Recovery of Sobolev functions restricted to iid sampling

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Abstract

We study $L_q$-approximation and integration for functions from the Sobolev space $W_{p}^{s}(\Omega)$ and compare optimal randomized (Monte Carlo) algorithms with algorithms that can only use iid sample points, uniformly distributed on the domain. The main result is that we obtain the same optimal rate of convergence if we restrict to iid sampling, a common assumption in learning and uncertainty quantification. The only exception is when $p = q = \infty$, where a logarithmic loss cannot be avoided.

Keywords: optimal recovery, rate of convergence, numerical integration, random information, interior cone condition

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We assume that $\Omega$ satisfies an interior cone condition. We study the problem of approximating a function $f$ from the Sobolev space $W_{p}^{s}(\Omega)$ in the $L_q(\Omega)$-norm based on function values $f(x_j)$ on a finite set of sampling points $P = \{x_1, \ldots, x_n\}$. This makes sense if $s > d/p$, in which case $W_{p}^{s}(\Omega)$ is compactly embedded into the space of bounded continuous functions $C_b(\Omega)$, and in the case $p = 1$ and $d = s$. We also study the problem of numerical integration.

There is a vast literature on the error for optimal sampling points. It is known that the rate of convergence of the worst case error of optimal deterministic algorithms is

$$n^{-s/d+(1/p-1/q)+}$$

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for the approximation problem and
\[ n^{-s/d} \]

for the integration problem, where \( a_+ := \max\{a, 0\}, a \in \mathbb{R} \). These are classical results for special domains like the cube, see Ciarlet [21, Chapter 3] and Heinrich [10, Section 6]. For general domains, we refer to Narcowich, Wendland and Ward [24] as well as Novak and Triebel [28], see also Remark 2.

In this paper we assume that we cannot choose the sampling points. Instead they are given to us as realizations of independent random variables which are uniformly distributed on the domain. That is, we get our data \( f(x_j) \) for random sample points \( x_j \in \Omega \) which are not under our control.

Then one can still consider the “uniform” or “worst case” error on the unit ball of \( W^s_p(\Omega) \), as done by two of the authors in [16]. There it is proved that, in expectation (and thus with high probability), the worst case error of random points is asymptotically optimal for \( L_q \)-approximation whenever \( q < p \); otherwise \( n \) random points behave as well as \( n/\log n \) optimal points.

But it is natural to study also a different approach and error criterion. If \( P \subset \Omega \) is random, then also an algorithm is a randomized or Monte Carlo algorithm and one may compare the algorithm with optimal randomized algorithms based on the expected error for inputs from the unit ball of \( W^s_p(\Omega) \). So we switch from the expected worst case error to the worst case expected error.

The class of all randomized algorithms is very large since one may construct the random variable \( x_{k+1} \) based on the realization \((x_1, f(x_1), \ldots, x_k, f(x_k))\) in a sophisticated and complicated way. Nevertheless, using this weaker notion of error, one cannot improve the rate \( n^{-s/d+(1/p-1/q)/2} \) for the approximation problem, see Mathé [21] and Heinrich [10]. For the integration problem one now obtains the improved order \( n^{-s/d-1/2} \) if \( p \geq 2 \) and \( n^{-s/d-1+1/p} \) if \( 1 \leq p < 2 \), see Bakhvalov [2] and Novak [25].

In this paper we check what we can get with randomized (or Monte Carlo) algorithms when we restrict to independent uniformly distributed sampling. We will prove that we still obtain the optimal order of convergence for the approximation problem unless \( p = q = \infty \): In this limiting case there is a logarithmic loss. For the integration problem we obtain the optimal order for all \( p \).

We now describe our results in more detail. We assume that \( \Omega \subset \mathbb{R}^d \) is open and bounded and satisfies an interior cone condition. That is, there is some \( r > 0 \) and \( \theta \in (0, \pi] \) such that for every \( x \in \Omega \), we find a cone

\[ K(x) := \{x + \lambda y : y \in \mathbb{S}^{d-1}, \langle y, \xi(x) \rangle \geq \cos\theta, \lambda \in [0, r]\} \]
with apex \( x \), direction \( \xi(x) \) in the unit sphere \( \mathbb{S}^{d-1} \), opening angle \( \theta \) and radius \( r \) such that \( K(x) \subset \Omega \). Here, \( \langle \cdot, \cdot \rangle \) denotes the standard inner product. For \( s \in \mathbb{N} \) and \( 1 \leq p \leq \infty \) such that \( s > d/p \) or \( p = 1 \) and \( s = d \), we consider the Sobolev space

\[
W^s_p(\Omega) := \{ f : \Omega \to \mathbb{R} \mid D^\alpha f \in L_p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq s \}
\]

with semi-norm

\[
|f|_{W^s_p(\Omega)} := \left( \sum_{|\alpha| = s} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p}
\]

and norm

\[
\|f\|_{W^s_p(\Omega)} := \left( \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p},
\]

with the usual modification for \( p = \infty \). Note that \( W^s_p(\Omega) \) is continuously embedded into \( C_b(\Omega) \), the space of bounded continuous functions with the sup norm, and that the embedding is compact in the case \( s > d/p \), see e.g. Maz’ya [22, Section 1.4] for this fact on domains satisfying an interior cone condition. We study the problem of \( L_q(\Omega) \)-approximation \((1 \leq q \leq \infty)\) on \( W^s_p(\Omega) \).

For a random operator \( A : W^s_p(\Omega) \to L_q(\Omega) \) we define the Monte Carlo error

\[
\Delta(A, W^s_p(\Omega), L_q(\Omega)) := \sup_{\|f\|_{W^s_p(\Omega)} \leq 1} \mathbb{E} \|f - A(f)\|_{L_q(\Omega)}.
\]

Later we will study a measurable algorithm \( A \), where the expectation exists, but we may use the notation \( \mathbb{E} \) for any mapping if we take the upper integral for the definition. Given a random point set \( P \subset \Omega \), we put

\[
\Delta(P, W^s_p(\Omega), L_q(\Omega)) := \inf_A \Delta(A, W^s_p(\Omega), L_q(\Omega)),
\]

where the infimum is taken over all random operators of the form \( A(f) = \varphi(f|_P) \) with a random mapping \( \varphi : \mathbb{R}^P \to L_q(\Omega) \). Note that \( f|_P = (f(x))_{x \in P} \) is the restriction of \( f \) to the point set \( P \) and we use this as our information. This is the smallest Monte Carlo error that can be achieved with the sampling point set \( P \). Moreover, we put

\[
\Delta(n, W^s_p(\Omega), L_q(\Omega)) := \inf_P \Delta(P, W^s_p(\Omega), L_q(\Omega)),
\]

where the infimum is taken over all random point sets of cardinality at most \( n \). This is the smallest Monte Carlo error that can be achieved with \( n \) optimally chosen sampling points. It is known, at least for special domains \( \Omega \), that

\[
\Delta(n, W^s_p(\Omega), L_q(\Omega)) \asymp n^{-s/d+(1/p-1/q)_+}, \tag{1}
\]
see Mathé [21] and Remark 1. The symbol \( \asymp \) means that the left hand side is bounded from above by a constant multiple of the right hand side for all \( n \in \mathbb{N} \) and vice versa; we use \( \preccurlyeq \) and \( \succcurlyeq \) for the one-sided relations. However, it is often not possible to choose the random sampling points to our liking. Here, we are interested in the smallest Monte Carlo error which can be achieved with \( n \) independent and uniformly distributed samples. The main result of our paper is the following.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded, satisfying an interior cone condition and let \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{N} \) such that \( s > d/p \) or \( p = 1 \) and \( s = d \). For every \( n \in \mathbb{N} \), let \( P_n \) be a set of \( n \) independent and uniformly distributed points on \( \Omega \).

\[
\Delta \left( P_n, W^s_p(\Omega), L^q(\Omega) \right) \asymp \begin{cases} 
(n/\log n)^{-s/d} & \text{if } p = q = \infty, \\
^{-s/d+(1/p-1/q)_+} & \text{else.}
\end{cases}
\]

This means that independent and uniformly distributed points are (asymptotically) as good as optimally selected (deterministic or random) sampling points in all cases except \( p = q = \infty \).

This answers the question for the power of independent uniformly distributed samples with respect to the Monte Carlo error criterion. On the other hand, one might also be interested in a stronger uniform error criterion. For a random operator \( A: W^s_p(\Omega) \to L^q(\Omega) \) the uniform error can be defined by

\[
\Delta^{\text{unif}} \left( A, W^s_p(\Omega), L^q(\Omega) \right) = \mathbb{E} \sup_{\|f\|_{W^s_p(\Omega)} \leq 1} \|f - A(f)\|_{L^q(\Omega)}.
\]

Note that the order of the supremum and the expected value is interchanged. Thus, while a small Monte Carlo error \( \Delta \) means that for every individual function, the error is small with high probability, a small uniform error \( \Delta^{\text{unif}} \) means that with high probability the error is small for every function. As before, given a random point set \( P \subset \Omega \), we put

\[
\Delta^{\text{unif}} \left( P, W^s_p(\Omega), L^q(\Omega) \right) := \inf_A \Delta^{\text{unif}} \left( A, W^s_p(\Omega), L^q(\Omega) \right),
\]

where the infimum is taken over all random operators of the form \( A(f) = \varphi(f|_P) \) with a random mapping \( \varphi: \mathbb{R}^P \to L^q(\Omega) \). This is thus the smallest uniform error that can be achieved with the random sampling point set \( P \). Moreover, we put

\[
\Delta^{\text{unif}} \left( n, W^s_p(\Omega), L^q(\Omega) \right) := \inf_P \Delta^{\text{unif}} \left( P, W^s_p(\Omega), L^q(\Omega) \right),
\]

where the infimum is taken over all random point sets of cardinality at most \( n \). It is known, at least for special domains, that

\[
\Delta^{\text{unif}} \left( n, W^s_p(\Omega), L^q(\Omega) \right) \asymp n^{-s/d+(1/p-1/q)_+},
\]

(2)
see again [24, 25] and Remark 2.

For the uniform error of independent uniformly distributed samples, the following result has been obtained in [16] for bounded convex domains. See also Ehler, Graef and Oates [7] and the survey [12] for earlier results in this direction.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded, satisfying an interior cone condition and let $1 \leq p, q \leq \infty$ and $s \in \mathbb{N}$ such that $s > d/p$ or $p = 1$ and $s = d$. For every $n \in \mathbb{N}$, let $P_n$ be a set of $n$ independent and uniformly distributed points on $\Omega$. Then

$$\Delta_{\text{unif}} \left( P_n, W^s_p(\Omega), L^q(\Omega) \right) \asymp \begin{cases} (n/\log n)^{-s/d+1/p-1/q} & \text{if } q \geq p, \\ n^{-s/d} & \text{if } q < p. \end{cases}$$

This means, with respect to the uniform error criterion, independent and uniformly distributed samples are as good as optimally chosen sampling points if and only if $q < p$. Moreover, if we are bound to independent and uniformly distributed samples, one can achieve the same rate for the uniform error as for the Monte Carlo error if and only if $q < p$ or $p = q = \infty$. In all other cases, the Monte Carlo error criterion provides a speed-up in comparison to the uniform error criterion.

All the upper bounds of Theorems 1 and 2 are achieved by the same algorithm. The algorithm works for all $p$ and $q$ and up to a given smoothness $s$. We describe the algorithm in Section 2.

Let us now turn to the integration problem

$$\text{INT}(f) = \int_{\Omega} f(x) \, dx .$$

We use a similar notation. For a random operator $A : W^s_p(\Omega) \to \mathbb{R}$ we define the Monte Carlo error

$$\Delta \left( A, W^s_p(\Omega), \text{INT} \right) := \sup_{\| f \|_{W^s_p(\Omega)} \leq 1} \mathbb{E} \left[ |\text{INT}(f) - A(f)| \right].$$

Given a random point set $P \subset \Omega$, we put

$$\Delta \left( P, W^s_p(\Omega), \text{INT} \right) := \inf_{A} \Delta \left( A, W^s_p(\Omega), \text{INT} \right),$$

where the infimum is taken over all random operators of the form $A(f) = \varphi(f|_P)$ with a random mapping $\varphi : \mathbb{R}^P \to \mathbb{R}$. This is the smallest Monte Carlo error that can be achieved with the sampling point set $P$. Moreover, we put

$$\Delta \left( n, W^s_p(\Omega), \text{INT} \right) := \inf_{P} \Delta \left( P, W^s_p(\Omega), \text{INT} \right),$$

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where the infimum is taken over all random point sets of cardinality at most \( n \). This is the smallest Monte Carlo error that can be achieved with \( n \) optimally chosen sampling points. It is known, at least for special domains \( \Omega \), that

\[
\Delta (n, W^s_p(\Omega), \text{INT}) \asymp n^{-s/d+(1/p-1/2)+1/2},
\]

see Bakhvalov [1, 2] and Novak [25]. Again, we are interested in the smallest Monte Carlo error which can be achieved with \( n \) independent and uniformly distributed sampling points. As a corollary to Theorem 1 we obtain the following.

**Corollary 1.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded, satisfying an interior cone condition and let \( 1 \leq p \leq \infty \) and \( s \in \mathbb{N} \) such that \( s > d/p \) or \( p = 1 \) and \( s = d \). For every \( n \in \mathbb{N} \), let \( P_n \) be a set of \( n \) independent and uniformly distributed points on \( \Omega \). Then

\[
\Delta (P_n, W^s_p(\Omega), \text{INT}) \asymp n^{-s/d+(1/p-1/2)+1/2}.
\]

This means that, with respect to the Monte Carlo error, independent and uniformly distributed points are (asymptotically) as good as optimally selected (deterministic or random) sampling points in all cases.

This answers the question for the power of independent uniformly distributed samples for numerical integration with respect to the Monte Carlo error criterion. Again, one might also be interested in a stronger uniform error criterion. For a random operator \( A: W^s_p(\Omega) \to \mathbb{R} \) the uniform error can be defined by

\[
\Delta_{\text{unif}} (A, W^s_p(\Omega), \text{INT}) = \mathbb{E} \sup_{\|f\|_{W^s_p(\Omega)} \leq 1} |\text{INT}(f) - A(f)|.
\]

Again the order of the supremum and the expected value is interchanged. As before, given a random point set \( P \subset \Omega \), we put

\[
\Delta_{\text{unif}} (P, W^s_p(\Omega), \text{INT}) := \inf_A \Delta_{\text{unif}} (A, W^s_p(\Omega), \text{INT}),
\]

where the infimum is taken over all random operators of the form \( A(f) = \varphi(f|_P) \) with a random mapping \( \varphi: \mathbb{R}^P \to \mathbb{R} \). This is thus the smallest uniform error that can be achieved with the random sampling point set \( P \). Moreover, we put

\[
\Delta_{\text{unif}} (n, W^s_p(\Omega), \text{INT}) := \inf_P \Delta_{\text{unif}} (P, W^s_p(\Omega), \text{INT}),
\]

where the infimum is taken over all random point sets of cardinality at most \( n \).

For the uniform error of independent uniformly distributed samples, the following result has been obtained in [16] for bounded convex domains.

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Corollary 2. Let $\Omega \subset \mathbb{R}^d$ be open and bounded, satisfying an interior cone condition and let $1 \leq p \leq \infty$ and $s \in \mathbb{N}$ such that $s > d/p$ or $p = 1$ and $s = d$. For every $n \in \mathbb{N}$, let $P_n$ be a set of $n$ independent and uniformly distributed points on $\Omega$. Then

$$\Delta^{\text{unif}} (P_n, W^s_p(\Omega), \text{INT}) \asymp \begin{cases} 
\frac{n}{\log n} & \text{if } p = 1, \\
^{-s/d} & \text{if } p > 1.
\end{cases}$$

This means, with respect to the uniform error criterion, independent and uniformly distributed samples are as good as optimally chosen sampling points if and only if $p > 1$. We also notice that for independent and uniformly distributed samples, the Monte Carlo error criterion provides a speed-up in comparison to the uniform error criterion for all $p$.

Remark 1. Different authors study different domains and possibly even different function spaces $W^s_p(\Omega)$. All the standard definitions of the Sobolev spaces coincide if $\Omega$ is a bounded Lipschitz domain; if the domain is not Lipschitz then different texts possibly use different spaces. If the Sobolev space on $\Omega$ is defined by restriction of the functions from $W^s_p(\mathbb{R}^d)$ then one obtains smaller spaces. Our results stay true for this altered definition since the lower bounds already hold for functions with compact support inside $\Omega$.

Remark 2. The upper bound for optimal points in (2), and therefore also the upper bound in (1), is known at least for Lipschitz domains. We refer to [24] and [28]. Here we consider more general domains and obtain (2) as a byproduct, see Remark 7. Hence, also for optimal points, we may have slightly more general results compared to the existing literature.

Remark 3. The lower bound for optimal points in (1), and therefore also the lower bound in (2), is known for the whole class of domains that we consider. In older texts the lower bounds are usually proved for Sobolev functions with a support inside the cube $[0,1]^d$ and then it is rather easy to prove the same lower bound for all domains that contain a cube, i.e., for all open sets. Because of a re-scaling we thus obtain different constants but the same order of convergence. The same is true for the lower bounds for random points as given in Theorem 2 and Corollary 2. They are known for functions with support in a cube from [16] and immediately transfer to the class of domains considered here. More generally, these lower bounds hold for all (non-empty) bounded open sets and even the boundedness is not needed. When we speak about uniformly distributed random points then, of course, we assume that $\Omega$ has a finite (Lebesgue) measure.

Remark 4. Our techniques can most likely be used to prove similar results for more general function spaces of isotropic smoothness like Triebel-Lizorkin or Besov spaces as well as Sobolev spaces on manifolds. Another interesting family of spaces are function
spaces of mixed smoothness as surveyed in [6]. Here, we are still quite far from understanding the power of independent and uniformly distributed sampling points, and even the power of optimal sampling points is not known in many cases. There are recent results in this direction for the special case of $L_2$-approximation on the Hilbert space $W^s_2(T^d)$ of functions with mixed smoothness $s$ on the $d$-torus. Namely, it is known that independent and uniformly distributed sampling points are optimal with respect to the Monte Carlo error [14] and optimal up to a logarithmic factor with respect to the uniform error [17, 34].

See also [27, 23, 32, 18, 5] for related results. This also implies that they are optimal up to logarithmic factors for the problem of integration on $W^s_2(T^d)$ with respect to both error criteria. We do not know whether the logarithmic loss can be avoided with uniformly distributed samples, as it is the case with isotropic smoothness. Known optimal sampling points for the integration problem on $W^s_2(T^d)$ have a very particular structure, see e.g. [6, Section 8.5] and the references therein for the uniform error and [15, 33] for the Monte Carlo error.

**Remark 5.** In numerical analysis and information-based complexity we usually want to find the optimal algorithm, based on the optimal point set $P = \{x_1, \ldots, x_n\} \subset \Omega$. Often it is not easy to find the optimal point set and even more often it is not possible to choose $P$, we simply have to use the information as it comes in. It is a very common assumption in learning theory and uncertainty quantification that the information comes in randomly, given by independent and identically distributed (iid) samples, see Berner, Grohs, Kutyniok and Petersen [3], Giles [8], Lugosi and Mendelson [20], Shalev-Shwartz and Ben-David [30], Steinwart and Christmann [31] and Zhang [37]. In the framework of information-based complexity, the power of iid information was recently studied and surveyed by Hinrichs, Krieg, Prochno and Ullrich [11, 12], by Huber and Jones [13] and by Kunsch, Novak and Rudolf [19]. Although we think that it is most natural to assume that the iid samples are uniformly distributed on the domain, it might also be interesting to study other distributions. The results of this paper stay valid, if the samples are iid with respect to a probability measure $\mu$ on $\Omega$ that satisfies $\mu(B) \geq c \text{vol}(B)$ for some constant $c > 0$ and all balls $B$ with center in $\Omega$ and radius smaller than a constant. But we believe that this condition is not necessary and it would be interesting to have a characterization of all measures for which iid information is optimal for $L_q(\Omega)$-approximation on $W^s_2(\Omega)$.

**Remark 6.** The main results of this paper are about the optimal order of convergence if we may use only iid sample points and we prove that one can achieve the same or almost the same order that one can achieve with optimal sample points. There are other ways to compare the power of different algorithms and, for example, one could study tractability properties of algorithms that are based on iid sample points.
It would be good to know more about the difference between optimal and iid sample points, depending on the problem, the dimension $d$ and the domain $\Omega \subset \mathbb{R}^d$. One possibility is to study the asymptotic constants, such as

$$\lim_{n \to \infty} \Delta \left(P_n, W_p^s(\Omega), L_q(\Omega)\right) \cdot n^{s/d-(1/p-1/q)+}.$$

We conjecture that all these asymptotic constants exist and possibly they do not depend on the shape of $\Omega$, only on its volume. The asymptotic constants are known only in very rare cases, see [26], though. Here we present an example from [12] that nicely shows the quality of iid samples. We study $L_1$-approximation for functions from the class

$$F_d = \{ f : [0,1]^d \to \mathbb{R} \mid |f(x) - f(y)| \leq d(x,y)\},$$

with the maximum metric on the $d$-torus, i.e.,

$$d(x,y) = \min_{k \in \mathbb{Z}} \|x + k - y\|_\infty.$$

Then

$$\lim_{n \to \infty} \Delta_{\text{unif}} \left(P_n, F_d, L_1(\Omega)\right) \cdot n^{1/d} = \frac{1}{2} \Gamma(1 + 1/d) \approx \frac{1}{2} - \frac{\gamma}{2d}$$

with the Euler number $\gamma \approx 0.577$, noting that $\Gamma'(1) = -\gamma$. This compares very well with the error of optimal methods that, for $n = m^d$, equals $\frac{d}{2d+2} n^{-1/d}$. To achieve the same error $\varepsilon$ in high dimension with random iid sample points, we have to multiply the number of optimal sample points by roughly $\exp(1 - \gamma) \approx 1.526$; this factor is quite small and does not increase with $d$.

2 The algorithm

Before we can formulate the algorithm, we state a result on polynomial reproduction using moving least squares. We use the following result of Wendland [35], see also Chapter 4 in his book [36]; in Lemma 7 of [16] it is shown that one can skip the well-separatedness of the points that appears in Wendland’s formulation. The covering radius of a finite point set $P \subset \mathbb{R}^d$ with respect to a bounded set $\Omega \subset \mathbb{R}^d$ will be denoted by

$$h_{P,\Omega} := \sup_{x \in \Omega} \operatorname{dist}(x, P),$$

where $\operatorname{dist}(x, P) = \min_{y \in P} \|x - y\|_2$. 

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Lemma 1. For every $\theta \in (0, \pi/5]$ and $s, d \in \mathbb{N}$ there are constants $c_1, c_2 > 0$ such that the following holds. For every cone $K \subset \mathbb{R}^d$ with opening angle $\theta$ and radius $\varrho > 0$ and every finite point set $P \subset K$ with covering radius $h_{P,K} \leq c_1 \varrho$, there are continuous functions $u_x: K \to \mathbb{R}$ for $x \in P$ such that the linear operator

$$S_{P,K}: C_0(K) \to C_0(K), \quad S_{P,K}f = \sum_{x \in P} f(x)u_x$$

is bounded with operator norm at most $c_2$ and equals the identity when restricted to the space of polynomials of degree at most $s$.

Implicitly, we used that cones themselves satisfy an interior cone condition, see e.g. Proposition 3.13 in [36], which we formulate as a lemma. Note that we can assume without loss of generality that $\Omega$ satisfies a cone condition with opening angle $\theta \in (0, \pi/5]$.

Lemma 2. Every cone with opening angle $\theta \in (0, \pi/5]$ and radius $\varrho > 0$ satisfies an interior cone condition with parameters $r' = \frac{4}{3} c_0 \varrho$ and $\theta' = \theta$, where $c_0 = \frac{\sin \theta}{1 + \sin \theta}$.

We add another lemma which will ensure that our algorithm is well defined. Recall that $\Omega \subset \mathbb{R}^d$ is open and bounded and satisfies an interior cone condition with parameters $r > 0$ and $\theta \in (0, \pi]$. That is, for every $x \in \Omega$ we find a cone $K(x)$ with apex $x$, opening angle $\theta$ and radius $r$ such that $K(x) \subset \Omega$. Without loss of generality, we may assume that $\theta \leq \pi/5$ and that the direction $\xi(x)$ of the cone $K(x)$ depends continuously on the apex $x$ for almost all $x \in \Omega$, see Lemma 7. We write $K(x, \varrho)$ for the cone with radius $\varrho > 0$ and the same vertex, direction and opening angle as $K(x)$. Then $K(x, \varrho)$ is contained in the closure of the Euclidean ball $B(x, \varrho) := \{ y \in \mathbb{R}^d : \| x - y \|_2 < \varrho \}$ intersected with $\Omega$.

Lemma 3. Let $c_0 = c_0 c_1 / 2$ with $c_0$ as in Lemma 2 and $c_1$ as in Lemma 1. If $P \subset \Omega$ satisfies $h_{P,\Omega} < c_0 r$, then we have $h_{P \cap K(x), K(x)} \leq c_1 r$ for all $x \in \Omega$.

Proof. Assume for a contradiction that there is some $x \in \Omega$ and some $y \in K(x)$ such that the ball $B(y, c_1 r / 2)$ is empty of $P \cap K(x)$. Using the cone condition satisfied by $K(x)$, we obtain a ball $B(z, c_0 r)$ that is contained in $B(y, c_1 r / 2) \cap K(x)$. See [16] Lemma 2], a direct consequence of [36] Lemma 3.7], for details. The ball $B(z, c_0 r)$ is thus empty of $P$, which contradicts $h_{P,\Omega} < c_0 r$. Note that in order to apply Lemma 2 from [16] the inequality $c_1 \leq \frac{4}{3} c_0$ was used implicitly, but this is justified since we may choose the constant $c_1$ in Lemma 1 small enough.

We now define the algorithm $A_P: C_b(\Omega) \to B(\Omega)$, which uses samples on an $n$-point set $P \subset \Omega$. Here, $B(\Omega)$ is the space of bounded functions with the supremum norm. The algorithm is independent of $p$ and $q$ and works for any smoothness up to $s$. We define the algorithm for general point sets $P$, although it will only be used with random points.
Algorithm 1. Let $\Omega \subset \mathbb{R}^d$ be open and bounded, satisfying an interior cone condition. Choose constants $r > 0$ and $\theta \in (0, \pi/5]$ and an almost everywhere continuous function $\xi : \Omega \to S^{d-1}$ such that for all $x \in \Omega$ and $\varrho \leq r$ the cone $K(x, \varrho)$ with apex $x$, opening angle $\theta$, radius $\varrho$ and direction $\xi(x)$ is contained in $\Omega$. Let $s \in \mathbb{N}$ and $c_0 = c_0(s, r, \theta, d)$ be the constant from Lemma 3. For all natural numbers $A$ we define the algorithm $m$ as follows. If no such cone exists, we take $m_A = \lfloor \log_2(rn^{1/d}) \rfloor$ and direction $\xi(x)$ and the polynomial reproducing map from Lemma 1. The cone $K$ ensures that for every $x \in \mathbb{R}^d$ a trivial output. For random points, Scenario 1 is exponentially unlikely. In Scenario 2, points are not suited to achieve an error smaller than a constant. We therefore return the region where we do not have any information about the target function. The sampling operator norm is bounded by the constant $c_2$ from Lemma 1. Further, it follows from the underlying construction in [36] that, for fixed $f$, the output $A_P f(x)$ is a measurable function of $(x, P) \in \Omega^{n+1}$. For this and other measurability issues, we refer the reader to Section 6. We will show the following.

Let us briefly comment on the structure of the algorithm. In Scenario 1, there is a large region where we do not have any information about the target function. The sampling points are not suited to achieve an error smaller than a constant. We therefore return a trivial output. For random points, Scenario 1 is exponentially unlikely. In Scenario 2, Lemma 3 ensures that for every $x \in \Omega$, there is a cone with apex $x$, opening angle $\theta$ and direction $\xi(x)$ such that sufficiently many points reside in this cone in order to apply the polynomial reproducing map from Lemma 1. The cone $K_P(x)$ with radius $r_P(x)$ is the smallest cone with this property. The condition $m \leq m_0$ leads to $r_P(x) \geq n^{-1/d}$ and imposes no essential restriction as this is asymptotically the quantity we would expect from an optimally distributed point set. The algorithm computes at each point $x$ an approximation based on the information given by the samples inside the cone $K_P(x)$ and one can compute the value $A_P f(x)$ numerically using an implementation of moving least squares.

For each point set $P$, the algorithm $A_P$ maps linearly from $C_b(\Omega)$ to $B(\Omega)$ and its operator norm is bounded by the constant $c_2$ from Lemma 1. Further, it follows from the underlying construction in [36] that, for fixed $f$, the output $A_P f(x)$ is a measurable function of $(x, P) \in \Omega^{n+1}$. For this and other measurability issues, we refer the reader to Section 6. We will show the following.
Theorem 3. Algorithm\footnote{1} obeys the following error bounds.

\[
\Delta \left( A_n, W^s_p(\Omega), L^q(\Omega) \right) \lesssim \begin{cases} 
(n/ \log n)^{-s/d} & \text{if } p = q = \infty, \\
- s/d + (1/p - 1/q) & \text{else.}
\end{cases} 
\]  

(4)

\[
\Delta^{\text{unif}} \left( A_n, W^s_p(\Omega), L^q(\Omega) \right) \lesssim \begin{cases} 
(n/ \log n)^{-s/d + 1/p - 1/q} & \text{if } q \geq p, \\
- s/d & \text{else.}
\end{cases} 
\]  

(5)

The next section forms the proof of these upper bounds. The corresponding lower bounds, which complete the proof of Theorems 1 and 2, are known, see Remark 3, except for the case \( p = q = \infty \), which will be proven in Section 4.

3 Proof of Theorem\footnote{3}

In Scenario 2, we get the following point-wise estimate for the error.

Lemma 4. There is a constant \( c_3 > 0 \) such that for all \( f \in W^s_p(\Omega) \) and all \( x \in \Omega \), we have in Scenario 2 that

\[
|f(x) - A_n f(x)| \leq c_3 r_n(x)^{s-d/p} ||f||_{W^s_p(K_n(x))}. 
\]

Proof. Using an affine map to a reference cone \( K \), together with the continuous embedding of \( W^s_p(K) \) into the continuous functions and the generalized Poincaré inequality from [22, Ch. 1.1.11] we find a constant \( c_4 > 0 \) independent of \( f, x \) and \( n \) and a polynomial \( \pi \) of degree at most \( s \) such that

\[
||f - \pi||_{L^\infty(K_n(x))} = ||f - \pi||_{L^\infty(K^c_n(x))} \leq c_4 r_n(x)^{s-d/p} ||f||_{W^s_p(K^c_n(x))}.
\]

Here, \( K^c_n(x) \) is the interior of \( K_n(x) \). By Lemma\footnote{1}

\[
|f(x) - A_n f(x)| \leq ||f - S_{P_n \cap K_n(x), K_n(x)} f||_{L^\infty(K_n(x))} \\
\leq ||f - \pi||_{L^\infty(K_n(x))} + ||S_{P_n \cap K_n(x), K_n(x)}(f - \pi)||_{L^\infty(K_n(x))} \\
\leq (1 + c_2) ||f - \pi||_{L^\infty(K_n(x))},
\]

which proves the statement.\hfill \Box
In Scenario 1, where we have $A_n = 0$, we clearly get the estimate
\[ \|f - A_n f\|_{L^q(\Omega)} \leq \text{vol}(\Omega)^{1/q} \|f\|_{C_b(\Omega)} \leq c_5 \|f\|_{W^{s,p}(\Omega)}. \]

Moreover, the probability of Scenario 1 is exponentially small in $n$. This can be shown by a simple net argument. We do not have to prove it since the probability estimate follows from [29, Theorem 2.1] for $\alpha = c N / \log N$ for a suitable constant $c > 0$. Together with Lemma 4, this implies that Theorem 3 follows once we prove the following estimates:

\[ \sup_{\|f\|_{W^{s,p}(\Omega)} \leq 1} \|r_n(x)^{s-d/p} \|_{W^{s,p}(\Omega)} \|_{L^q(\Omega)} \leq \begin{cases} 
(n/ \log n)^{-s/d} & \text{if } p = q = \infty, \\
n^{-s/d + (1/p - 1/q) +} & \text{else.} 
\end{cases} \quad (6) \]

\[ \mathbb{E} \sup_{\|f\|_{W^{s,p}(\Omega)} \leq 1} \|r_n(x)^{s-d/p} \|_{W^{s,p}(\Omega)} \|_{L^q(\Omega)} \leq \begin{cases} 
(n/ \log n)^{-s/d + 1/p - 1/q} & \text{if } q \geq p, \\
n^{-s/d} & \text{else.} 
\end{cases} \quad (7) \]

To prove these estimates, we have to study the random variable $r_n(x)$ for $x \in \Omega$, which is the radius of the approximation cone at $x$. In fact, it will be helpful to have upper bounds on the radius of the largest approximation cone containing a certain point $y \in \Omega$. We therefore introduce

\[ r^*_p(y) := \text{ess sup}_{x \in \Omega} r_p(x) 1(y \in K_p(x)) \quad (8) \]

and the random variable $r^*_n(y)$ obtained if $P$ is a set of $n$ independent and uniformly distributed points. Note that both $r_n(y)$ and $r^*_n(y)$ take values between $n^{-1/d}$ and $r$. We will prove the following.

**Proposition 1.** For all $y \in \Omega$ and $\alpha \geq 0$, we have

\[ \mathbb{E} r_n(y)^\alpha \asymp \mathbb{E} r^*_n(y)^\alpha \asymp n^{-\alpha/d}, \]

where the implied constants are independent of $y$ and $n$.

Our proof strategy for Proposition 1 involves suitably scaled grids which we now define. If $Q(y, \varrho) := \{ x \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i - y_i| \leq \varrho \}$ is a (closed) cube around a point $y \in \Omega$ and $\ell \in \mathbb{N}$, then we partition $Q(y, \varrho)$ into $\ell^d$ equally sized cubes of sidelength $2\varrho\ell^{-1}$. We collect all of the cubes which are fully contained in $\Omega$ into a set $\text{Grid}(y, \varrho, \ell)$. Obviously, its cardinality $\#\text{Grid}(y, \varrho, \ell)$ can be at most $\ell^d$. 

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Lemma 5. Let $\ell := \lceil \frac{\sqrt{d}}{c_0 r} \rceil$, where $c_0$ is as in Lemma 2. Let $y \in \Omega$ and $g_n \in \{r_n, r_n^*\}$. If $g_n(y) = 2^{-m} r$ for some $m < m_0$, then there is a cube $Q \in \text{Grid}(y, 2^{-m} r, \ell)$ with $Q \cap P_n = \emptyset$.

Proof. By definition of $K$ in Lemma 6. Let $y \in \Omega$ such that $g_n(y) = 2^{-m} r$ for some $m < m_0$. Then there exists a cube $Q(y, 2^{-m} r)$ with $Q(y, 2^{-m} r) \subseteq K(x, 2^{-m} r)$. By the definition of $r_n$, the cone $K = K(x, 2^{-m} r) \subseteq K_n(x)$ satisfies

$$h_{P_n \cap K, K} > c_1 2^{-m-1} r.$$

That is, we find $z \in K \subseteq \Omega$ such that $B(z, c_1 2^{-m} r)$ is empty of $P_n \cap K$. Arguing as in the proof of Lemma 3, we obtain a ball of radius $c_0 c_1 2^{-m} r$ contained in $K$ and empty of $P_n$. This ball contains a cube of sidelength $c 2^{-m} r$ with $c = c_0 c_1 / \sqrt{d}$, which in turn contains a cube of $\text{Grid}(y, 2^{-m} r, \ell)$ if

$$\ell^{-1} 2^{-m+2} r < c 2^{-m} r,$$

which is fulfilled for our choice of $\ell$. \hfill \Box

Lemma 6. There are constants $C, c > 0$ independent of $n$ such that for all $y \in \Omega$ and $t > 0$, and for $g_n \in \{r_n, r_n^*\}$ we have

$$\mathbb{P}(g_n(y) > t) \leq C \exp\left(-c t^d n\right).$$

Proof. For $t \geq r$, the probability is zero and the estimate is true for any $C, c > 0$. For $t < 2^{-m_0} r$, the estimate is ensured by taking $C \geq \exp(2^d c)$. Thus let $2^{-m_0} r \leq t < r$ from now on, i.e., $t \in [2^{-m_1} r, 2^{-m_1} r)$ for some $m_1 < m_0$. We have

$$\mathbb{P}(g_n(y) > t) = \sum_{m=0}^{m_1} \mathbb{P}(g_n(y) = 2^{-m} r).$$

By Lemma 3 we have

$$\mathbb{P}(g_n(y) = 2^{-m} r) \leq \mathbb{P}(\exists Q \in \text{Grid}(y, 2^{-m} r, \ell) : Q \cap P_n = \emptyset) \leq \#\text{Grid}(y, 2^{-m} r, \ell) (1 - \tilde{c}(2^{-m} r)^d)^n \leq \ell^d \exp\left(-\tilde{c}\cdot\lfloor(2^{-m} r)^d n\rfloor\right)$$

with a constant $\tilde{c} > 0$ depending only on $\ell$ and the domain $\Omega$. For $m \in \{0, \ldots, m_1\}$, the numbers $\lfloor(2^{-m} r)^d n\rfloor$ are distinct natural numbers greater than $t^d n/2$, such that we get

$$\mathbb{P}(g_n(y) > t) \leq \ell^d \sum_{k > t^d n/2} \exp(-\tilde{c} k) \leq C \exp(-\tilde{c} t^d n/2).$$

\hfill \Box
Now, we easily obtain Proposition \( \square \)

**Proof of Proposition** \( \square \). The statement is clear for \( \alpha = 0 \) and thus consider \( \alpha > 0 \). Let \( \varrho_n \in \{ r_n, r_n^* \} \). The estimate
\[
E \varrho_n(y)^\alpha \geq n^{-\alpha/d}
\]
is immediate. On the other hand, Lemma \( \square \) yields
\[
E \varrho_n(y)^\alpha \leq \int_0^\infty \mathbb{P}(\varrho_n(y)^\alpha > t) \, dt \leq C^\alpha \int_0^\infty \exp(-c \, t^{d/\alpha} \, n) \, dt = C'n^{-\alpha/d}
\]
where \( C' := C^\alpha \int_0^\infty \exp(-c \, u^{d/\alpha}) \, du \).

We are now ready to prove (6) and (7) which will conclude the proof of Theorem \( \square \). If not specified otherwise, \( f \) will always denote an element of the unit ball of \( W^s_p(\Omega) \).

3.1 **Proof of (6)**

We perform a case distinction in \( p \) and \( q \).

3.1.1 **The case** \( p = \infty \)

Here, we simply compute
\[
E \left\| r_n(x)^{s-d/p} \left| \mathbb{1}_{W^s_p(K_n(x))} \right| \right\|_{L_q(\Omega)} \leq E \| r_n(x)^s \|_{L_q(\Omega)}.
\]
In the case \( q < \infty \), we arrive with Jensen and Fubini at
\[
\left( E \| r_n(x)^s \|_{L_q(\Omega)}^q \right)^{1/q} = \left( \int_{\Omega} E r_n(x)^{sq} \, dx \right)^{1/q}
\]
and it follows by Proposition \( \square \) that this is dominated by \( n^{-s/d} \), as desired. In the case \( q = \infty \), we use that for any realization \( r_n(x) \leq c_6 h_{P_n,\Omega} \) with \( c_6 > 0 \) independent of both \( x \) and the point set, which follows from the proof of Lemma \( \square \) so that
\[
E \| r_n(x)^s \|_{L_q(\Omega)} \leq E h_{P_n,\Omega}^s \leq \left( \log \frac{n}{n} \right)^{s/d},
\]
where the last estimate is known e.g. from Reznikov and Saff [29, Theorem 2.1], using the cone condition of \( \Omega \). \( \square \)
3.1.2 The case $p = q < \infty$

Due to Jensen and Fubini

\[
\mathbb{E} \left\| r_n(x)^{s - d/p} |f| W^p_r(K_n(x)) \right\|_{L^p(\Omega)} \leq \left( \mathbb{E} \int_{\Omega} r_n(x)^{sp - d} |f|^p W^p_r(K_n(x)) \, dx \right)^{1/p} \\
= \left( \sum_{|\alpha| = s} \int_{\Omega} |D^\alpha f(y)|^p \cdot \int_{\Omega} \mathbb{E} \left| r_n(x)^{sp - d} 1(y \in K_n(x)) \right| \, dx \, dy \right)^{1/p}.
\]

Therefore, the desired upper bound \(6\) follows, once we prove

\[
\int_{\Omega} \mathbb{E} \left[ r_n(x)^{sp - d} 1(y \in K_n(x)) \right] \, dx \leq n^{-sp/d}
\]

with a constant independent of \(y\). We use Hölder’s inequality and Proposition \(1\) to obtain

\[
\mathbb{E} \left[ r_n(x)^{sp - d} 1(y \in K_n(x)) \right] \leq \left[ \mathbb{E} r_n(x)^{2(sp - d)} \right]^{1/2} \cdot \mathbb{P}(y \in K_n(x))^{1/2} \\
\leq n^{-sp/d + 1} \cdot \mathbb{P}(y \in K_n(x))^{1/2}.
\]

From Lemma \(3\) we deduce for any \(y \in \Omega\)

\[
\mathbb{P}(y \in K_n(x)) \leq \mathbb{P}(r_n(x) \geq \|x - y\|_2) \leq \exp \left( -c \|x - y\|_2^d n \right),
\]

since the cone \(K_n(x)\) has radius \(r_n(x)\). Here, implicit constants are independent of \(y\). Therefore,

\[
\int_{\Omega} \mathbb{P}(y \in K_n(x))^{1/2} \, dx \leq \int_{\mathbb{R}^d} \exp \left( -\frac{c}{2} \|x - y\|_2^d \right) \, dx = \frac{1}{n} \int_{\mathbb{R}^d} \exp \left( -\frac{c}{2} \|u\|_2^d \right) \, du,
\]

from which the desired estimate \(9\) follows.

\(\square\)

3.1.3 The case $p < q = \infty$

Again due to Jensen and Fubini and the fact that the essential supremum of the integral is smaller than the integral of the essential supremum, we compute

\[
\mathbb{E} \left\| r_n(x)^{s - d/p} |f| W^p_r(K_n(x)) \right\|_{L^\infty(\Omega)} \leq \left( \mathbb{E} \left\| r_n(x)^{sp - d} |f|^p W^p_r(K_n(x)) \right\|_{L^\infty(\Omega)} \right)^{1/p} \\
= \left( \mathbb{E} \text{ess sup}_{x \in \Omega} \sum_{|\alpha| = s} \int_{\Omega} |D^\alpha f(y)|^p \cdot r_n(x)^{sp - d} 1(y \in K_n(x)) \, dy \right)^{1/p} \\
\leq \left( \sum_{|\alpha| = s} \int_{\Omega} |D^\alpha f(y)|^p \cdot \mathbb{E} \left[ \text{ess sup}_{x \in \Omega} r_n(x)^{sp - d} 1(y \in K_n(x)) \right] \, dy \right)^{1/p}
\]

and the desired estimate \(10\) is obtained from Proposition \(1\). \(\square\)
3.1.4 The case $p < q < \infty$

This follows by interpolation from the previous cases. Namely,

$$\|f - A_n(f)\|_{L_q(\Omega)} \leq \|f - A_n(f)\|_{L_p(\Omega)}^{p/q} \cdot \|f - A_n(f)\|_{L_\infty(\Omega)}^{1-p/q}$$

and by Hölder’s inequality with conjugates $q/p$ and $(1 - p/q)^{-1}$, we get

$$E \|f - A_n(f)\|_{L_q(\Omega)} \leq (E \|f - A_n(f)\|_{L_p(\Omega)})^{p/q} \cdot (E \|f - A_n(f)\|_{L_\infty(\Omega)})^{1-p/q}.$$ 

Inserting the corresponding upper bounds for $q = \infty$ and $q = p$, we see that the desired bound for $q \in (p, \infty)$ follows. □

3.1.5 The case $q < p < \infty$

This follows from the case $q = p < \infty$. □

3.2 Proof of (7)

Again, we perform a case distinction.

3.2.1 The case $q < p = \infty$

Here, we observe

$$E \sup_{\|f\|_{W^s_p(K^\infty_n(x))} \leq 1} \left\| r_n(x)^{s-d/p} |f|_{W^s_p(K^\infty_n(x))} \right\|_{L_q(\Omega)} \leq E \left\| r_n(x)^s \right\|_{L_q(\Omega)}$$

and proceed as in Section 3.1.1. □

3.2.2 The case $q < p < \infty$

We choose $\delta > 0$ arbitrary and split the integrands of

$$\left\| r_n(x)^{s-d/p} |f|_{W^s_p(K^\infty_n(x))} \right\|_{L_q(\Omega)}^q = \int_\Omega r_n(x)^{(s-d/p)q} |f|_{W^s_p(K^\infty_n(x))}^q \, dx$$

into

$$r_n(x)^{sq+\delta} \quad \text{and} \quad r_n(x)^{-dq/p-\delta} |f|_{W^s_p(K^\infty_n(x))}^q$$

and apply Hölder’s inequality with conjugate indices $r = (1 - q/p)^{-1}$ and $p/q$ to obtain

$$\left\| r_n(x)^{s-d/p} |f|_{W^s_p(K^\infty_n(x))} \right\|_{L_q(\Omega)}^q \leq \left( \int_\Omega r_n(x)^{(sq+\delta)\frac{q}{r}} \, dx \right)^{1/r} \left( \int_\Omega r_n(x)^{-d-\delta/pq} \, dx \right)^{q/p}.$$
The right-hand integral satisfies by virtue of Fubini’s theorem
\[
\int_{\Omega} r_n(x)^{-d-\delta p/q} |f|^p_{W_p^s(K_n(x))} \, dx
\]
\[
= \sum_{|\alpha|=s} \int_{\Omega} |D^\alpha f(y)|^p \int_{\Omega} r_n(x)^{-d-\delta p/q} 1(y \in K_n(x)) \, dx \, dy
\]
\[
\leq \sup_{y \in \Omega} \int_{\Omega} r_n(x)^{-d-\delta p/q} 1(y \in K_n(x)) \, dx
\]
since \(f\) has semi-norm at most one.

Let \(y \in \Omega\) be arbitrary. Note that \(y \in K_n(x)\) implies \(\|y - x\| \leq r_n(x)\), and thus, since \(r_n(x) \geq n^{-1/d}\),
\[
\int_{\Omega} r_n(x)^{-d-\delta p/q} 1(y \in K_n(x)) \, dx \leq \int_{\Omega} \frac{1}{\max\{n^{-1/d}, \|y - x\|\}^{d+\delta p/q}} \, dx
\]
\[
\leq \int_{\|y-x\| \leq n^{-1/d}} n^{1+\delta p/q} d\,x + \int_{\|y-x\| > n^{-1/d}} \|y-x\|^{-d-\delta p/q} \, dx.
\]
where we extended the integral to the whole of \(\mathbb{R}^d\). The first integral on the right is \(\omega_d n^{\delta p/(dq)}\), where \(\omega_d\) is the volume of the \(d\)-dimensional unit ball, and the right integral transforms, by integration in spherical coordinates and subsequent substitution \(v = n^{1/d} u\), to
\[
d\omega_d \int_{n^{-1/d}}^{\infty} u^{-d-\delta p/q} u^{d-1} \, du = d\omega_d n^{\delta p/(dq)} \int_1^{\infty} v^{-1-\delta p/q} \, dv \leq n^{\delta p/(dq)}.
\]
Note that the surface area of \(S^{d-1}\) equals \(d\omega_d\). This shows
\[
\int_{\Omega} r_n(x)^{-d-\delta p/q} 1(y \in K_n(x)) \, dx \leq n^{\delta p/(dq)}
\]
where the implicit constant depends only on \(d, \delta, p\) and \(q\). Consequently, taking the supremum over all \(y \in \Omega\) yields
\[
\int_{\Omega} r_n(x)^{-d-\delta p/q} |f|^p_{W_p^s(K_n(x))} \, dx \leq n^{\delta p/(dq)}.
\]
Therefore,
\[
\sup_{\|f\|_{W_p^s(\Omega)} \leq 1} \left\| r_n(x)^{s-d/p} |f|_{W_p^s(K_n(x))} \right\|_{L_q(\Omega)} \leq \left( \int_{\Omega} r_n(x)^{(sq+)r} \, dx \right)^{1/r \frac{1}{n^{\delta/(dq)}}}.
\]
To establish (7) we note that by Jensen’s inequality and Fubini’s theorem we have
\[
\mathbb{E} \sup_{\|f\|_{W_p^s(\Omega)} \leq 1} \left\| r_n(x)^{s-d/p} |f|_{W_p^s(K_n(x))} \right\|_{L_q(\Omega)} \leq \left( \int_{\Omega} \mathbb{E} r_n(x)^{(sq+)r} \, dx \right)^{1/r \frac{1}{n^{\delta/(dq)}}}.
\]
By Proposition 1 we have that \(\mathbb{E} r_n(x)^{(sq+)r} \leq n^{-sqr/(d-\delta r/d)}\) with a constant independent of \(x\) and this completes the proof in this case. 

3.2.3 The case \( q \geq p \)

We use \( r_n(x) \leq c_6 h_{P_n, \Omega} \), so that

\[
\left\| r_n(x)^{-d/p} \left| f\right| \mathcal{W}^s_p(K_n^a(x)) \right\|_{L_q(\Omega)} \leq \left( c_6 h_{P_n, \Omega} \right)^{s-d/p} \left\| f\mathcal{W}^s_p(K_n^a(x)) \right\|_{L_q(\Omega)} \\
\leq (c_6 h_{P_n, \Omega})^{s-d/p} \left\| f\mathcal{W}^s_p(K_n^a(x)) \right\|_{L_p(\Omega)}^{p/q} \cdot \left\| f\mathcal{W}^s_p(K_n^a(x)) \right\|_{L_\infty(\Omega)}^{1-p/q} \leq 1.
\]

Now, since \( K_n(x) \subset B(x, c_6 h_{P_n, \Omega}) \), we have that

\[
\left\| f\mathcal{W}^s_p(K_n^a(x)) \right\|_{L_p(\Omega)}^{p/q} \leq \left( \sum_{|\alpha| = s} \int_{\Omega} \int_{\Omega} \left| D^\alpha f(y) \right|^p 1(y \in B(x, c_6 h_{P_n, \Omega})) \, dy \, dx \right)^{1/q} \\
= \left( \sum_{|\alpha| = s} \int_{\Omega} \left| D^\alpha f(y) \right|^p \int_{\Omega} 1(x \in B(y, c_6 h_{P_n, \Omega})) \, dx \, dy \right)^{1/q} \approx h_{P_n, \Omega}^d.
\]

Together,

\[
\sup_{\|f\|_{\mathcal{W}^s_p(\Omega)} \leq 1} \left\| r_n(x)^{-d/p} \left| f\mathcal{W}^s_p(K_n^a(x)) \right\|_{L_q(\Omega)} \leq h_{P_n, \Omega}^{s-d/p+d/q}.
\]

Taking the expected value, the estimate (7) follows from [29, Theorem 2.1]. Again we use the cone condition of \( \Omega \).

**Remark 7.** We obtained in the case \( q \geq p \) the upper bound

\[
\sup_{\|f\|_{\mathcal{W}^s_p(\Omega)} \leq 1} \left\| f - A_n(f) \right\|_{L_q(\Omega)} \leq h_{P_n, \Omega}^{s-d/p+d/q},
\]

which holds true for any realization of the point set \( P_n \) satisfying Scenario 2. By inserting a point set with covering radius of order \( n^{-1/d} \) instead of iid points, we obtain the order of convergence \( n^{-s/d-1/p+1/q} \). We thus obtain the order of convergence of optimal sampling points as stated in [2] for all open and bounded domains with the cone condition as a byproduct. Note that the upper bound in the case \( q < p \) is implied by [5], and the corresponding lower bounds are known, see Remark 3.

### 4 Lower bounds

In this section we complete the proof of Theorem 1 by providing the corresponding lower bound. As noted above, see Remark 3, the only case that is open is the lower bound for \( p = q = \infty \). We use a technique of Bakhvalov: bump functions, the average error with respect to a discrete measure and the theorem of Fubini.
There are bump functions $f_1, \ldots, f_m$ with support in disjoint balls contained in $\Omega$ of volume of the order $m^{-1}$ such that $\|f_i\|_{W^{s}_\infty(\Omega)} = 1$ and $\|f_i\|_\infty \geq m^{-s/d}$; we obtain such functions by a simple scaling. Now we consider the finite set

$$F := \left\{ \sum_{i=1}^{m} \varepsilon_i f_i : \varepsilon_i \in \{-1, 1\} \right\}.$$ 

Then $\|f\|_{W^{s}_\infty(\Omega)} = 1$ for all $f \in F$. To distinguish between $f$ and $g$ for each pair of distinct functions $f, g \in F$ we would need at least $m$ well chosen function values and here we lose a log term since we have to use iid information.

Let $m \approx n/2 \log n$. For every algorithm $S_{P_n}$ that uses the random sampling point set $P_n$, we have

$$\Delta(S_{P_n}, W^{s}_\infty(\Omega), L_\infty(\Omega)) = \sup_{\|f\|_{W^{s}_\infty} \leq 1} \mathbb{E} \|f - S_{P_n}(f)\|_\infty \geq \sup_{f \in F} \mathbb{E} \|f - S_{P_n}(f)\|_\infty \geq \mathbb{E} \frac{1}{2m} \sum_{f \in F} \|f - S_{P_n}(f)\|_\infty.$$ 

With constant probability, $P_n$ misses one of the balls (recall the coupon collector’s problem) and $S_{P_n}$ cannot determine the sign $\varepsilon_i$ of the corresponding bump $f_i$. Thus

$$\|f - S_{P_n}(f)\|_\infty \geq \|f_i\|_\infty$$

for at least half of the functions $f \in F$ and

$$\frac{1}{2m} \sum_{f \in F} \|f - S_{P_n}(f)\|_\infty \geq \frac{1}{2} \|f_i\|_\infty.$$ 

This yields

$$\Delta(S_{P_n}, W^{s}_\infty, L_\infty) \geq m^{-s/d}.$$ 

\[\square\]

5 Integration

The results on the integration problem are obtained as follows.

Corollary 1 is a direct consequence of (the upper bound of) Theorem 1, together with the lower bound that holds for all randomized algorithms. We use the algorithm for $n$ function values for $L_2$-approximation, as in the proof of the theorem, to construct an approximation $f_n$. Then we use another $n$ function values for the standard Monte Carlo method for $(f - f_n)$, to obtain the additional order $n^{-1/2}$. This algorithm \(\text{control}
variants” or “separation of the main part”) clearly uses iid random points, uniformly distributed in \( \Omega \).

Corollary 2 is a direct consequence of Theorem 2, together with the lower bounds of \([16]\). We use the algorithm for \( n \) function values for \( L_1 \)-approximation, as in the proof of the theorem, to construct an approximation \( f_n \). Then we output the integral of \( f_n \).

6 Measurability

In order to guarantee the measurability of our algorithm \( A_n \), we need that the direction \( \xi(x) \) of the cone \( K(x) \) in the definition of the interior cone condition of \( \Omega \) depends continuously on the apex \( x \) at almost every point. The following lemma provides us with such a choice if we take slightly smaller cones.

**Lemma 7.** Let \( \Omega \) be open and bounded satisfying an interior cone condition with parameters \( r > 0 \) and \( \theta \in (0, \pi) \). Then we can find for every \( x \in \Omega \) a cone with apex \( x \), radius \( (c_\theta/2)r \), angle \( 2 \arcsin(c_\theta/4) \) and direction \( \xi(x) \in \mathbb{S}^{d-1} \) such that \( \xi \) is continuous almost everywhere on \( \Omega \). Here, \( c_\theta \) is as in Lemma 2.

**Proof.** We find finitely many sets \( \Omega_k \), \( k = 1, \ldots, K \), each star-shaped with respect to a ball \( B_k = B(z_k, r_k) \subset \Omega_k \) (i.e., the straight line connecting any \( x \in \Omega_k \) with any \( y \in B_k \) is contained in \( \Omega_k \)), such that their union is \( \Omega \). For the existence of such a family of sets see e.g. \([22, 1.1.9, \text{Lemma 1}] \) or \([36, \text{Lemma 11.31}] \).

Fix some \( k \) as above. We can define \( \xi(x) \) for every \( x \in \Omega_k \) such that \( \xi \) is continuous almost everywhere on the interior of \( \Omega_k \). Namely, from the proof of \([36, \text{Lemma 11.31}] \) (as well as the proofs of Proposition 11.26 and Lemma 3.10 there) one can deduce that for every \( k \) one can choose \( r_k = (c_\theta/2)r \) and for every \( x \in \Omega_k \setminus \{z_k\} \) a cone with apex \( x \), radius \( r_0 \), angle \( 2 \arcsin(c_\theta/4) \) and direction \( \xi(x) = (z_k - x)/\|z_k - x\|_2 \) which is continuous on the interior of \( \Omega_k \) except for the point \( z_k \). Note that \( 2 \arcsin(c_\theta/4) \leq \pi/3 \).

In order to define \( \xi \) on \( \Omega \), we set \( \Omega'_1 = \Omega_1 \) and \( \Omega'_k = \Omega_k \setminus \bigcup_{j<k} \Omega'_j \) for \( 2 \leq k \leq K \). Then any \( x \in \Omega \) belongs to exactly one \( \Omega'_k \). Using this, we can define \( \xi(x) \) everywhere on \( \Omega \). The function \( \xi \) is continuous on \( E_1^C \), where \( E_1 \) is the union of the points \( z_k \) and the boundaries \( \partial \Omega_k \). Since for every \( x \in \partial \Omega_k \) and every \( r < r_k \) the ball \( B(x, r) \) contains a ball of proportional volume which is contained in the interior of \( \Omega_k \) (and thus not in \( \partial \Omega_k \)), the density of \( \partial \Omega_k \) is smaller than one at every \( x \in \partial \Omega_k \). Thus, by Lebesgue’s density theorem, the sets \( \partial \Omega_k \) and \( E_1 \) are null sets. Note that \( E_1^C \) is also an open set.

Possibly modifying constants we assume from now on that \( \Omega \) satisfies an interior cone condition with parameters \( r \) and \( \theta \) such that the direction \( \xi(x) \) of the cone \( K(x) \) is continuous in \( x \in \Omega \) except from a set \( E_1 \subset \Omega \) with measure zero. We now prove the
measurability of the functions appearing in the previous sections. Recall the definitions of $m_P(x)$, $r_P(x)$, $r^*_P(x)$, $K_P(x)$ and $A_P f(x)$ from Algorithm 1 and (8). The measurability of the corresponding random quantities follows from the following proposition.

**Proposition 2.** Let $n \in \mathbb{N}$. Then the following mappings are measurable (with respect to the corresponding Lebesgue or Borel $\sigma$-algebras):

i) $m_P(x)$ and consequently $r_P(x)$ as functions of $(x, P) \in \Omega^{n+1}$,

ii) $r^*_P(x)$ as a function of $(x, P) \in \Omega^{n+1}$,

iii) $A_P f(x)$ as a function of $(x, P) \in \Omega^{n+1}$ for every $f \in W^s_p(\Omega)$,

iv) $\sup_{\|f\|_{W^s_p(\Omega)} \leq 1} \|f - A_P f\|_{L^q(\Omega)}$ as a function of $P \in \Omega^n$,

v) $|f|_{W^s_p(K(\cdot))}$ as a function of $(x, P) \in \Omega^{n+1}$ for every $f \in W^s_p(\Omega)$.

Note that we slightly abuse notation since we allow repeated points in the $n$-point set $P$ such that the functions are defined for all $(x, P) \in \Omega^{n+1}$. In the following, we view $\Omega^k, k \in \mathbb{N}$, as a subset of $\mathbb{R}^{dk}$ from which it inherits the Euclidean distance. Since, on a finite-dimensional space, all norms are equivalent, we can also use distances induced by other norms. Before we provide a proof we need a few lemmas. We consider $n \in \mathbb{N}$ arbitrary but fixed.

**Lemma 8.** The function $P \mapsto h_{P, \Omega}$ is continuous on $\Omega^n$.

**Proof.** The function is even Lipschitz-continuous. It is sufficient to check that

$$h_{\{x_1, \ldots, x_n\}, \Omega} \geq h_{\{y_1, \ldots, y_n\}, \Omega} - \max_{i=1, \ldots, n} \|x_i - y_i\|_2,$$

for all point sets $\{x_1, \ldots, x_n\} \subset \Omega$ and $\{y_1, \ldots, y_n\} \subset \Omega$. \hfill $\Box$

**Lemma 9.** The set $E_2$ is a null set, where

$$E_2 := \{(x, P) \in \Omega^{n+1} : P \cap \partial K(x, 2^{-m}r) \neq \emptyset \text{ for some } m \in \mathbb{N}\}.$$

**Proof.** Since $E_2$ is measurable, it is sufficient to show for any fixed $x \in \Omega$ and $m \in \mathbb{N}$ that the set of all $P \in \Omega^n$ with $P \cap \partial K(x, 2^{-m}r) \neq \emptyset$ is a null set. But this is evident from the fact that the boundary of a cone has measure zero. \hfill $\Box$

**Lemma 10.** For all $m \leq m_0$, the function

$$h_m: \Omega^{n+1} \to \mathbb{R}, \quad h_m(x, P) = h_{P \cap K(x, 2^{-m}r), K(x, 2^{-m}r)}$$

is continuous on $E_1^C \cap E_2^C$. In particular, the function is measurable.
Proof. Let \((x, P) \in E^C_1 \cap E^C_2\) and let \(\varepsilon > 0\). We can perturb both \(x\) and \(P\) by a small amount such that each point of \(P\) is delocated by less than \(\varepsilon/2\) without removing or adding any point of \(P\) to the cone \(K(x, 2^{-m_r})\) and thus the covering radius of the perturbed configuration cannot change by more than \(\varepsilon\). We provide the details. Since \((x, P) \in E^C_1\), we find \(\delta > 0\) such that for all \(n\)-point sets \(Q = \{y_1, \ldots, y_n\} \subset \Omega\) with \(\|y_i - x_i\| < \delta\) for all \(i\) we have that \(\text{dist}(Q, \partial K(x, 2^{-m_r})) > \delta\). Since \(\xi\) is continuous on \(E^C_1\), we find \(\delta' > 0\) small enough such that for every \(y \in B(x, \delta')\) the sets \(\partial K(x, 2^{-m_r})\) and \(\partial K(y, 2^{-m_r})\) are closer than \(\delta/2\) in Hausdorff distance.

By the triangle inequality, for every \((y, Q)\) as above we have that for every point in \(Q \cap K(y, 2^{-m_r})\) we find a point in \(P \cap K(x, 2^{-m_r})\) which has distance less than \(\delta\) from it, and vice versa. If \(\delta < \varepsilon/2\), then \(|h_m(y, Q) - h_m(x, P)| < \varepsilon\) concludes the proof. Note that we also showed that \(E^C_1 \cap E^C_2\) is open. \(\square\)

Lemma 11. For all \(m \leq m_0\) and \(c > 0\), the set \(\{h_m = c\}\) is a null set. In particular,
\[ E_3 := \{(x, P) \in \Omega^{n+1} : h_m(x, P) = c, 2^{-m_r} \text{ for some } m \leq m_0\} \]
is a null set. Moreover, the set \(\{P \in \Omega^n : h_{P, \Omega} = c_0 r\}\) is a null set.

Proof. If the level set \(\{h_m = c\}\) is empty, this is clear, so assume it is not empty. By Lemma 10 it is a measurable set and it is therefore sufficient to show that for every \(x \in \Omega\) the set
\[ E = \{P \in \Omega^n : h_m(x, P) = c\} \]
is a null set. For this we will use the Lebesgue density theorem as follows. Let \(P \in E\) be arbitrary with \(P = \{x_1, \ldots, x_n\}\). Choose \(\varepsilon_0 > 0\) small enough such that \(B(y, \varepsilon_0) \subset \Omega\) for every \(y \in P\). Then for any \(\varepsilon < \varepsilon_0\),
\[ U(P, \varepsilon) := B(x_1, \varepsilon) \times \cdots \times B(x_n, \varepsilon) \subset \Omega^n. \]
Since \(h_m(x, P) = c\) and \(h_m(x, P)\) is nothing but the maximal value of the continuous function \(\text{dist}(\cdot, P \cap K)\) on the compact set \(K := K(x, 2^{-m_r})\), we find a point \(z \in K\) such that \(\text{dist}(z, P \cap K) = c\). Thus, \(B(z, c)\) is empty of \(P \cap K\). For each \(i\), we set \(B_i = B(z, c)\) if \(x_i \in K\) and \(B_i = K\) if \(x_i \notin K\). In both cases, we have \(\text{vol}(B(x_i, \varepsilon) \setminus B_i) \geq 1/2 \cdot \text{vol}(B(x_i, \varepsilon))\). Moreover, every point of \(B(x_i, \varepsilon) \setminus B_i\) is either outside the cone \(K\) or at a distance greater than \(c\) from \(z\). Thus, for any point set
\[ Q \in (B(x_1, \varepsilon) \setminus B_1) \times \cdots \times (B(x_n, \varepsilon) \setminus B_n) \]
we have \(h_m(x, Q) > c\), meaning that \(Q \notin E\). This gives
\[ \frac{\text{vol}(U(P, \varepsilon) \cap E)}{\text{vol}(U(P, \varepsilon))} \leq 1 - 2^{-n} \text{ for all } 0 < \varepsilon < \varepsilon_0, \]

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Thus, the density of $E$ at $P$ is not equal to one. For the last statement, note that it does not matter whether the density is defined via the test sets $U(P, \varepsilon)$ or classical Euclidean balls. Since the density is not equal to one for all $P \in E$, the Lebesgue density theorem shows that $E$ and consequently $\{h_m = c\}$ is a null set. In the same way one can show that the level sets of the function $P \mapsto h_{P, \Omega}$ are null sets. The set $E_3$ is a null set, since it is a finite union of level sets $\{h_m = c\}$. 

**Lemma 12.** The function $m_P(x)$ is continuous at almost every point $(x, P) \in \Omega^{n+1}$.

*Proof.* Let $(x, P) \in E_1^C \cap E_2^C \cap E_3^C$. Then $h_m(x, P) \neq c_12^{-m}r$ for every $m \in \mathbb{N}$ and $h_m$ is continuous at $(x, P)$. We want to deduce from this that $m_Q(y) = m_P(x)$ for all $(y, Q)$ in a neighbourhood around $(x, P)$. This implies that $m_P(x)$ is locally constant and thus continuous at $(x, P)$, which yields the conclusion.

The condition $m_P(x) = 0$ is equivalent to $h_m(x, P) > c_12^{-m}r$ for all $m \in \{1, \ldots, m_0\}$. The continuity of the finitely many $h_m$, $m \in \{1, \ldots, m_0\}$, implies that also $m_Q(y) = 0$ for $(y, Q)$ in a neighbourhood of $(x, P)$.

The condition $m_P(x) = m \in \{1, \ldots, m_0\}$ is equivalent to $h_m(x, P) < c_12^{-m}r$ and $h_m(x, P) > c_12^{-m'}r$ for all $m' \in \{m + 1, \ldots, m_0\}$. For $m_P(x) = m_0$ the latter set is empty. Again, the continuity of the finitely many $h_m$, $m \in \{1, \ldots, m_0\}$, implies that also $m_Q(y) = m$ for $(y, Q)$ in a neighbourhood of $(x, P)$. 

We can now give the proof of Proposition 2.

*Proof of Proposition 2.* By Lemma 12, the function $m_P(x)$ and therefore also $r_P(x)$ is continuous almost everywhere. This proves (i). Since the radius $r_P(x)$ and the direction $\xi$ are continuous almost everywhere, also the function $(x, P) \mapsto K^*_P(x)$ is continuous in the Hausdorff distance. This proves (v).

We turn to (ii). For almost all $(y, x, P) \in \Omega^{n+2}$ we have that $y$ does not lie on boundary of $K_P(x)$ and that $K_P(x)$ is continuous at $(y, x, P)$ in the Hausdorff distance. Thus, the value of the indicator function $1(y \in K_P(x))$ is constant in a neighborhood of $(y, x, P)$. This means that the function $\Omega^{n+2} \to \mathbb{R}$, $(y, x, P) \mapsto 1(y \in K_P(x))$ is measurable. Moreover, we already know that $(y, x, P) \mapsto r_P(x)$ is measurable. Thus, also $r_P^*(y)$ is measurable as essential supremum of the product of these two functions over $x \in \Omega$.

Next we prove (iii). Let $f \in W^*_p(\Omega)$. We show that $A_Pf(x)$ is continuous almost everywhere as a function of $(x, P) \in \Omega^{n+1}$. If $(x, P)$ is such that $h_{P, \Omega} > c_0r$, the output $A_Pf(x)$ is continuous at $(x, P)$ since it equals zero in a whole neighborhood of $(x, P)$. The set of all $(x, P)$ with $h_{P, \Omega} = c_0r$ is a zero set and may be ignored. So let now $(x, P)$ be such that $h_{P, \Omega} < c_0r$. Then we are in Scenario 2 of our algorithm in a whole neighborhood of $(x, P)$. By 16, Lemma 7 and 36, Theorem 4.7, the output $A_Pf(x)$ can
be computed as follows: First we compute the solutions $a_j^*(x)$ to formulas (4.6) and (4.7) in [33, Corollary 4.4], which depend continuously on $x$, the involved point set and the parameter $\delta$. In our case, the involved point set is given as a certain subset $Q(x, P)$ of the point set $P \cap K_P(x)$, selected according to the proof of [16, Lemma 7], and the parameter $\delta = \delta(x, P)$ is a constant multiple of the covering radius of $Q(x, P)$ in $K_P(x)$. Note that the recursive selection procedure from [16, Lemma 7] is not completely specified, but it may easily be specified e.g. by choosing the point with the smallest index whenever there are multiple choices. Both $Q(x, P)$ and $\delta(x, P)$ depend continuously on $(x, P)$ for almost all $(x, P)$. Secondly, we put $A_P f(x) = \sum_j a_j^*(x)f(x_j)$, which depends continuously on $(x, P)$ whenever the $a_j^*$ do so.

We prove (iv). We can replace the supremum over the unit ball by a countable supremum and then the statement follows from (iii). We provide the details. There is a countable subset $S$ of the unit ball of $W_p^s(\Omega)$ which is dense with respect to the supremum norm: In all cases except $p = 1$ and $s = d$, the unit ball is relatively compact in $C_b(\Omega)$. Thus, for any $k \in \mathbb{N}$ it may be covered by finitely many balls with radius $1/k$ and center inside the unit ball, and we obtain $S$ as the union of the centers over all $k \in \mathbb{N}$. In the case $p = 1$ and $s = d$, the space $W_p^s(\Omega)$ is separable and we get a countable subset $S$ of the unit ball which is dense with respect to the Sobolev norm, and therefore also with respect to the supremum norm. Now, for every $f$ in the unit ball and every $\varepsilon > 0$, we find some $g \in S$ with $\|f - g\|_{C_b(\Omega)} < \varepsilon$. The linearity and boundedness of $A_P : C_b(\Omega) \to B(\Omega)$ give
\[
\|(f - A_P f) - (g - A_P g)\|_{B(\Omega)} \leq \|f - g\|_{C_b(\Omega)} + \|A_P(f - g)\|_{B(\Omega)} < (1 + c_2)\varepsilon
\]
and thus
\[
\|f - A_P f\|_{L_q(\Omega)} - \|g - A_P g\|_{L_q(\Omega)} \leq \text{vol}(\Omega)^{1/q}(1 + c_2)\varepsilon
\]
and hence the supremum over the unit ball may be replaced by the supremum over $S$. \qed

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