Effect of bias in a reaction diffusion system in two dimensions

Pratik Mullick and Parongama Sen

1Department of Physics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata 700009, India.

We consider a single species reaction diffusion system on a two dimensional lattice where the particles $A$ are biased to move towards their nearest neighbours and annihilate as they meet: $A + A \rightarrow \emptyset$. Allowing the bias to take both negative and positive values parametrically, any nonzero bias is seen to drastically affect the behaviour of the system compared to the unbiased (simple diffusive) case. For positive bias, a finite number of dimers, which are isolated pairs of particles occurring as nearest neighbours, exist while for negative bias, a finite density of particles survive. Both the quantities vanish in a power law manner close to the diffusive limit with different exponents. In addition, a discontinuity is observed at the fully positive bias limit. The persistence behaviour is also analysed for the system.

I. INTRODUCTION

Non-equilibrium systems of diffusing particles, which undergo reactions such as pairwise annihilation is a much studied problem in recent times [1]. The particles may represent molecules, biological entities, or even abstract quantities like opinions in societies or market commodities and such systems are widely used to describe the pattern-formation phenomena in a variety of biological, chemical and physical systems. In a lattice representation of the simple single species problem, the lattice sites are filled with a particle at time $t = 0$ with a certain probability. At each time step, the particles are allowed to move to a nearest neighbor site. The moves may take place parallelly or asynchronously. In the pure diffusive case, no preferred direction for the jump is assigned. The reaction takes place only when a certain number $k$ of the particles: $kA \rightarrow lA$ with $l < k$. Annihilating unbiased diffusive particles or random particles with $k = 2$ and $l = 0$ mimic the dynamics of voter models and the Glauber-Ising model in one dimension while coalescing random particles map into the voter model in all dimensions. Such systems have been studied in one dimension [1–8] as well as in higher dimensions [13–20]. The steady state of the process is rather simple. The number of particles left in the system at steady state will depend on the dynamics. For asynchronous dynamics with $k = 2, l = 0$, i.e., $A + A \rightarrow \emptyset$, the number of particles in the steady state will be zero or one for initial states having even or odd particles respectively. For parallel updates, this number is again one or zero but it does not necessarily depend on the initial state as $k \geq 2, l = 0$ (if more than one particle converge on a site, they are all killed; $k$ does not have a fixed value). The point of interest in all these analyses is how the system approaches the steady state. In particular, one intends to know how the number of particles decays with time.

Another quantity of interest in this context is the persistence probability $P(t)$, which accounts for the survival of a field up to time $t$. In many cases, persistence probability decays as a power law, $P(t) \sim t^{-\theta}$, with the persistence exponent $\theta$ unrelated to other static and dynamic exponents [21]. For the random walk problem or reaction diffusion systems, $P(t)$ is defined as the probability that a site in the lattice remains unvisited till $t$. In one dimension, using asynchronous dynamics, this system maps to the Ising model with Glauber dynamics at zero temperature. For spin models, the persistence probability is defined as the probability that a spin has not changed till time $t$. The persistence exponent $\theta = 3/8$ has been exactly obtained [22] in one dimension with asynchronous updates. For the two dimensional unbiased annihilating reaction diffusion system with asynchronous dynamics, $\theta = 1/2$, obtained using field theoretical renormalisation group method [18].

On introduction of a bias in the system, the dynamics can change drastically as was already found for the case when the particles obeying $A + A \rightarrow \emptyset$ show a bias to move towards their nearest neighbours in one dimension [22, 24]. The particular case when the updates were made asynchronously was considered so as to map to a previously proposed model for opinion dynamics [22].

In this paper, we consider the single species reaction diffusion system in two dimensions with a bias to move towards the nearest particle with parallel updates. The bias may be either positive or negative. The parallel dynamics with $A + A \rightarrow \emptyset$ effectively makes $k \geq 2; l = 0$ (in a two dimensional square lattice, $k$ can vary between 2 and 4 depending on the number of particles converging on the same site and not a pre-assigned constant). We are primarily interested in the time evolution of the density of particles and persistence probability. The persistence behaviour is expected to be different from [15] as parallel dynamics is used and it is of interest whether it is twice the value of that obtained using asynchronous dynamics as has been noted for the Ising Glauber model [20].

II. DYNAMICS AND SIMULATION METHODS

We consider two types of dynamical schemes in the simulation. In the first case, the particles have a bias to move towards their nearest neighbours with probability $\varepsilon$ and are diffusive otherwise (Case I). The unbiased case for $\varepsilon = 0$ has been considered earlier [15, 16, 19, 20]. In order to realise the case that the particles may strictly
avoid the nearest neighbour, one can introduce a second type of dynamics (Case II). Here the bias to move towards the nearest neighbour is \( \varepsilon' \) and with probability \( (1 - \varepsilon') \), they move to a direction away from the nearest neighbour. One can establish a relation between \( \varepsilon \) and \( \varepsilon' \) in the following way: in case I, the total probability of a particle to move one step towards its nearest particle is \( \varepsilon + \frac{1 - \varepsilon}{4} \), which for case II, is equal to \( \varepsilon' \). Therefore,

\[
\varepsilon + \frac{1 - \varepsilon}{4} = \varepsilon'
\]  

Putting \( \varepsilon' = 1/4 \) gives the diffusive (unbiased) limit in Case II, which, according to Eq. (1) correctly corresponds to the diffusive case \( \varepsilon = 0 \) for Case I. For the extreme bias limits, \( \varepsilon = \varepsilon' = 1 \), which is also obtained from Eq. (1).

In the simulation, one has to be careful as there may be more than one nearest particle. Also, in Case II, when the move towards the nearest particle is not chosen, one has to make sure that the actual move does not bring it closer to the nearest particle. This can happen when there are more than one particle which are at nearest distance and/or non-unique moves that brings it closer to the nearest particle. However, it may be expected that at long times when the density of particles becomes small due to annihilation, the nearest particle will be unique and also the possibility of non-unique moves bringing the particle closer to its nearest particle will become rare. As for larger system sizes the dynamics continues for a long time, it is expected that in the simulation, the above equality will hold in the thermodynamic limit.

Case II in principle includes Case I and allows both positive and negative bias as desired. However, Case I being easier to handle computationally, we perform simulation for Case I for \( 0 \leq \varepsilon \leq 1 \) and for Case II, mainly the region \( \varepsilon' \leq 0.25 \), which cannot be realised in Case I. The results for \( 0 \leq \varepsilon \leq 1 \) and \( \varepsilon' \leq 0.25 \) will constitute the entire regime of \( \varepsilon' \), i.e., Case II.

The studies were performed on \( L \times L \) square lattices with \( L \leq 256 \) for Case I and \( L \leq 128 \) for Case II (unless otherwise specified). Periodic boundary condition is used for the simulation. The dynamics is initiated with a randomly half filled lattice and the density of particles \( \rho(t) \) is scaled by the initial density such that \( \rho(0) = 1 \). We perform parallel updating of the particles, i.e., in one Monte Carlo Step (MCS) all the particles are allowed to move simultaneously. The annihilation takes place after the completion of one MCS. The simulations are performed up to a maximum time of \( 10^7 \) so that a steady state is obtained within the maximum number of iterations (exceptional cases have been discussed in detail in section IV). In the steady state the number of remaining particles attain a constant value. During the time evolution, if the number of remaining particles becomes 0 or 1, the simulations are terminated.

![Graph](image)

**FIG. 1:** \( \rho(t) \) multiplied by time \( t \) for \( \varepsilon = 0 \) as a function of time \( t \) for several system sizes. The data for \( L = 512 \) is fitted according to the function \( g(t) = 0.6795 + 0.6453 \ln(t) \).

### III. RESULTS FOR CASE I

The case where the particles move towards their nearest neighbours with probability \( \varepsilon \) and is diffusive otherwise is discussed in this section. If two or more particles are found on the same site after the completion of one MCS, all of them are annihilated. The limit \( \varepsilon = 0 \) corresponding to the purely diffusive case was considered earlier [15, 16, 17, 18, 19] and the variation of \( \rho(t) \), the particle density, was found to be

\[
\rho(t) = \frac{a + b \ln(t)}{t}.
\]  

In our simulation we recover the above variation with with \( a = 0.6795 \pm 0.00141 \) and \( b = 0.6453 \pm 0.00023 \) (Fig. 1) for \( L = 512 \), consistent with earlier estimates [27]. Fig. 1 shows a deviation from the above form beyond a value of \( \ln(t) \). This is because here \( \rho(t) \) attains a saturation value.

Eq. (2) shows that one can approximately regard the variation of \( \rho(t) \) as a power law decay. The dominating term at large times being proportional to \( \ln(t)/t \), the approximate value of the exponent would be somewhat less than 1. Fig. 2 shows that it is possible to collapse the data for different system sizes with appreciable accuracy with suitable scaling factors. Assuming that \( \rho(t) \) approximately decays as \( t^{-\mu} \), the data collapse occurs with the following scaling form:

\[
\rho(t, L) = L^{-\nu} f(t/L^z).
\]  

To regain the form \( \rho(t) \sim t^{-\mu} \) for large \( L \), the scaling function \( f(x) \) must behave as \( x^{-\mu} \) such that

\[
y = z \mu.
\]  

The exponents found were \( y \approx 2 \) and \( z \approx 2.25 \) for \( \varepsilon = 0 \), giving \( \mu \approx 0.9 \) (Fig. 2(a)).

As \( \varepsilon \) is made non-zero, three different regions emerge in the behavior of \( \rho(t) \). Initially there is a fast decay followed by a slower decay and at later stages, a saturation...
region appears (inset of Fig. 3). The intermediate region of slower decay is prominently observed for 0 < ε < 1, but vanishes at ε = 1 and 0. The form of ρ(t) for ε = 0 given by Eq. 4 is not seen to be followed for any nonzero value of ε. Nevertheless for ε = 1, an analysis similar to that made for ε = 0 can be attempted to obtain an effective exponent. We find a nice collapse for the data for different system sizes using the values y = 1.45 and z = 1.32 (Fig. 2(b)). Thus the effective decay exponent μ for ρ is ≈ 1.1 which is different from that of ε = 0. In fact the initial faster decay region for all ε > 0 appears to have an approximate power law behavior with a unique value of the exponent μ ≈ 1.1 (shown in inset of Fig. 3).

We next study ρsat, the saturation value of the density of the particles as a function of ε. In section I it was mentioned that one expects either zero or only one surviving particle at steady state for the unbiased case. This will make ρsat vanishingly small as the system size is increased. Indeed, for ε < 1, ρsat is found to be ≪ 1, vanishing as 1/L^2. However for ε = 1, ρsat is one order of magnitude higher than those for ε < 1 (inset of Fig. 3). This indicates that a much larger number of particles remain in the system at the steady state for ε = 1. Indeed, the total number of remaining particles ρsatL^2 shows an abrupt increase at ε = 1 as shown in Fig. 4 In fact for ε = 1, ρsatL^2 increases with system size while for ε ≠ 1, ρsatL^2 is fairly a constant.

At ε = 1, it can be easily understood that at long times when few particles are left, as two particles always approach each other, a situation will emerge when they may appear on neighboring sites with no other neighboring sites occupied. Fig. 4 shows the snapshots at different times for different values of ε and at large times for ε = 1 one notes that such pairs will survive indefinitely. This is because the subsequent dynamics will simply be a swapping of the positions of the two neighbouring particles. These two particles forms what we call a dimer. A dimer is defined to be a pair of particles occupying neighboring sites and both the particles do not have any other neighbour. As these particles will never get annihilated at ε = 1, it is expected that there will be a larger number of surviving particles. To probe this in more detail, we calculate the density of dimers in the system, D(t) for all ε values. D(t) is the number of dimers divided by L^2. The data shown in Fig. 5 clearly shows a larger value of D(t) for ε = 1 in the steady state compared to other ε values as expected. However, surprisingly, a non-zero value is found for other values of ε > 0 also. At ε = 0, the value of D(t) at large times becomes negligibly small (O(10^{-6})).

Studying the saturation values of the actual number of dimers, (D(t))L^2 (⟨...⟩) indicates a thermal average in the steady state against ε (Fig. 4(a)), we note that it has a nonzero finite value increasing with ε (independent of L for ε < 1) and a jump at ε = 1 (increasing with L). This shows that the number of surviving particles can be greater than 1 even for ε < 1, the reason being the formation of dimers. Close to ε = 0, L^2⟨D(t)⟩ shows
a power law behavior with \( \varepsilon \) of the form \( \sim \varepsilon^\kappa \) with \( \kappa = 1.67 \pm 0.04 \) (Fig. 5(b)).

For nonzero \( \varepsilon \) therefore, the steady state can have several possibilities: (i) Zero particles (ii) Only one particle (iii) Dimers (iv) Coexistence of dimers and isolated particles. For \( \varepsilon = 1 \), the last case will not arise. Let us, for the sake of argument, assume that it is possible to have a state comprising of a dimer and an isolated particle. The dimer will remain static in its position, while the isolated particle will travel to form a “trimmer”, which is unstable and will decay immediately into an isolated particle. When \( N_p(t) \), the number of particles present at time \( t \) is even, the maximum number of dimers that can be formed is \( N_p(t)/2 \). We hence define,

\[
d_r(t) = \frac{2D(t)L^2}{N_p(t)}
\]  

(5)
as the probability that a dimer is formed out of the remaining particles. We note that \( d_r(t) \to 1 \) in a finite time for \( \varepsilon = 1 \). Therefore, if an even number of particles survive for \( \varepsilon = 1 \), all of them will form dimers. Of course, states with either zero or a single surviving particle are also possible for \( \varepsilon = 1 \). These features have all been checked in the simulation.

To get an idea which kind of steady states are more prevalent, we calculate the fraction of cases \( C_{1p} \) for which the system is left with a single particle. \( C_{1p} \) is found to be nearly 0.5 for all \( \varepsilon \) (inset of Fig. 5(a)). We also calculate the probability \( \langle C_{1d} \rangle \) of getting at least one dimer which turns out to be an increasing function of \( \varepsilon \), showing a jump at \( \varepsilon = 1 \) (Fig. 7(a)). Although the probability is less than 0.5, still it is quite appreciable \( (O(10^{-1})) \).

Although \( P(t) \) was studied for the \( \varepsilon = 0 \) using asynchronous dynamics [18, 21], there does not exist any result for the persistence probability for the parallel dynamics to the best of our knowledge. To calculate the persistent probability \( P(t) \), the initially occupied sites are regarded as unvisited. For \( \varepsilon = 0 \) the persistence probability \( P(t) \) was calculated for a maximum size \( L = 512 \) and it shows a power law decay: \( P(t) \sim t^{-\theta} \) with \( \theta \approx 1 \) (inset of Fig. 8). This value of the persistence exponent obtained for parallel dynamics, is also, therefore approximately twice the value for asynchronous dynamics, as it was previously established for \( P(t) \) for the one dimensional Ising model 20. Finite size scaling analysis was performed using the scaling ansatz 28

\[
P(t, L)L^\alpha = g(t/L^z_p),
\]  

(6)
and a data collapse was obtained with \( \alpha = 2.15 \) and \( z_p = 2.27 \) such that \( \theta = \alpha/z_p = 0.95 \). The results are shown in Fig. 8. The value of \( z_p \) is close to \( z \) and we conjecture they are equal as there is expected to be a single timescale in the system. As \( \varepsilon \) is increased, \( P(t) \) deviates from the power law behavior and saturates to higher values. Since annihilations are enhanced for nonzero \( \varepsilon \), a higher saturation value is obtained as \( \varepsilon \) is increased.

FIG. 6: (a) Variation of \( \langle D(t) \rangle \) in the steady state multiplied by \( L^2 \) with \( \varepsilon \) for several system sizes. (b) Power law behavior of this quantity close to \( \varepsilon = 0 \).

FIG. 7: Variation of (a) \( \langle C_{1d} \rangle \) and \( C_{1p} \) with \( \varepsilon \) and that of (b) \( \langle C_{1d} \rangle \) and \( C_{1p} \) with \( \varepsilon' \).

FIG. 8: Data collapse of \( P(t) \) for \( \varepsilon = 0 \) using Eq. (6). Inset shows the unscaled data.
IV. RESULTS FOR CASE II

In the simulation for Case II dynamics, a particle is allowed to move one step towards its nearest particle with probability $\varepsilon'$ and with probability $1 - \varepsilon'$ it goes to one of the three remaining nearest neighboring positions. The annihilations take place in the usual manner. The region $0.25 < \varepsilon' < 1$ of Case II corresponds to $0 < \varepsilon < 1$ of Case I, where the particles have the tendency to approach their nearest particle on the lattice, i.e., the bias is positive.

We first check the validity of Eq. (1) using the Case II dynamics; specifically it is verified where the diffusive point lies. Here it should be mentioned that when the particle does not choose to move to a site that brings it closer to its nearest particle, it moves to one of the remaining three neighbouring sites. This implies that the fact that one of these movements may also take it closer to its nearest particle is ignored. Hence one can expect a validity of Eq. (1) only in the thermodynamic limit, as argued in section II.

As the annihilations become less probable for negative bias, the system takes a very long time to attain the steady state and in the simulations, for larger system sizes, the system may not attain the steady state within the maximum number of Monte Carlo steps. Nevertheless, some essential information may be extrapolated from results obtained in the nonequilibrium region and at the maximum time of iteration.

At $\varepsilon' = 0$, the particles are completely repulsive and even at very long times, one may expect a nonzero density of particles for small $\varepsilon'$. At the diffusion limit on the other hand, the number of remaining particles being either zero or 1, this density is zero. Fig. (a) shows that for $\varepsilon' < 0.25$, $\rho(t)$ does not attain a saturation value even at large times. We study the probability $C_{1p}$ of getting a single surviving particle at the maximum time of observation (inset of Fig. (b)) and find that $C_{1p}$ increases abruptly at a value of $\varepsilon'$ which approaches 0.25 as the system size increases. This is consistent with Eq. (11) that the diffusive point is at $\varepsilon' = 0.25$ (in the thermodynamic limit). We also study $(C_{1d})$, the probability to get at least one dimer and this also shows an abrupt increase at $\varepsilon' \approx 0.25$ (Fig. (b)).

From Fig. (a), it is clear that $\rho(t)$ does not follow any conventional behaviour like a power law decay for finite values of $t$ in the region $\varepsilon' < 0.25$. The behaviour of $\rho(t)$ as $t \to \infty$ as a function of $\varepsilon'$ can be extrapolated from its behaviour at different finite fixed values of time $t$ denoted by $t_{\text{fixed}}$. It is found that $\rho(t_{\text{fixed}}) = \gamma(\varepsilon' - \varepsilon)^{\beta}$ and as $t_{\text{fixed}}$ is increased the value of $\varepsilon'$ approaches 0.25 while the exponent $\beta$ is $\approx 2.0$ for all values of $t_{\text{fixed}}$. The fittings become more accurate as the system size is increased as $\varepsilon' \to \varepsilon'_{c}$. For the two largest values of $t_{\text{fixed}}$, $\gamma$ is also found to be fairly a constant. From this we conclude that for $t \to \infty$, $\rho$ vanishes with an exponent $\beta \approx 2$ at $\varepsilon' = 0.25$.

The persistence probability $P(t)$ for small $\varepsilon'$ values shows a fast decay and goes to zero. Above a certain value of $\varepsilon'$, $P(t)$ saturates to a non-zero value. From Fig. (b) we can see that the change in the nature of $P(t)$ happens around $\varepsilon' \approx 0.25$. This is consistent with the fact that in the small $\varepsilon'$ region the particles have a tendency to move away from from its nearest walker, thereby reducing the probability of annihilation and allowing the particles to roam around more. For $\varepsilon' > 0.25$, the particles annihilate at a faster rate, such that a large number of sites can remain unvisited.

V. DISCUSSIONS AND CONCLUDING REMARKS

In this paper we have studied the time evolution of a single species reaction diffusion system where the particles perform nearest neighbour hops on a lattice with a bias to move towards the nearest particle. The bias can be made both positive and negative (Case II) parametrically. The particles are annihilated if they are found on the same site after the completion of one Monte Carlo step. A diffusive motion occurs for individual particles when the parameter $\varepsilon' = 0.25$, above (below) this value the particles have attractive (repulsive)
motion.

The dynamics of the density of particles $\rho(t)$ is studied and as expected, for $\varepsilon' = 0.25$, it vanishes in the thermodynamic limit as the number of surviving particles is either one or zero. However, the behaviour of $\rho(t)$ as a function of $t$ is completely different for $\varepsilon' \neq 0.25$. An approximate power law fitting and finite size analysis can be done for $\varepsilon' = 0.25$ and $\varepsilon' = 1.0$, showing that the exponents are different. On the other hand, for $\varepsilon' < 0.25$, $\rho(t)$ remains finite and vanishes only at the diffusive limit. The variation occurs in in a power law manner: $\rho(t \to \infty) \propto (0.25 - \varepsilon')^2$ as extrapolated from the data.

Although $\rho(t)$ is still vanishingly small for $\varepsilon' > 0.25$, a discontinuity occurs at $\varepsilon' = 1$, the fully positively biased point where the particles invariably move towards the nearest particles. On probing deeper, we find a dimerisation taking place for any $\varepsilon' > 0.25$, the number of dimers as well as the probability shows a jump at $\varepsilon' = 1$. The variation of the number of dimers (although its density is vanishingly small) shows a continuous increase from zero at the diffusive point, $D(t)L^2 \propto (\varepsilon' - 0.25)^{1.07}$.

The diffusive behaviour therefore occurs at a single point of the parameter space. This is reminiscent of the generalised voter model in two dimensions where the voter model occurs at a single point of the two dimensional parameter space [20]. The diffusive point separates two regions marked by completely different features: to its left one has a nonzero value of $\rho(t)$ and to the right, nonzero number of dimers, both vanishing in a power law manner with different exponents. The fully positive bias point $\varepsilon' = 1$ shows a distinct discontinuity in terms of the number of dimers and their probability of occurrence, whereas for the fully negative bias case $\varepsilon' = 0$, no discontinuity could be detected. The persistence probability also shows different behaviour for $\varepsilon' < 0.25$ and $\varepsilon' > 0.25$.

It is tempting to interpret $\rho(t)$ as an order parameter as it vanishes continuously as $(0.25 - \varepsilon')^\beta$ and claim that a phase transition is taking place at the diffusive point. The number of dimers also vanish at this point coming from the other side of the parameter space. However, we do not find conventional finite size dependence in either $\rho(t)$ or $D(t)L^2$. There is also no signature of any diverging length/time scale; the time $\tau$ to reach the steady state could be computed for very small systems (Fig. 11) and it shows a ‘U’ shaped form with peaks occurring at two extreme points: $\varepsilon' = 0$ and very close to $\varepsilon' = 1$. Exactly at $\varepsilon' = 1$, this time shows a sudden drop showing the feature of discontinuity at $\varepsilon'$ once again.

Even though phase transitions are apparently not occurring here (presumably because of the absence of fluctuations), the regions $\varepsilon' < 0.25$ and $\varepsilon' > 0.25$ are of interest as they reveal non-conventional steady state structures compared to the much studied case corresponding to $\varepsilon' = 0.25$. We also obtain the estimates of a dynamical exponent $z$ which are considerably different for $\varepsilon' = 0.25$ and $\varepsilon' = 1$.

In general, asynchronous and parallel dynamics are expected to affect the dynamics differently: still, at least for the diffusive limit $\varepsilon' = 0.25$, the form of $\rho(t)$ is identical for both types of dynamics. However it was found that the persistence exponent for the parallel dynamics is twice of that of the asynchronous one, a result analogous to the case for Ising model in one dimension [20].

Acknowledgement: This work was inspired by a comment made by S. K. Das. Useful discussions with P. Ray and S. Biswas are acknowledged. P. Mullick thanks DST-INSPIRE (Sanction No. 2015/IF0673) for their financial support. P. Sen acknowledges SERB (Scheme no EMR/2016/005429, Government of India) for financial grant.

[1] See e.g. T. M. Liggett, Interacting Particle Systems, Springer-Verlag, New York, (1985); V. Privman, ed, Nonequilibrium Statistical Mechanics in One Dimension, Cambridge University Press, Cambridge (1997); P. L. Krapivsky, S. Redner and E. Ben-Naim, A Kinetic View of Statistical Physics, Cambridge University Press, Cambridge (2009) and the references therein.

[2] Z. Racz, Phys. Rev. Lett. 55, 1707 (1985).

[3] J. L. Spouge, Phys. Rev. Lett. 60, (1988) 871.

[4] J. G. Amar and F. Family, Phys. Rev. A 41, 3258 (1990).

[5] D. ben-Avraham, M. A. Burschka, and C. R. Doering, J. Stat. Phys. 60, 695 (1990).

[6] F. C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg, Ann. Phys. 230, 250 (1994).

[7] K. Krebs, M. P. Pfannmuller, B. Wehefritz and H. Hinrichsen, J. Stat. Phys. 78, 1429 (1995).

[8] J. E. Santos, G. M. Schutz and R. B. Stinchcombe, J. Chem. Phys. 105, 2399 (1996).

[9] G. M. Schutz, Z. Phys. B 104, 583 (1997).

[10] K. Sasaki and T. Nakagawa, J. Phys. Soc. Japan 69, 1341
[11] M. J. de Oliveira, Brazilian Journal of Physics 30, 128 (2000).
[12] D. Ben-Avraham and E. Brunet, J. Phys. A 38, 3247 (2005).
[13] K. Kang and S. Redner, Phys. Rev. A 30, 2833 (1984); 32, 435 (1985).
[14] G. Zumofen, A. Blumen and J. Klafter, J. Chem. Phys. 82, 3198 (1985).
[15] L. Peliti, J. Phys. A 19, L365 (1986).
[16] M. Droz and L. Sasvari, Phys. Rev. E 48, 2343 (1993).
[17] B. P. Lee, J. Phys. A 27, 2633 (1994).
[18] J. Cardy, J. Phys. A.: Math. Gen. 28, L19 (1995).
[19] J. L. Cardy and U. C. Tauber, J. Stat. Phys. 90, 1 (1998).
[20] G. Szabó and M. A. Santos, Phys. Rev. E 59, R2509 (1999).
[21] A. J. Bray, S. N. Majumdar, G. Schehr, Adv. Phys. 62, 225 (2013).
[22] B. Derrida, V. Hakim and V. Pasquier, Phys. Rev. Lett. 75, 751 (1995).
[23] S. Biswas, P. Sen and P. Ray, Journal of Physics: Conf. Series 297, 012003 (2011).
[24] P. Sen and P. Ray, Phys. Rev. E 92, 012109 (2015).
[25] S. Biswas and P. Sen, Phys. Rev. E 80, 027101 (2009).
[26] G. I. Menon, P. Ray and P. Shukla, Phys. Rev. E 64, 046102 (2001).
[27] In [20], Eq. (2) was written in terms of \( \log_{10} t \) instead of natural log. Making the necessary conversion, the value of \( b \) is found to be very close.
[28] G. Manoj and P. Ray, Phys. Rev. E 62, 7755 (2000).
[29] M. J. de Oliveira, J. F. F. Mendes and M. A. Santos, J. Phys. A: Math. Gen. 26, 2317 (1993).