Nonequilibrium Damping of Collective Motion of Homogeneous Cold Fermi Condensates with Feshbach Resonances

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Collisionless damping of a condensate of cold Fermi atoms, whose scattering is controlled by a Feshbach resonance, is explored throughout the BCS and BEC regimes when small perturbations on its phase and amplitude modes are turned on to drive the system slightly out of equilibrium. Using a one-loop effective action, we first recreate the known result that for a broad resonance the amplitude of the condensate decays as $t^{-1/2}$ at late times in the BCS regime whereas it decays as $t^{-3/2}$ in the BEC regime. We then examine the case of an idealized narrow resonance, and find that this collective mode decays as $t^{-3/2}$ throughout both the BCS and BEC regimes. Although this seems to contradict earlier results that damping is identical for both broad and narrow resonances, the breakdown of the narrow resonance limit restores this universal behaviour. More measureably, the phase perturbation may give a shift on the saturated value to which the collective amplitude mode decays, which vanishes only in the deep BCS regime when the phase and amplitude modes are decoupled.

I. INTRODUCTION

We now have remarkable experimental control over cold alkali atoms interacting through a Feshbach resonance. Manipulation of the binding energy through external magnetic fields enables us to evolve them continuously from the weakly coupled BCS-like behavior of Cooper pairs to the strongly coupled Bose-Einstein Condensation (BEC) of molecules [1]. The transition is characterised by a crossover in which, most simply, the s-wave scattering length $a_s$ diverges as it changes sign [2, 3].

Recently, a considerable theoretical effort has been expended on understanding how such a tunable superconductor/condensate responds to an initial non-equilibrium state, intimately related to the speed of sound, can be derived easily from the derivative expansion of the renormalized one-loop effective action. In particular, under long wavelength and small energy approximations this action can be used to obtain a hydrodynamic description of the BEC-BCS crossover from which the equation of state, intimately related to the speed of sound, can be derived. In this paper we particularly build on the work [9, 10] on the BEC/BCS crossover in the presence of a Feshbach resonance. In [9, 10] we have shown that many condensate properties at $T = 0$ can be derived easily from the derivative expansion of the renormalized one-loop effective action. In particular, under long wavelength and small energy approximations this action can be used to obtain a hydrodynamic description of the BEC-BCS crossover from which the equation of state, intimately related to the speed of sound, can be derived. In this paper we particularly build on the work of [9], creating an extended full one-loop effective action to explore the real-time evolution of the collective mode.

We find that the situation is more subtle than as presented in [4], with some differences in the relaxational dynamics of the amplitude mode for broad and narrow resonances in the BCS regime with positive chemical potential. For broad resonances we recreate the results of [4] directly. However, for idealised narrow resonances we find a $t^{-3/2}$ decay across both BCS and BEC regimes. Nonetheless, we anticipate that for more realistic resonances the long-time behaviour of [4] will be recovered in the deep BCS regime, albeit with $t^{-3/2}$ transients. A new result is that, even under linear perturbations, the amplitude mode is found to decay to a shifted value from its initial condition, due to the phase-amplitude coupling, irrespective of whether the resonance is broad or narrow. Such a shift is expected for non-linear perturbations but not for linear perturbations, when typically the system relaxes to its initial equilibrium state. These findings can be tested experimentally.
II. EFFECTIVE ACTIONS

We consider a condensate comprising a mixture of fermionic atoms and molecular bosons, in which the fermions \( \psi_0(x) \), with spin \( \sigma = \{\uparrow, \downarrow\} \), undergo self-interaction through an s-wave BCS-type term, and two fermions can be bound into a molecular boson \( \phi(x) \) through a Feshbach resonance. The general ‘two-channel’ microscopic action is given by \((U > 0, g \text{ fixed})\)

\[
S = \int dt d^3x \left\{ \sum_{\uparrow \downarrow} \psi^*_\sigma(x) \left[ i \partial_t + \frac{\nabla^2}{2m} + \mu + \frac{g}{24m} \right] \psi_\sigma(x) + U \psi^*_\uparrow(x) \psi^*_\downarrow(x) \psi_\downarrow(x) \psi_\uparrow(x) + \phi^*(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x) - g \left[ \phi^*(x) \psi_\downarrow(x) \psi_\uparrow(x) + \phi(x) \psi^*_\uparrow(x) \psi^*_\downarrow(x) \right] \right\}.
\]

Thus, the bound bosons of Feshbach resonance (field \( \phi \)) have twice the mass of the fermions, \( M = 2m \), and a tunable binding energy, \( \nu \).

On introducing the auxiliary field \( \Delta(x) = U \psi_\uparrow(x) \psi_\downarrow(x) \), a Hubbard-Stratonovich transformation leads to an effective Lagrangian density \( S \) quadratic in the Fermi fields. We then integrate them out \([9]\) to write \( S \) in the non-local form

\[
S_{NL} = -iTr \ln G^{-1} + \int dt dx \left\{ -\frac{1}{U} |\Delta|^2 + \phi^*(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x) \right\},
\]

where \( G^{-1} \) is the inverse Nambu Green function,

\[
G^{-1} = \begin{pmatrix} i\partial_t - \varepsilon & \Delta(x) \\ \Delta^*(x) & i\partial_t + \varepsilon \end{pmatrix}
\]

and represents the two-component combined condensate (and \( \varepsilon = -\nabla^2/2m - \mu \)).

The combined condensate amplitude and phase of \( \Delta(x) = |\Delta(x)| e^{i\theta_\Delta(x)} \) can be determined from those of \( \Delta(x) \) and \( \phi(x) \), defined respectively by \( \Delta(x) = |\Delta(x)| e^{i\theta_\Delta(x)} \) and \( \phi(x) = -|\phi(x)| e^{i\theta_\phi(x)} \). The \( U(1) \) invariance of the action under \( \theta_\Delta \rightarrow \theta_\Delta + \text{const.}, \theta_\phi \rightarrow \theta_\phi + \text{const.} \) is spontaneously broken. Thus, \( S_{NL} = 0 \) permits spacetime constant gap solutions \( |\Delta(x)| = |\Delta_0| \neq 0 \) and \( |\phi(x)| = |\phi_0| \neq 0 \) (whereby \( (|\Delta(x)| = |\Delta_0|) \neq 0 \)).

We now consider the fluctuations around the gap configurations and simultaneously perturb in the derivatives of \( \theta_\Delta \) and \( \theta_\phi \), small perturbations in the scalar condensate densities \([11]\) \( \delta|\Delta| = |\Delta| - |\Delta_0| \) and \( \delta|\phi| = |\phi| - |\phi_0| \) and their derivatives. To guarantee Galilean invariance we expand \([11]\) \( \delta G^{-1} = -G^{-1} \) where \( G^{-1} \) is obtained from \( \Delta(x) \) with the Galilean scalar \( |\tilde{\Delta}(x)| \). We construct the condensate effective action \( S_{eff} \) at second order in \( \Sigma \). Diagrammatically, this amounts to taking account of fermionic one-loop effects in the effective action.

\( S_{eff} \) comprises the integral of a local density together with non-local fermionic cut contributions. Later we shall restrict ourselves to homogeneous collective modes. For their dynamics we only need the quadratic part of the local density, while retaining the full cut contributions. The relevant \( S_{eff}^{(2)} \) then takes the form

\[
S_{eff}^{(2)} = \int d^4x \left\{ -\frac{1}{8m} \rho_0^\phi (\nabla \theta_\Delta)^2 - \frac{1}{8m} \rho_0^\phi (\nabla \theta_\phi)^2 - \frac{1}{2} \Omega^2 (\theta_\Delta - \theta_\phi)^2 - 2|\phi_0| \delta|\phi| \theta_\phi - \frac{1}{U}(\delta \theta_\Delta)^2 \right\} + \int d^4q \int d^4p \left\{ M_\theta \delta \Delta(q,p) \tilde{\Theta}_\Delta(q) \tilde{\Theta}_\Delta(-q) + M_\Delta \delta \Delta(q,p) \delta|\Delta|(q) \delta|\Delta|(-q) \right\}.
\]

for \( M \)-functions to be given later. In \([11]\) \( \delta |\tilde{\Delta}| = |\Delta| - |\Delta_0| \), with Fourier transform \( \delta|\Delta| \) and \( \tilde{\Theta}_\Delta \) the Fourier transform of the Galilean invariant \( \Theta_\Delta = \theta_\Delta + (\nabla \theta_\Delta)^2/4m \). The varying coupling strength (as \( \nu \) varies with external magnetic field) is \( U_{eff} = U + g^2/(2\mu - \nu) \). The phases of \( \Delta \) and \( \phi \) are coupled with strength \( \Omega^2 = (2g/U)|\phi_0||\Delta_0| \). Defining \( E_p^2 = \varepsilon_p^2 + |\Delta_0|^2 \) and \( \varepsilon_p = \mathbf{p}^2/2m - \mu \), the fermion number density is \( \rho_0 = \rho_0^\phi + \rho_0^B \), where \( \rho_0^\phi = \int d^4p/(2\pi)^3 \left[ 1 - \varepsilon_p/E_p^2 \right] \) is the explicit fermion density, and \( \rho_0^B = 2|\phi_0|^2 \) is due to molecules (two fermions per molecule). For more details see \([9]\).

The separation of the action \([5]\) into local and seemingly non-local parts is somewhat misleading. The \( M \)-functions have \( g \)-independent terms that lead to further contributions to the local part of the action (the effective local Lagrangian). These are proportional to \( \delta \Delta^2 \) and \( (\delta \Delta)^2 \) and have been displayed elsewhere \([9]\). The remainders of the double integrals comprise different non-local combinations of two-Fermion cuts. They are the source of the Landau damping, which occurs in a collisionless regime via direct dissipationless energy transfer from the collective mode to single particles. For the relatively simple \( M \)-problem for the problem in hand of a homogeneous system it is easier not to make this separation into local and non-local parts.

The dynamical equations of the collective modes can be obtained by taking the variation of the above effective action with respect to the associated field variables. In what follows, we will consider broad and narrow resonances in turn.
III. BROAD RESONANCE

In the case of a broad resonance the one-channel model, which is obtained by eliminating the molecular bosons from the above two-channel effective action, suffices to describe its dynamics [12, 13]. Thus, the corresponding action is given by setting \( \phi = 0 \) in the action (1) which, in turn, gives \( \Delta = \Delta \).

Consider a spatially homogeneous condensate. If, at time \( t = 0 \) we turn on a small instantaneous homogeneous change in the external field this will induce a homogeneous perturbation \( \delta \Delta \) in the gap parameter (condensate amplitude). On the other hand, since the gradient of the phase is the fluid velocity, a spontaneous translation (kick) of the system will induce a perturbation \( \delta \Delta \) in the phase, which we need to include for completeness. Although we have not included a trap to constrain the condensate in the analysis, to keep the results as simple as possible, this translation of the system could be implemented through a kick on the trap.

The linearized equations of motion are Fourier analyzed in terms of frequency \( \omega \) or, more conveniently, expressed in terms of the Laplace variable \( s = -i \omega \) as

\[
-s^2 \mathcal{M}_{\theta \Delta \theta \Delta}(s^2) \theta_{\Delta} + s \mathcal{M}_{\theta \Delta \Delta}(s^2) \delta |\Delta| = \delta_{\theta_{\Delta}}, \\
-s \mathcal{M}_{\theta \Delta \Delta}(s^2) \theta_{\Delta} + \mathcal{M}_{\Delta \Delta}(s^2) \delta |\Delta| = \delta_{\Delta} 
\]

(6)

(where we have dropped the tildes that normally represent transforms). After some straightforward but tedious calculation the \( \mathcal{M} \)s are found to be

\[
\mathcal{M}_{\theta \Delta \theta \Delta}(s^2) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_p s^2 + 4E_R^2},
\]

\[
\mathcal{M}_{\theta \Delta \Delta}(s^2) = \int \frac{d^3 p}{(2\pi)^3} \frac{2|\Delta_0|}{E_p s^2 + 4E_R^2},
\]

(7)

The solutions of the Laplace transform of the equations are

\[
\theta_{\Delta}(s) = \frac{-\mathcal{M}_{\Delta \Delta}(s^2) \delta_{\theta_{\Delta}} + s \mathcal{M}_{\theta \Delta \Delta}(s^2) \delta_{\Delta}}{s^2 \mathcal{D}(s^2)},
\]

\[
\delta |\Delta|(s) = \frac{s^2 \mathcal{M}_{\theta \Delta \Delta}(s^2) \delta_{\Delta} - s \mathcal{M}_{\theta \Delta \Delta}(s^2) \delta_{\theta_{\Delta}}}{s^2 \mathcal{D}(s^2)},
\]

(8)

where

\[
\mathcal{D}(s^2) = \mathcal{M}_{\theta \Delta}(s^2) \mathcal{M}_{\Delta \Delta}(s^2) - \mathcal{M}_{\theta \Delta \Delta}(s^2). 
\]

(9)

The real-time evolution of \( \theta(t) \) and \( \delta |\Delta|(t) \) can be obtained from carrying out the inverse Laplace transform along the contour in the \( s \)-plane shown in Fig.1 [14].

The singularities include the pole of the phonon mode at \( s = i0 \) as well as the branch cuts, extending from \( s = \pm iE_{th} \) to \( s = \pm i\infty \) due to the possibility of breaking a condensate into fermionic excitations. The solutions to equations (8) are obtained as

\[
\theta_{\Delta}(t) = \frac{\mathcal{M}_{\theta \Delta \Delta}(0)}{\mathcal{D}(0)} \delta_{\theta_{\Delta}} - \frac{2}{\pi} \int_{E_{th}}^{\infty} dw \frac{\mathcal{D}_R(\omega) \mathcal{M}_{\Delta \Delta \Delta}(\omega) - \mathcal{D}_I(\omega) \mathcal{M}_{\Delta \Delta R}(\omega)}{\mathcal{D}_R^2(\omega) + \mathcal{D}_I^2(\omega)} \sin[\omega t] \delta_{\theta_{\Delta}} \\
+ \frac{2}{\pi} \int_{E_{th}}^{\infty} dw \frac{\mathcal{D}_R(\omega) \mathcal{M}_{\Delta \Delta \Delta}(\omega) - \mathcal{D}_I(\omega) \mathcal{M}_{\Delta \Delta R}(\omega)}{\mathcal{D}_R^2(\omega) + \mathcal{D}_I^2(\omega)} \sin[\omega t] \delta_{\Delta},
\]

(10)

\[
\delta |\Delta|(t) = -\frac{\mathcal{M}_{\theta \Delta \Delta}(0)}{\mathcal{D}(0)} \delta_{\theta_{\Delta}} - \frac{2}{\pi} \int_{E_{th}}^{\infty} dw \frac{\mathcal{D}_R(\omega) \mathcal{M}_{\theta \Delta \Delta \Delta}(\omega) - \mathcal{D}_I(\omega) \mathcal{M}_{\theta \Delta \Delta R}(\omega)}{\mathcal{D}_R^2(\omega) + \mathcal{D}_I^2(\omega)} \sin[\omega t] \delta_{\Delta} \\
- \frac{2}{\pi} \int_{E_{th}}^{\infty} dw \frac{\mathcal{D}_R(\omega) \mathcal{M}_{\theta \Delta \Delta \Delta}(\omega) - \mathcal{D}_I(\omega) \mathcal{M}_{\theta \Delta \Delta R}(\omega)}{\mathcal{D}_R^2(\omega) + \mathcal{D}_I^2(\omega)} \sin[\omega t] \delta_{\theta_{\Delta}}.
\]

(11)

From Eq. (9) \( \mathcal{D}_{R,I} \) are found to be

\[
\mathcal{D}_R(\omega) = [\mathcal{M}_{\theta \Delta \Delta \Delta} R \mathcal{M}_{\Delta \Delta R} - \mathcal{M}_{\theta \Delta \Delta \Delta} I \mathcal{M}_{\Delta \Delta I}] \\
- \mathcal{M}_{\theta \Delta \Delta \Delta} R \mathcal{M}_{\theta \Delta \Delta I},
\]

\[
\mathcal{D}_I(\omega) = [\mathcal{M}_{\theta \Delta \Delta \Delta} R \mathcal{M}_{\Delta \Delta I} + \mathcal{M}_{\theta \Delta \Delta \Delta} I \mathcal{M}_{\Delta \Delta R}] \\
- 2 \mathcal{M}_{\theta \Delta \Delta \Delta} R \mathcal{M}_{\theta \Delta \Delta I}(\omega).
\]

(13)

Upon turning on the perturbation \( \delta \theta \) (\( \delta_{\Delta} \)), since the phase mode is generally coupled to the amplitude mode, it can drive not only the phase (amplitude) modes but

\[
\mathcal{M}(s = i\omega + \epsilon) = \mathcal{M}_R(\omega) \pm i \mathcal{M}_I(\omega) \\
\mathcal{D}(s = i\omega + \epsilon) = \mathcal{D}_R(\omega) \pm i \mathcal{D}_I(\omega).
\]

(12)
In addition to the pole at \( s = i0 \) due to the phonon, the cuts encompass all singularities in a counterclockwise way. In addition to the pole at \( s = i0 \) due to the phonon, the cuts are displayed by a wiggly line.

also the amplitude (phase) modes away from their initial equilibrium values. Because of this coupling the perturbed modes relax to their respective saturated values given above. However, since the phase fluctuations can couple to the density fluctuations, we can observe this collective phase mode from the spectrum of the density-density correlation function (see Ref. [15] for details). Here we merely focus on the real-time behavior of the collective amplitude mode that can be observed by experimentally time-resolved techniques [9]. The main results below follow directly from Eqs. (11).

A. The BCS regime

In the BCS regime for \( \mu > 0 \), the threshold energy is \( E_{th} = 2|\Delta_0| \). The real and imaginary parts of the \( \mathcal{M} \)s are summarized as follows:

\[
\mathcal{M}_{\theta \Delta \theta \Delta}(\omega) = \frac{m}{2\pi^2}(2m\epsilon_F)^\frac{1}{2}|\Delta_0|^2 I_1(\tilde{\omega}, |\tilde{\mu}|, |\tilde{\Delta}_0|); \\
\mathcal{M}_{\theta \Delta \theta \Delta}(\omega) = \frac{m}{2\pi^2}(2m\epsilon_F)^\frac{1}{2} \frac{|\Delta_0|}{2}\times(J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) - J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|)); \\
\mathcal{M}_{\theta \Delta \Delta}(\omega) = \frac{m}{2\pi^2}(2m\epsilon_F)^\frac{1}{2}\frac{|\Delta_0|}{2}\times(J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) - J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|)); \\
\mathcal{M}_{\Delta \Delta}(\omega) = 4 \left( \frac{\tilde{\omega}^2}{4|\Delta_0|^2} - 1 \right) \mathcal{M}_{\theta \Delta \theta \Delta}(\omega).
\]

where

\[
I_1(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) = \mathcal{P} \int_0^\infty dx \frac{-x^2}{(x - \tilde{\mu})^2 + |\tilde{\Delta}_0|^2 \sqrt{\tilde{\omega}^2 - 4((x - \tilde{\mu})^2 + |\tilde{\Delta}_0|^2)}},
\]

\[
J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) = -\pi \theta(\tilde{\omega} - 2|\tilde{\Delta}_0|) \left\{ \left( \tilde{\omega}/2 \right)^2 - |\tilde{\Delta}_0|^2 + \tilde{\mu} \right\}^{\frac{1}{2}} \pm \theta(2\sqrt{\tilde{\mu}^2 + |\tilde{\Delta}_0|^2} - \tilde{\omega}) \left[ \tilde{\mu} - \left( \tilde{\omega}/2 \right)^2 - |\tilde{\Delta}_0|^2 \right]^{\frac{1}{2}}
\]

\[
J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) = \frac{m}{2\pi^2}(2m\epsilon_F)^\frac{1}{2}\frac{|\Delta_0|}{2}\times(J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) - J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|));
\]

\[
\mathcal{M}_{\theta \Delta \Delta}(\omega) = \frac{m}{2\pi^2}(2m\epsilon_F)^\frac{1}{2}\frac{|\Delta_0|}{2}\times(J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|) - J_{-1}(\tilde{\omega}, \tilde{\mu}, |\tilde{\Delta}_0|));
\]

\[
\mathcal{M}_{\Delta \Delta}(\omega) = 4 \left( \frac{\tilde{\omega}^2}{4|\Delta_0|^2} - 1 \right) \mathcal{M}_{\theta \Delta \theta \Delta}(\omega).
\]

We define \( \tilde{\omega} = \omega/\epsilon_F, \tilde{\mu} = \mu/\epsilon_F, |\tilde{\Delta}_0| = |\Delta_0|/\epsilon_F, \) and the symbol \( \mathcal{P} \) means that the principal value of the integral is taken. The damping dynamics of the collective motion, particularly at late times, largely depends on the behavior of the \( \mathcal{M}(\omega)\)s for \( \omega \) near the threshold energy, i.e. \( \omega = E_{th} + 0^+ \). These are found to be

\[
\mathcal{M}_{\theta \Delta \theta \Delta}(\omega = E_{th} + 0^+) \simeq \tilde{\mathcal{M}}_{\theta \Delta \theta \Delta} R; \\
\mathcal{M}_{\theta \Delta \Delta}(\omega = E_{th} + 0^+) \simeq \tilde{\mathcal{M}}_{\theta \Delta \Delta} I \left( \frac{\omega}{E_{th}} - 1 \right)^{-\frac{1}{2}};
\]

\[
\mathcal{M}_{\Delta \Delta}(\omega = E_{th} + 0^+) \simeq 8\tilde{\mathcal{M}}_{\theta \Delta \theta \Delta} R \left( \frac{\omega}{E_{th}} - 1 \right)^{-\frac{1}{2}};
\]

\[
\mathcal{M}_{\Delta \Delta}(\omega = E_{th} + 0^+) \simeq 8\tilde{\mathcal{M}}_{\theta \Delta \theta \Delta} I \left( \frac{\omega}{E_{th}} - 1 \right)^{-\frac{1}{2}}.
\]
with

\[ \mathcal{M}_{\theta \Delta R} = -\frac{m}{2\pi}(2m\mu)^{\frac{1}{2}} \left( 1 + \frac{\sqrt{\mu^2 + |\Delta_0|^2}}{\mu} \right) ; \]

\[ \mathcal{M}_{\theta \Delta t} = -\frac{m}{2\pi}(2m\mu)^{\frac{1}{2}} \left( \sqrt{2} \ln \left[ \frac{1 + \sqrt{\mu^2 + |\Delta_0|^2}}{1 - \sqrt{\mu^2 + |\Delta_0|^2}} \right] \right) ; \]

\[ \mathcal{M}_{\theta \Delta R}^\Delta = -\frac{m}{2\pi}(2m\mu)^{\frac{1}{2}} \frac{\sqrt{2}}{8} \ln \left[ \frac{1 + \sqrt{\mu^2 + |\Delta_0|^2}}{1 - \sqrt{\mu^2 + |\Delta_0|^2}} \right] ; \]

\[ \tilde{\mathcal{M}}_{\theta \Delta R}^\Delta = -\frac{m}{2\pi}(2m\mu)^{\frac{1}{2}} \frac{\sqrt{2}}{8} \ln \left[ \frac{1 + \sqrt{\mu^2 + |\Delta_0|^2}}{1 - \sqrt{\mu^2 + |\Delta_0|^2}} \right] . \]

(18)

As expected, \( \mathcal{M}_{\theta \Delta R} \) and \( \mathcal{M}_{\theta \Delta t} \) vanish in the deep BCS regime in which \( \mu \gg |\Delta_0| \), resulting from particle-hole symmetry. The well-known divergence on the BCS density of the state at threshold energy \( E_{th} = 2|\Delta_0| \) [4] renders \( \mathcal{M}_{\theta \Delta t}(\omega = E_{th} + 0^+) \) singular. Together with the behavior of \( \mathcal{M}_{\Delta R}(I_t)(\omega = E_{th} + 0^+) \) given by Eqs. (17), which shows their vanishing at threshold energy, it gives \( D_{R,t} \), defined in the expressions [13], as follows:

\[ D_R(\omega = E_{th} + 0^+) \simeq D_R ; \]

\[ D_t(\omega = E_{th} + 0^+) \simeq D_t \left( \frac{\omega}{E_{th}} - 1 \right)^{\frac{1}{2}} , \]  

(19)

where

\[ \bar{D}_R = -8\bar{M}_{\theta \Delta R}^2 - \bar{M}_{\theta \Delta t}^2 . \]

As seen from Eq. (8), the pole at \( s = i0 \) seems fictitious with respect to the amplitude perturbation [4], only becoming a true pole when an additional phase perturbation is turned on. It is the terms involving the singular behaviour of \( \mathcal{M}_{\theta \Delta t} \) at \( \omega \) near the threshold energy that give the dominant contributions to the late-time dynamics of the amplitude mode. Thus, one finds that the integrands in Eq. (11) behave as

\[ \frac{D_R M_{\theta \Delta t}-D_t M_{\theta \Delta R}}{D_R^2 + D_t^2} \]

\[ \simeq \frac{\mathcal{M}_{\theta \Delta t} (\omega = E_{th} + 0^+)}{D_R} \left( \frac{\omega}{E_{th}} - 1 \right)^{\frac{1}{2}} , \]

(21)

\[ \frac{D_R M_{\theta \Delta t}-D_t M_{\theta \Delta R}}{D_R^2 + D_t^2} \]

\[ \simeq D_R M_{\theta \Delta t} - D_t M_{\theta \Delta R} \left( \frac{\omega}{E_{th}} - 1 \right)^{\frac{1}{2}} , \]

(22)

These two terms in turn will lead to the different damping behaviour after carrying out the integration over \( \omega \).

Putting all this together, we find the late-time solutions as

\[ \delta |\Delta| (t) \simeq -\frac{M_{\theta \Delta R}(0)}{D(0)} \delta_\theta - \frac{2}{2\sqrt{\pi}} \frac{E_{th}}{D_R} \frac{E_{th}^{1/2}}{t^{1/2}} \cos(E_{th}t - \pi/4) \delta_\Delta - \frac{1}{\sqrt{\pi}} \frac{\bar{D}_R \bar{M}_{\theta \Delta R}^\Delta - \bar{D}_t \bar{M}_{\theta \Delta R}}{\bar{D}_R^2} \frac{1}{E_{th}^{3/2}} \frac{E_{th}^{3/2}}{t^{3/2}} \cos(E_{th}t + \pi/4) \delta_\theta , \]

(23)

In the BCS regime, \( \delta |\Delta| (t) \) decays at late times dominantly as \( t^{-1/2} \) [4] due to the amplitude perturbation while it has a subdominant decay as \( t^{-3/2} \) given by the perturbation \( \delta_\theta \).

Additionally, if it were possible to implement small phase perturbations, \( \delta |\Delta| (t) \) decays to a non-zero value proportional to the perturbation \( \delta_\theta \), resulting from the amplitude-phase coupling. This looks to give a shift on the saturated value that the amplitude mode decays to from its initial value, but this becomes vanishingly small in the deep BCS regime where \( \mathcal{M}_{\theta \Delta R} \simeq 0 \). We might hope that this subdominant behaviour could be made visible by judicious choice of initial perturbations, but we do not know.

B. The BEC regime

In the BEC regime the chemical potential changes its sign, \( \mu < 0 \). The threshold energy now becomes \( E_{th} = \sqrt{2(|\mu|^2 + |\Delta_0|^2)} \). The real and imaginary part of the \( \mathcal{M} \)s obtained from Eqs. (12) are given by

\[ M_{\theta \Delta R}(\omega) = \frac{m}{2\pi^2} (2m\rho_F)^{\frac{1}{2}} |\Delta_0|^2 J_2(\hat{\omega} - |\hat{\mu}|, |\hat{\Delta_0}|) ; \]

\[ M_{\theta \Delta t}(\omega) = \frac{m}{2\pi^2} (2m\rho_F)^{\frac{1}{2}} \frac{|\hat{\Delta_0}|}{4|\hat{\omega}| \sqrt{(\hat{\omega}/2)^2 - |\hat{\Delta_0}|^2}} \times (J_2(\hat{\omega} - |\hat{\mu}|, |\hat{\Delta_0}|) - J_2(-\hat{\omega} - |\hat{\mu}|, |\hat{\Delta_0}|)) ; \]

\[ \bar{D}_t = 16\bar{M}_{\theta \Delta R} \bar{M}_{\theta \Delta t} - 2\bar{M}_{\Delta R} \bar{M}_{\Delta t} (20) \]
\[
\mathcal{M}_{\theta\Delta} R(\omega) = \frac{m}{2\pi^2} (2m\epsilon_F)^{\frac{3}{2}} |\Delta_0| I_2(\hat{\omega}, -|\mu|, |\Delta_0|) ;
\]
\[
\mathcal{M}_{\theta\Delta} \Delta(\omega) = \frac{m}{2\pi^2} (2m\epsilon_F)^{\frac{3}{2}} \frac{1}{2|\omega|} \left|\Delta_0\right| \times (J_2(\hat{\omega}, -|\mu|, |\Delta_0|) - J_2(-\hat{\omega}, -|\mu|, |\Delta_0|)) ;
\]
\[
\mathcal{M}_{\Delta} R(\omega) = 4 \left( \frac{\hat{\omega}^2}{4|\Delta_0|^2} - 1 \right) \mathcal{M}_{\theta\Delta} \Delta R(\omega) ,
\]
(24)

where
\[
J_2(\hat{\omega}, |\mu|, |\Delta_0|) = -\pi \theta [\hat{\omega} - 2\sqrt{|\mu|^2 + |\Delta_0|^2}] \times \left[ \sqrt{\left(\hat{\omega}/2\right)^2 - |\Delta_0|^2} - |\mu| \right]^{\frac{3}{2}}
\]
(25)
together with Eqs. (14), (16). For \( \omega \) near the threshold energy, the \( \mathcal{M}s \) can be simplified as
\[
\mathcal{M}_{\theta\Delta} \Delta R(\omega = E_{th} + 0^+) \simeq \mathcal{M}_{\theta\Delta} \Delta R ;
\]
\[
\mathcal{M}_{\theta\Delta} \Delta(\omega = E_{th} + \epsilon) \simeq \mathcal{M}_{\theta\Delta} \Delta(\omega = E_{th} + 1) \simeq \mathcal{M}_{\theta\Delta} \Delta R ;
\]
\[
\mathcal{M}_{\Delta} \Delta(\omega = E_{th} + 0^+) \simeq 4 \left|\frac{\mu^2}{\Delta_0^2}\right| \mathcal{M}_{\theta\Delta} \Delta R ;
\]
\[
\mathcal{M}_{\Delta} \Delta(\omega = E_{th} + \epsilon) \simeq 4 \left|\frac{\mu^2}{\Delta_0^2}\right| \mathcal{M}_{\theta\Delta} \Delta R ;
\]
(26)

which, in the deep BEC regime, becomes
\[
\mathcal{M}_{\theta\Delta} \Delta R \simeq \frac{m}{2\pi^2} (2m\mu)^{\frac{3}{2}} \left( 2 - \sqrt{2} \right) \frac{\pi}{8} \left|\frac{\Delta_0^2}{|\mu|^2}\right| ;
\]
\[
\mathcal{M}_{\theta\Delta} \Delta R \simeq \frac{m}{2\pi^2} (2m\mu)^{\frac{3}{2}} \frac{\pi}{8} \left|\frac{\Delta_0^2}{|\mu|^2}\right| ;
\]
\[
\mathcal{M}_{\theta\Delta} \Delta R \simeq \frac{m}{2\pi^2} (2m\mu)^{\frac{3}{2}} \frac{\pi}{4} \left|\frac{\Delta_0^2}{|\mu|^2}\right| .
\]
(27)

Although the behaviour of the above \( \mathcal{M}s \) seems different from that in the BCS regime, \( \mathcal{D}_{R(\omega)} \) follows Eq. (19) with \( \mathcal{D}_{R(\omega)} \) given respectively by
\[
\mathcal{D}_R = 4 \left|\frac{\mu^2}{\Delta_0^2}\right| \mathcal{M}_{\theta\Delta} \Delta R - \mathcal{M}_{\theta\Delta} \Delta R ;
\]
\[
\mathcal{D}_I = 8 \left|\frac{\mu^2}{\Delta_0^2}\right| \mathcal{M}_{\theta\Delta} \Delta R \mathcal{M}_{\theta\Delta} \Delta R - 2 \mathcal{M}_{\theta\Delta} \Delta R \mathcal{M}_{\theta\Delta} \Delta R \mathcal{M}_{\theta\Delta} \Delta R .
\]
(28)

So, as \( \omega \) is near the threshold energy, the integrands in Eq. (11) can be approximated by
\[
\frac{\mathcal{D}_R \mathcal{M}_{\theta\Delta} \Delta R - \mathcal{D}_I \mathcal{M}_{\theta\Delta} \Delta R}{\mathcal{D}_R^2 + \mathcal{D}_I^2} \left|\omega = E_{th} + 0^+\right|
\]
\[
\simeq \frac{\mathcal{D}_R \mathcal{M}_{\theta\Delta} \Delta R - \mathcal{D}_I \mathcal{M}_{\theta\Delta} \Delta R}{\mathcal{D}_R^2 + \mathcal{D}_I^2} \left( \frac{\omega}{E_{th} - 1} \right) \left|\omega = E_{th} + 0^+\right|
\]
\[
\simeq \frac{\mathcal{D}_R \mathcal{M}_{\theta\Delta} \Delta R - \mathcal{D}_I \mathcal{M}_{\theta\Delta} \Delta R}{\mathcal{D}_R^2 + \mathcal{D}_I^2} \left( \frac{\omega}{E_{th} - 1} \right) \left|\omega = E_{th} + 0^+\right|
\]
(29)

The same damping behaviour will be seen from the above two respective contributions.

Putting all these results together now gives the late-time result:
\[
\delta|\Delta| (t) \simeq - \frac{\mathcal{M}_{\theta\Delta} \Delta(0)}{\mathcal{D}(0)} \delta_{\theta} - \frac{1}{\sqrt{\pi}} \frac{\mathcal{D}_R \mathcal{M}_{\theta\Delta} \Delta R - \mathcal{D}_I \mathcal{M}_{\theta\Delta} \Delta R}{\mathcal{D}_R^2 - \mathcal{D}_I^2} \left( \frac{\omega}{E_{th} - 1} \right) \left|\omega = E_{th} + \pi/4\right| \delta_{\Delta}
\]
\[
- \frac{1}{\sqrt{\pi}} \frac{\mathcal{D}_R \mathcal{M}_{\theta\Delta} \Delta R - \mathcal{D}_I \mathcal{M}_{\theta\Delta} \Delta R}{\mathcal{D}_R^2 - \mathcal{D}_I^2} \left( \frac{\omega}{E_{th} - 1} \right) \left|\omega = E_{th} + \pi/4\right| \delta_{\Delta}.
\]
(31)

Note that \( E_{th} = \sqrt{2(|\mu|^2 + |\Delta_0|^2)} \) in the BEC regime. The amplitude mode decays as \( t^{-3/2} \) in both oscillating terms, but we note that the coefficient of \( \delta_{\Delta} \) is the larger in the deep BEC regime, as seen from Eqs. (27) in which \( \mathcal{M}_{\theta\Delta} \Delta \gg \mathcal{M}_{\theta\Delta} \Delta \) for \( |\mu| \gg |\Delta_0| \). Insofar as it is possible to implement an initial phase shift \( \delta|\Delta| (t) \) then decays to a saturated value with a nonzero shift from the initial condition. The significance of introducing the phase perturbation is seen from the two results above.

In summary, our calculation of the generic \( t^{-1/2} \) damping in the BCS superconductor regime and the \( t^{-3/2} \) damping in the BEC molecular regime reproduces the results of [4] obtained by independent methods. As in [4], we have seen that the transition between the two regimes of the different damping behaviours occurs at \( \mu = 0 \). However, for \( \delta_{\theta} \neq 0 \), there is a residual displace-
ment of $\delta|\Delta|(t)$ in the BEC regime.

IV. NARROW RESONANCE

The situation is rather different when the Feshbach resonance of bound bosons is very narrow. We now have a two-channel model in which the resonance has to be taken into account explicitly. To illustrate this, for simplicity we consider the idealised case of $U = 0$ in the above Lagrangian where the effects from the background scattering length $a_{\text{bg}}$ are ignored as long as the system is away from the deep BCS regime. It leads to $\Delta = -g\phi$ from Eq. (3), so the field variables of the collective modes then are the phase $\theta_\phi$ and the amplitude $\delta|\phi|$

On perturbing the system the collective phenomena (in the same notation as before) are described by the following equations:

$$
-s^2 M_{\theta_\phi\phi}(s^2) \theta_\phi + s M_{\theta_\phi\phi}(s^2) g \delta|\phi| = \delta \theta_\phi,
-s M_{\theta_\phi\phi}(s^2) \theta_\phi + M_{\phi\phi}(s^2) g \delta|\phi| = \delta \phi. \tag{32}
$$

All $\mathcal{M}$s here can be obtained from those of the broad resonance together with the identification of $|\Delta_0| = g|\phi_0|$ as follows:

$$
\mathcal{M}_{\theta_\phi\theta_\phi} = \mathcal{M}_{\theta_\phi\Delta}, \quad \mathcal{M}_{\theta_\phi\phi} = \mathcal{M}_{\theta_\phi\Delta} + 2|\phi_0|;
\mathcal{M}_{\phi\phi} = 2(2\mu - \nu) + \mathcal{M}_{\Delta\Delta}. \tag{33}
$$

The behaviour of $\mathcal{M}$s at $\omega$ near the threshold energy can then be read off from the results in the broad resonance case above and the corresponding behaviour of the added terms.

The presence of the constant inhomogeneous terms in $\mathcal{M}_{\theta_\phi\phi}$ and $\mathcal{M}_{\phi\phi}$ in (33) allows for different damping behaviour from that seen previously and we shall see that these differences are realized.

A. The BCS regime

For $\mu > 0$ in the BCS regime, together with Eqs. (17), $\mathcal{M}_{\phi\phi R}$ and $\mathcal{M}_{\theta_\phi\phi R}$ follow

$$
\mathcal{M}_{\phi\phi R}(\omega = E_{\text{th}} + 0^+) \simeq \tilde{\mathcal{M}}_{\phi\phi R} = 2(2\mu - \nu); \tag{34}
$$

whereas the behaviour of the other $\mathcal{M}$s remains the same as in Eqs. (17) and (18) in which the labels $\theta_\Delta$ and $\Delta$ are replaced respectively by the corresponding $\theta_\phi$ and $\phi$, and $|\Delta_0|$ is changed to $g|\phi_0|$. For $2\mu \neq \nu$ we have

$$
\mathcal{M}_{\theta_\phi\phi R}(\omega = E_{\text{th}} + 0^+) \neq 0. \tag{35}
$$

In particular, in addition to $\mathcal{M}_{\theta_\phi\theta_\phi I}$, $\mathcal{M}_{\phi\phi R}$ now becomes singular as $\omega$ is near the threshold energy. Thus, the terms involving either $\mathcal{M}_{\theta_\phi\theta_\phi I}$ or $\mathcal{M}_{\phi\phi R}$ will become dominant in determining the late-time dynamics of the amplitude mode. We then find that the integrands in the corresponding equation to Eq. (11), in terms of the variables $\theta_\phi$ and $\delta\phi$, can be given by

$$
\begin{align*}
\mathcal{D}_R \mathcal{M}_{\theta_\phi\theta_\phi I} - \mathcal{D}_I \mathcal{M}_{\theta_\phi\theta_\phi R} &\simeq \frac{\mathcal{D}_R \mathcal{M}_{\theta_\phi\theta_\phi I} - \mathcal{D}_I \mathcal{M}_{\theta_\phi\theta_\phi R}}{\mathcal{D}_R^2 + \mathcal{D}_I^2} \left( \frac{\omega}{E_{\text{th}}} - 1 \right)^{1/2} \tag{36}
\end{align*}
$$

In general, the damping behaviour of the amplitude mode at late times then becomes
That is, the decay of the amplitude mode in the narrow resonance follows the power-law $t^{-3/2}$, prior to reaching a saturated value in the BCS regime. This is different from the broad resonance result quoted earlier where the amplitude mode decays dominantly as $t^{-1/2}$ at late times instead. The only exception to this decay behaviour of the amplitude mode arises when the unitary limit (infinite scattering length) $2\mu = \bar{v}$ occurs in the regime of positive chemical potential $\mu > 0$. We then have $M_{\phi \theta R}(\omega)$ of (33) vanishing linearly at $\omega$ near the threshold energy $E_{th}$, i.e., $M_{\phi \theta R}(\omega = E_{th} + 0^+) \approx \left( \frac{\omega}{E_{th}} - 1 \right)$, which gives the same $t^{-1/2}$ behaviour as its general counterpart in the broad resonance given in Eqs. (17).

This finding of $t^{-3/2}$ damping behaviour in the BCS regime seems to contradict the conclusion in [4] in which the same damping dynamics is found in both broad (one-channel model) and narrow (two-channel model) resonances. However, our idealization of narrow resonances for which we have taken the contact interaction strength $U = 0$ in [1] cannot provide a valid description in the regime of deep BCS. For a more general two-channel model with a nonzero $U$, characterized by the background scattering length, $U_{off} \approx U$ in the deep BSC regime [9][10]. Thus, the damping behaviour of [4] will be recovered in the deep BCS regime with a power law $t^{-1/2}$ decay, the same result as for the one-channel model, although the details are much messier. The outcome would be that, although this term might have a small coefficient, it has the dominant long-time behaviour. It should be possible to see the otherwise subdominant $t^{-3/2}$ behaviour in the BSC regime if $U \ll g^2/(2\mu - \bar{v})$.

\[ g\delta|\phi|(t) \approx - \frac{M_{\theta \phi \theta}(0)}{D(0)} \delta_{\theta \phi} - \frac{1}{\sqrt{\pi}} \frac{D_R M_{\theta \phi \theta \phi} - D_I M_{\theta \phi \theta \phi}}{D_R^2} E_{th}^{3/2} t^{3/2} \cos[E_{th}t + \pi/4] \delta_{\theta \phi} - \frac{1}{\sqrt{\pi}} \frac{D_R M_{\theta \phi \theta \phi} - D_I M_{\theta \phi \theta \phi}}{D_R} E_{th}^{3/2} t^{3/2} \cos[E_{th}t + \pi/4] \delta_{\theta \phi}. \] (44)

The outcome, that the damping behaviour is $t^{-3/2}$, as for broad resonances, is consistent with the argument in [4] but, yet again, there is the possibility of the condensate relaxing to a displaced value.

V. CONCLUSIONS

In this paper we determined the time evolution of a condensate of cold Fermi atoms, whose interaction can be tuned by a Feshbach resonance throughout the BCS and BEC regimes, in response to (small) non-equilibrium perturbations of the phase and amplitude modes. Interaction between the collective modes and the constituent particles is key for our understanding of the relational dynamics, which occurs in a collisionless regime via the Landau damping effect.

In particular, we have focussed on the evolution of the collective amplitude mode. For a broad resonance the amplitude mode decays as $t^{-1/2}$ at late times in the BCS regime with positive chemical potential whereas it decays as $t^{-3/2}$ in the BEC regime with negative chemical potential. Our conclusions, although obtained by different means, agree with those in the literature [4]. However, when the resonance is very narrow we disagree with [4], which sees no difference between the behavior for broad or narrow resonances. We find that for an idealized narrow Feshbach resonance the collective mode decays as $t^{-3/2}$ throughout both the BCS and BEC regimes.
regimes. Nonetheless, we expect that the deviation from idealized narrow resonances restores the original longtime behaviour in the deep BCS regime. The $t^{-3/2}$ behavior can possibly be seen experimentally in the BCS regime when the effect of the background scattering length can be largely ignored. More robustly, beyond the results of [4] the condensate amplitude can decay to a value shifted from its initial condition, arising from the perturbation on the phase mode due to the amplitude-phase coupling. The results can be tested experimentally.

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[1] M. Greiner, C. A. Regal, and D. S. Jin, Nature 426, 537 (2003); S. Jochim et al. Science 302 (2003); M. W. Zwierlein et al., Phys. Rev. Lett. 91, 250401 (2003).
[2] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. 92, 040403 (2004); M. W. Zwierlein et al., Phys. Rev. Lett. 92, 120403 (2004);
[3] C. Chin et al., Science 305, 1128 (2004); Y. Shin et al., Nature (London) 451, 689 (2008).
[4] V. Gurarie, Phys. Rev. Lett. 103, 075301 (2009).
[5] A. F. Volkov and S. M. Kogan, Zh. Eksp. Teor. Fiz. 65, 2038 (1973) [Sov. Phys. JETP 38, 1018 (1974)].
[6] E. A. Yuzbashyan and M. Dzero, Phys. Rev. Lett. 96, 230404 (2006).
[7] R. A. Barankov and L. S. Levitov, Phys. Rev. Lett. 96, 230403 (2006).
[8] A. Bulgac and S. Yoon, Phys. Rev. Lett. 102, 085302 (2009).
[9] D-S. Lee, C-Y. Lin and R. J. Rivers, Phys. Rev. Lett. 98, 020603 (2007) and references therein.
[10] D-S. Lee, C-Y. Lin and R. J. Rivers, Phys. Rev.A 80, 043621 (2009).
[11] I. J. R. Aitchison, P. Ao, D. J. Thouless and X.-M. Zhu, Phys. Rev. B 51, 6531 (1995).
[12] S. Georgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys. 80 1215 (2008).
[13] C. A. R. Sa de Melo, M. Randeria and J. R. Engelbrecht, Phys. Rev. Lett. 71, 3202 (1993).
[14] S.-Y. Wang, D. Boyanovsky, D.-S. Lee, H.-L. Yu, and S. M. Alamoudi, Annal Phys. 300, 1 (2002).
[15] Y. Ohashi and A. Griffin, Phys. Rev. A 67, 063612 (2003).
[16] A.V. Andreev, V. Gurarie and L. Radzihovsky, Phys. Rev. Lett. 93, 130402 (2004).
[17] V. Gurarie and L. Radzihovsky, Ann. Phys. 322, 2 (2007).