REVERSES OF FÉJER’S INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. Let f be a convex function on I and a, b ∈ I with a < b. If p : [a, b] → [a, ∞) is Lebesgue integrable and symmetric, namely p(b + a − t) = p(t) for all t ∈ [a, b], then we show in this paper that

\[
0 \leq \frac{1}{2} \int_a^b \left( t - \frac{a + b}{2} \right) p(t) dt \left( f'_+ \left( \frac{a + b}{2} \right) - \frac{f(a) + f(b)}{2} \right)
\]

and

\[
0 \leq \frac{1}{2} \int_a^b \left( \frac{1}{2} (b - a) - \left| t - \frac{a + b}{2} \right| \right) p(t) dt \left( f'_+ \left( \frac{a + b}{2} \right) - \frac{f(a) + f(b)}{2} \right)
\]

1. Introduction

The following inequality holds for any convex function f defined on R

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}, \quad a, \ b \in \mathbb{R}, \ a < b.
\]

It was firstly discovered by Ch. Hermite in 1881 in the journal Mathesis (see [7]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite’s note in Mathesis [7]. Since (1.1) was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For a monograph devoted to this result see [5]. The recent survey paper [4] provides other related results.

Let f : [a, b] → R be a convex function on [a, b] and assume that f'_+ (a) and f'_+ (b) are finite. We recall the following improvement and reverse inequality for
Theorem 1. Consider the integral \( \int_a^b f(t) p(t) \, dt \), where \( f \) is a convex function in the interval \((a, b)\) and \( p \) is a positive function in the same interval such that

\[
p(a + t) = p(b - t), \quad 0 \leq t \leq \frac{1}{2}(b - a),
\]

i.e., \( y = p(t) \) is a symmetric curve with respect to the straight line which contains the point \( \left( \frac{1}{2}(a + b), 0 \right) \) and is normal to the t-axis. Under those conditions the following inequalities are valid:

\[
f\left(\frac{a + b}{2}\right) \int_a^b p(t) \, dt \leq \int_a^b f(t) p(t) \, dt \leq \frac{f(a) + f(b)}{2} \int_a^b p(t) \, dt.
\]

If \( f \) is concave on \((a, b)\), then the inequalities reverse in (1.4).

Clearly, for \( p(t) \equiv 1 \) on \([a, b]\) we get 1.1.

If we take \( p(t) = |t - \frac{a+b}{2}|, \ t \in [a, b] \) in Theorem 1, then we have

\[
\frac{1}{4} f\left(\frac{a + b}{2}\right) (b-a)^2 \leq \int_a^b \left| t - \frac{a+b}{2} \right| f(t) \, dt \leq \frac{f(a) + f(b)}{8} (b-a)^2,
\]

for any convex function \( f : [a, b] \to \mathbb{R} \).

We observe that, if we take \( p(t) = (b-t) (t-a), \ t \in [a, b] \), then \( p \) satisfies the conditions in Theorem 1, and by (1.4) we have the following inequality as well

\[
\frac{1}{6} f\left(\frac{a + b}{2}\right) (b-a)^3 \leq \int_a^b (b-t) (t-a) f(t) \, dt \leq \frac{f(a) + f(b)}{12} (b-a)^3,
\]

for any convex function \( f : [a, b] \to \mathbb{R} \).

Motivated by the above results, in this paper we obtain an improvement and a reverse for each inequality in (1.4) and therefore generalize the Hermite-Hadamard inequalities (1.2) and (1.3).
2. IMPROVEMENTS AND REVERSE OF FÉJER INEQUALITIES

Following Roberts and Varberg [8, p. 5], we recall that if $f : I \to \mathbb{R}$ is a convex function, then for any $x_0 \in \mathring{I}$ (the interior of the interval $I$) the limits

$$f'_-(x_0) := \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) := \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and $f'_-(x_0) \leq f'_+(x_0)$. The functions $f'_-$ and $f'_+$ are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval $I$ (see [8, p. 7]).

From the monotonicity of the lateral derivatives $f'_-$ and $f'_+$ we also have the gradient inequality

$$f'_-(x) (x - y) \geq f(x) - f(y) \geq f'_+(y) (x - y)$$

for any $x, y \in \mathring{I}$.

If $I = [a, b]$, then at the end points we also have the inequalities

$$f(x) - f(a) \geq f'_+(a) (x - a)$$

for any $x \in (a, b]$ and

$$f(y) - f(b) \geq f'_-(b) (y - b)$$

for any $y \in [a, b)$.

We have the following refinement and reverse of Fejér’s first inequality:

**Theorem 2.** Let $f$ be a convex function on $I$ and $a, b \in I$, with $a < b$. If $p : [a, b] \to [0, \infty)$ is Lebesgue integrable and symmetric, namely $p(b + a - t) = p(t)$ for all $t \in [a, b]$, then

$$0 \leq \frac{1}{2} \int_a^b \left[ t - \frac{a + b}{2} \right] p(t) dt \left[ f'_+(\frac{a + b}{2}) - f'_-(\frac{a + b}{2}) \right]$$

(2.1)

$$\leq \int_a^b p(t) f(t) dt - \left( \int_a^b p(t) dt \right) f\left(\frac{a + b}{2}\right)$$

$$\leq \frac{1}{2} \int_a^b \left[ t - \frac{a + b}{2} \right] p(t) dt \left[ f'_-(b) - f'_+(a) \right].$$

**Proof.** Let $a, b \in I$, with $a < b$. Using the integration by parts formula for Lebesgue integral, we have

$$\int_\frac{a+b}{2}^b \left( \int_{\frac{a+b}{2}}^b p(s) ds \right) f'(t) dt$$

$$= \left( \int_a^b p(s) ds \right) f\left(\frac{a + b}{2}\right) + \int_\frac{a+b}{2}^b p(t) f(t) dt$$

$$= - \left( \int_{\frac{a+b}{2}}^b p(s) ds \right) f\left(\frac{a + b}{2}\right) + \int_{\frac{a+b}{2}}^b p(t) f(t) dt$$
and
\[
\int_{a}^{a+b} \left( \int_{a}^{t} p(s) \, ds \right) f'(t) \, dt \\
= \left( \int_{a}^{t} p(s) \, ds \right) f(t) \bigg|_{a}^{a+b} - \int_{a}^{a+b} p(t) \, f(t) \, dt \\
= \left( \int_{a}^{a+b} p(s) \, ds \right) f \left( \frac{a+b}{2} \right) - \int_{a}^{a+b} p(t) \, f(t) \, dt.
\]

By subtracting the second identity from the first, we get
\[
\int_{a}^{a+b} \left( \int_{a}^{b} p(s) \, ds \right) f'(t) \, dt - \int_{a}^{a+b} \left( \int_{a}^{t} p(s) \, ds \right) f'(t) \, dt \\
= \int_{a}^{a+b} p(t) \, f(t) \, dt + \int_{a}^{a+b} p(t) \, f(t) \, dt \\
- \left( \int_{a}^{a+b} p(s) \, ds \right) f \left( \frac{a+b}{2} \right) - \left( \int_{a}^{a+b} p(s) \, ds \right) f \left( \frac{a+b}{2} \right).
\]

By the symmetry of \( p \) we get
\[
\int_{a}^{b} p(s) \, ds = \int_{a}^{a+b} p(s) \, ds = \frac{1}{2} \int_{a}^{b} p(s) \, ds
\]
and then we can state the following identity of interest in itself
\[
(2.2) \quad \int_{a}^{b} p(t) \, f(t) \, dt - f \left( \frac{a+b}{2} \right) \int_{a}^{b} p(s) \, ds \\
= \int_{a}^{a+b} \left( \int_{a}^{b} p(s) \, ds \right) f'(t) \, dt - \int_{a}^{a+b} \left( \int_{a}^{t} p(s) \, ds \right) f'(t) \, dt.
\]

By the monotonicity of the derivative we have
\[
f'_+ (a) \leq f'_- (a) \leq f'_- \left( \frac{a+b}{2} \right), \text{ for almost every } t \in \left( a, \frac{a+b}{2} \right)
\]
and
\[
f'_+ \left( \frac{a+b}{2} \right) \leq f'_- (t) \leq f'_- (b), \text{ for almost every } t \in \left( \frac{a+b}{2}, b \right).
\]

This implies
\[
f'_+ (a) \left( \int_{a}^{t} p(s) \, ds \right) \leq f'_- (t) \left( \int_{a}^{t} p(s) \, ds \right) \\
\leq f'_- \left( \frac{a+b}{2} \right) \left( \int_{a}^{t} p(s) \, ds \right), \quad t \in \left[ a, \frac{a+b}{2} \right]
\]
and

\[
\frac{f_+'}{(a + b)\left(\int_{a}^{b} p(s) \, ds\right)} \leq f_0' \left(\int_{a}^{b} p(s) \, ds\right) \leq f_0' \left(\int_{\frac{a + b}{2}}^{b} p(s) \, ds\right),
\]

and by integration

\[
f_0' \left(\int_{a}^{b} p(s) \, ds\right) \leq \int_{a}^{b} f_0' \left(\int_{a}^{b} p(s) \, ds\right) \, dt \leq f_0' \left(\int_{a}^{b} p(s) \, ds\right) \left(\int_{a}^{b} p(s) \, ds\right) \, dt,
\]

and

\[
-f_0' \left(\int_{a}^{b} p(s) \, ds\right) \leq -\int_{a}^{b} f_0' \left(\int_{a}^{b} p(s) \, ds\right) \, dt \leq -f_0' \left(\int_{a}^{b} p(s) \, ds\right) \left(\int_{a}^{b} p(s) \, ds\right) \, dt.
\]

If we add these inequalities, then we get

\[
f_0' \left(\int_{a}^{b} p(s) \, ds\right) \leq \int_{a}^{b} f_0' \left(\int_{a}^{b} p(s) \, ds\right) \, dt \leq f_0' \left(\int_{a}^{b} p(s) \, ds\right) \left(\int_{a}^{b} p(s) \, ds\right) \, dt,
\]

Integrating by parts in the Lebesgue integral, we have

\[
\int_{\frac{a + b}{2}}^{b} \left(\int_{a}^{b} p(s) \, ds\right) \, dt = \left(\int_{a}^{b} p(s) \, ds\right) \left. t \right|_{a}^{b} + \int_{\frac{a + b}{2}}^{b} tp(t) \, dt
\]

\[
= \int_{\frac{a + b}{2}}^{b} tp(t) \, dt - \frac{a + b}{2} \int_{\frac{a + b}{2}}^{b} p(s) \, ds
\]

\[
= \int_{\frac{a + b}{2}}^{b} \left( t - \frac{a + b}{2} \right) p(t) \, dt = \frac{1}{2} \int_{a}^{b} \left| t - \frac{a + b}{2} \right| p(t) \, dt,
\]

where for the last equality we used the symmetry of \( p \).
Similarly,

\[
\int_a^{a+b} \left( \int_a^t p(s) \, ds \right) \, dt = \left( \int_a^t p(s) \, ds \right) \bigg|_a^{a+b} - \int_a^{a+b} p(t) \, t \, dt
\]

\[
= \frac{a+b}{2} \int_a^{a+b} p(s) \, ds - \int_a^{a+b} p(t) \, t \, dt
\]

\[
= \int_a^{a+b} \left( \frac{a+b}{2} - t \right) p(t) \, dt = \frac{1}{2} \int_a^b \left| \frac{a+b}{2} - t \right| p(t) \, dt.
\]

Then by (2.3) we obtain the desired result (2.1). 

**Remark 1.** If we take \( p \equiv 1 \) in (2.1) and since \( \int_a^b \left| t - \frac{a+b}{2} \right| = \frac{1}{4} (b-a)^2 \), hence by (2.1) we recapture the inequalities (1.2) from Introduction.

We also have the following refinement and reverse of Fejer’s second inequality:

**Theorem 3.** Let \( f \) be a convex function on \( I \) and \( a, b \in I, \) with \( a < b. \) If \( p : [a,b] \to [0,\infty) \) is Lebesgue integrable and symmetric, namely \( p(b+a-t) = p(t) \) for all \( t \in [a,b], \) then

(2.4) \[
0 \leq \frac{1}{2} \int_a^b \left[ \frac{1}{2} (b-a) - \left| t - \frac{a+b}{2} \right| \right] p(t) \, dt \left[ f_+^r \left( \frac{a+b}{2} \right) - f_-^r \left( \frac{a+b}{2} \right) \right]
\]

\[
\leq \left( \int_a^b p(t) \, dt \right) \frac{f(a) + f(b)}{2} - \int_a^b p(t) f(t) \, dt
\]

\[
\leq \frac{1}{2} \int_a^b \left[ (b-a) - \left| t - \frac{a+b}{2} \right| \right] p(t) \, dt \left[ f_+^r (b) - f_+^r (a) \right].
\]

**Proof.** Using the integration by parts for Lebesgue integral, we have

\[
\int_a^b \left( \int_a^t p(s) \, ds - \frac{1}{2} \int_a^b p(s) \, ds \right) f(t) \, dt
\]

\[
= \left( \int_a^t p(s) \, ds - \frac{1}{2} \int_a^b p(s) \, ds \right) f(t) \bigg|_a^b - \int_a^b p(t) f(t) \, dt
\]

\[
= \left( \int_a^b p(s) \, ds - \frac{1}{2} \int_a^b p(s) \, ds \right) f(b) + \left( \frac{1}{2} \int_a^b p(s) \, ds \right) f(a)
\]

\[
- \int_a^b p(t) f(t) \, dt
\]

\[
= \left( \int_a^b p(t) \, dt \right) \frac{f(a) + f(b)}{2} - \int_a^b p(t) f(t) \, dt.
\]
We also have

\[
\int_a^b \left( \int_a^t p(s) \, ds - \frac{1}{2} \int_a^b p(s) \, ds \right) f'(t) \, dt
= \int_a^b \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f(t) \, dt
= \int_a^{a+b} \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f(t) \, dt
+ \int_{a+b}^b \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f(t) \, dt
= \int_a^{a+b} \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f(t) \, dt
- \int_a^{a+b} \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f(t) \, dt.
\]

Observe that

\[
\int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \geq 0 \text{ for } t \in \left[ \frac{a+b}{2}, b \right]
\]

and

\[
\int_a^{a+b} p(s) \, ds - \int_a^t p(s) \, ds \geq 0 \text{ for } t \in \left[ a, \frac{a+b}{2} \right].
\]

By the monotonicity of the derivative we have

\[
f'_+ \left( \frac{a+b}{2} \right) \int_{a+b}^b \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) dt
\leq \int_{a+b}^b \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f'(t) \, dt
\leq f'_- \left( b \right) \int_{a+b}^b \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) dt
\]

and

\[
- f'_- \left( \frac{a+b}{2} \right) \int_a^{a+b} \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) dt
\leq - \int_a^{a+b} \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) f'(t) \, dt
\leq - f'_+ \left( a \right) \int_a^{a+b} \left( \int_a^t p(s) \, ds - \int_a^{a+b} p(s) \, ds \right) dt.
\]
If we add these inequalities, then we get

\[
\begin{align*}
(2.5) & \quad \left[ f_+ \left( \frac{a + b}{2} \right) \int_a^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) \, dt \\
& \quad - f_- \left( \frac{a + b}{2} \right) \int_a^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) \, dt \right] \\
& \leq \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) f'(t) \, dt \\
& \quad - \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) f'(t) \, dt \\
& \leq f_+ (b) \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) \, dt \\
& \quad - f'_-(a) \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) \, dt.
\end{align*}
\]

Observe that

\[
\begin{align*}
& \quad \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds - \int_a^{\frac{a+b}{2}} p(s) \, ds \right) \, dt \\
& = \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds \right) \, dt - \frac{b - a}{2} \int_{\frac{a+b}{2}}^b p(s) \, ds \\
& = \left( \int_a^t p(s) \, ds \right)_t^{b} - \int_{\frac{a+b}{2}}^b tp(t) \, dt - \frac{b - a}{2} \int_{\frac{a+b}{2}}^b p(s) \, ds \\
& = b \int_a^b p(s) \, ds - \frac{a + b}{2} \int_{\frac{a+b}{2}}^b p(s) \, ds - \int_{\frac{a+b}{2}}^b tp(t) \, dt - \frac{b - a}{2} \int_{\frac{a+b}{2}}^b p(s) \, ds \\
& = b \int_a^b p(s) \, ds - b \int_{\frac{a+b}{2}}^b p(s) \, ds - \int_{\frac{a+b}{2}}^b tp(t) \, dt \\
& = b \int_{\frac{a+b}{2}}^b p(s) \, ds - \int_{\frac{a+b}{2}}^b tp(t) \, dt = \int_{\frac{a+b}{2}}^b (b - t) \, p(t) \, dt
\end{align*}
\]

and

\[
\begin{align*}
& \quad \int_{\frac{a+b}{2}}^b \left( \int_a^{\frac{a+b}{2}} p(s) \, ds - \int_a^t p(s) \, ds \right) \, dt \\
& = \frac{b - a}{2} \int_{\frac{a+b}{2}}^b p(s) \, ds - \int_{\frac{a+b}{2}}^b \left( \int_a^t p(s) \, ds \right) \, dt \\
& = \frac{b - a}{2} \int_{\frac{a+b}{2}}^b p(s) \, ds - \left( \int_a^t p(s) \, ds \right)_t^{\frac{a+b}{2}} - \int_{\frac{a+b}{2}}^b tp(t) \, dt.
\end{align*}
\]
\[ = \frac{b-a}{2} \int_a^{\frac{a+b}{t}} p(s) \, ds - \frac{a+b}{2} \int_a^{\frac{a+b}{t}} p(s) \, ds + \int_a^{\frac{a+b}{t}} tp(t) \, dt \]
\[ = \int_a^{\frac{a+b}{t}} tp(t) \, dt - \frac{a+b}{2} \int_a^{\frac{a+b}{t}} p(s) \, ds = \int_a^{\frac{a+b}{t}} (t-a) \, p(t) \, dt. \]

If we change the variable \( s = b + a - t \), then
\[ \int_a^{\frac{a+b}{t}} (t-a) \, p(t) \, dt = \int_a^{\frac{a+b}{t}} (b-s) \, p(b + a - s) \, ds = \int_a^{\frac{a+b}{t}} (b-s) \, p(s) \, ds. \]

Finally, observe that
\[ \frac{1}{2} \int_a^{\frac{a+b}{t}} \left[ \frac{1}{2} (b-a) - \left| t - \frac{a+b}{2} \right| \right] p(t) \, dt \]
\[ = \frac{1}{2} \int_a^{\frac{a+b}{t}} \left[ \frac{1}{2} (b-a) - \left| t - \frac{a+b}{2} \right| \right] p(t) \, dt + \frac{1}{2} \int_a^{\frac{a+b}{t}} (t-a) \, p(t) \, dt + \frac{1}{2} \int_a^{\frac{a+b}{t}} (b-t) \, p(t) \, dt \]
\[ = \frac{1}{2} \int_a^{\frac{a+b}{t}} (t-a) \, p(t) \, dt + \frac{1}{2} \int_a^{\frac{a+b}{t}} (b-t) \, p(t) \, dt \]
\[ = \frac{1}{2} \int_a^{\frac{a+b}{t}} (t-a) \, p(t) \, dt + \frac{1}{2} \int_a^{\frac{a+b}{t}} (b-t) \, p(t) \, dt = \int_a^{\frac{a+b}{t}} (t-a) \, p(t) \, dt \]
and by (2.5) we get (2.4).

**Remark 2.** Observe that for \( p \equiv 1 \) we recapture the inequalities (1.3) from Introduction.

If we consider the symmetric weight \( p(t) = \left| t - \frac{a+b}{2} \right| , t \in [a, b] \) we obtain from Theorem 2 that

\[ 0 \leq \frac{1}{24} (b-a)^3 \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] \]
\[ \leq \int_a^{\frac{a+b}{t}} \left| t - \frac{a+b}{2} \right| f(t) \, dt - \frac{1}{4} (b-a)^2 f \left( \frac{a+b}{2} \right) \]
\[ \leq \frac{1}{24} (b-a)^3 \left[ f'_- (b) - f'_+ (a) \right] \]
and from Theorem 3 that

\begin{equation}
    0 \leq \frac{1}{64} (b - a)^3 \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] \\
    \leq (b - a)^2 \frac{f(a) + f(b)}{8} - \int_a^b \left| t - \frac{a+b}{2} \right| f(t) dt \\
    \leq \frac{1}{48} (b - a)^3 \left[ f'_- (b) - f'_+ (a) \right],
\end{equation}

where \( f \) is convex on \([a, b]\). These provide refinements and reverses of the inequalities (1.5).

If we consider the symmetric weight

\[ p(t) = (t - a)(b - t), \quad t \in [a, b] \]

we obtain from Theorem 2 that

\begin{equation}
    0 \leq \frac{1}{64} (b - a)^4 \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] \\
    \leq \int_a^b (t - a)(b - t) f(t) dt - \frac{1}{5} (b - a)^3 f \left( \frac{a+b}{2} \right) \\
    \leq \frac{1}{64} (b - a)^4 \left[ f'_- (b) - f'_+ (a) \right]
\end{equation}

and from Theorem 3 that

\begin{equation}
    0 \leq \frac{5}{192} (b - a)^4 \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] \\
    \leq (b - a)^3 \frac{f(a) + f(b)}{12} - \int_a^b (t - a)(b - t) f(t) dt \\
    \leq \frac{5}{192} (b - a)^4 \left[ f'_- (b) - f'_+ (a) \right],
\end{equation}

where \( f \) is convex on \([a, b]\). These provide refinements and reverses of the inequalities (1.6).

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