PROOF OF GROTHENDIECK-SERRE CONJECTURE ON
PRINCIPAL BUNDLES OVER REGULAR LOCAL RINGS
CONTAINING A FINITE FIELD

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ABSTRACT. Let $R$ be a regular local ring, containing a finite field. Let $G$ be a reductive group scheme over $R$. We prove that a principal $G$-bundle over $R$ is trivial, if it is trivial over the fraction field of $R$. In other words, if $K$ is the fraction field of $R$, then the map of non-abelian cohomology pointed sets

$$H^1_{\text{et}}(R, G) \rightarrow H^1_{\text{et}}(K, G),$$

induced by the inclusion of $R$ into $K$, has a trivial kernel.

Certain arguments used in the present preprint do not work if the ring $R$ contains a characteristic zero field. In that case and, more generally, in the case when the regular local ring $R$ contains an infinite field this result is proved in [FP].

1. INTRODUCTION

Assume that $U$ is a regular scheme, $G$ is a reductive $U$-group scheme. Recall that a $U$-scheme $G$ with an action of $G$ is called a principal $G$-bundle over $U$, if $G$ is faithfully flat and quasi-compact over $U$ and the action is simple transitive, that is, the natural morphism $G \times_U G \rightarrow G \times_U G$ is an isomorphism, see [Gro3, Section 6]. It is well known that such a bundle is trivial locally in étale topology but in general not in Zariski topology. Grothendieck and Serre conjectured that $G$ is trivial locally in Zariski topology, if it is trivial generically. More precisely

Conjecture. Let $R$ be a regular local ring, let $K$ be its field of fractions. Let $G$ be a reductive group scheme over $U := \text{Spec } R$, let $G$ be a principal $G$-bundle. If $G$ is trivial over $\text{Spec } K$, then it is trivial. Equivalently, the map of non-abelian cohomology pointed sets

$$H^1_{\text{et}}(R, G) \rightarrow H^1_{\text{et}}(K, G),$$

induced by the inclusion of $R$ into $K$, has a trivial kernel.

The main result of this paper is a proof of this conjecture for regular semi-local domains $R$, containing a finite field. Our proof was inspired by the preprint [FP], where the conjecture is proven for semi-local regular domains containing an infinite field. Thus, the conjecture holds for semi-local regular domains containing a field.

The proof in the present preprint uses [Pan1, Thm.1.1], [Pan2, Thm.1.0.1], the key ideas of the paper [FP] and a Bertini type theorem from [Poo].

Our result implies that two principal $G$-bundles over $U$ are isomorphic, if they are isomorphic over $\text{Spec } K$ as proved in the next section. This result is new even...
for constant group schemes (that is, for group schemes coming from the ground field).
Recall that a part of the Gersten conjecture asserts that the natural homomorphism of $K$-groups $K_n(R) \to K_n(K)$ is injective. Very roughly speaking, the Grothendieck–Serre conjecture is a non-abelian version of this part of the Gersten conjecture.

1.1. **History of the topic.** Here is a list of known results in the same vein, corroborating the Grothendieck–Serre conjecture.

- The case, where the group scheme $G$ comes from an infinite ground field, is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, and M. S. Raghunathan in [CTO] and [Rag1][Rag2]; O. Gabber announced a proof for group schemes coming from arbitrary ground fields.
- The case of an arbitrary reductive group scheme over a discrete valuation ring or over a henselian ring is completely solved by Y. Nisnevich in [Nis1]. He also proved the conjecture for two-dimensional local rings in the case, when $G$ is quasi-split in [Nis2].
- The case, where $G$ is an arbitrary reductive group scheme over a regular semi-local domain containing an infinite field, was settled by R. Fedorov and I. Panin in [FP].
- The case, where $G$ is an arbitrary torus over a regular local ring, was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in [CTS].
- For some simple group schemes of classical series the conjecture is solved in works of the author, A. Suslin, M. Ojanguren, and K. Zainoulline; see [Oja1], [Oja2], [PS1], [OP], [Zai], [OPZ].
- Under an isotropy condition on $G$ and assuming that the ring contains an infinite field the conjecture is proved in a series of preprints [PSV] and [Pa2].
- The case of strongly inner simple adjoint group schemes of the types $E_6$ and $E_7$ is done by the second author, V. Petrov, and A. Stavrova in [PPS]. No isotropy condition is imposed there, however it is supposed that the ring contains an infinite field.
- The case, when $G$ is of the type $F_4$ with trivial $g_2$-invariant and the field is of characteristic zero, is settled by V. Chernousov in [Che]; the case, when $G$ is of the type $F_4$ with trivial $f_3$-invariant and the field is infinite and perfect, is settled by V. Petrov and A. Stavrova in [PS2].

1.2. **Acknowledgments.** The author thanks A. Suslin for his interest to the topic of the present preprint.

2. **Main results**

Let $R$ be a commutative unital ring. Recall that an $R$-group scheme $G$ is called reductive, if it is affine and smooth as an $R$-scheme and if, moreover, for each algebraically closed field $\Omega$ and for each ring homomorphism $R \to \Omega$ the scalar extension $G_\Omega$ is a connected reductive algebraic group over $\Omega$. This definition of a reductive $R$-group scheme coincides with [DG] Exp. XIX, Definition 2.7. A well-known conjecture due to J.-P. Serre and A. Grothendieck (see [Ser] Remarque, p.31), [Gro1] Remarque 3, p.26-27, and [Gro2] Remarque 1.11.a) asserts that
given a regular local ring \( R \) and its field of fractions \( K \) and given a reductive group scheme \( G \) over \( R \), the map
\[
H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),
\]
induced by the inclusion of \( R \) into \( K \), has a trivial kernel. The following theorem, which is the main result of the present paper, asserts that this conjecture holds, provided that \( R \) contains a finite field. If \( R \) contains an infinite field, then the conjecture is proved in [FP].

**Theorem 1.** Let \( R \) be a regular semi-local domain containing a finite field, and let \( K \) be its field of fractions. Let \( G \) be a reductive group scheme over \( R \). Then the map
\[
H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),
\]
induced by the inclusion of \( R \) into \( K \), has a trivial kernel. In other words, under the above assumptions on \( R \) and \( G \), each principal \( G \)-bundle over \( R \) having a \( K \)-rational point is trivial.

**Theorem 1** has the following
**Corollary.** Under the hypothesis of Theorem 1, the map
\[
H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),
\]
induced by the inclusion of \( R \) into \( K \), is injective. Equivalently, if \( G_1 \) and \( G_2 \) are two principal bundles isomorphic over \( \text{Spec} K \), then they are isomorphic.

**Proof.** Let \( G_1 \) and \( G_2 \) be two principal \( G \)-bundles isomorphic over \( \text{Spec} \ K \). Let \( \text{Iso}(G_1, G_2) \) be the scheme of isomorphisms. This scheme is a principal \( \text{Aut} G_2 \)-bundle. By Theorem 1 it is trivial, and we see that \( G_1 \cong G_2 \). □

Note that, while Theorem 1 was previously known for reductive group schemes \( G \) coming from the ground field (an unpublished result due to O.Gabber), in many cases the corollary is a new result even for such group schemes.

For a scheme \( U \) we denote by \( \mathbb{A}^1_U \) the affine line over \( U \) and by \( \mathbb{P}^1_U \) the projective line over \( U \). Let \( T \) be a \( U \)-scheme. By a principal \( G \)-bundle over \( T \) we understand a principal \( G \times_U T \)-bundle.

In Section 3 we deduce Theorem 1 from the following result of independent interest (cf. [PSV, Thm.1.3]).

**Theorem 2.** Let \( R \) be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over a finite field \( k \), set \( U = \text{Spec} R \). Let \( G \) be a simple simply-connected group scheme over \( U \) (see [DG] Exp. XXIV, Sect. 5.3 for the definition). Let \( \mathcal{E}_t \) be a principal \( G \)-bundle over the affine line \( \mathbb{A}^1_U = \text{Spec} R[t] \), and let \( h(t) \in R[t] \) be a monic polynomial. Denote by \( (\mathbb{A}^1_U)_h \) the open subscheme in \( \mathbb{A}^1_U \) given by \( h(t) \neq 0 \) and assume that the restriction of \( \mathcal{E}_t \) to \( (\mathbb{A}^1_U)_h \) is a trivial principal \( G \)-bundle. Then for each section \( s : U \to \mathbb{A}^1_U \) of the projection \( \mathbb{A}^1_U \to U \) the \( G \)-bundle \( s^* \mathcal{E}_t \) over \( U \) is trivial.

The derivation of Theorem 1 from Theorem 2 is based on [Pan2, Thm.1.0.1] and [Pan1, Thm.1.1].

Let \( Y \) be a semi-local scheme. We will call a simple \( Y \)-group scheme quasi-split if its restriction to each connected component of \( Y \) contains a Borel subgroup scheme.
Theorem 3. Let $R$, $U$, and $G$ be as in Theorem 2. Let $Z \subset \mathbb{P}^1_U$ be a closed subscheme finite over $U$. Let $Y \subset \mathbb{P}^1_U$ be a closed subscheme finite and étale over $U$ and such that (i) $G_Y := G \times_U Y$ is quasi-split, (ii) $Y \cap Z = \emptyset$ and $Y \cap \{\infty\} \times U = Z \cap \{\infty\} \times U$, (iii) for any closed point $u \in U$, one has $\text{Pic}(\mathbb{P}^1_u - Y_u) = 0$, where $Y_u := \mathbb{P}^1_u \cap Y$. Let $\mathcal{G}$ be a principal $G$-bundle over $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial. Then the restriction of $\mathcal{G}$ to $\mathbb{P}^1_U - Y$ is also trivial. In particular, the principal $G$-bundle $\mathcal{G}$ is trivial locally for the Zariski topology.

The proof of this result is inspired by [FP, Thm.3].

2.1. Organization of the paper. In Section 3, we reduce Theorem 1 to Theorem 2. In Section 4, we reduce Theorem 2 to Theorem 3. This reduction is based on [Pan2, Thm.1.0.1], [Pan1, Thm.1.1], on a theorem of D. Popescu [Pop] and on Proposition 4.1. The latter proposition is a new ingredient comparing with respecting arguments from [FP, Section 4].

In Section 5 we prove Theorem 3. We give an outline of the proof in Section 5.1. We use the technique of henselization.

In Section 6 we give an application of Theorem 1.

In the Appendix we recall the definition of henselization from [Gab, Section 0].

3. Reducing Theorem 1 to Theorem 2

In what follows “$G$-bundle” always means “principal $G$-bundle”. Now we assume that Theorem 2 holds. We start with the following particular case of Theorem 1.

Proposition 3.1. Let $R$, $U = \text{Spec} R$, and $G$ be as in Theorem 2. Let $\mathcal{E}$ be a principal $G$-bundle over $U$, trivial at the generic point of $U$. Then $\mathcal{E}$ is trivial.

Proof. Under the hypothesis of the proposition, the following data are constructed in [Pan1 Thm.1.1]:
(a) a principal $G$-bundle $\mathcal{E}_t$ over $\mathbb{A}^1_U$;
(b) a monic polynomial $h(t) \in R[t]$.
Moreover these data satisfies the following conditions:
(1) the restriction of $\mathcal{E}_t$ to $(\mathbb{A}^1_U)_h$ is a trivial principal $G$-bundle;
(2) there is a section $s : U \to \mathbb{A}^1_U$ such that $s^* \mathcal{E}_t = \mathcal{E}$.

Now it follows from Theorem 2 that $\mathcal{E}$ is trivial. □

Proposition 3.2. Let $U$ be as in Theorem 2. Let $G$ be a reductive group scheme over $U$. Let $\mathcal{E}$ be a principal $G$-bundle over $U$ trivial at the generic point of $U$. Then $\mathcal{E}$ is trivial.

Proof. Firstly, using [Pan2 Thm.1.0.1], we can assume that $G$ is semi-simple and simply-connected. Secondly, standard arguments (see for instance [PSV, Section 9]) show that we can assume that $G$ is simple and simply-connected. (Note that for this reduction it is necessary to work with semi-local rings.) Now the proposition is reduced to Proposition 3.1. □

Proof of Theorem 4. Let us prove a general statement first. Let $k'$ be a finite field, $X$ be a $k'$-smooth irreducible affine variety, $H$ be a reductive group scheme over $X$. Denote by $k'[X]$ the ring of regular functions on $X$ and by $k'(X)$ the field of rational functions on $X$. Let $\mathcal{H}$ be a principal $H$-bundle over $X$ trivial over $k'(X)$.
Let \( p_1, \ldots, p_n \) be prime ideals in \( k'[X] \), and let \( \mathcal{O}_{p_1, \ldots, p_n} \) be the corresponding semi-local ring.

**Lemma 3.3.** The principal \( H \)-bundle \( \mathcal{H} \) is trivial over \( \mathcal{O}_{p_1, \ldots, p_n} \).

**Proof.** For each \( i = 1, 2, \ldots, n \) choose a maximal ideal \( m_i \subset k'[X] \) containing \( p_i \). One has inclusions of \( k' \)-algebras

\[
\mathcal{O}_{m_1, \ldots, m_n} \subset \mathcal{O}_{p_1, \ldots, p_n} \subset k'(X).
\]

By Proposition 3.2 the principal \( H \)-bundle \( \mathcal{H} \) is trivial over \( \mathcal{O}_{m_1, \ldots, m_n} \). Thus it is trivial over \( \mathcal{O}_{p_1, \ldots, p_n} \). \( \square \)

Let us return to our situation. Let \( m_1, \ldots, m_n \) be all the maximal ideals of \( R \). Let \( \mathcal{E} \) be a \( G \)-bundle over \( R \) trivial over the fraction field of \( R \). Clearly, there is a non-zero \( f \in R \) such that \( \mathcal{E} \) is trivial over \( Rf \). Let \( k \) be the prime field of \( R \). Note that \( k \) is perfect. It follows from Popescu’s theorem ([Pop, Swa]) that \( R \) is a filtered inductive limit of smooth \( k \)-algebras \( R_\alpha \). Modifying the inductive system \( R_\alpha \) if necessary, we can assume that each \( R_\alpha \) is integral. There are an index \( \alpha \), a reductive group scheme \( G_\alpha \) over \( R_\alpha \), a principal \( G_\alpha \)-bundle \( \mathcal{E}_\alpha \) over \( R_\alpha \), and an element \( f_\alpha \in R_\alpha \) such that \( G = G_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R \), \( \mathcal{E} \) is isomorphic to \( \mathcal{E}_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R \) as principal \( G \)-bundle, \( f \) is the image of \( f_\alpha \) under the homomorphism \( \varphi_\alpha : R_\alpha \to R \), \( \mathcal{E}_\alpha \) is trivial over \( (R_\alpha)_{f_\alpha} \).

For each maximal ideal \( m_i \) in \( R \) \((i = 1, \ldots, n)\) set \( p_i = \varphi_\alpha^{-1}(m_i) \). The homomorphism \( \varphi_\alpha \) induces a homomorphism of semi-local rings \( (R_\alpha)_{p_1, \ldots, p_n} \to R \). By Lemma 3.3 the principal \( G_\alpha \)-bundle \( \mathcal{E}_\alpha \) is trivial over \( (R_\alpha)_{p_1, \ldots, p_n} \). Whence the \( G \)-bundle \( \mathcal{E} \) is trivial over \( R \). \( \square \)

### 4. Reducing Theorem 2 to Theorem 3

Now we assume that Theorem 3 is true. Let \( k, U \) and \( G \) be as in Theorem 2. Let \( u_1, \ldots, u_n \) be all the closed points of \( U \). Let \( k(u_i) \) be the residue field of \( u_i \). Consider the reduced closed subscheme \( u \) of \( U \), whose points are \( u_1, \ldots, u_n \). Thus \[
u \cong \prod_i \text{Spec } k(u_i).
\]

Set \( G_u = G \times_U u \). By \( G_{u_i} \), we denote the fiber of \( G \) over \( u_i \); it is a simple simply-connected algebraic group over \( k(u_i) \).

**Proposition 4.1.** Let \( Z \subset \mathbb{A}^1_U \) be a closed subscheme finite over \( U \). There is a closed subscheme \( Y \subset \mathbb{A}^1_U \) which is étale and finite over \( U \) and such that

(i) \( G_Y := G \times_U Y \) is quasi-split,

(ii) \( Y \cap Z = \varnothing \),

(iii) for any closed point \( u \in U \) one has \( \text{Pic}(\mathbb{P}^1_u - Y_u) = 0 \), where \( Y_u := \mathbb{P}^1_u \cap Y \).

(Note that \( Y \) and \( Z \) are closed in \( \mathbb{P}^1_U \) since they are finite over \( U \).)

**Proof.** For every \( u_i \in u \) choose a Borel subgroup \( B_{u_i} \) in \( G_{u_i} \). The latter is possible since the fields \( k(u_i) \) are finite. Let \( B \) be the \( U \)-scheme of Borel subgroup schemes of \( G \). It is a smooth projective \( U \)-scheme (see [DG, Cor. 3.5, Exp. XXVI]). The subgroup \( B_{u_i} \) in \( G_{u_i} \) is a \( k(u_i) \)-rational point \( b_i \) in the fibre of \( B \) over the point \( u_i \). Using a variant of Bertini theorem (see [Poo, Thm.1.2]), we can find a closed subscheme \( Y' \) of \( B \) such that \( Y' \) is étale over \( U \) and all the \( b_i \)'s are in \( Y \) (take an embedding of \( B \) into a projective space \( \mathbb{P}^N_U \) and intersect \( B \) with appropriately chosen family of hypersurfaces containing the points \( b_i \). Arguing as in the proof of
Lemma 4.2. Let $U$ be as in the Proposition. Let $Z \subset A^1_U$ be a closed subscheme finite over $U$. Let $Y' \to U$ be a finite étale morphism such that for any closed point $u_i$ in $U$ the fibre $Y'_u$ of $Y'$ over $u_i$ contains a $k(u_i)$-rational point (it is the point $b_i$).

To prove this Lemma note that it’s easy to find field extensions $k_1$ and $k_2$ subjecting (i) to (ii). To satisfy (iv) it suffices to require that for any closed point $u_i$ in $U$ and for $r = 1$ and $r = 2$ the number of closed points in $Y'_u \otimes_k k_r$ is the same as the number of closed points in $Y'_u$, and to require that for any integer $n > 0$ and any closed point $u_i$ in $U$ the number of points $y \in Y''_u$ with $[k(y) : k(u_i)] = n$ is not more than the number of points $x \in A^1_{u_i}$ with $[k(x) : k(u_i)] = n$. Clearly, these requirements can be satisfied, which proves the item (iv).

The condition (v) holds for any closed $U$-embedding $i : Y'' \hookrightarrow A^1_U$ from item (iv), since the property (iii). The condition (vi) holds since the property (i).

Now complete the proof of Proposition 4.1. Take the $U$-scheme $Y' \subset B$ as in the beginning of the proof. This $U$-scheme $Y'$ satisfies the assumption of Lemma 4.2. Take the closed subscheme $Y$ of $A^1_U$ as in the item (v) of the Lemma. For this $Y$ the conditions (ii) and (iii) of the Proposition are obviously satisfied. The condition (i) is satisfied too, since already it is satisfied for the $U$-scheme $Y'$. The Proposition follows.

Proof of Theorem 3. Set $Z := \{h = 0\} \cup s(U) \subset A^1_U$. Clearly, $Z$ is finite over $U$. Since the principal $G$-bundle $E_t$ is trivial over $(A^1_U)_h$, it is trivial over $A^1_U - Z$. Note that $\{h = 0\}$ is closed in $P^1_U$ and finite over $U$ because $h$ is monic. Further, $s(U)$ is also closed in $P^1_U$ and finite over $U$ because it is a zero set of a degree one monic polynomial. Thus $Z \subset P^1_U$ is closed and finite over $U$.

Since the principal $G$-bundle $E_t$ is trivial over $(A^1_U)_h$, and $G$-bundles can be glued in Zariski topology, there exists a principal $G$-bundle $G$ over $P^1_U$ such that

(i) its restriction to $A^1_U$ coincides with $E_t$;

(ii) its restriction to $P^1_U - Z$ is trivial.

Now choose $Y$ in $A^1_U$ as in Proposition 4.1. Clearly, $Y$ is finite étale over $U$ and closed in $P^1_U$. Moreover, $Y \cap \{\infty\} \times U = \varnothing = Z \cap \{\infty\} \times U$ and $Y \cap Z = \varnothing$. Applying Theorem 3 with this choice of $Y$ and $Z$, we see that the restriction of $G$ to $P^1_U - Y$ is a trivial $G$-bundle. Since $s(U)$ is in $A^1_U - Y$ and $G|_{A^1_U}$ coincides with $E_t$, we conclude that $s^*E_t$ is a trivial principal $G$-bundle over $U$. □
5. Proof of Theorem \[5\]

We will be using notation from Theorem \[3\]. Let \( u \) be as in Section \[4\]. For \( u \in u \) set \( G_u = G|_u \).

**Proposition 5.1.** Let \( E \) be a \( G \)-bundle over \( \mathbb{P}^1_U \) such that \( E|_{\mathbb{P}^1_U} \) is a trivial \( G_u \)-bundle for all \( u \in u \). Assume that there exists a closed subscheme \( T \) of \( \mathbb{P}^1_U \) finite over \( U \) such that the restriction of \( E \) to \( \mathbb{P}^1_U - T \) is trivial and \((\infty \times U) \cap T = \emptyset \). Then \( E \) is trivial.

**Proof.** This follows from Theorem 9.6 of [PSV], since \( E|_{(\infty \times U)} \) is a trivial \( G \)-bundle. \( \square \)

5.1. An outline of the proof of Theorem \[3\]. Our proof of this Theorem almost literally coincides with the proof of [FP] Thm.3. Our arguments are simpler at certain points.

An outline of the proof.

Denote by \( Y^h \) the henselization of the pair \((\mathbb{A}_U^1, Y)\), it is a scheme over \( \mathbb{A}_U^1 \). Let \( s : Y \to Y^h \) be the canonical closed embedding, see Section 5.2 for more details. Set \( \hat{Y}^h := Y^h - s(Y) \). Let \( \gamma' \) be a \( G \)-bundle over \( \mathbb{P}^1_U - Y \). Denote by \( \text{Gl}(\gamma', \varphi) \) the \( \gamma' \)-bundle over \( \mathbb{P}^1_U \) obtained by gluing \( \gamma' \) with the trivial \( G \)-bundle \( G \times_U Y^h \) via a \( G \)-bundle isomorphism \( \varphi : G \times_U Y^h \to \gamma'|_{Y^h} \).

Note that the \( G \)-bundle \( \gamma' \) can be presented in the form \( \text{Gl}(\gamma', \varphi) \), where \( \gamma' = \text{Gl}(\gamma'|_{Y^h}) \). The idea is to show that

There is \( \alpha \in \text{G}(\hat{Y}^h) \) such that the \( G_u \)-bundle \( \text{Gl}(\gamma', \varphi \circ \alpha)|_{\mathbb{P}^1_U} \) is trivial (here \( \alpha \) is regarded as an automorphism of the \( G \)-bundle \( G \times_U Y^h \) given by the right translation by the element \( \alpha \)).

If we find \( \alpha \) satisfying condition \((*)\), then Proposition 5.1 applied to \( T = Y \cup Z \), shows that the \( G \)-bundle \( \text{Gl}(\gamma', \varphi \circ \alpha) \) is trivial over \( \mathbb{P}^1_U \). On the other hand, its restriction to \( \mathbb{P}^1_U - Y \) coincides with the \( G \)-bundle \( \gamma' = \text{Gl}(\gamma'|_{\mathbb{P}^1_U - Y}) \). Thus \( \gamma'|_{\mathbb{P}^1_U - Y} \) is a trivial \( G \)-bundle.

To prove \((*)\) it suffices to show that

(i) the bundle \( \text{Gl}(\gamma'|_{\mathbb{P}^1_U - Y}) \) is trivial;

(ii) each element \( \gamma_u \in G_u(\hat{Y}^h_u) \) can be written in the form

\[
\alpha|_{\hat{Y}^h_u} \cdot \beta_u|_{\hat{Y}^h_u}
\]

for certain elements \( \alpha \in \text{G}(\hat{Y}^h) \) and \( \beta_u \in G_u(Y^h_u) \).

A realization of this plan in details is given below in the paper.

5.2. Henselization of affine pairs. We will use the theory of henselian pairs and, in particular, a notion of a henselization \( A^h_I \) of a commutative ring \( A \) at an ideal \( I \) (see Appendix and [Gab], Section 0). We refer to [FP] subsection 5.2 for the geometric counterpart. Let \( S = \text{Spec} A \) be a scheme and \( T = \text{Spec}(A/I) \) be a closed subscheme. Let \( (T^h, \pi : T^h \to S, s : T \to T^h) \) be the henselization of the pair \((S, T)\) (cf. Definition [A,3]). By definition the scheme \( T^h \) is affine and the composite morphism \( \pi \circ s : T \to S \) is the closed embedding \( T \hookrightarrow S \). Recall that the pair \((T^h, s(T))\) is henselian, which means that for any affine étale morphism \( \pi : Z \to T^h \), any section \( \sigma \) of \( \pi \) over \( s(T) \) uniquely extends to a section of \( \pi \) over \( T^h \). It is known that \( \pi^{-1}(T) = s(T) \).
In the notation of \[\text{Gab}\] Section 0 we have \(T^h = \text{Spec} A^h\), \(\pi : T^h \to S\) is induced by the structure of \(A\)-algebra on \(A^h\).

Recall three properties of henselization of affine pairs

(i) Let \(T\) be a semi-local scheme. Then the henselization commutes with restriction to closed subschemes. In more details, if \(S' \subset S\) is a closed subscheme, then there is a natural morphism \((T \times_S S')^h \to T^h \times_S S'\). This morphism is an isomorphism and the canonical section \(s' : T \times_S S' \to (T \times_S S')^h\) coincides under this identification with

\[s \times_S \text{Id}_{S'} : T \times_S S' \to T^h \times_S S'.\]

(ii) If \(T = \coprod_i T_i\) is a disjoint union, then \(T^h = \coprod_i T^h_i\).

(iii) If we replace in a pair \((S, T)\) the scheme \(S\) by an étale affine neighborhood of \(T\), then the \((T^h, \pi, s)\) remains the same. In more details, given a pair \((S, T)\) as above we write temporarily \((S^\text{hens}, \pi_{S,T}, s_{S,T})\) for \((T^h, \pi, s)\). If \(p : W \to S\) is an étale morphism and \(t : T \to W\) is such that \(p \circ t : T \to S\) coincides with the closed embedding \(T\) into \(S\), then there is a canonical isomorphism \(\rho : W^\text{hens}_{T} \to S^\text{hens}_T\) of the \(S\)-schemes \((W^\text{hens}_T, \pi_{W,T})\) and \((S^\text{hens}_T, \pi_{S,T})\) such that \(\rho \circ s_{W,T} = s_{S,T}\).

5.3. Gluing principal \(G\)-bundles. Recall that \(U = \text{Spec} R\), where \(R\) is the semi-local ring of finitely many closed points on an irreducible \(k\)-smooth affine variety over a finite field \(k\). Also, \(G\) is a simple simply-connected group scheme over \(U\), and \(Y\) is a closed subscheme of \(\mathbb{P}^1_U\) finite and étale over \(U\).

We will assume below in the preprint that \(Y \subset \mathbb{A}^1_U\) (as in the hypotheses of Theorem 3). Let \((Y^h, \pi, s)\) be the henselization of the pair \((\mathbb{A}^1_U, Y)\) and let \(\hat{Y}^h = Y^h - s(Y)\) and let \(\text{in} : \mathbb{A}^1_U \to \mathbb{P}^1_U\) be the open inclusion.

**Proposition 5.2.** \(\text{FP}\) The schemes \(Y^h\) and \(\hat{Y}^h\) are affine.

Let us make a general remark. Let \(\mathcal{F}\) be a \(G\)-bundle over a \(U\)-scheme \(T\). By definition, a trivialization of \(\mathcal{F}\) is a \(G\)-equivariant isomorphism \(G \times_U T \to \mathcal{F}\). Equivalently, it is a section of the projection \(\mathcal{F} \to T\). If \(\varphi\) is such a trivialization and \(f : T' \to T\) is a \(U\)-morphism, we get a trivialization \(f^*\varphi\) of \(f^*\mathcal{F}\). Sometimes we denote this trivialization by \(\varphi|_{T'}\). We also sometimes call a trivialization of \(f^*\mathcal{F}\) a trivialization of \(\mathcal{F}\) on \(T'\).

The main cartesian square we will work with is

\[
\begin{array}{ccc}
\hat{Y}^h & \longrightarrow & Y^h \\
\downarrow & & \downarrow \text{in}\circ \pi \\
\mathbb{P}^1_U - Y & \longrightarrow & \mathbb{P}^1_U.
\end{array}
\]

Let \(\mathcal{A}\) be the category of pairs \((\mathcal{E}, \psi)\), where \(\mathcal{E}\) is a \(G\)-bundle on \(\mathbb{P}^1_U\), \(\psi\) is a trivialization of \(\mathcal{E}\big|_{\hat{Y}^h} := (\text{in} \circ \pi)^*\mathcal{E}\). A morphism between \((\mathcal{E}, \psi)\) and \((\mathcal{E}', \psi')\) is an isomorphism \(\mathcal{E} \to \mathcal{E}'\) compatible with trivializations.

Similarly, let \(\mathcal{B}\) be the category of pairs \((\mathcal{E}, \psi)\), where \(\mathcal{E}\) is a \(G\)-bundle on \(\mathbb{P}^1_U - Y\), \(\psi\) is a trivialization of \(\mathcal{E}\big|_{\hat{Y}^h}\).

Consider the restriction functor \(\Psi : \mathcal{A} \to \mathcal{B}\).

**Proposition 5.3.** \(\text{FP}\) The functor \(\Psi\) is an equivalence of categories.
Construction 5.4. \[\text{FP}\] By Proposition 5.3 we can choose a functor quasi-inverse to \(\Psi\). Fix such a functor \(\Theta\). Let \(\Lambda\) be the forgetful functor from \(A\). Let \(\Lambda(\Theta, \psi)\) be the \(\Theta\)-principal \(G\)-bundle over \(\mathbb{P}_1\). For \((\mathcal{E}, \psi) \in \mathcal{B}\) set
\[
\text{GL}(\mathcal{E}, \psi) = \Lambda(\Theta(\mathcal{E}, \psi)).
\]
Note that \(\text{GL}(\mathcal{E}, \psi)\) comes with a canonical trivialization over \(Y^h\).

Conversely, if \(\mathcal{E}\) is a principal \(G\)-bundle over \(\mathbb{P}_1\) such that its restriction to \(Y^h\) is trivial, then \(\mathcal{E}\) can be represented as \(\text{GL}(\mathcal{E}, \psi)\), where \(\mathcal{E}' = \mathcal{E}|_{\mathbb{P}_1 - Y}\), \(\psi\) is a trivialization of \(\mathcal{E}'\) on \(Y^h\).

Let \(\mathfrak{u}\) be as in Section 3. \(Y_u := Y \times_U \mathfrak{u}\). Let \((Y_u^h, \pi_u, s_u)\) be the henselization of \((A_u, Y_u)\). Using property (i) of henselization, we get \(Y_u^h = Y^h \times_U \mathfrak{u}\). Thus we have a natural closed embedding \(Y_u^h \hookrightarrow Y^h\). Set \(\dot{Y}_u^h = Y_u^h - s_u(Y_u)\). We get a closed embedding
\[
Y_u^h \hookrightarrow \dot{Y}_u^h.
\]
Thus the pull-back of the cartesian square (1) by means of the closed embedding \(u \hookrightarrow U\) has the form
\[
\begin{array}{ccc}
\dot{Y}_u^h & \longrightarrow & Y_u^h \\
\downarrow & & \downarrow \text{in}_u \circ \pi_u \\
\mathbb{P}_1 - Y_u & \longrightarrow & \mathbb{P}_1._\mathfrak{u}
\end{array}
\]
where \(\text{in}_u : A_u \rightarrow \mathbb{P}_1._\mathfrak{u}\).

Similarly to the above, we can define categories \(A_u\) and \(B_u\) and an equivalence of categories \(\Psi_u : A_u \rightarrow B_u\). Let \(\Theta_u\) be a functor quasi-inverse to \(\Psi_u\) and \(\Lambda_u\) be the forgetful functor from \(A_u\) to the category of \(G_u\)-bundles over \(\mathbb{P}_1._\mathfrak{u}\). Let \(\mathcal{E}_u\) be a principal \(G_u\)-bundle over \(\mathbb{P}_1._\mathfrak{u} - Y_u\) and \(\psi_u\) be a trivialization of \(G_u\) on \(Y_u^h\). Set
\[
\text{GL}_u(\mathcal{E}_u, \psi_u) = \Lambda_u(\Theta_u(\mathcal{E}_u, \psi_u)).
\]

Lemma 5.5. \[\text{FP}\] Let \((\mathcal{E}, \psi) \in \mathcal{B}\), and let \(\text{GL}(\mathcal{E}, \psi)\) be the \(G\)-bundle obtained by Construction 5.4. Then
\[
\text{GL}_u(\mathcal{E}|_{\mathbb{P}_1._\mathfrak{u} - Y_u}, \psi|_{Y_u^h}) \text{ and } \text{GL}(\mathcal{E}, \psi)|_{\mathbb{P}_1._\mathfrak{u}}
\]
are isomorphic as \(G_u\)-bundles over \(\mathbb{P}_1._\mathfrak{u}\).

Lemma 5.6. \[\text{FP}\] For any \((\mathcal{E}_u, \psi_u) \in \mathcal{B}_u\) and any \(\beta_u \in G_u(Y_u^h)\) the \(G_u\)-bundles
\[
\text{GL}_u(\mathcal{E}_u, \psi_u) \text{ and } \text{GL}_u(\mathcal{E}_u, \psi_u \circ \beta_u|_{Y_u^h})
\]
are isomorphic (here \(\beta_u|_{Y_u^h}\) is regarded as an automorphism of the \(G_u\)-bundle \(G_u \times_u \dot{Y}_u^h\) given by the right translation by \(\beta_u|_{Y_u^h}\)).

5.4. Proof of Theorem 3 presentation of \(G\) in the form \(\text{GL}(G', \varphi)\). Let \(U\), \(G\), \(Z\), \(Y\) and \(\mathcal{G}\) be as in Theorem 3.

Proposition 5.7. \[\text{FP}\] The \(G\)-bundle \(\mathcal{G}\) over \(\mathbb{P}_1_U\) is of the form \(\text{GL}(G', \varphi)\) for the \(G\)-bundle \(G' := \mathcal{G}|_{\mathbb{P}_1_U - Y}\) and a trivialization \(\varphi\) of \(G'\) over \(\dot{Y}_u^h\).

Proof. In view of Construction 5.4 it is enough to prove that the restriction of the principal \(G\)-bundle \(\mathcal{G}\) to \(Y^h\) is trivial. Let us choose a closed subscheme \(Z' \subset A^1_U\) such that \(Z'\) contains \(Z\), \(Z' \cap Y = \emptyset\), and \(A^1_U - Z'\) is affine. Then \(A^1_U - Z'\) is an affine neighborhood of \(Y\). By the property (iii) from subsection 5.2 the henselization of
the pair \((A^1_U - Z', Y)\) coincides with the henselization of the pair \((A^1_U, Y)\). Since \(G\) is trivial over \(A^1_U - Z'\), its pull-back to \(Y^h\) is trivial too. The proposition is proved.

Our aim is to modify the trivialization \(\varphi\) via an element
\[
\alpha \in G(Y^h)
\]
so that the \(G\)-bundle \(Gl(G', \varphi \circ \alpha)\) becomes trivial over \(\mathbb{P}^1_U\).

5.5. Proof of Theorem \(\mathcal{K}\): proof of property (i) from the outline. Now we are able to prove property (i) from the outline of the proof. In fact, we will prove the following modification of [FP, Lemma 5.11].

**Lemma 5.8.** Let \(Gl(G', \varphi)\) be the presentation of the \(G\)-bundle \(G\) over \(\mathbb{P}^1_U\) given in Proposition \(\mathcal{K}\). Set \(\varphi_u := \varphi|_{Y^h_u}\). Then there is \(\gamma_u \in G_u(Y^h_u)\) such that the \(G_u\)-bundle \(Gl_u(G'|_{\mathbb{P}^1_u - Y_u}, \varphi_u \circ \gamma_u)\) is trivial.

**Proof.** We show first that \(G'|_{\mathbb{P}^1_u - Y_u}\) is trivial. One has
\[
\mathbb{P}^1_u = \bigsqcup_{u \in \mathbf{u}} \mathbb{P}^1_u
\]
For \(u \in \mathbf{u}\) set \(Y_u := Y \times_U u\), \(G_u := G \times_U u\), and \(G_u := G \times_U u\).

Take \(u \in \mathbf{u}\). By our assumption on \(Y\), \(\text{Pic}(\mathbb{P}^1_u - Y_u) = 0\). The \(G_u\)-bundle \(G_u\) is trivial over \(A^1_U - Z_u\). Thus, by [Gil1 Corollary 3.10(a)], it is trivial over \(\mathbb{P}^1_u - Y_u\).

We see that \(G'|_{\mathbb{P}^1_u - Y_u} = G'|_{\mathbb{P}^1_u - Y_u}\) is trivial. Choosing a trivialization, we may identify \(\varphi_u\) with an element of \(G_u(Y^h_u)\). Set \(\gamma_u = \varphi_u^{-1}\). By the very choice of \(\gamma_u\) the \(G_u\)-bundle \(Gl_u(G'|_{\mathbb{P}^1_u - Y_u}, \varphi_u \circ \gamma_u)\) is trivial.

5.6. Proof of Theorem \(\mathcal{K}\): reduction to property (ii) from the outline. The aim of this section is to deduce Theorem \(\mathcal{K}\) from the following

**Proposition 5.9.** [FP] Each element \(\gamma_u \in G_u(Y^h_u)\) can be written in the form
\[
\alpha|_{Y^h_u} \cdot \beta_u|_{Y^h_u}
\]
for certain elements \(\alpha \in G(Y^h)\) and \(\beta_u \in G_u(Y^h_u)\).

**Deduction of Theorem \(\mathcal{K}\) from Proposition \(\mathcal{K}\)** Let \(Gl(G', \varphi)\) be the presentation of the \(G\)-bundle \(G\) from Proposition \(\mathcal{K}\). Let \(\gamma_u \in G_u(Y^h_u)\) be the element from Lemma 5.8. Let \(\alpha \in G(Y^h)\) and \(\beta_u \in G_u(Y^h_u)\) be the elements from Proposition 5.9. Set
\[
G^{\text{new}} = Gl(G', \varphi \circ \alpha).
\]

**Claim.** The \(G\)-bundle \(G^{\text{new}}\) is trivial over \(\mathbb{P}^1_U\).

Indeed, by Lemmas 5.5 and 5.6 one has a chain of isomorphisms of \(G_u\)-bundles
\[
G^{\text{new}}|_{\mathbb{P}^1_u} \cong \text{Gl}_u(G'|_{\mathbb{P}^1_u - Y_u}, \varphi_u \circ \alpha|_{Y^h_u}) \cong \text{Gl}_u(G'|_{\mathbb{P}^1_u - Y_u}, \varphi_u \circ \alpha|_{Y^h_u} \circ \beta_u|_{Y^h_u}) = \text{Gl}_u(G'|_{\mathbb{P}^1_u - Y_u}, \varphi_u \circ \gamma_u),
\]
which is trivial by the choice of \(\gamma_u\). The \(G\)-bundles \(G|_{\mathbb{P}^1_U - Y}\) and \(G^{\text{new}}|_{\mathbb{P}^1_U - Y}\) coincide by the very construction of \(G^{\text{new}}\). By Proposition 5.1 applied to \(T = Z \cup Y\), the \(G\)-bundle \(G^{\text{new}}\) is trivial. Whence the claim.

The claim above implies that the \(G\)-bundle \(G|_{\mathbb{P}^1_U - Y} = G^{\text{new}}|_{\mathbb{P}^1_U - Y}\) is trivial. Theorem \(\mathcal{K}\) is proved.
5.7. End of proof of Theorem 5.13 proof of property (ii) from the outline. In the remaining part of Section 5 we will prove Proposition 5.12. This will complete the proof of Theorem 5.13.

By our assumption on $Y$, the group scheme $G_Y = G \times_U Y$ is quasi-split. Thus we can and will choose a Borel subgroup scheme $B^+$ in $G_Y$.

Since $Y$ is an affine scheme, by [DG, Exp. XXVI, Cor. 2.3, Th 4.3.2(a)] there is an opposite to $B^+$ Borel subgroup scheme $B^-$ in $G_Y$. Let $U^+$ be the unipotent radical of $B^+$, and let $U^-$ be the unipotent radical of $B^-$.}

**Definition 5.10.** We will write $E$ for the functor, sending a $Y$-scheme $T$ to the subgroup $E(T)$ of the group $G_Y(T) = G(T)$ generated by the subgroups $U^+(T)$ and $U^-(T)$ of the group $G_Y(T) = G(T)$.

**Lemma 5.11.** The functor $E$ has the property that for every closed subscheme $S$ in an affine $Y$-scheme $T$ the induced map $E(T) \to E(S)$ is surjective.

**Proof.** The restriction maps $U^\pm(T) \to U^\pm(S)$ are surjective, since $U^\pm$ are isomorphic to vector bundles as $Y$-schemes (see [DG, Exp. XXVI, Cor. 2.5]).

Recall that $(Y^h, \pi, s)$ is the henselization of the pair $(A_U^1, Y)$. Also, $in : A_U^1 \to \mathbb{P}_U^1$ is the embedding. Denote the projection $A_U^1 \to U$ by $pr$ and the projection $A_U^1 \to Y$ by $pr_Y$.

**Lemma 5.12.** [FP] There is a morphism $r : Y^h \to Y$ making the following diagram commutative

$$
\begin{array}{ccc}
Y^h & \longrightarrow & Y \\
\downarrow{in \circ \pi} & & \downarrow{pr|_Y} \\
\mathbb{P}_U^1 & \longrightarrow & U
\end{array}
$$

and such that $r \circ s = \text{Id}_Y$.

We view $Y^h$ as a $Y$-scheme via $r$. Thus various subschemes of $Y^h$ also become $Y$-schemes. In particular, $Y^h$ and $Y_u^h$ are $Y$-schemes, and we can consider

$$E(Y^h) \subset G(Y^h) \quad \text{and} \quad E(Y_u^h) \subset G(Y_u^h) = G_u(Y_u^{\text{h}}).$$

**Lemma 5.13.**

$$G_u(Y_u^h) = E(Y_u^h)G_u(Y_u^{\text{h}}).$$

**Proof.** Firstly, one has $Y_u = \coprod_{u \in u} \coprod_{y \in Y_u} y$. (Note that $Y_u$ is a finite scheme.) Thus by property (ii) of henselization, we have

$$
Y_u^h = \coprod_{u \in u} \coprod_{y \in Y_u} y^h, \quad Y_u^h = \coprod_{u \in u} \coprod_{y \in Y_u} y^h,
$$

where $(y^h, \pi_y, s_y)$ is the henselization of the pair $(A_u^1, y)$, $y^h := y^h - s_y(y)$. We see that $y^h$ and $y^h$ are subschemes of $Y^h$, so we can view them as $Y$-schemes, and $G_{y^h} := G_Y \times_Y y^h$ is quasi-split. Also, $E(y^h)$ makes sense as a subgroup of $G(y^h) = G_u(y^h) = G_{y^h}(y^h)$. 


One has
\[ G_u(Y^h_u) = \prod_{u \in u} \prod_{y \in Y_u} G_u(y^h) = \prod_{u \in u} \prod_{y \in Y_u} G_{y^h}(y^h), \]
\[ E(Y^h_u) = \prod_{u \in u} \prod_{y \in Y_u} E(y^h), \]
\[ G_u(Y^h_u) = \prod_{u \in u} \prod_{y \in Y_u} G_u(y^h) = \prod_{u \in u} \prod_{y \in Y_u} G_{y^h}(y^h). \]
Thus it suffices for each \( u \in u \) and each \( y \in Y_u \) to check the equality
\[ G_{y^h}(y^h) = E(y^h)G_{y^h}(y^h). \]
This equality holds by Fait 4.3 and Lemma 4.5 of [Gil2]. In fact, \( y^h = \text{Spec} \mathcal{O} \), where \( \mathcal{O} = k(u)[t] \) is a henselian discrete valuation ring, and \( m_y \subset k(u)[t] \) is the maximal ideal defining the point \( y \in A^1_u \). Further, \( y^h = \text{Spec} L \), where \( L \) is the fraction field of \( \mathcal{O} \). The lemma is proved. □

We have the closed embedding (2) and the scheme \( Y^h \) is affine by Proposition 5.2. Recall that we regard \( Y^h \) as a \( Y \)-scheme via the morphism \( r|_{Y^h} \). Thus by Lemma 5.11 the restriction map \( E(Y^h) \to E(Y^h_u) \) is surjective. Since \( E(Y^h) \subset G(Y^h) \), the proposition 5.9 follows. This completes the proof of Theorem 3.

6. An application

The following result is a straightforward consequence of Theorem 1 and an exact sequence for étale cohomology. Recall that by our definition a reductive group scheme has geometrically connected fibres.

**Theorem 4.** Let \( R \) be as in Theorem 1 and \( G \) be a reductive \( R \)-group scheme. Let \( \mu: G \to T \) be a group scheme morphism to an \( R \)-torus \( T \) such that \( \mu \) is locally in the étale topology on \( \text{Spec} R \) surjective. Assume further that the \( R \)-group scheme \( H := \text{Ker}(\mu) \) is reductive. Let \( K \) be the fraction field of \( R \). Then the group homomorphism
\[ T(R)/\mu(G(R)) \to T(K)/\mu(G(K)) \]
is injective.

This theorem extends all the known results of this form proven in [CTO], [PS1], [Zai], [OPZ].

**Appendix A.** [FP]

For a commutative ring \( A \) we denote by \( \text{Rad}(A) \) its Jacobson ideal. The following definition one can find in [Gab, Section 0].

**Definition A.1.** If \( I \) is an ideal in a commutative ring \( A \), then the pair \((A, I)\) is called henselian, if \( I \subset \text{Rad}(A) \) and for every two relatively prime monic polynomials \( \bar{g}, \bar{h} \in \bar{A}[t] \), where \( \bar{A} = A/I \), and monic lifting \( f \in A[t] \) of \( \bar{g}\bar{h} \), there exist monic liftings \( g, h \in A[t] \) such that \( f = gh \). (Two polynomials are called relatively prime, if they generate the unit ideal.)

**Lemma A.2.** [FP] Let \((A, I)\) be a henselian pair with a semi-local ring \( A \) and \( J \subset A \) be an ideal. Then the pair \((A/J, (I + J)/J)\) is henselian.
The following definition one can find in [Gab, Section 0].

**Definition A.3.** The henselization of any pair \((A, I)\) is the pair \((\hat{A}, \hat{I})\) (over \((A, I)\)) defined as follows

\[
(\hat{A}, \hat{I}) := \text{the filtered inductive limit over the category } \mathcal{N} \text{ of } (A', \ker(\sigma)),
\]

where \(\mathcal{N}\) is the filtered category of pairs \((A', \sigma)\) such that \(A'\) is an étale \(A\)-algebra and \(\sigma \in \text{Hom}_{A-alg}(A', A/I)\).

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