A generalized boundary condition applied to Lieb-Schultz-Mattis type ingappabilities

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Abstract

We introduce a new boundary condition which renders the flux-insertion argument for the Lieb-Schultz-Mattis type theorems in two or higher dimensions free from the specific choice of system sizes. It also enables a formulation of the Lieb-Schultz-Mattis type theorems in arbitrary dimensions in terms of the anomaly in field theories in $1 + 1$ dimensions with a bulk correspondence as a BF-theory in $2 + 1$ dimensions. Furthermore, we apply the anomaly-based formulation to the constraints on a half-filled spinless fermion on a square lattice with $\pi$ flux, utilizing a time-reversal, magnetic translations and an on-site internal $U(N)$ symmetries. This demonstrates the role of the time-reversal anomaly on the ingappabilities of a lattice model.

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I. INTRODUCTION

Quantifying various phases for quantum many-body systems is a central task in condensed matter and statistical physics. Recent decades have witnessed several significant phase classifications, e.g. topological ordered phases [1, 2] and symmetry-protected topological (SPT) phases beyond Landau’s symmetry-breaking pattern of strongly-correlated systems with non-perturbative interactions [3, 4]. Furthermore, symmetries together with filling fractions also constrain low-energy spectrums when critical phases are gapped, which induce the concept of symmetry-protected critical phases [5, 6] that nontrivial critical phases are ingappable with a unique ground state if the symmetries are respected by Hamiltonians.

One of the most important general principles in quantum many-body systems is Lieb-Schultz-Mattis (LSM) theorem [7] and its generalizations [8–11]. They show the interplay between the global $U(1)_Q$ charge and translation symmetries. The theorem states, under certain conditions, an “ingappability” of the system, that is, either the presence of gapless excitations above the ground states or a ground-state degeneracy in the limit of the large system size. It is valid for Hamiltonians with appropriate symmetries for arbitrary strong interactions, and thus is non-perturbative in nature.

Generally in physics, we expect that the bulk property would not depend on the choices of boundary conditions. If that is the case, we can use a boundary condition which is convenient for the calculation and infer physical results which would be valid independent of the boundary condition. In many cases, the periodic boundary condition is chosen as a boundary condition. A typical example is the band theory of electronic structures.

The original proof [7] of the LSM theorem was also based on the periodic boundary condition. The LSM theorem in higher dimensions turns out to be more subtle. While it is easy to see the failure of the original proof in higher dimensions, the original proof would still work [7, 12] in an “anisotropic” thermodynamic limit in which the ratio of the system sizes in each direction diverges. However, one might worry that the system is essentially one-dimensional in such a limit. An alternative argument based on an adiabatic flux insertion and gauge invariance, which does not need such an anisotropic limit, was proposed later [10]. It still depends on several nontrivial assumptions including the stability of the gap against the flux insertion, and special choices of system size as we will discuss later. It was followed by a more rigorous proof [11] of the LSM theorem in higher dimensions, and its further
Nevertheless, the flux insertion argument is still attractive in its simplicity, intuitiveness, and connections to other concepts in physics. Indeed, some of the recent extensions [14] of the LSM theorem is based on the flux insertion argument. Because of this, it would be valuable to improve the flux insertion approach to the LSM-type theorems. One of the subtleties in the flux insertion argument was that the system sizes must meet a special condition: the lengths in all but one direction must be coprime with the denominator of the filling fraction. Although one may consider the “thermodynamic limit” with a series of finite-size systems satisfying this condition keeping the ratio of length to be of order of 1, it is desirable to remove such a rather artificial condition.

In this work, we introduce a new class of boundary conditions, which we call tilted boundary conditions (TLBC), which is useful for derivation of the LSM-type theorems in dimensions higher than one. As we will demonstrate, with TLBC, the flux insertion argument can be applied without the artificial condition on the system sizes. This is also the case for the higher symmetry \((SU(N))\) generalizations [6, 8].

Furthermore, the TLBC reveals previously unnoticed relations between anomaly in field theory with the LSM-type theorems. While the LSM theorem has been well understood in the context of Tomonaga-Luttinger Liquids in one spatial dimension, field-theoretical understanding of LSM-type theorems in higher dimensions has been rather limited. The TLBC allows us to understand the LSM theorem as an anomaly manifestation. As a further application of the anomaly-based approach, we will discuss the implications of the time-reversal anomaly [15, 16] with a lattice interpretation. Taking the advantage of the bulk-boundary correspondence between SPT phase and the time-reversal anomaly on its boundary, the anomaly leads to an ingappability constraint for the half-filled \(\pi\)-flux system when time-reversal, on-site \(U(N)\) and magnetic translational symmetries are respected. Our constraint agrees with those derived for lattice models known for \(N = 1\) [17–20].
II. TLBC AND LSM THEOREM

A. Flux insertion with PBC

Let us first review the flux insertion argument [10] for the LSM theorem in $d \geq 2$ dimensions, and some of the problems in it. For example, for the (hyper)cubic lattice in $d$ dimensions with PBC

\[ \vec{r}' \sim \vec{r} + L_i \hat{x}_i, \quad i = 1, 2, \ldots, d, \]

(1)

where $L_i$ is the length along $i$-th unit vector $\hat{x}_i$, and the Hamiltonian is required to possess translational symmetry and $U(1)_Q$ symmetry with $\nu = p/q$ particles averagely per unit cell, where $p$ and $q$ are coprime. There is an additional charge quantization condition due to the fundamental degrees of freedom being charge one:

\[ \nu V \in \mathbb{N}, \]

(2)

where $V \equiv \prod_i L_i$ is the total volume, and this condition is also implied by the compactness of global $U(1)_Q$ symmetry.

Starting from a ground state, which is also a momentum eigenstate, as the initial state, we consider an adiabatic insertion of unit flux quantum $2\pi$ through the hole rounded by $L_1$, or equivalently, application of a time-dependent $U(1)_Q$ gauge field $A_1 = 2\pi t/(TL_1)$ with $T \to +\infty$. After the insertion of the unit flux quantum, the Hamiltonian is equivalent (under a large gauge transformation) to the original one without the flux. If there is an excitation gap above the ground state and the flux insertion process is indeed adiabatic, we may expect that the final state should be identical to the original ground state. However, for an incommensurate filling, there can be a nontrivial momentum shift due to the flux insertion. For instance, the lattice momentum change along $\hat{x}_1$ is

\[ \Delta P_1^{\text{PBC}} = 2\pi \frac{p}{q} L_2 L_3 \cdots L_d \mod 2\pi, \]

(3)

which is nonzero if $(L_2 \cdots L_d)$ is nondivisible by $q$. In this case, the final state cannot be identical to the initial ground state. Restricting the system sizes $L_2, L_3, \ldots, L_d$ to coprime with $q$, the statement of the LSM theorem is derived. Namely, the system is either gapless or has at least $q$ degenerate ground states below the gap. However, this artificial restriction on the system size is clearly undesired. Physically we would expect the same statement to hold for
FIG. 1. TLBC when $d = 2$.

generic (large) system sizes, even though the above argument does not lead to a nontrivial restriction if $L_2 L_3 \cdots L_d$ is divisible by $q$.

B. Flux insertion with tilted boundary condition

Here we propose to use, instead of the standard PBC, the following tilted boundary condition (TLBC)

\[
\begin{aligned}
\vec{r} + L_i \hat{x}_i &\sim \vec{r} + \hat{x}_{i+1}, \quad i = 1, \cdots, d - 1; \\
\vec{r} + L_d \hat{x}_i &\sim \vec{r}, \quad i = d,
\end{aligned}
\]

(4)

where $\hat{x}_i$ is the unit vector along $L_i$. We sketch the two-dimensional case of TLBC in FIG. (1). Geometrically, under such identifications, the space is a $d$-dimensional torus. Combined with the flux insertion and momentum counting, this leads to the LSM theorem without the artificial restriction on the system sizes, as we will discuss below. The TLBC is consistent with all the imposed symmetries—$U(1)_Q$ and translations. Now let us consider
the following flux insertion process \((t \in [0, T])\):

\[
A_i(t) = \frac{2\pi t}{T \prod_{k=1}^{d} L_k}, \quad i = 1, \cdots, d.
\] (5)

Such a composite flux insertion after time \(T\) can be eliminated by the following well-defined large gauge transformation:

\[
\hat{U}_0 = \exp \left( \sum_{\vec{r}} \frac{2\pi r_1 \hat{n}_{\vec{r}}}{V} \right),
\] (6)

where we fix the range of \(\vec{r}\) as

\[
\vec{r} = r_1 \hat{x}_1, \quad (r_1 = 1, \cdots, V),
\] (7)

exhausting all the lattice points. The lattice momentum change along \(\hat{x}_i\) during this insertion process is

\[
\Delta P_i = \frac{2\pi n}{\prod_{k=i}^{d} L_k} \mod \left( \frac{2\pi}{\prod_{k=i}^{d} L_k} \right)
\]

\[
= 2\pi \nu \left( \frac{V}{\prod_{k=i}^{d} L_k} \right) \mod \left( \frac{2\pi}{\prod_{k=i}^{d} L_k} \right), \quad i = 1, 2, \cdots, d
\] (8)

where \(n\) is the total number of particles and \(n/V = \nu\) is the filling fraction by definition, and the different periods of \(\Delta P_i\)'s results from the identification of translations as \(T_{i+1} = (T_i)^{L_i}\) due to TLBC.

A direct observation on Eq. (8) is that each component of \(\Delta \vec{P}\) gives exactly the same constraint on the ground-state degeneracies of gapped phases in contrast with the previous derivation in Eq. (3), and we arrive at the LSM theorem: if \(\nu = p/q\) with \(p\) and \((q > 1)\) coprime, a trivial insulating phase with a unique ground state is excluded in the thermodynamic limit, when \(U(1)_Q\) and translations are respected by the \(d\)-dimensional Hamiltonian. The result has a well-defined thermodynamic limit without any other system size requirement beyond the charge quantization condition of Eq. (2). It should be noted, however, the present argument still relies on the (nontrivial) assumption that the excitation gap does not collapse under the adiabatic flux insertion.

C. Anomaly manifestation of LSM theorem

The TLBC also reveals a connection between the LSM theorem in higher dimensions and anomaly in field theory in 1 + 1 dimensions.
Let us consider an electronic system, without loss of generality, on a square lattice with TLBC, the lattice point of which can be thought as that of unit cells on a general lattice with translational symmetries. First, we consider a non-interacting, tight-binding model:

\[ H_{T-B} = -t \sum_{\langle \vec{r}, \vec{r}' \rangle} c^\dagger_{\vec{r}} c_{\vec{r}'} - \mu \sum_{\vec{r}} c^\dagger_{\vec{r}} c_{\vec{r}}, \]  

(9)

where \( t \) is a real positive number with chemical potential \( \mu \) and \( c^\dagger_{\vec{r}} \) the creation operator of spinless fermion at lattice site \( \vec{r} \) with \( \langle \cdots \rangle \) denoting summations only over nearest-neighboring sites.

By a generalized 't Hooft anomaly matching \[6, 21–23\], nontrivial symmetry anomalies in a low-energy effective field theory of an arbitrarily fine-tuned lattice model implies an ingappability of general lattice models, constrained only by filling and symmetry structures. Therefore, the following analysis for the non-interacting model should be also valid for more general, interacting systems.

Under the TLBC, we can exhaust all the sites by translating a single site to one direction only, e.g. \( \vec{r} = j \hat{x}_1 \) with \( j = 1, 2, \cdots, V \). With this coordinate convention, we further define \( \Psi_j \equiv c_{j \hat{x}_1} \) which implies

\[ H_{T-B} = -t \sum_{j=1}^{V} \left[ \sum_{i=1}^{d} \Psi^\dagger_{j+V_i} \Psi_j + \text{h.c.} \right] - \mu \sum_{j=1}^{V} \Psi^\dagger_j \Psi_j; \]

(10)

with the boundary condition as \( \Psi_j = \Psi_{j+V} \) if the range of \( j \) is extended to all integers. Then we can express the Hamiltonian in the momentum space by

\[ \Psi_j = \frac{1}{\sqrt{V}} \sum_{k=0}^{V-1} \Psi(k) \exp \left( -i \frac{2\pi k}{V} j \right), \]

(11)

as

\[ H_{T-B} = \sum_{k=0}^{V-1} \left\{ -2t \left[ \sum_{i=1}^{d} \cos \left( \frac{2\pi V_i}{V} k \right) \right] - \mu \right\} \Psi^\dagger(k) \Psi(k) \]

\[ = \sum_{k \in \text{B.Z.}} \left\{ -2t \left[ \sum_{i=1}^{d} \cos \left( \frac{2\pi V_i}{V} k \right) \right] - \mu \right\} \Psi^\dagger(k) \Psi(k) \]

\[ \equiv \sum_{k \in \text{B.Z.}} \epsilon(k) \Psi^\dagger(k) \Psi(k), \]

(12)

where we have chosen the Brillouin zone which is symmetric at \( k = 0 \) so that the zeros of \( \epsilon(k) \) is symmetric around the origin. Thus we can label these zeros as \( \{ \pm K_c \}_{c=1, \cdots, N_0} \) due
to the even parity: $\epsilon(k) = \epsilon(-k)$. The low-energy Hamiltonian, which describes excitations near the zeros of $\epsilon(k)$, is given as

$$\mathcal{H} = \sum_{c=1}^{N_0} \int_{-\Lambda < k < \Lambda} v_c k \left( \Psi^\dagger_c(k) \Psi_c(k) - \Psi^\dagger_c(k) \Psi_c(k) \right).$$

(13)

Thus $N_0 \sim L_x \times L_y \times \cdots \times L_d$ represents the number of the one-dimensional channels ($1 + 1$-dimensional Dirac fermions). Here the ultraviolet cut-off $\Lambda \ll 2\pi/(N_0 a)$, where $a$ is the lattice constant, and $v_f$ is the value of fermi velocity not necessarily positive. The fermionic operator $\Psi_c(k) = \Psi(K_c + k)$ and $\bar{\Psi}_c(k) = \Psi(-K_c + k)$. It is essential to remark the role played by the charge quantization condition in Eq. (2). To do the continuum limit above where $k$ is taken to be a continuum variable, it is also necessary that the $k = 0$ modes or $\Psi(\pm K_c)$ modes exist in the Hilbert space of a finite system. In $d = 1$, Eq. (2) implies precisely such existence of $\Psi(\pm K_c)$ modes. Therefore, we expect this conclusion generalizes to arbitrary $d \geq 1$ and the charge quantization indeed permits us to do this continuum limit.

We absorb the $|v_c|$ into the following definition of two-component Dirac operator:

$$\psi_c(k) = \begin{cases} 
|v_c|\Psi_c(k), & v_c > 0; \\
|v_c|\bar{\Psi}_c(k), & v_c < 0.
\end{cases}$$

(14)

With the definition above, the real space formulation of Lagrangian density takes a compact form as

$$\mathcal{L}_{U(1)} = \sum_{c=1}^{N_0} \sum_{\mu=0}^{1} \bar{\psi}_c(t, x) i \gamma^\mu \partial_{\mu} \psi_c(t, x),$$

(15)

where $\gamma^0 = \sigma_2$ and $\gamma^1 = \sigma_1$ with $\vec{\sigma}$ Pauli matrices, and the chirality is $\gamma_3 = \sigma_3$.

Then we minimally couple it with an external gauge field as $A_1(t, x) = 2\pi t/(VT)$ and $A_0(t, x) = 0$: $\partial_{\mu} \rightarrow \partial_{\mu} - ieA_{\mu}$, which exactly corresponds to the flux insertion defined by Eq. (5) except for that we are left by only one spatial dimension since we have applied the coordinate convention $\vec{r} = j \hat{x}_1$ with $j = 1, \cdots, V$ before on the lattice.

Combining Eqs. (11,14), we obtain the lattice translation $T_1$ in $\hat{x}_1$ direction representation of $\psi_c(t, x)$ as:

$$\psi^{T_1}_{c}(t, x) \equiv T_1 \psi_c(t, x) T_1^{-1} = \exp \left[ i \text{sgn}(v_c) \pi \gamma_3 \frac{2K_c}{V} \right] \psi_c(t, x),$$

(16)
which is simply the chiral symmetry transformation, at low energy, appearing on-site. It is straightforward to apply Fujikawa’s method \[24, 25\] to calculate the global symmetry anomaly as the phase ambiguity of the fermionic partition function responding to the chiral transformation as Eq. (16) in the background gauge field configuration \(A_\mu(t, x)\).

\[
Z_{U(1)} \equiv \int \mathcal{D}(\bar{\psi}^{T_1}, \psi^{T_1}) \exp \left( -\int \mathcal{L}_{U(1)}[\bar{\psi}_c^{T_1}(\tau, x), \psi_c^{T_1}(\tau, x), A_\mu] \right)
\]

\[
\text{exp} \left[ i \sum_{c=1}^{N_0} 2 \text{sgn}(v_c) \frac{2\pi K_c}{V} \right]
\]

\[
= \text{exp}(i2\pi\nu), \tag{17}
\]

where the fermionic measure \(\mathcal{D}(\bar{\psi}^{T_1}, \psi^{T_1}) = J_{T_1} \cdot \mathcal{D}(\bar{\psi}, \psi)\) is calculated by Fujikawa’s \(U(1)\) gauge-invariant regularization \[24, 25\] and we have used the fact the filling fraction is \(\nu\) and \(\nu = \sum_c 2\text{sgn}(v_c)K_c/V\), and evaluate the formal path integral in the Euclidean signature: \(\tau = it\).

We define a lattice partition function ratio as:

\[
Z'_{U(1)} \equiv \frac{\text{Tr}_{\text{G.S.}} \left[ T_1 \hat{U}_0 \hat{U}_{\text{flux}} T_1^{-1} \right]}{\text{Tr}_{\text{G.S.}} \left[ \hat{U}_0 \hat{U}_{\text{flux}} \right]}
\]

\[
= \frac{\langle \text{G.S.} | T_1 \hat{U}_0 \hat{U}_{\text{flux}} T_1^{-1} | \text{G.S.} \rangle}{\langle \text{G.S.} | \hat{U}_0 \hat{U}_{\text{flux}} | \text{G.S.} \rangle}, \tag{18}
\]

where we denote by “Tr_{G.S.” taking the trace only within lowest energy states since the Wick rotation “\(\tau = it\)” in the definition of \(Z_{U(1)}\) implies that we should project out excited states when we define \(Z'_{U(1)}\). We also assume a unique gapped lattice ground state \(|\text{G.S.}\rangle\) and \(\hat{U}_{\text{flux}}\) denotes the unitary time evolution by flux insertion followed by the large gauge transformation \(\hat{U}_0\). The continuum-limit form of \(Z'_{U(1)}\) in Eq. (18) exactly coincides with the form of \(Z_{U(1)}\) in Eq. (17), where the large gauge transformation \(\hat{U}_0\) is implicitly presented in the path integrals within \(Z_{U(1)}\) since the inner product of wave functionals \(|\{\bar{\psi}, \psi\}A_\mu\rangle\)’s at the last time slice can be done only after fixing the gauge by the (large) gauge transformation \(\hat{U}_0\), which is in the same situation as the lattice model. The necessity of \(\hat{U}_0\) can be seen once one notices that the wave functional \(|\psi\rangle\) is an associated complex line sector of the underlying \(U(1)_Q\) principal bundle and \(\hat{U}_0\) in Eq. (6) is exactly the gluing transition function between the initial and the final time slices. It implies that \(Z_{U(1)}\) is the low-energy limit of \(Z'_{U(1)}\). Since \(Z_{U(1)}\) is a topological invariant, e.g. invariant along any symmetry-respecting
renormalization-group flow by the generalized 't Hooft anomaly-matching, we can evaluate $Z'_{U(1)}$ by its low-energy limit $Z_{U(1)}$:

$$Z'_{U(1)} = Z_{U(1)}.$$  \hspace{1cm} (19)

However, the unique ground state must be featureless hence a $T_1$-eigenstate, which implies $T_1|\text{G.S.}\rangle = \exp(iP)|\text{G.S.}\rangle$ thereby $Z_{U(1)} = 1$. This contradicts with $Z_{U(1)} = Z'_{U(1)} = \exp(i2\pi \nu)$ if $q \neq 1$, unless

$$\hat{U}_{\text{flux}}|\text{G.S.}\rangle \perp |\text{G.S.}\rangle,$$  \hspace{1cm} (20)

which still conflicts the unique gapped ground state. Thus the ground states must be degenerate for fractional fillings. Then we arrive at the LSM theorem with a well-defined anomaly-manifestation in a general thermodynamic limit. Furthermore, the physical interpretation of $Z_{U(1)}$ can be understood by its lattice partner $Z'_{U(1)}$ which exactly measures the momentum changes after the charge pumping. In this sense, we call that the chiral anomaly is lattice-realized. The anomaly we have considered is not an emergent anomaly which results from, e.g. the emergent low-energy properties of $T_1$, since we only use the global symmetry transformation of $T_1$ rather than gauging its low-energy representation with a finite cyclicity [21, 22]. The absence of such a potential emergent anomaly in our calculation can be explicitly attributed to the lattice-realizing formulation $Z'_{U(1)}$ of the anomaly $Z_{U(1)}$ since $T_1$ generates only $Z_V$ at the lattice scale restricting the general form of $Z'_{U(1)}$.

Finally, for $d = 0$, due to Eq. (2), the filling $\nu_0 \in \mathbb{N}$. Then, the low-energy effective response theory is simply the $(0 + 1)$-dimensional Chern-Simons theory with level $\nu_0$:

$$s_{U(1)} = \nu_0 \int dt A(t),$$  \hspace{1cm} (21)

by a minimal coupling observation. Such an effective action is well-defined since $\nu_0$ is an integer, and the zero-dimensional theory is $U(1)_Q$ anomaly-free thereby well-defined. It is applicable even for bosonic theory since there is no spatial coordinate to give one a choice of spin structures while the boundary condition along $t$ or $\tau$ for field operator is already fixed as, respectively, periodic and anti-periodic for bosonic and fermionic situations. Thus, $d = 0$ cases can be always trivially gapped if we only have a $U(1)_Q$ symmetry. This agrees with the trivial $d = 0$ version of the LSM theorem. While the statement is rather trivial, this is still a useful exercise to check the consistency of the anomaly argument. In the next section, we will see somewhat more non-trivial consistency check in $d = 0$ for a higher symmetry.
D. Bulk-boundary correspondences: LSM theorem with $U(1)_Q$

In this Subsection, based on our field-theory formulation related to the TLBC, we discuss the LSM theorem from the point of view of anomaly inflow. This enables us to construct a higher dimensional SPT bulk theory where the massless theory $\mathcal{L}_{U(1)}$ in Eq. (15) can be seen as a boundary theory attached to the SPT bulk. To simplify the discussion, in the following we first assume $N_0 = 1$. We will generalize the argument to arbitrary $N_0$ later.

The translation symmetry is reduced to:

$$\psi^{T_{\xi}} = \exp \left( i \gamma_3 \pi \nu \right) \psi = \exp \left( i \gamma_3 \pi \frac{p}{q} \right) \psi.$$  \hfill (22)

By an observation on the left-right decomposition of the Dirac spinor as $\psi = [\psi_L, \psi_R]$ and an integer quantum Hall system supports chiral edge modes, we can arrive at the conclusion that the bulk effective response theory is composed by two Chern-Simons theories with opposite levels, which correspond to these two chiral components $\psi_{L,R}$:

$$S_{\text{bulk},U(1)} = \int \left[ -\frac{i}{4\pi} A_L \wedge dA_L + \frac{i}{4\pi} A_R \wedge dA_R, \right]$$  \hfill (23)

where

$$A_L = A_{U(1)} + a/2;$$

$$A_R = A_{U(1)} - a/2.$$  \hfill (24)

Here $A_{U(1)}$ is the background $U(1)_Q$ gauge field and $a\gamma_3/2$ is the background gauge field for the translational symmetry in Eq. (22) and is extended to a $U(1)$ gauge field, where we have taken a “1/2” normalization convention so that $a$ has the same constraint as a $\mathbb{Z}_q$-gauge field:

$$\oint_{\text{closed loop}} a \in 2\pi\mathbb{Z},$$  \hfill (25)

with $da = 0$. Then,

$$S_{\text{bulk},U(1)} = -\int \frac{i}{2\pi} a \wedge dA_{U(1)},$$  \hfill (26)

which is exactly a BF-theory \[26, 27\] coupling the $\mathbb{Z}_q$-gauge field and the $U(1)_Q$-gauge field. To detect the nontrivial aspect of such an SPT bulk, we evaluate the bulk partition function
on a compact closed manifold, e.g. three-torus $T^3 = S^1 \times T^2$, with the following background
gauge field:

\[
\int_{S^1} a = 2\pi \nu, \quad (27) \\
\int_{T^2} \frac{dA_{U(1)}}{2\pi} = 1, \quad (28)
\]

where $a$ is a pull-back from a flat gauge field on $S^1$ and $A_{U(1)}$ is a pull-back from a gauge
field on $T^2$ with a unit Chern number.

Therefore,

\[
Z_{\text{bulk},U(1)} = \exp(-S_{\text{bulk},U(1)}) = \exp(i2\pi \nu), \quad (29)
\]

which exactly reproduces the anomaly factor in Eq. (17). $Z_{\text{bulk},U(1)}$ above characterizes the
higher dimensional extension dependence of the boundary theory $\mathcal{L}_{U(1)}$.

This bulk construction can be generalized to arbitrary $N_0$ in $\mathcal{L}_{U(1)}$ by replacements of
“1/2” in Eq. (24) by color-dependent coefficients “sgn($v_c$)$K_c/(V\nu)$” due to Eq. (16). Then,
after the color indices “c” is summed up and by $\nu = \sum_c 2\text{sgn}(v_c)K_c/V$, we arrive at the
same effective bulk response theory as Eq. (26).

III. LSM THEOREM FOR $SU(N)$ SPIN-ROTATION SYMMETRY

In the following, we will use the methods developed in the previous section to investigate
the higher symmetry generalization of LSM theorem, e.g. replacement of the global onsite
symmetry by $SU(N)$ spin-rotation symmetry, or more precisely a $PSU(N) = SU(N)/\mathbb{Z}_N$
global symmetry [6, 28] by the spin operator satisfying the following $su(N)$ Lie algebra
commutation relations:

\[
\left[ S^{\alpha}_{\vec{r},\beta}, S^{\gamma}_{\vec{r}',\delta} \right] = \delta^{\alpha}_{\vec{r},\vec{r}'} \left( \delta^{\gamma}_{\delta} S^{\alpha}_{\vec{r},\beta} - \delta^{\alpha}_{\beta} S^{\gamma}_{\vec{r},\delta} \right), \quad (30)
\]

where $\alpha$ and $\beta$ are the “spin” indices that take values among 1 to $N$. In particular, we
focus on the quantum-anomaly manifestation of the LSM theorem. Its advantage is that
the lattice ingappabilities can be detected at any fine-tuned critical point which simplifies
the calculation, thanks to the ’t Hooft anomaly matching.
A. "Spin"-quantization condition

Let us consider the most general situation that the total number of Young-tableaux boxes per unit cell is \( b \). Analogous to Eq. (2), we also take the following "spin"-quantization condition:

\[
\frac{bV}{N} \in \mathbb{N}.
\]  

(31)

The reason we assume this condition is different from the \( U(1)_Q \) LSM case where charge quantization is naturally imposed by fundamental degrees of freedom. To clarify this seemingly unnatural requirement, let us see the situation where Eq. (31) is not satisfied, namely \( bV/N \notin \mathbb{N} \). Then the total Young-tableaux boxes \( bV \) of the system is not divisible by \( N \). By the knowledge from representation theory, there is no \( SU(N) \)-singlet sector in the Hilbert space and by the global \( PSU(N) \) symmetry, the system is exactly degenerate, even at all excited states since the states within any non-singlet irreducible representation must have the same energy due to Schur’s lemma applied within any of these nontrivial irreducible sectors.

Such a rather trivial type of ingappabilities is essential to understand finite-system spectrum. Nevertheless, we are not interested in it since these ingappabilities are not a many-body effect and they depend on a specific choice of system sizes. A many-body ingappability is commonly considered as almost degeneracies, e.g. the ingappabilities make sense only at general thermodynamic limits. Therefore, in the following discussion for \( (d \geq 1) \)-dimensional system, we avoid such an exact degeneracy explicitly exposed above by imposing Eq. (31). Nevertheless, when stating the final theorem without loss of generality, we will also include the cases that \( bV/N \notin \mathbb{N} \) for which the ground states are exactly degenerate.

B. Generalized LSM theorem and anomaly manifestation

Since we are only interested in the low-energy spectrum of the lattice model, we can equivalently reconstruct its low-energy physical properties by coupling \( b \) copies of the lattice models each of which has one fundamental \( SU(N) \) degree of freedom within each unit cell. It is possible due to the group-theoretical knowledge that any \( SU(N) \) irreducible representation with \( b_0 \) Young-tableaux boxes is contained in the tensor products of \( b_0 \) of fundamental representations. Then we can project out all the undesired degrees of freedom in the analog
of Affleck-Kennedy-Lieb-Tasaki chain construction from spin-1/2 degrees of freedom [29].
Moreover, such a projection can be realized by a strong interaction dynamically, hence the
quantum anomaly factor of the original system can be obtained by a summation of the
anomalies of these $b$ of fundamental lattices. Then the problem is reduced to the anomaly
related to fundamental lattices we will calculate below.

Similarly to the generalized LSM theorem in one dimension, we do the following $N$-flavor
fermionization representing spin degrees of freedom:

$$S^\alpha_{\vec{r}, \beta} = \Psi^{\alpha \dagger}(\vec{r})\Psi_{\beta}(\vec{r}) - \frac{1}{N}\delta^\alpha_\beta$$

with the restriction of total particle number on every site “\vec{r}”

$$\sum_{\alpha=1}^{N} \Psi^{\alpha \dagger}(\vec{r})\Psi_{\alpha}(\vec{r}) = 1,$$

so that $S^\alpha_{\vec{r}, \beta}$’s defined in Eq. (32) satisfy Eq. (30).

Again, we impose the TLBC defined as Eq. (4). In the analog of $SU(2)$ cases, a fine-tuned
critical model can be the Hubbard model in the conductive phase:

$$H_{SU(N)} = -t \sum_{\langle \vec{r}, \vec{r}' \rangle} \sum_{\alpha=1}^{N} \Psi^{\alpha \dagger}(\vec{r})\Psi_{\alpha}(\vec{r}') + \text{h.c.} - U \left[ \sum_{\alpha=1}^{N} \Psi^{\alpha \dagger}(\vec{r})\Psi_{\alpha}(\vec{r'}) - 1 \right]^2,$$

where the Hund’s rule coupling $U \gg |t|$ realizes the particle number restriction per unit cell
in Eq. (33). By an observation on Eq. (34) and Eq. (9), we can see that: 1) $SU(N)$ case has
a stronger particle number restriction per unit cell; 2) each flavor has a filling fraction $1/N$
within a unit cell. With this comparison, it is straightforward to derive the effective theory
of the current $SU(N)$ model after we apply the coordinate system $\vec{r} = j\hat{x}_1$ with $j = 1, \cdots, V$
again:

$$\mathcal{L}_{SU(N)}[\mathcal{A}] = \sum_{c=1}^{N_0} \sum_{\alpha=1}^{N} \bar{\psi}_c^\alpha(t, x)i\gamma^\mu(\partial_\mu - i\mathcal{A}_\mu)\psi_{c,\alpha}(t, x),$$

$$\nu_{\text{eff}} = \frac{1}{N},$$

where the inclusion of a dynamical fluctuating $U(1)_Q$ gauge field is used to eliminate $U(1)_Q$
phase degrees of freedom. It is because $U(1)_Q$ is unphysical, e.g. the fundamental observable
$S^\alpha_{\vec{r}, \beta}$ is invariant under $U(1)_Q$. It can be also viewed as a realization of the particle number
constraint as Eq. (33) regularized by the Grassmann-number ordering ambiguity to make the Lagrangian explicitly $U(1)_Q$ gauge invariant:

$$\sum_{\alpha=1}^{N} \Psi^\dagger(\vec{r}) \Psi(\vec{r}) = 1,$$

$$\sum_{\alpha=1}^{N} \left\{ (1 - \frac{1}{N}) \Psi^\dagger(\vec{r}) \Psi(\vec{r}) - \frac{1}{N} \Psi^\dagger(\vec{r}) \Psi(\vec{r}) \right\} = 0,$$

$$\sum_{\alpha} \left\{ (1 - \frac{1}{N}) \psi^\dagger(t, x) \psi(t, x) - \frac{1}{N} \psi^\dagger(t, x) \psi(t, x) = 0 \right\},$$

which can be done by a Lagrangian multiplier $\sum_{\alpha} \mathcal{A}_0 \bar{\psi}^\dagger \gamma^0 \psi(\vec{r})$ consistent with gauge invariance. The zeros of Hamiltonian $\epsilon(\alpha)(k)$ for each flavor in momentum space in the same notations satisfy:

$$\sum_{c=1}^{N_0} 2 \text{sgn}(v_c) K_c / V = \nu_{\text{eff}}.$$  

Similarly to Eq. (2), the $U(1)_Q$-LSM case, Eq. (31) also guarantees the existence of the continuum limit of $k$ variables in $\Psi(\pm K_c + k)$. Then we couple the theory to a background $PSU(N)$ gauge field $A_{PSU(N)}$ which locally takes value in $su(N)$ algebra. In the following discussion, $t_{N^2-1} \equiv \text{diag}[1, 1, \cdots, 1, -(N - 1)]_{N \times N}$ is denoted as the matrix representation of the last $su(N)$ generator. First, let us analyze the current gauge group, which actually is not $U(1) \times SU(N)$ since its center $Z_N$ generated by $(\exp(i2\pi/N), \exp(-i2\pi t_{N^2-1}/N)) \in U(1) \times SU(N)$ does not transform the matter field $\{\psi(\vec{r})\}$. It implies that the gauge group is $[U(1) \times SU(N)] / Z_N \cong U(N)$, and thus only $A_{PSU(N)} + \mathcal{A}$, $NA_{PSU(N)}$ and $N\mathcal{A}$ are canonical 1-form connections for fermions, which allow well-defined curvatures.

We consider the following flux insertion which generalizes the $U(1)$ case:

$$A_{PSU(N)} + \mathcal{A} = \frac{2\pi t}{TVN}(1 - t_{N^2-1})_{N \times N} + \delta \mathcal{A},$$

where $\delta \mathcal{A}$ is a dynamical canonical $U(1)$ connection since $[2\pi t(1 - t_{N^2-1}) / (TVN)]$ is globally well-defined, and then the functional integration $\int \mathcal{D} \mathcal{A}$ over $u(1) / Z_N$ connection can be converted to $\int \mathcal{D} \delta \mathcal{A}$ over $u(1)$ connection. The physical interpretation of such a $PSU(N)$ “flux” insertion is that we adiabatically rotate the spin along $\hat{x}_1$ direction and this spatial
dependent rotation matrix projectively represented by $SU(N)$ matrix is multi-valued since it is identified up to a center of $SU(N)$ due to the local $U(1)$ gauge degrees of freedom by $\mathcal{A}$. Therefore, the accompany $U(1)$ twisting is simply to compensate this artificial ambiguity in the $SU(N)$ rotation matrix.

In a similar sense, we can calculate the anomaly factor by Fujikawa’s gauge-invariant regularization \cite{6,24} on the fermionic measure $\prod_{c,\alpha} \mathcal{D}(\bar{\psi}^{T_1}_{c,\alpha}, \psi^{T_1}_{c,\alpha}) = J(T_1) \prod_{c,\alpha} \mathcal{D}(\bar{\psi}_{c,\alpha}, \psi_{c,\alpha})$ below:

$$Z_{SU(N)} \equiv \int \mathcal{D}\mathcal{A} \mathcal{D}(\bar{\psi}^{T_1}, \psi^{T_1}) \exp(-\int \mathcal{L}_{SU(N)}[\bar{\psi}^{T_1}_{c,\alpha}(\tau, x), \psi^{T_1}_{c,\alpha}(\tau, x), \mathcal{A} + A_{PSU(N)}])$$

$$\int \mathcal{D}\mathcal{A} \mathcal{D}(\psi, \bar{\psi}) \exp(-\int \mathcal{L}_{SU(N)}[\bar{\psi}_{c,\alpha}(\tau, x), \psi_{c,\alpha}(\tau, x), \mathcal{A} + A_{PSU(N)}])$$

$$= \exp \left[ i \sum_{c=1}^{N_0} 2\text{sgn}(v_c) \frac{2\pi K_c}{V} \right]$$

$$= \exp \left( \frac{2\pi}{N} \right),$$

(39)

where we have made use of the same notation for the transformation rule of $\psi \to \psi^{T_1}$ as Eq. (16) except that here we have a specified filling factor $\nu_{\text{eff}} = 1/N$ for each flavor.

Thus for the general case of $b$ of Young-tableaux boxes per unit cell:

$$Z^{(b)}_{SU(N)} = (Z_{SU(N)})^b$$

$$= \exp \left( i \frac{2\pi b}{N} \right).$$

(40)

Similarly to the $U(1)$ case in LSM theorem, we can also define the lattice-realization of $Z^{(b)}_{SU(N)}$ by $Z^{(b)}_{SU(N)}$ analogous to Eq. (18), and by anomaly-matching: $Z^{(b)}_{SU(N)} = Z^{(b)}_{SU(N)}$.

C. LSM-type anomaly in 0 dimension

To complete the $SU(N)$ generalization of LSM theorem, we also study the case of $d = 0$, which is a problem in single-body quantum mechanics. Unlike in $d \geq 1$, no quasi-degeneracy (asymptotic degeneracy in the thermodynamic limit) can be defined from the energy spectrum. Exact ground-state degeneracy can still be defined for $d = 0$. However, the exact degeneracy is generally a direct consequence of Schur’s lemma. Namely, an $SU(N)$ spin in any nontrivial irreducible representation must have exact degeneracy for a $SU(N)$-symmetric Hamiltonian.

Here we consider the spectrum of Hamiltonian imposing only the discrete subgroup $\mathbb{Z}_N \times \mathbb{Z}_N$ of $PSU(N)$, instead of the full $PSU(N)$ symmetry. In this case, an irreducible
representation of $PSU(N)$ can become reducible and the degeneracy of the spectrum may be lifted. Whether the ground state can be made unique by such a level splitting is a question analogous to the “ingappability” discussed for $d \geq 1$ dimensions in the context of the LSM theorem.

While this setting looks rather different from the LSM theorem in $d \geq 1$ dimensions, this can be understood in terms of quantum anomaly of a “field theory” in $0 + 1$ dimension. Indeed, the partition function would have a phase ambiguity from the projective representation by an $SU(N)$ spin. Despite of the facts that Dijkgraaf-Witten theory provides a non-Lagrangian construction of such an anomaly [30] and the boundary state of Haldane phase provides a bosonic realization in the $N = 2$: $SU(N) = SU(2)$ case [3], we are still interested in manifesting it in our explicit fermionic Lagrangian representation.

Before the general discussion for $SU(N)$ case, we give several explicit examples for $N = 2, 3$ cases. If $N = 2$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $\exp(i\pi S_x/2)$ and $\exp(i\pi S_x/2)$, which, in the fundamental representation $\rho_f$, satisfy
\[
\rho_f[\exp(i\pi S_x/2)]\rho_f[\exp(i\pi S_x/2)]\rho_f^{-1}[\exp(i\pi S_x/2)]\rho_f^{-1}[\exp(i\pi S_x/2)] = -1. \tag{41}
\]
If $N = 3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is generated by $\exp[i\pi(t_8 - t_3)/3]$ and $\exp[-i2\pi(t_2 - t_5 + t_7)/(3\sqrt{3})]$ where, in the fundamental representation, $\rho_f[t_8] = \text{diag}[1,1,-2]$, $\rho_f[t_3] = \text{diag}[1,-1,0]$, $\rho_f[t_2] = i\delta_{t,2}\delta_{m,1} - i\delta_{t,1}\delta_{m,2}$, $\rho_f[t_5] = i\delta_{t,3}\delta_{m,1} - i\delta_{t,1}\delta_{m,3}$ and $\rho_f[t_7] = i\delta_{t,3}\delta_{m,2} - i\delta_{t,2}\delta_{m,3}$ and, by Eq. (A.5),
\[
\rho_f[\exp[i\pi(t_8 - t_3)/3]]\rho_f[\exp[-i2\pi(t_2 - t_5 + t_7)/(3\sqrt{3})]]. \tag{42}
\]
For the adjoint representations, the commutators in Eqs. (41,42) are trivial by Eqs. (A.7,A.8), and the constructions of the $\mathbb{Z}_N \times \mathbb{Z}_N$ generators for other $SU(N)$ cases can be found in [31].

A general representation with $b$ Young-tableaux boxes for $SU(N)$ cases yields a commutator as $\exp(i2\pi nb/N)$, which reflects the $H^2(\mathbb{Z}_N \times \mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$ classification of the projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$, where $H^2(\cdot, U(1))$ denotes the second group cohomology with a $U(1)$ coefficient ring. Moreover, two transformations with a nontrivial commutator as Eqs. (41,42) cannot share a single one-dimensional invariant sub-Hilbert-space. These results exactly imply that any $b$ Young-tableaux representation $SU(N)$ “spin” can be gapped with a unique ground state respecting $\mathbb{Z}_N \times \mathbb{Z}_N$ if and only if $N$ divides $b$. Let us discuss and derive such zero-dimensional (in)gappabilities in our framework of fermionic anomaly.
For general $SU(N)$ cases, we first study the fundamental representation case or $b = 1$ at a single point. The low-energy effective theory for the (fine-tuned) lattice Hamiltonian $H_{SU(N)} = 0$ is simply:

$$L_{SU(N)}[A] = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha(\tau, x) i(\partial_\tau - iA_\tau) \psi_\alpha(\tau, x),$$

and also the $U(1)_Q$ dynamical gauge field is again used to project out the unphysical Hilbert subspace, and the $U(1)_Q$ charge at the site is restricted to be 1 the same as Eq. (33). Let us first fix the $U(1)_Q$ gauge field, or we treat $A$ as a background gauge field temporarily. Then we look for a one-dimensional bulk effective response theory on $X$ for this zero-dimensional system with a background field $A$ and $\partial X$ the temporal $S^1$ of the zero-dimensional system. The reason that we do not directly write down the zero-dimensional response theory is that the bulk theory is well-defined when the boundary theory has an anomaly making itself ill-defined. The most general bulk theory in the low-energy limit is the 2-D theta-term

$$S_{\text{bulk}}[A, X] = i\theta \int_X \frac{\mathcal{F}}{2\pi},$$

and then we can regularize the zero-dimensional partition function as

$$z_{SU(N)}[A] = |z_{SU(N)}[A]| \exp(-S_{\text{bulk}}[A]),$$

$$z_{SU(N)} = \int \mathcal{D}A \, z_{SU(N)}[A].$$

However, in our current system, we have no symmetry constraint on the choice of $\theta$. Nevertheless, the “filling constraint” in Eq. (33) restricts the $U(1)_Q$ charge. Let us do a dimensional reduction on $S_{\text{bulk}}[A, X]$:

$$S_{\text{bulk}}[A, X] = i\theta \int_X \frac{dA}{2\pi}$$

$$= i\frac{\theta}{2\pi} \int_{\tau:S^1} A_0,$$

which exactly means that the charge is $\theta/2\pi$, and, by Eq. (33),

$$\theta = 2\pi.$$

Then we consider the quantization property of $S_{\text{bulk}} = i \int_X \mathcal{F}$. The Dirac quantization gives $S_{\text{bulk}} \in 2\pi i \mathbb{Z}$ for any compact manifold without boundary which however is, as we will see below, still reducible since we have also imposed $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry with generators as $V_N$.
and $W_N$ having the following properties. Their spinor representations on $\psi_\alpha$ are denoted by matrices $\rho(V_N)$ and $\rho(W_N)$. The concrete expressions of $\rho(V_N)$ and $\rho(W_N)$ are not essential and basis-dependent, but the only substantial relation is
\[
\rho(V_N)\rho(W_N)\rho^{-1}(V_N)\rho^{-1}(W_N) = \exp(i2\pi/N).
\]
Let us consider two kinds of bulk extension $X$ and $X'$ with $\tilde{X} = X \coprod (-X') = T^2$ where $X$ and $(-X')$ are pasted along their common boundary, and we will construct a $\mathbb{Z}_N \times \mathbb{Z}_N$ bundle in the following way.

First we set up a discrete gauge bundle:
\[
\psi(\tau + L_\tau, s) = -\rho(V_N)\psi(\tau, s); \quad (50)
\]
\[
\psi(\tau, s + L_s) = \rho(W_N)\psi(\tau, s), \quad (51)
\]
where $s$ is the extra dimension extending the dimension. However, such a discrete-symmetry gauge “bundle” is ill-defined because of Eq. (49). To obtain a consistent gauge bundle, we insert a $U(1)_Q$ flux:
\[
\int_{\tilde{X}} \mathcal{F} = -\frac{2\pi}{N}, \quad (52)
\]
which induces an Aharonov-Bohm phase exactly eliminate the phase ambiguity brought by the commutator in Eq. (49).

Then
\[
S_{\text{bulk}}|_{\tilde{X}} = -2\pi i \frac{1}{N}. \quad (53)
\]
Thus the phase ambiguity of $z_{SU(N)}[\mathcal{A}]$ due to the different dimension extension $X$ and $X'$ is $\exp(i2\pi/N)$. Furthermore, since such a phase is independent on the fluctuating gauge field $\mathcal{A}$ which is integrated within one gauge sector connected by small gauge transformations, we can extract this phase out of the integration $\int \mathcal{D}\mathcal{A}$. Thus this phase ambiguity denoted by $Z_{SU(N)}$ is also shared by $z_{SU(N)}$ after such a functional integration:
\[
Z_{SU(N)} = \exp(i2\pi/N). \quad (54)
\]
By additivity of anomaly, we have for general representations with total $b$ of Young-tableaux boxes:
\[
Z_{SU(N)}^{(b)} = (Z_{SU(N)})^b = \exp \left( i2\pi \frac{b}{N} \right). \quad (55)
\]
This precisely implies that the partition function is enjoying an anomaly resultant from a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$. Indeed, when $b$ is divisible by $N$, such as spin-$1$ (adjoint) representation of $su(2)$, the interaction $S_z^2$ can gap the ground state uniquely preserving $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $\exp(i\pi S_x/2)$ and $\exp(i\pi S_z/2)$. For the adjoint representation of $su(3)$, which is made of three Young-tableaux boxes, the ground state can be also gapped uniquely since the generators of $\mathbb{Z}_N \times \mathbb{Z}_N$ share at least one, actually eight, one-dimensional invariant sub-Hilbert-spaces by Eqs. (A.7,A.8).  

Including the exact degeneracies in $d \geq 1$ when $bV/N \notin \mathbb{N}$ as well, Eq. (40) together with Eq. (55) exactly gives the following generalized LSM theorem in any dimension of $d \geq 0$, which has a well-defined thermodynamic limit independent on the system size specification:

If a $(d > 0)$-dimensional Hamiltonian possesses $SU(N)$ spin-rotation and translation symmetries, or if a $(d = 0)$-dimensional Hamiltonian possesses $\mathbb{Z}_N \times \mathbb{Z}_N$, the system does not permit a trivially gapped phase when the total number $b$ of Young-tableaux boxes per unit cell is not divisible by $N$. The converse is true if $d = 0$.

It is an appropriate point to remark that the role played by filling constraint is different between $d = 0$ and $d > 0$. In the former case, the filling condition restricts the theta term of a possible bulk extension, while filling restricts the form of translation symmetry at the low-energy continuum limit in $d > 0$. Moreover, unlike LSM theorem in arbitrary dimensions discussed in Sec. II, our generalization of LSM does not have a compact form when $d = 0$ is included, because of the higher symmetry $PSU(N)$ than $U(1)_Q$.

D. Bulk-boundary correspondences: LSM theorem with $PSU(N)$

In this Subsection, we will construct the SPT "bulk" theory in one higher dimensions, which has the boundary theory with $PSU(N)$ symmetry with the LSM-type anomaly, in two different ways.

1. A weak SPT point of view

To check whether the lattice system has any other LSM-type anomaly beyond Eq. (40), we make use of a weak-SPT viewpoint on quantum anomaly calculated before. Since the quantum anomaly of a spatially $d$-dimensional system can be understood as that of the
boundary of a spatially \((d + 1)\)-dimensional bulk SPT phase. Therefore, the relevant SPT bulk that the LSM-type anomaly of system corresponds to is assumed to lie in the classes of \(H^{d+2}(PSU(N) \times \mathbb{Z}^d, U(1))\) which is the \(PSU(N)\) (group) cohomology group with coefficient ring as \(U(1)\) [3], where \(\mathbb{Z}^d\) is a direct product of \(d\) of lattice translations corresponding to the translations of its \(d\)-dimensional spatial boundary in the thermodynamical limit [28]. By Künneth formula,

\[
H^{d+2}(PSU(N) \times \mathbb{Z}^d) = H^{d+2}(PSU(N) \times \mathbb{Z}^{d-1}) \oplus H^{d+1}(PSU(N) \times \mathbb{Z}^{d-1}) = H^{d+2}(PSU(N) \times \mathbb{Z}^{d-1}) \oplus H^{d+1}(PSU(N) \times \mathbb{Z}^{d-2}) \oplus H^d(PSU(N) \times \mathbb{Z}^{d-2}) = \cdots = \bigoplus_{k=3}^{d+2} H^k(PSU(N) \times \mathbb{Z}^{k-3}) \oplus H^2(PSU(N)),
\]

where, for clearness, the coefficient ring \(U(1)\) is suppressed. Let us explain the first line of the equation above: \(H^{d+2}(PSU(N) \times \mathbb{Z}^{d-1})\) represents the \((d + 1)\)-spatially-dimensional SPT phases protected by \(PSU(N) \times \mathbb{Z}^{d-1}\) where the \(d\)-th \(\mathbb{Z}\) is neglected while \(H^{d+1}(PSU(N) \times \mathbb{Z}^{d-1})\) denotes the \((d + 1)\)-dimensional SPT phases constructed by stacking, uniformly in the \(d\)-th direction, \(d\)-dimensional SPT’s protected by \(PSU(N) \times \mathbb{Z}^{d-1}\), where the \(d\)-th translation \(\mathbb{Z}\) is the obstruction for the phase trivialized by “dimerization” along that direction. Similarly, we can continue these analyses straightforward and arrive at that the \(H^2(PSU(N))\) on the last line exactly means \((d + 1)\)-spatially-dimensional SPT phases stacked by copies of a one-dimensional SPT phase (or chain) protected by \(PSU(N)\) uniformly in all the \(d\) spatial dimensions while all the \(d\) of translations are obstruction for it to be “dimerized” (or trivialized). Therefore, one characterizing property of phases by \(H^2(PSU(N))\) is that lifting any of the \(d\) translations and \(PSU(N)\) out will trivialize the phase, while other components do not have this property. In this sense, \(H^2(PSU(N))\) is the maximally mixed anomaly of \(PSU(N) \times \mathbb{Z}^{d}\).

On the other hand of our current system, the generalized LSM also has the same “mixing” characteristic since any \(N\) of unit cells can be \(SU(N)\) singlet if any of the translations are permitted to be broken explicitly, and \(PSU(N)\) symmetry is obviously essential for the ingappability as well. Thus the LSM-type anomaly of our system is a class in \(H^2(PSU(N))\). Conversely, the anomaly factor in Eq. (40) exactly implies a \(\mathbb{Z}_N\) group structure and the generator is the fundamental case: \(b = 1\). Therefore, \(b = 1, \cdots, N\) represents the \(\mathbb{Z}_N\)
elements. Since \( H^2(PSU(N)) \cong \mathbb{Z}_N \) thereby saturated by \( b = 1, \cdots, N \), we can come to the conclusion that all possible LSM-type anomalies have been extracted out by Eq. (40) with \( b = 1, 2, \cdots, N \), which are exactly classified by \( H^2(PSU(N)) \).

2. A BF-theory approach

Similarly to the \( U(1) \) case studied in details in Sec. II D, we can also derive the bulk theory as the following BF theory:

\[
S_{\text{bulk,PSU}} = - \int \frac{i}{2\pi} \text{Tr} \left[ a \wedge dA_{PSU(N)} \right],
\]

with “Tr” taken on \( PSU(N) \) indices and

\[
\oint_{\text{closed loop}} a \in 2\pi \frac{b}{N} \mathbb{Z},
\]

and \( da = 0 \). We can take the following gauge bundle on \( T^3 = S^1 \times T^2 \):

\[
\int_{S^1} a = 2\pi \frac{b}{N},
\]

\[
\int_{T^2} \text{Tr} \left[ \frac{dA_{PSU(N)}}{2\pi} \right] = 1,
\]

where \( a \) is a pull-back from a flat gauge field on \( S^1 \) and \( A_{PSU(N)} \) is a pull-back from a gauge field on \( T^2 \) with a unit Chern number when we extend it to a \( U(N) \) bundle (See Eq. (38) or [6]). Therefore,

\[
Z_{\text{bulk,PSU}} = \exp(-S_{\text{bulk,U(1)}}) = \exp \left( i2\pi \frac{b}{N} \right),
\]

by which we rederive the same anomaly factor as Eq. (55).

IV. TLBC APPLIED TO THE INGAPPA BILITIES BY TIME REVERSAL

In this Section, we will discuss the application of TLBC to the ingappability constrained by a time reversal symmetry. As an example, let us consider the following half-filled \( N \)-flavor spinless fermion on a square lattice with \( \pi \)-flux per plaquette:

\[
H_{\pi} = \sum_{f=1}^{N} t \left( \sum_{\vec{r}} c_{(\vec{r}+\hat{x})f}^\dagger c_{\vec{r}f} + \text{h.c.} \right) + \sum_{f=1}^{N} t \left[ \sum_{\vec{r}} c_{(\vec{r}+\hat{y})f}^\dagger c_{\vec{r}f}(-1)^{\tau_1} + \text{h.c.} \right]
\equiv H_{\pi 1} + H_{\pi 2},
\]
where we have chosen a gauge such that $H_{\pi}$ is still lattice-translation symmetric along $\hat{y}$. Nevertheless, the translation symmetry along $\hat{x}$ seems to be broken to two sites. However, physically, the system should be still symmetric along $\hat{x}$ with one site, and such a translation symmetry and the original translation along $\hat{y}$ in the current gauge choice, called magnetic translations [32, 33] equally for all flavors $f$’s satisfies:

$$
T_1 c_{\vec{r}f} T_1^{-1} = c_{(\vec{r}+\hat{x})f} \exp(i\pi r_x),
$$

$$
T_2 c_{\vec{r}f} T_2^{-1} = c_{(\vec{r}+\hat{y})f}.
$$

(63)

We will use $T_{1,2}$ to denote the magnetic translations in the following discussion, instead of original lattice translations. To make $T_1$ a well-defined unitary transformation representing an exact symmetry, we need to imposed that $L_y \in 2\mathbb{N}$. Then the charge quantization in Eq. (2) is automatically fullfilled. It should be noted that the gauge-invariant nature of $T_{1,2}$ is encoded in their commutator:

$$
T_1 T_2 T_1^{-1} T_2^{-1} = -1.
$$

(64)

Here “half-filled” implies that the particle number per physical $1 \times 1$ unit cell is $1/2$. However, the LSM theorem, even after generalized to $U(N)$ cases, cannot say anything nontrivial for the present case with the magnetic translation symmetry. Indeed, the following interaction, respecting $U(N)$ and $T_{1,2}$, can open a gap with a unique ground state:

$$
\Delta H = \sum_{\vec{r}, f} t(-1)^{\eta_1} \left[ i c_{\vec{r}f}^\dagger c_{(\vec{r}+\hat{x}+\hat{y})f} + c_{\vec{r}f}^\dagger c_{(\vec{r}+3\hat{x}+\hat{y})f} \right] + \text{h.c.}.
$$

(65)

It should be noted, however, that $\Delta H$ breaks the time-reversal symmetry explicitly. Thus we may expect that imposing time-reversal symmetry in addition to the magnetic translation symmetry potentially obstructs the trivially gapping.

Indeed, it has been proposed that, in $N = 1$ case, the Hall conductance of the system when trivially gapped by a $U(1)_Q$ and $T_{1,2}$ symmetric interaction, must be odd [17–20]. This implies that, there cannot be a unique ground state with a non-vanishing excitation gap, when the time-reversal symmetry is additionally imposed. However, the arguments in Refs. [17–20] again rely on the special choice of the system sizes. Here we apply the TLBC to the systems invariant under the magnetic translation and the time reversal, which reveals an anomaly manifestation of the ingappability constraint due to the time-reversal symmetry in the context of field theory.
A. Low-energy effective theory and symmetries

Our strategy is to apply the TLBC to the low-energy effective field theory. As a preparation, we first use the PBC to define the momentum representation (Fourier components)

\[ c_{(2n_1,r_2)} = \sum_{\vec{k}} a_{\vec{k}} \exp(i k_1 n_1 + ik_2 r_2), \]
\[ c_{(2n_1+1,r_2)} = \sum_{\vec{k}} b_{\vec{k}} \exp(i k_1 n_1 + ik_2 r_2), \]

(66)

where \(\vec{k} \in (-\pi, \pi] \times (-\pi, \pi].\) We will introduce the TLBC in the next subsection. The momentum representation of the tight-binding Hamiltonian (62) reads

\[ H_{\pi_1} = t \sum a_{\vec{k}}^\dagger b_{\vec{k}} [1 + \exp(-ik_1)] + b_{\vec{k}}^\dagger a_{\vec{k}} [1 + \exp(ik_1)], \]
\[ H_{\pi_2} = t \sum 2 \left( a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}} \right) \cos k_2. \]

(67)

(68)

Therefore,

\[ H_{\pi} = t \sum (a_{\vec{k}}^\dagger, b_{\vec{k}}^\dagger) \begin{pmatrix} 2 \cos k_2 & 1 + \exp(-ik_1) \\ 1 + \exp(ik_1) & -2 \cos k_2 \end{pmatrix} \begin{pmatrix} a_{\vec{k}} \\ b_{\vec{k}} \end{pmatrix}. \]

(69)

The low-energy excitation momentum points are localized around

\[ K = (\pi, -\pi/2) \text{ and } K' = (\pi, +\pi/2). \]

(70)

The low-energy effective theory can be obtained as,

\[ H_{\pi} \approx t \sum_{l=\{1,2\};k} \psi_{k}^{(l)}(-2\sigma_3 k_2 - \sigma_2 k_1)\psi_{k}^{(l)}, \]

(71)

where

\[ \psi_{k}^{(1)} = \sigma_2 \begin{pmatrix} a_{k+\vec{K}} \\ b_{k+\vec{K}} \end{pmatrix} = \begin{pmatrix} -ib_{k+\vec{K}} \\ ia_{k+\vec{K}} \end{pmatrix}, \]
\[ \psi_{k}^{(2)} = \begin{pmatrix} a_{k+\vec{K}'} \\ b_{k+\vec{K}'} \end{pmatrix}. \]

(72)

(73)

We have the following symmetry representation:

- Flavor \(U(N)\) symmetry \(U_{\{\phi}\}\)
Conventionally, the global $U(N)$ symmetry is defined as

$$U_\phi c_\phi U^{-1}_\phi = \exp \left( \sum_{k=0}^{N^2-1} -i\phi_k t_k \right) c_\phi,$$  

(74)

where $t_k$'s: $t_{f,f'} = \delta_{f,f'}$ the $U(1)$ generator while $\{t_k : k = 1,2,\cdots,N^2-1\}$ are $SU(N)$ generators in the fundamental representation with a renormalization such that $\phi_k$'s are each compactified by $2\pi$. Equivalently,

$$U_\phi = \exp \left( \sum_{\bar{r},f,f'} \sum_{k=0}^{N^2-1} i c_{\bar{r}f}^{\dagger} t_{f,f'}^k \phi_k c_{\bar{r}f'} \right).$$

(75)

Since $U(1)$ and $SU(N)$ share a center, the definition above is not faithful and a more systematic approach is to impose a global structure both on $U(1)$ and $SU(N)$ parameters $\{\phi_k\}$ by a quotient over $\mathbb{Z}_N$ since $U(N) \cong [U(1) \times SU(N)]/\mathbb{Z}_N$.

For the low-energy degrees of freedom,

$$U_\phi \psi U^{-1}_\phi = \exp \left( \sum_{k=0}^{N^2-1} -i\phi_k t_k \right) \psi.$$  

(76)

In the following discussion, the flavor indices “$f$” will be suppressed for simplicity.

- Magnetic translation $T_1$

The lattice Hamiltonian is invariant under the magnetic translation along $\hat{x}$-axis:

$$T_1 c_\phi T^{-1}_1 = c_{\phi+\hat{x}} \exp(i\pi r_2),$$

(77)

which means

$$T_1 a_k T^{-1}_1 = b_{k+Q},$$

(78)

$$T_1 b_k T^{-1}_1 = a_{k+Q} \exp(i k_1).$$

(79)

In the low-energy field theory,

$$T_1 \psi T^{-1}_1 = i\tau_1 \psi,$$  

(80)

where $\tau$ matrices act on the valley components.
• Translation $T_2$

The Hamiltonian is also invariant under the conventional translation along $\hat{y}$-axis:

$$T_2 c_\vec{r} T_2^{-1} = c_{\vec{r}+\vec{y}}, \quad (81)$$

which gives

$$T_2 a_k T_2^{-1} = a_k \exp(ik_2), \quad (82)$$
$$T_2 b_k T_2^{-1} = b_k \exp(ik_2), \quad (83)$$

and

$$T_2 \psi T_2^{-1} = -i \tau_3 \psi. \quad (84)$$

• Time-reversal symmetry $\mathbb{Z}_2^T$

Since we are interested in the constraint brought by time-reversal symmetry $\mathbb{Z}_2^T$, we define the following (anti-unitary) time-reversal symmetry on our spinless fermions as

$$\Theta_0 i \Theta_0^{-1} = -i, \quad (85)$$
$$\Theta_0 c_\vec{r} \Theta_0^{-1} = c_{\vec{r}}. \quad (86)$$

Thus

$$\Theta_0 a_k \Theta_0^{-1} = a_{-k}, \quad (87)$$
$$\Theta_0 b_k \Theta_0^{-1} = b_{-k}, \quad (88)$$

and

$$\Theta_0 \psi_\vec{k} \Theta_0^{-1} = -i \tau_2 \otimes \sigma_2 \psi_{-\vec{k}}, \quad (89)$$

or

$$\Theta_0 \psi_{1,k} \Theta_0^{-1} = -\sigma_2 \psi_{2,-k},$$
$$\Theta_0 \psi_{2,k} \Theta_0^{-1} = \sigma_2 \psi_{1,-k}. \quad (90)$$
Let us do a trivial rescaling $k_2 \to k_2/2$ and $t \to 1$. Back to the real space, the Hamiltonian density becomes:

$$\mathcal{H} = \sum_{f=1}^{N} \sum_{l=1}^{2} i \bar{\psi}_{f}^{(l)} (\sigma_3 \partial_2 + \sigma_2 \partial_1) \psi_{f}^{(l)}$$

$$= -i \bar{\psi} ( -\gamma^0 \sigma_2 \partial_1 + \gamma^0 \sigma_3 \partial_2 ) \psi$$

$$= -i \bar{\psi} ( \gamma^1 \partial_1 + \gamma^2 \partial_2 ) \psi,$$

(91)

where, for clearness, we have suppressed the summation over flavor “$f$” and valley “$l$” indices. $\bar{\psi} \equiv \psi^{\dagger} \gamma^0$ and $\{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu \nu} = 2 \cdot \text{diag}(+,-,-)$, the Dirac algebra in $(1 + 2)$ dimensions. We can choose the following basis:

$$\begin{cases} 
\gamma^0 = \sigma_1, \\
\gamma^1 = -i \sigma_3, \\
\gamma^2 = i \sigma_2.
\end{cases}$$

(92)

Then the Lagrangian density is $\mathcal{L}_{\text{Dirac}} = \bar{\psi} i \gamma^{i} \partial_{i} \psi$ with the following symmetry representations:

$$\begin{align*}
\hat{T}_1 \psi(t, x, y) &= i \tau_1 \psi(t, x, y), \\
\hat{T}_2 \psi(t, x, y) &= -i \tau_3 \psi(t, x, y), \\
\hat{\Theta}_0 \psi(t, x, y) &= -\tau_2 \otimes \gamma^2 \psi^*( -t, x, y) \\
&= -\tau_2 (\Theta_E \psi)(t, x, y)
\end{align*}$$

(93)

where $\Theta_E = \gamma^2 K$ is the emergent relativistic time-reversal of Dirac spinor with $K$ the antilinear operator $Ki = -iK$. After a Wick rotation $\partial_t \to i \partial_\tau$, the eigenvalue problem of the corresponding Euclidean Dirac operator reads

$$\Gamma^{i} \partial_{i} \psi(\tau, x, y) = \lambda \psi(\tau, x, y),$$

(94)

in which $\{ \Gamma^{j}, \Gamma^{k} \} = -2 \delta^{jk} = -2 \cdot \text{diag}(+,+,+)$ and

$$\begin{cases} 
\Gamma^0 = i \sigma_1, \\
\Gamma^1 = -i \sigma_3, \\
\Gamma^2 = i \sigma_2.
\end{cases}$$

(95)
The symmetry field-configuration representations is analytically continued to

$$\hat{T}_1 \psi(\tau, x, y) = i \tau_1 \psi(\tau, x, y),$$
$$\hat{T}_2 \psi(\tau, x, y) = -i \tau_3 \psi(\tau, x, y),$$
$$\hat{\Theta}_0 \psi(\tau, x, y) = -i \tau_2 \otimes \Gamma^2 \Gamma_0 \psi^*(-\tau, x, y)$$

$$= -\tau_2 \otimes (C R_\tau \psi)(\tau, x, y), \quad (96)$$

with $C \psi(\tau, x, y) = \Gamma^2 \psi^*(\tau, x, y)$ and $R_\tau \psi(\tau, x, y) = i \Gamma^0 \psi(-\tau, x, y)$.

B. TLBC for the effective field theory

Let us now impose the TLBC for the low-energy degrees of freedom $\psi(\tau, x, y)$ as

$$\begin{cases}
\psi(\tau, x + L_{\tau}, y) = T_2 \psi(\tau, x, y); \\
\psi(\tau, x, y + L_y) = \psi(\tau, x, y).
\end{cases} \quad (97)$$

Then we do a flux insertion into the hole rounded by $L_x$:

$$\begin{cases}
A_\tau(\tau, x, y) = 0; \\
A_x(\tau, x, y) = \frac{\theta}{L_x} \tau_1 t_0; \\
A_y^0(\tau, x, y) = \frac{\theta}{L_y} \frac{1}{N} (t_0 - t_{N^2 - 1}),
\end{cases} \quad (98)$$

where $t_0$ is an $(N \times N)$ identity matrix and $t_{N^2 - 1} = \text{diag}[1, 1, \ldots, 1, -(N - 1)]$. For later use, we also formally introduce a static flux into the hole of $L_y$, but we take $\theta = 0 \mod 2\pi$ later, which is gauge equivalent to $A_y = 0$ by a large $U(N)$-gauge transformation as $\exp [i 2\pi y (t_0 - t_{N^2 - 1}) / (N L_y)]$. Afterwards, we further do a $T_1$ transformation. Then the Euclidean path-integral partition function related to this flux insertion followed by $T_1$ transformation takes the form as:

$$z(\theta) \equiv \int \mathcal{D}(\bar{\psi}_\alpha, \psi_\alpha) \exp \left( - \int \mathcal{L}_{\text{Dirac}} [A_x(\tau), A_y^0] \right) \quad (99)$$

with the space-time manifold as a four-dimensional torus $T^4$ and $\mathcal{T}$ the time-ordering operator with the boundary condition combined with Eq. (97) as

$$\begin{cases}
\psi(\tau + L_{\tau}, x, y) = -T_1 \psi(\tau, x, y); \\
\psi(\tau, x + L_x, y) = T_2 \psi(\tau, x, y); \\
\psi(\tau, x, y + L_y) = \psi(\tau, x, y),
\end{cases} \quad (100)$$
where the minus sign in the boundary condition along $\tau$ is due to the fermionic nature of the path integral.

TLBC by $T_{1,2}$ is obviously consistent with $\Theta_0$ since $[T_{1,2}, \Theta_0] = 0$. Nevertheless, one might have noticed that the flux-insertion by $A_x$ is only $\pi$ flux through the $\tau$-$x$ plane rather than $2\pi$. At the first glance, it gives an ill-defined transition function across the cut between $\tau = L_\tau$ and $\tau = 0$. However, it is actually canonical because the latter $T_1$ transformation at $\tau = L_\tau$ anticommutates with $T_2$ imposed in $\hat{x}$ direction: $T_1 T_2 T_1^{-1} T_2^{-1} = -1$, and it makes the boundary condition Eq. (100) appear inconsistent, either. Such an extra "$-1$" sign inconsistency in the boundary condition exactly compensates the transition-function sign ambiguity brought by a $\pi$-flux insertion by $A_x(\tau, x, y)$ in Eq. (98). A systematic construction of the gauge bundle above is given in Eq. (A.1).

Then let us take a look at the following ratio

$$Z_\pi = \frac{z(\theta = 2\pi)}{z(\theta = 0)},$$

(101)

which appears as 1 by the gauge equivalence of $A_y^{\theta=2\pi}$ and $A_y^{\theta=0}$. However, by an adiabatic transformation of $\theta$ from 0 to $2\pi$, we can canonically evaluate this phase ambiguity and we first give the result of which the explanation follows

$$Z_\pi = (-1)^{\mathcal{I}_{4D}},$$

(102)

where the index $\mathcal{I}_{4D}$ of a 4-D Dirac operator with $N$ flavors and 2 valleys is the number difference of zero modes between positive and negative chiralities, and it can be calculated by the Atiyah-Singer index theorem [34, 35] as

$$\mathcal{I}_{4D} = 2 \left[ -\frac{N}{48} \int_X \frac{\text{tr} R \wedge R}{(2\pi)^2} - \frac{1}{2} \frac{\text{Tr} F \wedge F}{(2\pi)^2} \right]$$

$$= -\frac{2}{32\pi^2} \int_{T^4} d^4x e^{ijkl} \text{Tr}(F_{ij} F_{kl})$$

$$= 1,$$

(103)

where "tr" is taken over the four spacetime indices and "Tr" over the flavors, and the gravitational contribution, even on generic orientable manifolds without boundary, takes value in $2 \times 2 \times N\mathbb{Z} = 4N\mathbb{Z}$ thereby irrelevant to $Z_\pi$. The fact that the index $\mathcal{I}_{4D}$ mod 2 can detect the phase (sign) ambiguity in Eq. (102) can be understood in the following way. Our $(2 + 1)$-dimensional two-flavor Dirac fermion cannot be quantized in a
symmetry-preserving manner. Let us break $\mathbb{Z}_2^T$, and then a Pauli-Villars regulator Lagrangian $\sum_{\alpha,f} i \bar{\xi}_{\alpha,f} [i \Gamma^i (\partial_i - iA_i) - m_{P,V}] \xi_{\alpha,f}$ is permitted to regularize the 3-D partition function, where $\xi_{\alpha}$ is a two-flavor bosonic spinor and the diagonal mass term $(m_{P,V} \delta^{\alpha\beta} \delta^f_{f'})$ preserves $U(N)$ and $T_{1,2}$. The partition function is regularized as $|Z_{3D}| \exp(-i\pi \eta_{3D}/2)$ explicitly breaking $\mathbb{Z}_2^T$, where $|Z_{3D}|$ is the absolute value of our massless theory partition function and $\eta_{3D} \equiv \lim_{\epsilon \to 0^+} \sum_{k} \text{sgn}(\lambda_k) \exp(-\epsilon \lambda_k^2)$ with $\{\lambda_k\}$ the spectrum of 3D two-flavor Dirac operator. However, $\mathbb{Z}_2^T$ can be restored when we attach a 4D bulk on it, or equivalently, add a 4-D counter-term supported by $X'$ whose boundary is our 3D manifold: $\sum_{\alpha=1}^2 \delta_{\alpha,\alpha} \left[ (\sum_{f} \delta_{f,f}/48) \int_{X'} \text{tr} R \wedge R/(2\pi)^2 + \int_{X'} (1/2) \text{Tr} F \wedge F/(2\pi)^2 \right]$. Thus the regularized partition function by Atiyah-Patodi-Singer index theorem [36] is

$$Z_{3D}(X') = |Z_{3D}| \exp \left\{ i\pi \left[ \frac{\eta_{3D}}{2} + \frac{2N}{48} \int_{X'} \text{tr} R \wedge R/(2\pi)^2 + \int_{X'} \text{Tr} F \wedge F/(2\pi)^2 \right] \right\} = |Z_{3D}| (-1)^{\mathcal{A}_{4D}(X')} ,$$

(104)

which explicitly respects time-reversal symmetry since $\mathcal{A}_{4D}$ is an integer. Nevertheless, $Z_{3D}(X')$ also potentially depends on the extension $X'$, which disables our massless system to be purely a 3D model, which exactly characterizes a quantum anomaly. Therefore, by gluing rule of the phase of $(-1)^{\mathcal{A}_{4D}}$, we can detect such an extension dependence or quantum anomaly by calculating $\mathcal{A}_{4D}(\tilde{X})$ mod 2 on any compact orientable manifold $\tilde{X}$ without boundary. Indeed, topologically equivalently, $Z_{\pi}$ in Eqs. (101,102) measures the phase ambiguity induced by two different extensions, with and without, respectively, a $2\pi$ of $(t_0 - t_{N^2-1})/N$-flux in the plane spanned by the extra dimension and $\hat{y}$ direction.

The anomaly we have obtained is fully a mixed type since it will be trivialized once we discard any of the required symmetries. This property is consistent with the ingappability on the lattice level, e.g. the magnetic-translation and $U(N)$ symmetric gapping term but time-reversal breaking $\Delta H$ in Eq. (65) trivially gaps the system. Thus we expect such a non-abelian symmetry anomaly is the desired LSM type. Moreover, two layers (or copies) of lattice systems can be trivially gapped respecting all the required symmetries since we can always set opposite chemical-potential term, e.g. $\sum_{f} \sum_{Z=1,2} (-1)^Z \sum_{\ell} \delta_{\ell \ell_c} c_{Z,f}^\ell \bar{c}_{\ell,f}$ with $Z$ labelling the layers so that one of two is fully-filled while the other empty, without breaking (total) filling fraction and symmetries. This aspect is also implied by and consistent with our $\mathbb{Z}_2$ anomaly classification. In addition, there is no purely $U(N) \rtimes \mathbb{Z}_2^T$ anomaly, which implies that without $T_{1,2}$ we can trivially gap the system, consistent with the lattice situation as
well. Furthermore, we do not use the emergent properties of $T_{1,2}$ at low-energy limit, such as their finite cyclicality, so the corresponding emergent anomalies, if any, are not included. Instead, we only make use of the gauge-invariant nontrivial commutator $T_1 T_2 T_1^{-1} T_2^{-1} = -1$ already held at the lattice level. Combining these observations above, this mixed anomaly in field theory suggests the ingappability in the presence of the time reversal symmetry together with the flavor and magnetic translation symmetries. This will be further confirmed in the next subsection, by defining a corresponding quantity on the lattice.

C. Lattice-realization of $\mathbb{Z}_2$-classifying anomaly

$Z_\pi = -1$ defined for the effective field theory in the previous subsection is a topological invariant, e.g. it cannot be changed along renormalization flow (RG) from the critical point as long as the interaction perturbing the system respects all the required symmetries. Here we look for a lattice quantity $Z'_\pi$ whose low-energy and continuum limit is $Z_\pi$. The ’t Hooft anomaly matching then suggests it would be a robust topological invariant which can be evaluated exactly by its infrared counterpart $Z_\pi$.

Let us assume that the system is trivially gapped under all the symmetries. It will lead to a contradiction, as we will demonstrate below. To do so, we first impose the following TLBC:

$$
\begin{align*}
    c_{x+L_x,y} &= c_{x,y+1}, \\
    c_{x,y+L_y} &= c_{x,y}.
\end{align*}
$$

(105)

The lattice translation $T_2$ is well-defined, but the magnetic translation $T_1$ is problematic since it is $y$-dependent and $y$ coordinate has a freedom to be adjusted by Eq. (105). Nevertheless, the existence of $T_1$ on the lattice with periodic boundary conditions implies that such a symmetry has a different form in our tilted case especially in the low-energy sense. It follows straightforwardly from the assumption of the gapped unique ground state that the ground state should go back to itself after an adiabatic insertion ($T \rightarrow +\infty$) of $\pi$ flux into the hole rounded by $L_x$:

$$
A_x(t,\vec{r}) = \pi \frac{t}{L_x} T t_0
$$

(106)

followed by a canonical lattice $T'_1$ transformation:

$$
T'_1 c_{x,y} T'_1^{-1} = c_{x+1,y}.
$$

(107)
It is due to that the following large gauge transformation $V$:

$$V' c_{x,y} V'^{-1} = \exp \left( i \frac{\pi}{L_x} x \right) \exp(i\pi y)c_{x,y},$$

(108)

with $(x, y) \in [1, L_x] \times [1, L_y]$ is well-defined since the original magnetic translation $T_1$ requires $L_y \in 2\mathbb{Z}$ as mentioned before, and it is able to restore the initial Hamiltonian after the preceding $\pi$-flux insertion and $T'_1$.

Let us consider the following quantity: ($\mathcal{T}$ is time-ordering)

$$z'(\theta) \equiv \text{Tr}_{G.S.} \left\{ \hat{T}' \mathcal{T}' \hat{V}' \exp \left\{ -i \int H[A_x(t), A^\theta_y]dt \right\} \right\}$$

$$= \left\langle \text{G.S.} \left| \hat{T}' \mathcal{T}' \hat{V}' \exp \left\{ -i \int H[A_x(t), A^\theta_y]dt \right\} \right| \text{G.S.} \right\rangle,$$

(109)

where $H$ is the time-dependent lattice Hamiltonian and $\left| \text{G.S.} \right\rangle$ is the certain presumed unique gapped ground state. We also set an artificial flat $U(1)_Q$ gauge field in $y$ direction by $A_y = \theta(t_0 - t_{N2-1})/(NL_y)$ where our case corresponds to $\theta = 0$. Then $z'(\theta)$ satisfies:

$$z'(\theta) = z'(\theta + 2\pi),$$

(110)

due to the large gauge transformation mentioned before.

Let us imagine a series of model with $\theta$ changing from 0 to $2\pi$ and consider

$$Z'_x \equiv \frac{z'[\theta = 2\pi]}{z'[\theta = 0]}.$$  

(111)

Before evaluating this ratio of partition functions, we first notice that $T'_1 V'$ acting on $c_{x,y}$ with $(x, y) \in [1, L_x] \times [1, L_y]$:

$$T'_1 V' c_{x,y} (T'_1 V')^{-1} = \exp \left( i \frac{\pi}{L_x} x \right) \exp(i\pi y)c_{x+1,y}$$

$$= T_1 V c_{x,y} (T_1 V)^{-1},$$

(112)

which has exactly the same effect as the $T_1$ on periodic lattice followed by a “half-large” gauge transformation $V c_{x,y} V^{-1} \equiv \exp(i\pi x/L_x)$ with $(x, y) \in [1, L_x] \times [1, L_y]$, which is well-defined according to the discussion below Eq. (100).

Therefore, the continuum limit of $z'$ is

$$z' = \text{Tr}_{G.S.} \left\{ \hat{T}' \mathcal{T}' \hat{V}' \exp \left\{ - \int H[A_x(\tau), A^\theta_y]d\tau \right\} \right\}$$

$$\rightarrow \text{Tr}_{G.S.} \left\{ \hat{T}_1 V \mathcal{T} \exp \left\{ - \int H[A_x(\tau), A^\theta_y]d\tau \right\} \right\},$$

(113)
which precisely reproduces the form of $z$ in Eq. (99) with the boundary condition as Eq. (100). Similarly as the $U(1)_Q$ case, the role played by $V$ in the path integral of $z$ is implicitly an assignment of a transition function between the last two time slices preceding the $\hat{T}_1$ gluing. Thus, we can make use of 't Hooft anomaly matching to evaluate $Z'_\pi$ by $Z_\pi$:

$$Z'_\pi = Z_\pi = -1.$$ (114)

Therefore, we have

$$z'(\theta = 0) = z'(\theta = 2\pi) = -z'(\theta = 0),$$ (115)

which is impossible unless

$$z'(\theta = 0) = 0.$$ (116)

It implies the following two equal-energy states are actually orthogonal:

$$\hat{T}_1 V' \mathcal{F} \exp \left\{-i \int_0^T H[A_x(t), A_y = 0] dt \right\} \langle \text{G.S.} | \perp | \text{G.S.} \rangle,$$ (117)

thereby contradicting the assumption of the unique $|\text{G.S.}\rangle$. It means the ground-state degeneracy must be nontrivial and we arrive at an LSM-type ingappablility.

V. CONCLUSIONS AND DISCUSSIONS

In this work, we introduced a generalized boundary condition TLBC for discussion of the LSM theorem and related problems. Under the TLBC, each lattice site can be reached by repeating the lattice translation in a particular direction. As an advantage of the TLBC, the LSM theorem in two or more dimensions can be derived by the flux insertion argument without an artificial restriction on the system sizes, which renders the thermodynamic limit more natural. Moreover, with the TLBC, the LSM theorem in arbitrary dimensions is related to the global chiral anomaly of Dirac fermion in $(1+1)$ dimensions in a unified manner. This is also extended to the LSM theorem with the larger $SU(N)$ symmetry, and to the similar ingappability constraint under the time reversal and the magnetic translation symmetries with the on-site $U(N)$ symmetry.

Despite of the usefulness of TLBC as we have demonstrated, it cannot be applied to exploit consequences of spatial reflection symmetries, which are inconsistent with the TLBC.
TLBC and anomaly based detections of such kinds of crystalline LSM-type ingappabilities can be future interest.

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Appendix: Matrix representation of $su(3)$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$

In this part, matrix conventions of $su(3)$ in fundamental representations and adjoint representations are given and the generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup are also given in both representations.

The unnormalized $su(3)$ generators in the fundamental representation are given as

\[
\begin{align*}
t_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
t_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
t_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
t_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
t_8 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{align*}
\]

We can normalize them in the Gell-Mann matrix basis as:

\[
L_i \equiv \begin{cases} t_i/2, & i = 1, \ldots, 7 \\ t_i/(2\sqrt{3}), & i = 8. \end{cases}
\]
Then the generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$ becomes

$$V_3 = \exp[i\pi(t_8 - t_3)/3] = \exp[i2\pi(\sqrt{3}L_8 - L_3)/3],$$  \hspace{1cm} (A.3)

$$W_3 = \exp[-i2\pi(t_2 - t_5 + t_7)/(3\sqrt{3})] = \exp[-i4\pi(L_2 - L_5 + L_7)/(3\sqrt{3})].$$  \hspace{1cm} (A.4)

Thus, in the fundamental representation,

$$\rho_f(V_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i2\pi/3) & 0 \\ 0 & 0 & \exp(i4\pi/3) \end{pmatrix}; \quad \rho_f(W_3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \hspace{1cm} (A.5)$$

In the adjoint representation,

$$2L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix}, \quad 2L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix},$$

$$2L_3 = \begin{pmatrix} 0 & -2i & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad 2L_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$2L_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad 2L_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
Then

\[
\rho_{\text{adj}}(V_3) = \begin{pmatrix}
-1/2 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3}/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/2 & \sqrt{3}/2 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{3}/2 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

(A.7)
and

\[
\rho_{\text{adj}}(W_3) = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2
\end{pmatrix},
\]  

(A.8)

which seem to share two common invariant one-dimensional subspaces, the third and the eighth dimensions by observation on Eqs. (A.7, A.8), and actually eight common invariant one-dimensional subspaces in total since \(\rho_{\text{adj}}(V_3)\) and \(\rho_{\text{adj}}(W_3)\) commute with each other:

\[
\rho_{\text{adj}}(V_3)\rho_{\text{adj}}(W_3)\rho_{\text{adj}}^{-1}(V_3)\rho_{\text{adj}}^{-1}(W_3) = 1.
\]  

(A.9)

**Appendix: Gauge bundle for the index calculation in Eq. (103)**

let us give a mathematically rigorous construction of the gauge bundle of all the symmetries. We first re-parametrize the torus by rescaling \(\beta, L_{x,y,z} \to 2\pi\), and then \(\tau, x, y, z \in \mathbb{R}/2\pi\) with \(\mathbb{R}\) the universal covering of each \(S^1\)-component of \(T^4\). The symmetry-group gauge bundle can be expressed by transition function between various patches. In order to do so, we need to cover the \(T^4\) by four open areas:

\[
\begin{align*}
\mathcal{A}_1 = \{ p \in T^4 | \theta_1(p) = (\tau_1, x_1, y_1, z_1) \in (-\epsilon, \pi + \epsilon) \times (-\epsilon, \pi + \epsilon) \times [0, 2\pi) \times (-\epsilon, \pi + \epsilon) \}, \\
\mathcal{A}_II = \{ p \in T^4 | \theta_{II}(p) = (\tau_{II}, x_{II}, y_{II}, z_{II}) \in (-\epsilon, \pi + \epsilon) \times (-\pi - \epsilon, +\epsilon) \times [0, 2\pi) \times (-\epsilon, \pi + \epsilon) \}, \\
\mathcal{A}_{III} = \{ p \in T^4 | \theta_{III}(p) = (\tau_{III}, x_{III}, y_{III}, z_{III}) \in (-\pi - \epsilon, \epsilon) \times (-\pi - \epsilon, \epsilon) \times [0, 2\pi) \times (-\epsilon, \pi + \epsilon) \}, \\
\mathcal{A}_{IV} = \{ p \in T^4 | \theta_{IV}(p) = (\tau_{IV}, x_{IV}, y_{IV}, z_{IV}) \in (-\pi - \epsilon, +\epsilon) \times (-\epsilon, \pi + \epsilon) \times [0, 2\pi) \times (-\epsilon, \pi + \epsilon) \},
\end{align*}
\]

(A.1)
where $\epsilon$ is a smaller positive number than $\pi/2$, e.g. $\epsilon = \pi/4$, and we have used the coordinates \( \theta_{I,II,III,IV}(p \in T^4) \) of the universal covering $\mathbb{R}^4$ to coordinate $T^4$.

We denote the transition function as, e.g. $\psi_I = t_{I,II} \psi_{II}$ and etc., and the following transition functions are defined:

- Transition function between $A_I$ and $A_{II}$ on $A_I \cap A_{II}$:
  \[
  t_{I,II} \circ \theta_{I}^{-1}(\tau_I, x_I, y_I, z_I) = \begin{cases} 
  T_2, & (x_I, \tau_I) \in (-\epsilon, +\epsilon) \times (-\epsilon, \pi + \epsilon), \\
  1, & (x_I, \tau_I) \in (\pi - \epsilon, \pi + \epsilon) \times (-\epsilon, \pi + \epsilon),
  \end{cases} \tag{A.2}
  \]
  where $\tau_I$ and $x_I$ are free parameters in their own domains.

- Transition function between $A_{IV}$ and $A_{III}$ on $A_{IV} \cap A_{III}$:
  \[
  t_{IV,III} \circ \theta_{IV}^{-1}(\tau_{IV}, x_{IV}, y_{IV}, z_{IV}) = \begin{cases} 
  T_2, & (x_{IV}, \tau_{IV}) \in (-\epsilon, +\epsilon) \times (-\pi - \epsilon, \epsilon), \\
  1, & (x_{IV}, \tau_{IV}) \in (\pi - \epsilon, \pi + \epsilon) \times (-\pi - \epsilon, \epsilon).
  \end{cases} \tag{A.3}
  \]

- Transition function between $A_{II}$ and $A_{III}$ on $A_{II} \cap A_{III}$:
  \[
  t_{II,III} \circ \theta_{II}^{-1}(\tau_{II}, x_{II}, y_{II}, z_{II}) = \begin{cases} 
  -T_1 \exp \left(-i \frac{t_{II}}{2} x_{II}\right), & (x_{II}, \tau_{II}) \in (-\pi - \epsilon, \epsilon) \times (-\epsilon, \epsilon), \\
  1, & (x_{II}, \tau_{II}) \in (-\pi - \epsilon, \epsilon) \times (\pi - \epsilon, \pi + \epsilon).
  \end{cases} \tag{A.4}
  \]

- Transition function between $A_I$ and $A_{IV}$ on $A_I \cap A_{IV}$:
  \[
  t_{I,IV} \circ \theta_{I}^{-1}(\tau_I, x_I, y_I, z_I) = \begin{cases} 
  -T_1 \exp \left(-i \frac{t_{I}}{2} x_I\right), & (x_I, \tau_I) \in (-\epsilon, \pi + \epsilon) \times (-\epsilon, \epsilon), \\
  1, & (x_I, \tau_I) \in (-\epsilon, \pi + \epsilon) \times (\pi - \epsilon, \pi + \epsilon).
  \end{cases} \tag{A.5}
  \]

- Transition functions between $A'_i$ and $A'_j$ on $A'_i \cap A'_j$ are naturally identified to that of $t_{i,j}$ through $t_{i,j} \circ \theta_{i}^{-1}(\tau', x', y', z') = t_{i,j} \circ \theta_{i}^{-1}(\tau', x', y', z' + \pi)$.

- Transition functions between $A_i$ and $A'_i$ on $A_i \cap A'_i$:
  \[
  t_{i,i'} \circ \theta_{i}^{-1}(\tau_i, x_i, y_i, z_i) = \begin{cases} 
  \exp \left(-i \frac{t_{i}}{N_{i-1}} y_i\right), & z_i \in (-\epsilon, \epsilon), \\
  1, & z_i \in (\pi - \epsilon, \pi + \epsilon).
  \end{cases} \tag{A.6}
  \]

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