Schur Algebras for the Alternating Group and Koszul Duality

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Abstract. The alternating Schur algebra $\text{AS}_F(n, d)$ is defined as the commutant of the action of the alternating group $A_d$ on the $d$-fold tensor power of an $n$-dimensional $F$-vector space. It contains the classical Schur algebra as a subalgebra. When $F$ is a field of characteristic different from 2, we find a basis of $\text{AS}_F(n, d)$ in terms of bipartite graphs. We give a combinatorial interpretation of the structure constants of $\text{AS}_F(n, d)$ with respect to this basis.

What additional structure does being a module for $\text{AS}_F(n, d)$ impose on a module for the classical Schur algebra? Our answer to this question involves the Koszul duality functor, introduced by Krause for strict polynomial functors. It leads to a simple interpretation of Koszul duality for modules of the classical Schur algebra. Krause’s work implies that derived Koszul duality functor is an equivalence when $n \geq d$. Our combinatorial methods prove the converse.

1. Introduction

1.1. Schur-Weyl Duality and its Variants. Frobenius determined the irreducible characters of the symmetric group $S_d$ over $C$, the field of complex numbers, in 1900 [11]. Building on this, Schur classified the irreducible polynomial representations of $GL_n(C)$ and computed their characters in his PhD thesis [26]. The group $GL_n(C)$ acts on the factors of $(C^n)^{\otimes d}$, while $S_d$ permutes the tensor factors. In 1927, Schur used these commuting actions to reprove the results of his dissertation [27]. Following Weyl’s expositions of this method [31, 32], it is known as Schur-Weyl duality.

Over the years, several variants of Schur-Weyl duality have emerged. Shrinking $GL_n(C)$ to the orthogonal group $O_n(C)$, Brauer obtained the duality between Brauer algebras $Br_d(n)$ and $O_n(C)$ [7]. Motivated by the Potts model in statistical mechanics, Jones [15] and Martin [19] further shrunk $O_n(C)$ down to $S_n$, obtaining the partition algebras $P_d(n)$. Bloss [5] reduced $S_n$ to $A_n$ to obtain an algebra $\text{AP}_d(n)$ which coincides with the partition algebra when $n \geq 2d + 2$. We take the smallest possible step in the opposite direction: we reveal what takes the place of the polynomial representations of $GL_n(C)$ when the action of the symmetric group $S_d$ is restricted to the alternating group $A_d$. The situation is summarized in Table [1]. The significance of this investigation lies in its connection with the

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Koszul duality functor on the category of homogeneous polynomial representations of $\text{GL}_n(\mathbb{C})$ of degree $d$.

1.2. Schur algebras for the alternating group. Motivated by Green [13] Theorem 2.6c, define the Schur algebra as

$$S_F(n, d) = \text{End}_{S_d}(F^n)^{\otimes d}$$

for any field $F$, and positive integers $n$ and $d$. When $F$ is infinite, then $S_F(n, d)$-modules are the same as homogeneous polynomial representations of $\text{GL}_n(F)$ of degree $d$ (see [13] Section 2.4 and [22] Section 6.2). Define the alternating Schur algebra $AS_F(n, d)$ by replacing $S_d$ by $A_d$ in the definition above:

$$AS_F(n, d) = \text{End}_{A_d}(F^n)^{\otimes d}.$$

When $F$ has characteristic different from 2, this algebra has a decomposition (Lemma 2.1)

$$AS_F(n, d) = S_F(n, d) \oplus S_F^-(n, d)$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Here $S_F^-(n, d) = \text{Hom}_{S_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn})$, where $\text{sgn}$ denotes the sign character of $S_d$. The subspace $S_F^-(n, d)$ is an $(S_F(n, d), S_F(n, d))$-bimodule.

When $n^2 < d$, then $S_F^-(n, d) = 0$, and $AS_F(n, d) = S_F(n, d)$, as observed by Regev [24] Theorem 1. But when $n^2 \geq d$, $S_F^-(n, d) \neq 0$, and in Lemma 2.2 we note that $S_F^-(n, d)$ is a full tilting left $S_F(n, d)$-module as studied by Donkin in [9] Section 3.

1.3. Bases and structure constants. By indexing Schur’s basis of $S_F(n, d)$ by bipartite multigraphs with $n + n$ vertices and $d$ edges, M´endez [21] (see also Geetha and Prasad [12]) gave a combinatorial interpretation of the structure constants. We describe a basis of $S_F^-(n, d)$ in terms of bipartite simple graphs in Theorem 2.24. So from the decomposition (1), a basis of $AS_F(n, d)$ is obtained. An enumerative interpretation of the structure constants of $AS_F(n, d)$ is given in Theorems 2.14 and 2.25.
1.4. Koszul duality and modules. The term Koszul duality is used for several constructions which interchange the roles of exterior and symmetric powers.

The earliest notion of Koszul duality was introduced by Priddy [23]. It applies to pre-Koszul algebras, which are also called quadratic algebras. A pre-Koszul algebra is a quotient of a tensor algebra $T(V) = \bigoplus_{n\geq 0} \otimes^n V$ by a two-sided ideal $I$ that is generated in degree two. Its Koszul dual is the algebra $T(V^*)/(I \cap V^* \otimes V^*)^d$; the quotient of the dual tensor algebra by the annihilator in degree two of $I$. In this setting the Koszul dual of the symmetric algebra of $V$ is the exterior algebra of $V^*$.

Bernstein, Gelfand, and Gelfand [3] introduced an equivalence between the derived categories of graded modules over symmetric and exterior algebras, which was called the Koszul duality functor by Beilinson, Ginsburg, and Schectman [4]. Friedlander and Suslin [10] introduced the category of strict polynomial functors of degree $d$ as the representations of the Schur category of degree $d$, for each non-negative integer $d$ (see Section 1.1). The category of strict polynomial functors of degree $d$ unifies the categories of homogeneous polynomial representations of $GL_n(F)$ of degree $d$ across all $n$. Standard examples of strict polynomial functors of degree $d$ are the $d$th tensor power functor $\otimes^d$, the $d$th symmetric power functor $\text{Sym}^d$, and the $d$th exterior power functor $\wedge^d$. Evaluating a strict polynomial functor of degree $d$ at $F^n$ gives an $S_F(n,d)$-module for each $n$. Friedlander and Suslin showed that this evaluation functor is an equivalence of categories when $n \geq 2$. Chalupnik [8] and Touzé [29] used the term Koszul duality to refer to a functor on the category of strict polynomial functors of degree $d$ which takes the Schur functor associated to the partition $\lambda$ of $d$ to the Weyl functor associated with the partition $\lambda'$ conjugate to $\lambda$. Krause [16] discovered an internal tensor product on the category of strict polynomial functors of fixed degree $d$. Given such a tensor product it was then natural for him to define Koszul duality in this category as tensor product with $\wedge^d$. He pointed out that the Koszul duality functors defined by Chalupnik and Touzé are not the same; they coincide with a duality defined by Ringel [25] using tilting modules for quasi-hereditary algebras. This tilting module was described by Donkin [9] in the case of Schur algebras.

We introduce the term abstract Koszul duality to refer to a very simple functor which makes sense for any $\mathbb{Z}/2\mathbb{Z}$-graded algebra $AS = S \oplus S^{-}$. The abstract Koszul dual of an $S$-module $V$ is defined as

$$D(V) = S^{-} \otimes_S V.$$  

The multiplication operation on $AS$ gives rise to an $(S, S)$-bimodule homomorphism $\phi : S^{-} \otimes_S S^{-} \to S$. We prove (Theorem 3.4) that the category of $AS$-modules is same as the category of pairs $(M, \theta_M)$ where $M$ is an $S$-module and $\theta_M : D(M) \to M$ is compatible with $\phi$ in the sense of [15].

In Section 4 we specialize to the case $AS_F(n,d) = S_F(n,d) \oplus S_F^-(n,d)$ to obtain a Koszul duality functor $D$ on the category of $S_F(n,d)$-modules. In Theorem 4.5 we show that the evaluation at $F^n$ of the Koszul duality functor of Krause is naturally isomorphic to our Koszul duality functor when $n \geq 2$. In this sense, our abstract Koszul duality functor on Schur algebras coincides with Krause’s Koszul duality.

Using our combinatorial interpretation of the structure constants of $AS_F(n,d)$ we show that, when $n \geq 2$ and when the characteristic of $F$ is 0 or greater than $d$, then abstract Koszul duality is an equivalence (Theorem 4.2).
Krause [16] showed that derived Koszul duality functor is an equivalence of categories when \( n \geq d \). Using our combinatorial methods, we obtain the converse (Theorem 4.5). This proof uses a criterion of Happel [14] for a tensor functor to be a derived equivalence. In the context of derived Koszul duality, this criterion requires that the canonical algebra homomorphism \( S_F(n,d) \rightarrow \End_{S_F(n,d)}(S_F(n,d)) \) is an isomorphism. Donkin [9], Proposition 3.7 proved this when \( n \geq d \). When the characteristic of \( F \) is not 2 we give a simple combinatorial proof of Donkin’s result, and also show that it fails when \( n < d \) (Theorem 4.7).

We conclude this paper by discussing a possible application of our techniques to Bloss’s alternating partition algebra, and a diagrammatic interpretation of the Schur category (Section 5).

2. The Alternating Schur Algebra

Let \( F \) be a field of characteristic different from 2, \( n \) and \( d \) be positive integers. The symmetric group \( S_d \) acts on the tensor space \((F^n)^{\otimes d}\) by permuting the tensor factors. The Schur algebra can be defined as

\[
S_F(n,d) := \End_{S_d}(F^n)^{\otimes d}.
\]

By restricting the action of \( S_d \) to the alternating group \( A_d \), define the alternating Schur algebra as

\[
AS_F(n,d) := \End_{A_d}(F^n)^{\otimes d}.
\]

Clearly, \( S_F(n,d) \) is a subalgebra of \( AS_F(n,d) \).

**Lemma 2.1.** For any representations \( V \) and \( W \) of \( S_d \),

\[
\text{Hom}_{A_d}(V,W) = \text{Hom}_{S_d}(V,W) \oplus \text{Hom}_{S_d}(V, W \otimes \text{sgn}).
\]

Here \( W \otimes \text{sgn} \) denotes the twist of \( W \) by the sign character \( \text{sgn} : S_d \rightarrow \{\pm 1\} \).

Define,

\[
S_F^-(n,d) := \text{Hom}_{S_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn}).
\]

Lemma 2.1 gives a \( \mathbb{Z}/2\mathbb{Z} \)-grading of \( AS_F(n,d) \) in the sense of Bourbaki [6] Chapter III, Section 3.1:

\[
AS_F(n,d) = S_F(n,d) \oplus S_F^-(n,d).
\]

The summand \( S_F^-(n,d) \) is an \( (S_F(n,d), S_F(n,d)) \)-bimodule. Recall that a weak composition of \( d \) with \( n \) parts is a vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of non-negative integers summing to \( d \). Let \( \Lambda(n,d) \) denote the set of weak compositions of \( d \) with \( n \) parts.

For each \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,d) \), define

\[
\wedge^\lambda F^n = \wedge^{\lambda_1} F^n \otimes \cdots \otimes \wedge^{\lambda_n} F^n.
\]

where, for a non-negative integer \( s \), \( \wedge^s F^n \) is the \( s \)th exterior power of \( F^n \). As a left \( S_F(n,d) \)-module, \( S_F^-(n,d) = \bigoplus_{\lambda \in \Lambda(n,d)} \wedge^\lambda F^n \).

**Lemma 2.2.** The left module \( S_F^-(n,d) \) is a full tilting module of \( S_F(n,d) \).
2.1. Twisted Permutation Representations. Let $X$ be a finite set on which a group $G$ acts on the right (henceforth called a $G$-set). The space $F[X]$ of $F$-valued functions on $X$ may be regarded as a representation of $G$:

$$\rho_X(g)f(x) = f(x \cdot g), \quad \text{for } x \in X, \ g \in G, \ \text{and } f \in F[X].$$

Let $\chi$ be a multiplicative character $G \rightarrow F^*$. One may twist the representation (5) by $\chi$:

$$\rho_X^\chi(g)f(x) = \chi(g)f(x \cdot g).$$

Denote the representation space of this twisted action as $F[X] \otimes \chi$.

Suppose that $X$ and $Y$ are finite $G$-sets. Given a function $\kappa : X \times Y \rightarrow F$, the integral operator $\xi_\kappa : F[Y] \rightarrow F[X]$ associated to $\kappa$ is defined as

$$\xi_\kappa f(x) = \sum_{y \in Y} \kappa(x,y)f(y), \quad \text{for } f \in F[Y].$$

The function $\kappa$ is known as the integral kernel of $\xi_\kappa$.

If $Z$ is another finite $G$-set, $\kappa' : X \times Y \rightarrow F$ and $\kappa'' : Y \times Z \rightarrow F$ are functions.

Then

$$\xi_{\kappa'} \otimes \xi_{\kappa''} = \xi_{\kappa' \cdot \kappa''},$$

where $\kappa' \cdot \kappa'' : X \times Z \rightarrow F$ is the convolution product

$$\kappa' \cdot \kappa''(x,z) = \sum_{y \in Y} \kappa'(x,y)\kappa''(y,z).$$

We have (see [22, Section 4.2]):

**Theorem 2.3.** For any finite $G$-spaces $X$ and $Y$, and any multiplicative character $\chi : G \rightarrow F^*$,

$$\text{Hom}_G(F[Y], F[X] \otimes \chi) = \{\xi_\kappa | \kappa : X \times Y \rightarrow F \text{ such that } \kappa(x \cdot g,y \cdot g) = \chi(g)\kappa(x,y)\}. $$

The identity (6) implies that

$$\dim \text{Hom}_G(F[Y], F[X] \otimes \chi) \leq |(X \times Y)/G|,$$

with equality holding if $\chi$ is the trivial character. However, if $g \in G$, and $(x,y) \in X \times Y$ are such that $(x \cdot g,y \cdot g) = (x,y)$, then if $\xi_\kappa \in \text{Hom}_G(F[Y], F[X] \otimes \chi)$,

$$\kappa(x,y) = \kappa(x \cdot g,y \cdot g) = \chi(g)\kappa(x,y),$$

so that either $\chi(g) = 1$ or $\kappa$ vanishes on the $G$-orbit of $(x,y)$.

**Definition 2.4 (Transverse Pair).** A pair $(x,y) \in X \times Y$ is said to be transverse with respect to $\chi$ if $G_x \cap G_y \subset \ker \chi$. If $(x,y)$ is a transverse pair with respect to $\chi$, we write $x \pitchfork y$.

If $(x,y)$ is a transverse pair, then

$$\kappa(x \cdot g,y \cdot g) := \chi(g)\kappa(x,y)$$

is a well-defined non-zero function on the $G$-orbit of $(x,y)$. Let

$$X \pitchfork Y = \{(x,y) \in X \times Y | x \pitchfork y\}.$$ 

Then $X \pitchfork Y$ is stable under the diagonal action of $G$ on $X \times Y$. We have (see [22, Theorem 4.2.3]):
THEOREM 2.5. Let $X$ and $Y$ be finite $G$-sets, and $\chi : G \to F^*$ be a multiplicative character. For each orbit $O \in (X \triangleleft Y)/G$, choose a base point $(x_O, y_O) \in O$. Define
\[
\kappa_O(x, y) = \begin{cases} 
\chi(g) & \text{if } x = x_O \cdot g \text{ and } y = y_O \cdot g \text{ for some } g \in G, \\
0 & \text{otherwise}. 
\end{cases}
\]
For simplicity, write $\xi_O$ for $\kappa_{x_O}$. Then the set
\[
\{\xi_O \ | \ O \in (X \triangleleft Y)/G\}
\]
is a basis for $\text{Hom}_G(F[Y], F[X] \otimes \chi)$. Consequently,
\[
\dim \text{Hom}_G(F[Y], F[X] \otimes \chi) = |(X \triangleleft Y)/G|.
\]
In the special case where $\chi$ is the trivial character, we get:

COROLLARY 2.6. Let $X$ and $Y$ be finite $G$-sets. For each orbit $O$ in $(X \times Y)/G$ define
\[
\kappa_O(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in O, \\
0 & \text{otherwise}. 
\end{cases}
\]
Write $\xi_O = \kappa_{x_O}$. Then the set
\[
\{\xi_O \ | \ O \in (X \times Y)/G\}
\]
is a basis for $\text{Hom}_G(F[Y], F[X])$. Consequently,
\[
\dim \text{Hom}_G(F[Y], F[X]) = |(X \times Y)/G|.
\]
Given a function $\kappa : X \times Y \to F$, define
\[
\kappa^*(y, x) = \kappa(x, y) \text{ for } x \in X, y \in Y.
\]
The following is easy to see:

LEMMA 2.7. For any $G$-set $X$, the map $\xi \mapsto \xi^*$ is an anti-involution on the algebra $\text{End}_G(F[X])$.

2.2. Structure Constants of the Schur Algebra. We recall the combinatorial interpretation of structure constants of the Schur algebra from [12]. Let $[n] = \{1, \ldots, n\}$ and
\[
I(n, d) = \{\underline{i} := (i_1, \ldots, i_d) \ | \ i_s \in [n]\}.
\]
An element $w \in S_d$ acts on $I(n, d)$ by permuting the coordinates:
\[
(i_1, \ldots, i_d) \cdot w = (i_{w(1)}, \ldots, i_{w(d)}).
\]
For $\underline{i} = (i_1, \ldots, i_d) \in I(n, d)$, define
\[
e_\underline{i} = e_{i_1} \otimes \cdots \otimes e_{i_d},
\]
where $e_i$ is the $i$th coordinate vector in $F^n$. The vector space $(F^n)^{\otimes d}$ has a basis
\[
\{e_\underline{i} \ | \ \underline{i} \in I(n, d)\}.
\]
and $w \in S_d$ acts on a basis vector $e_\underline{i}$ as follows:
\[
w \cdot e_\underline{i} = w \cdot (e_{i_1} \otimes \cdots \otimes e_{i_d}) = e_{i_{w^{-1}(1)}} \otimes \cdots \otimes e_{i_{w^{-1}(d)}}.
\]
Let $F[I(n, d)]$ denote the space of all $F$-valued functions on $I(n, d)$. Mapping $e_\underline{i}$ to the indicator function of $\underline{i} \in I(n, d)$ defines an isomorphism of $(F^n)^{\otimes d}$ onto $F[I(n, d)]$. Thus $(F^n)^{\otimes d}$ can be regarded as a permutation representation of $S_d$. 
Let \( B(n, d) \) denote the set of all configurations of \( d \) distinguishable balls, numbered \( 1, \ldots, d \) in \( n \) boxes, numbered \( 1, \ldots, n \). The symmetric group \( S_d \) acts on such configurations by permuting the \( d \) balls. An element of \( B(n, d) \) is a set partition

\[
\{1, \ldots, d\} = S_1 \coprod \cdots \coprod S_n,
\]

where \( S_i \) is the set of balls in the \( i \)th box.

**Lemma 2.8.** Given \( i \in I(n, d) \), let \( b(i) \) denote the balls-in-boxes configuration in \( B(n, d) \) where the \( i \)th box contains the balls \( \{s : i_s = i\} \). Then \( b : I(n, d) \to B(n, d) \) is an \( S_d \)-equivariant bijection of \( I(n, d) \) onto \( B(n, d) \).

By Corollary 2.6, a basis for \( S_F(n, d) \) is indexed by orbits for the diagonal action of \( S_d \) on \( B(n, d) \times B(n, d) \).

**Definition 2.9 (Labelled bipartite multigraph).** Let \( [n] = \{1, \ldots, n\} \) (as before) and \( [n'] = \{1', \ldots, n'\} \). A labelling of a bipartite multigraph with vertex set \( [n'] \coprod [n] \) and \( d \) edges is a function \( l : [d] \to [n'] \times [n] \) such that, for each \( (i', j) \in [n'] \times [n] \), the cardinality of \( l^{-1}(i', j) \) is the number of edges joining \( i' \) and \( j \). In other words, labels are assigned to edges without distinguishing between edges joining the same pair of vertices.

Given a pair \( S = (S_1, \ldots, S_n) \) and \( T = (T_1, \ldots, T_n) \) in \( B(n, d) \), define a labelled bipartite graph \( \gamma_{S, T} \) with multiple edges on the vertex set \( [n'] \coprod [n] \) as follows:

There are \(|S_j \cap T_i|\) edges between \( i' \) and \( j \), labelled by the numbers of the balls in \( S_j \cap T_i \).

The bipartite multigraph is always drawn in two rows, with the vertices from \([n']\) in the upper row and vertices from \([n]\) in the lower row, numbered from left to right. Since the vertices are always labelled in this manner, the vertex labels can be omitted in the drawing.

**Example 2.10.** When \( S = (\{1\}, \{2\}, \{3, 4, 5\}) \) and \( T = (\{1, 2, 3\}, \emptyset, \{4, 5\}) \), the associated labelled multigraph is:

![Bipartite Multigraph Example]

Clearly, \( (S, T) \mapsto \gamma_{S, T} \) is a bijection from \( B(n, d) \times B(n, d) \) onto the set of labelled bipartite multigraphs with vertex set \( [n'] \coprod [n] \) and \( d \) edges. The symmetric group \( S_d \) acts on \( B(n, d) \times B(n, d) \) by permuting labels. Therefore the \( S_d \) orbits in \( B(n, d) \times B(n, d) \) are obtained by simply forgetting the labels, leaving only the underlying bipartite multigraph. We write \( \Gamma_{S, T} \) for the bipartite multigraph underlying \( \gamma_{S, T} \). Such a graph can also be represented by its adjacency matrix (whose \((i, j)\)th entry is the number of edges joining \( i' \) and \( j \)), which is a matrix of non-negative integers that sum to \( d \).

In view of Corollary 2.6, we recover a result of \([12, 21]\):

**Theorem 2.11.** Let \( M(n, d) \) denote the set of all bipartite multigraphs with vertex set \( [n'] \coprod [n] \) and \( d \) edges. For each \( \Gamma \in M(n, d) \), define \( \xi_\Gamma \in S_F(n, d) = \text{End}_{S_d}(F[B(n, d)]) \) by

\[
\xi_\Gamma f(S) = \sum_{\{T : \Gamma_{S, T} = \Gamma\}} f(T).
\]
Then
\[ \{ \xi_\Gamma \mid \Gamma \in M(n,d) \} \]
is a basis for \( S_F(n,d) \).

**Remark 2.12.** If \((i, j)\) has image \( \Gamma \) under the composition \( I(n,d)^2 \to B(n,d)^2 \to M(n,d) \), then the basis element \( \xi_{i^r, j^r} \) of [13, Section 2.6] coincides with the basis element \( \xi_\Gamma \) of Theorem 2.11.

The structure constants \( c^\Gamma_{\Gamma_1, \Gamma_2} \) are defined by
\[ \xi_{\Gamma_1} \xi_{\Gamma_2} = \sum_{\Gamma \in M(n,d)} c^\Gamma_{\Gamma_1, \Gamma_2} \xi_\Gamma. \]

**Definition 2.13.** Let \( l, l_1, l_2 \) be labellings of graphs \( \Gamma, \Gamma_1, \) and \( \Gamma_2 \) in \( M(n,d) \), respectively. We say that \( (l_1, l_2) \) is compatible with \( l \) if, for all \( s = 1, \ldots, d \), if we write \( l_1(s) = (i_1', j_1) \) and \( l_2(s) = (i_2', j_2) \), then
\[
\begin{align*}
(2.13.a) & \quad i_2' = j_1, \\
(2.13.b) & \quad l(s) = (i_1', j_2).
\end{align*}
\]

**Theorem 2.14.** Let \( l \) be any labelling of \( \Gamma \). The structure constant \( c^\Gamma_{\Gamma_1, \Gamma_2} \) is the number of pairs \((l_1, l_2)\) of labellings of \( \Gamma_1 \) and \( \Gamma_2 \) that are compatible with \( l \).

Before giving a proof, we illustrate the theorem with a few examples.

**Example 2.15.** Let \( w \in S_d \) be a permutation, and assume that \( n \geq d \). Let \( \Gamma(w) \) denote the bipartite graph where \((i', j)\) is an edge if and only if \( 1 \leq i \leq d \) and \( w(i) = j \). Then, for all \( w_1, w_2 \in S_d \), \( \xi_{\Gamma(w_1)} \xi_{\Gamma(w_2)} = \xi_{\Gamma(w_1 w_2)} \).

**Example 2.16.** Consider
\[
\begin{align*}
\Gamma_1 & = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}, & \Gamma_2 & = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}, & \Gamma & = \Gamma_3 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}.
\end{align*}
\]

To find \( c^\Gamma_{\Gamma_1, \Gamma_2} \), with
\[
\begin{align*}
\Gamma & = \Gamma_3 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \Gamma & = \Gamma_3 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \Gamma & = \Gamma_3 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array},
\end{align*}
\]
choose any labelling of \( \Gamma \), such as
\[
\begin{align*}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array},
\end{align*}
\]
choose any labelling of \( \Gamma \), such as
\[
\begin{align*}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array},
\end{align*}
\]
For this there are clearly three pairs of compatible labellings of \( \Gamma_1 \) and \( \Gamma_2 \), namely, we can choose which of the first three balls ends up in the second box of the middle row:
\[
\begin{align*}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}, & \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array},
\end{align*}
\]
On the other hand, if
\[ \Gamma = \Gamma_4, \]
we may take the labelling:
\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]
(11)
for which the only compatible labellings of \( \Gamma_1 \) and \( \Gamma_2 \) are:
\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]
(12)
It turns out that for no other \( \Gamma \in \Gamma(n, d) \) is it possible to find even one compatible way of labelling \( \Gamma_1 \) and \( \Gamma_2 \), so we have:
\[ \xi_{\Gamma_1} \xi_{\Gamma_2} = 3 \xi_{\Gamma_3} + \xi_{\Gamma_4}. \]

**Example 2.17.** Let \( F_{n,n} \) denote the complete bipartite graph with vertex set \([n'] \coprod [n] \) where every vertex in \([n'] \) is connected to every vertex in \([n] \). Then the coefficient of \( \xi_{F_{n,n}} \) in \( \xi_{F_{n,n}} \xi_{F_{n,n}} \) is the number of Latin squares of order \( n \) [28 Sequence A002860].

To see this, let \( l \) be any labelling of the edges of \( F_{n,n} \). Given labellings \( l_1 \) and \( l_2 \) of \( F_{n,n} \) that are compatible with \( l \), define the \((i, j)\)th entry of the Latin square associated to \( (l_1, l_2) \) to be \( k \) if \( l^{-1}(i', j) = l^{-1}_2(k', j) = l^{-1}_1(i', k) \). Remarkably, the number of Latin squares of order \( n \) is known only for \( n = 1, \ldots, 11 \).

**Proof of Theorem 2.14.** Given a labelling \( l \) of \( \Gamma \), define:
\[ S_j = \bigcup_{i=1}^{n} l^{-1}_1(i', j), \quad U_i = \bigcup_{j=1}^{n} l^{-1}_2(i', j). \]
Then \( S = (S_1, \ldots, S_n) \), and \( U = (U_1, \ldots, U_n) \) are elements of \( B(n, d) \), and by construction \( \Gamma_{S, U} = \Gamma \). Now Equation (8) implies that
\[ c_{\Gamma_1, \Gamma_2}^\Gamma = \# \{ T \in B(n, d) \mid \Gamma_{S, T} = \Gamma_1 \text{ and } \Gamma_{T, U} = \Gamma_2 \}. \]
Given \( T \in B(n, d) \) contributing to the above count, define labellings \( l_1 \) and \( l_2 \) of \( \Gamma_1 \) and \( \Gamma_2 \) by:
\[ l^{-1}_1(i', j) = S_j \cap T_i, \quad l^{-1}_2(i', j) = T_j \cap U_i. \]
Then \( (l_1, l_2) \) is compatible with \( l \). Conversely, for every pair \( (l_1, l_2) \) compatible with \( l \), take \( T = (T_1, \ldots, T_n) \) where
\[ T_k = \bigcup_{i=1}^{n} l^{-1}_1(i', k), \quad l^{-1}_2(k', j). \]
Then \( T \) contributes to the count in (12). \( \square \)

**Example 2.18.** In Example 2.16, the three compatible pairs of labels in (10) correspond to taking \( T \) as \( (\{1, 2\}, \{3, 4\}), (\{1, 3\}, \{2, 4\}) \), and \( (\{2, 3\}, \{1, 4\}) \), respectively, and the compatible pair of labels in (11) corresponds to \( T = (\{1, 2\}, \{3, 4\}) \).
2.3. A basis for $S_F(n, d)$. By Theorem \ref{lem:2.5} a basis of $S_F(n, d)$ is indexed by orbits in $B(n, d) \cap B(n, d)/S_d$.

**Lemma 2.19.** A pair $(S, T) \in B(n, d)^2$ lies in $B(n, d) \cap B(n, d)$ if and only if $\gamma_{S, T}$ is a simple bipartite graph.

**Proof.** Let $S = (S_1, \ldots, S_n)$, $T = (T_1, \ldots, T_n)$. If $\gamma_{S, T}$ is not simple, then there exist indices $i$ and $j$ such that $S_j \cap T_i$ contains at least two elements, say $k$ and $l$. The transposition $(kl) \in S_d$ stabilized $(S, T)$ but has $\text{sgn}((kl)) = -1$, so $(S, T) \notin B(n, d) \cap B(n, d)$.

However, if $\gamma_{S, T}$ is simple, then the simultaneous stabilizer of $S$ and $T$ in $S_d$ is trivial, so $(S, T) \in B(n, d) \cap B(n, d)$. □

In order to specify a basis for $S_F(n, d)$ using Theorem \ref{lem:2.5} we need to choose a base point for each $S_d$-orbit in $B(n, d) \cap B(n, d)$. We do this using the following definition:

**Definition 2.20 (Standard labelling of a bipartite simple graph).** Given a bipartite simple graph $\Gamma$ with vertex set $[n'] \coprod [n]$, label each edge by its index when the edges $(i', j)$ are arranged in increasing lexicographic order, with priority given to the upper index, i.e., $(i', j) < (r', s)$ if either $i' < r'$ or $i' = r'$ and $j < s$.

**Example 2.21.** Take

\[ \Gamma = \begin{array}{ccc} \bullet & \times & \bullet \\ \bullet & & \bullet \end{array} \]

The edges, written in lexicographic order, are:

$(1', 2), (2', 1), (2', 2), (3', 2)$.

Therefore the standard labelling is:

\[ \begin{array}{ccc} \bullet & \times & \bullet \\ \bullet & & \bullet \end{array} \]

**Definition 2.22 (Sign of a labelling of a bipartite simple graph).** Let $l_0$ denote the standard labelling of a simple bipartite graph $\Gamma$ on $[n'] \coprod [n]$, label each edge by its index when the edges $(i', j)$ are arranged in increasing lexicographic order, with priority given to the upper index, i.e., $(i', j) < (r', s)$ if either $i' < r'$ or $i' = r'$ and $j < s$.

**Example 2.23.** For the graph from Example 2.21 the labellings:

\[ l_1 = \begin{array}{ccc} \bullet & \times & \bullet \\ \bullet & & \bullet \end{array} \] and \[ l_2 = \begin{array}{ccc} \bullet & \times & \bullet \\ \bullet & & \bullet \end{array} \]

give rise to permutations 4231 and 1342 respectively, so that $\epsilon(\Gamma, l_1) = -1$, and $\epsilon(\Gamma, l_2) = +1$.

Recall, from Section 2.2, that $\gamma_{S, T}$ is a labelled bipartite graph associated to $(S, T) \in B(n, d) \times B(n, d)$, whose underlying unlabelled graph is denoted by $\Gamma_{S, T}$. 
Theorem 2.24. Let $N(n,d)$ denote the set of all bipartite simple graphs with vertex set $[n] \coprod [n]$ and $d$ edges. For each $\Gamma \in N(n,d)$, define $\zeta_\Gamma \in \mathcal{S}_F(n,d) = \text{Hom}_S(F[B(n,d)], F[B(n,d)] \otimes \text{sgn})$ by
\[
\zeta_\Gamma f(S) = \sum_{\{T | \Gamma^{S,T} = \Gamma\}} \epsilon(\gamma_{S,T}) f(T).
\]
The set \{\zeta_\Gamma | \Gamma \in N(n,d)\} forms a basis of $\mathcal{S}_F(n,d)$.

Proof. Recall that we choose the pair $(S_0, T_0)$ corresponding to the standard labelling $l_0$ of $\Gamma$ as the base point of the orbit associated to $\Gamma$. A pair $(S, T)$ is in the orbit of $(S_0, T_0)$ if and only if $\Gamma^{S,T} = \Gamma$. And the sign of the permutation $w \in S_d$ such that $S = S_0.w$ and $T = T_0.w$ is the sign of the labelled bipartite graph $\gamma_{S,T}$. So the integral kernel $\kappa_\Gamma$ of the operator $\zeta_\Gamma$ is:
\[
\kappa_\Gamma(S, T) = \begin{cases} 
\epsilon(\gamma_{S,T}) & \text{if } \Gamma^{S,T} = \Gamma, \\
0 & \text{otherwise}.
\end{cases}
\]
So theorem follows from Theorem 2.5. \qed

Theorem 2.14 tells us how to multiply two elements of the subalgebra $S_F(n,d)$ of $\mathcal{A}_F(n,d)$. The remaining structure constants are given by the following theorem.

Theorem 2.25. The remaining structure constants are given as follows:

\begin{itemize}
\item[(2.25.a)] Given $\Gamma_1 \in M(n,d)$, and $\Gamma_2 \in N(n,d)$,
\[
\xi_{\Gamma_1} \zeta_{\Gamma_2} = \sum_{\Gamma \in N(n,d)} e_{\Gamma_1,\Gamma_2}^\Gamma \zeta_\Gamma,
\]
where
\[
e_{\Gamma_1,\Gamma_2}^\Gamma = \sum_{l_1, l_2} \epsilon(l_1, l_2),
\]
and the sum runs over all labellings $l_1$ and $l_2$ of $\Gamma_1$ and $\Gamma_2$ respectively, that are compatible with the standard labelling $l$ of $\Gamma$.

\item[(2.25.b)] Given $\Gamma_1 \in N(n,d)$, and $\Gamma_2 \in M(n,d)$,
\[
\zeta_{\Gamma_1} \xi_{\Gamma_2} = \sum_{\Gamma \in N(n,d)} e_{\Gamma_1,\Gamma_2}^\Gamma \xi_\Gamma,
\]
where
\[
e_{\Gamma_1,\Gamma_2}^\Gamma = \sum_{l_1, l_2} \epsilon(l_1, l_2),
\]
and the sum runs over all labellings $l_1$ and $l_2$ of $\Gamma_1$ and $\Gamma_2$ respectively, that are compatible with the standard labelling $l$ of $\Gamma$.

\item[(2.25.c)] Given $\Gamma_1 \in N(n,d)$, and $\Gamma_2 \in N(n,d)$,
\[
\zeta_{\Gamma_1} \zeta_{\Gamma_2} = \sum_{\Gamma \in M(n,d)} e_{\Gamma_1,\Gamma_2}^\Gamma \xi_\Gamma,
\]
where
\[
e_{\Gamma_1,\Gamma_2}^\Gamma = \sum_{l_1, l_2} \epsilon(l_1, l_2),
\]
and the sum runs over all labellings $l_1$ and $l_2$ of $\Gamma_1$ and $\Gamma_2$ respectively, that are compatible with the standard labelling $l$ of $\Gamma$.
\end{itemize}
and the sum runs over all labellings $l_1$ and $l_2$ of $\Gamma_1$ and $\Gamma_2$ respectively, that are compatible with a fixed labelling $l$ of $\Gamma$.

**Proof.** Given a labelling $l$ of $\Gamma$, construct $S$ and $U$ in $B(n,d)$ as in the proof of Theorem 2.14. Define $\kappa_{\Gamma_2} : B(n,d) \times B(n,d) \to F$ by

$$\kappa_{\Gamma_2}(T,U) = \epsilon(\gamma_{T,U}).$$

Then $\zeta_{\Gamma_2}$ is the integral operator $\xi_{\kappa_{\Gamma_2}}$, as in (7). Then, by Equation (8), the structure constant in Part (2.25.a) of the theorem is given by:

$$c_{\Gamma_1 \Gamma_2} = \sum_T \zeta_{\Gamma_2}(T,U),$$

where the sum runs over all $T \in B(n,d)$ such that $\Gamma_S,T = \Gamma_1$, and $\Gamma_T,U = \Gamma_2$. Defining labelling $l_1$ and $l_2$ of $\Gamma_1$ and $\Gamma_2$ as in the proof of Theorem 2.14 we find that $\zeta_{\Gamma_2}(T,U) = \epsilon(\Gamma_2,l_2)$, proving (2.25.a). The proofs of the remaining assertions are similar. □

**Definition 2.26.** Given $\Gamma \in M(n,d) \sqcup N(n,d)$, we define $\Gamma^*$ to be the horizontal reflection of $\Gamma$, i.e., $i'$ is connected to $j$ in $\Gamma^*$ if and only if $j'$ is connected to $i$ in $\Gamma$. The operation $*$ on the set $M(n,d) \sqcup N(n,d)$ is an involution.

**Lemma 2.27.** For every $\Gamma \in N(n,d)$, let $l_0$ denote its standard labelling. Let $l_0^\ast$ denote the labelling of $\Gamma^*$ given by $l_0^\ast(i',j) = l_0(j',i)$. Then the linear map $AS_F(n,d) \to AS_F(n,d)$ defined by:

$$\xi_{\Gamma} \mapsto \xi_{\Gamma^*} \text{ for } \Gamma \in M(n,d),$$

$$\zeta_{\Gamma} \mapsto \epsilon(\Gamma^*,l_0^\ast)\zeta_{\Gamma^*} \text{ for } \Gamma \in N(n,d)$$

is an anti-involution of $AS_F(n,d)$.

**Remark 2.28.** The above involution, when restricted to the Schur algebra, is the same as the one described by Green [13, Section 2.7].

**Proof.** We show that the linear map in Lemma 2.27 is the same as the anti-involution in Lemma 2.7 with $X = B(n,d)$ and $G = A_d$.

For $\Gamma \in M(n,d)$, $\xi_{\Gamma}$ is the integral operator with kernel:

$$\kappa_{\Gamma}(S,T) = \begin{cases} 1 & \text{if } \Gamma_S,T = \Gamma, \\ 0 & \text{otherwise}. \end{cases}$$

Since $\Gamma_{T,S} = \Gamma_{S,T}^\ast$, $\kappa_{\Gamma} = \kappa_{\Gamma^\ast}$.

For $\Gamma \in N(n,d)$, $\zeta_{\Gamma}$ is the integral operator with kernel:

$$\kappa_{\Gamma}(S,T) = \begin{cases} \epsilon(\gamma_{S,T}) & \text{if } \Gamma_S,T = \Gamma, \\ 0 & \text{otherwise}. \end{cases}$$

Thus, if $\gamma_{S,T} = (\Gamma,l_0)$, then $\gamma_{T,S} = (\Gamma^*,l_0^\ast)$. Therefore,

$$\kappa_{\Gamma^\ast}(T,S) = \epsilon(\gamma_{T,S}) = \epsilon(\Gamma^*,l_0^\ast)\kappa_{\Gamma}(S,T).$$

So the kernels $\kappa_{\Gamma^\ast}$ and $\epsilon(\Gamma^*,l_0^\ast)\kappa_{\Gamma}$ coincide at $(T,S)$, and hence on its entire $S_d$-orbit in $B(n,d)$. □

We illustrate the above results with an example that will be used in the proof of Lemma 4.1.
Example 2.29. Recall that $\Lambda(n, d)$ denotes the set of all weak compositions of $d$ with at most $n$ parts. For $\lambda \in \Lambda(n, d)$ with $n \geq d$, let $\Gamma_\lambda \in N(n, d)$ denote the bipartite graph where $i'$ is connected to $j$ if

$$\lambda_1 + \cdots + \lambda_{i'-1} < j \leq \lambda_1 + \cdots + \lambda_i.$$

Then we have:

$$\zeta_{\Gamma_\lambda} \zeta_{\Gamma_\lambda} = \lambda_1! \cdots \lambda_n! \xi_{\Gamma_0^n},$$

where $\Gamma_0^n \in M(n, d)$ is the bipartite multigraph where $i'$ is connected to $i$ by $\lambda_i$ edges. For example,

$$\Gamma_{(2,1)}^* = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad \text{and} \quad \Gamma_{(2,1)} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}$$

For a labelling $l_1 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}$ of $\Gamma_{(2,1)}^*$ only the labelling

$$l_2 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}$$

of $\Gamma_{(2,1)}$, and the labelling $l = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}$ of $\Gamma_{(2,1)}^0$ are such that $(l_1, l_2)$ are compatible with $l$. Moreover, $\epsilon(\Gamma_{(2,1)}^*, l_1) = \epsilon(\Gamma_{(2,1)}, l_2)$. Interchanging the labels $a$ and $b$ in $l_1$ and $l_2$, respectively, gives another pair of labels compatible with $l$, so that $\zeta_{\Gamma_\lambda} \zeta_{\Gamma_\lambda} = 2 \xi_{\Gamma_0^n}$.

The remaining results in this section help us understand the structure of $S_F(n, d)$ as an $S_F(n, d)$-module. Some of them will play an important role in understanding Koszul duality (Section 4).

The notion of standard labelling (Definition 2.20) of graphs in $N(n, d)$ can be extended to graphs in $M(n, d)$ as follows: when an edge $(i', j)$ occurs with multiplicity $m$, it is simply listed $m$ times when the edges are arranged in lexicographic order with priority given to the upper index. Example 2.10 is the standard labelling of its underlying graph.

Definition 2.30. For $n \geq d$, define the following simple bipartite graphs associated to $\Gamma \in M(n, d)$:

(2.30.a) Let $D(\Gamma) \in N(n, d)$ be the graph with edges $(i', s)$ for every edge $(i', j)$ with label $s$ under the standard labelling of $\Gamma$.

(2.30.b) Let $U(\Gamma) \in N(n, d)$ be the graph with edges $(s', j)$ for every edge $(i', j)$ with label $s$ under the standard labelling of $\Gamma$.

Example 2.31. Let $n = 5$, $d = 5$, and $\Gamma$ (with its standard labelling) is given by:

\begin{center}
\begin{tikzpicture}
\node at (0,0) {}; \\
\node at (1,1) {}; \\
\node at (1,-1) {}; \\
\node at (2,0) {}; \\
\node at (3,0) {}; \\
\node at (4,0) {}; \\
\node at (5,0) {}; \\
\node at (6,0) {}; \\
\end{tikzpicture}
\end{center}
Then

\[ D(\Gamma) = \begin{array}{c|ccccc}
2 & 3 & 4 & 5 \\
\end{array} \]

and

\[ U(\Gamma) = \begin{array}{c|ccccc}
1 & 3 & 4 & 5 \\
\end{array} \]

The significance of the elements \( U(\Gamma) \) and \( D(\Gamma) \), for \( \Gamma \in M(n,d) \), is elaborated in the following lemmas.

**Lemma 2.32.** Let \( n \geq d \) and \( \Gamma \in N(n,d) \). Then \( \zeta_\Gamma = \xi_{U(\Gamma)}\xi_{\Gamma|\lambda_0}\xi_{D(\Gamma)} \), where \( \lambda_0 = (1^d,0^{n-d}) \in \Lambda(n,d) \). Consequently, \( S_F(n,d) \) is a cyclic \( (S_F(n,d),S_F(n,d)) \)-bimodule.

**Proof.** This can be done in two steps. Firstly, \( \zeta_{\Gamma|\lambda_0}\xi_{D(\Gamma)} = \zeta_{\Gamma} \), and secondly \( \xi_{U(\Gamma)}\zeta_{D(\Gamma)} = \zeta_{\Gamma} \). We indicate the proof of the second identity (the first is similar): Let \( l_0, l_1, \) and \( l_2 \) be the standard labellings of \( \Gamma, U(\Gamma) \) and \( D(\Gamma) \) respectively. The labellings \( (l_1,l_2) \) of \( D(\Gamma) \) and \( U(\Gamma) \) are the only ones that are compatible with \( l_0 \). This is because, for the edge \((i',j)\) of \( \Gamma \) labelled \( s \), \( s \) is the unique vertex such \( i' \) is connected to \( s \) in \( D(\Gamma) \) and \( s' \) is connected to \( j \) in \( U(\Gamma) \). The identity now follows from \ref{2.25.2}.

Similarly, we have:

**Lemma 2.33.** For \( n \geq d \) and \( \Gamma \in N(n,d) \), we have \( \zeta_\Gamma = \xi_{U(\Gamma)}\xi_{D(\Gamma)} \) and \( \zeta_\Gamma = \xi_{U(\Gamma)}\xi_{D(\Gamma)} \).

**Corollary 2.34.** As a left \( S_F(n,d) \)-module, \( S_F(n,d) \) is generated by \( \{\zeta_{\Gamma|\lambda} \mid \lambda \in \Lambda(n,d)\} \), and as a right \( S_F(n,d) \)-module, it is generated by \( \{\zeta_{\Gamma|\lambda} \mid \lambda \in \Lambda(n,d)\} \). Here \( \Gamma_{\lambda} \) is the graph associated to \( \lambda \) in Example \ref{2.32}.

**Proof.** For any \( \Gamma \in M(n,d) \), \( D(\Gamma) \) is of the form \( \Gamma_{\lambda}^1 \) for some \( \lambda \in \Lambda(n,d) \), so the statement for left modules follows from the second identity in Lemma \ref{2.33}. The statement for right modules follows by applying Lemma \ref{2.27} to the first identity in Lemma \ref{2.33}.

**Lemma 2.35.** Let \( n \geq d \) and \( \Gamma \in M(n,d) \). Then, for \( \Gamma' \in M(n,d) \), the structure constant of \( \zeta_{U(\Gamma')} \zeta_{D(\Gamma)} \) in the product \( \xi_{U(\Gamma')}\xi_{D(\Gamma)} \) is \( \delta_{\Gamma',\Gamma} \).

**Proof.** The edge \((i',j)\) with label \( s \) in the standard labelling of \( \Gamma \) gives rise to an edge \((s',j)\) with standard label \( s \) in \( U(\Gamma) \). The graph \( D(\Gamma)^+ \) has only one edge originating at \( s' \), namely \((s',i)\). Therefore, for any compatible pair \((l_1,l_2)\) of labellings of \( D(\Gamma)^+ \) and \( \Gamma' \), this edge must have label \( s \). Thus \( \Gamma' \) must have an edge \((i',j)\) labelled \( s \). In other words, \( \Gamma' = \Gamma \) and \( l_2 \) is its standard labelling.

### 3. Abstract Koszul Duality

#### 3.1. The Algebra

Recall \cite{6}, Chapter III, Section 3.1 that a \( \mathbb{Z}/2\mathbb{Z} \) grading on a ring \( AS \) is a decomposition \( AS = S \oplus S^- \) into additive subgroups such that \( S \) is a subring, \( S^- \) is closed under left and right multiplication by elements of \( S \), and for any \( \alpha, \beta \in S^- \), \( \alpha \beta \in S \). This \( \mathbb{Z}/2\mathbb{Z} \)-grading gives rise to:
(a) an \((S, S)\)-bimodule structure on \(S^-\),
(b) and an \((S, S)\)-bimodule homomorphism \(\phi : S^- \otimes_S S^- \rightarrow S\) (induced by
the \(S\)-balanced bilinear map \((\alpha, \beta) \mapsto \alpha \beta\) for \(\alpha, \beta \in S^-\)).

Example 3.1. We may take \(AS = AS_F(n, d) = \text{End}_{A_d}(F^n) \otimes d\), and \(S = S_F(n, d) = \text{End}_{S_d}(F^n) \otimes d\), for any field \(F\) with characteristic different from 2.

3.2. Modules. Let \(M\) be an \(AS\)-module. The \(AS\)-module structure can be
viewed as a linear map:
\[
AS \otimes_Z M = (S \oplus S^-) \otimes_Z M = (S \otimes_Z M) \oplus (S^- \otimes_Z M) \rightarrow M.
\]
So \(M\) is an \(S\)-module, and restriction of the module action \(AS \otimes Z M \rightarrow M\) to
\(S^- \otimes_Z M\) induces an \(S\)-module homomorphism:
\[
\theta_M : S^- \otimes_S M \rightarrow M.
\]
Furthermore, this homomorphism \(\theta_M\) has the property that the diagram
\[
\begin{array}{ccc}
S^- \otimes_S S^- \otimes_S M & \xrightarrow{\phi \otimes \text{id}_M} & S \otimes_S M \\
\downarrow^\text{id}_{S^-} \otimes\theta_M & & \downarrow^\text{id}_{S} \otimes\theta_M \\
S^- \otimes_S M & \xrightarrow{\theta_M} & M
\end{array}
\]
commutes.

Definition 3.2. Given an \((S, S)\)-bimodule \(S^-\) and an \((S, S)\)-bimodule homo-
morphism \(\phi : S^- \otimes_S S^- \rightarrow S\), for an \(S\)-module \(N\), an \(S\)-module homomorphism
\(\theta : S^- \otimes_S N \rightarrow N\) is said to be compatible with \(\phi\) if the diagram
\[
\begin{array}{ccc}
S^- \otimes_S S^- \otimes_S N & \xrightarrow{\phi \otimes \text{id}_N} & S \otimes_S N \\
\downarrow^\text{id}_{S^-} \otimes\theta & & \downarrow^\theta \\
S^- \otimes_S N & \xrightarrow{\theta} & N
\end{array}
\]
commutes.

3.3. Duality. Let \(AS = S \oplus S^-\) be as before. Define a functor \(D : S\text{-Mod} \rightarrow
S\text{-Mod}\) by:
\[
D(M) = S^- \otimes_S M,
\]
for every \(S\)-module \(M\). Given \(S\)-modules \(M\) and \(N\), and an \(S\)-module homomor-
phism \(f : M \rightarrow N\), let \(D(f) = \text{id}_{S^-} \otimes f : D(M) \rightarrow D(N)\). We call the resulting
functor \(D : S\text{-Mod} \rightarrow S\text{-Mod}\) an abstract Koszul duality functor. In Section 3 it
will be shown that, in the setting of Example 3.1 (the alternating Schur algebra),
abstract Koszul duality is essentially the Koszul duality functor of Krause[16].

The commutative diagram \([15]\), defining the compatibility of \(\theta\) with \(\phi\), can be
rewritten in terms of abstract Koszul duality as:
\[
\begin{array}{ccc}
D^2(N) & \xrightarrow{\phi \otimes \text{id}_N} & N \\
\downarrow^D(\theta) & & \downarrow^\theta \\
D(N) & \xrightarrow{\theta} & N
\end{array}
\]
DEFINITION 3.3. Given an \((S,S)\)-bimodule \(S^-\) and an \((S,S)\)-bimodule homomorphism \(\phi : S^- \otimes_S S^- \rightarrow S\), let \((S,\phi)\)-Mod denote the category whose objects are pairs \((N,\theta)\), where \(N\) is an \(S\)-module, and \(\theta : D(N) \rightarrow N\) is compatible with \(\phi\). A morphism \((N,\theta) \rightarrow (N',\theta')\) is an \(S\)-module homomorphism \(f : N \rightarrow N'\) such that the diagram

\[
\begin{array}{ccc}
D(N) & \xrightarrow{\theta} & N \\
\downarrow D(f) & & \downarrow f \\
D(N') & \xrightarrow{\theta'} & N'
\end{array}
\]

commutes.

THEOREM 3.4. Given an \(AS\)-module \(M\), let \(\theta_M\) be as in (13). Then \(M \mapsto (\mathbf{S},\phi)\)-Mod is an isomorphism of categories \(AS\)-Mod \(\rightarrow\) \((\mathbf{S},\phi)\)-Mod.

PROOF. Given an object \((N,\theta)\) in \((S,\phi)\)-Mod, the compatibility of \(\theta\) with \(\phi\) allows the \(S\)-module structure on \(N\) to be extended to an \(AS\)-module structure. This constructs the inverse of the functor in the theorem. \(\square\)

Given an \(S\)-module \(N\), the morphism \(\phi : S^- \otimes_S S^- \rightarrow S\) gives rise to a natural transformation \(\eta_N : D^2 \rightarrow \text{id}_{\mathbf{SMod}}\), defined as the composition:

\[
D^2(N) \xrightarrow{\phi \otimes \text{id}_N} S \otimes_S N

\]

THEOREM 3.5. Let \(AS, S, S^-\) and \(\phi\) be as in Section 3.1. The following are equivalent:

(3.5.a) The map \(\phi : S^- \otimes_S S^- \rightarrow S\) is an isomorphism.

(3.5.b) The natural transformation \(\eta : D^2 \rightarrow \text{id}_{\mathbf{SMod}}\) is a natural isomorphism.

(3.5.c) For every object \((N,\theta)\) in \((S,\phi)\)-Mod, \(\theta : S^- \otimes_S N \rightarrow N\) is an isomorphism of \(S\)-modules.

PROOF. To see that (3.5.a) implies (3.5.b), observe from the diagram (17) that if \(\phi\) is an isomorphism, then \(\eta_N\) is an isomorphism for every \(N\). It follows that \(\eta\) is a natural isomorphism. For the converse, taking \(N = S\), the commutativity of (17) shows that \(\phi\) is an isomorphism.

To see that (3.5.a) implies (3.5.c), note that the commutativity of (15) implies that, if \(\phi\) is an isomorphism, then \(\theta\) is an epimorphism, and \(\text{id}_{S^-} \otimes \theta\) is a monomorphism. Since tensoring is a right-exact functor, it follows that \(\text{id}_{S^-} \otimes \theta\) is also an epimorphism, hence an isomorphism. Since \(\phi \otimes \text{id}_N\) is also an isomorphism the inverse of \(\theta\) can be constructed by reversing the arrows in (15). For the converse, just take \(N = S\) in (3.5.c). \(\square\)

3.4. Abstract Ringel duality. Let \(S^-\) be an \((S,S)\)-bimodule. Denote the left \(S\)-module \(S^-\) by \(\_S^-\). For a left \(S\)-module \(M\), the homomorphism space \(\text{Hom}_S(\_S^-, M)\) inherits the structure of a left \(S\)-module from the right \(S\)-module structure on \(S^-\). Motivated by [25, Section 6], we call the functor

\[\text{Hom}_S(\_S^-, \_ : \mathbf{SMod} \rightarrow \mathbf{SMod}\]
the abstract Ringel duality functor on S-Mod. It is clear that the abstract Koszul duality functor is the left adjoint of abstract Ringel duality functor.

3.5. Abstract Simple Modules. In general, it is not clear how simple AS-modules can be classified using simple S-modules and Koszul duality. In this section, we give some results in this direction. These are enough to give a complete solution in the semisimple case.

Let $M$ be a simple S-module. We consider the following cases:

3.5.1. $DM$ is isomorphic to $M$. If $\eta_M : D^2M \to M$ is zero, then $(M,0)$ (where 0 is the zero map from $DM \to M$) is the unique $\phi$-compatible morphism. Otherwise, any non-zero morphism $\theta : DM \to M$ is an isomorphism. Schur’s lemma implies that $\theta \circ D\theta = a\eta_M$ for some $a \in (\text{End}_S M)^*$ (the multiplicative group of non-zero elements in the division algebra $\text{End}_S M$). If $a$ has a square root in $(\text{End}_S M)^*$, then $\theta$ can be normalized to make it a $\phi$-compatible morphism. Moreover, after normalization, $\pm \theta$ are two $\phi$-compatible morphisms, leading to two non-isomorphic AS-modules. Also, in this case, $(M, \pm \theta)$ are simple, because their restrictions to $S$ are simple. On the other hand, if $a$ does not have a square root in $\text{End}_S(M)$, then there is no simple AS-module whose restriction to $S$ is isomorphic to $M$.

3.5.2. $DM = 0$. In this case $(M,0)$ is the unique AS-module whose restriction to $S$ is isomorphic to $M$.

3.5.3. $DM$ is simple, but not isomorphic to $M$, $\eta_M \neq 0$. Let $\tilde{M} = DM \oplus M$. We have $D\tilde{M} = D^2M \oplus DM$. Any morphism $\theta : DM \to \tilde{M}$ can be written in matrix form as

$$\theta = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where $X : D^2M \to DM$, $Y : DM \to DM$, $Z : D^2M \to M$, and $W : DM \to M$. By Schur’s lemma, $W = 0$. The compatibility of $\theta$ with $\phi$ becomes:

$$(X Y) \begin{pmatrix} DX & DY \\ DZ & 0 \end{pmatrix} = \begin{pmatrix} \eta_{DM} & 0 \\ 0 & \eta_M \end{pmatrix}. $$

Multiplying out the left hand side gives:

$$\begin{pmatrix} XDX + YDZ & XDY \\ ZDX & ZDY \end{pmatrix} = \begin{pmatrix} \eta_{DM} & 0 \\ 0 & \eta_M \end{pmatrix}. $$

Since $\eta_M \neq 0$, $DY \neq 0$, and so $Y \neq 0$. Since $DM$ is simple, by Schur’s lemma, $Y$ is invertible. Since $D$ is a functor, $DY$ is also invertible. Hence, equality of top right entries implies that $X = 0$. Moreover, $Z = \eta_M D^{-1} Y$. In other words, $\theta$ is of the form:

$$\theta_Y = \begin{pmatrix} 0 & Y \\ \eta_M D^{-1} Y & 0 \end{pmatrix}. $$

Lemma 3.6. For all $Y, Y' \in (\text{End}_S M)^*$, $\text{Hom}_{AS}((\tilde{M}, \theta_Y), (\tilde{M}, \theta_{Y'}))$ is non-zero, and $\text{End}_{AS}(\tilde{M}, \theta_Y)$ is a division ring.

Proof. Any AS-module morphism $(\tilde{M}, \theta_y) \to (\tilde{M}, \theta_{y'})$ can be written in matrix form as

$$\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix},$$

where $X \in \text{End}_S DM$ and $W \in \text{End}_S M$, 

$$\text{Hom}_{AS}((\tilde{M}, \theta_Y), (\tilde{M}, \theta_{Y'})) = \text{End}_{AS}(\tilde{M}, \theta_Y).$$
and must satisfy:
\[
\begin{pmatrix}
X & 0 \\
0 & W
\end{pmatrix}
\begin{pmatrix}
\eta_M D Y^{-1} \\
0
\end{pmatrix}
= 
\begin{pmatrix}
0 & Y' \\
DY^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{D} X & 0 \\
0 & \tilde{D} W
\end{pmatrix}
\]
We get \(XY = Y'DW\), and \(W \eta_M D Y^{-1} = \eta_M D Y'^{-1} DX\). Taking \(W = \text{id}_M\), and \(X = Y'Y^{-1}\) gives a non-zero element of \(\text{Hom}_{AS}(\tilde{M}, \theta_Y)\). When \(Y = Y'\), then we have \(X = YDWY^{-1}\), so that \(X\) is non-zero (and hence invertible) if and only if \(W\) is. It follows that every non-zero element of \(\text{End}_{AS}(\tilde{M}, \theta_Y)\) is invertible.

**Lemma 3.7.** Let \(M\) be an \(S\)-module. Then the \(AS\)-module \(AS \otimes_S M\) is isomorphic to \((DM \oplus M, \theta)\), where \(\theta\) is given by the matrix:

\[
\theta = \begin{pmatrix}
0 & \text{id}_M \\
\eta_M & 0
\end{pmatrix}.
\]

**Proof.** Note that \(AS \otimes_S M = (S^- \otimes_S M) \oplus (S \otimes_S M) = DM \oplus M\). The map \(\theta\) comes from the action of \(S^-\) on this \(AS\)-module, which gives \(\eta_M : D^2M \to M\) on the first summand, and \(\text{id}_M : DM \to DM\) on the second summand.

**Theorem 3.8.** The \(AS\)-module \((DM \oplus M, \theta_Y)\) defined above is isomorphic to \(AS \otimes_S M\) for every \(Y \in (\text{End}_{S} DM)^*\). Consequently, whenever \(M\) and \(DM\) are simple, non-isomorphic \(S\)-modules, and \(\eta_M \neq 0\), then \(AS \otimes_S M\) is, up to isomorphism, the unique simple \(AS\)-module whose restriction to \(S\) contains \(M\).

**Proof.** To see that \(AS \otimes_S M = DM \oplus M\) is simple, note that its only proper non-trivial \(S\)-submodules are \(M\) and \(DM\). But \(M\) is not \(AS\)-invariant because \(S^-\) maps \(M\) onto \(DM\). Also, \(DM\) is not \(AS\)-invariant, because \(S^-\) maps \(DM\) onto \(D^2M\). Since \(\eta_M \neq 0\), \(D^2M\) cannot be contained in \(DM\). The theorem now follows from Lemma 3.6.

3.5.4. The case where \(\phi\) is an isomorphism. When \(\phi : S^- \otimes_S S^- \to S\) is an isomorphism the preceding results, using Theorem 3.5, can be summarized in the following form:

**Theorem 3.9.** Suppose that \(AS\) is endowed with a \(\mathbb{Z}/2\mathbb{Z}\)-grading \(AS = S \oplus S^-\), and \(\phi : S^- \otimes_S S^- \to S\) (as defined in Section 3.1) is an isomorphism. Let \(M\) be a simple \(S\)-module. Then

(3.9.a) Suppose there exists an isomorphism \(\theta : DM \to M\). Then \(\theta\) can be scaled to become compatible with \(\phi\). There exist at most two isomorphism classes of simple \(AS\)-modules \((M, \pm \theta)\) whose restrictions to \(S\) are isomorphic to \(M\). If \((\text{End}_{S} M)^*\) is a 2-divisible group, then these two classes always exist.

(3.9.b) Otherwise, up to isomorphism, \(AS \otimes_S M\) is the unique simple \(AS\)-module whose restriction to \(S\) contains \(M\) as a submodule. Also, \(AS \otimes_S M\) and \(AS \otimes_S DM\) are isomorphic as \(AS\)-modules.

**Corollary 3.10.** Suppose \(F\) is an algebraically closed field of characteristic not equal to 2. Let \(AS = S \oplus S^-\) be a \(\mathbb{Z}/2\mathbb{Z}\)-graded \(F\)-algebra. A complete set of isomorphism classes of simple \(AS\)-modules is given by:
(3.10.c) \((M, \pm \theta)\) (defined in Section 3.5.1), as \(M\) runs over isomorphism classes of simple \(S\)-modules such that \(DM\) is isomorphic to \(M\),

(3.10.d) \(\text{AS} \otimes_S M\), as \(M\) runs over isomorphism classes of all unordered pairs \(\{M, M'\}\) of non-isomorphic mutually dual simple \(S\)-modules.

4. Koszul duality for Schur Algebras

In this section, let \(S\) denote the Schur algebra \(S_F(n, d)\), and let \(S^-\) denote the \((S, S)\)-bimodule \(S_F^{-}(n, d)\). We now use our combinatorial methods from Section 2 to determine when abstract Koszul duality is an equivalence.

**Lemma 4.1.** When the characteristic of \(F\) is 0 or greater than \(d\), and \(n \geq d\), the map \(\phi : S^- \otimes_S S^- \to S\) is an isomorphism.

**Proof.** For each \(\lambda \in \Lambda(n, d)\), let \(\Gamma^0_\lambda \in M(n, d)\) be the bipartite multigraph with \(\lambda_i\) edges from \(i'\) to \(i\) (and no other edges), as in Example 2.29 Then

\[ \text{id}_S = \sum_{\lambda \in \Lambda(n, d)} \xi_\lambda^* \lambda. \]

Therefore by Example 2.29

\[ \text{id}_S = \sum_{\lambda \in \Lambda(n, d)} \frac{1}{\lambda_1! \cdots \lambda_n!} \xi_\lambda^* \xi_\lambda^\dagger. \]

Therefore the image of \(\phi\), which is a two-sided ideal of \(S\), contains the identity element, and therefore is all of \(S\).

The injectivity of \(\phi\) can be proved using a dimension count. Let \(N_d\) (resp., \(N_d^\prime\)) denote the graphs in \(N(n, d)\) with upper (resp., lower) degree sequence \((1^d, 0^{n-d})\). Let \(\Gamma(w) \subseteq N(n, d)\) be as in Example 2.15. For \(\Gamma \subseteq N_d\), define \(\Gamma \cdot w \subseteq N_d\) by \(\xi_\Gamma \xi_{\Gamma(w)} = \xi_{\Gamma \cdot w}\). Similarly, for \(\Gamma \subseteq N_d^\prime\), define \(w \cdot \Gamma \subseteq N_d^\prime\) by \(\xi_{\Gamma(w)} \xi_\Gamma = \xi_{w \cdot \Gamma}\).

Consider the equivalence relation on \(N_d \times N_d\) where \((\Gamma, \Gamma') \sim (\Gamma \cdot w^{-1}, w \cdot \Gamma')\) for \(w \subseteq S_d\). Let \(N_d \times S_d N_d \to M(n, d)\) be a bijection.

Now, given \((\Gamma', \Gamma'') \subseteq N_d \times N_d\), define \(\Gamma = \Phi(\Gamma', \Gamma'') \subseteq M(n, d)\) to be the graph for which the number of edges joining \((i', j)\) is the number of indices \(1 \leq k \leq n\) such that \((i', k)\) is an edge of \(\Gamma''\) and \((k', j)\) is an edge of \(\Gamma'\). This map induces an injective function \(\Phi : N_d \times S_d N_d \to M(n, d)\). Moreover, \(\Gamma = \Phi(U(\Gamma), U(\Gamma'))\), so \(\Phi : N_d \times S_d N_d \to M(n, d)\) is a bijection.

The elements \(\xi_{\Gamma'} \otimes \xi_{\Gamma''}\), as \(\Gamma'\) and \(\Gamma''\) run over \(N(n, d)\), span \(S^- \otimes S^-\). We have:

\[ \xi_{\Gamma'} \otimes \xi_{\Gamma''} = \xi_{U(\Gamma')} \xi_{D(\Gamma')} \otimes \xi_{U(\Gamma'')} \xi_{D(\Gamma'')} \]

\[ = \xi_{U(\Gamma')} \otimes \xi_{D(\Gamma')} \xi_{U(\Gamma'')} \xi_{D(\Gamma'')} \]

Now \(U(\Gamma) \subseteq N_d\) and \(\xi_{D(\Gamma')} \xi_{U(\Gamma')} \xi_{D(\Gamma'')}\) lies in the span of \(\xi_{\Gamma''}\), for \(\Gamma'' \subseteq N_d\). Therefore \(\xi_{\Gamma'} \otimes \xi_{\Gamma''}\) span \(S^- \otimes S^-\) as \((\Gamma', \Gamma'') \subseteq N_d \times N_d\). Moreover, \(\xi_{\Gamma'} \otimes \xi_{\Gamma''} = \xi_{\Gamma'} \otimes \xi_{\Gamma''}\), so \(\text{dim} \ S^- \otimes_S S^- \leq |N_d \times S_d N_d| = |M(n, d)|\).

Now, using Theorem 3.5 we have established a direct combinatorial proof of the following theorem:

**Theorem 4.2.** For a field \(F\) of characteristic 0 or greater than \(d\), and \(n \geq d\), the Koszul duality functor \(D : S_{\text{Mod}} \to S_{\text{Mod}}\) is an equivalence of categories.
4.1. Strict Polynomial functor. Friedlander and Suslin [10] introduced strict polynomial functors in order to establish the finite generation of the full cohomology ring of a finite group scheme. They also showed that the strict polynomial functors of degree $d$ unify modules over the Schur algebras $S_F(n, d)$ across all $n$. In this section, we briefly recall the definition of strict polynomial functors and some useful functors on the category of strict polynomial functors.

Following [16, 30], define the Schur category (also known as the divided power category) $\Gamma^d_F$ as the category whose objects are finite dimensional vector spaces over $F$. For objects $V$ and $W$, the morphism space is:

$$\text{Hom}_{\Gamma^d_F}(V, W) := \text{Hom}_{S^d_F}(V \otimes^d W, W \otimes^d W).$$

The category, $\text{Rep}_{\Gamma^d_F}$ of strict polynomial functors is the functor category $\text{Func}((\Gamma^d_F, F\text{-Mod}))$. Thus it is an abelian, complete, and co-complete category.

**Example 4.3.** Let $V$ and $W$ be objects of $\Gamma^d_F$. Some examples of strict polynomial functors are:

1. The $d$th tensor power functor $\otimes^d : \Gamma^d_F \to F\text{-Mod}$. On objects, $\otimes^d(V) = V \otimes^d$. On the morphism space, the map $\text{Hom}_{\Gamma^d_F}(V \otimes^d W, W) \to \text{Hom}_{S^d_F}(V \otimes^d W, W)$ is the identity map.

2. The $d$th divided power functor $\Gamma^d : \Gamma^d_F \to F\text{-Mod}$. On objects $\Gamma^d(V) = (V \otimes^d)^{S_d}$ and on the morphism space, the map $\text{Hom}_{\Gamma^d_F}(V, W) \to \text{Hom}_{S^d_F}((V \otimes^d)^{S_d}, (W \otimes^d)^{S_d})$ is given by the restriction.

3. Similarly, the $d$th exterior power functor $\wedge^d : \Gamma^d_F \to F\text{-Mod}$, and the $d$th symmetric power functor $\text{Sym}^d : \Gamma^d_F \to F\text{-Mod}$ are strict polynomial functors of degree $d$.

4. Let $U$ be an object in $\Gamma^d_F$. Then define $h_U : \Gamma^d_F \to F\text{-Mod}$ as follows:

$$h_U(W) = \text{Hom}_{\Gamma^d_F}(U, W) = \text{Hom}_{S^d_F}(U \otimes^d W, W).$$

The functor $h_U \in \text{Rep}\Gamma^d_F$ is called a representable functor. The functor $h : U \to h_U$ is the contravariant Yoneda embedding.

**Theorem 4.4.** [10, Theorem 3.2] The functor $ev_n : \text{Rep}\Gamma^d_F \to S_F(n, d)\text{-Mod}$ is an equivalence of categories whenever $n \geq d$. 

Given a strict polynomial functor $X$, $X(F^n)$ inherits the structure of an $S_F(n, d)$-module. For every non-negative integer $n$, we have the evaluation functor $ev_n : \text{Rep}\Gamma^d_F \to S_F(n, d)\text{-Mod}$ as:

$$ev_n(X) = X(F^n) \text{ for } X \in \text{Rep}\Gamma^d_F.$$
4.2. Koszul duality of strict polynomial functors: In [16], Krause defined an internal tensor product (⊗) on the category of strict polynomial functors of a fixed degree \( d \). Kulkarni, Srivastava, and Subrahmanyam [17], and independently, Acquilino and Reischuk [1] showed that this internal tensor product, via the Schur functor, is related to the Kronecker tensor product of representations of the symmetric group \( S_d \). Krause used this internal tensor product to introduce Koszul duality as the functor \((\wedge^d \otimes -) : \text{Rep} \Gamma^d F \to \text{Rep} \Gamma^d F\). We can think about this functor as follows, for the representable functor \( h_V \in \text{Rep} \Gamma^d F \), we have, using the notation of (20),

\[
\wedge^d \otimes h_V = \wedge^d V.
\]

For arbitrary \( X \in \text{Rep} \Gamma^d F \), following [16], we exploit a theorem of Mac Lane [18, III.7, Theorem 1], namely:

\[
X = \operatorname{colim}_{h_V \to X} h_V.
\]

Using this we have:

\[
\wedge^d \otimes X = \operatorname{colim}_{h_V \to X} \wedge^d \otimes h_V = \operatorname{colim}_{h_V \to X} \wedge^d V.
\]

In the following theorem, we relate the abstract Koszul duality of Schur algebra with the Koszul duality of strict polynomial functors.

**Theorem 4.5.** Consider the functors,

\[
(S^- \otimes_S ev_n(-)) : \text{Rep} \Gamma^d F \to \text{S-Mod},
\]

\[
ev_n(\wedge^d \otimes -) : \text{Rep} \Gamma^d F \to \text{S-Mod}.
\]

Then there exists a natural transformation

\[
\eta : (S^- \otimes_S ev_n(-)) \to ev_n(\wedge^d \otimes -),
\]

which is an isomorphism when \( n \geq d \).

**Proof.** Let \( X = h_V \). Then,

\[
ev_n(\wedge^d \otimes h_V) = ev_n(\wedge^d V) \text{ by (21)}
\]

\[
= \wedge^d V(F^n)
\]

\[
= \wedge^d \text{Hom}_F(V, F^n) \text{ by Equation (20)}
\]

\[
\simeq \text{Hom}_{S_d}(V^\otimes d, (F^n)^\otimes d \otimes \text{sgn}).
\]

On the other hand,

\[
S^- \otimes_S ev_n(h_V) = S^- \otimes_S h_V(F^n)
\]

\[
= \text{Hom}_{S_d}((F^n)^\otimes d, (F^n)^\otimes d \otimes \text{sgn}) \otimes_S \text{Hom}_{S_d}(V^\otimes d, (F^n)^\otimes d).
\]

Using these identifications, \( \eta_{h_V} : (g_1 \otimes g_2) = g_1 \circ g_2 \), for \( g_1 \in S^- \) and \( g_2 \in ev_n(h_V) \), defines an \( S \)-linear map

\[
\eta_{h_V} : S^- \otimes_S ev_n(h_V) \to ev_n(\wedge^d \otimes h_V).
\]

For arbitrary \( X \in \text{Rep} \Gamma^d F \), we construct \( \eta_X \) using Equation (22):

\[
\eta_X = \operatorname{colim}_{h_V \to X} \eta_{h_V}.
\]
From the Yoneda lemma [18] Page 59, every morphism \( h_V \to h_W \) between the representable functors is of the form \( h_f \) for a unique morphism \( f \in \text{Hom}_{\mathcal{F}_d}(W,V) \). The following diagram commutes:

\[
\begin{array}{ccc}
S^- \otimes_S \text{ev}_n(h_V) & \xrightarrow{\eta h_V} & \text{ev}_n(\wedge^d \otimes h_V) \\
\downarrow \text{id}_{S^-} \otimes \text{ev}_n(h_V) & & \downarrow \text{ev}_n(\text{id}_A \otimes h_f) \\
S^- \otimes_S \text{ev}_n(h_W) & \xrightarrow{\eta h_W} & \text{ev}_n(\wedge^d \otimes h_V)
\end{array}
\]

Taking colimits then gives the naturality of \( \eta \).

If \( n \geq d \), \( h_{F_n} \) is a small projective generator of \( \text{Rep}_{\Gamma_d} F \), i.e., every object has a presentation by \( h_{F_n} \), (see [16]). Note that the map \( \eta_{h_{F_n}} \) is surjective because \( \eta_{h_{F_n}}(f \otimes \text{id}_S) = f \) for \( f \in S^- \) and hence an isomorphism because \( \text{ev}_n(\wedge^d \otimes h_{F_n}) \) is isomorphic to \( S^- \). By the construction of \( \eta_X \), this implies that each \( \eta_X \) is an isomorphism for \( X \in \text{Rep}_{\Gamma_d} F \).

\[\square\]

4.3. Derived abstract Koszul duality. For a finite dimensional associative algebra \( A \), let \( \mathcal{D}(A\text{-Mod}) \) be the unbounded derived category of \( A\text{-Mod} \). For a \((A, A)\)-bimodule \( M \), the functor \( (M \otimes_A -) \) is a right exact functor so the total left derived functor \( (M \otimes_L A -) : \mathcal{D}(A\text{-Mod}) \to \mathcal{D}(A\text{-Mod}) \) exists. From Happel \[14\], we recall necessary and sufficient conditions for the functor \( (M \otimes_L A -) \) to be an equivalence of categories.

For each \( x \in A \), let \( \psi_x \in \text{End}_A(M_A) \) be defined by

\[\psi_x(y) = xy.\]

Taking \( x \) to \( \psi_x \) gives rise to a homomorphism of algebras:

\[\psi : A \to \text{End}_A(M_A).\]

Recall

**Theorem 4.6** (Happel \[14\]). For a finite dimensional algebra \( A \) and a \((A, A)\)-bimodule \( M \), the functor \( (M \otimes_A -) : \mathcal{D}(A\text{-Mod}) \to \mathcal{D}(A\text{-Mod}) \) is an equivalence of categories if and only if

1. \( \text{The module } M_A \text{ admits a finite resolution by finitely generated projective right modules over } A. \)
2. \( \text{The canonical map } \psi : A \to \text{End}_A(M_A) \text{ is an isomorphism, and for } i \geq 1, \text{Ext}^i_A(M, M) = 0. \)
3. \( \text{There exists an exact sequence consisting of right } A\text{-modules:} \)

\[0 \to A \to M_1 \to \cdots \to M_l \to 0, \]

where for \( 1 \leq i \leq l \), \( M_i \) is a direct summand of finite direct sum of copies of \( M \).

**Theorem 4.7.** Let \( A = S \) and \( M \) be the \((S, S)\)-bimodule \( S^- \). Then the map \( \psi \) in Equation \[25\] is an isomorphism if and only if \( n \geq d \).

**Proof.** Suppose \( n < d \). Consider the following labelled bipartite multigraph,

\[\Gamma_1 = \begin{array}{ccc}
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\end{array}\]

Suppose \( n = d \). Consider the following labelled bipartite multigraph,
Let $\Gamma_2 \in N(n,d)$. Since $\Gamma_2$ is a simple bipartite graph therefore any labelling of $\Gamma_2$ which satisfies the condition 1 in Definition 2.13 requires $d$ balls to place into $d$ distinct boxes out of $n$. This is not possible as $n < d$. Thus we get that $\xi_{\Gamma_1}, \zeta_{\Gamma_2} = 0$. Since $\zeta_{\Gamma_1}$ for $\Gamma \in N(n,d)$ forms a basis of $S^-$ we get $\xi_{\Gamma_1}$ is in the kernel of $\psi$, and so $\psi$ is not injective.

For the converse, suppose $n \geq d$. Then the map $\psi$ is an isomorphism is known from [9, Proposition 3.7]. But we give a combinatorial proof here. For $\theta \in \text{End}_S(S^-_\Gamma)$, we denote the coefficient of $\zeta_{\Gamma_1}$ in $\theta(\zeta_{\Gamma_2})$ by $\langle \theta(\zeta_{\Gamma_2}), \zeta_{\Gamma_1} \rangle$.

To see that $\psi$ is injective, note that any element of $S$ is of the form $s = \sum_{\Gamma \in M(n,d)} \alpha_{\Gamma} \xi_{\Gamma}$. Now $\psi(s) = 0$ if and only if

$$\sum_{\Gamma \in M(n,d)} \alpha_{\Gamma} \xi_{\Gamma} \zeta_{\Gamma'} = 0,$$

for every $\Gamma' \in N(n,d)$. Fix $\Gamma_1 \in M(n,d)$ and let $\Gamma' = D(\Gamma_1)^*$. Then by Lemma 2.33

$$\alpha_{\Gamma_1} = \langle \sum_{\Gamma \in M(n,d)} \alpha_{\Gamma} \xi_{\Gamma} D(\Gamma_1)^*, \zeta_{\Gamma_1} \rangle.$$

Thus $\alpha_{\Gamma_1} = 0$.

To see that $\psi$ is surjective, we will show that $\dim_F \text{End}_S(S^-_\Gamma) \leq |M(n,d)|$. Firstly, by Lemma 2.33 $S^-_\Gamma$ is generated by $G = \{ \zeta_{\Gamma} | \Gamma \in N^d \}$. Recall that $N^d$ denotes the set of graphs in $N(n,d)$ with upper degree sequence $(1^d, 0^{n-d})$. Therefore any $\theta \in \text{End}_S(S^-_\Gamma)$ is determined by its values on this set. Since $\theta$ is an $S$-module homomorphism $\theta(\zeta_{\Gamma})$ again lies in the span of $G$. Therefore $\theta$ is completely determined by the values:

$$\{ \langle \theta(\zeta_{\Gamma}), \zeta_{\Gamma'} \rangle | \Gamma, \Gamma' \in N^d \}.$$

Moreover, for any $w \in S_d$,

$$\langle \theta(\zeta_{\Gamma}), \zeta_{\Gamma'} \rangle = \langle \theta(\zeta_{\Gamma-w}), \zeta_{\Gamma-w} \rangle.$$

Therefore $\dim_F \text{End}_S(S^-_\Gamma) \leq |(N^d \times N^d)/S_d| = |M(n,d)|$. \hfill \Box

**Theorem 4.8.** Let $F$ be any field of characteristic different from 2. The functor

$$\text{(26)} \quad (S^- \otimes_S -) : D(S\text{-Mod}) \to D(S\text{-Mod})$$

is an equivalence of categories if and only if $n \geq d$.

**Proof.** If $n \geq d$ then from Theorem 4.5 $S^- \otimes_S \text{ev}_n(-)$ is isomorphic to $\text{ev}_n(\wedge^d \otimes -)$. Since the functor $\text{ev}_n$ is exact so the total left derived functors of $S^- \otimes_S \text{ev}_n(-)$ and $\text{ev}_n(\wedge^d \otimes -)$ are isomorphic to $S^- \otimes_S \text{ev}_n(-)$ and $\text{ev}_n(\wedge^d \otimes -)$ respectively. From [16 Theorem 4.9], $(\wedge^d \otimes -)$ is an equivalence. From Theorem 4.4 $\text{ev}_n(-)$ is an equivalence. Therefore $S^- \otimes_S -$ is an equivalence. So $\text{ev}_n(-)$ being an equivalence forces $(S^- \otimes_S -)$ to be an equivalence.

For the converse, suppose that $(S^- \otimes_S -)$ is an equivalence. Then from Theorem 4.6 the map $\psi : S \to \text{End}_S(S^-)$ is an isomorphism. Theorem 4.7 now implies that $n \geq d$. \hfill \Box

The behavior of derived Koszul duality for different values of $n$ and $d$ is summarized in Figure 1.
5. Concluding Remarks

5.1. Towards alternating partition algebras. The centralizer algebra $\text{End}_{S_n}((F^n)^{\otimes d})$ is a quotient of the partition algebra $P_d(n)$ of Jones [15] and Martin [19]. Further restricting the action of $S_n$ to the alternating group $A_n$ we get the alternating partition algebra $\text{AP}_F(n,d) = \text{End}_{A_n}((F^n)^{\otimes d})$, which from the isomorphism (2) decomposes as follows:

$$\text{End}_{A_n}((F^n)^{\otimes d}) = \text{End}_{S_n}((F^n)^{\otimes d}) \oplus \text{Hom}_{A_n}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn})$$

Let $P_d^-(n) = \text{Hom}_{A_n}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \text{sgn})$. Then $P_d^-(n)$ becomes a $(P_d(n), P_d(n))$-bimodule by inflation. Bloss [5] showed $P_d^-(n)$ is non-zero if and only if $n < 2d + 2$. So we get an abstract Koszul duality $(P_d^-(n) \otimes P_d(n))$ on the category of modules over the partition algebra $P_d(n)$ when $n < 2d + 2$. Many of the ideas and techniques in this article can be used to study bases, structure constants, and abstract Koszul duality for partition algebras. For $F = \mathbb{C}$, the dimensions of simple modules of $\text{AP}_F(n,d)$ are given combinatorially by Benkart, Halverson, and Harman, see [2].

5.2. A diagrammatic interpretation of the Schur category. The following is one possible way to define the notion of a diagram category in the spirit of Martin’s discussion in [20].

**Definition 5.1 (Diagram Category).** A category $\mathcal{C}$ is called a diagram category if there exists a sequence $\{V_n\}_{n \geq 0}$ of objects which constitute a skeleton of $\mathcal{C}$, and

![Figure 1. Dependence of derived Koszul duality on $n$ and $d$](image-url)
for each pair \((m, n)\) of non-negative integers, a class of “diagrams” \(M(m, n, d)\), a basis

\[ B_{m,n} = \{ \xi_{\Gamma} \mid \Gamma \in M(m, n, d) \} \]

of \(\text{Hom}_C(V_n, V_m)\), and a combinatorial rule for computing the structure constants \(c_{\Gamma'}_{\Gamma''}^{\Gamma} \) that are defined by:

\[ \xi_{\Gamma'} \circ \xi_{\Gamma''} = \sum_{\Gamma} c_{\Gamma'}_{\Gamma''}^{\Gamma} \xi_{\Gamma} \]

for \(\Gamma' \in M(l, m, d)\), \(\Gamma'' \in M(m, n, d)\) and \(\Gamma \in M(l, n, d)\).

**Remark 5.2.** In the examples discussed by Martin [20], given diagrams \(\Gamma'\) and \(\Gamma''\), there can exist more than one diagram \(\Gamma\) such that \(c_{\Gamma'}_{\Gamma''}^{\Gamma} > 0\). This is not a requirement in the above definition.

Consider the Schur category \(\mathbf{\Gamma}^d_F\) defined in Section 4.1. Take \(V_n = F^n\). Define \(M(m, n, d)\) to be the set of all bipartite multigraphs with vertex set \([n] \coprod [m]\) with \(d\) edges. Mimicking the discussion in Section 2.2 one may endow the Schur category with the structure of a diagram category in the sense of Definition 5.1.

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