On the Existence and Temporal Asymptotics of Solutions for the Two and Half Dimensional Hall MHD

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Abstract. In this paper, we deal with the $2\frac{1}{2}$ dimensional Hall MHD by taking the velocity field $u$ and the magnetic field $B$ of the form $u(t,x,y) = (\nabla \perp \phi(t,x,y), W(t,x,y))$ and $B(t,x,y) = (\nabla \perp \psi(t,x,y), Z(t,x,y))$. We begin with the Hall equations (without the effect of the fluid part). In this case, we provide several results such as the long time behavior of weak solutions, weak-strong uniqueness, the existence of local and global in time strong solutions, decay rates of $(\psi, Z)$, the asymptotic profiles of $(\psi, Z)$, and the perturbation around harmonic functions. In the presence of the fluid field, the results, by comparison, fall short of the previous ones in the absence of the fluid part and we show the existence of local and global in time strong solutions.

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1. Introduction

The Magnetohydrodynamics equations (MHD in short) provide a macroscopic description of a plasma and provide a relevant description for fusion plasmas, the solar interior and its atmosphere, the Earth’s magnetosphere and inner core, etc. The governing equations for the incompressible and resistive MHD are

\begin{align}
\text{Momentum Equation:} & \quad u_t + u \cdot \nabla u - J \times B + \nabla p - \mu \Delta u = 0, \\
\text{Incompressibility:} & \quad \text{div } u = 0, \\
\text{Ampère’s Law:} & \quad \text{curl } B = \mu_0 J, \\
\text{Faraday’s Law:} & \quad \text{curl } E = -B_t, \\
\text{Ohm’s Law for resistive MHD:} & \quad E + u \times B = \nu J, \\
\text{Incompressibility:} & \quad \text{div } B = 0,
\end{align}

where $u$ is the velocity field, $p$ is the pressure, and $B$ is the magnetic field. $\mu$ and $\nu$ are the viscosity and the resistivity constants, respectively. The right-hand side of (1.1e) is called the collision term and $J \times B$ in (1.1a) is called the Lorentz force. However, (1.1) is deficient in many respect: for example, (1.1) does not explain magnetic reconnection on the Sun which is very important role in acceleration plasma by converting magnetic energy into bulk kinetic energy. For this reason, the generalized Ohm’s Law is required and we here take the following

\begin{equation}
E + u \times B = \nu J + \frac{1}{en} (J \times B - \nabla p_e),
\end{equation}

where $e$ is the elementary charge, $n$ is the number density, and $p_e$ is the electron pressure. The second term on the right-hand side of (1.2) is called the Hall term. In terms of $(u, \bar{p}, B)$, we have the Hall MHD with $\mu_0 = en = 1$ for simplicity:
\[ u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla \bar{p} - \mu \Delta u = 0, \]  
(1.3a)

\[ B_t + u \cdot \nabla B - B \cdot \nabla u + \text{curl} ((\text{curl} B) \times B) - \nu \Delta B = 0, \]  
(1.3b)

\[ \text{div} u = 0, \quad \text{div} B = 0, \]  
(1.3c)

where we use the following in (1.3a):

\[ J \times B = B \cdot \nabla B - \frac{1}{2} \nabla |B|^2, \quad \bar{p} = p + \frac{1}{2} |B|^2. \]  

The Hall-MHD is important in describing many physical phenomena [2,20,26,32,35,37,38,43]. The Hall-MHD recently has been studied intensively. The Hall-MHD can be derived from either two fluids model or kinetic models in a mathematically rigorous way [1]. Global weak solution, local classical solution, global solution for small data, and decay rates are established in [5–7]. There have been many follow-up results of these papers; see [8,9,13–16,19,24,25,29,31,34,39–42,44,45,47–49] and references therein.

1.1. 2\(\frac{1}{2}\) Dimensional Hall MHD

The Hall term, \(\text{curl} ((\text{curl} B) \times B)\), is dominant when analyzing (1.3). So, even if we deal with (1.3) in the \(2\frac{1}{2}\) dimensional case, the global regularity problem for (1.3) is still open. As a result, compared to MHD and the incompressible Navier–Stokes equations, there are only a few results dealing with the \(2\frac{1}{2}\) dimensional (1.3); partial regularity theory [10], global regularity with partial dissipations in (1.3a) [18], ill-posedness without resistivity [30], and irreducibility property [46].

In this paper, we take its \(2\frac{1}{2}\) dimensional form of (1.3) through

\[ u(t, x, y) = (\nabla \perp \phi(t, x, y), W(t, x, y)) = (\phi_y(t, x, y), -\phi_x(t, x, y), W(t, x, y)), \]  
(1.4a)

\[ B(t, x, y) = (\nabla \perp \psi(t, x, y), Z(t, x, y)) = (\psi_y(t, x, y), -\psi_x(t, x, y), Z(t, x, y)). \]  
(1.4b)

Due to the presence of the pressure in (1.3a), we take the curl to (1.3a) and we rewrite (1.3) as

\[ \psi_t - \Delta \psi = [\psi, Z] - [\psi, \phi], \]  
(1.5a)

\[ Z_t - \Delta Z = [\Delta \psi, \psi] - [Z, \phi] + [W, \psi], \]  
(1.5b)

\[ W_t - \Delta W = -[W, \phi] - [\psi, Z], \]  
(1.5c)

\[ \Delta \phi_t - \Delta^2 \phi = -[\Delta \phi, \phi] + [\Delta \psi, \psi], \]  
(1.5d)

where we set \(\mu = \nu = 1\) for simplicity and \([f, g] = \nabla f \cdot \nabla \perp g = f_x g_y - f_y g_x\). (1.5) is used to investigate the influence of the Hall-term on the island width of a tearing instability [26] and to show a finite-time collapse to a current sheet [3,27,28,33]. (1.5) is also used in [11] to study regularity of stationary weak solutions.

1.2. Hall Equation

Since the Hall term is dominant when we deal with (1.5), we will mainly concentrate on the Hall equations: (1.5) without the effect of the fluid part. Then, (1.5) is reduced to the following equations

\[ \psi_t - \Delta \psi = [\psi, Z], \]  
(1.6a)

\[ Z_t - \Delta Z = [\Delta \psi, \psi]. \]  
(1.6b)

We will explain several results of (1.6) from Sects. 1.3 to 1.6, but before we do, we briefly describe them.

(1) The existence of weak solutions and decay rates of (1.6) can be proved by following [5,7]. In Sect. 1.3, we restate the decay rate of weak solutions in [7] to (1.6) (Theorem 1.1) and establish weak-strong uniqueness (Theorem 1.2).
(2) In Sect. 1.4, we deal with strong solutions of (1.6). We first establish the existence of unique local-in-time solutions with large initial data and a blow-up criterion (Theorem 1.3). Having established the local in time results, we then proceed to extend the solution globally-in-time and to derive decay rates by imposing some smallness condition to initial data (Theorem 1.4). We also improve the decay rates of ψ by using the structure of the equation of ψ (Theorem 1.5).

(3) It is reasonable to study the asymptotic stability of temporally decaying solutions in Theorems 1.4 and 1.5. In Sect. 1.5, we intend to find asymptotic profiles of such solutions of (1.6) from the observation that constant multiples of the two dimensional heat kernel Γ are solutions of (1.6) (Theorem 1.6).

(4) The aim of Sect. 1.6 is to analyze (1.6) around harmonic functions. We take ψ = ρ + ω or Z = ω + Z, where ω and Z are harmonic functions. We show that there exists unique global-in-time solutions if ρ0 or ω0 are sufficiently small (Theorem 1.7 and Theorem 1.8). We emphasize that the size of ψ and Z are arbitrary.

1.3. Weak Solution of (1.6)

The existence and decay rate of a weak solution even for (1.3) are already proved in [5,7] with u0 ∈ L2 and B0 ∈ L2. We here restate these results to (1.6). We first give the weak formulation of (1.6): for χ ∈ C∞ _c ((0,∞) × R²) and ζ ∈ C∞ _c ((0,∞) × R²)

\[
\int_0^\infty \int_{R^2} (\psi \chi)_t dxdt - \int_0^\infty \int_{R^2} (\nabla \psi \cdot \nabla \chi) dxdt = \int_{R^2} \psi(0) \chi(0) dx - \int_0^\infty \int_{R^2} (|\psi| Z \chi) dxdt, \tag{1.7}
\]

\[
\int_0^\infty \int_{R^2} (Z \zeta)_t dxdt - \int_0^\infty \int_{R^2} (\nabla Z \cdot \nabla \zeta) dxdt = \int_{R^2} Z(0) \zeta(0) dx - \int_0^\infty \int_{R^2} (|\psi| \zeta \Delta \psi) dxdt, \tag{1.8}
\]

where we use (2.3d) for the last term of (1.8). We also note that we can derive the following:

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) + \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 = 0. \tag{1.9}
\]

Since \(H^1 \subset L^2_\loc\) in 2D, the first term on the left-hand side of (1.7) is well-defined. Since (1.9) gives the compact embedding of ψ in L²([0, T] : \(H^1\)), we can follow [5, Proof of Theorem 2.1] to show the existence of a weak solution of (1.6) with (\(\nabla \psi_0, Z_0\)) ∈ L². Moreover, we have temporal decay rates of weak solutions which is the two dimensional version of [7].

**Theorem 1.1.** Let (\(\nabla \psi_0, Z_0\)) ∈ L². Then, there is a weak solution of (1.6) satisfying

\[
\|\nabla \psi(t)\|_{L^2}^2 + \|Z(t)\|_{L^2}^2 + 2 \int_0^t \left( \|\Delta \psi(s)\|_{L^2}^2 + \|\nabla Z(s)\|_{L^2}^2 \right) ds \leq \|\nabla \psi_0\|_{L^2}^2 + \|Z_0\|_{L^2}^2,
\]

for all t > 0. If (\(\nabla \psi_0, Z_0\)) ∈ L² ∩ L¹, \(\nabla \psi\) and Z decay in time as

\[
\|\nabla \psi(t)\|_{L^2} + \|Z(t)\|_{L^2} \leq \frac{C_0}{\sqrt{1 + t}},
\]

where \(C_0\) depends on \(\|\nabla \psi_0\|_{L^2 \cap L^1}\) and \(\|Z_0\|_{L^2 \cap L^1}\).

As in the case of the three- dimensional incompressible Navier–Stokes equations, the uniqueness of weak solution of (1.6) is unknown. Weak-strong uniqueness is to find a path space \(\mathcal{P}\) of a strong solution \(B \in \mathcal{P}\) such that all weak solutions which share the same initial condition \(B_0\) equal \(B\). In this paper, we do not aim to derive very general weak–strong uniqueness results as in the case of the incompressible Navier–Stokes equations [22], but focus on Serrin-type results.

**Theorem 1.2.** Let \(B_1 = (\nabla ^\perp \psi_1, Z_1)\) and \(B_2 = (\nabla ^\perp \psi_2, Z_2)\) be weak solutions of (1.6) with the same initial data (\(\nabla \psi_0, Z_0\)) ∈ L². Then, \(B_1 = B_2\) on [0, T] if \(B_2\) satisfies the condition

\[
(\Delta \psi_2, \nabla Z_2) \in L^p ([0, T] : L^q), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty. \tag{1.11}
\]
Remark 1.1. In [5], weak-strong uniqueness is established with $B_2 \in L^2 ([0, T] : W^{1,\infty}(\mathbb{R}^3))$. By contrast, we derive weak-strong uniqueness with $B_2 \in L^p ([0, T] : L^q(\mathbb{R}^3))$.

1.4. Strong Solutions of (1.6)

In this paper, we establish the local in time existence of unique strong solutions of (1.6) with initial data $(\nabla \psi_0, Z_0) \in H^2$. Let

$$
M(t) = ||\nabla \psi(t)||_{L^2}^2 + ||Z(t)||_{L^2}^2, \quad M(0) = ||\nabla \psi_0||_{L^2}^2 + ||Z_0||_{L^2}^2,
$$

$$
N(t) = ||\nabla^2 \psi(t)||_{L^2}^2 + ||\nabla Z(t)||_{L^2}^2, \quad E(t) = M(t) + \int_0^t N(s)ds. \tag{1.12}
$$

When the energy method is applied, we observe that the terms with the highest derivative, that are unlikely to be handled by Laplacian’s regularity, disappear due to the properties of the commutator in (2.3). For example, see (3.3). So, we can derive the following inequalities:

$$
\frac{d}{dt}(1 + M) + N \leq C (1 + M)^3
$$

and this gives the first part of the following result. To derive a blow-up criterion, we re-estimates some terms in Sect. 3.1.1 from $L^2$ to $L^r$, $r \neq 2$. From now on, constants that depend on $M(0)$ are not specified each time we state our results, and we will use $E_0$ in common.

Theorem 1.3. Let $(\nabla \psi_0, Z_0) \in H^2$. There exists $T^* = T(E_0)$ such that there exists a unique solution of (1.6) with $E(T^*) < \infty$. Moreover, the maximal existence time $T^* < \infty$ if and only if

$$\lim_{T \nearrow T^*} \int_0^T ||\nabla Z(t)||_{L^q}^q dt = \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty. \tag{1.13}$$

Remark 1.2. Compared to [5], the regularity of initial data is the borderline case: $B_0 = (\nabla^\perp \psi_0, Z_0) \in H^2$ with $2 = \frac{d}{2} + 1 = \frac{2}{1} + 1$. Moreover, the blow-up criterion in (1.13) is stated only in terms of the third component of $B$. A similar blow-up criterion can be derived in terms of $\Delta \psi$:

$$T^* < \infty \iff \lim_{T \nearrow T^*} \int_0^T ||\Delta \psi||_{L^p}^q dt = \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty. \tag{1.14}$$

The condition (1.11) and the blow-up criteria (1.13) and (1.14) are related to the scaling invariant property of (1.6): if $(\psi(t, x, y), Z(t, x, y))$ is a solution of (1.6) on $[0, T]$, so is

$$(\psi_{\lambda}(t, x, y) = \lambda^{-1}\psi (\lambda^2 t, \lambda x, \lambda y), \quad Z_{\lambda}(t, x, y) = Z (\lambda^2 t, \lambda x, \lambda y) \quad \text{on } [0, \lambda^2 T]. \tag{1.15}$$

Since (1.6) is dissipative, we typically expect the global well-posedness and temporal decay rates of solutions under some smallness conditions. In Sect. 3, we will derive the followings

$$\frac{d}{dt}(||\Delta \psi||_{L^2}^2 + ||\nabla Z||_{L^2}^2) + ||\nabla \Delta \psi||_{L^2}^2 + ||\Delta Z||_{L^2}^2 \leq CS(t) \left( ||\nabla \Delta \psi||_{L^2}^2 + ||\Delta Z||_{L^2}^2 \right), \tag{1.16}$$

where $S(t) = ||\Delta \psi||_{L^2}^2 + ||\nabla Z||_{L^2}^2$. By imposing the smallness condition of the form

$$\epsilon_1 = ||\nabla \psi_0||_{L^2}^2 + ||\nabla Z_0||_{L^2}^2, \quad C\epsilon_1 < 1 \tag{1.17}$$

we can obtain global-in-time solutions and can find decay rates of the solution in Theorem 1.3.

Theorem 1.4. Let $(\nabla \psi_0, Z_0) \in H^2$ which satisfies (1.17). Then, we can take $T^* = \infty$ in Theorem 1.3. If $(\nabla \psi_0, Z_0) \in L^1$ in addition, $(\Delta \psi, \nabla Z)$ decays in time as follows

$$||\Delta \psi(t)||_{L^2} + ||\nabla Z(t)||_{L^2} \leq \frac{E_0}{1 + t^2}, \quad ||\nabla \Delta \psi(t)||_{L^2} + ||\Delta Z(t)||_{L^2} \leq \frac{E_0}{(1 + t)^{3/2}}. \tag{1.18}$$
Theorem 1.5. Let behaviors accordingly.

From the linear part of (1.6), we expect the decay rates of the form
\[ \| \Delta \psi(t) \|_{L^2} + \| \nabla Z(t) \|_{L^2} \leq \frac{\mathcal{E}_0}{1 + \sqrt{t}}, \quad \| \nabla \Delta \psi(t) \|_{L^2} + \| \Delta Z(t) \|_{L^2} \leq \frac{\mathcal{E}_0}{1 + t}, \]
but, these are improved to (1.18) by combining with (1.10).

(2) According to (1.15), \( \dot{H}^1 \) is a scaling-invariant space of \( (\nabla \psi_0, Z_0) \) and we use the smallness condition (1.17) in Theorem 1.4. In [6], the smallness condition is presented in the Besov space \( \dot{B}^{3/2}_{2,1} \). We here replace \( \dot{B}^{3/2}_{2,1} \) with \( \dot{H}^1 \). This is possible by two reasons: (i) we reduce the dimension from 3 to 2 and (ii) we are able to avoid the Littlewood-Paley theory by exploiting some cancellation properties of the commutator (2.3).

(3) In the Proof of Theorem 1.3, we obtain (3.4):
\[ \frac{d}{dt} (\| \Delta \psi(t) \|_{L^2}^2 + \| \nabla Z(t) \|_{L^2}^2) + \| \nabla \Delta \psi(t) \|_{L^2}^2 + \| \Delta Z(t) \|_{L^2}^2 \leq C \| \Delta \psi(t) \|_{L^2}^2 \| \nabla \Delta \psi(t) \|_{L^2}^2. \]

Combined with the uniqueness part in Sect. 3.1.2, we can show the existence of a unique solution globally-in-time with \( (\nabla \psi_0, Z_0) \in \dot{H}^1 \) under the smallness condition (1.17). But, we do not state this case separately because we are more interested in using Theorem 1.4 to prove Theorems 1.5 and 1.6.

The decay rates in Theorem 1.4 are obtained by treating \( \nabla \psi \) and \( Z \) together. But, we observe that we can improve the decay rates of \( \psi \) by using the structure of (1.6a) which is a dissipative transport equation, and this is also the reason why the same method cannot be applied to \( Z \).

Theorem 1.5. Let \( (\nabla \psi_0, Z_0) \in H^2 \) satisfy (1.17). If \( \psi_0 \in L^1 \cap L^2 \) in addition, \( \psi \) decays in time as follows:
\[ \| \nabla \psi(t) \|_{L^2} \leq \frac{\mathcal{E}_0}{(1 + t)^{3/2}} \quad \text{for all } t > 0. \]

1.5. Asymptotic Behaviors

Theorems 1.4 and 1.5 provide upper bounds of decay rates. In particular, we have
\[ \| \nabla \psi(t) \|_{L^2} \leq \frac{\mathcal{E}_0}{1 + t}, \quad \| \nabla Z(t) \|_{L^2} \leq \frac{\mathcal{E}_0}{1 + t}. \]

Although there is no embedding relationship between \( \dot{H}^1 \) and \( L^\infty \), we can expect similar decay results in \( L^\infty \) if we establish the asymptotic behavior of \( (\psi, Z) \) as \( t \to \infty \). One motivation for considering asymptotic behavior is that the asymptotic behavior of the vorticity of the incompressible Navier–Stokes equations in 2D is well-established: [4], [21], [23, Page 44]. And similar results can be obtained to more complicated models such as an aerotaxis model coupled to fluid equations [12]. Along this direction, we also want to find an asymptotic profile of the solutions of (1.6). To do so, we assume \( (\nabla \psi_0, Z_0) \in H^2 \) as before and we impose the following additional conditions
\[ \psi_0 \in L^1, \quad Z_0 \in L^1, \quad \langle x \rangle \psi_0 \in L^1, \quad \langle x \rangle Z_0 \in L^1, \]
\[ \int_{\mathbb{R}^2} \psi_0(x)dx = \gamma, \quad \int_{\mathbb{R}^2} Z_0(x)dx = \eta, \]
where \( \langle x \rangle = \sqrt{1 + |x|^2} \). Depending on which of one given by (1.20) we choose, we can describe asymptotic behaviors accordingly.
Theorem 1.6. Suppose \((\nabla \psi_0, Z_0) \in H^2\) and we assume (1.20a). Then, we obtain
\[
\psi(t, x) = \Gamma(t) \ast \psi_0 + O(t^{-3/2}), \quad Z(t, x) = \Gamma(t) \ast Z_0 + O(t^{-2})
\]
as \(t \to \infty\), where \(\Gamma\) is the two dimensional heat kernel. If we assume (1.20b) and (1.20c),
\[
\psi(t, x) = \gamma \Gamma(t, x) + O(t^{-3/2}), \quad Z(t, x) = \eta \Gamma(t, x) + O(t^{-3/2}).
\]

We note that the asymptotic behavior of \(\nabla \psi\) is the same as \(Z\) because \(\nabla (\nabla^\perp Z \cdot \nabla \psi) \simeq \text{div} (\nabla \psi \Delta \psi)\) in terms of regularity and decay rates. So, the asymptotic behavior of \(B\) is
\[
B(t, x) = (\gamma \nabla^\perp \Gamma(t, x), \eta \Gamma(t, x)) + O(t^{-3/2}).
\]

1.6. Perturbation Around Harmonic Functions

Theorem 1.4 is about the existence of a solution globally-in-time when the initial data is small enough around zero. We now perturb (1.6) around harmonic functions. Then, the newly generated terms are linear and so one may guess that a smallness condition on harmonic functions is also necessary. But, we emphasize that this is not the only case. As one can see from the statements of Theorem 1.7 and Theorem 1.8 or the proof of them in Sect. 6, we can absorb these terms to the left-hand side of the desired bounds by multiplying by a large constant depending on the harmonic function we choose.

1.6.1. Case 1. Let \(\overline{\psi}\) be a harmonic function such that \(\|\nabla^2 \overline{\psi}\|_{L^\infty} < \infty\). (For example, \(\overline{\psi}(x, y) = x^2 - y^2\) or \(\overline{\psi}(x, y) = xy\).) Let \(\psi = \rho + \overline{\psi}\). Then, we obtain the following equations of \((\rho, Z)\):
\[
\begin{align*}
\rho_t - \Delta \rho &= [\rho, Z] + [\overline{\psi}, Z], \\
Z_t - \Delta Z &= [\Delta \rho, \rho] + [\Delta \rho, \overline{\psi}].
\end{align*}
\tag{1.21a}
\]

Since the smallness condition is stated by combining \(\overline{\psi}\) and the regularity of the various level of the initial conditions as shown just below, we take
\[
F_1 = \|\nabla \rho\|_{L^2}^2 + \|Z\|_{L^2}^2, \quad F_2 = \|\Delta \rho\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2
\]
\[
F_3 = \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2, \quad F_4 = \|\Delta^2 \rho\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2
\]
\tag{1.22}

and let \(\epsilon_2 = C^2_1 F_1(0) + C_1 F_2(0) + F_3(0)\) and \(C_1 = k \|\nabla^2 \overline{\psi}\|_{L^\infty}^2\) with \(k\) fixed by (6.8).

Theorem 1.7. There exists a constant \(C\) such that if \(C \epsilon_2 < 1\), there exists a unique solution \((\rho, Z)\) of (1.21) satisfying
\[
C^2_1 F_1(t) + C_1 F_2(t) + F_3(t) + (1 - C \epsilon_1) \int_0^t (C_1 F_3(s) + F_4(s)) \, ds
\]
\[
\leq C^2_1 F_1(0) + C_1 F_2(0) + F_3(0) \quad \text{for all } t > 0.
\]

1.6.2. Case 2. Let \(\overline{Z}\) be a harmonic function such that \(\|\nabla \overline{Z}\|_{L^\infty} < \infty\). (For example, \(\overline{Z}(x, y) = ax + by\).) Let \(Z = \omega + \overline{Z}\). Then, we obtain the following equations of \((\psi, \omega)\):
\[
\begin{align*}
\psi_t - \Delta \psi &= [\psi, \omega] + [\overline{\psi}, \overline{Z}], \\
\omega_t - \Delta \omega &= [\Delta \psi, \psi].
\end{align*}
\tag{1.23a}
\]

For the same reason as Case 1, let
\[
K_1 = \|\nabla \psi\|_{L^2}^2 + \|\omega\|_{L^2}^2, \quad K_2 = \|\Delta \psi\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2,
\]
\[
K_3 = \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2, \quad K_4 = \|\Delta^2 \psi\|_{L^2}^2 + \|\nabla \Delta \omega\|_{L^2}^2
\]
\tag{1.24}

and let \(\epsilon_3 = C^2_2 \|\psi_0\|_{L^2}^2 + C^2_2 K_1(0) + C_2 K_2(0) + K_3(0)\) and \(C_2 = k \|\nabla^2 \overline{Z}\|_{L^\infty}^2\) with \(k\) fixed by (6.13).
Theorem 1.8. There exists a constant $C$ such that if $C \epsilon < 1$, there exists a unique solution $(\psi, \omega)$ of (1.23) satisfying

$$
C_2^3 \|\psi(t)\|_{L^2}^2 + C_2^2 K_1(t) + C_2 K_2(t) + K_3(t) + (1 - C_2) \int_0^t (C_2^2 K_2(s) + C_2 K_3(s) + K_4(s)) \, ds
\leq C_2^3 \|\psi(0)\|_{L^2}^2 + C_2^2 K_1(0) + C_2 K_2(0) + K_3(0) \quad \text{for all } t > 0.
$$

Remark 1.4. (1) Compared to Theorem 1.4, we are not able to derive decay rates in Theorem 1.7 and Theorem 1.8 due to the terms having $\psi$ and $\overline{Z}$ on the right-hand side of (6.4), (6.6), (6.10), and (6.11).

(2) If we take $\psi = \rho + \overline{\psi}$ and $Z = \omega + \overline{Z}$, we obtain the following equations:

$$
\begin{align*}
\rho_t - \Delta \rho &= [\rho, \omega] + [\rho, \overline{Z}] + [\overline{\psi}, \omega] + [\overline{\psi}, \overline{Z}], \\
\omega_t - \Delta \omega &= [\Delta \rho, \rho] + [\Delta \rho, \overline{\psi}].
\end{align*}
$$

To obtain a result like Theorems 1.7 and 1.8, we may start by choosing $\overline{\psi}$ and $\overline{Z}$ such that $[\overline{\psi}, \overline{Z}] = 0$. One specific choice is $\overline{\psi} = \overline{Z} = ax + by$. In this case, $[\rho, \overline{Z}] = b \rho_x - a \rho_y$, $[\overline{\psi}, \omega] = a \omega_y - b \omega_x$ and $[\Delta \rho, \overline{\psi}] = b \Delta \rho_x - a \Delta \rho_y$, and these will be cancelled out when we apply our method. So, (1.25) is equivalent to (1.6). But, we do not have results dealing with more general $\overline{\psi}$ and $\overline{Z}$ to handle (1.25) as (1.21) or (1.23).

1.7. Hall MHD

After considering the Hall equations, we consider in this section the $2 \frac{1}{2}$ dimensional Hall MHD given by (1.5). Due to the presence of the fluid part, results similar to Theorems 1.5, 1.6, 1.7, and 1.8 will not be presented in this paper. Instead, we begin with the existence and the decay rate of weak solutions of (1.5) which are again the two dimensional version of [5, 7].

Theorem 1.9. Let $(\nabla \psi_0, Z_0, \nabla \phi_0, Z_0) \in L^2$. Then, there is a weak solution of (1.5) satisfying

$$
\begin{align*}
\|\nabla \psi(t)\|_{L^2}^2 + &\|Z(t)\|_{L^2}^2 + \|\nabla \phi(t)\|_{L^2}^2 + \|W(t)\|_{L^2}^2 \\
+ 2 \int_0^t &\left(\|\Delta \psi(s)\|_{L^2}^2 + \|\nabla Z(s)\|_{L^2}^2 + \|\Delta \phi(s)\|_{L^2}^2 + \|\nabla W(s)\|_{L^2}^2\right) \, ds
\leq \|\nabla \psi(0)\|_{L^2}^2 + \|Z_0\|_{L^2}^2 + \|\nabla \phi(0)\|_{L^2}^2 + \|W_0\|_{L^2}^2
\end{align*}
$$

for all $t > 0$. If $(\nabla \psi_0, Z_0, \nabla \phi_0, W_0) \in L^2 \cap L^1$, $(\nabla \psi, Z, \nabla \phi, W)$ decay in time as

$$
\|\nabla \psi(t)\|_{L^2} + \|Z(t)\|_{L^2} + \|\nabla \phi(t)\|_{L^2} + \|W(t)\|_{L^2} \leq \frac{C_0}{\sqrt{1 + t}},
$$

where $C_0$ depends on $\|\nabla \psi_0\|_{L^2 \cap L^1}$, $\|Z_0\|_{L^2 \cap L^1}$, $\|\nabla \phi_0\|_{L^2 \cap L^1}$, and $\|W_0\|_{L^2 \cap L^1}$.

We now proceed, as in Sect. 1.4, towards the strong solutions of (1.5). We first show the existence of unique local-in-time solutions with large initial data and we derive a blow-up criterion. The function spaces that we introduce are similar to those used for (1.6)

$$
\begin{align*}
P(t) &= \|\nabla \psi(t)\|_{H^2}^2 + \|Z(t)\|_{H^2}^2 + \|\nabla \phi(t)\|_{H^2}^2 + \|W(t)\|_{H^2}^2,
Q(t) &= \|\Delta \psi(t)\|_{H^2}^2 + \|\nabla Z(t)\|_{H^2}^2 + \|\Delta \phi(t)\|_{H^2}^2 + \|\nabla W(t)\|_{H^2}^2,
E(t) &= P(t) + \int_0^t Q(s) \, ds.
\end{align*}
$$

As for the Hall equations, constants that depend on $P(0)$ are not specified each time when we state our results, and we will use $E_0$ in common.
Theorem 1.10. Let \((\nabla \psi_0, Z_0, \nabla \phi_0, W_0) \in H^2\). There exists \(T^* = \mathcal{T}(\mathcal{E}_0) > 0\) such that there exists a unique solution of (1.5) with \(\mathcal{E}(T^*) < \infty\). Moreover, the maximal existence time \(T^* < \infty\) if and only if
\[
\lim_{T \to T^*} \int_0^T \| \nabla Z(t) \|_{L^q}^q \, dt = \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty.
\] (1.28)

Remark 1.5. We emphasize that the blow-up criterion is specified only in terms of \(Z\) even when the fluid part enters. This is the same condition obtained in [36, Theorem 2.1]. We expect the blow-up condition (1.14) can be derived to (1.5) and so we have the same condition of [36, Theorem 2.2]. But, we do not proceed to derive the condition [36, Theorem 2.3].

In Sect. 7, we will derive inequalities similar to (1.16) but the smallness condition is expressed more complicated by
\[
\epsilon_4 = \| \nabla \psi_0 \|_{H^1}^2 + \| Z_0 \|_{H^1}^2 + \| \nabla \phi_0 \|_{H^1}^2 + \| W_0 \|_{H^1}^2, \quad C \epsilon_4 < 1.
\] (1.29)

Compared to Theorem 1.4, we need to modify the smallness condition as (1.29) because (1.5) does not have a scaling-invariant property. Suppose that \((u, B) = 0\) and \((u = 0, B)\) solves (1.3), respectively. Then, the same is true for rescaled functions: \(u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)\) and \(B_\lambda(t, x) = \lambda^2 B(\lambda^2 t, \lambda x)\) accordingly. So, \(u\) and \(B\) have different scaling. Since (1.3) and so (1.5) include both \(u\) and \(B\), the smallness condition is determined by a combination the scaling invariant quantities of \(u\) and \(B\).

Theorem 1.11. Let \((\nabla \psi_0, Z_0, \nabla \phi_0, W_0) \in H^2\) which satisfies (1.29). Then, we can take \(T = \infty\) in Theorem 1.10. \((\nabla \psi_0, Z_0, \nabla \phi_0, W_0) \in L^1\) in addition, \((\Delta \psi, \nabla Z, \Delta \phi, \nabla W)\) decay in time as follows
\[
\| \Delta \psi(t) \|_{L^2} + \| \nabla Z(t) \|_{L^2} + \| \Delta \phi(t) \|_{L^2} + \| \nabla Z(t) \|_{L^2} \leq \frac{\mathcal{E}_0}{1 + t},
\] (1.30)

The decay rate (1.30) can be easily derived by using (1.26) and the argument in Sect. 3.3, we will skip the process of proving (1.30).

Remark 1.6. After this work was completed, we were informed of the work [17] where the \(2 \frac{1}{2}\) dimensional Hall MHD is also considered. In [17, Theorem 2.6], weak-strong uniqueness is established with \(J = \text{curl } B \in L^2_t \tilde{H}^{1 \frac{1}{2}}\). Since the Hall term is dominant, we expect that Theorem 1.2 also holds for (1.5) which is the same condition as \(J = \text{curl } B \in L^2_t \tilde{H}^{1 \frac{1}{2}}\) in terms of the scaling-invariance of \(B: B(t, x) \rightarrow B_\lambda(t, x) = \lambda^2 B(\lambda^2 t, \lambda x)\). There are two smallness conditions to obtain unique global-in-time solutions in [17, Theorem 2.5]. We can deduce from (7.5) that (1.29) can be used to find a unique global solutions with initial data in \(H^1\). So, the second condition in [17, Theorem 2.5] is similar to (1.29), but we do not have a smallness condition like the first one in [17, Theorem 2.5] because we do not know the advantage of the equation of \(u - J\).

Remark 1.7. We compare Theorems 1.10 and 1.11 with [18,36]. In [18], the following \(2 \frac{1}{2}\) dimensional Hall MHD are investigated:
\[
\begin{array}{l}
\begin{align*}
\psi_t + u \cdot \nabla \psi - B \cdot \nabla B + \nabla \bar{p} = \mu_1 u_{xx} + \mu_2 u_{yy}, \\
B_t + u \cdot \nabla B - B \cdot \nabla u + \text{curl} (\text{curl } B) = \nu_1 B_{xx} + \nu_2 B_{yy}, \\
div u = 0, \\
div B = 0,
\end{align*}
\end{array}
\] (1.31a)

where \(u(t, x, y) = (u_1, u_2, u_3)(t, x, y)\) and \(B(t, x, y) = (B_1, B_2, B_3)(t, x, y)\). By assuming at least one of \((\mu_1, \mu_2)\) is zero, (1.31) is locally well-posed with large initial data in \(H^2\) (Theorems 1.1 and 1.2) and globally well-posed with small initial data in \(H^2\) (Theorems 1.3 and 1.4). So the results in [18] are better than our results. But, we emphasize that we have more results when the Hall equations are considered. Compared to Theorem 1.11 where we assume the smallness of initial data, the global well-posedness with large initial data is established in [36, Theorem 2.4] when \(\nu \Delta B\) is replaced by \(\nu_1 \Lambda^3(B_1, B_2, 0) + \nu_2 \Lambda^2(0, 0, B_3)\).
1.7.1. Organization of the paper. There are several results in this paper: to make it easy to find where the proofs of our results are, we list them as follows.

- The Hall Eq. (1.6)
  1. Section 3: Strong solutions (Proof of Theorem 1.3, Theorems 1.2, 1.4)
  2. Section 4: Improved the decay rate of $\psi$ (Proof of Theorem 1.5)
  3. Section 5: Asymptotic behaviors of $(\psi, Z)$ (Proof of Theorem 1.6)
  4. Section 6: Perturbation around harmonic functions (Proof of Theorem 1.7, Theorem 1.8)

- Section 7: Strong solutions (1.5) (Proof of Theorem 1.10, Theorem 1.11)

2. Preliminaries

All constants will be denoted by $C$ and we follow the convention that such constants can vary from expression to expression and even between two occurrences within the same expression.

We here provide some inequalities in 2D:

$$
\|f\|_{L^p} \leq C(p) \|f\|_{L^2}^{\frac{2}{p}} \|\nabla f\|_{L^2}^{1-\frac{2}{p}}, \quad 2 < p < \infty \quad (2.1a)
$$

$$
\|f\|_{L^\infty} \leq C \|f\|_{L^2}^\frac{1}{2} \|\Delta f\|_{L^2}^\frac{1}{2}. \quad (2.1b)
$$

We also use the following inequalities which hold in any dimensions:

$$
\|\nabla f\|_{L^2} \leq \|f\|_{L^2}^{\frac{1}{2}} \|\Delta f\|_{L^2}^{\frac{1}{2}}, \quad \|\nabla^2 f\|_{L^p} \leq C(p) \|\Delta f\|_{L^p}, \quad 1 < p < \infty. \quad (2.2)
$$

We now recall the commutator $[f, g] = \nabla f \cdot \nabla^\perp g = f_x g_y - f_y g_x$. Then, the commutator has the following properties:

$$
[f, f] = 0, \quad [f, g] = -[g, f] \quad (2.3a)
$$

$$
\Delta [f, g] = [\Delta f, g] + [f, \Delta g] + 2[f_x, g_x] + 2[f_y, g_y], \quad (2.3b)
$$

$$
\int f [f, g] = 0, \quad (2.3c)
$$

$$
\int [f, g, h] = \int [g, h, f]. \quad (2.3d)
$$

We will use (2.1)–(2.3) repeatedly when proving our results and we will not refer them every time when it is obvious to use them.

We also use the following lemma, when bounding solutions globally-in-time with small initial data, which can be proved by an iteration method.

**Lemma 2.1.** Let $f$ and $g$ be nonnegative functions, with $f$ being differentiable and $g$ being integrable on $(0, \infty)$, satisfying

$$
f' + g \leq Cf g.
$$

If $f(0)$ is sufficiently small such that $Cf(0) = \epsilon < 1$, we have

$$
f(t) + (1 - \epsilon) \int_0^t g(s) ds \leq f(0)
$$

for all $t > 0$.

2.1. How to prove our results

To prove our results, we may use a fixed point argument. But, in principle, the calculations used to derive a priori estimates are easily applied to the fixed point argument. So, we only provide a priori estimates for the existence part and show the uniqueness.
3. Proofs of Theorem 1.3, Theorems 1.2, and 1.4

In this section we establish the local-in-time existence of unique strong solutions of (1.6). The analysis given in this section will apply to (1.5) in Sects. 6 and 7. Since the computations used to prove Theorem 1.3 can be used to prove Theorem 1.2, we begin with Theorem 1.3.

3.1. Proof of Theorem 1.3

We first recall (1.6):

\begin{align}
\psi_t - \Delta \psi &= [\psi, Z], \\
Z_t - \Delta Z &= [\Delta\psi, \psi].
\end{align}

3.1.1. A Priori Estimates. We multiply (3.1a) by $-\Delta\psi$, (3.1b) by $Z$, and integrate over $\mathbb{R}^2$. By using (2.3d), we have

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \psi\|^2_{L^2} + \|Z\|^2_{L^2} \right) + \|\Delta \psi\|^2_{L^2} + \|\nabla Z\|^2_{L^2} = - \int \Delta \psi [\psi, Z] + \int Z [\Delta \psi, \psi] = 0.
\end{equation}

We next multiply (3.1a) by $\Delta^2 \psi$, (3.1b) by $-\Delta Z$ and integrate over $\mathbb{R}^2$. Then,

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|\Delta \psi\|^2_{L^2} + \|\nabla Z\|^2_{L^2} \right) + \|\nabla \Delta \psi\|^2_{L^2} + \|\Delta Z\|^2_{L^2} = \int \Delta^2 \psi [\psi, Z] - \int \Delta Z [\Delta \psi, \psi].
\end{equation}

Since

\begin{equation}
\int \Delta^2 \psi [\psi, Z] - \int \Delta Z [\Delta \psi, \psi] = 2 \int \Delta \psi ([\psi_x, Z_x] + [\psi_y, Z_y]) \leq C \|\nabla^2 Z\|_{L^2} \|\nabla^2 \psi\|^2_{L^4},
\end{equation}

we obtain

\begin{equation}
\frac{d}{dt} \left( \|\Delta \psi\|^2_{L^2} + \|\nabla Z\|^2_{L^2} \right) + \|\nabla \Delta \psi\|^2_{L^2} + \|\Delta Z\|^2_{L^2} \leq C \|\Delta \psi\|^2_{L^2} \|\nabla \Delta \psi\|^2_{L^2}.
\end{equation}

We finally multiply (3.1a) by $-\Delta^3 \psi$, (3.1b) by $\Delta^2 Z$ and integrate over $\mathbb{R}^2$. By noticing, as (3.3), the cancellation of the terms having the highest order derivative in the first equality below, we have

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \Delta \psi\|^2_{L^2} + \|\Delta Z\|^2_{L^2} \right) + \|\Delta^2 \psi\|^2_{L^2} + \|\nabla \Delta Z\|^2_{L^2} = - \int \Delta^3 \psi [\psi, Z] + \int \Delta^2 Z [\Delta \psi, \psi]
= - \int \Delta^2 \psi [\Delta \psi, Z] - 2 \int \Delta^2 \psi ([\psi_x, Z_x] + [\psi_y, Z_y]) - 2 \int \Delta \psi ([\psi_x, \Delta Z_x] + [\psi_y, \Delta Z_y])
= (I) + (II) + (III).
\end{equation}

We bound each term on the right-hand side as follows. By using the definition of the commutator,

\begin{equation}
(I) = - \int (\nabla^\perp Z \cdot \nabla \Delta \psi) \Delta^2 \psi = \int (\nabla^\perp Z \cdot \nabla \Delta \psi) \cdot \nabla \Delta \psi + \int (\nabla^\perp Z \cdot \nabla \nabla \Delta \psi) \nabla \Delta \psi
= \int (\nabla \nabla^\perp Z \cdot \nabla \Delta \psi) \cdot \nabla \Delta \psi - \frac{1}{2} \int \nabla \nabla^\perp Z |\nabla \Delta \psi|^2 \leq C \int |\nabla^2 Z| \|\nabla^2 \psi\|^2 \leq C \|\nabla^2 Z\|_{L^2} \|\nabla^3 \psi\|^2_{L^4}
\end{equation}

\begin{equation}
\leq C \|\Delta Z\|^2_{L^2} \|\nabla \Delta \psi\|^2_{L^2} + \frac{1}{4} \|\Delta^2 \psi\|^2_{L^2}.
\end{equation}

By moving one derivative in $\Delta^2 \psi$ to $([\psi_x, Z_x] + [\psi_y, Z_y])$ and by using (2.3d),

\begin{equation}
(II) + (III) \leq C \int |\nabla^2 Z| \|\nabla^3 \psi\|^2 + C \int |\nabla^2 \psi| \|\nabla^3 \psi\| |\nabla^3 Z|
\end{equation}

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with the second term estimated by
\[
\int |\nabla^2 \psi| |\nabla^3 \psi| |\nabla^3 Z| \leq \|\nabla^2 \psi\|_{L^4} \|\nabla^3 \psi\|_{L^4} \|\nabla^3 Z\|_{L^2} \\
\leq C \|\Delta \psi\|_{L^2}^2 \|\nabla \Delta \psi\|_{L^2} + \frac{1}{4} \|\Delta^2 \psi\|_{L^2}^2 + \frac{1}{2} \|\nabla \Delta Z\|_{L^2}^2 .
\]
With these estimates, we have
\[
\frac{d}{dt} \left( \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) + \|\Delta^2 \psi\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2 \\
\leq C \|\Delta Z\|_{L^2}^2 \|\nabla \Delta \psi\|_{L^2}^2 + C \|\Delta \psi\|_{L^2}^2 \|\nabla \Delta \psi\|_{L^2}^4 . \tag{3.7}
\]
By (3.2), (3.4), and (3.7), we derive the following inequality:
\[
\frac{d}{dt} (1 + M) + N \leq CM^2 + CM^3 \leq C(1 + M)^3 , \tag{3.8}
\]
where \(M\) and \(N\) are defined in (1.12). From this, we deduce
\[
M(t) \leq \sqrt{\frac{(1 + M(0))^2}{1 - 2Ct(1 + M(0))^2}} - 1 \quad \text{for all } t \leq T^* < \frac{1}{2C(1 + M(0))^2} . \tag{3.9}
\]
Integrating (3.8) and using (3.9), we finally derive \(E(T^*) < \infty\).

3.1.2. Uniqueness. Suppose there are two solutions \((\psi_1, Z_1)\) and \((\psi_2, Z_2)\). Let \(\psi = \psi_1 - \psi_2\) and \(Z = Z_1 - Z_2\). By subtracting the equations for \((\psi_1, Z_1)\) and \((\psi_2, Z_2)\), we have
\[
\psi_t - \Delta \psi = [\psi_1, Z] + [\psi, Z_2], \tag{3.10a}
Z_t - \Delta Z = [\Delta \psi, \psi_1] + [\Delta \psi_2, \psi]. \tag{3.10b}
\]
From this, we see that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) + \|\Delta \psi\|_{L^2}^2 + \|\nabla \Delta \psi\|_{L^2}^2 \\
= - \int \Delta \psi_t \psi + \int \Delta \psi \psi_1 + \int \Delta \psi \psi_2 + \int Z_t[Z, \psi_1] + \int Z[Z, \Delta \psi] + \int Z[Z, \Delta \psi_2] \\
= - \int \Delta \psi_t \psi_1 + \int \Delta \psi \psi_2 + \int Z_t[Z, \psi_1] - \int \Delta \psi_t \psi_2 + \int Z[Z, \Delta \psi_2, \psi] \\
\leq C \left( \|\nabla Z_1\|_{L^2}^2 \|\Delta Z_1\|_{L^2}^2 + \|\Delta \psi_1\|_{L^2}^2 \|\nabla \Delta \psi_1\|_{L^2}^2 \right) \|\nabla \psi\|_{L^2}^2 + \|\nabla \psi_1\|_{L^2}^2 \|\nabla \Delta \psi_1\|_{L^2}^2 \\
\leq C \left( \|\nabla Z_1\|_{L^2}^2 \|\Delta Z_1\|_{L^2}^2 + \|\Delta \psi_1\|_{L^2}^2 \|\nabla \Delta \psi_1\|_{L^2}^2 \right) \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) .
\]
which gives
\[
\frac{d}{dt} \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) \leq C \left( \|\nabla Z_1\|_{L^2}^2 \|\Delta Z_1\|_{L^2}^2 + \|\Delta \psi_1\|_{L^2}^2 \|\nabla \Delta \psi_1\|_{L^2}^2 \right) \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) .
\]
Since \(\|\nabla Z_1\|_{L^2}^2 \|\Delta Z_1\|_{L^2}^2 + \|\Delta \psi_1\|_{L^2}^2 \|\nabla \Delta \psi_1\|_{L^2}^2\) is integrable on \([0, T^*)\), the uniqueness follows using Gronwall’s lemma.

3.1.3. Blow-up Criterion. To obtain (1.13), we first bound the right-hand side of (3.3) by
\[
\left| \int \Delta \psi ([\psi_x, Z_x] + [\psi_y, Z_y]) \right| \leq C \|\nabla Z\|_{L_p} \|\nabla^2 \psi\|_{L_q} \|\nabla^3 \psi\|_{L^2} , \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}.
\]
By (2.1a), we have
\[
\|\nabla Z\|_{L_p} \|\Delta \psi\|_{L_q} \|\nabla^3 \psi\|_{L^2} \leq C \|\nabla Z\|_{L_p} \|\nabla^2 \psi\|_{L^2}^{\frac{2}{3}} \|\nabla \Delta \psi\|_{L^2}^{\frac{2}{3}} \|\nabla \Delta \psi\|_{L^2}^{\frac{2}{3}} \\
\leq C \|\nabla Z\|_{L_p} \|\Delta \psi\|_{L^2}^2 + \|\nabla \Delta \psi\|_{L^2}^2 . \tag{3.12}
\]
So, we can rewrite (3.3) as
\[
\frac{d}{dt} \left( \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \right) + \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \leq C \|\nabla Z\|_{L^p}^q \left( \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \right).
\]
Integrating this in time by using Gronwall’s inequality, we have
\[
\|\Delta \psi(t)\|_{L^2}^2 + \|\nabla Z(t)\|_{L^2}^2 + \int_0^t \left( \|\nabla \Delta \psi(s)\|_{L^2}^2 + \|\Delta Z(s)\|_{L^2}^2 \right) ds 
\leq \mathcal{E}_0 \exp \left[ C \int_0^t \|\nabla Z(s)\|_{L^p}^q ds \right].
\] (3.13)
We then integrate (3.7) in time, without including \( \|\Delta^2 \psi\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2 \), to obtain
\[
\|\nabla \Delta \psi(t)\|_{L^2}^2 + \|\Delta Z(t)\|_{L^2}^2 \leq \mathcal{E}_0 \exp \left[ C \int_0^t \left( \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta \psi\|_{L^2}^2 \|\Delta Z\|_{L^2}^2 \right) ds \right].
\]
By using (3.13) for the second term in the integrand of the right-hand side, we obtain
\[
\|\nabla \Delta \psi(t)\|_{L^2}^2 + \|\Delta Z(t)\|_{L^2}^2 \leq \mathcal{E}_0 \exp \left[ \mathcal{C} B(t) + \mathcal{C} B^2(t) \right], \quad B(t) = \int_0^t \|\nabla Z(s)\|_{L^p}^q ds.
\] (3.14)
This completes the Proof of Theorem 1.3.

3.2. Proof of Theorem 1.2

The Proof of Theorem 1.2 is very similar to the one in Sects. 3.1.2 and 3.1.3. Let \( B_1 = (\nabla^+ \psi_1, Z_1) \) and \( B_2 = (\nabla^+ \psi_2, Z_2) \) be the two weak solutions of (1.6). Let \( \psi = \psi_1 - \psi_2 \) and \( Z = Z_1 - Z_2 \). By (3.11) and (2.3d), and by using (3.12), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) + \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 = - \int \Delta \psi [\psi, Z_2] + \int \Delta \psi_2 [\psi, Z] 
\leq C \|\nabla Z_2\|_{L^p} \|\nabla \psi\|_{L^q} \|\Delta \psi\|_{L^2} + C \|\Delta \psi_2\|_{L^p} \|\nabla \psi\|_{L^q} \|\nabla Z\|_{L^2} 
\leq C \left( \|\nabla Z_2\|_{L^p}^q + \|\Delta \psi_2\|_{L^p}^q \right) \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) + \frac{1}{2} \|\Delta \psi\|_{L^2}^2 + \frac{1}{2} \|\nabla Z\|_{L^2}^2
\]
and so we obtain
\[
\frac{d}{dt} \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) + \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \leq C \left( \|\nabla Z_2\|_{L^p}^q + \|\Delta \psi_2\|_{L^p}^q \right) \left( \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 \right).
\]
By Gronwall inequality, we complete the Proof of Theorem 1.2.

3.3. Proof of Theorem 1.4

3.3.1. A Priori Estimates. We now show that the strong solutions provided by Theorem 1.3 are in fact defined for all \( t > 0 \) under the smallness condition (1.17). We first rewrite (3.4) as
\[
\frac{d}{dt} \left( \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \right) + \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \leq C S(t) \left( \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right),
\]
where \( S(t) = \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \). Let \( \epsilon_1 = \|\Delta \psi_0\|_{L^2}^2 + \|\nabla Z_0\|_{L^2}^2 \). By Lemma 2.1, if \( C \epsilon_1 < 1 \),
\[
\|\Delta \psi(t)\|_{L^2}^2 + \|\nabla Z(t)\|_{L^2}^2 + \left( 1 - C \epsilon_1 \right) \int_0^t \left( \|\nabla \Delta \psi(s)\|_{L^2}^2 + \|\Delta Z(s)\|_{L^2}^2 \right) ds \leq \epsilon_1
\] (3.15)
for all \( t > 0 \). We next proceed to bound (3.7) by estimating the two terms on the right-hand side of (3.6) in a different way. From the last expression of (3.5), we obtain

\[
C \| \nabla^2 Z \|_{L^2} \| \nabla^3 \psi \|_{L^2}^2 \leq C \| \Delta Z \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2}^2 + \frac{1}{2} \| \Delta^2 \psi \|_{L^2}^2
\leq C \| \nabla Z \|_{L^2}^2 \| \nabla \Delta Z \|_{L^2}^2 + C \| \Delta \psi \|_{L^2}^2 \| \Delta^2 \psi \|_{L^2}^2 + \frac{1}{2} \| \Delta^2 \psi \|_{L^2}^2. \tag{3.16}
\]

By (2.1b), we bound the second term on the right-hand side of (3.6) as

\[
\| \nabla^2 \psi \|_{L^\infty} \| \nabla^3 \psi \|_{L^2} \| \nabla^3 Z \|_{L^2} \leq C \| \Delta \psi \|_{L^2}^2 \| \Delta^2 \psi \|_{L^2}^2 \| \nabla^3 \psi \|_{L^2} \| \nabla^3 Z \|_{L^2} \leq C \| \Delta \psi \|_{L^2}^2 \| \Delta^2 \psi \|_{L^2}^2 + \frac{1}{2} \| \nabla^3 Z \|_{L^2}^2. \tag{3.17}
\]

So, (3.7) is replaced with

\[
\frac{d}{dt} \left( \| \nabla \Delta \psi \|_{L^2}^2 + \| \Delta Z \|_{L^2}^2 \right) + \| \Delta^2 \psi \|_{L^2}^2 + \| \nabla \Delta Z \|_{L^2}^2 \leq CS(t) \left( \| \Delta^2 \psi \|_{L^2}^2 + \| \nabla \Delta Z \|_{L^2}^2 \right). \tag{3.18}
\]

Then (3.15) gives

\[
\| \nabla \Delta \psi(t) \|_{L^2}^2 + \| \Delta Z(t) \|_{L^2}^2 + (1 - C \epsilon_1) \int_0^t \left( \| \Delta^2 \psi(s) \|_{L^2}^2 + \| \nabla \Delta Z(s) \|_{L^2}^2 \right) ds
\leq \| \nabla \Delta \psi_0 \|_{L^2}^2 + \| \Delta Z_0 \|_{L^2}^2
\]
for all \( t > 0 \). This completes the first part of Theorem 1.4.

3.3.2. Decay Rates. To conclude this Section and the Proof of Theorem 1.4, we now prove the decay rates (1.18) in Theorem 1.4. We first write (3.4) as

\[
\frac{d}{dt} \left( \| \Delta \psi \|_{L^2}^4 + \| \nabla Z \|_{L^2}^2 \right) + (1 - C \epsilon_1) \| \nabla \Delta \psi \|_{L^2}^2 + (1 - C \epsilon_1) \| \Delta Z \|_{L^2}^2 \leq 0. \tag{3.19}
\]

By (2.2) and (1.10), we have

\[
\| \Delta \psi \|_{L^2}^4 \leq C \| \nabla \psi \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2}^2 \leq \frac{\epsilon_0}{1 + t} \| \nabla \Delta \psi \|_{L^2}^2,
\]
\[
\| \nabla Z \|_{L^2}^4 \leq C \| Z \|_{L^2}^2 \| \Delta Z \|_{L^2}^2 \leq \frac{\epsilon_0}{1 + t} \| \Delta Z \|_{L^2}^2.
\]

Then, (3.19) becomes

\[
\frac{d}{dt} \left( \| \Delta \psi \|_{L^2}^2 + \| \nabla Z \|_{L^2}^2 \right) + \epsilon_0 (1 + t) \left( \| \Delta \psi \|_{L^2}^2 + \| \nabla Z \|_{L^2}^2 \right)^2 \leq 0.
\]

By solving this ODE, we derive the following inequality for \( t > 0 \):

\[
\| \Delta \psi(t) \|_{L^2}^2 + \| \nabla Z(t) \|_{L^2}^2 \leq \frac{2 \| \Delta \psi_0 \|_{L^2}^2 + 2 \| \nabla Z_0 \|_{L^2}^2}{2 + \epsilon_0 \left( \| \Delta \psi_0 \|_{L^2}^2 + \| \nabla Z_0 \|_{L^2}^2 \right) (1 + t)^2}.
\]

We next write (3.18) as

\[
\frac{d}{dt} \left( \| \nabla \Delta \psi \|_{L^2}^2 + \| \Delta Z \|_{L^2}^2 \right) + (1 - C \epsilon_1) \| \Delta^2 \psi \|_{L^2}^2 + (1 - C \epsilon_1) \| \nabla \Delta Z \|_{L^2}^2 \leq 0.
\]

By (2.2) and (1.10), we have

\[
\| \nabla \Delta \psi \|_{L^2}^2 \leq C \| \nabla \psi \|_{L^2}^2 \| \Delta^2 \psi \|_{L^2}^2 \leq \frac{\epsilon_0}{\sqrt{1 + t}} \| \Delta^2 \psi \|_{L^2}^2,
\]
\[
\| \Delta Z \|_{L^2}^3 \leq C \| Z \|_{L^2} \| \nabla \Delta Z \|_{L^2} \leq \frac{\epsilon_0}{\sqrt{1 + t}} \| \nabla \Delta Z \|_{L^2}.
\]
So, we obtain
\[
\frac{d}{dt} \left( \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) + \mathcal{E}_0 \sqrt{1 + t} \left( \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right)^{\frac{3}{2}} \leq 0.
\]
From this, we derive the following inequality:
\[
\|\nabla \Delta \psi(t)\|_{L^2}^2 + \|\Delta Z(t)\|_{L^2}^2 \leq \frac{36 \|\nabla \Delta \psi_0\|_{L^2}^2 + 36 \|\Delta Z_0\|_{L^2}^2}{(6 + \mathcal{E}_0 \sqrt{\|\nabla \Delta \psi_0\|_{L^2}^2 + \|\Delta Z_0\|_{L^2}^2 (1 + t)^{\frac{3}{2}}})^2}
\]
and thus concluding the Proof of Theorem 1.4.

4. Proof of Theorem 1.5

In this section, we want to improve the decay rate of \( \psi \) by using Theorem 1.4. We first recall the equation of \( \psi \):
\[
\psi_t + \nabla^\perp Z : \nabla \psi - \Delta \psi = 0 \tag{4.1}
\]
which is a dissipative transport equation with a fast decaying coefficient \( \nabla^\perp Z \). We begin with the \( L^1 \) bound of \( \psi \):
\[
\|\psi(t)\|_{L^1} \leq \|\psi_0\|_{L^1}. \tag{4.2}
\]
By applying Fourier splitting method in [7], we also obtain the \( L^2 \) bound:
\[
\|\psi(t)\|_{L^2} \leq \frac{\mathcal{E}_0}{\sqrt{1 + t}}. \tag{4.3}
\]
We now test \( \frac{\psi}{t} \) to (4.1). Then, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \frac{\|\psi\|_{L^2}^2}{t} + \frac{\|\nabla \psi\|_{L^2}^2}{t} \right) = 0. \tag{4.4}
\]
By (2.1a) and (4.2), we have
\[
\|\psi\|_{L^2}^4 \leq C \|\psi\|_{L^1}^2 \|\nabla \psi\|_{L^2}^2 \leq \mathcal{E}_0 \|\nabla \psi\|_{L^2}^2
\]
and so (4.4) can be replaced with
\[
\frac{1}{2} \frac{d}{dt} \left( \frac{\|\psi\|_{L^2}^2}{t} \right) + \frac{t}{\mathcal{E}_0} \left( \frac{\|\psi\|_{L^2}^2}{t} \right)^2 \leq 0. \tag{4.5}
\]
On the other hand, testing \( -\Delta \psi \) to (4.1), we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_{L^2}^2 + \|\Delta \psi\|_{L^2}^2 \leq \|\nabla Z\|_{L^4} \|\nabla \psi\|_{L^4} \|\Delta \psi\|_{L^2} \leq C \|\nabla Z\|_{L^4}^4 \|\nabla \psi\|_{L^2}^2 + \frac{1}{2} \|\Delta \psi\|_{L^2}^2
\]
and so we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_{L^2}^2 + \frac{1}{2} \|\Delta \psi\|_{L^2}^2 \leq C \|\nabla Z\|_{L^4}^4 \|\nabla \psi\|_{L^2}^2. \tag{4.6}
\]
By (2.1a) and (1.18),
\[
\|\nabla Z(t)\|_{L^4}^4 \leq C \|\nabla Z(t)\|_{L^2}^2 \|\Delta Z(t)\|_{L^2}^2 \leq \frac{\mathcal{E}_0}{(1 + t)^5}.
\]
By taking \( t \) sufficiently large, which is expressed by \( t > t_0 \) for the rest of the Proof of Theorem 1.5, and by combining (4.5) and (4.6), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \frac{\|\psi\|_{L^2}^2}{t} + \|\nabla \psi\|_{L^2}^2 \right) + \frac{1}{t} \|\nabla \psi\|_{L^2}^2 + \|\Delta \psi\|_{L^2}^2 \leq 0.
\]
By (2.2) and (4.3), we have
\[ \|\nabla \psi\|^2_{L^2} \leq C \|\psi\|^2_{L^2} \|\Delta \psi\|^2_{L^2} \leq \frac{\mathcal{E}_0}{1+t} \|\Delta \psi\|^2_{L^2} \leq \frac{\mathcal{E}_0}{t} \|\Delta \psi\|^2_{L^2} \]
when \( t > t_0 \). So, we derive the following inequality:
\[ \frac{1}{2} \frac{d}{dt} \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \|\nabla \psi\|^2_{L^2} \right) + \frac{t}{\mathcal{E}_0} \left( \frac{\|\psi\|^2_{L^2}}{t} + \|\nabla \psi\|^2_{L^2} \right)^2 \leq 0. \quad (4.7) \]
We now solve this ODE to find
\[ \frac{\|\psi(t)\|^2_{L^2}}{t^2} + \|\nabla \psi(t)\|^2_{L^2} \leq \frac{\mathcal{E}_0 \left( \frac{\|\psi(0)\|^2_{L^2}}{t_0} + \|\nabla \psi(0)\|^2_{L^2} \right)}{\mathcal{E}_0 + \left( \frac{\|\psi(0)\|^2_{L^2}}{t_0} + \|\nabla \psi(0)\|^2_{L^2} \right) (t^2 - t_0^2)}. \]
Since \( \|\psi(t)\|_{H^1} \leq \|\psi_0\|_{H^1} \) for all \( t > 0 \) by Theorem 1.4, we obtain
\[ \|\nabla \psi(t)\|_{L^2} \leq \frac{\mathcal{E}_0}{1+t}. \quad (4.8) \]
By modifying (4.7) with the extra \( t \)-factor in the denominator, we have
\[ \frac{1}{2} \frac{d}{dt} \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \|\nabla \psi\|^2_{L^2} \right) + \frac{t^2}{\mathcal{E}_0} \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \|\nabla \psi\|^2_{L^2} \right)^2 \leq 0. \quad (4.9) \]
We now test \( \Delta^2 \psi \) to (4.1). Then, we have
\[ \frac{1}{2} \frac{d}{dt} \|\Delta \psi\|^2_{L^2} + \|\nabla \Delta \psi\|^2_{L^2} = \int \Delta \psi \Delta [\psi, Z] = \int \Delta \psi [\psi, \Delta Z] + 2 \int \Delta \psi (\psi_x Z_x + [\psi_y, Z_y]) = (I) + (II). \]
We first bound (I) as follows
\[ (I) \leq \|\Delta Z\|_{L^2} \|\nabla \psi\|_{L^\infty} \|\nabla \Delta \psi\|_{L^2} \leq \|\Delta Z\|^2_{L^2} \|\nabla \psi\|^2_{L^\infty} + \frac{1}{8} \|\nabla \Delta \psi\|^2_{L^2} \]
\[ \leq C \|\Delta Z\|^2_{L^2} \|\nabla \psi\|^2_{L^2} + \frac{1}{4} \|\nabla \Delta \psi\|^2_{L^2} \leq \frac{\mathcal{E}_0}{(1+t)^4} \|\nabla \psi\|^2_{L^2} + \frac{1}{4} \|\nabla \Delta \psi\|^2_{L^2}. \]
(II) is bounded by
\[ (II) \leq C \|\Delta Z\|_{L^2} \|\Delta \psi\|^2_{L^4} \leq C \|\Delta Z\|^2_{L^2} \|\Delta \psi\|^2_{L^2} + \frac{1}{4} \|\nabla \Delta \psi\|^2_{L^2} \]
\[ \leq \frac{\mathcal{E}_0}{(1+t)^2} \|\Delta \psi\|^2_{L^2} + \frac{1}{4} \|\nabla \Delta \psi\|^2_{L^2}. \]
So, we obtain
\[ \frac{d}{dt} \|\Delta \psi\|^2_{L^2} + \|\nabla \Delta \psi\|^2_{L^2} \leq \frac{\mathcal{E}_0}{t^2} \|\nabla \psi\|^2_{L^2} + \frac{\mathcal{E}_0}{t^2} \|\Delta \psi\|^2_{L^2} \quad (4.10) \]
when \( t > t_0 \). Then, (4.9) and (4.10) give
\[ \frac{d}{dt} \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \frac{\|\nabla \psi\|^2_{L^2}}{t} \right) + \mathcal{E}_0 t^2 \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \frac{\|\nabla \psi\|^2_{L^2}}{t} \right)^2 + \|\nabla \Delta \psi\|^2_{L^2} \leq 0. \]
By (2.2) and (4.8), we have
\[ \|\Delta \psi\|^2_{L^2} \leq C \|\nabla \psi\|^2_{L^2} \|\Delta \psi\|^2_{L^2} \leq \frac{\mathcal{E}_0}{(1+t)^2} \|\nabla \Delta \psi\|^2_{L^2} \leq \frac{\mathcal{E}_0}{t^2} \]
and we derive the following inequality: for \( t > t_0 \)
\[ \frac{d}{dt} \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \frac{\|\nabla \psi\|^2_{L^2}}{t} + \|\Delta \psi\|^2_{L^2} \right) + \mathcal{E}_0 t^2 \left( \frac{\|\psi\|^2_{L^2}}{t^2} + \frac{\|\nabla \psi\|^2_{L^2}}{t} + \|\Delta \psi\|^2_{L^2} \right)^2 \leq 0. \quad (4.11) \]
Let $I_0 = \frac{\|\psi(t_0)\|_{L^2}^2}{t_0} + \frac{\|\nabla \psi(t_0)\|_{L^2}^2}{t_0} + \|\Delta \psi(t_0)\|_{L^2}^2$. By solving (4.11), we have

$$
\frac{\|\psi(t)\|_{L^2}^2}{t^2} + \frac{\|\nabla \psi(t)\|_{L^2}^2}{t} + \|\Delta \psi(t)\|_{L^2}^2 \leq \frac{3I_0}{3 + 3(t^3 - t_0^3)}.
$$

(4.12)

Since $\|\psi(t)\|_{H^2} \leq \|\psi_0\|_{H^2}$ for all $t > 0$ by Theorem 1.4, we obtain

$$
\|\Delta \psi(t)\|_{L^2} \leq \frac{E_0}{(1 + t)^{3/2}}
$$

which complete the Proof of Theorem 1.5.

5. Proof of Theorem 1.6

The purpose of this section is to establish the asymptotic behavior of $(\psi, Z)$ as $t \to \infty$. Let

$$
\Gamma(t, x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}
$$

be the two dimensional heat kernel. We first notice that we have the $L^p$ estimates of $\Gamma$ in two dimensions: for $1 \leq p \leq r \leq \infty$

$$
\|\Gamma(t) * f\|_{L^r} \leq C(p, r)t^{-\left(\frac{1}{p} - \frac{1}{2}\right)}\|f\|_{L^p},
$$

$$
\|\nabla \Gamma(t) * f\|_{L^r} \leq C(p, r)t^{-\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}}\|f\|_{L^p},
$$

$$
\|\nabla^2 \Gamma(t) * f\|_{L^r} \leq C(p, r)t^{-\left(\frac{1}{p} - \frac{1}{2}\right) - 1}\|f\|_{L^p},
$$

(5.1)

where $*$ is the convolution in the space variables. We also observe that constant multiples of $\Gamma$ are solutions of (1.6) because $\Gamma$ and $\Gamma_t$ are radial functions and so

$$
\nabla^\perp \Gamma \cdot \nabla \Gamma = 0, \quad \nabla^\perp \Gamma \cdot \nabla \Delta \Gamma = \nabla^\perp \Gamma \cdot \nabla \Gamma_t = 0.
$$

(5.2)

We are now in position to prove Theorem 1.6. Let $\tilde{\psi} = \psi - \gamma \Gamma$ and $\tilde{Z} = Z - \eta \Gamma$, where $\gamma$ and $\eta$ are defined in (1.20c). By using (5.2), we have

$$
\tilde{\psi}_t - \Delta \tilde{\psi} = -\nabla^\perp Z \cdot \nabla \psi, \quad \tilde{Z}_t - \Delta \tilde{Z} = -\nabla^\perp \psi \cdot \nabla \Delta \psi.
$$

(5.3)

So, there are two types of the integral forms of $(\psi, Z)$ from (1.6) and (5.3):

$$
\psi(t) = \Gamma(t) * \psi_0 - \int_0^t \Gamma(t-s) * (\nabla^\perp Z \cdot \nabla \psi)(s) ds,
$$

(5.4a)

$$
Z(t) = \Gamma(t) * Z_0 - \int_0^t \Gamma(t-s) * (\nabla^\perp \psi \cdot \nabla \Delta \psi)(s) ds
$$

(5.4b)

and $\psi = \tilde{\psi} + \gamma \Gamma$ and $Z = \tilde{Z} + \eta \Gamma$ with

$$
\tilde{\psi}(t) = \Gamma(t) * (\psi_0 - \gamma \delta_0) - \int_0^t \Gamma(t-s) * (\nabla^\perp Z \cdot \nabla \psi)(s) ds,
$$

(5.5a)

$$
\tilde{Z}(t) = \Gamma(t) * (Z_0 - \eta \delta_0) - \int_0^t \Gamma(t-s) * (\nabla^\perp \psi \cdot \nabla \Delta \psi)(s) ds,
$$

(5.5b)

where $\delta_0$ is the Dirac delta function supported at the origin. Since the time integrals of (5.4) and (5.5) are same, the only differences in the asymptotic behaviors are given by the linear parts. In particular, we need (1.20c) to handle the linear part of (5.5). We here estimate $(\tilde{\psi}, \tilde{Z})$ which also give the estimation of $(\psi, Z)$. 

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We now estimate $\tilde{\psi}$ in $L^\infty$ with $\nabla^\perp Z \cdot \nabla \psi = \text{div} \left( \nabla^\perp Z \psi \right)$:

$$
\left\| \tilde{\psi}(t) \right\|_{L^\infty} \leq \left\| \Gamma(t) * (\psi_0 - \gamma \delta_0) \right\|_{L^\infty} + \int_0^t \left\| \text{div} \Gamma(t-s) * (\nabla^\perp Z \psi)(s) \right\|_{L^\infty} \, ds \\
+ \int_{\frac{t}{2}}^t \left\| \text{div} \Gamma(t-s) * (\nabla^\perp Z \psi)(s) \right\|_{L^\infty} \, ds = (I) + (II) + (III),
$$

We begin with (I):

$$(I) = \left\| \int_{\mathbb{R}^2} (\Gamma(t, x-y) - \Gamma(t, x)) \psi_0(y) \, dy \right\|_{L^\infty} \leq \left\| \nabla \Gamma(t) \right\|_{L^\infty} \| \psi \|_{L^1} \leq \frac{\mathcal{E}_0}{t^{\frac{3}{2}}}.
$$

To bound (II), we use Theorem 1.4, (4.3), and (5.1):

$$(II) \leq C \int_{\frac{t}{2}}^t (t-s)^{-\frac{5}{2}} \left\| \nabla^\perp Z(s) \psi(s) \right\|_{L^1} \, ds \leq C \int_0^t (t-s)^{-\frac{5}{2}} \| \nabla Z(s) \|_{L^2} \| \psi(s) \|_{L^2} \, ds \\
\leq \frac{\mathcal{E}_0}{t^{\frac{3}{2}}} \int_{\frac{t}{2}}^t \frac{1}{(s+1)^{\sqrt{s+1}}} \, ds \leq \frac{\mathcal{E}_0}{t^{\frac{3}{2}}}.
$$

We also bound (III) by using Theorems 1.4, 1.5, (4.3), and (5.1):

$$(III) \leq C \int_{\frac{t}{2}}^t (t-s)^{-\frac{5}{2}} \left\| \nabla^\perp Z(s) \psi(s) \right\|_{L^3} \, ds \leq C \int_0^t (t-s)^{-\frac{5}{2}} \| \nabla Z(s) \|_{L^6} \| \psi(s) \|_{L^6} \, ds \\
\leq C \int_{\frac{t}{2}}^t (t-s)^{-\frac{5}{2}} \| \nabla Z(s) \|_{L^2}^\frac{1}{2} \| \Delta Z(s) \|_{L^2}^\frac{1}{2} \| \psi(s) \|_{L^2}^\frac{1}{2} \| \nabla \psi(s) \|_{L^2}^\frac{1}{2} \, ds \\
\leq \frac{\mathcal{E}_0}{t^{\frac{3}{2}}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{5}{2}} s^{-\frac{1}{2}} s^{-\frac{3}{2}} s^{-\frac{1}{2}} \, ds \leq \frac{\mathcal{E}_0}{t^{\frac{3}{2}}}.
$$

Taking all these bounds into account, we find two types of asymptotic behaviors of $\psi$

$$
\psi(t, x) = \gamma \Gamma(t, x) + \tilde{\psi}(t, x) = \gamma \Gamma(t, x) + O(t^{-3/2}), \\
\psi(t, x) = \gamma \Gamma(t, x) + \tilde{\psi}(t, x) = \Gamma(t) * \psi_0 + O(t^{-3/2}).
$$

We now derive the same kind of estimates for $\tilde{Z}$ in $L^\infty$ with $\nabla^\perp Z \cdot \nabla \Delta \psi = \text{div} \left( \nabla^\perp \psi \Delta \psi \right)$:

$$
\left\| \tilde{Z}(t) \right\|_{L^\infty} \leq \left\| \Gamma(t) * (Z_0 - \gamma \delta_0) \right\|_{L^\infty} + \int_0^t \left\| \text{div} \Gamma(t-s) * (\nabla^\perp \psi \Delta \psi)(s) \right\|_{L^\infty} \, ds \\
+ \int_{\frac{t}{2}}^t \left\| \text{div} \Gamma(t-s) * (\nabla^\perp \psi \Delta \psi)(s) \right\|_{L^\infty} \, ds = (IV) + (V) + (VI).
$$

(IV) is bounded exactly as (I):

$$(IV) \leq \| \nabla \Gamma(t) \|_{L^\infty} \| \langle y \rangle Z \|_{L^1} \leq \frac{\mathcal{E}_0}{t^{\frac{3}{2}}}.
$$
Before bounding (V), we rewrite $\text{div} \Gamma(t - s) * (\nabla^\perp \psi \Delta \psi)$ as
\[
\text{div} \Gamma(t - s) * (\nabla^\perp \psi \Delta \psi)(x) \\
= - \int \partial_1 \Gamma(y - x) \Delta \psi(y) \partial_2 \psi(y) dy + \int \partial_2 \Gamma(y - x) \Delta \psi(y) \partial_1 \psi(y) dy \\
= \int \partial_1 \partial_k \Gamma(y - x) \partial_k \psi(y) \partial_2 \psi(y) dy - \int \partial_2 \partial_k \Gamma(y - x) \partial_k \psi(y) \partial_1 \psi(y) dy \\
+ \int \partial_1 \Gamma(y - x) \partial_k \psi(y) \partial_2 \partial_k \psi(y) dy - \int \partial_2 \Gamma(y - x) \partial_k \psi(y) \partial_1 \partial_k \psi(y) dy.
\]

We then integrate the last two terms by parts
\[
\int \partial_1 \Gamma(y - x) \partial_k \psi(y) \partial_2 \partial_k \psi(y) dy - \int \partial_2 \Gamma(y - x) \partial_k \psi(y) \partial_1 \partial_k \psi(y) dy \\
= - \int \partial_1 \partial_2 \Gamma(y - x) \partial_k \psi(y) \partial_2 \psi(y) dy - \int \partial_2 \Gamma(y - x) \partial_2 \partial_k \psi(y) \partial_k \psi(y) dy \\
+ \int \partial_1 \partial_2 \Gamma(y - x) \partial_k \psi(y) \partial_2 \psi(y) dy + \int \partial_2 \Gamma(y - x) \partial_1 \partial_k \psi(y) \partial_k \psi(y) dy
\]
which gives
\[
\int \partial_1 \Gamma(y - x) \partial_k \psi(y) \partial_2 \partial_k \psi(y) dy - \int \partial_2 \Gamma(y - x) \partial_k \psi(y) \partial_1 \partial_k \psi(y) dy = 0
\]
and so we obtain
\[
\text{div} \Gamma(t - s) * (\nabla^\perp \psi \Delta \psi)(x) = \int \partial_1 \partial_k \Gamma(y - x) \partial_k \psi(y) \partial_2 \psi(y) dy \\
- \int \partial_2 \partial_k \Gamma(y - x) \partial_k \psi(y) \partial_1 \psi(y) dy.
\]

We now bound (V) using Theorem 1.5 and (5.1):
\[
(V) \leq C \int_0^\frac{\tau}{2} (t - s)^{-\frac{3}{2}} \|\nabla \psi(s)\|_{L^1} ds \leq C \int_0^\frac{\tau}{2} (t - s)^{-\frac{3}{2}} \|\nabla \psi(s)\|_{L^2} ds \\
\leq \mathcal{E}_0 \frac{\tau^2}{t^2} \int_0^\frac{\tau}{2} \frac{1}{(s + 1)^2} ds \leq \mathcal{E}_0 \frac{\tau^2}{t^2}.
\]

We finally bound (VI) using Theorem 1.4, 1.5 and (5.1):
\[
(VI) \leq C \int_\frac{\tau}{2}^t (t - s)^{-\frac{5}{2}} \|\nabla^\perp \psi(s) \Delta \psi(s)\|_{L^2} ds \leq C \int_\frac{\tau}{2}^t (t - s)^{-\frac{5}{2}} \|\nabla \psi(s)\|_{L^6} \|\Delta \psi(s)\|_{L^6} ds \\
\leq C \int_\frac{\tau}{2}^t (t - s)^{-\frac{5}{2}} \|\nabla \psi(s)\|_{L^2} \|\Delta \psi(s)\|_{L^2} \|\nabla^\perp \psi(s)\|_{L^2} \|\nabla \psi(s)\|_{L^2} ds \\
\leq \mathcal{E}_0 \int_\frac{\tau}{2}^t (t - s)^{-\frac{5}{2}} s^{-\frac{3}{2}} s^{-\frac{1}{2}} ds \leq \frac{\mathcal{E}_0}{t^3}.
\]
Taking all these bounds into account, we also obtain two types of asymptotic behaviors of $Z$:
\[
Z(t, x) = \eta \Gamma(t, x) + \tilde{Z}(t, x) = \eta \Gamma(t, x) + O(t^{-\frac{3}{2}}), \\
Z(t, x) = \eta \Gamma(t, x) + \tilde{Z}(t, x) = \Gamma(t) * Z_0 + O(t^{-2}).
\]
6. Proof of Theorem 1.7 and Theorem 1.8

This section is devoted to proving the global existence and the uniqueness of solutions of (1.6) around harmonic functions. The analysis here is very close to the one in Sect. 3.3, but we will take a different kind of smallness condition, and the existence of harmonic functions requires a bit more computation.

6.1. Proof of Theorem 1.7

We recall the equations of \( \rho \) and \( Z \):

\[
\begin{align*}
\rho_t - \Delta \rho &= [\rho, Z] + [\bar{\psi}, Z], \\
Z_t - \Delta Z &= [\Delta \rho, \rho] + [\Delta \rho, \bar{\psi}].
\end{align*}
\]

6.1.1. A Priori Estimates. By (2.3d), we have

\[
\frac{d}{dt} \left( \|\nabla \rho\|_{L^2}^2 + \|Z\|_{L^2}^2 \right) + 2 \|\Delta \rho\|_{L^2}^2 + 2 \|\nabla Z\|_{L^2}^2 = 0.
\]

We next multiply (6.1a) by \( \Delta^2 \rho \), (6.1b) by \( -\Delta Z \) and integrate over \( \mathbb{R}^2 \) to get

\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta \rho\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \right) + \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2

= \int \Delta^2 \rho [\rho, Z] - \int \Delta Z [\Delta \rho, \rho] + \int \Delta^2 \rho [\bar{\psi}, Z] - \int \Delta Z [\Delta \rho, \bar{\psi}] = (I)+(II)+(III)+(IV).
\]

Treating (I)+(II) as (3.3) with \( 1/4 \) not \( 1/2 \), we have

\[
(I)+(II) \leq C \|\Delta \rho\|_{L^2}^2 \|\nabla \Delta \rho\|_{L^2}^2 + \frac{1}{4} \|\Delta Z\|_{L^2}^2.
\]

Since

\[
(III)+(IV) = 2 \int \Delta \rho (\bar{\psi}_x, Z_x) + [\bar{\psi}_y, Z_y]) \leq C \int |\nabla^2 \bar{\psi}| |\nabla Z| |\nabla^2 \rho|

\leq C \|\nabla^2 \bar{\psi}\|_{L^\infty} \|\Delta \rho\|_{L^2}^2 + \frac{1}{4} \|\Delta Z\|_{L^2}^2,
\]

we obtain

\[
\frac{d}{dt} \left( \|\Delta \rho\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \right) + \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \leq C \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\nabla^2 \bar{\psi}\|_{L^\infty}^2 \right) \|\Delta \rho\|_{L^2}^2.
\]

We finally multiply (6.1a) by \( -\Delta^3 \rho \), (6.1b) by \( \Delta^2 Z \) and integrate over \( \mathbb{R}^2 \):

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) + \|\Delta^2 \rho\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2

= -\int \Delta^3 \rho [\rho, Z] + \int \Delta^2 Z [\Delta \rho, \rho] - \int \Delta^3 \rho [\bar{\psi}, Z] + \int \Delta^2 Z [\Delta \rho, \bar{\psi}] = (V)+(VI)+(VII)+(VIII).
\]

By following (3.16) and (3.17) in the Proof of Theorem 1.4 with \( \frac{1}{2} \) replaced by \( \frac{1}{4} \), we have

\[
(V)+(VI) \leq C \|\nabla Z\|_{L^2}^2 \|\nabla \Delta Z\|_{L^2}^2 + C \|\Delta \rho\|_{L^2}^2 \|\Delta^2 \rho\|_{L^2}^2 + \frac{1}{4} \left( \|\Delta^2 \rho\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2 \right).
\]

The last two terms add up in the following way

\[
(VII)+(VIII) = -2 \int \Delta \rho ([\bar{\psi}_x, \Delta Z_x] + [\bar{\psi}_y, \Delta Z_y]) - 2 \int \Delta^2 \rho ([\bar{\psi}_x, Z_x] + [\bar{\psi}_y, Z_y])

\leq C \int |\nabla^2 \bar{\psi}| |\nabla^3 Z| |\nabla^3 \rho| + C \int |\nabla^2 \bar{\psi}| |\nabla^2 Z| |\nabla^4 \rho|

\leq C \|\nabla^2 \bar{\psi}\|_{L^\infty}^2 \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) + \frac{1}{4} \left( \|\Delta^2 \rho\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2 \right).
\]
So, we arrive at
\[
\frac{d}{dt} \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) + \|\Delta^2 \rho\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2 \\
\leq C \|\nabla Z\|_{L^2}^2 \|\nabla \Delta Z\|_{L^2}^2 + C \|\Delta \rho\|_{L^2}^2 \|\Delta^2 \rho\|_{L^2}^2 + C \|\nabla^2 \psi\|_{L^\infty}^2 \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right). \tag{6.6}
\]

Let \( C_1 = k \|\nabla^2 \psi\|_{L^\infty}^2 \) with \( k \) large enough which is determined below. By multiplying (6.2) by \( C_1 \) and (6.4) by \( C_1 \) and adding the resulting equations to (6.6), we have
\[
\frac{d}{dt} (C_1^2 F_1 + C_1 F_2 + F_3) + 2C_1 \left( \|\Delta \rho\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 \right) + C_1 \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) + F_4 \\
\leq \tilde{C} C_1 \|\nabla^2 \psi\|_{L^\infty}^2 \|\Delta \rho\|_{L^2}^2 + \tilde{C} \|\nabla^2 \psi\|_{L^\infty}^2 \left( \|\nabla \Delta \rho\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 \right) \\
+ C_1 \|\Delta \rho\|_{L^2}^2 \|\nabla \Delta \rho\|_{L^2}^2 + C \|\nabla Z\|_{L^2}^2 \|\nabla \Delta Z\|_{L^2}^2 + C \|\Delta \rho\|_{L^2}^2 \|\Delta^2 \rho\|_{L^2}^2 , \tag{6.7}
\]

where we fix two constants by \( \tilde{C} \) to determine \( k \) and \( F_1, F_2, F_3, F_4 \) are defined in (1.22). We now choose \( k \) such that
\[
k > 2\tilde{C}, \quad C_1 > 1. \tag{6.8}
\]

Then, one can easily check that (6.7) can be reduced to
\[
\frac{d}{dt} (C_1^2 F_1 + C_1 F_2 + F_3) + C_1 F_3 + F_4 \leq C (C_1^2 F_1 + C_1 F_2 + F_3) (C_1 F_3 + F_4).
\]

By Lemma 2.1, if \( C \epsilon_2 = C (C_1^2 F_1(0) + C_1 F_2(0) + F_3(0)) < 1 \), we obtain the following for all \( t > 0 \):
\[
C_1^2 F_1(t) + C_1 F_2(t) + F_3(t) + (1 - C \epsilon_2) \int_0^t (C_1 F_3(s) + F_4(s)) ds \leq C_1^2 F_1(0) + C_1 F_2(0) + F_3(0).
\]

### 6.1.2. Uniqueness

Suppose there are two solutions \( (\rho_1, Z_1) \) and \( (\rho_2, Z_2) \). Let \( \rho = \rho_1 - \rho_2 \) and \( Z = Z_1 - Z_2 \). Then, \( (\rho, Z) \) satisfies the following equations
\[
\rho_t - \Delta \rho = [\rho_1, Z] + [\rho, Z_2] + [\psi, Z], \\
Z_t - \Delta Z = [\Delta \rho, \rho_1] + [\Delta \rho_2, \rho] + [\Delta \rho, \psi].
\]

Since
\[
- \int \Delta \rho [\psi, Z] + \int Z [\Delta \rho, \psi] = 0,
\]
the proof of the uniqueness is identical to the one in Sect. 3.1.2.

### 6.2. Proof of Theorem 1.8

We recall the equations of \( \psi \) and \( \omega \):
\[
\psi_t - \Delta \psi = [\psi, \omega] + [\psi, Z], \tag{6.9a}
\omega_t - \Delta \omega = [\Delta \psi, \psi], \tag{6.9b}
\]
6.2.1. A Priori Estimates. Compared to Theorem 1.7, we also need the $L^2$ bound of $\psi$ to complete the Proof of Theorem 1.8. So, we first have

$$ \frac{d}{dt}\|\psi\|^2_{L^2} + 2\|\nabla\psi\|^2_{L^2} = 0. $$

We next multiply (6.9a) by $-\Delta \psi$, (6.9b) by $\omega$, and integrate over $\mathbb{R}^2$. Then,

$$ \frac{1}{2} \frac{d}{dt}\left(\|\nabla\psi\|^2_{L^2} + \|\omega\|^2_{L^2}\right) + \|\Delta\psi\|^2_{L^2} + \|\nabla\omega\|^2_{L^2} = -\int \Delta\psi[\psi, Z] \leq C \|\nabla Z\|^2_{L^\infty} \|\nabla\psi\|^2_{L^2} + \frac{1}{2} \|\Delta\psi\|^2_{L^2}$$

and so we obtain

$$ \frac{d}{dt}\left(\|\nabla\psi\|^2_{L^2} + \|\omega\|^2_{L^2}\right) + \|\Delta\psi\|^2_{L^2} + \|\nabla\omega\|^2_{L^2} \leq C \|\nabla Z\|^2_{L^\infty} \|\nabla\psi\|^2_{L^2}. \tag{6.10} $$

We now multiply (6.9a) by $\Delta^2 \psi$, (6.9b) by $-\Delta \omega$, and integrate over $\mathbb{R}^2$. Then,

$$ \frac{1}{2} \frac{d}{dt}\left(\|\Delta\psi\|^2_{L^2} + \|\nabla\omega\|^2_{L^2}\right) + \|\nabla\Delta\psi\|^2_{L^2} + \|\Delta\omega\|^2_{L^2} = \int \Delta^2\psi[\psi, \omega] - \int \Delta\omega[\Delta\psi, \psi] + \int \Delta^2\psi[\psi, Z] = (I)+(II)+(III).$$

As (6.3), we bound (I)+(II) by

$$(I)+(II) \leq C \|\Delta\psi\|^2_{L^2} \|\nabla\Delta\psi\|^2_{L^2} + \frac{1}{2} \|\Delta\omega\|^2_{L^2}. $$

And we bound (III) as

$$(III) = \int \Delta\psi\Delta[\psi, Z] = 2 \int \Delta\psi\left([\psi_x, Z_x] + [\psi_y, Z_y]\right) \leq C \|\nabla Z\|^2_{L^\infty} \|\Delta\psi\|^2_{L^2} + \frac{1}{2} \|\nabla\Delta\psi\|^2_{L^2}. $$

So, we obtain

$$ \frac{d}{dt}\left(\|\Delta\psi\|^2_{L^2} + \|\nabla\omega\|^2_{L^2}\right) + \|\nabla\Delta\psi\|^2_{L^2} + \|\Delta\omega\|^2_{L^2} \leq C \left(\|\nabla^3\psi\|^2_{L^2} + \|\nabla Z\|^2_{L^\infty}\right) \|\Delta\psi\|^2_{L^2}. \tag{6.11} $$

We finally multiply (6.9a) by $-\Delta^3 \psi$, (6.9b) by $\Delta^2 \omega$, and integrate over $\mathbb{R}^2$. Then,

$$ \frac{1}{2} \frac{d}{dt}\left(\|\nabla\Delta\psi\|^2_{L^2} + \|\Delta\omega\|^2_{L^2}\right) + \|\Delta^2\psi\|^2_{L^2} + \|\nabla\Delta\omega\|^2_{L^2} = -\int \Delta^3\psi[\psi, \omega] + \int \Delta^2\omega[\Delta\psi, \psi] - \int \Delta^3\psi[\psi, Z] = (IV)+(V)+(VI).$$

Similar to (6.5), we bound (IV)+(V) by

$$(IV)+(V) \leq C \|\nabla\omega\|^2_{L^2} \|\nabla\Delta\omega\|^2_{L^2} + C \|\Delta\psi\|^2_{L^2} \|\Delta^2\psi\|^2_{L^2} + \frac{1}{2} \|\Delta^2\psi\|^2_{L^2} + \frac{1}{2} \|\nabla\Delta\omega\|^2_{L^2}. $$

To estimate (VI), we use

$$(VI) = -\int \Delta^2\psi[\Delta\psi, Z] - 2 \int \Delta^2\psi[\psi_x, Z_x] - 2 \int \Delta^2\psi[\psi_y, Z_y] = (VI)_{(1)} + (VI)_{(2)} + (VI)_{(3)}. $$

The first term is bounded as above:

$$(VI)_{(1)} \leq C \|\nabla Z\|^2_{L^\infty} \|\nabla\Delta\psi\|^2_{L^2} + \frac{1}{4} \|\Delta^2\psi\|^2_{L^2}. $$

We next estimate (VI)_{(2)} as

$$(VI)_{(2)} = -2 \int \Delta\psi\Delta[\psi, Z] = -2 \int \Delta\psi[\Delta\psi, Z] - 4 \int \Delta\psi[\psi_{xx}, Z_{xx}] - 4 \int \Delta\psi[\psi_{xy}, Z_{xy}] = (VI)_{(2a)} + (VI)_{(2b)} + (VI)_{(2c)};$$
where we use the fact that $Z_x$ is harmonic. (VI)$_{(2a)}$ is bounded as above:

$$\text{(VI)}_{(2a)} \leq C \left\| \nabla Z \right\|_{L^\infty}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 + \frac{1}{16} \left\| \Delta^2 \psi \right\|_{L^2}^2.$$  

By the integration by parts,

$$\text{(VI)}_{(2b)} = -4 \int Z_{xx}[\Delta \psi, \psi_{xx}] = 4 \int Z_x[\Delta \psi, \psi_{xx}] \leq C \left\| \nabla Z \right\|_{L^\infty}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 + \frac{1}{16} \left\| \Delta^2 \psi \right\|_{L^2}^2.$$  

Similarly, we obtain

$$\text{(VI)}_{(2c)} \leq C \left\| \nabla Z \right\|_{L^\infty}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 + \frac{1}{16} \left\| \Delta^2 \psi \right\|_{L^2}^2.$$  

Since (VI)$_{(2)}$ and (VI)$_{(2)}$ are of the same form, we obtain

$$\frac{d}{dt} \left( \left\| \nabla \Delta \psi \right\|_{L^2}^2 + \left\| \Delta \omega \right\|_{L^2}^2 \right) + \left\| \Delta^2 \psi \right\|_{L^2}^2 + \left\| \nabla \Delta \psi \right\|_{L^2}^2$$

$$\leq C \left\| \nabla \Delta \psi \right\|_{L^2}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 + C \left\| \Delta \psi \right\|_{L^2}^2 \left\| \Delta^2 \psi \right\|_{L^2}^2 + C \left\| \nabla Z \right\|_{L^\infty}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2.$$  

Let $C_2 = k \left\| \nabla Z \right\|_{L^\infty}^2$ with $k$ large enough which is determined below. By following the argument in Sect. 6.1.1, we obtain

$$\frac{d}{dt} \left( C_2^3 \left\| \psi \right\|_{L^2}^2 + C_2^2 \left( C_2 K_1 + C_2 K_2 + K_3 \right) + 2 \left( \left\| \Delta \psi \right\|_{L^2}^2 + \left\| \nabla \omega \right\|_{L^2}^2 \right) \right)$$

$$+ C_2 \left( \left\| \nabla \Delta \psi \right\|_{L^2}^2 + \left\| \Delta \omega \right\|_{L^2}^2 \right) + K_4$$

$$\leq C C_2^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 \left\| \nabla \psi \right\|_{L^2}^2 + C C_2 \left\| \nabla Z \right\|_{L^\infty}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 + C \left\| \nabla \Delta \psi \right\|_{L^2}^2 \left\| \Delta^2 \psi \right\|_{L^2}^2$$

$$+ C \left\| \Delta \psi \right\|_{L^2}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2 + C \left\| \Delta \psi \right\|_{L^2}^2 \left\| \Delta^2 \psi \right\|_{L^2}^2 + C \left\| \nabla \omega \right\|_{L^2}^2 \left\| \nabla \Delta \psi \right\|_{L^2}^2,$$

where we fix two constants by $\tilde{C}$ to determine $k$ and $K_1, K_2, K_3, K_4$ are defined in (1.24). We now choose $k$ such that

$$k > 2\tilde{C}, \quad C_2 > \max\{2\tilde{C}, 1\}.$$  

Then, (6.12) can be reduced to

$$\frac{d}{dt} \left( C_2^3 \left\| \psi \right\|_{L^2}^2 + C_2^2 \left( C_2 K_1 + C_2 K_2 + K_3 \right) + C_2^2 K_2 + C_2 K_3 + K_4 \right)$$

$$\leq C C_2 K_2 K_3 + C K_2 K_4 \leq C \left( C_2^3 \left\| \psi \right\|_{L^2}^2 + C_2^2 K_1 + C_2 K_2 + K_3 \right) \left( C_2^3 K_2 + C_2 K_3 + K_4 \right).$$  

By Lemma 2.1, if $C \epsilon_3 = C \left( C_2^3 \left\| \psi(0) \right\|_{L^2}^2 + C_2^2 K_1(0) + C_2 K_2(0) + K_3(0) \right) < 1$, we obtain

$$C_2^3 \left\| \psi(t) \right\|_{L^2}^2 + C_2^2 K_1(t) + C_2 K_2(t) + K_3(t) + (1 - C \epsilon_3) \int_0^t \left( C_2^2 K_2(s) + C_2 K_3(s) + K_4(s) \right) ds$$

$$\leq C_2^3 \left\| \psi(0) \right\|_{L^2}^2 + C_2^2 K_1(0) + C_2 K_2(0) + K_3(0)$$

for all $t > 0$.

**6.2.2. Uniqueness.** Suppose there are two solutions $(\psi_1, \omega_1)$ and $(\psi_2, \omega_2)$. Let $\psi = \psi_1 - \psi_2$ and $\omega = \omega_1 - \omega_2$. Then, $(\psi, \omega)$ satisfies the following equations

$$\psi_t - \Delta \psi = \left[ \psi_1, \omega_1 \right] + \left[ \psi_2, \omega_2 \right] + \left[ \psi, Z \right],$$  

$$\omega_t - \Delta \omega = \left[ \Delta \psi_1, \psi_1 \right] + \left[ \Delta \psi_2, \psi_2 \right].$$  

Compared to (3.10), there is one extra term $\left[ \psi, Z \right]$. When we multiply (6.14a) by $-\Delta \psi$, (6.14b) by $g$, and integrate over $\mathbb{R}^2$, this term can be bounded by

$$- \int \Delta \psi \left[ \psi, Z \right] \leq \left\| \nabla Z \right\|_{L^\infty} \left\| \nabla \psi \right\|_{L^2} \left\| \Delta \psi \right\|_{L^2} \leq C \left\| \nabla Z \right\|_{L^\infty}^2 \left\| \nabla \psi \right\|_{L^2}^2 + \frac{1}{2} \left\| \Delta \psi \right\|_{L^2}^2.$$  

\( \text{Birkhäuser} \)
By changing the constant from 1 to $\frac{1}{2}$ in front of $\|\Delta \psi\|_{L^2}$, we can follow Sect. 3.1.2 for the remaining part to complete the proof of the uniqueness.

7. Proof of Theorem 1.10 and Theorem 1.11

In this section, we deal with the $2\frac{1}{2}$ dimensional Hall MHD. We first recall (1.5):

$$\psi_t - \Delta \psi = [\psi, Z] - [\psi, \phi], \quad (7.1a)$$
$$Z_t - \Delta Z = [\Delta \psi, \psi] - [Z, \phi] + [W, \psi], \quad (7.1b)$$
$$W_t - \Delta W = -[W, \phi] - [\psi, Z], \quad (7.1c)$$
$$\Delta \phi_t - \Delta^2 \phi = -[\Delta \phi, \phi] + [\Delta \psi, \psi]. \quad (7.1d)$$

Proceeding as Theorem 1.3, we define the following norms:

$$P(t) = P_1(t) + P_2(t) + P_3(t), \quad Q(t) = P_2(t) + P_3(t) + P_4(t),$$

$$P_1 = \|\nabla \psi\|_{L^2}^2 + \|Z\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2 + \|W\|_{L^2}^2,$$

$$P_2 = \|\Delta \psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 + \|\Delta \phi\|_{L^2}^2 + \|\nabla W\|_{L^2}^2,$$

$$P_3 = \|\nabla \Delta \psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 + \|\nabla \Delta \phi\|_{L^2}^2 + \|\Delta W\|_{L^2}^2,$$

$$P_4 = \|\Delta^2 \psi\|_{L^2}^2 + \|\nabla \Delta Z\|_{L^2}^2 + \|\Delta^2 \phi\|_{L^2}^2 + \|\nabla \Delta W\|_{L^2}^2.$$

We remember that the Hall term is the most difficult term to deal with the Hall MHD. So while (7.1) and the corresponding spaces (7.2) look very complicated, the terms resulting from the Hall effect has already been handled in Sect. 3 and other terms can be treated similarly. Thus, rather than presenting the proof in great detail, we will present a proof with a few calculations omitted.

7.1. Proof of Theorem 1.10

7.1.1. A Priori Estimates. By multiplying $-\Delta \psi$, $Z$, $W$, $-\phi$ to (7.1a), (7.1b), (7.1c), (7.1d), respectively, we first obtain

$$\frac{1}{2} \frac{d}{dt} P_1 + P_2 = 0. \quad (7.3)$$

By multiplying $\Delta^2 \psi$, $-\Delta Z$, $-\Delta W$, $\Delta \phi$ to (7.1a), (7.1b), (7.1c), (7.1d), respectively, we have

$$\frac{1}{2} \frac{d}{dt} P_2 + P_3 = \int \Delta^2 \psi [\psi, Z] - \int \Delta Z [\Delta \psi, \psi] - \int \Delta^2 \psi [\psi, \phi] + \int \Delta Z [Z, \phi]$$

$$- \int \Delta Z [W, \psi] + \int \Delta W [W, \phi] + \int \Delta W [\psi, Z] + \int \Delta \phi [\Delta \psi, \psi] \quad (7.4)$$

Treating as (3.3) with $\frac{1}{2}$ replaced with $\frac{1}{8}$, we have

$$I(a) + I(b) \leq C \|\Delta Z\|_{L^2} \|\Delta \psi\|_{L^2} \|\nabla \Delta \psi\|_{L^2} \leq C \|\Delta \psi\|_{L^2}^2 + \frac{1}{8} \|\Delta Z\|_{L^2}^2.$$

After some reduction, $I(c) + I(h)$ is estimated as

$$I(c) + I(h) = -2 \int \Delta \psi ([\psi_x, \phi_x] + [\psi_y, \phi_y]) \leq C \|\Delta \phi\|_{L^2} \|\nabla^2 \psi\|_{L^4}$$

$$\leq C \|\Delta \phi\|_{L^2} \|\Delta \psi\|_{L^2} \|\Delta \psi\|_{L^2} \leq C \|\Delta \phi\|_{L^2}^2 \|\Delta \psi\|_{L^2}^2 + \frac{1}{8} \|\nabla \Delta \phi\|_{L^2}^2.$$
By the definition of the commutator and by using after integrating by parts, we have

\[ I(d) + I(f) = \int \partial_k Z \partial_k \nabla^\perp \phi \cdot \nabla Z + \int \partial_k W \partial_k \nabla^\perp \phi \cdot \nabla W \]

\[ \leq C \| \nabla Z \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 + \frac{1}{8} \| \Delta Z \|_{L^2}^2 + C \| \nabla W \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 + \frac{1}{8} \| \Delta W \|_{L^2}^2. \]

We finally bound \( I(e) + I(g) \) as

\[ I(e) + I(g) = \int Z[\Delta \psi, W] + 2 \int Z[\psi_x, W_x] + 2 \int Z[\psi_y, W_y] \leq C \| \nabla Z \|_{L^4} \| \nabla W \|_{L^4} \| \Delta \psi \|_{L^2} \]

\[ \leq C \| \nabla Z \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 + C \| \nabla W \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 + \frac{1}{8} \| \Delta Z \|_{L^2}^2 + \frac{1}{8} \| \Delta W \|_{L^2}^2. \]

So, we obtain

\[ \frac{d}{dt} P_2 + P_3 \leq C \| \Delta \phi \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 + C \| \nabla Z \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 + C \| \nabla W \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 \]

\[ + C \| \Delta \phi \|_{L^2}^2 \| \nabla W \|_{L^2}^2 \| \nabla Z \|_{L^2}^2 + C \| \Delta \phi \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2}^2. \]  

(7.5)

By multiplying \(-\Delta^3 \psi, \Delta^2 Z, \Delta^2 W, -\Delta^2 \phi \) to (7.1a), (7.1b), (7.1c), (7.1d), respectively, we have

\[ \frac{1}{2} \frac{d}{dt} P_3 + P_4 = -\int \Delta^3 \psi[\psi, Z] + \int \Delta^2 Z[\Delta \psi, \psi] + \int \Delta^3 \psi[\psi, \phi] - \int \Delta^2 Z[\phi] \]

\[ + \int \Delta^2 Z[W, \psi] - \int \Delta^2 W[W, \phi] - \int \Delta^2 W[\psi, Z] - \int \Delta^2 \phi[\Delta \psi, \psi] \]

\[ = II(a) + II(b) + II(c) + II(d) + II(e) + II(f) + II(g) + II(h). \]

(7.6)

By following the computations used for (3.7), we have

\[ II(a) + II(b) \leq C \| \Delta Z \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2}^2 + C \| \Delta \psi \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2}^4 + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2 + \frac{1}{4} \| \Delta^2 \psi \|_{L^2}. \]

And we bound \( II(c) + II(h) \) as follows

\[ II(c) + II(h) = \int \Delta^2 \psi[\Delta \psi, \phi] + 2 \int \Delta^2 \psi[\psi_x, \phi_x] + 2 \int \Delta^2 \psi[\psi_y, \phi_y] \]

\[ + 2 \int \Delta \psi[\psi_x, \Delta \phi_x] + 2 \int \Delta \psi[\psi_y, \Delta \phi_y] \]

\[ = \int \partial_k \nabla^\perp \phi \cdot \nabla \Delta \psi \partial_k \Delta \psi + 2 \int \Delta^2 \psi[\psi_x, \phi_x] + 2 \int \Delta^2 \psi[\psi_y, \phi_y] \]

\[ + 2 \int \Delta \psi[\psi_x, \Delta \phi_x] + 2 \int \Delta \psi[\psi_y, \Delta \phi_y] \]

\[ \leq C \| \Delta \phi \|_{L^2} \| \nabla \Delta \psi \|_{L^2}^2 + C \| \Delta \phi \|_{L^4} \| \Delta \phi \|_{L^4} \| \Delta^2 \psi \|_{L^2} + C \| \Delta \phi \|_{L^2}^2 \| \Delta^2 \psi \|_{L^2} \]

\[ \leq C \| \Delta \phi \|_{L^2}^2 \| \nabla \Delta \phi \|_{L^2}^2 + C \| \Delta \phi \|_{L^2} \| \nabla \Delta \phi \|_{L^2}^2 + C \| \Delta \psi \|_{L^2} \| \nabla \Delta \psi \|_{L^2} \]

\[ + \frac{1}{4} \| \Delta^2 \psi \|_{L^2} + \frac{1}{2} \| \Delta^2 \phi \|_{L^2}. \]

We also bound \( II(d) \) as

\[ II(d) = \int \Delta Z[\Delta \phi] + 2 \int \Delta Z[\psi_x, \phi_x] + 2 \int \Delta Z[\psi_y, \phi_y] \]

\[ \leq C \| \Delta \phi \|_{L^4} \| \nabla Z \|_{L^4} \| \Delta Z \|_{L^2} + C \| \Delta \phi \|_{L^2} \| \Delta Z \|_{L^4} \]

\[ \leq C \| \Delta \phi \|_{L^2}^2 \| \nabla \Delta \phi \|_{L^2}^2 + C \| \nabla Z \|_{L^2}^2 \| \Delta Z \|_{L^2} + C \| \Delta \phi \|_{L^2} \| \Delta Z \|_{L^2}^2 + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2. \]

\[ \| \Delta \phi \|_{L^2}^2 \| \nabla \Delta \phi \|_{L^2}^2 + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2. \]
Similarly, we bound II(f) as
\[ II(f) \leq C \|\Delta \phi\|^2_{L^2_{\psi}} + \|\nabla \Delta \phi\|^2_{L^2_{\psi}} + C \|\nabla W\|^2_{L^2} \|\Delta W\|^2_{L^2} + C \|\Delta \phi\|^2_{L^2_{\psi}} \|\Delta W\|^2_{L^2} + \frac{1}{4} \|\nabla \Delta W\|^2_{L^2}. \]

We finally bound II(e)+II(g) as
\[
II(e)+II(g) = \int Z[\Delta W, \Delta \psi] + 2 \int Z[\Delta W_x, \psi_x] + 2 \int Z[\Delta W_y, \psi_y] + \int \Delta Z[W, \Delta \psi] \\
+ 2 \int \Delta Z[W_x, \psi_x] + 2 \int \Delta Z[W_y, \psi_y] \\
\leq C \|\nabla Z\|_{L^4} \|\Delta \psi\|_{L^4} + \|\nabla \Delta W\|_{L^2} + C \|\nabla W\|_{L^4} \|\Delta \psi\|_{L^4} + \|\nabla \Delta Z\|_{L^2} \\
\leq C \|\nabla Z\|_{L^2} \|\Delta \psi\|_{L^2} + C \|\nabla W\|_{L^2} \|\Delta W\|_{L^2} + C \|\Delta \psi\|_{L^2} \|\Delta \psi\|_{L^2} \\
+ \frac{1}{4} \|\nabla \Delta W\|_{L^2}^2 + \frac{1}{6} \|\nabla \Delta \psi\|_{L^2}^2. \\
\]

Collecting all the bounds, we derive
\[
\frac{d}{dt} P_3 + P_4 \leq C \|\Delta \phi\|^2_{L^2_{\psi}} + C \|\nabla \Delta \phi\|^2_{L^2_{\psi}} + C \|\Delta \phi\|^2_{L^2_{\psi}} \|\nabla \Delta \phi\|^2_{L^2_{\psi}} + C \|\Delta \psi\|^2_{L^2} \|\Delta \psi\|^2_{L^2} \\
+ C \|\Delta Z\|^2_{L^2} \|\nabla \Delta \psi\|^2_{L^2} + C \|\nabla W\|^2_{L^2} \|\Delta W\|^2_{L^2} + C \|\Delta \phi\|^2_{L^2} \|\Delta W\|^2_{L^2} \\
+ C \|\nabla Z\|^2_{L^2} \|\Delta Z\|^2_{L^2} + C \|\Delta \phi\|^2_{L^2} \|\Delta Z\|^2_{L^2} + C \|\Delta \psi\|^2_{L^2} \|\nabla \Delta \psi\|^2_{L^2} \] \tag{7.7}

By (7.3), (7.5), and (7.7),
\[
(1 + P(t))^3 + Q(t) \leq CP^2(t) + CP^3(t) \leq C (1 + P(t))^3 \\
\tag{7.8}
\]
from which we deduce
\[
P(t) \leq \sqrt{\frac{(1 + P(0))^2}{1 - 2Ct(1 + P(0))^2}} - 1 \quad \text{for all } t \in (0, T) < \frac{1}{2C(1 + P(0))^2}. \tag{7.9}
\]

Integrating (7.8) and using (7.9), we finally derive
\[
P(t) + \int_0^t Q(s)ds < \infty, \quad 0 < t < T. \tag{7.10}
\]

### 7.1.2. Uniqueness
Suppose there are two solutions \((\psi_1, Z_1, \phi_1, W_1)\) and \((\psi_2, Z_2, \phi_2, W_2)\). Let \(\psi = \psi_1 - \psi_2, Z = Z_1 - Z_2, \phi = \phi_1 - \phi_2\) and \(W = W_1 - W_2\). Then, \((\psi, Z, \phi, W)\) satisfies the following equations
\[
\psi_t - \Delta \psi = [\psi_1, Z] + [\psi_2, Z] - [\psi_1, \phi] - [\psi_2, \phi], \\
Z_t - \Delta Z = [\Delta \psi, \psi_1] + [\Delta \psi, \psi_2] - [Z_1, \phi] - [Z_2, \phi] + [W_1, \psi] + [W, \psi_2], \\
W_t - \Delta W = -W_1, \phi - [W, \phi_2] - [\psi_1, Z] - [\psi_2, Z], \\
\Delta \phi_t - \Delta^2 \phi = -[\Delta \phi_1, \phi] - [\Delta \phi_2, \phi] + [\Delta \psi_1, \psi] + [\Delta \psi_2, \psi].
\]
From this, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \psi\|^2_{L^2} + \|\nabla \phi\|^2_{L^2} + \|W\|^2_{L^2} \right) + \|\Delta \psi\|^2_{L^2} + \|\Delta \phi\|^2_{L^2} + \|\nabla W\|^2_{L^2} \\
= -\int \Delta \psi_1, Z - \int \Delta \psi_2, \psi_2 + \int Z[\Delta \psi_1, \psi_1] + \int Z[\Delta \psi_2, \psi] \\
- \int Z[Z_1, \phi] + \int Z[W_1, \psi_1] + \int Z[W, \psi_2] - \int W[W_1, \phi] - \int \psi_1[Z_1, Z] - \int W[Z, \psi_2] \\
+ \int \Delta \psi_1, \phi + \int \Delta \psi_2, \phi_2 + \int \phi[\Delta \phi_1, \phi_2] - \int \phi[\Delta \psi_1, \psi_2].
\]
The first line on the right-hand side is bounded as (3.11)

\[
C \left( \| \nabla Z_2 \|^2_{L^2} \| \Delta Z_2 \|^2_{L^2} + \| \Delta \psi_2 \|^2_{L^2} \| \nabla \Delta \psi_2 \|^2_{L^2} \right) \| \nabla \psi \|^2_{L^2} + \frac{1}{3} \| \Delta \psi \|^2_{L^2} + \| \nabla Z \|^2_{L^2}
\]

and the second line is bounded by

\[
C \| \nabla Z_2 \|_{L^\infty} \| \nabla \phi \|_{L^2} + C \| \nabla W_1 \|_{L^\infty} \| \nabla Z \|_{L^2} \| \nabla \psi \|_{L^2} + C \| \nabla \psi_2 \|_{L^\infty} \| \nabla Z \|_{L^2} \| \nabla W \|_{L^2} + C \| \nabla W_1 \|_{L^\infty} \| \nabla Z \|_{L^2} \| \nabla \psi \|_{L^2}
\]

The third line except for the last one is bounded by

\[
C \| \nabla \phi \|_{L^2} \| \Delta \psi \|_{L^2} + C \| \nabla \phi \|_{L^2} \| \Delta \psi \|_{L^2} + C \| \nabla \phi \|_{L^2} \| \Delta \phi \|_{L^2}
\]

Thus, we arrive at the following inequality:

\[
\frac{d}{dt} \left( \| \nabla \psi \|^2_{L^2} + \| Z \|^2_{L^2} + \| \nabla \phi \|^2_{L^2} + \| W \|^2_{L^2} \right) \leq C \mathcal{I} \left( \| \nabla \psi \|^2_{L^2} + \| Z \|^2_{L^2} + \| \nabla \phi \|^2_{L^2} + \| W \|^2_{L^2} \right),
\]

where

\[
\mathcal{I} = \| \nabla Z_2 \|^2_{L^2} \| \Delta Z_2 \|^2_{L^2} + \| \nabla \psi_2 \|^2_{L^2} \| \Delta \psi \|^2_{L^2} + \| \nabla Z_2 \|_{L^\infty} + \| \nabla W_1 \|_{L^\infty} + \| \nabla \psi_1 \|_{L^\infty}
\]

Since \( \mathcal{I} \) is integrable in time by (7.10), we conclude the uniqueness of solutions.

### 7.1.3. Blow-up Criterion.

To find a blow-up criterion, we first use (3.12) to bound I(a)+I(b) as

\[
\left| \int \Delta \psi \left( [\psi_x, Z_x] + [\psi_y, Z_y] \right) \right| \leq C \| \nabla Z \|^q_{L^p} \| \Delta \psi \|^2_{L^2} + \epsilon \| \nabla \Delta \psi \|^2_{L^2}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}.
\]

So, we can rewrite (7.5) as

\[
\frac{d}{dt} P_2 + P_3 \leq C \| \Delta \phi \|^2_{L^2} \| \Delta \psi \|^2_{L^2} + C \| \nabla Z \|^2_{L^2} \| \Delta \psi \|^2_{L^2} + C \| \nabla W \|^2_{L^2} \| \Delta \psi \|^2_{L^2} + C \| \Delta \phi \|^2_{L^2} \| \nabla W \|^2_{L^2}
\]

Integrating this in time with the aid of (7.3), we obtain

\[
\| \Delta \psi(t) \|^2_{L^2} + \| \nabla Z(t) \|^2_{L^2} + \| \Delta \phi(t) \|^2_{L^2} \leq C \mathcal{E}_0 \exp \left[ C \int_0^t \| \nabla Z(s) \|^2_{L^p} ds \right] \]  

Using the idea in Sect. 3.1.3 to bound (3.7) by using (3.13), we bound (7.7) as follows

\[
\| \nabla \Delta \psi(t) \|^2_{L^2} + \| \nabla Z(t) \|^2_{L^2} + \| \Delta \phi(t) \|^2_{L^2} + \| \Delta W(t) \|^2_{L^2} \leq C \exp \left[ C B(t) + CB^2(t) \right],
\]

where \( B(t) \) is defined in (3.14). This completes the Proof of Theorem 1.10.
7.2. Proof of Theorem 1.11

To prove Theorem 1.11, we need to bound the quantities used in the Proof of Theorem 1.10 in different ways. We first bound each term on the right-hand side of (7.4) as follow

\[
\begin{align*}
I(a)+I(b) & \leq C \| \Delta Z \|_{L^2} \| \Delta \phi \|_{L^2} \| \Delta \psi \|_{L^2} \leq C \| \Delta \phi \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 + \frac{1}{8} \| \Delta Z \|_{L^2}^2, \\
I(c)+I(h) & \leq C \| \nabla \phi \|_{L^2}^2 \| \nabla \Delta \phi \|_{L^2}^2 + C \| \nabla \psi \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2}^2 + \frac{1}{8} \| \nabla \Delta \psi \|_{L^2}^2, \\
I(e)+I(g) & \leq C \| Z \|_{L^2}^2 \| \Delta Z \|_{L^2} + C \| W \|_{L^2}^2 \| \Delta W \|_{L^2} + C \| \nabla \phi \|_{L^2}^2 \| \nabla \Delta \psi \|_{L^2} \\
& \quad + \frac{1}{8} \| \Delta Z \|_{L^2}^2 + \frac{1}{8} \| \Delta W \|_{L^2}^2, \\
I(d)+I(f) & \leq C \| \nabla \phi \|_{L^2}^2 \| \nabla \Delta \phi \|_{L^2}^2 + C \| W \|_{L^2}^2 \| \Delta W \|_{L^2}^2 + C \| \Delta Z \|_{L^2}^2 \\
& \quad + \frac{1}{8} \| \Delta W \|_{L^2}^2 + \frac{1}{8} \| \Delta Z \|_{L^2}^2.
\end{align*}
\]

So, we can rewrite (7.5) as

\[
\frac{d}{dt} P_2 + P_3 \leq C \left( \| \nabla \phi \|_{L^2}^2 + \| \nabla \psi \|_{L^2}^2 + \| Z \|_{L^2}^2 + \| W \|_{L^2}^2 + \| \Delta \psi \|_{L^2}^2 \right) P_3. \tag{7.12}
\]

By (7.3) and (7.12),

\[
\frac{d}{dt}(P_1 + P_2) + P_2 + P_3 \leq C \left( P_1 + P_2 \right) \left( P_2 + P_3 \right). \tag{7.13}
\]

Let \( \epsilon_4 = \| \nabla \phi_0 \|_{H^1}^2 + \| \nabla \psi_0 \|_{H^1}^2 + \| Z_0 \|_{H^1}^2 + \| W_0 \|_{H^1}^2 \). By Lemma 2.1, if \( \epsilon_4 \) is sufficiently small such as \( C \epsilon_4 < 1 \), (7.13) implies the following inequality for all \( t > 0 \):

\[
P_1(t) + P_2(t) + (1 - C \epsilon_4) \int_0^t \left( P_2(s) + P_3(s) \right) ds \leq \epsilon_4. \tag{7.14}
\]

We also bound the right-hand side of (7.6) as follows

\[
\begin{align*}
I(a)+I(b) & \leq C \| \nabla Z \|_{L^2}^2 \| \nabla \Delta Z \|_{L^2}^2 + C \| \Delta \phi \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 + C \| \nabla \phi \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 \\
& \quad + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2 + \frac{1}{4} \| \Delta \phi \|_{L^2}^2, \\
I(c)+I(h) & \leq C \| \nabla \phi \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 + C \| \nabla \psi \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 + C \| \nabla \phi \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 \\
& \quad + \frac{1}{4} \| \nabla \Delta W \|_{L^2}^2 + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2, \\
I(e)+I(g) & \leq C \| Z \|_{L^2}^2 \| \nabla \Delta Z \|_{L^2}^2 + C \| W \|_{L^2}^2 \| \nabla \Delta W \|_{L^2}^2 + C \| \nabla \phi \|_{L^2}^2 \| \Delta \psi \|_{L^2}^2 \\
& \quad + \frac{1}{4} \| \nabla \Delta W \|_{L^2}^2 + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2, \\
I(d)+I(f) & \leq C \| \nabla \phi \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 + C \| W \|_{L^2}^2 \| \nabla \Delta W \|_{L^2}^2 + C \| Z \|_{L^2}^2 \| \nabla \Delta Z \|_{L^2}^2 \\
& \quad + C \| \nabla \phi \|_{L^2}^2 \| \nabla \Delta W \|_{L^2}^2 + C \| \nabla \phi \|_{L^2}^2 \| \Delta \phi \|_{L^2}^2 + \frac{1}{4} \| \nabla \Delta W \|_{L^2}^2 + \frac{1}{6} \| \nabla \Delta Z \|_{L^2}^2.
\end{align*}
\]

So, we rewrite (7.7) as

\[
\frac{d}{dt} P_3 + P_4 \leq C \left( \| Z \|_{L^2}^2 + \| \nabla Z \|_{L^2}^2 + \| W \|_{L^2}^2 + \| \nabla \psi \|_{L^2}^2 + \| \nabla \phi \|_{L^2}^2 + \| \Delta \psi \|_{L^2}^2 \right) P_4.
\]
By (7.14), we derive the following for all $t > 0$:

$$P_3(t) + (1 - C\epsilon_4) \int_0^t P_4(s)ds \leq P_3(0).$$  \hspace{1cm} (7.15)

Combining (7.3), (7.14), (7.15), we conclude that

$$P(t) + (1 - C\epsilon_0) \int_0^t Q(s)ds \leq P(0)$$

for all $t > 0$. This completes the Proof of Theorem 1.11.

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**References**

[1] Acheritogaray, M., Degond, P., Frouvelle, A., Liu, J.-G.: Kinetic formulation and global existence for the Hall- Magnetohydrodynamics system. Kinet. Relat. Models 4, 901–918 (2011)
[2] Balbus, S.A., Terquem, C.: Linear analysis of the Hall effect in protostellar disks. Astrophys. J. 552, 235–247 (2001)
[3] Brizard, A.J.: Comment on exact solutions and singularities of an X-point collapse in Hall magnetohydrodynamics. J. Math. Phys. 59, 061509 (2018)
[4] Carpio, A.: Asymptotic behavior for the vorticity equations in dimensions two and three. Comm. Partial Differ. Equ. 19(5–6), 827–872 (1994)
[5] Chae, D., Degond, P., Liu, J.G.: Well-posedness for Hall-magnetohydrodynamics. Ann. Inst. H. Poincaré Anal. Non Linéaire 31(3), 555–565 (2014)
[6] Chae, D., Lee, J.: On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. J. Differ. Equ. 256(11), 3835–3858 (2014)
[7] Chae, D., Schonbek, M.: On the temporal decay for the Hall-magnetohydrodynamic equations. J. Differ. Equ. 255, 3971–3982 (2013)
[8] Chae, D., Wan, R., Wu, J.: Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion. J. Math. Fluid Mech. 17(4), 627–638 (2015)
[9] Chae, D., Weng, S.: Singularity formation for the incompressible Hall-MHD equations without resistivity. Ann. Inst. H. Poincaré Anal. Non Linéaire 33(4), 1009–1022 (2016)
[10] Chae, D., Wolf, J.: On partial regularity for the 3D nonstationary Hall magnetohydrodynamics equations on the plane. SIAM J. Math. Anal. 48(1), 443–469 (2016)
[11] Chae, D., Wolf, J.: Regularity of the 3D stationary hall magnetohydrodynamic equations on the plane. Comm. Math. Phys. 354(1), 213–230 (2017)
[12] Chae, M., Kang, K., Lee, J.: Asymptotic behaviors of solutions for an aerotaxis model coupled to fluid equations. J. Korean Math. Soc. 53(1), 127–146 (2016)
[13] Dai, M.: Local well-posedness of the Hall-MHD system in $H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$. Math. Nachr. 293(1), 67–78 (2020)
[14] Dai, M.: Local well-posedness for the Hall-MHD system in optimal Sobolev spaces. J. Differ. Equ. 289, 159–181 (2021)
[48] Zhao, X.: Global well-posedness and decay characterization of solutions to 3D MHD equations with Hall and ion-slip effects. Z. Angew. Math. Phys. 71(3), 89 (2020)
[49] Zhao, X.: Space-time decay of solutions to three-dimensional MHD equations with Hall and ion-slip effects. J. Math. Phys. 62(6), 061507 (2021)

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