Functions conditionally of negative type on groups acting on regular trees

Antoine GOURNAY* and Pierre-Nicolas JOLISSAINT†

February 3, 2015

Abstract

Let $T_{q+1}$ be the $q + 1$-regular tree and let $G$ be a group of automorphisms acting transitively on the vertices and on the boundary of $T_{q+1}$. We give an upper bound for the growth of cocycles with values in any unitary representation of the group $G$. This bound is optimal by projecting the Haagerup cocycle onto the subspace of the kernel of the divergence. We also obtain a description of functions conditionally of negative type which are unbounded.

1 Introduction

For a locally finite, non-oriented graph $X = (V, E)$, we will denote by $\mathcal{L}$ the Laplace operator defined as

$$\mathcal{L}f(x) = \left(\frac{1}{\deg(x)} \sum_{y \sim x} f(y) \right) - f(x),$$

for any $f \in \ell^2(V)$ and $x \in V$. More generally, for any Hilbert space $\mathcal{H}$, this defines an operator on

$$\ell^2(V, \mathcal{H}) := \{ F : V \to \mathcal{H} : \sum_{v \in V} \|F(v)\|^2 < \infty \}.$$

It is therefore natural to call a map $F \in \ell^2(V, \mathcal{H})$ harmonic if $\mathcal{L}F$ is identically 0.

Let $q \geq 2$ and let $T_{q+1}$ be the homogeneous $(q + 1)$-regular tree and let $\partial T_{q+1}$ be its boundary. The first result of our paper is the following theorem.

*Supported by the ERC-StG 277728 “GeomAnGroup”.
†Supported by Swiss SNF project 20-137696.
Theorem 1.1. Let $G$ be a closed non compact subgroup of $\text{Aut}(T_{q+1})$, with $q \geq 2$. Suppose that $G$ acts transitively on the vertices and on the boundary of $T_{q+1}$. Assume furthermore that $G$ acts on a Hilbert space $H$ by affine isometries. Then, any $G$-equivariant map $F : V \to H$ such that $F(x_0) = 0$ for some $x_0 \in V$ satisfies:

$$\|F(x)\|^2 \leq Ad(x, x_0) - B + Cq^{1-d(x,x_0)},$$

where $x_1$ is any vertex adjacent to $x_0$,

$$A = \frac{(q + 1)\|F(x_1)\|^2}{q - 1}, \quad B = \frac{2q\|F(x_1)\|^2}{(q - 1)^2}, \quad \text{and} \quad C = \frac{2\|F(x_1)\|^2}{(q - 1)^2}.$$

Furthermore, if the $F$ is harmonic and non-constant, then the equality occurs.

The second result shows that it is optimal by proving the existence of such an equivariant and harmonic map.

Proposition 1.2. Let $G$ be a closed non compact subgroup of $\text{Aut}(T_{q+1})$, with $q \geq 2$. Suppose that $G$ acts transitively on the vertices and on the boundary of $T_{q+1}$. Then, there exists a Hilbert space $H$ endowed with an affine isometric $G$-action and a map $F : V \to H$ which is $G$-equivariant, non-constant and harmonic.

Combining this with a result of Nebbia, we deduce that this map is essentially unique and we give a concrete realization of this cocycle as the projection of the Haagerup cocycle onto an appropriate $G$-invariant closed subspace of $\ell^2_{\text{alt}}(E)$.

The paper is organised as follows: Theorem 1.1 is proved in section 2, and Proposition 1.2 in section 3. As a consequence, we discuss the asymptotic behaviour of pure elements of $\text{CL}(G)$, the convex, positive cone of negative type functions on $G$. This is done in subsection 4.1. Then, in subsection 4.2, we decompose the Haagerup cocycle onto the kernel of the divergence and its orthogonal subspace in $\ell^2_{\text{alt}}(E)$, and compute the norms of the projected cocycles, giving an alternative proof of Proposition 1.2. In the last subsection of this paper, we give an interesting family of examples of functions conditionally of negative type and prove when they are pure.

Acknowledgements: We are grateful to Alain Valette to have suggested to look at the projection of the Haagerup cocycle and to have shown Lemma 2.3 to us.
2 Proof of Theorem 1.1

We first fix some notations concerning cocycles and affine isometric actions (we refer to Chapter 2 of [BdlHV08] for a complete discussion). Let $G$ be a second countable, locally compact group and let $\pi$ be a continuous unitary representation on a Hilbert space $\mathcal{H}$. A **cocycle** with values in $\pi$ is a continuous function $b : G \to \mathcal{H}$ satisfying the so-called cocycle relation, that is, $b(gh) = \pi(g)b(h) + b(g)$, for every $g, h \in G$. A cocycle of the form $g \mapsto (1 - \pi(g))v$, for some vector $v \in \mathcal{H}$, is called a **coboundary**. We denote by $Z^1(G, \pi)$ (resp. $B^1(G, \pi)$) the space of cocycles (resp. coboundaries) with values in $\pi$. The first cohomology space of $\pi$ is defined as the quotient space $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$. Note that, a map $b$ satisfies the cocycle relation, if and only if, $G$ acts by affine isometries on $\mathcal{H}$ by $\alpha(g)v = \pi(g)v + b(g)$. It is clear from the definition that $b(g) = \alpha(g)0$ and that the affine action $\alpha$ has a fixed point if and only if $b$ is a coboundary. We will say that $\pi$ (resp. $b$) is the linear (resp. translation) part of the affine isometric action $\alpha$, and use the self-explanatory notation $\alpha = (\pi, b)$.

Now, we turn to the proof of Theorem 1.1. We start with a lemma.

**Lemma 2.1.** Let $X = (V, E)$ be a locally finite graph. Let $F : V \to \mathcal{H}$ be any map.

1. For any $x \in V$, the following identity holds:
   $$\mathcal{L}(\|F\|^2)(x) = \|\nabla_x F\|^2_{\mathcal{H}} + 2\Re(\mathcal{L}F(x), F(x))_{\mathcal{H}},$$
   where
   $$\|\nabla_x F\|^2 = \|\nabla_x F\|^2_{\mathcal{H}} = \frac{1}{\deg(x)} \sum_{y \sim x} \|F(x) - F(y)\|^2.$$

2. Let $G$ be a group acting on $X$ by automorphisms and on $\mathcal{H}$ by affine transformations. Then, for any $G$-equivariant map $F : V \to \mathcal{H}$, the quantity $\|\nabla_x F\|$ is constant along each orbit of $x \in V$, that is,
   $$\|\nabla_{gx} F\| = \|\nabla_x F\|, \ \forall x \in V, g \in G.$$

**Proof :** The proof of the first identity is a simple calculation. Since we have
   $$\|\nabla_x F\|^2 = \|F(x)\|^2 + \frac{1}{\deg(x)} \sum_{y \sim x} \|F(y)\|^2 - \frac{2}{\deg(x)} \sum_{y \sim x} \Re(F(y), F(x)).$$
and
\[ \langle \mathcal{L}F(x), F(x) \rangle = -\|F(x)\|^2 + \frac{1}{\text{deg}(x)} \sum_{y \sim x} \langle F(y), F(x) \rangle, \]
it readily follows that
\[ \|\nabla_x F\|^2 + 2\Re\langle \mathcal{L}F(x), F(x) \rangle = -\|F(x)\|^2 + \frac{1}{\text{deg}(x)} \sum_{y \sim x} \|F(y)\|^2 \]
\[ = \mathcal{L}\|F\|^2(x). \]

To prove the second statement, we recall that the equivariance implies
\[ \|F(gx) - F(y)\| = \|\alpha(g)F(x) - F(y)\| \]
\[ = \|F(x) - \alpha(g^{-1})F(y)\| \]
\[ = \|F(x) - F(g^{-1}y)\|, \]
for all \( g \in G, x, y \in V, \) where \( \alpha \) is the map corresponding to the \( G \)-action on \( \mathcal{H} \). Hence, we obtain
\[ \|\nabla_{gx} F\|^2 = \frac{1}{\text{deg}(x)} \sum_{y \sim gx} \|F(gx) - F(y)\|^2 \]
\[ = \frac{1}{\text{deg}(x)} \sum_{y \sim gx} \|F(x) - F(g^{-1}y)\|^2 \]
\[ = \|\nabla_x F\|^2. \]

**Lemma 2.2.** Let \( G \) be acting transitively on \( \partial T_{q+1} \) and let us denote by \( G_{x_0} \) the stabiliser of \( x_0 \). If the map \( F : V \to \mathcal{H} \) satisfies \( F(x_0) = 0 \) for some point \( x_0 \) and if \( F \) is \( G \)-equivariant with respect to some isometric affine action \( \alpha = (\pi, b) \), where \( b \) is zero on \( G_{x_0} \), then both \( \|F\| \) and \( \langle \mathcal{L}F, F \rangle \) are radial.

**Proof:** Let \( x, y \in V \) be two vertices at distance \( r \) from \( x_0 \). Since \( G_{x_0} \) acts transitively on any sphere about \( x_0 \), we can find \( h \in G_{x_0} \) so that \( hx = y \).

By the hypothesis on \( b \), \( \alpha \) acts by unitary operators when restricted to the stabilizer of \( x_0 \). Using the equivariance of \( F \), it is straightforward that
\[ \|F(y)\| = \|\pi(h)F(x)\| = \|F(x)\|. \]

Moreover, using the \( G \)-equivariance of \( \mathcal{L} \) with respect to \( \pi \), we obtain
\[ \langle \mathcal{L}F(y), F(y) \rangle = \langle \mathcal{L}F(hx), F(hx) \rangle \]
\[ = \langle \pi(h)\mathcal{L}F(x), \pi(h)F(x) \rangle \]
\[ = \langle \mathcal{L}F(x), F(x) \rangle. \]
For the rest of the section, we will assume that $G$, $F$ and $\alpha = (\pi, b)$ satisfy the hypothesis of Theorem 1.1. Select one vertex on each sphere of radius $n$ and denote it by $x_n$. We can make two remarks.

1. Since $G$ acts transitively on $V$, then $\|\nabla_x F\| = \|\nabla_{x_0} F\| = \|F(x_1)\|$.

2. We recall that a non compact closed subgroup of $\text{Aut}(\mathcal{T}_{q+1})$ acts transitively on $\partial \mathcal{T}_{q+1}$ if and only if there exists $y \in V$ so that the stabilizer $G_y$ acts transitively on $\partial \mathcal{T}_{q+1}$ (see Proposition 10.1 in [FTN91]). If the group $G$ acts transitively on $V$, then the latter condition is also equivalent to having all the vertex-stabilizers acting transitively on $\partial \mathcal{T}_{q+1}$. This means that under the hypotheses of Theorem 1.1, the subgroup $G_{x_0}$ acts transitively on $\partial \mathcal{T}_{q+1}$.

Now, set $\varphi(n) = \|F(x_n)\|^2$. For radial maps, the Laplace operator takes a very simple form:

$$L\varphi(0) = \varphi(1) - \varphi(0),$$

and

$$L\varphi(n) = \frac{q}{q+1} \varphi(n+1) - \varphi(n) + \frac{1}{q+1} \varphi(n-1),$$

for any $n \geq 1$. Using Lemma 2.1 and setting $R_F(n) = 2\Re\langle LF(x_n), F(x_n) \rangle$, we obtain the relations for all $n \geq 1$,

$$\frac{q}{q+1} \varphi(n+1) - \varphi(n) + \frac{1}{q+1} \varphi(n-1) = \varphi(1) + R_F(n),$$

(1)

with initial conditions $\varphi(0) = 0$ and $\varphi(1) = \|F(x_1)\|^2$. Let us find a general solution to this second order linear recurrence equation. Set

$$\psi(n+1) = \varphi(n+1) - \varphi(n),$$

for all $n \geq 0$. We can express the left-hand side of the relation (1) using $\psi$:

$$\frac{q}{q+1} \varphi(n+1) - \varphi(n) + \frac{1}{q+1} \varphi(n-1) = \frac{q}{q+1} (\varphi(n+1) - \varphi(n)) - \frac{1}{q+1} (\varphi(n) - \varphi(n-1)) = \frac{q}{q+1} \psi(n+1) - \frac{1}{q+1} \psi(n).$$

For all $n \geq 1$, we obtain the new relation:

$$\psi(n+1) = \frac{1}{q} \psi(n) + \frac{(q+1)\|F(x_1)\|^2}{q} + \frac{q+1}{q} R_F(n),$$

and setting $\psi(x_0) = 0$, we obtain $\psi(n+1) = \frac{(q+1)\|F(x_1)\|^2}{q} + \frac{q+1}{q} R_F(n)$.

For all $n \geq 1$, we obtain the new relation:
with initial condition $\psi(1) = \|F(x_1)\|^2$. By iterating this relation, we get

$$
\psi(n + 1) = \frac{(q + 1)\|F(x_1)\|^2}{q - 1} - \frac{2\|F(x_1)\|^2}{(q - 1)q^n} + (q + 1) \sum_{j=1}^{n} \frac{R_F(n + 1 - j)}{q^j}.
$$

To proceed, we need a crucial negativity result.

**Lemma 2.3.** Under the hypotheses of Theorem 1.1, we have

$$
\Re(LF(x), F(x)) \leq 0,
$$

for all $x \in V$.

Let us postpone the proof of this lemma to the end of the section and let us show how to finish the proof of Theorem 1.1. Firstly, Lemma 2.3 implies the following inequality

$$
\psi(n + 1) \leq \frac{(q + 1)\|F(x_1)\|^2}{q - 1} - \frac{2\|F(x_1)\|^2}{(q - 1)q^n},
$$

with equality if $F$ is harmonic. Replacing $\psi$ by $\varphi$, we get

$$
\varphi(n + 1) \leq \varphi(n) + \frac{(q + 1)\|F(x_1)\|^2}{q - 1} - \frac{2\|F(x_1)\|^2}{(q - 1)q^n}.
$$

Iterating this inequality, we obtain the desired upper bound. Once again, if $F$ is harmonic, then the equality occurs, and we are done. □

### 2.0.1 Proof of Lemma 2.3

The end of the present section is dedicated to the proof of Lemma 2.3. The main steps of the proof are described in the next lemma.

**Lemma 2.4.** We assume that the group $G$ and the map $F$ satisfy the hypotheses of Theorem 1.1. Also, we write $K$ for the compact subgroup $G_{x_0}$ and $dk$ for the normalised Haar measure on $G$ so that the subgroup $K$ has measure 1. Fix $x \in V$ and let $g$ and $s$ in $G$ be such that $x = g^{-1}x_0$ and $sx_0$ is adjacent to $x_0$.

1. For any vertex $y \in V$ adjacent to $x_0$, the map $F$ satisfies the following integral formula:

$$
\int_{K} F(g^{-1}ky)dk = \frac{1}{q + 1} \sum_{u \sim x} F(u).
$$
2. Furthermore, if we denote by $\alpha = (\pi, b)$ the affine $G$-action on $H$, then
\[ \mathcal{L}F(x) = \int_{K} \pi(g^{-1}k)b(s)dk. \]

3. We also have
\[ \langle \mathcal{L}F(x), F(x) \rangle = -\langle P_{K}b(s), b(g) \rangle, \]
where $P_{K} = \int_{K} \pi(k)dk$ is the orthogonal projection onto the space of $\pi(K)$-fixed vectors in $H$.

4. In particular, desintegrating $\pi$ and $b$ as direct integrals over some measure space $(Z, \nu)$, we obtain
\[ \langle \mathcal{L}F(x), F(x) \rangle = -\int_{Z} \langle P_{z}b_z(s), b_z(g) \rangle d\nu(z), \]
where, for almost all $z \in Z$, $\pi_z$ is an irreducible unitary representation of $G$, $b_z$ is a $\pi_z$-cocycle, and
\[ P_z = \int_{K} \pi_z(k)dk \]
is the orthogonal projection onto the space of $\pi_z(K)$-invariant vectors.

5. Finally, for any irreducible unitary representation $\sigma$ and for any cocycle $w \in Z^1(G, \sigma)$, we have
\[ \langle Pw(s), w(g) \rangle \geq 0, \]
where $P = \int_{K} \sigma(k)dk$ denotes the orthogonal projection onto the space of $\sigma(K)$-invariant vectors.

Clearly, Lemma 2.3 follows directly from the last two claims above.

**Proof of Claim [1]**: Fix $x \in V$. Since $G$ acts on $V$ transitively, there exists $g \in G$ so that $x = g^{-1}x_0$. Let $y_1, \ldots, y_{q+1}$ be the $q + 1$ neighbours of $x_0$. Using the fact that the action of $G_{x_0}$ on the sphere of radius 1 centered at $x_0$ is transitive, we can find $h_j \in G_{x_0}$ so that $h_j y_j = y_1$, for $j = 1, \ldots, q + 1$. We remark that the cosets given by $h_j^{-1}(G_{x_0} \cap G_{y_1})$ are all distinct and that the subgroup $G_{x_0} \cap G_{y_1}$ has index $q + 1$ in $G_{x_0}$. Normalising the Haar measure on
so that the compact subgroup $G_{x_0}$ has measure 1, we obtain the following relation:

$$\int_{G_{x_0}} F(g^{-1}y_1) dk = \sum_{j=1}^{q+1} \int_{G_{x_0} \cap G_{y_1}} F(g^{-1}h_j^{-1}y_1) dk$$

$$= \sum_{j=1}^{q+1} F(g^{-1}y_j) \int_{G_{x_0} \cap G_{y_1}} dk$$

$$= \frac{1}{q+1} \sum_{j=1}^{q+1} F(g^{-1}y_j)$$

$$= \frac{1}{q+1} \sum_{u \sim x} F(u).$$

In particular, we see that the integral on the left-hand side does not depend on the choice of the neighbour of $x_0$.

**Proof of Claim 2**: Again, by transitivity of the $G$-action, we can find $s \in G$ so that $sx_0 = y_1$. It is straightforward to see that $b$, the translation part of the $G$-action on $H$, factors through a $G$-equivariant map $G/K \to H$ and that it coincides with $F$. Namely, $F(hx_0) = b(h)$, for all $h \in G$. In particular, $F(x) = b(g^{-1})$. By the cocycle relation, we have

$$F(g^{-1}y_1) - F(g^{-1}x_0) = F(g^{-1}ksx_0) - F(g^{-1}x_0)$$

$$= b(g^{-1}ks) - b(g^{-1}k)$$

$$= \pi(g^{-1}k)b(s) + b(g^{-1}k) - b(g^{-1}k)$$

$$= \pi(g^{-1}k)b(s).$$

Hence, we obtain the desired integral formula for the Laplace operator applied to $F$:

$$\mathcal{L}F(x) = \left( \frac{1}{q+1} \sum_{y \sim x} F(y) \right) - F(x)$$

$$= \int_K F(g^{-1}ksx_0) dk - F(g^{-1}x_0)$$

$$= \int_K (F(g^{-1}ksx_0) - F(g^{-1}x_0)) dk$$

$$= \int_K \pi(g^{-1}k)b(s) dk.$$
Proof Claim 3: Hence, applying Claim 2 we directly get:

\[ \langle LF(x), F(x) \rangle = \left\langle \int_K \pi(g^{-1}k)b(s)dk, b(g^{-1}) \right\rangle \]

\[ = \int_K \langle \pi(g^{-1}k)b(s), b(g^{-1}) \rangle dk \]

\[ = \int_K \langle \pi(k)b(s), \pi(g)b(g^{-1}) \rangle dk \]

\[ = -\int_K \langle \pi(k)b(s), b(g) \rangle dk \]

\[ = -\left\langle \int_K \pi(k)b(s)dk, b(g) \right\rangle \]

\[ = -\langle P_K b(s), b(g) \rangle. \]

Proof of Claim 4: Now, let us desintegrate \( \pi \) and \( b \) as direct integrals. We write

\[ \pi = \int_Z \pi_z d\nu(z), \]

and

\[ b = \int_Z b_z d\nu(z), \]

for some measure space \((Z, \nu)\). Recall that, for almost all \( z \in Z \), \( \pi_z \) is an irreducible unitary representation of \( G \) and \( b_z \) is a \( \pi_z \)-cocycle. Thus, we deduce from Claim 3:

\[ \langle LF(x), F(x) \rangle = -\int_Z \langle P_z b_z(s_0), b_z(g) \rangle d\nu(z). \]

Proof of Claim 5: As \( G \) acts transitively on \( \partial T_{q+1} \), the couple \((G, K)\) forms a Gelfand pair (see chapter II, section 4 in [FTN91]) and we treat three cases. If the irreducible representation \( \sigma \) is not spherical, then \( P = 0 \) and the inner-product is 0. If \( \sigma = 1_G \) is the trivial representation, then \( w = 0 \) and the inner-product is again equal to 0. Indeed, under the assumptions
on $G$, we have that $H^1(G, 1_G) = \text{Hom}(G, \mathbb{C}) = 0$ (see p. 5 of [Neb12]). Finally, if we suppose that $\sigma$ is spherical and non-trivial, then there exists a $\sigma(K)$-invariant vector $\eta$ such that

$$w(h) = (\sigma(h) - 1)\eta, \forall h \in G.$$ 

Up to rescaling $w$, we can assume that $\eta$ has norm 1. As the space of $\sigma(K)$-invariant vectors is one-dimensional, $P$ is a rank-one operator. So, we can write

$$P\xi = \langle \xi, \eta \rangle \eta,$$

for all $\xi \in \mathcal{H}$. This yields

$$\langle Pw(s), w(g) \rangle = \langle P(\sigma(s) - 1)\eta, (\sigma(g) - 1)\eta \rangle$$
$$= \langle (\sigma(s) - 1)\eta, \eta \rangle \langle \eta, (\sigma(g) - 1)\eta \rangle$$
$$= (\langle \sigma(s)\eta, \eta \rangle - 1)(\langle \eta, \sigma(g)\eta \rangle - 1)$$
$$= (\phi(s) - 1)(\phi(g^{-1}) - 1),$$

where $\phi$ is the (normalised) positive-definite function of $\pi$ associated to $\eta$. In particular, $\phi$ is a spherical function, that is, a radial eigenfunction of the normalised adjacency operator on $\mathcal{T}_{q+1}$ and $\phi(x_0) = 0$. This function being positive-definite and radial, it is therefore real-valued. By Cauchy-Schwarz, we conclude that $\phi(h) - 1 \leq 0$, for all $h \in G$ and therefore, the scalar product we began with is always positive or null. This ends the proof of the Claim

3 Proof of Proposition 1.2

The existence of an equivariant and harmonic map follows from a general argument due to Shalom. We recall it here briefly and we refer to Chapter 3 of [BdlHV08] for more details. Let $G$ be a group as in Theorem 1.1. Fixing a based-vertex $x_0 \in V$, we recall that $(G, K)$ is a Gelfand pair, with $K = G_{x_0}$. Moreover, $G$ is locally compact and compactly generated. As in the proof of Lemma 2.3, $H^1(G, 1_G) = 0$. Since $G$ does not have property (T), there exists a non-trivial irreducible unitary representation $\pi$ admitting an unbounded,

\footnote{We recall the argument briefly for the reader’s convenience. Let $\phi \in \text{Hom}(G, \mathbb{C})$ be a continuous homomorphism. The group $G$ is generated by the vertex-stabilizer $G_{x_0}$ and by the edge-stabilizer $G_{[x_0, y_1]}$. Both subgroups being compact, their images under $\phi$ are compact subgroups of $\mathbb{C}$. Therefore, $\phi$ is identically 0.}

3 Proof of Proposition 1.2

The existence of an equivariant and harmonic map follows from a general argument due to Shalom. We recall it here briefly and we refer to Chapter 3 of [BdlHV08] for more details. Let $G$ be a group as in Theorem 1.1. Fixing a based-vertex $x_0 \in V$, we recall that $(G, K)$ is a Gelfand pair, with $K = G_{x_0}$. Moreover, $G$ is locally compact and compactly generated. As in the proof of Lemma 2.3, $H^1(G, 1_G) = 0$. Since $G$ does not have property (T), there exists a non-trivial irreducible unitary representation $\pi$ admitting an unbounded,
continuous cocycle \( b \) (see Corollaire 1 in \([LSV04]\)). As \( K \) is compact, we can find a cocycle vanishing on \( K \) which is in the cohomology class of \( b \). Hence, we can assume that \( b \) is identically 0 on \( K \) and that \( b \) factors through a \( G \)-equivariant map on \( G/K \), which we denote by \( F \). By Proposition 3.3.7 in \([BdlHV08]\), the map \( F \) satisfies the following mean value property:

\[
\int_K F(g_0 k g_0^{-1} g K) dk = F(g_0 K),
\]

for all \( g, g_0 \in G \). As \( b \) is not a coboundary, we remark that \( F \) is not constant. Seen as a map on \( V \), we claim that \( F \) is harmonic.

**Proof of Proposition 1.2:** Fix \( g_0 \in G \). For every \( g \in G \), we will write \( gF(x) = F(gx) \). For any \( x \in G/K \), we set

\[
\left( \int_K g_0 k F dk \right)(x) = \int_K (g_0 k F)(x) dk = \int_K F(g_0 k x) dk.
\]

For \( x \in G/K \) fixed, the integral exists, as we are just integrating a continuous map over a compact set. Hence, the map \( \int_K g_0 k F dk : G/K \to \mathcal{H} \) is well defined. We claim that it is constant. Indeed, by the mean value property, we have

\[
\left( \int_K g_0 k F dk \right)(x) = \int_K (g_0 k F)(x) dk = \int_K F(g_0 k x) dk = \int_K F(g_0 k g_0^{-1} (g_0 x)) dk = F(g_0 K).
\]

Since \( \mathcal{L} \) annihilates the constants, we deduce that \( \mathcal{L} \left( \int_K g_0 k F dk \right) \equiv 0 \). Now, using the linearity of \( \mathcal{L} \) and the fact that it commutes with the \( G \)-action, we get:

\[
\mathcal{L} \left( \int_K g_0 k F dk \right) = \int_K \mathcal{L} (g_0 k F) dk = \int_K g_0 k (\mathcal{L} F) dk.
\]

Therefore, one has
\[(\mathcal{L}F)(g_0x_0) = \int_K (\mathcal{L}F)(g_0kx_0)dk\]
\[= \int_K (\mathcal{L}F)(g_0kx_0)dk\]
\[= \mathcal{L}\left(\int_K g_0kFdk\right)(x_0)\]
\[= 0.\]

Since \(g_0 \in G\) is arbitrary and the action of \(G\) on \(V\) is transitive, we conclude that \(\mathcal{L}F\) vanishes everywhere on \(V\) and we are done. \(\square\)

In fact, the irreducible representation \(\pi\) is unique and \(H^1(G, \pi)\) has complex dimension 1 (see [Neb12], where this representation is denoted by \(\sigma^-\)). Furthermore, \(\pi\) being spherical, it has no non-trivial \(K\)-invariant vector. In particular, this implies that \(H^1(G, \pi)\) is isomorphic to \(Z^1_K(G, \pi)\), the space of 1-cocycles which vanish on the compact subgroup \(K\). Therefore, Proposition 1.2 gives an alternative description of the unique unbounded, cocycle which is identically 0 on \(K\) and which appears in Nebbia’s paper.

**Corollary 3.1.** The cocycle \(F\) of Proposition 1.2 is a representative of the only non-trivial cohomology class of the unique irreducible unitary representation of \(G\) which has a non-vanishing cohomology group in degree 1.

**4 Conditionally negative type functions on \(G\)**

In this section, we will see that Proposition 1.2 gives rise to an interesting example of a kernel which is conditionally of negative type on the set of vertices of a regular tree. In order to state the result, we recall a few facts about negative type kernels and negative type functions. We refer to the Appendix C of [BdlHV08] for the proofs and more details.

**4.1 Definitions and application of Proposition 1.2**

A kernel \(\Psi : W \times W \rightarrow \mathbb{R}\) on a set \(W\) is said to be conditionally of negative type if it satisfies the following properties:

(i) \(\Psi(x, x) = 0\), for all \(x \in W\);
(ii) $\Psi(x, y) = \Psi(y, x)$, for all $x, y \in W$;

(iii) For any $x_1, \ldots, x_n \in W$ and for any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ satisfying $
\sum_{i=1}^{n} \alpha_i = 0$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \Psi(x_i, x_j) \leq 0.$$

Examples of such kernels are given by $\Psi(x, y) = \|f(x) - f(y)\|^2$, where $f : W \to \mathcal{H}$. A continuous function $\psi : G \to \mathbb{R}$ on a topological group $G$ is said to be **conditionally of negative type** if the kernel on $G$ defined by $(g, h) \mapsto \psi(g^{-1} h)$ is conditionally of negative type. It is easy to see that if $b$ is a continuous cocycle for some unitary representation $\pi$ of $G$, then the function $g \mapsto \|b(g)\|^2$ is conditionally of negative type. This example is essentially universal. Namely, given $\psi$ a function conditionally of negative type on a group $G$, by the GNS construction, there exist $\pi_\psi$ a unitary representation of $G$ and a cocycle $b_\psi \in Z^1(G, \pi_\psi)$ satisfying $\psi(g) = \|b_\psi(g)\|^2$, for all $g \in G$. It is well-known that, for a group $G$, the set of such functions forms a convex, positive cone, denoted by $\text{CL}(G)$. We say that a function $\psi \in \text{CL}(G)$ is **pure** if it lies on an extremal ray of $\text{CL}(G)$. We have the following:

**Theorem 4.1.** (Théorème 1, [LSV04]) Let $G$ be a topological group.

(i) Let $\psi$ be a function conditionally of negative type on $G$ and let $(\pi_\psi, \mathcal{H}_\psi, b_\psi)$ be its associated GNS triple. If $\psi$ is pure, then the orthogonal representation $\pi_\psi$ is irreducible.

(ii) Let $\pi$ be an irreducible orthogonal representation and let $b$ be any 1-cocycle for the representation $\pi$. Then, the function of negative type $\psi$ associated with $b$ is pure.

This yields:

**Corollary 4.2.** (i) The kernel defined on the set of vertices of $\mathcal{T}_{q+1}$ by

$$\Psi : (x, y) \mapsto d(x, y) - \frac{2q}{q^2 - 1} + \frac{2q^{1-d(x,y)}}{q^2 - 1}$$

is conditionally of negative type.
Let $G$ be as in Theorem 1.1 and choose a basepoint $x_0 \in V$. For $g \in G$, we set $|g| := d(x_0, gx_0)$. Then, the function on $G$ defined by $g \mapsto \Psi(gx_0, x_0) = |g| + \frac{2q}{q^2 - 1} (q^{-|g|} - 1)$ is the unique (up to multiplication by a positive scalar) pure negative type function in $\text{CL}(G)$ which is unbounded on $G$ and identically 0 on $G_{x_0}$.

**Proof**: To prove the first claim, we simply remark that $\Psi(x, y) = \|F(x^{-1} y)\|^2$, where the map $F$ is as in Proposition 1.2 and is normalised so that $\|F(x)\|^2 = \frac{q - 1}{q + 1}$.

Clearly, the function here above is conditionally of negative type and pure, by Corollary 3.1 and Theorem 4.1. Let us check the uniqueness. Let $\psi$ be a function conditionally of negative type which vanishes on $G_{x_0}$ and which is unbounded on $G$. Then, by Theorem 4.1 we know that $b_\psi$, the cocycle associated with $\psi$, is unbounded, pure and that it vanishes on $G_{x_0}$. By Corollary 3.1 we conclude that $b_\psi$ is the unique non-trivial cocycle associated with $\pi$ and the result follows. □

### 4.2 Decomposition of the Haagerup cocycle

By way of contrast, let us give an example of a natural negative type function on $G$ which is not pure. For any group acting on a tree, one can define the so-called **Haagerup cocycle**. In order to define and to decompose the negative type function associated to the Haagerup cocycle, we need to introduce some notations (for all this, see p. 90 of [BdlHV08] and Chapter 1 of [Woe00]). Let $X = (V, E)$ be a locally finite graph, where $E$ denotes the set of oriented edges. Each edge $e \in E$ has a source $s(e) \in V$ and a range $r(e) \in V$. There is an obvious fixed-point free involution $e \mapsto \overline{e}$ on $E$ with $s(\overline{e}) = r(e)$ and $r(\overline{e}) = s(e)$, for all $e \in E$. The set of all pairs $\{e, \overline{e}\}$ is the set of geometric edges of the graph $X$. We denote by $\ell^2_{\text{alt}}(E)$ the real Hilbert space of those maps $\xi : E \to \mathbb{R}$ satisfying $\xi(\overline{e}) = -\xi(e)$ and such that $\sum_{e \in E} |\xi(e)|^2 < \infty$. This vector space is endowed with the inner product

$$\langle \xi, \eta \rangle = \frac{1}{2} \sum_{e \in E} \xi(e) \eta(e).$$
Let $\ell^2(V, m)$ be the Hilbert space of square-summable functions on $V$ endowed with the inner product

$$(f, g) = \sum_{x \in V} f(x)g(x)m(x),$$

where $m(x)$ is the degree of the vertex $x$. Now we can define two operators connecting these spaces. Let $\nabla : \ell^2(V, m) \to \ell^2_{\text{alt}}(E)$ be the gradient, defined by

$$(\nabla f)(e) = f(e_+) - f(e_-).$$

It is straightforward from the definition that $(\nabla f)(e) = -(\nabla f)(e)$, for all $e \in E$. We also define the divergence $\nabla^* : \ell^2_{\text{alt}}(E) \to \ell^2(V, m)$ as the adjoint of $\nabla$, that is,

$$(\nabla^* u)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} u(y, x),$$

for all $u \in \ell^2_{\text{alt}}(E)$ and $x \in V$. The link between the Laplace operator $\mathcal{L}$ and the operators $\nabla$ and $\nabla^*$ is given by the following formula:

$$\mathcal{L} = -\nabla^* \nabla.$$ 

**Lemma 4.3. (Poincaré lemma on trees)** Let $T = (V, E)$ be a tree and fix a vertex $x_0$. For any map $\xi : E \to \mathbb{R}$ such that $\xi(e) = -\xi(e)$, for all $e \in E$, there is a unique function $\tilde{\xi} : V \to \mathbb{R}$ such that $\nabla \tilde{\xi} = \xi$ and $\tilde{\xi}(x_0) = 0$.

**Proof:** Set $\tilde{\xi}(x_0) = 0$. Let $n \geq 1$. Let $x_n$ be a vertex at distance $n$ from $x_0$. Let $(x_j)_{j=0}^{n}$ be the chain of vertices forming the unique geodesic path linking $x_0$ to $x_n$ in $X$. Set

$$\tilde{\xi}(x_n) = \sum_{j=0}^{n-1} \xi(x_j, x_{j+1}).$$

It is easy to see that $\nabla \tilde{\xi} = \xi$. \qed

Recall that a function on the vertices $\eta : V \to \mathbb{R}$ is harmonic if $\mathcal{L}\eta = 0$. In the case where $X = T$ is a tree, using the previous lemma, a map $\xi \in \ell^2(E)$ belongs to $\ker \nabla^*$ if and only if $\xi$ is harmonic. Therefore, it is natural to say that a map on the edges $\xi \in \ell^2_{\text{alt}}(E)$ is harmonic if $\xi \in \ker \nabla^*$. This remark suggests the following orthogonal decomposition:

$$\ell^2_{\text{alt}}(E) = \ker \nabla^* \oplus \overline{\text{im} \nabla},$$

since $(\ker \nabla^*)^\perp = \overline{\text{im} \nabla}$. We can give a more precise description of the projection onto $\overline{\text{im} \nabla}$. Recall first that, if the graph is non amenable, then $\overline{\text{im} \nabla}$
is closed and the Laplace operator $L$ is invertible. Furthermore, its inverse is the operator $-G$, with $G$ being the Green kernel of $X$ (see p.14 of [Woe00] for the definition). Now, let $\xi = h + \nabla k$, with $h \in \text{ker } \nabla^*$ and $k \in l^2(V,m)$. Then, $\nabla^* \xi = \nabla^* \nabla k = -Lk$. Hence, we obtain $k = G\nabla^* \xi$. This forces to define $Q$, the orthogonal projection onto $\text{im}(\nabla)$, by

$$Q(\xi) = \nabla G \nabla^* \xi.$$ 

We can deduce that the "harmonic part" of an element $\xi \in l^2_{\text{alt}}(E)$ is given by $(1 - Q)\xi$. Now, we can introduce the Haagerup cocycle and study its decomposition along the two subspaces $\text{ker } \nabla^*$ and $\text{im }\nabla$.

Let $X = (V,E)$ be a graph. For two vertices $x,y \in V$, we define the signed characteristic function of the geodesic $[x,y]$ by

$$\chi_{x \to y}(e) = \begin{cases} 1, & \text{if } e \text{ is on } [x,y] \text{ and } e \text{ points from } x \text{ to } y, \\ -1, & \text{if } e \text{ is on } [x,y] \text{ and } e \text{ points from } y \text{ to } x, \\ 0, & \text{otherwise}, \end{cases}$$

In the case where $X = T$ is a tree, then a simple calculation shows that

$$\|\chi_{x \to y}\|_{l^2_{\text{alt}}(E)} = \sqrt{d(x,y)}.$$ 

Now, let $G$ be a closed subgroup of $\text{Aut}(T)$. Let $\pi$ be the orthogonal representation of $G$ on $l^2_{\text{alt}}(E)$ induced by the action of $G$ on $T = (V,E)$. Let $x_0 \in V$ be fixed. The Haagerup cocycle is defined by $b : G \to l^2_{\text{alt}}(E)$ with

$$b(g) = \chi_{x_0 \to gx_0},$$

It is easy to check that $b$ satisfies the cocycle relation with respect to $\pi$. By a previous observation, we have the following identity:

$$\|b(g)\|_{l^2_{\text{alt}}(E)} = \sqrt{d(x_0,gx_0)}.$$ 

This proves that $b$ is a proper cocycle and that $G$ has the Haagerup property. Since $l^2_{\text{alt}}(E)$ can be decomposed into an orthogonal sum of two (closed) $G$-invariant subspaces, then we can conclude that the representation $\pi$ is not irreducible. Hence, we obtain the following corollary.

**Corollary 4.4.** The negative type function $g \mapsto d(x_0,gx_0)$ is not pure on $G$.

**Remark 4.5.** Corollary 4.4 also follows from the uniqueness (up to scalar multiplication) of unbounded pure negative type functions on $G$, by Corollary 4.2.
We will show in the sequel that the projection of the cocycle \( b \) onto \( \ker \nabla^* \) is still proper. To do so, we will prove that \( \|Q\chi_{x\rightarrow y}\|_{\ell^2(V,m)} \) is bounded, independently of \( x \) and \( y \). Here is a useful lemma allowing us to estimate the operator norm of the Green kernel.

**Lemma 4.6.** Let \( X = (V,E) \) be a graph and let \( P \) be the normalised adjacency operator acting on \( \ell^2(V,m) \), namely, the operator whose matrix coefficients \( p(x,y) \) are

\[
p(x,y) = \begin{cases} \frac{1}{\deg(x)}, & \text{if } x \sim y, \\ 0, & \text{otherwise}, \end{cases}
\]

(i) If \( \|P\| < 1 \), then the series \( \sum_{n \geq 0} P^n \) defines a bounded operator and we have the equalities

\[
\mathcal{L}^{-1} = -\sum_{n \geq 0} P^n = -G.
\]

In particular, \( \|G\| \leq \frac{1}{1-\|P\|} \).

(ii) (Theorem (11.1), [Woe00]) If \( X \) is a graph with all vertices of valency bounded by \( q + 1 \), then \( \|P\| \geq \frac{2\sqrt{q}}{q+1} \), with equality if \( X = T_{q+1} \).

For \( x, y \in V \), let us estimate the norm of the harmonic part of \( \chi_{x\rightarrow y} \). Firstly, it is a general fact that, for any \( \xi \in \ell^2_{\text{alt}}(E) \), we have

\[
\|(1-Q)\xi\| \leq \|\xi\|,
\]

and

\[
\|(1-Q)\xi\|^2 = \|\xi\|^2 - \|Q(\xi)\|^2.
\]

Now, let us compute \( Q\chi_{x\rightarrow y} \). It is easy to check that

\[
\nabla^*(\chi_{x\rightarrow y}) = \frac{\delta_y}{\deg(y)} - \frac{\delta_x}{\deg(x)}.
\]

We get

\[
\|\nabla^*\chi_{x\rightarrow y}\|_{\ell^2(V,m)}^2 = \left\| \frac{\delta_y}{\deg(y)} - \frac{\delta_x}{\deg(x)} \right\|_{\ell^2(V,m)}^2 = \frac{1}{\deg(y)} + \frac{1}{\deg(x)}.
\]
Using Lemma 4.6, we obtain
\[
\| (1 - Q) \chi_{x \to y} \|^2 = \| \chi_{x \to y} \|^2 - \| Q(\chi_{x \to y}) \|^2 \\
\geq d(x, y) - \| \nabla \|^2 \| G \|^2 \| \nabla^* \chi_{x \to y} \|^2 \\
\geq d(x, y) - \frac{2}{(1 - P)} \left( \frac{1}{\deg(x)} + \frac{1}{\deg(y)} \right) \\
\geq d(x, y) - \frac{4}{(1 - P)^2},
\]
since \( \| \nabla \| = \| \nabla^* \| = \sqrt{2} \).

In particular, the orthogonal projection of the cocycle \( b \) onto \( \ker \nabla^* \) is still proper and its compression exponent is \( \frac{1}{2} \). Indeed, for any \( g \in \text{Aut}(\mathcal{T}) \), we have
\[
\| (1 - Q)b(g) \|^2 \geq d(x_0, gx_0) - \frac{4}{(1 - P)^2}.
\]

If \( \mathcal{T} = \mathcal{T}_{q+1} \) is the homogeneous \( (q+1) \)-regular tree, then one can compute \( Q(\chi_{x \to y}) \) explicitly in order to give precisely its norm and to find some examples of negative type functions on \( G \). We will prove the following lemma.

**Lemma 4.7.** Let \( q \geq 2 \) be an integer and let \( \mathcal{T}_{q+1} \) be the homogeneous \( (q+1) \)-regular tree. Then, for any \( x, y \in V \), we have:

(i) \( |(Q\chi_{x \to y})(e)| = \begin{cases} 
q^{-d(y, e)} + q^{-d(x, e)}, & \text{if } e \text{ is on the geodesic } [x, y], \\
\frac{q + 1}{q^{-d(y, e)} - q^{-d(x, e)}}, & \text{otherwise}, 
\end{cases} \)

(ii) Moreover,
\[
\| Q(\chi_{x \to y}) \|^2_{\ell^2(\mathbb{E})} = \frac{2q}{q^2 - 1} (1 - q^{-d(x, y)}).
\]

(iii) Let \( G \) be a group acting isometrically on \( \mathcal{T}_{q+1} \) and fix \( x_0 \in V \). Then, the function conditionally of negative type on \( G \) given by
\[
g \mapsto \frac{2q}{q^2 - 1} (1 - q^{-|g|})
\]

is bounded and it tends to \( \frac{2q}{q^2 - 1} \) as \( g \) goes to infinity.

We can deduce:

**Corollary 4.8.** Let \( G \) be a subgroup of \( \text{Aut}(\mathcal{T}_{q+1}) \) acting properly on \( \mathcal{T}_{q+1} \). Fix a basepoint \( x_0 \in V \), set \( |g| := d(x_0, gx_0) \), for \( g \in G \), and write \( \tilde{b} \) for the projection of the Haagerup cocycle onto the closed invariant subspace \( \ker \nabla^* \), that is, \( \tilde{b}(g) = (1 - Q)\chi_{x_0 \to gx_0} \). We have:
(i) The map $\tilde{b}$ is a proper cocycle with respect to a subrepresentation of the natural unitary representation acting on $\ell^2_{\text{alt}}(E)$. Moreover, the cocycle $\tilde{b}$ satisfies the following estimate

$$\|\tilde{b}(g)\| \geq |g|^{1/2} - \frac{2q}{q^2 - 1},$$

for all $g \in G$.

(ii) Assume furthermore that $G$ is a closed subgroup of $\text{Aut}(T_{q+1})$ acting transitively on both $T_{q+1}$ and $\partial T_{q+1}$. Then, the function conditionally of negative type associated to $\tilde{b}$

$$g \mapsto |g| + \frac{2q}{q^2 - 1}(q^{-|g|} - 1),$$

is pure in $\text{CL}(G)$.

(iii) The cocycle $\tilde{b}$ coincides with the harmonic proper cocycle $F$ of Proposition 1.2 and Corollary 3.1.

Proof of Corollary 4.8 using Lemma 4.7: The first claim is clear. The second one follows from the fact that this negative type function is a multiple of the negative type function appearing in Corollary 4.2 and from the uniqueness of the latter. The last claim is a direct consequence of (ii).

Before proving Lemma 4.7 we recall that the Green kernel $G$ takes a particularly simple form on $T_{q+1}$.

Lemma 4.9. (Lemma (1.23), [Woe00]) Let $q \geq 2$ and let $G$ be the Green kernel defined on the homogeneous $(q+1)$-regular tree $T_{q+1}$. If we denote by $(G(x,y))_{x,y \in V}$ the associated matrix of $G$, then

$$G(x,y) = \frac{q^{1-d(x,y)}}{q-1},$$

for all $x, y \in V$.

Proof of Lemma 4.7: Since $\nabla^* \chi_{x \rightarrow y} = \frac{1}{q+1}(\delta_y - \delta_x)$, we need to compute $\nabla G \delta_x$. Let $e \in E$ be an oriented edge. Using the description of the Green
kernel, we get
\[
(\nabla G_\delta)(e) = (G_\delta(e_+)) - (G_\delta(e_-)) = G(e_+, x) - G(e_-, x) = \frac{q}{q - 1} (q^{-d(e_+, x)} - q^{-d(e_-, x)}).
\]

Clearly, \(|d(e_+, x) - d(e_-, x)| = 1\). Setting \(d(x, e) = \min\{d(x, e_-), d(x, e_+))\) (this is simply the natural distance between \(x\) and the geometric edge associated with \(e\) in the geometric realisation of \(T_{q+1}\)), we immediately obtain:
\[
(\nabla G_\delta)(e) = \begin{cases} 
-q^{-d(x,e)}, & \text{if } d(x, e) = d(x, e_-), \\
q^{-d(x,e)}, & \text{if } d(x, e) = d(x, e_+).
\end{cases}
\]

It is easy to see that \((\nabla G_\delta)(e)\) and \((\nabla G_\delta)(e)\) have the same sign if and only if \(x\) and \(y\) belong to the same connected component of \(T_{q+1}\setminus\{\text{mid}(e)\}\), which happens exactly when \(e\) does not lie on the geodesic \([x,y]\). This shows the first claim.

From the first claim, we deduce:
\[
\|Q\chi_{x\to y}\|_{\ell^2_{alt}(E)}^2 = \frac{1}{2} \sum_{e\in E} |(Q\chi_{x\to y})(e)|^2 = \sum_{e\in E} |(Q\chi_{x\to y})(e)|^2 = \frac{1}{(q + 1)^2} \sum_{e\in E} |(\nabla G_\delta - \nabla G_\delta)(e)|^2.
\]

To compute the last sum, we will decompose the set of geometric edges. First of all, let \(m = d(x,y)\) and let \(\{z_j\}_{j=0}^m\) be the set of vertices describing the geodesic \([x,y]\), with \(z_0 = x\) and \(z_m = y\). Let \(T_0\) be the subgraph which is induced on the connected component of \(T_{q+1}\setminus\{z_1\}\) containing \(x\). For \(1 \leq j \leq m - 1\), let \(T_j\) be the subgraph which is induced on the connected component of \(T_{q+1}\setminus\{z_{j-1}, z_{j+1}\}\) containing \(z_j\). Finally, let \(T_m\) be the subgraph which is induced on the connected component of \(T_{q+1}\setminus\{z_{m-1}\}\) containing \(y\). We remark that for all \(j\), the graph \(T_j\) is a subtree of \(T_{q+1}\) with root \(z_j\). With these notations, a geometric edge \(e\) belongs either to one of the \(T_j\), for

\[\text{Here, mid}(e)\text{ denotes the median point of } e\text{ in the geometric realisation of } T_{q+1}.\]
some $j$, or $e$ lies on $[x, y]$. Thus, we get:

$$
\|Q_{X \to Y}\|_{\text{ahl}(E)}^2 = \frac{1}{(q + 1)^2} \left( \left( \sum_{e \in [x, y]} |q^{-d(y,e)} + q^{-d(x,e)}|^2 \right) + \left( \sum_{j=0}^{m} \sum_{e \in T_j} |q^{-d(y,e)} - q^{-d(x,e)}|^2 \right) \right).
$$

To compute the first sum, let us denote by $e_j$ the edge $(z_j, z_{j+1})$. Therefore, we have

$$
\sum_{e \in [x, y]} |q^{-d(y,e)} + q^{-d(x,e)}|^2 = \sum_{j=0}^{m-1} \left| q^{-d(y,e_j)} + q^{-d(x,e_j)} \right|^2
$$

$$
= \sum_{j=0}^{m-1} |q^{-(m-j-1)} + q^{-j}|^2
$$

$$
= \sum_{j=0}^{m-1} q^{-2j} + q^{-2(m-j-1)} + 2q^{m-1}
$$

$$
= \frac{2m}{q^{m-1}} + 2 \sum_{j=0}^{m-1} \frac{1}{q^{2j}}
$$

$$
= \frac{2m}{q^{m-1}} + \frac{2(1 - q^{-2m})}{1 - q^{-2}}.
$$

Secondly, let us compute the sum over the edges belonging to the subtree $T_0$. For any edge $e$ in $T_0$, we notice that $d(y,e) = m + d(x,e)$. Since the number of edges in $T_0$ which are at distance $k$ to $x$ is equal to $q^{k+1}$, for $k \geq 0$, we have:

$$
\sum_{e \in T_0} |q^{-d(y,e)} - q^{-d(x,e)}|^2 = \sum_{k \geq 0} \sum_{e \in T_0 \atop d(x,e)=k} |q^{-k} - q^{-k-m}|^2
$$

$$
= (1 - q^{-m})^2 \sum_{k \geq 0} q^{-k+1}
$$

$$
= (1 - q^{-m})^2 \frac{q}{1 - q^{-1}}.
$$

By symmetry, the same is true for the sum over $T_m$. That is:

$$
\sum_{e \in T_m} |q^{-d(y,e)} - q^{-d(x,e)}|^2 = (1 - q^{-m})^2 \frac{q}{1 - q^{-1}}.
$$
Finally, we need to compute the sum over the edges belonging to $T_j$ for $1 \leq j \leq m - 1$. For any edge $e$ in $T_j$, we notice that $d(x, e) = d(x, z_j) + d(z_j, e)$ and $d(y, e) = d(y, z_j) + d(z_j, e) = m - j + d(z_j, e)$. Since the number of edges in $T_j$ which are at distance $k$ to $z_j$ is equal to $(q - 1)^2$, for $k \geq 0$, we have:

$$\sum_{e \in T_j} |q^{-d(y,e)} - q^{-d(x,e)}|^2 = \sum_{k \geq 0} \sum_{e \in T_j; d(z_j, e) = k} |q^{-m+j-k} - q^{-j-k}|^2$$

$$= \sum_{k \geq 0} \sum_{e \in T_j; d(z_j, e) = k} q^{-2k} |q^{-m+j} - q^{-j}|^2$$

$$= (q^{-m+j} - q^{-j})^2 \sum_{k \geq 0} (q - 1)q^k q^{-2k}$$

$$= (q^{-m+j} - q^{-j})^2 (q - 1) \sum_{k \geq 0} q^{-k}$$

$$= q \left(q^{-2(m-j)} + q^{-2j} - 2q^{-m}\right).$$

We can compute the sum over all the $T_j$, for $1 \leq j \leq m - 1$:

$$\sum_{j=1}^{m-1} \sum_{e \in T_j} |q^{-d(y,e)} - q^{-d(x,e)}|^2 = \sum_{j=1}^{m-1} q \left(q^{-2(m-j)} + q^{-2j} - 2q^{-m}\right)$$

$$= \frac{2(m - 1)}{q^{m-1}} + 2q \sum_{j=1}^{m-1} q^{-2j}$$

$$= \frac{2(m - 1)}{q^{m-1}} + \frac{21 - q^{2m}}{q (1 - q^{-2})}$$

$$= \frac{2(m - 1)}{q^{m-1}} + \frac{2q^{-1} - q^{1-2m}}{1 - q^{-2}}$$

Since we have
\[
\frac{1}{2} \sum_{e \in E} |(\nabla G)(\delta_y - \delta_x)(e)|^2 = \frac{1}{q^{m-1}} + \frac{1-q^{-2m}+q^{-1}-q^{1-2m}}{1-q^{-2}} + \frac{q(1-q^{-m})^2}{1-q^{-1}}
\]

\[
= \frac{1}{q^{m-1}} + \frac{2+q^{-1}+q-2q^{-m}-2q^{1-m}}{1-q^{-2}}
\]

\[
= \frac{-1-2q^{-1}-q^{-2}+q^{-m-2}+2q^{m-1}+q^{m}}{(1-q^{-2})q^{m-1}}
\]

\[
= \frac{(q^{m}-1)(1+q^{-1})^2}{(1-q^{-1})(1+q^{-1})q^{m-1}}
\]

\[
= \frac{(q^{m}-1)(1+q^{-1})}{(1-q^{-1})q^{m-1}}
\]

\[
= \frac{(q^{m}-1)(q^{-1}+1)}{(q-1)q^{m-1}},
\]

we deduce finally that

\[
\|Q\chi_{x-y}\|^2 = \frac{1}{(q+1)^2} \sum_{e \in E} |(\nabla G)(\delta_y - \delta_x)|^2
\]

\[
= \frac{2}{(q+1)^2} \frac{(q^{m}-1)(q+1)}{(q-1)q^{m-1}}
\]

\[
= \frac{2(q^{m}-1)}{(q^2-1)q^{m-1}}
\]

\[
= \frac{2q}{q^2-1} (1-q^{-m}),
\]

which proves the second claim. The last claim being straightforward, the proof is done. \qed

4.3 More examples and classification of pure conditionally negative type functions

We start by giving an interesting family of examples of kernels conditionally of negative type on trees. Let \(T = (V, E)\) be any tree. It was shown by Valette (Theorem 1, \[Val92\]) that, for any function \(\psi : V \to [0,1]\) satisfying the condition

\[
\psi(x) \leq \frac{1}{\deg(x)},
\]
for all $x \in V$ (with the convention that $\psi(x) = 0$ if $\deg(x) = \infty$), then, the kernel defined by

$$
\Psi(x, y) = \begin{cases} 
0, & \text{if } x = y, \\
\frac{d(x, y) - \psi(x) + \psi(y)}{2}, & \text{if } x \neq y,
\end{cases}
$$

is negative definite on $V$.

We address the following question. Let $G$ be a subgroup of $\text{Aut}(T)$. When is $\Psi$ a $G$-invariant kernel?

We will answer this question in a special case.

**Proposition 4.10.** Let $q \geq 2$ and let $G$ be a closed subgroup of $\text{Aut}(\mathcal{T}_{q+1})$ acting transitively on both $V$ and $\partial \mathcal{T}_{q+1}$. Then, a kernel $\Psi$ defined as above is $G$-invariant if and only if the function $\psi$ used to construct $\Psi$ is constant.

**Proof:** Clearly, the kernel $\Psi$ is $G$-invariant if and only if the function $\psi$ satisfies the following condition

$$
\psi(gx) + \psi(gy) = \psi(x) + \psi(y),
$$

for any $x, y \in V$ and $g \in G$. Since the stabilizer of any vertex acts transitively on any sphere about any point, it is straightforward to see that $\psi$ has to take at most 2 values. Indeed, let us fix vertex $x_0$. Recall the standard bipartition of $V$ given by $V_e$ and $V_o$. The set $V_e$ (resp. $V_o$) consists of vertices which are at even (resp. odd) distance of $x_0$. Let $u, v$ be both in the same subset of the bipartition. Then, $d(u, v)$ is even and the median point of the geodesic $[u, v]$ is a certain vertex $z$. We can find $g \in \text{stab}(z)$ sending $u$ on $v$. By condition (2) we obtain

$$
\psi(u) - \psi(v) = \psi(u) - \psi(gu) = \psi(gz) - \psi(z) = 0,
$$

which implies that $\psi(u) = \psi(v)$. To finish the proof, consider any geodesic segment of length 2 formed by vertices $(v_j)_{j=0}^2$. Since $G$ acts doubly transitively on $V$, we can find an element $g$ such that $gv_j = v_{j+1}$, for $j = 0, 1$. We observe that $d(v_1, gv_2) = 2$ and this forces $\psi(gv_2) = \psi(v_1)$. Again, by condition (2), we have

$$
\psi(v_0) = \frac{1}{2}(\psi(v_0) + \psi(v_2)) = \frac{1}{2}(\psi(gv_0) + \psi(gv_2)) = \frac{1}{2}(\psi(v_1) + \psi(gv_2)) = \psi(v_1),
$$

24
and therefore, $\psi$ is constant.

\[ \square \]

**Corollary 4.11.** Let $G$ and $\Psi$ be as in Proposition 4.10. The only negative type functions on $G$ induced by the negative type kernels $\Psi$ are of the form

$$ g \mapsto d(x_0, gx_0) - \alpha,$$

for some constant $\alpha \in [0, \frac{1}{q+1}]$.

These functions are not pure in general, as we observe from the next two results. We summarize the content of this section in the following Corollary.

**Corollary 4.12.** Let $G$ be a closed noncompact subgroup of $\text{Aut}(\mathcal{T}_{q+1})$, with $q \geq 2$. Suppose that $G$ acts transitively on the vertices and on the boundary $\partial \mathcal{T}_{q+1}$. Let $\psi$ be a function conditionally of negative type on $G$. Suppose that $\psi$ is pure in $\text{CL}(G)$ and that it vanishes on the stabilizer of some vertex $x_0$. We have the following alternative:

1. The function $\psi$ is bounded on $G$ and then it is of the form

$$ \psi(g) = \|\xi\|^2 - \langle \pi_\psi(g)\xi, \xi \rangle,$$

where $\pi_\psi$ is the irreducible unitary representation associated with $\psi$ via the GNS construction, and $\xi$ is a $\pi_\psi(Gx_0)$-fixed vector (which is unique, up to scalar multiplication).

2. The function $\psi$ is unbounded and then it is of the form

$$ \psi(g) = C\left( |g| + \frac{2q}{q^2 - 1}(q^{-|g|} - 1) \right),$$

where $|g| := d(gx_0, x_0)$ and $C$ is a positive constant.

**Proof:** The only thing left to prove is the first claim. It is a general fact that $\psi(g) = \|\pi_\psi(g) - 1\xi\|^2$, for some $G_{x_0}$-invariant vector $\xi \in \mathcal{H}_\psi$, using the GNS construction. By developing the norm and remarking that the coefficient $\langle \pi_\psi(\cdot)\xi, \xi \rangle$ is real-valued, we get $\psi(g) = 2\|\xi\|^2 - 2\langle \pi_\psi(\cdot)\xi, \xi \rangle$. Finally, since the representation $\pi_\psi$ is spherical, then the space of $G_{x_0}$-invariant vectors has dimension one, and the result follows.

For the sake of completeness, we prove the following Proposition.

25
Proposition 4.13. Let \( q \geq 2 \) be an integer (possibly infinite). Let \( G \) be a closed non compact subgroup of \( \text{Aut}(T_{q+1}) \). Suppose that \( G \) acts transitively on the vertices and on the boundary \( \partial T_{q+1} \). Fix a vertex \( x_0 \), and set \( |g| := d(gx_0, x_0) \), for \( g \in G \). Then, the function \( g \mapsto |g| \) is pure in \( \text{CL}(G) \) if and only if \( q = \infty \).

Proof : Let us show that, in the case \( q = \infty \), then the representation \( \pi \) acting on \( \ell^2_{\text{alt}}(E) \) is irreducible. Let us fix a geometric edge \( a = \{a_0, a_1\} \in E \). Firstly, we note that \( \pi \) is equivalent to the quasi-regular representation \( \lambda_{G/G_a} \), where \( G_a \) is the stabilizer of \( a \). By a theorem of Macky (see Theorem 2.1 in [BdlH97]), we need to show that the commensurator of \( G_a \) in \( G \) is exactly \( G_a \). Recall that the commensurator of \( G_a \) in \( G \), denoted by \( \text{Com}_G(G_a) \), is the set of elements \( g \in G \) such that the subgroup \( G_a \cap G'_a \) has finite index in both \( G_a \) and \( G'_a \), where \( a' \) is the edge satisfying \( a' = ga \). Clearly, \( G_a \) is contained in \( \text{Com}_G(G_a) \).

To prove the other inclusion, let \( g \in G \setminus G_a \) and set \( a' = ga \) and \( a'_i = g a_i \), for \( i = 0, 1 \). We will see that \( G_a \cap G_{a'} \) has not finite index in \( G_a \). We can suppose that the geodesic \( [a_0, a'_0] \) is contained in the geodesic \( [a_1, a'_1] \). Since the tree is of infinite degree, then, for all \( k \geq 2 \), there exists \( a'_k \in V \) such that \( a'_0 \sim a'_k \) and \( a'_j \neq a'_i \), for all \( j \geq k - 1 \). Using the transitivity of the action on \( \partial T_{\infty} \), for every \( k \), we can find \( \tilde{g}_k \in G_{a_0} \) sending \( a' \) to the edge \( \{a'_0, a'_k\} \). It is easy to see that \( \tilde{g}_k \in G_{a_0} \), for all \( k \), and that the cosets \( \tilde{g}_k (G_a \cap G_{a'}) \) are pairwise different. This ends the proof. \( \square \)

References

[BdlH97] Marc Burger and Pierre de la Harpe. Constructing irreducible representations of discrete groups. Proc. Indian Acad. Sci. Math. Sci., 107(3):223–235, 1997.

[BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. Kazhdan’s property \((T)\), volume 11 of New Mathematical Monographs. Cambridge University Press, Cambridge, 2008.

[FTN91] Alessandro Figà-Talamanca and Claudio Nebbia. Harmonic analysis and representation theory for groups acting on homogeneous trees, volume 162 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1991.

[LSV04] Nicolas Louvet, Yves Stalder, and Alain Valette. Fonctions conditionnellement de type négatif, représentations irréductibles et propriété \((T)\). Enseign. Math. (2), 50(3-4):239–266, 2004.
[Neb12] Claudio Nebbia. Cohomology for groups of isometries of regular trees. *Expo. Math.*, 30(1):1–10, 2012.

[Val92] Alain Valette. Negative definite kernels on trees. In *Harmonic analysis and discrete potential theory (Frascati, 1991)*, pages 99–105. Plenum, New York, 1992.

[Woe00] Wolfgang Woess. *Random walks on infinite graphs and groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.

Authors addresses:
TU Dresden
Fachrichtung Mathematik
Institut für Geometrie
01062 Dresden

antoine.gournay@tu-dresden.de

Institut de Mathématiques - Unimail
11 Rue Emile Argand
CH-2000 Neuchâtel
Switzerland

pierre-nicolas.jolissaint@unine.ch