A HOMOGENEOUS GIBBONS–HAWKING ANSATZ AND BLASCHKE PRODUCTS

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Abstract. A homogeneous Gibbons–Hawking ansatz is described, leading to 4-dimensional hyperkähler metrics with homotheties. In combination with Blaschke products on the unit disc in the complex plane, this ansatz allows one to construct infinite-dimensional families of such hyperkähler metrics that are, in a suitable sense, complete. Our construction also gives rise to incomplete metrics on 3-dimensional contact manifolds that induce complete Carnot–Carathéodory distances.

1. Introduction

The aim of this paper is to present an intriguing construction of hyperkähler structures.

In [6, Theorem 10] we exhibited a global rigidity of 4-dimensional hyperkähler metrics: if such a metric $g$ admits a homothetic vector field with a compact transversal, then $g$ is flat. We proved this by an argument involving the integral formula for the signature of a compact 4-manifold, applied to a quotient of a neighbourhood of the transversal. That line of reasoning obviously suggests the question whether the compactness hypothesis can be weakened to a completeness condition.

In the present paper we define the natural notion of completeness for a Riemannian metric with a homothety (slice-completeness) and give a construction leading to non-flat, slice-complete hyperkähler structures. In Section 2 we discuss a homogeneous Gibbons–Hawking ansatz, which forms the basis for our construction. In Section 3 we derive explicit formulæ for various metrics that arise in this construction. The main part of the construction of our examples is contained in Section 4.

In view of a related incompleteness result from [6], see Theorem 8 below, it was to be expected (and is confirmed here) that the construction of such slice-complete examples would be quite delicate. So it is all the more surprising that our construction actually yields an infinite-dimensional family of isometry classes of such structures (Section 5). In Section 6 it is shown that our construction does indeed give rise to non-flat hyperkähler metrics. In Section 7 we relate our construction to the theory of taut contact spheres developed in [6]; this relation originally motivated the search for the hyperkähler metrics described here. In that context we describe another surprising phenomenon, namely, examples of incomplete Riemannian metrics giving rise to complete Carnot–Carathéodory distances.

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2. A homogeneous Gibbons–Hawking ansatz

The Gibbons–Hawking ansatz \[7\] allows one to construct hyperkähler metrics with an $S^1$-invariance. We want to study metrics arising from this ansatz, subject to an additional homogeneity property amounting to the existence of a homothetic vector field.

**Definition.** A vector field $Y$ is **homothetic** for the Riemannian metric $g$ if it satisfies $L_Y g = g$. The **canonical slice** corresponding to such a vector field is the subset defined by the equation $g(Y,Y) = 1$.

The canonical slice is a hypersurface transverse to $Y$ and, if the flow of $Y$ is complete, it intersects each orbit exactly once. For a cone metric $g = e^{2s}(ds^2 + g)$ and $Y = \partial_s$, the canonical slice is the hypersurface $\{s = 0\}$ orthogonal to $Y$.

Most homothetic fields, however, are not orthogonal to any hypersurface; in such situations our definition still gives a natural choice of transversal.

A Riemannian metric on a product $M \times \mathbb{R}$ with translation along the $\mathbb{R}$-factor as homotheties is necessarily incomplete in the $\mathbb{R}$-direction: proper paths of the form $\{p\} \times (-\infty, s_0] \subset M \times \mathbb{R}$ have finite length in such a metric. Therefore, the best one can aim for is completeness in the transverse directions.

**Definition.** A Riemannian metric on a product $M \times \mathbb{R}$ with translation along the $\mathbb{R}$-factor as homotheties is **slice-complete** if the canonical slice is complete in the induced metric.

For the construction of our examples, we shall be working on a 4-manifold $W$ of the form $W = \Sigma \times \mathbb{R}_t \times S^1_{\theta}$ with $\Sigma$ an open surface; the subscripts denote the respective coordinates. We look for hyperkähler structures $(g, \Omega_1, \Omega_2, \Omega_3)$ with the following properties:

(i) The flow of $\partial_{\theta}$ preserves the metric $g$, and $\partial_{\theta}$ is a Hamiltonian vector field for each of the symplectic forms $\Omega_i$.

(ii) The vector field $\partial_t$ satisfies $L_{\partial_t} \Omega_i = \Omega_i$, $i = 1, 2, 3$, hence also $L_{\partial_t} g = g$.

Notice that this is stronger than just being homothetic.

The partial differential equations for a hyperkähler structure linearise under condition (i) to the 3-dimensional Laplace equation. Under the additional condition (ii), one can reduce these equations further to the Cauchy–Riemann equations in real dimension 2. We next expand on these two claims.

Given a 3-manifold $M$, any hyperkähler structure on the product $M \times S^1_{\theta}$ satisfying condition (i) can be described by the Gibbons–Hawking ansatz. In this ansatz, one selects Hamiltonian functions $x_1, x_2, x_3$ such that $dx_i = \partial_{\theta} \mathcal{J} \Omega_i$. Then there exist a unique 1-form $\eta$ and a unique positive function $V$ giving the following expressions for the symplectic forms $(\Omega_1, \Omega_2, \Omega_3)$, where $(i, j, k)$ runs over the cyclic permutations of $(1, 2, 3)$:

(1) \[ \Omega_i = (d\theta + \eta) \wedge dx_i + V \, dx_j \wedge dx_k, \]

and the following one for the hyperkähler metric:

(2) \[ g = V^{-1} \cdot (d\theta + \eta)^2 + V \cdot (dx_1^2 + dx_2^2 + dx_3^2). \]

Here the forms $dx_1, dx_2, dx_3$ are a basis for the annihilator of $\partial_{\theta}$, and so $(x_1, x_2, x_3)$ are (at least locally) coordinates for the orbit space of $\partial_{\theta}$. The function $V$ satisfies $\partial_{\theta} V \equiv 0$ and is thus locally a function of only $(x_1, x_2, x_3)$. The 1-form $\eta$ annihilates
$\partial \theta$ and is invariant under its flow, so it is locally pulled back from the orbit space of $\partial \theta$. All this means that $\eta$ and $V$ are locally objects on $(x_1, x_2, x_3)$-space, and we shall treat them as such for the purpose of local calculations.

The projection onto the orbit space of $\partial \theta$ is a Riemannian submersion from the metric $(1/V) g = g(\partial \theta, \partial \theta) g$ to the Euclidean metric $dx_1^2 + dx_2^2 + dx_3^2$. The condition for (1) to define a triple of closed 2-forms is then

$$d\eta + \ast dV = 0,$$

where the Hodge star operator is in terms of that Euclidean metric and the orientation defined by $(dx_1, dx_2, dx_3)$. For the (local) existence of $\eta$ it is necessary and sufficient that $V$ be harmonic with respect to the Euclidean metric. We call $V$ the Gibbons–Hawking potential.

The systematic construction of a hyperkähler structure on $M \times S^3$ proceeds as follows. Start with a local diffeomorphism $x = (x_1, x_2, x_3) : M \rightarrow \mathbb{R}^3$, and consider the metric $x^* g_{\mathbb{R}^3}$, where $g_{\mathbb{R}^3}$ is the standard Euclidean metric on $\mathbb{R}^3$. Use $x$ also to pull the standard orientation of $\mathbb{R}^3$ back to $M$. Let now $\eta$ and $V$ be a 1-form and a function, respectively, defined on $M$ and satisfying (3) with respect to the metric $x^* g_{\mathbb{R}^3}$ and the pulled-back orientation. By lifting $x, \eta, V$ to $M \times S^3$ in the obvious way, and inserting them into the defining equations (1) and (2), we obtain a hyperkähler structure on $M \times S^3$ invariant under the flow of $\partial \theta$.

One may regard $(M, x^* g_{\mathbb{R}^3})$ as a non-schlicht domain in $\mathbb{R}^3$, and $\eta$ and $V$ as multiple-valued objects in $\mathbb{R}^3$. On $M$, however, they are perfectly well defined, and so equation (3) can be read as an identity on $M$.

We now restrict our attention to 3-manifolds $M$ of the form $M = \Sigma \times \mathbb{R}$, with $\Sigma$ an open surface, and impose both conditions (i) and (ii). The following definition is useful for describing the special features of this case.

**Definition.** A tensorial object $o$ (on $W = M \times S^3$ or on $M$) is called homogeneous of degree $k$ if $L_{\partial \theta} o = k \cdot o$.

Condition (ii) requires that $g$ and the symplectic forms be homogeneous of degree 1. Since $\partial \theta$ is invariant under the flow of $\partial \theta$, the potential $V = 1/g(\partial \theta, \partial \theta)$ must be homogeneous of degree $-1$. Condition (ii) provides a convenient choice for the Hamiltonian functions $x_i$, because one easily checks that $x_i := \Omega_i(\partial \theta, \partial \theta)$ satisfies $dx_i = \partial_{\theta_i} \Omega_i$ in this case. This choice has the virtue that the $x_i$ are homogeneous of degree 1, i.e. that $\partial_i x_i = x_i$. We call $x := (x_1, x_2, x_3) : \Sigma \times \mathbb{R} \rightarrow \mathbb{R}^3 \setminus \{0\}$ the momentum map. Then the equations $\partial_i x_i = x_i$ say that $\partial_i$ is $x$-related to the position vector field on $\mathbb{R}^3$.

The uniqueness of $\eta$ for given Hamiltonian functions $x_i$ implies that in the present situation both $\eta$ and the function $\eta(\partial_i)$ are homogeneous of degree 0. Consider now the function $\rho := |x| : \Sigma \times \mathbb{R} \rightarrow \mathbb{R}^+$. The 1-form $d\rho/\rho$ is homogeneous of degree 0 and satisfies $d\rho/\rho(\partial_t) \equiv 1$. Hence

$$\eta = \eta(\partial_t) \frac{d\rho}{\rho} + \xi,$$

where the 1-form $\xi$ is homogeneous of degree 0 (i.e. $\partial_t$-invariant), $\partial_{\theta}$-invariant, and it annihilates both $\partial_t$ and $\partial_{\theta}$. So $\xi$ is the pull-back of a 1-form on $\Sigma$, for which we continue to write $\xi$.

The map $x/|x| : \Sigma \times \mathbb{R} \rightarrow S^2$ is independent of $t$ and thus the pull-back of a unique map $\Phi : \Sigma \rightarrow S^2$. The latter is a local diffeomorphism. We endow the
unit sphere $S^2$ with the metric induced from the standard metric on $\mathbb{R}^3$, and with the orientation as boundary of the 3-ball. This determines on $S^2$ a holomorphic structure and a standard volume form $\text{Vol}_{S^2}$. We endow $\Sigma$ with the holomorphic structure $J$ lifted from $S^2$ by $\Phi$, i.e. the structure that turns $\Phi : \Sigma \to S^2$ into a local biholomorphism.

**Theorem 1** (The homogeneous Gibbons–Hawking ansatz). The function $V$ and the 1-form $\eta = \eta(\partial_t) \frac{\rho}{\rho} + \xi$ satisfy $\Box$ on $\Sigma \times \mathbb{R}_t$ if and only if the following two conditions are satisfied:

- $d\xi = \rho V \Phi^* \text{Vol}_{S^2}$,
- $\varphi := \eta(\partial_t) + i\rho V$ is the pullback to $\Sigma \times \mathbb{R}_t$ of a holomorphic function on $(\Sigma, J)$.

**Proof.** Let $\bar{u} + i\bar{v}$ be a local holomorphic coordinate on $S^2$, and let $u + iv$ be its pullback under $\Phi$. Then $u, v$ are homogeneous of degree 0 and $(\rho, u, v)$ are local coordinates on $\Sigma \times \mathbb{R}_t$ giving the same orientation as $(x_1, x_2, x_3)$.

Since $V$ is homogeneous of degree $-1$, we have $V_{\rho} = -\rho^{-1} V$, therefore

$$dV = -\rho^{-1} V d\rho + V_u du + V_v dv.$$  

On the other hand,

$$*d\rho = \rho^2 \Phi^* \text{Vol}_{S^2}, \quad *du = -d\rho \wedge dv, \quad *dv = d\rho \wedge du,$$

and so

$$*dV = -\rho V \Phi^* \text{Vol}_{S^2} - V_u d\rho \wedge dv + V_v d\rho \wedge du.$$

Since $\xi$ is homogeneous of degree 0 and annihilates $\partial_t$, we have $\xi = \xi_1(u, v) du + \xi_2(u, v) dv$ and

$$d\eta = (\xi_2 - \xi_1) du \wedge dv - \eta(\partial_t) u \frac{d\rho}{\rho} \wedge dv - \eta(\partial_t) v \frac{d\rho}{\rho} \wedge du.$$

Then $\Box$ is seen to be equivalent to the system

- $d\xi = \rho V \Phi^* \text{Vol}_{S^2}$,
- $\eta(\partial_t) u = \rho V_v = (\rho V)_v$,
- $\eta(\partial_t) v = -\rho V_u = -(\rho V)_u$.

The last two equations are the Cauchy–Riemann equations for $\varphi$. $\square$

In order to describe the systematic construction of hyperkähler structures satisfying (i) and (ii), it is convenient to fix the complex structure $J$ on $\Sigma$ in advance. We then use holomorphic data of the following kind to construct the desired structures:

- a local biholomorphism $\Phi : (\Sigma, J) \to S^2$,
- a holomorphic function $\varphi : (\Sigma, J) \to \mathbb{H}$ with values in the upper half-plane $\mathbb{H}$.

Being homogeneous of degree 1, the function $\rho$ must be of the form $\rho = e^t \rho_0$ with $\rho_0$ the pullback of a positive function on $\Sigma$. Given a choice of $\Phi, \varphi, \rho_0$, the construction is as follows. The momentum map is given by $x = \rho \Phi$. We take an antiderivative $\xi$ for $(\Im \varphi) \Phi^* \text{Vol}_{S^2}$ on $\Sigma$. Set $\eta = (\Re \varphi) \frac{d\rho}{\rho} + \xi$ and $V = (\Im \varphi) \rho^{-1}$, both pulled back to $W$. The hyperkähler structure is given by (1) and (2) with these values for $x, \eta, V$. 
Given any pair of functions $h_1, h_2: \Sigma \to \mathbb{R}$, the diffeomorphism of $W$ given by

$$(p, t, \theta) \mapsto (p, t + h_1(p), \theta + h_2(p))$$

preserves $\partial_t$ and $\partial_\theta$ and pulls the hyperkähler structure with data $(\Phi, \varphi, \rho_0, \xi)$ back to the one corresponding to $(\Phi, \varphi, e^{h_1} \rho_0, \xi + dh_2)$. This implies that if $\Sigma$ is simply-connected, then the triple

(hyperkähler structure, $\partial_\theta$, $\partial_t$)

is determined up to isomorphism by the holomorphic data $(\Phi, \varphi)$ alone, thus allowing us to make any choice for $\rho_0$ and $\xi$. For general $\Sigma$, the choice of $\xi$ matters, but the function $\rho_0$ can always be chosen freely. We shall presently establish a convenient choice for $\rho_0$.

3. Canonical metrics

We continue to consider structures satisfying (i) and (ii). Recall that the canonical slice is the hypersurface

$$S = \{ p \in W : g(\partial_t, \partial_t)_p = 1 \}.$$ 

The vector field $\partial_\theta$ is a Killing field for $g$ and commutes with $\partial_t$. So $\partial_\theta$ is tangent to $S$, and the restriction of $\partial_\theta$ to $S$ is a Killing field for the 3-dimensional metric $g_3$ induced on $S$ by the hyperkähler metric. That metric $g_3$, in turn, induces the quotient metric on $S/(\text{flow of } \partial_\theta)$ which makes the projection a Riemannian submersion.

We now introduce the holomorphic function $\psi = -1/\varphi$, which still takes values in the upper half-plane $\mathbb{H}$. The next lemma shows that the choice $\rho_0 = \text{Im } \psi$ is especially convenient, because then $\{ t = 0 \}$ is the canonical slice.

**Lemma 2.** The canonical slice is the product $G \times S^1_\theta$, where $G$ is the surface in $\Sigma \times \mathbb{R}_t$ described equivalently by any of the following equations:

- $V = |\varphi|^2$,
- $\rho = \text{Im } \psi$,
- $t = \log \text{Im } \psi - \log \rho_0$.

**Proof.** Formula (2) gives

$$g(\partial_t, \partial_t) = V^{-1} \cdot (\eta(\partial_t))^2 + V \cdot (x_1^2 + x_2^2 + x_3^2).$$

By Theorem 1, this can be written as

$$g(\partial_t, \partial_t) = V^{-1} \cdot (\text{Re } \varphi)^2 + V \cdot \rho^2.$$

So the canonical slice is given by the equation $V^{-1} \cdot (\text{Re } \varphi)^2 + V \cdot \rho^2 = 1$, which again by Theorem 1 transforms to

$$V = (\text{Re } \varphi)^2 + (\text{Im } \varphi)^2 = |\varphi|^2.$$

Using $\rho = (\rho V)/V$ and Theorem 1, this is seen to be equivalent to

$$\rho = \frac{\rho V}{|\varphi|^2} = \frac{\text{Im } \varphi}{|\varphi|^2} = \text{Im } \psi.$$

The third description of the canonical slice then follows from $\rho = e^t \rho_0$. □

The surface $G$ is the graph of a function $\Sigma \to \mathbb{R}_t$. Hence the map $(p, t, \theta) \mapsto p$ induces a diffeomorphism $\sigma: S/(\text{flow of } \partial_\theta) \to \Sigma$. 

Lemma 3. The diffeomorphism $\sigma$ sends the quotient metric to the metric
\begin{equation}
\sigma \colon g_{\Sigma} := \frac{1}{|\psi|^2} \left[ (d\text{Im}\,\psi)^2 + (\text{Im}\,\psi)^2 \Phi^* g_{S^2} \right]
\end{equation}
on $\Sigma$, with $g_{S^2}$ denoting the standard metric on $S^2$.

Proof. It follows from (2) that the orthogonal complement to $\partial_\theta$ is described by the equation $d\theta + \eta = 0$, both on $W$ and on $S$. The restriction of the hyperkähler metric to this complement is then the same as the restriction of the quadratic form $q := V \cdot (dx_1^2 + dx_2^2 + dx_3^2)$. This $q$ is $\partial_\theta$-invariant and has $\partial_\theta$ as an isotropic direction, therefore its restriction to the canonical slice $S$ is the pullback of the quotient metric under the quotient projection.

The standard metric of $\mathbb{R}^3$ equals $dr^2 + r^2 g_{S^2}$ in spherical coordinates. This and the equations $\rho = \text{Im}\,\psi$ and $V = |\varphi|^2$ from Lemma 2 imply that the restriction of $q$ to $S$ is given by the right-hand side of (4), with $\Phi : \Sigma \to S^2$ replaced by $x/\rho : S \to S^2$. It is then clear that $\sigma$ pulls $g_{\Sigma}$ given by (4) back to the quotient metric. \hfill $\Box$

4. A slice-complete example

The orbits of the Killing field $\partial_\theta$ in the canonical slice are circles (of variable length $2\pi V^{-1/2} = 2\pi |\psi|$). So it is clear that the induced metric $g_3$ is complete if and only if the quotient metric is complete; the latter in turn is isometric to $g_{\Sigma}$. In the sequel the Riemann surface $(\Sigma, J)$ will be the unit disc $D \subset \mathbb{C}$.

Theorem 4. There are holomorphic data $(\Phi, \psi)$ on $D$ such that formula (4) defines a complete metric $g_D$ on $D$. Thus, the pair $(\Phi, \psi)$ gives rise to a slice-complete hyperkähler metric on $D \times \mathbb{R} \times S^1_{\theta}$.

The construction of the pair $(\Phi, \psi)$ will take up the rest of this section.

Saying that $g_D$ is complete means that any proper path $\gamma : [0, +\infty) \to D$ has infinite length in this metric. Intuitively, for the metric $\Phi^* g_{S^2}$ to be complete the map $\Phi : D \to S^2$ would have to wrap $D$ around $S^2$ so as to push the boundary of $D$ infinitely far away. Such a map, however, would have to be a covering. The 2-sphere being simply connected, this is impossible. Still, most of that boundary can be pushed infinitely far away. This is achieved as follows.

Equip $D$ with the Poincaré metric and consider the conformal covering projection
$$
\Phi : D \longrightarrow S^2 \setminus \{p_1, p_2, p_3\}
$$
of the sphere with three punctures by the unit disc. Let us recall the construction of such a covering map. Take a hyperbolic triangle with its three vertices on $\partial D$ (a so-called ideal hyperbolic triangle), tessellate $D$ by this triangle and the infinitely many images under successive reflection on the sides (Figure 1), then consider the quotient $D/\Gamma$ where $\Gamma$ is the group consisting of the hyperbolic translations within the group generated by those reflections. This quotient space $D/\Gamma$ is the result of gluing corresponding sides of two copies of the original triangle, hence a sphere with three punctures. As a consequence of the uniformisation theorem, there is a biholomorphism
$$
\mathbb{D}/\Gamma \longrightarrow S^2 \setminus \{p_1, p_2, p_3\};
$$
see [12], in particular pp. 117 and 147–149.

The map $\Phi$ is the composition
$$
D \longrightarrow D/\Gamma \longrightarrow S^2 \setminus \{p_1, p_2, p_3\},
$$
and it has rank 2 everywhere.

There is a distinguished sequence \((z_n)\) of points on the unit circle \(\partial \mathbb{D}\), namely the vertices of all triangles in the tessellation. For \(j = 1, 2, 3\) choose a small closed metric ball \(B_j\), centred at the puncture \(p_j\) in \(S^2\), such that the balls \(2B_1, 2B_2, 2B_3\) of double radius have disjoint closures. Their preimages under \(\Phi\) are shown in Figure 2.

For each \(n\) let \(D_n\) be the connected component of \(\Phi^{-1}(B_1 \cup B_2 \cup B_3)\) having \(z_n\) as limit point, and let \(2D_n\) be the same for \(\Phi^{-1}(2B_1 \cup 2B_2 \cup 2B_3)\). Then both \((D_n)\) and \((2D_n)\) form a sequence of pairwise disjoint horodisc-like regions.

**Lemma 5.** Let \(\gamma : [0, +\infty) \to \mathbb{D}\) be any proper path. Either \(\gamma\) has infinite length in the metric \(\Phi^* g_{S^2}\), or it has an end \(\gamma([t_0, +\infty))\) contained in some region \(2D_{n_0}\).

**Remark.** If \(\gamma\) is proper in \(\mathbb{D}\) and contained in \(2D_{n_0}\), we must of course have \(\gamma(t) \to z_{n_0}\) as \(t \to +\infty\). So the lemma says that a path of finite length in the metric \(\Phi^* g_{S^2}\) can only escape the disc through one of the points \(z_n\). Intuitively, the covering map \(\Phi\) wraps around the punctured sphere so as to push the boundary of the disc infinitely far away, with the exception of the countable set \(\{z_n : n \in \mathbb{N}\}\).
Proof. Suppose first that the path visits $\bigcup_{n=1}^{\infty} D_n$ only finitely often. By deleting an initial segment, we may then assume that the trace of $\gamma$ is disjoint from all the $D_n$.

Each triangle $T_k$ of the tessellation intersects exactly three of the $D_n$. Let $T'_k$ be the compact region obtained by removing from $T_k$ the interior of those three intersections. With the metric $\Phi^* g_{S^2}$ the $T'_k$ are pairwise congruent spherical regions in the shape of a hexagon, with three sides coming from the boundaries $\Phi^{-1}(\partial B_j)$, $j = 1, 2, 3$ (call these the odd sides), and three sides coming from the sides of $T_k$ (call these the even sides). The path $\gamma$ is proper and contained in the union $\bigcup_{k=1}^{\infty} T'_k$, therefore it must visit infinitely many different regions $T'_k$, and among those there must be infinitely many $T'_k$ where $\gamma$ enters on one (necessarily even) side and exists on another (also even) side. If $c_1 > 0$ is the minimum spherical distance between even sides, which is the same for all $T'_k$, then the length of $\gamma$ in the metric $\Phi^* g_{S^2}$ is bounded from below by the series $c_1 + c_1 + \cdots$ and therefore infinite.

Suppose now that $\gamma$ visits the disjoint union $\bigcup_{n=1}^{\infty} D_n$ infinitely often and there are at least two regions $D_{n_1}$ and $D_{n_2}$ which the path visits infinitely many times. Then the length of $\gamma$ in the metric $\Phi^* g_{S^2}$ is bounded below by the series $c_2 + c_2 + \cdots$, where $c_2$ is the minimum spherical distance between the balls $B_1, B_2, B_3$, and hence infinite.

There remains the case when $\gamma$ visits but a single region $D_{n_0}$ infinitely often. If it has an end $\gamma([t_0, +\infty))$ contained in $2D_{n_0}$, we are done. Otherwise $\gamma$ takes infinitely many journeys from $\partial D_{n_0}$ to $\partial(2D_{n_0})$ and the length of $\gamma$ in the metric $\Phi^* g_{S^2}$ is bounded below by the series $c_3 + c_3 + \cdots$, where $c_3$ is the radius of the metric ball $B_j = \Phi(D_{n_0})$, and so again this length is infinite. □

A path in $2B_j \setminus \{p_j\}$ may well have finite spherical length and limit $p_j$. Thus $2D_{n_0}$ contains paths with $z_{n_0}$ as limit and finite length in the metric $\Phi^* g_{S^2}$. To ensure that $g_0$ be a complete metric, we need to choose the function $\psi$ so that the integral of $|d\text{Im }\psi|/|\psi|$ along such paths is infinite. Such a $\psi$ can be found with the help of so-called Blaschke products. For the reader’s convenience we first review this notion.

Given $a \in \mathbb{D}$ with $a \neq 0$, let $F_a(z)$ be the orientation-preserving isometry for the Poincaré metric that exchanges 0 with $a$. The line segment joining 0 to $a$ is a Poincaré geodesic, and $F_a$ is the $180^\circ$ Poincaré rotation about the Poincaré midpoint of that segment (Figure 3). It is given by $F_a(z) = \frac{a - z}{1 - \bar{a}z}$.

![Figure 3. Poincaré rotation.](image-url)
The **Blaschke factor** $B_a(z)$ is the function

$$B_a(z) = \frac{\overline{a} - z}{|a| (1 - \overline{a}z)},$$

i.e. the result of multiplying the isometry $F_a$ by a unitary constant so that $B_a(0)$ is a positive real number. The purpose of this normalisation is to get a simple convergence condition for **Blaschke products**, which are finite or infinite products of the form

$$B(z) = z^m \prod_k B_{a_k}(z).$$

It is well known [9, Section 6.7] that under the condition $\sum_k (1 - |a_k|) < \infty$ this product converges to a holomorphic function $B : \mathbb{D} \to \mathbb{D}$.

**Theorem 6 (1 Theorem 1)**. Let $(z_n)$ be a sequence of pairwise distinct points on $\partial \mathbb{D}$. Let $D \subset \mathbb{D}$ be a convex open domain with its boundary interior to $\mathbb{D}$ except for the point $1 \in \partial \mathbb{D}$. Then there exists a Blaschke product $B : \mathbb{D} \to \mathbb{D}$ such that for each $n$ we have a finite number $\lambda_n$ with

$$\frac{B(z) - 1}{z - z_n} \to \lambda_n \text{ as } z \to z_n, \ z \in \zeta_n D := \{ z \zeta : \zeta \in D \}. \quad \square$$

We are now going to apply this theorem to the sequence $(z_n)$ of vertices in our hyperbolic tessellation.

In the Euclidean metric on $\mathbb{D}$, the size of the regions $2D_n$ is bounded above by the size of the three of those horodisc-like regions at the vertices of the hyperbolic triangle containing the origin $0 \in \mathbb{D}$. Thus we can choose a circular disc $D \subset \mathbb{D}$ whose boundary is interior to $\mathbb{D}$ except for 1, and such that for each $n$ the rotated image $z_n D$ contains $2D_n$. By the theorem above we have a Blaschke product $B : \mathbb{D} \to \mathbb{D}$ such that for all $n$

$$B(z) \to 1 \text{ as } z \to z_n, \ z \in 2D_n.$$

The holomorphic function $1 - B$ maps $\mathbb{D}$ into the open disc of radius 1 and centre 1. Moreover, for each $n$ this function has limit 0 as $z$ approaches $z_n$ inside $2D_n$.

The open disc of radius 1 and centre 1 is contained in the right half-plane $\{ z : \text{Re } z > 0 \}$, on which there is a well-defined holomorphic square root with values in the quadrant $Q_1 := \{ z : \arg z \in (-\pi/4, \pi/4) \}$. So there is a holomorphic function $\sqrt{1 - B} : \mathbb{D} \to Q_1$ mapping $\mathbb{D}$ into the region bounded by a half lemniscate symmetric about the positive real axis. This function likewise has limit 0 as $z$ approaches $z_n$ inside $2D_n$.

Finally, we set $\psi = i \cdot \sqrt{1 - B}$. This holomorphic function maps $\mathbb{D}$ into the quadrant

$$Q_2 = \{ z : \arg z \in (\pi/4, 3\pi/4) \} = \{ u + iv : |u| < |v| \} \subset \mathbb{H}.$$

In fact, $\psi(\mathbb{D})$ is contained in the region bounded by a half lemniscate symmetric about the positive imaginary axis (Figure [4]). Moreover, $\psi(z)$ converges to 0 as $z$ approaches $z_n$ inside $2D_n$.

This finishes the construction of the pair $(\Phi, \psi)$. We are now going to verify that the metric $g_D$ defined by this pair as in equation (4) is indeed complete. This will conclude the proof of Theorem [4].
Since $\psi$ satisfies $|\Re \psi| < |\Im \psi|$, we have $|\Im \psi|/|\psi| > 1/\sqrt{2}$. By Lemma 5, a proper path in $\mathbb{D}$ is infinitely long in the metric $(\Im \psi/|\psi|)^2 \cdot \Phi^* g_{S^2}$, hence also in the metric $g_{\mathbb{D}}$, unless it has an end contained in some region $2D_{n_0}$.

But if a proper path $\gamma: [t_0, +\infty) \to \mathbb{D}$ is contained in $2D_{n_0}$, then we must have $\gamma(t) \to z_{n_0}$ and $\psi(\gamma(t)) \to 0$ as $t \to +\infty$.

Writing $\psi(\gamma(t)) = u(t) + i v(t)$, we have

$$\int_{\gamma} \frac{|d\Im \psi|}{|\psi|} > \int_{t_0}^{+\infty} \frac{|v'(t)|}{\sqrt{2}v(t)} \ dt = \frac{1}{\sqrt{2}} \text{ (total variation of log} v(t)).$$

Since $\log v(t) \to \log 0 = -\infty$ as $t \to +\infty$, we deduce that the integral of $|d\Im \psi|/|\psi|$ along $\gamma$ is infinite, and so is the length of $\gamma$ in the metric $g_{\mathbb{D}}$.

5. AN INFINITE-DIMENSIONAL FAMILY OF EXAMPLES

The preceding construction goes through if we replace $\psi$ by $\psi_\mu = \mu \circ \psi$, where $\mu$ is any holomorphic function whose domain contains $\{0\} \cup \psi(\mathbb{D})$ and such that $\mu(0) = 0$ and $\mu(\psi(\mathbb{D})) \subset Q_2$, see Figure 5. With $\Phi$ the covering map described in Section 4, we thus obtain an infinite-dimensional family of holomorphic data $(\Phi, \psi_\mu)$ that yield slice-complete hyperkähler structures in $\mathbb{D} \times \mathbb{R}_t \times S^1_\theta$.

We now want to show that the family of pairs $(\Phi, \psi_\mu)$ gives rise to an infinite-dimensional family of slice-complete hyperkähler metrics. Consider triples $(H, X, Y)$ made of a hyperkähler structure $H = (g, J_1, J_2, J_3)$, a vector field $X$ preserving the entire structure, and a vector field $Y$ such that $L_Y \Omega_i = \Omega_i$ and $[X, Y] \equiv 0$. If two
such triples are isomorphic, then the image $x(S)$ of the canonical slice of $Y$ under the momentum map is the same for both. For a triple in our family, the image $x(S)$ is the radial graph in $\mathbb{R}^3$ of the multi-valued function $(\text{Im}\psi_\mu) \circ \Phi^{-1}: S^2 \to \mathbb{R}^+$. As $\mu$ varies, these graphs form an infinite-dimensional family of immersed surfaces in $\mathbb{R}^3$, so we have an infinite-dimensional family of isomorphism classes of triples. Then the metrics also constitute an infinite-dimensional family of isometry classes, because for a given hyperkähler metric $g$ the number of degrees of freedom for $J_1, J_2, J_3, X, Y$ is bounded a priori by a finite constant: the complex structures $J_1, J_2, J_3$ are parallel with respect to the Riemannian connection; the Lie algebra of Killing vector fields is finite-dimensional [10, Thm. VI.3.3]; any two homothetic vector fields differ by a Killing field.

6. Non-flatness

The following proposition implies, as we shall see below, that the hyperkähler metrics in our examples are non-flat. This proposition may be regarded as a converse of [6, Thm. 10].

**Proposition 7.** If a metric with a homothetic vector field is slice-complete and flat, then the canonical slice is compact.

**Proof.** Consider a tubular neighbourhood $U$ of the canonical slice $S$. Endow the universal cover $\tilde{U}$ with the lifted flat metric. The lift of the homothetic vector field to $\tilde{U}$ is a vector field $\tilde{Y}$ homothetic for the lifted metric, and its canonical slice is the inverse image $\tilde{S}$ of $S \subset U$ under the covering map $\tilde{U} \to U$. This $\tilde{S}$ is connected, because its tubular neighbourhood $\tilde{U}$ is connected. We conclude that $\tilde{S}$ is a universal cover of $S$ under the restricted projection $\tilde{S} \to S$, which is a local isometry for the induced metrics. Since $\tilde{S}$ is complete with the induced metric, so is $\tilde{S}$.

There is a local isometry $F: \tilde{U} \to \mathbb{E}^n$ into Euclidean space. If $U' \subset \tilde{U}$ is an open domain on which $F$ is injective, then on the image $F(U')$ we have the vector field $F_*\tilde{Y}$ that is homothetic for the Euclidean metric of $\mathbb{E}^n$ restricted to $F(U')$. Then there is a vector field $Y_0$, defined on all of $\mathbb{E}^n$ and homothetic for the Euclidean metric, that coincides with $F_*\tilde{Y}$ on $F(U')$. On $\tilde{U}$ we have the homothetic vector fields $\tilde{Y}$ and $F^*Y_0$, and they have to be identical because they coincide on $U'$. This means that the local isometry $F$ sends $\tilde{Y}$ to $Y_0$, hence it maps the canonical slice $\tilde{S}$ of $Y$ to the canonical slice $S_0$ of $Y_0$. Since $\tilde{S}$ is complete, the restriction $F: \tilde{S} \to S_0$ must be a covering. But homothetic vector fields in $\mathbb{E}^n$ are linear, and $S_0$ is thus an ellipsoid. This forces $\tilde{S}$ to be diffeomorphic with $S^{n-1}$ and in particular compact. A fortiori, the slice $S$ must be compact. \hfill \Box

Hyperkähler metrics are Ricci flat [2, p. 284], and thus in particular Einstein metrics. Regularity results of DeTurck–Kazdan [3], cf. [2, Sections 5.E–F], say that Einstein metrics (in dimension at least 3) are real analytic in harmonic coordinates (the reason being that the equations for an Einstein metric form a quasi-linear elliptic system in such coordinates). Thus, if such a metric is flat in some domain, then it must be flat everywhere. With Proposition 7 we conclude that the hyperkähler metrics described in Sections 4 and 5 are non-flat on every open set, because the canonical slice of each of these metrics is complete but non-compact.
According to [6] Prop. 34], 4-dimensional hyperkähler metrics that are cone metrics are necessarily flat, so the metrics constructed above are not cone metrics in any open set, i.e. none of their homothetic vector fields are hypersurface orthogonal, not even locally.

7. A Carnot–Carathéodory phenomenon

In this section we show how the examples constructed in Sections 4 and 5 lead in a natural way to incomplete Riemannian metrics that nonetheless induce complete Carnot–Carathéodory distances. We begin by explaining these concepts.

A 1-form $\alpha$ on a 3-manifold $M$ is a contact form if $d(e^t \alpha)$ is a symplectic form on $M \times \mathbb{R}$. This is equivalent to $\alpha \wedge d\alpha$ being a volume form on $M$. The pair $(M, \alpha)$ is then called a contact manifold.

It is well known [4, Section 3.3] that any two points on a contact manifold $(M, \alpha)$ can be joined by a Legendre path, i.e. a path $\gamma(s)$ such that $\alpha(\gamma'(s)) \equiv 0$. Given a Riemannian metric $g$ on $M$, the induced Carnot–Carathéodory distance is, for each pair of points $p_1, p_2 \in M$, the infimum of the lengths of Legendre paths joining $p_1$ to $p_2$, see [5] or [3]. Notice that this is bounded below by the Riemannian distance, hence it is certainly a complete distance if $g$ happens to be complete.

A triple of contact forms $(\alpha_1, \alpha_2, \alpha_3)$ on $M$ is called a contact sphere if for each $c = (c_1, c_2, c_3) \in \mathbb{R}^3 \setminus \{0\}$ the linear combination $\alpha_c = c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3$ is a contact form. The contact sphere is called taut if all volume forms $\alpha_c \wedge d\alpha_c$ with $|c| = 1$ are equal, see [6]. Multiplying the three 1-forms of a taut contact sphere by the same non-vanishing function $w$ gives a new taut contact sphere, thanks to the identity $\alpha_c \wedge d(\omega w) = w^2 \alpha \wedge d\alpha$.

We now recall from [6] a correspondence between hyperkähler structures on $M \times \mathbb{R}_t$, satisfying condition (ii) of Section 2 and taut contact spheres on $M$. Given the hyperkähler structure, consider the symplectic forms $\Omega_c = c_1 \Omega_1 + c_2 \Omega_2 + c_3 \Omega_3$ for $c \in S^2$. We know that the volume form $\Omega_c^2$ is the same for all unitary $c$. On the other hand, the equation $L_{\partial_t} \Omega_c = \Omega_c$ is equivalent to $\Omega_c = d(\partial_t \cdot \Omega_c)$, and this implies

$$\partial_t \cdot (\Omega_c^2) = 2 (\partial_t \cdot \Omega_c) \wedge d(\partial_t \cdot \Omega_c).$$

Since the 3-form $\partial_t \cdot (\Omega_c^2)$ does not depend on $c$, the family $(\partial_t \cdot \Omega_c)_{c \in S^2}$ induces a contact sphere on any transversal for $\partial_t$. If $(\alpha_1, \alpha_2, \alpha_3)$ is the contact sphere induced on the transversal $\{t = 0\}$, the conditions $L_{\partial_t} (\partial_t \cdot \Omega_1) = \partial_t \cdot \Omega_1$ and $(\partial_t \cdot \Omega_i)(\partial_t) = 0$ lead to the expressions

$$\partial_t \cdot \Omega_i = e^t \alpha_i, \quad i = 1, 2, 3,$$

where the $\alpha_i$ have been pulled back to $M \times \mathbb{R}_i$. Then also $\Omega_i = d(e^t \alpha_i)$.

Conversely, if we are given a taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ on $M$, then — subject only to the sign condition $(\alpha_1 \wedge d\alpha_1)/(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) > 0$, which will be satisfied after a suitable permutation of the $\alpha_i$ — the symplectic forms $\Omega_i = d(e^t \alpha_i)$ determine a hyperkähler structure on $M \times \mathbb{R}_i$ that obviously satisfies condition (ii) of Section 2. These two processes, passing from a hyperkähler structure to a taut contact sphere and vice versa, are inverses of each other.

Now we can define the natural metric mentioned at the beginning of this section.

**Definition.** For a hyperkähler structure satisfying condition (ii) of Section 2, we write $(\omega_1, \omega_2, \omega_3)$ for the taut contact sphere induced by the 1-forms $\partial_t \cdot \Omega_i$ on the canonical slice and call $g_s := \omega_1^2 + \omega_2^2 + \omega_3^2$ the short metric on the canonical slice.
Here is how the short metric $g_s$ and the canonical metric $g_3$ defined in Section 3 are related. Any taut contact sphere $(\alpha_1, \alpha_2, \alpha_3)$ on a 3-manifold satisfies the following structure equations, where $(i, j, k)$ ranges over the cyclic permutations of $(1, 2, 3)$, with $\beta_0$ a unique 1-form and $\Lambda_0$ a unique function:

$$\begin{align*}
\alpha_i &= \beta_0 \wedge \alpha_i + \Lambda_0 \alpha_j \wedge \alpha_k.
\end{align*}$$

(5)

If we have a hyperkähler structure satisfying (ii) on $M \times \mathbb{R}_t$, and $(\alpha_1, \alpha_2, \alpha_3)$ is the contact sphere induced by the 1-forms $\partial_t \Omega_t$ on the transversal $\{t = 0\}$, then the function $\Lambda_0$ from the structure equations is positive and we have the formula

$$\Omega_i = d(e^i \alpha_i) = e^i (\Lambda_0^{-1/2} (dt + \beta_0) \wedge \Lambda_0^{1/2} \alpha_i + \Lambda_0^{1/2} \alpha_j \wedge \Lambda_0^{1/2} \alpha_k).$$

It follows that the hyperkähler metric is given by

$$g = e^i (\Lambda_0^{-1} \cdot (dt + \beta_0)^2 + \Lambda_0 \cdot (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)).$$

The canonical slice is thus given by the equation $t = \log \Lambda_0$.

The taut contact sphere $(\omega_1, \omega_2, \omega_3)$ induced on the canonical slice $S$ is given by $\omega_i = e^i \alpha_i |_{TS}$. With $\beta := (dt + \beta_0)|_{TS}$ we have

$$g_3 := g|_{TS} = \beta^2 + \omega_1^2 + \omega_2^2 + \omega_3^2.$$ Clearly, $g_s$ is shorter than $g_3$. The short metric is incomplete in all the examples described in Sections 3 and 5 because of the following result.

**Theorem 8** ([6, Theorem 24]). If a hyperkähler structure on $\Sigma \times \mathbb{R}_t \times S^1_\theta$ satisfies conditions (i) and (ii) of Section 3 and the canonical slice is non-compact, then the short metric is incomplete. 

From the definitions of $\omega_i$ we have $d\omega_i = d(e^i \alpha_i)|_{TS}$. A straightforward computation yields the following structure equations for the taut contact sphere $(\omega_1, \omega_2, \omega_3)$ on the canonical slice:

$$d\omega_i = \beta \wedge \omega_i + \omega_j \wedge \omega_k.$$ 

(7)

These structure equations can be used to give an alternative definition of the 1-form $\beta$.

The next proposition is the last ingredient we need in order to display the announced Carnot–Carathéodory phenomenon.

**Proposition 9.** Given a Riemann surface $(\Sigma, J)$ and holomorphic data $(\Phi, \varphi)$, consider the corresponding hyperkähler structure on $W = \Sigma \times \mathbb{R}_t \times S^1_\theta$, as explained in Section 3. Let $(\omega_1, \omega_2, \omega_3)$ be the taut contact sphere induced on the canonical slice $S$, and let $\beta$ be the 1-form on $S$ defined by the structure equations (7). If $\varphi$ is non-constant, then $\beta$ vanishes only along a discrete set of orbits of the $S^1$-action on $S$, and it is a contact form in the rest of $S$, defining the opposite orientation to that defined by the contact forms $\omega_i$. Therefore, any two points on $S$ can be joined by a path tangent to $\ker \beta$.

The proof of this proposition is given in the appendix. We can use the paths tangent to $\ker \beta$ to define Carnot–Carathéodory distances on $S$. The one induced by $g_3$ is a complete distance, because $g_3$ is a complete metric. Now the relation $g_3 = g_s + \beta^2$ tells us that the incomplete metric $g_s$ coincides with $g_3$ on those paths and thus induces exactly the same Carnot–Carathéodory distance as $g_3$. So we have an incomplete metric $g_s$ inducing a complete Carnot–Carathéodory distance. The geometric interpretation of this fact is that the proper paths tangent to $\ker \beta$
are so wrinkled that they always have infinite length in the incomplete metric \( g_s \). This phenomenon is interesting because such paths can be arbitrarily \( C^0 \)-close to any given path — so \( g_s \) is, in some sense, very close to being complete.

**APPENDIX**

Here we prove Proposition 9. We first derive an explicit formula for \( \beta \), from which all the properties claimed in Proposition 5 can then be deduced.

In analogy to Section 2, we define the momentum map \( x = (x_1, x_2, x_3) : W \to \mathbb{R}^3 \) by

\[
x_1 = \Omega_{i}(\partial_{\theta_i}, \partial_{t}) = -e^{i} \alpha_{i}(\partial_{\theta_i}),
\]

and we define \( \rho : S \to \mathbb{R} \) by \( \rho = |x| \).

Observe that \( x \) is \( \partial_{\theta_i} \)-invariant, which allowed us in Section 2 to view it as a function on \( \Sigma \times \mathbb{R}_t \). Below, however, we want to consider the \( x_i \) as functions on \( S \), where they equal \(-\omega_{i}(\partial_{\theta_i})\). On \( TS \) we then have the identity \( \sum_{i=1}^{3} dx_i = \partial_{\theta_i} \mathcal{F} \omega_i \).

We also regard the functions \( \varphi \) and \( \psi = -1/\varphi \) as functions on the canonical slice \( S \), by first pulling them back to \( \Sigma \times \mathbb{R}_t \times S^3 \) and then restricting them to \( S \).

**Lemma 10.** On \( S \) we have the identity \( \psi = -\beta(\partial_{\theta}) + 1\rho \).

**Proof.** Lemma 2 states that on \( S \) the imaginary part of \( \psi \) is \( \rho \). In order to determine the real part, we use the following alternative expressions for the hyperkähler metric \( g \) on \( W \):

\[
V^{-1} \cdot (dt + \eta)^2 + V \cdot (dx_1^2 + dx_2^2 + dx_3^2) = e^{t} \left( \Lambda_0^{-1} \cdot (dt + \beta_0)^2 + \Lambda_0 \cdot (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \right).
\]

Computing \( g(\partial_{\theta_i}, \partial_{t}) \) in the two possible ways, we get \( V^{-1} \eta(\partial_{t}) = e^{t} \Lambda_0^{-1} \beta_0(\partial_{\theta_i}) \).

On the canonical slice we have \( e^{t} = \Lambda_0 \) and, by Lemma 2, \( V = |\varphi|^2 \). Moreover, \( \eta(\partial_{t}) = \text{Re} \varphi \) by the definition of \( \varphi \) in Theorem 11. So on \( S \) we have

\[
\beta(\partial_{\theta}) = \beta_0(\partial_{\theta}) = \frac{\text{Re} \varphi}{|\varphi|^2} = -\text{Re} \psi,
\]

as claimed. \( \square \)

From (8) we find the following alternative expressions for the metric \( g_3 \) induced on \( S \):

\[
|\varphi|^{-2} \cdot (dt + \eta)^2 + |\varphi|^{2} \cdot (dx_1^2 + dx_2^2 + dx_3^2) = \beta^2 + \omega_1^2 + \omega_2^2 + \omega_3^2.
\]

Introduce the auxiliary 1-form \( \gamma := \sum_{i=1}^{3} x_i \omega_i \) on \( S \). By taking the interior product with \( \partial_{\theta} \) in (9) we find, with Lemma 10,

\[
|\varphi|^{-2} \cdot (dt + \eta) = \beta(\partial_{\theta}) \beta - \gamma = -(\text{Re} \psi) \beta - \gamma.
\]

With the structure equations (7) for \( d\omega_i \), we obtain from \( dx_i = \partial_{\theta} \mathcal{F} \omega_i \) the equations

\[
dx_i = \beta(\partial_{\theta}) \omega_i + x_j \beta - x_j \omega_k + x_k \omega_j.
\]

This yields

\[
\rho d\rho = \sum_{i=1}^{3} x_i dx_i = \rho^2 \beta + \beta(\partial_{\theta}) \gamma = \rho^2 \beta - (\text{Re} \psi) \gamma.
\]
Formulae (10) and (11) constitute a linear system for $\beta$ and $\gamma$. We solve it for $\beta$, observing that from Lemma 10 we have $\beta(\partial_\theta)^2 + \rho^2 = |\psi|^2$, to obtain

$$\beta = |\psi|^{-2} \cdot (\rho \, d\rho - (\text{Re}\, \psi)|\varphi|^{-2} \cdot (d\theta + \eta))$$

$$= |\psi|^{-2} \rho \, d\rho - (\text{Re}\, \psi)(d\theta + \eta).$$

Using the expression $\eta = (\text{Re}\, \varphi) \frac{d\varphi}{\rho} + \xi = -|\psi|^{-2}(\text{Re}\, \psi) \frac{d\varphi}{\rho} + \xi$ from Theorem 1 as well as Lemma 10 and the identities in Lemma 2, we transform the last equality as follows:

$$\beta = |\psi|^{-2} \cdot (\rho^2 + (\text{Re}\, \psi)^2) \frac{d\rho}{\rho} - (\text{Re}\, \psi)(d\theta + \xi)$$

$$= \frac{d\rho}{\rho} - (\text{Re}\, \psi)(d\theta + \xi)$$

$$= d\log \text{Im}\, \psi - (\text{Re}\, \psi)(d\theta + \xi).$$

This is the desired explicit expression for $\beta$.

We next want to determine the subset of $S$ where $\beta$ vanishes. Write the canonical slice as $S = G \times S_\theta^1$ as in Lemma 2 and points in $G$ as $(p, t(p))$ with $p \in \Sigma$. We then see that $\beta$ vanishes precisely along the circles $\{(p, t(p)) \} \times S_\theta^1$ for those points $p \in \Sigma$ where $d\text{Im}\, \psi$ and $\text{Re}\, \psi$ vanish simultaneously. It is easy to verify that for a non-constant holomorphic function $\psi$ such points form a discrete subset of $\Sigma$. Therefore $\beta$ is non-zero outside a discrete set of circles, all of which are orbits of the $S^1$-action on the canonical slice.

Finally, we want to prove that $\beta$ is a contact form outside the described vanishing set. From $d(d\omega_i) = 0$ and the structure equations (7) one gets

$$d\beta \wedge \omega_i + \beta \wedge \omega_j \wedge \omega_k = 0.$$ 

Write $\beta = b_1 \omega_1 + b_2 \omega_2 + b_3 \omega_3$. Then

$$d\beta = -b_1 \omega_2 \wedge \omega_3 - b_2 \omega_3 \wedge \omega_1 - b_3 \omega_1 \wedge \omega_2,$$

and so

$$\beta \wedge d\beta = -(b_1^2 + b_2^2 + b_3^2) \omega_1 \wedge \omega_2 \wedge \omega_3 = -(b_1^2 + b_2^2 + b_3^2) \omega_1 \wedge d\omega_1.$$ 

So $\beta$ is indeed a contact form where it is non-zero, and we also observe that the induced orientation is opposite to the one defined by the $\omega_i$.

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