A GENERAL FIBRE THEOREM FOR MOMENT PROBLEMS AND SOME APPLICATIONS

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Abstract. The fibre theorem [11] for the moment problem on closed semi-algebraic subsets of \( \mathbb{R}^d \) is generalized to finitely generated real unital algebras. As an application two new theorems on the rational multidimensional moment problem are proved. Another application is a characterization of moment functionals on the polynomial algebra \( \mathbb{R}[x_1, \ldots, x_d] \) in terms of extensions. Finally, the fibre theorem and the extension theorem are used to reprove basic results on the complex moment problem due to Stochel and Szafraniec [12] and Bisgaard [2].

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1. Introduction

This paper deals with the classical multidimensional moment problem. A useful result on the existence of solutions is the fibre theorem [11] for closed semi-algebraic subsets of \( \mathbb{R}^d \). The crucial assumption for this theorem is the existence of sufficiently many bounded polynomials on the semi-algebraic set. The proof given in [11] was based on the decomposition theory of states on \( \ast \)-algebras. An elementary proof has been found by T. Netzer [7], see also M. Marshall [6, Chapter 4].

In the present paper we generalize the fibre theorem from the polynomial algebra \( \mathbb{R}[x_1, \ldots, x_d] \) to arbitrary finitely generated unital real algebras and we add further statements (Theorem 3). This general result and these additional statements allow us to derive a number of new applications. The first application developed in Section 3 is about the rational moment problem on semi-algebraic subsets of \( \mathbb{R}^d \). Pioneering work on this problem was done by J. Cimpric, M. Marshall, and T. Netzer in [5]. The main assumption therein is that the preordering is Archimedean. We essentially use the fibre theorem to go beyond the Archimedean case and derive two basic results on the rational multidimensional moment problem (Theorems 7 and 8). The second application given in Section 4 concerns characterizations of moment functionals in terms of extension to some appropriate larger algebra. We prove an extension theorem (Theorem 19) that provides a necessary and sufficient condition for a linear functional on \( \mathbb{R}[x_1, \ldots, x_d] \) being a moment functional. In the case \( d = 2 \) this result leads to a theorem of J. Stochel and F. H. Szafraniec [12] on the complex moment problem (Theorem 20). A third application of the general fibre theorem is a very short proof of a theorem of T.M. Bisgaard [2] on the two-sided complex moment problem (Theorem 21).

Let us fix some definitions and notations which are used throughout this paper. A complex \( \ast \)-algebra \( \mathcal{B} \) is a complex algebra equipped with an involution, that is, an antilinear mapping \( \mathcal{B} \ni b \mapsto b^\ast \in \mathcal{B} \) satisfying \((bc)^\ast = c^\ast b^\ast \) and \((b^\ast)^\ast = b^\ast \) for \( b, c \in \mathcal{B} \). Let \( \mathcal{B}^2 \) denote the set of finite sums \( \sum_j b_j^* b_j \) of hermitian squares \( b_j^* b_j \), where \( b_j \in \mathcal{B} \). A linear functional \( L \) on \( \mathcal{B} \) is called positive if \( L(\cdot \ast \cdot) \geq 0 \) for all \( \cdot \in \mathcal{B} \).
A *-semigroup is a semigroup \( S \) with a mapping \( s^* \to s \) of \( S \) into itself, called involution, such that \((st)^* = t^*s^* \) and \((s^*)^* = s \) for \( s, t \in S \). The semigroup *-algebra \( C[S] \) of \( S \) is the complex *-algebra which is the vector space of all finite sums \( \sum_{s \in S} \alpha_s s \), where \( \alpha_s \in \mathbb{C} \), with product and involution
\[
(\sum_{s} \alpha_s s)(\sum_{t} \beta_t t) := \sum_{s,t} \alpha_s \beta_t st, \quad (\sum_{s} \alpha_s s)^* := \sum_{s} \overline{\alpha_s} s^*.
\]
The polynomial algebra \( R[x_1, \ldots, x_d] \) is abbreviated by \( R_\ast[x] \). By a finite real algebra we mean a real algebra which is finite as a real vector space.

2. A GENERALIZATION OF THE FIBRE THEOREM

Throughout this section \( A \) is a finitely generated commutative real unital algebra. By a character of \( A \) we mean an algebra homomorphism \( \chi : A \to \mathbb{R} \) satisfying \( \chi(1) = 1 \). We equip the set \( \mathcal{A} \) of characters of \( A \) with the weak topology.

Let us fix a set \( \{f_1, \ldots, f_d\} \) of generators of the algebra \( A \). There is an algebra homomorphism \( \pi : \mathbb{R}^d \to A \) such that \( \pi(x_j) = f_j, \ j = 1, \ldots, d \). If \( \mathcal{J} \) denotes the kernel of \( \pi \), then \( \mathcal{A} \) is isomorphic to the quotient algebra \( R_\ast[x]/\mathcal{J} \). Further, each character \( \chi \) is completely determined by the point \( \chi := (\chi(f_1), \ldots, \chi(f_d)) \) of \( \mathbb{R}^d \). For simplicity we will identify \( \chi \) with \( \chi_\ast \) and write \( \chi(x) := \chi(f) \) for \( f \in A \). Then \( \mathcal{A} \) becomes a real algebraic subvariety of \( \mathbb{R}^d \). In the special case \( A = \mathbb{R}[x_1, \ldots, x_d] \) we can take \( f_1 = x_1, \ldots, f_d = x_d \) and obtain \( \mathcal{A} \cong \mathbb{R}^d \).

**Definition 1.** A preordering of \( A \) is a subset \( T \) of \( A \) such that \( T \cdot T \subseteq T, \ T + T \subseteq T, \ 1 \in T, \ a^2 T \subseteq T \) for all \( a \in A \).

Let \( \sum A^2 \) denote the set of finite sums \( \sum a_i^2 \) of squares of elements \( a_i \in A \). Since \( A \) is commutative, \( \sum A^2 \) is invariant under multiplication and hence the smallest preordering of \( A \).

For a preordering \( T \) of \( A \), we define
\[
\mathcal{K}(T) = \{ x \in \hat{A} : f(x) \geq 0 \text{ for all } f \in T \}.
\]

The main concepts are introduced in the following definition, see [11] or [6].

**Definition 2.** A preordering \( T \) of \( A \) has the
- \( \bullet \) moment property (MP) if each \( T \)-positive linear functional \( L \) on \( A \) is a moment functional, that is, there exists a positive Borel measure \( \mu \in \mathcal{M}(\hat{A}) \) such that
  \[
  L(f) = \int_{\hat{A}} f(x) d\mu(x) \quad \text{for all } f \in A,
  \]
- \( \bullet \) strong moment property (SMP) if each \( T \)-positive linear functional \( L \) on \( A \) is a \( \mathcal{K}(T) \)-moment functional, that is, there is a positive Borel measure \( \mu \in \mathcal{M}(\hat{A}) \) such that \( \text{supp} \mu \subseteq \mathcal{K}(T) \) and (1) holds.

To state our main result (Theorem [5]) we need some preparations.

Suppose that \( T \) a finitely generated preordering of \( A \) and let \( f = \{f_1, \ldots, f_k\} \) be a sequence of generators of \( T \). We consider a fixed \( m \)-tuple \( h = (h_1, \ldots, h_m) \) of elements \( h_k \in A \). Let \( h(\mathcal{K}(T)) \) denote the closure of the subset \( h(\mathcal{K}(T)) \subseteq \hat{A} \) in \( \hat{A} \), where \( h(\mathcal{K}(T)) \) is defined by
\[
h(\mathcal{K}(T)) = \{(h_1(x), \ldots, h_m(x)) : x \in \mathcal{K}(T)\}.
\]
For \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^m \) we denote by \( \mathcal{K}(T)_\lambda \) the semi-algebraic set
\[
\mathcal{K}(T)_\lambda = \{ x \in \mathcal{K}(T) : h_1(x) = \lambda_1, \ldots, h_m(x) = \lambda_m \}
\]
and by \( \mathcal{T}_\lambda \) the preordering generated by the sequence
\[
f(\lambda) := \{f_1, \ldots, f_k, h_1 - \lambda_1, \lambda_1 - h_1, \ldots, h_m - \lambda_m, \lambda_m - h_m\} \]
Then \( K(T) \) is the disjoint union of fibre sets \( K(T)_{\lambda} = K(T\lambda) \), where \( \lambda \in h(K(T)) \).

Let \( I_{\lambda} \) be the ideal of \( A \) generated by \( h_1 - \lambda_1, \ldots, h_m - \lambda_m \). Then we have \( T_{\lambda} := T + I_{\lambda} \) and the preordering \( T_{\lambda}/I_{\lambda} \) of the quotient algebra \( A/I_{\lambda} \) is generated by
\[
\pi_{\lambda}(f) := \{\pi_{\lambda}(f_1), \ldots, \pi_{\lambda}(f_k)\},
\]
where \( \pi_{\lambda} : A \to A/I_{\lambda} \) denotes the canonical map.

Further, let \( \hat{I}_{\lambda} := I(\hat{Z}(I_{\lambda})) \) denote the ideal of all elements \( f \in A \) which vanish on the zero set \( \hat{Z}(I_{\lambda}) \) of \( I_{\lambda} \). Clearly, \( I_{\lambda} \subseteq \hat{I}_{\lambda} \) and \( \hat{Z}(I_{\lambda}) = \hat{Z}(\hat{I}_{\lambda}) \). Set \( \hat{T}_{\lambda} := T + \hat{I}_{\lambda} \). Then \( \hat{T}_{\lambda}/\hat{I}_{\lambda} \) is a preordering of the quotient algebra \( A/\hat{I}_{\lambda} \).

Note that in general we have \( I_{\lambda} \neq \hat{I}_{\lambda} \) and equality holds if and only if the ideal \( I_{\lambda} \) is real. The latter means that \( \sum_j a_j^2 \in I_{\lambda} \) for finitely many elements \( a_j \in A \) always implies that \( a_j \in I_{\lambda} \) for all \( j \).

**Theorem 3.** Let \( A \) be a finitely generated commutative real unital algebra and let \( T \) be a finitely generated preordering of \( A \). Suppose that \( h_1, \ldots, h_m \) are elements of \( A \) that are bounded on the set \( K(T) \). Then the following are equivalent:
(i) \( T \) has property (SMP) (resp. (MP)) in \( A \).
(ii) \( T_{\lambda} \) satisfies (SMP) (resp. (MP)) in \( A \) for all \( \lambda \in h(K(T)) \).
(iii) \( \hat{T}_{\lambda}/I_{\lambda} \) has (SMP) (resp. (MP)) in \( A/I_{\lambda} \) for all \( \lambda \in h(K(T)) \).

Proposition 4(i) gives the implication (i)\( \rightarrow \) (ii), Proposition 4(ii) yields the equivalences (ii)\( \leftrightarrow \) (iii) and (ii)\( ' \leftrightarrow \) (iii)', and Proposition 4(iii) implies equivalence (ii)\( \leftrightarrow \) (ii)'.

The main assertion of Theorem 3 is the implication (ii)\( \rightarrow \) (i). Its proof is lengthy and technically involved. In the proof given below we reduce the general case to the case \( R_d[z] \).

**Proposition 4.** Let \( I \) be an ideal and \( T \) a finitely generated preordering of \( A \). Let \( \hat{I} \) be the ideal of all \( f \in A \) which vanish on the zero set \( \hat{Z}(I) \) of \( I \).
(i) If \( T \) satisfies (SMP) (resp. (MP)) in \( A \), so does \( T + I \).
(ii) \( T + I \) satisfies (SMP) (resp. (MP)) in \( A \) if and only if \( (T + I)/I \) does in \( A/I \).
(iii) \( T + I \) obeys (SMP) (resp. (MP)) in \( A \) if and only if \( T + \hat{I} \) does.

**Proof.** (i): See e.g. [10], Proposition 4.6.
(ii) See e.g. [10], Lemma 4.5.
(iii): It suffices to show that both preorderings \( T + I \) and \( T + \hat{I} \) have the same positive characters and the same positive linear functionals on \( A \). For the sets of characters, using the equality \( \hat{Z}(I) = \hat{Z}(\hat{I}) \) we obtain
\[
K(T + I) = K(T) \cap Z(I) = K(T) \cap Z(\hat{I}) = K(T + \hat{I}).
\]

Since \( T + I \subseteq T + \hat{I} \), a \( (T + \hat{I}) \)-positive functional is trivially \( (T + I) \)-positive.

Conversely, let \( L \) be a \( (T + \hat{I}) \)-positive linear functional on \( A \) and let \( f \in \hat{I} \).

Recall that \( A \cong \mathbb{R}[x]/\mathcal{I} \) as noted above. Take \( \hat{f} \in \mathbb{R}_d[\hat{x}] \) such that \( \pi(\hat{f}) = f \).

Clearly, \( \hat{I} := \pi^{-1}(\hat{I}) \) is an ideal of \( \mathbb{R}_d[z] \) and we have \( Z(\hat{I}) = Z(\mathcal{I}) \cap Z(\hat{I}) \).

Hence the polynomial \( \hat{f} \) vanishes on \( Z(\hat{I}) \). Therefore, by the real Nullstellensatz [5, Theorem 2.2.1, (3)], there are \( m \in \mathbb{N} \) and \( g \in \sum \mathbb{R}_d[z]^2 \) such that \( p := (\hat{f})^{2m} + g \in \mathcal{I}' \). Upon multiplying \( p \) by some even power of \( \hat{f} \) we can assume that \( 2m = 2k \) for some \( k \in \mathbb{N} \). Then
\[
\pi(p) = \hat{f}^{2k} + \pi(g) \in \mathcal{I}, \quad \pi(g) \in \sum A^2.
\]

Being \( (T + I) \)-positive, \( L \) annihilates \( I \) and is nonnegative on \( \sum A^2 \). Therefore, by [3]
\[
0 = L(p) = L(\hat{f}^{2k}) + L(\pi(g)), \quad L(\pi(g)) \geq 0, \quad L(\hat{f}^{2k}) \geq 0.
\]
Hence \( L(f^2) = 0 \). Since \( L \) is nonnegative on \( \sum \mathcal{A}^2 \), the Cauchy-Schwarz inequality holds. By a repeated application of this inequality we derive
\[
|L(f)|^2^k \leq L(f^2)^{2^k-1}L(1)^{2^k-1} \leq L(f^4)^{2^k-2}L(1)^{2^k-2+2^k-1} \leq \ldots \leq L(f^{2^k})L(1)^{1+\ldots+2^k-1} = 0.
\]
Thus \( L(f) = 0 \). That is, \( L \) annihilates \( \hat{T} \). Hence \( L \) is \((T + \hat{T})\)-positive which completes the proof of (iii). \( \square \)

**Proof of the implication (ii) \( \rightarrow \) (i) of Theorem 5**

As noted at the beginning of this section there is an algebra homomorphism \( \pi : \mathbb{R}[x] \to \mathcal{A} \) such that \( \pi(x_j) = f_j, j = 1, \ldots, d, \) and \( \mathcal{A} \) is isomorphic to the quotient algebra \( R_d [x]/\mathcal{J} \), where \( \mathcal{J} \) is the kernel of \( \pi \). We choose polynomials \( \tilde{h}_j \in R_d [x] \) such that \( \pi(\tilde{h}_j) = h_j \). Clearly, \( \hat{T} := \pi^{-1}(T) \) is a preordering of \( R_d [x] \) such that \( \hat{T} = T + \mathcal{J} \) and \( K(\hat{T}) = K(T) \). Hence each polynomial \( \tilde{h}_j \) is bounded on \( K(\hat{T}) \) and \( \tilde{h}(K(\hat{T})) = h(K(T)) \). Further, \( \tilde{T}_x := \tilde{T} + I_x = T_x + J \) is also a preordering of \( R_d [x] \). By Proposition 4(ii), \( \tilde{T}_x \) resp. \( \hat{T}_x \) has \((SMP) \) (resp. \((MP) \)) in \( R_d [x] \) if and only \( T_x \) does in \( A = R_d [x]/\mathcal{J} \). Therefore, the assertion (ii) \( \rightarrow \) (i) of Theorem 5 follows from the corresponding result for the preorderings \( \tilde{T} \) and \( \tilde{T}_x \) of the algebra \( R_d [x] \) proved in 11. \( \square \)

We state the special case \( \mathcal{Q} = \sum \mathcal{A}^2 \) of Proposition 4 separately as

**Corollary 5.** If \( \mathcal{I} \) is a ideal of \( \mathcal{A} \), then \( \mathcal{I} + \sum \mathcal{A}^2 \) obeys \((MP) \) (resp. \((SMP) \)) in \( \mathcal{A} \) if and only if \( \sum(\mathcal{A}/\mathcal{I})^2 \) does in \( \mathcal{A}/\mathcal{I} \).

Our applications to the rational moment problem in Section 3 are based on the following corollary.

**Corollary 6.** Let us retain the notation of Theorem 5. Suppose that for each \( \lambda \in h(K(T)) \) there exist a finitely generated real unital algebra \( B_\lambda \) and a surjective algebra homomorphism \( \rho_\lambda : B_\lambda \to \mathcal{A}/\mathcal{I}_x \) (or \( \rho_\lambda : B_\lambda \to \mathcal{A}/\tilde{T}_x \) ) such that \( \sum B_\lambda^2 \) obeys \((MP) \) in \( B_\lambda \). Then \( T \) has \((MP) \) in \( \mathcal{A} \).

**Proof.** Assume that \( \rho_\lambda : B_\lambda \to \mathcal{A}/\mathcal{I}_x \) for \( \lambda \in h(K(T)) \). Let \( J^\lambda \) be the kernel of the homomorphism \( \rho_\lambda \). Since \( \sum B_\lambda^2 \) has \((MP) \) in \( B_\lambda \), obviously \( I^\lambda + \sum B_\lambda^2 \) has in \( B_\lambda \) and therefore \( \sum(B_\lambda/ J^\lambda)^2 \) has in \( B_\lambda/ J^\lambda \) by Corollary 5. Since the algebra homomorphism \( \rho_\lambda \) is surjective, the algebra \( B_\lambda/ J^\lambda \) is isomorphic to \( \mathcal{A}/ \mathcal{I}_x \). Hence \( \sum(\mathcal{A}/ \mathcal{I}_x)^2 \) satisfies \((MP) \) in \( \mathcal{A}/ \mathcal{I}_x \) as well. Consequently, \( \mathcal{I}/ \mathcal{I}_x \subseteq \sum(\mathcal{A}/ \mathcal{I}_x)^2 \) obeys \((MP) \) in \( \mathcal{A}/ \mathcal{I}_x \). Then \( \mathcal{T} \) has \((MP) \) by Theorem 5(iii) \( \rightarrow \) (i).

The proof under the assumption \( \rho_\lambda : B_\lambda \to \mathcal{A}/\tilde{T}_x \) is almost the same; in this case the implication (iii) \( \rightarrow \) (i) of Theorem 5 is used. \( \square \)

Theorem 5 is formulated for a commutative unital real algebra \( \mathcal{A} \). In Sections 5 and 6 we are concerned with commutative complex semigroup \( * \)-algebras. This case can be easily reduced to Theorem 5 as we discuss in what follows.

If \( \mathcal{B} \) is a commutative complex \( * \)-algebra, its hermitian part
\[
\mathcal{A} = B_h := \{ b \in \mathcal{B} : b = b^* \}
\]
is a commutative real algebra.

Conversely, suppose that \( \mathcal{A} \) is a commutative real algebra. Then its complexification \( \mathcal{B} := \mathcal{A} + i\mathcal{A} \) is a commutative complex \( * \)-algebra with involution \( (a_1 + ia_2)^* := a_1 - ia_2 \) and scalar multiplication
\[
(a + i\beta)(a_1 + ia_2) := aa_1 - \beta a_2 + i(\alpha a_2 + \beta a_1), \quad \alpha, \beta \in \mathbb{R}, a_1, a_2 \in \mathcal{A},
\]
and \( A \) is the hermitian part \( B_h \) of \( B \). Let \( b \in B \). Then we have \( b = a_1 + ia_2 \) with \( a_1, a_2 \in A \) and since \( A \) is commutative, we get
\[
(4) \quad b^*b = (a_1 - ia_2)(a_1 + ia_2) = a_1^2 + a_2^2 + i(a_1a_2 - a_2a_1) = a_1^2 + a_2^2.
\]
Hence, if \( T \) is a preorder of \( A \), then \( b^*b \, T \subseteq T \) for \( b \in B \). In particular, \( \sum B^2 = \sum A^2 \).

Further, each \( R \)-linear functional \( L \) on \( A \) has a unique extension \( \hat{L} \) to a \( C \)-linear functional on \( B \). By (4), \( \hat{L} \) is nonnegative on \( \sum A^2 \) if and only of \( \hat{L} \) is nonnegative on \( \sum B^2 \), that is, \( \hat{L} \) is a positive linear functional on \( B \).

3. Rational moment problems in \( \mathbb{R}^d \)

We begin with some notation and some preliminaries to state the main results. For \( q \in \mathbb{R}_{d}[x] \) and a subset \( D \subseteq \mathbb{R}_{d}[x] \) we put
\[
Z(q) := \{ x \in \mathbb{R}^d : q(x) = 0 \}, \quad Z(D) := \{ x \in \mathbb{R}^d : q(x) = 0 \text{ for } q \in D \}.
\]
Let \( D(\mathbb{R}_{d}[x]) \) denote the family of all multiplicative subsets \( D \) of \( \mathbb{R}_{d}[x] \) such that \( 1 \in D \) and \( 0 \notin D \). The real polynomials in a single variable \( y \) are denoted by \( \mathbb{R}[y] \).

Let \( \{ f_1, \ldots, f_k \} \) be a \( k \)-tuple of polynomials of \( R_{d}[x] \) and \( D \in D(\mathbb{R}_{d}[x]) \). Then
\[
A := D^{-1}\mathbb{R}_{d}[x]
\]
is a real unital algebra which contains \( R_{d}[x] \) as a subalgebra. Let \( T \) be the preordering of \( A \) generated by \( f_1, \ldots, f_k \). Further, we fix an \( m \)-tuple \( h = \{ h_1, \ldots, h_m \} \) of elements \( h_j \in A \). For \( \lambda \in \mathbb{R}^d \) let \( \mathcal{I}_\lambda \) be the ideal of \( A \) generated by \( h_j - \lambda_j \cdot 1 \), \( j = 1, \ldots, m \). Recall that \( h(K(T)) \) is defined by (2).

We consider the following assumptions:

(i) The functions \( h_1, \ldots, h_m \in A \) are bounded on the set \( h(K(T)) \).

(ii) For all \( \lambda \in h(K(T)) \) there is a finitely generated set \( E_\lambda \in D(\mathbb{R}[y]) \), a finite commutative unital real algebra \( C_\lambda \), and a surjective homomorphism
\[
\rho_\lambda : E_\lambda^{-1}\mathbb{R}[y] \otimes C_\lambda \to D^{-1}\mathbb{R}_{d}[x]/\mathcal{I}_\lambda \equiv A/\mathcal{I}_\lambda.
\]
The following two theorems are the main results of this section.

**Theorem 7.** Suppose that the multiplicative set \( D \in D(\mathbb{R}_{d}[x]) \) is finitely generated and assume (i) and (ii). Then the (finitely generated) real algebra \( A \) obeys (MP). For each \( T \)-positive linear functional \( L \) on \( A \) there is a positive regular Borel measure \( \mu \) on \( \hat{A} \equiv \mathbb{R}^d/\{ Z(D) \} \) such that \( L(f) = \int f_{\hat{A}} f \, d\mu \) for all \( f \in A \).

If the set \( D \) is not finitely generated, a number of technical difficulties appear: In general the algebra \( A \) is no longer finitely generated and the character space \( \hat{A} \) is not locally compact in the corresponding weak topology. Recall that the fibre theorem requires a finitely generated algebra, because it is based on the Krivine-Stengle Positivstellensatz. However, circumventing these technical problems we have the following general result concerning the multidimensional rational moment problem.

**Theorem 8.** Let \( D_0 \in D(\mathbb{R}_{d}[x]) \) and let \( \{ f_1, \ldots, f_k \} \) be a \( k \)-tuple of polynomials of \( \mathbb{R}_{d}[x] \). Suppose that for each finitely generated subset \( D \in D(\mathbb{R}_{d}[x]) \) of \( D_0 \) there exists an \( m \)-tuple \( h = \{ h_1, \ldots, h_m \} \) of elements \( h_j \in A := D^{-1}\mathbb{R}_{d}[x] \) such that \( D, A, \) and the preordering \( T \) of \( A \) generated by \( f_1, \ldots, f_k \) satisfy assumptions (i) and (ii).

Let \( T_0 \) be the preordering of the algebra \( A_0 := D_0^{-1}\mathbb{R}_{d}[x] \) generated by \( f_1, \ldots, f_k \). Then for each \( T_0 \)-positive linear functional \( L_0 \) on \( A_0 \) there exists a regular positive
Borel measure $\mu$ on $\mathbb{R}^d$ such that $f \in L^1(\mathbb{R}^d,\mu)$ and

$$L_0(f) = \int_{\mathbb{R}^d} f \, d\mu \quad \text{for all} \quad f \in \mathcal{A}_0. \quad (5)$$

The proofs of these theorems require a number of preparatory steps. The following two results for $d = 1$ are crucial and they are of interest in itself.

**Proposition 9.** Suppose that $\mathcal{E} \in \mathcal{D}(\mathcal{R}[y])$. Then for each positive linear functional $L$ on the real algebra $\mathcal{B} := \mathcal{E}^{-1} \mathcal{R}[y]$ there exists a positive regular Borel measure $\mu$ on $\mathbb{R}$ such that $f \in L^1(\mathbb{R}, \mu)$ and

$$L(f) = \int f(x) \, d\mu(x) \quad \text{for} \quad f \in \mathcal{B}. \quad (6)$$

If $\mathcal{E}$ is finitely generated, so is the algebra $\mathcal{B}$ and $\sum \mathcal{B}^2$ satisfies (MP) in $\mathcal{B}$. In this case, $\mathcal{Z}(\mathcal{E})$ is a finite set and $\mu(\mathcal{Z}(\mathcal{E})) = 0$.

The proof of Proposition 9 is given below. First we derive the following corollary.

**Corollary 10.** Suppose that $\mathcal{E} \in \mathcal{D}(\mathcal{R}[y])$ is finitely generated and $\mathcal{C}$ is a finite commutative real unital algebra. Then $\sum (\mathcal{E}^{-1} \mathcal{R}[y] \otimes \mathcal{C})^2$ obeys (MP) in the algebra $\mathcal{E}^{-1} \mathcal{R}[y] \otimes \mathcal{C}$.

**Proof.** Throughout this proof we abbreviate $\mathcal{A} := \mathcal{E}^{-1} \mathcal{R}[y] \otimes \mathcal{C}$ and $\mathcal{B} := \mathcal{E}^{-1} \mathcal{R}[y]$.

Suppose that $L$ is a positive linear functional on the algebra $\mathcal{E}^{-1} \mathcal{R}[y] \otimes \mathcal{C}$. Let $\mathcal{I}$ denote the ideal of elements of $\mathcal{C}$ which vanish on all characters of $\mathcal{C}$. Since $\mathcal{C}$ is a finite algebra, the character set $\hat{\mathcal{C}}$ is finite and each positive linear functional on $\mathcal{C}$ is a linear combination of characters, so it vanishes on $\mathcal{I}$. Let $f \in \mathcal{B}$. Clearly, $L(f^2 \otimes \cdot)$ is a positive linear functional on $\mathcal{C}$. Thus $\mathcal{C}$ has a positive linear functional, $\mathcal{C}$ is not empty, say $\hat{\mathcal{C}} = \{\eta_1, \ldots, \eta_n\}$ with $n \in \mathbb{N}$. Further, $L(f^2 \otimes e) = 0$ for $e \in \mathcal{I}$. Since the unital algebra $\mathcal{C}$ is spanned by its squares, we therefore have $L(g \otimes e) = 0$ for all $g \in \mathcal{B}$ and $e \in \mathcal{C}$. We choose elements $e_1, \ldots, e_n \in \mathcal{C}$ such that $\eta_j(e_k) = \delta_{jk}$ for $j, k = 1, \ldots, n$ and define linear functionals $L_k$ on the algebra $\mathcal{B}$ by $L_k(\cdot) = L(\cdot \otimes e_k)$. Since $\eta_j(e_k^2 - e_k) = 0$ for all $j$, we conclude that $e_k^2 - e_k \in \mathcal{I}$. Hence

$$L_k(f^2) = L((f^2 \otimes e_k) = L((f \otimes e_k)^2) \geq 0, \quad f \in \mathcal{B}.$$ 

That is, $L_k$ is a positive linear functional on $\mathcal{B}$. Therefore, by Proposition 9 there exists a positive regular Borel measure $\mu_k$ on $\mathbb{R}$ such that $L_k(f) = \int f \, d\mu_k$ for $f \in \mathcal{B}$ and $\mu_k(\mathcal{Z}(\mathcal{E})) = 0$.

From Lemma 12 below we obtain $\hat{\mathcal{B}} = \{\chi_t; t \in \mathbb{R} \backslash \mathcal{Z}(\mathcal{E})\}$. It is easily verified that $\hat{\mathcal{A}} = \cup_{t=1}^{n} \{\chi_t \otimes \eta_j; t \in \mathbb{R} \backslash \mathcal{Z}(\mathcal{E})\}$. We define a positive regular Borel measure $\mu$ on $\hat{\mathcal{A}}$ by $\mu(\sum_j M_j \otimes \eta_j) = \sum_j \mu_j(M_j)$ for Borel sets $M_j \subseteq \mathbb{R} \backslash \mathcal{Z}(\mathcal{E})$.

Let $f \in \mathcal{B}$ and $e \in \mathcal{C}$. Since the element $c - \sum_j \eta_j(e_j) e_j$ is annihilated by all $\eta_k$, it belongs to $\mathcal{I}$. Therefore, $L(f \otimes e) = \sum_j \eta_j(e_j) L(f \otimes e_j)$ and hence

$$L(f \otimes e) = \sum_{j=1}^{n} \eta_j(e_j) L(f \otimes e_j) = \sum_{j} \eta_j(e_j) \int_{\mathcal{B}} f \, d\mu_j = \sum_{j} \eta_j(e_j) \int_{\hat{\mathcal{A}}} (f \otimes e_j) \, d\mu = \int_{\hat{\mathcal{A}}} (f \otimes e) \, d\mu,$$

that is, $L$ is given by the integral with respect to the measure $\mu$. This shows that $\sum \mathcal{A}^2$ obeys (MP) in the algebra $\mathcal{A}$.

In the proof of Proposition 9 we use the following lemmas. We retain the notation established above.

\[\square\]
Lemma 11. Let \( B \) be as in Proposition 3 and let \( f = \frac{p}{q} \in B \), where \( q \in \mathcal{E} \) and \( p \in \mathbb{R}[y] \). Then \( f(y) \geq 0 \) for all \( y \in \mathbb{R} \setminus H(q) \) if and only if \( f \in \sum B^2 \).

Proof. Suppose that \( f \geq 0 \) on \( \mathbb{R} \setminus H(q) \). Then \( q^2 f = pq \geq 0 \) on \( \mathbb{R} \setminus H(q) \) and hence on the whole real line, since \( H(q) \) is empty or finite. Since each nonnegative polynomial in one \((1)\) variable is a sum of two squares in \( \mathbb{R}[y] \), we have \( pq = p_1^2 + p_2^2 \) with \( p_1, p_2 \in \mathbb{R}[y] \). Therefore, since \( q \in \mathcal{E} \), we get \( f = (\frac{p_1}{q})^2 + (\frac{p_2}{q})^2 \in \sum B^2 \). The converse implication is obvious. \( \square \)

Lemma 12. For \( t \in \mathbb{R}^d \setminus H(D) \), \( p \in \mathbb{R}_d[X] \), and \( q \in D \), we define \( \chi_t(Y) = \frac{q(t)}{q(0)} \). Then \( A = \{ \chi_t; t \in \mathbb{R}^d \setminus H(D) \} \).

Proof. First we note that for any \( t \in \mathbb{R}^d \setminus H(D) \), \( \chi_t \) is a well-defined character on \( A \), that is, \( \chi_t \in A \).

Conversely, suppose that \( \chi \in A \). Put \( t_j = \chi(x_j) \) for \( j = 1, \ldots, d \). Then \( t = (t_1, \ldots, t_d) \in \mathbb{R}^d \) and \( \chi(p(x)) = p(\chi(x_1), \ldots, \chi(x_d)) = p(t) \) for \( p \in \mathbb{R}_d[X] \). For \( q \in D \) we have \( 1 = \chi(1) = \chi(q^{-1}) = \chi(q) \chi(q^{-1}) \). Hence \( \chi(t) \neq 0 \) for all \( t \in \mathbb{R}^d \setminus H(D) \). Further, \( \chi(q^{-1}) = q(t)^{-1} \) and therefore \( \chi(q^{-1}) = p(t) q(t)^{-1} = \chi_t(Y) \). Thus \( \chi = \chi_t \). \( \square \)

Set \( E := A + C_c(\mathbb{R}^d) \). Let \( f = \frac{p}{q} \in A \), where \( p \in \mathbb{R}_d[X] \), \( q \in D \), and \( \varphi \in C_c(\mathbb{R}^d) \). Then \( f + \varphi \in E \) and we define

\[
(7) \quad f + \varphi \geq 0 \text{ if } f(x) + \varphi(x) \geq 0 \text{ for } x \in \mathbb{R}^d \setminus H(q).
\]

Since \( H(q) \) is nowhere dense in \( \mathbb{R}^d \), \( (E, \geq) \) is a real ordered vector space.

In the proofs of the following two lemmas we modify some arguments from Choquet’s approach to the moment problem based on adapted spaces, see [3] or [1].

Lemma 13. For each positive linear functional \( L \) on the ordered vector space \( (E, \geq) \) there is a regular positive Borel measure \( \mu \) on \( \mathbb{R}^d \) such that

\[
(8) \quad L(f) = \int f \, d\mu \quad \text{for } f \in A.
\]

Proof. Fix \( p \in \mathbb{R}_d[X] \) and \( q \in D \). Put \( g = \frac{p^2 + 1}{q^2} \). We define \( g \) on the whole space \( \mathbb{R}^d \) by setting \( g = +\infty \) for \( x \in \mathbb{R} \setminus H(q) \) and abbreviate \( \|x\|^2 = x_1^2 + \cdots + x_d^2 \). Let \( \epsilon > 0 \) be arbitrary. Obviously, the set

\[
K_\epsilon := \{ x \in \mathbb{R}^d : \|x\|^2 + 1 + g \leq \epsilon^{-1} \}
\]

is compact and \( q^2(x) \geq \epsilon \) for \( x \in K_\epsilon \). Hence the compact set \( K_\epsilon \) and the closed set \( \mathcal{U}(q) := \{ x \in \mathbb{R}^d : q^2(x) \leq \epsilon/2 \} \) are disjoint, so by Urysohn’s lemma there exists a function \( \eta_\epsilon \in C_c(\mathbb{R}^d) \) such that \( \eta_\epsilon = 1 \) on \( K_\epsilon \), \( \eta_\epsilon = 0 \) on \( \mathcal{U}(q) \) and \( 0 \leq \eta_\epsilon \leq 1 \) on \( \mathbb{R}^d \). Then we have \( g \eta_\epsilon \in C_c(\mathbb{R}^d) \) and

\[
(9) \quad g(x) \leq g(x) \eta_\epsilon(x) + \epsilon([\|x\|^2 + 1]g(x) + g^2(x)) \quad \text{for } x \in \mathbb{R}^d \setminus H(q).
\]

(Indeed, by the definitions of \( \eta_\epsilon \) and \( K_\epsilon \) we have \( g(x) = g(x) \eta_\epsilon(x) \) if \( x \in K_\epsilon \) and \( g(x) < \epsilon([\|x\|^2 + 1]g(x) + g^2(x)) \) if \( x \notin K_\epsilon \).)

Since the restriction of \( L \) to \( C_c(\mathbb{R}^d) \) is a positive linear functional on \( C_c(\mathbb{R}^d) \), by Riesz’ theorem there exists a positive regular Borel measure \( \mu \) on \( \mathbb{R}^d \) such that

\[
(10) \quad L(\varphi) = \int \varphi \, d\mu \quad \text{for } \varphi \in C_c(\mathbb{R}).
\]
Using that $L$ is a positive functional with respect to the order relation $\geq$ on $E$ and inequality (13) and applying (14) with $\varphi = g\eta$, we derive

$$L(g) \leq L(g\eta) + \varepsilon L((\|x\|^2 + 1)g + g^2)$$

$$= \int g\eta d\mu + \varepsilon L((\|x\|^2 + 1)g + g^2) \leq \int gd\mu + \varepsilon L((\|x\|^2 + 1)g + g^2).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $L(g) \leq \int gd\mu$.

To prove the converse inequality, let $U_\eta$ be the set of $\eta \in C_c(\mathbb{R}^d)$ such that $0 \leq \eta \leq 1$ on $\mathbb{R}^d$ and $\eta$ vanishes in some neighborhood of $Z(q)$. Since $Z(q)$ is nowhere dense in $\mathbb{R}^d$ and using again by the positivity of $L$ and (14) we obtain

$$\int gd\mu = \sup_{\eta \in U_\eta} \int g\eta d\mu = \sup_{\eta \in U_\eta} L(\eta) \leq L(g).$$

Therefore, $L(g) = \int gd\mu$.

Thus we have proved the equality in (8) for elements of the form $g = \frac{g^2 + 1}{q^2}$. Setting $p = 0$, (8) holds for all elements $\frac{g}{q}$, where $q \in \mathcal{D}$, and hence for $\frac{g}{q}$ by linearity. Since $\mathcal{A}$ is spanned by its squares, the equality (8) holds for all $f \in \mathcal{A}$. \qed

**Lemma 14.** Let $(\mu_i)_{i \in I}$ be a net of positive regular Borel measures on $\mathbb{R}^d$ which converges vaguely to a positive regular Borel measure $\mu$ on $\mathbb{R}^d$. Let $L$ is a linear functional on $\mathcal{A}$ such that $L(f) = \int f d\mu_i$ for $i \in I$ and $f \in \mathcal{A}$. Then $L(f) = \int f d\mu$ for $f \in \mathcal{A}$.

**Proof.** Let $f \in E$, $g \geq 0$ and $f = \frac{f}{\eta}$, where $q \in \mathcal{D}$. Let $U_\eta$ be as in the preceding proof. For $\eta \in U_\eta$ we have $f\eta \in C_c(\mathbb{R}^d)$ and hence lim$_i \int f\eta d\mu_i = \int f\eta d\mu$. Then

$$\int f d\mu = \sup_{\eta \in U_\eta} \int f\eta d\mu = \sup_{\eta \in U_\eta} \lim_{i} \int f\eta d\mu_i \leq \lim_{i} \int f d\mu_i = L(f),$$

that is, $f \in L^1(\mu)$. Since $E$ is spanned by its squares, $E \subseteq L^1(X; \mu)$.

Now we use notation and facts from this proof of Lemma 13 and take an element $g = \frac{g^2 + 1}{q^2}$ as therein. Setting $h = (\|x\|^2 + 1)g + g^2$, inequality (13) means that

$$g \leq g\eta + \varepsilon h.$$

Using this inequality and (11), first with $f = g$ and then with $f = h$, we derive

$$\left| L(g) - \int gd\mu \right| = L(g) - \int gd\mu = \int gd\mu - \int gd\mu = \int (g - g\eta) d\mu_i - \int (g - g\eta) d\mu + \int g\eta d\mu_i - \int g\eta d\mu$$

$$\leq \varepsilon \left( \int h d\mu_i + \int h d\mu + \int g\eta d\mu_i - \int g\eta d\mu \right)$$

$$= \varepsilon \left( L(h) + L(h) \right) + \int g\eta d\mu_i - \int g\eta d\mu.$$

Since $g\eta \in C_c(\mathbb{R}^d)$, lim$_i \int g\eta d\mu_i = \int g\eta d\mu$ by the vague convergence. Therefore, taking lim$_i$ in the preceding inequality yields $|L(g) - \int gd\mu| \leq 2\varepsilon L(h)$. Hence $L(g) = \int gd\mu$ by letting $\varepsilon \to +0$. Arguing as in the proof of Lemma 13 this implies that $L(f) = \int f d\mu$ for all $f \in \mathcal{A}$. \qed

**Proof of Proposition 9.** Set $E := B + C_c(R)$. Let $(E, \geq)$ be the real ordered vector space defined above, see (17). Let $f \in B$. By Lemma 11 we have $f \geq 0$ if and only if $f \in \sum B^2$. Hence $L(f) \geq 0$. Since $R[|y|] \subseteq B$, $\mathcal{A}$ is a cofinal subspace of the ordered vector space $(E, \geq)$. Therefore, $L$ can be extended to a positive linear
functional \( \tilde{L} \) on \((E, \geq)\). Applying Lemma \( \ref{lemma:2} \) with \( d = 1 \) and \( A \) replaced by \( B \), to the functional \( \tilde{L} \) yields \( \bigcirc \).

Suppose in addition that \( E \) is finitely generated. Let \( q_1, \ldots, q_r \) be generators of \( E \). Then the algebra \( B \) is generated by \( y, q_1^{-1}, \ldots, q_r^{-1} \), so \( B \) is finitely generated. Further, \( Z(E) = \cap_i Z(q_i) \), so the set \( Z(E) \) is finite. If \( q \in E \), then \( q^{-2} \in A \) and hence \( L(q^{-2}) = \int q^{-2} \, d\mu < \infty \). Therefore, \( \mu(Z(q)) = 0 \), so that \( \mu(Z(E)) = 0 \). Hence, by Lemma \( \ref{lemma:1} \) the integral in \( \bigcirc \) is over the set \( \tilde{A} \cong \mathbb{R} \setminus Z(E) \) and \( \mu \) is a positive regular Borel measure on \( \tilde{A} \). Thus, \( \sum B^2 \) has property (MP). \( \square \)

Remarks. 1. It is possible that there is no nontrivial positive linear functional on the algebra \( B \) in Proposition \( \ref{proposition:2} \) for instance, this happens if \( Z(E) = \mathbb{R} \).

2. Let \( B \) be a real unital algebra of rational functions in one variable and consider the following conditions:

(i) All positive linear functionals on \( B \) can be represented as integrals with respect to some positive regular Borel measure on \( \mathbb{R} \) (or on the character set \( \tilde{B} \)),

(ii) \( f \in B \) and \( \chi(f) \geq 0 \) for all \( \chi \in \tilde{B} \) implies that \( f \in \sum B^2 \).

It seems to be of interest to characterize those algebras \( B \) for which (i) or (ii) holds. By Proposition \( \ref{proposition:1} \) and Lemma \( \ref{lemma:3} \) the algebra \( \mathcal{E}^{-1} \mathbb{R}[y] \) satisfies (i) and (ii). Since \( B := \mathbb{R}[\frac{1}{y+1}, \frac{1}{y-1}] \) is isomorphic to the polynomial algebra \( \mathbb{R}[x_1, x_2] \), (i) does not hold for \( B \). For \( \mathcal{E} := \mathbb{R}[x, \frac{1}{x+1}] \) condition (i) is true, while (ii) is not fulfilled.

Proof of Theorem \( \ref{theorem:7} \). First we note that the algebra \( A \) is finitely generated, because \( D \) is finitely generated by assumption. Let us fix \( \lambda \in h(K(\mathcal{T})) \) and abbreviate \( B_\lambda := \mathcal{E}^{-1} \mathcal{E}[y] \otimes \mathcal{A}_\lambda \). Then, by Corollary \( \ref{corollary:10} \) \( \sum B_\lambda^2 \) has (MP) in the algebra \( B_\lambda \). Therefore, Corollary \( \ref{corollary:1} \) applies and shows that \( \mathcal{T} \) satisfies (MP) in \( A \). The description of \( A \) was given in Lemma \( \ref{lemma:2} \).

Proof of Theorem \( \ref{theorem:5} \). Let \( I \) denote the net of all finitely generated multiplicative subsets \( D \) of \( \mathbb{R}[y] \) such that \( D \subseteq D_0 \). Fix \( D \in I \). Since \( D \) satisfies the assumptions (i) and (ii), Theorem \( \ref{theorem:7} \) applies to \( D \) and \( L_0[A_D] \). Hence there exists a regular positive Borel measure \( \mu_D \) on \( \mathbb{R}^d \) such that \( L_0(f) = \int f \, d\mu_D \) for \( f \in A_D \). Since \( f \mapsto \int f \, d\mu_D = \mu_D(f) \in L_0(\mathbb{R}^d) = L_0(1) \) for all \( D \in I \), the set \( \{ \mu_D; D \in I \} \) is vaguely compact. Hence there is a subnet of the net \( \{ \mu_D; D \in I \} \) which converges vaguely to some regular positive Borel measure \( \mu \) on \( \mathbb{R}^d \). For notational simplicity let us assume that the net \( \{ \mu_D; D \in I \} \) itself has this property. Now we fix \( F \in I \) and apply Lemma \( \ref{lemma:1} \) to the net \( \{ \mu_D; D \in \mathcal{D} \mathcal{F} \} \), the algebra \( A_F \) and the functional \( L = L_0[A_F] \) to conclude that \( L_0(f) = \int f \, d\mu \) for \( f \in A_F \). Since each function \( f \in A_F \) is contained in some algebra \( A_{\mathcal{D} \mathcal{F}} \) with \( F \in I \), (5) is satisfied. This completes the proof.

The general fibre Theorem \( \ref{theorem:5} \) fits nicely to the multidimensional rational moment problem, because in general the corresponding algebra \( A \) contains more bounded functions on the semi-algebraic set \( K(\mathcal{T}) \) than in the polynomial case. We illustrate the use of our Theorems \( \ref{theorem:7} \) and \( \ref{theorem:5} \) by two simple examples. The ideas therein can be combined to treat more involved examples.

Example 15. First let \( d = 2 \). Suppose that \( D \) contains \( x_1 - \alpha \) and the semi-algebraic set \( K(\mathcal{T}) \) is a subset of \( \{ (x_1, x_2) : |x_1 - \alpha| \geq c \} \) for some \( \alpha \in \mathbb{R} \) and \( c > 0 \). Then \( h_1 := (x_1 - \alpha)^{-1} \) is in \( A \) and bounded on \( K(\mathcal{T}) \), so assumption (i) holds. Let \( \lambda \in h_1(K(\mathcal{T})) \). Then \( x_1 = \lambda^{-1} + \alpha \) in \( A_\lambda \), so \( A_\lambda \) consists of rational functions in \( x_2 \) with denominators from some set \( \mathcal{E}_\lambda \in \mathcal{D}(\mathbb{R}[x_2]) \). Hence assumption (ii) is satisfied with \( C_\lambda = \mathbb{C} \). Thus Theorem \( \ref{theorem:7} \) applies if \( D \) is finitely generated. Replacing \( D \) by \( D_0 \) and \( \mathcal{D} \text{ by } \mathcal{D}_0 \) in the preceding, the assertion of Theorem \( \ref{theorem:5} \) holds as well.

The above setup extends at once to arbitrary \( d \in \mathbb{N}, d \geq 2 \), if we assume that \( x_j - \alpha_j \in D \) and \( |x_j - \alpha_j| \geq c \) on \( K(\mathcal{T}) \) for some \( \alpha_j \in \mathbb{R}, c > 0 \), and \( j = \ldots, d - 1 \).
Lemma 17. The set
\[ (13) \]
\[
S \text{ of the unit sphere } x \text{ for } t \in \mathbb{R}^d \text{ evaluation at } \mathcal{C} \text{ be the quotient algebra of } \mathbb{R}[x_1] \text{ by the ideal generated by } p(x_1)^2 - \lambda - \alpha.
\]
Then \( \mathcal{A}_\lambda \) is generated by two subalgebras which are isomorphic to \( \mathcal{C} \) and \( \mathcal{E}^{-1}\mathbb{R}[x_2] \), respectively. Hence assumptions (i) and (ii) are fulfilled, so \( \sum_i \mathcal{A}_i^2 \) has (MP) by Theorem 7 if \( \mathcal{D} \) is finitely generated. In the general case Theorem 8 applies.

Remark. Assumption (ii) is needed to ensure that the fibre preorderings \( T_\lambda \) satisfy (MP). The crucial result for this is Proposition 9 which states that the assertion holds for several other algebras of rational functions as well; a sample is \( \mathbb{R}[x_1, x_2, \frac{1}{x_1^2 + x_2^2}] \). Replacing \( \mathcal{E}^{-1}\mathbb{R}[y] \) by such an algebra the fibre theorem will lead to further results on the multidimensional rational moment problem.

4. An extension theorem

In this section we derive a theorem which characterizes moment functional on \( \mathbb{R}^d \) in terms of extensions.

Throughout let \( \mathcal{A} \) denote the real algebra of functions on \( (\mathbb{R}^d)^\times := \mathbb{R}^d \setminus \{0\} \) generated by the polynomial algebra \( \mathbb{R}[x]_d^d \) and the functions
\[ (12) \quad f_{kl}(x) := x_k x_l (x_1^2 + \cdots + x_d^2)^{-1}, \text{ where } k, l = 1, \ldots, d, x \in \mathbb{R}^d \setminus \{0\}. \]

Clearly, these functions satisfy the identity
\[ (13) \quad \sum_{k,l=1}^d f_{kl}(x)^2 = 1. \]

That is, the functions \( f_{kl}, k, l = 1, \ldots, d, \) generate the coordinate algebra \( C(S^{d-1}) \) of the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \). The next lemma describes the character set \( \hat{\mathcal{A}} \) of \( \mathcal{A} \).

Lemma 17. The set \( \hat{\mathcal{A}} \) is parameterized by the disjoint union of \( \mathbb{R}^d \setminus \{0\} \) and \( S^{d-1} \). For \( x \in \mathbb{R}^d \setminus \{0\} \) the character \( \chi_x \) is the evaluation of functions at \( x \) and for \( t \in S^{d-1} \) the character \( \chi^t \) acts by \( \chi^t(f_{kl}) = f_{kl}(t) \), where \( j, k, l = 1, \ldots, d \).

Proof. It is obvious that for any \( x \in \mathbb{R}^d \setminus \{0\} \) the point evaluation \( \chi_x \) at \( x \) is a character on the algebra \( \mathcal{A} \) satisfying \( (\chi(x_1), \ldots, \chi(x_d)) \neq 0 \).

Conversely, let \( \chi \) be a character of \( \mathcal{A} \) such that \( x := (\chi(x_1), \ldots, \chi(x_d)) \neq 0 \). Then the identity \( (x_1^2 + \cdots + x_d^2) f_{kl} = x_k x_l \) implies that
\[ (\chi(x_1)^2 + \cdots + \chi(x_d)^2) \chi(f_{kl}) = \chi(x_k) \chi(x_l) \]
and therefore
\[ \chi(f_{kl}) = (\chi(x_1)^2 + \cdots + \chi(x_d)^2)^{-1} \chi(x_k) \chi(x_l) = f_{kl}(x). \]

Thus \( \chi \) acts on the generators \( x_j \) and \( f_{kl} \), hence on the whole algebra \( \mathcal{A} \), by point evaluation at \( x \), that is, we have \( \chi = \chi_x \).

Next let us note that the quotient of \( \mathcal{A} \) by the ideal generated by \( \mathbb{R}_d[x]_d^d \) is (isomorphic to) the algebra \( C(S^{d-1}) \). Therefore, if \( \chi \) is a character of \( \mathcal{A} \) such that \( (\chi(x_1), \ldots, \chi(x_d)) = 0 \), then it gives a character on the algebra \( C(S^{d-1}) \). Clearly, each character of \( C(S^{d-1}) \) comes from a point of \( S^{d-1} \). Conversely, each point \( t \in S^{d-1} \) defines a unique character of \( \mathcal{A} \) by \( \chi^t(f_{kl}) = f_{kl}(t) \) and \( \chi^t(x_j) = 0 \) for all \( k, l, j \). \( \square \)
Theorem 18. The preordering $\sum A^2$ of the algebra $A$ satisfies (MP), that is, for each positive linear functional $L$ on $A$ there exist positive Borel measures $\nu_0$ on $S^{d-1}$ and $\nu_1 \in \mathcal{M}(R^d \setminus \{0\})$ such that for all polynomials $g$ we have

\begin{equation}
L(g(x, f_{11}(x), \ldots, f_{ad}(x))) = \int_{S^{d-1}} g(0, f_{11}(t), \ldots, f_{ad}(t)) \, d\nu_0(t) + \int_{R^d \setminus \{0\}} g(x, f_{11}(x), \ldots, f_{ad}(x)) \, d\nu_1(x).
\end{equation}

Proof. It suffices to prove that $\sum A^2$ has (MP). The assertions follow then from the definition of the property (MP) and the explicit form of the character set given in Lemma 17.

From the description of $\hat{A}$ it is obvious that the functions $f_{kl}$, $k, l = 1, \ldots, d$, are bounded on $\hat{A}$, so we can take them as polynomials $h_j$ in Theorem 3. Consider a non-empty fibers for $\lambda = (\lambda_{kl})$, where $\lambda_{kl} \in R$, and let $\chi \in \hat{A}$ be such that $\chi(f_{kl}) = \lambda_{kl}$ for all $k, l$. If $\chi = \chi'$, then $\chi(x_j) = 0$ for all $j$ and hence $A/I_\lambda = C \cdot 1$ has trivially (MP). Now suppose that $\chi = \chi_x$ for some $x \in R^d \setminus \{0\}$. Then $\chi_x(f_{kl}) = f_{kl}(x) = \lambda_{kl}$. Since $1 = \sum_k f_{kk}(x) = \sum_k \lambda_{k, k}$, there is a $k$ such that $\lambda_{kk} \neq 0$. From the equality $\lambda_{kk} = f_{kk}(x) = x_k^2 + \cdots + x_d^2$ follows that $x_k \neq 0$. Hence $\frac{d}{dx_k} = \frac{f_{kk}(x)}{f_{kk}(x)} = \frac{d}{dx_k}$, so that $x_l = \frac{d}{dx_k} x_k$ for all $l = 1, \ldots, d$. This implies that the quotient algebra $A/I_\lambda$ of $A$ by the fiber ideal $I_\lambda$ is an algebra of polynomials, the single variable $x_k$. Therefore, by Hamburger’s theorem on the solution of the one-dimensional moment problem, the preordering $\sum (A/I_\lambda)^2$ satisfies (MP). Hence $A$ itself obeys (MP) by Theorem 3.

The main result of this section is the following extension theorem.

Theorem 19. A linear functional $L$ on $R_d[x]$ is a moment functional if and only if it has an extension to a positive linear functional $\mathcal{L}$ on the larger algebra $A$.

Proof. Assume first that $L$ has an extension to a positive linear functional $\mathcal{L}$ on $A$. By Theorem 18 the functional $\mathcal{L}$ on $A$ is of the form described by equation (14). We define a positive Borel measure $\mu$ on $R^d$ by

$$\mu(\{0\}) = \nu_0(S^{d-1}), \quad \mu(M \setminus \{0\}) = \nu_1(M \setminus \{0\}).$$

Let $p \in R_d[x]$. Setting $g(x, 0, \ldots, 0) = p(x)$ in the equation of Theorem 18 we get

$$L(p) = \mathcal{L}(p) = \nu_0(\{0\})p(0) + \int_{R^d \setminus \{0\}} g(x, 0, \ldots, 0) \, d\nu_1(x) = \int_{R^d} p(x) \, d\mu(x).$$

Thus $L$ is moment functional on $R_d[x]$ with representing measure $\mu$.

Conversely, suppose that $L$ is a moment functional on $R_d[x]$ and let $\mu$ be a representing measure. Since $f_{kl}(t, 0, \ldots, 0) = \delta_{k1} \delta_{l1}$ for $t \in R$, $t \neq 0$, we have $\lim_{t \to 0} f_{kl}(t, 0, \ldots, 0) = \delta_{k1} \delta_{l1}$. Hence there is a well-defined character on the algebra $A$ given by

$$\chi(f) = \lim_{t \to 0} f(t, 0, \ldots, 0), \quad f \in A,$$

and $\chi(p) = p(0)$ for $p \in R_d[x]$. Then, for $f \in R_d[x]$, we have

\begin{equation}
L(f) = \mu(\{0\}) \chi(f) + \int_{R^d \setminus \{0\}} f(x) \, d\mu(x).
\end{equation}

For $f \in A$ we define $\mathcal{L}(f)$ by the right-hand side of (16). Then $\mathcal{L}$ is a positive linear functional on $A$ which extends $L$. 

\[\square\]
Remarks. 1. The problem of characterizing moment sequence in terms of extensions has been studied in several papers such as [12], [9], and [4].

2. Another type of extension theorems has been derived in [9]. The main difference to the above theorem is that in [9], see e.g. Theorem 2.5, a function

\[ h(x) := (1 + x_1^2 + \cdots + x_n^2 + p_1(x)^2 + \cdots + p_k(x)^2)^{-1} \]

is added to the algebra, where \( p_1, \ldots, p_k \in \mathbb{R}_d[z] \) are fixed. Then \( h(x) \) is bounded on the character set and so are \( x_j h \) and \( x_j x_k h \) for \( j, k = 1, \ldots, d \). The existence assertions of the results in [9] follow also from Theorem 3. Note that in this case the representing measure for the extended functional is unique (see [9], Theorem 2.5).

2. The measure \( \nu_1 \) in Theorem [15] and the representing measure \( \mu \) for the functional \( \mathcal{L} \) in Theorem [19] are not uniquely determined by \( \mathcal{L} \). (A counter-example can be easily constructed by taking an appropriate measure supported by a coordinate axis.) Let \( \mu_{rad} \) denote the measure on \([0, +\infty)\) obtained by transporting \( \mu \) by the mapping \( x \rightarrow ||x||^2 \). Then, as shown in [8, p. 2964, Nr 2.], if \( \mu_{rad} \) is determinate on \([0, +\infty)\), then \( \mu \) is uniquely determined by \( \mathcal{L} \). Thus, Theorem [19] fits nicely to the determinacy results via disintegration of measures developed in [8, Section 8].

5. Application to the complex moment problem

Given a complex 2-sequence \( s = (s_{m,n})_{(m,n)\in \mathbb{N}_0^2} \), the complex moment problem asks when does there exist a positive Borel measure \( \mu \) on \( \mathbb{C} \) such that the function \( z^m \overline{z}^n \) on \( \mathbb{C} \) is \( \mu \)-integrable and

\[ s_{m,n} = \int \! z^m \overline{z}^n \, d\mu(z) \quad \text{for all} \quad (m,n) \in \mathbb{N}_0^2. \]

The semigroup \( \ast \)-algebra \( \mathbb{C}[\mathbb{N}_0^2] \) of the \( \ast \)-semigroup \( \mathbb{N}_0^2 \) with involution \( (m,n) := (n,m) \in \mathbb{N}_0^2 \), is the \( \ast \)-algebra \( \mathbb{C}[z, \overline{z}] \) with involution given by \( z^* = \overline{z} \). If \( L \) denotes the linear functional on \( \mathbb{C}[z, \overline{z}] \) defined by

\[ L(z^m \overline{z}^n) = s_{m,n}, \quad (m,n) \in \mathbb{N}_0^2 \]

then \( (17) \) means that

\[ L_s(p) = \int \! p(z, \overline{z}) \, d\mu(z), \quad p \in \mathbb{C}[z, \overline{z}]. \]

Clearly, \( \mathbb{N}_0^2 \) is a subsemigroup of the larger \( \ast \)-semigroup

\[ \mathcal{N}_+ = \{(m,n) \in \mathbb{Z}^2 : m + n \geq 0\} \quad \text{with involution} \quad (m,n)^* = (n,m). \]

The following fundamental theorem was proved by J. Stochel and F.H. Szafraniec [12].

**Theorem 20.** A linear functional \( L \) on \( \mathbb{C}[z, \overline{z}] \) is a moment functional if and only if \( L \) has an extension to a positive linear functional \( \mathcal{L} \) on the \( \ast \)-algebra \( \mathbb{C}[\mathcal{N}_+] \).

In [12] this theorem was stated in terms of semigroups:

A complex sequence \( s = (s_{m,n})_{(m,n)\in \mathbb{N}_0^2} \) is a moment sequence on \( \mathbb{N}_0^2 \) if and only if there exists a positive semidefinite sequence \( \tilde{s} = (\tilde{s}_{m,n})_{(m,n)\in \mathcal{N}_+} \) on the \( \ast \)-semigroup \( \mathcal{N}_+ \) such that \( \tilde{s}_{m,n} = s_{m,n} \) for all \( (m,n) \in \mathbb{N}_0^2 \).

In order to prove Theorem 20 we first describe the semigroup \( \ast \)-algebra \( \mathbb{C}[\mathcal{N}_+] \).

Clearly, \( \mathbb{C}[\mathcal{N}_+] \) is the complex \( \ast \)-algebra generated by the functions \( z^m \overline{z}^n \) on \( \mathbb{C}\setminus\{0\} \), where \( m, n \in \mathbb{Z} \) and \( m + n \geq 0 \). If \( r(z) \) denotes the modulus and \( u(z) \) the phase of \( z \), then \( z^m \overline{z}^n = r(z)^{m+n}u(z)^{m-n} \). Setting \( k = m + n \), it follows that

\[ \mathbb{C}[\mathcal{N}_+] = \text{Lin} \{ r(z)^k u(z)^{2m-k} : k \in \mathbb{N}_0, m \in \mathbb{Z} \} . \]
The functions $r(z)$ and $u(z)$ itself are not in $C[N_r]$, but $r(z)u(z) = z$ and $v(z) := u(z)^2 = z\overline{z}^{-1}$ are in $C[N_r]$ and they generate the $*$-algebra $C[N_r]$. Writing $z = x_1 + ix_2$ with $x_1, x_2 \in \mathbb{R}$, we get

$$1 + v(z) = 1 + \frac{x_1 + ix_2}{x_1 - ix_2} = \frac{\overline{x}_1^2 + x_1 x_2}{\overline{x}_1^2 + x_2^2}, \quad 1 - v(z) = 2 \frac{x_1^2 - ix_1 x_2}{x_1^2 + x_2^2}.$$

This implies that the complex algebra $C[N_r]$ is generated by the five functions

$$(18)\quad x_1, \ x_2, \ \frac{x_1^2}{x_1^2 + x_2^2}, \ \frac{x_2^2}{x_1^2 + x_2^2}, \ \frac{x_1 x_2}{x_1^2 + x_2^2}.$$

Obviously, the hermitian part $C[N_r]_h$ of the complex $*$-algebra $C[N_r]$ is just the real algebra generated by the functions (18). This real algebra is the special case $d = 2$ of the $*$-algebra $A$ treated in Section 4. Therefore, if we identify $C$ with $\mathbb{R}^2$, the assertion of Theorem 20 follows at once from Theorem 19.

6. Application to the Two-Sided Complex Moment Problem

The two-sided complex moment problem is the moment problem for the $*$-semigroup $\mathbb{Z}^2$ with involution $(m, n) := (n, m)$. Given a sequence $s = (s_{m,n})_{(m,n) \in \mathbb{Z}^2}$ it asks when there exist a positive Borel measure $\mu$ on $C^\times := \mathbb{C}\setminus\{0\}$ such that the function $z^m \overline{z}^n$ on $C^\times$ is $\mu$-integrable and

$$s_{m,n} = \int_{C^\times} z^m \overline{z}^n \, d\mu(z) \quad \text{for all } (m,n) \in \mathbb{Z}^2.$$

Note that this requires conditions for the measure $\mu$ at infinity and at zero.

The following basic result was obtained by T.M. Bisgaard [2].

**Theorem 21.** A linear functional $L$ on $C[\mathbb{Z}^2]$ is a moment functional if and only if $L$ is a positive functional, that is, $L(f^* f) \geq 0$ for all $f \in C[\mathbb{Z}^2]$.

In terms of $*$-semigroups the main assertion of this theorem says that each positive semidefinite sequence on $\mathbb{Z}^2$ is a moment sequence on $\mathbb{Z}^2$. This result is somewhat surprising, since $C^\times$ has dimension 2 and no additional condition (such as strong positivity or some appropriate extension) is required.

First we reformulate the semigroup $*$-algebra $C[\mathbb{Z}^2]$. Clearly, $C[\mathbb{Z}^2]$ is generated by the functions $z, \overline{z}, z^{-1}, \overline{z}^{-1}$ on the complex plane, that is, $C[\mathbb{Z}^2]$ is the $*$-algebra $C[z, \overline{z}, z^{-1}, \overline{z}^{-1}]$ of complex Laurent polynomials in $z$ and $\overline{z}$. A vector space basis of this algebra is $\{z^k \overline{z}^l; k, l \in \mathbb{Z}\}$. Writing $z = x_1 + ix_2$ with $x_1, x_2 \in \mathbb{R}$ we have

$$z^{-1} = \frac{x_1 - ix_2}{x_1^2 + x_2^2} \quad \text{and} \quad \overline{z}^{-1} = \frac{x_1 + ix_2}{x_1^2 + x_2^2}.$$

Hence $C[\mathbb{Z}^2]$ is the complex unital $*$-algebra generated by the hermitian functions

$$(19)\quad x_1, \ x_2, \ y_1 := \frac{x_1}{x_1^2 + x_2^2}, \ y_2 := \frac{x_2}{x_1^2 + x_2^2}$$
on $\mathbb{R}^2 \setminus \{0\}$. All four functions are unbounded on $\mathbb{R}^2 \setminus \{0\}$ and we have

$$(20)\quad (y_1 + iy_2)(x_1 - ix_2) = 1.$$

**Proof of Theorem 21:**

As above we identify $C$ and $\mathbb{R}^2$ in the obvious way. As discussed at the end of Section 2 the hermitian part of the complex $*$-algebra $C[\mathbb{Z}^2]$ is a real algebra $A$. First we determine the character set $\hat{A}$ of $A$. Obviously, the point evaluation at each point $x \in \mathbb{R}^2 \setminus \{0\}$ defines uniquely a character $\chi_x$ of $A$. From (20) it follows at once that there is no character $\chi$ on $A$ for which $\chi(x_1) = \chi(x_2) = 0$. Thus,

$$\hat{A} = \{\chi_x; x \in \mathbb{R}^2, x \neq 0\}.$$

The three functions
\[ h_1(x) = x_1y_1 = \frac{x_1^2}{x_1^2 + x_2^2}, \quad h_2(x) = x_2y_2 = \frac{x_2^2}{x_1^2 + x_2^2}, \quad h_3(x) = 2x_1y_2 = \frac{2x_1x_2}{x_1^2 + x_2^2} \]

are elements of \( \mathcal{A} \) and they are bounded on \( \hat{\mathcal{A}} \cong \mathbb{R}^2 \setminus \{0\} \). Therefore, arguing as in the proof of Theorem 18 it follows that the preorderings \( \sum (\mathcal{A}/\mathcal{I}_\lambda)^2 \) for all fibers satisfy (MP) and so does \( \sum \mathcal{A}^2 \) by Theorem 3. Since \( \hat{\mathcal{A}} \cong \mathbb{R}^2 \setminus \{0\} = \mathbb{C}^\times \), this gives the assertion. \( \square \)

**Remark.** The algebra \( \mathcal{A} \) generated by the four functions \( x_1, x_2, y_1, y_2 \) on \( \mathbb{C}^\times \) is an interesting structure: The generators satisfy the relations

\[ x_1y_1 + x_2y_2 = 1 \quad \text{and} \quad (x_1^2 + x_2^2)(y_1^2 + y_2^2) = 1 \]

and there is a \(*\)-automorphism \( \Phi \) of the real algebra \( \mathcal{A} \) (and hence of the complex \(*\)-algebra \( \mathbb{C}[\mathbb{Z}^2] \)) given by \( \Phi(x_j) = y_j \) and \( \Phi(y_j) = x_j, j = 1, 2 \).

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**References**

[1] Berg, C., Christensen, J.P.R. and P. Ressel, *Harmonic Analysis on Semigroups*, Graduate Texts in Math., Springer-Verlag, Berlin, 1984.

[2] Bisgaard, T.M., The two-sided complex moment problem, Ark. Mat. 27 (1989), 23–28.

[3] Choquet, G., *Lectures on Analysis*, vol. III, Benjamin, Reading, 1969.

[4] Cichon, C., Stochel, J. and Szafraniec, F.H., Extending positive definiteness. Trans. Amer. Math. 363 (2010), 545–577.

[5] Cimpric, J., Marshall, M. and Netzer, T., On the real multidimensional rational K-moment problem, Trans. Amer. Math. Soc. 363 (2011), 5773–5788.

[6] Marshall, M., *Positive polynomials and sums of squares*, Math. Surveys and Monographs 146, Amer. Math. Soc., 2008.

[7] Netzer, T., An elementary proof of Schmüdgen’s theorem on the moment problem of closed semi-algebraic sets, Proc. Amer. Math. Soc. 136 (2008), 529–537.

[8] Putinar, M. and Schmüdgen, K., Multivariate determinateness, Indiana Univ. Math. J. 57 (2008), 2931–2968.

[9] Putinar, M. and Vasilescu, F.-H., Solving the moment problem by dimension extension, Ann. Math. 149 (1999), 1087–1069.

[10] Scheiderer, C., Non-existence of degree bounds for weighted sums of squares representations, J. Complexity 21 (2005), 823–844.

[11] Schmüdgen, K., On the moment problem of closed semi-algebraic sets, J. Reine Angew. Math. 558 (2003), 225–234.

[12] Stochel, J. and Szafraniec, F.H., The complex moment problem and subnormality: a polar decomposition approach. J. Funct. Anal. 159 (1998), 432–491.

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