LOCAL DECAY ESTIMATES

AVY SOFFER AND XIAOXU WU

Abstract. We give a proof of local decay estimates for Schrödinger-type equations, which is based on the knowledge of Asymptotic Completeness (AC). This approach extends to time dependent potential perturbations, as it does not rely on Resolvent Estimates or related methods. Global in time Strichartz estimates follow for quasi-periodic time-dependent potentials from our results.

Contents

1. Introduction 1
   1.1. Main results 2
   1.2. Applications 5
   1.3. Outline of the proof 5
   1.4. Challenges of the Analysis of Time-Dependent Hamiltonians 7
2. Phase Space Decompositions 7
   2.1. Incoming/outgoing waves 7
   2.2. Quasi-periodic evolution operators 9
3. The time-independent problem 10
   3.1. Compactness of \( C \) 11
   3.2. Decomposition of the Operator \( C \) 14
   3.3. Proof of Theorem 1.3 20
4. Time-dependent Problems 20
   4.1. Compactness of \( C(t) \) and its decomposition 20
   4.2. Properties of the operators \( C_r(t) \) 24
   4.3. Proof of Theorems 1.1 and 1.2 34
   4.4. Strichartz estimates 35
Appendix A. Proof of free estimates 36
References 40

1. Introduction

Local decay estimates are a priori estimates of the solutions of dispersive equations. It states that the solution (for an initial condition associated with scattering) decays at least in an integrable rate in time, in every compact region of space, for a dense set, in the scattering subspace (see Eq. (1.9)), of initial conditions. An equivalent statement is that the resolvent of the Hamiltonian of the dynamics is bounded on properly weighted \( L^2 \) space.

Such estimates play a crucial role in scattering theory, as they imply the existence and completeness of the Möller wave operators. Moreover, local decay and other propagation estimates have important applications beyond the proof of asymptotic completeness (AC). For example, they are essential in linear and nonlinear time-dependent resonance theory [24,25]. Notable consequences

Date: January 17, 2025.
of local decay include Strichartz estimates and the propagation of regularity for nonlinear dispersive equations [31]. These estimates provide insights into the rate of convergence with asymptotic dynamics.

Proving such estimates for an interacting system is indeed challenging, as we do not have direct application of the method of stationary phase. However, it is possible to prove asymptotic completeness (AC) without using local decay. This was first shown by Enss in 1978 [2]. The Enss method was applicable to both two-particle and three-particle scattering in quantum mechanics. However, proving AC for four or more particles required the use of local decay [20]. This was achieved using the Mourre estimate combined with Mourre’s method of differential inequalities [4, 15–17], or the method of propagation estimates developed by Sigal and Soffer [21]. See also [7, 9, 10, 22].

Recently, Liu and the first author introduced a general approach to proving asymptotic completeness (AC) [13, 14]. Their method applies to both linear and nonlinear dispersive equations and requires localization assumptions on the interaction terms. Specifically, their approach necessitates radial symmetry for nonlinear equations.

Subsequently, the authors improved Liu and Soffer’s method by constructing the free channel wave operator in an adapted way [27–29]. This enhanced method does not require radial symmetry or localization assumptions on the interaction terms. It is applicable to both linear and nonlinear dispersive equations, including those with time-dependent interaction terms and Klein-Gordon type equations.

Neither method directly employs local decay. However, proving local decay in these cases is not straightforward. In this work, we use the knowledge of asymptotic completeness (AC) obtained from [29] to derive local decay estimates. We first establish this for the Schrödinger equation with a time-independent potential that is localized in space, and then for a potential that is quasi-periodic in time and also localized in space.

Time-dependent potentials have been studied previously. In particular, Rodnianski and Schlag [19] established \( L^p \) estimates for time-dependent potentials under the assumption of smallness on the size of the potential. Recent progress in this direction can also be found in [1, 3, 6, 8, 26, 32].

1.1. Main results.

1.1.1. The time-dependent result. Let \( H_0 := -\Delta_x \) and \( \langle \cdot \rangle : \mathbb{R}^n \to \mathbb{R}, x \mapsto \sqrt{|x|^2 + 1}, n \geq 1 \). We consider the Schrödinger equation with a time-dependent potential \( V(x,t) \)

\[
\begin{cases}
  i \partial_t \psi(x,t) = (H_0 + V(x,t))\psi(x,t) \\
  \psi(x,0) = \psi_0 \in L^2_x(\mathbb{R}^n)
\end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad n \geq 5,
\]

where \( V(x,t) \) satisfies

**Assumption 1.1.** \( V(x,t) \) is real-valued and satisfies the condition \( \langle x \rangle^\sigma V(x,t) \in L^\infty_{x,t}(\mathbb{R}^{n+1}) \) for some \( \sigma > 6 \), where \( n \geq 5 \). Additionally, \( V(x,t) \) can be expressed as

\[
V(x,t) = \sum_{j=1}^N V_j(x,t), \quad N \in \mathbb{N}^+,
\]

where each \( V_j(x,t) \), for \( j = 1, \ldots, N \), is continuous and periodic in \( t \) in the sense of

\[
\lim_{s \to t} \|V_j(x,t) - V_j(x,s)\|_{L^\infty(\mathbb{R}^n)} = 0.
\]

This potential \( V \) is quasi-periodic in \( t \). The definition of a quasi-periodic function is provided below.

**Definition 1.1** (Quasi-periodic functions). A function \( f(t) \) is quasi-periodic if there exist \( N \) tori \( T_j := [0, T_j), j = 1, \cdots, N(N \in \mathbb{N}^+) \) and a function \( g : T_1 \times \cdots \times T_N \to \mathbb{C} \) such that whenever

\[
s_j = t \mod T_j, \quad j = 1, \cdots, N,
\]

\[
f(t) = g(s_1, \cdots, s_N).
\]
We say that two quasi-periodic functions are of the same type if the domains of their corresponding functions $g$ are the same.

An example of quasi-periodic functions which are not periodic is $f(t) = \sin(t) + \sin(\sqrt{2}t)$, where the periods of $\sin(t)$ and $\sin(\sqrt{2}t)$ are $2\pi$ and $\sqrt{2}\pi$, respectively. Since the ratio of these periods is $\sqrt{2}/2\pi = \sqrt{2}$, which is irrational, the function does not repeat itself over any common period, making it quasi-periodic.

Throughout this paper, $H^k(\mathbb{R}^n)$, for $k \geq 0$, denotes the Sobolev space and $A \lesssim B$ means $A \leq CB$ for some constant $C = C(a) > 0$. We also use the notation

\begin{equation}
\| \cdot \|_2 \quad \| \cdot \|_{L^2(\mathbb{R}^n)}
\end{equation}

depending on the context. Sometimes we use $\| \cdot \|_{2-2}$ to denote $\| \cdot \|_{L^2(\mathbb{R}^n)}$.

Under Assumption 1.1, $H = H_0 + V(x, t)$ is self-adjoint on the domain of $H_0$, i.e. $V(x, t)$ is $H_0$-bounded with the relative bound $< 1$: for all $u \in H^2_x(\mathbb{R}^n)$,

\begin{equation}
\|Vu\| \leq a\|H_0u\| + b\|u\|, \quad \exists 0 \leq a < 1, b > 0,
\end{equation}

and the evolution operators $U(t, s), t, s \in \mathbb{R}$, generated by $H$ is unitary on $L^2_x(\mathbb{R}^n)$. Indeed, for all initial data $\psi_0 \in L^2_x(\mathbb{R}^n)$, the solution $\psi(x, t)$ exists in $L^2_x(\mathbb{R}^n)$ for all $t \in \mathbb{R}$. See e.g. Theorem X.70 of [13].

In this note, we stick to the case when $t \geq 0$. We refer to the initial state $\psi_0 \in L^2_x(\mathbb{R}^n)$ as a scattering state if $\psi(x, t)$ evolves like a free flow asymptotically, that is, for some $\psi_+ \in L^2_x(\mathbb{R}^n)$,

\begin{equation}
\|\psi(x, t) - e^{-itH_0}\psi_+\| \to 0, \quad \text{as } t \to \infty.
\end{equation}

The space of all scattering states is a subspace of $L^2_x(\mathbb{R}^n)$ and such space varies as the initial time $t_0$ varies. We introduce the projection on the space of all scattering states at the initial time $t_0$ below. For $m > 0$, let $F(k \geq m) \equiv F(\frac{k}{m})(m > 0)$ denote a smooth characteristic function with

\begin{equation}
F(\lambda) = \begin{cases} 
1 & \text{when } \lambda > 1 \\
0 & \text{when } \lambda < 1/2 
\end{cases}
\end{equation}

and let $F(k < m) = 1 - F(k \geq m)$ denote the complement of $F(k \geq m)$. The projection on the space of all scattering states is given by, with $p := -i\nabla_x$, for $\alpha \in (0, 1 - \frac{2}{n})$, $n \geq 3$ and $t_0 \in \mathbb{R}$,

\begin{equation}
P_\alpha(t_0) := \lim_{t \to \infty} U(t_0, t_0 + t)F\left(\frac{|x - 2tp|}{t^2}\right) < 1)U(t_0 + t, t_0), \quad \text{on } L^2_x(\mathbb{R}^n),
\end{equation}

which is constructed in [29] provided that $V(x, t) \in L^\infty_x L^2(\mathbb{R}^{n+1}), n \geq 3$. It is worth noting that the r.h.s. of Eq. (1.11) is independent of $\alpha$ (See Eq. (1.12) below and Eq. (2.13) of [29]) and $P_\alpha(t_0)$ defined in Eq. (1.11) satisfies $P_\alpha(t_0) = \Omega_+(t)\Omega_+^*(t)$, where

\begin{equation}
\Omega_+(t) := s- \lim_{s \to \infty} U(t, t + s)e^{-iH_0} \quad \text{on } L^2_x(\mathbb{R}^n)
\end{equation}

and $\Omega_+^*(t)$ is the conjugate of $\Omega_+(t)$.

Throughout this paper, $C$ will denote a constant and may vary from one line to another. We write $\lesssim$ or $\gtrsim$ whenever $A \leq CB$ or $CA \geq B$ for some constant $C > 0$. We write $A \lesssim_a B$ or $A \gtrsim_a B$ if $A \leq C_a B$ or $C_a A \geq B$ for some constant $C_a > 0$ which depends on parameter $a$. The Fourier transform and its inverse are given by

\begin{equation}
\hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} f(x)dx
\end{equation}

and

\begin{equation}
f(x) := \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} \hat{f}(\xi)d\xi
\end{equation}
for \( f \in L^2_x(\mathbb{R}^n) \).

We prove first that, provided Assumption \( 1.1 \) holds true, there exists \( p_0 \in [2, \infty) \) such that

\[
(1.15) \quad \left( \int_0^\infty \| \langle x \rangle^{-\eta} U(t + t_0, t_0) P_c(t_0) \psi_0 \|^p dt \right)^{\frac{1}{p}} \lesssim \| \psi_0 \|
\]

for all \( \eta > \frac{3}{2} \) and \( p_0 \leq p < \infty \).

**Theorem 1.1.** If Assumption \( 1.1 \) is satisfied, then with \( p_0 = 2 \), \( 1.15 \) holds true for all \( p_0 \leq p < \infty \), when \( n \geq 8 \).

We refer to estimate \( 1.15 \) as \( L^p \) local decay estimate. It is known that the free flow satisfies \( L^2 \) local decay:

\[
(1.16) \quad \int_0^\infty \| \langle x \rangle^{-\delta} e^{-itH_0} \psi_0 \|^2 dt \lesssim \| \psi_0 \|^2, \quad \forall \delta > 1.
\]

So the conclusion of Theorem \( 1.1 \) is not sharp. Next, we explain what additional conditions are needed to obtain the \( L^2 \) local decay estimate

\[
(1.17) \quad \int_0^\infty \| \langle x \rangle^{-\eta} U(t + t_0, t_0) P_c(t_0) \psi_0 \|^2 dt \lesssim \| \psi_0 \|^2, \quad \text{for some } \eta > 1.
\]

We need extra properties of \( P_b(t_0) \equiv 1 - P_c(t_0) \). The existence of \( P_c(t_0) \) implies the existence of \( P_b(t_0) \). To be precise, we need

**Assumption 1.2.** Either

\[
(1.18) \quad \sup_{t_0 \in \mathbb{R}} \| P_b(t_0) \langle x \rangle^\delta \| < \infty \quad \text{for some } \delta > \frac{3}{2}
\]

or

\[
(1.19) \quad \sup_{t_0 \in \mathbb{R}} \| P_b(t_0) |p|^{-1/2} \langle x \rangle^{1/2 + \epsilon} \| < \infty \quad \text{for some } \epsilon > 0
\]

holds true.

Indeed, in the proof of \( L^2 \) local decay, we have to estimate \( P_b(t)e^{-itH_0} f, f \in L^2_x(\mathbb{R}^n) \). If we write

\[
P_b(t)e^{-itH_0} f = (P_b(t) \langle x \rangle^\delta) \left( \langle x \rangle^{-\delta} e^{-itH_0} f \right) = \left( P_b(t) |p|^{-1/2} \langle x \rangle^{1/2 + \epsilon} \right) \left( \langle x \rangle^{-1/2 - \epsilon} |p|^{1/2} e^{-itH_0} f \right),
\]

then, intuitively, Conditions \( 1.18 \) and \( 1.19 \) are corresponding to \( L^2 \) local decay estimate and

\[
(1.20) \quad \text{(\( L^2 \) local smoothing estimate)} \quad \int_0^\infty \| \langle x \rangle^{-1/2 - \epsilon} |p|^{1/2} e^{-itH_0} \psi_0 \|^2 dt \lesssim \| \psi_0 \|^2,
\]

respectively. We believe Assumption \( 1.2 \) is a necessary condition based on our experience with the time-independent potential. When \( H = H_0 + V \) is time-independent and \( V \) satisfies, for example, \( \langle x \rangle^{1+\epsilon} |V(x)| \lesssim 1 \) for some \( \epsilon > 0 \), \( P_b(t_0) \) is the projection onto the space of the discrete spectrum of \( H \). It is known that, additionally, if 0 is neither an eigenvalue nor a resonance of \( H \), \( H \) has finitely many eigenvalues, each with finite multiplicity. It is also known that in 5 or higher space dimensions, there is no resonance. See, for example, \( 19 \) and the references therein. In the case of \( 1.1 \), we prove in Proposition \( 4.7 \) that for all \( \delta \in [0, \frac{3}{2} - 2) \), \( n \geq 5 \),

\[
(1.21) \quad \sup_{t_0 \in \mathbb{R}} \| P_b(t_0) \langle x \rangle^\delta \| \lesssim_{\delta, n} 1,
\]

under Assumption \( 1.1 \). In the proof of Proposition \( 4.7 \) we observe that the localization of non-scattering states is influenced by their low-frequency component, which corresponds to the 0 threshold energy.

**Theorem 1.2.** If Assumptions \( 1.1 \) and \( 1.2 \) are satisfied, then \( 1.17 \) holds true.
1.1.2. The time-independent result. When the potential is time-independent, we can write \( U(t, 0) = e^{-itH} \). In this case, \( P_c(t_0) \equiv P_c \) is equal to the projection onto the continuous spectrum of \( H \), see, for example, page 2 of [19].

**Assumption 1.3.** \( V(x) \) is real-valued and satisfies \( \langle x \rangle^4 V(x) \in L^\infty_2(\mathbb{R}^3) \).

We prove that under Assumption 1.3 \( e^{-itH} P_c \) satisfies the local decay estimate

\[
\int_{-\infty}^{\infty} dt \| \langle x \rangle^{-\eta} e^{-itH} F(H \geq c) P_c \psi_0 \|^2 \lesssim_{\eta, c} \| \psi_0 \|^2
\]

for all \( \psi_0 \in L^2_2(\mathbb{R}^3) \), \( c > 0 \) and any \( \eta > 1 \).

**Theorem 1.3.** Assume \( V(x) \) satisfies Assumption 1.3. Then for any \( \eta > 1 \) and \( c > 0 \), (1.22) holds for all initial states \( \psi_0 \in L^2_2(\mathbb{R}^3) \).

**Remark 1.1.** We believe that a similar result can be proved in four or higher space dimensions by this method.

1.2. Applications. As an application, we obtain Strichartz estimates globally in \( t \). Strichartz estimates state that

\[
\| U(t, 0) P_c(0) f \|_{L^4_2 L^6(\mathbb{R}^{n+1})} \leq C_q \| f \|_{L^2_x(\mathbb{R}^n)}
\]

for \( 2 \leq r, q \leq \infty, \frac{2}{r} + \frac{3}{q} = \frac{3}{2} \), and \( (q, r, n) \neq (2, \infty, 2) \).

**Theorem 1.4.** If \( V(x, t) \) satisfies Assumptions 1.1 and 1.2, then Strichartz estimates (See (1.23)) are valid for all admissible \( (q, r, n) \) with \( n \geq 5 \).

1.3. Outline of the proof. The proof of Theorem 1.4 differs from that of Theorem 1.3 though the essential ideas remain the same. In the time-independent case, we address the issue arising from the zero energy threshold by applying an energy cut-off \( F(H \geq c) \) to the initial data. In contrast, in the time-dependent case, we resolve this issue by considering space dimensions \( n \geq 5 \). We provide an outline of the proof of Theorem 1.4 here.

We take \( t_0 = 0 \) and write \( P \equiv P_c(0) \) for simplicity. The case when \( t_0 \neq 0 \) can be treated similarly. The proof is based on a new compactness argument, the notion of ‘incoming/outgoing waves’ (see Definition 2.1) and the knowledge of AC (the identity \( P_c(t) = \Omega_+(t) \Omega_+^*(t), t \in \mathbb{R} \)). It consists of three steps.

We introduce some notions before showing the steps. Let \( P^\pm \) denote the projections onto the incoming/outgoing waves. We present the optimized (adjoint) Möller wave operators as introduced in [29]: for \( \alpha \in (0, 1 - \frac{2}{n}) \), \( n \geq 3 \),

\[
\Omega^\pm_\alpha(t) := s- \lim_{s \to \pm \infty} e^{itH_0} F\left( \frac{|x - 2sp|}{|s|^{\alpha}} < 1 \right) U(s + t, t), \quad \text{on } L^2_x(\mathbb{R}^n).
\]

Here, we use notations \( \Omega^\pm_\alpha(t) \) since \( \Omega^\pm_\alpha(t) \) are independent of \( \alpha \). See Eq. (2.13) of [29]. \( \Omega^\pm_\alpha(t) \) satisfy the intertwining property:

\[
\Omega^\pm_\alpha(t) U(t, 0) = e^{-itH_0} \Omega^\pm_\alpha(0).
\]

First, we write \( \psi(t) \) as the sum of a ‘free flow’ and a compact operator acting on \( \psi(t) \). Decomposing \( \psi(t) \) into the sum of \( P^+ \psi(t) \) and \( P^- \psi(t) \) and using \( P^\pm \Omega^\pm_\alpha(t) \psi(t) \) to approximate \( P^\pm \psi(t) \), respectively, we obtain with \( \psi(t) \equiv U(t, 0) P_c(0) \psi_0 \),

\[
\psi(t) = \psi_f(t) + C(t) \psi(t),
\]

where \( \psi_f(t) \) is given by, by Eq. (1.23),

\[
\psi_f(t) := P^+ \Omega^+_\alpha(t) \psi(t) + P^- \Omega^-_\alpha(t) \psi(t) = P^+ e^{-itH_0} \Omega^+_\alpha(0) \psi_0 + P^- e^{-itH_0} \Omega^-_\alpha(0) \psi_0
\]
and \( C(t) \) is defined by
\[
C(t) := P^+ (1 - \Omega^*_+(t)) + P^- (1 - \Omega^*_-(t)).
\] (1.28)

We note that \( C(t) \), \( t \in \mathbb{R} \), are compact on \( L^2(\mathbb{R}^n) \) and satisfies, by estimates \( \|P^\pm\| \leq 1 \) and \( \|\Omega^*_\pm(t)\| \leq 1 \),
\[
\sup_{t \in \mathbb{R}} \|C(t)\| \leq 4.
\] (1.29)

It is also worth noting that \( \psi_f(t) \) behaves like a free flow and satisfies \( L^2 \) local decay estimate and \( L^2 \) local smoothing estimate.

Next, we use \( P_c(t) = \Omega_+(t)\Omega^*_+(t) \) to decompose \( C(t) \) further. We define
\[
F_M(x,p) := F(|x| < M)F(|p| \geq \frac{1}{M})
\] (1.30)
and write with \( P_b(t) := 1 - P_c(t) \),
\[
C(t) = C_M(t) + C_r(t) + C(t)P_b(t),
\] (1.31)
where operators \( C_M(t) \) and \( C_r(t) \) are given by
\[
C_M(t) := C(t)\Omega_+(t)F_M(x,p)\Omega^*_+(t)
\] (1.32)
and
\[
C_r(t) := C(t)\Omega_+(t)(1 - F_M(x,p))\Omega^*_+(t).
\] (1.33)

Then Eq. (1.26) is equivalent to
\[
\psi(t) = \psi_f(t) + C_M(t)\psi(t) + C_r(t)\psi(t).
\] (1.34)

We prove in Proposition 4.1 that there exists a constant \( M_0 > 1 \) such that whenever \( M \geq M_0 \),
\[
\sup_{t \in \mathbb{R}} \|C_r(t)\| < \frac{1}{2}.
\] (1.35)

The proof of Proposition 4.1 relies on the condition that \( V \) is quasi-periodic in \( t \) and a standard compactness argument. It is also worth noting that \( C(t) \) is quasi-periodic in \( t \) by Assumption 1.1 (see also Corollary 2.4) and
\[
\int_0^\infty \|F_M(x,p)\Omega^*_+(t)\psi(t)\|^2 dt = \int_0^\infty \|F_M(x,p)e^{-itH_0}\Omega^*_+(0)\psi_0\|^2 dt
\] (1.36)
which together with (1.29), implies, with \( \psi(t) = P_c(t)\psi(t) \),
\[
\|C_M(t)\psi(t)\| = \|C_M(t)P_c(t)\psi(t)\| \leq \sup_{s \in \mathbb{R}} \|C(s)\|\|F_M(x,p)\Omega^*_+(t)\psi(t)\| \in L^2_t[0, \infty).
\] (1.37)

In the third step, by employing (1.35), we find that \( (1 - C_r(t))^{-1} \) is bounded on \( L^2(\mathbb{R}^n) \). Moving \( C_r(t)\psi(t) \) to the left-hand side of Eq. (1.34) and then applying \( (1 - C_r(t))^{-1} \) to both sides, we arrive at
\[
\psi(t) = (1 - C_r(t))^{-1}\psi_f(t) + (1 - C_r(t))^{-1}C_M(t)\psi(t).
\] (1.38)

We prove in Proposition 4.4 that
\[
\|\langle x \rangle^{-\eta}(1 - C_r(t))^{-1}\psi_f(t)\|_{L^2_x(\mathbb{R}^n \times \mathbb{R}^+)} \lesssim \|\psi(0)\|, \quad n \geq 5
\] (1.39)
for all \( \eta > \frac{3}{2} \). This together with estimate (1.37) and Eq. (1.38) yields the estimate (1.22).
1.4. Challenges of the Analysis of Time-Dependent Hamiltonians. Time Dependent Hamiltonians cannot be treated directly by the standard spectral theory methods. The spectral properties of \( H(t) \) for each fixed \( t \) cannot be directly related to the solution of the the Schrödinger equation with Hamiltonian \( H(t) \). We focus here on the problem where the interaction term is given by a localized in space potential, in 3 or higher dimensions. We are interested in the basic questions about the behavior of the solutions: localized (bound) states, scattering states, decay estimates for the scattering states etc...

1.4.1. Previous methods. There are classical methods of proving dispersive estimates for linear Schrödinger equations: Mourre’s Method and Resolvent estimates are general powerful such methods. Resolvent estimates, going back many decades ago, are based on constructing a good estimate on the resolvent of the full Hamiltonian in terms of the approximate free Hamiltonian. This approach is pretty general, but limited to localized and time-periodic interactions. Its application to N-body problems is very limited. It has no direct analog for dealing with time dependent potentials. A more general theory is the abstract Mourre’s method. It begins with the construction of a self-adjoint operator which has a positive commutator, at least when the Hamiltonian is localized near favorable points in the continuous spectrum of \( H \). This is an abstract condition. Together with extra domain assumptions, such an estimate implies Local Decay estimates, via the method of differential inequalities developed by Mourre \[15, 16\]. A more general theory, \[10\] allows to prove besides Local Decay, the optimal minimal and maximal velocity bounds, which are time dependent bounds. This approach does not work at the thresholds of the Hamiltonian. It does not have a generalization to time dependent potentials in general. When \( V \) is time-periodic, we use Floquet theory to reduce the problem to the one with time-independent Hamiltonian \[5\]. We refer to such Hamiltonian as the Floquet operator. If we have a potential that is periodic in time, the corresponding Floquet Operator is time independent, with infinitely many thresholds points of the form \( n\omega, n \in \mathbb{Z} \), where \( \omega := \frac{2\pi}{T} \) is the time frequency. However, in the quasi-periodic case the set of threshold points of the associated Floquet operator is dense in \( \mathbb{R} \) and so one cannot use Mourre’s Method at any point. Our goal in this work is to introduce a new method that will cover the case of quasi-periodic time dependent potentials (in 5 or more dimensions).

1.4.2. Connecting Quasi-Periodic Time Dependence to General Time Dependence. If the Fourier transform of \( V(x, t) \) in \( t \) is a finite measure, then by general principles based on Wiener’s theorem, \( V \) can be written as a sum of almost periodic potential and a part that decays in time. The decaying part will be small eventually and the main part of the almost periodic potential is quasi-periodic, so we focus on the quasi periodic case.

2. Phase Space Decompositions

In this section, we introduce the notion of incoming and outgoing waves, prove estimates for the free wave, and present properties of the evolution operator \( U(t, s) \).

2.1. Incoming/outgoing waves. We define the incoming/outgoing wave decompositions, inspired by Mourre \[15\], based on the dilation generator \( A \), given by

\[
A := \frac{1}{2}(x \cdot p + p \cdot x).
\]

**Definition 2.1** (Incoming/outgoing waves). The projection on outgoing waves is defined in \[23\]:

\[
P^+ := (\tanh(\frac{A}{R}) + 1)/2
\]

with \( R = 100 \). Here, 100 stands a sufficiently large number. We define the projection onto incoming waves as the complement of \( P^+ \)

\[
P^- := 1 - P^+.
\]
In what follows, we take \( R = 100 \). We could commute through \( P^\pm \) with \(|p|^a, |x|^a\) and \( \langle x \rangle^{-1/2-\epsilon}|p|^{1/2} \) for \( a \in (0, \pi R/2) \) and \( \epsilon \in (0, \frac{1}{2}) \) in the sense of the lemma listed below.

**Lemma 2.1.** For all \( a \in (0, \pi R/2) \), \( \epsilon \in (0, \frac{1}{2}) \), and for all \( f \in H^a(\mathbb{R}^n) \), \( P^\pm f \) satisfy the following:

\[
\| |p|^a P^\pm f \| \lesssim_{\pi/2-a, a/R} \| |p|^a f \|,
\]

\[(2.4)\]

\[
\| |x|^a P^\pm f \| \lesssim_{\pi/2-a, a/R} \| |x|^a f \|
\]

and

\[
\| \langle x \rangle^{-1/2-\epsilon} |p|^{1/2} P^\pm f \| \lesssim_{\pi/2-a, a/R} \| \langle x \rangle^{-1/2-\epsilon} |p|^{1/2} f \|.
\]

\[(2.6)\]

**Proof.** We define a function \( g \) on \( \mathbb{R} \):

\[
g(k) := (\tanh(k) + 1)/2, \quad k \in \mathbb{R}.
\]

We estimate \( \| |p|^a P^+ f \| \), \( \| |p|^a P^- f \| \) can be treated similarly. To estimate \( \| |p|^a P^+ f \| \), we find

\[
\| |p|^a P^+ f \| \leq \| |p|^a P^+ f \| + \| P^+ |p|^a f \|.
\]

\[(2.8)\]

For the first term of the right-hand side of (2.8), we first use the Fourier inversion theorem to express \( P^+ \) as

\[
P^+ = (\tanh(A) + 1)/2 = \frac{1}{\sqrt{2\pi}} \int e^{i w A/R} \hat{g}(w) dw,
\]

where \( \hat{g} \) denotes the Fourier transform of \( g \). Plugging (2.9) into \( \| |p|^a, P^+ |f \| \) and using relation

\[
|p|^a e^{i w A/R} = \frac{1}{e^{aw/R}} e^{i w A/R} |p|^a,
\]

we obtain

\[
\| |p|^a, P^+ |f \| = \frac{1}{\sqrt{2\pi}} \int \hat{g}(w) |p|^a e^{i w A/R} |f| dw
\]

\[
= \frac{1}{\sqrt{2\pi}} \int \hat{g}(w) \left( \frac{1}{e^{aw/R}} - 1 \right) e^{i w A/R} |p|^a f dw.
\]

\[(2.11)\]

In view of

\[
\hat{g}(w) = \frac{i}{2\sqrt{2\pi}} \frac{1}{\sinh(\pi w/2)} + \frac{1}{2\sqrt{2\pi}} \delta(w)
\]

and

\[
\int \delta(w) \left( \frac{1}{e^{aw/R}} - 1 \right) e^{i w A/R} |p|^a f dw = 0 \quad \forall a > 0,
\]

we plug (2.12) into (2.11) to obtain

\[
\| |p|^a, P^+ |f \| \leq \frac{1}{4\pi} \int \frac{1}{\sinh(\pi w/2)} \left( \frac{1}{e^{aw/R}} - 1 \right) |p|^a f dw.
\]

\[(2.14)\]

By estimates, for all \( a \in (0, \pi R/2) \),

\[
\int_{|w| > 1} \frac{1}{\sinh(\pi w/2)} \left| \frac{1}{e^{aw/R}} - 1 \right| dw \leq \int_{|w| > 1} \frac{2e^{-\pi w^2/2}}{1 - e^{-\pi /2}} (e^{a|w|/R} + 1) dw \lesssim_{\pi/2-a, a/R} 1
\]

and

\[
\int_{|w| \leq 1} \frac{1}{\sinh(\pi w/2)} \left| \frac{1}{e^{aw/R}} - 1 \right| dw \leq 2 \sup_{|w| \leq 1} \frac{w}{\sinh(\pi w/2)} \sup_{|w| \leq 1} \left| \frac{e^{-aw/R} - 1}{w} \right| \lesssim_{a/R} 1,
\]

\[(2.16)\]

this yields

\[
\| |p|^a, P^+ |f \| \lesssim_{\pi/2-a, a/R} \| |p|^a f \| \quad \forall a \in (0, \pi R/2).
\]

\[(2.17)\]
Estimate (2.17) together with estimates \(|P^+|p|^n f| \lesssim |p|^n f| and (2.8) yields (2.4) for
\(P^+\). Similarly, we have (2.4) for \(P^-\), and proceeding as in (2.8)-(2.17) in the dual
space (Fourier space), we obtain (2.5). By proceeding as in (2.8)-(2.17) in the dual space and subsequently in the configuration
space, we obtain (2.6).

The incoming/outgoing projections \(P^\pm\) and the free flows \(e^{\pm itH_0}, t \geq 0\), satisfy the estimate
\[
(2.18) \quad \|P^\pm e^{\pm itH_0} F(|p| \geq c) e^{\pm itH_0} \langle x \rangle^{-\delta}\| \lesssim_{n,c,\delta} \frac{1}{(t)^{\delta}}, \quad \forall \delta \geq 0 \text{ and } \forall t \geq 0.
\]

Estimate (2.18) follows by Mourre estimate, see [10, 15, 16]. By (2.18), we obtain estimates related
to \(P^\pm e^{\pm itH_0}\) listed in Lemma 2.2 below. These estimates will be the main components in proving
Theorems 1.1, 1.2 and 1.3.

**Lemma 2.2.** When \(R = 100\), the operators \(P^\pm e^{\pm itH_0}, t > 0\), satisfy: given \(t > 0\), (recall that \(n\) is the space dimension)

1. **High Energy Estimate** For all \(\delta > 0\) and \(c > 0\),
\[
(2.19) \quad \|P^\pm F(|p| \geq c) e^{\pm itH_0} \langle x \rangle^{-\delta}\| \lesssim_{n,c,\delta} \frac{1}{(t)^{\delta}}.
\]

2. **Pointwise Smoothing Estimate** For \(\delta > 0\), \(l \in (0, \delta)\), \(M \geq 1\) and \(t > 0\) with \(tM^2 \geq 1\),
\[
(2.20) \quad \|P^\pm F(|p| \geq M) e^{\pm itH_0} |p|^l \langle x \rangle^{-\delta}\| \lesssim_{n,\delta-l} \frac{1}{M^{\delta-l} t^{\delta}}.
\]

3. **Time Smoothing Estimate** For \(\delta > 2\), \(l = 1, 2,\)
\[
(2.21) \quad \int_0^1 t^2 \|P^\pm F(|p| \geq 1) e^{\pm itH_0} |p|^l \langle x \rangle^{-\delta}\| dt \lesssim_{n,\delta} 1.
\]

4. **Near Threshold Estimate** For all \(\delta > 0\) and all \(\epsilon \in (0, 1/2), \alpha \in [0, \pi R/2],\)
\[
(2.22) \quad \| \frac{|p|^{\alpha}}{(p)^{\alpha}} P^\pm e^{\pm itH_0} \langle x \rangle^{-\delta}\| \lesssim_{n,\epsilon,\alpha,R} \frac{1}{(t)^{\min(\frac{\alpha}{2}+\epsilon,1/2-\epsilon)(\alpha+\min(\frac{\alpha}{2},\epsilon,\delta))}},
\]
and in particular, when \(n \geq 5\), \(\delta > 2\) and \(\epsilon \in (0, \frac{n}{4} - 1)\), we have
\[
(2.23) \quad \|P^\pm e^{\pm itH_0} \langle x \rangle^{-\delta}\| \in L^1_t(\mathbb{R}^+).
\]

5. **Pointwise Local Smoothing Estimate** For all \(\epsilon \in (0, \frac{1}{4})\), \(\delta \in [0, 2], n \geq 5, l = 1, 2, j = 1, \ldots, n\) and \(t \geq 1\),
\[
(2.24) \quad \| \frac{1}{(x)^{2-\delta}} P^\pm e^{\pm itH_0} p_j^{(l)} \langle x \rangle^{-\frac{l+1}{2}+2-\delta}\| \lesssim_{n,\epsilon,\delta} \frac{1}{(t)^{\frac{1}{2}+l+2-\delta} - \epsilon}.
\]

**Proof.** See Appendix A for its proof.

### 2.2. Quasi-periodic evolution operators

We present several properties related to the quasi-periodicity of the evolution operators \(U(t,s)\), for \(t, s \in \mathbb{R}\), as well as the projections \(P_j(t)\) and the operators \(C(t)\).

**Lemma 2.3.** If Assumption 1.1 holds true, then for each \(t \in \mathbb{R}\), \(U(t+s,s)\) is an evolution operator
that is quasi-periodic in \(s\), with the same type as \(V\).

**Proof.** Let
\[
(2.25) \quad \mathcal{K}_t(V) := e^{itH_0} Ve^{-itH_0}, \quad t \geq 0
\]
and for a family of operators on \(L^2_\mathfrak{a}(\mathbb{R}^n), \{A_j\}_{j=1}^J, J \in \mathbb{N}^+,\) we define
\[
(2.26) \quad \prod_{j=1}^0 A_j = 1 \quad \text{and} \quad \prod_{j=1}^J A_j = A_1 \cdots A_J.
\]
By the Duhamel’s principle and iterating it for infinitely many times, we obtain

\[(2.27)\quad U(t + s, s) = \sum_{j=0}^{\infty} I_j(t, s),\]

where \(I_0(t, s) = 1\) and \(I_j(t, s), j = 1, \cdots\), are given by

\[(2.28)\quad I_j(t, s) := (-i) \int_{t_0}^{t} \int_{t_{j-1}}^{t_1} \cdots \int_{t_0}^{t_{j-1}} e^{-i t H_0} \prod_{k=1}^{j} \mathcal{K}_{t_k}(V(x, t_k + s)) \, dt_j \cdots dt_1.\]

By Definition 1.1 we conclude that operators \(I_j(t, s), j = 1, \cdots\), are quasi-periodic in \(s\) with the same type as \(V\). Hence, \(U(t + s, s)\) is quasi-periodic in \(s\) with the same type as \(V\).

Recall that

\[(2.29)\quad \Omega^*_\pm(t) := \lim_{s \to \pm \infty} e^{isH_0} F \left( \frac{|x - 2sp|}{|s|^{\alpha}} < 1 \right) U(s + t, t), \quad \text{on } L^2_x(\mathbb{R}^n),\]

\[(2.30)\quad P_c(t) := \lim_{s \to \pm \infty} U(t, t + s) F \left( \frac{|x - 2sp|}{|s|^{\alpha}} < 1 \right) U(s + t, t), \quad \text{on } L^2_x(\mathbb{R}^n),\]

\[(2.31)\quad P_0(t) := 1 - P_c(t)\]

and

\[(2.32)\quad C(t) := P^+ (1 - \Omega^*_+(t)) + P^- (1 - \Omega^*_-(t)).\]

**Corollary 2.1.** If Assumption 1.1 holds true, then for each \(t \in \mathbb{R}\), \(\Omega^*_\pm(t), \Omega^*_\pm, P_c(t), P_0(t)\) and \(C(t)\) are bounded operators on \(L^2_x(\mathbb{R}^n)\) uniformly in \(t\) and quasi-periodic in \(t\) with the same type as \(V\).

**Proof.** The boundedness of \(P_c(t)\) on \(L^2_x(\mathbb{R}^n)\) is proved in Theorem 2.1 of [29]. The boundedness of \(P_0(t)\) on \(L^2_x(\mathbb{R}^n)\) follows from the boundedness of \(P_c(t)\), since \(P_0(t) = 1 - P_c(t)\). The boundedness of \(C(t)\) on \(L^2_x(\mathbb{R}^n)\) follows from the existence of \(\Omega^*_\pm(t)\) and the estimate \(\|P^\pm\| \leq 1\). \(P_c(t), P_0(t)\), and \(\Omega^*_\pm(t)\) are quasi-periodic in \(t\) with the same type as \(V\) by Lemma 2.3. Therefore, so is \(C(t)\).

3. **The time-independent problem**

In this section, we prove Theorem 1.3. The proof is based on the properties of operator \(C(t)\) defined in Eq. (1.28). When \(V\) is time-independent, \(C \equiv C(t)\), is time-independent. That is,

\[(3.1)\quad C = P^+ (1 - \Omega^*_+) + P^- (1 - \Omega^*_-).\]

\(P_c \equiv P_c(t)\) is also time-independent and we can express \(U(t + s, s) = e^{-itH}\) for all \(s, t \in \mathbb{R}\). We first derive a representation of \(\psi(t) = e^{-itH} F(H \geq c) P_c \psi(0)\) analogous to Eq. (1.29). Let \(F_{c,1}(z) := F(z \geq \frac{t}{100})\) and \(F_{c,2}(z) := F(z \geq \frac{t}{1000})\). It is worth noting that Eq. (1.26) holds true with \(\psi(0) = F(H \geq c) \psi(0)\) when \(V\) is time-independent. Using

\[(3.2)\quad F(H \geq c) = F_{c,1}(H) F(H \geq c)\]

\[(3.3)\quad \psi(t) = \psi_f(t) + C_c \psi(t),\]

where operator \(C_c\) is given by

\[(3.4)\quad C_c := CF_{c,1}(H)\]

and \(\psi_f(t)\) reads, according to Eq. (1.27),

\[(3.5)\quad \psi_f(t) = P^+ e^{-itH_0} \Omega^*_+ \psi(0) + P^- e^{-itH_0} \Omega^*_- \psi(0).\]
In what follows in this section, we prove Theorem 1.3 by showing the compactness of $C_c$ and employing a new compactness argument based on AC. We also assume in this section that $V$ satisfies Assumption 1.3 and $n = 3$ when the context is clear.

### 3.1. Compactness of $C_c$

We prove the compactness of $C_c$ in this subsection.

**Proposition 3.1.** $C_c$ is a compact operator on $L^2_x(\mathbb{R}^3)$ for all $c > 0$.

The proof of Proposition 3.1 requires the well-known result Lemma 3.1

**Lemma 3.1.** $F_{c,1}(H) - F_{c,1}(H_0)$ is compact on $L^2_x(\mathbb{R}^3)$ for all $c > 0$.

**Proof.** See Appendix A for its proof. □

**Proposition 3.2.** For all $t \geq 1$ the estimates

\[(3.6) \quad \| P^\pm F_{c,1}(H_0)e^{\pm itH_0}\Omega^*_{\pm} \langle x \rangle^{-2} \| \lesssim_c \frac{1}{(t)^2} \]

are valid.

**Proof.** Let $O^\pm(t) \equiv P^\pm F_{c,1}(H_0)e^{\pm itH_0}\Omega^*_{\pm} \langle x \rangle^{-2}$. We estimate $O^+(t)$. The estimate for $O^-(t)$ can be derived similarly. By the Duhamel’s principle, $O^+(t)$ reads

\[(3.7) \quad O^+(t) = O^+_1(t) + O^+_2(t),\]

where the operators $O^+_j(t), j = 1, 2$, are given by

\[(3.8) \quad O^+_1(t) := P^+ F_{c,1}(H_0)e^{itH_0} \langle x \rangle^{-2} \]

and

\[(3.9) \quad O^+_2(t) := (-i) \int_0^\infty P^+ F_{c,1}(H_0)e^{i(t+s)H_0} V e^{-isH} \langle x \rangle^{-2} ds.\]

By estimate (2.19), $O^+_1(t)$ and $O^+_2(t)$ satisfy

\[(3.10) \quad \| O^+_1(t) \| \lesssim_c \frac{1}{(t)^2} \]

and

\[(3.11) \quad \| O^+_2(t) \| \lesssim_c \frac{1}{(t)^2} \int_0^\infty \| P^+ F_{c,1}(H_0)e^{i(t+s)H_0} \langle x \rangle^{-3} \| \| \langle x \rangle^3 V \|_{2 \to 2} \| e^{-isH} \| ds \]

\[\lesssim_c \frac{1}{(t)^2} \| \langle x \rangle^3 V \|_{L^\infty_x(\mathbb{R}^3)} ds\]

where we used the unitarity of $e^{-isH}$ and estimate $\| \langle x \rangle^3 V \|_{2 \to 2} \leq \| \langle x \rangle^3 V \|_{L^\infty_x(\mathbb{R}^3)}$. These together with Eq. (3.7) imply

\[(3.12) \quad \| O^+(t) \| \lesssim_c \frac{1}{(t)^2}.\]

Similarly, we have

\[(3.13) \quad \| O^-(t) \| \lesssim_c \frac{1}{(t)^2}.\]

□

**Proposition 3.3.** The operators $P^\pm e^{\pm itH_0}\Omega^*_\pm V(x)e^{\mp itH_0}, t > 0$, are compact on $L^2_x(\mathbb{R}^3)$. 

Proof. We prove the compactness of $O(t) \equiv P^+ e^{itH_0} \Omega^+ \nu(x) e^{-itH_0}$. $P^- e^{-itH_0} \Omega^- \nu(x) e^{itH_0}$ can be treated similarly. We decompose $P^+ e^{itH_0} \Omega^+ \nu(x) e^{-itH_0}$ into the sum of the low frequency part and the high frequency part
\[ O(t) = O_l(t) + O_h(t), \]
where $O_l(t)$ and $O_h(t)$ are given by
\begin{align}
O_l(t) &:= P^+ F(|p| < \frac{t^2 + 1}{t}) e^{itH_0} \Omega^+ \nu(x) e^{-itH_0} \\
O_h(t) &:= P^+ F(|p| \geq \frac{t^2 + 1}{t}) e^{itH_0} \Omega^+ \nu(x) e^{-itH_0}.
\end{align}

By the intertwining property
\[ F(|p| < \frac{t^2 + 1}{t}) \Omega^+ = \Omega^+ F(\sqrt{H} \frac{|x|}{t} < \frac{t^2 + 1}{t}), \]
$O_l(t)$ reads
\[ O_l(t) = P^+ e^{itH_0} \Omega^+ F(\sqrt{H} \frac{|x|}{t} < \frac{t^2 + 1}{t}) e^{-itH_0}. \]
Since $F(\sqrt{H} \frac{|x|}{t} < \frac{t^2 + 1}{t})$ is compact, due to Assumption 1.3, $F(\sqrt{H} \frac{|x|}{t} < \frac{t^2 + 1}{t}) V$ is compact and therefore (3.18) implies the compactness of $O(t)$ on $L^2_t(\mathbb{R}^n)$.

Next, we prove the compactness of $O_h(t)$. It requires estimates
\[ \| P^\pm F(|p| \geq \frac{t^2 + 1}{t}) e^{it(t+s)H_0} |p|^{-2} \| \lesssim_n \frac{t}{(t^2 + 1)(t+s)^2}, \quad \forall s \geq 0, \]
which follow from (2.20). To prove the compactness of $O_h(t)$, by the Duhamel principle, $O_h(t)$ reads
\[ O_h(t) = O_{h1}(t) + O_{h2}(t), \]
where $O_{hj}(t)$, $j = 1, 2$, are given by
\begin{align}
O_{h1}(t) &:= P^+ e^{itH_0} F(|p| \geq \frac{t^2 + 1}{t}) V \\
O_{h2}(t) &:= (-i) \int_0^\infty P^+ e^{i(t+s)H} F(|p| \geq \frac{t^2 + 1}{t}) V e^{-isH} V ds.
\end{align}

Since $\langle x \rangle^2 |p|^{-1} F(|p| \geq \frac{1}{2}) \langle x \rangle^{-3}$ is bounded on $L^2(x) \in \mathbb{R}^3$ and compact, by writing
\[ O_{h1}(t) = P^+ e^{itH_0} F(|p| \geq \frac{t^2 + 1}{t}) F(|p| \geq \frac{1}{8}) V \\
= P^+ e^{itH_0} F(|p| \geq \frac{t^2 + 1}{t}) |p|^{-2} \langle x \rangle^2 |p|^{-1} F(|p| \geq \frac{1}{8}) V \]
and using estimate (3.19), we conclude that $O_{h1}(t)$ is compact. Similarly, $P^+ e^{i(t+s)H} F(|p| \geq \frac{t^2 + 1}{t}) V e^{-isH}$ is compact. By estimate (3.19), we also obtain
\[ \| P^+ e^{i(t+s)H} F(|p| \geq \frac{t^2 + 1}{t}) V e^{-isH} \| \in L^1_{s}[0, \infty). \]

This together with the compactness of $P^+ e^{i(t+s)H_0} F(|p| \geq \frac{t^2 + 1}{t}) V e^{-isH}$ yields the compactness of $O_{h2}(t)$. The compactness of $O_{h1}(t)$ and $O_{h2}(t)$ implies the compactness of $O_h(t)$. And the compactness of $O_h(t)$ and $O_l(t)$ implies the compactness of $O(t)$. \qed

We also need the following Corollary for the proof of Theorem 1.1.
Corollary 3.1. If \((x)^2 V(x, t) \in L_{x,t}^{\infty}(\mathbb{R}^{n+1}), n \geq 1\), then \(P_{\pm} e^{itH_0} V(x, t + s), s > 0, t \in \mathbb{R}\), are compact operators on \(L_2^2(\mathbb{R}^n)\).

Proof. It suffices to show that \(B_{\pm} \equiv P_{\pm} e^{itH_0} \langle x \rangle^{-2}\) are compact. Since \(P_{\pm} F(|p| < \frac{s^2 + 1}{s}) e^{itH_0} \langle x \rangle^{-2}\) are compact, it suffices to show the compactness of \(P_{\pm} F(|p| \geq \frac{s^2 + 1}{s}) e^{itH_0} \langle x \rangle^{-2}\). For this, according to estimate (3.19), we have

\[
\| P_{\pm} F(|p|) \| \geq \frac{s^2 + 1}{s} e^{itH_0} \langle x \rangle^{-2} \lesssim_n \frac{s}{(s^2 + 1)s^2}.
\]

This together with

\[
P_{\pm} F(|p|) \geq \frac{s^2 + 1}{s} e^{itH_0} \langle x \rangle^{-2}
\]

and the compactness of \(\langle x \rangle^{-2} F(|p| \geq \frac{s^2 + 1}{10s}) \langle x \rangle^{-2}\) yields the compactness of \(B_{\pm}\). \(\square\)

Proof of Proposition 3.3. Note that \(C_c = C_{c,+} + C_{c,-}\), where the operators \(C_{c,\pm}\) are given by

\[
C_{c,\pm} := P_{\pm}(1 - \Omega_{\pm}) F_{c,1}(H).
\]

We prove that \(C_{c,+}\) is compact on \(L_2^2(\mathbb{R}^3)\) and the compactness of \(C_{c,-}\) will follow similarly. To prove the compactness of \(C_{c,+}\), we note by intertwining,

\[
C_{c,+} = P^+ F_{c,1}(H) - P^+ F_{c,1}(H_0) \Omega_{\pm}^*.
\]

By Lemma 3.1, it suffices to show that \(P^+ F_{c,1}(H_0)(1 - \Omega_{\pm}^*)\) is compact. For this, using relation \(1 = \Omega_{\pm}^* \Omega_+\) and the Duhamel principle to expand \(\Omega_+\), we obtain for all \(f \in L_2^2(\mathbb{R}^3)\),

\[
P^+ F_{c,1}(H_0) - P^+ F_{c,1}(H_0) \Omega_{\pm}^* = -i \int_0^\infty P^+ F_{c,1}(H_0) e^{itH_0} \Omega_{\pm}^* V(x) e^{-itH_0} dt.
\]

By Proposition 3.3 and the compactness of \(P^+ e^{itH_0} \Omega_{\pm}^*(1 - F_{c,1}(H)) \langle x \rangle^{-2}\), we conclude that

\[
\int_0^M P^+ F_{c,1}(H_0) e^{itH_0} \Omega_{\pm}^* V(x) e^{-itH_0} dt \text{ is compact for each } M > 0.
\]

On the other hand, by Proposition 3.2 and the unitarity of \(e^{-itH_0}\), we obtain

\[
\left\| \int_0^M P^+ F_{c,1}(H_0) e^{itH_0} \Omega_{\pm}^* V(x) e^{-itH_0} dt \right\| \leq \int_0^\infty \| P^+ F_{c,1}(H_0) e^{itH_0} \Omega_{\pm}^* \langle x \rangle^{-2} \| \| \langle x \rangle^2 V(x) \|_2 \| e^{-itH_0} \| dt \lesssim_c \| \langle x \rangle^3 V(x) \|_2 \int_0^\infty \frac{1}{|t|^2} dt \lesssim_c \frac{1}{M} \| \langle x \rangle^3 V(x) \|_2 \rightarrow 0,
\]

as \(M \rightarrow \infty\). Hence, we conclude that \(P^+ F_{c,1}(H_0) - P^+ F_{c,1}(H_0) \Omega_{\pm}^*\) is compact and therefore, \(C_{c,+}\) is compact. Similarly, \(C_{c,-}\) is compact. Thus, we conclude the compactness of \(C_c\). \(\square\)
3.2. Decomposition of the Operator $C_c$. Recall that

\begin{equation}
F_M(x,p) = F(|x| < M)F(|p| \geq \frac{1}{M})
\end{equation}

and

\begin{equation}
C_c = \left( P^+ (1 - \Omega^+_+(t)) + P^- (1 - \Omega^-_-(t)) \right) F_{c,1}(H).
\end{equation}

We define

\begin{equation}
C_{M,c} := C_c \Omega^+_+ F_M(x,p) \Omega^+_+
\end{equation}

and

\begin{equation}
C_{r,c} := C_c (1 - F_M(x,p)) \Omega^+_+.
\end{equation}

Since $F(H \geq c) = F(H \geq c)P_c$, then we have

\begin{equation}
C_c = C_{M,c} + C_{r,c}.
\end{equation}

In this section, we show that there exists $M_0 \geq 1$ such that for all $\psi_0 \in L^2_x(\mathbb{R}^3)$,

\begin{equation}
\| C_{M,c} e^{-itH} P_c \psi_0 \|_{L^2_t(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim_M \| \psi_0 \|,
\end{equation}

and for all $\eta > 1$ and $f \in L^2_x(\mathbb{R}^3)$,

\begin{equation}
\| (x)^{-\eta}(1 - C_{r,c})^{-1} P^\pm e^{-itH_0} f \|_{L^2_t(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim_\eta \| f \|.
\end{equation}

**Proposition 3.4.** There exists $M_0 \geq 1$ such that for all $M \geq M_0$, with the decomposition $C_c = C_{M,c} + C_{r,c}$, estimates (3.38)–(3.40) hold true for all $\eta > 1$.

The proof of Proposition 3.4 requires the well-known result Lemma 3.2.

**Lemma 3.2.** For all $c > 0$ and $\epsilon \in (0, 1/2)$, when $n \geq 3$,

\begin{equation}
\|F_{c,1}(H)(1 - F_{c,2}(H_0))|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim c 1
\end{equation}

and

\begin{equation}
\|F_{c,1}(H) - F_{c,1}(H_0))F_{c,2}(H_0)|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim c 1.
\end{equation}

**Proof.** See Appendix A.

**Lemma 3.3.** The estimate (3.38) holds true for all $M \geq 1$.

**Proof.** By the intertwining property and Eq. (3.35), $C_{M,c} e^{-iH} P_c \psi_0$ reads

\begin{equation}
C_{M,c} e^{-iH} P_c \psi_0 = C_c \Omega^+_+ F_M(x,p) e^{-iH_0} \Omega^+_+ \psi_0.
\end{equation}

By $L^2$ local decay estimate and the existence of $\Omega^+_+$, we note

\begin{equation}
\| F_M(x,p) e^{-iH_0} \Omega^+_+ \psi_0 \|_{L^2_t(\mathbb{R}^{3+1})} \lesssim_M \| \psi_0 \|.
\end{equation}

By the boundedness of $C$, $F(H \geq c)$, $\Omega^+_+$ and $\Omega^+_+(\text{see Corollary 2.1})$ for the boundedness of $C$, this together with Eq. (3.38) implies

\begin{equation}
\int \| C_{M,c} e^{-iH} P_c \psi_0 \|^2 dt \lesssim \int \| F_M(x,p) e^{-iH_0} \Omega^+_+ \psi_0 \|^2 dt
\end{equation}

\begin{equation}
\lesssim \int \| F(|x| < M) e^{-iH_0} F(|p| \geq \frac{1}{M}) \Omega^+_+ \psi_0 \|^2 dt
\end{equation}

\begin{equation}
\lesssim_M \| \psi_0 \|^2,
\end{equation}

that is,

\begin{equation}
\| C_{M,c} e^{-iH} P_c \psi_0 \|_{L^2_t(\mathbb{R}^{3+1})} \lesssim_M \| \psi_0 \|.
\end{equation}
**Proposition 3.5.** There exists $M_0 \geq 1$ large enough such that the estimate (3.39) holds true for all $M \geq M_0$.

**Proof.** Let $P_b := 1 - P_c$ be the projection on the space of discrete spectrum of $H$. By Proposition 3.1, $C_c$ is compact and can therefore be approximated by a finite-rank operator: with $C_cP_b = CF(H \geq c) P_b = 0$,

\begin{equation}
\|C_c - P_{c,N}\| \leq \frac{1}{1000},
\end{equation}

where the finite-rank operator $P_{c,N}$, for some $N \in \mathbb{N}^+$, is given by

\begin{equation}
P_{c,N} := \sum_{l=1}^{N} c_l(\phi_l(x),\cdot)_{L_x^2(\mathbb{R}^3)} \psi_l(x)
\end{equation}

with constants $c_l \in \mathbb{C} - \{0\}, j = 1, \cdots, N$, and $\psi_l, \phi_l \in L_x^2(\mathbb{R}^3)$ satisfying

\begin{equation}
\begin{cases}
\|\psi_l\|_{L_x^2(\mathbb{R}^3)} = 1, \\
\phi_l = P_c \phi_l, \\
\|\phi_l\|_{L_x^2(\mathbb{R}^3)} = 1.
\end{cases}
\end{equation}

Estimate (3.47) implies

\begin{equation}
\| (C_c - P_{c,N}) \Omega_+ (1 - F_M(x,p)) \Omega_+^* \| \leq \|C_c - P_{c,N}\| \leq \frac{1}{1000}.
\end{equation}

Using that

\begin{equation}
\text{s- \lim}_{M \to \infty} 1 - F_M(x,p) = 0, \quad \text{on } L_x^2(\mathbb{R}^3),
\end{equation}

$N \in \mathbb{N}^+$ and $AC$ (i.e. $\phi_l = P_c \phi_l = \Omega_+ \Omega_+^* \phi_l, l = 1, \cdots, N$), we obtain that there exists $M_0 > 0$ such that for all $M \geq M_0$,

\begin{equation}
\|\Omega_+ (1 - F_M(x,p)) \Omega_+^* \phi_l\| = \|(1 - F_M(x,p)) \Omega_+^* \phi_l\| < \frac{1}{1000 N |c_l|}, \quad \forall l = 1, \cdots, N.
\end{equation}

This together with (3.48) yields

\begin{equation}
\|P_{c,N} \Omega_+ (1 - F_M(x,p)) \Omega_+^* \| \leq \frac{1}{1000}.
\end{equation}

Estimates (3.50) and (3.53) together with Eq. (3.36) yield (3.39) for all $M \geq M_0$. \qed

**Proposition 3.6.** For all $M \geq 1$ and $c > 0$, we have

\begin{equation}
\|C_{c,M}|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim_M 1
\end{equation}

holds true for all $\epsilon \in (0, 1/2)$.

and

**Proposition 3.7.** Let $M_0$ be as in Proposition 3.5. The estimate (3.40) holds true for all $M \geq M_0$.

The proof of Propositions 3.5 and 3.7 requires

**Proposition 3.8.** Recall $F_{c,1}(H_0) = F(H_0 \geq \frac{c}{10})$ and $F_{c,2}(H_0) = F(H_0 \geq \frac{c}{100}), c > 0$. The estimates

\begin{equation}
\|P^\pm (\Omega_+^* - 1) F_{c,1}(H) |p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim_{c, \epsilon} 1
\end{equation}

hold true for all $c \in (0, 1/2)$.

The proof of Proposition 3.8 requires
Lemma 3.4.
\begin{equation}
\|F_{c,1}(H)(1 - F_{c,2}(H_0))|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim_{c,\epsilon} 1
\end{equation}
holds true for all $\epsilon \in (0,1/2)$ and $c > 0$.

Lemma 3.5.
\begin{equation}
\|(F_{c,1}(H) - F_{c,1}(H_0))F_{c,2}(H_0)|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim_{c,\epsilon} 1
\end{equation}
holds true for all $\epsilon \in (0,1/2)$ and $c > 0$.

Lemma 3.6. Let
\begin{equation}
\mathcal{O}(t) := \langle x \rangle^{-3/2}e^{-itH_0}|p|^{-1/2}(x)^{1/2+\epsilon}.
\end{equation}
For all $\epsilon \in (0,1/2)$, $c > 0$, $n \geq 3$ and $f \in \mathcal{L}_2^2(\mathbb{R}^n)$, $\mathcal{O}(t)$ satisfies
\begin{equation}
\left( \int |t|^{-2}\|\mathcal{O}(t)f\|^2dt \right)^{1/2} \lesssim_{n,c} \|f\|.
\end{equation}

Proof. To estimate $\mathcal{O}_\chi(|x| \leq 1)$, we note that by Hardy-Littlewood-Sobolev inequality,
\begin{equation}
\|\mathcal{O}_\chi(|x| \leq 1)\| \leq \|\langle x \rangle^{-3/2}|x|^{1/2}\|\|x|^{-1/2}\|\|\langle x \rangle^{1/2+\epsilon}\chi(|x| \leq 1)\| \lesssim 1.
\end{equation}
Next, we estimate $\mathcal{O}_\chi(|x| > 1)$. To estimate $\mathcal{O}_\chi(|x| > 1)$, we decompose $\mathcal{O}_\chi(|x| > 1)$ into $n$ pieces
\begin{equation}
\mathcal{O}_\chi(|x| > 1) = \sum_{j=1}^{n} \mathcal{O}_j,
\end{equation}
where operators $\mathcal{O}_j, j = 1, \cdots, n$, are given by
\begin{equation}
\mathcal{O}_j := \mathcal{O}_\chi(|x| > 1)F_j(x), \quad j = 1, \cdots, n,
\end{equation}
with $\{F_j\}_{j=1}^{n}$ a partition of unity, satisfying
\begin{equation}
|x_j| \geq \frac{1}{n}|x| \quad \text{for all } x \in \text{supp}(F_j), \quad j = 1, \cdots, n.
\end{equation}
We estimate $\mathcal{O}_1$ and the estimates for $\mathcal{O}_j, j = 2, \cdots, n$, can be derived similarly. Using equation
\begin{align}
e^{-itH_0}|p|^{-1/2}(x)^{1/2+\epsilon}F_1(x) \\
= x_1e^{-itH_0}|p|^{-1/2}(x)^{1/2+\epsilon}F_1(x) \\
- 2t\frac{p_1}{|p|^{1/2}}e^{-itH_0}(x)^{1/2+\epsilon}F_1(x) \\
+ \|p|^{-1/2}, x_1 \rangle e^{-itH_0}(x)^{1/2+\epsilon}F_1(x)
\end{align}
we decompose $\mathcal{O}_1$ into three parts
\begin{equation}
\mathcal{O}_1 = \sum_{j=1}^{3} \mathcal{O}_{1j},
\end{equation}
where operators $\mathcal{O}_{1j}, j = 1, 2, 3$, are given by
\begin{align}
\mathcal{O}_{11} := \langle x \rangle^{-3/2}x_1e^{-itH_0}|p|^{-1/2}(x)^{1/2+\epsilon}F_1(x) \\
\mathcal{O}_{12} := -2t(x)^{-3/2}\frac{p_1}{|p|^{1/2}}e^{-itH_0}(x)^{1/2+\epsilon}F_1(x) \\
\mathcal{O}_{13} := \langle x \rangle^{-3/2}|p|^{-1/2}, x_1 \rangle e^{-itH_0}(x)^{1/2+\epsilon}F_1(x)
\end{align}
and
\begin{align}
\mathcal{O}_{14} := \langle x \rangle^{-3/2}|p|^{-1/2}, x_1 \rangle e^{-itH_0}(x)^{1/2+\epsilon}F_1(x).
\end{align}
Similarly, we have
\[
\|O_{11}\| \leq \sup_{t \in \mathbb{R}} \|\langle x \rangle^{-3/2} |x|^{1/2} \| |x|^{-1/2} \| |e^{-itH_0}\| \|\langle x \rangle^{1/2+\epsilon} \frac{F_1(x)}{x_1} \chi(|x| > 1)\| \lesssim_n 1
\]
and with \( n \geq 3 \),
\[
\|O_{13}\| \leq \sup_{t \in \mathbb{R}} \|\langle x \rangle^{-3/2} |x|^{1+\epsilon} \| |x|^{-1-\epsilon} \| |p|^{-1-\epsilon} \| |\langle x \rangle^{-1/2+\epsilon} \frac{F_1(x)}{x_1} \chi(|x| > 1)\| \lesssim_{c,n} 1.
\]
By \( L^2 \) local smoothing estimate, \( O_{1,2} \) satisfies
\[
\int \langle t \rangle^{-2} \|O_{12} f\|^2 dt \lesssim \int \|\langle x \rangle^{-3/2} |p|^{1/2} e^{-itH_0} \frac{p_1}{|p|} \langle x \rangle^{1/2+\epsilon} \frac{F_1(x)}{x_1} \chi(|x| > 1) f\|^2 dt \lesssim \|\frac{p_1}{|p|} \langle x \rangle^{1/2+\epsilon} \frac{F_1(x)}{x_1} \chi(|x| > 1) f\|^2 \lesssim_n \| f\|^2
\]
for all \( f \in L^2_+(\mathbb{R}^n) \). This together with estimates (3.69) and (3.70) yields
\[
\left( \int \langle t \rangle^{-2} \|O_{1} f\|^2 dt \right)^{1/2} \lesssim_{c,n} \| f\|, \quad f \in L^2_+(\mathbb{R}^n).
\]
Similarly, we have
\[
\left( \int \langle t \rangle^{-2} \|O_{j} f\|^2 dt \right)^{1/2} \lesssim_{c,n} \| f\|, \quad f \in L^2_+(\mathbb{R}^n), \quad j = 2, 3.
\]
Hence, we conclude (3.69). \qed

and

**Lemma 3.7.** Recall that \( F_{c,1}(H_0) = F(H_0 \geq \frac{c}{10}) \) and \( F_{c,2}(H_0) = F(H_0 \geq \frac{c}{100}) \) for \( c > 0 \). The estimates
\[
\| P^\pm F_{c,1}(H_0)(\Omega^*_+ - 1) F_{c,2}(H_0) |p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \| \lesssim_{c, \epsilon} 1
\]
hold true for all \( c > 0 \) and \( \epsilon \in (0, \frac{1}{2}) \).

**Proof.** We write \( F_{c,1} = F_{c,1}(H_0) \) and \( F_{c,2} = F_{c,2}(H_0) \) in this proof if the context is clear. We estimate \( B^+ \equiv P^+ F_{c,1}(\Omega^*_+ - 1) F_{c,2} |p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \). The case of \( B^- \equiv P^- F_{c,1}(\Omega^*_+ - 1) F_{c,2} |p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \) can be treated similarly. For this, by the Duhamel principle, \( B^+ \) reads
\[
B^+ = (-i) \int_0^\infty P^+ F_{c,1} e^{itH_0} \Omega^*_+ V e^{-itH_0} F_{c,2} |p|^{-1/2} \langle x \rangle^{1/2+\epsilon} dt.
\]
By Proposition 3.3.2 this implies for all \( f \in L^2_+(\mathbb{R}^3) \),
\[
\|B^+ f\| \lesssim_c \int_0^\infty \| P^+ F_{c,1} e^{itH_0} \Omega^*_+ \langle x \rangle^{-2} \| \| \langle x \rangle^{7/2} V \|_{L^2_2} \| \| \langle x \rangle^{-3/2} F_{c,2} \langle x \rangle^{3/2} \| \|O(t) f\|| dt \lesssim_c \int_0^\infty \frac{1}{|t|^{7/2}} \| \langle x \rangle^{7/2} V \|_{L^2_2} \|O(t) f\|| dt.
\]
where operator $O(t)$ is defined in Eq. (3.58). Applying Lemma 3.6 and Cauchy-Schwarz inequality to (3.76), we arrive at
\[
\|B + f\| \lesssim c \int_{\mathbb{R}^3} (x)^{7/2} V(x) \left\|O(t) f\right\| dt
\]
\[
\lesssim c \| (x)^{7/2} V(x) \|_{L^\infty}\left( \int \frac{1}{(t)^2} \|O(t) f\|^2 dt \right)^{1/2}
\]
(3.77)
\]

Next, we prove Proposition 3.8.

**Proof of Proposition 3.8.** Let
\[
B^\pm := \mathcal{P}_\pm (\Omega^*_\pm - \mathbb{1}) F_c(1 - F_{c,2}(H_0)) |p|^{-1/2}(x)^{1/2+\epsilon}.
\]
We estimate $B^+$ and the estimate for $B^-$ can be derived similarly. We decompose $B^+$ into two parts
\[
B^+ = B^+_1 + B^+_2,
\]
(3.79)
where the operators $B^+_j, j = 1, 2,$ are given by
\[
B^+_1 := \mathcal{P}_+ (\Omega^*_+ - \mathbb{1}) F_{c,1}(1 - F_{c,2}(H_0)) |p|^{-1/2}(x)^{1/2+\epsilon}
\]
and
\[
B^+_2 := \mathcal{P}_+ (\Omega^*_+ - \mathbb{1}) F_{c,1}(H_0) |p|^{-1/2}(x)^{1/2+\epsilon},
\]
respectively. By Lemma 3.4, we obtain
\[
\|B^+_1\| \leq 2 \|F_{c,1}(1 - F_{c,2}(H_0)) |p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim c 1.
\]
Then it suffices to estimate $B^-_2$. For this, by intertwining, we write $B^-_2$ as
\[
B^-_2 = B^-_{21} + B^-_{22},
\]
(3.83)
where $B^-_{2j}, j = 1, 2,$ are given by
\[
B^-_{21} := \mathcal{P}_+ F_{c,1}(H_0)(\Omega^*_+ - \mathbb{1}) F_{c,2}(H_0) |p|^{-1/2}(x)^{1/2+\epsilon}
\]
and
\[
B^-_{22} := \mathcal{P}_+ (F_{c,1}(H_0) - F_{c,1}(H_0)) F_{c,2}(H_0) |p|^{-1/2}(x)^{1/2+\epsilon}.
\]
By Lemma 3.7, we have
\[
\|B^-_{21}\| \lesssim c, 1.
\]
(3.86)
By Lemma 3.5, we have
\[
\|B^-_{22}\| \lesssim c, 1.
\]
(3.87)
Estimates (3.86) and (3.87) yield
\[
\|B^-\| \lesssim c, 1.
\]
(3.88)
This together with estimate (3.82) yields
\[
\|B^-\| \lesssim c, 1.
\]
(3.89)
Similarly, we have
\[
\|B^-\| \lesssim c, 1.
\]
(3.90)
Now we prove Propositions 3.6 and 3.7.
Proof of Proposition 3.6. Recall that
\[ C_{M,c} = C_c \Omega_+ F_M(x,p) \Omega^*_+. \]
To estimate \( C_{M,c} \), by the Duhamel’s principle, \( C_{M,c} \) reads
\[ C_{M,c} = C_{M,c1} + C_{M,c2}, \]
where
\[ C_{M,c1} := C_c \Omega_+ F_M(x,p) |p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \]
and
\[ C_{M,c2} := (-i) \int_0^\infty C_c \Omega_+ F_M(x,p) e^{itH_0} \Omega^*_+ V(x) e^{-itH_0} |p|^{-1/2} \langle x \rangle^{1/2+\epsilon}. \]

By the definition of \( F_M(x,p) \) (see Eq. (3.33)) and the boundedness of \( \Omega_+ \) and \( C \) (see Corollary 2.1), \( C_{M,c1} \) satisfies
\[ \| C_{M,c1} \| \lesssim \| F_M(x,p) \| |p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \| \lesssim_M 1. \]

Next, we estimate \( C_{M,c2} \). To estimate \( C_{M,c2} \), we note
\[ \| F_M(x,p) e^{itH_0} \Omega^*_+ \langle x \rangle^{-2} \| \lesssim_M \frac{1}{|t|^2}. \]

Indeed we have
\[ \| F_M(x,p) e^{itH_0} \langle x \rangle^{-3} \| \lesssim_M \frac{1}{|t|^3} \quad \forall t \in \mathbb{R}. \]

Using this estimate and the unitarity of \( e^{-isH} \), we obtain estimate
\[ \| \int_0^\infty F_M(x,p) e^{i(t+s)H_0} V e^{-isH} ds \| \lesssim_M \int_0^\infty \frac{1}{(t+s)^3} \| \langle x \rangle^3 V \|_{L^2} \| e^{-isH} \| ds \]
\[ \lesssim_M \frac{1}{|t|^2} \| \langle x \rangle^3 V \|_{L^2} \]
for all \( t \geq 0 \). This together with
\[ \| F_M(x,p) e^{itH_0} \langle x \rangle^{-2} \| \lesssim_M \frac{1}{|t|^2} \quad \forall t \in \mathbb{R} \]
and the Duhamel’s principle yields (3.96).

Together with the boundedness of \( C \) (see Corollary 2.1) and \( \Omega_+ \) and Eq. (3.94) yields
\[ \| C_{M,c2} \| \leq \int_0^\infty \| F_M(x,p) e^{itH_0} \Omega^*_+ \langle x \rangle^{-2} \| \| \langle x \rangle^7/2 V \|_{L^2} \| O(t) \| dt \]
\[ \lesssim_M \int_0^\infty \frac{1}{|t|^2} \| \langle x \rangle^7/2 V \|_{L^2} \| O(t) \| dt. \]

Here, recall that
\[ O(t) = \langle x \rangle^{-3/2} e^{-itH_0} |p|^{-1/2} \langle x \rangle^{1/2+\epsilon}. \]

Applying Lemma 3.6 and Cauchy-Schwarz inequality to estimate (3.100), we arrive at
\[ \| C_{M,c2} \| \lesssim_{M,\epsilon} \| \langle x \rangle^7/2 V \|_{L^2}. \]

This together with estimate (3.95) and Eq. (3.92) yields
\[ \| C_{M,c} \| \lesssim_{M,\epsilon} 1. \]

We complete the proof. \( \square \)
Proof of Proposition 3.7. Using estimate (3.55) and Eqs. (3.27) and (3.28), we obtain

$$||C_c||^{-1/2}_p (x)^{1/2} < 1$$

for all $\epsilon \in (0, 1/2)$. This together with Proposition 3.6 and $C_{r,c} = C_c - C_{M,c}$ yields

$$||C_{r,c}P^\pm e^{-itH_0}f||_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R}^+)} \leq \left( \int_0^\infty ||C_{r,c}P^\pm e^{-itH_0}f||^2 dt \right)^{1/2}$$

(3.105)

$$\lesssim_{M,c} \left( \int_0^\infty \langle x \rangle^{-1/2-\epsilon} p^{1/2} ||C_{r,c}P^\pm e^{-itH_0}f||^2 dt \right)^{1/2}.$$

By estimates (2.26) and (2.28), this yields

$$||C_{r,c}P^\pm e^{-itH_0}f||_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim ||f||,$$

which leads to, by $L^2$ local decay estimate and $L^2$ local smoothing estimate,

$$||C_{r,c}P^\pm e^{-itH_0}f||_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim ||f||.$$  

(3.106)

Therefore, using $(1 - C_{r,c})^{-1} = 1 + (1 - C_{r,c})^{-1}C_{r,c}$ and subsequently

$$||\langle x \rangle^{-\eta}(1 - C_{r,c})^{-1}P^\pm e^{-itH_0}f|| \lesssim ||\langle x \rangle^{-\eta}P^\pm e^{-itH_0}f|| + ||\langle x \rangle^{-\eta}(1 - C_{r,c})^{-1}C_{r,c}P^\pm e^{-itH_0}f||$$

(3.107)

$$\lesssim ||\langle x \rangle^{-\eta}e^{-itH_0}f|| + ||C_{r,c}P^\pm e^{-itH_0}f||,$$

we arrive at

$$||\langle x \rangle^{-\eta}(1 - C_{r,c})^{-1}P^\pm e^{-itH_0}f||_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R}^+)} \lesssim ||\langle x \rangle^{-\eta}e^{-itH_0}f||_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R}^+)} + ||C_{r,c}P^\pm e^{-itH_0}f||_{L^2_{x,t}(\mathbb{R}^3 \times \mathbb{R}^+)}$$

(3.109)

for all $\eta > 1.$

3.3. Proof of Theorem 1.3.

Proof for Theorem 1.3. Using Eqs. (3.33), (3.34) and (3.36) and that $C_cP_0 = 0$, we obtain

$$\psi(t) = \psi_f(t) + C_{M,c}\psi(t) + C_{r,c}\psi(t).$$

(3.110)

Using (3.39), we have $(1 - C_{r,c})^{-1}$ is bounded on $L^2_x(\mathbb{R}^3)$. Then moving $C_{r,c}\psi(t)$ to the left-hand side of the Eq. (3.110) and applying $(1 - C_{r,c})^{-1}$ to both sides of the Eq. (3.110), we arrive at

$$\psi(t) = (1 - C_{r,c})^{-1}\psi_f(t) + (1 - C_{r,c})^{-1}C_{M,c}\psi(t).$$

(3.111)

This together with estimates (3.38) and (3.40) yields (1.22). □

4. Time-dependent Problems

In this section, we prove Theorems 1.1 and 1.2. We take the space dimension $n \geq 5$ in this section.

4.1. Compactness of $C(t)$ and its decomposition. Recall that

$$C(t) := P^+(1 - \Omega^+_+(t)) + P^- (1 - \Omega^-_-(t)).$$

We prove the compactness of $C(t)$ in this subsection.

Lemma 4.1. If $\langle x \rangle^3 V(x, t) \in L^\infty_x(\mathbb{R}^{n+1})$, then $C(t)$ is compact on $L^2_x(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.

Proof. It suffices to show the compactness of $P^\pm (1 - \Omega^+_+(t))$. We prove the compactness of $P^+(1 - \Omega^+_+(t))$ and the compactness of $P^- (1 - \Omega^-_-(t))$ can be derived similarly. By the Duhamel’s principle, $P^+(1 - \Omega^+_+(t))$ reads

$$P^+(1 - \Omega^+_+(t)) = i \int_0^\infty P^+ e^{isH_0} V(x, s + t) U(t + s, t) ds.$$
To prove the compactness of $P^+\left(\mathbb{I} - \Omega^+_+(t)\right)$, by Corollary 3.1 and Assumption 1.1 we obtain the compactness of $i \int_0^M P^+ e^{i s H_0} V(x, s + t) U(t + s, t) ds$. This together with estimate

$$
\| \int_0^\infty P^+ e^{i s H_0} V(x, s + t) U(t + s, t) ds \| \leq \int_0^\infty \| P^+ e^{i s H_0} \|^{\frac{3}{2}} \| V(x, t) \|_{\mathcal{L}_r^{\infty}(\mathbb{R}^{n+1})} ds
$$

as $M \to \infty$, yields the compactness of $P^+\left(\mathbb{I} - \Omega^+_+(t)\right)$. Here, in [13], we used estimate (2.22), with $\alpha = 0$, $n \geq 5$ and $\delta = 3$, and the unitarity of $U(t + s, t)$. \qed

Next, we prove the decomposition property of $C(t)$. Similar to the time-independent case, we note that $C(t)$ can be expressed as the sum of three operators

$$
C(t) = C_M(t) + C_r(t) + C(t) P_b(t),
$$

where the operators $C_M(t)$ and $C_r(t)$ are given by for $M > 0$,

$$
C_M(t) := C(t) \Omega_+(t) F_M(x, p) \Omega^*_+(t)
$$

and

$$
C_r(t) := C(t) \Omega_+(t) (\mathbb{I} - F_M(x, p)) \Omega^*_+(t).
$$

Similarly, we use the intertwining

$$
\Omega^*_+(t) U(t, 0) = e^{-it H_0} \Omega^*_+(0), \quad \text{on } \mathcal{L}_r^2(\mathbb{R}^n)
$$

to obtain

$$
C_M(t) U(t, 0) f = C(t) \Omega_+(t) F_M(x, p) e^{-it H_0} \Omega^*_+(0) f.
$$

We show in what follows that there exists $M_0 > 0$ such that whenever $M \geq M_0$,

$$
\sup_{t \in \mathbb{R}} \| C_r(t) \| < \frac{1}{2}.
$$

**Proposition 4.1.** If Assumption 1.1 is satisfied, then there exists $M_0 > 0$ such that whenever $M \geq M_0$, (4.9) holds true.

The proof of Proposition 4.1 relies on Corollary 2.1. By Corollary 2.1 we have $P_b(t)$, $U(T + t, t)$, $\Omega_+(t)$ and $\Omega^*_+(t)$ are quasi-periodic in $t$ with the same type as $V$. So we could use finitely many parameters $s_j$, $j = 1, \cdots, N$ (the same as those for $V$), which are in compact set $\overline{T}_1 \times \cdots \times \overline{T}_N$ to express these operators: with $\tilde{s} = (s_1, \cdots, s_N)$,

$$
\tilde{P}_b(\tilde{s}) := P_b(t), \quad \tilde{U}_x(T, 0) := U(T + t, t), \quad \tilde{\Omega}_+(\tilde{s}) := \Omega_+(t) \quad \text{and} \quad \tilde{\Omega}^*_+(\tilde{s}) := \Omega^*_+(t)
$$

for

$$
s_j = t \mod T_j, \quad j = 1, \cdots, N.
$$

We prove Proposition 4.1 by showing

**Proposition 4.2.** For all $\tilde{s} \in \overline{T}_1 \times \cdots \times \overline{T}_N$,

$$
\lim_{\tilde{u} \to \tilde{s}} \tilde{P}_b(\tilde{u}) = \tilde{P}_b(\tilde{s}).
$$

The proof of Proposition 4.2 requires two lemmas below.

**Lemma 4.2.** For each pair $\tilde{s}, \tilde{u} \in \overline{T}_1 \times \cdots \times \overline{T}_N$,

$$
dim \tilde{P}_b(\tilde{s}) = dim \tilde{P}_b(\tilde{u}) < \infty.
$$
Proof. By (11), we obtain that the kernel of $\Omega_+^*(t)$ is equal to the space of all eigenfunctions of the Floquet operator $K$ with $s = t$. This yields $\text{Ran}(\Omega_+^*(t)) = \text{Ran}(\Omega_-^*(t))$. Therefore for each $\psi \in \text{Ran}(P_b(t))$, we have $\Omega_+^*(t)\psi = \Omega_-^*(t)\psi = 0$, which implies
\begin{equation}
\psi = P^+(1 - \Omega_+^*(t))\psi + P^-(1 - \Omega_-^*(t))\psi = C(t)\psi.
\end{equation}
By Lemma 4.1, this implies that $\psi$ is an eigenfunction of the compact operator $C(t)$ associated with an eigenvalue $1$. Hence, $\dim(P_b(t)) < \infty$ for each $t \in \mathbb{R}$. This together with equation
\begin{equation}
P_b(t) = 1 - \Omega_+^*(t)\Omega_+^*(t) = U(t, 0) (1 - \Omega_+^*(0)\Omega_+^*(0)) U(0, t) = U(t, 0)P_b(0)U(0, t) \quad \forall t \in \mathbb{R}
\end{equation}
leads to
\begin{equation}
dim(P_b(t)) = \dim(P_b(0)) < \infty,
\end{equation}
which together with Eq. (4.10) and the connection between the evolution operators and the corresponding Floquet operators implies (4.13). $\square$

Lemma 4.3. If Assumption [L1] is satisfied, then
\begin{equation}
\lim\limits_{\tilde{u} \to \tilde{s}} \| (\tilde{\Omega}_+^*(\tilde{s}) - \tilde{\Omega}_+^*(\tilde{u})) F_M(x, p) \| = \lim\limits_{\tilde{u} \to \tilde{s}} \| F_M(x, p)(\tilde{\Omega}_+^*(\tilde{s}) - \tilde{\Omega}_+^*(\tilde{u})) \| = 0.
\end{equation}
Proof. We prove
\begin{equation}
\lim\limits_{\tilde{u} \to \tilde{s}} \| (\tilde{\Omega}_+^*(\tilde{s}) - \tilde{\Omega}_+^*(\tilde{u})) F_M(x, p) \| = 0
\end{equation}
and
\begin{equation}
\lim\limits_{\tilde{u} \to \tilde{s}} \| F_M(x, p)(\tilde{\Omega}_+^*(\tilde{s}) - \tilde{\Omega}_+^*(\tilde{u})) \| = 0
\end{equation}
can be proven similarly. (4.18) follows from
\begin{equation}
\lim\limits_{\tilde{u} \to \tilde{s}} \| F_M(x, p)(e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0) - e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0)) \| = 0
\end{equation}
and
\begin{equation}
\lim\limits_{T \to \infty} \sup\limits_{\tilde{u} \in \mathbb{T} \times \cdots \times \mathbb{T}} \| F_M(x, p)(\tilde{\Omega}_+^*(\tilde{u}) - e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0))f \| = 0.
\end{equation}
Let $\tilde{V}_\tilde{u}(x, T) := V(x, t + T)$ whenever $\tilde{s}$ and $t$ satisfy the condition (4.11). To prove (4.20), we use the Duhamel’s principle to write
\begin{equation}
F_M(x, p)(e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0) - e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0)) = (-i) \int_0^T F_M(x, p)e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, t)(\tilde{V}_\tilde{u}(x, t) - \tilde{V}_\tilde{u}(x, t))\tilde{U}_\tilde{u}(t, 0) dt.
\end{equation}
This together with Condition (4.10) and the unitarity of $e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, t)$ and $\tilde{U}_\tilde{u}(t, 0)$ yields (4.20).
To prove (4.21), by the Duhamel’s principle, we obtain
\begin{equation}
F_M(x, p)(e^{iT\hat{H}_0}\tilde{\Omega}_+^*(\tilde{u}) - e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0)) = (-i) \int_T^{\infty} F_M(x, p)e^{iT\hat{H}_0}\tilde{V}_\tilde{u}(x, t)\tilde{U}_\tilde{u}(t, 0) dt.
\end{equation}
This together with estimate
\begin{equation}
\| F_M(x, p)e^{iT\hat{H}_0}\tilde{\langle x \rangle}^{-2} \| \lesssim_M \frac{1}{(t)^2}
\end{equation}
and the unitarity of $\tilde{U}_\tilde{u}(t, 0)$ yields
\begin{equation}
\| F_M(x, p)(e^{iT\hat{H}_0}\tilde{\Omega}_+^*(\tilde{u}) - e^{iT\hat{H}_0}\tilde{U}_\tilde{u}(T, 0)) \| \lesssim_M \int_T^{\infty} \frac{1}{(t)^2} \| \tilde{\langle x \rangle}^2 \tilde{V}(x, w) \|_{L_{\infty,1}^\infty(\mathbb{R}^{n+1})} dt.
\end{equation}
This leads to (4.21). $\square$

Next, we prove Proposition 4.2.
Proof of Proposition 4.2. We first claim that for all \( \mathcal{S} \in \mathbb{R}^N \),
\[
\text{s-lim}_{\tilde{u} \rightarrow \mathcal{S}} \hat{P}_c(\tilde{u}) \hat{P}_s(\mathcal{S}) = \hat{P}_c(\mathcal{S}) \quad \text{on } L^2_s(\mathbb{R}^n)
\]
and prove the claim in the end. (4.26) implies
\[
\text{s-lim}_{\tilde{u} \rightarrow \mathcal{S}} \hat{P}_b(\tilde{u}) \hat{P}_c(\mathcal{S}) = 0 \quad \text{on } L^2_s(\mathbb{R}^n),
\]
which in turn implies
\[
\text{s-lim}_{\tilde{u} \rightarrow \mathcal{S}} \tilde{P}_b(\tilde{u}) - \hat{P}_b(\tilde{u}) \hat{P}_b(\mathcal{S}) = 0 \quad \text{on } L^2_s(\mathbb{R}^n).
\]
By (4.33), we arrive at
\[
\text{s-lim}_{\tilde{u} \rightarrow \mathcal{S}} \hat{P}_c(\tilde{u}) - \hat{P}_b(\tilde{u}) \hat{P}_b(\mathcal{S}) = 0 \quad \text{on } L^2_s(\mathbb{R}^n).
\]
Now we prove Proposition 4.1. Following the proof of Proposition 3.5, we note that it suffices to show that for each
\[
\text{s-lim}_{\tilde{u} \rightarrow \mathcal{S}} \tilde{P}_c(\tilde{u}) - \Omega_+(\mathcal{S}) \Omega_+(\mathcal{S}) f = 0,
\]
holds true. By Corollary 2.1, this is equivalent to
\[
\text{s-lim}_{\tilde{u} \rightarrow \mathcal{S}} \text{sup } \| \tilde{P}_c(\tilde{u}) - \Omega_+(\mathcal{S}) \Omega_+(\mathcal{S}) f \| = 0,
\]
where the operator $\tilde{\Omega}^*_+(\tilde{s})$ is given by
\begin{equation}
\tilde{\Omega}^*_+(\tilde{s}) = \Omega^*_+(t)
\end{equation}
for
\begin{equation}
\begin{aligned}
s_j &= t \mod T_j, \\
j &= 1, \ldots, N.
\end{aligned}
\end{equation}
We note that for each $\tilde{s} \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_N$,
\begin{equation}
\lim_{M \to \infty} \| (1 - F_M(x,p)) \tilde{\Omega}^*_+(\tilde{s}) f \| = 0.
\end{equation}
By Lemma 4.3 and (4.42), we obtain
\begin{equation}
\lim_{M \to \infty} \lim_{\tilde{s} \to \tilde{s}} \| F_M(x,p) \tilde{\Omega}^*_+(\tilde{u}) \tilde{F}_c(\tilde{s}) f \| = \lim_{M \to \infty} \| F_M(x,p) \tilde{\Omega}^*_+(\tilde{s}) f \| = \| \tilde{F}_c(\tilde{s}) f \|,
\end{equation}
which yields
\begin{equation}
\lim_{M \to \infty} \lim_{\tilde{s} \to \tilde{s}} \| (1 - F_M(x,p)) \tilde{\Omega}^*_+(\tilde{u}) \tilde{F}_c(\tilde{s}) f \| = 0.
\end{equation}
This together with (4.42) yields
\begin{equation}
\lim_{M \to \infty} \lim_{\tilde{s} \to \tilde{s}} \| (1 - F_M(x,p)) (\tilde{\Omega}^*_+(\tilde{u}) - \tilde{\Omega}^*_+(\tilde{s})) \tilde{F}_c(\tilde{s}) f \| = 0,
\end{equation}
where we used $\tilde{\Omega}^*_+(\tilde{s}) = \tilde{\Omega}^*_+(\tilde{s}) \tilde{F}_c(\tilde{s})$. This together with Proposition 4.2 yields
\begin{equation}
\lim_{M \to \infty} \lim_{\tilde{s} \to \tilde{s}} \| (1 - F_M(x,p)) \tilde{\Omega}^*_+(\tilde{u}) f - (1 - F_M(x,p)) \tilde{\Omega}^*_+(\tilde{s}) f \| = 0.
\end{equation}
Then this together with (4.42), compactness of $\mathbb{T}_1 \times \cdots \times \mathbb{T}_N$ and the standard compactness argument, yields (14.39).

### 4.2. Properties of the operators $C_r(t)$

In this section, we assume that $M \geq M_0$, where $M_0$ is given in Proposition 4.1 and prove

**Proposition 4.3.** If Assumption [1.1] holds true, then
\begin{equation}
\left( \int_0^\infty \| (x)^{-\eta}(1 - C_r(t))^{-1} P^\pm e^{-itH_0} f \|^2 dt \right)^{1/2} \lesssim_{n,\eta} \| f \|, \\
f \in L^2_\omega(\mathbb{R}^n)
\end{equation}
holds true for all $\eta > \frac{3}{2}$ and $n \geq 8$, and

**Proposition 4.4.** Under Assumptions [1.1] and [1.2] we have
\begin{equation}
\int_0^\infty \| (x)^{-\eta}(1 - C_r(t))^{-1} P^\pm e^{-itH_0} f \| dt \lesssim_{n,\eta} \| f \| \\
f \in L^2_\omega(\mathbb{R}^n)
\end{equation}
for all $\eta > \frac{3}{2}$ and $n \geq 5$.

Let
\begin{equation}
I^\pm(t) := \int_0^\infty P^\pm e^{\pm isH_0} V(x, t + s) U(t + s, t) ds, \\
t \in \mathbb{R}.
\end{equation}
The proof of Propositions 4.3 and 4.4 uses the lemma and two propositions listed below.

**Lemma 4.4.** For all $M \geq 1$, we have
\begin{equation}
\sup_{t \in \mathbb{R}} \| C_M(t) |p|^{-1/2}(x)^{1/2 + \epsilon} \| \lesssim_{M, \epsilon} 1
\end{equation}
and
\begin{equation}
\sup_{t \in \mathbb{R}} \| C_M(t) \frac{|p|}{|p|^{\alpha}} \| \lesssim_{M, \alpha} 1
\end{equation}
for all $\epsilon \in (0, 1/2)$ and $\alpha \in [0, \frac{4}{3})$ when $n \geq 4$. 
Proof. (4.51) follows by a proof similar to that of Proposition 3.6. To obtain (4.51), we use the Duhamel’s principle to obtain
\begin{equation}
C_M(t) \frac{\langle p \rangle}{|p|} = C(t) \Omega_+^s(t) F_M(x, p) \Omega_+^s(t) \frac{\langle p \rangle}{|p|} = C_{M, 1}(t) + C_{M, 2}(t),
\end{equation}
where the operators $C_{M, 1}(t)$ and $C_{M, 2}(t)$ are given by
\begin{equation}
C_{M, 1}(t) := C(t) \Omega_+^s(t) F_M(x, p) \frac{\langle p \rangle}{|p|}
\end{equation}
and
\begin{equation}
C_{M, 2}(t) := (\pm i) \int_0^\infty C(t) \Omega_+^s(t) F_M(x, p) e^{i s H_0} V(x, t + s) U(t + s, t) \frac{\langle p \rangle}{|p|} ds.
\end{equation}
By Eq. (4.30) and Corollary 2.1, $C_{M, 1}(t)$ satisfies
\begin{equation}
\sup_{t \in \mathbb{R}} \|C_{M, 1}(t)\| \lesssim_{M, \alpha} 1.
\end{equation}
To estimate $U(t + s, t) \frac{\langle p \rangle}{|p|}$, we note that by the Duhamel’s principle, Hardy-Littlewood Sobolev inequality and the unitarity of $U(t + s, t + u)$, $(U(t + s, t) - e^{-i s H_0}) \frac{\langle p \rangle}{|p|}$ satisfies for $s \geq 0$,
\begin{equation}
\| (U(t + s, t) - e^{-i s H_0}) \frac{\langle p \rangle}{|p|} \| \leq \int_0^s \| U(t + s, t + u) V(x, t + u) e^{-i u H_0} \frac{\langle p \rangle}{|p|} \| du \lesssim_s \sup_{u \in \mathbb{R}} \| \langle x \rangle^{\alpha} V(x, u) \| \mathcal{L}_x^\infty(\mathbb{R}^n).
\end{equation}
This together with Eq. (4.51), Hardy-Littlewood Sobolev inequality and estimate
\begin{equation}
\| F_M(x, p) e^{i s H_0} (-\beta) \|_{2 \rightarrow 2} \lesssim_{M, \beta} \frac{1}{(s)^{\beta}} \quad \beta > 0,
\end{equation}
yields
\begin{equation}
\| C_{M, 2}(t) \frac{\langle p \rangle}{|p|} \| \leq \int_0^\infty \| C(t) \Omega_+^s(t) \| \| F_M(x, p) e^{i s H_0} \langle x \rangle^{-5/4} \| \langle x \rangle^3 V(x, t + s) \| \mathcal{L}_x^\infty(\mathbb{R}^n) \| \langle x \rangle^{-\frac{7}{4}} \frac{\langle p \rangle}{|p|} e^{-i s H_0} \| ds \\
+ \int_0^\infty \| C(t) \Omega_+^s(t) \| \| F_M(x, p) e^{i s H_0} \langle x \rangle^{-9/4} \| \langle x \rangle^9 V(x, t + s) \| \mathcal{L}_x^\infty(\mathbb{R}^n) \| (U(t + s, t) - e^{-i s H_0}) \frac{\langle p \rangle}{|p|} \| ds \\
\lesssim \int_0^\infty \frac{1}{(s)^{3/4}} \sup_{u \in \mathbb{R}} \| \langle x \rangle^3 V(x, u) \| \mathcal{L}_x^\infty(\mathbb{R}^n) ds + \int_0^\infty \frac{8}{(s)^{9/4}} \| \langle x \rangle^9 V(x, u) \| \mathcal{L}_x^\infty(\mathbb{R}^n) ds
\end{equation}
for $\alpha \in [0, \frac{1}{4})$. This together with (4.55) and Eq. (4.61) yields (4.51). \qed

Proposition 4.5. Let $n \geq 5$ and $\alpha > 3/2$ be two fixed numbers. If $\langle x \rangle^{\frac{2}{\alpha} + \alpha} V(x, t) \in L^\infty_{x,t}(\mathbb{R}^{n+1})$, then for all $\epsilon \in (0, \min\{\alpha/2 - 3/4, 1/2\})$,
\begin{equation}
\| \frac{\langle p \rangle}{|p|} F^t(t) |p|^{-1/2} (x)^{1/2 + \epsilon} \| \leq C \max_{j = 1, 2} \| \langle x \rangle^{\frac{2}{\alpha} + \alpha} V(x, s) \|_{L^\infty_{x,t}(\mathbb{R}^{n+1})},
\end{equation}
where the constant $C = C(\epsilon, n, \alpha) > 0$ stands for a constant depending on $\alpha, n$ and $\epsilon$.

Proposition 4.6. There exists $\beta \in (0, 1/8]$ such that for all $\alpha \in (0, \frac{1}{2})$, with $\alpha_+: = \max\{\alpha - \beta, 0\}$, we have
\begin{equation}
\| \frac{\langle p \rangle^{\alpha + \beta}}{|p|^{\alpha + \beta}} F^t(t) \| \leq C,
\end{equation}
where the constant $C = C(\epsilon, n)$ stands for a constant depending on $n$ and $\epsilon$.

We prove Proposition 4.5 first. The proof of Proposition 4.5 requires the following lemma.

**Lemma 4.5.** If $\langle x \rangle^s V(x, t) \in L^\infty_t(L^r_x([R^n+1])$ for some $s \geq 2$, then for all $n \geq 5$ and $\epsilon \in (0, 1/3)$,

\[
\int_0^\infty \langle s \rangle^{-2-3\epsilon} \langle x \rangle^{-2} U(t + s, t)|p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \|ds \leq C \max_{j=0,1} \|\langle x \rangle^s V(x, s)\|_{L^\infty_t(L^r_x([R^n+1])},
\]

where $C = C(n, \epsilon) > 0$ stands for a constant depending on $n$ and $\epsilon$.

We defer the proof of Lemma 4.5 after the proof of Proposition 4.5.

**Proof of Proposition 4.5.** We estimate $\|\langle p \rangle^\alpha (p_i^\alpha \rho_j^\alpha I^+(t)|p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \|$ and $\|\langle p \rangle^\alpha (p_i^\alpha \rho_j^\alpha I^-(t)|p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \|$ can be treated similarly. By estimate (2.22), we obtain, for all $\epsilon_1 > 0$ satisfying condition $\frac{3}{4} + \alpha/2 - \epsilon_1 > 2$,

\[
\|\langle p \rangle^\alpha (p_i^\alpha \rho_j^\alpha I^+(t)|p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \| \\
\leq \int_0^\infty \|\langle p \rangle^\alpha (p_i^\alpha \rho_j^\alpha I^+(t)|p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \| ds \\
\leq \int_0^\infty \|\langle p \rangle^\alpha (p_i^\alpha \rho_j^\alpha I^+(t)|p|^{-1/2} \langle x \rangle^{1/2+\epsilon} \| ds.
\]

This together with Lemma 4.5 (take $\epsilon = \frac{3}{4} \left(-\frac{3}{2} + \frac{9}{2} - \epsilon_1 \right) > 0$) implies (4.69).

Now we prove Lemma 4.5. The proof of Lemma 4.5 requires

**Lemma 4.6.** For all $n \geq 5$,

\[
\|\langle x \rangle^{2|p|^{-1/2} \langle x \rangle^{3/2}} \|_n \leq 1.
\]

**Proof.** Let

\[
E := \|\langle x \rangle^{2|p|^{-1/2} \langle x \rangle^{3/2}} \|.
\]

We write $\langle x \rangle^{3/2}$ as $\langle x \rangle^{2} \langle x \rangle^{-1/2}$ and note that

\[
E \leq \|\langle x \rangle^{-2} \langle x \rangle^{2} \langle x \rangle^{-1/2} \| + \|\langle x \rangle^{-1/2} \langle x \rangle^{-1/2} \|
\]

By Hardy-Littlewood-Sobolev inequality,

\[
\|\langle x \rangle^{-1/2} \langle x \rangle^{-1/2} \| \leq \|\langle x \rangle^{-1/2} \langle x \rangle^{-1/2} \| \|\langle x \rangle^{-1/2} \langle x \rangle^{-1/2} \| \leq 1,
\]

for all $n \geq 2$. To estimate $\|\langle x \rangle^{-2} \langle x \rangle^{2} \langle x \rangle^{-1/2} \|$, we also note that

\[
\|\langle x \rangle^{-2} \langle x \rangle^{2} \langle x \rangle^{-1/2} \| \leq \sum_{j=1}^n \left( \|\langle x \rangle^{-2} \partial_{p_j} \langle x \rangle^{1/2} \| + 2 \|x_j \langle x \rangle^{-2} \partial_{p_j} \langle p \rangle^{-1/2} \langle x \rangle^{-1/2} \| \right),
\]

which implies

\[
\|\langle x \rangle^{-2} \langle x \rangle^{2} \langle x \rangle^{-1/2} \| \leq \sum_{j=1}^n \left( \|\langle x \rangle^{-2} \langle p \rangle^{-5/2} \langle x \rangle^{-1/2} \| + \|\langle x \rangle^{-2} \langle p \rangle^{-9/2} \langle x \rangle^{-1/2} \| \right)
\]

\[
+ \sum_{j=1}^n \|x_j \langle x \rangle^{-2} \langle p \rangle^{-5/2} \langle x \rangle^{-1/2} \|.}

\]
By employing estimate $\|p_j/|p|\| \leq 1$ and Hardy-Littlewood-Sobolev inequality, we have, for $n \geq 5$,

\[
\|x^{-2}|p|^{-5/2}x^{-1/2}\| \leq \|x^2x^{-\sigma/2}\||x|^{-2}|p|^{-2}\|p|^{-1/2}|x|^{-1/2}\||x|^{1/2}x^{-1/2}\|
\]
(4.69)

\[
\leq 1,
\]

(4.70)

and

\[
\|x_jx^{-2}p_j|p|^{-5/2}x^{-1/2}\| \leq \|x_jx^{-2}|x|^{-1}|p|^{-1}\||p_j/|p|\|^2p|^{-1/2}|x|^{-1/2}\||x|^{1/2}x^{-1/2}\|
\]
(4.71)

These yield

\[
\|x_jx^{-2}|p|^{-1/2}\rangle \|x_jx^{-2}\| \leq_n 1.
\]

This together with (4.65) and (4.66) yields (4.63). \hfill \Box

**Lemma 4.7.** For all $n \geq 4$,

\[
\|x^{-3/2}|p|^{-1/2}x_j\| \leq 1, \quad j = 1, \cdots, n.
\]

**Proof.** We note that

\[
\|x^{-3/2}|p|^{-1/2}x_j\| \leq \|x_jx^{-3/2}|p|^{-1/2}\| + \|x^{-3/2}[x_j, |p|^{-1/2}]\|
\]
(4.74)

By Hardy-Littlewood-Sobolev inequality and $\|p_j/|p|\| \leq 1$, we obtain

\[
\|x_jx^{-3/2}|p|^{-1/2}\| \leq \|x_jx^{-3/2}|x|^{1/2}\||x|^{-1/2}|p|^{-1/2}\| \leq 1
\]
and

\[
\|x^{-3/2}p_j|p|^{-5/2}\| \leq \|\langle x^{-3/2}\rangle\| |x|^{-3/2}\||x|^{-3/2}|p|^{-3/2}\||p_j/|p|\| \leq 1.
\]

These together with (4.74) imply (4.73). \hfill \Box

**Lemma 4.8.** For all $n \geq 3$ and $\epsilon \in (0, 1/2)$,

\[
\int (s)^{-1-2\epsilon}\|\langle x^{-2}e^{-isH_0}|p|^{-1/2}\rangle\|^{1/2+\epsilon}ds \leq C,
\]

where $C = C(n, \epsilon) > 0$ stands for a constant depending on $n$ and $\epsilon$.

**Proof.** By endpoint Strichartz estimate, Lemma 4.6 and Hölder’s inequality, we have

\[
\int (s)^{-1-2\epsilon}\|\langle x^{-2}e^{-isH_0}|p|^{-1/2}\rangle\|^{1/2+\epsilon}ds \leq C
\]
(4.77)

\[
\leq \int (s)^{-1-2\epsilon}\|\langle x^{-2}\rangle\|^{-1/2}\|\langle x^{-3/2}\rangle\|\|e^{-isH_0}\|ds
\]
\[
\leq \|\langle s^{-1/2-\epsilon}\|L_2^2(\mathbb{R})\|e^{-isH_0}\|L_2^2(\mathbb{R}^{n+1})\|^{2\epsilon/1-2\epsilon}ds
\]
(4.78)

\[
\leq_{c, n, 1}
\]
for $n \geq 3$ and $\epsilon \in (0, 1/2)$. Next, we estimate

\[
Q := \int (s)^{-1-2\epsilon}\|\langle x^{-2}e^{-isH_0}|p|^{-1/2}\rangle\|^{1/2+\epsilon}ds.
\]

To estimate $Q$, we let $\{F_j\}_{j=1}^n$ denote a partition of the unity satisfying

\[
|\langle x_j x^{-2}\rangle| \geq \frac{|p|}{n}, \quad \text{for all } x_j \in \supp(F_j).
\]

(4.80)
This implies

\begin{equation}
F_j \chi(|x| > \langle s \rangle) = F_j \chi(|x| > \langle s \rangle) \chi(|x_j| > \frac{\langle s \rangle}{n})
\end{equation}

and

\begin{equation}
\|F_j(x_j)^{-1/2} \langle x \rangle^{1/2+\epsilon} \chi(|x_j| > 1)\| \lesssim_n 1.
\end{equation}

By triangle inequality and estimate \(4.82\), we find

\begin{equation}
Q \lesssim_n \sum_{j=1}^n Q_j,
\end{equation}

where \(Q_j, j = 1, \cdots, n\), are defined by

\begin{equation}
Q_j := \int \langle s \rangle^{-1-2\epsilon} \|\langle x \rangle^{-2} e^{-isH_0} [p]^{-1/2} \langle x_j \rangle^{1/2+\epsilon} \chi(|x_j| > \frac{\langle s \rangle}{n})\| ds.
\end{equation}

We write

\begin{equation}
e^{-isH_0} x_j = (x_j - 2sp_j) e^{-isH_0}
\end{equation}

to obtain

\begin{equation}
Q_j \leq Q_{j1} + Q_{j2} + Q_{j3},
\end{equation}

where

\begin{equation}
Q_{j1} := \int \langle s \rangle^{-1-2\epsilon} \|\langle x \rangle^{-2} x_j \| [p]^{-1/2} \langle x_j \rangle^{1/2+\epsilon} \chi(|x_j| > \frac{\langle s \rangle}{n})\| ds,
\end{equation}

\begin{equation}
Q_{j2} := 2 \int |s| \langle s \rangle^{-1-2\epsilon} \|\langle x \rangle^{-2} e^{-isH_0} \| [p]^{-1/2} p_j \langle x_j \rangle^{1/2+\epsilon} \chi(|x_j| > \frac{\langle s \rangle}{n})\| ds
\end{equation}

and

\begin{equation}
Q_{j3} := \int \langle s \rangle^{-1-2\epsilon} \|\langle x \rangle^{-2} e^{-isH_0} [p]^{-1/2} \langle x_j \rangle^{1/2+\epsilon} \chi(|x_j| > \frac{\langle s \rangle}{n})\| ds.
\end{equation}

By employing estimate

\begin{equation}
\| \langle x_j \rangle^{1/2+\epsilon} x_j \chi(|x_j| > \frac{\langle s \rangle}{n})\| \lesssim_n \frac{1}{\langle s \rangle^{1/2-\epsilon}}, \quad \forall \epsilon \in (0, 1/2),
\end{equation}

the unitarity of \(e^{-isH_0}\) and Hardy-Littlewood Sobolev inequality, we have for all \(n \geq 3\),

\begin{equation}
Q_{j1} \leq \int \langle s \rangle^{-1-2\epsilon} \|\langle x \rangle^{-2} x_j \| [p]^{-1/2} \|e^{-isH_0}\| \|\langle x_j \rangle^{1/2+\epsilon} \chi(|x_j| > \frac{\langle s \rangle}{n})\| ds
\end{equation}

\begin{equation}
\lesssim \int \langle s \rangle^{-1-\epsilon} ds \lesssim \epsilon 1.
\end{equation}

By employing Hölder’s inequality and estimates \(4.30\), \(\|p_j^\perp\| \leq 1\) and

\begin{equation}
\int \|\langle x \rangle^{-2} [p]^{-1/2} e^{-isH_0} f\|^2 ds \lesssim \|f\|^2, \quad f \in L^2({\mathbb{R}}^n),
\end{equation}

This implies

\begin{equation}
F_j \chi(|x| > \langle s \rangle) = F_j \chi(|x| > \langle s \rangle) \chi(|x_j| > \frac{\langle s \rangle}{n})
\end{equation}

and

\begin{equation}
\|F_j(x_j)^{-1/2} \langle x \rangle^{1/2+\epsilon} \chi(|x_j| > 1)\| \lesssim_n 1.
\end{equation}
Lemma 4.9. For all $\alpha \in (0, \frac{7}{4})$ and $n \geq 5$, 
\begin{equation}
\| \langle x \rangle^{-\alpha} U(t + s, t) \|_{L^p_x} \lesssim_n \langle s \rangle^{\alpha/2}
\end{equation}
holds true.

**Remark 4.1.** The upper bound $\frac{7}{4}$ for $\alpha$ is not sharp but sufficient for Proposition 4.6.
Proof. Let $B(t, s) \equiv (x)^{-\alpha}U(t + s, t)^{(p)^{\alpha}}_{|p|^{\alpha}}$. We deal with the case when $s \geq 0$. The case when $s < 0$ can be treated similarly. We break $B(t, s)$ into two pieces
\begin{equation}
B(t, s) = B_1(t, s) + B_2(t, s),
\end{equation}
where the operators $B_j(t, s), j = 1, 2,$ are given by
\begin{equation}
B_1(t, s) := B(t, s)\chi(|p| > \frac{1}{\langle s \rangle^{1/2}})
\end{equation}
and $B_2(t, s) := B(t, s) - B_1(t, s)$. Using
\begin{equation}
\|\langle p \rangle^{\alpha} |\chi(|p| > \frac{1}{\langle s \rangle^{1/2}})\| \lesssim \langle s \rangle^{\alpha/2}
\end{equation}
and the unitarity of $U(t + s, t)$, we obtain
\begin{equation}
\|B_1(t, s)\langle p \rangle^{\alpha} |p|^{\alpha}\| \lesssim \langle s \rangle^{\alpha/2}.
\end{equation}
Next, we estimate $B_2(t, s)\langle p \rangle^{\alpha} |p|^{\alpha}$. Using
\begin{equation}
\|\langle x \rangle^{-\alpha} \langle p \rangle^{\alpha} e^{-iuH_0} |\chi(|p| \leq \frac{1}{\langle s \rangle^{1/2}})\| \lesssim 1, \quad \forall u \in \mathbb{R}, \quad \alpha \in \left[0, \frac{7}{4}\right],
\end{equation}
by the Duhamel’s principle, the unitarity of $U(t + s, t + u)$ and Hardy-Littlewood-Sobolev inequality, we arrive at, for $s \geq 0$,
\begin{align}
\|B_2(t, s)\langle p \rangle^{\alpha} |p|^{\alpha}\| & \leq \|\langle x \rangle^{-\alpha} e^{-iuH_0} \langle p \rangle^{\alpha} |\chi(|p| \leq \frac{1}{\langle s \rangle^{1/2}})\| \\
& \quad + \int_0^s \|\langle x \rangle^{-\alpha} U(t + s, t + u)V(x, t + u)e^{-iuH_0} \langle p \rangle^{\alpha} |\chi(|p| \leq \frac{1}{\langle s \rangle^{1/2}})\| du \\
& \lesssim 1 + \int_0^s \sup_{u \in \mathbb{R}} \|\langle x \rangle^{2}V(x, u)\|_{L^\infty(\mathbb{R}^d)} \|x|^{-2}\| \langle p \rangle^{\alpha}|p|^{2-\alpha} \chi(|p| \leq \frac{1}{\langle s \rangle^{1/2}})\| du \\
& \lesssim \langle s \rangle^{\alpha/2}.
\end{align}
This together with estimate (4.103) and Eq. (4.100) yields
\begin{equation}
\|B(t, s)\langle p \rangle^{\alpha} |p|^{\alpha}\| \lesssim \langle s \rangle^{\alpha/2}.
\end{equation}
\hfill\Box

Now we prove Proposition 4.6. By Lemma 4.9 and estimate (2.22), we obtain for $\alpha \in (0, \frac{7}{4})$,
\begin{equation}
\|\frac{|p|^{|\alpha\beta_+|}}{|p|^{|\alpha\beta_+|}} I^{\pm}(t)\langle p \rangle^{\alpha} |\| \lesssim_{\epsilon, n} \int_0^\infty \frac{1}{\langle s \rangle^{5/4+\alpha\beta_+/2-\epsilon}} \sup_{u \in \mathbb{R}} \|\langle x \rangle^{2\alpha+5/2}V(x, u)\|_{L^\infty(\mathbb{R}^d)} \langle s \rangle^{\alpha/2} ds.
\end{equation}
Then with $\beta = \frac{1}{3}$ and $\epsilon \in (0, 1/8)$, we obtain
\begin{align}
5/4 + \alpha\beta_+/2 - \epsilon - \alpha/2 &= 5/4 + \max\{\alpha - \beta, 0\}/2 - \epsilon - \alpha/2 \\
&> 5/4 - \frac{1}{16} - \frac{1}{8} > 1.
\end{align}
This together with (4.107) yields for all $\epsilon \in (0, \frac{1}{8})$,
\begin{equation}
\|\frac{|p|^{|\alpha\beta_+|}}{|p|^{|\alpha\beta_+|}} I^{\pm}(t)\langle p \rangle^{\alpha} |\| \lesssim_{\epsilon, n} 1.
\end{equation}
\hfill\Box
The proof of Proposition 4.3 also requires the following proposition.

**Proposition 4.7.** If Assumption (4.7) is satisfied, then for all \( \delta \in [0, \min\{\frac{n}{2} - 2, 2\}] \), \( n \geq 5 \),

\[
(4.110) \quad \sup_{t \in \mathbb{R}} \left\| P_b(t) \right\|_{\delta} \lesssim_{\delta, n} 1.
\]

**Proof.** It suffices to fix \( \delta \in [\max\{0, \frac{n}{2} + 4 - \sigma\}, \min\{\frac{n}{2} - 2, 2\}] \). It suffices to show that

\[
(4.111) \quad \sup_{t \in \mathbb{R}} \left\| P_b(t) \frac{x^2}{(x^2 - \sigma)} \right\|_{\delta, n} 1, \quad j = 1, \ldots, n.
\]

We estimate the case when \( j = 1 \) and the case when \( j = 2, \ldots, n \), can be treated similarly. To estimate \( P_b(t) \frac{x^2}{(x^2 - \sigma)} \), we break \( P_b(t) \frac{x^2}{(x^2 - \sigma)} \) into two parts

\[
(4.112) \quad P_b(t) \frac{x^2}{(x^2 - \sigma)} = P_b(t) \frac{x^2}{(x^2 - \sigma)} P^+ + P_b(t) \frac{x^2}{(x^2 - \sigma)} P^-.
\]

We note that with \( P_b(t) \Omega_+(t) = 0 \), by the Duhamel’s principle, the operator \( P_b(t) \frac{x^2}{(x^2 - \sigma)} P^+ \) reads

\[
(4.113) \quad (-i) \int_0^\infty P_b(t) U(t, t + s) V(x, t + s) e^{-isH^0} \frac{x^2}{(x^2 - \sigma)} P^+ \frac{1}{(x^2 - \sigma)} ds.
\]

Moving \( x^2 \) through \( e^{-isH^0} \), we obtain

\[
(4.114) \quad \sup_{t \in \mathbb{R}} \left\| P_b(t) \right\|_{\delta} \lesssim_{\delta, n} 1 \quad \sup_{t \in \mathbb{R}} \left\| P_b(t) \frac{x^2}{(x^2 - \sigma)} P^+ \right\|_{\delta} \lesssim_{\delta, n} 1.
\]

By estimates \( \left\| P_b(t) \right\| \leq 1 \), (2.21), (2.22) and (2.24) and the unitarity of \( U(t, t + s) \), this implies

\[
\sup_{t \in \mathbb{R}} \left\| P_b(t) \frac{x^2}{(x^2 - \sigma)} P^+ \right\|_{\delta} \lesssim_{\epsilon, n} \sup_{u \in \mathbb{R}} \left\| \frac{x^2}{(x^2 - \sigma)} V(x, u) \right\|_{L^\infty_x (\mathbb{R}^n)} \left( \int_1^\infty \left( \frac{1}{(s^{3/2} - \sigma) \frac{1}{(s^{3/2} - \sigma)}} + \frac{1}{(s^{3/2} - \sigma) \frac{1}{(s^{3/2} - \sigma)}} \right) ds + 1 \right)
\]

(4.115)

\[
\lesssim_{\epsilon, n} \sup_{u \in \mathbb{R}} \left\| \frac{x^2}{(x^2 - \sigma)} V(x, u) \right\|_{L^\infty_x (\mathbb{R}^n)}
\]

for all \( \epsilon \in (0, \frac{n}{2} - \frac{\delta}{2} - 1) \). This yields with \( \epsilon = \frac{n}{2} - \frac{\delta}{2} - \frac{1}{2} \),

\[
(4.116) \quad \sup_{t \in \mathbb{R}} \left\| P_b(t) \frac{x^2}{(x^2 - \sigma)} P^+ \right\|_{\delta, n} \sup_{u \in \mathbb{R}} \left\| \frac{x^2}{(x^2 - \sigma)} V(x, u) \right\|_{L^\infty_x (\mathbb{R}^n)}.
\]
By [11], we note that
\begin{equation}
|P_{b}(t)| = \lim_{s \to -\infty} U(t, t + s)|F_{c}(\frac{|x - 2sp|}{s^{\alpha}}) \geq 1|U(t + s, t), \quad \text{on } L^{2}_{x}(\mathbb{R}^{n})
\end{equation}
for all $\alpha \in (0, 1 - \frac{2}{n})$, $n \geq 3$. Hence, similarly, we have
\begin{equation}
\sup_{t \in \mathbb{R}} \|P_{b}(t)x^{2}P_{a}(\frac{1}{\langle x \rangle^{2 - \sigma}}) \| \lesssim \sup_{u \in \mathbb{R}} \|\langle x \rangle^{\frac{2}{3} + 4 - \delta} V(x, u)\|_{L^{\infty}_{x}(\mathbb{R}^{n})}.
\end{equation}

This together with (4.116) and Eq. (4.112) yields
\begin{equation}
\sup_{t \in \mathbb{R}} \|P_{b}(t)x^{2}P_{a}(\frac{1}{\langle x \rangle^{2 - \sigma}}) \| \lesssim \sup_{u \in \mathbb{R}} \|\langle x \rangle^{\frac{2}{3} + 4 - \delta} V(x, u)\|_{L^{\infty}_{x}(\mathbb{R}^{n})}.
\end{equation}

Similarly, we obtain
\begin{equation}
\sup_{t \in \mathbb{R}} \|P_{b}(t)x^{2}P_{a}(\frac{1}{\langle x \rangle^{2 - \sigma}}) \| \lesssim \sup_{u \in \mathbb{R}} \|\langle x \rangle^{\frac{2}{3} + 4 - \delta} V(x, u)\|_{L^{\infty}_{x}(\mathbb{R}^{n})}, \quad j = 2, \ldots, n.
\end{equation}

Thus, we conclude (4.110).

Next, we prove Proposition 4.3.

**Proof of Proposition 4.3.** Take $f \in L^{2}_{x}(\mathbb{R}^{n})$. We estimate $\langle x \rangle^{-\eta}(\mathbb{1} - C_{r}(t))^{-1}P^{+}e^{-itH_{0}}f$ and $\langle x \rangle^{-\eta}(\mathbb{1} - C_{r}(t))^{-1}P^{-}e^{-itH_{0}}f$ can be treated similarly. We write $(\mathbb{1} - C_{r}(t))^{-1}P^{+}e^{-itH_{0}}f$ as
\begin{equation}
(\mathbb{1} - C_{r}(t))^{-1}P^{+}e^{-itH_{0}}f = f_{1}(t) + f_{2}(t),
\end{equation}
where $f_{j}(t), j = 1, 2$, are given by
\begin{equation}
f_{1}(t) := P^{+}e^{-itH_{0}}f
\end{equation}
and
\begin{equation}
f_{2}(t) := (\mathbb{1} - C_{r}(t))^{-1}C_{r}(t)P^{+}e^{-itH_{0}}f.
\end{equation}

By estimate (2.1b) and the $L^{2}$ local decay of the free flow, $f_{1}(t)$ satisfies
\begin{equation}
\left(\int_{0}^{\infty} \|\langle x \rangle^{-\eta}f_{1}(t)\|^{2}dt\right)^{1/2} \leq \left(\int_{0}^{\infty} \|\langle x \rangle^{-\eta}P^{+}\langle x \rangle^{3/2}\|^{2}\|\langle x \rangle^{-3/2}e^{-itH_{0}}f\|^{2}dt\right)^{1/2} \lesssim \|f\|
\end{equation}
for all $\eta > \frac{3}{2}$. For $f_{2}(t)$, we write $C_{r}(t) = C(t) - C(t)P_{b}(t) - C_{M}(t)$ to decompose $f_{2}(t)$ further
\begin{equation}
f_{2}(t) = \sum_{j=1}^{3} f_{2j}(t),
\end{equation}
where $f_{2j}(t), j = 1, 2, 3$, read
\begin{equation}
f_{21}(t) := -(\mathbb{1} - C_{r}(t))^{-1}C(t)P_{b}(t)P^{+}e^{-itH_{0}}f,
\end{equation}
\begin{equation}
f_{22}(t) := -(\mathbb{1} - C_{r}(t))^{-1}C_{M}(t)P^{+}e^{-itH_{0}}f
\end{equation}
and
\begin{equation}
f_{23}(t) := (\mathbb{1} - C_{r}(t))^{-1}C(t)P^{+}e^{-itH_{0}}f.
\end{equation}
By Proposition 4.7, Corollary 2.1, estimate (2.5), and the $L^2$ local decay of the free flow, $f_{21}(t)$ satisfies for all $n \geq 8$ and $\eta \geq 0$,

\[
\left( \int_0^\infty \| \langle x \rangle^{-n} f_{21}(t) \|^2 dt \right)^{1/2} \lesssim \left( \int_0^\infty \| P_b(t) P^+ e^{-itH_0} f \|^2 dt \right)^{1/2} \lesssim \left( \int_0^\infty \| P_b(t) \langle x \rangle^{3/2} \| \| \langle x \rangle^{-3/2} P^+ \langle x \rangle^{3/2} \| \| \langle x \rangle^{-3/2} e^{-itH_0} f \| \|^2 dt \right)^{1/2} \lesssim \| f \|
\]

(4.129)

By Proposition 4.11, estimates (4.50) and (2.6), and $L^2$ local smoothing estimate of the free flow, $f_{22}(t)$ satisfies

\[
\left( \int_0^\infty \| \langle x \rangle^{-n} f_{22}(t) \|^2 dt \right)^{1/2} \lesssim \left( \int_0^\infty \| \mathcal{C}(t) P^+ e^{-itH_0} f \|^2 dt \right)^{1/2} \lesssim \left( \int_0^\infty \| C(t) \| p^{-1/2} \| \langle x \rangle^{-1/2-\varepsilon} \| p \|^{1/2} + \| p \|^{-1/2} \| \langle x \rangle^{1/2+\varepsilon} \| p \|^{1/2} e^{-itH_0} f \| \|^2 dt \right)^{1/2} \lesssim_M, \| f \|
\]

(4.130)

for all $\varepsilon \in (0, \frac{1}{2})$. Next, we estimate $\langle x \rangle^{-n} f_{23}(t)$. For this, we note that by Proposition 4.6 and Eqs. (4.11) and (4.49), for all $\alpha \in [0, \frac{7}{2})$ and $n \geq 8$, with $(\alpha - \frac{1}{2})_+ := \max\{0, \alpha - \frac{1}{2}\}$,

\[
\| \frac{\langle p \rangle^{\alpha} C(t)}{\langle \langle p \rangle \rangle^{\alpha-\frac{1}{2}}} \|_{n,1} \lesssim 1.
\]

(4.131)

This together with Corollary 2.2, Proposition 4.7, and estimates (4.51) and $\| \langle p \rangle^{\alpha} \| \leq 1$ implies

\[
\| \frac{\langle p \rangle^{\alpha-\frac{1}{2}} C(t)}{\langle \langle p \rangle \rangle^{\alpha-\frac{1}{2}}} \|_{n,1} \lesssim \| \langle p \rangle^{\alpha-\frac{1}{2}} C(t) \|_{n,1} + \| \langle p \rangle^{\alpha-\frac{1}{2}} C(t) P_b(t) \|_{n,1} \lesssim_{n,1} + \| \langle p \rangle^{\alpha} P_b(t) \|_{n,1} \lesssim_{n,1}
\]

(4.132)

where we also used Hardy-Littlewood-Sobolev inequality. This together with Hardy-Littlewood-Sobolev inequality and Neumann series yields for all $\eta > \frac{1}{2}$ and $\alpha \in (\frac{7}{2}, \min\{\eta, \frac{7}{2}\})$,

\[
\| \langle x \rangle^{-\eta} (I - C(t))^{-1} \|_{n,1} \lesssim_{n,\eta,1}
\]

(4.133)

We also note that Proposition 4.3 and Eq. (4.11) yields

\[
\sup_{t \in \mathbb{R}} \| \frac{\langle p \rangle^{\alpha} C(t)}{\langle \langle p \rangle \rangle^{\alpha}} \|_{n,1} \lesssim_{n,1}
\]

(4.134)
holds true for all $n \geq 5$, $\epsilon \in (0, \frac{1}{2})$ and $\alpha \in \left(\frac{3}{2}, \frac{7}{4}\right)$. This together with Corollary 2.1 and estimates 
\[ \| \frac{p}{\langle p \rangle^\alpha} \| \leq 1, \alpha \geq 0 \text{ and } (4.50), \]
yields
\[ \sup_{t \in R} \| \frac{p}{\langle p \rangle^\alpha} C_r(t) \| \leq \sup_{t \in R} \| \frac{p}{\langle p \rangle^\alpha} C(t) \| + \sup_{t \in R} \| \frac{p}{\langle p \rangle^\alpha} C_M(t) \| \]
\[ + \sup_{t \in R} \| \frac{p}{\langle p \rangle^\alpha} C_\alpha(t) \| \]
\[ \lesssim \epsilon, n, M + \sup_{t \in R} \| P_\alpha(t) \| \leq \epsilon, n, M + \sup_{t \in R} \| P_\alpha(t) \| \]
\[ \lesssim \epsilon, n, M. \]

This together with Proposition 4.7 and estimate 4.73 yields for $n \geq 8$, 
\[ \sup_{t \in R} \| \frac{p}{\langle p \rangle^\alpha} C_r(t) \| \leq \epsilon, n, M + \sup_{t \in R} \| P_\alpha(t) \| \leq \epsilon, n, M. \]

We fix $M = M_0$. This together with estimates 4.129 and 4.6 and Neumann series yields 
\[ \sup_{t \in R} \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P_\alpha \| \leq \epsilon, n, M. \]

for all $\epsilon \in (0, \frac{1}{2})$. Here we used the Restricted $L^2$ local decay of the free flow, we conclude 
\[ \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P_\alpha e^{-itH_0} f \|^2 dt \right)^{1/2} \lesssim \epsilon, n, M \| f \|. \]

Similarly, we have 
\[ \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P_\alpha e^{-itH_0} f \|^2 dt \right)^{1/2} \lesssim \epsilon, n, M \| f \|. \]

Now we prove Proposition 4.7.

Proof of Proposition 4.7. Using Assumption 1.2 instead of Proposition 4.7 and following a similar proof of Proposition 4.3 we obtain (4.48).

4.3. Proof of Theorems 1.1 and 1.2. Now we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By Proposition 4.3, we obtain that there exists $M_0 > 0$ such that whenever $M \geq M_0$, 
\[ \left( \int_0^\infty \| F_M(x, p) e^{-itH_0} f \|^2 dt \right)^{1/2} \lesssim M \| f \|. \]

for all $f \in L^2_r(\mathbb{R}^n)$.
yields

\[
\left( \int_0^\infty \| \langle x \rangle^{-\eta} U(t,0) f \|^2 dt \right)^{1/2} \leq \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P^+ e^{-itH_0} \Omega^*_n(0) f \|^2 dt \right)^{1/2} \\
+ \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P^- e^{-itH_0} \Omega^*_n(0) f \|^2 dt \right)^{1/2} \\
+ \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} C_M(t) U(t,0) f \|^2 dt \right)^{1/2}
\]

(4.142)

\[
\lesssim_{\eta,n,M} \| f \|
\]

for all \( \eta > \frac{2}{3} \) and \( f \in L^2_x(\mathbb{R}^n) \) with \( f = P_r(0)f \) and \( n \geq 8 \).

**Proof of Theorem 1.2.** Following a similar argument of Theorem 1.1 and using Proposition 4.4 instead of Proposition 4.3, we arrive at

\[
\left( \int_0^\infty \| \langle x \rangle^{-\eta} U(t,0) f \|^2 dt \right)^{1/2} \leq \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P^+ e^{-itH_0} \Omega^*_n(0) f \|^2 dt \right)^{1/2} \\
+ \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} P^- e^{-itH_0} \Omega^*_n(0) f \|^2 dt \right)^{1/2} \\
+ \left( \int_0^\infty \| \langle x \rangle^{-\eta} (1 - C_r(t))^{-1} C_M(t) U(t,0) f \|^2 dt \right)^{1/2}
\]

(4.143)

\[
\lesssim_{\eta,n,M} \| f \|
\]

for all \( \eta > \frac{2}{3} \) and \( f \in L^2_x(\mathbb{R}^n) \) with \( f = P_r(0)f \) and \( n \geq 8 \).

**4.4. Strichartz estimates.** The Strichartz estimates for the free flow (see \[30\], \[31\] for Strichartz estimates and \[12\] for endpoint Strichartz estimates) state that for \( 2 \leq r, q \leq \infty, \frac{2}{r} + \frac{2}{q} = \frac{2}{d} \), and \( (q, r, n) \neq (2, \infty, 2) \), the following estimates hold:

1. **Homogeneous Strichartz Estimate:**

\[
\| e^{-itH_0} f \|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \lesssim_{\eta,n,q,r} \| f \|_{L_x^2(\mathbb{R}^n)}
\]

2. **Dual Homogeneous Strichartz Estimate:**

\[
\left\| \int_\mathbb{R} dse^{-isH_0} F(s) \right\|_{L_x^q(\mathbb{R}^n)} \lesssim_{n,q,r} \| F \|_{L_t^q L_x^r(\mathbb{R}^n)}
\]

3. **Inhomogeneous Strichartz Estimate:**

\[
\left\| \int_{s<t} dse^{-i(t-s)H_0} F(s) \right\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \lesssim_{n,q,r} \| F \|_{L_t^q L_x^r(\mathbb{R}^{n+1})}
\]

where \((r, r')\) and \((q, q')\) are conjugate pairs.

For a perturbed system, the corresponding Strichartz estimates are

\[
\| U(t,0) P_r(0)f \|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \leq C_q \| f \|_{L_x^2(\mathbb{R}^n)}
\]

for \( 2 \leq r, q \leq \infty, \frac{2}{r} + \frac{2}{q} = \frac{2}{d} \), and \((q, r, n) \neq (2, \infty, 2) \). In this subsection, we prove that our local decay estimates imply Strichartz estimates by utilizing the inhomogeneous Strichartz estimate for the free flow.
Proof of Theorem 1.4. It suffices to check the endpoint Strichartz estimates, that is, the case when \((q, r, n) = (2, \frac{2n}{n-2}, n), n \geq 5\). By the Duhamel’s principle,

\[
U(t, 0) P_c(0) \psi = e^{-it H_0} \psi + \left( -i \right) \int_0^t \delta s e^{-i(t-s) H_0} V(x, s) U(s, 0) P_c(0) \psi \quad t>0.
\]

(4.144)

Similarly, we have

\[
\psi_1(t) \text{ enjoys Strichartz estimates of the free flow. For } \psi_2(t), \text{ by the inhomogeneous Strichartz estimate of the free flow, the bound}
\]

\[
\|\langle x \rangle^{-2} \|_{L^2_x(R^n) \rightarrow L^2_{x,t}^{\frac{2n}{n-2}}(R^n \times R^+)} \leq 1
\]

(4.146)

and Theorem 1.2 we have

\[
\|\psi_2(t)\|_{L^2_t L^{\frac{2n}{n-2}}_{x,t}(R^n \times R^+)} \lesssim \|V(x, t) U(t, 0) P_c(0) \psi\|_{L^2_t L^{\frac{2n}{n-2}}_{x,t}(R^n \times R^+)}
\]

(4.147)

\[
\lesssim \|\langle x \rangle^2 V(x, t)\|_{L^\infty_x L^{\infty}_{x,t}(R^n \times R^+)} \|\langle x \rangle^{-2} U(t, 0) P_c(0) \psi\|_{L^2_{x,t}(R^n \times R^+)}
\]

(4.148)

\[
\lesssim \|\langle x \rangle^2 V(x, t)\|_{L^\infty_x L^{\infty}_{x,t}(R^n \times R^+)} \|\psi\|_{L^2_x(R^n)}
\]

(4.149)

This together with the estimate for \(\psi_1(t)\) yields

\[
\|\psi(t)\|_{L^2_t L^{\frac{2n}{n-2}}_{x,t}(R^n \times R^+)} \lesssim \|\langle x \rangle^2 V(x, t)\|_{L^\infty_x L^{\infty}_{x,t}(R^n \times R^+)} \|\psi\|_{L^2_x(R^n)}.
\]

(4.150)

This completes the proof of endpoint Strichartz estimates.

\[
\square
\]

Appendix A. Proof of Estimate

Proof of Lemma 2.2: Proof of (2.19): It suffices to check the case when \(t > 1\). Let

\[
F_{2^j}(|p|) := F(|p| < 2^j) - F(|p| \geq 2^j), \quad j = 1, 2, \ldots
\]

Then

\[
\sum_{j=1}^{\infty} F_{2^j}(|p|) = F(|p| \geq 2).
\]

(4.151)

By estimate (2.18), it suffices to prove

\[
\| \sum_{j=1}^{\infty} P_{\pm} F_{2^j}(|p|) e^{\pm i t H_0} \langle x \rangle^{-\delta} \| \lesssim_n \frac{1}{(t)^{\delta}}.
\]

(4.152)

Indeed, using dilation transformation, we have

\[
\| P_{\pm} F_{2^j}(|p|) e^{\pm i t H_0} \langle x \rangle^{-\delta} \| = \| P_{\pm} F_{2^j}(|p|) e^{\pm i 2^j t H_0} \langle x/2^j \rangle^{-\delta} \|
\]

\[
\leq \| P_{\pm} F_{2^j}(|p|) e^{\pm i 2^j t H_0} \langle x/2^j \rangle^{-\delta} \| \times \| \langle x \rangle^\delta \langle x/2^j \rangle^{-\delta} \|
\]

\[
\lesssim_n \frac{1}{(2^j t)^{\delta}} \times 2^{\delta j} \lesssim_n \frac{1}{2^{j\delta}} \times \frac{1}{(t)^{\delta}},
\]

which implies (A.3).

Proof of (2.20): Using the dilation transformation,

\[
\| P_{\pm} F(|p| \geq M) e^{\pm i t H_0} |p|^\delta \langle x \rangle^{-\delta} \| = M^\delta \| P_{\pm} F(|p| > 1) e^{\pm i M^2 t H_0} |p|^\delta \langle x/M \rangle^{-\delta} \|.
\]

(4.153)
Proceeding as (A.4), we have for $M^2 t \geq 1$,

\[(A.6) \quad \|P^\pm F(|p| \geq M) e^{\pm iH_0} |p|^l \langle x \rangle^{-\delta} \| \leq \sum_{j=0}^{\infty} M^j \|P^\pm F_{2^j}(|p|) e^{\pm iM^2 t H_0} |p|^l \langle x/M \rangle^{-\delta} \|
\]

\[(A.7) \quad \lesssim \sum_{j=0}^{\infty} 2^{jM^4} \times \frac{1}{2^{j5} (M^2 t)^{\delta}} M^\delta
\]

\[(A.8) \quad \lesssim_n t^{-\delta} M^{8 - t \delta}.
\]

**Proof of (2.21):** We note that

\[(A.9) \quad \int_0^1 t^2 dt \|P^\pm F_{2^j}(|p|) e^{\pm it H_0} |p|^2 \langle x \rangle^{-\delta} \| \leq \int_0^{\frac{1}{2^{j/4}}} t^2 dt \|P^\pm F_{2^j}(|p|) e^{\pm it H_0} |p|^2 \langle x \rangle^{-\delta} \| + \int_{\frac{1}{2^{j/4}}}^1 t^2 dt \|P^\pm F_{2^j}(|p|) e^{\pm it H_0} |p|^2 \langle x \rangle^{-\delta} \| =: A_{j,1} + A_{j,2}.
\]

For $A_{j,1}$, we have

\[(A.10) \quad A_{j,1} \lesssim \int_0^{\frac{1}{2^{j/4}}} \, dt t^2 \lesssim 2^{-j/4}.
\]

For $A_{j,2}$, using dilation to replace $|p|$ with $2^j |p|$, we have

\[(A.11) \quad A_{j,2} = 2^{2j} \int_{\frac{1}{2^{j/4}}}^1 \, dt t^2 \|P^\pm F_1(|p|) e^{\pm i2^{2j} t H_0} |p|^2 \langle x/2^j \rangle^{-\delta} \|
\]

\[(A.12) \quad \lesssim_n 2^{2j} \int_{\frac{1}{2^{j/4}}}^1 \, dt t^2 \times \frac{1}{(t 2^j)^{\delta}} \times 2^{j \delta}
\]

\[(A.13) \quad \lesssim_n \frac{1}{2^{j(\delta-2)}}.
\]

This together with estimate (A.10) and Eq. (A.9) yields

\[\int_0^1 t^2 dt \|P^\pm F(|p| \geq 1) e^{\pm it H_0} |p|^2 \langle x \rangle^{-\delta} \| \leq \sum_{j=0}^{\infty} \int_0^1 t^2 dt \|P^\pm F_{2^j}(|p|) e^{\pm it H_0} |p|^2 \langle x \rangle^{-\delta} \|
\]

\[(A.14) \quad \lesssim_n \sum_{j=0}^{\infty} \left(2^{-j/4} + 2^{-j(\delta-2)} \right)
\]

\[\lesssim_n 1.
\]

**Proof of (2.22):** It suffices to estimate (2.22) when $t \geq 1$. By estimate (2.20) with $l = 0$ and $M = \frac{1}{(t)\frac{1}{2-\epsilon}}$, we have

\[(A.15) \quad \| |p|^\alpha (p)^{\alpha} P^\pm F(|p| \geq \frac{1}{(t)\frac{1}{2-\epsilon}} e^{\pm it H_0} \langle x \rangle^{-\delta} \| \lesssim_n \frac{1}{(p)\frac{1}{(4\epsilon)\delta}}.
\]
We also note that by estimate (2.4), we have
\[
\|p \|^\alpha \|P^\pm F(|p|) < \frac{1}{\langle t \rangle^{1/2 - \epsilon}} e^{\pm \text{i}t \mathcal{H}_0} (x) \|^{-\delta} \| \lesssim_{\alpha, R} \|p \|^\alpha F(|p|) < \frac{1}{\langle t \rangle^{1/2 - \epsilon}} e^{\pm \text{i}t \mathcal{H}_0} (x) \|^{-\delta} \|
\]
(A.16)
These yield (2.22).

Proof of (2.22): We note that similarly, for \( \epsilon \in (0, 1/2) \),
\[
\| \frac{1}{\langle x \rangle^{2-\delta}} P^\pm e^{\pm \text{i}t \mathcal{H}_0} F(|p|) \geq \frac{1}{\langle t \rangle^{1/2 - \epsilon}} P^\pm (x) (\frac{2}{2-\delta}) |p| \lesssim_{n, \delta} \frac{1}{\langle t \rangle^{2-\delta_1 + 2-\epsilon}} \|
\]
(A.17) and
\[
\| \frac{1}{\langle x \rangle^{2-\delta}} P^\pm e^{\pm \text{i}t \mathcal{H}_0} F(|p|) \leq \frac{1}{\langle t \rangle^{1/2 - \epsilon}} P^\pm (x) (\frac{2}{2-\delta}) |p| \lesssim_{n, \delta} \frac{1}{\langle t \rangle^{2-\delta_1 + 2-\epsilon}} \|
\]
(A.18)
These yield (2.24). \( \square \)

Proof of Lemma 3.1: Let \( \tilde{F}_{c,1}(z) := 1 - F_{c,1}(z) \) denote the complement of \( F_{c,1}(z) \). Then
\[
F_{c,1}(H) - F_{c,1}(H_0) = \tilde{F}_{c,1}(H_0) - \tilde{F}_{c,1}(H).
\]
Using the Fourier inversion Theorem, we have
\[
\tilde{F}_{c,1}(H_0) - \tilde{F}_{c,1}(H) = \frac{1}{\sqrt{2\pi}} \int \hat{\tilde{F}}_{c,1}(w) (e^{\text{i}wH_0} - e^{\text{i}wH}) \, dw \nonumber
\]
(A.19)
where \( \hat{\tilde{F}}_{c,1}(w) \) denotes the Fourier transform of \( F_{c,1}(z) \). By the Duhamel’s principle, this implies
\[
\tilde{F}_{c,1}(H_0) - \tilde{F}_{c,1}(H) = \frac{-i}{\sqrt{2\pi}} \int \hat{\tilde{F}}_{c,1}(w) \int_0^w e^{\text{i}(w-u)H_0} V e^{\text{i}uH} \, du \, dw.
\]
(A.20)
Since \( \langle w \rangle \tilde{F}_{c,1}(w) \in L^1_w(\mathbb{R}) \), it suffices to show that
\[
\int_0^w e^{\text{i}(w-u)H_0} V e^{\text{i}uH} \, du \text{ is compact for all } w \in \mathbb{R}.
\]
(A.21)
Since \( F(|p| \leq 1) \langle x \rangle^{-1} \) is compact, it suffices to show that
\[
\mathcal{O} := \int_0^w F(|p| \geq 1) e^{\text{i}(w-u)H_0} V e^{\text{i}uH} \, du \text{ is compact for all } w \in \mathbb{R}.
\]
(A.22)
For this, we write \( \mathcal{O} \) as
\[
\mathcal{O} = \int_0^w |p|^{1/2} e^{\text{i}(w-u)H_0} \langle x \rangle^{-1} \langle x \rangle |p|^{-1/2} F(|p| \geq 1) V e^{\text{i}uH} \, du.
\]
We note that
\[
\langle x \rangle |p|^{-1/2} F(|p| \geq 1) V = |p|^{-1/2} F(|p| \geq 1) \langle x \rangle V + |\langle x \rangle| |p|^{-1/2} F(|p| \geq 1) V \text{ is compact.}
\]
and therefore, \( \langle x \rangle |p|^{-1/2} F(|p| \geq 1) V \) is compact. This together with the Restricted \( L^2 \) local decay estimate yields the compactness of \( \mathcal{O} \). \( \square \)
Proof of Lemma 3.2. We estimate $F_{c,1}(H)(1-F_{c,2}(H_0))|p|^{-1/2}(x)^{1/2+\epsilon}$ first. Let $\tilde{F}_{c,j} := 1-F_{c,j}, j = 1, 2$. Using the Fourier inversion Theorem and the Duhamel’s principle, we obtain

\begin{equation}
F_{c,1}(H)\tilde{F}_{c,2}(H_0) = (-i) \int_{0}^{\infty} \hat{\tilde{F}}_{c,2}(w) F_{c,1}(H) e^{i(w-s)H} V e^{isH_0} dsdw,
\end{equation}

where we also used $F_{c,1}(H)\tilde{F}_{c,2}(H) = 0$. This together with (4.77) yields

\begin{equation}
\|F_{c,1}(H)\tilde{F}_{c,2}(H_0)|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim \int \langle w \rangle^{1+2\epsilon} |\hat{\tilde{F}}_{c,2}(w)||\langle x \rangle^2 V(x)||L^\infty(R^3)
\times \int \langle s \rangle^{-1-2\epsilon} |\langle x \rangle^{-2} e^{-isH_0}|p|^{-1/2}(x)^{1/2+\epsilon}|dsdw
\lesssim \|\langle x \rangle^2 V(x)||L^\infty(R^3).
\end{equation}

Similarly, we have (3.32). □

Proof of Lemma 3.1. Let $\bar{F}_{c,1}(z) := 1-F_{c,1}(z)$ denote the complement of $F_{c,1}(z)$. Then

\begin{equation}
F_{c,1}(H) - F_{c,1}(H_0) = \bar{F}_{c,1}(H_0) - \bar{F}_{c,1}(H).
\end{equation}

Using the Fourier inversion Theorem, we have

\begin{equation}
\bar{F}_{c,1}(H_0) - \bar{F}_{c,1}(H) = \frac{1}{\sqrt{2\pi}} \int \hat{\bar{F}}_{c,1}(w) (e^{iwH_0} - e^{iwH}) dw
\end{equation}

where $\hat{\bar{F}}_{c,1}(w)$ denotes the Fourier transform of $F_{c,1}(z)$. By the Duhamel’s principle, this implies

\begin{equation}
\bar{F}_{c,1}(H_0) - \bar{F}_{c,1}(H) = \frac{-i}{\sqrt{2\pi}} \int \hat{\bar{F}}_{c,1}(w) \int_{0}^{\infty} e^{i(w-u)H_0} V e^{iuH} du dw.
\end{equation}

Since $\langle w \rangle \hat{\bar{F}}_{c,1}(w) \in L^1_w(\mathbb{R})$, it suffices to show that

\begin{equation}
\int_{0}^{\infty} e^{i(w-u)H_0} V e^{iuH} du \text{ is compact for all } w \in \mathbb{R}.
\end{equation}

Since $F(|p| \leq 1)\langle x \rangle^{-1}$ is compact, it suffices to show that

\begin{equation}
\mathcal{O} := \int_{0}^{\infty} F(|p| \geq 1)e^{i(w-u)H_0} V e^{iuH} du \text{ is compact for all } w \in \mathbb{R}.
\end{equation}

For this, we write $\mathcal{O}$ as

\begin{equation}
\mathcal{O} = \int_{0}^{\infty} [p]^{1/2} e^{i(w-u)H_0} \langle x \rangle^{-1} [p]^{-1/2} F(|p| \geq 1) V e^{iuH} du.
\end{equation}

We note that

\begin{equation}
\langle x \rangle [p]^{-1/2} F(|p| \geq 1)V = [p]^{-1/2} F(|p| \geq 1) \langle x \rangle V + [(x), [p]^{-1/2} F(|p| \geq 1)] V
\end{equation}

and therefore, $\langle x \rangle [p]^{-1/2} F(|p| \geq 1) V$ is compact. This together with the Restricted $L^2$ local decay estimate yields the compactness of $\mathcal{O}$. □

Proof of Lemma 3.2. We estimate $F_{c,1}(H)(1-F_{c,2}(H_0))|p|^{-1/2}(x)^{1/2+\epsilon}$ first. Let $\tilde{F}_{c,j} := 1-F_{c,j}, j = 1, 2$. Using the Fourier inversion Theorem and the Duhamel’s principle, we obtain

\begin{equation}
F_{c,1}(H)\tilde{F}_{c,2}(H_0) = (-i) \int_{0}^{\infty} \hat{\tilde{F}}_{c,2}(w) F_{c,1}(H) e^{i(w-s)H} V e^{isH_0} dsdw,
\end{equation}

where we also used $F_{c,1}(H)\tilde{F}_{c,2}(H) = 0$. This together with (4.77) yields

\begin{equation}
\|F_{c,1}(H)\tilde{F}_{c,2}(H_0)|p|^{-1/2}(x)^{1/2+\epsilon}\| \lesssim \int \langle w \rangle^{1+2\epsilon} |\hat{\tilde{F}}_{c,2}(w)||\langle x \rangle^2 V(x)||L^\infty(R^3)
\times \int \langle s \rangle^{-1-2\epsilon} |\langle x \rangle^{-2} e^{-isH_0}|p|^{-1/2}(x)^{1/2+\epsilon}|dsdw
\lesssim \|\langle x \rangle^2 V(x)||L^\infty(R^3).
\end{equation}

Similarly, we have (3.32). □
where we also used $F_{c,1}(H)\hat{F}_{c,2}(H) = 0$. This together with (4.77) yields
\[
\|F_{c,1}(H)\hat{F}_{c,2}(H)\|p^{-1/2}\|x\|^{1/2+\epsilon}\|\| (w)^{1+2\epsilon}\|\hat{F}_{c,2}(w)\|\|x\|^2V(x)\|L^p(\mathbb{R}^3)
\times \int \langle s \rangle^{-1-2\epsilon} \|x\|^{-2} e^{-isH_0}p^{-1/2}\|x\|^{1/2+\epsilon}dsw
\]
(A.36)
\[
\lesssim \epsilon\|x\|^2V(x)\|L^p(\mathbb{R}^3).
\]
Similarly, we have \(\Box\).

Acknowledgment. A.S. is partially supported by NSF-DMS-220931. X.W. is partially supported by ARC-FL220100072, NSF-DMS-220931 and NSERC Grant NA7901. The authors thank Maxime Van de Moortel for careful reading and for useful discussions.

Parts of this work were done while the third author was at the Fields Institute for Research in Mathematical Sciences, Toronto, Texas A&M University, Rutgers University and University of Toronto.

\section*{References}

[1] M. Beceanu. New estimates for a time-dependent Schrödinger equation. \textit{Duke Math. J.}, 159(3):417–477, 2011.

[2] V. Enss. Asymptotic completeness for quantum mechanical potential scattering. I. Short range potentials. \textit{Comm. Math. Phys.}, 61(3):285–291, 1978.

[3] M. B. Erdoğan and W. R. Green. Dispersive estimates for the Schrödinger equation for $C^{-\frac{1}{p}}$ potentials in odd dimensions. \textit{Int. Math. Res. Not. IMRN}, (13):2532–2565, 2010.

[4] R. Froese and I. Herbst. A new proof of the Mourre estimate. \textit{Duke Math. J.}, 49(4):1075–1085, 1982.

[5] A. Galtbayar, A. Jensen, and K. Yajima. Local time-decay of solutions to Schrödinger equations with time-periodic potentials. \textit{J. Statist. Phys.}, 116(1-4):231–282, 2004.

[6] J. Gell-Redman, S. Gomes, and A. Hassell. Propagation of singularities and fredholm analysis for the time-dependent Schrödinger equation. \textit{arXiv preprint arXiv:2201.03140}, 2022.

[7] C. Gérard. A proof of the abstract limiting absorption principle by energy estimates. \textit{J. Funct. Anal.}, 256(3):718–746, 2009.

[8] M. Goldberg. Strichartz estimates for the Schrödinger equation with time-periodic $L^n/2$ potentials. \textit{J. Funct. Anal.}, 256(3):718–746, 2009.

[9] G. M. Graf. Asymptotic completeness for $N$-body short-range quantum systems: a new proof. \textit{Comm. Math. Phys.}, 132(1):73–101, 1990.

[10] W. Hunziker, I. M. Sigal, and A. Soffer. Minimal escape velocities. \textit{Comm. Partial Differential Equations}, 24(11-12):2279–2295, 1999.

[11] H. Jauslin and J. Lebowitz. Spectral and stability aspects of quantum chaos. \textit{Chaos: An Interdisciplinary Journal of Nonlinear Science}, 1(1):114–121, 1991.

[12] M. Keel and T. Tao. Endpoint Strichartz estimates. \textit{Amer. J. Math.}, 120(5):955–980, 1998.

[13] B. Liu and A. Soffer. The Large Times Asymptotics of NLS type equations. \textit{submitted}, 2021.

[14] B. Liu and A. Soffer. The large time asymptotic solutions of nonlinear Schrödinger type equations. \textit{Applied Numerical Mathematics}, 2023.

[15] E. Mourre. Absence of singular continuous spectrum for certain selfadjoint operators. \textit{Comm. Math. Phys.}, 78(3):391–408, 1980/81.

[16] P. Perry, I. M. Sigal, and B. Simon. Spectral analysis of $N$-body Schrödinger operators. \textit{Ann. of Math. (2)}, 114(3):519–567, 1981.

[17] M. Reed and B. Simon. \textit{Methods of modern mathematical physics. II. Fourier analysis, self-adjointness}. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[18] I. Rodnianski and W. Schlag. Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. \textit{Invent. Math.}, 155(3):451–513, 2004.

[19] I. M. Sigal and A. Soffer. The $N$-particle scattering problem: asymptotic completeness for short-range systems. \textit{Ann. of Math. (2)}, 126(1):35–108, 1987.

[20] I. M. Sigal and A. Soffer. Local decay and propagation estimates for time-dependent and time-independent hamiltonians. \textit{Preprint Princeton University}, 2(11):1, 1988.

[21] E. Skibsted. Propagation estimates for $N$-body Schroedinger operators. \textit{Comm. Math. Phys.}, 142(1):67–98, 1991.

[22] A. Soffer. Monotonic local decay estimates. \textit{arXiv preprint arXiv:1110.6549}, 2011.
24. A. Soffer and M. I. Weinstein. Time dependent resonance theory. *Geom. Funct. Anal.*, 8(6):1086–1128, 1998.
25. A. Soffer and M. I. Weinstein. Theory of nonlinear dispersive waves and selection of the ground state. *Physical review letters*, 95(21):213905, 2005.
26. A. Soffer and X. Wu. $L^p$ Boundedness of the Scattering Wave Operators of Schrödinger Dynamics with Time-dependent Potentials and applications. *arXiv preprint* [arXiv:2012.11356], 2020.
27. A. Soffer and X. Wu. On the Existence of Self-Similar solutions for some Nonlinear Schrödinger equations. *arXiv preprint* [arXiv:2205.14767], 2022.
28. A. Soffer and X. Wu. On The large Time Asymptotics of Klein-Gordon type equations with General Data-I. *arXiv preprint* [arXiv:2204.11261], 2022.
29. A. Soffer and X. Wu. On The large Time Asymptotics of Schrödinger type equations with General data. *arXiv preprint* [arXiv:2203.00724], 2022.
30. R. S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.
31. T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.
32. A. Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). *Invent. Math.*, 194(2):381–513, 2013.

(Avy Soffer)
**Department of Mathematics,**
**Rutgers University, New Brunswick, NJ 08903 USA.**
*Email address: soffer@math.rutgers.edu*

(Xiaoxu Wu)
**Mathematical Sciences Institute,**
**Australia National University, Acton, ACT 2601, Australia**
*Email address: Xiaoxu.Wu@anu.edu.au*