Shortest-Path Fractal Dimension for Percolation in Two and Three Dimensions

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We carry out a high-precision Monte Carlo study of the shortest-path fractal dimension \( d_{\text{min}} \) for percolation in two and three dimensions, using the Leath-Anderson-Wieczorek method which grows a cluster from an active seed site. A variety of quantities are sampled as a function of the chemical distance, including the number of activated sites, a measure of the radius, and the survival probability. By finite-size scaling, we determine \( d_{\text{min}} = 1.13077(2) \) and \( 1.3756(6) \) in two and three dimensions, respectively. The result in 2D rules out the recently conjectured value \( d_{\text{min}} = 217/192 \) [Phys. Rev. E 82, 020102(R) (2010)].

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As a standard model of disordered systems \([1, 2]\), percolation has been intensively studied over the last 50 years and applied to many other fields due to its richness in both mathematic and physics. The nature of phase transition of percolation has been well established. In particular, within the two-dimensional (2D) university class, there are only few critical exponents left to be expressed exactly, among which is the shortest-path fractal dimension \( d_{\text{min}} \), defined by

\[
\langle \bar{r} \rangle \sim r^{d_{\text{min}}},
\]

where \( r \) is the Euclidean distance between two sites belonging to the same cluster, and \( \ell \) is the shortest path.

The shortest path \( \ell \) between two sites in a cluster is the minimum number of steps on a path of occupied bonds/sites in the cluster, and was first studied independently by several groups in the early 1980’s \([3, 4]\). The length \( \ell \) is also called the chemical distance \([10]\). A related quantity is the spreading dimension \( d_{\ell} \) \([11]\), which describes the scaling of the mass of a critical cluster within a chemical distance \( \ell \) as \( N \sim \ell^{d_{\ell}} \), and is related to the fractal dimension \( d_{\ell} \) of the cluster by \( d_{\ell} = d_{f}/d_{\text{min}} \).

In percolation, the shortest-path naturally occurs during epidemic growth or burning algorithms. Previous measures of \( d_{\text{min}} \) in 2D include \( d_{\text{min}} = 1.18(4) \) \([6]\), \( 1.118(15) \) \([7]\), \( 1.15(3) \) \([10]\), \( 1.102(13) \) \([12]\), \( 1.132(4) \) \([13]\), \( 1.130(2) \) \([14]\), \( 1.1306(3) \) \([15]\) and \( 1.1303(8) \) \([16]\). A summary of the early work is given in Ref. \([13]\).

In 1984, Havlin and Nossal \([10]\) conjectured \( d_{\text{min}} = d_{f} - 1/\nu = 91/48 - 3/4 = 55/48 = 1.145833 \), which was soon shown to be too large \([12, 14]\). In 1987 Larsson speculated that \( d_{\text{min}} \) could be 17/16 or even 1, but these are both excluded. In 1988 Herrmann and Stanley conjectured \( d_{\text{min}} = 2 - d_{B} + d_{\text{red}} \), where \( d_{\text{red}} = 1/\nu = 3/4 \) is the “red”-bond dimension and \( d_{B} \) is the backbone dimension. Using Deng, Blöte, and Neinhuis’s result \( d_{B} = 1.6434(2) \) \([17]\) (see also \([18, 19]\) ), we find this prediction gives \( d_{\text{min}} = 2 - 1.6434(2) + 0.75 = 1.1066(2) \), which is too small. In 1989, Tzschichholz, Bunde, and Havlin \([20]\) considered \( d_{\text{min}} = 53/48 = 1.104166... \), which is also below measured values.

In 1998, Porto et al. conjectured that \( d_{\text{min}} \) is related to a pair-connectivity scaling exponent \( g_{1} \) by \( d_{\text{min}} = g_{1} + \beta/\nu \) where \( \beta = 5/36 \) for 2D. However, \( g_{1} \) was later shown to have the exact value \( g_{1} = 25/24 \) \([13, 21]\), which implies \( d_{\text{min}} = 55/48 = 1.145833 \), identical to Havlin and Nossal’s earlier conjecture \([10]\).

In 2010, one of us (Deng) and his coauthors conjectured an exact expression \([16]\) of \( d_{\text{min}} \) for the 2D critical and tricritical random-cluster model: \( d_{\text{min}} = (g + 2)/(g + 18)/32g \), where \( g \) is the Coulomb-gas coupling constant, related to the random cluster fugacity \( q \) by relation \( q = (2/\pi) \cos^{-1}(q/2 - 1) \). This conjecture is numerically correct up to the third or fourth decimal place for all values of \( q \) studied in Ref. \([16]\). For the \( q \to 1 \) limit—i.e., standard bond percolation—the predicted value \( d_{\text{min}} = 217/192 = 1.130208 \) was consistent with the numerical results in previous works \([6, 7, 10, 12, 15]\). In addition, the conjectured formula exhibits other good properties. It reproduces the exact results for the critical uniform spanning tree (\( q \to 0 \)) as well as for the tricritical \( q \to 0 \) Potts model; at the tricritical \( q \to 0 \) point, the derivative with respect to \( q \) is also correct.

The main goal of the present work is to carry out a high-precision Monte Carlo test of the conjecture in Ref. \([16]\) in the context of 2D percolation. A numerical estimate of \( d_{\text{min}} \) for 3D percolation is also provided. Some preliminary results of this work were reported in a recent paper on biased directed percolation \([22]\).

We simulate bond percolation on the square and the simple-cubic lattice by the Leath-Alexandrowicz algorithm \([6, 23]\), which grows a percolation cluster starting from a seed site. For each neighboring edge of the seed
site an occupied bond is placed with occupation probability $p$, and the neighboring site is activated and added into the growing cluster. After all the neighboring edges of the seed site have been visited, the growing procedure is continued from the newly added sites. This proceeds until no more new sites can be added into the cluster (the procedure dies out) or the initially set maximum time step $\ell_{\text{max}}$ is reached.

The above procedure is also called breadth-first growth, and $\ell$ is equal to the shortest-path length between the seed site and any activated sites at time step $\ell$. We set $\ell = 1$ for the beginning of the growth, and measure the number of activated sites $N(\ell)$ as a function of $\ell$. In addition, we record the Euclidean distance $r_i$ of each activated site $i$ to the seed site, and define a radius by

$$R(\ell) = \begin{cases} 0 & \text{if } N(\ell) = 0 \\ \sqrt{\sum_{i=1}^{N(\ell)} r_i^2} & \text{if } N(\ell) \geq 1. \end{cases}$$

(2)

The statistical averages, $N(\ell) \equiv \langle N(\ell) \rangle$ and $R(\ell) \equiv \langle R(\ell) \rangle$, and the associated error bars are calculated. We also sample the survival probability $P(\ell)$ that at time step $\ell$, the growing procedure still survives.

At criticality, one expects scaling behavior

$$N(\ell) \sim \ell^{Y_N}, \quad R(\ell) \sim \ell^{Y_R}, \quad P(\ell) \sim \ell^{-Y_P},$$

(3)

where critical exponents $Y_P$, $Y_N$, and $Y_R$ are related to $\beta, \nu$ and $d_{\text{min}}$ by

$$Y_N = \frac{\gamma}{\nu d_{\text{min}}} - 1, \quad Y_P = \frac{\beta}{\nu d_{\text{min}}},$$

$$Y_R = \frac{\gamma + 2\nu}{\nu d_{\text{min}}} - 1,$$

(4)

with $\gamma = d\nu - 2\beta$ and $d$ equal to the spatial dimensionality. In terms of exponents of epidemic processes 24, these quantities correspond to $\delta = Y_P$, $\eta = Y_N$, and $1/z - \delta = Y_R$.

To eliminate the unknown non-universal constants in front of the scaling behaviors 24, we define ratio $Q_O(\ell) = O(2\ell)/O(\ell)$ for $O = N, R$ and $P$. In the $\ell \to \infty$ limit, one has

$$Y_N = \log_2(Q_N), \quad Y_R = \log_2(Q_R), \quad Y_P = -\log_2(Q_P).$$

(5)

In 2D, one has the exactly known exponents $\beta = 5/36, \nu = 4/3$, and $\gamma = 43/18$ 25 26 27. In 3D, the exact values are unknown, and are numerically found to be $\beta/\nu = 0.4774(1)$ and $\nu = 0.8764(7)$ 24 26 28.

**Initial estimate of $d_{\text{min}}$ for 2D.** We first carried out simulations at the critical point $p = 1/2$ for bond percolation on the square lattice with time step up to $\ell_{\text{max}} = 1024$ and the number of samples about $2 \times 10^6$.

The asymptotic behavior of the observables $N, R$ and $P$ is expected to follow the form

$$O(\ell) = \ell^{Y}(a_0 + b_1\ell^{\eta_1} + b_2\ell^{\eta_2}),$$

(6)

where higher-order corrections are neglected and the critical exponent $Y$ is given by Eq. (3). The leading finite-size correction exponent is known to be $y_1 = -0.96(6) \approx -1$ 22. A least-squares criterion was used to fit the data assuming the above form. With $y_1$ being fixed at $-1$, the fit of $P$ gives $d_{\text{min}} = 1.1308 \pm 0.0002$ and $b_1 = 0.045(5)$, and the fit of $N$ yields $d_{\text{min}} = 1.1308 \pm 0.0002$.

As an illustration, we plot $P \ell^{-Y_P} - 0.045\ell^{-1}$ in Fig. 1 and $N\ell^{-Y_N}$ in Fig. 2 both vs. $\ell^{-1}$, where the $d_{\text{min}}$ value is set at a series of values in range $[1.1302, 1.1312]$ in steps of 0.0002, including the above estimate $d_{\text{min}} = 1.1308$. The term $-0.045\ell^{-1}$ is included in Fig. 1 to remove the overall slope seen in the data of $P$; we did not do this to the $N$ data (Fig. 2), and there the slope is evident. The values of $Y_P$ and $Y_N$ are obtained from Eq. (4), using the exactly known values of $\beta$ and $\nu$. Because the leading corrections have been subtracted in Fig. 1, it is expected that the curve for the correct $d_{\text{min}}$ value should asymptotically become flat and reach a constant. Figure 1 shows that as $\ell$ increases, the curve for the conjectured value 217/192 = 1.1302 is bending up while the curve for 1.1302 and 1.1312 are bending down. This implies that the correct $d_{\text{min}}$ value should fall somewhere in between. A similar behavior is seen in Fig. 2 where the curve for 1.1308 is approximately straight while those for 1.1302 and 1.1312 are bending down and up, respectively.

**Further simulations for 2D.** Although the conjectured number 217/192 seems to be ruled out by the data shown in Figs. 1 and 2, a more careful analysis is still desirable. The above analysis makes an assumption that the leading correction is governed by $\ell^{-1}$, but the physical origin of this term is unclear as the leading irrelevant thermal scaling field has exponent $y_1 = -2$. It is conceivable that more slowly convergent corrections exist but are not detected by the simulations up to $\ell_{\text{max}} = 1024$. In particular, percolation can be regarded as a special case of biased-directed percolation with the symmetry between spatial and temporal directions restored 22. In this case, multiplicative and/or additive logarithmic corrections can occur in principle such that the scaling behavior of $N, R$
and $P$ is modified as

$$O(\ell) \sim |\log(\ell/\ell_0)|^{y_m} \ell^{y_c} \left(1 + 1/|\log(\ell/\ell_1)|^{y_c}\right),$$

(7)

where $\ell_0$ and $\ell_1$ are constants, and $y_m$ and $y_c$ are the associated correction exponents. Corrections of the log log form are also possible. We note that due to cancellation associated correction exponents. Corrections of the log log form are also possible. We note that due to cancellation between nominator and denominator, the multiplicative logarithmic correction will not explicitly appear in ratio $Q_O(\mathcal{O} = \mathcal{N}, \mathcal{R}, P)$, for which the scaling behavior is modified as

$$Q_O(\ell) = 2^y \left(1 + 1/|\log(\ell/\ell_1)|^{y_c}\right),$$

(8)

where $y_c'$ can be equal to $y_c$ or $|y_m|$, depending on the relative amplitudes of the terms associated with them.

To investigate this, we carried out more extensive simulations up to $\ell_{\text{max}} = 16384$. The number of samples was $4.5 \times 10^{10}$ for $\ell \leq 1024$, $5 \times 10^{9}$ for $1024 < \ell \leq 4096$, $10^9$ for $4096 < \ell \leq 8192$, and $3 \times 10^8$ for $\ell > 8192$.

From the $Q_O$ data, we calculate the $d_{\text{min}}(\ell)$ value by Eqs. (1) and (5). Table I displays the resulting values of $d_{\text{min}}(\ell)$ from the ratios $Q_{\mathcal{N}} = Q_{\mathcal{R}}$ and $Q_P$. It can be clearly seen that for $\ell \leq 3072$, the $d_{\text{min}}(\ell)$ values that derive from $\mathcal{N}$ and $\mathcal{R}$ increase monotonically as $\ell$ increases. For clarity, these data are plotted in Fig. 2. The conjecture $d_{\text{min}} = 217/192$ would mean that the monotonically increasing curves for $d_{\text{min}}^{(\mathcal{N})}$ and $d_{\text{min}}^{(\mathcal{R})}$ must bend downward as $\ell$ become larger, and thus a very rapid drop would occur near origin $(1/\ell \rightarrow 0)$ in the inset of Fig. 2 which seems very unlikely. The $d_{\text{min}}^{(P)}$ data are less accurate and not shown in Fig. 2.

We fit the $d_{\text{min}}(\ell)$ data by

$$d_{\text{min}}(\ell) = d_{\text{min}} + b_1 \ell^{y_1} + b_2 \ell^{-2},$$

(9)

using a least-squares criterion. The data for small $\ell < \ell_{\text{min}}$ were gradually excluded to see how the residual

\[\chi^2\text{ changes with respect to } \ell_{\text{min}}\]. Table II lists the fitting results for $d_{\text{min}}^{(\mathcal{N})}$, $d_{\text{min}}^{(\mathcal{R})}$ and $d_{\text{min}}^{(P)}$. From these fits, we obtain $d_{\text{min}}^{(\mathcal{N})} = 1.13077(3)$, $d_{\text{min}}^{(\mathcal{R})} = 1.13077(2)$ and $d_{\text{min}}^{(P)} = 1.13066(15)$, where the estimates of the error margins are rather conservative. We note that the coefficient $b_2$ cannot be determined well in the fits for $\ell_{\text{min}} > 32$. Thus, fits with $b_2 = 0$ were also carried out, and the results agree with our above estimates of $d_{\text{min}}$. We also simulated critical site percolation on an $L \times L$ triangular lattice with periodic boundary conditions; this system is known to have zero amplitude of the leading irrelevant scaling field with exponent $y_1 = -2$. A row of lattice sites was chosen, and all the occupied sites on this
shown in Fig. 4, where the exponent 0.7 reflects the value of $y_1$. 

In conclusion, we determined the shortest-path fractal dimension $d_{\text{min}}$ for percolation in 2D and 3D to be $1.3077(2)$ and $1.3756(6)$ respectively. For the 2D value, we use the result which follows from $R(\ell)$ and has the smallest error bars. To our knowledge, these are the most accurate values currently available. The conjectured value in 2D, $d_{\text{min}} = 217/192$, appears to be ruled out with a high probability. The 3D result represents a substantial increase in precision over the previous values of $1.34(1)$ and $1.374(4)$.

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