ABSTRACT. The work of Bernstein-Zelevinsky and Zelevinsky gives a good understanding of irreducible subquotients of a reducible principal series representation of $GL_n(F)$, $F$ a $p$-adic field (without specifying their multiplicities which is done by a Kazhdan-Lusztig type conjecture). In this paper we make a proposal of a similar kind for principal series representations of $GL_n(\mathbb{R})$.

Although representation theory of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ is well understood, and so are the Langlands parameters for these groups, the author has not found a place which discusses Langlands parameters of subquotients of principal series representations on these groups. In fact, there is no explicit reference relating the Langlands parameters of the two components, one finite dimensional, and the other a discrete series representation of $GL_2(\mathbb{R})$ which appear inside a reducible principal series representation of $GL_2(\mathbb{R})$. The paper was conceived in the hope that this simple question may shed some light on possible Langlands parameters of reducible principal series representations of $GL_n(\mathbb{R})$, or even more generally $G(\mathbb{R})$, for $G$ a reductive group over $\mathbb{R}$, and how such questions on real and $p$-adic groups may be related. It is well-known that reducibility of non-unitary principal series of $G(F)$, $F$ a $p$-adic field, is intimately connected with Langlands parameters involving the Weil-Deligne group. For example, there is the well-known conjecture (usually attributed to Tom Haines) that for $F$ a $p$-adic field, the Langlands parameters of all the subquotients of a principal series representation induced from a cuspidal representation, have the same restriction to $W_F$ when $W_F$ is embedded in $W'_F = W_F \times SL_2(\mathbb{C})$ in such a way that the homomorphism of $W_F$ into $SL_2(\mathbb{C})$ lands inside the diagonal subgroup, and is the pair of characters $(\nu^{1/2}, \nu^{-1/2})$ where $\nu : F^\times \to \mathbb{C}^\times$ is the normalized absolute value of $F^\times$, treated also as a character of $W_F$.

As an introduction to Langlands parameters for $GL_n(\mathbb{R})$, we recall that $GL_n(\mathbb{R})$ has a discrete series representation if and only if $n \leq 2$. Further, any tempered representation of $GL_n(\mathbb{R})$ is irreducibly induced from a (unitary) principal series representation of a Levi subgroup. Thus the Langlands parameter $\sigma_\pi$ of any irreducible admissible representation $\pi$ of $GL_n(\mathbb{R})$ is of the form:

$$\sigma_\pi = \sum_1^\infty \sigma_i,$$
where $\sigma_i$ are irreducible representations of $W_\mathbb{R}$ of dimension $\leq 2$. Further, the map $\pi \to \sigma_\pi$ is a bijective correspondence between irreducible admissible representations of $GL_n(\mathbb{R})$ and (semi-simple) representations of $W_\mathbb{R}$ of dimension $n$.

We now begin with $GL_2(\mathbb{R})$. The following well-known proposition (in this form) is due to Jacquet-Langlands; in this, and in the rest of the paper, we denote by $\omega_\mathbb{R}$, the unique quadratic character of $\mathbb{R}^\times$.

**Proposition 1.** For a pair of characters $\chi_1, \chi_2 : \mathbb{R}^\times \to \mathbb{C}^\times$, let $Ps(\chi_1, \chi_2)$ be the corresponding principal series representation of $GL_2(\mathbb{R})$. Then this principal series representation is reducible if and only if $\chi_1 \cdot \chi_2^{-1}(t) = t^p \omega_\mathbb{R}(t)$ for some integer $p \neq 0$.

If the principal series is reducible, then it has two Jordan-Hölder factors, one finite dimensional (of dimension $|p|$), and the other infinite dimensional which is a discrete series representation of $GL_2(\mathbb{R})$ (by which we always mean up to a twist, or what's also called 'essentially discrete series').

**Remark 1.** One can reformulate this proposition, or more generally the work of Speh on reducibility of principal series representations of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$, see [Sp] as well as [Mo], so that it applies uniformly to all $GL_n(F)$, $F$ any local field. There is the conjecture 2.6 formulated in [GP] on when a representation $\pi$ is reducible if and only if $\chi_1 \cdot \chi_2^{-1}(t) = t^p \omega_\mathbb{R}(t)$ for some integer $p \neq 0$.

Recall that $W_\mathbb{R} = C^\times \cdot (j)$ with $j^2 = -1, jzj^{-1} = z$ for $z \in C^\times$. Since $W_\mathbb{R}^{ab} \cong \mathbb{R}^\times$ in which the map from $W_\mathbb{R}$ to $\mathbb{R}^\times$ when restricted to $C^\times \subset W_\mathbb{R}$ is just the norm mapping from $C^\times$ to $\mathbb{R}^\times$, characters of $W_\mathbb{R} \to C^\times$ can therefore be identified to characters of $\mathbb{R}^\times$.

Note that the characters of $\mathbb{R}^\times$ are of the form $\omega_\mathbb{R}^{0,1}(t)|t|^\nu$ where $s \in C^\times$. On the other hand, characters of $C^\times$ can be written as,

$$z \mapsto z^\mu \cdot z^\nu = z^{\mu - \nu} \cdot (zz)^\nu, \quad \mu, \nu \in C, \mu - \nu \in \mathbb{Z}.$$

**Proposition 2.** For a pair of characters $\chi_1, \chi_2 : \mathbb{R}^\times \to \mathbb{C}^\times$ with $\chi_1 \cdot \chi_2^{-1}(t) = t^p \omega_\mathbb{R}(t)$ for some integer $p \neq 0$, let $Ps(\chi_1, \chi_2)$ be the corresponding principal series representation of $GL_2(\mathbb{R})$ with $F(\chi_1, \chi_2)$ the corresponding finite dimensional subquotient of $Ps(\chi_1, \chi_2)$, and $Ds(\chi_1, \chi_2)$ the discrete series component. Then the Langlands parameter of $F(\chi_1, \chi_2)$ is $\chi_1 + \chi_2$, and that of $Ds(\chi_1, \chi_2)$ is the induction to $W_\mathbb{R}$ of the following character of $C^\times \subset W_\mathbb{R}$:

$$z \mapsto \tilde{\chi}_1(z) \cdot \tilde{\chi}_2(z) = \tilde{\chi}_1 \tilde{\chi}_2^{-1}(z) \tilde{\chi}_2(zz) = z^p \tilde{\chi}_2(zz),$$

where $\tilde{\chi}_1, \tilde{\chi}_2$ are characters of $C^\times$ extending the restriction of $\chi_1, \chi_2$ which are characters of $\mathbb{R}^\times$ to $\mathbb{R}^{\geq 0}$ with the property that $\tilde{\chi}_1 \tilde{\chi}_2^{-1}(z) = z^p$ for all $z \in C^\times$.

**Proof.** Observe $GL_2(\mathbb{R}) = SL_2(\mathbb{R})^{\pm} \times \mathbb{R}^{>0}$, where $SL_2(\mathbb{R})^{\pm}$ is the subgroup of $GL_2(\mathbb{R})$ consisting of matrices with determinant $\pm 1$. So any irreducible representation of $GL_2(\mathbb{R})$ restricted to $SL_2(\mathbb{R})^{\pm}$ remains irreducible, and the Langlands parameter of an irreducible representation of $GL_2(\mathbb{R})$ can be read-off from that of $SL_2(\mathbb{R})^{\pm}$. It
1. The Langlands parameters of these Jordan-H"older factors are obtained by writing
representation has 2 representations to analyze for GL$^n$.

Remark 3. Since $\chi$ is nonzero in the proposition, the character $\chi \in C^\times$ induces an irreducible 2-dimensional representation of $W_R$. This holds in our case by the following calculation on the determinant of an induced representation

$$\det \left[ \text{Ind}_{C^\times \chi}^{W_R} \right] = \chi|_{C^\times} \cdot \omega_R = t^p \omega_R(t) \chi_2(t^2) = \chi_1(t) \cdot \chi_2(t),$$

where in the last equality we used $\chi_1 \chi_2^{-1}(t) = t^p \omega_R(t)$.

Remark 2. Observe that for a discrete series representation $D$ of $GL_2(R)$, $D \otimes \omega_R \cong D$, and therefore if a principal series representation $\chi_1 \times \chi_2$ of $GL_2(R)$ is reducible, and is $F + D$ up to semi-simplification, where $F$ is finite dimensional and $D$ is a discrete series representation of $GL_2(R)$, then the principal series representation $\chi_1 \omega_R \times \chi_2 \omega_R$ of $GL_2(R)$ is also reducible, and is up to semi-simplification is $\omega_R F + D$; i.e., a discrete series representation lies in two distinct principal series representations of $GL_2(R)$, whereas a finite dimensional representation lies in only one; this is in marked contrast with $p$-adics! (This difference in reals and $p$-adics is at the source of not having a theory of ‘cuspidal supports’ for $GL_n(R)$.)

According to Bernstein-Zelevinsky and Zelevinsky, the simplest principal series representations to analyze for $GL_n(F)$, $F$ a $p$-adic field, is the principal series representation $P_s(\chi)$ induced from a character $\chi : (F^\times)^n \to C^\times$ with the property $\chi^w \neq \chi$ for $w \neq 1$. These principal series representations have Jordan-H"older factors of multiplicity 1 (this is a consequence of calculations with the Jacquet-module which is not as developed for real groups), and can be explicitly described in terms of Zelevinsky classification, cf. [Ze]. Also, the Langlands parameters of all subquotients can be explicitly described, see for instance [Ku]. The corresponding questions for $GL_n(R), GL_n(C)$ have to my knowledge not been attempted, although one does know — by the thesis work of B. Speh [Sp] — exactly when a principal series representation of $GL_n(R), GL_n(C)$ is reducible. For an exposition of the work of Speh, see [Mo]. For $p$-adic fields, Langlands parameters of all subquotients of a principal series representation $P_s(\chi)$ are intimately linked with the notion of Weil-Deligne group $W'_F = W_F \times SL_2(C)$. This group is not available to us for real groups. The Langlands parameters of the subquotients of the simplest principal series representation realized on functions of $G(R)/B(R)$ do not seem to be known.

The space of functions on $GL_n(F)/B(F)$ is the principal series representation which is $v^{-(n-1)/2} \times v^{-(n-3)/2} \times \cdots \times v^{(n-3)/2} \times v^{(n-1)/2}$. If $F$ is $p$-adic, this principal series representation has $2^{n-1}$ many Jordan-H"older factors, each appearing with multiplicity 1. The Langlands parameters of these Jordan-H"oder factors are obtained by writing
the interval \([-\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\)] as disjoint union of \(r\) many non-empty intervals for all \(0 \leq r \leq n\). (These Jordan-Hölder factors are parametrized by all parabolics \(P\) containing a given Borel subgroup \(B\), and are given by a Steinberg-like construction by considering functions on \(G/P\) modulo functions on \(G/Q\) for all parabolics \(Q\) strictly containing \(P\). Irreducibility of such representations is a theorem of Casselman, cf. [Ca].)

For each such interval \([i, i+1, \ldots, i+r]\), we have the representation of the Weil-Deligne group \(W_F \times \text{SL}_2(\mathbb{C})\) which is given by \(\nu^{r_i/2+i} \otimes \text{Sym}^{r_i}(\mathbb{C}^2)\), and the Langlands parameter of the corresponding representation of \(\text{GL}_n(F)\) to be:

\[
\sum \nu^{r_i/2+i} \otimes \text{Sym}^{r_i}(\mathbb{C}^2).
\]

**What about \(\text{GL}_n(\mathbb{R})\)?**

It would be most natural to expect that there are still \(2^{n-1}\) many Jordan-Hölder factors each appearing with multiplicity 1, and that the the Langlands parameters of the Jordan-Hölder factors have similar structure. For this, let \(D_2\) be the Langlands parameter of the lowest discrete series representation of \(\text{PGL}_2(\mathbb{R})\) which is given by Proposition 2 as \(\text{Ind}_{C^\times}^{W_\mathbb{R}} \chi\) for \(\chi : C^\times \to C^\times\) given by \(z \mapsto z/|z|\). We propose that the Langlands parameters of the subquotients of the principal series representation of \(\text{GL}_n(\mathbb{R})\) obtained on the space of functions of \(\text{GL}_n(\mathbb{R})/\text{B}(\mathbb{R})\) is parametrized by writing the interval \([-\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\)] as disjoint union of \(r\) many non-empty intervals for all \(0 \leq r \leq n\). For each such interval \([i, i+1, \ldots, i+r]\), associate the representation of \(W_\mathbb{R}\) to be \(\nu^{r_i/2+i} \otimes \text{Sym}^{r_i}(D_2)\), and the Langlands parameter of the corresponding representation of \(\text{GL}_n(\mathbb{R})\) to be:

\[
\sum \nu^{r_i/2+i} \otimes \text{Sym}^{r_i}(D_2);
\]

it may be mentioned here that since irreducible representations of \(W_\mathbb{R}\) are of dimension \(\leq 2\), \(\text{Sym}^{r_i}(D_2)\), \(r_i \geq 2\), are necessarily reducible (but tempered) representations of \(W_\mathbb{R}\). In fact, one has,

\[
\text{Sym}^{r}(D_2) = D_{r+1} + D_{r-1} + \cdots + D_2 \quad \text{(with } D_2 \text{ replaced by } \omega_\mathbb{R} \text{ if } r \text{ is even),}
\]

where \(D_\ell, \ell \geq 2\) are the 2 dimensional irreducible representations of \(W_\mathbb{R}\) which are induced from \(C^\times \subset W_\mathbb{R}\) from the character \(z = re^{i\theta} \mapsto e^{(\ell-1)i\theta}\).

**Remark 4.** Assuming that the Steinberg-like construction for \(\text{GL}_n(\mathbb{R})\) for any parabolic in \(\text{GL}_n(\mathbb{R})\) produces an irreducible representation — which we expect, just as for \(p\)-adic groups (which as mentioned earlier is a theorem due to Casselman) — the assertions above on Jordan-Hölder factors and their Langlands parameters reduce to an assertion on the Langlands parameter of these Steinberg like representations of \(\text{GL}_n(\mathbb{R})\).

**Steinberg representation:** There is an analogy between the representation \(D_2 : W_\mathbb{R} \to \text{SL}_2(\mathbb{C})\) associated to the lowest discrete series representation of \(\text{PGL}_2(\mathbb{R})\), and the defining representation of \(\text{SL}_2(\mathbb{C})\) of dimension 2 in the Deligne part of the Weil-Deligne group \(W'_F = W_F \times \text{SL}_2(\mathbb{C})\) for \(F\) non-archimedean local field. For this, we consider the Langlands parameter of the Steinberg representation \(S_{a\mathbb{C}}\) of \(G(\mathbb{R})\) obtained on functions on \(G/B\) for \(B\) a minimal parabolic in \(G\) modulo functions on \(G/Q\)
for $Q$ parabolics of $G$ strictly containing $B$. It is known that $St_G$ is a tempered representation of $G(\mathbb{R})$ of finite length which may not be irreducible unlike for $p$-adic fields (as is the case already for $SL_2(\mathbb{R})$; perhaps fine for adjoint groups?). Since the Steinberg representation is trivial on the center of the group, and hence in considering the Steinberg representation, it is best to assume that $G$ is an adjoint group, with dual group $\hat{G}(\mathbb{C})$ simply connected. If $F$ is a non-archimedean local field, then the Steinberg representation is supposed to have the Langlands parameter which is the representation of $SL_2(\mathbb{C})$ corresponding to the regular unipotent element in the dual group $\hat{G}(\mathbb{C})$ by the Jacobson-Morozov theorem with $W_F$ acting trivially, whereas for $F = \mathbb{R}$, the parameter of $St_G$ uses the representation $D_2 : W_\mathbb{R} \rightarrow SL_2(\mathbb{C})$ associated to the lowest discrete series representation of $PGL_2(\mathbb{R})$, and then the same embedding of $SL_2(\mathbb{C})$ in $\hat{G}$ associated to a regular unipotent element. We will give a proof of this proposal for $G(\mathbb{R}) = GL_n(\mathbb{R})$ below in which case $\hat{G}(\mathbb{C})$ is $GL_n(\mathbb{C})$, and the representation of $SL_2(\mathbb{C})$ associated by the Jacobson-Morozov to a regular unipotent element in $GL_n(\mathbb{C})$ is $\text{Sym}^{n-1}(\mathbb{C}^2)$.

**Definition 1.** For a pair of characters $\{\chi_1, \chi_2\}$ of $\mathbb{R}^\times$ and hence of $W_\mathbb{R}$, such that the principal series representation $\chi_1 \times \chi_2$ of $GL_2(\mathbb{R})$ is reducible, denote by $j(\chi_1 + \chi_2)$ the parameter of the discrete series representation of $GL_2(\mathbb{R})$ which appears as a sub-quotient in the principal series representation $\chi_1 \times \chi_2$ of $GL_2(\mathbb{R})$ and whose parameter is given in proposition 2.

**Lemma 1.** Let $\chi_1, \chi_2, \chi_3$ be characters of $\mathbb{R}^\times$ (neither assumed to be unitary). Assume that $\chi_1 \times \chi_2$ as well as $\chi_1 \times \chi_3$ are reducible principal series representations of $GL_2(\mathbb{R})$, and that $j(\chi_1 + \chi_2) = \sigma$ and $j(\chi_1 + \chi_3) = \tau$ are essentially discrete series representations of $GL_2(\mathbb{R})$. Assume further that the exponents (absolute value of central characters) of $\chi_2$ and that of $\tau$ are the same, in particular, $\chi_2 \times \tau$ is an irreducible representation of $GL_3(\mathbb{R})$, then the representation $\chi_2 \times \tau$ is a sub-quotient of $\sigma \times \chi_3$.

**Proof.** We will use standard methods of Whittaker model, well-known for $p$-adic fields, and surely also known for archimedean fields.

The principal series representation $\sigma \times \chi_3$ of $GL_3(\mathbb{R})$ is a sub-quotient of $\chi_1 \times \chi_2 \times \chi_3$, and the generic component of $\chi_1 \times \chi_2 \times \chi_3$ is contained inside $\sigma \times \chi_3$. However, the generic component of $\chi_1 \times \chi_2 \times \chi_3$ is also contained inside $\chi_2 \times \tau$. Since by assumption, $\chi_2 \times \tau$ is an irreducible representation, $\sigma \times \chi_3$ must contain $\chi_2 \times \tau$ as a sub-quotient. \hfill \Box

**Example 1.** The principal series representation $\nu \times 1 \times \nu^{-1}$ of $GL_3(\mathbb{R})$ contains the irreducible tempered representation $\omega_\mathbb{R} \times D_3$ where $D_3$ is the discrete series representation of $GL_2(\mathbb{R})/\mathbb{R}^{>0}$ contained in the principal series representation $\nu \times \omega_\mathbb{R}^{-1}$, in conformity with the suggestion earlier that the Steinberg representation of $GL_n(\mathbb{R})$ has parameter

$$\text{Sym}^{n-1}(D_2) = D_n + D_{n-1} + \cdots + D_2 \quad \text{(with $D_2$ replaced by $\omega_\mathbb{R}$ if $n$ is odd)},$$

thus in our case, the Steinberg representation of $GL_3(\mathbb{R})$ has the parameter, $\omega_\mathbb{R} + D_3$.

The essence of the previous lemma and this example is that the principal series representation $\nu \times \nu^{-1}$ of $GL_2(\mathbb{R})$ is irreducible, and it is the principal series representation $\nu \times \omega_\mathbb{R}^{-1}$ of $GL_2(\mathbb{R})$ which is reducible; however, the principal series
representation $\nu \times 1$ contains the discrete series $\nu^{1/2}D_2$, but this discrete series is also contained in the principal series $\omega R\nu \times \omega R$, so a certain part of the principal series representation $\nu \times 1 \times \nu^{-1}$ of $GL_3(\mathbb{R})$ is as if we are analyzing the principal series representation $\omega R\nu \times \omega R \times \nu^{-1}$ of $GL_3(\mathbb{R})$.

**Remark 5.** The analysis of the previous example proves that the Langlands parameter of $GL_n(\mathbb{R})$ is indeed

$$\text{Sym}^{n-1}(D_2) = D_n + D_{n-1} + \cdots + D_2 \quad (\text{with } D_2 \text{ replaced by } \omega R \text{ if } n \text{ is odd}).$$

More precisely, the previous argument proves that there is an irreducible tempered generic representation of $GL_n(\mathbb{R})$ contained as a sub-quotient in the principal series representation

$$\nu^{-(n-1)/2} \times \nu^{-(n-3)/2} \times \cdots \times \nu^{(n-3)/2} \times \nu^{(n-1)/2},$$

whose $L$-parameter is:

$$\text{Sym}^{n-1}(D_2) = D_n + D_{n-1} + \cdots + D_2 \quad (\text{with } D_2 \text{ replaced by } \omega R \text{ if } n \text{ is odd}).$$

Since the Steinberg representation of $GL_n(\mathbb{R})$ is the unique generic component of this principal series, our argument in fact proves temperedness too of the Steinberg representation.

It may be noted that there is a way to come up with the parameter of the Steinberg representation of $GL_n(\mathbb{R})$ since it is built (as for any other representation) from parameters for $GL_1(\mathbb{R})$ and $GL_2(\mathbb{R})$, unlike the case of $p$-adics, where the parameter of the Steinberg representation is a ‘brand-new’ parameter!

**Paraphrasing the work of Speh:** We now recall the thesis work of B. Speh [Sp] on reducibility of principal series representations of $GL_n(\mathbb{R})$, and then rephrase it using a language of segments similar to that due to Bernstein-Zelevinsky for $GL_n(F)$, for $F$, $p$-adics. The work of Speh is not published; we will follow the exposition of Moeglin [Mo] on Speh’s work closely.

For $j = 1, \cdots, t$, let $n_j = 1$ or $2$ be integers, with $\sum n_j = n$, and $s_j \in \mathbb{C}$. If $n_j = 1$, fix also a character $\sigma_j$ of order $1$ or $2$. If $n_j = 2$, fix an integer $p_j \geq 1$, and let $\sigma_j$ be the discrete series representation contained in the reducible principal series representation $\nu^{p_j/2} \times \nu^{-p_j/2} \omega R^{p_j+1}$.

Write $\chi$ for the collection of triples,

$$\chi = \{(n_j, s_j, \sigma_j), j = 1, \cdots, t\}.$$ 

Denote by $I(\chi)$ the representation of $GL_n(\mathbb{R})$ induced from a parabolic subgroup of $GL_n(\mathbb{R})$ with Levi subgroup which is $GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_t}(\mathbb{R})$ of the representation:

$$\bigotimes \sigma_j \otimes \nu^{s_j}.$$ 

**Theorem 1.** If $\chi$ is as above, $I(\chi)$ is irreducible if and only if for all $i, j \in [1, t]$ with $n_i \geq n_j$, either $s_i - s_j \not\in \overline{\mathbb{R}}$, or the appropriate one of the following conditions is satisfied:

(1) If $n_i = n_j = 1$, $|s_i - s_j|$ is not an even (resp. odd integer) nonzero integer if and only if $\sigma_i \neq \sigma_j$ (resp. $\sigma_i = \sigma_j$).

(2) If $n_i = 2$ and $n_j = 1$ (so that $p_i$ is defined), $-p_i/2 + |s_i - s_j| \not\in \{1, 2, 3, \cdots\}$. 

(3) If $n_i = n_j = 2$ (so that $p_i, p_j$ are defined), $-|p_i - p_j|/2 + |s_i - s_j| \not\in \{1, 2, 3, \ldots\}$. 

We now rephrase this theorem using a language of segments similar to that of Bernstein-Zelevinsky. For this, we associate to the (essentially) discrete series representation which appears in the reducible principal series $\nu^{s_i}(v_{p_i/2} \times v_{p_j/2} \omega_R^{p_i+1})$ of $GL_2(R)$, the segment $s_i + I_i = [s_j - p_j/2, s_j - p_j/2 + 1, \ldots, s_j + p_j/2 - 1, s_j + p_j/2]$ (the segment $I_i$ consists of integer translates of half-integers), and to a representation $\sigma_i v^{s_i}$ of $R^\times$ where $\sigma_i$ is a character of order 1 or 2, the single point $s_i$ also to be written as the segment $s_i + I_i$ with $I_i = 0$; for uniformity of statements, we take $I_i$ as $[-p_i/2, p_i/2]$ with $p_i = 0$. It may be noted that we have segments of arbitrary length already for $GL_2(R)$, whereas for $F$ a $p$-adic field, to have segments of arbitrary length, we must $GL_m(F)$ for $m$ large enough.

It is hoped that the following paraphrase of the previous theorem is simpler to use.

**Theorem 2.** If $\chi$ is as earlier, $I(\chi)$ is irreducible if and only if for all pairs $i, j \in [1, t]$ with $i \neq j$, the segments $s_i + I_i, s_j + I_j$ have the property that (assuming both $p_i, p_j$ are not 0) either

1. $s_i - s_j + (p_i - p_j)/2$ is non-integral, i.e., does not belong to $Z$. Or,
2. $s_i - s_j + (p_i - p_j)/2$ is integral, and one of the two segments: $s_i + I_i, s_j + I_j$ is contained in the other.

If $p_i = p_j = 0$, then either $s_i - s_j = 0$, or is not integral. If $s_i - s_j \in Z \setminus 0$, the quadratic characters $\sigma_i$ and $\sigma_j$ of $R^\times$ have the property that $\sigma_i/\sigma_j = \omega_R^{s_j - s_i}$.

Here is a proposition in which this ‘geometric’ point of view is useful.

**Proposition 3.** (a) A reducible principal series representation $\sigma_1 \times \chi_2$ of $GL_3(R)$ induced from an essentially discrete series representation of the $(2, 1)$ parabolic, so with $s_1 - s_2 + p_1/2 \in Z$, gives rise to three integers, $s_1 - s_2 - p_1/2, s_1 - s_2 + p_1/2$ and 0. This principal series is reducible if

$s_1 - s_2 - p_1/2 < s_1 - s_2 + p_1/2 < 0$, or $0 < s_1 - s_2 - p_1/2 < s_1 - s_2 + p_1/2$.

There is a natural way to construct an irreducible principal series induced from an essentially discrete series representation of the $(2, 1)$ parabolic using this data in which the discrete series on $GL_2(R)$ is constructed using the end points $s_1 - s_2 - p_1/2 < 0$ (or, $0 < s_1 - s_2 - p_1/2$ as the case may be), and the representation of $R^\times$ constructed from the middle term $s_1 - s_2 + p_1/2$ (or, $s_1 - s_2 - p_1/2$ as the case may be). (There is a possible sign character too which can be determined using the central character of representations of $GL_3(R)$.)

(b) A reducible principal series representation $\sigma_1 \times \sigma_2$ of $GL_4(R)$ induced from an essentially discrete series representation of the $(2, 2)$ parabolic, so with $s_1 - s_2 + (p_1 - p_2)/2 \in Z$, has the property that neither of the intervals $s_1 + [-p_1/2, p_1/2]$ and $s_2 + [-p_2/2, p_2/2]$ is contained in the other.

In this case, there is a natural way to construct an irreducible principal series induced from an essentially discrete series representation of the $(2, 2)$ parabolic using this data in which one of the discrete series on $GL_2(R)$ is constructed using the far end points of the union of these two intervals, and the other discrete series on $GL_2(R)$ is constructed using the other two points; i.e., if the intervals had end points which are $t_1 < t_2 < t_3 < t_4$, then the two discrete series representations arise from interval $t_1 < t_4$ and the interval $t_2 < t_3$. 
Proof. The idea of the proof of part (a) of the proposition is the same as lemma 1: note that in the proof of lemma 1, what was important was that in the notation of that lemma \( \chi_2 \times \tau \) is an irreducible representation of \( GL_3(\mathbb{R}) \) and not that it is essentially tempered (essentially tempered was used only for getting a proof of irreducibility). The irreducibility of the analogue of \( \chi_2 \times \tau \) follows in the generality of part (a) by Speh’s theorem.

Part (b) follows the same proof using existence and uniqueness of Whittaker model in the principal series representation \( \sigma_1 \times \sigma_2 \). In both the cases, the irreducible principal series representation induced from a discrete series representation of a maximal parabolic is generic, and is the unique generic component of a certain principal series representation induced from a character of the Borel subgroup.

Definition 2. For a pair of irreducible representations \( \{ \sigma_1, \sigma_2 \} \) of \( W_\mathbb{R} \) of dimension \( n_i = \text{dim} \sigma_i \leq 2 \) so that \( \sigma_1 + \sigma_2 \) is of dimension \( n = n_1 + n_2 \leq 4 \), let \( j(\sigma_1 + \sigma_2) \) be the Langlands parameter of the irreducible generic representation of \( GL_n(\mathbb{R}) \) contained in the principal series representation \( \pi(\sigma_1) \times \pi(\sigma_2) \) of \( GL_n(\mathbb{R}) \) constructed in the last proposition as a full induced representation from an essentially discrete series representation of a Levi subgroup, where \( \pi(\sigma_i) \) is the irreducible representation of \( GL_{n_i}(\mathbb{R}) \) with Langlands parameter \( \sigma_i \).

Conjecture 1. Consider the principal series representation \( Ps(\chi) = \chi_1 \times \cdots \times \chi_n \) of \( GL_n(\mathbb{R}) \) where \( \chi_i \) are characters on \( \mathbb{R}^\times \) (not necessarily unitary). Then an irreducible admissible representation \( \pi \) of \( GL_n(\mathbb{R}) \) appears in \( Ps(\chi) \) as a Jordan-Hölder factor if and only if its Langlands parameter \( \sigma_\pi \) is obtained from \( \tau_0 = \chi_1 + \chi_2 + \cdots + \chi_n \) by a sequence of operations starting with \( \tau_0 \) in which one goes from \( \tau_i \) to \( \tau_{i+1} \) by replacing a summand of \( \tau_i \) of the form \( \sigma_1 + \sigma_2 \), both \( \sigma_1, \sigma_2 \) irreducible, by \( j(\sigma_1 + \sigma_2) \) as in the previous definition.

Remark 6. Our proposal above is based on the assumption that the principal series representation \( \sigma_1 \times \chi_2 \) of \( GL_3(\mathbb{R}) \), and \( \sigma_1 \times \sigma_2 \) of \( GL_4(\mathbb{R}) \) (with \( \sigma_1 \) essentially discrete series representations of \( GL_2(\mathbb{R}) \)) have length 2. The author does not know if this is indeed the case.

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