Tetravalent $s$-Transitive Graphs of Order $4p^2$

Mohsen Ghasemi · Jin-Xin Zhou

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Abstract Let $s$ be a positive integer. A graph is $s$-transitive if its automorphism group is transitive on $s$-arcs but not on $(s + 1)$-arcs. Let $p$ be a prime. Zhou (Discrete Math 309:6081–6086, 2009) classified tetravalent $s$-transitive graphs of order $4p$. In this article a complete classification of tetravalent $s$-transitive graphs of order $4p^2$ is given.

Keywords $s$-Transitive graphs · Symmetric graphs · Cayley graphs

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1 Introduction

In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph $X$ we use $V(X)$, $E(X)$ and Aut$(X)$ to denote its vertex set, edge set and its full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to $u$ and $v$ in $X$, and $N(u)$ is the neighborhood of $u$ in $X$, that is, the set of vertices adjacent to $u$ in $X$. A graph $X$ is locally primitive if for any vertex $v \in V(X)$, the stabilizer Aut$(X)_v$ of $v$ in Aut$(X)$ is primitive on $N(v)$. An $s$-arc in a graph is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. For a subgroup $G \leq$ Aut$(X)$, a graph $X$ is said to be $(G, s)$-arc-transitive or $(G, s)$-regular if $G$ acts...
transitively or regularly on the set of $s$-arcs of $X$, respectively. A $(G, s)$-arc-transitive graph is said to be $(G, s)$-transitive if it is not $(G, s + 1)$-arc-transitive. In particular, an $(\text{Aut}(X), s)$-arc-transitive, $(\text{Aut}(X), s)$-regular or $(\text{Aut}(X), s)$-transitive graph is simply called an $s$-arc-transitive, $s$-regular or $s$-transitive graph, respectively. Note that 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A graph is edge-transitive if $\text{Aut}(X)$ is transitive on $E(X)$. A vertex- and edge-transitive graph is said to be $\frac{1}{2}$-arc-transitive if it is not arc-transitive.

Edge-transitive graphs or $s$-transitive graphs of small valencies have received considerable attention in the literature. For instance, Tutte [24] initiated the investigation of cubic $s$-transitive graphs by proving that there exist no cubic $s$-transitive graphs for $s \geq 6$. Gardiner and Praeger [12,13] generally explored the tetravalent symmetric graphs by considering the automorphism groups. Let $p$ be a prime. Conder [5] showed that for a fixed integer $n$ and any integer $s > 1$, there are only finitely many cubic $s$-transitive graphs of order $np$. Li [16] generalized this result to connected symmetric graphs of any valency, and he also posed the following problem: for small values $n$ and $k$, classify vertex-transitive locally primitive graphs of order $np$ and valency $k$.

In this paper we classify all symmetric graphs of order $np$ and valency $k$ for certain values of $n$ and $k$. The classification of $s$-transitive graphs of order $np$ and of valency 3 or 4 can be obtained from [3,4,25], where $1 \leq n \leq 3$. Feng et al. [7–10] classified cubic $s$-transitive graphs of order $np$ with $n = 4, 6, 8$ or 10. Recently, Feng et al. [11] classified tetravalent $\frac{1}{2}$-arc-transitive graphs of order $4p$, and Zhou and Feng [28,29] classified tetravalent $s$-transitive graphs of order $4p$ or $2p^2$. In this paper, a complete classification of tetravalent $s$-transitive graphs of order $4p^2$ is given.

2 Preliminaries

In this section, we introduce some notations and definitions as well as some preliminary results which will be used later in the paper.

For a regular graph $X$, use $d(X)$ to represent the valency of $X$, and for any subset $B$ of $V(X)$, the subgraph of $X$ induced by $B$ will be denoted by $X[B]$. Let $X$ be a connected vertex-transitive graph, and let $G \leq \text{Aut}(X)$ be vertex-transitive on $X$. For a $G$-invariant partition $B$ of $V(X)$, the quotient graph $X_B$ is defined as the graph with vertex set $B$ such that, for any two vertices $B$, $C \in B$, $B$ is adjacent to $C$ if and only if there exist $u \in B$ and $v \in C$ which are adjacent in $X$. Let $N$ be a normal subgroup of $G$. Then the set $B$ of orbits of $N$ in $V(X)$ is a $G$-invariant partition of $V(X)$. In this case, the symbol $X_B$ will be replaced by $X_N$.

For a positive integer $n$, denote by $\mathbb{Z}_n$ the cyclic group of order $n$ as well as the ring of integers modulo $n$, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to $n$, by $D_{2n}$ the dihedral group of order $2n$, and by $C_n$ and $K_n$ the cycle and the complete graph of order $n$, respectively. We call $C_n$ an $n$-cycle.

For two groups $M$ and $N$, $N \rtimes M$ denotes a semidirect product of $N$ by $M$. For a subgroup $H$ of a group $G$, denote by $C_G(H)$ the centralizer of $H$ in $G$ and by $N_G(H)$ the normalizer of $H$ in $G$. Then $C_G(H)$ is normal in $N_G(H)$.

**Proposition 2.1** [15, Chapter I, Theorem 4.5] The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(H)$ of $H$. 

\[\text{ Springer}\]
Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is semiregular on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if $G$ is transitive and semiregular. For any $g \in G$, $g$ is said to be semiregular if $\langle g \rangle$ is semiregular. The following proposition gives a characterization of Cayley graphs in terms of their automorphism groups.

**Proposition 2.2** [2, Lemma 16.3] A graph $X$ is isomorphic to a Cayley graph on a group $G$ if and only if its automorphism group has a subgroup isomorphic to $G$, acting regularly on the vertex set of $X$.

Let $X_N$ be the quotient graph of $X$. If $X_N$ and $X$ have the same valency, then $X$ is called a normal cover of $X_N$. Let $X$ be a connected tetravalent symmetric graph and $N$ an elementary abelian $p$-group. A classification of connected tetravalent symmetric graphs was obtained when $N$ has at most two orbits in [12] and a characterization of such graphs was given when $X_N$ is a cycle in [13].

The following proposition is due to Praeger et al., refer to [12, Theorem 1.1] and [20].

**Proposition 2.3** Let $X$ be a connected tetravalent $(G, 1)$-arc-transitive graph. For each normal subgroup $N$ of $G$, one of the following holds:

1. $N$ is transitive on $V(X)$;
2. $X$ is bipartite and $N$ acts transitively on each part of the bipartition;
3. $N$ has $r \geq 3$ orbits on $V(X)$, the quotient graph $X_N$ is a cycle of length $r$, and $G$ induces the full automorphism group $D_{2r}$ on $X_N$;
4. $N$ has $r \geq 5$ orbits on $V(X)$, $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is a connected tetravalent $G/N$-symmetric graph, and $X$ is a $G$-normal cover of $X_N$.

Moreover, if $X$ is also $(G, 2)$-arc-transitive, then case (3) cannot happen.

The following proposition characterizes the vertex stabilizer of the connected tetravalent $s$-transitive graphs, which can be deduced from [19, Lemma 2.5], or [18, Proposition 2.8], or [17, Theorem 2.2].

**Proposition 2.4** Let $X$ be a connected tetravalent $(G, s)$-transitive graph. Let $G_v$ be the stabilizer of a vertex $v \in V(X)$ in $G$. Then $s = 1, 2, 3, 4$ or $7$. Furthermore, $G_v$ is a 2-group for $s = 1$; $G_v$ is isomorphic to $A_4$ or $S_4$ for $s = 2$; $G_v$ is isomorphic to $A_4 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_3 \rtimes S_4$, or $S_3 \rtimes S_4$ for $s = 3$; $G_v$ is isomorphic to $\mathbb{Z}_3^2 \rtimes \text{GL}(2, 3)$ for $s = 4$; and $G_v$ is isomorphic to $[3^5] \rtimes \text{GL}(2, 3)$ for $s = 7$, where $[3^5]$ represents an arbitrary group of order $3^5$.

For the definitions of the graphs in the following proposition we refer the reader to [6]

**Proposition 2.5** [6, Theorem 5.1] Let $p$ be a prime. Then a tetravalent graph of order $4p^2$ is one-regular if and only if it is isomorphic to one of the following graphs. Furthermore, all the graphs are pairwise non-isomorphic.
(1) $X \cong \mathcal{B}W_{12}[5, 1, 5], |V(X)| = 36$, and $\text{Aut}(X) \cong G_{36} \rtimes \mathbb{Z}_2^2$;
(2) $X \cong \mathcal{G}PS2[4, 3, (01): (12)], |V(X)| = 36, |\text{Aut}(X)| = 144$;
(3) $X \cong \mathcal{N}C_{4p^2}^0, |V(X)| = 4p^2, p > 7, p \equiv \pm 1 \pmod{8}, \text{Aut}(X)$ is given in [13, Lemma 8.4];
(4) $X \cong \mathcal{N}C_{4p^2}^1, |V(X)| = 4p^2, p > 7, or p \equiv \pm 1 or 3 \pmod{4} \text{Aut}(X)$ is given in [13, Lemma 8.7];
(5) $X \cong \mathcal{C}A_{4p^2}^0, |V(X)| = 4p^2, p \equiv 1 \pmod{4}, \text{Aut}(X) \cong (\mathbb{Z}_{2p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_4$;
(6) $X \cong \mathcal{C}A_{4p^2}^1, |V(X)| = 4p^2, p > 2, \text{Aut}(X) \cong (\mathbb{Z}_{4p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$;
(7) $X \cong \mathcal{C}A_{4p^2}^2, |V(X)| = 4p^2, p \equiv 1 \pmod{4}, \text{Aut}(X) \cong G_{4p^2}^3 \rtimes \mathbb{Z}_4$.

3 Examples

In this section, we introduce several families of connected tetravalent symmetric graphs. The first example is the lexicographic product of $C_{2p^2}$ and $2K_1$.

Example 3.1 Let $p$ be a prime. The lexicographic product $C_{2p^2}[2K_1]$ is defined as the graph with vertex set $V(C_{2p^2}) \times V(2K_1)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(C_{2p^2}[2K_1])$, $u$ is adjacent to $v$ in $C_{2p^2}[2K_1]$ whenever $[x_1, x_2] \in E(C_{2p^2})$.

It can be obtained from [23, Theorem] and [27, Table 1] that $C_{2p^2}[2K_1]$ is a tetravalent 1-transitive graph and $\text{Aut}(C_{2p^2}[2K_1]) \cong (\mathbb{Z}_{2p^2}^2) \rtimes D_{4p^2}$.

Proposition 3.2 [1, Theorem 1.2] Let $X$ be a connected tetravalent Cayley graph on an abelian group $G$ of order $4p^2$, where $p > 2$ is a prime. Then either $G \leq A$, or $X \cong C_{2p^2}[2K_1]$.

The following graph was first defined by Praeger and Xu, see [21, Definition 2.1 (b)].

Example 3.3 Let $p$ be an odd prime. The graph $C(2; p^2, 2)$ has vertex set $\mathbb{Z}_{p^2} \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and its edges are defined by

$$\{(i, (x, y)), (i + 1, (y, z))\} \in E(C(2; p^2, 2))$$

for all $i \in \mathbb{Z}_{p^2}$ and $x, y, z \in \mathbb{Z}_2$. By Praeger and Xu [21, Lemma 2.12], $\text{Aut}(C(2; p^2, 2)) \cong \mathbb{Z}_{p^2}^2 \rtimes D_{2p^2}$ and hence $C(2; p^2, 2)$ is 1-transitive.

The following two examples are defined by Gardiner and Praeger, see [13, Definitions 2.2.2.3].

Example 3.4 Let $p > 3$ be a prime. The graphs $C^{\pm 1}(p; 4, 2)$ has vertex set $\mathbb{Z}_p^2 \times \mathbb{Z}_4$. For any $(i, j) \in \mathbb{Z}_p^2, (i, j; 0) \sim (i \pm 1, j; 1), (i, j; 1) \sim (i, j \pm 1; 2), (i, j; 2) \sim (i \pm 1, j; 3), (i, j; 3) \sim (i, j \pm 1; 0)$.
In this section, we classify tetravalent -transitive graphs of order 4 for each prime p. To do so, we need the following lemma.

Lemma 4.1 Let p be a prime and let n > 1 be an integer. Let X be a connected tetravalent graph of order 4p^n. If G ≤ Aut(X) is transitive on the arc set of X, then every minimal normal subgroup of G is solvable.
Proof Let $v \in V(X)$. Since $G$ is arc-transitive on $X$, by Proposition 2.4, $G_v$ either is a 2-group or has order dividing $2^4 \cdot 3^6$. It follows that $|G| \mid 2^6 \cdot 3^6 \cdot p^n$ or $|G| = 2^m + 2 \cdot p^n$ for some integer $m$. Let $N$ be a minimal normal subgroup of $G$.

Suppose that $N$ is non-solvable. Then $p > 3$ and $|G| \mid 2^6 \cdot 3^6 \cdot p^n$ because a $[2, p]$-group is solvable by a theorem of Burnside [22, Theorem 8.5.3]. It follows that $X$ is $(G, 2)$-arc-transitive. Let $X(v)$ be the neighborhood of $v$ in $X$. For any $v \in V(X)$, 3 $| |N_v|$ because $p > 3$, and the 2-arc-transitivity of $G$ implies that $N_v$ acts transitively on $X(v)$ because $N_v \trianglelefteq G_v$. By Proposition 2.3, $N$ has at most two orbits on $V(X)$. Hence, $2p^n$ divides $|N|$. Since $N$ is minimal, it is a product of isomorphic non-abelian simple groups. Since $|N| \mid 2^6 \cdot 3^6 \cdot p^n$, by Gorenstein [14, pp.12–14], each direct factor of $N$ is one of the following:

$$A_5, A_6, \text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSL}(2, 17), \text{PSL}(3, 3), \text{PSU}(3, 3) \text{ and } \text{PSU}(4, 2).$$

(2)

An inspection of the orders of such groups gives $n = 2$ and $N \cong A_5 \times A_5, A_6 \times A_6, \text{PSL}(2, 7) \times \text{PSL}(2, 7)$ or $\text{PSL}(2, 8) \times \text{PSL}(2, 8)$. Let $N \cong A_5 \times A_5$ or $A_6 \times A_6$. Then $p = 5$ and $|X| = 100$. However, from [26] we know that all tetravalent arc-transitive graphs of order 100 are 1-transitive, a contradiction. Let $N \cong \text{PSL}(2, 7) \times \text{PSL}(2, 7)$ or $\text{PSL}(2, 8) \times \text{PSL}(2, 8)$. If $N$ is transitive on $V(X)$, then $X$ must be $(N, 2)$-transitive. Clearly, a direct factor $T$ of $N$ has at least $p(= 5, 7)$ orbits on $V(X)$. This forces that $T$ is semiregular on $V(X)$ which is impossible because $|V(X)| = 4p^n$. Thus, $N$ has exactly two orbits on $V(X)$. Then $|N_v| = 2^5 \cdot 3^2$ or $2^5 \cdot 3^4$, which is impossible by Proposition 2.4. Thus, $N$ is solvable.

□

Lemma 4.2 The graphs $C^{±1}(p; 4, 2)$ and $C^{±ε}(p; 4, 2)$ are connected tetravalent 1-transitive graphs.

Proof Set $X = C^{±1}(p; 4, 2)$ and $A = \text{Aut}(X)$. By Gardiner and Praeger [13, Definition 2.2], $X$ is a connected tetravalent arc-transitive graph. Let $G = \langle α, β, σ, τ, γ \rangle$ which is given in Eq. (1). Then $G$ acts transitively but not regularly on the arc set of $X$. Since $p > 3$, $N = \langle α, β \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is a Sylow $p$-subgroup of $A$.

Suppose that $A$ has no non-trivial normal $p$-subgroup. Let $L$ be a minimal normal subgroup of $A$. By Lemma 4.1, $L$ is an elementary abelian 2-group. Let $M$ be the maximal normal 2-subgroup of $A$. Consider the quotient graph $X_M$ of $X$ relative to the orbit set of $M$, and let $K$ be the kernel of $A$ acting on $V(X_M)$. Since $p > 3$, every orbit of $M$ has length 2 or 4, and hence $|X_M| = p^2$ or $2p^2$. By the symmetry of $X$, every orbit of $M$ contains no edges and by Proposition 2.3, $d(X_M) = 2$ or 4.

Let $d(X_M) = 2$. Then $X_M \cong C_p^2$ or $C_2p^2$. It follows that $A/K \cong D_{2p^2}$ or $D_{4p^2}$. This forces that every Sylow $p$-subgroup of $A$ is cyclic, contrary the fact that $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Let $d(X_M) = 4$. If $|X_M| = 2p^2$ then by Proposition 2.3, $M = K \cong \mathbb{Z}_2$. By Zhou and Feng [29, Theorem 3.3], $A/M$ is a $(2, p)$-group and hence it is solvable. Since $M$ is a maximal normal 2-subgroup of $A$, we can take a maximal normal $p$-subgroup, say $T/M$ in $A/M$ with $T/M \cong \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. By Sylow Theorem, $T = M \times L$ where $L$ is a Sylow $p$-subgroup of $T$. Then $L$ is characteristic in $T$ and hence it is normal in
A because $T \trianglelefteq A$, a contradiction. Let $|X_M| = p^2$. If $d(X_M) = 4$ then by Proposition 2.3, $K = M$ is semiregular on $V(X_M)$. Therefore, $K = M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4$. Let $P$ be a Sylow $p$-subgroup of $A$. Since $p > 3$, $PM = P \times M$ is abelian. Clearly, $PM$ is transitive on $V(X)$. Then $PM$ is regular on $V(X)$ because $|PM| = 4p^2$. By Proposition 2.2, $X$ is a Cayley graph on $PM$. If $PM \trianglelefteq A$, then $P \trianglelefteq A$, a contradiction. If $PM$ is not normal in $A$, then by Proposition 3.2, $X \cong C_{2p^2}[2K_1]$ ($p > 3$), and hence $A \cong \mathbb{Z}_2^{2p^2} \rtimes D_{4p^2}$. This forces that every Sylow $p$-subgroup of $A$ is cyclic, contrary the fact that $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Therefore, $A$ has a minimal normal $p$-subgroup, say $U$. If $U \cong \mathbb{Z}_p$, then $U \leq N$ and $U$ is normal in $G$. However, by the argument following Example 3.4, $N$ is a minimal normal subgroup of $G$, a contradiction. Thus, $U = N \cong \mathbb{Z}_p^{2p^2}$ is minimal in $A$. Since $X$ is not one-regular, by Proposition 3.6, $|A| = 32p^2 = |G|$. Thus, $A = G = \langle \alpha, \beta, \sigma, \tau, \gamma \rangle$ and hence $X$ is 1-transitive.

Similarly, we can show that $\text{Aut}(C_{\pm \varepsilon}(p; 4, 2)) = \langle \alpha, \beta, \sigma', \tau, \gamma \rangle$, and hence $C_{\pm \varepsilon}(p; 4, 2)$ is also 1-transitive. □

In what follows, the notation “$C_{4}[n, m]$” will refer to the $m$th graph of order $n$ in the Wilson et al.’s census of all tetravalent edge-transitive graphs of order up to 150, and for their constructions, one may see [26].

**Theorem 4.3** Let $p$ be a prime and let $X$ be a connected tetravalent graph of order $4p^2$. Then $X$ is $s$-transitive for some positive integer $s$ if and only if it is isomorphic to one of the graphs in Table 1. Furthermore, all graphs in Table 1, are pairwise non-isomorphic.

### Table 1 Tetravalent $s$-transitive graphs of order $4p^2$

| $X$ | $s$-Transitivity | $\text{Aut}(X)$ |
|-----|------------------|----------------|
| $C_{4}[16, 2]$ | 2-Transitive | For construction see [26] |
| $C_{4}[16, 1]$ | 1-Transitive | For construction see [26] |
| $C_{4}[36, m]$ ($1 \leq m \leq 6$) | 1-Transitive | For construction see [26] |
| $C_{4}[100, m]$ ($1 \leq m \leq 6, m = 11$) | 1-Transitive | For construction see [26] |
| $C_{2p^2}[2K_1]$ | 1-Transitive | $\mathbb{Z}_2^{2p^2} \rtimes D_{4p^2}$ |
| $C(2; p^2, 2)$ | 1-Transitive | $\mathbb{Z}_2^{p^2} \rtimes D_{2p^2}$ |
| $C_{\pm 1}(p; 4, 2)$ | 1-Transitive | For construction see [13] |
| $C_{\pm \varepsilon}(p; 4, 2)$ | 1-Transitive | For construction see [13] |
| $BW_{12}[5, 1, 5]$ | 1-Transitive | $G_{36} \times \mathbb{Z}_2^2$ |
| $G_2\mathcal{P}S2[4, 3, (01) : (12)]$ | 1-Transitive | For construction see [6] |
| $\mathcal{N}C_{4p^2}$ | 1-Transitive | For construction see [13] |
| $\mathcal{N}C_{4p^2}$ | 1-Transitive | For construction see [13] |
| $\mathcal{C}A_{4p^2}$ | 1-Transitive | $(\mathbb{Z}_2^{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_4$ |
| $\mathcal{A}A_{4p^2}$ | 1-Transitive | $(\mathbb{Z}_4 \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^{2p^2}$ |
| $\mathcal{C}A_{4p^2}$ | 1-Transitive | $G_3 \times \mathbb{Z}_2^{p^2}$ |
Proof Let \( X \) be a tetravalent \( s \)-transitive graph of order \( 4p^2 \) for a positive integer \( s \). To finish the proof it suffices to show that \( X \) is one of the graphs listed in Table 1. If \( p \leq 5 \), then \( |X| = 16, 36 \) or 100 and by Wilson and Potočnik [26], \( X \) is isomorphic either to 2-transitive \( C4[16, 2] \), or to one of the following 1-transitive graphs: \( C4[16, 1], C4[36, m] (1 \leq m \leq 6) \) and \( C4[100, m] (1 \leq m \leq 6 \text{ or } m = 11) \). If \( X \) is one-regular then \( X \) is one of the graphs in Proposition 2.5. In what follows, we assume that \( p > 5 \) and that \( X \) is not one-regular. Set \( A = \text{Aut}(X) \) and let \( P \) be a Sylow \( p \)-subgroup. First we prove two claims.

Claim I If \( P \leq A \) then \( X \cong C^{\pm_1}(p; 4, 2) \) or \( C^{\pm_2}(p; 4, 2) \).

If \( P \) is a minimal normal subgroup of \( A \) then by Proposition 3.6, \( X \cong C^{\pm_1}(p; 4, 2) \) or \( C^{\pm_2}(p; 4, 2) \). Suppose that \( P \) contains a non-trivial subgroup, say \( N \), which is normal in \( A \). Consider the quotient graph \( X_N \) of \( X \) relative to the orbit set of \( N \), and let \( K \) be the kernel of \( A \) on \( V(X_N) \). Since \( p > 5 \), one has \( |X_N| = 4p \), and hence \( d(X_N) = 2 \) or 4 by Proposition 2.3.

Let \( d(X_N) = 2 \). Then \( X_N \cong C_{4p} \) and hence \( A/K \cong \text{Aut}(C_{4p}) \cong D_{8p} \). Let \( \Delta \) and \( \Delta' \) be two adjacent orbits of \( N \) in \( V(X) \). Then the subgraph \( X[\Delta \cup \Delta'] \) of \( X \) induced by \( \Delta \cup \Delta' \) has valency 2. Since \( p > 5 \), one has \( X[\Delta \cup \Delta'] \cong C_{2p} \). The subgroup \( K^* \) of \( K \) fixing \( \Delta \) pointwise also fixes \( \Delta' \) pointwise. The connectivity of \( X \) and the transitivity of \( A/K \) on \( V(X_N) \) imply that \( K^* = 1 \), and consequently, \( K \cong \text{Aut}(X[\Delta \cup \Delta']) \cong D_{4p} \). Since \( K \) fixes \( \Delta \), one has \( |K| \leq 2p \). It follows that \( |A| = |A/K||K| \leq 16p^2 \), and hence \( X \) is one-regular, a contradiction.

Let \( d(X_N) = 4 \). Then \( N = K \) and \( X_N \) is \( A/N \)-symmetric. Clearly, \( P/N \leq A/N \). Consider the quotient graph \( X_{P/N} \) of \( X_N \) relative to the orbit set of \( P/N \) in \( V(X_N) \). Let \( H/N \) be the kernel of \( A/N \) acting on \( V(X_{P/N}) \). Then \( |X_{P/N}| = 4 \) and by Proposition 2.3, \( X_{P/N} \cong C_4 \) and \( (A/N)/(H/N) \cong D_8 \). Let \( \Delta \) and \( \Delta' \) be two adjacent orbits of \( P/N \) in \( V(X_N) \), and let \( X_N[\Delta \cup \Delta'] \) be the subgraph of \( X_N \) induced by \( \Delta \cup \Delta' \). Since \( X_N \) is \( A/N \)-symmetric, each orbit of \( P/N \) has no edges, implying that \( X_N[\Delta \cup \Delta'] \cong C_{2p} \). Since \( p > 3 \), \( H/N \) acts faithfully on \( \Delta \). Therefore, \( H/N \) can be regarded as a group of automorphisms of \( C_{2p} \). It follows that \( |H/N| \leq 2p \) and hence \( |A/N| = 16p \) because of the symmetry of \( X \). Thus, \( |A| = 16p^2 \) and hence \( X \) is one-regular, a contradiction.

Claim II If \( A \) has a non-trivial normal 2-subgroup, then \( X \) is isomorphic to \( C^{\pm_1}(p; 4, 2), C^{\pm_2}(p; 4, 2), C_{2p^2}[2K_1], \) or \( C(2; p^2, 2) \).

Let \( M \) be the maximal normal 2-subgroup of \( A \) and assume \( M > 1 \). Consider the quotient graph \( X_M \) of \( X \) relative to the orbit set of \( M \), and let \( K \) be the kernel of \( A \) acting on \( V(X_M) \). Since \( p > 5 \), every orbit of \( M \) has length 2 or 4, and hence \( |X_M| = p^2 \) or \( 2p^2 \). By the symmetry of \( X \), every orbit of \( M \) contains no edges and by Proposition 2.3, \( X_M \) is of valency 2 or 4.

Let \( |X_M| = 2p^2 \). If \( d(X_M) = 2 \) then \( X \cong C_{2p^2}[2K_1] \). Let \( d(X_M) = 4 \). Then \( K = M \cong \mathbb{Z}_2 \) and by Zhou and Feng [29, Theorem 3.3], \( A/M \) is a \((2, p)\)-group and hence it is solvable. Since \( M \) is a maximal normal 2-subgroup of \( A \), we can take a maximal normal \( p \)-subgroup, say \( T/M \) in \( A/M \) with \( T/M \cong \mathbb{Z}_p \) or \( \mathbb{Z}_p \times \mathbb{Z}_p \). By Sylow Theorem, \( T = M \times L \) where \( L \) is a Sylow \( p \)-subgroup of \( T \). Then \( L \) is characteristic in \( T \) and hence it is normal in \( A \) because \( T \leq A \). Clearly, \( L \cong \mathbb{Z}_p \times \mathbb{Z}_p \) or \( \mathbb{Z}_p \)
$\mathbb{Z}_p$. For the former, by Claim I, $X \cong C^{\pm 1}(p; 4, 2)$ or $C^{\pm \varepsilon}(p; 4, 2)$. Assume $L \cong \mathbb{Z}_p$. Then $T \cong \mathbb{Z}_{2p}$. Let $C = C_A(T)$. Then $T \leq C$ and by Proposition 2.1, $A/C \leq \text{Aut}(T) \cong \mathbb{Z}_{p-1}$. Clearly, $PC/C \leq A/C$, where $P$ is a Sylow $p$-subgroup of $A$. If $C = T$ then $PT = P \times M \trianglelefteq A$, and hence $PM/M \trianglelefteq A/M$. This is contrary to the fact that $L/M$ a maximal normal $p$-subgroup of $A/M$. Thus, $C \triangleright T$. Let $H/T$ be a minimal normal subgroup of $A/T$ contained in $C/T$. Then $H = H_1 \times H_2$, where $H_1$ and $H_2$ are Sylow 2-subgroup and Sylow $p$-subgroup of $H$, respectively. It is easy to see that $H_i \unlhd A$ with $i = 1$ or 2. Since $M$ is a maximal normal 2-subgroup of $A$, one has $H_1 = M$. Then $H_2 = P$ because $H \triangleright T$. By Claim I, $X \cong C^{\pm 1}(p; 4, 2)$ or $C^{\pm \varepsilon}(p; 4, 2)$.

Let $|X_M| = p^2$. If $d(X_M) = 4$ then by Proposition 2.3, $K = M$ is semiregular on $V(X_M)$. Therefore, $K = M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4$. Since $p > 3$, $PM = P \times M$ is abelian. Clearly, $PM$ is transitive on $V(X)$. Then $PM$ is regular on $V(X)$ because $|PM| = 4p^2$. By Proposition 2.2, $X$ is a Cayley graph on $PM$. If $PM$ is not normal in $A$, then by Proposition 3.2, $X \cong C_{2p^2}[K_4](p > 3)$. If $PM \trianglelefteq A$, then $P \trianglelefteq A$ because $P$ is characteristic in $PM$. By Claim I, $X \cong C^{\pm 1}(p; 4, 2)$ or $C^{\pm \varepsilon}(p; 4, 2)$.

Assume $d(X_M) = 2$. Then $X_M \cong C_{p^2}$ and $A/K \cong D_{2p^2}$. It is easy to see that $K$ is a 2-group. By the maximality of $M$, one has $M = K$. Let $\Delta$ and $\Delta'$ be two adjacent orbits of $M$ in $V(X)$. Then the subgraph $X[\Delta \cup \Delta']$ of $X$ induced by $\Delta \cup \Delta'$ has valency 2. Thus $X[\Delta \cup \Delta'] \cong C_8$ or $2C_4$. Let $X[\Delta \cup \Delta'] \cong C_8$. Since $p > 5$, $K$ acts faithfully on $\Delta$. Therefore, $K \leq \text{Aut}(X[\Delta \cup \Delta']) \cong D_{16}$. Since $K$ fixes $\Delta$, one has $K \cong D_8$. Clearly, $PK/K \leq A/K$, namely, $PK \leq A$. Since $p > 5$ and $K \cong D_8$, one has $P \leq PK$. Then $P$ is characteristic in $PK$ and hence $P$ is normal in $A$. By Claim I, $X \cong C^{\pm 1}(p; 4, 2)$ or $C^{\pm \varepsilon}(p; 4, 2)$. Now let $X[\Delta \cup \Delta'] \cong 2C_4$. Since $A/K \cong D_{2p^2}$ and $p > 5$, $P$ is cyclic of order $p^2$. Set $P = \langle \alpha \rangle$. Without loss of generality, let

$$
\alpha = \left( x_{0,0}^0 x_{0,0}^0 \ldots x_{0,0}^{p^2-1} \right) \left( x_{1,0}^0 x_{1,0}^1 \ldots x_{1,0}^{p^2-1} \right) \left( x_{0,1}^0 x_{0,1}^1 \ldots x_{0,1}^{p^2-1} \right).$
$$

Consider a 4-cycle $C$ in the induced subgraph $X[\Delta \cup \Delta']$ and let $n$ be the number of edges of $C$ which are on some orbit of $\alpha$. Then $n = 0$, 1 or 2 and, consequently, $X[\Delta \cup \Delta']$ is one of the three cases:
It is easy to see that for Case III, \( X \cong 2C_{p^2}[2K_1] \), contrary to the connectivity of \( X \). For Case I, we have \( X \cong C_{2p^2}[2K_1] \), and for Case II, we have \( X \cong C(2; p^2, 2) \).

Now we are ready to complete the proof. By Claim II, if \( A \) has a non-trivial normal 2-subgroup, then \( X \cong C_{\pm 1}(p; 4, 2), C_{\pm 2}(p; 4, 2), C_{2p^2}[2K_1] \), or \( C(2; p^2, 2) \). In the remainder of proof, assume that \( A \) has no non-trivial normal 2-subgroups. Let \( M \) be a maximal normal \( p \)-subgroup of \( A \). It follows from Lemma 4.1 that \( |M| = p \) or \( p^2 \).

If \( |M| = p^2 \) then \( M = P \) is a Sylow \( p \)-subgroup of \( A \). By Claim I, \( X \cong C_{\pm 1}(p; 4, 2) \) or \( C_{\pm 2}(p; 4, 2) \). Suppose that \( |M| = p \). Consider the quotient graph \( X_M \) of \( X \) relative to the orbit set of \( M \), and let \( K \) be the kernel of \( A \) acting on \( V(X_M) \). Then \( X_M \) is a tetravalent symmetric graph of order \( 4p \). Let \( C = C_A(M) \). Then \( M \leq C \) and \( A/C \leq \text{Aut}(M) \cong \mathbb{Z}_{p-1} \). If \( C = M \), then \( A/K \) is abelian. This forces that \( A/K \) is regular on \( V(X_M) \), contrary to the arc-transitivity of \( A/K \) on \( X_M \). Thus, \( C > M \). Let \( T/M \) be a minimal normal subgroup of \( A/M \) contained in \( C/M \). If \( T/M \) is solvable, then the maximality of \( M \) implies that \( T/M \) is an elementary abelian 2-subgroup. Since \( T \leq C \), one has \( T = M \times Q \), where \( Q \) is Sylow 2-subgroup of \( C \). Then \( Q \) is characteristic in \( C \) and hence normal in \( A \), a contradiction. Thus, \( T/M \) is non-solvable. By Zhou [28, Theorem 4.1], \( A/M \leq \text{Aut}(G_{28}) \cong \text{PGL}(2,7) \times \mathbb{Z}_2 \). It follows that \( T/M \cong \text{PSL}(2,7) \). Let \( T' \) be the derived subgroup of \( T \). If \( T = T' \), then \( T \) is a covering group of \( \text{PSL}(2,7) \). However, the Schur multiplier of \( \text{PSL}(2,7) \) is \( \mathbb{Z}_2 \), a contradiction. Thus, \( M' < M \), and hence \( T = M' \times M \cong \text{PSL}(2,7) \times \mathbb{Z}_p \). Then \( M' \) is characteristic in \( T \), and hence it is normal in \( A \). Consider the quotient graph \( X_{M'} \), and let \( L \) be the kernel of \( A \) acting on \( V(X_{M'}) \). If \( X_{M'} \) has valency 4 then by Proposition 2.3, \( M' \) is semiregular, a contradiction. If \( X_{M'} \) has valency 2, then the induced subgraph by any two adjacent orbits of \( M' \) is a cycle of length \( 2p \). Since \( p > 5 \), \( L \) acts faithfully on each orbit. It follows that \( M' \leq \text{Aut}(C_{2p}) \cong D_{4p} \), and hence \( M' \) is solvable, a contradiction. \( \square \)

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