DEGENERATION OF THE MODIFIED DIAGONAL CYCLE

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Abstract. In this paper, we study the degeneration of the modified diagonal cycle $\Delta_e \in \text{CH}_1(C \times C \times C)$ defined by Gross-Schoen. When $C$ is non-hyperelliptic and of genus three, the degeneration is shown to correspond to a higher Chow cycle $\Delta_{e,1} \in \text{CH}^2(C' \times C', 1) \otimes \mathbb{Q}$ and it is indecomposable. Here $C'$ is hyperelliptic of genus two. The degeneration is different from the (pullback of) Beilinson cycle, on $C' \times C'$ when $C'$ is a generic bielliptic curve. Furthermore, we study a variant of this procedure, to obtain a degeneration into a higher Chow cycle in $\text{CH}^2(D \times D', 1) \otimes \mathbb{Q}$, which is indecomposable on a product of distinct (non-rational) curves $D, D'$, satisfying $g(D), g(D') \leq 2$.

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1. Introduction

The modified diagonal cycle

$$\Delta_e := \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3 \in \text{CH}_1(C \times C \times C)$$

is a one-dimensional cycle on a triple product of a smooth connected projective curve $C$ over a field $k$ which is an alternating sum of diagonal type cycles. We assume $k = \mathbb{C}$. It was shown in [Gr-Sc], that $\Delta_e$ gives a non-trivial element in the Griffiths group of one cycles modulo algebraic equivalence, if the curve $C$ is non-hyperelliptic.

We would like to understand the variation of the modified diagonal cycle, when a genus three non-hyperelliptic curve degenerates to a single nodal curve. The philosophy of S. Bloch [B1] says that the degeneration is a higher Chow cycle. This type of degeneration was

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first looked at by A. Collino [Co] in the case of a Jacobian of a smooth non-hyperelliptic
curve. He studied the variation of the Ceresa’s cycle, when the curve degenerates to a
nodal curve. If the normalization is a hyperelliptic curve $C$, he associated a higher Chow
cycle in $CH^2(J(C), 1)$, and proved it to be indecomposable.

In this paper, we look at the degeneration of the modified diagonal cycle. It turns out
that the outcome is very close to Collino’s computations. Suppose $C \to B$ is family of
projective curves of genus three over $B := \text{Spec}(R)$. Here $R$ is a discrete valuation ring,
and the generic fibre is smooth, the special fibre is a single nodal curve $C'_0$. Assume there
is a section $\tilde{e}$ for this family and $C' \to C'_0$ is the normalization. Then by [Gr-Sc, p.664],
there is a good family of triple products $\pi : Y \to B$ together with the modified diagonal
cycle $\Delta_e$ on $Y$. We show the following.

**Theorem 1.1.** The specialization of the cycle $\Delta_e$ on the special fibre of $\pi$ corresponds to
a higher Chow cycle $\Delta_{e,1}$ in $CH^2(C' \times C', 1) \otimes \mathbb{Q}$. It is indecomposable, if $C'$ is generic.

See §5, Theorem 5.6. Note that a higher Chow cycle on $CH^2(C' \times C', 1) \otimes \mathbb{Q}$ is indecomposable, if it is not in the image of the natural map

$$\text{Pic}(C' \times C')_\mathbb{Q} \otimes \mathbb{C}^* \longrightarrow CH^2(C' \times C', 1)_\mathbb{Q}.$$  

The method and the proof can be extended to a further degeneration of the family $C \to B$, where the normalization of the special fibre is an elliptic curve $E$, to obtain a higher Chow cycle $\Delta_{e,2}$ in $CH^2(E, 2) \otimes \mathbb{Q}$. This is a multiple of the Collino’s degenerated cycle, up to algebraic equivalence. See Proposition 5.8. We even further degenerate $\Delta_{e,2}$ to a cycle $\Delta_{e,3}$ in $CH^2(\text{Spec}(\mathbb{C}), 3) \otimes \mathbb{Q}$. This is related to Collino’s work in [Co, Remark 7.13].

The technical tools to make the degeneration precise are provided by specialization maps
(see Proposition 4.1) and Levine’s relative $K_0$ groups (see [Le2], §2.3). We briefly recall
them and its isomorphism with higher Chow groups in §2.3.

It is also useful to look at the natural cycles on product of curves. This will give a better
understanding, of the structure of higher Chow groups and related to the underlying ge-
ometry. In this respect, we investigate the relationship of the cycle $\Delta_{e,1}$ with the Beilinson
cycle $B_{C'} \in CH^2(C' \times C', 1) \otimes \mathbb{Q}$, see [Be]. When the curve $C'$ is generic and bielliptic, we
show that the (pullback of) Beilinson cycle does not occur as the degeneration, and the
pushforward on the Jacobian $J(C')$, is different from Collino’s cycle, even up to algebraic
equivalence. See Proposition 5.9. It raises some questions, which we mention in Remark
5.10.

Finally, we investigate a variant of the modified diagonal cycle on triple products $C \times C' \times C''$, where the curves may be different. We show

**Proposition 1.2.** Suppose $C, C'$ and $C''$ are smooth projective curves admitting finite
morphisms $\phi' : C \to C''$ and $\phi'' : C \to C''$. There is a modified diagonal cycle $\Delta_{e,X} \in
CH^2(X) \otimes \mathbb{Q}$, for a point $e \in C$. Here $X := C \times C' \times C''$. Assume that $g(C) = 3$ and $C$ is
non-hyperelliptic. Then the cycle $\Delta_{e,X}$ corresponds to a non-trivial cycle in the Griffiths...
group $A^2(X) \otimes \mathbb{Q}$. The degeneration of this cycle, corresponds to a higher indecomposable Chow cycle in $CH^2(D \times D', 1) \otimes \mathbb{Q}$, where $D, D'$ are generic non-rational curves with genus $\leq 2$.

This gives new higher indecomposable Chow cycles on double products of curves of low genus.

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2. Preliminaries

We assume that the base field is $k = \mathbb{C}$. In this section, we collect some facts and terminologies, we will need in the proofs.

Suppose $X$ is a smooth projective variety of dimension $d$ over $\mathbb{C}$. We denote the rational higher Chow groups $CH^r(X, s) := CH^r(X, s) \otimes \mathbb{Q}$. When $s = 0$, $CH^r(X, 0) = CH^r(X)$ is the usual Chow group of $X$. We refer to Bloch’s definition [Bl] of higher Chow groups and Levine’s cubical version [Le1].

2.1. Indecomposable cycles. There is a product map:

$$CH^{r-m}(X, s-m) \otimes CH^1(X, 1) \xrightarrow{\otimes m} CH^r(X, s).$$

The cokernel $\frac{CH^r(X, s)}{\text{Image}(\epsilon)}$ is the group of indecomposable cycles.

Let $K_j$ denote Quillen’s $K$-theory of coherent sheaves and $\mathcal{K}_{j, X}$ denote the corresponding Zariski sheaf on $X$. Then we have the isomorphism:

$$CH^r(X, s) \simeq H^{r-s}(X, K_r, X).$$

An element of $CH^r(X, 1)$ is written as a finite sum $\sum_i W_i \otimes h_i$, where $W_i \subset X$ are irreducible subvarieties of codimension $r - 1$ and the rational functions $h_i \in \mathbb{C}(W_i)^*$ satisfy $\sum_i \text{div}(h_i) = 0$ as a cycle on $X$.

2.2. Regulator indecomposable cycles. For a subring $A \subset \mathbb{R}$, denote $A(r) := (2\pi i)^r A \subset \mathbb{C}$ and the Deligne Cohomology $H_D^2(X, A(r))$ is the hypercohomology of the complex:

$$A(r)_D : A(r) \rightarrow O_X \rightarrow \Omega^1_X \rightarrow \cdots \rightarrow \Omega^{r-1}_X.$$

There is a short exact sequence:

$$0 \rightarrow \frac{H^{2r-2}(X, C)}{F^r H^{2r-2}(X, C) + F^{r-1} H^{2r-1}(X, C)} \rightarrow H^{2r-1}_D(X, A(r)) \rightarrow C^r \rightarrow 0.$$

Here $C^r := H^{2r-1}(X, A(r)) \cap F^r H^{2r-1}(X, C) = H^{2r-1}(X, A(r))_{\text{torsion}}.$
With $\mathbb{R}$-coefficients, we have
\[
H^{2r-1}_D(X, \mathbb{R}(r)) = \frac{H^{2r-2}(X, \mathbb{C})}{F^r H^{2r-2}(X, \mathbb{C}) + H^{2r-2}(X, \mathbb{R}(r))} = H^{2r-1}(X, \mathbb{R}(r-1)) \cap F^{r-1} H^{2r-2}(X, \mathbb{C}) = H^{r-1,r-1}(X, \mathbb{R}(r-1)).
\]

There is a cycle class map, resp. regulator
\[
c_{r,s} : CH^r(X, s) \rightarrow H^{2r-s}_D(X, \mathbb{Z}(r)).
\]
Extending the coefficients to $\mathbb{R}$, we have the real regulator map:
\[
c_{r,s} : CH^r(X, s) \rightarrow H^{2r-s}_D(X, \mathbb{R}(r)).
\]
When $s = 1$, we can make this explicit: given a cycle $W \in CH^r(X, 1)$,
\[
c_{r,1}(W) \in H^{2r-1}_D(X, \mathbb{R}(r)) \simeq H^{r-1,r-1}(X, \mathbb{R}(r-1))
\]
and it is the class of the current:
\[
\omega \mapsto (2\pi i)^{r-1-d} \sum_i \int_{W_i - W_i^{\text{sing}}} \omega \log |h_i|,
\]
for $\omega \in H^{d-r+1,d-r+1}(X, \mathbb{R}(d - r + 1))$.

**Definition 2.1.** A cycle $W \in CH^r(X, 1)$ is said to be real regulator indecomposable if there exists a differential form
\[
\omega \in (\text{Hodge}^{r-1}(X)_{\mathbb{R}})^\perp \subset H^{d-r+1,d-r+1}(X, \mathbb{R}(d - r + 1))
\]
such that $c_{r,1}(W)(\omega) \neq 0$. Here $\text{Hodge}^{r-1}(X)$ is the subspace of Hodge classes, and $\perp$ is the orthogonal subspace, for the intersection pairing.

It follows that if $W$ is regulator indecomposable then $W$ is indecomposable.

**2.3. Levine’s $K$-groups with supports.** In this subsection, we recall Levine’s $K$-groups [Le2, §8, p.48-51] with supports, and their relation to higher Chow groups.

Let $X$ be a quasi-projective scheme over a field $k$ and $K(X)$ denote the Quillen $K$-theory space (or spectrum) of $X$. The $K$-groups $K_j(X)$ are the $j$-th homotopy groups of $K(X)$.

1) There is a natural $\lambda$-ring structure on $K_*(X)$.

2) If $U \subset X$ is an open subset, then we have the $K$-theory with supports $K_X^W$ in $W := X - U$, defined as the homotopy fiber of the natural map:
\[
K(X) \rightarrow K(U).
\]

3) There is a long exact sequence:
\[
K_p^W(X) \rightarrow K_p(X) \rightarrow K_p(X - W) \rightarrow K_{p-1}(X) \rightarrow ...
\]
The relative $K$-theory space $K(X; Y_1, ..., Y_n)$ is defined as the homotopy fiber of $K(X; Y_1, ..., Y_n) := \text{holim}_{[0,1]^n} (K(X; Y_1, ..., Y_n)_*)$.

Here holim denotes the homotopy limit.

5) If $U \subset X$ is an open subset and $W = X - U$, the relative $K$-theory space with supports, $K^W(X; Y_1, ..., Y_n)$, is defined as the homotopy fiber of $K(X; Y_1, ..., Y_n) \rightarrow K(U; Y_1 \cap U, ..., Y_n \cap U)$.

6) There is a natural $\lambda$-operation on the relative $K$-groups with support $K^W(X, Y_1, ..., Y_n)$ which satisfy the special $\lambda$-identities. The resulting Adams operations are denoted $\psi^k$.

7) Recall the isomorphism, when $X$ is smooth and quasi-projective over a field:

$$CH^q(X, p)_Q \simeq K_p(X)^{(q)}_Q.$$  

The RHS is the $q$-graded piece for the Adams operation $\psi^k$, on the $\gamma$-filtration. In fact Levine obtains the above isomorphism, by inverting $(d + p - 1)!$, where $d := \dim(X)$.

8) Let $\square^n$ be the affine space $\mathbb{A}^n_k$, let $\partial\square^n$ be the collection of divisors

$$D^\epsilon_i : t_i = \epsilon, i = 1, ..., n; \epsilon = 0, 1,$$

and let $\partial_0\square^n = \partial\square^n - D^0_n$.

9) Let $K^q_r(X \times \square^p; X \times \partial\square^p)$ be the direct limit

$$K^q_r(X \times \square^p; X \times \partial\square^p) := \lim_{\to} K^W_r(X \times \square^p; X \times \partial\square^p)$$

over closed subsets $W$ of $X \times \square^p$ of codimension $q$, such that each component of $W$ intersects each face of $X \times \square^p$ in codimension $q$. Similarly, $K^q_r(X \times \square^p; X \times \partial_0\square^p)$ is the direct limit as above with $\partial$ replaced by $\partial_0$.

10) There are canonical isomorphisms:

$$K^q_0(X \times \square^p; X \times \partial\square^p)^{(q)}_Q \simeq Z_p(Z^q(X, *)_Q)$$

$$K^q_0(X \times \square^{p+1}; X \times \partial_0\square^{p+1})^{(q)}_Q \simeq Z_p(Z^q(X, p + 1)_Q).$$

Here $Z^q$ denotes the Levine’s cubical complex, and $Z_p$ denotes the (homological) $p$-dimensional cycles.

11) The natural map

$$K^q_0(X \times \square^p; X \times \partial\square^p)^{(q)}_Q \rightarrow K_0(X \times \square^p; X \times \partial\square^p)^{(q)}_Q$$
combined with 10), gives the map:

\[ Z_p(\mathcal{Z}^q(X,*)_\mathbb{Q}) \to K_0(X \times \Box^p; X \times \partial \Box^p)^{(q)}_\mathbb{Q}. \]

12) The above map descends to give the isomorphism:

\[ c^{q,p} : CH^q(X,p)_\mathbb{Q} \cong \to K_0(X \times \Box^p; X \times \partial \Box^p)^{(q)}_\mathbb{Q} \cong K_p(X)^{(q)}_\mathbb{Q}. \]

In fact with denominators \( \frac{1}{(d+p-1)!} \).

3. Modified diagonal cycle on a triple product of a curve

Suppose \( X \) is a smooth projective connected curve of genus \( g \) defined over the complex numbers. Consider the triple product \( Y = X \times X \times X \). Gross and Schoen [Gr-Sc] defined the modified diagonal cycle as follows: fix a closed point \( e \in X \). Define the following subvarieties of codimension 2 on \( Y \).

\[
\begin{align*}
\Delta_{123} & := \{ (x,x,x) : x \in X \} \\
\Delta_{12} & := \{ (x,x,e) : x \in X \} \\
\Delta_{13} & := \{ (x,e,x) : x \in X \} \\
\Delta_{23} & := \{ (e,x,x) : x \in X \} \\
\Delta_1 & := \{ (x,e,e) : x \in X \} \\
\Delta_2 & := \{ (e,x,e) : x \in X \} \\
\Delta_3 & := \{ (e,e,x) : x \in X \}
\end{align*}
\]

The modified diagonal cycle is defined as follows:

\[ \Delta_e := \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3. \]

It was shown in [Gr-Sc, Proposition 3.1,p.654] that the cycle \( \Delta_e \) is homologous to zero. Furthermore, if the genus \( g(X) = 0 \), then \( \Delta_e \) is rationally equivalent to zero and when the curve \( X \) is hyperelliptic and the point \( e \) is a Weierstrass point then \( \Delta_e \) is a torsion element in \( CH_1(Y) \) of order 6 [Gr-Sc, Proposition 4.8,p.658].

3.1. A projector \( P_e \) on the product variety. Fix a closed point \( e \in X \). A projector \( P_e \) is defined in [Gr-Sc, p.652] which acts on the Chow group of \( Y = X \times X \times X \). For any ordered subset \( T \subseteq \{1,2,3\} \), let \( T' \) be the complementary subset \( \{1,2,3\} - T \), and \( |T| \) denote the cardinality of \( T \). Write \( p_T : X \times X \times X \to X^{[T]} \) for the usual projection and let \( q_T : X^{[T]} \to X \times X \times X \) be the inclusion, with \( e \) inserted at the missing coordinates. Let \( P_T \) be the graph of the morphism \( q_T \circ p_T : Y \to Y \), viewed as a cycle of codimension 3 on \( Y \times Y \). Define

\[ P_e = \sum_T (-1)^{|T|} P_T \in Z^3(Y \times Y). \]

In other words,

\[ P_e = P_{123} - P_{12} - P_{13} - P_{23} + P_1 + P_2 + P_3 \]
Then we have

**Lemma 3.1.** The projector \((P_e)_*\) annihilates the cohomology groups \(H^6(Y), H^5(Y), H^4(Y)\) and maps \(H^3(Y)\) onto the Künneth summand \(H^1(X) \otimes H^1(X) \otimes H^1(X)\).

*Proof.* See [Gr-Sc, Corollary 2.6, p.654]. □

The modified diagonal cycle can now be written as:

\[
\Delta_e = (P_e)_* \Delta_X
\]

where \(\Delta_X \subset X \times X \times X\) is the diagonal \(\{(x,x,x) : x \in X\}\).

Consider the natural morphism, fixing a basepoint \(p \in X\).

\[
f : Y \rightarrow Sym^3X \rightarrow J(X) \quad (a,b,c) \mapsto a + b + c \rightarrow a + b + c - 3p
\]

The Ceresa cycle in the Jacobian \(J(X)\) is the cycle \(X_e - X_e^-\), fixing the base point \(e\) for the embedding \(X \hookrightarrow J(X)\).

**Proposition 3.2.** There is an equality of the push-forward cycle with the Ceresa cycle,

\[
f_* \Delta_e = 3(X_e - X_e^-)
\]

in the group of algebraic cycles modulo algebraic equivalence.

*Proof.* See [Cb, Proposition 2.9]. □

### 4. Specialization and Mayer-Vietoris exact sequence

S. Bloch [Bl4] has defined specialization maps for higher Chow groups over a DVR \(O\) with residue field \(k\) and fraction field \(F\). M. Levine [Le1] has defined higher Chow groups \(CH^*(X,\_)*\) for any variety \(f : X \rightarrow B\) over a one-dimensional regular Noetherian scheme \(B\). We will assume that \(B = \text{Spec}(O)\) for some local \(k\)-algebra and DVR \(O\) with residue field \(k\), and combine both results to obtain specialization maps in Levine’s context.

Higher Chow groups are defined as homology groups of a certain complex \(Z^*(X,\_)*\) defined in [Le1]. Levine shows a localization sequence (distinguished triangle)

\[
0 \rightarrow Z^{p-r}(Z,\_)* \rightarrow Z^p(X,\_)* \rightarrow Z^p(U,\_)*
\]

where \(Z \subset X\) is a closed subscheme of pure codimension \(r\) and \(U\) the open complement. Here, one has to be careful about the notion of dimension of a cycle, see [Le1]. The induced coboundary map will be denoted by

\[
\partial : CH^*(U,\_)* \rightarrow CH^{*-1}(Z,\_)* - 1.
\]

Let \(0 \in B\) be the closed point and \(\pi\) a uniformizing parameter of \(B\). We will apply this result in the case where \(Z = f^{-1}(0)\) is the closed fiber of \(f\), and \(r = 1\).
There is a canonical class \([\pi] \in f^*CH^1(B \setminus \{0\}, 1) \subset CH^1(U, 1)\) induced by the isomorphism

\[ CH^1(Y, 1) = H^0(Y, \mathcal{O}_Y^*) \]

for the smooth variety \(Y = B \setminus \{0\} \).

We define the specialization morphism

\[ sp : CH^*(U, *) \to CH^*(Z, *) \]

via

\[ sp(W) := \partial(W \cdot [\pi]). \]

The following is due to S. Bloch [Bl4] and W. Fulton [Fu, pg. 398]:

**Proposition 4.1.** The specialization map \(sp\) is a \(k\)-algebra homomorphism.

Note that \(sp\) agrees with Fulton’s specialization map \(\sigma\) for \(n = 0\), and that for all cycles on \(U\) which are restrictions of cycles from \(X\), specialization is simply given by taking the restriction from \(X\) to the closed fiber over \(Z\).

**Proof.** We follow Bloch [Bl4]. Denote by \(N^*_Z, N^*_X\) and \(N_U\) the cycle DGA’s \(Z^*(Z, *)\), \(Z^*(X, *)\) and \(Z^*(U, *)\) respectively. \(N_X\) also has a multiplicative structure, but only in the derived category, since one uses a moving lemma. There are natural restriction maps

\[ \alpha : N^*_X \to N^*_U, \quad i^* : N^*_X \to N^*_Z. \]

There is also a pullback homomorphism \(p^* : N^*_Z \to N^*_X\) and the coboundary \(\partial : N^*_U \to N^*_Z[-1]\). First one defines a section \(\vartheta_\pi\) to \(\partial\) by setting

\[ \vartheta_\pi := (\text{mult. with } [\pi]) \circ \alpha \circ p^* : N^*_Z[-1] \to N^*_U. \]

This satisfies \(\vartheta_\pi \circ \partial = id\) on \(N^*_Z[-1]\) and induces an additive splitting

\[ \alpha \oplus \vartheta_\pi : N^*_X \oplus N^*_Z[-1] \to N^*_U \]

in the derived category. As in [Bl4] it follows that

\[ sp \circ \alpha = i^* : N^*_X \to N^*_Z. \]

To finish the proof, one verifies the following formulas as in [Bl4]:

\[ \partial(x \cdot y) = sp(x) \cdot \partial(y) + (-1)^{\deg(y)} \partial(x) \cdot sp(y), \]

\[ \partial(\alpha(x) \cdot \vartheta_\pi(y)) = i^*(x) \cdot y = sp(\alpha(x)) \cdot \partial(\vartheta_\pi(y)). \]

\(\square\)
4.1. **Mayer-Vietoris spectral sequence.** Let $Z$ be a union of $m$ components $Z_1, ..., Z_m$ of the same dimension. Define

$$Z^{[0]} = \bigsqcup_{1 \leq i \leq m} Z_i \text{ and } Z^{[1]} = \bigsqcup_{1 \leq i < j \leq m} Z_i \cap Z_j.$$ 

Then there is an inclusion $i : Z^{[1]} \hookrightarrow Z^{[0]}$.

**Proposition 4.2.** We have the exact sequence

$$CH^r(Z^{[0]}, p) \to CH^r(Z, p) \to \ker(CH^{r-1}(Z^{[1]}, p-1)) \to 0,$$

**Proof.** This is part of the lower term sequence for the 3rd quadrant Mayer-Vietoris spectral sequence $[SM] E_1^{a,b} = CH^{a+r}(Z^{[a]}, -b) \Rightarrow CH^r(Z, -a - b)$. \hfill \Box

5. **Degeneration of the modified diagonal cycle**

We recall that good models of families of triple products of curves have been constructed in [Gr-Sc, p.664].

5.1. **Good family of triple products.** Suppose $C \to B := \text{Spec}(R)$ is a stable family of genus 3 curves over a DVR $R$, with a special singular fibre $C_0$. Then a good family of triple products

$$\pi : \mathcal{Y} \to B$$

exists. If $C$ is irreducible of genus three, with one node, then the normalization $C'$ is a smooth, hyperelliptic curve of genus 2. We assume that the inverse image of the node in $C'$ are Weierstrass points. The special fibre $C_0$ of $C \to B$ is $C' \cup \mathbb{P}^1$, such that $C' \cap \mathbb{P}^1$ consists precisely of the two Weierstrass points.

The special fibre $\mathcal{Y}_0 := \pi^{-1}(0)$ has 8 components [Gr-Sc p.666, Example 6.15]:

$$Y_1 = C' \times C' \times C'$$
$$Y_2 = C' \times C' \times \mathbb{P}^1$$
$$Y_3 = C' \times \mathbb{P}^1 \times C'$$
$$Y_4 = C' \times \mathbb{P}^1 \times \mathbb{P}^1$$
$$Y_5 = \mathbb{P}^1 \times C' \times C'$$
$$Y_6 = \mathbb{P}^1 \times C' \times \mathbb{P}^1$$
$$Y_7 = \mathbb{P}^1 \times \mathbb{P}^1 \times C'$$
$$Y_8 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

The normalization is given by $V_0 = \mathcal{Y}_0^{[0]} = \sqcup_{i=1}^8 Y_i$. 
5.2. Extending the modified diagonal cycle. Assume that the family of curves $\mathcal{X} \to B$ has a section $\tilde{e}$ over $B$, such that on the special fibre it corresponds to a Weierstrass point on the component $C'$. Let $Y = X \times X \times X$ be the generic fibre of $\mathcal{Y} \to B$. The naive extension $\overline{\Delta}_e$ is obtained by taking the closures of each irreducible component $\Delta_i, \Delta_{i,j}, \Delta_{i,j,k}$.

Denote $U$ the complement $Y - Y_0$. Consider the specialization map $(1)$:

$$sp : CH^2(U) \to CH^2(Y_0).$$

The image of $\Delta_e$ under the map $sp$ is the same as the restriction of $\overline{\Delta}_e$ to $Y_0$.

In our situation, the Mayer-Vietoris sequence in Proposition 4.2 gives a right exact sequence:

$$\to CH^2(V_0) \to CH^2(Y_0) \to Ker(CH^1(Y_0^{[1]}, -1) \overset{i}{\to} CH^2(V_0, -1)) \to 0.$$}

Since the right term is zero, the specialization cycle $\Delta^0_e$ lifts to a cycle on $V_0$. We denote this cycle again by $\Delta^0_e$.

5.3. Higher Chow cycle. Now note that the cycle $\overline{\Delta}_e$ is written as:

$$\overline{\Delta}_e := \overline{\Delta}_{123} - \overline{\Delta}_{12} - \overline{\Delta}_{13} - \overline{\Delta}_{23} + \overline{\Delta}_1 + \overline{\Delta}_2 + \overline{\Delta}_3.$$}

Here

\begin{align*}
\overline{\Delta}_{123} & := \{x, x, x : x \in \mathcal{X}\} \\
\overline{\Delta}_{12} & := \{(x, x, \tilde{e}) : x \in \mathcal{X}\} \\
\overline{\Delta}_{13} & := \{(x, \tilde{e}, x) : x \in \mathcal{X}\} \\
\overline{\Delta}_{23} & := \{\tilde{e}, x, x : x \in \mathcal{X}\} \\
\overline{\Delta}_1 & := \{(x, \tilde{e}, \tilde{e}) : x \in \mathcal{X}\} \\
\overline{\Delta}_2 & := \{\tilde{e}, x, \tilde{e} : x \in \mathcal{X}\} \\
\overline{\Delta}_3 & := \{\tilde{e}, \tilde{e}, x : x \in \mathcal{X}\}.
\end{align*}

We have the smooth families:

$$\mathcal{U} \to W, \mathcal{J}(\mathcal{C}_W) \to W$$

associated to the smooth projective family $\mathcal{C}_W \to W := B - \{0\}$ of genus three curves. The first one is the family of triple products and the second is the smooth Jacobian family. There is a proper morphism

$$f : \mathcal{U} \to \mathcal{J}(\mathcal{C}_W)$$

over $W$, with respect to the section $\tilde{e}$. On the special fibre, the morphism extends and factors via the triple product of the singular nodal curve $C$ whose normalization is $C''$ (see (5.1)), to the Jacobian of the single nodal curve. This is a generalised Jacobian, i.e., an extension of $J(C')$ by $\mathbb{C}^*$. The compactification is a $\mathbb{P}^1$-bundle over $J(C')$. Since the
node on $C$ corresponds to two Weierstrass points on $C'$, we assume that the Jacobian extension is trivial (see [Co]). The special fibre of the Jacobian family is $J(C') \times \mathbb{P}^1$.

One has a diagram of pushforward maps:

$$
\begin{array}{ccc}
Z^2(\mathcal{U}_Q) & \xrightarrow{sp} & Z^2(\mathcal{Y}^*)_Q \\
f_* \downarrow & & \downarrow g_* \\
Z^2(J(\mathcal{C}_W))_Q & \xrightarrow{sp} & Z^2(J(C')_Q \
\end{array}
$$

Here $Z^2(\mathcal{Y}^*)$ denotes the group of codimension two cycles on the simplicial set $\mathcal{Y}^*$.

**Lemma 5.1.** The diagram commutes, up to rational equivalence.

*Proof.* The pushforward map $g_*$ is induced by a finite morphism $V_0 = \mathcal{Y}^0_0 \rightarrow J(C') \times \mathbb{P}^1$. On the component $Y_1$ it is given by $f' : C' \times C' \rightarrow J(C')$ and the hyperelliptic map $C' \rightarrow \mathbb{P}^1$. On the $Y_2, Y_3$ and $Y_5$ it is $f' \times \text{id}$, and degenerate on the other components, since there is no non-constant morphism from $\mathbb{P}^1$ to $J(C')$. Both $f_*$ and $g_*$ make sense in families and the diagram commutes with specialization up to rational equivalence. $\square$

**Definition 5.2.** $\Delta_{e,1} := \text{sp}(\Delta_e)$.

**Lemma 5.3.** The specialization $\Delta_{e,1}$ is supported on $Y_1, Y_2, Y_3$ and $Y_5$. The restriction to $Y_1$ vanishes in $CH^2(Y_1)_Q$, if $e$ is a Weierstrass point on $C'$.

*Proof.* The specialization is a codimension two cycle on the simplicial set $\mathcal{Y}^*$. However, it is supported on the skeleton $\mathcal{Y}^0$ of the simplicial set. Furthermore, by the Mayer-Vietoris sequence in [5.2] it lies in the normalization $V_0$. But $V_0$ is a disjoint union of eight components $Y_1, ..., Y_8$, see [5.1]. Hence, we consider the specialization on each component $Y_i$, as follows.

The specialization of $\Delta_{e,1}$ to $Y_1$ is the restriction of (sum of) the closure in $\mathcal{Y}$ of each diagonal component of the modified diagonal cycle on $\mathcal{U}$, to $Y_1$. The restrictions of the closures correspond to the diagonal component of the modified diagonal cycle $\Delta_e$ on $C' \times C' \times C'$, and $e$ is a Weierstrass point on $C'$. In other words, the cycle $\text{sp}(\Delta_{e,1})$ is the modified diagonal cycle on $C' \times C' \times C'$. Hence by [Gr-Sc] Proposition 4.8, p.658], we conclude that this class is rationally equivalent to zero. The restrictions to $Y_4, Y_6, Y_7$ and $Y_8$ are zero as $g$ is constant on these components. More concretely, $g_*$ of the specialization on these components is the zero map, since there are no non-constant maps from $\mathbb{P}^1$ to $J(C')$. Furthermore, $g^*g_*$ on $\Delta_{e,1}$ is a multiple of $\Delta_{e,1}$. Hence triviality of $g_*$ on $Y_4, Y_6, Y_7, Y_8$ implies that the specialization on these components is torsion. $\square$

**Lemma 5.4.** The pushforward cycle $g_*\Delta_{e,1}$ has a representative in the subgroup

$$
Z^2(\mathcal{U}_Q) := \ker \left( (i_0^*, i_\infty^* : Z^2(J(C') \times \mathbb{P}^1)_Q \rightarrow \oplus^2 Z^2(J(C'))_Q \right).
$$

Therefore, $g_*\Delta_{e,1}$ defines a higher Chow cycle in $CH^2(J(C'), 1)_Q$. 

Proof. This is because the pushforward of $\Delta_{e,1}$ is supported on the generalised Jacobian, which is a trivial $\mathbb{C}^*$-extension of $J(C')$.

We will now use the following results, which are essentially due to Colombo and van Geemen [Cb, Cb-vG].

**Proposition 5.5.**

(a) The cycle $g_*\Delta_{e,1}$ is a degeneration of Ceresa’s cycle up to algebraic equivalence.

(b) The Abel-Jacobi class of Ceresa’s cycle specializes to the higher Abel-Jacobi invariant of Collino’s cycle $(C'_p, h_p) + (C'_q, h_q)$ in $CH^2(C' \times C', 1)_Q$, as defined in [Co].

Proof. (a) By [Cb-vG], $f_*\Delta_e$ is 3 times the Ceresa cycle up to algebraic equivalence. Hence, $sp(f_*\Delta_e) = g_*\Delta_{e,1}$ is a degeneration of the Ceresa cycle up to algebraic equivalence.

(b) The Abel-Jacobi invariant of Ceresa’s cycle $C - C^{-}$, is most conveniently described as an integration current $\int \Gamma$, where $\Gamma$ is a 3-chain on $J(C)$, with $\partial \Gamma = C - C^{-}$. This is a current in the relative intermediate Jacobian $D^3(J(C))$ (fiberwise) of the family $J(C_W) \to W$, resp. a current in $D^3(U)$ which can be evaluated on 3-forms with compact support. Colombo [Cb, pg. 788-789] has remarked that the specialization map on the level of currents is given by $sp(f_*\Gamma) = \tilde{f}_*\Gamma$, where $\tilde{\Gamma}$ is a 3-chain on $C' \times C' \times C'$.

Choosing test forms $\alpha \wedge \frac{dz}{2}$ with compact support, this defines a 2-current in $D^2(J(C'))$ which is the regulator value for Collino’s cycle as Colombo indicates in loc. cit..

We can now prove our main result (see Theorem 1.1 in the introduction):

**Theorem 5.6.** The degeneration $\Delta_{e,1}$ is a higher Chow cycle in $CH^2(C' \times C', 1)_Q$. It is indecomposable, if $C'$ is generic, and its regulator image is non-zero.

Proof. Replace Chow groups by (relative) $K_0$. We refer to Levine’s paper [Le2, §8, p.48-51], see [2.3]. We use the isomorphism of higher Chow group with the relative Chow group with supports, see [2.3, 12]. Then we note that in fact, $g_*\Delta_{e,1}$ is in the image of

$$K_0(J(C') \times \Box^1, J(C') \times \partial \Box^1)^{(2)}_Q \to K_0(J(C') \times \Box^1)^{(2)}_Q.$$  

Denote the cycle in $CH^2(J(C'), 1)_Q \simeq K_0(J(C') \times \Box^1, J(C') \times \partial \Box^1)^{(2)}_Q$ by $W$, which maps to $g_*\Delta_{e,1}$.

Then we use the commutative diagram

$$
\begin{array}{ccc}
K_0(C' \times C' \times \Box^1, C' \times C' \times \partial \Box^1)^{(2)}_Q & \to & K_0(C' \times C' \times \Box^1)^{(2)}_Q \\
f_* \downarrow & & \downarrow g_* \\
K_0(J(C') \times \Box^1, J(C') \times \partial \Box^1)^{(2)}_Q & \to & K_0(J(C') \times \Box^1)^{(2)}_Q.
\end{array}
$$

Since $f$ is a finite proper morphism, the inverse image $(f_*)^{-1}(W)$ is a multiple of $\Delta_{e,1}$, and it lies in the relative $K_0^*$ group of $C' \times C'$. Now we use the isomorphism [2.3, 12]:

$$CH^2(C' \times C', 1)_Q \cong K_0(C' \times C' \times \Box^1, C' \times C' \times \partial \Box^1)^{(2)}_Q,$$

to deduce that $\Delta_{e,1}$ corresponds to a higher Chow cycle in $CH^2(C' \times C', 1)_Q$. 

The indecomposability and non-triviality of the regulator image of $\Delta_{e,1}$ follows, because the pushforward $g_*\Delta_{e,1}$ is a multiple of Collino’s regulator indecomposable higher Chow cycle on $J(C')$, up to algebraic equivalence.

**Remark 5.7.** One can also look at Griffiths’ infinitesimal invariant of Ceresa’s cycle. It can be computed from the cohomology class in $H^4(U)$ of $C - C^-$. The specialization of this class is not homologous to zero on the closed fiber $V_0$. In Gross-Schoen [Gr-Sc] this non-zero class is explained. It is the image of the infinitesimal invariant under specialization, and hence the infinitesimal invariant of $\Delta_{e,1}$ is non-zero.

5.4. **Second degeneration higher Chow cycle $\Delta_{e,2}$**. One can iterate this procedure, to obtain the following.

**Proposition 5.8.** The second degeneration of the cycle $\Delta_{e,1} \in CH^2(C' \times C', 1)_\mathbb{Q}$ corresponds to an indecomposable higher Chow cycle $\Delta_{e,2} \in CH^2(E, 2)_\mathbb{Q}$. Here $E$ is a generic elliptic curve and is the normalization of the degeneration, a single nodal curve, of a family of smooth hyperelliptic genus two curves over Spec($R$), where $R$ is a DVR.

**Proof.** The proof is similar to the first degeneration $\Delta_{e,1}$. Hence we only indicate the proof, without details. We consider a smooth family of genus two curves $C' \to B - \{0\}$, $B :=$ Spec($R$), where $R$ is a DVR. The special fibre is a single nodal curve whose normalization is an elliptic curve $E$, with the inverse image of the node being two Weierstrass points. A good model $U$ of the double product smooth family is considered as earlier, together with the higher Chow cycle in $CH^2(C' \times C', 1)_\mathbb{Q}$. This is same as a cycle in $Z^2_\partial(J(C') \times \mathbb{P}^1)_\mathbb{Q}$, see Lemma [5.4]. Using the specialization map $sp$, we obtain the degeneration $\Delta_{e,2} \subseteq CH^2(E \times \mathbb{P}^1, 1)_\mathbb{Q}$. In fact, using the relative $K_0$-groups of Levine (see proof Theorem [5.6]), it corresponds to a higher Chow cycle in $CH^2(E, 2)_\mathbb{Q}$. This is a multiple of Collino’s degenerated cycle, up to algebraic equivalence. Hence, it is indecomposable with non-zero regulator image.

5.5. **Third degeneration higher Chow cycle $\Delta_{e,3}$**. We can even further degenerate $\Delta_{e,2}$ to a cycle $\Delta_{e,3} \in CH^2(C, 3) \otimes \mathbb{Q}$ by the same method. This is related to Collino’s work [Co, Remark 7.13]. There also a formula for the regulator is indicated.

5.6. **Beilinson’s higher Chow cycle on product of two curves**. Suppose $C'$ is a smooth projective curve over the complex numbers. Assume that $C'$ is a hyperelliptic curve of genus two, and $f : C' \to \mathbb{P}^1$ be the hyperelliptic map. Let $p, q \in C'$ be any two Weierstrass points and the divisor of $f$ is $div(f) = 2p - 2q$. Consider the higher Chow cycle on the product $C' \times C'$:

$$(\Delta_{C'}, f) - (C' \times \{p\}, f) - (\{q\} \times C', f) \in CH^2(C' \times C', 1).$$

This cycle was earlier considered in [Be]. We want to understand the regulator image of this cycle, denoted $B_{C'}$, and its relationship with the degeneration $\Delta_{e,1}$, in the previous section.
Consider the real regulator map:
\[ c_{2,1} : CH^2(C' \times C', 1)_Q \to H^2_D(C \times C', \mathbb{R}(2)). \]

We say that a higher Chow cycle \( Z \) is real regulator decomposable if the real regulator image lies in \( NS(C' \times C') \otimes \mathbb{R} \).

It was pointed out by R. Sreekantan that the pushforward of the Beilinson cycle \( B_{C'} \) is the Collino cycle, using the map \( C' \times C' \to J(C') \), \( (x, y) \mapsto x - y \). We note that in [Co-Fd], the real regulator image of Collino cycle is computed when the curve \( C' \) is bielliptic and generic. Furthermore, we use the sum map (3.1) in the degeneration. Hence we investigate the pullback of the Beilinson cycle \( B_E \), where \( E \) is an elliptic curve and its relationship with the degeneration.

Suppose \( C' \to E \) is a bielliptic curve of genus two. Denote \( \pi : C' \times C' \to E \times E \). We show:

**Proposition 5.9.** The pullback cycle \( \pi^* B_E \) of the Beilinson cycle \( B_E \) is real regulator decomposable. In particular, the pullback cycle is different from the degeneration \( \Delta_{e,1} \), up to algebraic equivalence, when the curve \( C' \) is a generic bielliptic curve of genus two.

**Proof.** Consider the product map: \( \pi : C' \times C' \to E \times E \), and this gives the pullback map
\[ \pi^* : CH^2(E \times E, 1)_Q \to CH^2(C' \times C', 1)_Q. \]

Hence it suffices to compute the real regulator image of \( B_E \).

We now see that the real Deligne cohomology \( H^3_D(E \times E, \mathbb{R}(2)) \) is three-dimensional, and is generated by the Neron-Severi group \( NS(E \times E)_{\mathbb{R}} \). The generators are \( E \times pt, pt \times E \) and the diagonal \( \Delta_E \).

This implies that the real regulator image \( c_{2,1}(B_E) \) lies in \( NS(E \times E)_{\mathbb{R}} \). In other words, \( B_E \) is real regulator decomposable. Hence the same holds for the pullback cycle \( B_{C'} \).

However, by [Co-Fd] Proposition 1.3, Lemma 1.5], the real regulator image of Collino’s cycle is non-zero and indecomposable, when the curve \( C' \) is a generic bielliptic curve. Since \( g_* \Delta_{e,1} \) is a multiple of Collino’s cycle, up to algebraic equivalence (see Theorem 5.6), we deduce that the real regulator image of \( \Delta_{e,1} \) is non-zero and indecomposable, when \( C' \) is generic.

Hence the pullback of the Beilinson cycle does not occur as the degeneration, and is different from Collino’s cycle, even up to algebraic equivalence. \( \square \)

**Remark 5.10.** The above proposition raises the following questions. We know that the Beilinson cycle \( B_{C'} \) is real regulator indecomposable, since it pushforwards to Collino’s cycle, under the difference map. Therefore, it is interesting to understand the pushforward of \( B_{C'} \) also under the sum map, and to compare it with the pullback cycle \( \pi^* B_E \), when \( C' \) is bielliptic and generic. These natural geometric cycles on \( C' \times C' \), when \( C' \) is bielliptic, give additional information on the structure of \( CH^2(C' \times C', 1) \).
6. **Higher Chow cycle on double products** \( D \times D' \), as degeneration

In this section, we show the existence of a higher indecomposable Chow cycle on a product of curves \( D \times D' \), where \( g(D), g(D') \leq 2 \), \( D \) and \( D' \) are distinct. Moreover, we would like to obtain them as degenerations as in previous section. For this purpose, we first consider a variant of the modified diagonal cycle on triple products \( C \times C' \times C'' \), and use degeneration techniques as in the previous section.

### 6.1. A variant of the modified diagonal cycle.

In this subsection, we consider a variant of the modified diagonal cycle on a triple product of smooth projective curves \( X := C \times C' \times C'' \), together with finite morphisms \( \phi' : C \to C' \) and \( \phi'' : C \to C'' \). Here \( g(C) = 3 \).

Fix a base point \( e \in C \). Define the diagonals:

- \( \Delta_{123,X} := \{ (x, \phi'(x), \phi''(x)) : x \in X \} \)
- \( \Delta_{12,X} := \{ (x, \phi'(x), \phi''(e)) : x \in X \} \)
- \( \Delta_{13,X} := \{ (x, \phi'(e), \phi''(x)) : x \in X \} \)
- \( \Delta_{23,X} := \{ (e, \phi'(x), \phi''(x)) : x \in X \} \)
- \( \Delta_{1,X} := \{ (x, \phi'(e), \phi''(e)) : x \in X \} \)
- \( \Delta_{2,X} := \{ (e, \phi'(x), \phi''(e)) : x \in X \} \)
- \( \Delta_{3,X} := \{ (e, \phi'(e), \phi''(x)) : x \in X \} \)

Denote the one-cycle on \( X \):

\[
\Delta_{e,X} := \Delta_{123,X} - \Delta_{12,X} - \Delta_{23,X} - \Delta_{13,X} + \Delta_{1,X} + \Delta_{2,X} + \Delta_{3,X}.
\]

We note the following lemma.

**Lemma 6.1.** The one-cycle \( \Delta_{e,X} \) is a nontrivial cycle in the Griffiths group \( A^2(X)_\mathbb{Q} \), if \( C \) is a non-hyperelliptic curve. When \( C \) is hyperelliptic, we assume that \( e \) is not a Weierstrass point, to have the same conclusion.

**Proof.** First, we consider the finite morphism:

\[
\pi : C \times C \times C \to C \times C' \times C'', \text{ given by } \pi := id \times \phi' \times \phi''.
\]

Then the cycle \( \Delta_{e,X} = \pi_*(\Delta_e) \in CH_1(X)_\mathbb{Q} \). Using [Gr-Sz], we deduce that \( \Delta_{e,X} \) lies in the Griffiths group \( A^2(X)_\mathbb{Q} \). It remains to check the non-triviality.

Consider the commutative diagram:

\[
\begin{array}{ccc}
C \times C \times C & \xrightarrow{A_{J'}} & J(C) \times J(C) \times J(C) \\
\downarrow \pi & & \downarrow Nm \\
C \times C' \times C'' & \xrightarrow{A_{J''}} & J(C) \times J(C') \times J(C'').
\end{array}
\]
Here \( Nm := Id \times Nm(\phi') \times Nm(\phi'') \), the product of Norm maps on the second and third factor. Now consider the dual of Norm maps which commutes with the isomorphisms, induced by theta polarization:

\[
J(C) \times J(C) \times J(C) \xrightarrow{\phi_\alpha^3} J(C)^* \times J(C)^* \times J(C)^*
\]

\[
\uparrow \pi^* \quad \uparrow Nm^* \quad \downarrow
\]

\[
J(C) \times J(C') \times J(C'') \xrightarrow{\phi_\theta \times \phi_\theta \times \phi_\theta} J(C)^* \times J(C')^* \times J(C'')^*.
\]

Hence there is a commutative diagram:

\[
\begin{array}{ccc}
J(C) \times J(C) \times J(C) & \xrightarrow{+} & J(C) \\
\uparrow \pi^* & = & \uparrow \pi^* \\
J(C) \times J(C') \times J(C'') & \xrightarrow{+ \circ \pi^*} & J(C).
\end{array}
\]

At the level of Chow groups, we obtain a map:

\[
\text{CH}_1(C \times C \times C)_Q \to \text{CH}_1(J(C))_Q,
\]

given by \( \gamma_* := (+ \circ \pi^*)_* \circ (AJ'')_* \circ \pi_* \).

The maps on the Chow groups, \((+ \circ AJ)_*\) and \(\gamma_*\) induce the same map (up to a multiple) on the rational Griffiths group.

Since by [Cb] and for \( C \) and \( e \) as in statement of lemma, we know that \((+ \circ AJ)_*(\Delta_e)\) is non-trivial, it follows that \(\gamma_*(\Delta_e)\) is also non-trivial in the Griffiths group of one-cycles.

This implies that the cycle \(\Delta_{e,X} = \pi_*(\Delta_e)\) is non-trivial in the Griffiths group of one cycles on \(X = C \times C' \times C''\).

\(\square\)

In the next subsection, we will look at the degeneration of the cycle \(\Delta_{e,X}\).

6.2. **Degeneration of the cycle \(\Delta_{e,X}\).** The proof will involve suitable degenerations of family of curves. Consider a family of projective curves \(C \to B\), \(C' \to B\) and \(C'' \to B\), together with finite morphisms over \(B := \text{Spec}(R)\), for some DVR \(R\), \(\phi' : C \to C'\), \(\phi'' : C \to C''\). The generic fibre of these families are smooth and irreducible.

The generic fibre of the triple product \(X := C \times_B C' \times_B C'' \to B\) is \(C \times C' \times C''\), where \(g(C) = 3\).

The special fibre of \(C \to B\) is \(D_0\). The special fibre of \(C' \to B\) is \(D'_0\), and similarly the special fibre of \(C'' \to B\) is \(D''_0\). Here \(D_0, D'_0, D''_0\) are irreducible with one node. They are not rational curves.

The triple product family \(X \to B\) and its normalization \(X' \to B\) admits a finite morphism \(\psi : Y \to X'\) over \(B\), where \(Y\) is the normalization of \(C \times_B C \times_B C \to B\). The morphism \(\psi\) is \(id \times \phi' \times \phi''\) (on the generic fibre and which extends over the normalizations). The special fibre of \(Y \to B\) is \(Y_0\), \(X' \to B\) is \(X_0\), and the open complements are \(U_Y := Y - Y_0\) and \(U_{X'} := X' - X_0\).
Fix a section $\tilde{e}$ of $C \to B$, so that the section intersects the special fibre at a smooth point of $D$.

Consider the specialization maps, which commutes over finite pushforwards:

\[
\begin{array}{ccc}
CH^2(U_Y)_\mathbb{Q} & \xrightarrow{sp} & CH^2(Y_0)_\mathbb{Q} \\
\downarrow \psi_* & & \downarrow \psi_0^0 \\
CH^2(U_{X'})_\mathbb{Q} & \xrightarrow{sp} & CH^2(X_0)_\mathbb{Q}.
\end{array}
\]

This diagram commutes further with pushward to respective Jacobian families over $B$, and to specialization of Jacobian of the special components. Consider $\Delta_{e,1} = sp(\Delta_e)$. In particular we have the equality

\[\psi_0^0(\Delta_{e,1}) = sp(\psi_* \Delta_e).\]

Notice that $\psi_* \Delta_e$ is the cycle $\Delta_{e,X}$ defined in the previous subsection.

Consider the commutative diagram:

\[
\begin{array}{ccc}
CH^2(Y_0)_\mathbb{Q} & \xrightarrow{h_*} & CH_1(J(Y_0))_\mathbb{Q} \\
\downarrow \psi_*^0 & & \downarrow k_* \\
CH^2(X_0)_\mathbb{Q} & \xrightarrow{h_*^0} & CH_1(J(X_0))_\mathbb{Q}.
\end{array}
\]

We note the following.

**Lemma 6.2.** The cycle $\psi_*^0(\Delta_{e,1})$ is non-zero.

*Proof.* We notice that $k_*$ takes the specialization of the degenerate Ceresa cycle (see proof of Theorem 5.6) to a non-trivial cycle, using the norm map and its dual, as in the proof of Lemma 6.1. More precisely, there is a pushforward $l_* : CH_1(J(Y_0))_\mathbb{Q} \to CH_1(J(Y_0))_\mathbb{Q}$ such that the composed map $l_* \circ k_* \circ h_*$ and $h_*$ induce the same map (up to a multiple) on the rational Griffiths group $A^2(J(Y_0))_\mathbb{Q}$. Since we already deduced in Theorem 5.6 the non-triviality of $\Delta_{e,1}$ whose image under $h_*$ is just a multiple of the degenerate Ceresa cycle (up to algebraic equivalence), we deduce that the cycle $\psi_*^0(\Delta_{e,1})$ is non-zero. \(\square\)

**Corollary 6.3.** The cycle $\psi_*^0(\Delta_{e,1})$ corresponds to an indecomposable higher Chow cycle in $CH^2(D \times D', 1)_\mathbb{Q}$.

*Proof.* We obtain a Chow cycle $\psi_*^0(\Delta_{e,1})$ in $CH^2(X_0)_\mathbb{Q}$ and which is supported on its individual components which look like a product of curves $D \times D' \times \mathbb{P}^1$. We apply Lemma 5.4 to the finite morphism $Y_0 \to X_0$, to obtain a higher Chow cycle in components of $X_0$, i.e., in $CH^2(D \times D', 1)_\mathbb{Q}$. To deduce indecomposability, see proof of Theorem 5.6 applied to the cycle $\psi_*^0(\Delta_{e,1})$. \(\square\)
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