THE BRST OPERATOR FOR THE LARGE $N = 4$
SUPERCONFORMAL ALGEBRA

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Abstract

We review the detailed structure of the large $N = 4$ superconformal algebra, and construct its BRST operator which constitutes the main object for analyzing $N = 4$ strings. We then derive the general condition for the nilpotency of the BRST operator and show that there exists a line of critical $N = 4$ string theories.
1 Introduction

It has long been known that bosonic strings and superstrings share many common features. For example, it is often observed that any calculation that can be done in bosonic strings can also be repeated in superstrings with just a few additional technical details. It is tempting then to speculate that all these string theories are deeply related and could be considered as different phases of a single unified theory. In fact, it has been shown that string theories with $N = 0, 1$ supersymmetries can be embedded into those with $N = 1, 2$, respectively [1-5]. It has also been shown that $N = 2$ superstrings can be regarded as spontaneously broken phases of $N = 4$ strings [3], and, more recently, that there exists a hierarchy of embeddings in which the $N$ string is embedded into the $N + 1$ string [7]. These $N$ strings are the ones based on the superconformal algebras (SCA) found by Ademollo et al. [8]. Out of these algebras the most intriguing is the $N = 4$, which admits two independent central extensions [6] (and can be reduced to the so-called small $N = 4$, admitting only one central extension and possessing an $SU(2)$ Kac-Moody algebra). For $N > 4$ there are no possible central extensions and the corresponding BRST charge is automatically nilpotent. It is the purpose of this paper to analyze instead the BRST operator for the large $N = 4$ SCA (i.e. the one with two central extensions), which is the key object to examine the physical states and to construct BRST invariant vertices for the $N = 4$ superstring. This is, in fact, the $N = 4$ superstring that enters the hierarchy constructed in ref. [7]. Therefore, we are going to summarize the detailed structure of the large $N = 4$ SCA in terms of components as well as in $N = 2$ superfields. We present also the explicit connection between the two descriptions. We then construct the BRST operator for the corresponding string theory and obtain the condition for its nilpotency. This analysis shows the existence of a one-parameter family of critical $N = 4$ string theories. As an aside, we present a useful technique which allows us to immediately write down the BRST operator once the operator algebra is given. We believe that our results will be useful in further studies of the $N = 4$ string theory.

The paper is organized as follows. In sect. 2, we review the component description of the large $N = 4$ SCA and derive its BRST operator. Our method of derivation is explained in appendix A. We then present in sect. 3 the equivalent results in terms of
$N = 2$ superfields. The nilpotency condition for the BRST operator is examined in sect. 4. Finally, sect. 5 reviews how to recognize the small $N = 4$ SCA as a subalgebra of the large $N = 4$ SCA using $N = 2$ superfields. This may be useful in trying to construct an embedding of the small $N = 4$ string into the large one. The ghost generators which appear in the BRST charges are collected in appendix B.

2 Large $N = 4$ SCA and its BRST operator in components

In this section, we first summarize the large $N = 4$ SCA and construct its BRST operator. The algebra is well known \[8, 9, 10\], but we try to make the group structure clearer. This also serves to establish our notation.

The large $N = 4$ SCA consists of the energy-momentum tensor $T$, four supersymmetry generators $G_a$, ($a = (1, 1), (2, 1), (1, 2), (2, 2)$), two commuting sets of $SU(2)$ currents $J^{A,i}$, ($A = (+, -)$ distinguishes between the two sets of currents and $i = (+, -, 3)$ is the index for the $SU(2)$ generators in the Cartan basis), four fermionic generators $F_a$ and one $U(1)$ current $J$. The fermionic generators $G_a$ and $F_a$ are doublets with respect to the two $SU(2)$ currents and the first and second entries in their suffices refer to these two $SU(2)$. The operator products (OPEs) for the $N = 4$ algebra in these components reads

\[
T(z)T(w) \sim \frac{1}{2}c \left( \frac{1}{(z-w)^2} + \frac{2}{(z-w)^3} + \frac{\partial T(w)}{(z-w)^2} \right),
\]

\[
T(z)\mathcal{O}(w) \sim \frac{h \mathcal{O}(w)}{(z-w)^2} + \frac{\partial \mathcal{O}(w)}{(z-w)},
\]

\[
J^\pm,i(z)J^\pm,j(w) \sim \frac{1}{2}k^\pm g^ij \left( \frac{1}{(z-w)^2} + \frac{f^ij, k^\pm, k(w)}{(z-w)} \right),
\]

\[
J^\pm,i(z)G_a(w) \sim \mp \frac{2k^\pm}{k} R^\pm,i_a b F_b(z) + \frac{R^\pm,i_a b G_b(w)}{(z-w)},
\]

\[
J^\pm,i(z)F_a(w) \sim \frac{R^\pm,i_a b F_b(w)}{(z-w)},
\]

\[
G_a(z)G_b(w) \sim \frac{2}{3} c \delta_{ab} \left( \frac{1}{(z-w)^3} + \frac{2M_{ab}(w)}{(z-w)^2} + \frac{2\eta_{ab}T(w) + \partial M_{ab}(w)}{(z-w)} \right),
\]

\[
F_a(z)G_b(w) \sim \frac{2R^\pm,i_a J^-_i(w) - 2R^+,i_a J^+_i(w) + \eta_{ab} J(w)}{(z-w)},
\]
\begin{align}
F_a(z)F_b(w) & \sim -\frac{1}{2}k\eta_{ab}\frac{1}{(z-w)}, \\
J(z)G_a(w) & \sim \frac{F_a(w)}{(z-w)^2}, \\
J(z)J(w) & \sim -\frac{1}{2}k\frac{1}{(z-w)^2},
\end{align}

(2.1)

where $O$ stands for the generators $G, J, F$ with their dimensions given by $h_O$:

\begin{align}
h_O &= \begin{cases}
\frac{3}{2} & \text{for } G_a, \\
1 & \text{for } J^{A,i}, J, \\
\frac{1}{2} & \text{for } F_a,
\end{cases}
\end{align}

(2.2)

\begin{align}
M_{ab} &= 4\frac{k^-}{k}R_{ab}^{+,i}J_i^+ + 4\frac{k^+}{k}R_{ab}^{-,i}J_i^-,
\end{align}

(2.3)

\begin{align}
k &= k^+ + k^-,
\end{align}

(2.4)

In the Cartan basis we have the following components for the $SU(2)$ Killing metric and structure constants

\begin{align}
g^{+-} &= 2, \quad g^{33} = 1; \quad f^{+-} = 2, \quad f^{3\pm} = \pm 1,
\end{align}

(2.5)

while other components not related by symmetry are zero. Using the double index notation $a = (\alpha, \bar{\alpha})$, the $SU(2)$ representation matrices $R_{ab}^{\pm,i \alpha \beta}$ can be written as follows

\begin{align}
R_{\alpha \bar{\alpha}}^{+,i (\alpha, \bar{\alpha}) (\beta, \bar{\beta})} &= \begin{cases}
\frac{1}{2}\sigma^{i\beta}_\alpha & \text{if } \bar{\alpha} = \bar{\beta} = 1 \\
\frac{1}{2}\sigma^{i\beta}_\alpha & \text{if } \bar{\alpha} = \bar{\beta} = 2 \\
0 & \text{otherwise}
\end{cases},
R_{\alpha \bar{\alpha}}^{-,i (\alpha, \bar{\alpha}) (\beta, \bar{\beta})} &= \begin{cases}
\frac{1}{2}\sigma^{i\beta}_{\bar{\alpha}} & \text{if } \alpha = \beta = 1 \\
\frac{1}{2}\sigma^{i\beta}_{\bar{\alpha}} & \text{if } \alpha = \beta = 2 \\
0 & \text{otherwise}
\end{cases}
\end{align}

(2.6)

where $\sigma^i = (\sigma^3, \sigma^+, \sigma^-)$ are the Pauli matrices in the Cartan basis and $\bar{\sigma}^i = (\sigma^3, -\sigma^+, -\sigma^-)$. Finally, the invariant tensor $\eta_{ab}$ is given in the double index notation by

\begin{align}
\eta_{(\alpha \bar{\alpha})(\beta \bar{\beta})} = \frac{1}{2}\bar{\eta}_{\alpha \beta}\bar{\eta}_{\bar{\alpha} \bar{\beta}}, \quad \text{with } \bar{\eta}_{12} = \bar{\eta}_{21} = 1, \quad \bar{\eta}_{11} = \bar{\eta}_{22} = 0.
\end{align}

(2.7)

Indices are raised and lowered with the tensors $g^{ij}$, $\eta_{ab}$ and their inverse. This notation is basically the same as the one given in refs. [10], but we find that the above double index notation makes the $SU(2) \otimes SU(2)$ group structure more manifest.
The BRST current for the $N = 4$ SCA can be derived directly from the OPEs (2.1) by the method described in appendix A, and is given by

$$J_{BRST}(z) = cT + dJ + c^\pm J^{\pm,i} + \gamma^a G_a + \delta^a F_a + bc \partial c + d \partial (ca) + c^\pm \partial (cb^{\pm,i}) + \partial c(\frac{3}{2} \beta_a \gamma^a + \frac{1}{2} \alpha_a \delta^a) + c(\partial \beta_a \gamma^a + \partial \alpha_a \delta^a) - \frac{1}{2} f^{ij} k \epsilon_i c_j b^{\pm,k}$$

$$= 2 \frac{k^\pm}{k} R^{\pm,i} a b \partial c_i^\pm \alpha_b \gamma^a + R^{\pm,i} a b c_i^\pm (\beta_b \gamma^a + \alpha_b \delta^a) - \frac{1}{2} R^{\pm,i} b_i^\pm \partial \gamma^a \gamma^b - \frac{1}{2} k \epsilon_i c_j b^{\pm,k} + c^\pm \partial (cb^{\pm,i}),$$

(2.8)

where $(c, b), (d, a), (c^\pm, b^\pm), (\gamma^a, \beta_a), (\delta^a, \alpha_a)$ are the reparametrization, $U(1)$ current, $SU(2)$ currents, supersymmetry and spin-$\frac{1}{2}$ fermion ghosts, respectively, with correlations

$$c(z)b(w) \sim d(z)a(w) \sim \frac{1}{z-w}, \quad c^\pm_i(z)b^\pm_j(w) \sim \frac{g_{ij}}{z-w}, \gamma^a(z)\beta_b(w) \sim \delta^a(z)\alpha_b(w) \sim \frac{\delta^a_b}{z-w}. \quad (2.9)$$

We will discuss the nilpotency condition for the BRST operator in sect. 4. It can be shown that the BRST operator can be cast into the form

$$Q = \oint \frac{dz}{2\pi i} \left[ c \left( T + \frac{1}{2} T_{gh} \right) + d \left( J + \frac{1}{2} J_{gh} \right) + c_i^A \left( J^{A,i} + \frac{1}{2} J_{gh}^{A,i} \right) + \gamma^a \left( G_a + \frac{1}{2} G_{gh,a} \right) + \delta^a \left( F_a + \frac{1}{2} F_{gh,a} \right) \right],$$

(2.10)

where the generators with subscript $gh$ are those for the ghosts. The explicit forms of these generators are given in appendix B.

3 Large $N = 4$ SCA and its BRST operator in $N = 2$ superfields

The results in the previous section are rather complicated to deal with if we write all components explicitly. It is much simpler, in general, to write everything in terms of $N = 2$ superfields, as done in ref. [11], even though this makes less manifest the $SU(2) \otimes SU(2)$ group structure. In this section, we describe the equivalent results in superfields and make an explicit connection with the components.
Our superspace conventions are as follows: \(Z \equiv (z, \theta, \bar{\theta})\) denotes the super-coordinates,

\[
D = \partial_{\theta} - \frac{1}{2} \bar{\theta} \partial_z, \quad \bar{D} = \partial_{\bar{\theta}} - \frac{1}{2} \theta \partial_{\bar{z}}, \quad \{D, \bar{D}\} = -\partial_z
\]

are the supercovariant derivatives in the \(N = 2\) superspace and

\[
z_{12} = z_1 - z_2 + \frac{1}{2}(\theta_1 \bar{\theta}_2 + \bar{\theta}_1 \theta_2), \quad \theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2,
\]

\[
D_1 z_{12} = D_2 z_{12} = -\frac{1}{2} \bar{\theta}_{12}, \quad \bar{D}_1 z_{12} = \bar{D}_2 z_{12} = -\frac{1}{2} \theta_{12}.
\]

The \(N = 4\) SCA in \(N = 2\) superfields is \(^1\)

\[
T(Z_1)T(Z_2) \sim \frac{1}{3} c + \theta_{12} \bar{\theta}_{12} T \left( \frac{z_{12}^2}{z_{12}^2} \right) + \theta_{12} DT + \bar{\theta}_{12} \bar{D}T + \theta_{12} \bar{\theta}_{12} \partial T,
\]

\[
T(Z_1)G(Z_2) \sim \frac{1}{2} \theta_{12} \bar{\theta}_{12} G \left( \frac{z_{12}^2}{z_{12}^2} \right) - \theta_{12} DG + \bar{\theta}_{12} \bar{D}G + \theta_{12} \bar{\theta}_{12} \partial G + xG,
\]

\[
T(Z_1)\bar{G}(Z_2) \sim \frac{1}{2} \theta_{12} \bar{\theta}_{12} \bar{G} \left( \frac{z_{12}^2}{z_{12}^2} \right) - \theta_{12} D\bar{G} + \bar{\theta}_{12} \bar{D}\bar{G} + \theta_{12} \bar{\theta}_{12} \partial \bar{G} - x\bar{G},
\]

\[
T(Z_1)J(Z_2) \sim -\theta_{12} DJ + \bar{\theta}_{12} \bar{D}J + \theta_{12} \bar{\theta}_{12} \partial J,
\]

\[
G(Z_1)\bar{G}(Z_2) \sim \frac{k x}{2} \theta_{12} \bar{\theta}_{12} \left( \frac{z_{12}^2}{z_{12}^2} \right) - k - \theta_{12} DJ + \bar{\theta}_{12} \bar{D}J + \theta_{12} \bar{\theta}_{12} (-T + \frac{1}{2} \partial J + \frac{x}{2} [D, \bar{D}] J),
\]

\[
J(Z_1)\bar{G}(Z_2) \sim -\theta_{12} \bar{\theta}_{12} \bar{G}, \quad J(Z_1)G(Z_2) \sim \theta_{12} \bar{\theta}_{12} G,
\]

\[J(Z_1)J(Z_2) \sim -2k \ln z_{12}, \quad \text{(3.3)}\]

where it is understood that all the operators on the right hand side are evaluated at the point \(Z_2\) and we have defined

\[
x = \frac{k^+ - k^-}{k}, \quad c = \frac{3k}{2} (1 - x^2). \quad \text{(3.4)}
\]

The parameter \(x\) measures the asymmetry between the two \(SU(2)\) current algebras.

Using the general procedure described in appendix A, we find that the BRST operator for this algebra is given by

\[
Q = \oint \frac{dz d\bar{z} d\theta d\bar{\theta}}{2\pi i} \left[ C_i T + C_j J + C_g G + C_{\bar{g}} \bar{G} + C_t \left( \frac{1}{2} \partial C_t B_t + \frac{1}{2} (DC_t)(\bar{D}B_t) \right) \right]
\]

\(^1\) We have absorbed the factor \(\sqrt{1 - 4\alpha^2}\) into the normalization of the current \(J\) compared with ref. \cite{[1]}. Here our \(x\) corresponds to their \(2\alpha\).
\begin{equation}
\begin{aligned}
&+ \frac{1}{2} (\bar{D} C_t)(D B_t) - \frac{1}{2} \partial (C_g B_g) + (D C_g)(\bar{D} B_g) + (\bar{D} C_g)(D B_g) - \frac{1}{2} \partial (C_g B_g) \\
&+ (D C_g)(\bar{D} B_g) + (D C_g)(D B_g) + (D C_j)(\bar{D} B_j) + (\bar{D} C_j)(D B_j)) \\
&+ (C_g B_g - C_g B_g) \left( C_j + \frac{x}{2} [D, \bar{D}] C_t \right) + C_g C_g \left( B_t - \frac{x}{2} [D, \bar{D}] B_j \right) \\
&+ C_g \left( -\frac{1}{2} C_g \partial B_j + (D C_g)(\bar{D} B_j) + (\bar{D} C_g)(D B_j) \right) ,
\end{aligned}
\end{equation}

where the four sets of fields \((C_t, B_t)\), \((C_g, B_g)\), \((\bar{C}_g, \bar{B}_g)\) and \((C_j, B_j)\) with spins \((-1, 1)\), \((-\frac{1}{2}, \frac{1}{2})\), \((-\frac{1}{2}, \frac{1}{2})\) and \((0, 0)\) are the reparametrization, supersymmetry and current ghosts in \(N = 2\) superfields for the large \(N = 4\) superstrings with the correlations

\begin{equation}
C(Z_1)B(Z_2) \sim \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}}.
\end{equation}

It can also be shown that the BRST operator \((3.5)\) for this \(N = 4\) superstring can be written as

\begin{equation}
Q = \oint dz d^2 \theta \frac{2\pi i}{2} \left[ C_t \left( T + \frac{1}{2} T_{gh} \right) + C_g \left( G + \frac{1}{2} G_{gh} \right) \\
+ C_g \left( \bar{G} + \frac{1}{2} \bar{G}_{gh} \right) + C_j \left( J + \frac{1}{2} J_{gh} \right) \right] ,
\end{equation}

where the generators with subscript \(gh\) are those for ghosts. Their explicit forms are reported in appendix B. We will examine the nilpotency condition for this BRST charge in the next section.

The relation between the superfields for generators in \((3.3)\) and the components in sect. 2 may be worked out. We find that the exact correspondence is as follows:

\begin{equation}
T = (1 + x) J^{-3} - (1 - x) J^{+3} + \theta G_{(2, 1)} + \bar{\theta} G_{(1, 2)} + \theta \bar{\theta} T_B ,
\end{equation}
\begin{equation}
G = 2 F_{(1, 1)} + \theta J^{-+} + \bar{\theta} J^{++} + \theta \bar{\theta} \left( G_{(1, 1)} + x \partial F_{(1, 1)} \right) ,
\end{equation}
\begin{equation}
G = 2 F_{(2, 2)} + \theta J^{+-} + \bar{\theta} J^{--} - \theta \bar{\theta} \left( G_{(2, 2)} + x \partial F_{(2, 2)} \right) ,
\end{equation}
\begin{equation}
J = 2 \int dz J_B - 2 \theta F_{(2, 1)} + 2 \bar{\theta} F_{(1, 2)} - \theta \bar{\theta} (J^{+3} + J^{-3}) ,
\end{equation}

where we have put the subscript \(B\) on the component energy-momentum tensor and \(U(1)\) current to avoid confusion. This result tells us that the components of the ghost fields are embedded in the ghost superfields as follows

\begin{equation}
C_t = c + \theta \gamma_{(1, 2)} - \bar{\theta} \gamma_{(2, 1)} + \theta \bar{\theta} \frac{c^3 - c^3_3}{2} ,
\end{equation}
\[ C_g = \gamma_{(1,1)} - \theta c_{1}^{+} + \bar{\theta} c_{1}^{-} + \frac{1}{2} \theta \bar{\theta} \left( \delta_{(1,1)} + x \partial \gamma_{(1,1)} \right), \]
\[ C_g = -\gamma_{(2,2)} - \theta c_{2}^{+} + \bar{\theta} c_{2}^{-} + \frac{1}{2} \theta \bar{\theta} \left( \delta_{(2,2)} + x \partial \gamma_{(2,2)} \right), \]
\[ 2C_j = -(1 + x)c_{3}^{+} - (1 - x)c_{3}^{-} + \theta \delta_{(1,2)} + \bar{\theta} \delta_{(2,1)} - \theta \bar{\theta} \partial d, \] 

in order to reproduce the structure for the BRST current given in eq. (2.8). The components of antighosts are then determined from the correlators (3.6) to be

\[ B_t = (1 + x)b_{3}^{+} - (1 - x)b_{3}^{+} - \theta \beta_{(1,2)} + \bar{\theta} \beta_{(1,2)} + \theta \bar{\theta} b, \]
\[ B_g = 2 \theta \alpha_{(1,1)} - \theta b_{1}^{-} - \bar{\theta} b_{1}^{+} + \theta \bar{\theta} \left( \beta_{(1,1)} + x \partial \alpha_{(1,1)} \right), \]
\[ B_g = 2 \theta \alpha_{(2,2)} - \theta b_{2}^{+} - \bar{\theta} b_{2}^{-} - \theta \bar{\theta} \left( \beta_{(2,2)} + x \partial \alpha_{(2,2)} \right), \]
\[ B_j = 2 \int dz a + 2 \theta \alpha_{(2,1)} - 2 \bar{\theta} \alpha_{(1,2)} - \theta \bar{\theta} (b_{3}^{+} + b_{3}^{-}) \] .

(3.10)

Although \( J \) and \( B_j \) contain integrals, i.e. \( J \) contains the integral of the bosonic \( U(1) \) current \( J_B \) and \( B_j \) the integral of the corresponding antghost \( \alpha \), this does not cause any problem because these fields always appear with derivatives in physical quantities.

A perhaps better way to present the large \( N = 4 \) algebra in \( N = 2 \) superfields is to use the chiral and antichiral fields \( H \equiv DJ \) and \( \bar{H} \equiv \bar{D}J \), instead of the general superfield \( J \), as independent fields. In this way no logarithm will appear in the OPEs and we will never need to introduce the integral of the bosonic \( U(1) \) current \( J_B \). The relevant OPEs in addition to the first three lines of eq. (3.3) are as follows:

\[ T(Z_1)H(Z_2) \sim \left( \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{2}} + \frac{2}{z_{12}} \right) \frac{1}{2} H + \frac{\theta_{12} \bar{D}H + \theta_{12} \bar{\theta}_{12} \partial H}{z_{12}}, \]
\[ T(Z_1)\bar{H}(Z_2) \sim \left( \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{2}} - \frac{2}{z_{12}} \right) \frac{1}{2} \bar{H} + \frac{-\theta_{12} \bar{D}\bar{H} + \theta_{12} \bar{\theta}_{12} \partial \bar{H}}{z_{12}}, \]
\[ G(Z_1)\bar{G}(Z_2) \sim \frac{kx \theta_{12} \bar{\theta}_{12}}{2z_{12}^{2}} + \frac{-k - \theta_{12} H + \bar{\theta}_{12} \bar{H} - \theta_{12} \bar{\theta}_{12}(T + \frac{1 + \gamma}{2} \bar{D}H + \frac{1 + \gamma}{2} D\bar{H})}{z_{12}}, \]
\[ H(Z_1)G(Z_2) \sim -\frac{\theta_{12}}{z_{12}} G, \quad H(Z_1)\bar{G}(Z_2) \sim \frac{\bar{\theta}_{12}}{z_{12}} \bar{G}, \]
\[ \bar{H}(Z_1)G(Z_2) \sim \frac{\theta_{12}}{z_{12}} G, \quad \bar{H}(Z_1)\bar{G}(Z_2) \sim -\frac{\bar{\theta}_{12}}{z_{12}} \bar{G}, \]
\[ H(Z_1)\bar{H}(Z_2) \sim -\frac{k}{2} \left( \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{2}} - \frac{2}{z_{12}} \right). \]

(3.11)

The BRST charge can now be written using the chiral and antichiral ghost superfields
\( (C_h, B_h) \) and \( (C\bar{h}, B\bar{h}) \) with correlations
\[
C_h(Z_1)B_h(Z_2) \sim \frac{\bar{\theta}_{12}}{z_{12}}, \quad C\bar{h}(Z_1)B\bar{h}(Z_2) \sim \frac{\theta_{12}}{z_{12}},
\]
(3.12)
and reads as follows:
\[
Q = \oint \frac{dzd\theta}{2\pi i} \left[ C_tT + C_gG + C\bar{g}\bar{G} \right] + \oint \frac{dzd\bar{\theta}}{2\pi i} C_h\bar{H} + \oint \frac{dzd\theta}{2\pi i} C\bar{h}H \\
+ \oint \frac{dzd^2\theta}{2\pi i} \left[ C_t \left( \frac{1}{2} \partial C_tB_t + \frac{1}{2}(DC_t)(DB_t) + \frac{1}{2}(DC_{\bar{t}})(\bar{D}B_{\bar{t}}) - \frac{1}{2}\partial(C_gB_g) \right) \\
+ (DC_g)(\bar{D}B_g) + (\bar{D}C_g)(DB_g) - \frac{1}{2}\partial(C_gB_g) + (DC\bar{g})(\bar{D}B\bar{g}) + (\bar{D}C\bar{g})(DB\bar{g}) \right) \\
+ (\bar{D}C_h)B_hC_t - (DC_h)B_hC_t + (C\bar{g}B\bar{g} - C_gB_g) \left( C_h - C\bar{h} + \frac{x}{2}[D, \bar{D}]C_t \right) \\
- C\bar{g} \left( (DC\bar{g})B\bar{h} + (DC\bar{g})B\bar{h} \right) + C_gC\bar{g} \left( B_t + \frac{1-x}{2}DB_h - \frac{1+x}{2}\bar{D}B_h \right) \].
\]
(3.13)

4 Nilpotency of the BRST operator

So far we have revealed the general structure of the large \( N = 4 \) superconformal theory and constructed the BRST operator. The most important property of the BRST charge is its nilpotency. In this section, we will examine what is the necessary and sufficient condition for this property to hold.

In order for the square of the BRST charge to vanish, the first order pole of the OPE of the BRST current (2.8) with itself must be zero up to total derivatives. Actually, we have computed the first order pole of this OPE and obtained quite a large number of terms in components. We have checked that they are indeed total derivatives for the case of \( k^+ = k^- \) and \( c = 0 \), which is the \( N = 4 \) supersymmetry realized in the \( N = 2 \) superstring \([2, 3]\). It is cumbersome to check the nilpotency of the BRST operator for the general case by this method. A much simpler way to check this is to compute the double commutators of the BRST operator with all the fundamental fields in the theory. The necessary and sufficient condition for the nilpotency of the charge is that they all vanish because this means that the square of the BRST charge vanishes in the Hilbert space of the theory.\(^2\)

\(^2\)In general, one can tell if an expression is a total derivative or not by checking that it has no simple pole in its OPEs with all the fundamental fields. We find that this method is quite useful.
We have computed the double commutators and the results are as follows. The double commutators with the ghosts and with the matter generators all vanish without any restriction. On the other hand, those with the antighosts are nonvanishing in general:

\[
[Q, \{Q, b\}] = \frac{k}{8}(1 - x^2)\partial^3 c, \quad [Q, \{Q, a\}] = -\frac{k}{2}\partial d, \\
[Q, [Q, \alpha]] = \frac{k}{2}(1 - x^2)\partial^2 \gamma, \quad [Q, [Q, \alpha]] = -\frac{k}{4}\delta, \\
[Q, \{Q, b^+\}] = \frac{k}{4}(1 + x)\partial c^+, \quad [Q, \{Q, b^-\}] = \frac{k}{4}(1 - x)\partial c^-.
\]

(4.1)

These results show that the necessary and sufficient condition for the nilpotency of the BRST charge for the large \(N = 4\) superstring is \(k = 0\) and \(x\) arbitrary. This implies that \(c = 0\) \([13]\), but we see that there is a line of critical \(N = 4\) string theories depending on the parameter \(x\). Note also that the converse is not true: \(c = 0\) allows solutions \(x = \pm 1\) with arbitrary \(k\). It is the string theory with \(x = 0\) that appears in the hierarchy of ref. \([7]\). We have also reproduced these results using \(N = 2\) superfields.

5 Discussions and conclusions

We discuss in this section how to obtain the small \(N = 4\) SCA as a subalgebra of the large \(N = 4\) SCA. While it is known that the small \(N = 4\) may be obtained from the large one, the precise procedure of how to derive it in the \(N = 2\) superfield notation has never been given. Here we fill this gap since it may be useful for recognizing possible embeddings of the small \(N = 4\) string into the large one.

The small \(N = 4\) SCA is a subalgebra of the large \(N = 4\) when \(x = \pm 1\). Let us take the case \(x = 1\). By setting \(k = 2\hat{k}(1 - x)^{-1}\) and taking the limit \(x \to 1\), one recognizes from eq. (3.3) the following small \(N = 4\) subalgebra

\[
T(Z_1)T(Z_2) \sim \frac{1}{6} c + \frac{\theta_{12}\bar{\theta}_{12} T}{z_{12}^2} + -\frac{-\theta_{12} DT + \bar{\theta}_{12} \bar{D}T + \theta_{12}\bar{\theta}_{12} \partial T}{z_{12}}, \\
T(Z_1)G_c(Z_2) \sim \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} + \frac{2}{z_{12}}\right)G_c + \frac{-\bar{\theta}_{12} \bar{D}G_c + \theta_{12}\bar{\theta}_{12} G_c}{z_{12}}, \\
T(Z_1)G_a(Z_2) \sim \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} - \frac{2}{z_{12}}\right)G_a + \frac{-\theta_{12}DG_a + \theta_{12}\bar{\theta}_{12}DG_a}{z_{12}}, \\
G_c(Z_1)G_a(Z_2) \sim -\frac{c}{6} \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^3} - \frac{1}{z_{12}^2}\right) - \frac{1}{2} \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} - \frac{2}{z_{12}}\right)T + \frac{\bar{\theta}_{12}}{z_{12}} \bar{D}T, \quad (5.1)
\]
where \( G_c \equiv DG \) and \( G_a \equiv \bar{D}G \) are chiral and antichiral superfields, respectively, and \( c = 6\hat{k} \). The parameter \( \hat{k} \) is the level of the remaining \( SU(2) \) current algebra contained in the small \( N = 4 \) SCA. One can proceed similarly for the case \( x = -1 \). The BRST charge in the \( N = 2 \) superfield notation can also be easily constructed. Using the ghosts with correlators

\[
C_t(Z_1)B_t(Z_2) \sim \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}}, \quad C_c(Z_1)B_c(Z_2) \sim \frac{\bar{\theta}_{12}}{z_{12}}, \quad C_a(Z_1)B_a(Z_2) \sim \frac{\theta_{12}}{z_{12}},
\]

(5.2)

where \( (C_t, B_t) \), \( (C_c, B_c) \) and \( (C_a, B_a) \) are general, chiral and antichiral superfields, respectively, one can write the BRST charge as follows

\[
Q = \oint dz d\bar{z} \frac{d\theta}{2\pi i} C_t T + \oint dz d\bar{z} \frac{d\bar{\theta}}{2\pi i} C_c G_c + \oint dz d\theta \frac{d\bar{\theta}}{2\pi i} C_a G_a \\
+ \oint dz d\bar{z} \left[ C_t \left( \frac{1}{2} \partial C_t B_t + \frac{1}{2} DC_t \bar{D} B_t + \frac{1}{2} \bar{D} C_t DB_t \right) + 2 DC_a B_a + C_a DB_a - 2 \bar{D} C_c B_c - C_c \bar{D} B_c \right] - B_t C_c C_a \].
\]

(5.3)

We checked that it is nilpotent for \( c = -12 \).

For completeness, we review also how the \( N = 3 \) SCA is contained into the large \( N = 4 \) SCA. The \( N = 3 \) subalgebra is recognized for \( x = 0 \) by defining \( \hat{G} = G - \bar{G} \), and reads

\[
T(Z_1)T(Z_2) \sim \frac{1}{3}c + \theta_{12}\bar{\theta}_{12} T + \frac{\theta_{12} DT + \bar{\theta}_{12} \bar{D} T + \theta_{12}\bar{\theta}_{12} \partial T}{z_{12}}, \\
T(Z_1)\hat{G}(Z_2) \sim \frac{1}{2} \theta_{12}\bar{\theta}_{12} \hat{G} + \frac{-\theta_{12} D\hat{G} + \bar{\theta}_{12} \bar{D} \hat{G} + \theta_{12}\bar{\theta}_{12} \partial \hat{G}}{z_{12}}, \\
\hat{G}(Z_1)\hat{G}(Z_2) \sim \frac{4}{3}c + 2 \theta_{12}\bar{\theta}_{12} T .
\]

(5.4)

It is clear from this derivation that the small \( N = 4 \) SCA does not contain the \( N = 3 \) SCA as its subalgebra. Obviously both contain the \( N = 2 \) SCA whose OPE is given by the first line of (5.1).

To conclude, we recall that we have constructed the BRST operator for the large \( N = 4 \) SCA and showed that its nilpotency requires the vanishing of the usual Virasoro central charge, but it is consistent with the existence of a one-parameter family of string theories labeled by a parameter \( x \). It would be interesting to explicitly construct and analyze some of these string theories, even though they do not seem to possess an interesting
space-time interpretation. As far as the search for an “universal” string theory started by Berkovits and Vafa, it seems possible that the small \( N = 4 \) strings can also be obtained from the large \( N = 4 \) ones at the values \( x = \pm 1 \), and that the large \( N = 4 \) string theories at arbitrary values of \( x \) may be embedded into the \( N = 5 \) string. If this can be achieved one has the result that all string theories based on the linear SCAs belong to a chain of embeddings, thus identifying the \( N = \infty \) string theory as the master theory which generates all the other ones by successive symmetry breaking.

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Appendix

A General method for constructing the BRST operator

In this appendix, we describe how to obtain the BRST operator directly from the operator algebra. Let us start from a graded Lie algebra. Suppose we are given bosonic generators $T^a$ and fermionic generators $G^\alpha$ satisfying the algebra

\[
\begin{align*}
[T^a, T^b] &= f^{ab} c^c, \\
[T^a, G^\alpha] &= f^{a\alpha} \beta G^\beta, \\
\{G^\alpha, G^\beta\} &= f^{\alpha\beta} a^a T^a.
\end{align*}
\] (A.1)

To construct the BRST operator we introduce ghosts and antighosts, i.e. anticommuting fields $(c_a, b^a)$ for the bosonic generators $T^a$ and commuting fields $(\gamma^\alpha, \beta^\alpha)$ for the fermionic generators $G^\alpha$. They have the following (anti)commutation relations

\[
\{c_a, b^b\} = \delta^b_a, \quad [\gamma^\alpha, \beta^\beta] = \delta^\beta_\alpha.
\] (A.2)

Using the Jacobi identities, it is easy to show that the BRST operator given by

\[
Q = c_a T^a + \gamma_\alpha G^\alpha + \frac{1}{2} f^{ab} c^b c_a + f^{a\alpha} \beta^\beta \gamma_\alpha c_a - \frac{1}{2} f^{a\beta} a^a \gamma_\beta \gamma_\alpha,
\] (A.3)

is nilpotent. This rule can be rewritten as follows. Take the terms containing the structure constants in eq. (A.3). They may be written as

\[
- \frac{1}{2} \frac{\partial_r}{\partial \Lambda} c_a [T^a, T^b] c_b + \frac{\partial_r}{\partial \Lambda} c_a [T^a, G^\alpha] \gamma_\alpha - \frac{1}{2} \frac{\partial_r}{\partial \Lambda} \gamma_\alpha \{G^\alpha, G^\beta\} \gamma_\beta,
\] (A.4)

where it is understood that the generators appearing in the (anti)commutators are replaced by $b^c \Lambda$ for $T^c$ and $\beta^\gamma \Lambda$ for $G^\gamma$, with an anticommuting parameter $\Lambda$ removed from the right by $\frac{\partial_r}{\partial \Lambda}$.

This result can be used to derive the BRST operator directly from an operator algebra. Suppose we are given the OPEs among $T^a(z)$ and $G^\alpha(z)$. We introduce ghost and antighost fields with correlators

\[
c_a(z) b^b(w) \sim \frac{\delta^b_a}{(z - w)}, \quad \gamma_\alpha(z) \beta^\beta(w) \sim \frac{\delta^\beta_\alpha}{(z - w)}.
\] (A.5)
The leading term in the BRST charge is then given by
\[ \oint (c_a T^a + \gamma_\alpha G^\alpha), \] (A.6)
while the terms corresponding to the structure constants are obtained from the above prescription and read
\[ -\frac{1}{2} \frac{\partial_r}{\partial\Lambda} \oint \oint c_a(z) T^a(z) T^b(w) c_b(w) + \frac{\partial_r}{\partial\Lambda} \oint \oint c_a(z) T^a(z) G^\alpha(w) \gamma_\alpha(w) \]
\[ -\frac{1}{2} \frac{\partial_r}{\partial\Lambda} \oint \oint \gamma_\alpha(z) G^\alpha(z) G^\beta(w) \gamma_\beta(w), \] (A.7)
where it is understood that the products of operators are replaced by their OPEs, and those generators appearing in the OPEs are replaced by the products of antighosts and \( \Lambda \).

As a simple example, let us consider the Virasoro algebra,
\[ T(z) T(w) \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \] (A.8)
We get
\[ -\frac{1}{2} \frac{\partial_r}{\partial\Lambda} \oint \oint c(z) \left[ \frac{2b(w)\Lambda}{(z-w)^2} + \frac{\partial b(w)\Lambda}{(z-w)} \right] c(w), \] (A.9)
which after the \( z \) integration gives correctly
\[ \oint \partial c b c. \] (A.10)

It is easy to show that the same prescription is valid for the \( N = 2 \) superfields if we write everything in terms of superfields and replace integrals by superspace integrals. It would be interesting to investigate if this method could be extended to the case of nonlinear algebras [14].

B  Ghost generators

In this appendix, we summarize all the ghost generators. In components we have
\[ T_{gh} = -2b \partial c - \partial b c - a \partial d - b^\pm \partial c^\pm - \frac{3}{2} \beta_a \partial \gamma^a - \frac{1}{2} \beta_a \gamma^a - \frac{1}{2} \alpha_a \partial \delta^a + \frac{1}{2} \partial \alpha_a \delta^a, \]
\[ J_{gh} = \partial (ca - \alpha_a \gamma^a), \]
\[ J_{gh}^{+i} = \partial (c b^{+i}) - f^{ij} k c^i_j c^{+i} b^{+k} + 2 \frac{k^+}{k} R^{+i}_a b^a \partial (\alpha b^{+b}) + R^{+i}_a b^a (\beta b^{+b} + \alpha b^{+a}) \, , \]

\[ J_{gh}^{-i} = \partial (c b^{-i}) - f^{ij} k c^i_j b^{-k} - 2 \frac{k^-}{k} R^{-i}_a b^a \partial (\alpha b^{+a}) + R^{-i}_a b^a (\beta b^{+a} + \alpha b^{+b}) \, , \]

\[ G_{gh,a} = \frac{3}{2} \partial c \beta_a + c \partial \beta_a + 2 \frac{k^\pm}{k} R^\pm_i a b^a \partial c^i \alpha_b + R^\pm_i a b^a c^i \beta_b + 4 \frac{k^\pm}{k} R_{ab}^\pm \partial c^i \beta_b \, , \]

\[ F_{gh,a} = \frac{1}{2} \partial c \alpha_a + c \partial \alpha_a + R^\pm_i a b^a c^i \alpha_b + 2 R_{ab}^\pm \partial c^i \beta_b - \gamma_a a \, . \] (B.1)

It is interesting to note that the \( U(1) \) current \( J_{gh} \) is a total derivative. It is also easy to check that these generators satisfy the OPE (2.1) with \( c = 0 \).

The ghost generators in \( N = 2 \) superfields are as follows

\[ T_{gh} = - \partial (B C_t) + (\bar{D} C_t)(D B_t) + (D C_t)(\bar{D} B_t) + (\bar{D} C_t)(D B_t) + (D C_t)(\bar{D} B_t) \]

\[ - \frac{1}{2} \partial (B C_g) + (1 + x)(\bar{D} C_g)(D B_g) + (1 - x)(D C_g)(\bar{D} B_g) \]

\[ - \frac{1}{2} \partial (B C_g) + (1 - x)(\bar{D} C_g)(D B_g) + (1 + x)(D C_g)(\bar{D} B_g) \]

\[ - \frac{x}{2} \left( [D, \bar{D}] C B_g + C_g[D, \bar{D}] B_g \right) + \frac{x}{2} \left( [D, \bar{D}] C B_g + C_g[D, \bar{D}] B_g \right) \, , \]

\[ G_{gh} = \frac{1}{2} C g \partial B_j + C_t \partial B_g + \frac{1}{2} C t B_g + C g B_t + (\bar{D} C_t)(D B_g) + (D C_t)(\bar{D} B_g) \]

\[ - C g B_g - (D C_g)(D B_g) - (D C_g)(\bar{D} B_g) - \frac{x}{2} \left( B_g[D, \bar{D}] C_t + C_g[D, \bar{D}] B_j \right) \, , \]

\[ \bar{G}_{gh} = - \frac{1}{2} C g \partial B_j + C_t \partial B_g + \frac{1}{2} C t B_g + C g B_t + (D C_t)(\bar{D} B_g) + (\bar{D} C_t)(D B_g) \]

\[ + C g B_g + (D C_g)(D B_g) + (D C_g)(\bar{D} B_g) + \frac{x}{2} \left( B_g[D, \bar{D}] C_t - C_g[D, \bar{D}] B_j \right) \, , \]

\[ J_{gh} = C t \partial B_j - C g B_g + C g B_g + (D C_t)(\bar{D} B_g) + (\bar{D} C_t)(D B_g) \] (B.2)

It is easy to check that these generators satisfy the OPE (3.3) with \( c = k = 0 \).
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