AN ALGEBRA OF DISTRIBUTIONS RELATED TO A STAR PRODUCT WITH SEPARATION OF VARIABLES

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Abstract. Given a star product with separation of variables \( \star \) on a pseudo-Kähler manifold \( M \) and a point \( x_0 \in M \), we construct an associative algebra of formal distributions supported at \( x_0 \). We use this algebra to express the formal oscillatory exponents of a family of formal oscillatory integrals related to the star product \( \star \).

1. Introduction

Berezin and Berezin-Toeplitz quantizations on a Kähler manifold \( M \) which depend on a certain small numerical parameter \( h \) produce deformation quantizations with separation of variables on \( M \) of the anti-Wick and Wick type, respectively, via an asymptotic procedure as \( h \to 0 \) (see \([2], [3], [4], [5], [6], [10], [14]\)). In the deformation quantization formalism, the small asymptotic parameter \( h \) is replaced with the formal parameter \( \nu \). Deformation quantizations with separation of variables exist on arbitrary pseudo-Kähler manifolds and admit a bijective parametrization by \( \nu \)-formal pseudo-Kähler forms.

Both Berezin and Berezin-Toeplitz quantizations are based upon an integral operator, the Berezin transform, which maps contravariant symbols to the corresponding covariant symbols. The \( h \)-dependent Berezin transform admits an asymptotic expansion as \( h \to 0 \) which gives a \( \nu \)-formal differential operator on \( M \), the formal Berezin transform. Any deformation quantization with separation of variables has the corresponding formal Berezin transform from which it can be completely recovered. A formal Berezin transform can be expressed in terms of what we call a formal oscillatory integral.

It was shown in \([14]\) and \([13]\) that there exists a formal algebraic counterpart of an oscillatory integral with a complex phase function on a manifold \( M \). It is called a formal oscillatory integral (FOI). A FOI is given by a fixed point \( x_0 \in M \) and a formal oscillatory integral kernel, 

\[
\exp(\varphi) \cdot \rho,
\]

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where $\varphi = \nu^{-1}\varphi_{-1} + \varphi_0 + \ldots$ is a $\nu$-formal complex phase function on $M$ such that $x_0$ is a nondegenerate critical point of $\varphi_{-1}$ with zero critical value, $\varphi_{-1}(x_0) = 0$, and $\rho = \rho_0 + \nu\rho_1 + \ldots$ is a $\nu$-formal complex volume form on $M$ such that $\rho_0$ does not vanish at $x_0$. We call $(\varphi, \rho)$ a phase-volume form pair at $x_0$. A FOI associated with the formal oscillatory kernel (1) is a $\nu$-formal distribution $\Lambda$ supported at $x_0$ which actually depends only on the jet of (1) at $x_0$. A FOI is described by algebraic axioms in terms of its oscillatory kernel. Heuristically, the $\nu$-formal distribution $\Lambda$ gives an interpretation of the formal expression

$$\Lambda(f) = \nu^{-\frac{n}{2}} \int_M e^{\varphi} f \rho,$$

where $n = \dim M$ and $f$ is an amplitude supported near $x_0$.

Let $\star$ be a star product of the anti-Wick type on a pseudo-Kähler manifold $M$, $I$ be its formal Berezin transform, $\mu$ be its formal trace density, and $x_0$ be a point in $M$. It was shown in [14] and [13] that the $\nu$-formal distribution

$$K^{(l)}(f_1 \otimes \ldots \otimes f_l) := (I f_1 \star \ldots \star I f_l)(x_0)$$

on $M^l$ is a FOI at the diagonal point $(x_0)^l := (x_0, \ldots, x_0) \in M^l$. Its formal oscillatory kernel is

$$\exp \left( F^{(l)} \right) \cdot \mu^{\otimes l},$$

where the jet of $F^{(l)}$ at $(x_0)^l$ is expressed via what we call a cyclic formal $(l + 1)$-point Calabi function of the star product $\star$ (see details in the main body of the paper).

In this paper we describe an associative algebra of $\nu$-formal distributions supported at $x_0$. For each $l \geq 1$, the jet of the oscillatory exponent $\exp F^{(l)}$ at $(x_0)^l$ is naturally expressed in terms of this algebra.

Many constructions in this paper depend on jets of functions at a given point but are stated in terms of functions which represent these jets. These representatives exist by Borel’s theorem.

2. Formal oscillatory integrals

Formal oscillatory integrals were introduced in [14] and developed further in [13]. Given a vector space $V$, we denote by $V((\nu))$ the space of $\nu$-formal vectors

$$v = \nu^r v_r + \nu^{r+1} v_{r+1} + \ldots,$$

All jets of functions considered in this paper are jets of infinite order given by the full Taylor series.
where \( r \in \mathbb{Z} \) and \( v_k \in V \) for all \( k \geq r \). The subspace \( V[[\nu]] \subset V((\nu)) \) consists of the vectors \([4]\) with \( r = 0 \).

Let, as above, \( M \) be a manifold, \( x_0 \) be a fixed point in \( M \), and \((\varphi, \rho)\) be a phase-volume form pair at \( x_0 \). Two pairs, \((\varphi, \rho)\) and \((\hat{\varphi}, \hat{\rho})\), at \( x_0 \) are called equivalent if there exists a formal function \( u = u_0 + \nu u_1 + \ldots \) on a neighborhood of \( x_0 \) such that

\[
\hat{\varphi} = \varphi + u \quad \text{and} \quad \hat{\rho} = e^{-u} \rho.
\]

Hence, it is natural to write the equivalence class of a pair \((\varphi, \rho)\) as \([1]\).

**Definition 2.1.** Given a pair \((\varphi, \rho)\) on a manifold \( M \) at \( x_0 \in M \), a formal distribution \( \Lambda = \Lambda_0 + \nu \Lambda_1 + \ldots \) on \( M \) supported at \( x_0 \) is called a formal oscillatory integral (FOI) associated with the pair \((\varphi, \rho)\) if \( \Lambda_0 \neq 0 \) and

\[
\Lambda (vf + (v \varphi + \text{div}_\rho v)f) = 0
\]

for any vector field \( v \) and any function \( f \) on \( M \).

Here \( \text{div}_\rho v = \mathbb{L}_v \rho / \rho \) is the divergence of the vector field \( v \) with respect to \( \rho \) and \( \mathbb{L}_v \) is the Lie derivative with respect to \( v \). As shown in \([13]\),

\[
\Lambda_0 = \alpha \delta_{x_0},
\]

where \( \alpha \) is a nonzero complex constant and \( \delta_{x_0} \) is the Dirac distribution at \( x_0 \), \( \delta_{x_0}(f) = f(x_0) \). For any pair \((\varphi, \rho)\) there exists an associated FOI which is determined up to a formal multiplicative constant \( c(\nu) = c_0 + \nu c_1 + \ldots \), where \( c_0 \neq 0 \). In particular, there is a unique such FOI \( \Lambda \) for which \( \Lambda(1) = 1 \). If a FOI is associated with a pair \((\varphi, \rho)\), then it is associated with any equivalent pair. If \( \Lambda \) is a FOI at \( x_0 \) associated with a pair \((\varphi, \rho)\) and \( \{x^i\} \) are local coordinates on a coordinate chart \( U \) containing \( x_0 \), then the pair \((\varphi, \rho)\) is equivalent to some pair \((\psi, dx)\) on \( U \), where \( dx = dx^1 \wedge \ldots \wedge dx^n \). In terms of the pair \((\psi, dx)\), condition \([5]\) can be stated as follows,

\[
\Lambda \left( \hat{\partial f} \hat{\partial x^i} + \hat{\partial \psi} \hat{\partial x^i} f \right) = 0
\]

for any \( i \) and any function \( f \), because \( \text{div}_{dx}(\hat{\partial} / \hat{\partial x^i}) = 0 \).

It is clear from the definition that a FOI at \( x_0 \) associated with a pair \((\varphi, \rho)\) depends only on the jets of \( \varphi \) and \( \rho \) at \( x_0 \). It was shown in \([13]\) that if a FOI \( \Lambda \) at \( x_0 \) is associated with pairs \((\varphi, \rho)\) and \((\hat{\varphi}, \hat{\rho})\) with the same volume form \( \rho \), then the jet of \( \hat{\varphi} - \varphi \) at \( x_0 \) is a \( \nu \)-formal constant.
This result is based on the following important statement. Given a FOI $\Lambda$ at $x_0$, consider a pairing on $C^\infty(M)[[\nu]]$ given by the formula

\[(f, g)_\Lambda := \Lambda(f \cdot g).\]

This pairing depends only on the jets of $f$ and $g$ at $x_0$ and therefore it induces a pairing on the space $\mathcal{F}$ of $\nu$-formal jets at $x_0$. The induced pairing will be denoted by the same notation $(\cdot , \cdot)_\Lambda$.

**Lemma 2.1.** For any FOI $\Lambda$ at $x_0$ the pairing [7] on $\mathcal{F}$ is nondegenerate.

We will give a shorter and more conceptual proof of this lemma than the one given in [13].

**Proof.** Let $\Lambda$ be a FOI at $x_0$. Fix a coordinate chart $U$ containing $x_0$ with local coordinates $\{x^i\}$. The FOI $\Lambda$ is associated with some pair $(\psi, dx)$ on $U$. For any functions $f, g$, we have from (6) that

\[
\left( \frac{\partial f}{\partial x^i}, g \right)_\Lambda + \left( f, \frac{\partial g}{\partial x^i} + \frac{\partial \psi}{\partial x^i}g \right)_\Lambda = \Lambda \left( \frac{\partial f}{\partial x^i}g + f \frac{\partial g}{\partial x^i} + \frac{\partial \psi}{\partial x^i}fg \right) = \Lambda \left( \frac{\partial (fg)}{\partial x^i} + \frac{\partial \psi}{\partial x^i}fg \right) = 0.
\]

It means that the transpose of the operator $\partial/\partial x^i$ with respect to the pairing [7] is

\[
\left( \frac{\partial}{\partial x^i} \right)^{\dagger} = -\frac{\partial}{\partial x^i} - \frac{\partial \psi}{\partial x^i}.
\]

The transpose of the multiplication operator by a function $f$ with respect to the pairing [7] is the same operator, $f^{\dagger} = f$. We see that any formal differential operator of finite order $A$ on $U$ has a transpose $A^{\dagger}$ with respect to this pairing. Suppose that a formal function $f \in C^\infty(M)[[\nu]]$ lies in the kernel of the pairing [7], that is,

\[(f, g)_\Lambda = 0\]

for any formal function $g$. For any differential operator $A$ on $U$ we have

\[(8) \quad \Lambda(Af) = \Lambda(Af \cdot 1) = (Af, 1)_\Lambda = (f, A^{\dagger}1)_\Lambda = 0.\]

Assume that the jet of $f = f_0 + \nu f_1 + \ldots$ at $x_0$ is nonzero. Let $r$ be the least nonnegative integer such that the jet of $f_r$ at $x_0$ is nonzero. Since $\Lambda_0 = \alpha \delta_{x_0}$, where $\alpha$ is a nonzero constant, we see from (8) that

\[(Af_r)(x_0) = 0\]

for any differential operator $A$ which does not depend on $\nu$. It contradicts the assumption that the jet of $f_r$ at $x_0$ is nonzero. Therefore, the pairing [7] induced on $\mathcal{F}$ is nondegenerate. \qed
Formal oscillatory integrals should naturally appear in the framework of deformation quantization because many star products are obtained from asymptotic expansions of oscillatory integrals. In this paper we are concerned with the family (2) of FOIs related to a star product with separation of variables.

3. Deformation quantization

Let $M$ be a Poisson manifold equipped with a Poisson bracket $\{\cdot,\cdot\}$. A formal deformation quantization on $M$ is given by a $\nu$-linear associative product on the space $C^\infty(M)[[\nu]]$ of formal functions,

$$(9) \quad f \star g = fg + \sum_{r=1}^{\infty} \nu^r C_r(f, g),$$

where $C_r$ are bidifferential operators on $M$ and

$$C_1(f, g) - C_1(g, f) = i\{f, g\}.$$ 

The product $\star$ is called a star product. It is assumed that the unit constant is the identity for a star product, $f \star 1 = 1 \star f = f$ for any $f$. The product (9) naturally extends to the space $C^\infty(M)((\nu))$.

Two star products $\star$ and $\tilde{\star}$ on a Poisson manifold $(M, \{\cdot,\cdot\})$ are called equivalent if there exists a formal differential operator $T = 1 + \nu T_1 + \ldots$ on $M$ such that

$$f \tilde{\star} g = T^{-1}(T f \star T g).$$

The operator $T$ is called an equivalence operator between the star products $\star$ and $\tilde{\star}$.

If a star product $\star$ on a manifold $M$ is fixed, we denote by $L_f$ and $R_f$ the left and the right star multiplication operators by a function $f$, respectively, so that $L_f g = f \star g = R_g f$. It follows from the associativity of the star product that $[L_f, R_g] = 0$ for any functions $f, g$.

Since a star product $\star$ on $M$ is given by bidifferential operators, it can be restricted to any open subset of $M$. Moreover, it induces a product on the space of formal jets at a given point. We will retain the same notation $\star$ for these induced products.

If $M$ is a symplectic manifold, then for each star product $\star$ on $M$ there exists a $\nu$-formal trace density $\mu$ globally defined on $M$ such that

$$\int_M f \star g \mu = \int_M g \star f \mu$$

if $f$ or $g$ is compactly supported (see [17]).

The concept of deformation quantization was introduced in [1]. Kontsevich showed in [15] that star products exist on arbitrary Poisson manifolds and gave an explicit parametrization of their equivalence classes.
On symplectic manifolds Fedosov constructed star products in each equivalence class in [7] and [8].

A star product (9) is called *natural* in [9] if, for every \( r \), the bidifferential operator \( C_r \) is of order not greater than \( r \) in each argument. Many important star products are natural, e.g., the Fedosov’s star products (see [16]).

We call a formal differential operator \( N = N_0 + \nu N_1 + \ldots \) *natural* if the order of the differential operator \( N_r \) is not greater than \( r \) for \( r \geq 0 \). A star product \( \star \) on \( M \) is natural if and only if the operators \( L_f \) and \( R_f \) are natural for every \( f \in C^\infty(M)[[\nu]] \). We denote by \( \mathfrak{N} \) the space of natural operators on \( M \). It is an associative algebra. It is also a Lie algebra with respect to the usual commutator \( A, B \mapsto [A, B] \).

Denote by \( \mathfrak{E} \) the group of formal differential operators on \( M \) of the form

\[
\exp(\nu^{-1}N),
\]

where \( N = \nu^2 N_2 + \nu^3 N_3 + \ldots \in \mathfrak{N} \). Observe that

\[
\exp(\nu^{-1}N) = 1 + \nu N_2 \pmod{\nu^2}.
\]

We call the operators from \( \mathfrak{E} \) the operators of exponential type.

**Lemma 3.1.** If \( S \in \mathfrak{E} \) and \( A \in \mathfrak{N} \), then \( SAS^{-1} \in \mathfrak{N} \).

**Proof.** If \( S = \exp(\nu^{-1}N) \) and \( N = \nu^2 N_2 + \ldots \in \mathfrak{N} \), then

\[
SAS^{-1} = \exp(\text{ad}(\nu^{-1}N))A = \sum_{r=0}^{\infty} \frac{1}{n!} (\nu^{-1} \text{ad}(N))^r A,
\]

where the series converges in the \( \nu \)-adic topology. Since \( \nu^{-1} \text{ad}(N) \) leaves \( \mathfrak{N} \) invariant, we see that \( SAS^{-1} \in \mathfrak{N} \). \( \square \)

In [9] the following important theorem was proved.

**Theorem 3.1.** (S. Gutt and J. Rawnsley)

Any equivalence operator between two equivalent natural star products is of exponential type.

4. The algebra \( \mathfrak{B} \)

In what follows we will use functions on formal neighborhoods of embedded submanifolds. Let \( Y \) be an embedded submanifold of a manifold \( X \) and let \( I_Y \) be the vanishing ideal of \( Y \) in \( C^\infty(X) \). We call

\[
C^\infty(X, Y) := C^\infty(X)/ \cap_{k=1}^{\infty} (I_Y)^k
\]

the space of functions on the formal neighborhood of \( Y \) in \( X \).
Given a manifold $M$, we identify the diagonal of $M^l$ with $M$ (thus assuming that $M \subset M^l$ for any $l$). An $l$-differential operator $C(f_1, \ldots, f_l)$ on $M$ defines a mapping

$$C : C^\infty(M^l, M) \to C^\infty(M).$$

Let $\mathbb{A} := (C^\infty(M)[[\nu]], \star)$ be a star algebra on a Poisson manifold $M$ with natural star product $\star$. Denote by $\widetilde{M}$ a copy of $M$ with the opposite Poisson structure. The opposite product $f \star_{\text{opp}} g := g \star f$ is a star product on $\widetilde{M}$. The product

$$\odot := \star \otimes \star_{\text{opp}}$$

is a natural star product on $M \times \widetilde{M}$. For $f, g, u, v \in \mathbb{A}$ we have

$$(f \otimes g) \odot (u \otimes v) = (f \star u) \otimes (v \star g).$$

Here $\otimes$ is a tensor product over the ring $\mathbb{C}[[\nu]]$. The product $\odot$ induces a product on $C^\infty(M \times M, M)[[\nu]]$ which will be denoted by the same symbol. We introduce the algebra $\mathbb{B} := (C^\infty(M \times M, M)[[\nu]], \odot)$.

We call an element of $\mathbb{B}$ factorizable if it is induced by a formal function $f \otimes g \in \mathbb{A} \otimes \mathbb{A}$, and use the same notation $f \otimes g$ for this element. There exists a homomorphism $F \mapsto N_{F}$ from $\mathbb{B}$ to $\mathfrak{N}$ given on the factorizable elements by

$$N_{fg} = L_{f}R_{g}.$$

We will prove that if $M$ is symplectic, then this mapping is an isomorphism. To this end, we need to recall several definitions and facts from [11].

If $A$ is a differential operator of order $r$ on a manifold $M$, then its principal symbol $\text{Symb}_r(A)$ is a fiberwise polynomial function of degree $r$ on the cotangent bundle $T^*M$. Given a natural operator $N = N_0 + \nu N_1 + \ldots$ on $M$, we call the formal series

$$\sigma(N) := \sum_{r=0}^{\infty} \text{Symb}_r(N_r)$$

the sigma symbol of $N$. It can be interpreted as a function on the formal neighborhood of the zero section $Z$ of $T^*M$,

$$\sigma(N) \in C^\infty(T^*M, Z).$$

The mapping $N \mapsto \sigma(N)$ is a surjective homomorphism from $\mathfrak{N}$ onto $C^\infty(T^*M, Z)$ whose kernel is $\nu \mathfrak{N}$. It follows that the sigma symbol $\sigma(N_{F})$ of $F = F_0 + \nu F_1 + \ldots \in C^\infty(M \times M, M)[[\nu]]$ depends only on $F_0$. It was proved in [11] that if $M$ is symplectic, then the mapping

$$C^\infty(M \times M, M) \ni F_0 \mapsto \sigma(N_{F_0})$$
is an isomorphism of $C^\infty(M \times M, M)$ onto $C^\infty(T^*M, Z)$.

**Theorem 4.1.** If $\ast$ is a natural star product on a symplectic manifold $M$, then the mapping $F \mapsto N_F$ is an isomorphism of the algebra $\mathbb{B}$ onto $\mathfrak{N}$.

**Proof.** We will construct the inverse mapping of the mapping $F \mapsto N_F$. Let $N$ be an arbitrary natural operator on $M$. There exists a unique element $F_0 \in C^\infty(M \times M, M)$ such that $\sigma(N_{F_0}) = \sigma(N)$.

Then $\nu^{-1}(N - N_{F_0}) \in \mathfrak{N}$. Let $F_1$ denote the unique element of $C^\infty(M \times M, M)$ such that $\sigma(N_{F_1}) = \sigma(\nu^{-1}(N - N_{F_0}))$.

Hence, $\nu^{-2}(N - N_{F_0} - \nu N_{F_1}) \in \mathfrak{N}$. Continuing this process, we produce a unique element $F = F_0 + \nu F_1 + \ldots \in \mathbb{B}$ such that $N = N_F$. □

We say that a formal distribution $\Lambda = \Lambda_0 + \nu \Lambda_1 + \ldots$ on $M$ supported at a point $x_0 \in M$ is *natural* if the order of the distribution $\Lambda_r$ is not greater than $r$ for every $r$. We denote the set of all such distributions by $\mathcal{N}$.

**Lemma 4.1.** A formal distribution $\Lambda$ supported at $x_0$ is natural if and only if there exists a natural operator $N \in \mathfrak{N}$ such that

$$\Lambda = \delta_{x_0} \circ N,$$

i.e., $\Lambda(f) = (Nf)(x_0)$ for any function $f$.

**Proof.** It is clear that if $N \in \mathfrak{N}$, then $\Lambda = \delta_{x_0} \circ N \in \mathcal{N}$. Conversely, given $\Lambda \in \mathcal{N}$, one can fix local coordinates around $x_0$ and find the unique formal differential operator with constant coefficients $C$ such that $\Lambda = \delta_{x_0} \circ C$. Then $C$ is natural. It can be extended to a natural operator on $M$ by multiplying it by an appropriate cutoff function. □

Denote by $\tau$ the involution on $\mathbb{B}$ such that $\tau(f \otimes g) = g \otimes f$. It is an antiautomorphism of $\mathbb{B}$. The algebra $\mathbb{B}$ acts on $\mathcal{N}$ so that an element $F \in \mathbb{B}$ maps $\Lambda \in \mathcal{N}$ to $\Lambda \circ N_{\tau(F)} \in \mathcal{N}$. Given $F \in \mathbb{B}$ and $x_0 \in M$, we set

$$\Lambda_F := \delta_{x_0} \circ N_{\tau(F)}.$$

**Lemma 4.2.** For $f, g \in \mathfrak{A}$ we have $\Lambda_{f \otimes g}(h) = (g \ast h \ast f)(x_0)$.

**Proof.**

$$\Lambda_{f \otimes g}(h) = (N_{\tau(f \otimes g)}h)(x_0) = (N_{g \otimes f}h)(x_0) = (L_g R_f h)(x_0) = (g \ast h \ast f)(x_0).$$ □
The formal distribution $\Lambda_F$ depends only on the jet of $F$ at the diagonal point $(x_0, x_0) \in M \times M$. We denote by $\mathcal{F}^{(2)}$ the space of $\nu$-formal jets on $M \times M$ at $(x_0, x_0)$.

**Lemma 4.3.** If $\ast$ is a star product on a symplectic manifold $M$, then the corresponding mapping $F \mapsto \Lambda_F$ induces a surjective mapping

$$\lambda : \mathcal{F}^{(2)} \to \mathcal{N}.$$

**Proof.** This statement follows from Theorem 4.1 and Lemma 4.1. □

Given a factorizable element $f \otimes g \in \mathcal{F}^{(2)}$, we get from Lemma 4.2 that

$$(10) \quad \langle \lambda(f \otimes g), h \rangle = (g \ast h \ast f)(x_0).$$

5. Star products with separation of variables

Berezin described in [2] and [3] a quantization procedure on Kähler manifolds which leads to star products with the property of separation of variables (see, e.g., [4], [5], [6], [10], [14]). It is natural to consider the star products with this property on pseudo-Kähler manifolds. Recall that an almost-Kähler manifold is a complex manifold equipped with a real symplectic form of type $p_1, 1$ with respect to the complex structure.

**Definition 5.1.** A star product (4) on a pseudo-Kähler manifold $M$ has the property of separation of variables of the Wick type if the operators $C_r, r \geq 1$, differentiate the first argument in holomorphic directions and the second argument in antiholomorphic ones. A star product is of the anti-Wick type if $C_r, r \geq 1$, differentiate the first argument in antiholomorphic directions and the second argument in holomorphic ones.

**Remark.** Observe that if $\ast$ is a star product of the anti-Wick type on a pseudo-Kähler manifold $M$, then the opposite product $f \ast_{\text{opp}} g := g \ast f$ is a product of the Wick type on the manifold $M$ with the same complex structure but with the opposite symplectic structure. Also, $\ast$ is a product of the Wick type on the manifold $\bar{M}$, which is a copy of $M$ with the opposite complex structure but with the same symplectic structure.

Let $\ast$ be a product of the anti-Wick type on $M$. If $a$ is a holomorphic function and $b$ is an antiholomorphic function locally defined on $M$, then for any function $f$ we have

$$a \ast f = af \quad \text{and} \quad f \ast b = bf,$$

i.e., $L_a = a$ and $R_b = b$ are pointwise multiplication operators.
Throughout this paper we will denote the pointwise multiplication operator by a function $f$ by the same symbol $f$.

Let $\omega_{-1}$ be a pseudo-Kähler form on $M$ (which determines a symplectic structure on $M$). In [10] it was shown that the star products of the anti-Wick type on $M$ are bijectively parametrized (not only up to equivalence) by the formal closed (1,1)-forms

$$\omega = \nu^{-1}\omega_{-1} + \nu_0 + \nu\omega_1 + \ldots$$
on $M$. We will briefly recall this parametrization.

Suppose that $\omega$ is fixed. Let $U$ be a contractible coordinate chart on $M$ with holomorphic coordinates $\{z^k\}$. There exists a formal potential $\Phi = \nu^{-1}\Phi_0 + \nu\Phi_1 + \ldots$ of $\omega$ on $U$, so that $\omega = i\partial\bar{\partial}\Phi$. As shown in [10], there exists a unique star product of the anti-Wick type $\star$ on $M$ such that on every contractible chart $U$ and for any potential $\Phi$ of $\omega$ on $U$,

$$L_{\frac{\partial\Phi}{\partial z^k}} = \frac{\partial\Phi}{\partial z^k} + \frac{\partial}{\partial z^k}$$

and

$$R_{\frac{\partial\Phi}{\partial \bar{z}^l}} = \frac{\partial\Phi}{\partial \bar{z}^l} + \frac{\partial}{\partial \bar{z}^l}.$$  

The formal form $\omega$ is called the classifying form of the star product $\star$. Every star product of the anti-Wick type has a unique classifying form.

Given a star product $\star$ of the anti-Wick type on $M$, there exists a $\nu$-formal differential operator $I = 1 + \nu I_1 + \nu^2 I_2 + \ldots$ globally defined on $M$ such that for any local holomorphic function $a$ and local antiholomorphic function $b$,

$$I(ab) = b \star a.$$  

It is called the formal Berezin transform of the star product $\star$. Observe that $Ia = a$ and $Ib = b$. It is proved in [12] that

$$L_b = I \circ b \circ I^{-1} \text{ and } R_a = I \circ a \circ I^{-1}.$$  

One can recover the product $\star$ from the operator $I$ using that

$$(ab) \star (a'b') = aI(a'b)b'$$

where the functions $a, a'$ are local holomorphic and $b, b'$ are local antiholomorphic. The equivalent star product

$$f \star' g := I^{-1}(If \star Ig)$$

on $M$ is a star product with separation of variables of the Wick type (see [12]).

**Lemma 5.1.** The formal Berezin transform $I$ of a star product of the anti-Wick type $\star$ is of exponential type.
Proof. The star products with separation of variables $\ast$ and $\ast'$ are natural (see [16]). Since $I$ is an equivalence operator between the products $\ast$ and $\ast'$, it is of exponential type according to Theorem 3.1.

It was shown in [14] and [13] that for any point $x_0 \in M$ and any integer $l \geq 1$ the functional
\[
K^{(l)}(f_1, \ldots, f_l) = I(f_1 \ast' \ldots \ast' f_l)(x_0) = (I f_1 \ast \ldots \ast I f_l)(x_0)
\]
on $M'$ is a FOI at $(x_0)^l \in M'$. Below we give a phase-volume form pair associated with $K^{(l)}$, which was found in [14] and [13].

Let $U$ be a contractible neighborhood in $M$ and $\Phi$ be a local potential of the classifying form $\omega$ of the product $\ast$ on $U$. Let $\hat{U}$ denote a copy of $U$ equipped with the opposite complex structure. One can find a function $\tilde{\Phi}(x, y)$ on $U \times \hat{U}$ such that $\tilde{\Phi}(x, x) = \Phi(x)$ and
\[
\bar{\partial}_{U \times \hat{U}} \tilde{\Phi}
\]
has zero of infinite order at every point of the diagonal of $U \times \hat{U}$. The function $\tilde{\Phi}(x, y)$ is called an almost analytic extension of $\Phi$ (see details in [13]). For each $l \geq 1$ we introduce a function $G^{(l)}$ on $U^l$ by the formula
\[
G^{(l)}(x_1, \ldots, x_l) := \tilde{\Phi}(x_1, x_2) + \tilde{\Phi}(x_2, x_3) + \ldots + \tilde{\Phi}(x_l, x_1)
\]
\[
- (\Phi(x_1) + \Phi(x_2) + \ldots + \Phi(x_l)).
\]
This function defines an element of $\nu^{-1}C^\infty(U^l, U)[[\nu]]$, where $U$ is identified with the diagonal of $U^l$. This element does not depend on the choice of the potential $\Phi$ and of the almost analytic extension of $\Phi$. Thus, taking such functions for every contractible neighborhood in $M$, we get a global element of $\nu^{-1}C^\infty(M^l, M)[[\nu]]$. We call it a cyclic formal $l$-point Calabi function of the classifying form $\omega$.

Now suppose that $x_0 \in U$ and consider the function
\[
F^{(l)}(x_1, \ldots, x_l) := G^{(l+1)}(x_0, x_1, \ldots, x_l)
\]
on $U^l$. The jet of $F^{(l)}$ at $(x_0)^l \in U^l$ is determined by the jet of $G^{(l+1)}$ at $(x_0)^{l+1} \in U^{l+1}$, which is the jet of the formal $(l + 1)$-point Calabi function of $\omega$ at $(x_0)^{l+1}$.

It was shown in [14] and [13] that the FOI $K^{(l)}$ at $(x_0)^l$ is associated with the pair $(F^{(l)}, \mu \otimes^l)$ on $U^l$, where $\mu$ is a trace density of the star product $\ast$.

\[\text{\footnotesize\textsuperscript{2}}\text{One can check that } \tilde{\Phi} \text{ defines a function on the formal neighborhood of the diagonal of } U \times \hat{U} \text{ which does not depend on the choice of the almost analytic extension of } \Phi.\]
The main goal of this paper is to develop an algebraic framework which will incorporate the jet of the formal oscillatory exponent \( \exp G^{(l)} \) at \((x_0)^l \in M^l\) for every \( l \geq 1 \).

6. The algebra \( \mathcal{C} \)

Let \( \ast \) be a star product of the anti-Wick type on a pseudo-Kähler manifold \( M \), \( I \) be its formal Berezin transform, and \( x_0 \) be a fixed point in \( M \). We choose a coordinate chart \( U \) containing \( x_0 \) with coordinates \( \{ z^k, \bar{z}^l \} \) such that \( z^k(x_0) = \bar{z}^l(x_0) = 0 \) for all \( k, l \). We consider various jet spaces on \( M \) at the point \( x_0 \) and on \( M \times M \) at the point \((x_0, x_0)\). We identify these spaces with spaces of formal series in local coordinates. Denote by \( \mathcal{F} = \mathbb{C}[[\nu, z, \bar{z}]] \) the space of \( \nu \)-formal jets on \( M \) at \( x_0 \) and by \( \mathcal{A} = (\mathcal{F}, \ast) \) the algebra on \( \mathcal{F} \) with the induced product \( \ast \). Denote by \( \mathcal{F}^{(2)} = \mathbb{C}[[\nu, z, \bar{z}, w, \bar{w}]] \) the space of \( \nu \)-formal jets on \( M \times M \) at \((x_0, x_0)\), where \( \{ z^k, \bar{z}^l \} \) and \( \{ w^k, \bar{w}^l \} \) are the coordinates on the first and the second factors of the chart \( U \times U \). For the involutive mapping \( \tau : \mathcal{F}^{(2)} \to \mathcal{F}^{(2)} \) such that \( \tau(f \otimes g) = g \otimes f \) for \( f, g \in \mathcal{F} \), one has \( \tau(z^k) = w^k \) and \( \tau(\bar{z}^l) = \bar{w}^l \).

Since \( \ast \) is a natural star product on \( M \) and \( M \) is symplectic, one can construct a bijection \( F \mapsto N_F \) from \( \mathcal{F}^{(2)} \) onto the space \( \mathfrak{N} \) of natural operators on \( M \) as in Section 4. For a factorizable element \( f \otimes g \in \mathcal{F}^{(2)}, N_{f \otimes g} = L_f R_g \). There is a mapping \( \lambda : \mathcal{F}^{(2)} \to \mathcal{N} \) to the space \( \mathcal{N} \) of natural distributions at \( x_0 \),

\[ \lambda(F) = \delta_{x_0} \circ N_{\tau(F)}, \]

which is surjective by Lemma 4.3. On factorizable elements \( f \otimes g \in \mathcal{F}^{(2)} \) the mapping \( \lambda \) is given by formula (10).

The space \( \mathcal{J} = \mathbb{C}[[z, \bar{z}]] \) of jets on \( M \) at \( x_0 \) has a descending filtration \( \mathcal{J} = F_0 \mathcal{J} \supset F_1 \mathcal{J} \supset \ldots \), where \( F_r \mathcal{J} \) is the space of jets which have zero of order at least \( r \) at \( x_0 \). We assume that \( F_r \mathcal{J} = \mathcal{J} \) for \( r < 0 \). We introduce a filtration

\[ \mathcal{F} = F_0 \mathcal{F} \supset F_1 \mathcal{F} \supset \ldots \]
on the space of formal jets \( \mathcal{F} = \mathcal{J}[[\nu]] \) which agrees with the filtration on \( \mathcal{J} \) and for which the filtration degree of \( \nu \) is 2,

\[ F_r \mathcal{F} = F_r \mathcal{J} + \nu F_{r-2} \mathcal{J} + \nu^2 F_{r-4} \mathcal{J} + \ldots. \]

We call it the standard filtration. Observe that \( \mathcal{F}/F_r \mathcal{F} \) is a finite dimensional vector space over \( \mathbb{C} \). One can check that

\[ \mathcal{F}^{(2)} = \lim_{r \to 0} (\mathcal{F} \otimes \mathcal{F})/F_r(\mathcal{F} \otimes \mathcal{F}), \]
where the subspaces
\[ F_r(\mathcal{F} \otimes \mathcal{F}) := \sum_{i+j=r} F_i \mathcal{F} \otimes F_j \mathcal{F} \]
form the standard filtration on \( \mathcal{F} \otimes \mathcal{F} \). Here \( \otimes \) is the tensor product over the ring \( \mathbb{C}[[\nu]] \).

**Lemma 6.1.** The algebra \( A = (\mathcal{F}, \star) \) is a filtered algebra with respect to the standard filtration.

*Proof.* Since \( \star \) is a natural star product (see [16]), the bidifferential operator \( C_r \) in (9) is of order not greater than \( r \) in each argument. Therefore, if \( f \in F_i \mathcal{F} \) and \( g \in F_j \mathcal{F} \), then \( C_r(f, g) \in F_{i+j-2r} \mathcal{F} \) and \( \nu^r C_r(f, g) \in F_{i+j} \mathcal{F} \), whence the lemma follows. \( \square \)

Lemma 6.1 allows to extend various mappings of the space \( \mathcal{F} \otimes \mathcal{F} \) to its completion \( \mathcal{F}(2) = \mathcal{F} \hat{\otimes} \mathcal{F} \) with respect to the topology associated with the standard filtration. We will tacitly assume that these extensions can be justified with the use of this lemma.

We define a filtered associative algebra \( \mathcal{C} := (\mathcal{F}(2), \circ) \), where the product \( \circ \) is given on the factorizable elements by the formula
\[(g_1 \otimes h_1) \circ (g_2 \otimes h_2) := (h_1 \circ g_2)(x_0) \cdot (g_1 \otimes h_2).\]

We introduce a trace on \( \mathcal{C} \) given on the factorizable elements by the formula
\[(14) \quad \text{tr}(f \otimes g) := (g \circ f)(x_0).\]

One can check the trace property on factorizable elements,
\[
\text{tr}((g_1 \otimes h_1) \circ (g_2 \otimes h_2)) = (h_1 \circ g_2)(x_0) \cdot (h_2 \otimes g_1)(x_0) = \\
\text{tr}((g_2 \otimes h_2) \circ (g_1 \otimes h_1)).
\]

**Lemma 6.2.** For \( F \in \mathcal{C} \), the following identity holds,
\[ \text{tr} F = \langle \lambda(F), 1 \rangle. \]

*Proof.* Given a factorizable element \( f \otimes g \in \mathcal{F}(2) \), we get from formula (10) that
\[ \text{tr}(f \otimes g) = (g \circ f)(x_0) = \langle \lambda(f \otimes g), 1 \rangle, \]
whence the lemma follows. \( \square \)

We introduce a splitting of \( \mathcal{C} \),
\[(15) \quad \mathcal{C} = \mathcal{G} \oplus \mathcal{H}, \]

where \( \mathcal{G} \) and \( \mathcal{H} \) are suitable subspaces.
where $G = \mathbb{C}[[\nu, z, \bar{w}]]$ and $\mathcal{H}$ is generated by $\bar{z}^l$ and $w^k$ for all $k, l$, i.e., any $H \in \mathcal{H}$ can be represented as

$$H = \bar{z}^l A_l + w^k B_k$$

for some $A_l, B_k \in \mathcal{C}$. This splitting does not depend on the choice of local holomorphic coordinates used in its definition. We will show that in the splitting the subspace $G$ is a subalgebra of $\mathcal{C}$ and $\mathcal{H}$ is a two-sided ideal of $\mathcal{C}$.

**Lemma 6.3.** The subspace $\mathcal{H} \subset \mathcal{C}$ is a two-sided ideal of the algebra $\mathcal{C}$ which lies in the kernel of the mapping $\lambda$.

**Proof.** It suffices to check the statement of the lemma on the generators

$$U^l = (\bar{z}^l u) \otimes v = (u \ast \bar{z}^l) \otimes v$$

and factorizable $F = f \otimes g \in \mathcal{H}$, where $u, v, f, g \in \mathcal{F}$ are arbitrary. We have

$$F \ast U^l = (f \otimes (u \ast \bar{z}^l)) \otimes v = (g \ast (u \ast \bar{z}^l))(x_0) \cdot (f \otimes v) = 0,$$

because $\bar{z}(x_0) = 0$. Then we see that

$$U^l \ast F = ((\bar{z}^l u) \otimes v) \ast (f \otimes g) = (v \ast f)(x_0) \cdot ((\bar{z}^l u) \otimes g) \in \mathcal{H}.$$

One can check similarly that $F \ast V_k \in \mathcal{H}$ and $V_k \ast F = 0$. It follows that $\mathcal{H}$ is a two-sided ideal of $\mathcal{C}$. We get from formula (10) that for any $h \in \mathcal{F}$,

$$\langle \lambda(U^l), h \rangle = (v \ast h \ast u \ast \bar{z}^l)(x_0) = ((v \ast h \ast u) \bar{z}^l)(x_0) = 0,$$

because $\bar{z}(x_0) = 0$. Thus, $\lambda(U^l) = 0$. One can similarly check that $\lambda(V^k) = 0$, which implies the second statement of the lemma. \hfill $\Box$

**Lemma 6.4.** The subspace $G \subset \mathcal{C}$ is a subalgebra of $\mathcal{C}$ isomorphic to the algebra $\mathcal{C} \setminus \mathcal{H}$.

**Proof.** The space $G$ is topologically generated by the elements $a \otimes b$, where $a \in \mathbb{C}[[\nu, z]]$ is formally holomorphic and $b \in \mathbb{C}[[\nu, \bar{z}]]$ is anti-holomorphic. We have

$$(a_1 \otimes b_1) \ast (a_2 \otimes b_2) = (b_1 \ast a_2)(x_0) \cdot (a_1 \otimes b_2) \in G.$$  

Therefore, $G$ is a subalgebra of $\mathcal{C}$. Clearly, it is isomorphic to the algebra $\mathcal{C} \setminus \mathcal{H}$. \hfill $\Box$

Let $\alpha : C^\infty(U)[[\nu]] \to \mathcal{F}$ be the mapping that maps $f$ to its jet at $x_0$. It is surjective by Borel’s theorem. We define a mapping

$$\gamma : C^\infty(U)[[\nu]] \to G$$
as follows. Given \( f \in C^\infty(U[[\nu]]) \), let \( \tilde{f} \in C^\infty(U \times \bar{U})[[\nu]] \) be an almost analytic extension of \( f \). We set \( \gamma(f) \) equal to the jet of \( \tilde{f} \) at \((x_0, x_0)\). This jet lies in \( \mathcal{G} \) and does not depend on the choice of the almost analytic extension of \( f \). The mapping \( \gamma \) is surjective. There is a bijection \( \beta : \mathcal{F} \rightarrow \mathcal{G} \) such that \( \gamma = \beta \circ \alpha \). In coordinates,

\[
\beta : f(z, \bar{z}) \mapsto f(z, \bar{w}).
\]

**Lemma 6.5.** Given \( g \in C^\infty(U[[\nu]]) \), the following formula holds,

\[
\lambda(\gamma(g)) = \delta_{x_0} \circ I \circ g \circ I^{-1}.
\]

**Proof.** Let \( g = ab \), where \( a \) is a holomorphic and \( b \) is an antiholomorphic function on \( U \). Then \( \gamma(g) = a \otimes b \). Using formula (11), we get that

\[
\lambda(\gamma(g)) = \delta_{x_0} \circ N_{\tau(a \otimes b)} = \delta_{x_0} \circ N_{b \otimes a} = \delta_{x_0} \circ (R_a L_b) =
\]

\[
\delta_{x_0} \circ I \circ (ab) \circ I^{-1} = \delta_{x_0} \circ I \circ g \circ I^{-1}.
\]

For a generic \( g \in C^\infty(U[[\nu]]) \), the distribution \( \delta_{x_0} \circ I \circ g \circ I^{-1} \) depends only on the jet of \( g \) at \( x_0 \) and the space \( \mathcal{F} \) is topologically generated by the elements \( \alpha(ab) \). Therefore, the lemma follows from the calculation above.

**Lemma 6.6.** The restriction of the mapping \( \lambda \) to \( \mathcal{G} \), \( \lambda|_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{N} \), is injective.

**Proof.** Let \( G \) be an arbitrary element of \( \mathcal{G} \) which lies in the kernel of \( \lambda \). There exists \( g \in C^\infty(U[[\nu]]) \) such that \( G = \gamma(g) \). Then for any \( h \in C^\infty(U[[\nu]]) \) we have from Lemma 6.5 that

\[
I(g \cdot I^{-1}h)(x_0) = \langle \lambda(\gamma(g)), h \rangle = \langle \lambda(G), h \rangle = 0.
\]

It was proved in [14] that the distribution \( f \mapsto (I f)(x_0) \) is a FOI at \( x_0 \). By Lemma 2.1, the pairing \( u, v \mapsto I(u \cdot v)(x_0) \) on \( C^\infty(U[[\nu]]) \) induces a nondegenerate pairing on \( \mathcal{F} \). Since \( I^{-1}h \) is an arbitrary element of \( C^\infty(U[[\nu]]) \), we see from (16) that the jet of \( g \) at \( x_0 \) is zero. Therefore, \( G = 0 \), whence the lemma follows.

**Corollary 6.1.** The ideal \( \mathcal{H} \) is the kernel of the mapping \( \lambda \) and the mapping \( \lambda|_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{N} \) is bijective.

**Proof.** The mapping \( \lambda \) is surjective by Lemma 4.3. It was proved in Lemma 6.5 that \( \mathcal{H} \) lies in the kernel of \( \lambda \). The corollary follows from the splitting (15) and Lemma 6.6.
7. The algebra of distributions

Corollary 6.1 implies that one can transfer the product $\ast$ from the algebra $\mathcal{C}$ to $\mathcal{N}$. We denote the resulting product on $\mathcal{N}$ by $\bullet$.

**Theorem 7.1.** The algebra $(\mathcal{N}, \bullet)$ is isomorphic to the algebra $(\mathcal{G}, \ast) \cong \mathcal{C}/\mathcal{H}$. The mapping

$$N \ni u \mapsto \langle u, 1 \rangle$$

is a trace on the algebra $(\mathcal{N}, \bullet)$. Its pullback via the mapping $\lambda$ is the trace $\text{tr}$ on $\mathcal{C}$.

**Proof.** The theorem follows from Lemmas 6.4 and 6.2. □

In the rest of the paper we will express the trace of the product of $l$ elements of the algebra $(\mathcal{N}, \bullet)$ in terms of the formal $l$-point Calabi function of the star product $\ast$.

The standard filtration on $\mathcal{F}$ induces a filtration on the formal differential operators on $\mathcal{F}$, which we also call standard. If $A$ is a differential operator of order $r$ which does not depend on $\nu$, its filtration degree is at least $-r$. We denote by $\mathfrak{N}_{x_0}$ the algebra of natural operators on $\mathcal{F}$. These operators are induced by the operators from $\mathfrak{N}$. Observe that if $N = N_0 + \nu N_1 + \ldots$ is a natural operator, then the filtration degree of $\nu^r N_r$ is at least $r$.

In the remainder of this section $\varphi = \nu^{-1} \varphi_1 + \varphi_0 + \ldots$ is a formal function on $M$ such that $x_0$ is a critical point of $\varphi_1$ with zero critical value, $\varphi_1(x_0) = 0$. We do not assume that the critical point $x_0$ is nondegenerate. Observe that the filtration degree of $\varphi$ is a least zero.

**Lemma 7.1.** If $N \in \mathfrak{N}_{x_0}$, then $e^{-\varphi}Ne^\varphi \in \mathfrak{N}_{x_0}$.

**Proof.** Assume that $N = N_0 + \nu N_1 + \ldots \in \mathfrak{N}_{x_0}$. Then for each $r \geq 0$ the formal differential operator

$$e^{-\varphi}(\nu^r N_r)e^\varphi = \sum_{k=0}^{r} \frac{1}{k!}(-\text{ad} \varphi)^k(\nu^r N_r)$$

is of order not greater than $r$. The operator (18) is natural and its $\nu$-filtration degree is at least zero. Its standard filtration degree is at least $r$. Therefore, the series

$$e^{-\varphi}Ne^\varphi = \sum_{r=0}^{\infty} e^{-\varphi}(\nu^r N_r)e^\varphi$$

converges in the topology associated with the standard filtration to an element of $\mathfrak{N}_{x_0}$. □
Below we define an action $e^\varphi : u \mapsto u \circ e^\varphi$ on $\mathcal{N}$ which behaves like a composition. However, the multiplication operator by the formal oscillatory exponent $e^\varphi$ is not a natural operator, because the Taylor series of $e^\varphi$ at $x_0$ contains negative powers of $\nu$.

Given $u \in \mathcal{N}$, there exists $N \in \mathfrak{N}_{x_0}$ such that $u = \delta_{x_0} \circ N$. We set

$$u \circ e^\varphi := e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi}Ne^\varphi).$$

Since $\varphi^{-1}(x_0) = 0$, we see that $e^{\varphi(x_0)} \in \mathbb{C}[[\nu]]$. By Lemma 7.2, $u \circ e^\varphi$ is an element of $\mathcal{N}$. We will show that it does not depend on the choice of $N$.

**Lemma 7.2.** If $u$ has two different representations $u = \delta_{x_0} \circ N = \delta_{x_0} \circ \tilde{N}$ for $N, \tilde{N} \in \mathfrak{N}_{x_0}$, then $e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi}Ne^\varphi) = e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi} \tilde{N}e^\varphi)$.

**Proof.** We have $\delta_{x_0} \circ (N - \tilde{N}) = 0$. Therefore, in coordinates, one can write $N - \tilde{N} = z^k A_k + z^l B_l$ for some $A_k, B_l \in \mathfrak{N}_{x_0}$. We need to show that

$$e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi}(N - \tilde{N})e^\varphi) = 0,$$

which follows from the observation that

$$e^{-\varphi}(N - \tilde{N})e^\varphi = z^k e^{-\varphi} A_k e^\varphi + z^l e^{-\varphi} B_l e^\varphi$$

and the fact that $z^k(x_0) = z^l(x_0)$ for all $k, l$. \hfill $\square$

**Lemma 7.3.** Let $\varphi = \nu^{-1}\varphi^{-1} + \varphi_0 + \ldots$ and $\psi = \nu^{-1}\psi^{-1} + \psi_0 + \ldots$ be formal functions on $M$ such that $x_0$ is a critical point of $\varphi^{-1}$ and $\psi^{-1}$ with zero critical value, $\varphi^{-1}(x_0) = \psi^{-1}(x_0) = 0$. Then for any $u \in \mathcal{N}$ one has

$$(u \circ e^\varphi) \circ e^\psi = u \circ e^{\varphi+\psi}.$$ 

**Proof.** Let $N \in \mathfrak{N}_{x_0}$ be such that $u = \delta_{x_0} \circ N$. Then

$$(u \circ e^\varphi) \circ e^\psi = (e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi}Ne^\varphi)) \circ e^\psi = e^{\varphi(x_0)+\psi(x_0)} \delta_{x_0} \circ (e^{-\psi}e^{-\varphi}Ne^\varphi e^\psi) = u \circ e^{\varphi+\psi}.$$ 

\hfill $\square$

We introduce a $\nu$-linear functional $K : \mathcal{N} \to \mathbb{C}[[\nu]]$,

$$K(u) := \langle u \circ e^\varphi, 1 \rangle.$$ 

If $A$ is a differential operator on $M$, we denote by $A^t$ its transpose that acts on a distribution $u$ as $A^t u := u \circ A$. Let $v$ be a vector field on $M$. Since $\nu v$ and $\nu \nu \varphi$ are natural operators, then for $u \in \mathcal{N}$ we get that $(\nu v - \nu \nu \varphi)^t u \in \mathcal{N}$.

**Lemma 7.4.** For any $u \in \mathcal{N}$, $((\nu v - \nu \nu \varphi)^t u) \circ e^\varphi = (u \circ e^\varphi) \circ (\nu v)$. 

Proof. Assume that \( u = \delta_{x_0} \circ N \) for some \( N \in \mathfrak{N}_{x_0} \). Then

\[
(\nu v - \nu v \varphi)^t u = \delta_{x_0} \circ N \circ (\nu v - \nu v \varphi).
\]

Therefore,

\[
((\nu v - \nu v \varphi)^t u) \circ e^\varphi = e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi}(N \circ (\nu v - \nu v \varphi)) e^\varphi) = e^{\varphi(x_0)} \delta_{x_0} \circ (e^{-\varphi} N e^\varphi) \circ (\nu v) = (u \circ e^\varphi) \circ (\nu v),
\]

because \( e^{-\varphi} \circ (v - \nu \varphi) \circ e^\varphi = v \).

Corollary 7.1. For any \( u \in \mathcal{N} \), \( K((\nu v - \nu v \varphi)^t u) = 0 \).

Proof. We have by Lemma 7.4 that

\[
K((\nu v - \nu v \varphi)^t u) = \langle ((\nu v - \nu v \varphi)^t u) \circ e^\varphi, 1 \rangle = \\
\langle (u \circ e^\varphi) \circ (\nu v), 1 \rangle = \langle u \circ e^\varphi, (\nu v) 1 \rangle = 0.
\]

\[\square\]

Theorem 7.2. Let \( S : \mathcal{N} \to \mathbb{C}[[\nu]] \) be a \( \nu \)-linear functional such that the equality

\[
S((\nu v - \nu v \varphi)^t u) = 0
\]

holds for any vector field \( v \) and any \( u \in \mathcal{N} \). Then there exists a formal constant \( c(\nu) \in \mathbb{C}[[\nu]] \) such that

\[
S(u) = c(\nu) \langle u \circ e^\varphi, 1 \rangle.
\]

Proof. Consider a functional \( T : \mathcal{N} \to \mathbb{C}[[\nu]] \) given by the formula

\[
T(u) := S(u \circ e^{-\varphi}).
\]

We will show that \( T((\nu v)^t u) = 0 \) for any vector field \( v \) and any \( u \in \mathcal{N} \). Let \( N \in \mathfrak{N}_{x_0} \) be such that \( u = \delta_{x_0} \circ N \). Given a vector field \( v \) and \( u \in \mathcal{N} \), we have

\[
T((\nu v)^t u) = S(((\nu v)^t u) \circ e^{-\varphi}) = S((\delta_{x_0} \circ (N \circ (\nu v)) \circ e^{-\varphi}) = \\
S(e^{-\varphi(x_0)} \delta_{x_0} \circ (e^\varphi(N \circ (\nu v)) e^{-\varphi})) = \\
S(e^{-\varphi(x_0)} \delta_{x_0} \circ (e^\varphi(N e^{-\varphi} \circ (\nu v - \nu \varphi))) = \\
S((u \circ e^{-\varphi}) \circ ((\nu v - \nu \varphi))) = S((\nu v - \nu v \varphi)^t (u \circ e^{-\varphi})) = 0.
\]

In local coordinates one can write any operator \( \mathcal{N} \in \mathfrak{N}_{x_0} \) as

\[
N = f + A^p \circ \left( \nu \frac{\partial}{\partial z^p} \right) + B^q \circ \left( \nu \frac{\partial}{\partial z^q} \right),
\]
where \( f = N1 \in \mathbb{C}[[\nu, z, \bar{z}]] \) and \( A^p, B^q \in \mathfrak{M}_{x_0} \). Then for \( u = \delta_{x_0} \circ N \) we have

\[
T(u) = T \left( \delta_{x_0} \circ \left( f + A^p \circ \left( \nu \frac{\partial}{\partial z^p} \right) + B^q \circ \left( \nu \frac{\partial}{\partial z^q} \right) \right) \right) = f(x_0)T(\delta_{x_0}) + T \left( \left( \nu \frac{\partial}{\partial z^p} \right)^t (\delta_{x_0} \circ A^p) \right) + T \left( \left( \nu \frac{\partial}{\partial z^q} \right)^t (\delta_{x_0} \circ B^q) \right) = f(x_0)T(\delta_{x_0}).
\]

It follows that \( T(u) = T(\delta_{x_0})\langle u, 1 \rangle \). We set \( c(\nu) := T(\delta_{x_0}) \). Using Lemma 7.3 we get that

\[
S(u) = T(u \circ e^\nu) = c(\nu)\langle u \circ e^\nu, 1 \rangle.
\]

Let \( x_0 \) be a point in \( M, U \) be a contractible coordinate chart with coordinates \( \{z^p, \bar{z}^q\} \) such that \( z^p(x_0) = 0 \) for all \( p, q \), and \( \Phi \) be a potential of the classifying form \( \omega \) of the star product \( \ast \) on \( U \). We choose an almost analytic extension \( \hat{\Phi} \) of \( \Phi \) on \( U \times U \). In Section 5 we introduced the cyclic function

\[
G^{(l)}(x_1, \ldots, x_l) = \hat{\Phi}(x_1, x_2) + \ldots + \hat{\Phi}(x_l, x_1) = (\Phi(x_1) + \ldots + \Phi(x_l))
\]
on the neighborhood \( U^l \) of the diagonal point \((x_0)^l\) of \( M^l \). The jet of the function \( G^{(l)} \) at \((x_0)^l\) \( \in M^l \) is given in local coordinates by the formula

\[
G^{(l)}(z, \bar{z}) = \Phi(z_1, \bar{z}_1) + \Phi(z_2, \bar{z}_2) + \ldots + \Phi(z_l, \bar{z}_l) = (\Phi(z_1, \bar{z}_1) + \ldots + \Phi(z_l, \bar{z}_l)) \in \nu^{-1} \mathbb{C}[[\nu, z_1, \ldots, z_l, \bar{z}_l]],
\]

where we have used the notations \( z = (z_1, \ldots, z_l) \), \( z_i = (z_i^1, \ldots, z_i^m) \), \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_l) \), \( \bar{z}_j = (\bar{z}_j^1, \ldots, \bar{z}_j^m) \), and \( m = \dim_\mathbb{C} M \). This is the jet of the formal \( l \)-point Calabi function of \( \omega \) at \((x_0)^l\).

**Lemma 7.5.** The diagonal point \((x_0)^l \in M^l \) is a critical point of the function \( G^{(l)} \) with zero critical value.

**Proof.** Clearly, \( G^{(l)}((x_0)^l) = 0 \). In local coordinates,

\[
\frac{\partial G^{(l)}}{\partial z^i} = \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_{i+1}) - \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_i) \quad \text{and} \quad \frac{\partial G^{(l)}}{\partial \bar{z}^q} = \frac{\partial \Phi}{\partial z^q}(z_{j-1}, \bar{z}_j) - \frac{\partial \Phi}{\partial z^q}(z_j, \bar{z}_j),
\]
where we identify $\tilde{z}_{l+1}$ with $\tilde{z}_1$ and $z_0$ with $z_l$. Therefore,
\[
\frac{\partial G^{(l)}}{\partial \overline{z}_i^l}(x_0)^l = 0 \quad \text{and} \quad \frac{\partial G^{(l)}}{\partial \overline{z}_j^l}(x_0)^l = 0.
\]

\[\square\]

Remark. The point $(x_0)^l \in M^l$ is a degenerate critical point of the function $G^{(l)}$, but it is a nondegenerate critical point of the function $G^{(l)}$. The point $M$ is a natural distribution on $\mathcal{N}$. The following identity holds for any natural distributions $u_1, \ldots, u_m \in \mathcal{N}$,
\[
(20) \quad \langle u_1 \bullet \ldots \bullet u_l, 1 \rangle = \langle (u_1 \otimes \ldots \otimes u_l) \circ \exp G^{(l)}, 1 \rangle.
\]

Remark. Observe that the left-hand side of (20) is the trace of the product $u_1 \bullet \ldots \bullet u_l$ in the algebra $(\mathcal{N}, \bullet)$, which agrees with the fact that $G^{(l)}$ is cyclic. The distribution $u_1 \otimes \ldots \otimes u_l$ on the right-hand side is a natural distribution on $M^l$ supported at $(x_0)^l$.

Proof. We introduce a functional $W^{(l)}$ on the space of natural distributions on $M^l$ supported at the point $(x_0)^l$ by the formula
\[
W^{(l)}(u_1 \otimes \ldots \otimes u_l) := \langle u_1 \bullet \ldots \bullet u_l, 1 \rangle.
\]

Suppose that $u_i = \lambda(f_i \otimes g_i)$ for $1 \leq i \leq l$, where $f_i, g_i \in F$ are arbitrary. Then, by formula (10),
\[
W^{(l)}(u_1 \otimes \ldots \otimes u_l) = (g_1 \ast f_2)(x_0) \cdot (g_2 \ast f_3)(x_0) \cdots \cdot (g_l \ast f_1)(x_0).
\]

Observe that $\delta_{x_0} = \lambda(1 \otimes 1)$ and $\delta_{x_0} \bullet \delta_{x_0} = \delta_{x_0}$. Clearly,
\[
W^{(l)}(\delta_{x_0}) = 1 \quad \text{and} \quad \langle \delta_{x_0}, 1 \rangle = 1,
\]

where we have used that $\delta_{x_0} = \delta_{x_0} \otimes \ldots \otimes \delta_{x_0}$ and $G^{(l)}((x_0)^l) = 0$. According to Theorem 7.2 in order to prove formula (20) it remains to verify that for any $i, j, p, q$,
\[
(21) \quad W^{(l)} \circ \left( \nu \frac{\partial}{\partial z_j^l} - \nu \frac{\partial G^{(l)}}{\partial z_j^l} \right)^t = 0 \quad \text{and} \quad W^{(l)} \circ \left( \nu \frac{\partial}{\partial \overline{z}_j^l} - \nu \frac{\partial G^{(l)}}{\partial \overline{z}_j^l} \right)^t = 0.
\]

We will check the first equality on the elements $u_1 \otimes \ldots \otimes u_l$ with $u_i = \lambda(f_i \otimes g_i)$, which topologically generate $\mathcal{N}$. We use formula (19) to calculate the action of
\[
\left( \nu \frac{\partial}{\partial z_j^l} - \nu \frac{\partial G^{(l)}}{\partial z_j^l} \right)^t = \left( \nu \frac{\partial}{\partial z_j^l} - \nu \frac{\partial \Phi}{\partial \overline{z}_j^l}(z_i, \overline{z}_{i+1}) + \nu \frac{\partial \Phi}{\partial \overline{z}_j^l}(z_i, \overline{z}_i) \right)^t
\]
on $u_1 \otimes \ldots \otimes u_l$. The operator
\[
\left( \nu \frac{\partial}{\partial \overline{z}_j^l} + \nu \frac{\partial \Phi}{\partial \overline{z}_j^l}(z_i, \overline{z}_i) \right)^t
\]
acts only on the factor $u_i$ in $u_1 \otimes \ldots \otimes u_l$. We have
\[
\left\langle \left( \nu \frac{\p}{\p z^p} + \nu \frac{\p \Phi}{\p z^p} \right)^t u_i, h \right\rangle = \left\langle u_i, \nu \frac{\p \Phi}{\p z^p} * h \right\rangle =
\left( g_i * \nu \frac{\p \Phi}{\p z^p} * h * f_i \right)(x_0) = \left\langle \lambda \left( f_i \otimes \left( g_i * \nu \frac{\p \Phi}{\p z^p} \right) \right), h \right\rangle.
\]
Therefore,
\[
\hat{u}_i := \left( \nu \frac{\p}{\p z^p} + \nu \frac{\p \Phi}{\p z^p} \right)^t u_i = \lambda \left( f_i \otimes \left( g_i * \nu \frac{\p \Phi}{\p z^p} \right) \right).
\]
We get
\[
W^{(l)}(u_1 \otimes \ldots \otimes \hat{u}_i \otimes \ldots \otimes u_l) = \left\langle u_1 \cdot \ldots \cdot \hat{u}_i \cdot \ldots \cdot u_l, 1 \right\rangle =
(g_1 * f_2)(x_0) \cdot (g_2 * f_3)(x_0) \cdot \ldots \cdot (g_{i-1} * f_i)(x_0) \cdot
\left( g_i * \nu \frac{\p \Phi}{\p z^p} * f_{i+1} \right)(x_0) \cdot (g_{i+1} * f_{i+2})(x_0) \cdot \ldots \cdot (g_l * f_1)(x_0).
\]
It remains to calculate
\[
W^{(l)} \left( \left( \nu \frac{\p \Phi}{\p z^p}(z_i, \bar{z}_{i+1}) \right)^t (u_1 \otimes \ldots \otimes u_l) \right).
\]
The jet $\nu \frac{\p \Phi}{\p z^p}(z_i, \bar{z}_{i+1})$ can be expressed as the following series convergent in the topology associated with the standard filtration,
\[
\nu \frac{\p \Phi}{\p z^p}(z_i, \bar{z}_{i+1}) = \sum_{\alpha} a_\alpha(z_i) b_\alpha(\bar{z}_{i+1}).
\]
We have
\[
\left\langle \left( \nu \frac{\p \Phi}{\p z^p}(z_i, \bar{z}_{i+1}) \right)^t (u_1 \otimes \ldots \otimes u_l), h_1 \otimes \ldots \otimes h_l \right\rangle =
\left\langle u_1 \otimes \ldots \otimes u_l, \nu \frac{\p \Phi}{\p z^p}(z_i, \bar{z}_{i+1})(h_1 \otimes \ldots \otimes h_l) \right\rangle =
\left\langle u_1 \otimes \ldots \otimes u_l, \left( \sum_{\alpha} a_\alpha(z_i) b_\alpha(\bar{z}_{i+1}) \right)(h_1 \otimes \ldots \otimes h_l) \right\rangle =
\sum_{\alpha} \left( g_1 * h_1 * f_1)(x_0) \cdot \ldots \cdot (g_i * a_\alpha(z_i) * h_i * f_i)(x_0) \cdot
(g_{i+1} * h_{i+1} * b_\alpha(\bar{z}_{i+1}) * f_{i+1})(x_0) \cdot \ldots \cdot (g_l * h_l * f_i)(x_0) \right)
\left\langle \sum_{\alpha} u_1 \otimes \ldots \otimes \hat{u}_i \otimes \hat{u}_{i+1} \otimes \ldots \otimes u_l, h_1 \otimes \ldots \otimes h_l \right\rangle,
\]
where \( \dot{u}_{i\alpha} = \lambda(f_i \otimes (g_i \ast a_\alpha)) \) and \( \dot{u}_{i+1\alpha} = \lambda((b_\alpha \ast f_{i+1}) \otimes g_{i+1}) \). We have thus proved that
\[
\left( \nu \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_{i+1}) \right)^t (u_1 \otimes \ldots \otimes u_l) = \sum_{\alpha} u_1 \otimes \ldots \otimes \dot{u}_{i\alpha} \otimes \dot{u}_{i+1\alpha} \otimes \ldots \otimes u_l.
\]
Now,
\[
W^{(l)} \left( \left( \nu \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_{i+1}) \right)^t (u_1 \otimes \ldots \otimes u_m) \right) =
\]
\[
W^{(l)} \left( \sum_{\alpha} u_1 \otimes \ldots \dot{u}_{i\alpha} \otimes \dot{u}_{i+1\alpha} \otimes \ldots \otimes u_m \right) =
\]
\[
\sum_{\alpha} \left( (g_i \ast f_2)(x_0) \cdot \ldots \cdot (g_i \ast f_i)(x_0) \cdot (g_i \ast a_\alpha \ast b_\alpha \ast f_{i+1})(x_0) \cdot (g_{i+1} \ast f_{i+2})(x_0) \cdot \ldots \cdot (g_{l} \ast f_1)(x_0) \right).
\]
We see that
\[
\sum_{\alpha} (g_i \ast a_\alpha \ast b_\alpha \ast f_{i+1})(x_0) = (g_i \ast \nu \frac{\partial \Phi}{\partial z^p} \ast f_{i+1})(x_0),
\]
because \( a_\alpha \ast b_\alpha = a_\alpha b_\alpha \). Hence,
\[
W^{(l)} \left( \left( \nu \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_{i+1}) \right)^t (u_1 \otimes \ldots \otimes u_l) \right) =
\]
\[
W^{(l)} \left( \left( \nu \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_{i+1}) + \nu \frac{\partial \Phi}{\partial z^p}(z_i, \bar{z}_i) \right)^t (u_1 \otimes \ldots \otimes u_l) \right),
\]
which proves the first equality in (21). The second one can be checked similarly. \( \square \)

Formula (20) allows to express the jet of \( \exp G^{(l)} \) at \( (x_0)^l \) in terms of the algebra \( (\mathcal{N}, \ast) \) for every \( l \geq 1 \).

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