THE QUILLEN METRIC, ANALYTIC TORSION AND TUNNELING FOR HIGH POWERS OF A HOLOMORPHIC LINE BUNDLE

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ABSTRACT. Let $L$ be a line bundle over a compact complex manifold $X$ (possibly non-Kähler) and denote by $h_L$ and $h_X$ fixed Hermitian metrics on $L$ and $TX$, respectively. We generalize the asymptotics for the induced Quillen metric on the determinant line associated to a higher tensor power of $L$ to the non-Kähler setting. In the case when $L$ is ample we also obtain the leading asymptotics for the Ray-Singer analytic torsion of a (possibly non-positively curved) metric on $L$, without assuming $h_X$ is Kähler. The key point of the proofs is to relate the asymptotics of the torsions above to “tunneling”, i.e. to the distribution of the exponentially small eigenvalues of the corresponding Dolbeault-Kodaira Laplacians. The proof thus avoids the use of the exact (i.e. non-asymptotic) deep results of Bismut-Gillet-Soulé for the Quillen metric, which are only known to hold under the assumption that $h_X$ be Kähler. Accordingly the proofs are comparatively simple also in the Kähler case. A brief comparison with the tunneling effect for Witten Laplacians and large deviation principles for fermions is also made.

1. INTRODUCTION

1.1. Setup. Let $L \to X$ be a holomorphic line bundle over a compact complex manifold $X$ and let $h_L$ be a smooth metric on the line bundle $L$ with normalized curvature form $\omega$ (the most interesting case will be when $\omega$ is not semi-positive). The normalization is made so that $\omega$ is real and defines an integer cohomology class: $[\omega] \in H^2(X, \mathbb{Z})$. It will be convenient to use the weight notation for $h_L$, i.e. fixing a local trivialization of $L$ we may locally write $\|s\|^2_{h_L} := e^{-\phi}$ so that $\omega = dd^c\phi := \frac{i}{2\pi} \partial \bar{\partial} \phi$ where $\phi$ will be called a weight on $L$. We also fix a metric $h_X$ on $X$ and denote its volume form by $dV$. These metrics induce, in the standard way, Hermitian products on the space $\Omega^{0,q}(X, L)$ of smooth $(0,q)$-forms with values in $L$. The corresponding analytic torsion, introduced by Ray-Singer, is defined as

$$T(h_L, h_X) := \prod_{q=1}^{n} (\det \Delta^{0,q}_\partial)^{-q+1}$$

where

$$\Delta^{0,q}_\partial := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \partial$$

is the Dolbeault-Kodaira Laplacian acting on $\Omega^{0,q}(X, L)$ and $\log \det \Delta^{0,q}_\partial := -\frac{\partial \zeta^{0,q}}{\partial s}|_{s=0}$, where $\zeta^{0,q}(s) = \sum_i (\lambda_i^{0,q})^{-s}$ is the meromorphic continuation to $\mathbb{C}$ of the zeta function for the strictly positive eigenvalues $\{\lambda_i^{0,q}\}$ of $\Delta^{0,q}_\partial$ (see [5, 17] and references).

Key words and phrases. Holomorphic line bundle, determinant bundle and analytic torsion, supersymmetry and quantum mechanics (MSC 2010: 32L05, 58J52, 81Q60).
therein). In the particular case of a Riemann surface

\[ T(h_L, h_X) := (\det \Delta_{\partial}^{0,1}) = (\det \Delta_{\partial}^{0,0}) \]

Given a positive number \( \lambda \) we will also consider a “truncated” version \( T_{[0, \lambda]}(h, \omega) \) of the analytic torsion \( T(h_L, h_X) \) obtained by replacing the regularized determinants with the product of all positive eigenvalues strictly smaller than \( \lambda \). Using functional calculus we may hence write

\[ \log T_{[0, \lambda]}(h, \omega) := -\sum_{q=1}^{n} q(-1)^q \log \det(\Delta_{\partial}^{0,q}) \]

We will be concerned with the limit when \( L \) is replaced by a large tensor power, written in our additive notation as \( kL \). In this context we will say that a sequence \( \lambda_k \) is exponentially small, if

\[ \lim \inf_{k \to \infty} (\log \lambda_k)/k < 0. \]

The asymptotics will be expressed in terms of the following bi-functional on the space of all weights on \( L \):

\[ E(\phi, \phi') := \frac{1}{(n+1)!} \int_X \sum_{j=0}^{n} (dd^c \phi)^{n-j} \wedge (dd^c \phi')^j \left( = \tilde{c}h(h, h') \right) \]

where \( \tilde{c}h(h, h') \) is (up to a multiplicative constant) the Bott-Chern class attached to the first Chern class [17].

Given a holomorphic line bundle \( L \) over \( X \) the corresponding determinant line \( DET(L) \) is the one-dimensional complex vector space defined as the following tensor product of \( \bar{\partial} \)-cohomology groups:

\[ DET(L) := \bigotimes_{q=1}^{n} \det(H^q(X, L))((-1)^q, \]

where \( \det(W) \) denotes the top exterior power of a complex vector space \( W \). Given Hermitian metrics \( h_L = e^{-\phi_L}I \) and \( h_X \) on \( L \) and \( X \), respectively, the corresponding Quillen metric on \( DET(kL) \) that we will denote by \( Q(h^{\otimes k}_L, h_X) \) is defined as the \( L^2 \)-metric multiplied by the analytic torsion, where the \( L^2 \)-metric is the one induced from the isomorphisms between the vector spaces \( H^q(X, L) \) and the kernel of \( \Delta_{\partial}^{0,q} \).

1.2. Statement of the main results.

Theorem 1.1. Let \( L \to X \) be a line bundle holomorphic line bundle over a compact complex manifold (not necessarily Kähler). Fix an Hermitian metric \( h_X \) on \( X \) and two metrics \( h_L \) and \( h'_L \) on \( L \). Then the following asymptotic anomaly formula for the corresponding Quillen metrics on the determinant lines \( DET(kL) \) holds

\[ \frac{1}{k^{n+1}} \log \left( \frac{Q(h^{\otimes k}_L, h_X)}{Q(h^{\otimes k}_{L'}, h_X)} \right) \to \tilde{c}h(h_L, h'_L) \]

as \( k \to \infty \).

In the case when \( h_X \) is a Kähler metric the previous theorem is a direct consequence of the exact anomaly formula for the Quillen metric of Bismut-Gillet-Soule for the determinant line applied to any fixed tensor power of \( L \) (see [17], Theorem 0.3
stated in part I). However, the exact formula is not known in the non-Kähler case. As explained below the key point of the proof is to first obtain the asymptotics for the “truncated” Quillen metric obtained by replacing the analytic torsions with suitable truncations (compare Theorem 1.2 below). It should be pointed out that many applications of the exact formula of Bismut-Gillet-Soulé, notably in arithmetic (Arakelov) geometry [17], only rely on its asymptotic version. Hence, the present proof should be of interest also in the Kähler setting as it is comparatively simple.

By standard arguments the previous formula can be used to obtain an asymptotic version of Theorem 0.1 in [7] in the case when \( \pi: X \to S \) is a submersion of a base \( S \) and a pair \((h_L, h_X)\) as above is fixed. Then the curvature of the Quillen metric on \( \text{DET}(kL) \), seen as a line bundle over \( S \), converges when divided by \( k^n + 1 \) to the push-forward to \( S \) of the top exterior power of the curvature form of \( h_L \).

In the case when \( L \) is an ample line bundle and \( \phi \) is a given smooth weight on \( L \) we will write \( P\phi \) for the semi-positively curved weight defined as the following upper envelope:

\[
P\phi = \sup\{\psi : \psi \leq \phi\}
\]

where \( \psi \) ranges over all smooth weights on \( L \) with positive curvature. Then the functional \( \mathcal{E}(P\phi, \phi) \) is still well-defined by basic pluripotential theory, using that \( P\phi \) is semi-positively curved and locally bounded (see [5] and references therein). Alternatively, by the regularity result in [3] \( P\phi \) is locally \( C^{1,1} \)—smooth and hence the current \( dd^c(P\phi) \) has locally bounded coefficients and exists point-wise almost everywhere on \( X \). As a consequence \( \mathcal{E}(P\phi, \phi) \) can be defined as a standard Lebesgue integral over \( X \).

**Theorem 1.2.** Let \( L \to X \) be an ample line bundle equipped with a smooth Hermitian metric \( h_L \) and let \( h_X \) be a fixed smooth Hermitian metric on \( X \) (which is not assumed to be Kähler). Let \( \lambda_k \) be a sequence such that \( \lambda_k = o(k) \) and such that \( \lambda_k \) is not exponentially small. Then the large \( k \) limits of the corresponding analytic torsions \( \frac{1}{k^n} \log T(h_L^\otimes k, h_X) \) and their “truncations” \( \frac{1}{k^n} \log T_{0, \lambda_k}(h_L^\otimes k, h_X) \) exist and both coincide with \( \mathcal{E}(P\phi_L, \phi_L) \).

In fact, it will be clear that the only contribution to the truncated analytic torsions in the previous theorem comes from the exponentially small eigenvalues (in an appropriate sense). In the case when \( h_X \) is a Kähler metric the asymptotics of the analytic torsion in the previous theorem were deduced in [5] from the exact anomaly formula in [7] for the Quillen metric (refered to above) combined with the asymptotics for the \( L^2 \)—part of the Quillen metric proved in [5].

As a consequence of the previous theorem we get the following geometric criterium for the existence of exponentially small eigenvalues of \( \bar{\partial} \)—Laplacians:

**Corollary 1.3.** Let \( L \) be a line bundle and \( h_L \) and \( h_X \) metrics as in the previous theorem. Suppose that

\[
\mathcal{E}(P\phi_L, \phi_L) > 0
\]

Then, for some \( q \), there is a sequence of exponentially small eigenvalues of the corresponding \( \bar{\partial} \)—Laplacian acting on \( (0, q) \)—forms with values in \( kL \). In the particular case of a line bundle of positive degree over a Riemann surface equipped with a metric \( h_L \) whose curvature is not semi-positive everywhere, the condition above is...
always satisfied. More precisely,

\[
\frac{1}{k^2} \log \sum_{i} \lambda_i^{(k)} \to -\frac{1}{2} \|d(P\phi - \phi)\|_X^2
\]

where \((\lambda_i^{(k)})\) are the “small” positive eigenvalues of the \(\bar{\partial} - \)Laplacian on the space of smooth sections with values in \(kL\) (i.e the eigenvalues are in \([0, k^{1-\varepsilon}]\)

Before continuing we briefly sketch the proofs of the theorems stated above. The starting point is the observation that differential \(\frac{1}{k^n+1} d\phi \log Q(e^{\varphi}, h_X)\) of the normalized logarithm of the “truncated” Quillen metric on \(\text{DET}(kL)\), seen as a functional on the space of all Hermitian metrics (or rather weights) on \(L\), is represented by the alternating sum of the one-point measures associated to the spaces of “low-energy” \((0, q)-\)forms. Using the large \(k\) asymptotics of the latter measures obtained in [2] gives

\[
\frac{1}{k^{n+1}} d\phi \log Q(e^{-k\phi}, h_X) \to (dd^c \phi)^n/n!
\]

and integrating between the line segment connecting \(\phi_0\) and \(\phi_1\) in the space of all weights on \(L\) then concludes the proof of Theorem 1.1 for the “truncated” Quillen metrics. Adapting arguments in [8] to the present setting we also show that the analytic torsion coincides with a suitable truncation to the leading order, concluding the proof of Theorem 1.1. Theorem 1.2 is then obtained by using the asymptotics for the \(H^0(X, kL)\)-part of the Quillen metrics for a general smooth (but possibly non-positively curved) metric in [5] (or alternatively the asymptotics for the corresponding one-point (Bergman) measures in [3]).

1.3. Comparison with the tunneling effect for Witten Laplacians. It may be illuminating to compare the asymptotics of the truncated analytic torsions in Theorem 1.2 with the well-known results concerning the tunneling effect for Witten’s deformation of the De Rham complex, which appeared in Witten’s heuristic approach to Morse theory based on supersymmetric quantum mechanics [18] (see [13] for rigorous results based on semi-classical analysis). Geometrically this latter setting corresponds to letting \(L \to X\) be the trivial line bundle over a real manifold \(X\) and replacing the operator \(\bar{\partial}\) with the exterior derivative \(d\). Any given global function \(\phi\) on \(X\) induces a Hermitian metrics \(h_{kL}\) on \(kL\) represented as \(h_{kL} = e^{-k\phi}\) in the standard global trivialization (i.e. \(s = 1\)) of \(L\), where now \(k\) makes sense for any positive number and where the semi-classical parameter \(h := 1/k\) corresponds to Planck’s constant in the quantum mechanical picture. To this setting one associates as before a Laplacian on \((0, q)-\)forms, \(\Delta_k^{(q)}\), depending on \(k\phi\), which may be identified with the Witten Laplacian (in a unitary frame). Assuming that \(\phi\) is a Morse function we denote by \(X(q)\) the finite set of all critical points of \(\phi\) where the Hessian of \(\phi\) has index \(q\). As shown in [13] the set of all “small” eigenvalues \((\lambda_{i,q}^{(k)})\) of the Witten Laplacian \(\Delta_k^{(q)}\) is in a one to one correspondence with the finite set \(X(q)\). Moreover, all the non-zero eigenvalues are actually exponentially small:

\[
\lim_{k \to \infty} \frac{1}{k} \log \lambda_{i,q}^{(k)} = -c_{i,q}(\phi)
\]

where the positive number \(c_{i,q}(\phi)\) may be expressed in terms of an Agmond distance between critical points. In the present complex geometric setting of the
\(\bar{\partial} - \text{Laplacian}\) the number of "small" eigenvalues \(N^{(k)}_q\) depends on \(k\) and is of the order \(O(k^n)\), more precisely:

\[
N^{(k)}_q = k^n \int_{X(q)} (-1)^q (dd^c \phi)^n/n! + o(k^n)
\]

where now \(X(q)\) is the subset of \(X\) where the curvature form \(dd^c \phi\) (i.e. the complex Hessian of \(\phi\)) has index \(q\) (as shown by Demailly in his proof of his holomorphic Morse inequalities [11]). Hence the asymptotics of the truncated analytic torsions in Theorem 1.2 may be interpreted as an averaged version of 1.2. This analogy becomes particularly striking in the Riemannian surface case (compare the asymptotics 1.1).

The approach in [13] to study the asymptotics of the exponentially small eigenvalues, i.e. of the quantum mechanical tunneling effect, uses that the Witten Laplacians are elliptic when viewed as semi-classical elliptic differential operators. In the particular case when \(q = 0\) there is also a probabilistic approach to the corresponding Witten Laplacian \(\Delta^{(0)}_k\) which refines the information in 1.2 with asymptotics for pre-factors under certain genericity assumptions (see [9] and references therein for the case of Euclidean domains). The point is that the operator \(\Delta^{(0)}_k\) is the Feller semigroup generator for the stochastic differential equation on the Riemannian manifold \(X\) describing a Brownian particle in the gradient field of \(-\phi\) at temperature \(1/k\). The invariant probability measure of the process is

\[
\mu_{k\phi} := e^{-k\phi}dV/Z_{k\phi}
\]

and its relation to the exponentially small eigenvalues was studied in [15] in connection to simulated annealing. As explained in [9] each local minima of \(\phi\) corresponds to a metastable state and the corresponding eigenvalue to the inverse life time of the state (i.e. the inverse mean exit time). See also the very recent work [16] for refinements of 1.2 for general degrees \(q\).

There are also quantum mechanical, as well as probabilistic motivations for considering the present complex geometric setting. For example in the Riemann surface case the \(\bar{\partial} - \text{Laplacian}\) is the Pauli Hamiltonian for a single quantum particle with spin up \((q = 0)\) and spin down \((q = 0)\) subject to the magnet field with two-form \(\mathcal{F}_{kA} := kd\phi\). Moreover, probabilistically the asymptotics of the corresponding truncated analytic torsions appear in the large deviation principle for the determinantal random point process on \(X\) with \(N_k\) particles defined by the following probability measure on \(X^{N_k}\):

\[
\mu^{(N_k)}_{k\phi} := \frac{|\det \Psi|^2(x_1,\ldots,x_{N_k})e^{-k(\phi_1(x_1)+\ldots+\phi_1(x_{N_k}))}dV\otimes N_k}{Z_{k\phi}}
\]

where \(N_k\) is the dimension of the space \(H^0(X,kL)\) of global holomorphic sections of \(kL\) and \(\det \Psi\) is any generator of the complex line \(\det H^0(X,kL)\), identifying \(\det \Psi\) with a holomorphic section over \(X^{N_k}\) [4]. Physically, this is the maximally filled fermionic many-particle ground state and \(\det \Psi\) is the corresponding Slater determinant. Note that in the setting of the Witten Laplacian the role of \(H^0(X,kL)\) is played by the one-dimensional kernel of \(d\) acting on the space smooth functions, inducing the one particle point-process [13].

The main analytical difference here is that the \(\bar{\partial} - \text{Laplacian}\) on \((0,q)\)-forms with values in \(kL\) is not an elliptic operator, when viewed as a semi-classical differential
operator (see [3]). There appears to be very few results concerning the tunneling ef-
flect for non-elliptic semi-classical operators (see however [14] and references therein for
the study of Kramers Fokker-Planck type operators generalizing the Witten Laplacian for
\( q = 0 \)).

Acknowledgments. The author is grateful to Johannes Sjöstrand and Frédéric
Faure for their interest in this work and for stimulating discussions related to the

2. Proofs of the main results

In the following we will fix the metric \( h_X \) on \( TX \). Then any given weight \( \phi \) (which
we will eventually vary) induces Hermitian products \( \langle \cdot, \cdot \rangle_\phi \) on the space \( \Omega^{0,q}(X,L) \)
using the metrics \( (e^{-\phi}, h_X) \) on \( L \) and \( TX \), respectively. Given \( \phi \) we now denote the
associated \( \bar{\partial} - \)Laplacians on \( \Omega^{0,q}(X,L) \) by \( \Delta^{0,q} \) (and sometimes by \( \Delta^{0,q}_\phi \) to indicate
the dependence on \( \phi \)). We will denote by \( \mathcal{H}^{0,q}_{|0,\lambda|} \) the subspace of \( \Omega^{0,q}(X,L) \) spanned
the eigenforms of \( \Delta^{0,q} \) with eigenvalues in \( ]0,\lambda[ \) (sometimes called the space of low-
energy \((0,q)-\)forms). To this space we associate its (non-normalized) one-point
measure, i.e.

\[
\mathbb{B}^{0,q}_{|0,\lambda|} := \sum_i |\Psi^{0,q}_i|^2 dV,
\]

where \( (\Psi^{0,q}_i) \) is any orthonormal base for \( \mathcal{H}^{0,q}_{|0,\lambda|} \). We will also use the notation
\( \mathcal{H}^{0,q} := \text{ker } \Delta^{0,q} \) and write \( \mathbb{B}^{0,q} \) for the corresponding one-point measure.

2.1. Variational formulae. If \( F(\phi) \) is a (Gateaux differentiable) functional on
the affine space of all weights on \( L \) we will write \( dF \) for its differential, which is a
one-form on the space of all weights. We will identify the linear functional \( dF|_\phi \) on
\( \mathcal{C}^\infty(X) \) with a measure in the usual way. Concretely, this means that

\[
\frac{d}{dt} F(\phi_t) = \int_X (dF|_{\phi_t}) \frac{d\phi_t}{dt}
\]

Given a pair of weights \( \phi, \phi' \) on \( L \) we now let

\[
\mathcal{L}^{0,q}(\phi, \phi') := -\log \det(\langle \Psi^{0,q}_i, \Psi^{0,q}_j \rangle_\phi)
\]

where \( (\Psi^{0,q}_i)_i \) is a base in ker \( \Delta^{0,q}_\phi \) which is orthonormal wrt \( \langle \cdot, \cdot \rangle_\phi \). We then have the following basic

Lemma 2.1. Fix a weight \( \phi' \) on \( L \) and write \( \mathcal{L}^{0,q}(\phi) := \mathcal{L}^{0,q}(\phi, \phi') \) and \( \mathcal{E}(\phi) := \mathcal{E}(\phi, \phi') \). Then

\[
d\mathcal{E}|_{\phi} = \frac{(dd^c \phi)^n}{n!}, \quad d\mathcal{L}^{0,q}|_{\phi} = \mathbb{B}^{0,q}
\]

Proof. The first point was first shown by Mabuchi (see also [3] and references therein). As for the second point it holds in a general setting where \( \Pi_\phi \) is the or-
the projection (wrt the Hermitian product \( \langle \cdot, \cdot \rangle_\phi \)) on a given \( N \)-dimensional
subspace of \( \mathcal{C}^\infty(X, L \otimes E) \) where \( E \) is a given Hermitian complex vector bundle
(here \( E = \Lambda^{0,q}(X, h_X) \)). Indeed, we can then compute the time derivative of the
corresponding functional \( \mathcal{L}(\phi_t, \phi') \) using the basic formula
may find a path \( \Psi(2.4) \) \( \text{wrt} \) for any given base in \( H \) vanishes identically. To fix ideas we start with \( \Psi_{\lambda} \).

In the proof we will repeatedly use that \( \Pi \phi \) is given by Lemma 2.2. By the cocycle property \( L(\phi_0, \phi_1)+L(\phi_1, \phi_2)+L(\phi_2, \phi_0) = 0 \) we have that \( \frac{d}{dt}L(\phi_t, \phi) \) is independant of \( \phi' \) and hence we can set \( \phi' = \phi_0 \) above so that \( H(0) = I \), which gives the desired formula. □

We will also have great use for the following lemma, whose proof is inspired by some arguments in [17]:

**Lemma 2.2.** The differential of the functional \( \phi \mapsto \tau_\lambda(\phi) := \log T_{\leq \lambda}(e^{-\phi}, h_X) \) at \( \phi \) is given by

\[
d\tau_\phi = \sum_{q=1}^{n} (-1)^q B_{t}^{0,q} [0,\lambda]
\]

for a generic number \( \lambda \).

**Proof.** In the proof we will repeatedly use that \( \bar{\partial} \) commutes with \( \Delta(= \Delta_{\bar{\partial}}) \) and hence if \( \Psi_{0,q} \) is an eigenform of \( \Delta^{0,q} \) then \( \bar{\partial} \Psi_{0,q} \) is an eigenform of \( \Delta^{0,q+1} \) unless it vanishes identically. To fix ideas we start with

The case \( n = 1 \):

Since, \( \Delta = \bar{\partial}^* \bar{\partial} \) on \( \Omega^{0,q}(X, L) \) it follows immediately from the definition of the determinant that

\[
det(1_{[0,\lambda]}(\Delta_{\bar{\partial}}^{(0)})) = det(\langle \Psi_i, \Psi_j \rangle)^{-1} det(\langle \bar{\partial} \Psi_i, \bar{\partial} \Psi_j \rangle)
\]

for any given base in \( H^{0,q}_{[0,\lambda]} \) (with \( q = 0 \)). Next we note that, given the path \( \phi_t \), we may find a path \( \langle \Psi_{i,q}(t) \rangle(t) \) of bases in \( H^{0,q}_{[0,\lambda]}(t) \) such that \( \langle \Psi_{i,q}(t) \rangle(t) \) is orthonormal wrt \( \phi_0 \) for \( t = 0 \) and for any fixed index \( i \) \( \langle \Psi_{i,q}^{0,q}(0) \rangle \) is an eigenform for \( \Delta_{\phi_0} \) and

\[
\left\langle \frac{d}{dt}_{t=0} (\Psi_{i,q}^{0,q}(t), H^{0,q}_{[0,\lambda]}(t)) \right\rangle_{\phi_0} = 0
\]

To see this we first recall that, as a well-known consequence of the ellipticity of Laplacians, for a generic \( \lambda \) the family \( H^{0,q}_{[0,\lambda]}(t) \) defines a vector bundle over \( \{t\} := ] - \epsilon, \epsilon[ \) (see Prop 1 on p. 123 in [17]). Moreover, since, for any \( t \), \( \ker \Delta^{0,q}_{\bar{\partial}}(t) \) is isomorphic to the Dolbeault cohomology group \( H^{0,q}_{\bar{\partial}}(X, L) \) it follows that \( F := H^{0,q}_{[0,\lambda]}(t) \) is also a vector bundle. Hence, we can start with an arbitrary smooth curve \( (s_i)(t) \) of bases in \( F_i \) satisfying the first requirements above at \( t = 0 \). Then, for \( t \) sufficiently small, \( \langle \Psi_{i,q}(t) \rangle(t) := s_i(t) - t \Pi_t(\frac{d}{dt}_{t=0}(s_i)(t)) \) has the desired properties, since \( \Pi_t(\frac{d}{dt}_{t=0}(\Psi_{i,q}^{0,q}(t))) = 0 \), where \( \Pi_t \) denotes the orthogonal projection onto \( H^{0,q}_{[0,\lambda]}(t) \).

Now we can decompose

\[
\frac{d}{dt}_{t=0} \log det(1_{[0,\lambda]}(\Delta_{\bar{\partial}}^{(0)})) = -\frac{d}{dt}_{t=0} \log det(\langle \Psi_i(t), \Psi_j(t) \rangle) + \frac{d}{dt}_{t=0} \log det(\langle \bar{\partial} \Psi_i(t), \bar{\partial} \Psi_j(t) \rangle)
\]

Using formula formula (2.3) and the fact that

\[
\frac{d}{dt}_{t=0} (\langle \Psi_i(t), \Psi_j(t) \rangle) = \left\langle -\frac{d\phi_t}{dt} \Psi_i(t), \Psi_j(t) \right\rangle + 0
\]
Using the bijection \(2.6\) we can now repeat the arguments used above (when \(q\sum\)) to deduce that the eigenvalues. In particular, 

\[
\alpha\]

The second term is computed similarly, by using that \(\alpha_i := \tilde{\partial}\Psi_i/\|\tilde{\partial}\Psi_i\|_{\phi_0}\) is an orthonormal base in \(H_{0|0,\lambda|}^0(t)\) for \(t = 0\) and that the relation \(2.4\) still holds with \(\Psi_i\) replaced by \(\alpha_i\) (indeed, since \(\tilde{\partial}\) commutes with \(\frac{d}{dt}\) and \(\Delta_{\tilde{\partial}}\) we get \(\tilde{\partial}\Pi_{0|0,\lambda|}^0(\frac{d}{dt}\Psi) = \tilde{\partial}\Pi_{0|0,\lambda|}^0(\frac{d}{dt}\Psi) = 0\).

Next, we note that \(\tilde{\partial}\) induces a bijection, intertwining the corresponding restricted Laplacians, such that

\[
\tilde{\partial}: N_q \rightarrow M_{q+1}, \quad \Delta_{N_q} = \tilde{\partial}^* \tilde{\partial}
\]

and hence \(\det(\Delta_{M_q}) = \det(\Delta_{N_{q-1}})\) so that \(\det(1_{0|0,\lambda|}(\Delta_{\tilde{\partial}}^q)) = \det(\Delta_{N_{q-1}})\det(\Delta_{N_q})\). The latter relation implies, since \(\det(\Delta_{N_{q-1}}) := 1 = \det(\Delta_{N_q})\) that

\[
-\sum_{q=1}^{n} q(-1)^q \log \det(1_{0|0,\lambda|}(\Delta_{\tilde{\partial}}^q)) = \sum_{q=1}^{n} (-1)^q \log \det(\Delta_{N_q})
\]

Using the bijection \(2.6\) we can now repeat the arguments used above (when \(n = 1\)) to deduce that

\[
\frac{d}{dt} \log(\det(\Delta_{N_q})) = \int_X \left(\mathbb{B}_{0|0,\lambda|}^0 \cap N_q - \sum_{q=1}^{n} \mathbb{B}_{0|0,\lambda|}^0 \cap M_{q+1}\right) \frac{d\phi_i}{dt}
\]

where the intersection with \(N_q\) indicates that we have replaced \(H_{0|0,\lambda|}^0\) with its subspace \(N_q\) in the definition \(2.1\) (and similarly for \(M_{q+1}\)). Hence, taking the alternating sum over \(q\) and using \(2.7\) proves the lemma in general dimensions. \(\square\)

2.2. Asymptotics. Next we recall the following asymptotics from [2] (which can be seen as a local version of Demailly’s strong holomorphic Morse inequalities):

**Proposition 2.3.** Let \(\lambda_k\) be a sequence of positive numbers such that \(\lambda_k = O(k^{1-\epsilon})\) for some \(\epsilon < 1/2\). Then the sequence \(k^{-n} \mathbb{B}_{0|0,\lambda_k|}^0 dV\) is uniformly bounded and

\[
k^{-n} \mathbb{B}_{0|0,\lambda_k|}^0 \rightarrow (-1)^q 1_{X(q)}(d\tilde{\partial} \phi)^q / n!
\]

weakly as \(k \rightarrow \infty\), where \(X(q)\) is the subset of \(X\) where \(d\tilde{\partial} \phi\) has exactly \(q\) negative eigenvalues. In particular,

\[
\text{dim } H_{0|0,\lambda_k|}^0 = k^n \int_X (-1)^q 1_{X(q)}(d\tilde{\partial} \phi)^q / n! + o(k^n).
\]
Proof. As shown in [2] (Prop 5.1) the upper bound in the convergence above holds for any sequence \( \lambda_k = k \mu_k \) with \( \mu_k \to 0 \). As for the lower bound it was shown to hold as long as \( \delta_k / \mu_k \to 0 \) for \( \delta_k \) a certain non-explicit sequence tending to zero (Prop 5.3). This latter sequence appears in Lemma 5.2 in [2] and the proof given there actually shows that \( \delta_k \) can be taken as \( \delta_k = \frac{1}{k^{1/2-\delta}} \) for any \( \delta > 0 \) (since the radius \( R_k \) appearing in that proof is equal to \( \log k \)). A further refinement of this argument will be considered in Prop 3.1. □

The previous asymptotics will allow us to obtain the asymptotics of truncated analytic torsions (and hence of exponentially small eigenvalues). Using the next proposition we will then deduce the corresponding asymptotics for the usual analytic torsions.

Proposition 2.4. The following asymptotics holds for any line bundle \( L \to X \) and smooth Hermitian metrics on \( L \) and \( X \):

\[
\log(\det(\Delta_k^{(q)})) = \log(\det(1_{[0,\lambda_k]}(\Delta_k^{(q)}))) + O(k^{n+\epsilon})
\]

if \( \lambda_k = O(k^{1-\epsilon}) \) for a given \( \epsilon \in ]0, 1[ \).

Proof. We will adapt to our setting some arguments from [8], where it was among other things shown that \( \log(\det(\Delta_k^{(q)})) = O(k^n) \log k \) if the metric on \( L \) is positively curved (the point being that the smallest positive eigenvalue of \( \Delta_k^{(q)} \) is then of the order \( k \)). As follows immediately from the definition (see below) the statement of the proposition to be proved is equivalent to

\[
\zeta_{1_{[\lambda_k]}}^{(\Delta^{(q)})}(0) = O(k^{n+\epsilon})
\]

To prove the latter asymptotics we first recall some general facts about spectral zeta functions (following chapter V in [17]). If \( D \) is an Hermitian non-negative operator on a Hilbert space then its spectral zeta function is defined by

\[
\zeta_D(s) = \frac{1}{\Gamma(s)} \int_0^\infty (t^s(\text{Tr}(e^{-tD}) - \Pi_D)) \frac{dt}{t}
\]

where \( \Pi_D \) denotes the orthogonal projection on the kernel of \( D \) (we assume that \( \zeta_D(s) \) exists for \( s \) a complex number of sufficiently negative real part and is then analytically contained to other values for \( s \)). In other words \( \zeta_D(s) \) is the Mellin transform of the spectral theta function

\[
\Theta_D(t) := (\text{Tr}(e^{-tD}) - \Pi_D) = \sum_{\nu_i > 0} e^{-t\nu_i}
\]

summing over the positive eigenvalues of \( D \). Next we assume that

- \( \Theta_D(t) \) converges for \( t > 0 \)
- For every positive integer \( M \) there are real numbers \( a_i = 0 \) with \( a_i = 0 \) for \( i < -n \) such that

\[
\Theta_D(t) = \sum_{j=-n}^{j=M} a_j t^j + O(t^{M+1})
\]

uniformly when \( t \to 0 \).
Under these assumptions one obtains (see formula 11 on p. 99 in [17])

\[(2.10) \quad \zeta_D(0) = a_0, \quad \zeta_D'(0) = \int_1^\infty \Theta(t) \frac{dt}{t} + \left( \gamma a_0 + \sum_{j < 0} \frac{a_j}{j} + \int_0^1 \rho_0(t) \frac{dt}{t} \right)\]

where \(\rho_0(t)\) is the \(O(t^{-M+1})\) term in [2.9] for \(M = 0\) and \(\gamma\) denotes Euler’s constant. We also define a scaled version of \(\zeta_D\) by setting \(\tilde{\zeta}_D := \kappa^{-n} \zeta_{k^{-1}D}\) so that the derivative \(\tilde{\zeta}_D(0)\) at \(s = 0\) satisfies

\[(2.11) \quad \kappa^{-n} \tilde{\zeta}_D'(0) = -\log k \tilde{\zeta}_D(0) + \tilde{\zeta}_D'(0)\]

(compare formula 40 in [8]).

We now come back to the present complex geometric setting. As shown in [8] (Thm 2), for any positive integer \(M\), there is an asymptotic expansion

\[(2.12) \quad k^{-n} \text{Tr}(e^{-tk^{-1}\Delta(q)}) \frac{dt}{t} = \sum_{j=-n}^{M} a_j^q(k) t^j + O(t^{M+1})\]

when \(t \to 0\) uniformly for \(t \in [0, 1]\) and moreover

\[(2.13) \quad a_j^q(k) = a_j^q + O(k^{-1/2})\]

as \(k \to \infty\). Next, we note that, for \(t \in [0, 1]\)

\[(2.14) \quad k^{-n} \text{Tr}(e^{-tk^{-1}\Delta(q)}) - k^{-n} \text{Tr}(e^{-tk^{-1}j_{k^{-1}}(\Delta(q))}) = \int (1 + X_{\mathbb{Q}}(dt \phi))^n / n! + o(1)\]

uniformly when \(k \to \infty\) (strictly speaking these asymptotics only hold when \(\epsilon < 1/2\) but in general the argument will show that the lhs above is uniformly bounded which is all that will be used to deduce the lemma). Indeed, the lhs above may be written and estimated from below and above as

\[e^{-tk^{-1}} k^{-n} \dim \mathcal{H}_{0, \lambda k}^0 \leq k^{-n} \sum_j e^{-tk^{-1}j_{k^{-1}}} \leq \kappa^{-n} \dim \mathcal{H}_{0, \lambda k}^0\]

which combined with Prop 2.3 proves the previous formula. Now combining 2.12 and 2.13 gives an expansion

\[(2.15) \quad k^{-n} \text{Tr}(e^{-tk^{-1}\Delta(q)}) = \sum_{j=-n}^{M} b_j^q k^j + O(t^{M+1}) + o(1)\]

uniformly for \(t \in [0, 1]\) as \(k \to \infty\) (where \(b_j^q = a_j^q\) for \(j \neq 0\)). Accordingly, it follows from 2.10 that

\[(2.16) \quad \tilde{\zeta}_{\Delta(q)}(0) = -b_0^q(k) = -b_0^q + o(1), \quad \tilde{\zeta}_{k, j_{k^{-1}}}(\Delta(q))(0) = -(b_0^q - \int X_{\mathbb{Q}}(-1)^q (dt \phi)^n / n!) + o(1)\]

Next, we apply the derivative formula in 2.10 to \(D = \kappa^{-1} j_{k^{-1}}(\Delta(q))\) to get

\[\tilde{\zeta}_{D}(0) = \int_1^\infty k^{-n} \text{Tr}(e^{-tk^{-1}\Delta(q)}) \frac{dt}{t} + O(1)\]

also using 2.15. From the basic estimate

\[k^{-n} \text{Tr}(e^{-tk^{-1}\Delta(q)}) \leq e^{-(t-1)k^{-1}\lambda_k(k^{-n} \text{Tr}(e^{-k^{-1}\Delta(q)})))}\]
we deduce, since the second factor is uniformly bounded according to 2.12 applied to \( t = 1 \) and since we have assumed \( \lambda_k = O(k^{1-\epsilon}) \) the bound

\[
\int_1^\infty k^{-n} \text{Tr}(e^{-tk^{-1}[k^{1-\epsilon},\omega])} \frac{dt}{t} \leq C \int_1^\infty e^{-(t-1)k^{1-\epsilon}} \frac{dt}{t} \leq C'k^\epsilon
\]

All in all this means that \( \tilde{\zeta}'_{1[k^{1-\epsilon},\omega]}(0) = O(k^\epsilon) \) which combined with 2.11 and 2.16 proves 2.8 and hence finishes the proof of the proposition. \( \square \)

2.3. Proof of Theorem 1.1. We start with a given \( \phi \) and take a smooth path \( \phi_t \) such that \( \phi_1 = \phi \) and \( \phi_0 \) has positive curvature. Letting

\[
f_k(t) := k^{-(n+1)}(\log T_{[0,\lambda_k]}(e^{-k\phi_t}, h_X)) + \sum_{q=0}^n (-1)^q L^{0,q}(k\phi_t)
\]

we have to prove that

\[
\frac{1}{V} \lim_{k \to \infty} (f_k(1) - f_k(0)) = \mathcal{E}(\phi_1) - \mathcal{E}(\phi_0) := \mathcal{E}(\phi_1, \phi_0)
\]

To this end, note that combining Lemma 2.1 and Lemma 2.2 gives \( \frac{df_k(t)}{dt} = k^{-n} \int_X \sum_{q=1}^n (-1)^q (\mathbb{E}^{0,q}_{\phi_t})_0,\lambda_k + \mathbb{E}^{0,1}_{\phi_t} \frac{d\phi_t}{dt} = k^{-n} \int_X \sum_{q=1}^n (-1)^q \mathbb{E}^{0,q}_{[0,\lambda_k]} \frac{d\phi_t}{dt}
\]

Using Prop 2.3 together with the dominated convergence theorem hence gives

\[
\frac{1}{V} \lim_{k \to \infty} (f_k(1) - f_k(0)) = \lim_{k \to \infty} \int_0^1 \frac{df_k(t)}{dt} dt = \frac{1}{Vn!} \int_0^1 \int_X (dd^c \phi_t)^n \frac{d\phi_t}{dt} dt
\]

According to the variational characterization of \( \mathcal{E} \) in Lemma 2.1 (or by directly computing the integral) this finishes the proof of 2.18. Finally, by the asymptotics 2.4 this finishes the proof of the theorem.

2.4. Proof of Theorem 1.2. By the Kodaira vanishing theorem we have that \( H^{0,q}(X, kL) = \{0\} \) for \( k >> 1 \) (since \( L \) is assumed ample) and hence \( L^{0,q}(k\phi_t) = 0 \) for \( q > 0 \). By Theorem 2 in [5] \( (\mathcal{L}(\phi_1) - \mathcal{L}_k(\phi_0))/k^{(n+1)} \) converges to \( (\mathcal{E}(P\phi_1) - \mathcal{E}(P\phi_0)) \) and hence the previous theorem (or rather its proof) gives

\[
\lim_{k \to \infty} (\log T_{[0,\lambda_k]}(e^{-k\phi_1}, \omega_0) - \log T_{[0,\lambda_k]}(e^{-k\phi_0}, \omega_0)) = (\mathcal{E}(\phi_1) - \mathcal{E}(\phi_0)) - (\mathcal{E}(P\phi_1) - \mathcal{E}(P\phi_0))
\]

Since, \( \phi_0 \) is assumed positively curved we have on one hand that \( P\phi_0 = \phi_0 \) so that the terms involving \( \phi_1 \) in the rhs above cancel. On the other hand, the smallest positive eigenvalue \( \lambda_k^{0,q} \) of \( \Delta_{k\phi_0}^{0,q} \) satisfies \( \lambda_k^{0,q} \geq Ck \) (see below) and hence the second term in the lhs above also vanishes. All in all this gives the convergence of the truncated analytic torsion in Theorem 1.2 and by Prop 2.4 the same asymptotics then hold for the analytic torsions.

As for the eigenvalue estimate used above it is a standard consequence of the Kodaira-Nakano identity [12] in the case when \( \omega \) is Kähler. In the non-Kähler case it was shown in [8] (Theorem 1).

Remark 2.5. The eigenvalue estimate refered to above can also be deduced from the Kähler case as follows: first one notes that the first positive eigenvalue on the subspace \( \ker \tilde{\partial} \) of the \( \tilde{\partial} \)—Laplacian associated to \( (k\phi, h_X) \) is the supremum over all
constants $C_k^{(q)}(h_X)$ such that for any $f$, a $\bar{\partial}$—closed $(0,1)$—form with values in $L$, the inhomogenous $\bar{\partial}$—equation $\bar{\partial}u = f$ for can be solved with an estimate
\[
\|u\|_{(k\phi, h_X)}^2 \leq \frac{1}{C_k^{(q)}(\omega)} \|f\|_{(k\phi, h_X)}^2
\]

But since $X$ is compact there is a constant $A$ such that $A^{-1}\omega_0 \leq h_X \leq A\omega_0$ for $\omega_0$ a fixed Kähler metric and hence replacing $h_X$ with $\omega_0$ only distorts $C_k^{(q)}(h_X)$ with a multiplicative constant independent of $k$. Finally, using the bijection 2.4 (and since $H^{0,q}(X, kL) = \{0\}$ for $k \gg 1$) and the fact that $\bar{\partial}$ commutes with $\Delta$ this gives the desired lower bound for the first positive eigenvalue on all of $\Omega^{0,q}(X, kL)$ for all $q$.

2.5. **Proof of Cor 1.3.** Let $\lambda_k = k^{1-\epsilon}$ for some $\epsilon \in ]1/2, 1[$. By Prop 2.3 the number of terms in the sum
\[
(2.19) \quad k^{-(n+1)} \log \det(1_{0,\lambda_k}(\Delta_{\bar{\partial}}^{(q)})) = \frac{1}{k^n} \sum \frac{1}{k} \log \lambda^{(q)}_{i,k}
\]
grows as constant times $k^n$. Hence, if there were no sequence of eigenvalues as in the statement of the corollary then the previous sum would converge to zero for any $q$ contradicting the positivity assumption in the corollary (according to the previous theorem). As for the final statement we first note that, when $n = 1$:
\[
\mathcal{E}(P\phi, \phi) = \frac{1}{2} \|d(P\phi - \phi)\|_X^2
\]
which follows from integration by parts, using the “orthogonality relation” $\int (P\phi - \phi)^p d\nu(P\phi) = 0$. Hence $\mathcal{E}(P\phi, \phi) \geq 0$ with equality iff $P\phi = \phi + c$ for some constant $c$. But then it follows from the definition of $P\phi$ that $c = 0$, i.e. $\phi$ has semi-positive curvature.

3. **Remarks on “small” vs. “exponentially small” eigenvalues**

Fix a number $\epsilon \in ]0, 1/2[$. For a given positive integer $k$ we say that an eigenvalue $\nu_k$ of the $\bar{\partial}$—Laplacian $\Delta_{\bar{\partial}}$ associated to $kL$ is “small” if $\nu_k \leq k^{1-\epsilon}$. By Prop 2.3 the number of “small” eigenvalues is of the order $O(k^n)$. Hence, as explained above the “small” eigenvalues which are not “exponentially small” make no contribution to the large $k$ limit of the sum 2.19. But it is tempting to ask whether all “small” eigenvalues are actually “exponentially small”?. As recalled in section 1.3 this is indeed the case in the setting of the Witten Laplacian under the assumption that $\phi$ be a Morse function. However in the present setting there are no non-degeneracy assumptions on the weight $\phi$ of the metric and hence it seems unlikely that the answer to the question above is yes, in general. Still when $q = 0$ the number of “small” and “exponentially small” small eigenvalues are the same to the leading order:

**Proposition 3.1.** Given any $\delta > 0$ there is a constant $C_k$ such that the number $N_{\delta,k}$ of eigenvalues in $[0, e^{-C_kK}]$ of the $\bar{\partial}$—Laplacian $\Delta_{\bar{\partial}}$ acting on smooth sections of $kL$ satisfies
\[
\int_{X(0)} (dd^c\phi)^n/n! - \delta \leq N_{\delta,k} \leq N_k \leq \int_{X(0)} (dd^c\phi)^n/n! + \delta
\]
for $k$ sufficiently large, where $N_k$ is the number of “small” eigenvalues.
Proof. Given $\delta > 0$ take $K_\delta$ a compact subset of the open set $X(0) := \{dd^c \phi > 0\}$ such that
\[
\int_{X(0)} (dd^c \phi)^n/n! - \delta \leq \int_{K_\delta} (dd^c \phi)^n/n!
\]
By Fatou’s lemma it will be enough to prove that
\[
\liminf_{k \to \infty} k^{-n} \int_{[0,e^{-c_k} K]} (dd^c \phi)^n(x) \geq (dd^c \phi)^n(x)/n!
\]
for any given $x \in K_\delta$. As shown in [2] (Prop 5.3) it is enough to, given a point $x \in K_\delta$, find a smooth section $\alpha_k$ of $kL$ such that
\[
(i) \liminf_{k \to \infty} k^{-n} \int_{X} |\alpha_k|^2 e^{-k \phi} (dd^c \phi)^n(x)/n! \geq (dd^c \phi)^n(x)/n!, 
(ii) k^{-1} \langle \Delta \partial \alpha_k, \partial \alpha_k \rangle_{k \phi} \leq \delta_k
\]
for $\delta_k = e^{-2C_\delta}$. To this end we simply take $\alpha_k = \chi s$ where $\chi$ is a smooth cut-off function which is equal to 1 close to $x$ and $s$ is local holomorphic frame for $L$ on a neighbourhood $x$ such that the corresponding local weight $\phi(z)$ is given by $\phi(z) = \sum_{i=1}^{n} \mu_i |z_i|^2 + O(|z|^3)$ (where $\mu_i > 0$ since $dd^c \phi > 0$ at $x$). Then (i) above follows from computing a Gaussian integral. Moreover,
\[
\langle \Delta \partial \alpha_k, \partial \alpha_k \rangle_{k \phi} = \langle \partial \alpha_k, \partial \alpha_k \rangle_{k \phi} = \int \langle (\partial \chi) |^2 e^{-k \phi} dV \leq e^{-2C_\delta} k^n
\]
By the compactness of $K_\delta$ the constant $C_\delta$ can be taken to be independent of $x$. This finishes the proof of the lower bound on $N_{\delta,k}$ and the upper bound is a special case of Prop [2,3].

It would be interesting to know if the analogue for $q > 0$ of the previous proposition is also valid? If one replaces $e^{-k \phi}$ with a sequence $\delta_k$ of the form $\delta_k = O(k^{-\infty})$, i.e. $\delta_k \leq C_m k^{-m}$ for any $m > 0$ then this is indeed case. This follows from taking $\alpha_k$ in the previous proof to be the given by $\Pi_k(x, \cdot)$ where $\Pi_k(x, y)$ is the local projector for $(0, q)$-forms constructed in [6], defining a local Fourier integral operator with a complex phase (in fact, only the phase function constructed in [6] is needed together with the leading term in the symbol expansion).

REFERENCES

[1] Berline, N; Getzler, E; Vergne, M: Heat kernels and Dirac operators. Corrected reprint of the 1992 original. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004.
[2] Berman ,R.J.: Bergman kernels and local holomorphic Morse inequalities. Math. Z., 248(2):325{344, 2004.
[3] Berman, R.J.: Bergman kernels and equilibrium measures for line bundles over projective manifolds. 35 pages. The American J. of Math., Vol 131, Nr. 5, 2009
[4] Berman, R.J.: Large deviations and entropy for determinantal point processes on complex manifolds. arXiv:0812.4224
[5] Berman, R.J.; Boucksom, S: Growth of balls of holomorphic sections and energy at equilibrium. Invent. Math. Vol. 181, Issue 2 (2010), p. 337
[6] Berman R.J; Sjoestrand, J: Asymptotics for Bergman-Hodge kernels for high powers of complex line bundles. Ann. de la Fac. des Sciences de Toulouse Vol. XIV (2007) no. 4, p. 719–771
[7] Bismut J.-M., Gillet H., Soulé C.: Analytic torsion and holomorphic determinant bundles. I,II,III Comm. Math. Phys. 115 (1988), 49–78, 79–126, 301–351
[8] Bismut, J.-M.; Vasserot, E.: The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle. Comm. Math. Phys. 125 (1989), no. 2, 355–367
[9] Bovier, A; Gayrard, V; Klein, M: Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. JEMS 7 (1), pp. 69-99 (2004).
[10] Demailly, J-P: Complex analytic and algebraic geometry. Available at www-fourier.ujf-grenoble.fr/~demailly/books.html
[11] Demailly, J-P: Champs magnetiques et inegalite de Morse pour la d’-cohomologie., Ann Inst Fourier, 355 (1985,185-229)
[12] Griffiths, P; Harris, J: Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.
[13] Helffer, B; Sjöstrand. Puits multiples en limite semi-classique IV -Etude du complexe de Witten -. Comm.in PDE, 10(3), p. 245-340 (1985).
[14] Herau, F; Hitrik, M; Sjöstrand, J: Tunnel effect and symmetries for Kramers Fokker-Planck type operators. \texttt{arXiv:1007.0838}
[15] Holley, R; Kusuoka, R; Stroock, D: Asymptotics of the spectral gap with applications to the theory of simulated annealing, J. Funct. Anal. 83-2 (1989), pp 333–347.
[16] Le Peutrec, D; Nier, F; Viterbo, C: Precise Arrhenius law for p-forms: The Witten Laplacian and Morse-Barannikov complex. \texttt{arXiv:1105.6007}
[17] Soulé, C. (1992), Lectures on Arakelov geometry, Cambridge Studies in Advanced Mathematics, 33, Cambridge: Cambridge University.
[18] Witten, E: Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661–692.

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