ON THE MOORE-PENROSE INVERSE, EP BANACH SPACE OPERATORS, AND EP BANACH ALGEBRA ELEMENTS

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Abstract. The main concern of this note is the Moore-Penrose inverse in the context of Banach spaces and algebras. Especially attention will be given to a particular class of elements with the aforementioned inverse, namely EP Banach space operators and Banach algebra elements, which will be studied and characterized extending well-known results obtained in the frame of Hilbert space operators and $C^*$-algebra elements.

Keywords: Moore-Penrose inverse, hermitian idempotents, EP Banach space operators, EP Banach algebra elements.

1. Introduction

The Moore-Penrose inverse is a notion that was introduced for matrices, see [21], and whose development has produced a wide literature. In the presence of an involution, in a Hilbert space or in a $C^*$-algebra, it is natural to extend and study the main properties of the aforesaid notion, see for example [4], [6], [8], [11], [12], [15], [18], and [19].

With the extension to general Banach space operators and more generally Banach algebra elements of the concept of hermitian element, see [2], [3], [5], [9], [20], [25], and [26], came also an extended notion of Moore-Penrose inverse, due to V. Rakocevic, see [22] and [23].

In particular, the class of $C^*$-algebra elements which commute with their Moore-Penrose inverse, the so-called EP elements, has been of especial interest. In fact, several results characterizing these elements and when the product of two EP elements is again EP have been obtained, see [1], [6], [8], [13], [14], and [15].

In this work the Moore-Penrose inverse in Banach spaces and algebras will be studied. In first place, several basic facts, as the relationships between the Moore-Penrose inverse and, on the one hand, closed invariant subspaces and, on the other, the adjoint of an operator will be considered. In addition, several characterizations of the aforementioned inverse will be proved.

On the other hand, EP Banach space operators and Banach algebra elements will be studied. In fact, they will be characterized extending...
to this context well-known characterizations of Hilbert space operators and of $C^*$-algebra elements, see [6], [12], and [15]. Furthermore, the problem of characterizing when the product of two EP elements is again EP will be considered extending results of [13], [8], [14], and [15], see also [1].

The work is organized as follows. In section 2 several preliminary definitions and results will be recalled. In section 3 the Moore-Penrose inverse in Banach spaces and algebras will be studied. In section 4 EP elements will be characterized, and in section 5 it will be considered the problem of determining when the product of two EP elements is again EP.

2. Preliminary Definitions and Results

From now on, $X$ will denote a Banach space, and $L(X,Y)$ the Banach algebra of all bounded and linear maps defined on $X$ with values in the Banach space $Y$. As usual, when $X = Y$, $L(X,Y)$ will be denoted by $L(X)$, the Banach algebra of all operators with domain $X$. In addition, if $T \in L(X)$, then $N(T)$ and $R(T)$ will stand for the null space and the range of $T$, respectively.

On the other hand, $A$ will denote a unital Banach algebra, that is a Banach algebra with a unit element $e$ such that $\|e\| = 1$. If $a \in A$, then $L_a: A \to A$ and $R_a: A \to A$ will denote the map defined by left and right multiplication, respectively:

$$L_a(x) = ax, \quad R_a(x) = xa,$$

where $x \in A$. Moreover, the following notation will be used:

$$N(L_a) = a^{-1}(0), \quad R(L_a) = aA,$$

$$N(R_a) = a_{-1}(0), \quad R(R_a) = Aa.$$

Recall that an element $a \in A$ is called regular, if it has a generalized inverse, namely if there exists $b \in A$ such that

$$a = aba.$$

Furthermore, a generalized inverse $b$ of a regular element $a \in A$ will be called normalized, if $b$ is regular and $a$ is a generalized inverse of $b$, equivalently,

$$a = aba, \quad b = bab,$$

see for example [11], [12], and [18].
Note that if $b$ is a generalized inverse of $a$, then $c = bab$ is a normalized generalized inverse of $a$.

Next follows the key notion in the definition of the Moore-Penrose inverse in context of Banach spaces and algebras.

**Definition 1.** Given a unital Banach algebra $A$, an element $a \in A$ is said to be hermitian, if $\| \exp(ita) \| = 1$, for all $t \in \mathbb{R}$.

As regard equivalent definitions and the main properties of hermitian Banach algebra elements and Banach space operators, see for example [2], [3], [20], [9], [26], [25], and [5]. In the following remark some relevant facts will be considered.

**Remark 2.** In the conditions of the previous definition, recall that if $A$ is a $C^*$-algebra, then $a \in A$ is hermitian if and only if $a$ is self-adjoint, see Proposition 20, section 12, Chapter I of [5]. Furthermore, $\mathcal{H} = \{ a \in A : a \text{ is hermitian} \} \subseteq A$ is a closed linear vector space over the real field, see [26] and Chapter 4 of [9]. Since $A$ is unital, $e \in \mathcal{H}$, which implies that $a \in \mathcal{H}$ if and only if $e - a \in \mathcal{H}$.

Observe also that necessary and sufficient for $a \in A$ to be hermitian is that $\| \exp(ita) \| \leq 1$, $\forall t \in \mathbb{R}$. In fact, if this condition holds, then

$$1 = \| \exp(ita) \exp(-ita) \| \leq \| \exp(ita) \| \| \exp(-ita) \| \leq 1.$$  

On the other hand, recall that there is an equivalent definition of hermitian element in terms of numerical range, see [17], [20], Chapter 4 of [9], section 10, Chapter I of [5], and also [2].

When $A = L(X)$, $X$ a Banach space, the hermitian idempotents will be of particular interest. Following [3], the set of all these elements will be denoted by $\mathcal{E}(X)$, that is

$$\mathcal{E}(X) = \{ P \in L(X) : P^2 = P, \text{ and } P \text{ is a hermitian operator} \}.$$  

Recall that, according to Theorem 2.2 of [20], $P = Q$ if and only if $R(P) = R(Q)$, where $P$ and $Q$ belong to $\mathcal{E}(X)$.

In addition, following the Definitions of page 108 and 110 in [3], $\mathcal{M}(X)$ will denote the collection of all subspaces $M$ of $X$ such that there is a (necessarily unique) $P \in \mathcal{E}(X)$ with $M = R(P)$. In this case, for $M \in \mathcal{M}(X)$, the unique $P \in \mathcal{E}(X)$ such that $R(P) = M$ will be denoted by $P_M$. Further, $N(P_M)$ will be denoted by $M'$.

Now the central notion of the present note will be considered.

Let $A$ be a Banach algebra and $a \in A$. The main concern of this work consists in the elements $a$ for which there exists $x \in A$ satisfying the following conditions:
(i) $a = axa$, \quad x = xax,
(ii) $ax$ and $xa$ are hermitian.

In Lemma 2.1 of [22] it was proved that given $a \in A$, there exists at most one $x$ such that the previous conditions hold. This fact led to the following definition

**Definition 3.** Let $A$ be a unital Banach algebra and $a \in A$. If there exists $x \in A$ such that the previous conditions are satisfied, the element $x$ will be called the Moore-Penrose inverse of $a$, and it will be denoted by $a^\dagger$.

As regard the Moore-Penrose inverse in Banach algebra, see [22], where this concept was introduced, [23], the continuation of [22], and see also [19], where Moore-Penrose inverse of Banach space operators were considered. For the original definition of the Moore-Penrose inverse for matrices, see [21].

**Remark 4.** In the same conditions of Definition 3, it is clear that if $a \in A$ has a Moore-Penrose inverse, then $a^\dagger$ also has one and $(a^\dagger)^\dagger = a$. Furthermore, invertible elements and hermitian idempotents have a Moore-Penrose inverse.

Note that the norm is a fundamental notion involved in Definitions 1 and 3. Actually, even when the underlying space is the same and two equivalent norms are considered, it is possible that an operator is hermitian (resp. has a Moore-Penrose inverse) in one of the norms but not in the other.

In fact, define $T(x, y) = (\frac{1}{2}x - \frac{1}{2}y, -\frac{1}{2}x + \frac{1}{2}y)$, where $(x, y) \in \mathbb{C}^2$. Then, it is not difficult to prove that, on the one hand, $T$ is a hermitian idempotent with the Euclidean norm, while on the other, $T$ is not hermitian in $(\mathbb{C}, \| \cdot \|_1)$, where $\| (x, y) \|_1 = |x| + |y|$, $(x, y) \in \mathbb{C}^2$.

In addition, if $S(x, y) = (x - y, 0)$, $(x, y) \in \mathbb{C}^2$, then, while in the Hilbert space norm $S^\dagger$ exists, a rather painful calculation shows that $S$ does not have a Moore-Penrose inverse in $(\mathbb{C}, \| \cdot \|_1)$.

On the other hand, recall that according to Theorem 6 of [13], in a $\mathbb{C}^*$-algebra the set of the elements with a Moore-Penrose inverse coincide with the one of the regular elements. However, in a Banach algebra this result does not hold in general any more. In fact, $S \in A = L((\mathbb{C}, \| \cdot \|_1))$ is an idempotent without a Moore-Penrose inverse. Moreover, in Theorem 4.6 of [19] it was proved that necessary and sufficient for a Banach space of dimension greater than 3 to be a Hilbert space is that the set of all regular operators coincides with the one of all bounded and linear maps with a Moore-Penrose inverse.
In the following section the Moore-Penrose inverse in Banach spaces and algebras will be studied.

3. Basic Properties of the Moore-Penrose Inverse

In this section several characterizations concerning the Moore-Penrose inverse in Banach spaces and algebras will be given. In particular, the relationships between the aforesaid concept and, on the one hand, the adjoint of a Banach space operator and, on the other, closed invariant subspaces will be considered.

For $T \in L(X)$, the definitions of the Moore-Penrose inverse as an operator and as an element of the algebra $A = L(A)$ coincide. On the other hand, the starting point of the next theorem will be the Banach algebra setting, in which the notion under consideration will be related to the Moore-Penrose inverse of Banach space operators.

**Theorem 5.** Let $A$ be a unital Banach algebra, and $a \in A$.

(i) An element $b \in A$ is a normalized generalized inverse of $a$ if and only if $L_b$ (resp. $R_b$) is a normalized generalized inverse of $L_a$ (resp. $R_a$) in $L(A)$.

(ii) For a regular element $a \in A$, necessary and sufficient for $b \in A$ to be the Moore-Penrose inverse of $a$ in $A$ is that $L_b$ (resp. $R_b$) is the Moore-Penrose inverse of $L_a$ (resp. $R_a$) in $L(A)$. Moreover, in this case $(L_a)\dagger = L_a\dagger$ (resp. $(R_a)\dagger = R_a\dagger$).

**Proof.** The first statement is clear.

As regard the second statement, a straightforward calculation shows that

$$\exp(itd) = L_{\exp(itd)},$$

where $d \in A$, and $t \in \mathbb{R}$. However, since $\| e \| = 1$, $\| L_d \| = \| d \|$. Therefore, for $t \in \mathbb{R}$,

$$\| \exp(itd) \| = \| L_{\exp(itd)} \| = \| \exp(itd) \| .$$

Next consider $b \in A$, a normalized generalized inverse of $a$. Since $L_{ab} = L_aL_b$ and $L_{ba} = L_bL_a$, the above equality shows that $b$ is the Moore-Penrose inverse of $a$ if and only if $L_b$ is the Moore-Penrose inverse of $L_a$.

In a similar way, it can be proved the second statement considering $R_a$ instead of $L_a$.

It is well known that necessary and sufficient for $T \in L(X)$ to be regular is that $N(T)$ and $R(T)$ are closed and complemented subspaces of $X$, see Theorem 1 of [7] or Theorem 3.8.2 of [10]. In the following
theorem, the corresponding characterization for Banach space operators and Banach algebra elements with a Moore-Penrose inverse will be given.

**Theorem 6.** Let $X$ be a Banach space and $T \in L(X)$. Then, the following statements are equivalent:

(i) $T$ has a Moore-Penrose inverse,

(ii) there exist $P$ and $Q$ in $E(X)$, such that $N(P) = N(T)$ and $R(Q) = R(T)$.

Let $A$ be a unital Banach algebra, and consider an element $a \in A$. Then, the following statements are equivalent:

(i) The map $L_a$ (resp. $R_a$) has a Moore-Penrose inverse in $L(A)$,

(ii) there exist $P$ and $Q$ in $E(A)$, such that $N(P) = a^{-1}(0)$ and $R(Q) = aA$ (resp. $N(P) = a^{-1}(0)$ and $R(Q) = Aa$.

Furthermore, if such $P$ and $Q$ exist, then they are unique.

**Proof.** If $T \in L(X)$ has a Moore-Penrose inverse $T^\dagger$, then consider $P = T^\dagger T$ and $Q = TT^\dagger$.

In order to verify the converse implication, consider the invertible operator $T' \in L(R(P), R(T))$,

$$T' = T |_{R(P)}^{R(T)} : R(P) \to R(T),$$

and define $S \in L(X)$ as follows:

$$S |_{N(Q)} \equiv 0, \quad S |_{R(P)}^{R(T)} = (T')^{-1} \in L(R(T), R(P)).$$

A straightforward calculation proves that $S$ is a normalized generalized inverse of $T$. On the other hand, since $TS$ and $Q$ are idempotents of $L(X)$ such that

$$R(TS) = R(T) = R(Q), \quad N(TS) = N(S) = N(Q),$$

it is clear that $TS = Q$. In particular, $TS \in E(X)$. Similarly, $ST \in E(X)$. Therefore, $S$ is the Moore-Penrose inverse of $T$.

Moreover, if $P'$ and $Q'$ are two other hermitian idempotents that satisfy the above conditions, then $R(Q) = R(Q')$ and $R(I - P) = R(I - P')$. Then, according to Theorem 2.2 of [20], or to Remark 2, $P = P'$ and $Q = Q'$.

The rest of the Theorem can be deduced from what has been proved.

Next follows the relationships between the Moore-Penrose inverse of a Banach space operator and a closed invariant subspace.
Proposition 7. Let $X$ be a Banach space, and $T \in L(X)$ such that there exists $T^\dagger$. Let $Y$ be a closed subspace of $X$ invariant under $T$ and $T^\dagger$.

(i) If $T'$ (resp. $T'^\dagger$) denotes the operator induced by $T$ (resp. by $T^\dagger$) on $Y$, then $T'^\dagger$ is the Moore-Penrose inverse of $T'$ on $Y$.

(ii) If $T_Y$ (resp. $T'^\dagger_Y$) denotes the operator induced by $T$ (resp. by $T^\dagger$) on the quotient Banach space $X/Y$, then $T'^\dagger_Y$ is the Moore-Penrose inverse of $T_Y$ on $X/Y$.

On the other hand, consider $X_i$, $i = 1, 2$, two Banach spaces, and $T_i \in L(X_i)$, $i = 1, 2$, two operators such that there exist $T_i^\dagger \in L(X_i)$, $i = 1, 2$. Define the Banach space $X = X_1 \oplus X_2$, with the norm $\| x_1 \oplus x_2 \| = \max\{ \| x_1 \|, \| x_2 \| \}$. Then $T_1^\dagger \oplus T_2^\dagger$ is the Moore-Penrose inverse of $T = T_1 \oplus T_2$ on $X$.

Proof. It is clear that $T'^\dagger$ is a normalized generalized inverse of $T'$. Moreover, according to the first statement of Proposition 4.12 of [9], $T'^\dagger$ is the Moore-Penrose inverse of $T$.

The second statement can be proved in a similar way, using the statement (ii) of Proposition 4.12 of [9].

As regard the last part of the Proposition, an easy calculation shows that $S = T_1^\dagger \oplus T_2^\dagger$ is a normalized generalized inverse of $T$. In addition, given Banach space operators $S_i \in L(X_i)$, $i = 1, 2$, it is not difficult to prove that,

$$
exp(S_1 \oplus S_2) = exp(S_1) \oplus exp(S_2),
$$

which implies that $\| exp(S_1 \oplus S_2) \| \leq \max\{ \| exp(S_1) \|, \| exp(S_2) \| \}$. In particular, if $S_i = itT_i^\dagger$, $i = 1, 2$, then $\| exp(itTS) \| \leq 1$, where $t \in \mathbb{R}$.

Similarly, $\| exp(itST) \| \leq 1$. Now well, according to Remark 2, $S = T_1^\dagger \oplus T_2^\dagger$ is the Moore-Penrose inverse of $T$. \hfill \blacksquare

In the setting of Banach algebras, Proposition 7 can be generalized in the following way.

Proposition 8. Let $A$ be a unital Banach algebra, and $a \in A$ such that there exists $a^\dagger$.

(i) If $B$ is a subalgebra of $A$ such that $a$ and $a^\dagger$ belong to $B$, then $a^\dagger$ is the Moore-Penrose inverse of $a$ in $B$.

(ii) If $J$ is a proper and closed bilateral ideal of $A$, then $(\bar{a})^\dagger = a^\dagger$, where if $d \in A$, $\bar{d}$ denotes the quotient class of $d$ in $A/J$.

Proof. The first statement is clear.
An easy calculation shows that \( \tilde{a}^\dagger \) is a normalized generalized inverse of \( \tilde{a} \) in the unital Banach algebra \( A/J \).

Next consider the map \( L_a \in L(A) \). It is clear that \( L_a(J) \subseteq J \). Moreover, following the notation of Proposition 7, \( L_aJ = L_a \in L(A/J) \). Similarly, \( L_a^\dagger J = L_a^\dagger \in L(A/J) \). Now well, according to the second statement of Theorem 5 and Proposition 7,

\[
(L_a)^\dagger = L_a^\dagger, \quad (L_aJ)^\dagger = L_a^\dagger J.
\]

Therefore,

\[
(L_a)^\dagger = (L_aJ)^\dagger = L_a^\dagger J = L_a^\dagger,
\]

with which, according again to the second statement of Theorem 5, the proof is concluded.

In the last Theorem of the present section, the relationship between the Moore-Penrose inverse and the adjoint in Banach spaces will be studied. First of all some notation will be recalled. If \( X \) is a Banach space, then its dual will be denoted by \( X^* \). In addition, if \( T \in L(X) \), the adjoint map of \( T \) will be denoted by \( T^* \). Next follows a necessary remark.

**Remark 9.** Let \( X \) and \( Y \) be two Banach spaces and \( U \in L(Y) \). Consider \( F: X \to Y \) an isometric isomorphism. Then, an easy calculation shows that

\[
\exp(F^{-1}UF) = F^{-1}\exp(U)F.
\]

In particular, \( F^{-1}UF \) is hermitian in \( L(X) \) if and only if \( U \) is hermitian in \( L(Y) \).

Furthermore, if \( U \) has a Moore-Penrose inverse in \( Y \), then it is not difficult to prove that \( F^{-1}U^\dagger F \) is the Moore-Penrose inverse of \( F^{-1}UF \) in \( X \).

**Theorem 10.** Let \( X \) be a Banach space and \( T \in L(X) \).

(i) If there exists \( T^\dagger \), then \( T^* \) has a Moore-Penrose inverse and \( (T^*)^\dagger = (T^\dagger)^* \).

(ii) Suppose that there exist \( (T^*)^\dagger \in L(X^*) \) and \( S \in L(X) \) such that \( S^* = (T^*)^\dagger \), then there exists \( T^\dagger \) and \( S = T^\dagger \).

In particular, if \( X \) is a reflexive space, then necessary and sufficient for \( T \in L(X) \) to have a Moore-Penrose inverse is that \( (T^*)^\dagger \) exists.

**Proof.** If \( T^\dagger \) exists, then \( (T^\dagger)^* \) is a normalized generalized inverse of \( T^* \). Moreover, according to Proposition 4.11 of [9], \( (T^\dagger)^* \) is the Moore-Penrose inverse of \( T^* \).

As regard the second part of the Proposition, according to what has been proved, there exists \( ((T^*)^*)^\dagger \) and \( ((T^*)^*)^\dagger = (S^*)^* \in L((X^*)^*) \).
Next consider $\tilde{T}$ (resp. $\tilde{S}$), the restriction of $(T^*)^*$ (resp. $(S^*)^*$) to the closed invariant subspace $\Lambda(X) \subseteq (X^*)^*$, where

$$\Lambda: X \to \Lambda(X) \subseteq (X^*)^*, \quad \Lambda(x)(f) = f(x), \quad (x \in X \text{ and } f \in X^*),$$

is the canonical isometric identification of $X$ with a closed subspace of $(X^*)^*$. Then, according to the first statement of Proposition 7, there exists $\tilde{T}^\dagger$ and $\tilde{T}^\dagger = \tilde{S}$. Now well, according to Remark 9, there exists $T^\dagger$ and $T^\dagger = S$.

The last statement is a consequence of the fact that, when $X$ is a reflexive space, $L(X^*) = \{S^*: S \in L(X)\}$.

4. EP Banach Space Operators and Banach Algebra Elements

In this section a particular case of the elements with a Moore-Penrose inverse will be considered, namely EP Banach space operators and Banach algebra elements. As in the previous section, the basic properties of these objects will be studied. Furthermore, several characterizations will be given extending well-known results obtained in the frame of Hilbert spaces and $C^*$-algebras, see [6], [12], and [15]. In first place, the definition of the aforementioned notion will be introduced.

**Definition 11.** Given a unital Banach algebra $A$, $a \in A$ will be said an EP element, if there exists $a^\dagger$, and $aa^\dagger = a^\dagger a$.

Observe that the name of the elements introduced in Definition 11 derives from the fact that the idempotents, projectors in the Banach space operator context, $aa^\dagger$ and $a^\dagger a$ are equal. On the other hand, in the following remark some of the basic results regarding EP Banach algebra elements will be collected. Each of them can be proved with a direct argument.

**Remark 12.** In the same conditions of Definition 11, consider $a \in A$ such that $a^\dagger$ exists. Observe that since $(a^\dagger)^\dagger = a$, $a$ is EP if and only if $a^\dagger$ is. In addition, a direct calculation proves that if $a \in A$ is EP, then so is $a^k$, for $k \geq 1$. Moreover, since according to Theorem 5, $(L_a)^\dagger = L_{a^\dagger}$, $a \in A$ is EP if and only if $L_a \in L(A)$ is. A similar equivalence can be obtained for the map $R_a \in L(A)$.

Recall that the group inverse of $a \in A$ is an element $b \in A$, such that $a = aba$, $b = bab$, and $ab = ba$, see for example [24]. Note that when $a$ is an EP element, then $a$ has a group inverse, which coincides with the Moore-Penrose inverse.

On the other hand, given a Banach space $X$ and $T \in L(X)$ and EP operator, if $Y \subseteq X$ is a closed subspace of $X$ which is invariant under
Next follows a characterization of EP Banach space operators and Banach algebra elements.

**Proposition 13.** Let $X$ be a Banach space and $T \in L(X)$. Then, the following statements are equivalent:

(i) $T$ is an EP operator,
(ii) there exists $P \in \mathcal{E}(X)$ such that $N(P) = N(T)$ and $R(P) = R(T)$.

Let $A$ be a unital Banach algebra, and consider a regular element $a \in A$ such that $a^\dagger$ exists. Then, the following statements are equivalent:

(i) $a$ is EP,
(ii) there exists $P \in \mathcal{E}(A)$ such that $N(P) = a^{-1}(0)$ and $R(P) = aA$,
(iii) there exists $P \in \mathcal{E}(A)$ such that $N(P) = a_{-1}(0)$ and $R(P) = Aa$.

Furthermore, if such $P$ exists, then it is unique.

*Proof.* If $T$ is an EP operator, then the idempotent $P = TT^\dagger = T^\dagger T$ complies with the required property.

On the other hand, according to Theorem 6, $T^\dagger$ exists. In addition, since $R(I - T^\dagger T) = N(T) = N(P) = R(I - P)$, and since $I - P$ is an hermitian idempotent, see Remark 2, [26] or Chapter 4 of [9], according to Theorem 2.2 of [20], $TT^\dagger = P = T^\dagger T$.

Consider a Banach algebra $A$, and $a \in A$ a regular element such that $a^\dagger$ exists. If $a$ is EP, then, according to Theorem 5 and the proof of Theorem 6, $P = L_a a^\dagger$ satisfies the desired condition.

In order to prove the converse implication, note that according to Theorem 5, $L_a$ has a Moore-Penrose inverse in $L(A)$. Moreover, thanks to what has been proved, $L_a \in L(A)$ is an EP operator, which according to Remark 12, is equivalent to the fact that $a$ is EP.

The third statement can be proved in a similar way using $R_a$ instead of $L_a$.

The last statement is a consequence of Theorem 6. 

In the following proposition, the relationship between EP operators and the adjoint will be studied.

**Proposition 14.** Let $X$ be a Banach space, and $T \in L(X)$ such that $T^\dagger$ exists. Then, necessary and sufficient for $T$ to be EP is that $T^* \in L(X^*)$ is EP.

*Proof.* Since, according to Proposition 10, $(T^*)^\dagger = (T^\dagger)^*$, it is clear that if $T$ is EP, then $T^*$ is EP.
On the other hand, if $T^*$ is EP, then, according to what has been proved, $(T^*)^*$ also is. Consider, as in Proposition 10, the isometric isomorphism $\Lambda: X \to \Lambda(X) \subseteq (X^*)^*$ and $\tilde{T}$, the restriction of $(T^*)^*$ to $\Lambda(X)$. Then, according to the last paragraph of Remark 12, $\tilde{T}$ is EP. Now well, since $\tilde{T}^\dagger$ is the restriction of $((\tilde{T}^\dagger)^*)^*$ to $\Lambda(X)$ and $\Lambda T T^\dagger = \tilde{T} \tilde{T}^\dagger \Lambda$, $\Lambda T^\dagger T = \tilde{T}^\dagger \tilde{T} \Lambda$,

$$TT^\dagger = T^\dagger T.$$ 

In [6] a well-known characterization of EP Hilbert space operators was stated. This result was firstly extended to $C^*$-algebras in [12], and in second place in [15], where other equivalent statements were proved and the main concept used was the Drazin inverse. The most important results of this section consist in the extension and reformulation of the aforesaid characterizations in Banach spaces and algebras, using instead of the adjoint, the Moore-Penrose inverse and the properties of hermitian projectors as developed in [3] and [20]. In order to begin with this subject, the characterization of [6] is considered.

Remark 15. Let $H$ be a Hilbert space and let $A \in L(H)$ be an operator with closed range. Recall that if $A^\dagger$ is the Moore-Penrose of $A$, then $N(A^\dagger) = N(A^*)$ and $R(A^\dagger) = R(A^*)$. Moreover, considering the $C^*$-algebra $L(H)$, according to the proof of Theoreme 10 in [12], see Remark 19, there are invertible operators $P_1$ and $P_2$ defined on $H$ such that $A^* = P_1 A^\dagger = A^\dagger P_2$. Therefore, the relevant information contained in the characterization of [6] consists in the fact that $A$ is EP if and only if $N(A) = N(A^\dagger)$, $R(A) = R(A^\dagger)$, or $A^\dagger = \tilde{P} A$, where $\tilde{P}$ is an invertible operator defined on $H$. In the following theorem, in the context of Banach spaces, the characterization of [6] will be reformulated using the range and the null space of the Moore-Penrose inverse instead of the corresponding spaces of the adjoint. Furthermore, such characterization will be obtained precisely thanks to Theorem 2.2 of [20], which states that hermitian idempotents in Banach spaces have properties similar to the ones of orthogonal projectors in Hilbert spaces.

Theorem 16. Let $X$ be a Banach space, and consider $T \in L(X)$ such that $T^\dagger$ exists. Then, the following statements are equivalent:

(i) $T$ is an EP operator,

(ii) $N(T) = N(T^\dagger)$,

(iii) $R(T) = R(T^\dagger)$,

(iv) there exists an invertible operator $P \in L(X)$ such that $T^\dagger = PT$. 


Proof. Recall that since $T^\dagger$ exists, $TT^\dagger$ (resp. $T^\dagger T$) is a hermitian idempotent such that $R(T) = R(TT^\dagger)$ (resp. $R(T^\dagger) = R(T^\dagger T)$). Furthermore, according to Theorem 2.2 of [20], $R(T) = R(T^\dagger)$ if and only if $TT^\dagger = T^\dagger T$.

Similarly, since $T^\dagger$ exists, according Remark 2 or to Theorem 4.4 of [9], $I - TT^\dagger$ (resp. $I - T^\dagger T$) is a hermitian idempotent such that $N(T^\dagger) = R(I - TT^\dagger)$ (resp. $N(T) = R(I - T^\dagger T)$). Consequently, according again to Theorem 2.2 of [20], $T$ is an EP operator if and only if $N(T) = N(T^\dagger)$.

Next suppose that there exists an invertible operator $P \in L(X)$ such that $T^\dagger = PT$. It is clear that $N(T) = N(T^\dagger)$. In particular, $T$ is an EP operator.

On the other hand, if $T$ is an EP operator, it is not difficult to prove that $X = N(T) \oplus R(T)$, and $\tilde{T} = T \mid_{R(T)}: R(T) \to R(T)$ is an invertible operator. Define the operator $P'$ in the following way:

$$P' \mid_{N(T)} \equiv I_{N(T)}, \quad P' \mid_{R(T)} = \tilde{T}^2,$$

where $I_{N(T)}$ denotes the identity map on $N(T)$.

Clearly, $P'$ is invertible, and a straightforward calculation shows that $T = P'T^\dagger$. In order to conclude the proof, define $P = (P')^{-1}$.

Remark 17. In the same conditions of Theorem 16, note that a straightforward calculation proves the following identities: $T^\dagger = PT = TP$.

On the other hand, observe that $T$ is an EP operator if and only if there exists an invertible operator $Q \in L(X)$ such that $T = QT^\dagger = T^\dagger Q$. In fact, if such identities hold, then $N(T) = N(T^\dagger)$. Consequently, according to the second statement of Theorem 16, $T$ is an EP operator. In order to prove the converse implication, note that $T$ is an EP operator if and only if $T^\dagger$ is. Then, according to the last statement of Theorem 16 and what has been proved, there is an invertible operator $Q$ such that $T = QT^\dagger = T^\dagger Q$.

Recall that in the Hilbert space context, the operator $A$ has a Moore-Penrose inverse if and only if $A^\ast$, $A^\dagger$ and $(A^\dagger)^\ast$ also have one. Since in [6] and in Theorem 16 the condition of being EP has been characterized using a relationship between $A$ and $A^\ast$ and between $A$ and $A^\dagger$, more equivalent statements can be deduced considering $A^\ast$, $A^\dagger$, and $(A^\dagger)^\ast$, and their corresponding adjoints or Moore-Penrose inverses.

Finally, note that for any closed range operator $A$ defined on a Hilbert space, $N(A) = N((A^\dagger)^\ast)$ and $R(A) = R((A^\dagger)^\ast)$. Therefore,
no characterization of the EP condition involving $A$ and $\left(A^\dagger\right)^{\ast}$ can be obtained.

Next follows the characterization of EP Banach algebra elements. As in Theorem 16, using the left and right multiplication representations and ideas of [12] and [15], this result reformulates and extends Theorem 10 of [12] and Theorem 3.1 of [15] to Banach algebras.

**Theorem 18.** Let $A$ be a unital Banach algebra, and consider $a \in A$ such that $a^\dagger$ exists. Then, the following statements are equivalent:

(i) $a$ is an EP Banach algebra element,
(ii) $L_a \in L(A)$ is an EP operator,
(iii) $a^{-1}(0) = (a^\dagger)^{-1}(0),$
(iv) $aA = a^\dagger A,$
(v) there exists an invertible operator $P \in L(A)$ such that $L_{a^\dagger} = PL_a = L_a P,$
(vi) there exists an invertible operator $Q \in L(A)$ such that $L_a = QL_{a^\dagger} = L_{a^\dagger} Q,$
(vii) $R_a \in L(A)$ is an EP operator,
(viii) $a^{-1}_1(0) = (a^\dagger)^{-1}_1(0),$
(ix) $Aa = Aa^\dagger,$
(x) there exists an invertible operator $U \in L(A)$ such that $R_{a^\dagger} = UL_a = R_a U,$
(xi) there exists an invertible operator $V \in L(A)$ such that $R_a = VR_{a^\dagger} = R_{a^\dagger} V,$
(xii) $a^2 a^\dagger = a = a^\dagger a^2,$
(xiii) $a \in a^\dagger A \cap A a^\dagger,$
(xiv) $a^\dagger \in a A \cap A a,$
(xv) $a \in a^\dagger A^{-1} \cap A^{-1} a^\dagger,$
(xvi) $a^\dagger \in a A^{-1} \cap A^{-1} a,$
(xvii) $a A^{-1} = a^\dagger A^{-1},$
(xviii) $A^{-1} a = A^{-1} a^\dagger.$

**Proof.** Since $A$ is a unital Banach algebra, according to Theorem 5 and Remark 12, the first statement is equivalent to the second and the seventh.

In addition, according to Theorem 16, the second, third, fourth, fifth and sixth statements are equivalent. Furthermore, according to the same theorem, the seventh, eighth, ninth, tenth and eleventh statements are equivalent.

On the other hand, recovering an argument used in Theorem 10 of [12], it is possible to prove that the first and the twelfth statements are equivalent.
The equivalence between the twelfth and the thirteenth statements can be proved as in Theorem 3.1 of [15]. In addition, in order to prove that the first and the fourteenth statements are equivalent, use what has been proved and the fact that \( a \) is EP if and only if \( a^\dagger \) is.

Recovering another argument used in Theorem 10 of [12], it is clear that the first statement implies the fifteenth. On the other hand, the later statement implies the third and the eighth. Moreover, in order to prove that the first and the sixteenth statements are equivalent, use what has been proved and the fact that \( a \) is EP if and only if \( a^\dagger \) is.

If \( a \) is an EP element, then according to the proof of Theorem 10 of [12], there exist two invertible elements \( c \) and \( d \), such that \( a^\dagger = ac \) and \( a = a^\dagger d \), which, according to a straightforward calculation proves that \( aA^{-1} = a^\dagger A^{-1} \). On the other hand, if the seventeenth statement holds, then it is not difficult to prove that \( a_{-1}(0) = (a^\dagger)_{-1}(0) \).

Finally, if \( a \) is an EP element, then according again to the proof of Theorem 10 of [12], the invertible elements of the previous paragraph \( c \) and \( d \) also satisfy that \( a^\dagger = ca \) and \( a = da^\dagger \), which, according to a straightforward calculation proves that \( A^{-1}a = A^{-1}a^\dagger \). On the other hand, if the last statement holds, then it is not difficult to prove that \( a^{-1}(0) = (a^\dagger)^{-1}(0) \).

\[ \text{Remark 19.} \] Let \( A \) be a \( C^* \)-algebra and \( a \in A \). Recall that according to Theorem 10 of [12], if \( a^\dagger \) exists, then

\[
\begin{align*}
\ast \in A^{-1}a^\dagger & \quad \text{and} \quad a^\dagger \in a^\dagger A^{-1}.
\end{align*}
\]

According to these facts, it is not difficult to prove that

\[
\begin{align*}
(a^\dagger)^{-1}(0) & = (a^\dagger)^{-1}(0), & a_{-1}(0) & = a_{-1}(0), \\
a^\ast A & = a^\dagger A, & Aa^\ast & = Aa^\dagger, \\
a^\ast A^{-1} & = a^\dagger A^{-1}, & A^{-1}a^\ast & = A^{-1}a^\dagger.
\end{align*}
\]

Consequently, as in the case of a closed range Hilbert space operator, the relevant equivalences in Theorem 10 of [12] and Theorem 3.1 of [15] are the following identities:

\[
\begin{align*}
a^{-1}(0) & = (a^\dagger)^{-1}(0), & a_{-1}(0) & = a_{-1}(0), \\
aA & = a^\dagger A, & Aa & = Aa^\dagger, \\
aA^{-1} & = a^\dagger A^{-1}, & A^{-1}a & = A^{-1}a^\dagger.
\end{align*}
\]

On the other hand, in Theorem 18, Theorem 10 of [12] and Theorem 3.1 of [15] it has been characterized the condition of being EP using
a relationship between $a$ and $a^\dagger$ or between $a$ and $a^*$. Since $a \in A$ is an EP element if and only if $a^*, a^\dagger$, and $(a^\dagger)^*$ also are, it is possible to obtain more equivalent statements for an element to be EP, applying the mentioned results to $a^*$, $a^\dagger$, and $(a^\dagger)^*$, and to their corresponding adjoints and Moore-Penrose inverse.

Observe that, as in the case of a closed range Hilbert space operator, according to Proposition 2.4 of [4], it is not difficult to prove that

$$a^{-1}(0) = ((a^\dagger)^*)^{-1}(0), \quad a_{-1}(0) = (a^\dagger)_{-1}^*(0),$$

$$aA = (a^\dagger)^*A, \quad Aa = A(a^\dagger)^*.$$

Therefore, in order to characterize EP elements, no characterization involving $a$ and $(a^\dagger)^*$ can be obtained.

Finally, if $X$ is a Banach space and $T \in L(X)$ is an operator with a Moore-Penrose inverse, Theorem 18 provides a set of characterizations for $T$ to be EP, considering it as an element of the Banach algebra $A = L(X)$.

5. The Product of Two EP Elements

In the present section it will be studied when the product of two EP Banach space operators or Banach algebra elements is again EP. In first place, a remark is considered.

Remark 20. The problem of characterizing when the product of two EP matrices is again EP was first posed in [1], and solved in Theorem 1 of [13], where it was formulated using the row space of a matrix. In [14] a simple proof of the latter Theorem was given using an operator theoretical language.

For closed range Hilbert space operators, in [8] it was proved that the first and the fourth statements of Theorem 2 of [14] are equivalent, while the Example in section 3, page 115, of [16] shows that the third and the fourth statements of Theorem 2 of [14] are not any more equivalent. On the other hand, in Theorem 4.3 and Corollary 4.4 of [15] the problem under consideration was studied in the context of $C^*$-algebras and closed range Hilbert space operators. In the following theorems, using hermitian idempotents and reformulating an idea of Theorem 4.3 of [15], it will be characterized when the product of two Banach algebra elements or Banach space operators is again EP.

Two more observations. Recall that if $X$ is a Banach space, then an operator $T \in L(X)$ is called upper semi-Fredholm (resp. lower semi-Fredholm), if $R(T)$ is closed and $N(T)$ is finite dimensional (resp. $R(T)$
has finite codimension). If \( T \in L(X) \) is EP, then necessary and sufficient for \( T \) to be a Fredholm operator is that \( T \) is upper semi-Fredholm (resp. lower semi-Fredholm). Moreover, in this case \( \text{ind}(T) = 0 \), where \( \text{ind} \) denotes the index of \( T \). On the other hand, if \( T \in L(X) \) is an operator such that \( T^\dagger \) exists, then \( I - T^\dagger T \) is the hermitian idempotent onto \( N(T) \), see Theorem 2 of [14].

**Theorem 21.** Let \( X \) be a Banach space, and let \( S \) and \( T \) be two EP Banach space operators defined on \( X \) such that the Moore-Penrose inverse of \( ST \) exists.

(i) If \( N(ST) = N(S) + N(T) \) and \( R(ST) = R(S) \cap R(T) \), then \( ST \) is EP.

(ii) If \( ST \) is an EP Banach space operator, then \( R(ST) \subseteq R(T) \) and \( N(S) \subseteq N(ST) \).

(iii) Necessary and sufficient for \( R(ST) \subseteq R(T) \) (resp. \( N(S) \subseteq N(ST) \)) is that \( (I - TT^\dagger)ST = 0 \) (resp. \( ST(I - S^\dagger S) = 0 \)).

(iv) If \((I - T^\dagger T)ST = 0, ST(I - S^\dagger S) = 0, \) and \( S \) and \( T \) are Fredholm operators, then \( N(ST) = N(S) + N(T) \) and \( R(ST) = R(S) \cap R(T) \). In particular, \( ST \) is EP.

**Proof.** Suppose that \( N(ST) = N(S) + N(T) \) and \( R(ST) = R(S) \cap R(T) \). Define \( M = N(S) \) and \( N = N(T) \). Then, \( M \) and \( N \) belong to \( \mathcal{M}(X) \), see Remark 2. In fact, \( P_M = I - S^\dagger S \) and \( P_N = I - T^\dagger T \). In addition, \( M' = N(P_M) = R(S) \) and \( N' = N(P_N) = R(T) \).

Now well, since \( ST(ST)^\dagger \) is a hermitian idempotent such that

\[
R(ST(ST)^\dagger) = R(ST) = R(S) \cap R(T) = M' \cap N',
\]

\( M' \cap N' \in \mathcal{M}(X) \) and \( P_{M' \cap N'} = ST(ST)^\dagger \).

On the other hand, since \( (ST)^\dagger ST \) is an hermitian idempotent such that

\[
R(I - (ST)^\dagger ST) = N(ST) = N(S) + N(T) = M + N,
\]

\( M + N \) belongs to \( \mathcal{M}(X) \) and \( P_{M+ N} = I - (ST)^\dagger ST \).

Therefore, according to Theorem 2.28 of [3],

\[
P_{M+ N} = I - P_{M' \cap N'}.
\]

However, \( M + N = N(ST) \) and \( M' \cap N' = R(ST) \). Consequently,

\[
I - (ST)^\dagger ST = I - ST(ST)^\dagger,
\]
equivalently, \( (ST)^\dagger ST = ST(ST)^\dagger \).

Next suppose that \( ST \) is an EP Banach space operator. Define \( M_1 = R(ST) \) and \( N_1 = R(T) \). Then \( M_1 \) and \( N_1 \) belong to \( \mathcal{M}(X) \), \( P_{M_1} = ST(ST)^\dagger \), \( P_{N_1} = TT^\dagger \), \( M_1' = N((ST)^\dagger) \) and \( N_1' = N(T^\dagger) \).
Now well, according to the observation that precedes Theorem 2.28 of [3], equivalent for \( R(ST) \) to be contained in \( R(T) \) is that \( N(T^\dagger) \) is contained in \( N((ST)^\dagger) \). However, since \( T \) and \( ST \) are EP Banach space operators, according to Theorem 16, \( N(T^\dagger) = N(T) \subseteq N(ST) = N((ST)^\dagger) \).

On the other hand, consider \( M_2 = R((ST)^\dagger) \) and \( N_2 = R(S) \). As before, \( M_2 \) and \( N_2 \) belong to \( \mathcal{M}(X) \), \( P_{M_2} = (ST)^\dagger ST \), \( P_{N_2} = SS^\dagger \), \( M_2^* = N(ST) \), and \( N_2^* = N(S^\dagger) \).

Now well, according again to the observation that precedes Theorem 2.28 of [3], necessary and sufficient for \( N(S) \) to be contained in \( N(ST) \) is that \( R((ST)^\dagger) \) is contained in \( R(S^\dagger) \). However, due to the fact that \( S \) and \( ST \) are EP Banach space operators, according to Theorem 16, \( R((ST)^\dagger) = R(ST) \subseteq R(S) = R(S^\dagger) \).

As regard the third statement, in order to prove the equivalence it is enough to observe that \( R(T) = R(TT^\dagger) = N(I - TT^\dagger) \) and \( N(S) = N(S^\dagger S) = R(I - S^\dagger S) \).

In order to prove the last statement, suppose that
\[
ST(I - S^\dagger S) = 0 \quad \text{and} \quad (I - TT^\dagger)ST = 0.
\]

Next observe that, since \( S \) and \( T \) are EP Banach space operators,
\[
(I - S^\dagger S)ST = 0 \quad \text{and} \quad ST(I - TT^\dagger) = 0.
\]

Therefore,
\[
STS^\dagger S = S^\dagger SST = ST \quad \text{and} \quad STTT^\dagger = TT^\dagger ST = ST.
\]

Consequently, \( R(ST) \) and \( N(ST) \) are closed invariant subspaces for \( S^\dagger S \), \( TT^\dagger \), \( I - S^\dagger S \), and \( I - TT^\dagger \). In particular, according to Proposition 4.12 of [9], \( TT^\dagger \) and \( I - TT^\dagger \) are hermitian idempotents when restricted to \( R(ST) \) and \( N(ST) \), and due to the fact that \( (I - TT^\dagger)(X) = (I - TT^\dagger)(N(ST)) = N(T) \), it is clear that
\[
R(ST) = TT^\dagger(R(ST)) \quad \text{and} \quad N(ST) = TT^\dagger(N(ST)) \oplus N(T).
\]

Define the linear and bounded operator \( U \in L(N(ST)) \) in the following way: \( U \mid N(T) = I_{N(T)} \), the identity map of \( N(T) \), and \( U = T^\dagger T(N(ST)) : TT^\dagger(N(ST)) \to T(N(ST)) \). Note that \( T(N(ST)) \subseteq R(T) \cap N(S) \subseteq N(S) \subseteq N(ST) \). Moreover, a straightforward calculation proves that \( T(N(ST)) = R(T) \cap N(S) \). What is more, since \( TT^\dagger(N(ST)) \subseteq R(T) \) and \( N(T) \cap R(T) = 0 \), \( U : N(ST) \to R(T) \cap N(S) \oplus N(T) \) is a Banach space isomorphism.

Now well, if \( N(S) \) and \( N(T) \) are finite dimensional, then \( R(U) = N(ST) = R(T) \cap N(S) \oplus N(T) \). Therefore, \( N(ST) = N(S) + N(T) \).
In order to prove that \( R(ST) = R(S) \cap R(T) \), a duality argument will be considered.

Observe that, according to the hypothesis of the last statement of the Theorem, \( S^* \) and \( T^* \) are Fredholm operators defined on \( X^* \), the dual space of \( X \). Moreover, according to Proposition 14 and Theorem 10, \( S^* \) and \( T^* \) are EP, and \( T^*S^* \) has a Moore-Penrose inverse. In addition, it is clear that

\[
(I - S^*(S^*)^\dagger)T^*S^* = 0 \quad \text{and} \quad T^*S^*(I - (T^*)^\dagger)T^* = 0.
\]

Therefore, according to what has been proved, \( N(T^*S^*) = N(T^*) + N(S^*) \). Moreover, since \( ST \) is a Fredholm operator, its range is a closed subspace of \( X \) and

\[
R(ST) = \perp^\perp N((ST)^*) = \perp^\perp (N(S^*) + N(T^*)) = \perp^\perp N(S^*) \cap \perp^\perp N(T^*)
= R(S) \cap R(T),
\]

where if \( V \) is a subspace of \( X^* \), then \( \perp^\perp V = \{ x \in X : f(x) = 0, \forall f \in V \} \).

In the following Theorem it will be studied when the product of two Banach algebra elements is EP.

**Theorem 22.** Let \( A \) be a unital Banach algebra. Consider \( a \) and \( b \), two EP elements of \( A \) such that \( ab \) has a Moore-Penrose inverse.

(i) If \((ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0) \) and \( abA = aA \cap bA \), then \( ab \) is EP.

(ii) If \( ab \) is an EP Banach algebra element, then \( abA \subseteq bA \) and \( a^{-1}(0) \subseteq (ab)^{-1}(0) \).

(iii) Necessary and sufficient for \( abA \subseteq bA \) (resp. \( a^{-1}(0) \subseteq (ab)^{-1}(0) \)) is that \((e - bb^\dagger)ab = 0 \) (resp. \( ab(e - a^\dagger a) = 0 \)).

(iv) If \((e - bb^\dagger)ab = 0 \), \( ab(e - a^\dagger a) = 0 \), and \( a^{-1}(0) \) and \( b^{-1}(0) \) are finite dimensional, then \((ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0) \) and \( abA = aA \cap bA \). In particular, \( ab \) is EP.

Similarly,

(v) If \((ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0) \) and \( Aab = Aa \cap Ab \), then \( ab \) is EP.

(vi) If \( ab \) is an EP Banach algebra element, then \( Aab \subseteq Aa \) and \( b^{-1}(0) \subseteq (ab)^{-1}(0) \).

(vii) Necessary and sufficient for \( Aab \subseteq Aa \) (resp. \( b^{-1}(0) \subseteq (ab)^{-1}(0) \)) is that \( ab(e - aa^\dagger) = 0 \) (resp. \( e - b^\dagger b)ab = 0 \)).

(viii) If \( ab(e - aa^\dagger) = 0 \), \( e - b^\dagger b)ab = 0 \), and \( a^{-1}(0) \) and \( b^{-1}(0) \) are finite dimensional, then \((ab)^{-1}(0) = a^{-1}(0) + b^{-1}(0) \) and \( Aab = Aa \cap Ab \). In particular, \( ab \) is EP.

**Proof.** Consider \( L_a \) (resp. \( R_a \)), \( L_b \) (resp. \( R_b \)), and \( L_{ab} \) (resp. \( R_{ab} \)), and then apply Theorem 21, and use Theorem 5 and Remark 12.
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