A TDOA technique with Super-Resolution based on the Volume Cross-Correlation Function

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Abstract

TDOA (Time Difference of Arrival) is an important and widely used wireless localization technique. Among the enormous approaches of TDOA, high resolution TDOA algorithms have drawn much attention for its ability to resolve closely spaced signal delays in multipath environment. However, the state-of-art high resolution TDOA algorithms still have performance weakness on resolving time delays in a wireless channel with dense multipath effect, as well as difficulties of implementation for their high computation complexity. In this paper, we propose a novel TDOA algorithm with super resolution based on certain kind of multi-dimensional cross-correlation function: the volume cross-correlation function (VCC). The proposed TDOA algorithm has excellent time resolution performance in multipath environment, and it also has a much lower computational complexity. Our algorithm does not require priori knowledge about the waveform or power spectrum of transmitted signals, therefore has great potential of usage in various passive wireless localization systems. Numerical simulations is also provided to demonstrate the validity of our conclusion.

Index Terms

Time Difference of Arrival (TDOA), General Cross-Correlation (GCC), Volume Cross-Correlation Function, super resolution, multipath environment

I. INTRODUCTION

The problem of source localization has drawn enormous attention in various areas of signal processing. Common localization techniques include methods based on Time of Arrival (TOA), Time Difference of Arrival (TDOA), and Direction of Arrival (DOA), etc [1,2,3,4,5]. Among these approaches, TDOA based methods requires no timing synchronization between sources and receivers and are widely used in wireless communication [12,67,8], indoor microphone positioning [9,10], wireless sensor network [34], passive localization system [11,12, and sonar [13,14].

As well known, the classic and most widely used TDOA estimation algorithm is Generalized Cross-Correlation algorithm (GCC) [15]. Because the GCC algorithm is based on the time cross-correlation function of received signals from two different base stations, its time resolution is restricted by the reciprocal of signal’s bandwidth. As a result, GCC method has limited time resolution and can only resolve the TDOA of multipath signals that are well separated in arrival time. For this reason, recently some high resolution TDOA algorithms have been proposed to deal with the scenario where signals from different paths have close delays. There are mainly three branches of TDOA algorithms that have high resolution: one is the optimal maximal likelihood (ML) time delay estimators using techniques like expectation maximization (EM) [16], or importance sampling [17,18]; another branch is the super resolution TDOA algorithms based on subspace methods [19,20,21]; the third branch is the high resolution TDOA estimation methods using sparse recovery algorithms based on $\ell_1$ optimization [22,23]. Except for those main branches, some delay estimation techniques that have super resolution and ability of dealing with multipath environment, such as the technique of time delay estimation from low-rate samples over a union of subspaces [24] can also be adapted to TDOA estimation. In this paper, we are going to propose a highly efficient super resolution TDOA algorithm, which has the ability to resolve TDOAs with respect to dense multipath components.

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As we know, the TDOA corresponding to the direct path (or Line of Sight, LOS) is what we only need in order to precisely localize signal sources, but cannot be determined unless we could resolve all the TDOAs caused by every pair of multipath components. Therefore, improving the time resolution and enhancing the ability of identifying each multipath TDOA are two major tasks concerned in design of TDOA techniques.

In this paper, we develop a new TDOA estimation algorithm based on a certain kind of multi-dimensional cross-correlation function, named the Volume Cross-Correlation function (VCC). This VCC function takes two matrices (which represent subspaces), instead of two vectors, as arguments. It calculates the geometrical volume of the high dimensional parallelepiped spanned by column vectors of these two matrices. It can be considered as a generalized distance measure, which describes the linear dependence between the two subspaces spanned by columns of each input matrix. In our proposed TDOA method, the received signal is formulated as a deterministic object with unknown linear subspace structure plus random noise. Then this unknown subspace will be extracted from noise through singular value decomposition of the data matrix, which is a commonly used denoising method in modern signal processing. Afterwards the VCC function is calculated with inputs being the basis of the estimated subspace. At last the corresponding TDOA estimation is indicated by the zeros (or equivalently, the peaks of its reciprocal) of our VCC function.

In order to analyze the performance of the proposed TDOA algorithm, we choose the passive localization system as a typical application scenario. In our analysis, the received signals commonly encountered in passive localization systems are divided into two different categories: the slowly changing subspace signal and the fast changing subspace signal. The so-called slowly changing subspace signal means the subspace structure of the signal remains unchanged during the time interval of a large amount of observations. For these signals, multiple observation data are used to calculate a sample covariance matrix, then eigen-decomposition methods can be used to estimate the basis of the corresponding signal subspace, these bases are then used as the input of our VCC function to estimate TDOA. As for the fast changing subspace signal, contrast to the term ”slowly changing”, it refers to the circumstance that the subspace structure are changing among different observations; therefore there is only a single observation available to estimate the current signal subspace. For this model, a Hankel matrix \(^{25}\) is constructed and the basis of signal subspace is extracted through singular value decomposition of this matrix, before it is put into the calculation of VCC function. Analysis on the performance of our TDOA algorithm are carried on these two categories of received signals. The two signal categories covers most wireless signals encountered in passive localization sytems, therefore our TDOA technique has a potential of wide usage in passive localization systems, such as passive radar, passive sonar and localization systems in wireless communication.

The rest of this paper is organized as follows. In section II, we give the problem formulation of multipath TDOA, as well as the definition and property of VCC function. In section III and section IV, we propose and analyze our proposed TDOA algorithm based on two categories of signals, respectively. The performance of our TDOA method is demonstrated through numerical simulations in section V, and the advantages of our method are stated in section VI. The last two sections discuss some future work and conclude our paper.

II. PRELIMINARY MATERIAL

A. The problem of TDOA estimation in multipath environment

In a typical TDOA localization system, the transmitted signal from a remote signal source is received by two separated sensors. If there is only one propagation path in each channel, the received signal from these two receivers, denoted by \(x_1(t)\) and \(x_2(t)\), will be:

\[
\begin{align*}
 x_1(t) & = s(t - \tau_1) + w_1(t), \\
 x_2(t) & = s(t - \tau_2) + w_2(t),
\end{align*}
\]

where \(s(t)\) is the transmitted signal from the source, \(\tau_1\) and \(\tau_2\) represents the propagation delay of the signal \(s(t)\) arriving at the two receivers, \(w_1(t)\) and \(w_2(t)\) are noises on receiver 1 and 2. The primary mission of TDOA is to estimate the difference of
propagation time $|\tau_1 - \tau_2|$ from the received signals $x_1(t)$ and $x_2(t)$; this time difference is then used to determine the location of signal source.

However, in true environments, there might be multipath effect in the wireless channel, especially in urban areas where buildings and vehicles will lead to significant scattering of wireless signals. The received signal in (1) and (2) in a multipath environment will become:

$$x_1(t) = \sum_{l=1}^{L_1} \alpha_{1,l} \tau_1(t - \tau_{1,l}) + w_1(t), \quad (3)$$

$$x_2(t) = \sum_{l=1}^{L_2} \alpha_{2,l} \tau_2(t - \tau_{2,l}) + w_2(t), \quad (4)$$

where $\alpha_{1,l}$ and $\alpha_{2,l}$ are the propagation gains (also known as the channel coefficients) of the $l$th path along which the signal transmitting from source to receiver 1 and 2, respectively, $\tau_{1,l}$ and $\tau_{2,l}$ represents the corresponding path delays, $L_1$ and $L_2$ are the number of channel paths.

From (3) and (4), it can be seen that in a multipath channel, there are theoretically multiple TDOAs which can be resolved. Denote these TDOAs by

$$\Delta \tau_{1,2} := \tau_{2,2} - \tau_{1,1}, \quad l_1 = 1, \ldots, L_1, \quad \text{and} \quad l_2 = 1, \ldots, L_2,$$  \hspace{1cm} (5)$$

we call these multiple TDOAs as *multipath TDOA*. Although in source localization systems, the direct path TDOA is the only concerned, which is $\Delta \tau_{1,1} = \tau_{2,1} - \tau_{1,1}$, precise estimation of the direct path TDOA $\Delta \tau_{1,1}$ actually requires resolution of every multipath TDOA in (5). In other words, because the channel path delays and propagation gains are basically unknown at the receivers, we cannot tell the difference between direct path TDOA and other indirect path TDOAs merely from the received signals. Therefore, we need to resolve every multipath TDOA, before we pick the direct path TDOA and continue the localization process. If we failed to find the direct path TDOA at the first stage, the following localization treatment won’t work. From this point of view, the primary goal of TDOA localization in multipath environment is to precisely resolve every multipath TDOA shown in (5). Therefore, algorithms with a sufficiently high resolution to identify multipath TDOAs are preferred in the TDOA localization problem in multipath environment.

**B. The Volume Cross-Correlation Function**

In this paper, we will introduce a TDOA estimation algorithm based on novel cross-correlation function called the Volume Cross-Correlation Function, or VCC. Before we give the precise definition of VCC function, we first introduce some background knowledge essential for understanding VCC.

The basic relationship between linear subspaces are generally described by principal angles in mathematical literature. The rigorous definition for principal angles are stated as follows:

**Definition 1:** Consider linear subspaces $X_1$ and $X_2$, with dimensions $\dim(X_1) = d_1, \dim(X_2) = d_2$, denote $m = \min(d_1,d_2)$. The principal angles between subspaces $X_1$ and $X_2$, denoted by $0 \leq \theta_i \leq \cdots \leq \theta_m \leq \pi/2$, are defined recursively as

$$\cos \theta_i = \max_{u_i \in X_1, v_i \in X_2} u_i^T v_i,$$

subject to

$$\|u_i\|_2 = \|v_i\|_2 = 1,$$

$$u_i^T u_j = 0, \quad v_i^T v_j = 0,$$  \hspace{1cm} (6)$$

where $i = 1, \ldots, m, \quad j = 1, \ldots, i - 1$.

The principal angle is an important mathematical tool to depict the relationship between subspaces. As a matter of fact, principal angle plays a key role in deriving the geodesic distance in the theory of Grassmann manifold, in which the entire linear subspace is regarded as a point on manifold. Except for the geodesic distance, various distance metrics of linear subspaces can also be defined using principal angle. VCC function in this paper is also related with the principal angle.
The geometrical volume of a matrix $X \in \mathbb{R}^{P \times d}$ (which represents a subspace whose basis vectors are the columns of $X$) with dimension $d \ (d < P)$ is defined as \cite{29}:
\begin{equation}
\text{vol}_d(X) := \prod_{i=1}^{d} \sigma_i, 
\end{equation}
where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0$ are singular values of matrix $X$. As a matter of fact, $\text{vol}_d(X)$ is the $d$ dimensional parallelotope spanned by the column vectors of matrix $X$. Another equivalent expression of volume is \cite{28} \cite{30}:
\begin{equation}
\text{vol}_d(X) = \sqrt{\det(X^T X)}. 
\end{equation}

The relation between volume and principal angles is depicted by the next lemma from \cite{26}:
\begin{lemma}
Consider two linear subspaces $X_1$ and $X_2$ in $\mathbb{R}^N$, their dimensions are $\dim(X_1) = d_1, \dim(X_2) = d_2$, and their basis matrices are $X_1 \in \mathbb{R}^{N \times d_1}$ and $X_2 \in \mathbb{R}^{N \times d_2}$, then we have
\begin{equation}
\frac{\text{vol}_{d_1+d_2}([X_1, X_2])}{\text{vol}_{d_1}(X_1) \text{vol}_{d_2}(X_2)} = \prod_{j=1}^{\min(d_1, d_2)} \sin \theta_j(X_1, X_2),
\end{equation}
where $0 \leq \theta_j(X_1, X_2) \leq 2\pi, 1 \leq j \leq \min(d_1, d_2)$ are the principal angles between $X_1$ and $X_2$.
\end{lemma}

The Volume Cross-Correlation (VCC) function is defined as
\begin{equation}
\text{vcc}(X_1, X_2) := \frac{\text{vol}_{d_1+d_2}([X_1, X_2])}{\text{vol}_{d_1}(X_1) \text{vol}_{d_2}(X_2)}. 
\end{equation}
From \cite{9} we have that the definition of VCC function \cite{10} is actually the product of sines of the principal angles between subspaces $X_1$ and $X_2$. Therefore \cite{10} can be regarded as a kind of distance measure of subspaces $X_1$ and $X_2$. Intuitively, if the subspace $X_1$ and $X_2$ are linearly dependent, then $\dim(X_1 \cap X_2) > 0$. According to the definition of principal angles, there must be a vanishing principal angle $\theta_j(X_1, X_2)$. That is
\begin{equation}
\frac{\text{vol}_{d_1+d_2}([X_1, X_2])}{\text{vol}_{d_1}(X_1) \text{vol}_{d_2}(X_2)} = 0 
\end{equation}
On the other hand, if $X_1$ is perpendicular to $X_2$, then
\begin{equation}
\frac{\text{vol}_{d_1+d_2}([X_1, X_2])}{\text{vol}_{d_1}(X_1) \text{vol}_{d_2}(X_2)} = 1
\end{equation}
holds obviously. As a matter of fact, VCC function measures the extent of linear dependency between subspaces \cite{31}, and will be used to derive our TDOA algorithm.

\section{III. Estimating the TDOA of slowly changing subspace signals using VCC function}

\subsection{A. The slowly-changing subspace signal}
We choose the passive localization system as a typical application scenario to demonstrate and analyze our TDOA algorithm. In the passive localization system, information about the wireless channel as well as the source signals are generally unknown by the receivers. Hence TDOA technique is quite suitable for this kind of localization system \cite{32,33}. Firstly, we focus on the category of signals that have a slowly changing subspace structure.

A typical type of signals we encounter in passive localization systems are radar signals radiated by non-cooperative radar systems. The common pulse radar waveform can be expressed as the following expression:
\begin{equation}
s(t) = \sum_{m=-\infty}^{+\infty} \sqrt{P_s} g(t - mT_p),
\end{equation}
where $P_s > 0$ is the transmitting power of the radar, and $g(t) \in \mathbb{C}$ is the general form of the radar pulse waveform, $T_p$ is the pulse repetition interval (PRI).

In the multipath environment, the received signal from the $i$’th receiver would be
\begin{equation}
x_i(t) = \sum_{l=1}^{L_i} \alpha_{i,l} \sum_{m=-\infty}^{+\infty} s(t - mT_p - \tau_{i,l}) + w_i(t), \quad i = 1, 2, \cdots
\end{equation}
where $\alpha_{i,l}$ and $\tau_{i,l}$ are the corresponding channel coefficients and path delays, $w_i(t)$ is the Gaussian white noise of $i$’th receiver. For convenience, only the received signal in a single PRI is considered, i.e.,

$$x_i(t) = \sum_{l=1}^{L_i} \alpha_{i,l} \sqrt{P_g} g(t - \tau_{i,l}) + w_i(t), \quad t \in (0, T_p), \quad i = 1, 2, \cdots$$  \hspace{1cm} (15)

after sampling the received signal with the rate $1/T_s$, the discrete received signal will be

$$y_i(kT_s) = \sum_{l=1}^{L_i} \alpha_{i,l} \sqrt{P_g} g(kT_s - \tau_{i,l}) + w_i(kT_s),$$  \hspace{1cm} (16)

take $d_{i,l} := [\tau_{i,l}/T_s]$ as the integral channel path delay, and ignore the non-integral part of the ratio $\tau_{i,l}/T_s$, then (16) can be approximately expressed as

$$y_i(kT_s) \approx \sum_{l=1}^{L_i} \alpha_{i,l} \sqrt{P_g} g((k - d_{i,l})T_s) + w_i(kT_s), \quad i = 1, 2, \cdots.$$  \hspace{1cm} (17)

Rewrite (17) as a vector, we have

$$y_i = \sqrt{P_g} C_i \alpha_i + w_i, \quad i = 1, 2, \cdots, \hspace{1cm} (18)$$

where $y_i := [y_i(0), y_i(T_s), \cdots, y_i((N - 1)T_s)] \in \mathbb{C}^N$, and

$$C_i = \begin{bmatrix}
g((0 - d_{1,1})T_s) & g((0 - d_{1,2})T_s) & \cdots & g((0 - d_{1,L_i})T_s) 
g((1 - d_{1,1})T_s) & g((1 - d_{1,2})T_s) & \cdots & g((1 - d_{1,L_i})T_s) 
\vdots & \vdots & \ddots & \vdots 
g((N - 1 - d_{1,1})T_s) & g((N - 1 - d_{1,2})T_s) & \cdots & g((N - 1 - d_{1,L_i})T_s)
\end{bmatrix} \in \mathbb{C}^{N \times L_i}, \hspace{1cm} (19)$$

where $\alpha_i = [\alpha_{i,1}, \cdots, \alpha_{i,L_i}] \in \mathbb{C}^{L_i}$ is the channel coefficient vector composed of the corresponding channel’s path gains, and $w_i$ is the noise vector. As is shown in (18), the received radar signal in a multipath channel generally has a deterministic subspace structure with the corresponding subspace $\text{span}(G_i)$, i.e., spanned by different time-shift versions of radar waveform $g(t)$.

Except for radar signals, the common linearly modulated wireless communication signals such as DS-CDMA, OFDM, QAM, and others that carry symbols on some periodic pulse shapes, can also be modeled as the signal with a subspace structure in [18] [34,35,36,37]. The subspace signal structure is mainly related with two main parameters describing the channel’s state: the channel’s path delays $d_{i,l}$, and the channel’s path gains $\alpha_{i,l}$. As a matter of fact, channel delays are caused by different distances between receivers and signal sources (or reflective objects), and generally signal sources and reflective objects seldom have extremely high velocities, therefore channel delays can be generally seen to be constant in a short time. On the other hand, the wireless channel’s path gains fluctuates with time, which is caused by channel fading effect. This fact also means that for a time interval long enough for the receiver to obtain relatively large samples of the received signal (according to chapter 2 of [38], the time scale of this interval can be up to 20s in a typical channel scenario), these sample data can be formulated as

$$y_i^{(j)} = G_i \alpha_i^{(j)} + w_i^{(j)}, \quad j = 1, 2, \cdots, \hspace{1cm} (20)$$

where $j$ indicates different observation segment of time. Although the channel coefficient vector $\alpha^{(j)}$ is fluctuating with $j$, the subspace structure determined by matrix $G_i$, will remain almost unchanged. We call this category of signals the slowly changing subspace signal, meaning that the subspace structure in (18) changes slowly with time, and can be treated as invariant in the observation interval.

B. Estimating the signal subspace using multiple observation data

As is mentioned, received signals in the form of (18) have unknown subspace structure. In order to make use of this subspace structure and estimate TDOA using the VCC function introduced earlier, we need to extract the subspace $\text{span}(G_i)$ from the received signals first.
Typically, when the received signals in a passive localization system are radar signals, the estimation of the subspace \( \text{span}(G_i) \) is very simple. Because a typical radar transmits a pulse waveform repeatedly with a PRI of \( T_p \), we can receive these multiple radar pulses according to \( T_p \) as in (20) and estimate the corresponding signal subspace using the well-known subspace methods like MUSIC, ESPRIT, etc. Even though there are modern radars transmitting pulse waveforms with a cycled or randomized PRI, or even transmitting agile pulse waveforms, some new technologies, such as pulse deinterleaving and recognition, and PRI identification, will ensure the correct recognition of radar PRI. According to the estimated PRI, multiple radar pulse signals in the form of (20) can be obtained [39,40,41]. Without lossing generality, we assume the PRI has been clearly recognized in the receiver. The process of sampling multiple radar signals can be demonstrated in figure 1, the gradient change of the background color in figure 1 represents the fluctuation of channel coefficients \( \alpha_{i,l} \), they are reasonably assumed to take independent values among different pulses’ durations.

Denote the multiple sample data of the received radar signal by
\[
y^{(1)}_i, \ldots, y^{(m)}_i,
\]
and use them to calculate the sampled covariance matrix
\[
\hat{R}_i = \frac{1}{m} \sum_{j=1}^{m} y^{(j)}_i (y^{(j)}_i)^H,
\]
Just as well-known facts on subspace methods, the signal subspace can be estimated through eigen-decomposition of \( \hat{R}_i \), i.e.,
\[
\hat{R}_i = U_{i,s} \Lambda_{i,s} U_{i,s}^H + U_{i,n} \Lambda_{i,n} U_{i,n}^H,
\]
where the matrix \( U_{i,s} \) is the estimated basis matrix of the signal subspace \( \text{span}(G_i) \). It has also been proved that \( \text{span}(U_{i,s}) \) approximates the signal subspace \( \text{span}(G_i) \) asymptotically for a sufficiently large \( m \) [42,43].

![Radar Transmitting Signal](image)

**Fig. 1: Getting multiple observations of radar signals (the gradient change of color demonstrates the fluctuation of channel coefficients)**

**C. TDOA estimation of slowly changing subspace signal using VCC function: Main idea**

In this part, we will use the estimated basis of signal subspace, i.e., \( U_{i,s} \), to estimate TDOA using VCC function. Similar to the traditional TDOA methods, we first manually delay the received signals from receiver 1 by \( \Delta d \cdot T_s \),
\[
y_1((k - \Delta d)T_s) = \sum_{l=1}^{L_1} \alpha_{1,l} \sqrt{P_s} g((k - \Delta d - d_{1,l})T_s) + w_1((k - \Delta d)T_s).
\]
The delayed basis of signal subspace from receiver 1, denoted by $G_{1}^{[\Delta d \ T_s]}$, is

$$
G_{1}^{[\Delta d \ T_s]} = 
\begin{bmatrix}
  g((0 - \Delta d - d_{1,1})T_s) & g((0 - \Delta d - d_{1,2})T_s) & \cdots & g((0 - \Delta d - d_{1,L_1})T_s) \\
  g((1 - \Delta d - d_{1,1})T_s) & g((1 - \Delta d - d_{1,2})T_s) & \cdots & g((1 - \Delta d - d_{1,L_1})T_s) \\
  \vdots & \vdots & \ddots & \vdots \\
  g((N - 1 - \Delta d - d_{1,1})T_s) & g((N - 1 - \Delta d - d_{1,2})T_s) & \cdots & g((N - 1 - \Delta d - d_{1,L_1})T_s)
\end{bmatrix},
$$

(25)
in addition, the basis of signal subspace from receiver 2 is (without manual delay)

$$
G_2 = 
\begin{bmatrix}
  g((0 - d_{2,1})T_s) & g((0 - d_{2,2})T_s) & \cdots & g((0 - d_{2,L_2})T_s) \\
  g((1 - d_{2,1})T_s) & g((1 - d_{2,2})T_s) & \cdots & g((1 - d_{2,L_2})T_s) \\
  \vdots & \vdots & \ddots & \vdots \\
  g((N - 1 - d_{2,1})T_s) & g((N - 1 - d_{2,2})T_s) & \cdots & g((N - 1 - d_{2,L_2})T_s)
\end{bmatrix}.
$$

(26)

It can be easily seen that, if and only if

$$\Delta d = d_{2,i_2} - d_{1,i_1}, \quad i_2 = 1, \cdots, L_2, \quad i_1 = 1, \cdots, L_1,$$

(27)
the subspace $\text{span}(G_{1}^{[\Delta d \ T_s]})$ and subspace $\text{span}(G_2)$ are linearly dependent. In other words, we have

$$\dim(\text{span}(G_{1}^{[\Delta d \ T_s]}) \cap \text{span}(G_2)) > 0.$$

(28)

In order to estimate the unknown subspaces $\text{span}(G_{1}^{[\Delta d \ T_s]})$ and $\text{span}(G_2)$, the received signals from receiver 1 are delayed manually as follows

$$y_{1}^{(j)}[\Delta d T_s] := [y_{1}^{(j)}((0 - \Delta d)T_s), \cdots, y_{1}^{(j)}((N - 1 - \Delta d)T_s)]^T, \quad j = 1, 2, \cdots, m,$$

(29)
simultaneously the received signals from receiver 2 remains as

$$y_{2}^{(j)} := [y_{2}^{(j)}(0 \cdot T_s), \cdots, y_{2}^{(j)}((N - 1)T_s)]^T, \quad j = 1, 2, \cdots, m,$$

(30)
then the sampled covariance matrices for both of them could be written as

$$\hat{\mathbf{R}}_{1}^{[\Delta d \ T_s]} = \frac{1}{m} \sum_{j=1}^{m} y_{1}^{(j)}[\Delta d T_s] (y_{1}^{(j)}[\Delta d T_s])^H,$$

(31)
$$\hat{\mathbf{R}}_2 = \frac{1}{m} \sum_{j=1}^{m} y_{2}^{(j)} (y_{2}^{(j)})^H.$$

(32)
The signal subspaces and noise subspaces are separated as follows

$$\hat{\mathbf{R}}_{1}^{[\Delta d \ T_s]} = \mathbf{U}_{1,s}^{[\Delta d \ T_s]} \Lambda_{1,s}^{[\Delta d \ T_s]} (\mathbf{U}_{1,s}^{[\Delta d \ T_s]})^H + \mathbf{U}_{1,n}^{[\Delta d \ T_s]} \Lambda_{1,n}^{[\Delta d \ T_s]} (\mathbf{U}_{1,n}^{[\Delta d \ T_s]})^H,$$

(33)
$$\hat{\mathbf{R}}_2 = \mathbf{U}_{2,s} \mathbf{A}_2 \mathbf{U}_{2,s}^H + \mathbf{U}_{2,n} \mathbf{A}_2 \mathbf{U}_{2,n}^H,$$

(34)
the matrices $\mathbf{U}_{1,s}^{[\Delta d \ T_s]}$ and $\mathbf{U}_{2,s}$ are the estimated basis of $\text{span}(G_{1}^{[\Delta d \ T_s]})$ and span $G_2$.

Therefore, when $\Delta d = d_{2,i_2} - d_{1,i_1}$, the subspaces $\text{span}($$\mathbf{U}_{1,s}^{[\Delta d \ T_s]}$$)$ and $\text{span}(\mathbf{U}_{2,s})$ are approximately linearly dependent. As a result, there must be a principal angle between $\text{span}(\mathbf{U}_{1,s}^{[\Delta d \ T_s]})$ and $\text{span}(\mathbf{U}_{2,s})$ that vanishes $[26]$, i.e., approximates 0.

Taking the estimated basis matrices of the signal subspaces from receiver 1 and receiver 2 as inputs and using Lemma [1] the VCC function will satisfy that

$$\frac{\text{vol}_{\mathbf{U}_{1,s} \mathbf{U}_{2,s}}((\mathbf{U}_{1,s}^{[\Delta d \ T_s]}, \mathbf{U}_{2,s}))}{\text{vol}_{\mathbf{U}_{1,s}}((\mathbf{U}_{1,s}^{[\Delta d \ T_s]}))} \approx 0,$$

(35)
if and only if the manual delay $\Delta d$ satisfies $[27]$. The symbol $\approx$ means approximation for a sufficiently large number of observations $m$. The corresponding multipath TDOAs are indicated by peak values of the reciprocal of left side of (35). Because $\mathbf{U}_{1,s}^{[\Delta d \ T_s]}$ and $\mathbf{U}_{2,s}$ are unitary matrices, the denominator of (35) actually equals 1 and can be ignored.
D. The TDOA algorithm based on VCC function for the slowly changing subspace signals: Algorithms

Algorithm 1.

For \( \Delta d = -N + 1, \ldots, N - 1 \), Loop:

1) Obtain delayed multiple observation data

\[
y_i^{(j)}[\Delta d \cdot T_s] = [y_i^{(j)}((0 - \Delta d) T_s), \ldots, y_i^{(j)}((N - 1 - \Delta d) T_s)]^T, \quad j = 1, \ldots, m,
\]

and non-delayed observation data

\[
y_2^{(j)} = [y_2^{(j)}(0 \cdot T_s), \ldots, y_2^{(j)}((N - 1) T_s)]^T, \quad j = 1, \ldots, m,
\]

calculate the sampled covariance matrices \( \hat{R}_1^{[\Delta d \cdot T_s]} \) and \( \hat{R}_2 \) according to (33) and (35).

2) Estimate the dimensions \( L_1, L_2 \) as well as the basis of the signal subspaces by eigenvalue decomposition of the sampled covariance matrix:

\[
\hat{R}_1^{[\Delta d \cdot T_s]} = U_{1,s}^{[\Delta d \cdot T_s]} \Lambda_{1,s}^{[\Delta d \cdot T_s]} (U_{1,s}^{[\Delta d \cdot T_s]})^H + U_{1,n}^{[\Delta d \cdot T_s]} \Lambda_{1,n}^{[\Delta d \cdot T_s]} (U_{1,n}^{[\Delta d \cdot T_s]})^H,
\]

\[
\hat{R}_2 = U_{2,s} A_{2,s} U_{2,s}^H + U_{2,n} A_{2,n} U_{2,n}^H,
\]

where \( U_{1,s}^{[\Delta d \cdot T_s]} \in \mathbb{C}^{N \times L_1} \) and \( U_{2,s} \in \mathbb{C}^{N \times L_2} \) span the estimated signal subspaces from receiver 1 and receiver 2, respectively.

3) Calculate the reciprocal of VCC function:

\[
r_{\text{vol}}(\Delta d \cdot T_s) := 1/\text{vol}_{L_1 + L_2}([U_{1,s}^{[\Delta d \cdot T_s]}, U_{2,s}]),
\]

End Loop.

Find the peaks of \( r_{\text{vol}}(\Delta d \cdot T_s) \) and corresponding value of \( \Delta d \).

Below are some remarks:

1) Generally, before we estimate the basis of signal subspaces, i.e., \( U_{1,s}^{[\Delta d \cdot T_s]} \) and \( U_{2,s} \), the dimensions of the signal subspaces is needed to be known first. In our model these dimensions are actually the number of channel propagation paths, i.e., \( L_i \) in (14), which are generally unknown at the receiver. Luckily there are a large amount algorithms to estimate the dimension of signal subspace from a sampled covariance matrix, popular methods include Akaike Information Criterion (AIC) [44], Minimum Description Length (MDL) [45], Bayesian Information Criterion (BIC) [46], Predictive Description Length (PDL) [47] and so on. Because the estimation of dimension \( L_i \) is another topic, and has many mature algorithms with high precisions, so in this paper we just assume the number of channel paths \( L_i \) is precisely estimated, and focus our analysis on the estimation of signal subspaces and its influence on TDOA estimation.

2) As in the traditional methods, we test our VCC function with every possible delays, therefore the value of \( \Delta d \) is taken from \(-N + 1\) to \(N - 1\), the minus values just mean adding manual delays to the received signals from receiver 2.

3) One of the key points in our algorithm is the reciprocal of our VCC function, which is \( r_{\text{vol}}(\Delta d \cdot T_s) = 1/\text{vol}_{L_1 + L_2}([U_{1,s}^{[\Delta d \cdot T_s]}, U_{2,s}]) \). As is mentioned early, \( r_{\text{vol}}(\Delta d \cdot T_s) \) will reach its peak if and only if \( \Delta d = d_{2,l_1} - d_{1,l_2} \). Therefore, theoretically we can resolve every multipath TDOA, i.e.,

\[
\Delta d_{l_2,l_1} := d_{2,l_2} - d_{1,l_1}, \quad \text{for all} \quad l_1 = 1, \ldots, L_1; \text{and} \quad l_2 = 1, \ldots, L_2.
\]

It is another problem with independent interest that how can find the TDOA corresponding to the direct channel path, i.e., \( \Delta d_{1,1} = |d_{2,1} - d_{1,1}| \) using all of the multipath TDOA. It is called disambiguation of TDOA [48,49]. There exists lots of strategies to solve this problem and the adaptation of our TDOA algorithm to these disambiguation strategies will be left for a future work.
IV. Estimating the TDOA of fast changing subspace signals using VCC function

A. The fast changing subspace signal

Contrast to the slowly changing subspace signal model, there are also a large category of signals that don’t have a steady subspace structure as in \([13]\). For example, in passive localization systems, FM radio transmitters, TV broadcast stations are usually the signal sources to localize, or are used as the illuminators-of-opportunity to localize a reflective target. Because this category of signals are randomly varying with time and have no repeating waveforms, we cannot get multiple observations of the usually the signal sources to localize, or are used as the illuminators-of-opportunity to localize a reflective target. Because this discussion.

The form of:

\[
X = \begin{bmatrix}
    x(0) & x(1) & \cdots & x(K - 1) \\
    x(1) & x(2) & \cdots & x(K) \\
    \vdots & \vdots & \ddots & \vdots \\
    x(M - 1) & x(M) & \cdots & x(N - 1)
\end{bmatrix},
\] (36)

where \(1 < M < N, K = N - M + 1\). Large numbers of literatures are focusing on the left singular vectors of the Hankel matrix \(X\), because they contain important information about the signal \(x\) \([25]\). Therefore the subspace spanned by a subset of these left singular vectors is called "the signal subspace" (generally the left singular vectors corresponding to larger singular values will be chosen). As a matter of fact, this signal subspace extracted from the Hankel matrix can be used to perform noise reduction, signal forecasting, and change point detection, etc \([50,51,52,53]\). This methodology has been widely applied in areas like time series analysis, multivariate statistics, multivariate geometry, dynamic systems, and signal processing. The Hankel matrix technique can be used to analyze a wide variety of signals, like wireless signals, seismologic, meteorological, geophysical time series as well as economic time series. In particular, no statistical assumption concerning the signal is needed while performing the subspace extraction from Hankel matrices, thus this methodology is suitable to deal with the fast changing subspace signal and develop our subspace-based TDOA algorithm.

B. Estimating the signal subspace using a single observation data

Similar to the previous analysis, for fast changing subspace signal, the received baseband signal from the \(i\)th receiver is in the form of:

\[
x_i(t) = \sum_{l=1}^{K_i} \alpha_{i,l}{s}(t - \tau_{i,l}) + w_{i}(t), \quad i = 1, 2, \cdots,
\] (37)

where \(\alpha_{i,l}\) represents the channel’s path gain corresponding to the \(l\)th path, and \(\tau_{i,l}\) is the channel’s path delay. The original transmitted signal \(s(t)\) can be FM, PSK or AM signals, etc. For convenience, \(s(t)\) is assumed to be FM signals in the following discussion.

Given a sampling rate \(T_s\), for a sampled signal vector \(x_i(0), x(T_s), \cdots, x_i((N - 1) \cdot T_s)\) with length \(N\), the corresponding Hankel matrix is

\[
X_i = \begin{bmatrix}
    x_i(0 \cdot T_s) & x_i(1 \cdot T_s) & \cdots & x_i((N - M) \cdot T_s) \\
    x_i(1 \cdot T_s) & x_i(2 \cdot T_s) & \cdots & x_i((N - M + 1) \cdot T_s) \\
    \vdots & \vdots & \ddots & \vdots \\
    x_i((M - 1) \cdot T_s) & x_i(M \cdot T_s) & \cdots & x_i((N - 1) \cdot T_s)
\end{bmatrix} \in \mathbb{C}^{M \times (N - M + 1)},
\] (38)

where \(1 < M < N\). The signal subspace corresponding to the data \(x_i(0), x(T_s), \cdots, x_i((N - 1) \cdot T_s)\) is extracted by taking certain number of (denote this number by \(K_i\)) left singular vectors of the Hankel matrix \(X_i\), with respect to the \(K_i\) largest singular values of \(X_i\). The signal subspace is then represented by the basis matrix whose column vectors are those singular vectors of \(K_i\).
if we take \( d_{i,t} := [\tau_{i,t}/T_s] \), the Hankel matrix \( X_i \) can also be approximately written as
\[
X_i \approx \sum_{l=1}^{L_i} \alpha_{i,l} S^{[d_{i,t} T_s]} + W_i,
\]
where
\[
S^{[d_{i,t} T_s]} := \begin{bmatrix}
    s((0 - d_{i,t}) T_s) & s((1 - d_{i,t}) T_s) & \cdots & s((N - M - d_{i,t}) T_s) \\
    s((1 - d_{i,t}) T_s) & s((2 - d_{i,t}) T_s) & \cdots & s((N - M + 1 - d_{i,t}) T_s) \\
    \vdots & \vdots & \ddots & \vdots \\
    s((M - 1 - d_{i,t}) T_s) & s((M - d_{i,t}) T_s) & \cdots & s((N - 1 - d_{i,t}) T_s)
\end{bmatrix},
\]
is the Hankel matrix of the sampled transmitted signal \( s(kT_s) \), delayed by \( d_{i,t} \cdot T_s \). \( W_i \) is the Hankel matrix of the noise \( w_i(kT_s) \). Before the proposed VCC function is used, some definitions and propositions from the Singular Spectrum Analysis (SSA) is needed for clarity [53][54].

According to the theory of SSA, most kinds of wireless signals that is generated from linear systems can be modeled as (or approximated by) time series of a finite rank [54], which is defined by:

**Definition 2:** If a time series with length \( N \), denoted by \( s(0), \cdots, s(N-1) \), have a finite rank, then for sufficiently large dimensions \( M \) and \( N-M \), the Hankel matrix
\[
S = \begin{bmatrix}
    s(0) & s(1) & \cdots & s(N-M) \\
    s(1) & s(2) & \cdots & s(N-M+1) \\
    \vdots & \vdots & \ddots & \vdots \\
    s(M-1) & s(M) & \cdots & s(N-1)
\end{bmatrix} \in \mathbb{C}^{M \times (N-M+1)}
\]
must have a low rank, i.e., \( \text{rank}(S) < \min\{M, N-M+1\} \).

According to Definition 2 and analysis in [54], no matter what form the transmitted wireless signal \( s(kT_s) \) has, the Hankel matrix \( S \) will generally have a much low rank (or can be approximated by a Hankel matrix with a low rank). Because in this case, we are interested in a sum of several Hankel matrices as in [59], another proposition is derived by us to continue our analysis.

**Proposition 1:** Consider two time series of low rank, i.e., \( s_1(0), \cdots, s_1(N-1) \) and \( s_2(0), \cdots, s_2(N-1) \), their Hankel matrices, \( S_1 \) and \( S_2 \), with ranks \( r_1 \) and \( r_2 \); if the singular value decomposition of \( S_1 \) and \( S_2 \) is:
\[
S_1 = \sum_k r_1 \sigma_{1,k} u_{1,k} v_{1,k}^H, \quad S_2 = \sum_m r_2 \sigma_{2,m} u_{2,m} v_{2,m}^H,
\]
then the sum of these Hankel matrices \( S := S_1 + S_2 \) will satisfy
\[
S = \sum_k r_1 \sigma_{1,k} u_{1,k} v_{1,k}^H + \sum_m r_2 \sigma_{2,m} u_{2,m} v_{2,m}^H,
\]
in addition, \( \text{rank}(S) = r_1 + r_2 \).

The result of Proposition 1 is quite obvious and we will skip the proof. The result in (42) implies that if we take the left singular vectors of the Hankel matrix \( S \) corresponding to its \( r_1 + r_2 \) largest singular values, which are \( u_1, \cdots, u_{r_1+r_2} \), the following relation will hold:
\[
\text{span}(u_1, \cdots, u_{r_1+r_2}) = \text{span}(u_{1,1}, \cdots, u_{1,r_1}) \oplus \text{span}(u_{2,1}, \cdots, u_{2,r_2})
\]
Combining Lemma 3 and Proposition 1. We can now immediately get another proposition on the Hankel matrix we are interested in, i.e.,
\[
S = \sum_{l=1}^{L_i} \alpha_{i,l} S^{[d_{i,t} T_s]}.
\]
For each series’ Hankel matrix \( U^{[d_{i,l}, T_s]} \) \( \in \mathbb{C}^{M \times (N-M+1)} \), which have a low rank \( r_{i,l} := \text{rank}(U^{[d_{i,l}, T_s]}) \), if we denote \( U^{[d_{i,l}, T_s]} \) \( \in \mathbb{C}^{M \times r_{i,l}} \) by the matrix whose columns are left singular vectors corresponding to the \( r_{i,l} \) largest singular values of \( U^{[d_{i,l}, T_s]} \); then for the linear combination

\[
S = \sum_{i=1}^{L_i} \alpha_{i,l} U^{[d_{i,l}, T_s]},
\]

the matrix, \( U_i \in \mathbb{C}^{M \times K_i} \), whose columns are left singular vectors corresponding to \( S \)'s \( K_i \) largest singular values, will satisfy

\[
\text{span}(U_i) = \bigoplus_{i=1}^{L_i} \text{span}(U^{[d_{i,l}, T_s]}),
\]

where \( K_i := \sum_{l=1}^{L_i} r_{i,l} \).

Proposition 2 provides theoretical foundation for the application of our VCC function. It is also intuitive to infer that the relation (45) also holds for \( X_i \) in (39) where Gaussian noises are presented, because the singular vectors of \( X_i \) are chosen corresponding to the \( K_i \) largest singular values. This is a natural noise reduction procedure, that is, the singular vectors corresponding to the noise will be discarded. Overall, in the subspace estimation step, the basis matrix of the signal subspace, denoted by \( U_i \), is extracted from the singular vectors of Hankel matrix \( X_i \).

C. TDOA estimation of fast changing subspace signals using VCC function: Main idea

According to the analysis in the last section, for a fast changing subspace signal, the basis matrix \( U_i \) of signal subspace is extracted from the Hankel matrix \( X_i \). In this section, different \( U_i \) from different receivers will be used as the input arguments of VCC function. Similarly, we manually delay the sampled received signal \( x_1(kT_s) \) by \( \Delta d \cdot T_s \), then the delayed version of Hankel matrix \( X_i^{[\Delta d, T_s]} \) will be:

\[
X_i^{[\Delta d, T_s]} = \begin{bmatrix}
x_1((0 - \Delta d) \cdot T_s) & x_1((1 - \Delta d) \cdot T_s) & \cdots & x_1((N - M - \Delta d) \cdot T_s) \\
x_1((1 - \Delta d) \cdot T_s) & x_1((2 - \Delta d) \cdot T_s) & \cdots & x_1((N - M - \Delta d + 1) \cdot T_s) \\
\vdots & \vdots & \ddots & \vdots \\
x_1((M - \Delta d - 1) \cdot T_s) & x_1((M - \Delta d) \cdot T_s) & \cdots & x_1((N - \Delta d - 1) \cdot T_s)
\end{bmatrix},
\]

(46)

The basis matrix \( U_i^{[\Delta d, T_s]} \) \( \in \mathbb{C}^{M \times K_i} \) satisfies

\[
\text{span}(U_i^{[\Delta d, T_s]}) = \bigoplus_{i=1}^{L_i} \text{span}(U^{[\Delta d + d_{1,l}, T_s]}),
\]

(47)

where \( U_i^{[\Delta d + d_{1,l}, T_s]} \) is the basis matrix of signal subspace from Hankel matrix

\[
S^{[\Delta d + d_{1,l}, T_s]} = \begin{bmatrix}
s((0 - \Delta d - d_{1,l}) \cdot T_s) & s((1 - \Delta d - d_{1,l}) \cdot T_s) & \cdots & s((N - M - \Delta d - d_{1,l}) \cdot T_s) \\
s((1 - \Delta d - d_{1,l}) \cdot T_s) & s((2 - \Delta d - d_{1,l}) \cdot T_s) & \cdots & s((N - M + 1 - \Delta d - d_{1,l}) \cdot T_s) \\
\vdots & \vdots & \ddots & \vdots \\
s((M - 1 - \Delta d - d_{1,l}) \cdot T_s) & s((M - \Delta d - d_{1,l}) \cdot T_s) & \cdots & s((N - 1 - \Delta d - d_{1,l}) \cdot T_s)
\end{bmatrix}.
\]

(48)

For the same reason, Hankel matrix from receiver 2 is

\[
X_2 = \begin{bmatrix}
x_2(0 \cdot T_s) & x_2(1 \cdot T_s) & \cdots & x_2((N - M) \cdot T_s) \\
x_2(1 \cdot T_s) & x_2(2 \cdot T_s) & \cdots & x_2((N - M + 1) \cdot T_s) \\
\vdots & \vdots & \ddots & \vdots \\
x_2((M - 1) \cdot T_s) & x_2(M \cdot T_s) & \cdots & x_2((N - 1) \cdot T_s)
\end{bmatrix},
\]

(49)
The basis matrix of signal subspace $U_2 \in \mathbb{R}^{M \times K_2}$ satisfies
\[
\text{span}(U_{2,s}) = \bigoplus_{i=1}^{L_2} \text{span}(U_{2,s}^{[d_{2,i}, T_s]}),
\]
where $U_{2,s}^{[d_{2,i}, T_s]}$ is the basis matrix of signal subspace from Hankel matrix
\[
\begin{bmatrix}
s((0 - d_{2,1}) \cdot T_s) & s((1 - d_{2,1}) \cdot T_s) & \cdots & s((N - M - d_{2,1}) \cdot T_s) \\
s((1 - d_{2,1}) \cdot T_s) & s((2 - d_{2,1}) \cdot T_s) & \cdots & s((N - M - d_{2,1} + 1) \cdot T_s) \\
\vdots & \vdots & \ddots & \vdots \\
s((M - 1 - d_{2,1}) \cdot T_s) & s((M - d_{2,1}) \cdot T_s) & \cdots & s((N - 1 - d_{2,1}) \cdot T_s)
\end{bmatrix}.
\]
(51)

The next step is the same as before, we can infer that, if and only if
\[
\Delta d = d_{2,i_2} - d_{1,i_1},
\]
(52)
the component $\text{span}(U_{1,s}^{[(\Delta d + d_{1,1}) \cdot T_s]})$ from the signal subspace $\text{span}(U_{1}^{[\Delta d \cdot T_s]})$ and the component $\text{span}(U_{2,s}^{[d_{2,i_2} \cdot T_s]})$ from $\text{span}(U_{2})$ become linearly dependent. Therefore, the VCC function:
\[
\text{vol}_{K_1 + K_2}([U_1^{[\Delta d \cdot T_s]}, U_2])
\]
(53)
will tend to zero when $\Delta d = d_{2,i_2} - d_{1,i_1}$.

D. The TDOA algorithm based on VCC function for the slowly changing subspace signals: Algorithm

Algorithm 2.

For $\Delta d = -M + 1, \cdots, -1$, Loop:

1) Construct delayed Hankel matrix
\[
X_1^{[\Delta d \cdot T_s]} = \begin{bmatrix}
x_1((0 - \Delta d) \cdot T_s) & x_1((1 - \Delta d) \cdot T_s) & \cdots & x_1((N - M - \Delta d) \cdot T_s) \\
x_1((1 - \Delta d) \cdot T_s) & x_1((2 - \Delta d) \cdot T_s) & \cdots & x_1((N - M - \Delta d + 1) \cdot T_s) \\
\vdots & \vdots & \ddots & \vdots \\
x_1((M - 1 - d_{1,1}) \cdot T_s) & x_1((M - d_{1,1}) \cdot T_s) & \cdots & x_1((N - d_{1,1}) \cdot T_s)
\end{bmatrix},
\]
and non-delayed Hankel matrix
\[
X_2 = \begin{bmatrix}
x_2(0 \cdot T_s) & x_2(1 \cdot T_s) & \cdots & x_2((N - M) \cdot T_s) \\
x_2(1 \cdot T_s) & x_2(2 \cdot T_s) & \cdots & x_2((N - M + 1) \cdot T_s) \\
\vdots & \vdots & \ddots & \vdots \\
x_2((M - 1) \cdot T_s) & x_2((M - 1) \cdot T_s) & \cdots & x_2((N - 1) \cdot T_s)
\end{bmatrix},
\]
where $1 \leq M \leq N$.

2) Perform singular value decomposition of $X_1^{[\Delta d \cdot T_s]}$ and $X_2$, i.e.,
\[
X_1^{[\Delta d \cdot T_s]} = \sum_{j=1}^{\min(M,N-M+1)} \sigma_{1,j}^{[\Delta d \cdot T_s]} \cdot u_{1,j}^{[\Delta d \cdot T_s]} \cdot v_{1,j}^{[\Delta d \cdot T_s]^H},
\]
\[
X_2 = \sum_{j=1}^{\min(M,N-M+1)} \sigma_{2,j} \cdot u_{2,j} \cdot v_{2,j}^H,
\]
where $\sigma_{1,j}, u_{1,j}, v_{1,j}, u_{2,j}, v_{2,j}$ are the corresponding singular values, left and right singular vectors of matrices $X_1^{[\Delta d \cdot T_s]}$ and $X_2$, take $K_1$ and $K_2$ as the dimensions of signal subspaces, then the bases of signal subspaces are
\[
U_1^{[\Delta d \cdot T_s]} := [u_{1,1}^{[\Delta d \cdot T_s]}, \cdots, u_{1,K_1}^{[\Delta d \cdot T_s]}] \in \mathbb{C}^{N \times K_1}
\]
and $$U_2 := [u_{2,1}, \cdots u_{2,K_2}] \in \mathbb{C}^{N \times K_2}.$$ 

3) Calculate the reciprocal of VCC function:

$$r_vol(\Delta d \cdot T_s) := 1/\text{vol}_{K_1+K_2}([U_1^{T_1 \cdot \Delta d}, U_2]),$$

End Loop.

Find the peaks of $$r_vol(\Delta d \cdot T_s)$$ and corresponding value of $$\Delta d.$$

Below are some necessary notations.

1) There are two important parameter when constructing the Hankel matrix, i.e., the dimensions $$M$$ and $$N - M + 1$$. It is a hard problem that how to choose these two dimensions in order to meet different requirements in diverse applications [53]. Because optimization on the dimensions $$M$$ and $$N - M + 1$$ is another complicated topic, in this paper we just choose these two dimensions empirically based on the experiments and simulations.

2) Another important parameter affecting the extraction of signal subspace and calculation of VCC function is the dimension of signal subspaces, i.e., $$K_i$$. Also it is difficult and another topic to determine the rank of an unknown wireless signal. This parameter will be also determined empirically. Actually, in the numerical simulation which will be shown in the next section, $$K_i$$ is chosen to be 3 times of $$L_i$$.

V. NUMERICAL SIMULATIONS

In this section, numerical simulations are carried out to demonstrate and validate the performance of our proposed TDOA algorithm. In the simulations, we have experimented on both the two categories of signals discussed earlier in this paper.

A. TDOA estimation using Linear Frequency Modulation (LFM) radar waveforms

In this simulation, a simple linear frequency modulation waveform is chosen as a typical slowly changing subspace signal. The radar waveform in (13) is generated with a sample rate $$1MHz$$, its length are 2048, and the frequency sweeps linearly from $$50kHz$$ to $$500kHz$$. The multipath channel are manually generated, and the multipath delay are chosen arbitrarily to be $$\{d_{1,i,1}\}^L_{i=1} = \{40, 75, 200\}$$ and $$\{d_{2,i,2}\}^L_{i=1} = \{50, 100, 185, 250\}$$. The multiple observations in the form of (20) are directly generated by Monte-Carlo method, in which the channel coefficients $$\alpha_i^{(j)} = [\alpha_{i,1}^{(j)}, \cdots, \alpha_{i,L}^{(j)}]$$ with respect to different $$j$$ are generated independently from complex Gaussian distributions in order to simulate the channel fading effect. In addition, the mean value of $$|\alpha_{i,1}|$$ is greater than the mean value of $$|\alpha_{i,l}|, l > 1$$, meaning that the direct path has a greater propagation gain than the reflective path. The length $$N$$ of each observation vector is 512, and totally 512 observation data are generated.

In the simulation, we compare our proposed TDOA algorithm with the publicly known super resolution MUSIC-Type TDOA algorithm proposed by Fengxiang Ge in [20], because both algorithms have super resolution and can make use of multiple observation data. Since the simulation focuses on demonstrating the ability of resolving multipath TDOA, we just assume the dimensions of signal subspaces in both algorithms, i.e., the number of channel paths $$L_i$$, have been accurately estimated. The normalized TDOA estimation results of both algorithms are plotted in figure 2 where the signal-to-noise ratio $$SNR$$, defined as the power ratio of signal and noise, is set to be 0dB. According to the simulation setting, there should be peaks at $$\Delta d = -150, -100, -25, -15, 10, 25, 50, 60, 110, 145, 175, 210$$ in the TDOA estimation results. The position of these peaks are labeled in the figure. As shown in figure 2 at higher SNR, both algorithms shows clear peaks at these TDOA positions. But when the SNR is low, as shown in figure 3 the MUSIC-Type algorithm fails to reveal most of the peaks of multipath TDOA, but our VCC algorithm can still show clear peaks. In addition, because Ge’s algorithms requires to solve a quadratic optimization for every delay parameter $$\Delta d$$, the computational complexity of our VCC algorithm is significantly lower. As a whole, the proposed TDOA algorithm based on VCC function has excellent super resolution as well as high computational efficiency.
Comparison of Music-Type algorithm and VCC function for TDOA estimation, with SNR=0

Discrete path delays for receiver 1: [40 75 200]
Discrete path delays for receiver 2: [50 100 185 250]

Fig. 2: Comparison of Ge’s MUSIC-type algorithm and our VCC algorithm for TDOA estimation

Comparison of Music-Type algorithm and VCC function for TDOA estimation, with SNR=−10

Discrete path delays for receiver 1: [40 75 200]
Discrete path delays for receiver 2: [50 100 185 250]

Fig. 3: Comparison of Ge’s MUSIC-type algorithm and our VCC algorithm for TDOA estimation at a low SNR
B. TDOA estimation using Frequency Modulation (FM) broadcast signals

In this part of simulation, we chose the commonly seen frequency modulation (FM) broadcast signals as one example of the fast changing subspace signal, to demonstrate the TDOA estimation performance of our proposed method. The FM signals used here are baseband complex-valued signals from a real world radio broadcast station, and are gathered from several remote located radio receivers. Since the raw data collected from real world do not have multipath components, we simulate the multipath environment by manually interleaving the real world FM signal. The simulations for both the real world single path and the simulated multipath channels are carried out.

1) Real world FM signal, the channel has only one single path: In the simulation, the FM signals of a radio station are received from two differently located radio receivers, the sample rate of the received baseband signals is 256kHz, with a length of 4096.

In order for a better time resolution, we first increase the original sample rate through interpolating the raw signals by a factor of 4. A part of the waveform in time domain and the frequency spectrum of these two baseband signals are plotted in figure 4. From the waveform of both signals, we can see that the corresponding discrete time TDOA is from 14 to 16.

In the simulation, we compared our VCC algorithm with the traditional GCC-PHAT method, the high resolution $\ell_1$ regularization algorithm, and also the super resolution MUSIC-Type algorithm by Ge. In the simulation of $\ell_1$ regularization algorithm, the power spectrum of the transmitted signal is required to be known, while the other three algorithms don’t use knowledge of the power spectrum. In our algorithm, the parameters $N$, $M$ and $K_i$ are chosen empirically to be $N = 544$, $M = 512$, $K_1 = K_2 = 3$.

The normalized TDOA estimation results of GCC-PHAT, $\ell_1$ regularization, MUSIC-type as well as VCC algorithm are shown in figure 5.

It can be seen that in a channel with only a single path, both our proposed VCC algorithm and Ge’s MUSIC-Type algorithm outperform the traditional GCC-PHAT and the $\ell_1$ regularization algorithms; because the latter two methods give a much wider peak, and also reveal too many false peaks except for the real TDOA peak. Although our VCC method and MUSIC-Type algorithm have similar super resolution ability, the computational complexity of our method is much lower.

2) Real world FM signal, the multipath channel is manually simulated: In order to validate the ability to resolve multipath TDOAs of our proposed VCC algorithm, we have manually simulated a multipath environment, the received signals from two receivers are simulated according to the following expression:

$$
y_1(kT_s) = \alpha_{1,1}s(kT_s) + \alpha_{1,2}s((k - 60)T_s) + \alpha_{1,3}s((k - 120)T_s),$$

$$
y_2(kT_s) = \alpha_{2,1}s((k - 25)T_s) + \alpha_{2,2}s((k - 100)T_s) + \alpha_{1,3}s((k - 195)T_s).$$

Fig. 4: A part of the time domain waveform and the frequency spectrum of the two baseband signals from different radio receivers
Demonstration of different TDOA estimation techniques, with no multipath component

\[\Delta d\]

Fig. 5: Comparison of different TDOA techniques in a single path channel environment

The \(s(kT_s)\) here is a real world original FM signal mentioned before, which is also one among the two signals plotted in figure 4. The channel coefficients \(\alpha_{i,j}, i = 1, 2, j = 1, 2, 3\) are also generated to simulate a Rician fading channel, among these coefficients the mean value of \(|\alpha_{i,1}|\) is greater than that of the other coefficients. In the simulation, the parameters \(N, M\) and \(K_i\) are also chosen empirically to be \(N = 896, M = 768, K_1 = K_2 = 9\). The TDOA estimation results of GCC-PHAT, \(\ell_1\) regularization, MUSIC-type and our method are shown in figure 6.

For convenience, we also labeled the theoretical peaks of multipath TDOAs in the simulation results. As is seen, both Ge’s MUSIC-Type algorithm and our VCC algorithm outperforms the other two methods. However, Ge’s MUSIC-Type method and our VCC algorithm have their advantages and disadvantages at different aspect. We can see that the MUSIC-Type method has a much sharper peak, but fails to resolve every multipath TDOA, and still has some false peaks, while our VCC method may not have such sharp peaks, but successfully reveals every TDOA peak precisely with no false peak. In addition, our VCC algorithm is much better on computational efficiency.

VI. THE ADVANTAGE OF OUR PROPOSED TDOA ALGORITHM BASED ON VCC FUNCTION OVER THE OTHER ALGORITHMS

We have introduced and demonstrated our novel TDOA algorithm based on the Volume Cross-Correlation function, the proposed method exploits the subspace structures from the received signals, and employs a Volume Cross-Correlation function which measures the linear dependence of subspaces, to estimate the time difference between two signals. In this section, we will summarize the advantage of our proposed TDOA algorithm.

As is seen in the previous sections, the proposed TDOA algorithm outperforms the other state-of-art high resolution algorithms, especially the MUSIC-Type algorithm of Ge [20]. Although our proposed method also involves similar steps to Ge’s MUSIC-type method such as the estimation of signal subspace, the methodology of our algorithm is quite different with the MUSIC-Type algorithm. Let us firstly recall the original methodology of the MUSIC algorithm, the goal of MUSIC algorithm is to estimate a parameter vector \(\theta := [\theta_1, \cdots, \theta_p], p \geq 1\), from the received signal with a subspace structure:

\[
y = A(\theta)x + w.
\]

Here the matrix \(A := [a(\theta_1), \cdots, a(\theta_p)]\) is the basis matrix of the signal subspace, and the vectors \(a(\theta_1), \cdots, a(\theta_p)\) are affected by the parameter vector \(\theta = [\theta_1, \cdots, \theta_p]\). In order to estimate the parameter vector \(\theta\), we search in the range of the
Variable $\hat{\theta}$ for peaks of the function

$$P(\hat{\theta}) = \frac{1}{\| a(\hat{\theta})^H U_n a(\hat{\theta}) \|_2},$$

which measures the perpendicularity between the reference vector $a(\hat{\theta})$ and the noise subspace $\text{span}(U_n)$. This methodology requires knowledge of the reference vector $a(\hat{\theta})$, or knowledge of the signal subspace. However, in passive localization problems, knowledge about the transmitted signal is usually impossible. For this reason, Ge’s MUSIC-type method uses an extra quadratic optimization to explore the best unknown $a(\hat{\theta})$ for given $\theta$, and continues the search for peaks with respect to the parameter $\hat{\theta}$. This optimization step dramatically increases the computational complexity and also decrease the performance.

However, our proposed method circumvents this difficulty by estimating the unknown signal subspaces first, then the linear dependency of the estimated subspaces are measured by the reciprocal of VCC function,

$$r_{\text{vol}}(\Delta d \cdot T_s) := 1/\text{vol}_{L_1 + L_2}(U_{1,s}^{[\Delta d \cdot T_s]}, U_{2,s}),$$

instead of measuring perpendicularity between known signal and estimated noise subspace. From this point of view, one advantage of our TDOA algorithm is that it exploits a looser correlation function, which measures the linear dependency of estimated subspaces, and this methodology brings more accuracy and high efficiency. This has been validated by the simulation results that, compared with the MUSIC-Type TDOA algorithm, our VCC algorithm has strong ability to completely resolve multipath delays caused by multipath signals.

As for the other state-of-art super resolution TDOA methods, they all have some practical disadvantages. As a matter of fact, all these earlier mentioned methods, especially these ML estimation approaches, have extremely high computational complexity, thus have implementation difficulties. Both the $f_1$ optimization method and the unions of subspaces technique requires priori knowledge of the transmitted pulse shape, which is difficult to be known at the receiver in practice. On the other hand, our VCC algorithm has much lower computational complexity, and also requires no priori knowledge of the transmitted signal’s pulse.

Fig. 6: Comparison of different TDOA techniques in a simulated multipath channel environment
shape or spectrum. This is another important advantage of our proposed TDOA algorithm. Indeed, we can deal with wireless signals which are repeatedly transmitted waveforms, or repeated waveforms modulated by arbitrary symbols, by using the slowly changing subspace signal model; on the other hand, we deal with all other signals, that don’t have any repeated structure, based on the fast changing subspace signal model. Therefore, these two categories of signal models requires less assumption and priori knowledge of the transmitted signal, and can cover all kinds of wireless signals encountered in the passive localization system. Therefore our proposed TDOA algorithm have the potential of wide application in passive localization systems.

VII. BRIEF DISCUSSIONS ABOUT THE DISAMBIGUATION OF MULTIPATH TDOA ESTIMATION

It has been demonstrated that our proposed TDOA algorithm outperforms the other state-of-art super resolution algorithms for its ability to resolve multipath TDOAs. And resolving every multipath TDOAs is the first step to precisely localize the signal source. Given the multipath TDOA estimations, i.e., $d_{2,l_2} - d_{1,l_1}$, it is still an important task to pick out the direct path (or Line of Sight, LOS) TDOA estimation. This problem is another topic and goes far beyond the scope of this paper, therefore we leave this disambiguation problem as a future work, and only provide a brief discussion here.

1) Firstly, it is a common sense that, the propagation gain of the direct path (or LOS path) is generally greater than the propagation gain of the reflective path (or NLOS path). Therefore, from the perspective of channel fading, the channel coefficient of the direct path, i.e., $\alpha_{i,l}$, will obey a different distribution with that of the reflective path, i.e., $\alpha_{i,l}, l > 1$. Generally the amplitude $|\alpha_{i,1}|$ obeys Rician distribution, meaning a nonzero mean; while $|\alpha_{i,l}|, l > 1$ obeys Rayleigh distribution, meaning a zero mean. From this point of view, in the received signal $x_l(t) = \sum_{i=1}^{L} \alpha_{i,l} s(t - \tau_{i,l}) + n_{l}(t)$, the direct-path signal component $\alpha_{i,1} s(t - \tau_{i,1})$ will have a larger power than the reflective-path signal components. We can intuitively expect that the estimated TDOA peak corresponding to the direct path will dominant the other peaks. This intuition has been validated by our numerical simulations, where direct-path TDOA really has the highest peak. This property can be used to tell the direct-path TDOA from the others when the channel has a dominant direct path gain.

2) When the channel does not have a dominant direct path gain, or every channel coefficient $\alpha_{i,l}$ obeys the Rayleigh distribution with a zero mean, another method called TDOA disambiguation can be used. This method was proposed by Scheuing and Yang, but haven’t drawn much attention because it is based on traditional GCC algorithm, which has no sufficient resolution for disambiguation. But their method can be adapted to our localization algorithm. Actually, they are making use of the auto-correlation functions of the received signals, which explores the differences among the channel’s delays from the received signal itself. In another word, the auto-correlation can reveal time differences like $d_{1,l_1} - d_{1,l_1}'$ and $d_{2,l_2} - d_{2,l_2}'$. And these differences are related with the multipath TDOA in a manner of $d_{1,l_1} - d_{1,l_1}' = \Delta d_{1,l_1} - \Delta d_{1,l_1}'$ and $d_{2,l_2} - d_{2,l_2}' = \Delta d_{2,l_2} - \Delta d_{2,l_2}'$. This relation is called the Raster Condition [48,49], and can be used to find the direct-path TDOA. Their auto-correlation can be directly replaced by our VCC function, with the input argument being the delayed and non-delayed basis of signal subspaces from the same received signal. As is said, it is another complicated problem, and beyond the scope of this paper, we will leave this as future work.

VIII. CONCLUSION

In this paper, a super resolution TDOA estimation technique using the Volume Cross-Correlation function is proposed. This technique firstly estimates the unknown signal subspace from the received signal, and estimate the time difference through the novel VCC function, which calculates the linear dependency of these subspaces. We analyzed the performance of our TDOA estimation algorithm upon two typical categories of signals, i.e., the slowly changing subspace signal and the fast changing subspace signal. Both categories can cover various kinds of wireless signals encountered in passive localization systems. Analysis and numerical simulations have demonstrated that our algorithm has both a low computational complexity and excellent ability of super resolution for TDOA estimation in a multipath environment. As a whole, the proposed TDOA estimation technique is a highly efficient algorithm with super resolution to deal with multipath environment.
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