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ABOUT BREZIS-MERLE PROBLEM WITH LIPSCHITZ CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give blow-up analysis for a Brezis and Merle’s problem with Dirichlet condition. As an application we have a proof of a compactness result under Lipschitz condition on the prescribed scalar curvature and a weaker assumption on the regularity of the domain (smooth domain or $C^{2,\alpha}$ domain, $1 \geq \alpha > 0$).

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1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = -(\partial_{11} + \partial_{22})$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth (or $C^{2,\alpha}$, $\alpha > 0$) boundary.

We consider the following equation:

$$(P) \begin{cases} \Delta u = Ve^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial \Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b < +\infty, \ e^u \in L^1(\Omega) \text{ and } u \in W^{1,1}_0(\Omega).$$

We can see in [7] a nice formulation of this problem $(P)$ in the sense of the distributions. This Problem arises from geometrical and physical problems see for example [1, 2, 18, 19]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-19], where one can find some existence and compactness results. In [6] we have the following important Theorem,

Theorem A (Brezis-Merle [6]). For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to $(P)$ with,

$$0 < a \leq V_i \leq b < +\infty$$

then, for all compact subset $K$ of $\Omega$ it holds,

$$\sup_K u_i \leq c,$$

with $c$ depending on $a, b, K$ and $\Omega$.

One can find in [6] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of $e^{u_i}$, namely, we have:
Theorem B (Brezis-Merle [6]). For \((u_i)_i,\) and \((V_i)_i\) two sequences of functions relative to the problem \((P)\) with,
\[
0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,
\]
then, for all compact subset \(K\) of \(\Omega\) it holds:
\[
\sup_K u_i \leq c,
\]
with \(c\) depending on \(b, C, K\) and \(\Omega\).

We look to the uniform boundedness on all \(\bar{\Omega}\) of sequences of solutions of the Problem \((P)\). Remark that, when \(a = 0\) the boundedness of \(\int_\Omega e^{u_i}\) is a necessary condition in the problem \((P)\) as showed in [6] by the following counterexample.

**Theorem C (Brezis-Merle [6]).** There are two sequences \((u_i)_i\) and \((V_i)_i\) of the problem \((P)\) with,
\[
0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,
\]
such that,
\[
\sup_{\Omega} u_i \to +\infty.
\]

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [6]) obtained by an approximation argument with the Fatou’s lemma and they applied the maximum principle in \(W^{1,1}_0(\Omega)\) which arises from Kato’s inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [5] for a topic about the weak form of the maximum principle.

**Remarks:**
1) Theorem 1 of [6], can be obtained by the usual maximum principle and Agmon regularity theorem which require \(C^2\) regularity on the domain.

2) The duality Theorem which we use require \(C^2\) regularity on the domain, see Gilbarg-Trudinger books.

Note that for the problem \((P)\), by using the Pohozaev identity, we can prove that \(\int_\Omega e^{u_i}\) is uniformly bounded when \(0 < a \leq V_i \leq b < +\infty\) and \(\|\nabla V_i\|_{L^\infty} \leq A\) and \(\Omega\) starshaped, when \(a = 0\) and \(\nabla \log V_i\) is uniformly bounded, we can bound uniformly \(\int_\Omega V_i e^{u_i}\). In [16] Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [9] Chen-Li have proved that if \(a = 0\) and \(\nabla \log V_i\) is uniformly bounded, then the functions are uniformly bounded near the boundary.

In [9] Chen-Li have proved that if \(a = 0\) and \(\int_\Omega e^{u_i}\) is uniformly bounded and \(\nabla \log V_i\) is uniformly bounded, then we have the compactness result directly. Ma-Wei in [16], extend this result in the case where \(a > 0\).

If we assume \(V\) more regular, we can have another type of estimates called \(\sup + \inf\) type inequalities. It was proved by Shafrir see [17] that, if \((u_i)_i, (V_i)_i\) are two sequences of functions solutions of the previous
equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

One can see in [10] an explicit value of $C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with $A$ its Lipschitz constant then $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$ see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [10]. Also, one can see in [14] an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [15] explicit form, $(8\pi m, m \in \mathbb{N}^*)$ exactly, for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8] we have some a priori estimates on the 2 and 3-spheres $S_2, S_3$.

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem with Lipschitz condition.

The Brezis-Merle Problem (see [6]) is:

**Problem.** Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i$. Also, we consider a sequence of solutions $(u_i)$ of $(P)$ relative to $(V_i)$ such that,

$$\int_{\Omega} e^{u_i} dx \leq C,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C?$$

Here, we give a characterization of the behavior of the blow-up points on the boundary and also, in particular we extend Chen-Li theorems, indeed, the result of Chen-Li holds for analytic domains and our result holds for smooth of $C^{2,\alpha}$ domains. For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b.$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary, but for the proof of the compactness for the Brezis-Merle problem we assume that:

$$||\nabla V_i||_{L^\infty} \leq A.$$

Our main results are:

**Theorem 1.1.** Assume that $\max_{\Omega} u_i \to +\infty$, where $(u_i)$ are solutions of the problem $(P)$ with:
then, after passing to a subsequence, there is a function \( u \), there is a number \( N \in \mathbb{N} \) and there are \( N \) points \( x_1, \ldots, x_N \in \partial \Omega \), such that,

\[
\partial_{\nu} u_i \to \partial_{\nu} u + \sum_{j=1}^{N} \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi,
\]

in the sense of measures on \( \partial \Omega \).

\( u_i \to u \) in \( C_{\text{loc}}^{1}(\bar{\Omega} - \{x_1, \ldots, x_N\}) \).

**Theorem 1.2.** Assume that \( (u_i) \) are solutions of \( (P) \) relative to \( (V_i) \) with the following conditions:

\[
0 \leq V_i \leq b, \quad ||\nabla V_i||_{L^{\infty}} \leq A \quad \text{and} \quad \int_{\Omega} e^{u_i} \leq C,
\]

we have,

\[
||u_i||_{L^{\infty}} \leq c(b, A, C, \Omega).
\]

In the previous theorem we have a proof of the global a priori estimate which concern the problem \( (P) \).

The proof of Chen-Li and Ma-Wei \([9,16]\), use the moving-plane method for the case \( \nabla \log V_i \) uniformly bounded near the boundary (and \( C^{2,\alpha} \) domain, \( 1 \geq \alpha > 0 \)) and for analytic domain for the case \( \nabla V_i \) uniformly bounded.

To prove Theorem 1.2, we argue by contradiction and use Theorem 1.1.

2. Proof of the theorems

**Proof of theorem 1.1:**

We have:

\[
u_i \in W_{0}^{1,1}(\Omega).
\]

Since \( e^{u_i} \in L^{1}(\Omega) \) by the corollary 1 of Brezis-Merle’s paper (see [6]) we have \( e^{u_i} \in L^{k}(\Omega) \) for all \( k > 2 \) and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

\[
u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\Omega).
\]

We denote by \( \partial_{\nu} u_i \) the inner normal derivative. By the maximum principle we have, \( \partial_{\nu} u_i \geq 0 \).

By the Stokes formula we have,

\[
\int_{\partial \Omega} \partial_{\nu} u_i d\sigma \leq C,
\]
We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure $\mu$ such that,

$$\int_{\partial\Omega} \partial_t u_i \varphi d\sigma \to \mu(\varphi), \quad \forall \, \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. For $\epsilon > 0$ small enough set $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$. We choose a function $\eta_\epsilon$ such that,

$$\eta_\epsilon \equiv 1, \text{ on } I_\epsilon, \quad 0 < \epsilon < \delta/2,$$

$$\eta_\epsilon \equiv 0, \text{ outside } I_{2\epsilon},$$

$$0 \leq \eta_\epsilon \leq 1,$$

$$||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{align*}
\Delta \tilde{\eta}_\epsilon &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\
\tilde{\eta}_\epsilon &= \eta_\epsilon \quad \text{in } \partial\Omega.
\end{align*}$$

**Remark:** We use the following steps in the construction of $\eta_\epsilon$:

We take a cutoff function $\eta_0$ in $B(0, 2)$ or $B(x_0, 2)$:

1. We set $\eta_0(x) = \eta_0(|x - x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2. Or, in the general case: we use a chart $(f, \Omega = f(B_r(0)))$, for $r > 0$ small enough and $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets $I_\epsilon$ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because $f, f^{-1}$ are Lipschitz, $|f(x) - x_0| \leq k_2|x| \leq 1$ for $|x| \leq 1/k_2$ and $|f(x) - x_0| \geq k_1|x| \geq 2$ for $|x| \geq 2/k_1 > 1/k_2$, the support of $\eta$ is in $I_{(2/k_1)\epsilon}$.

$$\begin{align*}
\eta_\epsilon &\equiv 1, \quad \text{on } f(I_{(1/k_1)\epsilon}), \quad 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \quad \text{outside } f(I_{(2/k_1)\epsilon}), \\
0 &\leq \eta_\epsilon \leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(I_{(2/k_1)\epsilon})} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{align*}$$

3. Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0), f(B_1(0)) = D_1(x_0)$ and $f(B_1^+) = D_1^+(x_0)$ with $f$ and $f^{-1}$ smooth diffeomorphism.

$$\begin{align*}
\eta_\epsilon &\equiv 1, \quad \text{on the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \quad \text{outside } J_\epsilon = f(I_{2\epsilon}), \\
0 &\leq \eta_\epsilon \leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(J_\epsilon)} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{align*}$$

And, $H_1(J_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, because $f$ is Lipschitz. Here $H_1$ is the Hausdorff measure.
We solve the Dirichlet Problem:

\[
\begin{cases}
\Delta \bar{\eta}_e = \Delta \eta_e & \text{in } \Omega \subset \mathbb{R}^2, \\
\bar{\eta}_e = 0 & \text{in } \partial \Omega.
\end{cases}
\]

and finally we set \( \tilde{\eta}_e = -\bar{\eta}_e + \eta_e \). Also, by the maximum principle and the elliptic estimates we have:

\[
||\nabla \tilde{\eta}_e||_{L^\infty} \leq C(||\eta_e||_{L^\infty} + ||\nabla \eta_e||_{L^\infty} + ||\Delta \eta_e||_{L^\infty}) \leq \frac{C_1}{e^2},
\]

with \( C_1 \) depends on \( \Omega \).

We use the following estimate, see [3, 7, 19],

\[
||\nabla u_i||_{L^q} \leq C_q, \forall \ i \ and \ 1 < q < 2.
\]

We deduce from the last estimate that, \( (u_i) \) converge weakly in \( W_{0}^{1,q}(\Omega) \), almost everywhere to a function \( u \geq 0 \) and \( \int_{\Omega} e^u < +\infty \) (by Fatou’s lemma). Also, \( V_i \) weakly converge to a nonnegative function \( V \) in \( L^\infty \). The function \( u \) is in \( W_{0}^{1,q}(\Omega) \) solution of:

\[
\begin{cases}
\Delta u = V e^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\
u = 0 & \text{in } \partial \Omega.
\end{cases}
\]

According to the corollary 1 of Brezis-Merle result, see [6], we have \( e^{ku} \in L^1(\Omega) \), \( k > 1 \). By the elliptic estimates, we have \( u \in C^1(\bar{\Omega}) \).

For two vectors \( v, w \) of \( \mathbb{R}^2 \) we denote by \( v \cdot w \) the inner product of \( v \) and \( w \).

We can write,

\[
\Delta((u_i - u)\tilde{\eta}_e) = (V_i e^{u_i} - V e^{u})\tilde{\eta}_e - 2\nabla(u_i - u) \cdot \nabla \tilde{\eta}_e. \tag{1}
\]

We use the interior estimate of Brezis-Merle, see [6],

**Step 1:** Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between \( \tilde{\eta}_e \) and \( u \), we obtain,

\[
\int_{\Omega} V e^u \tilde{\eta}_e dx = \int_{\partial \Omega} \partial_n u \tilde{\eta}_e \leq C' ||\partial_n u||_{L^\infty} = C' \epsilon
\]

We have,

\[
\begin{cases}
\Delta u_i = V_i e^{u_i} \in \Omega \subset \mathbb{R}^2, \\
u_i = 0 & \text{in } \partial \Omega.
\end{cases}
\]

We use the Green formula between \( u_i \) and \( \tilde{\eta}_e \) to have:
\[ \int_{\Omega} V_i e^{\eta_i} \eta_i \, dx = \int_{\partial \Omega} \partial_i u_i \eta_i \, d\sigma \rightarrow \mu(\eta_i) \leq \mu(J_i') \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3) \]

From (2) and (3) we have for all \( \epsilon > 0 \) there is \( i_0 = i_0(\epsilon) \) such that, for \( i \geq i_0 \),

\[ \int_{\Omega} |(V_i e^{\eta_i} - V e^{\eta}) \eta_i| \, dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4) \]

**Step 2**: Estimate of integral of the second term of the right hand side of (1).

Let \( \Sigma_\epsilon = \{ x \in \Omega, d(x, \partial \Omega) = \epsilon^3 \} \) and \( \Omega_{\epsilon^3} = \{ x \in \Omega, d(x, \partial \Omega) \geq \epsilon^3 \}, \epsilon > 0 \). Then, for \( \epsilon \) small enough, \( \Sigma_\epsilon \) is a manifold.

The measure of \( \Omega - \Omega_{\epsilon^3} \) is \( k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3 \). Here \( \mu_L \) is the Lebesgue measure.

**Remark**: for the unit ball \( \bar{B}(0, 1) \), our new manifold is \( \bar{B}(0, 1 - \epsilon^3) \).

(Proof of this fact; let’s consider \( d(x, \partial \Omega) = d(x, z_0), z_0 \in \partial \Omega \), this imply that \( (d(x, z_0))^2 \leq (d(x, z))^2 \) for all \( z \in \partial \Omega \) which it is equivalent to \( (z - z_0) \cdot (2x - z - z_0) \leq 0 \) for all \( z \in \partial \Omega \), let’s consider a chart around \( z_0 \) and \( \gamma(t) \) a curve in \( \partial \Omega \), we have:

\[ (\gamma(t) - \gamma(t_0) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0 \] and it is clear that, \( \gamma'(t_0) \cdot (x - \gamma(t_0)) = 0 \), which imply that \( x = z_0 - sv_0 \) where \( v_0 \) is the outward normal of \( \partial \Omega \) at \( z_0 \))

With this fact, we can say that \( S = \{ x, d(x, \partial \Omega) \leq \epsilon \} = \{ x = z_0 - sv_0, z_0 \in \partial \Omega, -\epsilon \leq s \leq \epsilon \} \). It is sufficient to work on \( \partial \Omega \). Let’s consider a charts \( z, D = B(z, 4\epsilon^3), \gamma_z \) with \( z \in \partial \Omega \) such that \( \cup_k B(z_k, \epsilon_k) \) is cover of \( \partial \Omega \). One can extract a finite cover \( \{ B(z_k, \epsilon_k) \}, k = 1, ..., m \), by the area formula the measure of \( S \cap B(z_k, \epsilon_k) \) is less than a \( k\epsilon \) (a \( \epsilon \)-rectangle). For the reverse inequality, it is sufficient to consider one chart around one point on the boundary).

We write,

\[ \int_{\Omega} |\nabla (u_i - u) \cdot \nabla \eta| \, dx = \int_{\Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \eta| \, dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \eta| \, dx. \quad (5) \]

**Step 2.1**: Estimate of \( \int_{\Omega - \Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \eta| \, dx \).

First, we know from the elliptic estimates that \( \| \nabla \eta \| \leq C_1/\epsilon^2 \), \( C_1 \) depends on \( \Omega \)

We know that \( \| \nabla u_i \| \) is bounded in \( L^q, 1 < q < 2 \), we can extract from this sequence a subsequence which converge weakly to \( h \in L^q \). But, we know that we have locally the uniform convergence to \( |\nabla u| \) (by Brezis-Merle’s theorem), then, \( h = |\nabla u| \) a.e. Let \( q' \) be the conjugate of \( q \).

We have, \( \forall f \in L^{q'}(\Omega) \)

\[ \int_{\Omega} |\nabla u_i| f \, dx \rightarrow \int_{\Omega} |\nabla u| f \, dx \]

If we take \( f = 1_{\Omega - \Omega_{\epsilon^3}} \), we have:
for $\epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}$, $i \geq i_1$, $\int_{\Omega-\Omega,3} |\nabla u_i| \leq \int_{\Omega-\Omega,3} |\nabla u| + \epsilon^3$.

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega-\Omega,3} |\nabla u_i| \leq \text{meas}(\Omega - \Omega,3)||\nabla u||_{L^\infty} + \epsilon^3 = \epsilon^3(k_1||\nabla u||_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega-\Omega,3} |\nabla (u_i - u) \cdot \nabla \tilde{\eta}_\epsilon | dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 1) \quad \text{(6)}$$

The constant $C_1$ does not depend on $\epsilon$ but on $\Omega$.

**Step 2.2:** Estimate of $\int_{\Omega,3} |\nabla (u_i - u) \cdot \nabla \tilde{\eta}_\epsilon | dx$.

We know that, $\Omega_\epsilon \subset \subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_\epsilon)$. We have,

$$||\nabla (u_i - u)||_{L^\infty(\Omega,3)} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega,3} |\nabla (u_i - u) \cdot \nabla \tilde{\eta}_\epsilon | dx \leq ||\nabla (u_i - u)||_{L^\infty(\Omega,3)}||\nabla \tilde{\eta}_\epsilon||_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla (u_i - u) \cdot \nabla \tilde{\eta}_\epsilon | dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2) \quad \text{(7)}$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon] | dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^\infty} + 2 + C) \quad \text{(8)}$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\left\{ \begin{array}{ll} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial \Omega. \end{array} \right.$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$.

We can use Theorem 1 of [6] to conclude that there is $q \geq \tilde{q} > 1$ such that:
\[ \int_{V_\epsilon(x_0)} e^{\tilde{q}u_i - u_i} dx \leq \int_{\Omega} e^{\tilde{q}u_i - \tilde{q}u} dx \leq C(\epsilon, \Omega). \]

where, \( V_\epsilon(x_0) \) is a neighborhood of \( x_0 \) in \( \bar{\Omega} \). Here we have used that in a neighborhood of \( x_0 \) by the elliptic estimates, \( 1 - C \epsilon \leq \tilde{q} \leq 1 \). (We can take, \( f(B_{\epsilon^3}(0)) \) and we have \( B_{k_2 \epsilon^3}(x_0) \subset f(B_{\epsilon^3}(0)) \subset B_{k_1 \epsilon^3}(x_0) \) for a chart \( (f, B_1(0)) \) around \( x_0 \)).

Thus, for each \( x_0 \in \partial \Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \) there is \( \epsilon_{x_0} > 0 \), \( q_{x_0} > 1 \) such that:

\[ \int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}u_i} dx \leq C, \quad \forall \ i. \] (9)

By the elliptic estimates (see [13]) \((u_i)_i\) is uniformly bounded in \( W^{2,q_1}(V_\epsilon(x_0)) \) and also, in \( C^1(V_\epsilon(x_0)) \).

Finally, we have, for some \( \epsilon > 0 \) small enough,

\[ \|u_i\|_{C^1,B(x_0,\epsilon)} \leq c_3 \quad \forall \ i. \]

We have proved that, there is a finite number of points \( \bar{x}_1, \ldots, \bar{x}_m \) such that the sequence \((u_i)_i\) is locally uniformly bounded (in \( C^{1,0}(0, \bar{\Omega}) \)) in \( \Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \).

**Proof of theorem 1.2:**

We know that:

\[ u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}). \]

We can do integration by parts. The first Pohozaev identity applied around each blow-up point see for example [16] gives:

\[ \int_{\partial\Omega_{x_k}} [ (\partial_{\nu} u_i) \nabla u_i - \frac{1}{2} |\nabla u_i|^2 \nu] dx = \int_{\Omega_{x_k}} \nabla V_i e^{u_i} - \int_{\partial\Omega_{x_k}} V_i e^{u_i} \nu, \] (10)

Here \( \Omega_{x_k} \) is a neighborhood of \( x_k \) on which we can use the integration by part obtained by a chart around \( x_k \).

We use the boundary condition on \( \Omega \) and the boundedness of \( u_i \) and \( \partial_j u_i \) outside the \( x_k \), to have:

\[ \int_{\partial\Omega} (\partial_{\nu} u_i)^2 dx \leq c_0(b, A, C, \Omega). \] (11)

Thus we can use the weak convergence in \( L^2(\partial\Omega) \) to have a subsequence \( \partial_{\nu} u_i \), such that:

\[ \int_{\partial\Omega} \partial_{\nu} u_i \varphi dx \to \int_{\partial\Omega} \partial_{\nu} u \varphi dx, \quad \forall \ \varphi \in L^2(\partial\Omega), \] (12)

Thus, \( \alpha_j = 0, j = 1, \ldots, N \) and \((u_i)\) is uniformly bounded.

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