NONVANISHING COHOMOLOGY
AND CLASSES OF GORENSTEIN RINGS

DAVID A. JORGENSEN AND LIANA M. ŞEGA

Abstract. We give counterexamples to the following conjecture of Auslander: given a finitely generated module $M$ over an Artin algebra $\Lambda$, there exists a positive integer $n_M$ such that for all finitely generated $\Lambda$-modules $N$, if $\text{Ext}^i_\Lambda(M, N) = 0$ for all $i \gg 0$, then $\text{Ext}^i_\Lambda(M, N) = 0$ for all $i \geq n_M$. Some of our examples moreover yield homologically defined classes of commutative local rings strictly between the class of local complete intersections and the class of local Gorenstein rings.

Introduction

In this paper we give examples on the vanishing of Ext and Tor which simultaneously disprove a conjecture of Auslander and identify new classes of commutative local (meaning also Noetherian) rings lying between the class of local complete intersections and the class of local Gorenstein rings.

The following conjecture of Auslander appears in [1, p. 795] and [11]: let $\Lambda$ be an Artin algebra. For every finitely generated $\Lambda$-module $M$ there exists an integer $n_M$ such that for all finitely generated $\Lambda$-modules $N$, if $\text{Ext}^i_\Lambda(M, N) = 0$ for all $i \gg 0$, then $\text{Ext}^i_\Lambda(M, N) = 0$ for all $i \geq n_M$.

Auslander’s conjecture is known to hold, for example, when $\Lambda$ is a group ring of a finite group over a field, by [8, 2.4], or when $\Lambda$ is a local complete intersection (see the discussion later in the introduction). Part (1) of our main theorem below gives a counterexample to Auslander’s Conjecture over a commutative selfinjective Koszul $k$-algebra. Part (2) is relevant in the context of recent research on refinements of the Gorenstein condition, as we shall explain shortly.

Theorem. Let $k$ be a field which is not algebraic over a finite field. Then there exist commutative finite dimensional selfinjective Koszul $k$-algebras $A$ with Hilbert series $\sum (\text{rank}_k A_i) t^i = 1 + 5t + 5t^2 + t^3$ and finitely generated graded $A$-modules $M$ with linear resolution and $\sum (\text{rank}_k M_i) t^i = 2t + 8t^2 + 2t^3$ such that the following hold:

1. For each positive integer $q$ there exists a finitely generated graded $A$-module $N_q$ with linear resolution and $\sum (\text{rank}_k (N_q)_i) t^i = 1 + t$ satisfying $\text{Ext}^i_A(M, N_q) \neq 0$ if and only if $i = 0, q - 1, q$.

Date: November 8, 2018.

This work was done while the first author was a member and the second author was a postdoctoral fellow at the Mathematical Sciences Research Institute, Berkeley, CA. They are both grateful to MSRI for its generous support.
(2) There exists a finitely generated graded $A$-module $V$ with linear resolution and

$$\sum (\text{rank}_k V_i) t^i = 2 + 2t$$

satisfying

$$\text{Tor}^A_i(M, V) = 0 \quad \text{for all} \quad i > 0 \quad \text{and} \quad \text{Ext}^A_i(M, V) \neq 0 \quad \text{for all} \quad i > 0.$$

It is easy to see that Auslander’s Conjecture holds when $A$ is a commutative local ring with maximal ideal $m$ satisfying $m^2 = 0$. We show in a corollary of our main theorem (Corollary 3.3.(2)) that the conjecture already fails for a commutative local ring $(B, m)$ with $m^3 = 0$.

The rings $A$ in our main theorem and the rings $B$ in the corollary are constructed by Gasharov and Peeva in [10] to give a counterexample to an unrelated conjecture of Eisenbud. We turned to these rings since they admit modules of infinite complete intersection dimension (see [6] for the definition), which is a necessary property of any module yielding a counterexample to Auslander’s Conjecture (cf. [7]).

Recall that a local complete intersection is a local ring $R$ whose completion with respect to the maximal ideal $m$ is a quotient of a regular local ring by a regular sequence. Let $CI$ denote the class of all such rings. Let $TE$ represent the class of commutative local rings $R$ which have the following property: $\text{Tor}^R_i(M, N) = 0$ for all $i \gg 0$ implies $\text{Ext}^R_i(M, N) = 0$ for all $i > \dim R$, for all finitely generated $R$-modules $M$ and $N$. One of the main theorems of Avramov and Buchweitz in [3] gives the first inclusion in the chain

$$CI \subseteq TE \subseteq GOR.$$  

(The second inclusion is clear: just take $N = R$ in the definition of the class $TE$.) In [14], the authors remark that forty years of research in commutative algebra have not produced a class of local rings intermediate between $CI$ and $GOR$, and ask whether either inclusion above is strict. In a recent paper, Huneke and Jorgensen [14] prove that the first inclusion is strict. Part (2) of our main theorem shows that so is the second.

In [14] an $AB$ ring is defined to be a local Gorenstein ring $R$ with the property that $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ implies $\text{Ext}^i_R(M, N) = 0$ for all $i > \dim R$, for all finitely generated $R$-modules $M$ and $N$. Let $AB$ denote the class of all $AB$ rings. It is shown in [3] (cf. also [14]) that $CI \subseteq AB$, and subsequently in [14] (cf. also [21]) it is shown that this inclusion is strict. Note that the condition defining $AB$ rings is a strengthening of Auslander’s Conjecture. Therefore part (1) of our main theorem also shows that $AB$ lies properly between $CI$ and $GOR$.

The paper is organized as follows. In Section 1 we give some positive results on Auslander’s Conjecture for commutative non-Gorenstein rings. In particular, we prove that Golod rings, and commutative local rings which are “small” in various senses satisfy (a strong form of) Auslander’s Conjecture. The rings $A$ and $B$ and the corresponding modules are defined in Section 2. Here we also explain our method for computing homology and cohomology.

The main theorem above is an immediate consequence of Corollary 3.3.1) and Proposition 5.1 proved in Section 3. We also give there the corresponding statements for the ring $B$, and compare these with the results of Section 1, noting that our examples involving the ring $B$ are “smallest” in various senses where one can expect Auslander’s Conjecture to fail.

In the final Section 4 we discuss classes of homologically defined local Gorenstein rings, including the ones described above. We give local Gorenstein rings which
are known to satisfy Auslander’s Conjecture, and we compare these rings to our examples from Section 3.

1. SOME COMMUTATIVE RINGS FOR WHICH AUSLANDER’S CONJECTURE HOLDS

In this section $R$ denotes a commutative local ring, with maximal ideal $m$ and residue field $k$.

As is evidenced by results of [3] and [14], Auslander’s Conjecture is relevant and interesting in the context of commutative local rings (of possibly nonzero Krull dimension). It turns out that all the commutative local rings for which Auslander’s Conjecture is known to hold actually satisfy a stronger condition, which we call the Uniform Auslander Condition:

(uac) There exists an integer $n$ such that for all finitely generated $R$-modules $M$ and $N$, if $\text{Ext}_{R}^{i}(M, N) = 0$ for all $i \gg 0$ then $\text{Ext}_{R}^{i}(M, N) = 0$ for all $i \geq n$.

In this section we prove that (uac) holds for certain rings which are small in various senses. Let $\dim edim R$ denote the minimal number of generators of $m$ and $\lambda(R)$ denote the length of $R$.

1.1. Proposition. The local ring $(R, m)$ satisfies (uac) if any one of the following conditions holds.

1. $m^2 = 0$.
2. $\dim edim R - \dim R \leq 2$.
3. $m^3 = 0$ and $\dim edim R = 3$.
4. $m^3 = 0$ and $\lambda(R) \leq 7$.

Recall that the the Poincaré series of $M$ over $R$ is the formal power series

$$P_{M}^{R}(t) = \sum_{i=0}^{\infty} b_{i}(M)t^{i} \in \mathbb{Z}[t]$$

where $b_{i}(M) = \text{rank}_{k} \text{Tor}_{i}^{R}(M, k)$ are the Betti numbers of $M$.

Since some of the results existing in the literature are stated in terms of Tor rather than Ext, we remind the reader of the following:

1.2. Assume that $R$ is artinian and let $E$ denote the injective hull of $k$. For an $R$-module $M$ we set $M^{\vee} = \text{Hom}_{R}(M, E)$. By Matlis duality, for all finitely generated $R$-modules $M$ and $N$ and all $i$ we have:

$$\text{Tor}_{i}^{R}(M, N^{\vee}) \cong \text{Ext}_{R}^{i}(M, N)^{\vee}.$$ 

In some cases one can actually prove a property much stronger than (uac). We call it trivial vanishing and it states:

(tv) For any pair $(M, N)$ of finitely generated $R$-modules, if $\text{Ext}_{R}^{i}(M, N) = 0$ for all $i \gg 0$, then either $M$ has finite projective dimension or $N$ has finite injective dimension.

Proof of Proposition 1.1. Let $M$ and $N$ be finitely generated $R$-modules such that $\text{Ext}_{R}^{i}(M, N) = 0$ for all $i \gg 0$ and assume that $M$ is not free.

(1) The first syzygy $\text{Syz}_{1}(M)$ in a minimal free resolution of $M$ is annihilated by $m$, hence it is a finite sum of copies of $k$. Since $\text{Ext}_{R}^{i}(M, N) = 0$ for some $i > 1$ implies $\text{Ext}_{R}^{i-1}(\text{Syz}_{1}(M), N) = 0$, we conclude that $N$ has finite injective dimension. The ring therefore satisfies (tv), and hence (uac).
(2) By Scheja [22], \( R \) is either a complete intersection or a Golod ring (see 1.63). If it is a complete intersection, apply [3] 4.7] (cf. also the last section). If it is Golod, then apply Proposition 1.4 below.

(3) If \( N \) is injective, then \( \text{Ext}_R^i(M, N) = 0 \) for all \( i > 0 \). Therefore assume that \( N \) is not injective, hence \( N^\vee \) is non-free. From 1.24 we have \( \text{Tor}_i^R(M, N^\vee) = 0 \) for all \( i \gg 0 \). By taking syzygies, we may assume that there exist finitely generated non-zero \( R \)-modules \( X \) and \( Y \) such that \( \text{Tor}_i^R(X, Y) = 0 \) for all \( i > 0 \) and \( m^2X = m^2Y = 0 \). We conclude from [15, 2.5] that there exist positive integers \( u, v \) such that \( u + v = 3 \) and \( b_{i+1}(X) = ub_i(X), b_{i+1}(Y) = vb_i(Y) \) for all \( i \geq 0 \). It follows that one of the modules \( X \) or \( Y \) has constant Betti numbers (because either \( u = 1 \) or \( v = 1 \)), hence one of the modules \( M \) or \( N^\vee \) has eventually constant Betti numbers. Using a result of Avramov [2, 1.6] we conclude that one of the modules \( \text{Syz}_2^R(M), \text{Syz}_2^R(N^\vee) \) has a periodic resolution of period 2. The hypothesis then implies \( \text{Tor}_i^R(M, N^\vee) = 0 \) for all \( i > 1 \), hence \( \text{Ext}_R^i(M, N) = 0 \) for all \( i > 1 \).

(4) By (3), we may assume that \( \text{edim} R \geq 4 \). The ring \( R \) then satisfies the condition in the hypothesis of [15, 3.1], hence, in view also of [12, 3.6] it satisfies (tv). □

1.3. Serre proved a coefficientwise inequality

\[
P_k^R(t) \leq \frac{(1 + t)^{\text{edim} R}}{1 - \sum_{j=1}^{\infty} \text{rank} H_j(K^R)t^{j+1}}
\]

of formal power series, where \( K^R \) denotes the Koszul complex on a minimal set of generators of \( m \). If equality holds, then \( R \) is said to be a Golod ring.

1.4. Proposition. If \( R \) is a Golod ring, then it satisfies (tv), and hence (uac).

For the proof we need some considerations on complexes. We refer to [3] Appendix A] for the basic notions. The Poincaré series of a complex is the extension of the corresponding notion for modules, cf. [3] §7, for example.

Vanishing of homology over Golod rings was studied by Jorgensen [17, 3.1]. His result was extended in [21, 8.3] to complexes with finite homology as follows:

1.5. Let \( R \) be a Golod ring and \( M \) and \( N \) complexes with finite homology.

If \( \text{Tor}_i^R(M, N) = 0 \) for all \( i > 0 \), then \( P_k^R(t) \) or \( P_N^R(t) \) is a Laurent polynomial.

1.6. A complex of \( R \)-modules \( D \) is said to be dualizing if it has finite homology and there is an integer \( d \) such that \( \text{Ext}_R^i(k, D) \cong k \) and \( \text{Ext}_R^i(k, D) = 0 \) for \( i \neq d \). (By [12] V, 3.4], this definition agrees with the one given by Hartshorne in [12].)

Any quotient of a local Gorenstein ring has a dualizing complex. In particular, a complete local ring has a dualizing complex.

1.6.1. For a complex \( G \) we set \( G^\dagger = \text{RHom}_R(G, D) \). As noted in [12] V, §2, if \( G \) has finite homology, then so does \( G^\dagger \).

\textbf{Proof of Proposition 1.4} We may assume that \( R \) is complete, hence it has a dualizing complex \( D \). Let \( M \) and \( N \) be finitely generated \( R \)-modules such that \( \text{Ext}_R^i(M, N) = 0 \) for all \( i > 0 \). By [12] V, 2.6(b)] we have:

\[
\text{RHom}_R(M, N)^\dagger \cong M \otimes_K N^\dagger
\]

By hypothesis, \( \text{RHom}_R(M, N) \) has finite homology, and by 1.6.1 so does \( M \otimes_K N^\dagger \). This means that \( \text{Tor}_i^R(M, N^\dagger) = 0 \) for all \( i > 0 \), hence \( P_k^R(t) \) or \( P_N^R(t) \) is a Laurent polynomial, from [12]. It follows that \( M \) has finite projective dimension or \( N \) has finite injective dimension (see [12] V, 2.6(a)). □
Several classes of local Gorenstein rings are known to satisfy (uac). They will be discussed in Section 3.

2. Constructions

Let \( k \) be a field and let \( \alpha \in k \) be a nonzero element. In this section we describe the rings \( A_\alpha \) and \( B_\alpha \) constructed in [10] by Gasharov and Peeva, we define the modules \( M \) and \( L \) of our main theorem and its corollary, and we discuss our method for computing homology and cohomology.

The ring \( A = A_\alpha \).

Consider the polynomial ring \( k[X_1, X_2, X_3, X_4, X_5] \) in five (commuting) variables and set \( A = k[X_1, X_2, X_3, X_4, X_5]/I_\alpha \), where \( I_\alpha \) is the ideal generated by the following quadric relations:

\[
\begin{align*}
\alpha X_1 X_3 + X_2 X_5, & \quad X_1 X_4 + X_2 X_4, \quad X_2 + \alpha X_1 X_5 - X_2 X_5, \\
X_4^2 + X_1 X_5 - X_2 X_5, & \quad X_2^2, \quad X_2^3, \quad X_3 X_4, \quad X_4 X_5, \quad X_5^2.
\end{align*}
\]

By [10], \( A_\alpha \) is a local Gorenstein ring with Hilbert series \( \text{Hilb}_{A_\alpha}(t) = 1 + 5t + 5t^2 + t^3 \).

As a \( k \)-vector space, it has a basis consisting of the 12 elements

\[
1, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_1 x_2, \quad x_1 x_3, \quad x_1 x_4, \quad x_1 x_5, \quad x_2 x_5, \quad x_1 x_2 x_5,
\]

where \( x_i \) denotes the residue class of \( X_i \) modulo \( I_\alpha \).

The ring \( B = B_\alpha \).

Set \( B_\alpha = A_\alpha/(x_5) \). As noted in [10], \( B_\alpha \) is a local ring with Hilbert series \( \text{Hilb}_{B_\alpha}(t) = 1 + 4t + 3t^2 \). As a \( k \)-vector space, it has a basis formed by the images in \( B_\alpha \) of the following 8 elements in \( A_\alpha \):

\[
1, \quad x_1, \quad x_2, \quad x_3 x_4, \quad x_1 x_2, \quad x_1 x_3, \quad x_1 x_4, \quad x_1 x_5.
\]

When there is no danger of confusion we will suppress \( \alpha \) from the notation and simply write \( A \) or \( B \) for \( A_\alpha \) or \( B_\alpha \).

2.1. One may check that the set of generators of \( I_\alpha \) listed above is itself a Gröbner basis for \( I_\alpha \) with respect to the reverse lexicographic term order. Since all of these generators are quadrics, by [9, Section 4] we have that \( A = A_\alpha \) is Koszul. Similarly, the generators

\[
\alpha x_1 x_3 + x_2 x_3, \quad x_1 x_4 + x_2 x_4, \quad x_3^2, \quad x_4^2, \quad x_5^2, \quad x_2, \quad x_3 x_4
\]

of the ideal defining \( B_\alpha \) form a Gröbner basis of their ideal with respect to the reverse lexicographic term order, and so \( B = B_\alpha \) is also Koszul.

Modules with nonperiodic (or periodic of period \( \neq 2 \)) minimal resolutions having constant Betti numbers equal to 2 were given in [10] over the rings \( A_\alpha \) and \( B_\alpha \) with \( \alpha \neq \pm 1 \). We wanted a module with nonperiodic resolution and constant Betti numbers, but we found that the modules in [10] did not provide counterexamples using our technique.

The modules \( M \) and \( L \).

Let \( M \) be the image of the map \( d_0: A^2 \to A^2 \) given in the standard basis of \( A^2 \) as a free \( A \)-module by the matrix

\[
\begin{pmatrix}
x_1 & x_3 \\
x_4 & x_2
\end{pmatrix}.
\]
Set \( L = M \otimes_A B \).

For any ring \( R \) we let \(-^*\) denote the \( R\)-module \( \text{Hom}_R(-, R) \).

### 2.2. Lemma

Consider the sequence of homomorphisms

\[ C : \cdots \rightarrow A^2 \xrightarrow{d_{i+1}} A^2 \xrightarrow{d_i} A^2 \xrightarrow{d_{i-1}} A^2 \rightarrow \cdots, \]

where each map \( d_i \) is given in the standard basis of \( A^2 \) over \( A \) by the matrix

\[
\begin{pmatrix}
  x_1 & \alpha^i x_3 \\
  x_4 & x_2
\end{pmatrix}.
\]

Then \( C \) is an exact complex. Moreover, the complexes \( C^*, C \otimes_A B \), and \( C^* \otimes_A B \) are also exact.

We use the subscript of the differential to keep track of degrees within our complexes.

### 2.3. Remark

Let \( W \) be a finitely generated module over a Noetherian ring \( R \). A **complete resolution** of \( W \) is a complex \( T \) of finite projective \( R \)-modules such that \( H_i(T) = H_i(T^*) = 0 \) for all \( i \in \mathbb{Z} \), and \( T_{\geq r} = P_{\geq r} \) for some projective resolution \( P \) of \( W \) and some \( r \).

If \( W \neq 0 \), then its **G-dimension** is the shortest length of a resolution by modules \( G \) with \( G \cong G^* \) and \( \text{Ext}^i(G, R) = \text{Ext}^i_q(G^*, R) = 0 \) for all \( i > 0 \); it is denoted \( \text{G-dim}_R W \). By a basic result of Auslander and Bridger [5], the ring \( R \) is Gorenstein if and only if every finite \( R \)-module \( W \) has \( \text{G-dim}_R W < \infty \).

By [3], 4.4.4, \( W \) has a complete resolution if and only if \( \text{G-dim}_R W < \infty \). Lemma 2.2 shows that \( C \) is a complete resolution of the \( A \)-module \( M \) and \( C \otimes_A B \) is a complete resolution of the \( B \)-module \( L \). In particular, \( L \) has finite G-dimension. (Since G-dimension is bounded by the dimension of the ring, we actually have \( \text{G-dim}_B L = 0 \).)

#### Proof of Lemma 2.2

It is immediate to check from the defining equations of \( A \) that \( d_id_{i+1} = 0 \).

As a \( k \)-vector space, \( A^2 \) is 24-dimensional. We let \((a, b)\) denote an element of \( A^2 \) written in the standard basis of \( A^2 \) as a free \( A \)-module. It is easy to see that for each \( i \), the following elements in \( \text{Im} d_i \) are linearly independent over \( k \):

\[
\begin{align*}
  d_i(1, 0) &= (x_1, x_4) & d_i(0, 1) &= (\alpha^i x_3, x_2) \\
  d_i(0, 1) &= (\alpha^i x_3, x_2) & d_i(0, x_1) &= (\alpha^i x_1 x_3, x_1 x_2) \\
  d_i(x_1, 0) &= (0, x_4) & d_i(0, x_3) &= (0, -\alpha x_1 x_3) \\
  d_i(x_2, 0) &= (x_1 x_2, -x_1 x_4) & d_i(0, x_5) &= (0, x_2 x_5) \\
  d_i(x_3, 0) &= (x_1 x_3, 0) & d_i(x_2 x_5, 0) &= (x_1 x_2 x_5, 0) \\
  d_i(x_4, 0) &= (x_1 x_4, x_2 x_5 - x_1 x_5) & d_i(0, x_1 x_5) &= (0, x_1 x_2 x_5)
\end{align*}
\]

As a consequence, \( \text{rank}_k(\text{Im} d_i) \geq 12 \) for each \( i \). On the other hand, we have

\[
\text{rank}_k(\text{Ker} d_i) = \text{rank}_k(A^2) - \text{rank}_k(\text{Im} d_i) \leq 12
\]

for each \( i \). It follows that \( \text{rank}_k(\text{Im} d_{i+1}) = \text{rank}_k(\text{Ker} d_i) = 12 \) for each \( i \), hence the complex \( C \) is exact. Since \( A \) is selfinjective, the complex \( C^* \) is exact as well.

Similar computations over the ring \( B \) show that the complexes \( C \otimes_A B \) and \( C^* \otimes_A B \) are exact. \(\square\)
The proof of the lemma shows that
\[ \text{Hilb}_M(t) = \text{Hilb}_{M^*}(t) = 2t + 8t^2 + 2t^3, \]
and one may verify that
\[ \text{Hilb}_L(t) = \text{Hilb}_{L^*}(t) = 2t + 6t^2. \]
Every finitely generated graded module \( W \) over a standard graded local ring \( R \) with \( R_0 = k \) has a minimal graded free resolution. Consequently, the modules \( \text{Tor}^R_i(W, k) \) inherit a structure of graded \( R \)-modules. The bigraded Poincaré series of \( W \) is the formal power series in two variables:
\[ P^R_W(t, z) = \sum_{i,j} \text{rank}_k \text{Tor}^R_i(W, k) t^i z^j, \]
where \( j \) is the index of degree. The usual Poincaré series is obtained by letting \( z = 1 \).

The module \( W \) is said to have a linear resolution if all its minimal generators are in the same degree \( p \) and \( W \) has a minimal graded free resolution in which all the entries of the matrices defining the differentials have degree 1. This is equivalent to \( \text{Tor}_i(W, k) = 0 \) for all \( i \neq p \).

By definition, the \( k \)-algebra \( R \) is Koszul if the \( R \)-module \( k \) has a linear resolution. In this case, it is known that
\[ P^R_k(t, z) = \frac{1}{\text{Hilb}_R(-tz)}. \]

It is clear from Lemma 2.2 that the \( A \)-modules \( M \) and \( M^* \), as well as the \( B \)-modules \( N \) and \( N^* \), have a linear resolution, and
\[ P^A_M(t, z) = P^A_{M^*}(t, z) = P^B_L(t, z) = P^B_{L^*}(t, z) = \frac{2z}{1 - tz}. \]
Note that all the syzygies of these modules have the same Poincaré series. Since the rings \( A \) and \( B \) are Koszul, the Poincaré series of \( k \) over \( A \) and \( B \) are
\[ P^A_k(t, z) = \frac{1}{1 - 5tz + 5t^2z^2 - t^3z^3} \quad \text{and} \quad P^B_k(t, z) = \frac{1}{1 - 4tz + 3t^2z^2}. \]

Next we describe our approach to calculating (co)homology over the rings \( A \), respectively \( B \), when one of the modules is \( M \), respectively \( L \).

### Computing Ext and Tor

We set \( F = C_{\geq 0} \) and \( G = C_{<0} \). Lemma 2.2 shows that \( F \) is a minimal free resolution of \( M \) over \( A \) and \( F \otimes_A B \) is a minimal free resolution of \( L \) over \( B \). Also, \( G^* \) is a minimal free resolution of \( M^* \) over \( A \) and \( G^* \otimes_A B \) is a minimal free resolution of \( L^* \) over \( B \).

Let \( N \) be a finitely generated \( A \)-module and let \( P \) be a finitely generated \( B \)-module.

#### 2.6.1. The module \( \text{Tor}^A_i(M, N) \) is the \( i \)th homology of the complex
\[ F \otimes_A N : \cdots \rightarrow N^2 \xrightarrow{d_i \otimes_A N} N^2 \xrightarrow{d_{i-1} \otimes_A N} N^2 \rightarrow \cdots \rightarrow N^2 \xrightarrow{d_1 \otimes_A N} N^2 \rightarrow 0, \]
that is,
\[ \text{Tor}^A_i(M, N) = \text{Ker}(d_i \otimes_A N) / \text{Im}(d_{i+1} \otimes_A N). \]

Similarly, \( \text{Tor}^B_i(L, P) \) is the \( i \)th homology of the complex \( (F \otimes_A B) \otimes_B P = F \otimes_A P \).
2.6.2. The module $\text{Ext}^i_A(M, N)$ is the $(-i)$th homology of the complex $\text{Hom}_A(F, N)$ and this complex can be identified with

$$F^* \otimes_A N : 0 \to N^2 \xrightarrow{d_i^* \otimes A N} N^2 \to \cdots \to N^2 \xrightarrow{d_i^* \otimes A N} N^2 \to \cdots$$

where $F^* = \text{Hom}_A(F, A)$, and for each $i$ the map $d_i^*$ is given in the standard basis of $A^2$ by the transpose of the matrix corresponding to $d_i$. Thus

$$\text{Ext}^i_A(M, N) = \text{Ker}(d_{i+1}^* \otimes A N)/\text{Im}(d_i^* \otimes A N).$$

Similarly, $\text{Ext}^i_B(L, P)$ is the $(-i)$th homology of the complex $F^* \otimes_A P$.

2.6.3. The module $\text{Ext}^i_A(N, M)$ is isomorphic to $\text{Tor}^i_A(M^*, N)^*$, and $\text{Tor}^i_A(M^*, N)$ is the $i$th homology of the complex

$$G^* \otimes_A N : \cdots \to N^2 \xrightarrow{d_{i+1}^* \otimes A N} N^2 \xrightarrow{d_{i+1}^* \otimes A N} N^2 \to \cdots \to N^2 \xrightarrow{d_{i}^* \otimes A N} N^2 \to 0.$$

Thus

$$\text{Ext}^i_A(N, M) = (\text{Ker}(d_{i-1}^* \otimes A N)/\text{Im}(d_{i-1}^* \otimes A N))^*.$$  

Similarly, $\text{Tor}^i_B(L^*, P)$ is the $i$th homology of the complex $G^* \otimes_A P$.

3. Results on vanishing

In this section we prove the main results stated in the introduction. Our method and constructions were inspired by the paper [13] of Heitmann.

We fix $\alpha \in k$ and use the notation introduced in Section 2.

For each integer $q$ we set $T_q = A/J_q$, where $J_q$ is the ideal of $A = A_\alpha$ generated by the linear relations

$$x_1 - x_2, x_1 - \alpha^q x_3, x_1 - x_4, x_5.$$  

Note that $T_q$ is also a $B$-module.

We let $o(\alpha)$ denote the order of $\alpha$ in the group of units of $k$, and set

$$s = \begin{cases} 
0 & \text{if } o(\alpha) = \infty \\
o(\alpha) & \text{otherwise.} 
\end{cases}$$

3.1. Proposition. The following hold for every integer $q$ and every $i > 0$:

(a) $\text{Tor}^i_A(M, T_q) \neq 0$ if and only if $i \equiv q - 1, q \mod (s)$.

(b) $\text{Ext}^i_A(M, T_q) \neq 0$ if and only if $i \equiv q - 1, q \mod (s)$.

(c) $\text{Ext}^i_A(T_q, M) \neq 0$ if and only if $i \equiv -q, -q - 1 \mod (s)$.

3.2. Proposition. The following hold for every integer $q$ and every $i > 0$:

(a) $\text{Tor}^i_B(L, T_q) \neq 0$ if and only if $i \equiv q - 1, q \mod (s)$.

(b) $\text{Ext}^i_B(L, T_q) \neq 0$ if and only if $i \equiv q - 1, q \mod (s)$.

(c) $\text{Tor}^i_B(L^*, T_q) \neq 0$ if and only if $i \equiv -q, -q - 1 \mod (s)$.

3.3. Corollary. If $o(\alpha) = \infty$, then the following hold for any integer $q > 0$:

(1) $\text{Ext}^i_A(M, T_q) \neq 0$ if and only if $i = 0, q - 1, q$.

(2) $\text{Ext}^i_B(L, T_q) \neq 0$ if and only if $i = 0, q - 1, q$.

3.4. Remark. The corollary shows that, when $o(\alpha) = \infty$, the rings $A = A_\alpha$ and $B = B_\alpha$ provide counterexamples to Auslander’s Conjecture. In view also of the results below, the first part gives part (1) of the main theorem in the introduction.
3.5. Let \( m \) denote the maximal ideal of \( B \). The expression for \( \operatorname{Hilb}_B(t) \) given above indicates that \( m^3 = 0 \), edim \( B = 4 \) and \( \lambda(B) = 8 \). Comparing these numerical data with the results stated in Proposition 1.1, we see that our examples are minimal primarily with respect to the invariant sup\( \{ n \mid m^n = 0 \} \) and secondarily with respect to the invariants edim and length.

3.6. Remark. Let \( R \) be a Noetherian ring and let \( W \) and \( N \) be finitely generated \( R \)-modules. Assume that \( W \) has a complete resolution \( T \), as defined in 2.3. For each \( i \) the Tate (co)homology groups are defined by

\[
\operatorname{Ext}^i_R(W, N) = H_{−i} \operatorname{Hom}(T, N) \quad \text{and} \quad \operatorname{Tor}^R_i(W, N) = H_i(T \otimes_R N).
\]

If \( r \) is as in 2.3 then it is clear that for all \( i > r \) one has

\[
\operatorname{Ext}^i_R(W, N) \cong \operatorname{Ext}^i_R(W, N) \quad \text{and} \quad \operatorname{Tor}^R_i(W, N) \cong \operatorname{Tor}^R_i(W, N).
\]

Also, when \( \text{G-dim}_W = 0 \), one has

\[
\operatorname{Ext}^{-i−1}_R(W, N) \cong \operatorname{Tor}^R_i(W^*, N).
\]

In terms of Tate (co)homology, the propositions above can be formulated as follows. Let \( q \) be an integer. Then for all \( i \) we have:

1. \( \operatorname{Ext}^i_A(M, T_q) = 0 \) if and only if \( i \equiv q − 1, q \mod (s) \).
2. \( \operatorname{Tor}^i_A(M, T_q) = 0 \) if and only if \( i \equiv q − 1, q \mod (s) \).
3. \( \operatorname{Ext}^i_B(L, T_q) = 0 \) if and only if \( i \equiv q − 1, q \mod (s) \).
4. \( \operatorname{Tor}^i_B(L, T_q) = 0 \) if and only if \( i \equiv q − 1, q \mod (s) \).

3.7. Remark. It is now clear that, using the modules in the propositions, one can obtain a wide variety of distributions of nonzero (co)homology. For example, when \( s = 0 \) one can construct arbitrarily large intervals of either vanishing or nonvanishing (co)homology: for integers \( a \) and \( b \) satisfying \( 0 \leq a < b \) there exist finitely generated \( A \)-modules \( N_{a,b} \) and \( Z_{a,b} \) such that

1. \( \operatorname{Ext}^i_A(M, N_{a,b}) \neq 0 \) if and only if \( a \leq i \leq b \) (and \( i = 0 \)), and
2. \( \operatorname{Ext}^i_A(M, Z_{a,b}) = 0 \) for all \( a < i < b \) and \( \operatorname{Ext}^i_A(M, Z_{a,b}) \neq 0 \) for \( i = a, b \) (and \( i = 0 \)).

Indeed, one may take \( N_{a,b} = \bigoplus_{q=a+1}^b T_q \) and \( Z_{a,b} = T_a \bigoplus T_{b+1} \).

When \( s \) is positive, we obtain recurring intervals of vanishing/nonvanishing cohomology. For example, assume that \( s = 4 \) and set \( T = T_0 \). We have then

\[
\operatorname{Ext}^i_A(M, T) = 0 \quad \text{for all} \quad i \equiv 1, 2 \mod (4),
\]

\[
\operatorname{Ext}^i_A(M, T) \neq 0 \quad \text{for all} \quad i \equiv 0, 3 \mod (4).
\]

We give only the proof of Proposition 3.1. The proof of Proposition 3.2 is similar.

**Proof of Proposition 3.1.** We let overbars denote residue classes modulo \( J_q \) and we perform computations of Ext and Tor as explained in 2.3. We only give the proof of (1); the other arguments are similar.

1. The differential \( \partial_q = d_i \otimes A \) of the complex \( F \otimes_A T_q \) is given in the standard basis of \( T_q \) over \( T_q \) by the matrix

\[
\begin{pmatrix}
\bar{x}_1 & \alpha^{i−q} \bar{x}_1 \\
\bar{x}_1 & \bar{x}_1
\end{pmatrix}.
\]
As a \( k \)-vector space, \( T_q \) has a basis consisting of \( 1, x_1 \) and for each \( i \geq 0 \)
\[
\text{rank}_k(\text{Im} \, \partial_i) = \begin{cases} 
1 & \text{if } i \equiv q \mod (s) \\
2 & \text{otherwise}.
\end{cases}
\]

Since \( \dim_k(\text{Ker} \, \partial_i) = \dim_k(T_q^2) - \dim_k(\text{Im} \, \partial_i) \), we then have
\[
\text{rank}_k(\text{Ker} \, \partial_i) = \begin{cases} 
3 & \text{if } i \equiv q \mod (s) \\
2 & \text{otherwise}.
\end{cases}
\]

By Proposition 3.1 and 2.5 there exists an \( A \)-module \( W \) with
\[
P^A_{W \otimes_A T_q}(t, z) = 2z(1 - tz)^{-1}
\]
and \( \text{Tor}^A_i(M, V) = 0 \) for all \( i > 0 \) and \( W \otimes_R T_q \) is a sum of 2 copies of \( k \) each generated in degree 1. Indeed, if \( q \leq 0 \) then take \( W \) to be a first syzygy of \( M \), and if \( q > 0 \) then take \( W \) to be a first syzygy of \( M^* \), and use for example [15, 1.4(2)] to see that \( W \otimes_R T_q \) is annihilated by the maximal ideal of \( A \).

The bigraded version of a usual computation, cf. [19, 1.1] for example, gives
\[
P_{W \otimes_A T_q}(t, z) = P^A_W(t, z)P^A_{T_q}(t, z)
\]
and \( W \otimes_R T_q \) is a sum of copies of \( k \), we may use the formula for \( P^A_k(t, z) \) given in 2.5 to conclude that
\[
P^A_{T_q}(t, z) = \frac{1}{1 - 4tz + t^2z^2}.
\]

The expansion of this fraction as a power series shows that the \( A \)-module \( T_q \) has a linear resolution. Similar computations show that \( T_q \) has a linear resolution as a \( B \)-module as well.

Now let \( U \) be the cokernel of the map \( A^6 \to A^2 \) given in the standard bases of \( A^6 \), respectively \( A^2 \), over \( A \) by the matrix
\[
\begin{pmatrix}
x_3 & 0 & x_1 & x_4 & x_2 & 0 \\
-x_2 & x_3 & -x_4 & 0 & 0 & x_1
\end{pmatrix},
\]
and set \( V = U \otimes_A (m^2, x_5) \). Note that \( V \) is also a \( B \)-module.

The next proposition gives part (2) of the main theorem in the introduction. (See also 3.12 below).

3.9. Proposition. With the notation above, the following hold:

(a) \( \text{Tor}^A_i(M, V) = 0 \) for all \( i > 0 \).

(b) \( \text{Ext}^i_A(M, V) \neq 0 \) for all \( i > 0 \).

3.10. Remarks. We can obtain an example of vanishing Exts and nonvanishing Tors just by replacing \( V \) with \( V^* \). (Since \( A \) is zero-dimensional and Gorenstein, one has \( -^* = -^* \). Recall then from 1.2 that \( \text{Tor}^A_i(M, V^*) \cong \text{Ext}^i_A(M, V)^* \) and \( \text{Ext}^i_A(M, V^*) \cong \text{Tor}^A_i(M, V)^* \).)

One can replace \( A \) with \( B \) and \( M \) with \( L \) in the statement of Theorem 3.9. The proof is similar.
Proof of Proposition 3.9. (1). Recall from 2.6.1 that \( \text{Tor}^A_i(M, V) \) is the \( i \)th homology of the complex \( F \otimes_A V \), where the differential \( d_i: A^2 \to A^2 \) of \( F \) is given in the standard basis of \( A^2 \) by the matrix

\[
\begin{pmatrix}
x_1 & \alpha^i x_3 \\
x_4 & x_2
\end{pmatrix}.
\]

The module \( V \) is the quotient of \( A^2 \) by \((m^2, x_3)\) and the relations \((x_3, -x_2), (0, x_3), (x_1, -x_4), (x_4, 0), (x_2, 0), (0, x_1)\). We let \( \text{cls} a \) denote the residue class in \( V \) of \( a \in A^2 \). As a \( k \)-vector space, \( V \) has a basis formed by the 4 elements \( v_1 := \text{cls} (1, 0), v_2 := \text{cls} (0, 1), v_3 := \text{cls} (x_1, 0), v_4 := \text{cls} (0, x_2) \).

Given this basis for \( V \), we then make the obvious choice for a basis of \( V^2 \).

We set \( \delta_i = d_i \otimes_A V: V^2 \to V^2 \). For each \( i \), the following elements in \( \text{Im}(\delta_i) \) can be easily seen to be linearly independent over \( k \):

\[
\begin{align*}
\delta_i(v_1, 0) &= (v_3, 0) \\
\delta_i(v_2, 0) &= (0, v_3) \\
\delta_i(0, v_1) &= \alpha^i(v_4, 0) \\
\delta_i(0, v_2) &= (0, v_4).
\end{align*}
\]

As a consequence, \( \text{rank}_k(\text{Im}(\delta_i)) \geq 4 \). On the other hand,

\[
\text{rank}_k(\text{Ker}(\delta_i)) = \text{rank}_k(V^2) - \text{rank}_k(\text{Im}(\delta_i)) \leq 8 - 4 = 4
\]

It follows that \( \text{Ker}(\delta_i) = \text{Im}(\delta_{i+1}) \) for each \( i > 0 \), hence \( H_i(F \otimes_A V) = 0 \).

(2). Recall from 2.6.2 that \( \text{Ext}^A_i(M, V) \) is the \((-i)\)th homology of the complex \( F^* \otimes_A V \), where the differential \( d_i^*: A^2 \to A^2 \) of \( F^* \) is given in the standard basis of \( A^2 \) by the matrix

\[
\begin{pmatrix}
x_1 & x_4 \\
\alpha^i x_3 & x_2
\end{pmatrix}.
\]

Set \( \delta_i^* = d_i^* \otimes_A V \). Similar computations as above show that only two elements of the basis of \( V^2 \) over \( k \) are not in \( \text{Ker}(\delta_i^*) \). Their images are

\[
\begin{align*}
\delta_i^*(v_1, 0) &= (v_3, \alpha^i v_4) \\
\delta_i^*(0, v_2) &= (v_3, v_4)
\end{align*}
\]

We conclude

\[
\text{rank}_k(\text{Im}(\delta_i)) \leq 2 \quad \text{and} \quad \text{rank}_k(\text{Ker}(\delta_i)) \geq 8 - 2 = 6 \quad \text{for all} \quad i.
\]

We thus have \( H_i(F^* \otimes_A V) \neq 0 \) for all \( i > 0 \).

3.11. Remark. Formulated in terms of Tate (co)homology, the proof of Proposition 3.9 shows that

(a) \( \hat{\text{Tor}}^A_i(M, V) = 0 \) for all \( i \).

(b) \( \hat{\text{Ext}}^A_i(M, V) \neq 0 \) for all \( i \).

3.12. The proof of the proposition shows that the module \( V \) has

\[
\text{Hilb}_V(t) = 2 + 2t.
\]

Similar computations as in 3.8 show that \( V \) has a linear resolution, both as an \( A \)-module, and as a \( B \)-module.
For each finitely generated $A$-module $N$ we set
\[ c(N) = \text{rank}_k(N) - \text{rank}_k(\text{socle } N). \]

The next proposition shows that when $o(\alpha) = \infty$, $M$ is rather rigid.

**3.13. Proposition.** When $o(\alpha) = \infty$ the following hold for any finitely generated $A$-module $N$:

1. If $\text{Tor}^A_i(M, N) = 0$ for some $j > 0$, then $\text{Tor}^A_i(M, N) \neq 0$ for at most $2c(N)$ values of $i > 0$.
2. If $\text{Ext}^A_i(M, N) = 0$ for some $j > 0$, then $\text{Ext}^A_i(M, N) \neq 0$ for at most $2c(N)$ values of $i > 0$.
3. If $\text{Ext}^A_i(N, M) = 0$ for some $j > 0$ then $\text{Ext}^A_i(N, M) \neq 0$ for at most $2c(N)$ values of $i > 0$.
4. $\text{Ext}^A_i(M, N) = 0$ for all $i \gg 0$ if and only if $\text{Ext}^A_i(N, M) = 0$ for all $i \gg 0$.

**3.14. Remark.** The statements (1) and (2) remain valid, with similar proofs, when replacing $A$ by $B$, and $M$ by $L$.

**Proof.** We will show that if $H_j(C^* \otimes_A N) = 0$ for some $j$, then $H_i(C^* \otimes_A N) \neq 0$ for at most $2c(N)$ values of $i$. In view of 2.6.2 and 2.6.3, this proves (2), (3) and (4). The proof of (1) is along similar lines, using the complex $F \otimes_A N$ and 2.6.1 and it is omitted.

For every $i$ set
\[ u_i = \text{rank}_k \left( \text{Im}(d^i_{i+1} \otimes_A N) \right) \quad \text{and} \quad v_i = \text{rank}_k \left( \text{Ker}(d^i_{i+1} \otimes_A N) \right). \]

Since $C^* \otimes_A N$ is a complex, we have $u_i \leq v_i$ for all $i$, with equality if and only if $H_i(C^* \otimes_A N) = 0$.

Assume that $u_j = v_j$ for some $j \in \mathbb{Z}$. We need to show that $u_i \neq v_i$ for at most $2c(N)$ values of $i \in \mathbb{Z}$. Set
\[ r = \max \{ u_i \mid i \in \mathbb{Z} \}. \]

Since $u_i + v_{i+1} = 2 \text{rank}_k N$ and $u_i \leq v_i$ for all $i$, we conclude that $u_i + u_{i-1} \leq 2 \text{rank}_k N$, with equality if and only if $u_i = v_i$. Taking $i = j$ we obtain that $u_j + u_{j-1} = 2 \text{rank}_k N$. Since $u_i \leq r$ for all $i$, we have $\text{rank}_k N \leq r$.

**Claim.** $u_i \neq r$ for at most $c(N)$ values of $i \in \mathbb{Z}$.

Assuming the claim for the moment, we finish the proof.

As noted above, we need to show that $u_i + u_{i-1} \neq 2 \text{rank}_k N$ for at most $2c(N)$ values of $i \in \mathbb{Z}$. We have
\[ \{ i \in \mathbb{Z} \mid u_i + u_{i-1} \neq 2 \text{rank}_k N \} = \{ i \in \mathbb{Z} \mid u_i \neq r \} \cup \{ i \in \mathbb{Z} \mid u_{i-1} \neq r \}, \]
and the claim shows that these latter sets have at most $2c(N)$ elements.

**Proof of claim.** Let $E$ be a basis of $N$ over $k$ and set $e = \text{rank}_k N$. Let $\chi_i$ be the $e \times e$ matrix which represents in the basis $E$ the map $N \to N$ given by multiplication by $x_i$, for $i = 1, \ldots, 4$.

With the obvious choice for the basis of $N^2$ over $k$, the map $d_i \otimes_A N : N^2 \to N^2$ is represented by the $2e \times 2e$ matrix
\[ \Omega_i = \begin{pmatrix} \chi_1 & \alpha^i \chi_3 \\ \chi_4 & \chi_2 \end{pmatrix}. \]
Let \( k[y] \) be the polynomial ring over \( k \) in a single variable \( y \). We consider the following \( 2e \times 2e \) matrix with elements in \( k[y] \):

\[
\Omega(y) = \begin{pmatrix}
\chi_1 & y\chi_3 \\
\chi_4 & \chi_2
\end{pmatrix}.
\]

Since \( r \) is the maximum of \( \{ \text{rank} \Omega_i \} \), there exists a nonzero \( r \times r \) minor \( \Delta_\ell \) of \( \Omega_\ell \) for some \( \ell \). Let \( \Delta(y) \) denote the \( r \times r \) minor of \( \Omega(y) \) corresponding to \( \Delta_\ell \). Then \( \Delta(y) \) is a polynomial in \( y \). Since \( \Delta(\alpha^\ell) = \Delta_\ell \) is nonzero, \( \Delta(y) \) is a nonzero polynomial. Note that it has degree at most \( c(N) \), and therefore it has at most \( c(N) \) roots in \( k \). In conclusion, the \( r \times r \) minor \( \Delta_i = \Delta(\alpha_i) \) of \( \Omega_i \) is zero for at most \( c(N) \) values of \( i \). □

3.15. Remark. Formulated in terms of Tate cohomology, cf. 3.6, the proofs of parts (1) and (2) of Proposition 3.13 actually show the following:

(1) If \( \tilde{\text{Tor}}^j_A(M, N) = 0 \) for some \( j \), then \( \text{Tor}^A_i(M, N) \neq 0 \) for at most \( 2c(N) \) values of \( i \).

(2) If \( \text{Ext}^j_A(M, N) = 0 \) for some \( j \), then \( \text{Ext}^i_A(M, N) \neq 0 \) for at most \( 2c(N) \) values of \( i \).

Similar statements can be given for the ring \( B \) and the module \( L \).

4. Classes of Gorenstein Rings

In this section we discuss homologically defined classes of local Gorenstein rings, introduced in [3] and [14], and show using the examples in the previous section that these classes of local rings lie properly between the class of local complete intersections and the class of local Gorenstein rings.

Throughout this section \( R \) is a local ring with maximal ideal \( m \). Let (ci) denote the condition that \( R \) is a local complete intersection, and (gor) the condition that \( R \) is a local Gorenstein ring. We further consider the following properties (cf. [3, 6.3]):

(te) \( \text{Tor}^R_i(M, N) = 0 \) for all \( i \gg 0 \) implies \( \text{Ext}^i_R(M, N) = 0 \) for all \( i \gg 0 \),

(et) \( \text{Ext}^i_R(M, N) = 0 \) for all \( i \gg 0 \) implies \( \text{Tor}^i_R(M, N) = 0 \) for all \( i \gg 0 \),

(ee) \( \text{Ext}^i_R(M, N) = 0 \) for all \( i \gg 0 \) implies \( \text{Ext}^i_R(N, M) = 0 \) for all \( i \gg 0 \),

where \( M \) and \( N \) range over all finitely generated \( R \)-modules.

4.1. Note that by taking \( N = R \), the property (te) implies \( R \) is Gorenstein; by taking \( M = R \), the property (ee) implies \( R \) is Gorenstein.

4.2. Avramov and Buchweitz prove in [3, 6.1] that if \( R \) is a local complete intersection, then it satisfies both (et) and (te). This gives the implication (1) in the following diagram, reproduced from [3, 6.3]; the remaining implications are clear.

![Diagram](attachment:image.png)
In [3, 6.4] the question is raised whether any of these implications can be reversed. We first note that (2) is reversible:

4.3. Proposition. The following statements are equivalent.

1. $R$ satisfies (te).
2. $R$ is Gorenstein and satisfies (et).

Proof. We give only the proof of $(1) \implies (2)$. The reverse implication can be proved similarly.

Assume that $R$ satisfies (te). By 4.1, the ring $R$ is Gorenstein. By taking syzygies, it suffices to prove that (et) holds for maximal Cohen-Macaulay $R$-modules $M$ and $N$. Note that such modules are in particular reflexive, that is they are isomorphic to their double dual.

Let $M$ and $N$ be maximal Cohen-Macaulay $R$-modules such that $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$. By [14, 2.1] we have then $\text{Tor}^i_R(M, N^*) = 0$ for all $i \gg 0$. Since (te) holds, this implies $\text{Ext}^i_R(M, N^*) = 0$ for all $i \gg 0$. Using again [14, 2.1] we conclude that $\text{Tor}^i_R(M, N) = 0$ for all $i \gg 0$. \hfill $\Box$

In [14], $R$ is said to be an AB ring whenever it satisfies the following condition:

(ab) $R$ is Gorenstein and there exists an integer $n$ such that for all pairs $(M, N)$ of finitely generated $R$-modules $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ implies $\text{Ext}^i_R(M, N) = 0$ for all $i > n$.

It is shown in [14] that if $R$ is an AB ring, then the integer $n$ above can be taken to be $\dim R$, but not less. Note that the condition (ab) is the Uniform Auslander Condition (uac) from Section 1 together with the requirement that $R$ is Gorenstein.

Another property of a local Gorenstein ring $R$ is defined in [14]. We say that $\text{Ext}^i_R(M, N)$ has a gap of length $g$ if there exists an $n > 0$ such that $\text{Ext}^i_R(M, N) \neq 0$ for $i = n - 1$ and $i = n + g$ and $\text{Ext}^i_R(M, N) = 0$ for all $n \leq i \leq n + g - 1$. Set

$$\text{Ext-gap}(R) := \sup\{g \mid \text{Ext}^i_R(M, N) \text{ has a gap of length } g \},$$

where $M$ and $N$ range over all finitely generated $R$-modules. Similarly, one can define the notion of Tor-gap. It is proved in [14, 3.3(2)] that over a Gorenstein ring Ext-gap is finite if and only if Tor-gap is finite.

We define the property finite gap as follows:

(gap) $R$ is Gorenstein and Ext-gap($R$) is finite.

4.4. The following implications are known to hold:

$$(\text{ci}) \implies (3) \implies (\text{gap}) \implies (4) \implies (\text{ab}) \implies (5) \implies (\text{ee})$$

The implication (3) is proved in [20, 1.6], cf. also [16, 2.3] for a more precise version. The implication (4) is given by [14, 3.3(3)] and (5) by [14, 4.1].

In [14], it is also proved that the implication (3) above and (1) in 4.2 are not reversible. The details of this are as follows.

Every local Gorenstein ring $R$ (which is not a complete intersection) has multiplicity at least $\text{edim } R - \dim R + 2$. A local Gorenstein ring $R$ is said to have minimal multiplicity if its multiplicity is equal to $\text{edim } R - \dim R + 2$. When $R$ is artinian,
minimal multiplicity just means $m^4 = 0$. A local Gorenstein ring $R$ of minimal multiplicity is not a complete intersection precisely when $\text{edim } R - \dim R \geq 3$.

4.5. Let $R$ be a local Gorenstein ring of minimal multiplicity. By [14 3.6] and [14 3.2(3)], $R$ satisfies $(\text{gap})$, and hence $(\text{ab})$.

By [14 3.6], if $R$ is a Gorenstein ring of minimal multiplicity, but not a complete intersection, then $R$ satisfies the property $(\text{tv})$ introduced in Section 4. Thus all Gorenstein rings of minimal multiplicity also satisfy $(\text{te})$.

The main theorem stated in the introduction and proved in the previous section, shows that

(a) there exist local Gorenstein rings which are not AB rings.

(b) there exist local Gorenstein rings which do not satisfy $(\text{te})$.

4.6. The facts discussed above are summarized in the following refinement of the diagrams in 4.2 and 4.4.

\begin{center}
\begin{tikzpicture}
\node (et) at (0,0) {$(\text{et}) \& (\text{te})$};
\node (te) at (2,0) {$(\text{te})$};
\node (ci) at (-2,-2) {$(\text{ci})$};
\node (ee) at (0,-2) {$(\text{ee})$};
\node (gor) at (2,-2) {$(\text{gor})$};
\node (gap) at (-2,-4) {$(\text{gap})$};
\node (ab) at (2,-4) {$(\text{ab})$};
\draw[->] (et) -- (te);
\draw[->] (et) -- (ci);
\draw[->] (et) -- (ee);
\draw[->] (et) -- (gor);
\draw[->] (et) -- (gap);
\draw[->] (et) -- (ab);
\draw[->] (te) -- (6,0);
\draw[->] (ci) -- (-6,-2);
\draw[->] (ee) -- (6,-2);
\draw[->] (gor) -- (6,-4);
\draw[->] (gap) -- (4,-4);
\draw[->] (ab) -- (4,-4);
\end{tikzpicture}
\end{center}

In conclusion, the following classes of local rings lie strictly between the class of the local complete intersections and that of local Gorenstein rings:

(a) the local rings satisfying $(\text{te})$.

(b) the AB rings.

(c) the local rings satisfying $(\text{gap})$.

4.7. Our examples in the previous section are local Gorenstein rings $R$ with $m^4 = 0$, $\text{edim } R = 5$, and $\lambda(R) = 12$ which satisfy neither $(\text{ab})$ nor $(\text{te})$. By [14 3.6], these examples are minimal with respect to the invariant $\inf \{n \mid m^n = 0\}$. Other aspects of the minimality of these examples can be deduced from [21] as we now describe.

Let $R$ be a Gorenstein local ring. By [21 3.4] and [14 3.1], if $\text{edim } R - \dim R \leq 4$, then $R$ satisfies $(\text{ab})$. Our examples of Gorenstein rings not satisfying $(\text{ab})$ are thus minimal with respect to embedding dimension.

If $R$ is standard graded with $\lambda(R) < 12$, then it follows that $m^3 = 0$ or $\text{edim } R \leq 4$, hence $(\text{ab})$ or the above considerations apply. In particular, any such ring satisfies $(\text{ab})$, and so in the standard graded case our examples of rings not satisfying $(\text{ab})$ are minimal with respect to length.

4.8. There is a question remaining:

Are there further implications between the properties displayed in 4.6?

Acknowledgements

The authors thank Luchezar Avramov for his comments and suggestions on earlier versions of this paper. They used the computer algebra package Macaulay 2 [18] for testing some preliminary examples.
References

[1] M. Auslander, *Selected works*, I. Reiten, S. O. Smalø, and Ø. Solberg, Editors, AMS, Collected Works, vol. 10, 1999.
[2] L. L. Avramov, *Homological asymptotics of modules over local rings*, Commutative Algebra (Berkeley, 1987), MSRI Publ. 15, Springer, New York 1989; pp. 33–62.
[3] L. L. Avramov, R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000), 285–318.
[4] L. L. Avramov, R.-O. Buchweitz, L. M. Šega, *Extensions of a dualizing complex by its ring: commutative versions of a conjecture of Tachikawa*, J. Pure Appl. Algebra (to appear).
[5] M. Auslander, M. Bridger, *Stable Module Theory*, Memoirs of the A.M.S. 94 (1969), American Math Society, Providence, R.I.
[6] L. L. Avramov, V. N. Gasharov, I. V. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. 86 (1997), 67–114.
[7] T. Araya, Y. Yoshino, *Remarks on a depth formula, a grade inequality and a conjecture of Auslander*, Comm. Alg. 26 (1998), 3793–3806.
[8] D. J. Benson, J. F. Carlson, G. R. Robinson, *On the vanishing of group cohomology*, J. Algebra 131 (1990), 40–73.
[9] R. Froberg, *Koszul algebras*, Advances in commutative ring theory (Fez, 1997), Lecture Notes in Pure and Appl. Math., 205, Dekker, New York 1999; pp. 337–350.
[10] V. Gasharov, I. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. 320 (1999), 569–580.
[11] D. Happel, *Homological conjectures in representation theory of finite-dimensional algebras*, preprint, 1990.
(Available at: http://www.math.ntnu.no/~oyvinso/Nordfjordeid/Program/references.html)
[12] R. Hartshorne, *Residues and duality*, Lecture Notes in Math., 20, Springer, Berlin 1971.
[13] R. Heitmann, *A counterexample to the rigidity conjecture for rings*, Bull. Amer. Math. Soc. 29 (1993), 94–97.
[14] C. Huneke and D. A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, Math. Scand. (to appear)
[15] C. Huneke, L. M. Šega, A. N. Vraciu, *Vanishing of Ext and Tor over Cohen-Macaulay local rings*, preprint, 2003.
[16] D. A. Jorgensen, *Vanishing of (co)homology over commutative rings*, Comm. Algebra 29 (2001), 1883-1898.
[17] D. A. Jorgensen, *A generalization of the Auslander-Buchsbaum formula*, J. Pure Appl. Algebra 144 (1999), 145–155.
[18] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2
[19] C. Miller, *Complexity of tensor products of modules and a theorem of Huneke-Wiegand*, Proc. Amer. Math. Soc. 126 (1998), 53–60.
[20] M. P. Murthy, *Modules over regular local rings*, Ill. J. Math. 7 (1963), 558-565.
[21] L. M. Šega, *Vanishing of cohomology over Gorenstein rings of small codimension*, Proc. Amer. Math. Soc. 131, (2003), 2313-2323.
[22] C. Scheja, *Über die Bettizahlen lokaler Ringe*, Math. Ann. 155 (1964), 155–172.

Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019
E-mail address: djorgens@math.uta.edu

Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720-5070
E-mail address: lsega@msri.org