Dynamics of a self gravitating light-like shell with spherical symmetry

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Abstract

A novel Hamiltonian description of the dynamics of a spherically symmetric, light-like, self-gravitating shell is presented. It is obtained via the systematic reduction of the phase space with respect to the Gauss–Codazzi constraints, model and rare procedure in the canonical gravity. The Hamiltonian of the system (numerically equal to the value of the Arnowitt–Deser–Misner mass) is explicitly calculated in terms of the gauge-invariant ‘true degrees of freedom’, i.e. as a function on the reduced phase space. A geometric interpretation of the momentum canonically conjugate to the shell’s radius is given. Models of matter compatible with the shell dynamics are found. A transformation between the different time parameterizations of the shell is calculated. The presented model may become a new toy model of quantum gravity.

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1. Introduction

Thin shells of matter play an important role in many fields of gravitational physics. They are used as building blocks in a wide range of cosmological models [1–5] and provide a convenient laboratory for testing properties of interaction between gravitation and matter. Recently, shells of light-like matter have been used to construct a toy model of quantum gravity [6]. In particular, the quantum version of the gravitational collapse has been analyzed in this way. Most of these models possess either spherical or cylindrical symmetry, which enables us to analytically solve the dynamical equations of the system. In the quantum version of the theory, spherical symmetry ensures the existence of a well-defined ‘position’ (configuration) operator which leads to a simple description of asymptotic states [7–9].

The complete ‘quantization’ of these gravitational toy models cannot be obtained unless we are able to decipher the underlying canonical structure of the corresponding phase space.
and find the Hamiltonian function generating its dynamics as a function of appropriately chosen canonical variables. Such a reduction of the theory—an elimination of gauge degrees of freedom and a formulation of the dynamics in terms of the ‘true degrees of freedom’—was proposed, e.g. by J A Wheeler and B deWitt.

The research on a thin matter shell was started by Werner Israel. In his seminal papers [10, 11] he considered the dynamics of a self-gravitating shell of matter with the aim to find a simple model of gravitational collapse. The dynamics of such an object reduces to a proper tailoring of the two different spacetimes (so-called Israel’s junction conditions), describing the two sides of the shell. The classical equations of motion of a shell carrying self-gravitating light-like matter were derived by Barrabès and Israel twenty years later [12].

However, in many investigations—such as creating a toy model for quantum gravity—a variation principle or a proper Hamiltonian is needed. Many approaches to this problem consisted in guessing a Hamiltonian or a super-Hamiltonian of the system from its equations of motion (e.g. [13–15]). Precisely, in [13, 14] Berezin et al studied a quantum black hole modeled by a thin shell of dust matter. In this paper the equations of motion for such a shell were calculated from Israel’s junction condition, and the Hamiltonian of the system was simply set to the derived total mass (energy) of the system. In [15] Hájíček investigated the quantum gravitational collapse in a simple case of a spherically symmetric thin shell of dust matter with a fixed rest mass, interacting with its own gravitational field. Here the Hamiltonian of the system was just guessed in a form that reproduced the equations of motion for the shell.

Nonetheless, equations of motion do not uniquely determine an action or a Hamiltonian they are generated by (the so-called inverse problem of variational principle). Some attempts to derive a Hamiltonian structure in a spherically symmetric case were made [16–18], yet they based on an intermediate variational principle. Kraus and Wilczek [16] studied the back reaction in the Hawking effect, i.e. the self-gravitational interaction of the radiation and its interaction with the hole. For this purpose a complete dynamical description of a self-gravitating particle was needed, which was implied by a Hamiltonian derived from a variational principle guessed for a spherically symmetric system of the dust shell and its gravitational field. Friedman, Louko et al analyzed the Hamiltonian dynamics of a massive [17] and a null [18] spherical dust shell. Their Hamiltonian was, similarly as in [16], obtained from a postulated variational principle. In a massive case, with the interior mass fixed, Friedman and Louko’s Hamiltonian reduces to the one found by Kraus and Wilczek in the limit of a massless shell. Unfortunately the results were very complicated and gauge-dependent Hamiltonians—functions of non local momenta—were not useful to serve as a framework for quantization.

The turn in research on deriving dynamics of thin matter shells was made by Hájíček [19, 20] who looked for a simple ‘toy model’ for quantum gravity and quantum collapse. In [19] the dynamics of the spherically symmetric shell is obtained from a super-Hamiltonian on which a set of conditions was imposed. The work [20] by Hájíček and one of us (JK) contains derivation of both the super Hamiltonian and the symplectic structure for discontinuous fluid. It was obtained from the Einstein–Hilbert variational principle for ideal fluid, without assuming any additional symmetries.

We have done a considerable simplification of this theory by applying the theory of distributions (in [21] for a light-like shell and [22] for a massive shell). The work [21] on the dynamics of a self-gravitating null shell of matter is the first one using fully gauge-invariant, intrinsic geometric objects encoding physical properties of both the shell and the light-like matter living on the shell. Moreover, null matter was treated in a fully dynamical (and not phenomenological) way. All its properties were encoded in a matter Lagrangian, without assuming any of its additional properties (such as being dust or ideal fluid). In this paper we apply these results to the case of a spherically symmetric shell of null matter. The results...
obtained here provide a successful implementation of the programme started by Wheeler and deWitt, continued by Hájíček, and can be used as a simple ‘toy model’ for quantum gravity. In general, a matter shell may be considered as a singularity of the spacetime geometry arising when two different, smooth four-geometries are tailored together along a three-dimensional hypersurface \( S \). In this paper we consider the case when \( S \) is a null (‘light-like’) surface. Here, ‘tailoring’ means that the (properly understood) internal three-geometries of \( S \), induced from each of the two four-geometries, do coincide [12, 21, 23–25]. The four-dimensional connection coefficients \( \Gamma^\mu_{\nu\lambda} \) may, however, be discontinuous across \( S \). The curvature tensor density of such spacetime contains derivatives of those discontinuities and, whence, is singular. Such singularity may be nicely described in the sense of distributions as \( G^a_b = G^a_b \delta_S \), where \( \delta_S \) is the Dirac delta distribution concentrated on \( S \) and \( G^a_b \) is a smooth tensor-density field living on \( S \). Contrary to the time-like shell, where the tensor-density \( G^a_b \) may be decomposed into a product of the three-volume element on \( S \) and the tensor \( G^a_b \), here such a decomposition is meaningless because of the degeneracy of the three-metric. Hence, the ‘energy–momentum tensor’ \( G \) cannot be defined. Nevertheless, we have shown in [21] how to construct the ‘energy–momentum tensor-density’ \( G \) in a nice, geometric way. Using this quantity, the dynamics of the composed ‘null-matter-shell + gravity’ system was derived from first principles, in both the Hamiltonian and the Lagrangian approach, and for general null matter.

For this purpose an appropriate Lagrangian \( L = L_{\text{grav}} + L_{\text{matter}} \) was used, where \( L_{\text{grav}} = \frac{1}{16\pi} \sqrt{|g|} R \). Here \( R = R_{\text{reg}} + R_{\text{sing}} \) is the four-dimensional curvature scalar, \( R_{\text{reg}} \) denotes its regular part and \( \sqrt{|g|} R_{\text{sing}} = -\text{sing}(\mathcal{G}) = -G^{\mu\nu} g_{\mu\nu} \delta_S \) is its singular part.

Assuming spherical symmetry of the shell, we obtain a simple Hamiltonian system in this way. In the case of a massive shell its complete description was given in [26–28]. Here, we are going to present similar results for the case of a null shell. Similarly as was done in [26], a spherical symmetry condition is imposed at the Hamiltonian level only. (On the other hand, one could also impose this symmetry already at the level of variational (Lagrangian) formulation, as was done in [27, 28].) As a result, we obtain the Hamiltonian dynamics of the system described in terms of gauge-independent variables and their conjugate momenta.

The fundamental difference between the null and the massive case consists in the fact that there are no massive shells without matter. On the other hand, gravitational ‘shock waves’, i.e. null shells without any matter, are perfectly allowed by the theory. Nevertheless, our theory also admits null-like matter, which couples consistently to gravity. We have analyzed several models of such matter. In most cases it couples to an isolated horizon only, which is rather a trivial case. However, other interesting and self-consistent models have been found (see appendix A). They lead to non trivial dynamical systems.

The space of initial data of the ‘matter + gravity’ system can be parameterized by the following space of functions:

\[
P : = \{(g_{kl}, P^{kl}, z^K, p_K)\},
\]

where \( g_{kl} \) is a three-dimensional metric on a space-like Cauchy surface \( \mathcal{C} \), and \( P^{kl} \) are appropriate Arnowitt–Deser–Misner (ADM) momenta [29] describing the external curvature of this surface. Moreover, \( z^K \) describe configuration variables of the matter and \( p_K \) describe their conjugate momenta. We limit ourselves to the topologically trivial case \( \mathcal{C} \simeq \mathbb{R}^3 \) and assume that the geometry of \( \mathcal{C} \) is asymptotically flat at infinity.

The above phase space is equipped with the canonical pre-symplectic structure:

\[
\Omega := \frac{1}{16\pi} \int_{\mathcal{C}} \left( \delta P^{kl}(x) \wedge \delta g_{kl}(x) \right) d^3x + \int_{\mathcal{C}} \left( \delta p_K(x) \wedge \delta z^K(x) \right) d^3x.
\]
Assuming that the matter is concentrated on a shell 2-surface \( S_t = S \cap C \subset C \), we obtain

\[
\Omega := \frac{1}{16\pi} \int_C \left[ \delta P^{kl}(x) \wedge \delta g_{kl}(x) \right] d^3x + \int_{S_t} \left[ \delta \rho_K(x) \wedge \delta z^K(x) \right] d^2x.
\] (2)

The structure of the paper is following. In section 2 we analyze the above pre-symplectic form and reduce it with respect to both the spherical symmetry and the Gauss–Codazzi constraints fulfilled by the ADM data. It turns out that the radial component of the constraint equations gives us an ‘equation of state’ which has been previously postulated in [18]. The exact solution of the constraint equations is obtained in section 3. Using these results, both the gravitational part (section 4.1) and the matter part (section 4.3) of the symplectic form are reduced. Then, an exact form of the Hamiltonian of the system is derived which encodes its equations of motion. These equations are solved with respect to the Schwarzschild time in section 5. Next, we perform a transition from the Schwarzschild to the Minkowski time (section 6), and solve the resulting equations of motion. In appendix A we discuss different models of null matter.

This paper is a continuation of [21] and [30]. Most of the results presented here are contained in the PhD thesis [31] of one of us (EC).

2. Spherical symmetry

Spherical symmetry of the system implies the existence of spherical coordinates \((x^1, x^2, x^3) = (\theta, \phi, r)\) on the Cauchy surface \(C\), so that initial data \((g_{kl}, P^{kl}, z^K, p_K)\) are invariant with respect to rotations. For calculational purposes, we may choose the radial coordinate in such a way that it is constant on the shell (i.e. the history of the shell is given by the equation \(r = x^3 \equiv \zeta = \text{const}\)). This is not a physical assumption, but merely a choice of a gauge, as legal as any other gauge fixing conditions (e.g. conditions used in [20, 26, 32]). It considerably simplifies our calculations (e.g. we have \(\delta S = \Delta(x^3)\) in this coordinate system, where \(\Delta\) denotes the one-dimensional Dirac delta).

Another gauge condition which we use in this paper is the continuity of all the 10 components of the metric. In many approaches only the internal geometry of the shell is supposed to be continuous across the shell. Our condition is stronger. Again, this is not a physical restriction imposed on the theory but merely a gauge fixing condition which allows us to use the theory of distributions to calculate the (singular!) curvature tensor. In particular, our condition implies an equivalence of the affine structure of null geodesics on the shell, as seen from both sides of the shell. Without our gauge condition, this equivalence must be imposed separately (see [27, 28] for further discussion of the role of this gauge condition).

Due to spherical symmetry the two four-geometries, which we tailor together across the shell must be Schwarzschild. Moreover, due to the topological triviality of the Cauchy surface \(C\), the internal portion of the spacetime must be Minkowski.

We assume that asymptotically, for \(r \to \infty\) and \(r \to 0\), the radial coordinate is ‘asymptotically flat’, i.e. equals to the Schwarzschild radius outside and the Minkowski radius inside the shell.

Any spherically symmetric three-metric on \(C\) has the following form:

\[
g_{kl} = \begin{pmatrix}
I(r) \gamma_{AB} & 0 \\
0 & n^2(r)
\end{pmatrix},
\] (3)
where \( l \) and \( n \) are the functions of a radial coordinate \( r \), and \( \gamma_{AB} \) (where \( A, B = 1, 2 \) label angular coordinates) denotes the standard metric on the unit sphere:

\[
\gamma_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.
\] (4)

Spherical symmetry also implies that the components \( P^{kl} \) of ADM momenta assume the following form:

\[
P^{AB} = \frac{1}{2} u(r) \gamma^{AB} \sqrt{\det \gamma},
\]

\[
P^{33} = \frac{f(r)}{n(r)} \sqrt{\det \gamma},
\]

\[
P^{3A} = 0.
\] (5)

where \( u \) and \( f \) are again the functions of the radial coordinate. These functions are piecewise smooth outside the sphere \( r = \zeta \), whereas the metric coefficients \( n \) and \( l \) are also supposed to be continuous at \( r = \zeta \).

The Einstein equations imply constraints which must be satisfied by the above data, namely the Gauss–Codazzi equations for the components \( G^{0}_{\mu} \) of the Einstein tensor density. The standard decomposition of \( G^{0}_{\mu} \) into the spatial (tangent to \( C \)) part and the time-like (normal to \( C \)) part, respectively, gives us

\[
G^{0}_{l} = -P^{k}_{|k},
\] (6)

\[
2G^{0}_{\mu}n^{\mu} = -sg\left(\frac{\partial}{\partial x^{3}} P_{3i} - \frac{1}{2} P^{2}\right) \frac{1}{g},
\] (7)

where \( n \) is the future-oriented vector, orthonormal to the Cauchy surface \( C \) and \( s \) denotes the sign of \( g^{30} \). In [33] (see also [21]) we have shown how to decompose these equations into the regular and the singular (proportional to Dirac delta distribution on \( S_t = C \cap S \)) parts. The regular part of the vector constraint reads

\[
\text{reg}(P_{|k}^{k}) = 0,
\] (8)

whereas the regular part of the scalar constraint reduces to

\[
\text{reg} \left( g^{-\frac{3}{2}} R - \left( P^{ij} P_{ij} - \frac{1}{2} P^{2} \right) \frac{1}{g} \right) = 0.
\] (9)

The singular part of the divergence of the ADM momentum \( P_{kl} \) contains derivatives in the direction of \( x^3 \):

\[
s\left( P_{|k}^{i} \right) = s\left( \partial_{y} P_{i}^{3} \right) = \Delta(x^{3}) \left[ P_{i}^{3} \right],
\]

where the square brackets \([ ]\) denote the jump of \( P_{i}^{3} \) between the two sides of the singular surface. Hence, the right-hand side of (6) has the form

\[
P_{l|k}^{i} = \left[ P_{i}^{3} \Delta(x^{3}) \right].
\] (10)

In [21] we have proved that the components \( P^{kl} \) are regular, whereas the singular part of the three-dimensional curvature scalar reduces to the jump of the external curvature \( k \) of the two-dimensional surface \( S_t \subset C \) across the shell:

\[
s\left( g^{-\frac{3}{2}} R \right) = 2g \sqrt{\tilde{g}^{33}} [k] \Delta(x^{3}) = 2 \sqrt{\gamma l} [k] \Delta(x^{3}).
\]

Hence, the right-hand side of (7) has the form

\[
s\left( g^{-\frac{3}{2}} R - s \left( P^{ij} P_{ij} - \frac{1}{2} P^{2} \right) \frac{1}{g} \right) = 2 \sqrt{\gamma l} [k] \Delta(x^{3}).
\] (11)
Due to the Einstein equations, these objects must match the matter energy–momentum tensor. *A priori*, there are serious difficulties concerning the definition of such an object in case of the matter living exclusively on the null surface $S$. We have shown in [21] that such a quantity may be consistently defined. It is a three-dimensional *tensor density* denoted by $\tau^a b$ (one three-dimensional index $a = 0, 1, 2$ up and one index down!). We stress that, due to the degeneracy ($\sqrt{\det g_{ab}} = 0$) of the metric tensor $g_{ab}$ on $S$, it is impossible to find any tensor representation of this quantity. Also, the otherwise trivial ‘rising of indices’ is forbidden, whereas the ‘lowering’ becomes a non-invertible (losing information) procedure!

Naively, one could expect Gauss–Codazzi constraints in the form

$$G^0 \mu = 8\pi \tau^0 \mu.$$  

We stress, however, that there is no way to define the right-hand side as components of any well defined four-dimensional object! In our notation, where $x^3$ is constant on $S$, only $\tau^0 b$ makes sense! Fortunately, we have shown in [21] (see also [33]) that, due to the null character of $S$, the singular part of the constraint equations contains only three conditions. Indeed, the orthogonal (to $S$) part of the constraint equation coincides with one of the tangent parts of these equations, namely the component along the null vector on $S$. (The fourth constraint, existing in a non-degenerate case, is replaced here by the degeneracy condition ‘$\det g_{ab} = 0$’ for the metric on $S$.) The tangent (to $S$) part of $G^0 b$ splits into the two-dimensional part tangent to $S_t$ and the transversal part (along null rays). The first one gives us

$$\left[ P^3 b \right] = -8\pi \tau^0 b.$$  

In the spherically symmetric case both left- and right-hand side must vanish and the above constraint is fulfilled automatically. The remaining null tangent part of the Einstein equations reduces, as we have shown in [21], to the following constraint:

$$\left[ \frac{P^{33}}{\sqrt{g}} + \sqrt{\det g_{AB}} k \right] = 0.$$  

In our notation it can be rewritten in the form which has been postulated in [18]:

$$-s \frac{P}{n} + e = 0,$$

where we have denoted

$$p := \frac{1}{8\pi} \text{sing} \left( P^{33} ; 3 \right) = \frac{1}{8\pi} [P^{33}],$$

$$e := \frac{1}{16\pi} \text{sing} \left( g ; (3) R - \left( \frac{1}{2} P^{kl} P_{kl} - \frac{1}{2} \right) g \right) = \frac{1}{8\pi} \sqrt{\gamma} l[k].$$

The ‘equation of state’ (14) is a consequence of the ‘nullness’ of matter and does not come from its internal properties.

The equations (12) and (13), together with (8) and (9), provide the complete set of constraints fulfilled by the initial data $(l(r), n(r), u(r), f(r))$ of the theory.

To reduce the phase space with respect to these constraints we will proceed as in [26], but with some modification. Equation (15), together with the vector constraint (8), can be written in terms of momenta (5):

$$-8\pi s \frac{P}{n} \Delta(r - \zeta) = f' - \frac{1}{2} \frac{u}{n} l',$$

where prime denotes the radial derivative $\partial / \partial r$ and $\Delta$ is the one-dimensional Dirac delta. Equation (16), together with (9), may be rewritten in the analogous way:

$$-8\pi e(p, l, n) \Delta(r - \zeta) = \left( \frac{l'}{n} \right)' - n - \frac{1}{4} \frac{(l')^2}{ln} + \frac{1}{4} \frac{n}{l} f^2 - \frac{1}{2} fu.$$
The above constraints generate two-dimensional group of spacetime reparameterizations, where the variables \((t, r)\) may be replaced by the new variables \((\tilde{t}, \tilde{r})\), preserving the spherical symmetry of our system. Gauge transformations enter as the degeneracy directions of the symplectic structure \(P_{\text{sym}}\), obtained by the restriction of the form \(\Omega\) from \(P\) to the space of the spherically symmetric data which we denote by \(P_{\text{sym}}\). In order to calculate this restriction let us consider (2) in our specific case and integrate over angular variables \(x_1\) and \(x_2\). Hence, we obtain the following symmetric structure in \(P_{\text{sym}}\):

\[
\Omega = \int_0^\infty \left( \frac{1}{4} \delta u(r) \wedge \delta l(r) + \frac{1}{2} \delta f(r) \wedge \delta n(r) \right) dr + 4\pi \delta(P_{K}l(\zeta)) \wedge \delta z^K,
\]

where we have denoted \(p_K =: \sqrt{\det g_{AB}P_K}\). Moreover, for purely calculational reasons it is convenient to write the 2-form \(\Omega\) as an exterior derivative \(\Omega = \delta \Theta\) of the following 1-form:

\[
\Theta = \int_0^\infty \left( \frac{1}{4} u(r) \delta l(r) - \frac{1}{2} n(r) \delta f(r) \right) dr + 4\pi l(\zeta) P_K \delta z^K.
\]  

As already mentioned, the Cauchy surface \(\mathcal{C}\) is tailored from two pieces: a piece of a 3-surface in the flat Minkowski spacetime for \(r < \zeta\) and a piece of a 3-surface in the Schwarzschild spacetime for \(r > \zeta\).

3. Solution of the constraint equations

In order to solve the constraint equation we have to impose a gauge condition which enables us to uniquely fix the time coordinate. For purely technical reasons we start with the family of \(\beta\)-gauge conditions proposed in [34] in order to prove the positivity of the ADM mass. When written in the spherical symmetric case, the condition reduces to: \(\beta P^{33}g_{33} + P^{AB}g_{AB} = 0\), where \(\beta\) is a fixed constant. Hence, we have

\[
u = -\beta \frac{f}{l}.
\]  

Assume that \(\beta < -1\). Inserting the above relation to the vector constraint (17) we see that, outside the shell, our initial data must fulfill the following equation:

\[
f' + \frac{1}{2} \beta \frac{l'}{l} f = 0.
\]  

This implies that the function \(\log(f l^{\frac{\beta}{2}})\) is constant everywhere outside of the shell. Hence, there are two constants \(A_+\) and \(A_-\) such that

\[
f = \begin{cases} 
A_+ l^{\frac{\beta}{2}} & \text{for } r > \zeta, \\
A_- l^{\frac{\beta}{2}} & \text{for } r < \zeta.
\end{cases}
\]

The jump \((A_+ - A_-)\) is determined by the constraint equation (17)—i.e. the only singular term on the right-hand side of that equation is produced by the jump of \(f\) and is equal to \((A_+ - A_-)l(\zeta)^{\frac{\beta}{2}} \Delta(r - \zeta)\). Denote the physical radius of the shell by \(R\):

\[
R := \sqrt{l(\zeta)},
\]

and the normalized radial component of the momentum by \(U\):

\[
U := \frac{4\pi s p}{n(\zeta)\sqrt{l(\zeta)}} = \frac{4\pi s p}{n(\zeta)R}.
\]

Equation (17) implies

\[
A_+ - A_- = -2U R^{1+\beta}.
\]
To completely solve the constraint equation, the boundary conditions must be imposed, i.e. for $r \to \infty$ the initial data must be asymptotically flat, and for $r \to 0$ must be regular. This implies that $l$ must behave like $r^2$ both at zero and at infinity, whereas $f$ must vanish at infinity. Consequently, for $\beta < -1$, we obtain $A_\pm = 0$:

$$f = \begin{cases} 0 & \text{for } r > \zeta, \\
2UR^{1+\beta}l^{-\frac{2}{1+\beta}} & \text{for } r < \zeta. \end{cases} \ (26)$$

Now, we are going to solve the scalar constraint equation (18). Using the gauge condition (21) and the vector constraint (23), we obtain the following equation which is valid outside and inside of the shell:

$$0 = \left( \frac{l'}{n} \right)' - n - \frac{1}{4} \left( \frac{l'}{n} \right)^2 + \frac{1 + 2\beta}{4l^{1+\beta}} n A_\pm^2. \ (27)$$

Consider the following quantity:

$$k := \frac{l'}{n4\sqrt{l}}. \ (28)$$

We see that (27) may be rewritten as follows:

$$0 = \frac{\sqrt{l}}{n} k' - 1 + \frac{1 + 2\beta}{4l^{1+\beta}} A_\pm^2. \ (29)$$

or, equivalently,

$$0 = \left( k^2 - 4\sqrt{l} \left( 1 + \frac{A_\pm^2}{4l^{1+\beta}} \right) \right)'. \ (30)$$

This means that the function in the brackets must be piecewise constant. The regularity of the metric at $r = 0$ implies that for a small $r$ the function $l$ shall behave like $n^2 r^2$. This implies that $k^2$ behaves like $4\sqrt{l}$ and, consequently, the corresponding internal constant vanishes. Denoting the remaining external constant by $' -8H'$, we have, due to $A_\pm = 0$:

$$\frac{l'}{n} = \begin{cases} \pm2\sqrt{l}\sqrt{1 - \frac{2H}{\sqrt{l}}} & \text{for } r > \zeta, \\
2\sqrt{l}\sqrt{1 + \frac{A_\pm^2}{4l^{1+\beta}}} & \text{for } r < \zeta, \end{cases} \ (31)$$

where the sign ‘$\pm$’ appearing in the first line depends upon the sign of $l'$. Outside of the shell this sign may change at the points where the expression under the square root vanishes. On the contrary, inside of the shell this sign is always positive, because for a space-like hypersurface in Minkowski’s spacetime $l$ is always increasing.

The value of $H$ is equal to the ADM energy calculated at $r \to \infty$. This is implied by the fact that, for large values of $r$, we have $l(r) \sim r^2$. Consequently, due to (31), we obtain

$$g_{33} = n^2 \sim \frac{1}{1 - \frac{2H}{\sqrt{l}}}. \ (32)$$

We expect that the ADM mass $H$ will play the role of the Hamiltonian of the total ‘gravity + matter’ system (cf [35–37]). The numerical value of $H$ may be obtained from the singular part of (18). Since the singular part of the right-hand side of this equation is equal to the jump $\frac{l'}{n}$, we obtain

$$-8\pi e = 2R \left( \sqrt{1 - \frac{2H}{R}} - 1 + \frac{A_\pm^2}{4R^{2+2\beta}} \right), \ (33)$$
where $\epsilon$ denotes the sign of $l'$ outside the shell. Now we insert the value of $A_\pm$ calculated from (25) (where $A_\pm = 0$), our ‘equation of state’ (14) and the value of $p$ calculated from (24). In this way we obtain

$$
-U = \epsilon \sqrt{1 - \frac{2H}{R}} - \sqrt{1 + U^2},
$$

(34)

and, consequently

$$
H(R, U) = \frac{R}{2} \{1 - (\sqrt{1 + U^2} - U)^2\}.
$$

(35)

The value of $\epsilon$ can be obtained from (34):

$$
\epsilon = \text{sgn}(\sqrt{1 + U^2} - U) = 1.
$$

(36)

In the next section we prove that the reduced phase space for the gravitational degrees of freedom $\tilde{P}$ may be parameterized globally by the two variables $R$ and $U$, whereas the function $l$ and the constants $\zeta$, $\beta$ play the role of gauge parameters. Given the point $(R, U)$ of this space, a specific choice of the gauge parameters allows us to reconstruct the entire Cauchy data, with the values of $A_\pm$ and $H$ given uniquely by (26) and (35). However, gauge parameters are not completely arbitrary. One condition is obvious: $l(\zeta) = R^2$. Moreover, inside the shell, $l$ must increase monotonically from 0 to $R^2$. Because the sign ‘$\epsilon$’ given by (36) is positive, outside of the shell $l$ must increase and, due to the asymptotical flatness, must behave like $r^2$ at infinity. Any function $l$ is allowed provided it fulfills these conditions. Equations (31), (26) and (21) enable us to completely reconstruct the data $(n, l, f, u)$. Within our gauge subspace, given by condition (21), all states may be obtained in this way. In the following we show that the states of the system obtained for other values of $\zeta$ and $l$, are equivalent up to a gauge transformation.

For negative values of $\beta$ equation (26) implies that the entire external curvature vanishes outside the shell: $P_{\mu \nu} = 0$. Due to the fact that the external portion of spacetime is Schwarzschild with the mass $H$, the only surfaces satisfying this condition are standard surfaces $\{t = \text{const}\}$. The Cauchy surfaces corresponding to different negative values of $\beta$ coincide, therefore, outside of the shell and differ only inside of the shell.

### 4. Canonical structure of the reduced phase space

The dynamics of our system will be given uniquely by the Hamiltonian (35), if we only know the reduced symplectic structure expressed in terms of the gravitational variables $(R, U)$ and, possibly, additional matter variables.

It is possible that there is no matter at all, and the matter Lagrangian is equal to zero. Yet, the spacetime may still exhibit singularity and our ‘null shell’ becomes a shock wave of pure gravitation. The phase space is then two-dimensional and the symplectic form consists of the gravitational part only. Its construction is given below.

#### 4.1. Reduction of gravitational part of the symplectic form

We restrict the gravitational part of the symplectic form $\Omega$ given by (19) to the gauge space (21) and express it in terms of parameters $(R, U, l, \zeta)$. It turns out that this form does not depend upon $l$ and $\zeta$. This proves that the latter are, indeed, gauge variables. For technical reasons it is easier to work with the 1-form $\Theta$ given by (20), because the restriction to the gauge space commutes with the exterior derivative of the form.
Formula (20) may be rewritten as follows:

\[
\Theta = \int_0^\infty \left( \frac{1}{4} \frac{u}{n} + \beta \frac{l}{l} \right) \delta l - \frac{1}{2n} \delta l \frac{f(l\hat{z})}{l} + 4\pi R^2 P_k \delta z^K. \tag{37}
\]

Due to gauge condition (21), the first term in the above formula vanishes. Equation (26) implies

\[
f(l\hat{z}) = 2U R^{1+\beta} B(\xi - r), \tag{38}
\]

where \(B\) denotes the Heaviside function. Then

\[
\{\delta(f(l\hat{z}))\}(r) = 2B(\xi - r)\delta(U R^{1+\beta}), \tag{39}
\]

and, consequently

\[
-\frac{1}{2} \int_0^\infty n l^{-\frac{2}{3}} \{\delta(f(l\hat{z}))\}(r) \, dr = -\left\{ \int_0^\xi (n l^{-\frac{2}{3}})(r) \, dr \right\} \delta(U R^{1+\beta}), \tag{40}
\]

because of the condition \(l(\xi) = R^2\) and due to the vanishing of \(f\) outside of the shell. Finally, we obtain

\[
\Theta = -w\delta(U R^{1+\beta}) + 4\pi R^2 P_K \delta z^K, \tag{41}
\]

where \(w\) denotes

\[
w := \int_0^\xi (n l^{-\frac{2}{3}})(r) \, dr. \tag{42}
\]

Using (31) we obtain

\[
w = \int_0^\xi \frac{n}{l} l^{-\frac{2}{3}} l' \, dr = \int_0^\xi \frac{l^{\frac{1+n}{2}}}{2\sqrt{1 + \left\{ U \left( \frac{\xi}{R} \right)^{-(1+\beta)} \right\}^2}} l' \, dr. \tag{43}
\]

The function \(l\) is monotonic over the interval \([0, \xi]\). Moreover, we have \(-(1 + \beta) > 0\). Hence, we may calculate the integral with respect to the following variable:

\[
\xi := U \left( \frac{\sqrt{1 + \xi^2}}{R} \right)^{-(1+\beta)}, \tag{44}
\]

over the interval \([0, U]\). We have

\[
w = -\frac{1}{1 + \beta} \left( U R^{1+\beta} \right)^{\frac{1}{2\beta+1}} \int_0^U \xi^{\frac{1}{2\beta+1}} \sqrt{1 + \xi^2} \, d\xi. \tag{45}
\]

The last integral may be denoted as \(F(U)\), where \(F\) is the indefinite integral. It turns out that the specific form of \(F\) will not be needed. The first (gravitational) part of (41) reads

\[
\frac{1}{1 + \beta} F(U) (U R^{1+\beta})^{\frac{1}{2\beta+1}} \delta(U R^{1+\beta}) \tag{46}
\]

\[
= \frac{1}{2} F(U) \delta(U R^{1+\beta}) R^2 \tag{47}
\]

\[
= \frac{1}{2} \delta(F(U) U R^{1+\beta} R) - \frac{1}{2} R^2 U R^{1+\beta} F'(U) \delta(U) \tag{48}
\]

\[
= \frac{1}{2} \delta(F(U) U R^{1+\beta} R^2) - \frac{1}{2} R^2 \frac{1}{\sqrt{1 + U^2}} \delta(U). \tag{49}
\]
The first term in the above formula is a complete (variational) derivative. Hence, it vanishes under the exterior derivative, when we calculate the symplectic form $\Omega = \delta \Theta$. Therefore, the second term alone gives us an equally good primitive form $\tilde{\Theta}$ for $\Omega$: $\delta \Theta = \delta \tilde{\Theta} = \Omega$. Taking into account that
\[
\frac{1}{\sqrt{1 + U^2}} \delta(U) = \delta \text{arsinh } U,
\]
we obtain the following formula:
\[
\tilde{\Theta} = -\frac{1}{2} \rho \delta \mu + 4\pi \rho P^K K \delta z^K,
\]
where by $\mu$ we denote the momentum canonically conjugate to the variable $\rho := \rho^2$:
\[
\mu := \text{arsinh } U.
\]
(48)

Consequently, we have
\[
\frac{1}{2} \delta \mu \wedge \delta \rho + 4\pi \delta (\rho P^K K) \wedge \delta z^K.
\]
(50)

Finally the Hamiltonian (35) may also be expressed in terms of variables $(\mu, \rho)$:
\[
H(\mu, \rho) = \frac{1}{2} \sqrt{1 - e^{-2\mu}}.
\]
(51)

For the shock wave, when matter degrees of freedom vanish from the very beginning, the symplectic form is of the following form:
\[
\Omega = \frac{1}{2} \delta \mu \wedge \delta \rho,
\]
(52)

and the Hamiltonian is given by formula (51), with $\mu$ and $\rho$ being now canonical variables.

### 4.2. Geometric interpretation of the momentum $\mu$

We are going to prove that the quantity $\mu$ may be interpreted as a hyperbolic angle between the vector normal to the external Schwarzschild surface $\{t^{\text{Schw}} = \text{const.}\}$ and the vector normal to the internal Minkowski surface $\{t^{\text{Mink}} = \text{const.}\}$. The angle $\alpha(u, v)$ between the two normalized vectors $u, v$ is defined by their (hyperbolic) scalar product:
\[
\cosh \alpha(u, v) := (u|v).
\]
(53)

Similarly as in the Euclidean geometry, we call this quantity the angle between the two surfaces: the Schwarzschild one and the Minkowski one. To prove this interpretation of $\mu$, it is sufficient to use formula (31). On the internal side of the shell we use the second part of the formula and put $l = R^2$. We then obtain
\[
\frac{l'}{2\sqrt{l}} = \sqrt{1 + U^2}.
\]
(54)

But inside of the shell the geometry of $C$ is given by a three-dimensional spherically symmetric surface in the Minkowski space. It is a matter of straightforward calculations to prove that in the Minkowski spacetime the quantity on the left-hand side of the above equation is equal to $\cosh \alpha$, where $\alpha$ is precisely the angle between such a subspace and the Minkowski flat surface $\{t^{\text{Mink}} = \text{const.}\}$. This implies that $U = \sinh \alpha$. But our surface is a smooth extension of the external side of the Schwarzschild space $\{t^{\text{Schw}} = \text{const.}\}$. This finally proves that $\mu = \alpha$ is the angle between the leaves of the three-dimensional foliations of the Schwarzschild and the Minkowski spaces.
4.3. Reduction of material part of the symplectic form

Let us first consider a simple non-trivial Lagrangian of the spinorial (Dirac) type (see appendix A for the discussion of the properties of null matter):

\[ L = \sqrt{\det g_{AB}(z^2 \dot{z}^1 - z^1 \dot{z}^2)}. \]

It implies the second type constraints

\[ P_1 = z^2, \quad P_2 = -z^1, \]

which inserted into the symplectic form (50) reduce it to the following form:

\[ \Omega_1 = \frac{1}{2} \delta \mu \wedge \delta \rho + 8\pi \delta (\sqrt{\rho} z^2) \wedge \delta (\sqrt{\rho} z^1). \]  

(55)

Moreover, denoting

\[ p := \sqrt{8\pi \rho} z^2, \]

\[ q := \sqrt{8\pi \rho} z^1, \]

leads to the following symplectic structure

\[ \Omega = \frac{1}{2} \delta \mu \wedge \delta \rho + \delta p \wedge \delta q, \]

(58)

in a four-dimensional phase space parameterized by variables: (p, q, µ, ρ). Because the Hamiltonian (51) does not depend on the first pair of variables; hence, their evolution is trivial: they remain constant in time. And the evolution equations for ‘geometric variables’ (µ, ρ) generated by the Hamiltonian (51) are of the following form:

\[ \dot{\mu} = -\frac{1}{2} \frac{1}{\sqrt{\rho}} [1 - e^{-2\mu}], \]

(60)

the same as in the case of the gravitational shock wave described in the previous section.

It turns out that a similar reduction is valid for the generic Lagrangian dependent on the two degrees of freedom (A, 2), but, typically, the momentum µ canonically conjugated to ρ shall be modified by a function σ(p, q) giving the new momentum ν := µ − σ(p, q).

Consequently, the Hamiltonian (51) depends upon the variables p and q via µ = ν + σ(p, q) and the dynamics of the material variables is no longer trivial. We can see that after performing the reduction of the material part of the symplectic form (50) for the two degrees of freedom K = 1, 2, it has the following form:

\[ \Omega = \frac{1}{2} \delta \mu \wedge \delta \rho + 4\pi \delta (\rho P_1) \wedge \delta z^K. \]  

(61)

From (A, 2) we have that

\[ P_K = F_K(z^1, z^2); \]

(62)

hence,

\[ \Omega = 4\pi \delta \rho \wedge F_K \delta z^K + 4\pi \rho F_{K,L} \delta z^L \wedge \delta z^K + \frac{1}{2} \delta \mu \wedge \delta \rho. \]  

(63)

The assumption dF ≠ 0 implies that F_{2,1} − F_{1,2} ≠ 0. It is then either positive or negative, and the form \( F_{K,L} \delta z^L \wedge \delta z^K \) is a symplectic form in two dimensions. Besides, the Darboux theorem implies that there exists such a coordinate system \( \xi = \xi(z^1, z^2) \) and \( \eta = \eta(z^1, z^2) \) in which

\[ F_{K,L} \delta z^L \wedge \delta z^K = \delta (F_K \delta z^K) = \frac{1}{8\pi} \delta (\xi \delta \eta - \eta \delta \xi). \]  

(64)
Therefore,
\[ F_K \delta z^K = \frac{1}{8\pi} (\xi \delta \eta - \eta \delta \xi) + \frac{1}{8\pi} \delta \sigma, \]
(65)
where \( \xi = \sigma(\xi, \eta) \) is a function of variables \((\xi, \eta)\). Inserting the above two equations to formula (63) we thus obtain \( \Omega \) in the following form:
\[ \Omega = \delta(\sqrt{\rho}) \wedge \delta(\sqrt{\rho}) + \frac{1}{2} \delta(\mu - \sigma) \wedge \delta \rho. \]
(66)

We also introduce the new canonical variables
\[ p := \sqrt{\rho} \xi, \]
(67)
\[ q := \sqrt{\rho} \eta, \]
(68)
\[ \nu := \mu - \sigma \]
(69)
in order to simplify the form \( \Omega \):
\[ \Omega = \delta p \wedge \delta q + \frac{1}{2} \delta \nu \wedge \delta \rho. \]
(70)

The Hamiltonian of the system, still equal to the same expression (51), can be expressed in terms of the new variables as follows:
\[ H(\nu, \rho, p, q) = \frac{1}{2} \sqrt{\rho} \left(1 - e^{-2\nu(\xi, \eta)}\right). \]
(71)

This is a universal form of a Hamiltonian for a spherically symmetric self-gravitating shell of null matter coupled to two matter fields. The properties of any specific model of such matter are uniquely implied by the function \( \sigma \), dependent upon two material variables \((\xi, \eta)\).

This function is uniquely determined by the two functions \( F_K \) contained in the Lagrangian.

**Example.** Consider the following Lagrangian density (see appendix B):
\[ L = \sqrt{\text{det} g_{AB}} \left((z^2 + 2z^1 z^2)\dot{z}^1 + ((z^1)^2 - z^1)\dot{z}^2\right). \]

Rewriting this Lagrangian in the new variables \( \xi \) and \( \eta \) such that \( z^1 = \frac{1}{2}(\xi + \eta) \) and \( z^2 = \frac{1}{2}(\xi - \eta) \) we obtain
\[ \frac{1}{2} \sigma = (z^1)^2 z^2 = \frac{1}{8} (\xi + \eta)(\xi^2 - \eta^2). \]

The above equation expressed in terms of the canonical variables \( p = \sqrt{\rho} \xi = \sqrt{\rho}(z^1 + z^2), \)
\( q = \sqrt{\rho} \eta = \sqrt{\rho}(z^1 - z^2) \) and \( \rho \) takes the following form:
\[ \frac{1}{2} \sigma = \frac{1}{8} \rho^{-3/2} (p + q)(p^2 - q^2). \]

The Hamiltonian of the whole system reduces to
\[ H(\nu, \rho, p, q) = \frac{1}{2} \sqrt{\rho} \left(1 - e^{-2\nu(\xi, \eta)}(p+q)(p^2-q^2)\right). \]
5. Dynamics: reconstruction of spacetime geometry

The Hamiltonian (71) uniquely generates the dynamics of our system in the form of the following the Hamilton equations for the canonical variables \((\rho, \nu)\) and \((p, q)\):

\[
\frac{1}{2} \dot{\rho} = \frac{\partial H}{\partial \nu} = \sqrt{\rho} e^{-2(\nu + \sigma)}, \tag{72}
\]

\[
\frac{1}{2} \dot{\nu} = -\frac{1}{4} \frac{1}{\sqrt{\rho}} (1 - e^{-2(\nu + \sigma)}) - \sqrt{\rho} e^{-2(\nu + \sigma)} \frac{\partial \sigma(\rho, p, q)}{\partial \rho}, \tag{73}
\]

\[
\dot{q} = \frac{\partial H}{\partial p} = \sqrt{\rho} e^{-2(\nu + \sigma)} \frac{\partial \sigma(\rho, p, q)}{\partial p}, \tag{74}
\]

\[
\dot{p} = -\frac{\partial H}{\partial q} = -\sqrt{\rho} e^{-2(\nu + \sigma)} \frac{\partial \sigma(\rho, p, q)}{\partial q}. \tag{75}
\]

The first two equations can be written in term of \(\mu = \nu + \sigma\) and \(\rho\) only. Hence, (72) takes the universal form, identical to (59):

\[
\dot{\rho} = 2 \sqrt{\rho} e^{-2\mu}. \tag{76}
\]

Combining it with (73), (74) and (75) leads to

\[
\dot{\sigma} = -\frac{1}{\rho} e^{-2(\nu + \sigma)} \left( p \frac{\partial \sigma}{\partial (\frac{p}{\sqrt{\rho}})} + q \frac{\partial \sigma}{\partial (\frac{p}{\sqrt{\rho}})} \right) = 2 \sqrt{\rho} e^{-2(\nu + \sigma)} \frac{\partial \sigma}{\partial \rho}. \tag{77}
\]

Again, (73) and (77) imply the universal equation, identical to (60):

\[
\dot{\mu} = -\frac{1}{2} \frac{1}{\sqrt{\rho}} (1 - e^{-2\mu}). \tag{78}
\]

We conclude that the dynamics of the ‘geometrical’ variables \((\rho, \mu)\) does not depend upon the choice of a model of matter and in the case \(\mu = \nu\) is identical to the one from the above example. The dynamics is also identical to the case of a gravitational shock wave in empty spacetime with no matter.

Once we know the dynamics of variables \((\rho, \mu)\) we can uniquely (up to the gauge) reconstruct the spacetime in which the dynamics is realized. Suppose that we have an explicit solution of (76) and (78). Choose a gauge \(\beta < -1\) and, separately for each moment of time, gauge variables \(\zeta\) and \(l\). This enables us to entirely reconstruct the set of the Cauchy data at each instant of time separately. To reconstruct the whole geometry of spacetime we also need lapse and shift functions. For this purpose let us write the Einstein equations in terms of canonical variables \(g_{kl}\) and \(P^{kl}\). Because those objects are already known (as well as their time derivatives) at each moment of time, in result we obtain elliptic equations for the lapse and the shift. In this way for the lapse function we obtain the second-order equation in variable \(r\) as a condition for preserving of the \(\beta\)-gauge in time. This equation has to be solved with the following boundary conditions: \(N = 1\) at infinity and \(\frac{dN}{dr} = 0\) at \(r = 0\). In order to calculate the shift function we have to use the equation for a time derivative of a three-dimensional metric. This is an equation of the first order with respect to the shift function and enables us to reconstruct it uniquely.
5.1. Solution of the Hamilton equations

From (76) and (78):

\[ \dot{\rho} = 2\sqrt{\rho} e^{-2\mu}, \]  
\[ \dot{\mu} = -\frac{1}{2} \frac{1}{\sqrt{\rho}} (1 - e^{-2\mu}), \]  

there follows

\[ \frac{d\rho}{d\mu} = -4\rho \frac{e^{-2\mu}}{1 - e^{-2\mu}}. \]  

Solution of the above equation is of the form

\[ \sqrt{\rho} = \frac{1}{1 - e^{-2\mu}}. \]  

Therefore, we have the equation for the time evolution of \( \mu \):

\[ \dot{\mu} = -\frac{1}{2} \frac{1}{\sqrt{\rho_0}} (1 - e^{-2\mu})^2, \]  

whose solution reads

\[ \log(e^{2\mu} - 1) - \frac{1}{e^{2\mu} - 1} = -\frac{1}{\sqrt{\rho_0}} (t - t_0). \]  

The constant \( \rho_0 \) may be expressed in terms of the whole energy of the system: \( \sqrt{\rho_0} = 2E \). The solution of (79) for the time evolution of \( \rho \) is as follows:

\[ \sqrt{\rho} - E \log \rho = t - t_0. \]  

6. Transition from the Schwarzschild time to the Minkowski time

Equations (76) and (78) are the same as the Hamilton equations for variables \((\mu, \rho)\) in case when \(\sigma = 0\). Then, \( v = \mu \) and \((\mu, \rho)\) are canonical variables. Because of the identical form of the dynamical equations, in further calculations we limit ourselves to this particular case.

Up to now, we have described the evolution of our system with respect to the Schwarzschild time which coincides with the Minkowski time at the spatial infinity. We can also describe our evolution with respect to any other time variable, corresponding to a different (3+1)-foliation of spacetime. The leaves \( S_G \) of the new foliation do not have to coincide with the previous hypersurfaces \( S_t = \{ t = \text{const.} \} \), but we assume that asymptotically (at space-infinity) they do.

A change of the time variable is not a standard transformation in classical mechanics. Here, we use the approach introduced by one of us in [32], based on the notion of a contact manifold.

Suppose that the leaves \( S_G \) of the new foliation intersect the shell at the Schwarzschild time \( t + v(\rho, \mu) \). The quantity \( v \) will be called a retardation of the new time variable with respect to the previous one, calculated on the shell. Suppose that the value of this retardation depends upon the actual dynamical situation, i.e. upon the position and the momentum, but it does not depend explicitly upon the time variable. This means that the new gauge condition is intrinsic, depending only on initial data. The function \( v = v(\rho, \mu) \) contains the entire information about the transition between the old variables \((\rho(t), \mu(t))\) and the new ones \((\rho^G(t), \mu^G(t))\) because
once we know \((\rho(t), \mu(t))\) we can solve the equations generating dynamics of a system and set
\[
\rho^G(t) := \rho(t + v(\rho(t), \mu(t))), \tag{86}
\]
\[
\mu^G(t) := \mu(t + v(\rho(t), \mu(t))). \tag{87}
\]

For an arbitrary, not constant, function \(v\), such a transformation is, in general, not canonical. We will show in the following how to specify the canonical structure of our reduced phase space in terms of these new variables. For that purpose it will be convenient to use the language of contact manifolds. Observe that the entire information about the dynamics of a system may be retrieved from the three-dimensional contact space, defined as the surface \(\{E = H(\rho, \mu)\}\) in the four-dimensional space \([t, E, \rho, \mu]\) equipped with the standard contact form:
\[
\Psi := \frac{1}{2} \delta \mu \wedge \delta \rho - \delta E \wedge \delta t. \tag{88}
\]

This symplectic form in the four-dimensional phase space becomes degenerate when restricted to the surface \(\{E = H(\rho, \mu)\}\). Trajectories of the system are defined uniquely as those whose tangent vector belongs to this degeneracy. To prove this it is sufficient to parameterize our subspace in terms of three variables \((t, \rho, \mu)\), and rewrite the form in the following way:
\[
\Psi := \frac{1}{2} \delta \mu \wedge \delta \rho - \left( \frac{\partial H}{\partial \mu} \delta \mu + \frac{\partial H}{\partial \rho} \delta \rho \right) \wedge \delta t. \tag{89}
\]

We see that the vector annihilating the above form must be proportional to the vector
\[
Z := \frac{\partial}{\partial t} + \frac{\partial}{\partial \rho} \dot{\rho} + \frac{\partial}{\partial \mu} \dot{\mu}, \tag{90}
\]
where \(\dot{\rho}\) and \(\dot{\mu}\) are given by the Hamilton equations (76) and (78).

In [32] it was shown that the choice of the new gauge condition \(G\) is equivalent to the choice of the variable \(T\):
\[
T := t - v \tag{91}
\]
as a new time variable. Let us rewrite, therefore, our symplectic form \(\Psi\) in terms of the new time \(T\). It will be convenient to have the energy \(E\) instead of \(\mu\) as an independent parameter and treat \(\mu\) as a function \(\mu = \mu(E, \rho)\) obtained from solving (51). Therefore, we have
\[
\Psi = \frac{1}{2} \delta \mu \wedge \delta \rho - \delta E \wedge (\delta T - \delta E) = \frac{1}{2} \left( \delta \mu - 2 \frac{\partial v}{\partial \rho} \delta \rho \right) \wedge \delta \rho - \delta E \wedge \delta T. \tag{92}
\]

Now define
\[
V(E, \rho) := 2 \int \frac{\partial v}{\partial \rho}(E, \rho) \, d\rho + a(\rho) = \int \frac{1}{R} \frac{\partial v}{\partial R}(E, R) \, d\rho + a(\rho), \tag{93}
\]
where \(a(\rho)\) is arbitrary. Then, we have
\[
\Psi = \frac{1}{2} \delta \tilde{\mu} \wedge \delta \rho - \delta E \wedge \delta T, \tag{94}
\]
where
\[
\tilde{\mu} := \mu - V(H(\rho, \mu), \rho) \tag{95}
\]
plays the role of the momentum canonically conjugate to \(\rho^G(t)\). Differentiating (95) with respect to \(E\) provides
\[
\frac{\partial}{\partial E}(\tilde{\mu} - \mu) = -2 \frac{\partial v}{\partial \rho}(E, \rho) = -\frac{1}{R} \frac{\partial v}{\partial R}(E, R). \tag{96}
\]
It is interesting to note (cf [32]) that any choice of the function \( \tilde{\mu} = \tilde{\mu}(\mu, \rho) \) leads to a certain gauge condition, i.e. to the definition of a new time parameter on the shell because we may always reconstruct the retardation function \( v \) from (96). As an example, consider \( \tilde{\mu} = U n(\zeta) = 4\pi p/R \) (see formula (24)) which corresponds to the Minkowski time calculated at the internal side of the shell. Indeed, in the Minkowski spacetime the momentum canonically conjugate to the shell’s position \( R \) is equal to the kinetic momentum \( p \). The change of the variables from \( R \) to \( \rho \) implies a transformation of momenta according to the identity
\[
p \, dR = \frac{1}{2} \frac{p}{R} \, d\rho,
\]
which, finally, gives \( \frac{1}{2} \tilde{\mu} d\rho \) when integrated over the shell. We are going to show in the following that the evolution defined in this way corresponds, indeed, to the Minkowski time. For this purpose we express the new momentum in terms of the old one using (49):
\[
\tilde{\mu} = \sinh \mu \sqrt{1 - \frac{2E}{R}} = \sinh \mu e^{-\mu} = \frac{1}{2} (1 - e^{-2\mu}),
\]
(97)
where the energy \( E \) is equal to the value of the Hamiltonian \( H \). But, from (51), we obtain
\[
\sqrt{1 - \frac{2E}{R}} = e^{-\mu},
\]
(98)
and, consequently,
\[
\tilde{\mu} = \frac{E}{R}; \quad \frac{\partial \tilde{\mu}}{\partial E} = \frac{1}{R}.
\]
(99)
On the other hand, from (98) we get
\[
\frac{\partial \mu}{\partial E} = \frac{1}{(R - 2E)}.
\]
(100)
Thus, (96) implies
\[
\frac{\partial v}{\partial R} = \frac{2E}{R - 2E}.
\]
(101)
To prove that the above quantity describes retardation between the Minkowski time \( T \) inside the shell and the extrinsic Schwarzschild time \( T \), differentiate (91) over \( t \):
\[
\dot{T} = 1 - \frac{\partial v}{\partial R} \dot{R},
\]
(102)
where we have used the energy conservation \( \dot{E} \equiv 0 \). The derivative \( \dot{R} \) can be calculated from (76), and relation (98), leading to
\[
\frac{\partial v}{\partial R} \dot{R} = \frac{2E}{R - 2E};
\]
and finally
\[
\dot{T} = 1 - \frac{2E}{R}.
\]
(104)
Integrating (101) we obtain a retardation between the Minkowski time \( T \) and the Schwarzschild time \( t \):
\[
v(E, R) = 2E \log(R - 2E);
\]
(105)
hence,
\[
T = t + 2E \log(R - 2E) = t + 2R \tilde{\mu} \log R (1 - 2\tilde{\mu}),
\]
(106)
and $R > 2E$, otherwise $\mu$ would not be defined (equation (97)). The Hamiltonian may be expressed in terms of the Minkowski variables $(\rho, \bar{\mu})$:

$$H = \sqrt{\rho} \bar{\mu}$$

(107)

and the Hamilton equations take the form

$$\frac{1}{2} \frac{d}{dt} \rho = \sqrt{\rho},$$

(108)

$$\frac{1}{2} \frac{d}{dt} \bar{\mu} = -\frac{1}{2} \frac{\bar{\mu}}{\sqrt{\rho}}.$$  

(109)

Using the relations $\frac{d}{dt} = \frac{d}{\tau} \frac{d}{d\tau}$ and (106) between $\tau$ and $T$ we can reconstruct (79) and (80).

The dynamical equations (108) and (109) may be explicitly solved:

$$\sqrt{\rho} = T - T_0 + \sqrt{\rho_0},$$  

(110)

$$\bar{\mu} = \bar{\mu}_0 \frac{\sqrt{\rho_0}}{T - T_0 + \sqrt{\rho_0}}.$$  

(111)

Keeping in mind that $\sqrt{\rho} = R$, the above solution for $R$ (linear propagation with velocity equal to 1) is implied by the fact that the only null-like, spherically symmetric surfaces in the Minkowski space are light-cones. Hence, the evolution cannot be global, it ends at the cone’s vertex. An interesting case of the global (in the Minkowski time) evolution is obtained when the two cones: the future oriented one and the past oriented one, cross each other. The theory of crossing shells was thoroughly analyzed in [38].

7. Conclusions

We have derived from first principles the Hamiltonian dynamics of a spherically symmetric shell of null matter. For this purpose we have imposed a strong condition: continuity of all ten components of the metric across the shell. This is not a physical restriction, but only a gauge condition imposed on possible coordinate systems which can be used. Due to this gauge condition we were able to define a singular (Dirac-delta like) Riemann tensor of a non-continuous spacetime connection $\Gamma$ in terms of the standard formulae of differential geometry, where derivatives are understood in the sense of distributions. Imposing spherical symmetry leads to an effective reduction of the phase space of the system. The resulting ‘reduced phase space’ is obtained as a quotient of an infinite-dimensional space of Cauchy data with respect to the (again infinite-dimensional) group of gauge transformations generated by the Gauss–Codazzi constraints. Such a complete reduction which, miraculously, leads to a finite dimensional phase space, has not been successfully performed up to now. In particular, the radial component of the constraint equations produces the ‘equation of state’ (14): a relation between the radial component of the momentum $p$ and the surface energy density $\epsilon$ on $S$. In our approach the relation is not postulated but is derived as a consequence of the nullness of matter.

Choosing $\rho = R^2$ (where $R$ is the physical radius of the shell) as a configuration parameter of the shell, the corresponding canonical momentum was shown to be the (hyperbolic) angle $\mu$ between the two foliations: the Schwarzschild foliation outside of the shell and the Minkowski foliation inside of it. The ADM energy calculated at space infinity, expressed in terms of the canonical variables $(\rho, \mu)$ was proved to be the Hamiltonian of the complete ‘gravity +
matter’ system. It generates the dynamics of the canonical variables and enables us to uniquely reconstruct (up to an arbitrary choice of the gauge) the spacetime dynamics.

We have also discussed the dependence of the above picture upon the choice of the time variable. To implement such a transformation at the level of the Hamiltonian dynamics, we have proposed a new method based on the contact structure of the enlarged phase space. To illustrate this method we have shown how to transform the entire theory from the Schwarzschild time, measured at space infinity, to the Minkowski time, measured inside the shell.

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Appendix A. Null matter

Because the shell metric $g_{ab}$, $a, b = 0, 1, 2$; is degenerate, we have a priori no standard scalar density on $S$ which can be used in the definition of the Lagrangian. We must, therefore, manufacture such a density from the following ingredients: (i) the matter fields $z^k$, (ii) their derivatives $z^k_\alpha$ along the shell, (iii) the degeneracy field $X$ of $g_{ab}$ on $S$ and (iv) the two-dimensional volume form $\lambda := \sqrt{\det g_{AB}}$ on each surface \( \{x^0 = \text{const.}\} \). Choose the field $X$ (otherwise given up to a multiplicative constant) in such a way that \( \langle dx^0; X \rangle = 1 \). The following object: $L = \sum_K \lambda X^a \dot{z}^K_\alpha F_K(z_L)$, (\( K = 1, \ldots, n \)), is a scalar density and may be taken as a matter Lagrangian. A more sophisticated example is given by $L = \lambda \left( X^a \dot{z}^K_\alpha \sigma_{KL}(z^A) \right)^\alpha \left( X^c \dot{\xi}^c \right)\beta$, where $\xi$ is a scalar field, $2\alpha + \beta = 1$ and $\sigma_{KL}$ denotes any metric tensor in the space of field variables. However, in the case of spherical symmetry we have $X^a = \delta^a_0$ and all these models lead to the trivial Hamiltonian $H = 0$, which implies the equation $\dot{\lambda} = 0$. The shell surface must be, therefore, an isolated horizon (cf [33], equation (6.2)).

We are interested in examples of null-matter which may non-trivially couple to a generic null shell, not necessarily an isolated horizon. Such a nontrivial example can be obtained if one considers a thin shell of matter described by the Lagrangian density with at least two degrees of freedom which depends not only on velocities, but also on the configuration variables. Consider the Lagrangian density of the type

\[
L = L(z^K; \dot{z}^K_\alpha) = \lambda X^a \dot{z}^K_\alpha F_K(z^L), \quad (A.1)
\]

where $K, L = 2 \ldots n$, and $F_K$ is a covector field defined on the space of material variables. We show in a following that this leads to a non-trivial model already for the two degrees of freedom: $K, L = 1, 2$. In a spherically symmetric case this Lagrangian takes the form

\[
L = \lambda \dot{z}^K F_K(z^L). \quad (A.2)
\]

The Euler–Lagrange equations for this system read

\[
\dot{\lambda} F_1 = \lambda (F_{2,1} - F_{1,2}) \dot{z}^2, \quad (A.3)
\]

\[
\dot{\lambda} F_2 = -\lambda (F_{2,1} - F_{1,2}) \dot{z}^1, \quad (A.4)
\]

If the quantity $(F_{2,1} - F_{1,2})$ is equal to zero, i.e. if $F$ is closed: $dF = 0$, then $\dot{\lambda} = 0$, and matter described by this Lagrangian density couples again to an isolated horizon. Assume that $dF \neq 0$. Then the Euler–Lagrange equations imply the following constraints equations:

\[
F_1 \dot{z}^1 + F_2 \dot{z}^2 = 0. \quad \square
\]
Appendix B. Examples of non-trivial matter Lagrangians

1. A simple example of a Lagrangian of the type (A.1) may be the following one:

\[ L = \lambda (z^2 \dot{z}^1 - z^1 \dot{z}^2), \]  

(B.1)

whose structure resembles the Dirac Lagrangian for spinor fields. This Lagrangian implies the following constraints:

\[ \lambda \mathcal{P}_1 = \mathcal{P}_1 = \frac{\partial L}{\partial \dot{z}^1} = \lambda z^2 \]  

(B.2)

\[ \lambda \mathcal{P}_2 = \mathcal{P}_2 = \frac{\partial L}{\partial \dot{z}^2} = -\lambda z^1. \]  

(B.3)

These are second type constraints. Inserting them into the symplectic form (50) we obtain

\[ \Omega = \frac{1}{2} \delta \mu \wedge \delta \rho + 8\pi \delta(\sqrt{\rho z^2}) \wedge \delta(\sqrt{\rho z^1}). \]  

(B.4)

Similarly as in the spinor theory, one configuration variable (in electrodynamics it is the imaginary part of spinor in the Majorana representation) becomes momentum canonically conjugated to the second variable (the real part in the Majorana representation).

2. Another example

\[ L = \lambda (z^2 + 2z^1 z^2) \dot{z}^1 + ((z^1)^2 - z^1) \dot{z}^2) \]  

(B.5)

is used to reduce the symplectic form described in section 4.3.

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