UNIFORM ATTRACTORS OF STOCHASTIC THREE-COMPONENT GRAY-SCOTT SYSTEM WITH MULTIPlicative NOISE

JUNWEI FENG AND HUI LIU
School of Mathematical Sciences, Qufu Normal University
Qufu, Shandong 273165, China

JIE XIN∗
School of Mathematical Sciences, Qufu Normal University
Qufu, Shandong 273165, China
College of Information Science and Engineering, Shandong Agricultural University
Taian, Shandong 271018, China
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Abstract. In a bounded domain, we study the long time behavior of solutions of the stochastic three-component Gray-Scott system with multiplicative noise. We first show that the stochastic three-component Gray-Scott system can generate a non-autonomous random dynamical system. Then we establish some uniform estimates of solutions for stochastic three-component Gray-Scott system with multiplicative noise. Finally, the existence of uniform and cocycle attractors is proved.

1. Introduction. We consider the following stochastic three-component Gray-Scott system with multiplicative noise from [17]

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} - d_1 \Delta \tilde{u} + (F + k) \tilde{u} - \tilde{u}^2 \tilde{v} + G \tilde{u}^3 - N \tilde{w} &= \sigma_1(x, t) + \alpha \tilde{u} \circ \frac{d\omega}{dt}, \\
\frac{\partial \tilde{v}}{\partial t} - d_2 \Delta \tilde{v} + F \tilde{v} + \tilde{u}^2 \tilde{v} - G \tilde{u}^3 &= \sigma_2(x, t) + \alpha \tilde{v} \circ \frac{d\omega}{dt}, \\
\frac{\partial \tilde{w}}{\partial t} - d_3 \Delta \tilde{w} + (F + N) \tilde{w} - k \tilde{u} &= \sigma_3(x, t) + \alpha \tilde{w} \circ \frac{d\omega}{dt},
\end{align*}
\]

with initial data

\[
\tilde{u}(x, 0) = \tilde{u}_0(x), \quad \tilde{v}(x, 0) = \tilde{v}_0(x), \quad \tilde{w}(x, 0) = \tilde{w}_0(x), \quad x \in \mathcal{O},
\]

and boundary condition

\[
\tilde{u}(x, t)|_{\partial \mathcal{O}} = 0, \quad \tilde{v}(x, t)|_{\partial \mathcal{O}} = 0, \quad \tilde{w}(x, t)|_{\partial \mathcal{O}} = 0, \quad x \in \partial \mathcal{O},
\]

on a bounded domain \( \mathcal{O} \subset \mathbb{R}^n \), for \( t \geq 0 \). Here \( F, G, N, k, \alpha \) and \( d_i (i = 1, 2, 3) \) are positive constants. Besides, \( \sigma_i \in \Sigma \), where \( i = 1, 2, 3 \). \( \circ \) denotes the Stratonovich

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* Corresponding author: Jie Xin.
sense of the stochastic term and $\omega$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see below) [1, 7].

A non-autonomous random dynamical system (NRDS) is a measurable mapping $\varphi : \mathbb{R}^+ \times \Omega \times \Sigma \times X \to X$; see [12, 13, 26]. Furthermore, $\varphi$ has two base flows $\{\vartheta_t\}_{t \in \mathbb{R}}$ and $\{\theta_t\}_{t \in \mathbb{R}}$ working on $\Omega$ and $\Sigma$, respectively. $X$ is a phase space, $\mathbb{R}^+ = [0, \infty)$ as well as $\Sigma$ is a symbol space constructed by time-dependent terms, called the symbol of the system [5].

The Gray-Scott equation is a crucial part of reaction-diffusion system. When $\bar{\omega} = 0, G = 0, \alpha = 0, \sigma_1 = \sigma_3 = 0, \sigma_2 = F$, the system (1) reduces to the two-component Gray-Scott system. The three-component reversible Gray-Scott equation was first studied in [22]. Then the global attractor of Gray-Scott equations was considered in [32]. In [28], Wang investigated the existence of a pullback global attractor of a non-autonomous reaction-diffusion equations on $\mathbb{R}^{n}$. He also proved the pullback asymptotic compactness of solutions by applying uniform prior estimates. For stochastic reaction-diffusion equation with multiplicative noise on unbounded domains, the existence of a random attractor was proved in [29]. In [18], by constructing skew product flow, Gu, Zhou and Wang discussed the existence and the structure of uniform attractor for a non-autonomous three-component reversible Gray-Scott system. Interested readers can refer to [17, 19, 30, 31, 32, 33] for more information on Gray-Scott equations.

The cocycle attractor for NRDS was first proposed in [10]. Then it got greatly developed in [26, 27]. For other different kinds of attractors, such as random attractors, pullback attractors, global attractors, and so on, we can refer to [2, 3, 4, 6, 8, 9, 14, 15, 21, 23, 25]. In [4], Bates, Lu and Wang proved the existence of random attractors for a stochastic reaction-diffusion equation. [17] investigated upper semicontinuity of random attractors for a stochastic three-component reversible Gray-Scott system.

[13] investigated the existence of uniform and cocycle attractors of a stochastic reaction-diffusion equation with additive noise. Compared with [13], we study the existence of uniform and cocycle attractors of a stochastic three-component Gray-Scott system with multiplicative noise. The noise studied is completely different. Moreover, the equation we studied is more complicated than that in [13]. Thus the proof process is more difficult. Our research is also of great significance and value.

The rest of this article is as follows. In Section 2, some concepts about uniform attractor, cocycle attractor, and NRDS are provided. In Section 3, we will get the stochastic three-component Gray-Scott system that generates a NRDS. In Section 4, some uniform prior estimates of solutions are established. In Section 5, the existence of uniform attractor is proved.

Denote by $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$ the norm and inner product in $L^2(\mathcal{O})$ or $[L^2(\mathcal{O})]^3$. Define $U = [L^6(\mathcal{O})]^3$, $V = [H^1(\mathcal{O})]^3$. Applying $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^1}$ to express the norm in $L^p(\mathcal{O})$ and $H^1(\mathcal{O})$. Assume that $c$ is a positive constant.

2. Preliminaries. In this section, we provide some useful concepts needed later, including uniform attractor, cocycle attractor and NRDS, see [11, 13, 5] for more details.

Denote by $(X, d)$ a Polish metric space. For non-empty sets in $X$, define the Hausdorff semi-metric

$$\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b), \quad A, B \in 2^X \setminus \emptyset.$$
Let $\mathcal{B}(M)$ be the Borel sigma-algebra of any metric space $M$. Suppose that $(\Sigma, d_\Sigma)$ is a compact Polish space with

$$\theta_t \Sigma = \Sigma, \quad \forall t \in \mathbb{R},$$

here $\theta$ is a smooth translation operator.

Throughout this essay, let $\mathcal{D}$ be some class of random sets in $X$. Moreover, $\mathcal{D}$ is neighborhood-closed and inclusion-closed (see [11, Section 2.1, p. 1232]).

For any $B \in \mathcal{D}$, define the omega limit set of $B$

$$\mathcal{W}(\omega, \Xi, B) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi(t, \theta_{-t} \omega, \theta_{-t} \Xi, B(\theta_{-t} \omega)), \quad \forall \Xi \subset \Sigma, \omega \in \Omega.$$

**Definition 2.1.** Suppose that $G_1, \ G_0$ are Banach spaces, define an operator $A_{\sigma(s)}(\cdot) : G_1 \rightarrow G_0$ for any $s \in \mathbb{R}$, where $\sigma(s)$ reflects the dependence on time, then the function $\sigma(s)$ is called the symbol of the system (1).

**Definition 2.2.** If $\Sigma$ contains all $\sigma(\cdot)$ and satisfies

$$\theta_t \sigma(\cdot) := \sigma(\cdot + t), \quad \forall t \in \mathbb{R},$$

then the set $\Sigma$ is called the symbol space of the system (1).

**Property 2.3 ([13]).** $\{ \theta_t \}_{t \in \mathbb{R}}$ is the smooth translation operator satisfying

(i) $\theta_0 = \text{identity operator on } \Sigma$;

(ii) $\theta_s \circ \theta_t = \theta_{t+s}, \quad \forall t, s \in \mathbb{R}$;

(iii) $(t, \sigma) \mapsto \theta_t \sigma$ is continuous.

**Property 2.4([13]).** $\{ \vartheta_t \}_{t \in \mathbb{R}}$ is the base flow of $\phi$ with

(i) $\vartheta_0 = \text{identity operator on } \Omega$;

(ii) $\vartheta_t \Omega = \Omega, \quad \forall t \in \mathbb{R}$;

(iii) $\vartheta_s \circ \vartheta_t = \vartheta_{t+s}, \quad \forall t, s \in \mathbb{R}$;

(iv) $(t, \omega) \mapsto \vartheta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable;

(v) $\mathcal{P}$-preserving: $\mathcal{P}(\vartheta_t \mathcal{F}) = \mathcal{P}(\mathcal{F}), \quad \forall t \leq 0, \mathcal{F} \in \mathcal{F}$.

**Definition 2.5([13]).** A mapping $\phi(t, \omega, \sigma, x) : \mathbb{R}^+ \times \Omega \times \Sigma \times X \to X$ is called a NRDS on $X$ if

(i) $\phi$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\Sigma) \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\phi(0, \omega, \sigma, \cdot)$ is the identity on $X, \quad \forall \sigma \in \Sigma, \omega \in \Omega$;

(iii) for any fixed $\sigma \in \Sigma, \ x \in X$ and $\omega \in \Omega$, $\phi$ satisfies

$$\phi(t + s, \omega, \sigma, x) = \phi(t, \theta_s \omega, \theta_s \sigma) \circ \phi(s, \omega, \sigma, x), \quad \forall t, s \in \mathbb{R}^+.$$

For any $t \in \mathbb{R}^+, \ \omega \in \Omega$ and $x \in X$, if $\sigma \mapsto \phi(t, \omega, \sigma, x)$ is continuous, then the NRDS $\phi$ is continuous in $\Sigma$. The continuity of $\phi$ in $X$ can be defined in like manner.

From now on, suppose that $\phi$ is a NRDS.

**Definition 2.6([13]).** An (autonomous) random set $A \in \mathcal{D}$ is called the (random) $\mathcal{D}$-uniform attractor for a NRDS $\phi$ if

(i) $A$ uniformly (pullback) attracts every $D \in \mathcal{D}$, i.e.,

$$\lim_{t \to \infty} \sup_{\sigma \in \Sigma} \text{dist}(\phi(t, \theta_{-t} \omega, \theta_{-t} \sigma, D(\theta_{-t} \omega)), A(\omega)) = 0, \quad \forall \omega \in \Omega;$$

(ii) $A$ is the minimal compact (autonomous) random set satisfying (i).

**Definition 2.7([13]).** A non-autonomous random set $A = \{ A_\sigma(\omega) \}_{\sigma \in \Sigma, \omega \in \Omega}$ is referred to as the (random) $\mathcal{D}$-cocycle attractor for a NRDS $\phi$ if

(i) $A(\cdot)$ is a compact random set in $X$.
(ii) A pullback attracts every $D \in \mathcal{D}$, i.e.,
\[
\lim_{t \to \infty} \text{dist}(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), A_\sigma(\omega)) = 0, \quad \forall \omega \in \Omega, \sigma \in \Sigma;
\]
(iii) $A$ is invariant, i.e.,
\[
\phi(t, \omega, \sigma, A_\sigma(\omega)) = A_{\theta_t \sigma}(\vartheta_{t}\omega), \quad \forall t \in \mathbb{R}^+;
\]
(iv) $A$ is the minimal compact non-autonomous random set satisfying (ii).

**Theorem 2.8** ([13]). Suppose that NRDS $\phi$ is continuous in both $\Sigma$ and $X$, then uniform attractor $A$ and cocycle attractor $\pi$ for $\phi$ have the following relation
\[
A(\omega) = \bigcup_{\sigma \in \Sigma} A_\sigma(\omega), \quad \forall \omega \in \Omega.
\]

**Theorem 2.9** ([13]). For a NRDS $\phi$, suppose that $\pi : \mathbb{R}^+ \times \Omega \times X \to X$ is a mapping
\[
\pi(t, \omega, \{\sigma \times \{x\}) = \{\theta_t \sigma \} \times \{\phi(t, \omega, \sigma, x)\}.
\]

Then $\pi$ is a (random) cocycle satisfying
\begin{enumerate}[(i)]  
  \item $\pi$ is $(\mathcal{B}(\mathbb{R}^+)) \times \mathcal{F} \times \mathcal{B}(\mathbb{X}))$-measurable;
  \item $\pi(0, \omega, \chi) = \chi, \quad \forall \omega \in \Omega, \chi \in \mathbb{X}$;
  \item $\pi(t + s, \omega, \chi) = \pi(t, \vartheta_s, \pi(s, \omega, \chi)), \quad \forall t, s \in \mathbb{R}^+, \omega \in \Omega, \chi \in \mathbb{X}$.
\end{enumerate}

**Proposition 2.10** ([13]). If a random set $A$ is uniformly $\mathcal{D}$-pullback attracting under NRDS $\phi$, then $A$ is forward uniformly attracting in probability, i.e.,
\[
\lim_{t \to \infty} \mathcal{P}\{\omega \in \Omega : \sup_{\sigma} \text{dist}(\phi(t, \omega, \sigma, B(\omega)), A(\vartheta_t \omega)) > \varepsilon\} = 0, \quad \forall \varepsilon > 0, B \in \mathcal{D}.
\]

**Theorem 2.11** ([13]). Assume that NRDS $\phi$ is continuous in $\Sigma$ and $X$. Let $\Xi \in \Sigma$ be dense. If $\phi$ possesses a compact uniformly $\mathcal{D}$-attracting set $K$ and a closed uniformly $\mathcal{D}$-absorbing set $B \in \mathcal{D}$, then $A$ has a unique random uniform attractor $A \in \mathcal{D}$
\[
A(\omega) = \mathcal{W}(\omega, \Sigma, B) = \mathcal{W}(\omega, \Xi, B), \quad \forall \omega \in \Omega.
\]

Besides, $A$ is negatively semi-invariant
\[
A(\vartheta_t \omega) \subseteq \phi(t, \omega, \Sigma, A(\omega)), \quad \forall t \geq 0, \omega \in \Omega.
\]

**Proposition 2.12** ([13]). Assume that NRDS $\phi$ is continuous in $\Sigma$ and $X$. Let $\mathcal{U}$ be a random uniform attractor. If $\phi$ possesses a $\mathcal{D}$-random uniform attractor $A$, and $\mathcal{U}$ uniformly attracts deterministic compact sets, then
\[
\mathcal{P}(A = \mathcal{U}) = 1.
\]

### 3. NRDS generated by stochastic three-component Gray-Scott system

In this section, we give the change of form for the system (1)-(3) similar to [13, 17, 30, 31, 32, 33]. In addition, we prove that the system (1)-(3) generates a NRDS.

Let $\bar{g} = (\bar{u}, \bar{v}, \bar{w})^T$, the system (1)-(3) can be rewritten as
\[
\frac{\partial \bar{g}}{\partial t} - A \bar{g} + H(\bar{g}) = \sigma(x, t) + \alpha \bar{g} \circ \frac{d\omega}{dt}, \quad t > 0,
\]
\[
\bar{g}(x, 0) = \bar{g}_0(x), \quad x \in \mathcal{O},
\]
\[
\bar{g}(x, t)|_{\partial \mathcal{O}} = 0,
\]
where
\[
\sigma(x, t) = (\sigma_1(x, t), \sigma_2(x, t), \sigma_3(x, t))^T,
\]
Let $T$ here and $H(\bar{g}) = \begin{pmatrix} (F+k)\bar{u} - \bar{u}^2\bar{v} + G\bar{u}^3 - N\bar{w} \\ F\bar{v} + \bar{u}^2\bar{v} - G\bar{u}^3 \\ (F+N)\bar{w} - k\bar{u} \end{pmatrix}$, here $T$ denotes the transposition.

Define $\Omega = \{ \omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0 \}$ as a probability space. Let $F$ be a Borel sigma-algebra. Let $\mathcal{P}$ be the two-sided Wiener measure on $(\Omega, F)$. $\vartheta$ is a translation operator satisfying

$$\vartheta_t \omega = \omega(\cdot + t) - \omega(t), \quad \forall t \in \mathbb{R}, \quad \omega \in \Omega.$$ 

Hence $\mathcal{P}$ is ergodic and invariant under $\vartheta$ [16, 20]. Let

$$\beta(t, \omega) = e^{-\alpha \omega(t)}, \quad \forall \omega \in \Omega.$$ 

We find that $\beta(t, \omega)$ is the stationary solution of the Ornstein-Uhlenbeck equation

$$\frac{d\beta}{dt} + \alpha \beta \frac{d\omega}{dt} = 0.$$ 

Let

$$u(t) = \beta(t, \omega)\bar{u}(t), \quad v(t) = \beta(t, \omega)\bar{v}(t), \quad w(t) = \beta(t, \omega)\bar{w}(t),$$

the system (1)-(3) becomes

$$\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u + (F+k)u - \beta^{-2}(t, \omega)u^2 v + G\beta^{-2}(t, \omega)u^3 - Nw &= \beta(t, \omega)\sigma_1, \\
\frac{\partial v}{\partial t} - d_2 \Delta v + Fv + \beta^{-2}(t, \omega)u^2 v - G\beta^{-2}(t, \omega)u^3 &= \beta(t, \omega)\sigma_2, \\
\frac{\partial w}{\partial t} - d_3 \Delta w + (F+N)w - ku &= \beta(t, \omega)\sigma_3, 
\end{align*}$$

(5)

with following conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad u(x, t)_{|\partial \Omega} = 0, \quad v(x, t)_{|\partial \Omega} = 0, \quad w(x, t)_{|\partial \Omega} = 0.$$ 

(6)

Let $g = (u, v, w)^T$, the system (5)-(6) can be rewritten as

$$\frac{\partial g}{\partial t} - Ag + H(g, \omega) = \beta(t, \omega)\sigma(x, t), \quad t > 0,$$

$$g(x, 0) = g_0(x), \quad x \in \mathcal{O},$$

$$g(x, t)_{|\partial \mathcal{O}} = 0,$$

where

$$\begin{align*}
\sigma(x, t) &= (\sigma_1(x, t), \sigma_2(x, t), \sigma_3(x, t))^T, \\
A &= \begin{pmatrix} d_1 \Delta & 0 & 0 \\ 0 & d_2 \Delta & 0 \\ 0 & 0 & d_3 \Delta \end{pmatrix}, \\
H(g, \omega) &= \begin{pmatrix} (F+k)u - \beta^{-2}(t, \omega)u^2 v + G\beta^{-2}(t, \omega)u^3 - Nw \\ Fv + \beta^{-2}(t, \omega)u^2 v - G\beta^{-2}(t, \omega)u^3 \\ (F+N)w - ku \end{pmatrix},
\end{align*}$$

here $T$ denotes the transposition.
In [5, 24], there exists a standard method of solving the system (5). Clearly, when \( g_0 \in [L^2(\Omega)]^3 \), the system (5) possesses a unique solution \( g(t, \omega, \sigma, g_0) \in C([0, \infty); [L^2(\Omega)]^3) \cap L^2_{loc}((0, \infty); [H^1(\Omega)]^3) \), where \( g(0, \omega, \sigma, g_0) = g_0 \).

For any \( t \geq 0, \omega \in \Omega \), \( \sigma \in \Sigma \) and \( g_0 \in [L^2(\Omega)]^3 \), define
\[
\phi(t, \omega, \sigma, g_0) = g(t, \omega, \sigma, g_0)\beta^{-1}(t, \omega).
\]
Obviously, \( \phi(t, \omega, \sigma, g_0) \) is the solution of (1) satisfying Definition 2.5. Thus, the NRDS created by (4) is continuous in initial data and symbols.

The tempered uniform and cocycle attractors of (1) can be investigated. Define an attraction universe \( \mathcal{D} \),
\[
\mathcal{D} = \{ D : D \text{ is a bounded random set in } [L^2(\Omega)]^3 \text{ satisfying } \lim_{t \to \infty} e^{-Ft}\|D(\vartheta - \omega)\|^2 = 0, \forall \omega \in \Omega \}.
\]
We find that \( \mathcal{D} \) is inclusion-closed and neighborhood-closed.

4. Uniform estimates of solutions. In this section, in order to prove the existence of uniform attractor, we need to uniformly estimate the solutions of (5) in the same way as [17].

Lemma 4.1. For any \( D \in \mathcal{D} \) and \( \omega \in \Omega \), there is a time \( T = T(D, \omega) > 1 \) such that, for any \( \sigma \in \Sigma \),
\[
\|g(t, \vartheta - \omega, \vartheta - t, \sigma, g_0)\|^2 + \int_0^t e^{F(s-t)}\|\nabla g(s)\|^2 ds \\
\leq c\beta^{-2}(-t, \omega) \int_{-\infty}^0 e^{Fs} \beta^2(s, \omega)\|\sigma(s)\|^2 ds + c
\]
holds uniformly in \( g_0 \in D \) and \( t \geq T \).

Proof. Let \( W(x, t) = \frac{N}{k}w(x, t), \mu = \frac{k}{N} \), then (5) becomes
\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u + (F + k)u - \beta^{-2}(t, \omega)u^2v + G\beta^{-2}(t, \omega)u^3 - kW &= \beta(t, \omega)\sigma_1, \\
\frac{\partial v}{\partial t} - d_2 \Delta v + Fv + \beta^{-2}(t, \omega)u^2v - G\beta^{-2}(t, \omega)u^3 &= \beta(t, \omega)\sigma_2, \\
\frac{\partial W}{\partial t} - \mu d_3 \Delta W + (\mu F + k)W - ku &= \beta(t, \omega)\sigma_3.
\end{align*}
\]
Taking the inner product \( (\partial u/\partial t, Gu) \), arriving at
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt}(\|u(t, \omega, \sigma, u_0)\|^2) + d_1 G\|\nabla u\|^2 + G(F + k)\|u\|^2 \\
- G\beta^{-2}(t, \omega)\int_\Omega u^3 v dx + G^2 \beta^{-2}(t, \omega)\int_\Omega u^4 dx - kG\int_\Omega uW dx \\
= G\int_\Omega \beta(t, \omega)u\sigma_1(t) dx.
\end{align*}
\]
Taking the inner product \( (\partial v/\partial t, v) \) to obtain
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt}(\|v(t, \omega, \sigma, v_0)\|^2) + d_2 \|\nabla v\|^2 + F\|v\|^2 + \beta^{-2}(t, \omega)\int_\Omega u^2 v^2 dx \\
- G\beta^{-2}(t, \omega)\int_\Omega u^3 v dx = \int_\Omega \beta(t, \omega)v\sigma_2(t) dx.
\end{align*}
\]
Taking the inner product \((\mu (\partial W / \partial t), GW)\) to have
\[
\frac{1}{2} \frac{d}{dt} (\mu G W(t, \omega, \sigma, W_0)^2) + \mu Gd_3 \|\nabla W\|^2 + G(\mu F + k)\|W\|^2 \\
- kG \int_O uW dx = G \int_O \beta(t, \omega) W \sigma_3(t) dx.
\] (11)

Summing up (9)-(11), we obtain
\[
\frac{d}{dt} (G\|u\|^2 + \|v\|^2 + \mu G\|W\|^2) + 2d_1 G\|\nabla u\|^2 + 2d_2 \|\nabla v\|^2 \\
+ 2\mu Gd_3 \|\nabla W\|^2 + 2G(F + k)\|u\|^2 + 2F\|v\|^2 + 2G(\mu F + k)\|W\|^2 \\
= 4kG \int_O uW dx - 2\beta^{-2}(t, \omega) \int_O (Gu^2 - uv)^2 dx + 2G \int_O \beta(t, \omega) u\sigma_1(t) dx \\
+ 2 \int_O \beta(t, \omega) v\sigma_2(t) dx + 2G \int_O \beta(t, \omega) W \sigma_3(t) dx.
\] (12)

Using Young’s inequality, we get
\[
2G(F + k)\|u\|^2 + 2F\|v\|^2 + 2G(\mu F + k)\|W\|^2 - 4kG \int_O uW dx \\
\geq 2FG\|u\|^2 + 2F\|v\|^2 + 2\mu FG\|W\|^2,
\] (13)

and
\[
2G \int_O \beta(t, \omega) u\sigma_1(t) dx + 2 \int_O \beta(t, \omega) v\sigma_2(t) dx + 2G \int_O \beta(t, \omega) W \sigma_3(t) dx \\
\leq (FG\|u\|^2 + F\|v\|^2 + \mu FG\|W\|^2) + \frac{\beta^2(t, \omega)}{F} (G\|\sigma_1\|^2 + \|\sigma_2\|^2 + \frac{G}{\mu} \|\sigma_3\|^2).
\] (14)

By (13)-(14), we have
\[
\frac{d}{dt} (G\|u\|^2 + \|v\|^2 + \mu G\|W\|^2) + 2d_1 G\|\nabla u\|^2 + 2d_2 \|\nabla v\|^2 \\
+ 2\mu Gd_3 \|\nabla W\|^2 + F(G\|u\|^2 + \|v\|^2 + \mu G\|W\|^2) \\
\leq \frac{\beta^2(t, \omega)}{F} (G\|\sigma_1\|^2 + \|\sigma_2\|^2 + \frac{G}{\mu} \|\sigma_3\|^2).
\] (15)

Let
\[
d = \min\{d_1, d_2, d_3\}, \quad C_1 = \max\{1, G, \frac{G}{\mu}\} \\
\] and
\[
then (15) becomes
\[
\frac{d}{dt} \|g\|^2 + F\|g\|^2 + 2d\|\nabla g\|^2 \leq C_1 \beta^2(t, \omega)\|\sigma\|^2.
\] (16)

Multiplying (16) by \(e^{Ft}\) and then integrating over \((0, t)\), we get
\[
\|g(t, \omega, \sigma, g_0)\|^2 + 2d \int_0^t e^{F(s-t)} \|\nabla g(s, \omega, \sigma, g_0)\|^2 ds \\
\leq e^{-Ft}\|g_0\|^2 + C_1 \int_0^t e^{F(s-t)} \beta^2(s, \omega)\|\sigma(s)\|^2 ds.
\] (17)
Replacing \( \omega \) and \( \sigma \) with \( \vartheta_{-t}\omega \) and \( \vartheta_{-t}\sigma \) to obtain

\[
\|g(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 + 2d \int_0^t e^{F(s-t)}\|\nabla g(s)\|^2 ds \\
\leq e^{-Ft}\|g_0\|^2 + C_1 \int_0^t e^{F(s-t)}\beta^2(s, \vartheta_{-t}\omega)\|\sigma(s-t)\|^2 ds.
\]

Since \( g_0 \in D(\vartheta_{-t}\omega) \), there is a \( T = T(\omega, D) > 1 \) satisfying

\[
e^{-Ft}\|g_0\|^2 \leq 1, \quad \forall t \geq T.
\]

So, we obtain

\[
\|g(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 + 2d \int_0^t e^{F(s-t)}\|\nabla g(s)\|^2 ds \\
\leq c + C_1\beta^{-2}(-t, \omega) \int_{-\infty}^0 e^{Fs}\beta^2(s, \omega)\|\sigma(s)\|^2 ds,
\]

where and hereafter \( c \) is a positive constant, moreover, \( c \) can be changed if needed. In this way, the proof is completed.

**Lemma 4.2.** For any \( D \in \mathcal{D} \) and \( \omega \in \Omega \), there is a \( T = T(D, \omega) > 1 \) such that, for any \( \sigma \in \Sigma \),

\[
\int_{t-1}^t \|\nabla g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 ds \leq c + c\beta^{-2}(-t, \omega) \int_{-\infty}^0 e^{Fs}\beta^2(s, \omega)\|\sigma(s)\|^2 ds
\]

holds uniformly in \( g_0 \in D \) and \( t \geq T \).

**Proof.** Applying Lemma 4.1, for any \( t \geq T \), we obtain

\[
\int_{t-1}^t \|\nabla g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 ds \\
\leq e^F \int_{t-1}^t e^{F(s-t)}\|\nabla g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 ds \\
\leq e^F \int_0^t e^{F(s-t)}\|\nabla g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 ds \\
\leq c + C_1\beta^{-2}(-t, \omega) \int_{-\infty}^0 e^{Fs}\beta^2(s, \omega)\|\sigma(s)\|^2 ds,
\]

so that is the proof.

**Lemma 4.3.** For any \( D \in \mathcal{D} \) and \( \omega \in \Omega \), there is a \( T = T(D, \omega) > 1 \) such that, for any \( \sigma \in \Sigma \),

\[
\int_{t-1}^t \|g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|_{L^6}^6 ds \leq c + c\beta^{-6}(-t, \omega) \int_{-\infty}^0 e^{Fs}\beta^6(s, \omega)\|\sigma(s)\|_{L^6}^6 ds
\]

holds uniformly in \( g_0 \in D \) and \( t \geq T \).

**Proof.** Let

\[
V(x, t) = \frac{v(x, t)}{G},
\]
then (8) becomes
\[
\frac{\partial u}{\partial t} - d_1 \Delta u + (F + k)u - G \beta^{-2}(t, \omega)u^2V + G \beta^{-2}(t, \omega)u^3 - kW = \beta(t, \omega)\sigma_1,
\]
\[
\frac{\partial W}{\partial t} - d_2 \Delta V + FV + \beta^{-2}(t, \omega)u^2V - \beta^{-2}(t, \omega)u^3 = \frac{1}{G}\beta(t, \omega)\sigma_2,
\]
\[
\mu \frac{\partial W}{\partial t} - \mu d_3 \Delta W + (\mu F + k)W - ku = \beta(t, \omega)\sigma_3.
\]
Taking the inner product \((\partial u/\partial t, u^3)\) to get
\[
\frac{1}{6} \frac{d}{dt} \|u(t, \omega, \sigma, u_0)\|_{L^6}^6 + 5d_1 \|u^2\nabla u\|_{L^6}^2 + (F + k) \|u\|_{L^6}^6
\]
\[-G \beta^{-2}(t, \omega) \int_{\Omega} u^7Vdx + G \beta^{-2}(t, \omega) \int_{\Omega} u^3dx - k \int_{\Omega} u^5Wdx
\]
\[= \int_{\Omega} \beta(t, \omega)u^5\sigma_1(t)dx. \tag{22}\]
Taking the inner product \((\partial V/\partial t, GV^5)\) to obtain
\[
\frac{1}{6} \frac{d}{dt} (G\|V(t, \omega, \sigma, V_0)\|_{L^6}^6) + 5d_2G \|V^2\nabla V\|_{L^6}^2 + GF \|V\|_{L^6}^6
\]
\[+ G\beta^{-2}(t, \omega) \int_{\Omega} u^3V^6dx - G \beta^{-2}(t, \omega) \int_{\Omega} u^3V^5dx
\]
\[= \int_{\Omega} \beta(t, \omega)V^5\sigma_2(t)dx. \tag{23}\]
Taking the inner product \((\mu(\partial W/\partial t), W^5)\) to find
\[
\frac{1}{6} \frac{d}{dt} (\mu\|W(t, \omega, \sigma, W_0)\|_{L^6}^6) + 5\mu d_3 \|W^2\nabla W\|_{L^6}^2 + (\mu F + k) \|W\|_{L^6}^6
\]
\[-k \int_{\Omega} uW^5dx = \int_{\Omega} \beta(t, \omega)W^5\sigma_3(t)dx. \tag{24}\]
Summing up (22)-(24), arriving at
\[
\frac{d}{dt}(\|u\|_{L^6}^6 + G\|V\|_{L^6}^6 + \mu\|W\|_{L^6}^6) + 10d_1 \|u^2\nabla u\|_{L^6}^2 + 10d_2G \|V^2\nabla V\|_{L^6}^2
\]
\[+ 10\mu d_3 \|W^2\nabla W\|_{L^6}^2 + (F + k) \|u\|_{L^6}^6 + 6GF \|V\|_{L^6}^6 + 6(\mu F + k) \|W\|_{L^6}^6
\]
\[= 6k \int_{\Omega} u^5Wdx + 6k \int_{\Omega} uW^5dx - 6G \beta^{-2}(t, \omega) \int_{\Omega} (u^8 - u^7V - u^3V^5 + u^2V^6)
\]
\[+ 6 \int_{\Omega} \beta(t, \omega)u^5\sigma_1(t)dx + 6 \int_{\Omega} \beta(t, \omega)V^5\sigma_2(t)dx + 6 \int_{\Omega} \beta(t, \omega)W^5\sigma_3(t)dx. \tag{25}\]
Applying Young’s inequality, we have
\[
6(F + k) \|u\|_{L^6}^6 + 6GF \|V\|_{L^6}^6 + 6(\mu F + k) \|W\|_{L^6}^6
\]
\[-6k \int_{\Omega} u^5Wdx + 6k \int_{\Omega} uW^5dx
\]
\[\geq 6F(\|u\|_{L^6}^6 + G\|V\|_{L^6}^6 + \mu\|W\|_{L^6}^6), \tag{26}\]
\[-6G \beta^{-2}(t, \omega) \int_{\Omega} (u^8 - u^7V - u^3V^5 + u^2V^6) \leq 0, \tag{27}\]
and
\[ 6 \int_0^t \beta(t, \omega) u^5 \sigma_1(t) \, dx + 6 \int_0^t \beta(t, \omega) V^5 \sigma_2(t) \, dx + 6 \int_0^t \beta(t, \omega) W^5 \sigma_3(t) \, dx \]
\[ \leq 5 F \| u \|_{L^6}^6 + 5GF \| V \|_{L^6}^6 + 5\mu F \| W \|_{L^6}^6 \]
\[ + \frac{\beta(t, \omega)}{F^5} (\| \sigma_1 \|_{L^6}^6 + \frac{1}{G^5} \| \sigma_2 \|_{L^6}^6 + \frac{1}{\mu^5} \| \sigma_3 \|_{L^6}^6). \]  
(28)

By (26)-(28), we obtain
\[ \frac{d}{dt} (\| u \|_{L^6}^6 + G \| V \|_{L^6}^6 + \mu \| W \|_{L^6}^6) + F (\| u \|_{L^6}^6 + G \| V \|_{L^6}^6 + \mu \| W \|_{L^6}^6) \]
\[ \leq \frac{\beta(t, \omega)}{F^5} (\| \sigma_1 \|_{L^6}^6 + \frac{1}{G^5} \| \sigma_2 \|_{L^6}^6 + \frac{1}{\mu^5} \| \sigma_3 \|_{L^6}^6). \]  
(29)

Let
\[ C_2 = \frac{\max\{1, \frac{1}{G^5}, \frac{1}{\mu^5}\}}{F^5 \min\{1, \frac{1}{G^5}, \frac{1}{\mu^5}\}}, \]
then (29) becomes
\[ \frac{d}{dt} \| g \|_{L^6}^6 + F \| g \|_{L^6}^6 \leq C_2 \beta(t, \omega) \| \sigma \|_{L^6}^6. \]  
(30)

For \( \zeta \geq 0 \), multiplying (30) by \( e^{F \zeta} \) and then integrating over \((0, \zeta)\), we get
\[ \| g(\zeta, \omega, \sigma, g_0) \|_{L^6}^6 \leq e^{-F \zeta} \| g_0 \|_{L^6}^6 + C_2 \int_0^\zeta e^{F(s-\zeta)} \beta(s, \omega) \| \sigma(s) \|_{L^6}^6 \, ds. \]  
(31)

For every \( t \geq T \), integrating (31) with respect to \( \zeta \) over \((t-1, t)\) to obtain
\[ \int_{t-1}^t \| g(\zeta, \omega, \sigma, g_0) \|_{L^6}^6 \, d\zeta \]
\[ \leq \frac{1}{F} e^{-F(t-1)} \| g_0 \|_{L^6}^6 + C_2 \int_0^t e^{F(s-t)} \beta(s, \omega) \| \sigma(s) \|_{L^6}^6 \, ds. \]  
(32)

Replacing \( \omega \) and \( \sigma \) with \( \vartheta_{-t} \omega \) and \( \vartheta_{-t} \sigma \) to get
\[ \int_{t-1}^t \| g(\zeta, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, g_0) \|_{L^6}^6 \, d\zeta \]
\[ \leq \frac{1}{F} e^{-F(t-1)} \| g_0 \|_{L^6}^6 + C_2 \int_0^t e^{F(s-t)} \beta(s, \vartheta_{-t} \omega) \| \sigma(s) \|_{L^6}^6 \, ds. \]  
(33)

Since \( g_0 \in D(\vartheta_{-t} \omega) \), there is a \( T = T(\omega, D) > 1 \) satisfying
\[ \frac{1}{F} e^{-F(t-1)} \| g_0 \|_{L^6}^6 \leq 1, \quad \forall t \geq T. \]  
(34)

So, we obtain
\[ \int_{t-1}^t \| g(\zeta, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, g_0) \|_{L^6}^6 \, d\zeta \leq c + C_2 \beta^{-6} (-t, \omega) \int_{-\infty}^0 e^{F(s)} \beta^6(s, \omega) \| \sigma(s) \|_{L^6}^6 \, ds. \]
Lemma 4.4. For any $D \in \mathcal{D}$ and $\omega \in \Omega$, there is a $T = T(D, \omega) > 1$ such that, for any $\sigma \in \Sigma$,
\[
\| \nabla g(t, \partial_t \omega, \partial_t \sigma, g_0) \|^2 \\
\leq c \beta^2(-t, \omega) \int_{-\infty}^{0} (1 + e^{F_s}) \beta^2(s, \omega) \| \sigma(s) \|^2 ds \\
+ c \sup_{0 \leq t \leq 1} \{ \beta^4(-t + 1, \omega) \} \beta^2(-t, \omega) \int_{-\infty}^{0} e^{F_s} \beta^6(s, \omega) \| \sigma(s) \|_{L^\infty}^6 ds + c
\]
holds uniformly in $g_0 \in D$ and $t \geq T$.

Proof. For (8), taking the inner product $(\partial u/\partial t, -\Delta u)$ to get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u(t, \omega, \sigma, u_0) \|^2 + d_1 \| \Delta u \|^2 + (F + k) \| \nabla u \|^2 \\
+ \beta^2(t, \omega) \int_{\Omega} u^2 v \Delta u dx - G \beta^2(t, \omega) \int_{\Omega} u^3 \Delta u dx + k \int_{\Omega} W \Delta u dx
\]
\[
= - \int_{\Omega} \beta(t, \omega) \sigma_1(t) \Delta u dx.
\]
(35)

Taking the inner product $(\partial v/\partial t, -\Delta v)$ to find
\[
\frac{1}{2} \frac{d}{dt} \| \nabla v(t, \omega, \sigma, v_0) \|^2 + d_2 \| \Delta v \|^2 + F \| \nabla v \|^2 \\
- \beta^2(t, \omega) \int_{\Omega} u^2 v \Delta v dx + G \beta^2(t, \omega) \int_{\Omega} u^3 \Delta v dx
\]
\[
= - \int_{\Omega} \beta(t, \omega) \sigma_2(t) \Delta v dx.
\]
(36)

Taking the inner product $(\mu(\partial W/\partial t), -\Delta W)$, we get
\[
\frac{1}{2} \frac{d}{dt} (\mu \| \nabla W(t, \omega, \sigma, W_0) \|^2) + d_3 \| \Delta W \|^2 + (\mu F + k) \| \nabla W \|^2 \\
+ k \int_{\Omega} u \Delta W dx = - \int_{\Omega} \beta(t, \omega) \sigma_3(t) \Delta W dx.
\]
(37)

Summing up (35)-(37), we obtain
\[
\frac{d}{dt}(\| \nabla u(t, \omega, \sigma, u_0) \|^2 + \| \nabla v(t, \omega, \sigma, v_0) \|^2 + \mu \| \nabla W(t, \omega, \sigma, W_0) \|^2)
\]
\[
+ 2(F + k) \| \nabla u \|^2 + 2F \| \nabla v \|^2 + 2(\mu F + k) \| \nabla W \|^2
\]
\[
+ 2d_1 \| \Delta u \|^2 + 2d_2 \| \Delta v \|^2 + 2d_3 \| \Delta W \|^2
\]
\[
= -2 \beta^2(t, \omega) \int_{\Omega} u^2 v \Delta u dx + 2G \beta^2(t, \omega) \int_{\Omega} u^3 \Delta u dx + 2 \beta^2(t, \omega) \int_{\Omega} u^2 v \Delta v dx
\]
\[
- 2G \beta^2(t, \omega) \int_{\Omega} u^3 \Delta v dx - 2k \int_{\Omega} W \Delta u dx - 2k \int_{\Omega} u \Delta W dx
\]
\[
- 2 \int_{\Omega} \beta(t, \omega) \sigma_1(t) \Delta u dx - 2 \int_{\Omega} \beta(t, \omega) \sigma_2(t) \Delta v dx - 2 \int_{\Omega} \beta(t, \omega) \sigma_3(t) \Delta W dx.
\]
(38)
By Young’s inequality, we obtain
\[
2(F + k)\|\nabla u\|^2 + 2F\|\nabla v\|^2 + 2(\mu F + k)\|\nabla W\|^2 \\
+ 2k \int_\Omega W\Delta u dx + 2k \int_\Omega u\Delta W dx \\
\geq 2F\|\nabla u\|^2 + 2F\|\nabla v\|^2 + 2\mu F\|\nabla W\|^2,
\]
(39)

and
\[
-2\beta^{-2}(t, \omega) \int_\Omega u^2 v\Delta u dx + 2G\beta^{-2}(t, \omega) \int_\Omega u^3\Delta u dx \\
+ 2\beta^{-2}(t, \omega) \int_\Omega u^2 v\Delta v dx - 2G\beta^{-2}(t, \omega) \int_\Omega u^3\Delta v dx \\
\leq (c\beta^{-4}(t, \omega) \int_\Omega u^4 v^2 dx + \frac{1}{2}d_1\|\Delta u\|^2) \\
+ (c\beta^{-4}(t, \omega) \int_\Omega u^6 dx + \frac{1}{2}d_1\|\Delta u\|^2) \\
+ (c\beta^{-4}(t, \omega) \int_\Omega u^4 v^2 dx + \frac{1}{2}d_2\|\Delta v\|^2) \\
+ (c\beta^{-4}(t, \omega) \int_\Omega u^6 dx + \frac{1}{2}d_2\|\Delta v\|^2) \\
\leq c\beta^{-4}(t, \omega)(\|u\|_{L^6}^6 + \|v\|_{L^6}^6) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2.
\]
(41)

By (39)-(41), we get
\[
\frac{d}{dt}(\|\nabla u(t, \omega, \sigma, u_0)\|^2 + \|\nabla v(t, \omega, \sigma, v_0)\|^2 + \mu\|\nabla W(t, \omega, \sigma, W_0)\|^2) \\
+ 2F(\|\nabla u\|^2 + \|\nabla v\|^2 + \mu\|\nabla W\|^2) \\
\leq c\beta^{-4}(t, \omega)(\|u\|_{L^6}^6 + \|v\|_{L^6}^6) + c\beta^2(t, \omega)(\|\sigma_1\|^2 + \|\sigma_2\|^2 + \|\sigma_3\|^2),
\]
(42)

and then get
\[
\frac{d}{dt}\|\nabla g(t, \omega, \sigma, g_0)\|^2 + 2F\|\nabla g\|^2 \leq c\beta^{-4}(t, \omega)\|g\|_{L^6}^6 + c\beta^2(t, \omega)\|\sigma\|^2.
\]
(43)

Let \(t \geq T\) and \(s \in (t - 1, t)\). Integrating (43) over \((s, t)\), we obtain
\[
\|\nabla g(t, \omega, \sigma, g_0)\|^2 + 2F\int_s^t \|\nabla g(\tau)\|^2 d\tau \\
\leq \|\nabla g(s, \omega, \sigma, g_0)\|^2 + c \int_s^t \beta^{-4}(\tau, \omega)\|g(\tau)\|_{L^6}^6 d\tau \\
+ c \int_s^t \beta^2(\tau, \omega)\|\sigma(\tau)\|^2 d\tau.
\]
(44)
Then integrating (44) with respect to \(s\) over \((t - 1, t)\) to get
\[
\|\nabla g(t, \omega, \sigma, g_0)\|^2 \leq \int_{t-1}^t \|\nabla g(s, \omega, \sigma, g_0)\|^2 ds + c \int_{t-1}^t \beta^{-4}(s, \omega)\|g(s)\|_{L^6}^6 ds + c \int_{t-1}^t \beta^2(s, \omega)\|\sigma(s)\|^2 ds.
\]

Replacing \(\omega\) and \(\sigma\) with \(\vartheta_{-t}\omega\) and \(\vartheta_{-t}\sigma\) to have
\[
\|\nabla g(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 \leq \int_{t-1}^t \|\nabla g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 ds + c \int_{t-1}^t \beta^{-4}(s, \vartheta_{-t}\omega)\|g(s)\|_{L^6}^6 ds + c \int_{t-1}^t \beta^2(s, \vartheta_{-t}\omega)\|\sigma(s - t)\|^2 ds
\]
\[
\leq \int_{t-1}^t \|\nabla g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 ds + c \beta^4(-t, \omega) \sup_{0 \leq t \leq 1} \{\beta^{-4}(-t + 1, \omega)\} \int_{t-1}^t \|g(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|_{L^6}^6 ds + c \beta^{-2}(-t, \omega) \int_{-\infty}^0 \beta^2(s, \omega)\|\sigma(s)\|^2 ds.
\]

By Lemma 4.2 and Lemma 4.3 to obtain
\[
\|\nabla g(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 \leq c + c \beta^{-2}(-t, \omega) \int_{-\infty}^0 e^{Fs} \beta^2(s, \omega)\|\sigma(s)\|^2 ds
\]
\[
+ c \beta^4(-t, \omega) \sup_{0 \leq t \leq 1} \{\beta^{-4}(-t + 1, \omega)\}(c + c \beta^{-6}(-t, \omega) \int_{-\infty}^0 e^{Fs} \beta^0(s, \omega)\|\sigma(s)\|_{L^6}^6 ds)
\]
\[
+ c \beta^{-2}(-t, \omega) \int_{-\infty}^0 \beta^2(s, \omega)\|\sigma(s)\|^2 ds
\]
\[
\leq c + c \beta^{-2}(-t, \omega) \int_{-\infty}^0 (1 + e^{Fs})\beta^2(s, \omega)\|\sigma(s)\|^2 ds
\]
\[
+ c \beta^{-4}(-t + 1, \omega) \beta^{-2}(-t, \omega) \int_{-\infty}^0 e^{Fs} \beta^6(s, \omega)\|\sigma(s)\|_{L^6}^6 ds + c.
\]

so it finishes the proof.

Combining (4) with Lemma 4.4, we can uniformly estimate the solutions of (1).

**Corollary 4.5.** For every \(D \in \mathcal{D}\) and \(\omega \in \Omega\), there is a \(T = T(D, \omega) > 1\) and a \(L > 0\) such that, for any \(\sigma \in \Sigma\),
\[
\|\nabla \phi(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2
\]
\[
\begin{align*}
&\leq L\beta^{-2}(-t,\omega)\int_{-\infty}^{0} (1 + e^{F_s})\beta^2(s,\omega)\|\sigma(s)\|^2 ds \\
&+ L \sup_{0 \leq t \leq 1} \{\beta^{-4}(-t+1,\omega)\beta^{-2}(-t,\omega)\int_{-\infty}^{0} e^{F_s}\beta^6(s,\omega)\|\sigma(s)\|^6 ds + L\}
\end{align*}
\]
holds uniformly in \( g_0 \in D \) and \( t \geq T \).

5. **Uniform and cocycle attractors.** In this section, we prove the existence of uniform and cocycle attractors [13].

For any \( \omega \in \Omega \) and \( \sigma \in \Sigma \), define
\[
E(\omega) = \left\{ g \in V : \|\nabla g\|^2 \leq L\beta^{-2}(-t,\omega)\int_{-\infty}^{0} (1 + e^{F_s})\beta^2(s,\omega)\|\sigma(s)\|^2 ds \\
+ L \sup_{0 \leq t \leq 1} \{\beta^{-4}(-t+1,\omega)\beta^{-2}(-t,\omega)\int_{-\infty}^{0} e^{F_s}\beta^6(s,\omega)\|\sigma(s)\|^6 ds + L\} \right\},
\]
(48)
here \( L \) is from Corollary 4.5. By Sobolev compactness embeddings, \( E \) is a compact random set in \([L^2(\Omega)]^3\). Moreover, by Corollary 4.5, \( E \) is a uniformly \( D \)-pullback absorbing set for \( \phi \).

**Theorem 5.1.** Suppose that (1) creates a NRDS \( \phi \), then \( \phi \) has a unique \( D \)-cocycle attractor \( A = \{A_\sigma(\cdot)\}_{\sigma \in \Sigma} \)
\[
A_\sigma(\omega) = W(\omega, \sigma, E), \quad \forall \sigma \in \Sigma, \ \omega \in \Omega,
\]
(49)
where \( E \) is a random set from (48). In addition, \( A \) satisfies the following properties:
(i) \( A \) is upper semi-continuous, i.e., for any \( \omega \in \Omega \),
\[
\text{dist}_{H}(A_\sigma(\omega), A_{\sigma_0}(\omega)) \to 0, \quad \forall \sigma \to \sigma_0;
\]
(ii) \( A \) is uniformly compact, i.e., for any \( \omega \in \Omega \), \( \cup_{\sigma \in \Sigma} A_\sigma(\omega) \) is compact in \([L^2(\Omega)]^3\);
(iii) \( A \) is described by \( D \)-complete trajectories of \( \phi \), i.e.,
\[
A_\sigma(\omega) = \{\xi(\omega, 0) : \xi \text{ is a } D\text{-complete trajectory of } \phi\}, \quad \forall \sigma \in \Sigma, \ \omega \in \Omega.
\]

By Theorem 2.8, Theorem 2.11, Proposition 2.10 and Proposition 2.12, we can change Theorem 5.1 to the following.

**Theorem 5.2.** Suppose that (1) creates a NRDS \( \phi \), then \( \phi \) has a \( D \)-cocycle attractor \( A \) and a \( D \)-uniform attractor \( \mathcal{A} \in D \) such that
\[
\mathcal{A}(\omega) = W(\omega, \Sigma, E)
= \cup_{\sigma \in \Sigma} A_\sigma(\omega)
= \{\xi(\omega, 0) : \xi \text{ is a } D\text{-complete trajectory of } \phi\}, \quad \forall \omega \in \Omega,
\]
(50)
where \( E \) is a random set from (48). \( A \) is forward-attracting in probability. Besides, \( \mathcal{A} \) is upper semi-continuous in symbols.

6. **Conclusion.** In this paper, by using the uniform prior estimates of solutions, we obtain the existence of uniform and cocycle attractors of the stochastic three-component Gray-Scott system with multiplicative noise. Firstly, we deduce that the stochastic three-component Gray-Scott system can generate a non-autonomous random dynamical system. Then we establish some uniform estimates of solutions for
stochastic three-component Gray-Scott system with multiplicative noise. Finally, the existence of uniform and cocycle attractors is proved.

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REFERENCES

[1] L. Arnold, Random Dynamical Systems, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[2] A. V. Babin and M. I. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.
[3] P. W. Bates, H. Lisei and K. Lu, Attractors for stochastic lattice dynamical systems, Stoch. Dyn., 6 (2006), 1–21.
[4] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, J. Differential Equations, 246 (2009), 845–869.
[5] V. V. Chepyzhov and M. I. Vishik, Attractors for Equations of Mathematical Physics, American Mathematical Society Colloquium Publications, 49, AMS, Providence, RI, 2002.
[6] A. Cheskidov and L. Kavlie, Pullback attractors for generalized evolutionary systems, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), 749–779.
[7] I. Chueshov, Monotone Random Systems Theory and Applications, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2002.
[8] H. Crauel, A. Debussche and F. Flandoli, Random attractors, J. Dynam. Differential Equations, 9 (1997), 307–341.
[9] H. Crauel and F. Flandoli, Attractors for random dynamical systems, Probab. Theory Related Fields, 100 (1994), 365–393.
[10] H. Crauel, P. E. Kloeden and M. Yang, Random attractors of stochastic reaction-diffusion equations on variable domains, Stoch. Dyn., 11 (2011), 301–314.
[11] H. Cui, M. M. Freitas and J. A. Langa, On random cocycle attractors with autonomous attraction universes, Discrete Contin. Dyn. Syst. Ser. B, 22 (2017), 3379–3407.
[12] H. Cui and P. E. Kloeden, Invariant forward attractors of non-autonomous random dynamical systems, J. Differential Equations, 265 (2018), 6166–6186.
[13] H. Cui and J. A. Langa, Uniform attractors for non-autonomous random dynamical systems, J. Differential Equations, 263 (2017), 1225–1268.
[14] X. Ding and J. Jiang, Random attractors for stochastic retarded reaction-diffusion equations on unbounded domains, Abstr. Appl. Anal., 1 (2013), 16pp.
[15] X. Ding and J. Jiang, Randoms attractors for stochastic retarded lattice dynamical systems, Abstr. Appl. Anal., 2 (2012), 27pp.
[16] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise, Stochastics Stochastics Rep., 59 (1996), 21–45.
[17] A. Gu and H. Xiang, Upper semicontinuity of random attractors for stochastic three-component reversible Gray-Scott system, Appl. Math. Comput., 225 (2013), 387–400.
[18] A. Gu, S. Zhou and Z. Wang, Uniform attractor of non-autonomous three-component reversible Gray-Scott system, Appl. Math. Comput., 219 (2013), 8718–8729.
[19] X. Jia, J. Gao and X. Ding, Random attractors for stochastic two-compartment Gray-Scott equations with a multiplicative noise, Commun. Math. Sci., 16 (2018), 97–122.
[20] K. Lu and B. Wang, Global attractors for the Klein-Gordon-Schrödinger equation in unbounded domains, J. Differential Equations, 170 (2001), 281–316.
[21] M. Mhara, N. J. Suematsu, T. Yamaguchi, K. Ohgane, Y. Nishiura and M. Shimomura, Three-variable reversible Gray-Scott model, J. Chem. Phys., 121 (2004), 8968–8972.
[22] M. Ochs, Weak Random Attractors, Citeseer, 1999.
[23] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edition, Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997.
[24] B. Wang, Attractors for reaction-diffusion equations in unbounded domains, Phys. D, 128 (1999), 41–52.
[25] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, J. Differential Equations, 253 (2012), 1544–1583.
B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **34** (2014), 269–300.

B. Wang, Pullback attractors for non-autonomous reaction-diffusion equations on $\mathbb{R}^n$, *Front. Math. China*, **4** (2009), 563–583.

Z. Wang and S. Zhou Random attractor for stochastic reaction-diffusion equation with multiplicative noise on unbounded domains, *J. Math. Anal. Appl.*, **384** (2011), 160–172.

Y. You, Dynamics of two-compartment Gray-Scott equations, *Nonlin. Anal.*, **74** (2011), 1969–1986.

Y. You, Dynamics of three-compartment reversible Gray-Scott model, *Disc. Cont. Dynam. Syst. Ser. B*, **14** (2010), 1671–1688.

Y. You, Global attractor of the Gray-Scott equations, *Commun. Pure. Appl. Anal.*, **7** (2008), 947–970.

Y. You, Robustness of global attractors for reversible Gray-Scott systems, *J. Dynam. Differential Equations*, **24** (2012), 495–520.

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E-mail address: fdxinjie@sina.com, fdxinjie@qfnu.edu.cn
E-mail address: liuhuinanshi@qfnu.edu.cn
E-mail address: lyfengjunwei@sina.com