AN EXPLICIT GROSS-ZAGIER FORMULA RELATED TO THE
SYLVESTER CONJECTURE

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1. Introduction

A nonzero rational number is called a cube sum if it is of the form $a^3 + b^3$ with $a, b \in \mathbb{Q}^\times$. For any $n \in \mathbb{Q}^\times$, let $E_n$ be the elliptic curve over $\mathbb{Q}$ defined by $x^3 + y^3 = n$. If $n$ is a cube, then the Mordell-Weil group $E_n(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$ and $n$ is not a cube sum. If $n$ is twice a cube, then $E_n(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and $n$ is a cube sum. Otherwise, the torsion group $E_n(\mathbb{Q})_{tor}$ is trivial. Therefore, if $n$ is not twice a cube, then $n$ is a cube sum if and only if $\text{rank}_{\mathbb{Z}} E_n(\mathbb{Q}) > 0$. A famous conjecture concerning the cube sums, attributed to Sylvester, is the following

**Conjecture 1.1** (Sylvester [Syl79], Selmer [Sel51]). *Any prime number $p \equiv 4, 7, 8 \text{ mod } 9$ is a cube sum.*

For a good summary of this conjecture, please refer to [DV09, DV17]. For odd prime $p$, the explicit 3-descent [Sat86, DV09] gives that

$$\text{rank}_{\mathbb{Z}} E_p(\mathbb{Q}) \leq \begin{cases} 
0, & p \equiv 2, 5 \text{ mod } 9; \\
1, & p \equiv 4, 7, 8 \text{ mod } 9; \\
2, & p \equiv 1 \text{ mod } 9.
\end{cases}$$

And

$$\epsilon(E_p) = \begin{cases} 
-1, & p \equiv 4, 7, 8 \text{ mod } 9; \\
+1, & \text{otherwise},
\end{cases}$$

where $\epsilon(E_p)$ is the sign in the functional equation of the Hass-Weil L-function $L(s, E_p)$. Then the Birch and Swinnerton-Dyer (BSD) conjecture implies the Sylvester conjecture. In 1994,
Elkies announced a proof of Conjecture 1.1 for all primes \( p \equiv 4, 7 \mod 9 \), unfortunately, without any detailed publication. However, Dasgupta and Voight [DV17] proved the following weaker theorem using method substantially different from that of Elkies.

**Theorem 1.2.** Let \( p \equiv 4, 7 \mod 9 \) be a rational prime number such that \( 3 \mod p \) is not a cubic residue. Then \( p \) and \( p^2 \) are cube sums.

Dasgupta and Voight proved the above theorem by establish the nontriviality of certain related Heegner points. By the work of Gross-Zagier [GZ86] and Kolyvagin [Ko90], the rank part of the BSD conjecture for \( E_p \) is true. If \( \ell \nmid 6p \) is a prime, then \( E_p \) has good reduction at \( \ell \). Then Perrin-Riou [PR87] and Kobayashi [Kob13] proved that the \( \ell \)-part full BSD conjecture holds for \( E_p \). Since \( E_p \) has potential good ordinary reduction at \( p \), the \( p \)-part full BSD conjecture of \( E_p \) is also true by the work of Li-Liu-Tian [LLT]. To summerize, the following theorem is known.

**Theorem 1.3.** Let \( p \equiv 4, 7 \mod 9 \) be a rational prime number such that \( 3 \mod p \) is not a cubic residue. Then

1. \( \text{ord}_{s=1} L(s, E_p) = \text{rank}_Z E_p(\mathbb{Q}) = 1 \);
2. The Tate-Shafarevich groups \( \text{Sha}(E_p) \) is finite, and for any prime \( \ell \nmid 6 \), the \( \ell \)-part of \( \text{Sha}(E_p) \) is as predicted by the Birch and Swinnerton-Dyer conjecture for \( E_p \).

For \( \ell = 2, 3 \), there is no general theory for the \( \ell \)-part full BSD conjecture of \( E_p \) which we can follow. In this paper, we adopt a similar method as in [CST17] to prove the following result by the comparison of the full BSD conjecture and an explicit Gross-Zagier formula.

**Theorem 1.4.** Let \( p \equiv 4, 7 \mod 9 \) be a rational prime number such that \( 3 \mod p \) is not a cubic residue. Then the 3-part of \( \text{Sha}(E_p) \cdot \text{Sha}(E_{3p^2}) \) is as predicted by the Birch and Swinnerton-Dyer conjecture for \( E_p \) and \( E_{3p^2} \).

We explain the proof of Theorem 1.4 and the outline of the paper as follows. Let \( K = \mathbb{Q}(\sqrt{-3}) \) be an imaginary quadratic field with \( \mathcal{O}_K = \mathbb{Z}[\omega] \) its ring of integers, \( \omega = \frac{-1 + \sqrt{-3}}{2} \). For any \( n \in \mathbb{Q}^\times \), the elliptic curve \( E_n \) has Weierstrass equation \( y^2 = x^3 - 2^4 \cdot 3^3 \cdot n^2 \), and has complex multiplication by \( \mathcal{O}_K \) over \( K \). We fix the complex multiplication \( [\ ] : \mathcal{O}_K \simeq \text{End}_{\mathbb{Q}}(E_n) \) by \( [\omega](x, y) = (\omega x, y) \). In Section 2, we realize the elliptic curve \( E_9 \) as a modular curve and study explicitly its modular automorphisms. By Shimura’s reciprocity law, the Galois actions of CM points on \( E_9 \) can be obtained by the modular actions.

Let \( p \equiv 4, 7 \mod 9 \) be a prime. Let \( \chi : G_K \to \mathcal{O}_K^\times \) be the character given by \( \chi(\sigma) = (\sqrt[3]{3p})^\sigma - 1 \). The base change L-function \( L(s, E_9, \chi) \) has sign \(-1\) and has a decomposition

\[
L(s, E_9, \chi) = L(s, E_p) \cdot L(s, E_{3p^2}).
\]

By the Gross-Zagier formula, the central value of the derivative of \( L(s, E_9, \chi) \) is related to certain Heegner points on \( E_9 \). By carefully exploring the coordinates of these Heegner points, Dasgupta and Voight [DV17] proved the nontriviality of these Heegner points under the assumption that \( 3 \mod p \) is not a cubic residue, which will be carried out in Section 3.1 for our convenience, and hence \( L(s, E_9, \chi) \) has vanishing order 1 at \( s = 1 \).

Section 4 is devoted to the explicit Gross-Zagier formula. We carefully embed \( K \) into \( M_2(\mathbb{Q}) \) with fixed point \( \tau = (2p\omega - 9)/(9p\omega - 36) \in \mathcal{H} \). The Heegner point \( f(\tau) \), via the natural modular parametrization \( f : X_0(3^5) \to E_9 \), is defined over \( H_{9p} \). The morphism \( f \) is
a test vector for $E_9$ and $\chi$, i.e. there is a nontrivial relation between the central value of the derivative of $L(s, E_9, \chi)$ and the height of the Heegner cycle

$$P_\chi(f) = \sum_{\sigma \in \text{Gal}(H_{9p}/H)} f(\tau^\sigma) \otimes \chi(\sigma) \in E_9(H_{9p}) \otimes \mathbb{Q} K.$$ 

This test vector differs from the admissible one in [CST14, Definition 1.4]. With the aid of [CST14, Theorem 1.6], we have an explicit height formula relating $L'(1, E_9, \chi)$ and the Heegner cycle $P_\chi(f)$ (see Theorem 4.4).

**Theorem 1.5.** For primes $p \equiv 4, 7 \mod 9$, we have the following explicit height formula of Heegner cycles:

$$\frac{L'(1, E_p)L(1, E_{3p^2})}{\Omega_p \Omega_{3p^2}} = 2^\alpha \cdot \langle P_\chi(f), P_{\chi^{-1}}(f) \rangle_{K,K}$$

where $\alpha = 0$ if $p \equiv 4 \mod 9$ and $\alpha = -1$ if $p \equiv 7 \mod 9$, $\Omega_p, \Omega_{3p^2}$ denote the minimal real periods of the elliptic curves $E_p$ and $E_{3p^2}$ respectively, and $\langle \cdot, \cdot \rangle_{K,K}$ denotes the $K$-linear Néron-Tate height pairing of $E_9$ over $K$ (see page 2531, [CST14]).

Comparing this explicit Gross-Zagier formula with the full BSD conjectures for $E_p$ and $E_{3p^2}$, Theorem 1.4 follows from the $\sqrt{-3}$-nondivisibility of the corresponding Heegner points. This is carried out in Section 3 and Section 5.

We remark two points in our work. One is the high ramification of the related automorphic representations which makes the local computations in Section 4.2 very complicated and cannot be covered by the general computations developed by Yueke Hu [Hu16] and others. The other is that the action of complex conjugation on the Heegner point is not modular any more when $p \equiv 4 \mod 9$, while it is always modular in the classical Heegner hypothesis case. This makes the discussions of the non-divisibility of the Heegner points in Section 3.2 more special and technical.

For any integer $c \geq 1$, let $O_c$ be the order of $K$ of conductor $c$ and let $H_c$ be the ring class field of conductor $c$. We will using the following field extension diagram (all the extensions are Galois over $K$):

```
\begin{center}
\begin{tikzpicture}

\node (H9p) at (0,0) {$H_{9p}$};
\node (H3p) at (-2,-1) {$H_{3p}$};
\node (L3p) at (2,-1) {$L(3p)$};
\node (Lp) at (0,-2) {$L(p)$};
\node (3p) at (4,0) {$H_9$};
\node (3) at (-4,0) {$H_{3p}(\sqrt[3]{3})$};
\node (K) at (2,-4) {$K(\sqrt[3]{3})$};
\node (Kp) at (-2,-4) {$K(\sqrt[3]{p})$};
\node (Q) at (0,-4) {$\mathbb{Q}$.}

\path[->,thick]
(H9p) edge node [left] {3} (H3p)
(H9p) edge node [right] {} (3p)
(L3p) edge node [left] {3} (H3p)
(L3p) edge node [right] {} (Lp)
(Lp) edge node [left] {3} (L3p)
(Lp) edge node [right] {} (Kp)
(Lp) edge node [left] {3} (K)
(K) edge node [right] {} (Kp)
(K) edge node [left] {3} (Q);
\end{tikzpicture}
\end{center}
```

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2. The Modular Curve \( X_\Gamma \)

Let \( X \) be an algebraic curve defined over \( \mathbb{Q} \) and \( F \) a field extension of \( \mathbb{Q} \). Denote by \( \text{Aut}_F(X) \) the group of algebraic automorphisms of \( X \) which are defined over \( F \). Let

\[
\mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}
\]

be the Poincaré upper half plane. The group \( \text{GL}_2(\mathbb{Q})^+ \) acts on \( \mathcal{H} \) by linear fractional transformation.

Let \( U_0(3^5) \) be the open compact subgroup of \( \text{GL}_2(\hat{\mathbb{Z}}) \) consisting of matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( c \equiv 0 \mod 3^5 \), and let \( \Gamma_0(3^5) = \text{GL}_2(\mathbb{Q})^+ \cap U_0(3^5) \). Let \( X_0(3^5) \) be the modular curve over \( \mathbb{Q} \) of level \( \Gamma_0(3^5) \) whose underlying Riemann surface is

\[
X_0(3^5)(\mathbb{C}) = \text{GL}_2(\mathbb{Q})^+ \backslash \left( \mathcal{H} \bigsqcup \mathbb{P}^1(\mathbb{Q}) \right) \times \text{GL}_2(\mathbb{A}_f)/U_0(3^5) \simeq \Gamma_0(3^5) \backslash \mathcal{H} \bigsqcup \Gamma_0(3^5) \backslash \mathbb{P}^1(\mathbb{Q}).
\]

Define \( N \) to be the normalizer of \( \Gamma_0(3^5) \) in \( \text{GL}_2^+(\mathbb{Q}) \). It follows from [KM88, Theorem 1] that the linear fractional action of \( N \) on \( X_0(3^5) \) induces an isomorphism

\[
N/\mathbb{Q}^\times \Gamma_0(3^5) \simeq \text{Aut}_\mathbb{Q}(X_0(3^5)).
\]

Moreover, all the algebraic automorphisms in \( \text{Aut}_\mathbb{Q}(X_0(3^5)) \) are defined over \( K \). We identify \( \text{Aut}_\mathbb{Q}(X_0(3^5)) \) with \( N/\mathbb{Q}^\times \Gamma_0(3^5) \) by linear fractional transformations. By [AL70, Theorem 8], and [Ogg80], the quotient group \( N/\mathbb{Q}^\times \Gamma_0(3^5) \simeq S_3 \rtimes \mathbb{Z}/3\mathbb{Z} \), where \( S_3 \) denotes the symmetric group with 3 letters which is generated by the Atkin-Lehner operator \( W = \begin{pmatrix} 0 & 1 \\ -3^5 & 0 \end{pmatrix} \) and the matrix \( A = \begin{pmatrix} 28 & 1/3 \\ 34 & 1 \end{pmatrix} \), and the subgroup \( \mathbb{Z}/3\mathbb{Z} \) is generated by the matrix \( B = \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix} \).

Put

\[
U = \langle U_0(3^5), W, A \rangle \subset \text{GL}_2(\mathbb{A}_f).
\]

Then \( \mathbb{Q}^\times \backslash \mathbb{Q}^\times U \) is an open compact subgroup of \( \mathbb{Q}^\times \backslash \text{GL}_2(\mathbb{A}_f) \). Put

\[
\Gamma = \text{GL}_2(\mathbb{Q})^+ \cap U = \langle \Gamma_0(3^5), W, A \rangle,
\]

and let \( X_\Gamma \) be the modular curve over \( \mathbb{Q} \) of level \( \Gamma \) whose underlying Riemann surface is

\[
X_\Gamma(\mathbb{C}) = \text{GL}_2(\mathbb{Q})^+ \backslash \left( \mathcal{H} \bigsqcup \mathbb{P}^1(\mathbb{Q}) \right) \times \text{GL}_2(\mathbb{A}_f)/U \simeq \Gamma \backslash \mathcal{H} \bigsqcup \Gamma \backslash \mathbb{P}^1(\mathbb{Q}).
\]

Then \( X_\Gamma \) is a smooth projective curve over \( \mathbb{Q} \) of genus 1, and \( X_\Gamma \) has three cusps

\[
\Gamma \backslash \mathbb{P}^1(\mathbb{Q}) = \{ [\infty], [1/9], [2/9] \}.
\]

The cusp \([\infty] \) is rational over \( \mathbb{Q} \), and the cusps \([1/9] \) and \([2/9] \) are both defined over \( K \). We take \( X_\Gamma \) to be an elliptic curve over \( \mathbb{Q} \) with \([\infty] \) as its zero element. Let \( N_\Gamma \) be the normalizer of \( \Gamma \) in \( \text{GL}_2(\mathbb{Q})^+ \). Then we have a natural embedding

\[
\Phi : N_\Gamma / \mathbb{Q}^\times \Gamma \hookrightarrow \text{Aut}_\mathbb{Q}(X_\Gamma).
\]
The matrices
\[ B = \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 9 \\ -3^3 & -2 \end{pmatrix}. \]
lie in \( N_\Gamma \), and hence induce automorphisms of \( X_\Gamma \).

We will always take the simple Weierstrass equation \( y^2 = x^3 - 2^4 \cdot 3 \) for the elliptic curve \( E_9 \), unless special clarification.

**Proposition 2.1.**

1. The elliptic curve \( (X_\Gamma, [\infty]) \) is isomorphic to \( E_9 \) over \( \mathbb{Q} \), and we identify \( X_\Gamma \) with \( E_9 \) so that the cusp \([1/9]\) has coordinates \((0, 4\sqrt{-3})\).

2. We have an embedding
\[ \Phi : N_\Gamma/\mathbb{Q}^\times \Gamma \hookrightarrow \mathcal{O}_K^\times \rtimes \Gamma \backslash \mathbb{P}^1(\mathbb{Q}). \]

Moreover, for any point on \( P \in X_\Gamma \), we have
\[ \Phi(B)(P) = [\omega^2] P, \quad \Phi(C)(P) = [\omega^2] P + (0, 4\sqrt{-3}). \]

In particular, the automorphisms \( \Phi(B) \) and \( \Phi(C) \) are defined over \( K \).

**Proof.** It is known from [DV17] that \( E_9 \) is the natural quotient of \( X_0(3^5) \) by the finite group \( S_3 \). Since the automorphic group of the elliptic curve \( E_9 \) is isomorphic to \( \mathcal{O}_K^\times \), we have
\[ \text{Aut}_\mathbb{Q}(X_\Gamma) \cong \mathcal{O}_K^\times \rtimes X_\Gamma(\overline{\mathbb{Q}}). \]

Then for any \( M \in N_\Gamma \) and \( P \in X_\Gamma \), \( \Phi(M)(P) = [\alpha] P + S \), where \( \alpha \in \mathcal{O}_K^\times \), \( S \in X_\Gamma(\overline{\mathbb{Q}}) \). Taking \( P = [\infty] \), we see \( S = \Phi(M)([\infty]) \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \). The formulae for \( \Phi(B) \) and \( \Phi(C) \) are taken from [DV17], which can also be verified numerically using SageMath. \( \square \)

Let \( V \subset U_0(3^5) \) be the subgroup consisting of matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( a \equiv d \mod 3 \), and put \( U_0 = \langle V, W, A \rangle \). Let \( X^0_\Gamma \) be the modular curve over \( \mathbb{Q} \) whose underlying Riemann surface is
\[ X^0_\Gamma(\mathbb{C}) = \text{GL}_2(\mathbb{Q})^\times \backslash \left( \mathcal{H} \bigcup \mathbb{P}^1(\mathbb{Q}) \right) \times \text{GL}_2(\mathbb{A}_f)/U_0. \]

Under class field theory, \( \mathbb{Q}^\times \hat{\mathbb{Z}} / \mathbb{Q}^\times \det(U_0) \cong \text{Gal}(K/\mathbb{Q}) \). Noting that \( \text{GL}_2(\mathbb{Q})^\times \cap U_0 = \Gamma \), we see that the modular curve \( X^0_\Gamma \) is isomorphic to \( X_\Gamma \times_{\mathbb{Q}} K \) as a curve over \( \mathbb{Q} \) (cf. [Shi94, Chapter 6]). Put
\[ U/U_0 = \langle \epsilon \rangle, \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The non-trivial Galois action of \( \text{Gal}(K/\mathbb{Q}) \) on \( X^0_\Gamma \) is given by the right translation of \( \epsilon \) on \( X^0_\Gamma \). We have
\[ \text{Aut}_{\mathbb{Q}}(X^0_\Gamma) = \text{Aut}_K(X_\Gamma) \rtimes \text{Gal}(K/\mathbb{Q}) \cong (X_\Gamma(K) \times \mathcal{O}_K^\times) \rtimes \text{Gal}(K/\mathbb{Q}). \]

Let \( N_{\text{GL}_2(\mathbb{A}_f)}(U_0) \) be the normalizer of \( U_0 \) in \( \text{GL}_2(\mathbb{A}_f) \). Then there is a natural homomorphism
\[ N_{\text{GL}_2(\mathbb{A}_f)}(U_0)/U_0 \rightarrow \text{Aut}_{\mathbb{Q}}(X^0_\Gamma) \]
induced by right translation on \( X^0_\Gamma \). The curve \( X^0_\Gamma \) is not geometrically connected and has two connected components over \( \mathbb{C} \). An element \( g \in N_{\text{GL}_2(\mathbb{A}_f)}(U_0) \) maps one component of
$X_1^0$ onto the other if and only if it has image $-1$ under the composition of the following morphisms:

$$\text{GL}_2(\mathbb{A}_f) = \text{GL}_2(\mathbb{Q}) + \text{GL}_2(\hat{\mathbb{Z}})^{\text{det}} \xrightarrow{\mathbb{Q}_1^\infty \hat{\mathbb{Z}}^\times} \mathbb{Z}_3^\times/(1 + 3\mathbb{Z}_3),$$

where the last morphism is the projection from $\hat{\mathbb{Z}}^\times$ to its $3$-adic factor.

Let $p \equiv 4, 7 \mod 9$ be a rational prime number. Let $\rho : K \to M_2(\mathbb{Q})$ be the normalised embedding with fixed point $\tau = (2p\omega - 9)/(9p\omega - 36) \in \mathcal{H}$, i.e. we have

$$\rho(t) \left( \frac{\tau}{1} \right) = t \left( \frac{\tau}{1} \right), \quad \text{for any } t \in K.$$

Note that

$$\tau = M\omega, \quad M = \begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Then the embedding $\rho : K \to M_2(\mathbb{Q})$ is explicitly given by

$$\rho(\omega) = M \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} 2p + 8 + 36/p & -4p/9 - 2 - 9/p \\ 9p + 36 + 144/p & -2p - 9 - 36/p \end{pmatrix}.$$

Let $R_0(3^5)$ bet the standard Eichler order of discriminant $3^5$ in $M_2(\mathbb{Q})$. Then $K \cap R_0(3^5) = \mathcal{O}_p$. Let $\mathcal{O}_{K,3}$ be the completion of $\mathcal{O}_K$ at the unique place above $3$. We have

$$\mathcal{O}_{K,3}^\times / \mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3}) = \langle \omega_3 \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 + 3\omega_3 \rangle^{\mathbb{Z}/3\mathbb{Z}},$$

where $\omega_3$ is the $3$-local component of $\omega$. It is straightforward to verify that $\omega_3$ and $1 + 3\omega_3$ normalize $U_0$, and hence we have an embedding

$$\mathcal{O}_{K,3}^\times / \mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3}) \hookrightarrow \text{Aut}_\mathbb{Q}(X_1^0).$$

If $p \equiv 7 \mod 9$, the element

$$w = M \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} M^{-1} = \begin{pmatrix} -2p - 17 & 4p + 17 \\ -9p - 72 & 2p + 17 \end{pmatrix},$$

is a nontrivial normalizer of $K^\times$ in $\text{GL}_2(\mathbb{Q})$ and $w$ normalizes $U_0$, and hence also induces an automorphism of $X_1^0$.

**Theorem 2.2.**

1. For any point $P \in X_1^0$, we have

$$P^{1 + 3\omega_3} = [\omega^2]P,$$

and

$$P^{\omega_3} = \begin{cases} [\omega^2]P + (0, 4\sqrt{-3}), & p \equiv 4 \mod 9; \\ [\omega]P + (0, 4\sqrt{-3}), & p \equiv 7 \mod 9. \end{cases}$$

2. Suppose $p \equiv 7 \mod 9$. For any point $P \in X_1^0$, we have

$$P^{w_\infty} = [\omega^{w_\infty}]P - (0, 4\sqrt{-3}).$$

**Proof.** Since $\omega_3$ and $1 + 3\omega_3$ and $w_\infty$ all have determinant $\equiv 1$ mod $3$, as elements in $\text{Aut}_\mathbb{Q}(X_1^0)$, they lie in the subgroup $\text{Aut}_K(X_1)$. Suppose $P = [z, 1], z \in \mathcal{H}$, be a point on $X_1^0$. We have

$$B(1 + 3\omega_3)A^2 = \begin{pmatrix} 60p + 837/p + 214 & 2p/3 + 9/p + 7/3 \\ 5130p + 71145/p + 18252 & 57p + 765/p + 199 \end{pmatrix} \in V,$$
where the subscript 3 denotes the 3-adic component of the adelic matrices. Then

$$P^{1 + 3\omega_3} = \Phi(B)(P) = [\omega^2]P.$$  

If \( p \equiv 4 \mod 9 \), then

$$C \omega_3 A^2 = \left( \begin{array}{cc} 867p + 11635/p + 2685 & 31p/3 + 416/3p + 32 \\ -20808p - 281925/p - 64440 & -248p - 3360/p - 768 \end{array} \right)_3, \quad CA^2 \right) \in V.$$  

and hence

$$P^{\omega_3} = \Phi(C)(P) = [\omega^2]P + (4, \sqrt{-3}).$$

If \( p \equiv 7 \mod 9 \), then

$$BC \omega_3 A^2 = \left( \begin{array}{cc} 867p + 11635/p + 2685 & 31p/3 + 416/3p + 32 \\ 49419p + 660510/p + 153045 & 589p + 7872/p + 1824 \end{array} \right)_3, \quad BCA^2 \right) \in V.$$  

and hence

$$P^{\omega_3} = \Phi(BC)(P) = [\omega]P + (0, 4\sqrt{-3}).$$

Suppose \( p \equiv 7 \mod 9 \). We have

$$\left\{ \begin{array}{ll} BC^2 w\epsilon A^2 \in V, & \frac{p - 7}{9} \equiv 0 \mod 3; \\ C^2 w\epsilon A^2 \in V, & \frac{p - 7}{9} \equiv 1 \mod 3; \\ B^2 C^2 w\epsilon A^2 \in V, & \frac{p - 7}{9} \equiv 2 \mod 3. \end{array} \right.$$  

Hence the second assertion follows.

\( \square \)

Let \( \tilde{K}^\times \rightarrow \text{Gal}(K^{ab}/K) \) be the Artin reciprocity law and we denote by \( \sigma_t \) the image of \( t \in \tilde{K}^\times \). Let \( P_0 = [\tau, 1] \) be the CM point on \( X^0_1 \).

**Corollary 2.3.** 1. The point \( P_0 \in X^0_1(H_{9p}) \) satisfies

$$P_0^{\sigma_{1 + 3\omega_3}} = [\omega^2]P_0,$$

and

$$P_0^{\omega_3} = \left\{ \begin{array}{ll} [\omega^2]P_0 + (0, 4\sqrt{-3}), & p \equiv 4 \mod 9 \\ [\omega]P_0 + (0, 4\sqrt{-3}), & p \equiv 7 \mod 9 \end{array} \right.$$  

2. Suppose \( p \equiv 7 \mod 9 \). We have

$$P_0 = [\omega^{\frac{\sqrt{-3}}{3}}]P_0 - (0, 4\sqrt{-3}).$$

**Proof.** By Shimura’s reciprocity law, we have

$$P_0^{\sigma_{t}} = P_0^{t} = [\tau, t], \quad t \in \tilde{K}^\times.$$ 

Since \( \tilde{K}^\times \cap \mathcal{U}_0 = \hat{O}_{0p}^\times \), we see \( P_0 \) is defined over the ring class field \( H_{9p} \), and the Galois actions of \( \sigma_{\omega_3} \) and \( \sigma_{1 + 3\omega_3} \) are clear. \( \square \)

**Proposition 2.4.** 1. The field \( H_{9p} = H_{3p}(\sqrt[3]{3}) \) with Galois group \( \text{Gal}(H_{9p}/H_{3p}) \simeq \langle 1 + \frac{3\omega_3}{2} \rangle \), and

$$\left( \frac{\sqrt[3]{3}}{3} \right)^{\sigma_{1 + 3\omega_3}^{-1}} = \omega^2.$$
2. We have \((\sqrt[3]{3})^{\omega_3 - 1} = 1\) and
\[
(\sqrt[p]{3})^{\omega_3 - 1} = \begin{cases} \omega^2, & p \equiv 4 \mod 9 \\ \omega, & p \equiv 7 \mod 9 \end{cases}
\]

Proof. The ideal \(7\mathcal{O}_K = (1 + 3\omega)(1 + 3\omega^2)\) and let \(v\) be the place corresponding to the prime ideal \((1 + 3\omega)\). Then the local-global principle, we have
\[
\left(\sqrt[3]{3}\right)^{\sigma_{1 + 3\omega_3 - 1} = 1 + 3\omega, \ 3} = (1 + 3\omega_3, 3)^{-1} = 3^{-2} \mod (1 + 3\omega) = \omega^2,
\]
where \((\ , \ )_w\) denotes the 3-rd Hilbert symbol over \(K_w\).

Since \(p \equiv 1 \mod 3\), the prime \(p\) splits in \(K\) and let \(v\) and \(\overline{v}\) be the two places of \(K\) above \(p\). Then similarly
\[
\left(\sqrt[p]{3}\right)^{\omega_3 - 1} = (\omega_v, p)^{-1} \cdot (\omega_{\overline{v}}, p)^{-1} = \omega^{-\frac{p-1}{3}}.
\]

The elliptic curve \(E_1\) has Weierstrass equation \(y^2 = x^3 - 2^4 \cdot 3^3\). Consider the isomorphism \(\phi : E_9 \longrightarrow E_1, \ (x, y) \mapsto ((\sqrt[3]{3})^2 x, 3y)\).

We have the following commutative diagram:
\[
\begin{array}{ccc}
E_9(H_{9p})^\sigma_{1 + 3\omega_3 - \omega_2} & \xrightarrow{\text{Tr}_{H_{9p}/(L(3,p))}} & E_9(L_{(3,p)})^\sigma_{1 + 3\omega_3 - \omega_2} \\
\phi \downarrow & & \phi \\
E_1(H_{3p}) & \xrightarrow{\text{Tr}_{H_{3p}/(L(p))}} & E_1(L_{(p)})
\end{array}
\]

Put \(Q = \phi(P_0)\) and \(R = \text{Tr}_{H_{3p}/(L(p))}Q\).

Corollary 2.5. The point \(R \in E_1(L_{(p)})\) and satisfies
\[
R^{\sigma_{\omega_3}} = \begin{cases} [\omega^2]R + (0, 12\sqrt{-3}), & p \equiv 4 \mod 9 \\ [\omega]R + (0, -12\sqrt{-3}), & p \equiv 7 \mod 9 \end{cases}
\]
and if \(p \equiv 7 \mod 9\),
\[
R = [\omega^{\frac{p-1}{3}}]R + (0, 12\sqrt{-3}).
\]

It follows from [DV17, Lemma 5.2.9] that
\[
E_1(L_{(p)})_{\text{tor}} = E_1(K) = \{O, (12\omega^i, \pm 36), (0 \pm 12\sqrt{-3})\}_{i=0,1,2}.
\]

The complex multiplication on the torsion points are given as follows:

| \(T\) | \(-\omega T\) | \(1 - \omega T\) |
|-------|--------|--------|
| \((12\omega^i, \pm 36)\) | \((12\omega^{i+1}, \pm 36)\) | \((0, \pm 12\sqrt{-3})\) |
| \((0, \pm 12\sqrt{-3})\) | \((0, \mp 12\sqrt{-3})\) | \(O\) |

and

| \(T\) | \(-\omega^2 T\) | \(1 - \omega^2 T\) |
|-------|--------|--------|
| \((12\omega^i, \pm 36)\) | \((12\omega^{i+2}, \mp 36)\) | \((0, \mp 12\sqrt{-3})\) |
| \((0, \pm 12\sqrt{-3})\) | \((0, \mp 12\sqrt{-3})\) | \(O\) |
The torsion point \( T_i = (12\omega^i, -36) \) satisfies
\[
\overline{T_i} = [\omega^i]T_i = [\omega^{i+2}]T_i + (0, 12\sqrt{-3}).
\]
So the Galois action \( \sigma_{\omega_3} \) and the complex conjugation can not distinguish \( R \) from torsion points.

3. Heegner points

3.1. Non-triviality of Heegner points. This subsection is a rewrite of Section 5 of [DV17], where Dasgupta and Voight analyze the behavior of the coordinates of the Heegner points \( P_0 \) modulo \( p \). Comparing these coordinates to those of torsion points, they conclude that the Heegner points \( P_0 \) are non-trivial.

Let \( \eta \) be the Dedekind eta-function. We have the following transformation law of eta functions:
\[
\eta(\tau + 1) = \exp(2\pi i/24)\eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau).
\]
In terms of eta-products, we have a modular parametrization:
\[
\Phi : X_0(243) \longrightarrow E_9 : y^2 + 3y = x^3 - 3, \quad z \mapsto (x(z), y(z)),
\]
where
\[
\begin{align*}
x(z) &= \frac{\eta(9z)\eta(27z)}{\eta(3z)\eta(81z)}, \\
y(z) &= \frac{\eta(9z)^4 + 9\eta(9z)\eta(81z)^3}{\eta(27z)^4 - 3\eta(9z)\eta(81z)^3}.
\end{align*}
\]
Put
\[
M = \begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix}.
\]
Straight calculation yields the following:
\[
\begin{align*}
81M(z) &= \begin{pmatrix} 18 & -1 \\ 1 & 0 \end{pmatrix} \left( \frac{9z - 4}{9} \right) = T^{18}S \left( \frac{9z - 4}{9} \right), \\
27M(z) &= \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \left( \frac{9z - 4}{3} \right) = T^6S \left( \frac{9z - 4}{3} \right), \\
9M(z) &= \begin{pmatrix} 2 & -9 \\ 1 & -4 \end{pmatrix} (9z) = T^2ST^{-4}(9z), \\
3M(z) &= \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix} (3z) = TST^3ST^{-1}(3z).
\end{align*}
\]
Then we have
\[
\begin{align*}
\eta(81M(\omega p/9)) &= e^{3\pi i/2} \sqrt{-i(\omega p - 4)/9}\eta(\omega p - 4)/9) \\
\eta(27M(\omega p/9)) &= e^{\pi i/2} \sqrt{-i(\omega p - 4)/3}\eta(\omega p - 4)/3) \\
\eta(9M(\omega p/9)) &= e^{-\pi i/6} \sqrt{-i(\omega p - 4)}\eta(\omega p) \\
\eta(3M(\omega p/9)) &= e^{\pi i/4} \sqrt{-(\omega p - 4)}\eta(\omega p/3).
\end{align*}
\]
Thus
\[
\begin{align*}
x(\tau) &= e^{-\pi i/6} \sqrt{3} \frac{\eta((\omega p - 4)/3)\eta(\omega p)}{\eta((\omega p - 4)/9)\eta(\omega p/3)}. \\
\end{align*}
\]
By Ligozat’s criterion [Lig70, Proposition 3.2.1], \( f(z) = \eta(27z)/\eta(3z) \) is a modular function on \( \Gamma_0(81) \). Let \( \overline{\mathbb{Z}} \) be the integral closure of \( \mathbb{Z} \) in \( \overline{\mathbb{Q}} \).
Case 1. In case $p \equiv 4 \mod 9$, we write $k = \frac{-p^4}{9} \in \mathbb{Z}$, we have
\[
\frac{\omega p - 4}{9} = \frac{p(\omega - 1)}{9} - k.
\]
Since $p + k \equiv 4 \mod 8$ and $p + k \equiv k + 1 \mod 3$, $p + k \equiv 4 - 8k \mod 24$,
\[
\frac{\eta(\omega p)}{\eta((\omega p - 4)/9)} = e^{\pi i (p+k)/12} \frac{\eta(p(\omega - 1))}{\eta(p(\omega - 1)/9)} = e^{\pi i/3} \eta(p(\omega - 1)/9).
\]
Similarly,
\[
\frac{\eta((\omega p - 4)/3)}{\eta(3\omega p)} = e^{-\pi i (k+3p)/6} \frac{\eta(p(\omega - 1)/3)}{\eta(3p(\omega - 1)/3)} = -\frac{\eta(p(\omega - 1)/3)}{\eta(3p(\omega - 1)/3)}.
\]
Put all things together, we get
\[
x(\tau) = -e^{-5\pi i/6} \omega^{-k} \sqrt{3} \frac{\eta(p(\omega - 1)/3)}{\eta(3p(\omega - 1)/3)} \frac{\eta(p(\omega - 1)/9)}{\eta(3p(\omega - 1)/9)} f(p(\omega - 1)/27) f(p\omega/9)
\]
Twisting to $E_1 : y^2 + y = 3x^3 - 1$, we get
\[
x(Q) = -e^{-5\pi i/6} \omega^{-k} \sqrt{3} \frac{f(p(\omega - 1)/27) f(p\omega/9)}{f(p(\omega - 1)/9)}.
\]
By [DV17, Proposition 5.2.1],
\[
x(Q)^p \equiv -e^{-5\pi i/6} \omega^{-k} 3^{p/6} \frac{f((\omega - 1)/27) f(\omega/9)}{f((\omega - 1)/9)} \mod p\mathbb{Z}.
\]
Note that
\[
f \left( z - \frac{2}{3} \right) = e^{-4\pi i/3} f(z).
\]
So
\[
f((\omega - 1)/9) = e^{4\pi i/3} f((\omega - 7)/9).
\]
By the same method in [DV17, Lemma 5.2.4], we can compute that
\[
f((\omega - 1)/27) = -\omega \frac{1}{\sqrt{3}}.
\]
It is computed by Dasgupta and Voight that
\[
f(\omega/9) = e^{-\pi i/6} \frac{1}{\sqrt{3}}, \quad f((\omega - 7)/9) = -\omega^2 \frac{1}{\sqrt{9}}, \quad f((\omega - 7)/27) = -\omega \frac{1}{\sqrt{3}}.
\]
Then
\[
\frac{f((\omega - 1)/27) f(\omega/9)}{f((\omega - 1)/9)} = 3^{-1/6} e^{-\pi i/6}.
\]
As a result,
\[
x(Q)^p \equiv e^{(5-5p)\pi i/6} \omega^{-p} 3^{(p-1)/6} \mod p\mathbb{Z}.
\]
Note that $p \equiv 4 \mod 9$ is prime, so $(p - 1)/6$ is an integer.
We consider the reduction map 

\[ p \mod \frac{7p+4}{9} \in \mathbb{Z}, \text{ we have} \]

\[
\frac{\omega p - 4}{9} = \frac{p(\omega - 7)}{9} - k.
\]

Since \(7p + k \equiv 4 \mod 8\) and \(7p + k \equiv k + 1 \mod 3\), \(7p + k \equiv 4 - 8k \mod 24\),

\[
\frac{\eta(\omega p)}{\eta((\omega p - 4)/9)} = e^{\pi i (7p + k)/12} \frac{\eta(p(\omega - 7))}{\eta(p(\omega - 7)/9)} = e^{\pi i/3} \frac{\eta(p(\omega - 7))}{\eta(p(\omega - 7)/9)}
\]

Similarly,

\[
\frac{\eta((\omega p - 4)/3)}{\eta(3\omega p)} = e^{-\pi i (k + 7p)/4} \frac{\eta(p(\omega - 7)/3)}{\eta(3p(\omega - 7))} = -\frac{\eta(p(\omega - 7)/3)}{\eta(3p(\omega - 7))}.
\]

Again, we get

\[
x(\tau) = -e^{-5\pi i/6} \omega^{-k} \frac{\eta(p(\omega - 7))}{\eta(p(\omega - 7)/9)} \frac{\eta(p(\omega - 7)/3)}{\eta(3p(\omega - 7))} \frac{\eta(3p(\omega - 7))}{\eta(3p(\omega - 7)/3)} = -e^{-5\pi i/6} \omega^{-k} \frac{\eta(p(\omega - 7)/3)}{\eta(3p(\omega - 7)/3)} \eta(p(\omega - 7)/9) \eta(3p(\omega - 7)/9) + \frac{\eta(3p(\omega - 7)/3)}{\eta(3p(\omega - 7)/9)} \eta(p(\omega - 7)/9).
\]

Twisting to \(E_1 : y^2 + y = 3x^3 - 1\), we get

\[
x(Q) = -e^{-5\pi i/6} \omega^{-k} \sqrt{3} f(p(\omega - 7)/27) f(p \omega/9) f(p(\omega - 7)/9).
\]

By [DV17, Proposition 5.2.1],

\[
x(Q)^p \equiv -e^{-5\pi i/6} \omega^{-pk} 3^{p/6} f((\omega - 7)/27) f(\omega/9) f((\omega - 7)/9) \mod p\mathbb{Z}.
\]

Now

\[
\frac{f((\omega - 7)/27) f(\omega/9)}{f((\omega - 7)/9)} = -3^{-1} e^{\pi i/6}.
\]

So

\[
x(Q)^p \equiv e^{(5 - 5p)\pi i/6} \omega^{-pk - 13(p - 1)/6} \mod p\mathbb{Z}.
\]

Note that \(p \equiv 7 \mod 9\) is prime, so \((p - 1)/6\) is an integer.

By the assumption that \(3 \mod p\) is not a cubic residue, we decompose \(p\mathcal{O}_K = p\mathfrak{p} \mathfrak{P}\) so that

\[
3^{\frac{p-1}{4}} \equiv \omega \mod \mathfrak{p}, \quad 3^{\frac{p-1}{4}} \equiv \omega^2 \mod \mathfrak{P}.
\]

Since \(H_{3p}/K\) is totally ramified, let \(\mathfrak{p}\) and \(\mathfrak{P}\) be the primes of \(H_{3p}\) above \(p\) and \(\mathfrak{P}\) respectively. We have

\[
\mathcal{O}_K / p\mathcal{O}_K \cong \mathcal{O}_K / \mathfrak{p} \bigoplus \mathcal{O}_K / \mathfrak{P} \subset \mathcal{O}_{H_{3p}} / \mathfrak{p} \bigoplus \mathcal{O}_{H_{3p}} / \mathfrak{P}.
\]

We consider the reduction map

\[
\text{Red} : E_1(L(\rho)) \setminus \{O\} \longrightarrow \mathcal{O}_{H_{3p}} / \mathfrak{p} \bigoplus \mathcal{O}_{H_{3p}} / \mathfrak{P}, \quad T \mapsto (x(T) \mod \mathfrak{p}, x(T) \mod \mathfrak{P}).
\]

By [DV17, Lemma 5.2.9], we have \(E_1(L(\rho))_{\text{tor}} = E_1(K)_{\text{tor}}\). Let \(D\) be the image of \(E_1(L(\rho))_{\text{tor}} \setminus \{O\}\) under the reduction map. Then

\[
D = \{(0, 0), (12\omega^i, 12\omega^j)_{i=0,1,2} \} \subset \mathcal{O}_K / \mathfrak{p} \bigoplus \mathcal{O}_K / \mathfrak{P}.
\]
Proposition 3.1 (Dasgupta-Voight). If $3$ is not a cube modulo $p$, the reduction $\text{Red}(R)$ doesn’t lie in $\text{Red}(E_1(L_{(p)})_{\text{tor}}) = D \cup \{O\}$. In particular, $R$ is nontorsion.

Proof. It follows from (3.1) and (3.2) that if $p \equiv 4 \mod 9$, $k = (4 - p)/9$, then we have
\[
x(Q)^p \equiv \omega^{-k}(-3)^{p-1} \mod p\mathbb{Z},
\]
and if $p \equiv 7 \mod 9$, $k = (4 - 7p)/9$, then we have
\[
x(Q)^p \equiv \omega^{-k-1}(-3)^{p-1} \mod p\mathbb{Z}.
\]
Transferring to $E_1: y^2 = x^3 - 2^4 \cdot 3^3$, the $x$-coordinate of $Q = \phi(P_0)$ satisfies
\[
x(Q)^p \equiv 12^p\omega^i(-3)^{p-1} \mod p\mathbb{Z}
\]
for some $i = 0, 1, 2$. Then we have
\[
x(Q)^2 \equiv x(Q)^{2p} \equiv \begin{cases} (12\omega^j)^2 \text{ mod } \mathfrak{p}, \\ (12\omega^j)^2 \text{ mod } \overline{\mathfrak{p}}, \end{cases}
\]
and hence
\[
x(Q) \equiv \begin{cases} \pm 12\omega^{i+2} \text{ mod } \mathfrak{p}, \\ \pm 12\omega^{i+1} \text{ mod } \overline{\mathfrak{p}}. \end{cases}
\]
(3.3)

Since $p \equiv 1 \mod 3$ and the extension $H_{3p}/K$ is totally ramified at the places above $p$, by Corollary 2.3, we have
\[
[1 - \omega^\alpha]Q \equiv (0, 12\sqrt{-3}) \mod \mathfrak{p} \text{ (resp. } \overline{\mathfrak{p}}),
\]
where $\alpha = 2$ if $p \equiv 4 \mod 9$ and $\alpha = 1$ if $p \equiv 7 \mod 9$. This implies that $(Q \text{ mod } \mathfrak{p})$ and $(Q \text{ mod } \overline{\mathfrak{p}})$ both belongs to $E_1(\mathbb{F}_p)[3]$. It follows from the tables below Corollary 2.5 that
(3.4)

\[
Q \equiv \begin{cases} (12\omega^{i+2}, (-1)^{\alpha-1} \cdot 36) \text{ mod } \mathfrak{p}, \\ (12\omega^{i+1}, (-1)^{\alpha-1} \cdot 36) \text{ mod } \overline{\mathfrak{p}}. \end{cases}
\]

Since the extension $H_{3p}/L_{(p)}$ is totally ramified at the places above $p$, we have
(3.5)

\[
R = \text{Tr}_{H_{3p}/L_{(p)}}Q \equiv \frac{p - 1}{3}Q \equiv [a]Q \mod p\mathbb{Z},
\]
where $a = 1$ if $p \equiv 4$ and $a = 2$ if $p \equiv 7 \mod 9$. It follows from (3.3) that the reduction of $R$ doesn’t lie in $D$, and hence $R$ is not torsion. □

3.2. Non-divisibility of Heegner points. It follows from (3.4) and (3.5) that
\[
R \equiv \begin{cases} (12\omega^{i+2}, -36) \text{ mod } \mathfrak{p}, \\ (12\omega^{i+1}, -36) \text{ mod } \overline{\mathfrak{p}}. \end{cases}
\]
Put $T = (12\omega^{i+2}, -36)$ which satisfies the relations
\[
T = \begin{cases} [\omega^2]T + (0, 12\sqrt{-3}), \\ [\omega]T + (0, -12\sqrt{-3}). \end{cases}
\]
By Corollary 2.5, the point \( Y = R - T \) belongs to \( E_1(L(p))^{\sigma_3=\omega^a} \) which is identified with \( E_p(K) \) under the isomorphism \((x,y) \mapsto ((\sqrt{P})^2 x, py)\) defined over \( L(p) \). Then the point \( Y + \overline{Y} \) is identified with an element in \( E_p(\mathbb{Q}) \).

**Proposition 3.2.** The point \( Y \) is not divisible by \( \sqrt{-3} \) in \( E_1(L(p))^{\sigma_3=\omega^a} \).

**Proof.** Suppose \( Y = \sqrt{-3}X + S \) for some \( X \in E_1(L(p))^{\sigma_3=\omega^a} \) and \( S \in E_1(L(p))^{\sigma_3=\omega^a} \). From the formulae
\[
Y^{\sigma_3} = [\omega]^a Y, \quad X^{\sigma_3} = [\omega]^a X, \quad S^{\sigma_3} = [\omega]^a S,
\]
we have
\[
[\sqrt{-3}]Y = [\sqrt{-3}]X = [\sqrt{-3}]S \equiv O \mod \mathfrak{p} \text{ (resp. } \mod \overline{\mathfrak{p}}).\]

By our choice of \( T \), we have
\[
S \equiv Y \equiv \begin{cases} O, & \text{mod } \mathfrak{p}, \\ [1 - \omega](12\omega^{i+1}, -36), & \text{mod } \overline{\mathfrak{p}}, \end{cases}
\]
which implies that \((x(S) \mod \mathfrak{p}, x(S) \mod \overline{\mathfrak{p}})\) doesn’t lie in the subset \( D \), and this contradicts with the fact that \( S \) is a torsion point. This proves that \( Y \) is not divisible by \( \sqrt{-3} \) in \( E_1(L(p))^{\sigma_3=\omega^a} \).

\[\square\]

**Proposition 3.3.** The point \( Y + \overline{Y} \in E_p(\mathbb{Q}) \) is not divisible by 3.

**Proof.** If \( p \equiv 7 \mod 9 \), it follows from Corollary 2.5 that
\[
\overline{Y} = [\omega^{\frac{p-7}{2}}]Y \mod \text{torsion,}
\]
and hence we have
\[
\hat{h}_Q(Y + \overline{Y}) = \hat{h}_Q(1 + \omega^{\frac{p-7}{2}} Y) = \hat{h}_Q(Y).
\]

Then the proposition in case \( p \equiv 7 \mod 9 \) follows from Proposition 3.2.

We present a uniform proof of the proposition in both the case \( p \equiv 4 \mod 9 \) and the case \( p \equiv 7 \mod 9 \) in the following.

Since the Heegner point \( R \) is not torsion, by the work of Gross-Zagier [GZ86] and Kolyvagin [Kol90], we know that
\[
\text{rank}_\mathbb{Z} E_p(\mathbb{Q}) = \text{ord}_{s=1} L(s, E_p) = 1, \quad \text{rank}_\mathbb{Z} E_{3p^2}(\mathbb{Q}) = \text{ord}_{s=1} L(s, E_{3p^2}) = 0.
\]
Hence the free component of \( E_p(K) \) has rank 1 over \( \mathcal{O}_K \) and let \( Q_0 \) be a generator of the free component of \( E_p(K) \). Then for some \( \beta, \beta' \in \mathcal{O}_K \) and \( S, S' \in E_1(L(p))_{\text{tor}}, \),
\[
R = [\beta]Q_0 + S \text{ and } R = [\beta']Q_0 + S'.
\]
Since \( \hat{h}_Q(R) = \hat{h}_Q(R) \), we have \( N_{K/\mathbb{Q}}(\beta) = N_{K/\mathbb{Q}}(\beta') \). We may assume
\[
\beta = \gamma \delta, \quad \beta' = \overline{\gamma} \delta u,
\]
where the primes dividing \( \gamma \) are all split in \( K \), and the primes dividing \( \delta \) are all inert or ramified in \( K \), and \( u \in \mathcal{O}_K^\times \).

Put
\[
Z = [\delta]Q_0 + S_1, \quad S_1 \in E_1(L(p))_{\text{tor}},
\]
so that \( R = [\gamma]Z \). Then
\[
\overline{Z} = [\delta u]Q_0 + S_2, \quad S_2 \in E_1(L(p))_{\text{tor}}.
\]

\[13\]
We first prove the reduction \( \text{Red}(Z) \) doesn’t lie in \( D \). Suppose \( \text{Red}(Z) \in D \). Then the order of the point \( Z \) mod \( \mathfrak{P} \) (resp. \( Z \) mod \( \overline{\mathfrak{P}} \)) is 3 or \( \sqrt{-3} \) according to \( \text{Red}(Z) = (12\omega^j, 12\omega^j) \), \( j = 0, 1, 2 \) or \( \text{Red}(Z) = (0, 0) \). From \( R = [\gamma]Z \) and the order of \( R \) mod \( \mathfrak{P} \) (resp. \( R \) mod \( \overline{\mathfrak{P}} \)) is 3, we see \( \text{Red}(Z) = (12\omega^j, 12\omega^j) \). It follows from \( R = [\gamma]Z \) and \( \text{Red}(R) \notin D \) that

\[
\begin{align*}
Z &\equiv \begin{cases} 
(12\omega^j, b) \text{ mod } \mathfrak{P}, \\
(12\omega^j, -b) \text{ mod } \overline{\mathfrak{P}}, 
\end{cases} \\
R = [\gamma]Z &\equiv \begin{cases} 
[\gamma](12\omega^j, b) \text{ mod } \mathfrak{P}, \\
[-\gamma](12\omega^j, b) \text{ mod } \overline{\mathfrak{P}}, 
\end{cases}
\end{align*}
\]

where \( b = 36 \) or \(-36\). Then

\[
R = [\gamma]Z \equiv \begin{cases} 
(12\omega^{j+2}, -36), \text{ mod } \mathfrak{P}, \\
(12\omega^{j+1}, -36), \text{ mod } \overline{\mathfrak{P}},
\end{cases}
\]

which contradicts with the fact

\[
R \equiv \begin{cases} 
(12\omega^{j+2}, -36), \text{ mod } \mathfrak{P}, \\
(12\omega^{j+1}, -36), \text{ mod } \overline{\mathfrak{P}},
\end{cases}
\]

Next we prove that the unit \( u \) is a third root of unity. Note for some \( S_3 \in E_1(L(\mu))_{\text{tor}} \), we have

\[
\overline{Z} = [u]Z + S_3,
\]

and hence

\[
\overline{Z} - Z = [u - 1]Z = S_3.
\]

Then \( \text{Red}(Z) \notin D \) and \( \text{Red}(S_3) = \text{Red}([u - 1]Z) \in D \) forces \( u = \omega^k \) for some \( k = 0, 1, 2 \). Then

\[
\overline{R} + R = [(\gamma + \overline{\gamma}u)\delta]Q_0 + \text{torsion}.
\]

By Proposition 3.2, we see \( \beta \) is a 3-unit, and hence both \( \gamma \) and \( \delta \) are 3-units. Since \( u = \omega^k \), \( k = 0, 1, 2 \),

\[
\gamma + \overline{\gamma}u = \gamma + \overline{\gamma} + (u - 1)\overline{\gamma}
\]

is also a 3-unit. Then \( (\gamma + \overline{\gamma}u)\delta \) is a 3-unit. Finally, we see

\[
Y + Y = \overline{R} + R + \text{torsion} = [(\gamma + \overline{\gamma}u)\delta]Q_0 + \text{torsion}.
\]

Hence the proposition follows.

\[\square\]

4. The explicit Gross-Zagier formulae

4.1. Test vectors and the explicit Gross-Zagier formulae. Let \( \pi \) be the automorphic representation of \( \text{GL}_2(\mathbb{A}) \) corresponding to \( E_9/\mathbb{Q} \). Then \( \pi \) is only ramified at 3 with conductor \( 3^5 \). For \( n \in \mathbb{Q}^\times \), let \( \chi_n : \text{Gal}(K^{ab}/K) \to \mathbb{C}^\times \) be the cubic character given by \( \chi_n(\sigma) = (\sqrt[3]{n})^{\sigma^{-1}} \). Define

\[
L(s, E_9, \chi_n) = L(s - 1/2, \pi_K \otimes \chi_n), \quad \epsilon(E_9, \chi) = \epsilon(1/2, \pi_K \otimes \chi_n),
\]

where \( \pi_K \) is the base change of \( \pi \) to \( \text{GL}_2(\mathbb{A}_K) \).

Proposition 4.1. For \( m \in \mathbb{Q}^\times \), we have

\[
L(s, E_m, \chi_n) = L(s, E_{mn^2})L(s, E_{mn}).
\]
Proof. Note $\chi_n \in H^1(\text{Gal}(\mathcal{O}/K), \text{Aut}_\mathbb{Q}(E_m))$ and $E_{mn}$ is the twist of $E_m$ by $\chi_n$ over $K$. Let $\phi_m$ be the Hecke character associated to $E_{mn}/K$. Then by CM theory, we have $\phi_{mn} = \chi_n^{-1}\phi_m$. By [Sil94, Theorem 10.5], $L(s, E_{mn}/K) = L(s, \phi_m)L(s, \overline{\phi_m})$. Then

$$L(s, E_m, \chi_n) = L(s, E_{mn}/K)$$

$$= L(s, \chi_n\phi_m)L(s, \chi_n\overline{\phi_m})$$

$$= L(s, \chi_n\phi_m)L(s, \chi_n^{-1}\phi_m)$$

$$= L(s, E_{mn})L(s, E_{mn}^*).$$

Here we use the fact that both $\phi_m$ and $\chi_n$ are equivariant under complex conjugation, and hence that $L(s, \chi_n\phi_m) = L(s, \chi_n^{-1}\phi_m)$. \hfill \qed

Let $p \equiv 4, 7 \mod 9$ be a prime number, and put $\chi = \chi_{3p}$. By Proposition 4.1, we have

$$L(s, E_{9p}, \chi) = L(s, E_p)L(s, E_{3p}).$$

By [Liv95], we have the epsilon factors $\epsilon(E_{9p}) = +1$ and $\epsilon(E_p) = -1$, and hence the epsilon factor $\epsilon(E_9, \chi) = -1$. For a quaternion algebra $B/\mathbb{A}$, we define its ramification index $\epsilon(B_v) = +1$ for any place $v$ of $\mathbb{Q}$ if the local component $B_v$ is split and $\epsilon(B_v) = -1$ otherwise.

**Proposition 4.2.** The incoherent quaternion algebra $B$ over $\mathbb{A}$, which satisfies

$$\epsilon(1/2, \pi_v, \chi_v) = \chi_v(-1)\epsilon_v(B)$$

for all places $p$ of $\mathbb{Q}$, is only ramified at the infinity place.

*Proof.* Since $\pi$ is unramified at finite places $v \dagger 3$ and $\chi$ is unramified at finite places $v \neq 3p$ and $p$ is split in $K$, by [Gro88, Proposition 6.3], we get $\epsilon(1/2, \pi_v, \chi_v) = +1$ for all finite $v \neq 3$. Again by [Gro88, Proposition 6.5], we also know that $\epsilon(1/2, \pi_\infty, \chi_\infty) = -1$. Since $\epsilon(1/2, \pi, \chi) = -1$, we see $\epsilon(1/2, \pi_3, \chi_3) = +1$. Since $\chi$ is a cubic character, $\chi_v(-1) = 1$ for any $v$. Hence $B$ is only ramified at the infinity place. \hfill \qed

Let $B^\times_f = \text{GL}_2(A_f)$ be the finite part of $B^\times$. For any open compact subgroup $U \subset B^\times_f$, the Shimura curve $X_U$ associated to $B$ of level $U$ is the usual modular curve with complex uniformization

$$X_U(\mathbb{C}) = \text{GL}_2(\mathbb{Q})^+ \backslash \left( \mathcal{H} \bigcup \mathbb{P}^1(\mathbb{Q}) \right) \times \text{GL}_2(A_f)/U.$$ 

Let

$$\pi_{E_9} = \lim_{\longrightarrow U} \text{Hom}_{\xi_U}^0(X_U, E_9),$$

where $\text{Hom}_{\xi_U}^0(X_U, E_9)$ denotes the morphisms in $\text{Hom}_{\mathbb{Q}}(X_U, E_9) \otimes \mathbb{Q}$ using the Hodge class $\xi_U$ as a base point. Then $\pi_{E_9}$ is an automorphic representation of $B^\times$ over $\mathbb{Q}$ and $\pi$ is the Jacquet-Langlands correspondence of $\pi_{E_9} \otimes \mathbb{Q} \otimes \mathbb{C}$ on $\text{GL}_2(\mathbb{A})$. By Proposition 4.2 and a theorem of Tunell-Saito [YZZ13, Theorem 1.4.1], the space

$$\text{Hom}_{\mathbb{A}}(\pi_{E_9} \otimes \chi, \mathbb{C}) \otimes \text{Hom}_{\mathbb{A}}(\pi_{E_9} \otimes \chi^{-1}, \mathbb{C})$$

is one-dimensional with a canonical generator $\beta^0_\chi = \otimes_\chi \beta^0_v$ where, for each place $v$ of $\mathbb{Q}$, the bilinear form

$$\beta^0_v : \pi_{E_9, v} \otimes \pi_{E_9, v} \rightarrow \mathbb{C}$$
Let \( \phi \) be the Petersson norm of \((4.2) \Omega \)

\[
\text{For more details we refer to [YZZ13, Section 1.4] and [CST14, Section 3].}
\]

The elliptic curve \( E_9 \) has conductor \( 3^5 \) and let \( f : X_0(3^5) \to E_9 \) be a nontrivial modular parametrization which sends the infinity cusp \([\infty]\) to the zero element \(O\). Let

\[
\mathcal{R} = \left( \hat{\mathbb{Z}} \sqcup \hat{\mathbb{Z}} \right) \subset \mathbb{B}_f(\hat{\mathbb{Z}}) = M_2(\hat{\mathbb{Z}})
\]

be the Eichler order of discriminant \( 3^5 \). Then \( U_9(3^5) = \mathcal{R}^\times \), and by the newform theory [Cas73], the invariant subspace \( \pi_{E_9}^{R,\times} \) has dimension 1 and is generated by \( f \).

**Proposition 4.3.** The modular parametrization \( f : X_0(3^5) \to E_9 \) is a test vector for the pair \((\pi_{E_9}, \chi)\), i.e. \( \beta_0(f, f) \neq 0 \).

**Proof.** Let \( \mathcal{R}' \) be the admissible order for the pair \((\pi_{E_9}, \chi)\) in the sense of [CST14, Definition 1.3]. Since \( \mathcal{R}' \) and \( \mathcal{R} \) only differs at 3. It suffices to verify that \( \beta_3(f_3, f_3) \neq 0 \). We delay this verification to the next subsection Proposition 4.6. \( \square \)

Let \( \omega_{E_n} \) be the invariant differential on the minimal model of \( E_n \). Define the minimal real period \( \Omega_n \) of \( E_n \) by

\[
\Omega_n = \int_{E_n(\mathbb{R})} |\omega_{E_n}|.
\]

By [ZK87, Formula (9)], we have

\[
\Omega_p \Omega_{3p^2} = (3p)^{-1} \Omega_9^2
\]

Using Sage we compute that \( \{\Omega_9, \Omega_9 \cdot (\frac{1}{2} + \sqrt{-3} \times \frac{1}{2})\} \) is a \( \mathbb{Z} \)-basis of the period lattice \( L \) of the minimal model of \( E_9 \). So

\[
\sqrt{3} \Omega_9^2 = 2 \int_{\mathbb{C}/L} dxdy = \int_{E(\mathbb{C})} |\omega_{E_9} \wedge \omega_{E_9}| = \frac{1}{6} \cdot 8\pi^2 (\phi, \phi)_{\Gamma_0(3^5)},
\]

where \( \phi \) is the newform of level \( 3^5 \) and weight 2 associated to \( E_9 \), and \((\phi, \phi)_{\Gamma_0(3^5)} \) is the Petersson norm of \( \phi \) defined by

\[
(\phi, \phi)_{\Gamma_0(3^5)} = \int_{\Gamma_0(3^5)} |\phi(z)|^2 dxdy, \quad z = x + iy.
\]

Recall \( \tau = (2p\omega - 9)/(9p\omega - 36) \in \mathcal{H} \) and let \( P_1 = [\tau, 1]_{\Gamma_0(3^5)} \) be the CM point on \( X_0(3^5)(H_{3p}) \) and note that \( f(P_1) = P_0 \). Define the Heegner point

\[
R_1 = Tr_{H_{3p}/L(3,p)} P_0 \in E_9(L(3,p)).
\]

**Theorem 4.4.** For primes \( p \equiv 4, 7 \mod 9 \), we have the following explicit formula of Heegner points:

\[
\frac{L'(1, E_p) L(1, E_{3p^2})}{\Omega_p \Omega_{3p^2}} = 2^\alpha \cdot 9 \cdot \hat{\mathcal{h}}_{Q}(R_1),
\]

where \( \alpha = 0 \) if \( p \equiv 4 \mod 9 \) and \( \alpha = -1 \) if \( p \equiv 7 \mod 9 \).
Proof. Let \( R' \) be the admissible order for the pair \((\pi_{E_9}, \chi)\) and let \( f' \neq 0 \) be a test vector in \( V(\pi_{E_9}, \chi) \) which is defined in [CST14, Definition 1.4]. The newform \( f \) only differs from \( f' \) at the local place 3. Define the Heegner cycle

\[
P^0_\chi(f) = \frac{\#\text{Pic}(O_p)}{\text{Vol}(K^\times/K^\times \mathbb{Q}^\times, dt)} \int_{K^\times \mathbb{Q}^\times \setminus \mathbb{R}^\times} f(P_1)^{\alpha} \chi(t) dt,
\]

and define \( P^0_{\chi^{-1}}(f) \) similarly as in [CST14, Theorem 1.6]. It follows from Proposition 4.6 that

\[
\frac{\rho^0_3(f'_3, f'_3)}{\rho^0_3(f_3, f_3)} = 2^{\alpha+2},
\]

where \( \alpha = 0 \) if \( p \equiv 4 \mod 9 \) and \( \alpha = -1 \) if \( p \equiv 7 \mod 9 \). By [CST14, Theorem 1.6], We have

\[
L'(1, E_9, \chi) = 2^{\alpha+1} \cdot \frac{(8\pi^2) \cdot (\phi, \phi)_{\Gamma_0(3^5)}}{\sqrt{3p} \cdot (f, f)_{R'}} \cdot \langle P^0_\chi(f), P^0_{\chi^{-1}}(f) \rangle_{K, K},
\]

where \((\cdot, \cdot)_{R'}\) is the the pairing on \( \pi_{E_9} \times \pi_{E_9}^\vee \) defined as in [CST14, page 789], and \((\cdot, \cdot)_{K, K}\) is a pairing from \( E_9(K)_\mathbb{Q} \times_K E_9(K)_\mathbb{Q} \) to \( \mathbb{C} \) such that \((\cdot, \cdot)_{K, K} = \text{Tr}_{\mathbb{C}/\mathbb{R}}(\cdot, \cdot)_{K, K}\) is the Neron-Tate height over the base field \( K \), see [CST14, page 790]. In our case, by [CST14, Lemma 2.2, Lemma 3.5],

\[
(f, f)_{R'} = \frac{\text{Vol}(X_{R^\times})}{\text{Vol}(X_{R^\times})} \deg f = 6 \cdot \frac{\text{Vol}(X_{R^\times})}{\text{Vol}(R^\times)} = 4.
\]

So, we get

\[
(4.3) \quad L'(1, E_9, \chi) = 2^{\alpha-1} \frac{(8\pi^2) \cdot (\phi, \phi)_{\Gamma_0(3^5)}}{\sqrt{3p^2}} \cdot \langle P^0_\chi(f), P^0_{\chi^{-1}}(f) \rangle_{K, K}.
\]

On the other hand

\[
P^0_\chi(f) = \frac{\#\text{Pic}(O_p)}{\#\text{Pic}(O_{9p})} \sum_{t \in \text{Pic}(O_{9p})} f(P_1)^{\alpha} \chi(t).
\]

Since

\[
\frac{\#\text{Pic}(O_p)}{\#\text{Pic}(O_{9p})} = [K^\times \mathbb{Q}^\times : K^\times \mathbb{Q}^\times]^{-1} = \frac{1}{9},
\]

we have

\[
P^0_\chi(f) = \frac{1}{9} \sum_{t \in \text{Pic}(O_{9p})} f(P_1)^{\alpha} \chi(t).
\]

If we put

\[
R_2 = \sum_{\sigma \in \text{Gal}(H_{9p}/L_{3p})} f(P_1)^{\sigma} \chi(\sigma) = 3R_1 \in E_0(L_{3p}),
\]

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then
\[
\langle P^0_\chi(f), P^0_{\chi^{-1}}(f) \rangle_{K,K} = \frac{1}{92} \left( \sum_{\sigma \in \text{Gal}(L_{(3p)}/K)} R_2^0 \chi^{-1}(\sigma), \sum_{\sigma \in \text{Gal}(L_{(3p)}/K)} R_2^0 \chi(\sigma) \right)_{K,K}
\]
\[
= \frac{1}{27} \left( R_2, \sum_{\sigma \in \text{Gal}(L_{(3p)}/K)} R_2^\sigma \chi(\sigma) \right)_{K,K}
\]
\[
= \frac{1}{27} \left( \langle R_2, R_2 \rangle_{K,K} - \langle R_2, R_2^\sigma \chi^{1+3\omega_3} \rangle_{K,K} \right),
\]
where \( i = 1 \) if \( p \equiv 4 \mod 9 \) and \(-1 \) if \( p \equiv 7 \mod 9 \). By Corollary 2.3, \( R_2^{\omega_3 \chi^{1+3\omega_3}} = [\omega] R_2 \),

\[
\langle R_2, R_2^\sigma \chi^{1+3\omega_3} \rangle_{K,K} = \frac{1}{2} \left( \hat{h}_K([1 + \omega] R_2) - \hat{h}_K([\omega] R_2) - \hat{h}_K(R_2) \right).
\]
Since \(|1 + \omega| = |\omega| = 1\), by definition, \( \hat{h}_K([1 + \omega] R_2) = \hat{h}_K([\omega] R_2) = \hat{h}_K(R_2) \). Then

\[
\langle R_2, R_2^\sigma \chi^{1+3\omega_3} \rangle_{K,K} = -\frac{1}{2} \hat{h}_K(R_2),
\]
and hence

\[
\langle P^0_\chi(f), P^0_{\chi^{-1}}(f) \rangle_{K,K} = \frac{1}{18} \hat{h}_K(R_2) = \frac{1}{9} \hat{h}_Q(R_2) = \hat{h}_Q(R_1).
\]

Finally, combining (4.1)-(4.4), we get

\[
\frac{L'(1, E_p) L(1, E_{3p^2})}{\Omega_p \Omega_{3p^2}} = 2^\alpha \cdot 9 \cdot \hat{h}_Q(R_1).
\]

Recall there is an isomorphism

\[
\phi : E_9 \longrightarrow E_1, \quad (x, y) \mapsto \left( (\sqrt{3})^2 x, 3y \right),
\]
and we have the following commutative diagram:

\[
E_9(H_{9p})^{\sigma_1, 3\omega_3 = \omega^2} \xrightarrow{\text{Tr}_{H_{9p}/L_{(3p)}}} E_9(L_{(3p)})^{\sigma_1, 3\omega_3 = \omega^2}.
\]

In particular, we have \( \phi(R_1) = R \), and hence the following

**Corollary 4.5.** For primes \( p \equiv 4, 7 \mod 9 \), we have

\[
\frac{L'(1, E_p) L(1, E_{3p^2})}{\Omega_p \Omega_{3p^2}} = 2^\alpha \cdot 9 \cdot \hat{h}_Q(R),
\]

where \( \alpha = 0 \) if \( p \equiv 4 \mod 9 \) and \( \alpha = -1 \) if \( p \equiv 7 \mod 9 \).

4.2. Local computations.
4.2.1. Reduction to local integrals. Let \( R' \) be the admissible order for the pair \((\pi_{E_9}, \chi)\) and let \( f' \in V(\pi_{E_9}, \chi) \) be the admissible test vector in the sense of [CST14]. Then Theorem 1.5 in [CST14] gives the explicit height formula for the Heeger cycle associated to the morphism \( f' \) and character \( \chi \). In order to get the explicit height formula of the Heegner cycles \( P^0(\chi) \) associated to the morphism \( f \in \pi_{E_9} \times \chi \) and the character \( \chi \), we will apply the variation theorem [CST14, Theorem 1.6], where

\[
\mathcal{R} = \left( \frac{\hat{\mathbb{Z}}}{3^a \cdot \mathbb{Z}}, \frac{\hat{\mathbb{Z}}}{\mathbb{Z}} \right) \subset \mathbb{B}_f(\hat{\mathbb{Z}}) = M_2(\hat{\mathbb{Z}})
\]

is the Eichler order of discriminant \( 3^a \). Note the orders \( R' \) and \( R \) only differs at the 3-local place, and as a consequence, the morphisms \( f \) and \( f' \) only differs at the 3-local place in the sense of [CST14, page 2534]. Then by [CST14, Theorem 1.6], Theorem 4.4 follows immediately from the following

Proposition 4.6. Let \( f_3 \) resp. \( f'_3 \) denote the 3-adic component of \( f \) resp. \( f' \). We have

\[
\frac{\beta_3^0(f'_3, f'_3)}{\beta_3^0(f_3, f_3)} = 2^{\alpha + 2},
\]

where \( \alpha = 0 \) if \( p \equiv 4 \mod 9 \) and \( \alpha = -1 \) if \( p \equiv 7 \mod 9 \).

Since \( f'_3 \) is a \((K^\times_3, \chi_3^{-1})\)-eigenform, it is easy to see the local integral

\[
\beta_3^0(f'_3, f'_3) = \text{Vol}(K^\times_3 / Q^\times_3).
\]

Then it suffices to compute \( \beta_3^0(f_3, f_3) \). Since \( \chi \) has conductor \( 9p \), we have

\[
\beta_3^0(f_3, f_3) = \text{Vol}(K^\times_3 / Q^\times_3) \sum_{t \in K^\times_3 / Q^\times_3 \mathcal{O}_{9p,3}^\times} \Phi(t) \chi_3(t)
\]

\[
= \frac{\text{Vol}(K^\times_3 / Q^\times_3)}{18} \sum_{t \in S \cup S'} \Phi(t) \chi_3(t),
\]

where

\[
\Phi(t) = \frac{(\pi_3(t)f_3, f_3)}{(f_3, f_3)}.
\]

and

\[
S = \{1 + y\sqrt{-3} | y \in \mathbb{Z}/9\mathbb{Z}\}, \quad S' = \{3y + \sqrt{-3} | y \in \mathbb{Z}/9\mathbb{Z}\}.
\]

Then \( S \cup S' \) is a complete system of representatives of \( K^\times_3 / Q^\times_3 \mathcal{O}_{9p,3}^\times \). The rest of the section is devoted to the computation of the local matrix coefficients \( \Phi(t), t \in S \cup S' \).

4.2.2. Supercuspidal representation. We lay out the basic facts about supercuspidal representations we need in this subsection, following [Sai93] and [Hu16]. Let \( \psi \) be the additive character such that \( \psi(x) = e^{2\pi i \lambda(x)} \) where \( \iota : \mathbb{Q}_3 \to \mathbb{Q}_3 / \mathbb{Z}_3 \subset \mathbb{Q} / \mathbb{Z} \) is the map given by \( x \mapsto -x \mod \mathbb{Z}_3 \). Let \( dx \) be the Haar measure on \( \mathbb{Q}_3 \) which is self-dual with respect to \( \psi \), and we fix a Haar measure \( d^*x \) on \( \mathbb{Q}_3^\times \) such that \( \text{Vol}(\mathbb{Z}_3^\times) = 1 \). The local representation \( \pi_3 = \pi_{E_9,3} \) is supercuspidal, and the Kirillov model \( \mathcal{H}(\pi_3, \psi) \) is the unique realization of \( \pi_3 \) on the Schwartz function space \( S(\mathbb{Q}_3^\times) \) such that

\[
\pi \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) \phi(x) = \psi(bx)\phi(ax), \quad \phi \in S(\mathbb{Q}_3^\times).
\]
The GL$_2(\mathbb{Q}_3)$-invariant pairing on $\pi_3 \times \pi_3$ is given by

$$(\phi_1, \phi_2) = \int_{\mathbb{Q}_3^x} \phi_1(x) \overline{\phi_2(x)} d^x x.$$ 

Put

$$1_{\nu,n}(x) = \begin{cases} \nu(u), & \text{if } x = u3^n \text{ for } u \in \mathbb{Z}_3^x, \\ 0, & \text{otherwise}, \end{cases}$$

where $\nu$ is a character of $\mathbb{Z}_3^x$. Then $\{1_{\nu,n}(x)\}_{\nu,n}$ is an orthogonal basis of $S(\mathbb{Q}_3^x)$ with respect to the paring $(\cdot, \cdot)$. For $\phi(x) \in S(\mathbb{Q}_3^x)$, we have the Fourier expansion

$$\phi(x) = \sum_{\nu} \sum_n \hat{\phi}_n(\nu^{-1}) 1_{\nu,n},$$

where

$$\hat{\phi}_n(\nu^{-1}) = \int_{\mathbb{Z}_3^x} \phi(3^n x) \nu^{-1}(x) d^x x.$$ 

The action of $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $1_{\nu,n}$ can be described as follows:

$$\pi_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} 1_{\nu,n} = C_\nu 1_{\nu^{-1},-n+n_\nu},$$

where

$$C_\nu = \epsilon(1/2, \pi \otimes \nu^{-1}, \psi),$$

and $n_\nu = -\max\{5, 2i\}$, where $i$ is the conductor of $\nu$. From $w^2 = -1$, one can see that

$$n_\nu = n_{\nu^{-1}}, \quad C_\nu C_{\nu^{-1}} = 1.$$ 

It is well known that $1_{1,0}$ is the normalized local new form, and hence, is parallel to $f_3$.

4.2.3. Local root numbers. In this subsection, we compute the local root numbers $C_\nu$ for characters $\nu$ of conductor $\leq 3$. Let $\theta : K^\times \backslash A_K^\times \to \mathbb{C}^\times$ be the unitary Hecke character associated to the CM elliptic curve $E_{9/K}$, i.e. the product of the algebraic Hecke character associated to $E_{9/K}$ and the norm character $| \cdot |_{K}^{1/2}$. Then $\theta$ has conductor $9\mathcal{O}_K$. For any place $v$ of $K$, let $\theta_v$ be the local component of $\theta$ at the place $v$. Note

$$\mathcal{O}_{K,3}^\times/(1 + 9\mathcal{O}_{K,3}) \simeq \langle \pm 1 \rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle 1 + \sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 - \sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 + 3\sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}}.$$ 

Lemma 4.7. The local character $\theta_3$ is given explicitly by

$$\theta_3(-1) = -1, \quad \theta_3(1 + \sqrt{-3}) = -\frac{1 - \sqrt{-3}}{2},$$

$$\theta_3(1 - \sqrt{-3}) = -\frac{1 + \sqrt{-3}}{2}, \quad \theta_3(1 + 3\sqrt{-3}) = -\frac{1 + \sqrt{-3}}{2}.$$

Proof. Note

$$\theta_\infty(-1)\theta_3(-1) = 1, \quad \theta_\infty(-1) = -1.$$
Let \( p = (\pi) \) be a prime of \( K \) relatively prime to 6, with unique generator \( \pi \equiv 2 \mod 3 \). By [Sil92, Example 10.6], we have
\[
\theta(p) = -N_p^{-1/2} \left( \frac{-3}{\pi} \right)_6 \pi.
\]
If \( p = (5) \), then
\[
\theta(5) = -\left( \frac{-3}{5} \right)_6 = -1.
\]
Let \( v, \varpi \) be the places above 7 with corresponding prime ideals \( p_v = \left( \frac{1 + 3\sqrt{-3}}{2} \right) \) and \( p_\varpi = \left( \frac{1 - 3\sqrt{-3}}{2} \right) \). Then
\[
\theta_v(1 + 3\sqrt{-3}) = -\sqrt{7} \cdot \left( \frac{-3}{1+3\sqrt{-3}} \right)_6 \cdot 1 + 3\sqrt{-3} = -\sqrt{7} \cdot -1 \cdot \sqrt{-3} \cdot 1 + 3\sqrt{-3}.
\]
By
\[
\theta_\infty(10)\theta_2(10)\theta_3(10)\theta_5(10) = 1,
\]
we get \( \theta_2(2) = -1 \). Since \( \theta_2 \) is unramified, we see \( \theta_2(1 \pm \sqrt{-3}) = -1 \). By
\[
\theta_\infty(1 \pm \sqrt{-3})\theta_2(1 \pm \sqrt{-3})\theta_3(1 \pm \sqrt{-3}) = 1, \quad \theta_\infty(1 \pm \sqrt{-3}) = \frac{2}{1 \pm \sqrt{-3}},
\]
we get
\[
\theta_3(1 \pm \sqrt{-3}) = -\frac{1 \mp \sqrt{-3}}{2}.
\]
By
\[
\theta_\infty(1 + 3\sqrt{-3})\theta_2(1 + 3\sqrt{-3})\theta_3(1 + 3\sqrt{-3})\theta_v(1 + 3\sqrt{-3}) = 1, \quad \theta_\infty(1 + 3\sqrt{-3}) = \frac{2\sqrt{7}}{1 + 3\sqrt{-3}},
\]
we get
\[
\theta_3(1 + 3\sqrt{-3}) = -\frac{1 + \sqrt{-3}}{2}.
\]
\[\square\]

Let \( \psi_{K_3}(x) = \psi_3 \circ \text{Tr}_{K_3/Q_3}(x), x \in K_3 \), be an additive character of \( K_3 \). For an arbitrary character \( \chi : K_3^\times \rightarrow \mathbb{C}^\times \), denote by \( \pi_\chi \) the representation of \( GL_2(Q_3) \) constructed from \( \chi \) via the Weil representation. Let \( \eta_3 \) be the local character associated to the field extension \( K_3/Q_3 \) under local class field theory. By [JL70, Theorem 4.7], we have
\[
(4.6) \quad \epsilon(1/2, \pi_\chi, \psi_{K_3}) = \epsilon(1/2, \eta_3, \psi_3)\epsilon(1/2, \chi, \psi_{K_3}).
\]
And it is straightforward to compute
\[
\epsilon(1/2, \eta_3, \psi_3) = -i.
\]
If \( \chi \) is ramified, then
\[
\epsilon(s, \chi, \psi_{K_3}) = \int_{\mathbb{C}_{K_3} \setminus \mathbb{C}_{K_3}^{\times}} x^{-1}(y) |y|^{-s} \psi_{K_3}(y) dy,
\]
where $|\cdot|$ is the normalized norm on $K_3$ and $dy$ is the self-dual Haar measure on $K_3$ w.r.t. $\psi_{K_3}$.

The supercuspidal representation $\pi_3 = \pi_{\theta_3}$ is the representation of $GL_2(\mathbb{Q}_3)$ constructed from $\theta_3$ via the Weil representation. For any character $\nu : \mathbb{Q}_3^\times \to \mathbb{C}^\times$, $\pi_3 \otimes \nu = \pi_{\theta_3\nu_{K_3}}$, $\nu_{K_3} = \nu \circ N_{K_3/\mathbb{Q}_3}$. The character $\theta_3$ has conductor 4 and $\psi_{K_3}$ has conductor 1. Let $\nu : \mathbb{Q}_3^\times \to \mathbb{C}^\times$ be a character of conductor $\leq 2$, then $\nu_{K_3}$ has conductor $\leq 3$. Hence $\theta_3\nu_{K_3}^{-1}$ has conductor 4. Then

\[(4.7) \quad \epsilon(1/2, \theta_3\nu_{K_3}^{-1}, \psi_{K_3}) = \theta_3\nu_{K_3}^{-1}(\sqrt{-3})^{5} \cdot 3^{-2} \sum_{\alpha \in \mathcal{O}_{K_3,3}^\times/(1+9\mathcal{O}_{K,3})} (\theta_3\nu_{K_3}^{-1})^{-1}(\alpha)\psi_{K_3} \left( \frac{\alpha}{\sqrt{-3}} \right).\]

Let $\nu : \mathbb{Q}_3^\times \to \mathbb{C}^\times$ be a character of conductor 3, then $\nu_{K_3} = \nu \circ N_{K_3/\mathbb{Q}_3}$ has conductor 5. Hence $\theta_3\nu_{K_3}^{-1}$ has conductor 5. Then

\[(4.8) \quad \epsilon(1/2, \theta_3\nu_{K_3}^{-1}, \psi_{K_3}) = \theta_3(\sqrt{-3})^{6}3^{3-\frac{2}{3}} \sum_{\alpha \in \mathcal{O}_{K_3,3}^\times/(1+9\sqrt{-3}\mathcal{O}_{K,3})} (\theta_3\nu_{K_3})^{-1}(\alpha)\psi_{K_3} \left( -\frac{\alpha}{27} \right).\]

We have

$\mathbb{Z}_3^\times / (1+9\mathbb{Z}_3) \simeq \langle -1 \rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle -2 \rangle^{\mathbb{Z}/3\mathbb{Z}}$.

Let $\nu_2 : \mathbb{Z}_3^\times \to \mathbb{C}^\times$ be the character of conductor 2 given by

$\nu_2(-1) = 1$ and $\nu_2(-2) = \psi \left( \frac{1}{3} \right)$.

The 4 characters of $\mathbb{Z}_3^\times$ of conductor 2 are $\nu_i^1\nu_2^{\pm 1}$ with $i = 0, 1$. Also

$\mathbb{Z}_3^\times / (1+27\mathbb{Z}_3) \simeq (\mathbb{Z}/27\mathbb{Z})^\times$

is cyclic with generator 2. Let $\nu_3 : \mathbb{Z}_3^\times \to \mathbb{C}^\times$ be the character of conductor 3 given by

$\nu_3(-1) = 1, \quad \nu_3(-2) = \psi \left( \frac{1}{9} \right)$.

Then $\nu_3^3 = \nu_2$. There are 12 characters of conductor 3, namely, $\nu_i^j\nu_3^j$ with $i = 0, 1$ and $j \in (\mathbb{Z}/9\mathbb{Z})^\times$.

From the decomposition

$\mathcal{O}_{K,3}^\times/(1+9\sqrt{-3}\mathcal{O}_{K,3}) \simeq \langle -1 \rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle 1+\sqrt{-3} \rangle^{\mathbb{Z}/9\mathbb{Z}} \times \langle (1+\sqrt{-3})(1-\sqrt{-3}) \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1+3\sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}}$,

and the formulae (4.6), (4.7) and (4.8), a straight-forward computation gives the following table of local root numbers $C_\nu$ with $\nu$ of conductor $\leq 3$.

| $\nu$ | $\nu_1^1\nu_3$ | $\nu_1^0\nu_3$ | $\nu_1^1$ | $\nu_1^0$ | $\nu_1^3$ | $\nu_1^2$ | $\nu_1^3$ | $\nu_1^2$ | $\nu_1^3$ | $\nu_1^2$ | $\nu_1^3$ |
|------|----------------|----------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $C_\nu$ | $\psi \left( \frac{1}{11} \right)$ | $\psi \left( \frac{1}{29} \right)$ | $\psi \left( \frac{21}{31} \right)$ | $\psi \left( \frac{21}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ | $\psi \left( \frac{14}{31} \right)$ |
4.2.4. *Local matrix coefficients*. Note that $\Phi(t)$ is bi-invariant under $U_0(3^5)_3$ and hence induces a function on the double cosets $ZU_0(3^5)_3\backslash GL_2(\mathbb{Q}_3)/U_0(3^5)_3$, where $Z$ denotes the center of $GL_2(\mathbb{Q}_3)$. Consider the natural projection map

$$
\text{pr} : GL_2(\mathbb{Q}_3) \rightarrow ZU_0(3^5)_3 \backslash GL_2(\mathbb{Q}_3)/U_0(3^5)_3.
$$

Under the embedding $\rho : K \hookrightarrow M_2(\mathbb{Q})$, we write

$$
\sqrt{-3} = \begin{pmatrix} 4p + 17 + 72/p & -8p/9 - 4 - 18/p \\ 18p + 72 + 288/p & -4p - 17 - 72/p \end{pmatrix} = \begin{pmatrix} a & 3^{-2}b \\ 3^3c & -a \end{pmatrix}
$$

with $a \equiv 0 \mod 3$, $b \equiv p \mod 9$ and $c \equiv -1 \mod 9$. Then $\det(\sqrt{-3}) = -a^2 - 3bc = 3$. We have the following decomposition of matrix:

$$
\begin{pmatrix} a & 3^ib \\ 3^i c & d \end{pmatrix} = \begin{pmatrix} (ac^{-1} - bd^{-1}3^{i+i}) & d^{-1}b3^{i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3^i & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.
$$

Let $i < 5$ be an integer. From

$$
\begin{pmatrix} 1 & 0 \\ 3^i & 1 \end{pmatrix} = -w \begin{pmatrix} 1 & -3^i \\ 0 & 1 \end{pmatrix} w,
$$

we have

$$
\pi_3 \left( \begin{pmatrix} 1 & -3^i \\ 0 & 1 \end{pmatrix} w \right) 1_{1,0}(x) = C_1 \psi^{-}(3^i x) 1_{1,5}(x) = C_1 \sum_{c(\nu) = 5 - i} \hat{\psi}^{-i-5}(\nu^{-1}) 1_{\nu^{-1}-5}(x),
$$

$$
\pi_3 \left( \begin{pmatrix} 1 & 0 \\ 3^i & 1 \end{pmatrix} \right) 1_{1,0} = \sum_{c(\nu) = 5 - i} C_1 C_{\nu} \hat{\psi}^{-i-5}(\nu^{-1}) 1_{\nu^{-1}, \min(0, 2i-5)},
$$

that is

$$
\begin{align*}
\pi_3 \left( \begin{pmatrix} 1 & 0 \\ 3^i & 1 \end{pmatrix} \right) 1_{1,0} &= \begin{cases} 
\sum_{c(\nu) = 5 - i} C_1 C_{\nu} \hat{\psi}^{-i-5}(\nu^{-1}) 1_{\nu^{-1}, 2i-5}, & \text{if } i \leq 2; \\
\sum_{c(\nu) = 5 - i} C_1 C_{\nu} \hat{\psi}^{-i-5}(\nu^{-1}) 1_{\nu^{-1}, 0}, & \text{if } i = 3, \\
C_1 \psi^{-1}(1) 1_{1,0} + C_1 C_{\nu} \hat{\psi}^{-1}(\nu_{1}) 1_{\nu_{1}, 0}, & \text{if } i = 4.
\end{cases}
\end{align*}
$$

*Formula of $\Phi(1 + y \sqrt{-3})$.*

$$
1 + y \sqrt{-3} = \begin{pmatrix} 1 + ay & 3^{-2}by \\ 3^i cy & 1 - ay \end{pmatrix}.
$$

From the decomposition (4.10) we see

$$
\text{pr}(1 + y \sqrt{-3}) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & y = 0; \\
\begin{pmatrix} 1 & 3^{-2}by \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3^3 & 1 \end{pmatrix}, & y \in (\mathbb{Z}/9\mathbb{Z})^\times; \\
\begin{pmatrix} 1 & 3^{-1}by \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix}, & y = 3, 6;
\end{cases}
$$
where
\[
b_y = \begin{cases} 
\left( \frac{1 - a^2 y^2}{b c y^2} - 3 \right)^{-1}, & y \in (\mathbb{Z}/9\mathbb{Z})^\times; \\
\left( \frac{9(1 - a^2 y^2)}{b c y^2} - 3 \right)^{-1}, & y = 3, 6.
\end{cases}
\]

Note that \(b_y \equiv b_{-y} \mod 9\) if \(y \in (\mathbb{Z}/9\mathbb{Z})^\times\) and \(b_3 \equiv b_6 \mod 3\), we have
\[
\Phi(1 + y\sqrt{-3}) = \Phi(1 - y\sqrt{-3}), \quad y \in \mathbb{Z}/9\mathbb{Z}.
\]

First we note \(\Phi(1) = 1\). From (4.11) we see that
\[
\pi_3 \left( \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right) 1_{1,0} = \hat{\psi}_{-1}(1) 1_{1,0} + C_1 C_{\nu_1} \hat{\psi}_{-1}(\nu_1) 1_{\nu_1,0},
\]
and hence
\[
\pi_3 \left( 1 + 3\sqrt{-3} \right) 1_{1,0}(x) = \psi \left( \frac{b_3 x}{3} \right) \left[ \hat{\psi}_{-1}(1) 1_{1,0}(x) + C_1 C_{\nu_1} \hat{\psi}_{-1}(\nu_1) 1_{\nu_1,0}(x) \right].
\]
By noting
\[
\hat{\psi}_n(\nu) = \nu(-1) \hat{\psi}_n \text{ and } b_3 \equiv -1 \mod 3,
\]
wWe have
\[
\Phi(1 + 3\sqrt{-3}) = \hat{\psi}_{-1}(1) 1_{1,0}(x) + \nu_1(b_3) C_1 C_{\nu_1} \hat{\psi}_{-1}(\nu_1) 1_{\nu_1,0}(x)
\]
\[
= \hat{\psi}_{-1}(1)^2 + C_1 C_{\nu_1} \hat{\psi}_{-1}(\nu_1)^2.
\]

By (4.11), we have
\[
\pi_3 \left( \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right) 1_{1,0} = C_1 \sum_{c(\nu) = 2} C_\nu \hat{\psi}_{-2}(\nu^{-1}) 1_{\nu^{-1},0},
\]
and hence
\[
\pi_3(1 + y\sqrt{-3}) 1_{1,0}(x) = \psi \left( \frac{b_y x}{9} \right) \left[ \sum_{c(\nu) = 2} C_\nu \hat{\psi}_{-2}(\nu^{-1}) 1_{\nu^{-1},0}(x) \right].
\]
If \(y \in (\mathbb{Z}/9\mathbb{Z})^\times\), then
\[
\Phi(1 + y\sqrt{-3}) = \sum_{c(\nu) = 2} C_1 C_{\nu^{-1}} \nu(-b_y^{-1}) \hat{\psi}_{-2}(\nu)^2.
\]

Note that \(\mathbb{Z}_3^\times / (1 + 9\mathbb{Z}_3) \simeq \langle -1 \rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle -2 \rangle^{\mathbb{Z}/3\mathbb{Z}}\).

Recall that \(\nu_2 : \mathbb{Z}_3^\times \rightarrow \mathbb{C}^\times\) be the character of conductor 2 given by
\[
\nu_2(-1) = 1 \text{ and } \nu_2(-2) = \psi \left( \frac{1}{3} \right).
\]
The 4 characters of \(\mathbb{Z}_3^\times\) of conductor 2 are \(\nu_i^i\nu_2^{\pm 1}\) with \(i = 0, 1\). Then
\[
\Phi(1 + y\sqrt{-3}) = C_1 C_{\nu_2} \nu_2(-b_y^{-1}) \psi_{-2}(\nu_2)^2 + C_1 C_{\nu_2} \nu_2(-b_y^{-1}) \psi_{-2}(\nu_2^{-1})^2
\]
\[
+C_1 C_{\nu_2}(-b_y^{-1}) \psi_{-2}(\nu_1 \nu_2)^2 + C_1 C_{\nu_2}(-b_y^{-1}) \psi_{-2}(\nu_1 \nu_2^{-1})^2.
\]
Note $b_1^{-1} \equiv -1 \mod 9$, $b_2^{-1} \equiv 2 \mod 9$ and $b_4^{-1} \equiv -4 \mod 9$. We conclude
\[
\sum_{y=1,2,4} \Phi(1 + y\sqrt{-3}) = 0.
\]

**Formula of $\Phi(3y + \sqrt{-3})$.**

\[
3y + \sqrt{-3} = \left(\frac{3y + a}{3^3} 3^{-2b} \right).
\]
We have the decomposition
\[
\begin{aligned}
(3y + a & \quad 3^{-2b}) = \\
&= \left(\frac{9y^2 - a^2 - 3bc}{c(3y-a)} & \quad \frac{b}{3^2(3y-a)}\right) \left(\begin{array}{cc}
1 & 0 \\
0 & 3^3 & 1
\end{array}\right) \left(\begin{array}{cc}
c & 0 \\
0 & 3y - a
\end{array}\right).
\end{aligned}
\]

Let $\alpha = \text{ord}_3(3y - a)$. Then
\[
\begin{aligned}
\text{pr} \left(\begin{array}{cc}
3y + a & 3^{-2b} \\
3^3 & 3y - a
\end{array}\right) &= \left(\frac{9y^2 - a^2 - 3bc}{3^2c(3y-a)} & \quad \frac{b}{3^2(3y-a)}\right) \left(\begin{array}{cc}
1 & 0 \\
0 & 3^{3-\alpha} & 1
\end{array}\right) \\
&= \left(\begin{array}{cc}
A_y & B_y \\
0 & 3^{3-\alpha}
\end{array}\right) \left(\begin{array}{cc}
1 & 0 \\
0 & 3^{3-\alpha}
\end{array}\right)
\end{aligned}
\]
where $A_y = \frac{9y^2 - a^2 - 3bc}{3^{3-\alpha}c(3y-a)}$ and $B_y = \frac{3\alpha b}{3y-a}$ are units. If $\alpha = \text{ord}_3(3y - a) \geq 1$ and $i = 3 - \alpha \leq 2$, then by (4.11)
\[
(4.19) \pi_3(3y + \sqrt{-3})_{1,0}(x) = \psi \left(\frac{B_yx}{3^{\alpha+2}}\right) \sum_{c(\nu) = 5-i} C_1 C_\nu \tilde{\psi}_{i-5}(\nu^{-1})_{1,\nu-1,2i-5} \left(\frac{A_yx}{3^{2\alpha-1}}\right) \\
= \psi \left(\frac{B_yx}{3^{\alpha+2}}\right) \sum_{c(\nu) = \alpha+2} \nu^{-1}(A_y) C_1 C_\nu \tilde{\psi}_{\alpha-2}(\nu^{-1})_{1,\nu-1,0}(x)
\]

Then
\[
(4.20) \Phi(3y + \sqrt{-3}) = \sum_{c(\nu) = \alpha+2} C_1 C_\nu \nu^{-1}(\tilde{b}_y) \tilde{\psi}_{\alpha-2}(\nu^{-1})^2,
\]
where
\[
(4.21) \tilde{b}_y = \frac{9y^2 - a^2}{3bc} - 1.
\]

Then we see
\[
(4.22) \Phi(3y + \sqrt{-3}) = \Phi(-3y + \sqrt{-3}), \quad y \in \mathbb{Z}/9\mathbb{Z}.
\]

**Remark 4.1.** Although (4.20) is a general formula, if we use it to compute the $\Phi(3y + \sqrt{-3})$, it will be very complicated since it involves high-level characters. In the following, we will use some decomposition trick to get some formula involving small-level characters.

**Case** $p \equiv 4 \mod 9$. Then $a \equiv -3 \mod 9$. By (4.19),
\[
\begin{aligned}
(4.20) \Phi(3y + \sqrt{-3}) &= \sum_{c(\nu) = \alpha+2} C_1 C_\nu \nu^{-1}(-\tilde{b}_y) \tilde{\psi}_{\alpha-2}(\nu^{-1})^2 \\
&= \sum_{c(\nu) = \alpha+2} C_1 C_\nu \psi_{\alpha-2}(\nu^{-1})^2.
\end{aligned}
\]
\[
\pi_3(\sqrt{-3})_{1,0}(x) = \psi \left( \frac{B_0x}{3^{\alpha+2}} \right) \sum_{c(\nu)=3} C_1 C_\nu \hat{\psi}_{-3}(\nu^{-1}) 1_{\nu^{-1},-1} \left( \frac{A_0x}{3^{2\alpha-1}} \right) \\
= \psi \left( \frac{B_0x}{3^3} \right) \sum_{c(\nu)=3} \nu^{-1}(A_0) C_1 C_\nu \hat{\psi}_{-3}(\nu^{-1}) 1_{\nu^{-1},0}(x).
\]

\[
\pi_3(\sqrt{-3})_{1,0}(x) = \psi \left( -\frac{B_0x}{3^3} \right) \sum_{c(\nu)=3} \nu(A_0) C_1 C_\nu \hat{\psi}_{-3}(\nu) 1_{\nu,0}(x) \\
= \psi \left( -\frac{B_0x}{3^3} \right) \sum_{c(\nu)=3} \nu^{-1}(-A_0) C_1 C_\nu \hat{\psi}_{-3}(\nu^{-1}) 1_{\nu^{-1},0}(x).
\]

Then

\[
\Phi(\sqrt{-3}) = \langle \pi_3(\sqrt{-3})_{1,0}, 1_{1,0} \rangle = \sum_{c(\nu)=3} \nu^{-1}(-A_0 B_0^{-1}) C_1 C_\nu \hat{\psi}_{-3}(\nu^{-1})^2.
\]

If \( y \in (\mathbb{Z}/9\mathbb{Z})^\times \), then by \((4.14)\),

\[
\pi_3(1 - y \sqrt{-3})_{1,0}(x) = \psi \left( \frac{b_{-y}x}{9} \right) \left[ C_1 \sum_{c(\nu)=2} C_\nu \hat{\psi}_{-2}(\nu^{-1}) 1_{\nu^{-1},0}(x) \right].
\]

Hence

\[(*) \quad \Phi(3y + \sqrt{-3}) = \langle \pi_3(1 - y \sqrt{-3})_{1,0}, \pi_3(\sqrt{-3})_{1,0} \rangle
\]

\[
= \int_{\mathbb{Z}_3^x} \left( \sum_{c(\mu)=3 \atop c(\nu)=2} C_\mu C_\nu \mu^{-1}(-A_0) \hat{\psi}_{-2}(\nu^{-1}) \hat{\psi}_{-3}(\mu^{-1}) \psi \left( \frac{(-B_0 + 3b_{-y})x}{27} \right) (\mu\nu)^{-1}(x) \right) d^x x
\]

\[
= \sum_{c(\mu)=3 \atop c(\nu)=2} C_\mu C_\nu \mu^{-1}(-A_0)(\mu\nu)(B_0 - 3b_y) \hat{\psi}_{-2}(\nu^{-1}) \hat{\psi}_{-3}(\mu^{-1}) \hat{\psi}_{-3}((\mu\nu))
\]

\[
= \sum_{c(\mu)=3 \atop c(\nu)=2} C_\mu C_{\nu^{-1}} \mu^{-1}(-A_0)(\mu^{-1}\nu^{-1})(B_0 - 3b_y) \hat{\psi}_{-2}(\nu) \hat{\psi}_{-3}(\mu) \hat{\psi}_{-3}((\mu\nu)).
\]

If \( y = 3, 6 \), by \((4.22)\), it suffices to compute \( \Phi(3 \cdot 6 + \sqrt{-3}) \). By \((4.13)\),

\[
\pi_3(1 + 3 \sqrt{-3})_{1,0}(x) = \psi \left( \frac{b_3x}{3} \right) \left[ \hat{\psi}_{-1}(1)_{1,0}(x) + C_1 C_\nu \hat{\psi}_{-1}(\nu_1) 1_{\nu_1,0}(x) \right].
\]
Hence

\[(**)
\Phi(3 \cdot 6 + \sqrt{-3}) = \langle \pi_3(1 - 6\sqrt{-3})1_{1,0}, \pi_3(\sqrt{-3})1_{1,0} \rangle
\]

\[= \sum_{\nu=1,\nu_1} \sum_{c(\mu)=3} C_{\mu} C_{\nu} \mu^{-1}(-A_0) \psi_{-1}(\nu) \psi_{-3}(\mu^{-1}) \int_{\mathbb{Z}/3} \psi \left( \frac{(-B_0 + 9b_3)x}{27} \right) (\mu\nu)^{-1} (x)d^\times x
\]

\[= \sum_{c(\mu)=3} C_{\mu} C_{\nu} \mu^{-1}(-A_0) (\mu\nu) (B_0 - 9b_3) \psi_{-1}(\nu^{-1}) \psi_{-3}(\mu^{-1}) \psi_{-3}((\mu\nu)^{-1})
\]

\[= \sum_{c(\mu)=3} C_{\mu} C_{\nu}^{-1} \mu(-A_0) (\mu^{-1}\nu^{-1}) (B_0 - 9b_3) \psi_{-1}(\nu) \psi_{-3}(\mu) \psi_{-3}((\mu\nu))
\]

Case \( p \equiv 7 \mod 9 \). Then \( 9|a \). When \( y \in (\mathbb{Z}/9\mathbb{Z})^\times \), \( \alpha = \text{ord}_3(3y - a) = 1 \). By (4.20),

\[\Phi(3y + \sqrt{-3}) = \sum_{c(\nu)=3} C_{1} C_{\nu} \nu^{-1}(-\tilde{b}_y) \psi_{-3}(\nu^{-1})^2,
\]

where

\[\tilde{b}_y = \frac{9y^2 - a^2}{3bc} - 1.
\]

When \( y = 0, 3, 6 \), we use the decomposition

\[\Phi(3y + \sqrt{-3}) = \langle \pi_3(1 - y\sqrt{-3})1_{1,0}, \pi_3(\sqrt{-3})1_{1,0} \rangle
\]

\[= \langle \pi_3(1 - (y + 1)\sqrt{-3})1_{1,0}, \pi_3(3 + \sqrt{-3})1_{1,0} \rangle.
\]

By (4.19),

\[\pi_3(3 + \sqrt{-3})1_{1,0}(x) = \psi \left( \frac{B_1 x}{3^3} \right) \sum_{c(\nu)=3} \nu^{-1}(A_1) C_{1} C_{\nu} \psi_{-3}(\nu^{-1}) 1_{\nu^{-1},0}(x),
\]

\[\pi_3(3 + \sqrt{-3})1_{1,0}(x) = \psi \left( \frac{-B_1 x}{3^3} \right) \sum_{c(\nu)=3} \nu(A_1) C_{1} C_{\nu^{-1}} \psi_{-3}(\nu) 1_{\nu,0}(x)
\]

\[= \psi \left( \frac{-B_1 x}{3^3} \right) \sum_{c(\nu)=3} \nu^{-1}(-A_1) C_{1} C_{\nu} \psi_{-3}(\nu^{-1}) 1_{\nu^{-1},0}(x).
\]

By (4.14),

\[\pi_3(1 - (y + 1)\sqrt{-3})1_{1,0} = \psi \left( \frac{b_{y+1} x}{9} \right) \left[ C_1 \sum_{c(\nu)=2} C_{\nu} \psi_{-2}(\nu^{-1}) 1_{\nu^{-1},0}(x) \right].
\]
Then
\[\Phi(3y + \sqrt{-3}) = \langle \pi_3(1 - (y + 1)\sqrt{-3})1_{1,0}, \pi_3(3 + \sqrt{-3})1_{1,0} \rangle\]

\[= \int_{Z_3^\times} \left( \sum_{c(\mu)=3} C_\mu C_\nu \mu^{-1}(-A_1) \hat{\psi}_{-2}(\nu^{-1}) \hat{\psi}_{-3}(\mu^{-1}) \psi \left( \frac{(-B_1 + 3b_{y+1})x}{27} \right) (\mu\nu)^{-1}(x) \right) dx\]

\[= \sum_{c(\mu)=3} C_\mu C_\nu \mu^{-1}(-A_1) (\mu\nu) (B_1 - 3b_{y+1}) \hat{\psi}_{-2}(\nu^{-1}) \hat{\psi}_{-3}(\mu^{-1}) \hat{\psi}_{-3}((\mu\nu)^{-1})\]

\[= \sum_{c(\mu)=3} C_\mu^{-1} C_{\nu^{-1}} \mu(-A_1) (\mu^{-1}\nu^{-1}) (B_1 - 3b_{y+1}) \hat{\psi}_{-2}(\nu) \hat{\psi}_{-3}(\mu) \hat{\psi}_{-3}((\mu\nu)) .\]

Finally, we have for \(i = 0, 1\) and \(j \in (Z/3Z)^\times\)
\[\hat{\psi}_{-2}(\nu_1^i \nu_2^j) = \frac{1}{6} \sum_{\alpha \in (Z/9Z)^\times} \nu_1(\alpha)^i \psi \left( \frac{\alpha(1 + 3j)}{9} \right)\]

\[= \frac{1}{6} \sum_{k \in Z/9Z} \psi \left( \frac{(-2)^k + 3jk}{9} \right) + (-1)^i \psi \left( \frac{(-2)^k + 3jk}{9} \right) .\]

And for \(i = 0, 1\) and \(j \in (Z/9Z)^\times\)
\[\hat{\psi}_{-3}(\nu_1^i \nu_3^j) = \frac{1}{18} \sum_{\alpha \in (Z/27Z)^\times} \nu_1(\alpha)^i \psi \left( \frac{\alpha(1 + 3j)}{27} \right)\]

\[= \frac{1}{18} \sum_{k \in Z/9Z} \psi \left( \frac{(-2)^k + 3jk}{27} \right) + (-1)^i \psi \left( \frac{(-2)^k + 3jk}{27} \right) .\]

Combining all these together, we have

**Proposition 4.8.** If \(p \equiv 4 \text{ mod } 9\), then
\[\Phi(1 \pm 1\sqrt{-3}) = -\frac{1}{2}, \ \Phi(1 \pm 2\sqrt{-3}) = 1,\]
\[\Phi(1 \pm 3\sqrt{-3}) = -\frac{1}{2}, \ \Phi(1 \pm 4\sqrt{-3}) = -\frac{1}{2},\]
\[\Phi(3y \pm \sqrt{-3}) = 0, \ y \in Z/9Z.\]

If \(p \equiv 7 \text{ mod } 9\), then
\[\Phi(1 \pm 1\sqrt{-3}) = -\frac{1}{2}, \ \Phi(1 \pm 2\sqrt{-3}) = -\frac{1}{2},\]
\[\Phi(1 \pm 3\sqrt{-3}) = -\frac{1}{2}, \ \Phi(1 \pm 4\sqrt{-3}) = 1,\]
\[\Phi(\sqrt{-3}) = 1, \ \Phi(3 \pm \sqrt{-3}) = -\frac{1}{2}, \ \Phi(6 \pm \sqrt{-3}) = -\frac{1}{2},\]
\[\Phi(9 + \sqrt{-3}) = -\frac{1}{2}, \ \Phi(12 + \sqrt{-3}) = 1.\]
Proof. Assume $p \equiv 4 \mod 9$. Then $a, b, c \mod 9$ in (4.9) is independent of $p$. From (4.12), we can see that $b \mod 9$ is independent of $p$. Then by (4.14), we see that $\Phi(1 + y\sqrt{-3})$ is independent of $p$. It follows from expressions (*) and (**) that, for each $y \in \mathbb{Z}/9\mathbb{Z}$, the value of $\Phi(3y + \sqrt{-3})$ only depend on $p \mod 27$. Using SageMath, we get all the values of $\Phi$. The case $p \equiv 7 \mod 9$ is similar. □

The local character $\chi_3$ has conductor $\mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3})$, and hence it is in fact a character of the quotient group $\mathcal{O}_{K,3}^\times/\mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3})$. Note that

$\mathcal{O}_{K,3}^\times/\mathbb{Z}_3^\times(1 + 9\mathcal{O}_{K,3}) \cong \langle 1 + \sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 + 3\sqrt{-3}\rangle^{\mathbb{Z}/3\mathbb{Z}}$.

We have the following

Lemma 4.9. The local character $\chi_3$ is given explicitly by the following table:

| $p \mod 9$ | $\chi_3(1 + \sqrt{-3})$ | $\chi_3(1 + 3\sqrt{-3})$ | $\chi_3(1 + 2\sqrt{-3})$ | $\chi_3(1 + 4\sqrt{-3})$ | $\chi_3(\sqrt{-3})$ |
|------------|--------------------------|---------------------------|---------------------------|---------------------------|--------------------------|
| 4          | $\omega$                 | $\omega$                  | 1                         | $\omega^2$                | 1                        |
| 7          | $\omega^2$               | $\omega$                  | $\omega^2$                | 1                         | 1                        |

Proof. This follows directly from the explicit local class field theory, see the proof of Proposition 2.4. We just remark that $1 + 9\mathcal{O}_{K,3} \subset K_3^\times 3$, which follows easily by considering the 3-adic valuation of the terms of the binomial expansion:

$$(1 + 9x)^{1/3} = 1 + \sum_{n \geq 1} \frac{1/3(1/3 - 1) \cdots (1/3 - n + 1)}{n!} (9x)^n$$

where $x \in \mathcal{O}_{K,3}$. The 3-adic valuation of the $n$-th term

$$\frac{1/3(1/3 - 1) \cdots (1/3 - n + 1)}{n!} (9x)^n$$

is greater than $n/3$ for sufficiently large $n$. Then the right-hand side of (4.23) is convergent, and we see $1 + 9\mathcal{O}_{K,3} \subset K_3^\times 3$.

Hence for any $t \in K_3^\times$,

$$\chi_3(t) = \left(\sqrt[3]{3p}\right)^{\sigma_t - 1} = (t, 3p)^3_3 = \begin{cases} (t, 12)_3, & p \equiv 4 \mod 9; \\ (t, 21)_3, & p \equiv 7 \mod 9. \end{cases}$$

Recall $\sigma_t$ is the image of $t$ under the the Artin map, and $(\cdot, \cdot)_3$ denotes the 3-rd Hilbert symbol over $K_3^\times$. Using the local and global principal, it is straight-forward to compute the values of $\chi_3$ as in the above table. □

Proof of Proposition 4.6. Note $f_3'$ is $\chi_3$-eigen under $K_3^\times$, we have

$$\beta_3^0(f_3', f_3') = \text{Vol}(K_3^\times / \mathbb{Q}_3^\times).$$

By (4.5), Proposition (4.8) and Lemma (4.9), we get

$$\beta_3^0(f_3, f_3) = \frac{1}{2\alpha + 2} \text{Vol}(K_3^\times / \mathbb{Q}_3^\times),$$

where $\alpha = 0$ if $p \equiv 4 \mod 9$ and $\alpha = -1$ if $p \equiv 7 \mod 9$. □
5. The 3-part of the Birch and Swinnerton-Dyer conjectures

Let $F$ be a number field. Let $\phi : A \to A'$ be an isogeny of elliptic curves over $F$ of degree $m$ and $\phi'$ is its dual isogeny. The commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & A[\phi] & \to & A & \to & A' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A[m] & \to & A & \to & A' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A'[{\phi'}] & \to & A' & \to & A & \to & 0 \\
\end{array}
\]

induces the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & A'[\phi'](F)/\phi A[m](F) & \to & A'[\phi'](F)/\phi A[m](F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A'(F)/\phi A(F) & \to & \text{Sel}_\phi(A/F) & \to & \text{III}(A/F)[\phi] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A(F)/mA(F) & \to & \text{Sel}_m(A/F) & \to & \text{III}(A/F)[m] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A(F)/\phi' A'(F) & \to & \text{Sel}_{\phi'}(A'/F) & \to & \text{III}(A'/F)[\phi'] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Sel}_{\phi'}(A'/F)/\phi \text{Sel}_m(A/F) & \to & \text{III}(A'/F)[\phi']/\phi \text{III}(A/F)[m] & \to & 0 \\
\end{array}
\]

From this diagram, we immediately have the following

**Lemma 5.1.** Let $A, A'$ and $\phi, \phi'$ be as above.

\[
|\text{Sel}_m(A/F)| = \frac{|\text{Sel}_\phi(A/F)| |\text{Sel}_{\phi'}(A'/F)|}{|A'[\phi'](F)/\phi A[m](F)| |\text{III}(A'/F)[\phi']/\phi \text{III}(A/F)[m]|}.
\]

Let $n$ be a positive cube-free integer and $E_n'$ be the elliptic curve given by Weierstrass equation $y^2 = x^3 + (4n)^2$. Then there is an unique isogeny $\phi_n : E_n \to E'_n$ of degree 3 up to $[\pm 1]$ and denote $\phi'_n$ its dual isogeny.

**Proposition 5.2.** Let $p \equiv 4, 7 \pmod{9}$ be primes such that $3 \pmod{p}$ is not a cubic residue. Then

\[
\dim_{\mathbb{F}_3} \text{Sel}_3(E_p(\mathbb{Q})) \leq 1, \quad \dim_{\mathbb{F}_3} \text{Sel}_3(E_{3p^2}(\mathbb{Q})) = 0.
\]
Proof. By [Sat86, Theorem 2.9], we know that
\[ \text{Sel}_{\phi_p}(E_p(\mathbb{Q})) = \text{Sel}_{\phi'_p}(E_p(\mathbb{Q})) = \mathbb{Z}/3\mathbb{Z}, \]
and
\[ \text{Sel}_{\phi_{3p^2}}(E_p(\mathbb{Q})) = \mathbb{Z}/3\mathbb{Z}, \quad \text{Sel}_{\phi'_{3p^2}}(E'_p(\mathbb{Q})) = 0. \]

Note that \( E_p[3](\mathbb{Q}) \) and \( E_{3p^2}[3](\mathbb{Q}) \) are trivial and \( |E'_p[\phi'_p](\mathbb{Q})| = |E'_{3p^2}[\phi'_{3p^2}](\mathbb{Q})| = 3. \) By Lemma 5.1, the proposition follows.

\[ \square \]

Proof of Theorem 1.2. By [ZK87, Table 1], we know that \( c_p(E_p) = 3, c_3(E_p) = 1 \) or 2 depending on \( p \) congruent to 4 or 7 modulo 9 respectively, and \( c_\ell(E_p) = 1 \) for primes \( \ell \neq 3, p, \) while \( c_p(E_{3p^2}) = 3, c_\ell(E_{3p^2}) = 1 \) for primes \( \ell \neq p. \) Since the Heegner point \( R_1 \) is not torsion, by the work of Gross-Zagier [GZ86] and Kolyvagin [Kol90], we know that
\[ \text{rank}_2 E_p(\mathbb{Q}) = \text{ord}_{s=1} L(s, E_p) = 1, \quad \text{rank}_2 E_{3p^2}(\mathbb{Q}) = \text{ord}_{s=1} L(s, E_{3p^2}) = 0. \]

Let \( P \) be the generator of the free part of \( E_p(\mathbb{Q}). \) Then the BSD conjecture predicts that
\[ \frac{L'(1, E_p)}{\Omega_p} = 2^m \cdot 3 \cdot |\text{III}(E_p)| \cdot \hat{h}_Q(P), \]
where \( m = 0 \) if \( p \equiv 4 \mod 9 \) and 1 otherwise, and
\[ \frac{L(1, E_{3p^2})}{\Omega_{3p^2}} = 3 \cdot |\text{III}(E_{3p^2})|. \]

Combining these two, we get
\[ \frac{L'(1, E_p)}{\Omega_p} \cdot \frac{L(1, E_{3p^2})}{\Omega_{3p^2}} = 2^m \cdot 9 \cdot |\text{III}(E_p)| \cdot |\text{III}(E_{3p^2})| \cdot \hat{h}_Q(P). \]

By Proposition 5.2 we know
\[ |\text{III}(E_p)[3^\infty]| = |\text{III}(E_{3p^2})[3^\infty]| = 1. \]

By Theorem 4.4 and Corollary 4.5, in order to prove the 3-part of (5.1), it suffices to prove
\[ \hat{h}_Q(P) = u\hat{h}_Q(R_1) = u\hat{h}_Q(R) \]
for some \( u \in \mathbb{Z}_3^\times \cap \mathbb{Q}. \) However, it follows from Proposition 3.2 and 3.3 that
\[ \hat{h}_Q(P) = u\hat{h}_Q(Y + Y) = w\hat{h}_Q(R + R) = u\hat{h}_Q(R) \]
for some \( u, w \in \mathbb{Z}_3^\times \cap \mathbb{Q}. \)

\[ \square \]

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