AFFINE TRANSLATION SURFACES IN THE ISOTROPIC 3-SPACE

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Abstract. The isotropic 3-space $I^3$ is a real affine 3-space endowed with the metric $dx^2 + dy^2$. In this paper we describe Weingarten and linear Weingarten affine translation surfaces in $I^3$. Further we classify the affine translation surfaces in $I^3$ that satisfy certain equations in terms of the position vector and the Laplace operator.

1. Introduction

It is well-known that a surface is called translation surface in a Euclidean 3-space $\mathbb{R}^3$ if it is the graph of a function $z(x, y) = f(x) + g(y)$ for the standard coordinate system of $\mathbb{R}^3$. One of the famous minimal surfaces of $\mathbb{R}^3$ is Scherk’s minimal translation surface which is the graph of the function

$$z(x, y) = \frac{1}{c} \log \frac{\cos (cx)}{\cos (cy)}, \ c \in \mathbb{R}^* := \mathbb{R} - \{0\}.$$

In order for more generalizations of the translation surfaces to see in various ambient spaces we refer to [4, 5, 7, 12, 16, 19, 20, 24, 26].

In 2013, H. Liu and Y. Yu [14] defined the affine translation surfaces in $\mathbb{R}^3$ as the graph of the function

$$z(x, y) = f(x) + g(y + ax), \ a \in \mathbb{R}^*$$

and described the minimal affine translation surfaces which are given by

$$z(x, y) = \frac{1}{c} \log \frac{\cos (c\sqrt{1 + a^2}x)}{\cos (c[y + ax])}, \ a, c \in \mathbb{R}^*.$$

These are called affine Scherk surface. Then H. Liu and S.D. Jung [15] classified the affine translation surfaces in $\mathbb{R}^3$ of arbitrary constant mean curvature.

In the isotropic 3-space $I^3$, there exist three different classes of translation surfaces given by (see [15, 25])

$$\begin{cases} z(x, y) = f(x) + g(y), \\ y(x, z) = f(x) + g(z), \\ x(y, z) = \frac{1}{2} \left( f\left(\frac{y+z}{2}\right) + g\left(\frac{y-z}{2}\right)\right), \end{cases}$$

where $x, y, z$ are the standard affine coordinates in $I^3$. These surfaces are respectively called translation surfaces of Type 1,2,3 in $I^3$. Such surfaces of constant isotropic Gaussian and mean curvature were obtained in [15] as well as Weingarten ones.

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The translation surfaces of Type 1 in \( \mathbb{I}^3 \) that satisfy the condition
\[
\Delta I,II r_i = \lambda_i r_i, \quad \lambda_i \in \mathbb{R}, \ i = 1, 2, 3,
\]
were presented in [13], where \( r_i \) is the coordinate function of the position vector and \( \Delta I,II \) the Laplace operator with respect to the first and second fundamental forms, respectively. This condition is natural, being related to the so-called submanifolds of finite type, introduced by B.-Y. Chen in the late 1970’s (see [8, 9, 11]). More details of translation surfaces in the isotropic spaces can be found in [2, 3, 6].

In this paper, we investigate the affine translation surfaces of Type 1 in \( \mathbb{I}^3 \), i.e. the graphs of the function
\[
z (x, y) = f (ax + by) + g (cx + dy), \quad ad - bc \neq 0
\]
and classify ones of Weinagarten type. Moreover we describe the affine translation surfaces of Type 1 that satisfy the condition \( \Delta I,II r_i = \lambda_i r_i \).

2. Preliminaries

The isotropic 3-space \( \mathbb{I}^3 \) is a real affine space defined from the projective 3-space \( P (\mathbb{R}^3) \) with an absolute figure consisting of a plane \( \omega \) and two complex-conjugate straight lines \( f_1, f_2 \) in \( \omega \) (see [1][10][17], [21][23]). Denote the projective coordinates by \( (X_0, X_1, X_2, X_3) \) in \( P (\mathbb{R}^3) \). Then the absolute plane \( \omega \) is given by \( X_0 = 0 \) and the absolute lines \( f_1, f_2 \) by \( X_0 = X_1 + iX_2 = 0, X_0 = X_1 - iX_2 = 0 \). The intersection point \( F (0 : 0 : 0 : 1) \) of these two lines is called the absolute point. The group of motions of \( \mathbb{I}^3 \) is a six-parameter group given in the affine coordinates \( x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}, z = \frac{X_3}{X_0} \) by
\[
(x, y, z) \mapsto (x', y', z') : \begin{cases} 
  x' = a_1 + x \cos \phi - y \sin \phi, \\
  y' = a_2 + x \sin \phi + y \cos \phi, \\
  z' = a_3 + a_4x + a_5y + z,
\end{cases}
\]
where \( a_1, ..., a_5, \phi \in \mathbb{R} \).

The metric of \( \mathbb{I}^3 \) is induced by the absolute figure, i.e. \( ds^2 = dx^2 + dy^2 \). The lines in \( z \)-direction are called isotropic lines. The planes containing an isotropic line are called isotropic planes. Other planes are non-isotropic.

Let \( M \) be a surface immersed in \( \mathbb{I}^3 \). We call the surface \( M \) admissible if it has no isotropic tangent planes. Such a surface can get the form
\[
r : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{I}^3 : (x, y) \rightarrow (r_1 (x, y), r_2 (x, y), r_3 (x, y)).
\]

The components \( E, F, G \) of the first fundamental form \( I \) of \( M \) can be calculated via the metric induced from \( \mathbb{I}^3 \).

Denote \( \Delta I \) the Laplace operator of \( M \) with respect to \( I \). Then it is defined as
\[
\Delta \phi = \frac{1}{\sqrt{|W|}} \left\{ \frac{\partial}{\partial x} \left( G \phi_x - F \phi_y \right) \frac{1}{\sqrt{|W|}} - \frac{\partial}{\partial y} \left( F \phi_x - E \phi_y \right) \frac{1}{\sqrt{|W|}} \right\},
\]
where \( \phi \) is a smooth function on \( M \) and \( W = EG - F^2 \).

The unit normal vector field of \( M^2 \) is completely isotropic, i.e. \( (0, 0, 1) \). Moreover, the components of the second fundamental form \( II \) are
\[
L = \frac{\det (r_{xx}, r_{x}, r_y)}{\sqrt{EG - F^2}}, \quad M = \frac{\det (r_{xy}, r_x, r_y)}{\sqrt{EG - F^2}}, \quad N = \frac{\det (r_{yy}, r_x, r_y)}{\sqrt{EG - F^2}},
\]
where \( r_{xy} = \frac{\partial^2 r}{\partial x \partial y} \), etc.
The relative curvature (so-called the isotropic curvature or isotropic Gaussian curvature) and the isotropic mean curvature are respectively defined by

\[(2.3) \quad K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + LG}{2(EG - F^2)}.\]

Assume that nowhere \( M \) has parabolic points, i.e. \( K \neq 0 \). Then the Laplace operator with respect to \( II \) is given by

\[(2.4) \quad \Delta^II \phi = -\frac{1}{\sqrt{|w|}} \left\{ \frac{\partial}{\partial x} \left( \frac{N\phi_x - M\phi_y}{\sqrt{|w|}} \right) - \frac{\partial}{\partial y} \left( \frac{M\phi_x - L\phi_y}{\sqrt{|w|}} \right) \right\}\]

for a smooth function \( \phi \) on \( M \) and \( w = LN - M^2 \).

In particular, if \( M \) is a graph surface in \( \mathbb{I}^3 \) of a smooth function \( z(x, y) \) then the metric on \( M \) induced from \( \mathbb{I}^3 \) is given by \( dx^2 + dy^2 \). Thus its Laplacian turns to

\[(2.5) \quad \Delta^I = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.\]

The matrix of second fundamental form \( II \) of \( M \) corresponds to the Hessian matrix \( \mathcal{H}(z) \), i.e.,

\[
\begin{pmatrix}
L & M \\
M & N
\end{pmatrix} = \begin{pmatrix}
z_{xx} & z_{xy} \\
z_{xy} & z_{yy}
\end{pmatrix}.
\]

Accordingly, the formulas (2.3) reduce to

\[(2.6) \quad K = \det(\mathcal{H}(z)), \quad H = \frac{\text{trace}(\mathcal{H}(z))}{2}.\]

3. Weingarten affine translation surfaces

Let \( M \) be the graph surface in \( \mathbb{I}^3 \) of the function \( z(x, y) = f(u) + g(v) \), where

\[(3.1) \quad u = ax + by, \quad v = cx + dy.\]

If \( ad - bc \neq 0 \), we call the surface \( M \) affine translation surface of Type 1 in \( \mathbb{I}^3 \) and the pair \((u, v)\) affine parameter coordinates.

In the particular case \( a = d = 1 \) and \( b = c = 0 \) (or \( a = d = 0 \) and \( b = c = 1 \)), such a surface reduces to the translation surface of Type 1 in \( \mathbb{I}^3 \). Let us fix some notations to use remaining part:

\[
\frac{\partial f}{\partial x} = a \frac{df}{du} = af', \quad \frac{\partial f}{\partial y} = bf', \quad \frac{\partial g}{\partial x} = c \frac{dg}{dv} = cg', \quad \frac{\partial g}{\partial y} = dg',
\]

and so on. By (2.6), the relative curvature \( K \) and the isotropic mean curvature \( H \) of \( M \) turn to

\[(3.2) \quad K = (ad - bc)^2 f'' g'' \quad \text{and} \quad 2H = (a^2 + b^2) f'' + (c^2 + d^2) g''.
\]

Now we can state the following result to describe the Weingarten affine translation surfaces of Type 1 in \( \mathbb{I}^3 \) that satisfy the condition

\[(3.3) \quad K_x H_y - K_y H_x = 0,
\]

where the subscript denotes the partial derivative.
Theorem 3.1. Let $M$ be a Weingarten affine translation surface of Type 1 in $\mathbb{H}^3$. Then one of the following occurs:

(i) $M$ is a quadric surface given by

$$z(x, y) = c_1 u^2 + c_1 (a^2 + b^2) v^2 + c_2 u + c_3 v + c_4, \ c_1, ..., c_4 \in \mathbb{R}.$$ 

(ii) $M$ is of the form either

$$z(x, y) = f(u) + c_1 v^2 + c_2 v + c_3, \ f''' \neq 0, \ c_1, c_2, c_3 \in \mathbb{R}$$
or

$$z(x, y) = g(v) + c_1 u^2 + c_2 u + c_3, \ g''' \neq 0, \ c_1, c_2, c_3 \in \mathbb{R},$$

where $(u, v)$ is the affine parameter coordinates given by (3.1).

Remark 3.1. We point out that a quadric surface in $\mathbb{H}^3$ is the set of the points satisfying an equation of the second degree.

Proof. It follows from (3.2) and (3.3) that

$$(a^2 + b^2) f'' - (c^2 + d^2) g'' = 0.$$ 

To solve (3.4), we have several cases:

Case (a). $(a^2 + b^2) f'' = (c^2 + d^2) g''$. Then we derive

$$z(x, y) = c_1 u^2 + c_1 (a^2 + b^2) v^2 + c_2 u + c_3 v + c_4, \ c_1, ..., c_4 \in \mathbb{R},$$

which gives the statement (i) of the theorem.

Case (b). $(a^2 + b^2) f'' \neq (c^2 + d^2) g''$. Then, by (3.4), the surface has the form either

$$z(x, y) = g(v) + c_1 u^2 + c_2 u + c_3, \ g''' \neq 0, \ c_1, c_2, c_3 \in \mathbb{R}$$
or

$$z(x, y) = f(u) + c_4 v^2 + c_5 v + c_6, \ f''' \neq 0, \ c_1, ..., c_6 \in \mathbb{R}.$$ 

This implies the second statement of the theorem. Therefore the proof is completed. $\square$

Now we intend to find the linear Weingarten affine translation surfaces of Type 1 in $\mathbb{H}^3$ that satisfy

$$(a^2 + b^2) f'' + (c^2 + d^2) g'' = 0, \ (a, b, c, d) \neq (0, 0, 0, 0).$$ 

Without lose of generality, we may assume $\alpha \neq 0$ in (3.5) and thus it can be rewritten as

$$(a^2 + b^2) f'' + (c^2 + d^2) g'' = 0, \ (a, b, c, d) \neq (0, 0, 0, 0).$$ 

Hence the following result can be given.

Theorem 3.2. Let $M$ be a linear Weingarten affine translation surface of Type 1 in $\mathbb{H}^3$ that satisfies (3.6). Then we have:

(i) $M$ is a quadric surface given by

$$z(x, y) = c_1 u^2 + c_2 v^2 + c_3 u + c_4 v + c_5, \ c_1, ..., c_5 \in \mathbb{R};$$
Weingarten.

This surface plotted as in Fig. 1 satisfies the conditions to be Weingarten and linear

\[ I(r) (4.1) \]

parameterization on such a surface as follows

Theorem 4.1. Let

\[ \triangle I \]

Thus we first give the following result.

\[ \square \]

the theorem. Otherwise, we have the second statement of the theorem. This proves

the theorem.

Example 3.1. Consider the affine translation surface of Type 1 in \( \mathbb{I}^3 \) with

\[ z (x, y) = \cos (x - y) + (x + y)^2, \quad -\frac{\pi}{6} \leq x, y \leq \frac{\pi}{6}. \]

This surface plotted as in Fig. 1 satisfies the conditions to be Weingarten and linear

Weingarten.

4. AFFINE TRANSLATION SURFACES SATISFYING \( \triangle^I r_i = \lambda_i r_i \)

This section is devoted to classify the affine translation surfaces of Type 1 in \( \mathbb{I}^3 \) that satisfy the conditions \( \triangle^I r_i = \lambda_i r_i, \lambda_i \in \mathbb{R} \). For this, we get a local parameterization on such a surface as follows

\[ r (x, y) = (r_1 (x, y), r_2 (x, y), r_3 (x, y)) \]

\[ = (x, y, f (ax + by) + g (cx + dy)). \]

Thus we first give the following result.

Theorem 4.1. Let \( M \) be an affine translation surface of Type 1 in \( \mathbb{I}^3 \) that satisfies

\[ \triangle^I r_i = \lambda_i r_i. \]

Then it is congruent to one of the following surfaces:

(i) \( (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0) \)

\[ z (x, y) = c_1 u^2 - \frac{c_1 (a^2 + b^2)}{c^2 + d^2} v^2 + c_3 u + c_4 v + c_5; \]

(ii) \( (\lambda_1, \lambda_2, \lambda_3) = (0, 0, \lambda > 0) \)

\[ z (x, y) = c_1 e^{\sqrt{\frac{-\lambda}{a^2 + b^2} u} + c_2 e^{-\sqrt{\frac{-\lambda}{a^2 + b^2} u} + c_3 e^{\sqrt{\frac{-\lambda}{c^2 + d^2} v} + c_4 e^{-\sqrt{\frac{-\lambda}{c^2 + d^2} v}}; \}

(iii) \( (\lambda_1, \lambda_2, \lambda_3) = (0, 0, \lambda < 0) \)

\[ z (x, y) = c_1 \cos \left( \sqrt{\frac{-\lambda}{a^2 + b^2} u} \right) + c_2 \sin \left( \sqrt{\frac{-\lambda}{a^2 + b^2} u} \right) + c_3 \cos \left( \sqrt{\frac{-\lambda}{c^2 + d^2} v} \right) + c_4 \sin \left( \sqrt{\frac{-\lambda}{c^2 + d^2} v} \right), \]

where \((u, v)\) is the affine parameter coordinates given by (3.1) and \( c_1, ..., c_5 \in \mathbb{R} \).
Proof. It is easy to compute from (2.5) and (4.1) that
\[(4.2) \quad \Delta^I r_1 = \Delta^I r_2 = 0\]
and
\[(4.3) \quad \Delta^I r_3 = (a^2 + b^2) f'' + (c^2 + d^2) g'' .\]
Assuming \(\Delta^I r_i = \lambda_i r_i, \ i = 1, 2, 3,\) in (4.2) and (4.3) yields \(\lambda_1 = \lambda_2 = 0\) and
\[(4.4) \quad (a^2 + b^2) f'' + (c^2 + d^2) g'' = \lambda (f + g), \ \lambda_3 = \lambda.\]
If \(\lambda = 0\) in (4.4), then we derive
\[f(u) = c_1 u^2 + c_2 u + c_3\]
and
\[g(v) = -c_1 \frac{(a^2 + b^2)}{(c^2 + d^2)} v^2 + c_4 v + c_5, \ c_1, ..., c_5 \in \mathbb{R},\]
which proves the statement (i) of the theorem.

If \(\lambda \neq 0\) then (4.4) can be rewritten as
\[(4.5) \quad (a^2 + b^2) f'' - \lambda f = \mu = -(c^2 + d^2) g'' + \lambda g, \ \mu \in \mathbb{R}.\]
In the case \(\lambda > 0,\) by solving (4.5) we obtain
\[
\begin{cases}
  f(u) = c_1 \exp \left( \frac{\lambda}{a^2 + b^2} u \right) + c_2 \exp \left( -\frac{\lambda}{a^2 + b^2} u \right) + \frac{\mu}{\lambda}, \\
  g(v) = c_3 \exp \left( \frac{\lambda}{c^2 + d^2} v \right) + c_4 \exp \left( -\frac{\lambda}{c^2 + d^2} v \right) - \frac{\mu}{\lambda},
\end{cases}
\]
where \(c_1, ..., c_4 \in \mathbb{R}.\) This gives the statement (ii) of the theorem.

Otherwise, i.e., \(\lambda < 0,\) then we derive
\[
\begin{cases}
  f(u) = c_1 \cos \left( \frac{\lambda}{a^2 + b^2} u \right) + c_2 \sin \left( \frac{\lambda}{a^2 + b^2} u \right) + \frac{\mu}{\lambda}, \\
  g(v) = c_3 \cos \left( \frac{\lambda}{c^2 + d^2} v \right) + c_4 \sin \left( \frac{\lambda}{c^2 + d^2} v \right) - \frac{\mu}{\lambda},
\end{cases}
\]
for \(c_1, ..., c_4 \in \mathbb{R}.\) This completes the proof. \(\square\)

Example 4.1. Take the affine translation surface of Type 1 in \(\mathbb{I}^3\) with
\[z(x, y) = \cos(x + y) + \sin(x - y), \ -\pi \leq x, y \leq \pi.\]
Then it satisfies \(\Delta^I r_i = \lambda_i r_i\) for \(\lambda_1 = \lambda_2 = 0, \ \lambda_3 = -2\) and can be drawn as in Fig. 2.

Next, we consider the affine translation surface of Type 1 in \(\mathbb{I}^3\) that satisfies
\(\Delta^I r_i = \lambda_i r_i, \ \lambda_i \in \mathbb{R}.\) Then its Laplace operator with respect to the second fundamental form \(II\) has the form
\[(4.6) \quad \Delta^I \phi = \frac{(f'' g')^{-1}}{2(ad - bc)} \left[ (-b \phi_x + a \phi_y) (f'')^2 g''' + (d \phi_x - c \phi_y) f''' (g'')^2 \right] \]
\[+ \frac{(f'' g')^{-1}}{ad - bc} \left[ (2ab \phi_{xy} - b^2 \phi_{xx} - a^2 \phi_{yy}) f'' + (2cd \phi_{xy} - d^2 \phi_{xx} - c^2 \phi_{yy}) g'' \right]
\]
for a smooth function \(\phi\) and \(f'' g'' \neq 0.\) Hence we have the following result.
Theorem 4.2. Let $M$ be an affine translation surface of Type 1 in $\mathbb{F}^3$ that satisfies $\Delta^I r_i = \lambda_i r_i$. Then it is congruent to one of the following surfaces:

(i) $(\lambda_1 \neq 0, \lambda_2 \neq 0, 0)$

$$z(x, y) = \ln \left( x^{\frac{1}{\lambda_1}} y^{\frac{1}{\lambda_2}} \right) + c_1, \ c_1 \in \mathbb{R};$$

(ii) $(\lambda \neq 0, 0, 0)$

$$z(x, y) = \ln \left( (uv)^{\frac{1}{\lambda}} \right) + c_1, \ c_1 \in \mathbb{R},$$

where $(u, v)$ is the affine parameter coordinates given by (3.1).

Proof. Let us assume that $\Delta^I r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$. Then, from (4.1) and (4.6), we state the following system

(4.7) $$\frac{d f'''}{(f'')}^2 - b \frac{g'''}{(g'')}^2 = 2 (ad - bc) \lambda_1 x,$$

(4.8) $$- c \frac{f'''}{(f'')}^2 + a \frac{g'''}{(g'')}^2 = 2 (ad - bc) \lambda_2 y,$$

(4.9) $$\frac{f''' f'}{(f'')}^2 + \frac{g''' g'}{(g'')}^2 - 4 = 2 \lambda_3 (f + g).$$

In order to solve above system we have to distinguish two cases depending on the constants $a, b, c, d$ for $ad - bc \neq 0$.

Case (a). Two of $a, b, c, d$ are zero. Without loss of generality we may assume that $b = c = 0$ and $a = d = 1$. Then the equations (4.7) and (4.8) reduce to

(4.10) $$\frac{f'''}{(f'')}^2 = 2 \lambda_1 x$$

and

(4.11) $$\frac{g'''}{(g'')}^2 = 2 \lambda_2 y.$$ 

If $\lambda_1 = \lambda_2 = 0$ then we obtain a contradiction from (4.9) since $f, g$ are non-constant functions. Thereby we need to consider the remaining cases:

Case (a.1). $\lambda_1 = 0$, i.e. $f''' = 0$. Then substituting (4.10) and (4.11) into (4.9) implies $\lambda_3 = 0$ and

$$g(y) = \frac{2}{\lambda_2} \ln y + c_1, \ c_1 \in \mathbb{R}.$$ 

Substituting it in (4.11) gives a contradiction.

Case (a.2). $\lambda_2 = 0$, i.e. $g''' = 0$. Hence we can similarly obtain $\lambda_3 = 0$ and

$$f(x) = \frac{2}{\lambda_1} \ln x + c_1, \ c_1 \in \mathbb{R},$$

which gives a contradiction by considering it into (4.10).

Case (a.3). $\lambda_1 \lambda_2 \neq 0$. By substituting (4.10) and (4.11) into (4.9) we deduce

(4.12) $$\lambda_1 x f' + \lambda_2 y g' - 2 = \lambda_3 (f + g).$$

Case (a.3.1). If $\lambda_3 = 0$, then (4.12) reduces to

(4.13) $$\lambda_1 x f' + \lambda_2 y g' = 2.$$
By solving (4.13) we find

\[(4.14) \quad f(x) = \frac{\xi}{\lambda_1} \ln x + c_1 \text{ and } g(v) = \frac{2 - \xi}{\lambda_2} \ln y + c_2, \quad c_1, c_2 \in \mathbb{R}, \ \xi \in \mathbb{R}^+.\]

Substituting (4.14) into (4.10) and (4.11) yields \(\xi = 1\). This proves the first statement of the theorem.

**Case (a.3.2).** If \(\lambda_3 \neq 0\) in (4.12) then we can rewrite it as

\[(4.15) \quad \lambda_1 x f' - \lambda_3 f - 2 = c_1 = -\lambda_2 y g' + \lambda_3 g, \quad c_1 \in \mathbb{R}.\]

After solving (4.15), we conclude

\[(4.16) \quad f(x) = -\frac{2 + c_1}{\lambda_3} + c_2 x^{\frac{1}{\lambda_3}}\]

and

\[(4.17) \quad g(y) = \frac{c_1}{\lambda_3} + c_3 y^{\frac{\lambda_3}{2}}, \quad c_2, c_3 \in \mathbb{R}.\]

By considering (4.16) and (4.17) into (4.10) and (4.11), respectively, we conclude \(\lambda_3 = 0\), which implies that this case is not possible.

**Case (b).** At most one of \(a, b, c, d\) is zero. Suppose that \(\lambda_1 = 0\) in (4.7). It follows from (4.7) that

\[(4.18) \quad \frac{f'''}{(f'')}^2 = \frac{c_1}{d} \text{ and } \frac{g'''}{(g'')}^2 = \frac{c_1}{b}, \quad c_1 \in \mathbb{R},\]

where we may assume that \(b \neq 0 \neq d\) since at most one of \(a, b, c, d\) can vanish. If \(c_1 = 0\) then we derive a contradiction from (4.9) since \(f''y'\) \(\neq 0\). Considering (4.18) into (4.8) yields \(\frac{c_1}{d^2} = 2\lambda_2 y\), which is no possible since \(y\) is an independent variable. This implies that \(\lambda_1\) is not zero and it can be similarly shown that \(\lambda_2\) is not zero. Hence from (4.7) and (4.8) we can write

\[(4.19) \quad \frac{f'''}{(f'')}^2 = 2 (\lambda_1 ax + \lambda_2 by)\]

and

\[(4.20) \quad \frac{g'''}{(g'')}^2 = 2 (\lambda_1 cx + \lambda_2 dy).\]

Compatibility condition in (4.19) or (4.20) gives \(\lambda_1 = \lambda_2\). Put \(\lambda_1 = \lambda_2 = \lambda\). By substituting (4.19) and (4.20) into (4.9) we deduce

\[(4.21) \quad \lambda uf' + \lambda vg' - 2 = \lambda_3 (f + g),\]

where \((u, v)\) is the affine parameter coordinates given by (3.1).

**Case (b.1).** If \(\lambda_3 = 0\), then (4.21) reduces to

\[(4.22) \quad \lambda uf' + \lambda vg' = 2.\]

By solving (4.22) we find

\[(4.23) \quad f(u) = \frac{\xi}{\lambda} \ln u + c_1 \text{ and } g(v) = \frac{2 - \xi}{\lambda} \ln v + c_2, \quad c_1, c_2 \in \mathbb{R}, \ \xi \in \mathbb{R}^+.\]

Substituting (4.23) into (4.19) and (4.20) yields \(\xi = 1\). This proves the second statement of the theorem.

**Case (b.2).** If \(\lambda_3 \neq 0\) in (4.11) then we can rewrite it as

\[(4.24) \quad \lambda uf' - \lambda_3 f - 2 = c_1 = -\lambda vg' + \lambda_3 g, \quad c_1 \in \mathbb{R}.\]
After solving (4.24), we deduce
\begin{equation}
(4.25) \quad f(u) = -\frac{2 + c_1}{\lambda_3} + c_2 u \lambda_3
\end{equation}
and
\begin{equation}
(4.26) \quad g(v) = \frac{c_1}{\lambda_3} + c_3 v \lambda_3, \quad c_2, c_3 \in \mathbb{R}.
\end{equation}
Considering (4.25) and (4.26) into (4.19) and (4.20), respectively, we find \( \lambda_3 = 0 \), however this is a contradiction.

\textbf{Example 4.2.} Given the affine translation surface of Type 1 in \( \mathbb{I}^3 \) as follows
\[ z(x,y) = \ln (2x + y) + \ln (x - y), \quad (u,v) \in [3,5] \times [1,2]. \]
Then it holds \( \Delta^I r_i = \lambda_i r_i \) for \( (\lambda_1, \lambda_2, \lambda_3) = (1,1,0) \) and we plot it as in Fig. 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A (linear) Weingarten affine translation surface of Type 1.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{An affine translation surface of Type 1 with \( \Delta^I r_i = \lambda_i r_i \), \( (\lambda_1, \lambda_2, \lambda_3) = (0,0,2) \).}
\end{figure}

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Figure 3. An affine translation surface of Type 1 with $\Delta^I r_i = \lambda_i r_i$, $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$.

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