Gauge Fixing and BFV Quantization

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Abstract

Nonsingularity conditions are established for the BFV gauge-fixing fermion which are sufficient for it to lead to the correct path integral for a theory with constraints canonically quantized in the BFV approach. The conditions ensure that anticommutator of this fermion with the BRST charge regularises the path integral by regularising the trace over non-physical states in each ghost sector. The results are applied to the quantization of a system which has a Gribov problem, using a non-standard form of the gauge-fixing fermion.

1 Introduction

This paper investigates the Fradkin-Vilkovisky theorem, which establishes the validity of the path-integral expression used in canonical BRST quantization according to the BFV scheme, particularly addressing the question of gauge-fixing. The main idea is to show that the path integral gives a trace over physical states, with the gauge-fixing term regularising the trace on the non-physical states by ensuring that these terms cancel out, using the relation between supertrace and cohomology.

It has long been recognised that in the quantization of gauge systems some mechanism is required for the cancellation of redundancy arising from gauge equivalence. The standard method uses a Lagrangian modified by gauge-fixing and Faddeev-Popov ghost terms, which contribute to the path integral a factor which nicely divides out the gauge redundancy. The Faddeev-Popov method was extended by Batalin, Fradkin, Fradkina and Vilkovisky in a series of papers [1, 2, 3, 4, 5] to allow relativistic gauge fixing. (The method
developed in these papers will be referred to as the BFV approach.) Henneaux subsequently gave an interpretation of the BFV approach in terms of BRST cohomology \[6\]. There is a detailed account of these ideas in the book of Henneaux and Teitelboim \[7\].

The central idea of these methods is that the original Hamiltonian \( H_c \) must be adjusted by a term \( i[\Omega, \chi] \) where \( \Omega \) is the BRST charge of the theory \( \chi \) is the gauge fixing fermion and \([\Omega, \chi]\) is the super commutator \( \Omega \chi + \chi \Omega \). However exactly which class of gauge-fixing fermions leads to correct results has not been entirely clear, as is demonstrated by the interesting work of Govaerts \[8\] and Govaerts and Troost \[9\]. The main result of this paper is to establish the precise conditions which the gauge fixing fermion \( \chi \) must satisfy if the procedure is to give the correct result. These conditions (which will be referred to as the nonsingularity conditions) are given fully in section \[3\] together with a proof of the main result. Two examples are given of the use of this theorem; one is the simple example of a system with translational invariance in one direction, the corresponding constraint being that the component of momentum in this direction is zero. In this case well-known results are recovered. The second is an example where a Gribov ambiguity might be expected to prevent application of any technique involving gauge-fixing; it is shown that in fact a gauge fixing fermion can be chosen so that the the BFV formalism is valid.

The proof of the main result depends on the fact (pointed out by Schwarz \[10\]) that the supertrace (or, in the terminology of Schwarz' paper, the Lefschetz trace)of a BRST-invariant operator is equal to the alternating sum of the traces of this operator on cohomology classes. This is precisely the trace over physical states if the only non-trivial BRST cohomology is at ghost number zero. Then, provided it can be shown that the path integral expression for the vacuum generating function does give the supertrace of the evolution operator, the theorem is established. However, since even in the quantum-mechanical setting taking a trace involves infinite sums, care must be taken that operators have well-defined traces. It will be shown that the operator \( \exp [i[\Omega, \chi]t] \), which is equal to the identity operator on the space of physical states, regularises the trace on the non-physical states in such a way that contributions from these states cancel out.

The paper begins with a brief description of the quantization method set up by BFV and extended by Henneaux. A simple but important lemma is then proved, which establishes that the first nonsingularity condition ensures that the only states with non-trivial BRST cohomology have ghost number
zero. A study of the eigenstates and eigenvalues of \( \exp(i[\Omega, \chi]t) \) is then used to show (by arguments similar to those of McKean and Singer \([11]\)) that the path integral defined by the extended Hamiltonian \( \tilde{H}_t + i[\Omega, \chi] \) gives the correct generating functional. (Here \( \tilde{H}_t \) is a BRST-invariant extension of the original Hamiltonian \( H_c \).) The main result is then applied to two specific systems; the second of these has a Gribov problem \([12]\) which makes standard gauge-fixing methods unworkable but can be handled by the methods of this paper. The difficulties of gauge-fixing in canonical quantization when there is a Gribov problem have been analysed by McMullan \([13]\).

2 BFV Quantization

The starting point of the BFV approach is the unconstrained 2n-dimensional phase space \( \mathbb{R}^{2n} \), (with local coordinates \((p_i, q^i)\), where \( i = 1, \ldots, n \)) together with a set of \( m \) first class constraints \( T_a(p, q) = 0, a = 1, \ldots, m \) (with \( m < n \)) and first class Hamiltonian \( H_c(p, q) \). The true (reduced) phase space of this system is then the space \( B = C/G \) where \( C \) is the submanifold of \( \mathbb{R}^{2n} \) on which the constraints hold and \( G \) denotes the group generated by the constraints (which acts naturally on \( C \), with symplectic structure given by the Dirac bracket. (Finite-dimensional language is used here, but in principle the ideas can be extended, at least formally, to field theory.) A set of gauge-fixing functions \( X^a, a = 1, \ldots, m \) are also introduced; the nature of these functions is a central point of the discussion of this paper.

Now, as is shown in \([14]\), the generating functional is given by the Faddeev formula

\[
Z = \int \mathcal{D}p \mathcal{D}q \left[ \prod_t \left( \prod_{a=1}^m \left[ \delta \left( T_a(p(t), q(t)) \right) \delta \left( X^a(p(t), q(t)) \right) \right] \right) \right. \\
\times \det \left( \{ T_a(p(t), q(t)), X^b(p(t), q(t)) \} \right) \\
\times \exp \left( i \int_0^t p_a(t) \dot{q}^a(t) - H_c(p(t), q(t)) dt \right),
\]

where the integration is over paths \( p(t), q(t) \) in the unconstrained phase space which begin and end at the same point. (Standard quantization of the unconstrained phase space in the Schrödinger picture is used.) In establishing this expression, further assumptions have to be made about the gauge-fixing
functions $X^a$, including the requirement that the matrix $\{ T_a, X^b \}$ (where $\{ , \}$ denotes the Poisson bracket) must satisfy the condition
\[
\det(\{ T_a, X^b \}) \neq 0
\] (2)
at all points $(p, q)$ in $\mathbb{R}^{2m}$. When it is not possible to find a set of gauge-fixing functions satisfying these conditions there is a Gribov problem [12], and it is not possible to define a unique representative of each class in the reduced phase space $C/B$ by picking out the zeroes of a set of gauge-fixing functions. (An example where this occurs is considered in section 5.)

The key idea in the BFV approach is to extend the phase space by including Lagrange multipliers $l^a, a = 1, \ldots, m$ for the constraints together with their canonically conjugate momenta $k_a$, a set of $m$ ghosts, $\eta^a$ together with their conjugate momenta $\pi_a$ and a set of $m$ antighosts and corresponding momenta $\phi_a$ and $\theta_a$. An extended Hamiltonian $H_{ext} = H_c + H_h + H_g$ is then defined, with $H_g$ taking a form prescribed by the constraints, their commutators, and the gauge-fixing functions, while $H_h$ (which is often zero) depends on the commutator of $H_c$ with the constraints. The extended Hamiltonian is quite complicated, but it is shown in [5], by clever manipulation of path integrals, both that the corresponding generating functional
\[
Z = \int \mathcal{D}p \mathcal{D}q \mathcal{D}k \mathcal{D}l \mathcal{D}\pi \mathcal{D}\eta \mathcal{D}\phi \mathcal{D}\theta \\
\times \exp \left( i \int_0^t p(t) \dot{q}(t) + k(t) \dot{l}(t) + \pi(t) \dot{\eta}(t) + \phi(t) \dot{\theta}(t) \right.

\left. - H_{ext}(p(t), q(t), k(t), l(t), \pi(t), \eta(t), \phi(t), \theta(t)) dt \right)
\]
(3)
is independent of the choice of gauge-fixing functions and that it is equal to the Faddeev formula [4] for the generating functional for the original Hamiltonian on the reduced phase space. This result is known as the Fradkin-Vilkovisky Theorem.

3 BRST Cohomology and the Fradkin-Vilkovisky theorem

A significant insight of Henneaux [6] was an interpretation of the BFV Hamiltonian $H_{ext} = H_c + H_h + H_g$ in terms of the cohomology of the BRST operator.
Ω corresponding to the $2(m+n)$-dimensional phase space (with typical point $(p, q, l, k)$) subject to the constraints $T_a(p, q) = 0$, $k_a = 0$. Henneaux showed that the term $H_h$ (which contains ghosts) led to a modified Hamiltonian $\hat{H}_t = H_c + H_h$ satisfying $[\hat{H}_t, \Omega] = 0$, while the term $H_g$ could be expressed as $i[\Omega, \chi]$ with a “gauge-fixing fermion” $\chi$ of ghost number $-1$ constructed from the gauge-fixing functions $X^a, a = 1, \ldots, m$ in a prescribed way. Henneaux also demonstrated the correspondence between observables on the reduced phase space and operators which commute with $\Omega$, and the related correspondence between states for the reduced system and $\Omega$-cohomology classes.

Henneaux proves the Fradkin-Vilkovisky theorem by showing that the path integral (3) is invariant under infinitesimal change of gauge-fixing fermion. However this is not a full proof, first, because (as observed by Govaerts [8]) the space of orbits of the gauge group may not be connected, and second because it must be shown that the space of possible gauge-fixing fermions includes one which gives the correct generating functional. (This is achieved in the longer proof in [5] by showing that the path integral (3) reduces to the Faddeev formula (1) after integrating out the fermions and the Lagrange multiplier variables while rescaling the gauge-fixing functions.)

Now the Faddeev formula (1) gives a useful expression for the reduced phase space generating functional, but involves gauge-fixing functions of a kind which may not exist for a general system. However the existence of the reduced phase space is quite independent of the existence of such gauge-fixing functions. Indeed, as was shown quite generally by Kostant and Sternberg [15], the space of observables on the reduced phase space is isomorphic to the zero $\Omega$ cohomology on the extended phase space, so that the space of physical states can be identified with the zero $\Omega$ cohomology group of states for the extended phase space. This suggests that it might be possible to prove the Fradkin-Vilkovisky theorem without reference to the Faddeev formula, and hence without recourse to gauge-fixing functions satisfying (2), and it is just such a programme which will be carried out here, showing that there is a connection between the supertrace of an observable and the BRST cohomology of the states which leads to the Fradkin-Vilkovisky theorem, with precise criteria which the gauge-fixing fermion $\chi$ must satisfy.

In the BRST cohomology approach to gauge quantization in the Schrödinger picture, the full space of states $\mathcal{H}$ consists of functions $f(q, l, \eta, \phi)$ of the configuration space variables. (Precisely which functions are included in $\mathcal{H}$ depends on the system under consideration, as will be seen below; since the inner product on states is not positive definite the usual $L^2$ prescription
will not be valid.) The ghost number of a state is the degree in the ghost variables \( \eta \) less the degree in the antighost variables \( \phi \), and physical states are identified as elements of \( H^0(\Omega) \), (where for \( i = -m, \ldots, m \), \( H^i(\Omega) \) denotes the space of cohomology classes of \( \Omega \) at ghost number \( i \)). Observables are operators on \( \mathcal{H} \) which commute with \( \Omega \), and thus have a well defined action on the cohomology classes \( H^i(\Omega) \).

A new proof of the Fradkin-Vilkovisky theorem, with explicit conditions which the gauge-fixing fermion must satisfy, will now be given. The nonsingularity conditions are that

(i) the only states which are zero eigenstates of the operator \([\Omega, \chi]\) are states of ghost number zero which are not \( \Omega \)-exact;

(ii) on each ghost and \( \tilde{H}_t \) sector the real part of the eigenvalues \( \lambda_n \) of \( i[\Omega, \chi] \) tend to infinity with \( n \);

(iii) the Hamiltonian \( \tilde{H}_t \) must have finite trace on the space of zeroes of \([\Omega, \chi] \), on which it acts modulo \( \Omega \).

As will be seen, the purpose of the first condition is to ensure that \( \Omega \) only has cohomology at ghost-number zero, the second condition ensures that \( i[\Omega, \chi] \) regularises the traces of non-physical states involved in the path integral and the third ensures that the system does possess a generating functional.

The first stage of the proof of the Fradkin-Vilkovisky theorem given in this section is to show (by two lemmas) that the supertrace of \( \exp[iHt] \) gives the trace of this operator over physical states, that is, the desired generating functional.

For a given operator \( A \) on \( \mathcal{H} \), the supertrace is defined by

\[
\text{Str}A = \text{Tr}((-1)^g A)
\]

where \( g \) denotes ghost number, so that \((-1)^g f = f \) when \( f \) is a state of even ghost number and \((-1)^g f = -f \) when \( f \) is a state of odd ghost number. (We assume for the time being that the various infinite sums here are well defined; it will be seen below that this is ensured by the nonsingularity conditions.) Following Henneaux and Teitelboim (and the earlier work of Schwarz [10]), it is shown that taking the supertrace of an observable is equivalent to taking the alternate sum of the traces over the cohomology classes \( H^i(\Omega) \).
Lemma 3.1 Suppose that $A$ is an observable on $\mathcal{H}$, so that $[A, \Omega] = 0$. Then, assuming that the necessary traces exist,

$$\text{Str} A = \sum_{i=-m}^{m} (-1)^i \text{Tr}_{H^i(\Omega)} A. \quad (5)$$

Proof We make the following decomposition of the space of states $\mathcal{H}$:

$$\mathcal{H} = G \oplus F \oplus E \quad (6)$$

where $G = \text{Im} \Omega$ and $G \oplus E = \text{Ker} \Omega$. Also, for $i = -m, \ldots, m$, let $E_i, F_i$ and $G_i$ denote the subspaces of $E, F$ and $G$ respectively with ghost number $i$. Then $G_{i+1} = \Omega F_i$ for $i = -m, \ldots, m - 1$, while $G_{-m}$ and $F_m$ are empty since the operator $\Omega$ raises ghost number by one and annihilates all states of top ghost number.

Now if $f$ is an eigenstate of $A$ with eigenvalue $\lambda$, then $\Omega f$ is either zero or also an eigenstate of $A$ with the same eigenvalue; thus the map $\Omega : F_i \rightarrow G_{i+1}$ is an isomorphism, so that

$$\text{Tr}_{F_i} A = \text{Tr}_{G_{i+1}} A, \quad (7)$$

which means that the contributions to the supertrace from $F_i$ and $G_{i+1}$ cancel giving

$$\text{Str} A = \sum_{i=-m}^{m} (-1)^i \text{Tr}_{E_i} A \quad (8)$$

as required. \hfill \blacksquare

It is next shown that, if $\chi$ can be chosen so that $[\Omega, \chi]$ has a certain property, then $\Omega$ only has cohomology at ghost number zero.

Lemma 3.2 If $\chi$ can be chosen so that the operator $[\Omega, \chi]$ is invertible except on states of zero ghost number which are not $\Omega$-exact, then $H^p(\Omega)$ is trivial except when $p$ is zero.

Proof Let $V$ denote the subspace of states with non-zero ghost number. Suppose that $f$ is a state in $V$ such that $\Omega f = 0$ and that $h = [\Omega, \chi] f$. Then

$$f = [\Omega, \chi]^{-1} h = [\Omega, \chi][\Omega, \chi]^{-2} h. \quad (9)$$
Now, since \([\Omega, \chi]\) is invertible on all states with non-zero ghost number and exact states of zero ghost number,
\[
\Omega[\Omega, \chi]^{-2}h = [\Omega, \chi]^{-1}\Omega[\Omega, \chi]^{-1}h = [\Omega, \chi]^{-1}\Omega f = 0, \tag{10}
\]
so that
\[
f = [\Omega, \chi][\Omega, \chi]^{-2}h \\
   = \Omega\chi[\Omega, \chi]^{-2}h. \tag{11}
\]
Thus \(f\) is cohomologically trivial.

The proof of the Fradkin-Vilkovisky theorem is completed by the observation that (by standard arguments) the path integral expression (3) gives the supertrace of the evolution operator \(\exp[iH_{\text{ext}}]\), which by the arguments above must be the desired generating functional \(\text{Tr}_{\text{phys}}\exp[i\tilde{H}t]\). The second non-singularity condition ensures that the sums in the supertrace may be reordered, in other words the additional term \(i[\Omega, \chi]\) in the Hamiltonian regularises the trace.

4 A simple example of the theorem

The first application of the theorem is to the simple situation in which the unconstrained phase space is \(\mathbb{R}^4\) (with coordinates \(p_1, p_2, q^1, q^2\)) and there is a single first-class constraint \(p_1 = 0\). Now it is clear that including the gauge-fixing condition \(q^1 = 0\) gives the reduced phase space with coordinates \(p_2, q^2\); the constraint simply means that one of the two dimensions in the configuration space was redundant. It will now be seen how this emerges from the BFV formalism in a way which is useful in more complex situations, following the “quartet mechanism” of Henneaux and Teitelboim [7] which shows that in this case \(i[\Omega, \chi]\) is a number operator which is clearly invertible on states of non-zero ghost number, as well as on exact states of zero ghost number.

The extended phase space for this system has dimension \((6, 4)\) with local coordinates \(p_1, p_2, q^1, q^2, k, l, \eta, \theta, \phi\). Quantization is carried out using a space of states which are functions of the variables \(q^1, q^2, l, \eta, \theta, \phi\), while observables are built from the operators \(\hat{p}_1, \hat{p}_2, \hat{q}^1, \hat{q}^2, k, l, \hat{\eta}, \hat{\theta}, \hat{\phi}\) with canonical graded commutation relations
\[
[\hat{p}_i, \hat{q}^j] = -i\delta^j_i, \quad [\hat{k}, \hat{l}] = -i, \quad [\hat{\eta}, \hat{\pi}] = -i \text{ and } [\hat{\theta}, \hat{\phi}] = i. \tag{12}
\]
The operators $\hat{p}_i$ and $\hat{q}^i$ are represented in the standard way, while the action of $\hat{k}, \hat{l}, \hat{\eta}, \hat{\theta}, \hat{\phi}$ is defined by

\[
\begin{align*}
\hat{f}(q^1, q^2, l, \eta, \phi) &= -\frac{\partial}{\partial l} f(q^1, q^2, l, \eta, \phi) \\
\hat{f}(q^1, q^2, l, \eta, \phi) &= i l f(q^1, q^2, l, \eta, \phi) \\
\hat{f}(q^1, q^2, l, \eta, \phi) &= \eta f(q^1, q^2, l, \eta, \phi) \\
\hat{f}(q^1, q^2, l, \eta, \phi) &= -i \frac{\partial}{\partial \eta} f(q^1, q^2, l, \eta, \phi) \\
\hat{f}(q^1, q^2, l, \eta, \phi) &= -i \phi f(q^1, q^2, l, \eta, \phi) \\
\hat{f}(q^1, q^2, l, \eta, \phi) &= \frac{\partial}{\partial \phi} f(q^1, q^2, l, \eta, \phi).
\end{align*}
\] (13)

The inner product is defined by

\[
(f, g) = \int dq^1 dq^2 dl d\eta d\phi \tilde{f}(q^1, q^2, -l, \eta, \phi) g(q^1, q^2, l, \eta, \phi)
\] (14)

where $\tilde{\cdot}$ denotes reversal of order of anticommuting terms together with complex conjugation of complex coefficients so that $\hat{p}, \hat{q}, \hat{k}, \hat{l}, \hat{\eta}$ and $\hat{\theta}$ are Hermitian while $\hat{\phi}$ and $\hat{\pi}$ are antihermitian. As always in BRST quantization, since the Hermitian operator $\Omega$ has zero square, an indefinite inner product is required. The BRST operator and gauge-fixing fermion take the standard forms

\[
\begin{align*}
\Omega &= \hat{p}_1 \hat{\eta} + \hat{k} \hat{\theta}, \\
\chi &= \hat{l} \hat{\pi} + \hat{q}^1 \hat{\phi}
\end{align*}
\] (15)

and the commutator is

\[
[\Omega, \chi] = -i (\hat{p}_1 \hat{l} - \hat{k} \hat{q}^1 + \hat{\eta} \hat{\phi} + \hat{\theta} \hat{\pi}).
\] (16)

Defining creation and annihilation operators

\[
\begin{align*}
a &= \frac{1}{\sqrt{2}} (\hat{p}^1 + i \hat{k}) \\
b &= \frac{1}{\sqrt{2}} (\hat{l} - i \hat{q}^1) \\
a^* &= \frac{1}{\sqrt{2}} (\hat{p}^1 - i \hat{k}) \\
b^* &= \frac{1}{\sqrt{2}} (\hat{l} + i \hat{q}^1) \\
\gamma &= \frac{1}{\sqrt{2}} (\hat{\eta} + i \hat{\theta}) \\
\gamma^* &= \frac{1}{\sqrt{2}} (\hat{\phi} + i \hat{\pi}) \\
\overline{\gamma} &= \frac{1}{\sqrt{2}} (\hat{\phi} - i \hat{\pi}) \\
\overline{\gamma}^* &= \frac{1}{\sqrt{2}} (-\hat{\phi} + i \hat{\pi})
\end{align*}
\] (17)

which satisfy commutation relations

\[
\begin{align*}
[a, b^*] &= [b, a^*] = 1 \\
[\gamma, \overline{\gamma}^*] &= [\overline{\gamma}, \gamma^*] = 1
\end{align*}
\] (18)
(with all other commutators zero), the full space of states is constructed from vacuum states \( \Psi_0^u, u = 1, 2, \ldots \) which are normalized states annihilated by \( a, b, \gamma \) and \( \gamma^* \). (Explicitly, the vacuum states are

\[
\Psi_0^u = \delta \left( q - l \right) \exp(\eta \phi) f_u(q^2)
\]

where \( \{ f_u \} \) is an orthonormal basis of \( L^2(\mathbb{R}) \).) The full basis of states \( \Psi_{mnrs}^u, u, m, n = 1, 2, \ldots, r, s = 0, 1 \) is defined by

\[
\Psi_{mnrs}^u = \frac{1}{\sqrt{m!n!}} a^m b^n \gamma^r \gamma^s \Psi_0^u
\]

gives the orthonormality relation

\[
(\Psi_{mnrs}^u, \Psi_{m'n'r's'}^{u'}) = \delta_{mn} \delta_{m'n'} \delta_{rs} \delta_{r's'} \delta_{u'u'}
\]

The space of states \( \mathcal{H} \) is then defined to be the space of states of the form

\[
\sum_{m,n,r,s,u} a_{mnrsu} \Psi_{mnrs}^u \text{ where } \sum_{m,n,r,s,u} |a_{mnrsu}|^2 \text{ is finite.}
\]

Observing that the operator \( i[\Omega, \chi] \) can be expressed as

\[
i[\Omega, \chi] = a^* b + b^* a + \gamma^* \gamma^* + \gamma \gamma^\dagger
\]

which has eigenvalue \( m + n + r + s \) on the state \( \Psi_{mnrs}^u \), that \( \gamma \gamma^* + \gamma^* \gamma \) is the operator which gives the sum of the ghost and antighost number and that all terms in \( \Omega \) contain ghost or antighost creation operators it is clear that the only zero eigenstates of \( i[\Omega, \chi] \) have zero ghost number and are not \( \Omega \)-exact. Hence by Lemma 3.2 the BRST charge \( \Omega \) has nontrivial cohomology only at ghost number zero, as required; moreover, provided that the classical Hamiltonian \( H_c \) is chosen so that \( \exp(iH_c t) \) has a well defined trace on the reduced phase space, the operator \( \exp(i(H + i[\Omega, \chi])t \) will have a well-defined supertrace, and this will be calculated by the path integral (3).

To show how these ideas may be extended to more complex situations it is useful to introduce the operators \( P_{mn}(\hat{k}, \hat{l}), m, n = 1, \ldots \) defined inductively by

\[
P_{00}(\hat{k}, \hat{l}) = 1, \quad P_{m+1,n}(\hat{k}, \hat{l}) = \hat{k}P_{mn}(\hat{k}, \hat{l}) + P_{mn}(\hat{k}, \hat{l})\hat{k}, \quad P_{m,n+1}(\hat{k}, \hat{l}) = \hat{l}P_{mn}(\hat{k}, \hat{l}) + P_{mn}(\hat{k}, \hat{l})\hat{l}.
\]

(It is easily verified that this does consistently define \( P_{mn} \) for all non-negative integers \( m \) and \( n \).) Then, since the vacuum is annihilated by \( a \) and \( b \),

\[
P_{mn}(\hat{k}, \hat{l})\Psi^u_0 = i^m (a^*)^m (b^*)^n \Psi^u_0
\]
and also

\[ i[\Omega, \chi]P_{mn}(\hat{k}, \hat{l})\Psi_0^u = i(\hat{k}P_{mn}(\hat{k}, \hat{l})\hat{l} - \hat{l}P_{mn}(\hat{k}, \hat{l})\hat{k})\Psi_0^u \]  

(25)

so that

\[ i(\hat{k}P_{mn}(\hat{k}, \hat{l})\hat{l} - \hat{l}P_{mn}(\hat{k}, \hat{l})\hat{k}) = (m + n + 1)P_{mn}(\hat{k}, \hat{l}) \]  

(26)

as can also be shown algebraically directly from the definition of \( P_{mn}(\hat{k}, \hat{l}) \).

An alternative approach to the space of states for this system, also developing from the quartet mechanism, is given by Marnelius and Sandström [16] in their interesting discussion of states in BFV quantization, and extended to a wide variety of systems.

5 An example to show gauge-fixing in the presence of the Gribov problem

An interesting toy example is the system described by Henneaux and Teitelboim [7] whose initial phase space is \( \mathbb{R}^4 \) (with coordinates \( p_1, p_2, q^1, q^2 \)) on which the single first class constraint

\[ T \equiv T_1 + T_2 - \frac{1}{4} = 0 \]  

(27)

with \( T_1 = \frac{1}{4}((p_1)^2 + (q^1)^2) \) and \( T_2 = \frac{1}{4}((p_2)^2 + (q^2)^2) \), is imposed. In this case the constraint surface \( C \) is the three-dimensional sphere \( S^3 \). The group action on \( C \) generated by \( T \) is most easily analysed by defining \( z_1 = q^1 + ip_1 \), \( z_2 = q^2 + ip_2 \). Then (with the standard prescription that \( \delta \epsilon(\cdot) = \epsilon\{T, \cdot\} \)) \( T \) acts as the infinitesimal \( U(1) \) transformation \( \delta z_i = (1 + \frac{1}{2}i\epsilon z_i) \), giving as true (reduced) phase space of the system the Hopf fibration \( S^2 = S^3/U(1) \), which is well known to be non-trivial [17], so that the system has a Gribov problem. This means that no bosonic gauge-fixing function \( X \) can be found to directly ensure the validity of the Faddeev formula [1]. However the approach developed in this paper does enable a gauge-fixing procedure to be set up.

To carry out the BFV quantization of this system the phase space is extended to \( S^1 \times \mathbb{R}^{(5,4)} \) with coordinates \( l, k, p_1, p_2, q^1, q^2, \eta, \pi, \theta \) and \( \phi \), with quantization of these variables as before, but with a different space of states.

The BRST operator and gauge-fixing fermion are

\[ \Omega = T\hat{\eta} + k\hat{\theta}, \quad \chi = (\hat{\phi}\hat{\theta} - \hat{\theta}\hat{\phi})\hat{\pi}\sinh \hat{l} - (X + T \cosh \hat{l})\hat{\phi} \]  

(28)
where \( X = \frac{1}{4}((p_1)^2 - (q_1)^2) \).

The space of states is constructed from the states

\[
P_{mn}(\hat{t}, \hat{k})\gamma^r\gamma^s\Psi_0^u, \quad m, n, u = 1, 2 \ldots, r, s = 0, 1
\]

where \( \hat{t} \) is the operator \( \tanh\frac{1}{4} \hat{l} \) and the vacuum states \( \Psi_0^u \) are

\[
\Psi_0^u = \exp(\hat{l}T)g_u(q^2)(1 + i\eta\phi)
\]

where \( g_u(q^2), u = 1, 2 \ldots \) are eigenstates of \( T_2 - \frac{1}{4} \) with eigenvalues \( \frac{1}{4}, \frac{3}{4}, \ldots \) with normalisation factors set so that the \( \Psi_{0,u} \) are orthonormal. The vacuum states are all annihilated by \( \Omega \) and \( \chi \), so that they are also annihilated by the commutator \([\Omega, \chi]\).

Some algebra shows that (if rows are labeled first by \( m - n \) and then by \( m + n \)) the commutator \( i[\Omega, \chi] \) is upper triangular with diagonal elements having real part \( \sim (m + n + r + s) \). Essentially the same arguments as in the previous example then shows that the commutator \([\Omega, \chi]\) does satisfy the nonsingularity conditions. Then, provided that a suitable Hamiltonian is used, that is, one which has a finite trace on the space of physical states, the path integral with Hamiltonian extended by \( i[\Omega, \chi] \) will give the correct supertrace, and thus the desired trace over physical states.

6 Conclusion

This paper establishes a criterion for the admissibility of a gauge-fixing fermion which can be applied to systems where the usual criteria are inadequate. In particular it is shown that, by introducing a gauge-fixing term which is not constructed by the standard prescription from a bosonic gauge-fixing function, it is possible to fix the gauge of a system with a Gribov problem.

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