Generation of uniform magnetic field using a spheroidal helical coil structure

Yavuz Öztürk¹,² and Bekir Aktaş¹
¹Gebze Technical University (GTU), Physics Department, Gebze/Kocaeli, Türkiye.
²TÜBİTAK BILGEM, Gebze/Kocaeli, Türkiye.
E-mail: yavuz.ozturk@tubitak.gov.tr

Abstract. Uniformity of magnetic fields are of great importance especially in magnetic resonance studies, namely in magnetic resonance spectroscopy applications (NMR, FMR, ESR, EPR etc.) and magnetic resonance imaging applications (MRI, FMRI). Field uniformity is also required in some other applications such as eddy current probes, magnetometers, magnetic traps, particle counters etc. Here we proposed a coil winding regime, which follows the surface of a spheroid (an ellipsoid of rotation); in light of previous theoretical studies suggesting perfect uniformity for a constant ampere per turn in the axial direction thereof. We demonstrated our theoretical results from finite element calculations suggesting 0.15% of field uniformity for the proposed structure, which we called a Spheroidal Helical Coil.

1. Introduction
Maintaining high magnetic field uniformity is the key feature for many magnetic monitoring applications. Magnetic Resonance Imaging (MRI) instruments need extreme field intensities (over 1 Tesla) with high field uniformity. The former is required for the sake of increasing signal-to-noise ratio of the weak resonance signal without increasing the acquisition rate beyond reasonable time scales. The latter is needed to address the signals from the body under examination. Since using permanent magnets is not practical both in the sense of manufacturing and utilization of the device, magnetic fields are generated using coils in typical imaging machines. Modern MRI instruments depend on superconducting wires and cryogenic technology to reach extreme intensities, and shimming techniques to maintain high field uniformity [1].

Helmholtz coils are the most practical and widely used winding structures for obtaining field uniformity in a relatively small region [2,3]. Alternative coil structures like, tetra coils [4,5], saddle coils [6,7] or planar coils [8] are also used to reach high field uniformity. Ellipsoidal structures, which offer theoretically perfect uniform magnetic fields, are studied as well. J. C. Maxwell was maybe the first to work out the demagnetization constants of ellipsoidal structures [9]. Later, E. C. Stoner [10] and J. A. Osborn [11] investigated demagnetization in ellipsoidal structures and published detailed analysis and tables in their respective articles. Similar calculations can also be seen in the famous books by J. A. Stratton [12] and L. D. Landau [13]. In his paper “on the problem of the ellipsoid” G. E. Marsh showed that a linear current function defined on the ellipsoidal surface can generate a homogeneous magnetic field inside along its principle axis [14].
2. Theory

In case the so-called current function ($\Phi$) is linear along the $z$ axis, it will have the form:

$$\Phi = -Kz$$

(1)

Marsh proved that this assumption in (1) will result in a homogeneous field inside an ellipsoid [14]. The relation between the current density and the current function is proven to be:

$$j = \nabla \Phi \times \hat{n}$$

(2)

where $\hat{n}$ is normal to the surface of the ellipsoid [14].

The derivation of the magnetic field is started using the discontinuity of the vector potential ($A$) at the boundary (the surface):

$$\frac{\partial A^+}{\partial n} - \frac{\partial A^-}{\partial n} = -\mu_0 j$$

(3)

The vector potential also obeys the Laplace equation ($\nabla^2 A_i = 0$). The Laplace equation in ellipsoidal coordinates for $A$ is as follows:

$$(\eta - \zeta) R_\lambda \frac{\partial}{\partial \lambda} \left( R_\lambda \frac{\partial \phi}{\partial \lambda} \right) + (\zeta - \lambda) R_\eta \frac{\partial}{\partial \eta} \left( R_\eta \frac{\partial \phi}{\partial \eta} \right) + (\lambda - \eta) R_\zeta \frac{\partial}{\partial \zeta} \left( R_\zeta \frac{\partial \phi}{\partial \zeta} \right) = 0$$

(4)

where

$$R_\lambda = \left[ (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \right]^{1/2}$$

(5)

In case the family of confocal ellipsoids –which are parameterized by $\lambda$ - are equipotential lines, one must have:

$$\frac{\partial}{\partial \lambda} \left( R_\lambda \frac{\partial \phi}{\partial \lambda} \right) = 0$$

(6)

From here Marsh showed that, $\phi_0 = \text{const} \cdot \int_0^\infty d\lambda/R_\lambda$ being the elliptic integral and $\hat{z}$ being the unit vector along the $z$ direction, the magnetization can be derived to be:

$$B = \mu_0 abcK \left( \frac{d\phi_0}{da^2} + \frac{d\phi_0}{db^2} \right) \hat{z}$$

(7)

which is constant along $z$ depending only on ellipsoidal parameters [14].

When the current is calculated by using the $\Phi = -Kz$ current function, one can write down:

$$j = \nabla \Phi \times \hat{n} = -K (\nabla z) \times \hat{n} = K (\hat{n} \times \hat{z})$$

(8)

From here we may conclude that in order to have constant current, $(\hat{n} \times \hat{z})$ has to be constant as well. Since $|\hat{n} \times \hat{z}|$ is also constant, $\hat{n} \cdot \hat{z} = \text{constant} \equiv d$ can be defined. Thus one can obtain
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{1}{\frac{d^2}{c^2} - 1} \frac{z^2}{c^2} = 0 \tag{9}
\]

which is the equation of a real quadric cone. The intersection of this ellipsoid with the unit ellipsoid
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{10}
\]
is the desired winding curve. So using (10) to eliminate \(z\) in (9) one will have:
\[
\left( \frac{c^2}{a^2} \frac{d^2}{1 - d^2} + 1 \right) \frac{x^2}{a^2} + \left( \frac{c^2}{b^2} \frac{d^2}{1 - d^2} + 1 \right) \frac{y^2}{b^2} = 1 \tag{11}
\]

This is actually the equation of an ellipse with semi-axes:
\[
a \left( \frac{c^2}{a^2} \frac{d^2}{1 - d^2} + 1 \right)^{-1/2} \quad \text{and} \quad b \left( \frac{c^2}{b^2} \frac{d^2}{1 - d^2} + 1 \right)^{-1/2} \tag{12}
\]

From here, the introduction of \(\hat{n} \cdot \hat{z} = \cos \theta\) and parameterization of the ellipsoid with
\[
\begin{align*}
x^2 &= a^2 \sin^2 \alpha \cos^2 \beta, \\
y^2 &= a^2 \sin^2 \alpha \sin^2 \beta, \\
z^2 &= a^2 \cos^2 \alpha
\end{align*} \tag{13}
\]
yield
\[
\cot^2 \alpha = c^2 \cot^2 \theta \left( \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{a^2} \sin^2 \beta \right) \tag{14}
\]

Here for the case of a spheroid \((a = b)\), \(z\) variation must vanish, thus the parameter \(\alpha\) has to be a constant giving
\[
\cot^2 \alpha = \frac{c^2}{a^2} \cot^2 \theta = \frac{c^2}{a^2} \frac{1 - \sin^2 \theta}{\sin^2 \theta} = \frac{c^2}{a^2} \left( \frac{1}{\sin^2 \theta} - 1 \right) \tag{15}
\]

This corresponds to the case of a spheroid where the internal field will be uniform if the current is proportional to:
\[
I \propto \sin \theta = \left( 1 + \frac{a^2}{c^2} \cot \alpha \right)^{-1/2} \tag{16}
\]

This is equivalent, in terms of windings, to having a constant number of ampere per turns per unit length along the \(z\) axis [14]. This result is also confirmed by the earlier studies of L. J. Laslett [15] and J. E. Everett [16] for the special case of a sphere. A similar study was pursued by Živaljević and Aleksić on prolate and oblate spheroids. They approached the problem by defining a scalar magnetic potential for solving the Laplace equation using the suitable boundary conditions [17]. They have used both prolate and oblate spheroidal coordinates respectively (Figure 1).
Using prolate spheroidal coordinates they wrote down the electric and magnetic field vectors as follows:

\[
E(u, v) = E_w(u, v)\hat{w}
\]

\[
H(u, v) = H_u(u, v)\hat{u} + H_v(u, v)\hat{v}
\] (17) (18)

where \(\hat{w}, \hat{u}, \hat{v}\) are prolate spheroidal coordinate vectors and in terms of Cartesian unit vectors:

\[
\hat{u} = \frac{c}{h_u} (\cosh u \sin v \cos w \hat{x} + \cosh u \sin v \sin w \hat{y} + \sinh u \cos v \hat{z})
\]

\[
\hat{v} = \frac{h_v}{c} (\sinh u \cos v \cos w \hat{x} + \sinh u \cos v \sin w \hat{y} + \cosh u \sin v \hat{z})
\]

\[
\hat{w} = -\sin w \hat{x} + \cos w \hat{y}
\] (19)

Here \(h\) factors are called Lame coefficients and for prolate coordinates they are defined as:

\[
h_u = h_v = h = c(\sinh^2 u + \sin^2 v)^{1/2}
\]

\[
h_w = c \sinh u \sin v
\] (20)

where \(a\) and \(b\) are elliptic constants for \(a > b\) and \(c = \sqrt{a^2 - b^2}\) is the eccentricity [18]. Solving the Laplace equation, the magnetic field is found to be:

\[
H(u, v) = C_0 J_z (\cot v \sinh u \hat{u} - \cosh u \hat{v})
\] (21)

where \(C_0\) is a constant of spheroid and equals to:

\[
C_0 = \frac{\sinh^2 u_0}{2} \ln \frac{\cosh u_0 + 1}{\cosh u_0 - 1} - \cosh u_0
\] (22)
while \( J_s \) is the surface current \([17]\). Using (19) one may write down:

\[
H(u, v) = \frac{cC_0 J_s}{h} (\cosh \sinh u \cos v \cos w - \cosh \sinh u \cos v \cos w) \hat{x}
+ \frac{cC_0 J_s}{h} (\cosh \sinh u \cos v \cos w - \cosh \sinh u \cos v \cos w) \hat{y}
+ \frac{cC_0 J_s}{h \sin v} (\cos^2 v \sinh^2 u - \sin^2 v \cosh^2 u) \hat{z}
\]

\[
H(u, v) = \frac{cC_0 J_s}{h \sin v} ((1 - \sin^2 v) \sinh^2 u - \sin^2 v \cosh^2 u) \hat{z}
\]

\[
H(u, v) = \frac{cC_0 J_s}{h \sin v} (\sinh^2 u + \sin^2 v) \hat{z}
\]

\[
H(u, v) = \frac{cJ_0 h}{c \sin v} \hat{z}
\]

From here it is deduced that the surface current \((J_s)\) has to be of the form:

\[
J_s \propto \sin v / h
\]

in order to have uniform magnetic field inside the spheroid. Consequently the magnetic field inside will be:

\[
H = \frac{C_0 J_0}{\sqrt{a^2 - b^2}} \hat{z}
\]

where \( J_0 \) is the constant of proportionality in (25) \([17]\). Živaljević and Aleksić also calculated the magnetic field outside the spheroid.

**Figure 2.** Prolate ellipsoidal coil magnetostatic field distribution and equipotential lines for \( a/b = 2 \) and \( a/b = 3 \) ratios respectively \([17]\).
Their results for both inside and outside for $a/b = 2$ and $a/b = 3$ can be seen in Figure 2. As a further step using (25) they deduced a winding function $N$ (a coil number density) to be as follows:

$$N = \frac{N_0 \sin \nu}{h}$$  \hspace{1cm} (27)

Their results (as in Figure 2) are consistent with the previous literature [9-12], however the so-called winding function does not say much about the actual winding of the coil. In fact it can be shown that the result in equation (25) can produce the current function of Marsh [14] in equation (1). Hence the derivations made by Živaljević and Aleksić [17] can be reconciled with the well-known constant ampere per turn result. The derivation connecting the number density in (27) to the current function in (1) is carried out elsewhere [19].

3. Calculations and results

In order to confirm the conclusion of “constant number of ampere per turns per unit length along the $z$ axis”, we set up some Finite Element Analysis (FEA) calculations using CST EM Studio© software. We defined a hundred current rings of which radii obeying a spheroid having dimensions $a = 320 \text{ mm}$ and $b = 160 \text{ mm}$. A unit ampere of current was assumed to flow through each current carrying ring. The rings were defined to be of copper wires 0.9 mm in diameter. Simulation results were plotted on the principle ($z$) and lateral axes (which can be taken as $x$ axis) (Figure 3). Defined structure and cross-section of a few meshed wires can be seen in Figure 4. Our results from FEA calculations along principle and lateral axes are given in Figure 5 and Figure 6, respectively. Field uniformities along both axes were calculated to be 0.03%. Clearly seen that these results confirm the theory.

That kind of a ring assembly is not fully realizable, unless one has current preserving superconducting rings. Yet there have been some experimental works applying similar methods. Since current has to be supplied to the system, there has to be some input and output junctions. Namely, current has to be introduced to the coils and has to leave them by some means. In order to avoid using multiple current supplies, open ring structures which are progressively soldered to each other were preferred in previous works [16,19]. On the other hand, these works were addressing respectively large structures where the effects and defects from solders and jumpers can be omitted.
Figure 4. Modeled spheroidal current rings structure is on the left. Small arrows symbolize the current flow on the rings. Cross-sectional view of the meshed circular rings is on the right. The rings are defined to be of copper wires having 0.9 mm of diameter.

For small scale structures this method is expected to sensibly affect the magnetic field uniformity. In order to attain an experimentally feasible structure in which the field uniformity can be maintained, we proposed a coil winding regime which was described by the following formulas.

\[
\begin{align*}
    x &= b \sqrt{1 - \frac{z^2}{a^2}} \cos \left( \frac{2\pi z}{d} \right) \\
y &= b \sqrt{1 - \frac{z^2}{a^2}} \sin \left( \frac{2\pi z}{d} \right)
\end{align*}
\]

(28)

where \( a > b, -a \leq z \leq a \) and \( d \) is the \( z \) distance between windings and calculated by dividing the principle axis length to the desired number of windings \((d = \frac{2a}{N_{\text{windings}}})\). The coil structure proposed above is a spheroidal helical coil structure.

Figure 5. Magnitude of the H-field along principle axis is plotted. The principle field uniformity was calculated to be 0.03%.
Magnitude of the Magnetic Field along Lateral Axis

![Graph showing Magnitude of the Magnetic Field along Lateral Axis](image)

**Figure 6.** Magnitude of the H-field along lateral axis is plotted. The lateral field uniformity was calculated to be 0.03%.

Calculations were conducted using CST EM Studio© as before and the coil was again accepted to be of a 0.9 mm copper wire. Current to the coil was introduced by a current path which was assumed to be a perfect wire. The path curled around the coil far enough not to influence the results. Our results from FEA calculations along principle and lateral axes can be seen in Figure 8 and 9 respectively. Field uniformities along both axes were calculated to be %0.15 which are considerably successful results. The spikes at the ends of both figures were considered to be artifacts caused partly by the inconvenience of meshing procedure, which was not good enough to resolve the magnetic field just above the coils. Another reason may be the slight mismatch between the coil and the current path which introduced artificial discontinuities in the first derivative of the current.

![Some spheroidal helical coil structures](image)

**Figure 7.** Some spheroidal helical coil structures.
4. Conclusion

In this work, theoretical studies for ellipsoidal structures are shortly reviewed and their results which showed constant number of ampere per turns per unit length along the $z$ axis are confirmed via FEA calculations. As a further step a continuous winding structure on an ellipsoid of rotation – which we called spheroidal helical coil structure – have been proposed and also confirmed by FEA calculations and proven to be a good candidate for the generation of a uniform magnetic field within a closed volume.
References

[1] Kuperman V 2000 Magnetic Resonance Imaging: Physical Principles and Applications (San Diego: Academic Press) p 9–26
[2] Magdaleno-Adame S, Olivares-Galvan J C, Campero-Littlewood E, Escarela-Perez R, Blanco-Brisset E 2010 Coil Systems to Generate Uniform Magnetic Field Volumes Excerpt from the Proceedings of the COMSOL Conference 2010 Boston
[3] Bell G B and Marino A A 1989 Exposure systems for production of uniform magnetic fields J. of Bioelectricity 8 (2) 147–58
[4] Merritt R, Purcell C and Stroink G 1983 Uniform magnetic field produced by three, four, and five square coils Rev. Sci. Instrum. 54 879–82
[5] Gottardi G, Mesirca P, Agostini C, Remondini D and Bersani F, 2003 A four coil exposure system (tetracoil) producing a highly uniform magnetic field Bioelectromagnetics 24 125–33
[6] Bonetto F, Anoardo E and Polello M 2006 Saddle coils for uniform static magnetic field generation in NMR experiments Concepts Magn. Reson. Part B 29 B1 9–19
[7] Dinale J and Vrbancich J 2014 Generation of long prolate volumes of uniform magnetic field in cylindrical saddle-shaped coils Meas. Sci. Technol. 25 035903 1–13
[8] Sasada I and Nakashima Y 2006 Planar coil system consisting of three coil pairs for producing a uniform magnetic field J. Appl. Phys. 99 08D904
[9] Maxwell J C 1904 Electricity and Magnetism ed 3 vol 2 (Oxford: The Clarendon Press) p 66–70
[10] Stoner E C 1945 Demagnetizing factors for ellipsoids Phil. Mag. 36 p 803–21
[11] Osborn J A 1945 Demagnetizing factors of the general ellipsoid Phys. Rev. 6 7352–57.
[12] Stratton J 1941 Electromagnetic Theory (McGraw-Hill Book Company, Inc.: New York) p 209–13
[13] Landau L D and Lifshitz E M 1960 Electrodynamics of Continuous Media (Oxford: Pergamon Press) p 20–27
[14] Marsh G E 1975 Uniform magnetic field in deflection coil design and the problem of ellipsoid J. Appl. Phys. 46 3178–81
[15] Laslett L J 1966 An equivalent distribution of surface currents for the generation of a prescribed static magnetic field within the enclosed volume J. Appl. Phys. 37 2361–63
[16] Everett J E and Osemeikhian J E 1966 Spherical coils for uniform magnetic fields J. Sci. Instrum. 43 470–74
[17] Živaljevič D U and Aleksič S R 2007 Generating homogeneous magnetostatic field inside prolate and oblate ellipsoidal coil Int. J. Electron. Commun. (AEÜ) 61 637–44
[18] Arfken G 1970 Mathematical methods for physicists (2nd ed.) (New York: Academic Press) p 103–8
[19] Öztürk Y and Aktaş B to be published
[20] Clark J W 1938 A new method for obtaining a uniform magnetic field. Rev. Sci. Instrum. 9 320–22