HAUSDORFF DIMENSIONS OF LEVEL SETS RELATED TO MOVING DIGIT MEANS

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Abstract. In this paper, we will introduce and study the lower moving digit mean \( \underline{M}(x) \) and the upper moving digit mean \( \overline{M}(x) \) of \( x \in [0,1] \) in \( p \)-adic expansion, where \( p \geq 2 \) is an integer. Moreover, the Hausdorff dimension of level set

\[
B(\alpha, \beta) = \{x \in [0,1]: \underline{M}(x) = \alpha, \overline{M}(x) = \beta\}
\]

is determined for each pair of numbers \( \alpha \) and \( \beta \) satisfying with \( 0 \leq \alpha \leq \beta \leq p - 1 \).

1. Introduction

To determine the Hausdorff dimensions of sets of numbers in which the distributions of the digits are of specific characters for some representation is a fundamental and important problem in number theory and multifractal analysis. It has a long history and there are a great many classic work on this topic, such as Besicovitch [1], Eggleston [6], Billingsley [3], Barreira et al. [2] and Fan et al. [8, 9], in which the sets are associated with frequencies of digits of numbers in different expansions, long-term time averages and ergodic limits, etc. Actually, the expressions in these sets are of a common feature, i.e., they are usually described by the arithmetic mean. Different from this, in this paper we would like to study the moving digit means of numbers and investigate the corresponding level sets related to it in the unit interval.

The moving digit mean is derived from the concept of tangential dimension at a point for measure studied by Guido and Isola [11, 12]. To be precise, if the measure is a Bernoulli measure, then the tangential dimension is of a linear relation with the moving digit mean (see [5]). More significantly, when we explore the multifractal behavior at a point of a measure, the tangential dimension is more sensitive than the local dimensions of measures, which enables the tangential dimension to provide more information than the local dimension. Thus, compared with the arithmetic mean, the moving digit mean used in the description of level sets can also provide more information on the distributions of digits of numbers. In the following, we introduce the corresponding concepts and notations.

Let \( p \geq 2 \) be an integer and \( A = \{0, 1, \ldots, p-1\} \) be the alphabet with \( p \) elements. It is known that each number \( x \in I = [0,1] \) can be expanded into an infinite non-terminating expression

\[
\sum_{n=1}^{\infty} \frac{x_n}{p^n} = 0.x_1x_2x_3\ldots, \quad \text{where } x_n \in A, \ n \geq 1,
\]

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which is called the \( p \)-adic expansion of \( x \). Denoted by \( S_n(x) = \sum_{i=1}^{n} x_i, n \geq 1 \), the \( n \)-th partial sum of \( x \). Let \( T : I \to I \) be the shift operator defined by
\[
Tx = 0.x_2x_3x_4 \ldots, \quad \text{for any } x = 0.x_1x_2x_3 \ldots \in I.
\]

Let \( x \in I \), we call
\[
(1.1) \quad M(x) = \lim_{n \to \infty} \lim_{m \to \infty} \frac{S_n(T^m x)}{n} \quad \text{and} \quad \bar{M}(x) = \lim_{n \to \infty} \lim_{m \to \infty} \frac{S_n(T^m x)}{n}
\]
the lower and upper moving digit means of \( x \) respectively.

Remark 1.1. Let \( n \geq 1 \). Denote by
\[
M_n(x) = \lim_{m \to \infty} \frac{S_n(T^m x)}{n} \quad \text{and} \quad \bar{M}_n(x) = \lim_{m \to \infty} \frac{S_n(T^m x)}{n}
\]
the \( n \)-th lower and upper moving digit means respectively. It is easy to check that, for any \( x \in I \), the two sequences \( \{ -M_n(x) \}_{n \geq 1} \) and \( \{ \bar{M}_n(x) \}_{n \geq 1} \) are both subadditive since they satisfy the inequalities:
\[
-M_{m+n}(x) \leq -M_m(x) - M_n(x) \quad \text{and} \quad \bar{M}_{m+n}(x) \leq \bar{M}_m(x) + \bar{M}_n(x), \quad m, n \geq 1.
\]
Thus, the limits of \( M_n(x) \) and \( \bar{M}_n(x) \) always exist as \( n \to \infty \). So, the definitions of \( M(x) \) and \( \bar{M}(x) \) are both reasonable.

To investigate the influence of the lower and upper moving digit means on the distribution of digits of numbers, define the level set
\[
(1.2) \quad B(\alpha, \beta) = \{ x \in I : \ M(x) = \alpha, \bar{M}(x) = \beta \}, \quad \text{where } 0 \leq \alpha \leq \beta \leq p - 1,
\]
which may be called Banach set with lower level \( \alpha \) and upper level \( \beta \). In this paper, we would like to determine the Hausdorff dimension of set \( B(\alpha, \beta) \) for any \( p \geq 2 \), which is a non-trivial and meaningful generalization of the work in [5]. First, we give the definition of \( p \)-adic entropy function below.

Let \( 0 \leq \alpha \leq p - 1 \) and \( \delta > 0 \). Denote
\[
H(\alpha, n, \delta) = \left\{ x_1x_2 \cdots x_n \in \mathbb{A}^n : n(\alpha - \delta) < \sum_{i=1}^{n} x_i < n(\alpha + \delta) \right\}
\]
and \( h(\alpha, n, \delta) = \text{Card} H(\alpha, n, \delta) \). Here and in the sequel, the symbol Card is the cardinality of some finite set. Furthermore, define the \( p \)-adic entropy function as
\[
(1.3) \quad h(\alpha) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h(\alpha, n, \delta)}{(\log p)n}, \quad 0 \leq \alpha \leq p - 1.
\]
It is evident that \( 0 \leq h(\alpha) \leq 1 \) for any \( 0 \leq \alpha \leq p - 1 \).

Denote by \( \dim_H \) the Hausdorff dimension of some set. Then we have

**Theorem 1.2.** For any \( 0 \leq \alpha \leq \beta \leq p - 1 \), we have
\[
(1.4) \quad \dim_H B(\alpha, \beta) = \sup_{\alpha \leq t \leq \beta} h(t).
\]

In particular, let \( 0 \leq \alpha \leq p - 1 \) and write
\[
(1.5) \quad B(\alpha) := B(\alpha, \alpha) = \{ x \in I : M(x) = \bar{M}(x) = \alpha \}.
\]
Then we may obtain immediately that

**Corollary 1.3.** For any \( 0 \leq \alpha \leq p - 1 \), we have \( \dim_H B(\alpha) = h(\alpha) \).
Note that we will discuss another set $B^*(\alpha)$ which is related to the moving digit mean $M(x)$ in the end of this paper. As a result, $B(\alpha)$ is different from and larger than $B^*(\alpha)$ in the sense of Hausdorff dimension.

Moreover, in the case $p = 2$, we can give an explicit expression to the binary entropy function $h_2$ by the following calculation:

\begin{equation}
 h_2(\alpha) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \sum_{i=\lceil n(\alpha-\delta) \rceil}^{\lceil n(\alpha+\delta) \rceil} \binom{n}{i}}{(\log 2)n} = -\alpha \log \alpha - (1-\alpha) \log(1-\alpha),
\end{equation}

where $\binom{n}{i}$ is the binomial coefficient. Thus, the following conclusion can be easily obtained by Theorem 1.2 and Corollary 1.3.

**Corollary 1.4.** Let $p = 2$. If $0 \leq \alpha \leq \beta \leq 1$, then $\dim H B^*(\alpha, \beta) = \sup_{\alpha \leq t \leq \beta} h_2(t)$. If $0 \leq \alpha \leq 1$, then $\dim H B(\alpha) = h_2(\alpha)$.

In the present paper, the readers are assumed to know well the definitions and basic properties of Hausdorff dimension and Hausdorff measure. For these and more related theory, one can refer to Falconer’s book [7].

The structure of this paper is as follows. In the next section, the concepts of $p$-adic lower and upper entropy functions are introduced, their relations to the $p$-adic entropy function are also shown. In Section 3, the concepts Besicovitch sets in $p$-adic expansion are introduced. Moreover, their Hausdorff dimensions are also determined, which generalizes an early work of Besicovitch. In Section 4, we will introduce some special Moran sets and determine their Hausdorff dimension for ready use. The last section is devoted to the proof of Theorem 1.2; some further discussion are also presented there.

## 2. $p$-adic Entropy Function

In this section, we will introduce the definitions of $p$-adic lower entropy function $h(\alpha)$ and $p$-adic upper entropy function $\bar{h}(\alpha)$, and then present some properties about the two functions and the $p$-adic entropy function $h(\alpha)$.

Let $0 \leq \alpha \leq p-1$, $n \geq 1$ and $\delta > 0$. Denote

$$\bar{H}(\alpha, n, \delta) = \left\{ x_1x_2 \cdots x_n \in A^n : \sum_{i=1}^{n} x_i < n(\alpha + \delta) \right\}$$

and $\bar{h}(\alpha, n, \delta) = \text{Card} \bar{H}(\alpha, n, \delta)$. Define the $p$-adic upper entropy function

\begin{equation}
 \bar{h}(\alpha) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \bar{h}(\alpha, n, \delta)}{(\log p)n}.
\end{equation}

Oppositely, denote

$$H(\alpha, n, \delta) = \left\{ x_1x_2 \cdots x_n \in A^n : \sum_{i=1}^{n} x_i > n(\alpha - \delta) \right\}$$

and $h(\alpha, n, \delta) = \text{Card} H(\alpha, n, \delta)$. Define the $p$-adic lower entropy function

\begin{equation}
 h(\alpha) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h(\alpha, n, \delta)}{(\log p)n}.
\end{equation}

Note that we have $0 \leq h(\alpha), \bar{h}(\alpha) \leq 1$ and the limits in (2.1) and (2.2) both exist since $\bar{h}(\alpha, n, \delta)$ and $h(\alpha, n, \delta)$ are increasing for $\delta > 0$. Moreover, for the two functions $h(\alpha)$ and $\bar{h}(\alpha)$, we have the following relation between them.
Theorem 2.1. $h(p - 1 - \alpha) = h(\alpha)$.

Proof. For any $n$-word $x_1 \cdots x_n \in A^n$, it is easy to check that the word $(p - 1 - x_1) \cdots (p - 1 - x_n) \in H(\alpha)$ if and only if the word $x_1 \cdots x_n \in H(p - 1 - \alpha)$. So, there is a one-to-one corresponding between the two sets $H(\alpha)$ and $H(p - 1 - \alpha)$.

Recall the definition of $p$-adic function $h(\alpha)$ in (1.3). Actually, we may have

$$h(\alpha) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h(\alpha, n, \delta)}{(\log p)n} = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h(\alpha, n, \delta)}{(\log p)n},$$

according to Proposition 4.2 in [13]. Moreover, we can also discover the following properties of $p$-adic function.

Theorem 2.2. For the function $h(\alpha)$ defined on $[0, p - 1]$, we have

1. $h(0) = h(p - 1) = 0$;
2. $h(\alpha)$ is concave and continuous on $[0, p - 1]$;
3. $h(\alpha)$ is symmetric with respect to the line $\alpha = (p - 1)/2$. That is, we have $h(\alpha) = h(p - 1 - \alpha)$ for any $0 \leq \alpha \leq p - 1$. It follows that $h(\alpha)$ is increasing on $[0, (p - 1)/2]$ and decreasing on $[(p - 1)/2, p - 1]$;
4. If $0 \leq \alpha \leq (p - 1)/2$, then $h(\alpha) = h(\alpha)$; if $(p - 1)/2 \leq \alpha \leq p - 1$, then $h(\alpha) = h(\alpha)$.

Proof. (1) The conclusion $h(0) = 0$ is followed by the the estimation

$$h(0) \leq \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \frac{n^{\alpha \delta}}{(n\delta - 1)!}}{(\log p)n} \leq \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \frac{\delta \log \frac{n^{n \delta}}{(n\delta - 1)!}}{(\log p)n} = \lim_{\delta \to 0} \frac{\delta \log \frac{n^{n \delta}}{(n\delta - 1)!}}{(\log p)n} = 0.$$

Here, the third equality is followed by the well-known Stirling’s approximation and $y! = y(y - 1) \cdots (y - [y])$ if $y > 0$. The other conclusion $h(p - 1) = 0$ can be deduced similarly.

(2) Let $m \geq 1$ and take $m$ $n$-words $X_1, X_2, \ldots, X_m \subset H(\alpha, n, \delta)$. It is obvious that the concatenation of these words satisfies $X_1X_2 \cdots X_m \subset H(\alpha, nm, \delta)$. Thus,

$$(h(\alpha, n, \delta))^m \leq h(\alpha, nm, \delta).$$

Let $\alpha, \beta \in (0, p - 1)$ and $s, t$ be two positive integers. Then

$$(h(\alpha, n, \delta))^s (h(\beta, n, \delta))^t \leq h(\alpha, ns, \delta)h(\beta, nt, \delta) \leq h\left(\frac{s\alpha + t\beta}{p + q}, n(s + t), \delta\right).$$

Hence,

$$\frac{s}{s + t} h(\alpha) + \frac{t}{s + t} h(\beta) \leq h\left(\frac{s\alpha + t\beta}{s + t}, n(s + t), \delta\right).$$

Since $s$ and $t$ are two arbitrary positive integers, we have

$$\lambda h(\alpha) + (1 - \lambda) h(\beta) \leq h(\lambda \alpha + (1 - \lambda) \beta)$$

for any $0 < \lambda < 1$. It means that the function $h(\alpha)$ is rational concavity.

By the definition of $h(\alpha)$, for any $\eta > 0$, there exists $\delta_0 > 0$ such that

$$\lim_{n \to \infty} \frac{\log h(\alpha, n, \delta)}{(\log p)n} < h(\alpha) + \frac{\eta}{2}.$$

[13]
for any \(0 < \delta < \delta_0\). Take a number \(\gamma\) which satisfies \(|\alpha - \gamma| < \delta/2\). Then, we have \(H(\gamma, n, \delta/2) \subset H(\alpha, n, \delta)\). It yields that \(h(\gamma, n, \delta/2) \leq h(\alpha, n, \delta)\). Moreover, by the definition of \(h(\gamma)\), there exists some \(\delta_1\) satisfying \(0 < \delta_1 < \delta_0\) such that

\[
\lim_{n \to \infty} \frac{\log h(\gamma, n, \delta/2)}{(\log p)n} + \frac{\eta}{2} \leq \lim_{n \to \infty} \frac{\log h(\alpha, n, \delta)}{(\log p)n} + \frac{\eta}{2},
\]

for any \(0 < \delta < \delta_1\). This, together with (2.5), yields that \(h(\gamma) < h(\alpha) + \eta\) if \(|\alpha - \gamma| < \delta/2\), where \(0 < \delta < \delta_1\). It implies that the function \(h(\alpha)\) is upper semi-continuous. So, \(h(\alpha)\) is concave and then continuous on \(0, p - 1\).

Next, we show that \(h(\alpha)\) is continuous at \(\alpha = 0\) and \(\alpha = p - 1\). Since \(h(\alpha, n, \delta) \leq \tilde{h}(\alpha, n, \delta)\), similar to the estimation (2.4) we have

\[
\lim_{\alpha \to 0^+} h(\alpha) \leq \lim_{\alpha \to 0^+} \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \left(\alpha^{\lfloor n(\alpha + \delta)\rfloor} / [n(\alpha + \delta)!]\right)}{(\log p)n} \leq \lim_{\alpha \to 0^+} \frac{(\alpha + \delta) \log \frac{\varepsilon}{(\alpha + \delta)}}{\log p} = \lim_{\alpha \to 0^+} \frac{\alpha \log \frac{\varepsilon}{\alpha}}{\log p} = 0.
\]

It follows that \(\lim_{\alpha \to 0^+} h(\alpha) = 0 = h(0)\). Thus, \(h(\alpha)\) is continuous at \(\alpha = 0\). Similarly, the continuity of \(h(\alpha)\) at \(\alpha = p - 1\) holds as well. Thus, \(h(\alpha)\) is concave and continuous on \([0, p - 1]\).

(3) The proof of the property \(h(\alpha) = h(p - 1 - \alpha)\) is similar to the discussion for Theorem 2.1. The monotonicity of \(h(\alpha)\) is followed by the concavity of \(h(\alpha)\) in property (2).

(4) The former part is followed by the increasing property of \(h(\alpha)\) in (3) and the inequality

\[
h(\alpha, n, \delta) \leq \tilde{h}(\alpha, n, \delta) \leq 2 \left(\left\lfloor \frac{\alpha + \delta}{2\delta} \right\rfloor + 1\right) h(\alpha, n, \delta),
\]

where \(\alpha \leq (p - 1)/2\). The second part can be dealt with in a similar way. \(\Box\)

3. Besicovitch sets

In this section, we will determine the Hausdorff dimensions of Besicovitch sets \(E(\alpha)\) and \(\tilde{E}(\alpha)\) and present at last a further property of the \(p\)-adic entropy function for ready use.

Let \(x \in I\). Denote by

\[
\bar{A}(x) = \lim_{n \to \infty} S_n(x) / n \quad \text{and} \quad \bar{A}(x) = \lim_{n \to \infty} S_n(x) / n
\]

the lower and upper digit means of \(x\), respectively. Let \(0 \leq \alpha \leq p - 1\). Define the level sets

(3.1) \(E(\alpha) = \{x \in I: A(x) \geq \alpha\}\) and \(\tilde{E}(\alpha) = \{x \in I: \bar{A}(x) \leq \alpha\}\),

which are called the Besicovitch sets in this paper. Note that the Hausdorff dimensions of \(E(\alpha)\) and \(\tilde{E}(\alpha)\) are determined by Besicovitch [1] in the case of binary expansion. For the present general case, we have

**Theorem 3.1.** Let \(0 \leq \alpha \leq p - 1\). Then

(3.2) \(\dim_H E(\alpha) = \begin{cases} 1, & 0 \leq \alpha < (p - 1)/2; \\ h(\alpha), & (p - 1)/2 \leq \alpha \leq p - 1, \end{cases}\)
and

\[
\dim_H E(\alpha) = \begin{cases} 
    h(\alpha), & 0 \leq \alpha < (p-1)/2; \\
    1, & (p-1)/2 \leq \alpha \leq p-1.
\end{cases}
\]

To prove this theorem, we would like to introduce a lemma about the Hausdorff dimensions of homogeneous Moran sets. Here, it is assumed the readers are familiar with the definition and structure of homogeneous Moran sets, for which one can see [10] for more details.

Let \(\{N_k\}_{k \geq 1}\) be a sequence of integers and \(\{c_k\}_{k \geq 1}\) be a sequence of positive numbers satisfying \(N_k \geq 2, 0 < c_k < 1\) and \(N_k c_k \leq 1\). Let \(M = M(I, \{N_k\}_{k \geq 1}, \{c_k\}_{k \geq 1})\) be the homogeneous Moran set determined by the sequences \(\{N_k\}_{k \geq 1}\) and \(\{c_k\}_{k \geq 1}\).

Denote

\[
s = \lim_{k \to \infty} \frac{\log(N_1 N_2 \cdots N_k)}{-\log(c_1 c_2 \cdots c_{k+1} N_{k+1})}.
\]

Then we have

\textbf{Lemma 3.2} (See Theorem 2.1 and Corollary 2.1 in [10]). Let \(M\) be a homogeneous Moran set, then \(\dim_H M \geq s\). Moreover, if \(\inf_{k \geq 1} c_k > 0\), then \(\dim_H M = s\).

\textbf{Proof of Theorem 3.1.} We will give only the proof of (3.3) since the conclusion (3.2) can be dealt with in a similar way.

For the second part of (3.3), it is obvious that \(\{x \in I : \lim_{n \to \infty} S_n(x)/n = (p-1)/2\} \subset \overline{E}(\alpha)\). Since \(\lim_{n \to \infty} S_n(x)/n = (p-1)/2\) for almost all \(x \in I\) by the ergodic theorem, we have

\[
\dim_H \overline{E}(\alpha) \geq \dim_H \left\{ x \in I : \lim_{n \to \infty} \frac{S_n(x)}{n} = \frac{p-1}{2} \right\} = 1.
\]

It follows that \(\dim_H \overline{E}(\alpha) = 1 \) as \((p-1)/2 \leq \alpha \leq p-1\).

For the first part of (3.3), first we will show the upper bound of Hausdorff dimension of \(\overline{E}(\alpha)\) is \(h(\alpha)\) as \(0 \leq \alpha < (p-1)/2\). For any \(\delta > 0\), we have

\[
\overline{E}(\alpha) \subset \bigcap_{l=1}^{\infty} \bigcup_{n=l}^{\infty} \bigcup_{x_1, \ldots, x_n \in I} I(x_1 \cdots x_n),
\]

where the cylinder \(I(x_1 \cdots x_n) = \{y = 0, y_1 y_2 \cdots \in I : y_1 = x_1, \ldots, y_n = x_n\}\). By the definition of \(\overline{h}(\alpha)\), for any \(\eta > 0\), there exists an integer \(N\) such that

\[
\overline{h}(\alpha, n, \delta) < p^{\eta (\overline{h}(\alpha) + \frac{\delta}{2})}, \quad \forall n > N.
\]

Then, for any \(l > N\), the \((\overline{h}(\alpha) + \eta)\)-Hausdorff measure of \(\overline{E}(\alpha)\) satisfies

\[
\mu_{p^{\eta l}}^{\overline{h}(\alpha) + \eta}(\overline{E}(\alpha)) \leq \sum_{n=l}^{\infty} \overline{h}(\alpha, n, \delta)(p^{-n})^s < \sum_{n=l}^{\infty} (p^{-\frac{\delta}{2}})^n < \infty.
\]

This implies that \(\dim_H \overline{E}(\alpha) \leq \overline{h}(\alpha) + \eta\). Thus, \(\dim_H \overline{E}(\alpha) \leq \overline{h}(\alpha) = h(\alpha)\) by the arbitrariness of \(\eta\) and [11] of Theorem 2.2.

Next, we turn to show that the lower bound of the Hausdorff dimension of \(\overline{E}(\alpha)\) is \(h(\alpha)\). For this, we will prove \(\dim_H \overline{E}(\alpha) \geq \tau\) for any \(0 < \tau < h(\alpha)\).

Since \(\tau < h(\alpha)\), we can take two sequences, one is an increasing integer sequence \(\{n_j\}_{j \geq 1}\) and the other is a decreasing positive sequence \(\{\delta_j\}_{j \geq 1}\) satisfying \(\lim_{j \to \infty} \delta_j = 0\), such that

\[
h(\alpha, n_j, \delta_j) > p^{n_j \tau}.
\]
Let $j \geq 1$ and write

$$F_j(\alpha) = \left\{ x_1 x_2 \cdots x_n \in A^n : \left| \frac{\sum_{i=1}^{n_j} x_i}{n_j} - \alpha \right| < \delta_j \right\}.$$ 

Take a positive integer sequence $\{m_i\}_{i \geq 1}$ satisfies

$$\lim_{j \to \infty} \frac{n_{j+1}}{\sum_{i=1}^{j} m_i n_i} = 0.$$ 

Denote by

$$q_j = m_1 n_1 + m_2 n_2 + \cdots + m_j n_j, \quad j \geq 1.$$ 

Based on the sequence of sets $\{F_j(\alpha)\}_{j \geq 1}$, construct the Moran set

$$F(\alpha) = \left\{ 0, x_1 x_2 \cdots x_{q_i} \cdots x_{q_{i+1}} \in F_i(\alpha)^{m_i}, \forall i \geq 1 \right\}$$

$$= 0 \cdot \prod_{i=1}^{\infty} F_i(\alpha)^{m_i}.$$ 

Here and in the sequel, if $F$ is a set of words with equal length and $m$ is a positive integer, then we use the notation $F^m$ to denote the set in which every word is the concatenations of $m$ words in the set $F$, and $F^{\infty}$ the set in which every sequence is the concatenations of infinite words in the set $F$. For the sequence of sets of words $\{F_i\}_{i \geq 1}$, $\prod_{i=1}^{\infty} F_i$ denotes the set in which every sequence is the successive concatenations of words in the set $F_i$ according to the order of natural numbers.

It is easy to see that

$$m_1 n_1 (\alpha - \delta_1) + \cdots + m_j n_j (\alpha - \delta_j) \leq S_n(x) q_j \leq m_1 n_1 (\alpha + \delta_1) + \cdots + m_j n_j (\alpha + \delta_j).$$

Since $m_i n_i \to \infty$ and $\delta_j \to 0$ as $j \to \infty$, we have

$$\lim_{j \to \infty} \frac{S_n(x)}{q_j} = \alpha$$

by the squeeze theorem. This implies the upper limit of $S_n(x)/n$ is $\alpha$. So, we have

$$F(\alpha) \subset E(\alpha)$$

and then $\dim_H E(\alpha) \geq \dim_H F(\alpha)$.

For any integer $n$ large enough, there exist two integers $j \geq 1$ and $b$ such that

$$0 \leq b < m_{k+1} \quad \text{and} \quad \sum_{i=1}^{j} m_i n_i + b n_{j+1} \leq n < \sum_{i=1}^{j} m_i n_i + (b + 1) n_{j+1}.$$ 

Then, by the first assertion of Lemma 3.3, we have

$$\dim_H F(\alpha) \geq \lim_{j \to \infty} \frac{\left( \sum_{i=1}^{j} m_i n_i + b n_{j+1} \right) r \log p}{\left( \sum_{i=1}^{j} m_i n_i + (b + 1) n_{j+1} \right) \log p - n_{j+1} r \log p} = \tau.$$ 

Thus, we obtain that $\dim_H E(\alpha) \geq \tau$, which shows the first part of (3.3).

The proof is completed now. $\square$

Moreover, denote by

$$A(x) = \lim_{n \to \infty} \frac{S_n(x)}{n}, \quad x \in I,$$

the arithmetic digit mean of $x$ if the limit exists and define the level set related to it as

$$E(\alpha) = \{ x \in I : A(x) = \alpha \}.$$
Then, by the same technique used in the proof of Theorem 3.1, we may get

**Theorem 3.3.** For any $0 \leq \alpha \leq p - 1$, we have that $\dim H E(\alpha) = h(\alpha)$.

**Corollary 3.4.** $h\left(\left\lfloor \frac{p - 1}{2} \right\rfloor\right) = 1$.

**Proof.** Since $A(x) = \left\lfloor \frac{p - 1}{2} \right\rfloor$ for almost all $x \in I$, by Theorem 3.3 we have

$$1 = \dim H \left\{ x \in I : A(x) = \frac{p - 1}{2} \right\} = \dim H E\left(\frac{p - 1}{2}\right) = h\left(\frac{p - 1}{2}\right).$$

It ends the proof. □

**Remark 3.5.** By Corollary 3.4, Theorem 1.2 can be restated in details as follows: if $0 \leq \alpha \leq \beta < \left\lfloor \frac{p - 1}{2} \right\rfloor$, then $\dim H B(\alpha, \beta) = h(\beta)$; if $0 \leq \alpha \leq \left\lfloor \frac{p - 1}{2} \right\rfloor \leq \beta \leq 1$, then $\dim H B(\alpha, \beta) = 1$; if $\left\lfloor \frac{p - 1}{2} \right\rfloor < \alpha \leq \beta \leq p - 1$, then $\dim H B(\alpha, \beta) = h(\alpha)$.

### 4. Some Moran sets

In this section, we will introduce some Moran sets constructed by sets of words with bounded digit sums and then determine their Hausdorff dimensions. Based on them, we will construct suitable subsets to achieve the lower bound of Hausdorff dimension of $B(\alpha, \beta)$ in the last section.

Let $M \geq 1$ be an integer. Take two integers $P$ and $Q$ satisfying $0 \leq P \leq Q \leq (p - 1)M$. Write

$$W([P, Q], M) := \left\{ x_1x_2 \cdots x_M \in A^M : P \leq \sum_{i=1}^{M} x_i \leq Q \right\}.$$

Then define the Moran set

$$\mathcal{W}([P, Q], M) := 0.W([P, Q], M)^\infty.$$

For the size of the set $\mathcal{W}([P, Q], M)$, by the second assertion in Lemma 3.2, we can get immediately that

**Lemma 4.1.** Let $0 \leq P \leq Q \leq (p - 1)M$ and $M \geq 1$. Then

$$\dim H \mathcal{W}([P, Q], M) = \frac{\log \text{Card} W([P, Q], M)}{(\log p)M}.$$

Here and in the sequel, if $P = Q$, then write $W([P, Q], M)$ as $W(P, M)$ and $\mathcal{W}([P, Q], M)$ as $W(P, M)$ respectively for simplicity.

Let $\alpha$ be a real number and $0 \leq \alpha \leq p - 1$. Let $n \geq 1$. Define the function

$$\omega(\alpha, n) = \text{Card} W([\alpha n], n).$$

Then the corresponding properties in the following lemma is evident.

**Lemma 4.2.** Let $0 \leq \alpha \leq p - 1$ and $n \geq 1$. Then

1. For each $\alpha$, $\omega(\alpha, n)$ is increasing with respect to $n$;
2. For each $n$, $\omega(\alpha, n)$ is constant on $[(k - 1)/n, k/n)$, where $1 \leq k \leq (p - 1)n$, with respect to $\alpha$;
3. For each $n$, $\omega(\alpha, n)$ is increasing on $[0, (p - 1)/2 + 1/n)$ and decreasing on $[(p - 1)/2 + 1/n, p - 1]$ with respect to $\alpha$.

Moreover, we have
Lemma 4.3. Let \( 0 \leq \alpha \leq p - 1 \). Then
\[
\lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor \alpha n \rfloor, n)}{(\log p)n} = h(\alpha).
\]

**Proof.** We first show that
\[
(4.1) \quad \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor \alpha n \rfloor, n)}{(\log p)n} = h(\alpha).
\]
The proof is divided into three cases: \( 0 \leq \alpha < (p - 1)/2 \), \( \alpha = (p - 1)/2 \) and \( (p - 1)/2 < \alpha \leq p - 1 \). Here, we give only the proof of the first case.

Take \( \delta > 0 \) such that \( \alpha + \delta < (p - 1)/2 \). Since
\[
\text{Card } W(\lfloor (\alpha + \delta)n \rfloor, n) \leq \frac{n^{\alpha\delta}}{(\delta - 1)!} \text{Card } W(\lfloor \alpha n \rfloor, n),
\]
by the Stirling’s approximation we have
\[
\lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor (\alpha + \delta)n \rfloor, n)}{(\log p)n} \leq \lim_{n \to \infty} \frac{\log n^{\alpha\delta}}{(\log p)n} + \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor \alpha n \rfloor, n)}{(\log p)n} = \frac{\delta \log \frac{p}{e}}{\log p} + \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor \alpha n \rfloor, n)}{(\log p)n}.
\]
Let \( \delta \to 0 \) in both sides, since \( \lim_{\delta \to 0} (\delta \log \frac{p}{e})/\log p = 0 \), we have
\[
(4.2) \quad \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor (\alpha + \delta)n \rfloor, n)}{(\log p)n} \leq \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor \alpha n \rfloor, n)}{(\log p)n}.
\]
Moreover, by the properties of function \( \omega(\alpha, n) \) in Lemma 4.2, we have
\[
\text{Card } W(\lfloor (\alpha + \delta)n \rfloor, n) \leq \text{Card } W(\lfloor \lfloor (\alpha - \delta)n \rfloor + 1, \lfloor (\alpha + \delta)n \rfloor \rfloor, n) = h(\alpha, n, \delta) \leq (|\alpha\delta| + 1) \text{Card } W(\lfloor (\alpha + \delta)n \rfloor, n).
\]
It follows that
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor (\alpha + \delta)n \rfloor, n)}{(\log p)n} = \lim_{\delta \to 0} \frac{\log h(\alpha, n, \delta)}{(\log p)n} = h(\alpha).
\]
This, together with (4.2), yields that
\[
h(\alpha) \leq \lim_{n \to \infty} \frac{\log \text{Card } W(\lfloor \alpha n \rfloor, n)}{(\log p)n}.
\]
On the other hand, the inequality for the opposite direction is apparently true. So, the equality (4.1) is established.

Since the equality \( \lim_{n \to \infty} \log \text{Card } W(\lfloor \alpha n \rfloor, n)/(\log p)n = h(\alpha) \) can be proved as the above way, the proof of this lemma is finished now. \( \square \)

In the sequel, we will construct a Moran set \( W_M(\alpha) \), where \( 0 \leq \alpha < p - 1 \), to obtain the lower bound of Hausdorff dimension of \( B(\alpha, \beta) \) in Theorem 1.2. At first, we construct recursively two sequences of sets of words \( \{W_n(\alpha, M)\}_{n=1}^{\infty} \) and \( \{V_n(\alpha, M)\}_{n=1}^{\infty} \) below.

Let \( M \) be sufficiently large such that \( |\alpha M| + 1 < (p - 1)M \). For brevity, write
\[
W_1(\alpha, M) = W(\lfloor \alpha M \rfloor, M), \quad V_1(\alpha, M) = W(\lfloor \alpha M \rfloor + 1, M).
\]
Lemma 4.5. Then we have

\[
W_{n+1}(\alpha, M) = \{x_1 \cdots x_{2^n} M \in W(\alpha^{2^n} M, 2^n M) : \\
\quad x_{2^n-1} M_{i+1} \cdots x_{2^n-1} M_{(i+1)} \in W_n(\alpha, M) \cup V_n(\alpha, M), i = 0, 1\}, \\
V_{n+1}(\alpha, M) = \{x_1 \cdots x_{2^n} M \in W(\alpha^{2^n} M + 1, 2^n M) : \\
\quad x_{2^n-1} M_{i+1} \cdots x_{2^n-1} M_{(i+1)} \in W_n(\alpha, M) \cup V_n(\alpha, M), i = 0, 1\}.
\]

The above definitions are valid since the estimation
\[
(4.3) \quad 2[\alpha^k M] < [\alpha^{k+1} M] + 1 \leq 2 ([\alpha^k M] + 1)
\]
holds for any \(0 < \alpha < p - 1\) and \(k \geq 0\).

Remark 4.4. For any \(0 \leq i \leq n - 1\), we can decompose uniquely each word in \(W_n(\alpha, M)\) and \(V_n(\alpha, M)\) into successive concatenations of \(2^i M\)-words, the sum of elements in each \(2^i M\)-word is \([\alpha^{2^i} M]\) or \([\alpha^{2^i} M] + 1\).

Based on the family of sets of \(\alpha\)-words \(\{W_n(\alpha, M)\}_{n=1}^{\infty}\), define the Moran set
\[
W_M(\alpha) := 0. \prod_{n=1}^{\infty} W_n(\alpha, M).
\]

Then we have

Lemma 4.5. Let \(0 \leq \alpha < p - 1\), then
\[
(4.4) \quad \lim_{M \to \infty} \dim_H W_M(\alpha) = h(\alpha).
\]

Proof. For the case \(0 \leq \alpha < (p - 1)/2\), take \(M\) to be large enough such that \([\alpha M] + 1 < [(p - 1) M/2]\). By the monotonicity of function \(\omega\) in Remark 4.4 we have

\[
\text{Card } W([\alpha M], M) = \omega(\alpha, M) \leq \omega(\alpha + 1/M, M) = \text{Card } W([\alpha M] + 1, M).
\]

From this and the structures of words in \(W_M(\alpha)\) in Remark 4.4 we know that
\[
(4.5) \quad \dim_H W([\alpha M], M) \leq \dim_H W_M(\alpha) \leq \dim_H W([\alpha M], [\alpha M] + 1, M).
\]

Moreover, by Lemma 4.3 we have

\[
\dim_H W([\alpha M], M) = \frac{\log \text{Card } W([\alpha M], M)}{(\log p) M}
\]

and

\[
\dim_H W([\alpha M], [\alpha M] + 1, M) = \frac{\log (\text{Card } W([\alpha M], M) + \text{Card } W([\alpha M] + 1, M))}{(\log p) M}.
\]

Thus, by Lemma 4.3 we may obtain that
\[
\lim_{M \to \infty} \dim_H W([\alpha M], M) = \lim_{M \to \infty} \dim_H W([\alpha M], [\alpha M] + 1, M) = h(\alpha).
\]

This, together with (4.5), leads to the conclusion (4.4).
On the other hand, for the case \((p - 1)/2 \leq \alpha < p - 1\), we can get similarly that
\[
\dim_H \mathcal{W}([\alpha M] + 1, M) \leq \dim_H \mathcal{W}_M(\alpha) \leq \dim_H \mathcal{W}([\alpha M], [\alpha M] + 1, M)
\]
for sufficiently large \(M\). The proof of the remainder is similar to the first case. We omit the details here. The proof is completed now. \(\square\)

5. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2. First, we would like to present the following lemma to reveal the relations among the lower and upper digit means of \(x \in I\) and the lower and upper moving digit means of \(x \in I\), which will be used to achieve the upper bound of Hausdorff dimension of \(B(\alpha, \beta)\).

**Lemma 5.1.** For any \(x \in I\), we have
\[
(5.1) \quad M(x) \leq A(x) \leq \bar{A}(x) \leq \bar{M}(x).
\]

*Proof.* It suffices to show that \(\bar{A}(x) \leq \bar{M}(x)\). Write \(\bar{M}(x) = \beta \in [0, p - 1]\). Then, for any \(\epsilon > 0\), there exists an integer \(N > 0\), such that
\[
\lim_{m \to \infty} \frac{S_n(T^m x)}{n} < \beta + \epsilon, \quad \forall n \geq N.
\]
Furthermore, there exists an integer \(Q = Q(n) \geq 1\) such that
\[
\frac{S_n(T^m x)}{n} < \beta + \epsilon, \quad \text{i.e.,} \quad \sum_{i=1}^{n} x_{m+i} \leq \lfloor n(\beta + \epsilon) \rfloor
\]
for any \(m \geq Q\). Suppose \(t \geq Q\) and \(t = Q + r + kn\), where \(0 \leq r \leq n - 1\), then
\[
\frac{S_i(x)}{t} \leq \frac{Q + r + kn(\beta + \epsilon)}{Q + r + kn}.
\]
Let \(t \to \infty\), then \(k \to \infty\) and
\[
\lim_{i \to \infty} \frac{S_i(x)}{t} \leq \frac{\lfloor n(\beta + \epsilon) \rfloor}{n} \leq \beta + \epsilon.
\]
Since \(\epsilon\) is arbitrary, we have \(\bar{A}(x) \leq \beta = \bar{M}(x)\). \(\square\)

Next, we would like to present a lemma for the computation of lower bound of the Hausdorff dimension of \(B(\alpha, \beta)\). Let \(M\) be a subset of \(\mathbb{N}\). We say the set \(M\) is of density \(\rho \in [0, 1]\) in \(\mathbb{N}\) if
\[
\lim_{n \to \infty} \frac{\text{Card}\{i \in M: i \leq n\}}{n} = \rho.
\]
Write \(\mathbb{N} \setminus M = \{n_i\}_{i \geq 1}\) where \(n_i < n_{i+1}\) for all \(i \geq 1\). Define a mapping \(\varphi_M: I \to I\) by
\[
0.x_1x_2 \ldots \mapsto 0.x_{n_1}x_{n_2} \ldots .
\]
Under the mapping \(\varphi_M\), for any given subset \(D \subset I\), we may obtain another set \(\varphi_M(D) = \{\varphi_M(x): x \in D\}\). Moreover, we have

**Lemma 5.2** (See Lemma 2.3 in [4]). Suppose that the set \(M\) is of density zero in \(\mathbb{N}\). Then for any set \(D \subset I\) we have \(\dim_H D = \dim_H \phi_M(D)\).
Lemma 5.2 implies that for a set $D$, its Hausdorff dimension is invariant after deleting the digits, for which the set of their positions is of density zero in $\mathbb{N}$, from the sequences of numbers in $D$.

Now, we are ready to give the proof of Theorem 1.2

**Proof of Theorem 1.2.** According to Remark 3.5, the proof is divided into three parts:

1. For the upper bound, by Lemma 5.1, we have $\dim_H B(\alpha, \beta) < h(\beta)$ if $0 \leq \alpha \leq \beta < (p-1)/2$;
2. For the lower bound, construct the set $W_M(\alpha, \beta) = \{x \in W_n(\alpha, M) \times W_n(\beta, M)^n \}$.
   Then we have: (a) $W_M(\alpha, \beta) \subset B(\alpha, \beta)$; (b) $\dim_H W_M(\alpha, \beta) = \dim_H W_M(\beta)$.
   For the proof of (a), note that for any $i \geq 1$, each word in the set
   $$\prod_{n=1}^{i} (W_n(\alpha, M) \times W_n(\beta, M)^n),$$
   is of length $i2^iM$ and the length of words in $W_n(\alpha, M)$ and $W_n(\beta, M)$, $n > i$, are of common lengths $2^{n-1}M$. So, we may decompose every number $x \in W_M(\alpha, \beta)$ into successive concatenations of $2^iM$-words. Take $n$ to be sufficiently large and write $n = k2^iM + r$, $0 \leq r \leq 2^iM - 1$. Then
   $$\frac{(k-1)[\alpha 2^iM]}{k2^iM + r} \leq M_n(x) \leq \frac{(k+2)([\alpha 2^iM] + 1)}{k2^iM + r}.$$ 
   Let $n \to \infty$, then $k \to \infty$. It yields that
   $$\frac{[\alpha 2^iM]}{2^iM} \leq M(x) \leq \frac{[\alpha 2^iM] + 1}{2^iM}.$$
   Since this inequality holds for all $i \geq 1$, we have $M(x) = \alpha$ by letting $i \to \infty$.
   The other conclusion $M(x) = \beta$ can be deduced in a similar manner. So, we have $W_M(\alpha, \beta) \subset B(\alpha, \beta)$.

3. For the second assertion (b), the set of positions occupied by the words in sets $W_n(\alpha, M)$, $n \geq 1$, is of density zero for any $x \in W_M(\alpha, \beta)$. By deleting all these words in the sequences of numbers in $W_M(\alpha, \beta)$, we obtain the set $W_M(\beta)$. Then, (b) is established by Lemma 5.2.

By (a) and (b), we obtain that
   $$\dim_H B(\alpha, \beta) \geq \dim_H W_M(\alpha, \beta) = \dim_H W_M(\beta).$$

Let $M \to \infty$, it yields that $\dim_H B(\alpha, \beta) \geq h(\beta)$ according to Lemma 4.3. The proof of this part is finished.

(2) This part is split into four cases for consideration: i) $\alpha < (p-1)/2 < \beta$; ii) $\alpha = (p-1)/2 < \beta$; iii) $\alpha < \beta = (p-1)/2$; iv) $\alpha = \beta = (p-1)/2$. 


Case i): \( \alpha < (p - 1)/2 < \beta \). Since \( \log ((p - 1)/2) = 1 \), for \( \epsilon \) small enough, there exists \( \delta_0 > 0 \) and \( n_0 > 0 \) such that

\[
\alpha < \frac{p - 1}{2} - \delta_0 < \frac{p - 1}{2} + \delta_0 < \beta \quad \text{and} \quad \log h(\frac{p - 1}{2}, n_0, \delta_0) > 1 - \epsilon.
\]

Based on the set

\[
H\left(\frac{p - 1}{2}, n_0, \delta_0\right) = \left\{ x_1 \cdots x_{n_0} \in A^{n_0} : n_0\left(\frac{p - 1}{2} - \delta_0\right) < \sum_{i=1}^{n_0} x_i < n_0\left(\frac{p - 1}{2} + \delta_0\right) \right\},
\]

define

\[
(5.2) \quad \mathcal{H}\left(\frac{p - 1}{2}, n_0, \delta_0\right) = 0.H\left(\frac{p - 1}{2}, n_0, \delta_0\right) \to \infty.
\]

Then, by Lemma 4.1, we have

\[
\dim_H \mathcal{H}\left(\frac{p - 1}{2}, n_0, \delta_0\right) = \log \text{Card} H\left(\frac{p - 1}{2}, n_0, \delta_0\right) = \log h(\frac{p - 1}{2}, n_0, \delta_0) > 1 - \epsilon.
\]

Now, construct the set

\[
(5.3) \quad \mathcal{W}_{M, n_0, \delta_0}(\alpha, \beta) = 0. \prod_{n=1}^{\infty} \left( W_n(\alpha, M) \times H\left(\frac{p - 1}{2}, n_0, \delta_0\right) \right) \times W_n(\beta, M).
\]

Similar to the proof of the foregoing part, we can also deduce that

\[ \mathcal{W}_{M, n_0, \delta_0}(\alpha, \beta) \subset B(\alpha, \beta) \quad \text{and} \quad \dim_H \mathcal{W}_{M, n_0, \delta_0}(\alpha, \beta) = \dim_H \mathcal{H}(\frac{p - 1}{2}, n_0, \delta_0). \]

Thus, we have

\[
\dim_H B(\alpha, \beta) \geq \dim_H \mathcal{H}(\frac{p - 1}{2}, n_0, \delta_0) > 1 - \epsilon.
\]

It proves this case since \( \epsilon \) is arbitrary.

Case ii): \( \alpha = (p - 1)/2 < \beta \). In this case, take \( \delta_0 \) satisfying \( (p - 1)/2 < (p - 1)/2 + \delta_0 < \beta \). Put

\[
H'\left(\frac{p - 1}{2}, n_0, \delta_0\right) = \left\{ x_1 \cdots x_{n_0} \in A^{n_0} : \frac{p - 1}{2} n_0 < \sum_{i=1}^{n_0} x_i < (\frac{p - 1}{2} + \delta_0) n_0 \right\}
\]

and

\[
(5.3) \quad \mathcal{H}'\left(\frac{p - 1}{2}, n_0, \delta_0\right) = 0.H'\left(\frac{p - 1}{2}, n_0, \delta_0\right) \to \infty.
\]

Then, similar to the proof of Lemma 4.3, we have

\[
\lim_{\delta_0 \to 0} \lim_{n_0 \to \infty} \frac{\log \text{Card} H'(\frac{p - 1}{2}, n_0, \delta_0)}{(\log p) n_0} = \lim_{\delta_0 \to 0} \lim_{n_0 \to \infty} \frac{\log \text{Card} H(\frac{p - 1}{2}, n_0, \delta_0)}{(\log p) n_0} = 1.
\]

So, for any \( \epsilon > 0 \), there exists \( n_0 \) and \( \delta_0 \) such that

\[
\dim_H \mathcal{H}'(\frac{p - 1}{2}, n_0, \delta_0) > 1 - \epsilon.
\]

Next, construct the Moran set \( \mathcal{W}'_{M, n_0, \delta_0}(p - 1)/2, \beta) \) as \( \mathcal{W}_{M, n_0, \delta_0}(\alpha, \beta) \) in (5.3) by replacing \( H((p - 1)/2, n_0, \delta_0) \) with \( H'(p - 1)/2, n_0, \delta_0) \). For the remaining proof of this case, it is just similar to the above discussion in Case i).
Case iii): $\alpha < \beta = (p-1)/2$. It can be proved as that of Case ii).

Case iv): $\alpha = \beta = (p-1)/2$. In this case, take $n_0$ to be even and consider the set

$$W_{n_0} \left( \frac{p-1}{2} \right) = 0.W \left( \frac{p-1}{2}, n_0 \right),$$

where

$$W \left( \frac{p-1}{2}, n_0 \right) = \left\{ x_1 \cdots x_{n_0} \in A^{n_0} : \sum_{i=1}^{n_0} x_i = \frac{p-1}{2} n_0 \right\}.$$

Then we have

$$\lim_{n_0 \to \infty} \text{dim}_H W_{n_0} \left( \frac{p-1}{2} \right) = h \left( \frac{p-1}{2} \right) = 1$$

by Lemma 4.3. It is evident that

$$W_{n_0} \left( \frac{p-1}{2} \right) \subset B \left( \frac{p-1}{2}, \frac{p-1}{2} \right) = B \left( \frac{p-1}{2} \right).$$

Thus, $\text{dim}_H B((p-1)/2) \geq \text{dim}_H W_{n_0}((p-1)/2)$. Let $n_0 \to \infty$, then we have $\text{dim}_H B((p-1)/2) = 1$.

(3) Three cases will be considered in this part.

Case 1): $(p-1)/2 < \alpha < \beta < p-1$. This case can be proved as that of part (1) and we omit the details here.

Case 2): $(p-1)/2 < \alpha < \beta = p-1$. First, by Lemma 5.1 we may obtain that $B(\alpha, p-1) \subset E(\alpha)$. So, $\text{dim}_H B(\alpha, p-1) \leq \text{dim}_H E(\alpha) = h(\alpha)$.

Second, construct the set

$$W_M(\alpha, p-1) = 0. \prod_{n=1}^{\infty} (W_n(\alpha) \times (p-1)^n),$$

where $(p-1)^n$ means the word $(p-1) \cdots (p-1)$ of length $n$. Then we can deduce similarly that

$$W_M(\alpha, p-1) \subset B(\alpha, p-1) \quad \text{and} \quad \text{dim}_H W_M(\alpha, p-1) = \text{dim}_H W_M(\alpha).$$

Thus, $\text{dim}_H B(\alpha, p-1) \geq \text{dim}_H W_M(\alpha) = h(\alpha)$.

The above two assertions imply that $\text{dim}_H B(\alpha, p-1) = h(\alpha)$.

Case 3): $\alpha = \beta = p-1$. Since $B(p-1, p-1) \subset E(p-1)$ and $\text{dim}_H E(p-1) = h(p-1) = 0$, we obtain that $\text{dim}_H B(p-1, p-1) = 0 = h(p-1)$.

The proof is finished now. \qed

At last, it should be pointed out that we can even study the moving digit mean of $x$:

$$M(x) = \lim_{n \to \infty} \lim_{m \to \infty} \frac{S_n(T^m x)}{n}, \quad x \in I.$$

Moreover, define the level sets related to it as

$$B^*(\alpha) = \{ x \in I : M(x) = \alpha \}, \quad 0 \leq \alpha \leq p-1.$$

In this situation, the set $B^*(\alpha)$ is somewhat trivial because we have

**Theorem 5.3.** Let $0 \leq \alpha \leq p-1$. If $\alpha = 0, 1, \ldots, p-1$, then $B^*(\alpha)$ is a countable set. Otherwise, $B^*(\alpha)$ is an empty set. Hence, we always have

$$\text{dim}_H B^*(\alpha) = 0$$

for any $0 \leq \alpha \leq p-1$.  

Proof. Let $\alpha = i$, where $i = 0, 1, \ldots, p-1$. Then each number in $B^*(\alpha)$ is ultimately 1-periodic ending with $i^\infty$, that is

\[ B^*(\alpha) = \{ x \in I : x = 0.x_1x_2 \ldots x_n \cdot i^{\infty}, n \geq 1 \}. \]

Thus, $B^*(\alpha)$ is countable. On the other hand, if $i < \alpha < i + 1$, where $i = 0, 1, \ldots$, or $p - 2$, then for any $x \in B^*(\alpha)$ and $n \geq 1$ the limit $\lim_{m \to \infty} S_n(T^m x)/n$ does not exist according to the proof by contradiction and Cauchy’s criterion for convergence of sequences. It follows that $B^*(\alpha) = \emptyset$. \qed

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