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Renormalization of an effective Light-Cone QCD-inspired theory for the Pion and other Mesons

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Abstract

The renormalization of the effective QCD-Hamiltonian theory for the quark-antiquark channel is performed in terms of a renormalized or fixed-point Hamiltonian that leads to subtracted dynamical equations. The fixed point-Hamiltonian brings the renormalization conditions as well as the counterterms that render the theory finite. The approach is renormalization group invariant. The parameters of the renormalized effective QCD-Hamiltonian comes from the pion mass and radius, for a given constituent quark mass. The 1s and excited 2s states of $\bar{u}q$ are calculated as a function of the mass of the quark $q$ being s, c or b, and compared to the experimental values.
I. INTRODUCTION

The effective mass operator equation for the lowest Light-Front Fock-state component of a bound system of a constituent quark and antiquark of masses \(m_1\) and \(m_2\), obtained in the effective one-gluon-exchange interaction approximation, which is our starting point, was reviewed in Ref. [1]. The breakthrough to simplify this equation in the spin zero channel was achieved in [2] by formulating the ↑↓-model, which is reduced to

\[
M^2 \psi(x, \vec{k}_\perp) = \left[ \frac{\vec{k}_\perp^2 + m_1^2}{x} + \frac{\vec{k}_\perp^2 + m_2^2}{1-x} \right] \psi(x, \vec{k}_\perp) \\
- \int dx' d\vec{k}'_\perp \theta(x') \theta(1-x') \left( \frac{4m_1m_2}{3\pi^2} \frac{\alpha}{Q^2} - \lambda \right) \psi(x', \vec{k}'_\perp),
\]

(1)

where \(M\) is the mass of the bound-state and \(\psi\) is the projection of the light-front wavefunction in the quark-antiquark Fock-state. The mean four-momentum transfer is [3]

\[
Q^2(x, \vec{k}_\perp, x', \vec{k}'_\perp) = -\frac{1}{2} \left[ (k_1 - k'_1)^2 - (k_2 - k'_2)^2 \right].
\]

(2)

The coupling constant \(\alpha\) for the Coulomb-like potential and \(\lambda\) is the bare coupling constant of the Dirac-delta hyperfine interaction. For the purposes of this work, we leave out the energy transfer in Eq.(2), that leads to aditional singularities for the tridimensional momentum going to infinity, that we are not going to treat here. The effective mass operator of Eq.(1) acting in the quark-antiquark sector, has also been obtained by iterated resolvents, which were derived in [4] and presented in greater detail in [5], that allows to express systematically the higher Fock-state components of the wave-function in functionals of the lower ones. In this way, the higher Fock-state components can be retrieved from the \(q\bar{q}\) projection, and the full complexity of the QCD theory is in principle described by the effective Hamiltonian acting in the lowest Fock-state component [4].

For convenience, the Sawicki tranformation, first derived for equal masses [6] and consistently formulated for unequal masses in Ref. [3], is applied to Eq.(1) which allows to rewrite it in the instant-form momentum basis. It is useful in this case since the momentum transfer is approximated by a rotational invariant form given by Eq.(2):
\[ x(k_z) = \frac{(E_1 + k_z)}{E_1 + E_2}, \quad (3) \]

and the Jacobian of the transformation of \((x, \vec{k}_\perp)\) to \(\vec{k}\) is:

\[ dx d\vec{k}_\perp = \frac{x(1-x)}{m_r A(k)} d\vec{k}, \quad (4) \]

with the dimensionless function

\[ A(k) = \frac{1}{m_r} \frac{E_1 E_2}{E_1 + E_2}, \quad (5) \]

and the reduced mass \(m_r = m_1 m_2 / (m_1 + m_2)\). The individual energies are \(E_i = \sqrt{m_i^2 + k^2} (i=1,2)\) and \(k \equiv |\vec{k}|\).

The mass operator equation in instant form momentum variables is given by:

\[ M^2 \varphi(\vec{k}) = [E_1 + E_2]^2 \varphi(\vec{k}) - \int d\vec{k}' \left( \frac{4m_s}{3\pi^2} \frac{\alpha}{\sqrt{A(k)A(k')Q^2}} - \frac{\lambda}{m_r \sqrt{A(k)A(k')}} \right) \varphi(\vec{k}'), \quad (6) \]

where \(m_s = m_1 + m_2\), the phase-space factor is included in the factor \(1/\sqrt{A(k)A(k')}\) and \(\sqrt{x(1-x)} \psi(x, \vec{k}_\perp) = \sqrt{A(k)} \varphi(\vec{k})\).

The mass operator equation (6) needs to be regularized and renormalized in order to give physical results. In Ref. [2], the delta function was smeared out to a Yukawa form, regularizing Eq.(6) and the parameters found from the pion mass and radius. In principle the size parameter in momentum space of the Yukawa potential should be let to infinity while the physical input is kept constant. To make vanish the dependence on the size parameter is a non trivial task, which is the purpose of this work. Here we obtain a renormalized form of the equation for the bound state mass, which is  

i) invariant under renormalization group transformations,  

ii) the physical input is given by the pion mass and radius  

iii) no regularization parameter.

We are going to apply the recent renormalization techniques developed in the context non-relativistic Hamiltonian theory [7–9] to Eq.(6), since it defines an effective Hamiltonian for the quark-antiquark dynamics. The renormalized T-matrix is the solution of a subtracted scattering equation, which the physical input is given by the T-matrix at some reference scale.
\( \mu \). The scheme is invariant under the change of the arbitrary scale \( \mu \) and consequently the inhomogeneous term of the subtracted scattering equation satisfies a renormalization group equation, which expresses the matching of the theories at the scales \( \mu \) and \( \mu + d\mu \) \(^8\). We will find the mass of the bound state from the pole of the renormalized scattering matrix defined from the mass operator Eq.(6).

This work is organized as follows. In sec.II, the operators for the Coulomb-like and for the singular interactions appearing in the mass operator are defined, and the Lippman-Schwinger equation for the T-matrix related to the given mass operator is written. In sec. III, we show how to renormalized the effective theory defined by the mass operator in sec.II through the definition of a renormalized or fixed point singular interaction. This procedure is equivalent to use subtracted scattering equations. We discuss the renormalization group invariance of the method. The explicit form of the renormalized T-matrix is obtained (appendix A) and the physical input is the pion mass which determines the excited states as well as the mass of the other mesons. The physical observables of the renormalized effective theory does not depend on the subtraction point. In sect.IV, the calculation of the pion charge radius is discussed and although we have simplified the spin dependence in the dynamical equation it is important in the evaluation of the radius. For this purpose we have used an effective pseudo-scalar Lagrangian to construct the spin part of the pion wave-function, and turning off the Coulomb-like interaction we retrieve a well known result valid in the soft pion limit \(^{14}\). To gain insight we also write down the expressions where the quark spin is neglected.

The numerical results are presented in sec. V. We have solved a nonrelativistic example for a Coulomb plus a Dirac-delta interaction compared to a model in which the Dirac-delta is substituted by a Yukawa. We also show how this effective theory, including a Dirac-delta, can mimic a finite range theory, calibrated to the pion mass. Then, we present results for the pion charge radius and for the mass of the excited state, that we identified with the isovector vector mesons. Finally, in sec. VI, we present our conclusions.
II. DEFINITIONS

For our purposes it is convenient to work in an operator form of Eq.(6):

\[
(M_0^2 + V + V^\delta) |\varphi> = M^2 |\varphi>,
\]

and the free mass operator \( M_0 \) is the sum of the energies of quark 1 and 2, \( V \) the Coulomb-like potential and \( V^\delta \) is the Dirac-delta interaction in the non-relativistic limit. The matrix elements of these operators are given by:

\[
<k|V|k'> = -\frac{4m_s\alpha}{3\pi^2\sqrt{A(k)Q^2}\sqrt{A(k')}} ,
\]

and the short-range singular interaction

\[
<k|V^\delta|k'> = <\vec{k}|\chi> \frac{\lambda}{m_r} <\chi|\vec{k}'> = \frac{\lambda}{m_r} \frac{1}{\sqrt{A(k)}} \frac{1}{\sqrt{A(k')}} .
\]

The phase-space factor \( A(k) \) is defined by Eq.(3), the square momentum transfer \( Q^2 \) comes from Eq.(2). For convenience in the formal manipulations of the next section, the form-factor of the separable singular interaction is introduced and defined by \(<\vec{k}|\chi> = 1/\sqrt{A(k)}.\)

The T-matrix is obtained from the Lippman-Schwinger equation:

\[
T(M^2) = V + V^\delta + (V + V^\delta)G_0^{(+)}(M^2)T(M^2) ,
\]

where \( M \) is the mass of the scattering state and the Green’s function with outgoing wave boundary condition is

\[
G_0^{(+)}(M^2) = \frac{1}{M^2 - M_0^2 + i\varepsilon} .
\]

In the next section we will obtain a renormalized form of \( T(M^2) \).

III. RENORMALIZATION OF THE EFFECTIVE THEORY

The ideas that have been developed in Refs. [7–9] to construct a renormalized T-matrix in non-relativistic Hamiltonian theory, can be applied in the case where the two-body interaction can be split in a regular potential of finite range \( (V) \) and a Dirac-delta singularity
as in Eq.(10). In this case, the physical information at the subtraction point will be introduced through the renormalized mass operator as well as all the counterterms that render finite the T-matrix equation (10). The physical information at the subtraction point is the T-matrix corresponding to the Dirac-delta interaction. This renormalization approach has been applied in the two-nucleon system to calculate the T-matrix of the one-pion-exchange potential, without the necessity of regularization or form-factors [7].

The renormalized interaction is given by:

\[ V_R = V + V_R^\delta, \tag{12} \]

where the renormalized Dirac-delta interaction is given by:

\[
V_R^\delta = \frac{1}{1 + T_R^\delta(\mu^2)G_0^{(+)}(\mu^2)}T_R^\delta(\mu^2)
= T_R^\delta(\mu^2)\frac{1}{1 + G_0^{(+)}(\mu^2)T_R^\delta(\mu^2)}
= T_R^\delta(\mu^2)\sum_{n=0}^{\infty} \left[ -G_0(\mu^2)T_R^\delta(\mu^2) \right]^n , \tag{13} \]

and \( T_R^\delta(\mu^2) \) is the renormalized T-matrix of the Dirac-delta interaction, with matrix elements given by:

\[
< \vec{p}|T_R^\delta(\mu^2)|\vec{q}> = < \vec{p}|\chi > \lambda_R(\mu^2) < \chi|\vec{q}> , \tag{14} \]

where \( \lambda_R(\mu^2) \) is the renormalized strength of the Dirac-delta interaction at the mass scale \( \mu^2 \). In the non-relativistic limit the form-factor \( \chi(q) = 1 \), and the renormalized interaction becomes the Dirac-delta. The scattering equation with the renormalized interaction appears in a subtracted form [7–9] after a little rearrangement of terms, in which all the divergent momentum integrals are removed and it is written as

\[
T_R(M^2) = T_R(\mu^2) + T_R(\mu^2) \left( G_0^{(+)}(M^2) - G_0^{(+)}(\mu^2) \right) T_R(M^2). \tag{15} \]

It is presented here in a general way, where we have dropped the \( \delta \) superscript, just to remind the reader that for a regular potential Eq.(15) is completely equivalent to the traditional Lippman-Schwinger scattering equation.
The renormalized interaction is independent of the subtraction point, i.e., the physics expressed by the renormalized interaction is invariant by changes in the arbitrary renormalization point, this physical requirement is given by:

\[
\frac{d}{d\mu^2} V^\delta_R = 0 ; \tag{16}
\]

qualifying the interaction as the fixed-point of Eq.(16), which implies that the T-matrix found from the solution of

\[
T_R(M^2) = V + V^\delta_R + \left(V + V^\delta_R\right) G^{(+)}_0(M^2) T_R(M^2) \), \tag{17}
\]

is invariant under dislocations of the subtraction point. Consequently, the renormalized coupling constant of the Dirac-delta interaction changes as the subtraction point moves, according to the Callan-Symanzik equation

\[
\frac{d}{d\mu^2} T^\delta_R(\mu^2) = -T^\delta_R(\mu^2) \frac{1}{(\mu^2 + i\varepsilon - M_0^2)^2} T^\delta_R(\mu^2) \), \tag{18}
\]

obtained from Eq.(17).

Although the renormalized interaction is not well defined for singular interactions, the resulting T-matrix obtained by solving Eq.(17) is finite. This gives a posteriori justification for the formal manipulations used in (13) and (17). The sum in the expression of the renormalized interaction (13) explicits all the counterterms which exactly cancels the infinities in the momentum integrals of the scattering equation (17), while introducing the physical information through the value of the renormalized strength of the Dirac-delta interaction. We choose \(\lambda_R(\mu^2)\), in the following, in accordance with the physical value of the pion mass. We observe that, instead of working formally with the operator \(V^\delta_R\), we could use an ultraviolet momentum cut-off (\(\Lambda\)), by defining in this way a regularized interaction. After the construction of the T-matrix regularized equation one could perform the limit \(\Lambda \to \infty\), arriving at the same results as the ones obtained directly with the use of the renormalized interaction.
The solution of the scattering equation (17) is found by using the two potential formula in terms of the T-matrix of the regular potential $V$, $T^V(M^2)$ and the renormalized T-matrix of the Dirac-Delta interaction, see Appendix A, which results in

$$T_R(M^2) = T^V(M^2) + \frac{\left(1 + T^V(M^2)G_0^{(+)}(M^2)\right)|\chi> <\chi|\left(G_0^{(+)}(M^2)T^V(M^2) + 1\right)}{\lambda^{-1}_R(\mu^2) - <\chi|\left(\frac{1}{M^2+i\varepsilon-M_0^2} - \frac{1}{\mu^2+i\varepsilon-M_0^2}\right)|\chi> - <\chi|G_0^{(+)}(M^2)T^V(M^2)G_0^{(+)}(M^2)|\chi>},$$

(19)

where the $T^V(M^2)$ is the solution of the Lippman-Schwinger equation

$$T^V(M^2) = V + VG_0^{(+)}(M^2)T^V(M^2).$$

(20)

The structure of Eq.(19) allows one more subtraction in the denominator, which turns faster the convergence of the momentum integral in the term where $T^V(M^2)$ is present. This subtraction is appropriate if the potential $V$ has a Coulomb or Yukawa form,

$$<\vec{p}|V|\vec{q}> = \frac{1}{\eta^2 + |\vec{p} - \vec{q}|^2}. \quad (21)$$

Thus, we define the renormalized strength of the Dirac-delta interaction at the subtraction point such that,

$$\lambda^{-1}_R(\mu^2) = \lambda^{-1}_R(\mu^2) + <\chi|G_0^{(+)}(\mu^2)T^V(\mu^2)G_0^{(+)}(\mu^2)|\chi> \quad ; \quad (22)$$

and introduce the physical information in the renormalized T-matrix (19), through the value of $\overline{\lambda}^{-1}_R(\mu^2)$.

Substituting Eq.(22) in (19), we obtain the renormalized T-matrix written as:

$$T_R(M^2) = T^V(M^2) + \left(1 + T^V(M^2)G_0^{(+)}(M^2)\right)|\chi> t_R(M^2) <\chi|\left(G_0^{(+)}(M^2)T^V(M^2) + 1\right),$$

(23)

where,

$$t^{-1}_R(M^2) = \overline{\lambda}^{-1}_R(\mu^2) - <\chi|(G^{(+)}(M^2) - G^{(+)}(\mu^2))|\chi>. \quad (24)$$
and the interacting Green’s function is

\[ G^{V(+)}(M^2) = G_0^{(+)}(M^2) + G_0^{(+)}(M^2)T^V(M^2)G_0^{(+)}(M^2) . \]

(25)

We use the renormalization condition that at the pion mass, \( M = m_\pi \), the T-matrix, for

\[ m_1 = m_2 \text{ and } m_1 = m_u = m_d \text{ has the bound-state pole, consequently} \]

\[ t_\mathcal{R}^{-1}(m_\pi^2) = 0 , \]

(26)

and choosing \( \mu = m_\pi \) for convenience, which implies that

\[ \overline{\lambda}_\mathcal{R}^{-1}(m_\pi^2) = 0 . \]

(27)

The invariance of the renormalized T-matrix \([23]\) under the dislocation of the subtraction point, just reads that

\[ \frac{d}{d\mu^2} t_\mathcal{R}(M^2) = 0 , \]

(28)

and from \([28]\)

\[ \overline{\lambda}_\mathcal{R}^{-1}(\mu^2) = \overline{\lambda}_\mathcal{R}^{-1}(\mu^2) - \langle \chi | \left( G^{V(+)}(\mu^2) - G^{V(+)}(\mu^2) \right) | \chi \rangle . \]

(29)

At the general subtraction point \( t_\mathcal{R}^{-1}(\mu^2) = \overline{\lambda}_\mathcal{R}^{-1}(\mu^2) \) and the renormalized T-matrix at \( \mu^2 \) is given by:

\[ T_\mathcal{R}(\mu^2) = T^V(\mu^2) + \left( 1 + T^V(\mu^2)G_0^{(+)}(\mu^2) \right) | \chi > \overline{\lambda}_\mathcal{R}(\mu^2) < \chi | \left( G_0^{(+)}(\mu^2)T^V(\mu^2) + 1 \right) . \]

(30)

The full renormalized interaction can be written in a form analogous to Eq.\([13]\):

\[ V_\mathcal{R} = T_\mathcal{R}(\mu^2) \sum_{n=0}^{\infty} \left[ -G_0^{(+)}(\mu^2)T_\mathcal{R}(\mu^2) \right] ^n , \]

(31)

and with that, one could obtain the equation for the renormalized T-matrix \([17]\) in the subtracted form \([13]\), displayed again below

\[ T_\mathcal{R}(M^2) = T_\mathcal{R}(\mu^2) + T_\mathcal{R}(\mu^2) \left( G_0^{(+)}(M^2) - G_0(\mu^2) \right) T_\mathcal{R}(M^2) . \]
We observe that one could equally well construct the Callan-Symanzik equation for \( \frac{d}{d\mu^2} T_R(\mu^2) \), by performing the limit of \( M \to \mu \) in (15), finding

\[
\frac{d}{d\mu^2} T_R(\mu^2) = -T_R(\mu^2) \frac{1}{(\mu^2 + i\varepsilon - M_0^2)^2} T_R(\mu^2) ,
\]

with the boundary condition given by Eq.(30). The solution of Eq.(32) gives the dependence of \( \bar{\lambda}_R \) on the subtraction point \( \mu \) as expressed by Eq.(29).

Now comes a subtle point: It is important to realize that the renormalization condition given by \( \bar{\lambda}_R^{-1}(m_\pi^2) = 0 \), and considering Eq.(24), the bare strength of the Dirac-delta interaction is given by

\[
m_{\text{bare}} \lambda_R^{-1} = \left[ \frac{1}{m_r} < \chi|G^V(m_\pi^2)|\chi> \right]_{m_u,m_d} - \left[ \frac{1}{m_r} < \chi|G^V(M_b^2)|\chi> \right]_{m_u,m_d} = 0 .
\]

The zeroes of \( t_R^{-1} \) give the position of the zero angular momentum states of the bound quark-antiquark systems, for different quark masses and excitation. It is easy to imagine that in the vicinity of a bound-state of the Coulomb-like potential, because of the presence of the pole in \( G^V \), the function \( t_R^{-1} \) is rapidly varying and it is infinity at the Coulomb bound state and because of the change in sign of \( G^V \) it will necessarily presents a zero, and a bound state of the whole potential.

**IV. PION CHARGE RADIUS**

The renormalized effective theory defined by Eqs.(23,24) with the bound state mass equation (34), once the pion mass is known has only one free interaction parameter, \( \alpha \), the
strength of the effective one-gluon-exchange potential. The value of $\alpha$ can be found from the pion charge radius. The pion wave-function in the effective theory comes from the residue of the T-matrix, Eq.(23), at the pion pole, such that

$$\psi_\pi(x, \vec{k}_\perp) = \sqrt{\frac{A(k)}{x(1-x)}} \varphi_\pi(\vec{k}),$$

(35)

where

$$\varphi_\pi(\vec{k}) = \langle \vec{k} | G_0(m_\pi^2) \left(1 + T^V(m_\pi^2)G_0(m_\pi^2)\right) | \chi >,$$

(36)

and

$$x(k_z) = \frac{1}{2} + \frac{k_z}{2E},$$

(37)

from Eq.(3). The absolute normalization of the $q\bar{q}$ Fock-component of the pion wave-function (35) is such that the asymptotic form is given by the first term of (36), with residue at the pion pole equal one. We impose such normalization condition to be consistent with the soft pion limit ($m_\pi=0$) for the electromagnetic form factor when the Coulomb type interaction goes to zero, as it will be shown below.

A. Including Quark Spin

The pion electromagnetic form-factor is obtained from the impulse approximation of the plus component of the current ($j^+ = j^0 + j^3$) in the Breit-frame with momentum transfer $q^+ = 0$ and $q^2 = -q^2$. The leptonic decay constant of $\pi^+$ ($f_\pi$) is a physical quantity which depends directly on the probability to find the quark-antiquark Fock-state in the pion wave-function and consequently properly normalizes it once the empirical value of $f_\pi$ is given. Computing the pion form-factor from an effective Lagrangian, described below, the value of $f_\pi$ gives the normalization of the form-factor. In this case the $q\bar{q}$ component of the pion wave-function is normalized such that, in the vanishing limit of the Coulomb-type interaction, it retrieves the asymptotic form Eq.(3). The coupling of the pion field to
the quark field are taken from an effective Lagrangian with pseudo-scalar coupling with the pion quark coupling constant given by the Goldberger-Treiman \cite{11} relation at the quark level,

\[ \mathcal{L}_{\text{eff}} = -i \frac{m}{f_\pi} \bar{q} \gamma^5 \tau \gamma q, \]  

(38)

and \( f_\pi = 93 \text{ MeV} \) is the pion weak decay constant.

The general structure of the \( q\bar{q} \) bound state forming the pion comes from the pseudo-scalar coupling, and we will use such spin structure in the computation of the Feynman triangle diagram which expresses the impulse approximation to compute the pion electromagnetic current,

\[
(p_{\pi}^\mu + p_{\pi}'^\mu) F_\pi(q^2) = i2 \frac{m^2}{f_\pi^2} N_c \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \frac{k + m}{k^2 - m^2 + i\varepsilon} \gamma^5 \right] \frac{k - p_{\pi} + m}{(k - p_{\pi}')^2 - m^2 + i\varepsilon} \gamma^\mu \left( k - p_{\pi} + m \right) \left( k - p_{\pi}' + m \right),
\]

(39)

where \( F_\pi(q^2) \) is the pion electromagnetic form-factor, the pion momentum in the initial and final states are defined by \( p_{\pi}^0 = p_{\pi}^0 \) and \( p_{\pi}^\perp = -\bar{p}_{\pi}^\perp = \frac{q^\perp}{2} \) in the Breit-Frame. \( N_c = 3 \) is the number of colors.

The + component of the current is calculated from (39). It is chosen because after the integration over \( k^- = k^0 - k^3 \) the suppression of the pair diagram is maximal for this component in the frame where \( q^+ = 0 \) and just the valence wave-function enters in the form-factor \cite{12,13}. Although we are going to compute the integration in the - component of the moment assuming a constant vertex, one can identify in the expression how the wave-function correspondent to the non-constant vertex of Eq.(35) should be introduced in the expression. The result is \cite{12}

\[
F_\pi(q^2) = \frac{2}{(2\pi)^3} \frac{M^2}{f_\pi^2} N_c \int_0^1 dx \int d^2K_\perp M_\perp^2 \left( 1 + \frac{(1 - x)q_\perp \cdot \vec{K}_\perp}{K_\perp^2 + M^2} \right) \psi_\pi(x, \vec{K}_\perp) \psi_\pi(x, \vec{K}'_\perp),
\]

(40)

where the momentum fraction \( x = (p^+ - k^+)/p^+ \). The relative transverse momentum is given by:

\[
\vec{K}_\perp = (1 - x) (\bar{p}_\perp - \vec{k}_\perp) - x\vec{k}_\perp,
\]

(41)
and $K'_\perp = \vec{K}_\perp + (1-x)\vec{q}_\perp$. The free mass operator for the $q - \vec{q}$ is written in terms of the momentum fraction and the relative perpendicular momentum

$$M'_0 = \frac{K^2_\perp + M^2}{x(1-x)},$$

and $M'_0$ is written as a function of $K'_\perp$. The expression for the pion form-factor gives the standard Drell-Yan formula once the bound-state wave-function of the constant vertex model, the asymptotic form, is recognized

$$\psi_\pi(x, \vec{K}_\perp) = \frac{1}{\sqrt{x(1-x)(m^2_\pi - M^2_0)}},$$

which is the first term of the pion wave-function in Eq. (36). The second term in (36) comes from the Coulomb-like potential, and implies in a contribution from the effective theory to the pion radius. The other factors in Eq. (40) comes from the Melosh rotations of the individual spin wave-function of the quarks.

The pion charge radius from Eq. (36) in the soft-pion limit with constant vertex, corresponding to the wave-function of Eq. (38) with $m_\pi = 0$, gives the well known result of $r_\pi = \left[6 \frac{d}{dq^2} F_\pi(q^2)|_{q^2=0}\right]^{1/2} = \sqrt{3}/(2f_\pi)$ from Ref. [14]. The form factor (40) in the soft-pion limit with constant vertex, for $q^2 = 0$, reduces to expression for $f_\pi [12]$.

$$f_\pi = \frac{2}{(2\pi)^3} \frac{M^2}{f_\pi} N_c \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \int d^2 K_\perp \psi_\pi(x, \vec{K}_\perp),$$

obtained from the computation of the leptonic decay transition amplitude of $\pi^+$ with the effective Lagrangian (38) and the wave function given by Eq. (43).

**B. Neglecting Quark Spin**

In order to gain insight on the importance of the inclusion of the quark spin in the computation of the form-factor, we have simplified the numerator of Eq. (40), which is the result from the Dirac algebra, by taking the limit of $m \to \infty$. In this way the quark spin is neglected and the form-factor reduces to the formula found for scalar particles, and
only the overall normalization, which depends on $f_\pi$ is maintained. The wave-function is in the form expressed by Eq.(13) with $A(k) = 1$. With the above approximations the pion electromagnetic form-factor is

$$F_\pi(q^2) = \frac{1}{\pi^3} \frac{M^4}{f_\pi^2} N_c \int_0^1 dx \int d^2 K_\perp \psi_\pi(x, \vec{K}_\perp) \psi_\pi(x, \vec{K}_\perp') ,$$  \hspace{1cm} (45)

with the relative transverse momentum

$$\vec{K}_\perp = (1-x)(\vec{p}_\perp - \vec{k}_\perp) - x\vec{k}_\perp ,$$  \hspace{1cm} (46)

and $\vec{K}_\perp' = \vec{K}_\perp + (1-x)\vec{q}_\perp$. In case of the $q\bar{q}$ wave-function being supposed to be the complete pion wave-function, the form factor is written as

$$F_\pi(q^2) = N \int_0^1 dx \int d^2 K_\perp \psi_\pi(x, \vec{K}_\perp) \psi_\pi(x, \vec{K}_\perp') ,$$  \hspace{1cm} (47)

and from $F_\pi(0) = 1$ the normalization $N$ is determined.

The difference between the form factors defined by Eq.(13) and Eq.(17) are the normalizations of the $q\bar{q}$ Fock-component of the pion wave-function. In the first case the normalization is defined by $f_\pi$, while for the second case the $q\bar{q}$ Fock-state component is normalized such that $F_\pi(0) = 1$. As we will see, Eq.(17) gives a too small pion radius, while from Eq.(15) or Eq.(40) the pion radius can be described reasonably. The absolute normalization of the form-factor computed with the inclusion of the quark spin as expressed by Eq.(10) is undetermined, since the integral diverges. However, the pion radius is finite with the magnitude of this Fock-component of the pion wave-function known from the empirical value of $f_\pi$, number of colors and the attributed constituent quark mass.

To close this section, we observe that our aim is to fit the strength $\alpha$ of the Coulomb-like potential using the pion charge radius. For this purpose, we introduce the pion wave-function from Eq.(35) in the form-factor expression, Eq.(10) and calculate the charge radius. Although the form-factor diverges, the charge radius is finite and for $m_\pi = 0$ and $\alpha = 0$ it retrieves the soft-pion limit. The pion mass and the effective Coulomb-like interaction give a correction to the soft pion limit, $r_\pi^{soft} = 0.58$ fm towards the experimental result of $0.67 \pm 0.02$ fm \([13]\).
V. NUMERICAL RESULTS

A. Test Case

We begin the section on the numerical calculations comparing our results for the s-wave bound-state energies which are consistent with [10] for the non-relativistic Coulomb plus repulsive Yukawa model

\[ [\epsilon - p^2] \varphi(p) = \frac{1}{\pi} \int_0^\infty dp' \frac{p'}{p} \left[ \ln \frac{(p - p')^2}{(p + p')^2} - \ln \frac{(p - p')^2 + \eta^2}{(p + p')^2 + \eta^2} \right] \varphi(p') . \]  

(48)

In our numerical procedure we checked the results with up to 200 Gaussian-Legendre quadrature points and the interval \(-1 < z < 1\) was transformed to \(0 < k < \infty\) through the variable transformation \(k = c(1 - z)/(1 + z)\) with \(c\) about 1.

In Figure 1, we show our results for the first excited state \(\epsilon^{(2)}\) as a function of the ground state \(\epsilon^{(1)}\) and compare with the calculation of [10] for \(\eta = 0.1, 1\) and 10. For these values of \(\eta\) we present results in Table I. Our precision is about 0.5%, that will suffice for our purposes. Also the calculation with the renormalized Coulomb plus Dirac-delta is shown in Figure 1 and Table I. The calculation with non-relativistic renormalized model corresponds to find the zeroes of the non-relativistic form of Eq.(34),

\[ \int d\vec{q} d\vec{p} <\vec{q}| (G^{V nr}(\epsilon^{(n)}) - G^{V nr}(\epsilon^{(1)})) |\vec{p}> = 0 , \]  

(49)

with the non-relativistic resolvent \(G^{V nr}(\epsilon)\) for negative energies obtained from the solution of

\[ G^{V nr}(\epsilon) = G^{nr}_0(\epsilon) + G^{nr}_0(\epsilon)V_C G^{V nr}(\epsilon) , \]  

(50)

where \(< \vec{p}|V_C|\vec{q}> = [\pi^2(|\vec{p} - \vec{q}|^2)]^{-1}\) and the free resolvent \(G^{nr}_0(\epsilon) = [\epsilon - k^2]^{-1}\).

In momentum space Eq.(49) is given by

\[ 4\pi \int_0^\infty dp \ p^2 \left[ \frac{1}{-\epsilon^{(n)} - p^2} - \frac{1}{-\epsilon^{(1)} - p^2} \right] + 8\pi^2 \int_0^\infty dq \ q^2 \int_0^\infty dp \ p^2 \left[ \frac{t_C(p, q; -\epsilon^{(n)})}{(-\epsilon^{(n)} - p^2)(-\epsilon^{(n)} - q^2)} - \frac{t_C(p, q; -\epsilon^{(1)})}{(-\epsilon^{(1)} - p^2)(-\epsilon^{(1)} - q^2)} \right] = 0 , \]  

(51)
and s-wave projected T-matrix in Eq. (51) is the solution of

\[ t_C(p, q; \epsilon) = \frac{1}{2\pi^2pq} \ln \frac{(p-q)^2}{(p+q)^2} + \frac{1}{\pi} \int_0^\infty dp'p' \ln \frac{(p-p')^2}{(p+p')^2} \frac{t_C(p', q; \epsilon)}{\epsilon - p'^2}. \]  

(52)

We solve Eq. (49) for \( n = 2 \) for each given \( \epsilon^{(1)} \). Now, it is clear the motivation for plotting Figure 1 the binding energy of the excited state against the ground state energy. In Table I as well as in Figure 1, the renormalized model reproduce with less then 10% accuracy the model results of Eq.(48). In the limit of \( \eta \to \infty \), the renormalized model should work better, however in the test case of Eq.(48) this limit corresponds to a vanishing Yukawa interaction, and the results tends trivially the Coulomb value of the first excited state of 0.25 for the ground state value of 1, what in fact is observed in Figure 1 and Table I.

B. Effective Pion Model

The effective model of [2] corresponds to use the non-relativistic phase-space \( A(k) = 1 \) in Eq.(41) and a smeared delta-interaction of a Yukawa form:

\[ m^2_\pi \varphi(\vec{k}) = \left[ 4m^2 + 4k^2 \right] \varphi(\vec{k}) - \frac{4}{3\pi^2\alpha} \int \frac{d\vec{k}'}{m} \left( \frac{2m^2}{(\vec{k} - \vec{k}')^2} + \frac{\eta^2}{\eta^2 + (\vec{k} - \vec{k}')^2} \right) \varphi(\vec{k}'), \]

(53)

which was solved with parameters adjusted to fit the pion mass and the rho-meson mass, resulting \( m = 406 \) MeV, \( \alpha = 0.6904 \), and \( \eta = 1330 \) MeV [2]. The value of \( \eta \) was found from the condition that the first excited bound state mass \( m^*_\pi \) fullfills the Strutinsk requirement at the extremum

\[ \frac{d}{d\eta} m^*_\pi = 0. \]

(54)

However, the pion charge radius calculated according to the non-relativistic formula in [2] is about one half of the empirical value. We will confirm this fact, in the renormalized effective model while using Eq.(17) and \( \varphi_\pi \) from Eq.(33) with \( A(k) = 1 \) to compute the pion radius.

In Figure 2, we present our results for the pion mass as a function of \( \alpha \) for \( \eta = 1330 \) MeV. Our agreement with the calculation of [2] is within 10%. In Figure 3, we show results for our calculation of the ground and excited states masses from Eq.(53), for \( \alpha \) varying
and \( \eta = 1330 \) MeV, compared to the renormalized model for the Coulomb plus Dirac-delta interaction. In the last case, the bound state masses of the pion ground and excited states in s-wave, are found numerically from the zeroes of Eq.(14) with the Green’s function of the Coulomb-like potential obtained from the solution of the integral scattering equation (20). In both equations, which define the non-relativistic renormalized model Coulomb plus Dirac-Delta interaction, is used \( A(k) = 1 \). We have disregarded the effect of the energy transfer in \( Q^2 \) of Eq.(2), as being a short-range effect parameterized by the value of the pion mass, which is input in this calculation. In momentum space Eq.(34) is given by

\[
4\pi \int_0^\infty dp \frac{p^2}{A(p)} \left[ \frac{1}{m_\pi^2 - 4m^2 - 4p^2} - \frac{1}{m^2 - 4m^2 - 4p^2} \right] \\
+ 8\pi^2 \int_0^\infty dq \frac{q^2}{A(q)} \int_0^\infty dp \frac{p^2}{A(p)} \left[ \frac{t^V(p, q; m_\pi^2)}{(m_\pi^2 - 4m^2 - 4p^2)(m_\pi^2 - 4m^2 - 4q^2)} - \frac{t^V(p, q; m_\pi^2)}{(m_\pi^2 - 4m^2 - 4p^2)(m_\pi^2 - 4m^2 - 4q^2)} \right] = 0 ; \tag{55}
\]

and s-wave projected T-matrix in Eq.(53) is

\[
t^V(p, q; M^2) = \int_{-1}^1 d\cos(\theta) < \vec{p} T^V(M^2) | \vec{q} > ; \tag{56}
\]

which is the solution of

\[
t^V(p, q; M^2) = \frac{4m}{3\pi^2 pq} \ln \frac{(p-q)^2}{(p+q)^2} \frac{\alpha}{\sqrt{A(p)A(q)}} \\
+ \frac{8m}{3\pi^2} \int_0^\infty \frac{dp'}{\sqrt{A(p)A(p')}} \frac{p'}{p} \ln \frac{(p-p')^2}{(p+p')^2} \frac{t^V(p', q; M^2)}{M^2 - 4m^2 - 4p^2} , \tag{57}
\]

the momentum space representation of s-wave projection of Eq.(20).

The agreement between the renormalized model and the smeared delta model is within a fraction of percent, which still improves as the weakly bound limit of the ground state is approached, as shown in Figure 3. The calculation of [2] is few percent below ours. The inclusion of the relativistic phase-space in the renormalized model of the Coulomb plus Dirac-delta interaction Eq.(14) make less bound the excited state for a given ground state mass. As the pion mass grows the relativistic phase-space effect tends to vanish.
In the other study performed, \( \alpha = 0.6904 \) was kept unchanged, while varying \( \eta \) between 350 and 1350 MeV. The results for the excited bound state mass against the ground-state mass are shown in Figure 4. The renormalized Coulomb plus Dirac-delta calculation agrees within a fraction of percent with the smeared Dirac-delta calculation. The relativistic phase-space in the renormalized interaction makes the excited state less bound for a given ground state mass.

C. Effective Meson Model

The bound state masses of the meson ground and excited states in s-wave, are found numerically from the zeroes of Eq.\((34)\) with the Green’s function of the Coulomb-like potential obtained from the solution of the integral scattering equation \((20)\). The energy transfer in \( Q^2 \) in Eq.\((2)\) was neglected.

The renormalized strength of the singular interaction, from Eq.\((22)\)
\[
\lambda^{-1}_R(m^2_\pi) = \langle \chi | G_0(m^2_\pi) T^V(m^2_\pi) G_0(m^2_\pi) | \chi \rangle; \quad (58)
\]
with the condition Eq.\((27)\) at the physical pion mass is shown in Figure 5 for \( m = 406 \) MeV and compared to the perturbative calculation
\[
\lambda^{-1}_R(m^2_\pi)|_{pert} = \langle \chi | G_0(m^2_\pi) V G_0(m^2_\pi) | \chi \rangle; \quad (59)
\]
with \( V \) from Eq.\((8)\) and \( T^V \) solution of Eq.\((21)\). For values of \( \alpha \) below 0.2 the agreement between the perturbative and nonperturbative calculation of the renormalized strength is quite good, giving confidence to our numerical calculations.

The wave-function \( \varphi_\pi \) of the pion from Eq.\((36)\) in momentum space is written as
\[
\varphi_\pi(p) = \frac{1}{m^2_\pi - 4m^2 - 4p^2} \left( 1 - A(p) \right) + 2\pi \int_0^\infty dq \frac{q^2}{A(q)} \frac{T^V(p, q; m^2_\pi)}{m^2_\pi - 4m^2 - 4p^2}; \quad (60)
\]
The first-order perturbative pion wave-function is
\[
\varphi_\pi(p)|_{pert} = \frac{1}{A(p)} \frac{1}{m^2_\pi - 4m^2 - 4p^2} \left( 1 + \frac{8m}{3\alpha} \int_0^\infty dq \frac{q}{A(q)} \frac{\ln \left( \frac{(p-q)^2}{(p+q)^2} \right)}{p m^2_\pi - 4m^2 - 4p^2} \right); \quad (61)
\]
and for $\alpha = 0$ the pion wave-function has the asymptotic form
\[
\varphi^{0}_\pi(p) = \frac{1}{\sqrt{A(p)m^2}} \frac{1}{m^2 - 4m^2 - 4p^2}.
\] (62)

In [2], the pion wave-function eigenfunction of the effective square mass operator of the Coulomb plus Yukawa model, Eq.(53), was approximated by the following analytical form:
\[
\varphi^a_\pi(p) = \frac{N}{(1 + p^2/p^2_a)^2},
\] (63)
where $N$ is an arbitrary normalization and the fit is performed for $\alpha = 0.6904$, $\eta = 1330$ MeV and $m=406$ MeV and $p_a=515$ MeV is adjusted to the numerical solution of Eq.(53).

The various non-relativistic models of the pion wave-function, with $A(k)=1$, are plotted in Figure 6. In this figure, the comparison between $\varphi_\pi$, $\varphi_\pi|_{\text{pert}}$, $\varphi^a_\pi$ and $\varphi^0_\pi$ are performed. The model parameters are $\alpha = 0.6904$ and $m = 406$ MeV and the normalization is arbitrary. The perturbative calculation reproduce $\varphi_\pi(p)$ for $p$ below $m$. The asymptotic wave-function $\varphi^0_\pi(p)$ overestimate $\varphi_\pi(p)$, as it should be for a repulsive Coulomb-type interaction, and for small $p$ it approaches the nonperturbative eigenfunction. The analytical approximation works quite good for momentum up to about $1.5m$. More results on the pion wave-function are shown in Figure 7, where the results for $A(k) = 1$ with the full calculation are compared. The effect of $A(p)$ diminishes the magnitude of $\varphi(p)$, as one could antecipate from Eq.(60). Also we plot $\varphi_\pi(p)$ for $\alpha = 0.18$ and $m = 386$ MeV, which is above the curve of previous case due to the decrease of the Coulomb attraction (this parameters fits the empirical pion radius). We point also that, for very high momentum, again the asymptotic form dominates, as the second term of Eq.(60) tends to zero faster than the first one.

We are going to compute the pion radius in the effective renormalized model of the pion using Eq.(31), Eq.(34) and Eq.(37). In the last two cases, the pion wave-function from (50) with $A(k) = 1$ is used. The pion wave-function is known from Eqs.(35) and (60) from which the charge radius including the effect of quark spin is obtained from integration by Gaussian quadrature of the difference
\[
\Delta F_\pi(q^2) = F_\pi(q^2) - F_\pi(0),
\] (64)
obtained from Eq. (40), and the pion radius is calculated from

\[ r_\pi = \left[ 6 \frac{d}{dq^2} \Delta F_\pi(q^2)|_{q^2=0} \right]^{\frac{1}{2}}. \]

In the case the quark spin is neglected, two possibilities of calculation of the pion radius are considered. One through Eq. (45) in which only the spin factors are simplified in the limit of the quark mass being infinity, while the normalization is defined as in Eq. (40), known from the empirical value of \( f_\pi \) and number of colors. The second possibility is to declare normalized to one the \( q\bar{q} \) Fock-component of the pion wave-function and use the formula for the form-factor in which the quark has no spin, Eq. (47).

The results for the pion charge radius as a function of the strength \( \alpha \) of the Coulomb-like interaction, Eq. (8), are shown in Figure 8. For \( r_\pi = 0.67 \text{fm} \) we found \( \alpha = 0.18 \) using Eq. (40). Our calculations obtains the pion charge radius from the relativistic expression (40) which is known to give about twice the non-relativistic radius [12]. The attractive Coulomb-like interaction increases the radius over the soft-pion limit with \( \alpha = 0 \) which is below its experimental value. In that sense a consistence is found with the effective theory which has an attractive Coulomb-like interaction. The repulsion would be completely inconsistent with the pion radius.

The pion charge radius obtained from first order perturbative calculation of the pion wave-function

\[ \psi_\pi(x, \vec{k}_\perp)|_{pert} = \sqrt{\frac{A(k)}{x(1-x)} \varphi_\pi(k)|_{pert}}, \]

is also shown in Figure 8. In consistency with the strength calculation presented in Figure 5, we observe that for \( \alpha \) below 0.2 the perturbative calculations match the nonperturbative results.

The results for the scalar quarks form-factors with \( \varphi \) from Eq. (60) calculated in the effective renormalized theory for \( A(k) = 1 \) are also shown in Figure 8. The calculation of \( r_\pi \) with Eq. (45) show values above the ones calculated with Eq. (47), which are two small compared to the empirical value, indicating the importance of the physical normalization.
of the $q\bar{q}$ Fock-component of the wave-function using $f_\pi$. We also performed the first order perturbative calculation of $r_\pi$ for the wave-function normalized to 1. The agreement between the perturbative nonperturbative calculation is reasonable for $\alpha$ below 0.2.

The plot of the pion charge radius against the mass of the first excited-state is shown in Figure 9. For decreasing values of $\alpha$, the pion charge radius diminishes and consistently the excited-state mass increases, i.e., this state becomes less bound. It is clear from this figure that to simultaneously fit the radius and the mass of the rho-meson (768 MeV) we are obliged to use a different quark mass from the value of 406 MeV. For comparison we also show the results from Eq. (45).

The experimental pion radius of $0.67\pm0.02$ fm is fitted with $m_u = m_\pi = 386$ MeV and $\alpha = 0.18$ resulting $r_\pi = 0.67$ fm and the mass of the singlet-2s state of 768 MeV, we remind that the pion mass is input in the renormalized model calculation. The singlet-2s excited state mass of the $\pi u$ system is identified with the $\rho$ meson mass in the present effective QCD model. The quark mass is varied to form mesons with one up antiquark together with the strange, charm or bottom quarks. The masses of the constituent quarks were within the range of 500 to 5000 MeV. The results are shown in Table II and compared to the experimental data. The singlet-2s states are identified with the lowest mass vector mesons states, since the Dirac-delta interaction is the effective hyperfine interaction, the reason for the splitting between the pseudo-scalar and vector mesons. Although, in the singlet channel the hyperfine interaction is attractive, which is not valid for the spin one mesons, we believe that the Dirac-delta interaction mock up short-range physics beyond that, by taking care of the empirical value of the pion mass.

It is clear that the split between the heavy meson masses will be not adjusted in the present calculation for $\alpha$ small. This is reasonable, since these mesons are weakly bound and the wave-function spreads out in the region where the confinement potential, not present in our model, should be important. In Figure 10, we study the difference of the excited and ground state masses as a function of the ground state mass. As we have seen in Table II, the difference is underestimated for $\alpha = 0.18$ above the kaon mass. For $\alpha = 0.5$ the
experimental data is reproduced. This is in fact reasonable if we think that $\alpha$ should on average increase with the size scale, indicating the confining behaviour, which is stronger for the heavy mesons since they are less bound than the pion and the kaon.

VI. CONCLUSION

The essential development made in this work is the renormalization of the effective QCD-inspired Hamiltonian theory with a singularity at zero range and its consequent application to the pion and other mesons. The method is an example of the Hamiltonian renormalization procedure and it is equivalent to a subtracted equation for the transition matrix. The physical renormalization condition is given to the two-particle model at the subtraction point and in the cases discussed here it is the ground state binding energy or mass. The treatment is shown to be renormalization group invariant, i.e., independent on the arbitrary subtraction point. This independence is expressed by a fixed-point Hamiltonian that brings the physical input to the theory, the pion mass or ground state binding energy, as well as the necessary counterterms that render all the momentum integrations finite.

First, we have studied in an example, the renormalization method applied to a two-body model with a Coulomb plus Dirac-delta interaction, where we have calculated the excited state energy for a given ground state binding energy. The results are compared to calculations with a Coulomb plus repulsive Yukawa interaction and the renormalized model reproduce with less then 10% accuracy the model results of Eq.(48). The success of this result drive us to the solution of the renormalization problem of the $\uparrow\downarrow$-model [2].

The effective mass operator equation in the $\uparrow\downarrow$-model Eq.(1), has as eigenstate the lowest Fock-state component ($\bar{qq}$) of the light-cone wave-function of a meson bound system of constituent quarks or dressed quarks (not to be confused with the bare quark). This model picks out one particular aspect of the gluon exchange between quarks, namely the strong attraction of the spin-spin interaction in the singlet channel. Previously [3], the renormalization has been carried by first regularizing the Dirac-delta interaction through a Yukawa
form and then its parameter was found by the Strutinsky requirement that the mass of the excited state is stationary in respect to variation of the regularization parameter. The step forward in this work was the use of the renormalization group invariant approach, in which the regularization parameter is not necessary to solve the model. All the short-range physics is parametrized by one parameter: the renormalized strength of the Dirac-delta interaction, which is determined by the mass of the pion. We showed that, the results for the mass of the excited state obtained with the renormalized model and the smeared delta regularized model are in agreement within fraction of percent, for the same ground state mass, which was varied either by changing the Coulomb interaction intensity or the Yukawa range. The concordance still improves as the weakly bound limit of the ground state is approached. The effect of the relativistic phase-space in the renormalized model of the Coulomb plus Dirac-delta interaction Eq. (34) makes the excited state, for a given ground state mass, less bound. This difference tends to vanish as the pion becomes weakly bound.

The various models of the $\mathbf{7}_q$ Fock-component of the pion light-cone wave-function, considering $A(k) = 1$ and the relativistic phase-space ($A(k)$ from Eq. (3)), were calculated and compared with the analytical form and perturbative result. We obtained the reduced wave-function $\varphi_{\pi} (60)$ solution of the renormalized $\uparrow\downarrow$-model, $\varphi_{\pi} \mid_{\text{pert}} (61)$, the analytical form $\varphi_{\pi}^0 (63)$ and the asymptotic form $\varphi_{\pi}^0 (64)$. We have used model parameters $\alpha = 0.6904$ and $m = 406$ MeV [2]. Although for for $p$ below $m$ all the calculations are reasonably consistent, the high momentum tail is dominated by the asymptotic wave-function $\varphi_{\pi}^0 (p)$, which over-estimate $\varphi_{\pi} (p)$, as it should be for a repulsive Coulomb-type interaction, and for small $p$ it approaches the nonperturbative eigenfunction, as well as for very high momentum. The analytical approximation works quite good for momentum up to about $1.5m$, however it does not have the asymptotic tail for high momentum. The relativistic phase-space diminishes the magnitude of $\varphi (p)$ as clearly seen in Eq. (61).

The calculation of the pion charge radius was performed in the renormalized effective QCD-inspired Hamiltonian theory with Eqs. (36) and (40) and although we have simplified the spin dependence in the dynamical equation it is important in the evaluation of the
radius. For this purpose we have used an effective pseudo-scalar Lagrangian to construct the spin part of the pion wave-function, it gives the absolute normalization of the lowest Fock-component of the light-cone wave-function in terms of the weak decay constant $f_\pi$, the constituent quark mass and number of colors. Turning off the Coulomb-like interaction, we retrieved the well known result valid in the soft pion limit [14]. We also compared the values for $r_\pi$ obtained with the expressions where the quark spin is neglected and the wave-function normalized to 1. We pointed out the necessity of the correct normalization of the wave-function of the $\uparrow\downarrow$-model according to the $f_\pi$ value to be able to fit $r_\pi$, which is reminiscent of the fact that $f_\pi$ and $r_\pi$ are closely related in the light-front phenomenology [12].

The experimental pion radius of 0.67±0.02 fm was fitted with $m_u = m_\pi = 386$ MeV and $\alpha = 0.18$ resulting $r_\pi = 0.67$ fm and the mass of the singlet-2s state of 768 MeV, we remind that the pion mass is input in the renormalized model calculation. The singlet-2s excited state mass of the $\overline{u}u$ system was identified with the $\rho$ meson mass in the present effective QCD model. We stress that in the singlet channel the hyperfine interaction is attractive, which is not valid for the spin one channel of the vector mesons, however, we believe that the Dirac-delta interaction mock up short-range physics beyond that, which is brought to the model by the empirical value of the pion mass. In essence, without being too naive, in order to fit the pion charge radius with success and with reasonable parameters, it was essential that i) the Coulomb-like interaction be attractive and ii) the normalization of the form-factor be consistent with the empirical $f_\pi$ value.

In respect to the masses of the ground state of the pseudo-scalar and vector mesons with one up antiquark together with the strange, charm or bottom quarks, which were calculated as a function of the constituent quark mass, the results were in qualitative agreement with the data for $\alpha = 0.18$. We tried a better fit of this data, since it was clear that the split between the heavy meson masses would not be adjusted in the present calculation for $\alpha$ small. This in fact, seems reasonable, since these mesons are weakly bound and the quarks can be found in a region where the confinement potential, not present in our model,
is important. The difference between the vector and pseudo-scalar mesons masses for the same $\bar{q}q$ pair is underestimated for $\alpha = 0.18$ above the kaon mass. We found that for $\alpha = 0.5$ the experimental data is reproduced. This gives us some hope that it is possible to refine the $\uparrow\downarrow$-model to include more physics than initially thought. In regard to this extension, it is reasonable to think that $\alpha$ should on average increase with the size scale, indicating the confining behaviour, which we found stronger for the heavy mesons since they are less bound than the pion and the kaon. In short, a reasonable description of the physics of the pion and other scalar and vector mesons were found, taking into account the simplicity of the renormalized effective Light-Cone QCD-inspired theory.

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APPENDIX A: DERIVATION OF THE RENORMALIZED T-MATRIX

In this Appendix, the solution of Eq. (17) to find the renormalized T-matrix, Eq. (19), is performed in detail. We want to solve Eq. (17)

\[ T_R(M^2) = V + V_R^\delta + (V + V_R^\delta) G_0^{(+)}(M^2) T_R(M^2) \] \hspace{1cm} (A1)

The regular part of the potential, \( V \), is defined by Eq. (8) and the renormalized singular interaction is given by Eq. (13), rewritten as a matrix equation

\[ V_R^\delta = |\chi > \lambda_R(\mu^2) < \chi| - |\chi > \lambda_R(\mu^2) < \chi| G_0^+(\mu^2) V_R^\delta ; \] \hspace{1cm} (A2)

which has the solution

\[ V_R^\delta = |\chi > v_R(\mu^2) < \chi| , \] \hspace{1cm} (A3)

with the function

\[ v_R(\mu^2) = \left[ \lambda_R(\mu^2) + < \chi | G_0^+(\mu^2) | \chi > \right]^{-1} . \] \hspace{1cm} (A4)

The function \( v_R(\mu^2) \) contains the divergences in the momentum integrals which exactly cancels such infinities in Eq. (A1). It is enough for the formal manipulations that will come. However, one could equally well introduce a cutoff in Eqs. (A1) and (A4), and performs the limit of the cutoff going to infinity just after the solution of Eq. (A1), in which all the necessary cancellations happens and the limit is finite.

Next, Eq. (A1) is rewritten as:

\[ \left( 1 - V G_0^{(+)}(M^2) \right) T_R(M^2) = V + V_R^\delta + V_R^\delta G_0^{(+)}(M^2) T_R(M^2) \] \hspace{1cm} (A5)

and inverting the operator in the left-hand using the regular T-matrix, \( T^V(M^2) \), solution of Eq. (20), one has

\[ T_R(M^2) = T^V(M^2) + \left( 1 + T^V(M^2) G_0^{(+)}(M^2) \right) |\chi > v_R(\mu^2) < \chi| \left( G_0^{(+)}(M^2) T_R(M^2) + 1 \right) . \] \hspace{1cm} (A6)
The "bra" function, \( \langle \chi \rvert G^{(+)}(M^2)T_R(M^2) \rangle \), has to be calculated in order to find the renormalized T-matrix. We multiply Eq.(A6) by \( \langle \chi \rvert G^{(+)}(M^2) \rangle \) on both sides, and solving it we get

\[
\langle \chi \rvert G^{(+)}(M^2)T_R(M^2) \rangle = \langle \chi \rvert G^{(+)}(M^2) \rangle + \langle \chi \rvert G^{V}(M^2) \rangle v_R(\mu^2) \langle \chi \rvert \langle \chi \rvert G^{V}(M^2) \rangle v_R(\mu^2) \rvert \langle \chi \rvert \rangle, \tag{A7}
\]

where regular potential resolvent is

\[
G^{V}(M^2) = G^{(+)}(M^2) + G^{(+)}(M^2)T_R(M^2)G^{(+)}(M^2). \tag{A8}
\]

The "bra" function of Eq.(A7) is introduced back in Eq.(A6), and with a little algebra one finds

\[
T_R(M^2) = T_V(M^2) + \frac{(1 + T_V(M^2)G^{(+)}(M^2)) \langle \chi \rvert \langle \chi \rvert G^{(+)}(M^2)T_V(M^2) + 1 \rangle}{v_R^{-1}(\mu^2) - \langle \chi \rvert G^{V}(M^2) \rangle v_R(\mu^2) - \langle \chi \rvert G^{(+)}(M^2) \rangle v_R(\mu^2)} \rvert \langle \chi \rvert \rangle, \tag{A9}
\]

which, after introducing Eq.(A4) and the explicit form of the resolvent of Eq.(A8), results in the renormalized T-matrix of Eq.(19).
TABLE I. Results for S-wave binding energies of the non-relativistic model of Eq.(48), our calculation with $N = 100$ [C-Y], compared to the effective Coulomb plus Dirac-delta interaction [C-$\delta$].

| $\eta$ | $\epsilon^{(1)}$ [C-Y] | $\epsilon^{(2)}$ [C-Y] | $\epsilon^{(2)}$ [C-$\delta$] |
|-------|-----------------|-----------------|-----------------|
| 0.1   | 0.1109          | 0.06781         | 0.06237         |
| 1     | 0.5119          | 0.1813          | 0.1736          |
| 10    | 0.9495          | 0.2449          | 0.2439          |

TABLE II. Results for S-wave Meson Masses: singlet-1s ($M_{b,th}^{(1s)}$) and singlet-2s ($M_{b,th}^{(2s)}$) and experimental values ($M_{b,exp}^{(1s,2s)}$). All masses in MeV. $\alpha = 0.18$.

| $\bar{q}q$ | $m_1$ | $m_2$ | $M_{b,th}^{(1s)}$ | $M_{b,exp}^{(1s)}$ | $M_{b,th}^{(2s)}$ | $M_{b,exp}^{(2s)}$ |
|------------|------|------|-----------------|------------------|-----------------|------------------|
| $\bar{u}d$ | 386  | 386  | 140             | $\pi^\pm$ : 140  | 768             | $\rho$ : 768     |
| $\bar{u}s$ | 386  | 500  | 511             | $K^\pm$ : 494    | 882             | $K^*$ : 892      |
| $\bar{u}c$ | 386  | 1500 | 1852            | $D^0$ : 1865     | 1882            | $D^{*0}$ : 2007  |
| $\bar{u}b$ | 386  | 5000 | 5375            | $B^\pm$ : 5279   | 5383            | $B^*$ : 5325     |
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Fig. 1 Excited state binding energy ($\epsilon^{(2)}$) as a function of the ground state binding energy ($\epsilon^{(1)}$) for the non-relativistic model. Attractive Coulomb plus repulsive Yukawa non-relativistic model of the Ref. [10] (full square). Our numerical calculation of Eq. (48) is given by the solid line, and the solution of the effective model Eq. (49) (Coulomb plus Dirac-Delta) is given by the dashed line.
Fig. 2 Pion mass as a function of the strength of the Coulomb interaction $\alpha$. Non-relativistic model for Coulomb plus Yukawa interaction, $m = 406$ MeV, solid line. Yukawa range parameter $\eta=1330$ MeV \cite{2}. The full square is the calculation of Ref. \cite{2}.
Fig. 3 Mass of the excited state as a function of the ground state pion mass. Non-relativistic model for Coulomb plus Yukawa interaction, $m = 406$ MeV, solid line. Strength of the Coulomb potential within $0 < \alpha < 0.7$ and Yukawa range parameter $\eta=1330$ MeV [2]. The full square is the calculation of Ref. [2]. The non-relativistic effective model (Coulomb plus Dirac-Delta) is given by the full dots. The relativistic effective model for the Coulomb plus Dirac-Delta interaction is given by the dashed-line.
Fig. 4 Mass of the excited state as a function of the ground state pion mass. Non-relativistic model for Coulomb plus Yukawa interaction, $m = 406$ MeV, solid line. Strength of the Coulomb potential $\alpha = 0.6904$ and Yukawa range parameter within $0 < \eta < 1350$ MeV [2]. The full square is the calculation of Ref. [2]. The non-relativistic effective model (Coulomb plus Dirac-Delta) is given by the full dots. The relativistic effective model for the Coulomb plus Dirac-Delta interaction is given by the dashed line.
Inverse renormalized strength $\lambda^{-1}_R(m^2_\pi)$ in units of $m_r$ as a function of the Coulomb intensity potential parameter $\alpha$ for a pion mass of 140 MeV. Nonperturbative calculation (solid line) and first-order perturbative calculation (dashed line), see text for the explanation.
Fig. 6 Pion wave-function ($\varphi$) with arbitrary normalization as a function of momentum in units of quark mass. Calculations performed with $A(k) = 1$. Pion model wave-function model of Eq.(60) for $\alpha = 0.6904$ and $m = 406$ MeV (solid line), first-order perturbative calculation from Eq.(61) (short-dashed line), asymptotic form Eq. (62) (dotted line) and fit from Eq.(63) (dashed line).
Fig. 7 Pion wave-function ($\varphi$) as a function of momentum in units of quark mass. Pion model for $\alpha = 0.6904$ and $m = 406$ MeV, calculation with Eq. (60) (dashed line); considering $A(k) = 1$ (solid line). Results for $\alpha = 0.18$ and $m = 386$ MeV (dotted line).
Fig. 8 Pion charge radius as a function of the strength $\alpha$ of the Coulomb potential. Quark mass of 406 MeV. Results for Eq. (40): nonperturbative calculation of the wave function (solid line) and first order perturbative calculation (dashed line). Results for Eq. (45) and wave-function obtained with $A(k) = 1$ (solid line with triangles). Results for Eq. (47) with the wave function obtained with $A(k) = 1$: nonperturbative calculation of the wave function (solid line with dots) and first order perturbative calculation (dashed line with dots).
Fig. 9 Pion charge radius as a function of the mass of the excited $q\bar{q}$ state for a constrained pion mass of 140 MeV. The strength $\alpha$ is within the range of Fig.6 and the quark mass is 406 MeV. Results for Eq. (40) (solid line) and for Eq. (45) with wave-function obtained considering $A(k) = 1$ (solid line with triangles).
Fig. 10 The difference between the masses of the excited and ground state of the $\pi q$ system as a function of the ground state mass in the effective relativistic model. The quark mass is varied in the range between 386 and 5000 MeV. The other quark mass is fixed to 386 MeV. Calculations with $\alpha = 0.18$ (solid line) and $\alpha = 0.5$ (dashed line). The experimental values from Table II are given by the full circles.