SYMMETRIES OF PARABOLIC CONTACT STRUCTURES

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Abstract. We generalize the concept of locally symmetric spaces to parabolic contact structures. We show that symmetric normal parabolic contact structures are torsion–free and some types of them have to be locally flat. We prove that each symmetry given at a point with non–zero harmonic curvature is involutive. Finally we give restrictions on number of different symmetries which can exist at such a point.

Affine locally symmetric spaces are well known and studied objects from differential geometry. The classical definition says that a local symmetry at $x$ on a manifold $M$ with an affine connection $\nabla$ is a locally defined affine transformation such that $s_x(x) = x$ and $T_x s_x = -\text{id}$ on $T_x M$. We can understand $\nabla$ as a geometric structure on $M$ such that the symmetry $s_x$ preserves this structure. There is a natural generalization of this concept: For a manifold with an arbitrary geometric structure, one can define a local symmetry as an automorphism of this geometric structure, which satisfies the two above conditions. Best known examples of this concept are Riemannian symmetric spaces, see [4], and projective symmetric spaces, see [6]. This concept also generalizes nicely to geometric structures which can be described as $\mathbb{R}$–graded parabolic geometries, see [9, 10].

In this article, we are interested in symmetries of contact manifolds endowed with some additional structures which can be described as parabolic geometries, the so–called parabolic contact structures, see [2]. Discussion of the Levi bracket implies that we cannot define a symmetry in the classical sense, see [9]. Motivated by the definition of a symmetry for Cauchy–Riemann structures from [3], we define a symmetry at $x$ as a morphism of the contact geometry such that $s_x(x) = x$ and $T_x s_x = -\text{id}$ on the contact distribution at $x$. This definition works nicely for all parabolic contact structures. Then, to study symmetries on parabolic contact structures, we can use general techniques known from theory of parabolic geometries, see [1, 2].

In this article, we discuss the curvature of symmetric parabolic contact geometries in detail. The theory of harmonic curvature for parabolic geometries allows us to prove that symmetric normal parabolic contact geometries must be torsion–free. Moreover, some types of them have to locally flat, if they are symmetric.

Contrary to the classical case, symmetries of parabolic contact structures are not involutive in general and we use Weyl structures to study this question. More precisely, we show that only locally flat geometries can carry non–involutive symmetries at each point. Moreover, for each involutive symmetry...
on a parabolic contact geometry, there exists an admissible affine connection which is invariant with respect to the symmetry. Finally we show that in many cases, there can exist at most one symmetry at points with non–zero curvature.

1. PARABOLIC CONTACT STRUCTURES

We remind here basic definitions and summarize properties of parabolic geometries. We discuss parabolic contact structures in detail. We introduce here Weyl structures which are our main tool to deal with parabolic contact geometries. We follow concepts and notation of [2, 1] and the reader can find all details and proofs therein.

1.1. Contact structures and parabolic geometries. Consider a manifold \( M \) endowed with a distribution \( H \) of \( TM \) of corank one. Then \( H \subset TM \) forms a filtration and on the graded bundle \( \text{gr}(TM) = H \oplus TM/H \), there is the Levi bracket \( \mathcal{L} : H \times H \rightarrow TM/H \) which is a bilinear bundle map induced by the Lie bracket of vector fields. The well known definition says that \( H \subset TM \) forms a contact structure on \( M \) if the Levi bracket is non–degenerate at each point. The subbundle \( H \) is then called contact distribution.

We will discuss here contact manifolds endowed with some additional structures which can be described as parabolic geometries. Let us remind that for a semisimple Lie group \( G \) and its parabolic subgroup \( P \), a parabolic geometry of type \((G, P)\) is a pair \((p : \mathcal{G} \rightarrow M, \omega)\) consisting of a principal \( P \)-bundle \( \mathcal{G} \rightarrow M \) and of a 1–form \( \omega \in \Omega^1(\mathcal{G}, \mathfrak{g}) \), called Cartan connection, which is \( P \)-equivariant, reproduces generators of fundamental vector fields and induces a linear isomorphism \( T_u \mathcal{G} \cong \mathfrak{g} \) for each \( u \in \mathcal{G} \). The Lie algebra \( \mathfrak{g} \) of \( G \) is then equipped (up to the choice of Levi factor \( \mathfrak{g}_0 \) in \( p \)) with a grading of the form \( \mathfrak{g} = -k \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \) such that the Lie algebra of \( P \) is exactly \( p = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \). There is the usual notation such that \( \mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \), \( \mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \) and \( P_+ \subset P \) is the subgroup corresponding to \( \mathfrak{p}_+ \). By \( G_0 \) we denote the subgroup in \( P \), with Lie algebra \( \mathfrak{g}_0 \), consisting of all elements whose \( \text{Ad} \)-action preserves the grading of \( \mathfrak{g} \). Each element \( g \) of \( P \) can be uniquely written as \( g_0 \exp Z_1 \cdots \exp Z_k \) for suitable \( g_0 \in G_0 \) and \( Z_i \in \mathfrak{g}_i \), thus \( \exp Z_1 \cdots \exp Z_k \in P_+ \). Let us remind that for each parabolic geometry, there is an element \( E \in \mathfrak{g}_0 \) with the property \( [E, X] = iX \) for each \( X \in \mathfrak{g}_i \), the so–called grading element. To study contact structures, we have to focus on a special case of \([2]–grading: A contact grading of a simple Lie algebra \( \mathfrak{g} \) is a grading \( \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) such that \( \mathfrak{g}_{-2} \) is one–dimensional and the Lie bracket \( \{–, –\} : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2} \) is non–degenerate. Let us remark that for each contact grading, the subspace \( \mathfrak{g}_{-2} \oplus \mathfrak{g}_2 \) coincides with the subspace generated by \( E \).

It is well known that the Cartan connection \( \omega \) provides an identification \( TM \cong \mathcal{G} \times_P \mathfrak{g}/p \). Suppose we have a parabolic geometry corresponding to a contact grading. Because each contact grading of \( \mathfrak{g} \) induces \( P \)-invariant filtration of the form \( \mathfrak{g} = \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 = \mathfrak{g}_2 \), the subspace \( \mathfrak{g}^{-1}/p \subset \mathfrak{g}/p \) defines a subbundle \( T^{-1}M := \mathcal{G} \times_P \mathfrak{g}^{-1}/p \) of corank one in \( TM \). There is the Levi bracket on \( \text{gr}(TM) = T^{-1}M \oplus TM/T^{-1}M \) and the geometry is called regular, if the Levi bracket corresponds to the Lie bracket.
\([-,-]\) : \(\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}\) under the above identification. Then for regular parabolic geometries corresponding to contact gradings, the underlying filtration \(T^{-1}M \subset TM\) defines a contact structure on \(M\), and each such geometry is called parabolic contact structure or parabolic contact geometry. Moreover, define \(G_0 := G/P_+\), which is a principal \(G_0\)-bundle over \(M\). This is the reduction of the natural frame bundle of \(\text{gr}(TM)\) with respect to \(\text{Ad} : G_0 \to GL(\mathfrak{g}_{-1})\) and in this way, we get an additional geometric structure on \(T^{-1}M\).

Let us remind some more facts on parabolic contact structures that will be needed: The \(P\)-bundle \(G \to G/P\) together with the (left) Maurer–Cartan form \(\omega_G \in \Omega^1(G, \mathfrak{g})\) forms a geometry which is called homogeneous model. A morphism between geometries of type \((G, P)\) from \((\mathcal{G} \to M, \omega)\) to \((\mathcal{G}' \to M', \omega')\) is a \(P\)-bundle morphism \(\varphi : \mathcal{G} \to \mathcal{G}'\) such that \(\varphi^* \omega' = \omega\). We will suppose that the maximal normal subgroup of \(G\) is contained in \(P\) is trivial. With this assumption, there is one–to–one correspondence between morphisms of parabolic geometries and their base morphisms. Let us remind that such geometries are called effective.

The curvature is described by \(P\)-equivariant mapping \(\kappa : \mathcal{G} \to \wedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}\), the so–called curvature function. The Maurer–Cartan equation implies that the curvature of the homogeneous model vanishes. Conversely, it can be proved that if the curvature of a geometry vanishes, then it is locally isomorphic to the homogeneous model of the same type. If \(\kappa\) has its values in a subbundle \(\wedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}\), we call the geometry torsion–free. The regular geometry is called normal, if the curvature satisfies \(\partial^* \circ \kappa = 0\), where \(\partial^*\) is the differential in the standard complex computing Lie algebra homology of \(\mathfrak{p}_+\) with coefficients in \(\mathfrak{g}\). Then we can define the harmonic curvature \(\kappa_H\) which is the composition of the curvature function with the projection \(\ker(\partial^*) \to \ker(\partial^*)/\text{im}(\partial^*)\). There is the following general statement, see \([2]\):

**Theorem.** On a regular normal parabolic geometry, the curvature \(\kappa\) vanishes if and only if the harmonic curvature \(\kappa_H\) vanishes.

It can be proved that \(\ker(\partial^*)/\text{im}(\partial^*)\) is a \(G_0\)-submodule of \(\wedge^2 \mathfrak{g}_{-1}^+ \otimes \mathfrak{g}\) and decomposes into the direct sum of components each of which is contained in some \(\mathfrak{g}_{-i}^+ \wedge \mathfrak{g}_{-j}^+ \otimes \mathfrak{g}_k\). According to this decomposition, \(\kappa_H\) decomposes into the sum of components of homogeneity \(\ell = i + j + k\) that we denote by \(\kappa^{(\ell)}\). One can use the Kostant’s version of the Bott–Borel–Weil theorem to find all components of \(\kappa_H\). For parabolic contact geometries, it turns out that there can exist only the following three types of components:

- \(\kappa^{(1)}\) valued in \(\mathfrak{g}_{-1}^+ \wedge \mathfrak{g}_{-1}^+ \otimes \mathfrak{g}_{-1}\),
- \(\kappa^{(2)}\) valued in \(\mathfrak{g}_{-1}^+ \wedge \mathfrak{g}_{-1}^+ \otimes \mathfrak{g}_0\) and
- \(\kappa^{(4)}\) valued in \(\mathfrak{g}_{-2}^+ \wedge \mathfrak{g}_{-1}^+ \otimes \mathfrak{g}_1\).

See Appendix for more detailed description of all contact gradings and corresponding geometries with their components of harmonic curvature.

### 1.2. Adjoint tractor bundles and Weyl structures

Let us introduce here briefly concept of adjoint tractor bundles which allows us to write formulas and make computations in a more convenient form. The adjoint tractor bundle is the natural bundle \(\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}\) corresponding to the
restriction of Ad–action of $G$ on $\mathfrak{g}$. For each parabolic contact geometry, the filtration of $\mathfrak{g}$ induces a filtration $\mathcal{A}M = A^{-2}M \supset A^{-1}M \supset A^{0}M \supset A^{1}M \supset A^{2}M$ such that $A^{i}M = \mathcal{G} \times_{\mathcal{P}} \mathfrak{g}^i$, and there is the associated graded bundle $\text{gr}(\mathcal{A}M) = \mathcal{A}_{-2}M \oplus \mathcal{A}_{-1}M \oplus A^{0}M \oplus A^{1}M \oplus A^{2}M$, where $A^{i}M = A^{i}M / A^{i+1}M$ equals to $\mathcal{G}_0 \times_{\mathcal{G}_0} \mathfrak{g}_i$. Clearly, $TM \simeq \mathcal{A}M / A^{0}M$ and $T^*M \simeq A^{1}M$. On the graded bundle $\text{gr}(\mathcal{A}M)$, there is the algebraic bracket $\{\cdot, \cdot\} : A_{i}M \times A_{j}M \to A_{i+j}M$ defined by means of the Lie bracket on $\mathfrak{g}$. Clearly, its part $\mathcal{A}_{-1}M \times \mathcal{A}_{-1}M \to \mathcal{A}_{-2}M$ on $\text{gr}(TM) = \mathcal{A}_{-2}M \oplus \mathcal{A}_{-1}M$ coincides with the Levi bracket thanks to the regularity. Since each fiber of $\text{gr}(\mathcal{A}M)$ is isomorphic to $\mathfrak{g}$, the grading element defines a unique element $E(x) \in \text{gr}_0(\mathcal{A}_xM)$ such that $\{E(x), \cdot\}$ is a multiplication by $i$ on $\text{gr}_1(\mathcal{A}_xM)$. In fact, these elements form a section $E$ of $\text{gr}_0(\mathcal{A}M)$ which is called the grading section. Let us also remark that we simultaneously get an action $\bullet$ of $\text{gr}(\mathcal{A}M)$ on arbitrary tensor products of $\text{gr}(\mathcal{A}M)$ which is given using the tensoriality of the algebraic bracket. In particular, the grading section acts on each homogeneous component of the tensor as multiplication by its homogeneity.

Now we should remind basic facts on Weyl structures. For any parabolic contact geometry $(\mathcal{G} \to M, \omega)$ with the underlying $G_0$–bundle $p_0 : \mathcal{G}_0 \to M$, a Weyl structure is a global smooth $G_0$–equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the canonical projection $\pi : \mathcal{G} \to \mathcal{G}_0$. Weyl structures always exist and for any two Weyl structures $\sigma$ and $\hat{\sigma}$, there are $G_0$–equivariant functions $\Upsilon_1 : \mathcal{G}_0 \to \mathfrak{g}_1$ and $\Upsilon_2 : \mathcal{G}_0 \to \mathfrak{g}_2$ such that

$$\hat{\sigma}(u_0) = \sigma(u_0) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)$$

for all $u_0 \in \mathcal{G}_0$. Clearly, $\Upsilon_i \in \Gamma(\mathcal{A}_iM)$ and $\Upsilon := (\Upsilon_1, \Upsilon_2)$ is a smooth section of $\text{gr}(T^*M)$. Moreover, the Campbell–Baker–Hausdorff formula implies $\exp \Upsilon = \exp \Upsilon_1 \exp \Upsilon_2 = \exp(\Upsilon_1 + \Upsilon_2) = \exp \Upsilon_2 \exp \Upsilon_1$, see [5].

For each Weyl structure $\sigma$, we can form the pullback $\sigma^* \omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$. This decomposes as $\sigma^* \omega = \sigma^* \omega_- + \sigma^* \omega_0 + \sigma^* \omega_+$ and the part $\sigma^* \omega_- \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-)$ is called the soldering form. Each Weyl structure $\sigma$ induces by means of its soldering form an isomorphism $TM \simeq \text{gr}(TM)$ which we write as $\xi \mapsto (\xi_-, \xi_+)$. If $\sigma$ and $\exp \Upsilon_1 \exp \Upsilon_2$ is another Weyl structure, the isomorphism changes as $\xi \mapsto (\xi_-, \xi_+ - (\Upsilon_1, \xi_-))$. In particular, $\sigma$ and $\exp \Upsilon_2$ induce the same isomorphism for an arbitrary $\Upsilon_2 : \mathcal{G}_0 \to \mathfrak{g}_2$.

The part $\sigma^* \omega_- \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ defines a principal connection on $p_0 : \mathcal{G}_0 \to M$ which we call the Weyl connection. This connection induces connections on all associated bundles. In particular, for each $\sigma$ we get a preferred connection on $\text{gr}(TM) = \mathcal{G}_0 \times_{\mathcal{G}_0} \mathfrak{g}_-$ and via the above isomorphism, we get a preferred connection on the tangent bundle, cotangent bundle and their tensor products. We call each such connection Weyl connection, too. For a Weyl structure $\sigma$, we denote the corresponding connection by $\nabla^\sigma$. For $\sigma$ and $\hat{\sigma} = \sigma \exp \Upsilon_1 \exp \Upsilon_2$ we have

$$\begin{align*}
\nabla^\sigma_\xi s &= \nabla^\hat{\sigma}_\xi s + \frac{1}{2} \{\Upsilon_1, \{\Upsilon_1, \xi_-\}\} - \{\Upsilon_2, \xi_-\} - \{\Upsilon_1, \xi_+\} \bullet s,
\end{align*}$$

where $\xi \in \mathfrak{X}(M)$ and $s$ is a section of an appropriate bundle. The positive part $\sigma^* \omega_+ \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$ is called rho–tensor and is denoted by $P^\sigma$. We will not need it explicitly, see [2] [1] for details.
Let us finally remind the so-called normal Weyl structures. A normal Weyl structure at \( u \) is the only \( G_0 \)-equivariant section \( \sigma_u : G_0 \rightarrow \mathcal{G} \) satisfying 
\[
\sigma_u \circ \pi \circ \text{Fl}_{t}^{-1}(X)(u) = \text{Fl}_{t}^{-1}(X)(u)
\]
where by \( \text{Fl}_{t}^{-1}(X)(u) \) we denote flows of constant vector fields \( \omega^{-1}(X) \in \mathfrak{X}(\mathcal{G}) \). Each normal Weyl structure \( \sigma_u \) is defined locally over some neighborhood of \( p(u) \) and depends only on the \( G_0 \)-orbit of \( u \in \mathcal{G} \), see \[2\].

2. Basic facts on symmetries

We formulate here the definition of a symmetry on a parabolic contact geometry and describe its basic properties. We study the action of symmetries on Weyl structures and we describe some interesting subclasses of them. We focus here on the question of involutivity of our symmetries.

2.1. Definitions. Let \( (\mathcal{G} \rightarrow M, \omega) \) be a parabolic contact structure. A (local) symmetry with the center at \( x \in M \) is a (locally defined) diffeomorphism \( s_x \) on \( M \) such that:

(1) \( s_x(x) = x \),
(2) \( T_x s_x = -\text{id} \) on \( T^{-1}M \),
(3) \( s_x \) is a base morphism of some (locally defined) automorphism \( \varphi \) of the parabolic contact geometry.

The geometry is called (locally) symmetric if there is a (local) symmetry at each point \( x \in M \).

Clearly, each symmetry is a local symmetry. In this article, we discuss local symmetries and local properties of locally symmetric geometries and we will shortly say ‘symmetry at \( x \)’ and ‘symmetric’ instead of ‘local symmetry at \( x \)’ and ‘locally symmetric’, respectively. Global symmetries and their systems we will discuss elsewhere. Moreover, we will also call the automorphism \( \varphi \) of \( \mathcal{G} \) and its underlying automorphism \( \varphi_0 \) of \( G_0 \) a ‘symmetry at \( x \)’.

2.2. Basic properties of symmetries. Let \( s_x \) be a symmetry on a parabolic contact geometry and let \( \varphi \) be as above. Since each symmetry \( s_x \) preserves \( x \), the (uniquely given) automorphism \( \varphi \) has to preserve the fiber over \( x \). Then for each frame \( u \in p^{-1}(x) \) we have \( \varphi(u) = u g_0 \exp Z = u g_0 \exp Z_1 \exp Z_2 \) for suitable \( g_0 \in G_0, Z_1 \in \mathfrak{g}_1, Z_2 \in \mathfrak{g}_2 \), where \( Z = Z_1 + Z_2 \). Let us describe the element \( g_0 \exp Z_1 \exp Z_2 \) in detail:

For each \( \xi(x) = [u, X] \in T^{-1}M \), i.e. for each \( X \in \mathfrak{g}^{-1}/\mathfrak{p} \), we have

\[
T_x s_x \xi(x) = [\varphi(u), X] = [u g_0 \exp Z_1 \exp Z_2, X] = [u, \text{Ad}_{\exp(-Z_2)} \text{Ad}_{\exp(-Z_1)} \text{Ad}_{g_0^{-1}} X]
\]

and simultaneously \( T_x s_x \xi(x) = -\xi(x) = [u, -X] \). All together, the element \( g_0 \exp Z_1 \exp Z_2 \) has to induce \( -\text{id} \) on \( \mathfrak{g}^{-1}/\mathfrak{p} \) by the \( \text{Ad} \)-action. Moreover, \( \exp Z_1 \exp Z_2 \) acts trivially on \( \mathfrak{g}^{-1}/\mathfrak{p} \). Indeed, there is the formula

\[
\text{Ad}_{\exp Z} X = \sum_{j=0}^{\infty} \frac{1}{j!} \text{ad}^{j}_Z X = X + [Z, X] + \frac{1}{2} [Z, [Z, X]] + \cdots
\]

for all \( X \in \mathfrak{g}_- \) and \( Z \in \mathfrak{p}_+ \) and if \( X \in \mathfrak{g}_{-1} \), all brackets on the right hand side belong to \( \mathfrak{p} \). In fact, along the fiber over \( x \), the \( \mathfrak{P}_+ \)-parts of the above elements are determined by \( \varphi \) and can be arbitrary in general, one only has
to impose the compatibility of \( \varphi \) with the right action of \( P \). Then the element \( g_0 \) has to cause the sign change on \( g^{-1}/p \simeq g_{-1} \). Since our geometries are effective, there can exist at most one element \( g_0 \in G_0 \) which gives \(-\text{id}\) on \( g_{-1} \) and it has to be the same element along the fiber. In particular, the underlying morphism \( \varphi_0 \) is of the form \( \varphi_0(u_0) = u_0g_0 \) for each \( u_0 \in p_0^{-1}(x) \). Clearly, the element \( g_0 \) has to induce identity on \( g_{-2} = [g_{-1}, g_{-1}] \).

One of basic properties of classical symmetries is their involutivity and there is a natural question on involutivity of our symmetries. Thus let us focus on \( s_x : s_x \circ s_x \). Let us first point out that \( \varphi \circ \varphi = \text{id}_G \) if and only if \( s_x \circ s_x = \text{id}_M \), which follows directly from effectivity. Clearly, \( \varphi_0 \) is then involutive, too. Thus it suffice to study the morphism \( \varphi^2 \). In the above notation, \( \varphi^2(u) = u g_0 \exp Z_1 \exp Z_2 g_0 \exp Z_1 \exp Z_2 \) holds and using the known fact \( \exp X g_0 = g_0 \exp(\text{Ad}_{g_0} X) \), we can rewrite this as \( u g_0 \exp(-Z_1) \exp Z_2 \exp Z_1 \exp Z_2 \). Moreover, \( g_0^2 \) acts as id on \( g_{-1} \) and thus on \( g \), because \( g_{-1} \) generates \( g_{-} \), \( p = g_{+}^* \) and \( g_0 \subset g_{+}^* \otimes g_{-} \). Thus it lies in the kernel of the Ad–action which coincides with the maximal normal subgroup of \( G \) which is contained in \( P \). Effectivity then gives \( g_0^2 = e \). All together, we have got \( \varphi^2(u) = u \exp 2Z_2 \). In another frame \( uh \) for \( h \in P \) we then have \( \varphi^2(uh) = uh \exp 2\text{Ad}_{h} Z_2 \). Thus we can simply view \( Z_2 \) as \( P \)-equivariant function \( Z_2 : p^{-1}(x) \to g_2 \). In fact, we have proved the following statement:

**Lemma.** On a parabolic contact geometry, each symmetry \( s_x \) at \( x \) defines uniquely a covector \( Z_2 \in T^*_x M \) through the equation

\[
\varphi^2(u) = u \exp 2Z_2(u)
\]

along the fiber over \( x \). The symmetry \( s_x \) is involutive if and only if the covector \( Z_2 \) equals to zero.

Thanks to the above observations, it is easy to describe the differential of each symmetry at its center.

**Proposition.** For each symmetry \( s_x \) at \( x \) on a parabolic contact geometry, the mapping \( T_x s_x : T_x M \to T_x M \) is involutive, thus \( T_x M \) decomposes into two eigenspaces with eigenvalues \(-1 \) and \( 1 \). The eigenspace corresponding to the eigenvalue \(-1 \) has to coincide with \( T_x^{-1} M \), the contact distribution, and there exists one–dimensional eigenspace corresponding to the eigenvalue \( 1 \).

**Proof.** For each \( \xi(x) = [u, X] \) from \( T_x M \) we have

\[
T_x s_x^2 \xi(x) = [\varphi^2(u), X] = [u \exp 2Z_2(u), X] = [u, \text{Ad}_{\exp(-2Z_2(u))} X] = [u, X] = \xi(x),
\]

which follows directly from formulas (4) and (5). The rest follows immediately from the definition of the symmetry. \( \square \)

2.3. Action of symmetries on Weyl structures. Let us now discuss relations of various Weyl structures to the symmetry \( s_x \). For each Weyl structure \( \sigma \) we can write

\[
\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)
\]

for each \( u_0 \in G_0 \) and for suitable functions \( \Upsilon_1 : G_0 \to g_1 \) and \( \Upsilon_2 : G_0 \to g_2 \) which are generally determined by \( \varphi \) and \( \sigma \). With the notation from the last
section, we have \( \Upsilon_2(u_0) = Z_2(\sigma(u_0)) \) in the fiber over \( x \). The Lemma \( \ref{lem:equivariance} \)
then shows that \( \Upsilon_2 \) does not depend on the choice of a Weyl structure \( \sigma \)
and coincides for all Weyl structures at \( x \). Clearly, \( \Upsilon_2 \) vanishes at \( x \) if and
only if \( s_x \) is involutive. The function \( \Upsilon_1 \) depends on the choice of a Weyl
structure \( \sigma \) at \( x \) and with the above notation, \( \Upsilon_1(u_0) = Z_1 \) for \( \sigma(u_0) = u \).

Let us now focus on the role of \( \Upsilon_1 \) for the isomorphism \( TM \simeq \text{gr}(TM) \)
given by the Weyl structure \( \sigma \): For a tangent vector \( \xi(x) = [\sigma(u_0), X_{-2} + \)

\( X_{-1} \] where \( X_i \in \mathfrak{g}_i \) we have \( T_x s_x, [\sigma(u_0), X_{-2} + X_{-1}] = [\sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0), X_{-2} + X_{-1}] \)
which follows from the fact that \( \varphi_0(u_0) = u_0 g_0 \) for \( g_0 \) giving \(-\text{id} \) on \( \mathfrak{g}_{-1} \) and
from formulas \( \ref{eq:comp} \) and \( \ref{eq:comp2} \). In particular, the isomorphism \( TM \simeq \text{gr}(TM) \)
given by a Weyl structure \( \sigma \) reflects the decomposition of \( T_x M \) into \( \pm 1 \)-
eigenspaces for \( T_x s_x \) if and only if the Weyl structure \( \sigma \) satisfies \( \Upsilon_1(u_0) = 0 \)
for each \( u_0 \) from the fiber over \( x \).

**Lemma.** On a parabolic contact geometry with a symmetry \( s_x \) at \( x \), there
are Weyl structures \( \hat{\sigma} \) such that \( \varphi(\hat{\sigma}(u_0)) = \hat{\sigma}(\varphi_0(u_0)) \exp \hat{\Upsilon}_1(u_0) \exp \hat{\Upsilon}_2(u_0) \)
holds for suitable \( \hat{\Upsilon}_1 \) such that \( \hat{\Upsilon}_1(u_0) = 0 \) for each \( u_0 \) from the fiber over \( x \).

**Proof.** Consider an arbitrary Weyl structure \( \sigma \) and let \((\Upsilon_1, \Upsilon_2)\) be determined
by \( \sigma \) as above. Let us verify that the Weyl structure
\( \hat{\sigma}(u_0) = \sigma(u_0) \exp(-\frac{1}{2} \Upsilon_1(u_0)) \)
satisfies the condition: The formula \( \ref{eq:comp} \) and the Campbell–Baker–Hausdorff
formula allow us to write
\begin{align*}
\varphi(\hat{\sigma}(u_0)) &= \varphi(\sigma(u_0)) \exp(-\frac{1}{2} \Upsilon_1(u_0)) \\
&= \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0) \exp(-\frac{1}{2} \Upsilon_1(u_0)) \\
&= \sigma(\varphi_0(u_0)) \exp \frac{1}{2} \Upsilon_1(u_0) \exp \Upsilon_2(u_0).
\end{align*}

Equivariance of \( \Upsilon_1 \) gives \( \Upsilon_1(\varphi_0(u_0)) = \Upsilon_1(u_0 g_0) = -\Upsilon_1(u_0) \) in the fiber
over \( x \) for \( g_0 \) giving \(-\text{id} \) on \( \mathfrak{g}_{-1} \) and we can rewrite the above expression as
\begin{align*}
\sigma(\varphi_0(u_0)) \exp(-\frac{1}{2} \Upsilon_1(\varphi_0(u_0))) \exp \Upsilon_2(u_0) &= \hat{\sigma}(\varphi_0(u_0)) \exp \Upsilon_2(u_0)
\end{align*}
in the fiber over \( x \). Thus \( \hat{\sigma} \) is the required Weyl structure. \( \square \)

Let us call each Weyl structure \( \hat{\sigma} \) satisfying the condition in the Lemma
an **almost \( s_x \)-invariant Weyl structure** at \( x \). All almost \( s_x \)-invariant Weyl
structures form a family of Weyl structures which is parametrized over \( \mathfrak{g}_2 \)
at \( x \). Really, all Weyl structures inducing the same isomorphism \( T_x M \simeq \)
\( \text{gr}(T_x M) \) as \( \hat{\sigma} \) are of the form \( \hat{\sigma} \exp F_1(u_0) \exp F_2(u_0) \) for arbitrary functions
\( F_2 : \mathfrak{g}_0 \to \mathfrak{g}_2 \) and \( F_1 : \mathfrak{g}_0 \to \mathfrak{g}_1 \) where \( F_1(u_0) = 0 \) in the fiber over \( x \), see \( \ref{prop:involutive} \).

Let us finally describe the involutivity of our symmetries in the language
of Weyl structures.

**Proposition.** On a parabolic contact geometry with a symmetry \( s_x \) at \( x \),
the following facts are equivalent:
(a) the symmetry $s_x$ is involutive,
(b) there exists a Weyl structure $\sigma$ such that $\varphi(\hat{\sigma}(u)) = \sigma(\hat{\varphi}(u))$ holds in the fiber over $x$,
(c) there exists a Weyl structure $\sigma_u$ such that $\varphi(\sigma_u(u_0)) = \sigma_u(\varphi_0(u_0))$ holds over some neighborhood of $x$.

Proof. (a) $\Rightarrow$ (b) Let $\hat{\sigma}$ be an arbitrary almost $s_x$–invariant Weyl structure. The Lemma 2.2 says that the involutivity implies vanishing of $\Upsilon$ in the fiber over $x$. Thus if $s_x$ is involutive, the almost $s_x$–invariant Weyl structure $\hat{\sigma}$ has to satisfy (b).

(b) $\Rightarrow$ (c) Let $\hat{\sigma}$ be an arbitrary Weyl structure satisfying $\varphi(\hat{\sigma}(u_0)) = \hat{\sigma}(\varphi_0(u_0))$ in the fiber over $x$. Consider the normal Weyl structure $\sigma_u$ such that $\sigma_u(u_0) = \hat{\sigma}(u_0)$ for $\varphi_0(u_0) = x$. The condition of the normality prescribes $\sigma_u$ uniquely on a normal neighborhood of $x \in M$, see 1.2 for definition. But then, because $\varphi(\sigma_u(u_0)) = \sigma_u(\varphi_0(u_0))$ holds in the fiber over $x$, it has to hold over some normal neighborhood of $x$ and $\sigma_u$ satisfies (c).

(c) $\Rightarrow$ (a) Consider an arbitrary Weyl structure $\sigma$ satisfying (c). This can be equivalently written as $\varphi^{-1}(\sigma(\varphi_0(u_0))) = \sigma(u_0)$ which means that the corresponding Weyl connection is invariant with respect to $s_x$. Since the isomorphism $T_x M \simeq \text{gr}(T_x M)$ reflects the decomposition of $T_x M$ into $\pm 1$–eigenspaces, we can describe $s_x$ on a neighborhood of $x$ nicely via geodesics of the invariant connection. Indeed, each vector $(\xi_{-2}(x), \xi_{-1}(x)) \in T_x M$ determines uniquely a geodesic at $x$, and the symmetry $s_x$ maps it on a geodesic at $x$, which is uniquely determined by a vector $(\xi_{-2}(x), -\xi_{-1}(x))$. This describes $s_x$ on a neighborhood of $x$ and one can see directly that it has to be involutive.

Let us call each Weyl structure satisfying the condition (b) of the Proposition $s_x$–invariant Weyl structure at $x$ and each Weyl structure satisfying the condition (c) of the Proposition $s_x$–invariant Weyl structure on a neighborhood of $x$.

3. Symmetries of homogeneous models

In this section, we focus on homogeneous models, which are simplest examples of parabolic contact geometries. We describe explicitly their symmetries and we give some concrete examples of homogeneous symmetric geometries.

3.1. Description of symmetries. Let $(G \rightarrow G/P, \omega_G)$ be a homogeneous model of a parabolic contact geometry of type $(G, P)$. It is well known that all automorphisms of the homogeneous model are just left multiplications by elements of $G$ and an analog of the Liouville theorem states that any local automorphism can be uniquely extended to a global one, see [7, 2]. Thus if the homogeneous model is locally symmetric, then it is symmetric. Moreover, because $G$ acts transitively on $G/P$, it suffices to find a symmetry at the origin to decide whether the homogeneous model is symmetric.

Proposition. All symmetries of the homogeneous model of a parabolic contact geometry centered at the origin $o = eP$ are given by left multiplications by elements $g_0 \exp Z_1 \exp Z_2 \in P$, where $Z_1 \in \mathfrak{g}_1$ and $Z_2 \in \mathfrak{g}_2$ are arbitrary
and \(g_0 \in G_0\) is such that \(\text{Ad}_{g_0} = -\text{id}\) on \(\mathfrak{g}_{-1}\). In particular, if there is one symmetry at a point, then there is an infinite amount of them.

Proof. For homogeneous models, \(T^{-1}(G/P) = G \times_P \mathfrak{g}^{-1}/\mathfrak{p}\). Then we can write each tangent vector \(\xi(o) \in T^{-1}_{o}(G/P)\) as \(\xi(o) = [e, X]\) for suitable \(X \in \mathfrak{g}^{-1}/\mathfrak{p}\). Since automorphisms of the homogeneous model are left multiplications \(\lambda_g\) by elements \(g \in G\), all symmetries at the origin are exactly left multiplications \(\lambda_g\) satisfying \(\lambda_g(o) = o\) and \(T_o \lambda_g \cdot \xi(o) = -\xi(o)\) for all contact vectors \(\xi(o)\). The first condition is equivalent to the fact that \(g \in P\). Then \(g\) can be written as \(g = g_0 \exp Z_1 \exp Z_2\) and the second condition means that

\[
T_o \lambda_g \exp Z_1 \exp Z_2 \cdot [e, X] = [g_0 \exp Z_1 \exp Z_2, X] = [e, \text{Ad}_{g_0}^{-1} \exp Z_1 \exp Z_2 \cdot X]
\]

and \(-\xi(o) = [e, -X]\) coincide for each \(X \in \mathfrak{g}^{-1}/\mathfrak{p}\). Thus we look for elements \(g \in P\) such that \(\text{Ad}_{\exp(-Z_2)} \text{Ad}_{\exp(-Z_1)} \text{Ad}_{g_0}^{-1} X = -X\) for all \(X \in \mathfrak{g}^{-1}/\mathfrak{p}\) and the rest follows immediately from observations in Section 2.2.

Let us finally discuss involutivity of these symmetries. The symmetry \(g_0 \exp Z_1 \exp Z_2\) is involutive if and only if the element \((g_0 \exp Z_1 \exp Z_2)^2\) induces identity on \(G/P\) and effectivity says that it has to be equal to \(e\). We have

\[
\begin{align*}
g_0 \exp Z_1 \exp Z_2 g_0 \exp Z_1 \exp Z_2 &= e \exp 2Z_2.
\end{align*}
\]

Thus involutive symmetries at the origin are left multiplications by elements \(g_0 \exp Z_1\) where \(g_0\) and \(Z_1\) are as above and \(Z_2\) has to be equal to zero. If \(Z_2\) is non-zero, then the symmetry is not involutive. In particular, there exist non-involutive symmetries on homogeneous models.

Clearly, if \(g\) induces (involutive) symmetry at the origin \(o = eP\), then \(hgh^{-1}\) induces (involutive) symmetry at the point \(hP\).

3.2. Examples. Let us introduce here some examples of parabolic contact structures and discuss symmetries on their homogeneous models, see [2] for detailed description.

Lagrangean contact structures. Let us start with \(\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})\), the split real form of \(\mathfrak{sl}(n+2, \mathbb{C})\), for \(n \geq 1\). This admits a contact grading which is given by the following decomposition into blocks of sizes 1, 1 and 1:

\[
\begin{pmatrix}
g_0 & g_1^L & g_2^L \\
g_0^L & g_0 & g_2^R \\
g_{-2} & g_{-1}^L & g_0
\end{pmatrix}.
\]

The splittings \(\mathfrak{g}_{\pm 1} = g_{\pm 1}^L \oplus g_{\pm 1}^R\) are \(g_0\)-invariant and \(g_L\) and \(g_R\) are isotropic for \([- , -] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}\). Let us choose \(G = PGL(n+2, \mathbb{R})\), the quotient of \(GL(n+2, \mathbb{R})\) by its center. Then \(P\) consists of classes of block upper triangular matrices and \(G_0\) of block diagonal matrices. In particular, \(G_0\) coincides by means of the \(\text{Ad}\)-action with the group of all automorphisms of graded Lie algebra \(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}\) which in addition preserve the decomposition \(\mathfrak{g}_{-1} = g_{-1}^L \oplus g_{-1}^R\). Thus for a parabolic contact geometry of type \((G, P)\), the underlying geometry consists of a contact distribution together with a fixed decomposition of the form \(T^{-1}M = L \oplus R\) into two subbundles (of rank \(n\)) each of which is isotropic with respect to \(L\). These geometries are known as Lagrangean contact structures. The homogeneous model is the flag manifold of lines in hyperplanes in \(\mathbb{R}^{n+2}\).
Let us now discuss symmetries at the origin of the homogeneous model. We look for an element \( g_0 \in G_0 \) such that \( \text{Ad}_{g_0}X = -X \) for each \( X \in \mathfrak{g}_{-1} \). Elementary matrix computation shows that there is a solution which is represented by the matrix of the form
\[
g_0 = \begin{pmatrix}
-1 & 0 & 0 \\
0 & E & 0 \\
0 & 0 & -1
\end{pmatrix},
\]
where \( E \) is the identity matrix, and thus the homogeneous model is symmetric. All symmetries at the origin are represented by matrices of the form
\[
\begin{pmatrix}
-1 & -V \gamma & E \\
0 & W & 0 \\
0 & 0 & -1
\end{pmatrix},
\]
where \( V, W \in \mathbb{R}^n \) and \( \gamma \in \mathbb{R} \) are arbitrary, and the involutive ones have to satisfy \( \gamma = -\frac{1}{2}V W \).

4. Curvature restrictions

In this section, we discuss restrictions on the curvature of a parabolic contact geometry, which are caused by the existence of a symmetry. We study the torsion of symmetric parabolic contact geometries in detail. We...
show that there are relations between the curvature of a symmetric geometry and involutivity of its symmetries.

4.1. **Torsion restrictions.** Let us work with normal parabolic contact geometries here. In fact, the normality assumption is only some technical restriction which plays no role, if we understand symmetries as morphisms of the underlying geometry, and this clearly is the most reasonable point of view. For such underlying geometry, there are various non–isomorphic parabolic geometries inducing this underlying structure and it can be proved that the normal one always exists, see [2]. Assuming normal geometry, we can discuss its components of harmonic curvature, which are easily computable and provide an information on the whole curvature of the parabolic geometry, see [1] and Appendix. Let us start with the harmonic torsion $\kappa^{(1)}$, which has its values in $g_{-1}^* \wedge g_{-1}^* \otimes g_{-1}$.

**Lemma.** If there is a symmetry $s_x$ at $x$ on a normal parabolic contact geometry, then $\kappa^{(1)}$ vanishes at $x$.

**Proof.** Let $\varphi$ be as usual. For $u \in p^{-1}(x)$ we have $\varphi(u) = u g_0 \exp Z$ for suitable $g_0 \in G_0$ and $Z \in p_+$. Then for each $X, Y \in g_{-1}$ we get

$$\kappa^{(1)}(\varphi(u))(X, Y) = \kappa^{(1)}(u g_0 \exp Z)(X, Y) = \exp(-Z) g_0^{-1} \cdot \kappa^{(1)}(X, Y)$$

where $\cdot$ denotes the induced $\text{Ad}$–action on $g_{-1}^* \wedge g_{-1}^* \otimes g_{-1}$. The action of $g_0^{-1}$ is of the form

$$g_0^{-1} \cdot \kappa^{(1)}(X, Y) = \text{Ad}_{g_0^{-1}}(\kappa^{(1)}(u)(\text{Ad}_{g_0} X, \text{Ad}_{g_0} Y)) =$$

$$-\kappa^{(1)}(u)(-X, -Y) = -\kappa^{(1)}(u)(X, Y)$$

since the element $g_0$ acts as $-\text{id}$ on $g_{-1}$, and the action of $\exp(-Z)$ is trivial. Because automorphisms preserve curvature, $-\kappa^{(1)}(X, Y)$ has to be equal to $\kappa^{(1)}(X, Y)$ in the fiber over $x$ and then it has to vanish at $x$. □

**Proposition.** Each normal symmetric parabolic contact geometry is torsion–free. Moreover, normal symmetric geometries have to be locally isomorphic to the homogeneous models of the same type.

**Proof.** Thanks to the regularity, the curvature satisfies $\kappa(u)(g^i, g^j) \subset g^{i+j+\ell}$ for all $u \in G$ and for some $\ell \geq 1$ in general. Moreover, it can be proved that the component of degree $\ell$ mapping $g_i \times g_j$ to $g_{i+j+\ell}$ corresponds to the component of $\kappa_H(u)$ of degree $\ell$, see [2]. The above Lemma shows that $\ell \geq 2$ for symmetric geometries. Moreover, if the component of degree 2 is non–zero, then the only possibility is that it maps $g_{-1} \times g_{-1}$ to $g_0$. Thus it has its values in $g_{-1}^* \wedge g_{-1}^* \otimes p$. It follows directly from the homogeneity reasons that components of degree $\geq 3$ have to have their values in this subbundle, too, and the geometry is torsion–free.

Finally, let us remind that vanishing of the harmonic curvature implies vanishing of the whole curvature, see Theorem [1.1]. This applies if $\kappa_H$ coincides with $\kappa^{(1)}$ which has to vanish for symmetric geometries, and they are locally isomorphic to homogeneous models. Now, it suffices to discuss components of harmonic curvature for concrete geometries, see Appendix. □
4.2. Obstructions to flatness and involutive symmetries. One can see from the discussion of the harmonic curvature that among all normal parabolic contact geometries, only

- contact projective structures,
- Lagrangean contact structures,
- partially integrable almost CR–structures of hypersurface type

can carry a symmetry at a point with non–zero harmonic curvature. For each such symmetric geometry, there is exactly one obstruction to being locally isomorphic to the homogeneous model of the same type. For three–dimensional almost CR–structures and three–dimensional Lagrangean contact structures, there is the harmonic curvature $\kappa^{(4)}$ valued in $\mathfrak{g}^{*}_{-1} \wedge \mathfrak{g}^{*}_{-2} \otimes \mathfrak{g}_{1}$. For the other ones, we have the harmonic curvature $\kappa^{(2)}$ valued in $\mathfrak{g}^{*}_{-1} \wedge \mathfrak{g}^{*}_{-1} \otimes \mathfrak{g}_{0}$.

Let us first focus on $\kappa^{(2)}$. Let $s_x$ be a symmetry at $x$ on a normal symmetric parabolic contact geometry and let $\varphi$ be as usual. For $u \in \rho^{-1}(x)$ and $X, Y \in \mathfrak{g}_{-1}$ we have

$$\kappa^{(2)}(\varphi(u))(X, Y) = \kappa^{(2)}(ug_0 \exp Z)(X, Y) = \exp(-Z)g_0^{-1} \cdot \kappa^{(2)}(u)(X, Y),$$

where $\cdot$ is the induced Ad–action on $\mathfrak{g}^{*}_{-1} \wedge \mathfrak{g}^{*}_{-1} \otimes \mathfrak{g}_{0}$. For the action of $g_0^{-1}$ we can write

$$g_0^{-1} \cdot \kappa^{(2)}(u)(X, Y) = \text{Ad}_{g_0^{-1}}(\kappa^{(2)}(u)(\text{Ad}_{g_0}X, \text{Ad}_{g_0}Y)) = \text{Ad}_{g_0^{-1}}(\kappa^{(2)}(u)(-X, -Y)) = \text{Ad}_{g_0^{-1}}(\kappa^{(2)}(u)(X, Y)).$$

Because $g_0$ is a subspace of $L(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \simeq \mathfrak{g}^{*}_{-1} \otimes \mathfrak{g}_{1}$, the element $g_0^{-1}$ has to act trivially on $g_0$ and thus on $\kappa^{(2)}(u)(X, Y)$ for each $X, Y$. Because also $\exp(-Z)$ acts trivially on $\kappa^{(2)}(u)$, we get no additional restriction. In fact, $\kappa^{(2)}$ is a tensor of type $\lambda^2 T^{-1}\kappa \otimes T^{-1} \kappa \otimes T^{-1} M$ which is invariant with respect to the symmetry $s_x$. Let us denote this tensor by $W$.

We try to differentiate $W$ with respect to various Weyl connections. We focus on connections corresponding to almost $s_x$–invariant Weyl structures, i.e. Weyl structures $\sigma$ satisfying $\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)$ for suitable $\Upsilon_2$ and $\Upsilon_1$ such that $\Upsilon_1 = 0$ at $x$, see [23] for details.

**Lemma.** On a symmetric normal parabolic contact geometry with a symmetry $s_x$ at $x$, let $\sigma$ be an arbitrary almost $s_x$–invariant Weyl structure and $\nabla^\sigma$ the corresponding Weyl connection. Then

(a) $\nabla^\sigma_\xi W = 0$ holds at $x$ for each $\xi$ from the contact distribution,

(b) $\{\Upsilon_2, \xi\} \cdot W = 0$ holds at $x$ for each $\xi$ such that $T_x s_x \xi(x) = \xi(x)$ and $\Upsilon_2$ is determined by $s_x$ at $x$.

**Proof.** In general, $\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)$ holds for each Weyl structure $\sigma$ and suitable $\Upsilon_1$ and $\Upsilon_2$, and this can be rewritten as

$$\varphi^{-1}(\sigma(\varphi_0(u_0))) = \sigma(u_0) \exp(-\Upsilon_1(u_0)) \exp(-\Upsilon_2(u_0))$$

or simply $\varphi^* \sigma = \sigma \exp(-\Upsilon_1) \exp(-\Upsilon_2)$. For corresponding Weyl connections we then have $\nabla^{\varphi^* \sigma} = \nabla^{\sigma} \exp(-\Upsilon_1) \exp(-\Upsilon_2)$ and if we apply this on $W$, we get

$$\nabla^{\varphi^* \sigma}_\xi W = \nabla^{\sigma}_\xi \exp(-\Upsilon_1) \exp(-\Upsilon_2) W$$

(6)
for each vector field \( \xi \). Because we suppose that \( \sigma \) is almost \( s_x \)-invariant Weyl structure, then moreover \( \Upsilon_1 \) equals to zero in the fiber over \( x \).

Let us discuss both sides of equation (6) at \( x \) in detail. We start with the left hand side. At the point \( x \) we have

\[
\nabla^\sigma_{\xi^*} W(\eta, \mu)(\nu) = (s_x^* \nabla^\sigma)_{\xi} W(\eta, \mu)(\nu) = \\
T_x s_x^{-1} \nabla^\sigma_{T_x s_x \xi} W(T_x s_x \eta, T_x s_x \mu)(T_x s_x \nu) = \\
(\xi 1) \nabla^\sigma_{T_x s_x \xi} W(\eta, \mu)(\nu) = \nabla^\sigma_{T_x s_x \xi} W(\eta, \mu)(\nu)
\]

for each \( \xi \in \mathfrak{X}(M) \) and \( \eta, \mu, \nu \in \Gamma(T^{-1}M) \) since \( T_x s_x \) gives \(-\text{id}\) on \( T_x^{-1}M \).

Now we focus on the right hand side of (6). We use the general formula for the change of Weyl connections, see formula (1) in 1.2. Because our Weyl structure \( \sigma \) satisfies \( \Upsilon_1 = 0 \) over \( x \), the right hand side of (6) simplifies to

\[
\nabla^\sigma_x \exp(-\Upsilon_1) \exp(-\Upsilon_2) W = \nabla^\sigma_x W + \{\Upsilon_2, \xi \} \cdot W
\]

in the fiber over \( x \). If we put the above observations together, we see that the equation (6) can be rewritten as

\[
\nabla^\sigma_{T_x s_x \xi} W = \nabla^\sigma_\xi W + \{\Upsilon_2, \xi \} \cdot W
\]

in the fiber over \( x \). Let us discuss some concrete choices of the vector \( \xi(x) \):

(a) Suppose \( \xi \) is contained in the contact distribution \( T^{-1}M \). In particular, \( \xi(x) = \xi_{-1}(x) \). Then \( T_x s_x \xi(x) = -\xi(x) \) and the equation (7) simplifies to

\[
\nabla^\sigma_\xi W = \nabla^\sigma_\xi W
\]

at \( x \). The algebraic bracket simply vanishes because \( \xi_{-2}(x) = 0 \) in this case. This implies \( \nabla^\sigma_\xi W = 0 \) at \( x \) for each \( \xi \) from the contact distribution.

(b) Let us now suppose that \( T_x s_x \xi(x) = \xi(x) \). Such vectors exist and for an almost \( s_x \)-invariant Weyl structure \( \sigma \), these are exactly the vectors satisfying \( \xi = \xi_{-2} \) at \( x \), see 2.3. Then the equation (7) simplifies to

\[
\nabla^\sigma_\xi W = \nabla^\sigma_\xi W + \{\Upsilon_2, \xi \} \cdot W
\]

and we get the restriction \( \{\Upsilon_2, \xi \} \cdot W = 0 \) in the fiber over \( x \).

The part (a) is not surprising. Actually, \( \nabla^\sigma W \) defines a tensor of type \( T^{-1}M \otimes \wedge^2 T^{-1}M \otimes T^{-1}s_x T^{-1}M \), i.e. of odd degree, which is invariant with respect to \( s_x \). The consequences of the part (b) we formulate in the following statement.

**Proposition.** On a symmetric normal parabolic contact geometry with a symmetry \( s_x \) at \( x \), suppose that \( W \) is non–zero at \( x \). Then each almost \( s_x \)-invariant Weyl structure has to be \( s_x \)-invariant.

**Proof.** With the above notation, we will discuss the formula (6) from the Lemma for an almost \( s_x \)-invariant Weyl structure \( \sigma \) and for some vector field \( \xi \) such that \( \xi(x) \) is non–zero and satisfies \( T_x s_x \xi(x) = \xi(x) \). Let us point out that such vectors exist and satisfy \( \xi(x) = \xi_{-2}(x) \) in the isomorphism \( TM \cong \text{gr}(TM) \) given by \( \sigma \), see 2.3. In some concrete frame \( u = \sigma(u_0) \) from the fiber over \( x \), we can write \( \xi(x) = [u, X] \) for suitable non–zero \( X \in \mathfrak{g}_{-2} \). Similarly, \( \Upsilon_2(x) = [u, Z] \) for suitable \( Z \in \mathfrak{g}_2 \) and the algebraic bracket \( \{\Upsilon_2, \xi \} \) corresponds to \( [u, [Z, X]] \) at \( x \). Moreover, if \( Z \neq 0 \), we can choose \( X \) such that \( [Z, X] \) is exactly the grading element \( E \), see 2.1. Then \( \{\Upsilon_2, \xi \} \)
corresponds to the grading section $E(x)$, see $[12]$. In particular, it acts by the algebraic action • on $W$ by its homogeneity. Because $W$ has homogeneity two, we get

$$\{\Upsilon_2, \xi\} \cdot W = 2W$$

in the fiber over $x$ and we have a restriction of the form $2W = 0$ at $x$. This is a contradiction with the assumption that $W$ is non-zero at $x$. Thus the only possibility is that $Z = 0$ and thus $\Upsilon_2$ has to vanish at $x$. But this means that almost $s_x$-invariant Weyl structure is $s_x$-invariant, see $[23]$.

Before we proceed further, let us return to $\kappa^{(4)}$ valued in $g^{s-1}_{-1} \otimes g^{s-2}_{-2} \otimes g_1$. The discussion of $\kappa^{(4)}$ is parallel to the discussion of $\kappa^{(2)}$ and we summarize it very briefly. If $\kappa^{(4)}$ is non-zero, then it defines a tensor of even degree which has to be invariant with respect to the symmetry $s_x$. Really, for each $X \in g_{-1}$ and $V \in g_{-2}$ we have

$$\kappa^{(4)}(\phi(u))(X, V) = \kappa^{(4)}(u g_0 \exp Z)(X, V) = \exp(-Z) g_0^{-1} \cdot \kappa^{(4)}(u)(X, V) = Ad_{g_0}^{1}(\kappa^{(4)}(u)(Ad_{g_0} X, Ad_{g_0} V)) = -\kappa^{(4)}(u)(-X, V) = \kappa^{(4)}(u)(X, V).$$

Let us denote this tensor by $Y$. Again, we can differentiate $Y$ with respect to a Weyl connections corresponding to almost invariant Weyl structures. For an almost $s_x$-invariant Weyl structure $\sigma$ we have the equation

$$\nabla^{\varphi^*\sigma}_X Y = \nabla^{\sigma}_{\varphi^* X} \exp(-\Upsilon_1) \exp(-\Upsilon_2) Y$$

for each vector field $\xi$ and suitable $\Upsilon_1$ and $\Upsilon_2$ corresponding to $\sigma$, where $\Upsilon_1$ vanishes at $x$. The left hand side can be rewritten as

$$\nabla^{\varphi^*\sigma}_X Y(\eta, \mu) = (s_x^* \nabla^{\sigma}_{\varphi^* \xi}) Y(\eta, \mu) = T_x s_x^{-1} \nabla^{\sigma}_{T_x s_x \xi} Y(T_x s_x \eta, T_x s_x \mu)$$

$$= (-1)^2 \nabla^{\sigma}_{s_x \xi \eta, \mu} Y(\eta, \mu) = \nabla^{\sigma}_{s_x \xi \eta, \mu} Y(\eta, \mu)$$

for each $\xi \in \mathfrak{X}(M)$, $\eta \in \Gamma(T^{-1}M)$ and $\mu \in \mathfrak{X}(M)$ such that $\mu = \mu_{-2}$ via the isomorphism given by $\sigma$ at $x$. Really, $T_x s_x$ gives $-\text{id}$ on $T^{-1}_x M$ and $\nabla^{\sigma}_{s_x \xi \eta, \mu}(x) = \mu(x)$. Thus we get the restriction of the form

$$\nabla^{\varphi^*\sigma}_{T_x s_x \eta, \mu} Y = \nabla^{\varphi^*\sigma}_{\xi, \eta} Y + \{\Upsilon_2, \xi_{-2}\} \cdot Y$$

and we have $\nabla^{\varphi^*\sigma}_{\xi_{-1}} Y = 0$ and $\{\Upsilon_2, \xi_{-2}\} \cdot Y = 0$ in the fiber over $x$. Because $Y$ is of homogeneity four, the same arguments as in the proof of the above Proposition shows that $\Upsilon_2$ vanishes at $x$ and then each $s_x$-invariant Weyl structure has to be $s_x$-invariant. All these observations together with the last Proposition give us the following statement.

**Theorem.** On a symmetric normal parabolic contact geometry with a symmetry $s_x$ at $x$, suppose that its harmonic curvature is non-zero at $x$. Then $s_x$ is involutive.

**Proof.** In such case, almost $s_x$-invariant Weyl structures have to be $s_x$-invariant and the rest follows from $[23]$.

**Corollary.** On a symmetric normal parabolic contact geometry with a symmetry $s_x$ at $x$, suppose that its harmonic curvature is non-zero at $x$. Then there are admissible affine connections which are invariant with respect to the symmetry $s_x$; We take Weyl connections corresponding to $s_x$-invariant Weyl structures.
5. Uniqueness of Symmetries

We discuss here the question how many different symmetries can exist at a point with non–zero curvature. We first give one general restriction and then some consequences for concrete geometries.

5.1. Algebraic restriction. Let $s_x$ and $\bar{s}_x$ are two different symmetries at $x$ on a symmetric normal parabolic contact geometry with non–zero harmonic curvature at $x$ and denote by $\varphi$ and $\bar{\varphi}$ corresponding automorphisms of the parabolic geometry. Clearly, $s_x \neq \bar{s}_x$ if and only if $\varphi \neq \bar{\varphi}$. Then symmetries $s_x$ and $\bar{s}_x$ are involutive and there exist $s_x$–invariant and $\bar{s}_x$–invariant Weyl structures, see \[2\] and \[3\].

**Lemma.** For each two different involutive symmetries $s_x$ and $\bar{s}_x$ at $x$ on a symmetric parabolic contact geometry, $s_x$–invariant and $\bar{s}_x$–invariant Weyl structures form two disjoint families of Weyl structures.

**Proof.** Suppose there is a Weyl structure $\sigma$ which is $s_x$–invariant and $\bar{s}_x$–invariant at $x$, i.e. $\varphi(\sigma(u_0)) = \sigma(\varphi(u_0))$ and simultaneously $\bar{\varphi}(\sigma(u_0)) = \sigma(\bar{\varphi}(u_0))$ in the fiber over $x$. Then, the corresponding Weyl connection $\nabla^\sigma$ is invariant with respect to both symmetries $s_x$ and $\bar{s}_x$. But similarly as in the last part of the proof of Proposition \[2\] the connection $\nabla^\sigma$ determines uniquely the symmetry via behavior of its geodesics at $x$. Consequently, $s_x = \bar{s}_x$ on a neighborhood of $x$. □

Let $\sigma$ be $s_x$–invariant Weyl structure and let $\bar{\sigma}$ be $\bar{s}_x$–invariant Weyl structure. Then $\bar{\sigma} = \sigma \exp \Upsilon_1 \exp \Upsilon_2$ holds for suitable $\Upsilon_1 : G_0 \to g_1$ and $\Upsilon_2 : G_0 \to g_2$. The last Lemma says that $\Upsilon_1$ has to be non–zero at $x$.

**Proposition.** Suppose there are two different involutive symmetries at $x$ on a symmetric normal parabolic contact geometry and let $\sigma$ and $\bar{\sigma}$ be corresponding invariant Weyl structures. For all $\xi$ from the contact distribution, the bracket $\{ \Upsilon_1, \xi \}$ acts trivially by the algebraic action on $W$ or $\bar{\Upsilon}$, respectively, at $x$.

**Proof.** Let us start with $W$. Let $\xi$ be an arbitrary vector field from the contact distribution, thus $\xi = \xi_{-1}$ for each Weyl structure. The Lemma \[1\] gives $\nabla^\sigma_{\xi_{-1}} W = 0$ and $\nabla^{\bar{\sigma}}_{\xi_{-1}} W = 0$ at $x$. Simultaneously, we have $\sigma = \sigma \exp \Upsilon_1 \exp \Upsilon_2$ and the formula (1) from \[1\] gives

$$\nabla^\sigma_{\xi_{-1}} W = \nabla^{\bar{\sigma}}_{\xi_{-1}} W + \{\xi_{-1}, \Upsilon_1 \} \cdot W$$

at $x$, since $\xi_{-2}(x) = 0$. Because both covariant derivatives vanish at $x$, we get the restriction of the form $\{\xi_{-1}, \Upsilon_1 \} \cdot W = 0$ at $x$ for each $\xi$ from the contact distribution. One can see from \[2\] that the same line of arguments works for $\bar{\Upsilon}$ and we get the restriction of the form $\{\xi_{-1}, \Upsilon_1 \} \cdot \bar{\Upsilon} = 0$ at $x$ for each $\xi$ from the contact distribution. □

**Remark.** Let us again point out that the existence of a non–involutive symmetry at $x$ causes vanishing of the harmonic curvature at $x$, see \[2\].

5.2. Examples. Let us now discuss the above restrictions for concrete types of geometries. The key point is to find sufficiently nice $\xi$ such that the action of the above algebraic bracket is easily understandable.
Lagrangian contact structures. Let us first point out that we use here the notation from [32]. The decomposition of the contact distribution into two isotropic subbundles $T^{-1}M = L \oplus R$ can be interpreted as a product structure on $T^{-1}M$, which an operator $J : T^{-1}M \to T^{-1}M$ satisfying $J^2 = \text{id}$. The subbundles $L$ and $R$ are simply eigenspaces of $J$. The Levi bracket $\mathcal{L} : T^{-1}M \times T^{-1}M \to TM/T^{-1}M$ is non-degenerate antisymmetric bilinear map, and then, $\mathcal{L}(-, J-) = \text{non-degenerate symmetric map which defines a conformal class of pseudometrics on } T^{-1}M$ of signature $(n, n)$. We denote the class by $g$. Each pseudometric is then given by the choice the identification $TM/T^{-1}M \simeq \mathbb{R}$. In particular, the question whether $g(\xi, \eta)$ equals to zero for some $\xi, \eta \in T^{-1}M$ makes sense, because the answer does not depend on the choice of the metric from the class.

Proposition. Suppose there are two different involutive symmetries at $x$ on a symmetric normal Lagrangian contact structure and denote by $\sigma$ and $\exp \Upsilon_1 \exp \Upsilon_2$ corresponding invariant Weyl structures. Identify $\Upsilon_1$ with its image in $T^{-1}M = L \oplus R$ via an isomorphism given by a metric from $g$ and denote by $\Upsilon^R_1$ and $\Upsilon^L_1$ corresponding components in $L$ and $R$. If $g(\Upsilon^R_1, \Upsilon^L_1) \neq 0$ at $x$, then the harmonic curvature vanishes at $x$.

Proof. We discuss the restriction from the Proposition 5.1 for Lagrangian contact structures in detail. Let us write $\Upsilon_1(x) = [u, Z]$ for suitable $Z = \left(\begin{array}{ccc} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \in \mathfrak{g}_1$, which has to be non-zero, see Lemma 5.1. Choose $\xi_1 \in \Gamma(T^{-1}M)$ such that $\xi_1(x) = [u, X]$ for $X$ of the form $X = \left(\begin{array}{ccc} 0 & 0 & 0 \\ T & 0 & 0 \\ 0 & S & 0 \end{array}\right) \in \mathfrak{g}_{-1}$. The bracket $\{\xi_1, \Upsilon_1\}$ then corresponds to $[u, [X, Z]]$ at $x$, where 

$$[X, Z] = \left(\begin{array}{ccc} -ST & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & ST \end{array}\right) \in \mathfrak{g}_0.$$ 

It is easy to verify that with this choice, $[X, Z]$ is simply a grading element multiplied by a non-zero number $-ST$. Then the bracket $\{\xi_1, \Upsilon_1\}$ is a non-zero multiple of the grading section $E(x)$. Via the identification given by the metric from $g$, the components $S$ and $T$ correspond to components $\Upsilon^R_1$ and $\Upsilon^L_1$ of $\Upsilon_1$ in subbundles $R$ and $L$ at $x$ and the fact that $ST \neq 0$ means that $g(\Upsilon^R_1, \Upsilon^L_1) \neq 0$ at $x$. Because the grading section acts on $W$ by its homogeneity, $\{\xi_1, \Upsilon_1\}$ acts trivially on $W$ if and only $W$ vanishes at $x$. Clearly, the same arguments work for $Y$. \□

Partially integrable almost CR–structures. Let us first point out that we use here the notation from [32]. Moreover, suppose that the geometry is oriented and then, we can speak about signature of the structure. Using the complex structure $J$ given on $T^{-1}M$, we can define a non–degenerate symmetric mapping $\mathcal{L}(-, J-)$, which defines a conformal class of pseudometrics on $T^{-1}M$. The signature is given by the signature of the structure. Let us denote the class by $g$. Each pseudometric from the class is given by the choice of the identification $TM/T^{-1}M \simeq \mathbb{R}$. In particular, the question
whether \(g(\xi, \xi) \neq 0\) for \(\xi \in T^{-1}M\) makes sense, because the answer does not depend on the choice of the pseudometric from the class.

**Proposition.** Suppose there are two different involutive symmetries at \(x\) on a symmetric normal partially integrable almost CR–structure and denote by \(\sigma\) and \(\exp \Upsilon_1 \exp \Upsilon_2\) corresponding invariant Weyl structures. Identify \(\Upsilon_1\) with its image in \(T^{-1}M\) via an isomorphism given by a metric from \(g\). If \(g(\Upsilon_1, \Upsilon_1) \neq 0\) at \(x\), i.e. if the length of \(\Upsilon_1\) is non–zero at \(x\), then the harmonic curvature vanishes at \(x\).

**Proof.** We discuss the restriction from the Proposition 5.1 for CR–structures in detail. Let us write \(\Upsilon_1(x) = [u, Z]\) for suitable \(Z = \left( \begin{array}{ccc} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & S & 0 \end{array} \right)\) which has to be non–zero, see Lemma 5.1. Choose \(\xi_1 \in T^{-1}M\) such that \(\xi_1(x) = [u, X]\) for \(X\) of the form

\[
X = \left( \begin{array}{ccc} 0 & 0 & 0 \\ S & 0 & 0 \\ 0 & -S & 0 \end{array} \right) \in \mathfrak{g}_1.
\]

The bracket \(\{\xi_1, \Upsilon_1\}\) then corresponds to \([u, [X, Z]]\) at \(x\), where

\[
\left( \begin{array}{ccc} -SIS^* & 0 & 0 \\ 0 & 0 & SIS^* \end{array} \right) \in \mathfrak{g}_0.
\]

It is easy to verify that with this choice, \([X, Z]\) is simply a grading element multiplied by a non–zero number \(-SIS^*\). Then the bracket \(\{\xi_1, \Upsilon_1\}\) is a non–zero multiple of the grading section \(E(x)\). Using the identification \(T^{-1}M \simeq T^{-1}M\) given by a metric from \(g\), \(-SIS^*\) corresponds to \(g(\Upsilon_1, \Upsilon_1)\) and \(SIS^* \neq 0\) means that \(g(\Upsilon_1, \Upsilon_1) \neq 0\). Because the grading section acts on \(W\) by its homogeneity, \(\{\xi_1, \Upsilon_1\}\) acts trivially on \(W\) if and only \(W\) vanishes at \(x\). Clearly, the same arguments work for \(Y\).

**Corollary.** Suppose there are two different involutive symmetries at \(x\) on a symmetric normal strictly pseudoconvex partially integrable almost CR–structure. Then the harmonic curvature vanishes at \(x\).

**Appendix: Contact gradings and corresponding geometries**

Let us sketch here briefly a classification of contact gradings of real semisimple Lie algebras. There is the well know classification of all (complex) semisimple Lie algebras in the language Dynkin diagrams and description of all their real forms in the language of Satake diagrams, see \([2, 8]\). It can be proved that if a Lie algebra admits a contact grading, then it has to be simple. It turns out that except \(\mathfrak{sl}(2, \mathbb{R})\), \(\mathfrak{sl}(n, \mathbb{H})\), \(\mathfrak{so}(n - 1, 1)\), \(\mathfrak{sp}(p, q)\) and some real forms of \(E_6\) and \(F_4\), any non–compact non–complex real simple Lie algebra admits a unique real contact grading, see \([2]\).

Let us start with real classical Lie algebras, i.e. real forms of Lie algebras of type \(A_\ell\), \(B_\ell\), \(C_\ell\) and \(D_\ell\). In the first row of the following table, we indicate a real simple Lie algebra which admits a contact grading. In the second row we specify the geometry, which corresponds to the unique contact grading and in the last row we write its components of harmonic curvature.
| $\mathfrak{sl}(3, \mathbb{R})$ | contact geometry | components of $\kappa_H$ |
|--------------------------|-----------------|--------------------------|
| Lagrangean contact structures of dimension 3 | $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$ | |
| $\mathfrak{sl}(n+2, \mathbb{R})$ for $n \geq 2$ | Lagrangean contact structures of dimension $2n+1$ | $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$ |
| $\mathfrak{su}(2,1)$ and $\mathfrak{su}(1,2)$ | partially integrable almost CR structures of dimension 3 | $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$ |
| $\mathfrak{su}(p+1, q+1)$ for $p+q \geq 2$ | partially integrable almost CR structures of dimension $2p+2q+1$ | $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$ |
| $\mathfrak{so}(p+2, q+2)$ for $p+q \neq 4$ | Lie contact structures of dimension $2p+2q+1$ | $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$ |
| $\mathfrak{so}(p+2, q+2)$ for $p+q = 4$ | Lie contact structures of dimension 9 | $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$ |
| $\mathfrak{sp}(n+2)$ for $n \geq 1$ | contact projective structures | $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_0$ |

Let us also give a brief overview of contact gradings corresponding to exotic Lie algebras. For types $G_2$ and $F_4$, there is exactly one real algebra admitting contact grading, the split real form. For $E_6$, there are three real forms which admit a contact grading, the split form and two $\mathfrak{su}$–algebras. For $E_7$, there are three different real forms and for $E_8$, there are two different real forms admitting a contact grading. The description of corresponding geometries can be found in [2]. All these geometries have harmonic curvatures only of type $\kappa^{(1)}$ valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$.

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