The Development of Napoleon’s Theorem on the Quadrilateral in Case of Outside Direction

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Abstract: In this article we discuss Napoleon’s theorem on the rectangles having two pairs of parallel sides for the case of outside direction. The proof of Napoleon’s theorem is carried out using a congruence approach. In the last section we discuss the development of Napoleon’s theorem on a quadrilateral by drawing a square from the midpoint of a line connecting each of the angle points of each square, where each of the squares is constructed on any quadrilateral and forming a square by using the row line concept.

Keywords: Napoleon’s Theorem, Napoleon’s Theorem on Rectangles, Outside Direction, Congruence

1. Introductions

Napoleon’s theorem was found by Napoleon Bonaparte (1769-1821), an Emperor of French who had a big interest in geometry [8]. Napoleon’s theorem on triangle is equilateral triangle constructed on each side of triangle that leads into outside. Furthermore on each equilateral triangle one can obtain midpoint that is angle from a new equilateral triangle [2]. The new equilateral triangle can be called as Napoleon’s triangle [6].

Napoleon’s theorem on the triangle can be proved by using congruence [7] and trigonometry algebra [1, 3 and 5]. Many authors have provided a wide range of alternative proofs as well as developed the Napoleon’s theorem, for example [6] develops on the inner Napoleon's theorem, whereas [14] describes it if an isosceles (right angle) of mother triangle is given, what properties must the external and internal Napoleon triangles have? What properties must the external and internal Napoleon’s triangles of a mother have to ensure that the mother is an isosceles triangle (right angle) triangle? While [2] developed for the case of the pedal triangle. Many other developments associated with the various theorems in geometry are discussed by a number of authors [3, 4, 5, 10, 11, 12 and 13]. The authors in [5, 6, 7, 8, 9 and 15] developed for the rectangles for the case of inside direction where the proofs are done using a very simple concept.

In this paper we give the development of Napoleon's theorem to the quadrilateral having two pairs of parallel sides and one form of development of Napoleon’s theorem for the case of parallelogram. Its development is to construct a quadrilateral formed by the lines connecting the midpoint of each square obtained from the construction. The main content is the process of proofs using a simple geometry concept (the same as in [1, 9] and 17). In the final part we give three corollaries in many cases.

2. Napoleon’s Theorem on Quadrilateral

Before discussing Napoleon's theorem on a quadrilateral, we first give an illustration of the Napoleon theorem for triangles fot the case of outside direction, the illustration can be seen in Figure 1. Similar conditions will also be applied to the quadrilateral
In [11] the development of the Napoleon theorem is carried out for the case of inside direction. While in this paper we do it for the case that leads out, so the development of Napoleon's theorem in this paper for quadrilateral is also done for the case that leads out. If the Napoleon's triangle theorem applied to any construct the triangle with equilateral triangles outward on each side. However, for any quadrilateral we cannot enforce Napoleon’s theorem to construct rectilinear direction out on any quadrilateral proficiency level (as an extreme example only create trapezium), it would not produce such a theorem Napoleon, as intended. But if taken rectangle, it is definitely Napoleon’s theorem applies. For the following will be made for cases of a parallelogram.

Napoleons’ theorem on rectangle will be discussed is on the quadrilateral that has two pairs parallell line, one of them is parallelogram in case of square that constructed outside direction. On AB side constructed $ABHG$ square, AD side constructed $ADEF$ square, CD side constructed $CDKL$ square, and BC side constructed $BCIJ$ square. Then every squares constructed in outside direction. Furthermore every midpoints are connected to shape the square that can be called as Napoleon’s quadrilateral in outside direction.

**Theorem 1.** Provide rectangle in shaping $ABCD$ parallelogram. On each side is constructed $ABHG$ square, $ADEF$ square, $CDKL$ square, and $BCIJ$ square in outside direction. For example $M$, $N$, $O$, and $P$ are midpoints of square that constructed in outside direction. If the four midpoints are connected they will be shaped $MNOP$ square.

**Proof: Alternatif 1.** To showing $MNOP$ is square, it will proved by $MN = NO$, and $\angle PMN = 90^\circ$, from $\Delta GAD$ and $\Delta BAF$, are obtained $AG = AB$, $\angle GAD = \angle FAB$, $AD = AF$, so $\Delta GAD \cong \Delta BAF$ [4]. Look at $\Delta GQT$ and $\Delta BAT$ on figure 4, $\angle TGQ = \angle TBA$ and $\angle GTQ = \angle BTA$, so $\angle GQT = \angle BAT = 90^\circ$.

Furthermore pull line $NP$ and $MO$ then cut it in one point, say point $R$. For example point $S$ cuts point $NP$ and $GD$, and $U$ cuts point $FB$ and $MO$ line. It will be shown that $BF$ is parallel with $PN$, $FN$ and $BP$ is a half diagonal square $ADFE$ and $BCIJ$, because of $ADFE$ square and $BCIJ$ are parallel then $FN$ is parallel with $BP$. Then pull FP line then it is obtained opposite angle those are $\angle BFP = \angle FPN$ and $\angle BPF = \angle PFN$ that caused $BF$ also parallel with $PN$. Because of $BF$ is parallel with $PN$ so $\angle GQT = \angle QSR = \angle MRN = 90^\circ$. To make it clearer look at Figure 3.
For example, $T$, $U$, and $V$ is the midpoint of the line $AD$, $CD$, $BC$ and $AB$. From $\triangle MVR$ and $\triangle NSR$, $MV = SR$, $\angle MVR = \angle NSR$, $VR = NS$, then $\angle MVR \approx \angle NSR$, causing $MR = RN$. Because $MR = RN$ and $\angle MVR = 90^\circ$, so $\angle MVR$ is an isosceles triangle, as well as $\angle NRS$ that causes $\angle MVR = \angle NRO = 45^\circ$. Then it is obtained that $\triangle MVR \approx \triangle RSN$, causing $MR = RN$. Because $MR = RN$ and $\angle MRN = 90^\circ$, so $\triangle MRN$ is an isosceles triangle, as well as $\triangle RNO$ that causes $\angle MNR = \angle RNO = 45^\circ$. Then it is obtained that $\angle MNO = 90^\circ$, in the same way $\angle OPM = \angle NOP = 90^\circ$. From $\triangle MRP$ and $\triangle NRO$, $MR = RO$, $\angle MRP = \angle NRO$, thus $\triangle MRP \approx \triangle NRO$. So that led to $MN = ON$, then in the same way then $MP = OP$. $\triangle MNOP$ it is clear that the quadrilateral is a square.

Alternative 2. $\triangle MNOP$ is a square will be proven by showing $MN = NO$, and $\angle PMN = \angle MNO = 90^\circ$ with trigonometric approach is by using the cosine rule and sine rule [6].

![Figure 4. Proof of Napoleon’s Theorem on a quadrilateral using trigonometry.](image)

In parallelogram $ABCD$, Let $AB = CD = a$, side $AC = BD = b$, use basic trigonometry to obtain

$$AM = MB = OC = OD = \frac{a}{\sqrt{2}}$$

$$AM = MB = OC = OD = \frac{b}{\sqrt{2}}$$

So that at $\triangle MAN$ using the cosine rule [5] apply

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - 2 \cdot \frac{1}{2}a \cdot \frac{1}{2}b \cdot \cos \angle MAN$$

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot \cos(270^\circ - \angle BAD)$$

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot \cos(270^\circ - \angle BAD)$$

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \sin \angle BAD$$

and on $\triangle ABD$ apply

$$\sin \angle BAD = \frac{2 \cdot \Delta ABD \cdot a \cdot b}{a \cdot b}$$

Substitute the equation (2) to (1) in order to obtain

$$MN^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \frac{2 \cdot \Delta ABD \cdot a \cdot b}{a \cdot b}$$

$$MN = \frac{1}{2}a^2 + \frac{1}{2}b^2 + 2 \cdot \Delta ABD$$

$$MN = \frac{1}{2}a^2 + \frac{1}{2}b^2 + L \cdot ABD$$

(3)

Subsequently the $\triangle NDO$ we obtained

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - 2 \cdot \frac{1}{2}a \cdot \frac{1}{2}b \cdot \cos \angle NDO$$

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot \cos(90^\circ - \angle ADC)$$

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - a \cdot b \cdot \cos(90^\circ - \angle ADC)$$

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \sin \angle ADC$$

(4)

If

$$t = \frac{L_{ABCD}}{b}$$

Then we have

$$\sin \angle ADC = \frac{L_{ABCD}}{a \cdot b}$$

(5)

Substitute the equation (5) to (4) we obtained

$$NO^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 + a \cdot b \cdot \frac{L_{ABCD}}{a \cdot b}$$

$$NO = \frac{1}{2}a^2 + \frac{1}{2}b^2 + L \cdot ABD$$

(6)

Based on the equation (5) and equation (6) it is clear that $MN = NO$. Furthermore, to show $\angle MNO = 90^\circ$ consider Figure 5.

![Figure 5. Proof of Napoleon’s Theorem on a quadrilateral using trigonometry.](image)
Then to show \( \angle MNO = 90^\circ \), must first be proven \( OQ = NQ \) with the cosine rule. Furthermore, by using the cosine rule on \( \triangle NOQ \) obtained \( \angle NOQ = 90^\circ \), so \( \triangle NOQ \) is an isosceles triangle that causes \( \angle ONQ \) = \( \angle NOQ \) = \( 45^\circ \), as well as on \( \triangle MNO \) the same way in order to obtain \( \angle MNO = \angle NMQ = \angle NMP = 90^\circ \), so it is evident that \( MNOP \) quadrilateral is a square.

3. The Development of Napoleon’s Theorem on Quadrilateral

Basically Napoleon’s theorem on the quadrilateral can only be developed in a quadrilateral that has two pairs of parallel sides. The following will be given the development of Napoleon’s theorem on a parallelogram, theorem developed based on a rectangle to a square case built leading to the outside which is a square formation by connecting endpoints to construct a square formed on the sides of the initial parallelogram.

**Theorem 2** Given a parallelogram quadrilateral \( ABCD \), and on each side of the square was built leading out. Then draw a line from point \( Q \) to point \( S \), and \( T \) is the midpoint of the fourth line. If the four points are connected, the square formed \(QRST\).

\[ \text{Proof.} \text{ Suppose point } Q, R, S, \text{ and } T \text{ is the midpoint of the line } FG, EL, KJ, \text{ and } HI. \text{ To show } QRST \text{ is square it will be proved } TQ = QR, \text{ and } \angle TQR = 90^\circ. \text{ Consider Figure 6, draw a line from point } Q \text{ to point } S \text{ and point } R \text{ to the point } Q. \text{ So the lines } QS \text{ and } RQ \text{ lines intersect at one point, said point } U. \text{ Before the show } TQ = QR, \text{ will at first show } UT = UR. \text{ Note Figure 4, } UY = UT, \text{ and } YT = VR \text{ so } UT = UR. \text{ Then Note figure 6, } UY = UT, \text{ and } YT = VR \text{ so } UT = UR. \text{ Then look at } \triangle QUT \text{ and } \triangle QUR, \text{ and } \angle TUR = \angle UTQ \text{ and } \angle UQR = \angle UQD \text{ then obtained } TQ = QR. \text{ From theorem 1, } VU = UY, \text{ and } \angle VUY = 90^\circ, \text{ therefore } \angle QTR = 90^\circ, \text{ so it proved } QRST \text{ quadrilateral is a square.}

Based on the Theorem 1 and Theorem 2, and by way of connecting the vertex and the lines are connected are three types of effect, that the evidence is not given here because it always can easily be shown, however, given the geometry illustration

**Corollary 1.** On the square due \( MNOP \) and \( TQRS \) formed parallel lines \( PQ // SN \) and \( MR // TO \), then the square formed \( VWZU \), and if formed parallel lines \( MS // QO \) and \( TN // PR \), then the square formed \( A_1B_1C_1D_1 \). The illustration is in Figure 7.

**Corollary 2.** On the square \( VWZU \) and \( A_1B_1C_1D_1 \) parallel lines \( UA_1 // VC_1 \) and \( ZD_1 // WB_1 \) are formed, then a square \( K_1N_1M_1H_1 \) is formed, and if parallel lines \( VB_1 // UD_1 \) and \( ZD_1 // WB_1 \) are formed, then a square \( O_1P_1Q_1R_1 \) is formed. The illustration is in Figure 8.
Corollary 3. On the Picture $K_1N_1M_1H_1$ and $O_1P_1Q_1R_1$ form parallel lines $K_1P_1//M_1R_1$ and $K_1Q_1//M_1O_1$, then they form a square $E_1F_1I_1G_1$, and if parallel lines $N_1R_1//M_1O_1$ and $K_1Q_1//M_1O_1$ are formed, then a square $J_1L_1T_1S_1$ is formed. Illustration is in Figure 9.
4. Conclusion

After several experiments on Napoleon's Theorem in the quadrilateral we thus observe that Napoleon's theorem applied only to the quadrilateral which possess two pairs of parallel sides like a square, rhombus, rectangle, and parallelogram. Napoleon's Theorem in the ranks of the cases led to the outside that is if the square was built on each side of the square center of the fourth point would be to the square Napoleon called quadrilateral outside. Proof of that is done by using the concept of congruence. Development of Napoleon on a quadrilateral theorem can be developed to form a square on the midpoint of the line so as to form a new square, and forming a square with the concept of parallel lines that intersect.

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