Control of a flexible inverse pendulum based on the singular perturbation method

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Abstract. In this article, a partial differential equation model for a flexible inverted pendulum system is derived by using the Hamilton principle. Specifically, problems of stabilization and the optimization of such a system are considered. In addition, the singular perturbation method has been used to divide the partial differential equation model for a fast and a slow subsystem. For a fast subsystem stabilization, the control algorithm proposed a boundary force applied at the free end of the beam which proved that the closed-loop subsystem is appropriate and exponentially stable. To stabilize the slow subsystem, a sliding mode control method was used to design the controller, while the method of linear matrix inequality was used in designing the sliding surface. In conclusion, it was shown that the slow subsystem is exponentially stable.

1. Introduction
In the present days, with the development of the mechanical system design technologies, especially in the aerospace field, lightweight materials are increasingly used which allows achieving a significant increase in speed and energy savings. However, the use of such materials often leads to uncontrolled vibrations, which, in turn, can negatively affect the desired dynamic modes of the corresponding systems. To improve reliability and manageability, reliable tools are needed to neutralize these vibrations or at least limit them within acceptable values. In this study, we consider a flexible inverse pendulum with a load at the top point, assuming small density. Here, in contrast to the classical model, the system under consideration takes into account the vibration components.

In recent years, several studies have been devoted to managing the flexible pendulum [1-7]. For instance, model equations were presented in [7] where a flexible pendulum control system was developed based on the classical frequency analysis method. [8-10] used the Lagrange method to develop a model of the dynamics of a flexible pendulum and a regulator was developed based on a linear method of analysis in the state space.

The method of singular perturbations has traditionally been used to model processes occurring at different time scales. Within this method, the occurrence of fast and slow movements was the result of external heterogeneous factors. This method often allows to simplify the model (if fast movements were neglected), while fast movements consideration is usually performed in a small boundary layer. The calculation on separate time scales helps to improve the approximation of the model [2].
In most of the developed control methods, the use of the singular perturbation method led to a model reduction, i.e., ignoring the high-frequency components of the system models [1,3]. As a result, multiple time scale methods were developed for various control algorithms [11], such as feedback, etc.

The singular perturbation method has proved useful for analyzing high-gain feedback systems and controlling dynamic networks (both linear and nonlinear dynamical systems). Singular perturbations and time scales in control theory alongside its applied aspects have also been the subject of serious research in [12-14].

Flexible manipulators are widely used in various engineering fields, since they are considered an integral part of various robotic and mechatronic systems. There are several concepts for managing these objects, including schemas which use a partial derivative-based control method and an adaptive control method. Within the framework of the relay control paradigm, sliding modes are also used. The paper [15] proposed a simple combined control system based on a dynamic model that consisted of partial differential equations and ordinary differential equations with certain geometric constraints.

In this study, we consider a flexible inverse pendulum where the dynamics are divided into slow and fast components using the method of singular perturbations. Next, the sliding mode control principle is used, developed as part of a more complex control mechanism for a slow subsystem [16,17]. In turn, a simple control algorithm [18] based on feedback principles is also proposed for the fast subsystem.

2. Model of dynamics of a flexible inverted pendulum

Figure 1 shows a flexible inverted pendulum consisting of an elastic pendulum, and the movement of which is carried out in a given plane. The following symbols are used:

- $f$ - force applied to the point load,
- $M$ - equivalent mass of the system in a stationary state,
- $\rho$ - density of the elastic pendulum,
- $m$ - mass of the load at the end of the inverted pendulum

\[ X - axis: P0x = r + x \sin \theta + u(x,t) \cos \theta, \]
\[ Y - axis: P0y = x \cos \theta - u(x,t) \sin \theta \]

![Figure 1. Model of a flexible inverted pendulum system.](image)

$L$ - Length of the elastic pendulum, $F$ - Control input signal; $E$ - Young's modulus, $\theta$ - Angular position of the link, $u(x, t)$ - External deviation from the tangent drawn at the attachment point, $J$ - Moment of inertia of the pendulum.

The kinematic model of an elastic pendulum is defined as follows:

The kinematic model of a stationary cart has the following form:
The kinetic energy of a system consisting of a pendulum and a cart is determined by the equation
\[ T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} \int_0^L \rho f \int_0^L \ddot{P} \dot{P} dx + \frac{1}{2} m \dot{P}_o \dot{P}_o. \] (1)

Formula for the potential energy of a flexible inverted pendulum:
\[ V = \rho g \int_0^L (x \cos \theta - u(x, t) \sin \theta) dx + \frac{1}{2} \int_0^L (u_{xx}(x, t))^2 dx + m g (L \cos \theta - u(L, t) \sin \theta) \] (2)

Non-conservative work of external forces:
\[ \delta W_{nc} = F \delta r + f \delta u(L, t) \] (3)

To build a dynamic model of a flexible inverted pendulum system, the Hamilton principle is applied:
\[ \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt = 0, \] (4)

where \( t_1, t_2 \) are time constants; \( t \in [t_1, t_2] \) is the operating time; and \( \delta T, \delta V, \delta W_{nc} \) are the increments of kinetic energy, potential energy, and non-conservative operation, respectively. Taking into account the Hamilton principle, the equations of dynamics will take the form:
\[ -\rho (\ddot{u}(x, t) + \dot{x} \dot{\theta} + \dot{r} - g \theta) - El u_{xxx}(x, t) = 0 \] (5)
\[ m (\ddot{u}(L, t) + L \ddot{\theta} + \dot{r} - g \theta) - El u_{xxx}(L, t) - f = 0 \] (6)
\[ u_{xx}(L, t) = 0 \] (7)
\[ \left( f + mL^2 + \rho L \frac{L^2}{3} \right) \ddot{\theta} + \left( mL + \rho L \frac{L}{2} \right) \ddot{r} - \left( mL + \rho L \frac{L}{2} \right) g \theta + m \{L \ddot{u}(L, t) - g u(L, t)\} + \rho \int_0^L (x \ddot{u}(x, t) - g u(x, t)) dx = 0 \] (8)
\[ (M + m + \rho L) \ddot{r} + \left( mL + \rho L \frac{L}{2} \right) \ddot{\theta} - m \ddot{u}(L, t) + \rho \int_0^L \ddot{u}(x, t) dx - F = 0 \] (9)
\[ u(0, t) = 0 \] (10)
\[ u_x(0, t) = 0 \] (11)

Note that in the dynamic model (5) - (11), the actual dynamics of the rigid pendulum is mainly represented by the relations (8) and (9), while the flexible one is described by partial differential equations with boundary conditions, given (6) - (7) and (10) - (11), where in turn (10) - (11) are the boundary conditions at the end of the links.

3. Dynamics of a single disturbance

System management is quite a complex task since its overall dynamics include both flexible/elastic and rigid components. Since traditional modeling methods increase the order of the model, we divide the system dynamics into slow and fast subsystems using the method of singular perturbations to reduce its complexity.

With the specified method, the perturbation parameter is selected. In this paper, parameter \( \varepsilon^2 \) is chosen which determines the system parameters.

Thus, the parameter \( \frac{E_l}{\rho} \) in equation (5) can be redefined as follows:
\[ \frac{E_l}{\rho} = aK, \] (12)

where \( a \ll K \).
The transverse vibrations formalized by the function \( u \) in the relative system occur in fast time. Therefore, it is necessary to enter a new variable \( u(x,t) \) in the same order of the system, which can be changed using the following equation:

\[
u(x,t) = \varepsilon^2 w(x,t),\]

where \( \varepsilon^2 = \frac{1}{\kappa^2} \) it is a perturbation parameter.

Using (13), we can rewrite the model of rigid oscillatory movements as follows:

\[
-\left(\varepsilon^2 \ddot{w}(x,t) + x \dddot{\theta} + \dddot{\varphi} - g\theta\right) - aw_{xxxx}(x,t) = 0
\]

\[
m\left(\varepsilon^2 \ddot{w}(L,t) + L \dddot{\theta} + \dddot{\varphi} - g\theta\right) - EI\varepsilon^2 w_{xxxxxx}(L,t) - f = 0
\]

\[
\varepsilon^2 w_{xx}(L,t) = 0
\]

\[
\varepsilon^2 w(0,t) = 0
\]

\[
\varepsilon^2 w_x(0,t) = 0
\]

\[
\left(\int_0^1 + mL^2 + \rho L^2 \frac{L^2}{3}\right) \dddot{\theta} + \left(mL + \rho L^2 \right) \dddot{\varphi} - \left(mL + \rho L^2 \right) g\theta + m\left(E\varepsilon^2 \ddot{w}(L,t) - ge^2 u(L,t)\right) + \rho \int_0^1 \left(x^2 \dddot{u}(x,t) - ge^2 u(x,t)\right) dx = 0
\]

\[
\left(M + m + \rho L \right) \dddot{\varphi} + \left(mL + \rho L^2 \right) \dddot{\theta} + m\left(E\varepsilon^2 \ddot{w}(L,t)\right) + \rho \int_0^1 \left(x^2 \dddot{w}(x,t)\right) dx = 0
\]

Equations (14) – (20) represent a form of singular perturbation which will be included in the system of dynamics equations (5) – (11).

### 3.1. Slow subsystem

The slow and fast subsystems presented in this section were obtained using standard approaches of the singular perturbation method. After applying the parameter \( \varepsilon^2 \), the complete system consists of two motion models at different time scales. Initially, the model of the dynamics of rigid vibrations of the elastic pendulum constructed by the method of singular perturbations, will be transformed into a model of rigid vibrations, which does not include any parameters corresponding to the flexible pendulum. Then, it will be included in equations (19) and (20), which form a slow subsystem. Similarly, after applying the usual procedures of the singular perturbation method, a fast subsystem will be obtained from equation (14).

By setting the value \( \varepsilon = \theta \) in equations (19) and (20), we obtain a model of rigid an elastic pendulum dynamics, represented in compact form as:

\[
\left(f + mL^2 + \rho L^2 \frac{L^2}{3}\right) \dddot{\theta} + \left(mL + \rho L^2 \right) \dddot{\varphi} - \left(mL + \rho L^2 \right) g\theta = 0
\]

\[
\left(M + m + \rho L \right) \dddot{\varphi} + \left(mL + \rho L^2 \right) \dddot{\theta} - F_s = 0
\]

The equation of motion for the transverse vibration of the elastic pendulum (14) takes the form:

\[
-\left(x \dddot{\theta} + \dddot{\varphi} - g\theta\right) - aw_{xxxx}(x,t) = 0
\]

And the boundary equation (boundary equation) takes the form:

\[
m(L \dddot{\theta} + \dddot{\varphi} - g\theta) - \rho aw_{xxxx}(L,t) - f_s = 0
\]

The slow subsystem defined in equations (22) and (23) is a model of hard movement without attracting any soft parameters.
3.2. Fast subsystem

Equation (14) is a perturbed flexible motion model obtained with the perturbation parameter \( \varepsilon \), which is very small and depends solely on \( E, \rho, \) and \( I \).

To analyze the dynamic behavior of a fast subsystem, the accelerated time scale is defined as \( V = \frac{t - t_0}{\varepsilon} \). It can be determined by making sure that slow variables on fast time scales remain constant. Using the standard procedures of the singular perturbation method, one can define the fast variable \( w_f \) as follows:

\[
    w_f = w - w_s
\]

where the index \( f \) is used to describe variables in the accelerated time scale/fast time.

Differentiation to the accelerated time scale/fast time gives:

\[
    \frac{dw}{dt} = \frac{1}{\varepsilon}
\]

Given (26) and (27), we get:

\[
    \ddot{w} = \ddot{w}_s + \frac{d}{dt} w_f = \ddot{w}_s + \frac{d}{dt} \dot{w}_f = \ddot{w}_s + \frac{1}{\varepsilon} \ddot{w}_f
\]

where \( \ddot{w}_f \) denotes the time derivative on the accelerated time scale. Furthermore, differentiating (28), we get:

\[
    \dddot{w} = \dddot{w}_s + \frac{d}{dt} \dddot{w}_f = \dddot{w}_s + \frac{d}{dt} \dddot{w}_f = \dddot{w}_s + \frac{1}{\varepsilon^2} \dddot{w}_f
\]

Taking into account (29), (14) takes the form:

\[
    \varepsilon^2 \left( \dddot{w}_s + \frac{1}{\varepsilon^2} \dddot{w}_f \right) + x \dddot{\theta} + \dddot{\gamma} - g \theta = a \left( w_{sxxxx}(x, t) + w_{fxxxx}(x, t) \right) = 0
\]

Taking into account (24), (30) is modified in the form:

\[
    - \left( \varepsilon^2 \left( \dddot{w}_s + \frac{1}{\varepsilon^2} \dddot{w}_f \right) \right) - a \left( w_{fxxxx}(x, t) \right) = 0
\]

However, the \( W_s \) slow variable is constantly on an accelerated time scale, which implies \( \dddot{w}_s = 0 \) and \( \varepsilon = 0 \). Therefore, dynamics on an accelerated time scale can be represented as

\[
    - \dddot{w}_f - aw_{fxxxx}(x, t) = 0
\]

Taking into account (29), from (15), we get:

\[
    m \left( \varepsilon^2 \left( \dddot{w}_f(L, t) + \frac{1}{\varepsilon^2} \dddot{w}_f(L, t) \right) + L \dddot{\theta} + \dddot{\gamma} - g \theta \right) - EI \varepsilon^2 \left( w_{sxxxx}(L, t) - w_{fxxxx}(L, t) \right) - (f_f + f_s) = 0
\]

Then:

\[
    m \left( \dddot{w}_f(L, t) \right) - \rho aw_{fxxx}(L, t) - f_f = 0
\]

The boundary conditions will look like:

\[
    w_{fxx}(L, t) = 0,
\]

\[
    w_f(0, t) = 0,
\]

\[
    w_{fx}(0, t) = 0.
\]

Equation (34) describes a fast subsystem on an accelerated time scale which takes into account the vibration of a flexible pendulum.
In this section, we have simplified the system based on the method of singular perturbations and the model of system dynamics is decomposed into a slow subsystem with rigid vibrations, expressed in terms of linear odes and a fast subsystem with vibrations, formalized by partial differential equations. Compared to the traditional model, the difficulty of managing the simplified model is drastically reduced.

4. Analysis and development of the management system
The key point in developing a control algorithm is that it must be able to work out the undefined parameters of a rigid model and an elastic pendulum. Besides, it should provide exponential stability of slow and fast subsystems [19, 22].

Considering these facts, this section develops a controller based on the principle of sliding mode control, which is part of a composite control scheme for a slow subsystem. A simple feedback control algorithm is proposed for a fast subsystem. An exponential analysis of the stability of slow and fast subsystems is proposed as a justification for the composite control scheme.

4.1. The control algorithm for the slow subsystem
Due to the presence of simulation errors and unknown interference, the controller must have high reliability. Sliding mode control is a typical method that provides reliable control. In sliding mode, the system is not sensitive to external interference and parametric uncertainties. Therefore, in this study, the development of the regulator was carried out based on the sliding mode control method. Discussing the equivalent matrix form, in which we take into account possible additive interference:

\[ \dot{x} = Ax + B(u + d(t)), \]  

where \( x = [\theta, \dot{\theta}, r, \dot{r}] \) and \(|d(t)| \leq D\). The sliding surface is defined as follows:

\[ s = B^T P x = 0 \]  

where \( P = P^T > 0 \). The controller is described as follows:

\[ u(t) = u_{eq} + u_n \]  

\[ u_{eq} = -(B^T P B)^{-1} B^T P A x(t) \]  

\[ u_n = -(B^T P B)^{-1} [B^T P D + kesi] sat(s), \]  

where: \( kesi = 0.5 \), \( sat(s) \) is a saturation function and is defined as:

\[ sat(s) = \begin{cases} 1 & s > \Delta \\ k s & |s| \leq \Delta, k = 1 / \Delta \\ -1 & s < -\Delta \end{cases} \]

The \( \Delta \) parameter defines the so-called “boundary layer”.

The Lyapunov function is selected as follows:

\[ V = \frac{1}{2} s^2, \]  

where:

\[ \dot{s} = B^T P x(t) = B^T P \left(Ax + B(u + d(t))\right) = B^T P A x(t) + B^T P B u + B^T P B d(t) \]

\[ = B^T P A x(t) + B^T P B (-(B^T P B)^{-1} B^T P A x(t) \]

\[ - (B^T P B)^{-1} [B^T P D + kesi] sat(s)) + B^T P B d(t) \]

\[ = - [B^T P B D + kesi] sat(s) + B^T P B d(t) \]

where therefore,

\[ \dot{V} = s \dot{s} = -[B^T P B D + kesi] s \cdot sat(s) \leq 0 \]  

(43)
To make sure of the algorithm stability, the sliding surface of a closed system, as well as the sliding surface function $P$, was constructed using the method of linear matrix inequalities.

Then the formula for the General control algorithm is entered:

$$u(t) = -K_0 x + v(t)$$

(44)

where

$$v(t) = K_0 x + u_{eq} + u_n$$

$$\dot{x}(t) = \dot{A}x(t) + B(v + d(t))$$

(45)

where $\dot{A} = A - BK_0$. In addition, $K_0$ are introduced to meet the Hurwitz stability requirement. Thus, a slow subsystem is stable as a closed system. Consider the function

$$V = x^T P x = 2x^T P \dot{A} x(t) + 2x^T PB(v + d(t))$$

(46)

The function describing the switching surface $s$ in a finite time tends to zero, that is, $s = 0$ and equation (46) takes the form:

$$\dot{V} = 2x^T P \dot{A} x = x^T (P \dot{A} + \dot{A}^T P)x$$

(47)

To ensure that $\dot{V} < 0$, the following condition should be met:

$$P \dot{A} + \dot{A}^T P < 0$$

(48)

The former inequality equals to the relations

$$(A - BK_0)P^{-1} + P^{-1}(A - BK_0)^T < 0$$

(49)

$$AP^{-1} + P^{-1}A^T < BL + L^T B^T$$

(50)

Inequality (50) is solved using standard linear matrix inequality methods.

4.2. The control algorithm for the slow subsystem

This section discusses the study of an elastic pendulum. It is assumed that the pendulum is fixed at one end and free at the other. A load is attached to the free end. The feedback controller is defined as follows:

$$f_f = -\alpha \hat{\omega}_f(L, v) + \rho \alpha \beta \hat{\omega}_{fxxx}(L, v)$$

(51)

To prove the stability of the fast subsystem according to the controller formula (51), the Lyapunov function is defined as follows:

Step 1. Consider the functions

$$E(v) = \int_0^1 (EI \ddot{w}_{fxx}^2 + \rho \ddot{\omega}_f^2) dx + K_1(\rho \ddot{w}_{fxxx}(L, v)) + \frac{m}{\beta} \hat{\omega}_f(L, v))^2$$

(52)

$$\dot{E}(v) = 2 \int_0^1 (w_{fxx} \ddot{w}_{fxx} + \ddot{\omega}_f \ddot{\omega}_f) dx + K_1(\rho \ddot{w}_{fxxx}(L, v)) + \frac{m}{\beta} \dot{\hat{\omega}}_f(L, v))$$

(53)

$$\left(-\rho \alpha \hat{\omega}_{fxxx}(L, v) + \frac{m}{\beta} \hat{\omega}_f(L, v)\right) = -\frac{2K_1(\rho \alpha)^2}{\beta} w_{fxxx}^2(L, v) - \frac{2K_1 \alpha m \beta^2}{\beta^2} \hat{\omega}^2(L, v) \leq 0$$

In turn, $K_1$ is defined as follows:

$$K_1 = \frac{\beta^2}{(m + a \beta) \rho} > 0$$

(54)

Step 2. Define the Lyapunov Function as follows:

$$V(v) = v E(v) + \int_0^L x \ddot{w}_f(x, v)W_{fx}(x, v) dx,$$

(55)
Differentiation (55) in fast time is obtained:
\[
\dot{V}(v) = E(v) + vE(v) + \int_0^L ax\omega_f(x,v)\omega_f(x,v) \, dx - \int_0^L xw_{fxxx}(x,v)w_f(x,v) \, dx = \\
\int_0^L (aw_{fxx}^2 + \omega_f^2) \, dx + K_1(-\rho ax_fxxx(L,v) + \frac{m}{\beta} \omega_f(L,v)^2) + K_1(-\rho ax_fxxx(L,v) + \\
\frac{m}{\beta} \omega_f(L,v)^2 + v \left( -\frac{2K_1(\rho a)^2}{\beta} w_{fxxx}^2(L,v) - \frac{2K_1(\rho a)^2}{\beta^2} \omega_f^2(L,v) \right) - Lw_f(L,v)w_{fxxx}(L,v) - \\
\frac{3}{2} \int_0^L w_{fxx}^2(x,v) \, dx + \frac{aL}{2} \omega_f^2(L,v) - \frac{a}{\delta_1} \int_0^L \omega_f^2(x,v) \, dx
\]
(56)
Using the Cauchy-Schwarz inequality, we obtain the following inequality:
\[
(-\rho ax_fxxx(L,v) + \frac{m}{\beta} \omega_f(L,v))^2 \leq 2((\rho a)^2 w_{fxxx}^2(L,v) + 2\frac{m^2}{\beta^2} \omega_f^2(L,v)
\]
(57)
\[
w_{fxx}^2(L,v) \leq \int_0^L w_{fxx}^2(x,v) \, dx
\]
(58)
\[
w_f(L,v)w_{fxxx}(L,v) \leq \delta_1 w_{fxx}^2(L,v) + \frac{1}{\delta_1} w_{fxxx}^2(L,v)
\]
(59)
Accordingly, using 57-59, (56) can be rewritten as:
\[
\dot{V}(v) \leq -(1 - \delta_1) \int_0^L (aw_{fxx}^2 + \omega_f^2) \, dx - \left( \frac{2K_1(\rho a)^2}{\beta} - K_1(\rho a)^2 - \frac{L}{\delta_1} \right) w_{fxxx}^2(L,v) - \\
\left( \frac{2K_1mav}{\beta^2} - \frac{aL}{2} - \frac{K_1m^2}{\beta^2} \right) \omega_f^2(L,v)
\]
Then select a parameter that satisfies the following inequality:
\[
\begin{align*}
1 - \delta_1 & \geq 0 \\
\frac{2K_1(\rho a)^2}{\beta} - K_1(\rho a)^2 & \geq 0 \\
\frac{2K_1mav}{\beta^2} - \frac{aL}{2} - \frac{K_1m^2}{\beta^2} & \geq 0
\end{align*}
\]
Then \( \dot{V}(v) \leq 0 \)
From (55) we can get:
\[
(v - C)E(v) \leq V(v) \leq (v - C)E(v),
\]
(60)
Where C is a positive constant. Then, (60) takes the form
\[
E(v) \leq V(v)(v - C) \leq (v + C)E(v)(v - C),
\]
(61)
\[
v > \max \left\{ \frac{\beta(\rho a)^2 + \frac{L}{\delta_1}}{2K_1(\rho a)^2}, \frac{\beta^2(\frac{aL}{2} + \frac{K_1m^2}{\beta^2})}{2K_1(\rho a)^2} \right\}
\]
where \( \dot{V}(v) \leq 0 \)
If \( T > v \), equation (61) takes the form:
\[
E(v) \leq \frac{V(T)}{(v-C)} \leq \frac{(T+C)E(0)}{(v-C)}
\]
(62)
By squaring and integrating, we get:
\[
\int_0^\infty E(v)^2 \, dv \leq \int_0^\infty \frac{(v+C)^2E(0)^2}{(v-C)^2} \, dv
\]
(63)
Which means exponential stability.

5. Conclusion

To solve the problem of controlling a flexible manipulator, the method of singular perturbation was used, by which the model was decomposed into slow and fast subsystems at different time scales.

- The slow subsystem was developed by the regulator, to stabilize the system.
- The fast subsystem was developed by the feedback controller.

The simulation results showed that the developed control strategy gives good results in a system with interference.

References

[1] Butikov E I 2011 An improved criterion for Kapitza’s pendulum stability *Journal of Physics A: Mathematical and General* 44(29) 1-17
[2] Butikov E I 2012 Oscillations of a simple pendulum with extremely large amplitudes *European Journal of Physics* 33(6) 1555-63
[3] Mikheev Y V, Sobolev V A and Fridman E M 1988 Asymptotic analysis of digital control systems *Autom Remote Control* 49(9) 1175-80
[4] Semenov M E, Grachikov D V, Rukavitsyn A G et al. 2014 On the state feedback control of inverted pendulum with hysteretic nonlinearity *MATEC Web of Conferences* 16 05009
[5] Semenov M E, Meleshenko P A, Nguyen H T T et al. 2014 Radiation of Inverted Pendulum with Hysteretic Nonlinearity (Guangzhou, China: PIERS Proceedings) p 1442-5
[6] Semenov M E, Matveev M G, Lebedev G N et al. 2017 Stabilization of the reverse flexible pendulum with hysteresis properties *Mechatronics, automation, control* 18(8) 516-25
[7] Butikov E I 2002 Subharmonic resonances of the parametrically driven pendulum *Journal of Physics A: Mathematical and Theoretical* 35(30) 6209-31
[8] Sun J Y, Huang X C, Liu X T et al. 2013 Study on the force transmissibility of vibration isolators with geometric nonlinear damping *Nonlinear Dynamics* 74(4) 1103-12
[9] Solovyov A M, Semenov M E, Meleshenko P A et al. 2017 Hysteretic nonlinearity and unbounded solutions in oscillating systems *Proceedia Engineering* 201(201) 578-83
[10] Semenov M E, Solovyov A M, Meleshenko P A et al. 2018 Nonlinear Damping: From Viscous to Hysteretic Dampers Recent Trends in Applied Nonlinear Mechanics and Physics, ed Mohamed Belhaq *Springer Proceedings in Physics* 199 259-75 https://doi.org/10.1007/978-3-319-639376_15
[11] Zhang Z Y and Miao X J 2010 Global existence and uniform decay forwave equation with dissipative term and boundary damping *Computers and Mathematics with Applications* 59(2) 1003-18
[12] Dadios E P, Fernandez P S and Williams D J 2006 Genetic algorithm on line controller for the flexible inverted pendulum problem *Journal of Advanced Computational Intelligence and Intelligent Informatics* 10(2) 155-60
[13] Luo Zheng-Hua and Bao-Zhu Guo 1997 Shear force feedback control of a single-link flexible robot with a revolute joint *IEEE Transaction on automatic control* 42(1) 53-65
[14] Osintsev M S and Sobolev V A 2014 Lowering the dimension of optimal estimation problems for dynamical systems with singular perturbations *Computing math and math. Phys* 54(1) 50-64
[15] Ryazhskikh V I, Semenov M E, Rukavitsyn A G et al. 2017 Stabilization of the reverse pendulum on a two-wheeled vehicle *Bulletin of the South Ural state University Series “Mathematics Mechanics Physics”* 9(3) 27-33
[16] Wang L and Ross J 1990 Synchronous neural networks of nonlinear threshold elements with hysteresis *Neurobiology* 87(3) 988-92
[17] Zhang Z Y, Liu Z H, Miao X J et al. 2011 Global existence and uniform stabilization of a
generalized dissipative Klein–Gordon equation type with boundary damping. Journal of Mathematical Physics 52(2) 023502-12

[18] Reshmin S A and Chernousko F L 2006 Optimal speed control of an inverted pendulum in the form of synthesis Izvestiya RAS Theory and control system 3 51-62

[19] Reshmin S A 2008 Search for the main bifurcation value of the maximum control moment in the problem of synthesis of optimal pendulum control Izvestiya RAS: Theory and control systems 2 5-20

[20] Anokhin N V 2012 Bringing a multi-link pendulum to the equilibrium position using a single control moment Izvestiya RAS Theory and control systems 5 44-53

[21] Formalsky A M 2006 On the stabilization of a double inverted pendulum using a single control moment Izvestiya RAS Theory and control systems 3 5-12

[22] Aranovsky S V, Biryuk A E, Nikulchev E V et al. 2019 Synthesis of the observer in the problem of stabilization of the reverse pendulum with an error in the position sensors Izvestiya RAS Theory and control systems 2 145-53