Stability of Some Positive Linear Operators on Compact Disk

M. Mursaleen, Khursheed J. Ansari and Asif Khan

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
Email: mursaleenm@gmail.com; ansari.jkhursheed@gmail.com; asifjnu07@gmail.com

Abstract. Recently, Popa and Raşa [18, 19] have shown the (in)stability of some classical operators defined on \([0, 1]\) and found best constant when the positive linear operators are stable in the sense of Hyers-Ulam. In this paper we show Hyers-Ulam (in)stability of complex Bernstein-Schurer operators, complex Kantrovich-Schurer operators and Lorentz operators on compact disk. In the case when the operator is stable in the sense of Hyers and Ulam, we find the infimum of Hyers-Ulam stability constants for respective operators.

1. Introduction

The equation of homomorphism is stable if every “approximate” solution can be approximated by a solution of this equation. The problem of stability of a functional equation was formulated by S.M. Ulam [23] in a conference at Wisconsin University, Madison in 1940: “Given a metric group \((G, \cdot, \rho)\), a number \(\varepsilon > 0\) and a mapping \(f : G \to G\) which satisfies the inequality \(\rho(f(xy), f(x)f(y)) < \varepsilon\) for all \(x, y \in G\), does there exist a homomorphism \(a\) of \(G\) and a constant \(k > 0\), depending only on \(G\), such that \(\rho(a(x), f(x)) \leq k\varepsilon\) for all \(x \in G\)?” If the answer is affirmative the equation \(a(xy) = a(x)a(y)\) of the homomorphism is called stable; see [5, 10]. The first answer to Ulam’s problem was given by D.H. Hyers [9] in 1941 for the Cauchy functional equation in Banach spaces, more precisely he proved: “Let \(X, Y\) be Banach spaces, \(\varepsilon\) a non-negative number, \(f : X \to Y\) a function satisfying \(\|f(x + y) - f(x) - f(y)\| \leq \varepsilon\) for all \(x, y \in X\), then there exists a unique additive function with the property \(\|f(x) - a(x)\| \leq \varepsilon\) for all \(x \in X\).” Due to the question of Ulam and the result of Hyers this type of stability is called today Hyers-Ulam stability of functional equations. A similar problem was formulated and solved earlier by G. Pólya and G. Szegő in [16] for functions defined on the set of positive integers. After Hyers result a large amount of literature was devoted to study Hyers-Ulam stability for various equations. A new type of stability for functional equations was introduced by T. Aoki [2] and Th.M. Rassias [20] by replacing \(\varepsilon\) in the Hyers theorem with a function depending on \(x\) and \(y\), such that the Cauchy difference can be unbounded. For other results on the Hyers-Ulam stability of functional equations one can refer to [3, 15, 21, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The Hyers-Ulam stability of linear operators was considered for the first time in the papers by Miura, Takahasi et al. (see [7, 8, 14]). Similar type of results are obtained in [22] for weighted composition operators on \(C(X)\), where \(X\) is a compact Hausdorff space. A result on the stability of a linear composition operator of the second order was given by J. Brzdek and S.M. Jung in [4].

Recently, Popa and Raşa obtained [17] a result on Hyers-Ulam stability of the Bernstein-Schnabl operators using a new approach to the Fréchet functional equation, and in [18, 19], they have shown the (in)stability of some classical operators defined on \([0, 1]\) and found the best constant for the positive linear operators in the sense of Hyers-Ulam.

Motivated by their work, in this paper, we show the (in)stability of some complex positive linear operators on compact disk in the sense of Hyers-Ulam. We find the infimum of the Hyers-Ulam stability constants for complex Bernstein-Schurer operators and complex Kantrovich-Schurer operators on compact disk. Further we show that Lorentz polynomials are unstable in the sense of Hyers-Ulam on a compact disk.
2. The Hyers-Ulam stability property of operators

In this section, we recall some basic definitions and results on Hyers-Ulam stability property which form the background of our main results.

Definition 2.1. Let $A$ and $B$ be normed spaces and $T$ a mapping from $A$ into $B$. We say that $T$ has the Hyers-Ulam stability property (briefly, $T$ is HU-stable) \[22\] if there exists a constant $K$ such that:

(i) for any $g \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\|Tf - g\| \leq \varepsilon$, there exists an $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$. The number $K$ is called a HUS constant of $T$, and the infimum of all HUS constants of $T$ is denoted by $K_T$. Generally, $K_T$ is not a HUS constant of $T$; see \[7, 8\].

Let now $T$ be a bounded linear operator with the kernel denoted by $N(T)$ and the range denoted by $R(T)$. Consider the one-to-one operator $\tilde{T}$ from the quotient space $A/N(T)$ into $B$:

$$
\tilde{T}(f + N(T)) = Tf, \quad f \in A,
$$

and the inverse operator $\tilde{T}^{-1} : R(T) \to A/N(T)$.

Theorem 2.2. (\[22\]). Let $A$ and $B$ be Banach spaces and $T : A \to B$ be a bounded linear operator. Then the following statements are equivalent:

(a) $T$ is HU-stable;
(b) $R(T)$ is closed;
(c) $\tilde{T}^{-1}$ is bounded.

Moreover, if one of the conditions (a), (b), (c) is satisfied, then $K_T = \|\tilde{T}^{-1}\|$.

Remark 2.3. (1) Condition (i) expresses the Hyers-Ulam stability of the equation $Tf = g$, where $g \in R(T)$ and $f \in A$ is unknown.

(2) If $T : A \to B$ is a bounded linear operator, then (i) is equivalent to:

(ii) for any $f \in A$ with $\|Tf\| \leq 1$ there exists an $f_0 \in N(T)$ such that $\|f - f_0\| \leq K$, (see \[13\]).

So, in what follows, we shall study the HU-stability of a bounded linear operator $T : A \to B$ by checking the existence of a constant $K$ for which (ii) is satisfied, or equivalently, by checking the boundedness of $\tilde{T}^{-1}$.

The main results used in our approach for obtaining, in some concrete cases, the explicit value of $K_T$ are the formula given above and a result by Lubinsky and Ziegler \[12\] concerning coefficient bounds in the Lorentz representation of a polynomial. Let $p \in \Pi_n$, where $\Pi_n$ is the set of all polynomials of degree at most $n$ with real coefficients. Then $p$ has a unique Lorentz representation of the form

$$
p(x) = \sum_{k=0}^{n} c_k x^k (1 - x)^{n-k}, \quad (2.1)
$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$. Remark that, in fact, it is a representation in Bernstein-Bézier basis. Let $T_n$ denote the usual $n$th degree Chebyshev polynomial of the first kind. Then the following representation holds (see \[12\]):

$$
T_n(2x - 1) = \sum_{k=0}^{n} d_{n,k} x^k (1 - x)^{n-k} (-1)^{n-k}, \quad (2.2)
$$
where
\[ d_{n,k} := \sum_{j=0}^{\min\{k,n-k\}} \binom{n}{2j} \binom{n-2j}{k-j} 4^j, \quad k = 0, 1, \ldots, n. \]

It is proved in [18] that \( d_{n,k} = \binom{2n}{2k} \), \( k = 0, 1, \ldots, n \). Therefore
\[ T_n(2x - 1) = \sum_{k=0}^{n} \binom{2n}{2k} (-1)^{n-k} x^k (1 - x)^{n-k}. \]

**Theorem 2.4.** (Lubinsky and Ziegler [12]). Let \( p \) have the representation (2.1), and let \( 0 \leq k \leq n \). Then
\[ |c_k| \leq d_{n,k} \|p\|_{\infty} \]
with equality if and only if \( p \) is a constant multiple of \( T_n(2x - 1) \) where \( \|p\|_{\infty} = \max_{x \in [a,b]} |P(x)| \).

Let \( C[0, 1] \) be the space of all continuous, real-valued functions defined on \([0, 1]\), and \( C_b[0, +\infty) \) the space of all continuous, bounded, real-valued functions on \([0, +\infty)\). Endowed with the supremum norm, they are Banach spaces.

Popa and Raşa have shown the Hyers-Ulam stability of the following operators:

**(i) Bernstein operators** ([18])

For each integer \( n \geq 1 \), the sequence of classical Bernstein operators \( B_n : C[0, 1] \to C[0, 1] \) is defined by (see [1])
\[ B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0, 1], \ n \geq 1. \]

It is stable in the Hyers-Ulam sense and the best Hyers-Ulam stability constant is given by
\[ K_{B_n} = \frac{2n}{\binom{n}{\lfloor n/2 \rfloor}}, \ n \in \mathbb{N}. \]

**(ii) Szász-Mirakjan operators** ([18])

The \( n \)th Szász-Mirakjan operator \( L_n : C_b[0, +\infty) \to C_b[0, +\infty) \) defined by (see [1], pp. 338)
\[ L_n f(x) = e^{-nx} \sum_{j=0}^{\infty} f\left(\frac{j}{n}\right) \frac{n^j}{j!} x^j, \quad x \in [0, +\infty) \]

is not stable in the sense of Hyers and Ulam for each \( n \geq 1 \).

**(iii) Beta operators** ([18])

For each \( n \geq 1 \), the Beta operator \( B_n : C[0, 1] \to C[0, 1] \) defined by [13]
\[ L_n f(x) := \frac{\int_0^1 t^{nx}(1 - t)^{n(1-x)} f(t) \, dt}{\int_0^1 t^{nx}(1 - t)^{n(1-x)} \, dt} \]
is not stable in the sense of Hyers and Ulam.

(iv) Stancu operators \([19]\)

Let \(C[0, 1]\) be the linear space of all continuous functions \(f : [0, 1] \to \mathbb{R}\), endowed with the supremum norm denoted by \(\|\|\|\), and \(a, b\) real numbers, \(0 \leq a \leq b\). The Stancu operator \([21]\) \(S_n : C[0, 1] \to \Pi_n\) is defined by

\[
S_n f(x) = \sum_{k=0}^{n} f\left(\frac{k + a}{n + b}\right) \binom{n}{k} x^k (1 - x)^{n-k},
\]

\(f \in C[0, 1]\). It is HU-stable and the infimum of the Hyers-Ulam constant is

\[
K_{S_n} = \frac{2n}{2\left[\frac{n}{2}\right]}/\left(\frac{n}{2}\right),
\]

for each \(n \geq 1\).

(v) Kantorovich operators \([19]\)

Let \(X = \{f; \ f : [0, 1] \to \mathbb{R}, \ \text{where} \ f \ \text{is bounded and Riemann integrable}\}\) be endowed with the supremum norm denoted by \(\|\|\|\). The Kantorovich operator defined by

\[
K_n f(x) = (n + 1) \sum_{k=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt \right) \binom{n}{k} x^k (1 - x)^{n-k},
\]

\(f \in X, x \in [0, 1]\) is stable in Hyers-Ulam sense and the best HUS constant is

\[
K_{S_n} = \frac{2n}{2\left[\frac{n}{2}\right]}/\left(\frac{n}{2}\right).
\]

3. Main Results

In this section, we show the Hyers-Ulam stability of some other operators. Let \(D_R\) denote the compact disk having radius \(R\), i.e., \(D_R = \{z \in \mathbb{C} : |z| \leq R\}\).

(i) Bernstein-Schurer Operators

Let \(X_{D_R} = \{f : D_R \to \mathbb{C} \ \text{be analytic in} \ D_R\}\) be the collection of all analytic functions endowed with the supremum norm denoted by \(\|\|\|\). The complex Bernstein-Schurer operator \([3]\) \(S_{n,p} : X_{D_R} \to \Pi_{n+p}\) is defined by

\[
S_{n,p}(f)(z) = \sum_{k=0}^{n+p} \binom{n+p}{k} z^k (1 - z)^{n+p-k} f\left(\frac{k}{n}\right), \ z \in \mathbb{C}, \ f \in X_{D_R}.
\]

We have \(N(S_{n,p}) = \{f \in X_{D_R} : f\left(\frac{k}{n}\right) = 0, \ 0 \leq k \leq n+p\}\), which is a closed subspace of \(X_{D_R}\), and \(R(S_{n,p}) = \Pi_{n+p}\). The operator \(\tilde{S}_{n,p} : X_{D_R}/N(S_{n,p}) \to \Pi_{n+p}\) is bijective, \(\tilde{S}_{n,p}^{-1} : \Pi_{n+p} \to X_{D_R}/N(S_{n,p})\) is bounded since \(\dim\Pi_{n+p} = 2(n+p+1)\). So according to Theorem 2.2 the operator \(S_{n,p}\) is Hyers-Ulam stable.

**Theorem 3.1.** For \(n \geq 1\)

\[
K_{S_{n,p}} = \|\tilde{S}_{n,p}^{-1}\| = \frac{2(n+p)}{2\left[\frac{n+p}{2}\right]}\left(\frac{n+p}{\left[\frac{n+p}{2}\right]}\right).
\]
Proof. Let \( q \in \Pi_{n+p}, \|q\| \leq 1 \), and its Lorentz representation

\[
q(z) = \sum_{k=0}^{n+p} c_k(q) z^k (1-z)^{n+p-k}, \quad |z| \leq R.
\]

Consider the constant function \( f_q \in X_{D_R} \) defined by

\[
f_q \left( \frac{k}{n} \right) = \frac{c_k(q)}{\binom{n+p}{k}}, \quad 0 \leq k \leq n + p.
\]

Then \( S_{n,p}f_q = q \) and \( \tilde{S}_{n,p}^{-1}(q) = f_q + N(S_{n,p}) \).

As usual, the norm of \( \tilde{S}_{n,p}^{-1} : \Pi_{n+p} \to X_{D_R}/N(S_{n,p}) \) is defined by

\[
\|\tilde{S}_{n,p}^{-1}\| = \sup_{\|q\| \leq 1} \|\tilde{S}_{n,p}^{-1}(q)\| = \sup_{\|q\| \leq 1} \inf_{h \in N(S_{n,p})} \|f_q + h\|.
\]

Clearly

\[
\inf_{h \in N(S_{n,p})} \|f_q + h\| = \|f_q\| = \max_{0 \leq k \leq n+p} |c_k(q)| \binom{n+p}{k}.
\]

Therefore

\[
\|\tilde{S}_{n,p}^{-1}\| = \sup_{\|q\| \leq 1} \max_{0 \leq k \leq n+p} |c_k(q)| \binom{n+p}{k} \leq \sup_{\|q\| \leq 1} \max_{0 \leq k \leq n+p} d_{n+p,k} \|q\| \binom{n+p}{k} = \max_{0 \leq k \leq n+p} d_{n+p,k} \binom{n+p}{k}.
\]

On the other hand, let \( r(z) = T_n(2z-1), \ |z| \leq R \). Then \( \|r\| = 1 \) and \( |c_k(r)| = d_{n+p,k}, \ 0 \leq k \leq n+p, \) according to Theorem 2.4. Consequently

\[
\|\tilde{S}_{n,p}^{-1}\| \geq \max_{0 \leq k \leq n+p} |c_k(r)| \binom{n+p}{k} = \max_{0 \leq k \leq n+p} d_{n+p,k} \binom{n+p}{k}
\]

and so

\[
\|\tilde{S}_{n,p}^{-1}\| = \max_{0 \leq k \leq n+p} d_{n+p,k} \binom{n+p}{k} = \max_{0 \leq k \leq n+p} \frac{2(n+p)}{2k} \binom{n+p}{k}.
\]

Let

\[
a_k = \frac{2(n+p)}{2k} \binom{n+p}{k}, \quad 0 \leq k \leq n+p.
\]

Then

\[
\frac{a_{k+1}}{a_k} = \frac{2n + 2p - 2k - 1}{2k + 1}, \quad 0 \leq k \leq n+p.
\]

The inequality \( \frac{a_{k+1}}{a_k} \geq 1 \) is satisfied if and only if \( k \leq \lceil \frac{n+p-1}{2} \rceil \), therefore

\[
\max_{0 \leq k \leq n+p} a_k = a_{\lceil \frac{n+p-1}{2} \rceil + 1} = \begin{cases} a_{\lceil \frac{n+p}{2} \rceil}, & \text{n+p is even;} \\ a_{\lfloor \frac{n+p}{2} \rfloor + 1}, & \text{n+p is odd.} \end{cases}
\]

Since \( a_{\lceil \frac{n+p}{2} \rceil + 1} \) if \( n+p \) is an odd number, we conclude that

\[
K_{S_{n,p}} = \|\tilde{S}_{n,p}^{-1}\| = \left( \frac{2(n+p)}{2\lceil \frac{n+p}{2} \rceil} \right) \left( \binom{n+p}{\lceil \frac{n+p}{2} \rceil} \right).
\]
This completes the proof of the theorem.

(ii) Kantrovich-Schurer Operators

Let $X_{D_R} = \{ f : D_R \to \mathbb{C} \text{ analytic in } D_R \}$ be the collection of all analytic functions endowed with the supremum norm denoted by $\| . \|$. The complex Kantrovich-Schurer operator (3) $L_{n,p} : X_{D_R} \to \Pi_{n+p}$ is defined by

$$L_{n,p}(f)(z) = (n + p + 1) \sum_{k=0}^{n+p} \binom{n+p}{k} z^k (1-z)^{n+p-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt, \quad z \in \mathbb{C}, \quad f \in X_{D_R}. $$

We have

$$N(L_{n,p}) = \{ f \in X_{D_R} : f(t) = 0, \quad t \in D_R \}. $$

The operators $L_{n,p}$ are Hyers-Ulam stable since their ranges are finite dimensional spaces.

**Theorem 3.2.** For $n \geq 1$

$$K_{L_{n,p}} = \| \tilde{T}_{n,p}^{-1} \| = \frac{(n + 1) \binom{2(n+p)}{n+p+1}}{(n + p + 1) \binom{n+p}{k}}. $$

**Proof.** Let $q \in \Pi_{n+p}, \| q \| \leq 1$, and its Lorentz representation

$$q(z) = \sum_{k=0}^{n+p} c_k(q) z^k (1-z)^{n+p-k}, \quad |z| \leq R. $$

Consider the constant function $f_q \in X_{D_R}$ defined by

$$f_q(t) = \frac{(n+1)c_k(q)}{(n+p+1)\binom{n+p}{k}}, \quad 0 \leq k \leq n+p, \quad t \in D_R. $$

Then $L_{n,p}f_q = q$ and $\tilde{L}_{n,p}^{-1}(q) = f_q + N(L_{n,p})$.

As usual, the norm of $\tilde{L}_{n,p}^{-1} : \Pi_{n+p} \to X_{D_R}/N(L_{n,p})$ is defined by

$$\| \tilde{L}_{n,p}^{-1} \| = \sup_{\| q \| \leq 1} \| \tilde{L}_{n,p}^{-1}(q) \| = \sup_{\| q \| \leq 1} \inf_{h \in N(L_{n,p})} \| f_q + h \|. $$

Clearly

$$\inf_{h \in N(L_{n,p})} \| f_q + h \| = \| f_q \| = \max_{0 \leq k \leq n+p} \frac{(n+1)|c_k(q)|}{(n+p+1)\binom{n+p}{k}}. $$

Therefore

$$\| \tilde{L}_{n,p}^{-1} \| = \sup_{\| q \| \leq 1} \max_{0 \leq k \leq n+p} \frac{(n+1)|c_k(q)|}{(n+p+1)\binom{n+p}{k}} \leq \sup_{\| q \| \leq 1} \max_{0 \leq k \leq n+p} \frac{(n+1)\| q \| d_{n+p,k}}{(n+p+1)\binom{n+p}{k}} = \max_{0 \leq k \leq n+p} \frac{(n+1)d_{n+p,k}}{(n+p+1)\binom{n+p}{k}}. $$
On the other hand, let \( r(z) = T_n(2z-1), \ |z| \leq R. \) Then \( \|r\| = 1 \) and \( |c_k(r)| = d_{n+p,k}, \ 0 \leq k \leq n+p, \) according to Theorem 2.4. Consequently

\[
\|\tilde{L}^{-1}_{n,p}\| \geq \max_{0 \leq k \leq n+p} \frac{(n+1)|c_k(r)|}{(n+p+1)\binom{n+p}{k}} = \max_{0 \leq k \leq n+p} \frac{(n+1)d_{n+p,k}}{(n+p+1)\binom{n+p}{k}}
\]

and so

\[
\|\tilde{L}^{-1}_{n,p}\| = \max_{0 \leq k \leq n+p} \frac{(n+1)d_{n+p,k}}{(n+p+1)\binom{n+p}{k}} = \max_{0 \leq k \leq n+p} \frac{(n+1)\binom{2(n+p)}{2k}}{(n+p+1)\binom{n+p}{k}}.
\]

Let

\[
a_k = \frac{(n+1)\binom{2(n+p)}{2k}}{(n+p+1)\binom{n+p}{k}}, \ 0 \leq k \leq n+p.
\]

Then

\[
\frac{a_{k+1}}{a_k} = \frac{2n + 2p - 2k - 1}{2k + 1}, \ 0 \leq k \leq n+p.
\]

The inequality \( \frac{a_{k+1}}{a_k} \geq 1 \) is satisfied if and only if \( k \leq \frac{n+p-1}{2} \), therefore

\[
\max_{0 \leq k \leq n+p} a_k = a_{\lfloor \frac{n+p+1}{2} \rfloor + 1} = \begin{cases} a_{\lfloor \frac{n+p}{2} \rfloor}, & \text{n+p is even;} \\ a_{\lfloor \frac{n+p+1}{2} \rfloor + 1}, & \text{n+p is odd.} \end{cases}
\]

Since \( a_{\lfloor \frac{n+p}{2} \rfloor + 1} = a_{\lfloor \frac{n+p}{2} \rfloor} \) if \( n+p \) is an odd number, we conclude that

\[
K_{L_{n,p}} = \|\tilde{L}^{-1}_{n,p}\| = \frac{(n+1)\binom{2(n+p)}{2\lfloor \frac{n+p}{2} \rfloor}}{(n+p+1)\binom{n+p}{\lfloor \frac{n+p}{2} \rfloor}}.
\]

This completes the proof of the theorem.
(iii) Lorentz Operators

The complex Lorentz polynomial \( L_n(f)(z) \) attached to any analytic function \( f \) in a domain containing the origin is given by

\[
L_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{z}{n} \right)^k f^{(k)}(0), \quad n \in \mathbb{N}.
\]

For \( R > 1 \) and denoting \( D_R = \{ z \in \mathbb{C}; |z| < R \} \), suppose that \( f : D_R \to \mathbb{C} \) is analytic in \( D_R \), i.e.,

\[
f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \forall z \in D_R.
\]

**Theorem 3.3.** For each \( n \geq 1 \), the Lorentz polynomial on compact disk is Hyers-Ulam unstable.

**Proof.** Let us denote \( e_j(z) = z^j \), then from Lorentz operators we can easily obtain that \( L_n(e_0)(z) = 1, L_n(e_1)(z) = e_1(z) \) and that for all \( j, n \in \mathbb{N}, \ j \geq 2 \), we have

\[
L_n(e_j)(z) = \binom{n}{j} j! \frac{z^j}{n^j}, \quad 1 \leq R_1 < R
\]

\[
= z^j \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{j-1}{n} \right).
\]

Also, since an easy computation shows that

\[
L_n(f)(z) = \sum_{j=0}^{\infty} c_j L_n(e_j)(z), \quad \forall |z| \leq R_1,
\]

and \( L_n(e_0)(z) = 1, L_n(e_1)(z) = e_1(z) \). It follows that for each \( j \geq 2, (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{j-1}{n}) \) is an eigen value of \( L_n \). It can be easily seen that \( L_n \) is injective. Therefore \( 1/(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{j-1}{n}) \) is an eigen value of \( L_n^{-1} \). Since

\[
\lim_{j \to \infty} \frac{1}{(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{j-1}{n})} = \lim_{j \to \infty} \frac{n^j}{(n-1)(n-2) \cdots (n-j+1)} = +\infty,
\]

we conclude that \( L_n^{-1} \) is unbounded and so \( L_n \) is HU-unstable.

This completes the proof of the theorem.
References

[1] F. Altomare and M. Campiti, Korovkin-Type Approximation Theory and its Applications, W. de Gruyter, Berlin, New York, 1994.

[2] T. Aoki, On the stability of linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64-66.

[3] G.A. Anastassiou and S.G. Gal, Approximation by complex Bernstein-Schurer and Kantorovich-Schurer polynomials in compact disks, Computers and Mathematics with Applications 58 (2009) 734-743.

[4] J. Brzdek and S.M. Jung, A note on stability of an operator linear equation of the second order, Abstr. Appl. Anal. (2011) 15. Article ID602713.

[5] J. Brzdek and Th.M. Rassias, Functional Equations in Mathematical Analysis, Springer, 2011.

[6] S.G. Gal, Approximation by complex Lorentz polynomials, Math. Commun., 16 (2011), 67-75.

[7] O. Hatori, K. Kobayasi, T. Miura, H. Takagi and S.E. Takahasi, On the best constant of Hyers-Ulam stability, J. Nonlinear Convex Anal. 5 (2004) 387-393.

[8] G. Hirasawa and T. Miura, Hyers-Ulam stability of a closed operator in a Hilbert space, Bull. Korean Math. Soc. 43 (2006) 107-117.

[9] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222-224.

[10] D.H. Hyers, G. Isac and Th.M. Rassias, Stability of Functional Equation in Several Variables, Birkhäuser, Basel, 1998.

[11] G.G. Lorentz, Bernstein polynomials, 2nd edition, Chelsea Publ., New York, 1986.

[12] D.S. Lubinsky and Z. Ziegler, Coefficients bounds in the Lorentz representation of a polynomial, Canad. Math. Bull. 33 (1990) 197-206.

[13] A. Lupaş, Die Folge der Betaoperatoren, Dissertation, Univ. Stuttgart, 1972.

[14] T. Miura, M. Miyajima and S.E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, Math. Nachr. 258 (2003) 90-96.

[15] M. Mursaleen and K.J. Ansari, Stability results in intuitionistic fuzzy normed spaces for a cubic functional equation. Appl. Math. Inform. Sci. 7(5), (2013) 1685-1692.

[16] G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, I, Springer, Berlin, 1925.

[17] D. Popa and I. Raşa, The Fréchet functional equation with applications to the stability of certain operators, J. Approx. Theory 1 (2012) 138-144.

[18] D. Popa and I. Raşa, On the stability of some classical operators from approximation theory, Expo. Math. 31(2013) 205-214.

[19] D. Popa and I. Raşa, On the best constant in Hyers-Ulam stability of some positive linear operators, Jour. Math. Anal. Appl. 412(2014) 103-108.
[20] Th.M. Rassias, On the stability of the linear mappings in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297-300.

[21] D.D. Stancu, Asupra unei generalizări a polinoamelor lui Bernstein, Stud. Univ. Babeş-Bolyai 14 (1969) 31-45.

[22] H. Takagi, T. Miura and S.E. Takahasi, Essential norms and stability constants of weighted composition operators on C(X), Bull. Korean Math. Soc. 40 (2003) 583-591.

[23] S.M. Ulam, A collection of Mathematical problems, Interscience, New York, 1960.

[24] Z. Gajda, On stability of additive mappings, Int. J. Math. Math. Sci. 14 (1991), 431-434.

[25] J. Brzdęk, Hyperstability of the Cauchy equation on restricted domains, Acta Math. Hungar. 141 (2013), 58-67.

[26] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, On some recent developments in Ulam’s type stability, Abstr. Appl. Anal. 2012 (2012), Article ID 716936, 41 pp.

[27] I.A. Rus, Remarks on Ulam stability of the operatorial equations, Fixed Point Theory 10 (2009), 305-320.

[28] C. Urs, Ulam-Hyers stability for coupled fixed points of contractive type operators, J. Nonlinear Sci. Appl. 6 (2013), no. 2, 124-136.

[29] W. Sintunavarat, Generalized Hyers-Ulam stability, well-posedness, and limit shadowing of fixed point problems for α-β-contraction mapping in metric spaces. The Scientific World Journal 2014, Article ID 569174, 7 pp.

[30] I.A. Rus, Ulam stability of operatorial equations, Functional Equations in Mathematical Analysis, 287-305, Springer, New York, 2012.

[31] M. Bota, T.p. Petru, G. Petruşel, Hyers-Ulam stability and applications in gauge spaces. Miskolc Math. Notes 14 (2013), no. 1, 41-47.

[32] M. Bota, E. Karapinar, O. Mleşniţe, Ulam-Hyers stability results for fixed point problems via α-χ-contractive mapping in (b)-metric space. Abstr. Appl. Anal. 2013, Art. ID 825293, 6 pp.

[33] A. Petruşel, G. Petruşel, C. Urs, Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators, Fixed Point Theory Appl. 2013, 2013:218, 21 pp.

[34] J. Brzdęk, L. Cădariu, K. Ciepliński, Fixed Point Theory and the Ulam stability, J. Function Spaces 2014 (2014), Article ID 829419, 16 pp.

[35] S. A. Mohiuddine, M. Mursaleen, Khursheed J. Ansari, On the Stability of Fuzzy Set-Valued Functional Equations, The Scientific World Journal, Volume 2014, Article ID 392943, 12pages.