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Original articles

Analysis of exact solution of stochastic sex-structured HIV/AIDS epidemic model with effect of screening of infectives

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Abstract

The world of uncertainty motivates the study of stochastic perturbation in the mathematical models of real life. The main objective of this paper is to study stochastic sex-structured HIV/AIDS epidemic model with effect of screening of infectives. We have shown that the proposed stochastic epidemic model with boundedness and permanence has a unique global positive solution. The selection of suitable Lyapunov functions provides sufficient conditions for investigating persistence and extinction of disease. Based on numerical experiments, the theoretical findings of this paper have been verified.

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1. Introduction

One of the most deadly viruses is the human immunodeficiency virus (HIV), which causes the acquired immune deficiency syndrome (AIDS). It infects the immune system’s cells and destroys the immune system’s ability to fight infections and diseases during the process. AIDS refers to advanced HIV infection stages. According to recent estimates, about 34 million people worldwide are infected with HIV/AIDS and most are living in low and middle income countries. Such an epidemic can be effectively studied using mathematical modeling. Advanced information and precautions for an epidemic can be provided by a proper mathematical model [18]. Several mathematical models have been studied to explain the dynamics of HIV spread due to the inherent strengths of the mathematical model. The mathematical approach to understanding the HIV epidemic and how to control the immune systems and their dynamics is discussed in [20,30]. In Lin et al. [22], the role of infections and their effects has been created by dividing the infectious stage into ‘r’ classes. Workowski and Berman et al. in [34] dealt with treatment and proper counseling that can control HIV transmission. Treatment of HIV/AIDS infected people may reduce transmission and infection rates. Cai et al. [7] studied the importance of treatment in the transmission of HIV/AIDS has been discussed. Individuals who do not know about HIV can spread/transmit the disease to society unknowingly. In

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Tripathi et al. [33], this was discussed. In Kaur et al. [17], a stage-structured HIV model was created and analyzed by incorporating awareness and treatment effects.

In the real world, however, epidemic systems are often subject to environmental noise, and the effects of a fluctuating environment are not incorporated by deterministic models. Stochastic differential equation models therefore play an important role in different branches of applied science, including infectious dynamics, as they provide some additional degree of realism compared to their deterministic counterpart [2,4]. Consequently, some authors have incorporated noise into HIV models and examined their dynamics [13,23]. For example, by approximating one of the variables by a mean reverting process Dalal et al. [9] analyzed a stochastic internal HIV model. Ji and Jiang [14] considered a cell-mediated immune response model of stochastic HIV-1 infection. They defined an appropriate condition in the large infection-free equilibrium for stochastic asymptotic stability. Since there is no infection equilibrium, they only explored the dynamics of the corresponding deterministic model around the two infection equilibria (one without triggering CTLs and the other with). Following the concept of [14], both Liu [24] and Liu et al. [24] demonstrated the asymptotic behavior of a nonlinear incidence stochastic delayed model of HIV-1 infection and a cell-to-cell model of HIV-1 infection. The asymptotic properties of a stochastic predator–prey system with functional response from Holling II was studied in Han et al. [11] and the asymptotic behavior of stochastically perturbed DI SIR epidemic models with saturated incidence were analyzed in Liu et al. [26]. A stochastic SIR model’s threshold behavior was discussed in Cai et al. [6]. Asymptotic behavior and stability of a stochastic AIDS transmission model have been discussed in Ding et al. [10]. With time delays influenced by stochastic perturbations, Edoardo et al. [5] discussed the epidemic model’s stability. The threshold of an imperfect vaccination stochastic SIS model has been analyzed in Liu et al. [25].

In this paper, motivated by the referred works, we will study the dynamics of the epidemic model of stochastic sex-structured HIV/AIDS with effect of screening of infectives which is considered in [3]:

\[
\begin{align*}
\frac{dS_m}{dt} &= \Gamma_1 - \mu S_m - a_1 S_m I_f, \\
\frac{dS_f}{dt} &= \Gamma_2 - \mu S_f - a_2 S_f I_m, \\
\frac{dI_m}{dt} &= a_1 S_m I_f - (\mu + \beta_1) I_m - b_1 I_m, \\
\frac{dI_f}{dt} &= a_2 S_f I_m - (\mu + \beta_2) I_f - b_2 I_f, \\
\frac{dA}{dt} &= b_1 I_m + b_2 I_f - (\mu + \beta_3) A,
\end{align*}
\]

where \(a_1 = \alpha(1 - \gamma_f), \ b_1 = \alpha(1 - \gamma_m), \ b_2 = [\xi_1 \tau \gamma_m + \xi_2 (1 - \tau) \gamma_m + \xi_3 (1 - \gamma_m)], \ b_2 = [\xi_1 \tau \gamma_f + \xi_2 (1 - \tau) \gamma_f + \xi_3 (1 - \gamma_f)], \ \Gamma_1 \) is the rate of recruitment in the susceptible males class, \( \Gamma_2 \) is the rate of recruitment in the susceptible females class, \( \mu \) is the rate of natural death rate, \( \beta_1 \) is the death rate due to infection in the male class, \( \beta_2 \) is the death rate due to infection in the female class, \( \beta_3 \) is the death rate due to AIDS, \( \gamma_m \) is the fraction of total infected males, who are screened, \( \gamma_f \) is the fraction of infected females, who are screened, \( \xi_1 \) is the AIDS progression rate in treated class, \( \xi_2 \) is the AIDS progression rate in untreated class, \( \tau \) is the fraction of aware infectives taking treatment, \( \alpha \) is the rate of transmission of HIV and \( S_m(t), \ S_f(t), \ I_m(t), \ I_f(t), \ A(t) \) denote susceptible males, susceptible females, infected males, infected females, AIDS-class respectively. Since the variable \( A \) does not appear explicitly in the first four equation of the system (1). So, we omit the last equation.

We assume that stochastic perturbations are white noise type which are directly proportional to \( S_m(t), \ S_f(t), \ I_m(t) \) and \( I_f(t) \). Then, the deterministic system (1) will be extended to the following system of stochastic differential equations of the form:

\[
\begin{align*}
\frac{dS_m}{dt} &= \left[ \Gamma_1 - \mu S_m - a_1 S_m I_f \right] dt + \sigma_{S_m} S_m dB_{S_m}(t), \\
\frac{dS_f}{dt} &= \left[ \Gamma_2 - \mu S_f - a_2 S_f I_m \right] dt + \sigma_{S_f} S_f dB_{S_f}(t), \\
\frac{dI_m}{dt} &= \left[ a_1 S_m I_f - (\mu + \beta_1) I_m - b_1 I_m \right] dt + \sigma_{I_m} I_m dB_{I_m}(t), \\
\frac{dI_f}{dt} &= \left[ a_2 S_f I_m - (\mu + \beta_2) I_f - b_2 I_f \right] dt + \sigma_{I_f} I_f dB_{I_f}(t),
\end{align*}
\]

where \( \sigma_{S_m}, \ \sigma_{S_f}, \ \sigma_{I_m}, \ \sigma_{I_f} \) are the intensities of the standard Gaussian white noises and \( B_{S_m}(t), \ B_{S_f}(t), \ B_{I_m}(t), \ B_{I_f}(t) \) are independent standard Brownian motions respectively.

The paper is organized as follows: In Section 2, we proved the existence and uniqueness of the global positive solutions, the stochastic boundedness and permanence of the system (2). In Section 3, the extinction of the proposed stochastic model is presented and in Section 4, the disease persistence of the stochastic system (2) is studied. We
investigated the suitable numerical simulations are presented in Section 5 to illustrate the theoretical results. Finally, the conclusion of this paper is discussed.

Let \((\Omega, \mathcal{F}, P)\) be the complete probability space with a filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions, (i.e., it is right continuous and increasing while \(\mathcal{F}_0\) contains all \(P\)-null sets). Let \(\mathbb{R}^4_+ = \{ Y \in \mathbb{R}^4, Y_i > 0 \ \forall \ 1 \leq i \leq 4 \}\) and \(Y(t) = (S_m(t), S_f(t), I_m(t), I_f(t))\) and denote \(C^{2,1}((0, \infty) \times \mathbb{R}^4; \mathbb{R}_+^4)\) as the family of all non-negative functions \(V(t, Y)\) defined on \((0, \infty) \times \mathbb{R}^4\) such that they are continuously twice differentiable in \(Y\) and once in \(t\).

We consider the following differential operator \(\mathcal{L}\) associated with 4-dimensional stochastic differential equation of the form

\[
dY(t) = f(t, Y)dt + g(t, Y)dB(t),
\]

where

\[
\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{4} f_i(t, Y) \frac{\partial}{\partial Y_i} + \frac{1}{2} \sum_{i,j=1}^{4} (g_i^T(t, Y)g_j(t, Y))_{ij} \frac{\partial^2}{\partial Y_i \partial Y_j}.
\]

If \(\mathcal{L}\) acts on a function \(V \in C^{2,1}(\mathbb{R}^4 \times (0, \infty))\), then

\[
\mathcal{L}V(t, Y) = V_t(t, Y) + V_Y(t, Y)f(t, Y) + \frac{1}{2} \text{trace}(g^T(t, Y)V_{YY}(t, Y)g(t, Y)),
\]

where \(V_t = \frac{\partial V}{\partial t}, V_Y = \left(\frac{\partial V}{\partial Y_1}, \frac{\partial V}{\partial Y_2}, \frac{\partial V}{\partial Y_3}, \frac{\partial V}{\partial Y_4}\right)\) and \(V_{YY} = \left(\frac{\partial^2 V}{\partial Y_i \partial Y_j}\right)_{4 \times 4}^T\).

**Theorem 1.1** ([19]). Let \(\mathcal{D} \subset \mathbb{R}^4_+, [0, \infty) \times \mathcal{D} = U\) be the domain containing the line \(y = y^*\) and assume there exists a function \(V(t, Y)\) two times continuously differentiable in \(U\) which is the positive define in Lyapunov sense and satisfies \(\mathcal{L}V \leq 0\) for \(y \neq y^*\). Then the solution \(Y(t) = y^*\) of stochastic differential equation (3) is stable in probability.

2. Analysis of solutions

In this section, we prove that the existence and uniqueness of the solution of the stochastic HIV/AIDS model (2). Next, we discuss one of the important concept of population dynamics, that is the stochastic ultimate boundedness of the solution of the model (2). Also, we investigate the long time survival in a population dynamics based on the concept of stochastic permanence. In accordance with the approaches indicated in [15,27,28,31,32]. Now let us begin with the existence and uniqueness of the solution of the stochastic HIV/AIDS model (2).

**Theorem 2.1.** If the initial value \((S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}^4_+\) of the solution of stochastic epidemic model (2) is given, then there exists a unique solution \((S_m(t), S_f(t), I_m(t), I_f(t))\) in \(\mathbb{R}^4_+\) for \(t \geq 0\) with probability one, that is \((S_m(t), S_f(t), I_m(t), I_f(t)) \in \mathbb{R}^4_+\ \forall \ t \geq 0\) almost surely.

**Proof.** For any given initial value \((S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}^4_+, \) there is a unique local solution \((S_m(t), S_f(t), I_m(t), I_f(t))\) on \([0, \tau_n),\) where \(\tau_n\) is the explosion time, because the stochastic epidemic model (2) satisfies the locally Lipschitz continuous conditions. We need to show that the solution of the stochastic model (2) is global. We need only to prove that \(\tau_n = \infty\) almost surely.

Choose a sufficiently large positive constant \(l_0 > 0\) such that \(S_m(0), S_f(0), I_m(0)\) and \(I_f(0)\) belong to \(\left[\frac{1}{l_0}, 0\right]\). Consider the following sequence of stopping times for each integer \(l \geq l_0\) as

\[
\tau_l = \inf\left\{t \in [0, \tau_n) : S_m(t) \notin \left(\frac{1}{l}, 1\right), S_f(t) \notin \left(\frac{1}{l}, 1\right), I_m(t) \notin \left(\frac{1}{l}, 1\right), I_f(t) \notin \left(\frac{1}{l}, 1\right)\right\}.
\]

For the empty set \(\emptyset,\) we set \(\inf \emptyset = \infty.\) Since \(\tau_l\) is non-decreasing as \(l \to \infty,\) we have

\[
\tau_\infty = \lim_{l \to \infty} \tau_l.
\]

Then \(\tau_\infty \leq \tau_n\) a.s. Now, we have to show that \(\tau_\infty = \infty\) a.s. If not, then there exist \(T > 0\) and \(\delta \in (0, 1)\) such that

\[
P[\tau_\infty \leq T] > \delta.
\]
There is an integer \( l_1 \geq l_0 \) such that
\[
P[\tau_l \leq T] \geq \delta, \quad \forall \ l \geq l_1. \tag{6}
\]
We define a function \( V : \mathbb{R}_+^4 \to \mathbb{R}_+ \) as follows
\[
V(S_m, S_f, I_m, I_f) = (S_m - 1 - \ln S_m) + (S_f - 1 - \ln S_f) + (I_m - 1 - \ln I_m) + (I_f - 1 - \ln I_f),
\]
then by applying Itô’s formula, we get
\[
dV(S_m, S_f, I_m, I_f) = \left[ \left( 1 - \frac{1}{S_m} \right) (\Gamma_1 - \mu S_m - a_1 S_m I_f) + \frac{1}{2} \sigma_{S_m}^2 \right] dt + \sigma_{S_m} (S_m - 1) dB_{S_m}
\]
\[
+ \left( 1 - \frac{1}{S_f} \right) (\Gamma_2 - \mu S_f - a_2 S_f I_m) + \frac{1}{2} \sigma_{S_f}^2 dt
\]
\[
+ \left( 1 - \frac{1}{I_m} \right) (a_1 S_m I_f - (\mu + \beta_1) I_m - b_1 I_m) + \frac{1}{2} \sigma_{I_m}^2 dt
\]
\[
+ \left( 1 - \frac{1}{I_f} \right) (a_2 S_f I_m - (\mu + \beta_2) I_f - b_2 I_f) + \frac{1}{2} \sigma_{I_f}^2 dt
\]
\[
\leq \Gamma_1 + \Gamma_2 + 4\mu + \beta_1 + \beta_2 + b_1 + b_2 + \frac{1}{2} \sigma_{S_m}^2 + \frac{1}{2} \sigma_{S_f}^2 + \frac{1}{2} \sigma_{I_m}^2 + \frac{1}{2} \sigma_{I_f}^2
\]
\[
=: M.
\]
Then, we have
\[
dV(S_m, S_f, I_m, I_f) \leq M dt + \sigma_{S_m} (S_m - 1) dB_{S_m} + \sigma_{S_f} (S_f - 1) dB_{S_f}
\]
\[
+ \sigma_{I_m} (I_m - 1) dB_{I_m} + \sigma_{I_f} (I_f - 1) dB_{I_f}.
\tag{7}
\]
By integration on both sides to (7) from 0 to \( \tau_l \wedge T \), we get
\[
\int_0^{\tau_l \wedge T} dV(S_m(u), S_f(u), I_m(u), I_f(u))
\]
\[
\leq \int_0^{\tau_l \wedge T} M du + \int_0^{\tau_l \wedge T} \left[ \sigma_{S_m} (S_m - 1) dB_{S_m} + \sigma_{S_f} (S_f - 1) dB_{S_f}
\right.
\]
\[
\left. + \sigma_{I_m} (I_m - 1) dB_{I_m} + \sigma_{I_f} (I_f - 1) dB_{I_f} \right].
\tag{8}
\]
where \( \tau_l \wedge T = \min \{ \tau_l, T \} \). Taking expectations on both sides of the above equation, we get
\[
\mathbb{E} V \left( S_m(\tau_l \wedge T), S_f(\tau_l \wedge T), I_m(\tau_l \wedge T), I_f(\tau_l \wedge T) \right)
\]
\[
\leq V \left( S_m(0), S_f(0), I_m(0), I_f(0) \right) + MT.
\tag{9}
\]
Setting $\Omega_l = \{\tau_l \leq T\}$ for $l \geq l_1$ and from (6), we have $\mathbb{P}(\Omega_l) \geq \delta$. For every $\zeta \in \Omega_l$, there is at least $S_m(\tau_l, \zeta)$ or $S_f(\tau_l, \zeta)$ or $I_m(\tau_l, \zeta)$ or $I_f(\tau_l, \zeta)$ equals either $l$ or $\frac{1}{l}$ and therefore

$$V \left( S_m(\tau_l, \zeta), S_f(\tau_l, \zeta), I_m(\tau_l, \zeta), I_f(\tau_l, \zeta) \right)$$

is no less than either $l - 1 - \ln l$, $\frac{1}{l} - 1 - \ln \left( \frac{1}{l} \right)$. Hence

$$V \left( S_m(0), S_f(0), I_m(0), I_f(0) \right) + MT \geq \mathbb{E} \left( \mathbb{I}_{\Omega_l(c)} V \left( S_m(\tau_l, \zeta), S_f(\tau_l, \zeta), I_m(\tau_l, \zeta), I_f(\tau_l, \zeta) \right) \right)$$

$$\geq \delta \min \left\{ l - 1 - \ln l, \frac{1}{l} - 1 - \ln \left( \frac{1}{l} \right) \right\}, \quad (10)$$

where $\mathbb{I}_{\Omega_l(c)}$ is the indicator function of $\Omega_l$. By letting $l \to \infty$ we obtain

$$\infty = V(S_m(0), S_f(0), I_m(0), I_f(0)) + MT < \infty$$

therefore, we have the contradiction. The proof is completed. $\square$

**Definition 2.1 ([21]).** Let $Y(t) = (S_m(t), S_f(t), I_m(t), I_f(t))$ be the solution of the stochastic epidemic model (2) is said to be stochastically ultimately bounded, when given any $\epsilon \in (0, 1)$, there exists a positive constant $\theta > 0$ such that the solution $Y(t)$ to the epidemic model (2) has the property that

$$\lim_{t \to \infty} \mathbb{P}(|Y(t)| > \theta) < \epsilon, \quad \text{(11)}$$

for any positive initial value $(S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}^4_+$. 

**Theorem 2.2.** For any positive initial value $(S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}^4_+$, the solutions of stochastic epidemic model (2) are stochastically ultimately bounded. 

**Proof.** Define

$$U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right) = S_m^v + S_f^v + I_m^v + I_f^v$$

for $(S_m(t), S_f(t), I_m(t), I_f(t)) \in \mathbb{R}^4_+$ and $v > 1$. By applying Itô’s formula to $e^t U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right)$,

$$d \left[ e^t U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right) \right]$$

$$= e^t U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right) + \frac{d}{dt} U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right)$$

$$= e^t \left[ S_m^v + S_f^v + I_m^v + I_f^v + v S_m^{-1} \left( I_1 - \mu S_m - a_1 S_f I_f \right) + v S_f^{-1} \left( I_2 - \mu S_f - a_2 S_f I_m \right) \right]$$

$$+ \frac{\nu}{2} \left[ \sigma_m^2 S_m^v + \sigma_f^2 S_f^v + \sigma_m^2 I_m^v + \sigma_f^2 I_f^v \right] dt + \int^t_0 \left[ \sigma_m S_m^v d B_m + \sigma_f S_f^v d B_f \right]$$

$$+ \int^t_0 \left[ \sigma_m I_m^v d B_m + \sigma_f I_f^v d B_f \right],$$

where $K > 0$ is suitable constant.

Integrating the above equality from 0 to $t$ and then taking expectation

$$\mathbb{E} \left( e^t U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right) \right) \leq U \left( S_m(0), S_f(0), I_m(0), I_f(0) \right) + K \mathbb{E} \int^t_0 e^s \ ds.$$

$$e^t \mathbb{E} U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right) \leq U \left( S_m(0), S_f(0), I_m(0), I_f(0) \right) + K(e^t - 1),$$

which implies

$$\mathbb{E} U \left( S_m(t), S_f(t), I_m(t), I_f(t) \right) \leq e^{-t} U \left( S_m(0), S_f(0), I_m(0), I_f(0) \right) + K.$$

Since

$$|Y(t)|^v = (S_m^v(t) + S_f^v(t) + I_m^v(t) + I_f^v(t))^v$$

$$\leq 4^v \max \left\{ S_m^v(t), S_f^v(t), I_m^v(t), I_f^v(t) \right\}$$

$$\leq 4^v (S_m^v + S_f^v + I_m^v + I_f^v),$$
we get
\[ \mathbb{E} |Y(t)|^\nu \leq 4^\frac{\nu}{2} (e^{-t} U (S_m(0), S_f(0), I_m(0), I_f(0)) + K), \]
which means
\[ \limsup_{t \to \infty} \mathbb{E} |Y(t)|^\nu \leq 4^\frac{\nu}{2} K < \infty. \]
This implies that there is a constant \( \theta_1 \) such that
\[ \limsup_{t \to \infty} \mathbb{E} \left| \sqrt{Y(t)} \right| < \theta_1. \]
Then, given any \( \epsilon > 0 \), choose \( \theta = \frac{\theta_1^2}{\epsilon^2} \), using Chebyshev’s inequality, we get
\[ \mathbb{P}(|Y(t)| > \theta |) \leq \frac{\mathbb{E} |\sqrt{Y(t)}|}{\sqrt{\theta}}. \]
Hence
\[ \limsup_{t \to \infty} \mathbb{P}(|Y(t)| > \theta |) \leq \frac{\theta_1}{\sqrt{\theta}} = \epsilon. \]
This completes the proof. \( \square \)

**Definition 2.2** ([21]). The solution \( Y(t) = (S_m(t), S_f(t), I_m(t), I_f(t)) \) of the stochastic epidemic model (2) is said to be stochastically permanent, if for any \( \epsilon \in (0, 1) \), there exists a pair of positive constants \( \theta \) and \( \Psi \) such that for any initial value \( (S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}_+^4 \), the solution \( Y(t) \) to stochastic model (2) has the properties
\[ \liminf_{t \to \infty} \mathbb{P}(|Y(t)| \leq \theta) \geq 1 - \epsilon, \quad (12) \]
\[ \liminf_{t \to \infty} \mathbb{P}(|Y(t)| \geq \Psi) \geq 1 - \epsilon. \quad (13) \]

**Theorem 2.3.** Let \( \mu < (\Gamma_1 + \Gamma_2) \) and for any positive initial value \( (S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}_+^4 \), the solution \( (S_m(t), S_f(t), I_m(t), I_f(t)) \) satisfies
\[ \limsup_{t \to \infty} \mathbb{E}(|Y(t)|^{-p}) \leq \varrho, \quad (14) \]
where \( p > 0 \) be a constant satisfying
\[ \frac{p + 1}{2} \max \left\{ \sigma^2_{S_m}, \sigma^2_{S_f}, \sigma^2_{I_m}, \sigma^2_{I_f} \right\} < (\Gamma_1 + \Gamma_2) - \mu, \quad (15) \]
\[ \varrho = 4^p \frac{4m_1 + m_2}{4\eta m_1} \max \left[ 1, \left( \frac{2m_1 + m_2 + \sqrt{m_2^2 + 4m_1 m_2}}{2m_1} \right)^{p - 2} \right], \quad (16) \]
in which \( \eta > 0 \) be a constant satisfying
\[ \eta < (\Gamma_1 + \Gamma_2) - \mu - \frac{p + 1}{2} \max \left( \sigma^2_{S_m}, \sigma^2_{S_f}, \sigma^2_{I_m}, \sigma^2_{I_f} \right), \quad (17) \]
\[ m_1 = (\Gamma_1 + \Gamma_2) - \mu - \frac{p + 1}{2} \max \left( \sigma^2_{S_m}, \sigma^2_{S_f}, \sigma^2_{I_m}, \sigma^2_{I_f} \right) - \eta, \quad (18) \]
\[ m_2 = \mu + \max \left( \sigma^2_{S_m}, \sigma^2_{S_f}, \sigma^2_{I_m}, \sigma^2_{I_f} \right) + 2\eta. \quad (19) \]

**Proof.** Let
\[ W(S_m, S_f, I_m, I_f) = \frac{1}{S_m + S_f + I_m + I_f}. \]
for \((S_m(t), S_f(t), I_m(t), I_f(t)) \in \mathbb{R}_+^4\). By using Itô’s formula, we get
\[
dW(S_m, S_f, I_m, I_f) = -W^2\left[\Gamma_1 + \Gamma_2 - \mu (S_m + S_f + I_m + I_f) - (\beta_1 + b_1)I_m - (\beta_2 + b_2)I_f\right]dt
- W^2\left[\sigma_S S_m dB_s + \sigma_S S_f dB_f + \sigma_I I_m dB_m + \sigma_I I_f dB_f\right]
+ W^3\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right]dt.
\]
By choosing a constant \(p > 0\) that satisfies (15) and using Itô’s formula, we have
\[
\mathcal{L}[\eta (1 + W)^p] = p (1 + W)^{p-1}\left[-W^2[\Gamma_1 + \Gamma_2 - \mu (S_m + S_f + I_m + I_f) - (\beta_1 + b_1)I_m - (\beta_2 + b_2)I_f]
+ W^3\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right]\right]
+ \frac{p(p - 1)}{2} W^4\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right],
\]
where
\[
Q = -W^2[\Gamma_1 + \Gamma_2 - \mu (S_m + S_f + I_m + I_f) - (\beta_1 + b_1)I_m - (\beta_2 + b_2)I_f]
- W^3[\Gamma_1 + \Gamma_2 - \mu (S_m + S_f + I_m + I_f) - (\beta_1 + b_1)I_m - (\beta_2 + b_2)I_f]
+ W^3\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right] + \frac{(p - 1)}{2} W^4\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right].
\]
Since
\[
W^3\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right] < \max \left(\sigma_S^2, \sigma_S^2, \sigma_I^2, \sigma_I^2\right) W,
\]
\[
W^4\left[\sigma_S^2 S_m^2 + \sigma_S^2 S_f^2 + \sigma_I^2 I_m^2 + \sigma_I^2 I_f^2\right] < \max \left(\sigma_S^2, \sigma_S^2, \sigma_I^2, \sigma_I^2\right) W^2,
\]
then
\[
Q \leq \mu + \max \left(\sigma_S^2, \sigma_S^2, \sigma_I^2, \sigma_I^2\right) W - \left[\Gamma_1 + \Gamma_2 - \mu - \frac{p + 1}{2} \max \left(\sigma_S^2, \sigma_S^2, \sigma_I^2, \sigma_I^2\right)\right] W^2.
\]
We choose a sufficiently small positive constant \(\eta\) such that it satisfies (17) and using by Itô’s formula, we get
\[
\mathcal{L}[\eta (1 + W)^p] = \eta \eta (1 + W)^p + \eta \eta \mathcal{L}[\eta (1 + W)^p]
= \eta ^{(1 + W)^p - 2} (\eta (1 + W)^2 + Q)
\leq \eta ^{(1 + W)^p - 2} (\eta - m_1 W^2 + m_2 W)
\leq \eta \eta ^{p - 2}.
\]
where
\[
\eta = \frac{4m_1 + m_2}{4m_1} \max \left[1, \left(\frac{2m_1 + m_2 + \sqrt{m_2^2 + 4m_1m_2}}{2m_1}\right)^{p - 2}\right]
\]
and \(m_1, m_2\) are already defined in the theorem. Then
\[
\mathbb{E}[\eta ^{(1 + W)^p}] \leq (1 + W(0)^p) + \frac{\eta_0}{\eta} \eta ^{p - 2}.
\]
Therefore
\[
\limsup_{t \to \infty} \mathbb{E}[W(t)^p] \leq \limsup_{t \to \infty} \mathbb{E}[1 + W] \leq \frac{Q_0}{\eta}.
\]
Also
\[
[S_m + S_f + I_m + I_f]^p \leq 4^p [S_m^2 + S_f^2 + I_m^2 + I_f^2] \leq 4^p |Y(t)|^p,
\]
consequently,
\[
\limsup_{t \to \infty} \mathbb{E}\left[\frac{1}{|Y(t)|^p}\right] \leq \limsup_{t \to \infty} \mathbb{E}[W(t)^p] \leq \frac{4^p Q_0}{\eta} = \varrho. \quad \blacksquare
\]

**Theorem 2.4.** Assume \(\max\left[\sigma_{S_m}^2, \sigma_{S_f}^2, \sigma_{I_m}^2, \sigma_{I_f}^2\right] < 2 \left((\Gamma_1 + \Gamma_2) - \mu\right)\), then the solutions of stochastic epidemic model (2) are stochastically permanent.

**Proof.** From Theorem 2.2, we have \(\mathbb{P}\{|Y(t)| > \theta\} \leq \epsilon\),
\[
\mathbb{P}\{|Y(t)| \leq \theta\} \geq 1 - \epsilon.
\]
This implies that
\[
\liminf_{t \to \infty} \mathbb{P}\{|Y(t)| \leq \theta\} \geq 1 - \epsilon.
\]
By Theorem 2.3, we get
\[
\limsup_{t \to \infty} \mathbb{P}\left[\frac{1}{|Y(t)|^p}\right] \leq \varrho.
\]
For any \(\epsilon > 0\), let \(\Psi = \frac{\epsilon^p}{\varrho^p}\), then
\[
\mathbb{P}\{|Y(t)| < \Psi\} = \mathbb{P}\left[\frac{1}{|Y(t)|^p} > \frac{1}{\Psi}\right] \leq \Psi^{\frac{1}{p}} \mathbb{E}\left(|Y(t)|^{-p}\right),
\]
hence,
\[
\limsup_{t \to \infty} \mathbb{P}\{|Y(t)| < \Psi\} \leq \Psi^{\frac{1}{p}} \varrho = \epsilon,
\]
which gives
\[
\limsup_{t \to \infty} \mathbb{P}\{|Y(t)| \geq \Psi\} \geq 1 - \epsilon.
\]
The proof is completed. \(\blacksquare\)

3. Extinction of the disease

There is a disease free equilibrium \(E_0\left(\frac{\Gamma_1}{\mu}, \frac{\Gamma_1}{\mu}, 0, 0\right)\) of the deterministic HIV/AIDS epidemic model (1) and it is globally stable if \(R_0 = \sqrt{\frac{a_1 \beta_1 \Gamma_1}{(\mu + \beta_1 + b_1)(\mu + \beta_2 + b_2)} < 1}\). This means that the disease will die out after a certain period of time. Hence, studying the disease-free equilibrium to control infectious disease is interesting. But the stochastic HIV/AIDS epidemic model (2) does not have disease free equilibrium and the stochastic solutions do not converge to \(E_0\). Therefore, we estimate the fluctuation around the disease free equilibrium \(E_0\) in this section to analyze whether or not the disease is going to die out.

**Theorem 3.1.** Let \(Y(t) = (S_m(t), S_f(t), I_m(t), I_f(t))\) be the solution of the stochastic model (2) with any given initial value \((S_m(0), S_f(0), I_m(0), I_f(0)) \in \mathbb{R}^4_+\). If \(R_0 \leq 1\), \(\sigma_{S_m}^2 + \frac{(2 \mu + \beta_1 + b_1)^2}{6(\mu + \beta_1 + b_1)} < \mu\), \(\sigma_{S_f}^2 + \frac{(2 \mu + \beta_2 + b_2)^2}{6(\mu + \beta_2 + b_2)} < \mu\), \(\sigma_{I_m}^2 < (\mu + \beta_1 + b_1)\), \(\sigma_{I_f}^2 < (\mu + \beta_2 + b_2)\) and the system satisfies the following condition
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left(6c\Gamma_2 \left(\mu - \frac{(2 \mu + \beta_1 + b_1)^2}{6(\mu + \beta_1 + b_1)} - \sigma_{S_m}^2\right) (S_m(s) - \frac{\Gamma_1}{\mu})^2 \right) ds \leq \frac{Q_0}{\eta}.
\]
+ 6\gamma_1 \left( \mu - \frac{(2\mu + \beta_2 + b_2)^2}{6(\mu + \beta_2 + b_2)} - \sigma_{S_j}^2 \right) \left( S_j - \frac{\Gamma_2}{\mu} \right)^2 \\
+ c\gamma_2 \left( \mu + \beta_1 + b_1 - \sigma_{I_m}^2 \right) I_m^2(s) + \gamma_1 \left( \mu + \beta_2 + b_2 - \sigma_{I_f}^2 \right) I_f^2(s) \right) \, ds \\
\leq \frac{6\gamma_1 \gamma_2}{\mu^2} \left( c\gamma_1 \sigma_{S_m}^2 + \gamma_2 \sigma_{S_j}^2 \right), \text{ a.s.} \tag{21} \end{align}

where $c \in \left[ \frac{\sigma_{S_m}}{\mu(\mu + \beta_1 + b_1)}, \frac{\mu(\mu + \beta_2 + b_2)}{\gamma_1} \right]$ is a positive number, i.e. the disease will die out with probability one.

**Proof.** For our proof we will use some ideas from [1,8,16,29,35]. Define a $C^2$-functions $V_{11}$, $V_{12}$, $V_{13}$, $V_{14}$, $V_{15}$ and $V_{16}$ defined for $(S_m, S_f, I_m, I_f) \in \mathbb{R}^4_+$ by

\begin{align*}
V_{11} &= \frac{1}{2} \left( S_m - \frac{\Gamma_1}{\mu} \right)^2, \quad V_{12} = \frac{1}{2} \left( S_f - \frac{\Gamma_2}{\mu} \right)^2, \quad V_{13} = I_m, \quad V_{14} = I_f, \quad V_{15} = \frac{1}{2} \left( S_m - \frac{\Gamma_1}{\mu} + I_m \right)^2, \\
V_{16} &= \frac{1}{2} \left( S_f - \frac{\Gamma_2}{\mu} + I_f \right)^2.
\end{align*}

By applying Itô’s formula, we have

\[ dV_{11} = \mathcal{L}V_{11} \, dt + \left( S_m - \frac{\Gamma_1}{\mu} \right) \sigma_{S_m} S_m \, dB_{S_m}(t). \]

where

\begin{align*}
\mathcal{L}V_{11} &= \left( S_m - \frac{\Gamma_1}{\mu} \right) \left( \Gamma_1 - \mu S_m - a_1 S_m I_f \right) + \frac{1}{2} \sigma_{S_m}^2 S_m^2, \\
&= -\mu \left( S_m - \frac{\Gamma_1}{\mu} \right)^2 - \mu \left[ a_1 I_f \left( S_m - \frac{\Gamma_1}{\mu} \right) \right] \left[ \left( S_m - \frac{\Gamma_1}{\mu} \right) \mu + \Gamma_1 \right] + \frac{1}{2} \sigma_{S_m}^2 S_m^2, \\
&\leq -\mu \left( S_m - \frac{\Gamma_1}{\mu} \right)^2 - \mu \Gamma_1 a_1 I_f \left( S_m - \frac{\Gamma_1}{\mu} \right) + \frac{1}{2} \sigma_{S_m}^2 S_m^2 \tag{22} \end{align*}

Similarly, we get

\begin{align*}
\mathcal{L}V_{12} &= \left( S_f - \frac{\Gamma_2}{\mu} \right) \left( \Gamma_2 - \mu S_f - a_2 S_f I_m \right) + \frac{1}{2} \sigma_{S_f}^2 S_f^2, \\
&= -\mu \left( S_f - \frac{\Gamma_2}{\mu} \right)^2 - \mu \left[ a_2 I_m \left( S_f - \frac{\Gamma_2}{\mu} \right) \right] \left[ \left( S_f - \frac{\Gamma_2}{\mu} \right) \mu + \Gamma_2 \right] + \frac{1}{2} \sigma_{S_f}^2 S_f^2, \\
&\leq -\mu \left( S_f - \frac{\Gamma_2}{\mu} \right)^2 - \mu \Gamma_2 a_2 I_m \left( S_f - \frac{\Gamma_2}{\mu} \right) + \frac{1}{2} \sigma_{S_f}^2 S_f^2 \tag{23} \end{align*}

\begin{align*}
\mathcal{L}V_{13} &= \frac{1}{\mu} \left[ a_1 I_f \left( S_m - \frac{\Gamma_1}{\mu} \right) \mu + \Gamma_1 \right] - \left( \mu + \beta_1 + b_1 \right) I_m, \\
&= a_1 I_f \left( S_m - \frac{\Gamma_1}{\mu} \right) + \frac{1}{\mu} a_1 \Gamma_1 I_f - \left( \mu + \beta_1 + b_1 \right) I_m \tag{24} \end{align*}

\begin{align*}
\mathcal{L}V_{14} &= \frac{1}{\mu} \left[ a_2 I_m \left( S_f - \frac{\Gamma_2}{\mu} \right) \mu + \Gamma_2 \right] - \left( \mu + \beta_2 + b_2 \right) I_f, \\
&= a_2 I_f \left( S_f - \frac{\Gamma_2}{\mu} \right) + \frac{1}{\mu} a_2 \Gamma_2 I_m - \left( \mu + \beta_2 + b_2 \right) I_f \tag{25} \end{align*}

\begin{align*}
\mathcal{L}V_{15} &= \left( S_m - \frac{\Gamma_1}{\mu} + I_m \right) \left[ -\mu \left( S_m - \frac{\Gamma_1}{\mu} \right) \mu + \beta_1 + b_1 \right] I_m + \frac{1}{2} \sigma_{S_m}^2 S_m^2 + \frac{1}{2} \sigma_{I_m}^2 I_m^2, \\
\end{align*}
Combining (22), (23), (24), (25), (26) and (27), we get

$$222$$

Integrating (28) from 0 to $$t$$, we get

$$222$$

Because of the basic reproduction number $$R_0 \leq 1$$, we get $$\frac{\alpha_2 \Gamma_2}{\mu (\mu + \beta_1 + b_1)} \leq \frac{\mu (\mu + \beta_2 + b_2)}{a_1 \Gamma_1}$$, then we choose a positive number $$c$$, so that $$\frac{\alpha_2 \Gamma_2}{\mu (\mu + \beta_1 + b_1)} \leq c \leq \frac{\mu (\mu + \beta_2 + b_2)}{a_1 \Gamma_1}$$. Define a $$C^2$$-function $$V_1 : \mathbb{R}_+^1 \to \mathbb{R}_4$$ by

$$V_1 = 4c \Gamma_2 \Gamma_2 V_1 + 4 \Gamma_1 \Gamma_2 + \frac{1}{\mu} \Gamma_1 \Gamma_2 V_13 + \frac{1}{\mu} \Gamma_1 \Gamma_1 V_14 + 2c \Gamma_2 V_15 + 2 \Gamma_1 V_16.$$

Combining (22), (23), (24), (25), (26) and (27), we get

$$222$$

Integrating (28) from 0 to $$t$$ and taking expectation, we get

$$222$$
∀ following conditions hold:

Lemma 4.1

Lemma before discussing the ergodic properties of the stochastic model (2).

Therefore, in this section, we demonstrate the existence of a unique ergodic equilibrium in stochastic model (2). This completes the proof. □

Remark 3.1. Theorem 3.1 shows that the solution of the system (2) fluctuates around the certain level which is relevant to $E_0$. The value of $\sigma_{s_m}$ and $\sigma_{s_f}$ decreasing, then the solution of the stochastic system (2) will be close to the disease free equilibrium $E_0$ of the system (1) for most of the time. Besides, if $\alpha_{s_m} = \alpha_{s_f} = 0$, then $E_0$ is also the disease free equilibrium of the stochastic model (2). We can get the proof of Theorem 3.1

$$L V_1 \leq -6c \Gamma_2 \left(\mu - \frac{(2\mu_1 + b_1)}{6(\mu_1 + b_1)} - \sigma^2_{s_m}\right) \left(S_m - \frac{\Gamma_1}{\mu} \right)^2 - 6\Gamma_1 \left(\mu - \frac{(2\mu + b_2)}{6(\mu + b_2)} - \sigma^2_{s_f}\right) \left(S_f - \frac{\Gamma_2}{\mu} \right)^2$$

$$+ c \Gamma_2 \left(\mu + b_1 + 1 - \sigma^2_{s_m}\right) I^2_m - \Gamma_1 \left(\mu + b_2 + 1 - \sigma^2_{s_f}\right) I^2_f.$$  

If the basic reproduction number $\mathcal{R}_0 \leq 1$, $\mu > \frac{(2\mu_1 + b_1)}{6(\mu_1 + b_1)}$, $\mu > \frac{(2\mu + b_2)}{6(\mu + b_2)}$, $\alpha_{s_m} < (\mu_1 + b_1)$ and $\alpha_{s_f} < (\mu + b_2)$, then we get $L V_1 \leq 0$. Consequently, the model solution (2) is stochastically asymptotically stable most of the time.

4. Disease persistence

We are interested in two things while studying epidemic dynamic disease models. One is when the disease dies, as seen in the above section, the other is when the disease prevails. We assume that $\mathcal{R}_0 > 1$, then there is a unique endemic equilibrium $E_1 = \left(S^*, S^*, I^*_m, I^*_f\right)$ where $S^*_m = \frac{\Gamma_1}{\mu + a_1 I^*_f}$, $S^*_f = \frac{\Gamma_2}{\mu + a_2 I^*_m}$, $I^*_m = \frac{a_1 I^*_m \Gamma f}{a_1 \mu (\mu + b_1)(\mu + b_2) + a_1 a_2 I^*_m (\mu + b_2)}$, $I^*_f = \frac{a_2 I^*_f \Gamma f}{a_2 \mu (\mu + b_1)(\mu + b_2) + a_1 a_2 I^*_m (\mu + b_2)}$ for the deterministic model (1) is globally stable but there is no endemic equilibrium in stochastic model (2). Therefore, in this section, we demonstrate the existence of a unique ergodic stationary distribution for the stochastic model (2), which shows that the disease will persist. We need the following Lemma before discussing the ergodic properties of the stochastic model (2).

Lemma 4.1 (I2.36). The solution of the stochastic model (2) has a unique ergodic stationary distribution, if the following conditions hold:

C1: For any bounded domain $U \subset \mathbb{R}^4_+$, there exists a positive constant $\Lambda$ such that $\sum_{h,k=1}^{4} a_{hk} \rho_h \rho_k \geq \Lambda |\rho|_2$, $\forall y \in U$ and $\rho \in \mathbb{R}^4_+$.

C2: There exist a neighborhood $\mathbb{D}$ and a non-negative $C^2-$function $V$ such that $L V$ is negative for any $y \in \mathbb{R}^4_+ \setminus \mathbb{D}$. 


Then, the Markov process \( Y(t) \) has a stationary distribution \( \pi(.) \) with density in \( \mathbb{R}^4_+ \) such that for any Borel set \( B \subset \mathbb{R}^4_+ \), \( \lim_{t \to \infty} P(t, y, B) = \pi(B) \) and
\[
\mathbb{P}_y \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(Y(t)) dt = \int_{\mathbb{R}^4_+} f(y) \pi(dy) \right\} = 1.
\]
\( \forall y \in \mathbb{R}^4_+ \), where \( f(y) \) is a function integrable with respect to the probability measure \( \pi \).

**Theorem 4.1.** Let \( (S_m(t), S_f(t), I_m(t), I_f(t)) \in \mathbb{R}^4_+ \) be the solution of the stochastic model (2) with any given initial value \((S_m(0), S_f(0), I_m(0), I_f(0))\). If \( R_0 > 1, \sigma_s > 0, \sigma_f > 0, \sigma_m > 0, \sigma_I > 0 \) and \( \min\{\kappa_S^2m^2, \kappa_f^2f^2, \kappa_I^2m^2, \kappa_I^2f^2\} > \kappa > 0 \), then the solution of the stochastic model (2) admits a unique ergodic stationary distribution. Here \( E_1 = (S_m^*, S_f^*, I_m^*, I_f^*) \) is the unique endemic equilibrium of the deterministic model (1). Especially, we have
\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \left\{ \kappa_1 (S_m(s) - S_m^*)^2 + \kappa_2 (S_f(s) - S_f^*)^2 + \kappa_3 (I_m(s) - I_m^*)^2 + \kappa_4 (I_f(s) - I_f^*)^2 \right\} ds \right] \leq \kappa.
\]
(32)
where
\[
\kappa_1 = \left[ \mu - \left( 1 + \frac{(2\mu + \beta_1 + b_1)^2}{2\mu (\mu + \beta_1 + b_1)} \right) \sigma^2_s \right] > 0, \quad \kappa_2 = \left[ \mu - \left( 1 + \frac{(2\mu + \beta_2 + b_2)^2}{2\mu (\mu + \beta_2 + b_2)} \right) \sigma^2_f \right] > 0,
\]
\[
\kappa_3 = \left[ \frac{\mu + \beta_1 + b_1 - \sigma^2_m}{2} \right] > 0, \quad \kappa_4 = \left[ \frac{\mu + \beta_2 + b_2 - \sigma^2_I}{2} \right] > 0,
\]
\[
\kappa = \left[ S_m^2 + \frac{(2\mu + \beta_1 + b_1)^2 S_m^2}{2\mu (\mu + \beta_1 + b_1)} + \frac{S_m^*}{2} \right] \sigma^2_s \left[ S_f^2 + \frac{(2\mu + \beta_2 + b_2)^2 S_f^2}{2\mu (\mu + \beta_2 + b_2)} + \frac{S_f^*}{2} \right] \sigma^2_f + \left[ I_m^2 + \frac{(2\mu + \beta_1 + b_1)^2 S_m^2 I_m^2}{4\mu (\mu + \beta_1 + b_1)} + \frac{I_m^*}{2} \right] \sigma^2_m + \left[ I_f^2 + \frac{(2\mu + \beta_2 + b_2)^2 S_f^2 I_f^2}{4\mu (\mu + \beta_2 + b_2)} + \frac{I_f^*}{2} \right] \sigma^2_I.
\]

**Proof.** We will use some methodology in our proof from [1,8,16,29,35]. When \( R_0 > 1 \), there is a unique endemic equilibrium point \( E_1 = (S_m^*, S_f^*, I_m^*, I_f^*) \) of the model (1) such that
\[
\Gamma_1 = \mu S_m^* + a_1 S_m^* I_f^*, \quad \Gamma_2 = \mu S_f^* + a_2 S_f^* I_m^*.
\]
\[
(\mu + \beta_1 + b_1) = \frac{a_1 S_m^* I_f^*}{I_m^*}, \quad (\mu + \beta_2 + b_2) = \frac{a_2 S_f^* I_m^*}{I_f^*}.
\]
Define the Lyapunov functions \( V_{21}, V_{22}, V_{23}, V_{24}, V_{25}, V_{26} \) and \( V_{27} \) defined for \((S_m, S_f, I_m, I_f) \in \mathbb{R}^4_+ \) by
\[
V_{21} = \frac{1}{2} \left[ (S_m - S_m^*) + (I_m - I_m^*) \right]^2, \quad V_{22} = \frac{1}{2} \left[ S_m - S_m^* \right]^2, \quad V_{23} = I_m - I_m^* - I_m^* \ln \frac{I_m}{I_m^*},
\]
\[
V_{24} = \frac{1}{2} \left[ (S_f - S_f^*) + (I_f - I_f^*) \right]^2, \quad V_{25} = \frac{1}{2} \left[ S_f - S_f^* \right]^2, \quad V_{26} = I_f - I_f^* - I_f^* \ln \frac{I_f}{I_f^*},
\]
\[
V_{27} = \left( S_m - S_m^* - S_m^* \ln \frac{S_m}{S_m^*} \right) + \left( I_m - I_m^* - I_m^* \ln \frac{I_m}{I_m^*} \right) + \left( S_f - S_f^* - S_f^* \ln \frac{S_f}{S_f^*} \right) + \left( I_f - I_f^* - I_f^* \ln \frac{I_f}{I_f^*} \right).
\]
By using Itô’s formula, we have
\[
dV_{21} = LV_{21}dt + (S_m - S_m^* + I_m - I_m^*) (\sigma_{S_m} S_m dB_{S_m}(t) + \sigma_{I_m} I_m dB_{I_m}(t)).
\]
\( \mathcal{L} V_{21} = \left[ (S_m - S_m^*) + (I_m - I_m^*) \right] \left[ I_1 - a_1 S_m I_f - \mu S_m + a_1 S_m I_f - (\mu + \beta_1 + b_1) I_m \right] + \frac{1}{2} \left( \sigma_{S_m}^2 S_m^* + \sigma_{I_m}^2 I_m^* \right) \\
= \left[ (S_m - S_m^*) + (I_m - I_m^*) \right] \left[ - \mu (S_m - S_m^*) - (\mu + \beta_1 + b_1) (I_m - I_m^*) \right] + \frac{1}{2} \left( \sigma_{S_m}^2 S_m^* + \sigma_{I_m}^2 I_m^* \right) \\
= - \mu (S_m - S_m^*)^2 - (\mu + \beta_1 + b_1) (I_m - I_m^*)^2 - \frac{1}{2} \sigma_{S_m}^2 S_m^* (I_m - I_m^*) \\
+ \frac{1}{2} \left( \sigma_{S_m}^2 S_m^* + \sigma_{I_m}^2 I_m^* \right) \\
\leq - \mu (S_m - S_m^*)^2 + \frac{2(\mu + \beta_1 + b_1)^2}{\mu + \beta_1 + b_1} (S_m - S_m^*)^2 - \frac{1}{2} \sigma_{S_m}^2 S_m^* (I_m - I_m^*) \\
+ \frac{1}{2} \left( \sigma_{S_m}^2 S_m^* + \sigma_{I_m}^2 I_m^* \right), 
\) 
(33)

the last inequality is derived from the fact \( 2ab \leq a^2 + b^2, \forall a, b \in \mathbb{R} \).

\( \mathcal{L} V_{22} = (S_m - S_m^*) \left[ I_1 - a_1 S_m I_f - \mu S_m \right] + \frac{1}{2} \sigma_{S_m}^2 S_m^* \\
= (S_m - S_m^*) \left[ a_1 S_m^* I_f^* + \mu S_m^* - a_1 S_m I_f - \mu S_m \right] + \frac{1}{2} \sigma_{S_m}^2 S_m^* \\
= - \mu (S_m - S_m^*)^2 - a_1 I_f (S_m - S_m^*)^2 - a_1 S_m (S_m - S_m^*) (I_f - I_f^*) + \frac{1}{2} \sigma_{S_m}^2 S_m^* \\
\leq - \mu (S_m - S_m^*)^2 - a_1 S_m (S_m - S_m^*) (I_f - I_f^*) + \frac{1}{2} \sigma_{S_m}^2 S_m^*, 
\) 
(34)

where the inequality in (34) is derived by \(-a_1 I_f (S_m - S_m^*)^2 \leq 0\).

\( \mathcal{L} V_{23} = \left( 1 - \frac{I_m^*}{I_m} \right) \left[ a_1 S_m I_f \frac{a_1 S_m^* I_f^*}{I_m^*} \right] + \frac{1}{2} \sigma_{I_m}^2 I_m^* \\
= a_1 (S_m - S_m^*) (I_f - I_f^*) + a_1 S_m^* I_f^* \left[ \frac{S_m}{S_m^*} \frac{I_f}{I_f^*} - \frac{I_m}{I_m^*} \frac{S_m I_f I_m^*}{S_m I_f I_m^*} \right] + \frac{1}{2} \sigma_{I_m}^2 I_m^* \\
\leq a_1 (S_m - S_m^*) (I_f - I_f^*) + a_1 S_m^* I_f^* \left[ 2 - \frac{S_m}{S_m^*} \frac{I_f}{I_f^*} - \frac{I_m}{I_m^*} \frac{S_m I_f I_m^*}{S_m I_f I_m^*} - 1 \right] + \frac{1}{2} \sigma_{I_m}^2 I_m^*, 
\) 
(35)

The second inequality is derived from the fact \( \log x \leq x - 1, \forall x \geq 0 \) and the last inequality implied by \( \left[ 2 - \frac{S_m}{S_m^*} \frac{I_f}{I_f^*} - \frac{I_m}{I_m^*} \frac{S_m I_f I_m^*}{S_m I_f I_m^*} - 1 \right] \leq 0 \). By the similar approach, we get

\( \mathcal{L} V_{24} = \left[ (S_f - S_f^*) + (I_f - I_f^*) \right] \left[ I_2 - a_2 S_f I_m - \mu S_f + a_2 S_f I_m - (\mu + \beta_2 + b_2) I_f \right] + \frac{1}{2} \left( \sigma_{S_f}^2 S_f^* + \sigma_{I_f}^2 I_f^* \right) \\
= \left[ (S_f - S_f^*) + (I_f - I_f^*) \right] \left[ - \mu (S_f - S_f^*) - (\mu + \beta_2 + b_2) (I_f - I_f^*) \right] + \frac{1}{2} \left( \sigma_{S_f}^2 S_f^* + \sigma_{I_f}^2 I_f^* \right) \\
= - \mu (S_f - S_f^*)^2 - (\mu + \beta_2 + b_2) (I_f - I_f^*)^2 - \frac{1}{2} \sigma_{S_f}^2 S_f^* (I_f - I_f^*) \\
+ \frac{1}{2} \left( \sigma_{S_f}^2 S_f^* + \sigma_{I_f}^2 I_f^* \right) \\
\leq - \mu (S_f - S_f^*)^2 + \frac{(2 \mu + \beta_2 + b_2)^2}{2 (\mu + \beta_2 + b_2)} (S_f - S_f^*)^2 - \frac{1}{2} \sigma_{S_f}^2 S_f^* (I_f - I_f^*) \\
+ \frac{1}{2} \left( \sigma_{S_f}^2 S_f^* + \sigma_{I_f}^2 I_f^* \right). 
\) 
(36)
and

\[
\mathcal{L}_{25} = \left(S_f - S_f^*\right) \left[I_f - a_2 S_f I_m - \mu S_f\right] + \frac{1}{2} \sigma_{S_f}^2 S_f
\]

\[
= \left(S_f - S_f^*\right) \left[a_2 S_f^* I_m^* + \mu S_f^* - a_2 S_f I_m - \mu S_f\right] + \frac{1}{2} \sigma_{S_f}^2 S_f
\]

\[
= -\mu \left(S_f - S_f^*\right)^2 - a_2 I_m \left(S_f - S_f^*\right)^2 - a_2 S_f^* \left(S_f - S_f^*\right) \left(I_m - I_m^*\right) + \frac{1}{2} \sigma_{S_f}^2 S_f^2
\]

\[
\leq -\mu \left(S_f - S_f^*\right)^2 - a_2 S_f^* \left(S_f - S_f^*\right) \left(I_m - I_m^*\right) + \frac{1}{2} \sigma_{S_f}^2 S_f^2, \quad (37)
\]

and

\[
\mathcal{L}_{26} = \left(1 - \frac{I_f^*}{I_f}\right) \left[a_2 S_f I_m - \frac{a_2 S_f^* I_m^*}{I_f}\right]
\]

\[
= a_2 \left(S_f - S_f^*\right) \left(I_m - I_m^*\right) + a_2 S_f^* I_m^* \left[\frac{S_f}{S_f^*} + \frac{I_m}{I_m^*} - \frac{I_f}{I_f^*} - \frac{S_f I_m I_f^*}{S_f^* I_m^* I_f}\right] + \frac{1}{2} \sigma_{I_f^*}^2 I_f^*
\]

\[
\leq a_2 \left(S_f - S_f^*\right) \left(I_m - I_m^*\right) - a_2 S_f^* I_m^* \left[2 - \frac{S_f}{S_f^*} - \frac{I_m}{I_m^*} - \frac{I_f}{I_f^*} \left(\frac{S_f^* I_m^*}{S_f I_m - 1}\right)\right] + \frac{1}{2} \sigma_{I_f^*}^2 I_f^*, \quad (38)
\]

and

\[
\mathcal{L}_{27} = \left(1 - \frac{S_m^*}{S_m}\right) \left(I_f - a_1 S_m I_f - \mu S_m\right) + \left(1 - \frac{I_m^*}{I_m}\right) \left(a_1 S_m I_f - (\mu + \beta_1 + b_1) I_m\right)
\]

\[
+ \left(1 - \frac{S_f^*}{S_f}\right) \left(I_f - a_2 S_f I_m - \mu S_f\right) + \left(1 - \frac{I_f^*}{I_f}\right) \left(a_2 S_f I_m - (\mu + \beta_2 + b_2) I_f\right)
\]

\[
+ \frac{1}{2} \left(\sigma_{S_m}^2 S_m^2 + \sigma_{S_f}^2 S_f^2 + \sigma_{I_m}^2 I_m^2 + \sigma_{I_f}^2 I_f^2\right)
\]

\[
= \frac{\mu S_m^*}{S_m} \left[2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m}\right] + a_1 S_m^* I_f^* \left[2 - \frac{S_m^*}{S_m - I_m - I_f} \left(\frac{S_m I_m}{S_m - I_m - I_f}\right)\right]
\]

\[
+ \mu S_f^* \left[2 - \frac{S_f}{S_f^*} - \frac{S_f^*}{S_f}\right] + a_2 S_f^* I_f^* \left[2 - \frac{S_f^*}{S_f - I_f} \left(\frac{S_f I_f}{S_f - I_f}\right)\right]
\]

\[
+ \frac{1}{2} \left(\sigma_{S_m}^2 S_m^2 + \sigma_{S_f}^2 S_f^2 + \sigma_{I_m}^2 I_m^2 + \sigma_{I_f}^2 I_f^2\right)
\]

\[
\leq \frac{\mu S_m^*}{S_m} \left[2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m}\right] + a_1 S_m^* I_f^* \left[2 - \frac{S_m^*}{S_m - I_m - I_f} \left(\frac{S_m I_m}{S_m - I_m - I_f}\right)\right]
\]

\[
+ \mu S_f^* \left[2 - \frac{S_f}{S_f^*} - \frac{S_f^*}{S_f}\right] + a_2 S_f^* I_f^* \left[2 - \frac{S_f^*}{S_f - I_f} \left(\frac{S_f I_f}{S_f - I_f}\right)\right]
\]

\[
+ \frac{1}{2} \left(\sigma_{S_m}^2 S_m^2 + \sigma_{S_f}^2 S_f^2 + \sigma_{I_m}^2 I_m^2 + \sigma_{I_f}^2 I_f^2\right), \quad (39)
\]

According to the fact that the arithmetic mean is greater than or equal to the geometric mean, it follows that

\[
2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m} \leq 0, \quad 2 - \frac{S_f}{S_f^*} - \frac{S_f^*}{S_f} \leq 0, \quad 2 - \frac{S_m}{S_m - I_m - I_f} \left(\frac{S_m I_m}{S_m - I_m - I_f}\right) \leq 0, \quad 2 - \frac{S_m}{S_m^*} - \frac{S_m^*}{S_m} \leq 0.
\]

By (34) and (35),

\[
\mathcal{L}_{22} + S_m \mathcal{L}_{23} \leq -\mu \left(S_m - S_m^*\right)^2 + \frac{1}{2} \sigma_{S_m}^2 S_m^2 + \frac{1}{2} \sigma_{I_m}^2 I_m^2, \quad (40)
\]
and taking (33) and (40), we have

\[
\mathcal{L}_{V_{21}} + \frac{(2\mu + \beta_1 + b_1)^2}{2\mu (\mu + \beta_1 + b_1)} (\mathcal{L}_{V_{22}} + S_m^2 \mathcal{L}_{V_{23}}) \\
\leq -\mu (S_m - S_m^e)^2 - \frac{1}{2} (\mu + \beta_1 + b_1) (I_m - I_m^e)^2 + \frac{1}{2} \sigma_m^2 S_m^2 + \frac{1}{2} \sigma_m^2 I_m^2 \\
+ \frac{(2\mu + \beta_1 + b_1)^2}{2\mu (\mu + \beta_1 + b_1)} (\sigma_m^2 S_m^2 + \sigma_m^2 S_m I_m^e).
\]

(41)

Similarly, we can obtain

\[
\mathcal{L}_{V_{24}} + \frac{(2\mu + \beta_2 + b_2)^2}{2\mu (\mu + \beta_2 + b_2)} (\mathcal{L}_{V_{25}} + S_j^e \mathcal{L}_{V_{26}}) \\
\leq -\mu (S_f - S_f^e)^2 - \frac{1}{2} (\mu + \beta_2 + b_2) (I_f - I_f^e)^2 + \frac{1}{2} \sigma_f^2 S_f^2 + \frac{1}{2} \sigma_f^2 I_f^2 \\
+ \frac{(2\mu + \beta_2 + b_2)^2}{2\mu (\mu + \beta_2 + b_2)} (\sigma_f^2 S_f^2 + \sigma_f^2 S_f I_f^e).
\]

(42)

Then

\[
\mathcal{L}_{V_2} = \mathcal{L}_{V_{21}} + \frac{(2\mu + \beta_1 + b_1)^2}{2\mu (\mu + \beta_1 + b_1)} (\mathcal{L}_{V_{22}} + S_m^2 \mathcal{L}_{V_{23}}) + \mathcal{L}_{V_{24}} + \frac{(2\mu + \beta_2 + b_2)^2}{2\mu (\mu + \beta_2 + b_2)} (\mathcal{L}_{V_{25}} + S_j^e \mathcal{L}_{V_{26}}) + \mathcal{L}_{V_{27}} \\
\leq -\mu (S_m - S_m^e)^2 - \frac{1}{2} (\mu + \beta_1 + b_1) (I_m - I_m^e)^2 + \frac{1}{2} \sigma_m^2 S_m^2 + \frac{1}{2} \sigma_m^2 I_m^2 \\
+ \frac{(2\mu + \beta_1 + b_1)^2}{2\mu (\mu + \beta_1 + b_1)} (\sigma_m^2 S_m^2 + \sigma_m^2 S_m I_m^e) \\
- \mu (S_f - S_f^e)^2 - \frac{1}{2} (\mu + \beta_2 + b_2) (I_f - I_f^e)^2 + \frac{1}{2} \sigma_f^2 S_f^2 + \frac{1}{2} \sigma_f^2 I_f^2 \\
+ \frac{(2\mu + \beta_2 + b_2)^2}{2\mu (\mu + \beta_2 + b_2)} (\sigma_f^2 S_f^2 + \sigma_f^2 S_f I_f^e)
\]

(43)

By using the inequality \(a^2 = 2(a - b)^2 + 2b^2, \forall a, b \in \mathbb{R}\), we have

\[
\mathcal{L}_{V_2} \leq -\left[ \mu - \left(1 + \frac{(2\mu + \beta_1 + b_1)^2}{2\mu (\mu + \beta_1 + b_1)}\right) \sigma_m^2 \right] (S_m - S_m^e)^2 \\
- \left[ \mu - \left(1 + \frac{(2\mu + \beta_2 + b_2)^2}{2\mu (\mu + \beta_2 + b_2)}\right) \sigma_f^2 \right] (S_f - S_f^e)^2 \\
- \left( \frac{\mu + \beta_1 + b_1}{2} - \sigma_m^2 \right) (I_m - I_m^e)^2 - \left( \frac{\mu + \beta_2 + b_2}{2} - \sigma_f^2 \right) (I_f - I_f^e)^2 \\
+ \left[ S_m^2 + \frac{(2\mu + \beta_1 + b_1)^2 S_m^2}{2\mu (\mu + \beta_1 + b_1)} + S_m^e \frac{S_m^2}{2} \right] \sigma_m^2 \\
+ \left[ S_f^2 + \frac{(2\mu + \beta_2 + b_2)^2 S_f^2}{2\mu (\mu + \beta_2 + b_2)} + S_f^e \frac{S_f^2}{2} \right] \sigma_f^2 \\
+ \left[ I_m^2 + \frac{(2\mu + \beta_1 + b_1)^2 S_m^2 I_m^e}{4\mu (\mu + \beta_1 + b_1)} + I_m^e \frac{I_m^2}{2} \right] \sigma_m^2 \\
+ \left[ I_f^2 + \frac{(2\mu + \beta_2 + b_2)^2 S_f^2 I_f^e}{4\mu (\mu + \beta_2 + b_2)} + I_f^e \frac{I_f^2}{2} \right] \sigma_f^2
\]

(44)
which can be simplified into
\[
\mathcal{L}V_2 \leq -\kappa_1 \left( S_m - S_m^* \right)^2 - \kappa_2 \left( S_f - S_f^* \right)^2 - \kappa_3 \left( I_m - I_m^* \right)^2 - \kappa_4 \left( I_f - I_f^* \right)^2 + \kappa, \tag{45}
\]
where \( \kappa_1, \kappa_2, \kappa_3, \kappa_4 \) and \( \kappa \) are defined in Theorem statement. Integrating (45) from 0 to \( t \) and taking expectation, we get
\[
0 \leq E \left[ V_2(S_m(t), S_f(t), I_m(t), I_f(t)) \right] \leq E \left[ V_2(S_m(0), S_f(0), I_m(0), I_f(0)) \right] \\
\leq -E \int_0^t \left\{ -\kappa_1 \left( S_m - S_m^* \right)^2 - \kappa_2 \left( S_f - S_f^* \right)^2 - \kappa_3 \left( I_m - I_m^* \right)^2 - \kappa_4 \left( I_f - I_f^* \right)^2 \right\} ds + \kappa t. \tag{46}
\]
Thus, taking the limit \( t \to \infty \) the above equation we get,
\[
\lim_{t \to \infty} \frac{1}{t} E \int_0^t \left\{ -\kappa_1 \left( S_m - S_m^* \right)^2 + \kappa_2 \left( S_f - S_f^* \right)^2 + \kappa_3 \left( I_m - I_m^* \right)^2 + \kappa_4 \left( I_f - I_f^* \right)^2 \right\} ds \leq \kappa. \tag{47}
\]
Note that if \( \kappa < \min \left\{ \kappa_1 S_m^2, \kappa_2 S_f^2, \kappa_3 I_m^2, \kappa_4 I_f^2 \right\} \), then
\[
-\kappa_1 \left( S_m - S_m^* \right)^2 - \kappa_2 \left( S_f - S_f^* \right)^2 - \kappa_3 \left( I_m - I_m^* \right)^2 - \kappa_4 \left( I_f - I_f^* \right)^2 + \kappa = 0 \tag{48}
\]
lies entirely in \( \mathbb{R}_+^4 \), thus there exist a positive constant \( z \) greater than 0 and a compact set \( \mathbb{U} \subset \mathbb{R}_+^4 \) such that, for any \( y \in \mathbb{U} \),
\[
\kappa_1 \left( S_m - S_m^* \right)^2 + \kappa_2 \left( S_f - S_f^* \right)^2 + \kappa_3 \left( I_m - I_m^* \right)^2 + \kappa_4 \left( I_f - I_f^* \right)^2 \geq \kappa + z. \tag{49}
\]
By taking (45) into account, for any \( y \in \mathbb{R}_+^4 \setminus \mathbb{D} \), \( \mathcal{L}V_2 \leq -z \), which implies that the property (C2) in Lemma 4.1 is satisfied. The diffusion matrix of the stochastic model (2) is given by
\[
\begin{bmatrix}
\sigma_{S_m}^2 S_m & 0 & 0 & 0 \\
0 & \sigma_{S_f}^2 S_f & 0 & 0 \\
0 & 0 & \sigma_{I_m}^2 I_m & 0 \\
0 & 0 & 0 & \sigma_{I_f}^2 I_f
\end{bmatrix}.
\]
There is a positive constant
\[
\Lambda = \min_{(S_m, S_f, I_m, I_f) \in \mathbb{U} \subset \mathbb{R}_+^4} \left\{ \sigma_{S_m}^2 S_m^2, \sigma_{S_f}^2 S_f^2, \sigma_{I_m}^2 I_m^2, \sigma_{I_f}^2 I_f^2 \right\}
\]
such that
\[
\sum_{h,k=1}^n \sum_{l} a_{hl}(y) a_{kl}(y) \rho_h \rho_k = \sigma_{S_m}^2 S_m^2 \rho_1^2 + \sigma_{S_f}^2 S_f^2 \rho_2^2 + \sigma_{I_m}^2 I_m^2 \rho_3^2 + \sigma_{I_f}^2 I_f^2 \rho_4^2 + \Lambda |\rho|^2, \quad \forall \rho \in \mathbb{R}_+.
\tag{50}
\]
Then, according to Lemma 4.1, we obtain the ergodic property of the stochastic model (2). This completes the proof. \( \square \)

**Remark 4.1.** Theorem 4.1 indicates that disease prevails, it is known that when \( R_0 > 1, \kappa_1 > 0, \kappa_2 > 0, \kappa_3 > 0, \kappa_4 > 0 \), the solution of the stochastic HIV/AIDS model (2) fluctuates around the endemic equilibrium. Particularly, when the intensities are equal to zero, the stochastic HIV/AIDS model (2) degenerated into the corresponding deterministic HIV/AIDS model (1). If the value of intensities \( \sigma_{S_m}, \sigma_{S_f}, \sigma_{I_m}, \sigma_{I_f} \) decreasing, then the difference between the solution of the stochastic HIV/AIDS model (2) and the endemic equilibrium \( E_1 \) is small to reflect that the disease will persist.
Fig. 1. Numerical simulations of the path $S_m(t)$, $S_f(t)$, $I_m(t)$, $I_f(t)$ for the deterministic model (1) and stochastic model (2) when $R_0 < 1$.

5. Numerical experiments

We perform some numerical examples to illustrate the analytical results of stochastic model (2). Then the system of Eqs. (2) can be rewritten as the following discretization equations:

$$
\begin{align*}
S_m(i + 1) &= S_m(i) + (I_1 - \mu S_m(i) - a_1 S_m(i) I_f(i)) \Delta t \\
&\quad + \sigma_{S_m} S_m(i) \sqrt{\Delta t} \chi(i) + \frac{\sigma_{S_m}^2}{2} S_m(i) (\chi(i)^2 - 1) \Delta t, \\
S_f(i + 1) &= S_f(i) + (I_2 - \mu S_f(i) - a_2 S_f(i) I_m(i)) \Delta t \\
&\quad + \sigma_{S_f} S_f(i) \sqrt{\Delta t} \chi(i) + \frac{\sigma_{S_f}^2}{2} S_f(i) (\chi(i)^2 - 1) \Delta t, \\
I_m(i + 1) &= I_m(i) + (a_1 S_m(i) I_f(i) - (\mu + \beta_1) I_m(i) - b_1 I_m(i)) \Delta t \\
&\quad + \sigma_{I_m} I_m(i) \sqrt{\Delta t} \chi(i) + \frac{\sigma_{I_m}^2}{2} I_m(i) (\chi(i)^2 - 1) \Delta t, \\
I_f(i + 1) &= I_f(i) + (a_2 S_f(i) I_m(i) - (\mu + \beta_2) I_f(i) - b_2 I_f(i)) \Delta t \\
&\quad + \sigma_{I_f} I_f(i) \sqrt{\Delta t} \chi(i) + \frac{\sigma_{I_f}^2}{2} I_f(i) (\chi(i)^2 - 1) \Delta t.
\end{align*}
$$

(51)

where $\chi(i)$, $i = 1, 2 \ldots n$ is the Gaussian random variables $N(0, 1)$.

We choose the intensities of the noise $\sigma_{S_m} = 0.03$, $\sigma_{S_f} = 0.01$, $\sigma_{I_m} = 0.02$, $\sigma_{I_f} = 0.01$ and the other parameter values of the stochastic model (2) are chosen as: $I_1 = 50$, $I_2 = 40$, $\mu = 0.016$, $\alpha = 0.00003$, $\gamma_m = 0.78$, $\gamma_f = 0.7$, $\sigma_1 = 0.005$, $\sigma_2 = 0.006$, $\tau = 0.75$, $\mu_1 = 0.0008$, $\mu_2 = 0.0009$, $\mu_3 = 0.003$ and the initial values are $S_m = 100$, $S_f = 200$, $I_m = 150$, $I_f = 400$. The values of the parameter are the same as [3]. The basic reproduction
Fig. 2. Histogram of $S_m(t)$ and $S_f(t)$ of extinction time for the stochastic model (2) when $R_0 < 1$.

Fig. 3. Numerical simulations of the path $S_m(t)$, $S_f(t)$, $I_m(t)$, $I_f(t)$ for the deterministic model (1) and stochastic model (2) when $R_0 > 1$.

number for these parameter values

$$R_0 = \sqrt{\frac{a_1a_2\Gamma_1\Gamma_2}{(\mu + \beta_1 + b_1)(\mu + \beta_2 + b_2)\mu^2}} = 0.96623 < 1.$$  

Theorem 3.1 and Remark 3.1 conditions

$$0.00090541 = \frac{\sigma^2_{S_m}}{6(\mu + \beta_1 + b_1)} < \mu = 0.016, \quad 0.00010549 = \frac{\sigma^2_{S_f}}{6(\mu + \beta_2 + b_2)} < \mu = 0.016,$$
The probability distribution histogram of the solution \((S_m(t), S_f(t), I_m(t), I_f(t))\) of the stochastic model (2). Fig. 4.

\[ 0.0004 = \sigma^2_{I_m} < (\mu + \beta_1 + b_1) = 0.022215, \quad 0.0001 = \sigma^2_{I_f} < (\mu + \beta_2 + b_2) = 0.022375, \]

\[ 0.01600 = \mu > \frac{(2\mu + \beta_1 + b_1)^2}{6(\mu + \beta_1 + b_1)} = 5.4071e - 6, \quad 0.01600 = \mu > \frac{(2\mu + \beta_2 + b_2)^2}{6(\mu + \beta_2 + b_2)} = 5.4917e - 6 \]

are satisfied and show that the stochastic model (2) disease free equilibrium point \(E_0(3125, 2500, 0, 0)\) is stochastically asymptotically stable. Fig. 1 shows that the deterministic model (1) and the stochastic model (2) have similar properties. Both of the model’s solutions converge to the disease free equilibrium point \(E_0\). The histogram of the probability distribution for the extinction time is shown in Fig. 2.

We choose the parameter values \(\Gamma_1 = 50, \Gamma_2 = 40, \mu = 0.016, \alpha = 0.00003, \gamma_m = 0.5, \gamma_f = 0.56, \sigma_1 = 0.005, \sigma_2 = 0.006, \tau = 0.85, \mu_1 = 0.0008, \mu_2 = 0.0009, \mu_3 = 0.003\) with intensities of white noise \(\sigma_{S_m} = 0.03, \sigma_{S_f} = 0.03, \sigma_{I_m} = 0.02, \sigma_{I_f} = 0.03\). Note that the basic reproduction number for these parameter values

\[ R_0 = \sqrt{\frac{a_1a_2\Gamma_1\Gamma_2}{(\mu + \beta_1 + b_1)(\mu + \beta_2 + b_2)\mu^2}} = 1.7558 > 1 \]

and the deterministic model (1) has a unique endemic equilibrium point

\[ E_1 = (S^*_m, S^*_f, I^*_m, I^*_f) = (1867.857, 1356.648, 898.9622, 815.8059). \]

Additionally,

\[ \kappa_1 = \left[ \mu - \left( 1 + \frac{(2\mu + \beta_1 + b_1)^2}{2\mu(\mu + \beta_1 + b_1)} \right) \sigma^2_{S_m} \right] = 0.0151 > 0, \]

\[ \kappa_2 = \left[ \mu - \left( 1 + \frac{(2\mu + \beta_2 + b_2)^2}{2\mu(\mu + \beta_2 + b_2)} \right) \sigma^2_{S_f} \right] = 0.0156 > 0, \]
We choose the following parameters as $\Gamma$, $E$, $\sigma$ and the initial values are $S_0$, $E_0$, $I_0$. Both models converge to the endemic equilibrium point $E_1$ around the endemic equilibrium $E_1$ of the deterministic model (1), and the average fluctuations around the endemic equilibrium $E_1$ are small due to the weak noise intensities that the disease reflects will persist. Then, Theorem 4.1 conditions are fulfilled. Therefore the stochastic model (2) solutions fluctuate for a long time around the positive unique endemic equilibrium $E_1$ of the deterministic model (1), and the average fluctuations around the endemic equilibrium $E_1$ are small due to the weak noise intensities that the disease reflects will persist.

The following shows the similarities and differences between the deterministic model (1) and the stochastic model (2) in four different cases of the screening parameters. We choose the following parameters as $\gamma = 0.85, \gamma_f = 0.56$ and $\sigma_{S_0} = 0.1, \sigma_{S_f} = 0.1, \sigma_{I_0} = 0.2, \sigma_{I_f} = 0.2$. Then the basic reproduction number $R_0 = 0.96818 < 1$ and the deterministic model (1) has the disease free equilibrium point $E_0 = (3125, 2500, 0, 0)$ which is stable but the conditions of Theorem 3.1 are not satisfied. However, the disease still die out for stochastic model (2) it is seen in Fig. 5, this indicates that the conditions given in Theorem 3.1 are not necessary.

**Case 1:** If we choose $\gamma = 0.75, \gamma_f = 0.56$ and $\sigma_{S_0} = 0.2, \sigma_{S_f} = 0.02, \sigma_{I_0} = 0.04, \sigma_{I_f} = 0.04$. Fig. 6 shows that the basic reproduction number $R_0 = 1.24751 > 1$ and the deterministic model (1) has the endemic equilibrium point $E_1 = (2459.908, 2040.694, 480.1569, 327.7247)$ which is stable and the conditions $\kappa_1 = 0.0156 > 0$, $\kappa_2 = 0.0156 > 0$, $\kappa_3 = 0.0094813 > 0$, $\kappa_4 = 0.009612 > 0$ of Theorem 4.1 are satisfied.
Fig. 6. Numerical simulations of the path $S_m(t)$, $S_f(t)$, $I_m(t)$, $I_f(t)$ for the deterministic model (1) and stochastic model (2) when $R_0 > 1$.

**Case 3:** If we choose $\gamma_f = 0.95$, $\gamma_m = 0.5$, and $\sigma_{S_m} = 0.02$, $\sigma_{S_f} = 0.02$, $\sigma_{I_m} = 0.04$, $\sigma_{I_f} = 0.04$. The remaining parameter values are same in the first case. Then the basic reproduction number $R_0 = 0.5963 < 1$ and the conditions of Theorem 3.1 are satisfied. It is shown in Fig. 7.

**Case 4:** If we choose $\gamma_f = 0.85$, $\gamma_m = 0.5$ and $\sigma_{S_m} = 0.1$, $\sigma_{S_f} = 0.1$, $\sigma_{I_m} = 0.2$, $\sigma_{I_f} = 0.2$. Note that $R_0 = 1.03088 > 1$ the conditions $\kappa_1 = 0.006 > 0$, $\kappa_2 = 0.006 > 0$, $\kappa_3 = -0.028813 < 0$, $\kappa_4 = -0.028911 < 0$ of Theorem 4.1 are not satisfied. Therefore the deterministic model (1) only has the endemic equilibrium $E_1 = (3062.931, 2400.129, 44.38456, 72.05186)$ which is stable (see Fig. 8). However, the disease still die out for the stochastic model (2), hence which indicates that the conditions given in Theorem 4.1 are sufficient but not necessary.

The relationship between the parameters $\gamma_m$, $\gamma_f$, $\xi_1$, $\xi_2$ and basic reproduction number is shown in Fig. 9. In addition, from Figs. 5 to 8 shows that the HIV screening parameters of male and female increase, the equilibrium of the male and female HIV infections may be reduced. This is shown more clearly in Fig. 10. In Fig. 11 shows path simulations of male and female infectives of the different rate of aware of infectives taking treatment.

6. Conclusion

The problems of the most real world are not deterministic. The stochastic effects that take place in the deterministic model give us a more practical way to build epidemic models. In this paper, we have studied the stochastic sex-structured HIV/AIDS epidemic model with effect of screening of infectives. Firstly, we have proved some qualitative properties, such as the existence of global positive solutions, boundedness and permanence solution of the proposed stochastic model (2). Secondly, by constructing suitable Lyapunov functions and applying Itô’s
formula, we have found that the stochastic model (2) has a disease free equilibrium point $E_0$ and it is globally asymptotically stable when the reproduction number $R_0$ does not exceed a critical level. We also shown that, when the white noise intensity is sufficiently small and the reproduction number $R_0$ is larger than a critical level, the stochastic model (2) has a unique stationary distribution, and any solution of the stochastic model (2) in the distribution concentrates around the unique endemic equilibrium $E_1$. Finally, to validate our theoretical studies, some computer-numerical simulations are provided. We conclude that the stochastic model (2) shows that disease extinction and persistence depends on the magnitude of the intensity of white noise as well as the parameters involved in the stochastic model (2). In addition, our analysis shows that screening with appropriate counseling is an effective way to reduce the prevalence of HIV/AIDS infections. Increasing the screening rates reduces the basic reproduction number and the magnitude of the infectious individuals in the population. We conclude that efforts should be made, by screening with appropriate counseling, in order to prevent the disease. Furthermore, some other topics of interest are worthy of further consideration. The approach used in this paper can also be used to investigate other epidemic models, such as TB, malaria, dengue, the current COVID-19 pandemic model, etc. We are interested to study analysis of extinction and persistence of the disease for stochastic HIV-TB co-infection epidemic model. In addition, this paper only analyzes the effect of white noise on the deterministic model, we can also add colored noise to the deterministic model and analyze the existence of an ergodic stationary distribution of the positive solutions to the model being considered. We are planning to study these in our future work.

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Fig. 8. Numerical simulations of the path $S_m(t)$, $S_f(t)$, $I_m(t)$, $I_f(t)$ for the deterministic model (1) and stochastic model (2) when $R_0 > 1$.

Fig. 9. The plot shows the effects of the parameters $\gamma_m$, $\gamma_f$, $\xi_1$, $\xi_2$ and the basic reproduction number.
Fig. 10. The plot shows the effects of the parameters $\gamma_m, \gamma_f$, on the equilibrium levels of $I_m$ and $I_f$.

Fig. 11. The plot shows the effect of the parameter $\tau$ on the equilibrium levels of $I_m$ and $I_f$.

References

[1] R.P. Agarwal, Q. Badshah, G. ur Rahman, S. Islam, Optimal control and dynamical aspects of a stochastic pine wilt disease model, J. Franklin Inst. 356 (2019) 3991–4025.
[2] L. Arnold, W. Horsthemke, J.W. Stucki, The influence of external real and white noise on the Lotka–Volterra model, Biom. J. 21 (1979) 451–471.
[3] S. Athithan, M. Ghosh, Analysis of a sex-structured HIV/AIDS model with the effect of screening of infectives, Int. J. Biomath. 7 (2014).
[4] J.R. Beddington, R.M. May, Harvesting natural populations in a randomly fluctuating environment, Science 197 (1977) 463–465.
[5] E. Beretta, V. Kolmanovskii, L. Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations, Math. Comput. Simulation 45 (1998) 269–277.
[6] Y. Cai, Y. Kang, Q. Wang, A stochastic epidemic model incorporating media coverage, Commun. Math. Sci. 14 (2016) 893–910.
[7] L. Cai, X. Li, M. Ghosh, B. Guo, Stability analysis of an HIV/AIDS epidemic model with treatment, J. Comput. Appl. Math. 229 (2009) 313–323.
[8] C. Chen, Y. Kang, The asymptotic behavior of a stochastic vaccination model with backward bifurcation, Appl. Math. Model. 40 (2016) 6051–6068.
[9] N. Dalal, D. Greenhalgh, X. Mao, A stochastic model for internal HIV dynamics, J. Math. Anal. Appl. 341 (2008) 1084–1101.
[10] Y.S. Ding, M. Xu, L.J. Hu, Asymptotic behavior and stability of a stochastic model for AIDS transmission, Appl. Math. Comput. 204 (2008) 99–108.
[11] Q.X. Han, D.Q. Jiang, C.Y. Ji, Analysis of a delayed stochastic predator–prey model in a polluted environment, Appl. Math. Model. 38 (2014) 3067–3080.
[12] R.Z. Hasminskii, Stochastic Stability of Differential Equations, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.
[13] Z. Huang, Q. Yang, J. Cao, Complex dynamics in a stochastic internal HIV model, Chaos Solitons Fractals 44 (2011) 954–963.
[14] C. Ji, D. Jiang, Dynamics of an HIV-1 infection model with cell-mediated immune response and stochastic perturbation, Int. J. Biomath. 5 (2012) 25.
[15] C.Y. Ji, D.Q. Jiang, Threshold behaviour of a stochastic SIR model, Appl. Math. Model. 38 (2014) 5067–5079.
[16] D. Jiang, J. Yu, C. Ji, N. Shi, Asymptotic behavior of global positive solution to a stochastic SIR model, Math. Comput. Model. 54 (2011) 221–232.
[17] N. Kaur, M. Ghosh, S.S. Bhatia, Modeling the spread of HIV in a stage structured population: Effect of awareness, Int. J. Biomath. 05 (2012) 18.
[18] W.O. Kermack, A.G. McKendrick, Contribution to the mathematical theory of epidemics, Proc. R. Soc. Lond. Ser. A. 115 (1927) 700–721.
[19] R. Khasminskii, Stochastic Stability of Differential Equations, in: Stochastic Modelling and Applied Probability, vol. 66, Springer Berlin Heidelberg, Berlin, Heidelberg, ISBN: 978-3-642-23279-4, 2012.
[20] D. Kirschner. Using mathematics to understanding HIV immune dynamics, Notices Amer. Math. Soc. 43 (1996) 191–202.
[21] X. Li, X. Mao, Population dynamical behaviour of non-autonomous Lotka–Volterra competitive system with random perturbations, Discrete Contin. Dyn. Syst. 24 (2009) 523–545.
[22] X. Lin, H.W. Hethcote, P. Van Den Driessche, An epidemiological model for HIV/AIDS with proportional recruitment, Math. Biosci. 118 (1993) 181–195.
[23] Q. Liu, Asymptotic behaviors of a cell-to-cell HIV-1 infection model perturbed by white noise, Phys. A 467 (2017) 407–418.
[24] Q. Liu, D. Jiang, T. Hayat, B. Ahmad, Asymptotic behavior of a stochastic delayed HIV-1 infection model with nonlinear incidence, Phys. A 486 (2017) 867–882.
[25] Q. Liu, D. Jiang, N. Shi, T. Hayat, A. Alsaedi, The threshold of a stochastic SIS epidemic model with imperfect vaccination, Math. Comput. Simulation 144 (2018) 78–90.
[26] H. Liu, Q.S. Yang, D.Q. Jiang, The asymptotic behavior of stochastically perturbed DI SIR epidemic models with saturated incidence, Automatica 48 (2012) 820–825.
[27] J. Lv, K. Wang, Asymptotic properties of a stochastic predator–prey system with Holling II functional response, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 4037–4048.
[28] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.
[29] Y. Pang, Y. Han, W. Li, The threshold of a stochastic SIQS epidemic model, Adv. Differential Equations 320 (2014).
[30] A.S. Perelson, Modelling Viral and Immune System Dynamics, Vol. 2, Macmillan Magazines Ltd, 2002, pp. 28–36.
[31] F. Rao, The complex dynamics of a stochastic toxic-phytoplankton zooplankton model, Adv. Differential Equations 22 (2014).
[32] F. Rao, Dynamics analysis of a stochastic SIR epidemic model, Abstr. Appl. Anal. (2014) 356013.
[33] A. Tripathi, R. Naresh, D. Sharma, Modelling the effect of screening of unaware infectives on the spread of HIV infection, Appl. Math. Comput. 184 (2007) 1053–1068.
[34] K.A. Workowski, S. Berman, Sexually transmitted diseases treatment guidelines, 2010, MMWR Recomm. Rep. 59 (RR-12) (2011) 1–110.
[35] Q. Yang, D. Jiang, N. Shi, C. Ji, The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence, J. Math. Anal. Appl. 388 (2012) 248–271.
[36] C. Zhu, G. Yin, Asymptotic properties of hybrid diffusion systems, SIAM J. Control Optim. 46 (2007) 1155–1179.