Learning the Einstein-Podolsky-Rosen correlations on a Restricted Boltzmann Machine

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We construct a hidden variable model for the EPR correlations using a Restricted Boltzmann Machine. The model reproduces the expected correlations and thus violates the Bell inequality, as required by Bell’s theorem. Unlike most hidden-variable models, this model does not violate the locality assumption in Bell’s argument. Rather, it violates measurement independence, albeit in a decidedly non-conspiratorial way.

INTRODUCTION

The Restricted Boltzmann Machine (RBM) is a machine learning model dating to the early 1980s [1] which has enjoyed renewed interest over the last decade for its utility in deep learning [2]. Inspired by physical models of locally interacting spins, Boltzmann machines are able to “learn” underlying patterns in data sets by systematically adjusting their weights in such a way that the equilibrium state of the network, a Boltzmann distribution over the network configurations, expresses the structural correlations in the data set. In this paper, we show that one can train a simple RBM to model the data from an EPR (Einstein-Podolsky-Rosen) experiment [3]. The hidden units of the machine correspond to a set of hidden variables, giving us a local, stochastic hidden variable model for the puzzling correlations seen in the experiment. (See also [4] for an earlier Ising-inspired model.)

First, we provide a brief look at the essentials of Boltzmann machines and the subset of these called Restricted Boltzmann Machines. Second, we review Bell’s theorem, which shows that any model of the EPR correlations having the property of measurement independence (also known as statistical independence) must be nonlocal. Third, we describe an RBM that generates EPR data and thereby provides a model of the EPR correlations. Finally, we discuss the essentially local nature of the model, and show the rather natural, unconstrained way in which measurement independence fails, yielding a local theory that violates the Bell inequality and reproduces the predictions of quantum mechanics.

BOLTZMANN MACHINES

Boltzmann machines [5, 6] are stochastic models inspired by Hopfield networks [7] and more generally Ising-type models, models of interacting two-state spin systems at a finite temperature. They provide a model of collective computation, one that may shed light on the way in which the brain processes information, since neurons can be idealized as simple two-state (on/off) systems or “units” massively interconnected with other similar units.

General Boltzmann machines can have an arbitrarily large number of binary units, connected in any topology one likes (see Figure 1). Once the topology is fixed, what distinguishes one instantiation from another are the weights of the links connecting the units (analogous to the interaction strengths in an Ising model) and the biases of the individual units (analogous to the coupling to an inhomogeneous external field). The idea behind a Boltzmann machine is that, given a function to define the energy of the network – a Hamiltonian – and an appropriate stochastic dynamics for updating the (binary) states of the units, the network will have a natural equilibrium distribution of configurations given by the Boltzmann distribution, and these distributions can thereby represent probability distributions over sets of hypotheses, states of the world, or what have you.

The total energy of a Boltzmann machine is the sum of the self-energies of each unit and the interaction energies between neighboring units:

\[ E = - \sum_i b_i s_i - \sum_{i < j} w_{ij} s_i s_j. \]  

(We write \( i < j \) in order to avoid double-counting the interaction energy between pairs of units.) In machine learning, the coefficients \( b_i \) are called biases, the interaction strengths \( w_{ij} \) are called weights, and the states \( s_i \)}
take values 1 or 0, rather than +1 or −1 as is conventional in the Ising model.

The dynamics of a Boltzmann machine are stochastic, as in the Ising model at finite temperature. The probability that a given unit $i$ will turn (or remain) on is a function of the difference in the total network energy resulting from the unit’s being on and off:

$$\Delta E_i = E_{s_i=0} - E_{s_i=1} = b_i + \sum_j w_{ij} s_j. \quad (2)$$

Thus the difference in energy $\Delta E_i$ between the two possible states of $i$ depends on the weights of the connections to other units, and on the bias of unit $i$. The update rule is

$$P(s_i = 1) = \frac{1}{1 + e^{-\Delta E_i}}. \quad (3)$$

(The temperature does not appear in this expression because we train the machine at fixed temperature, such that $kT = 1$). Updates are generally done in an asynchronous manner.

The state of the entire network at a given moment is given by a vector $s$. Given the energy function (Eq. 1) and update rule (Eq. 3), the probability that the network will be in configuration $s$ is given by the Boltzmann distribution

$$P(s) = \frac{e^{-E(s)}}{\sum_k e^{-E(s_k)}}, \quad (4)$$

where the index $k$ ranges over all possible states of the network.

For most purposes, we make a nominal distinction between visible and hidden units, so that $s = (v, h)$. The visible units represent observed (or observable) properties of the objects of interest, and the relative frequencies of 1s and 0s on these units encode the correlations in the world we are interested in. The hidden units encode the structural properties behind these correlations, structure that the machine learns so as to be able to generate and predict the observed properties. Training (or “learning,” in the parlance of the field) a Boltzmann machine is a procedure to get it to reproduce the correlations in the data as correlations on the visible units. It involves picking a more-or-less random set of weights and biases, checking the output (the frequencies of the various visible configurations), and making adjustments to the weights and biases so that the output approaches the target, i.e. so that it approximates the data in the training distribution.

Efficient training of large Boltzmann machines with unrestricted connections between the units is highly non-trivial in part because the partition function (the denominator of Eq. 1) is difficult to calculate, and in part because the visible units depend not only on the hidden units but on the other visible units. We modeled the EPR correlations using a Restricted Boltzmann Machine (RBM), a particular kind of Boltzmann machine in which the $m$ visible units and $n$ hidden units form two layers, with no intra-layer connections (see Figure 2). This is a bipartite, undirected graph, and the energy function (Eq. 1) above takes the form

$$E(v, h) = -\sum_{i=1}^{m} c_i v_i + \sum_{j=1}^{n} d_j h_j + \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} v_i h_j \quad (5)$$

where $c_i$ and $d_j$ are the biases for the visible and hidden units, respectively.

The functional form of Eq. 5 implies that the units within each layer are conditionally independent. This makes the machine considerably easier to train than a general Boltzmann machine [1,9]. The conditional probabilities $P(v|h)$ and $P(h|v)$ take the simple product form

$$P(v|h) = \prod_{i=1}^{m} P(v_i|h) \quad (6)$$

$$P(h|v) = \prod_{j=1}^{n} P(h_j|v). \quad (7)$$

The fact that the conditional probabilities factor in this way not only makes learning easier, but it also allows us to retain the idea that the detector settings in the EPR experiment can be regarded as independent degrees of freedom, which may thus be freely chosen.

**EPR AND BELL’S THEOREM**

Bell’s theorem is a demonstration that the predictions of quantum mechanics for the EPR experiment are incompatible with two assumptions, the most prominent of which is a particular kind of locality, sometimes called Bell-locality or strong locality [10].

In its modern guise due to Bohm [11], the EPR experiment involves spin measurements on a pair of particles that have been prepared in a maximally entangled state. There are two stations, which we will refer to as $A$ and $B$, and detectors with two possible settings at each station, the different settings corresponding to measurements of different components of spin. The detector settings $\alpha = \{a, a'\}$ and $\beta = \{b, b'\}$ and measurement outcomes $x_{\alpha} = \{+1, -1\}$ and $x_{\beta} = \{+1, -1\}$ are two-valued, so we can treat them as Bernoulli random variables. A theory or model of the experiment consists of one or more “states” $\lambda \in \Lambda$, each of which implies a joint probability distribution over $\alpha, \beta, x_{\alpha}$, and $x_{\beta}$.  

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1 Goodfellow et. al. [8] give examples that show that this interpretation of the significance of the hidden units should be taken with at least a grain of salt.
Bell’s locality criterion is intended to encode the assumption that the underlying theory that accounts for the correlations will be local in the sense that the outcome \( x_\alpha \) at A is independent of both the detector setting \( \beta \) at B (“parameter independence”) and the outcome \( x_\beta \) at B (“outcome independence”). This is equivalent to assuming that the joint probability distribution is equal to the product of the two marginal distributions, and is therefore often referred to as factorizability:

\[
P(x_\alpha, x_\beta | \alpha, \beta, \lambda) = P(x_\alpha | \alpha, \lambda)P(x_\beta | \beta, \lambda). \tag{8}
\]

There is an additional, often tacit assumption called measurement independence, which we will have occasion to discuss further. In brief, it is the assumption that the state \( \lambda \) is independent of the detector settings \( \alpha \) and \( \beta \), so that

\[
P(\lambda | \alpha, \beta) = P(\lambda). \tag{9}
\]

Bell showed that the assumption of locality, plus the assumption of measurement independence, imply that the correlations \( C(\alpha, \beta) \) between measurement results at various detector settings should satisfy an inequality known as the Bell-inequality, later generalized to the CHSH-Bell inequality [12]:

\[
S = |C(a, b) + C(a, b') + C(a', b) − C(a', b')| ≤ 2. \tag{10}
\]

For appropriate choices of the detector angles \( a, a', b, \) and \( b' \), this inequality is violated by quantum mechanics. If one takes the singlet state

\[
\psi = \frac{1}{\sqrt{2}} (|+-⟩−|--⟩) \tag{11}
\]

as \( \lambda \) and chooses \( a = 0, a' = \pi/2, b = \pi/4, \) and \( b' = −\pi/4 \) radians as orientations for the measuring apparatuses, quantum mechanics predicts that the correlation coefficients have values such that \( S = 2\sqrt{2} = 2.828 \), which violates the inequality (see the Theory column of Table I).

\[
\begin{array}{l|ccc}
\text{Theory} & \text{Data} & \text{Model} \\
C(a, b) & -0.707 & -0.713 & -0.711 \\
C(a, b') & -0.707 & -0.701 & -0.699 \\
C(a', b) & -0.707 & -0.714 & -0.713 \\
C(a', b') & 0.707 & 0.709 & 0.704 \\
\end{array}
\]

**TABLE I.** Correlation coefficients for detector settings \( a = 0, a' = \pi/2, b = \pi/4, \) and \( b' = −\pi/4. \)

As the quantum-mechanical predictions are borne out by experiment [13–15], any theory that accounts for the experimental data must violate at least one of the two assumptions used to derive the inequality. Quantum mechanics, as it happens, violates locality, as do Bohmian mechanics [16,17], Nelson’s stochastic mechanics [18,19], and the GRW spontaneous collapse model [21,22].

The RBM model does not violate locality; rather, it violates measurement independence. Models of this sort are far less common, as they have seemed to many (including Bell [23,24], Shimony et al. [25] and others) to preclude “free will” on the part of the experimenter, or to involve some sort of conspiracy or fine-tuning on the part of nature [20]. Many of these models are billed as retro-causal [27–30], but there are others, sometimes called “superdeterministic” [31–34], at least one of which [35,36] invokes a nonlocal, spacelike constraint of the sort suggested in Weinstein [37]. The RBM model, we will see, is different again.

**TRAINING THE EPR MACHINE**

Training a Boltzmann machine, restricted or otherwise, involves confronting it with data and getting it to revise its model of the data in such a way as to reproduce the correlations in the data as correlations on the visible units.

In our case, the data include both the outcomes of experiments and the detector settings, since we are interested in exhibiting correlations between detector settings and pairs of outcomes. Rather than use data from actual runs of an EPR-type experiment to train our machine, we simulated 100,000 runs using a simple Python script, deriving outcomes for randomly chosen detector settings using ordinary quantum mechanical calculations. The resulting correlation coefficients are in the Data column of Table I.

The RBM we constructed has four visible and four hidden units (see Figure 3). The units \( v_1 \) and \( v_2 \) represent the detector settings \( \alpha \) and \( \beta \), respectively, while \( v_3 \) and \( v_4 \) represent \( x_\alpha \) and \( x_\beta \), the outcomes of measurements at those detectors. For example, consider the

FIG. 3. RBM for EPR: Green is on (0); blue is off (1). In the configuration shown, \( \lambda = (1,1,0,1) \), \( \alpha = a' \), \( \beta = b \), and \( x_\alpha = x_\beta = −1 \).
correlation between the outcomes with settings \(a\) and \(b\). The observed value of \(C(a,b)\) in our training data was \(-0.713\), which means that when the detectors were set to measure \(a\) and \(b\), the results were different (perfectly anti-correlated) around 85.7% of the time and the same (perfectly correlated) around 14.3% of the time. The goal is to reflect this as a correlation between the on/off probabilities of the visible units \(v_1, v_2, v_3, v_4\) such that

\[
P(v_3 = v_4 | (v_1, v_2) = (0, 0)) \approx 0.143 \tag{12}
\]

\[
P(v_3 \neq v_4 | (v_1, v_2) = (0, 0)) \approx 0.857. \tag{13}
\]

Training the machine involves initializing the network with random weights and biases, and adjusting them in an iterative process so as to bring the distribution on the visible units in line with the data.

Because of the restricted topology of the network, the rule for adjusting the weights is both simple and local, despite the extensive interconnection of the units. Recall that we are trying to get the visible layer to align with the data, so we are especially interested in the \(\mathbf{v} = (v_1, v_2, v_3, v_4)\) portion of the overall configuration \((\mathbf{v}, \mathbf{h})\). From Eq. (3) and Eq. (5), it follows that

\[
\frac{\partial \log p(\mathbf{v})}{\partial w_{ij}} = \langle v_i h_j \rangle_{\text{data}} - \langle v_i h_j \rangle_{\text{model}}. \tag{14}
\]

As such, the weight update rule is of the form:

\[
\Delta w_{ij} = \epsilon(\langle v_i h_j \rangle_{\text{data}} - \langle v_i h_j \rangle_{\text{model}}), \tag{15}
\]

where \(\epsilon\) is a small, real-valued parameter colloquially known as the learning rate. Note that the expectation value \(\langle v_i h_j \rangle\) is simply the probability that both components will have the value 1, i.e., that they will both be on.

Because the layers are independent, we have

\[
P(h_j = 1 | \mathbf{v}) = \sigma(d_j + \sum_i v_i w_{ij}) \tag{16}
\]

\[
P(v_i = 1 | \mathbf{h}) = \sigma(c_i + \sum_j h_j w_{ij}), \tag{17}
\]

where \(\sigma(x) = 1/(1 + e^{-x})\). The training data from the simulation gives us a distribution over visible vectors \(\mathbf{v}\). Each visible vector determines a probability for the value of each hidden unit, and so we are able to determine \(\langle v_i h_j \rangle_{\text{data}}\) in a straightforward fashion.

The determination of \(\langle v_i h_j \rangle_{\text{model}}\) must be approximated for RBMs with more than a small number of units, as one otherwise needs to calculate the probability of every configuration explicitly, since there are no data to condition over. The number of configurations grows exponentially with the number of units, so this gets out of hand quickly. However, methods exist for efficient approximation \[35\], and our results were obtained in this manner.

**RESULTS**

Training the model on 100,000 trials using persistent contrastive divergence \[39\] yielded a Restricted Boltzmann Machine with the weights and biases in Table I. The correlation coefficients for this model are given in the

| \(h_1\)  | \(h_2\)  | \(h_3\)  | \(h_4\)  |
|--------|--------|--------|--------|
| \((-3.320\) | \((-1.015\) | \((-0.933\) | \((-3.753\) |

| \(v_1\)  | \(v_2\)  | \(v_3\)  | \(v_4\)  |
|--------|--------|--------|--------|
| \((-5.026\)  | \(3.527\)  | \(3.546\)  | \(-2.456\)  |
| \((-4.872\)  | \(3.575\)  | \(3.585\)  | \(2.471\)  |
| \((-3.467\)  | \(-5.587\)  | \(5.578\)  | \(3.717\)  |
| \((-3.464\)  | \(3.326\)  | \(5.577\)  | \(-5.592\)  | \(3.721\)  |

**TABLE II.** Weights \(w_{ij}\) of the connections between visible and hidden units. Biases for the individual units in parentheses.

*Model* column of Table II. They are a remarkably close fit to the actual data. The model could be further refined, but there is no point, as it would merely overfit the data. We have thus successfully modeled the key properties of simulated EPR data as the output of a Restricted Boltzmann Machine with four hidden units.

**DISCUSSION**

The RBM model is a stochastic hidden variable theory of a very interesting sort. The binary states of the four hidden units are the hidden variables in the model, giving rise to \(2^4 = 16\) hidden states \(\lambda\), each of which generates a distinct set of probabilities for the activations of the visible units. The Bell locality condition (Eq. 8) is not violated, since the probabilities of the outcomes \(x_\alpha\) at A and \(x_\beta\) at B are independent of the detector settings and outcomes at B and A, respectively, in virtue of the conditional independence of the visible units (Eq. 6). This is a direct result of the topology of the RBM: there are no connections between visible units (or between hidden units). It has a kind of connectivity that would easily allow it to be embedded in a relativistic spacetime, with the hidden units and visible units each mutually spacelike, and each of the hidden units timelike or lightlike to each of the visible units.

One might be forgiven for thinking of the RBM model as a retrocausal model, especially if one takes the topology of the model as suggesting a causal structure of this sort. That is, one might think that there are boundary conditions in the past (the state preparation – not modeled here) and the future (the detector settings), and that the future boundary conditions, which are after all freely specifiable, retroactively bring about changes in the prior state of the system.

But the RBM model described here is not that. The weights and biases of the network are independent of
the detector settings, and so is the probability distribution. Predictions are conditional probabilities, specifically probabilities for measurement outcomes conditioned on the states of the detectors. The model is atemporal and therefore acausal. There is no dynamics that propagates a change from the future to the past, or from the past to the future. Outcomes are generated not by the dynamical evolution of an initial condition, but in the way of equilibrium statistical mechanics. The dynamics lies in the update rule, Eq (3), which generates a stationary probability distribution. Experiments involve sampling from this distribution.

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