A WINTGEN TYPE INEQUALITY FOR SURFACES IN 4D NEUTRAL PSEUDO-RIEMANNIAN SPACE FORMS AND ITS APPLICATIONS TO MINIMAL IMMERSIONS

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Abstract. Let $M$ be a space-like surface immersed in a 4-dimensional pseudo-Riemannian space form $\mathbb{R}^4_{2c}$ with constant sectional curvature $c$ and index two. In the first part of this article, we prove that the Gauss curvature $K$, the normal curvature $K^D$, and mean curvature vector $H$ of $M$ satisfy the general inequality: $K + K^D \geq \langle H, H \rangle + c$. In the second part, we investigate space-like minimal surfaces in $\mathbb{R}^4_{2c}$ which satisfy the equality case of the inequality identically. Several classification results in this respect are then obtained.

1. Introduction.

Let $\mathbb{E}^m_t$ denote the pseudo-Euclidean $m$-space equipped with pseudo-Euclidean metric of index $t$ given by

$$g_t = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n} dx_j^2,$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}^m_t$.

We put

$$S^k_s(c) = \left\{ x \in \mathbb{E}^{k+1}_s : \langle x, x \rangle = \frac{1}{c} > 0 \right\},$$

$$H^k_s(c) = \left\{ x \in \mathbb{E}^{k+1}_s : \langle x, x \rangle = \frac{1}{c} < 0 \right\},$$

where $\langle , \rangle$ is the associated inner product. Then $S^k_s(c)$ and $H^k_s(c)$ are complete pseudo-Riemannian manifolds of constant curvature $c$ and with index $s$, which are known as pseudo-Riemannian k-sphere and the pseudo-hyperbolic k-space, respectively. The pseudo-Riemannian manifolds $\mathbb{E}^k_s, S^k_s(c)$ and $H^k_s(-c)$ are called pseudo-Riemannian space forms.

A vector $v$ is called space-like (respectively, time-like) if $\langle v, v \rangle > 0$ (respectively, $\langle v, v \rangle < 0$). A vector $v$ is called light-like if it is nonzero and it satisfies $\langle v, v \rangle = 0$. A surface $M$ in a pseudo-Riemannian manifold is called space-like if each nonzero tangent vector is space-like.

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Let $M$ be a space-like surface immersed in a 4-dimensional pseudo-Riemannian space form $R^2_4(c)$ with constant sectional curvature $c$ and index 2. In section 3, we recall a minimal immersion of $H^2(-\frac{1}{3})$ into the neutral pseudo-hyperbolic 4-space $H^4_2(-1)$ discovered recently by the author in [4]. In section 4, we prove that the Gauss curvature $K$, the normal curvature $K^D$, and mean curvature vector $H$ of $M$ in $R^2_4(c)$ satisfy the following general inequality:

\[ K + K^D \geq \langle H, H \rangle + c. \tag{1.4} \]

In this section, we also show that there exist many minimal space-like surfaces which satisfy the equality case of this inequality. In section 5, we investigate space-like minimal surfaces in the neutral pseudo-hyperbolic 4-space $H^4_2(-1)$ which satisfy the equality case of (1.4). Several classification results in this respect are then obtained.

2. Preliminaries.

2.1. Basic formulas and definitions. Let $R^2_4(c)$ denote the 4-dimensional neutral pseudo-Riemannian space form of constant curvature $c$ and with index two. Then the curvature tensor $\tilde{R}$ of $R^2_4(c)$ is given by

\[ \tilde{R}(X,Y)Z = c \left( \langle X, Z \rangle Y - \langle Y, Z \rangle X \right) \]

for vectors $X,Y,Z$ tangent to $R^2_4(c)$. Let $\psi : M \to R^2_4(c)$ be an isometric immersion of a space-like surface $M$ into $R^2_4(c)$. Denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections on $M$ and $R^2_4(c)$, respectively.

For vector fields $X,Y$ tangent to $M$ and vector field $\xi$ normal to $M$, the formulas of Gauss and Weingarten are given respectively by (cf. [1], [2], [9]):

\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \]
\[ \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \]

where $\nabla_X Y$ and $A_\xi X$ are the tangential components and $h(X,Y)$ and $D_X \xi$ are the normal components of $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$, respectively. These formulas define the second fundamental form $h$, the shape operator $A$, and the normal connection $D$ of $M$ in $R^2_4(c)$.

For each normal vector $\xi \in T_x^\perp M$, $A_\xi$ is a symmetric endomorphism of the tangent space $T_x M, x \in M$. The shape operator and the second fundamental form
are related by

\[ \langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle. \] (2.4)

The mean curvature vector \( H \) of \( M \) in \( \mathbb{R}^4_2(c) \) is defined by

\[ H = \left( \frac{1}{2} \right) \text{trace } h. \] (2.5)

The mean curvature of \( M \) in \( \mathbb{R}^4_2(c) \) is defined to be \( \sqrt{-\langle H, H \rangle} \).

The equations of Gauss, Codazzi and Ricci are given respectively by

\[ R(X,Y)Z = \langle X,Z \rangle Y - \langle Y,Z \rangle X + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \] (2.6)
\[ (\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \] (2.7)
\[ \langle R^D(X,Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle, \] (2.8)

for vector fields \( X,Y,Z \) tangent to \( M \) and \( \xi \) normal to \( M \), where \( \bar{\nabla}h \) is defined by

\[ (\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \] (2.9)

and \( R^D \) is the curvature tensor associated with the normal connection \( D \), i.e.,

\[ R^D(X,Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]} \xi. \] (2.10)

For a space-like surface \( M \) in \( \mathbb{R}^4_2(c) \), the normal curvature \( K^D \) is given by

\[ K^D = \langle R^D(e_1, e_2)e_3, e_4 \rangle. \] (2.11)

A surface \( M \) in \( \mathbb{R}^4_2(c) \) is called a parallel surface if \( \bar{\nabla}h = 0 \) holds identically. An immersion \( \psi \) of a surface \( M \) in a pseudo-hyperbolic 4-space \( \mathbb{R}^4_2(c) \) is called full if \( \psi(M) \) does not lies in any totally geodesic submanifold of \( \mathbb{R}^4_2(c) \).

The surface \( M \) in \( \mathbb{R}^4_2(c) \) is called totally umbilical if the second fundamental form \( h \) of \( M \) satisfies \( h(X,Y) = g(X,Y)\xi, \forall X,Y \in TM \), for some normal vector field \( \xi \).

For an immersion \( \psi: M \rightarrow H^4_2(-1) \) of \( M \) into \( H^4_2(-1) \), let

\[ \phi = \iota \circ \psi : M \rightarrow \mathbb{E}^5_3 \]

 denote the composition of \( \psi \) with the standard inclusion \( \iota : H^4_2(-1) \rightarrow \mathbb{E}^5_3 \) via (1.3).

Denote by \( \tilde{\nabla} \) and \( \nabla \) the Levi-Civita connections of \( \mathbb{E}^5_3 \) and of \( M \), respectively. Let \( h \) be the second fundamental form of \( M \) in \( H^4_2(-1) \). Since \( H^4_2(-1) \) is totally umbilical with one as its mean curvature in \( \mathbb{E}^5_3 \), we have

\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) + \phi \] (2.12)

for \( X,Y \) tangent to \( M \).
2.2. Connection forms. Let \( \{e_1, e_2\} \) be an orthonormal frame of the tangent bundle \( TM \) of \( M \). Then we have
\[
\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1, \quad \langle e_1, e_2 \rangle = 0.
\]
We may choose an orthonormal normal frame \( \{e_3, e_4\} \) of \( M \) in \( R^3_2(c) \) such that
\[
\langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = -1, \quad \langle e_3, e_4 \rangle = 0.
\]
For the orthonormal frame \( \{e_1, e_2, e_3, e_4\} \), we put
\[
\nabla_X e_1 = \omega^2_1(X)e_2, \quad D_X e_3 = \omega^3_3(X)e_4,
\]
where \( \omega^2_1 \) and \( \omega^3_3 \) are the connection forms of the tangent and the normal bundles.

The Gauss curvature \( K \) and the normal curvature \( K^D \) of \( M \) are related with the connection forms \( \omega^2_1 \) and \( \omega^3_3 \) by
\[
d\omega^2_1 = -K(\ast 1), \quad d\omega^3_3 = -K^D(\ast 1),
\]
where \( \ast \) is the Hodge star operator of \( M \).

2.3. Ellipse of curvature. The ellipse of curvature of a surface \( M \) in \( R^3_2(c) \) is the subset of the normal plane defined as
\[
\{h(v,v) \in T^\perp_p M : |v| = 1, v \in T_p M, \ p \in M\}.
\]
To see that it is an ellipse, we consider an arbitrary orthogonal tangent frame \( \{e_1, e_2\} \). Put \( h_{i,j} = h(e_i, e_j), i, j = 1, 2 \) and look at the following formula
\[
h(v,v) = H + \frac{h_{11} - h_{22}}{2} \cos 2\theta + h_{12} \sin 2\theta, \quad v = \cos \theta e_1 + \sin \theta e_2.
\]
As \( v \) goes once around the unit tangent circle, \( h(v,v) \) goes twice around the ellipse. The ellipse of curvature could degenerate into a line segment or a point.

The center of the ellipse is \( H \). The ellipse of curvature is a circle if and only if the following two conditions hold:
\[
|h_{11} - h_{22}|^2 = 4|h_{12}|^2, \quad \langle h_{11} - h_{22}, h_{12} \rangle = 0.
\]

3. A minimal immersion of \( H^2(-\frac{1}{4}) \) into \( H^4_2(-1) \).

In this section, we recall a minimal immersion of \( H^2(-\frac{1}{4}) \) into \( H^4_2(-1) \) discovered recently in [3].

Consider the map \( \phi : \mathbb{R}^2 \to \mathbb{R}^5_3 \) defined by
\[
\phi(s,t) = \left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}}, t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}} \right) e^{\sqrt{3}t}, \quad \left( \frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left( \frac{t^3}{3} + \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}}, \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{1}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}} \right).
\]
The position vector \( x \) of \( \phi \) satisfies \( \langle x, x \rangle = -1 \) and the induced metric via \( \phi \) is \( g = ds^2 + e^{\frac{2s}{\sqrt{3}}}dt^2 \). Thus, \( \phi \) defines an isometric immersion \( \psi_\phi : H^2(-\frac{1}{4}) \to H^4_2(-1) \).
of the hyperbolic plane \( H^2(\frac{-1}{3}) \) of constant curvature \(-\frac{1}{3}\) into \( H^4_2(\frac{-1}{3}) \). This surface satisfies \( K^D = 2K = -\frac{2}{3} \). So, we have \( K + K^D = -1 \).

It was proved in [4] that, up to rigid motions, \( \psi : H^2(\frac{-1}{3}) \to H^4_2(\frac{-1}{3}) \) is the only parallel minimal space-like surface lying fully in \( H^4_2(\frac{-1}{3}) \).

Recently, B.-Y. Chen and B. D. Suceavă proved in [5] the following classification theorem.

**Theorem 3.1.** Let \( \psi : M \to H^4_2(\frac{-1}{3}) \) be a minimal immersion of a space-like surface \( M \) into \( H^4_2(\frac{-1}{3}) \). If the Gauss curvature \( K \) and the normal curvature \( K^D \) of \( M \) are constant, then one of the following three statements holds.

1. \( K = -1, K^D = 0 \), and \( \psi \) is totally geodesic.

2. \( K = K^D = 0 \) and \( \psi \) is congruent to an open part of the minimal surface defined by

\[
L(u, v) = \frac{1}{\sqrt{2}} (\cosh u, \cosh v, 0, \sinh u, \sinh v).
\]

3. \( K^D = 2K = -\frac{2}{3} \) and \( \psi \) is congruent to an open part of the minimal surface \( \psi : H^2(\frac{-1}{3}) \to H^4_2(\frac{-1}{3}) \) induced from (3.1).

**Remark 3.1.** If \( M \) is a space-like totally geodesic surface in \( H^4_2(\frac{-1}{3}) \), then the surface is congruent to an open part of the surface in \( H^4_2(\frac{-1}{3}) \) induced from (3.3) via (1.3).

### 4. A Wintgen Type Inequality for Space-like Surfaces in \( R^4_2(c) \).

We need the following result for later use.

**Theorem 4.1.** Let \( M \) be a space-like surface in a 4-dimensional neutral pseudo-Riemannian space form \( R^4_2(c) \) of constant sectional curvature \( c \). Then we have

\[
K + K^D \geq \langle H, H \rangle + c
\]

at every point in \( M \).

The equality sign of (4.1) holds at a point \( p \in M \) if and only if, with respect to some suitable orthonormal frame \( \{e_1, e_2, e_3, e_4\} \) at \( p \), the shape operators at \( p \) take the forms:

\[
A_{e_3} = \begin{pmatrix} 2\gamma + \mu & 0 \\ 0 & \mu \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.
\]

**Proof.** Assume that \( \psi : M \to R^4_2(c) \) is an isometric immersion of a space-like surface \( M \) into a pseudo-Riemannian space form \( R^4_2(c) \) of constant sectional curvature \( c \).

If \( p \in M \) is totally geodesic point, i.e., \( h(p) = 0 \), then we have \( K(p) = -1 \) and \( K^D(p) = 0 \). So we have \( K + K^D = c \) at \( p \).
If \( p \in M \) is a non-totally geodesic point, then we may choose an orthonormal frame \( \{e_1, e_2, e_3, e_4\} \) at \( p \) such that the shape operators at \( p \) satisfy

\[
A_{e_3} = \begin{pmatrix} \alpha & 0 \\ 0 & \mu \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta & \gamma \\ \gamma & -\delta \end{pmatrix}
\]

for some functions \( \alpha, \gamma, \delta, \mu \), with respect to \( \{e_1, e_2, e_3, e_4\} \).

From (2.4), (2.13), (2.14) and (4.3) we know that the second fundamental form of \( \psi \) satisfies

\[
h(e_1, e_1) = -\alpha e_3 - \delta e_4, \quad h(e_1, e_2) = -\gamma e_4, \quad h(e_2, e_2) = -\mu e_3 + \delta e_4.
\]

(4.4)

It follows from (4.4) and the equation of Gauss that the Gauss curvature \( K \), the normal curvature \( K^D \) and the mean curvature vector \( H \) of \( M \) at \( p \) satisfy

\[
K(p) = -\alpha \mu + \gamma^2 + \delta^2 + c,
\]

(4.5)

\[
K^D(p) = \gamma(\mu - \alpha),
\]

(4.6)

\[
H(p) = \frac{\alpha + \mu}{2} e_3.
\]

(4.7)

From (4.5)-(4.7) we have

\[
K(p) + K^D(p) = (H(p), H(p)) + \frac{1}{4}(2\gamma - \alpha + \mu)^2 + \delta^2 + c
\]

(4.8)

\[
\geq (H(p), H(p)) + c.
\]

Consequently, we obtain inequality (4.1).

If the equality case of (4.1) holds at \( p \in M \), then (4.8) implies that we have \( \delta = 0 \) and \( \alpha = 2\gamma + \mu \). Hence, we derive (4.2) from (4.3).

Conversely, if we have (4.2) at \( p \in M \), then it is easy to verify that the equality sign of (4.1) holds at \( p \). \( \square \)

**Remark 4.1.** Inequality (4.1) is a pseudo-hyperbolic version of an inequality of P. Wintgen obtained in [10] (see, also [8]).

**Remark 4.2.** Every space-like totally umbilical surface in \( \mathbb{R}^{4}_{2}(c) \) satisfies the equality case of (4.1) identically.

**Remark 4.3.** It follows from Theorem 4.1 that if a space-like surface \( M \) in \( \mathbb{R}^{4}_{2}(c) \) satisfies the equality case of inequality (4.1) identically, then \( M \) is a Chen surface (in the sense of [6, 7]).

**Remark 4.4.** It follows from Theorem 4.1 and conditions in (2.18) that if a space-like surface \( M \) in \( \mathbb{R}^{4}_{2}(c) \) satisfies the equality case of inequality (4.1) identically, then it has circular ellipse of curvature.

**Remark 4.5.** The minimal surface given by \( \psi_{\phi} : H^{2}(-\frac{1}{3}) \rightarrow H_{2}^{4}(-1) \) discovered in [1] satisfies the equality case of (4.1) identically (with \( H = 0, c = -1 \)).
Remark 4.6. On the neutral pseudo-Euclidean 4-space $\mathbb{E}_4^2$ equipped with the metric (4.9)

$$g_2 = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

we may consider the canonical complex coordinate system $\{z_1, z_2\}$ with

$$z_1 = x_1 + ix_2, z_2 = x_3 + ix_4.$$

The complex structure on $\mathbb{E}_4^2$ obtained in this way is called the standard complex structure on $\mathbb{E}_4^2$. In this way, we can regard $\mathbb{E}_4^2$ as a Lorentzian complex plane $\mathbb{C}_1^2$.

Lemma 4.1. Every space-like holomorphic curve in $\mathbb{C}_1^2$ satisfies the equality case of inequality (4.1) identically (with $H = c = 0$).

Proof. Let $\psi : M \to \mathbb{C}_1^2$ be a holomorphic space-like curve in $\mathbb{C}_1^2$. Let $e_1$ be a unit tangent vector field of $M$. Then $e_2 = Je_1$ is a unit tangent vector field of $M$ which is perpendicular to $e_1$. Consider an orthonormal normal frame $\{e_3, e_4\}$ of $M$ in $\mathbb{C}_1^2$ with $e_4 = Je_3$. Then it follows from $\tilde{\nabla}_X J = 0$ that

$$A_{e_4}X = JA_{e_3}X, \ \forall X \in TM.$$

(4.10)

By applying (4.10) we know that the shape operator $A$ satisfies

$$A_{e_3} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} -b & a \\ a & b \end{pmatrix}$$

(4.11)

for some functions $a, b$, with respect to $\{e_1, e_2, e_3, e_4\}$.

By applying (4.11) we obtain $H = 0$ and $K = -K^D = 2(a^2 + b^2)$. Therefore, we obtain the equality case of (4.1) identically. \qed

5. AN APPLICATION TO MINIMAL SURFACES IN $H_4^2(-1)$.

Recall that a function $f$ on a space-like surface $M$ is called logarithm-harmonic, if $\Delta(\ln f) = 0$ holds identically on $M$, where $\Delta(\ln f) := *d*(\ln f)$ is the Laplacian of $\ln f$ and $*$ is the Hodge star operator. A function $f$ on $M$ is called subharmonic if $\Delta f \geq 0$ holds everywhere on $M$.

In this section, we establish the following simple geometric characterization of the minimal immersion $\psi_\phi : H_4^2(-\frac{1}{3}) \to H_4^2(-1)$ given in section 3.

Theorem 5.1. Let $\psi : M \to H_4^2(-1)$ be a non-totally geodesic, minimal immersion of a space-like surface $M$ into $H_4^2(-1)$. Then

$$K + K^D \geq -1$$

(5.1)

holds identically on $M$.

If $K + 1$ is logarithm-harmonic, then the equality sign of (5.1) holds identically if and only if $\psi : M \to H_4^2(-1)$ is congruent to an open portion of the immersion
\( \psi : H^2(-1/3) \to H^4_2(-1) \) which is induced from the map \( \phi : \mathbb{R}^2 \to \mathbb{E}_3^5 \) defined by

\[
\phi(s, t) = \left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2t}{\sqrt{3}}}, t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2t}{\sqrt{3}}} \right).
\]

(5.2)

Proof. Assume that \( \psi : \mathbb{M} \to H^4_2(-1) \) is a non-totally geodesic, minimal immersion of a space-like surface \( \mathbb{M} \) into \( H^4_2(-1) \). Then the mean curvature vector \( H \) vanishes identically. Thus, we obtain inequality (5.1) from (4.1).

From now on, let us assume that \( \mathbb{M} \) is a minimal space-like surface in \( H^4_2(-1) \) which satisfies the equality case of (5.1) identically. Then Theorem ?? implies that there exists an orthonormal frame \( \{ e_1, e_2, e_3, e_4 \} \) such that the shape operators take the following special forms:

\[
A_{e_3} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.
\]

(5.3)

Hence, after applying (2.4), (2.13) and (2.14), we obtain

\[
h(e_1, e_1) = -\gamma e_3, \quad h(e_1, e_2) = -\gamma e_4, \quad h(e_2, e_2) = \gamma e_3.
\]

(5.4)

It follows from (2.15), (5.3) and the equation of Codazzi that

\[
e_1 \gamma = -2\gamma \omega^2_1(e_2) + \gamma \omega^4_3(e_2),
\]

(5.5)

\[
e_2 \gamma = 2\gamma \omega^2_1(e_1) - \gamma \omega^4_3(e_1).
\]

(5.6)

Since the star operator satisfies

\[
*(d\gamma) = -(e_2 \gamma)\omega^1 + (e_1 \gamma)\omega^2,
\]

Eqs. (5.3) and (5.4) imply that

\[
*d\gamma = \gamma (\omega^4_3 - 2\omega_2^2).
\]

(5.7)

Thus, we find from (2.10) and (5.7) that

\[
\Delta \gamma = \gamma (2K - K^D) + \frac{*(d\gamma \wedge *d\gamma)}{\gamma},
\]

(5.8)

where \( \Delta \gamma \) is the Laplacian of \( \gamma \) defined by \( \Delta \gamma = *d*d\gamma \).

From (5.8) we deduce that

\[
\Delta \gamma = \gamma (2K - K^D) + \frac{|d\gamma|^2}{\gamma}.
\]

(5.9)

On the other hand, it follows from

\[
\Delta(\ln(K + 1)) = *d*d(\ln(K + 1)), \quad K = 2\gamma^2 - 1
\]
that
\[
\Delta(\ln(K + 1)) = \frac{(K + 1)\Delta K - * (dK \wedge *dK)}{(K + 1)^2}
= \frac{2\gamma^2(4|d\gamma|^2 + 4\gamma \Delta\gamma) - 16\gamma^2|d\gamma|^2}{(K + 1)^2}
= \frac{2\gamma \Delta\gamma - 2|d\gamma|^2}{\gamma^2}.
\]
(5.10)

Therefore, (5.9) and (5.10) yield
\[
(5.11) \Delta(\ln(K + 1)) = 2(2K - K^D).
\]

Now, let us assume that \(K + 1\) is a logarithm-harmonic function, then Eq. (5.11) gives \(K^D = 2K\). Hence, after combining this with the equality case of (5.1), we obtain that \(K^D = 2K = -\frac{2}{3}\). Therefore, by applying Theorem 3.1 we conclude that, up to rigid motions of \(H^2_2(-1)\), the minimal surface is an open portion of the minimal surface \(\psi_\phi : H^2_2(-\frac{1}{3}) \to H^2_2(-1)\) induced from the map (3.1).

The converse can be verified by direct computation. □

**Corollary 5.1.** Let \(\psi : M \to H^2_2(-1)\) be a minimal immersion of a space-like surface \(M\) of constant Gauss curvature into \(H^2_2(-1)\). Then the equality sign of (5.1) holds identically if and only if one of the following two statements holds.

1. \(K = -1, K^D = 0\), and \(\psi\) is totally geodesic.
2. \(K^D = 2K = -\frac{2}{3}\) and \(\psi\) is congruent to an open part of the minimal surface \(\psi_\phi : H^2_2(-\frac{1}{3}) \to H^2_2(-1)\) induced from (3.1).

**Proof.** Let \(\psi : M \to H^2_2(-1)\) be a minimal immersion of a space-like surface \(M\) into \(H^2_2(-1)\). If the Gauss curvature \(K\) is constant and the equality sign of (5.1) holds, then both \(K\) and \(K^D\) are constant. Therefore, by applying Theorem 3.1 we obtain either Case (1) or Case (2).

The converse is trivial. □

6. **Space-like minimal surfaces in \(E^4_2\) satisfying the equality.**

It follows from Lemma 4.1 that there exist infinitely many non-totally geodesic, minimal space-like surfaces in \(E^4_2\) which satisfy the equality case of inequality (4.1) identically (with \(H = c = 0\)).

On the other hand, we have the following.

**Proposition 6.1.** Let \(\psi : M \to E^4_2\) be a minimal immersion of a space-like surface \(M\) into the pseudo-Euclidean 4-space \(E^4_2\). Then
\[
(6.1) K \geq -K^D
\]
holds identically on \(M\).

If \(M\) has constant Gauss curvature, then the equality sign of (6.1) holds identically if and only if \(M\) is a totally geodesic surface.
Let $\psi : M \to \mathbb{E}^4_2$ be a minimal immersion of a space-like surface $M$ into $\mathbb{E}^4_2$. Then inequality \((6.1)\) follows immediately from Theorem \(\text{??}\).

Assume that the equality case of \((6.1)\) holds identically. Then Theorem \(\text{??}\) implies that there exists an orthonormal frame \(\{e_1, e_2, e_3, e_4\}\) such that the shape operator $A$ takes the special forms:

\begin{align*}
Ae_3 &= \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \\
Ae_4 &= \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.
\end{align*}

\(\text{(6.2)}\)

From \(\text{(6.2)}\) and the equation of Codazzi we find

\begin{align*}
e_1\gamma &= -2\gamma \omega^2_1(e_2) + \gamma \omega^3_3(e_2), \\
e_2\gamma &= 2\gamma \omega^2_1(e_1) - \gamma \omega^4_3(e_1).
\end{align*}

\(\text{(6.3)}\) \(\text{and} \ (6.4)\)

If the Gauss curvature $K$ is a nonzero constant, then the function $\gamma$ is a nonzero constant. In this case, \(\text{(6.3)}\) \(\text{and} \ (6.4)\) imply that

\begin{equation}
2\omega^2_1 = \omega^4_3.
\end{equation}

\(\text{(6.5)}\)

Thus, after taking exterior differentiation of \(\text{(6.3)}\) \(\text{and} \ (6.4)\), we obtain $2K = K^D$. Combining this with the equality of \(\text{(6.1)}\) yields $K = 0$, which is a contradiction. Therefore, we must have $K = 2\gamma^2 = 0$. Consequently, $M$ is totally geodesic in $\mathbb{E}^4_2$.

The converse is trivial. \(\square\)

**Proposition 6.2.** Let $\psi : M \to \mathbb{E}^4_2$ be a minimal immersion of a space-like surface $M$ into $\mathbb{E}^4_2$. We have

1. If the equality sign of \(\text{(6.1)}\) holds identically, then $K$ is a non-logarithm-harmonic function.

2. If $M$ contains no totally geodesic points and the equality sign of \(\text{(6.1)}\) holds identically on $M$, then $\ln K$ is subharmonic.

**Proof.** Assume that $M$ is a minimal space-like surface in $\mathbb{E}^4_2$ which satisfies the equality case of \(\text{(6.1)}\), i.e., $K = -K^D$ identically. Then Theorem \(\text{??}\) implies that there exists an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the shape operator $A$ takes the special forms given by \(\text{(6.2)}\).

From \(\text{(6.2)}\) and the equation of Codazzi we obtain \(\text{(6.3)}\) \(\text{and} \ (6.4)\). Thus, we may apply the same arguments as in section 5 to obtain that

\begin{equation}
\Delta(\ln K) = 2(2K - K^D)
\end{equation}

at each non-totally geodesic point. Hence, after combining this with $K = -K^D$, we obtain $K = 0$. But this is impossible, since in this case $\ln K$ is undefined. This proves statement (1).
Next, assume that $M$ contains no totally geodesic points and that the equality sign of (6.1) holds identically on $M$. Then, we find from (6.2), (6.6) and $K = -K^D$ that

$$\Delta(\ln K) = 6K = 12\gamma^2 > 0,$$

which implies that $\ln K$ is a subharmonic function. This proves statement (2). □

7. Space-like minimal surfaces in $S^4_2(1)$ satisfying the equality.

Now, we study space-like minimal surfaces in $S^4_2(1)$ satisfying the equality case of inequality (4.1).

Proposition 7.1. Let $\psi : M \to S^4_2(1)$ be a minimal immersion of a space-like surface $M$ into the neutral pseudo-sphere $S^4_2(1)$. Then

$$(7.1) \quad K + K^D \geq 1$$

holds identically on $M$.

If $M$ has constant Gauss curvature, then the equality sign of (7.1) holds identically if and only if $M$ is a totally geodesic surface.

Proof. Assume that $\psi : M \to S^4_2(1)$ is a minimal immersion of a space-like surface $M$ into $S^4_2(1)$. Then inequality (4.1) in Theorem ?? reduces to inequality (7.1).

Suppose that the equality case of (7.1) holds identically on $M$, then Theorem ?? implies that there exists an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the shape operator $A$ takes the following special forms:

$$(7.2) \quad A_{e_3} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.$$ 

Hence, by applying (2.4), (2.13) and (2.14), we know that the second fundamental form $h$ satisfies

$$(7.3) \quad h(e_1, e_1) = -\gamma e_3, \quad h(e_1, e_2) = -\gamma e_4, \quad h(e_2, e_2) = \gamma e_3.$$ 

It follows from (7.3) that the Gauss curvature of $M$ is given by $K = 1 + 2\gamma^2$. Now, let us assume that the Gauss curvature $K$ is constant. Then $\gamma$ is constant. Let us suppose that $M$ is non-totally geodesic in $S^4_2(1)$. Then, by applying (7.3) and the equation of Codazzi, we find

$$(7.4) \quad 2\omega_1^2 = \omega_3^2.$$ 

Thus, after taking exterior differentiation of (7.4) and applying (2.10), we obtain

$$(7.5) \quad 2K = K^D.$$ 

By combining (7.5) with the equality of (7.1), we get $K^D = \frac{2}{\gamma}$.

On the other hand, it follows from (2.11) and (7.2) that $K^D = -2\gamma^2 \leq 0$, which contradicts to $K^D = \frac{2}{\gamma}$. Consequently, $M$ must be totally geodesic in $S^4_2(1)$. 

Conversely, if $M$ is totally geodesic in $S^4_2(1)$, then we have $K = 1$ and $K^D = 0$. So, we get $K + K^D = 1$, which is exactly the equality case of (7.1). \qed

Finally, we prove the following.

**Proposition 7.2.** Let $\psi : M \to S^4_2(1)$ be a minimal immersion of a space-like surface $M$ into $S^4_2(1)$. We have

1. If the equality sign of (7.1) holds identically, then $K - 1$ is non-logarithm-harmonic.
2. If $M$ contains no totally geodesic points and if the equality case of (7.1) holds, then $\ln(K - 1)$ is subharmonic.

**Proof.** Assume that $M$ is a minimal space-like surface of $S^4_2(1)$ which satisfies the equality case of (7.1) identically. Then we have $K + K^D = 1$. Moreover, from Theorem 5 we know that there exists an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the shape operator $A$ satisfies

\begin{equation}
A_{e_3} = \begin{pmatrix}
\gamma & 0 \\
0 & -\gamma
\end{pmatrix},
A_{e_4} = \begin{pmatrix}
0 & \gamma \\
\gamma & 0
\end{pmatrix}.
\end{equation}

Hence, we may applying the same arguments as in section 5 to obtain that

\begin{equation}
\Delta(\ln(K - 1)) = 2(2K - K^D).
\end{equation}

If $K - 1$ is logarithm-harmonic, then Eq. (7.7) yields $K^D = 2K$. Thus, after combining this with the equality $K + K^D = 1$, we obtain

\begin{equation}
K^D = \frac{2}{3}.
\end{equation}

On the other hand, we find from (7.6) that $K^D = -2\gamma^2 \leq 0$, which contradicts to (7.8). Consequently, $K - 1$ cannot be a logarithm-harmonic function. This proves statement (1).

Next, assume that $M$ contains no totally geodesic points and if the equality case of (7.1) holds. Then, we find from (7.6) and (7.7) that

\begin{equation}
\Delta(\ln(K - 1)) = 4(3\gamma^2 + 1) > 0.
\end{equation}

Hence, $\ln(K - 1)$ is a subharmonic function. This proves statement (1). \qed

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