On Riemannian nonsymmetric spaces and flag manifolds

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March 30, 2022

2000 Mathematics Subject Classification. 53C20. 53C35

Keywords. Γ-symmetric spaces. Adapted riemannian metrics. Graded Lie algebras.

Abstract

In this work we study riemannian metrics on flag manifolds adapted to the symmetries of these homogeneous nonsymmetric spaces. We first introduce the notion of riemannian Γ-symmetric space when Γ is a general abelian finite group, the symmetric case corresponding to Γ = Z₂. We describe and study all the riemannian metrics on SO(2n + 1)/SO(r₁) × SO(r₂) × SO(r₃) × SO(2n + 1 − r₁ − r₂ − r₃) for which the symmetries are isometries. We consider also the lorentzian case and give an example of a lorentzian homogeneous space which is not a symmetric space.

1 Introduction

If M is a homogeneous symmetric space, then at each point x ∈ M we have a symmetry sₓ that is a diffeomorphism of M satisfying sₓ² = Id. It is equivalent to say that at every point x ∈ M we have a subgroup Γₓ of Diff(M) isomorphic to Z₂. The notion of Γ-symmetric space is a generalization of the classical notion of symmetric space by considering a general finite abelian group of symmetries Γ instead of Z₂. The case Γ = Zₖ was considered from the algebraic point of view by V. Kac and the differential geometric approach was carried

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out by A.J. Ledger, M. Obata [6] and O. Kowalski [3] in terms of $k$-symmetric spaces. A $k$-manifold is a homogeneous reductive space and the classification of these varieties is given by the corresponding classification of Lie algebras. The general notion of $\Gamma$-symmetric spaces was introduced by R. Lutz [5] and was algebraically reconsidered by Y. Bahturin and M. Goze [1]. In this last work the authors proved, in particular, that a $\Gamma$-symmetric space is a homogeneous space $G/H$ and the Lie algebra $\mathfrak{g}$ of $G$ is $\Gamma$-graded. They give also a classification of $\Gamma$-symmetric spaces when $G$ is a classical simple complex Lie algebra and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. We can see in particular that the flag manifold admits such a structure. The particular case of Grassmannian manifolds comes into the framework of symmetric manifolds. But for a general flag manifold, it is not the case. There is a great interest to study these manifolds, in an affine or riemannian point of view. For example, in loops groups theory we have to look complex algebraic homogeneous spaces $U_n$ and these spaces are Grassmannians or flag manifolds. We will describe symmetries which provide a flag manifold with a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure. We then study riemannian metrics adapted to this structure, that is riemannian metrics for which the riemannian connection is the canonical torsion free connection of a homogeneous space. We have to impose in addition that the symmetries are isometries (in the case of riemannian symmetric spaces this is a natural consequence of the very definition) We compute these metrics for flag manifolds and describe the associated riemannian invariants in some peculiar cases.

2 $\Gamma$-symmetric spaces

In this section we recall some basical notions (see [1] for more details).

2.1 Definition

Let $\Gamma$ be a finite abelian group. A $\Gamma$-symmetric space is a triple $(G, H, \Gamma_G)$ where $G$ is a connected Lie group, $H$ a closed subgroup of $G$ and $\Gamma_G$ an abelian finite subgroup of the group of automorphisms of $G$ isomorphic to $\Gamma$:

$$\Gamma_G = \{ \rho_\gamma \in Aut(G), \; \gamma \in \Gamma \}$$

such that $H$ lies between $G_\Gamma$ the closed subgroup of $G$ consisting of all elements left fixed by the automorphisms of $\Gamma_G$ and the identity component of $G_\Gamma$. The elements of $\Gamma_G$ satisfy :

$$\begin{align*}
\rho_{\gamma_1} \circ \rho_{\gamma_2} &= \rho_{\gamma_1 \gamma_2}, \\
\rho_e &= Id \; \text{where} \; e \; \text{is the unit element of} \; G, \\
\rho_\gamma(g) &= g, \forall \gamma \in \Gamma \iff g \in H.
\end{align*}$$

We also suppose that $H$ does not contain any proper normal subgroup of $G$. 

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2.2 Γ-symmetries on the homogeneous space $M = G/H$

Given a Γ-symmetric space $(G, H, \Gamma_G)$ we construct for each point $x$ of $M = G/H$ a subgroup $\Gamma_x$ of $\text{Diff}(M)$, the group of diffeomorphisms of $M$, isomorphic to $\Gamma$ which has $x$ as an isolated fixed point. We denote by $\tilde{g}$ the class of $g \in G$ in $M$ and $e$ the identity of $G$. We consider
\[ \Gamma_e = \{ s(\gamma, e) \in \text{Diff}(M), \ \gamma \in \Gamma \} \]
with $s(\gamma, e)(\tilde{g}) = \rho(\gamma)(g)$.

In another point $x = \tilde{g}_0$ of $M$ we have
\[ \Gamma_x = \{ s(\gamma, x) \in \text{Diff}(M), \ \gamma \in \Gamma \} \]
with $s(\gamma, \tilde{g}_0)(y) = g_0(s(\gamma, e))(g_0^{-1}y)$. All these subgroups $\Gamma_x$ of $\text{Diff}(M)$ are isomorphic to $\Gamma$.

Since for every $x \in M$ and $\gamma \in \Gamma$, the map $s(\gamma, x)$ is a diffeomorphism of $M$, such that $s(\gamma, x)(x) = x$ the tangent linear map $(T_s(\gamma, x))_x$ is in $GL(T_xM)$. Thus, for every $x \in M$, we obtain a linear representation
\[ S_x : \Gamma \rightarrow GL(T_xM) \]
defined by
\[ S_x(\gamma) = (T_s(\gamma, x))_x \]
and $S(\gamma)$ can be considered as a $(1,1)$-type tensor on $M$ satisfying
1. For every $\gamma \in \Gamma$, the map $x \in M \rightarrow S_x(\gamma)$ is of class $C^\infty$,
2. For every $x \in M$, $\{X_x \in T_x(M) \text{ such that } S_x(\gamma)(X_x) = X_x, \ \forall \gamma \}$ = $\{0\}$.

If we denote by
\[ \tilde{\Gamma}_x = \{ S_x(\gamma), \gamma \in \Gamma \} \]
then $\tilde{\Gamma}_x$ is a subgroup of $GL(n, T_x(M))$ isomorphic to $\Gamma$.

2.3 Γ-grading of the Lie algebra of $G$

Let $\mathfrak{g}$ be the Lie algebra of $G$. Each automorphism $\rho_\gamma$ of $G$ induces an automorphism $\tau_\gamma$ of $\mathfrak{g}$. Let $\hat{\Gamma}$ be the set of all these automorphisms $\tau_\gamma$. Then $\hat{\Gamma}$ is a finite abelian subgroup of $\text{Aut}(\mathfrak{g})$ isomorphic to $\Gamma$ and $\mathfrak{g}$ is graded by $\Gamma$ that is
\[ \mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma \]
with $[\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2}] \subset \mathfrak{g}_{\gamma_1 \gamma_2}$. The group $\hat{\Gamma}$ is canonically isomorphic to the dual group of $\Gamma$. Conversely every $\Gamma$-grading of $\mathfrak{g}$ defines a $\Gamma$-symmetric space $(G, H, \Gamma_G)$ where $G$ is a Lie group corresponding to $\mathfrak{g}$ and the Lie algebra of $H$ is the component $\mathfrak{g}_e$ corresponding to the identity of $\Gamma$. 
2.4 Canonical connections of a $\Gamma$-symmetric space

If $(G, H, \Gamma_G)$ is a $\Gamma$-symmetric space, the homogeneous space $M = G/H$ is reductive. In fact the Lie algebra $\mathfrak{g}$ being $\Gamma$-graded we have $\mathfrak{g} = \bigoplus \mathfrak{g}_\gamma = \mathfrak{g}_e \oplus \mathfrak{m}$ with $\mathfrak{m} = \bigoplus_{\gamma \in \Gamma, \gamma \neq e} \mathfrak{g}_\gamma$ and $[\mathfrak{g}_e, \mathfrak{m}] \subset \mathfrak{m}$. If we suppose $H$ connected, this last relation means that $\text{ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If $\text{ad}(H)(\mathfrak{m}) = \mathfrak{m}$, then any connection on $G/H$ invariant by left translations of $G$ is defined by the $\mathfrak{g}_e$-component $\omega$ of the canonical 1-form $\theta$ of $G$. In this case the curvature $\Omega$ is given by

$$\Omega(X, Y) = -\frac{1}{2}[X, Y]_{\mathfrak{g}_e}$$

for every $X, Y \in \mathfrak{m}$. Moreover the Lie algebra of the holonomy group in $\bar{e}$ is generated by all elements of the form $[X, Y]_{\mathfrak{g}_e}$, $X, Y \in \mathfrak{m}$. This connection is called [4] the canonical connection of the principal fibered bundle $G(G/H, H)$. Its torsion and curvature are given at the origin $\bar{e}$ of $G/H$ by

$$T(X, Y)_{\bar{e}} = -[X, Y]_\mathfrak{m}$$

$$R(X, Y)_{\bar{e}} = -[X, Y]_{\mathfrak{g}_e}$$

for all $X, Y \in \mathfrak{m}$.

If $\Gamma = \mathbb{Z}/2\mathbb{Z}$, that is if $(G, H, \Gamma_G)$ is a symmetric space, then the canonical connection $\nabla$ on $M = G/H$ is torsion free. In all the other cases, for example when $\Gamma$ is the Klein group, the torsion $T$ of $\nabla$ does not vanish. We consider then the connection $\nabla$ given by

$$\nabla = \nabla - T.$$ 

This connection is torsion free. Its curvature tensor writes

$$(R_{\nabla}(X, Y)(Z))_{\bar{e}} = \frac{1}{4}[X, [Y, Z]]_\mathfrak{m} - \frac{1}{4}[Y, [X, Z]]_\mathfrak{m} - \frac{1}{2}[[X, Y]_\mathfrak{m}, Z]_\mathfrak{m} - \frac{1}{2}[[X, Y]_{\mathfrak{g}_e}, Z]_\mathfrak{m}$$

for all $X, Y, Z \in \mathfrak{m}$ while the curvature of $\nabla$ is given by

$$(R_{\nabla}(X, Y)(Z))_{\bar{e}} = -[[X, Y]_{\mathfrak{g}_e}, Z]_\mathfrak{m}.$$ 

The geodesics of $\nabla$ and $\nabla$ are the same. The connection $\nabla$ is called the torsion-free canonical connection. We can note that the canonical connection satisfies also

$$\nabla T = 0$$

$$\nabla R_{\nabla} = 0.$$ 

Moreover the symmetries $S_{\bar{e}}(\gamma)$ are affine transformations with respect to $\nabla$. 

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3 Riemannian Γ-symmetric spaces

3.1 Riemannian symmetric space

Let $M = G/H$ a homogeneous symmetric space, where $G$ is a connected Lie group. We denote by 0 the coset $H$ of $M$, that is the class on $G/H$ of the identity 1 of $G$. The Lie algebra $\mathfrak{g}$ of $G$ is $\mathbb{Z}_2$-graded $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_a$ where $\mathbb{Z}_2 = \{ e, a \}$ and this decomposition is $ad(H)$-invariant. The Lie algebra of $H$ is $\mathfrak{g}_e$ and the tangent space at 0 $T_0 M$ is identified to $\mathfrak{g}_a$.

Every $G$-invariant metric $g$ on $G/H$ is given by an $ad(H)$-invariant non degenerate symmetric bilinear form $B$ on $\mathfrak{g}/\mathfrak{g}_e$ by $B(\overline{X}, \overline{Y}) = g(X, Y)$ for $X, Y \in \mathfrak{g}$ and $\overline{X}$ the class of $X$ in $\mathfrak{g}/\mathfrak{g}_e$. We identify $X \in \mathfrak{g}$ with the projection on $M$ of the associated left invariant vector field on $G$. Moreover $\mathfrak{g}$ is a riemannian metric if and only if $B$ is positive definite. The identification of $\mathfrak{g}/\mathfrak{g}_e$ with $\mathfrak{g}_a$ permits to consider $B$ as a non degenerate bilinear form on $\mathfrak{g}_a$. This form satisfies $B(X, [Y, Z]_{\mathfrak{g}_a}) = B(X, 0)$ for all $Y, Z \in \mathfrak{g}_a$ because $[\mathfrak{g}_e, \mathfrak{g}_a] \subseteq \mathfrak{g}_e$. Then $B(X, [Y, Z]_{\mathfrak{g}_a}) + B([Y, X], Z) = 0$ for all $X, Y, Z \in \mathfrak{m} = \mathfrak{g}_a$ and $M = (G/H, g)$ is naturally reductive. This means that the riemannian connection of $G$ coincides with the canonical torsion free connection $\nabla$ of $M$ and the symmetries $S_x \in \Gamma_x$ for all $x \in M$ are isometric. Conversely let $g$ be a metric on $G/H$ such that for each $x \in M S_x$ is an isometry. If $ad(H)$ is a compact subgroup of $GL(\mathfrak{g})$, then there exists an $ad(H)$-invariant inner product $\tilde{B}$ on $\mathfrak{g}$ such that

1) $\tilde{B}(\mathfrak{g}_e, \mathfrak{g}_a) = 0$
2) $\tilde{B}|_{\mathfrak{g}_a} = B$ induces the riemannian metric $g$ on $G/H$

Since $[\mathfrak{g}_a, \mathfrak{g}_a] \subseteq \mathfrak{g}_e$ the naturally reductivity is obvious and the riemannian connection coincides with $\nabla$. Recall that if $G$ is a semi-simple Lie group then $B$ is neither but the restriction to $\mathfrak{g}_a$ of the Killing-Cartan form $\tilde{B}$ on $G$ that is $B(X, Y) = tr(adX \circ adY)$ for all $X, Y \in \mathfrak{m}$.

3.2 Riemannian Γ-symmetric spaces

Let $\Gamma$ be a finite abelian group not isomorphic to $\mathbb{Z}_2$ and $g$ any $G$-invariant metric on a $\Gamma$-symmetric space $M = G/H$. Let us suppose that the symmetries $S_x$ are isometries for $g$. As $\Gamma$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, this property doesn’t imply in general the coincidence of the associated Levi-Civita connection and $\nabla$. 

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Definition 1 Let \((G, H, \Gamma_G)\) be a \(\Gamma\)-symmetric space and \(g\) a \(G\)-invariant metric on \(M\). We say that \((M, g)\) is a riemannian \(\Gamma\)-symmetric space if the symmetries \(S_x\) are isometries for all \(x \in M\).

Lemma 2 Let \((G, H, \Gamma_G)\) a \(\Gamma\)-symmetric space and \(g = \oplus_{\gamma \in \Gamma} g_{\gamma}\) the associated \(\Gamma\)-grading of the Lie algebra \(g\) of \(G\). Then for every \(\gamma \in \Gamma\)

\[\text{ad}(H)g_{\gamma} \subset g_{\gamma}\]

Proof. Let \(X\) be in \(g_{\gamma}\). For every \(\tau_\alpha \in \hat{\Gamma}\), we have \(\tau_\alpha(X) = \lambda(\gamma, \alpha)X\) with \(\lambda(\gamma, \alpha) = \pm 1\). Then

\[\tau_\alpha(\text{ad}(h)(X)) = \text{ad}(\rho_\alpha(h))(\tau_\alpha(X)) = \lambda(\gamma, \alpha)\text{ad}(h)(X)\]

because all the elements of \(H\) are invariant by the automorphisms \(\rho_\alpha\). This proves that \(\text{ad}(h)X \in g_{\gamma}\).

Proposition 3 If \(\text{ad}(H)\) is a compact subgroup of \(GL(g)\) and \(g\) a \(G\)-invariant metric on the \(\Gamma\)-symmetric space \(M = G/H\) then there exits an \(\text{ad}(H)\)-inner product \(\tilde{B}\) on \(g\) such that

1) \(\tilde{B}(g_{\gamma}, g_{\gamma'}) = 0\) for \(\gamma \neq \gamma'\) in \(\Gamma\)
2) \(\tilde{B}|_{g_{\alpha}} = B\) induces the riemannian metric \(g\) on \(G/H\)

Proof. Since each homogeneous component \(g_{\gamma}\) is invariant by \(\text{ad}(H)\), there exists an inner product \(B\) on \(g\) which is \(\text{ad}(g_{\gamma})\)-invariant and which defines \(g\). As the symmetries \(S(\gamma, x)\) are isometries, we deduce that the automorphisms \(\tau_\gamma\) are isometries for \(\tilde{B}\). If \(X \in g_{\gamma}, Y \in g_{\gamma'},\) there exits \(\alpha \in \Gamma\) such that

\[\tau_\alpha(X) = \lambda(\alpha, \gamma)X, \tau_\alpha(Y) = \lambda(\alpha, \gamma')Y\]

with \(\lambda(\alpha, \gamma)\lambda(\alpha, \gamma') = -1\). Thus

\[\tilde{B}(X, Y) = \tilde{B}(\tau_\alpha(X), \tau_\alpha(Y)) = -\tilde{B}(X, Y)\text{and} \tilde{B}(X, Y) = 0.\]

Example. Let us consider the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-symmetric space

\[(SO(5); SO(2) \times SO(2) \times SO(1), \Gamma_G)\]

where \(\Gamma_G\) is defined as follows. One writes a general element of \(so(5)\) by

\[
so(5) = \left\{ \begin{pmatrix}
0 & x_1 & a_1 & a_2 & b_1 \\
-x_1 & 0 & a_3 & a_4 & b_2 \\
-a_1 & -a_3 & 0 & x_2 & c_1 \\
-a_2 & -a_4 & -x_2 & 0 & c_2 \\
b_1 & -b_2 & -c_1 & -c_2 & 0
\end{pmatrix}, x_i, a_i, b_i, c_i \in \mathbb{R} \right\}.
\]
We put
\[ \mathfrak{g}_e = \{X \in \mathfrak{so}(5) / a_i = b_i = c_i = 0\}, \]
\[ \mathfrak{g}_a = \{X \in \mathfrak{so}(5) / x_i = b_i = c_i = 0\}, \]
\[ \mathfrak{g}_b = \{X \in \mathfrak{so}(5) / x_i = b_i = c_i = 0\}, \]
\[ \mathfrak{g}_c = \{X \in \mathfrak{so}(5) / x_i = a_i = b_i = 0\}. \]

If \( \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, a, b, c\} \), then \( \mathfrak{so}(5) = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c \) is a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-grading.

In this case
\[ \hat{\Gamma} = \{\tau_e, \tau_a, \tau_b, \tau_c\} \]
with \( \tau_e = id, \tau_a(X) = X \) for \( X \in \mathfrak{g}_e \oplus \mathfrak{g}_a, \tau_a(X) = -X \) for \( X \in \mathfrak{g}_b \oplus \mathfrak{g}_c, \tau_b(X) = X \) for \( X \in \mathfrak{g}_e \oplus \mathfrak{g}_b, \tau_c(X) = X \) for \( X \in \mathfrak{g}_e \oplus \mathfrak{g}_c, \tau_e(X) = -X \) for \( X \in \mathfrak{g}_a \oplus \mathfrak{g}_b \). Since \( G = SO(5) \) is connected, this grading gives a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric structure on \( M = SO(5)/SO(2) \times SO(2) \times SO(1) \) and \( \mathfrak{g}_e \) is the Lie algebra of \( H = SO(2) \times SO(2) \times SO(1) \). We denote by \( \{\{X_1, X_2\}, \{A_1, A_2, A_3, A_4\}, \{B_1, B_2\}, \{C_1, C_2\}\} \) the basis of \( \mathfrak{so}(5) \) where each big letter corresponds to the matrix of \( \mathfrak{so}(5) \) with the small letter equal to 1 and other coefficients are zero. This basis is adapted to the grading. Let us denote by \( \{\omega_1, \omega_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \gamma_1, \gamma_2\} \) the dual basis.

Every \( \text{ad}(H) \)-invariant symmetric bilinear form \( B \) on \( \mathfrak{m} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c \) such that \( B(\mathfrak{g}_a, \mathfrak{g}_a) = 0 \) for \( \gamma \neq \gamma' \) in \( \Gamma \) is written
\[ B = t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1\alpha_4 - \alpha_2\alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2). \]

In fact, since \( H \) is connected the bilinear product \( B \) is \( \text{ad}(H) \)-invariant if and only if
\[ B([X,Y],Z) + B(Y,[X,Z]) = 0 \]
for \( Y, Z \in \mathfrak{m} \) and \( X \in \mathfrak{h} = \mathfrak{g}_e \).

The brackets of \( \mathfrak{so}(5) \) with respect to the basis \( \{X_1, A_1, B_1, C_1\} \) are summarized in the following table

|   | \( X_1 \) | \( X_2 \) | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( B_1 \) | \( B_2 \) | \( C_1 \) | \( C_2 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( X_1 \) | 0 | 0 | \(-A_3\) | \(-A_4\) | \( A_1\) | \(-B_2\) | \( B_1\) | 0 | 0 |
| \( X_2 \) | 0 | 0 | \(-A_2\) | \( A_1\) | \(-A_4\) | \( A_3\) | 0 | 0 | \(-C_2\) | \( C_1\) |
| \( A_1 \) | 0 | \(-X_2\) | 0 | \(-C_2\) | 0 | 0 | \( B_1\) | 0 |
| \( A_2 \) | 0 | 0 | \(-X_1\) | \(-C_2\) | 0 | 0 | \( B_1\) | 0 |
| \( A_3 \) | 0 | \(-X_2\) | 0 | \(-C_1\) | \( B_2\) | 0 |
| \( A_4 \) | 0 | 0 | 0 | \(-C_2\) | 0 | \( B_2\) |
| \( B_1 \) | 0 | \(-X_1\) | \(-A_1\) | \(-A_2\) | 0 | \(-X_2\) |
| \( B_2 \) | 0 | \(-X_1\) | \(-A_1\) | \(-A_2\) | 0 | \(-X_2\) |
| \( C_1 \) | 0 | 0 | 0 | \(-C_2\) | 0 | \(-X_2\) |
| \( C_2 \) | 0 | 0 | 0 | \(-C_2\) | 0 | \(-X_2\) |

The identity \( B([X_1, A_j], A_j) = 0 \) implies
\[ B(A_1, A_3) = B(A_1, A_2) = B(A_2, A_4) = B(A_3, A_4) = 0, \]
The identity $B([X_2, A_i], A_j) + B(A_i, [X_2, A_j]) = 0$ gives for $i \neq j$

$$B(A_2, A_3) + B(A_1, A_4) = 0, \quad -B(A_3, A_3) + B(A_1, A_1) = 0 \quad -B(A_4, A_4) + B(A_2, A_2) = 0 \quad -B(A_2, A_2) + B(A_1, A_1) = 0$$

In the same way we find

$$B(B_1, B_1) = B(B_2, B_2), \quad B(C_1, C_1) = B(C_2, C_2)$$

this gives

$$B = t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1 \alpha_4 - \alpha_2 \alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2).$$

The metric $g$ on $S0(5)/SO(2) \times SO(2) \times SO(1)$

associated to $B$ is naturally reductive if and only if $t = v = w$ and $u = 0$. In fact, if $\mathfrak{g}$ is naturally reductive then $B$ satisfies

$$B(X, [Z, Y]_m) + B([Z, X]_m, Y) = 0$$

for every $X, Y, Z \in m$. In particular $B(A_1, [B_2, C_2]_m) + B([C_2, A_1]_m, B_2) - B(A_1, A_4) + B(0, B_2) = 0$ and $u = 0$. Similarly $B(A_1, [B_1, C_1]) + B([B_1, A_1], C_1) = 0$ gives $-B(A_1, A_1) + B(C_1, C_1) = 0$ that is $t = w$, and $B(B_1, [A_1, C_1]) + B([A_1, B_1], C_1) = 0$ gives $B(B_1, B_1) - B(C_1, C_1) = 0$ that is $v = w$.

**Proposition 4** The riemannian connection $\nabla_g$ of the metric $g$ on $S0(5)/SO(2) \times SO(2) \times SO(1)$ coincides with the canonical torsion free connection $\overline{\nabla}$ if and only if $B = \sum_{i=1}^4 \alpha_i^2 + \sum_{i=1}^2 \beta_i^2 + \sum_{i=1}^2 \gamma_i^2$.

**Remark** If $g$ is a $G$-invariant metric on $G/H$ such that its connection $\nabla_g$ is equal to $\overline{\nabla}$ the bilinear form $B$ is naturally reductive. In the previous example, since $G$ is a simple Lie group, this inner product $B$ is the restriction to $\mathfrak{m}$ of the Kiling-Cartan form $K$ of $G$.

$$B(X, Y) = K(X, Y) = tr(adX \circ adY).$$

Then the homogeneous component $\mathfrak{g}_\gamma$ are pairwise orthogonal and the $\tau_\gamma$ are isometries. But it is not the case in general.

Let us return to the general case.

**Definition 5** Let $(G, H, \Gamma_G; g)$ a riemannian $\Gamma$-symmetric space. We say that $g$ is adapted to the $\Gamma$-structure if the Levi-Civita connection coincides with the canonical one.
Proposition 6 Every riemannian $\Gamma$-symmetric space with adapted riemannian connection is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{m}$ with $\mathfrak{m} = \oplus \Gamma \neq \mathfrak{g}_c$.

Proof. Any $G$-invariant riemannian metric $g$ on a reductive homogeneous space $G/H$ with an $ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{m}$ corresponds to an $ad(H)$-invariant non degenerate symmetric bilinear form $B_m$ on $\mathfrak{m}$. Since $M = G/H$ is a riemannian $\Gamma$-symmetric space, its $G$-invariant riemannian metric $g$ is parallel with respect to the canonical torsionless connection $\nabla$. Then from [4] Theorem 3.3 the riemannian connection of $g$ and $\nabla$ coincides on $G/H$ if and only if $B_m$ satisfies

$$B_m(X, [Y, Z]_m) + B_m([Y, Z]_m, X) = 0$$

for all $X, Y, Z \in \mathfrak{m}$. This means that $(G/H, g)$ is naturally reductive.

3.3 Irreducible riemannian $\Gamma$-symmetric spaces

Let $(G, H, \Gamma_G)$ a $\Gamma$-symmetric space. Since $G/H$ is a reductive homogeneous space with an $adH$ invariant decomposition $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{m}$ then the Lie algebra of the holonomy group of $\nabla$ is spanned by the endomorphisms of $\mathfrak{m}$ given by $R(X, Y)_0$ for all $X, Y \in \mathfrak{m}$. Recall that $R(X, Y)_0 = -[[X, Y]_b, Z]$ for all $X, Y, Z \in \mathfrak{m}$. In particular we have $R(X, Y)_0 = 0$ as soon as $X \in \mathfrak{g}_{\gamma}, Y \in \mathfrak{z}_{\gamma}$ with $\gamma, \gamma' \neq e$. For example if $\Gamma = Z_2 \times Z_2$ then $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_a \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_c$ and $R(\mathfrak{g}_a, \mathfrak{g}_b)_0 = R(\mathfrak{g}_a, \mathfrak{g}_c)_0 = R(\mathfrak{g}_b, \mathfrak{g}_c)_0 = 0$.

Lemma 7 Let $\mathfrak{g}$ be a simple Lie algebra $Z_2 \times Z_2$-graded. Then

$$[\mathfrak{g}_a, \mathfrak{g}_a] \oplus [\mathfrak{g}_b, \mathfrak{g}_b] \oplus [\mathfrak{g}_c, \mathfrak{g}_c] = \mathfrak{g}_c.$$

Proof. Let $U$ denote $[\mathfrak{g}_a, \mathfrak{g}_a] \oplus [\mathfrak{g}_b, \mathfrak{g}_b] \oplus [\mathfrak{g}_c, \mathfrak{g}_c]$. Then $I = U \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is an ideal of $\mathfrak{g}$. In fact $X \in I$ is decomposed as $X_U + X_a + X_b + X_c$. The main point is to prove that $[X_U, Y]$ is in $I$ for any $Y \in \mathfrak{g}_c$. But $X_U$ is decomposed as $[X_a, Y_a] + [X_b, Y_b] + [X_c, Y_c]$. The Jacobi identity shows that $[[X_a, Y_a], Y] \in [\mathfrak{g}_a, \mathfrak{g}_a]$.

Remark that in any case , as soon as $\Gamma$ is not $Z_2$ the representation $ad \mathfrak{g}_c$ is not irreducible on $\mathfrak{m}$. In fact each component $\mathfrak{g}_\gamma$ is an invariant subspace of $\mathfrak{m}$.

Definition 8 The representation $ad \mathfrak{g}_c$ on $\mathfrak{m}$ is called $\Gamma$-irreducible if $\mathfrak{m}$ can not be written $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $\mathfrak{g}_c \oplus \mathfrak{m}_1$ and $\mathfrak{g}_c \oplus \mathfrak{m}_2$ are $\Gamma$-graded Lie algebras.

Example. Let $\mathfrak{g}_1$ be a simple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1$. Let $\sigma_1, \sigma_2, \sigma_3$ the automorphisms of $\mathfrak{g}$ given by

$$\left\{ \begin{array}{l}
\sigma_1(X_1, X_2, X_3, X_4) = (X_2, X_1, X_3, X_4), \\
\sigma_2(X_1, X_2, X_3, X_4) = (X_1, X_2, X_4, X_3), \\
\sigma_3 = \sigma_1 \circ \sigma_2.
\end{array} \right.$$
They define a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-gradation on \(\mathfrak{g}\) and we have \(\mathfrak{g}_e = \{(X, X, Y, Y)\}, \mathfrak{g}_a = \{(0, 0, Y, -Y)\}, \mathfrak{g}_b = \{(X, -X, 0, 0)\}\) and \(\mathfrak{g}_c = \{(0, 0, 0, 0)\}\) with \(X, Y \in \mathfrak{g}_1\).

In particular \(\mathfrak{g}_a\) is isomorphic to \(\mathfrak{g}_1\) so we have \([\mathfrak{g}_e, \mathfrak{g}_a] = \mathfrak{g}_a\) and since \(\mathfrak{g}_1\) is simple we can not have \([\mathfrak{g}_e, \mathfrak{g}_a] = \mathfrak{g}_a\) for \(i = 1, 2\). Then \(\mathfrak{g}\) is \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-graded and this decomposition is \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-irreducible.

Suppose now that \(\mathfrak{g}\) is a simple Lie algebra. Let \(K\) be the Killing-Cartan form of \(\mathfrak{g}\). It is invariant by all automorphisms of \(\mathfrak{g}\). In particular

\[K(\tau_\gamma X, \tau_\gamma Y) = K(X, Y)\]

for any \(\tau_\gamma \in \tilde{\Gamma}\). If \(X \in \mathfrak{g}_a\) and \(Y \in \mathfrak{g}_{\beta}, \alpha \neq \beta\) there exists \(\gamma \in \Gamma\) such that \(\tau_\gamma X = \lambda(\alpha, \gamma) X\) and \(\tau_\gamma Y = \lambda(\beta, \gamma) Y\) with \(\lambda(\alpha, \gamma) \lambda(\beta, \gamma) \neq 1\). Thus \(K(X, Y) = 0\) and the homogeneous components \(\mathfrak{g}_\gamma\) are pairwise orthogonal with respect to \(K\).

Moreover \(K_\gamma = K|_{\mathfrak{g}_\gamma}\) is a nondegenerate bilinear form. Since \(\mathfrak{g}\) is a simple Lie algebra, there exists an \(ad\mathfrak{g}_e\)-invariant inner product \(\tilde{B}\) on \(\mathfrak{g}\) such that the restriction \(B = \tilde{B}|_m\) to \(m\) defines a riemannian \(\Gamma\)-symmetric structure on \(G/H\).

This means that \(B(\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}) = 0\) for \(\gamma \neq \gamma' \in \Gamma\). We consider an orthogonal basis of \(B\). For each \(X \in \mathfrak{g}_e\), \(ad\mathfrak{g}_e\) is expressed by a skew-symmetric matrix \((a_{ij}(X))\) and \(K(X, X) = \sum a_{ij}(X) a_{ji}(X) < 0\). So \(K\) is negative-definite on \(\mathfrak{g}_e\).

Let \(K_\gamma\) and \(B_\gamma\) be the restrictions of \(K\) and \(B\) at the homogeneous component \(\mathfrak{g}_\gamma\). Let \(\beta \in m^*\) be such that

\[K_\gamma(X, Y) = B_\gamma(\beta_\gamma(X), Y)\]

for all \(X, Y \in \mathfrak{g}_\gamma\) and \(\beta_\gamma = \beta|_{\mathfrak{g}_e}\). Since \(B_\gamma\) is nondegenerate on \(\mathfrak{g}_\gamma\), the eigenvalues of \(\beta_\gamma\) are real and non zero. The eigenspaces \(\mathfrak{g}_{\gamma^1}, \ldots, \mathfrak{g}_{\gamma^p}\) of \(\beta_\gamma\) are pairwise orthogonal with respect to \(B_\gamma\) and \(K_\gamma\). But for every \(Z \in \mathfrak{g}_e\) we have

\[K_\gamma([Z, X], Y) = K_\gamma(X, [Z, Y]) = B_\gamma(\beta_\gamma(X), [Z, Y])\]

so \(B_\gamma(\beta_\gamma[Z, X], Y) = B_\gamma([Z, \beta_\gamma(X)], Y)\) for every \(Y \in \mathfrak{g}_\gamma\) and \(\beta_\gamma[Z, X] = [Z, \beta_\gamma(X)]\) that is \(\beta_\gamma \circ ad Z = ad Z \circ \beta_\gamma\) for any \(Z \in \mathfrak{g}_e\). This implies that \([\mathfrak{g}_e, \mathfrak{g}_{\gamma^i}] \subset \mathfrak{g}_{\gamma^i}\).

Now we shall examine the particular case corresponding to \(\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2\). The eigenvalues of the involutive automorphisms \(\tau_\gamma\) being real, the Lie algebra \(\mathfrak{g}\) admits a real \(\Gamma\)-decomposition \(\mathfrak{g} = \sum_{\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathfrak{g}_\gamma\). Then we can consider that \(\mathfrak{g}\) is a real Lie algebra.

Now if \(i \neq j\) then

\[K_\gamma([\mathfrak{g}_{\gamma^i}, \mathfrak{g}_{\gamma^j}], [\mathfrak{g}_{\gamma^i}, \mathfrak{g}_{\gamma^j}]) \subset K([\mathfrak{g}_{\gamma^i}, \mathfrak{g}_{\gamma^j}], \mathfrak{g}_e) \subset (\mathfrak{g}_{\gamma^i}, \mathfrak{g}_{\gamma^j}) = 0\]

and we have

\([\mathfrak{g}_{\gamma^i}, \mathfrak{g}_{\gamma^j}] = \{0\}\)

for \(i \neq j\).
Example. In the section 4, we study the riemannian homogeneous manifold \( SO(2l + 1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4) \). This manifold is \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric and the Lie algebra \( so(2l + 1) \) admits a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-grading. By referring to the study which follows we see that

\[
\mathfrak{g}_a = A_1 + A_2, \quad \mathfrak{g}_b = B_1 + B_2, \quad \mathfrak{g}_c = C_1 + C_2
\]

with \([A_1, A_2] = [B_1, B_2] = [C_1, C_2] = 0\) and we have

\[
K(A_1, A_2) = K(B_1, B_2) = K(C_1, C_2) = 0.
\]

So we have an orthogonal decomposition of each invariant space \( \mathfrak{g}_a, \mathfrak{g}_b, \mathfrak{g}_c \) but the graduation is \( \Gamma \)-irreductible. In fact we have \([A_1, B_1] = [A_2, B_2] = [C_1, C_2]\).

Let \{e, a, b, c\} be the elements of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with \( a^2 = b^2 = c^2 = e \) and \( ab = c \). Each component \( \mathfrak{g}_\gamma, \gamma \neq e \), satisfies \([\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \subset \mathfrak{g}_e\) and \( \mathfrak{g}_e \oplus \mathfrak{g}_\gamma \) is a symmetric Lie algebra. Endowed with the inner product \( \tilde{B} \), the Lie algebra \( \mathfrak{g}_e \oplus \mathfrak{g}_\gamma \) is an orthogonal symmetric Lie algebra. The Killing-Cartan form is not degarate on \( \mathfrak{g}_e \oplus \mathfrak{g}_\gamma \). Then \( \mathfrak{g}_e \oplus \mathfrak{g}_\gamma \) is semi-simple. It is a direct sum of orthogonal symmetric Lie algebras of the following two kinds:

i) \( \mathfrak{g} = \mathfrak{g}' + \mathfrak{g}' \) with \( \mathfrak{g}' \) simple

ii) \( \mathfrak{g} \) is simple.

The first case has been study above and the representation is \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-irreducible. In the second case \( ad[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \) is irreducible in \( \mathfrak{g}_\gamma \) and the representation is \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-irreducible on \( \mathfrak{m} \).

4 Flag manifolds

In this section we study riemannian properties of the oriented flag manifold

\[
M = SO(2l + 1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)
\]

associated to its \( \Gamma \)-symmetric structures.

For \( \mathfrak{g} \) classical complex simple Lie algebra of type \( B_l \), it is always possible to endow \( \mathfrak{g} \) with a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-grading such that

\[
\mathfrak{g}_e = so(r_1) \oplus ... \oplus so(r_4)
\]

with \( r_1 + r_2 + r_3 + r_4 = 2l + 1 \) [1]. The compact homogeneous space

\[
M = SO(2l + 1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)
\]

is a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric space. We suppose \( r_1 \leq r_2 \leq r_3 \leq r_4 \). In case \( r_1 r_2 \neq 0 \) and \( r_3 = r_4 = 0 \) then \( M \) is a symmetric space. The symmetric structure on the Grassmannian

\[
SO(2l + 1)/SO(r_1) \times SO(r_2)
\]

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is well known (see [4]). If $r_1 r_2 r_3 \neq 0$, then the homogeneous space $M$ can not be symmetric. In what follows we shall explicitly construct on $M$ a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-riemannian structure. Let us consider the decomposition of a matrix of $so(2l+1)$

\[
\begin{pmatrix}
X_1 & A_1 & B_1 & C_1 \\
-tA_1 & X_2 & C_2 & B_2 \\
-tB_1 & -tC_2 & X_3 & A_2 \\
-tC_1 & -tB_2 & -tA_2 & X_4
\end{pmatrix}
\]

with $A_1 \in \mathcal{M}(r_1, r_2), B_1 \in \mathcal{M}(r_1, r_3), C_1 \in \mathcal{M}(r_1, r_4), C_2 \in \mathcal{M}(r_2, r_3), B_2 \in \mathcal{M}(r_2, r_4), A_2 \in \mathcal{M}(r_3, r_4)$ and $X_i \in so(r_i), i = 1, ..., 4$. Let us consider the subspaces of $\mathfrak{g}$:

\[
\mathfrak{g}_c = \begin{pmatrix}
X_1 & 0 & 0 & 0 \\
0 & X_2 & 0 & 0 \\
0 & 0 & X_3 & 0 \\
0 & 0 & 0 & X_4
\end{pmatrix}, \quad \mathfrak{g}_a = \begin{pmatrix}
0 & A_1 & 0 & 0 \\
tA_1 & 0 & 0 & 0 \\
0 & 0 & 0 & A_2 \\
0 & 0 & -tA_2 & 0
\end{pmatrix}
\]

\[
\mathfrak{g}_b = \begin{pmatrix}
0 & 0 & B_1 & 0 \\
0 & 0 & 0 & B_2 \\
tB_1 & 0 & 0 & 0 \\
0 & -tB_2 & 0 & 0
\end{pmatrix}, \quad \mathfrak{g}_c = \begin{pmatrix}
0 & 0 & 0 & C_1 \\
0 & 0 & C_2 & 0 \\
0 & -tC_2 & 0 & 0 \\
-tC_1 & 0 & 0 & 0
\end{pmatrix}.
\]

Then $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading of $so(2l+1)$. This graduation defines the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space

\[(SO(2l+1); SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4), (\mathbb{Z}_2 \times \mathbb{Z}_2)_{so}).\]

Let $B$ be a $\mathfrak{g}_e$-invariant inner product on $\mathfrak{g}$. By hypothesis $B(\mathfrak{g}_a, \mathfrak{g}_b) = 0$ as soon as $\alpha \neq \beta$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$. This shows that $B$ is written $B = B_{\mathfrak{g}_c} + B_{\mathfrak{g}_a} + B_{\mathfrak{g}_b} + B_{\mathfrak{g}_c}$, where $B_{\mathfrak{g}_c}$ is an inner product on $\mathfrak{g}_c$. The restriction $B_{\mathfrak{g}_c}$ to $\mathfrak{g}_e$ is a biinvariant inner product. If $r_4 > 2$, all the components $so(r_i)$ are simple Lie algebras and $B_{\mathfrak{g}_c}$ is written

\[B_{\mathfrak{g}_c} = a_1 K_1 + a_2 K_2 + a_3 K_3 + a_4 K_4\]

where $K_i$ is the Killing-Cartan form of $so(r_i)$. If some components $so(r_i)$ are abelian from the index $i_0$, that is $r_i \leq 2$ for $i \geq i_0$ then $B_{\mathfrak{g}_c}$ is of the form $\Sigma_{j \leq a_0} a_j K_j + q$ where $q$ is a definite positive form on the abelian Lie algebra $\oplus_{j \geq i_0} so(r_j)$. Let us compute $B_{\mathfrak{g}_a}$. We denote by $A_1$ the subspace of $\mathfrak{g}_a$ whose vectors are

\[
\begin{pmatrix}
0 & A_1 & 0 & 0 \\
tA_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

In the same manner we define $A_2, B_1, B_2, C_1$ and $C_2$. For $1 \leq i \leq r_1$ and $r_1 + 1 \leq j \leq r_2$, let $A_{ij}$ be the corresponding elementary matrices of $A_1$ that is the $A_{ij} = (a_{r_1})$ with $a_{ij} = -a_{ji} = 1$ other coordinates being equal to 0. Similary
$X_{ij}$ denotes the elementary matrices of the diagonal block corresponding to $so(r_1), Y_{ij}$ to $so(r_2), Z_{ij}$ to $so(r_3)$ and $T_{ij}$ to $so(r_4)$. We have

\[
\begin{align*}
[X_{ij}, A_{jl}] &= A_{il}, & 1 \leq i < j \leq r_1, & r_1 + 1 \leq l \leq r_1 + r_2, \\
[X_{ij}, A_{il}] &= -A_{jl}, & 1 \leq i < j \leq r_1, & r_1 + 1 \leq l \leq r_1 + r_2,
\end{align*}
\]

and

\[
\begin{align*}
[Y_{ij}, A_{il}] &= A_{jl}, & r_1 + 1 \leq i < j \leq r_1 + r_2, & 1 \leq l \leq r_1, \\
[Y_{ij}, A_{jl}] &= -A_{il}, & r_1 + 1 \leq i < j \leq r_1 + r_2, & 1 \leq l \leq r_1.
\end{align*}
\]

The relation

\[B_{g_a}([X_{rs}, A_{ij}], A_{ij}) = 0\]

for all $X_{rs} \in so(r_1) \oplus so(r_2)$ implies

\[
\begin{align*}
B_{g_a}(A_{ij}, A_{il}) &= 0, & i, l \in 1, ..., r_1, i \neq l, & j = r_1 + 1, ..., r_1 + r_2, \\
B_{g_a}(A_{ij}, A_{il}) &= 0, & i = 1, ..., r_1, & j, l \in r_1 + 1, ..., r_1 + r_2, j \neq l.
\end{align*}
\]

From the identities

\[
\begin{align*}
B([X_{il}, A_{ij}], A_{ij}) + B(A_{ij}, [X_{il}, A_{ij}]) &= 0, \\
B([Y_{ij}, A_{ij}], A_{il}) + B(A_{ij}, [Y_{ij}, A_{il}]) &= 0,
\end{align*}
\]

we obtain

\[
\begin{align*}
B_{g_a}(A_{ij}, A_{il}) &= B_{g_a}(A_{ij}, A_{ij}), & i, l \in 1, ..., r_1, & j = r_1 + 1, ..., r_1 + r_2, \\
B_{g_a}(A_{li}, A_{il}) &= B_{g_a}(A_{ij}, A_{ij}), & l = 1, ..., r_1, & j, i = r_1 + 1, ..., r_1 + r_2.
\end{align*}
\]

We deduce that all the basis vectors of $A_1$ have the same norm with respect the inner product $B$. From the identity

\[B([X_{ij}, A_{jl}], A_{js}) + B(A_{jl}, [X_{ij}, A_{js}]) = 0\]

$1 \leq i < j \leq r_1, l, s \in [[r_1 + 1, ..., r_1 + r_2]],$ we obtain

\[B(A_{jl}, A_{js}) + B(A_{is}, A_{jl}) = 0.\]

Suppose that $r_1 \geq 3$. There exists $r, 1 \leq r \leq r_1$ which is not equal to $i$ or $j$. In this case we have

\[X_{ij}, A_{rs} = 0\]

and

\[B([X_{ij}, A_{jl}], A_{rs}) + B(A_{jl}, [X_{ij}, A_{rs}]) = 0\]

gives

\[B(A_{jl}, A_{rs}) = 0\]

for $r \neq i$. This implies that the vectors $A_{ij}$ are pairwise orthogonal as soon as $r_1 > 2$. It remains now to compute $B(A_1, A_2)$. The action of $so(r_1)$ is faithful on $A_1$ and trivial on $A_2$. Thus the $(ad_{so(r_1)})$-invariance of $B_{g_a}$ implies that

\[B_{g_a}(A_1, A_2) = 0.\]
All the previous identities implies, if \( r_4 > 2 \), that

\[
B_{g_0} = t_{A_1} \Sigma (\alpha_{ij}^1)^2 + t_{A_2} \Sigma (\alpha_{ij}^2)^2,
\]

where \( \{\alpha_{ij}^1, \alpha_{ij}^2\} \) is the dual basis of the basis of \( g_0 \) given respectively by the elementary matrices of \( A_1 \) and \( A_2 \) and \( t_{A_1} > 0, t_{A_2} > 0 \). All these computations can be extended to the other components \( g_0 \) and \( g_c \).

**Proposition 9** If \( r_4 > 2 \), then all \( g_c \)-invariant inner product on \( m = g_0 \oplus g_0 \oplus g_c \) is given by

\[
B = t_{A_1} \Sigma (\alpha_{ij}^1)^2 + t_{A_2} \Sigma (\alpha_{ij}^2)^2 + t_{B_1} \Sigma (\beta_{ij}^1)^2 + t_{B_2} \Sigma (\beta_{ij}^2)^2 + t_{C_1} \Sigma (\gamma_{ij}^1)^2 + t_{C_2} \Sigma (\gamma_{ij}^2)^2
\]

where \( \{\alpha_{ij}^1, \alpha_{ij}^2, \beta_{ij}^1, \beta_{ij}^2, \gamma_{ij}^1, \gamma_{ij}^2\} \) is the dual basis of the basis of \( A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus C_1 \oplus C_2 \) given by the elementary matrices and the parameters \( t_{A_1}, t_{A_2}, t_{B_1}, t_{B_2}, t_{C_1}, t_{C_2} \) being nonnegative.

It remains to examine the particular cases corresponding to some \( r_i \) equal to 2 or 1. This imply that \( \text{so}(r_i) \) is abelian (and not simple).

1. If \( r_1 = 2 \) and \( r_2 = 1 \) then \( r_3 = r_4 = 1 \) and the \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-grading of \( \text{so}(5) \) is given by

\[
\text{so}(5) = (\text{so}(2) \oplus \text{so}(1) \oplus \text{so}(1) \oplus \text{so}(1)) \oplus g_0 \oplus g_0 \oplus g_c
\]

with \( \dim g_0 = 3, \dim g_0 = 3, \dim g_c = 3 \) and the homogeneous space is isomorphic to \( \text{SO}(5)/\text{SO}(2) \).

Every \( \text{so}(2) \)-invariant metric on \( m \) is of type

\[
B = t_{A_1} ((\alpha_{13}^1)^2 + (\alpha_{23}^1)^2) + t_{A_2} ((\alpha_{13}^2)^2) + t_{B_1} ((\beta_{14}^1)^2) + t_{B_2} ((\beta_{14}^2)^2) + t_{C_1} ((\gamma_{15}^1)^2) + t_{C_2} ((\gamma_{15}^2)^2).
\]

2. If \( r_1 = r_2 = r_3 = 2 \) and \( r_4 = 1 \) then \( g = \text{so}(7) \). The corresponding \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric space is isomorphic to \( \text{SO}(7)/\text{SO}(2) \times \text{SO}(2) \times \text{SO}(2) \).

In this case the relation \( B(A_{il}, A_{ps}) = 0 \) is not valid. We deduce that every \((\text{so}(2) \oplus \text{so}(2) \oplus \text{so}(2))\)-invariant inner product on \( m \) is written

\[
B = t_{A_1} (\alpha_{13}^1)^2 + (\alpha_{23}^1)^2 + (\alpha_{14}^1)^2 + (\alpha_{24}^1)^2) + u_{A_1} (\alpha_{14}^1 \alpha_{24}^1 - \alpha_{14}^2 \alpha_{24}^2) + t_{A_2} (\alpha_{23}^1)^2 + u_{A_2} (\alpha_{23}^1 \alpha_{24}^1 - \alpha_{23}^2 \alpha_{24}^2) + t_{B_1} (\beta_{14}^1)^2 + t_{B_2} (\beta_{14}^2)^2
\]

The remaining cases correspond to \( r_1 = 2, r_2 = r_3 = r_4 = 1 \) which is treated in the example, to \( r_1 = 2, r_2 = 1, r_3 = r_4 = 0 \) and the homogeneous space is \( \text{SO}(3)/\text{SO}(2) \) and it is a symmetric space and to \( r_1 = r_2 = r_3 = 1, r_4 = 0 \) and
\(g_e = \{0\}\). So Proposition 7 and the previous results give all the metric on flag manifolds \(M\) which provide \(M\) with a riemannian \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric structure. In general, for these metrics the Levi-Civita connection is not adapted to symmetries. This connection corresponds to the canonical torsionfree connection \(\nabla\) of the \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric homogeneous space if and only if the metric is naturally reductive with respect to the \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-graduation. Recall that this means that
\[
B([X, Y]_m, Z) + B([X, Z]_m, Y) = 0.
\]
for all \(X, Y, Z \in m\). Applying this identity to a triple of vectors in \(A_1 \times B_1 \times C_2\) more precisely to a triple \((A_{r_1+1,1}, B_{r_2+1,1}, C_{r_1+1,r_2+1})\) we obtain that
\[
t_{A_1} = t_{B_1} = t_{C_2}.
\]
If we choose good triple in \(A_1 \times B_2 \times C_2\) and \(A_2 \times B_2 \times C_2\) we find
\[
t_{A_1} = t_{B_2} = t_{C_2}
\]
and
\[
t_{A_2} = t_{B_2} = t_{C_2}.
\]
Suppose now that the inner product corresponds to one of the particular cases that is there is \(i_0\) such that \(r_{i_0} = 2\). Thus in the expression of \(B\) some double products appear. For example in the second case, \(r_1 = r_2 = r_3 = 2\) and \(r_4 = 1\). As we have
\[
[B_{2,5}, C_{4,5}] = -A_{2,4}
\]
then
\[
B(A_{1,3}, [B_{2,5}, C_{4,5}]) + B([A_{1,3}, B_{2,5}], C_{4,5}) = 0
\]
gives
\[
B(A_{1,3}, A_{2,4}) = 0
\]
that is \(u_{A_1} = 0\). In the same way we find that all coefficients \(u\) are equal to 0.

**Proposition 10** Every invariant metric \(g\) on \(SO(2l + 1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)\) which is adapted to the \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-symmetric structure is given by an inner product \(B\) on \(m\) of type
\[
B = t(\Sigma(\alpha_{ij}^1)^2 + \Sigma(\alpha_{ij}^2)^2 + \Sigma(\beta_{ij}^1)^2 + \Sigma(\beta_{ij}^2)^2 + \Sigma(\gamma_{ij}^1)^2 + \Sigma(\gamma_{ij}^2)^2)
\]
with \(t > 0\).

**Example : The homogeneous manifold** \(SO(5)/SO(2) \times SO(2) \times SO(1)\)

In the previous section we have described the \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-graduation of the Lie algebra \(so(5)\) and we have computed the \(G\)-invariant metrics which are adapted to this graduation. Such a metric is given by an inner product \(B\) on \(so(5)\) which is written
\[
B = t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1 \alpha_4 - \alpha_2 \alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2).
\]
Proposition 11. Every inner product on so(5) for which the homogeneous components are pairwise orthogonal and which is adj.g.e.-invariant is written:

\[ B = q_1 + t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1\alpha_4 - \alpha_2\alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2) \]

where \( q_1 \) is any inner product on g.e and \( 4t^2 - u^2 > 0, \ t, v, w > 0 \). This inner product gives an adapted riemannian metric on \( SO(5)/SO(2) \times SO(2) \) if it is equal to

\[ B = q_1 + t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \beta_1^2 + \beta_2^2 + \gamma_1^2 + \gamma_2^2). \]

Remarks. 1) If \( q_1 = \omega_1^2 + \omega_2^2 + \omega_3^2 \) and \( t = 1 \), then \( -B \) coincides with the Killing-Cartan form of so(5). Its covariant operator \( \nabla_1 \) satisfies

\[ 2(\nabla_1)_XY = -[X, Y]. \]

2) Suppose that \( g \) is the metric \( B \) on so(5) corresponding to the inner product

\[ B = \sum_{i=1}^{3} \omega_i^2 + \sum_{i=1}^{4} \alpha_i^2 + \sum_{i=1}^{2} \beta_i^2 + \sum_{i=1}^{2} \gamma_i^2. \]

To simplify the notations, we shall put \( E_i = A_i, \ i = 1, 2, 3, 4, \) \( E_5 = B_1, \) \( E_6 = B_2, \) \( E_7 = C_1, \) \( E_8 = C_2. \) Then the sectional curvatures at the identity of \( \mathfrak{m} \) are given by

\[ g(R(X,Y)Y, X) = \frac{1}{4}B([X, Y]\mathfrak{m}, [X, Y]\mathfrak{m}) + B([X, Y]\mathfrak{g}_e, [X, Y]\mathfrak{g}_e) \]

and with respect to the orthonormal basis \( \{E_i\}_{i=1,...,8} \) we obtain

\[
\begin{align*}
R_{1221} &= R_{1331} = 1, \ R_{1551} = R_{1771} = 1/4 \\
R_{1441} &= R_{1661} = R_{1881} = 0 \\
R_{2442} &= 1, \ R_{2552} = R_{2882} = 1/4 \\
R_{2332} &= R_{2662} = R_{2772} = 0 \\
R_{3443} &= R_{3553} = R_{3883} = 0 \\
R_{3663} &= R_{3773} = 1/4 \\
R_{4554} &= R_{4774} = 0 \\
R_{4664} &= R_{4884} = 1/4 \\
R_{5665} &= 1, \ R_{5775} = R_{5885} = 1/4 \\
R_{6776} &= R_{6886} = 1/4 \\
R_{7887} &= 1.
\end{align*}
\]

So the sectional curvature is positive.

3) On the Ambrose-Singer tensor.

In [8] the authors classify the homogeneous riemannian spaces using the Ambrose-Singer tensor \( T \). The symmetric case corresponds to \( T = 0 \). The general riemannian homogeneous spaces are classified in 8 categories distinguished by algebraic properties of \( T \). For the riemannian nonsymmetric space \( M = SO(5)/SO(2) \times SO(2) \), this tensor corresponds to

\[ T = \nabla - \nabla. \]
If \( \{E_i\}_{i=1}^{8} \) is the orthonormal basis defined above, we consider the linear map on \( M \) given by

\[
c_{12}(T)(X) = \sum_{i=1}^{8} B_m(T(E_i, E_j), X).
\]

As \( T(E_i, E_j) = -T(E_j, E_i) \), we have \( c_{12}(T)(X) = 0 \) and \( B_m(T(X, Y), Z) = -B_m(T(Y, X), Z) \) and the tensor \( T \) is of type \( T_3 \) in the terminology of [8].

4) On the geodesics.

Following [4], if we set for each \( X \in \mathfrak{m} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c \), \( f_t = \exp(tX) \in SO(5) \) and \( x_t = f_t(0) \in M = SO(5)/SO(2) \times SO(2) \times SO(1) \) where \( 0 \) is the coset \( SO(2) \times SO(2) \times SO(1) \) in \( M \), then the curve \( x_t \) is a geodesic in \( M \). Conversely each geodesic starting from \( 0 \) is of the form \( \exp(tX)(0) \) for some \( X \in \mathfrak{m} \). It is not hard to see that for \( E \in \mathfrak{m} \), \( \exp(tE) = (I_8 + E^2 + \sin tE - \cos tE^2) \) where \( I_8 \) is the identity of rank 8.

Two points \( \exp(t_1)E \) and \( \exp(t_2)E \) of this \( 2\pi \)-periodic curve falls in the same coset of \( M \) if and only if \( t_2 - t_1 = 2k\pi \) for some \( k \in \mathbb{Z} \). This shows that \( f_t \) projects in a one-to-one manner in \( M \) and its image \( x_1 \) is a closed geodesic (of length \( 2\pi \)).

As an example one has

\[
\exp(tA_1) = \begin{pmatrix}
\cos t & 0 & \sin t & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\sin t & 0 & \cos t & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

5 On lorentzian \( (\mathbb{Z}_2 \times \mathbb{Z}_2) \)-symmetric structure

It is easy to generalize the notion of riemannian \( \Gamma \)-symmetric homogeneous space to the notion of semi-riemannian \( \Gamma \)-symmetric homogeneous space, in particular to a lorentzian metric. A lorentzian symmetric space \( M = G/H \) is determined by a nondegenerate \( ad_h \)-invariant bilinear form on \( \mathfrak{m} \) of signature \((1, n-1)\). In this case \( M \) the Riemann curvature tensor of the Levi-Civita connection is covariant constant.

**Definition 12** Let \((G, H, \Gamma_G)\) a \( \Gamma \)-symmetric space, \( g \) a semi-riemannian metric of signature \((1, n-1)\) where \( n = \dim M \) and \( B \) the corresponding \( ad_{g_e} \)-invariant symmetric bilinear form on \( \mathfrak{m} \). Then \( M = G/H \) is called a \( \Gamma \)-symmetric lorentzian space if the homogeneous components of \( \mathfrak{m} \) are pairwise orthogonal with respect to \( B \).

Since in the riemannian case, this does not imply that the riemannian connection \( \nabla_g \) of \( g \) coincides with \( \nabla \). If \( g \) satisfies this property, we will say that the connection \( \nabla_g \) is adapted to the \( \Gamma \)-symmetric structure.
From the classification of $ad_{g_e}$-invariant form on $so(2l + 1)$ given in Proposition 7, the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space $SO(2l + 1)/SO(r_1) \times \ldots \times SO(r_4)$ is lorentzian if and only if there exists one homogeneous component of $m$ of one dimensional. For example if we consider the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space $SO(5)/SO(2) \times SO(2) \times SO(1)$ the homogeneous components are of dimension 2 and every semi-riemannian metric is of signature $(2p, 8 - 2p)$ and cannot be a lorentzian metric. So $SO(5)/SO(2) \times SO(2) \times SO(1)$ can not be lorentzian. Nevertheless one can consider the grading of $so(5)$ given by

$$
\begin{pmatrix}
0 & a_1 & b_1 & b_2 & b_3 \\
-a_1 & 0 & c_1 & c_2 & c_3 \\
-b_1 & -c_1 & 0 & x_1 & x_2 \\
-b_2 & -c_2 & -x_1 & 0 & x_3 \\
-b_3 & -c_3 & -x_2 & -x_3 & 0
\end{pmatrix}
$$

where $g_e$ is parametrized by $x_1, x_2, x_3$, $g_0$ by $a_1$, $g_0$ by $b_1, b_2, b_3$ and $g_e$ by $c_1, c_2, c_3$. Let us denote by $\{X_1, X_2, X_3, A_1, B_1, B_2, B_3, C_1, C_2, C_3\}$ the corresponding graded basis. Here $g_e$ is isomorphic to $so(3) \oplus so(1) \oplus so(1)$ and we obtain the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric homogeneous space

$$
SO(5)/SO(3) \times SO(1) \times SO(1) = SO(5)/SO(3).
$$

Every nondegenerated symmetric bilinear form on $so(5)$ invariant by $g_e = so(3)$ is written

$$
q = t(\omega_1^2 + \omega_2^2 + \omega_3^2) + \omega_1^2 + v(\beta_1^2 + \beta_2^2 + \beta_3^2) + w(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)
$$

where $\{\omega_i, \alpha_i, \beta_i, \gamma_i\}$ is the dual basis of the basis $\{X_i, A_1, B_i, C_i\}$. In particular

**Proposition 13** The lorentzian inner product

$$
q = \omega_1^2 + \omega_2^2 + \omega_3^2 - \alpha_1^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2
$$

induces a structure of lorentzian $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure on the nonsymmetric homogeneous space $SO(5)/SO(3)$.

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