregated phase containing a pure bubble of impurities \( (n_B' = 0) \) is simply:
\[ \tilde{E}_{\text{pure}} = \frac{1}{2} v + \frac{1}{2v} \]  
(29)

where \( v = \frac{a_B}{n_B} \) is a reduced volume and \( \chi = \frac{a_B}{a} \) is a parameter characterising the relative strengths of interactions. One can easily find the minimum \( \tilde{E}_{\text{pure}} = \sqrt{\chi} \) at \( v = \sqrt{\chi} \) [10]. On the other hand, in the mixed phase \( n_B' = f n_B \) where there is inside the impurity volume \( V' \) a nonzero fraction \( f \) of the outside boson density, the reduced energy is given by:
\[ \tilde{E}_{\text{mixed}} = \frac{1}{2} v (1 - f)^2 + f + \frac{1}{2} (\chi - 1) + \frac{1}{4v^2 f} \]  
(30)

For \( \chi \geq 2 \), minimising this energy with respect to \( f \) and \( v \) gives \( \tilde{E}_{\text{mixed}} = 1 \), with \( f = 1 \) and \( v \to \infty \), corresponding to a completely mixed phase where impurities spread into the condensate and form a uniform mixture. For \( \chi < 2 \), however, a non trivial solution of finite volume is found corresponding to a bubble of polaron, i.e. impurities mixed with bosons. This bubble has a lower energy than a pure bubble of impurities because of the induced two-body attraction. Moreover, its stability results from the equilibrium between the induced three-body repulsion and the pressure from the outside bosons. At the particular value \( \chi = 1 \), the induced two-body interaction exactly cancels the effects of the direct repulsion between impurities, and the polaron forms a Bose gas interacting only with three-body interactions. Finally, at the critical value \( \chi_c = 9/16 = 0.5625 \), the energy of the polaron bubble becomes equal to \( E_{\text{pure}} \).

At this point, it is energetically more favourable for the polaron to collapse into a bubble of impurities devoid of any bosons, about twice smaller than the polaron bubble. This scenario is represented in Fig. 2.

It should be noted that in this simple mean-field theory, the kinetic energies of the bosons and impurities are neglected and densities are assumed to be uniform inside and outside the volume \( V' \). Taking into account the kinetic energies would smooth the densities near the bubble surface and add a small surface tension that would determine the bubble’s geometry - a sphere in the present case of a homogeneous Bose gas. Apart from these effects, the kinetic energies may be neglected for a sufficiently large number of impurities.

Indeed, the contribution of the kinetic energies to \( \tilde{E} \) is of the order of \( A_B - (N_I \pi a_0 / \xi)^{-1} \). Although \( a/\xi \) is assumed to be small, a mesoscopic number of impurities can easily make \( A_B \ll 1 \).

It is also worthwhile to point out that in this limit, the polaron bubble cannot exist as a metastable state for \( \chi \leq \chi_c \). Indeed, the activation energy for any local fluctuation in the polaron bubble to nucleate a region free of bosons is solely due to the kinetic energy cost of reducing the density of bosons to zero in that region, which is proportional to \( A_B \). For \( A_B \ll 1 \), there is therefore almost no energy cost, and the polaron bubble should quickly collapse to an impurity bubble. The smallness of \( A_B \) is also required to guarantee that the range \( \xi / \sqrt{J} \) of mediated interactions remains smaller than the average spacing between the impurities.

4 Ultracold atomic mixtures

Finally, let us consider how this physics could be observed with ultracold atoms. The basic requirements are a mixture of heavy and light atoms (for instance \( m/M = 0.05 \)) and a small ratio \( a/\xi \) (for instance 0.0001). One should then specify the ratio \( g_B/g \). A small ratio ensures that the impurities remain a minority among the bosons (for instance \( g_B/g = 0.2 \), giving \( n_I \approx 0.2 n_B' \)). Near the three-body-dominated regime \( \chi \approx 1 \) \((v \approx 2, f \approx 0.6)\), these values lead to the boson diluteness parameter \( n_B a_0^3 \approx 4 \times 10^{-12} \), the impurity 2-body and 3-body diluteness parameters \( \lambda_2 \approx 5 \times 10^{-5} (\chi - 1)^3 \) and \( \lambda_3 \approx 0.1 \). The Born conditions of Eq. (27) are also satisfied with \( \alpha_2 \approx 3 \times 10^{-3} \) and \( \alpha_3 \approx 2 \times 10^{-6} \). Taking a typical condensate density \( n_B = 10^{15} \text{cm}^{-3} \), and the lightest boson available (lithium-7), one finds \( A_B \approx 0.3 a_0, a \approx 3 a_0, \) and \( a_1 \approx 150 a_0 \), where \( a_0 \) designates the Bohr radius \( a_0 \approx 5.29 \times 10^{-11} \text{m} \). These are moderate values of atomic scattering lengths, for which atomic losses should remain small. Requiring \( A_B \) to be smaller than 0.01 sets the number of impurities to \( n_I \sim 10^6 \), which also corresponds to a realistic number. The formation of a bubble could be observed by imaging the impurity
density, as in previous experiments, and the transition from the impurity bubble to the polaron bubble could be evidenced by probing the change of boson density inside the bubble, for instance by shining a light causing inelastic collisions between bosons and impurities\(^1\).

It should be pointed out that the energy difference \(\Delta = (E_{\text{mixed}} - E_{\text{pure}})/N_I\) between the polaron bubble and the impurity bubble is only about 14 nK near \(\chi \approx 1\). Although it would seem that a very low temperature \(k_B T \ll \Delta\) is needed to observe the two phases distinctly, previous experiments [3, 5] have shown that such tiny interaction energies are enough to drive observable transitions even at temperatures of the order of 100 nK. Interestingly, it is possible to increase the energy difference by considering a higher ratio \(a/\xi\), for instance \(a/\xi = 0.01\), giving \(\Delta \approx 300\) nK. In this case, the polaron bubble becomes strongly interacting with \(\lambda_2 \approx 0.5(\chi - 1)^3\) and \(\lambda_3 \approx 1.0\), along with \(\alpha_2 \approx 0.35\) and \(\alpha_3 \approx 0.02\). Although the mean-field theory is not quantitative any more in this regime, its variational nature indicates that a dense bubble of polarons dominated by strong three-body interactions should exist near \(\chi = 1\). These strong three-body interactions may indeed overcome the inelastic three-body processes naturally occurring in ultracold atomic clouds. To account for these inelastic processes, the three-body coupling constant \(g_3\) may be generalised to a complex number \((4\pi\hbar^2/M) \times (L_4^3 - iL_4^3)\). Since all scattering lengths \(a_B, a,\) and \(L_2\) remain small near \(\chi = 1\), the three-body loss rates \(K = (\hbar/M)L_4^3\) should take an off-resonant value, typically \(\sim 10^{-28}\) cm\(^6\) s\(^{-1}\) [12], corresponding to \(L_3 \sim 10\) nm. On the other hand, \(L_3 \sim 100\) nm, which shows that elastic three-body collisions may dominate inelastic ones in this regime.

As a concrete illustration, the expected phases of a mixture of lithium-7 and caesium-133 atoms (in their respective ground hyperfine states) are shown in Fig. 3, as a function of applied magnetic field. The figure is obtained for \(N_I = 10^5\) caesium atoms, and a lithium density \(n_B \sim 10^{12}\) cm\(^{-3}\). The interaction parameter \(\chi = g_1g_B/g^2\) is obtained from the scattering lengths data for caesium-133 [81], lithium-7 [82], and caesium-133 – lithium-7 [83]. The polaron bubble phase is expected in a 100 gauss-wide window. Even for the relatively low density considered, the polaron bubble appears to be dense with respect to the induced three-body interactions, with \(\lambda_3\) approaching unity.

\[\text{Figure 3: Expected phases of caesium-133 atoms immersed in a condensate of lithium-7 atoms, as a function of applied magnetic field. Top panel: interaction parameter } \chi = g_1g_B/g^2. \text{ Bottom panel: various dimensionless parameters [see Eqs. (27) and (28)] characterizing the validity of the Hartree-Fock ansatz for a dilute system, obtained for a total number of caesium atoms } N_I \sim 10^5 \text{ and a density of lithium atoms } n_B \sim 10^{12} \text{ cm}^{-3}.\]

5 Conclusion

In summary, it has been shown that mixtures of bosons with repulsive interactions may not only fully mix or fully separate, but also form polaron bubbles. In contrast to usual dilute gases, these polaron bubbles can be rather dense and dominated by three-body interactions. These bubbles therefore provide a unique setting in the context of ultracold atomic gases to study strongly-interacting gases with elastic three-body interactions, a situation that has been so far out of experimental reach.

\[\text{\(^1\)This idea was suggested by Yoshiro Takahashi.}\]
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