Estimates for Robin $p$-Laplacian eigenvalues of convex sets with prescribed perimeter

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Abstract

In this paper, we prove an upper bound for the first Robin eigenvalue of the $p$-Laplacian with a positive boundary parameter and a quantitative version of the reverse Faber-Krahn type inequality for the first Robin eigenvalue of the $p$-Laplacian with negative boundary parameter, among convex sets with prescribed perimeter.

The proofs are based on a comparison argument obtained by means of inner sets, introduced by Payne, Weimberger [PW61] and Polya [P60].

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1 Introduction

Let $\Omega$ be a bounded, open and convex set in $\mathbb{R}^n$. We consider the following problem

$$
\begin{cases}
-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{p,\beta}(\Omega)|u|^{p-2}u & \text{in } \Omega \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \beta|u|^{p-2}u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\beta \in \mathbb{R}$.

The fundamental eigenvalue of the Robin Laplacian on $\Omega$ is defined by

$$
\lambda_{p,\beta}(\Omega) = \min_{v \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^p \, dx + \beta \int_{\partial \Omega} |v|^p \, d\mathcal{H}^{n-1}}{\int_{\Omega} |v|^p \, dx}
$$

and a minimizer $u$ in (1.2) satisfies the equation (1.1) in the weak form if

$$
\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \beta \int_{\partial \Omega} |u|^{p-2}u \varphi \, d\mathcal{H}^{n-1} = \lambda_{p,\beta}(\Omega) \int_{\Omega} |u|^{p-2}u \varphi \, dx, \quad \forall \varphi \in W^{1,p}(\Omega).
$$
1 INTRODUCTION

It is well-known (see for instance [Lin92] for the Dirichlet case) that the first eigenvalue \( (1.2) \) is simple and that in the case of a ball, the corresponding eigenfunction is radially symmetric ([Bha88]).

In this paper we want to study the different behaviour of eigenvalues in the case \( \beta > 0 \) and \( \beta < 0 \). For sake of completeness, we recall that, in the case of \( \beta = 0 \), we recover Neumann boundary condition, for which the first eigenvalue is zero and the associated eigenfunctions are constants.

It is well known that in the case \( \beta > 0 \), Bossel [Bos86] and Daners [Dan06] proved a Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian in two and higher dimensional case, respectively. In particular, they proved that among sets of given volume, the one which minimizes the first Robin-eigenvalue is the ball, i.e.

\[
\lambda_{2,\beta}(\Omega^\sharp) \leq \lambda_{2,\beta}(\Omega),
\]

where \( \Omega^\sharp \) is the ball, centered at the origin, having the same volume as \( \Omega \). This result was generalized by Bucur, Daners and Giacomini in [BD10, BG10, BG15] to the eigenvalues of the \( p \)-Laplacian with Robin boundary conditions.

This Faber-Krahn inequality for fixed volume and the following rescaling property [BFK17]

\[
\lambda_{p,\beta}(t\Omega) \leq \frac{1}{t} \lambda_{p,\beta}(\Omega) \leq \lambda_{p,\beta}(\Omega), \quad \forall t > 1,
\]

give a Faber-Krahn inequality for fixed perimeters, i.e.

\[
\lambda_{p,\beta}(\Omega^\star) \leq \lambda_{p,\beta}(\Omega),
\]

where \( \Omega^\star \) is the ball having the same perimeter as \( \Omega \).

Our aim is to give a continuity bound to the ratio

\[
\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},
\]

indeed we prove

**Theorem 1.1.** Let \( \beta \) be a positive parameter. Let \( \Omega \) be a bounded, open and convex set in \( \mathbb{R}^n \) and let \( \Omega^\star \) be the ball, centered at the origin, such that \( P(\Omega) = P(\Omega^\star) = \rho \). Let \( \lambda_{p,\beta}(\Omega) \) and \( \lambda_{p,\beta}(\Omega^\star) \) be the first eigenvalues of the \( p \)-Laplacian operator with Robin boundary conditions on \( \Omega \) and \( \Omega^\star \), let \( v \) be a positive eigenfunction associated to \( \lambda_{p,\beta}(\Omega^\star) \). Then

\[
\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)} \leq C(n, p, \beta, \rho) \left( 1 - \frac{n^{n-1} \omega_n^{\frac{n-1}{n}} |\Omega|}{P(\Omega)^{\frac{n-1}{n}}} \right),
\]

where \( \omega_n \) is the measure of the unitary ball in \( \mathbb{R}^n \), and

\[
C(n, p, \beta, \rho) = \frac{\|v\|^p_{\infty} |\Omega^\star|}{\|v\|^p_p}.
\]
1 INTRODUCTION

It is possible to give a uniform bound to the constant in (1.5) from above with a constant independent of the parameter $\beta$ and the perimeter, indeed it holds

$$C(n, p, \beta, \rho) \leq C(n, p) := \frac{\|v_\infty\|_p^p |\Omega^*|}{\|v_\infty\|_p^p},$$

(1.6)

where $v_\infty$ is the first Dirichlet eigenfunction. Thanks to the rescaling property of Dirichlet eigenvalues and eigenfunctions, $C(n, p)$ does not depend on the perimeter. For more details see Remark 3.1.

We observe that this result can be seen as a generalization to the Robin case of the result in [BNT10], which holds true in the case of Dirichlet eigenvalues of the $p$-Laplacian.

When $\beta$ is a negative parameter, the authors in [BFNT19] proved a reverse Faber-Krahn inequality for the first eigenvalue of the Dirichlet-Laplacian among convex sets of given perimeter. In particular, they proved that among convex sets of given perimeter the ball $\Omega^*$ maximizes the first Robin eigenvalue of the $p$-Laplacian, i.e.

$$\lambda_{p, \beta}(\Omega) \leq \lambda_{p, \beta}(\Omega^*).$$

(1.7)

For completeness’ sake, we quote that in [AFK17], the authors already proved that the disc maximizes the first eigenvalue under a perimeter constraint, among $C^2$ domains in $\mathbb{R}^2$, while the question remained open in arbitrary dimension.

This question is related to the conjecture of Bareket (see [Bar77]) claiming that the ball maximizes $\lambda_{p, \beta}(\Omega)$ among all Lipschitz sets with given volume. Freitas and Krejčiřík in [FK15] proved that the conjecture is false, giving a counter-example based on the asymptotic behavior of the eigenvalues on a disc and an annulus of the same area when $\beta \to -\infty$. They also proved that among sets of area equal to 1, the conjecture is true, provided $\beta$ is close to 0.

In [CL21] the authors proved a quantitative version of the reverse Faber-Krahn (1.7) in the case $p = 2$ among convex sets of fixed perimeter following the Fuglede’s approach introduced in [Fug89].

In this paper, we recover the result in [CL21] obtaining a quantitative version of (1.7) for all $p$, but using a different approach. Indeed, following the method introduced by Payne and Weinberger in [PW61], we establish a comparison using the so-called parallel coordinates method. In particular, we firstly prove a lower bound in terms of perimeter and measure of $\Omega$, that is

**Theorem 1.2.** Let $\beta$ be a negative parameter. Let $\Omega$ be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^*$ be the ball, centered at the origin, such that $P(\Omega) = P(\Omega^*) = \rho$. Let $\lambda_{p, \beta}(\Omega)$ and $\lambda_{p, \beta}(\Omega^*)$ be the first eigenvalues of the $p$-Laplacian operator with Robin boundary conditions on $\Omega$ and $\Omega^*$, let $v$ be a positive eigenfunction associated to $\lambda_{p, \beta}(\Omega^*)$. Then

$$\frac{\lambda_{p, \beta}(\Omega^*) - \lambda_{p, \beta}(\Omega)}{|\lambda_{p, \beta}(\Omega)|} \geq C(n, p, \beta, \rho) \left(1 - \frac{n^{-1} \omega_n^{-1} |\Omega|}{\rho^{-1}}\right),$$

(1.8)

where $\omega_n$ is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n, p, \beta, \rho) = \frac{v_m^{p - 1} |\Omega^*|}{\|v\|_p^p}$ with $v_m = \min_{\Omega^*} v$. 

In this case, the constant $C(n, p, \beta, \rho)$ cannot be replaced by a constant independent of $\beta$ and of the perimeter, as it is shown in Remark 3.2.

Then we prove the quantitative result as in [CL21].

\begin{theorem}

Let $n \geq 2$, $\rho > 0$ and $\beta < 0$. Then there exists two positive constants $C(n, p, \beta, \rho) > 0$ and $\delta_0(n, p, \beta, \rho) > 0$, such that, for all $\Omega \subset \mathbb{R}^n$ bounded and convex with $P(\Omega) = \rho$ and $\lambda_{p, \beta}(\Omega^*) - \lambda_{p, \beta}(\Omega) \leq \delta_0$, it holds
\begin{equation}
\lambda_{p, \beta}(\Omega^*) - \lambda_{p, \beta}(\Omega) \geq C(n, p, \beta, \rho) g(A_{\mathcal{H}}^*(\Omega)) \tag{1.9}
\end{equation}

where $\Omega^*$ is a ball with the same perimeter of $\Omega$, $A_{\mathcal{H}}^*$ is the Hausdorff asymmetry defined in (2.3) and $g$ is defined in (2.7).

\end{theorem}

The paper is organized as follows: in Section 2 we recall some preliminary results and useful tools for our aim; in Section 3 we provide the proof of our main results.

\section{Notations and Preliminaries}

Throughout this article, $| \cdot |$ will denote the Euclidean norm in $\mathbb{R}^n$, while $\cdot$ is the standard Euclidean scalar product for $n \geq 2$. By $\mathcal{H}^k(\cdot)$, for $k \in [0, n)$, we denote the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$.

The measure and the perimeter of $\Omega$ in $\mathbb{R}^n$ will be denoted by $|\Omega|$ and $P(\Omega)$, respectively, and, if $P(\Omega) < \infty$, we say that $\Omega$ is a set of finite perimeter. In our case, $\Omega$ is a bounded, open and convex set; this ensures us that $\Omega$ is a set of finite perimeter and that $P(\Omega) = \mathcal{H}^{n-1}(\partial \Omega)$. Moreover, if $\Omega$ is an open set with Lipschitz boundary, it holds

\begin{theorem} [Coarea formula]

Let $f : \Omega \to \mathbb{R}$ be a Lipschitz function and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then,
\begin{equation}
\int_\Omega u(x) |\nabla f(x)| \, dx = \int_\mathbb{R} dt \int_{\Omega \cap f^{-1}(t)} u(y) \, d\mathcal{H}^{n-1}(y). \tag{2.1}
\end{equation}

\end{theorem}

Some references for results relative to the sets of finite perimeter and for the coarea formula are, for instance [AFP00, Mag12].

\subsection{Asymmetry index $A^*(E)$}

In the case $\beta < 0$, we want to prove a quantitative result so we need an index that gives us information about the shape of $\Omega$. We will consider the Hausdorff asymmetry index, as already done in [CL21], so we recall some basic notions about the Hausdorff distance.

We recall that if $E, F$ are any two convex sets in $\mathbb{R}^n$ the Hausdorff distance between $E, F$ is defined as
\begin{equation}
d_H(E, F) := \inf\{\varepsilon > 0 : E \subset F + \varepsilon B, F \subset E + \varepsilon B\},
\end{equation}
where $B$ is the unitary ball centered at the origin and $F + \varepsilon B$ is the well-known Minkowski sum. For such set we define two isoperimetric deficit

$$D(E) := P(E) - P(E^\sharp), \quad M(E) := |E^\sharp| - |E|$$

where $E^\sharp$ and $E^\star$ are the ball with the same measure and the same perimeter of $E$, respectively.

Moreover, we will consider the following Hausdorff asymmetry indices

$$A_\sharp^\star_H(E) = \min_{x \in \mathbb{R}^n} \{d_H(E, B_r(x)), P(\Omega) = P(B_r(x))\}$$

and

$$A_\star^\sharp_H(E) = \min_{x \in \mathbb{R}^n} \{d_H(E, B_r(x)), |\Omega| = |B_r(x)|\}.$$ 

Lemma 2.9 in [GLPT20] tells us how these two indices are related one to the other.

**Lemma 2.2.** Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be a bounded, convex, with $D(E) \leq \delta$ then

$$A_\star^\star_H(E) \leq C(n) A_\sharp^\star_H(E).$$

With these definitions, we can recall the quantitative isoperimetric inequality proved in [Fug89, Fus15].

**Theorem 2.3 (Fuglede).** Let $n \geq 2$, and let $E$ be a bounded open and convex set with $|E| = \omega_n$. There exists $\delta, C$, depending only on $n$, such that if $D(E) \leq \delta$ then then

$$D(E) \geq Cg\left(A_\star^\star_H(E)\right),$$

where $g$ is defined by

$$g(s) = \begin{cases}  2^n & \text{if } n = 2 \\ f^{-1}(s^2) & \text{if } n = 3 \\ s^{n+1} & \text{if } n \geq 4 \end{cases}$$

and $f(t) = \sqrt{t \log(\frac{1}{t})}$ for $0 < t < e^{-1}$.

We are interested in a modified version of this theorem, in terms of $M(\Omega)$, so we have

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and convex set and let $\Omega^*$ be the ball satisfying $P(\Omega) = P(\Omega^*) = \rho$. Then, there exist $\delta, C$, depending only on $n$ and $\rho$, such that, if

$$M(\Omega) = |\Omega^*| - |\Omega| \leq \delta$$

then

$$M(\Omega) \geq Cg\left(A_\star^\star_H(\Omega)\right)$$

where $g$ is the function defined in (2.7).
Proof. Let us start by setting \( \delta < \frac{\|\Omega\|}{2} \), so
\[
|\Omega| > \frac{\|\Omega\|}{2} \tag{2.9}
\]
and let us assume \( |\Omega| = \omega_n \).

By (2.9) and the differentiability of the function \( h(t) = t^{\frac{n-1}{n}} \), we have
\[
|\Omega^*|^{\frac{n-1}{n}} - |\Omega|^{\frac{n-1}{n}} = h'(\xi) (|\Omega^*| - |\Omega|) \leq \frac{n-1}{\omega_n^n} (|\Omega^*| - |\Omega|) \cdot
\]
Hence, if \( M(\Omega) \leq \delta \), then
\[
|\Omega^*|^{\frac{n-1}{n}} - |\Omega|^{\frac{n-1}{n}} = \frac{P(\Omega)}{n\omega_n^{\frac{1}{n}}} - |\Omega|^{\frac{n-1}{n}} = \frac{D(\Omega)}{n\omega_n^{\frac{1}{n}}} \leq \delta(n, \rho).
\]
So we can apply (2.6)
\[
P(\Omega) \geq n\omega_n \left( 1 + \gamma(n)g\left(A^*_H(\Omega)\right) \right)
\]
to obtain
\[
|\Omega^*| - |\Omega| = \frac{P(\Omega)}{n\omega_n^{\frac{1}{n}}} - |\Omega| \geq \omega_n \left( 1 + \gamma(n)g\left(A^*_H(\Omega)\right) \right)^{\frac{1}{n-1}} - \omega_n
\]
\[
\geq \frac{n\omega_n\gamma(n)}{n-1} g\left(A^*_H(\Omega)\right),
\]
where in the last step we used Bernoulli’s inequality
\[
(1 + x)^r \geq 1 + rx \quad x \geq -1, \ r \geq 1.
\]
Applying Lemma 2.2 we finally have
\[
|\Omega^*| - |\Omega| \geq C(n)g\left(A^*_H(\Omega)\right).
\]
The general case will be recovered by rescaling and by the following inequality for \( Y = \omega_n|\Omega|^{-\frac{1}{n}} \)
\[
|Y^*| - |Y| \leq 2\omega_n^n |\Omega^*|^{-1} (|\Omega^*| - |\Omega|). \tag{2.10}\]

2.2 Quermassintegrals

Let us recall some basic facts about convex sets. Let \( K \subset \mathbb{R}^n \) be a non-empty, bounded, convex set, let \( B \) be the unitary ball centered at the origin and \( \rho > 0 \). We can write the Steiner formula for the Minkowski sum \( K + \rho B \) as
\[
|K + \rho B| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) \rho^i. \tag{2.11}\]
The coefficients $W_i(K)$ are known in literature as quermassintegrals of $K$. In particular, $W_0(K) = |K|$, $nW_1(K) = P(K)$ and $W_n(K) = \omega_n$ where $\omega_n$ is the measure of $B$.

If $K$ has $C^2$ boundary, the quermassintegrals can be written in terms of principal curvatures of $K$. More precisely, denoting with $H_j$ the $j$-th normalized elementary symmetric function of the principal curvature $\kappa_1, \ldots, \kappa_{n-1}$, i.e.

$$H_0 = 1, \quad H_j = \left( \frac{n-1}{j} \right)^{-1} \sum_{1 \leq i_1 < \ldots < i_j \leq n-1} \kappa_{i_1} \ldots \kappa_{i_j} \quad j = 1, \ldots, n-1,$$

then the quermassintegrals can be written as

$$W_i(K) = \frac{1}{n} \int_{\partial K} H_{i-1} \, d\mathcal{H}^{n-1} \quad i = 1, \ldots, n. \quad (2.12)$$

Moreover, the Steiner formula holds true also for quermassintegrals, that is

$$W_j(K + \rho B) = \sum_{i=0}^{n-j} \binom{n-j}{i} W_{j+i}(K) \rho^i \quad j = 0, \ldots, n-1. \quad (2.13)$$

For $j = 1$ we have

$$P(K + \rho B_1) = n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(K) \rho^i \quad \kappa_1 \ldots \kappa_j$$

$$= P(K) + n(n-1)W_2(K)\rho + \ldots + nW_n(K)\rho^n,$$

from which follows

$$\lim_{\rho \to 0} \frac{P(K + \rho B) - P(K)}{\rho} = n(n-1)W_2(K), \quad (2.14)$$

and if $\partial K$ is of class $C^2$ formula (2.12) gives

$$\lim_{\rho \to 0} \frac{P(K + \rho B) - P(K)}{\rho} = (n-1) \int_{\partial K} H_1 \, d\mathcal{H}^{n-1}.$$

Furthermore, Aleksandrov-Fenchel inequalities hold true

$$\left( \frac{W_j(K)}{\omega_n} \right)^{\frac{n-j}{n}} \geq \left( \frac{W_i(K)}{\omega_n} \right)^{\frac{n-i}{n}} \quad 0 \leq i < j \leq n-1, \quad (2.15)$$

where equality hold if and only if $K$ is a ball. When $i = 0$ and $j = 1$, formula (2.15) reduce to the classical isoperimetric inequality, i.e.

$$P(K) \geq n\omega_n^\frac{1}{n} |K|^{-\frac{n-1}{n}}.$$

In the following we will use (2.15) for $i = 1$ and $j = 2$, that is

$$W_2(K) \geq n^{-\frac{n-2}{n-1}} \omega_n^\frac{1}{n} P(K)^{\frac{n-2}{n-1}}. \quad (2.16)$$
2.3 Some useful lemmas

Let $\Omega \subset \mathbb{R}^n$ be a convex set, for $t \in [0, r_\Omega]$ we denote by

$$\Omega_t = \{ x \in \Omega : d(x) > t \}$$

where $d(x)$ is the distance of $x \in \Omega$ from the boundary of $\Omega$ and $r_\Omega$ is the inradius of $\Omega$. By the Brunn-Minkowski Theorem ([Sch13, Theorem 7.4.5]) and the concavity of the distance function, the map

$$t \mapsto P(\Omega_t)^{\frac{1}{n-1}}$$

is concave in $[0, r_\Omega]$, hence absolutely continuous in $(0, r_\Omega)$. Moreover, there exists its right derivative at 0 and it is negative, since $P(\Omega_t)^{\frac{1}{n-1}}$ is strictly monotone decreasing.

Lemma 2.5. Let $\Omega$ be a bounded, convex, open set in $\mathbb{R}^n$. Then for almost every $t \in (0, r_\Omega)$

$$-\frac{d}{dt} P(\Omega_t) \geq n(n-1)W_2(\Omega_t)$$

(2.17)

and the equality holds if $\Omega$ is a ball.

Proof. For every $\rho \in (0, t)$ it holds

$$\Omega_t + \rho B_1 \subset \Omega_{t-\rho}$$

and if $\Omega$ is a ball, the two sets coincide. The monotonicity of the perimeter with respect to the inclusion of convex sets and formula (2.14) give, for almost every $t \in (0, r_\Omega)$,

$$-\frac{d}{dt} P(\Omega_t) = \lim_{\rho \to 0^+} \frac{P(\Omega_{t-\rho}) - P(\Omega_t)}{\rho} \geq \lim_{\rho \to 0^+} \frac{P(\Omega_t + \rho B_1) - P(\Omega_t)}{\rho} = n(n-1)W_2(\Omega_t)$$

Combining the chain rule, the previous lemma and the fact that $|\nabla d(x)| = 1$ almost everywhere, we obtain

Lemma 2.6. Let $f : [0, +\infty) \to [0, +\infty)$ be a strictly increasing $C^1$ function with $f(0) = 0$. Set $u(x) = f(d(x))$ and

$$E_t = \{ x \in \Omega : u(x) > t \} = \Omega_{f^{-1}(t)}$$

then

$$-\frac{d}{dt} P(E_t) \geq (n-1)\frac{W_2(E_t)}{|\nabla u|_{u=t}}$$

(2.18)
2.4 Monotonicity of eigenfunctions

It is well-known (by definition) that for fixed $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$, the map

$$\beta \in \mathbb{R} \mapsto \lambda_{p,\beta}(\Omega)$$

is increasing and it holds (see for instance [Lê06])

$$\lambda_{p,N} < \lambda_{p,\beta} < \lambda_{p,D} \quad \forall \beta \in (0, +\infty)$$

and

$$\lim_{\beta \to 0^+} \lambda_{p,\beta} = \lambda_{p,N} = 0 \quad \lim_{\beta \to +\infty} \lambda_{p,\beta} = \lambda_{p,D}$$

where $\lambda_{p,N}$ and $\lambda_{p,D}$ are the first eigenvalue of the Neumann $p$-Laplacian and Dirichlet $p$-Laplacian respectively.

Now we want to observe that, if we consider a ball $B_R$ and we fix the value of the eigenfunction in the origin, the map

$$\beta \to v_{\beta}, \quad v_{\beta}(0) = C$$

is decreasing, in the sense that

$$\text{if } \beta_1 < \beta_2 \Rightarrow v_{\beta_1}(x) \geq v_{\beta_2}(x) \quad \forall x \in B_R$$

First of all, let us recall that the first eigenvalue of $p$-laplacian is simple and that the corresponding eigenfunction is radially symmetric, i.e. there exists $h: [0, R] \to \mathbb{R}$ such that $u(x) = h(|x|)$. In the following, we will denote with $v_{p,\gamma}$ both the eigenfunction and the function $h$.

Lemma 2.7. Let $0 < r < R$, $1 < p < +\infty$, $\beta \in \mathbb{R}$. Let us denote by $\lambda_{p,\beta}$ the first eigenvalue of the $p$-Laplacian on the ball $B_R$ defined in (1.2) with boundary parameter $\beta$ and let $v_{p,\beta}$ be the corresponding eigenfunction, then $v_{p,\beta}|_{B_r}$ is the first eigenfunction of the $p$-Laplacian on the ball $B_r$ with boundary parameter

$$\gamma = -\frac{|v'_{p,\beta}|^{p-2}v'_{p,\beta}(r)}{v_{p,\beta}^{p-1}(r)}.$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_\beta := \lambda_{2,\beta}$ and $v_\beta := v_{2,\beta}$.

By the radiality of $v_\beta$, we can infer that it is a solution to

$$\begin{cases} -\Delta v_\beta = \lambda_\beta v_\beta & \text{in } B_r \\ \frac{\partial v_\beta}{\partial v} + \gamma v_\beta = 0 & \text{on } \partial B_r. \end{cases}$$

Let us suppose by contradiction that $\lambda_\beta$ is not the first eigenvalue of the Robin Laplacian with boundary parameter $\gamma$. So we can choose

$$\lambda_{\gamma} < \lambda_\beta$$

(2.19)
and \( w_\gamma \) respectively first eigenvalue and the first eigenfunction of the Robin Laplacian with boundary parameter \( \gamma \), that is
\[
\lambda_\gamma = \frac{\int_{B_r} |\nabla w_\gamma|^2 \, dx + \gamma \int_{\partial B_r} w_\gamma^2 \, d\mathcal{H}^{n-1}}{\int_{B_r} w_\gamma^2 \, dx} = \min_{\psi \in W^{1,2}(B_r)} \frac{\int_{B_r} |\nabla \psi|^2 \, dx + \gamma \int_{\partial B_r} \psi^2 \, d\mathcal{H}^{n-1}}{\int_{B_r} \psi^2 \, dx}.
\]
(2.20)

We know that \( w_\gamma \) is unique up to a multiplicative constant, so we can choose the constant such that \( v_\beta = w_\gamma \) on \( \partial B_r \).

Let us consider the function
\[
f = \begin{cases} 
  w_\gamma(x) & \text{if } x \in B_r, \\
  v_\beta(x) & \text{if } x \in B_R \setminus B_r,
\end{cases}
\]
we can use it as test function in the definition of \( \lambda_\beta \). In particular
\[
\lambda_\beta \leq \frac{\int_{B_R} |\nabla f|^2 + \beta \int_{\partial B_r} f^2 \, d\mathcal{H}^{n-1}}{\int_{B_R} f^2 \, dx} = \frac{\int_{B_r} |\nabla w_\gamma|^2 + \int_{B_R \setminus B_r} |\nabla v_\beta|^2 + \beta \int_{\partial B_r} v_\beta^2 \, d\mathcal{H}^{n-1}}{\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}.
\]

If we add and subtract \( \gamma \int_{\partial B_r} w_\gamma \, d\mathcal{H}^{n-1} \) and \( \int_{B_r} |\nabla u|^2 \), we recall that \( v_\beta \) and \( w_\gamma \) are eigenfunction, so (1.2) hold, and we recall that \( v_\beta \) and \( w_\gamma \) coincides on \( \partial B_r \), we have
\[
\lambda_\beta \leq \frac{\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R} v_\beta^2 \, dx - \lambda_\beta \int_{B_r} v_\beta^2 \, dx}{\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}
\]
\[
= \frac{\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R \setminus B_r} v_\beta^2 \, dx}{\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}.
\]
\[
< \lambda_\beta \frac{\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} w_\gamma^2 \, dx}{\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} w_\gamma^2 \, dx} = \lambda_\beta,
\]
where in the last formula we use (2.19), and this is an absurd.\( \square \)
Proposition 2.8. Let $R > 0$, $1 < p < +\infty$ and $\beta_1 < \beta_2$. Let us denote by $\lambda_{p,\beta_1}$ and $\lambda_{p,\beta_2}$ the eigenvalues defined in (1.2) and let $v_{p,\beta_1}$ and $v_{p,\beta_2}$ be the corresponding eigenfunctions normalized such that $v_{p,\beta_1}(0) = v_{p,\beta_2}(0) > 0$, then

$$v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x) \quad \forall x \in B_R. \quad (2.21)$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_{\beta_i} := \lambda_{2,\beta_i}$ and $v_{\beta_i} := v_{2,\beta_i}$.

Since both $v_{\beta_1}$ and $v_{\beta_2}$ are radial, we can write the laplacian in polar coordinates, that is

$$r^{n-1} \Delta u(r) = \left(r^{n-1} u'(r)\right) \quad r \in [0, R]. \quad (2.22)$$

Therefore function $u_{\beta_i}$, for $i = 1, 2$, satisfies

$$v'_{\beta_i}(s) = -\frac{1}{r^{n-1}} \int_0^s s^{n-1} \lambda_{\beta_i} v_{\beta_i} \, ds. \quad (2.23)$$

Since $\lambda_{\beta_1} < \lambda_{\beta_2}$ and $v_{\beta_1}(0) = v_{\beta_2}(0)$, by continuity there exists $\delta > 0$ such that

$$\lambda_{\beta_1} v_{\beta_1}(s) < \lambda_{\beta_2} v_{\beta_2}(s) \quad \forall s \in (0, \delta),$$

and by (2.23)

$$v'_{\beta_1}(s) > v'_{\beta_2}(s) \quad s \in (0, \delta). \quad (2.24)$$

By classical ODE comparison result, we obtain

$$v_{\beta_1}(s) > v_{\beta_2}(s) \quad s \in (0, \delta). \quad (2.24)$$

Let us define

$$A = \{ r > \delta : v_{\beta_1}(s) = v_{\beta_2}(s) \},$$

we want to prove that $A$ is empty, so by (2.24) we get the claim. Let us suppose by contradiction that $A \neq \emptyset$, hence there exists

$$t = \inf A.$$

By continuity $v_{\beta_1}(t) = v_{\beta_2}(t)$, and this, combined with (2.24), leads to

$$v'_{\beta_1}(t) < v'_{\beta_2}(t). \quad (2.25)$$

Let us set

$$\gamma_i = -\frac{v'_{\beta_i}(t)}{v_{\beta_i}(t)},$$

by (2.25) we have $\gamma_1 > \gamma_2$.

By Lemma 2.7, $v_{\beta_1}$ and $v_{\beta_2}$ are the first eigenfunction of the Robin Laplacian in $B_t$ respectively with eigenvalue $\lambda_{\gamma_1}$ and $\lambda_{\gamma_2}$ with parameter $\gamma_1$ and $\gamma_2$. Therefore by monotonicity of eigenvalues with respect to the boundary parameter, we get

$$\lambda_{\beta_1} = \lambda_{\gamma_1} > \lambda_{\gamma_2} = \lambda_{\beta_2}$$

that is an absurd. \qed
3 PROOFS OF MAIN RESULTS

Remark 2.1. We highlight that, if we fix the value $v_{p,\beta}(0)$, (2.21) implies that

$$\beta \mapsto \|v_{p,\beta}\|_{L^p}$$

is non increasing, while the map

$$\beta \mapsto C(n, p, \beta, \rho) = \frac{v_{p,\beta}(0)|\Omega^*|}{\|v_{p,\beta}\|_p}$$

is non decreasing, for all $\beta \in \mathbb{R}$.

3 Proofs of main results

Proof of Theorem 1.1. The quantity

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^*)}{\lambda_{p,\beta}(\Omega)},$$

is bounded from above by 1, so inequality (1.5) is trivial when

$$\left(1 - \frac{n^{\frac{p-1}{n}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n-1}{n}}}\right) \geq \frac{1}{C(n, p, \beta, \rho)} = \frac{\|v\|_p^{\frac{p}{p}}|\Omega^*|}{\|v\|_p^{\frac{p}{p}}}.$$ 

We can assume

$$\left(1 - \frac{n^{\frac{p-1}{n}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n-1}{n}}}\right) < \frac{1}{C(n, p, \beta, \rho)}. \tag{3.1}$$

Let $v$ be the solution to

$$\begin{cases} -\Delta_p v = \lambda_{p,\beta}(\Omega^*)|v|^{p-2}v & \text{in } \Omega^* \\ |\nabla v|^{p-2} \frac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial \Omega^*, \end{cases} \tag{3.2}$$

it is well known that $v$ is positive and radially symmetric. So the function

$$g(t) = |\nabla v|_{v=t}$$

is well defined for all $t \in (v_m, v_M)$, where $v_m = \min_{\Omega^*} v$ and $v_M = \max_{\Omega^*} v$.

Let us define $u(x) = G(d(x))$, $x \in \Omega$, where

$$G^{-1}(t) = \int_{v_m}^t \frac{1}{g(s)} ds, \quad v_m < t < v_M.$$ 

By construction, $u \in W^{1,2}(\Omega)$ and

$$\min_{\Omega} u = G(0) = v_m,$$

$$\|u\|_{\infty} \leq v_M,$$

$$|\nabla u|_{u=t} = |G'(d(x))|_{G=t} = g(t) = |\nabla v|_{v=t}.$$
Let
\[ E_t = \{ x \in \Omega : u(x) > t \}, \quad B_t = \{ x \in \Omega^* : v(x) > t \}. \]

By Lemma 2.6 and formula (2.16) we have
\[ -\frac{d}{dt} P(E_t) \geq (n - 1) \frac{W_2(E_t)}{g(t)} \geq (n - 1) n^{-\frac{n-2}{2}} \frac{\omega_n}{\omega_n} \frac{(P(E_t))^{\frac{n-2}{n}}} {g(t)} \]
while for \( v \) it holds
\[ -\frac{d}{dt} P(B_t) = (n - 1) \frac{W_2(B_t)}{g(t)} = (n - 1) n^{-\frac{n-2}{2}} \frac{\omega_n}{\omega_n} \frac{(P(B_t))^{\frac{n-2}{n}}} {g(t)} \]
and \( P(E_0) = P(B_0) \). Then, by classical comparison theorems for differential inequalities,
\[ P(E_t) \leq P(B_t), \quad 0 \leq t \leq \| u \|_{\infty}. \tag{3.3} \]

Denoting by \( \mu(t) = |E_t| \) and by \( \nu(t) = |B_t| \), the coarea formula (2.1) ensures us that
\[ -\mu'(t) = \int_{u=t} 1_{|\nabla u|} d\mathcal{H}^{n-1} = \int_{v=t} \frac{P(E_t)}{g(t)} \leq \frac{P(B_t)}{g(t)} = -\nu'(t), \]
for \( t \in (0, \| u \|_{\infty}) \). The first equality holds true since \( |\nabla u| \neq 0 \) in \( \{ v_m < u < \| u \|_{\infty} \} \) (see [BZ88]). So the function \( \nu - \mu \) is decreasing in \([0, \| u \|_{\infty}]\), and
\[
\int_{\Omega} u^p \, dx = \int_0^{\| u \|_{\infty}} pt^{p-1} \mu(t) \, dt = \int_0^{v_M} pt^{p-1} \nu(t) \, dt - \int_0^{v_M} pt^{p-1} (\nu(t) - \mu(t)) \, dt \\
\geq \int_{\Omega^*} v^p \, dx - v_M^p (|\Omega^*| - |\Omega|). 
\]

Moreover, by (3.3), we get
\[
\int_{u=t} |\nabla u|^{p-1} d\mathcal{H}^{n-1} = g(t)^{p-1} P(E_t) \leq g(t)^{p-1} P(B_t) = \int_{v=t} |\nabla v|^{p-1} d\mathcal{H}^{n-1}, 
\]
so, if we integrate from 0 to \( \| u \|_{\infty} \),
\[
\int_{\Omega} |\nabla u|^p \, dx \leq \int_0^{\| u \|_{\infty}} \int_{v=t} |\nabla v|^{p-1} d\mathcal{H}^{n-1} \, dt \leq \int_{\Omega^*} |\nabla v|^p. 
\]

We also observe that by construction both \( u \) and \( v \) are constant on \( \partial \Omega \), so
\[
\beta \int_{\partial \Omega} u^p \, d\mathcal{H}^{n-1} = \beta v_M^p P(\Omega) = \beta \int_{\partial \Omega^*} v^p \, d\mathcal{H}^{n-1}.
\]
We finally get
\[ \lambda_{p,\beta}(\Omega) \leq \frac{\int_{\Omega} |\nabla u|^p \, dx + \beta \int_{\partial \Omega} u^p \, d\mathcal{H}^{n-1}}{\int_{\Omega} u^p \, dx} \leq \frac{\int_{\Omega^*} |\nabla v|^p \, dx + \beta \int_{\partial \Omega^*} v^p \, d\mathcal{H}^{n-1}}{\int_{\Omega^*} v^p \, dx - v_M^p(|\Omega^*| - |\Omega|)} \]

(3.4)

The claim follows from (3.1) as the quantity
\[ 1 - C(n, p, \beta, \rho) \left( 1 - \frac{|\Omega|}{|\Omega^*|} \right) \]

is non-negative.

\[ \square \]

**Remark 3.1.** The constant \( C(n, p, \beta, \rho) = \frac{\|v\|_{L^\infty}^p |\Omega^*|}{\|v\|_{L^p}^p} \) depends on the perimeter of the set \( \Omega \) and on \( \beta \). Thanks to Proposition 2.8 and Remark 2.1 it is possible to bound the constant \( C(n, p, \beta, \rho) \) from above with a constant independent of the perimeter and of \( \beta \). Indeed \( \forall \beta > 0 \), if we denote by \( v_{p,\infty} \) the first Dirichlet eigenfunction normalized in such a way \( v_{p,\beta}(0) = v_{p,\infty}(0) \), we have
\[ C(n, p, \beta, \rho) \leq \frac{v_{p,\infty}(0) |\Omega^*|}{\|v_{p,\infty}\|_{L^p}^p} =: C(n, p) \]

that is independent of the perimeter thanks to the rescaling properties of the Dirichlet \( p \)-Laplacian eigenfunction.

**Proof of Theorem 1.2.** Let \( v \) be a positive eigenfunction associated to \( \lambda_{p,\beta}(\Omega^*) \), then \( v \) is a \( p \) sub-harmonic function. We denote by \( v_m = v(0) = \min_{\Omega^*} v \) and by \( v_M = \max_{\Omega^*} v \).

Let us consider the function \( \tilde{u} = v_M - v \), that is a positive function with zero trace, and \( g(t) = |\nabla v|_{v=t}, \ v_m < t < v_M \). We set \( \tilde{u}(x) = G(d(x)), \ x \in \Omega \), where \( G^{-1}(t) = \int_0^t \frac{1}{g(s)} \, ds \) with \( 0 < t < v_M - v_m \). By construction, \( \tilde{u} \in W^{1,p}_0(\Omega) \). Now we can set \( u = v_M - \tilde{u} \), and we have:
\[ u_M = \max_{\Omega} u = v_M \]
\[ u_m = \min_{\Omega} u = v_M - \max_{\Omega^*} \tilde{u} \geq v_M - \max_{\Omega^*} v = v_m \]

\[ |\nabla u|_{u=t} = |\nabla v|_{v=t} = g(t) \quad u_m < t < u_M. \]

(3.5)

Let
\[ \tilde{E}_t = \{ x \in \Omega : \tilde{u}(x) > t \}, \quad \tilde{B}_t = \{ x \in \Omega : \tilde{u}(x) > t \}, \]
\[ E_t = \{ x \in \Omega : u(x) > t \} = \Omega \setminus \tilde{E}_{v_M-t}, \quad B_t = \{ x \in \Omega : v(x) > t \} = \Omega \setminus \tilde{B}_{v_M-t}. \]

(3.6)
3 PROOFS OF MAIN RESULTS

By Lemma 2.6 and formula (2.16) we have
\[- \frac{d}{dt} P(\tilde{E}_t) \geq (n - 1) \frac{W_2(\tilde{E}_t)}{g(t)} \geq (n - 1)n^{-\frac{n-2}{n-1}} \omega_n^{-\frac{n-2}{n-1}} \frac{P(\tilde{E}_t)}{g(t)}\]
while for \( v \) it holds
\[- \frac{d}{dt} P(\tilde{B}_t) = (n - 1) \frac{W_2(\tilde{B}_t)}{g(t)} = (n - 1)n^{-\frac{n-2}{n-1}} \omega_n^{-\frac{n-2}{n-1}} \frac{P(\tilde{B}_t)}{g(t)}\]
and \( P(E_0) = P(B_0) \). Then, by classical comparison theorems for differential inequalities,
\[ P(\tilde{E}_t) \leq P(\tilde{B}_t), \quad 0 \leq t \leq v_M - v_m. \] (3.7)

Hence
\[ P(E_t) = P(\tilde{E}_{v_M-t}) \leq P(\tilde{B}_{v_M-t}) = P(B_t), \quad v_m \leq t \leq v_M. \] (3.8)
Moreover, denoted by \( \tilde{\mu}(t) = |\tilde{E}_t| \) and \( \tilde{\nu}(t) = |\tilde{B}_t| \), the coarea formula (2.1) ensures us that
\[- \tilde{\mu}'(t) = \int_{\tilde{u}=t} 1_{|\nabla \tilde{u}|} \, d\mathcal{H}^{n-1} = \int_{\tilde{v}=t} 1_{|\nabla \tilde{v}|} \, d\mathcal{H}^{n-1} = -\tilde{\nu}'(t), \quad 0 \leq t < v_M - v_m. \]
Moreover, setting \( \mu(t) = |E_t| = |\Omega| - \tilde{\mu}(v_M-t) \) and \( \nu(t) = |B_t| = |\Omega^*| - \tilde{\nu}(v_M-t) \), we have
\[- \mu'(t) \leq -\nu'(t) \text{ in } [v_m, v_M]. \] So the function \( \nu - \mu \) is decreasing in \([v_m, v_M]\), and
\[ \int_\Omega u^p \, dx = \int_0^{v_M} pt^{p-1} \mu(t) \, dt = \int_0^{v_M} pt^{p-1} \nu(t) \, dt - \int_0^{v_M} pt^{p-1}(\nu(t) - \mu(t)) \, dt \]
\[ = \int_0^{v_M} pt^{p-1} \nu(t) \, dt - \int_0^{v_m} pt^{p-1}(\nu(t) - \mu(t)) \, dt - \int_{v_m}^{v_M} pt^{p-1}(\nu(t) - \mu(t)) \, dt \]
\[ \leq \int_{\Omega^*} v^p \, dx - v_m^p(|\Omega^*| - |\Omega|) = \int_{\Omega^*} v^p \, dx \left[ 1 - \frac{v_m^p}{\|v\|_p^p}(|\Omega^*| - |\Omega|) \right]. \]
Moreover, by (3.8), we get
\[ \int_{\tilde{u}=t} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1} = g(t)^{p-1} P(\tilde{E}_t) \leq g(t)^{p-1} P(\tilde{B}_t) = \int_{\tilde{v}=t} |\nabla \tilde{v}|^{p-1} \, d\mathcal{H}^{n-1}, \]
so, if we integrate from 0 to \( \|u\|_\infty \),
\[ \int_\Omega |u|^p \, dx = \int_\Omega |\nabla \tilde{u}|^p \, dx = \int_{\tilde{u}=t} \|\tilde{u}\|_\infty \int_{\tilde{u}=t} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1} \, dt \]
\[ \leq \int_{\tilde{u}=t} \|\tilde{u}\|_\infty \int_{\tilde{v}=t} |\nabla \tilde{v}|^{p-1} \, d\mathcal{H}^{n-1} \, dt = \int_{\Omega^*} |\nabla \tilde{v}|^p \, dx = \int_{\Omega^*} |\nabla \tilde{v}|^p. \]
We also observe that by construction both $u$ and $v$ are constant on $\partial \Omega$, so
\[
\beta \int_{\partial \Omega} u^p \, d\mathcal{H}^{n-1} = \beta u_M^p P(\Omega) = \beta v_M^p P(\Omega) = \beta \int_{\partial \Omega} v^p \, d\mathcal{H}^{n-1}.
\]
We finally get
\[
\lambda_{p,\beta}(\Omega) \leq \frac{\int_{\Omega} |\nabla u|^p \, dx + \beta \int_{\partial \Omega} u^p \, d\mathcal{H}^{n-1}}{\int_{\Omega} u^p \, dx} \leq \frac{\int_{\Omega^*} |\nabla v|^p \, dx + \beta \int_{\partial \Omega^*} v^p \, d\mathcal{H}^{n-1}}{\int_{\Omega^*} v^p \, dx \left[1 - \frac{v_m^p}{\|v\|_p^p}(|\Omega^*| - |\Omega|)\right]} \tag{3.9}
\]
Hence, by direct calculation
\[
\frac{\lambda_{p,\beta}(\Omega^*) - \lambda_{p,\beta}(\Omega)}{\lambda_{p,\beta}(\Omega)} \geq \frac{v_m^p}{\|v\|_p^p}(|\Omega^*| - |\Omega|) \tag{3.10}
\]

**Remark 3.2.** Unlike the constant in Theorem 1.1, the constant $C(n, p, \beta, \rho) = \frac{v_m^p |\Omega^*|}{\|v\|_p^p}$ cannot be bounded from below with a constant independent of the perimeter and of $\beta$. Indeed, for example if $n = p = 2$ and $P(\Omega) = 2\pi$, we have that
\[
v_\beta(x) = I_0 \left(\sqrt{-\lambda_\beta(B_1)} |x|\right),
\]
where $I_0$ is the modified Bessel function.

We recall that for $z$ sufficiently large (see [AS64, Section 9.7]), we have
\[
I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[1 + \frac{1}{8z} + \frac{9}{2! (8z)^2} + \ldots\right]
\]
therefore
\[
\|v_\beta(x)\|_{L^2} \sim \frac{e^\beta}{\beta^2} \xrightarrow{\beta \to -\infty} +\infty
\]
and
\[
C(2, 2, \beta, 2\pi) = \frac{(v_\beta)_m^2 |B_1|}{\|v_\beta\|_2^2} \sim \frac{\beta^2}{e^{-\beta}} \xrightarrow{\beta \to -\infty} 0.
\]

**Remark 3.3.** We want to highlight that the constant $C$ depends actually just on $n, p$ and $\rho \frac{1}{n \omega_n}$ $\beta$. Indeed, for all $\Omega \subset \mathbb{R}^n$ bounded and convex set with $P(\Omega) = \rho$, we can consider
\[
\Omega_1 = \left(\frac{n \omega_n}{\rho}\right) \overline{\Omega} \quad \Omega_1^* = \left(\frac{n \omega_n}{\rho}\right) \overline{\Omega^*} \quad t := \left(\frac{\rho}{n \omega_n}\right)^\frac{1}{n}
\]
so \(P(\Omega_1) = P(\Omega_1') = n\omega_n\) and we have

\[
\frac{\lambda_p,\beta(\Omega^*) - \lambda_p,\beta(\Omega)}{-\lambda_p,\beta(\Omega)} = \frac{\lambda_p,\beta(t\Omega_1^*) - \lambda_p,\beta(t\Omega_1)}{-\lambda_p,\beta(t\Omega_1)} = \frac{\lambda_p,\beta(t^{p-1}\beta)(\Omega_1^*) - \lambda_p,\beta(t^{p-1}\beta)(\Omega_1)}{-\lambda_p,\beta(\Omega_1)} \\
\geq C(n, p, n\omega_n, t^{p-1}\beta) \left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}ight) \\
= C(n, p, n\omega_n, \left(\frac{\rho}{n\omega_n}\right)^{\frac{n-1}{n}} \beta) \left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}ight) \\
= C(n, p, \left(\frac{\rho}{n\omega_n}\right)^{\frac{n-1}{n}} \beta) \left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}ight) .
\]

Proof of Theorem 1.3. Using (3.10), the isoperimetric inequality and

\[|\lambda_p,\beta(\Omega)| \geq |\beta| \frac{P(\Omega)}{|\Omega|},\]

we have

\[
\lambda_p,\beta(\Omega^*) - \lambda_p,\beta(\Omega) \geq |\lambda_p,\beta(\Omega)| \left\|v^p\right\|_p (|\Omega^*| - |\Omega|) \\
\geq |\beta| \frac{P(\Omega)}{|\Omega|} \left\|v^p\right\|_p (|\Omega^*| - |\Omega|) \\
\geq |\beta| \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}}{\rho^{\frac{n-1}{n}}} \left\|v^p\right\|_p (|\Omega^*| - |\Omega|). \tag{3.11}
\]

Now, if we suppose that \(\lambda_p,\beta(\Omega^*) - \lambda_p,\beta(\Omega) \leq \delta_0\), by (3.11) we have

\[|\Omega^*| - |\Omega| \leq K(n, \rho, p, \beta)\delta_0.\]

So by Lemma 2.4 we conclude

\[\lambda_p,\beta(\Omega^*) - \lambda_p,\beta(\Omega) \geq C(n, \rho, p, \beta) g(\mathcal{A}^*_{\mathcal{H}}(\Omega)). \]

\[\square\]

Remark 3.4. Our result, Theorem 1.3, applies only when

\[\lambda_p,\beta(\Omega^*) - \lambda_p,\beta(\Omega) \leq \delta_0.\]

It is possible to get rid of this constraint, obtaining a weaker result.

In order to obtain it, we need the quantitative version of the isoperimetric inequality proved in [FMP08]

\[P(\Omega) \geq P(\Omega^2) \left(1 + \gamma(n)\alpha(\Omega)^2\right) \quad \text{where } \alpha(\Omega) = \min \left\{ \frac{|\Omega \triangle B_r|}{|\Omega|} \left| \frac{|\Omega \triangle B_r|}{|\Omega|} \right| B_r = |\Omega| \right\}, \tag{3.12}\]
and the following result [EFT05, Lemma 4.2]: there exists a constant $C(n)$ such that if $C, W$ are open and convex sets such that $|C| = |W|$ and $|C \triangle W| < \frac{|C|}{2}$, it holds

$$d_H(C, W) \leq C(n) \left[ \text{diam}(C) + \text{diam}(W) \right] \left( \frac{|C \triangle W|}{|C|} \right)^{\frac{1}{n}}. \tag{3.13}$$

Moreover we have to recall that

$$|\Omega^*| = \frac{P(\Omega)^{\frac{1}{n-1}}}{n n^{n-1} \omega_n^{\frac{1}{n}}} \quad \text{and} \quad P(\Omega^2) = n \omega_n^{\frac{1}{n}} |\Omega|^{\frac{n-1}{n}},$$

and we have

$$1 - \frac{|\Omega|}{|\Omega^*|} = 1 - \frac{n^{n-1} \omega_n^{-1} |\Omega|}{P(\Omega)} \geq 1 - \frac{1}{\left(1 + \gamma(n) \alpha^2(\Omega)\right)^{\frac{1}{n-1}}} \geq 1 - \frac{1}{\left(1 + \gamma(n) \alpha^2(\Omega)\right)} = \frac{\gamma(n) \alpha^2(\Omega)}{1 + \gamma(n) \alpha^2(\Omega)},$$

where we used Bernoulli’s inequality

$$(1 + x)^r \geq 1 + rx \quad \forall x \geq -1, \forall r \geq 0.$$ 

Since $0 < \alpha^2(\Omega) < 4$ we have

$$1 - \frac{|\Omega|}{|\Omega^*|} \geq \frac{\gamma(n)}{1 + 4\gamma(n)} \alpha^2(\Omega) = C(n) \alpha^2(\Omega).$$

If $\Omega^2$ is a ball that realizes the minimum in (3.12), then using (3.13) we obtain

$$C(n) \alpha^2(\Omega) \geq C(n) \frac{d_H(\Omega, \Omega^2)^{2n}}{\left[\text{diam}(\Omega) + \text{diam}(\Omega^2)^2\right]^{2n}}$$

and so

$$\frac{\lambda_{p,\beta}(\Omega^*) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \geq C(n, p, \beta \rho) C(n) \frac{d_H(\Omega, \Omega^2)^{2n}}{\left[\text{diam}(\Omega) + \text{diam}(\Omega^2)^2\right]^{2n}}.$$
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