The UV Fate of Anomalous $U(1)$s

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ABSTRACT: In four dimensions, anomalous Abelian gauge theories need to be UV-completed at a finite scale below which massive chiral fermions become part of the spectrum. In these constructions, the limit of vanishing gauge boson mass is singular — the photon must have a mass. We show that this class of theories requires yet further UV-completion at a finite cutoff below which a radial mode must become part of the spectrum, and cannot be decoupled: a St"uckelberg limit does not exist. Further, a parametric separation of scales between the massive photon and the radial mode requires the presence of a large hierarchy of charges in the UV-completion, effectively probing the non-compact limit. If the low-energy spectrum contains a mixed $U(1)$-gravitational anomaly, such large charge ratio requires an equally large number of massive fermion species to be part of the spectrum. As a result, a tiny photon mass comes at the cost of a quantum gravity cutoff that lies parametrically below $M_{Pl}$. When the infrared fermion spectrum features both $U(1)^3$ and mixed gravitational anomalies, making the radial mode heavy correspondingly lowers the quantum gravity cutoff to coincide with the Weak Gravity Conjecture scale for values of the gauge coupling above a certain critical size.
1 Introduction

Within the realm of effective field theory (EFT), certain features of Abelian gauge theories starkly differ from those of their non-Abelian counterparts. Charge quantization is not ‘built-in’, and the gauge group may be taken to be $\mathbb{R}$ as much as $U(1)$. Even if ratios of charges are assumed to be integer, arbitrarily large values appear consistent without the need to introduce an equally large number of degrees of freedom. However, some of these features are not expected to survive further UV-completion. Several arguments suggest that in a theory of quantum gravity Abelian charges must be quantized, and the corresponding gauge group compact [1]. To the extent that theories featuring large integer charge ratios approximately realize the non-compact limit — abiding by the letter of the law but violating its spirit — a shadow of suspicion hangs over those constructions.

A further distinction between Abelian and non-Abelian gauge theories arises when the corresponding vector bosons are massive. A massive Abelian gauge theory coupled to a conserved current is renormalizable: the photon mass $m_{\gamma}$, and the typical gauge coupling strength $g$ are free parameters of the theory — ignoring Landau poles, such a theory may be valid up to arbitrarily high scales [2–4]. This is in contrast with massive Yang-Mills theory, where the breakdown of perturbation theory, manifest in the loss of perturbative unitarity in longitudinal gauge boson scattering, requires that the theory be UV-completed at scales of order $\Lambda \lesssim 4\pi m/g$ (up to group theoretic factors).

The above distinction plays a crucial role in the context of gauge theories featuring anomalous fermion content in the infrared. For both Abelian and non-Abelian groups, an
anomalous gauge theory can be consistently quantized in perturbation theory so long as
gauge bosons are massive, and that the theory be further UV-completed at some finite
cutoff scale [5–7]. For a non-Abelian theory the role of the anomaly is incidental: it forces
the gauge bosons to acquire a mass, but plays no role in the cutoff size. Instead, the non-
renormalizability of massive Yang-Mills already sets an upper bound on the EFT cutoff,
of order $\sim 4\pi m/g$. On the other hand, the fact that the Abelian non-linear sigma model
is renormalizable makes us expect that as the anomaly vanishes any upper bound on the
cutoff of the anomalous EFT must disappear. Indeed, the analysis of [7] reveals that a $U(1)$
gauge theory with anomalous fermion content may be valid up to scales $\gg 4\pi m_\gamma/g$, with
$g$ the typical gauge coupling of the infrared spectrum. In 4 dimensions, the cutoff of the
anomalous EFT corresponds to the scale below which massive fermions must appear, with
the appropriate charge assignments to cancel the anomalies of the low-energy spectrum.

In this paper, building on the seminal work of [7], we explore the ultraviolet fate of
anomalous Abelian gauge theories in four dimensions. The resolution of the anomaly by
the appearance of massive fermions focuses our attention on anomaly-free theories in which
anomaly-cancellation occurs due to fermions appearing at different scales. As illustrated in
Figure 1, these theories, which represent partial UV completions of the anomalous EFTs
that are the focus of [7], feature a variety of mass scales above the photon mass. When
gravitational effects are decoupled, the most relevant scales are the mass $M_f$ of heavy
fermions responsible for anomaly-cancellation, and a possible cutoff $\Lambda_*$ of the anomaly-free
theory. When gravitational effects are included, the Planck scale, and the quantum gravity
scale $\Lambda_{QG}$ (which may differ from $M_{Pl}$) become part of the picture. The appearance of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{scales.png}
\caption{Illustration of the different scales relevant to our discussion. At low energies,
the anomalous EFT contains a number of massless fermions ($\chi_i$), and a massive photon.
Massive fermions ($\psi_i$) responsible for cancelling the anomalies of the low-energy spectrum
appear at scale $M_f$. This anomaly-free extension may require further UV-completion at
some higher scale $\Lambda_*$. In the presence of gravity, the quantum gravity scale, which may be
below $M_{Pl}$, will play an interesting role in our discussion.}
\end{figure}
these scales raises a variety of questions: What is the relationship between the cutoff $\Lambda_*$ of the anomaly-free theory and the scale of the heavy fermions? Under what circumstances can the former be decoupled? In the presence of gravity, what is the relation between these scales and the quantum gravity cutoff? In addressing these questions, we show that:

(i) This class of massive Abelian gauge theories are themselves EFTs, and require further UV-completion at a finite scale $\Lambda_*$ above the scale of the heavy fermions. This cutoff scale corresponds to an upper bound on the mass of a radial mode that cannot be decoupled from the spectrum — in this class of theories, the Stückelberg limit does not exist.

(ii) The above result is a consequence of the presence of massive fermions with chiral charge assignments necessary to cancel the anomalies of the low-energy EFT. The loss of perturbative unitarity in fermion-anti-fermion scattering into a number of longitudinal photons at center of mass energies of order $\Lambda_*$ sets an upper bound on the scale of UV-completion required to recover a perturbative expansion.

(iii) The upper bound on the scale of UV-completion depends only logarithmically on the mass of the heavy fermions, and is essentially given by $\Lambda_* \sim 4\pi m_\gamma/g_0$, with $g_0$ the quantum of charge. This highlights how the regime $\Lambda_* \gg 4\pi m_\gamma/g$ is only accessible by introducing a large hierarchy of charges in the UV, regardless of the specific details of the UV-completion.

The consequences of probing the (morally) non-compact limit required to realize a parametrically large cutoff differ dramatically depending on whether the low-energy theory features a mixed $U(1)$-gravitational anomaly. We find that:

(iv) If the infrared fermion spectrum only contains a $U(1)^3$ anomaly, but no mixed gravitational anomaly, then a large ratio of charges may be introduced with impunity, and the limit $\Lambda_* \gg 4\pi m_\gamma/g$ realized in keeping with the results of [7]. However, in the presence of a mixed gravitational anomaly, a large ratio of charges necessarily requires the presence of an equally large number $N$ of massive fermion species, correspondingly lowering the quantum gravity scale down to $\Lambda_{QG} \sim 4\pi M_{Pl}/\sqrt{N}$ [8]. Thus probing the non-compact limit in the context of theories featuring a $U(1)$-gravitational anomaly in the infrared comes at the cost of a quantum gravity cutoff that is parametrically below $M_{Pl}$.

(v) If the anomalous EFT only contains a mixed gravitational anomaly, but no $U(1)^3$ anomaly, then $\Lambda_*$ may be pushed all the way up to the quantum gravity scale. When this limit is saturated:

$$\Lambda_* \sim \Lambda_{QG} \sim 4\pi \left(\frac{M_{Pl}^2 m_\gamma}{g}\right)^{1/3},$$

and $\Lambda_{QG} \rightarrow 0$ in the limit of vanishing photon mass. The right-hand-side of Eq.(1.1) parametrically coincides with the upper bound on the cutoff of the anomalous EFT.
obtained in [7, 9], below which the massive fermions responsible for cancelling the anomaly must appear. To the extent that theories with massive fermions appearing above this scale would belong in the Swampland [10] of consistent EFTs that are inconsistent in the presence of gravity, our result shows that such a possibility is self-consistently avoided by the lowering of the quantum gravity scale mandated by the presence of a large number of species.

(vi) In the more generic case where the low-energy fermion spectrum features both $U(1)^3$ and $U(1)$-gravitational anomalies, there exists a critical value of the gauge coupling, given by

$$g_* \sim \left( \frac{64\pi^3 m_\gamma}{M_{Pl}} \right)^{1/4}.$$  \hfill (1.2)

For $g \lesssim g_*$ the statement in (v) applies, whereas for $g \gtrsim g_*$ saturating the upper bound on $\Lambda_*$ comes at the cost of lowering the quantum gravity scale down to

$$\Lambda_{QG} \sim g M_{Pl},$$  \hfill (1.3)

which parametrically coincides with the Weak Gravity Conjecture (WGC) [11] scale as seen from the low-energy EFT. Moreover, in this regime, $g M_{Pl} \sim g_0^{1/3} M_{Pl}$, which is the version of the magnetic WGC scale advocated for in [12]. This provides a four-dimensional, field-theoretic example of a class of massive Abelian gauge theories where the WGC scale emerges in the role of a quantum gravity cutoff in a way that is tied to the presence of a large number of species, in turn required to realize the approximately non-compact limit.

This paper is organized as follows. In section 2 we briefly review the results of [7] from the point of view most relevant to our discussion. In 3 we show how the presence of massive fermions with chiral charge assignments leads to the breakdown of perturbation theory at high energies, calling for further UV-completion, and precluding a Stückelberg limit. In section 4 we focus on the implications of our results for Abelian gauge theories featuring anomalies in the infrared. We present our conclusions in section 5.

2 EFT cutoffs in anomalous Abelian gauge theories

Unlike massive Yang-Mills, a non-zero gauge boson mass in the context of an Abelian gauge theory does not lead to the breakdown of perturbation theory at high external momenta. Although not obvious in unitary gauge, where the gauge boson propagator falls off slower than $1/k^2$, it becomes apparent if we enlarge the theory so as to introduce a gauge redundancy by incorporating an additional degree of freedom $\theta$ transforming non-linearly under the gauge action. This allows us to rewrite the vector mass term as

$$\mathcal{L} \supset \frac{1}{2} m_\gamma^2 A_\mu^2 \rightarrow \frac{1}{2} (\partial_\mu \theta - m_\gamma A_\mu)^2,$$  \hfill (2.1)

which remains invariant under gauge transformations $A_\mu \rightarrow A_\mu + \frac{1}{g_0} \partial_\mu \alpha$, and $\theta \rightarrow \theta + \frac{m_\gamma}{g_0} \alpha$.

This is the so-called ‘Stückelberg trick’ — its crucial insight being that it is possible to
restore gauge invariance without introducing operators of dimension higher than 4, making the renormalizability of a massive Abelian gauge theory manifest [2]. This remains true if any fermionic current that $A_\mu$ couples to is vector-like, regardless of the fermion mass. In this case, $m_\gamma$ is a free parameter of the theory, and the limit $m_\gamma \to 0$ remains unproblematic.

This is no longer true if the current that $A_\mu$ couples to is not conserved, such as in the context of theories with anomalous fermion content. Nevertheless, for both Abelian and non-Abelian groups, gauge theories with anomalies can be consistently quantized, so long as the corresponding vector bosons are massive, and that the theory is treated as an EFT only valid up to a finite cutoff scale [5–7]. For an Abelian gauge theory, the upper bound on the EFT cutoff depends solely on the anomaly, and differs parametrically for $U(1)^3$ and mixed $U(1)$-gravitational anomalies. In the remainder of this section, we review the status of theories with Abelian gauge anomalies, closely following [7].

For illustration, we focus on a theory containing a single massless fermion, coupled to $A_\mu$ through a left-handed current. Allowing for a non-zero photon mass, the corresponding lagrangian reads

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \theta - m_\gamma A_\mu)^2 + \bar{\chi} i \gamma^\mu \chi + g A_\mu \bar{\chi} L \gamma^\mu \chi L ,$$  \hspace{1cm} (2.2)

where $g \equiv g_0 Q$. \footnote{Distinguishing between $g_0$ and $g$ may seem unnecessary at this point, but will be relevant in our subsequent discussion, where extensions of the anomalous EFT may feature a quantum of charge $g_0$ that differs from the typical gauge coupling of the infrared spectrum.} Although Eq.(2.2) remains invariant under gauge transformations, under which the fermions transform as $\chi_L \to \exp iQ_\alpha \chi_L$ and $\chi_R \to \chi_R$, the corresponding path integral is not due to the non-trivial jacobian of the fermionic functional determinant. Effectively, the presence of a $U(1)^3$ anomaly leads to an additional term in the lagrangian, of the form

$$\delta L = \frac{1}{3} \frac{g_0^2 Q^3}{16\pi^2} \alpha F \tilde{F} .$$  \hspace{1cm} (2.3)

At this point, one could try to restore gauge invariance by modifying the theory into an anomaly-free one — e.g. by introducing a coupling between $A_\mu$ and the right-handed fermion current with the same strength, rendering the entire interaction vector-like. Alternatively, one could choose to leave the theory as it is, i.e. anomalous, and instead build a gauge invariant version of the anomalous EFT by adding a term to the lagrangian proportional to $\theta F \tilde{F}$ with the appropriate coefficient to cancel Eq.(2.3):

$$L \supset -\frac{1}{3} \frac{g^3}{16\pi^2 m_\gamma} \theta F \tilde{F} .$$  \hspace{1cm} (2.4)

This is the Abelian version of the Wess-Zumino term [13, 14] — Eq.(2.2) extended with this new term provides a gauge invariant description of our anomalous EFT. However, it is apparent from Eq.(2.4) that gauge invariance of the anomalous theory has only been achieved at the cost of renormalizability. Moreover, the coefficient of the $\theta F \tilde{F}$ term diverges in the limit $m_\gamma \to 0$, which provides an easy way to see that the limit of a massless photon is indeed not allowed in an anomalous theory. The scale suppressing this operator
corresponds to the cutoff of the anomalous EFT. Following standard NDA counting [15], an upper bound on the scale of UV-completion as mandated by the presence of a $U(1)^3$ anomaly is given by [7]

$$\Lambda_{U(1)^3} \sim \frac{64\pi^3 m_\gamma}{g^2}.$$  \hspace{1cm} (2.5)

Turning on gravity, Eq.(2.3) is accompanied by an extra term $\propto \alpha R \tilde{R}$ due to the presence of a mixed $U(1)$-gravitational anomaly. As before, gauge invariance may be restored in perturbation theory by including the following term:

$$\mathcal{L} \supset -\frac{1}{24} \frac{g}{16\pi^2 m_\gamma} \sqrt{|g|\theta R \tilde{R}}.$$  \hspace{1cm} (2.6)

The scale suppressing this operator sets an upper bound on the scale of UV-completion as mandated by the presence of a mixed $U(1)$-gravitational anomaly. Parametrically [7, 9]:

$$\Lambda_{\text{grav}} \sim \frac{4\pi}{M_{Pl} m_\gamma g^3}.$$  \hspace{1cm} (2.7)

In a four-dimensional theory, Eq.(2.5) and (2.7) correspond to the scale below which massive fermions must appear, with charge assignments appropriate to cancel the corresponding anomaly. As depicted in Figure 2, the cutoff of the anomalous EFT may lie parametrically above $\sim 4\pi m_\gamma/g$ in the regime where $m_\gamma/M_{Pl} \lesssim g \lesssim 4\pi$. For a more general fermion content, Eq.(2.5) and (2.7) hold after the respective substitutions $g^3 \rightarrow g_0^3 |\sum Q_3^i|$, and $g \rightarrow g_0 |\sum Q_i|$. \hspace{1cm} (2.8)

However, in the absence of large hierarchies of charges in the low-energy spectrum, Eq.(2.5) and (2.7) will still apply, parametrically, for theories featuring the corresponding anomaly, and with $g$ understood as the typical size of the gauge coupling present in the low-energy EFT.

Eq.(2.5) and (2.7) were first obtained in [7] and [9] respectively, and, as sketched above, can be derived within the anomalous EFT alone. However, they can be readily understood by considering the effect of heavy fermions with mass $M_f$ that must be present in any four-dimensional UV-completion in order to render the full theory anomaly-free. Through the diagram depicted in Figure 3, the heavy fermions responsible for cancelling the $U(1)^3$ anomaly lead to a non-zero contribution to the photon mass, of the form $\sim \frac{g^3}{64\pi^3 M_f}$. The requirement that $m_\gamma^2 \gtrsim \delta m_\gamma^2$ yields Eq.(2.5), with the role of $\Lambda_{U(1)^3}$ played by the mass of the heavy fermions. Crucially, as discussed in [7], this is more than a statement about the natural size of $m_\gamma$ — fine-tuning the photon mass below $\delta m_\gamma$ would require fine-tuning the coefficients of an infinite number of higher-dimensional-operators, effectively signalling the breakdown of perturbation theory within the anomalous EFT. Identical considerations apply to a version of Figure 3 with gravitons propagating in the internal lines in theories with a mixed gravitational anomaly, and similarly lead to the scale in Eq.(2.7) being identified with an upper bound on the mass of the anomaly-cancelling fermions.

Throughout this paper, expressions of the form $\sum Q_i$, and $\sum Q_3^i$, are to be understood as sums over the charges of left-handed fermions.
Figure 2: Upper bound on the cutoff of a massive Abelian gauge theory with anomalous fermion content consisting of a massless fermion coupled to $A_\mu$ through a left-handed current. The solid blue and dashed orange lines correspond to the scale at which perturbation theory breaks down as a result of the presence of a $U(1)^3$ and a mixed $U(1)$-gravitational anomaly respectively, as given in Eq.(2.5) and (2.7). To the extent that theories living in the orange shaded region (above the orange dashed line but below the solid blue line) correspond to consistent non-gravitational EFTs that become inconsistent in the presence of gravity, the orange wedge represents a piece of Swampland [10]. The upper bound on the EFT cutoff implied by the anomalies lies above $4\pi m_\gamma/g$ for values of the gauge coupling $m_\gamma/M_{Pl} \lesssim g \lesssim 4\pi$. (Both axes are in a log scale.)

Figure 3: A non-zero radiative contribution to the photon mass, parametrically of the form Eq.(2.8), arises from this three-loop diagram, with massive fermions responsible for cancelling the $U(1)^3$ anomaly of the low-energy theory propagating inside the loops. If the low-energy theory also contains a mixed $U(1)$-gravitational anomaly, there is an analogous diagram with gravitons (instead of vectors) propagating in the internal lines.

3 UV-completing anomalous $U(1)$s with fermions

We now turn our attention to the ultraviolet fate of the anomalous Abelian gauge theories discussed in section 2, cancelling the anomalies of the low-energy theory by including massive fermions with appropriate quantum numbers, which is the only option in four
dimensions. For simplicity, we focus first on a single heavy fermion $\psi$ with mass $M_f$ and chiral $U(1)$ charges. The terms in the lagrangian involving $\psi$ read

$$L \supset \bar{\psi} i \gamma^\mu \partial_\mu \psi + g_0 A_\mu J_\psi^\mu - M_f (\bar{\psi} L \psi_R + h.c.),$$

(3.1)

where

$$J_\psi^\mu = Q_L \bar{\psi}_L \gamma^\mu \psi_L + Q_R \bar{\psi}_R \gamma^\mu \psi_R,$$

(3.2)

and $Q_L \neq Q_R$ in general. By assumption, the charge spectrum of the infrared fermion sector is such that the full theory is free of both $U(1)^3$ and mixed gravitational anomalies. Thus, when $Q_L = Q_R$ the massless sector itself is non-anomalous, and the heavy fermion couples to $A_\mu$ through a vector current. However, when $Q_L \neq Q_R$ the heavy fermion plays a role in anomaly cancellation, and the theory will appear anomalous below the scale $M_f$.

Having enlarged the fermion spectrum so as to make the theory anomaly-free, a gauge transformation leaves the fermionic functional determinant in the path integral unchanged. However, it is the fermion mass term that now breaks gauge invariance, which we may restore by introducing appropriate couplings to $\theta$, as follows:

$$L \supset -M_f \left( \epsilon^{\pm \theta / f} \bar{\psi}_L \psi_R + h.c. \right) = -M_f \left[ \cos \left( \frac{\theta}{f} \right) \bar{\psi} \psi \pm i \sin \left( \frac{\theta}{f} \right) \bar{\psi} \gamma^5 \psi \right],$$

(3.3)

where

$$f \equiv \frac{m_r}{g_0 |Q_L - Q_R|},$$

(3.4)

and the upper (lower) sign in Eq.(3.3) applies when $Q_L - Q_R > 0$ ($Q_L - Q_R < 0$). Expanding the sine and cosine in Eq.(3.3) as an infinite sum of higher-dimensional-operators suppressed by increasing powers of $f$ suggests that our attempt to restore gauge invariance has only been successful at the cost of renormalizability, and one might expect the theory to be valid only up to scales not much above $f$ itself. Indeed, any attempt to rewrite Eq.(3.3) in a way that involves only renormalizable interactions necessarily requires introducing additional degrees of freedom. For instance, if the theory is further embedded into a weakly coupled Abelian Higgs model with condensate charge $Q_L - Q_R$ and vev $f$, then a radial mode will be present below the scale $4\pi f$. The UV-completion of the anomalous EFT by the addition of massive fermions is itself an effective field theory. In what follows, we will refer to this partial UV-completion as the anomaly-free EFT. We now confront a nested set of EFTs: the anomalous EFT valid up to the scale $M_f$, and the non-anomalous EFT valid from the scale $M_f$ up to some cutoff $\Lambda_*$. The question then arises: what is the cutoff of the anomaly-free EFT? When does it coincide with the apparent cutoff of the anomalous EFT? And when, if ever, can we take $\Lambda_* \to \infty$?

It is clear from the form of Eq.(3.3) that there are at least some cases in which the limit $\Lambda_* \to \infty$ is allowed and the theory becomes fully renormalizable, regardless of the value of the photon mass: (i) when $M_f \to 0$, regardless of the left- and right-handed charge assignments, and (ii) when $Q_R \to Q_L$, regardless of the fermion mass. In both cases, any upper bound on $\Lambda_*$ due to Eq.(3.3) must decouple — a Stückelberg limit must exist.

However, to the extent that Eq.(3.3) involves irrelevant operators, one might wonder whether the description of the non-anomalous EFT should be enlarged to include additional
irrelevant operators compatible with the gauge symmetry, whose appearance might lead to an independent bound on $\Lambda_*$, and additional conditions on the realization of a St"uckelberg limit. The status of such operators can be readily ascertained from a chiral lagrangian analysis, which we carry out in section 3.1. In 3.2, we show how, whenever $M_f \neq 0$ and $Q_L \neq Q_R$, the loss of perturbative unitarity at high energies signals the breakdown of perturbation theory, and leads to an upper bound on $\Lambda_*$, beyond which the theory requires further UV completion. In this case, a St"uckelberg limit does not exist, as we elaborate on in section 3.3.

### 3.1 Chiral lagrangian analysis

Although the irrelevant operators appearing in Eq.(3.3) are the minimal set required to preserve gauge invariance, the symmetries allow (and one in general expects) a whole host of irrelevant operators to appear, any of which could point to the scale $\Lambda_*$ at which the anomaly-free EFT breaks down. While these could be enumerated by simply writing down the most general set of gauge-invariant irrelevant operators involving $A_\mu, \theta, \chi$, and $\psi$, this does not provide clear guidance as to the relative size of the various operators, and hence to the size of the corresponding cutoff $\Lambda_*$. Indeed, not all irrelevant operators in the anomaly-free EFT are created equal. This can be seen most clearly by considering the simplest UV completion of the anomaly-free EFT: the Abelian Higgs model. The operators in Eq.(3.3) can be obtained from a theory of a massless vector, massless fermions, and complex scalar $\Phi$ with Lagrangian

$$L = |D_\mu \Phi|^2 - \frac{\lambda}{2} \left( |\Phi|^2 - \frac{f^2}{2} \right)^2 - \left( y \Phi \bar{\psi} L \psi R + h.c. \right) + \mathcal{L}(A_\mu, \psi, \chi).$$  \hspace{1cm} (3.5)

The potential for $\Phi$ leads to spontaneous symmetry breaking, which can be conveniently parameterized in terms of a radial mode $\rho$ and goldstone $\theta$ via

$$\Phi = \frac{1}{\sqrt{2}} (f + \rho) e^{i \theta}. \hspace{1cm} (3.6)$$

The radial mode $\rho$ acquires a mass $m_\rho = \sqrt{\lambda} f$ and may be integrated out to give an effective lagrangian of the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial \theta)^2 + \frac{1}{2f^2(\lambda f^2)} (\partial \theta)^4$$

$$- \left( M_f e^{iT} \bar{\psi} L \psi_R - \frac{M_f}{f^2(\lambda f^2)} (\partial \theta)^2 \bar{\psi} L \psi_R \right) + \text{h.c.} + \ldots \hspace{1cm} (3.7)$$

This contains precisely the effective operator appearing in Eq.(3.3), as well as higher-derivative terms for $\theta$, and derivative couplings between $\theta$ and the fermions. The former operator arose even before integrating out $\rho$ and is correspondingly independent of $m_\rho$, while the latter are generated by integrating out $\rho$ and proportional to negative powers of $m_\rho$. While this is suggestive, one would like to understand the natural size of various operator coefficients independent of the specific UV completion.
The relative size of these irrelevant operators can be understood more generally by approaching the anomaly-free EFT from the perspective of the chiral symmetries weakly gauged by the vector field and broken by the fermion mass, allowing us to bring NDA [15] power counting to bear on the problem. To do so, we begin by framing the non-anomalous EFT as one in which the $U(1)_L \times U(1)_R$ chiral symmetry of $\psi_L, \psi_R$ is weakly gauged and spontaneously broken. Under the $U(1)_L \times U(1)_R$ global symmetry the fermion fields transform as $\psi_L \rightarrow L \psi_L$ and $\psi_R \rightarrow R \psi_R$. Parameterizing $L, R$ as $L = e^{i(\alpha + \beta)/2}$ and $R = e^{i(-\alpha + \beta)/2}$ for real $\alpha, \beta$, we have axial transformations $LR^\dagger = e^{i\alpha}$. The fermion mass arises due to unspecified (and possibly strong) interactions that break $U(1)_L \times U(1)_R \rightarrow U(1)_V$, giving one goldstone mode, which we can organize as $U = e^{i\pi f}$, \[ U = e^{i\pi f}, \tag{3.8} \] where $U$ transforms linearly under $U(1)_L \times U(1)_R$, $U \rightarrow U' = LUR^\dagger$. The goldstone $\pi$ correspondingly transforms under the shift $\pi \rightarrow \pi' = \pi + \alpha f$. (Although we will identify $\pi$ with $\theta$ momentarily, it is useful to differentiate the two for the time being.) From this perspective, the Abelian gauge symmetry can be thought of as gauging a particular subgroup of the vector and axial chiral symmetries, under which $\alpha = (Q_L - Q_R)\gamma$ and $\beta = (Q_L + Q_R)\gamma$. Clearly when $Q_L = Q_R$ we are gauging the vector symmetry preserved by the fermion mass, while for $Q_L \neq Q_R$ chiral symmetry breaking necessarily implies gauge symmetry breaking.

We can then construct the non-anomalous EFT as the most general one invariant under the local $U(1)_L \times U(1)_R$ symmetries. The leading derivative interaction allowed by these symmetries is

$$
\mathcal{L} \supset \frac{f^2}{2} |D_\mu U|^2 = \frac{1}{2} (\partial \pi)^2 - g_0 f (Q_L - Q_R) A_\mu \partial^\mu \pi + \frac{1}{2} g_0^2 f^2 (Q_L - Q_R)^2 A_\mu A^\mu. \tag{3.9}
$$

From this, it is clear that the chiral lagrangian can be matched to the anomaly-free EFT by making the identifications $\pi = \theta$ and $g_0 f |Q_L - Q_R| = m_\gamma$. Note the latter identification is a consequence of starting with the chiral symmetry breaking — in reality, it is possible for the gauge symmetry to be broken more strongly than the chiral symmetry (by e.g. a UV completion in which multiple scalars acquire vevs and contribute to the mass of the vector, while only one scalar couples to the fermions), but not visa versa. So one expects in full generality $m_\gamma \geq |Q_L - Q_R| g_0 f$ whenever $f$ denotes the scale of chiral symmetry breaking.

The chiral lagrangian formulation then allows us to use NDA power counting to enumerate irrelevant operators consistent with the symmetries and estimate the size of the corresponding Wilson coefficients in terms of their dependence on $g_*, \Lambda_*$, and $f$, where $\Lambda_* = g_* f$. The most interesting operators for our purposes include (with Hermitian conju-
where the coefficients $c_i$ are all $O(1)$ numbers. In general, the precise values of the $c_i$ are not fixed and depend on details of the UV completion. However, there is one exception: we must have $|c_y| = 1$, since $O_y$ defines the fermion mass $M_f$. Indeed, we recognize $O_y$ (plus its Hermitian conjugate) as giving the operators in Eq.(3.3), now reproduced via the chiral Lagrangian.

As for the remaining factors, NDA power counting cleanly distinguishes different operator classes. The operator $O_y$ is independent of $g_*$ and depends only on $M_f$ and $f$ (via $U$). In contrast, operators such as $O_n$ and $O_f$ depend on $g_*$ via $\Lambda_*$. This agrees precisely with the Abelian Higgs UV completion considered earlier, which corresponds to $\lambda = g_2^2$, $c_y = -1, c_2 = \frac{1}{2}, c_f = -1$, and $c_{L,R} = 0$.

In principle, any of these operators can lead to an upper bound on the scale $\Lambda_*$ at which the anomaly-free EFT breaks down. For example, the operators $O_n$ give the leading unitarity violation at large Mandelstam $s$, independent of $M_f$, implying $\Lambda_* \sim 4\pi f / \sqrt{c_n}$. Taking $M_f \to 0$ only decouples some of the irrelevant operators allowed by the symmetries of the non-anomalous EFT. This leads to a refinement of our criteria for achieving the St"uckelberg limit for arbitrary $m_\gamma$: (i) when $M_f \to 0$ and either $f \to \infty$ or $c_i \to 0 \forall i$, regardless of the left- and right-handed charge assignments, and (ii) when $Q_R \to Q_L$, regardless of $M_f, f, \text{ and the coefficients } c_i$.

In practice, however, when setting a quantitative upper bound on the scale $\Lambda_*$ it bears emphasizing that only the coefficient of $O_y$ is uniquely fixed in terms of the spectrum of the anomaly-free EFT, whereas the coefficients of all other irrelevant operators depend on the UV completion. This is in contrast to massive non-Abelian theories, where the same two-derivative operator that gives rise to the goldstone kinetic terms also induces derivative interactions with fixed relative coefficients that can be used to bound the cutoff. As such, only $O_y$ leads to a model-independent bound on $\Lambda_*$ in the Abelian case. As for the other operators, one might imagine a UV completion in which all of the $c_i$ except $c_y$ are suppressed (or tuned) to approach a St"uckelberg limit. This justifies proceeding with an analysis of the breakdown of perturbative unitarity using $O_y$, or, equivalently the operators in Eq.(3.3).

### 3.2 Loss of perturbative unitarity

To set a model-independent upper bound on $\Lambda_*$ in the anomaly-free EFT, we will study the high-energy behaviour of scattering amplitudes in a theory described by the following
At high energies, this contact interaction provides the leading contribution to the scattering process $\psi\bar{\psi} \to n\theta$.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}(\partial_\mu \theta - m_\gamma A_\mu)^2$$

$$+ \bar{\psi}i\gamma^\mu \psi + g_0 A_\mu J_\mu - M_f \left[ \cos \left( \frac{\theta}{\bar{\tau}} \right) \bar{\psi}\psi \pm i \sin \left( \frac{\theta}{\bar{\tau}} \right) \bar{\psi} \gamma^5 \psi \right]$$  \hspace{1cm} (3.14)

where $\mathcal{L}_{g.f.} = -\frac{1}{2}(\partial_\mu A_\mu + \xi m_\gamma \theta)^2$ is a standard gauge-fixing term, $J_\mu$ is as given in Eq.(3.2), and $\mathcal{L}_\chi$ refers to the terms in the lagrangian involving massless fermions (kinetic terms plus couplings to $A_\mu$), which will not be relevant for the subsequent discussion, other than noting that their charge assignments must be such that the full theory is anomaly-free. (As in Eq.(3.3), the $+ (-)$ sign corresponds to the case $Q_L - Q_R > 0 \ (Q_L - Q_R < 0).$)

A process that reflects the breakdown of perturbation theory in the context of the EFT described by Eq.(3.14) concerns scattering of a fermion–anti-fermion pair into a number of longitudinal vectors at high center of mass energy. Making use of the Goldstone equivalence theorem, we can obtain the leading high-energy contribution to the corresponding scattering amplitude by considering the process $\psi\bar{\psi} \to n\theta$ in a general gauge. Expanding Eq.(3.3) as a power series in $\theta/\bar{\tau}$, we see that it contains operators coupling two fermions to any number of $\theta$'s:

$$\mathcal{L} \supset -M_f \left[ \bar{\psi}\psi \sum_{\{n\ even\}} \frac{(-1)^{\frac{n}{2}}}{n!} \left( \frac{\theta}{\bar{\tau}} \right)^n \pm i \bar{\psi} \gamma^5 \psi \sum_{\{n\ odd\}} \frac{(-1)^{\frac{n+1}{2}}}{n!} \left( \frac{\theta}{\bar{\tau}} \right)^n \right].$$  \hspace{1cm} (3.15)

At high energies, $\sqrt{s} \gtrsim M_f$, the leading contribution is due to the contact operator depicted in Figure 4 — diagrams involving more vertices will contain further powers of $M_f$, and therefore feature a milder high-energy behaviour.

The observation that the presence of massive fermions with chiral charge assignments leads to the breakdown of perturbation theory at sufficiently high scales was first made in [17] in the context of the Standard Model without a dynamical mechanism responsible for generating fermion masses, and later refined in [18, 19]. Of course, in the ‘Higgsless’ Standard Model, a bound $\Lambda \lesssim 4\pi v$ already follows from the loss of perturbative unitarity in longitudinal gauge boson scattering. Thus, for a non-zero fermion mass to reliably indicate a cutoff scale parametrically above the weak scale, the mechanism responsible
for restoring perturbation theory at scales above $\sim 4\pi v$ needs to be introduced into the analysis. In contrast, for a massive $U(1)$, the two-derivative terms in the action involving only longitudinal gauge bosons do not imply the loss of perturbative unitarity. Although higher-derivative operators such as the $\mathcal{O}_n$ encountered in section 3.1 can lead to the breakdown of perturbation theory in longitudinal gauge boson scattering, the precise bound in this case depends on the UV-completion via the unknown operator coefficients $c_n$. We can, however, derive a model-independent upper bound on the cutoff scale that stems purely from the presence of massive chiral fermions.

As discussed in [20] (see also [21]), a reasonable estimate of the range of validity of perturbation theory within a unitary theory can be obtained by demanding that

$$
\langle \Psi | T \hat{T} | \Psi \rangle = \sum_X | \mathcal{M}(\Psi \rightarrow X) |^2 \lesssim \pi^2 ,
$$

(3.16)

for any unit-normalized state $|\Psi\rangle$, and where $iT = S - \hat{1}$ as usual. $\sum_X$ refers to a sum over all possible final states, with an integral over the corresponding phase space being implicit. For the case at hand, we will estimate the scale of perturbation theory breakdown when

$$
\sum_{n=2}^{\infty} \frac{1}{n!} \int_{\Pi_n} | \mathcal{M}(\psi \bar{\psi} \rightarrow n\theta) |^2 \left| \sqrt{\pi} = \Lambda^* \right. \sim \pi^2 ,
$$

(3.17)

where $\int_{\Pi_n}$ refers to the integral over the final $n$-body phase space, and the factor of $1/n!$ takes care of the fact that all $n$ particles in the final state are identical. The scale $\Lambda^*$ provides an upper bound on the cutoff scale at which the theory requires UV-completion.

Allowing for the possibility of several fermion species with the same $|Q_L - Q_R|$, we choose our initial state to be the $s$-wave flavor- and spin-singlet:

$$
|\psi \bar{\psi}\rangle \equiv \frac{1}{\sqrt{2N}} \sum_{a=1}^{N} |\psi^a_+ \bar{\psi}^a_+ - \psi^a_- \bar{\psi}^a_- \rangle_{l=0} .
$$

(3.18)

This specific choice is of course not necessary to derive a unitarity bound. However, choosing the initial state to be a spin-eigenstate is convenient since in that case only operators with $n$ being either even or odd lead to a non-vanishing contribution, simplifying our analysis. Including a (conveniently normalized) sum over fermion flavors allows us to keep track of factors of $N$, which will be important when $N \gg 1$. Choosing a spherical wave (as opposed to a plane wave) we ensure that our initial state is correctly normalized, and a bound of the form Eq.(3.17) applies.  

With our initial state as in Eq.(3.18), we have, for even $n$:

$$
i\mathcal{M}(\psi \bar{\psi} \rightarrow n\theta) = -\frac{i}{\sqrt{16\pi} \sqrt{2N}} \sum_{a=1}^{N} \left[ \bar{v}^a_+(p_2)u^a_+(p_1) - \bar{v}^a_-(p_2)u^a_-(p_1) \right] 
\simeq -\frac{i}{\sqrt{16\pi}} \sqrt{2N} \frac{M_f}{f^n} \sqrt{s} ,
$$

(3.19)

\footnote{Of course, any other choice of $l$ is equally valid. The only effect of choosing an $l = 0$ state is to make the scattering amplitude smaller by a factor of $\sqrt{16\pi}$ compared to the analogous plane-wave calculation.}
where we have used \( \bar{v}_\pm(p_2)u_\pm(p_1) \simeq \pm \sqrt{s} \) at large momentum. The asymptotic expression for the volume of the \( n \)-body phase space factor when all particles in the final state are massless [19, 22] is given by
\[
\int_{\Pi_n} \simeq \frac{2\pi}{(4\pi)^{2(n-1)}(n-1)!(n-2)!} s^{n-2}.
\] (3.20)

Up to overall \( \mathcal{O}(1) \) corrections, Eq.(3.17) can then be written as
\[
\sum_{n=2}^{\infty} \frac{1}{n!(n-1)!(n-2)!} \left( \frac{\Lambda_\ast}{4\pi f} \right)^{2(n-1)} \equiv \mathcal{F} \left( \frac{\Lambda_\ast}{4\pi f} \right) \sim \left( \frac{4\pi f}{\sqrt{N M_f}} \right)^2,
\] (3.21)

where \( \mathcal{F}(x) \equiv \frac{x^2}{2} F_5(\frac{1}{2}, 1, \frac{3}{2}, \frac{3}{2}, 2; \frac{x^4}{4f^2}), \) and \( F_q(b_1, \ldots, b_q; z) \) is a generalized hypergeometric function. We do not need the specific form of \( \mathcal{F}(x) \), except for noting that it is a monotonically increasing function, with asymptotic expansions at small and large \( x \) (up to irrelevant overall \( \mathcal{O}(1) \) factors):
\[
\mathcal{F}(x) \sim \begin{cases} x^2 & \text{for } x \lesssim 1 \\ e^{3x^{2/3}} & \text{for } x \gtrsim 1 \end{cases}
\] (3.22)

In the regime where \( M_f \lesssim 4\pi f/\sqrt{N} \), the right-hand-side of Eq.(3.21) is always \( \gtrsim 1 \), and so it is appropriate to expand \( \mathcal{F}(x) \) for \( x \gtrsim 1 \). Eq.(3.21) then reads
\[
x^{3x^{2/3}} \left|_{x=\frac{\Lambda_\ast}{4\pi f}} \right. \sim \left( \frac{4\pi f}{\sqrt{NM_f}} \right)^2.
\] (3.23)

Thus, up to \( \mathcal{O}(1) \) corrections, we have
\[
\Lambda_\ast \sim 4\pi f \left( \log \frac{4\pi f}{\sqrt{NM_f}} \right)^{3/2}.
\] (3.24)

An a priori weaker bound could be obtained by applying the perturbativity bound to each term in Eq.(3.17), instead of performing the full sum over \( n \), and this highlights how different values of \( n \) dominate the overall bound in the different regimes. For instance, the case \( n = 2 \) already implies a non-trivial bound:
\[
\Lambda_\ast^{(n=2)} \sim 4\pi f \frac{4\pi f}{\sqrt{NM_f}},
\] (3.25)

which agrees with Eq.(3.24) when \( M_f \sim 4\pi f/\sqrt{N} \), but leads to a much weaker bound in the limit of small fermion mass \( M_f \ll 4\pi f/\sqrt{N} \). In general, we may impose the perturbativity bound on the \( n \)-th term in Eq.(3.17). After expanding for large values of \( n \), using Stirling’s approximation \( n! \simeq (\frac{n}{e})^n \sqrt{2\pi n} \), we find
\[
\Lambda_\ast^{(n)} \sim 4\pi f \left( \frac{n-1}{e} \right)^3 \left\{ \left( 2\pi(n-1) \right)^{3/2} \left( \frac{4\pi f}{\sqrt{NM_f}} \right)^2 \right\}^{\frac{1}{n(n-1)}}.
\] (3.26)
The value of $n$ for which the bound is the strongest is given by

$$n^* - 1 \simeq \frac{2}{3} \log \frac{4\pi f}{\sqrt{NM_f}} \sim \log \frac{4\pi f}{\sqrt{NM_f}}. \tag{3.27}$$

which is large for $M_f \ll 4\pi f/\sqrt{N}$, justifying our earlier approximation. Evaluating Eq.(3.26) for $n = n^*$, we find

$$\Lambda_n^{(n=n^*)} \sim 4\pi f \left( \log \frac{4\pi f}{\sqrt{NM_f}} \right)^{3/2}, \tag{3.28}$$

which reproduces Eq.(3.24).

Our bound Eq.(3.24) applies so long as all massive fermion species appear roughly at the same scale $M_f$, and the value of $q_i \equiv |Q_L^{(i)} - Q_R^{(i)}|$ is parametrically of the same size for all $i = 1, \cdots, N$. If the $q_i$ are parametrically different, then the bound will be dominated by the species with the largest $q_i = q_{\text{max}}$. In this case, the strongest bound can be obtained by choosing an initial state that involves only the fermion with the largest $q_i$ (instead of summing over flavors, as in Eq.(3.18)), and the resulting bound is just Eq.(3.24) with $N \to 1$, and $f$ as given in Eq.(3.4) with $|Q_L - Q_R| \to q_{\text{max}}$.

### 3.3 No Stückelberg limit

The form of Eq.(3.24) is consistent with our expectations: $\Lambda_\ast \to \infty$ independently in the limits $|Q_L - Q_R| \to 0$, and $M_f \to 0$. In the former case, all massive fermions couple to the gauge field through a vector current, all (potentially) chiral fermions are massless, and a Stückelberg limit exists. The limit $M_f \to 0$ also allows for a Stückelberg limit but the cutoff scale decouples much slower than before. Rewriting Eq.(3.24) as

$$M_f \sim \frac{4\pi f}{\sqrt{N}} e^{-\left( \frac{\Lambda_\ast}{4\pi f} \right)^{2/3}} \tag{3.29}$$

it becomes apparent that a cutoff scale that is parametrically above $4\pi f$ requires the massive chiral fermions to be exponentially light. As noted earlier, in this case the existence of other UV-dependent irrelevant operators whose coefficients are independent of $M_f$ additionally requires $f \to \infty$ or $c_i \to 0 \forall i$ to prevent them from independently obstructing the Stückelberg limit.

More generally, whenever a $U(1)$ gauge theory contains massive fermions with chiral charge assignments, i.e. both $M_f \neq 0$ and $Q_L \neq Q_R$, a Stückelberg limit is not allowed, and the theory requires UV-completion at a finite scale $\Lambda_\ast$. For example, if the UV-completion is in the form of an Abelian Higgs model with vev $f$ and charge $q = Q_L - Q_R$, the radial mode will appear at scale $m_{\rho} \lesssim 4\pi f$. The derivative couplings between the radial mode and $\theta$ are crucial in order to restore perturbation theory in $\psi \bar{\psi}$ scattering at high energies. For instance, for $n = 2$, the right diagram on Figure 5 precisely cancels the high-energy piece of the left diagram, due to the term $\mathcal{L} \supset \frac{f}{\rho} (\partial_{\mu} \theta)^2$. 

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consistent, four-dimensional gauge theories with anomalous fermion content in the infrared
necessarily feature massive chiral fermions responsible for canceling the anomalies of the
low-energy spectrum. They are correspondingly subject to the general constraints obtained
in section 3. In this section, we discuss the implications of these constraints for various
classes of Abelian gauge theories that appear anomalous below a certain scale. We begin
with purely non-gravitational phenomena, focusing on the effect of the \( U(1)^3 \) anomaly
in section 4.1. We turn on gravity in section 4.2, and discuss the implications of a low-
energy fermion spectrum featuring a mixed \( U(1) \)-gravitational anomaly. Finally, in 4.3, we
consider the more generic case where both anomalies are present, and elaborate on their
surprising interplay.

4.1 \( U(1)^3 \) anomaly

Picking up where we left off in section 2, we focus first on an anomalous EFT that contains
a single massless fermion coupling to \( A_\mu \) through a left-handed current. As advertised, we
will first neglect gravitational interactions in our discussion, i.e. we ignore the presence of
a mixed gravitational anomaly, as well as any potential extra requirements stemming from
quantum-gravity consistency such as compactness of the gauge group. The \( U(1)^3 \) anomaly
of the anomalous EFT can be cancelled by introducing a single massive fermion with chiral
charge assignments \( Q_L \) and \( Q_R \) such that

\[
\sum_{i} Q_i^3 = Q^3 + Q_L^3 - Q_R^3 = 0 .
\] (4.1)

As per Fermat’s Last Theorem, the only integer solutions to this equation are either \( \{Q_R = Q, Q_L = 0\} \) or \( \{Q_R = 0, Q_L = -Q\} \). However, since we are ignoring gravitational effects for
the time being, we allow ourselves to entertain non-integer solutions to Eq.(4.1). WLOG,
we set \( q \equiv Q_R - Q_L = 1 \) in what follows. (Factors of \( q \) may be restored by performing the
simultaneous rescaling \( Q_i \to Q_i/q \) and \( g_0 \to g_0q \).)

An upper bound on the cutoff scale of this anomaly-free extension is given by our result
Eq.(3.24). So long as the mass of the heavy fermion is not exponentially small, we have

\[
\Lambda_* \sim \frac{4\pi m_*}{g_0} = \frac{4\pi m_*}{g} Q .
\] (4.2)
Probing the regime $\Lambda_* \gg 4\pi m_\gamma/g$ therefore requires $Q \gg 1$, and a solution to Eq.(4.1) in this limit implies $Q_L \simeq Q^3/\sqrt{3}$. Taking $Q$ large, we can then push $\Lambda_*$, as well as the mass of the heavy fermion, parametrically above $\sim 4\pi m_\gamma/g$. An upper bound on $Q$, however, stems from the requirement that the heavy fermion remains weakly coupled, parametrically:

$$\frac{g_0^2 (Q_L^2 + Q_R^2)}{16\pi^2} \sim \frac{g_0^2 Q^3}{16\pi^2} \lesssim 1 \quad \Rightarrow \quad Q \lesssim \frac{16\pi^2}{g^2}.$$  \hspace{1cm} (4.3)

With this requirement, we obtain an upper bound on the scale of UV-completion, of the form

$$\Lambda_* \lesssim \frac{64\pi^3 m_\gamma}{g^3},$$  \hspace{1cm} (4.4)

which coincides with $\Lambda_{U(1)^3}$ in Eq.(2.5). Since $M_f \lesssim \Lambda_*$, this allows us to saturate the upper bound on the cutoff of the anomalous EFT obtained in [7], at the cost of a further UV-completion incorporating a radial mode appearing roughly at the same scale.

Although we have focused the discussion on an anomalous EFT with a single left-handed charged fermion, our conclusion applies more generally in the context of anomalous EFTs with $U(1)^3$ anomalies so long as the infrared fermion spectrum does not feature wild hierarchies of charges. Even with gravity turned on, the above results apply parametrically so long as $\sum_{\text{IR}} Q_i = 0$, and $\sum_{\text{IR}} Q_i^3 \sim Q^3$, with $Q$ the typical charge in the low-energy spectrum. A concrete example is a theory with massless left-handed fermions carrying charges $Q$, $Q$, and $-2Q$. This theory has a vanishing $U(1)$-gravitational anomaly, whereas $\sum_{\text{IR}} Q_i^3 = -6Q^3 \sim -Q^3$. We can extend this theory into one free of anomalies by introducing two massive chiral fermions, with charges such that

$$Q_R^{(1)} - Q_L^{(1)} = -\left(Q_R^{(2)} - Q_L^{(2)}\right) = 1.$$  \hspace{1cm} (4.5)

(As before, we have set the right-hand-side above to unity WLOG.) With this charge assignment, the heavy fermions do not introduce a mixed gravitational anomaly. Since $N = 2 = O(1)$, the upper bound Eq.(4.2) also applies in this case, and $Q \gg 1$ is required to achieve a parametrically high cutoff. As in our previous example, an upper bound on $Q$ stems from the requirement that the massive fermions remain weakly coupled, while at the same time having appropriate charge assignments so as to cancel the $U(1)^3$ anomaly. The optimal choice of charges, i.e. allowing for the largest $Q$ while maintaining perturbativity, are such that $Q_L^{(2)} \gg Q_L^{(1)}$, in which case anomaly-cancelation requires $Q_L^{(2)} \simeq \sqrt{2}Q^{3/2} \gg 1$. (Unlike in our previous example, integer solutions to the anomaly equations now exist for values $Q \gg 1$.) The resulting upper bound on $Q$ from the requirement of weak coupling is, again, given by Eq.(4.3), and therefore $\Lambda_* \lesssim 64\pi^3 m_\gamma/g^3$ follows.

\subsection*{4.2 Mixed $U(1)$-gravitational anomaly}

We now focus on the implications of a mixed $U(1)$-gravitational anomaly by considering theories such that $\sum_{\text{IR}} Q_i^3 = 0$ but $\sum_{\text{IR}} Q_i \sim Q$, with $Q$ the typical charge of the low-energy spectrum. Although such charge assignments may seem non-generic, the purpose
of this section is purely to illustrate the effect of the mixed gravitational anomaly, without
distractions stemming from additional requirements imposed by the presence of a $U(1)^3$
anomaly. A specific example of this kind is the ‘taxicab number’ theory with massless
left-handed fermions carrying charges $-Q, -12Q, 9Q, \text{ and } 10Q$.

Cancelling the gravitational anomaly without introducing a $U(1)^3$ anomaly requires
more than a single massive fermion. In general, we may introduce a number $N$ of chiral
fermion species, with charge assignments

$$Q^{(i)}_R - Q^{(i)}_L \equiv q_i. \quad (4.6)$$

The requirement that the massive fermions cancel the mixed gravitational anomaly of the
low-energy spectrum can be written as

$$q_1 + \cdots + q_N = \sum_{\text{IR}} Q_i \sim Q. \quad (4.7)$$

If one, or an $O(1)$ number, of the $q_i$ is much larger than the rest, then satisfying Eq.\(4.7\)
requires $q_{\text{max}} \sim Q$. Eq.\(3.24\) then applies with $N \sim 1$ and $|Q_L - Q_R| \sim q_{\text{max}} \sim Q$, leading
to $\Lambda_* \lesssim 4\pi m_\gamma / g$ — a parametric separation of scales is not possible if an $O(1)$ number of
heavy fermions is responsible for cancelling the anomaly of the low-energy EFT. Instead,
probing the regime of a parametrically large cutoff in a theory with a mixed gravitational anomaly in the infrared requires the presence of a parametrically large number of massive fermion species, all of which contribute significantly to the anomaly.

The optimal charge assignment, i.e. requiring the smallest number of species for a
given $Q$, is such that the largest possible number of $q_i$ have equal sign, adding coherently
to cancel the anomaly of the low-energy EFT. The case with $q_i = q$ for all $i = 1, \cdots , N$
does not allow for the massive fermions to be themselves free of $U(1)^3$ anomalies. However,
it will generally be enough to have $q_i = q$ for an $O(1)$ fraction of all the fermion species,
with a small number of the $q_i$ having opposite sign. Setting $q = 1$ as usual, this essentially implies

$$N \sim Q. \quad (4.8)$$

Through the requirement that the $U(1)$-gravitational anomaly of the low-energy spectrum
is cancelled by the heavy fermions, the anomaly links a large ratio of charges, required to
achieve a large separation of scales, to a large number of species. Parametrically, the cutoff
of our anomaly-free extension can now be written as

$$\Lambda_* \lesssim \frac{4\pi m_\gamma}{g} Q \sim \frac{4\pi m_\gamma}{g} N. \quad (4.9)$$

An upper bound on $N$ immediately follows from the requirement that the scale of
UV-completion falls bellow the quantum gravity scale, $\Lambda_{QG} \sim 4\pi M_{Pl}/\sqrt{N}$. Demanding

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\(^4\)For our taxicab number theory, Eq.(4.7) reads $q_1 + \cdots + q_N = 6Q \sim Q$.

\(^5\)Again, for our taxicab number example, it is enough to have $q_i = 1$ for $i = 1, \cdots , N - 1$, and $q_N = -1$. In this case, integer solutions to the anomaly equations exist, and Eq.(4.7) reads $N - 2 = 6Q \Rightarrow Q = (N - 2)/6 \sim N$. 

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Figure 6: The cutoff $\Lambda_*$ of an anomaly-free theory that (partially) UV-completes an anomalous EFT with a mixed $U(1)$-gravitational anomaly can be taken large at the cost of increasing the number $N$ of massive fermion species responsible for rendering the theory anomaly-free. In turn, this lowers the quantum gravity scale parametrically below $M_{Pl}$. The cutoff of the anomaly-free theory may be taken to saturate the apparent cutoff of the anomalous EFT, $\Lambda_{\text{grav}}$, and, in that limit, all scales coincide: $\Lambda_* \sim \Lambda_{\text{grav}} \sim \Lambda_{\text{QG}}$.

that $\Lambda_* \lesssim \Lambda_{\text{QG}}$, we find

$$N \lesssim \left( \frac{g M_{Pl}}{m_{\gamma}} \right)^{2/3}. \quad (4.10)$$

Plugging this back in Eq.(4.9), we obtain an upper bound on the cutoff of the anomaly-free EFT, of the form

$$\Lambda_* \lesssim 4\pi \left( \frac{M_{Pl}^2 m_{\gamma}}{g} \right)^{1/3}, \quad (4.11)$$

which is precisely the upper bound on the cutoff scale of the anomalous EFT obtained in [7, 9], i.e. $\Lambda_{\text{grav}}$ in Eq.(2.7). More generally, it is illuminating to obtain upper bounds on $\Lambda_*$ and $\Lambda_{\text{QG}}$, which can be written in the following form

$$\Lambda_* \lesssim 4\pi \left( \frac{M_{Pl}^2 m_{\gamma}}{g} \right)^{1/3} \left( \frac{\Lambda_*}{\Lambda_{\text{QG}}} \right)^{2/3} \quad \text{and} \quad \Lambda_{\text{QG}} \lesssim 4\pi \left( \frac{M_{Pl}^2 m_{\gamma}}{g} \right)^{1/3} \left( \frac{\Lambda_{\text{QG}}}{\Lambda_*} \right)^{1/3}. \quad (4.12)$$

This highlights how pushing $\Lambda_*$ (morally, the mass of the radial mode) up comes at the cost of bringing the quantum gravity cutoff down to the same scale. This is further illustrated in Figure 6.

Our analysis shows how the upper bound on the cutoff of an anomalous EFT featuring a $U(1)$-gravitational anomaly obtained by [7, 9] is, in some sense, self-imposed by the theory itself. To the extent that the scale of UV-completion must lie below the quantum gravity scale, the lowering of $\Lambda_{\text{QG}}$ in theories with a large number of species self-consistently imposes the bound Eq.(2.7) on the scale of the heavy fermions. Referring back to Figure 2, this implies that the Swampland of theories living in the orange wedge cannot be accessed from the Landscape of theories lying below.
4.3 Both Abelian anomalies

We now turn to the more generic case where the low energy spectrum features both a $U(1)^3$ anomaly and a mixed gravitational anomaly. We will focus on the example of a theory with a massless left-handed fermion, but as before our conclusions will apply more generally so long as $\sum Q^3 \sim Q^3$ and $\sum Q_i \sim Q_i$. Introducing $N$ massive fermion species such that 

$$Q^R_i - Q^L_i \equiv 1 \quad \forall \ i = 1, \cdots, N \quad (4.13)$$

is now enough for the heavy fermions to cancel both anomalies. Solving both the gravitational and $U(1)^3$ anomalies requires, respectively:

$$Q = N, \quad \text{and} \quad (4.14)$$

$$Q^3 = N \left(3Q^2_L + 3Q_L + 1\right), \quad (4.15)$$

where for simplicity we have assumed that $Q^L_i = Q_L \forall i$. In the regime where $Q = N \gg 1$ necessary to allow for a large separation of scales, $Q_L \simeq N/\sqrt{3}$, and integer solutions exists for certain values of $N \gg 1$.

As in section 4.2, an upper bound on $N$ as given in Eq.(4.10) follows from the requirement that the theory be UV-completed below the quantum gravity scale. However, the requirement that the heavy fermions be weakly coupled now sets an additional upper bound on $N$, of the form

$$g^2 N \left( Q^2_L + Q^2_R \right) \gtrsim \frac{g^2 N^3}{16\pi^2} \lesssim 1 \quad \Rightarrow \quad N \lesssim \left( \frac{16\pi^2}{g^2} \right)^{1/3} \lesssim \frac{16\pi^2}{g^2}, \quad (4.16)$$

which is just Eq.(4.3) after setting $Q = N$.

Comparing Eq.(4.10) and Eq.(4.16), we identify a critical value of the gauge coupling:

$$g_* \sim \left( \frac{64\pi^3 m_c}{M_{Pl}} \right)^{1/4}. \quad (4.17)$$

When $g \lesssim g_*$, the upper bound on Eq.(4.10) is the most stringent, and the conclusions of section 4.2 apply in this regime: $\Lambda_*$ can saturate the cutoff scale of the anomalous EFT, which corresponds to $\Lambda_{grav} \sim 4\pi \left( M_{Pl}^2 m_c/g \right)^{1/3}$ in this regime, at the cost of bringing the quantum gravity cutoff down to the same scale. On the other hand, in the regime $g \gtrsim g_*$, the upper bound in Eq.(4.16) is dominant, in turn implying $\Lambda_* \lesssim 64\pi^3 m_c/g^3$, which again coincides with the upper bound on the cutoff of the anomalous EFT for this range of couplings. Overall: $\Lambda_*$ can be taken to saturate min\{\Lambda_{grav}, \Lambda_{U(1)^3}\}.

Intriguingly, in the regime where $g \gtrsim g_*$ the scale of quantum gravity $\Lambda_{QG}$ is also subject to a surprising constraint of its own. On the one hand, the upper bound on $N$ in Eq.(4.16) implies $\Lambda_{QG} \gtrsim gM_{Pl}$. On the other hand, after identifying $N \sim (4\pi M_{Pl}/\Lambda_{QG})^2$, Eq.(4.9) can be rewritten as an upper bound on $\Lambda_{QG}$. In combination, these bounds give

$$gM_{Pl} \lesssim \Lambda_{QG} \lesssim gM_{Pl} \sqrt{\frac{64\pi^3 m_c/g^3}{\Lambda_*}}. \quad (4.18)$$
Figure 7: Upper bound on $\Lambda_*$ (purple line; to be identified with the scale at which a radial mode appears), and corresponding behavior of the quantum gravity scale (green line), relative to the cutoffs of the anomalous EFT ($\Lambda_{\text{grav}}$ and $\Lambda_{U(1)^3}$) in a theory featuring both $U(1)^3$ and mixed gravitational anomalies. For $g \lesssim g_*$, the maximum value of $\Lambda_*$ simultaneously coincides with both $\Lambda_{\text{grav}}$ and $\Lambda_{\text{QG}}$. For $g \gtrsim g_*$, the upper bound on $\Lambda_*$ coincides with $\Lambda_{U(1)^3}$, while $\Lambda_{\text{QG}}$ matches the apparent WGC scale $g M_{\text{Pl}}$ in the same limit. (Both axes are in a log scale.)

Thus, saturating the upper bound on the scale of UV-completion $\Lambda_*$ entails bringing the quantum gravity cutoff down to coincide with $g M_{\text{Pl}}$ — the WGC scale as seen in the infrared. Moreover, in this regime $g M_{\text{Pl}} \sim g_0^{1/3} M_{\text{Pl}}$ (remember Eq.(4.16)), which is the form of the ‘magnetic’ WGC scale that has been advocated for in [12]. The upper bound on $\Lambda_*$, and the behavior of $\Lambda_{\text{QG}}$ in the various coupling regimes, are illustrated in Figure 7.

There are a variety of noteworthy features in this result. The appearance of the WGC scale $g M_{\text{Pl}}$ in association with the scale of quantum gravity was due not to any direct applications of the WGC, but rather a direct consequence of the large number of species required to probe effective non-compactness of the anomalous $U(1)$. Even more surprising is the fact that the WGC scale emerges in a largely field-theoretical example featuring a massive photon, where the direct applicability of the WGC remains conjectural [23]. Finally, although the WGC scale appearing in Eq.(4.18) plays the role of the quantum gravity cutoff only for $g \gtrsim g_*$, note that $g_* \to 0$ in the limit of a vanishing photon mass.

5 Conclusions

The study of chiral gauge theories — the Standard Model being a prime example — has provided deep insight into general aspects of quantum field theory, as well as concrete understanding of phenomena realised in nature. Moreover, it is in this context that the fundamental differences between Abelian and non-Abelian theories become most apparent. In particular, the existence of mixed $U(1)$-gravitational anomalies — absent in the non-
Abelian case — provides hope that further scrutiny of this class of theories may provide some insight into the properties of Abelian gauge theories consistently coupled to gravity.

In this paper, building on the seminal work of [7], we have focused on four-dimensional, massive Abelian gauge theories that are anomaly-free but for which anomaly-cancellation occurs due to fermions appearing at different scales. We have shown that the presence of massive chiral fermions leads to an upper bound on the scale below which a radial mode must become part of the spectrum, which cannot be decoupled: This is an example of a class of massive U(1) gauge theories for which a Stückelberg limit is not allowed.

Maximizing the separation of scales between the massive photon and the scale of the radial mode requires wandering into the (morally) non-compact limit, by introducing a parametrically large ratio of charges in the UV-completion. We have shown that when a U(1)-gravitational anomaly is present in the low-energy theory, such cavalier behaviour is automatically penalized by the quantum gravity scale appearing parametrically below $M_{Pl}$, and that the theory itself self-consistently precludes falling into the Swampland where the massive fermions responsible for cancelling the anomaly of the low-energy spectrum appear above the cutoff of the anomalous EFT. In this example, the Landscape and the Swampland are not smoothly connected. (See [24] for another example in a similar spirit.)

Even more strikingly, in the presence of both a U(1) and a mixed gravitational anomaly, there exists a critical value of the gauge coupling, $g_*$. In the regime $g \gtrsim g_*$, saturating the upper bound on the mass of the radial mode comes at the cost of lowering the quantum gravity scale down to $\Lambda_{QG} \sim g M_{Pl}$, which coincides with the WGC scale as seen from the low-energy EFT. Our work therefore provides a four-dimensional, field-theoretic example of the WGC scale emerging in the role of a quantum gravity cutoff, in a massive Abelian gauge theory, tied to the presence of a large number of species required to probe the non-compact limit.

Our results resonate with those of [23], where an attempt was made to understand the difficulties of realizing tiny photon masses in UV-completions that include gravity by studying the properties of string theory constructions where the photon mass is non-zero everywhere in field space, and which appear qualitatively different from standard Higgs-like models. Heuristic arguments involving the expectation that large number of degrees of freedom must become part of the low-energy theory when wandering over large distances in field space, as well as the corresponding lowering of the quantum gravity cutoff in a theory with a large number of species, make for a compelling argument that the limit of tiny photon masses is problematic. Drawing from various conjectures established to varying levels of rigour, [23] further suggests that in such constructions a radial mode must be part of the spectrum, and advocates for demanding that the theory satisfies the WGC even though the photon mass is non-vanishing. Our work provides motivation to further investigate the suggestions of [23], in the context of string theory constructions with massive photons that feature anomalous infrared fermion content [25–29], which we leave to future work.

Our work opens several avenues for further exploration. For example, in dimensions higher than four, there exist additional possibilities for cancelling gauge anomalies that do not require the presence of heavy fermions, such as the ten-dimensional Green-Schwarz mechanism [30] and variations thereof [25, 31]. Extending our work to study
UV-completions in dimensions higher than four could provide more insight into the structure of this class of massive $U(1)$s. Last but not least, we would be remiss to not mention the ubiquity of massive photons in various extensions of the Standard Model, most notably as mediators for dark matter, or as dark matter itself. If these theories were to contain an anomaly canceled by massive chiral fermions (see e.g. [32] for work along these lines), our result would apply and could restrict the validity of some of these approaches.

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