ON THE PATH INTEGRAL TREATMENT FOR AN
AHARONOV–BOHM FIELD ON THE HYPERBOLIC PLANE

Christian Grosche

II. Institut für Theoretische Physik
Universität Hamburg, Luruper Chaussee 149
22761 Hamburg, Germany

ABSTRACT

In this paper I discuss by means of path integrals the quantum dynamics of a charged particle on the hyperbolic plane under the influence of an Aharonov–Bohm gauge field. The path integral can be solved in terms of an expansion of the homotopy classes of paths. I discuss the interference pattern of scattering by an Aharonov–Bohm gauge field in the flat space limit, yielding a characteristic oscillating behavior in terms of the field strength. In addition, the cases of the isotropic Higgs-oscillator and the Kepler–Coulomb potential on the hyperbolic plane are shortly sketched.
1 Introduction

The Aharonov–Bohm gauge field has a long history, beginning in 1959 by a classical paper by Aharonov and Bohm [Aharonov and Bohm (1959)]. The effect has been well studied and well confirmed [Anandan and Safko (1994)], but not necessarily well understood. It describes the motion of charged particles, i.e. electrons, which are scattered by an infinitesimal thin solenoid. The magnetic vector potential $A$ of the solenoid produces a magnetic field which is essentially $\delta$-like, i.e., its support is an infinitesimal thin solenoid, and it is vanishing everywhere else. Geometrically this experimental set-up corresponds to the quantum motion of a particle (which we consider as spin-less) in $\mathbb{R}^2$, where a point has been removed with the consequence that topologically $\mathbb{R}^2$ becomes no longer connected. Since the solenoid is assumed impenetrable, the space of the particle motion $\mathcal{M}$ is the Euclidean plane minus the cross section of the solenoid. Everywhere in $\mathcal{M}$, $\nabla \times A = 0$ and hence $A = \nabla f(x)$, where $f(r)$ is an arbitrary scalar function of $r = |x|, x \in \mathbb{R}^2$. Classically, a charged particle is not affected at all by the solenoid. However, in quantum mechanics, the particle’s wave function picks up in a scattering experiment a phase factor according to

$$\Psi_\alpha(x) = \Psi_0(x) \exp \left( \frac{ie}{\hbar c} \int_{\text{path } \alpha} A \cdot dx \right),$$

(1)

where $\Psi_0(x)$ is the vector potential-free solution. The wave-function $\Psi$ effective to a measurement is the sum of solutions corresponding to inequivalent paths, i.e., $\Psi = \sum_\alpha \Psi_\alpha$. Topologically the paths $\alpha$ can be distinguished by their winding numbers $n$, thus giving rise to infinitely many homotopy classes designated by the number $n$.

Path integral treatments of the Aharonov–Bohm effect in the Euclidean plane are due to Bernido and Inomata [Bernido and Inomata (1980)], Gerry and Singh [Gerry and Singh (1979)], Liang [Liang (1988)], and Schulman [Schulman (1971)]. Harmonic interactions have been dealt with in [Kibler and Campigotto (1993)], the Coulomb–Kepler potential have been taken into account by, e.g. [Chetounai et al. (1989), Dragomir et al. (1992), Hoang et al. (1992), Kibler and Negadi (1987), Lin (1998), Park and Yoo (1998)], relativistic particles by, e.g. [Bernido (1993), Gamboa and Rivelles (1991), Hoang et al. (1992), Hoang and Giang (1993), Lin (1998), Park and Yoo (1998)], and a more comprehensive bibliography can be found in, e.g. [Anandan and Safko (1994), Grosche and Steiner (1998)].

Path integrals, e.g. [Feynman and Hibbs (1965), Grosche (1996), Grosche and Steiner (1998), Kleinert (1995), and Schulman (1981)] provide us with global information of the quantum motion, including the topological effects on the wave-function. If we want to study the Aharonov–Bohm effect by means of path integrals [Bernido and Inomata (1980), Gerry and Singh (1979), Liang (1988)] we consider the time evolution from $t = 0$ to $t = T$ of the wave-function of a particle according to

$$\Psi_\alpha(x''; T) = \sum_\beta \int K_{\alpha,\beta}(x'', x'; T) \Psi_\alpha(x'; 0) \, dx',$$

(2)

where

$$K_{\alpha,\beta}(x'', x'; T) = K_0(x'', x'; T) \exp \left[ \frac{ie}{\hbar c} \left( \int_{\text{path } \alpha} x'' - \int_{\text{path } \beta} x' \right) A \cdot dx \right],$$

(3)

and this leads us the the formal expression separating the sum over $\alpha$ and $\beta$ (under the assump-
tion the separation is well-defined)
\[
\sum_{\alpha,\beta} K_{\alpha\beta} \Psi_{\beta} = K \sum_\beta \Psi_{\beta} .
\] (4)

Provided the paths \(\alpha, \beta\) cover in an idealized experiment the whole range from minus infinity to plus infinity, we can express the separation of the time evolution of the particle according to
\[
K(x'', x'; T) = \sum_{n=-\infty}^{\infty} K_n(x'', x'; T) ,
\] (5)

where \(n = 0\) denotes the unperturbed case in \(\mathbb{R}^2\), i.e., we obtain the free propagator on the entire \(\mathbb{R}^2\). For the final result we obtain for the Feynman kernel the following form, e.g. [Berndio and Inomata (1980), Grosche and Steiner (1998), Liang (1988)]

\[
K(x'', x'; T) = \frac{m}{2 \pi i \hbar T} \exp \left( \frac{im}{2 \hbar T} (r^2 + r'^2) \right) \sum_{n=-\infty}^{\infty} e^{in(\varphi'' - \varphi')} I_{|n-\xi|} \left( \frac{mr''r'}{i \hbar T} \right) .
\] (6)

Here, two-dimensional polar coordinates \((r, \varphi)\) have been used, and \(\xi = e\Phi/2\pi \hbar c\) with \(\Phi = B \times \text{area the magnetic flux.}\)

## 2 Aharonov–Bohm Field on the Hyperbolic Plane

In this paper I would like to give a path integral treatment of the Aharonov–Bohm effect on the hyperbolic plane [Kuperin et al. (1994)], i.e., the scattering of (spin-less) electrons by an Aharonov–Bohm field on leaky tori. Such systems play an important rôle in the theory of quantum chaos, e.g. [Gutzwiller (1991)]. The hyperbolic plane, respectively Lobachevsky space, is defined as one sheet of the double sheeted hyperboloid

\[
u^2 = u_0^2 - u_1^2 - u_2^2 = R^2 , \quad u_0 > 0 .
\] (7)

The model of the upper-half plane \(U = \{ \Im(z) = y > 0 | z = x + iy \}\) endowed with the metric has the form (where I have set for simplicity \(R = 1\))

\[
ds^2 = \frac{dx^2 + dy^2}{y^2} , \quad x \in \mathbb{R}, y > 0 .
\] (8)

Alternatively I can also consider the unit disc model \(D = \{ z = re^{i\vartheta} | r < 1, \vartheta \in [0, 2\pi) \}\)

\[
ds^2 = 4 \frac{dr^2 + r^2 d\vartheta^2}{(1 - r^2)^2} , \quad r < 1 , \vartheta \in [0, 2\pi) ,
\] (9)

and the pseudosphere \(\Lambda = \{ z = i \tanh(\tau/2) e^{-i\varphi} \tau > 0, \varphi \in [0, 2\pi) \}\)

\[
ds^2 = d\tau^2 + \sinh^2 \tau d\varphi^2 , \quad \tau > 0, \varphi \in [0, 2\pi) .
\] (10)

\(U, D\) and \(\Lambda\) are three coordinate space representations out of nine of the hyperbolic plane [Grosche et al. (1996), Grosche (1996), Olevskiï(1950)]. Plane waves have the asymptotic representation \(\propto y^{1/2 \pm ik}\) (e.g. on \(U, k\) the wave-number), \(e^{-(-1/2 \pm i/2)\tau}\) (on \(\Lambda\), and the coordinate
The bound state solutions are given by hyperbolic plane are Möbius transformations corresponding to the symmetry group $\text{PSL}(2, \mathbb{R})$, and magnetic fields give rise to the consideration of automorphic forms in the theory of the Selberg trace formula \[\text{Hejhal (1976)}.\]

Constant magnetic fields on the hyperbolic plane have been studied in, e.g. \[\text{Comtet (1987)}, \text{Fay (1977)}, \text{Pnueli (1994)}\], and by means of path integrals in \[\text{Grosche (1988), Grosche (1990a)}\]. The path integral formulation for a particle on the hyperbolic plane subject to a constant magnetic field on $\Lambda$ has the form \[\text{Grosche (1990a)}\] (I implicitly assume that the constant negative curvature of the hyperbolic plane, i.e., the two-dimensional hyperboloid equals one, $\mathbf{u} \in \Lambda$)

\[
K(u'', u'; T) = K(\tau'', \tau', \varphi'', \varphi'; T) \\
= \int_{\tau(0) = \tau'}^{\tau(T) = \tau''} \mathcal{D}\tau(t) \sinh \tau \int_{\varphi(0) = \varphi'}^{\varphi(T) = \varphi''} \mathcal{D}\varphi(t) \\
\times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} (\dot{\tau}^2 + \sinh^2 \tau \varphi^2) - b(\cosh \tau - 1)\dot{\varphi} - \frac{\hbar^2}{2m} \left(1 - \frac{1}{\sinh^2 \tau}\right) \right] \, dt \right\} \\
= \exp \left( -\frac{i\hbar T}{8m} \lim_{N \to \infty} \left( \frac{m}{2\pi \hbar c} \right)^N \prod_{j=1}^{N-1} \int_0^{\infty} \sinh \tau_j \, d\tau_j \int_0^{2\pi} d\varphi_j \right) \\
\times \exp \left[ \frac{i}{\hbar} \sum_{j=1}^N \left( \frac{m}{2c} \left( \Delta^2 \tau_j + \sinh^2 \tau_j \Delta^2 \varphi_j \right) - b(\cosh \tau_j - 1)\Delta \varphi_j - \frac{\epsilon \hbar^2}{8m \sinh^2 \tau_j} \right) \right] \\
= \sum_{l=-\infty}^{N_{\text{max}}} \sum_{N=0}^{\left\lfloor \frac{T}{\hbar} \right\rfloor} \left[ \sum_{k=0}^{N} e^{-iE_N/T} \Psi_{Nl}^{\pm}(\tau'', \varphi'') \Psi_{Nl}^{\pm}(\tau', \varphi') + \int_0^\infty dk e^{-iE_k/T} \Psi_{kl}^{\pm}(\tau'', \varphi'') \Psi_{kl}^{\pm}(\tau', \varphi') \right].
\]

Here $b = eB/\hbar c$, with $B$ the strength of the magnetic field, $c$ denotes the velocity of light. For the magnetic field $\mathbf{B}$ I have chosen the gauge

\[
\mathbf{A} = \begin{pmatrix} A_\tau \\ A_\varphi \end{pmatrix} = B(\cosh \tau - 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Due to $d\mathbf{B} = (\partial_\tau A_\varphi - \partial_\varphi A_\tau) \, d\tau \wedge d\varphi = (m/2)B \sinh \tau d\tau \wedge d\varphi$, $d\mathbf{B}$ has the form constant $\times$ volume form and can thus interpreted indeed as a constant magnetic field. In the lattice formulation I have taken \[\text{Grosche (1996)}, \text{Grosche and Steiner (1998)}\] $\Delta q_j = q_j - q_{j-1}$, $q_j = q(t_j)$, $t_j = j\epsilon$, $j = 1, \ldots, N$, $\epsilon = T/N$, $N \to \infty$, $f^2(q_j) \equiv f(q_{j-1})f(q_j)$, for any function $f$ of the coordinates. The bound state solutions are given by

\[
\Psi_{N,l}^{\pm}(\tau, \varphi) = \left[ \frac{\Gamma(2b - |l|)\Gamma(2b - N + |l|)}{4\pi N! \Gamma(2b - N) \Gamma(2b - |l|)} \right]^{1/2} \\
\times e^{i\varphi} \left( \frac{\tanh \tau}{2} \right)^{|l|} \left( 1 - \tanh^2 \frac{\tau}{2} \right)^{b-N} P_N(|l|, 2b - 2N - 1) \left( 1 - 2 \tanh^2 \frac{\tau}{2} \right),
\]

\[
E_N = \frac{\hbar^2}{2m} \left[ b^2 + \frac{1}{4} - \left( b - N - \frac{1}{2} \right)^2 \right], \quad (N = 0, 1, \ldots \leq N_{\text{max}} < b - \frac{1}{2}).
\]
\[ P_n^{(a,b)}(x) \] are Jacobi polynomials \cite{Gradshteyn and Ryzhik (1980)}. The energy-levels \cite{14} are the Landau levels on the hyperbolic plane. This is in complete analogy to the flat space case, where the Landau levels are \( E_n = \hbar \omega (n + \frac{1}{2}) \) with \( \omega = eB/\hbar c \) the cyclotron frequency, and the bound states are described by Laguerre polynomials, e.g. \cite{Grosche and Steiner (1998)}. The flat space limit can be recovered \cite{Grosche et al. (1996)} by re-introducing the constant curvature \( k = 1/R \) \((R > 0)\), redefining \( E_N \to E_N/R^2, b \to bR^2 \) (note \( b(\cosh \tau - 1) \to br^2R^2/2, r > 0 \) the polar variable in \( \mathbb{R}^2 \), as \( R \to \infty \)), and considering the limit \( R \to \infty \).

For the continuous states the wave-functions and the energy spectrum, respectively, I obtain

\[
\Psi_{k,l}^b(\tau, \varphi) = \frac{1}{\pi|l|} \sqrt{\frac{k \sinh 2\pi k}{4\pi}} \Gamma\left(1 + \frac{i k}{2} + b + |l|\right) \Gamma\left(1 + \frac{i k}{2} - b\right) \\
\times e^{il\varphi} \left(\tanh \frac{\tau}{2}\right)^{|l|} \left(1 - \tanh^2 \frac{\tau}{2}\right)^{\frac{1}{2} + ik} \\
\times {}_2F_1\left(\frac{1}{2} - ik + b + |l|, \frac{1}{2} + ik - b; 1 + |l|; \tanh^2 \frac{\tau}{2}\right),
\]

\( E_k = \frac{\hbar^2}{2m} \left(k^2 + b^2 + \frac{1}{4}\right). \) \( \text{(15)} \)

\( {}_2F_1(a, b; c; z) \) is the hypergeometric function, and \( k > 0 \) denotes the wave-number. I note that a minimum strength of \( B \) is required in order that bound states can occur, and only a finite number of bound states can exist. For the case that the magnetic field vanishes I obtain \cite{Grosche and Steiner (1988)} (e.g. \cite{Gradshteyn and Ryzhik (1980)} for the relation of the Legendre functions to the hypergeometric function)

\[
\Psi_{k,l} = \sqrt{\frac{k \sinh \pi k}{2\pi}} \Gamma\left(\frac{1}{2} + ik + |l|\right) e^{i\varphi} \mathcal{P}_{-|l|/2}(\cosh \tau),
\]

\( E_k = \frac{\hbar^2}{2m} \left(k^2 + \frac{1}{4}\right). \) \( \text{(17)} \)

For instance, we have the relation \cite{Abramowitz and Stegun (1984)}

\[
\mathcal{P}_{-\nu/2}(\cosh \tau) = \frac{1}{\Gamma(1 - \nu)} 2^{2\mu}(1 - e^{-2\tau})^{-\mu} e^{-(\nu + 1/2)\tau} \\
\times {}_2F_1\left(\frac{1}{2} - \mu; \frac{1}{2} + \nu - \mu; 1 - 2\mu; 1 - e^{-2\tau}\right).
\]

\( \text{(19)} \)

However, for the vector potential for an Aharonov–Bohm gauge field, we need another Ansatz. According to \cite{Kuperin et al. (1994)} I take for \( A = Be_\varphi \) with \( B = \text{const} \). Therefore I get for the classical Hamiltonian

\[
\mathcal{H} = \frac{\hbar^2}{2m} \left[p^2 + \frac{1}{\sinh^2 \tau} \left(p_\varphi - \frac{eB}{\hbar c}\right)^2\right],
\]

\( \text{(20)} \)

and for the Lagrangian, respectively \((b = eB/\hbar c)\)

\[
\mathcal{L} = \frac{m}{2}(\dot{\varphi}^2 + \sinh^2 \tau \dot{\varphi}^2) + \frac{e}{c} A \cdot \left(\dot{\varphi}^2\right) = \frac{m}{2}(\dot{\varphi}^2 + \sinh^2 \tau \dot{\varphi}^2) + \xi \dot{\varphi}.
\]

\( \text{(21)} \)
Note that the vector potential in (12) vanishes at \( \tau = 0 \), which means that we can take any constant for \( A_\varphi \) depending on the gauge, and the requirement that it is non-zero. With the momentum operators \( p_\varphi = (\hbar/i)(\partial_\varphi + \coth \tau) \) and \( p_\varphi = (\hbar/i)\partial_\varphi \) we get for the quantum Hamiltonian (together with the quantum potential \( \propto \hbar^2/p_\varphi \))

\[
H = \frac{\hbar^2}{2m} \left[ p_\varphi^2 + \frac{1}{\sinh^2 \tau} \left( p_\varphi - \frac{eB}{\hbar c} \right)^2 \right] + \frac{\hbar^2}{8m} \left( 1 - \frac{1}{\sinh^2 \tau} \right).
\]

The angular variable \( \varphi \) varies in the interval \([0, 2\pi]\), and therefore we usually assume \( \varphi_j \in [0, 2\pi] \), \( \forall j \). However, the path can loop around the infinitesimal solenoid many times, which has the consequence that in our case \( \varphi_j \in \mathbb{R}, \forall j \). Therefore, the path integral, if calculated according to (11), gives only a partial propagator which belongs to a class of paths topologically constraint by \( \varphi_j \in [0, 2\pi], \forall j \). For the total propagator, we have to take into account all paths from all homotopically different classes. This can be done by considering the path integration over the angular variable \( \varphi_j \) remaining in the physical space \( \mathbb{M} \) with \( \Delta \varphi_j = \varphi_j - \varphi_{j-1} + 2\pi n \) \((\varphi_j \in [0, 2\pi], n \in \mathbb{Z})\), or alternatively switching to the covering space \( \mathbb{M}' \) with \( \Delta \varphi_j = \varphi_j - \varphi_{j-1} \), where \( \varphi_j \in \mathbb{R} \). I therefore incorporate the effect of the infinitesimal thin solenoid by a \( \delta \)-function constraint in the path integral, with an additional integration \( \int d\varphi \) [Berndio and Inomata (1980)], therefore I get (expanding the \( \delta \)-function, \( \xi = e\Phi/2\pi\hbar c \) with \( \Phi \) the magnetic flux.)

\[
K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) = \int d\varphi \int_{\tau(0) = \tau'}^{\tau(T) = \tau''} D\tau(t) \sinh \tau \int_{\varphi(0) = \varphi'}^{\varphi(T) = \varphi''} D\varphi(t) \delta \left( \varphi - \int_0^T \dot{\varphi} \, dt \right) \times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \left( \dot{\tau}^2 + \sinh^2 \tau \dot{\varphi}^2 \right) + b \dot{\varphi} - \frac{\hbar^2}{8m} \left( 1 - \frac{1}{\sinh^2 \tau} \right) \right] dt \right\}
\]

\[
= \exp \left( - \frac{itB}{8m} \right) \int \frac{d\varphi}{2\pi} \int d\lambda e^{i\lambda\varphi} \lim_{N \to \infty} \left( \frac{m}{2\pi i\hbar c} \right)^N \prod_{j=1}^{N-1} \int_0^{\pi} \sinh \tau_j \, d\tau_j \int_0^{2\pi} d\varphi_j
\]

\[
\times \exp \left[ \frac{i}{\hbar} \sum_{j=1}^N \left( \frac{m}{2\tau_j} \left( \Delta^2 \tau_j + \sinh^2 \tau_j \Delta^2 \varphi_j \right) + (\xi - \lambda) \Delta \varphi_j - \frac{\hbar^2}{8m \sinh^2 \tau_j} \right) \right]
\]

\[
= \int d\varphi \int \frac{d\lambda}{2\pi} e^{i\lambda\varphi} \sum_{l=-\infty}^{\infty} e^{il(\varphi''-\varphi')} K_{\lambda+l-\xi}(\tau'', \tau'; T),
\]

where

\[
K_{\lambda+l-\xi}(\tau'', \tau'; T) = e^{-itB/8m} \int_{\tau(0) = \tau'}^{\tau(T) = \tau''} D\tau(t) \exp \left[ \frac{i}{\hbar} \int_0^T \left( \frac{m}{2} \dot{\tau}^2 - \frac{\hbar^2}{2m} \left( \frac{\lambda + l - \xi}{\sinh^2 \tau} \right)^2 - 1/4 \right) \, dt \right].
\]

Using Poisson’s summation formula

\[
\sum_{l=-\infty}^{\infty} e^{il\theta} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\theta + 2\pi k),
\]
I obtain (by changing the integration variable $\lambda \to \lambda + \xi - l$)

$$K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) = \frac{1}{2\pi} \int_{\mathbb{R}} d\varphi \int_{\mathbb{R}} d\lambda e^{i\lambda\varphi} \sum_{l=-\infty}^{\infty} e^{i(\varphi'' - \varphi')} K_{\lambda+l-\xi}(\tau'', \tau', T)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} d\varphi \int_{\mathbb{R}} d\lambda e^{i(\varphi'' - \varphi' + i(\lambda + \xi))} K_{\lambda}(\tau'', \tau', T)$$

$$= \int_{\mathbb{R}} d\varphi \sum_{k=-\infty}^{\infty} \delta(\varphi'' - \varphi' - \varphi + 2\pi k) e^{i(\lambda + \xi)\varphi} \int_{\mathbb{R}} d\lambda K_{\lambda}(\tau'', \tau', T) . \quad (26)$$

$K_{\lambda}$ is now given by

$$K_{\lambda}(\tau'', \tau', T) = e^{-\frac{iHT}{8m}} \int_{\tau(0)=\tau''}^{\tau(T)=\tau'} D\tau(t) \exp \left[ \frac{1}{\hbar} \int_{0}^{T} \left( \frac{m}{2} \dot{r}^2 - \frac{\hbar^2}{2m} \lambda^2 - \frac{1}{4} \right) dt \right]$$

$$= \int_{0}^{\infty} dk e^{-\frac{iE_kT}{8m}} \Psi_{k,\lambda}(\tau'') \Psi^{*}_{k,\lambda}(\tau') . \quad (27)$$

The wave-functions and the energy spectrum are given by (17, 18), respectively, with $l \to \lambda$. Performing the $\varphi$-integration in (26) yields

$$K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) = \sum_{n=-\infty}^{\infty} e^{i\xi(\varphi'' - \varphi' + 2\pi n)} \int_{\mathbb{R}} d\lambda e^{i\lambda(\varphi'' - \varphi' + 2\pi n)} K_{\lambda}(\tau'', \tau', T) , \quad (28)$$

which displays the expansion into the winding numbers. For $\xi = 0$ the free Feynman kernel on $A$ is recovered.

If we want to study the effect of scattering by an Aharonov–Bohm solenoid we must consider interference terms according to

$$I_{nl} = K_{n}^{\mu} K_{l} + K_{l}^{\mu} K_{n} . \quad (29)$$

Unfortunately, a closed expression for the propagator (27) does not exist. We can either analyze (27) by means of an asymptotic expansion of the Legendre functions, i.e., $P_{\mu}^{-\frac{1}{2}}(z) \propto (\Gamma(ip)/\Gamma(1/2 + ip - \mu))/(2z)^{\frac{1}{2} - ip}/\sqrt{\pi} + c.c.$, as $|z| \to \infty$, which yields very complicated and analytically intractable integrals over $\Gamma$-functions. Alternatively I can use the formula $\lim_{v \to \infty} \nu^\mu P_{\nu}^{-\frac{1}{2}}(\cosh(z/\nu)) = I_{\nu}(z)$ [Gradshteyn and Ryzhik (1980)] which corresponds to the flat space limit of the hyperbolic space with constant curvature $R$. Restricting therefore the evaluation of $I_{nl}$ to the flat space limit $R \to \infty$ I re-introduce the constant curvature $R$ into the path integral (27) by means of $m r^2 \to m R^2 \frac{r^2}{r^2 + z^2} = m r^2$, and $m \sinh^2 \tau \to m R^2 \sinh^2 \tau \to m R^2 \tau^2 = m r^2$ ($r = R \tau$ is the radial variable in Euclidean polar coordinates), as $R \to \infty$ [Izvestev et al. (1997)]. This gives for $K_{\lambda}$ in this limit the usual free Feynman kernel in polar coordinates in $\mathbb{R}^{2}$ [Grosche and Steiner (1998), Peak and Inomata (1969)]

$$K_{\lambda}(\tau'', \tau', T) \simeq K_{\lambda}(\tau'', \tau', T) = \frac{m}{2\pi i T} \exp \left[ \frac{im}{2T} (r'^2 + r''^2) I_{\lambda} \left( \frac{m}{i T} \right) \right] . \quad (30)$$

Following [Berndio and Inomata (1980)] we can now evaluate $I_{nl}$. By means of the asymptotic formula ($|z| \to \infty$, $\Re(z) > 0$)

$$I_{\lambda}(z) \simeq \sqrt{\frac{i}{2\pi z}} \exp \left( \frac{z^2 - 1/4}{2z} \right) , \quad (31)$$

$$\text{6}$$
and a Gaussian integration we get the asymptotic expansion
\[
\int_{-\infty}^{\infty} d\lambda e^{i\lambda \Theta} I_\lambda(z) \simeq \exp \left( z + \frac{1}{8z} - \frac{z^2}{2} \Theta^2 \right). \tag{32}
\]

Hence I obtain for the partial propagator \( K_n \) (with \( z = mr''/i\hbar T \), the condition \( \Re(z) \) ignored, c.f. [Berndio and Inomata (1980), Grosche and Steiner (1998), Peak and Inomata (1969)])
\[
K_n(\tau'', \tau'; \varphi'', \varphi'; T) \simeq \frac{m}{2\pi i\hbar T} \exp \left[ \frac{imR^2}{2\hbar T}(\tau'' - \tau')^2 \right. \\
\left. + \frac{i\hbar T}{8mR^2\tau''} + i\xi(\varphi'' - \varphi' + 2\pi n) + \frac{imR^2\tau''}{2\hbar T}(\varphi'' - \varphi + 2\pi n) \right]. \tag{33}
\]

Consequently, I get for the interference term
\[
I_{nl} \simeq 2 \left( \frac{m}{2\pi i\hbar T} \right)^2 \\
\times \cos \left[ 2\pi(l - n) \left( \xi + \frac{mR^2\tau''}{\hbar T}(\varphi'' - \varphi' - \pi) \right) + 2\pi^2 \frac{mR^2\tau''}{\hbar T}(l - n)(l + n + 1) \right]. \tag{34}
\]

The principal feature of this result consists that the interference patterns does not depend only on the initial \((\tau', \varphi')\) and final points \((\tau'', \varphi'')\), but on the homotopy class numbers \(n\) and \(m\) as well which describe the windings around the infinitesimal thin solenoid. This flux dependent shift is a proper Aharonov–Bohm effect. The interference term vanishes for \(n = l\).

The maximum contribution to the Aharonov–Bohm effect on the (hyperbolic) plane is observed for the smallest non-vanishing value \(|n - l| = 1 > 0\). Therefore, the maximum effect is observed for the interference of the winding number \(l = 0\) and \(n = -1\), or vice versa, yielding the interference term
\[
I_{0,-1} = 2 \left( \frac{m}{2\pi i\hbar T} \right)^2 \cos(2\pi \xi). \tag{35}
\]

This is the standard result, e.g. [Feynman and Hibbs (1965)] and [Berndio and Inomata (1980)] and references therein.

3  Higgs-Oscillator and Kepler–Coulomb Potential

Obviously, we can incorporate potential terms in the radial path integration \(\tau\), e.g., we can include the Higgs-oscillator potential [Grosche et al. (1996), Higgs (1979)]
\[
V_{(\text{Higgs})}(u) = \frac{m}{2} \omega^2 R^2 \frac{u_1^2 + u_2^2}{u_0^2} = \frac{m}{2} \omega^2 R^2 \tanh^2 \tau, \tag{36}
\]
which is the analogue of the harmonic oscillator in a space of constant curvature, or the Kepler–Coulomb potential [Barut et al. (1990), Grosche (1990b), Grosche et al. (1996)], respectively
\[
V_{(\text{Coulomb})}(u) = -\frac{\alpha}{R} \frac{u_0}{\sqrt{u_1^2 + u_2^2}} - 1 = -\frac{\alpha}{R} (\coth \tau - 1). \tag{37}
\]
For clarity, I have included the dependence on the constant curvature $R$ explicitly. In these cases, the result (23) is more appropriate. The combined $d\varphi\,d\lambda$-integration yields $\lambda = 0$, and the total propagator becomes

$$K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) = \sum_{l=-\infty}^{\infty} e^{\imath(l\varphi'' - \varphi')} K_{[l-\xi]}(\tau'', \tau'; T),$$

and the effect of the solenoid exhibits in a modification of the angular momentum dependence of $K_{[l-\xi]}$. This feature, however, modifies the number of bound states of the system with respect to the quantum number $l$. For instance, for the Higgs-oscillator case this gives ($\nu^{2} = m^{2}\omega^{2}R^{4}/\hbar^{2} + 1/4$)

$$\Psi_{nl}^{(\text{Higgs})}(\tau, \varphi; R) = (2\pi \sinh \tau)^{-1/2} S_{n}^{(\nu)}(\tau; R) e^{\imath \varphi},$$

$$S_{n}^{(\nu)}(\tau; R) = \frac{1}{\Gamma(|l - \xi| + 1)} \left[ \frac{2(\nu - |l - \xi| - 2n - 1)\Gamma(n + |l - \xi| + 1)\Gamma(\nu - |l - \xi|)}{R^{2}\Gamma(\nu - |l - \xi| - n)n!} \right]^{1/2}$$

$$\times (\sinh \tau)^{|l - \xi| + 1/2} (\cosh \tau)^{n + 1/2 - \nu} F_{1}(-|l - \xi|, \nu - n; 1 + |l - \xi|; \tanh^{2} \tau),$$

with the discrete energy spectrum given by

$$E_{n}^{(\text{Higgs})} = - \frac{\hbar^{2}}{2mR^{2}} \left[ (2n + |l - \xi| - \nu + 1)^{2} - \frac{1}{4} \right] + \frac{m}{2} \omega^{2}R^{2}.$$  

(41)

Only a finite number exist with $N_{\text{max}} = [\nu - |l - \xi| - 1] \geq 0$ ($[x]$ denotes the integer value of $x \in \mathbb{R}$). The continuous wave-functions have the form

$$\Psi_{kl}^{(\text{Higgs})}(\tau, \varphi; R) = (2\pi \sinh \tau)^{-1/2} S_{k}^{(\nu)}(\tau; R) e^{\imath \varphi},$$

$$S_{k}^{(\nu)}(\tau; R) = \frac{1}{\Gamma(|l - \xi| + 1)} \sqrt{\frac{k \sinh \pi k}{2\pi^{2}R^{2}}} \frac{\Gamma(\nu - |l - \xi| + 1 - ik)}{\Gamma(\nu - |l - \xi| + 1 - ik)}$$

$$\times (\tanh \tau)^{|l - \xi| + 1/2} (\cosh \tau)^{ik}$$

$$\times 2F_{1}\left(\frac{\nu + |l - \xi| + 1 - ik}{2}, \frac{1}{2}; 1 + |l - \xi|; \tanh^{2} \tau\right),$$

(43)

with the continuous energy-spectrum given by

$$E_{p}^{(\text{Higgs})} = \frac{\hbar^{2}}{2mR^{2}} \left( k^{2} + \frac{1}{4} \right) + \frac{m}{2} \omega^{2}R^{2}.$$  

(44)

In the case of the Kepler–Coulomb problem on $\Lambda$ we obtain for the discrete energy spectrum ($\tilde{N} = N + |l - \xi| + \frac{1}{2}$, $N = 0, 1, 2, \ldots, N_{\text{max}} = [\sqrt{R/a} - |l - \xi| - \frac{1}{2}]$, $a = \hbar^{2}/m\alpha$ is the Bohr radius)

$$E_{N}^{(\text{Coulomb})} = \frac{\alpha}{R} - \hbar^{2} \frac{\tilde{N}^{2} - \frac{1}{4}}{2mR^{2}} - \frac{m\alpha^{2}}{2\hbar^{2}N^{2}}.$$  

(45)

The wave-functions I do not state, c.f. [Grosche et al. (1996)], and the continuous states are modified by their angular momentum dependence, i.e., $l \rightarrow l - \xi$. However, the effect of the Aharonov–Bohm field is not only restricted to a modification of the discrete spectrum, but the effect on the scattering states happens through an interference term $I_{nl}$ similarly to (29), for the Coulomb potential and the Higgs-oscillator as well. Again, a closed expression for the radial propagator does not exist and we are restricted to the investigation of the limiting case along the lines following (29). This I do not repeat once more.
4 Summary

I therefore have shown the admissibility of path integration of the Aharonov–Bohm effect on the hyperbolic plane. It can be studied in a straightforward manner yielding analogous results in comparison to the flat space case. For scattering states we find interference, due to the modification of the angular momentum dependence according to $l \rightarrow l - \xi$, giving a cos-like pattern in terms of the strength of the vector-potential, for the free motion, the Kepler–Coulomb problem, and the Higgs-oscillator (which is absent in the flat space case); the bound state wave-function and the corresponding energy levels are modified in their angular momentum dependence $l \rightarrow l - \xi$ as well, together including an alteration of the number of bound states. We found the usual expansion of the total propagator in terms of an expansion into the winding number $n$ of the homotopy class of paths. All these features are well-known form the corresponding flat-space cases. The complicated interference expression (29) could not be evaluated due the non-constant curvature features of the hyperbolic plane. This would involve an analytical intractable integration over Legendre functions with respect to the order. However, the investigation of the flat space-limit gave the well-known result. Therefore the effect of an Aharonov–Bohm gauge field on the hyperbolic plane, i.e., scattering on leaky tori, exhibits the same features as in the flat space case of $\mathbb{R}^2$.

References

[Abramowitz and Stegun (1984)] Abramowitz, M., Stegun, I.A. (Editors): *Pocketbook of Mathematical Functions*. Harry Deutsch, Frankfurt/Main, 1984.

[Aharonov and Bohm (1959)] Aharonov, Y., Bohm, D.: Significance of Electromagnetic Potentials in the Quantum Theory. *Phys. Rev.* 115 (1959) 485–491.

[Anandan and Safko (1994)] Anandan, J.S., Safko, J.L. (eds.): *Quantum Coherence and Reality. Proceedings of the International Conference on Fundamental Aspects of Quantum Theory in Celebration of the 60th Birthday of Yakir Aharonov*, Columbia, USA, 1992. World Scientific, Singapore, 1994.

[Barut et al. (1990)] Barut, A.O., Inomata, A., Junker, G.: Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature: II. Hyperbolic Space. *J. Phys. A: Math. Gen.* 23 (1990) 1179–1190.

[Bernido (1993)] Bernido, C.C.: Path Integral Treatment of the Gravitational Anyon in a Uniform Magnetic Field. *J. Phys. A: Math. Gen.* 26 (1993) 5461–5471.

[Berndio and Inomata (1980)] Bernido, C.C., Inomata, A.: Topological Shifts in the Aharonov–Bohm Effect. *Phys. Lett. A* 77 (1980) 394–396. Path Integrals with a Periodic Constraint: The Aharonov–Bohm Effect. *J. Math. Phys.* 22 (1981) 715–718.

[Chetounai et al. (1989)] Chetounai, L., Guechi, L., Hammann, T.F.: Exact Path Integral Solution of the Coulomb Plus Aharonov–Bohm Potential. *J. Math. Phys.* 30 (1989) 655–658.

[Comtet (1987)] Comtet, A.: On the Landau Levels on the Hyperbolic Plane. *Ann. Phys. (N.Y.)* 173 (1987) 185–209.

[Drăganescu et al. (1992)] Drăganescu, Gh.E., Campigotto, C., Kibler, M.: On a Generalized Aharonov–Bohm Plus Coulomb System. *Phys. Lett. A* 170 (1992) 339–343.
[Fay (1977)] Fay, J.D.: Fourier Coefficients of the Resolvent for a Fuchsian Group. *J. Reine und Angew. Math.* 293 (1977) 143–203.

[Feynman and Hibbs (1965)] Feynman, R.P., Hibbs, A.: *Quantum Mechanics and Path Integrals*. McGraw Hill, New York, 1965.

[Gamboa and Rivelles (1991)] Gamboa, J., Rivelles, V.O.: Quantum Mechanics of Relativistic Particles in Multiply Connected Spaces and the Aharonov–Bohm Effect. *J. Phys. A: Math. Gen.* 24 (1991) L659–L666.

[Gerry and Singh (1979)] Gerry, C.C., Singh, V.A.: Feynman Path-Integral Approach to the Aharonov–Bohm Effect. *Phys. Rev. D* 20 (1979) 2550–2554. Remarks on the Effects of Topology in the Aharonov–Bohm Effect. *Nuovo Cimento B* 73 (1983) 161–170. On the Experimental Consequences of the Winding Numbers of the Aharonov–Bohm Effect. *Phys. Lett. A* 92 (1982) 11–12.

[Gradshteyn and Ryzhik (1980)] Gradshteyn, I.S., Ryzhik, I.M.: *Table of Integrals, Series, and Products*. Academic Press, New York, 1980.

[Grosche (1988)] Grosche, C.: The Path Integral on the Poincaré Upper Half-Plane With a Magnetic Field and for the Morse Potential. *Ann. Phys. (N.Y.)* 187 (1988) 110–134.

[Grosche (1990a)] Grosche, C.: Path Integration on the Hyperbolic Plane With a Magnetic Field. *Ann. Phys. (N.Y.)* 201 (1990) 258–284.

[Grosche (1990b)] Grosche, C.: The Path Integral for the Kepler Problem on the Pseudosphere. *Ann. Phys. (N.Y.)* 204 (1990) 208–222.

[Grosche (1996)] Grosche, C.: *Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae*. World Scientific, Singapore, 1996.

[Grosche et al. (1996)] Grosche, C., Pogosyan, G.S., Sissakian, A.N.: Path-Integral Approach to Superintegrable Potentials on the Two-Dimensional Hyperboloid. *Phys. Part. Nucl.* 27 (1996) 244–278.

[Grosche and Steiner (1988)] Grosche, C., Steiner, F.: The Path Integral on the Pseudosphere. *Ann. Phys. (N.Y.)* 182 (1988) 120–156.

[Grosche and Steiner (1998)] Grosche, C., Steiner, F.: *Handbook of Feynman Path Integrals*. Springer, Berlin, Heidelberg, 1998.

[Gutzwiller (1991)] Gutzwiller, M.C.: *Chaos in Classical and Quantum Mechanics*. Springer, Berlin, Heidelberg, 1991.

[Hejhal (1976)] Hejhal, D.A.: *The Selberg Trace Formula for PSL(2,\(\mathbb{R}\))*. Lecture Notes in Physics 548. Springer, Berlin, Heidelberg, 1976.

[Higgs (1979)] Higgs, P.W.: Dynamical Symmetries in a Spherical Geometry. *J. Phys. A: Math. Gen.* 12 (1979) 309–323.

[Hoang and Giang (1993)] Hoang, L.V., Giang, N.T.: On the Green Function for a Hydrogen-Like Atom in the Dirac Monopole Field Plus the Aharonov–Bohm Field. *J. Phys. A: Math. Gen.* 26 (1993) 3333–3338.

[Hoang et al. (1992)] Hoang, L.V., Hai, L.X., Komarov, L.I., Romaova, T.S.: Relativistic Analogy of the Aharonov–Bohm Effect in the Presence of Coulomb Field and Magnetic Charge. *J. Phys. A: Math. Gen.* 25 (1992) 6461–6469.
[Izmest’ev et al. (1997)] Izmest’ev, A.A., Pogosyan, G.S., Sissakian, A.N., Winternitz, P.: Contractions of Lie Algebras and Separation of Variables. Two-Dimensional Hyperboloid. *Int. J. Mod. Phys.* 12 (1997) 53–61.

[Kibler and Campigotto (1993)] Kibler, M., Campigotto, C.: On a Generalized Aharonov–Bohm Plus Oscillator System. *Phys. Lett.* A 181 (1993) 1–6.

[Kibler and Negadi (1987)] Kibler, M., Negadi, T.: Motion of a Particle in a Coulomb Field Plus Aharonov–Bohm Potential. *Phys. Lett.* A 124 (1987) 42–46.

[Kleinert (1995)] Kleinert, H.: *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*. World Scientific, Singapore, 1995.

[Kuperin et al. (1994)] Kuperin, Yu.A., Romanov, R.V., Rudin, H.E.: Scattering on the Hyperbolic Plane in the Aharonov–Bohm Gauge Field. *Lett. Math. Phys.* 31 (1994) 271–278.

[Liang (1988)] Liang, J.Q.: Path Integrals in Multiply Connected Spaces and the Aharonov–Bohm Interference. *Physica B* 151 (1988) 239–244.

[Lin (1998)] Lin, D.-H.: Path Integral for a Relativistic Aharonov–Bohm–Coulomb System. *J. Phys. A: Math. Gen.* 31 (1998) 4785–4793.

[Olevskiı (1950)] Олевский, М.Н.: Триорготональные системы в пространствах постоянной кривизны, в которых уравнение $\Delta_2 u + \lambda u = 0$ допускает полное разделение переменных. *Mat.Cs.* 27 (1950) 379–426.

[Olevskiı, M.N.: Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables. *Math.Sb.* 27 (1950) 379–426 (in Russian)].

[Park and Yoo (1998)] Park, D.K., Yoo, S.-K.: Propagators for Spinless and Spin-1/2 Aharonov–Bohm–Coulomb Systems. *Ann. Phys. (N.Y.)* 263 (1998) 295–309.

[Peak and Inomata (1969)] Peak, D., Inomata, A.: Summation Over Feynman Histories in Polar Coordinates. *J. Math. Phys.* 10 (1969) 1422–1428.

[Pnueli (1994)] Pnueli, A.: Scattering Matrices and Conductances of Leaky Tori. *Ann. Phys. (N.Y.)* 231 (1994) 56–83.

[Schulman (1971)] Schulman, L.S.: Approximate Topologies. *J. Math. Phys.* 12 (1971) 304–308.

[Schulman (1981)] Schulman, L.S.: *Techniques and Applications of Path Integration*. John Wiley & Sons, New York, 1981.