Dielectric permeability tensor and linear waves in spin-1/2 quantum kinetics with non-trivial equilibrium spin-distribution functions

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A consideration of waves propagating parallel to the external magnetic field is presented. The dielectric permeability tensor is derived from quantum kinetic equations with non-trivial equilibrium spin-distribution functions in the linear approximation on amplitude of wave perturbations. It is possible to consider equilibrium spin-distribution functions with nonzero z-projection proportional to the difference of the spin distribution function while x- and y-projections are equal to zero. It is called trivial equilibrium spin-distribution functions. In general case, x- and y-projections of the spin-distribution functions are nonzero which is called the non-trivial regime. Corresponding equilibrium solution is found in [Phys. Plasmas 23, 062103 (2016)]. Contribution of the nontrivial part of the spin-distribution function appears in the dielectric permeability tensor in the additive form. It is explicitly found here. Corresponding modification in the dispersion equation for the transverse waves is derived. Contribution of nontrivial part of the spin-distribution function in the spectrum of transverse waves is calculated numerically. It is found that the term caused by the nontrivial part of the spin-distribution function can be comparable with the classic terms for the relatively small wave vectors and frequencies above the cyclotron frequency. In majority of regimes, the extra spin caused term dominates over the spin term found earlier, except the small frequency regime, where their contributions in the whistler spectrum are comparable. A decrease of the left-hand circularly polarized wave frequency, an increase of the high-frequency right-hand circularly polarized wave frequency, and a decrease of frequency changing by an increase of frequency at the growth of the wave vector for the whistler are found. A dramatic decrease of the spin wave frequency resulting in several times larger group velocity of the spin wave is found either. Found dispersion equations are used for obtaining of an effective quantum hydrodynamics reproducing these results. This generalization requires the introduction of corresponding equation of state for the thermal part of the spin current in the spin evolution equation.

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I. INTRODUCTION

Spin effects modify properties of plasmas [1, 2], as well as electron gas in other mediums [15–18]. They play a role for the degenerate and non-degenerate plasmas. However, the spin effects are more prominent for the degenerate plasmas since spin polarization clearly splits the Fermi step on two Fermi steps of different width for the spin-up and spin-down electrons [19–21]. It allows to distinguish two types of electrons and consider them as two species [21–23]. Consideration of the independent evolution of electrons with different spin projections leads to discovery of the spin-electron acoustic waves.

Spin evolution modifies hydrodynamic and kinetic properties of plasma [2, 24–26]. In both cases dynamical equations contain the force of spin-spin interaction: \( S^\beta(\mathbf{r}, t)\nabla_r B^\beta \) in hydrodynamics [27], \( \nabla_p S^\beta(\mathbf{r}, \mathbf{p}, t) \cdot \nabla_r B^\beta \) in kinetics [1, 28, 29], where \( B^\beta = B^\beta(\mathbf{r}, t) \) is the magnetic field, \( \nabla_r \) is the gradient in coordinate space, and \( \nabla_p \) is the gradient in momentum space. This force contains the spin density \( S \). It is the coordinate space density of spin in hydrodynamics \( S(\mathbf{r}, t) \) and it is the phase space density of spin (the spin distribution function) in kinetics \( S(\mathbf{r}, \mathbf{p}, t) \). Therefore, the complete model requires an equation of spin density evolution. The time evolution of spin density happens due to two mechanisms: kinematic mechanism, where the flow of spinning particles in and out of the vicinity of the point in space change the local spin density, and dynamical, where the change of spin happens due to the interparticle interaction. The kinematic mechanism gives the spin current. In hydrodynamics, the spin current \( J^{\alpha \beta} \) has the structure similar to the structure of the momentum current \( \Pi^{\alpha \beta} \) existing in Euler equation [2, 31]. Tensor \( \Pi^{\alpha \beta} \) contains the flow of the local center of mass \( n v^\alpha v^\beta \), the flow on the thermal velocities (the thermal pressure or the Fermi pressure for degenerate fermions) \( p^{\alpha \beta} \), and the quantum part which is usually called the quantum Bohm potential. The spin current \( J^{\alpha \beta} \) contains the flow of spin on the velocity of the local center of mass \( S^\alpha v^\beta \), the thermal part of the spin current \( J_\text{th}^{\alpha \beta} \) (or the Fermi spin current for degenerate fermions [31]) and the quantum part calculated by Takabayasi [27].

Majority of works on the spin evolution in plasmas are focused on interaction and drop the Fermi spin current [32–34]. The thermal spin current is not considered in the ferrofluids either [35]. Hence, usually, the Fermi spin...
current is assumed to be equal to zero. However, recently, an equation of state is derived for the Fermi spin current. More detailed study of physical effects similar to the Fermi spin current required kinetic modeling. Corresponding research is performed in Refs. [31, 33]. It is shown that the kinetic analysis can be done with non-zero equilibrium scalar distribution function $f_0$ and non-zero z-projection of the equilibrium spin distribution function $S_{0z}$ while $S_{0x} = S_{0y} = 0$ [31, 33]. However, the general model requires consideration of $S_{0x} \neq 0$ and $S_{0y} \neq 0$ which are found in Ref. [33]. Required analysis is performed in this paper for waves propagating parallel to the external magnetic field.

Influence of the spin on properties of magnetized plasmas is studied in many papers [1–4, 37, 39]. Quantum hydrodynamics [2, 3, 37, 40] and quantum kinetic equations for spin-1/2 plasmas are developed for the study of electrons as two fluids [21, 36]. The spin-electron acoustic waves found from these models are studied in different regimes [23, 25, 48, 54, 56]. Study of the nontrivial part of the equilibrium distribution functions continues this research.

This paper is organized as follows. In Sec. II, basic quantum kinetic equations for spin-1/2 plasmas are presented. In Sec. III, the equilibrium distribution functions are presented and the main structure of the dielectric permeability tensor is described. In Sec. IV, the dispersion equation and spectrum of the transverse waves propagating parallel to the external are studied under influence of extra spin effects caused by the non-trivial part of the equilibrium distribution functions. In Sec. V, an equation of state for the Fermi spin current entering the hydrodynamic spin evolution equation is deduced from spectrum derived from kinetic model. In Sec. VI, a summary of the obtained results is presented. In Sec. VII, Appendix A is presented, where the linearized kinetic equations and their solutions are found. In Sec. VIII, Appendix B is presented, where details of the the dielectric permeability tensor and some details of its calculation are described. In Sec. IX, Appendix C is presented, where an approximate form of the dielectric permeability tensor is found. In Sec. X, Appendix D is presented, where the dimensionless form of the dispersion equation is demonstrated.

## II. QUANTUM KINETIC MODEL FOR SPIN-1/2 PLASMAS

The quantum kinetics of spin-1/2 particles can be modeled by the distribution functions (the scalar function $f$ and the vector (spin) function $S$) defined in the six-dimensional phase space [1, 28, 29, 57].

Equation for the scalar distribution function $f = f(r, p, t)$ is the generalized Vlasov equation [1, 28, 29, 33, 39, 57]:

\[
\partial_t f + \mathbf{v} \cdot \nabla f + q_e \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_p f + \mu_e \nabla^\alpha B^\beta \cdot \nabla_p S^{\beta \gamma} = 0, 
\]

which contains an extra term (the last term) caused by the spin-spin interaction.

The kinetic equation for the vector distribution function $\mathbf{S} = \mathbf{S}(\mathbf{r}, \mathbf{p}, t)$ has the following form [1, 28, 29, 33, 57]:

\[
\partial_t S^\alpha + \mathbf{v} \cdot \nabla S^\alpha + q_e \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_p S^\alpha + \mu_e \nabla^\beta B^\alpha \cdot \nabla_p S^{\beta \gamma} - \frac{2 \mu_e}{\hbar} \epsilon^{\alpha \beta \gamma} S^\beta B^\gamma = 0.
\]

The last two terms are caused by the spin-spin interaction. Kinetic equations [1] and [2] contain the following notations: $\mathbf{E}$ and $\mathbf{B}$ are the electric fields, $q_e = -|e|$ is the charge of electrons, $\mu_e = -g \mu_B$ is the magnetic moment of electrons, $\mu_B = |e|/2mc$ is the Bohr magneton, $g = 1.00116$, $\mathbf{r} = m \mathbf{v}$ is the coordinate (momentum) space, $\mathbf{t}$ is time, $\partial_t$ is the time derivative, $\nabla \mathbf{r}$ ($\nabla \mathbf{p}$) is the gradient on the space coordinate (on the momentum), $\nabla^\alpha \mathbf{S}$ and $\nabla_p^\alpha \mathbf{S}$ are projections of described gradients on the coordinate axis, $\hbar$ is the reduced Planck constant, $c$ is the speed of light, $\epsilon^{\alpha \beta \gamma}$ is the antisymmetric symbol (the Levi-Civita symbol).

Kinetic equations are coupled to the Maxwell equations $\nabla \cdot \mathbf{E} = 4\pi \rho, \nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B}$, $\nabla \cdot \mathbf{B} = 0$, and

\[
\nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} + \frac{4\pi}{c} \mathbf{j} + 4\pi \nabla \times \mathbf{M},
\]

where $\rho = q_e \int f(r, \mathbf{p}, t) d\mathbf{p} + q_n \mu_0$, $\mathbf{j} = q_e \int \mathbf{v} f(r, \mathbf{p}, t) d\mathbf{p}$, $\mathbf{M} = \mu_e \int \mathbf{S}(r, \mathbf{p}, t) d\mathbf{p}$ is the magnetization.

## III. STRUCTURE OF THE DIELECTRIC PERMEABILITY TENSOR

Linear on the small perturbations kinetic equations are needed to be considered for the derivation of the dielectric permeability tensor. Assume that the following functions $f_0(p)$, $S_0(p, \phi)$, $\mathbf{B}_0 = \mathbf{B}_{ext} = \mathbf{B}_0 e_z$ have non-zero values in the equilibrium state. Moreover, the explicit forms
of the equilibrium distribution functions for the partially spin-polarized degenerate electrons appear as the sum or the difference of the Fermi steps for the spin-up and spin-down electrons:

\[ f_0(p) = \frac{1}{(2\pi\hbar)^3} \Theta(pF\uparrow - p) + \Theta(pF\downarrow - p), \]

and

\[ S_{0x} = \Sigma(p) \cos \varphi, \quad S_{0y} = \Sigma(p) \sin \varphi, \quad S_{0z} = \Sigma(p), \]

with

\[ \Sigma(p) = \frac{1}{(2\pi\hbar)^3} \left[ \Theta(pF\uparrow - p) - \Theta(pF\downarrow - p) \right]. \]

where \( \Theta \) is the step function, \( p = |p| \) is the module of the momentum, \( pF\uparrow \) and \( pF\downarrow \) are the Fermi momenta for the spin-up and spin-down electrons, \( v_{Fs} = p_{Fs}/m = (6\pi^2 n_0)^{1/3} \hbar/m, \ s = \uparrow, \downarrow, \) and \( \varphi \) is the polar angle in cylindrical coordinates in the momentum space, \( n_0 \) are the concentrations of the spin-up and spin-down electrons. Present small perturbations of the described equilibrium state as the plane waves. For instance, the scalar distribution function can be presented as \( f = f_0 + \delta f \), where \( \delta f = F e^{-\omega t + ikr} \) and \( F \) is an amplitude of perturbation. Moreover, the waves propagating parallel to the external magnetic field are under consideration \( k = \{0,0,k_z\} \). Below, the following notations are used for the charge cyclotron frequency \( \Omega_c = q dB_0/mc \), the magnetic moment cyclotron frequency \( \Omega_m = 2\mu_e B_0/\hbar \), and the partial Langmuir frequency \( \omega_{Ls}^2 = 4\pi e^2 n_0 / m \). Parameters \( \Omega_c = q dB_0/nc \) and \( \Omega_m = 2\mu_e B_0/\hbar \) are equal to each other if the anomalous part of magnetic moment of electron is neglected. Linearized kinetic equations and their solutions are presented in Appendix A.

Linear kinetic theory allows to calculate the dielectric permeability tensor which enters the equation for the small perturbations of the electric field:

\[ \left[ k^2 \delta^{\alpha\beta} - k^\alpha k^\beta - \frac{\omega^2}{c^2} \varepsilon^{\alpha\beta} (\omega) \right] \delta E^\beta = 0, \]

where the dielectric permeability tensor appears as

\[ \varepsilon^{\alpha\beta} (\omega) = \delta^{\alpha\beta} + \varepsilon_1^{\alpha\beta} (\omega) + \varepsilon_2^{\alpha\beta} (\omega), \]

where \( \delta^{\alpha\beta} \) is the Kronecker symbol, \( \varepsilon_1^{\alpha\beta} (\omega) \) is the dielectric permeability tensor caused by the current

\[ \varepsilon_1^{\alpha\beta} (\omega) \delta E^\beta = \frac{4\pi e q_e}{\omega m} \int p^\alpha \delta f dp, \]

and \( \varepsilon_2^{\alpha\beta} (\omega) \) is the dielectric permeability tensor caused by the curl of magnetization

\[ \varepsilon_2^{\alpha\beta} (\omega) \delta E^\beta = \frac{4\pi \mu_e}{\omega} \varepsilon^{\gamma\delta} k_z c \int \delta S^\gamma dp. \]

The explicit form of \( \varepsilon_1^{\alpha\beta} \) and \( \varepsilon_2^{\alpha\beta} \) are presented in Appendix B. Meanwhile, consider structure of tensors \( \varepsilon_1^{\alpha\beta} \) and \( \varepsilon_2^{\alpha\beta} \). Tensor \( \varepsilon_1^{\alpha\beta} \) can be separated into three parts: \( \varepsilon_1^{\alpha\beta} = \varepsilon_{10}^{\alpha\beta} + \varepsilon_{12}^{\alpha\beta} + \varepsilon_{11}^{\alpha\beta} \). The first part is the quasi-classic term \( \varepsilon_{10}^{\alpha\beta} \) existing with no account of the spin evolution. Tensor \( \varepsilon_{10}^{\alpha\beta} \) consists of two terms \( \varepsilon_{10}^{\alpha\beta} = - \sum_{s = \uparrow, \downarrow} f \sin \theta dP_{Fs}(\theta, s) \) since the spin-polarized plasma with the equilibrium scalar distribution function splited in two terms \( \Theta \) is considered, where \( \Theta \) is an angle of the spherical coordinates in the velocity space defined as \( \cos \theta = v_z/v \).

Despite this fact, tensor \( \varepsilon_2^{\alpha\beta} \) has well-known structure presented in textbooks (see for instance [58]). Tensor \( \varepsilon_2^{\alpha\beta} \) can be separated into two parts: \( \varepsilon_2^{\alpha\beta} = \varepsilon_{21}^{\alpha\beta} + \varepsilon_{22}^{\alpha\beta} \). Tensors \( \varepsilon_{11}^{\alpha\beta}, \varepsilon_{12}^{\alpha\beta}, \varepsilon_{21}^{\alpha\beta}, \varepsilon_{22}^{\alpha\beta} \) appear due to the spin evolution. Tensors \( \varepsilon_{11}^{\alpha\beta} \) and \( \varepsilon_{21}^{\alpha\beta} \) are found at the account of the trivial part of the equilibrium distribution functions. It means that \( f_0 \) and \( S_{0x} \) are given by equations (11) and (12) while \( S_{0x} = S_{0y} = 0 \). The account of non-zero \( S_{0x}, S_{0y} \) given by equations (13) (the non-trivial part of the equilibrium distribution functions) leads to existence of tensors \( \varepsilon_{12}^{\alpha\beta}, \varepsilon_{22}^{\alpha\beta} \). Calculation of tensors \( \varepsilon_{12}^{\alpha\beta} \) and \( \varepsilon_{22}^{\alpha\beta} \) and derivation of their contribution to the plasma properties are the main subjects of this paper.

The parts of the dielectric permeability tensor by \( \varepsilon_{10}^{\alpha\beta}, \varepsilon_{11}^{\alpha\beta}, \varepsilon_{21}^{\alpha\beta} \) are in accordance with the earlier developed models [5, 7, 21, 67], while tensors \( \varepsilon_{12}^{\alpha\beta} \) and \( \varepsilon_{22}^{\alpha\beta} \) are the generalization of the mentioned papers.

### IV. TRANSVERSE WAVES PROPAGATING PARALLEL TO THE EXTERNAL MAGNETIC FIELD

A partially explicit form of dispersion equation for the transverse waves appears as follows

\[ \frac{k^2 \omega^2}{c^2} = 1 - \sum_{s = \uparrow, \downarrow} \frac{3 \omega_{Ls}^2}{4 \omega k_z v_{Fs}} \left[ \frac{2(\omega + \Omega_c)}{k_z v_{Fs}} \right] + \frac{1 - (\omega + \Omega_c)^2}{(k_z v_{Fs})^2} \ln \left( \frac{\omega + k_z v_{Fs} + \Omega_c}{\omega - k_z v_{Fs} + \Omega_c} \right) + \Sigma_{\mp}, \]

for \( \delta E_x = \pm \delta E_y \) (left/right hand circular polarization) correspondingly, where \( \Sigma_{\mp} \) are the terms caused by the spin evolution presented in the nonexplicit form.

Consider new term in the long-wavelength regime \( k_z v_{Fs}/|\omega \pm \Omega_c| \ll 1 \). Start with regime \( \omega \pm |\Omega_c| > 0 \). In this regime, equation (11) has the following form

\[ \frac{k^2 \omega^2}{c^2} = 1 - \frac{\omega_{Lc}^2}{\omega(\omega \pm |\Omega_c|)} + \frac{\omega_{Lc}^2}{\omega^2} \left[ \frac{2n_{0\uparrow} - n_{0\downarrow}}{\omega \pm |\Omega_c|} \right] \]

\[ \mp \left( \frac{6\pi^2}{32\pi} \right) \frac{\omega_{Lc}^2}{\omega(\omega \pm |\Omega_c|)} \left[ \frac{n_{0\uparrow} - n_{0\downarrow}}{\omega \pm |\Omega_c|} \right] k_z, \]

where \( \omega_{Lc}^2 = \omega_{L\uparrow}^2 + \omega_{L\downarrow}^2 = 4\pi e^2 n_0 e/m \) is the Langmuir frequency for all electrons and \( n_0e = n_{0\uparrow} + n_{0\downarrow} \) is the concentration of all electrons.
Equation \( (12) \) contains coefficients proportional to \( \hbar k^2/m \). It resembles a similarity to the well-known quantum Bohm potential. However, it comes from the spin evolution \( \Delta_0 \). Equation \( (12) \) is a generalization of the equation which is well-known from spin-1/2 hydrodynamics \( \Delta_0 \).

The fourth term on the right-hand side of equation \( (12) \) is proportional to the difference of the Fermi energies for electrons with different spin projections \( \varepsilon_{F\uparrow} - \varepsilon_{F\downarrow} \sim n_{\uparrow0} - n_{\downarrow0} \) which is a signature of the Fermi spin current (see equation of state derived in \( (31) \), and discussion in introductions of Refs. \( [6, 7] \)).

Compare the fourth term on the right-hand side of equation \( (12) \) with \( k_\perp^2 c^2/\omega^2 \). Consider the ration of these terms and find \( \Xi = (3\pi^2)^{\frac{3}{2}} r \tau (n_{0e}/k_z) \omega / 8 (\omega \pm |\Omega_\mu|) \), where \( r = c^2 n_{0e}^\frac{1}{2} / mc^2 \) and \( \tau = [(1+\eta)^{\frac{3}{2}} - (1 - \eta)^{\frac{3}{2}}] \) with \( \eta = |n_\uparrow0 - n_\downarrow0| / n_{0e} \in [0, 1] \) is the spin polarization. Basically, this ratio is proportional to parameter \( r \). Parameter \( r \) is small even for \( n_0 = 10^{27} \text{ cm}^{-3} \) (\( r \approx 2 \times 10^{-4} \)). The ratio \( \Xi \) is not affected by frequency in the high-frequency regime \( \omega \gg |\Omega_\mu| \). It is decreased by \( \omega / |\Omega_\mu| \) at small frequencies while \( \Xi \) grows at the intermediate frequencies \( \omega \approx |\Omega_\mu| \) for the right-hand circular polarization. Small spin polarization \( \eta \ll 1 \) decreases parameter \( \Xi \). The ratio \( \Xi \) increases at the large spin polarization \( \eta \sim 1 \) and
small wave vectors $k_z \ll \sqrt{n_{0e}}$.

Next, compare the second term and the fourth term on the right-hand side of equation (12). Their ratio has the following form $\Lambda = (3\pi^2)^{2/3}k_z/n_{0e}^{1/3}r/32\pi$. Ratio $\Lambda$ grows with increase of wave vector $k_z$, but large values of $k_z$ cannot be considered since equation (12) is derived in the following limit $\omega \pm |\Omega_n| > k_zv_F$. Find $\Lambda \sim 10^{-2}$ at $k_z = 0.1n_{0e}^{1/3}, \eta = 0.5$, and $\omega > |\Omega_n|$. It is relatively small value, but it is a good value for a spin effect in plasma. In this regime $\Lambda > \Xi$.

Moreover, compare the third term and the fourth term on the right-hand side of equation (12). Both of them present contribution of spin effects while the fourth term is derived in this paper. Their ratio has the following form

$$\Pi = \frac{8\pi k_z n_{0e}^{2/3}}{(3\pi^2)^{2/3}n_{0e}^{1/3}m\omega}(1 + \eta)^{2/3} + (1 - \eta)^{2/3} + (1 - \eta)^{2/3}$$

(13)

It is decreased by factor $k_z/n_{0e}^{1/3}$, but for small frequencies $\omega$ parameter $n_{0e}^{2/3}/(m\omega)$ leads to increase of $\Pi$.

Consider the left-hand circularly polarized electromagnetic waves taking the upper sign in equation (12). This wave appears due to the classic terms in equation (12). Consider modifications of its properties arising due to the spin terms: the third and fourth terms on the right-hand side of equation (12). The third term ($\sim \hbar k_z^2$) is small in considering regime of the small wave vectors. Hence, this analysis is focused on the last term in equation (12). It has positive sign ($n_{0\uparrow} < n_{0\downarrow}$) while the classic term coming from the charge evolution (the second term on the right-hand side) is negative. Hence, they give opposite influences. The classic term gives a considerable increase of frequency in compare with the frequency of wave propagating in vacuum. So, the spin term decreases the frequency of the wave. This effect is presented in Fig. 4. The spin caused third and fourth terms have opposite signs in this regime.

Next, consider the right-hand circularly polarized electromagnetic waves. In this regime, the lower sign should be taken in equation (12) (for $\omega > |\Omega_e|$). Consider frequencies $\omega$ larger $1.1 |\Omega_e |$ and obtain the classic right-hand circularly polarized electromagnetic wave. Its analysis is similar to the presented above for the left-hand circularly polarized waves, but the sign of the last term in equation (12) is different. It is negative like the classic term caused by the electron motion. Therefore, both terms lead to increase of the frequency. The increase of the frequency of right-hand circularly polarized electromagnetic wave caused by the spin effects in compare to the classic regime is demonstrated in Fig. 2. The spin caused third and fourth terms have same sign in this regime.

The regime of right-hand circularly polarized waves demonstrates the spin wave solution at $0 < \omega < |\Omega_e | < 0.1 |\Omega_n |$.

Equation (12) corresponds to relatively large deviation of frequency $\omega$ from the cyclotron frequency $|\Omega_e |$ for the large variations of the wave vector, or the small deviations of frequency $\omega$ from the cyclotron frequency $|\Omega_e |$ for the small values of the wave vector. Hence, it allows consideration of area $\omega > |\Omega_n |$ and analysis of the spin waves with $\omega > |\Omega_n |$ for small wave vectors only. It is illustrated in Fig. 1. It shows that the nontrivial part of the equilibrium distribution functions increases the deviation of the spin wave frequency from the cyclotron frequency several times in compare with the results following from the third term on the right-hand side in equation (12).

Next, consider regime $\omega > |\Omega_n | < 0$, which is meaningful for the right-hand circularly polarized waves. As the result find, find the following dispersion equation:

$$\frac{k_z^2e^2}{\omega^2} = 1 - \frac{\omega^2_e}{\omega(\omega - |\Omega_e |)} + \frac{\hbar k_z^2}{2m n_{0e}} \frac{n_{0\uparrow} - n_{0\downarrow}}{\omega - |\Omega_e |}$$

(14)

Equation (14) demonstrates that the last term changes its sign for the right-hand circularly polarized waves at the transition to the small frequency regime $\omega > |\Omega_e |$ in compare with the large frequency regime $\omega > |\Omega_e |$ presented by equation (12) with the lower sign.

Considering the low-frequency limit of the dispersion equation for the right-hand circularly polarized waves (whistles) (14), find the following analytical solution:

$$\omega = |\Omega_e | \left[ \frac{k_z^2e^2}{\omega_{Le}^2} - \eta - \frac{\hbar k_z^2}{2m |\Omega_e |} \right]$$

$$+ \frac{k_z^2e^2}{\omega_{Le}^2} \frac{(3\pi^2)^{2/3}}{32\pi} \left( (1 + \eta)^{2/3} - (1 - \eta)^{2/3} \right) k_z n_{0e}^{1/3}.$$  

(15)

It is found by the iteration method assuming that spin contribution is relatively small. The first term in equation (15) is the classic term. The spin effects give two contributions in equation (15). The second term on the right-hand side is found earlier in literature (10, 11). In this model it comes from the trivial part of the equilibrium distribution functions. Its contribution gives a decrease of frequency of the whistler. The last term in equation (15) comes from the non-trivial part of the equilibrium distribution functions and gives an increase of the whistler frequency as it is demonstrated in Fig. 3.

As it is described above, at small frequencies the last term is decreased by factor $\omega/ |\Omega_e |$. Hence, it becomes comparable with the third term and competition between them is revealed in nonmonotoniol shift of the whistler spectrum (see Fig. 3).
V. HYDRODYNAMIC FERMI SPIN CURRENT

Analysis shows that a phenomenological generalization of the quantum hydrodynamic equations allows to derive the last term in equations (12), (14), and (16) which is the main spin correction appearing in the considered regime. It is found that an additional term should appear in the spin (magnetization) evolution equation. Moreover, it can be interpreted as the divergence of the spin current (spin flux). Therefore, an equation of state for the hydrodynamic Fermi spin current existing in the hydrodynamic spin evolution equation can be extracted from obtained results. The equation of state corresponds to the long-wavelength $k \to 0$ limits. In this limit, the Fermi spin current leading to the last terms in equations (12) and (14) can be captured for future study of the long-wavelength excitations. The magnetization evolution equation has the following structure $2, 31, 50$

$$n(\partial_t + u \cdot \nabla) \mu$$

$$- \frac{\hbar}{2m_\mu} \partial^3[n \mu \times \partial^3 \mu] + \mathfrak{F} = \frac{2 \mu_e}{\hbar} n[\mu \times B], \quad (16)$$

where $\mu = M/n$, $M = M(r, t)$ is the magnetization, $n = n(r, t)$ is the concentration of particles, $u(r, t)$ is the velocity field, and $\mathfrak{F}$ is the divergence of the thermal part of the spin current (which is called Fermi spin current for the degenerate electron gas), its explicit form is found in the following form

$$\mathfrak{F}_x = \frac{(6\pi^2)^2}{32\pi} \frac{\omega_{L_x}^2}{(\omega \pm |\Omega_\mu|)c} \frac{n_{\uparrow \downarrow} - n_{\downarrow \uparrow}}{n_0^2} (\Omega_\mu \delta E_x + \omega \delta E_y),$$

$$\mathfrak{F}_y = \frac{(6\pi^2)^2}{32\pi} \frac{\omega_{L_x}^2}{(\omega \pm |\Omega_\mu|)c} \frac{n_{\uparrow \downarrow} - n_{\downarrow \uparrow}}{n_0^2} (\omega \delta E_x - \Omega_\mu \delta E_y),$$

$$\mathfrak{F}_z = 0,$$

for left/right-hand polarized transverse waves correspondingly, at $\omega \pm |\Omega_\mu| > 0$. At $\omega \ll |\Omega_\mu|$, the lower sign should be chosen in the denominator and the upper sign should be chosen in front of the expression. This is a frequency dependent equation of state. Hence, it can be suitable for the linear or weakly-non-linear phenomena.

Advantage of this equation of state is the fact that it derived for the perturbation evolution while other equations of state derived in literature are derived for the equilibrium regimes $31, 36$. However, as it is mentioned above, the equation of state is found in small range of parameters.

VI. CONCLUSION

It has been demonstrated that the non-trivial part of the equilibrium distribution functions gives a considerable contribution in the long-wavelength limit. Spectra of all classic transverse waves propagating parallel to the external field are changed. In this regime, there are three classic waves and the spin wave. The spectra of classic waves are shifted while spin wave spectrum is modified dramatically. The well-known dispersion equation follows from the trivial part of the distribution functions leads to decreasing frequency as a function of the wave vector. The non-trivial part of the equilibrium distribution functions leads to further decrease of frequency as a function of the wave vector. Moreover, the module of group velocity $d\omega/dk_z$ increases several times.

It has been found that the dispersion equation is different for the right-hand circularly polarized waves with $\omega < |\Omega_\mu|$ while the hydrodynamic model or the earlier developed kinetic models give the same dispersion equation in both regimes. The modifications are caused by the by the non-trivial part of the equilibrium distribution functions. Corresponding generalization of hydrodynamic equations has been developed, where the frequency dependent equation of state for the spin current has been found and included.

An analytical expression is found for the spectrum of whistler. This spectrum explicitly presents the contribution of the spin effects including effects caused by the non-trivial equilibrium distribution functions.

All described above show that the developed kinetic model is necessary for the description of the spin effects in plasmas. This model is an essential generalization of existing models. The model allows to discover new spin related effects in linear and non-linear waves propagating parallel or perpendicular to the external magnetic field or for the oblique wave propagation. The equation of state for the hydrodynamic spin current has been extracted from the obtained results. Hence, the generalized hydrodynamic will provide a simple approximate description of phenomena related to the nontrivial part of the equilibrium distribution functions.

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VII. APPENDIX A: LINEAR SOLUTIONS OF KINETIC EQUATIONS

The following linearized Fourier transformed kinetic equations can be found from kinetic equations (1) and (2) at the consideration of small amplitude plane wave perturbations of the described equilibrium state:

$$-i \omega \delta f + i \mathbf{v} \cdot \mathbf{k} \delta f + \frac{q_e}{c} B_0 (\mathbf{v} \times \mathbf{e}_z) \cdot \nabla_\mathbf{p} \delta f$$

$$+ q_e \delta \mathbf{E} \cdot \nabla_\mathbf{p} f_0 + \mu_e (\mathbf{k} \cdot \nabla_\mathbf{p}) (\mathbf{S}_0 \cdot \delta \mathbf{B}) = 0, \quad (19)$$
and

\[-\omega \delta S + i(v \cdot k)\delta S + \frac{q_e}{e}B_0((v \times e_z) \cdot \nabla_p)\delta S \]

\[+ \frac{q_e}{e}((v \times \delta B) \cdot \nabla_p)S_0 + \mu_e(k \cdot \nabla_p)f_0\delta B \]

\[+ q_e(\delta E \cdot \nabla_p)S_0 + \frac{2\mu_e}{\hbar}(B_0 \times \delta S - S_0 \times \delta B) = 0, \quad (20) \]

where the wave vector has the following structure \(k = \{0, 0, k_z\}\) that corresponds to the propagation of waves parallel to the external magnetic field which leads to \(\delta B_z = 0\). Consider the following part of the Lorentz-like force in the spin evolution kinetic equation \(\frac{d}{dt}(v \times \delta B)\cdot \nabla_p S_0\) which is non-zero since \(S_{0x}\) and \(S_{0y}\) are non-isotropic functions. It can be represented in the following form \(-\frac{(q_e/mv)}{e^a\gamma}S_{0y}(v^2\delta B_z - v_z(v \cdot \delta B))/v_z^2\). Present more explicit form of the sixth term in the spin evolution kinetic equation \(q_e(\delta E \cdot \nabla_p)S_0 = q_e(\delta E \cdot v)\partial_z S_0 + (e_z \times S_0)(v \times \delta E) \cdot e_z)/mv_z^2\), where \(\partial_z\) is the derivative on kinetic energy \(\epsilon = p^2/2m\).

Solution of the linearized kinetic equations (19) and (20) leads to the following perturbations of the distribution functions

\[\delta f = \frac{1}{\Omega_e} \int_{C_0}^{\varphi} \left( q_e(v \cdot \delta E)\partial f_0 + \mu_e(k_z v_z) \left( \delta B \cdot \partial S_0 \right) \right) \exp \left( i \frac{(\omega - k_z v_z)}{\Omega_e}(\varphi' - \varphi) \right) d\varphi', \quad (21) \]

\[\delta S_z = \frac{1}{\Omega_e} \int_{C_3}^{\varphi} \left( q_e(v \cdot \delta E)\partial S_{0z} \right) \exp \left( i \frac{2\mu_e}{\hbar}(\delta B_x S_{0y} - \delta B_y S_{0x}) \right) \exp \left( i \frac{(\omega - k_z v_z)}{\Omega_e}(\varphi' - \varphi) \right) d\varphi', \quad (22) \]

\[\delta S_x = \frac{1}{2} \left\{ \int_{C_1}^{\varphi} \exp \left( -i \frac{\Omega_e}{\Omega_e}(\varphi' - \varphi) \right) (\Pi_x(\varphi') + i\Pi_y(\varphi')) d\varphi' \right\} \exp \left( -i \frac{(\omega - k_z v_z)}{\Omega_e}(\varphi) \right), \quad (23) \]

and

\[\delta S_y = \frac{1}{2} \left\{ \int_{C_1}^{\varphi} \exp \left( -i \frac{\Omega_e}{\Omega_e}(\varphi' - \varphi) \right) (\Pi_x(\varphi') + i\Pi_y(\varphi')) d\varphi' \right\} \exp \left( -i \frac{(\omega - k_z v_z)}{\Omega_e}(\varphi) \right), \quad (24) \]

Solutions for \(\delta S_x\) and \(\delta S_y\) (presented by equations (23) and (24)) contain the following functions

\[\Pi_x(\varphi) = \frac{1}{\Omega_e} \exp \left( i \frac{(\omega - k_z v_z)}{\Omega_e}(\varphi) \right) \]

\[\times \left( \mu_e(k \cdot \nabla_p)f_0\delta B_x + \frac{2\mu_e}{\hbar}S_{0x}\delta B_y + S_{0y}(v \cdot \delta B) \right) \frac{q_e}{e} \frac{v_z}{mv_z^2} + q_e(v \cdot \delta E)\partial_z S_{0x} - q_e S_{0y} \frac{(v \times \delta E) e_z}{mv_z^2} \right), \quad (25) \]

and

\[\Pi_y(\varphi) = \frac{1}{\Omega_e} \exp \left( i \frac{(\omega - k_z v_z)}{\Omega_e}(\varphi) \right) \]

\[\times \left( \mu_e(k \cdot \nabla_p)f_0\delta B_y - \frac{2\mu_e}{\hbar}S_{0z}\delta B_x + S_{0x}(v \cdot \delta B) \right) \frac{q_e}{e} \frac{v_z}{mv_z^2} + q_e(v \cdot \delta E)\partial_z S_{0y} + q_e S_{0x} \frac{(v \times \delta E) e_z}{mv_z^2} \right), \quad (26) \]

Constants \(C_0, C_1, C_2\) and \(C_3\) are chosen that distribution functions \(\delta f\) and \(\delta S\) are periodic functions of angle \(\varphi\): \(\delta f(\varphi + 2\pi) = \delta f(\varphi)\) and \(\delta S(\varphi + 2\pi) = \delta S(\varphi)\).
VIII. APPENDIX B: EXPLICIT FORM OF DIELECTRIC PERMEABILITY TENSOR

Tensor $\tilde{\Pi}^{\alpha\beta}_{\text{Cl}}(\theta, s)$ has the following explicit form:

$$
\tilde{\Pi}^{\alpha\beta}_{\text{Cl}}(\theta, s) = \frac{3\omega^2}{2\omega} \left( \begin{array}{cccc}
\frac{1}{4} & \frac{\sin^2 \theta}{\sin^2 \theta} & \frac{\sin^2 \theta}{\sin^2 \theta} & 0 \\
-\frac{1}{4} & \frac{\sin^2 \theta}{\sin^2 \theta} & \frac{\sin^2 \theta}{\sin^2 \theta} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\cos^2 \theta}{\cos^2 \theta}
\end{array} \right).
$$

(27)

Tensors $\varepsilon^{\alpha\beta}_{11}$ and $\varepsilon^{\alpha\beta}_{21}$ are calculated in Refs. [3], [4] and have the following form: $\varepsilon^{\alpha\beta}_{11} = 0$,

$$
\varepsilon^{\alpha\beta}_{21,a} = -\frac{m^2 \mu^2 c^2}{\pi \hbar^2 2\omega^2} \times
\sum_{s=\uparrow, \downarrow} \int \sin \theta d\theta \sum_{r=+,-} \frac{\mu^2 c^2}{\omega - k_z v_F s \cos \theta + r \Omega}.
$$

(28)

and

$$
\varepsilon^{\alpha\beta}_{21,b} = m^3 \mu^2 c^2 \times
\sum_{s=\uparrow, \downarrow} \int \sin \theta d\theta \sum_{r=+,-} \frac{\mu^2 c^2 (-1)^{s} \omega^2 dv}{\omega - k_z v \cos \theta + r \Omega}.
$$

(29)

where $\omega^2 = 4\pi e^2 n_0 / m$, $\tilde{\Pi}^{\alpha\beta}_{\text{Cl}}(\theta, s)$ is similar to the traditional result for degenerate electrons presented in many textbooks (see for instance [38]), but it also includes the spin separation effect.

Elements of the dielectric permeability tensor caused by the nontrivial part of $\delta F$ have the following structure

$$
\varepsilon_{12} = \frac{4\pi}{\omega} \left( \begin{array}{cccc}
\alpha_+ - \alpha_- & -i(\alpha_+ + \alpha_-) & 0 & 0 \\
i(\alpha_+ + \alpha_-) & \alpha_+ - \alpha_- & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
$$

(31)

where

$$
\alpha_{\pm} = \frac{1}{2\Omega} \int \frac{dz}{\omega - k_z v_{F \pm} z} \left\{ -i G_{\pm} \frac{\partial \Sigma(p)}{\partial p} \right\} dp.
$$

(32)

and

$$
\Sigma(p) = \frac{1}{(2\pi \hbar)^3} \left[ \Theta(p F_{\uparrow} - p) - \Theta(p F_{\downarrow} - p) \right].
$$

(33)

Elements of the dielectric permeability tensor caused by the nontrivial part of $\delta S$ have the following structure

$$
\varepsilon_{22} = \varepsilon_{22,a} + \varepsilon_{22,b} + \varepsilon_{22,c}:
$$

$$
\varepsilon_{22,a} = \left( \begin{array}{cccc}
\kappa_- - \kappa_+ & i(\kappa_+ + \kappa_-) & 0 & 0 \\
i(\kappa_+ + \kappa_-) & \kappa_- - \kappa_+ & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
$$

(34)

$$
\varepsilon_{22,b} = \left( \begin{array}{cccc}
\chi_+ - \chi_- & -i(\chi_+ + \chi_-) & 0 & 0 \\
i(\chi_+ + \chi_-) & \chi_+ - \chi_- & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
$$

(35)

and

$$
\varepsilon_{22,c} = \frac{4\pi}{\omega} \left( \begin{array}{cccc}
\beta_+ - \beta_- & -i(\beta_+ + \beta_-) & 0 & 0 \\
i(\beta_+ + \beta_-) & \beta_+ - \beta_- & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
$$

(36)

Further integration in the dielectric permeability tensor gives the following result:

$$
\varepsilon^{\alpha\beta}_{10} = -\sum_{s=\uparrow, \downarrow} \tilde{\Pi}^{\alpha\beta}_{\text{Cl}}(s),
$$

(40)

with

$$
\tilde{\Pi}^{\alpha\beta}_{\text{Cl}}(s) = \frac{3\omega^2}{2\omega} \times
\left( \begin{array}{cccc}
\frac{1}{4}[G_- + G_+] & \frac{i}{4}[G_- - G_+] & 0 & 0 \\
-\frac{i}{4}[G_- - G_+] & \frac{1}{4}[G_- + G_+] & 0 & 0 \\
0 & 0 & 0 & \frac{G_{zz}}{k_z v_F}
\end{array} \right)
$$

(41)

where

$$
G_\pm = G(\omega \pm \Omega_{e}) = \frac{1}{k_z v_F} \left[ \frac{2(\omega \pm \Omega_{e})}{k_z v_F} z \right],
$$

(42)

$$
+ \left( 1 - \frac{(\omega \pm \Omega_{e})^2}{(k_z v_F)^2} \right) \ln \left( \frac{\omega + k_z v_F s \pm \Omega_{e}}{\omega - k_z v_F s \pm \Omega_{e}} \right).
$$
and
\[
G_{zz} = \frac{\omega}{k_z v_{Fs}} \left[ \frac{\omega}{k_z v_{Fs}} \ln \left( \frac{\omega + k_z v_{Fs}}{\omega - k_z v_{Fs}} \right) - 2 \right].
\] (43)

Tensors \( \varepsilon^{\alpha \beta}_{11} \) and \( \varepsilon^{\alpha \beta}_{21} \) are caused by contribution of \( f_0 \) and \( S_{02} \) in the spin evolution. They are derived in [6] and [7]. Their structure is described in the following form:
\[
\varepsilon_{21,a} = k_z^2 \begin{pmatrix}
\gamma_+ + \gamma_- & -i(\gamma_+ - \gamma_-) & 0 \\
i(\gamma_+ - \gamma_-) & \gamma_+ + \gamma_- & 0 \\
0 & 0 & 0
\end{pmatrix},
\] (44)
and
\[
\varepsilon_{21,b} = k_z^2 \begin{pmatrix}
\delta_+ - \delta_- & -i(\delta_+ + \delta_-) & 0 \\
i(\delta_+ + \delta_-) & \delta_+ - \delta_- & 0 \\
0 & 0 & 0
\end{pmatrix}.
\] (45)

Next, present the explicit forms of elements \( \gamma_\pm \) and \( \delta_\pm \)
\[
\gamma_\pm = \sum_{s=\uparrow,\downarrow} m^2 v_{Fs} \frac{1}{\pi \hbar^2} \frac{\mu_c^2 e^2}{2 \omega^2} \times
\]
\[
\times \left( 2 - \frac{\omega + \Omega_\mu}{k_z v_{Fs}} \ln \left( \frac{\omega + k_z v_{Fs} + \Omega_\mu}{\omega - k_z v_{Fs} + \Omega_\mu} \right) \right),
\] (46)
and
\[
\delta_\pm = \sum_{s=\uparrow,\downarrow} m^3 \frac{\mu_c^2 e^2}{\pi \hbar^3} \frac{(-1)^s}{k_z} \left( v_{Fs} (\omega \pm \Omega_\mu) \right)
\]
\[
- \frac{(\omega \pm \Omega_\mu)^2 - (k_z v_{Fs})^2}{2 k_z} \ln \left( \frac{\omega + k_z v_{Fs} + \Omega_\mu}{\omega - k_z v_{Fs} + \Omega_\mu} \right).
\] (47)
The integral in \( \alpha_\pm \) contains the Dirac delta function of the momentum module. So, it can be easily presented as an integral over angle \( \theta \):
\[
\alpha_\pm = -\frac{\pi q_e \mu_e}{2(2\pi \hbar)^3} \frac{k_z^2 c}{\omega} \sum_s (-1)^s \times
\]
\[
\times \frac{m^2 v_{Fs}}{k_z v_{Fs}} \int \frac{\sin^2 \theta \cos \theta d\theta}{\omega - k_z v_{Fs} \cos \theta + \Omega_\mu}.
\] (48)

After taking the last integral, the explicit form of \( \alpha_\pm \) is found:
\[
\alpha_\pm = -\frac{q_e \mu_e}{16 \pi \hbar^4} \frac{k_z c}{\omega} \sum_s (-1)^s m^2 v_{Fs} \left[ \frac{1}{2} + \left( \frac{\omega \pm \Omega_\mu}{k_z v_{Fs}} \right)^2 \right]
\]
\[
- \frac{\omega \pm \Omega_\mu}{k_z v_{Fs}} \left( 1 - \frac{\omega \pm \Omega_\mu}{k_z v_{Fs}} \right) \sqrt{\frac{\omega \pm \Omega_\mu + k_z v_{Fs}}{\omega \pm \Omega_\mu - k_z v_{Fs}}}. \] (49)

Using the explicit form of \( \Sigma(p) \) represent \( \beta_\pm \) in the following form:
\[
\beta_\pm = \frac{k_z c}{\omega} \frac{q_e \mu_e}{16 \pi \hbar^3} \sum_s (-1)^s \times
\]
\[
\times \frac{1}{\pi} \int_0^\pi d\theta \int_0^{P_{Fs}} dp \frac{\cos \theta}{p \pm \Omega_\mu \pm k_z v_{Fs} - \cos \theta}.
\] (50)

Taking integral over the angle \( \theta \) find the following:
\[
\beta_\pm = \frac{k_z c}{\omega} \frac{q_e \mu_e}{16 \pi \hbar^3} \sum_s (-1)^s \int_0^{P_{Fs}} dp \left( -1 \right)
\]
\[
+ \left( \frac{\omega \pm \Omega_\mu}{\omega \pm \Omega_\mu + k_z v_{Fs}} \sqrt{\omega \pm \Omega_\mu + k_z v_{Fs}} \right). \] (51)

Finally, taking integral over the momentum module find the explicit form of function \( \beta_\pm \):
\[
\beta_\pm = \frac{q_e \mu_e}{16 \pi \hbar^3} \frac{k_z c}{\omega} \sum_s (-1)^s \left( -\frac{1}{2} \right) \times
\]
\[
+ \frac{m^2}{k_z^2} (\omega \pm \Omega_\mu) \left( (\omega \pm \Omega_\mu) - \sqrt{(\omega \pm \Omega_\mu)^2 - k_z^2 v_{Fs}^2} \right) \] (52)
for \( \omega + \Omega_\mu > 0 \) and \( \omega + \Omega_\mu < -k_z v_{Fs} \), or
\[
\beta_+ = \frac{q_e \mu_e}{16 \pi \hbar^3} \frac{k_z c}{\omega} \sum_s (-1)^s \left( -\frac{1}{2} \right) \times
\]
\[
+ \frac{m^2}{k_z^2} (\omega \pm \Omega_\mu) \left( (\omega \pm \Omega_\mu) + \sqrt{(\omega + \Omega_\mu)^2 - k_z^2 v_{Fs}^2} \right) \] (53)
for \( -k_z v_{Fs} < \omega + \Omega_\mu < 0 \).

The final form of the functions \( \kappa_\pm \) can be found after integration over the angle in equation (48). It appears as follows
\[
\kappa_\pm = \frac{q_e \mu_e}{16 \pi \hbar^3} \frac{k_z c}{\omega} \frac{m^2}{k_z^2 v_{Fs}} \sum_s (-1)^s v_{Fs} \times
\]
\[
\times \left[ \frac{\omega \pm \Omega_\mu}{k_z v_{Fs}} + 1 - \sqrt{\frac{\omega \pm \Omega_\mu + k_z v_{Fs}}{\omega \pm \Omega_\mu - k_z v_{Fs}}} \right]. \] (54)

To find the final form of functions \( \chi_\pm \) take integrals over angles \( \varphi \) and \( \theta \) and then take integral over the velocity module and obtain the following results:
\[
\chi_\pm = \frac{q_e \mu_e}{4 \pi \hbar^3} \frac{m^2}{\omega} \sum_s (-1)^s \times
\]
\[
\int_0^{v_{FS}} dv \frac{1}{1 + \frac{\omega \Omega_\mu}{k_z v}} \sqrt{\frac{\omega + \Omega_\mu + k_z v}{\omega + \Omega_\mu - k_z v}} = q_e \mu_e \frac{k_z e}{\omega} \frac{m^2}{k_z^2 2\hbar^3} \sum_s (-1)^s \left( \frac{\omega \pm \Omega_\mu}{\omega \pm \Omega_\mu - k_z^2 v_{FS}} \right).
\]

(55)

The dispersion equation appears in the following form at the application of the found structure of the dielectric permeability tensor:

\[
det \begin{pmatrix}
\varepsilon_{xx} - \frac{k_z^2 c^2}{\omega^2} & \varepsilon_{xy} & 0 \\
\varepsilon_{yx} & \varepsilon_{yy} - \frac{k_z^2 c^2}{\omega^2} & 0 \\
0 & 0 & \varepsilon_{zz}
\end{pmatrix} = 0,
\]

(56)

with \(\varepsilon_{xx} = \varepsilon_{yy} \equiv \epsilon\), and \(\varepsilon_{yx} = \varepsilon_{xy} = i\varepsilon\), where \(\epsilon = k_z^2 (\gamma_+ - \gamma_- + \delta_+ - \delta_-) + \frac{4\pi}{\omega} (\alpha_+ - \alpha_- + \beta_+ - \beta_-) + \kappa_- + \kappa_+ + \chi_+ + \chi_- - \sum_s \frac{3\omega_s^2}{\omega} (G_+ + G_-).\) All functions \(\alpha_\pm, \beta_\pm, \gamma_\pm, \delta_\pm, \chi_\pm, \kappa_\pm, \chi_c, G_{\pm} \) are described above, these functions are related to the following elements of the dielectric permeability tensor \(\varepsilon_{12}, \varepsilon_{22, c}, \varepsilon_{21, a}, \varepsilon_{21, b}, \varepsilon_{22, a}, \varepsilon_{22, b}, \varepsilon_{10}\) correspondingly, where \(\varepsilon_{21} = \varepsilon_{21, a} + \varepsilon_{21, b}, \varepsilon_{22} = \varepsilon_{22, a} + \varepsilon_{22, b} + \varepsilon_{22, c}\) and \(\varepsilon_{11} = 0\) for the waves propagating parallel to the external field. Tensor \(\varepsilon_{21, a}\) comes from 

\[
\frac{\mu_e}{\varepsilon_{FS}} \frac{\partial b}{\partial p} \cdot \frac{\partial b}{\partial t} f_0
\]

in equation (2). Tensor \(\varepsilon_{22, b}\) comes from torque-like term in equation (2): \(-2(\mu_e/h)S_{0z}(r, p, t)e_z \times \delta \mathbf{B}(r, t)\) .

Tensors \(\varepsilon_{22, a}\) and \(\varepsilon_{22, b}\) appear from \(\varepsilon_{22, a}(\partial (\mathbf{E} \cdot \mathbf{v}) \tau) S_{0x}\) and \(\varepsilon_{22, b}(\partial (\delta \mathbf{E} \cdot \mathbf{v}) \tau) S_{0y}\) in equation (2). Tensor \(\varepsilon_{22, a}\) comes from \(\varepsilon_{22, a}(\partial (\partial \mathbf{E} \cdot \mathbf{v}) \tau) S_{0x}\) and \(\varepsilon_{22, b}(\partial (\delta \mathbf{E} \cdot \mathbf{v}) \tau) S_{0y}\). Tensor \(\varepsilon_{22, b}\) comes from \(|e_e \times S_0|/((\mathbf{v} \times \mathbf{E}) \cdot e_c)/m^2\). Tensor \(\varepsilon_{22, c}\) appears from \(\varepsilon_{22, c}(\mathbf{v} \times \delta \mathbf{B}) \tau) S_0\). All \(= \pm \varepsilon\) presented above describe parts of terms containing \(\omega + \Omega_\mu\) or \(\omega - \Omega_\mu\), where \(a = e, \mu\). The dielectric permeability tensor \(\varepsilon_{FS}\) contains superposition of these terms.

The dispersion equation splits on three equations, one equation is for the longitudinal waves \(\varepsilon_{zz} = 0\) which is discussed in several papers [30, 31] and two equations are for the transverse waves

\[
\frac{k_z^2 c^2}{\omega^2} = \epsilon \pm \Xi,
\]

(57)

where

\[
\epsilon + \Xi = 2k_z^2 (\gamma_- - \delta_-) - \frac{8\pi}{\omega} (\alpha_- + \beta_-) + 2\kappa_- + 2\chi_- - \sum_s \frac{3\omega_s^2}{\omega} G_-,
\]

(58)

and

\[
\epsilon - \Xi = 2k_z^2 (\gamma_+ + \delta_+) + \frac{8\pi}{\omega} (\alpha_+ + \beta_+)
\]

Coefficient \(\pm\) in equation (57) appears as solution of a quadratic equation. Hence, this coefficient is independent from all \(\pm \) and \(\mp\) presented above. In equation (57) \(\epsilon\) corresponds to transverse waves with different circular polarizations \(\delta E_x = \pm \delta E_y\). Here and below \(\pm \) and \(\mp\) are produced by \(\pm \) in equation (57).

Presented above leads to the following structure of function \(\Sigma_{\mp}\) introduced in equation (11):

\[
\Sigma_{\mp} = 2k_z^2(\gamma_+ + \delta_+) \pm \frac{8\pi}{\omega} (\alpha_+ + \beta_+) \pm 2\kappa_+ \pm 2\chi_.
\]

(60)

IX. APPENDIX C: APPROXIMATE FORM OF FUNCTIONS \(\alpha_\pm, \beta_\pm, \gamma_\pm, \delta_\pm\)

Consider the small frequency and small wave vector limit, where \(k_z \nu_{FS}/\omega \gg 1\).

Consider approximate forms of functions \(\alpha_\pm, \beta_\pm, \gamma_\pm, \delta_\pm\) which are elements of dispersion equation (11):

\[
\tilde{\varepsilon}_{10} = 1 - \sum_{s=\uparrow, \downarrow} \frac{\omega_s^2}{\nu_{FS}} \left( \frac{k_z v_{FS}}{\omega \pm |\Omega_\mu|} + \frac{1}{5} \left( \frac{k_z v_{FS}}{\omega \pm |\Omega_\mu|} \right)^3 \right)
\]

(61)

\[
\tilde{\varepsilon}_{21, a} = 2k_z^2 \gamma_+ = -\frac{\omega_s^2}{\omega^2} \frac{4 \pi}{m^2} \frac{1}{(\omega \pm |\Omega_\mu|)^2},
\]

(62)

\[
\tilde{\varepsilon}_{21, b} = \mp 2k_z^2 \delta_+ = \frac{1}{(\omega \pm |\Omega_\mu|)^2},
\]

(63)

\[
= \sum_{s=\uparrow, \downarrow} \omega_s^2 \frac{\omega_s^2}{\omega^2} \frac{h k_z}{2m v_{FS}} \left( \frac{k_z v_{FS}}{\omega \pm |\Omega_\mu|} + \frac{1}{5} \left( \frac{k_z v_{FS}}{\omega \pm |\Omega_\mu|} \right)^3 \right)
\]

(64)
at \( \omega \pm \Omega_\mu > 0 \) and \( \omega + \Omega_\mu < 0 \), \( \varepsilon_{22} \) and \( \varepsilon_{22,c} \) are given by equations (64) and (65), have same form, but differs by the different cyclotron frequencies entering their expressions,

\[
\varepsilon_{22,a} = \pm 2\kappa_\mp = \pm 2q_{e}\mu_e \frac{k_z c_\mp m^2}{\omega} \frac{k_z}{8h^3} \sum_s (-1)^i v_{Fs} \times
\]

\[
\left[ \frac{k_z v_{Fs}}{\omega \pm |\Omega_\mu|} + \frac{1}{4} \frac{k_z^2 v_{Fs}^2}{(\omega \pm |\Omega_\mu|)^3} \right]
\]

\[
\approx \mp \frac{(6\pi^2)^{\frac{3}{2}}}{32\pi} \frac{\omega_k^2 n_{0u}^2 - n_{0d}^2}{\omega_k} k_z, \quad (66)
\]

If \( \omega \pm |\Omega_\mu| > 0 \), the functions \( \chi_\mp \) give the following assumptions:

\[
\varepsilon_{22,b} = \mp 2\chi_\mp = \mp 2q_{e}\mu_e \frac{k_z c_\mp m^2}{\omega} \frac{k_z}{4h^3} \sum_s (-1)^i v_{Fs} \times
\]

\[
\frac{k_z v_{Fs}}{\omega \pm |\Omega_\mu|} \left( 1 + \frac{1}{4} \frac{k_z^2 v_{Fs}^2}{(\omega \pm |\Omega_\mu|)^3} \right)
\]

\[
\approx \mp \frac{(6\pi^2)^{\frac{3}{2}}}{16\pi} \frac{\omega_k^2 n_{0u}^2 - n_{0d}^2}{\omega_k} k_z \approx \mp 4\kappa_\mp. \quad (67)
\]

If \( \omega - |\Omega_\mu| < 0 \), the function \( \chi_+ \) give the following assumption:

\[
\varepsilon_{22,b} = 2\chi_+ = \frac{k_z c \mu_e m^2}{\omega} \frac{k_z}{k_z} \times
\]

\[
\sum_s (-1)^i \left( 2(\omega - |\Omega_\mu|) - \frac{1}{2} \frac{k_z^2 v_{Fs}^2}{\omega - |\Omega_\mu|} \right) \approx 0. \quad (68)
\]

Functions \( \alpha_\pm, \beta_\pm \) and \( \gamma_\pm \) gives no contribution in equation (12). Functions \( \delta_\pm \) give the third term on the right-hand side. Functions \( \kappa_\pm \) and \( \chi_\pm \) lead to the last term. Earlier result can be found for instance in Ref. [4]. The fourth term is an extra term in compare with earlier papers.

X. APPENDIX D: DIMENSIONLESS FORM OF THE DISPERSION EQUATION FOR TRANSVERSE WAVES

Dimensionless variables are used for numerical analysis of the obtained results \( \xi = \omega/\omega_L, \kappa = k_c c/\omega_L, f = |\Omega_\|/\omega_L, g = 1.00116 \). Equation (12) is presented in the described dimensionless variables:

\[
\xi^2 - \kappa^2 - \frac{\xi}{\xi + f} - \eta \kappa^2 \frac{1}{\xi + g f} = 0 \quad (69)
\]

for \( \xi + f > 0 \), where \( R = \sqrt{n_{0e}/\omega_L}, \Upsilon = \hbar \omega_L/m c^2 \), for \( n_{0e} = 10^{27} \text{ cm}^{-3} \) find \( R \approx 16, 7, \Upsilon \approx 2.4 \times 10^{-3} \). Signs in equation (69) corresponds to the circular polarization of the plane wave \( \delta E_x = \pm i \delta E_y \). If \( \xi - f < 0 \) equation (69) changes to

\[
\xi^2 - \kappa^2 - \frac{i}{\xi - f} - \eta \kappa^2 \frac{1}{\xi - g f} = 0 \quad (70)
\]
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