A Ramsey-type result for geometric $\ell$-hypergraphs

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Abstract

Let $n \geq \ell \geq 2$ and $q \geq 2$. We consider the minimum $N$ such that whenever we have $N$ points in the plane in general position and the $\ell$-subsets of these points are colored with $q$ colors, there is a subset $S$ of $n$ points all of whose $\ell$-subsets have the same color and furthermore $S$ is in convex position. This combines two classical areas of intense study over the last 75 years: the Ramsey problem for hypergraphs and the Erdős-Szekeres theorem on convex configurations in the plane. For the special case $\ell = 2$, we establish a single exponential bound on the minimum $N$, such that every complete $N$-vertex geometric graph whose edges are colored with $q$ colors, yields a monochromatic convex geometric graph on $n$ vertices.

For fixed $\ell \geq 2$ and $q \geq 4$, our results determine the correct exponential tower growth rate for $N$ as a function of $n$, similar to the usual hypergraph Ramsey problem, even though we require our monochromatic set to be in convex position. Our results also apply to the case of $\ell = 3$ and $q = 2$ by using a geometric variation of the stepping up lemma of Erdős and Hajnal. This is in contrast to the fact that the upper and lower bounds for the usual 3-uniform hypergraph Ramsey problem for two colors differ by one exponential in the tower.

1 Introduction

The classic 1935 paper of Erdős and Szekeres [13] entitled A Combinatorial Problem in Geometry was a starting point of a very rich discipline within combinatorics: Ramsey theory (see, e.g., [16]). The term Ramsey theory refers to a large body of deep results in mathematics which have a common theme: “Every large system contains a large well-organized subsystem.” Motivated by the observation that any five points in the plane in general position must contain four members in convex position, Esther Klein asked the following.

Problem 1.1. For every integer $n \geq 2$, determine the minimum $f(n)$, such that any set of $f(n)$ points in the plane in general position, contains $n$ members in convex position.

Celebrated results of Erdős and Szekeres [13] [14] imply that

\[ 2^{n-1} + 1 \leq f(n) \leq \binom{2n-4}{n-2} \leq 2^{2n(1-o(1))}. \]
They conjectured that \( f(n) = 2^{n-1} + 1 \), and Erdős offered a \$500 reward for a proof. Despite much attention over the last 75 years, the constant factors in the exponents have not been improved.

In the same paper \[13\], Erdős and Szekeres gave another proof of a classic result due to Ramsey \[24\] on hypergraphs. An \( \ell \)-uniform hypergraph \( H \) is a pair \((V, E)\), where \( V \) is the vertex set and \( E \subset \binom{V}{\ell} \) is the set of edges. We denote \( K_\ell^n = (V, E) \) to be the complete \( \ell \)-uniform hypergraph on an \( n \)-element set \( V \), where \( E = \binom{V}{\ell} \). When \( \ell = 2 \), we write \( K^2_n = K_n \). Motivated by obtaining good quantitative bounds on \( f(n) \), Erdős and Szekeres looked at the following problem.

**Problem 1.2.** For every integer \( n \geq 2 \), determine the minimum integer \( r(K_n, K_n) \), such that any two-coloring on the edges of a complete graph \( G \) on \( r(K_n, K_n) \) vertices, yields a monochromatic copy of \( K_n \).

Erdős and Szekeres \[13\] showed that \( r(K_n, K_n) \leq 2^{2n} \). In \[10\], Erdős gave a construction showing that \( r(K_n, K_n) > 2^{n/2} \). Despite much attention over the last 65 years, the constant factors in the exponents have not been improved.

Generalizing Problem 1.2 to \( q \)-colors and \( \ell \)-uniform hypergraphs has also been studied extensively. Let \( r(K_\ell^n; q) \) be the least integer \( N \), such that any \( q \)-coloring on the edges of a complete \( N \)-vertex \( \ell \)-uniform hypergraph \( H \), yields a monochromatic copy of \( K_\ell^n \). We will also write

\[
r(K_\ell^n; q) = r(K_\ell^n, K_\ell^n, ..., K_\ell^n) \text{ \text{\( q \) times}}.
\]

Erdős, Hajnal, and Rado \[11, 12\] showed that there are positive constants \( c \) and \( c' \) such that

\[
2^{cn^2} < r(K_3^n, K_3^n) < 2^{c'n^2}.
\]

They also conjectured that \( r(K_3^n, K_3^n) > 2^{cn^2} \) for some constant \( c > 0 \), and Erdős offered a \$500 reward for a proof. For \( \ell \geq 4 \), there is also a difference of one exponential between the known upper and lower bounds for \( r(K_\ell^n, K_\ell^n) \), namely,

\[
twr_{\ell-1}(cn^2) \leq r(K_\ell^n, K_\ell^n) \leq twr_\ell(c'n),
\]

where \( c \) and \( c' \) depend only on \( \ell \), and the tower function \( twr_\ell(x) \) is defined by \( twr_1(x) = x \) and \( twr_{i+1} = 2^{twr_i(x)} \). As Erdős and Rado have shown \[12\], the upper bound in equation (3) easily generalizes to \( q \) colors, implying that \( r(K_\ell^n; q) \leq twr_\ell(c'n) \), where \( c' = c'(\ell, q) \). On the other hand, for \( q \geq 4 \) colors, Erdős and Hajnal (see \[16\]) showed that \( r(K_\ell^n; q) \) does indeed grow as a \( \ell \)-fold exponential tower in \( n \), proving that \( r(K_\ell^n; q) = twr_\ell(\Theta(n)) \). For \( q = 3 \) colors, Conlon, Fox, and Sudakov \[6\] modified the construction of Erdős and Hajnal to show that \( r(K_\ell^n, K_\ell^n, K_\ell^n) > twr_\ell(c\log^2 n) \).

Interestingly, both Problems 1.1 and 1.2 can be asked simultaneously for geometric graphs, and a similar-type problem can be asked for geometric \( \ell \)-hypergraphs. A geometric \( \ell \)-hypergraph \( H \) in the plane is a pair \((V, E)\), where \( V \) is a set of points in the plane in general position, and \( E \subset \binom{V}{\ell} \) is a collection of \( \ell \)-tuples from \( V \). When \( \ell = 2 \) \((\ell = 3) \), edges are represented by straight line segments (triangles) induced by the corresponding vertices. The sets \( V \) and \( E \) are called the vertex set and edge set of \( H \), respectively. A geometric hypergraph \( H \) is convex, if its vertices are in convex position.

Geometric graphs \((\ell = 2)\) have been studied extensively, due to their wide range of applications in combinatorial and computational geometry (see \[23, 18, 19\]). Complete convex geometric
graphs are very well understood, and are some of the most well-organized geometric graphs (if not the most). Many long standing problems on complete geometric graphs, such as its crossing number [2], number of halving-edges [27], and size of crossing families [9], become trivial when its vertices are in convex position. There has also been a lot of research on geometric 3-hypergraphs in the plane, due to its connection to the $k$-set problem in $\mathbb{R}^3$ (see [22], [25], [8]). In this paper, we study the following problem which combines Problems 1.1 and 1.2.

**Problem 1.3.** Determine the minimum integer $g(K_n^\ell;q)$, such that any $q$-coloring on the edges of a complete geometric $\ell$-hypergraph $H$ on $g(K_n^\ell;q)$ vertices, yields a monochromatic convex $\ell$-hypergraph on $n$ vertices.

Another chromatic variant of the Erdős-Szekeres convex polygon problem was studied by Devillers et al. [7], where they considered colored points in the plane rather than colored edges.

We will also write

$$g(K_n^\ell;q) = g(K_n^\ell,\ldots,K_n^\ell).$$

Clearly we have $g(K_n^\ell;q) \geq \max\{r(K_n^\ell;q), f(n)\}$. An easy observation shows that by combining equations (1) and (3), we also have

$$g(K_n^\ell;q) \leq f(r(K_n^\ell;q)) \leq \text{twr}_{\ell+1}(cn),$$

where $c = c(\ell, q)$. Our main results are the following two exponential improvements on the upper bound of $g(K_n^\ell;q)$.

**Theorem 1.4.** For geometric graphs, we have

$$2^{q(n-1)} < g(K_n^\ell;q) \leq 2^{8qn^2\log(qn)}.$$

The argument used in the proof of Theorem 1.4 above extends easily to hypergraphs, and for each fixed $\ell \geq 3$ it gives the bound $g(K_n^\ell;q) < \text{twr}_\ell(O(n^2))$. David Conlon pointed out to us that one can improve this slightly as follows.

**Theorem 1.5.** For geometric $\ell$-hypergraphs, when $\ell \geq 3$ and fixed, we have

$$g(K_n^\ell;q) \leq \text{twr}_\ell(cn),$$

where $c = O(q \log q)$.

By combining Theorems 1.4, 1.5 and the fact that $g(K_n^\ell;q) \geq r(K_n^\ell;q)$, we have the following corollary.

**Corollary 1.6.** For fixed $\ell$ and $q \geq 4$, we have $g(K_n^\ell;q) = \text{twr}_\ell(\Theta(n))$.

As mentioned above, there is an exponential difference between the known upper and lower bounds for $r(K_n^3,K_n^3)$. Hence, for two-colorings on geometric 3-hypergraphs in the plane, equation [2] implies

$$g(K_n^3,K_n^3) \geq r(K_n^3,K_n^3) \geq 2^{cn^2}.$$
Our next result establishes an exponential improvement in the lower bound of \( g(K^3_n, K^3_n) \), showing that \( g(K^3_n, K^3_n) \) does indeed grow as a 3-fold exponential tower in a power of \( n \). One noteworthy aspect of this lower bound is that the construction is a geometric version of the famous stepping up lemma of Erdős and Hajnal [11] for sets. The lemma produces a \( q' \)-coloring \( \chi' \) of \( \binom{[q]}{\ell+1} \) from a \( q \)-coloring \( \chi \) of \( \binom{[n]}{\ell} \) for appropriate \( q' > q \), where the largest monochromatic clique of \( \ell \)-sets under \( \chi' \) is not too much larger than the largest monochromatic clique of \( \ell \)-sets under \( \chi \) (see [10] for more details about the stepping up lemma). While it is a major open problem to apply this method to \( r(K^3_n, K^3_n) \) and improve the lower bound in equation (2), we are able to achieve this in the geometric setting as shown below.

**Theorem 1.7.** For geometric 3-hypergraphs in the plane, we have

\[
g(K^3_n, K^3_n) \geq 2^{2^{cn}},
\]

where \( c \) is an absolute constant. In particular, \( g(K^3_n, K^3_n) = \text{twr}_3(\Theta(n)) \).

We systemically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. All logarithms are in base 2.

## 2 Proof of Theorems 1.4 and 1.5

Before proving Theorems 1.4 and 1.5, we will first define some notation. Let \( V = \{p_1, ..., p_N\} \) be a set of \( N \) points in the plane in general position ordered from left to right according to \( x \)-coordinate, that is, for \( p_i = (x_i, y_i) \in \mathbb{R}^2 \), we have \( x_i < x_{i+1} \) for all \( i \). For \( i_1 < \cdots < i_t \), we say that \( X = (p_{i_1}, ..., p_{i_t}) \) forms an \( t \)-cup (\( t \)-cap) if \( X \) is in convex position and its convex hull is bounded above (below) by a single edge. See Figure 1. When \( t = 3 \), we will just say \( X \) is a cup or a cap.

![Figure 1: A 4-cup and a 5-cap.](image)

**Proof of Theorem 1.4.** We first prove the upper bound. Let \( G = (V, E) \) be a complete geometric graph on \( N = 2^{8q^2 n + 1} \) vertices, such that the vertices \( V = \{v_1, ..., v_N\} \) are ordered from left to right according to \( x \)-coordinate. Let \( \chi \) be a \( q \)-coloring on the edge set \( E \). We will recursively construct a sequence of vertices \( p_1, ..., p_t \) from \( V \) and a subset \( S_t \subset V \), where \( t = 0, 1, ..., qn^2 \) (when \( t = 0 \) there are no vertices in the sequence), such that the following holds.

1. for any vertex \( p_i \), all pairs \( (p_i, p) \) where \( p \in \{p_j : j > i\} \cup S_t \) have the same color, which we denote by \( \chi'(p_i) \),

2. for every pair of vertices \( p_i \) and \( p_j \), where \( i < j \), either \( (p_i, p_j, p) \) is a cap for all \( p \in \{p_k : k > j\} \cup S_t \), or \( (p_i, p_j, p) \) is a cup for all \( p \in \{p_k : k > j\} \cup S_t \).
3. the set of points $S_t$ lies to the right of the point $p_t$, and
4. $|S_t| \geq \frac{N}{q} t - t$.

We start with no vertices in the sequence, and set $S_0 = V$. After obtaining vertices $\{p_1, ..., p_t\}$ and $S_t$, we define $p_{t+1}$ and $S_{t+1}$ as follows. Let $p_{t+1} = (x_{t+1}, y_{t+1}) \in \mathbb{R}^2$ be the smallest indexed element in $S_t$ (the left-most point), and let $H$ be the right half-plane $x > x_{t+1}$. We define $t$ lines $l_1, ..., l_t$ such that $l_i$ is the line going through points $p_i, p_{t+1}$. Notice that the arrangement $\bigcup_{i=1}^{t} l_i$ partitions the right half-plane $H$ into $t + 1$ cells. See Figure 2. Since $V$ is in general position, by the pigeonhole principle, there exists a cell $\Delta \subset H$ that contains at least $(|S_t| - 1)/(t + 1)$ points of $S_t$.

Let us call two elements $v'_1, v'_2 \in \Delta \cap S_t$ equivalent if $\chi(p_{t+1}, v'_1) = \chi(p_{t+1}, v'_2)$. Hence, there are at most $q$ equivalence classes. By setting $S_{t+1}$ to be the largest of those classes, we have the recursive formula

$$|S_{t+1}| \geq \frac{|S_t| - 1}{(t + 1)q}.$$ 

Substituting in the lower bound on $|S_t|$, we obtain the desired bound

$$|S_{t+1}| \geq \frac{N}{(t + 1)!q} - (t + 1).$$

This shows that we can construct the sequence $p_1, ..., p_{t+1}$ and the set $S_{t+1}$ with the desired properties. For $N = 2^{8qn^2 \log(qn)}$, we have

$$|S_{qn^2}| \geq \frac{2^{8qn^2 \log(qn)}}{(qn^2)!q^{n^2}} - qn^2 \geq 1.$$ (4)

Hence, $P_1 = \{p_1, ..., p_{qn^2}\}$ is well defined. Since $\chi'$ is a $q$-coloring on $P_1$, by the pigeonhole principle, there exists a subset $P_2 \subset P_1$ such that $|P_2| = n^2$, and every vertex has the same color. By construction of $P_2$, every pair in $P_2$ has the same color. Hence these vertices induce a monochromatic geometric graph.

Now let $P_2 = \{p'_1, ..., p'_{n^2}\}$. We define partial orders $<_1, <_2$ on $P_2$, where $p'_i <_1 p'_j$ ($p'_i <_2 p'_j$) if and only if $i < j$ and the set of points $P_2 \setminus \{p'_1, ..., p'_j\}$ lie above (below) the line going through points $p'_i$ and $p'_j$. See Figure 3. By construction of $P_2$, $<_1, <_2$ are indeed partial orders and every
two elements in \( P_2 \) are comparable by either \( \prec_1 \) or \( \prec_2 \). By a corollary to Dilworth’s Theorem \([9]\) (see also Theorem 1.1 in \([15]\)), there exists a chain \( p_1^*, ..., p_n^* \) of length \( n \) with respect to one of the partial orders. Hence \( (p_1^*, ..., p_n^*) \) forms either an \( n \)-cap or an \( n \)-cup. Therefore, these vertices induce a monochromatic convex geometric graph.

![Figure 3: Partial orders \( \prec_1, \prec_2 \).](image)

For the lower bound, we proceed by induction on \( q \). The base case \( q = 1 \) follows by taking the complete geometric graph on \( 2^{n-1} \) vertices, whose vertex set does not have \( n \) members in convex position. This is possible by the construction of Erdős and Szekeres \([14]\). Let \( G_0 \) denote this geometric graph. For \( q > 1 \), we inductively construct a complete geometric graph \( G = (V, E) \) on \( 2^{(q-1)(n-1)} \) vertices, and a coloring \( \chi : E \to \{1, 2, ..., q-1\} \) on the edges of \( G \), such that \( G \) does not contain a monochromatic convex geometric graph on \( n \) vertices. Now we replace each vertex \( v_i \in G \) with a small enough copy \( 2 \) of \( G_0 \), which we will denote as \( G_i \), where all edges in \( G_i \) are colored with the color \( q \), and all edges between \( G_i \) and \( G_j \) have color \( \chi(v_i v_j) \). Then we have a complete geometric graph \( G' \) on

\[
2^{(q-1)(n-1)}2^{n-1} = 2^{q(n-1)}
\]

vertices, such that \( G' \) does not contain a monochromatic convex graph on \( n \) vertices.

By following the proof above, one can show that \( g(K_n^3; q) \leq \text{twr}(O(n^2)) \). However, the following short argument due to David Conlon gives a better bound. The proof uses an old idea of M. Tarsi (see \([22]\) Chapter 3) that gave an upper bound on \( f(n) \).

**Lemma 2.1.** For geometric 3-hypergraphs, we have \( g(K_n^3; q) \leq r(K_n^3; 2q) \leq 2^{2cn} \), where \( c = O(q \log q) \).

**Proof.** Let \( H = (V, E) \) be a complete geometric 3-hypergraph on \( N = r(K_n^3; 2q) \) vertices, and let \( \chi \) be a \( q \) coloring on the edges of \( H \). By fixing an ordering on the vertices \( V = \{v_1, ..., v_N\} \), we say that a triple \( (v_{i_1}, v_{i_2}, v_{i_3}) \), \( i_1 < i_2 < i_3 \), has a *clockwise (counterclockwise)* orientation, if \( v_{i_1}, v_{i_2}, v_{i_3} \) appear in clockwise (counterclockwise) order along the boundary of \( \text{conv}(v_{i_1} \cup v_{i_2} \cup v_{i_3}) \). Hence by Ramsey’s theorem, there are \( n \) points from \( V \) for which every triple has the same color and the same orientation. As observed by Tarsi (see Theorem 3.8 in \([26]\)), these vertices must be in convex position.

\(^2\)Obtained by an affine transformation.
Lemma 2.2. For $\ell \geq 4$ and $n \geq 4^\ell$, we have $g(K^\ell_n; q) \leq r(K^\ell_n; q + 1) \leq twr(\ell cn)$, where $c = O(q \log q)$.

Proof. Let $H = (V, E)$ be a complete geometric $\ell$-hypergraph on $N = r(K^\ell_n; q + 1)$ vertices, and let $\chi$ be a $q$ coloring on the $\ell$-tuples of $V$ with colors $1, 2, \ldots, q$. Now if an $\ell$-tuple from $V$ is not in convex position, we change its color to the new color $q + 1$. By Ramsey’s theorem, there is a set $S \subset V$ of $n$ points for which every $\ell$-tuple has the same color. Since $n \geq 4^\ell$, by the Erdős-Szekeres Theorem, $S$ contains $\ell$ members in convex position. Hence, every $\ell$-tuple in $S$ is in convex position, and has the same color which is not the new color $q + 1$. Therefore $S$ induces a monochromatic convex geometric $\ell$-hypergraph.

Theorem 1.5 now follows by combining Lemma 2.1 and 2.2.

3 A lower bound construction for geometric 3-hypergraphs

In this section, we will prove Theorem 1.7, which follows immediately from the following lemma.

Lemma 3.1. For sufficiently large $n$, there exists a complete geometric 3-hypergraph $H = (V, E)$ in the plane with $2^{2[n/2]}$ vertices, and a two-coloring $\chi'$ on the edge set $E$, such that $H$ does not contain a monochromatic convex 3-hypergraph on $2n$ vertices.

Proof. Let $G$ be the complete graph on $2n/2$ vertices, where $V(G) = \{1, \ldots, 2n/2\}$, and let $\chi'$ be a red-blue coloring on the edges of $G$ such that $G$ does not contain a monochromatic complete subgraph on $n$ vertices. Such a graph does indeed exist by a result of Erdős [10], who showed that $r(K_n, K_n) > 2^{n/2}$. We will use $G$ and $\chi'$ to construct a complete geometric 3-hypergraph $H$ on $2^{n/2}$ vertices, and a coloring $\chi'$ on the edges of $H$, with the desired properties.

Set $M = 2^{n/2}$. We will recursively construct a point set $P_t$ of $2^t$ points in the plane as follows. Let $P_1$ be a set of two points in the plane with distinct $x$-coordinates. After obtaining the point set $P_t$, we define $P_{t+1}$ follows. We inductively construct two copies of $P_t$, $L = P_t$ and $R = P_t$, and place $L$ to the left of $R$ such that all lines determined by pairs of points in $L$ go below $R$ and all lines determined by pairs of points of $R$ go above $L$. Then set $P_{t+1} = L \cup R$. See Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Constructing $P_{t+1}$ from $P_t$.}
\end{figure}

Let $P_M = \{p_1, \ldots, p_{2^M}\}$ be the set of $2^M$ points in the plane, ordered by increasing $x$-coordinate, from our construction. Notice that $P_M$ contains $2^{M-t}$ disjoint copies of $P_t$. For $i < j$, we define

$$\delta(p_i, p_j) = \max\{t : p_i, p_j \text{ lies inside a copy of } P_t = L \cup R, \text{ and } p_i \in L, p_j \in R\}.$$ 

Notice that
**Property A:** $\delta(p_i, p_j) \neq \delta(p_j, p_k)$ for every triple $i < j < k$.

**Property B:** for $i_1 < \cdots < i_n$, $\delta(p_{i_1}, p_{i_n}) = \max_{1 \leq j \leq n-1} \delta(p_{i_j}, p_{i_j+1})$.

Now we define a red-blue coloring $\chi'$ on the triples of $P_M$ as follows. For $i < j < k$,

$$\chi'(p_i, p_j, p_k) = \chi(\delta(p_i, p_j), \delta(p_j, p_k)).$$

Now we claim that the geometric 3-hypergraph $H = (P_M, E)$ does not contain a monochromatic convex 3-hypergraph on $2n$ vertices. For sake of contradiction, let $S = \{q_1, \ldots, q_{2n}\}$ be a set of $2n$ points from $P_M$, ordered by increasing $x$-coordinate, that induces a red convex 3-hypergraph. Set $\delta_i = \delta(q_i, q_{i+1})$.

**Case 1.** Suppose that there exists a $j$ such that $\delta_j, \delta_{j+1}, \ldots, \delta_{j+n-1}$ forms a monotone sequence. First assume that

$$\delta_j > \delta_{j+1} > \cdots > \delta_{j+n-1}.$$ 

Since $G$ does not contain a red complete subgraph on $n$ vertices, there exists a pair $j \leq i_1 < i_2 \leq j + n - 1$ such that $(\delta_{i_1}, \delta_{i_2})$ is blue. But then the triple $(q_{i_1}, q_{i_2}, q_{i_2+1})$ is blue, a contradiction. Indeed, by Property B,

$$\delta(q_{i_1}, q_{i_2}) = \delta(q_{i_1}, q_{i_1+1}) = \delta_{i_1}.$$ 

Therefore, since $\delta_{i_1} > \delta_{i_2}$ and $(\delta_{i_1}, \delta_{i_2})$ is blue, the triple $(q_{i_1}, q_{i_2}, q_{i_2+1})$ must also be blue. A similar argument holds if $\delta_j < \delta_{j+1} < \cdots < \delta_{j+n-1}$.

**Case 2.** Suppose we are not in Case 1. For $2 \leq i \leq 2n$, we say that $i$ is a local minimum if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a local maximum if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a local extremum if it is either a local minimum or a local maximum. This is well defined by Property A.

**Observation 3.2.** For $2 \leq i \leq 2n$, $i$ is never a local minimum.

**Proof.** Suppose $\delta_{i-1} > \delta_i < \delta_{i+1}$ for some $i$, and suppose that $\delta_{i-1} \geq \delta_{i+1}$. We claim that $q_{i+1} \in \text{conv}(q_{i-1}, q_i, q_{i+2})$. Indeed, since $\delta_{i-1} \geq \delta_{i+1} > \delta_i$, this implies that $q_{i-1}, q_i, q_{i+1}, q_{i+2}$ lies inside a copy of $P_{i-1} = L \cup R$, where $q_{i-1} \in L$ and $q_i, q_{i+1}, q_{i+2} \in R$. Since $\delta_{i+1} > \delta_i$, this implies that $q_i, q_{i+1}, q_{i+2}$ lie inside a copy $P_{i+1} = L' \cup R' \subset R$, where $q_i, q_{i+1} \in L'$ and $q_{i+2} \in R'$.

Notice that all lines determined by $q_i, q_{i+1}, q_{i+2}$ go above the point $q_{i-1}$. Therefore $q_{i+1}$ must lie above the line that goes through the points $q_{i-1}, q_{i+2}$, and furthermore, $q_{i+1}$ must lie below the line that goes through the points $q_{i-1}, q_i$. Since $\delta_{i+1} > \delta_i$, the line through $q_i, q_{i+1}$ must go below the point $q_{i+2}$, and therefore $q_{i+1} \in \text{conv}(q_{i-1}, q_i, q_{i+2})$. See Figure 5. If $\delta_{i-1} < \delta_{i+1}$, then a similar argument shows that $q_i \in \text{conv}(q_{i-1}, q_{i+1}, q_{i+2})$.

Since $\delta_1, \ldots, \delta_{2n}$ does not have a monotone subsequence of length $n$, it must have at least two local extrema. Since between any two local maximums there must be a local minimum, we have a contradiction by Observation 3.2. This completes the proof.
4 Concluding remarks

For $q \geq 4$ colors and $\ell \geq 3$, we showed that $g(K_n^{\ell}; q) = \text{twr}_{\ell}(\Theta(n))$. Our bounds on $g(K_n^{\ell}; q)$ for $q \leq 3$ can be summarized in the following table.

| $\ell$ | $q = 2$ | $q = 3$ |
|--------|---------|---------|
| 2      | $2^{\Omega(n)} < g(K_n, K_n) \leq 2^{O(n^2 \log n)}$ | $2^{\Omega(n)} < g(K_n; 3) \leq 2^{O(n^2 \log n)}$ |
| 3      | $g(K_n^3, K_n^3) = 2^{2^{\Theta(n)}}$ | $g(K_n^3; 3) = 2^{2^{\Theta(n)}}$ |
| $\geq 4$ | twr_{\ell-1}(\Omega(n^2)) \leq g(K_n^{\ell}, K_n^{\ell}) \leq \text{twr}_{\ell}(O(n)) | twr_{\ell}(\Omega(\log^2 n)) \leq g(K_n^{\ell}, 3) \leq \text{twr}_{\ell}(O(n)) |

**Off-diagonal.** The Ramsey number $r(K_s, K_n)$ is the minimum integer $N$ such that every red-blue coloring on the edges of a complete $N$-vertex graph $G$, contains either a red clique of size $s$, or a blue clique of size $n$. The off-diagonal Ramsey numbers, i.e., $r(K_s, K_n)$ with $s$ fixed and $n$ tending to infinity, have been intensively studied. For example, it is known \[11, 13, 15, 21\] that $R_2(3, n) = \Theta(n^2 / \log n)$ and, for fixed $s > 3$,

\[
c_1 (\log n)^{1/(s-2)} \left( \frac{n}{\log n} \right)^{(s+1)/2} \leq r(K_s, K_n) \leq c_2 \frac{n^{s-1}}{\log^{s-2} n}. \tag{5}
\]

Another interesting variant of Problem \[13\] is the following off-diagonal version.

**Problem 4.1.** Determine the minimum integer $g(K_s, K_n)$, such that any red-blue coloring on the edges of a complete geometric graph $G$ on $g(K_s, K_n)$ vertices, yields either a red convex geometric graph on $s$ vertices, or a blue convex geometric graph on $n$ vertices.

For fixed $s$, one can show that $g(K_s, K_n)$ grows single exponentially in $n$. In particular

\[
2^{n-1} + 1 \leq g(K_s, K_n) \leq 4^s n.
\]

The lower bound follows from the fact that $g(K_s, K_n) \geq f(n)$. The upper bound follows from the inequalities
Indeed, by the Erdős-Szekeres theorem, if $G$ contains a red-clique of size $4^s$, then there must be a red convex geometric graph on $s$ vertices. Likewise, if $G$ contains a blue clique of size $4^n$, then there must be a blue convex geometric graph on $n$ vertices.

**Higher dimensions.** Generalizing Problem [1.1] to higher dimensions has also been studied. Let $f_d(n)$ be the smallest integer such that any set of at least $f_d(n)$ points in $\mathbb{R}^d$ in general position contains $n$ members in convex position. The following upper and lower bounds were obtained by Károlyi [17] and Károlyi and Valtr [20] respectively,

$$2^{cn^{1/(d-1)}} \leq f_d(n) \leq \left(\frac{2n - 2d - 1}{n - d}\right) + d = 2^{2n(1-o(1))}.$$

A geometric $\ell$-hypergraph $H$ in $d$-space is a pair $(V, E)$, where $V$ is a set of points in general position in $\mathbb{R}^d$, and $E \subset \binom{V}{\ell}$ is a collection of $\ell$-tuples from $V$. When $\ell \leq d + 1$, $\ell$-tuples are represented by $(\ell - 1)$-dimensional simplices induced by the corresponding vertices.

**Problem 4.2.** Determine the minimum integer $g_d(K_\ell^n; q)$, such that any $q$-coloring on the edges of a complete geometric $\ell$-hypergraph $H$ in $d$-space on $g_d(K_\ell^n; q)$ vertices, yields a monochromatic convex $\ell$-hypergraph on $n$ vertices.

When $d = 2$, we write $g_2(K_\ell^n; q) = g(K_\ell^n; q)$. Clearly $g_d(K_\ell^n; q) \geq \max\{f_d(n), R(K_\ell^n; q)\}$. One can also show that $g_d(K_\ell^n; q) \leq g(K_\ell^n; q)$. Indeed, for any complete geometric $\ell$-hypergraph $H = (V, E)$ in $d$-space with a $q$-coloring $\chi$ on $E(H)$, one can obtain a complete geometric $\ell$-hypergraph in the plane $H' = (V', E')$, by projecting $H$ onto a 2-dimensional subspace $L \subset \mathbb{R}^d$ such that $V'$ is in general position in $L$. Thus we have

$$g_d(K_\ell^n; q) \leq g(K_\ell^n; q) \leq 2^{cn^2 \log n},$$

where $c = O(q \log q)$, and for $\ell \geq 3$

$$g_d(K_\ell^n; q) \leq g(K_\ell^n; q) \leq \text{twr}_\ell(c' n^2),$$

where $c' = c'(q, \ell)$.

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**References**

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