The Schrödinger-Virasoro type Lie bialgebra: a twisted case

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Abstract. In this paper we investigate Lie bialgebra structures on a twisted Schrödinger-Virasoro type algebra $\mathfrak{L}$. All Lie bialgebra structures on $\mathfrak{L}$ are triangular coboundary, which is different from the relative result on the original Schrödinger-Virasoro type Lie algebra. In particular, we find for this Lie algebra that there are more hidden inner derivations from itself to $\mathfrak{L} \otimes \mathfrak{L}$ and we develop one method to search them.

Key words: Lie bialgebras, Yang-Baxter equation, twisted Schrödinger-Virasoro algebras.

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§1 Introduction

In order to investigate the free Schrödinger equations, the original Schrödinger-Virasoro Lie algebra was introduced by [5] in the context of non-equilibrium statistical physics. It is the first case of this type Lie algebras. Since then there appeared some other cases, whose structure and representation theories were investigated in the corresponding references. All the cases of the Schrödinger-Virasoro type Lie algebra are closely related to the Schrödinger algebra and the (generalized) Virasoro algebra, which play important roles in many areas of mathematics and physics. The twisted deformation of the original one was introduced by [14], whose representation theory and cohomological theory were investigated there and its derivation algebra and automorphism group were determined in [7]. The Harish-Chandra modules and those of intermediate series on both original and twisted cases were partially classified in [6]. The deforming cases of the Schrödinger algebra were introduced in [18], whose 2-cocycles were completely given in [9]. The extended case of this type algebra was also introduced in [14], whose structure theory were considered in [3]. Generalized Schrödinger-Virasoro algebras were introduced in [17] and their automorphisms also Verma modules were studied and completely determined there. Recently, the Whittaker modules and bialgebra structures on the original case were investigated respectively in [24] and [4].

Now we introduce the Schrödinger-Virasoro algebra $\mathfrak{L}$, which is an infinite-dimensional Lie algebra over the complex field $\mathbb{C}$ with basis $\{L_n, W_n, Z_n | n \in \mathbb{Z}\}$ and the following non-vanishing Lie brackets:

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n}, & [L_m, W_n] &= -(m + n)W_{m+n}, \\
[L_m, Z_n] &= -(3m + n)Z_{m+n}, & [W_m, W_n] &= (m - n)Z_{m+n}.
\end{align*}
\]

Indeed, $\mathfrak{L}$ is a twisted case of the deforming Schrödinger-Virasoro algebras, which factually was investigated under the physics background before the Schrödinger-Virasoro type algebras

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appeared. According to our observations and computations, we find that the central extensions and Leibniz central extensions are not consistent with each other and the results on Harish-Chandra modules and those of intermediate series are also different to many other cases of the Schrödinger-Virasoro type Lie algebras. That is also our motivation to concentrate on this case of deformative Schrödinger-Virasoro algebras in this paper.

It is well known that Drinfeld introduced the notion of Lie bialgebras in 1983 to solve of the Yang-Baxter equation (see [1]). After that, many types of bialgebra were considered on different algebras. However, different algebra backgrounds, different difficulties and also there are no uniform methods to deal with such problems on all algebras. Moreover, sometimes, different algebra backgrounds, maybe different results. The Witt and Virasoro Lie bialgebras were initially investigated in [16] and classified in [13]. During the recent years, Lie bialgebras of generalized Witt types, generalized Virasoro-like type, generalized Weyl type, Hamiltonian type and Block type were considered respectively in [15, 20, 23, 21, 8], most of which parallel to that given in [13]. Lie superbialgebra structures on the Ramond $N=2$ super Virasoro algebra were considered and determined in [23]. Recently, the Lie bialgebra structures on the original Schrödinger-Virasoro algebra were considered by [4], which are different from that given in [13]. In this paper we shall investigate Lie bialgebra structures on $L$. Compared to the case considered in [4], it is more complicated according to the different basis and brackets between them and interesting based on the analysis presented above.

Some relative definitions and concepts on Lie bialgebras are collected and presented here. For any vector space $L$, denote $\xi$ and $\tau$ respectively the cyclic map of $L \otimes L \otimes L$ and the twist map of $L \otimes L$, which imply $\xi(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_3 \otimes x_1$ and $\tau(x_1 \otimes x_2) = x_2 \otimes x_1$ for any $x_1, x_2, x_3 \in L$. Based on them, one can reformulate the definitions of a Lie algebra, a Lie coalgebra and furthermore a Lie bialgebra as follows.

For a vector space $L$ and two bilinear maps $\delta : L \otimes L \to L$ and $\Delta : L \to L \otimes L$, the pair $(L, \delta)$ becomes a Lie algebra if the following conditions satisfy:

$$\text{Ker}(1 - \tau) \subset \text{Ker} \delta, \quad \delta \cdot (1 \otimes \delta) \cdot (1 + \xi + \xi^2) = 0,$$

and the pair $(L, \Delta)$ becomes a Lie coalgebra if the following conditions satisfy:

$$\text{Im} \Delta \subset \text{Im}(1 - \tau), \quad (1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta = 0.$$

And then the triple $(L, \delta, \Delta)$ becomes a Lie bialgebra if $(L, \delta)$ is a Lie algebra, $(L, \Delta)$ is a Lie coalgebra, and the following compatible condition holds:

$$\Delta \delta(x \otimes y) = x \cdot \Delta y - y \cdot \Delta x, \quad \forall x, y \in L, \quad (1.2)$$

where the symbol "·" to stand for the diagonal adjoint action:

$$x \cdot (\sum_i a_i \otimes b_i) = \sum_i ([x, a_i] \otimes b_i + a_i \otimes [x, b_i]).$$
Denote 1 the identity element of \( U \), which is the universal enveloping algebra of \( L \). For any \( r = \sum a_i \otimes b_i \in L \otimes L \), we introduce the following notations

\[
r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i.
\]

Then we can define \( c(r) \) to be elements of \( U \otimes U \otimes U \) by

\[
c(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]
\]

\[
= \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].
\]

**Definition 1.1** (1) A coboundary Lie bialgebra is a 4-tuple \((L, \delta, \Delta, r)\), where \((L, \delta, \Delta)\) is a Lie bialgebra and \(r \in \text{Im}(1 - \tau) \subset L \otimes L\) such that \(\Delta = \Delta_r\) is a coboundary of \(r\), i.e.,

\[
\Delta_r(x) = x \cdot r \quad \text{for} \quad x \in L.
\]

(2) A coboundary Lie bialgebra \((L, \delta, \Delta, r)\) is called triangular if it satisfies the following classical Yang-Baxter Equation

\[
c(r) = 0.
\]

(3) \(r \in \text{Im}(1 - \tau) \subset L \otimes L\) is said to satisfy the modified Yang-Baxter equation if

\[
x \cdot c(r) = 0, \quad \forall \ x \in L.
\]

Regard \( \mathfrak{U} = L \otimes L \) as an \( L \)-module under the adjoint diagonal action. Denote by \( \text{Der}(L, \mathfrak{U}) \) the set of derivations \( D : L \to \mathfrak{U} \), namely, \( D \) is a linear map satisfying

\[
D([x, y]) = x \cdot D(y) - y \cdot D(x),
\]

and \( \text{Inn}(L, \mathfrak{U}) \) the set consisting of the derivations \( v_{\text{inn}}, v \in \mathfrak{U} \), where \( v_{\text{inn}} \) is the inner derivation defined by \( v_{\text{inn}} : x \mapsto x \cdot v \). Then it is well known that \( H^1(L, \mathfrak{U}) \cong \text{Der}(L, \mathfrak{U})/\text{Inn}(L, \mathfrak{U}) \), where \( H^1(L, \mathfrak{U}) \) is the first cohomology group of the Lie algebra \( L \) with coefficients in the \( L \)-module \( \mathfrak{U} \).

The main result of this paper can be formulated as follows.

**Theorem 1.2** Every Lie bialgebra \((L, [\cdot, \cdot], \Delta)\) is triangular coboundary.

### §2 Proof of the main result

Throughout the paper we denote by \( \mathbb{Z}^* \) the set of all nonnegative integers and \( \mathbb{C}^* \) the set of all nonnegative complex numbers. For any subset \( \Omega \) of \( \mathbb{Z} \), denote \( \mathbb{Z} \setminus \Omega = \{ x \in \mathbb{Z} \mid x \notin \Omega \} \).

Firstly, the results of the following lemma can be found in the references or obtained by using the similar arguments as those given therein (e.g. [1, 2, 13, 20]).
Lemma 2.1  

(i) Denote $\mathcal{L}^\otimes n$ the tensor product of $n$ copies of $\mathcal{L}$ and regard it as an $\mathcal{L}$-module under the adjoint diagonal action of $\mathcal{L}$. If $x \cdot r = 0$ for some $r \in \mathcal{L}^\otimes n$ and all $x \in \mathcal{L}$, then $r = 0$.

(ii) $r$ satisfies (1.4) if and only if it satisfies (1.5).

(iii) Let $L$ be a Lie algebra and $r \in \text{Im}(1 - \tau) \subset L \otimes L$, then

$$(1 + \xi + \xi^2) \cdot (1 \otimes \Delta_r) \cdot \Delta_r(x) = x \cdot c(r), \quad \forall \ x \in L,$$

and the triple $(L, [\cdot, \cdot], \Delta_r)$ is a Lie bialgebra if and only if $r$ satisfies (1.4).

A derivation $D \in \text{Der}(\mathcal{L}, \mathcal{V})$ is homogeneous of degree $\alpha \in \mathbb{Z}$ if $D(L_p) \subset \mathcal{V}_{\alpha + p}$ for all $p \in \mathbb{Z}$. Denote $\text{Der}(\mathcal{L}, \mathcal{V})_\alpha = \{D \in \text{Der}(\mathcal{L}, \mathcal{V}) \mid \deg D = \alpha\}$ for $\alpha \in \mathbb{Z}$. Let $D$ be an element of $\text{Der}(\mathcal{L}, \mathcal{V})$. For any $\alpha \in \mathbb{Z}$, define the linear map $D_\alpha : \mathcal{L} \rightarrow \mathcal{V}$ as follows: For any $\mu \in \mathcal{L}_q$ with $q \in \mathbb{Z}$, write $D(\mu) = \sum_{p \in \mathbb{Z}} \mu_p$ with $\mu_p \in \mathcal{V}_p$, then we set $D_\alpha(\mu) = \mu_{q + \alpha}$. Obviously, $D_\alpha \in \text{Der}(\mathcal{L}, \mathcal{V})_\alpha$ and we have

$$D = \sum_{\alpha \in \mathbb{Z}} D_\alpha,$$  \hspace{1cm} (2.1)

which holds in the sense that for every $u \in \mathcal{L}$, only finitely many $D_\alpha(u) \neq 0$, and $D(u) = \sum_{\alpha \in \mathbb{Z}} D_\alpha(u)$ (we call such a sum in (2.1) summable).

Proposition 2.2  $\text{Der}(\mathcal{L}, \mathcal{V}) = \text{Inn}(\mathcal{L}, \mathcal{V})$.

This proposition follows from a series of claims.

Claim 1  If $\alpha \in \mathbb{Z}^*$, then $D_\alpha \in \text{Inn}(\mathcal{L}, \mathcal{V})$.

For $\alpha \neq 0$, denote $\gamma = -\alpha^{-1} D_\alpha(L_0) \in \mathcal{V}_\alpha$. Then for any $x_n \in \mathcal{W}_n$, applying $D_\alpha$ to $[L_0, x_n] = -nx_n$, and using $D_\alpha(x_n) \in \mathcal{V}_{n+\alpha}$, we obtain

$$-(\alpha + n)D_\alpha(x_n) - x_n \cdot D_\alpha(L_0) = L_0 \cdot D_\alpha(x_n) - x_n \cdot D_\alpha(L_0) = -nD_\alpha(x_n),$$  \hspace{1cm} (2.2)

i.e., $D_\alpha(x_n) = \gamma_{\text{inn}}(x_n)$. Thus $D_\alpha = \gamma_{\text{inn}}$ is inner.

Claim 2  $D_0(L_0) = 0$.

For any $n \in \mathbb{Z}$ and $x_n \in \mathcal{W}_n$, applying $D_0$ to $[L_0, x_n] = -nx_n$, one has $x_n \cdot D_0(L_0) = 0$. Thus by Lemma 2.1(i), $D_\alpha(L_0) = 0$.

Claim 3  Replacing $D_0$ by $D_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, one can suppose $D_\alpha(\mathcal{L}) = 0$. 

\hspace{1cm} 4
For any \( n \in \mathbb{Z} \), one can write \( D_0(L_n), D_0(W_n) \) and \( D_0(Z_n) \) as follows

\[
D_0(L_n) = \sum_{i \in \mathbb{Z}} (a_{n,i}L_i \otimes L_{n-i} + b_{n,i}L_i \otimes W_{n-i} + c_{n,i}L_i \otimes Z_{n-i} + d_{n,i}W_i \otimes Z_{n-i} + b_{n,i}^\dagger W_i \otimes L_{n-i}) \\
+ e_{n,i}Z_i \otimes L_{n-i} + d_{n,i}^\dagger Z_i \otimes W_{n-i} + e_{n,i}W_i \otimes W_{n-i} + f_{n,i}Z_i \otimes Z_{n-i}),
\]

\[
D_0(W_n) = \sum_{i \in \mathbb{Z}} (g_{n,i}L_i \otimes L_{n-i} + h_{n,i}L_i \otimes W_{n-i} + p_{n,i}L_i \otimes Z_{n-i} + q_{n,i}W_i \otimes Z_{n-i} + h_{n,i}^\dagger W_i \otimes L_{n-i}) \\
+ p_{n,i}^\dagger Z_i \otimes L_{n-i} + q_{n,i}^\dagger Z_i \otimes W_{n-i} + s_{n,i}W_i \otimes W_{n-i} + t_{n,i}Z_i \otimes Z_{n-i}),
\]

\[
D_0(Z_n) = \sum_{i \in \mathbb{Z}} (\alpha_{n,i}L_i \otimes L_{n-i} + \beta_{n,i}L_i \otimes W_{n-i} + \gamma_{n,i}L_i \otimes Z_{n-i} + \mu_{n,i}W_i \otimes Z_{n-i} + \beta_{n,i}^\dagger W_i \otimes L_{n-i}) \\
+ \gamma_{n,i}^\dagger Z_i \otimes L_{n-i} + \mu_{n,i}^\dagger Z_i \otimes W_{n-i} + \nu_{n,i}W_i \otimes W_{n-i} + \omega_{n,i}Z_i \otimes Z_{n-i}),
\]

where all coefficients of the tensor products are in \( \mathbb{C} \), and the sums are all finite. For any \( n \in \mathbb{Z} \), the following identities hold,

\[
L_1 \cdot (L_n \otimes L_{-n}) = (1 - n)L_{n+1} \otimes L_{-n} + (n + 1)L_n \otimes L_{1-n},
\]

\[
L_1 \cdot (L_n \otimes W_{-n}) = (1 - n)L_{n+1} \otimes W_{-n} + (n - 1)L_n \otimes W_{1-n},
\]

\[
L_1 \cdot (L_n \otimes Z_{-n}) = (1 - n)L_{n+1} \otimes Z_{-n} + (n - 3)L_n \otimes Z_{1-n},
\]

\[
L_1 \cdot (W_n \otimes Z_{-n}) = -(n + 1)W_{n+1} \otimes Z_{-n} + (n - 3)W_n \otimes Z_{1-n},
\]

\[
L_1 \cdot (W_n \otimes L_{-n}) = -(n + 1)W_{n+1} \otimes L_{-n} + (n + 1)W_n \otimes L_{1-n},
\]

\[
L_1 \cdot (Z_n \otimes L_{-n}) = -(n + 3)Z_{n+1} \otimes L_{-n} + (n + 1)Z_n \otimes L_{1-n},
\]

\[
L_1 \cdot (Z_n \otimes W_{-n}) = -(n + 3)Z_{n+1} \otimes W_{-n} + (n - 1)Z_n \otimes W_{1-n},
\]

\[
L_1 \cdot (W_n \otimes W_{-n}) = -(n + 1)W_{n+1} \otimes W_{-n} + (n - 1)W_n \otimes W_{1-n},
\]

\[
L_1 \cdot (Z_n \otimes Z_{-n}) = -(n + 3)Z_{n+1} \otimes Z_{-n} + (n - 3)Z_n \otimes Z_{1-n}.
\]

Let \( \Omega \) denote the set consisting of 9 symbols \( a, b, c, d, b^\dagger, c^\dagger, d^\dagger, e, f \). For each \( x \in \Omega \) we define \( M_x = \max\{ |p| \mid x_{1,p} \neq 0 \} \). Using the induction on \( \sum_{x \in \Omega} M_x \) in the above identities, and replacing \( D_0 \) by \( D_0 - u_{\text{inn}} \), where \( u \) is some linear combination of \( L_p \otimes L_{-p}, L_p \otimes W_{-p}, L_p \otimes Z_{-p}, W_p \otimes L_{-p}, W_p \otimes Z_{-p}, Z_p \otimes L_{-p}, Z_p \otimes W_{-p}, W_p \otimes W_{-p}, W_p \otimes L_{-p} \) and \( Z_p \otimes Z_{-p} \) with \( p \in \mathbb{Z} \), we can suppose \( a_{1,i} = b_{1,j} = c_{1,k} = d_{1,m} = b_{1,n}^\dagger = c_{1,p}^\dagger = d_{1,q}^\dagger = e_{1,s} = f_{1,t} = 0 \), for any \( i \in \mathbb{Z} \setminus \{-1, 2\}, \)

\( j \in \mathbb{Z} \setminus \{1, 2\}, k \in \mathbb{Z} \setminus \{2, 3\}, m \in \mathbb{Z} \setminus \{0, 3\}, n \in \mathbb{Z} \setminus \{0, 1\}, p \in \mathbb{Z} \setminus \{-2, -1\}, q \in \mathbb{Z} \setminus \{-2, 1\}, s \in \mathbb{Z} \setminus \{0, 1\} \) and \( t \in \mathbb{Z} \setminus \{-2, 3\} \). Thus the expression of \( D_0(L_1) \) can be simplified as

\[
D_0(L_1) = a_{1,-1}L_{-1} \otimes L_2 + a_{1,2}L_2 \otimes L_{-1} + b_{1,1}L_1 \otimes W_0 + b_{1,2}L_2 \otimes W_{-1} \\
+ c_{1,2}L_2 \otimes Z_{-1} + c_{1,3}L_3 \otimes Z_{-2} + d_{1,0}W_0 \otimes Z_1 + d_{1,3}W_3 \otimes Z_{-2} \\
+ b_{1,-1}W_{-1} \otimes L_2 + b_{1,0}W_0 \otimes L_1 + c_{1,-2}Z_{-2} \otimes L_3 + c_{1,-1}Z_{-1} \otimes L_2 \\
+ d_{1,-2}Z_{-2} \otimes W_3 + d_{1,1}Z_1 \otimes W_0 + e_{1,0}W_0 \otimes W_1 + e_{1,1}W_1 \otimes W_0 \\
+ f_{1,-2}Z_{-2} \otimes Z_3 + f_{1,3}Z_3 \otimes Z_{-2}.
\]
Applying $D_0$ to $[L_1, L_{-1}] = 2L_0$, we obtain

\[
\sum_{i \in \mathbb{Z}} \left( (i+2)a_{-1,i} - (i-2)a_{-1,1-i} \right) L_i \otimes L_{-i} + (ib_{-1,i} - (i-2)b_{-1,i-1}) L_i \otimes W_{-i} + ((i-2)c_{-1,i} - (i-2)c_{-1,1-i}) L_i \otimes Z_{-i} + ((i-2)d_{-1,i} - id_{-1,1-i}) W_i \otimes Z_{-i} + ((i+2)d_{-1,i} + ib_{-1,i}) W_i \otimes L_{-i} + ((i+2)c_{-1,i} - (i+2)c_{-1,1-i}) Z_i \otimes L_{-i},
\]

for any $i \in \mathbb{Z}$. By the following facts

\[
\begin{align*}
&c_{0,1} = -13/5c_{-1,2} = 1/5c_{-1,-1}, 
&-3/5c_{1,-3} = 3c_{1,-2} = 3c_{1,1} = c_{1,-2}, 
&d_{1,0} = d_{1,1} = d_{1,1} = -3/5d_{1,-1} = -5d_{1,-3},
&f_{1,-2} = -f_{1,-3},
&f_{1,3} = -f_{1,2},
\end{align*}
\]

which together with the fact that the set $\{i \in \mathbb{Z} | a_{-1,i} \neq 0\}$ is finite, forces

\[
a_{-1} = a_{-1,-1} + a_{-1,0} = 3a_{1,-1} + 3a_{1,2} + a_{1,1} = 3a_{1,2} + a_{1,0} + 3a_{-1,1} = 0, \quad \forall i \in \mathbb{Z} \setminus \{\pm 1\}.
\]

Similarly, comparing the coefficients of $L_i \otimes L_{-i}$ in the above identity, one has

\[
(i-2)a_{-1,i} = (i+2)a_{-1,i}, \quad \forall i \in \mathbb{Z} \setminus \{\pm 1\},
\]

for any $i \in \mathbb{Z} \setminus \{1, 2\}$, $i_3 \in \mathbb{Z} \setminus \{\pm 1, 0, 2\}$, $i_4 \in \mathbb{Z} \setminus \{2, -1\}$, $i_5 \in \mathbb{Z} \setminus \{3, -2\}$, $i_6 \in \mathbb{Z} \setminus \{3, -2, -1, 0\}$, $i_7 \in \mathbb{Z} \setminus \{1, 0\}$ and $i_8 \in \mathbb{Z} \setminus \{3, 2\}$.

Based on the following facts

\[
L_1 \cdot (W_{-1} \otimes L_1) = 0, \quad L_{-1} \cdot (W_{-1} \otimes L_1) = 2(W_{-2} \otimes L_1 - W_{-1} \otimes L_0),
\]

one can replace $D_0$ by $(D_0 + \frac{b_{-1,1}}{2}(W_{-1} \otimes L_1))(L_{-1})$, and then assume $b_{-1,1} = 0$. Then
Thus $D_0(L_1)$ can be rewritten as
\[
D_0(L_1) = a_{-1,1}L_1 \otimes L_0 + a_{-1,2}L_2 \otimes L_1 - a_{-1,1}L_0 \otimes L_{-1} + a_{-1,1}L_1 \otimes L_{-2} + c_{-1,1}L_1 \otimes Z_{-2} - \frac{5}{3} c_{-1,2}L_2 \otimes Z_{-3} + d_{-1,1}W_{-1} \otimes Z_0 + 3d_{-1,1}W_0 \otimes Z_{-1} - 3d_{-1,1}W_1 \otimes Z_{-2} + 5d_{-1,1}W_2 \otimes Z_{-3} - \frac{5}{3} c_{-1,2}Z_{-2} \otimes L_2 + c_{-1,2}Z_{-2} \otimes L_1 + d_{1,-3}Z_{-3} \otimes W_2 - \frac{3}{5} d_{1,-3}Z_{-2} \otimes W_1 + \frac{3}{5} d_{1,-3}Z_{-3} \otimes W_0 + \frac{1}{5} d_{1,-3}Z_0 \otimes W_{-1} + e_{-1,1}W_0 \otimes W_0 \otimes W_{-1} + f_{-1,3}Z_{-3} \otimes Z_2 + f_{-1,2}Z_2 \otimes Z_{-3}.
\]

Meanwhile, one also can deduce the following identity:
\[
D_0(L_1) = \left(\frac{1}{3}a_{-1,1} - a_{-1,2}\right)L_2 \otimes L_{-1} - (a_{-1,2} + \frac{1}{3}a_{-1,1})L_{-1} \otimes L_2 + \frac{1}{3} c_{-1,1}(L_2 \otimes Z_{-1} + L_3 \otimes Z_{-2}) - 3d_{-1,1}(W_0 \otimes Z_1 + W_3 \otimes Z_{-2}) + \frac{1}{3} c_{-1,2}(Z_{-2} \otimes L_3 + Z_{-1} \otimes L_2) - \frac{3}{5} d_{-1,3}(Z_{-2} \otimes W_3 + Z_1 \otimes W_0) - e_{-1,1}W_0 \otimes W_1 - e_{-1,0}W_1 \otimes W_0 - f_{-1,3}Z_{-2} \otimes Z_3 - f_{-1,2}Z_3 \otimes Z_2.
\]

Applying $D_0$ to $[L_2, L_{-1}] = 3L_1$, one has
\[
\sum_{i \in \mathbb{Z}} \left( (i-3)a_{2,i} - (i+2)a_{2,i+1} \right) L_i \otimes L_{-1} + \left( (i-1)b_{2,i} - (i+2)b_{2,i+1} \right) L_i \otimes W_{1,i} + \left( (i+1)c_{2,i} - (i+2)c_{2,i+1} \right) L_i \otimes Z_{1,i} + \left( (i-3)b_{2,i} - (i+2)b_{2,i+1} \right) W_i \otimes L_{1,i} + \left( (i-3)c_{2,i} - (i+2)c_{2,i+1} \right) Z_i \otimes L_{1,i} + \left( (i-3)c_{2,i} - (i+2)c_{2,i+1} \right) W_i \otimes W_{1,i} + \left( (i+1)f_{2,i} - (i+2)f_{2,i+1} \right) Z_i \otimes Z_{1,i} = a_{-1,2}(4L_0 \otimes L_1 + L_{-2} \otimes L_3) + 3a_{-1,1}(L_1 \otimes L_0 - L_0 \otimes L_1) + a_{-1,1}(L_3 \otimes L_{-2} + 4L_1 \otimes L_0) + 3(a_{-1,2} + a_{-1,1})L_{-1} \otimes L_2 - (2a_{-1,1} - 3a_{-1,1})L_2 \otimes L_{-1} - 4c_{-1,1}(L_1 \otimes Z_0 + Z_0 \otimes L_1) + 4c_{-1,1}L_2 \otimes Z_{-1} - 6c_{-1,1}L_2 \otimes L_{-1} - 2e_{-1,1}(W_0 \otimes W_1 - W_1 \otimes W_0) + 2e_{-1,0}(W_1 \otimes W_0 - W_2 \otimes W_{-1}) + d_{-1,1}(3W_1 \otimes Z_0 - 2W_2 \otimes Z_{-1} - 6W_2 \otimes Z_{-3} - 6W_0 \otimes Z_1 + 18W_3 \otimes Z_{-2}) + 1/5 d_{-1,1}(-20Z_3 \otimes W_4 - 21Z_1 \otimes W_2 - 6Z_2 \otimes W_{-1} + 11Z_0 \otimes W_1 + 18Z_2 \otimes W_3 - 6Z_1 \otimes W_0) - f_{-1,3}(8Z_{-3} \otimes Z_4 - 3Z_{-2} \otimes Z_3 + 3Z_{-1} \otimes Z_2) - f_{-1,2}(3Z_2 \otimes Z_{-1} - 3Z_3 \otimes Z_{-2} + 8Z_4 \otimes Z_{-3}).
\]

Comparing the coefficient of $L_i \otimes L_{1-i}$ and recalling that $\{i \mid a_{2,i} \neq 0\}$ is finite, one has
\[
0 = a_{2,i} = a_{-1,2} = a_{-1,1} \quad \text{for} \quad i \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\},
\]
\[
0 = a_{2,0} + (3a_{-1,1} + 4a_{2,-1}) = a_{2,1} - (6a_{-1,1} + a_{2,-1}) = a_{2,2} + (5a_{-1,1} + 4a_{2,-1}) = a_{2,3} = (2a_{-1,1} + a_{2,-1}).
\]
Similarly, one can obtain the following identities

\[ b_{2,i_1} = c_{2,i_2} = d_{2,i_3} = b^\dagger_{2,i_4} = c^\dagger_{2,i_5} = d^\dagger_{2,i_6} = e_{2,i_7} = f_{2,i_8} = 0, \]
\[ b_{2,1} - b_{2,-1} = b_{2,0} + 2b_{-2,-1} = b^\dagger_{2,2} + 2b^\dagger_{2,1} = b^\dagger_{2,3} - b^\dagger_{2,1} = 0, \]
\[ -1/6d_{2,0} = -1/4d_{2,1} = 2/3d_{2,3} = -1/6d_{2,2} = 2/5d_{2,1} = d_{-1,-1}, \]
\[ -5/4d^\dagger_{2,-2} = 10/3d^\dagger_{2,-1} = -5/6d^\dagger_{2,0} = 2d^\dagger_{2,1} = -5/6d^\dagger_{2,2} = d^\dagger_{-1,-3}, \]
\[ b_{2,3} = b^\dagger_{2,-1} = e_{-1,1} = c^\dagger_{-1,-2} = 0, \]
\[ e_{2,0} = -2e_{-1,0} = e_{2,2} = -2e_{-1,-1}, \]
\[ -5/8f_{2,-2} = 20/7f_{2,-1} = -f_{2,0} = 2f_{2,1} = -f_{2,2} = 20/7f_{2,3} = -5/8f_{2,4} = f_{-1,2} = f_{-1,-3}, \]

for any \( i_1 \in \mathbb{Z}\{\pm 1, 0, 1, 3\}, i_2 \in \mathbb{Z}\{-1\}, i_3 \in \mathbb{Z}\{0, 1, 2, 3, 4\}, i_4 \in \mathbb{Z}\{-1, 2, 3\}, i_5 \in \mathbb{Z}\{3\}, i_6 \in \mathbb{Z}\{\pm 2, \pm 1, 0\}, i_7 \in \mathbb{Z}\{0, 1, 2\} \) and \( i_8 \in \mathbb{Z}\{\pm 2, \pm 1, 0, 3, 4\}. \)

From the equation \([L_1, L_{-2}] = 3L_{-1}\), we obtain

\[
\sum_{p \in \mathbb{Z}} \left( (i+3)a_{-2,i} - (i-2)a_{-2,-i} \right) L_i \otimes L_{-1-i} + \left( (i+1)b_{-2,i} - (i-2)b_{-2,-i} \right) L_i \otimes W_{-1-i} \\
+ \left( (i-1)c_{-2,i} - (i-2)c_{-2,-i} \right) L_i \otimes Z_{-1-i} + \left( (i-1)d_{-2,i} - id_{-2,-i} \right) W_i \otimes Z_{-1-i} + \left( i+3b^\dagger_{-2,i} \right) \\
- \left( i+1 \right) a^\dagger_{-2,i} - (i+2)b^\dagger_{-2,-i} \right) Z_i \otimes L_{-1-i} + \left( (i+1)d^\dagger_{-2,i} - (i+2)d^\dagger_{-2,-i} \right) \\
\times Z_i \otimes W_{-1-i} + \left( (i+1)e_{-2,i} - ie_{-2,-i} \right) W_i \otimes W_{-1-i} + \left( (i-1)f_{-2,i} - (i+2)f_{-2,-i} \right) Z_i \otimes Z_{-1-i} \\
= 1/3a_{-1,-1} \left( L_3 \otimes L_2 + 13L_{-1} \otimes L_0 - 13L_0 \otimes L_{-1} - L_2 \otimes L_{-3} \right) + 3b_{-1,0} \left( L_0 \otimes W_{-1} - L_1 \otimes W_{-2} \right) \\
+ 1/5d_{-1,-3} \left( 3Z_0 \otimes W_{-1} - 14Z_4 \otimes W_3 - 6Z_1 \otimes W_0 - 6Z_2 \otimes W_2 - 15Z_3 \otimes W_2 - 6Z_2 \otimes W_1 \right) \\
+ 3d_{-1,-3} \left( W_{-1} \otimes Z_0 - 2W_{-2} \otimes Z_1 - 2W_0 \otimes Z_1 - 8W_3 \otimes Z_4 - 2W_1 \otimes Z_2 + 5W_2 \otimes Z_3 \right) \\
+ e_{-1,-1} \left( 3W_{-1} \otimes W_0 + 3W_0 \otimes W_{-1} - 2W_{-2} \otimes W_1 - W_0 \otimes W_{-1} - W_1 \otimes W_0 - 2W_1 \otimes W_{-2} \right) \\
+ f_{-1,-3} \left( 3Z_3 \otimes Z_2 - 8Z_4 \otimes Z_3 - 3Z_2 \otimes Z_1 - 3Z_1 \otimes Z_2 - 8Z_3 \otimes Z_4 + 3Z_2 \otimes Z_3 \right) .
\]

Comparing the coefficients of the tensor products of the above formula, we can deduce

\[ a_{-2,i_1} = b_{-2,i_2} = d_{-2,i_3} = b^\dagger_{-2,i_4} = d^\dagger_{-2,i_5} = e_{-2,i_6} = f_{-2,i_7} = 0, \]
\[ a_{-1,-1} = 0, -1/4a_{-2,-2} = 1/6a_{-2,-1} = -1/4a_{-2,0} = a_{-2,-3}, \]
\[ 1/2d_{-2,-2} = -2d_{-2,-1} = 1/6d_{-2,0} = -2/7d_{-2,1} = 1/8d_{-2,2} = d_{-1,-1}, \]
\[ 5/8d^\dagger_{-2,-4} = -10/7d^\dagger_{-2,-3} = 5/6d^\dagger_{-2,-2} = -10d^\dagger_{-2,-1} = 5/2d^\dagger_{-2,0} = d^\dagger_{-1,-3}, \]
\[ b_{-2,0} = -2b_{-2,-1} = 2b_{-2,1}, e_{-2,-2} = 2e_{-2,-1} = e_{-2,0}, b^\dagger_{-2,-2} = -2b^\dagger_{-2,-3} = -2b^\dagger_{-2,-1}, \]
\[ 5/8f_{-2,-4} = -20/7f_{-2,-3} = f_{-2,-2} = f_{-2,0} = -20/7f_{-2,1} = 5/8f_{-2,2} = -2f_{-2,-1} = f_{-1,-3}, \]

for all \( i_1 \in \mathbb{Z}\{-3, -2, 0, \pm 1\}, i_2 \in \mathbb{Z}\{\pm 0, \pm 1\}, i_3 \in \mathbb{Z}\{\pm 2, \pm 1, 0\}, i_4 \in \mathbb{Z}\{-3, -2, -1\}, \)
\( i_5 \in \mathbb{Z}\{-4, -3, -2, -1, 0\}, i_6 \in \mathbb{Z}\{-2, 0, 1\} \) and \( i_7 \in \mathbb{Z}\{-4, -3, \pm 2, 1\}. \)
Applying $D_0$ to $[L_2, L_{-2}] = 4L_0$, which combined with the relations obtained above, we can obtain the following identities:

$$a_{2,-1} + a_{-2,-3} = b_{-2,-1} = b_{2,-1} = c_{2,-1} = c_{2,3} = b_{2,1} = b_{-2,-3} = c_{2,1} + e_{-2,-1} = 0.$$  

Thus we can rewrite $D_0(L_{\pm 1})$ as follows:

$$D_0(L_1) = -3d_{-1,1}(W_0 \otimes Z_1 + W_3 \otimes Z_{-2}) - \frac{3}{5}d_{-1,-3}(Z_{-2} \otimes W_3 + Z_1 \otimes W_0) - e_{-1,1}(W_0 \otimes W_1 + W_1 \otimes W_0) - f_{-1,-3}(Z_{-2} \otimes Z_3 + Z_3 \otimes Z_{-2}),$$

$$D_0(L_{-1}) = d_{-1,1}(W_0 \otimes Z_0 + 3W_0 \otimes Z_{-1} - 3W_1 \otimes Z_{-2} + 5W_2 \otimes Z_{-3})$$

$$+ \frac{1}{5}d_{-1,-3}(5Z_{-3} \otimes W_2 - 3Z_{-2} \otimes W_1 + 3Z_{-1} \otimes W_0 + Z_0 \otimes W_{-1})$$

$$+ e_{-1,1}(W_1 \otimes W_0 + W_0 \otimes W_{-1}) + f_{-1,-3}(Z_{-3} \otimes Z_2 + Z_2 \otimes Z_{-3}).$$

Noticing that

$$L_{-1} \cdot (W_0 \otimes W_0) = W_{-1} \otimes W_0 + W_0 \otimes W_{-1},$$

$$L_1 \cdot (W_0 \otimes W_0) = -(W_1 \otimes W_0) - (W_0 \otimes W_1),$$

and then replacing $D_0$ by $D_0 - e_{-1,1}(W_0 \otimes W_0)_{inn}$, one can assume $e_{-1,1} = 0$.

According to the following identities,

$$L_{-1} \cdot (Z_{-2} \otimes Z_2 + Z_2 \otimes Z_{-2}) = 5Z_{-3} \otimes Z_2 + Z_{-2} \otimes Z_1 + Z_1 \otimes Z_{-2} + 5Z_2 \otimes Z_{-3},$$

$$L_1 \cdot (Z_{-2} \otimes Z_2 + Z_2 \otimes Z_{-2}) = -Z_{-1} \otimes Z_2 - 5Z_{-2} \otimes Z_3 - 5Z_3 \otimes Z_{-2} - Z_2 \otimes Z_{-1},$$

$$L_{-1} \cdot (Z_{-1} \otimes Z_1 + Z_1 \otimes Z_{-1}) = 4Z_{-2} \otimes Z_1 + 2Z_{-1} \otimes Z_0 + 2Z_0 \otimes Z_{-1} + 4Z_1 \otimes Z_{-2},$$

$$L_1 \cdot (Z_{-1} \otimes Z_1 + Z_1 \otimes Z_{-1}) = -2Z_0 \otimes Z_1 - 4Z_{-1} \otimes Z_2 - 4Z_2 \otimes Z_{-1} - 2Z_1 \otimes Z_0,$$

$$L_{-1} \cdot (Z_0 \otimes Z_0) = 3Z_{-1} \otimes Z_0 + 3Z_0 \otimes Z_{-1},$$

$$L_1 \cdot (Z_0 \otimes Z_0) = -3Z_1 \otimes Z_0 - 3Z_0 \otimes Z_1,$$

and then replacing $D_0$ by

$$D_0 - \frac{1}{5}f_{-1,-3}(Z_{-2} \otimes Z_2 + Z_2 \otimes Z_{-2})_{inn} + \frac{1}{20}f_{-1,-3}(Z_{-1} \otimes Z_1 + Z_1 \otimes Z_{-1})_{inn} - \frac{1}{30}f_{-1,-3}(Z_0 \otimes Z_0)_{inn},$$

one can safely assume $f_{-1,-3} = 0$.

**Remark** (i) Judging from the appearance, it is hard to believe one can assume $f_{-1,-3} = 0$. We realize this by subtracting three inner derivations of both $L_{-1}$ and $L_1$ continuously and simultaneously after a deep observation.

(ii) These inner derivations do not affect $D_0(L_{\pm 2})$ recalling the actions of $D_0$ on both $[L_1, L_{-2}] = 3L_{-1}$ and $[L_2, L_{-1}] = 3L_1$.  

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Thus $D_0(L_{\pm 1})$ can be rewritten as

\[
D_0(L_1) = -3d_{-1,-1}(W_0 \otimes Z_1 + W_3 \otimes Z_{-2}) - \frac{3}{5} d_{-1,3}^t(Z_2 \otimes W_3 + Z_1 \otimes W_0),
\]

\[
D_0(L_{-1}) = d_{-1,-1}(W_{-1} \otimes Z_0 + 3W_0 \otimes Z_{-1} - 3W_1 \otimes Z_{-2} + 5W_2 \otimes Z_{-3})
\]

\[
+ \frac{1}{5} d_{-1,3}^t(5Z_{-3} \otimes W_2 - 3Z_{-2} \otimes W_1 + 3Z_{-1} \otimes W_0 + Z_0 \otimes W_{-1}).
\]

And $D_0(L_2)$ can be rewritten as follows:

\[
a_{2,-1}(L_{-1} \otimes L_3 - 4L_0 \otimes L_2 + 6L_1 \otimes L_1 - 4L_2 \otimes L_0 + L_3 \otimes L_{-1}) + e_{2,1}W_1 \otimes W_1
\]

\[
+ d_{-1,-1}(-6W_0 \otimes Z_2 + \frac{5}{2} W_1 \otimes Z_1 - 6W_2 \otimes Z_0 + \frac{3}{2} W_3 \otimes Z_{-1} - 4W_4 \otimes Z_{-2})
\]

\[
+ d_{-1,3}^t(-\frac{4}{5} Z_{-2} \otimes W_4 + \frac{3}{10} Z_{-1} \otimes W_3 - \frac{6}{5} Z_0 \otimes W_2 + \frac{1}{2} Z_1 \otimes W_1 - \frac{6}{5} Z_2 \otimes W_0),
\]

while $D_0(L_{-2})$ can be rewritten as follows:

\[
-a_{2,-1}(L_{-3} \otimes L_1 - 4L_{-2} \otimes L_0 + 6L_{-1} \otimes L_{-1} - 4L_0 \otimes L_{-2} + L_1 \otimes L_{-3}) - e_{2,1}W_{-1} \otimes W_{-1}
\]

\[
+ d_{-1,-1}(2W_{-2} \otimes Z_0 - \frac{1}{2} W_{-1} \otimes Z_{-1} + 6W_0 \otimes Z_{-2} - \frac{7}{2} W_1 \otimes Z_{-3} + 8W_2 \otimes Z_{-4})
\]

\[
+ d_{-1,3}^t(\frac{8}{5} Z_{-4} \otimes W_2 - \frac{7}{10} Z_{-3} \otimes W_1 + \frac{6}{5} Z_{-2} \otimes W_0 - \frac{1}{10} Z_{-1} \otimes W_{-1} + \frac{2}{5} Z_0 \otimes W_{-2}).
\]

Set $u = L_{-1} \otimes L_1 - 2L_0 \otimes L_0 + L_1 \otimes L_{-1}$. Observing that $L_{\pm 1} \cdot u = 0$, one can assume $a_{2,-1} = 0$, when $D_0$ is replaced by $D_0 - a_{2,-1}(L_{-1} \otimes L_1 - 2L_0 \otimes L_0 + L_1 \otimes L_{-1})$. Thus

\[
D_0(L_2) = d_{-1,-1}(-6W_0 \otimes Z_2 + \frac{5}{2} W_1 \otimes Z_1 - 6W_2 \otimes Z_0 + \frac{3}{2} W_3 \otimes Z_{-1} - 4W_4 \otimes Z_{-2})
\]

\[
+ d_{-1,3}^t(-\frac{4}{5} Z_{-2} \otimes W_4 + \frac{3}{10} Z_{-1} \otimes W_3 - \frac{6}{5} Z_0 \otimes W_2 + \frac{1}{2} Z_1 \otimes W_1 - \frac{6}{5} Z_2 \otimes W_0)
\]

\[
+ e_{2,1}W_1 \otimes W_1,
\]

and

\[
D_0(L_{-2}) = d_{-1,-1}(2W_{-2} \otimes Z_0 - \frac{1}{2} W_{-1} \otimes Z_{-1} + 6W_0 \otimes Z_{-2} - \frac{7}{2} W_1 \otimes Z_{-3} + 8W_2 \otimes Z_{-4})
\]

\[
+ d_{-1,3}^t(\frac{8}{5} Z_{-4} \otimes W_2 - \frac{7}{10} Z_{-3} \otimes W_1 + \frac{6}{5} Z_{-2} \otimes W_0 - \frac{1}{10} Z_{-1} \otimes W_{-1} + \frac{2}{5} Z_0 \otimes W_{-2})
\]

\[
- e_{2,1}W_{-1} \otimes W_{-1}.
\]
From the equation \([L_{-1}, W_1] = 0\), we obtain
\[
\sum_{p \in \mathbb{Z}} \left( (i - 3) g_{1,i} - (i + 2) g_{1,i+1} \right) L_i \otimes L_{-i} + (i h_{1,i} - (i + 2) h_{1,i+1}) L_i \otimes W_{-i} + ((i + 2) p_{1,i} \right) L_i \otimes Z_{-i} + ((i + 2) q_{1,i} - i q_{1,i+1}) W_i \otimes Z_{-i} + ((i - 2) h_{1,i}^\dagger - i h_{1,i+1}^\dagger) W_i \otimes L_{-i} + ((i - 2) p_{1,i}^\dagger - (i + 2) p_{1,i+1}^\dagger) Z_i \otimes L_{-i} + (i q_{1,i}^\dagger - (i - 2) q_{1,i+1}^\dagger) Z_i \otimes W_{-i} + (i s_{1,i} - i s_{1,i+1}) W_i \otimes W_{-i} + ((i + 2) t_{1,i} - (i - 2) t_{1,i+1}) Z_i \otimes Z_{-i}\right)
= d_{-1,-1}(2W_0 \otimes Z_0 - 3W_1 \otimes Z_1 - 5W_3 \otimes Z_3) + d_{-1,-3}(2/5Z_0 \otimes W_0 - Z_3 \otimes W_3 + 3/5Z_{-1} \otimes W_1).\]

From the above formula it follows that
\[
g_{1,i_1} = h_{1,i_2} = p_{1,i_3} = q_{1,i_4} = h_{1,i_5}^\dagger = p_{1,i_6}^\dagger = q_{1,i_7}^\dagger = s_{1,i_8} = t_{1,i_9} = d_{-1,-1} = 0,
\]
\[
d_{-1,-3} = g_{1,0} + 3g_{1,-1} = g_{1,1} - 3g_{1,-1} = g_{1,2} + g_{1,-1} = h_{1,0} + h_{1,-1} = h_{1,2}^\dagger + h_{1,1}^\dagger = 0,
\]
for all \(i_1 \in \mathbb{Z} \setminus \{\pm 1, 0, 1, 2\}\), \(i_2 \in \mathbb{Z} \setminus \{-1, 0\}\), \(i_5 \in \mathbb{Z} \setminus \{1, 2\}\) and \(i_j \in \mathbb{Z}\) for \(j \in \{3, 4, 6, 7, 8, 9\}\).

Thus \(D_0(W_1)\) can be rewritten as
\[
D_0(W_1) = g_{1,-1}(L_{-1} \otimes L_2 - 3L_0 \otimes L_1 + 3L_1 \otimes L_0 - L_2 \otimes L_{-1}) + h_{1,-1}(L_{-1} \otimes W_2 - L_0 \otimes W_1) + h_{1,1}^\dagger(W_1 \otimes L_0 - W_2 \otimes L_{-1}).
\]

Then \(D_0(L_{\pm 1})\) and \(D_0(L_{\pm 2})\) can be rewritten as follows:
\[
D_0(L_{-1}) = D_0(L_1) = 0,
\]
\[
D_0(L_2) = e_{2,1}W_1 \otimes W_1, \quad D_0(L_{-2}) = -e_{2,1}W_{-1} \otimes W_{-1}.
\]

Applying \(D_0\) to \([L_2, [L_1, W_1]] = 0\), we obtain
\[
g_{1,-1}(L_{-3} \otimes L_3 - 4L_{-2} \otimes L_2 + 5L_{-1} \otimes L_1 - 5L_1 \otimes L_{-1} - L_3 \otimes L_{-3} + 4L_2 \otimes L_{-2}) + h_{1,1}^\dagger(8W_2 \otimes L_{-2} - W_{-1} \otimes L_1 - 6W_1 \otimes L_{-1} - 3W_3 \otimes L_{-3}) - 6e_{2,1}(W_1 \otimes W_{-1} + W_{-1} \otimes W_1)
= h_{1,-1}(8L_{-2} \otimes W_2 - 3L_{-3} \otimes -6L_{-1} \otimes W_1 - L_1 \otimes W_{-1}).
\]

From the above formula it follows that
\[
e_{2,1} = g_{1,-1} = h_{1,-1} = h_{1,1}^\dagger = 0.
\]

Then one can deduce
\[
D_0(W_1) = D_0(L_{\pm 1}) = D_0(L_{\pm 2}) = 0.
\]
Applying $D_0$ to $[L_1, W_0] = -W_1$, we obtain

$$
\sum_{i \in \mathbb{Z}} \left( \left( (i+1)g_{0,i} - (i-2)g_{0,i-1} \right) L_i \otimes L_{1-i} + \left( (i-1)h_{0,i} - (i-2)h_{0,i-1} \right) L_i \otimes W_{1-i} + \left( (i-3)p_{0,i} \right) \right)
\begin{align*}
&\sum_{i \in \mathbb{Z}} \left( (i+1)g_{0,i} - (i-2)g_{0,i-1} \right) L_i \otimes Z_{1-i} + \left( (i-3)g_{0,i} - iq_{0,i-1} \right) W_i \otimes Z_{1-i} + \left( (i+1)h_{0,i} + ih_{0,i-1} \right) W_i \otimes L_{1-i} + \\
&\sum_{i \in \mathbb{Z}} \left( (i+1)p_{0,i} - (i+2)p_{0,i-1} \right) Z_i \otimes L_{1-i} + \left( (i-1)q_{0,i} - (i+2)q_{0,i-1} \right) Z_i \otimes W_{1-i} + \left( (i-1)s_{0,i} \right) \right)
\end{align*}

from which it follows that

$$
g_{0,i} = h_{0,i} = p_{0,i} = q_{0,i} = h_{0,i}^\dagger = p_{0,i}^\dagger = q_{0,i}^\dagger = s_{0,i} = t_{0,i} = 0,$$

for all $i \in \mathbb{Z} \setminus \{\pm 1\}$, $i \in \mathbb{Z} \setminus \{-1\}$, $i \in \mathbb{Z} \setminus \{\pm 1\}$ and $i \in \mathbb{Z}$ for $j \in \{3, 4, 6, 7, 8, 9\}$.

Thus $D_0(W_0)$ can be rewritten as

$$D_0(W_0) = g_{0,0} \left( L_{-1} \otimes L_1 - 2L_0 \otimes L_0 + L_1 \otimes L_{-1} \right) + h_{0,0} \left( L_{-1} \otimes \right),$$

Applying $D_0$ to $[L_1, W_{-1}] = 0$, we obtain

$$\sum_{i \in \mathbb{Z}} \left( (i+2)g_{-1,i} - (i-2)g_{-1,i-1} \right) L_i \otimes L_{-i} + \left( (i+1)h_{-1,i} - (i-2)h_{-1,i-1} \right) L_i \otimes W_{-i} + \left( (i-2)p_{-1,i} \right) \right)
\begin{align*}
&\sum_{i \in \mathbb{Z}} \left( (i+2)g_{-1,i} - (i-2)g_{-1,i-1} \right) L_i \otimes Z_{-i} + \left( (i-2)q_{-1,i} - iq_{-1,i-1} \right) W_i \otimes Z_{-i} + \left( (i+1)h_{-1,i} + ih_{-1,i-1} \right) W_i \otimes L_{-i} + \\
&\sum_{i \in \mathbb{Z}} \left( (i+2)p_{-1,i} - (i+2)p_{-1,i-1} \right) Z_i \otimes L_{-i} + \left( (i+2)q_{-1,i} - (i+2)q_{-1,i-1} \right) Z_i \otimes W_{-i} + \left( (i-2)s_{-1,i} \right) \right)
\end{align*}

from which it follows that

$$g_{-1,i} = h_{-1,i} = p_{0,i} = q_{0,i} = h_{0,i}^\dagger = p_{0,i}^\dagger = q_{0,i}^\dagger = s_{0,i} = t_{0,i} = 0,$$

for all $i \in \mathbb{Z} \setminus \{-2, \pm 1, 0\}$, $i \in \mathbb{Z} \setminus \{0, 1\}$, $i \in \mathbb{Z} \setminus \{-2, -1\}$ and $i \in \mathbb{Z}$ for $j \in \{3, 4, 6, 7, 8, 9\}$.

Thus $D_0(W_{-1})$ can be rewritten as

$$D_0(W_{-1}) = g_{-1,-2} \left( L_{-2} \otimes L_1 - 3L_{-1} \otimes L_0 + 3L_0 \otimes L_{-1} - L_1 \otimes L_{-2} \right) + h_{-1,0} \left( L_0 \otimes W_{-1} - L_1 \otimes W_{-2} \right) + h_{-1,-2} \left( W_{-2} \otimes L_1 - W_{-1} \otimes L_0 \right).$$

Applying $D_0$ to $[L_{-1}, W_0] = -W_{-1}$ and comparing the coefficients of the tensor products, we obtain

$$g_{0,-1} = g_{-1,-2} = h_{-1,0} = h_{-1,-2} = 2h_{0,-1} = 0.$$
Thus we can rewrite

\[ D_0(W_0) = h_{0,1}L_1 \otimes W_{-1} + h_{0,-1}^\dagger W_{-1} \otimes L_1, \]
\[ D_0(W_{-1}) = h_{-1,0}(L_0 \otimes W_{-1} - L_1 \otimes W_{-2}) + h_{-1,-2}^\dagger(W_{-2} \otimes L_1 - W_{-1} \otimes L_0). \]

Applying \( D_0 \) to \([L_2, W_{-1}] = -W_1\) and comparing the coefficients of the tensor products, we obtain \( h_{0,1} = h_{0,-1}^\dagger = 0 \). Thus

\[ D_0(W_{-1}) = D_0(W_0) = 0, \]

which together with

\[ D_0(W_1) = D_0(L_{\pm 1}) = D_0(L_{\pm 2}) = 0, \]

and the Lie brackets of \( \mathfrak{L} \), implies

\[ D_0(L_n) = D_0(W_n) = D_0(Z_n) = 0, \quad \forall \ n \in \mathbb{Z}. \]

It is easy to check that the following claim also holds for the algebra \( \mathfrak{L} \) here.

**Claim 4** For any \( D \in \text{Der}(\mathfrak{L}, \mathfrak{Y}) \), (2.1) is a finite sum.

Then finally the proposition follows. \( \Box \)

The following lemma is still true for \( \mathfrak{L} \) by employing the technique of Lemma 2.5 in [4].

**Lemma 2.3** Suppose \( v \in \mathfrak{Y} \) such that \( x \cdot v \in \text{Im}(1 - \tau) \) for all \( x \in \mathfrak{L} \). Then \( v \in \text{Im}(1 - \tau) \).

**Proof of Theorem 1.2** Let \((\mathfrak{L}, [\cdot, \cdot], \Delta)\) be a Lie bialgebra structure on \( \mathfrak{L} \). By Proposition 2.2, \( \Delta = \Delta_r \) for some \( r \in \mathfrak{L} \otimes \mathfrak{L} \). Combining \( \text{Im} \Delta \subset \text{Im}(1 - \tau) \) and Lemma 2.3, one can deduce \( r \in \text{Im}(1 - \tau) \). Then Lemma 2.1 shows that \( c(r) = 0 \). Hence Definition 1.1 says that \((\mathfrak{L}, [\cdot, \cdot], \Delta)\) is a triangular coboundary Lie bialgebra. \( \Box \)

**References**

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