Existence and regularity estimates for quasilinear equations with measure data: the case $1 < p \leq \frac{3n-2}{2n-1}$

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Abstract

We obtain existence and global regularity estimates for gradients of solutions to quasilinear elliptic equations with measure data whose prototypes are of the form $-\text{div}(|\nabla u|^{p-2}\nabla u) = \delta |\nabla u|^q + \mu$ in a bounded $\Omega \subset \mathbb{R}^n$ potentially with non-smooth boundary. Here either $\delta = 0$ or $\delta = 1$, $\mu$ is a finite signed Radon measure in $\Omega$, and $q$ is of linear or super-linear growth, i.e., $q \geq 1$. Our main concern is to extend earlier results to the strongly singular case $1 < p \leq \frac{3n-2}{2n-1}$. In particular, in the case $\delta = 1$ which corresponds to a Riccati type equation, we settle the question of solvability that has been raised for some time in the literature.

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1 Introduction and main results

This paper can be viewed as a continuation of our earlier work \cite{19,22} in which we studied gradient regularity of solutions to quasilinear elliptic equations with measure data

\[
\begin{aligned}
-\text{div}(A(x, \nabla u)) &= \mu \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

and applied it to obtain sharp existence results for the Riccati type equation

\[
\begin{aligned}
-\text{div}(A(x, \nabla u)) &= |\nabla u|^q + \mu \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1.2)

Here \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), \( n \geq 2 \), and \( \mu \) is a finite signed Radon measure in \( \Omega \). The principal operator \( \text{div}(A(x, \nabla u)) \) is modeled after the \( p \)-Laplacian defined by \( \Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) \). In the papers \cite{22,23} and \cite{19} the case \( 2 - \frac{1}{n} < p \leq n \) and the case \( \frac{3n-2}{n} < p \leq 2 - \frac{1}{n} \) were considered, respectively. In this paper, we consider the remaining 'strongly singular' case \( 1 < p \leq \frac{3n-2}{2n-1} \), which eventually settle the question of solvability (raised in \cite{4} pages 13–14) for \( (1.2) \) for all \( 1 < p \leq n \) and \( q \geq 1 \).

More precisely, in \( (1.1)-(1.2) \), the nonlinearity \( A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is a Carathéodory vector valued function, i.e., \( A(x, \xi) \) is measurable in \( x \) and continuous with respect to \( \xi \) for a.e. \( x \). Moreover, for a.e. \( x \), \( A(x, \xi) \) is continuously differentiable in \( \xi \) away from the origin and satisfies

\[
|A(x, \xi)| \leq \Lambda|\xi|^{p-1}, \quad |\nabla_\xi A(x, \xi)| \leq \Lambda|\xi|^{p-2},
\]

(1.3)

\[
\langle \nabla_\xi A(x, \xi) \eta, \eta \rangle \geq \Lambda^{-1}|\eta|^2|\xi|^{p-2},
\]

(1.4)

for every \( (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0,0\} \) and a.e. \( x \in \mathbb{R}^n \), where \( \Lambda \) is a positive constant.

As for \( p \) in \( (1.3)-(1.4) \), we shall restrict ourselves to the range:

\[
1 < p \leq \frac{3n-2}{2n-1}.
\]

We shall also require that \( A(x, \xi) \) satisfy a smallness condition of BMO type in the \( x \)-variable. Such a condition is called the \((\delta, R_0)\)-BMO condition defined below (see, e.g., \cite{BMO} and \cite{19,22}). This condition allows \( A(x, \xi) \) has discontinuity in \( x \) and it can be used as a substitute for the Sarason \cite{BMO} VMO condition.

**Definition 1.1** We say that \( A(x, \xi) \) satisfies a \((\delta, R_0)\)-BMO condition for some \( \delta, R_0 > 0 \) if

\[
[A]_{R_0} := \sup_{y \in \mathbb{R}^n, 0 < r \leq R_0} \int_{B_r(y)} \Theta(A, B_r(y))(x) dx \leq \delta,
\]

where

\[
\Theta(A, B_r(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|A(x, \xi) - \overline{A}_{B_r(y)}(\xi)|}{|\xi|^{p-1}},
\]

and \( \overline{A}_{B_r(y)}(\xi) \) denotes the average of \( A(\cdot, \xi) \) over the ball \( B_r(y) \), i.e.,

\[
\overline{A}_{B_r(y)}(\xi) := \frac{1}{|B_r(y)|} \int_{B_r(y)} A(x, \xi) dx = \frac{1}{|B_r(y)|} \int_{B_r(y)} A(x, \xi) dx.
\]
As far as the regularity of the boundary of $\Omega$ is concerned, we require that it be sufficiently flat in the sense of Reifenberg [24]. Namely, at each boundary point and every scale, we ask that the boundary of $\Omega$ be trapped between two hyperplanes separated by a distance that depends on the scale. This class of domains includes $C^1$ domains and Lipschitz domains with sufficiently small Lipschitz constants (see [26]). Moreover, they also include certain domains with fractal boundaries and thus allow for a wide range of potential applications.

**Definition 1.2** Given $\delta \in (0, 1)$ and $R_0 > 0$, we say that $\Omega$ is a $(\delta, R_0)$-Reifenberg flat domain if for every $x \in \partial \Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{z_1, z_2, \ldots, z_n\}$, which may depend on $r$ and $x$, so that in this coordinate system $x = 0$ and that

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}.$$ 

In this paper, all solutions of (1.1) and (1.2) with a finite signed measure $\mu$ will be understood in the renormalized sense (see [6]). For instance, we use the following one:

$$\text{For } u \in W^{1,p}_\text{loc}(\Omega), \text{ we define the usual two-sided truncation operator } T_k \text{ by}$$

$$T_k(s) = \max\{\min\{s, k\}, -k\}, \quad s \in \mathbb{R}.$$ 

For our purpose, the following notion of gradient is needed. If $u$ is a measurable function defined in $\Omega$, finite a.e., such that $T_k(u) \in W^{1,p}_\text{loc}(\Omega)$ for any $k > 0$, then there exists a measurable function $v : \Omega \to \mathbb{R}^n$ such that $\nabla T_k(u) = v\chi_{\{|u| < k\}}$ a.e. in $\Omega$ for all $k > 0$ (see [1] Lemma 2.1). In this case, we define the gradient $\nabla u$ of $u$ by $\nabla u := v$. It is known that $v \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$ if and only if $u \in W^{1,1}_\text{loc}(\Omega)$ and then $v$ is the usual weak gradient of $u$. On the other hand, for $1 < p \leq 2 - \frac{1}{n}$, by looking at the fundamental solution we see that in general distributional solutions of (1.1) may not even belong to $u \in W^{1,1}_{\text{loc}}(\Omega)$.

The notion of renormalized solutions is a generalization of that of entropy solutions introduced in [1] and [3], where the right-hand side is assumed to be in $L^1(\Omega)$ or in $W_0^1(\Omega)$. Several equivalent definitions of renormalized solutions were given in [6]. Here we use the following one:
Definition 1.3 Let $\mu = \mu_0 + \mu_s \in M_b(\Omega)$, with $\mu_0 \in M_0(\Omega)$ and $\mu_s \in M_s(\Omega)$. A measurable function $u$ defined in $\Omega$ and finite a.e. is called a renormalized solution of (1.1) if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{n}{n-1}$, and $u$ has the following additional property. For any $k > 0$ there exist nonnegative Radon measures $\lambda^+_k, \lambda^-_k \in M_0(\Omega)$ concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\mu^+_k \to \mu^+_s$, $\mu^-_k \to \mu^-_s$ in the narrow topology of measures and that

$$\int_{\{u < k\}} (A(x, \nabla u), \nabla \varphi) \, dx = \int_{\{u < k\}} \varphi \, d\mu_0 + \int_{\Omega} \varphi \, d\lambda^+_k - \int_{\Omega} \varphi \, d\lambda^-_k,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Here we recall that a sequence $\{\mu_k\} \subset M_0(\Omega)$ is said to converge in the narrow topology of measures to $\mu \in M_0(\Omega)$ if $\lim_{k \to \infty} \int_{\Omega} \varphi \, d\mu_k = \int_{\Omega} \varphi \, d\mu$, for every bounded and continuous function $\varphi$ on $\Omega$.

It is known that if $\mu \in M_0(\Omega)$ then there is one and only one renormalized solution of (1.1) (see [3, 6]). However, to the best of our knowledge, for a general $\mu \in M_0(\Omega)$ the uniqueness of renormalized solutions of (1.1) is still an open problem.

Recall that the Hardy-Littlewood maximal function $M$ is defined for each locally integrable function $f$ in $\mathbb{R}^n$ by

$$M(f)(x) = \sup_{\rho > 0} \int_{B_\rho(x)} |f(y)| \, dy \quad \forall x \in \mathbb{R}^n.$$  

For a signed measure $\mu$ in $\mathbb{R}^n$, the first order fractional maximal function of $\mu$, $M_1(\mu)$, is defined by

$$M_1(\mu)(x) := \sup_{\rho > 0} \frac{\mu(B_\rho(x))}{\rho^{n-1}} \quad \forall x \in \mathbb{R}^n.$$  

A nonnegative function $w \in L^1_{loc}(\mathbb{R}^n)$ is said to be an $A_\infty$ weight if there are two positive constants $C$ and $\nu$ such that

$$w(E) \leq C \left( \frac{|E|}{|B|} \right)^\nu w(B),$$

for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $E \subset B$. The pair $(C, \nu)$ is called the $A_\infty$ constants of $w$ and is denoted by $[w]_{A_\infty}$. It is well-known that

$$A_\infty = \bigcup_{q > 1} A_q,$$

where we say that a nonnegative function $w \in L^1_{loc}(\mathbb{R}^n)$ belongs to the Muckenhoupt $A_q$ class, $q > 1$, if

$$[w]_{A_q} := \sup_{\text{balls } B \subset \mathbb{R}^n} \left( \int_B w \, dx \right) \left( \int_B w^{\frac{1}{q-1}} \, dx \right)^{q-1} < +\infty.$$  

Our first result concerns with a weighted ‘good-$\lambda$’ type inequality for renormalized solutions of (1.1).
Theorem 1.4 Let \( w \in A_\infty, \mu \in M(\Omega), 1 < p \leq \frac{3n-2}{2n-1}, \) and \( \gamma_1 \in \left(0, \frac{p-1}{n} \right). \) For any \( \varepsilon > 0, R_0 > 0, \) one can find constants \( \delta_1 = \delta_1(n,p,\Lambda,\gamma_1,\varepsilon,\|w\|_{A_\infty}) \in (0,1), \) \( \delta_2 = \delta_2(n,p,\Lambda,\gamma_1,\varepsilon,\|w\|_{A_\infty},\text{diam}(\Omega)/R_0) \in (0,1), \) and \( \Lambda_0 = \Lambda_0(n,p,\Lambda,\gamma_1) > 1 \) such that if \( \Omega \) is \((\delta_1,R_0)-\text{Reifenberg flat}\) and \( |A|_{R_0} \leq \delta_1 \) then for any renormalized solution \( u \) to (1.1) with \( |\nabla u| \in L^{2-p}(\Omega), \) we have

\[
\begin{align*}
w((\{M(|\nabla u|^{\gamma_1})^{1/\gamma_1} > \Lambda_0 \lambda, \langle M_1(\mu) \rangle \}^{1/\gamma_1} \leq \delta_2 \lambda \}) \cap U_{\epsilon,\lambda} \cap \Omega) \\
\leq C_\varepsilon w((\{M(|\nabla u|^{\gamma_1})^{1/\gamma_1} > \lambda \}) \cap \Omega),
\end{align*}
\]  

(1.5) for any \( \lambda > 0. \) Here \( U_{\epsilon,\lambda} = \{M((\nabla u)^{2-p})^{\frac{1}{2-p}} \leq \varepsilon^{-1} \lambda \} \) and the constant \( C \) depends only on \( n,p,\Lambda,\text{diam}(\Omega)/R_0, \) and \( |w|_{A_\infty}. \)

The presence of the set \( U_{\epsilon,\lambda} \) in (1.5) makes Theorem 1.4 different from [19] Theorem 1.5 in which the case \( \frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n} \) was treated. However, Theorem 1.4 can be used to obtain the following existence and regularity of solutions to (1.1), which extends the results of [19, 22] to the case \( 1 < p \leq \frac{3n-2}{2n-1}. \)

Theorem 1.5 Let \( \mu \in M(\Omega) \) and \( 1 < p \leq \frac{3n-2}{2n-1}. \) For any \( 2 - p < q < \infty \) and \( w \in A_{\frac{2p}{2-p}}, \) we can find \( \delta = \delta(n,p,\Lambda,q,\|w\|_{A_{\frac{2p}{2-p}}}) \in (0,1) \) such that if \( \Omega \) is \((\delta,R_0)-\text{Reifenberg flat}\) and \( |A|_{R_0} \leq \delta \) for some \( R_0 > 0, \) then there exists a renormalized solution \( u \) to (1.1) such that

\[
||\nabla u||_{L^p(\Omega)} \leq C||M_1(\mu)||^{\frac{1}{2-p}}_{L^p(\Omega)}.
\]  

(1.6)

Here the constant \( C \) depends only on \( n,p,\Lambda,q,\|w\|_{A_{\frac{2p}{2-p}}}, \) and \( \text{diam}(\Omega)/R_0. \)

Remark 1.6 By uniqueness of renormalized solutions with data in \( M(\Omega), \) we see that (1.6) indeed holds for any renormalized solution \( u \) to (1.1) with datum \( \mu \in M(\Omega). \)

Remark 1.7 Theorem 1.5 also holds if we replace the weighted Lebesgue space \( L^q_w(\Omega) \) with the more general weighted Lorentz space \( L^{q,s}_w(\Omega), \) \( 0 < s \leq \infty, \) (see, e.g., [19, 22]).

We now describe our results in regard to the Riccati type equation (1.2). For this, we shall need the notion of capacity associated to the Sobolev space \( W^{1,s}(\mathbb{R}^n), \) \( 1 < s < +\infty. \) For a compact set \( K \subseteq \mathbb{R}^n, \) we define

\[
\text{Cap}_{1,s}(K) = \inf \left\{ \int_{\mathbb{R}^n} (|\nabla \varphi|^s + \varphi^s) \, dx : \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq \chi_K \right\}.
\]

As [19] Theorem 1.9 and [22] Theorem 1.6, we obtain the following sharp existence result but now for the case \( 1 < p \leq \frac{3n-2}{2n-1}. \)

Theorem 1.8 Let \( 1 < p \leq \frac{3n-2}{2n-1} \) and \( q \geq 1. \) There exists a constant \( \delta = \delta(n,p,\Lambda,q) \in (0,1) \) such that the following holds. Suppose that \( |A|_{R_0} \leq \delta \) and \( \Omega \) is \((\delta,R_0)-\text{Reifenberg flat}\) for some \( R_0 > 0. \) Then there exists \( c_0 = c_0(n,p,\Lambda,q,\text{diam}(\Omega),R_0) > 0 \) such that if \( \mu \) is a finite signed measure in \( \Omega \) with

\[
|\mu|(K) \leq c_0 \text{Cap}_{1,s,q,s}^{\frac{q}{q-1}}(K)
\]  

(1.7)
for all compact sets $K \subset \Omega$, then there exists a renormalized solution $u \in W^{1,q}_0(\Omega)$ to the Riccati type equation (1.2) such that
\[
\int_K |\nabla u|^q \leq C \text{Cap}_{1, q-1}(K)
\]
for all compact sets $K \subset \Omega$. Here the constant $C$ depends only on $n, p, \Lambda, q, \text{diam}(\Omega)$, and $R_0$.

There is a vast literature on equations with a power growth in the gradient of the form (1.2). We only mention here the pioneering work [11] which originally used capacity to treat the 'linear' case (1.2). We only mention here the pioneering work [11] which originally used capacity to treat (1.2) in the 'linear' case $p = 2$. For other contributions, see, e.g., the references in [19].

It is known that condition (1.7) is sharp in the sense that if (1.2) has a solution with $\mu$ being nonnegative and compactly supported in $\Omega$ then (1.7) holds with a different constant $c_0$ (see [11, 20]). It is more general than the Marcinkiewicz space condition $\mu \in L^{n(q-1)}(\Omega)$, $q > \frac{n(p-1)}{p-1}$, (with a small norm), or the Fefferman-Phong type condition involving Morrey spaces (see, e.g., [20]). Moreover, Theorem 1.8 implies that any compact set $K \subset \Omega$ that is removable for the equation $-\text{div}(A(x, \nabla u)) = |\nabla u|^q$ must be small in the sense that $\text{Cap}_{1, q-1}(K) = 0$ (see [20] Theorem 3.9).

The paper is organized as follows. In Section 2 we obtain some important comparison estimates that are needed for the proof of Theorem 1.4. The proofs Theorems 1.4, 1.5, and 1.8 are given in Sections 3, 4, and 5 respectively.

2 Comparison estimates

Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a solution of (1.1). For each ball $B_{2R} = B_{2R}(x_0) \subset \subset \Omega$, we consider the unique solution $w \in W^{1,p}_{\text{loc}}(B_{2R}) + u$ to the equation
\[
\begin{cases}
- \text{div} (A(x, \nabla w)) = 0 & \text{in} \quad B_{2R}, \\
\quad w = u & \text{on} \quad \partial B_{2R}.
\end{cases}
\]

(2.1)

Then we have the following estimate for the difference $\nabla u - \nabla w$ in terms of the total variation of $\mu$ in $B_{2R}$ and the norm of $\nabla u$ in $L^{2-p}(B_{2R})$. This estimate holds true for all $1 < p \leq \frac{3n-2}{2n-1}$. For earlier results of this type for $p > \frac{3n-2}{2n-1}$, we refer to [16, 17, 7, 19].

Lemma 2.1 Let $u$ and $w$ be as in (2.1), and assume that $1 < p \leq \frac{3n-2}{2n-1}$. Then
\[
\left( \int_{B_{2R}} |\nabla(u - w)|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} \leq C \left( \frac{|\mu|(B_{2R})}{R^n-1} \right)^{\frac{p-1}{p-1}} + \frac{|\mu|(B_{2R})}{R^n-1} \int_{B_{2R}} |\nabla u|^{2-p},
\]
for any $0 < \gamma_1 < \frac{n(p-1)}{p-1}$.

Proof. By scaling invariance, we may assume that $|\mu|(B_{2R}) = 1$ and $B_{2R} = B_2$. For $k > 0$, using $\varphi = T_{2k}(u - w)$ as a test function, we have
\[
\int_{B_2 \cap \{|u - w| < 2k\}} g(u, w) dx \leq Ck, \quad \text{with} \quad g(u, w) = \frac{|\nabla(u - w)|^2}{(|\nabla w| + |\nabla u|)^{2-p}}.
\]
Set $E_k = B_2 \cap \{|k - u| < 2k\}$, and $F_k = B_2 \cap \{|u - w| > k\}$. Using Hölder’s inequality yields

\[
k\{x : |k - u| > 2k \cap B_2\}^{\frac{n-1}{n}} \leq C \left(\int_{B_2} |T_{2k}(u - w) - T_k(u - w)|^{\frac{n-1}{n}}\right)^{\frac{n-1}{n}}
\]

\[
\leq C \int_{E_k} |\nabla(u - w)|
\]

\[
\leq C \int_{E_k} g(u, w)^{1/p} + g(u, w)^{1/2}|\nabla u|^{\frac{2-p}{p}}
\]

\[
\leq C|E_k|^{\frac{n-1}{n}} \left(\int_{E_k} g(u, w)^{1/p} + \left(\int_{E_k} g(u, w)^{1/2}\right)^{1/2} \left(\int_{E_k} |\nabla u|^{2-p}\right)^{\frac{1}{2-p}}\right).
\]

Here in the third inequality we used that

\[
|\nabla(u - w)| \leq C \left(g(u, w)^{1/p} + g(u, w)^{1/2}|\nabla u|^{\frac{2-p}{p}}\right),
\]  

(2.2)

which holds provided $1 < p < 2$. As $\int_{E_k} g(u, w) \leq Ck$, we thus find

\[
k^{1/2}|F_{2k}|^{\frac{1}{p}} \leq Ck^{-1/2 + 1/p}|F_k|^{\frac{1}{p}} + C Q_1^{2-p},
\]

where we set $Q_1 = ||\nabla u||_{L^2-\nu(B_2)}$.

Note that we can write for $\epsilon \geq 0$,

\[
k^{1/2}|F_{2k}|^{\frac{1}{p} + \epsilon} \leq Ck^{-1/2 + 1/p}|F_k|^{\frac{1}{p} + \epsilon} + C Q_1^{2-p},
\]

which implies

\[
||u - w||_{L^{\frac{1}{1/2 + \epsilon}}}^{1/2} \leq C||u - w||_{L^{\frac{1}{p-1/2 + \epsilon}}}^{1/2} + C Q_1^{2-p}.
\]

Choosing

\[
\epsilon = \frac{3n - 2p(2n - 1)}{2(p - 1)n},
\]

we have

\[
\frac{1}{2(\frac{1}{p} + \epsilon)} = \frac{1/p - 1/2}{\frac{1}{p} + \epsilon} = \frac{n(p - 1)}{n - p}.
\]

Thus, using Holder’s inequality we obtain

\[
||u - w||_{L^{\frac{n(p - 1)}{n - p}}}^{(p - 1)} \leq C + C Q_1^{2-p}.
\]  

(2.3)

For $k, \lambda \geq 0$, and $q = \frac{n(p - 1)}{n - p}$, we have

\[
\{x : g(u, w) > \lambda\} \cap B_2 \leq \{x : |u - w| > k\} \cap B_2 +
\]

\[
\quad + \frac{1}{\lambda} \int_0^\lambda \{x : |u - w| \leq k, g(u, w) > s\} \cap B_2|ds
\]

\[
\leq Ck^{-q}||u - w||_{L^{q}}^{q} \leq (B_2) + \frac{1}{\lambda} \int_{B_2} \{x : |u - w| \leq k\} g(u, w)dx
\]

\[
\leq Ck^{-q}||u - w||_{L^{q}}^{q} \leq (B_2) + \frac{Ck}{\lambda}.
\]
Then choosing

\[ k = \left[ \lambda \|u - w\|_{L^{q,\infty}(B_2)}^{q} \right]^{\frac{1}{q+1}}, \]

we obtain

\[ \lambda^{\frac{q}{q+1}} |\{ x : g(u, w) > \lambda \} \cap B_2 | \leq C \|u - w\|_{L^{q,\infty}(B_2)}^{q} \]

for all \( \lambda > 0 \). This means

\[ \|g(u, w)\|_{L^{q,\infty}(B_2)} \leq C \|u - w\|_{L^{q,\infty}(B_2)}^{q}. \]

Let \( \gamma_1 \in (0, \frac{n(p-1)}{n-1}) \). By (2.2) and Holder’s inequality with exponents \( \frac{2}{p} \) and \( \frac{2}{2-p} \):

\[
\int_{B_2} |\nabla (u - w)|^{\gamma_1} \leq C \int_{B_2} \left[ g(u, w)^{\gamma_1/p} + g(u, w)^{\gamma_1/2} |\nabla u|^{\gamma_1(2-p)/2} \right]^{\frac{2}{2-p}} \leq C \|g(u, w)\|_{L^{q,\infty}(B_2)}^{\gamma_1/p} + C \left( \int_{B_2} |\nabla u|^{\gamma_1} \right)^{\frac{2}{2-p}} \leq C \|g(u, w)\|_{L^{q,\infty}(B_2)}^{\gamma_1/p} + C \|g(u, w)\|_{L^{q,\infty}(B_2)}^{\gamma_1/2} \cdot Q_{1}^{\frac{(2-p)\gamma_1}{2}}. \]

Here we used the fact that

\[ \gamma_1 < \frac{q}{q+1}, \quad \gamma_1 \leq 2-p. \]

Combining this with (2.3) yields

\[
\int_{B_2} |\nabla (u - w)|^{\gamma_1} \leq C + CQ_{1}^{(2-p)\gamma_1} \]

which implies the result.

Just as [19, Proposition 2.3], we also obtain from Lemma 2.1 the following result.

**Proposition 2.2** Let \( 0 < \gamma_1 < \frac{(p-1)n}{n-1} \). There exists \( v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2}) \) such that for any \( \varepsilon > 0 \),

\[
\|\nabla v\|_{L^{\infty}(B_{R/2})} \leq C_{\varepsilon} \left[ \frac{|\mu|(B_{2R})}{R^{n-1}} \right]^{\frac{1}{p-1}} + C \left( \int_{B_{2R}} |\nabla u|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \varepsilon \left( \int_{B_{2R}} |\nabla u|^{2-p} \right)^{\frac{1}{2-p}},
\]

and

\[
\left( \int_{B_R} |\nabla u - \nabla v|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C_{\varepsilon} \left[ \frac{|\mu|(B_{2R})}{R^{n-1}} \right]^{\frac{1}{p-1}} + C((|A|_{R_0})^\kappa + \varepsilon) \left( \int_{B_{2R}} |\nabla u|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \varepsilon \left( \int_{B_{2R}} |\nabla u|^{2-p} \right)^{\frac{1}{2-p}},
\]

for some \( C_{\varepsilon} = C(n, p, \Lambda, \varepsilon) > 0 \). Here \( \kappa \) is a constant in \( (0, 1) \).
Lemmas 2.1 and Proposition 2.2 can be extended up to boundary. We recall that $\Omega$ is $(\delta_0, R_0)$-Reifenberg flat with $\delta_0 < 1/2$. Fix $x_0 \in \partial \Omega$ and $0 < R < R_0/10$. With $u \in W^{1,p}_0(\Omega)$ being a solution to (1.1), we now consider the unique solution $w \in W^{1,p}_0(\Omega_{10R}(x_0)) + u$ to the following equation

$$\begin{align*}
- \text{div}(A(x, \nabla w)) &= 0 \quad \text{in} \quad \Omega_{10R}(x_0), \\
\n &= u \quad \text{on} \quad \partial \Omega_{10R}(x_0),
\end{align*}$$

(2.4)

where we define $\Omega_{10R}(x_0) = \Omega \cap B_{10R}(x_0)$.

Then we have the following analogue of Lemmas 2.1.

**Lemma 2.3** Let $0 < \gamma_1 < \frac{(p-1)n}{n-1}$, $1 < p \leq \frac{3n-2}{2n-1}$, and let $u, w$ be as in (2.4). Then we have

$$\left( \int_{B_{10R}(x_0)} |\nabla (u - w)|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C \left[ \frac{|\mu|(B_{10R}(x_0))}{R^{n-1}} \right]^{\frac{1}{p-1}} + C \left( \int_{B_{10R}(x_0)} |\nabla u|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \varepsilon \left( \int_{B_{10R}(x_0)} |\nabla u|^{2-p} \right)^{\frac{1}{2-p}},$$

and

$$\left( \int_{B_{10R}(x_0)} |\nabla (u - V)|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C \left[ \frac{|\mu|(B_{10R})}{R^{n-1}} \right]^{\frac{1}{p-1}} + C \left( \int_{B_{10R}(x_0)} |\nabla u|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \varepsilon \left( \int_{B_{10R}(x_0)} |\nabla u|^{2-p} \right)^{\frac{1}{2-p}},$$

for some $C_\varepsilon = C(n, p, \Lambda, \varepsilon) > 0$. Here $\kappa$ is a constant in $(0,1)$ and the balls are centered at $x_0$.

**3 Proof of Theorem 1.4**

The proof of Theorem 1.4 is based on Propositions 2.2 and 2.4 and the following technical lemma (see [13]).
Lemma 3.1 Let $\Omega$ be a $(\delta, R_0)$-Reifenberg flat domain with $\delta < 1/4$ and let $w$ be an $A_\infty$ weight. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^L$ with centers $y_i \in \overline{\Omega}$ and radius $r \leq R_0/4$ covers $\Omega$. Let $E \subset F \subset \Omega$ be measurable sets for which there exists $0 < \varepsilon < 1$ such that

1. $w(E) < \varepsilon w(B_r(y_i))$ for all $i = 1, \ldots, L$, and
2. for all $x \in \Omega$, $\rho \in (0, 2r]$, we have $w(E \cap B_\rho(x)) \geq \varepsilon w(B_\rho(x)) \implies B_\rho(x) \cap \Omega \subset F$.

Then $w(E) \leq C \varepsilon w(F)$ for a constant $C$ depending only on $n$ and $[w]_{A_\infty}$.

Proof of Theorem 1.4 The proof is reminiscent of that of [18, Theorem 1.5] (see also [22, Theorem 1.4], [17, Theorem 8.4], and [18, Theorem 3.1]).

Let $R = \text{diam}(\Omega)$. Suppose that $0 < \gamma < \frac{n(p-1)}{n-1}$ and $u$ is a renormalized solution of (1.1) such that $|\nabla u| \in L^{2,p}(\Omega)$. By [6, Theorem 4.1] we have

$$\|\nabla u\|_{L^{\frac{n}{n-1}, \infty}(\Omega)} \leq C |[\mu](\Omega)|^{\frac{1}{p-1}},$$

which implies that

$$\left(\frac{1}{R^n} \int_{\Omega} |\nabla u|^{\gamma_1}\right)^{\frac{1}{\gamma_1}} \leq C \gamma_1 \left[\frac{|\mu|(\Omega)}{R^n}\right]^{\frac{1}{p-1}}.$$

For $k > 0$, let $\mu_0, \lambda_0^+, \lambda_0^-$ be as in Definition 1.3. Let $u_k \in W^{1,p}_0(\Omega)$ be the unique solution of the equation

$$\begin{cases}
-\text{div}(A(x, \nabla u_k)) = \mu_k & \text{in } \Omega, \\
u_k = 0 & \text{on } \partial\Omega,
\end{cases}$$

where we set $\mu_k = \chi_{\{|u|<k\}} \mu_0 + \lambda_0^+ - \lambda_0^-$. Note that we have $u_k = T_k(u)$ and $\mu_k \to \mu$ in the narrow topology of measures (see [6, Remark 2.32]). Thus,

$$\nabla u_k \to \nabla u \text{ in } L^{\gamma_1}(\Omega) \cap L^{2-p}(\Omega). \quad (3.1)$$

Let us set

$$F_\lambda = \{(M(|\nabla u|^{\gamma_1}))^{1/\gamma_1} > \lambda\} \cap \Omega,$$

and

$$E_{\lambda, \lambda_2} = \{(M(|\nabla u|^{\gamma_1}))^{1/\gamma_1} > \lambda_0 \lambda, (M(\mu))^{1/\gamma_1} \leq \delta_2 \lambda\} \cap U_{\epsilon, \lambda} \cap \Omega,$$

where

$$U_{\epsilon, \lambda} = \{(M(|\nabla u|^{2-p}))^{1/\gamma_1} \leq \varepsilon^{-1} \lambda\},$$

and $\delta_2 \in (0, 1)$, $\lambda > 0$. The constant $\lambda_0$ depends only on $n, p, \gamma_1, \Lambda$ and is to be chosen later.

Also, let $\{y_i\}_{i=1}^L \subset \Omega$ and a ball $B_0$ with radius $2R$ such that

$$\Omega \subset \bigcup_{i=1}^L B_{r_0}(y_i) \subset B_0, \quad \text{where } r_0 = \min\{R_0/1000, R\}.$$
As in the proof of [19, Theorem 1.5], we have
\[ w(E_{\lambda, \delta_2}) \leq \varepsilon w(B_{r_0}(y_i)) \quad \forall \lambda > 0, \forall i = 1, 2, \ldots, L, \]  
(3.2)
provided \( \delta_2 = \delta_2(n, p, \Lambda, \epsilon, [w]_{A_\infty}, R/R_0) > 0 \) is small enough.

In order to apply Lemma 3.1, we now verify that for all \( x \in \Omega, r \in (0, 2r_0] \), and \( \lambda > 0 \) we have
\[ w(E_{\lambda, \delta_2} \cap B_r(x)) \geq \varepsilon w(B_r(x)) \implies B_r(x) \cap \Omega \subset F_\lambda, \]  
(3.3)
provided \( \delta_2 \) is small enough depending on \( n, p, \Lambda, \gamma_0, \epsilon, [w]_{A_\infty}, R/R_0 \).

Indeed, take \( x \in \Omega \) and \( 0 < r \leq 2r_0 \). By contraposition, assume that \( B_r(x) \cap \Omega \cap F_\lambda^c \neq \emptyset \) and \( E_{\lambda, \delta_2} \cap B_r(x) \neq \emptyset \), i.e., there exist \( x_1, x_2 \in B_r(x) \cap \Omega \) such that
\[ [M(|\nabla u|^{\gamma_1})(x_1)]^{1/\gamma_1} \leq \lambda, \]  
(3.4)
and
\[ M(|\nabla u|^{2-p})(x_2) \leq (\varepsilon^{-1})^{2-p}, \quad M_1(\mu)(x_2) \leq (\delta_2 \lambda)^{p-1}. \]  
(3.5)
We need to prove that
\[ w(E_{\lambda, \delta_2} \cap B_r(x)) < \varepsilon w(B_r(x)). \]  
(3.6)
It follows from (3.3) that
\[ M(|\nabla u|^{\gamma_1})(y)^{1/\gamma_1} \leq \max\{ [M(\chi_{B_{2r}(x)}|\nabla u|^{\gamma_1})(y)]^{1/\gamma_1}, 3^n \lambda \} \quad \forall y \in B_r(x). \]

Therefore, for all \( \lambda > 0 \) and \( \Lambda_0 \geq 3^n \), we find
\[ E_{\lambda, \delta_2} \cap B_r(x) = \left\{ M(\chi_{B_{2r}(x)}|\nabla u|^{\gamma_1})^{1/\gamma_1} > \Lambda_0 \lambda, (M_1(\mu))^{p-1} \leq \delta_2 \lambda \right\} \cap U_{\epsilon, \lambda} \cap \Omega \cap B_r(x). \]  
(3.7)
To prove [3.6], we separately consider the case \( B_{8r}(x) \subset \subset \Omega \) and the case \( B_{8r}(x) \cap \Omega^c \neq \emptyset \).

1. **The case \( B_{8r}(x) \subset \subset \Omega \):** Applying Proposition 2.2 to \( u = u_k \in W^{1,p}_0(\Omega), \mu = \mu_k \) and \( B_{2r} = B_{8r}(x) \), there is a function \( v_k \in W^{1,p}(B_{2r}(x)) \cap W^{1,\infty}(B_{2r}(x)) \) such that for any \( \eta > 0 \),
\[ \|\nabla v_k\|_{L^\infty(B_{2r}(x))} \leq C_\eta \left[ \frac{\|\mu_k\|_{L^{\infty}(B_{8r}(x))}}{\eta^{p-1}} \right]^{\frac{1}{p-1}} \]
\[ + C \left( \int_{B_{2r}(x)} |\nabla u_k|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \eta \left( \int_{B_{2r}(x)} |\nabla u_k|^{2-p} \right)^{\frac{1}{2-p}}, \]
and
\[ \left( \int_{B_{4r}} |\nabla u_k - \nabla v_k|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C_\eta \left[ \frac{\|\mu_k\|_{L^{\infty}(B_{8r}(x))}}{\eta^{p-1}} \right]^{\frac{1}{p-1}} \]
\[ + C(\|A_{R_0}\|^{\gamma_1} + \eta) \left( \int_{B_{8r}} |\nabla u_k|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \eta \left( \int_{B_{8r}(x)} |\nabla u_k|^{2-p} \right)^{\frac{1}{2-p}}, \]

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for some $\kappa \in (0, 1)$.

Notice that using (3.4), (3.5), and property (3.1), we get

$$\limsup_{k \to \infty} \|\nabla v_k\|_{L^\infty(B_{2r}(x))} \leq C_\eta \left[ \frac{|\mu| B_{2r}(x)}{p^{n-1}} \right]^{\frac{1}{p-1}} + C \left( \int_{B_{2r}(x)} |\nabla u|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \eta \left( \int_{B_{2r}(x)} |\nabla u|^{2-p} \right)^{\frac{1}{2-p}}$$

$$\leq C_\eta [M_1(\mu)(x_2)]^{\frac{1}{p-1}} + C [M(|\nabla u|^{\gamma_1})(x_1)]^{\frac{1}{\gamma_1}} + C_\eta [M(|\nabla u|^{2-p})(x_2)]^{\frac{1}{2-p}}$$

$$\leq [C_\eta \delta_2 + C + C\eta \epsilon^{-1}] \lambda \leq C_1 \lambda,$$

provided $C_\eta \delta_2, \eta \epsilon^{-1} \leq 1$, and

$$\limsup_{k \to \infty} \left( \int_{B_{4r}(x)} |\nabla u_k - \nabla v_k|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C_\eta \left[ \frac{|\mu| B_{2r}(x)}{p^{n-1}} \right]^{\frac{1}{p-1}}$$

$$+ C((|A| R_0)^n + \eta) \left( \int_{B_{2r}(x)} |\nabla u|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \eta \left( \int_{B_{2r}(x)} |\nabla u|^{2-p} \right)^{\frac{1}{2-p}}$$

$$\leq C_\eta [M_1(\mu)(x_2)]^{\frac{1}{p-1}} + C((|A| R_0)^n + \eta) [M(|\nabla u|^{\gamma_1})(x_1)]^{\frac{1}{\gamma_1}}$$

$$+ C_\eta [M(|\nabla u|^{2-p})(x_2)]^{\frac{1}{2-p}}$$

$$\leq C (C_\eta \delta_2 + \delta_1^n + \eta \epsilon^{-1}) \lambda.$$

Here we also used that $\mu_k \to \mu$ in the narrow topology of measures and $|A| R_0 \leq \delta_1$. Thus there exists $k_0 > 1$ such that for all $k \geq k_0$ we have

$$\|\nabla v_k\|_{L^\infty(B_{2r}(x))} \leq 2C_1 \lambda,$$

and

$$\left( \int_{B_{4r}(x)} |\nabla u_k - \nabla v_k|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C (C_\eta \delta_2 + \delta_1^n + \eta \epsilon^{-1}) \lambda.$$  \hspace{1cm} (3.9)

Note that by (3.7) we find

$$|E_{\lambda, \delta_2} \cap B_r(x)| \leq |\{M_1(\chi_{B_{2r}(x)} |\nabla (u_k - v_k)|^{\gamma_1}) \}^{\frac{1}{\gamma_1}} > \Lambda_0 \lambda / 9 \} \cap B_r(x)|$$

$$+ |\{M_1(\chi_{B_{2r}(x)} |\nabla (u - u_k)|^{\gamma_1}) \}^{\frac{1}{\gamma_1}} > \Lambda_0 \lambda / 9 \} \cap B_r(x)|$$

$$+ |\{M_1(\chi_{B_{2r}(x)} |\nabla v_k|^{\gamma_1}) \}^{\frac{1}{\gamma_1}} > \Lambda_0 \lambda / 9 \} \cap B_r(x)|.$$ \hspace{1cm} (3.10)

On the other hand, in view of (3.8) we see that for $\Lambda_0 \geq \max \{3^n, 20C_1 \}$ (C_1 is the constant in (3.8)) and $k \geq k_0$, it holds that

$$|\{M_1(\chi_{B_{2r}(x)} |\nabla v_k|^{\gamma_1}) \}^{\frac{1}{\gamma_1}} > \Lambda_0 \lambda / 9 \} \cap B_r(x)| = 0.$$

Thus, we deduce from (3.9) and (3.10) that for any $k \geq k_0$,

$$|E_{\lambda, \delta_2} \cap B_r(x)| \leq \frac{C}{\lambda^{\gamma_1}} \left[ \int_{B_2(x)} |\nabla (u_k - v_k)|^{\gamma_1} + \int_{B_2(x)} |\nabla (u - u_k)|^{\gamma_1} \right]$$

$$\leq \frac{C}{\lambda^{\gamma_1}} \left( (C_\eta \delta_2 + \delta_1^n + \eta \epsilon^{-1}) \lambda^{\gamma_1} r^n + \int_{B_2(x)} |\nabla (u - u_k)|^{\gamma_1} \right).$$
Then letting $k \to \infty$ we get

$$|E_{\lambda, \delta_2} \cap B_r(x)| \leq C \left( C_{\eta} \delta_2 + \delta_1^\kappa + \eta \varepsilon^{-1} \right)^{\gamma_1} |B_r(x)|.$$  

This gives

$$w(E_{\lambda, \delta_2} \cap B_r(x)) \leq c \left( \frac{|E_{\lambda, \delta_2} \cap B_r(x)|}{|B_r(x)|} \right)^{\nu} w(B_r(x)) \leq c \left( C_{\eta} \delta_2 + \delta_1^\kappa + \eta \varepsilon^{-1} \right)^{\gamma_1 \nu} w(B_r(x)) \leq \varepsilon w(B_r(x)),$$

where $\eta, \delta_1 \leq C(n, p, \Lambda, \gamma_1, \varepsilon, |w|_{A_\infty})$ and $\delta_2 \leq C(n, p, \Lambda, \gamma_1, \varepsilon, |w|_{A_\infty}, R/R_0)$.

2. The case $B_{8r}(x) \cap \Omega^c \neq \emptyset$: Let $x_3 \in \partial \Omega$ such that $|x_3 - x| = \text{dist}(x, \partial \Omega)$. We have

$$B_{2r}(x) \subset B_{10r}(x_3) \subset B_{100r}(x_3) \subset B_{109r}(x) \subset B_{109r}(x_1),$$

and

$$B_{100r}(x_3) \subset B_{109r}(x) \subset B_{109r}(x_2).$$

Applying Proposition 2.4 to $u = u_k \in W_0^{1,p}(\Omega), \mu = \mu_k$ and $B_{10R} = B_{100r}(x_3)$, for any $\eta > 0$ there exists $\delta_0 = \delta_0(n, p, \Lambda, \eta)$ such that the following holds. If $\Omega$ is a $(\delta_0, R_0)$-Reifenberg flat domain, there exists a function $V_k \in W^{1,\infty}(B_{10r}(x_3))$ such that

$$||\nabla V_k||_{L^\infty(B_{10r}(x_3))} \leq C_{\eta} \left[ \frac{|\mu_k|(B_{100r}(x_3))}{r^{n-1}} \right]^{\frac{1}{p-1}} + C \left( \int_{B_{100r}(x_3)} |\nabla u_k|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \eta \left( \int_{B_{100r}(x_3)} |\nabla u_k|^{2-p} \right)^{\frac{1}{2-p}},$$

and

$$\left( \int_{B_{10r}(x_3)} |\nabla(u_k - V_k)|^{\gamma_1} dx \right)^{\frac{1}{\gamma_1}} \leq C_{\eta} \left[ \frac{|\mu_k|(B_{100r}(x_3))}{r^{n-1}} \right]^{\frac{1}{p-1}} + C((|A|_{R_0})^\kappa + \eta) \left( \int_{B_{100r}(x_3)} |\nabla u_k|^{\gamma_1} \right)^{\frac{1}{\gamma_1}} + \eta \left( \int_{B_{100r}(x_3)} |\nabla u_k|^{2-p} \right)^{\frac{1}{2-p}},$$

for some $\kappa \in (0, 1)$. As above, we also obtain

$$|E_{\lambda, \delta_2} \cap B_r(x)| \leq C \left( C_{\eta} \delta_2 + \delta_1^\kappa + \eta \varepsilon^{-1} \right)^{\gamma_1} |B_r(x)|,$$

and thus

$$w(E_{\lambda, \delta_2} \cap B_r(x)) \leq c \left( \frac{|E_{\lambda, \delta_2} \cap B_r(x)|}{|B_r(x)|} \right)^{\nu} w(B_r(x)) \leq c \left( C_{\eta} \delta_2 + \delta_1^\kappa + \eta \varepsilon^{-1} \right)^{\gamma_1 \nu} w(B_r(x)) \leq \varepsilon w(B_r(x)).$$

where $\eta, \delta_1 \leq C(n, p, \Lambda, \gamma_1, \varepsilon, |w|_{A_\infty})$ and $\delta_2 \leq C(n, p, \Lambda, \gamma_1, \varepsilon, |w|_{A_\infty}, R/R_0)$.

Using [3.2] and [3.3], we can now apply Lemma 3.1 with $E = E_{\lambda, \delta_2}$ and $F = F_\lambda$ to complete the proof of the theorem.
4 Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. Our main tools are good-$\lambda$ type bounds obtained in Theorem 1.4 and stability results of renormalized solutions obtained in [3] Theorem 3.2.

Proof of Theorem 1.5. Let $R_0 > 0$ and fix a number $\gamma_1 \in \left(0, \frac{(p-1)n}{n-1}\right)$. Suppose for now that $u$ is a renormalized solution of (1.1) such that $|\nabla u| \in L_{2,\infty}^q(\Omega)$, $q > 2-p$.

By Theorem 1.4, for any $\epsilon > 0$, $R_0 > 0$ one can find $\delta = \delta(n, p, \Lambda, \epsilon, [w]_{A_\infty}) \in (0, 1/2)$, $\Lambda_0 = \Lambda_0(n, p, \Lambda) > 1$ such that if $\Omega$ is a $(\delta, R_0)$-Reifenberg flat domain and $[A]_{R_0} \leq \delta$ then

\[
\omega\left(\left\{(M(|\nabla u|^{1/\gamma_1})^{\frac{1}{\gamma_1}} > \Lambda_0 \lambda, M(|\nabla u|^{2-p})^{\frac{1}{2-p}} \leq \epsilon^{-1} \lambda, (M(\mu))^{\frac{1}{p-1}} \leq \delta_2 \lambda\right\} \cap \Omega\right)
\leq C\omega\left(\left\{(M(|\nabla u|^{1/\gamma_1})^{\frac{1}{\gamma_1}} > \lambda\right\} \cap \Omega\right),
\]

for all $\lambda > 0$. Here the constant $C$ depends only on $n, p, \Lambda, [w]_{A_\infty}$, and $diam(\Omega)/R_0$.

Thus, we find

\[
w\left(\left\{(M(|\nabla u|^{1/\gamma_1})^{\frac{1}{\gamma_1}} > t\right\} \cap \Omega\right) \leq \omega\left(\left\{(M(\mu))^{\frac{1}{p-1}} > \delta_2 \Lambda_0 \right\} \cap \Omega\right)
\]

\[
+ w\left(\left\{(M(|\nabla u|^{2-p})^{\frac{1}{2-p}} > \epsilon^{-1} \Lambda_0 \right\} \cap \Omega\right) + C\omega\left(\left\{(M(|\nabla u|^{1/\gamma_1})^{\frac{1}{\gamma_1}} > \frac{t}{\Lambda_0} \right\} \cap \Omega\right)
\]

for all $t > 0$. This gives,

\[
\|M(|\nabla u|^{1/\gamma_1})^{1/\gamma_1} \|_{L_{2,\infty}^q(\Omega)} \leq C\delta_2^{-1}\|M(\mu)^{1/(p-1)} \|_{L_{2,\infty}^q(\Omega)}
\]

\[
+ C\epsilon\|M(|\nabla u|^{2-p})^{1/(2-p)} \|_{L_{2,\infty}^q(\Omega)} + C\epsilon\|M(|\nabla u|^{1/\gamma_1})^{1/\gamma_1} \|_{L_{2,\infty}^q(\Omega)}.
\]

Using the boundedness of $M$ on $L_{q/(2-p)}^q(\mathbb{R}^n)$, where $q/(2-p) > 1$ and $w \in A_{\frac{q}{2-p}}$, and choosing $\epsilon < \frac{1}{4C}$ in the last inequality we deduce

\[
\|M(|\nabla u|^{1/\gamma_1})^{1/\gamma_1} \|_{L_{2,\infty}^q(\Omega)} \leq 2C\delta_2^{-1}\|M(\mu)^{1/(p-1)} \|_{L_{2,\infty}^q(\Omega)} + C\epsilon\|\nabla u \|_{L_{2,\infty}^q(\Omega)}.
\]

Thus with $\epsilon = \frac{1}{4C+C'}$ we conclude that

\[
\|\nabla u \|_{L_{2,\infty}^q(\Omega)} \leq \widetilde{C}\|M(\mu)^{1/(p-1)} \|_{L_{2,\infty}^q(\Omega)}.
\]

To show existence, let $B_{R_1}(x_1)$ be a ball such that $\Omega \subseteq B_{R_1}(x_1)$ and extend $\mu$ by zero outside $\Omega$. Then we can write

\[
\mu = f - \text{div}F + \mu^+ - \mu^-,
\]

as distributions in $B_{R_1+1}(x_1)$, where $f \in L^1(B_{R_1+1}(x_1))$, $F \in L_{\infty}^{p,\alpha}(B_{R_1+1}(x_1), \mathbb{R}^n)$, and $\mu_\pm$ is concentrated on a set of zero $p$-capacity. Let $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$ where $\rho \in C_0^\infty(B_1(0))$ is a nonnegative radial function with $\|\rho\|_{L^1(\mathbb{R}^n)} = 1$. Then for any $\epsilon \in (0, 1)$ we have

\[
\rho_\epsilon \ast \mu = \rho_\epsilon \ast f - \text{div}(\rho_\epsilon \ast F) + \rho_\epsilon \ast \mu^+ - \rho_\epsilon \ast \mu^-.
\]
as distributions in $B_{R_1}(x_1)$.

Let $u_\epsilon \in W^{1,p}_0(\Omega)$ be the unique solution of

\[
\begin{cases}
-\text{div}(A(x,\nabla u)) = \rho_\epsilon * \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Then we can deduce from [14] Theorem 1.10 that $|\nabla u_\epsilon| \in L^p(\Omega)$ provided $\delta = \delta(n,p,\Lambda,q,s,[w]_{A^{\frac{2}{p-1}}})$ is sufficiently small. Thus we may apply [14] and get

\[
\|\nabla u_\epsilon\|_{L^p(\Omega)} \leq C\|\mu_1(\mu_2 * \mu)\|^\frac{1}{p-1} \|L^p(\Omega)
\]

\[
\leq C\|\mu_1(\mu_2)\|^\frac{1}{p-1} \|L^p(\Omega)
\]

The theorem now follows from the stability result of [6] Theorem 3.2.

\section{Proof of Theorem 1.8}

We will need the following important compactness result.

\begin{lemma}
Suppose that $1 < p \leq \frac{2n-2}{n-1}$. For each $j > 0$, let $\mu_j \in \mathfrak{M}_0(\Omega)$ and $u_j$ be the solution of (1.1) with datum $\mu = \mu_j$ in $\Omega$. Assume that $\{|\mu_1(\mu_2)\|^\frac{1}{p-1}\}_{j}$, $q > 2 - p$, is a bounded and equi-integrable subset of $L^1(\Omega)$. Then, there exists a subsequence $\{u_{j'}\}_{j'}$ and a finite a.e. function $u$ with the property that $T_k(u) \in W^{1,p}_0(\Omega)$ for all $k > 0$, $u_{j'} \to u$ a.e., and

\[
\nabla u_{j'} \to \nabla u \text{ strongly in } L^q(\Omega, \mathbb{R}^n).
\]

\end{lemma}

\begin{proof}
By de la Vallée-Poussin Lemma on equi-integrability, there exists a strictly increasing and convex function $G : [0, \infty) \to [0, \infty)$ with $G(0) = 0$ such that $\lim_{t \to \infty} G(t)/t = \infty$ and

\[
\sup_j \int_\Omega G(|\mu_1(\mu_2)|)^\frac{1}{p-1})wdx \leq C.
\]

Moreover, we may assume that $G$ satisfies a moderate growth condition (see [15]): there exists $c_1 > 1$ such that

\[
G(2t) \leq c_1 G(t) \quad \forall t \geq 0.
\]

Let $\Phi(t) := G(t^q)$, where $q > 2 - p > p - 1$. Then applying (1.1) with $w = 1$ and with $\Phi^{-1}(t)$ in place of $t$ we find

\[
\left|\left\{\Phi(\mu_1(\nabla u_j \mid \nabla u_j)) > t\right\} \cap \Omega\right| \leq \left|\left\{\Phi(\mu_1(\nabla u_j \mid \nabla u_j)) > t\right\} \cap \Omega\right|
\]

\[
+ \left|\left\{\Phi(\mu_2(\nabla u_j \mid \nabla u_j)) > t\right\} \cap \Omega\right|
\]

\[
+ C \varepsilon \left|\left\{\Phi(\mu_3(\nabla u_j \mid \nabla u_j)) > t\right\} \cap \Omega\right|
\]

\end{proof}
for any $\epsilon > 0$. Here $\delta_2$ depends on $\epsilon$, but $\Lambda_0$ and $C$ do not.

Then arguing as in the proof of [19, Theorem 1.4] we find

$$
\int_\Omega \Phi(\|\nabla u_j\|^{\gamma_1}) \frac{1}{\gamma_1} dx \leq H(\epsilon) \int_\Omega \Phi(\|M_1(\mu_j)\|^{\frac{1}{p-1}}) dx \\
+ 2 \int_\Omega \Phi(\|\epsilon \Lambda_0(M(||\nabla u_j||^{2-p})\|^{\frac{1}{p}}) dx
$$

for all sufficiently small $\epsilon > 0$.

Note that by approximation as in the proof of Theorem 1.5 and by uniqueness (see Remark 1.6), we may assume that $\mu_j \in C^\infty(\Omega)$. Thus by the result of [14, Theorem 1.10], we may assume that

$$
\int_\Omega \Phi(|\nabla u_j|) dx < +\infty. \tag{5.3}
$$

Now as the function $t \mapsto \Phi(t^{\frac{1}{p-1}}) = G(t^{\frac{p}{p-1}})$ satisfies the $\nabla_2$ condition (see [21]), we deduce (see, e.g., [10, 2]) that

$$
2 \int_\Omega \Phi(\|\epsilon \Lambda_0(M(||\nabla u_j||^{2-p})\|^{\frac{2}{p-1}}) dx \leq C \Lambda_0 \epsilon \int_\Omega \Phi(|\nabla u_j|) dx. \tag{5.4}
$$

Combining (5.2), (5.3), (5.4), and choosing $\epsilon$ sufficiently small we arrive at

$$
\int_\Omega G(|\nabla u_j|^{\gamma}) dx \leq C \int_\Omega G(\|M_1(\mu_j)\|^{\frac{2}{p-1}}) dx \leq C.
$$

Thus by de la Vallée-Poussin Lemma the set $\{|\nabla u_j|^{\gamma}\}$ is also bounded and equi-integrable in $L^1(\Omega)$.

On the other hand, it follows from the proof of [6, Theorem 3.4] that there exists a subsequence $\{u_{j'}\}_{j'}$ converging a.e. to a function $u$ such that $|u| < \infty$ a.e., $T_k(u) \in W^{1,p}_0(\Omega)$ for all $k > 0$, and moreover

$$
\nabla u_{j'} \to \nabla u \text{ a.e. in } \Omega.
$$

At this point, applying Vitali Convergence Theorem we obtain the strong convergence (5.1) as desired.

Proof of Theorem 1.8. The proof of Theorem 1.8 is based on Schauder Fixed Point Theorem using Lemma 5.1 and Theorem 1.5. Indeed, with these results at hand, the proof is similar to that of [19, Theorem 1.9], and thus we omit the details.

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