Fueter’s theorem for the biregular functions of Clifford analysis

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Abstract

In this paper we present a generalization of the Fueter’s theorem for monogenic functions to the case of the biregular functions.

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1 Introduction

Let $\mathbb{R}_{0,m}$ be the $2^m$-dimensional real Clifford algebra generated by the standard basis $\{e_1, \ldots, e_m\}$ of the Euclidean space $\mathbb{R}^m$ (see [5]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations

$$e_je_k + e_ke_j = -2\delta_{jk}, \quad j, k = 1, \ldots, m$$

where $\delta_{jk}$ denotes the Kronecker delta. A basis for the algebra is then given by the elements $e_A = e_{j_1} \cdots e_{j_k}$ where $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$ and $j_1 < \cdots < j_k$ ($e_{\emptyset} = 1$ is the identity element). A general element $a$ of $\mathbb{R}_{0,m}$ may thus be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R}$$
and its conjugate $\overline{a}$ is defined by

$$\overline{a} = \sum_A a_A \overline{e}_A, \quad \overline{e}_A = e_{j_k} \ldots e_{j_1}, \quad \overline{e}_j = -e_j, \ j = 1, \ldots, m.$$  

Suppose now that $\Omega$ is an open subset of $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ and let $f$ be an $\mathbb{R}_{0,m}$-valued function defined in $\Omega$. Then $f$ is of the form

$$f(x, y) = \sum_A f_A(x, y)e_A, \quad (x, y) = (x_0, \ldots, x_m, y_0, \ldots, y_m) \in \Omega$$

where the functions $f_A$ are $\mathbb{R}$-valued. The variables $x$ and $y$ will also be identified with the paravectors $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_j e_j$ and $y = y_0 + \underline{y} = y_0 + \sum_{j=1}^m y_j e_j$, respectively. Next, we introduce the generalized Cauchy-Riemann operators

$$\partial_x = \partial_{x_0} + \sum_{j=1}^m e_j \partial_{x_j}, \quad \partial_y = \partial_{y_0} + \sum_{j=1}^m e_j \partial_{y_j},$$

which factorize the Laplacian in the variables $x$ and $y$, respectively, i.e.

$$\Delta_x = \sum_{j=0}^m \partial_{x_j}^2 = \partial_x \partial_{\overline{x}} = \overline{\partial}_x \partial_x, \quad \Delta_y = \sum_{j=0}^m \partial_{y_j}^2 = \partial_y \partial_{\overline{y}} = \overline{\partial}_y \partial_y.$$  

In this paper we shall deal with the so-called biregular functions, which are defined as follows.

**Definition 1** Assume that $f \in C^1(\Omega)$. Then the function $f$ is called biregular in $\Omega$ if it fulfills in $\Omega$ the system $\partial_x f = f \partial_y = 0$.

The biregular functions were introduced in the 1980s by Brackx and Pincket as an extension to two higher dimensional variables of the standard monogenic functions, i.e. $C^1$ functions $f : \Omega \subset \mathbb{R}^{m+1} \rightarrow \mathbb{R}_{0,m}$ satisfying $\partial_x f = 0$ (or $f \partial_x = 0$). Some of the main properties of the biregular functions may be found in e.g. [2, 3, 4, 18, 19]. For a detailed study of the monogenic functions we refer the reader to [1, 8, 10].

An important technique to generate monogenic functions is the so-called Fueter’s theorem. Discovered by R. Fueter in the setting of quaternionic analysis (see [9]), this technique has been extended to $\mathbb{R}^{m+1}$ within the framework of Clifford analysis in [17, 20] ($m$ odd) and in [15] ($m$ even). For other works on this topic we refer the reader to e.g. [6, 7, 11, 12, 13, 14, 16].

For $m$ odd, Fueter’s theorem states the following:
Theorem 1 Let \( u + iv \) be a holomorphic function in some open subset \( \Xi \) of the upper half of the complex plane \( \mathbb{C} \) and assume that \( P_k(\bar{x}) \) is a homogeneous monogenic polynomial of degree \( k \) in \( \mathbb{R}^m \). Put \( \omega = x/r \), with \( r = |x| \). If \( m \) is odd, then the function

\[
\Delta_{x}^{k+\frac{m-1}{2}} \left[ (u(x_0, r) + \omega v(x_0, r))P_k(\bar{x}) \right]
\]

is (left) monogenic in \( \Omega = \{ x \in \mathbb{R}^{m+1} : (x_0, r) \in \Xi \} \).

Fueter’s theorem discloses a remarkable connection existing between the classical holomorphic functions and its higher dimensional counterpart (i.e. the monogenic functions). The purpose of this paper is to generalize this important result to the case of the biregular functions. We shall see how a similar relationship exists between the latter and the holomorphic functions of two complex variables.

2 Auxiliary results

We group together here a series of lemmas which will be needed in the proof of our main result.

For an \( \mathbb{R} \)-valued \( C^1 \) function \( \phi \) and an \( \mathbb{R}_{0, m} \)-valued \( C^1 \) function \( g \), it is clear that

\[
\partial_{x}(\phi g) = (\partial_{x}\phi)g + \phi(\partial_{x}g),
\]

and

\[
(\phi g)\partial_{x} = g(\partial_{x}\phi) + \phi(g\partial_{x}).
\]

Moreover, for a vector-valued \( C^1 \) function \( f = \sum_{j=1}^{m} f_j e_j \), we have

\[
\partial_{x}(f g) = (\partial_{x}f) g - f(\partial_{x}g) - 2 \sum_{j=1}^{m} f_j (\partial_{x}g).
\]

Indeed,

\[
\partial_{x}(f g) = \sum_{j=1}^{m} e_j \left( (\partial_{x}f_j)g + f_j(\partial_{x}g) \right) = (\partial_{x}f) g + \sum_{j=1}^{m} e_j f(\partial_{x}g),
\]

which results in (3) on account of the equality

\[
e_j f = -f e_j - 2f, \quad j = 1, \ldots, m.
\]

In the same spirit we can also prove:

\[
(g f)\partial_{y} = g(f \partial_{y}) - (g f)(\partial_{y}) - 2 \sum_{j=1}^{m} f_j (\partial_{y}g).
\]
Lemma 1 Suppose that $f(t_1, \ldots, t_d)$ is an $\mathbb{R}$-valued $C^\infty$ function on $\mathbb{R}^d$ and that $D_{t_j}$ and $D^{t_j}$ are differential operators defined by

$$D_{t_j}(n)\{f\} = \left(\frac{1}{t_j} \partial_{t_j}\right)^n f, \quad j = 1, \ldots, d,$$

$$D^{t_j}(n)\{f\} = \partial_{t_j} \left( \frac{D^{t_j}(n-1)\{f\}}{t_j} \right), \quad j = 1, \ldots, d,$$

for $n \geq 1$ and $D_{t_j}(0)\{f\} = D^{t_j}(0)\{f\} = f$. Then one has

(i) $\partial^2_{t_j} D_{t_j}(n)\{f\} = D_{t_j}(n)\{\partial^2_{t_j} f\} - 2nD_{t_j}(n+1)\{f\}$,

(ii) $\partial_{t_j} D_{t_j}(n-1)\{f/t_j\} = D^{t_j}(n)\{f\}$,

(iii) $D^{t_j}(n)\{\partial_{t_j} f\} = \partial_{t_j} D_{t_j}(n)\{f\}$,

(iv) $D_{t_j}(n)\{\partial_{t_j} f\} - \partial_{t_j} D^{t_j}(n)\{f\} = 2n/t_j D^{t_j}(n)\{f\}$,

(v) $\partial^2_{t_j} D^{t_j}(n)\{f\} = D^{t_j}(n)\{\partial^2_{t_j} f\} - 2nD^{t_j}(n+1)\{f\}$.

Proof. We prove (i) by induction. When $n = 1$, we have

$$\partial^2_{t_j} D_{t_j}(1)\{f\} = \frac{\partial^3_{t_j} f}{t_j} - \frac{2 \partial^2_{t_j} f}{t_j^2} + \frac{2 \partial_{t_j} f}{t_j^3} = D_{t_j}(1)\{\partial^2_{t_j} f\} - 2D_{t_j}(2)\{f\}$$

as desired.

Now we proceed to show that when (i) holds for a positive integer $n$, then it also holds for $n + 1$. Indeed,

$$\partial^2_{t_j} D_{t_j}(n+1)\{f\} = D_{t_j}(1)\{\partial^2_{t_j} D_{t_j}(n)\{f\}\} - 2D_{t_j}(2)\{D_{t_j}(n)\{f\}\}$$

$$= D_{t_j}(1)\left\{D_{t_j}(n)\{\partial^2_{t_j} f\} - 2nD_{t_j}(n+1)\{f\}\right\}$$

$$- 2D_{t_j}(n+2)\{f\}$$

$$= D_{t_j}(n+1)\{\partial^2_{t_j} f\} - 2(n+1)D_{t_j}(n+2)\{f\}.$$

Statement (ii) easily follows from the definition of $D^{t_j}(n)\{f\}$. Next, using (ii), we obtain (iii) as

$$D^{t_j}(n)\{\partial_{t_j} f\} = \partial_{t_j} D_{t_j}(n-1)\{\partial_{t_j} f/t_j\} = \partial_{t_j} D_{t_j}(n)\{f\}.$$
To obtain (iv) we use (i) and (ii):

\[
D_{t_j}(n)\{\partial_j f\} - \partial_j D^{t_j}(n)\{f\} = D_{t_j}(n)\{\partial_j f\} - \partial^2_{t_j} D_{t_j}(n-1)\{f/t_j\} = D_{t_j}(n)\{\partial_j f\} - D_{t_j}(n-1)\{\partial^2_{t_j} f/t_j\} + 2(n-1)D_{t_j}(n)\{f/t_j\} = 0.
\]

Finally, from (i)-(iii) it follows that

\[
\partial^2_{t_j} D^{t_j}(n)\{f\} = \partial^2_{t_j} D_{t_j}(n-1)\{f/t_j\} = \partial_{t_j} D_{t_j}(n)\{\partial_j f\} - 2n\partial_{t_j} D_{t_j}(n)\{f/t_j\} = D^{t_j}(n)\{\partial^2_{t_j} f\} - 2nD^{t_j}(n+1)\{f\},
\]

thus proving (v). \(\Box\)

Throughout this paper we denote by \(P_{k,l}(x,y)\) a homogeneous biregular polynomial in \(\mathbb{R}^m \times \mathbb{R}^m\) of degree \(k\) in \(x\) and degree \(l\) in \(y\), i.e.

\[
\partial_x P_{k,l}(x,y) = P_{k,l}(x,y) \partial_x = 0, \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^m, \\
P_{k,l}(t_1x, t_2y) = t_1^k t_2^l P_{k,l}(x,y), \quad t_1, t_2 \in \mathbb{R}.
\]

Put

\[\omega = x/r, \quad \nu = y/\rho\]

where \(r = |x|\) and \(\rho = |y|\).

**Lemma 2** Let \(h(x_0, r, y_0, \rho)\) be an \(\mathbb{R}\)-valued \(C^\infty\) function on \(\mathbb{R}^4\) such that

\[
\partial^2_{x_0} h + \partial^2_{y_0} h + \partial^2_{\rho} h = 0.
\]

Then

\[
\Delta^n_{\omega}(h P_{k,l}) = \prod_{j=1}^n (2k + m - (2j - 1))D_r(n)\{h\} P_{k,l}, \\
\Delta^n_{\nu}(h \omega P_{k,l}) = \prod_{j=1}^n (2k + m - (2j - 1))D_r(n)\{h\} \omega P_{k,l}, \\
\Delta^n_{\rho}(h P_{k,l}) = \prod_{j=1}^n (2l + m - (2j - 1))D_r(n)\{h\} P_{k,l},
\]

5
$$\Delta_g^n(h P_{k,l}) = \prod_{j=1}^{n} (2l + m - (2j - 1)) D^\rho(n) \{ h \} P_{k,l} \nu.$$ 

**Proof.** We first prove that for any $\mathbb{R}$-valued $C^2$ function $g(x_0, r, y_0, \rho)$ in the variables $x_0, r, y_0$ and $\rho$ the following equalities hold

$$\Delta_x(g P_{k,l}) = (\partial^2_{x_0} g + \partial^2_r g + (2k + m - 1) D_r(1) \{ g \}) P_{k,l},$$

$$\Delta_x(g \omega P_{k,l}) = (\partial^2_{x_0} g + \partial^2_r g + (2k + m - 1) D_r(1) \{ g \}) \omega P_{k,l},$$

$$\Delta_y(g P_{k,l}) = (\partial^2_{y_0} g + \partial^2_\rho g + (2l + m - 1) D_\rho(1) \{ g \}) P_{k,l},$$

$$\Delta_y(g P_{k,l} \nu) = (\partial^2_{y_0} g + \partial^2_\rho g + (2l + m - 1) D_\rho(1) \{ g \}) P_{k,l} \nu.$$ 

In fact, it follows that

$$\partial_{x_j} g = \sum_{j=1}^{m} e_j \partial_{x_j} g = \sum_{j=1}^{m} e_j (\partial_r g)(\partial_{x_j} r) = \omega \partial_r g,$$

and

$$\Delta_x \omega = -\partial^2_x \omega = (m - 1) \partial_x \left( \frac{1}{r} \right) = -\frac{(m - 1)}{r^2} \omega,$$

$$\Delta_x g = \partial^2_{x_0} g + \Delta_x g = \partial^2_{x_0} g - \partial_x (\omega \partial_r g)$$

$$= \partial^2_{x_0} g + \partial^2_r g + \frac{m - 1}{r} \partial_r g.$$ 

Therefore

$$\Delta_x(g P_{k,l}) = (\Delta_x g) P_k + g(\Delta_x P_{k,l}) + 2 \sum_{j=1}^{m} (\partial_{x_j} g)(\partial_{x_j} P_{k,l})$$

$$= \left( \partial^2_{x_0} g + \partial^2_r g + \frac{m - 1}{r} \partial_r g \right) P_{k,l} + 2 \frac{\partial g}{r} \sum_{j=1}^{m} x_j (\partial_{x_j} P_{k,l})$$

$$= \left( \partial^2_{x_0} g + \partial^2_r g + \frac{2k + m - 1}{r} \partial_r g \right) P_{k,l}$$

6
where we have also used Euler’s theorem for homogeneous functions. Moreover,

\[ \Delta_x(g\omega P_{k,l}) = (\Delta_x \omega) g P_{k,l} + \omega \Delta_x(g P_{k,l}) + 2 \sum_{j=1}^{m} (\partial_{x_j} \omega)(\partial_{x_j} (g P_{k,l})) \]

\[ = -(m - 1) g \omega P_{k,l} + \omega \Delta_x(g P_{k,l}) \]
\[ + 2 \sum_{j=1}^{m} \left( \frac{e_j}{r} - \frac{x_j}{r^2} \omega \right) \left( \frac{x_j}{r} (\partial_r g) P_{k,l} + g(\partial_{x_j} P_{k,l}) \right) \]
\[ = \left( \partial^2_{x_0} g + \partial^2_r g + (2k + m - 1) \left( \frac{\partial_r g}{r} - \frac{g}{r^2} \right) \right) \omega P_{k,l} \]

In a similar way we can prove the other two equalities.

The proof now follows by induction using the previous equalities together with statements (i) and (v) of Lemma 1.

It is clear that the lemma is true in the case \( n = 1 \). Assume that the formulae hold for a positive integer \( n \); we will prove them for \( n + 1 \).

We thus get

\[ \Delta_x^{n+1}(h P_{k,l}) = \prod_{j=1}^{n} (2k + m - (2j - 1)) \Delta_x(D_r(n)\{h\} P_{k,l}). \]

But

\[ \Delta_x(D_r(n)\{h\} P_{k,l}) \]
\[ = (\partial^2_{x_0} D_r(n)\{h\} + \partial^2_r D_r(n)\{h\} + (2k + m - 1) D_r(n+1)\{h\}) P_{k,l} \]
\[ = (D_r(n)\{\partial^2_{x_0} h + \partial^2_r h\} + (2k + m - (2n + 1)) D_r(n+1)\{h\}) P_{k,l} \]

yielding

\[ \Delta_x^{n+1}(h P_{k,l}) = \prod_{j=1}^{n+1} (2k + m - (2j - 1)) D_r(n+1)\{h\} P_{k,l}, \]

which establishes the first formula. The others may be proved similarly. □

**Lemma 3** Assume that \( A, B, C \) and \( D \) are \( \mathbb{R} \)-valued \( C^1 \) functions on \( \mathbb{R}^4 \).
Then the function

\[ F(x, y) = A(x_0, r, y_0, \rho) P_{k,l}(x, y) + B(x_0, r, y_0, \rho) \omega P_{k,l}(x, y) \]
\[ + C(x_0, r, y_0, \rho) P_{k,l}(x, y) \nu + D(x_0, r, y_0, \rho) \omega P_{k,l}(x, y) \nu, \]
is biregular if the following Vekua-type systems are satisfied
\[
\begin{align*}
\partial_{x_0} A - \partial_r B &= \frac{2k + m - 1}{r} B \\
\partial_{x_0} B + \partial_r A &= 0 \\
\partial_{y_0} A - \partial_\rho C &= \frac{2l + m - 1}{\rho} C \\
\partial_{y_0} C + \partial_\rho A &= 0
\end{align*}
\]
\[
\begin{align*}
\partial_{x_0} C - \partial_r D &= \frac{2k + m - 1}{r} D \\
\partial_{x_0} D + \partial_r C &= 0 \\
\partial_{y_0} B - \partial_\rho D &= \frac{2l + m - 1}{\rho} D \\
\partial_{y_0} D + \partial_\rho B &= 0.
\end{align*}
\]

Proof. It is not hard to check, using the Leibniz rules (1), (3) and Euler’s theorem for homogeneous functions, that the following equality holds
\[
\partial_x F(x, y) = \partial_r A \omega P_{k,l} - \left( \partial_r B + \frac{2k + m - 1}{r} B \right) P_{k,l} + \partial_c C \omega P_{k,l} - \left( \partial_r D + \frac{2k + m - 1}{r} D \right) P_{k,l}.
\]

Therefore
\[
\partial_x F(x, y) = \left( \partial_{x_0} A - \partial_r B - \frac{2k + m - 1}{r} B \right) P_{k,l} + \left( \partial_{x_0} B + \partial_r A \right) \omega P_{k,l}
\]
\[
\left( \partial_{x_0} C - \partial_r D - \frac{2k + m - 1}{r} D \right) P_{k,l} + \left( \partial_{x_0} D + \partial_r C \right) \omega P_{k,l}.
\]

Similarly, using (2), (4) and Euler’s theorem for homogeneous functions, we can also obtain that
\[
F(x, y)\partial_y = \left( \partial_{y_0} A - \partial_\rho C - \frac{2l + m - 1}{\rho} C \right) P_{k,l}
\]
\[
+ \left( \partial_{y_0} B - \partial_\rho D - \frac{2l + m - 1}{\rho} D \right) \omega P_{k,l} + \left( \partial_{y_0} D + \partial_\rho B \right) \omega P_{k,l}.
\]

which completes the proof. □

3 Fueter’s theorem

Suppose that \( u_j(x_1, y_1, x_2, y_2) \) and \( v_j(x_1, y_1, x_2, y_2), \ j = 1, 2, \) are \( \mathbb{R} \)-valued \( C^1 \) functions which satisfy the following Cauchy-Riemann systems
\[
\begin{align*}
\partial_{x_1} u_1 &= \partial_{y_1} v_1 \\
\partial_{y_1} u_1 &= -\partial_{x_1} v_1 \\
\partial_{x_1} u_2 &= \partial_{y_1} v_2 \\
\partial_{y_1} u_2 &= -\partial_{x_1} v_2
\end{align*}
\]
\[ \begin{align*}
\partial_{x_2} u_1 &= \partial_{y_2} v_1 \\
\partial_{y_2} u_1 &= -\partial_{x_2} v_1
\end{align*} \]

in some open subset \( \Xi \subset \mathbb{R}^2_+ \times \mathbb{R}^2_+ \) = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_1, y_2 > 0\}.

In other words, the functions \( u_1 + iv_1 \) and \( u_2 + iv_2 \) are holomorphic with respect to the complex variable \( z_1 = x_1 + iy_1 \) while \( u_1 + iv_2 \) and \( v_1 + iv_2 \) are holomorphic with respect to \( z_2 = x_2 + iy_2 \).

Remark: Note that a simple way to obtain these types of functions is by using holomorphic functions of two complex variables. Indeed, if \( u + iv \) is a holomorphic function in \( \mathbb{C}^2 \), then we can put \( u_1 = u \), \( v_1 = u_2 = v \) and \( v_2 = -u \).

Finally, for the biregular functions we propose the following result.

**Theorem 2** Let \( u_j, v_j \ (j = 1, 2) \) be as above. If \( m \) is odd, then the function

\[ \Delta^k \frac{m-1}{2} \Delta^l \frac{m-1}{2} \left( u_1(x_0, r, y_0, \rho)P_{k,l}(z, y) + v_1(x_0, r, y_0, \rho) \bar{\omega} P_{k,l}(z, y) \\
+ u_2(x_0, r, y_0, \rho)P_{k,l}(z, y) \bar{\nu} + v_2(x_0, r, y_0, \rho) \bar{\omega} P_{k,l}(z, y) \bar{\nu} \right) \]

is biregular in \( \Omega = \{(x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} : (x_0, r, y_0) \in \Xi\} \).

**Proof.** By Lemma 2 we get that

\[ \Delta^k \frac{m-1}{2} \Delta^l \frac{m-1}{2} \left( u_1 P_{k,l} + v_1 \bar{\omega} P_{k,l} + u_2 \bar{\nu} P_{k,l} + v_2 \bar{\omega} P_{k,l} \right) \]

\[ = (2k + m - 1)!!(2l + m - 1)!! \left( AP_{k,l} + B \bar{\omega} P_{k,l} + C \bar{\nu} P_{k,l} + D \bar{\omega} P_{k,l} \right), \]

with

\[ A = D_r \left( k + \frac{m-1}{2} \right) D_\rho \left( l + \frac{m-1}{2} \right) \{u_1\}, \]

\[ B = D^r \left( k + \frac{m-1}{2} \right) D_\rho \left( l + \frac{m-1}{2} \right) \{v_1\}, \]

\[ C = D_r \left( k + \frac{m-1}{2} \right) D^\rho \left( l + \frac{m-1}{2} \right) \{u_2\}, \]

\[ D = D^r \left( k + \frac{m-1}{2} \right) D^\rho \left( l + \frac{m-1}{2} \right) \{v_2\}. \]

The task is now to prove that \( A, B, C \) and \( D \) satisfy the Vekua-type systems of Lemma 3. In order to do that, it will be necessary to use the assumptions on \( u_j \) and \( v_j \ (j = 1, 2) \) and Lemma 1.
Indeed, by the statements (iii) and (iv) of Lemma 1 and using the fact that $u_1 + iv_1$ is holomorphic with respect to the complex variable $z_1 = x_1 + iy_1$, it follows that

\[
\partial_{x_0} A - \partial_r B = D \rho \left( l + \frac{m-1}{2} \right) \left\{ D_r \left( k + \frac{m-1}{2} \right) \{\partial_{x_0} u_1\} - \partial_r D^r \left( k + \frac{m-1}{2} \right) \{v_1\} \right\} \\
= D \rho \left( l + \frac{m-1}{2} \right) \left\{ D_r \left( k + \frac{m-1}{2} \right) \{\partial_r v_1\} - \partial_r D^r \left( k + \frac{m-1}{2} \right) \{v_1\} \right\} \\
= \frac{2k + m-1}{r} D^r \left( k + \frac{m-1}{2} \right) D \rho \left( l + \frac{m-1}{2} \right) \{v_1\} \\
= \frac{2k + m-1}{r} B
\]

and

\[
\partial_{x_0} B + \partial_r A = D \rho \left( l + \frac{m-1}{2} \right) \left\{ D^r \left( k + \frac{m-1}{2} \right) \{\partial_{x_0} v_1\} + \partial_r D_r \left( k + \frac{m-1}{2} \right) \{u_1\} \right\} \\
= D \rho \left( l + \frac{m-1}{2} \right) \left\{ D^r \left( k + \frac{m-1}{2} \right) \{\partial_{x_0} v_1\} + D^r \left( k + \frac{m-1}{2} \right) \{\partial_r u_1\} \right\} \\
= D^r \left( k + \frac{m-1}{2} \right) D \rho \left( l + \frac{m-1}{2} \right) \{\partial_{x_0} v_1 + \partial_r u_1\} \\
= 0.
\]

In a similar way we can verify the other systems of Lemma 3. □

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References

[1] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*. Research Notes in Mathematics, 76, Pitman (Advanced Publishing Program), Boston, MA, 1982.

[2] F. Brackx and W. Pincket, *A Bochner-Martinelli formula for the biregular functions of Clifford analysis*. Complex Variables Theory Appl. 4 (1984), no. 1, 39–48.
[3] F. Brackx and W. Pincket, *Two Hartogs theorems for nullsolutions of overdetermined systems in Euclidean space*. Complex Variables Theory Appl. 4 (1985), no. 3, 205–222.

[4] F. Brackx and W. Pincket, *Series expansions for the biregular functions of Clifford analysis*. Simon Stevin 60 (1986), no. 1, 41–55.

[5] W. K. Clifford, *Applications of Grassmann’s Extensive Algebra*. Amer. J. Math. 1 (1878), no. 4, 350–358.

[6] F. Colombo, I. Sabadini and F. Sommen, *The Fueter mapping theorem in integral form and the F-functional calculus*. Math. Methods Appl. Sci. 33 (2010), no. 17, 2050–2066.

[7] F. Colombo, I. Sabadini and F. Sommen, *The inverse Fueter mapping theorem*. Commun. Pure Appl. Anal. 10 (2011), no. 4, 1165–1181.

[8] R. Delanghe, F. Sommen and V. Souček, *Clifford algebra and spinor-valued functions*. Mathematics and its Applications, 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[9] R. Fueter, *Die funktionentheorie der differentialgleichungen \( \Delta u = 0 \) und \( \Delta \Delta u = 0 \) mit vier variablen*. Comm. Math. Helv. 7 (1935), 307–330.

[10] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers*. Wiley and Sons Publications, Chichester, 1997.

[11] K. I. Kou, T. Qian and F. Sommen, *Generalizations of Fueter’s theorem*. Methods Appl. Anal. 9 (2002), no. 2, 273–289.

[12] D. Peña Peña and F. Sommen, *Monogenic Gaussian distribution in closed form and the Gaussian fundamental solution*. Complex Var. Elliptic Equ. 54 (2009), no. 5, 429-440.

[13] D. Peña Peña and F. Sommen, *A note on the Fueter theorem*. Adv. Appl. Clifford Algebr. 20 (2010), no. 2, 379–391.

[14] D. Peña Peña and F. Sommen, *Fueter’s theorem: the saga continues*. J. Math. Anal. Appl. 365 (2010) 29–35.

[15] T. Qian, *Generalization of Fueter’s result to \( \mathbb{R}^{n+1} \)*. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 8 (1997), no. 2, 111–117.
[16] T. Qian and F. Sommen, *Deriving harmonic functions in higher dimensional spaces*. Z. Anal. Anwendungen 22 (2003), no. 2, 275–288.

[17] M. Sce, *Osservazioni sulle serie di potenze nei moduli quadratici*. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 220–225.

[18] F. Sommen, *Plane waves, biregular functions and hypercomplex Fourier analysis*. Proceedings of the 13th winter school on abstract analysis (Srní, 1985). Rend. Circ. Mat. Palermo (2) Suppl. No. 9 (1985), 205–219 (1986).

[19] F. Sommen, *Martinelli-Bochner type formulae in complex Clifford analysis*. Z. Anal. Anwendungen 6 (1987), no. 1, 7582.

[20] F. Sommen, *On a generalization of Fueter’s theorem*. Z. Anal. Anwendungen 19 (2000), no. 4, 899–902.