Abstract. Consider the dynamics of a gas bubble in an inviscid, compressible liquid with surface tension. Kinematic and dynamic boundary conditions couple the bubble surface deformation dynamics with the dynamics of waves in the fluid. This system has a spherical equilibrium state, resulting from the balance of the pressure at infinity and the gas pressure within the bubble. We study the linearized dynamics about this equilibrium state in a center of mass frame. We prove that the velocity potential and bubble surface perturbation satisfy point-wise in space exponential time-decay estimates. The time-decay rate is governed by the imaginary parts scattering resonances. These are characterized by a non-selfadjoint spectral problem or as pole singularities in the lower half plane of the analytic continuation of a resolvent operator from the upper half plane, across the real axis into the lower half plane. The time-decay estimates are a consequence of resonance mode expansions for the velocity potential and bubble surface perturbations. The weakly compressible case (small Mach number, $\epsilon$), is a singular perturbation of the incompressible limit. The scattering resonances which govern the remarkably slow time-decay, are Rayleigh resonances, associated with capillary waves, due to surface tension, on the bubble surface, which impart their energy slowly to the unbounded fluid. Rigorous results, asymptotics and high-precision numerical studies, indicate that the Rayleigh resonances which are closest to the real axis satisfy $|\Im(\lambda(\epsilon))| = O(\exp(-\kappa \text{ We} \epsilon^{-2})), \ k > 0$. Here, $\text{We}$ denotes the Weber number, a dimensionless ratio comparing inertia and surface tension. To obtain the above results we prove a general result estimating the Neumann to Dirichlet map for the wave equation, exterior to a sphere.

Key words.

AMS subject classifications.

1. Introduction. Consider a gas bubble, surrounded by an unbounded, inviscid and incompressible fluid with surface tension. This system has a family of equilibria, consisting of translates of a spherical gas bubble, whose radius is set by a balance of pressure at infinity with pressure inside the bubble. In this paper we prove pointwise in space time-decay estimates for the linearized evolution near this family of equilibria. We also obtain very precise information on the rate of decay.

Background: The dynamics of gas bubbles in a liquid play an important role in many fields. Examples include underwater explosion bubbles [10] (15 cm), seismic wave-producing bubbles in magma [39], bubbly flows behind ships and propellers (1 cm), bubbles at the ocean surface [19] (0.015 – 0.5 cm), microfluidics [46] (50 $\mu$m), bubbles used as contrast agents in medical imaging [9] (2 $\mu$m), and sonoluminescence [3, 27] (0.1 – 10 $\mu$m). For a discussion of these and other applications of bubble dynamics, see the excellent review articles [16, 25] and references cited therein.

The dynamics of a bubble in a liquid are governed by the compressible Navier-Stokes equations in the liquid external to the bubble, a description of the gas within the bubble, and boundary conditions (kinematic and dynamic) which couple the fluid and gas. We assume the gas inside the bubble to be at a uniform pressure throughout and to satisfy a thermodynamic law relating the bubble pressure to the bubble volume. Rayleigh initiated the study of bubble dynamics and derived an equation for the radial oscillations of a spherically symmetric gas bubble in an incompressible, inviscid liquid with surface tension [13, 28]. This problem has a spherical equilibrium, balancing the

*School of Natural Sciences, University of California, Merced, CA
†Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY
pressure at infinity and the pressure within the bubble. A general (asymmetric) perturbation of this spherical equilibrium will excite all harmonics and Rayleigh showed, in the linear approximation, that each harmonic executes undamped time-periodic oscillations [13]. With viscosity present, the pulsating bubble oscillations are damped and the spherical equilibrium is approached.

A second very important damping mechanism, energy preserving in nature, is due to compressibility of the fluid. In a non-viscous and compressible fluid, a perturbation of the equilibrium bubble will, due to coupling at the gas-liquid interface, generate acoustic waves in the fluid which, in an unbounded region, propagate to infinity. The bubble dynamics acquire an effective damping, due to energy transfer to the fluid, and its propagation to infinity. If surface tension is included asymmetric modes should damp and the bubble shape should approach that of a sphere as time advances.

Keller and co-workers [8,10,11] modeled slight compressibility of the fluid by the linear wave equation, exterior to the bubble. Both the bubble interior pressure law and the boundary conditions are kept as in the incompressible case. In the spherically symmetric setting this leads to an ODE, which captures acoustic radiation damping. A systematic derivation of the model of Keller et al., in the spherically symmetric setting, was presented by Prosperetti-Lezzi [17, 26].

We expect, for the inviscid, compressible problem with surface tension, that the family of translates of the spherical equilibrium is (locally) nonlinearly asymptotically stable. Specifically, a small perturbation of the equilibrium spherical bubble, will induce translational motion of the bubble and deformation of its surface. We expect that, in a frame of reference which moves with the bubble center of mass, \( \xi_{cm}(t) \), a small perturbation of the spherical bubble will damp toward a spherical equilibrium shape. As in many studies of asymptotic stability for coherent structures in spatially-extended conservative nonlinear PDEs, linear decay estimates of the type established in this work can be expected to play a role in the estimation of the convergence to equilibrium for the nonlinear dynamics.

### 1.1. PDEs for a gas bubble in a compressible liquid.

Consider a gas bubble, occupying a bounded region \( B(t) \), surrounded by an inviscid and compressible fluid with surface tension. We consider the boundary of the bubble, \( \partial B(t) \), to be parametrized by a function \( R : (\alpha, t) \mapsto R(\alpha, t) \in \partial B(t) \subset \mathbb{R}^3 \), with parameter \( \alpha \), e.g. spherical coordinates.

The time-evolution of the fluid is governed by the system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p(\rho) &= 0, & x \in \mathbb{R}^3 \setminus B(t) \quad (1.1a) \\
\partial_t \rho + \nabla \cdot (\rho u) &= 0, & x \in \mathbb{R}^3 \setminus B(t) \quad (1.1b) \\
(u \circ R) \cdot \hat{n} = \partial_t R \cdot \hat{n}, & \partial B(t) \quad (1.1c) \\
p_{\text{bubble}}|_{\partial B(t)} - p_{\text{fluid}}|_{\partial B(t)} = 2\sigma H[R] & \partial B(t), \quad (1.1d)
\end{align*}
\]

where \( H[R] = \frac{1}{2} \nabla \cdot \hat{n} \) denotes the mean curvature at location \( R \in \partial B(t) \); see, for example, [13] (Article 275, Equation 5), or [7] (Section 3-3).

The pressure within the fluid is assumed to obey an equation of state: \( p = p(\rho) \). Equations (1.1a) and (1.1b) express conservation of momentum and mass. Equation (1.1c) is the kinematic boundary condition, i.e. the normal velocity, \( \partial_t R \cdot \hat{n} \), of the material point on the bubble surface moves with the normal velocity of the fluid. The Young-Laplace boundary condition, also called the dynamic boundary
condition, (1.1d), expresses that the jump in pressure at the fluid bubble interface is proportional to the mean curvature [15]:

\[ p_{\text{bubble}}|_{\partial B(t)} - p_{\text{fluid}}|_{\partial B(t)} = \text{Surface Tension} \times (2 \times \text{Mean Curvature}). \quad (1.2) \]

We assume that the pressure within the bubble is spatially uniform and given by the polytropic gas law [19]:

\[ p_{\text{bubble}}|_{\partial B(t)} = P_B = k \left( \frac{4 \pi}{3} a^3 \right) \gamma, \quad \gamma > 1. \quad (1.3) \]

The pressure within the fluid is assumed to satisfy a general smooth relation of the form: \( p = p(\rho) \), so that the system is determined by only one state variable.

Finally we assume that the initial velocity is irrotational, \( \nabla \land u_0 = 0 \). It follows that the velocity field remains irrotational for all \( t \geq 0 \); see, for example, [15], Article 33. Thus, there is a single-valued velocity potential \( \Phi \), such that \( u(x, t) = \nabla \Phi \).

**Equilibrium solutions:** Equations (1.1) have a spherically symmetric equilibrium solution:

\[ u = 0, \quad R = a \hat{r} = a \frac{x - x_0}{|x - x_0|}, \quad p = p_\infty, \quad \rho = \rho_\infty. \quad (1.4) \]

The equilibrium bubble radius, \( a \), is uniquely determined via the dynamic boundary condition

\[ \frac{k}{\left( \frac{4 \pi}{3} a^3 \right) \gamma} - p_\infty = \frac{2 \sigma}{a} \quad \text{or} \quad P_{eq} = p_\infty + \frac{2 \sigma}{a}. \quad (1.5) \]

We consider these dynamics in the linear approximation. Introduce spherical coordinates, \((r, \theta, \phi) = (r, \Omega)\), with the origin chosen to be the bubble center of mass, \( \xi_{\text{cm}}(t) \). We express a small perturbation of the spherical bubble as:

\[ \Phi(r, \Omega, t) = \text{constant} + \Psi(r, \Omega, t), \quad r = |x - \xi_{\text{cm}}(t)|, \quad (1.6) \]

\[ R(t, \Omega) = (1 + \beta(\Omega, t)) \frac{x - \xi_{\text{cm}}(t)}{|x - \xi_{\text{cm}}(t)|}, \quad (1.7) \]

In Appendix C we re-write the system (1.1), relative to coordinates centered at \( \xi_{\text{cm}}(t) \). In the linear approximation, \( \xi_{\text{cm}}(t) = \xi_{\text{cm}}(0) \) and the nondimensional system of equations, linearized about the spherical equilibrium bubble are:

\[ \epsilon^2 \partial_t^2 \Psi - \Delta \Psi = 0 \quad (r > 1, \quad (1.8a)) \]

\[ \Psi_r = \beta_t \quad (r = 1, \quad (1.8b)) \]

\[ \Psi_t = 3 \gamma \left( \frac{C_a}{2} + \frac{2}{W_e} \right) \left\langle \beta, Y_0^m \right\rangle Y_0^m - \frac{1}{W_e} (2 + \Delta_s) \beta \quad (r = 1, \quad (1.8c)) \]

\[ \left\langle \beta, Y_1^m \right\rangle_{L^2(S^2)} = 0. \quad (|m| \leq 1, \quad (1.8d)) \]

\^At nonlinear order, \( \xi_{\text{cm}}(t) \) will typically evolve.
Here, \( \epsilon \) denotes the Mach number (\( M = \epsilon \) is used in the derivation of the non-dimensional equations in Appendix \( C \)), \( Ca \), the Cavitation number, and \( We \), the Weber number.

We shall focus on the initial-boundary value problem for (1.8) with data corresponding to an initial perturbation of only the bubble surface:

\[
\Psi(r, \Omega, t = 0) = \partial_t \Psi(r, \Omega, t = 0) = 0, \quad \beta(t = 0, \Omega) \text{ given and sufficiently smooth.}
\]

(1.9)

\( \Delta_S \) denotes the Laplacian on \( S^2 \), in spherical coordinates given by (A.6). The spherical harmonics, \( Y_{m}^{l}(\Omega) \), are eigenfunctions:

\[
- \Delta_S Y_{m}^{l}(\Omega) = l(l+1)Y_{m}^{l}, \quad l \geq 0, \quad |m| \leq l,
\]

forming a complete orthonormal set in \( L^2(S^2) \) with respect to the inner product \( \langle \alpha_1, \alpha_2 \rangle_{L^2(S^2)} \); see Section \( D \). The orthogonality conditions (1.8d), derived in Appendix \( C \), express our choice of coordinates (in the linearized approximation) placing the origin at the bubble center of mass.

1.2. Overview of results and discussion. We conclude this section with an overview and discussion of results.

1. **Time-decay of solutions to wave equation on \( \mathbb{R}^3 \setminus \{|x| \leq 1\} \) with time-dependent Neumann data:** Theorem 4.2 is a general result, of independent interest, on the time-decay and resonance expansion of solutions to the initial-boundary value problem for the wave equation on \( \mathbb{R}^3 - S^2 \). The data prescribed on \( S^2 \) are assumed sufficiently smooth and exponentially decaying with time. Theorem 4.2 generalizes the results of \( 14, 45 \); see also \( 14, 39 \).

2. **Exponential time-decay estimates for the bubble surface perturbation (Theorem 5.1):** Theorem 4.2 on the Neumann to Dirichlet map, together with the detailed information we obtain on the locations of scattering resonances in the lower half plane, is used to prove that the solution of the initial value problem (1.8) with initial data (1.9) decays exponentially to zero, pointwise, at a rate \( O(e^{-\Gamma(\epsilon)t}) \), \( \Gamma(\epsilon) > 0 \), as \( t \) tends to infinity. Moreover, the linearized velocity potential, \( \Psi(r, \Omega, t) \) and bubble surface perturbation, \( \beta(\Omega, t) \), satisfy resonance expansions in terms of outwardly radiating states of the scattering resonance problem. The expansion converges in \( C^2(K \times \mathbb{R}_+) \), where \( K \) denotes any compact subset of \( \{|x| \geq 1\} \).

3. **Scattering resonances and radiation damping:** The exponential rate of decay, \( \Gamma(\epsilon) = |\Im \lambda_{l,k}^{\pm}(\epsilon)| \), is determined via the scattering resonance problem, \( \psi \), a non-selfadjoint spectral problem associated with the time-harmonic solutions of the linearized compressible Euler equations + boundary conditions on \( |x| = 1 \), with outgoing radiation conditions at infinity. Two families of scattering resonances, associated with the Helmholtz equation \( \Delta \Psi + \omega^2 \Psi = 0 \) for \( |x| > 1 \), appear in the linearized problem. Rigid resonances, \( \{\omega_{l,k}(\epsilon)\}_{l \geq 0, 1 \leq k \leq l+1} \) (Theorem 3.1), are associated with Neumann (sound soft) boundary conditions imposed on the sphere, and outgoing radiation conditions at infinity and deformation resonances, \( \{\lambda_{l,k}(\epsilon)\}_{l \geq 0, 1 \leq k \leq l+2} \) (Theorem 6.1), associated with the hydrodynamic boundary conditions on the sphere, responsible for the deformation of the bubble-fluid interface, and outgoing radiation conditions at infinity. Theorem 6.3 implies that for each fixed \( \epsilon > 0 \), there is a strip containing the real axis, in which there are no scattering resonances; hence \( |\Im \lambda_{l,k}^{\pm}(\epsilon)| > 0 \).

4. **Asymptotics of deformation resonances and Fermi’s Golden Rule:** In the incompressible limit, \( \epsilon \downarrow 0 \), all resonances (rigid and deformation) have imaginary parts which tend to minus infinity, except for the sub-family of deformation
resonances called Rayleigh resonances, \( \{ \lambda_{l}^{\pm}(\epsilon) \}_{l \geq 0} \). Rayleigh resonances are scattering resonances in the lower half plane to which the real Rayleigh eigenfrequencies (of undamped oscillations in the incompressible problem, \( \epsilon = 0 \)) perturb upon inclusion of small compressibility, \( \epsilon > 0 \). Their detailed asymptotics for \( \epsilon \) small is given in Theorem 6.2. In particular, we find for the Imaginary parts of the Rayleigh Deformation Resonances:

\[
\Im \lambda_{l=0}^{\pm}(\epsilon) = -\epsilon \left[ \frac{1}{2} \left( \frac{3\gamma}{2} C_{a} + 2(3\gamma - 1) \frac{1}{\text{We}} \right) \right],
\]

\[
\Im \lambda_{l}^{\pm}(\epsilon) = -\frac{1}{\epsilon} \left[ \frac{1}{2} (l + 2)(l - 1) t^{i+1} (l + 1)^{l} \left[ \frac{2^l l^l}{(2l)!} \right]^{2} \left( \frac{\epsilon^2}{\text{We}} \right)^{l+1} + O \left( \frac{\epsilon^2}{\text{We}} \right)^{l+2} \right]
\]

for \( l = 2, 3, \ldots \). The proof of (1.10) relies on use of detailed properties of spherical Hankel function and, in particular, a subtle result on the Taylor expansion of the function \( z \mapsto G_{l}(z) = z \partial h_{l}^{(1)}(z)/h_{l}^{(1)}(z) \) in a neighborhood of \( z = 0 \); see Proposition 8.4.

The infinite set of pairs of (real) Rayleigh eigenfrequencies \( \{ \lambda_{l}^{\pm}(0) \}_{l=0,2,\ldots} \) may be viewed as: embedded eigenvalues in the continuous spectrum of the unperturbed \( (\epsilon = 0, \text{incompressible}) \) spectral problem. The negative imaginary part in (1.10) is an instance of the Fermi Golden Rule. An expression coined originally in the context quantum electrodynamics [3], it refers to the induced damping of an “excited state” (the bubble perturbation), due to coupling of an “atom” (the deforming bubble) to a field (wave equation); see also, for example, [29, 33, 44].

5. Scaling behavior of the decay rate, \( |\Im \lambda_{l}^{\pm}(\epsilon)| \) (Section 7): Theorem 5.2 asymptotics and high-precision numerics show that the exponential rate of decay is given by \( |\Im \lambda_{l}^{\pm}(\epsilon)| \approx |\Im \lambda_{l}^{\pm}(\epsilon)| \geq 0 \), where \( \lambda_{l}^{\pm}(\epsilon) \) are the scattering resonance energies closest to the real axis, and in the lower half plane. These resonances \( \lambda_{l}^{\pm}(\epsilon) \sim \lambda_{l}(\epsilon) \), where \( \lambda_{l}(\epsilon) = O \left( \epsilon^{-2} \text{We} \right) \). Moreover, an asymptotic study of the results of Theorem 6.2 yields:

\[
\Re \lambda_{l}^{\pm}(\epsilon) = \frac{1}{\epsilon} O \left( \frac{\text{We}}{\epsilon^2} \right), \quad \Im \lambda_{l}^{\pm}(\epsilon) = \frac{1}{\epsilon} O \left( \frac{\text{We}}{\epsilon^2} e^{-\kappa \sqrt{\text{We}}} \right), \quad \kappa > 0.
\]

In contrast, the monopole (spherically symmetric) resonance has an imaginary part which is \( O(\epsilon) \). This remarkably slow rate of decay is related to the scattering resonance problem being a singular perturbation problem in the (incompressible) limit of small \( \epsilon \); the wave equation \( \epsilon^2 \partial_{t}^{2} \Phi = \Delta \Phi \) reduces to Laplace’s equation \( \Delta \Phi = 0 \) as \( \epsilon \to 0 \).

Figure 14 displays, for a particular small choice of \( \epsilon \), a range of resonance energies including those located closest to the real axis. Physically, these very slowly decaying bubble shape modes are associated with capillary waves on the bubble surface, are excited only by asymmetric perturbations, and very slowly transfer their energy to the infinite fluid. [4]

---

2We remark on an interesting example, where scattering resonances converge to the real axis. Stefanov and Vodev [36] show, for the system of linear elasticity on the exterior of a ball in \( \mathbb{R}^3 \), that the scattering resonances converge to the real axis exponentially fast, as the spherical harmonic index, \( l \), tends to infinity. Consequently, the time-decay of solutions is not exponentially fast; see [35] and references cited therein. The modes associated with the resonances which come ever closer to the real axis are called Rayleigh surface waves, a decay rate limiting mechanism analogous to the capillary surface waves (Rayleigh resonance modes) of the bubble problem we consider in this article.
6. Monopole vs. Multipole radiation: A discussion of emission of acoustic radiation in the physics literature (see [19] and references cited therein) is based on a heuristic energy balance argument, which assumes that the dominant emission of acoustic radiation is through the monopole \((l = 0, \text{purely radial})\) mode. This assumption implies decay of the bubble perturbation on the time scale of order \(\tau_{\text{monopole}}(\epsilon) \sim \epsilon^{-1}\). Our results demonstrate that there are indeed non-symmetric vibrational modes of a bubble, which have the much longer lifetime

\[
\tau(\epsilon) \sim \epsilon (\epsilon^{-2} \text{We}) \exp (\kappa \epsilon^{-2} \text{We}) \gg \tau_{\text{monopole}}(\epsilon), \quad \kappa > 0.
\]

Of course, in many systems one would expect viscosity (not included in the modeling) to contribute a dominant correction to the imaginary part.

We note a relation of these results to those in [18] and [21], who used energy flux arguments to study the radiation of acoustic energy in compressible atmospheres.

Some future directions: In addition to the question of nonlinear asymptotic stability of the family of spherical equilibrium bubbles, we remark that there is a rich class of solutions in the radially symmetric setting. In the case of radial symmetry, Rayleigh showed that the dynamics of a spherical bubble in an incompressible liquid exactly reduce to a nonlinear ordinary differential equation for the time-evolution of the radius. This equation admits the spherical equilibrium (constant radius) state as well as time-periodic (radially pulsating) states. Plesset [24] extended the analysis in the spherically symmetric case to include viscosity. It is of interest to study the dynamics near the spherically symmetric time-periodic states of the Rayleigh-Plesset equations. We expect that these would be unstable; periodic oscillations would couple to continuous spectral modes, resulting in radiation-damped “breather” oscillations; see, for example, [34] and references cited therein. The period map or monodromy operator associated with the linearization about such a state, for \(\epsilon = 0\), would have an embedded Floquet multiplier on the unit circle, which would perturb to an unstable Floquet multiplier outside the unit circle [20,32].

Finally, we remark that the problem we consider is one of a very large class, involving an infinite dimensional conservative system comprised of two coupled subsystems: one subsystem has a discrete number of degrees of freedom (here the mode amplitudes of the spherical harmonics of the bubble surface) and one has a continuum of degrees of freedom, the velocity potential governed by the wave equation. While
there has been significant progress on such systems, where the number of “soliton”
degrees of freedom of the discrete subsystem is finite (for example, see [12, 34, 43] and
references cited therein), systems like the bubble-fluid system, which involve the cou-
ppling of infinite dimensional systems, are central in physical and engineering science
and are an important future direction.

Acknowledgements: The authors thank D. Attinger, J.B. Keller, A. Soffer, E.
Spiegel, J. Xu and M. Zworski for stimulating discussions. This research was sup-
ported in part by US National Science Foundation grant DMS-07-07850 and NSF
grant DGE-0221041. The Ph.D. thesis research of the first author was supported, in
part, by the IGERT Joint Program in Applied Mathematics and Earth and Environ-
mental Science at Columbia University. Some of the discussion of this paper, appears
in greater detail in [31].

1.3. Some oft used notation.
• $\langle \alpha, \beta \rangle_{L^2(S^2)}$ = inner product on $L^2(S^2)$; Section D.
• $\Delta_S$ = Laplacian on $S^2$; Equation (A.6), $Y^m_l(\theta, \phi) = Y^m_l(\Omega)$, spherical har-
monic; Appendix B.
• Dimensionless quantities: $W_e = Weber$ number, $C_a = Cavitation$ number ,
$\epsilon = M = Mach$ number; Appendix C

\begin{equation}
\begin{aligned}
    r_l &= \begin{cases}
        \frac{3\gamma}{2} C_a + 2(3\gamma - 1) \frac{1}{W_e}, & l = 0 \\
        \frac{1}{W_e} (l + 2)(l - 1), & l \geq 1,
    \end{cases} \quad (1.11)
\end{aligned}
\end{equation}

2. Conservation of Energy. An important role in our analysis of time-decay
of solutions of the initial-boundary value problem for (1.8) is played by the following
conservation law. We use it to establish the scattering resonance frequencies as lying
in the lower half plane; see the proof of Proposition 6.1.

**Proposition 2.1.** Let $(\Psi(x, t), \beta(\Omega, t))$ denote a smooth complex-valued solu-
tion of the linearized equations.

(i) Then if $\Psi$ decays sufficiently rapidly as $|x| \to \infty$, then the following functional is time-invariant

\begin{equation}
    \mathcal{E} = \int_{|x| \geq 1} \left( \epsilon^2 |\partial_t \Psi|^2 + |\nabla \Psi|^2 \right) dx + 3\gamma \left( \frac{C_a}{2} + \frac{2}{W_e} \right) |\langle \beta, Y^0_0 \rangle|^2 + \frac{1}{W_e} \langle (-\Delta_S - 2)\beta, \beta \rangle, \quad (2.1)
\end{equation}

where $\langle \alpha, \beta \rangle$ denotes the inner product on $L^2(S^2)$.

(ii) The energy, $\mathcal{E}$, can also be expressed as:

\begin{equation}
    \mathcal{E} = \int_{|x| \geq 1} \left( \epsilon^2 |\partial_t \Psi|^2 + |\nabla \Psi|^2 \right) dx + \left( \frac{3\gamma}{2} C_a + 2(3\gamma - 1) \frac{1}{W_e} \right) |\langle \beta, Y^0_0 \rangle|^2 + \frac{1}{W_e} \sum_{l \geq 2} \sum_{|m| \leq l} (l + 2)(l - 1) |\langle \beta, Y^m_l \rangle|^2. \quad (2.2)
\end{equation}

Furthermore, $\mathcal{E}$ is positive definite on solutions of the linearized equations (1.8) since
$\gamma > 1$.

**Proof.** Part (i): Multiplying the wave equation (1.8a) by $\partial_t \Psi$ and taking the real part of the resulting equation yields

\begin{equation}
    \partial_t e(\Psi, \partial_t \Psi) + \nabla \cdot \left( -\partial_t \Psi \nabla \Psi - \partial_t \Psi \nabla \Psi \right) = 0, \quad (2.3)
\end{equation}
where \( e(\Psi, \partial_t \Psi) = \varepsilon^2 |\partial_t \Psi|^2 + |\nabla \Psi|^2 \). Integration of the local conservation law \( (2.3) \) over the region \(|x| \geq 1\) yields
\[
\frac{d}{dt} \int_{|x| \geq 1} e(\Psi, \partial_t \Psi) \, dx + \int_{|x|=1} (\partial_t \Psi \nabla \Psi \cdot \hat{n} + \partial_t \Psi \nabla \Psi \cdot \hat{n}) \, dS = 0, \tag{2.4}
\]
where \( \hat{n} \) denotes the unit normal to \( S^2 \), pointing into the ball. The theorem no follows from re-expressing the second term in \( (2.4) \) using the boundary conditions:
\[
\int_{|x|=1} (\partial_t \Psi \nabla \Psi \cdot \hat{n} + \partial_t \Psi \nabla \Psi \cdot \hat{n}) \, dS
= \frac{d}{dt} \left[ 3\gamma \left( \frac{\omega_0^2}{2} + \frac{\omega_0^2}{\kappa} \right) \left| \langle \beta, Y_0^0 \rangle \right|^2 + \frac{1}{\kappa} \left| (\nabla S_2 - 2) \beta, \beta \right| \right] \tag{2.5}
\]

Part (ii): We show now, using the center of mass constraint, \( (1.8d) \), that the energy is positive definite on solutions of the linearized system, \( (1.8) \).

By \( (1.8d) \) we can expand \( \beta \) in spherical harmonics as \( \beta = \langle \beta, Y_0^0 \rangle Y_0^0 + \beta_{l \geq 2} \), where \( \beta_{l \geq 2} = \sum_{l \geq 2} \sum_{|m| \leq l} \langle \beta, Y_l^m \rangle Y_l^m \). Consider the expression \( \langle (\nabla S_2 - 2) \beta, \beta \rangle \).

We have, by orthogonality
\[
\langle (\nabla S_2 - 2) \beta, \beta \rangle = -2 \left| \langle \beta, Y_0^0 \rangle \right|^2 + \langle (\nabla S_2 - 2) \beta_{l \geq 2}, \beta_{l \geq 2} \rangle
= -2 \left| \langle \beta, Y_0^0 \rangle \right|^2 + \sum_{l \geq 2} \sum_{|m| \leq l} (l+2)(l-1) \left| \langle \beta, Y_l^m \rangle \right|^2 \tag{2.6}
\]

Use of \( (2.6) \) in the expression for the energy \( 2.1 \) yields \( 2.2 \), which is clearly positive definite.

3. The Scattering Resonance Problem. Our results on time-decay of the velocity potential and bubble surface perturbations (Theorems \( 4.2 \) and \( 5.1 \)) are proved using their representation as inverse Laplace transforms of a resolvent operator applied to the initial data; see Equations \( (4.19) \) and \( (5.12) \), in which the contour of integration is a horizontal line in the upper half of the complex frequency plane. Time decay (and a resonance mode expansion) is deduced by analytically continuing the resolvent kernel (Green’s function) and deforming the integration contour across the real axis (essential spectrum) into the lower half plane. The time-decay is determined by poles of this analytically continued resolvent kernel. These poles are called scattering resonances or scattering poles. An alternative characterization of scattering resonances is as complex frequencies in the lower half plane for which there are non-trivial time-harmonic solutions of the linearized equations, satisfying an\( \text{outgoing radiation} \) (non-self-adjoint) boundary condition at infinity.

In this section we use the alternative (spectral) characterization as solutions of a non-selfadjoint eigenvalue problem. We consider two classes of scattering resonances:

1. Rigid resonances, associated with the wave equation on the exterior of the (rigid) unit sphere with Neumann boundary conditions, and

2. Deformation resonances, associated with the linearized system \( (1.8) \).

Both families of resonances, for \( \epsilon > 0 \), lie in the lower half plane. Of particular interest is the resonance in each family with the smallest (in magnitude) imaginary part. Denote by \( \omega_\epsilon(\epsilon) \) and \( \lambda_\epsilon(\epsilon) \) the rigid and deformation resonances of smallest imaginary parts. We find for \( \epsilon \) small:
\[
|\Im \omega_\epsilon(\epsilon)| = O(\epsilon^{-1}) \text{, } |\Im \lambda_\epsilon(\epsilon)| = O(\epsilon^{-3} e^{-\frac{\kappa}{\epsilon^2}}), \text{ } \kappa > 0 \Rightarrow |\Im \lambda_\epsilon(\epsilon)| \ll |\Im \omega_\epsilon(\epsilon)|. \tag{3.1}
\]

\(^3\)Here, and in later sections, \( \lambda_\epsilon^\pm(\epsilon) \) may be denoted simply \( \lambda_\epsilon(\epsilon) \).
3.1. Rigid - Neumann scattering resonances. We consider the scattering resonance problem, associated with the wave equation \( \epsilon^2 \partial_t^2 \Psi = \Delta \Psi \) in the exterior of the unit sphere with Neumann boundary conditions. Seeking time-harmonic solutions, \( \Psi(r, \Omega, t) = e^{-i\omega t}\Psi_\omega \), which are outgoing as \( r \to \infty \), we arrive the following eigenvalue problem

\[
\left( \Delta + (\epsilon \omega)^2 \right) \Psi_\omega (r, \Omega) = 0, \quad r > 1; \quad \partial_r \Psi_\omega (r, \Omega) = 0, \quad r = 1.
\] (3.2)

Outgoing solutions are spanned by functions of the form \( h_l^{(1)}(\epsilon \omega r) Y_l^m(\Omega) \). Thus, \( \omega \) is a Neumann scattering resonance if and only if \( \partial h_l(\epsilon \omega) = 0 \). The following result summarizes results in [22,41]; see also [1].

**Theorem 3.1.** Fix an arbitrary \( \epsilon > 0 \) and arbitrary.

(i) For each \( l \geq 0 \), the equation \( \partial h_l^{(1)}(\epsilon \omega) = 0 \) has a family of solutions \( \{\omega_{l,k}(\epsilon) = \epsilon^{-1} \omega_{l,k} : k = 1, \ldots, l + 1\} \).

(ii) The set of all Neumann scattering resonances is a discrete subset of the lower half complex plane and is uniformly bounded away from the real axis. In particular, there exists \( l_0 \geq 0, 1 \leq k_* \leq l_0 + 1 \) such that \( \omega_{l_0}(\epsilon) = \text{det} \omega_{l_0,k_*}(\epsilon) \) is a resonance whose imaginary part is of minimal magnitude, i.e.

\[
\Im \omega_l(k) \geq \Im \omega_s(k) > 0, \quad \text{for all } l \geq 0, \quad k = 1, \ldots, l + 1
\] (3.3)

(iii) There exist constants \( C_1, C_2 > 0 \), such that for all \( l \geq 0 \)

\[
C_1 \epsilon^{-1} l^{1/3} \leq |\Im \omega_l(k)|, \quad l \gg 1 \quad \text{and} \quad |\omega_l(k)| = \epsilon^{-1} |\omega_l| \leq C_2 \epsilon^{-1} l.
\]

This result is essentially due to Tokita [41] and uses fundamental results on the asymptotics of Hankel and Airy functions; see Olver [1,22]. Explicit approximations to zeros of \( \partial H_\nu \) are given in Equation (E.17).

3.2. Deformation resonances. We seek time-harmonic solutions of the linearized perturbation equations [18]: \( \Psi = e^{-i\lambda t} \Psi_\lambda (r, \Omega), \beta = e^{-i\lambda t} \beta_\lambda (\Omega) \). Substitution into (3.8) yields the following Helmholtz eigenvalue problem:

\[
\left( \Delta + (\lambda \epsilon)^2 \right) \Psi_\lambda = 0, \quad r > 1
\] (3.4a)

\[
\partial_r \Psi_\lambda = -i\lambda \beta_\lambda, \quad r = 1
\] (3.4b)

\[
-i \lambda \Psi_\lambda = 3\gamma \left( \frac{C_0}{2} + \frac{2}{\omega \epsilon} \right) \delta \beta_\lambda Y_0^0 Y_0^0 - \frac{1}{\omega \epsilon} (2 + \Delta S) \beta_\lambda, \quad r = 1
\] (3.4c)

\[
\Psi_\lambda \quad \text{outgoing} \quad r \to \infty.
\] (3.4d)

If \( \lambda \) is such that (3.4) has a non-trivial solution, then we call \( \lambda \) a (deformation) scattering resonance energy or scattering frequency, and \( (\Psi_\lambda, \beta_\lambda) \) a corresponding scattering resonance mode.

Since outgoing solutions of the three-dimensional Helmholtz equation are linear combinations of solutions of the form \( h_l^{(1)}(r)Y_l^m(\Omega), \quad |m| \leq l \), where \( h_l^{(1)} \) denotes the outgoing spherical Hankel function of order \( l \), we seek solutions of the boundary value problem (3.4) of the form: \( \Psi_\lambda(r, \Omega) = A Y_l^m(\Omega) h_l^{(1)}(\epsilon \lambda r), \beta_\lambda(\Omega) = B Y_l^m(\Omega), \quad r \geq 1, \quad \Omega \in S^2 \), where \( A \) and \( B \) are constants to be determined. This Ansatz automatically solves the Helmholtz equation and satisfies the outgoing radiation condition. To impose the boundary conditions at \( r = 1 \) we substitute we substitute the expressions for \( \Psi_\lambda \) and \( \beta_\lambda \) into (3.4) and obtain the following two linear homogeneous equations...
for the constants $A$ and $B$. This system has a non-trivial solution if and only if $\lambda$ solves the equation:

$$r_l \epsilon \lambda \partial h_l^{(1)}(\epsilon \lambda) + \lambda^2 h_l^{(1)}(\epsilon \lambda) = 0,$$

(3.5)

where $r_l$ is given by (1.11). We shall be interested in the character of solutions to (3.5) for small, positive and fixed $\epsilon$. By (D.11)

$$\lim_{z \to 0} \frac{z \partial h_l^{(1)}(z)}{h_l^{(1)}(z)} = \lim_{z \to 0} \frac{p_l(z)}{(l+1)p_l(z) - p_{l+1}(z)} = -(l+1).$$

(3.6)

Thus, we write (3.5) in the form:

$$\lambda^2 + r_l G_l(\epsilon \lambda) = 0,$$

(3.7)

we call the solutions of (3.7) deformation resonances. Their properties are summarized in the following result, proved in Section 6.

**Theorem 3.2 (Deformation resonances).** Fix $\epsilon > 0$ and arbitrary.

(i) There are $l+2$ solutions of Equation (3.7) denoted $\{\lambda_{l,j}(\epsilon)\}$, for $l = 0, 2, 3, \ldots$ and $j = 1, \ldots, l+2$. These are the deformation resonance energies.

(ii) The set of deformation resonance energies is a discrete subset of the lower half complex plane and is uniformly bounded away from the real axis. That is, for some $l^{\ast}(\epsilon), k^{\ast}(\epsilon)$,

$$\exists \lambda_{l,j}(\epsilon) \leq \exists \lambda_{l^{\ast},j^{\ast}}(\epsilon) \equiv \exists \lambda_{\ast}(\epsilon) < 0, \quad \text{all} \quad l \geq 0, \quad |j| \leq l+2.$$  

(3.8)

(iii) $\lambda_{l,j}(\epsilon) = O(l)$ as $l \to \infty$.

(iv) In the incompressible limit, $\epsilon \to 0^+$, the imaginary parts of all resonances tend to $-\infty$ except for the family of Rayleigh resonances with real frequencies:

$$\lambda_{l}^{\pm}(0) = \pm \sqrt{\frac{3\gamma}{2}} Ca + \frac{2}{We}(3\gamma - 1), \quad \lambda_{l}^{\pm}(0) = \pm \sqrt{\frac{1}{We}(l+2)(l+1)(l-1)}, \quad l \geq 2.$$  

(3.9)

**4. A theorem on resonance expansions and time-decay for the exterior Neumann problem for the wave equation.** Our strategy for solving the initial-boundary value problem (1.8), (1.9) is to (1) construct the solution to the wave equation (1.8a) with kinematic boundary condition (1.8b), which specifies Neumann boundary data on the unit sphere. This is the Neumann to Dirichlet map $\partial_t \beta \mapsto \Psi = NtD[\partial_t \beta]$. Then, (2) substitute $\Psi = \partial_t \beta \to NtD[\partial_t \beta]$ into the dynamic boundary condition (1.8c) and then study the closed nonlocal equation for $\beta$ on the unit sphere.

Denote by $U$ the exterior of the unit sphere in $\mathbb{R}^3$:

$$U = \{x : |x| > 1\}.$$  

In particular, we consider the general initial-boundary value problem

$$
\begin{align*}
(e^{-2\partial_t^2} - \Delta) u &= 0, & \text{in} \ U \times (0, \infty), \\
 u &= 0, & \partial_t u = 0, & \text{on} \ U \times \{t = 0\}, \\
 \partial_t u &= \partial_t f, & \text{on} \ \partial U \times (0, \infty).
\end{align*}
$$

(4.1a) (4.1b) (4.1c)
where the function $\partial_t f : \partial U \times (0, \infty) \to \mathbb{R}$ is a prescribed boundary forcing function, whose properties (anticipating those of the bubble shape perturbation, $\partial_t \beta$) are prescribed below.

We introduce a norm of functions, $\partial_t f = \partial_t f(\Omega, t)$, defined on the sphere, $S^2$, which encodes smoothness in $\Omega \in S^2$ and decay in time, $t$.

**Definition 4.1.**

(i) For a function $J : [0, \infty) \to \mathbb{R}$, define for any $\kappa \in \mathbb{R}$,

$$[J]_\kappa = \sup_{t \geq 0} e^{\kappa t} |J(t)|. \quad (4.2)$$

(ii) For a function $g : \mathbb{R} \times S^2_\Omega \to \mathbb{R}$, with spherical harmonic expansion:

$$g(t, \Omega) = \sum_{l \geq 0} \sum_{|m| \leq l} g^m_l(t) Y^m_l(\Omega),$$

define, for $q, \kappa \in \mathbb{R}$, the norm

$$[g]_{q, \kappa} = \sum_{l \geq 0} \sum_{|m| \leq l} (1 + l)^q \sup_{t \geq 0} e^{\kappa t} \left( |g^m_l(t)| + |\partial_t g^m_l(t)| \right)$$

$$= \sum_{l \geq 0} \sum_{|m| \leq l} (1 + l)^q \left( |g^m_l|_\kappa + |\partial_t g^m_l|_\kappa \right). \quad (4.3)$$

The main result of this section is the following theorem, a generalization [41, 45].

**Theorem 4.2.** Consider the Neumann initial-boundary value problem for the wave equation (4.1) on the exterior of the unit sphere.

(i) **Time Decay Estimate:** There exists $\alpha > 0$, such that if $[\partial_t f]_{\alpha, \eta} < \infty$, with $\eta > 0$, then there exists a unique solution of the initial-boundary value problem equations (4.1) satisfying the bound

$$|u(x, t)| \leq \begin{cases} C \frac{1}{|\eta|} e^{-\min\{\eta, |3\omega_*|\}(t - \frac{|x| - 1}{ct})} |\partial_t f|_{\alpha, \eta}, & 1 < |x| < 1 + ct \\ 0, & \text{if } |x| \geq 1 + ct \end{cases} \quad (4.4)$$

Here, $\omega_* (3\omega_* < 0)$ denotes the scattering resonance of the exterior wave equation with Neumann boundary condition, which is closest to the real axis; see Theorem 3.1.

(ii) **Resonance Expansion:** Let $\alpha$ be as above. Suppose $f : S^2 \times \mathbb{R}_+ \to \mathbb{C}$ has the spherical harmonic expansion in the sense that

$$\lim_{L \to \infty} [\partial_t f - \partial_t f^{(L)}]_{\alpha, \eta} = 0, \quad \text{where } f^{(L)}(\Omega, t) \equiv \sum_{l=0}^L \sum_{|m| \leq l} f^m_l(t) Y^m_l(\Omega). \quad (4.5)$$

Denote by $u^{(L)}(r, \Omega, t)$ the resonance expansion partial sum:

$$u^{(L)} = i \sum_{l=0}^L \sum_{|m| \leq l} \left[ \sum_{k=1}^{l+1} \int_0^{(r-c)/(1/c)} e^{-i\omega_k^l(t-s)} \partial_s f^m_l(s) ds \frac{h^{(1)}_l(\omega_s k r/c)}{(\omega_s k/c)^2} \partial^2 h^{(1)}_l(\omega_s k/c) \right] Y^m_l(\Omega). \quad (4.6)$$

Here $h^{(1)}_l(z)$ is the outgoing spherical Hankel function of order $l$ and

$$\{\omega_k^l\}$$

are the solutions of $\partial h^{(1)}_l(\omega/c) = 0$;

see Theorem 3.1. The limit $u(x, t) = \lim_{L \to \infty} u^{(L)}(x, t)$ exists and converges to the unique solution of the initial-boundary value problem (4.1). Furthermore, we have
the following error estimate for the resonance expansion. For \(0 \leq |a| + |b| \leq 2\):

\[
\left| \partial_t^2 \partial_x^2 (u - u^{(L)})(x,t) \right| \\
\leq \begin{cases} 
C [\partial_t f - \partial_x f^{(L)}]_{\omega, \eta} \frac{1}{|\omega|} e^{-\min(\eta, |3\omega|)}(t - \frac{|x|}{c})^{1}, & 1 < |x| < 1 + ct \\
0, & |x| \geq 1 + ct.
\end{cases} 
\tag{4.7}
\]

4.1. Proof of Theorem 4.2. We first note that the decay estimate of Part 1 is a consequence of the resonance expansion of Part 2. Indeed, suppose Part 2 holds. Then, to prove the decay estimate (4.4), note that \(|u(x,t)| \leq |u^{(L)}(x,t)| + |u(x,t) - u^{(L)}(x,t)|\).

The first term is a finite sum, each of whose terms satisfies the decay estimate, while the second is controlled by (4.7).

We now turn to the proof of the resonance expansion, Part 2 of Theorem 4.2. We first derive and, by Laplace transform methods, solve the equations for the spherical harmonic coefficients \(u^m(r,t)\); see Equation (4.24). Then, we form the partial sum for \(u^{(L)}(r,\Omega, t)\) and prove convergence. In proving Theorem 4.2 we use the following two estimates:

**Proposition 4.3.** Fix \(r > 1\) and \(l \geq 0\). Then we have the bound

\[
Q_l(\omega, r) = \frac{h_l^{(1)}(\omega r/c)}{(\omega/c) \partial h_l^{(1)}(\omega/c)} e^{i\omega(r-1)/c}
\tag{4.8}
\]

\[
= \frac{1}{\omega r/c} \left( 1 + O_{l,r} \left( \frac{1}{|\omega|} \right) \right) \text{ as } |\omega| \to \infty.
\tag{4.9}
\]

**Proof.** We use facts about \(h_l(z)\), documented in Appendix D.1. Note that \(h_l^{(1)}(z) = z^{-l-1}p_l(z)e^{iz}\), where \(p_l(z)\) is a polynomial of degree \(l\). Furthermore, we have the following recursion relating \(h_l\) and \(\partial h_l\) derivatives

\[
\partial h_l^{(1)}(z) = z^{-1}h_l^{(1)}(z) - h_l^{(1)}(z);
\tag{4.10}
\]

see Appendix D.3. Therefore,

\[
Q_l(\omega, r) = \frac{p_l(\omega r/c)}{r^{l+1} [p_l(\omega/c) - p_{l+1}(\omega/c)]}.
\]

Now fix \(r > 1\) and \(l \geq 0\). The asymptotics (4.9) follows using that \(p_l(z) \sim d_l^z\) for large \(|z|\) and that \(|d_l^z|/|d_{l+1}^z| = 1\).

**Proposition 4.4.** Let \(0 < \eta < |3\omega_s| = \min_{l,k} |3\omega_{l,k}|\). Then,

\[
\int_0^t e^{i\omega_{l,k}(t-s)} e^{-\eta s} ds e^{-3\omega_{l,k}(t-c)} \leq \frac{C}{|3\omega_s| - \eta} e^{-\eta(t - \frac{c}{c})}, \quad r < 1 + ct.
\]

We now embark on the proof of Theorem 4.2. Substitution of the expansion

\[
u(r, \Omega, t) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} u_l^m(r, t)Y_l^m(\Omega)
\tag{4.11}
\]

into equation (4.13), expressing the Laplacian as \(\Delta_r + r^{-2} \Delta_S\) (radial and spherical parts) and using that \(-\Delta_S Y_l^m = l(l+1)Y_l^m\), we obtain the following equations for
From the boundary conditions equation (4.13) we have
\[ \partial_r \tilde{u}_l^m(r, p) = \partial_t \tilde{f}_l^m(p), \quad \text{at} \; r = 1 \] (4.15)

Equation (4.14) is solved in terms of the outgoing spherical hankel function \( h_l^{(1)} \); see Appendix D:
\[ \tilde{u}_l^m(r, p) = A_l^m(p) h_l^{(1)}(ipr/c), \] (4.16)
where \( A_l^m(p) \) is to be determined. Imposing the boundary condition (4.15) yields
\[ A_l^m(p) = \frac{\partial \tilde{f}_l^m(p)}{(ip/c) \partial h_l^{(1)}(ip/c)}. \]
Inversion of the Laplace transform yields:
\[ u_l^m(r, t) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{\omega t} \frac{\partial \tilde{f}_l^m(p) h_l^{(1)}(ipr/c)}{(ip/c) \partial h_l^{(1)}(ip/c)} dp, \] (4.17)

where the contour is chosen in the right half plane, with \( \Re p = \mu > 0 \) sufficiently large so that all poles lie to left.

We find it convenient to work with a rotated (horizontal) contour and therefore make the change of variable
\[ \omega = ip. \] (4.18)

Then,
\[ u_l^m(r, t) = \frac{1}{2\pi} \int_{-\infty + i\mu}^{\infty + i\mu} e^{-i\omega t} \frac{\partial \tilde{f}_l^m(-i\omega) h_l^{(1)}(\omega r/c)}{(\omega/c) \partial h_l^{(1)}(\omega/c)} d\omega \] (4.19)
\[ = \frac{1}{2\pi} \int_{-\infty + i\mu}^{\infty + i\mu} e^{-i\omega(t-(r-1)/c)} Q_l(\omega, r) \, d\omega, \quad \mu > 0, \]
where \( Q_l(\omega, r) \) is defined in (4.8).

We first consider, \( r = |x| \) fixed with \( r > 1 + ct \) and prove the finite propagation speed result:
\[ u_l^m(r, t) = 0, \quad r > 1 + ct. \] (4.20)
Proof of Equation (4.20). By Theorem 3.3, the poles of \(Q_l(\omega, r)\) are in the lower half \(\omega-\) plane. Therefore, for any \(\rho > 0\)

\[
\frac{1}{2\pi} \left( \int_{-\rho+i\mu}^{\rho+i\mu} e^{-\omega(t-(r-1)/c)} Q_l(\omega, r) d\omega \right) = 0, \quad \mu > 0,
\]

(4.21)

where \(C_\rho = \{ \omega : |\omega - i\mu| = \rho, 3\omega > \mu \}\). It follows from (4.21) that for \(r > 1 + ct\)

\[
u_l^m(r, t) = \lim_{\rho \to \infty} \frac{1}{2\pi} \int_{-\rho+i\mu}^{\rho+i\mu} e^{i\omega((r-1)/c - t)} Q_l(\omega, r) d\omega,
\]

(4.22)

Note that the integrand is analytic in the upper half plane, continuous up to the real line and, by Proposition 4.3, \(\max_{\theta \in [0, \pi]} |Q_l(\rho e^{i\theta}, r)| \to 0 \text{ as } \rho \to \infty\). By Jordan’s Lemma the limit in (4.22) vanishes, proving (4.20).

For \(r \leq 1 + ct\), we decompose the Laplace transform of the initial data, \(\hat{\partial_t f_l^m}\):

\[
\hat{\partial_t f_l^m}(-i\omega) = \int_0^{t-(r-1)/c} e^{i\omega t} \hat{\partial_s f_l^m}(s) ds + \int_{t-(r-1)/c}^{\infty} e^{i\omega t} \hat{\partial_s f_l^m}(s) ds
\]

and the corresponding solution

\[
u_l^m(r, t) = \frac{1}{2\pi} \int_{-\infty+i\mu}^{\infty+i\mu} e^{-i\omega t} \mathcal{F}_{l,m}^{(1)}(\omega; t) \frac{h_l^{(1)}(\omega r/c)}{(\omega/c) \partial h_l^{(1)}(\omega/c)} d\omega
\]

(4.23)

\[\equiv U_{l,m}^{(1)}(r, t)\]

\[\left. + \frac{1}{2\pi} \int_{-\infty+i\mu}^{\infty+i\mu} e^{-i\omega t} \mathcal{F}_{l,m}^{(2)}(\omega; t) \frac{h_l^{(1)}(\omega r/c)}{(\omega/c) \partial h_l^{(1)}(\omega/c)} d\omega \right. \]

(4.24)

\[\equiv U_{l,m}^{(2)}(r, t)\]

In the following two propositions, we evaluate \(U_{l,m}^{(1)}\) and \(U_{l,m}^{(2)}\).

**Proposition 4.5.** Let \(\omega_{l,k} = \omega_{l,k}(c^{-1})\), \(k = 1, 2, \ldots, l + 1\) denote the solutions of \(\partial h_l^{(1)}(\omega/c) = 0\); see Theorem 3.3 with \(c = c^{-1}\). For \(r < 1 + ct\), \(U_{l,m}\) can be expressed as a sum over residues:

\[
U_{l,m}^{(1)}(r, t) = \sum_{k=1}^{l+1} \int_0^{t-(r-1)/c} \frac{e^{-i\omega t} \partial_s f_l^m(s) ds}{(\omega/c) \partial h_l^{(1)}(\omega/c)} \text{ Res}_{\omega=\omega_{l,k}^{-1}(c)} \frac{h_l^{(1)}(\omega r/c)}{(\omega/c) \partial h_l^{(1)}(\omega/c)}
\]

(4.25)

\[= \sum_{k=1}^{l+1} \int_0^{t-(r-1)/c} \frac{e^{-i\omega t} \partial_s f_l^m(s) ds}{(\omega/c) \partial h_l^{(1)}(\omega/c)} \frac{h_l^{(1)}(\omega_{l,k} r/c)}{(\omega_{l,k} c^2) \partial^2 h_l^{(1)}(\omega_{l,k}/c)}
\]

\[= \sum_{k=1}^{l+1} \int_0^{t-(r-1)/c} \frac{e^{-i\omega t} \partial_s f_l^m(s) ds}{(\omega/c) \partial h_l^{(1)}(\omega/c)} \frac{h_l^{(1)}(\omega_{l,k} r/c)}{(\omega_{l,k} c^2) \partial^2 h_l^{(1)}(\omega_{l,k}/c)}
\]

The notation of Theorem 3.3 suggests the following 

For ease of presentation, in this self-contained section, we mildly abuse notation here and refer to the solutions as simply \(\omega_{l,k}\).
Proposition 4.6. \( U_{l,m}^{(2)}(r, t) = 0 \).

Proof of Proposition 4.6. We evaluate the contour integral representation of \( U_{l,m}^{(1)} \) by the method of residues. Consider the clockwise-traversed rectangular contour \( \Gamma^{M,µ,γ} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \), where

\[
\begin{align*}
\Gamma_1 &= \{ x + i\mu : -M \leq x \leq M \}, \quad \Gamma_2 = \{ M - iy : -\mu \leq y \leq \gamma \} \\
\Gamma_3 &= \{ -x - i\gamma : -M \leq x \leq M \}, \quad \Gamma_4 = \{ -M + iy : -\gamma \leq y \leq \mu \}.
\end{align*}
\]

By Theorem 3.1, \( \partial h_1^{(1)}(\omega/c) \) has \( l+1 \) zeros in the lower half plane and we can therefore choose \( M > 0 \) and \( \gamma > 0 \) such that \( \Gamma^{M,µ,γ} \) encircles these zeros.

Claim 1: \( \int_{\Gamma_2} + \int_{\Gamma_4} \to 0 \) as \( M \to \infty \)

This claim is a direct application of the asymptotics (4.9) on the Hankel function ratio in the integrand. Namely,

\[
\left| \int_{\Gamma_{2,4}} \int_0^{t-(r-1)/c} e^{-i\omega(t-s)} \partial_s f_l^m e^{i\omega/c} ds \frac{h_1^{(1)}(\omega r/c)}{\omega/c} \partial h_1^{(1)}(\omega/c) d\omega \right| \leq \left| \int_{\Gamma_{2,4}} \int_0^{t-(r-1)/c} e^{\mu(t-s)} |\partial_s f_l^m| ds e^{-i\omega} \frac{2}{|\omega r/c|} \right| \leq \frac{C}{r M} e^{-\omega r/c} \int_0^{t-(r-1)/c} e^{\mu(t-s)} |\partial_s f_l^m| ds \to 0 \text{ as } M \to \infty.
\]

We therefore have

\[
U_{l,m}^{(1)}(r, t) = \sum_{k=0}^{l+1} \int_0^{t-(r-1)/c} e^{-i\omega_1, k(t-s)} \partial_s f_l^m(s) ds \text{ Res }_{\omega = \omega_1, k} \frac{h_1^{(1)}(\omega r/c)}{\omega/c} \partial h_1^{(1)}(\omega/c) + \frac{1}{2\pi i} \int_{\Gamma_3} \int_0^{t-(r-1)/c} e^{-i\omega(t-s)} \partial_s f_l^m(s) ds \frac{h_1^{(1)}(\omega r/c)}{\omega/c} \partial h_1^{(1)}(\omega/c) d\omega, \quad (4.26)
\]

where \( M > 0 \) sufficiently large. The size of \( M \) depends on \( l \), since the contour must enclose all \( l+1 \) poles of the integrand. This completes the proof of Claim 1.

Our goal is to deform the contour \( \Gamma_3 = \Gamma_3^{M,γ} \) downward, by sending \( \gamma \to \infty \), and to establish the following claim, from which Proposition 4.6 follows immediately.

Claim 2: The \( d\omega \) integral in (4.26) satisfies the estimate

\[
\sup_{M > 0} \left| \int_{\Gamma_3^{M,γ}} \cdots d\omega \right| \to 0, \quad \text{as } \gamma \to \infty. \quad (4.27)
\]

To prove Claim 2, we integrate by parts in \( s \), exploiting the oscillatory character of the integrand, in order to obtain absolutely convergent \( d\omega \) integrals.

\[
\int_{\Gamma_3^{M,γ}} \cdots d\omega = -\frac{1}{2\pi i} \left[ I_1(r, t) + I_2(r, t) + I_3(r, t) \right] \quad (4.28)
\]
where

\[ I_1(r, t) = \int_{-M-i\gamma}^{M-i\gamma} \int_0^{t-(r-1)/c} e^{-i\omega(t-s)} \partial_s^2 f_1^m(s) \, ds \frac{h_1^{(1)}(\omega r/c)}{(\omega^2/c) \partial h_1^{(1)}(\omega/c)} \, d\omega \quad (4.29a) \]

\[ I_2(r, t) = -\partial_t f_1^m(t - (r - 1)/c) \int_{-M-i\gamma}^{M-i\gamma} e^{-i\omega(r-1)/c} \frac{h_1^{(1)}(\omega r/c)}{(\omega^2/c) \partial h_1^{(1)}(\omega/c)} \, d\omega \quad (4.29b) \]

\[ I_3(r, t) = \partial_t f_1^m(0) \int_{-M-i\gamma}^{M-i\gamma} e^{-i\omega t} \frac{h_1^{(1)}(\omega r/c)}{(\omega^2/c) \partial h_1^{(1)}(\omega/c)} \, d\omega \quad (4.29c) \]

We conclude the proof of Claim 2, and therewith Proposition 4.5, by now showing for \( j = 1, 2, 3 \),

\[ \sup_{M>0} |I_j(r, t)| = O(\gamma^{-1}) \to 0, \text{ as } \gamma \to \infty. \quad (4.30) \]

**Estimation of \( I_1(r, t) \):** Choose \( \gamma > 2\eta \). Using the bound \( 4.28 \), we have

\[ |I_1(r, t)| \leq \int_{-M-i\gamma}^{M-i\gamma} \int_0^{t-(r-1)/c} e^{-i\omega(t-s)} e^{i\omega s} \partial_s^2 f_1^m(s) \, ds \frac{h_1^{(1)}(\omega r/c)}{(\omega^2/c) \partial h_1^{(1)}(\omega/c)} \, d\omega \]

\[ \leq \int_0^{t-(r-1)/c} e^{-\gamma(t-(r-1)/c) - \eta s} \, ds \int_{-M-i\gamma}^{M-i\gamma} \frac{1}{|\omega|^2} \, d\omega \]

\[ \leq \frac{8C}{\gamma r} e^{-\eta(t-(r-1)/c)} |\partial_t f|_{\alpha, \eta}. \]

**Estimation of \( I_2(r, t) \):**

\[ |I_2(r, t)| \leq |\partial_t f_1^m(t - (r - 1)/c)| \int_{-M-i\gamma}^{M-i\gamma} \frac{h_1^{(1)}(\omega r/c) e^{-i\omega(r-1)/c}}{(\omega/c) \partial h_1^{(1)}(\omega/c)} \, d\omega \]

\[ \leq \frac{2C}{\gamma r} e^{-\eta(t-(r-1)/c)} \int_{-M-i\gamma}^{M-i\gamma} \frac{1}{|\omega|^2} \, d\omega \]

\[ \leq \frac{2C}{\gamma r} e^{-\eta(t-(r-1)/c)} |\partial_t f|_{\alpha, \eta}. \]

**Estimation of \( I_3(r, t) \):** Lastly,

\[ |I_3(r, t)| \leq |\partial_t f_1^m(0)| \int_{-M-i\gamma}^{M-i\gamma} e^{-i\omega t} \frac{h_1^{(1)}(\omega r/c) e^{-i\omega(r-1)/c}}{(\omega/c) \partial h_1^{(1)}(\omega/c)} \, d\omega \]

\[ \leq \frac{2C}{\gamma r} |\partial_t f_1^m(0)| e^{-\gamma(t-(r-1)/c)} \int_{-M-i\gamma}^{M-i\gamma} \frac{1}{|\omega|^2} \, d\omega \]

\[ \leq \frac{2C}{\gamma r} e^{-\gamma(t-(r-1)/c)} |\partial_t f|_{\alpha, \eta}. \]

**Proof of Proposition 4.7** We must prove that \( U_{1,m}^{(2)}(r, t) \equiv 0 \). Note the following bound, for \( 3\omega > 0 \) and \( r < 1 + ct \):

\[ |F_{1,m}^{(2)}(\omega, t)| = \int_{t-(r-1)/c}^{\infty} e^{i\omega s} \partial_s f_1^m(s) \, ds \leq C|\omega|^{-1} e^{-(3\omega + \eta)(t-(r-1)/c)} |\partial_t f|_{\alpha, \eta}. \quad (4.31) \]

This follows from direct substitution and integration of the bound

\[ |e^{i\omega s} \partial_s f_1^m(s)| \leq e^{-3\omega s} e^{-\eta s} |\partial_t f|_{\alpha, \eta}. \]
We now use (4.31) to establish that \( U_{l,m}^{(2)}(r,t) \equiv 0 \). Indeed,
\[
|U_{l,m}^{(2)}(r,t)| = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega t} F_{l,m}^{(2)}(\omega,t) \frac{h_i^{(1)}(\omega r/c)}{\omega c} d\omega \right| \\
\leq \frac{2CC}{\pi r} \int_{-\infty}^{\infty} e^{\nu t} \frac{e^{-\nu (t-1/c)} e^{-(\nu+n)(t-1/c)}}{|\omega|^2} d\omega \left| \partial_i f \right|_{\alpha,\eta} \leq 2C \int_{-\infty}^{\infty} \frac{dz}{z^2 + \nu^2} dz \to 0
\]
as \( \nu \to \infty \), and thus, \( U_{l,m}^{(2)}(r,t) = 0 \). This completes the proof of Claim 2 and thus Proposition 4.5.

**Proof of the Resonance Expansion** (4.5). The resonance partial sum, defined as
\[
u^{(L)}(r, \Omega, t) = \sum_{l=0}^{L} \sum_{|m| \leq 1} u_{l,m}^{(r,t)} Y_{l,m}^{(1)}(\Omega)
\]
is
\[
u^{(L)} = i \sum_{l=0}^{L} \sum_{|m| \leq 1} \left[ \int_{0}^{t} e^{-\omega t} \partial_s f_i^{(r,t)}(s) ds \right] Y_{l,m}^{(1)}(\Omega)
\]
which gives Equation (4.6).

To study the convergence of \( \nu^{(L)}(r, \Omega, t) \) as \( L \) tends to infinity, consider the difference \( \nu^{(N)}(r, \Omega, t) - \nu^{(L)}(r, \Omega, t) \), where \( N \) and \( L \) are large with \( N > L \) and prove:

**Proposition 4.7.** Let \( |\Im \omega| = \min_{k,l} |\Im \omega_{l,k}| \) and pick \( \eta \in (0, |\Im \omega|) \). Assume \( \left| \partial_\alpha f \right|_{\alpha,\eta} \leq C \). Then, for \( |a| + |b| \leq 2 \),
\[
\left| \partial_\alpha \partial_b \left( \nu^{(N)}(x,t) - \nu^{(L)}(x,t) \right) \right| \leq C \left| \partial_\alpha f^{(N)} - \partial_\alpha f^{(L)} \right|_{\alpha,\eta} e^{-\eta (t-1/\omega)}
\]
which gives Equation (4.5).

Therefore, for any compact subset \( I_t \times \Omega_x \subset \mathbb{R}_t \times \mathbb{R}_x \), the set \( \{ \nu^{(L)} \} \) is a Cauchy sequence in \( C^2(I_t \times \Omega_x) \) and converges uniformly on \( I_t \times \Omega_x \) to a limit \( \nu(x,t) \), which is a classical solution of the initial-boundary value problem (4.1).

**Proof.** We begin by estimating the difference \( \nu^{(N)}(x,t) - \nu^{(L)}(x,t) \):
\[
\left| \nu^{(N)}(x,t) - \nu^{(L)}(x,t) \right| \leq \sum_{l=0}^{L} \sum_{|m| \leq 1} \left| Y_{l,m}^{(1)}(\Omega) \right| \left| \int_{0}^{t} e^{-\omega t} \partial_s f_i^{(r,t)}(s) ds \right| \frac{h_i^{(1)}(\omega l r/c)}{\omega c} \leq \sum_{l=0}^{L} \sum_{|m| \leq 1} \left| Y_{l,m}^{(1)}(\Omega) \right| \left| \partial f_i^{(r,t)} \right|_{\infty,\eta} \int_{0}^{t} e^{-\eta (t-1)} e^{-\eta s} ds \frac{h_i^{(1)}(\omega l r/c)}{\omega c} \leq C
\]
To complete the proof of Proposition 4.7, we require Proposition 4.3 and the following two propositions:
Proposition 4.8. There is a constant $C > 0$, independent of $r$, such that for $0 \leq a \leq 2$, and any $r > 1$,

$$\left| \frac{\partial_r^a \left( h_1^{(1)}(\omega_{l,k} r/c) \right)}{(\omega_{l,k}/c^2) \partial^2 h_1^{(1)}(\omega_{l,k}/c)} \right| \leq C \frac{(1 + l)^q \cdot l^a}{r}. \quad (4.34)$$

Proposition 4.9. $|\partial_\theta^a \partial_\phi^b Y_l^m(\theta, \phi)| = O(l^{1/2+n})$, for $|a| + |b| = n, \ n = 0, 1, 2$.

Proposition 4.8 is proved using the uniform asymptotic expansions for Hankel of large argument and large order and the asymptotic locations of the resonances $\{\omega_{l,k}\}$. The arguments of the type used in [41] and very close to our proof of Proposition 4.7 presented in Appendix F. Proposition 4.9 is a classical result of Calderon and Zygmund [5]. We now complete the proof using Propositions 4.4, 4.8 and 4.9. First, we bound the $(l, m)$ term of (4.33) as follows:

$$|Y_l^m(\Omega)| \cdot |\partial f_l^m|_{\infty, \eta} \leq \sum_{l=0}^{l+1} \int_0^{t-r^{-1}} e^{\omega_{l,k}(t-s)} e^{-\eta s} ds \left| \frac{h_1^{(1)}(\omega_{l,k} r/c)}{(\omega_{l,k}/c^2) \partial^2 h_1^{(1)}(\omega_{l,k}/c)} \right| \leq C l^{1/2} \times |\partial f_l^m|_{\infty, \eta} \times \left( \frac{l+1}{|3\omega_* - \eta|} e^{-\eta(t-r^{-1})} \right) \times \frac{(1 + l)^q}{r}.
$$

Summing this estimate over $l$ and $m$ yields:

$$\left| u^{(N)}(x, t) - u^{(L)}(x, t) \right| \leq \frac{C}{|3\omega_* - \eta|} \left[ \partial_t f^{(N)} - \partial_t f^{(L)} \right]_{q, \eta} \cdot r^{-1} e^{-\eta(t-r^{-1})}.
$$

Similarly, we have for any $a, b$ with $|a| + |b| \leq 2$, we obtain (4.32). This completes the proof of Proposition 4.7. Theorem 4.2 now follows by passing to the limit $N \to \infty$, which proves the resonance expansion bound on $u - u^{(L)}$, 4.7.

5. Linearized bubble dynamics: solution of the initial-boundary value problem, decay estimates & resonance expansion. We next state our main results on decay estimates and resonance expansions for solutions of the initial-boundary value problem (1.1), (1.2). A key tool is the general result, Theorem 4.2, for the Neumann to Dirichlet map.

Theorem 5.1. Fix $\epsilon > 0$ and arbitrary. Suppose $\|\beta(t = 0)\| = \sum_{l=0}^\infty \sum_{m=0}^\infty |(1 + l)^{2+\epsilon} |\beta_l^m(0)| < \infty$. There exists a unique solution $\Psi(r, \Omega, t)$, $\beta(\Omega, t)$, defined for $r > 1, \ \Omega \in S^2$, which solves the initial value problem (1.1).

(i) Decay Estimates: The solution satisfies the following decay estimates:

$$|\beta(\Omega, t)| \leq C \|\beta(t = 0)\| e^{-|3\lambda_*(\epsilon)|t}, \quad \Omega \in S^2, \quad (5.1)$$

$$|\Psi(x, t)| \leq \begin{cases} C \frac{1}{|x|} e^{-|3\lambda_*(\epsilon)|t - \min_{|3\omega_*|} \{|3\omega_*| \right| \right| t \leq (1 + \epsilon^{-1} t), \quad |x| \leq 1 + \epsilon^{-1} t, \\ 0, \quad |x| > 1 + \epsilon^{-1} t. \end{cases} \quad (5.2)
$$

Here, $3\lambda_* < \epsilon < 0$ and $3\omega_* < \epsilon < 0$ are, respectively, imaginary parts of the deformation and rigid scattering resonances, which are close to the real axis; see Theorems 4.7 and 5.3.
(ii) **Resonance Expansion of $\beta$:** Define the resonance partial sum for the bubble surface perturbation:

$$
\beta^{(L)}(\Omega, t) = \sum_{l=0}^{L} \sum_{|m| \leq l} \eta^m_l(t) Y_l^m(\Omega)
$$

$$
= \sum_{l=0}^{L} \sum_{|m| \leq l} \eta^m_l(0) \sum_{j=1}^{l+2} \lambda_{l,j} e^{-i\lambda_{l,j} t} \text{Res}_{\lambda=\lambda_{l,j}} \left[ \lambda^2 + \rho_l G_l(\epsilon \lambda) \right]^{-1} Y_l^m(\Omega),
$$

(5.3)

where $\{\lambda_{l,j}\}$, the solutions of $\lambda^2 + \rho_l G_l(\epsilon \lambda) = 0$, are the deformation scattering resonances; see [3,7] and Theorem 3.2. The limit $\beta(\Omega, t) = \lim_{L \to \infty} \beta^{(L)}(\Omega, t)$ exists and converges uniformly in $C^2(S^2)$.

(iii) **Resonance Expansion of $\Psi$:** Given $\beta$, defined in (ii), define

$$
\Psi^{(L)}(r, \Omega, t) = \sum_{l=0}^{L} \sum_{|m| \leq l} \lambda^{m}(r, \Omega) \sum_{k=1}^{l+1} \int_0^{t \pm (r-1)} e^{-i\omega_{l,k}(t-s)} \partial_s \eta^m_l(s) \, ds \frac{h_l^{(1)}(i\omega_{l,k}r)}{(e^{2\omega_{l,k}} - \partial^2 h_l^{(1)}(i\omega_{l,k}))},
$$

(5.4)

where $\{\omega_{l,k}\}$ denotes the set of rigid Neumann scattering resonances; see Theorem 3.1. The limit $\Psi(r, \Omega, t) = \lim_{L \to \infty} \Psi^{(L)}(r, \Omega, t)$ exists and converges in $C^2$ on any compact subset $\Omega \subset \{|x| > 1\}$.

**Proof of Theorem 5.7.** We apply the results of Section 4 with $u(r, \Omega, t) = \Psi(r, \Omega, t)$, $f(\Omega, t) = \beta(\Omega, t)$ and $c = \epsilon^{-1}$, with $\partial_r \Psi(1, \Omega, t) = \partial_r \beta(\Omega, t)$ as Neumann data, coming from the linearized kinematic boundary condition and obtain:

$$
\Psi(r, \Omega, t) = \sum_{l=0, l \neq 1}^{\infty} \sum_{|m| \leq l} \Psi_l^m(r, t) Y_l^m(\Omega),
$$

(5.5)

$$
\Psi_l^m(\Omega, r, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ip} \partial_t \eta^m_l(p) \frac{h_l^{(1)}(ipr)}{(ip) \partial h_l^{(1)}(ip)} \, dp, \quad l = 0, 2, 3, \ldots, |m| \leq l
$$

(5.6)

Setting $\omega = ip$ in (5.6) and applying Proposition 4.3 we obtain

$$
\Psi_l^m(\Omega, r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \int_0^{t \pm (r-1)} e^{i\omega s} \partial_s \eta^m_l(s) \, ds \frac{1}{G_l(\epsilon \omega)} \, d\omega.
$$

(5.7)

Our goal is to obtain a closed equation for $\beta$. This we obtain by substitution of (5.5), (5.7) into the linearized dynamic boundary condition (1.8c). Evaluating (5.7) at $r = 1$ and expressing it as a time convolution, we get, for each $l = 0, 2, 3, \ldots$ and $|m| \leq l$,

$$
\partial_t \Psi_l^m(1, \Omega, t) = \partial_t \int_0^t \left( \mathcal{L}^{-1} \frac{1}{G_l(\epsilon s)} \right) (t-s) \partial_s \eta^m_l(s) \, ds,
$$

(5.8)

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Now taking the Laplace transform of equation (5.8) we obtain for the Laplace transform of the linearized dynamic boundary condition (1.8c):

$$
\frac{1}{G_l(\epsilon p)} \left[ p^2 \tilde{\beta}_l^m(p) - p \tilde{\beta}_l^m(0) \right] = \left[ 3 \gamma \left( \frac{C_a}{\pi} + \frac{2}{\omega_0} \right) \delta_{l0} + \frac{1}{\omega_0} (l+2)(l-1) \right] \tilde{\beta}_l^m(p),
$$

(5.9)
for \( l = 0, 2, 3, \ldots, \) and \(|m| \leq l\). In obtaining (5.9), we have used well-known relations for the Laplace transform of a time-derivative and a time-convolution, and that 
\[-\Delta s Y_l^m = l(l + 1)Y_l^m.\]
Next, solving (5.12) for \( \beta_l^m(p) \) yields
\[\beta_l^m(p) = \beta_l^m(0) \frac{1}{p^2 - r_l G_l(iep)} \]  
(5.10)
where \( r_l = \frac{1}{\omega_e}(l + 2)(l - 1) + 3\gamma \left( \frac{C_0}{\omega_e} + \frac{2}{\omega_e} \right) \delta_{00}. \) Inverting the Laplace transform and changing variables to \( \lambda = ip, \) we have
\[\beta_l^m(t) = \beta_l^m(0) \frac{1}{2\pi} \int_{-\infty + i\mu}^{\infty + i\mu} \frac{ie^{i\lambda t}}{\lambda^2 + r_l G_l(i\lambda)} d\lambda, \quad \mu > 0. \]  
(5.11)
Thus,
\[\beta(t) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} \beta_l^m(0) \frac{1}{2\pi} \int_{-\infty + i\mu}^{\infty + i\mu} \frac{ie^{i\lambda t}}{\lambda^2 + r_l G_l(i\lambda)} d\lambda Y_l^m(\Omega), \quad \mu > 0. \]  
(5.12)
We will show (1) each term in the series (5.12) can be expressed as a finite sum of residues at poles in the lower half \( \lambda \)-plane and (2) under suitable regularity hypotheses on \( \beta(\Omega, 0) \), the series converges uniformly in \( C^2(S^1_\Omega \times \mathbb{R}^+) \).

Evaluation of \( \beta_l^m(t) \), in terms of residues: The poles of the integrand of (5.11) are precisely the deformation resonances, described in Theorem 4.3. In particular, there are \( l + 2 \) poles, zeros of \( \lambda^2 + r_l G_l(i\lambda) \).

Choose \( \Gamma^{M, \mu, \gamma} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \) a rectangular contour of the type used in the proof of Proposition 4.5 with \( M, \gamma, \mu \) so that \( \Gamma^{M, \mu, \gamma} \) encircles these \( l + 2 \) poles in the complex \( \lambda \)-plane. By the residue theorem
\[
\sum_{j=1}^{l+2} i\lambda_{l,j} e^{-i\lambda_{l,j} t} \text{Res}_{\lambda = \lambda_{l,j}} [\lambda^2 + r_l G_l(i\lambda)]^{-1} = -\frac{1}{2\pi i} \int_{\Gamma^{M, \mu, \gamma}} \frac{i\lambda e^{-i\lambda t}}{\lambda^2 + r_l G_l(i\lambda)} d\lambda. \]  
(5.13)
Note that
\[\beta_l^m(t) = \beta_l^m(0) \frac{1}{2\pi} \lim_{M \to \infty} \int_{\Gamma^{M, \mu, \gamma}} \cdots d\lambda, \]  
(5.14)
where \( \cdots \) denotes the integrand in (5.13). Thus, we would have
\[\beta_l^m(t) = \sum_{j=1}^{l+2} i\lambda_{l,j} e^{-i\lambda_{l,j} t} \text{Res}_{\lambda = \lambda_{l,j}} [\lambda^2 + r_l G_l(i\lambda)]^{-1}, \]  
(5.15)
as well as the expansion (5.3), if we can prove \( \int_{\Gamma_2, \cdots} d\lambda \to 0 \) as \( M \to \infty \), and \( \int_{\Gamma_j} \cdots d\lambda \to 0 \) as \( \gamma \to \infty \).

To prove \( \int_{\Gamma_2, \cdots} d\lambda \to 0 \) as \( M \to \infty \) we begin by noting (Appendix D.1) the identity
\[G_l(i\lambda) = -(l + 1) + i\epsilon \lambda + \frac{\epsilon \lambda p_l'(i\lambda)}{p_l(i\lambda)}. \]  
(5.16)
Thus \( G_l(i\lambda) \) to be asymptotically linear in \( \epsilon \lambda \). Thus \( M \) can be chosen large enough so that
\[
\frac{1}{|\lambda^2 + r_l G_l(i\lambda)|} \leq \frac{C_l}{|\lambda|^2}, \quad \text{for} \ \Re \lambda > M.
\]  
(5.17)
Therefore, we have
\[
\left| \int_{\Gamma_2} e^{-i\lambda t} \frac{\frac{i\lambda}{\lambda^2 + r_l G_l(\epsilon \lambda)}}{d\lambda} \right| \leq \int_{\Gamma_2} \left| e^{-i\lambda t} \frac{|C_l|}{|\lambda|^2} \right| d\lambda \leq C_l(\mu + \gamma) e^{\mu t} \frac{1}{M} \to 0 \text{ as } M \to \infty.
\]
A similar estimate holds for the integral along \( \Gamma_4 \).

We next bound the integral: \( \lim_{M \to \infty} \int_{\lambda=\gamma} \ldots d\lambda = \int_{\lambda=\gamma} \ldots d\lambda \) as \( \gamma \to \infty \). For \( \gamma = \Re \lambda \) sufficiently large,
\[
\left( \lambda + \frac{r_l G_l(\epsilon \lambda)}{\lambda} \right)^{-1} = \left( \lambda + \frac{r_l}{\lambda} \left[ -(l+1) + i\epsilon \lambda + \frac{\epsilon \lambda p_l(\epsilon \lambda)}{r_l(\epsilon \lambda)} \right] \right)^{-1}
\]
(5.18)

Therefore, \( \partial_{\lambda} \left( \lambda + \frac{r_l G_l(\epsilon \lambda)}{\lambda} \right)^{-1} = -\lambda^{-2} + O(\lambda^{-3}) \). Using these asymptotics and integration by parts we have
\[
\left| \int_{M - i\gamma}^{M - i\gamma} e^{-i\lambda t} \frac{\frac{i\lambda}{\lambda^2 + r_l G_l(\epsilon \lambda)}}{d\lambda} \right| \leq \frac{1}{t} \left| \int_{M - i\gamma}^{M - i\gamma} e^{-i\lambda t} \left( -\frac{1}{\lambda^2} + O(\lambda^{-3}) \right) d\lambda \right| + e^{-\gamma t} \frac{1}{\lambda} (1 + \epsilon r_l(\lambda^{-1})) \left| \int_{M - i\gamma}^{M - i\gamma} \right|
\]
(5.19)
\[
\leq \frac{\kappa_l e^{-\gamma t}}{t} \int_{M - i\gamma}^{M - i\gamma} \frac{1}{|\lambda|^2} d\lambda + O(M^{-1}) \leq \frac{\kappa_l e^{-\gamma t}}{t}, \quad t > 0.
\]

Recall that
\[
\int_{\lambda=\gamma} \ldots d\lambda = \lim_{M \to \infty} \int_{M - i\alpha}^{M - i\alpha} e^{-i\lambda t} \frac{\frac{i\lambda}{\lambda^2 + r_l G_l(\epsilon \lambda)}}{d\lambda}.
\]
Thus, passing to the limit in (5.19), first \( M \to \infty \) and then \( \gamma \to \infty \), we obtain that the contribution from the contour \( \Gamma_3 \) can be made arbitrarily small. This proves (5.15), the expression for \( \beta^m(t) \) as a sum over residues. Furthermore, multiplication of expression (5.15) by \( Y^m_l(\Omega) \) and summing over \( l \) from 0 up to \( L \) yields the expression for \( \beta^{(L)}(\Omega, t) \), the resonance expansion partial sum for the bubble surface perturbation, \( \beta(\Omega, t) \).

We now discuss the convergence of \( \beta^{(L)}(\Omega, t) \) as \( L \to \infty \). Let \( N > L \) and consider the difference:
\[
\beta^{(N)}(\Omega, t) - \beta^{(L)}(\Omega, t) = \sum_{l=L}^{N} \sum_{|n| \leq l} \beta^m_l(0) \sum_{j=1}^{l+2} e^{-i\lambda_l j t} \Res_{\lambda=\lambda_i, j} \frac{\lambda}{\lambda^2 + r_l G_l(\epsilon \lambda)} Y^m_l(\Omega).
\]
(5.20)
We require the following bound, proved in Appendix F:

**Proposition 5.2.** For \( l \) sufficiently large,

\[
\text{Res}_{\lambda = \lambda_l} \left( \frac{\lambda}{\lambda^2 + r_l G_l(\epsilon \lambda)} \right) = O \left( l^{-4/3} \right).
\]

(5.21)

To bound the difference in (5.20) we use Proposition 5.2 and Theorem 3.2 characterizing and estimating the deformation resonances, \( \lambda_{l,j} \) and the pointwise bounds on \( Y_{\nu}^m(\Omega) \) of Proposition 1.9. Let \( \partial_{\Omega}^\alpha \) denote any differential operator of order \( |\alpha| \), acting tangentially to \( S^2 \), i.e. \( \partial_{\Omega}^\alpha = \partial_{\Omega}^{\alpha_1} \partial_{\Omega}^{\alpha_2} \), \( |\alpha| = \alpha_1 + \alpha_2 \). Then,

\[
\left| \partial_{\Omega}^\alpha \left( \beta^{(N)}(\Omega, t) - \beta^{(L)}(\Omega, t) \right) \right| \leq C \ e^{-t \left| \Im \lambda_{\star}(\epsilon) \right|} \sum_{l=L}^N \sum_{\alpha \leq \ell} |\beta_m^\alpha(0)| \sum_{j=1}^{l+2} (1 + l)^{-\frac{4}{3}} l^{\alpha_1 + \frac{1}{3}}
\]

\[
\leq C \ e^{-t \left| \Im \lambda_{\star}(\epsilon) \right|} \sum_{l=L}^N \sum_{\alpha \leq \ell} (1 + l)^{2 + \frac{2}{3}} |\beta_m^\alpha(0)| \cdot e^{-|\Im \lambda_{\star}| t}.
\]

(5.22)

Thus, if \( \sum_{l=L}^N \sum_{\alpha \leq \ell} (1 + l)^{2 + \frac{2}{3}} |\beta_m^\alpha(0)| < \infty \), we have that \( \beta^{(L)}(\Omega, t) \) and derivatives in \( \Omega \) and \( t \) up to order two, converge uniformly in \( S^2 \times \mathbb{R}^+ \). Moreover, the \( \lim_{L \to \infty} \beta(\Omega, t) \) is in \( C^2(S^2 \times \mathbb{R}^+) \) and satisfies the decay estimate

\[
\left| \beta(\Omega, t) \right| \leq \sum_{l=L}^N \sum_{\alpha \leq \ell} (1 + l)^{2 + \frac{2}{3}} |\beta_m^\alpha(0)| \cdot e^{-|\Im \lambda_{\star}| t}.
\]

(5.23)

Now apply our result on the Neumann to Dirichlet map, Theorem 4.2, to conclude for \( |x| > 1, \ t > 0 \) that

\[
|\Psi \left( r, \Omega, t \right) | \leq C \ e^{-\min \{|\Im \lambda_{\star}(\epsilon)|, |\Im \omega_{\star}(\epsilon)|\} t} \left| \partial_t \beta_{\alpha, \star}(\epsilon \lambda) \right|
\]

\[
\leq C e^{-\min \{|\Im \lambda_{\star}(\epsilon)|\} t} \sum_{l=L}^N \sum_{\alpha \leq \ell} (1 + l)^{2 + \frac{2}{3}} |\beta_m^\alpha(0)| .
\]

(5.24)

The second inequality in (5.24) follows since \( |\Im \lambda_{\star}(\epsilon)| < |\Im \omega_{\star}(\epsilon)| = O(\epsilon^{-1}) \) for \( \epsilon \) sufficiently small. Moreover, \( (\Psi \left( r, \Omega, t \right), \beta(\Omega, t)) \) is a classical solution of the initial value problem.

6. Scattering Resonance Energies — Statements of Detailed Results.

As discussed in Section 5 two families of scattering resonance energies arise in the analysis of the linearized bubble dynamics near the spherical equilibrium state:

1. Rigid resonance energies, \( \{ \omega_{l,k} \} \), solutions of \( \partial h_{l(1)}^{(1)}(\epsilon \omega) = 0 \).
2. Deformation resonance energies, \( \{ \lambda_{l,k} \} \), solutions of

\[
\lambda^2 + r_l G_l(\epsilon \lambda) = 0, \quad G_l(z) \equiv \frac{z \partial h_{l(1)}^{(1)}(z)}{h_{l(1)}^{(1)}(z)}, \quad r_l \text{ is given by } (1.11).
\]

(6.1)

Both classes of resonances depend on \( \epsilon \). The incompressible limit, \( \epsilon \to 0 \), is a singular limit. Indeed our results show that for each \( l = 0, 2, 3, \ldots \), as \( \epsilon \) tends to zero, the imaginary parts of all \( l + 1 \) rigid resonances and of \( l \) of the \( l + 2 \) deformation
resonances lie near circular arcs in the lower half plane and have imaginary parts which tend to minus infinity. The remaining two deformation resonances are just below and converge to the real axis as \( \epsilon \to 0 \); see Figure 1.1 of the Introduction. The imaginary parts are \( \mathcal{O}(\epsilon^{-3} \exp(\kappa \epsilon^{-2})) \) as \( \epsilon \) tends to zero.

6.1. Deformation resonances; \( \{ \lambda_{l,j}(\epsilon) \} \), such that \( \lambda^2 + r_l G_l(\epsilon \lambda) = 0 \).

**Theorem 6.1.** Fix \( \epsilon > 0 \) and arbitrary.

(i) There are \( l + 2 \) solutions of

\[
\lambda^2 + r_l G_l(\epsilon \lambda) = 0,
\]

denoted \( \{ \lambda_{l,j}(\epsilon) \} \), for \( l = 0, 2, 3, \ldots \) and \( j = 1, \ldots, l + 2 \). These are the deformation resonance energies.

(ii) \( \{ \lambda_{l,j}(\epsilon) \} \) are symmetric about the imaginary axis and satisfy \( \Im \lambda_{l,j} < 0 \).

**Theorem 6.2** (Asymptotics of Rayleigh deformation resonances, \( \lambda^\pm_l(\epsilon) \) for small Mach number, \( \epsilon \)).

(i) For \( l = 0 \), the two deformation resonances are located in the lower half plane, a distance \( \mathcal{O}(\epsilon) \) from the real axis, are given by:

\[
\lambda_0^\pm(\epsilon) = \pm \sqrt{r_0 - \frac{r_0^2}{4} - i \frac{r_0}{2} \epsilon}, \quad r_0 = \frac{3\gamma}{2} C a + 2(3\gamma - 1) \frac{1}{\text{We}}. \tag{6.3}
\]

(ii) Fix \( L_\ast > 2 \). There exists \( \epsilon^*(L_\ast) \) such that for \( \epsilon \leq \epsilon^* \) and \( 2 \leq l \leq L_\ast \), two of the \( l + 2 \) deformation resonances (see Theorem 6.1), are located very slightly below the real axis and given by: \( \lambda^\pm_l = \Re \lambda^\pm_l(\epsilon) + i\Im \lambda^\pm_l(\epsilon) \), where

\[
\Re \lambda^\pm_l(\epsilon) = \pm \sqrt{\frac{1}{\text{We}} (l + 2)(l - 1)(l + 1)} \left[ 1 - \frac{(l + 2)(l - 1)}{2(2l - 1)} \left( \frac{\epsilon}{\sqrt{\text{We}}} \right)^2 + \mathcal{O}_{l, \text{real}}(\epsilon^4) \right],
\]

\[
\Im \lambda^\pm_l(\epsilon) = -\frac{1}{\epsilon} \frac{1}{2} [(l + 2)(l - 1)]^{l+1} (l + 1)^l \left[ \frac{2l!}{(2l)!} \right]^2 \left( \frac{\epsilon}{\sqrt{\text{We}}} \right)^{2l+2} + \mathcal{O}_l \left( [\epsilon \text{We}^{-1/2}]^{2l+4} \right). \tag{6.4}
\]

(iii) In the incompressible limit, \( \epsilon \to 0^+ \), the deformation resonance equation (6.2), which has \( l + 2 \) roots in the lower half plane for \( \epsilon > 0 \), reduces to the quadratic equation:

\[
\lambda^2 + r_l G_l(0) = \lambda^2 - 3\gamma \left( \frac{C a}{2} + \frac{2}{\text{We}} \right) \delta_0 - \frac{1}{\text{We}} (l + 2)(l + 1)(l - 1) = 0, \quad l = 0, 2, 3, \ldots
\]

and has two real roots:

\[
\lambda^\pm_l(0) = \pm \sqrt{3\gamma \left( \frac{C a}{2} + \frac{2}{\text{We}} \right) \delta_0 + \frac{1}{\text{We}} (l + 2)(l + 1)(l - 1), \quad l = 0, 2, 3, \ldots \tag{6.5}
\]

which are real frequencies. The corresponding solutions of the linearized time-dependent perturbation equations:

\[
\Psi^\pm_l = A^\pm_l e^{-i\lambda^\pm_l(0)t} Y^m_l(\Omega) \rho^{-l-1}, \quad \beta^\pm_l = B^\pm_l e^{-i\lambda^\pm_l(0)t} Y^m_l(\Omega), \quad l = 0, 2, \ldots
\]

are undamped and oscillatory solutions.
Remark 6.1. In Section 7, we present an asymptotic analysis and numerical computations offering strong evidence for the scattering resonance energy nearest to the real axis being a distance of order

$$O\left(\epsilon^{-1} e^{-\kappa \text{We}^{-2}}\right), \quad \kappa > 0,$$

occurring at order $$l_*(\epsilon) = O(\text{We}^{-2}).$$ (6.6)

Finally, the following theorem establishes that for any fixed $$\epsilon$$ and sufficiently large $$l$$, the deformation resonances (excluding one) are distributed near a circular arc. In particular, this implies that for any fixed $$\epsilon$$ scattering resonances are in the lower half plane and uniformly bounded away from the real axis.

Theorem 6.3. Fix $$\epsilon > 0$$ and arbitrary. The $$l + 2$$ deformation resonances are approximated as follows:

(i) There are $$l + 1$$ resonances near zeros of $$\partial H_{\nu}^{(1)}(\epsilon \lambda)$$; see Equation (6.17). Precisely, there exists $$K_1 > 0$$ such that for $$l \geq K_1 (\epsilon^{-2} \text{We})^{3/(1-3q)}$$, $$0 < q < 1/3$$, those in the fourth quadrant can be expressed as:

$$\lambda_{l,s}(\epsilon) = \frac{l + 1/2}{\epsilon} \left[ 1 + 2^{-1/3} e^{-2\pi i/3(l + 1/2)^{-2/3}} |\eta_s^l| + 3 2^{-1/3} e^{-4\pi i/3(l + 1/2)^{-4/3}} |\eta_s^l|'' + \mathcal{O}\left(s^{-1/3(l + 1/2)^{-2/3-q}}\right) \right],$$

(6.7)

for $$s = 1, 2, \ldots, \lfloor (l + 1)/2 \rfloor + 1$$. Here, $$\eta_s^l$$ is the $$s$$th zero of $$\partial_z \text{Ai}(z)$$; $$0 > \eta_1 > \eta_2 > \cdots$$. Note the error is uniform in $$\epsilon$$. These resonances come in pairs, symmetric about the imaginary axis, $$\{\lambda_{l,s}, -\lambda_{l,s}\}$$. For even $$l$$, $$\lambda_{l,(l+2)/2}$$ is a simple resonance on the negative imaginary axis.

(ii) The remaining deformation resonance is located on the negative imaginary axis and can be expressed, for some $$K_2$$ sufficiently large and $$l \geq K_2 e^{-2} \text{We}$$, as

$$\lambda_{l,l+2}(\epsilon) = \frac{-l + 1/2}{\epsilon} \left[ c^2(l + 1/2) \text{We} + \frac{1}{2} \frac{\text{We}}{c^2(l + 1/2)} + \mathcal{O}\left(c^{-1}(l + 1/2)^{-3}\right) \right].$$

(6.8)

(iii) The set of deformation resonance energies is uniformly bounded away from $$\mathbb{R}$$. That is, for some $$l_*(\epsilon), j_*(\epsilon),

$$\Im \lambda_{l,j}(\epsilon) \leq \Im \lambda_{l-j,\epsilon}(\epsilon) \equiv \Im \lambda_*(\epsilon) < 0, \quad \text{all } l \geq 0, \quad |j| \leq l + 2.$$ (6.9)

(iv) Moreover, there exists $$C > 0$$ such that as $$l \to \infty$$,

$$\lambda_{l,j}(\epsilon) = e^{-1} \mathcal{O}(l), \quad C^{-1/3} \leq |\Im \lambda_{l,j}(\epsilon)|.$$

We conclude this section with a proof of Theorem 6.1. Theorem 6.2 is proved in Section 8. Theorem 6.3 is proved in Section 9. To prove Theorem 6.1 we use the following rational function representation of $$G_{i}(z) = \partial h_{1}^{(1)}(z)/h_{1}^{(1)}(z)$$, which follows from (D.5):

Proposition 6.4. $$G_{i}(z) = -(l + 1) + iz + \frac{zp_{l}(iz)}{p_{l}(iz)}$$.

Proof of Theorem 6.1.

(i) We first observe that there $$l + 2$$ scattering resonance energies. To see this we observe, by Proposition 6.4, that $$\lambda^2 + r_{l} G_{i}(\epsilon \lambda) = 0$$ is equivalent to

$$\lambda^2 + r_{l} \left[ -(l + 1) + i\epsilon \lambda + \frac{\epsilon \lambda p_{l}'(\epsilon \lambda)}{p_{l}(\epsilon \lambda)} \right] = 0,$$

(6.10)
which clearly has \( l + 2 \) roots since \( p_l(z) \) is a polynomial of degree \( l \). For \( \epsilon = 0 \), the equation reduces to \( \lambda^2 = r_l(l + 1) \), yielding two real frequencies, corresponding to the non-decaying Rayleigh modes of the incompressible limit problem [13].

(ii) We now prove that if \( \lambda \) is a solution of (6.10), then so is \( -\overline{\lambda} \). By (D.4)

\[
\lambda^2 + r_l G_l(\epsilon \lambda) = 0 \iff \lambda^2 + r_l \left[ -\frac{1}{2} + \frac{\epsilon \lambda \partial H^{(1)}_{\nu}(\epsilon \lambda)}{H^{(1)}_{\nu}(\epsilon \lambda)} \right] = 0, \quad \nu = l + 1/2. \tag{6.11}
\]

Using (D.10) we have \( \lambda^2 + r_l \left[ -\frac{1}{2} + \frac{\epsilon \lambda}{H^{(1)}_{\nu}(\epsilon \lambda)} \left( \nu \frac{H^{(1)}_{\nu+1}(\epsilon \lambda)}{\epsilon \lambda} - H^{(1)}_{\nu}(\epsilon \lambda) \right) \right] = 0. \) Taking the complex conjugate and using the analytic continuation formulæ (D.7) and (D.8) we obtain

\[
0 = \overline{\lambda}^2 + r_l \left[ -\frac{1}{2} + \epsilon \nu - \epsilon \lambda \frac{H^{(1)}_{\nu+1}(\epsilon \lambda)}{H^{(1)}_{\nu}(\epsilon \lambda)} \right] = \left( -\overline{\lambda} \right)^2 + r_l \left[ -\frac{1}{2} + \epsilon \nu + \epsilon \lambda \frac{H^{(1)}_{\nu+1}(\epsilon (-\overline{\lambda}))}{H^{(1)}_{\nu}(\epsilon (-\overline{\lambda}))} \right]. \tag{6.12}
\]

Thus, \( -\overline{\lambda} \) satisfies (6.11) and is therefore a scattering resonance.

We now prove that all resonances lie in the open lower half plane, \( \Im \lambda < 0 \). We begin, following [11], by showing that there are no resonances along the real axis. Let \( \nu = l + 1/2 \). Using the definition of \( H^{(1)}_{\nu} \) in terms of Bessel functions, equation (6.11) for the resonances can be written

\[
0 = \left( \frac{\lambda^2}{r_l} - \frac{1}{2} \right) \left[ J_{\nu}(\epsilon \lambda) + i Y_{\nu}(\epsilon \lambda) \right] + (\epsilon \lambda) \left[ \partial J_{\nu}(\epsilon \lambda) + i \partial Y_{\nu}(\epsilon \lambda) \right]. \tag{6.13}
\]

Now assume \( \lambda = x/\epsilon \), where \( x \) is real and nonzero, is a resonance and we show that this leads to a contradiction. Indeed,

\[
0 = \left( \frac{1}{\epsilon^2 r_l} - \frac{1}{2} \right) \left[ J_{\nu}(x) + i Y_{\nu}(x) \right] + x \left[ J'_{\nu}(x) + i Y'_{\nu}(x) \right] \tag{6.14}
\]

\[
= \left[ \left( \frac{1}{\epsilon^2 r_l} - \frac{1}{2} \right) J_{\nu}(x) + x J'_{\nu}(x) \right] + i \left[ \left( \frac{1}{\epsilon^2 r_l} - \frac{1}{2} \right) Y_{\nu}(x) + x Y'_{\nu}(x) \right]. \tag{6.15}
\]

Since \( x \in \mathbb{R} \) both quantities in braces equal zero. Therefore, we have

\[
\left( \frac{1}{\epsilon^2 r_l} - \frac{1}{2} \right) = -\frac{x J'_{\nu}(x)}{J_{\nu}(x)} = -\frac{x Y'_{\nu}(x)}{Y_{\nu}(x)}
\]

implying \( -x \left[ J'_{\nu}(x)Y_{\nu}(x) - Y'_{\nu}(x)J_{\nu}(x) \right] = 0. \) The expression in square bracket is the Wronskian of two linearly independent solutions of Bessel’s equation and satisfies the following identity [11]:

\[
J'_{\nu}(x)Y_{\nu}(x) - Y'_{\nu}(x)J_{\nu}(x) = \frac{2}{\pi x} \neq 0. \tag{6.16}
\]

Hence, for \( x \neq 0 \), we have a contradiction and thus \( \Im \lambda \neq 0 \).

Next, we claim that there are no resonances in the upper half plane, and therefore \( \Im \lambda < 0 \). For if \( \lambda \) is a scattering resonance with \( \Im \lambda > 0 \), this would give rise to a solution of the initial-boundary value problem for the linearized bubble perturbation
system (1.8) of the form: \( \Psi(r, \Omega) = e^{-i\lambda t} h^{(1)}(\epsilon \lambda r) \). Since \( \Im \lambda > 0 \), this solution would decay to zero exponentially as \( r = |x| \to \infty \) (\( h^{(1)}(z) \) is \( e^{iz} \) times a rational function of \( z \)), be smooth and be exponentially growing in \( t \). This contradicts conservation of energy, Proposition 2.1. Thus, \( \Im \lambda \leq 0 \) and by the above discussion \( \Im \lambda < 0 \).

7. Longest-lived mode; formal asymptotics and precision numerics.

From Theorems 6.1 to 6.3 we know that for any finite (not necessarily small) \( \epsilon \), there is a strip:

\[
\Im \lambda^\pm(\epsilon) < \Im \lambda < 0,
\]

in which there are no scattering resonances. Moreover, the incompressible limit spectral problem, \( \epsilon = 0 \), has eigenvalues only at the real Rayleigh frequency pairs, given in Equations (6.3) and (6.5), and none in the lower half plane.

Question: How thin is the strip? That is, can we estimate \( \Im \lambda^\pm(\epsilon) \), for \( \epsilon \) small?

Figure 1.1 displays the distribution of numerically computed Rayleigh resonances for a fixed \( \epsilon > 0 \) and several values of \( l \). The figure shows the resonance in the lower half plane (a pair, \( \lambda^\pm(\epsilon) \) by \( \lambda \mapsto -\lambda \) symmetry), which is closest to the real axis. These resonances correspond to the slowest time-decaying term in the resonance expansion and the resulting decay rate in Theorem 4.2.

Let \( l_*(\epsilon) \) denote the mode index of the resonances nearest the real axis and \( \lambda^\pm_*(\epsilon) = \lambda^\pm(l_*(\epsilon)) \) the resonance itself. In Equation (6.4) we display the leading order behavior of the \( \Im \lambda^\pm(\epsilon) \), for \( l \) not too large, and \( \epsilon \) small:

\[
\Im \lambda^\pm(\epsilon) \sim -i \mathcal{M}(l, \epsilon)
\]

where

\[
\mathcal{M}(l, \epsilon) = \frac{1}{2\epsilon} [(l + 2)(l - 1)]^{l+1} (l + 1)^l \left[ \frac{2^{l-1}}{(2l)!} \right]^2 \left( \frac{\epsilon}{\sqrt{\epsilon We}} \right)^{2(l+1)}
\]

We attempt to compute the \( l_*(\epsilon) \) and \( \Im \lambda^\pm_*(\epsilon) \sim -\mathcal{M}(l_*(\epsilon), \epsilon) \), by minimization of \( \mathcal{M}(l, \epsilon) \) over \( l \), treated as a continuous variable, while keeping \( \epsilon \) \( \sqrt{\epsilon We} \) fixed. The calculation presented below in Remark 7.2 suggests the following conjecture:

**Longest lived mode index:** There exists \( A, \epsilon_* > 0 \) such that for \( \epsilon \leq \epsilon_* \)

\[
l_*(\epsilon) \approx A \epsilon^{-2} \sqrt{\epsilon We}.
\]

**Real part and imaginary part (“reciprocal lifetime”) of longest lived resonance mode:** There exists \( r_1, r_1, r_2 > 0 \) such that \( \lambda^\pm_*(\epsilon) = \Re \lambda^\pm_*(\epsilon) + i \Im \lambda^\pm_*(\epsilon) \), where

\[
\Re \{ \lambda^\pm_*(\epsilon) \} \approx \pm \frac{r_1}{\epsilon} (\epsilon^{-2} \sqrt{\epsilon We}), \quad \Im \{ \lambda^\pm_*(\epsilon) \} \approx -\frac{r_2}{\epsilon} (\epsilon^{-2} \sqrt{\epsilon We}) \exp(-i_2 \epsilon^{-2} \sqrt{\epsilon We})
\]

**Remark 7.1.** \( |\Im \lambda^\pm_*(\epsilon)| \), the distance from \( \lambda^\pm_*(\epsilon) \) to the real axis, tends to zero very rapidly (beyond all orders in \( \epsilon \)) as \( \epsilon \to 0 \). In contrast, the radial resonance, \( \lambda_0(\epsilon) \), tends to the real axis only linearly in \( \epsilon \); see Theorem 6.2.
Numerical evidence supporting (7.3) and (7.4) is obtained by numerically computing the resonances for fixed $\epsilon$ and various $l$, and determining the optimal mode index, $l_*(\epsilon)$, and resonance $\lambda_*^{\text{numerical}}(\epsilon)$. The Mathematica function FindRoot with options AccuracyGoal -> 20, PrecisionGoal -> 20, WorkingPrecision -> 2000, and MaxIterations -> 1500 were used to compute the resonances. Figure 7.1 shows the numerically computed optimal mode along with a numerical fit of the data. Note the expressions in Equations (7.3) and (7.4) are the leading terms of the numerical fit functions.

**Remark 7.2.** Formally minimizing $M(l, \epsilon)$ over $l \geq 2$ for fixed $\epsilon \text{We}^{-1}$, will approximately determine the longest lived mode and its lifetime. The optimization can be performed by treating $l$ as a continuous index and applying Stirling’s formula, $n! \approx \sqrt{2\pi n} (n/e)^n$, $n \gg 1$. This leads to a formal approximation of $l_*(\epsilon)$, the angular momentum index of the most slowly decaying mode, and $\lambda_*^{\pm}(\epsilon) = \Re \lambda_*^{\pm}(\epsilon) + i\Im \lambda_*^{\pm}(\epsilon)$, $\Im \lambda_*^{\pm} < 0$, the corresponding scattering resonance energies. In particular, we obtain: $l_*(\epsilon) \sim \frac{1}{e^2} \epsilon^{-2} \text{We}$, $\Re \lambda_*^{\pm}(\epsilon) \sim \pm 1 \left( \frac{4}{\pi^3} \right)^{1/3} (\epsilon^{-2} \text{We})$, $\Im \lambda_*^{\pm}(\epsilon) \sim -\frac{1}{\epsilon} (\epsilon^{-2} \text{We}) \exp(-\frac{4}{\pi^3} \epsilon^{-2} \text{We})$.

**8. Rayleigh deformation resonance analysis; Proof of Theorem 6.2.** In this section we prove the assertions of Theorem 6.2 concerning the Rayleigh resonances, $\lambda_*^{\pm}(\epsilon)$, $l = 0, 2, 3, \ldots$ appearing very close to the real axis for $\epsilon$ small.

The main ingredients of the proof of Theorem 6.2 are the following several propositions, which establish the detailed structure of the Taylor expansion of $\lambda^2 + r_t G(\epsilon \lambda) = 0$ for $\lambda$ bounded and $\epsilon$ sufficiently small. This will enable the explicit determination of the Taylor expansion of the Rayleigh resonances, $\lambda_*^{\pm}(\epsilon)$, and in particular their
imaginary parts.

**Proposition 8.1.** For $\varepsilon$ small, the equation $\lambda^2 + r_l G_l(\varepsilon \lambda) = 0$ can be written as

$$
\lambda^2 + r_l \left[ -(l + 1) + \sum_{j=1}^{\infty} \frac{(\varepsilon \lambda)^{2j}}{(2j)!} G_l^{(2j)}(0) + \sum_{j=1}^{\infty} \frac{(\varepsilon \lambda)^{2j+1}}{(2j+1)!} G_l^{(2j+1)}(0) \right] = 0, \quad (8.1)
$$

where $A(\lambda)$ is even and real for real $\lambda$ and $B(\lambda)$ is odd and purely imaginary for real $\lambda$.

**Proof.** This proposition follows by Taylor expansion of $G_l(z)$ about $z = 0$ and the following:

**Claim 8.2.** For all $j = 0, 1, 2, \ldots$

$$
\Im \left\{ G_l^{(2j)}(0) \right\} = 0, \quad \Re \left\{ G_l^{(2j+1)}(0) \right\} = 0. \quad (8.2)
$$

This claim follows directly from the observation that

$$
G_l(z) \text{ is real on the imaginary axis.} \quad (8.3)
$$

To prove $\text{(8.2)}$, we consider $G_l(iy), \ y \in \mathbb{R}$. By Proposition 6.4, it suffices to show that the ratio $p_l(iy)/p_l(y)$ is purely imaginary. By (8.2) we have

$$
p_l(iy) = \sum_{n=0}^{l} \frac{(2l-n)!}{2^{l-n}(l-n)!} y^n, \quad (8.4)
$$

which is purely imaginary,

$$
p_l'(iy) = -\sum_{n=0}^{l} \frac{(2l-n)!}{2^{l-n}(l-n)!} y^{n-1}, \quad (8.4)
$$

which is real.

The ratio is therefore purely imaginary, which implies that $G_l(iy)$ is real for $y$ real.

We now prove Claim 8.2 by induction. Consider the rational representation of $G_l$ in Proposition 6.4. As noted in (8.3), $G_l(0) = -(l + 1)$. Moreover, for $y \in \mathbb{R}$

$$
\partial G_l(0) = \lim_{y \to 0} \frac{G_l(iy) - G_l(0)}{iy} = -i \lim_{y \to 0} \frac{G_l(iy) - G_l(0)}{y} \in i\mathbb{R},
$$

since $G_l(iy) \in \mathbb{R}$ for $y \in \mathbb{R}$.

Next, we make the induction hypothesis:

$$
\Im \left( \frac{d^{2j-2}G_l}{dz^{2j-2}}(0) \right) = 0, \quad \Re \left( \frac{d^{2j-1}G_l}{dz^{2j-1}}(0) \right) = 0 \quad (8.5)
$$

and seek to prove the assertion (8.5) with $j$ replaced by $j + 1$.

From the Taylor series of $G_l(z)$ we have:

$$
\frac{d^{2j}G_l}{dz^{2j}}(0) = \lim_{y \to 0} \frac{(2j)!}{(iy)^{2j}} \left[ G_l(iy) - \sum_{k=1}^{j} \frac{(iy)^{2k-2}}{(2k-2)!} \frac{d^{2k-2}G_l}{dz^{2k-2}}(0) - \sum_{k=1}^{j} \frac{(iy)^{2k-1}}{(2k-1)!} \frac{d^{2k-1}G_l}{dz^{2k-1}}(0) \right]. \quad (8.6)
$$

Thus, $\partial^{2j}G_l(0) = 0$ is real by induction hypothesis (8.5).
Similarly, we have
\[
\frac{d^{2j+1} G_l}{dz^{2j+1}}(0) = \lim_{y \to 0} \frac{(2j+1)!}{(iy)^{2j+1}} \left[ G_l(iy) - \sum_{k=1}^{j} \frac{(iy)^{2k} d^{2k} G_l}{d z^{2k}}(0) - \sum_{k=1}^{j} \frac{(iy)^{2k-1} d^{2k-1} G_l}{d z^{2k-1}}(0) \right]
\]
(8.7)

which is purely imaginary by induction hypothesis and since \( \partial^2 G_l(0) = 0 \). This completes the proof of Claim 8.2 and thus Proposition 8.1. \( \square \)

**Proposition 8.3**. There is a positive integer \( 2 \leq q_* < \infty \), for which Equation (8.1) is of the form:
\[
\lambda^2 + r_l \left[ -(l + 1) + \sum_{j=1}^{\infty} \frac{(\varepsilon \lambda)^{2j}}{(2j)!} G_l^{(2j)}(0) + \frac{(\varepsilon \lambda)^{2q_*+1}}{(2q_* + 1)!} G_l^{(2q_*+1)}(0) + \mathcal{O}(\varepsilon^{2q_*+3}) \right] = 0,
\]
where \( \mathcal{A}(\lambda) \) is even and real-valued for real \( \lambda \) and \( \mathcal{B}(\lambda) \) is odd and imaginary-valued for real \( \lambda \).

**Proof of Proposition 8.3**. We begin by observing that \( G_l(z) \) is not an even function of \( z \). This is a direct consequence of the asymptotic behavior of \( G_l(x) \) as \( x \to \pm \infty \):
\[
\begin{align*}
G_l(x) & \approx -(l + 1) + ix + l = -1 + ix, \quad x \gg 1 \\
G_l(-x) & \approx -(l + 1) - ix - l = -(2l + 1) - ix, \quad x \gg 1.
\end{align*}
\]
Therefore, there is a first non-zero odd derivative, Taylor coefficient. We call the order of this coefficient, \( q_* \). The next result asserts that \( q_* = l \). \( \square \)

**Proposition 8.4**.
\[
G_l^{(2k+1)}(0) = \begin{cases} 
    i (2l + 1)! \left( \frac{x^l}{(2l)!} \right)^2, & k = l, \\
    0, & 0 \leq k \leq l - 1.
\end{cases}
\]
(8.10)

**Proof of Proposition 8.4**. Since we are interested in the odd derivatives of \( G_l \) at the origin we calculate:
\[
\begin{align*}
G_l(z) - G_l(-z) &= -(l + 1) + i z + \frac{z p'_l(z)}{p_l(z)} + (l + 1) + i z - \frac{-z p'_l(-z)}{p_l(z)} \\
&= 2iz + z \left[ \frac{p'_l(z)}{p_l(z)} + \frac{p'_l(-z)}{p_l(-z)} \right] \\
&= z \left[ 2ip_l(z)p_l(-z) + p'_l(z)p_l(-z) + p'_l(-z)p_l(z) \right].
\end{align*}
\]
(8.11)

By (10.6), \( a^l_{l-1} = i^{-l} \) and therefore \( 2ip_l(z)p_l(-z) = 2ia^l_{l-1} a^l_{l-1} + O(z^{2l-1}) = -2iz^{2l} + O(z^{2l-1}) \). This suggests separating out this term and writing
\[
G_l(z) - G_l(-z) = z \left[ \frac{-2iz^{2l} + R_l(z)}{p_l(z)p_l(-z)} \right],
\]
where
\[
R_l(z) = 2ip_l(z)p_l(-z) + p'_l(z)p_l(-z) + p'_l(-z)p_l(z) - (-2iz^{2l})
\]
(8.13)
The expression (8.12) simplifies due to a subtle cancellation which follows from Lemma 8.5.

\[ R_l(z) = 0 \]  
(8.14)

and therefore

\[ G_l(z) - G_l(-z) = -2iz^{2l+1} \left[ \frac{1}{p_l(z)p_l(-z)} \right] \]

We prove (8.14) below, but first note its consequences. For \( z \) small we have

\[ G_l(z) - G_l(-z) = \frac{-2iz^{2l+1}}{(a_0^l)^2 + \mathcal{O}(z)} = 2k \left[ \frac{2^j!}{(2l)!} \right]^2 z^{2l+1} + \mathcal{O}(z^{2l+3}) \]  
(8.15)

where we have used that

\[ a_0^l = -i \frac{(2l)!}{2^l l!}. \]

Since

\[ G_l(z) - G_l(-z) = 2 \sum_{k=0}^{\infty} \frac{G_l^{(2k+1)}(0)}{(2k+1)!} z^{2k+1}, \]  
(8.16)

matching terms with Equation (8.15) implies \( q = l \) and Equation (8.10).

To complete the proof of Proposition 8.4 we now give the proof of Lemma 8.5. The idea is to re-express the polynomial expression for \( R_l(z) \) in terms of spherical Hankel functions and to achieve simplification by using the Wronskian identity governing \( h^{(1)}_l \) and \( h^{(2)}_l \). Recall

\[ p_l(z) = z^{l+1}h^{(1)}_l(z)e^{-iz} \]  
(8.17)

\[ p_l'(z) = z^l \left[ z h^{(1)}_l(z) + (l + 1 - iz) h^{(1)}_l(z) \right] e^{-iz}. \]  
(8.18)

Substituting into (8.13) yields

\[
R_l(z) = 2i \left[ z^{l+1}h^{(1)}_l(z)e^{-iz} \right]
\]

\[
+ z^l \left[ i z \partial h^{(1)}_l(z) + (l + 1 - iz) h^{(1)}_l(z) \right] e^{-iz} - (1)^{l+1} z^{l+1} h^{(1)}_l(-z) e^{iz} + 2iz
\]

\[
= 2iz^2 \left[ \left( -1 \right)^{l+1} z^2 h^{(1)}_l(z) h^{(1)}_l(-z) + 1 \right]
\]

\[
+ \left( -1 \right)^{l+1} z^{2l} \left[ z^2 \partial h^{(1)}_l(z) h^{(1)}_l(-z) + (l + 1 - iz) z h^{(1)}_l(z) h^{(1)}_l(-z) \right]
\]

\[
+ \left( -1 \right)^{l+1} z^{2l} \left[ z^2 \partial h^{(1)}_l(-z) h^{(1)}_l(z) - (l + 1 + iz) z h^{(1)}_l(-z) h^{(1)}_l(z) \right]
\]

\[
= 2iz^2 \left[ \left( -1 \right)^{l+1} z^2 h^{(1)}_l(z) h^{(1)}_l(-z) + 1 \right]
\]

\[
+ \left( -1 \right)^{l+1} z^{2l} \left[ z^2 ( \partial h^{(1)}_l(z) h^{(1)}_l(-z) + \partial h^{(1)}_l(-z) h^{(1)}_l(z) ) - 2iz^2 h^{(1)}_l(z) h^{(1)}_l(-z) \right]
\]

\[
= z^{2l} \left[ 2i + \left( -1 \right)^{l+1} z^2 \left( \partial h^{(1)}_l(z) h^{(1)}_l(-z) + \partial h^{(1)}_l(-z) h^{(1)}_l(z) \right) \right] = z^{2l} [0] = 0 .
\]  
(8.19)
In the last line we used a Wronskian relation $h_i^{(1)}(z)\partial h_i^{(2)}(z) - h_i^{(2)}(z)\partial h_i^{(1)}(z) = -2iz^{-2}$ with the analytic continuation formulae

\[ h_i^{(1)}(z) = (-1)^i h_i^{(1)}(-z) \quad \text{and} \quad \partial h_i^{(2)}(z) = (-1)^{i+1}\partial h_i^{(1)}(-z). \]

This completes the proof of the key Lemma 8.5 and thus Proposition 8.4.

We now complete the proof of Theorem 6.2. Recall that deformation scattering resonance energies satisfy the equation $\lambda^2 + r_l G_l(\epsilon \lambda) = 0$. In the (incompressible) limit, $\epsilon \to 0$, we have the equation $\lambda^2 + r_l G_l(0) = 0$, which has two real solutions, the Rayleigh frequencies, to which there are associated undamped oscillatory modes of the incompressible linearized equations about the spherical bubble equilibrium.

We are interested in where these Rayleigh frequencies perturb for positive and small $\epsilon$. By Theorem 6.1, they necessarily perturb to the open lower half plane. Propositions 8.1 and 8.4 enable us to compute, in detail, the location of the Rayleigh scattering resonances to which these real frequencies perturb.

We first consider the case: $l = 0$. Using properties of Hankel functions (see Appendix D.1) we have $G_0(\epsilon \lambda) = i\epsilon \lambda - 1$ and the equation for the resonance energies is:

\[ \lambda^2 + i r_0 \epsilon \lambda - r_0 = 0 \]

yielding the solution in Equation (6.2). This completes the proof of Part 1 of Equation (6.2).

For general $l \geq 2$, we can, by Propositions 8.3 and 8.4, write the resonance equation in the form:

\[ \lambda^2 + r_l \left[ -(l + 1) + \sum_{j=1}^l \frac{(\epsilon \lambda)^{2j}}{(2j)!} G_i^{(2j)}(0) + \frac{(\epsilon \lambda)^{2l+1}}{(2l + 1)!} G_i^{(2l+1)}(0) + O(\epsilon^{2l+2}) \right] = 0, \quad (8.20) \]

where $A(\lambda)$ is even and real-valued for real $\lambda$ and $B(\lambda)$ is odd and purely imaginary for real $\lambda$ and $r_l > 0$ is given in (1.11). It is clear from equation (8.20) that the lowest order term contributing an imaginary part is at $O(\epsilon^{2l+1})$. Taking $2l + 1$ derivatives in $\epsilon$ and setting $\epsilon = 0$ yields

\[ 2\lambda(0)\lambda^{(2l+1)}(0) + r_l G_i^{(2l+1)}(0)\lambda(0)^{2l+1} = 0. \]

Again, using Proposition 8.4 we find $\lambda^{(2l+1)}(0) = -\frac{r_l}{2} G_i^{(2l+1)}(0)\lambda(0)^{2l}$ and

\[ 3\lambda^{(2l+1)}(0) = -\frac{r_l}{2} [r_l(l + 1)]^l (2l + 1)! \left[ \frac{2l!}{(2l)!} \right]^2 = -2^{2l-1} r_l^{l+1}(l + 1)^l(2l + 1)! \left[ \frac{l!}{(2l)!} \right]^2. \]

This proves Part 2 of Theorem 6.2.

9. Estimation of deformation resonances for $l \geq K_1 \epsilon^{-6-\delta}$, $\delta > 0$; Proof of Theorem 6.3. In Theorem 6.1 we established that there are $l + 2$ deformation resonances in the lower half plane. These are solutions of $\lambda^2 + r_l G_l(\epsilon \lambda) = 0$; see Equations (1.11) and (6.1) or

\[ \frac{(\epsilon \lambda)^2}{\epsilon^2 We^{-1}} h_i^{(1)}(\epsilon \lambda) + \epsilon \lambda \partial h_i^{(1)}(\epsilon \lambda) = 0. \quad (9.1) \]

Equivalently, in terms of standard Hankel functions:

\[ \frac{(\epsilon \lambda)^2}{\epsilon^2 We^{-1} (\nu^2 - (3/2)^2)} - \frac{1}{2} H^{(1)}_\nu(\epsilon \lambda) + \epsilon \lambda \partial H^{(1)}_\nu(\epsilon \lambda) = 0, \quad \nu = l + 1/2. \quad (9.2) \]
In this section, we estimate the location of these resonances in the parameter regime where the second term dominates. In Section 9.1, we will show that \( l + 1 \) deformation resonances are near the \( l + 1 \) zeros of \( \partial H^{(1)}_\nu(\epsilon \lambda) \) by applying Rouché’s Theorem. See Appendix E.3 for the locations of the zeros of Hankel functions. In Section 9.2, the remaining resonance is found by the method of dominant balance.

The proof of Theorem 6.3 is based on an application of Rouché’s Theorem [2]: Suppose \( f \) and \( g \) are analytic inside and on a simple, closed curve \( C \) and satisfy \(|g| < |f|\) on \( C \). Then \( f \) and \( f + g \) have the same number of zeros (counting multiplicities) inside \( C \).

9.1. Proof of Theorem 6.3, Part 1. Let

\[ \nu z = \epsilon \lambda \]  

(9.3)

and rewrite (9.2) as

\[
\frac{\partial H^{(1)}_\nu(\nu z)}{f(z)} + \frac{1}{\nu z} \left( \frac{(\nu z)^2}{\epsilon^2 \text{We}^{-1}(\nu^2 - 9/4)} - \frac{1}{2} \right) H^{(1)}_\nu(\nu z) = 0, \quad \nu = l + 1/2. \tag{9.4}
\]

We control the zeros of (9.4) in terms of the zeros of \( \partial H^{(1)}_\nu \) for \( l \) sufficiently large. Here, \( f \) and \( g \) are defined for the application of Rouché’s Theorem to the locations of all \( l + 2 \) resonances.

We prove that the \( l + 1 \) roots of Equation (9.4), denoted \( z_{\nu,1}, z_{\nu,2}, \ldots, z_{\nu,l+1} \), are asymptotically (large) close to zeros of \( \partial H^{(1)}_\nu(\nu z) \). This implies, via (9.3), that there are resonances at the locations:

\[ \lambda_{l,j}(\epsilon) = \nu z_{\nu,j}/\epsilon \]  

(9.5)

Let \( y_{\nu,j} \) denote a solution of \( \partial H^{(1)}_\nu(y) = 0 \) in the fourth quadrant; see Equation (E.17). We claim that \( (f + g)(z) \) has a unique simple zero, \( z_{\nu,j} \), in the interior of \( \gamma_z \), a small curve about the point \( z = \nu^{-1}y_{\nu,j} \). The curve \( \gamma_z \) will be the image of a small circle defined in (9.10). To prove this, we show that for all \( z \in \gamma_z \),

\[ \left| \frac{g(z)}{f(z)} \right| = \left| \frac{1}{\nu z} \left( \frac{(\nu z)^2}{\epsilon^2 \text{We}^{-1}(\nu^2 - 9/4)} - \frac{1}{2} \right) \frac{H^{(1)}_\nu(\nu z)}{\partial H^{(1)}_\nu(\nu z)} \right| < 1, \tag{9.6}
\]

for \( \nu \geq K_1(\epsilon^{-2} \text{We})^{3/(1-3q)} \), where \( 0 < q < 1/3 \).

The key estimate, (9.6), is a consequence of the following two claims. Below, we use \( C \) to denote a generic positive constant, independent of \( \nu, \epsilon, \text{We} \).

Claim 9.1.

(i) Choose \( \epsilon > 0 \) and \( 0 < c_1 < c_2 \). Assume \( c_1 \leq |z| \leq c_2 \). Then, there exists \( C > 0 \), depending only on \( c_1 \) and \( c_2 \), and \( \nu_0 = O(1) > 0 \) such that

\[
\left| \frac{1}{\nu z} \left( \frac{(\nu z)^2}{\epsilon^2 \text{We}^{-1}(\nu^2 - 9/4)} - \frac{1}{2} \right) \right| \leq \frac{C}{\epsilon^2 \text{We}^{-1} \nu}, \quad \nu \geq \nu_0. \tag{9.7}
\]

(ii) There exists \( \nu_0 > 0 \) and \( c_3 > 0 \) such that for all \( \nu \geq \nu_0 \) and \( |\arg z| < \pi \) with \( |z| \geq c_3 \),

\[
\left| \frac{H^{(1)}_\nu(\nu z)}{\partial H^{(1)}_\nu(\nu z)} \right| \leq C \nu^{1/3} \left| \frac{\text{Ai}(\eta)}{\partial \text{Ai}(\eta)} \right|, \quad \text{where} \tag{9.8}
\]
\[ \eta = e^{2 \pi i / 3} \nu^{2/3} \zeta, \quad \frac{2}{3} \zeta^{3/2} = \int_{z}^{1} \frac{\sqrt{1 - t^2}}{t} dt = \log \frac{1 + \sqrt{1 - z^2}}{z} - \sqrt{1 - z^2}; \quad (9.9) \]

see also Sections E.3 and E.4.

**Claim 9.2.** Let \( q \in (0, 1/3) \) and denote by \( \eta'_q \) the \( q \)-th solution of \( \partial A_i(\eta) = 0 \); see Section E.2. There exists \( \nu_1 > 0, \kappa > 0 \) such that if \( \nu \geq \nu_1 \) and \( s \in \{1, 2, \ldots, [(l + 1)/2] + 1\} \), then

\[ \left| \frac{A_i(\eta)}{\partial A_i(\eta)} \right| \leq C \nu^{1/3 + q}, \quad \text{on the circle } C_{\eta} = \left\{ \eta : |\eta - \eta'_q| = \frac{k}{2} \right\} \quad (9.10) \]

chosen to enclose exactly one zero of \( \partial A_i \); see Equation (9.22).

Inequality (9.10) is a direct consequence of (9.7), (9.8), and (9.10) of Claims 9.1 and 9.2.

Assuming Claims 9.1 and 9.2, we now prove the assertion of Part 1 of Theorem 6.3 concerning \( l + 1 \) of the \( l + 2 \) resonances. (The \( (l + 2) \)nd is covered by a separate argument below; see Section 9.2.)

First, fix the parameters \( q, \nu_1, \kappa \) as in Claim 9.2 and choose \( \bar{\eta} \in C_{\eta} \), so that the upper bound in (9.10) holds. For such \( \bar{\eta} \), by the change of variables (9.9), \( z(\bar{\eta}) \) varies in an annulus \( 0 < c_1 < |z| < c_2 \), where \( 0 < c_1 < c_2 < \infty \) depend on \( \kappa \) and \( \nu_1 \).

By (9.10), we write \( \bar{\eta} \in C_{\eta} \) as

\[ \bar{\eta} = \eta'_s + \frac{\kappa}{2} s^{-1/3} \nu^{-q} e^{i\theta}, \quad 0 \leq \theta < 2\pi. \quad (9.11) \]

Under the mapping \( \eta \mapsto \zeta \), (9.9), the circle \( C_{\eta} \) is transformed into the circle \( C_{\zeta} \):

\[ \zeta(\bar{\eta}) = \nu^{-2/3} e^{-2\pi i / 3} \left( \eta'_s + \frac{\kappa}{2} s^{-1/3} \nu^{-q} e^{i\theta} \right) \in C_{\zeta}. \quad (9.12) \]

Finally, we map \( C_{\zeta} \) to the curve \( \gamma_z \) in the \( z \)-plane using a formula from [22]

\[ z(\zeta) = 1 - 2^{-1/3} \zeta + \frac{3}{19} 2^{-2/3} \zeta^2 + \frac{1}{700} \zeta^3 + \zeta^4 \mathcal{O}(1), \quad \zeta \text{ small}. \quad (E.15) \]

Thus, to leading order, \( \gamma_z \) is a still a circle:

\[ \gamma_z = z(C_{\zeta}) = 1 - 2^{-1/3} C_{\zeta} + \mathcal{O}\left( \left[ \eta'_e \nu^{-2/3} \right]^2 \right) \quad (9.13) \]

with radius the same as that of \( 2^{-1/3} C_{\zeta} \), namely \( 2^{-4/3} \kappa s^{-1/3} \nu^{-2/3} \). Since this radius is decreasing in both \( s \) and \( \nu \), it can be bounded uniformly. Using this uniform bound, we can find fixed \( c_1 \) and \( c_2 \) defining an annulus which encloses \( \gamma_z \) for all values of \( s \) and \( \nu \).

Additionally, if we take \( y'_{s, k} \) to be a zero of \( \partial H'_{s, k} \) corresponding to \( \eta'_s \), and approximated by Equation (E.17) with \( k = s \), then \( \nu^{-1} y'_{s, k} \) is inside the curve \( \gamma_z \) defined by Equation (9.13). It is equivalent to show that when mapped into the \( \zeta \) variable, the image of \( \nu^{-1} y'_{s, k} \) lies in \( C_{\zeta} \). From Equation (E.17), we see the image of \( \nu^{-1} y'_{s, k} \) lies in a disk of radius \( \sim \nu^{-4/3} \) which is smaller than the radius of \( C_{\zeta} \). With this choice of \( c_1 \) and \( c_2 \), we have, using the definitions in (9.7) and Claims 9.1 and 9.2 that

\[ \left| \frac{g(z)}{f(z)} \right| \leq C \frac{\nu}{1 + \nu^{-q}}, \]

where
Therefore, there exists \( K_1 > 0 \) such that for any \( q \in (0, 1/3) \), if \( l \geq K_1 (e^{-2\text{We}})^{3/(1-3q)} \) then \( |f(z)/g(z)| < 1 \) on the required curve \( \gamma_z \). The largest range of \( l \)-values is obtained by taking \( q = \delta > 0 \) and small and thus \( l \geq K_1 (e^{-2\text{We}})^{3+O(\delta)} \).

Since we have shown \( \partial H^{(1)}_{\nu}(\nu z) = 0 \) has one zero inside \( \gamma_z \), by Rouché’s Theorem there is also a single solution, \( z_{\nu,s} \), of \((\ref{E.3})\), inside \( \gamma_z \). To finish the proof of Theorem \((\ref{E.3})\) Part 1, we compute \( z_{\nu,s} \) and map it to the \( \lambda \)-plane. Since \( z_{\nu,s} \) is inside \( \gamma_z \), we use Equation \((\ref{E.15})\) evaluated at \((\ref{9.12})\) for the approximation. We find:

\[
z_{\nu,s} = 1 + 2^{-1/3} e^{-2\pi i/3} (l + 1/2)^{-2/3} |\eta_s'| + \frac{2}{3} 2^{-1/3} e^{-4\pi i/3} \nu^{-4/3} |\eta_s'|^2 + O(\nu^{-2/3} - q).
\]

\[
(9.14)
\]

Mapping back to \( \lambda \), using \((\ref{9.5})\) completes the proof.

**Proof of Claim \((9.6)\)** Estimate \((9.7)\) is straightforward. To prove estimate \((9.8)\) we use the following leading order asymptotics of Hankel functions, uniform in \( z \) and valid for large \( \nu \) (see Appendix \((E.3)\))

\[
H^{(1)}_{\nu}(\nu z) \sim 2 e^{-\pi i/3} \left( \frac{4\zeta}{1 - z^2} \right)^{1/4} \frac{Ai(\eta)}{\nu^{1/3}}, \quad |\arg z| < \pi,
\]

\[
\partial H^{(1)}_{\nu}(\nu z) \sim -4 e^{\pi i/3} \frac{1}{z} \left( \frac{1 - z^2}{4\zeta} \right)^{1/4} \frac{\partial Ai(\eta)}{\nu^{2/3}}, \quad |\arg z| < \pi.
\]

\[
(9.9)
\]

\[
(9.10)
\]

In particular, for any fixed \( 0 < \delta < 1 \) and sufficiently large \( \nu \)

\[
\left| \frac{H^{(1)}_{\nu}(\nu z)}{\partial H^{(1)}_{\nu}(\nu z)} \right| \leq \frac{1 + \delta}{1 - \delta} \nu^{1/3} \left| z \left( \frac{\zeta}{1 - z^2} \right)^{1/2} \frac{Ai(\eta)}{\partial Ai(\eta)} \right|.
\]

\[
(9.15)
\]

To show this is bounded near \( z = 1 \), we use the Taylor series of \( \zeta(z) \) about \( z = 1 \), as given in Appendix \((E.3)\)

\[
\zeta = -e^{\pi i/3} 2^{1/3} (z - 1) [1 + O(z - 1)],
\]

\[
(9.16)
\]

so, due to the boundedness of \( z \left| z \left( \frac{\zeta}{1 - z^2} \right)^{1/2} \right| \leq C. \) This completes the proof of \((9.8)\), and therewith Claim \((9.6)\). \(\square\)

**Proof of Claim \((9.7)\)** Since \( \partial Ai(\eta_s') = 0 \), Taylor expansion of \( Ai \) and \( \partial Ai \) about \( \eta = \eta_s' \) implies the existence of analytic functions \( A_1 \) and \( A_2 \), such that

\[
Ai(\eta) = Ai(\eta_s') + A_1(\eta)(\eta - \eta_s')^2
\]

\[
\partial Ai(\eta) = \eta_s' Ai(\eta_s')(\eta - \eta_s') + A_2(\eta)(\eta - \eta_s')^2.
\]

\[
(9.17)
\]

\[
(9.18)
\]

To obtain \((9.18)\) we also used that \( \partial^2 Ai(\eta) = \eta Ai(\eta) \). It follows that

\[
\frac{Ai(\eta)}{\partial Ai(\eta)} = \frac{Ai(\eta_s') + A_1(\eta)(\eta - \eta_s')^2}{\eta_s' Ai(\eta_s')(\eta - \eta_s') + A_2(\eta)(\eta - \eta_s')^2} \equiv \frac{1}{\eta_s'(\eta - \eta_s')} \partial'_\nu(\eta)
\]

\[
(9.19)
\]

Thus, we now estimate \( \partial'_\nu(\eta) \). Using the domain specified in Equation \((9.10)\), we have the bound

\[
\left| \frac{1}{\eta_s'(\eta - \eta_s')} \right| \leq C \nu^{1/3} \nu.
\]

\[
(9.20)
\]
We complete the proof of Claim 9.2 by showing \( |\eta'_{s'}(\eta)| \leq C \) on a circle around \( \eta'_{s'} \) that is small enough such that it encloses no other \( \partial \text{Ai}(\eta) \) zero. The specific circle used is defined by

\[
|\eta - \eta'_{s'}| = C s^{-1/3} \nu^{-q}
\]

and is determined by the spacing between consecutive zeros of \( \partial \text{Ai}(\eta) \); see Equations (9.22) and (9.27). There are two cases to study:

- First, consider the case \( s' \leq s'_{1} \leq s'_{0} \geq 1 \) (where \( s' \) and \( s'_{1} \) independent of \( l \)).

Using (E.8), giving the locations of the larger \( \partial \text{Ai}(\eta) \) zeros, \( \{\eta'_{s'} \mid s \geq s'_{0}\} \), we estimate the spacing between the zeros as

\[
|\eta'_{s+1} - \eta'_{s'}| = \frac{2}{3} \left( \frac{3\pi}{2} \right)^{2/3} s^{-1/3} + \mathcal{O}(s^{-4/3}) \quad (9.21)
\]

so that the spacing satisfies the lower bound

\[
|\eta'_{s+1} - \eta'_{s'}| \geq \kappa s^{-1/3}, \quad s \geq s'_{1}
\]

where \( \kappa = (3\pi/2)^{2/3}/3 \approx 0.93693 \). So, first, we consider the open disk \( |\eta - \eta'_{s'}| < \frac{\kappa}{s} s^{-1/3} \), which clearly contains only one zero of \( \partial \text{Ai} \).

Now we estimate \( A_{1} \) and \( A_{2} \), appearing in (9.19). Using the Cauchy integral formula and \( A_{1}(\eta) \) obtained from (9.17) we have

\[
A_{1}(\eta) = \frac{1}{2\pi i} \oint_{|t - \eta_{s'}| = \kappa s^{-1/3}} \frac{A_{1}(t)}{t - \eta} dt = \frac{1}{2\pi i} \oint_{|t - \eta_{s'}| = \kappa s^{-1/3}} \frac{A(t) - A(\eta_{s'})}{(t - \eta_{s'})^2(t - \eta)} dt.
\]

Using the asymptotic expansion of \( \text{Ai}(\eta) \) on the negative real axis from Appendix E.1 and the location of the \( \eta'_{s} \) from (E.7) we have

\[
C_{s} s^{-1/6} \leq \sup_{|\eta - \eta'_{s'}| \leq \kappa s^{-1/3}} |\text{Ai}(\eta)| \leq C_{s} s^{-1/6}, \quad s \geq s'_{1}. \quad (9.24)
\]

This allows the estimate for \( |\eta - \eta'_{s'}| < \kappa s^{-1/3} \):

\[
|A_{1}(\eta)| \leq \frac{1}{2\pi} \sup_{|\eta - \eta'_{s'}| = \kappa s^{-1/3}} \frac{2|\text{Ai}(\eta)|}{(\kappa s^{-1/3})^2} \oint_{|t - \eta_{s'}| = \kappa s^{-1/3}} \frac{1}{|t - \eta|} |dt|,
\]

\[
\leq C \left( \frac{s^{-1/6}}{(\kappa s^{-1/3})^2} \right) \oint_{|t - \eta_{s'}| = \kappa s^{-1/3}} \frac{1}{|t - \eta|} |dt|,
\]

\[
\leq C \left( \frac{s^{-1/6}}{(\kappa s^{-1/3})^2} \right) \frac{1}{\kappa s^{-1/3}} \int_{|t - \eta_{s'}| = \kappa s^{-1/3}} |dt|,
\]

\[
\leq C \left( \frac{s^{-1/6}}{\kappa s^{-1/3}} \right) \frac{1}{\kappa s^{-1/3} - |\eta - \eta'_{s'}|} \int_{|t - \eta_{s'}| = \kappa s^{-1/3}} |dt|,
\]

Next, we apply the last inequality to the smaller circle \( |\eta - \eta'_{s'}| = C s^{-1/3} \nu^{-q} \) we can estimate

\[
|A_{1}(\eta)| \leq C \left( \frac{s^{-1/6}}{(\kappa s^{-1/3})^2} \right) \frac{1}{1 - |\eta - \eta'_{s'}/(\kappa s^{-1/3})|} \leq C s^{1/2} \frac{1}{1 - C \nu^{-q}} \leq C s^{1/2}. \quad (9.25)
\]
Similarly,

\[ |A_2(\eta)| \leq Cs^{5/6}. \quad (9.26) \]

The full numerator of (9.19) can be estimated, using (9.24) and (9.25),

\[ |1 + Ai(\eta_0')^{-1}A_1(\eta - \eta_0')^2| \leq 1 + |Ai(\eta_0')^{-1}A_1(\eta)| |\eta - \eta_0'|^2 \]
\[ \leq 1 + C s^{1/6}C s^{1/2} (Cs^{-1/3} \nu^{-q})^2 \leq 1 + C \nu^{-2q}. \]

For the denominator, using (E.8), (9.24) and (9.26),

\[ \left| \left[ \eta_0' Ai(\eta_0') \right]^{-1} A_2(\eta)(\eta - \eta_0') \right| \leq |\eta_0'|^{-1} |Ai(\eta_0')^{-1}A_2(\eta)| |\eta - \eta_0'| \]
\[ \leq Cs^{-2/3}Cs^{1/6}Cs^{5/6}Cs^{-1/3} \nu^{-q} \leq C \nu^{-q}. \]

Therefore, \( \left| 1 + [\eta_0' Ai(\eta_0')]^{-1} A_2(\eta)(\eta - \eta_0') \right|^{-1} \leq 1 + C \nu^{-q} \). Combining these estimates, we find \( |a_\nu'| \leq C \).

• The case \( 1 \leq s < s_1' \) is handled similarly. Since we do not have an explicit approximate formula for the zeros of \( \partial \overline{A_i} \) zeros, we introduce the among the first \( s_1' \) zeros

\[ 2\delta' = \min_{1 \leq s \leq s_1'} |\eta_{s+1}' - \eta_s'|. \quad (9.27) \]

For \( 1 \leq s < s_1' \), define \( A_1^s \) and \( A_2^s \) via (9.17) and (9.18). Then,

\[ a_\nu' = \frac{1 + Ai(\eta_0')^{-1}A_1^s(\eta)(\eta - \eta_0')^2}{1 + \eta_0'Ai(\eta_0')^{-1}A_2^s(\eta)(\eta - \eta_0')} \quad (9.28) \]

In the disk \( |\eta - \eta_0'| \leq \delta' \nu^{-q} \) (in the theorem \( Cs^{-1/3} = \delta' \)) we use the Cauchy integral formula to make the estimates \( |A_1^s(\eta)| \leq C \) and \( |A_2^s(\eta)| \leq C \), which give \( |a_\nu'| \leq C \).

This completes the proof of Claim 9.2 and therewith, Part 1 of Theorem 6.3.

9.2. Proof of Theorem 6.3 Part 2. We now discuss the single resonance, away from the arc. As before, we begin by seeking solutions of Equation (6.1) for large \( l \) using the asymptotics in Equations (6.9) and (6.10). However, we now seek zeros in a domain not associated with zeros of \( \partial \overline{A_i} \). In particular, we focus where \( |\eta| \) is large and \( |\arg \eta| < \pi \). Using the leading term of the asymptotic approximation to Airy functions in Equations (E.1) and (E.2) and Equation (9.9) we obtain the approximate equation

\[ \nu^2 z^2 - \frac{\epsilon^2}{2 \text{ We}} \left( \frac{\nu^2 - \frac{9}{4}}{4} \right) = -\epsilon^2 \nu \frac{1}{\text{ We}} \left( \frac{\nu^2 - \frac{9}{4}}{4} \right) \left( 1 - z^2 \right)^{1/2} \quad (9.29) \]

where \( \lambda = \nu \epsilon / \epsilon \). For \( \nu \gg 3/2 \), we can drop the terms involving \( 9/4 \), square both sides, and write an approximate polynomial equation for the resonances in \( z \):

\[ z^4 + \nu^2 \epsilon^4 \left( \frac{1}{\text{ We}} \right)^2 z^2 - \epsilon^2 \left( \frac{1}{\text{ We}} \right) z^2 - \nu^2 \epsilon^4 \left( \frac{1}{\text{ We}} \right)^2 = 0. \quad (9.30) \]

All neglected terms are of higher order in \( \epsilon \) and will be absorbed into the final error.

We identify the large parameter \( N = \frac{\epsilon^2 \nu}{\text{ We}} \), set \( \alpha = \frac{\epsilon^2}{\text{ We}} \) and obtain:

\[ z^4 + N^2 z^2 - \alpha^2 z^2 - N^2 = 0. \quad (9.31) \]
In the regime, $\text{le}^2 \text{We}^{-1} \geq K_2$, we have $N \geq K_2$, and the dominant balance is between the first two terms. Choosing a solution in the lower half plane leads to $z \approx -iN$.

Accordingly, we look for a solution in the form

$$z(N) = -iN + z_0 + z_1 N^{-1} + z_2 N^{-2} + \mathcal{O}(N^{-3}).$$

(9.32)

Iteratively, we compute $z_0 = 0$, $z_{-1} = i(\alpha - 1)/2$, and $z_{-2} = 0$. Changing variables back to $\lambda$ leads to the desired result. This completes the proof of Theorem 6.3.

**Appendix A. Spherical coordinates.**

Spherical coordinates in $\mathbb{R}^3$:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \quad (A.1)$$

**Standard orthonormal frame**

$$\hat{r} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}, \quad \hat{\theta} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix} = -\partial_\theta \hat{r}, \quad \hat{\phi} = \begin{pmatrix} -\sin \theta \sin \phi \\ \sin \theta \cos \phi \\ 0 \end{pmatrix} = \partial_\phi \hat{r} \quad (A.2)$$

**grad, div, $\Delta$ in spherical coordinates**: 

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi} \quad (A.3)$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta F_\theta) + \frac{1}{r \sin \theta \sin \phi} \frac{\partial}{\partial \phi} F_\phi \quad (A.4)$$

$$\text{div } \text{grad } f = \Delta f = \Delta_r f + \frac{1}{r^2} \Delta_S f$$

$$\Delta_r f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right), \quad \text{radial Laplacian} \quad (A.5)$$

$$\Delta_S f = \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right], \quad \text{Laplacian on } S^2 \quad (A.6)$$

**Surface and Volume elements**: $d\Omega = \sin \theta \ d\theta \ d\phi, \ dx = r^2 \sin \theta \ dr \ d\theta \ d\phi = r^2 \ dr \ d\Omega$

**Appendix B. Spherical Harmonics.** $Y_l^m(\theta, \phi) = Y_l^m(\Omega)$, $l \geq 0$, $|m| \leq l$ denote the spherical harmonics. These are eigenfunctions of the Laplacian on $S^2$:

$$-\Delta S Y_l^m(\theta, \phi) = l(l+1) Y_l^m(\theta, \phi). \quad (B.1)$$

The functions $\{Y_l^m\}$ form a complete orthonormal system in $L^2(S^2)$:

$$\langle Y_l^m, Y_{l'}^{m'} \rangle_{L^2(S^2)} = \delta_{ll'} \delta_{mm'} \quad \text{with } L^2(S^2) \text{ inner product:}$$

$$\langle f, g \rangle_{L^2(S^2)} = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \overline{g(\theta, \phi)} \sin \theta \ d\theta \ d\phi = \int_{S^2} f(\Omega) \overline{g(\Omega)} \ d\Omega \quad (B.2)$$

We shall make use of the following explicit formulae:

$$Y_0^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{1}{\pi}} \quad (B.3)$$

$$Y_{-1}^1(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \ e^{-i\phi}, \quad Y_1^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta, \quad Y_1^1(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta \ e^{i\phi} \quad (B.4)$$
Appendix C. PDEs for a bubble in a compressible fluid and linearization. The inviscid, compressible fluid phase is governed by the Euler and conservation of mass equations with normal velocity and stress boundary conditions:

\[ 0 = \partial_t u + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p(\rho), \quad \mathbf{x} \in \mathbb{R}^3 \setminus B(t) \]  
\[ 0 = \partial_t \rho + \nabla \cdot (\rho u), \quad \mathbf{x} \in \mathbb{R}^3 \setminus B(t) \]  
\[ u \cdot \hat{n} = \partial_t R \cdot \hat{n}, \quad \partial B(t) \]  
\[ p_{\text{bubble}}|_{\partial B(t)} - p_{\text{fluid}}|_{\partial B(t)} = \sigma \nabla \cdot \hat{n}, \partial B(t). \]

The pressure within the fluid is assumed to obey an equation of state: \[ p = p(\rho). \]  The pressure of the gas within the bubble is assumed to be constant and to vary inversely with bubble-volume:

\[ p_{\text{bubble}} = \frac{k}{B(t)^{\gamma}}. \]

For air (composed of mostly diatomic gases), the ratio of specific heats \( \gamma = 1.4. \) Equation (C.1d) is called the Young-Laplace equation \[15\] and states that the discontinuity in pressure at the interface is proportional to the mean curvature \( H \) due to the relation \( \nabla \cdot \hat{n} = 2H; \) see Lamb \[13\] (Article 275, Equation 5), or do Carmo \[7\] (Section 3-3, Example 5).

Equations (C.1) have a spherically symmetric equilibrium solution:

\[ u = 0, \quad R(\theta, \phi) = a \hat{r}(\theta, \phi) \]  
\[ p = p_{\infty}, \quad \rho = \rho_{\infty}, \quad P_B = P_{eq} \]

where the equilibrium bubble radius, \( a \) and pressure, \( P_{eq} \) are given by \[15\] .

To make the fluid compressibility more explicit, we define a variable sound speed \( c^2 = \frac{d p}{d \rho} \) and rewrite the continuity equation (C.1b) in terms of the pressure, \( p = p(\rho) \):

\[ \partial_t p + \nabla p \cdot u + c^2 \rho \nabla \cdot u = 0. \]

C.1. Nondimensionalization of the compressible Euler equations. We shall use the equilibrium bubble radius, \( a \), as a length scale, and a typical bubble wall velocity, \( U \), as a velocity scale to nondimensionalize the compressible Euler equations (C.1). In this section, we denote the nondimensional variables with subscript asterisks. Thus, the non-dimensional spatial coordinate is denoted \( \mathbf{x}_* \). Now, introduce non-dimensional variables:

\[ B = a B_*, \quad \mathbf{x} = a \mathbf{x}_*, \quad t = \frac{a}{U} t_*, \quad u = U u_*, \quad p = \rho_{\infty} U^2 p_*, \]

\[ \partial B = a \partial B_*, \quad \nabla_* = \frac{1}{a} \nabla \mathbf{x}_*, \quad \partial_t = \frac{U}{a} \partial_{t_*}, \quad \rho = \rho_{\infty} \rho_*, \quad P_B = \rho_{\infty} U^2 P_{eq} B_*. \]

In terms of these non-dimensional variables, the first two equations of the Euler system become:

\[ \partial_t u_* + (u_* \cdot \nabla \mathbf{x}_*) u_* + \frac{1}{\rho_*} \nabla \mathbf{x}_* p_*(\rho_*) = 0, \quad \mathbf{x}_* \in \mathbb{R}^3 \setminus B_*(t_*) \]  
\[ \partial_t \rho_* + \nabla \mathbf{x}_* \cdot (\rho_* u_* ) = 0, \quad \mathbf{x}_* \in \mathbb{R}^3 \setminus B_*(t_*,). \]
Here, we have introduced the non-dimensional speed $c$ that $p$ above we have the expression for in terms of the non-dimensional pressure, $p_*$ as:
\[
\partial_t p_* + \nabla_x p_* \cdot u_* + c_*^2 \rho_* \nabla_x \cdot u_* = 0. \tag{C.10}
\]
Here, we have introduced the non-dimensional speed: $c_*^2 = \left( \frac{\phi}{\pi} \right)^2 = \frac{dp_*}{d\rho_*}(\rho_*)$. Note that $p_*$, $u_*$, and $\rho_*$ are all functions of $(x_*, t_*)$.

Next, we nondimensionalize the boundary conditions. Using the substitutions above we have
\[
u_* \cdot \n^* = \partial_t R_* \cdot \n^*, \quad x_* \in \partial B_*(t_*), \tag{C.11}
\]
\[
P_{*B_*} - p_* = \frac{1}{We} \nabla_x \cdot \n^*, \quad x_* \in \partial B_*(t_*) \tag{C.12}
\]
where $We$ denotes the Weber number, denoted by
\[
We = \frac{\rho_\infty aU^2}{\sigma} = \frac{\text{INERTIA}}{\text{CURVATURE}}. \tag{C.13}
\]
We now express the dynamic boundary condition more explicitly. First note $|B(t)| = |aB_*(t_*)| = a^3|B_*(t_*)|$. Furthermore, the bubble pressure at equilibrium is given by
\[
P_{eq} = \left( \frac{k}{(\frac{4\pi}{3})a^3} \right) \Rightarrow k = P_{eq} \left( \frac{4\pi}{3}a^3 \right)^\gamma \Rightarrow P_{eq} = P_{eq} \left( \frac{4\pi}{3}a^3 \right)^\gamma. \tag{C.14}
\]
Thus, using the expression for $P_{eq}$ in (1.5), we have
\[
P_{*B_*} = \frac{P_B}{\rho_\infty U^2} = \frac{P_{eq}}{\rho_\infty U^2} \left( \frac{4\pi}{3} \right)^\gamma \left( \frac{a^3}{|B_*(t_*)|} \right) = \left( \frac{Ca}{2} + \frac{2}{We} \right) \left( \frac{4\pi}{3} \right)^\gamma \tag{C.14}
\]
Here, $Ca$ denotes the Cavitation number
\[
Ca = \frac{p_\infty}{\frac{\gamma}{2} \rho_\infty U^2} = \frac{\text{EXTERNAL PRESSURE}}{\text{KINETIC ENERGY PER VOLUME}}. \tag{C.15}
\]
The nondimensional system is given by Equations (C.8), (C.9), (C.11) and (C.12).

Finally, we conclude this section by displaying the spherical bubble equilibrium solution in non-dimensional variables:
\[
u_* = 0, \quad R_* = \hat{r}(\theta, \phi), \quad p_* = \frac{1}{2}Ca, \quad \rho_* = 1, \quad P_{*B_*} = \frac{Ca}{2} + \frac{2}{We}. \tag{C.16}
\]

C.2. Nondimensional Euler equations in Center of Mass Coordinates.

The bubble’s dimensional center of mass is defined
\[
\xi_{\text{cm}}(t) := \int_{B(t)} x \ dx \tag{C.17}
\]
and is nondimensionalized by $\xi_{\text{cm}}(t) = a \xi_{\text{cm}*}(t_*)$. In this section, variables and operators defined with respect to the moving coordinate system are denoted with a prime ('). Let the moving coordinates be defined
\[
x'_*(t_*) := x_* - \xi_{\text{cm}*}(t_*) \tag{C.18}
\]
such that

$$\int_{B'_*(t_*)} x'_s \, dx'_s = 0.$$  \hfill (C.19)

We will use the following substitutions.

$$\nabla x_* = \nabla x'_s, \quad \hat{n} = \hat{n}' , \quad \hat{R}_* = \hat{R}'_* + \xi_{cm*}(t_*)$$

$$u_*(x_*, t_*) = u'_s(x_* - \xi_{cm*}(t_*), t_*) = u'_s(x'_s, t_*)$$

$$\rho_*(x_*, t_*) = \rho'_s(x'_s, t_*) ,$$

$$p_*(x_*, t_*) = p'_s(x'_s, t_*) P_{B*}(\hat{R}_*, t_*) = P'_{B*}(\hat{R}'_* , t_*) , c_*(x_* , t_*) = c'_s(x'_s , t_*) .$$  \hfill (C.20)

The resulting system of nonlinear equations is:

$$0 = ( \partial_{t_*} - \partial_{\theta_*} \xi_{cm*} \cdot \nabla x'_s ) u'_s + ( u'_s \cdot \nabla x'_s ) u'_s + \frac{1}{\rho'_s} \nabla x'_s p'_s, \quad x'_s \in \mathbb{R}^3 \setminus B'_*(t_*)$$  \hfill (C.21a)

$$0 = ( \partial_{t_*} - \partial_{\theta_*} \xi_{cm*} \cdot \nabla x'_s ) p'_s + \nabla x'_s \rho'_s \cdot u'_s + c'_2 \rho'_s \nabla x'_s \cdot u'_s, \quad x'_s \in \mathbb{R}^3 \setminus B'_*(t_*)$$  \hfill (C.21b)

$$u'_s \cdot \hat{n}' = \partial_t ( \rho'_s + \xi_{cm*} ) \cdot \hat{n}' , \quad x'_s \in \partial B'_*(t_*)$$  \hfill (C.21c)

$$P'_{B*} - p'_s = \frac{1}{\rho'_s} \nabla x'_s , \quad x'_s \in \partial B'_*(t_*)$$  \hfill (C.21d)

$$0 = \int_{B'_*} x'_s \, dx'_s .$$  \hfill (C.21e)

### C.3. Linearization of Euler equations about the spherical equilibrium.

To simplify the notation, we drop asterisks and primes in Equations (C.21), and linearize about the nondimensional equilibrium (C.16) by taking:

$$u = 0 + \delta u_1 + \mathcal{O}(\delta^2), \quad R(\theta, \phi, t) = R(\theta, \phi, t) \hat{r} = \left[ 1 + \delta R_1(\theta, \phi, t) + \mathcal{O}(\delta^2) \right] \hat{r}$$  \hfill (C.22)

$$p = \frac{C}{2} + \delta \rho_1 + \mathcal{O}(\delta^2) ; \quad p = p(\rho), \quad \rho = 1 + \delta \rho_1 + \mathcal{O}(\delta^2)$$  \hfill (C.23)

$$\xi_{cm}(t) = 0 + \delta \xi_{cm,1}(t) + \mathcal{O}(\delta^2), \quad \xi_{cm,1}(t) = \xi_{1,\hat{x}}(t) \hat{x} + \xi_{1,\hat{y}}(t) \hat{y} + \xi_{1,\hat{z}}(t) \hat{z}$$  \hfill (C.24)

Consider initial conditions that are perturbations from equilibrium of size $\delta, \delta \leq 1$. We expand quantities in Equations (C.21) in powers of $\delta$: $c^2 = \frac{dp}{d\rho}(1 + \delta \rho_1 + \ldots) = \frac{1}{M^2} + \delta c_1^2 + \mathcal{O}(\delta^2)$, where $M$ denotes the Mach number

$$M = \left( \frac{dp}{d\rho} (1) \right)^{-1} = \frac{U}{c_\infty} = \left[ \frac{\text{INERTIA}}{\text{COMPRESSIBILITY}} \right]^{1/2} .$$  \hfill (C.25)

In addition to the above expansions in powers of $\delta$, we require the implied expansions for $\hat{n}$ and $\text{div} \, \hat{n}$. From the relation

$$F(r, \theta, \phi, t) \equiv r - R(\theta, \phi, t) = 0$$

which defines the bubble surface, $\partial B(t)$, we have using the polar coordinate represent-
\[ \begin{align*}
\n\text{RADIATIVE DECAY OF BUBBLE OSCILLATIONS IN A COMPRESSIBLE FLUID} & \quad 41 \\
\text{tions, (A.3) and (A.4), for grad and div} & \\
\n\hat{n}\big|_{r=R} &= \frac{\text{grad } F}{|\text{grad } F|} \bigg|_{r=R} = \frac{\hat{\mathbf{r}} - \frac{1}{R} \partial_\theta R \hat{\theta} - \frac{1}{R \sin \theta} \partial_\phi R \hat{\phi}}{(1 + \frac{1}{R^2}(\partial_\theta R)^2 + \frac{1}{R \sin \theta} (\partial_\phi R)^2)^{\frac{1}{2}}} \\
&= 1 \cdot \hat{\mathbf{r}} - \delta \left( \partial_\theta R_1 \hat{\theta} + \frac{1}{\sin \theta} \partial_\phi R_1 \hat{\phi} \right) + O(\delta^2), \\
(\text{div } \hat{n})|_{r=R} &= \left[ \text{div} \left( 1 \cdot \hat{\mathbf{r}} - \delta \partial_\theta R_1 \hat{\theta} - \delta \frac{1}{\sin \theta} \partial_\phi R_1 \hat{\phi} + O(\delta^2) \right) \right] \bigg|_{r=R} \\
&= 2 - \delta (2 + \Delta_S) R_1 + O(\delta^2) \\
\n\text{It follows that} & \\
\mathbf{u} \cdot \hat{n} &= 0 + \delta \partial_r u_1 + O(\delta^2) \\
\partial_t \xi_{\text{cm}} \cdot \hat{n} &= 0 + \delta \left( \partial_t \xi_{1,x}(t) \hat{x} + \partial_t \xi_{1,y}(t) \hat{y} + \partial_t \xi_{1,z}(t) \hat{z} \right) \cdot \hat{\mathbf{r}} + O(\delta^2) \\
&= \delta \left( \partial_t \xi_{1,x}(t) \sin \theta \cos \phi + \partial_t \xi_{1,y}(t) \sin \theta \sin \phi + \partial_t \xi_{1,z}(t) \cos \theta \right) + O(\delta^2) \\
\partial_t \mathbf{R} \cdot \hat{n} &= 0 + \delta \partial_t R_1 + O(\delta^2) \\
\n\text{For the bubble pressure, we use the following expansion of the volume, } |B(t)|: & \\
|B(t)| &= \int_0^\pi d\phi \int_0^{2\pi} d\theta \sin \theta \int_0^{R(\theta, \phi, t)} r^2 dr = \frac{4}{3} \pi + \delta \int_0^\pi d\phi \int_0^{2\pi} R_1(\theta, \phi, t) \sin \theta d\theta + O(\delta^2) \\
\n\text{Therefore,} & \\
\left( \text{\frac{4}{3} \pi}{|B(t)|} \right)^\gamma &= 1 - 3 \gamma \delta \langle Y^0_0, R_1 \rangle_{L^2(S^2)} Y^0_0 + O(\delta^2), \quad \text{where } Y^0_0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{1}{\pi}}. \\
\n\text{The center of mass constraint can be expanded is follows:} & \\
0 &= \int_{B(t)} \mathbf{x} \, d\mathbf{x} = \int_0^{2\pi} d\theta \int_0^\pi d\phi \int_0^{R(\theta, \phi, t)} r \, \hat{\mathbf{r}}(\theta, \phi) r^2 \sin \theta \, dr \\
&= \frac{1}{4} \int_0^{2\pi} \int_0^\pi R^4(\theta, \phi, t) \left( \sin \theta \cos \phi \over \sin \theta \sin \phi \over \cos \theta \right) \sin \theta d\theta d\phi \\
\n\text{Since } R = 1 + \delta R_1 + O(\delta^2), \text{ we have at order } \delta: & \\
\int_0^{2\pi} \int_0^\pi R_1(\theta, \phi, t) \left( \sin \theta \cos \phi \over \sin \theta \sin \phi \over \cos \theta \right) \sin \theta d\theta d\phi &= 0 \\
\n\text{Equivalently, we can express these three orthogonality conditions in terms of spherical harmonics of degree } l = 1: Y^1_0(\theta, \phi), Y^0_0(\theta, \phi) \text{ and } Y^{-1}_1(\theta, \phi). \quad \text{Indeed, by Equation (B.3)} & \\
\sqrt{\frac{3}{2\pi}} \sin \theta \cos \phi &= Y^{-1}_1 - Y^1_1, \quad \sqrt{\frac{3}{2\pi}} \sin \theta \sin \phi = i (Y^{-1}_1 + Y^1_1), \quad \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta = Y^0_0. \\
\end{align*} \]
Therefore, the three orthogonality conditions of Equation (C.29) are equivalent to
\[ \langle R_1(\cdot, t), Y^m_1(\cdot) \rangle_{L^2(S^2)} = 0, \ m = -1, 0, 1. \] We can also re-express \( \partial_t \xi_{\text{cm}} \cdot \hat{n} \), given above, in terms of \( Y^m_1 \) as follows
\[
(\partial_t \xi_{\text{cm}} \cdot \hat{n}) (\theta, \phi, t) = \delta (\partial_t \xi_{\text{cm}} \cdot \hat{r}) (\theta, \phi, t) + O(\delta^2)
\]
\[
= \sqrt{\frac{\pi}{3}} \left( \sqrt{2} \partial_t \xi_{1,x} (t) (Y^1_{-1} - Y^1_{1}) + \sqrt{2} i \partial_t \xi_{1,y} (t) (Y^1_{-1} + Y^1_{1}) + 2 \partial_t \xi_{1,z} (t) Y^0_1 \right) + O(\delta^2).
\]
(C.31)

Substitution of these expansions into Equations (C.21) yields the following equations for the linearized perturbations to the fluid phase:
\[
0 = \partial_t u_1 + \nabla p_1, \quad 0 = \partial_t p_1 + \frac{1}{M^2} \nabla \cdot u_1 \quad \text{(C.32)}
\]
Assuming the flow is irrotational, we have \( u = \delta u_1 + \delta^2 u_2 + O(\delta^3) = \delta \nabla \varphi_1 + \delta^2 \nabla \varphi_2 + O(\delta^3) \). Since \( u_1 = \nabla \varphi_1 \), we have \( 0 = \nabla (\partial_t \varphi_1 + p_1) \) and therefore
\[
0 = \partial_t \varphi_1 + p_1, \quad 0 = \partial_t p_1 + \frac{1}{M^2} \Delta \varphi_1
\]
implying the wave equation for \( \varphi_1 \):
\[
M^2 \partial_t^2 \varphi_1 - \Delta \varphi_1 = 0, \quad r > 1, \ t > 0 \quad \text{(C.33)}
\]
The linearized boundary conditions and center of mass constraint are
\[
\partial_r \varphi_1 = \partial_t R_1 + (\partial_t \xi_{\text{cm}} \cdot \hat{r}) (\theta, \phi, t), \quad r = 1, \ t \geq 0 \quad \text{(C.34)}
\]
\[
\partial_r \varphi_1 = 3\gamma \left( \frac{Ca}{2} + \frac{2}{We} \right) \langle Y^0_0, R_1 \rangle Y^0_0 - \frac{1}{We} (2 + \Delta_S) R_1, \quad r = 1, \ t \geq 0 \quad \text{(C.35)}
\]
\[
\langle R_1(\cdot, t), Y^m_1 \rangle = 0, \quad |m| \leq 1 \quad \text{(C.36)}
\]
As initial conditions we take:
\[
R_1(\theta, \phi, t) \big|_{t=0} \text{ sufficiently smooth}
\]
\[
\varphi_1(r, \theta, \phi, t) \big|_{t=0}, \quad \partial_t \varphi_1(r, \theta, \phi, t) \big|_{t=0} \equiv 0, \quad r > 1,
\]
(C.37)
\[
\xi_{\text{cm},1}(t) \big|_{t=0} = 0.
\]

**Proposition C.1.** Let \( \varphi_1(r, \theta, \phi, t), \ R_1(\theta, \phi, t), \ \xi_{\text{cm},1}(t) \) denote a sufficiently regular solution of the initial-boundary value problem for the wave equation (C.33) on the region \( r > 1 \) with linearized kinematic and dynamic boundary conditions (C.34) and (C.35), linearized coordinate normalization (center of mass) condition, and initial conditions (C.37). Then,
\[
\xi_{\text{cm},1}(t) = 0, \quad t \geq 0.
\]
(C.38)
It follows that the linearized perturbation: \( \Psi = \varphi_1, \ \beta = R_1 \) satisfies the initial-boundary value problem (1.38), (1.39) of the Introduction.

To prove Proposition C.1 we first introduce the projection operators onto the \( |m| = 1 \) spherical harmonics
\[
P^m_1 = \langle \cdot, Y^m_1 \rangle_{L^2(S^2)} Y^m_1.
\]
(C.39)
Since $P^m_1$ commutes with $\partial_t$ and $\Delta$, for any $|m| \leq 1$, the function $U = P^m_1 \partial_t \varphi_1$ satisfies the wave equation

$$(m^2 \partial_t^2 - \Delta) U = 0, \quad r > 1$$

Moreover, applying $P^m_1$ to the dynamic boundary condition (C.35) and using (C.36) yields the Dirichlet boundary condition

$$U = 0, \quad r = 1.$$  

It follows that $U = P^m_1 \partial_t \varphi_1$ is identically zero on $r > 1$. Therefore,

$$P^m_1 \varphi_1 (r, \theta, \phi, t) = F(r) Y^m_1(\theta, \phi).$$

Since $P^m_1 \varphi_1$ is a time-independent solution of the wave equation for $r > 1$, clearly $\Delta P^m_1 \varphi_1 = 0$, and therefore $(\Delta_r - 2r^{-2}) F(r) = 0$. Thus, $F(r) = c_1 j_1(r) + c_2 y_1(r)$ for some constants $c_1, c_2$ and therefore

$$P^m_1 \varphi_1 (r, \theta, \phi, t) = (c_1 j_1(r) + c_2 y_1(r)) Y^m_1(\theta, \phi),$$

$$\partial_t P^m_1 \varphi_1 (r, \theta, \phi, t) = (c_1 \partial_r j_1(r) + c_2 \partial_r y_1(r)) Y^m_1(\theta, \phi).$$

Evaluating (C.42) at $r = 1$ and $t = 0$ we have, using the initial condition $\varphi_1 (r, \theta, \phi, t = 0) = 0$ for $r > 1$,

$$c_1 j_1(1) + c_2 y_1(1) = 0, \quad c_1 \partial_r j_1(1) + c_2 \partial_r y_1(1) = 0.$$  

implying $c_1 = c_2 = 0$, by linear independence of $\{j_1(r), y_1(r)\}$. Now applying $P^m_1$ to the kinematic boundary condition (C.34) and using (C.36), we obtain

$$\langle Y^m_1, (\partial_t \xi_{cm,1}(t) \cdot \hat{r}) \rangle_{L^2(S^2)} = 0, \quad |m| \leq 1.$$

We now claim that $\partial_t \xi_{cm,1}(t) = 0$, $t \geq 0$ and therefore $\xi_{cm,1}(t) = 0$, $t \geq 0$. To see this, observe that

$$\partial_t \xi_{cm,1}(t) \cdot \hat{r} = \partial_t \xi_{1x}(t) \sin \theta \cos \theta + \partial_t \xi_{1y}(t) \sin \theta \sin \phi + \partial_t \xi_{1z}(t) \cos \theta$$

$$= \sqrt{\frac{2\pi}{3}} \left[ \partial_t \xi_{1x}(t)(Y^1_{1-1} - Y^1_1) + i \partial_t \xi_{1x}(t)(Y^1_{1-1} + Y^1_1) + \frac{1}{\sqrt{2}} \partial_t \xi_{1z}(t)Y^0_1 \right].$$

Taking inner product with $Y^m_1$, $|m| \leq 1$, yields $\partial_t \xi_{1x}(t) = \partial_t \xi_{1y}(t) = \partial_t \xi_{1z}(t) = 0$, $t \geq 0$. This completes the proof.

**Appendix D. Hankel Functions.** Separation of variables, applied to the to the wave equation in $\mathbb{R}^3$, in spherical coordinates, leads to the spherical Bessel’s equation

$$r^2 R''(r) + 2r R'(r) + (r^2 - l(l + 1)) R(r) = 0.$$  

(D.1)

Two linearly independent solution are the spherical Bessel functions

$$j_l(r) = \sqrt{\frac{\pi}{2r}} J_{l+1/2}(r), \quad y_l(r) = \sqrt{\frac{\pi}{2r}} Y_{l+1/2}(r)$$

where $J_{l+1/2}$ and $Y_{l+1/2}$ are Bessel functions of order $l + 1/2$. The outgoing spherical Hankel function $h_l^{(1)}(z)$, $z \in \mathbb{C}$, is obtained by taking a linear combination of $j_l$ and $y_l$:

$$h_l^{(1)}(z) = j_l(z) + iy_l(z).$$  

(D.3)
In terms of the Hankel functions, \( H_{l+1/2}^{(1)}(z) = J_{l+1/2}(z) + iY_{l+1/2}(z) \) and

\[
\frac{\pi}{2z} H_{l+1/2}^{(1)}(z).
\]

**D.1. Polynomial Representation.** We can represent the spherical Hankel functions as in [40] by

\[
h_{l}^{(1)}(z) = z^{-l-1}p_l(z)e^{iz} \quad \text{(D.5)}
\]

where \( p_l(z) \) is the polynomial of order \( l \) given by

\[
p_l(z) = i^{l-1} \sum_{k=0}^{l} \left( \frac{i}{2} \right)^k \frac{(l+k)!}{k!(l-k)!} z^{l-k} \sum_{n=0}^{l-n-1} \frac{(2l-n)!}{(l-n)!n!} z^n = \sum_{n=0}^{l} a_l^n z^n \quad \text{(D.6)}
\]

**D.2. Analytic continuation.** The Hankel functions have the following analytic continuations and symmetries

\[
H_{\nu}^{(1)}(ze^{\pi i}) = -e^{-\nu \pi i} H_{\nu}^{(2)}(z) 
\]

\[
H_{\nu}^{(2)}(ze^{\pi i}) = -e^{-\nu \pi i} H_{\nu}^{(1)}(z) \quad \text{(D.7)}
\]

**D.3. Derivatives.** The derivative of a spherical Hankel function can be expressed in terms of spherical Hankel functions in a variety of ways:

\[
\frac{d}{dz} h_{l}^{(1)}(z) = \frac{lh_{l}^{(1)}(z)}{z} - h_{l+1}^{(1)}(z) = h_{l-1}^{(1)}(z) - \frac{(l+1)h_{l}^{(1)}(z)}{z} \quad \text{(D.9)}
\]

These relations are also valid for \( j_l, y_l, h_l^{(2)} \). Similarly,

\[
\frac{d}{dz} H_{\nu}^{(1)}(z) = \frac{\nu H_{\nu}^{(1)}(z)}{z} - H_{\nu+1}^{(1)}(z) = H_{\nu-1}^{(1)}(z) - \frac{\nu H_{\nu}^{(1)}(z)}{z} \quad \text{(D.10)}
\]

These relations are also valid for the Bessel functions \( J_l, Y_l \), and the incoming Hankel function \( H_l^{(2)} \).

Finally, a limit we require is a consequence of the above recursions and the polynomial representation of \( h_l^{(1)} \):

\[
l \frac{z \partial h_{l}^{(1)}(z)}{h_{l}^{(1)}(z)} = l - \frac{z h_{l+1}^{(1)}(z)}{h_{l}^{(1)}(z)} = l - \frac{p_{l+1}(z)}{p_l(z)} \to l - (2l + 1) = -(l + 1), \quad z \to 0
\]

**Appendix E. Asymptotic Expansions of Special Functions.**

**E.1. Airy function asymptotics.** The Airy function and its derivative can be approximated for large argument by (Olver [23])

\[
Ai(\tau) = \frac{1}{2} \pi^{-1/2} \tau^{-1/4} e^{-\mu} \left[ 1 + O(|\mu|^{-1}) \right], \quad |\arg \tau| < \pi \quad \text{(E.1)}
\]

\[
\partial Ai(\tau) = -\frac{1}{2} \pi^{-1/2} \tau^{-1/4} e^{-\mu} \left[ 1 + O(|\mu|^{-1}) \right], \quad |\arg \tau| < \pi \quad \text{(E.2)}
\]

\[
Ai(-\tau) = \pi^{-1/2} \tau^{-1/4} \cos(\mu - \pi/4) \left[ 1 + \tan(\mu - \pi/4)O(\mu^{-1}) \right], \quad |\arg \tau| < 2\pi/3 \quad \text{(E.3)}
\]

\[
Ai'(-\tau) = \pi^{-1/2} \tau^{1/4} \cos(\mu - 3\pi/4) \left[ 1 + \tan(\mu - 3\pi/4)O(\mu^{-1}) \right], \quad |\arg \tau| < 2\pi/3 \quad \text{(E.4)}
\]
where \( \mu = \frac{2}{3} \tau^{3/2} \). Additionally, we have the inequalities \[23\]

\[
|\text{Ai}(\tau)| < C \left( 1 + |\tau|^{1/4} \right)^{-1} |e^{-\mu}|, \quad |\arg \tau| \leq \pi \tag{E.5}
\]
\[
|\partial \text{Ai}(\tau)| < C \left( 1 + |\tau|^{1/4} \right) |e^{-\mu}|, \quad |\arg \tau| \leq \pi. \tag{E.6}
\]

E.2. Zeros of Airy Functions. All of the zeros of the Airy function and its derivative lie on the negative real axis. The larger zeros have the following approximations \[22, 41, 42\].

**Asymptotics for zeros of \( \text{Ai}(z) \):**

\[
\eta_s = -\left[ \frac{3}{8} \pi (4s - 1) \right]^{2/3} + \mathcal{O}(s^{-4/3}), \quad s = s_0, \ldots, \left\lfloor \frac{l + 1}{2} \right\rfloor. \tag{E.7}
\]

**Asymptotics for zeros of \( \partial \text{Ai}(z) \):**

\[
\eta'_s = -\left[ \frac{3}{8} \pi (4s - 3) \right]^{2/3} + \mathcal{O}(s^{-4/3}), \quad s = s'_0, \ldots, \left\lfloor \frac{l + 1}{2} \right\rfloor + 1. \tag{E.8}
\]

E.3. Hankel function asymptotics. The Hankel functions can be approximated for large order \( \nu \), uniformly in \( z \) in the sector \( |\arg z| < \pi \) by

\[
H_{\nu}^{(1)}(\nu z) \sim 2e^{-\pi i/3} \left( \frac{4\zeta}{1 - z^2} \right)^{1/4} \frac{\text{Ai}(e^{2\pi i/3} \nu^{2/3} \zeta)}{\nu^{1/3}} \tag{E.9}
\]
\[
\partial H_{\nu}^{(1)}(\nu z) \sim -4e^{-\pi i/3} \frac{1}{z} \left( \frac{1 - z^2}{4\zeta} \right)^{1/4} \frac{\text{Ai}(e^{2\pi i/3} \nu^{2/3} \zeta)}{\nu^{1/3}} \tag{E.10}
\]

where

\[
\frac{2}{3} \zeta^{3/2} = \int_{z}^{1} \frac{\sqrt{1 - t^2}}{t} dt = \log \frac{1 + \sqrt{1 - z^2}}{z} - \sqrt{1 - z^2}. \tag{E.11}
\]

We note the useful Taylor series

\[
\frac{2}{3} \zeta^{3/2} = (- \log(z) + \log(2) - 1) + \frac{z^2}{4} + O(z^4) \quad \text{as} \; z \to 0, \tag{E.12}
\]
\[
=-\frac{2}{3} i \sqrt{2} (z - 1)^{3/2} + \frac{3i(z - 1)^{5/2}}{5\sqrt{2}} + O\left((z - 1)^{7/2}\right) \quad \text{as} \; z \to 1, \tag{E.13}
\]
\[
=-iz + \frac{i\pi}{2} - \frac{i}{2z} - \frac{1}{24} \frac{1}{z^3} + O\left(\frac{1}{z^4}\right) \quad \text{as} \; z \to \infty. \tag{E.14}
\]

which yield the limits \( \lim_{z \to 0} \zeta(z) = \infty, \lim_{z \to 1} \zeta(z) = 0, \lim_{z \to \infty} \zeta(z) = \infty \). For \( \zeta \) close to zero, \( E.11 \) is inverted to yield

\[
z(\zeta) = 1 - 2^{-1/3} \zeta + \frac{1}{10} 2^{-2/3} \zeta^2 + \frac{1}{100} \zeta^3 + \zeta^4 + O(1). \tag{E.15}
\]

E.4. Zeros of Hankel functions. Concerning the zeros of Hankel functions a consequence of the analysis in \[22\] is the following:

**Theorem E.1.** Let \( \nu = l + 1/2, \) where \( l \geq 2 \) is an integer. The \( l + 1 \) zeros of \( \partial H_{\nu}^{(1)}(y) \) and the \( l \) zeros of \( H_{\nu}^{(1)}(y) \) lie near arcs in the lower half plane.
The detailed locations of the previous zeros given in the following

**Proposition E.2.** The $l$ zeros of $H_{\nu}^{(1)}(y)$ are

$$y_{\nu,k} = \nu \left[ 1 + 2^{-1/3} e^{-2\pi i/3} \nu^{-2/3} |\eta_k| \right] + \frac{3}{10} 2^{-2/3} e^{-4\pi i/3} \nu^{-4/3} |\eta_k|^2 + \mathcal{O}\left( \nu^{-4/3} \right),$$

(E.16)

for $k = 1, \ldots, [(l+1)/2]$, where $\eta_k$ is the $k$-th zero of $Ai$.

The $l+1$ zeros of $\partial H_{\nu}^{(1)}(y)$ are

$$y'_{\nu,k} = \nu \left[ 1 + 2^{-1/3} e^{-2\pi i/3} \nu^{-2/3} |\eta'_k| \right] + \frac{3}{10} 2^{-2/3} e^{-4\pi i/3} \nu^{-4/3} |\eta'_k|^2 + \mathcal{O}\left( \nu^{-4/3} \right),$$

(E.17)

for $k = 1, \ldots, [(l+1)/2] + 1$ where $\eta'_k$ is the $k$-th zero of $\partial Ai$.

We comment on approximating the zeros of $\partial H_{\nu}^{(1)}(\nu z)$ by Equation (E.17), which is closely related to the analysis of Olver [23]. The proof is analogous for $H_{\nu}^{(1)}(\nu z)$. Introduce the variables

$$\zeta = \zeta(z), \quad \eta = \eta(\zeta) = e^{2\pi i/3} \nu^{2/3} \zeta; \quad \text{see Equation (E.11).}$$

(E.18)

Using the uniform asymptotic approximation for $\partial H_{\nu}^{(1)}(\nu z)$ in (E.10) we can rewrite the equation $\partial H_{\nu}^{(1)}(y) = 0$, for $\nu = l + 1/2$ large as

$$\partial Ai(\eta) + \frac{e^{-2\pi i/3}}{\nu^{2/3}} Ai(\eta) \frac{\partial \eta}{\partial \nu} \left[ 1 + \mathcal{O}(\nu^{-2}) \right] = 0.$$  

(E.19)

For $\nu$ large, the zeros, $\eta$, are well approximated by zeros, $\{\eta'_k\}$, of $\partial Ai(\eta)$; see (22). These lie on the negative real axis; see Figure E.1 (left). Then, the asymptotics of

![Complex mappings](image)

**Fig. E.1.** Complex mappings of some $\partial Ai(\eta)$ zeros (left plot) to approximate zeros of $\partial H_{\nu}^{(1)}(\nu z)$ (right plot) by transformations $\eta \mapsto \zeta \mapsto z \mapsto \nu z$. The first map, $\eta \mapsto \zeta$, is given by inverting Equation (E.18). The second map, $\zeta \mapsto z$, is computed using Equation (E.19).

Airy functions and the properties of the mapping $\eta \mapsto \zeta \mapsto z \mapsto \nu z$ can be used to show give approximations to the zeros of $\partial H_{\nu}(y)$ and to establish their location near an arc in the lower half plane.

**Appendix F. Proof of Proposition 5.2.** In this section we prove Proposition 5.2, which states:

$$\text{Res}_{\lambda = \lambda_{k,j}} \frac{\lambda}{\lambda^2 + 1} G_1(\epsilon \lambda) = \mathcal{O}\left( \nu^{-4/3} \right).$$  

(F.1)

The main tool of this analysis is the uniform asymptotics of Hankel functions.
Specifically, \(\epsilon\lambda\) for the Hankel terms in (F.2), letting \(\lambda\) be the approximation for these resonances from Theorem 6.3:

\[
\eta \quad \text{There will be two cases to consider: resonances corresponding to Airy prime zeros for } s = 1, \ldots, s_0 \text{ which are } O(1) \text{ and for } s \approx l/2 \text{ which we will see are } O(l^{2/3}).
\]

In particular, we have used

\[
\frac{h^{(1)\prime}_l(z)}{h^{(1)}_l(z)} = -\frac{1}{2z} + H^{(1)\prime}_\nu(z) H^{(1)}_\nu(z), \quad \nu = l + 1/2.
\]

Now we expand the terms in the last line of (F.2) about \(\lambda_{l,j}\) for large \(l\). We recall the approximation for these resonances from Theorem 6.3:

\[
\lambda_{l,s} = \frac{l + 1/2}{\epsilon} \left[ 1 - 2^{-1/3} (l + 1/2)^{-2/3} e^{-2\pi i/3} \eta_s + O(l^{-1}) \right], \quad s = 1, 2, \ldots, \left\lfloor \frac{l + 1/2}{2} \right\rfloor + 1.
\]

There will be two cases to consider: resonances corresponding to Airy prime zeros \(\eta_s\) for \(s = 1, \ldots, s_0\) which are \(O(1)\) and for \(s \approx l/2\) which we will see are \(O(l^{2/3})\). Specifically,

\[
\eta_s = -\left( \frac{3\pi}{8} \right)^{2/3} (4s - 3)^{2/3} + \cdots, \quad s = s_0, \ldots, \left\lfloor \frac{l + 1}{2} \right\rfloor + 1
\]

\[
= -\left( \frac{3\pi}{2} \right)^{2/3} \nu^{2/3} + O(\nu^{-2/3}), \quad s \approx l/2 \to \infty.
\]

Since

\[
\frac{l(l+1)}{(\epsilon \lambda)^2} = \frac{(\nu - 1/2)(\nu + 1/2)}{\nu^2 \left[ 1 - e^{-2\pi i/3} 2^{-1/3} \nu^{-2/3} \eta_s \right]}
\]

we have

\[
\frac{l(l+1)}{(\epsilon \lambda)^2} = \begin{cases} 
1 + O(\nu^{-2/3}), & s \text{ small,} \\
O(1), & s \text{ large.}
\end{cases}
\]

For the Hankel terms in (F.2), letting \(\epsilon \lambda = \nu z\) and using Appendix E.3 we have

\[
\frac{H^{(1)\prime}_\nu(\nu z)}{H^{(1)}_\nu(\nu z)} \sim e^{2\pi i/3} \frac{1}{z} \left( \frac{1 - z^2}{\zeta} \right)^{1/2} \frac{Ai'(\tau)}{Ai(\tau)} \nu^{-1/3}, \quad \tau = e^{2\pi i/3} \nu^{2/3} \zeta.
\]

We simplify this ratio for \(z \approx 1\) using the Taylor expansion (E.13), which yields

\[
\zeta = 2^{1/3} e^{-\pi i/3} (z - 1) + O((z - 1)^2).
\]
Thus, \( \zeta^{-1}(1 - z^2) = -(1 + z)\frac{e^{1/\zeta}}{\zeta} = -\frac{2}{\pi} e^{-2\pi i/3} + \cdots \) and 
\( e^{2\pi i/3} \zeta^2 e^{2\pi i/3} = e^{2\pi i/3} \eta + \cdots \), where the dots indicate higher order terms. Now for the Airy functions 
\( \frac{\partial A_i(\tau)}{A_i(\tau)} = O(1) \) for small \( s \). From Appendix [5.1]

\[
\frac{\partial A_i(\tau)}{A_i(\tau)} \sim -\tau^{1/2}, \quad \tau \text{ large, } |\text{arg } \tau| < \pi.
\] (F.8)

So, for large \( s \)

\[
\left( \frac{\partial A_i(\tau)}{A_i(\tau)} \right)_{\tau = e^{2\pi i/3}, \nu^{2/3} \zeta} \sim - e^{-\pi i/3} \left( \frac{3\pi}{2} \right)^{2/3} \nu^{2/3} \left[ e^{-\pi i/6} \left( \frac{3\pi}{2} \right)^{1/3} \nu^{1/3} \right].
\]

Combining results

\[
\left( \frac{\partial A_i(\tau)}{A_i(\tau)} \right)_{\tau = e^{2\pi i/3}, \nu^{2/3} \zeta} = \begin{cases} 
O(1), & s \text{ small, } \\
\nu^{1/3} + O(\nu^{-2/3}), & s \approx \frac{1}{2} \text{ large.}
\end{cases}
\]

We can now conclude the proof. For \( s \) large

\[
- \frac{1}{2\epsilon \lambda} + \frac{H_{\nu}^{(1)}(\epsilon \lambda)}{H_{\nu}^{(1)}(\epsilon \lambda)} = - \frac{1}{2\nu} + O(1) = O(1).
\] (F.10)

Thus,

\[
\operatorname{Res}_{\lambda = \lambda_{i,j}, \lambda^2 + \tau G_l(\epsilon \lambda)} = \lim_{\lambda \to \lambda_{i,j}} \left[ 1 + \epsilon^2 r_l \left( -1 + O(1) - \frac{2}{\nu} O(1) - O(1^2) \right) \right]^{-1} = O(\nu^{-2}).
\] (F.11)

For \( s \) small,

\[
\operatorname{Res}_{\lambda = \lambda_{i,j}, \lambda^2 + \tau G_l(\epsilon \lambda)} = \lim_{\lambda \to \lambda_{i,j}} \left[ 1 + \epsilon^2 r_l \left( O(\nu^{-4/3}) + O(\nu^{-2/3}) \right) \right]^{-1} = O(\nu^{-4/3}).
\] (F.12)

The asymptotic formula (F.11) is a consequence of (F.11) and (F.12). \( \square \)

REFERENCES

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, New York, (1965).
[2] L. V. Ahlfors, Complex Analysis, McGraw-Hill Book Co., 2nd edition ed., 1966.
[3] M. P. Brenner, D. Lohse, and T. F. Dupont, Bubble shape oscillations and the onset of sonoluminescence, Phys. Rev. Lett., 75 (1995), pp. 954–957.
[4] N. Burq and M. Zworski, Resonance expansions in semi-classical propagation, Communications in Mathematical Physics, 223 (2001), pp. 1–12.
[5] A. P. Calderon and A. Zygmund, Singular integral operators and differential equations, American Journal of Mathematics, 79 (1957), pp. 901–921.
[6] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Atom-Photon Interactions, Wiley-Interscience, 1992.
[7] M.P. do Carmo, Differential geometry of curves and surfaces, Prentice-Hall Englewood Cliffs, NJ, 1976.
[8] D. Epstein and J.B. Keller, Expansion and contraction of planar, cylindrical, and spherical underwater gas bubbles, The Journal of the Acoustical Society of America, 52 (1972), p. 975.
[9] K. Ferrara and M. Pollard, R. Borden, Ultrasound microbubble contrast agents: Fundamentals and application to gene and drug delivery, Annual Review of Biomedical Engineering, 9 (2007), pp. 415–447.

[10] J. B. Keller and I. I. Kolodner, Damping of underwater explosion bubble oscillations, Journal of Applied Physics, 72 (1956), pp. 1152–1161.

[11] J. B. Keller and M. Miksis, Bubble oscillations of large amplitude, The Journal of the Acoustical Society of America, 68 (1980), pp. 628–633.

[12] A. Komech, M. Kunze, and H. Spohn, Effective dynamics for a mechanical particle coupled to a wave field, Communications in Mathematical Physics, 203 (1999), pp. 1–19.

[13] H. Lamb, Hydrodynamics, Cambridge University Press, 6 ed., 1993.

[14] P. D. Lax and R. S. Phillips, Scattering Theory, Academic Press, 1989.

[15] L.G. Leal, Advanced Transport Phenomena: Fluid mechanics and Convective Transport Processes, Cambridge Univ Press, 2007.

[16] TG Leighton, From seas to surgeries, from babbling brooks to baby scans: the acoustics of gas bubbles in liquids, International Journal of Modern Physics B, 18 (2004), pp. 3267–3314.

[17] A Lezzi and A Prosperetti, Bubble dynamics in a compressible liquid, part 2: second-order theory, Journal of Fluid Mechanics Digital Archive, 185 (1987), pp. 289–321.

[18] M.J. Lighthill, The Asymptotic Expansion of Bessel Functions of Large Order, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 267 (1962), pp. 147–182.

[19] M. S. Longuet-Higgins, Monopole emission of sound by asymmetric bubble oscillations, part I: normal modes, Journal of Fluid Mechanics Digital Archive, 201 (1989), pp. 525–541.

[20] P. D. Miller, A. Soffer, and M. I. Weinstein, Metastability of breather modes of time dependent potentials, Nonlinearity, 13 (2000), pp. 567–568.

[21] D.W. Moore and E.A. Spiegel, The generation and propagation of sound in a compressible atmosphere, J. Appl. Phys., 139 (1964), pp. 48.

[22] FWJ Olver, The Asymptotic Expansion of Bessel Functions of Large Order, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 247 (1954), pp. 326–368.

[23] The Asymptotic Solution of Linear Differential Equations of the Second Order for Large Values of a Parameter, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 247 (1954), pp. 307–327.

[24] M. S. Plesset, The dynamics of cavitation bubbles, J. Appl. Mech., 16 (1949), p. 277.

[25] A. Prosperetti, Bubbles, Physics of Fluids, 16 (2004), pp. 1852–1865.

[26] A. Prosperetti and A. Lezzi, Bubble dynamics in a compressible liquid, part 1: first-order theory, Journal of Fluid Mechanics Digital Archive, 168 (1986), pp. 457–478.

[27] S. J. Putterman and P. H. Roberts, Comment on “bubble shape oscillations and the onset of sonoluminescence”, Phys. Rev. Lett., 80 (1998), pp. 3666–3667.

[28] L. Rayleigh, On the pressure developed in a liquid during the collapse of a spherical cavity, Phil. Mag., 34 (1917), pp. 94–98.

[29] M. Reed and B. Simon, Methods of Modern Mathematical Physics, vol. IV, Academic Press, 1978.

[30] M. Ripepe and E. Gordeev, Gas bubble dynamics model for shallow volcanic tremor at Stromboli, J. Geophys. Res., 104 (1999).

[31] A. Shaw-Krauss, Radiative Decay of Bubble Oscillations in a Compressible Fluid, PhD thesis, Columbia University, New York, NY, USA, 2010.

[32] I. M. Sigal, Nonlinear wave and Schrödinger equations i. Instability of time- periodic and quasiperiodic solutions, Communications in Mathematical Physics, 153 (1993), p. 297.

[33] A. Soffer and M. I. Weinstein, Time dependent resonance theory, Geometric and Functional Analysis, 8 (1998).

[34] Resonances, radiation damping, and instability of Hamiltonian nonlinear waves, Inventiones Mathematicae, 136 (1999).

[35] P. Stefanov, Resonance expansions and Rayleigh waves, Math. Res. Lett., 8 (2001), pp. 105–124.

[36] P. Stefanov and G. Vodev, Distribution of resonances for the Neumann problem in linear elasticity outside a ball, Annales de l’institut Henri Poincaré (A) Physique théorique, 60 (1994), pp. 303–321.

[37] P. Stefanov and G. Vodev, Distribution of resonances for the Neumann problem in linear elasticity outside a strictly convex body, Duke Math. J., 78 (1995), pp. 677–714.

[38] Neumann resonances in linear elasticity for an arbitrary body, Communications in Mathematical Physics, 176 (1996), pp. 645–659.

[39] S.H. Tang and M. Zworski, Resonance expansions of scattered waves, Communications on
[40] M. Taylor, *Partial differential equations I, II, III*, Bull. Amer. Math. Soc. 35 (1998), 175-177., (1998).

[41] T. Tokita, *Exponential decay of solutions for the wave equation in the exterior domain with spherical boundary*, J. Math. Kyoto Univ. 12 (1972), pp. 413–430.

[42] G. N. Watson, *A Treatise on the Theory of Bessel Function*, Cambridge University Press, Cambridge, 1952.

[43] M. I. Weinstein, *Resonance Problems in Photonics*, in Frontiers of Applied Mathematics, World Scientific, 2007.

[44] V. Weisskopf and E. Wigner, *Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie*, Z. Phys., 63 (1930).

[45] C. Wilcox, *The initial-boundary value problem for the wave equation in an exterior domain with spherical boundary*, Amer. Math. Soc. Not., Abstract, 6 (1959), pp. 869–870.

[46] J. Xu and D. Attinger, *Acoustic excitation of superharmonic capillary waves on a meniscus in a planar microgeometry*, Physics of Fluids, 19 (2007), p. 108107.