How to quantise probabilities while preserving their convex order

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Abstract
We introduce an algorithm which, given probabilities $\mu \leq_{cx} \nu$ in convex order and defined on a separable Banach space $B$, constructs finitely-supported approximations $\mu_n \to \mu, \nu_n \to \nu$ which are in convex order $\mu_n \leq_{cx} \nu_n$. We provide upper-bounds for the speed of convergence, in terms of the Wasserstein distance. We discuss the (dis)advantages of our algorithm and its link with the discretisation of the Martingale Optimal Transport problem, and we illustrate its implementation with numerical examples. We study the operation which, given $\mu/\nu$ and some (finite) partition of $B$, outputs $\mu_n/\nu_n$, showing that applied to a probability $\gamma$ and to all partitions it outputs the set of all probabilities $\zeta \leq_{cx} \gamma$.

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1. Introduction

The main contribution of this paper is the introduction of an algorithm which, given probabilities $\mu \leq_{cx} \nu$ in convex order and with finite first moment, defined on a separable Banach space $B$, constructs finitely-supported approximations $\mu_n, \nu_n$ which are in convex order $\mu_n \leq_{cx} \nu_n$ and which converge to $\mu, \nu$ in Wasserstein distance.

Obtaining a quantisation algorithm which preserves the convex order is important because of the link with the MOT (Martingale Optimal Transport) problem, a version of the OT problem in which one minimises not over the set $\Pi(\mu, \nu)$ of all transports $\pi$. 

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from $\mu$ to $\nu$, but rather over the subset $\mathcal{M}(\mu, \nu)$ of those transports which additionally satisfy the martingale constraint, i.e. which are the joint laws of $(X, Y)$ such that

$$X \sim \mu, \quad Y \sim \nu, \quad \mathbb{E}[Y|X] = X.$$ 

The link is provided by Strassen’s theorem, which states that $\mathcal{M}(\mu, \nu)$ is non-empty if and only if $\mu \leq_{cx} \nu$. It is important to quantise the probabilities since, in the case of finitely-supported measures, the MOT problem is a linear program, and is thus easily solved numerically.

This problem has already been considered by several authors, all of whom consider probabilities in the space $\mathcal{P}_p(B)$ of measures on $B$ with finite $p^{\text{th}}$ moment (where $p \in [1, \infty)$), endowed with the $p$-Wasserstein distance $W_p$, in the setting of finite-dimensional $B$.

The first result, due to Baker [6], considers the case where the measures are defined on the real line $B = \mathbb{R}$. In this case, Baker provides an algorithm which, given $\mu, \nu \in \mathcal{P}_1$ and $n \in \mathbb{N}$, returns a measure $\mu_n \Rightarrow U_n(\mu)$ supported on at most $n$ points and such that $U_n(\mu) \Rightarrow \mu$ in $\mathcal{P}_1$, and $\mu \leq_{cx} \nu$ implies $U_n(\mu) \leq_{cx} U_n(\nu)$. While Baker’s $U$-quantisation $U_n(\mu)$ provides a great solution to the problem at hand, it is only defined in dimension one, and it turns out that finding a quantisation which preserves the convex order, just like most questions about MOT, is a lot more challenging in dimension higher than 1; indeed, as remarked by [10] ‘it is a highly intriguing challenge to extend the martingale transport theory to the case where $\mu, \nu$ are supported on $\mathbb{R}^d, d > 1$’.

The dual quantisation proposed in [34], though defined for general $B = \mathbb{R}^N$, preserves the convex order only in dimension $N = 1$ [3]. A variant is considered in [23], who applied the optimal quadratic quantisation to the first marginal $\mu$, and the dual quantisation to the second marginal $\nu$. This scheme does preserve the convex order in any dimension; however, both of the algorithms in [34, 23] are only defined for probabilities $\mu, \nu$ with compact support, and do not generalise to the case of several (not just two) marginals.

Another two discretisation techniques have been considered in [2, 3]. Given $\mu \leq_{cx} \nu$ and arbitrary finitely-supported probabilities $(\mu_n, \nu_n)$ converging to $(\mu, \nu)$ in $\mathcal{P}_p$, the metric projection $\alpha_n$ of $\mu_n$ on $\{\alpha : \alpha \leq_{cx} \nu_n\}$ exists and it satisfies $\alpha_n \rightarrow \mu$ in $\mathcal{P}_p$ (and, trivially, $\alpha_n \leq_{cx} \nu_n$ and $\alpha_n$ is finitely-supported). Analogously, the metric projection $\beta_n$ of $\nu_n$ on $\{\beta : \mu_n \leq_{cx} \beta\}$ exists and satisfies $\beta_n \rightarrow \nu$ in $\mathcal{P}_p$ (and, trivially, $\mu_n \leq_{cx} \beta_n$ and $\beta_n$ is finitely-supported). While this method is conceptually very neat, it does have the disadvantage that $\alpha_n, \beta_n$ cannot be explicitly computed, though this issue admits the following partial workaround. If $\mu_n$ is the empirical measure $\frac{1}{n} \sum_{i=1}^n \delta_{X_i}$, corresponding to IID random variables $X_i \sim \mu$ and analogously for $\nu_n$, then the corresponding random measure $\alpha_n$ solves a quadratic optimisation problem with linear constraints, and so it can be computed numerically (not so for $\beta_n$) and $(\alpha_n, \nu_n) \rightarrow (\mu, \nu)$ a.s.

An entirely different approach is investigated by [21]. Their numerical scheme relies on an arbitrary discretisation of the measures $\mu, \nu$, along with an appropriate relaxation of the martingale constraint, which only requires that

$$\mathbb{E} \left[ \left| \mathbb{E}[Y|X] - X \right| \right] \leq \epsilon, \quad (1)$$
holds for a $\epsilon \geq 0$, where $X, Y$ have laws $\mu, \nu$; the case $\epsilon = 0$ corresponds to the martingale constraint.

The algorithm which we propose constructs $\mu_n$ in a way which generalises Baker’s $\mathcal{U}$-quantisation (though this link is not obvious), and it is what we call a proper barycentric quantisation (see Definition 29); it turns out that this more general construction, unlike the proper quantisation, allows to obtain all (and only) the finitely-supported probabilities $\nu_n \leq_{cx} \nu$. To build a barycentric quantisation $\nu_n$ of $\nu$ requires the knowledge not just of $\nu$, but rather of a martingale transport $\pi$ from $\mu$ to $\nu$, and the choice of two partitions $\Pi^{\mu}_1 = (P^{\mu}_1, \ldots, P^{\mu}_n)$, $\Pi^{\nu}_2 = (P^{\nu}_2, \ldots, P^{\nu}_n)$ of $B$; if $\Pi^{\nu}_1$ is chosen to be equal to the partition $\Pi^{\mu}$ used to build $\mu_n$, and $\mu \leq_{cx} \nu$, then $\nu_n \leq_{cx} \nu$.

If $(\Pi^{\mu}_1)_n, (\Pi^{\nu}_2)_n$ are refining then $(\mu_n)_n$ and $(\nu_n)_n$ are increasing in convex order, and if moreover $\nu_n \sigma(\Pi^{\mu}_1)$ and $\nu_n \sigma(\Pi^{\nu}_2)$ equal the Borel $\sigma$-algebra $\mathcal{B}(B)$ of $B$ then the corresponding $(\mu_n)_n, (\nu_n)_n$ converge to $\mu, \nu$. We also prove the upper-bounds

$$W_2(\mu_n, \mu) \leq \sum_i (\text{diam } (P^{\mu}_1))^p \mu (\mathcal{P}^{\mu}_1), \quad W_2(\nu_n, \nu) \leq \sum_j (\text{diam } (P^{\nu}_2))^p \nu (\mathcal{P}^{\nu}_2).$$

Needing a $\pi \in \mathcal{M}(\mu, \nu)$ is a con of our approach, because only $\mu, \nu$ (not $\pi$) are inputs of the MOT problem and while the existence of such a $\pi$ is guaranteed by Strassen’s Theorem, algorithms which produce a $\pi \in \mathcal{M}(\mu, \nu)$ given arbitrary $\mu \leq_{cx} \nu$ are so far only known in dimension one [11, 24]. We also mention the constructions reported in [22, Section 4.2] e.g. Bass’s construction [7] related to the Skorokhod embedding problem and the local variance gamma model of [17] motivated by the application of MOT to obtain an arbitrage-free measure consistent with given market option prices. We leave to separate papers the difficult task of developing algorithms which work in the multi-dimensional setting.

On the other hand, our approach has many pros: it leads to explicit expressions for $\mu_n, \nu_n$ which can easily be calculated numerically by evaluating integrals, it outputs non-random $\mu_n \leq_{cx} \nu_n$, it works when $B$ is an arbitrary separable Banach space and the proofs turn out to be easy, as soon as one has the right idea. The idea is to consider an analogous statement for random variables, just like Skorokhod’s representation Theorem allows to draw a parallel between weak convergence of measures and convergence in probability of random variables. This is made possible by the equivalent characterisation of the convex order provided by Strassen’s Theorem.

As we mentioned, finding a quantisation which preserves the convex order is of interest when one wants to numerically solve the MOT problem; however, this is only one piece
of the puzzle, since to carry out this solution it is necessary to identify conditions under which \( \text{MOT}_n \to \text{MOT} \), i.e., the MOT problem with marginals \((\mu_n, \nu_n)\) converges to the MOT problem with marginals \((\mu, \nu)\) (in an appropriate sense) when
\[
\mu_n \to \mu, \quad \nu_n \to \nu, \quad \mu_n \leq_{\text{cx}} \nu_n, \quad \mu \leq_{\text{cx}} \nu.
\]
(2)

Solutions to this problem have been provided from different perspectives in \([5, 24, 41]\), who essentially showed that the MOT problem is stable, i.e., no conditions other than (2) are required to obtain the convergence of the MOT problems. However, all these papers consider only the one-dimensional setting and, as usual, the situation in the multi-dimensional setting is a lot more complicated; in particular, the MOT problem lacks stability in this setting \([16]\). While we leave for a separate paper the work of finding conditions under which \( \text{MOT}_n \to \text{MOT} \), in this paper we do provide a statement in this direction, which in particular shows that if the transport \( \pi \) which one uses to build \((\nu_n)_n\) already happens to be optimal for the MOT problem (with inputs \((\mu, \nu)\)), then \( \text{MOT}_n \to \text{MOT} \). Of course, this is a rather weak and not particularly useful statement, because the whole point of using \( \text{MOT}_n \) to approximate MOT is that one does not know a priori the optimiser of MOT. We also carry out numerical examples in which we illustrate the implementation of the proposed construction.

In the last section, we define in somewhat more abstract terms what is a barycentric quantisation and we study the properties of such construction. In particular, we show that the (proper) barycentric quantisations, which we define separately for measures and for random variables, are closely connected and that one can characterise which measures \( \zeta \) such that \( \zeta \leq_{\text{cx}} \gamma \) are finitely-supported (resp. supported on at most \( n \) points) using barycentric quantisations of \( \gamma \), but not using only proper barycentric quantisations.

Finally, we recall that it is is natural to consider a more general OT problem, in which there are multiple marginals, rather than just two as described above. In this paper we concentrate on the case of two marginals, because the extension of our results to the case of finitely many marginals is trivial, though it complicates the notation.

The rest of this article is structured as follows. In Section 2, we briefly introduce the OT problem and its several variants with particular focus on the MOT problem. In Section 3, we discuss the discretisation technique for an OT problem with additional linear (or convex) constraints and state several questions which we will then address in this paper in the setting of MOT. The core of our paper is Section 4, in which we introduce our quantisation algorithm which preserves the convex order, providing both an explicit representation, bounds on the speed of convergence and a remark about stability. In Section 5, we provide several examples in which we show the implementation of the proposed martingale quantisation and the resolution of the corresponding discretised MOT problem. Finally, in Section 6 we introduce the notion of barycentric quantisation and study its properties.
2. Optimal Transport problems

In this section, we recall the OT problem, briefly mention several variants thereof (among which the MOT) and introduce the notation and setting used in this paper.

2.1. The classic Optimal Transport problem

Consider probability measures $\mu, \nu$ on Polish spaces $X, Y$ endowed with their Borel $\sigma$-algebras, and a non-negative measurable cost function $c : X \times Y \to [0, \infty]$. In the 1781 article \cite{31}, Monge considered (a special case of) the OT problem

$$\inf_{T \# \mu = \nu} \int_X c(x, T(x))d\mu(x), \quad (3)$$

where the infimum runs over all measurable function $T : X \to Y$, called transport maps, such that $T \# \mu := \mu \circ T^{-1}$ equals $\nu$. This constraint ensures that a feasible transport map is one that pushes forward $\mu$ into $\nu$, i.e. it does not waste (or generate) mass during the transportation. This formulation of the OT problem has drawbacks, mostly notably the non-linearity of the constraint makes it very complex.

In 1942, Kantorovich \cite{25, 26} formulated the OT problem from a different perspective, which has become the standard one. Instead of minimising over transport maps $T$, he used as variables transport plans, defined as those probability measures $\pi \in \mathcal{P}(X \times Y)$ on the product space $X \times Y$ (endowed with its Borel $\sigma$-algebra $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$), which have marginals $\mu$ and $\nu$, i.e.

$$\pi(\cdot \times Y) = \pi \circ P_x^{-1} = \mu, \quad \pi(Y \times \cdot) = \pi \circ P_y^{-1} = \nu. \quad (4)$$

where $P_x$ and $P_y$ denote the projections on the first and second coordinate.

Denoting with $\Pi(\mu, \nu)$ the set of such transport plans, Kantorovich’s formulation of OT problem can be stated as

$$P(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y)d\pi(x, y), \quad (OT)$$

or, in probabilistic jargon, as

$$P(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[c(X, Y)],$$

the infimum being over the set of (joint laws $\pi$ of) random variables $X, Y$ with laws $\mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu$. We always denote with $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space, and $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. If we want to highlight the fact that $(X, Y)$ have law $\pi$, we write $\mathbb{E}[c(X, Y)]$ for $\mathbb{E}[f(X, Y)]$ (we use analogous notations for the conditional expectation).

Since the objective function of (OT) is linear in $\pi$ and $\Pi(\mu, \nu)$ is defined by linear inequalities, this formulation renders the OT problem a linear optimisation problem, thus making it much more tractable. Moreover, $\Pi(\mu, \nu)$ is (non-empty\footnote{It always contains the product measure $\mu \times \nu$.} and) compact.
with respect to the weak topology of probability measures, and this ensures the existence of an optimal solution under very mild assumptions.

Clearly, the two formulations are closely related and the concept of transport plan introduced by Kantorovich extends the transport map, in the sense that it allows mass positioned at \( x \) to be reallocated to multiple positions in \( Y \). In particular, if \( \pi^* \) is an optimal transport plan and is of the form \( \pi^* = (\text{Id}, T^*) \# \mu \) for some map \( T^* \), then \( T^* \) is an optimal transport map.

OT theory permits to define a most useful distance on the space of probability measures. If \( X = Y \) is a Polish space equipped with a metric \( d \), \( p \in [1, \infty) \), consider the OT problem with cost function \( c := d^p \), i.e.

\[
W_p(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y)^p \, d\pi(x, y) \right)^{1/p}.
\]  

(5)

The quantity \( W_p \) defines a distance on the space\(^2\)

\[ P_p(X) := \left\{ \mu \in \mathcal{P}(X) : \int d(x_0, x)^p \mu(dx) < \infty \right\}, \]

of probability measures on \( X \) with finite moments of order \( p \), often called the \( p \)-Wasserstein distance.

**Remark 1.** The following are equivalent

1. \( W_p(\mu_n, \mu) \to 0 \)

2. \[
\int f d\mu_n \to \int f d\mu
\]

(\( \ast \))

holds for all \( f \in C_0^{\text{lip}}(X) \) i.e all continuous functions \( f : X \to \mathbb{R} \) for which there exists \( C \in \mathbb{R} \) such that

\[
|f(x)| \leq C (1 + d(x, x_0)^p)
\]

for some (or equivalently, for any) \( x_0 \in X \).

3. \( \mu_n \to \mu \) weakly (i.e. \( \ast \) holds for \( f : X \to \mathbb{R} \) bounded and continuous) and \( \ast \) holds for \( f(x) := 1 + d(x, x_0)^p \) for some (or, equivalently, for any) \( x_0 \in X \).

In particular, \( W_1(\mu_n, \mu) \to 0 \) if and only if \( \int f d\mu_n \to \int f d\mu \) for every continuous \( f : X \to \mathbb{R} \) with sub-linear growth, and when \( d \) is bounded, \( W_p \) metrises the weak convergence. Finally, \( P_p(X) \), metrised as always by \( W_p \), is complete and separable [40, Theorem 6.18].

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\(^2\)Though a priori the definition of \( P_p(X) \) seems to depend on the choice of \( x_0 \in X \), it is easy to show that it does not.
2.2. Constrained OT

Interest in OT has been growing over the past two decades due to its wide applications to different fields (e.g. mathematical finance, economics, machine learning and image processing). It is then not surprising that many interesting variants of the original OT problem has been introduced and studied by many different authors. For example, in entropic OT, one replaces the linear cost functional $\pi \mapsto \int cd\pi$ with a strictly convex one, which much improves the speed of convergence of numerical algorithms to solve the problem. Alternatively, one can keep the cost functional unchanged, but require the transports to satisfy additional constraints. In this case we talk of the constrained OT problem ($cOT$), defined as

$$P_c(\mu, \nu) := \inf_{\pi \in \Pi_c(\mu, \nu)} \int cd\pi,$$

thus substituting in (OT) the set $\Pi(\mu, \nu)$ of transport plans by a set $\Pi_c(\mu, \nu)$ of transport plans which satisfy some additional constraints. As long as $\Pi_c(\mu, \nu)$ is convex, ($cOT$) is a convex optimisation problem, and thus remains tractable. Several instances of ($cOT$) with linear constraints are found in the literature, e.g.:

1. [27] considers capacity constraints, i.e. those transports $\pi \ll \mathcal{L}^{2N}$ whose density with respect to the Lebesgue measure $\mathcal{L}^{2N}$ is bounded above by a fixed constant.

2. Working in a multi-marginal setting, [4] consider causal transports (or the closely-related bi-causal transports). These are those joint laws $\pi$ of $(X,Y)$, with $X = (X_i)_{i=1}^N, Y = (Y_i)_{i=1}^N$ with values in $\mathbb{R}^N$, for which the conditional law of $Y_{1:t} := (Y_1, \ldots, Y_t)$ given $(X_1, \ldots, X_N)$ is the same as the conditional law of $Y_{1:t}$ given $(X_1, \ldots, X_t)$, for every $t = 1, \ldots, N$.

3. Many authors considered the martingale constraint $\mathbb{E}^\pi[Y|X] = X$, of which we talk in more detail in the next section.

4. [42] considers the general OT problem with linear constraints, and then discuss in more detail the two cases of transports which satisfy the martingale constraint, and of transports invariant under the action of a (product) group.

Unlike the classic OT problem, for which $\Pi(\mu, \nu)$ is always non-empty, it can happen that $\Pi_c(\mu, \nu) = \emptyset$, in which case ($cOT$) is of no interest; this leads us to the following definition.

**Definition 2.** We say that $(\mu, \nu)$ is viable (for ($cOT$)) if $\Pi_c(\mu, \nu) \neq \emptyset$.

It is of course of interest to characterise when $(\mu, \nu)$ is viable; this depends on the specific nature of the constraints. In this regard, it is interesting to recall Tchakaloff’s Theorem [8], which states that given $\mu \in \mathcal{P}_1(B)$, if $B = \mathbb{R}^d$ and $f^1, \ldots, f^m \in L^1(\mathbb{P}, B)$ represent the linear constraints $\int f^i d\mu = \int f^i d\hat{\mu}, i = 1, \ldots, m$ to be verified by some looked-for finitely-supported $\hat{\mu}$, then such a quantisation $\hat{\mu}$ of $\mu$ always exists. Such
Theorem also admits a martingale version [12], and this is not at all obvious since the martingale constraint corresponds to infinitely many linear constraints. However, these theorems only provide the existence of such quantisations, they do not describe a procedure to construct them explicitly.

2.3. Martingale Optimal Transport

The first formulation of the MOT problem appeared in [9] motivated by the model-independent pricing problem in Mathematical Finance. The aim of this problem is to find lower and upper bounds for the price of an exotic pay-off resulting by all arbitrage-free models consistent with some market specifications. The no-arbitrage condition translates into the requirement that \((X, Y)\) form a martingale (under the pricing measure), i.e. \(\mathbb{E}[Y|X] = X\), or equivalently
\[
\mathbb{E}[(Y - X)g(X)] = 0, \quad \text{for all } g \in C^0_b(B),
\]
where \(C^0_b(B)\) denotes the set of continuous bounded functions from \(B\) to \(\mathbb{R}\). In this case, we need to ask that \(B := X = Y\) has an additional vector space structure, and thus we assume that \(B\) is a separable Banach space. Moreover, since \(X, Y\) need to be integrable, we must ask that their laws \(\mu, \nu\) have finite first moments, i.e. \(\mu, \nu \in \mathcal{P}_1(B)\).

As usual, \(L^1(\mathbb{P}, B)\) denotes the set of integrable \(B\)-valued random variables. Recall that the conditional expectation of a \(B\)-valued random variable is defined and studied in [38, Section 5.3], making use of Bochner’s theory of integration.

In MOT one considers as the set \(\Pi_c(\mu, \nu)\) of constrained transports the set \(\mathcal{M}(\mu, \nu)\) of martingale couplings (a.k.a. martingale transports) between \(\mu\) and \(\nu\), i.e. laws of \((X, Y)\) such that \((X, Y)\) is a martingale, \(X \sim \mu, Y \sim \nu\). In more analytic terms,
\[
\mathcal{M}(\mu, \nu) = \left\{ \pi \in \Pi(\mu, \nu) : \int g(x)(y - x)d\pi(x, y) = 0 \quad \text{for all } g \in C^0_b(B) \right\}.
\]
The existence of a martingale coupling is not always guaranteed as in classical OT; however, several equivalent characterisations of viability are known; to introduce the most important one, we need a definition.

**Definition 3.** We say that probabilities \(\mu, \nu\) on \(B\) with finite first moments are increasing in convex order (denoted by \(\mu \leq_{cvx} \nu\)) if
\[
\int_B f(x)d\mu(x) \leq \int_B f(x)d\nu(x),
\]
for holds every Lipschitz\(^3\) convex function \(f: B \to \mathbb{R}\).

\(^3\)One can equivalently ask such inequality to hold for all continuous convex functions, because every such function \(f\) is the increasing limit of Lipschitz convex functions \(f_n\) (\(f_n\) can be obtained by inf-convolution between \(f\) and the map \(x \mapsto n\|x\|_B\)). However, we prefer using Lipschitz functions because in this case the integrals \(\int f d\mu, \int f d\nu\) cannot take the value \(\infty\).
An application of Jensen’s inequality implies that \( \mu \leq \epsilon \nu \) is necessary to ensure that \((\mu, \nu)\) is a viable input for the MOT problem. Moreover, sufficiency was proved by many authors in different settings; the case of \( B \) separable Banach space is due to Strassen [37]. To summarise, we have the following

**Theorem 4** (Strassen). Given probabilities \( \mu, \nu \in \mathcal{P}_1(B) \) on a separable Banach space \( B \), \( \mathcal{M}(\mu, \nu) \neq \emptyset \) holds if and only if \( \mu \leq \epsilon \nu \).

Since the seminal papers [9], a sizeable stream of papers has been devoted to studying MOT and its applications in mathematical finance [22] and references therein.

### 3. Discrete methods for OT

#### 3.1. Discretisation of classic OT

Since the OT problem (OT) is an infinite-dimensional LP (Linear Program), a tempting idea for solving it numerically is to reduce to the case in which \( \mu, \nu \) are finitely-supported, so as to make (OT) a finite-dimensional LP, which can then be easily solved numerically with several algorithms. With this in mind, one wants to construct a sequence of finitely-supported measures \((\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n \in \mathbb{N}}\) converging to the target measures \( \mu, \nu \), in order to calculate \( P(\mu, \nu) \) as the limit of \( P(\mu_n, \nu_n) \), since the latter can be calculated numerically. This procedure relies on the fundamental stability result in OT [40, Theorem 5.20], which ensures the continuity of the map \((\mu, \nu) \mapsto P(\mu, \nu)\) with respect to the Wasserstein distance.

Let’s explicitly formulate the discrete OT problem (i.e. the OT problem relative to finitely-supported measures \( \mu, \nu \)) as an LP. When \( \mu, \nu \) are finitely-supported, they can be written as convex combinations of Dirac measures. If they are supported on at most \( n \in \mathbb{N} \setminus \{0\} \) points, we can write

\[
\mu_n = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \quad \text{where} \quad \alpha_i \geq 0 \quad \forall i, \quad \sum_{i=1}^{n} \alpha_i = 1, \\
\nu_n = \sum_{j=1}^{n} \beta_j \delta_{y_j}, \quad \text{where} \quad \beta_j \geq 0 \quad \forall j, \quad \sum_{j=1}^{n} \beta_j = 1.
\]

In this case, the cost function is defined as the matrix \( c = (c_{i,j})_{i,j=1}^{n} \) where \( c_{i,j} := c(x_i, y_j) \in \mathbb{R}_+ \), and the feasible set is the polytope

\[
\Pi(\mu_n, \nu_n) := \left\{ p \in \mathbb{R}_{++}^{n \times n} : \sum_{j=1}^{n} p_{i,j} = \alpha_i \quad \forall i, \quad \sum_{i=1}^{n} p_{i,j} = \beta_j \quad \forall j, \quad 1, ..., n \right\}.
\]

Notice, that we are denoting the discretised version of the transport (previous denoted by \( \pi \)) with the matrix \( p \in \mathbb{R}_{++}^{n \times n} \).
The discrete OT is then the following finite-dimensional LP

$$\min \left\{ \sum_{i,j=1}^{n} p_{i,j} c_{i,j} : p \in \mathbb{R}^{n \times n}_{\geq 0}, p \mathbb{1}_n = \alpha, p^T \mathbb{1}_n = \beta \right\},$$

(7)

where $\mathbb{1}_n \in \mathbb{R}^{n \times 1}$ denotes the column vector with all entries equal to one, $\alpha = (\alpha_1, ..., \alpha_n)^T$ and $\beta = (\beta_1, ..., \beta_n)^T$. At this stage, computational methods for LPs can be applied to solve (7), see [35]. Among these methods, we quote the well known network simplex [1], hungarian [28] and the auction [14] algorithms. As the above methods have complexity (at least) $O(n^3)$, this leaves the door open to faster and more efficient numerical methods. For example, a well-known technique to tackle with optimisation problems, it is to add a regularisation term to the original objective function in order to obtain an approximating version of the problem which has an improved general structure and it is relatively easier to solve. In the setting of OT, [13] (see also [18] and [35]) proposed an entropic regularisation which they showed to be solved by a fast and simple numerical scheme, namely the iterative Bregman algorithm [15]. The very same regularisation can also be applied to the MOT problem although the algorithm can not be obtained in closed form (due to the additional martingale constraint). Indeed, [19] proposed an extension of the iterative Bregman algorithm based on the dual formulation of the problem in a general multidimensional setting. We also mention that there are several other unrelated numerical methods to solve (OT) [30] which do not rely on the discretisation of $\mu, \nu$.

3.2. Discretisation of constrained OT

Let’s now move on to the constrained OT problem framework. Consider the problem (cOT), where $\Pi_c(\mu, \nu)$ is the set of transports $\pi \in \Pi_c(\mu, \nu)$ from $\mu$ to $\nu$ which additionally satisfy and some linear (or convex) constraints. Mimicking the discrete method from the classical OT framework, we are interested in approximating $P_c(\mu, \nu)$ by $P_c(\mu_n, \nu_n)$, where $(\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n \in \mathbb{N}}$ are sequences of finitely-supported measures converging to $\mu, \nu$ respectively.

We are interested in whether it is possible to somehow adapt the discretisation method used in OT to the more general setting of cOT, and in particular in the following questions:

Q.1 If $(\mu, \nu)$ is viable, does there exist a sequence $(\mu_n, \nu_n)_{n}$ of finitely-supported probabilities such that $(\mu_n, \nu_n)_{n}$ converges to $(\mu, \nu)$ (in some appropriate sense, e.g., in the weak topology)?

Q.2 Can $(\mu_n, \nu_n)$ be explicitly computed? How?

Q.3 Given $(\mu_n, \nu_n) \to (\mu, \nu)$ as in Q.1, does $P_c(\mu_n, \nu_n) \to P_c(\mu, \nu)$?

In the following sections, we provide some possible answers to the questions above, in the setting of the MOT problem in arbitrary dimension. We point out that in this
case the answers are quite delicate, since the map \((\mu, \nu) \mapsto P_c(\mu, \nu)\) is not continuous (unless \(B = \mathbb{R}\)) [16], and it can happen that \(\Pi_c(\mu, \nu) \neq \emptyset\) whereas \(\Pi_c(\mu_n, \nu_n) = \emptyset\) for some \(n\), for some sequence \((\mu_n, \nu_n)\) converging to \((\mu, \nu)\) [6, Theorem 2.1].

4. Discretisations preserving the convex order

In the setting of MOT, Theorem 4 shows that question Q.1 becomes: is it possible to approximate \((\mu, \nu)\) with finitely-supported \((\mu_n, \nu_n)\) while preserving the convex order? This question is highly non-trivial, and has already been considered by several authors, as discussed in the introduction. In the next subsection, we discuss in some detail Baker’s approach, so that we are later able to show in Remark 12 that it is a special case of our approach; we also introduce and develop our new approach, providing an explicit construction, bounds on the speed of convergence, and even some (rather weak) stability result.

4.1. The \(U\)-quantisation

In [6], Baker considered the case \(B = \mathbb{R}\), and showed that in general \(\mathcal{M}(\mu_n, \nu_n) = \emptyset\) when the approximating measure \(\mu_n, \nu_n\) are defined as the \(L^2\)-quantiser of \(\mu\) and \(\nu\), even if \(\mathcal{M}(\mu, \nu) \neq \emptyset\). He then proposed an approximation, called \(U\)-quantisation, which preserves the convex order. Given \(\mu, \nu \in \mathcal{P}_2(\mathbb{R})\) such that \(\mu \leq_{cx} \nu\), let \(F_\mu(x) = \mu((-\infty, x])\) be the (cumulative) distribution function of \(\mu\), and recall that the quantile function of \(\mu\) is the generalised inverse

\[
F_\mu^{-1} : [0, 1] \to [-\infty, \infty] : p \mapsto \inf \{ x \in \mathbb{R} : p \leq F_\mu(x) \}.
\]

Then, for \(n \in \mathbb{N} \setminus \{0\}\), the \(U\)-quantisation of \(\mu\) of order \(n\) is defined by

\[
\hat{\mu}_n := U_n(\mu) := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}, \quad \text{where} \quad x_i := n \int_{i-1}^{i} F_\mu^{-1}(u) du,
\]

and it is a probability supported on \(n\) points. The \(U\)-quantisation operators \(U_n : \mathcal{P}_2 \to \mathcal{P}_2\) preserve the convex order and converge pointwise to the identity, meaning that

1. \(\mu \leq_{cx} \nu\) implies \(U_n(\mu) \leq_{cx} U_n(\nu)\) for all \(n\) [6, Theorem 3.5].
2. \(U_n(\mu)\) converge to \(\mu\) in \(4\) \(\mathcal{W}_1\) as \(n \to \infty\) for all \(\mu \in \mathcal{P}_1(\mathbb{R})\) [6, Theorem 3.6].

We point out the curious formal analogy between \(U_n(\mu)\) in (8) and the empirical measure \(\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}\) considered in [2, 3].

\footnote{To be precise, [6] only proves weak convergence; the stronger convergence in \(\mathcal{W}_1\) follows from our Remark 12 and Theorem 5.}
4.2. The martingale quantisation

Our proposed solution to Q.1 for the MOT problem is summarised in the subsequent Theorem 5. We recall that B is assumed to be a separable Banach space throughout the paper.

**Theorem 5.** Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_p(B)$ such that $\mu \leq \mu \nu$. Then, there exist finitely-supported probability measures $\mu_n, \nu_n \in \mathcal{P}_p(B)$, $n \in \mathbb{N}$, such that $\mu_n \leq \mu \nu_n$ for all $n \in \mathbb{N}$ and $\mu_n \to \mu, \nu_n \to \nu$ in $\mathcal{P}_p$. Moreover, one can additionally obtain that $\mu_n \leq \mu \nu$ for all $n \in \mathbb{N}$.

We will prove Theorem 5 as a simple corollary of the following analogous result involving random variables. We now introduce some notations and facts used without further notice throughout the paper.

Given $\Pi, \Pi_1, \Pi_2$ finite partitions of a set $E$, and $X, Y : \Omega \to E$, we write

$$\#\Pi := \text{card}(\Pi), \quad X^{-1}(\Pi) := \{X^{-1}(P) : P \in \Pi\}, \quad \Pi_1 \times \Pi_2 := \{M \times N, \ M \in \Pi_1, \ N \in \Pi_2\},$$

so that $\#\Pi$ is the number of elements of $\Pi$ and $W^{-1}(\Pi)$ is a finite partition of $\Omega$. Clearly $\Pi_1 \times \Pi_2$ is a finite partition of $E \times E$ and

$$(X, Y)^{-1}(\Pi_1 \times \Pi_2) = \{X^{-1}(M) \cap Y^{-1}(M) : M \in \Pi_1, N \in \Pi_2\}.$$

In particular, taking $\Pi_2$ to be the trivial partition $\{E\}$ we get

$$X^{-1}(\Pi_1) = (X, Y)^{-1}(\Pi_1 \times \{E\}).$$

If $E$ is endowed with a $\sigma$-algebra $\mathcal{E}$, and $n \in \mathbb{N} \setminus \{0\}$, we will consider the family

$$\left\{\Pi = (P_i)_{i=1}^k \subseteq \mathcal{E} : k \leq n, P_i \cap P_j = \emptyset \forall i \neq j, \cup_{i=1}^k P_i = E\right\},$$

of ($\mathcal{E}$-measurable) partitions of $E$ which are made of at most $n$ elements. Clearly, if $X$ is a $E$-random variable, then $X^{-1}(\Pi)$ is a partition of $\Omega$ made of at most $n$ elements. We recall that if $\mathcal{G} \subseteq \mathcal{F}$ is a $\sigma$-algebra on $\Omega$, $G \in \mathcal{G}$ is called an atom of $\mathcal{G}$ if for every $H \in \mathcal{G}$ either $G \cap H = G$ or $G \cap H = \emptyset$, and that $\mathcal{G}$ is finite if and only if it is of the form $\mathcal{G} = \sigma(H)$ for some finite partition $H \subseteq \mathcal{F}$, and in this case $H$ is the set of atoms of $\mathcal{G}$, and $\mathcal{G}$ is the family of all possible unions of sets in $H$ [32, Proposition I.2.1].

**Theorem 6.** For $i = 1, 2, \ n \in \mathbb{N}$, let $\Pi_i^n \subseteq \mathcal{B}(B)$ be finite partitions of $B$ which satisfy\(^5\)

$$\sigma\left(\Pi_i^n, n \in \mathbb{N}\right) = \mathcal{B}(B), \quad \Pi_i^n \subseteq \Pi_i^{n+1} \quad (9)$$

\(^5\)Of course such partitions exist, since $B$ is separable. For example, choose countably many Borel sets $(A_n)_{n \in \mathbb{N}}$ which generate $\mathcal{B}(B)$ (e.g. the balls of radius $1/(n+1)$, $n \in \mathbb{N}$, centred at points in a countable dense set), and then take $\Pi_1^n = \Pi_2^n$ as the family of the atoms of $\sigma \left( (A_i)_{i \leq n} \right)$.
Let $p \in [1, \infty)$ and given $X, Y \in L^p(\mathbb{P}, B)$ such that $\mathbb{E}[Y|X] = X$, consider the ‘martingale quantisation’ $(X_n, Y_n)$ of $(X, Y)$, defined by

$$X_n := \mathbb{E} \left[ X | \sigma^n_X \right], \quad Y_n := \mathbb{E} \left[ Y | \sigma^n_{X,Y} \right],$$

where

$$\sigma^n_X := \sigma(X^{-1}(\Pi^n_1)), \quad \sigma^n_{X,Y} := \sigma(X^{-1}(\Pi^n_1)) \lor \sigma(Y^{-1}(\Pi^n_2))$$

i.e. $\sigma^n_X, \sigma^n_{X,Y}$ are the $\sigma$-algebras generated by the partitions $X^{-1}(\Pi^n_1)$ and $X^{-1}(\Pi^n_1) \lor Y^{-1}(\Pi^n_2)$.

Then $X_n$ (resp. $Y_n$) is supported on at most $\#\Pi^n_1$ (resp. $\#\Pi^n_2$) points, and

$$\mathbb{E} \left[ Y_n | \sigma^n_X \right] = \mathbb{E}[Y_n|X_n] = X_n, \quad X_n = \mathbb{E}[X|X_n], \quad Y_n = \mathbb{E}[Y|Y_n],$$

(12a) $X_n \to X$ a.s. and in $L^p$, $Y_n \to Y$ a.s. and in $L^p$. (12b)

**Proof.** If a $\sigma$-algebra $\mathcal{G}$ is generated by a partition made of $n$ sets and $Z : \Omega \to B$ is $\mathcal{G}$-measurable, then $Z$ only takes at most $n$ values, since it is constant on every atom of $\mathcal{G}$; thus, $X_n$ (resp. $Y_n$) is supported on at most $\#\Pi^n_1$ (resp. $\#\Pi^n_2$) points.

The assumptions (9) imply that $(\sigma^n_X)_n, (\sigma^n_{X,Y})_n$ are filtrations on $(\Omega, F)$ such that $\forall_n \sigma^n_X = \sigma(X), \forall_n \sigma^n_{X,Y} = \sigma(X,Y)$, and so

$$\mathbb{E} [X | \forall_n \sigma^n_X] = X, \quad \mathbb{E} [Y | \forall_n \sigma^n_{X,Y}] = \mathbb{E}[Y|X,Y] = Y.$$  

(13)

By definition $(X_n)_n$ (resp. $(Y_n)_n$) is a martingale with respect to the filtration $(\sigma^n_X)_n$ (resp. $(\sigma^n_{X,Y})_n$) and is closed by $X$ (resp. $Y$). Thus, (13) and the martingale convergence theorems [36, Theorems 1.5, 1.14] yield (12b). The tower property gives

$$X_n = \mathbb{E} \left[ X | \sigma_X^n \right] | X_n = \mathbb{E}[X|X_n], \quad Y_n = \mathbb{E} \left[ Y | \sigma_{X,Y}^n \right] | Y^n = \mathbb{E}[Y|Y_n].$$

(14)

Let us now prove $\mathbb{E}[Y_n|\sigma^n_X] = \mathbb{E}[Y_n|X_n] = X_n$. Let $\{A^n_i\}_i = X^{-1}(\Pi^n_1)$ be the family of atoms of $\sigma^n_X$. Then, $A^n_i \in \sigma^n_X \subseteq \sigma^n_{X,Y} \cap \sigma(X)$ and so

$$\mathbb{E} \left[ Y_n 1_{\{X \in A_i\}} \right] = \mathbb{E} \left[ Y 1_{\{X \in A_i\}} \right] = \mathbb{E} \left[ X 1_{\{X \in A_i\}} \right] = \mathbb{E} \left[ X_n 1_{\{X \in A_i\}} \right].$$

In other words, the measures $\mathbb{E} \left[ Y_n 1_{\{X \in \cdot\}} \right]$ and $\mathbb{E} \left[ X_n 1_{\{X \in \cdot\}} \right]$ coincide on the $\pi$-system\(^{6}\) $\{\emptyset, A^n_i\}$, which generates $\sigma^n_X$, and so they coincide on $\sigma^n_X$, which means that $\mathbb{E}[Y_n|\sigma^n_X] = X_n$. It follows that $\sigma(X_n) \subseteq \sigma_X^n$, and so taking $\mathbb{E}[\cdot|X_n]$, the tower property gives

$$X_n = \mathbb{E}[X_n|X_n] = \mathbb{E} \left[ X_n | \sigma_X^n \right] | X_n = \mathbb{E}[Y_n|X_n].$$

\(^{6}\)Trivially $\{\emptyset, A^n_i\}$, is closed under intersections since the atoms are all disjoint.
Lemma 7. If $X, Y$ are $B$-valued random variables with laws $\mu, \nu \in \mathcal{P}_p$ then

$$W_p(\mu, \nu) \leq \|X - Y\|_{L^p(P, B)}.$$

Proof. If $(X, Y)$ have joint law $\pi$ then

$$\|X - Y\|^p_{L^p(P, B)} = \int |x - y|^p d\pi(x, y),$$

so the thesis follows trivially from the definition of $W_p(\mu, \nu)$. \qed

Proof of Theorem 5. By Theorem 4 there exist $X, Y \in L^p$ such that $\mathbb{E}[Y|X] = X$. Applying Theorem 6 to such $X, Y$ yields some $X_n, Y_n$, whose laws $\mu_n, \nu_n$ satisfy the desired properties by Lemma 7 and Theorem 4. \qed

Remark 8. Importantly, the proofs of Theorems 5 and 6 provide an algorithm to construct the discrete random variables $X_n, Y_n$ which satisfy the martingale condition, and thus the discretisation $(\mu_n, \nu_n)$ of $(\mu, \nu)$ which preserves the convex order, thus providing an answer to both question Q.1 and Q.2.

Remark 9. The previous construction can be applied without any change to countable partitions of $B$, in which case one obtains sequences of random variables $X_n, Y_n$ and measures $\mu_n, \nu_n$, each of which is countably supported.

Given the above results, we can provide the following definition which summarises the proposed answer to Q.1 in the setting of MOT.

Definition 10. We call the sequence of probability measures $(\mu_n, \nu_n)_{n \in \mathbb{N}}$ (or, equivalently, of random variables $(X_n, Y_n)_{n \in \mathbb{N}}$) a martingale quantisation of the couple $(\mu, \nu)$ (resp. $(X, Y)$).

Remark 11. It is important to underline that $X_n$ only depends on the target random variable $X$; on the other hand, the quantised random variable $Y_n$ depends both on $X$ and $Y$. This dependence seems to be necessary to ensure that $X_n$ and $Y_n$ satisfy the martingale condition in this general setting.

Remark 12. There is a close link between the $U$-quantisation, and the martingale quantisation. Indeed, given a probability $\mu$ on $\mathbb{R}$, its quantile function $X := F_{\mu}^{-1}$, seen as a measurable function on the probability space $\Omega := (0, 1)$ endowed with the Lebesgue measure $\mathcal{L}^1 = \mathbb{P}$ on the Borel $\sigma$-algebra $\mathcal{B}((0, 1))$, is a random variable with law $\mu$. Thus, if one considers the partition $\Pi_n$ of $\mathbb{R}$ given by the intervals

$$\left(0, \frac{1}{n}\right], \left(\frac{i - 1}{n}, \frac{i}{n}\right] \text{ for } i = 1, \ldots, n-1, \left(\frac{n - 1}{n}, 1\right],$$

then the $\sigma$-algebra $X^{-1}(\sigma(\Pi_n))$ appearing in (11) is the one generated by the partition

$$\left(0, \frac{1}{n}\right], \left(\frac{1}{n}, \frac{2}{n}\right], \ldots, \left(\frac{n - 1}{n}, 1\right) \quad (15)$$

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of $\Omega = (0,1)$, and thus $X_n := \mathbb{E}[X|\sigma_X]$ as defined in (10) takes the value $\int_{-\frac{1}{n}}^{\frac{1}{n}} F_\mu^{-1}(u) du$ on the interval $\left(\frac{i-1}{n}, \frac{i}{n}\right] \cap (0,1)$, and so the law of $X_n$ is the probability $\mathcal{U}_n(\mu)$ defined in (8). Thus, the $\mathcal{U}$-quantisation is a special case of the martingale quantisation, in which one chooses $X$ as being the quantile function of $\mu$, and $X^{-1}(\sigma(\Pi_1^\mu))$ as being the finite $\sigma$-algebra whose atoms are listed in (15), so that in particular all the weights $\alpha_i$ in the representation $\mu = \sum_i \alpha_i \delta_{x_i}$ are equal to $\frac{1}{n}$. However, notice that the martingale quantisation $Y_n$ of $Y \sim \nu$ involves also $X$, and so its law is unrelated to $\mathcal{U}_n(\nu)$; moreover, given $\mu \leq_{cv} \nu$, while the quantile functions $X := F_\mu^{-1}$ and $Y := F_\nu^{-1}$ have laws $\mu$ and $\nu$, they do not$^7$ in general form a martingale (on $(0,1)$ endowed with $\mathcal{L}^1$).

4.3. Explicit representation of the martingale quantisation

In this section, we will provide an explicit integral expression for the martingale quantisation, i.e. for the random variables $X_n, Y_n$ which appear in Theorem 6, and their laws $\mu_n, \nu_n$ which appear in Theorem 5, thus positively answering question Q.2.

**Proposition 13.** Let $X \in L^p(\mathbb{P}, B)$, $X \sim \mu$ and $\hat{X} := \mathbb{E}[X|\hat{\sigma}_X]$ where $\Pi_1 = (P_{1,i})_i$ is a finite partition of $B$ and $\hat{\sigma}_X := \sigma(X^{-1}(\Pi_1))$. Then,

$$\hat{X} = \sum_i x_i 1_{P_{1,i}}, \quad (16)$$

so that its law is given by

$$\hat{\mu} = \mathcal{L}(\hat{X}) = \sum_i \mu(P_{1,i}) \delta_{x_i}, \quad (17)$$

where the sum is over $i$ such that $\mu(P_{1,i}) > 0$ and $x_i$ can be expressed as

$$x_i := \frac{1}{\pi(P_{1,i} \times B)} \int_{P_{1,i} \times B} x \pi(dx, dy) = \frac{1}{\mu(P_{1,i})} \int_{P_{1,i}} x \mu(dx), \quad (18)$$

so that $x_i$ is the barycentre of the probability $\mu_i := \frac{1_{P_{1,i}}}{\mu(P_{1,i})} \cdot \mu$.

Assume moreover that $Y \in L^p(\mathbb{P}, B)$, $Y \sim \nu$ satisfies $\mathbb{E}[Y|X] = X$, and define $\hat{Y} := \mathbb{E}[Y|\hat{\sigma}_{X,Y}]$ where $\Pi_1 = (P_{1,i})_i$, $\Pi_2 = (P_{2,j})_j$ are finite partitions of $B$ and $\hat{\sigma}_{X,Y} := \sigma(X^{-1}(\Pi_1)) \lor \sigma(Y^{-1}(\Pi_2))$.

Then,

$$\hat{Y} = \sum_{i,j} y_{i,j} 1_{P_{1,i} \times P_{2,j}}, \quad (19)$$

$^7$Indeed, if $\mu$ has no atoms then $F_\mu$ has no jumps, so $F_\mu^{-1}$ is strictly increasing, and so $\sigma(F_\mu^{-1})$ equals the Borel $\sigma$-algebra $\mathcal{B}((0,1))$. Thus $F_\nu^{-1}$ is $\sigma(F_\mu^{-1})$-measurable, and so $(F_\mu^{-1}, F_\nu^{-1})$ is martingale only if $F_\mu^{-1} = F_\nu^{-1}$, i.e. only if $\mu = \nu$. 15
so that its law is given by

\[ \hat{\nu} = \mathcal{L} \left( \hat{Y} \right) = \sum_{i,j} \pi \left( P_{1,i} \times P_{2,j} \right) \delta_{y_{i,j}}, \quad (20) \]

where \( \pi = \mathcal{L}(X,Y) \) and the sum is over \( i,j \) such that \( \pi \left( P_{1,i} \times P_{2,j} \right) > 0 \). In addition, \( y_{i,j} \) can be expressed as

\[ y_{i,j} := \frac{1}{\pi \left( P_{1,i} \times P_{2,j} \right)} \int_{P_{1,i} \times P_{2,j}} y \pi (dx, dy) = \int_{B} y \nu_{i,j} (dy), \quad (21) \]

so that \( y_{i,j} \) is the barycentre of the law \( \nu_{i,j} := \pi_{i,j} \circ P_{y}^{-1} \) of \( P_{y} \) under \( \pi_{i,j} \), where

\[ \pi_{i,j} := \frac{1}{\pi \left( P_{1,i} \times P_{2,j} \right)} \pi \quad (22) \]

In particular, the \( X_{n}, Y_{n} \) which appear in Theorem 6 admit the explicit representation given by eqs. (16) (18) (19) and (21), in which the quantities \( P_{1,i}, P_{2,j}, x_{i}, y_{i,j} \) also depend on the index \( n \).

**Remark 14.** Notice that the above expressions for \( \hat{\mu}, \hat{\nu} \) do not depend on the random variables \( X, Y \), they only depend on their joint law \( \pi \in \mathcal{M}(\mu, \nu) \). However, such law is not an input of the MOT problem (only \( \mu, \nu, c \) are), so it not always known in advance; when no such martingale coupling between \( \mu \) and \( \nu \) is known, one cannot apply the previous formulas to explicitly compute the laws \( \mu_{n}, \nu_{n} \). As discussed in the introduction, this is a disadvantage of our algorithm, since currently algorithms which output a \( \pi \in \mathcal{M}(\mu, \nu) \) given \( \mu \leq_{\text{cx}} \nu \) are only known in dimension 1.

To prove the above proposition, we will need the following simple lemma.

**Lemma 15.** If \( (E, \mathcal{E}) \) is a measurable space, \( \Pi = \{ P_{j} \}_{j} \subseteq \mathcal{E} \) a finite partition of \( E \), \( W \) a \( E \)-valued random variable, \( G \in L^{1}(\mathbb{P}, B) \), and \( \theta = \mathcal{L}(W,G) \) then

\[ \mathbb{E}[G|\sigma(W^{-1}(\Pi))] = \sum_{j: \mathbb{P}(W \in P_{j}) > 0} 1_{P_{j}}(W) \frac{\mathbb{E}[1_{P_{j}}(W)G]}{\mathbb{P}(W \in P_{j})} \]

has law \( \zeta \) given by the formula

\[ \zeta = \sum_{j} \theta(P_{j} \times B) \delta_{z_{j}}, \quad \text{with} \quad z_{j} := \frac{1}{\theta(P_{j} \times B)} \int_{P_{j} \times B} g \theta(e, g). \]

**Proof.** The formula for \( \mathbb{E}[G|\sigma(W^{-1}(\Pi))] \) is simply the definition of conditional expectation with respect to the \( \sigma \)-algebra generated by the partition \( W^{-1}(\Pi) \). That \( \zeta \) are as stated follows from the fact that the random variable \( \sum_{i} 1_{H_{i}} w_{i} \) has law \( \sum_{i} \mathbb{P}(H_{i}) \delta_{w_{i}} \). \( \square \)
Proof of Proposition 13. Applying Lemma 15 with $G = W \sim \mu$, we get that $d\theta(e, g) = d\mu(e)d\delta_v(g)$, and so $\hat{\mu}$ is given by eqs. (17) and (18). Applying instead Lemma 15 with

$$E := B \times B, \quad \Pi := \Pi_1 \times \Pi_2, \quad W = (X, Y) \sim \pi,$$

writing $e \in E$ as $e = (x, y)$ and using $\theta(\cdot \times B) = \pi$ gives

$$\hat{\nu} = \sum_{i,j} \pi(P_{i,j} \times P_{2,j}) \delta_{y_{i,j}}, \quad \text{with } y_{i,j} := \frac{1}{\pi(P_{i,j} \times P_{2,j})} \int_{P_{i,j} \times P_{2,j} \times B} g \ d\theta(x, y, g).$$

If then one takes $G = Y$, it follows that $d\theta(x, y, g) = d\pi(x, y) d\delta_y(g)$, and so we get that $\hat{\nu}$ is given by eqs. (20) and (21).

4.4. Bounds on the speed of convergence

We will now show that the average diameter of the the elements of $\Pi_1$ (resp. $\Pi_2$) with respect to $\mu$ (resp. $\nu$) is an upper-bound for the Wasserstein-1 distance $W_1(\mu, \hat{\mu})$ (resp. $W_1(\nu, \hat{\nu})$) between the laws of $X$ and $\hat{X}$ (resp. $Y$ and $\hat{Y}$) which appear in Proposition 13. We remind the reader that the diameter of $E \subseteq B$ is defined as

$$\text{diam } E := \sup_{x,y \in E} \|x - y\| \in [0, \infty].$$

Theorem 16. Under the assumptions of Proposition 13, the following estimates hold:

$$W_p(\mu, \hat{\mu}) \leq \left\| X - \hat{X} \right\|_{L^p(P, B)} \leq \sum \left( \text{diam } (P_{1,i}) \right)^p \mu (P_{1,i}) \quad (23a)$$

$$W_p(\nu, \hat{\nu}) \leq \left\| Y - \hat{Y} \right\|_{L^p(P, B)} \leq \sum \left( \text{diam } (P_{2,j}) \right)^p \nu (P_{2,j}) \quad (23b)$$

where the sum over $i$ (resp. $j$) is over the $i$ such that $\mu(P_{1,i}) > 0$ (resp. $j$ such that $\nu(P_{2,j}) > 0$). In particular, in the setting of Theorem 6, calling $\mu, \nu, \mu_n, \nu_n$ the laws of $X, Y, X_n, Y_n$, we obtain the bounds

$$W_p(\mu, \mu_n) \leq \sum \left( \text{diam } (P_{1,i}^n) \right)^p \mu (P_{1,i}^n), \quad W_p(\nu, \nu_n) \leq \sum \left( \text{diam } (P_{2,j}^n) \right)^p \nu (P_{2,j}^n). \quad (24)$$

To prove Theorem 16, we need a lemma, which uses the well known fact that the convex hull of $E$, denoted with $\text{co } E$, can be obtained as $\text{co } E = \cup_{n \in \mathbb{N}} \text{co }^n E$, where $\text{co }^n E$ is defined by induction as follows:

$$\text{co }^1 E := \{ tx + (1-t)y : x, y \in E, t \in [0, 1] \}, \quad \text{co }^{n+1} E := \text{co }^1 (\text{co }^n E), n \in \mathbb{N}.$$ 

We denote with $\overline{\text{co } E}$ the closed convex hull of $E$, which coincides with the closure $\overline{\text{co } E}$ of $\text{co } E$.

Lemma 17. If $F \subseteq E \subseteq B$ then $\text{diam } (\overline{\text{co } E}) = \text{diam } E \geq \text{diam } F$. 

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Proof. Trivially, \( F \subseteq E \subseteq B \) implies \( \text{diam}(E) \geq \text{diam}(F) \). This implies that \( (\text{co}^n(E))_n \) is increasing, and so \( \text{diam}(\text{co}(E)) = \sup_n \text{diam}(\text{co}^n(E)) \). Since the norm is continuous, \( \text{diam}(E) = \text{diam}(\bar{E}) \), so the rest of the thesis follows once we prove the inequality \( \text{diam}(E) \geq \text{diam}(\text{co}^1(E)) \). To prove it, given \( \varepsilon > 0 \) choose \( z^1, z^2 \in \text{co}^1(E) \) such that \( \|z^1 - z^2\| \geq \text{diam}(\text{co}^1(E)) - \varepsilon \). For \( i = 1, 2 \) write \( z^i \) as

\[
z^i = t^i x^i + (1 - t^i)y^i, \quad \text{where } x^i, y^i \in E, t^i \in [0, 1].
\]

Choose \( a, b \in F := \{x^1, x^2, y^1, y^2\} \) such that

\[
\|a - b\| = \text{diam}(F) =: d \leq \text{diam}(E).
\]

If \( c := \frac{a + b}{2} \) is the mid-point between \( a \) and \( b \), and \( r := d/2 \), then \( F \) is a subset of \( B_r(c) := \{z : |z - c| \leq r\} \). Since \( B_r(c) \) is convex and \( F \subseteq B_r(c) \), we have \( z^1, z^2 \in B_r(c) \), and so \( \|z^1 - z^2\| \leq d \). We have proved that

\[
\text{diam}(E) \geq d \geq \|z^1 - z^2\| \geq \text{diam}(\text{co}^1(E)) - \varepsilon, \quad \text{for all } \varepsilon > 0,
\]

and taking \( \varepsilon \downarrow 0 \), we conclude. \( \square \)

In the course of the following proof, we will repeatedly make use of the fact that, if \( C \subseteq B \) is closed and \( \mu \) is a probability on \( C \) then its barycentre \( \text{bar}(\mu) := \int x \mu(dx) \) belongs to \( C \); this well known fact easily follows from Hanh-Banach’s theorem, as in [29, Proposition 2.39].

Proof of Theorem 16. If \( P_y \) denotes the projection of \( B \times B \) onto its second coordinate, the explicit expression for \( \hat{Y} \) gives that

\[
\|Y - \hat{Y}\|_{L^p(P,B)} = \sum_{i,j} \int_{P_{1,i} \times P_{2,j}} \|y - P_y(\text{bar} \ (\pi_{i,j}))\|^p \pi(dx,dy). \tag{25}
\]

Since \( \pi_{i,j} ((P_{1,i} \times P_{2,j})^c) = 0 \), we have that \( \text{bar} \ (\pi_{i,j}) \in \text{co}(P_{1,i} \times P_{2,j}) \), and since \( \text{co}(E \times F) = \text{co}(E) \times \text{co}(F) \), for all \( E, F \subseteq B \), we get that \( P_y(\text{bar} \ (\pi_{i,j})) \in \text{co}(P_{2,j}) \), and so

\[
\|y - P_y(\text{bar} \ (\pi_{i,j}))\| \leq \text{diam}(\text{co}(P_{2,j})), \quad (x, y) \in P_{1,i} \times P_{2,j}
\]

from which, using Proposition 17, we get

\[
\|y - P_y(\text{bar} \ (\pi_{i,j}))\| \mathbb{1}_{P_{1,i} \times P_{2,j}}(x, y) \leq \text{diam} (P_{2,j}) \mathbb{1}_{P_{1,i} \times P_{2,j}}(x, y)
\]

Taking \( p \)-powers, summing over \( i, j \) and integrating with respect to \( \pi \), using (25), we finally get (23b). Finally, (23a) follows from applying (23b) not to \( X, Y, \Pi_1, \Pi_2 \), but rather to

\[
X' := \mathbb{E}[X], \quad Y' := X, \quad \Pi'_1 := \{B\}, \quad \Pi'_2 := \Pi_1,
\]

which we can do since\(^8\) \( \mathbb{E}[Y'|X'] = X' \). \( \square \)

\(^8\)If this reasoning seems puzzling, it should become clear after reading section 6, in particular Theorem 35.
Notice that, although the bounds proved in Theorem 16 have the pleasing quality of being easy to compute numerically for partitions made ‘simple’ sets, they are not sharp: it is possible to find $\Pi^1, \Pi^2$ such that the upper-bounds do not converge to 0 even if $(\mu_n, \nu_n)$ converge to $\mu, \nu$, and the upper-bounds can even be vacuous (i.e. they equal $+\infty$) for some $\pi, \Pi^1, \Pi^2$.

Remark 18. We could have worked identically if considering partitions which are countable, instead of finite. This can be important, because $B$ admits a countable partitions made of bounded sets, in fact for each $\epsilon > 0$ there exists a countable partition made of sets of diameter at most $\epsilon$. Using such partitions is useful since one can also replace the bounds (23) with

$$\mathcal{W}_p(\mu, \hat{\mu}) \leq \sup_i (\text{diam}(P_{1,i}))^p, \quad \mathcal{W}_p(\nu, \hat{\nu}) \leq \sup_j (\text{diam}(P_{2,j}))^p,$$

which are less precise, but have the interesting feature that the upper-bounds do not depend on $\mu, \nu$. Of course, one could achieve the same result working with finite partitions whose elements which intersect the supports of the measures are bounded, and such partitions always exist when the measures are compactly supported. In particular, if $\mu, \nu$ are compactly supported probabilities on $B = \mathbb{R}^N$, and the partitions are made of sets whose diameter converges to zero (e.g. hypercubes with sides of length $1/2^n$), the given bounds always go to 0.

4.5. Stability

Question Q.3 addresses the so-called stability result for a general cOT problem. As already remarked, in the setting of MOT, this question is highly non trivial. Indeed, a positive answer in this direction has been provided on the real line setting from different perspectives [5, 24, 41] whereas only the very recent work of [16] shows that the stability result does not hold for the MOT problem on $\mathbb{R}^d, d \geq 2$. Therefore, we reformulate question Q.3 as follows: if $(\mu_n, \nu_n) \to (\mu, \nu)$ under which additional assumptions do we have that

$$P_n := \inf_{M(\mu_n, \nu_n)} \mathbb{E}[c(X_n, Y_n)] \to \inf_{M(\mu, \nu)} \mathbb{E}[c(X, Y)] =: P ?$$

Here, we provide a rather weak answer to this question leaving a more structured result for further investigation. In particular, this is linked with the fact that the martingale quantisation algorithm needs as an input an arbitrary (i.e. not necessarily an optimal) $\tilde{\pi} = L(X,Y) \in M(\mu, \nu)$ and outputs $\pi_n = L(X_n, Y_n)$. One can choose such $\tilde{\pi}, \pi_n$ as shown in Lemma 20.

Recall that $f : B \times B \to \mathbb{R}^+$ is said to have sub-linear growth if

$$|f(x,y)| \leq C(1 + \|x\| + \|y\|)$$

for some $C \in \mathbb{R}$, i.e. if $f \in C^0_p(B \times B)$ with $p = 1$.

The following result is the martingale analogue of a well-known result in OT.
Lemma 19. Let $\tilde{\pi} \in \mathcal{M}(\mu, \nu)$, $\mu_n \to \mu$, $\nu_n \to \nu$, $\pi_n \in \mathcal{M}(\mu_n, \nu_n)$ with $\pi_n \to \tilde{\pi}$. Then, the sequence $(\pi_n)_n$ is tight so that it admits accumulation points. Any accumulation point belongs to $\mathcal{M}(\mu, \nu)$.

Proof. By Prokhorov’s Theorem [39, Theorem 5.2], $(\mu_n)_n$ and $(\nu_n)_n$ are (uniformly) tight, therefore, for every $\epsilon > 0$ there exist compact sets $K^\mu_\epsilon$, $K^\nu_\epsilon \subseteq B$ such that $\mu_n (B \setminus K^\mu_\epsilon) \leq \epsilon/2$, $\nu_n (B \setminus K^\nu_\epsilon) \leq \epsilon/2$. Then,

$$\pi_n (B \times B \setminus K^\mu_\epsilon \times K^\nu_\epsilon) \leq \mu_n (B \setminus K^\mu_\epsilon) + \nu_n (B \setminus K^\nu_\epsilon) \leq \epsilon,$$

so that $(\pi_n)_n$ is tight and there exist $\pi \in \mathcal{P}(B \times B)$ such that $\pi_n$ weakly converges to $\pi$ (up to taking a subsequence without relabelling). In particular, for any $f \in C^0_0(B \times B)$, we have

$$\int f(x, y)\pi_n(dx, dy) \to \int f(x, y)\pi(dx, dy). \quad (29)$$

Let $g, h \in C^0_0(B)$. Choosing $f(x, y) = g(x)$ and $f(x, y) = h(y)$ in $(29)$, we obtain that $\pi \in \Pi(\mu, \nu)$. Moreover,

$$\int \|(x, y)\|\pi_n(dx, dy) = \int \|x\|\mu_n(dx) + \int \|y\|\nu_n(dy) \to \int \|x\|\mu(dx) + \int \|y\|\nu(dy)$$

$$= \int \|(x, y)\|\pi(dx, dy),$$

and so, by Remark 1, $\pi_n \to \pi$. Finally, since the map $(x, y) \mapsto g(x)(y - x)$ has sub-linear growth, we have that

$$0 = \int (y - x)g(x)\pi_n(dx, dy) \to \int (y - x)g(x)\pi(dx, dy),$$

so that $\pi \in \mathcal{M}(\mu, \nu)$. \hfill \Box

Lemma 20. Under the assumptions of Lemma 19, if $c : B \times B \to \mathbb{R}^+$ is a continuous cost function with sub-linear growth, then

$$E^\pi [c(X, Y)] \geq \limsup_n P_n \geq \liminf_n P_n \geq P. \quad (30)$$

In particular, $(27)$ holds along a minimising subsequence if

$$\pi = \pi^* \in \arg\min_{\pi \in \mathcal{M}(\mu, \nu)} E [c(X, Y)]. \quad (31)$$

Proof. For $n \in \mathbb{N}$, let $\pi_n$ be a martingale quantisation which can be obtained starting from the (sub-optimal) transport $\tilde{\pi}$. Then, by the continuity of $c$ and assumption $(28)$, it follows that

$$E^{\pi_n} [c(X,Y)] \to E^\pi [c(X,Y)] \geq \min_{\pi \in \mathcal{M}(\mu, \nu)} E^\pi [c] = P.$$
Let \( \pi^*_n \in \mathcal{M}(\mu_n, \nu_n) \) be the minimiser. Then, by Lemma 19, \((\pi^*_n)_n\) is tight and there exists \( \bar{\pi} \in \mathcal{M}(\mu, \nu) \) such that \( \pi^*_n \to \bar{\pi} \) (up to taking a subsequence without relabelling). It then follows that

\[
\mathbb{E}^{\pi_n}[c(X,Y)] \geq \min_{\pi \in \mathcal{M}(\mu, \nu)} \mathbb{E}^{\pi}[c(X,Y)] = P_n = \mathbb{E}^{\pi^*_n}[c(X,Y)] \to \mathbb{E}^{\bar{\pi}}[c(X,Y)],
\]

Therefore,

\[
\mathbb{E}^{\bar{\pi}}[c(X,Y)] \geq \min_{\pi \in \mathcal{M}(\mu, \nu)} \mathbb{E}^{\pi}[c(X,Y)].
\]

Putting the pieces together, we get the claim.

If particular, if (31) holds, i.e. \( \bar{\pi} = \pi^* \) is the minimiser, then

\[
\limsup_n P_n \leq \limsup_n \mathbb{E}[c(X_n, Y_n)]
\]

where: the first inequality follows by definition; the equality follows from \( c(X_n, Y_n) \to c(X, Y) \) \( \mathbb{P} \)-a.s. and\(^9\) in \( L^1(B, \mathbb{P}) \) since \( c \) is assumed to be continuous and with sub-linear growth and finally, the last inequality follows from (31).

\[\Box\]

**Remark 21.** The existence of the optimiser \( \pi^* \) is proved in [9, Theorem 1.1] on the real-line and in [42] in the general setting of Banach spaces.

**Remark 22.** Clearly, Lemma 20 has limited usefulness, as it states that in order to approximate the optimal value one should already know the optimal \( \pi^* \), in which case to calculate the optimal value it is simpler to compute \( \mathbb{E}^{\pi^*}[c(X,Y)] \) rather than solving \( \inf_{\mathcal{M}(\mu, \nu)} \mathbb{E}[c(X,Y)] \) and computing its liminf. However, Lemma 20 suggests a recursive scheme which might approximate the optimal value \( P \) of the MOT problem and, at least, obtains smaller and thus more precise optimal values.

Indeed, starting from the given transport \( \pi^{(0)} := \bar{\pi} \in \mathcal{M}(\mu, \nu) \), by Lemma 20 we can construct \( \pi^{(1)} := \bar{\pi} \in \mathcal{M}(\mu, \nu) \) such that \( P^{(1)} := \mathbb{E}^{\pi^{(1)}}[c(X,Y)] \leq \mathbb{E}^{\pi^{(0)}}[c(X,Y)] =: P^{(0)} \). Reiterating this step, one could obtain a sequence \( \{P^{(l)}\}_{l \in \mathbb{N}} \) satisfying

\[
P^{(0)} \geq \ldots \geq P^{(l)} \geq P^{(l+1)} \geq \ldots \geq P, \quad l \geq 1.
\]

5. Numerical examples

In this section, we provide some examples in which we implement the proposed martingale quantisation scheme and solve the corresponding discretised MOT, illustrating cases in which (27) seems to hold.

\[\text{9Because } \{c(X_n, Y_n), n \in \mathbb{N}\} \text{ is uniformly integrable.}\]
Example 23. Consider an MOT problem on the real line between two uniform marginal distributions
\[ \mu(dx) = \frac{1}{2} \mathbb{1}_{[-1,1]}(x) dx \]
\[ \nu(dy) = \frac{1}{4} \mathbb{1}_{[-2,2]}(y) dy, \]
and cost function given by \( c(x, y) := |y - x|^\rho \), \( \rho = 2.3 \), i.e.
\[ P := \min_{\pi \in \mathcal{M}(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |y - x|^\rho \pi(dx, dy). \] (32)

For this problem, it is known [2, Example 6.1] the expression of the martingale optimal transport
\[ \pi^*(dx, dy) = \frac{1}{2} \mathbb{1}_{[-1,1]}(x) \left( \frac{1}{4} \delta_{x+1}(dy) + \frac{3}{4} \delta_{x-1}(dy) \right) dx, \] (33)
so that the corresponding optimal value of the problem is \( P = \int_{\mathbb{R} \times \mathbb{R}} |y - x|^\rho \pi^*(dx, dy) = 1. \)

Firstly, we apply our martingale quantisation algorithm using as initial martingale transport \( \tilde{\pi} \) the (sub-optimal) left-curtain coupling\(^{11}\)
\[ \pi^{lc}(dx, dy) := \frac{1}{2} \mathbb{1}_{[-1,1]}(x) \left( \frac{3}{4} \delta_{x+1}(dy) + \frac{1}{4} \delta_{x-1}(dy) \right) dx. \]

and two \( n \)-partitions \( \{ P_{i,j} \}_{i,j=1}^n \) of \( \text{supp}(\mu) = [-1,1] \) and \( \text{supp}(\nu) = [-2,2] \), respectively. In particular, since for any integrable function \( f \) and \( P_i, P_j \subseteq \mathbb{R} \) it holds
\[ \int_{P_i \times P_j} f(x, y) \pi(dx, dy) = \int_{P_i} \frac{1}{2} \left\{ \frac{3}{4} f \left( x, \left( -\frac{x}{2} - \frac{3}{2} \right) \right) \mathbb{1}_{P_j} \left( -\frac{x}{2} - \frac{3}{2} \right) + \frac{1}{4} f \left( x, \left( \frac{3}{2} x + \frac{1}{2} \right) \right) \mathbb{1}_{P_j} \left( \frac{3}{2} x + \frac{1}{2} \right) \right\} dx, \]

\(^{10}\)The same holds as long as \( \rho > 2 \). Analogously, one can consider the case \( \rho < 2 \) and the MOT problem in (32) with max in place of min.

\(^{11}\)Identical results can be obtained using the right-curtain coupling
\[ \pi^{rc}(dx, dy) := \frac{1}{2} \mathbb{1}_{[-1,1]}(x) \left( \frac{3}{4} \delta_{x+1}(dy) + \frac{1}{4} \delta_{x-1}(dy) \right) dx. \]
we have that

\[ \omega_{X,i}^n = \mu \left( P_{1,i}^n \right) = \int_{P_{1,i}^n} \mu(dx), \]

\[ x_i^n = \frac{1}{\omega_{X,i}^n} \int_{P_{1,i}^n} x \mu(dx), \]

\[ \omega_{Y,i,j}^n = \tilde{\pi} \left( P_{1,i}^n \times P_{2,j}^n \right) = \int_{P_{1,i}^n} \frac{1}{2} \left( \mathbb{1}_{P_{2,j}^n} \left( \frac{3x}{2} + \frac{1}{2} \right) + \mathbb{1}_{P_{2,j}^n} \left( \frac{3x}{2} - \frac{1}{2} \right) \right) \mu(dx), \]

\[ y_{i,j}^n = \frac{1}{\omega_{Y,i,j}^n} \int_{P_{1,i}^n} \int_{P_{2,j}^n} y \tilde{\pi}(dx, dy) \]

\[ = \frac{1}{\omega_{Y,i,j}^n} \int_{P_{1,i}^n} \int_{P_{2,j}^n} \left( \frac{1}{2} \left( \frac{3x}{2} - \frac{1}{2} \right) \mathbb{1}_{P_{2,j}^n} \left( \frac{3x}{2} - \frac{1}{2} \right) + \frac{3}{4} \mathbb{1}_{P_{2,j}^n} \left( \frac{3x}{2} + \frac{1}{2} \right) \right) \mu(dx). \]

This allows us to generate finitely-supported \((\mu_n)_n, (\nu_n)_n\) and subsequently solve the primal LP corresponding to the discretised MOT \(P(\mu_n, \nu_n)\) as shown in Appendix A.

In Figure 1a, we plot the values \(P_n := P(\mu_n, \nu_n)\) of the primal LP as a function of quantisation step \(n\) which shows the numerical convergence of \(P_n\) to \(P = 1\). The heat map of the optimiser is displayed in Figure 1b for \(n = 100\). As expected, the optimiser \(p_n\) (i.e. the solution of the LP) is entirely concentrated on the lines \(y = x \pm 1\) which represent the support of the martingale optimal transport \(\pi^*\) in (33).

![Figure 1a](image)

![Figure 1b](image)

Figure 1.: Results of Example 23: (a) Values of \(P_n\) for \(5 \leq n \leq 200\); (b) Heat map of the optimiser \(p_n\) for \(n = 100\) (the two red lines \(y = x \pm 1\) represent the support of the optimal transport \(\pi^*\)).

**Example 24.** As a second example, we still consider an MOT problem on the real line with same cost function as in the previous example but with marginals given by two gaussian distributions \(X \sim N(0, 1)\) and \(Y \sim N(0, 2)\). In this case, we do not know a priori a martingale transport \(\tilde{\pi}\) between \(\mu\) and \(\nu\) in order to start our martingale quantisation scheme. However, we can easily obtain such one by exploiting the fact that
if $Z \sim N(0,1)$ independent of $X$, we have that $Y = X + Z$ so that the law of $Y$ is given by the convolution between the law of $X$ and the law of $Z$. A martingale transport can be thus obtained as
\[ \pi(dx, dy) = \mu(dx)\eta(x + dy) := \int_{\mathbb{R}} \delta_{x+z}(dy)\eta(dz), \]
where $\eta$ stands for the law of $Z$. As before, we first apply our martingale quantisation scheme and then solve the corresponding LP. We choose to quantise $\mu$ using the Voronoi ($L^2$-) quantisation so that $\{P^n_{1,i}\}$ is given by optimal quantisation grid of the standard normal distribution with size $n^{12}$. Moreover, choosing $\{P^n_{2,j}\} = \{P^n_{1,i}\}$, the expression for $\nu$ can be easily obtained from
\[
\omega^n_{i,j} := \pi(P^n_{1,i} \times P^n_{2,j}) = \int_{P^n_{1,i}} \mu(dx) \int_{P^n_{2,j}} \eta(x + dy) = \int_{P^n_{1,i}} \mu(dx)\eta(x + P_{2,j})
\]
\[
y^n_{i,j} = \frac{1}{\omega^n_{i,j}} \int_{P^n_{1,i} \times P^n_{2,j}} y\pi(dx, dy) = \frac{1}{\omega^n_{i,j}} \int_{P^n_{1,i}} \mu(dx) \int_{P^n_{2,j}} \eta(x + dy)y
\]
Analogously as before, the results are presented in Figure 2.

![Figure 2: Results of Example 24](image)

(a) Values of $P_n$ for $10 \leq n \leq 100$; (b) Heat map of the optimiser $p_n$ for $n = 100$.

**Example 25.** We now consider the equivalent of Example 23 but with marginal distributions supported on $\mathbb{R}^2$. This MOT problem was also studied in [3, Example 5.2]. Consider $\mu, \nu$ be two uniform distribution on $[-1,1]^2$ and $[-2,2]^2$, respectively and the MOT problem with cost function $c(x, y) = |x_1 - y_1|^\rho + |x_2 - y_2|^\rho$ where $x = (x_1, x_2), y = (y_1, y_2)$ and $\rho = 2.3$. It can been shown that the martingale optimal transport is given by
\[
\pi^* (dx, dy) = \mu(dx) \sum_z \delta_{(x+z)}(dz) \tag{34}
\]

---

12Precomputed quantisation grids for $N(0, I_d)$ for $d = 1, \ldots, 10$ and $n = 1, \ldots, 104$ can be downloaded from [www.quantize.maths-fi.com/](http://www.quantize.maths-fi.com/).
where \( z \) is a Rademacher distribution on \( \mathbb{R}^2 \) i.e. each \( z_i \) is such that \( z_i = 1 \) or \( z_i = -1 \) with probability \( 1/2 \). Although it might seem trivial, we used the optimal transport \( \pi^* \) as initial transport for the martingale quantisation and then solved the corresponding LP. In Figure 3, we exhibit the heat-map of the points \((x_2 - x_1, y_2 - y_1)\) under the optimiser \( p_n \) for \( n = 100 \). The red lines \((y = \pm 2, y = x)\) represent the theoretical support of the projection of the optimal transport \( \pi^* \) on \((x_2 - x_1, y_2 - y_1)\).

![Figure 3: Results of Example 25: heat map of the projection of the optimiser \( p_n \) on \((x_2 - x_1, y_2 - y_1)\) for \( n = 100 \) (the red lines \( y = x \pm 2, y = x \) represent the theoretical support).](image)

### 6. Barycentric quantisation

In the previous section, we saw how, given \( \mu \leq_{cx} \nu \) and partitions \( \Pi_1, \Pi_2 \) of \( B \), we can explicitly build some \( \hat{\mu}, \hat{\nu} \) as laws of some \( \hat{X}, \hat{Y} \), which we build starting from \( X \sim \mu, Y \sim \nu \) such that \( \mathbb{E}[Y|X] = X \). If for \( i = 1, 2 \) we choose \( \Pi_i = \Pi_i^n \) for some filtration \( (\Pi_i^n)_n \) which generates \( \mathcal{B}(B) \), the above construction provides the martingale quantisation. Since the constructions \((\mu, \nu) \mapsto (\hat{\mu}, \hat{\nu})\) and \((X, Y) \mapsto (\hat{X}, \hat{Y})\) have proved to be useful, in this section we define them in somewhat more abstract terms as ‘barycentric quantisations’, studying their properties and showing in Proposition 36 and 38 that one can characterise which measures \( \zeta \) such that \( \zeta \leq_{cx} \gamma \) are finitely-supported (resp. supported on at most \( n \) points) using barycentric quantisations of \( \gamma \), but not using only proper barycentric quantisations (see Example 28).
6.1. Preliminaries on quantisation theory

Quantisation theory aims to represent a continuous signal with a discrete one trying to loose less information as possible. When the signal is described by a random variable $G$, and $n \in \mathbb{N}$, a $n$-quantisation of $G$ is a $\sigma(G)$-measurable random variable $Z$ which takes at most $n$ values. Equivalently $Z = q(G)$ for some Borel-measurable function $q$ which takes at most $n$ values, i.e. $q = \sum_{j=1}^n z_j 1_{P_j}$ for some $z_j \in B$ and a partition $P_1, \ldots, P_n$ of $B$ made of Borel sets. We refer to [20] for a comprehensive treatment of the quantisation theory in the finite-dimensional case (also known as vector quantisation theory) and [33] for its extension to the infinite-dimensional setting.

If the signal is represented by a probability $\gamma$, we say that a probability $\zeta$ is a $n$-quantisation of $\gamma$ if $\zeta = \sum_{j=1}^n \gamma(P_j) \delta_{z_j}$ for some $z_j \in B$ and a partition $P_1, \ldots, P_n$ of $B$ made of Borel sets.

**Remark 26.** These two notions are of course closely linked:

1. If $Z$ is a $n$-quantisation of $G$, the law of $Z$ is a quantisation of the law of $G$.
2. If $\zeta$ is a $n$-quantisation of $\gamma$, then there exist $G$ with law $\gamma$, and $Z$ which is a $n$-quantisation of $G$ and whose law is $\zeta$.

6.2. Barycentric quantisation

The quantisation $\zeta = \sum_j \gamma(P_j) \delta_{z_j}$ of $\gamma$ satisfies $\zeta \leq_{\text{lex}} \gamma$ if $z_j$ is the barycentre of the probability $\gamma_j := \frac{1}{\gamma(P_j)} \cdot \gamma$, for each $j$. Indeed, Jensen inequality states that $\delta_{z_j} \leq_{\text{lex}} \gamma_j$, from which $\zeta \leq_{\text{lex}} \gamma$ follows trivially by linearity of the integral. This leads us to the following definition.

**Definition 27.** We will say that $\zeta$ is the proper barycentric quantisation of $\gamma \in \mathcal{P}_1(B)$ with respect to $\Pi := (P_j)_{j=1}^n$ if $\Pi \subseteq B(B)$ is a partition of $B$ and $\zeta$ is of the form

$$
\zeta = \sum_j \gamma(P_j) \delta_{z_j}, \quad \text{with} \quad z_j := \frac{1}{\gamma(P_j)} \int_{P_j} y \gamma(dy),
$$

(35)

where the sum runs over those values of $j = 1, \ldots, n$ for which $\gamma(P_j) > 0$ (and for such values $z_j$ is defined). Notice that in this case $\zeta \in \mathcal{P}_1(B)$, since $\zeta$ is finitely-supported.

Consider a probability measure $\gamma \in \mathcal{P}_1(B)$ and the set

$$
\mathcal{P}_{\text{cx}}^{(n)}(\gamma) := \{ \zeta \in \mathcal{P}_1(B) : |\text{supp}(\zeta)| \leq n \in \mathbb{N}, \; \zeta \leq_{\text{lex}} \gamma \},
$$

of probability measures, supported on at most $n \in \mathbb{N}$ points, which are smaller than $\gamma$ in convex order. The following example shows that a generic element of $\mathcal{P}_{\text{cx}}^{(n)}(\gamma)$ cannot be obtained as a quantisation of $\gamma$.

\text{[13]} Just take $Z = q(X)$ for $q = \sum_{j=1}^n z_j 1_{P_j}$, where $z_j, P_j$ are given by the identity $\zeta = \sum_j \gamma(P_j) \delta_{z_j}$. 

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Example 28. Consider the following probability measures on $\mathcal{B}(\mathbb{R})$

$$
\mu(dx) = \frac{1}{4}\delta_{(-1)}(dx) + \frac{1}{2}\delta_0(dx) + \frac{1}{4}\delta_1(dx),
$$

$$
\nu(dy) = \frac{1}{2}\left(\delta_{(-1)}(dy) + \delta_1(dy)\right).
$$

and the kernel $\kappa_x(dy)$ defined by

$$
\kappa_x(dy) = \begin{cases} 
\delta_{(-1)}(dy) & \text{if } x = -1 \\
\frac{1}{2}\left(\delta_{(-1)}(dy) + \delta_1(dy)\right) & \text{if } x = 0 \\
\delta_1(dy) & \text{if } x = 1.
\end{cases}
$$

Since $\nu$ can be obtained from $\mu$ by splitting the mass concentrated at $x = 0$ and sending it in equal parts to the points $x = \pm 1$ (and not moving the mass concentrated at $x = \pm 1$) we have that $\nu(dy) = \int_{\mathbb{R}} \mu(dx)\kappa_x(dy)$. Since the measure $\kappa_x(dy)$ has barycentre $x$ for all $x$, and Jensen inequality gives $\delta_x \leq cx \kappa_x$, integrating in $\mu(dx)$ we get that $\mu \leq cx \nu$ (by linearity of the integral), and so $\mu \in \mathcal{P}_c^3(\nu)$. However, $\mu$ can not be obtained as a quantisation of $\nu$, since $\mu$ is supported on strictly more points than $\nu$.

Luckily, the above example suggests a construction which allows to obtain all of $\mathcal{P}_c^{(3)}(\gamma)$. Indeed, notice that if

$$
\pi(dx, dy) := \mu(dx)\kappa_x(dy), \quad P_1 = (-\infty, 0), \quad P_2 = \{0\}, \quad P_3 = (0, \infty),
$$

then $\mu = \sum_{j=1}^3 \pi_j\delta_{y_j}$, where $\pi_j := \pi(P_j \times \mathbb{R}) = \mu(P_j)$ and $y_1 = \{-1\}, y_2 = \{0\}, y_3 = \{1\}$ can be expressed as

$$
y_j := \frac{1}{\pi_j} \int_{P_j \times \mathbb{R}} y \pi(dx, dy), \quad j = 1, 2, 3.
$$

This leads us to the following definition.

**Definition 29.** Given $\gamma, \zeta \in \mathcal{P}_1(\mathcal{B})$, with $\zeta$ finitely-supported (and thus of the form $\zeta = \sum_{i=1}^n w_i \delta_{z_i}$), we will call $\zeta$ the barycentric quantisation of $\gamma$ with respect to $(\pi, \Pi_1, \Pi_2)$ if $\pi$ is a probability on $\mathcal{B} \times \mathcal{B}$ whose second marginal is $\gamma$ and $\zeta$ is of the form

$$
\zeta = \sum_{i \in I, j \in J: \pi_{i,j} > 0} \pi_{i,j} \delta_{y_{i,j}},
$$

where $\pi_{i,j} := \pi(P_{i,i} \times P_{2,j})$ for all $i \in I, j \in J$, and

$$
y_{i,j} := \frac{1}{\pi_{i,j}} \int_{P_{i,i} \times P_{2,j}} y \pi(dx, dy), \quad \text{for } i \in I, j \in J: \pi_{i,j} > 0,
$$

for some finite partitions $\Pi_1 := (P_{i,i})_{i \in I}, \Pi_2 := (P_{2,j})_{j \in J} \subseteq \mathcal{B}(\mathcal{B})$ of $\mathcal{B}$. 

27
As will we see in Remark 37, the concept of barycentric quantisation generalises that of proper barycentric quantisation. The two concepts are not equivalent since, as we saw, \( \mu \) in Example 28 is a barycentric quantisation of \( \nu \), but it is not a proper barycentric quantisation of \( \nu \).

In Remark 26 we saw that the notion of quantisation for measures admits a corresponding notion of quantisation for random variables. As we will now see, the same is true for the barycentric quantisation; this link is worth developing because, even if ultimately we are interested in the measures themselves, in proofs it is often easier to work with random variables. This is of course the same reason why Strassen’s Theorem, and Skorokhod’s representation Theorem, are so useful.

We will need the following simple result.

**Lemma 30.** \( H \subseteq \mathcal{F} \) is a finite partition of \( \Omega \) if and only if \( H = W^{-1}(\Pi) \) for some finite partition \( \Pi \) of \( B \) and \( B \)-valued random variable \( W \). It is always possible to choose \( \Pi, W \) such that \( W \) takes exactly one value in each element of \( \Pi \); in this case \( H \) and \( \Pi \) have the same number of elements, and \( H \) is the family of sets of the form \( \{ W = w \} \), where \( w \) ranges over the image of \( W \).

If \( G \) is a \( B \)-valued random variable which takes finitely many values and \( H \subseteq \mathcal{F} \) a finite partition of \( \Omega \), then \( H \subseteq \sigma(G) \) if and only if \( H = G^{-1}(\Pi) \) for some finite partition \( \Pi \subseteq \mathcal{B}(B) \) of \( B \).

**Proof.** Trivially, if \( \Pi \) is a finite partition of \( B \) and \( W \) a \( B \)-valued random variable then \( H := W^{-1}(\Pi) \) is a partition of \( \Omega \), and if \( W \) takes exactly one value in each element of \( \Pi \) then \( H = \{ \{ W = w \} : w \in \text{Im}(W) \} \), and so \( H \) has as many elements as \( \Pi \).

Conversely, given a partition \( H = \{ H_i \}_{i=1}^n \subseteq \mathcal{F} \) of \( \Omega \), one can build as follows a finite partition \( \Pi \) of \( B \) and a \( W \) which takes exactly one value in each element of \( \Pi \) and is such that \( H = W^{-1}(\Pi) \): choose distinct points \( b_1, \ldots, b_n \in B \) and a partition \( \Pi := \{ P_i \}_{i=1}^n \subseteq \mathcal{B}(B) \) of \( B \) such that \( b_i \in P_i \) for all \( i \) (for example one can take \( P_i = \{ b_i \} \) for \( i < n \), and \( P_n = B \\setminus \{ b_1, \ldots, b_{n-1} \} \)), and define \( W := \sum_{i=1}^n b_i 1_{P_i} \).

Now consider the second statement: let \( G \) takes finitely-many values and \( H \subseteq \mathcal{F} \) be a finite partition. If \( H = G^{-1}(\Pi) \) for some finite partition \( \Pi \) of \( B \) then \( H \subseteq \sigma(G) \) follows from \( \Pi \subseteq \mathcal{B}(B) \). Conversely, if \( H \subseteq \sigma(G) \) then each element \( H_i \) of \( H = \{ H_i \}_{i=1}^n \) is a union of atoms of \( \sigma(G) \), and thus is of the form \( \bigcup_{g \in S_i} \{ G = g \} = \{ G \in S_i \} \) for some \( S_i \subseteq \text{Im}(G) \). Since \( H_i \cap H_j = \emptyset \) for \( i \neq j \), if \( P_i := S_i \setminus \bigcup_{j<i} S_j \) for \( i < n \) and \( P_n = B \setminus \bigcup_{i<n} P_i \), then \( \{ G \in S_i \} = \{ G \in P_i \} \), and so \( \Pi := \{ P_i \}_{i=1}^n \) is a partition of \( B \) such that \( H = G^{-1}(\Pi) \).

**Definition 31.** Given \( Z, G \in L^1(\mathbb{P}, B) \), \( n \in \mathbb{N} \), we will call \( Z \) a barycentric quantisation of \( G \) with respect to \( \mathcal{G} \) if \( Z \) is of the form

\[
Z = \mathbb{E}[G|\mathcal{G}], \tag{38}
\]

for some finite \( \sigma \)-algebra \( \mathcal{G} \subseteq \mathcal{F} \). We will say that a barycentric quantisation \( Z \) of \( G \) with respect to \( \mathcal{G} \) is proper if \( \mathcal{G} \subseteq \sigma(G) \).
Remark 32. If $Z$ a barycentric quantisation of $G$ with respect to $\mathcal{G}$ then it is also a barycentric quantisation of $G$ with respect to $\sigma(Z)$ since

$$Z = \mathbb{E}[Z|\sigma(Z)] = \mathbb{E}[\mathbb{E}[G|\mathcal{G}]|\sigma(Z)] = \mathbb{E}[G|\sigma(Z)],$$

Since $Z$ takes\textsuperscript{14} at most as many values as the number of atoms of $\mathcal{G}$ (because $Z = \mathbb{E}[G|\mathcal{G}]$ is constant on each atom of $\mathcal{G}$), $\sigma(Z)$ has at most as many atoms as $\mathcal{G}$ (since the atoms of $\sigma(Z)$ are the sets of the form $\{Z = z\}$, for $z$ in the support of $L(Z)$).

The two definitions of barycentric quantisation (for probabilities on $B$, and for $B$-valued random variables) are strictly connected, as we now explain first considering a general barycentric quantisation, and then considering a proper one.

Remark 33. Let $Z$ be the barycentric quantisation of $G$ with respect to the finite $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$. If $H$ is the set of atoms of $\mathcal{G}$, Lemma 30 gives that $H = W^{-1}(\Pi)$ for some $\Pi$ and $W$, so $\mathcal{G} = \sigma(W^{-1}(\Pi))$, and so by Lemma 15 the law $\zeta$ of $Z$ is a barycentric quantisation of the law $\gamma$ of $G$ with respect to $\gamma = (\pi, \Pi, \{B\})$, where $\pi$ is the law of $(W, G)$. Moreover, when we applied Lemma 30 to get $\Pi$ and $W$, we could have chosen to have that $W$ takes one value in each element of $\Pi$ and thus $\Pi$ has as many element as the atoms of $\mathcal{G}$.

Conversely, let $\zeta$ be a barycentric quantisation of $\gamma$ with respect to $\gamma = (\pi, \Pi_1, \Pi_2)$. If $X, G$ are the projections on the first and second coordinates, defined on the probability space $B \times B$ endowed with the probability $\pi$, and $Z := \mathbb{E}[G|\mathcal{G}]$ where $\mathcal{G} := \sigma(\{P_{1,i} \times P_{2,j}\})$, then trivially $G$ has law $\gamma$ and $Z$ is a barycentric quantisation of $G$ with respect to $\mathcal{G}$; moreover $Z$ has law $\zeta$, since

$$Z = \sum_{i,j} 1_{P_{i,j}} \frac{\mathbb{E}[1_{P_{i,j}}(X)1_{P_{2,j}}(G)|\mathcal{G}]}{\mathbb{P}(P_{1,i} \times P_{2,j})} = \sum_{i,j} 1_{P_{i,j}} y_{i,j},$$

where $y_{i,j}$ are as in (37).

Remark 34. If $Z$ is a proper barycentric quantisation of $G$ with respect to $\mathcal{G}$ then, calling $H$ the family of atoms of $\mathcal{G}$, we can write $\mathcal{G} = \sigma(G^{-1}(\Pi))$ for some partition $\Pi \subseteq B(B)$ of $B$ using Lemma 30, and then the law $\zeta$ of $Z$ is a barycentric quantisation of the law $\gamma$ of $G$ with respect to $\Pi$ by Lemma 15.

Conversely, if $\zeta$ is a proper barycentric quantisation of $\gamma$ with respect to the partition $\Pi \subseteq B(B)$ of $B$, and $G$ has law $\gamma$, then $Z := \mathbb{E}[G|\sigma(G^{-1}(\Pi))]$ is a proper barycentric quantisation of $G$ with respect to $\mathcal{G} := \sigma(G^{-1}(\Pi))$ since $\sigma(G^{-1}(\Pi)) \subseteq \sigma(G)$, and $Z$ has law $\zeta$ by Lemma 15.

Remark 35. Remarks 33 and 34 imply that the estimates of the speed of convergence provided in Theorem 16 hold whenever $\tilde{\nu}/\tilde{Y}$ is a barycentric quantisation (resp. $\tilde{\mu}/\tilde{X}$ is a proper barycentric quantisation) of $\nu/Y$ (resp. of $\mu/X$). Since any proper barycentric quantisation is a barycentric quantisation, the last lines of the proof of Theorem 16 should now appear obvious.

\textsuperscript{14}Of course $Z$ is an equivalence class, so by ‘$Z$ takes at most $n$ values’ we mean that it admits a representative which takes at most $n$ values, or equivalently that the law of $Z$ is supported on at most $n$ points.
Theorem 36. If \( \zeta, \gamma \in \mathcal{P}_1(B) \) then t.f.a.e.:

1. \( \zeta \leq_{cx} \gamma \) and \( \zeta \) is finitely-supported

2. \( \zeta \) is a barycentric quantisation of \( \gamma \) with respect to \((\pi, \Pi_1, \Pi_2)\), for some \( \pi \in \mathcal{P}_1(B) \) with second marginal \( \gamma \) and \( \Pi_1, \Pi_2 \subseteq \mathcal{B}(B) \) finite partitions of \( B \).

3. \( \zeta \) is a barycentric quantisation of \( \gamma \) with respect to \((\pi, \Pi_1, \{B\})\), for some \( \pi \in \mathcal{P}_1(B) \) with second marginal \( \gamma \) and \( \Pi_1 \subseteq \mathcal{B}(B) \) finite partition of \( B \).

Proof. Trivially, item 3 implies item 2. If item 2 holds, by item 33 there exist \( Z \sim \zeta, G \sim \gamma \) and a finite \( \sigma \)-algebra \( \mathcal{G} \subseteq \mathcal{F} \) such that \( \mathbb{E}[G|\mathcal{G}] = Z \), and so item 1 follows from the easy half of Strassen’s theorem (Theorem 4). Finally, if item 1 holds then the hard half of Strassen’s theorem implies the existence of \( Z \sim \zeta, G \sim \gamma \) such that \( \mathbb{E}[G|Z] = Z \); since \( \zeta \) is finitely supported, \( \mathcal{G} := \sigma(Z) \) is finite and so by Remark 33 \( \zeta \) is a barycentric quantisation of \( \gamma \) with respect to \((\pi, \Pi_1, \{B\})\) for some \( \pi, \Pi_1 \).

Remark 37. Notice that the Proposition 36 shows that, in the definition of barycentric quantisation of \( \gamma \), we could equivalently have demanded that the first marginal of \( \pi \) has finite support and \( \Pi_2 = \{B\} \). However, doing this is not a good idea. Indeed, allowing for a more general \( \Pi_2 \) allows to more easily to consider settings which naturally involve two partitions, for example making it clear that if \( \zeta \) is a proper barycentric quantisation of \( \gamma \) then \( \zeta \) is a barycentric quantisation of \( \gamma \); moreover, the choice of \( \Pi_2 \) strongly affect the upper bounds we discussed in Theorem 35, so in this regard \( \Pi_2 \) should rather be chosen to have sets as small (in diameter) as possible, instead of being composed of only one set \( B \) (of infinite diameter).

The next proposition shows that the situation of Example 28 is standard, and makes Proposition 36 slightly more precise, as it keeps track of the number of points in the support of a finitely-supported measure.

Proposition 38. The set \( \mathcal{P}_{cx}^{(n)}(\gamma) \) coincides with the set of barycentric quantisations of \( \gamma \) with respect to \((\pi, \Pi_1, \{B\})\), where \( \Pi_1 \) spans all partitions of \( B \) made of at most \( n \) elements and \( \pi \) spans all probabilities on \( B \times B \) whose second marginal is \( \gamma \).

Proof. We first show that the stated barycentric quantisations are in \( \mathcal{P}_{cx}^{(n)}(\gamma) \); then we prove the converse, i.e. that every \( \zeta \in \mathcal{P}_{cx}^{(n)}(\gamma) \) comes from a barycentric quantisation of the kind considered in the statement.

If \( \zeta \) is a barycentric quantisations of \( \gamma \) with respect to \((\pi, \Pi_1, \{B\})\), by definition the number of points in the support of \( \zeta \) equals the number of \( P_i \in \Pi_1 \) such that \( \pi(P_i \times B) > 0 \); in particular, if \( \Pi_1 \) has at most \( n \) elements then \( \zeta \) is supported on at most \( n \) points. Since Proposition 36 shows that \( \zeta \leq_{cx} \gamma \), we proved \( \zeta \in \mathcal{P}_{cx}^{(n)}(\gamma) \).

Conversely, if \( \zeta \in \mathcal{P}_{cx}^{(n)}(\gamma) \), by Strassen’s Theorem (i.e. Theorem 4) there exists \( G, Z \) with laws \( \gamma, \zeta \) and such that \( \mathbb{E}[G|Z] = Z \), so \( Z \) is a barycentric quantisation of \( G \) with respect to \( \mathcal{G} := \sigma(Z) \). If \( \{z_i\}_{i=1}^k \) is the support of \( \zeta \) (so that \( k \leq n \)), the set of atoms \( \{Z = z_i\}_{i=1}^k \) of \( \mathcal{G} \) equals \( Z^{-1}(\Pi_1) \), where \( \Pi_1 = (P_i)_{i=1}^k \) is the partition of \( B \) given by
Given the link between barycentric quantisations for measures and for random variables expressed in Remark 33, it is natural to expect a version of Proposition 36 for random variables. The reason why such version is not immediately obvious, is that we have so far not seen a definition of convex order for random variables analogous to that for measures, so we now introduce it.

**Definition 39.** Given $Z, G \in L^1(B, \mathbb{P})$, we say that $Z \leq_{cx} G$ if there exists a $\sigma$-algebra $G \subseteq \mathcal{F}$ such that $f(Z) \leq E[f(G)|G]$ holds for every $f : B \to \mathbb{R}$ Lipschitz and convex.

The following simple fact is the analogue for random variables of the fact that $\mu \leq_{cx} \nu$ if and only if there exists a kernel $K$ such that $\nu = \mu K$ and $\text{bar}(K_x) = x$ for $\mu$ a.e. $x$.

**Lemma 40.** Given $Z, G \in L^1(B, \mathbb{P})$, t.f.a.e.:

1. $Z \leq_{cx} G$
2. $Z = E[G|G]$ for some $\sigma$-algebra $G \subseteq \mathcal{F}$

If the above conditions hold, then $f(Z) \leq E[f(G)|Z]$ holds for every $f : B \to \mathbb{R}$ Lipschitz and convex, and in particular $Z = E[G|Z]$.

**Proof.** Assume item 2 holds. The conditional Jensen inequality implies item 1, with the same $G$; since the tower property of conditional expectation shows that $Z = E[G|Z]$, one can take w.l.o.g. $G = \sigma(Z)$ also in (the definition used in) item 1. Conversely, applying the inequality $f(Z) \leq E[f(G)|G]$ to $f(x) = x$ and to $f(x) = -x$, gives $Z = E[G|G]$. 

Analogously we can state the equivalent of Proposition 36 for random variables.

**Corollary 41.** Given $Z, G \in L^1(B, \mathbb{P})$, t.f.a.e.:

1. $Z \leq_{cx} G$ and $Z$ only takes finitely many values
2. $Z$ is barycentric quantisation of $G$

**Proof.** If item 1 holds then Theorem 40 shows that $Z = E[G|Z]$; since $\sigma(Z)$ takes finitely many values, item 2 holds. Conversely, if item 2 holds then $Z$ takes finitely many values (since it is $G$-measurable), and Theorem 40 implies $Z \leq_{cx} G$.

**A. Discrete MOT - primal LP**

Consider the MOT problem

$$\inf_{\pi \in M(\mu, \nu)} E[c(X, Y)]$$
For a given $n \in \mathbb{N}$, a (sub-optimal) martingale transport $\tilde{\pi} \in \mathcal{M}(\mu, \nu)$ and two finite partitions $\Pi^1_n, \Pi^2_n$ of $B$, we can apply the proposed martingale quantisation scheme so that to reduce the above MOT to the following LP

$$
\begin{align*}
\min_{p \in \mathbb{R}^{n_\mu \times n_\nu}_{+}} & \sum_{i=1}^{n_\mu} \sum_{j=1}^{n_\nu} p_{i,j} c_{i,j} \\
\text{subject to } & \sum_{j} p_{i,j} = \omega_{X,i}^{(n_\mu)} \quad i = 1, \ldots, n_\mu \\
& \sum_{i} p_{i,j} = \omega_{X,i}^{(n_\nu)} \quad j = 1, \ldots, n_\nu \\
& \sum_{j} p_{i,j} y_j^{(n_\nu)} = \omega_{X,i}^{(n_\mu)} x_i^{(n_\mu)} \quad i = 1, \ldots, n_\mu
\end{align*}
$$

(MOT-LP)

where that $n_\mu := \#\text{supp}(\mu_n) \leq \#\Pi^1_n$, $n_\nu := \#\text{supp}(\nu_n) \leq \#\Pi^1_n \cdot \#\Pi^2_n$.

Problem (MOT-LP) can be readily implemented once recasted in the canonical form

$$
\begin{align*}
\min_{x} & \quad d^T x \\
\text{subject to } & \quad A x = b \\
& \quad x \geq 0
\end{align*}
$$

(39)

where $d$ and $x$ are (column) vectors (whereas $p$ and $c$ are matrices). To do so, it is sufficient to define $x := \text{vec}(p)$ and $d := \text{vec}(c)$, i.e. the vectorisations of the transport $p$ and the cost matrix $c$. The constraint matrix is given by $A := [A_{C_1}, A_{C_2}, A_{C_3}]^T$ where the matrices $A_{C_1}, A_{C_2}$ and $A_{C_3}$ (whose dimensions are: $n_\mu \times n_\mu n_\nu$, $n_\nu \times n_\mu n_\nu$ and $n_\mu \times n_\mu n_\nu$, resp.) are defined by
whereas $b := \begin{bmatrix} \omega_X^{(n_\mu)}, \omega_Y^{(n_\mu)}, \omega_X^{(n_\mu)} \cdot x^{(n_\mu)} \end{bmatrix}^T$.

**Remark 42.** Problem (MOT-LP) recasted in canonical form has $n_\mu n_\nu$ variables and $2n_\mu + n_\nu$ constraints.

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