Topological Mass Generation
in
Four-Dimensional Gauge Theory

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Abstract

The Lagrangian of non-Abelian tensor gauge fields describes the interaction of the Yang-Mills and massless tensor bosons of increasing helicities. We have found a metric-independent gauge invariant density which is a four-dimensional analog of the Chern-Simons density. The Lagrangian augmented by this Chern-Simons-like invariant describes massive Yang-Mills boson, providing a gauge-invariant mass gap for a four-dimensional gauge field theory. We present invariant densities which can provide masses to the high rank tensor bosons.
1 Introduction

Several mechanisms are currently known for generating massive vector particles that are compatible with the gauge invariance. One of them is the spontaneous symmetry breaking mechanism, which generates masses and requires the existence of the fundamental scalar particle - the Higgs boson. The scalar field provides the longitudinal polarization of the massive vector boson and ensures unitarity of its scattering amplitudes \[1, 2\].

The argument in favor of a pure gauge field theory mechanism was a dynamical mechanism of mass generation proposed by Schwinger \[3\], who was arguing that the gauge invariance of a vector field does not necessarily lead to the massless spectrum of its excitations and suggested its realization in (1+1)-dimensional gauge theory\[4\].

Compatibility of gauge invariance and mass term in (2+1)-dimensional gauge field theory was demonstrated by Deser, Jackiw and Templeton \[18, 19\] and Schonfeld \[20\], who added to the YM Lagrangian a gauge invariant Chern-Simons density:

\[
L_{YMC} = -\frac{1}{2} Tr G_{ij} G_{ij} + \frac{\mu}{2} \varepsilon_{ijk} \text{Tr} (A_i \partial_j A_k - ig^2/3 A_i A_j A_k),
\]

where \(G_{ij}\) is a field strength tensor. The mass parameter \(\mu\) carries dimension of \([mass]\). The corresponding free equation of motion for the vector potential \(A_i = e_i e^{ikx}\) has the form

\[
(-k^2 \eta_{ij} + k_i k_j)e_j + i\mu \varepsilon_{ijl} k_j e_l = 0
\]

and shows that the gauge field excitation becomes massive.

In this article we suggest a similar mechanism that generates masses of the YM boson and tensor gauge bosons in (3+1)-dimensional space-time at the classical level. As we shall see, in non-Abelian tensor gauge theory \[21, 22, 23\] there exists a gauge invariant, metric-independent density \(\Gamma\) in five-dimensional space-time\[2\]:

\[
\Gamma = \varepsilon_{lmnpq} Tr G_{lm} G_{np,q} = \partial_l \Sigma_l, \tag{1}
\]

which is the derivative of the vector current \(\Sigma_l\) (1,...=0,1,...,4). This invariant in five dimensions has many properties of the Chern-Pontryagin density \(P = \partial_{\mu} C_{\mu}\) in four-dimensional YM theory, which is a derivative of the Chern-Simons topological vector current \(C_{\mu}\). Considering the fifth component of the vector current \(\Sigma_4 \equiv \Sigma\) and fields which are independent on the fifth spacial coordinate \(x_4\), one can get a gauge invariant density which is defined in four-dimensional space-time\[3\]:

\[
\Sigma = \varepsilon_{\mu\nu\rho\lambda} Tr G_{\mu\nu} A_{\rho\lambda}. \tag{2}
\]

Its dimensionality is \([mass]\), therefore in order to get dimensionless functional in four dimensions we should multiply it by the parameter \(m\) which has dimensionality \([mass]\). Adding this term to the Lagrangian of non-Abelian tensor gauge fields leaves intact its gauge invariance, and to lowest order in coupling constant the equations of motion for the YM field \(A_{\mu} = \epsilon_{\mu} e^{ikx}\) of helicities \(\lambda = \pm 1\) and for the antisymmetric part \(B_{\mu\nu} = b_{\mu\nu} e^{ikx}\) of the rank-2 gauge field \(A_{\mu\nu}\), which carries helicity zero \(\lambda = 0\) state, can be written in the following form:

\[
(-k^2 \eta_{\mu\nu} + k_{\mu} k_{\nu}) e_{\mu} + im \varepsilon_{\nu\mu\lambda\rho} k_{\lambda} b_{\rho} = 0,
\]

\[
(-k^2 \eta_{\nu\rho} + k_{\nu} k_{\rho}) e_{\nu} + \frac{2m}{3} \varepsilon_{\nu\lambda\mu} k_{\nu} e_{\mu} = 0.
\]

\[\text{1Extended discussion and references can be found in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].}\]

\[\text{2The definition of the higher-rank field strength tensors is given by formula (5).}\]

\[\text{3We are using Greek letters to numerate four-dimensional coordinates.}\]
These field equations describe massive state of the vector particle of the mass

\[ M^2 = \frac{4}{3} m^2. \]  

(3)

Thus at the classical level the YM vector boson becomes massive. The anti-symmetric tensor \( B_{\mu \nu}^a \), which carries zero helicity state, provides the longitudinal polarization of the massive vector meson, suggesting an alternative mechanism for mass generation in non-Abelian gauge field theories in four-dimensional space-time. Because both of the fields, the vector and the antisymmetric tensor, are in the adjoint representation, it follows that all vector fields \( A_{\mu}^a, a = 1, ..., \dim G \) acquire the same mass \( M \). At this stage the symmetric part \( A_{\mu \nu}^S \) of the rank-2 gauge field, which carries helicities \( \lambda = \pm 2 \), remains massless.

As a next step we shall demonstrate that in five-dimensional space-time there actually exists an infinite series of invariants \( \Gamma_s \ (s = 1, 3, ...) \) which are constructed by means of the totally antisymmetric Levi-Civita epsilon tensor \( \varepsilon_{lmnpq} \) in combination with the generalized field strength tensors \( G_{mn,l1...ls} \). These invariants can be represented as total derivatives of the vector currents \( \Sigma^s_l \):

\[ \Gamma_s = \partial_l \Sigma^s_l, \]

where the vector currents \( \Sigma^s_l \) involve a free index \( l \) carried by the Levi-Civita epsilon tensor. Considering the fifth component of the vector current \( \Sigma^4_4 \equiv \Sigma_s \) one can see that the remaining indices will not repeat the external index. Furthermore, if all dependence of the tensor gauge fields on the fifth spatial coordinate \( x_4 \) is suppressed, we are left with the invariant densities which are defined in four-dimensional space-time. Their dimensionality is \([\text{mass}]^3\) therefore in order to get dimensionless functional in (3+1)-dimensions we should multiply them by the parameters \( m_s \) which have units \([\text{mass}]^1\):

\[ m_s \int_{M_4} \Sigma_s. \]

Adding these densities to the Lagrangian of non-Abelian tensor fields keeps intact its gauge invariance, up to total divergence terms, so that the Lagrangian takes the following form:

\[ \mathcal{L}_m = \mathcal{L}_{YM} + \sum_s (\mathcal{L}_{s+1} + \frac{2s}{s+1} \mathcal{L}'_{s+1}) + \sum_s m_s \Sigma_s. \]  

(4)

The natural appearance of the mass parameters hints at the fact that the theory turns out to be a massive theory.

In the next section we present a short introduction into the theory of non-Abelian tensor gauge fields defining their gauge transformations, fields strength tensors and the Lagrangian [21, 22, 23]. In section three we derive different properties of the metric-independent and gauge invariant density \( \Gamma \) and of the corresponding vector current \( \Sigma^s_l \) and its reduction to four dimensions. In section four we overview the helicity content of the massless tensor gauge fields before including the topological mass term \( m \Sigma \) into the Lagrangian. In section five we analyze how the spectrum of the theory is changing when we add the invariant \( m \Sigma \) to the Lagrangian. As we shall see, a massive spin-1 YM boson with its three spin polarizations, \( \lambda = \pm 1, 0 \) appears in the spectrum, while the antisymmetric field has been absorbed as its longitudinal polarization. In the sixth section we are presenting infinite series of dimensionful invariants \( \Sigma_s \ (s = 1, 3, ...) \) which exist in four-dimensional space-time and can be added to Lagrangian in order to generate masses of the tensor bosons. In section seven we are presenting topological invariants in six dimensions.
2 Non-Abelian Gauge Fields

In our recent approach the gauge fields are defined as rank-$(s+1)$ tensors [21, 22, 23]

$$A_{\mu \lambda_1...\lambda_s}^a(x),$$

which are totally symmetric with respect to the indices $\lambda_1...\lambda_s$. The number of symmetric indices $s$ runs from zero to infinity $^4$. The index $a$ numerates the generators $L^a$ of an appropriate Lie algebra. The extended non-Abelian gauge transformation $\delta_\xi$ of the tensor gauge fields is defined in the Appendix and comprises a closed algebraic structure. The generalized field strength tensors are defined as follows [21, 22, 23]:

$$G_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu A_\nu],$$
$$G_{\mu \nu, \lambda} = \partial_\mu A_{\nu \lambda} - \partial_\nu A_{\mu \lambda} - ig( [A_\mu A_{\nu \lambda}] + [A_{\mu \lambda} A_\nu] ),$$
$$G_{\mu \nu, \lambda \rho} = \partial_\mu A_{\nu \lambda \rho} - \partial_\nu A_{\mu \lambda \rho} - ig( [A_\mu A_{\nu \lambda \rho}] + [A_{\mu \lambda \rho} A_\nu] + [A_{\mu \lambda \rho} A_\nu] ),$$

and transform homogeneously with respect to the extended gauge transformations $\delta_\xi$. The tensor gauge fields are in the matrix representation $A^a_{\mu \lambda_1...\lambda_s} = (L_c)^{ab}A^a_{\mu \lambda_1...\lambda_s}$ and $f^{abc}$ are the structure constants of the Lie algebra.

Using field strength tensors one can construct two infinite series of forms $L_s$ and $L'_s$ invariant with respect to the transformations $\delta_\xi$. They are quadratic in field strength tensors. The first series is given by the formula [21, 22, 23]

$$L_{s+1} = - \frac{1}{4} \left( \sum_{i=0}^{2s} a_i^s \ G_{\mu \nu, \lambda_1...\lambda_i}^a G_{\mu \nu, \lambda_{i+1}...\lambda_{2s}}^a \right) \left( \sum_p \eta_{\lambda_1 \lambda_2} \eta_{\lambda_{2s-1} \lambda_{2s}} \right),$$

where the sum $\sum_p$ runs over all nonequal permutations of $\lambda_i$'s and $a_i^s = \frac{s!}{i!(2s-i)!}$. The second series of gauge invariant quadratic forms is given by the formula [21, 22, 23, 30, 32]

$$L'_{s+1} = \frac{1}{4} \left( \sum_{i=1}^{2s+1} \frac{a_{i-1}^s}{s} \ G_{\mu \lambda_1...\lambda_i}^a G_{\mu \lambda_{i+2}...\lambda_{2s+2}}^a \eta_{\lambda_1 \lambda_2} \eta_{\lambda_{2s+1} \lambda_{2s+2}} \right),$$

where the sum $\sum_p$ runs over all nonequal permutations of $\lambda_i$'s, with exclusion of the terms which contain $\eta_{\lambda_1 \lambda_{2s+2}}$.

These forms contain quadratic kinetic terms, as well as cubic and quartic terms describing nonlinear interaction of gauge fields with dimensionless coupling constant $g$. In order to make all tensor gauge fields dynamical one should add all these forms in the Lagrangian [21, 22, 23, 30, 32]:

$$\mathcal{L} = \mathcal{L}_{YM} + L_2 + L'_2 + ... + g_{s+1}(L_{s+1} + \frac{2s}{s+1}L'_{s+1}) + ....$$

The coupling constants $g_3, g_4, ...$ remain arbitrary because each term is separately invariant with respect to the extended gauge transformations $\delta_\xi$ and still leaves these coupling constants undetermined. The Lagrangian $\mathcal{L}$ is well defined in any dimension.

$^4$A priori the tensor fields have no symmetries with respect to the first index $\mu$. The free field theory of totally symmetric tensors of high rank were constructed in [24, 25, ?, ?, 26, ?, ?, 8, 12, 17, 28, 45, ?, 29].
3 Metric-Independent Density $\Gamma$

Let us consider a new invariant in five-dimensional space-time ($4 + 1$), which can be constructed by means of the totally antisymmetric Levi-Civita epsilon tensor $\varepsilon_{\mu \nu \lambda \rho \sigma}$ ($\mu, \nu, \ldots = 0, 1, 2, 3, 4$) in combination with the generalized field strength tensors

$$\Gamma = \varepsilon_{\mu \nu \lambda \rho \sigma} Tr G_{\mu \nu} G_{\lambda \rho, \sigma} = 2 \varepsilon_{\mu \nu \lambda \rho \sigma} G^{a}_{\mu \nu} G^{a}_{\lambda \rho, \sigma}. \tag{9}$$

We shall demonstrate that this invariant in five dimensions has many properties of the Chern-Pontryagin density

$$\mathcal{P} = \frac{1}{4} \varepsilon_{\mu \nu \lambda \rho \sigma} Tr G_{\mu \nu} G_{\lambda \rho} = \partial_{\mu} C_{\mu} \tag{10}$$

in Yang-Mill theory in four dimensions, where

$$C_{\mu} = \varepsilon_{\mu \nu \lambda \rho \sigma} Tr (A_{\nu} \partial_{\lambda} A_{\rho} - i \frac{2}{3} g A_{\nu} A_{\lambda} A_{\rho}) \tag{11}$$

is the Chern-Simons topological current. Indeed, $\Gamma$ is obviously diffeomorphism-invariant and does not involve a space-time metric. It is gauge invariant because under the gauge transformation $\delta_{\xi}$ (56) it vanishes:

$$\delta_{\xi} \Gamma = \varepsilon_{\mu \nu \lambda \rho \sigma} Tr (\delta G_{\mu \nu} G_{\lambda \rho, \sigma} + G_{\mu \nu} \delta G_{\lambda \rho, \sigma}) = -ig\varepsilon_{\mu \nu \lambda \rho \sigma} Tr ((G_{\mu \nu} \xi) G_{\lambda \rho, \sigma} + G_{\mu \nu} (\left[ G_{\lambda \rho, \sigma} \xi \right] + [G_{\lambda \rho} \xi_{\sigma}])) = 0. \tag{12}$$

The variation of its integral over the gauge fields $A^{a}_{\mu}$ and $A^{a}_{\mu \lambda}$ gives:

$$\delta_{A} \int_{M_{5}} d^{5}x \ \Gamma = \varepsilon_{\mu \nu \lambda \rho \sigma} \int d^{5}x Tr ((\nabla_{\mu} \delta A_{\nu} - \nabla_{\nu} \delta A_{\mu}) G_{\lambda \rho, \sigma} + G_{\mu \nu} (\nabla_{\lambda} \delta A_{\rho} - \nabla_{\rho} \delta A_{\lambda}) + ig[A_{\mu \sigma} G_{\lambda \rho}] - ig[A_{\lambda \rho} \delta A_{\mu}]]) =$$

$$+ \ 2\varepsilon_{\mu \nu \lambda \rho \sigma} \int d^{5}x Tr ((\nabla_{\mu} G_{\lambda \rho, \sigma} - \nabla_{\sigma} G_{\lambda \rho, \mu}) \delta A_{\nu} + (\nabla_{\lambda} G_{\mu \nu}) \delta A_{\rho})$$

$$+ \ 2\varepsilon_{\mu \nu \lambda \rho \sigma} \int d^{5}x Tr (\nabla_{\mu} (G_{\lambda \rho, \sigma} \delta A_{\nu}) + \nabla_{\lambda} (G_{\mu \nu} \delta A_{\rho})). \tag{13}$$

Recalling the Bianchi identity in YM theory and the generalized Bianchi identities for higher-rank field strength tensor $G_{\nu \lambda, \rho}$ presented in the Appendix, one can see that $\Gamma$ gets contribution only from the boundary terms and vanishes when the fields vary in the bulk of the manifold:

$$\delta_{A} \int_{M_{5}} d^{5}x \ \Gamma = 2\varepsilon_{\mu \nu \lambda \rho \sigma} \int_{M_{5}} d^{5}x \ \partial_{\mu} Tr (G_{\lambda \rho, \sigma} \delta A_{\nu} + G_{\nu \lambda} \delta A_{\rho}) =$$

$$= 2\varepsilon_{\mu \nu \lambda \rho \sigma} \int_{\partial M_{5}} Tr (G_{\lambda \rho, \sigma} \delta A_{\nu} + G_{\nu \lambda} \delta A_{\rho}) d\sigma_{\mu} = 0. \tag{14}$$

Therefore $\Gamma$ is insensitive to the local variation of the fields. It became obvious that $\Gamma$ is a total derivative of some vector current $\Sigma_{\mu}$. Indeed, simple algebraic computation gives

$$\Gamma = \varepsilon_{\mu \nu \lambda \rho \sigma} Tr G_{\mu \nu} G_{\lambda \rho, \sigma} = \partial_{\mu} \Sigma_{\mu}, \tag{12}$$

where

$$\Sigma_{\mu} = 2\varepsilon_{\mu \nu \lambda \rho \sigma} Tr (A_{\nu} \partial_{\lambda} A_{\rho} - \partial_{\lambda} A_{\nu} A_{\rho} - 2igA_{\nu} A_{\lambda} A_{\rho}). \tag{13}$$

$^{5}$The trace of the commutators vanishes: $Tr([A_{\mu}; G_{\lambda \rho, \sigma} \delta A_{\nu}] + [A_{\lambda}; G_{\mu \nu} \delta A_{\rho}]) = 0.$
After some rearrangement and taking into account the definition of the field strength tensors (5) we can get the following form of the vector current:

$$\Sigma_\mu = \varepsilon_{\mu\nu\lambda\rho\sigma} \text{Tr} G_{\nu\lambda} A_{\rho\sigma}. \quad (14)$$

It is instructive to compare the expressions (9), (10) and (11), (14). Both entities $P$ and $\Gamma$ are metric-independent, are insensitive to the local variation of the fields and are derivatives of the corresponding vector currents $C_\mu$ and $\Sigma_\mu$. The difference between them is that the former is defined in four dimensions, while the latter in five. This difference in one unit of the space-time dimension originates from the fact that we have at our disposal high-rank tensor gauge fields to build new invariants. The same is true for the Chern-Simons topological current $C_\mu$ and for the current $\Sigma_\mu$, where the latter is defined in five dimensions. It is also remarkable that the current $\Sigma_\mu$ is linear in YM field strength tensor and in the rank-2 gauge field, picking up only its antisymmetric part.

While the invariant $\Gamma$ and the vector current $\Sigma_\mu$ are defined on a five-dimensional manifold, we may restrict the latter to one lower, four-dimensional manifold. The restriction proceeds as follows. Let us consider the fifth component of the vector current $\Sigma_\mu$:

$$\Sigma_\equiv \Sigma_4 = \varepsilon_{4\nu\lambda\rho\sigma} \text{Tr} G_{\nu\lambda} A_{\rho\sigma}. \quad (15)$$

Considering the fifth component of the vector current $\Sigma_\equiv \Sigma_4$ one can see that the remaining indices will not repeat the external index and the sum is restricted to the sum over indices of four-dimensional space-time. Therefore we can reduce this functional to four dimensions. This is the case when the gauge fields are independent on the fifth coordinate $x_4$. Thus the density $\Sigma$ is well defined in four-dimensional space-time and, as we shall see, it is also gauge invariant up to the total divergence term. Therefore we shall consider its integral over four-dimensional space-time$^6$:

$$\int_{M_4} d^4 x \Sigma = \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} d^4 x \text{Tr} G_{\nu\lambda} A_{\rho\sigma}. \quad (16)$$

This entity is an analog of the Chern-Simons secondary characteristic

$$CS = \varepsilon_{ijk} \int_{M_3} d^3 x \text{Tr} (A_i \partial_j A_k - ig \frac{2}{3} A_i A_j A_k), \quad (17)$$

but, importantly, instead of being defined in three dimensions it is now defined in four dimensions. Thus the non-Abelian tensor gauge fields allow to build a natural generalization of the Chern-Simons characteristic in four-dimensional space-time.

As we claimed this functional is gauge invariant up to the total divergence term. Indeed, its gauge variation under $\delta \xi$ (54), (56) is

$$\delta_\xi \int_{M_4} d^4 x \Sigma = \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} \text{Tr} (-ig[G_{\nu\lambda} \xi] A_{\rho\sigma} + G_{\nu\lambda}(\nabla_\rho \xi_\sigma - ig[A_{\rho\sigma} \xi])) d^4 x =$$

$$= \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} \partial_\rho \text{Tr} (G_{\nu\lambda} \xi_\sigma) d^4 x = \varepsilon_{\nu\lambda\rho\sigma} \int_{\partial M_4} \text{Tr} (G_{\nu\lambda} \xi_\sigma) d\sigma_\rho = 0. \quad (18)$$

Here the first and the third terms cancel each other and the second one, after integration by part and recalling the Bianchi identity (58), leaves only the boundary term, which vanishes when the gauge parameter $\xi_\sigma$ tends to zero at infinity.

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$^6$Below we are using the same Greek letters to numerate now the four-dimensional coordinates. There should be no confusion because the dimension can always be recovered from the dimension of the epsilon tensor.
It is interesting to know whether or not the invariant $\Sigma$ is associated with some new topological characteristic of the gauge fields. If the YM field strength $G_{\nu \lambda}$ vanishes, then the vector potential is equal to the pure gauge connection $A_\mu = U - \partial_\mu U$. Inspecting the expression for the invariant $\Sigma$ one can get convinced that it vanishes on such fields because there is a field strength tensor $G_{\nu \lambda}$ in the integrant. Therefore it does not differentiate topological properties of the gauge function $U$, like its winding number. Both "small" and "large" gauge transformations have zero contribution to this invariant. It may distinguish fields which are falling less faster at infinity and have nonzero field strength tensor $G_{\nu \lambda}$ and the tensor gauge field $A_{\rho \sigma}$.

The dimension of this functional is not difficult to calculate. In four dimensions the gauge fields have dimension of $[mass]^1$, therefore if we intend to add this new density to the Lagrangian we should introduce the mass parameter $m$:

$$m \Sigma = m \varepsilon_{\nu \lambda \rho \sigma} Tr \ G_{\nu \lambda} A_{\rho \sigma}, \ (19)$$

where parameter $m$ has units $[mass]^1$. Adding this term to the Lagrangian of non-Abelian tensor gauge fields keeps intact its gauge invariance and our aim is to analyze the particle spectrum of this gauge field theory. The natural appearance of the mass parameters hints at the fact that the theory turns out to be a massive theory. We shall see that the YM vector boson becomes massive, suggesting an alternative mechanism for mass generation in gauge field theories in four-dimensional space-time.

We have to notice that the Abelian version of the invariant $\Sigma$ was investigated earlier in [34, 36, 37, 38, 39, 40, 41, 42, 43, 44]. Indeed, if one considers instead of a non-Abelian group the Abelian group one can see that the invariant $\Sigma$ reduces to the $\varepsilon_{\nu \lambda \rho \sigma} F_{\nu \lambda} B_{\rho \sigma}$ and when added to the Maxwell Lagrangian provides a mass to the vector field [34, 36, 33, 37, 38, 39, 40]. Attempts at producing a non-Abelian invariant in a similar way have come up with difficulties because they involve non-Abelian generalization of gauge transformations of antisymmetric fields [39, 41, 42]. Let us compare the formulas (2.16) and (2.17) suggested in [39, 40] for the transformation of antisymmetric field with the gauge transformation $\delta_\xi$ (54). For lower-rank fields the latter can be written in the following way [21, 22, 23]:

$$\delta_\xi A_\mu = \partial_\mu \xi - ig [A_\mu, \xi], \quad \delta_\xi A_\mu = 0,$$

$$\delta_\xi A_{\mu \nu} = -ig [A_{\mu \nu}, \xi], \quad \delta_\xi A_{\mu \nu} = \partial_\mu \xi_\nu - ig [A_{\mu}, \xi_\nu].$$

The antisymmetric part of this transformation coincides with the one suggested in [39] if one takes also the auxiliary field $A^i_\mu$ of [39] equal to zero. The crucial point is that the gauge transformations of non-Abelian tensor gauge fields $\delta_\xi$ (54) cannot be limited to a YM vector and antisymmetric field $B^{a}_{\mu \nu}$. Instead, antisymmetric field is augmented by a symmetric rank-2 gauge field, so that together they form a gauge field $A^{a}_{\mu \nu}$ which transforms as it is given above and is a fully propagating field. It is also important that one should include all high-rank gauge fields in order to be able to close the group of gauge transformations and to construct invariant Lagrangian (8).

Let us shortly overview the helicity content of the massless tensor gauge fields before including a massive term into the Lagrangian [21, 22, 23, 31, 32].

4 Helicity Content of Massless Tensor Gauge Fields

Let us first recapitulate the analysis of the particle spectrum before including new massive terms into the Lagrangian [21, 22, 23]. In the Yang-Mills theory

$$\mathcal{L}_{YM} = -\frac{1}{4} G^{a}_{\mu \nu} G^{a}_{\mu \nu},$$
the free equation of motion is
\[ \partial_\mu F^a_{\mu\nu} = 0, \]
or, in terms of vector gauge field,
\[ (\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^a_\nu = 0, \]
and it describes the propagation of massless gauge boson of helicity \( \lambda = \pm 1 \).

The second term in (8) of the Lagrangian \( \mathcal{L} \) defines the kinetic operator and the interactions of the rank-2 gauge field \( A^a_{\mu\lambda} \): \[ \mathcal{L}_2 + \mathcal{L}_2' = - \frac{1}{4} G^a_{\mu\nu,\lambda} G^a_{\mu\nu,\lambda} - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu,\lambda\lambda} + \frac{1}{4} G^a_{\mu\nu,\lambda} G^a_{\mu\nu,\nu\lambda} + \frac{1}{2} G^a_{\mu\nu} G^a_{\mu\nu,\nu\lambda}. \]
Its free equation of motion is \[ \partial_\mu F^a_{\mu\nu,\lambda} - \frac{1}{2} (\partial_\mu F^a_{\mu\lambda,\nu} + \partial_\nu F^a_{\mu\lambda,\mu} + \partial_\lambda F^a_{\mu\nu,\mu} + \eta_{\nu\lambda} \partial_\mu F^a_{\mu\rho,\rho}) = 0, \]
where \( F^a_{\mu\lambda,\nu} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} \), or, in terms of tensor gauge field, it takes the form
\[ \partial^2 (A^a_{\nu\lambda} - \frac{1}{2} A^a_{\lambda\nu}) - \partial_\nu \partial_\lambda (A^a_{\mu\lambda} - \frac{1}{2} A^a_{\lambda\mu}) - \partial_\lambda \partial_\mu (A^a_{\nu\mu} - \frac{1}{2} A^a_{\nu\mu}) + \partial_\nu \partial_\lambda (A^a_{\mu\nu} - \frac{1}{2} A^a_{\mu\nu}) + \frac{1}{4} \eta_{\nu\lambda} (\partial_\mu \partial_\rho A^a_{\mu\rho} - \partial^2 A^a_{\mu\rho}) = 0 \]
and is invariant with respect to the group of gauge transformations
\[ \delta A^a_{\mu\lambda} = \partial_\mu \xi^a_\lambda + \partial_\lambda \zeta^a_\mu, \]
where \( \xi^a_\mu \) and \( \zeta^a_\mu \) are gauge parameters. This free equation describes the propagation of massless modes of \textit{helicity-two and helicity-zero}, \( \lambda = \pm 2, 0 \), \textit{charged gauge bosons} \[21, 22, 23, 31, 32\]. This can be seen by decomposition of the rank-2 gauge field into symmetric \( A^S_{\mu\lambda} \) and antisymmetric parts \( B_{\mu\lambda} \). For the symmetric tensor gauge fields \( A^S_{\mu\lambda} \) the equation reduces to the free Einstein and Fierz-Pauli equation
\[ \partial^2 A^S_{\nu\lambda} - \partial_\nu \partial_\mu A^S_{\mu\lambda} - \partial_\lambda \partial_\mu A^S_{\mu\nu} + \partial_\nu \partial_\lambda A^S_{\mu\mu} + \eta_{\nu\lambda} (\partial_\mu \partial_\rho A^S_{\mu\rho} - \partial^2 A^S_{\mu\rho}) = 0, \]
which describes the propagation of massless gauge boson of helicity two, \( \lambda = \pm 2 \). For the antisymmetric part of the tensor field it reduces to the equation \[33, 34, 36\]
\[ \partial^2 B_{\nu\lambda} - \partial_\nu \partial_\mu B_{\mu\lambda} + \partial_\lambda \partial_\mu B_{\mu\nu} = 0 \]
and describes the propagation of helicity-zero state, \( \lambda = 0 \). Let us now see how the spectrum is changing when we add new invariant \( \Sigma \) to the Lagrangian.

\footnote{It has sixteen components in the four-dimensional space-time.}
5 Particle Spectrum with Topological Mass Term

With the new mass term we have to consider the Lagrangian
\[
\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_2 + \mathcal{L}_2' + \frac{m}{4} \Sigma = \frac{1}{4} \mathcal{F}_{\mu \nu}^a \mathcal{F}^{a\mu \nu} - \\
\frac{1}{4} G_{\mu \nu, \lambda} G^{a\mu \nu, \lambda} + \frac{1}{4} G_{\mu \lambda, \nu} G^{a\mu \lambda, \nu} + \frac{1}{4} G_{\nu \mu, \lambda} G^{a\nu \mu, \lambda} - \frac{1}{4} G_{\mu \nu, \lambda} G^{a\mu \nu, \lambda} + \frac{1}{2} G_{\mu \nu} G_{\lambda \nu\lambda} + \\
+ \frac{m}{2} \varepsilon_{\nu \lambda \rho \sigma} G_{\nu \lambda} A^a_{\rho \sigma}.
\]

(25)

The equations of motion which follow for the YM and rank-2 gauge fields are³:
\[
\nabla_{\mu} G_{\mu \nu}^{ab} + \frac{m}{2} \varepsilon_{\nu \lambda \rho \sigma} G_{\mu \lambda, \rho}^{a} + \\
+ g f^{abc} A_{\mu \lambda}^{b} G_{\mu \nu, \lambda}^{c} - \frac{1}{2} g f^{abc} (A_{\mu \lambda}^{b} G_{\mu \nu, \lambda}^{c} + A_{\lambda \mu}^{b} G_{\nu \mu, \lambda}^{c} + A_{\lambda \nu}^{b} G_{\mu \lambda, \nu}^{c} - A_{\nu \lambda}^{b} G_{\mu \nu, \mu}^{c}) = 0,
\]

(26)

\[
\nabla_{\mu} G_{\mu \nu}^{ab} + \frac{m}{2} \varepsilon_{\nu \lambda \rho \sigma} G_{\mu \lambda, \rho}^{a} - \\
- \frac{1}{2} (\nabla_{\mu} G_{\mu \lambda, \nu}^{b} + \nabla_{\mu} G_{\lambda \nu, \mu}^{b} + \nabla_{\nu} G_{\mu \lambda, \mu}) + \eta_{\nu \lambda} \nabla_{\mu} G_{\mu \nu, \rho}^{b} + \\
+ g f^{abc} A_{\mu \lambda}^{b} G_{\mu \nu, \lambda}^{c} + \frac{1}{2} g f^{abc} (A_{\mu \lambda}^{b} G_{\mu \nu, \lambda}^{c} + A_{\lambda \mu}^{b} G_{\nu \mu, \lambda}^{c} + A_{\lambda \nu}^{b} G_{\mu \lambda, \nu}^{c} - \eta_{\nu \lambda} A_{\mu \nu, \mu}^{b} G_{\mu \nu, \rho}^{c}) = 0.
\]

Let us consider the free equations when the coupling constant g is equal to zero:
\[
\partial_{\mu} F_{\mu \nu}^{a} + \frac{m}{2} \varepsilon_{\nu \lambda \rho} F_{\mu \lambda, \rho}^{a} = 0,
\]

(27)

\[
\partial_{\mu} F_{\mu \nu, \lambda}^{a} - \frac{1}{2} (\partial_{\mu} F_{\mu \lambda, \nu}^{a} + \partial_{\mu} F_{\mu \nu, \lambda}^{a} + \partial_{\nu} F_{\mu \mu, \lambda}^{a} + \eta_{\nu \lambda} \partial_{\mu} F_{\mu \rho, \rho}^{a}) + \frac{m}{2} \varepsilon_{\nu \lambda \rho} F_{\mu \rho}^{a} = 0,
\]

where \(F_{\mu \nu}^{a} = \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a}, \quad F_{\mu \nu, \lambda}^{a} = \partial_{\mu} A_{\nu, \lambda}^{a} - \partial_{\nu} A_{\mu, \lambda}^{a}. \) This is a coupled system of equations which involved the vector YM field and antisymmetric part of the rank-2 gauge field. This form of the equations clearly demonstrates why in non-Abelian tensor gauge field theory it is possible to have equations in four dimensions which include gauge invariant mass term. In the first equation the free index \(\nu\) is attached to the epsilon tensor and its last three indices are contracted to the field strength tensor of rank-2 gauge field \(F_{\mu \lambda, \rho}^{a}. \)

Obviously in vector gauge theory there are no such objects to contract indices in four dimensions. In the second equation the free indices \(\nu, \lambda\) are attached to the epsilon tensor and its last two indices are contracted to the YM field strength tensor \(F_{\mu \rho}^{a}. \)

In order to analyze the particle content of the free equations (27) it is convenient to derive them in terms of dual field strength tensors
\[
F_{\mu \nu}^{a} = \frac{1}{2} \varepsilon_{\mu \nu \lambda \rho} F_{\lambda \rho}^{a}, \quad F_{\mu}^{a} = \frac{1}{2} \varepsilon_{\mu \nu \lambda \rho} F_{\nu \lambda, \rho},
\]

(28)

so that the equations will take the form⁹
\[
\frac{1}{2} \varepsilon_{\nu \mu \lambda \rho} \partial_{\mu} F_{\lambda \rho}^{a} - m F_{\nu}^{a} = 0,
\]

\[
\frac{3}{2} \varepsilon_{\nu \lambda \rho} \partial_{\mu} F_{\rho}^{a} - 2m F_{\nu}^{a} = 0.
\]

(29)

³ At this stage we keep only YM and rank-2 gauge fields in the field equations, the rank-3 gauge field is inessential for our analysis of the mass spectrum of the lower-rank gauge fields. In the next section we shall include higher-rank gauge fields as well.

⁹ Thus the second free equation can be written in terms of the dual field strength tensor \(F_{\mu}^{a}. \) Notice that the full interacting Lagrangian \(\mathcal{L}_2 + \mathcal{L}_2’\) cannot be written in terms of this dual tensor because of the term \(\frac{1}{4} G_{\mu \nu, \rho}^{a} G_{\mu \lambda, \lambda}^{a}. \)
From the first equation it follows that
\[ \partial_\mu F^*_{\lambda \rho} + \partial_\lambda F^*_{\mu \rho} + \partial_\rho F^*_{\mu \lambda} - m \varepsilon_{\mu \lambda \rho} F^*_{\nu} = 0, \]
then taking derivative and using Bianchi identity \( \partial_\mu F^*_{\nu} = 0 \) yields
\[ \partial^2 F^*_{\lambda \rho} - m \varepsilon_{\lambda \rho \mu \nu} \partial_\mu F^*_{\nu} = 0, \]
and using the second equation in (29) we can get
\[ (\partial^2 + \frac{4}{3} m^2) F^*_{\lambda \rho} = 0, \]
which describes a particle of the mass
\[ k^2 = \frac{4}{3} m^2 = M^2. \]

In a similar manner from the second equation it follows that
\[ \partial_\mu F^*_{\nu} - \partial_\nu F^*_{\mu} + \frac{2}{3} m \varepsilon_{\mu \nu \lambda \rho} F^*_{\lambda \rho} = 0, \]
then taking derivative and using Bianchi identity \( \partial_\mu F^*_{\nu} = 0 \) yields
\[ \partial^2 F^*_{\nu} - \frac{2}{3} m \varepsilon_{\nu \mu \lambda \rho} \partial_\mu F^*_{\lambda \rho} = 0, \]
and using the first equation in (29) we can get the same result:
\[ (\partial^2 + \frac{4}{3} m^2) F^*_{\nu} = 0. \]

The above consideration does not tell us how many propagating modes describes the system of equations (34) on the mass-shell (31). In order to understand better the structure and the number of propagating modes one should analyze the corresponding free equations (27) in terms of fields
\[ \partial^2 A^a_\nu - \partial_\nu \partial_\mu A^a_\mu + m \varepsilon_{\nu \mu \lambda \rho} \partial_\mu A^a_\lambda \rho = 0, \]
\[ \partial^2 (A^a_\nu \lambda - \frac{1}{2} A^a_\lambda \nu) - \partial_\nu \partial_\mu (A^a_\mu \lambda - \frac{1}{2} A^a_\lambda \mu) - \partial_\lambda \partial_\mu (A^a_\nu \mu - \frac{1}{2} A^a_\mu \nu) + \partial_\lambda \partial_\mu (A^a_\nu \mu - \frac{1}{2} A^a_\mu \nu) \]
\[ + \partial_\nu \partial_\lambda (A^a_\mu \mu - \frac{1}{2} A^a_\mu \mu) + \frac{1}{2} \eta_{\nu \lambda} (\partial_\mu \partial_\rho A^a_{\nu \rho} - \partial^2 A^a_{\nu \mu} + m \varepsilon_{\nu \lambda \mu \rho} \partial_\mu A^a_\rho = 0. \]

As we already mentioned, only the antisymmetric part \( B_{\mu \lambda} \) of the rank-2 gauge field \( A_{\nu \lambda} \) interacts through the mass term, the symmetric part \( A^S_{\nu \lambda} \) completely decouples from both equations, therefore we arrive at the following system of equations:
\[ \partial^2 A_{\nu} - \partial_\nu \partial_\mu A_{\mu} + m \varepsilon_{\nu \mu \lambda \rho} \partial_\mu B_{\lambda \rho} = 0, \]
\[ \partial^2 B_{\nu \lambda} - \partial_\nu \partial_\mu B_{\mu \lambda} + \partial_\lambda \partial_\mu B_{\nu \mu} + \frac{2 m}{3} \varepsilon_{\nu \lambda \mu \rho} \partial_\mu A_{\rho} = 0. \]

The symmetric part \( A^S_{\nu \lambda} \) fulfils the massless equation (24) and therefore it is not influenced by the given mass term\(^{10}\).

\(^{10}\)As we shall see in the next section the symmetric field can acquire a mass when we include the next invariant mass term \( m_3 \Sigma_3 \).
One can find the structure and the number of propagating modes calculating the rank of the system (34) when it is written in the momentum representation\(^{11}\):

\[
\begin{align*}
(-k^2 \eta_{\nu\mu} + k_\nu k_\mu) e_\mu + i m \, \varepsilon_{\nu\mu\lambda\rho} k_\mu b_{\lambda\rho} &= 0, \\
(-k^2 \eta_{\nu\mu} \eta_{\lambda\rho} + k_\nu k_\mu \eta_{\lambda\rho} - \eta_{\nu\mu} k_\lambda k_\mu) b_{\mu\rho} + i \frac{2m}{3} \varepsilon_{\nu\lambda\rho\mu} k_\mu e_\rho &= 0.
\end{align*}
\] (35)

When \(k^2 \neq M^2\) the system (35) is off mass-shell and we have four pure gauge field solutions:

\[
\begin{align*}
& e_\mu = k_\mu, \quad b_{\nu\lambda} = 0; \\
& e_\mu = 0, \quad b_{\nu\lambda} = k_\nu \xi_\lambda - k_\lambda \xi_\nu.
\end{align*}
\] (36)

When \(k^2 \neq M^2\) the system (35) has seven solutions. These are four pure gauge solutions (36) and additional three solutions representing propagating modes:

\[
\begin{align*}
e_{\mu}^{(1)} &= (0, 1, 0, 0), & \quad b_{\gamma\gamma}^{(1)} &= \frac{1}{i} \frac{M}{\sqrt{k^2 + M^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
e_{\mu}^{(2)} &= (0, 0, 1, 0), & \quad b_{\gamma\gamma}^{(2)} &= -\frac{1}{i} \frac{M}{\sqrt{k^2 + M^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\
e_{\mu}^{(3)} &= (0, 0, 0, \frac{M}{\sqrt{k^2 + M^2}}), & \quad b_{\gamma\gamma}^{(3)} &= \frac{1}{i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\] (37)

These propagating modes cannot be factorized into separately vector or separately tensor solutions as it happens for the pure gauge solutions (36). It is a genuine superposition of vector and tensor fields. Let us consider the limit \(M \to 0\). The above solutions will factorize, into two massless vector modes \(e_{\mu}^{(1)}\), \(e_{\mu}^{(2)}\), of helicities \(\lambda = \pm 1\) and helicity \(\lambda = 0\) mode \(b_{\gamma\gamma}^{(3)}\) of antisymmetric tensor - massless modes described in the previous section. But when \(M \neq 0\), in the rest frame \(k^2 = 0\), these solutions represent three spin-1 polarizations.

The above analysis suggests the following physical interpretation. A massive spin-1 particle appears here as a vector field of helicities \(\lambda = \pm 1\) which acquires an extra polarization state absorbing antisymmetric field of helicity \(\lambda = 0\), or as antisymmetric field of helicity \(\lambda = 0\) which absorbs helicities \(\lambda = \pm 1\) of the vector field. It is sort of "dual" description of massive spin-1 particle.

In order to fully justify this phenomenon of superposition of polarizations one should develop quantum-mechanical description of tensor fields [40, 41]. It is a challenging problem because in this field theoretical model we have infinite number of fields. Not much is known about how to deal with such systems. One can develop "naive" Feynman rules for transition amplitudes, but there is a need for deeper understanding of the corresponding path integral which is not only over fields at infinitely many space-time points, but also over infinitely many fields. It is impossible to make truncation to finite number of fields without breaking their gauge symmetries.

\(^{11}\)We are using the method developed in [31, 32].
6 High-Rank Mass Terms

Let us consider now the next invariant in five-dimensional space-time \((4+1)\) which can be constructed using totally antisymmetric Levi-Civita epsilon tensor in combination with the generalized field strength tensors. It has the following form:

\[
\Gamma_3 = \varepsilon_{\mu\nu\lambda\rho\sigma} Tr\{G_{\mu\nu}G_{\lambda\rho,\sigma\alpha} + 2G_{\mu\nu,\alpha}G_{\lambda\rho,\sigma\alpha} + G_{\mu\nu,\sigma}G_{\lambda\rho,\alpha\alpha}\}. \tag{38}
\]

As one can easily check this entity is also gauge invariant, because under the gauge transformation (56) its variation vanishes:

\[
\delta_{\xi}\Gamma_3 = 0.
\]

It is not a metrically independent density, because not all Lorentz indices are contracted by the epsilon tensor, part of them are contracted by the space-time metric. In this respect it differs from the density \(\Gamma\), but it keeps other important properties of density \(\Gamma\) which we shall explore here. Indeed, the density \(\Gamma_3\) can be represented as a derivative of the vector current:

\[
\Gamma_3 = \partial_\mu \Xi_\mu, \quad \Xi_\mu = \varepsilon_{\mu\nu\lambda\rho\sigma} Tr\{G_{\nu\lambda}A_{\rho\sigma\alpha} + 2G_{\nu\lambda,\alpha}A_{\rho\sigma\alpha} + G_{\nu\lambda,\alpha\alpha}A_{\rho\sigma}\}. \tag{39}
\]

Considering the fifth component of the vector current \(\Xi_\mu\)

\[
\Xi \equiv \Xi_4 = \varepsilon_{4\nu\lambda\rho\sigma} Tr\{G_{\nu\lambda}A_{\rho\sigma\alpha} + 2G_{\nu\lambda,\alpha}A_{\rho\sigma\alpha} + G_{\nu\lambda,\alpha\alpha}A_{\rho\sigma}\}, \tag{40}
\]

we shall reduce it to four dimensions\(^{12}\). This is the case when the gauge fields are independent on the fifth coordinate \(x_4\). The density \(\Xi\) is well defined in four-dimensional space-time and it is gauge invariant up to a total divergence term. Indeed, its integral over the four-dimensional space-time\(^{13}\)

\[
\Xi = \varepsilon_{\nu\lambda\rho\sigma} \int d^4x Tr\{G_{\nu\lambda}A_{\rho\sigma\alpha} + 2G_{\nu\lambda,\alpha}A_{\rho\sigma\alpha} + G_{\nu\lambda,\alpha\alpha}A_{\rho\sigma}\} \tag{41}
\]

changes under the gauge variation (54), (56) as follows

\[
\delta_{\xi}\Xi = \varepsilon_{\nu\lambda\rho\sigma} \int Tr(-ig[G_{\nu\lambda} \xi]A_{\rho\sigma\alpha} + G_{\nu\lambda}(\nabla_\rho \xi_{\sigma\alpha} - ig[A_{\rho\sigma\alpha} \xi_{\alpha\sigma}] - 2ig[A_{\rho\sigma\alpha} \xi_{\alpha\sigma}\xi_{\sigma\alpha}])
\]

\[
- 2ig[A_{\rho\sigma\alpha} \xi_{\alpha\sigma}] - ig[A_{\rho\sigma\alpha} \xi_{\sigma\alpha}] + 2(-ig[G_{\nu\lambda,\alpha} \xi] - ig[G_{\nu\lambda} \xi_{\alpha}\xi_{\alpha}])A_{\rho\sigma}
\]

\[
+ 2G_{\nu\lambda,\alpha}(\nabla_\rho \xi_{\sigma\alpha} - ig[A_{\rho\sigma\alpha} \xi_{\alpha\sigma}] - ig[A_{\rho\sigma\alpha} \xi_{\alpha\sigma}])A_{\rho\sigma}
\]

\[
+ (-ig[G_{\nu\lambda,\alpha} \xi] - 2ig[G_{\nu\lambda,\alpha} \xi_{\alpha}] - ig[G_{\nu\lambda} \xi_{\alpha\alpha}])A_{\rho\sigma}
\]

\[
+ G_{\nu\lambda,\alpha\alpha}(\nabla_\rho \xi_{\sigma} - ig[A_{\rho\sigma} \xi_{\sigma}])d^4x = \varepsilon_{\nu\lambda\rho\sigma} \int \partial_\mu(G_{\nu\lambda}\xi_{\sigma\alpha} + 2G_{\nu\lambda,\alpha}\xi_{\sigma\alpha} + G_{\nu\lambda,\alpha\alpha}\xi_{\sigma}) d^4x = 0,
\]

and vanishes because terms in front of \(\xi, \xi_{\alpha}\) and \(\xi_{\alpha\alpha}\) cancel each other, the others after integration by part and recalling the Bianchi identities (58), (59) reduce to the boundary terms which vanish when the gauge parameters \(\xi_{\sigma\alpha}, \xi_{\alpha\sigma}\) and \(\xi_{\sigma}\) tend to zero at infinity.

\(^{12}\)Here the index \(\alpha\) can repeat the external index \(\mu\). Therefore we should separately consider the term \(\varepsilon_{\nu\lambda\rho\sigma} Tr\{G_{\nu\lambda}A_{\rho\sigma44} + 2G_{\nu\lambda,4}A_{\rho\sigma4} + G_{\nu\lambda,44}A_{\rho\sigma}\}\) as an additional expression in \(\Xi_4\). We have additional tensor fields in four-dimensions: \(A_{\mu4} = A_{\mu}, A_{\mu44} = \tilde{A}_{\mu}, A_{\mu44} = \tilde{A}_{\mu} A_{\mu4} = \tilde{A}_{\mu}, A_{\mu44} = \tilde{A}_{\mu}\), and for them the above invariant reduces to the form \(\varepsilon_{\nu\lambda\rho\sigma} Tr\tilde{G}_{\nu\lambda}A_{\rho\sigma}\), which we already studied in the previous sections. In the following consideration we shall take all these additional fields equal to zero.

\(^{13}\)Below we are using the same Greek letters to numerate now four-dimensional coordinates.
The dimension of this functional is not difficult to calculate, in four dimensions the gauge fields have dimension of \([\text{mass}]^4\), therefore if we intend to add this new term to the action we should introduce the next mass parameter \(m_3\):

\[
\frac{m_3}{2} \equiv \frac{m_3}{2} \varepsilon_{\nu\lambda\rho\sigma} \int d^4 x \{ G_{\nu\lambda} A_{\rho\sigma} + 2 G_{\nu\lambda,\alpha} A_{\rho\sigma\alpha} + G_{\nu\lambda,\alpha\beta} A_{\rho\sigma\beta} \} d^4 x, \tag{42}
\]

where \(m_3\) has dimension of \([\text{mass}]^4\). To study the influence of this term on the particle spectrum of the theory we have to consider quadratic on gauge fields terms of the Lagrangian. The free equations of motion will take the following form\(^{14}\):

\[
\begin{align*}
\partial_\mu F_{\mu\nu}^a + \frac{1}{2} \partial_\mu (F_{\mu\nu,\lambda\lambda} + F_{\nu\lambda,\mu\lambda} + F_{\lambda\mu,\nu\lambda}) + \frac{m_3}{2} \varepsilon_{\nu\mu\lambda\rho} F_{\mu\lambda,\rho} &= 0, \\
\partial_\mu F_{\mu\nu,\lambda\lambda} - \frac{1}{2} (\partial_\mu F_{\mu\nu,\lambda\nu} + \partial_\mu F_{\nu\lambda,\mu\mu} + \partial_\lambda F_{\mu\nu,\mu} + \eta_{\nu\lambda} \partial_\mu F_{\mu\nu,\rho\rho}) + m_3 \varepsilon_{\nu\mu\rho\lambda} F_{\mu\rho,\sigma\sigma} + m_3 \varepsilon_{\nu\mu\rho\lambda} F_{\mu\rho,\sigma\sigma} &= 0, \\
\partial_\mu F_{\mu\nu,\lambda\rho} - \frac{1}{3} \partial_\mu F_{\nu\lambda,\mu\lambda} - \frac{1}{3} \partial_\mu F_{\mu\nu,\lambda\rho} + \frac{1}{3} \partial_\mu F_{\nu\lambda,\mu\lambda} + \frac{1}{3} \partial_\mu F_{\nu\lambda,\mu\lambda} - \\
- \eta_{\nu\rho} (\partial_\mu F_{\mu\nu,\rho\sigma\sigma} + \frac{1}{6} \partial_\mu F_{\mu\nu,\rho\sigma\sigma}) - \eta_{\nu\rho} (\frac{1}{3} \partial_\mu F_{\mu\nu,\rho\sigma\sigma} + \frac{1}{6} \partial_\mu F_{\nu\rho,\sigma\sigma}) + \\
+ \eta_{\nu\rho} (\frac{1}{3} \partial_\mu F_{\nu\rho,\sigma\sigma} - \frac{1}{3} \partial_\mu F_{\nu\rho,\sigma\sigma} + \frac{1}{3} \partial_\mu F_{\nu\rho,\sigma\sigma}) + \\
+ \eta_{\nu\rho} \partial_\mu F_{\nu\rho,\sigma\sigma} - \frac{1}{2} (\eta_{\nu\rho} \partial_\mu F_{\nu\rho} + \eta_{\nu\rho} \partial_\mu F_{\nu\rho} + \frac{1}{2} (\partial_\mu F_{\nu\rho} + \partial_\rho F_{\nu\rho}) + \\
+ m_3 (\varepsilon_{\nu\mu\rho\sigma} F_{\mu\rho,\sigma\sigma} + \varepsilon_{\nu\mu\rho\sigma} F_{\mu\rho,\sigma\sigma} + \varepsilon_{\nu\mu\gamma\sigma} \eta_{\rho\sigma} F_{\mu\gamma,\sigma}) &= 0.
\end{align*}
\]

From the second equation for the rank-2 gauge field it follows that now its symmetric part \(A_{\nu\lambda}^S\) interacts through the second mass term with the antisymmetric part of the rank-3 gauge field

\[
\frac{m_3}{2} (\varepsilon_{\nu\mu\rho\sigma} F_{\mu\rho,\sigma\sigma} + \varepsilon_{\nu\mu\rho\sigma} F_{\mu\rho,\sigma\sigma}) = m_3 (\varepsilon_{\nu\mu\rho\sigma} \partial_\mu A_{\rho\sigma\lambda}^a + \varepsilon_{\nu\mu\rho\sigma} \partial_\mu A_{\rho\sigma\nu}^a)
\]

and from the third equation - that the rank-3 gauge field interacts with the rank-2 gauge field, so that together they form a coupled system of equations similar to the one considered in the previous section and can produce massive particle of spin-2. In general it is a complicated system of coupled linear equations and full understanding of its solutions requires detailed analysis which we shall provide elsewhere.

At the end of this section we shall present the general form of the invariant which can be constructed in terms of higher-rank field strength tensors and epsilon tensor in four dimensions

\[
\Sigma_{2s+1} = \varepsilon_{\mu\nu\rho\sigma} \int d^4 x \{ G_{\mu\nu} A_{\rho\sigma\lambda_1...\lambda_{s+1}}^a + ... + G_{\mu\nu,\lambda_1...\lambda_{s+1}} A_{\rho\sigma} \}. \tag{44}
\]

As we already suggested, it can be added to the massless Lagrangian (8) with different mass parameters \(m_{2s+1}\) as in (4). The consequences of this extension on the particle spectrum is not so easy to analyze and some general method should be developed to resolve the particle spectrum at subsequent levels.

\(^{14}\)We keep only the YM, rank-2 and rank-3 gauge fields in the free field equations. The rank-4 and high-rank gauge fields should be considered at subsequent levels.
7 Topological Density in Six Dimensions

In the previous sections we considered the densities in five and four dimensions. It is also possible to construct invariants in higher dimensions. First let us consider metric-independent density in six dimensions:

\[ \Delta = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} Tr G_{\mu\nu,\lambda} G_{\rho\sigma,\kappa}, \]  

(45)

which is a gauge invariant entity, because under transformation \( \delta \xi \) (56) it vanishes:

\[ \delta_{\xi} \Delta = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} Tr (\delta G_{\mu\nu,\lambda} G_{\rho\sigma,\kappa} + G_{\mu\nu,\lambda} \delta G_{\rho\sigma,\kappa}) \]

\[ = -ig\varepsilon_{\mu\nu\lambda\rho\sigma\kappa} Tr ([G_{\mu\nu,\lambda} \xi] + [G_{\mu\nu} \xi_{\lambda}] G_{\rho\sigma,\kappa}) + G_{\mu\nu,\lambda} ( [G_{\rho\sigma,\kappa} \xi] + [G_{\rho\sigma} \xi_{\kappa}] ) = 0. \]

The \( \Delta \) is obviously diffeomorphism-invariant and does not involve a space-time metric. It is also true that \( \Delta \) is a total derivative of a vector current \( \Pi_{\mu} \). Indeed, simple algebraic computation gives

\[ \Delta = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} Tr G_{\mu\nu,\lambda} G_{\rho\sigma,\kappa} = 2 \partial_{\mu} \Pi_{\mu}, \]  

(46)

where

\[ \Pi_{\mu} = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} Tr G_{\nu\lambda,\rho} A_{\sigma\kappa}. \]  

(47)

While the invariant \( \Delta \) and the vector current \( \Pi_{\mu} \) are defined on a six-dimensional manifold, we may restrict the latter to one lower, five-dimensional manifold. Considering the sixth component of the vector current \( \Pi_{\mu} \)

\[ \Pi \equiv \Pi_{5} = \varepsilon_{5\nu\lambda\rho\sigma\kappa} Tr G_{\nu\lambda,\rho} A_{\sigma\kappa}. \]  

(48)

one can see that the remaining indices will not repeat the external index and the sum is restricted to the sum over indices of five-dimensional space-time. Therefore we can reduce this functional to five dimensions. This is the case when the gauge fields are independent on the sixth coordinate \( x_{5} \). Thus the density \( \Pi \) is well defined in five-dimensional space-time and, as we shall see, it is also gauge invariant up to the total divergence term. Therefore we shall consider its integral over five-dimensional space-time:

\[ \int_{M_{5}} d^{5}x \Pi = \varepsilon_{\nu\lambda\rho\sigma\kappa} \int_{M_{5}} d^{5}x Tr G_{\nu\lambda,\rho} A_{\sigma\kappa}. \]  

(49)

This functional is gauge invariant up to the total divergence term. Its gauge variation under \( \delta_{\xi} \) (54), (56) is

\[ \delta_{\xi} \int_{M_{5}} d^{5}x \Pi = \varepsilon_{\nu\lambda\rho\sigma\kappa} \int_{M_{5}} \partial_{\sigma} Tr (G_{\nu\lambda,\rho} \xi_{\kappa}) d^{5}x = \varepsilon_{\nu\lambda\rho\sigma\kappa} \int_{\partial M_{5}} Tr (G_{\nu\lambda,\rho} \xi_{\kappa}) d\sigma = 0, \]

where the boundary term vanishes when the gauge parameter \( \xi_{\kappa} \) tends to zero at infinity.

One can construct higher-rank extension of the above densities. The first in this infinite series is

\[ \Delta_{3} = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} G_{\mu\nu,\lambda\alpha} G_{\rho\sigma,\kappa\alpha}, \]  

(50)

the second has two varieties

\[ \Delta_{5} = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} G_{\mu\nu,\lambda\alpha\beta} G_{\rho\sigma,\kappa\alpha\beta}, \]

\[ \Delta_{5}^{\prime} = \varepsilon_{\mu\nu\lambda\rho\sigma\kappa} G_{\mu\nu,\lambda\alpha\alpha} G_{\rho\sigma,\kappa\beta\beta}. \]  

(51)

and so on. These invariants may be important for the physics of D-branes in corresponding dimensions.
Note Added

In the higher-spin literature and, in particular, in the work of Metsaev and others, it was shown that the consistency of the interaction vertices with the Poincaré symmetry requires that the cubic interaction vertices should contain a number of derivatives greater or equal to \( s_1 + s_2 + s_3 - 2s_{\text{min}} \), where \( s_a, a = 1, 2, 3 \) are the spins of the interacting particles. This result seems to be in contrast with the form of the interaction vertices in the generalized Yang-Mills theory [21, 22, 23], in which all interaction vertices between high-spin fields have dimensionless coupling constants in four-dimensional space-time. That is, the cubic interaction vertices have only first order derivatives, there is no self-interaction cubic vertices and that the quartic vertices have no derivatives.

Let us see first why the generalized Yang-Mills theory predicts that the cubic interaction vertices have only first order derivatives and that it avoids self-interaction cubic vertices. The general structure of the vertices is defined by the gauge and Lorentz invariant Lagrangian (4), which is quadratic in the field strength tensors (6) and (7). The field strength tensors themselves are quadratic in high-spin fields (5), therefore the Lagrangian (4) contains only quadratic, cubic and quartic vertices. The cubic vertices appear in the product of the derivative terms and quadratic terms of the field strength tensors (5) and have the following general structure: \( g \partial A_{s_1} A_{s_2} A_{s_3} \), while there is no self-interaction vertices with \( s_1 = s_2 = s_3 \). The quartic vertices appear in the product of quadratic terms \( g^2 A_{s_1} A_{s_2} A_{s_3} A_{s_4} \). These structure of the vertices is a consequence of the gauge and Lorentz invariance of the Lagrangian (4), which has been proven explicitly by using formulas (54) and (56) [21, 22, 23].

Next, let us see that:

A) from the work of the Göteborg group it also follows that there exists a large class of Poincaré invariant cubic vertices for high spin fields which have only first order derivatives, in agreement with the cubic vertices of the generalized Yang-Mills theory,

B) in the work of Metsaev there is a place where the author has made assumption which leads him to the conclusion that the cubic vertices should have higher derivatives, in contrast with the results of Göteborg group,

C) from spinor representation of the cubic vertices for high spin fields it also follows that there are dimensionless Poincaré invariant cubic vertices [54, 55].

D) dimensionless cubic vertices for high spin fields appear in open string theory with Chan-Paton charges when one compute tree-level scattering amplitudes [56, 57, 58, 59].

A) Let us review the results of the Göteborg group which are published in [45, 46, 47]. In the light-front formulation of relativistic dynamics used in [45, 46, 47] the massless particles of spin-\( s \) are described by a complex function \( \phi_s \) which encodes two physical helicities \( h = \pm s \) of the massless particles. In this approach there are no auxiliary fields and questions associated with the gauge invariance, because it permits to work with the physical fields \( \phi_s \) exclusively. What one should be concerned of is the relativistic invariance of the scattering amplitudes. The Poincaré group is realized here non-linearly and one should derive the self-interaction cubic vertices as a non-linear realization of the Poincaré group [45, 46]. The authors came to the conclusion that there are \( s \) derivatives in the cubic self-interaction vertices and the coupling constant has dimension of \([\text{mass}]^{1-s}\). This result of Lars Brink and his collaborators raised expectations that a consistent interacting theory might exist in flat space-time, because this approach demonstrated the existence of physically non-trivial interaction of high spin particles.

The next important step has been made in [47] where the authors derived the cubic
vertices for all massless bosonic representations of the Poincaré group which includes interactions between different spins \( s_1, s_2, s_3 \). For the cubic vertices in four dimensions they found the following expressions (see formulas (A1.4-6) in [47]):

\[
M_3 = \int D\beta_1^2 \beta_2^2 \beta_3^2 \bar{P}^{(s_2+s_3-s_1)} \phi_{s_1}(1) \phi_{s_2}(2) \phi_{s_3}(3) + CC, \quad \text{if} \quad s_2 + s_3 > s_1, \quad (A1.4)
\]

\[
M_3 = \int D\beta_1^2 \beta_2^2 \beta_3^2 \bar{P}^{(s_1-s_2-s_3)} \phi_{s_1}(1) \phi_{s_2}(2) \phi_{s_3}(3) + CC, \quad \text{if} \quad s_1 > s_2 + s_3, \quad (A1.5)
\]

\[
M_3 = \int D\beta_1^2 \beta_2^2 \beta_3^2 \bar{P}^{(s_1+s_2+s_3)} \phi_{s_1}(1) \phi_{s_2}(2) \phi_{s_3}(3) + CC, \quad (A1.6)
\]

where \( \beta_a = 2p_a^+ \) and \( a = 1, 2, 3 \) numerates the interacting particles, \( D \) denotes the momentum integration and momentum delta functions. The transverse momentum is \( P = \frac{1}{3} \sum_a \beta_a p_a, \beta_a = \beta_{a+1} - \beta_{a+2} \). Here transverse momenta \( p_a \), fields \( \phi_s \) and all vectors \( A \) are defined as complex variables \( A = A_1 + iA_2, \bar{A} = A_1 - iA_2 \). If one takes the spins of the scattered particles such that

\[
s_2 + s_3 - s_1 = 1, \quad \text{or} \quad s_1 - s_2 - s_3 = 1, \quad \text{or} \quad s_1 + s_2 + s_3 = 1, \quad (52)
\]

then one can get the cubic vertices (A1.4-6) which are linear in momentum \( P \), in agreement with the cubic vertices of the generalized Yang-Mills theory [21, 22, 23].

B) It seems to me that the following assumption in the work of Metsaev [48, 49], who follows the light-front formulation of the Göteborg group, is not necessary and therefore leads him to a different conclusion. His general formula for the cubic vertex (5.9) in [48] is

\[
M_3 \sim Z^{(s_1+s_2+s_3-k)/2} \prod B_a^{s_1+(k-s_1-s_2-s_3)/2}, \quad (5.9)
\]

where \( Z \) and \( B_a, a = 1, 2, 4 \) are linear in momenta functions and the vertex has therefore \( k \) powers of the transverse momenta. It is assumed that: "The powers of the forms \( B_a \) and \( Z \) in (5.9) must be nonnegative integers." This leads the author to the conclusion that the number of derivatives in the vertex should be greater or equal to \( s_1 + s_2 + s_3 - 2s_{min} \). But as it was demonstrated in [47] the ratio of two polynomials of momenta can be reducible in four dimensions, therefore one should allow negative integer powers as well. This leads the authors of [47] to the vertices (A1.4-6).

C) As it was demonstrated in [47, 50, 51, 52, 53], the "morphology" of the available invariant vertices is much richer when the interaction between different spins is allowed (52). The main difficulty here is to derive or to guess the genuine form of the full Lagrangian which is behind the perturbative constructions, that is, to extend the results to the second and higher orders in the deformation parameter. There is a need here to understand the structure of high spin interactions beyond the perturbation theory. The spinor representation of the scattering amplitudes may offer such a solution. One can get the dimensionless cubic vertices using the results of the Benincasa and Cachazo [54], they are [55]:

\[
M_3 = f < 1, 2 >^{-2h_1-2h_2-1} < 2, 3 >^{2h_1+1} < 3, 1 >^{2h_2+1}, \quad h_3 = -1 - h_1 - h_2,
\]

\[
M_3 = k [1, 2]^{2h_1+2h_2-1} [2, 3]^{-2h_1+1} [3, 1]^{-2h_2+1}, \quad h_3 = 1 - h_1 - h_2. \quad (53)
\]

and are identical to the conditions (52). The formulas (53) give a general expression for the cubic vertices in terms of two independent helicities \( h_1 \) and \( h_2 \). It allows to choose any \( h_1 \) and \( h_2 \) and then to find out \( h_3 \) for which the three-particle interaction vertex in four-dimensional space-time will have dimensionless coupling constants \( f \) and \( h \). The details can be found in [54, 55].
9 Appendix

The extended non-Abelian gauge transformation $\delta_\xi$ of the tensor gauge fields is defined by the equations [21, 22, 23]:

$$\delta_\xi A_\mu = \partial_\mu \xi - ig [A_\mu, \xi]$$
$$\delta_\xi A_{\mu\nu} = \partial_{\mu\nu} \xi - ig [A_{\mu\nu}, \xi]$$
$$\delta_\xi A_{\mu\nu\lambda} = \partial_{\mu\nu\lambda} \xi - ig [A_{\mu\nu\lambda}, \xi] - ig [A_{\mu\nu}, \xi_\lambda] - ig [A_{\mu\lambda}, \xi_\nu] - ig [A_{\nu\lambda}, \xi_\mu], \quad (54)$$

where $\xi_{\lambda_1...\lambda_s}(x)$ are totally symmetric gauge parameters, and comprises a closed algebraic structure. The tensor gauge fields are in the matrix representation $A^a_{\mu\lambda_1...\lambda_s} = (L_c)^{ab} A^c_{\mu\lambda_1...\lambda_s} = if^{abc} A^c_{\mu\lambda_1...\lambda_s}$ and $f^{abc}$ are the structure constants. The generalized field strength tensors are defined as follows [21, 22, 23]:

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu],$$
$$G_{\mu\nu,\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} - ig ([A_\mu A_{\nu\lambda}] + [A_{\mu\lambda} A_\nu]),$$
$$G_{\mu\nu,\lambda\rho} = \partial_\mu A_{\nu\lambda\rho} - \partial_\nu A_{\mu\lambda\rho} - ig ([A_\mu A_{\nu\lambda\rho}] + [A_{\mu\lambda\rho} A_\nu] + [A_{\mu\rho} A_{\nu\lambda}] + [A_{\mu\lambda} A_{\nu\rho}]),$$

and transform homogeneously with respect to the extended gauge transformations $\delta_\xi$:

$$\delta G^a_{\mu\nu} = -ig [G^a_{\mu\nu}, \xi],$$
$$\delta G^a_{\mu\nu,\lambda} = -ig ([G^a_{\mu\nu,\lambda}, \xi] + [G^a_{\mu\nu}, \xi_\lambda]),$$
$$\delta G^a_{\mu\nu,\lambda\rho} = -ig ([G^a_{\mu\nu,\lambda\rho}, \xi] + [G^a_{\mu\nu,\lambda}, \xi_\rho] + [G^a_{\mu\nu,\lambda}, \xi_\rho] + [G^a_{\mu\lambda}, \xi_\nu]),$$

The field strength tensors fulfil the Bianchi identities [31]. In the YM theory the Bianchi identity is

$$[\nabla_\mu, G_{\nu\lambda}] + [\nabla_\nu, G_{\lambda\mu}] + [\nabla_\lambda, G_{\mu\nu}] = 0, \quad (56)$$

and for the higher-rank field strength tensors $G_{\nu\lambda,\rho}$ and $G_{\nu\lambda,\rho\sigma}$ they are:

$$[\nabla_\mu, G_{\nu\lambda,\rho}] - ig [A_{\mu\rho}, G_{\nu\lambda}] + [\nabla_\nu, G_{\lambda\mu,\rho}] - ig [A_{\nu\rho, G_{\lambda\mu}}] + [\nabla_\lambda, G_{\mu\nu,\rho}] - ig [A_{\lambda\rho}, G_{\mu\nu}] = 0, \quad (57)$$
$$[\nabla_\mu, G_{\nu\lambda,\rho\sigma}] - ig [A_{\mu\rho, G_{\nu\lambda,\sigma}}] - ig [A_{\mu\sigma, G_{\nu\lambda,\rho}}] - ig [A_{\mu\rho, G_{\nu\lambda,\sigma}}] + cyc.perm.(\mu\nu\lambda) = 0 \quad (58)$$

and so on.

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