**Implications of higher order tensor in Einstein field equations on vacuum condition**

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**Abstract.** We investigated the implications of introducing the higher order tensor in the Einstein field equations for the Schwarzschild metric on the vacuum condition and non de Sitter. Since the components of this tensor contain the fourth-order derivative, the solution of the Einstein field equations was solved by using the Frobenius method. To see the implications, we tested two cases, i.e., the deviation of light and the time delay of light. We found that the deviation of light has no correction while the time delay of light needs a correction.

**1. Introduction**

The behavior of the macroscopic object can be well studied by means of the general relativity theory to complete the Newtonian mechanics. At the beginning of 20th century, this theory was verified by the observations, such as the perihelion of mercury and the deviation of light. This theory states that all the physical laws should be valid for all the observers both in the inertial system and in the non-inertial system. At the very beginning, Einstein introduced the curve space-time concept, which is caused by the massive object. Then, he formulated the Einstein field equations in the tensor formulation, which connect the curved space-time tensor and the energy-momentum tensor.

Later, the astronomical observations open the new questions, which need the answers. One of the important questions is referred to why the inflation of the universe occurs in the observation whereas it should not exist. Later, de Sitter, who inserted the cosmological constant to the Einstein field equations, believed that this constant is responsible for the inflation. This constant is assumed to be very small to link the classical observation, such as the perihelion of mercury or the deviation of light. However, this additional term does not give the satisfied answers to explain the mechanism of the inflation.

The attempts to give the complete answer for the inflation drive some physicists to explore their idea. Since all the tensors in the Einstein field equation obey the free divergence, the new tensors can be included as long as the latter condition is fulfilled, a good review on the new introduced tensors can be found in Ref. [1]. One of the new tensors was proposed by Deser and Tekin by including the fourth-order derivative [2]. The explicit form of the tensor was derived by minimizing the Lagrangian density, which is a function of the quadratic Ricci tensor and the quadratic Ricci scalar. Then, they inserted this tensor to the Einstein field equations. This modification then becomes an interesting topic in astrophysics and cosmology [2-6]. Some similar treatments can also be found in Refs. [7-9].

The purpose of the paper is to investigate the consequences of adding this tensor to the classical tests, i.e., the deviation of light and the time delay of light using the perturbative solution. In section 2, we give the components of all available tensors in the Einstein field equations for the vacuum condition and
derive the solution for the Schwarzschild metric. Then, in section 3 we apply the solution to see the consequences for the deviation of light and the time delay of light. The conclusions of this paper will be given in section 4. Note that the discussion can be extended to investigate the dynamics of universe including the inflation of universe via the Robertson-Walker metric.

2. Method and material

This section concerns with how to obtain the perturbative solution from the previous study. In their paper, Deser and Tekin proposed a Lagrangian density, which is a function of the quadratic Ricci scalar $R$ and the Ricci tensor $R_{\mu\kappa \nu \lambda} = \alpha R^2 + \beta R_{\mu\kappa}^2$ (2). Those two functions are linearly independent while $\alpha$ and $\beta$ are the arbitrary constants. In this case, the Greek index runs from 0 to 3, which represents the space-time index. Therefore, by minimizing the action function $S = \int d^4 x \sqrt{-g} L$ with respect to $g_{\mu\kappa}$, where $g$ is a determinant of the metric tensor $g_{\mu\kappa}$, the tensor containing the higher order derivative can be given as

$$
\Phi_{\mu\kappa} = 2\alpha R \left( R_{\mu\kappa} - \frac{1}{4} R g_{\mu\kappa} \right) + (2\alpha + \beta) \left( g_{\mu\kappa} \square - \nabla_\mu \nabla_\kappa \right) R + \beta \left( R_{\mu\kappa} - \frac{1}{2} R g_{\mu\kappa} \right)
$$

$$+ 2\beta \left( R_{\mu\kappa\sigma\rho} - \frac{1}{4} g_{\mu\kappa} R_{\sigma\rho} \right) R^{\sigma\rho},$$

(1)

where $\square = \nabla^\mu \nabla_\mu$ is d’Alembertian operator. The tensor in Eq. (1) is the new curvature tensor, which has fourth-order derivative. In addition, the free divergence of this tensor $\nabla^\kappa \Phi_{\mu\kappa} = 0$ is satisfied, as hold for all the tensors in the Einstein field equations.

The modified Einstein field equations with the cosmological constant (de Sitter model) then are written as

$$G_{\mu\kappa} + \Lambda g_{\mu\kappa} + \Phi_{\mu\kappa} = \lambda T_{\mu\kappa}.$$  

(2)

All the tensors in the left-hand side denote the curvature tensors while the tensor in the right-hand side describes the energy-momentum tensor. In Eq. (2), the cosmological constant $\Lambda$ is positive for de Sitter model and negative for anti de Sitter model. Moreover, the coupling constant $\lambda$ is related to the Newton’s gravitational constant in the classical mechanics. Note that the vacuum condition can be satisfied if all the components of the energy-momentum tensor vanish, i.e., $T_{\mu\kappa} = 0$.

To get our intention, we initially consider the static Schwarzschild metric with the convention of the metric tensor $(+ - - -)$

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2.$$  

(3)

The non-zero components of the Ricci tensor, the scalar Ricci, and the tensor containing the higher order derivative for this metric can be given as (10)

$$R_{00} = e^{\nu-\lambda} \left( \frac{\nu^2}{4} - \frac{1}{4} \lambda^2 \nu'' + \frac{\nu'}{2} + \nu \right),$$  

$$R_{11} = -\frac{\nu'}{2} + \frac{1}{4} \lambda^2 \nu'' + \frac{\nu'}{4} + \frac{\lambda'}{r},$$  

$$R_{22} = \left\{ -\frac{1}{2} \nu' \nu + \frac{1}{2} \lambda \nu - 1 \right\} e^{-\lambda} + 1,$$

$$R_{33} = R_{22} \theta,$$

$$R = e^{-\lambda} \left( \frac{1}{2} r^2 \nu'^2 - \frac{\lambda'}{2} \nu' \nu'' + 2 \nu' + \nu - 2 \frac{\lambda'}{2} \nu - \frac{2}{r^2} \nu'' \right),$$

$$\Phi_{00} = \alpha \left\{ e^{\nu-2\lambda} \left( \frac{1}{8} \nu^4 - \frac{1}{4} \lambda' \nu^3 + \frac{1}{2} \nu' \nu'' + \frac{\nu'}{2} - \frac{11}{8} \nu'^2 \lambda' + \frac{3}{2} \nu' \nu' \lambda' + \frac{3}{2} \nu'^2 \right) \right\}$$

$$+ \left\{ \frac{3}{8} \nu'^2 \nu''^2 - 2 \frac{\nu'}{2} \nu'' + 2 \frac{\nu'^2}{r^2} - 8 \frac{\lambda'}{r^3} - \frac{10 \nu'}{3} + \frac{\lambda'}{2} \nu'' \lambda' - \frac{11}{2} \nu'^2 \lambda' \right\}$$

$$- \frac{8 \nu'^2 \lambda'}{r} + 6 \frac{\nu'}{2} \nu'' - 2 \frac{\nu'^2}{r} + 16 \frac{\lambda'}{r^3} + 16 \frac{\lambda'}{r^4} + 2 \frac{\nu'}{r^2} + 14 \frac{\nu'^2}{r^2} + 6 \frac{\lambda'}{r^2}$$

$$- \frac{8}{r^2} + 4 \frac{\lambda'}{r^2} + 4 \nu' \lambda' - 2 \nu' \nu'' + \frac{\lambda'}{r^2} - 2 \nu'' + \frac{\lambda'}{r^2} + 4 \frac{\lambda'}{r^2}$$

$$+ 12 \frac{\nu'^2 - 2 e^{\nu-\lambda}}{r^2}.$$
where

\[ e^{-\lambda} \left( \frac{1}{8} v'^4 - 3 \frac{1}{8} v'^3 r - 4 \lambda v'' + \frac{17}{8} v''^2 r^2 - 7 v''^2 \lambda r^2 + \frac{1}{2} v''^3 r - 4 \lambda v'' r - 4 v''^2 r \right) \]

and

\[ e^{-\lambda} \left( \frac{1}{4} v'^4 - 3 \frac{1}{4} v'^3 r - 4 \lambda v'' + \frac{15}{4} v''^2 r^2 - 3 v''^2 \lambda r^2 + \frac{3}{4} v''^3 r - 3 \lambda v'' r - 3 v''^2 r \right) \]

Here, the prime (') symbol describes the derivative with respect to the radius \( r \). By writing the solution \( v(r) = v_0(r) + v_1(r) + \lambda_0(r) + \lambda_1(r) \), where \( v_0 \) and \( \lambda_0 \) are the non-perturbative solutions, and expanding the perturbative terms

\[ v_1(r) = \sum_{n=0}^{\infty} a_n r^{n+s}, \quad \lambda_1(r) = \sum_{n=0}^{\infty} b_n r^{n+s}, \]

the case of \( a \neq 0 \) and \( b = 0 \) leads to the perturbative solution using the Frobenius method for \( v(r) \) and \( \lambda(r) \), respectively [10].

\[ v(r) = \ln \left( 1 - \frac{2m}{r} \right) + a_0, \quad \lambda(r) = - \ln \left( 1 - \frac{2m}{r} \right), \]

where \( a_0 \) is the zeroth order of the expansion series in the Frobenius method. The Frobenius method can be found in some textbooks, such in Refs. (11-12). Thus, the static Schwarzschild solution is given by

\[ ds^2 = C \left( 1 - \frac{2m}{r} \right) dt^2 - \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]
3. Results and discussions
This section discusses two implications caused by the higher order tensor for the solution obeying Eq. [15]. In this case, we take the deviation of light passing through the gravitational field of the sun and the time delay of light.

3.1. The deviation of light
To consider the deviation of light, let us consider the null geodesic, which describes the dynamics of light in the general relativity. That geodesic is obtained by differentiating Eq. (15) with respect to the proper time $\tau$

$$K = C \left(1 - \frac{2m}{r}\right) t^2 - \left(1 - \frac{2m}{r}\right)^{-1} r^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2 = 0, \quad (16)$$

with the dot ( . ) symbol represents the derivative of $\tau$. Then, the dynamics of object in the general relativity theory is represented by the geodesic equation

$$\frac{\partial K}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial K}{\partial x^\mu} \right) = 0. \quad (17)$$

Taking $\mu = 0, 2, 3$ in Eq. (17), we obtain three equations of motion

$$\frac{d}{d\tau} \left[ (1 - 2m/r)\dot{t} \right] = 0, \quad (18)$$
$$\frac{d}{d\tau} (r^2 \dot{\theta}) - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0, \quad (19)$$
$$\frac{d}{d\tau} (r^2 \sin^2 \theta \dot{\phi}) = 0. \quad (20)$$

By observing Eqs. (18-20), we deduce that the time delay of light has no correction since the constant $C$ does not appear. For the further discussion, by imposing the boundary condition for the equatorial plane $\theta = \pi/2$ and $\dot{\theta} = 0$, and using the perturbative solution, the deviation of light can be given as $\delta = 4m/R$, where $R$ is the minimum distance between the null coordinates and the path of light, the complete derivation can be found in the textbooks of general relativity theory [13-17].

3.2. The time delay of light
This concept was originally deduced by Shapiro with the same condition in the deviation of light. According to the concept, the curved space-time caused by the gravitational field of the massive object leads to the longer time than the flat time-space. Let us consider the null geodesic on the equatorial plane $\theta = \pi/2$

$$K = C \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 = 0. \quad (21)$$

After that, we define the nearest radius from the sun as $r \sin \varphi = R$. Differentiating the both sides and replacing $\varphi$ with $r$ lead Eq. (21) to

$$dt^2 \approx \frac{dr^2}{C r} \left[ \left(1 - \frac{2m}{r}\right)^{-2} + \left(1 - \frac{2m}{r}\right)^{-1} \frac{R^2}{r^2 - R^2} \right]. \quad (22)$$

The next step is to expand the right-hand side in Eq. (22) in terms of $m/r$ for the first order

$$dt \approx \frac{r}{\sqrt{C} \sqrt{r^2 - R^2}} \left[ 1 + \frac{2m}{r} - \frac{m R^2}{r^3} \right] dr. \quad (23)$$

By integrating Eq. (21) both sides, we find

$$t(r, R) = \int_R^r \frac{r}{\sqrt{C} \sqrt{r^2 - R^2}} \left[ 1 + \frac{2m}{r} - \frac{m R^2}{r^3} \right] dr$$
$$= \frac{1}{\sqrt{C}} \left( r - 2m R^2 + 2m \ln \left[ \frac{r + \sqrt{r^2 - R^2}}{r} \right] \right). \quad (24)$$

It is clear that the tensor containing the higher order derivative gives the correction $1/\sqrt{C}$ in the total time delay of light. For the next study, we can extend the discussion to investigate the effect of this tensor for the inflation of the universe.
4. Conclusions
We find that adding the tensor containing the higher order derivative in the Einstein field equation gives the correction for the time delay of light while there is no correction for the deviation of light. The next question is how to find the value of $C$ in the case of the time delay of light. We suppose that determining this value needs some appropriate conditions.

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