GROUP ACTIONS ON DG-MANIFOLDS AND THEIR RELATION TO EQUIVARIANT COHOMOLOGY

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Abstract. Let $G$ be a Lie group acting by diffeomorphisms on a manifold $M$ and consider the image of $T[1]G$ and $T[1]M$ of $G$ and $M$ respectively in the category of differential graded manifolds. We show that the obstruction to lift the action of $T[1]G$ on $T[1]M$ to an action on a $\mathbb{R}[n]$-bundle over $T[1]M$ is measured by the $G$ equivariant cohomology of $M$. We explicitly calculate the differential graded Lie algebra (dgla) of the symmetries of the $\mathbb{R}[n]$-bundle over $T[1]M$ and we use this dgla to understand which actions are hamiltonian.

1. Introduction

A differential graded manifold (dg-manifold) is a non-negatively graded supermanifold endowed with a homology vector field; they are also known in the literature by the name of NQ-manifolds (cf. [28, 1]). The category of dg-manifolds provides a framework on which one can study geometrical structures on manifolds, and at the same time it incorporates the tools, methods and scope of rational homotopy theory (cf. [31]).

For several geometrical structures, the use of the category of dg-manifolds to study them, might be a matter of the researcher’s personal taste. But in the case of Courant algebroids over manifolds, the use of the category of dg-manifolds to study them has provided us with the appropriate framework to understand their properties [24].

Of particular interest are the exact Courant algebroids, bundles $E$ over a manifold $M$ which sit in the middle of the exact sequence $0 \to T^*M \to E \to TM \to 0$ and that are provided with a nondegenerate inner product $\langle \cdot, \cdot \rangle$ and a bracket $[\cdot, \cdot]$ satisfying some coherence conditions generalizing the ones of a Lie algebroid (see [26]). These structures were developed by Courant in order to study symplectic, Poisson and foliation structures from the point of view of Dirac structures, i.e. maximal isotropic subbundles $L \subset E$ which are involutive with respect to the bracket (see [11]). Later on, the complex counterpart of a Dirac structure was carefully studied by [9, 15, 17] and the Generalized Complex manifolds were born.

Several authors [7, 13, 22, 20] studied hamiltonian group actions on generalized complex manifolds and furthermore proved that the appropriate concept of reduction holds for this type of geometrical structure. One of the key points of the construction of the reduction procedure was to find the conditions on which a group acts by hamiltonian symmetries on a generalized complex manifold. In
the papers cited above, the conditions were given for the action of a compact Lie group, but the general framework in order to study more general type of actions was not developed. We believe that the present paper provides the appropriate framework to understand the actions of Lie groups (not necessarily compact) on Exact Courant algebroids.

The key observation that triggered the study of the symmetries of $\mathbb{R}[n]$-bundles over $T[1]M$ is the following: If $E$ is an exact Courant algebroid with a splitting $E \cong T^*M \oplus TM$ whose curvature form is the degree three closed form $H$, then the differential graded Lie algebra of symmetries of $E$, is isomorphic to the differential graded Lie algebra $\mathfrak{sym}^*(P, Q)$ of symmetries of the homology vector field $Q = d + H\partial_t$ in the dg-manifold $(\mathbb{R}[2] \oplus T[1]M, Q)$.

The previous fact is not difficult to prove if one uses the approach developed by Roytenberg in [24] to understand Courant algebroids. In his description, the information that defines a Courant algebroid structure over the Euclidean vector bundle $E$ over $M$, is encoded in a cubic Hamiltonian $\Theta$ on the minimal symplectic realisation of $E[1]$ that satisfies the master equation $\{\Theta, \Theta\}$ with respect to the Poisson bracket. The symmetries of such cubic Hamiltonian $\Theta$ becomes the differential graded algebra generated by the degree 0 and degree 1 functions, together with the degree 2 functions whose bracket with $\Theta$ is zero, with differential $\{\cdot, \cdot\}$ and bracket $\{\cdot, \cdot\}$. In the case of an exact Courant algebroid with curvature form $H$, the dgla of symmetries of the associated cubic Hamiltonian $\Theta$ is indeed isomorphic to the symmetries $\mathfrak{sym}^*(P, Q)$ of $Q = d + H\partial_t$ in the dg-manifold $(\mathbb{R}[2] \oplus T[1]M, Q)$ (see section 3). Therefore, to understand Lie group actions on exact Courant algebroids is equivalent to understand Lie group actions on $\mathbb{R}[n]$ bundles over $T[1]M$ in the category of dg-manifolds.

In this paper we decided to focus our efforts on studying the properties of Lie group actions on $\mathbb{R}[n]$-bundles over $T[1]M$, and we have left the applications to the symmetries of Exact Courant algebroids to a forthcoming publication. The reason on which we based this decision was the fact that the study of the symmetries of $\mathbb{R}[n]$-bundles over $T[1]M$ became interesting on its own, and that the results that we present on this paper deserved to be presented in an independent manner.

The main result of this paper is Definition 3.18 on which we encode the conditions under which a Lie group $G$ acts by symmetries on a $\mathbb{R}[n]$-bundle over $T[1]M$. This definition is based on the result of Theorem 3.17 where we show that the differential forms of the model for equivariant cohomology defined by Getzler [14] encode the appropriate conditions for the group $G$ to act globally, and the conditions on its Lie algebra to act by infinitesimal symmetries on the $\mathbb{R}[n]$-bundle over $T[1]M$. To prove this theorem we first calculate the dgla $\mathfrak{sym}^*(P, Q)$ of symmetries of $P$, a $\mathbb{R}[n]$-bundle over $T[1]M$, with homology vector field $Q = d + H\partial_t$, and then we show that isomorphism classes of $L_\infty$ maps from the Lie algebra $(\mathfrak{g}[1] \to \mathfrak{g})$ of $T[1]G$ to $\mathfrak{sym}^*(P, Q)$, are in 1-1 correspondence with elements in the cohomology group $H^{n+1}(\mathfrak{g}[1] \to \mathfrak{g}, \Omega^\bullet M)$ (Proposition 3.15). We prove then that the cohomology of $(\mathfrak{g}[1] \to \mathfrak{g})$ with coefficients in $\Omega^\bullet M$ is precisely the image under the Van Est type map of the equivariant cohomology, and therefore we conclude that the equivariant cohomology of Getzler encodes the appropriate conditions in order to define an action of $G$ on $\mathbb{R}[n]$-bundles over $T[1]M$. 
In section 4 we study the infinitesimal information of the action of $T^1 G$ on a $\mathbb{R}[n]$-bundle, and we propose two approaches in order to study Hamiltonian actions. The first approach says that the action is Hamiltonian whenever the infinitesimal map $(g[1] \to g) \to \text{sym}^\ast(P, Q)$ is a strict map of dgla’s; the second approach says that the action is hamiltonian whenever the induced map from $g$ to the derived algebra of $\text{gsym}^\ast(P, Q)$ is a map of Leibniz algebras. We write the equations that each approach must satisfy and we show some examples were they have been used. We remark that both approaches are equivalent when $n = 1$, and if $H$ is symplectic, they are equivalent to the existence of a moment map.

We included section 2 in order to put in a topological framework the situations studied in sections 3 and 4. That is, we study actions of groups on principal $K$-bundles, whenever the homotopy type of the group $K$ is the one of a $K(\mathbb{Z}, n)$. We introduce the set $\text{Bun}_G(K; M)$ of isomorphism classes of $G$-equivariant principal $K$-bundles over $M$ and we compare it to the set $[EG \times_G M, BK]$ of isomorphism classes of principal $K$-bundles over $M$ with a homotopy action of $G$; this comparison we do via a canonical map $\Phi : \text{Bun}_G(K; M) \to [EG \times_G M, BK]$. Note that the homotopy type of $BK$ is the one of a $K(\mathbb{Z}, n + 1)$, and therefore $[EG \times_G M, BK] = H^{n+1}(EG \times_G M, \mathbb{Z})$. We see that the isomorphism classes of principal $K$-bundles over $M$ with a homotopy action of $G$ are classified by the $n + 1$ $G$-equivariant cohomology of $M$.

We show that in the case when $K = S^1$, the map $\Phi$ is an isomorphism for $G$ compact and connected (section 2.3.1); and when $K = PU(\mathcal{H})$ we show that $\Phi$ is surjective, and it becomes an isomorphism if one only considers certain $G$-equivariant principal $PU(\mathcal{H})$-bundles on which the local action of the isotropy groups are certain projective representations (Proposition 2.8). We remark that the results of this first section have appeared before in [21, 3] but the exposition and the proofs that we provide are different.

2. Symmetries of principal bundles

When studying actions of groups on topological spaces, one encounters situations on which one would like to lift an action from the base of a principal bundle to the total space. These liftings could be performed (whenever possible) in different manners depending on what one wants to achieve.

In this section we will focus our attention in two types of liftings. The first are the strict liftings, namely the ones on which the group action can be lifted to an action on the total space by principal bundle morphisms. The second are the liftings up to homotopy, on which the action on the base of each element of the group is lifted to a principal bundle morphism on the total space, but whose lifts do not preserve the composition law of the group and discrepancy in the composition of the lifts satisfy some coherence conditions are measured by homotopies.

2.1. Strict symmetries of principal bundles. Let $K$ be a topological group and $P \to M$ a principal $K$ bundle over the compact manifold $M$. The group of symmetries of the principal bundle $P$ will be the subgroup of elements of the homeomorphism group $\text{Homeo}(P)$ that are $K$ equivariant, i.e.

$$\text{Sym}(P) := \{F : \in \text{Homeo}(P) | F \text{ is } K \text{ equivariant}\}.$$
A $K$-equivariant homeomorphism $F$ induces a homeomorphism $f$ on the base space making the following diagram commute

\[
\begin{array}{ccc}
P & \xrightarrow{F} & P \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & M.
\end{array}
\]

Conversely, for any map $f : M \to M$ such that there is an isomorphism $\phi : P \cong f^*P$ of principal $K$-bundles, one can construct a $K$-equivariant homeomorphism $F : P \to P$, $F := \tilde{f} \circ \phi$, where $\tilde{f}$ is the canonical map $\tilde{f} : f^*P \to P$ induced by the pullback that lifts the map $f$.

Therefore the group $\text{Sym}(P)$ sits in the middle of the short exact sequence

\[
1 \longrightarrow \text{Aut}(P) \longrightarrow \text{Sym}(P) \longrightarrow \text{Homeo}_P(M) \longrightarrow 1
\]

where $\text{Aut}(P)$ is the set of symmetries that leave the base fixed, namely the group of gauge transformations, and $\text{Homeo}_P(M)$ is the group of homeomorphisms $f : M \to M$ such that there exists a principal $K$-bundle isomorphism $f^*P \cong P$.

**Definition 2.1.** Let $G$ be a topological group acting on $M$ by homeomorphisms $\phi : G \to \text{Homeo}(M)$. We say that a principal $K$-bundle $P \to M$ is a $G$-equivariant principal $K$-bundle whenever the action of $G$ can be lifted to a homomorphism $\tilde{\phi} : G \to \text{Sym}(P)$ making following diagram commute

\[
\begin{array}{ccc}
\text{Sym}(P) & \xrightarrow{\tilde{\phi}} & \text{Homeo}_P(M) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\phi} & \text{Homeo}_P(M).
\end{array}
\]

Two $G$-equivariant principal $K$-bundles $P, P'$ are isomorphic whenever there is a $G$ equivariant isomorphism $P \cong P'$ of principal $K$-bundles.

**Definition 2.2.** Denote the set of isomorphism classes of $G$-equivariant principal $K$-bundles over $M$ by

\[
\text{Bun}_G(M; K).
\]

This set $\text{Bun}_G(M; K)$ of isomorphism classes is well understood for some specific choices of $K$ and $G$. In the next section we will study the case on which the $G$ action can be lifted to an action on the total space but up to homotopy, and then we will see the relations among the two types of liftings.

**2.2. Symmetries up to homotopy of principal bundles.** In the case that the group $G$ acts by homeomorphisms in $M$ living in $\text{Homeo}_P(M)$ it is easy to see that there are many ways to find maps $\tilde{\phi} : G \to \text{Sym}(P)$ that lift the maps $\phi$, but some of these lifts may fail to be homeomorphism of groups. The discrepancy of a lift to be a homeomorphism could be measured by the gauge transformations that define the compositions

\[
\Phi_{g,h} := \tilde{\phi}(g) \circ \tilde{\phi}(h) \circ \tilde{\phi}((gh)^{-1}) \in \text{Aut}(P)
\]

for all pairs of elements in $G$. But we would like that these gauge transformations would satisfy some sort of cocycle property with respect to some homotopy associated to a triple $(g, h, k)$ of elements in $G$. In the model $\text{Sym}(P)$ of symmetries that
we have previously defined it is not an easy task to carry out the definition of the higher homotopies; instead we change the model for the symmetries for principal $K$-bundles and we take one which is suited for defining higher homotopies.

Let us alternatively consider a principal $K$-bundle as a map $a : M \to BK$ to the classifying space $BK$ of principal $K$-bundles. If the group $G$ acts by homeomorphisms in $\text{Homeo}_p(M)$ then we can find for every $g \in G$ a homotopy between the maps $a$ and $a \circ g$; this information produces a map

$$F_1 : G \to BK^M \times \Delta^1 = \text{Maps}(M \times \Delta^1, BK).$$

But we would like that the homotopies $F_1(g) \circ (Id \times h)$, $F_1(h)$ and $F_1(gh)$ associated to the pair $(g, h)$ be the boundary of a map of the two simplex

$$F_2(g, h) \in BK^M \times \Delta^2$$

in such a way that we are measuring via a homotopy the error by which the lifting of the action of $G$ is not a homomorphism of groups. The discrepancies for the pairs are assembled in a map

$$F_2 : G^2 \to BK^M \times \Delta^2$$

that must satisfy coherence conditions determined by a map

$$F_3 : G^3 \to BK^M \times \Delta^3$$

and so on.

With the previous description in mind we can define when the action of $G$ could be lifted to a homotopy action on the principal $K$-bundle $a : M \to BK$.

**Definition 2.3.** The group $G$ acts on the principal bundle $a : M \to BK$ by homotopies whenever there exists a map of simplicial spaces

$$F_* : N_* G \to \text{Sing}(BK^M)$$

from the nerve $N_* G$ of the group $G$, to the singular cells of $BK^M$, consisting of maps

$$F_0 : \ast \to BK^M \quad \ast \mapsto a$$

$$F_1 : G \to BK^M \times \Delta^1$$

$$F_2 : G^2 \to BK^M \times \Delta^2$$

$$F_n : G^n \to BK^M \times \Delta^n$$

such that $F_0 = a$ and $F_1(h)$ is a homotopy between $a$ and $oh$.

It is easy to see that any such map $F_i$ defines a map

$$\tilde{F}_i : G^i \times M \times \Delta^i \to BK$$

which assembles to a map of topological spaces

$$\tilde{F} : |N_* (G \times M)| \to BK$$

providing a map

$$\tilde{F} : EG \times_G M \to BK.$$

Two principal bundles $a^0, a^1 : M \to BG$ with homotopy actions of $G$ defined by the simplicial maps $F^0$ and $F^1$ respectively, are equivalent whenever the induced maps on spaces

$$\tilde{F}^0, \tilde{F}^1 : EG \times_G M \to BK$$
are homotopy equivalent.

We have then that the equivalence classes of principal $K$-bundles with homotopy actions of $G$ are in 1-1 correspondence with the homotopy classes of maps

$$[EG \times_G M, BK] := \pi_0(\text{Maps}(EG \times_G M, BK)).$$

2.3. **Strict actions vs. homotopy actions.** In general there is no canonical map from the set $\text{Bun}_G(M, K)$ of $G$ equivariant $K$-principal bundles over $M$ to the set $[EG \times_G M, BK]$ of homotopy classes of homotopy actions on principal $K$-bundles.

However in the case of interest for this section, namely when $G$ is a compact Lie group and $K$ has the homotopy type of an Eilenberg-MacLane space $K(\mathbb{Z}, n)$, there is a way to construct the desired map, let us see how.

Because $G$ is a compact Lie group, we can find finite dimensional $G$-manifolds $EG_1 \subset \cdots \subset EG_k \subset EG_{k+1} \subset \cdots$ where $G$ acts freely (constructed from Stiefel manifolds), and such that the universal principal $G$-bundle $EG$ is obtained by the direct limit

$$EG := \lim_{\rightarrow} EG_k.$$  

We have then a canonical map

$$\text{Bun}_G(M, K) \to \lim_{\rightarrow} \text{Bun}(EG_k \times_G M, K)$$

$$(P \to M) \mapsto \lim_{\rightarrow} (EG_k \times_G P \to EG_k \times_G M)$$

obtained by considering the principal $K$ bundles $EG_k \times_G P \to EG_k \times_G K$ for all values of $k$. Furthermore, as the sets $EG_k \times_G M$ are all paracompact (because $M$ is compact and the $EG_k$’s can be endowed with a Riemannian metric), by the classification of isomorphism classes of principal bundles we have the correspondence

$$\text{Bun}(EG_k \times_G M, K) \cong [EG_k \times_G M, BK]$$

which is moreover compatible with the inclusions; hence

$$\lim_{\rightarrow} \text{Bun}(EG_k \times_G M, K) \cong \lim_{\rightarrow} [EG_k \times_G M, BK].$$

Now, $K$ is of the homotopy type of a $K(\mathbb{Z}, n)$ and therefore $BK$ is of the homotopy type of a $K(\mathbb{Z}, n+1)$. Therefore we have the correspondence

$$\lim_{\rightarrow} [EG_k \times_G M, BK] \cong \lim_{\rightarrow} H^{n+1}(EG_k \times_G M, \mathbb{Z}).$$

In this case the inverse limit of the cohomologies is isomorphic to the cohomology of $EG \times_G M$

$$\lim_{\rightarrow} H^{n+1}(EG_k \times_G M, \mathbb{Z}) \cong H^{n+1}(EG \times_G M, \mathbb{Z})$$

because the cohomology group of the right hand side of (2.1) is finitely generated, and therefore $\lim_{\rightarrow} \cong \lim_{\rightarrow}$ vanishes.

We have then that

$$\lim_{\rightarrow} \text{Bun}(EG_k \times_G M, K) \cong [EG \times_G M, BK]$$

and therefore we get the desired map

$$\Phi : \text{Bun}_G(M, K) \to [EG \times_G M, BK].$$

When $G$ is the trivial group the map $\Phi$ is bijective and it is precisely the classifying map for principal bundles, but when $G$ is not trivial there is no reason to expect that the map $\Phi$ is injective nor surjective. The failure of $\Phi$ being injective
would imply that non-isomorphic $G$-equivariant principal $K$-bundles can become homotopic when seen as principal bundles with homotopy actions; i.e. that there exists higher homotopies that may rectify the error of the actions for not being isomorphic. And if $\Phi$ were surjective, we would have a set theoretical inverse that would allow to rectify any homotopy action on a principal bundle into a strict action.

Considering the case on which $M$ is a point might be clarifying.

The domain of the function $\Phi$ becomes

$$\text{Bun}_G(*, K) \cong \text{Hom}_c(G, K)/K$$

where $\text{Hom}_c$ denotes continuous homomorphisms of groups and the quotient by $K$ denotes that two homomorphisms are equivalent whenever they are conjugate by an element in $K$.

The map $\Phi$ becomes

$$\Phi : \text{Hom}_c(G, K)/K \rightarrow [BG, BK]$$

and if we take the groups to be $G = K = S^1$ we have that $\Phi$ is an isomorphism because

$$\text{Hom}_c(S^1, S^1) = \mathbb{Z}$$

measured by the winding number,

$$[BS^1, BS^1] = H^2(BS^1, \mathbb{Z}) = \mathbb{Z}$$

measured by the Chern class, and $\Phi$ maps the identity map $S^1 \rightarrow S^1$ to the generator of $H^2(\mathbb{Z}, \mathbb{Z})$.

The case on which $G$ is compact and $K$ is some explicit model for $K(\mathbb{Z}, 1)$ and $K(\mathbb{Z}, 2)$ is very interesting because the objects of study are related to equivariant line bundles and equivariant gerbes, which are of particular importance in the study of Twisted Equivariant K-theory. These two cases were studied carefully by Atiyah and Segal in [3, Prop. 6.3] and here we will borrow some of their ideas for studying the map $\Phi$.

2.3.1. $G$ compact and $K = S^1$. In the case that $G$ is a compact group and $K$ is the model of $K(\mathbb{Z}, 1)$ given by the topological group $S^1$, Atiyah and Segal proved that the map

$$\Phi : \text{Bun}_G(M, S^1) \xrightarrow{\cong} [EG \times_G M, BS^1] \cong H^2(EG \times_G M, \mathbb{Z})$$

is an isomorphism. When $G$ acts freely on $M$, the isomorphism follows from the classification of isomorphism classes of line bundles, as we have that in this case

$$\text{Bun}_G(M, S^1) \cong \text{Bun}(M/G, S^1) \cong H^2(M/G, \mathbb{Z})$$

Let us show that the map $\Phi$ is bijective for the other extreme case, namely when $M$ is a point. Then we need to show that $\text{Hom}_c(G, S^1)$ is in bijective correspondence with $H^2(BG, \mathbb{Z})$.

The idea of the proof consists on defining some hypercohomology groups

$$\mathbb{H}^n(N\mathbb{G}, sh(S^1))$$
on the nerve $N\bullet G$ of the simplicial space defined by the group $G$ with coefficients in the sheaf $sh(S^1)$, which are isomorphic to $H^{n+1}(BG, \mathbb{Z})$; and moreover which can be associated a spectral sequence that permits to show that

$$H^1(N\bullet G, S^1) \cong \text{Hom}(G, S^1).$$

For a simplicial space $X\bullet$ and a topological abelian group $A$, the hypercohomology $H^*(X\bullet, sh(A))$ with coefficients in the sheaf $sh(A)$ of continuous $A$-valued functions, is the cohomology of the double complex $C^{p,*}$ where for each $p \geq 0$ the cochain complex $C^p,$ calculates the sheaf cohomology groups $H^*(X, sh(A))$.

Filtering the double complex $C^{p,*}$ with the filtration $F_i = C^{p>i,*}$ we obtain a spectral sequence that abuts to $H^*(X\bullet, sh(A))$ and whose first page becomes

$$E^{p,q}_1 = H^q(X, sh(A)).$$

with the differential obtained by the alternating sum of the pullback of the faces.

In the case that $A = S^1$ and $X\bullet = N\bullet (G \rtimes M)$, the standard short exact sequence of sheaves

$$1 \rightarrow sh(\mathbb{Z}) \rightarrow sh(\mathbb{R}) \rightarrow sh(S^1) \rightarrow 1,$$

together with the fact that the continuous cohomology of a compact Lie group $G$ with values in a representation $V$ is given by $H^0_c(G, V) = V^G$ and $H^*_{c>0}(G, V) = 0$, permits to show (see [3, lemma 6.5]) that there is an isomorphism of groups

$$H^n(N\bullet (G \rtimes M), sh(S^1)) \cong H^{n+1}(EG \times_G M, \mathbb{Z}).$$

Let us use the spectral sequence defined above to calculate $H^1(N\bullet G, sh(S^1))$. In this case we have that $E^{p,q}_1 = 0$ for $q > 0$ because $N_0G = *$, and moreover we have that the second page page for $q = 0$ becomes isomorphic to the continuous cohomology of $G$ with coefficients in $S^1$

$$E^{p,0}_2 = H^p(G, S^1).$$

We can conclude that

$$H^1(N\bullet G, sh(S^1)) = E^{1,0}_2 = H^1(G, S^1) = \text{Hom}_c(G, S^1)$$

as we know that the first continuous cohomology consists of maps $f : G \rightarrow S^1$ such that for all pairs $g, h$ in $G^2$ we have that

$$\delta f(g, h) = f(g) f(h) f((gh)^{-1}) = 1,$$

namely continuous homomorphisms of groups.

We have then that there is a bijective correspondence

$$\Phi : \text{Hom}(G, S^1) \xrightarrow{\cong} H^2(BG, \mathbb{Z})$$

whenever $G$ is a compact topological group.

The proof above is essentially the same one that Lashof, May and Segal [21] use to show that whenever $G$ and $K$ are both compact Lie groups with $K$ abelian, the map

$$\Phi : \text{Bun}_G(M, K) \xrightarrow{\cong} [EG \times_G M, BK]$$

is an isomorphism.

The next interesting case is when we take the structural group $K$ has the homotopy type of a $K(\mathbb{Z}, 2)$. Let us see this case in more detail.
2.3.2. G compact and K = PU(H). By Kuiper’s theorem \cite{20} we know that the
topological group U(H) of unitary operators on a separable Hilbert space endowed
with the norm topology is contractible. Acting by scalars in S^1 on H gives a
homomorphism S^1 → U(H) whose cokernel is denoted by PU(H) and is called the
group of projective unitary operators. Because U(H) is contractible we have that
PU(H) ≃ B S^1 = K(Z, 2) and therefore BPU(H) ≃ K(Z, 3).

Principal PU(H)-bundles, also called projective unitary bundles, are used to
define the twisted K-theory groups in the following way. The unitary group U(H)
acts by conjugation on the space Fred(H) of Fredholm operators on H; as the
scalars act trivially under conjugation, this action factors through an action of the
projective unitary group PU(H). For any projective unitary bundle P → M over
M we can define the associated bundle \( M \times_{PU(H)} Fred(H) \) → M and the twisted
K-theory of M with respect to P is defined as the group

\[ K^0(M; P) := \Gamma[M, P \times_{PU(H)} Fred(H)] \]

the set of homotopy classes of sections of the associate Fredholm bundle.

To define the Equivariant version of the Twisted K-theory groups one needs
to endow the projective unitary bundle P → M with a G action satisfying some
stability condition. The reason for this extra condition will become clear once we
study the map \( \Phi \) when M is a point

\[ \Phi : \text{Hom}(G, PU(H))/PU(H) → [BG, BP(U(H))] ≃ H^3(BG, \mathbb{Z}); \]

let us study this case with detail.

Using the hypercohomology groups defined above we have that

\[ H^2(N_\bullet G, sh(S^1)) \cong H^3(BG, \mathbb{Z}). \]

By studying the spectral sequence that abuts to the hypercohomology, Atiyah and
Segal \cite{3, Prop. 6.3] proved that

\[ \text{Ext}_c(G, S^1) = H^2(N_\bullet G, sh(S^1)) \]

where \( \text{Ext}_c(G, S^1) \) is the set of isomorphism classes of continuous central extensions

\[ 1 → S^1 → \tilde{G} → G → 1. \]

On the left hand side of \cite{23} any homomorphism \( a : G → PU(H) \) determines an
S^1 central extension of G by the pullback of the projective unitary bundle \( a^*U(H) \).

Therefore we have that the map \( \Phi \) can be seen as the map

\[ \Phi : \text{Hom}(G, PU(H))/PU(H) → \text{Ext}(G, S^1) \]

(2.3)

\[ (\phi : G → PU(H)) ↦ (\tilde{G} := a^*U(H)). \]

Lemma 2.4. If G is a compact topological group then the map \( \Phi \) is surjective.

Proof. Take an S^1 central extension \( \tilde{G} \) of G and let us show that there is a homo-
morphism \( \bar{a} : \tilde{G} → U(H) \) such that \( \bar{a} \) restricted to \( S^1 \) acts on \( H \) by multiplication
of scalars, or equivalently, that the map \( \bar{a} \) fits into the diagram

\[
\begin{array}{ccccc}
1 & \longrightarrow & S^1 & \longrightarrow & \tilde{G} \\
\text{=} & & \downarrow{\bar{a}} & & \text{=} \\
1 & \longrightarrow & S^1 & \longrightarrow & U(H) \longrightarrow PU(H) \longrightarrow 1.
\end{array}
\]
By Peter-Weyl’s theorem we know that $L^2(\widetilde{G})$ contains all irreducible representations of $\widetilde{G}$. Let us take the subspace $V_{sc}(\widetilde{G})$ of $L^2(\widetilde{G})$ spanned by the vectors on which $S^1 \subset \widetilde{G}$ acts by multiplication of scalars. Let us choose any isomorphism of Hilbert spaces

$$\mathcal{H} \cong \mathcal{H} \otimes V_{sc}(\widetilde{G})$$

and endow $\mathcal{H}$ with the action of $\widetilde{G}$ defined by the unitary representation of $\widetilde{G}$ on $V_{sc}(\widetilde{G})$. It is clear that this action provides us with the desired homomorphism

$$\widetilde{a} : \widetilde{G} \to U(\mathcal{H})$$

that projects to $a : G \to PU(\mathcal{H})$. □

The proof of the lemma 2.4 implicitly shows the impossibility of the map $\widetilde{\Phi}$ of (2.4) to be injective. If we have two homomorphisms $a, b : G \to PU(\mathcal{H})$ with $a^*U(\mathcal{H}) \cong b^*U(\mathcal{H}) \cong \widetilde{G}$, the only way on which the homomorphisms $\widetilde{a}, \widetilde{b} : \widetilde{G} \to U(\mathcal{H})$ where to be conjugate is when the induced actions on $\mathcal{H}$ are isomorphic as $\widetilde{G}$ representations; this happens when both representations can be written as a sum of the same number of irreducible representations for each irreducible representation that appear in $V_{sc}(\widetilde{G})$.

**Definition 2.5.** A homomorphism $a : G \to PU(\mathcal{H})$ is called stable if the unitary representation defined by $\widetilde{a} : \widetilde{G} = a^*U(\mathcal{H}) \to U(\mathcal{H})$ contains each of the irreducible representations of $\widetilde{G}$ that appears in $V_{sp}(\widetilde{G})$ an infinitely number of times. We call the subspace of stable homomorphisms by

$$\text{Hom}_{st}(G, PU(\mathcal{H})) \subset \text{Hom}(G, PU(\mathcal{H}))$$

By Peter-Weyl’s theorem we have that any two stable homomorphisms $a, b : G \to PU(\mathcal{H})$ such that $a^*U(\mathcal{H}) \cong b^*U(\mathcal{H})$ are automatically conjugate (simple change of base). Therefore we have that the map $\widetilde{\Phi}$ is bijective on stable homomorphisms

$$\widetilde{\Phi} : \text{Hom}_{st}(G, PU(\mathcal{H}))/PU(\mathcal{H}) \xrightarrow{\cong} \text{Ext}(G, S^1)$$

and we can conclude that

**Proposition 2.6.** The set of stable homomorphisms from $G$ to $PU(\mathcal{H})$ for $G$ compact, is in 1-1 correspondence with the third integral cohomology group of $BG$

$$\Phi : \text{Hom}_{st}(G, PU(\mathcal{H}))/PU(\mathcal{H}) \xrightarrow{\cong} H^3(BG, \mathbb{Z}).$$

The stability condition could be formulated for $G$-equivariant projective unitary bundles in the following way:

**Definition 2.7.** A $G$-equivariant projective unitary bundle $P \to M$ is called stable, if for all $m \in M$ with isotropy $G_m$ there exists a $G_m$-equivariant neighborhood $U_m$ of $m$ and an isomorphism of bundles with $G_m$ action

$$P|_{U_m} \cong U_m \times PU(\mathcal{H})$$

such that on the right hand side $G_m$ acts on $PU(\mathcal{H})$ via a stable homomorphism. The set of stable $G$-equivariant projective unitary bundles will be denoted by

$$\text{Bun}_{st}^G(M, PU(\mathcal{H})).$$

The $G$-equivariant projective unitary bundles defined here are equivalent to the $G$ projective bundles defined in [R, condition (i) page 29], and in [R, Prop. 6.3] it was proved that
Proposition 2.8. If $G$ is a compact group, then the map
\[ \Phi : \text{Bun}_G(M, PU(H)) \to [EG \times_G M, BP_U(H)] = H^3(EG \times_G M, \mathbb{Z}) \]
is an isomorphism.

This result implies that for any $G$ action up to homotopy over a projective unitary bundle $P \to M$, one can “rectify” the $G$ action to a strict one, and moreover the $G$-equivariant projective unitary bundle can be taken to be stable.

We have seen that when the structural group $K$ equals the circle or the projective unitary group, there is a way to relate the $G$-equivariant principal $K$-bundles with the homotopy $G$ actions on principal $K$-bundles.

It would be interesting to find out in general what are the properties of the map
\[ \Phi : \text{Bun}_G(M, K(Z, n)) \to H^{n+1}(EG \times_G M, \mathbb{Z}) \]
whenever the Eilenberg-MacLane spaces $K(Z, n)$ are endowed with some group structure.

In the next section we will study the symmetries of principal $K(Z, n)$-bundles from the perspective of differential graded manifolds, and we will see how the equivariant cohomology naturally appears when one wants to find a lift to an action from the base to the total space.

3. Infinitesimal Symmetries of $\mathbb{R}[n]$-Bundles over $T[1]M$

In this section we will study the infinitesimal symmetries of principal $K(Z, n)$-bundles by studying the symmetries of $\mathbb{R}[n]$-bundles over $T[1]M$ in the category of differential graded manifolds.

We will argue that a $G$ equivariant $\mathbb{R}[n]$ bundle over the dg-manifold $T[1]M$ is characterized by an $n + 1$ equivariant cohomology class on $M$, through a similar argument to the one explained in the previous section.

We will start with a quick review of the category of dg-manifolds and then we will calculate the differential graded Lie algebra of infinitesimal symmetries of $\mathbb{R}[n]$ bundles over $T[1]M$. Then we will see how a Lie group $G$ could act on a $\mathbb{R}[n]$-bundle and we will see its relation to the equivariant cohomology.

3.1. dg-manifolds. Let us start with some notational conventions. Let $M$ be a differentiable (super)manifold and by $\mathcal{O}_M$ let us denote its sheaf of smooth functions. For $P = \{P_k\}_{k \in \mathbb{Z}}$ a graded vector bundle over $M$, $S(P)$ will denote the the sheaf of graded commutative $\mathcal{O}_M$-algebras freely generated by $P$; the locally ringed space $(M, S(P^*))$ will also be denoted by $P$ where $P^*$ is the dual vector bundle. For an integer $k$, $P[k]$ denotes the shifted vector bundle with $P[k]:= P_{k+l}$. To keep the notation simple, we will usually denote a vector bundle and its $\mathcal{O}$-module of sections with the same symbol.

Definition 3.1. A (non-negatively) graded manifold is a locally ringed space $P = (M, \mathcal{O}_P)$, which is locally isomorphic to $(U, \mathcal{O}_U \otimes S(P^*))$, where $U \subset \mathbb{R}^{n|m}$ is an open domain of $M$ and $P = \{P_i\}_{-n \leq i \leq 0}$ is a finite dimensional negatively graded (super)vector space. The number $n$ is called the degree of the graded manifold $P$. 
The global sections of $P$ will be called the functions on $P$ and they will be denoted by $C(P)$, and the derivations of $C(P)$ will be the vector fields of $P$ and they will be denoted $\text{Vect}^*(P)$.

**Definition 3.2.** A differential graded manifold (dg-manifold) is a graded manifold $P$ equipped with a degree 1 vector field $Q$ of $\text{Vect}^1(P)$ satisfying $[Q, Q]/2 = Q^2 = 0$ (a homology vector field).

Morphisms of dg-manifolds are morphisms of locally ringed spaces respecting the homology vector field. We recommend [35, 24] for an introduction to the theory of differential graded manifolds.

If $M$ is a differentiable manifold, the odd tangent bundle $T[1]M = (M, \Omega^*(M))$ is a graded manifold and the dg-structure is given by the De Rham differential $Q = d$.

The functor $M \mapsto (T[1]M, d)$ from manifolds to dg-manifolds is a full and faithful functor.

A dg-manifold over a point of degree $n$ is the same as an $L^\infty$-algebra of degree $n$, also called Lie $n$-algebra. A dg-manifold of degree $n$ is what is known as a “Lie $n$ algebroid”.

### 3.2. Symmetries of dg-manifolds.

A homological vector field $Q$ on the graded manifold $P$ is the same as a Maurer-Cartan element in the graded Lie algebra $\text{Vect}^*(P)$, that is

$$Q \in \text{Vect}^1(P) \text{ such that } \frac{1}{2}[Q, Q] = 0.$$ 

Any vector field $\alpha \in \text{Vect}^0(P)$ of degree 0 may define another Maurer-Cartan element by taking the action on $Q$ of the exponential of the adjoint action of $\alpha$

$$Q \mapsto e^{(\text{ad}_{\alpha})}Q := Q + [\alpha, Q] + \frac{1}{2}[[\alpha, [\alpha, Q]] + \cdots$$

whenever we know that the series above converge. The infinitesimal version of this action is given by the adjoint action of $\alpha$ on $Q$ and therefore the action is trivial whenever $[\alpha, Q] = 0$. We say then that the infinitesimal symmetries of the Maurer-Cartan element are given by vector fields $\alpha$ of degree 0 such that the adjoint action of $\alpha$ on $Q$ vanishes, i.e. $[\alpha, Q] = 0$. Note that these infinitesimal symmetries of $Q$ become a Lie algebra with respect to the brackets of $\text{Vect}^0(P)$, as we have that for $\alpha_1$ and $\alpha_2$ commuting with $Q$, the equality $[[\alpha_1, \alpha_2], Q] = 0$ follows from the Jacobi identity and the fact that $Q$ is a homology vector field.

Furthermore note that for any vector field $\beta$ of degree -1, the degree 0 vector field $[\beta, Q]$ commutes with $Q$ (again because of the Jacobi identity) and therefore it gives an infinitesimal symmetry of $Q$. This means that we have to see the symmetries of the dg-manifold $P$ as a differential graded Lie algebra, where the differential is defined by the operator $[Q, -]$ and the bracket is the one of vector fields.

**Definition 3.3.** Let $P$ be a dg-manifold with homological vector field $Q$. The (infinitesimal) symmetries of the dg-manifold $P$ with homology vector field $Q$ is the differential graded algebra $\text{sym}^*(P, Q)$ with

$$\text{sym}^q(P, Q) = \begin{cases} \text{Vect}^q(P) & \text{for } q < 0 \\ \{\alpha \in \text{Vect}^0(P) | [\alpha, Q] = 0\} & \text{for } q = 0 \\ 0 & \text{for } q > 0 \end{cases}$$

whose differential is $[Q, -]$ and the bracket is the bracket of vector fields.
Example 3.4. Let $M$ be a differentiable manifold and let us consider the dg-manifold $T[1]M$ with homology vector field $d$ the De Rham differential. By the Fr"olicher-Nijenhuis theorem the derivations of degree 0 of the algebra of differential forms are generated by Lie derivatives with respect to vector fields, and by contractions with respect to vector valued 1-forms. A vector valued 1-form $\sigma \in \Omega^1(M, TM)$ acts trivially on functions, and on exact 1-forms acts as follows:

$$(\iota_\sigma df)(X) = df(\sigma(X)) = \sigma(X)f$$

where $f$ is a function on $M$ and $X \in \mathfrak{X}M$ is a vector field on $M$. Therefore the commutator $[\iota_\sigma, d]$ is zero if and only if the vector valued 1-form $\sigma$ is trivial. Then we see that the degree 0 symmetries of $(T[1]M, d)$ are given by Lie derivatives with respect to vector fields (they commute with $d$).

The degree -1 derivations of the algebra of differential forms are given by contractions with respect to vector fields and there are no degree $\ast < 1$ derivations.

Then the dgla $\mathfrak{sym}^{-1}(T[1]M, d) \xrightarrow{\iota_X} \mathfrak{sym}^0(T[1]M, d)$ of symmetries of the dg-manifold $(T[1]M, d)$ is isomorphic to the dgla $\mathfrak{X}M[1] \xrightarrow{=} \mathfrak{X}M$ of vector fields of $M$ in degree 0 and -1, with the identity map as differential and with the brackets of vector fields: $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ and $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$.

Example 3.5. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The Lie group $G$ acts by left translation on the symmetries of the dg-manifold $(T[1]G, d)$ and therefore we may consider the Lie algebra of $(T[1]G, d)$ as the symmetries of $(T[1]G, d)$ that are invariant under left translation

$$\text{Lie}(T[1]G) := \left((\mathfrak{X}G)^G[1] \xrightarrow{=} (\mathfrak{X}G)^G\right)$$

which becomes isomorphic to the dgla $\mathfrak{g}[1] \xrightarrow{=} \mathfrak{g}$ where the brackets are the ones induced from the bracket of the Lie algebra $\mathfrak{g}$.

3.2.1. $\mathbb{R}[n]$ bundles over $T[1]M$. It is well known that the cohomology with real coefficients of the Eilenberg-Maclane space $K(\mathbb{Z}, n)$ is isomorphic to the symmetric graded algebra generated by one variable of degree $n$. Therefore we could consider the dg-manifold over a point $\mathbb{R}[n] := (*, S(\mathbb{R}^*[-n]))$ as a representative of $K(\mathbb{Z}, n)$ in the category of dg-manifolds.

Following [28] we can consider a bundle $P$ with fiber $\mathbb{R}[n]$ over $T[1]M$ in the category of dg-manifolds. As a graded manifold $P$ is nothing else as $T[1]M \oplus \mathbb{R}[n] = (M, \Omega^*(M) \otimes S(\mathbb{R}^*[-n]))$, but as a dg-manifold we would need to choose a connection in $P$, namely a homology vector field $Q$ on $P$ that projects to the De Rham differential on the base.

If we take $t$ to be a variable of degree $n$ and $S[t]$ denotes the graded symmetric polynomial algebra on $t$, then the functions on $P$ are isomorphic to the algebra

$$C(P) = \Omega^*(M) \otimes S[t].$$
A derivation of degree 1 lifting the De Rham differential is of the form

\[ Q = d + H \partial_t \]

where \( H \) is a \( n + 1 \)-form on \( M \), and is moreover a homological vector field if and only if \( dH = 0 \). This implies that the choices of homological vector fields are the same as closed \( n + 1 \) forms, which also could be described as dg-manifold maps

\[ T[1]M \to \mathbb{R}[n+1]. \]

A gauge transformation on the homological vector field \( Q \) is given by any map of graded manifolds

\[ T[1]M \to \mathbb{R}[n], \]

namely an \( n \)-form \( B \in \Omega^n(M) \), and maps the homological vector field \( Q \) to

\[ Q \mapsto Q' = d + (H + dB)\partial_t. \]

We can conclude

**Lemma 3.6.** The isomorphism classes of \( \mathbb{R}[n] \) bundles over \( T[1]M \) are in 1-1 correspondence with the cohomology group \( H^{n+1}(M; \mathbb{R}) \).

The gauge transformations amount for vertical automorphisms of the dg-manifold \( P \), but as we argued in chapter 2 we also want to study symmetries of \( P \) that are horizontal. This we will do by calculating the differential graded Lie algebra of symmetries of the connection \( Q \).

### 3.3. Symmetries of \( \mathbb{R}[n] \) bundles over \( T[1]M \)

Let us describe explicitly the dgla of symmetries in the case that \( P = T[1]M \oplus \mathbb{R}[n] \) and \( Q = d + H \partial_t \).

Any derivation of the algebra \( \Omega^* M \otimes S[t] \) is generated by its action on \( \Omega^* M \) and on the variable \( t \). Then, if we restrict a derivation of \( \Omega^* M \otimes S[t] \) to the domain \( \Omega^* M \otimes 1 \), and we project its image to \( \Omega^* M \otimes 1 \), we obtain a derivation of the algebra of differential forms. Conversely, any derivation of the algebra of differential forms induces a derivation of the algebra \( \Omega^* M \otimes S[t] \) by sending \( t \mapsto 0 \). This implies that we have a surjective map that splits

\[ \text{sym}^*(P, Q) \to \text{sym}^*(T[1]M, d) \]

whose kernel is generated by the derivations of the algebra \( \Omega^* M \otimes S[t] \) of the form \( A \partial_t \) for \( A \) a differential form of degree less or equal to \( n \).

If we denote the symmetries of degree 0 by

\[ \mathcal{L}_X + B \partial_t \quad \text{for} \quad X \in \mathfrak{X} M, B \in \Omega^n M, \]

the symmetries of degree -1 by

\[ \iota_X + \alpha \partial_t \quad \text{for} \quad X \in \mathfrak{X} M, \alpha \in \Omega^{n-1} M \]

and the rest of the symmetries by

\[ \eta \partial_t \quad \text{for} \quad \eta \in \Omega^{n-p} M, \]

we see that

\[
\begin{align*}
\text{sym}^0(P, Q) & = \{ \mathcal{L}_X + B \partial_t | \mathcal{L}_X H - dB = 0 \} \\
\text{sym}^{-1}(P, Q) & \cong \mathfrak{X} M \oplus \Omega^{n-1} M \\
\text{sym}^{-q}(P, Q) & \cong \Omega^{n-q} M \quad (\text{for} \ q > 1). 
\end{align*}
\]
Simple calculations show us that the differential in \( \text{sym}^\ast(P, Q) \) becomes
\[
\begin{align*}
[Q, \mathcal{L}_X + B\partial_t] &= 0 \\
[Q, \iota_X + \alpha\partial_t] &= \mathcal{L}_X + (d\alpha + \iota_X H)\partial_t \\
[Q, \eta\partial_t] &= (d\eta)\partial_t,
\end{align*}
\]
and the brackets become
\[
\begin{align*}
[\mathcal{L}_X + B\partial_t, \mathcal{L}_Y + C\partial_t] &= \mathcal{L}_{[X,Y]} + (\mathcal{L}_X C - \mathcal{L}_Y B)\partial_t \\
[\mathcal{L}_X + B\partial_t, \iota_Y + \beta\partial_t] &= \iota_{[X,Y]} + (\mathcal{L}_X \beta - \iota_Y B)\partial_t \\
[\mathcal{L}_X + B\partial_t, \eta\partial_t] &= (\mathcal{L}_X \eta)\partial_t \\
[\iota_X + \alpha\partial_t, \iota_Y + \beta\partial_t] &= (\iota_X \beta + \iota_Y \alpha)\partial_t \\
[\iota_X + \alpha\partial_t, \eta\partial_t] &= (\iota_X \eta)\partial_t.
\end{align*}
\]
Note that when the \( n + 1 \) form \( H = 0 \), the dgla structure defined above is the same one that was defined by Dorfman in [12].

As a complex we can see that \( \text{sym}^\ast(P, Q) \) is isomorphic to the complex
\[
\Omega_0^\ast \to \Omega_1^\ast 
\cdots \to \Omega_{n-2}^\ast \to \mathfrak{X}M \oplus \Omega_{n-1}^\ast \to \text{sym}^0(P, Q)
\]
where the differentials are obtained from the operator \([Q, \_]\) and whose formulas can be seen in (3.1).

**Lemma 3.7.** Consider the homology vector fields \( Q = d + H\partial_t \) and \( Q' = d \) on \( P = T[1]\mathbb{M} \oplus \mathbb{R}[n] \). Then \( \text{sym}^\ast(P, Q) \) and \( \text{sym}^\ast(P, d) \) are isomorphic as complexes.

**Proof.** Consider the map
\[
F : \text{sym}^\ast(P, Q) \to \text{sym}^\ast(P, d)
\]
to be the identity on the derivations of degree less or equal than 1, and
\[
F(\mathcal{L}_X + B\partial_t) = \mathcal{L}_X + (B - \iota_X H)\partial_t,
\]
on derivations of degree 0. The map \( F \) commutes with the differentials as it can be easily seen from the commutativity of the following diagrams:
\[
\begin{align*}
\iota_X + \alpha\partial_t & \quad \xrightarrow{[Q, \_]} \quad \mathcal{L}_X + (d\alpha + \iota_X H)\partial_t \\
\quad & \quad \xrightarrow{F} \quad \mathcal{L}_X + (d\alpha)\partial_t \\
\iota_X + \alpha\partial_t & \quad \xrightarrow{[d, \_]} \quad \mathcal{L}_X + (d\alpha)\partial_t
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{L}_X + B\partial_t & \quad \xrightarrow{[Q, \_]} \quad (dB - \mathcal{L}_X H)\partial_t \\
\quad & \quad \xrightarrow{F} \quad (dB - \mathcal{L}_X H)\partial_t
\end{align*}
\]

The previous isomorphism of complexes implies that

**Corollary 3.8.** The cohomology of the complex \( \text{sym}^\ast(P, Q) \) is given by
\[
H^{-p}(\text{sym}^\ast(P, Q)) = \begin{cases} 
H^{n-p}(M) & \text{if } -p < 0 \\
\mathfrak{X}M \oplus H^n(M) & \text{if } p = 0.
\end{cases}
\]
Proof. The complex $\mathfrak{sym}^*(P, d)$ is isomorphic to the complex
\[ \Omega^0 M \overset{d}{\to} \Omega^1 M \overset{d}{\to} \cdots \overset{d}{\to} \Omega^{n-2} M \overset{d}{\to} \mathfrak{X} M \oplus \Omega^{n-1} M \overset{id \times d}{\to} \mathfrak{X} M \oplus \Omega^n M. \]
The result follows. □

Remark 3.9. The map $F : \mathfrak{sym}^*(P, Q) \to \mathfrak{sym}^*(P, d)$ is not an isomorphism of graded Lie algebras as a simple calculation shows that
\[ [F(L_X), F(L_Y)] = F([L_X Y]) + (d_Y \cdot_Y H) \partial_t. \]
Nevertheless let us point out that the dgla $\mathfrak{sym}^*(P, Q)$ can be made isomorphic to $\mathfrak{sym}^*(P, d)$ if we add a term depending on $H$ to the bracket for elements in $\mathfrak{sym}^0(P, d)$; if we call this bracket $[,]_H$ the bracket would be
\[ [L_X + B \partial_t, L_Y + C \partial_t]_H = [L_X, L_Y] + (L_X C - L_Y B + d_Y L_X H) \partial_t. \]

We will postpone the study of some of the properties of the dgla $\mathfrak{sym}^*(P, Q)$ to the next chapter and we will concentrate now on defining the $G$-equivariant $\mathbb{R}[n]$ bundles over $T[1]M$.

3.4. $G$ equivariant $\mathbb{R}[n]$ bundles over $T[1]M$. Let us suppose that $M$ is provided with the action of a Lie group $G$, and that we have a $\mathbb{R}[n]$ bundle $P$ over $T[1]M$ with homology vector field $Q = d + H \partial_t$. We would like to see whether or not the $G$ action on $M$ could be lifted to a symmetry on $P$.

If the group $G$ acts on $P$ by bundle symmetries, then we have that for all $g \in G$ the homology vector fields
\[ d + H \partial_t \quad \text{and} \quad d + g^* H \partial_t \]
must be gauge equivalent; and they are so if there exists an $n$-form $B_g$ such that
\[ H - g^* H = dB_g. \]

We then have an assignment $g \mapsto B_g$ for all elements in $G$, but we would like this assignment to depend continuously on $G$. This can be made precise if we consider the map
\[ \pi : G \times M \to M \]
which projects onto the second coordinate and we consider the $n$-forms $B_g$ to be a section $B \in \Gamma(\pi^* \Lambda^n T^* M)$ of the bundle
\[ \pi^* \Lambda^n T^* M \to G \times M \]
such that for all $g \in G$
\[ B_g = B|_{\{g\} \times M}. \]

But as we have seen in the first chapter, the composition of the gauge transformations for $g$ and $h$ may not be the one of $gh$, and they may differ by a $n-1$-form, i.e. for $(g, h) \in G \times G$ we have
\[ B_h - B_{gh} + g^* B_h = dA_{g,h} \]
with $A_{g,h} \in \Omega^{n-1} M$.

The $A_{g,h}$'s may also satisfy higher coherence properties for elements in $G^3$ and this may continue until we exhaust all possibilities by going down in the degree in $\Omega^* M$.

All this information can be made precise using differentiable cohomology $[33, 34]$. 
3.4.1. Differentiable cohomology. The differentiable cohomology of a group with values on a module has been studied in detail by Van Est [33, 34] and generalized to continuous groups by Segal in [27]. In this particular case, the differentiable cohomology

\[ H^*_d(G, \Omega^\bullet M) \]

can be defined as the cohomology of the total complex associated to the double complex \( C^{p,q} \) where \( C^{p,q} \) is the vector space of sections of the bundle

\[ \pi^* \Lambda^q T^* M \to G^p \times M \]

over \( G^p \times M \) where \( \pi : G^p \times M \to M \) is the projection on the last coordinate. The differential in the second parameter

\[ d : C^{p,q} \to C^{p,q+1} \]

is the De Rham differential and the differential

\[ \delta : C^{p,q} \to C^{p+1,q} \]

is the alternating sum of the pullbacks of the face maps

\[ G^p \times M \to G^{p+1} \times M \]

of the simplicial set that is obtained from the action groupoid \( G \ltimes M \).

**Definition 3.10.** The differentiable cohomology of \( G \) with values in \( \Omega^\bullet M \) is the cohomology

\[ H^*_d(G, \Omega^\bullet M) := H_* (C^{\ast,*}, \delta \pm d) \]

of the double complex \( C^{p,q} \).

**Remark 3.11.** With the differentiable cohomology in hand, and by the argument of section 3.4, we see that in order for the Lie group \( G \) to act on the dg-manifold \( P = (T[1]M \oplus \mathbb{R}[n], Q = d + H\partial_t) \) it is a necessary condition that the closed \( n+1 \)-form \( H \) may be lifted to a closed differentiable cohomology form of \( G \) with values in the differential forms of \( M \).

We can filter the double complex \( C^{p,q} \) by the degree of the first coordinate \( C^{>p,q} \), and we get a spectral sequence converging to \( H^*_d(G, \Omega^\bullet M) \) whose second page is the differentiable cohomology of \( G \) with respect of the cohomology of \( M \)

\[ E_2 \cong H^0_d(G, H^q(M)) \]

When the Lie group \( G \) is connected, we have that \( G \) acts trivially on the cohomology of \( M \), and therefore the second page becomes

\[ E_2 \cong H^0_d(G, \mathbb{R}) \otimes H^q(M); \]

furthermore if the Lie group is compact we have that \( H^*_d(G, \mathbb{R}) = \mathbb{R} \), and therefore the spectral sequence collapse at level 2 showing that the differentiable cohomology of \( G \) with values in differential form is isomorphic to the cohomology of \( M \).

**Lemma 3.12.** If \( G \) is connected and compact, we have that

\[ H^*_d(G, \Omega^\bullet M) \cong H^*(M). \]
The previous lemma could be rephrased by saying that any closed differential form is cohomologous to an invariant one. Therefore if we have a homology vector field $d + H\partial_t$ and if the group $G$ is compact and connected, there exists a gauge transformation $B$ such the form

$$\overline{H} := H + dB$$

is $G$ invariant; therefore one could use the homology vector field $d + \overline{H}\partial_t$ on $T[1]M \oplus \mathbb{R}[n]$ so that all higher coherence for the action may be taken to be zero.

3.4.2. Van Est map. The infinitesimal version of the lift of the $n + 1$-form $H$ to the differentiable cohomology group $H^d_{n+1}(G, \Omega^\bullet M)$ can be obtained by the use of the van Est map. By differentiating the differentiable cohomology classes van Est \cite{33, 34} defined a map

$$H^*_d(G, \Omega^\bullet M) \to H^*(g, \Omega^\bullet M)$$

from the differentiable cohomology with values in the $G$-module of differentiable forms on $M$, to the Lie algebra cohomology of the Lie algebra $g$ of $G$ with values in the $g$ module of differentiable forms. This Lie algebra cohomology is defined in similar fashion to the differentiable cohomology through a double complex

$$D^{p,q} = \text{Hom}_G(\Lambda^p g^*, \Omega^q M)$$

where the differential in the second coordinate is the De Rham differential and the differential in the first one is given by the Chevalley-Eilenberg differential in Lie algebra cohomology defined in \cite{10}.

Alternatively the cohomology $H^*(g, \Omega^\bullet M)$ could be defined by taking the cohomology of the differential graded algebra defined on the vector space

$$\Lambda^* g^* \otimes \Omega^* M$$

where the differential $\delta = \delta_1 \otimes 1 + 1 \otimes \delta_2 + 1 \otimes d$ is generated by the dual of the bracket of the Lie algebra

$$\delta_1 : g^* \to g^* \wedge g^*,$$

the map

$$\delta_2 : \Omega^* M \to g^* \otimes \Omega^* M$$

induced by the Lie algebra representation

$$g \otimes \Omega^* M \to \Omega^* M,$$

and the De Rham differential $1 \otimes d$.

If $\theta^a$ denotes the dual of $a \in g$ and $X_a \in \mathfrak{X}M$ is the vector field defined by $a \in g$, then the differentials are

$$\delta_1 \theta^a = -\frac{1}{2} f^a_{bc} \theta^b \theta^c$$

where $[b, c] = f^a_{bc} a$ and

$$\delta_2 \sigma = \theta^a L_{X_a} \sigma + d\sigma.$$ 

We have then the isomorphism

$$H^*(g, \Omega^\bullet M) \cong H^*(\Lambda^* g^* \otimes \Omega^* M, \delta).$$

This infinitesimal lift in $H^d_{n+1}(g, \Omega^\bullet M)$ can be also understood as an $L_\infty$ map from the Lie algebra $g$ to the dgla of infinitesimal symmetries of $Q$

$$g \to \text{sym}^*(P, Q)$$
such that the map lifts the canonical map $\mathfrak{g} \to \mathfrak{X}M$: $a \mapsto X_a$ given by the infinitesimal action of the Lie group

$$\mathfrak{sym}^*(P, Q) \xrightarrow{\sigma} \mathfrak{X}M.$$ 

In this particular case the $L_\infty$ map is given by a graded map of degree -1

$$\sigma : \Lambda^j \mathfrak{g} \to \mathfrak{sym}^*(P, Q) \quad (3.2)$$

which is defined by the maps

$$\sigma_0 = H \in \Omega^{n+1} M,$$

and

$$\sigma_j : \Lambda^j \mathfrak{g} \to \Omega^{n+1-j} M \quad \text{for } 0 < j \leq n + 1$$

satisfying the equations

$$0 = dH$$

$$a(\sigma_0) = L_{X_a} H = d\sigma_1(a)$$

$$a(\sigma_1(b)) - b(\sigma_1(a)) - \sigma_1([a, b]) = -d\sigma_2(a, b)$$

$$(\delta \sigma_j)(a_0 \wedge \cdots \wedge a_j) = d\sigma_{j+1}(a_0 \wedge \cdots \wedge a_j)$$

$$(\delta \sigma_{n+1}) = 0$$

where the differential $\delta$ is the Chevaley-Eilenberg differential.

If we take the canonical projection map

$$\pi : H^{n+1}(\mathfrak{g}, \Omega^\bullet M) \to H^{n+1}(M),$$

we see that the action of the Lie algebra $\mathfrak{g}$ on $M$ could be lifted to an action on $(T[1]M \oplus \mathbb{R}[n], Q)$ whenever the De Rham closed form $H$ could be lifted to a closed $n + 1$ form on the cohomology of $\mathfrak{g}$ with values in the differential forms $\Omega^\bullet M$. We have then that

**Proposition 3.13.** The Lie algebra map $\mathfrak{g} \to \mathfrak{X}M$ could be lifted to a map of $L_\infty$ algebras $\mathfrak{g} \to \mathfrak{sym}(P, Q)$ whenever the closed form $H$ could be lifted to a closed form in the complex that calculates $H^{n+1}(\mathfrak{g}, \Omega^\bullet M)$. The equivalence classes of lifts are in 1-1 correspondence with the cohomology classes in $\pi^{-1}(H)$.

This infinitesimal version of the the action $\mathfrak{g} \to \mathfrak{sym}^*(P, Q)$ only takes into account the infinitesimal contributions of the elements in the differentiable cohomology $H^*_d(G, \Omega^\bullet M)$. This differentiable cohomology is built out from the vector spaces of sections

$$C^{p, q} = \Gamma(\pi^* \Lambda^p T^* M, G^p \times M)$$

which correspond to the subspace of differentiable forms of

$$\Omega^q(G^p \times M)$$

which do not depend on the vectors of the tangent space $T G^p$.

In order to make the action to depend also on the tangent space of $G$ we need to take into account the full space of differentiable forms on $G^p \times M$ and not just $C^{p, q}$. This can be achieved by considering the full dgla of infinitesimal symmetries of the Lie group $T[1]G$, namely the Lie algebra of $T[1]G$, together with its action on $P$. 
3.4.3. Lie algebra of $T[1]G$. If we want to lift the action of $T[1]G$ on $(T[1]M, d)$ to an action on the bundle $(T[1]M \oplus \mathbb{R}[n], Q)$ we need to lift the map

$$\text{Lie}(T[1]G) \to \text{sym}^*(T[1]M, d)$$

to an $L_\infty$ map $\tilde{\sigma} : \text{Lie}(T[1]G) \to \text{sim}^*(P, Q)$ of dgla's

$$\xymatrix{ \text{sim}^*(P, Q) \ar[d] \ar[r]^\sim & \mathcal{X}M[1] \ar[r] & \mathcal{X}M. }$$

The map $\tilde{\sigma}$ can be described by a graded map

$$\tilde{\sigma} : \Lambda g \otimes S g \to \Omega^{* - n - 1} M$$

where

$$\Lambda g \otimes S g = S \left( \left(g[1] \to g\right)[1] \right),$$

satisfying certain cocycle conditions, and such that once restricted to $\Lambda^* g$ agrees with the map $\sigma$ defined in (3.2), i.e.

$$\tilde{\sigma}|_{\Lambda g \otimes 1} = \sigma.$$

The map $\tilde{\sigma} : \text{Lie}(T[1]G) \to \text{sim}^*(P, Q)$ is an $L_\infty$ map whenever $\tilde{\sigma}$ becomes a cocycle in the complex that calculates the cohomology of $g[1] \to g$ with coefficients in the module $\Omega^* M$

$$H^{n+1}(g[1] \to g, \Omega^* M).$$

The cohomology groups $H^*(g[1] \to g, \Omega^* M)$ could be calculated from the complex

$$\text{Hom}^*_g(\Lambda g \otimes S g, \Omega^* M)$$

together with the appropriate Chevalley-Eilenberg differential, but this description is cumbersome. Instead we define the cohomology ring

$$H^*(g[1] \to g, \Omega^* M)$$

as the cohomology of the differential graded algebra $S((g[1] \to g)[1])^* \otimes \Omega^* M$ whose differential is generated by the Chevalley-Eilenberg differential on $S((g[1] \to g)[1])^*$, the De Rham differential, and the differential

$$\Omega^* M \to \Lambda g^* \otimes S g^* \otimes \Omega^* M$$

that is induced by the action of $(g[1] \to g)$ on the differential forms. Let us be more precise.

3.4.4. Cohomology of $(g[1] \to g)$ with values on $\Omega^* M$. To the dgla $g[1] \to g$ could be associated the dga whose underlying algebra is

$$S((g[1] \to g)[1])^* = \Lambda g^* \otimes S g^*$$

and whose differential is defined on generators as the dual of the structural maps

$$[.] : g \wedge g \to g$$

$$g[1] \to g$$

$$[.] : g[1] \otimes g \to g[1].$$
If $\theta^a$ and $\Omega^a$ are respectively the generators of $\Lambda g^*$ and $Sg^*$ associated to $a \in g$, the duals of the structural maps become

$$\theta^a \mapsto -\frac{1}{2} f_{bc}^a \theta^b \theta^c,$$

$$\theta^a \mapsto \Omega^a,$$

and therefore the differential $\delta_1$ on the complex $\Lambda g^* \otimes Sg^*$ is defined on generators by the equations

$$\delta_1 \theta^a = \Omega^a - \frac{1}{2} f_{bc}^a \theta^b \theta^c,$$

$$\delta_1 \Omega^a = f_{bc}^a \Omega^b \theta^c.$$

We can see that the dga associated to $(g[1] \to g)$ is precisely the Weil algebra $(\Lambda g^* \otimes Sg^*, \delta_1)$.

The map of dgla’s Lie$(T[1]G) \to \mathfrak{sym}^*(T1M, d)$ define derivations

$$g[1] \otimes \Omega^1 M \to \Omega^{1-1} M$$

$$a \otimes \omega \mapsto \iota_{X_a} \omega$$

and

$$g \otimes \Omega^0 M \to \Omega^1 M$$

$$a \otimes \omega \mapsto L_{X_a} \omega$$

whose adjoints induce a degree 1 map

$$\delta_2 : \Omega^1 M \to \Lambda g^* \otimes Sg^* \otimes \Omega^0 M$$

$$\omega \mapsto \Omega^0 \iota_{X_a} \omega + \theta^a L_{X_a} \omega.$$

It is a simple calculation to show that the algebra

$$\Lambda g^* \otimes Sg^* \otimes \Omega^0 M$$

together with the derivation that the degree 1 map

$$\delta := \delta_1 \otimes 1 + 1 \otimes \delta_2 + 1 \otimes d$$

defines, becomes a dga.

**Definition 3.14.** The cohomology ring

$$H^\ast(g[1] \to g, \Omega^\ast M)$$

is the cohomology of the dga whose underlying algebra is

$$\Lambda g^* \otimes Sg^* \otimes \Omega^\ast M$$

and whose differential is

$$\delta := \delta_1 \otimes 1 + 1 \otimes \delta_2 + 1 \otimes d.$$

Taking again the canonical projection map

$$\tilde{\pi} : H^{n+1}(g[1] \to g, \Omega^* M) \to H^{n+1}(M),$$

we see that the action of the dgla $(g[1] \to g)$ on $T[1]M$ could be lifted to an action on $(T[1]M \oplus \mathbb{R}[n], Q)$ whenever the De Rham closed form $H$ could be lifted to a closed $n+1$ form on the cohomology of $(g[1] \to g)$ with values in the differential forms $\Omega^* M$. Therefore we have that
Proposition 3.15. The map of dga’s \((\mathfrak{g}[1] \rightarrow \mathfrak{g}) \rightarrow (\mathfrak{X}M[1] \rightarrow \mathfrak{X}M)\) could be lifted to a map of \(L_\infty\) algebras \(\tilde{\sigma} : (\mathfrak{g}[1] \rightarrow \mathfrak{g}) \rightarrow \text{sym}^*(P, Q)\) making the following diagram commutative

\[
\begin{array}{c}
\sim^*(P, Q) \\
\text{sim}^*(P, Q)
\end{array}
\]

\[
\begin{array}{c}
\tilde{\sigma} \\
\text{sim}^*(P, Q)
\end{array}
\]

\[
\begin{array}{c}
(\mathfrak{g}[1] \rightarrow \mathfrak{g}) \\
(\mathfrak{X}M[1] \rightarrow \mathfrak{X}M)
\end{array}
\]

whenever the closed form \(H\) could be lifted to a closed form in the complex that calculates \(H^{n+1}(\mathfrak{g}[1] \rightarrow \mathfrak{g}, \Omega^\bullet M)\).

The equivalence classes of lifts are in 1-1 correspondence with the cohomology classes in \(\tilde{\pi}^{-1}(H)\).

Coming back to the main question, we see by Remark 3.11 and Propositions 3.13 and 3.15 that in order to lift the \(G\) action on \(M\) to an action on the bundle \(P = (T[1]M \oplus \mathbb{R}[n], Q = d \oplus H\partial_t)\) we need to lift the closed form \(H\) to closed forms in the complexes that calculate \(H^{n+1}(G, \Omega^\bullet M)\), \(H^{n+1}(\mathfrak{g}, \Omega^\bullet M)\) and \(H^{n+1}(\mathfrak{g}[1] \rightarrow \mathfrak{g}, \Omega^\bullet M)\). These lifts are related in the sense that they fit into the diagram

\[
\begin{array}{c}
H^{n+1}(\mathfrak{g}[1] \rightarrow \mathfrak{g}, \Omega^\bullet M) \\
H^{n+1}(\mathfrak{g}, \Omega^\bullet M) \\
H^{n+1}(G, \Omega^\bullet M)
\end{array}
\]

where the upper diagonal arrow is given at the level of dga’s by the forgetful map

\[
\Lambda \mathfrak{g}^* \otimes S \mathfrak{g}^* \otimes \Omega^\bullet M \rightarrow \Lambda \mathfrak{g}^* \otimes \Omega^\bullet M, \quad \Omega^\bullet \rightarrow 0
\]

and the lower diagonal arrow is given by the Van Est map.

But it would be conceptually easier if we could find a cohomology ring that would fit in the upper left corner of the diagram

\[
\begin{array}{c}
? \\
H^{n+1}(\mathfrak{g}[1] \rightarrow \mathfrak{g}, \Omega^\bullet M)
\end{array}
\]

\[
\begin{array}{c}
H^{n+1}(G, \Omega^\bullet M) \\
H^{n+1}(\mathfrak{g}, \Omega^\bullet M)
\end{array}
\]

in such a way that the vertical maps were forgetful functors where the component of \(S \mathfrak{g}^*\) gets mapped to 1, and where the horizontal maps were obtained by generalizations of the Van Est map by differentiating the group cohomology to Lie algebra cohomology.

Fortunately the cohomology ring that fits in the upper left corner of the diagram has been already defined by Getzler in [14] where it was denoted by

\[
H^*(G, \Omega^\bullet M \otimes S \mathfrak{g}^*);
\]

Getzler has shown that this cohomology ring is isomorphic to the cohomology of the homotopy quotient \(M \times_G EG\) and therefore provides a differentiable model for the equivariant cohomology of \(M\).
With the cohomology ring defined by Getzler in hand, the problem of finding
the lift of the \(G\) action on \(M\) to \(P\) could be reduced to finding a lift of the closed
\(n + 1\) form to

\[ H^{n+1}(G, \Omega^* M \otimes S\mathfrak{g}^*). \]

In the next section we will reproduce the definition of the cohomology ring defined
by Getzler and we will show that it fits in the upper corner of the diagram (3.3).

3.4.5. Getzler’s model of Equivariant cohomology. By the works of Bott-Shulman-
Stasheff and Getzler [6, 14] we know that one way to calculate the cohomology of
the double complex of differentiable forms

\[ \Omega^* (N_\mathbf{e}(G \ltimes M)) \]
on the nerve of the groupoid \(G \ltimes M\), is through the differentiable cohomology groups

\[ H^*_d(G, \Omega^* M \otimes S\mathfrak{g}^*) \]
of the group \(G\) with values on differentiable forms of \(M\), tensor the symmetric algebra of the dual of the Lie algebra; Getzler has shown that there is an isomorphism
of rings to the equivariant cohomology

\[ H^*_d(G, \Omega^* M \otimes S\mathfrak{g}^*) \cong H^*(EG \times_G M; \mathbb{R}). \]

The model for equivariant cohomology of Getzler, that uses differentiable coho-
mology, is very well suited to our purpose of relating the equivariant cohomology
to the symmetries of principal \(\mathbb{R}[n]\) bundles over \(T[1]M\). Let us first recall the
definition of the cohomology groups (see [14])

\[ H^*_d(G, \Omega^* M \otimes S\mathfrak{g}^*) \]
and then we will explain the relation to the symmetries of \(\mathbb{R}[n]\) bundles over \(T[1]M\).

Consider the complex \(C^k(G, \Omega^* M \otimes S\mathfrak{g}^*)\) with elements smooth maps

\[ f(g_1, \ldots, g_k|X) : G^k \times \mathfrak{g} \to \Omega^* M \]
which vanish if any of the arguments \(g_i\) equals the identity of \(G\). The operators \(d\) and \(\iota\) are defined by the formulas

\[ (df)(g_1, \ldots, g_k|X) = (-1)^k df(g_1, \ldots, g_k|X) \quad \text{and} \]
\[ (\iota f)(g_1, \ldots, g_k|X) = (-1)^k \iota(X)f(g_1, \ldots, g_k|X), \]
as in the case of the differential in Cartan’s model for equivariant cohomology [13, 15].

The coboundary \(\bar{d} : C^k \to C^{k+1}\) is given by the formula

\[ (\bar{d}f)(g_0, \ldots, g_k|X) = f(g_1, \ldots, g_k|X) + \sum_{i=1}^{k} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k|X) \]
\[ + (-1)^{k+1} g_kf(g_0, \ldots, g_{k-1}|Ad(g_k^{-1})X), \]
and the extra contraction \(\bar{\iota} : C^k \to C^{k-1}\) is given by the formula

\[ (\bar{\iota}f)(g_1, \ldots, g_{k-1}|X) = \sum_{i=0}^{k-1} (-1)^i \frac{\partial}{\partial t} f(g_1, \ldots, g_i, e^{tX_i}, g_{i+1}, \ldots, g_{k-1}|X) \]
where \(X_i = Ad(g_{i+1} \ldots g_{k-1})X\).

If the image of the map

\[ f : G^k \to \Omega^* M \otimes S\mathfrak{g}^* \]
is a homogeneous polynomial of degree \( l \), then the total degree of the map \( f \) equals 
\[ \text{deg}(f) = k + l. \]
It follows that the structural maps \( d, \iota, \bar{d} \) and \( \bar{\iota} \) are degree 1 maps, and the operator 
\[ d_G = d + \iota + \bar{d} + \bar{\iota} \]
becomes a degree 1 map that squares to zero.

**Definition 3.16.** The cohomology of the complex 
\[ (C^*(G, \Omega^* M \otimes S\mathfrak{g}^*), d_G) \]
is denoted by 
\[ H^*_G(G, \Omega^* M \otimes S\mathfrak{g}^*) \]
and we will call it the model of Getzler for equivariant cohomology.

In [14] it was shown that the complex \((C^*(G, \Omega^* M \otimes S\mathfrak{g}^*), d_G)\) together with the cup product 
\[ (a \cup b)(g_1, ..., g_{k+l}|X) = (-1)^{|a||b|}a(g_1, ..., g_k|\text{Ad}(\gamma^{-1})X)b(g_{k+1}, ..., g_{k+l}|X) \]
for \( \gamma = g_{k+1}...g_{k+l} \), becomes a differential graded algebra, and moreover that there is a canonical isomorphism of rings 
\[ H^*_G(G, \Omega^* M \otimes S\mathfrak{g}^*) \cong H^*(M \times_G EG, \mathbb{R}) \]
with the cohomology of the homotopy quotient.

Let us now show that the equivariant cohomology of Getzler is the cohomology theory that fits in the upper left corner of diagram (3.3).

**Theorem 3.17.** The equivariant cohomology of Getzler makes the following diagram of graded algebras commutative

\[
\begin{array}{ccc}
H^*(G, \Omega^* M \otimes S\mathfrak{g}^*) & \rightarrow & H^*(\mathfrak{g}[1] \rightarrow \mathfrak{g}, \Omega^* M) \\
\downarrow & & \downarrow \\
H^*(G, \Omega^* M) & \rightarrow & H^*(\mathfrak{g}, \Omega^* M)
\end{array}
\]

where the vertical arrows are forgetful maps \((\Omega^a \mapsto 0)\) and the horizontal maps are given by Van Est type maps (differentiating group cohomology to Lie algebra cohomology).

**Proof.** The Van Est type of map is defined on homogeneous elements
\[ \overline{R} : C^k(G, \Omega^* M \otimes S\mathfrak{g}^*) \rightarrow \text{Hom}_R(\Lambda^k \mathfrak{g}, \Omega^* M \otimes S\mathfrak{g}^*) \]
by the formula
\[ (\overline{R} f)(a_1 \wedge ... \wedge a_k) = (-1)^{kl} \sum_{\sigma} (-1)^{|\sigma|} \partial_{a_{\sigma(k)}}|_{t_k=0} ... \partial_{a_{\sigma(1)}}|_{t_1=0} f(e^{t_{\sigma(k)}} \ldots e^{t_{\sigma(1)}}) \]
when \( k > 0 \) and where \( k + l \) is the total degree of the homogeneous map \( f \) (see [2, Def. 3.3]). The map \( \overline{R} \) is the identity when \( k = 0 \).

Taking the adjoint map on the right hand side we have the map
\[ R : C^k(G, \Omega^* M \otimes S\mathfrak{g}^*) \rightarrow \Lambda^k \mathfrak{g}^* \otimes \Omega^* M \otimes S\mathfrak{g}^* \]
\[ f \mapsto \theta^{a_1} \ldots \theta^{a_k} (\overline{R} f)(a_1 \wedge ... \wedge a_k) \]
where the elements \( a_1, ..., a_k \) run over a base of the Lie algebra \( \mathfrak{g} \).
It follows from the definition of the map $\overline{R}$ that the map

$$R : C^*(G, \Omega^* M \otimes Sg^*) \to \Lambda^* g^* \otimes \Omega^* M \otimes Sg^*$$

is surjective, and moreover by Proposition 3.6 in [2] we have that the map $R$ becomes a map of algebras.

Now, in order to show that the map $R$ induces the desired algebra homomorphism

$$H^*(G, \Omega^* M \otimes Sg^*) \to H^*(g[1] \to g, \Omega^* M)$$

we need to show that the differential $R(d_G)$ induced by $R$ is precisely the differential $\delta$ of definition [3, 14]. Again, by Proposition 3.6 in [2] we know that $R$ defines a map of differential graded algebras from $C^*(G, \Omega^* M \otimes Sg^*)$ to the induced dga structure on $\Lambda^* g^* \otimes \Omega^* M \otimes Sg^*$; therefore it is enough to show that the induced differential of $R$ on the generators of the algebra $\Lambda^* g^* \otimes \Omega^* M \otimes Sg^*$ match the differential on the generators that define $\delta$.

The differential $\delta$ was defined by the following differentials on generators

$$\theta^a \overset{\delta}{\mapsto} \Omega^a - \frac{1}{2} f^a_{bc} \theta^b \theta^c$$

$$\omega \overset{\delta}{\mapsto} \theta^a \mathcal{L}_{X_a} \omega + \Omega^a \iota_{X_a} \omega$$

where we know that the terms $\theta^a \mathcal{L}_{X_a} \omega$ and $f^a_{bc} \Omega^b \theta^c$ come from the action of $g$ on $\Omega^* M$ and $g[1]$ respectively, and that the term $-\frac{1}{2} f^a_{bc} \theta^b \theta^c$ comes from the Lie bracket on $g$.

Let us compare the differentials $R(d_G)$ and $\delta$. First of all note that the differential $d + \iota$ in

$$C^0(G, \Omega^* M \otimes Sg^*) = \Omega^* M \otimes Sg^*$$

agrees with the terms of the differential $\delta_2$ given by $\Omega^a \iota_{X_a} \omega + d\omega$. Therefore we need to see what happens with the maps $\overline{\tau}$ and $\overline{d}$ when we apply the functor $R$.

For the map $\overline{\tau}$ it is enough to take a map $f : G \to \Omega^* M \otimes Sg^*$ and to calculate the induced map $R(\overline{\tau}) : R(f) \to R(\overline{\tau}f)$. We have then that

$$R(f) = \theta^a \overline{\mathcal{L}} f(a) = -\theta^a \partial_t |_{t=0} f(e^{-ta})$$

where $a$ runs over a base of $g$, and

$$R(\overline{\tau}f)(X) = \overline{\tau} f(X) = \partial_t f(e^{tX}X)$$

once we evaluate in $X \in g[2]$. It follows then that

$$R(\overline{\tau})R(f) = R(\overline{\tau}) \left( \theta^a \partial_t |_{t=0} f(e^{ta}) \right) = \Omega^a \partial_t |_{t=0} f(e^{ta})$$

and hence the differential induced by $R(\overline{\tau})$ is precisely the one that promotes the connection forms $\theta^a$ to the curvature forms $\Omega^a$, i.e. the dual of the identity map $g[1] \overset{\iota}{\to} g$.

The last map we need to study is the map $\overline{d}$. This map restricted to the functions $G^k \to \mathbb{R}$ is the differential in the cohomology ring $H^*_d(G, \mathbb{R})$ and it was proved by Van Est [33] that the image of the Van Est map is precisely $H^*(g, \mathbb{R})$. This implies that the induced differential $R(\overline{d})$ act on the connection forms by the standard map

$$R(\overline{d}) (\theta^a) = -\frac{1}{2} f^a_{bc} \theta^b \theta^c.$$
In order to see which other maps induce $R(d)$ it is enough to take a map $h \in C^0(G, \Omega^\bullet M \otimes Sg^*)$ and to calculate the induced map $R(d) : Rh \to R(dh)$. We can think of $h$ as an element in $\Omega^\bullet M \otimes Sg^*$ and therefore $R(h) = h$. We have that

$$\nabla h(g|X) = h(X) - g^*h(\text{Ad}(g^{-1})X)$$

and differentiating we get

$$R\nabla h(a|X) = -\partial_t|_{t=0} (e^{-ta})^* h(\text{Ad}(e^{ta})X) = \mathcal{L}_{X_a} h(X) + h([X, a]).$$

Therefore if $h$ is only a differential form of $M$, we have that

$$(R\nabla)h = \theta^a \mathcal{L}_{X_a} h$$

and if $h = \Omega^a$ we have that

$$(R\nabla)\Omega^a(c|b) = \Omega^a([b, c]) = f_{bc}^{\alpha}$$

and hence we see that

$$R(R\nabla)\Omega^a = f_{bc}^{\alpha} \Omega^b \theta^c.$$  

We can see now that

$$\delta_1 \theta^a = (R(d) + R(\iota)) \theta^a \quad \delta_1 \Omega^a = R(d) \Omega^a$$

and therefore the image of the Van Est type map $R$ is the cohomology of $\mathfrak{g}[1] \to \mathfrak{g}$ with coefficients in $\Omega^\bullet M$

$$H^*(G, \Omega^\bullet M \otimes Sg^*) \to H^*(\mathfrak{g}[1] \to \mathfrak{g}, \Omega^\bullet M).$$

The lower horizontal map of diagram (3.4) is defined as the restriction of the map $R$ when one takes $\Omega^a \mapsto 0$ and the vertical maps are the natural forgetful maps. We can conclude that the equivariant cohomology theory of Getzler makes the diagram of rings (3.4) commute. This finishes the proof of the theorem. \[\Box\]

With Theorem 3.17 in hand, we can now say when a Lie group act by symmetries on the dg-manifold $P = (T[1]M \oplus \mathbb{R}[n], Q = d + H\partial_t)$.

**Definition 3.18.** The group $G$ acts by symmetries on $P = (T[1]M \oplus \mathbb{R}[n], Q = d + H\partial_t)$ whenever the $n+1$-form $H$ can be lifted to an $n+1$-equivariant cohomology class in

$$H^{n+1}_d(G, \Omega^\bullet M \otimes Sg^*).$$

And therefore the equivalence classes of $G$ actions on the dg-manifold $P = (T[1]M \oplus \mathbb{R}[n], Q = d + H\partial_t)$ are in one to one correspondence with elements in $H^{n+1}_d(G, \Omega^\bullet M \otimes Sg^*)$.

Now that we have characterized the group actions on $\mathbb{R}[n]$-bundles over $T[1]M$, we turn to the questions of figuring out which actions are Hamiltonian. The next section is devoted to this.
4. Hamiltonian symmetries of $\mathbb{R}[n]$ bundles over $T[1]M$

We have seen that the infinitesimal symmetries of the dg-manifold $P = (T[1]M \oplus \mathbb{R}[n], Q = d + H \partial_t)$ are organized in the dgla $\mathfrak{sy}m^*(P, Q)$. It was noticed by Dorfman [12], that in the case of $H = 0$ and $n = 2$, the antisymmetrization of the derived bracket of $\mathfrak{sy}m^*(P, Q)$ recovered the Courant bracket of the exact Courant algebroids [11]. The same procedure of deriving the brackets can be performed on $\mathfrak{sy}m^*(P, Q)$ leading to a very interesting Leibniz algebra, that in the case of $n = 2$, it recovers the twisted Courant-Dorfman bracket of the exact Courant algebroids [29].

In order to understand the hamiltonian actions on $\mathbb{R}[n]$-bundles over $T[1]M$ we need to introduce the derived algebra associated to a dgla.

4.1. The derived algebra of $\mathfrak{sy}m^*(P, Q)$. The construction of the derived bracket and the derived algebraic structure that can be defined from a dgla has been extensively studied by several authors, among them [19, 36, 37]. In our explicit case we have

**Definition 4.1.** The derived algebra $D_{\mathfrak{sy}m}^*(P, Q)$ of $\mathfrak{sy}m^*(P, Q)$ is the complex

$$D_{\mathfrak{sy}m}^*(P, Q) := \mathfrak{sy}m^{*-0}(P, Q)[1]$$

together with the differential $\delta := [Q, -]$ and the derived bracket

$$[a, b] := [[Q, a], b].$$

It is a simple calculation to show that $D_{\mathfrak{sy}m}^*(P, Q)$ becomes a dg-Leibniz algebra; namely that $\delta$ and $[,]$ satisfy the properties

$$\delta[a, b] = [\delta a, b] + (-1)^{\|a\|} b \delta a$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{\|a\|\|b\|} [b, [a, c]]$$

where $\|a\|$ denotes the degree of $a$ in $D_{\mathfrak{sy}m}^*(P, Q)$, and therefore $\|a\| = |a| + 1$ where $|a|$ is the degree of $a$ in $\mathfrak{sy}m^*(P, Q)$.

The derived algebra is then

$$D_{\mathfrak{sy}m}^k(P, Q) \cong \begin{cases} \mathfrak{X}M \oplus \Omega^{n-1}M & \text{if } k = 0 \\ \Omega^{n-1-k}M & \text{if } k < 0 \end{cases}$$

where the differential $[Q, -]$ becomes the De Rham differential

$$\Omega^0 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2} M \xrightarrow{d} \mathfrak{X}M \oplus \Omega^{n-1}M,$$

and the brackets are given by the formulas

$$[\iota_X + \alpha \partial_t, \iota_Y + \beta \partial_t] = \iota_{[X,Y]} + (\mathcal{L}_X \beta - \iota_Y d\alpha - \iota_Y \iota_X H) \partial_t$$

$$[\iota_X + \alpha \partial_t, \eta \partial_t] = \mathcal{L}_X \eta \partial_t$$

$$[\mu \partial_t, \eta \partial_t] = 0.$$

**Remark 4.2.** In the case of $n = 2$ and $H$ a closed three form, the derived algebra $D_{\mathfrak{sy}m}^*(P, Q)$ becomes the complex

$$\Omega^0 M \xrightarrow{d} \mathfrak{X}M \oplus \Omega^1 M$$

where the bracket $[,]$ is precisely the $H$-twisted Courant-Dorfman bracket of the exact Courant algebroid $TM \oplus T^*M$ (cf. [7, 24]).
4.2. Hamiltonian symmetries of $Q = d + H \partial_t$. Let us recall that the degree zero symmetries of $Q$ in $\mathfrak{sym}^*(P,Q)$ consist of vector fields and $n$-forms $L_X + B \partial_t$ that commute with $Q$, and this happens whenever $L_X H - dB = 0$. This implies that if $H$ is not invariant in the direction of the vector field $X$, the error of not being invariant is parameterized by the forms such that $dB = L_X H$.

Let us now consider the sub-dgla $\mathfrak{gsym}^*(P,Q)$ of $\mathfrak{sym}^*(P,Q)$ which in degree zero consists only of the vector fields of $M$ that leave $H$ fixed, i.e.

$$\mathfrak{gsym}^0(P,Q) = \{ L_X + B \partial_t \in \mathfrak{gsym}^0(P,Q) | L_X H = 0 = B \},$$

that in degree $-1$ the anticommutator with $Q$ has no component with $\partial_t$, i.e.

$$\mathfrak{gsym}^{-1}(P,Q) = \{ \iota_X + \alpha \partial_t \in \mathfrak{gsym}^{-1}(P,Q) | d\alpha + \iota_X H = 0 \},$$

and that $\mathfrak{gsym}^k(P,Q) = \mathfrak{sym}^k(P,Q)$ for $k < -1$.

It is a simple calculation to show that indeed $\mathfrak{gsym}^*(P,Q)$ is a dgla; and we should think of it as the infinitesimal symmetries of $P$ obtained from the geometrical symmetries of $M$ (vector fields of $M$ which leave $H$ fixed) together with their higher homotopies.

Note that in the case that $n = 1$ and $H$ is a symplectic form, the elements in $\mathfrak{gsym}^{-1}(P,Q)$ are precisely pairs $\iota_X + f \partial_t$ of vector fields in $M$ and functions on $M$ such that

$$df + \iota_X H = 0.$$

This equation implies that $X$ is the Hamiltonian vector field that the function $f$ defines with respect to the symplectic form.

Moreover, if we consider the derived bracket on $\mathfrak{gsym}^{-1}(P,Q)$ we have that

$$[\iota_X + f \partial_t, \iota_Y + g \partial_t] = [L_X, \iota_Y + g \partial_t] = \iota_{[X,Y]} + (L_X g) \partial_t,$$

which in particular implies that the derived bracket recovers the Poisson bracket on functions because

$$[\iota_X + f \partial_t, \iota_Y + g \partial_t] = \iota_{[X,Y]} + \{f, g\}$$

where $\{f, g\}$ is the Poisson bracket with respect to the symplectic form $H$.

If we take the projection map

$$\mathfrak{gsym}^{-1}(P,Q) \to C^\infty M \quad \iota_X + f \partial_t \mapsto f$$

we see, due to the nondegeneracy of the symplectic form, that we have an isomorphism of Lie algebras from the derived algebra $\mathfrak{gsym}^{-1}(P,Q)$ to the Lie algebra $(C^\infty M, \{\}, \)$. Having the previous example in mind, one can generalize the algebra of Hamiltonian symmetries of forms $H$ of higher degree, by simply taking the derived algebra of the dgla $\mathfrak{gsym}^*(P,Q)$:

**Definition 4.3.** Let $H$ be a closed $n+1$ form on a manifold $M$. Denote by $\mathfrak{Ham}^*(H)$ the Hamiltonian algebra of symmetries of the form $H$ and let this algebra be the derived algebra of the dgla $\mathfrak{gsym}^*(P,Q)$ whenever $Q = d + H \partial_t$ is a homology vector field over $P = T[1]M \oplus \mathbb{R}[n]$; that is

$$\mathfrak{Ham}^*(H) := D\mathfrak{gsym}^*(P,Q).$$

We have then that $\mathfrak{Ham}^*(H)$ is a dg-Leibniz algebra that as a complex is

$$\Omega^0 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2} M \xrightarrow{d} \mathfrak{gsym}^0(H).$$
with $\mathcal{Ham}^0(H) = \{\iota_X + \alpha \partial_t \in \mathfrak{sym}^{-1}(P, Q) | d\alpha + \iota_X H = 0\}$, and whose brackets become

$$\begin{align*}
[\iota_X + \alpha \partial_t, \iota_Y + \beta \partial_t] &= \iota_{[X,Y]} + (\mathcal{L}_X \beta) \partial_t \\
[\iota_X + \alpha \partial_t, \eta \partial_t] &= (\mathcal{L}_X \eta) \partial_t \\
[\eta \partial_t, \iota_X + \alpha \partial_t] &= (-1)^{n-|\eta|} (\iota_X d\eta) \partial_t \\
[\eta \partial_t, \mu \partial_t] &= 0.
\end{align*}$$

The dg-Leibniz algebra $\mathcal{Ham}^*(H)$ is by construction a sub-dg Leibniz algebra of $\mathcal{Dsym}^*(P, Q)$, and therefore in the case that $H$ is a three form we have that $\mathcal{Ham}^*(H)$ can be seen as a sub-dg Leibniz algebra of the Courant-Dorfman algebra of the exact Courant algebroid twisted by $H$.

### 4.2.1. $n$-plectic structures.

In the case that the closed $n+1$-form $H$ is non-degenerate in the sense that

$$\forall v \in T_x M, \iota_v H = 0 \Rightarrow v = 0,$$

the form $H$ has been called $n$-plectic [4, 5]. In the $n$-plectic case, the degree zero part of the dg-Leibniz algebra of hamiltonian symmetries is isomorphic to the $n-1$ forms on $M$

$$\mathcal{Ham}^0(H) \xrightarrow{\cong} \Omega^{n-1} M \quad (\iota_X + \alpha \partial_t) \mapsto \alpha,$$

as we know that in this case the equation $\iota_X H + d\alpha = 0$ determines $X$ uniquely; let us then denote by $X_\alpha$ the vector field on $M$ such that

$$d\alpha + \iota_X H = 0.$$

The complex $\mathcal{Ham}^*(H)$ is then isomorphic to

$$\Omega^0 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2} M \xrightarrow{d} \Omega^{n-1} M$$

whose bracket becomes

$$\begin{align*}
[\alpha, \beta] &= \mathcal{L}_{X_\alpha} \beta \\
[\alpha, \eta] &= \mathcal{L}_{X_\alpha} \eta \\
[\eta, \alpha] &= (-1)^{n-|\eta|} \iota_{X_\alpha} d\eta \\
[\eta, \mu] &= 0.
\end{align*}$$

with $\alpha, \beta$ forms of degree $n-1$, and $\eta, \mu$ forms of degree less than $n-1$.

**Remark 4.4.** When $H$ is a symplectic form $\mathcal{Ham}^*(H)$ is isomorphic to the Poisson algebra $(C^\infty M, \{\cdot, \cdot\})$.

**Remark 4.5.** When $H$ is a non degenerate closed form, the brackets on $\mathcal{Ham}^*(H)$ has been already defined by [4, Def. 3.3] where it has been called the hemi-bracket in the sense of Roytenberg [25], and it has been further studied by Rogers [23, Section 6].

We note that the bracket that we have defined in $\mathcal{Ham}^*(H)$ differs from the one defined in [4, 23] in the case that $\alpha$ is an $n-1$ form and $\eta$ is a form of degree $k < n-1$; in our work the bracket $[\eta, \alpha] = (-1)^{n-|\eta|} \iota_{X_\alpha} d\eta$ and in the works of [4, 23] the bracket is zero.

**Remark 4.6.** In [23, Theorem 1] it has been shown that there is an alternative definition for the brackets in $\mathcal{Ham}^*(H)$. If we use the notation of [23] and we denote the complex $L_\infty(M, H) := \mathcal{Ham}^*(H)$, the brackets that make it into an $L_\infty$ algebra are

$$[\alpha_1, \ldots, \alpha_k] := \pm \iota_{(X_{\alpha_1} \wedge \cdots \wedge X_{\alpha_k})} H$$

when all the $\alpha$’s are forms of degree $n-1$, and zero otherwise.
We believe that the $L_\infty$ algebra $L_\infty(M, H)$ must be isomorphic to the dg-Leibniz algebra $\mathfrak{Ham}^*(H)$ in the category of strongly homotopic Leibniz algebras. We point out that as $\mathfrak{Ham}^*(H)$ was defined via a derived bracket from a dgla, it comes provided with higher brackets defined by the formula

$$[a_1, ..., a_k] := \ldots [[Q, a_1], a_2]..., a_k].$$

It was shown by Uchino [32] that these higher derived brackets satisfy the higher coherence of a strongly homotopic Leibniz algebra. Therefore both $L_\infty(M, H)$ and $\mathfrak{Ham}^*(H)$ are strongly homotopic Leibniz algebras, and as such they must be isomorphic.

We finally point out that in [4, Theorem 4.6] it was shown that, when $H$ is a non-degenerate closed three form, the algebras $L_\infty(M, H)$ and $\mathfrak{Ham}^*(H)$ are isomorphic as Lie 2-algebras in the sense of Roytenberg [25].

### 4.2.2. Hamiltonian group actions.

Following the discussion above, we have two different approaches for determining when the action of a group $G$ into the dg-manifold $P = (T[1]M \oplus R[n], Q = d + H \partial t)$ is given by hamiltonian symmetries; we either work with the dgla $\mathfrak{sym}^*(P, Q)$ or with the (sh)-Leibniz algebra $\mathfrak{Ham}^*(H)$.

**Approach 1.** We say that the group $G$ acts with hamiltonian symmetries on the dg-manifold $P = (T[1]M \oplus R[n], Q = d + H \partial t)$ whenever the infinitesimal action induces a strict map of dgla’s

$$(\mathfrak{g}[1] \rightarrow \mathfrak{g}) \rightarrow \mathfrak{sym}^*(P, Q).$$

A strict map of dgla’s

$$\Phi^*: (\mathfrak{g}[1] \rightarrow \mathfrak{g}) \rightarrow \mathfrak{sym}^*(P, Q)$$

consists of maps

$$\Phi^0 : \mathfrak{g} \rightarrow \mathfrak{sym}^0(P, Q) \quad \quad \Phi^{-1} : \mathfrak{g}[1] \rightarrow \mathfrak{sym}^{-1}(P, Q)$$

$$a \mapsto \mathcal{L}_{X_a} \quad \quad a \mapsto \iota_{X_a} + \alpha_a \partial t$$

satisfying the equations

$$[\Phi^{-1}(a), \Phi^{-1}(b)] = \Phi^{-2}([a, b])$$

$$\Phi^0(a) = [Q, \Phi^{-1}(a)]$$

$$[\Phi^0(a), \Phi^{-1}(b)] = \Phi^{-1}([a, b]),$$

which are equivalent to the equations

$$(4.1) \quad \quad \iota_{X_a} \alpha_b + \iota_{X_b} \alpha_a = 0$$

$$\quad \quad d\alpha_a + \iota_{X_a} H = 0$$

$$\quad \quad \mathcal{L}_{X_a} \alpha_b = \alpha_{[a, b]}.$$

These strict maps of dgla’s can be alternatively understood as some closed and invariant elements in the Cartan model of equivariant cohomology, let us see how:

**Lemma 4.7.** The strict maps of dgla’s $(\mathfrak{g}[1] \rightarrow \mathfrak{g}) \rightarrow \mathfrak{sym}^*(P, Q)$ are in 1-1 correspondence with invariant and equivariantly closed elements of degree $n + 1$ of the form

$$H + \xi_a \Omega^a$$

in the Cartan model for equivariant cohomology $\Omega^* M \otimes S\mathfrak{g}^*$. 
Proof. Recall first that the equivariant differential in the Cartan model for equivariant cohomology is the operator $d + \Omega^a \iota_{X_a}$ (see [16]).

Therefore the form $H + \xi_a \Omega^a$ is equivariantly closed if

$$(d + \Omega^a \iota_{X_a})(H + \xi_b \Omega^b) = dH + (d\xi_a + \iota_{X_a} H)\Omega^a + (\iota_{X_a} \xi_b + \iota_{X_a} \xi_a)\Omega^a \Omega^b = 0;$$

furthermore, the form $H + \xi_a \Omega^a$ is invariant if $L_{X_b} H = 0$ and

$$L_{X_b} (\xi_a \Omega^a) = L_{X_b} \xi_a \Omega^a + \xi_a L_b \Omega^a$$

$$= L_{X_b} \xi_a \Omega^a - \xi_a f_{bc}^a \Omega^c$$

$$= L_{X_b} \xi_a \Omega^a - \xi_{[b,c]} \Omega^c$$

$$= (L_{X_b} \xi_a - \xi_{[b,a]}) \Omega^a = 0.$$

We can see that the form $H + \xi_a \Omega^a$ is invariant and equivariantly closed if and only if the forms $H$ and $\alpha_a = \xi_a$ satisfy the equations described in (4.1).

□

Note that in the case that $n \leq 2$, all the equivariant forms of degree $n + 1$ are of the form $H + \xi_a \Omega^a$, and therefore the strict maps of dgl'a's $(g[1] \rightarrow g) \rightarrow \mathfrak{sym}^*(P, Q)$ are in 1-1 correspondence with degree $n + 1$ invariant and equivariantly closed forms in the Cartan model for equivariant cohomology.

Approach 2. We say that the group $G$ acts with hamiltonian symmetries on the dg-manifold $P = (T[1]M \oplus R[n], Q = d + H \partial_t)$ whenever the infinitesimal action given by the (non-necessarily strict) map of dgl'a's

$$(g[1] \rightarrow g) \rightarrow \mathfrak{sym}^*(P, Q)$$

induces a map of Leibniz algebras

$$g \rightarrow \mathfrak{ham}^*(H)$$

at the level of the derived algebras associated to $g[1] \rightarrow g$ and $\mathfrak{sym}^*(P, Q)$ respectively.

A map of Leibniz algebras $\Psi : g \rightarrow \mathfrak{ham}^*(H)$ is given by a degree zero map

$$\Psi : g \rightarrow \mathfrak{ham}^0(H)$$

$$a \rightarrow \iota_{X_a} + \alpha_a \partial_t$$

with $d\alpha_a + \iota_{X_a} H = 0$, and such that

$$\Psi([[a, b]]) = [\Psi(a), \Psi(b)].$$

The derived bracket on $g$ induced from the structure on $g[1] \rightarrow g$ is clearly the Lie bracket. Therefore the fact that $\Psi$ preserves the bracket is equivalent to the equations

$$\iota_{X_{[a, b]}} + \alpha_{[a, b]} \partial_t = [\iota_{X_a} + \alpha_a \partial_t, \iota_{X_b} + \alpha_b \partial_t] = \iota_{[X_a, X_b]} + L_{X_{[a, b]}} \alpha_b \partial_t$$

which imply that $L_{X_{a}} \alpha_b = \alpha_{[a, b]}$ for all $a, b \in g$.

The maps of Leibniz algebras $\Psi : g \rightarrow \mathfrak{ham}^*(H)$ can also be characterized by invariant forms in the Cartan model of equivariant cohomology as follows:

Lemma 4.8. The maps of Leibniz algebras $g \rightarrow \mathfrak{ham}^*(H)$ are in 1-1 correspondence with degree $n + 1$ invariant forms of the type

$$H + \xi_a \Omega^a$$
in the Cartan model of equivariant cohomology, such that
\((d + \Omega^b \iota_{X_b})(H + \xi_a \Omega^a) = \frac{1}{2} c_{a,b} \Omega^a \Omega^b \)
with \(c_{a,b}\) constant functions.

**Proof.** We have seen that a map of Leibniz algebras \(\Psi : g \to \mathfrak{ham}^*(H), a \mapsto \iota_{X_a} H + \alpha_a \partial_t\) determines the equations \(d\alpha_a + \iota_{X_a} H = 0\) and \(\mathcal{L}_{X_a} \alpha_b = \alpha_{[a,b]}\). From the proof of Lemma 4.7 we know that the equations \(\mathcal{L}_{X_a} H = 0\) and \(\mathcal{L}_{X_a} \alpha_b = \alpha_{[a,b]}\) are equivalent to the statement that the \(n + 1\) form \(H + \alpha_a \Omega^a\) is invariant.

Now, applying the operator \(\iota_{X_b}\) to both sides of the equation \(d\alpha_a + \iota_{X_a} H = 0\) we get that \(\iota_{X_b} d\alpha_a + \iota_{X_a} \iota_{X_b} H = 0\); hence we have that
\[(4.2) \quad \iota_{X_b} d\alpha_a + \iota_{X_a} d\alpha_b = 0.\]
Moreover we have that
\(\mathcal{L}_{X_a} \alpha_b = \alpha_{[a,b]} = -\mathcal{L}_{X_b} \alpha_a\)
and therefore \(\mathcal{L}_{X_a} \alpha_b + \mathcal{L}_{X_b} \alpha_a = 0\), which together with the equation (4.2) implies that
\(d(\iota_{X_a} \alpha_b + \iota_{X_b} \alpha_a) = 0\)
and therefore the functions
\(c_{a,b} := \iota_{X_a} \alpha_b + \iota_{X_b} \alpha_a\)
are constant.

We have then that the information encoded in a map of Leibniz algebras \(\Psi\) is the same as the information encoded in the invariance of the \(n + 1\) form \(H + \alpha_a \Omega^a\) together with the equation
\((d + \Omega^b \iota_{X_b})(H + \xi_a \Omega^a) = \frac{1}{2} c_{a,b} \Omega^a \Omega^b \)
with \(c_{a,b}\) constant functions. \(\square\)

Following lemmas 4.7 and 4.8 we see that any strict map of dgla’s \((g[1] \to g) \to \mathfrak{ham}^*(P,Q)\) induces a map of Leibniz algebras \(g \to \mathfrak{ham}^*(H)\), but the converse is only true whenever the constant functions \(c_{a,b}\) are all zero. At this point it is unclear which of the two approaches presented before for hamiltonian actions is the more appropriate one.

**Remark 4.9.** When \(n = 1\), namely when \(H\) is a closed 2-form, both approaches to Hamiltonian groups actions described above are clearly equivalent.

When \(n = 2\), namely when \(H\) is a closed 3-form, the second approach to Hamiltonian actions described above is the one that has been used in [7, Thm. 2.13] when considering trivially extended \(G\)-actions on exact Courant algebroids.

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