DARBOUX TRANSFORMS AND SIMPLE FACTOR DRESSING OF
CONSTANT MEAN CURVATURE SURFACES

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Abstract. We define a transformation on harmonic maps \( N : M \rightarrow S^2 \) from a Riemann surface \( M \) into the 2–sphere which depends on a parameter \( \mu \in \mathbb{C}^* \), the so–called \( \mu \)–Darboux transformation. In the case when the harmonic map \( N \) is the Gauss map of a constant mean curvature surface \( f : M \rightarrow \mathbb{R}^3 \) and \( \mu \) is real, the Darboux transformation of \( -N \) is the Gauss map of a classical Darboux transform of \( f \). More generally, for all parameter \( \mu \in \mathbb{C}^* \) the transformation on the harmonic Gauss map of \( f \) is induced by a (generalized) Darboux transformation on \( f \). We show that this operation on harmonic maps coincides with simple factor dressing, and thus generalize results on classical Darboux transforms of constant mean curvature surfaces [HJP97], [Bur06], [IK05]: every \( \mu \)–Darboux transform is a simple factor dressing, and vice versa.

1. Introduction

By the Ruh–Vilms Theorem [RV70] a constant mean curvature surface \( f : M \rightarrow \mathbb{R}^3 \) of a Riemann surface \( M \) into \( \mathbb{R}^3 \) is characterized by the harmonicity of its Gauss map \( N : M \rightarrow S^2 \). This fact allows to use integrable system methods for constant mean curvature surfaces: using the harmonic Gauss map one can introduce a spectral parameter \( \lambda \) to obtain the associated \( \mathbb{C}^* \)–family \( d_{\lambda} \) of flat connections on the trivial \( \mathbb{C}^2 \) bundle over \( M \). This family is unitary on the unit circle where it describes the associated family of harmonic maps on the universal cover \( \tilde{M} \) of \( M \). More generally, we will use families of flat connections to construct new harmonic maps: we consider a \( \mathbb{C}^* \) family of flat connections \( d_{\lambda} \) so that \( d_{\lambda=1} = d \) is the trivial connection and \( d_{\lambda} \) satisfies a reality condition. Assuming that the map \( \lambda \rightarrow d_{\lambda}^{(0,1)} \) (which sends \( \lambda \in \mathbb{C}^* \) to the \((0,1)\)–part of \( d_{\lambda} \)) can be extended to a map on \( \mathbb{CP}^1 \) which is meromorphic in \( \lambda \) with only a simple pole at zero we see that \( d_{\lambda} \) is of the form \( d_{\lambda} = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)} \) with \( \omega^{(1,0)} \) and \( \omega^{(0,1)} \) of type \((1,0)\) and \((0,1)\) respectively. If in addition \( \omega^{(1,0)} \) is nilpotent then \( d_{\lambda} \) is the associated family of a harmonic map. From our point of view, dressing of a harmonic map [Uhl89, TU00] is thus the gauge of \( d_{\lambda} \) by an appropriate dressing matrix \( r_{\lambda} \): we give conditions on \( r_{\lambda} \) such that \( \hat{d}_{\lambda} = r_{\lambda} \cdot d_{\lambda} \) is the associated family of a harmonic map. Fixing \( \mu \in \mathbb{C}_* \), a simple factor dressing is then given by a dressing matrix \( r_{\lambda} \) which has a simple pole at \( \bar{\mu}^{-1} \) if \( \mu \notin S^1 \), and which depends on the choice of a \( d_{\mu} \)–parallel line subbundle of the trivial \( \mathbb{C}^2 \) bundle over the universal cover \( \tilde{M} \) of \( M \). We emphasize that both the associated family and the dressing are given by an operation on the harmonic map, and only the Sym–Bobenko formula [Bob91] then induces a transformation on constant mean curvature surfaces. We compare our definition of a simple factor dressing with the simple factor dressing in [DK05] which is defined on the frame of the constant mean curvature surface: indeed both transformations agree up to a rigid motion.

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In contrast to dressing, the classical Darboux transformation is originally a transformation on the level of surfaces: geometrically, two conformal immersions $f, \hat{f} : M \to \mathbb{R}^3$ form a Darboux pair if there exists a sphere congruence enveloping both $f$ and $\hat{f}$. In this case, both $f$ and $\hat{f}$ are isothermic. In particular, the classical Darboux transformation can be applied to a constant mean curvature surface $f : M \to \mathbb{R}^3$: the map $\hat{f} : M \to \mathbb{R}^3$ is a classical Darboux transform of $f$ if and only if $T = \hat{f} - f$ is a solution of a certain Riccati equation. However, $\hat{f}$ has constant mean curvature only if $T$ additionally satisfies an initial condition.

In [BLPP08] the classical Darboux transformation is generalized to a transformation on conformal immersions $f : M \to S^4$ by weakening the enveloping condition. It turns out that this Darboux transformation is also a key ingredient for integrable systems methods in surface theory: in the case when $M = T^2$ is a 2–torus the spectral curve of a conformal torus $f : T^2 \to S^2$ is essentially the set of all Darboux transforms $\hat{f} : T^2 \to S^4$ of $f$.

In this paper, we are interested in a (local) transformation theory for general constant mean curvature surfaces $f : M \to \mathbb{R}^3$, and thus have to allow the Darboux transforms $\hat{f} : \hat{M} \to \mathbb{R}^4$ to be defined on the universal cover $\hat{M}$ of $M$. To preserve the constant mean curvature property we will only consider so–called $\mu$–Darboux transforms [CLP10]: for $\mu \in \mathbb{C}_\ast$ a $\mu$–Darboux transform $\hat{f}$ of a constant mean curvature surface $f$ is constructed by using a parallel section of $d_\mu$ where $d_\mu$ is the associated family of the Gauss map $N$ of $f$. In this case, the difference $T = \hat{f} - f$ between $f$ and $\hat{f}$ also satisfies a Riccati type equation which generalizes the aforementioned equation. It turns out $\hat{f}$ is a classical Darboux transform if and only if $\mu \in S^1 \cup \mathbb{R}_\ast$. In all cases $T$ satisfies the required initial condition to preserve the constant mean curvature property: every $\mu$–Darboux transform of a constant mean curvature surface has constant mean curvature. Thus, the $\mu$–Darboux transformation induces a transformation on the harmonic Gauss map.

We extend this latter transformation to a $\mu$–Darboux transformation on harmonic maps $N : M \to S^2$: using the associated family of flat connections of $N$ and a $d_\mu$–parallel section, we present an algebraic operation to obtain a new harmonic map on the universal cover $\hat{M}$ of $M$. It turns out, that a simple factor dressing of a harmonic map $N$ coincides with a $\mu$–Darboux transform of $N$: the $d_\mu$–parallel bundle which is used to define the simple factor dressing matrix $r_\lambda$ is spanned by the $d_\mu$–parallel section which gives the $\mu$–Darboux transform. In particular, we obtain a generalization of results on the classical Darboux transformation [HJP97, Bur06, IK05]: a $\mu$–Darboux transform of a constant mean curvature surface $f : M \to \mathbb{R}^3$ is given by a simple factor dressing of the parallel constant mean curvature surface of $f$, and vice versa.

Many other surface classes are also linked to harmonicity, e.g., Hamiltonian stationary Lagrangians $f : M \to \mathbb{C}^2$. In this case, the so–called left normal $N : M \to S^2$ of $f$ is harmonic, and we can apply again both a simple factor dressing and the $\mu$–Darboux transformation on the harmonic map $N$. The $\mu$–Darboux transformation on the harmonic left normal is induced by a transformation on the level of surfaces [LR10]. In particular, though there is no Sym–Bobenko formula for Hamiltonian stationary Lagrangians, we now have an interpretation of simple factor dressing of a harmonic left normal $N$ on the level of surfaces: the left normal of a $\mu$–Darboux transform of $f$ is the simple factor dressing of $-N$. Moreover, in [Qui08] a dressing on (constrained) Willmore surfaces is introduced and it is shown that the simple factor dressing with simple pole at $\mu \in \mathbb{R}_\ast \cup S^1$ coincides with the Darboux transformation on the conformal Gauss map of a (constrained) Willmore surface as defined in [BFL+02]. Indeed, the latter transformation can be extended [Les10]...
in a way similar to what we discuss in this paper to the case when \( \mu \in \mathbb{C}_* \), and we expect that the simple factor dressing of a (constrained) Willmore surface \( f : M \to S^4 \) is the \( \mu \)-Darboux transformation on the conformal Gauss map of \( f \).

2. Harmonic maps and family of flat connections

We first recall the well-known link \([Uhl89]\) between a harmonic map into the 2-sphere and a \( \mathbb{C}_* \)-family of flat connections. In the following we will always identify the Euclidean 4-space \( \mathbb{R}^4 \) with the quaternions \( \mathbb{H} \). In particular, we identify \( \mathbb{R}^3 = \text{Im} \mathbb{H} \) with Euclidean product \( \langle a, b \rangle = -\text{Re} \langle ab \rangle \), \( a, b \in \text{Im} \mathbb{H} \), and \( S^2 = \{ n \in \text{Im} \mathbb{H} \mid n^2 = -1 \} \). A map \( N : M \to S^2 \) from a Riemann surface \( M \) into the 2-sphere is harmonic if it is a critical point of the energy functional, that is \([FLPP01]\), if

\[
d * dN = NdN \wedge *dN.
\]

Here we write for a 1-form \( \omega \)

\[
* \omega(X) = \omega(J_M X), \quad X \in TM,
\]

where \( J_M \) denotes the complex structure of the Riemann surface \( M \). In other words, \( * \) is the negative Hodge star operator.

We decompose \( dN \) into \((1,0)\) and \((0,1)\)-parts

\[
(dN)' = \frac{1}{2}(dN - N * dN), \quad (dN)'' = \frac{1}{2}(dN + N * dN)
\]

with respect to \( N \). The harmonicity condition is now expressed by the condition that \((dN)'\), or equivalently \((dN)''\), is closed.

Every smooth map \( N : M \to S^2 \) induces a quaternionic linear endomorphism \( J \in \Gamma(\text{End}(\mathbb{H})) \) on the trivial bundle \( \mathbb{H} = M \times \mathbb{H} \) over \( M \) by setting

\[
J \phi = N \phi, \quad \phi \in \Gamma(\mathbb{H}).
\]

If we define the Hopf fields of the complex structure \( J \) (with respect to the flat connection \( d \)) as

\[
A = \frac{J(dJ) + *dJ}{4} \quad \text{and} \quad Q = \frac{J(dJ) - *dJ}{4}
\]

then the closedness of \((dN)'\) can be rephrased in terms of the complex structure \( J \):

**Lemma 2.1.** A smooth map \( N : M \to S^2 \) is harmonic if and only if the Hopf field \( A \) of the associated complex structure \( J \in \Gamma(\text{End}(\mathbb{H})) \) satisfies

\[
d * A = 0.
\]

Note that the derivative of \( J \) is given in terms of the Hopf fields by

\[
dJ = 2(*Q - *A),
\]

and therefore, the condition \( d * A = 0 \) is equivalent to \( d * Q = 0 \). Since

\[
A = \frac{1}{2} * (dJ)' \quad \text{and} \quad Q = -\frac{1}{2} * (dJ)''
\]

the Hopf fields both anti-commute with the complex structure \( J \) and have type \((1,0)\) and \((0,1)\) with respect to \( J \), that is

\[
* A = JA = -AJ, \quad *Q = -JQ = QJ.
\]
In particular, by type considerations this implies
\begin{equation}
A \wedge Q = Q \wedge A = 0.
\end{equation}

Moreover, if we decompose the trivial connection \( d \) on \( \mathbb{H} \) into \( J \) commuting and anti-commuting parts \( d = d_+ + d_- \) then
\begin{equation}
d_- = A + Q
\end{equation}
where we used \( d_- = \frac{1}{2} J (dJ) \) and the equations \((2.1)\) and \((2.3)\).

To introduce a spectral parameter \( \lambda \in \mathbb{C}^* \) we consider \( \mathbb{H} \) as a complex \( \mathbb{C}^2 \) via the splitting \( \mathbb{H} = \mathbb{C} + j \mathbb{C} \) with \( \mathbb{C} = \text{span}\{1, i\} \). In other words, if we define the complex structure \( I \) by right–multiplication by \( i \in \mathbb{H} \), then \( \mathbb{C}^2 \) can be identified with \((\mathbb{H}, I)\). Under this identification \( I \in \text{End}_\mathbb{C}(\mathbb{C}^2) \) becomes a complex linear endomorphism. For simplicity of notation we will use the same symbol for the endomorphism \( A \) given by \((2.8)\) im \( I \).

Moreover, if we decompose the trivial connection \( d \) on \( \mathbb{H} \) into \( J \) commuting and anti–commuting parts \( d = d_+ + d_- \) then
\begin{equation}
d_- = A + Q
\end{equation}
where we used \( d_- = \frac{1}{2} J (dJ) \) and the equations \((2.1)\) and \((2.3)\).

If we denote by \( E \) and \( E^\perp = Ej \) the \( \pm i \) eigenspaces of the complex structure \( J \) on \( \mathbb{H} \) respectively then the orthogonal projections with respect to the splitting \( \mathbb{H} = E \oplus E^\perp \) are given by
\begin{align*}
\pi_E &= \frac{1}{2} (1 - IJ), \quad \pi_{E^\perp} = \frac{1}{2} (1 + IJ).
\end{align*}

Since \( J \) is quaternionic linear \( J \) commutes with \( I \), and so does \( A \). Recalling \((2.6)\) that \( A \) anti–commutes with \( J \), we see
\begin{equation}
A^{(1,0)} = A \pi_{E^\perp} = \pi_E A, \quad \text{and} \quad A^{(0,1)} = A \pi_E = \pi_{E^\perp} A,
\end{equation}
in particular \((A^{(1,0)})^2 = (A^{(0,1)})^2 = 0\), and
\begin{equation}
\text{im} A^{(1,0)} \subset E \subset \ker A^{(1,0)}, \quad \text{im} A^{(0,1)} \subset E^\perp \subset \ker A^{(0,1)}.
\end{equation}

Since \( E^\perp = Ej \) we have \( \pi_E (\phi j) = (\pi_{E^\perp} \phi) j \) and we obtain
\begin{equation}
A^{(1,0)} (\phi j) = (A^{(0,1)}) j, \quad \phi \in \Gamma(\mathbb{H}).
\end{equation}
Moreover, \( \lambda (\phi j) = (\lambda \phi) j \) for \( \lambda \in \mathbb{C} \) so that the reality condition
\begin{equation}
d_\lambda (\phi j) = (d_{-\lambda}) j, \quad \phi \in \Gamma(\mathbb{H}),
\end{equation}

holds for the complex connection $d_\lambda$. In particular, $d_\lambda$ is a quaternionic connection if and only if $\lambda \in S^1$.

To compute the curvature of $d_\lambda$ we first observe that $I$ commutes with $J$, and thus also with $A$, since $J$ is quaternionic linear. Denoting by
\[(2.11) \quad \alpha_\lambda = (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)} \]
the connection form of $d_\lambda$ we see with (2.7) that
\[\alpha_\lambda \wedge \alpha_\lambda = (2 - \lambda - \lambda^{-1})A \wedge A.\]

On the other hand, we write with (2.3)
\[(2.12) \quad A^{(1,0)} = \ast A \frac{J - I}{2}, \quad A^{(0,1)} = \ast A \frac{J + I}{2},\]
and recall (2.1) and (2.4) to obtain
\[d\alpha_\lambda = (d \ast A) \left( (\lambda - 1)\frac{J - I}{2} + (\lambda^{-1} - 1)\frac{J + I}{2} \right) + (\lambda + \lambda^{-1} - 2) \ast A \wedge \ast A.\]

Since $A \wedge A = \ast A \wedge \ast A$ the curvature of $d_\lambda$ is therefore given by
\[R_\lambda = (d \ast A) \left( (\lambda - 1)\frac{J - I}{2} + (\lambda^{-1} - 1)\frac{J + I}{2} \right),\]

Lemma 2.1 now yields the familiar link between harmonic maps and $\mathbb{C}_\ast$–families of flat connections:

**Theorem 2.2.** A smooth map $N : M \to S^2$ is harmonic if and only if the associated family of complex connections
\[d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}\]
on the trivial bundle $\mathbb{C}^2$ over $M$ is flat for all $\lambda \in \mathbb{C}_\ast$.

**Remark 2.3.** Up to gauge equivalence, $d_\lambda$ is the family of flat connections used by [Hit90] to construct the spectral curve of a harmonic torus in the 2–sphere, see [CLP10].

The family of flat connections induces a $S^1$–family of harmonic maps. This family is given by $d_\lambda$–parallel sections and thus is only defined on the universal cover $\tilde{M}$ of $M$. We denote by $\tilde{\mathbb{H}} = \tilde{M} \times \mathbb{H}$ and $\tilde{\mathbb{C}}^2 = \tilde{M} \times \mathbb{C}^2$ the trivial bundles over $\tilde{M}$.

**Theorem 2.4.** Let $N : M \to S^2$ be a harmonic map from a Riemann surface $M$ into the 2–sphere, $J$ the corresponding complex structure, and $d_\lambda$ the associated family of flat connections. For $\mu \in \mathbb{C}_\ast$ the Hopf field $A_\mu = \frac{J(d_\mu J) + \ast d_\mu J}{4}$ of $J$, taken with respect to the flat connection $d_\mu$, is co–closed with respect to $d_\mu$ for $\mu \in \mathbb{C}_\ast$, that is
\[d_\mu \ast A_\mu = 0.\]

In particular, if $\varphi \in \Gamma(\tilde{\mathbb{H}})$ is a $d_\mu$–parallel section for $\mu \in S^1$, then $N_\varphi = \varphi^{-1}N \varphi : \tilde{M} \to S^2$ is harmonic with respect to $d$. Furthermore, denoting by $\Phi$ the endomorphism given by left multiplication by $\varphi$, the associated $\mathbb{C}_\ast$ family of flat connections of $N_\varphi$ is given by the gauge
\[d_{\varphi,\lambda} = \Phi^{-1} \cdot d_{\lambda \mu}.\]
Proof. Write \( d_\mu = d + \alpha_\mu \) with connection form \( \alpha_\mu \) given by (2.11). The equations (2.3) and (2.12) show \(*\alpha_\mu = J_\alpha_\mu = -\alpha_\mu J\) where we also used that \([I, J] = 0\). From type arguments we therefore obtain

\[
\alpha_\mu \wedge (dJ)^\prime = (dJ)^\prime \wedge \alpha_\mu = 0.
\]

Using \( d_\mu J = dJ + [\alpha_\mu, J] \) we also see with \([I, J] = 0\)

\[
(\mu d_\mu J)' = (dJ)' + [\alpha_\mu, J],
\]

that is, the Hopf field of \( J \) with respect to \( d_\mu \) is

\[
A_\mu = \mu A^{(1,0)} + \mu^{-1} A^{(0,1)}.
\]

Since \( J \) is harmonic with respect to \( d \) and \( d\alpha_\mu + \alpha_\mu \wedge \alpha_\mu = 0 \) by the flatness of \( d_\mu \), the equations (2.13) and (2.14) give

\[
d_\mu(d_\mu J)' = d((dJ)' + [\alpha_\mu, J]) + [\alpha_\mu \wedge ((dJ)' + [\alpha_\mu, J])] = 0.
\]

This shows that \(*A_\mu = -\frac{1}{2}(d_\mu J)'\) is closed with respect to \( d_\mu \). For \( \mu \in S^1 \) the connection \( d_\mu \) is quaternionic, and is given by the gauge \( d_\mu = \Phi \circ d \circ \Phi \). Furthermore, the complex structure of \( N_\varphi \) is given by \( J_\varphi = \Phi^{-1} J_\Phi \), and we obtain \( dJ_\varphi = \text{Ad}(\Phi^{-1})(d_\mu J) \) since \( \Phi \) is parallel with respect to \( d_\mu \). Thus, \( J_\varphi \) has Hopf field

\[
A_\varphi = \Phi^{-1} A_\mu \Phi,
\]

and \( 0 = d_\mu (*A_\mu \circ \Phi) = \Phi \circ (d \circ A_\varphi) \) shows that \( N_\varphi \) is harmonic with respect to \( d \). Finally, from (2.15) we see that

\[
d_\mu + (\lambda - 1)A^{(1,0)}_\mu + (\lambda^{-1} - 1)A^{(0,1)}_\mu = d + (\lambda \mu - 1)A^{(1,0)} + ((\lambda \mu)^{-1} - 1)A^{(0,1)} = d_\lambda,
\]

and gauging \( d_\varphi, \lambda = d + (\lambda - 1)A^{(1,0)}_\varphi + (\lambda^{-1} - 1)A^{(0,1)}_\varphi \) by \( \Phi \) we get

\[
\Phi \cdot d_\varphi, \lambda = d_\lambda.
\]

\[ \Box \]

Remark 2.5. Since \( d_\mu \) is quaternionic for \( \mu \in S^1 \), the section \( \varphi \) is unique up to a quaternionic constant and thus \( N_\varphi \) is uniquely given by \( \mu \) up to an orthogonal map. The family \( N_\varphi \) is called the associated family of \( N \).

3. Dressing of a harmonic map into the 2-sphere

We have seen that a harmonic map from a Riemann surface \( M \) into the 2-sphere gives rise to an associated \( \mathbb{C}_s \)-family of flat connections \( d_\lambda \) on the trivial \( \mathbb{C}^2 \) bundle over \( M \) and the associated family of harmonic maps on the universal cover \( \tilde{M} \) of \( M \). To construct new harmonic maps from given ones via dressing, one uses again the family of flat connections: Gauging \( d_\lambda \) by an appropriate \( \lambda \)-dependent dressing matrix, one can reconstruct a harmonic map from the new family of flat connections.

**Theorem 3.1.** Let \( M \) be Riemann surface and assume that for every \( \lambda \in \mathbb{C}_s \) the connection

\[
d_\lambda = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)}, \quad \lambda \in \mathbb{C}_s,
\]

on \( \mathbb{C}^2 \) is flat where

\[
\omega^{(1,0)} \in \Gamma(K \text{End}_\mathbb{C}(\mathbb{C}^2)), \quad \omega^{(0,1)} \in \Gamma(\overline{K} \text{End}_\mathbb{C}(\mathbb{C}^2))
\]
are non–trivial endomorphism–valued 1–forms on the trivial bundle $\mathbb{C}^2 = (\mathbb{H}, I)$ over $M$ of type $(1, 0)$ and $(0, 1)$ with respect to $I$. If $\omega^{(1,0)}$ is in addition nilpotent

$$
(\omega^{(1,0)})^2 = 0
$$

and $\omega^{(1,0)}$ and $\omega^{(0,1)}$ satisfy the reality condition

$$
\omega^{(1,0)}(\phi j) = (\omega^{(0,1)}\phi) j, \quad \phi \in \Gamma(\mathbb{C}^2),
$$

then $d\lambda$ is the associated family of a harmonic map $N : M \to S^2$.

Proof. The kernel of $\omega^{(1,0)}$ defines a line bundle $E = \ker \omega^{(1,0)}$ away from the zeros of $\omega^{(1,0)}$. To show that $E$ extends smoothly across the zeros of $\omega^{(1,0)}$, we equip the bundle $K \text{End}_\mathbb{C}(\mathbb{C}^2)$ with a complex holomorphic structure $D$ such that $\omega^{(1,0)}$ is a holomorphic section in $(K \text{End}_\mathbb{C}(\mathbb{C}^2), D)$.

First note that a section $\sigma \in \Gamma(\mathbb{K})$ can be identified with the 2–form $\hat{\sigma} \in \Omega^2(M)$ by setting $\hat{\sigma}(X, Y) = \sigma(X, Y) - \sigma(Y, X)$ for $X, Y \in \Gamma(TM)$. Under this identification, a complex holomorphic structure on the canonical bundle $K$ is a complex linear operator $D : \Gamma(K) \to \Omega^2(M)$ satisfying the Leibniz rule

$$
D(\omega \lambda) = (D\omega)\lambda - \omega \wedge d\lambda
$$

for $\omega \in \Gamma(K)$ and $\lambda : M \to \mathbb{C}$. In particular, when tensoring the canonical bundle $K$ with $\text{End}_\mathbb{C}(\mathbb{C}^2)$, we see that for every $\eta \in \Gamma(K \text{End}_\mathbb{C}(\mathbb{C}^2))$ the operator $D : \Gamma(K \text{End}_\mathbb{C}(\mathbb{C}^2)) \to \Omega^2(\text{End}_\mathbb{C}(\mathbb{C}^2))$.

$$
D\omega = d\omega - [\eta \wedge \omega], \quad \omega \in \Gamma(K \text{End}_\mathbb{C}(\mathbb{C}^2)),
$$

is a complex holomorphic structure on $K \text{End}_\mathbb{C}(\mathbb{C}^2)$.

Since $d\lambda$ is flat for all $\lambda \in \mathbb{C}$ and $d$ is the trivial connection we have

$$
0 = R_\lambda = (\lambda - 1) d\omega^{(1,0)} + (\lambda^{-1} - 1) d\omega^{(0,1)} + (\lambda^{-1} - 1)(\lambda - 1) [\omega^{(0,1)} \wedge \omega^{(1,0)}]
$$

where we used that $(\omega^{(1,0)})^2 = (\omega^{(0,1)})^2 = 0$ by (3.2) and (3.3). In particular, taking the $\lambda$–coefficient we have

$$
0 = d\omega^{(1,0)} - [\omega^{(0,1)} \wedge \omega^{(1,0)}],
$$

that is, $\omega^{(1,0)} \in \Gamma(K \text{End}_\mathbb{C}(\mathbb{C}^2))$ is a holomorphic section with respect to the holomorphic structure $D = d - \omega^{(0,1)}$. But then $\ker \omega^{(1,0)}$ can be extended across the zeros of $\omega^{(1,0)}$ into a holomorphic line bundle $E$ over $M$.

We now define a complex structure $J \in \Gamma(\text{End}(\mathbb{H}))$ on $\mathbb{H} = E \oplus E^\perp$ by setting $J|_E = I$ and $J|_{E^\perp} = -I$. Note that $J$ is quaternionic linear since $E^\perp = E_j$. If we decompose $d = d_+ + d_-$ into $J$ commuting and anti–commuting parts, then $d_- = \frac{1}{2} J(dJ)$, and $E$ and $E^\perp$ are $d_+$ stable, whereas $d_-$ maps $E$ to $E^\perp$ and vice versa. For $\phi \in \Gamma(E)$ we have

$$
0 = (d - \omega^{(0,1)})(\omega^{(1,0)} \phi) = -\omega^{(1,0)} \wedge (d\phi - \omega^{(0,1)} \phi)
$$

where we used that $\omega^{(1,0)}$ is holomorphic with respect to $d - \omega^{(0,1)}$ and $E \subset \ker \omega^{(1,0)}$. Decomposing $d\phi = (d\phi)^{(1,0)} + (d\phi)^{(0,1)}$ into $(1, 0)$ and $(0, 1)$–part with respect to $I$, we see by type arguments that

$$
\omega^{(1,0)} \wedge (d\phi)^{(1,0)} = 0
$$

and $(d\phi)^{(0,1)} - \omega^{(0,1)} \phi$ is a 1–form with values in $E$. Now (3.2) shows $\text{im} \omega^{(1,0)} \subset E$ and the reality condition (3.3) gives $\text{im} \omega^{(0,1)} \subset E^\perp$. Since $E$ is $d_+$ stable we thus see for $\phi \in \Gamma(E)$ that

$$
0 = \pi_{E^\perp}((d\phi)^{(0,1)} - \omega^{(0,1)} \phi) = (\pi_{E^\perp} (d_- \phi)^{(0,1)}) - \omega^{(0,1)} \phi.
$$
Recall (2.5) that $d_- = A + Q$ and observe that
\[ Q^{(0,1)}\phi = \frac{1}{2}(Q + *Q)\phi = 0 \]
since $J|_E = I$ and (2.3) holds. Substituting into (3.4) we see
\[ \omega^{(0,1)} = A^{(0,1)} \]
on $E$. Since $E \subset \ker \omega^{(1,0)}$ the reality condition (3.3) shows $E^\perp \subset \ker \omega^{(0,1)}$ and from (2.8) we thus see that $\omega^{(0,1)} = A^{(0,1)}$ on $\mathbb{C}^2 = E \oplus E^\perp$. Using the reality conditions (3.3) and (2.9) we also have $\omega^{(1,0)} = A^{(1,0)}$ on $\mathbb{C}^2$. In other words, $d_\lambda$ is the associated family of complex connections (2.6) of the map $N : M \to S^2$ which is given by the complex structure $J$. In particular, since $d_\lambda$ is flat for all $\lambda \in \mathbb{C}_*$ the map $N$ is harmonic by Theorem 2.2.

**Remark 3.2.** Let $d_\lambda$ be a family of flat connections satisfying the assumptions of Theorem 3.1. From the previous proof we see that the associated harmonic map $N$ of $d_\lambda$ has complex structure $J$ with $J|_E = I$ and $J_{E^\perp} = -I$. Here $E$ is the line bundle defined by the kernel of $\omega^{(1,0)}$.

To obtain families of flat connections $d_\lambda$ of the form (3.1) we observe:

**Lemma 3.3.** Let $d_\lambda$, $\lambda \in \mathbb{C}_*$, be a family of connections on $\mathbb{C}^2$ satisfying

(i) the reality condition (2.10)
\[ d_\lambda(\phi j) = (d_{\lambda^{-1}}\phi) j \quad \text{for} \quad \phi \in \Gamma(\mathbb{C}^2), \]

(ii) the $(0,1)$–part of $d_\lambda$ can be extended to a meromorphic map $\lambda \mapsto d_\lambda^{(0,1)}$ on $\mathbb{CP}^1$ which is holomorphic on $\mathbb{C}_* \cup \infty$ and has a simple pole at 0, and

(iii) $d_{\lambda=1} = d$.

Then the family of connections $d_\lambda$ is of the form
\[ d_\lambda = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)} \]
with $\omega^{(1,0)} \in \Gamma(K\operatorname{End}_\mathbb{C}(\mathbb{C}^2))$ and $\omega^{(0,1)} \in \Gamma(\bar{K}\operatorname{End}_\mathbb{C}(\mathbb{C}^2))$.

**Proof.** Write $d_\lambda = d + \omega_\lambda$ with connection form $\omega_\lambda \in \Omega^1(\operatorname{End}_\mathbb{C}(\mathbb{C}^2))$. The conditions (i) and (ii) imply that the $(0,1)$–part of $d_\lambda$ is given by $d^{(0,1)} + (\lambda^{-1} - 1)\omega_\lambda^{(0,1)}$ with $\omega_\lambda^{(0,1)} \in \Gamma(\bar{K}\operatorname{End}_\mathbb{C}(\mathbb{C}^2))$ for $\lambda \in \mathbb{CP}^1$. Now $\lambda \mapsto \omega_\lambda^{(0,1)}$ is holomorphic on the compact $\mathbb{CP}^1$ thus $\omega^{(0,1)} = \omega_\lambda^{(0,1)}$ is independent of $\lambda$. Finally, by the reality condition (2.10) the $(1,0)$–part of $d_\lambda$ is given by $d^{(1,0)} + (\lambda - 1)\omega^{(1,0)}$ with $\omega^{(1,0)}(\phi j) = (\omega^{(0,1)}\phi) j$ for $\phi \in \Gamma(\mathbb{C}^2)$.

By combining the previous results we can construct new harmonic maps by gauging the associated family of flat connections of a given harmonic map $N : M \to S^2$ with an appropriate $\lambda$–dependent map. From this point on we will assume that $N$ is non–trivial: if $N$ is not a constant harmonic map, then we can assume without loss of generality, by passing to $-N$ if necessary, that the associated family (2.6) of flat connections $d_\lambda \neq d$ is non–trivial.
Theorem 3.4 (Dressing). Let \( N : M \to S^2 \) be a non–trivial harmonic map from a Riemann surface \( M \) into the 2–sphere and \( d_\lambda \) the associated family \((2.6)\) of flat connections. If \( r_\lambda : M \to \text{GL}(2, \mathbb{C}) \) is a smooth map into the regular matrices with

(i) \( r_1 = \text{id} \) is the identity matrix for \( \lambda = 1 \),

(ii) \( \lambda \to r_\lambda \) is meromorphic on \( \mathbb{C}P^1 \) and holomorphic at 0 and \( \infty \),

(iii) \( r_\lambda \) satisfies the (generalized) reality condition

\[
(\lambda \phi ) j \sigma_\lambda = r_{\lambda^{-1}}(\phi j), \quad \phi \in \mathbb{C}^2,
\]

with \( \sigma_\lambda \in \mathbb{C}_* \), and

(iv) the map \( \lambda \to \hat{d}_\lambda \) is holomorphic on \( \mathbb{C}_* \) where \( \hat{d}_\lambda = r_\lambda \cdot d_\lambda \) is the connection obtained by gauging \( d_\lambda \) by the dressing matrix \( r_\lambda \),

then \( \hat{d}_\lambda \) is the associated family \((2.6)\) of a harmonic map \( \hat{N} : M \to S^2 \). The harmonic map \( \hat{N} \) is called the dressing of \( N \) by \( r_\lambda \).

Proof. We first show that \( \hat{d}_\lambda \) satisfies the assumptions of Lemma 3.3. Since the reality condition (iii) gives \( r_{\lambda^{-1}}(\phi j) = (r_{\lambda^{-1}}(\phi j) \sigma_{\lambda j} \) we obtain, together with \( r_\lambda \cdot d_\lambda = r_\lambda \circ d_\lambda \circ r_{\lambda^{-1}} \) and the reality condition \((2.10)\) for \( d_\lambda \), that \( \hat{d}_\lambda(\phi j) = (\hat{d}_{\lambda^{-1}}(\phi) j) \).

From (iv) we see that \( \lambda \to \hat{d}_\lambda(0,1) \) is holomorphic for \( \lambda \in \mathbb{C}_* \), and (ii) shows that this map extends holomorphically at \( \infty \) and has a simple pole at \( \lambda = 0 \). Finally \( \hat{d}_{\lambda=1} = d_{\lambda=1} = d \) by (i), and we can apply Lemma 3.3 to get

\[
\hat{d}_\lambda = d + (\lambda - 1)\omega(1,0) + (\lambda^{-1} - 1)\omega(0,1)
\]

with \( \omega(1,0) \in \Gamma(K\text{End}_\mathbb{C}(\mathbb{C}^2))\), \( \omega(0,1) \in \Gamma(K\text{End}_\mathbb{C}(\mathbb{C}^2))\). Since \( \lim_{\lambda \to \infty} \lambda^{-1} d_\lambda = A(1,0) \) and \( r_\lambda \) is holomorphic at \( \infty \) we see that

\[
\omega(1,0) = \lim_{\lambda \to \infty} \lambda^{-1} \hat{d}_\lambda = \lim_{\lambda \to \infty} r_\lambda \cdot \lambda^{-1} d_\lambda = \text{Ad}(r_\infty)A(1,0)
\]

and, similarly, \( \omega(0,1) = \text{Ad}(r_0)A(0,1) \). In particular, this shows \( (\omega(0,1))^2 = (\omega(1,0))^2 = 0 \), and since \( \hat{d}_\lambda \) satisfies the reality condition \((2.10)\) we also have the reality condition \((3.3)\) for the 1–forms \( \omega(1,0) \) and \( \omega(0,1) \). Therefore, the family of flat connections \( \hat{d}_\lambda \) defines with Theorem 3.1 a harmonic map \( \hat{N} : M \to S^2 \). \(\square\)

Remark 3.5. By Remark 3.2 the associated complex structure \( \hat{J} \) of the family of flat connections \( \hat{d}_\lambda \) is given by the quaternionic extension of \( \hat{J}|_E = I \) where \( E \) is given by the kernel of \( \omega(1,0) \). Equation \((3.5)\) shows \( \hat{E} \subset \ker(\text{Ad}(r_\infty)A(1,0)) \), and recalling that the \(+i\) eigenspace \( E \) of \( J \) satisfies \( \hat{E} \subset \ker A(1,0) \), with equality away from the zeros of \( \ker A(1,0) \), we see

\[
\hat{E} = r_\infty E.
\]

Thus for \( \hat{\phi} = r_\infty \phi \in \hat{E} \) we obtain

\[
\hat{J}\hat{\phi} = r_\infty \phi i = r_\infty J\phi,
\]

in other words, the complex structure \( \hat{J} \) is given by extending \( \hat{J}|_E = \text{Ad}(r_\infty)J|_E \) quaternionically.

A particular dressing is given by prescribing that \( r_\lambda \) has only a simple pole in \( \lambda \) on \( \mathbb{C}_* \):
Example 3.6 (Simple factor dressing). Let \( N : M \to S^2 \) be a non–trivial harmonic map from a Riemann surface into the 2–sphere with associated family \( d_\lambda \) of flat connections \((2.6)\). Fix \( \mu \in \mathbb{C}_* \) and let \( M_\mu \) be a \( d_\mu \) parallel line subbundle of the trivial \( \mathbb{C}^2 = \tilde{M} \times \mathbb{C}^2 \) bundle over the universal cover \( \tilde{M} \) of \( M \). Denoting by \( \pi_\mu \) and \( \pi_\mu^\perp \) the projections on \( M_\mu \) and \( M_\mu^\perp \) with respect to the splitting \( \mathbb{C}^2 = M_\mu \oplus M_\mu^\perp \) we define
\[
 r_\lambda = \pi_\mu \circ \gamma_\lambda + \pi_\mu^\perp
\]
where \( \gamma_\lambda \) is the complex linear endomorphism given by
\[
 \gamma_\lambda = \frac{1 - \bar{\mu}^{-1}}{1 - \mu} \frac{\lambda - \mu}{\lambda - \bar{\mu}^{-1}} \in \text{End}_\mathbb{C}(\mathbb{C}^2).
\]
Note that \( r_\lambda = \text{id} \) for \( \mu \in S^1 \) so that \( r_\lambda \) trivially satisfies the conditions of Theorem 3.4. Therefore, we will from now on assume that \( \mu \notin S^1 \). Since \( \pi_\mu(\varphi j) = (\pi_\mu^\perp \varphi) j \) and \( \tilde{\gamma}_\lambda^{-1} = \gamma_{\lambda^{-1}} \) we see that the reality condition (iii) in Theorem 3.4
\[
 (r_\lambda \phi) j \tilde{\gamma}_{\lambda^{-1}} = r_{\lambda^{-1}}(\phi j)
\]
holds for \( \phi \in \mathbb{C}^2 \). As we have seen before this implies that the reality condition \((2.10)\) holds for \( \tilde{d}_\lambda = r_\lambda \cdot d_\lambda \). Moreover, \( \lambda \mapsto r_\lambda \) is meromorphic on \( \mathbb{C}_* \) with simple zero at \( \lambda = \mu \) and simple pole at \( \lambda = \bar{\mu}^{-1} \), and \( \lambda \mapsto r_\lambda \) is holomorphic at 0 and \( \infty \) which in particular shows (ii) of Theorem 3.4.

The condition (i) of Theorem 3.4 that is \( r_1 = \text{id} \), trivially holds. Therefore, it only remains to verify the holomorphicity of \( \tilde{d}_\lambda \), see (iv) of Theorem 3.4. The only issue is at \( \lambda = \mu \) and \( \lambda = \bar{\mu}^{-1} \) but, since \( \tilde{d}_\lambda \) satisfies the reality condition \((2.10)\), it is enough to consider \( \lambda = \mu \). We express \( d_\lambda \) in terms of \( d_\mu \) as
\[
 d_\lambda = d_\mu + (\lambda - \mu) A^{(1,0)} + (\lambda^{-1} - \mu^{-1}) A^{(0,1)}
\]
so that
\[
 \tilde{d}_\lambda = r_\lambda \cdot d_\mu + (\lambda - \mu) \text{Ad}(r_\lambda) A^{(1,0)} + \frac{\mu - \lambda}{\mu \lambda} \text{Ad}(r_\lambda) A^{(0,1)}.
\]
We observe that \( \text{Ad}(r_\lambda) \) has only a simple pole at \( \mu \) which shows that \( (\lambda - \mu) \text{Ad}(r_\lambda) A^{(1,0)} + \frac{\mu - \lambda}{\mu \lambda} \text{Ad}(r_\lambda) A^{(0,1)} \) is holomorphic at \( \lambda = \mu \). Finally, we decompose \( d_\mu \) with respect to the splitting \( \mathbb{C}^2 = M_\mu \oplus M_\mu^\perp \)
\[
 d_\mu = D + \beta
\]
into a differential operator \( D \) which leaves \( M_\mu \) and \( M_\mu^\perp \) parallel, and a tensor \( \beta \) mapping \( M_\mu \) to \( M_\mu^\perp \) and vice versa. By assumption \( M_\mu \) is \( d_\mu \)–parallel, and thus \( \beta|_{M_\mu} = 0 \) and \( \text{Ad}(r_\lambda) \beta = \gamma_\lambda \beta \).

Since \( M_\mu \) and \( M_\mu^\perp \) are \( D \) stable, we see \( r_\lambda \cdot D = D \) so that
\[
 \lambda \mapsto r_\lambda \cdot d_\mu = D + \gamma_\lambda \beta
\]
is holomorphic in \( \lambda = \mu \). This shows that \( \lambda \mapsto \tilde{d}_\lambda \) is holomorphic in \( \mathbb{C}_* \).

Therefore, \( r_\lambda \) satisfies for all \( \mu \in \mathbb{C}_* \), the assumptions of Theorem 3.3. In particular, we obtain for every \( \mu \in \mathbb{C}_* \) and every choice of \( d_\mu \)–parallel line bundle \( M_\mu \) a harmonic map \( \tilde{N} : \tilde{M} \to S^2 \), the simple factor dressing of \( N \) given by \( \mu \) and \( M_\mu \). Note that for \( \mu \in S^1 \) the simple factor dressing \( \tilde{N} = N \) is trivial.
4. Simple factor dressing of constant mean curvature surfaces

By the Ruh–Vilms Theorem \[\text{RV70}\] an immersion \(f : M \to \mathbb{R}^3\) from a Riemann surface into \(\mathbb{R}^3\) has constant mean curvature if and only if its Gauss map \(N : M \to S^2\) is harmonic. In particular, for a given constant mean curvature surface \(f\) the Gauss map \(N\) gives a harmonic complex structure \(J\) on the trivial \(\mathbb{H}\)–bundle \(\mathbb{H}\) and we have an associated \(\mathbb{C}^*\)–family of flat complex connections \[\text{BFL}+02\]

\[
\frac{dN}{df} = -df N.
\]

The splitting of \(dN = (dN)' + (dN)''\) into \((1, 0)\) and \((0, 1)\)–part of \(dN\) with respect to \(N\) is the decomposition of the shape operator into trace and trace-free parts \[\text{BFL}+02\] so that

\[
(dN)' = -Hdf
\]

where \(H\) is the mean curvature of \(f\). We will assume from now on without loss of generality that constant mean curvature surfaces have mean curvature \(H = 1\). In this case \(A\) is the left multiplication by \(-\frac{dN}{df}\) since

\[
-2 * A\phi = (dJ)'\phi = (dN)'\phi = -df\phi, \quad \phi \in \Gamma(\mathbb{H}).
\]

**Theorem 4.1** (Sym–Bobenko formula, \[\text{Bob91}\]). Let \(f : M \to \mathbb{R}^3\) be a constant mean curvature surface with Gauss map \(N : M \to S^2\). If \(d_\lambda\) is the associated family \[\text{BFL}+02\] of complex flat connections of \(N\) then \(f\) is locally given, up to translation, by

\[
f = -2 \left( \frac{\partial}{\partial t} \varphi^{-1} \right)_{t=0} \varphi^{-1}
\]

where \(\varphi_\lambda \in \Gamma(\mathbb{H})\) are \(d_\lambda\)–parallel sections on the universal cover \(\tilde{M}\) of \(M\), depending smoothly on \(\lambda = e^{it} \in S^1\). Conversely, every non–trivial harmonic map \(N : M \to S^2\) from a simply connected Riemann surface \(M\) into the 2–sphere gives by \[\text{BFL}+02\] a constant mean curvature surface \(f : M \to \mathbb{R}^3\) on the universal cover \(\tilde{M}\) of \(M\).

**Proof.** Let \(d_\lambda\) be the associated family of flat connections of a non–trivial harmonic map \(N : M \to S^2\) and \(\varphi_\lambda \in \Gamma(\mathbb{H})\) a smooth family of sections with \(d_\lambda\varphi_\lambda = 0\). With \(\lambda = e^{it} \in S^1\) we obtain

\[
d \left( \frac{\partial}{\partial t} \varphi_\lambda \right)_{t=0} = \frac{\partial}{\partial t} d\varphi_\lambda \bigg|_{t=0} = I(A^{(0,1)} - A^{(1,0)})\varphi_1 = - * A\varphi_1.
\]

Now, if \(f\) is a constant mean curvature surface then its Gauss map \(N\) has \((dN)' \neq 0\) and \(2 * A\) is the left multiplication by \(df\). Thus \(f\) is given, up to translation, by \[\text{BFL}+02\].

Conversely, if \(N\) is a non–trivial harmonic map then we may assume that \((dN)' \neq 0\), and the above computation shows that the map \(f\) defined by \[\text{BFL}+02\] satisfies \(df\varphi = 2 * A\varphi\) where \(df\) does not vanish identically. By \[\text{BFL}+02\] we have \(*df = Ndf = -dfN\) so that \(N\) is the Gauss map of the (branched) conformal immersion \(f\) and \(f\) has constant mean curvature by the Ruh–Vilms theorem. \(\square\)
In particular, the associated family and the dressing of a harmonic map \( N : M \to S^2 \) also induce new constant mean curvature surfaces.

**Corollary 4.2.** Let \( f : M \to \mathbb{R}^3 \) be a constant mean curvature surface and let \( d_\lambda \) be the associated family of its Gauss map \( N \). For \( \mu \in S^1 \) let \( \varphi \in \Gamma(\mathbb{H}) \) be a \( d_\mu \)-parallel section and \( N_\varphi = \varphi^{-1}N\varphi \) the associated harmonic map. If \( \varphi_\lambda \) is a smooth \( S^1 \)-family of \( d_\lambda \)-parallel sections with \( \varphi_\lambda = \varphi \) then

\[
 f_\varphi = -2\varphi^{-1}\left( \frac{\partial}{\partial t} \varphi_{\lambda=1} \right)|_{t=0} = -2\varphi^{-1}\left( \frac{\partial}{\partial t} \varphi_{\lambda=1} \right)|_{t=s}.
\]

is a constant mean curvature surface \( f_\varphi : \tilde{M} \to \mathbb{R}^3 \) with Gauss map \( N_\varphi \) where \( \mu = e^{is} \in S^1 \).

**Proof.** Recall from Theorem 2.3 that \( d_{\varphi,\lambda} = \Phi^{-1} \cdot d_{\lambda \mu} \) is the associated family of \( N_\varphi \) where \( \Phi \) is the endomorphism given by left multiplication by \( \varphi \). In particular, \( \varphi_\lambda^{\mu} := \varphi^{-1}\varphi_\lambda \mu \) is \( d_{\varphi,\lambda} \)-parallel, and the Sym–Bobenko formula shows

\[
 f_\varphi = -2 \left( \frac{\partial}{\partial t} \varphi_{\lambda=1} \right)|_{t=0} = -2\varphi^{-1}\left( \frac{\partial}{\partial t} \varphi_{\lambda=1} \right)|_{t=s}.
\]

\( \square \)

**Corollary 4.3.** Let \( f : M \to \mathbb{R}^3 \) be a constant mean curvature surface, and let \( N : M \to S^2 \) be the Gauss map of \( f \). Then the dressing \( \tilde{N} \) of \( N \) by \( r_\lambda \) is the Gauss map of the constant mean curvature surface

\[
 \tilde{f} = f - 2 \left( \frac{\partial}{\partial t} r_{\lambda=1} \right)|_{t=0} (\cdot) \varphi^{-1}_{\lambda=1},
\]

where \( \varphi_\lambda \in \Gamma(\mathbb{H}) \) are \( d_\lambda \)-parallel sections, depending smoothly on \( \lambda = e^{it} \in S^1 \).

**Proof.** First note that \( \left( \frac{\partial}{\partial t} r_{\lambda=1} \right)|_{t=0} \) is in general not quaternionic linear. We recall that the associated family of \( \tilde{N} \) is given by Theorem 3.3 by \( \tilde{d}_\lambda = r_\lambda \cdot d_\lambda \). Thus for a smooth \( S^1 \)-family \( \varphi_\lambda \) of \( d_\lambda \)-parallel sections we see that \( \tilde{\varphi}_\lambda = r_\lambda \varphi_\lambda \) is \( d_\lambda \)-parallel. Now the Sym–Bobenko formula (4.3) and \( r_1 = \text{id} \) give the claim. \( \square \)

Identifying \( \mathbb{R}^4 = \mathbb{H} \) with \( \text{gl}(2, \mathbb{C}) \)-matrices of the form \( \left\{ \begin{pmatrix} a & -\bar{b} \\ b & a \end{pmatrix} \right| a, b \in \mathbb{C} \} \) via

\[
 a_0 + ja_1 \mapsto \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix}
\]

the inner product in \( \mathbb{R}^3 \) is given by \( < v, w > := -\frac{1}{2} \text{tr}(vw) \) for \( v, w \in \mathbb{R}^3 \), and the coordinate frame of an immersion \( f : M \to \mathbb{R}^3 \) is under this identification the unique (up to sign) smooth map \( F : \tilde{M} \to \text{SU}(2, \mathbb{C}) \) with

\[
 e^{-\frac{z}{2}} f_x = -iF\sigma_1 F^{-1}, \quad e^{-\frac{z}{2}} f_y = -iF\sigma_2 F^{-1}, \quad N = -iF\sigma_3 F^{-1}
\]

where \( z = x + iy \) is a conformal coordinate, \( e^u \) is the induced metric, and \( \sigma_l \) are the Pauli–matrices

\[
 \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
in particular, \( f_z = \frac{1}{2}(f_x - if_y) \) and \( f_{\bar{z}} = \frac{1}{2}(f_x + if_y) \) are given by

\[
\begin{align*}
  f_z &= -ie^\frac{u}{2}Fe_-F^{-1} \\
  f_{\bar{z}} &= -ie^\frac{u}{2}Fe_+F^{-1} \\
  N &= -iF\sigma_3F^{-1}
\end{align*}
\]

(4.4)

with \( e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Since the metric and the mean curvature of an immersion \( f \) are given by \( e^u = 2 < f_z, f_{\bar{z}} > \) and \( H = 2e^{-u} < f_{zz}, N > \) respectively, the frame \( F \) of a constant mean curvature surface \( f \) with mean curvature \( H = 1 \) and Hopf differential \( Qdz, Q = < f_{zz}, N > \), satisfies with \( < f_{\bar{z}}, f_z > = < f_{\bar{z}}, f_{\bar{z}} > = 0 \) the equations

\[
\begin{align*}
  F^{-1}F_z &= \begin{pmatrix} -\frac{1}{2}u_z & Qe^{-\frac{u}{2}} \\
  -\frac{i}{2}e^\frac{u}{2} & \frac{1}{2}u_z \end{pmatrix} \\
  F^{-1}F_{\bar{z}} &= \begin{pmatrix} \frac{1}{2}u_{\bar{z}} & \frac{1}{2}e^\frac{u}{2} \\
  -Qe^{-\frac{u}{2}} & -\frac{1}{2}u_{\bar{z}} \end{pmatrix}.
\end{align*}
\]

In particular, the Gauss–Codazzi equations for the constant mean curvature surface \( f \) are satisfied if and only if

\[
d^F := d + F^{-1}dF
\]

is a flat connection. Again, we can introduce the spectral parameter \( \lambda \in \mathbb{C}_* \):

**Lemma 4.4.** Let \( f : M \to \mathbb{R}^3 \) be a constant mean curvature surface with Gauss map \( N \) and coordinate frame \( F \) and let \( F_\lambda : M \to \text{GL}(2, \mathbb{C}), \lambda \in \mathbb{C}_* \), be an extended frame of \( f \), that is a solution of

\[
\begin{align*}
  F_\lambda^{-1}(F_\lambda)_z &= \begin{pmatrix} -\frac{u}{2} & Qe^{-\frac{u}{2}} \\
  -\frac{i}{2}e^\frac{u}{2} & \frac{1}{2}u \end{pmatrix} \\
  F_\lambda^{-1}(F_\lambda)_{\bar{z}} &= \begin{pmatrix} \frac{u}{2} & \lambda^{-1}e^\frac{u}{2} \\
  -Qe^{-\frac{u}{2}} & -\frac{u}{2} \end{pmatrix}
\end{align*}
\]

(4.5) (4.6)

with \( F_{\lambda=1} = F \). Then \( F_\lambda \) gives the associated family (2.6) of flat connections of the Gauss map \( N \) of \( f \) by

\[
d_\lambda = F \cdot dF_\lambda
\]

(4.7)

where \( dF_\lambda = d + F_\lambda^{-1}dF_\lambda \).

**Proof.** Recalling (1.12) that \( A \) is the left multiplication by \( -\frac{df}{2} \) and \( d_\lambda = d + \alpha_\lambda \) with

\[
\alpha_\lambda = (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}
\]

we see from (4.4)

\[
\alpha_\lambda^{(1,0)} = -\frac{\lambda - 1}{2}e^\frac{u}{2}Fe_-F^{-1}dz, \quad \alpha_\lambda^{(0,1)} = \frac{\lambda^{-1} - 1}{2}e^\frac{u}{2}Fe_+F^{-1}d\bar{z}.
\]

Putting \( S_\lambda = F_\lambda F^{-1} \) we have

\[
S_\lambda^{-1}dS_\lambda = F(F_\lambda^{-1}dF_\lambda - F^{-1}dF)F^{-1}
\]

and using (4.5) and (4.6) we get \( S_\lambda^{-1}dS_\lambda = \alpha_\lambda \). Thus \( d_\lambda = S_\lambda^{-1} \cdot d \) which shows the claim. \( \square \)
From the previous lemma we see that for a constant $v \in \mathbb{C}^2 = \mathbb{H}$ the section $\varphi_\lambda = FF_\lambda^{-1}v$ is $d_\lambda$–parallel, and we obtain from (4.3)
\[ f = 2 \left( \frac{\partial}{\partial t} F_{\lambda}^{\nu} \biggm|_{t=0} F^{-1} \right) + \text{const.} \]
Note that this coincides with the usual Sym–Bobenko formula for the extended frame: writing $U_\lambda = F_\lambda^{-1}(F_\lambda)v$ we see
\[ \left( \frac{\partial}{\partial t} F_{\lambda}^{\nu} \biggm|_{t=0} F^{-1} \right)_z = \left( \frac{\partial}{\partial t} (F_{\lambda}^{\nu})_z \biggm|_{t=0} F^{-1} \right) - \frac{\partial}{\partial t} F_{\lambda}^{\nu} \biggm|_{t=0} F^{-1} F_z F^{-1} \]
\[ = F \left( \frac{\partial}{\partial t} U_\lambda^{\nu} \biggm|_{t=0} F^{-1} \right) \]
\[ = \frac{1}{2} f_z \]
where we used (4.5) and (4.4). A similar argument gives $\left( \frac{\partial}{\partial t} F_{\lambda}^{\nu} \biggm|_{t=0} F^{-1} \right)_z = \frac{1}{2} f_z$. 

We now connect the simple factor dressing on the extended frame [TU00, DK05] with the frame independent definition in Example 3.6. We fix $\mu \in \mathbb{C}_*$ and recall that the simple factor dressing matrix is given by
\[ r_\lambda = \pi_\mu \circ \gamma_\lambda + \pi_\mu^1 \]
where $M_\mu$ is a $d_\mu$–parallel bundle, $\pi_\mu$ and $\pi_\mu^1$ denote the projections onto $M_\mu$ and $M_\mu^1$ respectively, and $\gamma_\lambda$ is the complex linear endomorphism given by
\[ \gamma_\lambda = \frac{1 - \bar{\mu}^{-1}}{1 - \mu} \lambda - \frac{\mu}{\bar{\mu}^{-1}} \in \text{End}_\mathbb{C}(\mathbb{C}^2). \]
In particular, the simple factor dressing $\hat{N}$ of $N$ by $r_\lambda$ has associated family of flat connections $\hat{d}_\lambda = r_\lambda \cdot d_\lambda$ and gives by Corollary 4.3 a constant mean curvature surface $\hat{f}$. We denote the extended frame of $\hat{f}$ by $\hat{F}_\lambda$. Then Lemma 4.4 shows that
\[ \hat{d}_\lambda = \hat{F}_\lambda \cdot d \hat{F}_\lambda \]
with $\hat{F} = \hat{F}_{\lambda=1}$. Writing $\hat{S}_\lambda = \hat{F}_\lambda \hat{F}^{-1}$, we thus have $r_\lambda \cdot d_\lambda = \hat{S}_\lambda^{-1} \cdot d$, and $d_\lambda = F \cdot d \hat{F}_\lambda$ gives
\[ \hat{S}_\lambda = s_\lambda \circ S_\lambda \circ r_\lambda^{-1} \]
with $S_\lambda = F_\lambda F^{-1}$, and a $z$–independent $s_\lambda$. On the other hand, [DK05] give a simple factor dressing $\tilde{f}$ of a constant mean curvature surface $f$ with extended frame $F_\lambda$: the extended frame $\tilde{F}_\lambda$ of $\tilde{f}$ is given by
\[ h_\lambda \circ F_\lambda = \tilde{F}_\lambda \circ g_\lambda \]
where $h_\lambda = \pi_{M_0} \tau_\lambda + \pi_{M_0^1}$ is given by the choice of a constant line $M_0 \subset \mathbb{C}^2$ and
\[ \tau_\lambda = \frac{\lambda - \mu}{\bar{\mu}(\lambda - \bar{\mu}^{-1})} \in \text{End}_\mathbb{C}(\mathbb{C}^2). \]
Moreover, $g_\lambda$ is obtained from $h_\lambda$ by replacing the constant line $M_0$ by the line bundle $F_\mu^{-1} M_0$ given by the extended frame, that is
\[ g_\lambda = \pi_{F_\mu^{-1} M_0} \tau_\lambda + \pi_{(F_\mu^{-1} M_0)^1}. \]
Putting $s_\lambda = h^{-1} h_\lambda$, $h = h_{\lambda=1}$, we get
\[ s_\lambda = \pi_{M_0} \gamma_\lambda + \pi_{M_0^1}. \]
since $\tau^{-1}_1 \tau_\lambda = \gamma \lambda$. By (4.7) the line $M_\mu = FF_\mu^{-1}M_0$ is $d_\mu$–parallel, and the simple factor dressing matrix $r_\lambda$ of $M_\mu$ is given by

$$r_\lambda = Fg^{-1}g_\lambda F^{-1}.$$

By definition of $S_\lambda = F_\lambda F^{-1}$ and $s_\lambda = h^{-1}h_\lambda$ this shows with (4.9)

$$s_\lambda \circ S_\lambda = (h^{-1}F_\lambda) \circ (h^{-1}F^{-1})^{-1} \circ r_\lambda.$$

Plugging into (4.8) we see that $\hat{F}_\lambda = h^{-1}F_\lambda W$ where $W : M \to \text{GL}(2, \mathbb{C})$ is independent of $\lambda$. The Sym–Bobenko formula then yields that $\hat{f}$ and $\hat{f} = h^{-1}\tilde{f}h$ coincide up to translation.

5. Darboux transforms of harmonic maps into the 2–sphere

The classical Darboux transformation on isothermic surfaces can be extended to a transformation on conformal maps $f : M \to S^2$ from a Riemann surface into the 4-sphere, [BLPP08]. In the case when $f$ is a constant mean curvature surface, one obtains a genuine generalization of the classical Darboux transformation [CLP10]. Here we consider a special case of the general Darboux transformation, the so–called $\mu$–Darboux transforms. These have constant mean curvature, but are only classical Darboux transforms for special spectral parameter $\mu$. In particular, we obtain an induced transformation on harmonic maps $N : M \to S^2$.

**Theorem 5.1** ([CLP10]). Let $f : M \to \mathbb{R}^3$ be a constant mean curvature surface in $\mathbb{R}^3$ with Gauss map $N$ and associated family $d_\lambda$ of flat connections. For $\mu \in \mathbb{C}_*$ and $d_\mu$–parallel section $\varphi \in \Gamma(\mathbb{H})$, define

$$T = \frac{1}{2}(N\varphi(a - 1)\varphi^{-1} + \varphi b \varphi^{-1})$$

where $a = \frac{\mu + \mu^{-1}}{2}, b = i\frac{\mu^{-1} - \mu}{2}$. Then $T$ is nowhere vanishing if $\mu \neq 1$, and the map $\hat{f} : M \to \mathbb{R}^4 = \mathbb{H}$,

$$\hat{f} = f + T^{-1},$$

has constant real part. Moreover, $\text{im} \hat{f}$ is a constant mean curvature surface with Gauss map

$$\hat{N} = -T^{-1}NT.$$ 

The map $\hat{f}$ is called a $\mu$–Darboux transform of $f$.

In other words, a $\mu$–Darboux transform is, up to a translation, a constant mean curvature surface in $\mathbb{R}^3$. Note that $\hat{f}$ depends on the choice of the $d_\mu$–parallel section $\varphi \in \Gamma(\mathbb{H})$.

The Darboux transformation is a key ingredient [BLPP08] for integrable systems methods in surface theory. In the case when $M = T^2$ is a 2–torus the spectral curve of a conformal torus $f : T^2 \to S^2$ is essentially the set of all Darboux transforms $\hat{f} : T^2 \to S^4$ of $f$. If $f : T^2 \to \mathbb{R}^3$ is a constant mean curvature torus this general spectral curve is biholomorphic [CLP10] to the spectral curve of the harmonic Gauss map $N$ of $f$: the spectral curve of $N$ is given [Hit90] by the compactification of the Riemann surface which is given by the eigenlines of the holonomies of $d_\lambda$, $\lambda \in \mathbb{C}_*$. The eigenlines are exactly given by parallel sections with multipliers, that is $d_\mu$–parallel sections $\varphi$, $\mu \in \mathbb{C}_*$, of the trivial $\mathbb{C}^2$ bundle over $\mathbb{C}$ which satisfy $\gamma^\ast \varphi = \varphi h_\gamma$ with $h_\gamma \in \mathbb{C}_*$ for $\gamma \in \pi_1(T^2)$. On the other hand, parallel sections give $\mu$–Darboux transforms, and the multiplier condition then implies that the $\mu$–Darboux transform is a conformal map on the torus.
Here we are interested in (local) transformation theory of general constant mean curvature surfaces \( f : M \to \mathbb{R}^3 \) and will allow the \( \mu \)-Darboux transforms to be defined on the universal cover \( \tilde{M} \) of \( M \). Note that \( a, b \in \mathbb{C} \) in the above theorem satisfy \( a^2 + b^2 = 1 \), however, \( a, b \in \mathbb{R} \) if and only if \( \mu \in S^1 \). In this case, \( T \) is independent of the choice of the \( d_\mu \)-parallel section \( \varphi \) and \( T^{-1} = N + \frac{b}{1-a} \). In particular, \( \hat{f} = g + \frac{b}{1-a} \) is a translate of the parallel constant mean curvature surface \( g = f + N \) of \( f \). On the other hand, for \( \mu \in \mathbb{R}_+ \) we see that \( a \in \mathbb{R} \) so that \( T \), and thus \( \hat{f} \), takes values in \( \mathbb{R}^3 \).

**Theorem 5.2** ([CLP10]). Let \( f : M \to \mathbb{R}^3 \) be a constant mean curvature surface. Then a constant mean curvature surface \( \tilde{f} : \tilde{M} \to \mathbb{R}^4 \) is a classical Darboux transform of \( f \) if and only if \( \tilde{f} \) is a \( \mu \)-Darboux transform of \( f \) with \( \mu \in \mathbb{R}_+ \cup S^1 \).

By Theorem 5.1 the \( \mu \)-Darboux transformation preserves the harmonicity of the Gauss map. More generally:

**Theorem 5.3.** Let \( N : M \to S^2 \) be a non-trivial harmonic map from a Riemann surface into the 2-sphere and \( d_\lambda \) the associated family of flat connections \((2.7)\). Define for \( \mu \in \mathbb{C}_+ \) and \( d_\mu \)-parallel section \( \varphi \in \Gamma(\mathbb{R}^2) \) the map \( T : \tilde{M} \to \mathbb{H} \)

\[
(5.1) \quad T = \frac{1}{2}(N\varphi(a-1)\varphi^{-1} + \varphi b \varphi^{-1})
\]

where \( a = \frac{\mu + \mu^{-1}}{2}, b = i \frac{\mu^{-1} - \mu}{2} \). Then \( T \) is nowhere vanishing if \( \mu \neq 1 \), and

\[
\hat{N} = T^{-1}NT
\]

is harmonic. We call \( \hat{N} \) a \( \mu \)-Darboux transform of \( N \).

**Remark 5.4.** Again, we emphasize that \( \hat{N} \) depends in general on the choice of the \( d_\mu \)-parallel section \( \varphi \). However, if \( \mu \in S^1 \), then \( a, b \in \mathbb{R} \) and \( T \) is independent of \( \varphi \). But then \([T, N] = 0\) gives \( \hat{N} = N \) for \( \mu \in S^1 \).

In particular, our choice of sign for a \( \mu \)-Darboux transform is so that it coincides with the sign of the simple factor dressing of a harmonic map on \( S^1 \). However note that with this choice the \( \mu \)-Darboux transform of the Gauss map of a constant mean curvature surface \( f \) is the negative Gauss map of the \( \mu \)-Darboux transform of \( f \).

**Proof.** We essentially follow the proof in [CLP10] for the analogue statement for the Gauss map of a constant mean curvature surface. Putting \( \hat{z} = \varphi z \varphi^{-1} \) for \( z \in \mathbb{C} \) we write \( T = \frac{1}{2}(N(\hat{\varphi} - 1) + b) \) and, if \( \mu \neq 1 \), then

\[
(5.2) \quad 2T(1 - \hat{\varphi})^{-1} \varphi + N = \frac{\hat{b}}{1 - \hat{\varphi}}.
\]

Since \( N^2(p) = -1 \) and \( \frac{\hat{b}^2}{(1 - \hat{\varphi})^2} = \frac{1 + \hat{\varphi}}{1 - \hat{\varphi}} \neq -1 \) for all \( \mu \in \mathbb{C}_+, \mu \neq 1 \), this shows that \( T(p) \neq 0 \) for all \( p \in M \). Next we observe with \((2.6)\) and \((2.12)\) that

\[
(5.3) \quad d_\mu = d + \ast A(J(a - 1) + b)
\]

which shows with \((2.2)\) that

\[
0 = d_\mu \varphi = d\varphi - (dN)\prime T \varphi
\]

and thus \( d\hat{z} = [(dN)\prime T, \hat{z}] \) for \( z \in \mathbb{C} \). Differentiating \((5.1)\) gives \( \hat{a}^2 + \hat{b}^2 = 1 \) the Riccati type equation

\[
(5.4) \quad dT = (dN)\prime \frac{\hat{a} - 1}{2} - T(dN)\prime T,
\]
which shows

\[ N dT - * dT = - ((dN)'(N(\dot{a} - 1) + (NT + TN)(dN)'T) \].

Since \( d\tilde{N} = [\tilde{N}, T^{-1}dT] + T^{-1}dNT \) we thus obtain

\[ d\tilde{N} + \tilde{N}d* \tilde{N} = \frac{1}{2} T^{-1}(dN + N*dN)(\dot{b} - (\dot{a} - 1)\dot{N}) \]

Now (5.1) gives \(-(\dot{a} - 1) + T\dot{N}(\dot{a} - 1) - T\dot{b} = 0\), that is

\[ \dot{N} = T^{-1} + \frac{\dot{b}}{\dot{a} - 1} \]

and using the Riccati type equation (5.4) we obtain

\[ d* \tilde{Q} = d* A \]

for the Hopf fields of \( \tilde{N} \) and \( N \). This shows that \( \tilde{N} \) is harmonic. \( \square \)

Note that for \( \mu \in \mathbb{R} \), the equation (5.4) is independent of the choice of the parallel section \( \varphi \). In particular, if \( N \) is the Gauss map of a constant mean curvature surface \( f : M \rightarrow \mathbb{R}^3 \) then the solutions of the Riccati equation (5.4) give \([BFL02]\) the classical Darboux transforms of \( f \). The condition (5.2) then guarantees that \( \tilde{f} = f + T^{-1} \) has constant mean curvature.

We can now generalize the results on \( \mu \)–Darboux transforms for constant mean curvature surfaces \([CLP10]\) and Hamiltonian stationary Lagrangians in \([LR10]\): for a conformal immersion \( f : M \rightarrow \mathbb{R}^4 \) from a Riemann surface \( M \) into 4–space, the Gauss map \( \nu : M \rightarrow \text{Gr}_2(\mathbb{R}^4) \) is a map from \( M \) into the Grassmannian of 2–planes in \( \mathbb{R}^4 \). Identifying \( \text{Gr}_2(\mathbb{R}^4) = S^2 \times S^2 \) the Gauss map \( \nu \) gives rise to two maps \( N, R : M \rightarrow S^2 \) satisfying

\[ *d\phi = Ndf = -dfR. \]

\( N \) and \( R \) are called the left and right normal of \( f \). From \([BFL02]\) we know that the \((1,0)\)–part of \( dN \) with respect to \( N \) is given by \((dN)' = -dfH \) for some quaternion valued function \( H : M \rightarrow \mathbb{H} \) which satisfies \( RH = HR \).

Examples of surfaces with harmonic left normal are constant mean curvature surfaces in 3–space, minimal surfaces in 3–space, or Hamiltonian stationary Lagrangian immersions in \( \mathbb{C}^2 = \mathbb{R}^4 \). All surfaces with harmonic left normal are constrained Willmore \([LR10]\). If a surface \( f : M \rightarrow \mathbb{R}^4 \) has harmonic left normal we can associate again a family of flat connections \( d\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)} \) on \( \mathbb{H} = \mathbb{C}^2 \) where \( A \) is the Hopf field of the associated complex structure \( J \) of \( N \). Note that the family \( d\lambda \) is trivial if and only if \( f \) is a minimal surface.

**Theorem 5.5.** Let \( f : M \rightarrow \mathbb{R}^4 \) be a conformal immersion with harmonic left normal \( N : M \rightarrow S^2 \) which is not a minimal surface, so that \((dN)' = -dfH \) with non–trivial \( H : M \rightarrow \mathbb{H} \). For \( \mu \in \mathbb{C} \), let \( \varphi \in \Gamma(\mathbb{H}) \) be a \( d\mu \)–parallel section of the associated family of flat connections of \( N \). For \( \mu \neq 1 \) put \( T = \frac{1}{\mu}(N\varphi(a - 1)\varphi^{-1} + \varphi b\varphi^{-1}) \) with \( a = \frac{\mu + \mu^{-1}}{2}, b = i\mu^{-1} - \mu \), and

\[ \hat{f} = f + (HT)^{-1} \]

away from the (isolated) zeros of \( H \).

Then the map \( \hat{f} \) is either constant, or a (branched) conformal immersion with harmonic left normal \( \hat{N} = -T^{-1}NT \).
Proof. As before (5.3) we have \( d_\mu = d + *A(J(a - 1) + b) \), so that with \( (dN)' = -dfH \) for a \( d_\mu \)-parallel section \( \varphi \in \Gamma(\mathbb{H}) \)

\[
d\varphi = -dfHT\varphi.
\]

Putting \( \beta = HT\varphi \) this gives \( 0 = df \wedge d\beta \) which implies \( *d\beta = -Rd\beta \) by type arguments. In particular, \( d\beta \) has only isolated zeros if \( \beta \) is not constant \([\text{FLPP01}]\). From Theorem (5.3) we see that \( T \) has no zeros so that \( (HT)^{-1} \) is defined away from the zeros of \( H \), and

\[
d\hat{f} = df + d(HT)^{-1} = df + d(\varphi\beta^{-1}) = -(HT)^{-1}d\beta\varphi^{-1}(HT)^{-1}
\]

shows that \( \hat{f} \) is either constant, or a branched conformal immersion with

\[
*\hat{f} = -(HT)^{-1}R(HT)d\hat{f}.
\]

Using \( RH = HN \) we see that in the latter case \( \hat{f} \) has left normal

\[
\hat{N} = -T^{-1}NT.
\]

Theorem (5.3) therefore shows that the left normal \( \hat{N} \) of \( \hat{f} \) is harmonic. \( \square \)

Remark 5.6. Since \( d_\mu \)-parallel sections are holomorphic, the arguments in \([\text{LR10}]\) for the special case of Hamiltonian stationary Lagrangians show that \( \hat{f} \) as defined in the above theorem is a generalized Darboux transform of \( f \). We call \( \hat{f} \) a \( \mu \)-Darboux transform of \( f \) as it arises from a \( d_\mu \)-parallel section \( \varphi \) for \( \mu \in \mathbb{C}_* \).

Similarly, a \( \mu \)-Darboux transformation is defined on the conformal Gauss map of a (constrained) Willmore surface \( f : M \to S^4 \) and an analogue of Theorem (5.3) holds \([\text{LZS10}]\).

6. Darboux transformation and simple factor dressing

We show that the \( \mu \)-Darboux transformation and the simple factor dressing of a harmonic map coincide. In particular, a \( \mu \)-Darboux transform of a constant mean curvature surface \( f : M \to \mathbb{R}^4 \) is given by a simple factor dressing of the Gauss map of the parallel surface \( g \) of \( f \), and vice versa. This generalizes results for classical Darboux transformations \([\text{HJP97}, \text{Bur06}, \text{IK05}]\). Moreover, since the \( \mu \)-Darboux transformation is defined for all surfaces \( f : M \to \mathbb{R}^4 \) with harmonic left normal, the simple factor dressing on the harmonic left normal can thus also be given an interpretation on the level of surfaces.

Theorem 6.1. Let \( N : M \to S^2 \) be a non-trivial harmonic map. Then every \( \mu \)-Darboux transform of \( N \) is given by a simple factor dressing, and vice versa.

More precisely, if we denote by \( d_\lambda \) the associated family of flat connections of \( N \) and put \( M_\mu = \varphi \mathbb{C} \) for a \( d_\mu \)-parallel section \( \varphi \in \Gamma(\mathbb{H}) \), \( \mu \in \mathbb{C}_* \), then the simple factor dressing \( \hat{N} \) of \( N \) with respect to \( M_\mu \) is the \( \mu \)-Darboux transform of \( N \) with respect to \( \varphi \), that is

\[
\hat{N} = T^{-1}NT
\]

with \( T = \frac{1}{2}(N\varphi(a - 1)\varphi^{-1} + \varphi b \varphi^{-1}) \) and \( a = \frac{b + \mu^{-1}}{2}, b = i\mu^{-1} - \mu \).

Proof. In this proof we adapt the arguments in \([\text{Qui08}]\) to the case of harmonic maps \( N : M \to S^2 \), and generalize her setting from \( \mu \in \mathbb{R}_* \cup S^1 \) to the general case \( \mu \in \mathbb{C}_* \): As before, we denote by \( \hat{z} = \varphi z \varphi^{-1} \) for \( z \in \mathbb{C} \), and recall that \( \hat{a}^2 + \hat{b}^2 = 1 \). Let \( J \) be the complex structure of \( N \) and \( E \) the \( +i \) eigenspace of \( J \). Putting \( \rho = \frac{1 + \mu}{2} \) and \( \hat{T} = T\hat{\rho}^{-1} \) we first show that the \( +i \) eigenspace of the complex structure \( \hat{J} \) of \( \hat{N} = T^{-1}NT \) is given by \( \hat{E} = \hat{T}E \): the equation (5.2) shows \( (\hat{T} + N)^2 = -1 + \hat{\rho}^{-1} \), that is,

\[
\hat{T}^2 + \hat{TN} + \hat{NT} = \hat{\rho}^{-1}.
\]
From this we see that \(N\) commutes with
\[
de \hat{T}^{-2} = 1 + N \hat{T}^{-1} + \hat{T}^{-1} N
\]
and thus \([T^2 \de^{-1}, N] = 0\). For \(\hat{\phi} = T \phi, \phi \in E\) we therefore obtain
\[
N \hat{\phi} = \hat{T} N \phi = \hat{\phi} i,
\]
and \(\hat{E}\) is the \(+i\) eigenspace of \(\hat{J}\).

Since \(N\) is completely determined by the \(+i\) eigenspace of \(\hat{J}\) it is enough to show by Remark 3.5 that \(\hat{E} = r_\infty E\) where \(r_\lambda = \pi_\mu \circ \gamma_\lambda + \pi_\mu^\perp\). Here \(\pi_\mu\) and \(\pi_\mu^\perp\) are the projections onto \(M_\mu\) and \(M_\mu^\perp\) respectively, and \(\gamma_\lambda = \frac{1 - \bar{\mu}}{1 - \mu} \frac{2i}{\lambda - \bar{\mu}},\) We first observe that \(a - 1 = \frac{\mu - 1}{2}\) and \(b = \mu^2 - 1\) so that
\[
\frac{b}{1 - a} = \frac{\mu + 1}{\mu - 1}.
\]

Since \(\varphi \in \Gamma(M_\mu)\) we have \(r_\infty \varphi = \varphi \frac{1 - \bar{\mu}}{1 - \mu}\) and \(r_\infty (\varphi j) = \varphi j\), so that (5.2) shows
\[
(\hat{T} + N - I) \varphi = \varphi \frac{2i}{\mu - 1} = - (r_\infty) \frac{2I}{1 - \mu^{-1}} \varphi
\]
and
\[
(\hat{T} + N - I)(\varphi j) = - \varphi j \frac{2i}{1 - \mu^{-1}} = - (r_\infty) \frac{2I}{1 - \mu^{-1}} \varphi j,
\]
in other words,
\[
\hat{T} + N - I = - r_\infty \circ \frac{2I}{1 - \mu^{-1}}.
\]

Finally, for \(\phi \in E\) we have \((N - I) \phi = 0\) since \(E\) is the \(+i\) eigenspace of \(J\), and thus
\[
\hat{T} \phi = - r_\infty \phi \frac{2i}{1 - \mu^{-1}}.
\]

This proves that \(\hat{T} E = r_\infty E\), and thus \(\hat{N}\) is the simple factor dressing of \(N\) by \(r_\lambda\).

As an immediate consequence of Theorem 6.1 and Theorem 5.1 simple factor dressing and the \(\mu\)-Darboux transformation are essentially the same for constant mean curvature surfaces:

**Theorem 6.2.** The Gauss map of a \(\mu\)-Darboux transform \(\hat{f}\) of a constant mean curvature surface \(f: M \to \mathbb{R}^3\) is a simple factor dressing of the Gauss map of the parallel surface \(g = f + N\) of \(f\), and vice versa.

More generally, if \(f\) is a surface with harmonic left normal \(N\), then Theorem 6.1 and Theorem 5.5 show that a simple factor dressing of \(N\) is induced by a transformation on the surface \(f\):

**Theorem 6.3.** Let \(f: M \to \mathbb{R}^4\) be a conformal immersion with harmonic left normal \(N: M \to S^2\) which is not a minimal surface. Then a simple factor dressing of \(-N\) is the left normal of a \(\mu\)-Darboux transform of \(f\), and vice versa.
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