On initial-boundary value problem of the stochastic Navier–Stokes equations in the half space

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ABSTRACT
We study the initial-boundary value problem of the stochastic Navier–Stokes equations in half-space. We prove the existence of weak solutions in standard Besov space-valued random processes when the initial data belong to the critical Besov space.

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1. Introduction
In this article, we study the stochastic Navier–Stokes equations

\[ du(t, x) = \left( \Delta u(t, x) - \nabla p(t, x) - \text{div}(u \otimes u)(t, x) \right) dt + g(t, x) dB_t, \quad \text{div } u = 0 \]  

(1.1)

for \((t, x) \in (0, \infty) \times \mathbb{R}^n_+, n \geq 2\), with the boundary condition \(u(t, x) = 0\) for \((t, x) \in (0, \infty) \times \mathbb{R}^{n-1}_+\), the initial condition \(u(0, x) = u_0\) for \(x \in \mathbb{R}^n_+\), and the random noise \(g(t, x) dB_t\). Here \(\mathbb{R}^n_+\) denotes the half space and \(\{B_t(\omega) : t \geq 0, \omega \in \Omega\}\) denotes an \(n\)-dimensional Wiener process defined on a probability space \(\Omega\). The Navier–Stokes system is considered a reliable model in science and engineering. Despite the considerable effort made by countless mathematicians, fundamental mathematical issues related to uniqueness, regularity, and turbulence in fluid motions remain open questions. Although turbulent phenomena can be frequently observed in many real-world situations, it is complicated and unclear to define or characterize them mathematically. Because chaotic behavior in fluid motions may be interpreted as a presence of randomness, much interest in stochastic Navier–Stokes equations has been increased.

The study of the stochastic Navier–Stokes equations started around the early 70s', for example, by Bensoussan and Temam [1], and Foias [2]. Many mathematicians made essential contributions. We list a few of them here. Capiński and Gatarek [3] obtained an existence theorem for the stochastic Navier–Stokes equations in a Hilbert space setting. Mikulevicius and Rozovskii [4] proved the existence of a global weak (martingale) solution of the stochastic Navier–Stokes equation and the global existence of a probabilistically strong solution in the 2D case. Kim [5] established the existence of local strong solutions to the stochastic Navier–Stokes equations in \(\mathbb{R}^3\) when the initial data are sufficiently small and slightly more regular.
Taniguchi [6] studied the energy solutions to the stochastic Navier–Stokes equations in two-dimensional unbounded domains. For a recent overview, we refer the reader to the papers by Flandoli and Gatarek [7], Flandoli and Romito [8], and the lecture notes or monographs by Flandoli [9], Kuksin and Shirikyan [10], and Breit, Feireisl, Hofmanová [11] and the references cited therein.

For the deterministic Navier–Stokes equations, Fujita and Kato [12] proved the existence of mild solutions. Many papers continued to study such solutions with a broader class of initial data, which should be in some particular classes due to the scaling structure of the Navier–Stokes equations. By adapting harmonic analysis tools like the Littlewood–Paley theory, the Fujita–Kato result extended to the refined function space setting like Besov spaces. For this, we refer the reader to the monograph by Bahouri, Chemin, and Danchin [15]. After that, Koch and Tataru [14] extends the class of initial data to $BMO^{-1}$. This line of development is well illustrated in the monograph by Cannone [13].

Recently, Du and Zhang [16] prove the existence of solutions to the stochastic Navier–Stokes equations in the whole space. Due to a technical reason, they found solutions in the sense Definition 1.2 such that $\int_{\mathbb{R}^n_+} u(x,t) \Phi(x) dx - \langle u_0, \Phi \rangle = \int_0^t \int_{\mathbb{R}^n_+} (u \cdot \Delta \Phi + (u \otimes u) : \nabla \Phi) dx dt + \int_0^t \langle g, \Phi \rangle dB_t$ for all $t \leq \tau$ with probability 1 for every $\Phi \in C^\infty_0(\mathbb{R}^n_+)$ with $\text{div}_x \Phi = 0$, $\Phi|_{x_n=0} = 0$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

**Theorem 1.1.** Let $2 \leq n$ and $n < p < \infty$, $2 < q < \infty$, and $2\alpha = 1 - \frac{n}{p} - \frac{2}{q} \geq 0$. Assume $u_0 \in B_{pq0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)$ with $\text{div} u_0 = 0$ and $g \in L^q_{\alpha_2}(\Omega \times (0, \infty), \mathcal{P}; L^p(\mathbb{R}^n_+))$ with $\text{div} g = 0$, where $p_2$ and $\alpha_2$ satisfy

$$p_2 \leq p, \quad \alpha \leq \alpha_2, \quad \left(\frac{n}{p_2} - \frac{n}{p}\right) + 2(\alpha_2 - \alpha) = 1.$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$\|u_0\|_{B_{pq0}^{-1+\frac{n}{p}}(\mathbb{R}^n_+)} + \|g\|_{L^q_{\alpha_2}(\Omega \times (0, \infty), \mathcal{P}; L^p(\mathbb{R}^n_+))} < \delta,$$

then there is a unique local weak solution $(u, \tau)$ of the initial boundary value problem (1.1) in the sense Definition 1.2 such that $\mathbb{P}(\tau > 0) = 1$ and $\mathbb{P}(\tau = \infty) \geq 1 - \varepsilon$.

The exact notation and definition of function spaces will be given in Section 2. Now, we introduce the definition of a local weak solution of the Equation (1.1).

**Definition 1.2.** (Local solution for Navier-Stokes equations). Let $p$, $q$, $p_2$, $\alpha$, $\alpha_2$ be given numbers in Theorem 1.1 and $u_0$ and $g$ be given datum given in Theorem 1.1. For a vector field $u$ and a stopping time $\tau$, we say that $(u, \tau)$ is a local weak solution for the stochastic Navier-Stokes equations (1.1) if $u \in L^q_{\alpha}(((0, \tau[), \mathcal{P}; L^p(\mathbb{R}^n_+))$ is a progressively measurable process and

$$\int_{\mathbb{R}^n_+} u(x,t) \Phi(x) dx - \langle u_0, \Phi \rangle = \int_0^t \int_{\mathbb{R}^n_+} (u \cdot \Delta \Phi + (u \otimes u) : \nabla \Phi) dx dt + \int_0^t \langle g, \Phi \rangle dB_t$$

for all $t \leq \tau$ with probability 1 for every $\Phi \in C^\infty_0(\mathbb{R}^n_+)$ with $\text{div}_x \Phi = 0$, $\Phi|_{x_n=0} = 0$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.
Theorem 1.3. In addition, if \( g \in \mathcal{L}^q(\Omega \times (0, \infty), \mathcal{P}; \dot{B}^{-2\alpha-1}_{pq0}(\mathbb{R}^n_+)) \), then the solution \( u \) obtained in Theorem 1.1 belongs to \( L^q(0, \tau(\omega); \dot{B}^{-2\alpha-1}_{pq0}(\mathbb{R}^n_+)) \) almost surely.

Theorem 1.4. Let \( p, p_2, q, \alpha, \alpha_2 \) be numbers in Theorem 1.1. Moreover, if \( \frac{n}{2p} + \frac{2}{q} < \frac{1}{\alpha}, \frac{n}{p_2} - \frac{n}{p} < 1 - \frac{2}{q}, 2\alpha_2 < 1 - \frac{2}{q} \) and \( g \in \mathcal{M}^{p,q}_\alpha(\Omega \times (0, \infty), \mathcal{P}; \mathcal{L}^2(\mathbb{R}^n_+)) \), then the solution \( u \) obtained in Theorem 1.1 belongs to \( C(0, \infty; \mathcal{L}^p(\mathbb{R}^n_+)) \).

We list here a few remarks on our results.

Remark 1.5. (1) The standard Besov spaces are used throughout the article (cf. [16]).
(2) The regularity exponent of the initial data is negative.
(3) The exponents in the main theorems are rather complicated and restricted due to the Hardy–Littlewood-type embedding and scaling structure of governing equations.

The article is organized as follows. Section 2 presents definitions of function spaces. In Section 3, we give a few estimates for the Newtonian potential and the Heat kernel and a representation of the Helmholtz projection operator. In Section 4, we prove the existence of the solutions to the stochastic Stokes equations in half-space. In Sections 5, 6, and 7, we prove Theorems 1.1, 1.3, and 1.4, respectively. In Appendices, we gather proofs of technical lemmas for the reader’s convenience. Although they can be seen as variants used in different kinds of literature, we present them for completeness.

2. Function spaces

To fix notations, we recall some standard definitions. The Fourier transform of a Schwartz function \( f \in S(\mathbb{R}^n) \) is given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.
\]

The definition of the Fourier transform is extended to tempered distributions \( f \in S'(\mathbb{R}^n) \) by duality.

We fix a function \( \Psi \in S(\mathbb{R}^n) \) whose Fourier transform is nonnegative and satisfies

\[
\chi_{\{1 \leq |\xi| \leq 2^{-2/7}\}}(\xi) \leq \widehat{\Psi}(\xi) \leq \chi_{\{1-1/7 \leq |\xi| \leq 2\}}(\xi)
\]

so that for \( \xi \neq 0 \),

\[
\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j} |\xi|) = 1.
\]

For \( j \in \mathbb{Z} \), we define the associated Littlewood-Paley operator by

\[
\Delta_j f(x) = \Psi_{2^{-j}} * f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \widehat{\Psi}(2^{-j} \xi) \hat{f}(\xi) d\xi.
\]

For \( 1 \leq p \leq \infty \), we denote the Lebesgue spaces by \( L^p(\mathbb{R}^n) \) with the norm

\[
\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.
\]
Now, we define three types of the Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$, $\dot{B}^s_{p,q}(\mathbb{R}^n_+)$, and $\dot{B}^s_{p,q}(\mathbb{R}^n_+)$). We usually omit $\mathbb{R}^n$ and $\mathbb{R}^n_+$ in these notations because the symbol $+$ represents the half-space $\mathbb{R}^n_+$.

**Definition 2.1.** (Besov spaces on $\mathbb{R}^n$). Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. We define the homogeneous Besov space to be the set $\dot{B}^s_{p,q}(\mathbb{R}^n)$ of all $f \in S'(\mathbb{R}^n)$ with the finite norm

$$\|f\|_{\dot{B}^s_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \Delta^j f \right\|_p^q \right)^{1/q}, \quad 1 \leq q < \infty,$$

$$\|f\|_{\dot{B}^s_{p,\infty}} = \sup_{j \in \mathbb{Z}} 2^{jsq} \left\| \Delta^j f \right\|_p$$

with the obvious supremum modification when $p, q = \infty$.

**Definition 2.2.** (Besov spaces on $\mathbb{R}^n_+$). Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. We define the homogeneous Besov space in $\mathbb{R}^n_+$ to be the set

$$\dot{B}^s_{p,q}(\mathbb{R}^n_+) = \left\{ f = F|_{\mathbb{R}^n_+} : F \in \dot{B}^s_{p,q}(\mathbb{R}^n) \right\}$$

of all restrictions of $F \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ with the finite norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)} = \inf \left\{ \|F\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} : F \in \dot{B}^s_{p,q}(\mathbb{R}^n), F|_{\mathbb{R}^n_+} = f \right\}.$$

The space $\dot{B}^s_{p,q}(\mathbb{R}^n_+)$ is the closure of $C^\infty_0(\mathbb{R}^n_+)$ in the norm of $\dot{B}^s_{p,q}(\mathbb{R}^n_+)$.\n
**Remark 2.3.** Let $0 \leq s < \infty$ and $1 \leq p, q \leq \infty$. Then

$$(\dot{B}^s_{p,q}(\mathbb{R}^n_+))' = \dot{B}^{-s}_{p',q'}(\mathbb{R}^n_+) \quad \text{and} \quad (\dot{B}^s_{p,q}(\mathbb{R}^n_+))' = \dot{B}^{-s}_{p',q'}(\mathbb{R}^n_+),$$

where $p'$ denotes the Hölder conjugates of $p$.

**Definition 2.4.** (Weighted Bochner spaces). Let $X$ be a Banach space. For $0 \leq s < \infty$ and $1 \leq r \leq \infty$, we denote by $L^s_t(I;X)$ the weighted Bochner space with the norm $\|f\|_{L^s_t(I;X)}$ given by

$$\|f\|_{L^s_t(I;X)} = \int_I t^{sr} \|f(t)\|_X^r \, dt.$$ 

In particular, if $s = 0$, it is the same as the usual Bochner space, i.e., $L^0_t(I;X) = L^r(I;X)$.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. Let $\{\mathcal{G}_t : t \geq 0\}$ be a filtration of $\sigma$-fields $\mathcal{G}_t \subset \mathcal{G}$ with $\mathcal{G}_0$ containing all $\mathbb{P}$-null subsets of $\Omega$. Assume that $W(t)$ is a one-dimensional $\{\mathcal{G}_t\}$-adapted Wiener process defined on $(\Omega, \mathcal{G}, \mathbb{P})$. We denote the expectation of a random variable $X$ by

$$\mathbb{E}X = \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega).$$

Following the standard convention, we omit the argument $\omega$ of random variables $X(\omega)$.

We note that the solution $u$ and data $(u_0, g)$ in (1.1) are random variables. We construct suitable spaces for them using the Besov spaces. We shall use two different types of spaces. The first type emphasizes the regularity in $x$, whereas the second type does the joint regularity in $(t, x)$. We can consider $u$ and $g$ as Banach space-valued stochastic processes. Hence
where $(\Omega \times (0, \infty), \mathcal{P}, \mathbb{P} \otimes \ell(0, \infty))$ is a suitable choice for their common domain, where $\mathcal{P}$ is the predictable $\sigma$-field generated by $\{\mathcal{G}_t : t \geq 0\}$ (see, for instance, pp. 84–85 of [17]) and $\ell(0, \infty)$ is the Lebesgue measure on $(0, \infty)$.

**Definition 2.5.** (Stochastic spaces). We define $L^p(\mathbb{R}_+^n)$ and and $B^{-1+\frac{p}{q}}_{p0}(\mathbb{R}_+^n)$ as sets of $L^p(\mathbb{R}_+^n)$-valued $\mathcal{F}_0$-measurable random variables and $B^{-1+\frac{p}{q}}_{p0}(\mathbb{R}_+^n)$-valued $\mathcal{F}_0$-measurable random variables with the norms

$$\|f\|_{L^p(\mathbb{R}_+^n)} = \left(\mathbb{E}\|f\|^p_{L^p(\mathbb{R}_+^n)}\right)^{\frac{1}{p}}, \quad \|f\|_{B^{-1+\frac{p}{q}}_{p0}(\mathbb{R}_+^n)} = \left(\mathbb{E}\|f\|^q_{B^{-1+\frac{p}{q}}_{p0}(\mathbb{R}_+^n)}\right)^{\frac{1}{q}}.$$

For stopping time $\tau$, we denote $(0, \tau] := \{\omega \in \Omega \times (0, \infty) \mid 0 < t \leq \tau(\omega)\}$. For a Banach space $X$ and $\alpha \in \mathbb{R}$, we define the stochastic Banach spaces $\mathcal{L}_\alpha((0, \tau], \mathcal{P}; X)$ and $\mathcal{M}_\alpha^{\gamma_1, \gamma_2}((0, \tau], \mathcal{P}; X)$ to be the space of $X$-valued processes with the norms

$$\|f\|_{\mathcal{L}_\alpha((0, \tau], \mathcal{P}; X)} = \mathbb{E}\int_0^\tau s^\alpha r\|f(s, \cdot)\|_X ds,$$

$$\|f\|_{\mathcal{M}_\alpha^{\gamma_1, \gamma_2}((0, \tau], \mathcal{P}; X)} = \mathbb{E}\left(\int_0^\tau s^\gamma r\|f(s, \cdot)\|_X ds\right)^{\frac{\gamma_2}{\gamma_1}}.$$

In particular, if $\tau = \infty$, then we denote

$$\mathcal{L}_\alpha((0, \infty], \mathcal{P}; X) = \mathcal{L}_\alpha((0, \infty), \mathcal{P}; X),$$

$$\mathcal{M}_\alpha^{\gamma_1, \gamma_2}((0, \infty], \mathcal{P}; X) = \mathcal{M}_\alpha^{\gamma_1, \gamma_2}((0, \infty), \mathcal{P}; X).$$

Note that the elements of $\mathcal{L}_\alpha((0, \tau], \mathcal{P}; X)$ and $\mathcal{M}_\alpha^{\gamma_1, \gamma_2}((0, \tau], \mathcal{P}; X)$ are treated as functions rather than distributions or classes of equivalent functions.

### 3. Kenel estimates in half spaces

If $y = (y', y_n) \in \mathbb{R}^{n-1} \times (0, \infty)$, then we denote by $y^* = (y', -y_n)$ the reflection point of $y \in \mathbb{R}_+^n$. From now on we simply write $\|f\|_p$ instead of $\|f\|_{L^p(\mathbb{R}^n)}$.

**Definition 3.1.** The fundamental solution of the Laplace equation in $\mathbb{R}^n$ is given by

$$N(x) = \begin{cases} \frac{1}{\sigma_n (2-n) |x|^{n-2}} & \text{if } n \geq 3 \\ \frac{1}{2\pi} \ln |x| & \text{if } n = 2 \end{cases}$$

where $\sigma_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

**Lemma 3.2.** If $1 < p < \infty$, $1 \leq q \leq \infty$, and $0 < s < 1/p' = 1 - 1/p$, then

$$\left\| \nabla_x^2 \int_{\mathbb{R}_+^n} N(\cdot - y)f(y)dy \right\|_{\dot{\mathcal{B}}^{-s}_{p0}(\mathbb{R}_+^n)} \lesssim \left\| f \right\|_{\dot{\mathcal{B}}^{s}_{p0}(\mathbb{R}_+^n)}$$

$$\left\| \nabla_x^2 \int_{\mathbb{R}_+^n} N(\cdot - y^*)f(y)dy \right\|_{\dot{\mathcal{B}}^{-s}_{p0}(\mathbb{R}_+^n)} \lesssim \left\| f \right\|_{\dot{\mathcal{B}}^{s}_{p0}(\mathbb{R}_+^n)}.$$

**Proof.** See Appendix A. \qed
Definition 3.3. The fundamental solution of the heat equation in $\mathbb{R}^n$ is given by

$$\Gamma_t(x) = \Gamma(x, t) = \begin{cases} (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} & \text{if} \ t > 0 \\ 0 & \text{if} \ t \leq 0. \end{cases}$$

We define

$$\Gamma_t * f(x) = \int_{\mathbb{R}^n} \Gamma(x-y, t)f(y)dy,$$

$$\Gamma_t^* * f(x) = \int_{\mathbb{R}^n} \Gamma(x-y^*, t)f(y)dy.$$

Lemma 3.4. There is $c > 0$ such that for all $1 \leq p \leq \infty$, $t > 0$, and $j \in \mathbb{Z}$

$$\|\Delta_j(\Gamma_t * f)\|_p \lesssim \exp(-ct2^j)\|\Delta_j f\|_p.$$ 

Proof. See Lemma 2.4 in [15].

Lemma 3.5. If $1 \leq p, q \leq \infty$ and $\beta \in \mathbb{R}$, then

$$\|\Gamma_t * f\|_{L^q(0, \infty; B^\beta_{pq}(\mathbb{R}^n))} + \|\Gamma_t^* * f\|_{L^q(0, \infty; B^\beta_{pq}(\mathbb{R}^n))} \lesssim \|f\|_{B^\beta_{pq}(\mathbb{R}^n)}.$$ 

Proof. See Appendix B.

Lemma 3.6. If $1 \leq p, q \leq \infty$ and $0 < \alpha < \infty$, then

$$\|\Gamma_t * f\|_{L^q(0, \infty; L^p(\mathbb{R}^n))} + \|\Gamma_t^* * f\|_{L^q(0, \infty; L^p(\mathbb{R}^n))} \lesssim \|f\|_{B^{2\alpha - \frac{3}{q}}_{pq}(\mathbb{R}^n)}.$$ 

Proof. See Appendix C.

Definition 3.7. We define

$$\mathcal{U}F(x, t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x-y, t-s)F(y, s)dyds,$$

$$\mathcal{U}^*F(x, t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x-y^*, t-s)F(y, s)dyds.$$ 

and

$$\mathcal{V}g(x, t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x-y, t-s)g(y, s)dydB_s,$$

$$\mathcal{V}^*g(x, t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x-y^*, t-s)g(y, s)dydB_s.$$ 

Lemma 3.8. If $p, q, \alpha, p_1, q_1, \alpha_1$ satisfy $0 < \frac{1}{2} - \frac{1}{2} \left(\frac{n}{p_1} - \frac{n}{p}\right) - \alpha < 1$, $1 < p_1 < p < \infty$, $1 < q_1 \leq q < \infty$, $0 \leq \alpha_1 < 1 - \frac{1}{q_1}$, and $\left(\frac{n}{p_1} - \frac{n}{p}\right) + \left(\frac{2}{q_1} - \frac{2}{q}\right) + 2(\alpha_1 - \alpha) = 1$, then

$$\|D_2\mathcal{U}F\|_{L^q(0, \infty; B^{-2\alpha}_{pq}(\mathbb{R}^n_+))} + \|D_2\mathcal{U}^*F\|_{L^q(0, \infty; B^{-2\alpha}_{pq}(\mathbb{R}^n_+))} \lesssim \|F\|_{L^{\alpha_1 q_1}(0, \infty; L^p(\mathbb{R}^n_+))}.$$ (3.1)
Proof. See Appendix D. □

Lemma 3.9. Let \( \tilde{g} \) be a zero extension of \( g \) over \( \mathbb{R}^n \). For \( j \in \mathbb{N} \),
\[
E \int_0^\infty 2^j \| \Delta_j \tilde{g}(t) \|^2_p \, dt \lesssim E \int_0^\infty \| \Delta_j \tilde{g}(t) \|^2_p \, dt,
\] (3.2)

Proof. See Appendix E. □

Here we recall the Helmholtz projection operator. See Section 3 of \[18\] for more details.

Lemma 3.10. Let \( \mathcal{F} = (F_{kl})_{k,l=1}^n \) with
\[
F_{kl} = F_{lk} \quad \text{and} \quad F_{mk}|_{x_n = 0} = 0.
\]
If \( f = \text{div} \mathcal{F} \), then the Helmholtz projection operator \( \mathbb{P}f \) in \( \mathbb{R}^n_+ \) is given by
\[
\mathbb{P}f = \text{div} \mathcal{F}', \quad (\mathcal{F}')_{mn}|_{x_n = 0} = 0
\]
with
\[
\| \mathcal{F}' \|_{L^p(\mathbb{R}_+^n)} \lesssim \| \mathcal{F} \|_{L^p(\mathbb{R}_+^n)} \quad 1 < p < \infty.
\] (3.3)

Proof. See Section 3 in \[19\]. □

Lemma 3.11. Let \( 0 < \lambda < 1 \) and define for \( t > 0 \)
\[
I_\lambda f(t) = \int_0^t (t - s)^{-\lambda} f(s) \, ds.
\]
Then the operator \( I_\lambda : L^p_\alpha(0, \infty) \to L^q_\beta(0, \infty) \) is bounded if \( 1 < p \leq q < \infty \) and \( \beta \leq \alpha \) satisfy
\[
0 < \alpha \beta + 1 < p, \quad 0 < \beta q + 1 < q, \quad 1 + \frac{1}{q} + \beta = 1 + \lambda + \alpha.
\]

Proof. See Theorem 1.2 in \[20\]. □

4. Stochastic stokes equations

In this section, we consider the initial-boundary value problem of the stochastic Stokes equations in \( \Omega \times (0, \infty) \times \mathbb{R}^n_+ \),
\[
dw(t, x) = (\Delta w(t, x) - \nabla \pi(t, x) + \text{div} \mathcal{F}(t, x)) \, dt + g(t, x) \, dB_t
\] (4.1)
with the boundary condition \( w|_{x_n = 0} = 0 \) and the initial condition \( w|_{t=0} = u_0 \).

Definition 4.1. (Weak solution for stochastic Stokes equations). Let \( u_0, \mathcal{F} \) and \( g \) satisfy the assumption in Proposition 4.4. We say that \( w \) is a weak solution for the stochastic Stokes equations (4.1) if \( w \in L^3_\alpha(\Omega \times (0, \infty), \mathcal{P}; L^p(\mathbb{R}_+^n)) \) is a progressively measurable process and
\[
\int_{\mathbb{R}_+^n} w(t, x) \cdot \Phi(x) \, dx = \int_{\mathbb{R}_+^n} u_0(x) \cdot \Phi(x) \, dx
\]
\[
= \int_0^t \int_{\mathbb{R}_+^n} \left( w \cdot \Delta \Phi + \mathcal{F} : \nabla \Phi \right) \, dx \, dt + \int_0^t \langle g, \Phi \rangle \, dB_t
\]
for all \( t < \infty \) with probability 1 for every \( \Phi \in C_0^\infty(\mathbb{R}^n_+) \) with \( \text{div}_x \Phi = 0, \Phi|_{x_n=0} = 0 \), where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing.

In Solonnikov [21], it was shown that if \( \mathcal{F}, u_0 \) and \( g \) are sufficiently smooth with respect to \((t,x)\), then the solution \( w \) of (4.1) is represented by

\[
\begin{align*}
  w(t,x) &= \int_{\mathbb{R}^n_+} K(x,y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n_+} K(x,y,t-s)\mathbb{P}\text{div}\mathcal{F}(y,s)dyds \\
  & \quad + \int_0^t \int_{\mathbb{R}^n_+} K(x,y,t-s)g(y,s)dydB_s \\
  &= w_1(t,x) + w_2(t,x) + w_3(t,x),
\end{align*}
\]

where \( \mathbb{P} \) is Helmholtz projection operator in \( \mathbb{R}^n_+ \) and the kernel \( K = (K_{ij}) \) is given by

\[
K_{ij}(x,y,t) = \delta_{ij} \left( \Gamma(x-y,t) - \Gamma(x-y^*,t) \right) \\
+ 4(1 - \delta_{jn})D_j \int_{\mathbb{R}^{n-1}_+} D_y N(x-z) \Gamma(z-y^*,t)dz,
\]

where \( y^* = (y', -y_n) \). In Section 3.1 of [22], it was shown that the kernel can be decomposed as

\[
K_{ij}(x,y,t) = \delta_{ij} \left( \Gamma(x-y,t) - \Gamma(x-y^*,t) \right) \\
+ 4(1 - \delta_{jn})D_j \int_{\mathbb{R}^{n-1}_+} D_y N(x-z) - N(x-z^*) \Gamma(z-y^*,t)dz \\
- 2(1 - \delta_{jn})D_j \int_{\mathbb{R}^{n-1}_+} D_y N(x-z) - N(x-z^*)R_j \Gamma(z-y^*,t)dz \\
- 4(1 - \delta_{jn})\delta_{jn} R_j \Gamma(x-y^*,t),
\]

where \( R_j, 1 \leq j \leq n-1 \) are \( n-1 \) dimensional Riesz transform.

**Lemma 4.2.** Let \( 1 < p < \infty, 2 < q \leq \infty, \) and \( 0 \leq 2\alpha < 1 - \frac{1}{p} - \frac{2}{q} \) Assume \( u_0 \in B^{2\alpha - \frac{2}{q}}_{pq0}(\mathbb{R}^n_+) \) with \( \text{div} u_0 = 0, \mathcal{F} \in L^p_{q_1}(0,\infty;L^p(\mathbb{R}^n_+)) \) for some \( p_1, q_1, \) and \( \alpha_1 \) satisfying

\[
q_1 \leq q, \quad \alpha \leq \alpha_1 < 1 - \frac{1}{q_1}, \quad \left( \frac{n}{p_1} - \frac{n}{p} \right) + \left( \frac{2}{q_1} - \frac{2}{q} \right) + 2(\alpha_1 - \alpha) = 1.
\]

Then \( w_1, w_2 \in L^p_{q_1}(0,\infty;L^p(\mathbb{R}^n_+)) \) satisfy

\[
\begin{align*}
  \|w_1\|_{L^p_{q_1}(0,\infty;L^p(\mathbb{R}^n_+))} & \lesssim \|u_0\|_{B^{2\alpha - \frac{2}{q}}_{pq0}(\mathbb{R}^n_+)}, \\
  \|w_2\|_{L^p_{q_1}(0,\infty;L^p(\mathbb{R}^n_+))} & \lesssim \|\mathcal{F}\|_{L^p_{q_1}(0,\infty;L^p(\mathbb{R}^n_+))}.
\end{align*}
\]

**Proof.** See Theorem 1.2 in [23] for \( q < \infty \) and Theorem 1.3 in [24] for \( q = \infty \). \( \square \)

**Lemma 4.3.** Let \( g \in L^q_{\alpha_2}((\Omega \times (0,\infty), \mathcal{P};L^p(\mathbb{R}^n_+)) \) for some \( p_2 \) and \( \alpha_2 \) satisfying

\[
p_2 \leq p, \quad \alpha \leq \alpha_2, \quad \left( \frac{n}{p_2} - \frac{n}{p} \right) + 2(\alpha_2 - \alpha) = 1.
\]

\[
\text{Lemma 4.2.} \quad \text{Lemma 4.3.}
\]
Then,

$$\|w_3\|_{L^q_0(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))} \lesssim \|g\|_{L^q_{\alpha_2}(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))}.$$  

**Proof.** See Appendix F. \[\square\]

**Proposition 4.4.** Let $2 \leq p < \infty$, $2 < q \leq \infty$, and $0 \leq 2\alpha < 1 - \frac{1}{p} - \frac{2}{q}$. Assume $u_0 \in \mathcal{F}_{p,q}^0(\mathbb{R}^n_+)$ with $\text{div } u_0 = 0$, $\mathcal{F} \in L^q_{\alpha_1}(\Omega \times (0,\infty), \mathcal{P}; L^{p_1}(\mathbb{R}^n_+))$ for some $p_1$, $q_1$, and $\alpha_1$ satisfying (4.5) and $g \in L^q_{\alpha_2}(\Omega \times (0,\infty), \mathcal{P}; L^{p_2}(\mathbb{R}^n_+))$ for some $p_2$ and $\alpha_2$ satisfying (4.7) with $\text{div } g = 0$. Then there is a weak solution $w \in L^q_{\alpha}(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))$ to the stochastic Stokes equations (4.1) satisfying

$$\|w\|_{L^q_0(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))} \leq C_1 \left( \|u_0\|_{\mathcal{F}_{p,q}^0(\mathbb{R}^n_+)} + \|g\|_{L^q_{\alpha_2}(\Omega \times (0,\infty), \mathcal{P}; L^{p_2}(\mathbb{R}^n_+))} + \|\mathcal{F}\|_{L^q_{\alpha_1}(\Omega \times (0,\infty), \mathcal{P}; L^{p_1}(\mathbb{R}^n_+))} \right).$$

(4.8)

Moreover, $u$ is unique in $L^q_{\alpha}(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))$ when $0 \leq \alpha < 1 - \frac{1}{q}$.

**Proof.** It is a direct consequence of Lemma 4.2 and Lemma 4.3, so we omit the detail. \[\square\]

### 5. Proof of Theorem 1.1

Define the Banach space $\mathcal{X}$ and its closed subset $C_R$ as

$$\mathcal{X} := L^q_0(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+)) \quad \text{and} \quad C_R := \{ u \in \mathcal{X} : \|u\|_{\mathcal{X}} \leq R \},$$

where $0 < R \leq 1$ is to be chosen later. Define for $s \geq 0$

$$\theta(s) = \max \{ 0, \min \{ 2 - s/R, 1 \} \}.$$  

For $v \in \mathcal{X}$, we define the map $\chi_{v(\omega)} : [0,\infty) \to [0,1]$ by

$$\chi_{v(\omega)}(t) := \theta(\|v(\omega)\|_{L^q_0(0,t; L^p(\mathbb{R}^n_+))}).$$

Note that for $\omega \in \Omega$, if $\chi_{v(\omega)}(t) > 0$, then $\|v\|_{L^q_0(0,t; L^p(\mathbb{R}^n_+))} \leq 2R$ and if $\chi_{v(\omega)}(t) = 0$, then $\|v\|_{L^q_0(0,t; L^p(\mathbb{R}^n_+))} > 2R$, and so

$$\|\chi_{v}v\|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))} \leq 2R.$$  

Fix $u_0 \in \mathcal{F}_{p,q}^0(\mathbb{R}^n_+)$ with $\text{div } u_0 = 0$. Given $v \in C_R$, we consider the problem

$$dw_v = (\Delta w_v + \nabla P_v - \text{div} (\chi_{v}^2 v \otimes v))dt + gdB_t \quad \text{and} \quad \text{div } w = 0 \quad (5.1)$$

with the boundary condition $w_v|_{\partial\Omega} = 0$ and the initial condition $w_v|_{t=0} = u_0$. From the solvability of this problem, we can define the solution map by

$$S(v) = w_v.$$  

We shall show the existence of a solution by using the Banach fixed point theorem.
5.1. S is bounded

Let $2\alpha = 1 - \frac{n}{p} - \frac{2}{q} > 0$. Choose $q_1 = \frac{q}{2}, p_1 = \frac{p}{2}$, and $\alpha_1 = 2\alpha$ so that

$$\left(\frac{n}{p_1} - \frac{n}{p}\right) + \left(\frac{2}{q_1} - \frac{2}{q}\right) + 2(\alpha_1 - \alpha) = 1.$$  

By Proposition 4.4, (5.1) has a solution $S(v)$ with an estimate

$$\|S(v)\|_{L^q_v(\Omega \times (0,\infty), \mathcal{P}; L^q_v(\mathbb{R}^n_+))} \leq C_1\left(\|u_0\|_{\overline{B}_{p_0}^{q_0 - \frac{2}{q}}(\mathbb{R}^n_+)} + \|g\|_{L^q_v(\Omega \times (0,\infty), \mathcal{P}; L^p_v(\mathbb{R}^n_+))} + \|\chi_v^2 v \otimes v\|_{\mathcal{M}^q_{2q}(\Omega \times (0,\infty), \mathcal{P}; L^q_v(\mathbb{R}^n_+))}\right).$$  

(5.2)

For every $\omega \in \Omega$, the Hölder inequality gives

$$\|\chi_v^2 v \otimes v\|_{L^q_v(0,\infty; L^q_v(\mathbb{R}^n_+))} \leq \|\chi_v v\|_{L^q_v(0,\infty; L^p_v(\mathbb{R}^n_+))} \leq 4R^2.$$  

(5.3)

If $v \in C_R$, then by Proposition 4.4 and assumption of Theorem 1.1,

$$\|S(v)\|_{\mathcal{X}} = \|w\|_{\mathcal{X}} \leq C_1(\|u_0\|_{\overline{B}_{p_0}^{q_0 - \frac{2}{q}}(\mathbb{R}^n_+)} + \|g\|_{L^q_v(\Omega \times (0,\infty), \mathcal{P}; L^p_v(\mathbb{R}^n_+))} + 4R^2) \leq C_1(\delta + 4R^2).$$

We can take $\delta = R^2$ so that

$$\|S(v)\|_{\mathcal{X}} \leq 5C_1R^2.$$  

(5.4)

If $R < \min\left(\frac{1}{5C_1}, 1\right)$, then $S : C_R \to C_R$ is bounded operator.

5.2. S is contractive

The following auxiliary result will be needed to prove that $S$ is contractive.

Lemma 5.1. For $\omega \in \Omega$, $t > 0$ and $u, v \in C_R$,

$$|\chi_u(t) - \chi_v(t)| \leq R^{-1}\|u - v\|_{L^q_v(0,\infty; L^q_v(\mathbb{R}^n_+))}.$$  

(5.5)

Proof. See Appendix G.  

We shall show that there is $C_2 > 0$ such that for $u, v \in C_R$,

$$\|S(u) - S(v)\|_{\mathcal{X}} \leq C_2R\|u - v\|_{\mathcal{X}}.$$  

(5.6)

Note that $V = S(u) - S(v)$ and $P = P_u - P_v$ satisfy

$$V_t - \Delta V + \nabla P = \text{div}\left(\chi_u^2 u \otimes u - \chi_v^2 v \otimes v\right), \quad \text{div} V = 0$$

with $V|_{t=0} = 0$ and $V|_{x_u=0} = 0$. By Proposition 4.4,

$$\|S(u) - S(v)\|_{L^q_v(\Omega \times (0,\infty), \mathcal{P}; L^q_v(\mathbb{R}^n_+))} \lesssim \|\chi_u u \otimes \chi_v u - \chi_v v \otimes \chi_v v\|_{\mathcal{M}^q_{2q}(\Omega \times (0,\infty), \mathcal{P}; L^q_v(\mathbb{R}^n_+))}.$$  

(5.7)
Since
\[
\chi_u u \otimes \chi_u u - \chi_v v \otimes \chi_v v = (\chi_u - \chi_v) \chi_u (u \otimes u) + \chi_u \chi_v u \otimes (u - v) + \chi_u \chi_v (u - v) \otimes v + (\chi_u - \chi_v) \chi_v (v \otimes v),
\]
we use (5.3) and (5.5) to get
\[
\| (\chi_u - \chi_v) \chi_u (u \otimes u) \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))} \leq R^{-1} \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))} \| \chi_u u \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))}^2 \\
\leq 4R \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))},
\]
\[
\| (\chi_u - \chi_v) \chi_v (v \otimes v) \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))} \leq R^{-1} \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))} \| \chi_v v \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))}^2 \\
\leq 4R \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))},
\]
\[
\| \chi_u \chi_v u \otimes (u - v) \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))} \leq \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))} \| \chi_u u \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))} \\
\leq 4R \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))},
\]
\[
\| \chi_u \chi_v (u - v) \otimes v \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))} \leq \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))} \| \chi_v v \|_{L^q_{2\alpha}(0,\infty; L^p_T(\mathbb{R}^n_+))} \\
\leq 4R \| u - v \|_{L^q_0(0,\infty; L^p(\mathbb{R}^n_+))}.
\]
Thus, combining these estimates with (5.7), we obtain that
\[
\| S(u) - S(v) \|_{L^q_0(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))} \leq C_2 R \| u - v \|_{L^q_0(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))}.
\]
This proves (5.6). Moreover, if \( R < \frac{1}{C_2} \), then \( S : C_R \to C_R \) is contractive.

### 5.3. Existence

By the Banach fixed point theorem, we have obtained a unique solution \( u \) to (5.1) in \( C_R \). Now, we define the stopping time,
\[
\tau(\omega) = \inf \left\{ 0 \leq T \leq \infty : \| u \|_{L^q_0(0,T; L^p(\mathbb{R}^n_+))} \geq R \right\}.
\]
Then, \( (u, \tau) \) is the unique solution in the sense of Definition 1.2.

Notice from definition of \( \tau \) that for all \( h \in (0,\infty) \),
\[
\{ \omega : \tau(\omega) \leq h \} = \{ \omega : \| u \|_{L^q_0(0,h; L^p(\mathbb{R}^n_+))} \geq R \}.
\]
Using the Chebyshev inequality, we obtain that for all \( h \in (0,\infty) \),
\[
\mathbb{P}(\tau = 0) \leq \mathbb{P}(\{ \tau \leq h \}) \leq \frac{1}{R^q} \mathbb{E}\| u \|_{L^q_0(0,h; L^p(\mathbb{R}^n_+))}^q.
\]
Since \( \| u \|_{L^q_0(0,h; L^p(\mathbb{R}^n_+))} \to 0 \) as \( h \to 0 \) almost surely, we have
\[
\mathbb{P}(\tau = 0) = 0 \quad \text{and} \quad \mathbb{P}(\tau > 0) = 1.
\]
By the definition of the stopping time \( \tau \) and (5.4),
\[
\mathbb{P}(\tau < \infty) = \mathbb{E}1_{\{\tau < \infty\}} \leq \mathbb{E}\left(1_{\{\tau < \infty\}} \frac{\|u\|^q_{L^q_2(0,\infty;L^p(R^n_+))}}{R^q}\right)
\leq \frac{\|u\|^q_{L^q_2(0,\infty;L^p(R^n_+))}}{R^q} \leq \frac{(5C_1R^2)^q}{R^q} = (5C_1R)^q.
\]
If \( R < \frac{1}{5C_1+C_2} \wedge \frac{\epsilon_1}{5C_1} \), then
\[
\mathbb{P}(\tau = \infty) = 1 - \mathbb{P}(\tau < \infty) \geq 1 - \epsilon.
\]
This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.3

The solution \( u \) constructed in Theorem 1.1 can be decomposed as a sum
\[
u = w_1 + V + w_3,
\]
where \( w_1 \) and \( w_3 \) are introduced in (4) and
\[
V(x, t) = -\int_0^t \int_{\mathbb{R}^n_+} K(x, y, t - s)\mathbb{P}(\text{div}(\chi_2 u u)(y, s))dyds
\]
and \( K \) is defined in (4.3). To conclude Theorem 1.3, it suffices to show that \( w_1, V, \) and \( w_3 \) satisfy some estimates, namely, Lemma 6.1, Lemma 6.2, and Lemma 6.3.

6.1. Estimate of \( w_1 \)

Lemma 6.1. For \( 1 < p < \infty, 1 \leq q \leq \infty, \) and \( \alpha > 0, \)
\[
\|w_1\|_{L^q(0,\infty;\hat{B}_{pq}^{-2\alpha-2/q}(R^n_+))} \lesssim \|u_0\|_{\hat{B}_{pq0}^{-2\alpha-2/q}(R^n_+)}. \]

Proof. We define \( \tilde{u}_0 \in \hat{B}_{pq}^{-2\alpha-2/q}(\mathbb{R}^n) \) by
\[
\langle \tilde{u}_0, f \rangle = \left(u_0, f|_{\mathbb{R}^n_+}\right)
\]
for \( f \in \hat{B}_{pq}^{2\alpha+2/q}(\mathbb{R}^n). \) Then it is the zero extension of \( u_0 \in \hat{B}_{pq0}^{-2\alpha-2/q}(R^n_+). \) Note that \( \|u_0\|_{\hat{B}_{pq0}^{-2\alpha-2/q}(R^n_+)} \) is comparable to \( \|\tilde{u}_0\|_{\hat{B}_{pq}^{-2\alpha-2/q}(\mathbb{R}^n)}. \) From (4.4), we can write
\[
w_1 = w_{11} + w_{12} + w_{13},
\]
where
\[
w_{11}(x, t) = \Gamma_t \ast \tilde{u}_0(t, x) - \Gamma_t^* \ast \tilde{u}_0(t, x),
\]
\[
w_{12}(x, t) = 4 \frac{\partial}{\partial x_i} \int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} (N(x-y) - N(x-y^*)) \Gamma_t^* \ast u_0(t, y)dy
\]
and
\[
w_{13}(x, t) = \frac{\partial}{\partial x_i} \int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} (N(x-y) - N(x-y^*)) \Gamma_t^* \ast (u_0(t, y) - \tilde{u}_0(t, x))dy
\]
Using the estimates (6.1), (6.2), and Lemma 3.5, we obtain that

\[ w_{13}(x, t) = -4\delta_{in} \sum_{j=1}^{n-1} R_j^I \Gamma^*_t * \tilde{u}_0(x). \]

Since \( R_j^I \) is an \( L^p(\mathbb{R}^{n-1}) \)-multiplier, the operator \( R_j^I \) is bounded in \( L^p(\mathbb{R}^n) \). By Fubini’s Theorem, it is also bounded in \( \dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n) \). Thus,

\[ \| w_{11}(t) \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)} + \| w_{13}(t) \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)} \lesssim \| \Gamma_t * \tilde{u}_0 \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)} + \| \Gamma_t^* * \tilde{u}_0 \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)}. \]  

(6.1)

From Lemma 3.2, we have

\[ \| w_{12}(t) \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)} \lesssim \| \Gamma_t^* * \tilde{u}_0 \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)}. \]  

(6.2)

Using the estimates (6.1), (6.2), and Lemma 3.5, we obtain that

\[ \| w_1 \|_{L^q(0, \infty; \dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n))} \lesssim \| \Gamma_t * \tilde{u}_0 \|_{L^q(0, \infty; \dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n))} + \| \Gamma_t^* * \tilde{u}_0 \|_{L^q(0, \infty; \dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n))} \lesssim \| u_0 \|_{\dot{B}^{-2\alpha-\frac{n}{q}}_{pq} (\mathbb{R}^n)}. \]

\[ \square \]

### 6.2. Estimate of \( V \)

**Lemma 6.2.** For \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), and \( \alpha > 0 \),

\[ \| V \|_{L^q(0, \infty; \dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n))} \lesssim \| u \|_{L^p(0, \tau(\omega); L^p(\mathbb{R}^n))}^2. \]

**Proof.** From (4.4) and Lemma 3.10, we can write

\[ V = V_1 + V_2 + V_3 \]

where

\[ V_1(x, t) = D_x(UF'(x, t) - U^* F'(x, t)) \]

\[ V_2(x, t) = 4 \frac{\partial}{\partial x_i} \int_{\mathbb{R}_+^n} \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} (N(x - y) - N(x - y^*)) D_j U^* F'_j(y, t) dy \]

\[ - 2 \frac{\partial}{\partial x_i} \int_{\mathbb{R}_+^n} \frac{\partial}{\partial y_n} (N(x - y) - N(x - y^*)) \sum_{j=1}^{n-1} R_j^I D_j U^* F'_j(y, t) dy, \]

\[ V_3(x, t) = -4\delta_{in} \sum_{j=1}^{n-1} R_j^I D_j U^* F'_j(x, t). \]
with $F_{ij} = \chi_{u_i} u_i u_j$. From Lemma 3.2, Lemma 3.8, and (3.3), we have
\[
\|V_k(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} \leq \|V_1(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} + \|V_2(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} + \|V_3(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)}
\]
\[
< \|D_\delta U(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} + \|D_\delta U(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)}
\]
\[
< \|F\|_{L_{t}^{q_1}(0, \infty; L^{p_1}(\mathbb{R}^n_+))}.
\]
Finally, we can take $q_1 = \frac{q}{2}$, $p_1 = \frac{p}{2}$, and $\alpha_1 = 2\alpha$ so that the exponents $p, q, \alpha, p_1, q_1, \alpha_1$ satisfy every condition in Lemma 3.8. Thus,
\[
\|F\|_{L_{t}^{q_1}(0, \infty; L^{p_1}(\mathbb{R}^n_+))} \lesssim \|u\|_{L_{t,x}^{p,q}(\mathbb{R}_+ \times \mathbb{R}^n)}.
\]

## 6.3. Estimate of $w_3$

**Lemma 6.3.** For $1 < p < \infty$, $1 \leq q \leq \infty$, and $\alpha > 0$,
\[
\mathbb{E} \int_0^\infty \|w_3(t)\|^q_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} dt \lesssim \mathbb{E} \int_0^\infty \|g(t)\|^q_{\dot{B}^{-2\alpha-1}_{pq,0}(\mathbb{R}^n_+)} dt.
\]

**Proof.** We can write
\[
w_{3i} = w_{31i} + w_{32i} + w_{33i},
\]
where
\[
w_{31i}(x, t) = \mathcal{V} \tilde{g}(x, t) - \mathcal{V} \tilde{g}(x, t),
\]
\[
w_{32i}(x, t) = 4 \frac{\partial}{\partial x_i} \int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} (N(x - y) - N(x - y^n)) \mathcal{V} \tilde{g}(y, s) dy
\]
\[
- 2 \frac{\partial}{\partial x_i} \int_{\mathbb{R}^n_+} \frac{\partial}{\partial y_n} (N(x - y) - N(x - y^n)) \sum_{j=1}^{n-1} R_j \mathcal{V} \tilde{g}(y, s) dy,
\]
\[
w_{33i}(x, t) = -4 \delta_{ia} \sum_{j=1}^{n-1} R_j \mathcal{V} \tilde{g}(x, t).
\]
By Lemma 3.2,
\[
\sum_{k=1}^{3} \|w_{3k}(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} \lesssim \|\mathcal{V} \tilde{g}(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} + \|\mathcal{V} \tilde{g}(t)\|_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)}.
\]
By Lemma 3.9,
\[
\mathbb{E} \int_0^\infty \|\mathcal{V} \tilde{g}(t)\|^q_{\dot{B}^{-2\alpha}_{pq}(\mathbb{R}^n_+)} dt = \mathbb{E} \int_0^\infty \sum_{-\infty < j < \infty} 2^{-2\alpha a j} \|\Delta_j \mathcal{V} \tilde{g}(t)\|^q_p dt
\]
\[
\lesssim \mathbb{E} \int_0^\infty \sum_{-\infty < j < \infty} 2^{-2\alpha a j} \|\Delta_j \mathcal{V} \tilde{g}(t)\|^q_p dt
\]
\[
= \mathbb{E} \int_0^\infty \|\tilde{g}(t)\|^q_{\dot{B}^{-2\alpha-1}_{pq,0}(\mathbb{R}^n_+)} dt.
\]
Thus, we get the result of Lemma 6.3. \qed
This completes the proof of Theorem 1.3.

7. Proof of Theorem 1.4

In this section, we assume \( g \in \mathcal{M}_{\alpha}^{p,q}(\Omega \times (0, \infty), \mathcal{P}; L^2(\mathbb{R}^n_+)) \), where
\[
\frac{n}{2p} + \frac{2}{q} < 1, \quad \frac{n}{p_2} - \frac{n}{p} < 1 - \frac{2}{q}, \quad 2\alpha_2 < 1 - \frac{2}{q}.
\]

Similar to the proof of Theorem 1.3, we decompose
\[
u = w_1 + V + w_3,
\]
where
\[
w_1(x, t) = T_K u_0(x)
\]
\[
V(x, t) = \int_0^t T_K \|\nabla u\|^2 (u \otimes u)(x, t, s) ds
\]
\[
w_3(x, t) = \int_0^t T_K g(x, t, s) dB_s,
\]
and
\[
T_K u_0(x) = \int_{\mathbb{R}^n_+} K(x, y, t) u_0(y) dy,
\]

\[
T_K g(x, t, s) = \int_{\mathbb{R}^n_+} K(x, y, t - s) g(y, s) dy.
\]

By (4.4), Lemma 3.2, Lemma 3.6, and Besov inclusion, we have
\[
\sup_{0 < t < \infty} t^{2s} \| T_K u_0 \|_{L^p} \lesssim \| u_0 \|_{\dot{B}^{-2\alpha}_p} \lesssim \| u_0 \|_{\dot{B}^{-2\alpha}_{pq}}.
\]

We recall the following estimates.

Lemma 7.1. For \( 1 \leq r < \infty \) and \( 0 < s, t \),
\[
\int_{\mathbb{R}^n_+} |\Gamma(x, t) - \Gamma(x, s)|^r dx \leq c \begin{cases} s^{-\frac{n+2}{2}} (t-s)^r, & t < 2s, \quad r \geq 1, \\ s^{-\frac{n}{2}} r^{\frac{1}{2}}, & t \geq 2s, \quad r > 1. \end{cases}
\]
\[
\int_{\mathbb{R}^n_+} |D_x \Gamma(x, t) - D_x \Gamma(x, s)|^r dx \leq c \begin{cases} s^{-\frac{n+2}{2}} (t-s)^r, & t < 2s, \quad r \geq 1, \\ s^{-\frac{n+1}{2}} r^{\frac{3}{2}}, & t \geq 2s, \quad r > 1. \end{cases}
\]

Proof. See Appendix H.

Let \( 0 < s < t < \frac{3}{2}s \). Since \( T_K \) is a semi-group,
\[
T_{K_t} (T_{K_s} u_0) = T_{K_{t+s}} u_0.
\]
By (4.4) and the singular integral theorem,
\[
\mathbb{E}\|T_{K, u_0} - T_{K, u_0}\|^p_{L^p(\mathbb{R}^n_+)} = \mathbb{E}\|T_{K_{t-\frac{1}{2}s}^{\frac{1}{2}s}}(T_{K_1^{\frac{1}{2}s}}u_0) - T_{K_1^{\frac{1}{2}s}}(T_{K_1^{\frac{1}{2}s}}u_0)\|^p_{L^p(\mathbb{R}^n_+)}
\leq c\mathbb{E}\int_{\mathbb{R}^n_+} |(\Gamma_{t-\frac{1}{2}s} - \Gamma_{\frac{1}{2}s})^\ast(T_{K_1^{\frac{1}{2}s}}u_0)(x)|^p \, dx
\]
\[+ \mathbb{E}\int_{\mathbb{R}^n_+} |(\Gamma_{t-\frac{1}{2}s}^\ast - \Gamma_{\frac{1}{2}s}^\ast)^\ast(T_{K_1^{\frac{1}{2}s}}u_0)(x)|^p \, dx.
\]
By Young's inequality and (7.3),
\[
\mathbb{E}\|T_{K, u_0} - T_{K, u_0}\|^p_{L^p(\mathbb{R}^n_+)} \leq c\mathbb{E}\left(\int_{\mathbb{R}^n_+} |\Gamma(x, t - \frac{1}{2}s) - \Gamma(x, \frac{1}{2}s)|^p \, dx\right)^\frac{p}{q}
\|
T_{K_1^{\frac{1}{2}s}}u_0\|^p_{L^p(\mathbb{R}^n_+)}
\]
\[+ c \mathbb{E}\left(\int_{\mathbb{R}^n_+} |\Gamma^\ast(x, t - \frac{1}{2}s) - \Gamma^\ast(x, \frac{1}{2}s)|^p \, dx\right)^\frac{p}{q}
\|
T_{K_1^{\frac{1}{2}s}}u_0\|^p_{L^p(\mathbb{R}^n_+)}
\]
\[\leq c s^{-p}(t - s)^p \mathbb{E}\|T_{K_1^{\frac{1}{2}s}}u_0\|^p_{L^p(\mathbb{R}^n_+)}
\]
\[\leq c s^{-p-\alpha} (t - s)^p \|u_0\|^p_{L^{p_2}(\mathbb{R}^n_+)}.\]
This implies that \(T_{K}u_0 \in C(0, \infty; L^p(\mathbb{R}^n_+)).\)
Using the Burkholder–Davis–Gundy inequality, Minkowski's integral inequality, (4.4), and Young's inequality for \(p_2 < p\) and \(\frac{n}{p_2} - \frac{n}{p} < 1 - \frac{2}{q}\), we have
\[
\mathbb{E}\int_{\mathbb{R}^n_+} \left| \int_s^t T_{K_{t-n}} gdB_\eta \right|^p \, dx
\]
\[\leq c \mathbb{E}\int_{\mathbb{R}^n_+} \left( \int_s^t |T_{K_{t-n}} g|^2 \, d\eta \right)^\frac{p}{2} \, dx
\]
\[\leq c \mathbb{E}\left( \int_s^t \left( \int_{\mathbb{R}^n_+} |T_{K_{t-n}} g|^p \, dx \right)^\frac{p}{q} \, d\eta \right)^\frac{1}{p}
\]
\[\leq c \mathbb{E}\left( \int_s^t (t - \eta)^{-\frac{n}{p_2} + \frac{n}{p} \frac{q}{q-2} \eta^{-2\alpha_2 + \frac{q}{q-2}} \, d\eta \right)^\frac{1}{q-2}
\|
\int_s^t \eta^{\alpha_2} \|g(\eta)\|_{L^q}^q \, d\eta \right)^\frac{p}{q}
\]
\[\leq c s^{-\alpha_2} (t - s)^{-\frac{n}{p_2} + \frac{n}{p} \frac{q}{q-2} \frac{1}{q} + \frac{1}{q} - \frac{2}{q}} \mathbb{E}\left( \int_s^t \eta^{\alpha_2} \|g(\eta)\|_{L^q}^q \, d\eta \right)^\frac{p}{q}
\]
and
\[
\mathbb{E}\int_{\mathbb{R}^n_+} \left| \int_0^s (T_{K_{t-n}} - T_{K_{t-n}}) gdB_\eta \right|^p \, dx
\]
\[\leq c \mathbb{E}\int_{\mathbb{R}^n_+} \left( \int_0^s |(T_{K_{t-n}} - T_{K_{t-n}}) g|^2 \, d\eta \right)^\frac{p}{2} \, dx
\]
\[\leq c \mathbb{E}\left( \int_0^s \left( \int_{\mathbb{R}^n_+} |(T_{K_{t-n}} - T_{K_{t-n}}) g|^p \, dx \right)^\frac{p}{2} \, d\eta \right)^\frac{p}{q}.
\]
\[ \begin{align*}
&\leq c \mathbb{E} \left( \int_0^s \left( \int_{\mathbb{R}_+^n} \left| (\Gamma_{t-\eta} - \Gamma_{s-\eta}) \ast g \right|^\frac{p}{q} \, dx \right)^\frac{q}{p} \, d\eta \right)^\frac{p}{q} \\
&\quad + c \mathbb{E} \left( \int_0^s \left( \int_{\mathbb{R}_+^n} \left| (\Gamma_{t-\eta}^+ - \Gamma_{s-\eta}^+) \ast g \right|^\frac{p}{q} \, dx \right)^\frac{q}{p} \, d\eta \right)^\frac{p}{q} \\
&\leq c(t-s)^p \mathbb{E} \left( \int_0^{2s-t} (s-\eta)^{-\frac{m}{p}+\frac{n}{p}+2} \|g(\eta)\|^2_{L^2_2(\mathbb{R}_+^n)} \, d\eta \right)^\frac{p}{q} \\
&\quad + \mathbb{E} \left( \int_0^s (s-\eta)^{-\frac{m}{p}+\frac{n}{p}+2} \|g(\eta)\|^2_{L^2_2(\mathbb{R}_+^n)} \, d\eta \right)^\frac{p}{q} \\
&:= I_1 + I_2. \tag{7.5}
\end{align*} \]

Since \( 2\alpha_2 \frac{q}{q-2} < 1 \), we have

\[ I_1^\frac{p}{q} \leq (t-s)^2 \left( \int_0^{2s-t} (s-\eta)^{-\frac{m}{p}+\frac{n}{p}+2} \frac{q}{q-2} \eta^{-2\alpha_2 \frac{q}{q-2}} \, d\eta \right)^\frac{1}{q} \leq (t-s)^2 \left( \int_0^{2s-t} (s-\eta)^{-\frac{m}{p}+\frac{n}{p}+2} \eta^{-2\alpha_2 \frac{q}{q-2}} \, d\eta \right)^\frac{1}{q} \leq (t-s)^2 \left( \int_0^\infty \eta^{-2\alpha_2 \frac{q}{q-2}} \|g\|^q_{L^2_2(\mathbb{R}_+^n)} \, d\eta \right)^\frac{1}{q}. \]

Since \( 2s-t > \frac{1}{2} s \) for \( s < t < \frac{3}{2} s \), we have

\[ \int_0^{2s-t} (s-\eta)^{-\frac{m}{p}+\frac{n}{p}+2} \frac{q}{q-2} \eta^{-2\alpha_2 \frac{q}{q-2}} \, d\eta \]

\[ \leq c s^{-\frac{m}{p}+\frac{n}{p}+1-\frac{2}{q}-2\alpha_2} + s^{-2\alpha_2} (t-s)^{-\frac{m}{p}+\frac{n}{p}+1-\frac{2}{q}} \left( \mathbb{E} \int_0^\infty \eta^{2\alpha_2 q} \|g(\eta)\|^q_{L^2_2(\mathbb{R}_+^n)} \, d\eta \right)^\frac{1}{q}. \tag{7.6} \]

Hence,

\[ I_1^\frac{p}{q} \leq c(t-s)^2 \left( s^{-\frac{m}{p}+\frac{n}{p}+1-\frac{2}{q}-2\alpha_2} + s^{-2\alpha_2} (t-s)^{-\frac{m}{p}+\frac{n}{p}+1-\frac{2}{q}} \right) \left( \mathbb{E} \int_0^\infty \eta^{2\alpha_2 q} \|g(\eta)\|^q_{L^2_2(\mathbb{R}_+^n)} \, d\eta \right)^\frac{1}{q}. \]

Since \( -\frac{m}{p} - \frac{n}{p} < 1 - \frac{2}{q} \), we have

\[ I_2^\frac{p}{q} \leq \left( \int_0^s (s-\eta)^{-\frac{m}{p}+\frac{n}{p}+1-\frac{2}{q}-2\alpha_2} \frac{q}{q-2} \eta^{-2\alpha_2 \frac{q}{q-2}} \, d\eta \right)^\frac{1}{q} \leq c s^{-2\alpha_2} (t-s)^{-\frac{m}{p}+\frac{n}{p}+1-\frac{2}{q}} \left( \mathbb{E} \int_0^\infty \eta^{2\alpha_2 q} \|g(\eta)\|^q_{L^2_2(\mathbb{R}_+^n)} \, d\eta \right)^\frac{1}{q}. \tag{7.7} \]
From (7.4), (7.6), and (7.7), we have
\[
\mathbb{E}\left\| \int_0^t T_{K_{t-s}} g dB_t - \int_0^s T_{K_{t-s}} g dB_t \right\|_{L^p_x}\n\]
\[
\leq c\left( \mathbb{E} \int_0^t \int_0^t |T_{K_{t-s}} g dB_t - \int_0^s T_{K_{t-s}} g dB_t|^p \right)^{\frac{1}{p}}.
\]

Hence, \( \int_0^t T_{K_{t-s}} g dB_t \in C(0, \infty; \mathbb{L}^p_x) \).

Using Definition 2.12 and (H.2), we have for \( \frac{1}{2} + \frac{n}{2p} < 1 - \frac{2}{q} \),
\[
\mathbb{E}\| \int_0^t T_{K_{t-s}} \div \chi^2(u \otimes u) \|_{L^p_x}\n\]
\[
\leq \mathbb{E} \int_0^t \| T_{K_{t-s}} \div \chi^2(u \otimes u) \|_{L^p_x} d\eta
\]
\[
\leq \mathbb{E} \int_0^t (t - \eta)^{-\frac{1}{2} - \frac{n}{p} + \frac{n}{2p}} \| u(\eta) \otimes u(\eta) \|_{L^2_x}^2 d\eta
\]
\[
\leq c \mathbb{E} \int_0^t (t - \eta)^{-\frac{1}{2} - \frac{n}{p} + \frac{n}{2p}} \| u(\eta) \|_{L^p_x}^2 d\eta
\]
\[
\leq c \left( \int_0^t (t - \eta)^{-\frac{1}{2} - \frac{n}{p} + \frac{n}{2p}} \eta^\alpha \frac{q}{q - 2} d\eta \right)^{\frac{1}{2}} \left( \int_0^\infty \eta^\alpha \| u(\eta) \|_{L^p_x}^q d\eta \right)^{\frac{2}{q}}
\]
\[
\leq c(t - s)^{\frac{1}{2} - \frac{n}{p} - \frac{2}{3} + \frac{n}{p}} s^{-2\alpha} \left( \int_0^\infty \eta^\alpha \| u(\eta) \|_{L^p_x}^q d\eta \right)^{\frac{2}{q}},
\]
and
\[
\mathbb{E}\| \int_0^s (T_{K_{t-s}} - T_{K_{t-s-n}}) \mathbb{P} \div \chi^2(u \otimes u) \|_{L^p_x}\n\]
\[
\leq c \mathbb{E} \int_0^s \| (T_{K_{t-s}} - T_{K_{t-s-n}}) \mathbb{P} \div \chi^2(u \otimes u) \|_{L^p_x} d\eta
\]
\[
\leq c \mathbb{E} \int_0^s \left( \int_{\mathbb{R}_+^n} \left( |(\Gamma_{t-s} - \Gamma_{t-s-n}) \ast \mathbb{P} \div \chi^2(u \otimes u) |^p dx \right)^{\frac{1}{p}} \right)^{\frac{1}{2}} d\eta
\]
\[
\leq c(t - s) \int_0^{2s-t} (s - \eta)^{-\frac{3}{2} - \frac{n}{p} + \frac{n}{2p}} \mathbb{E} \| u(\eta) \|_{L^p_x}^2 d\eta
\]
\[
\leq c(t - s)^{\frac{1}{2} - \frac{n}{p} + \frac{n}{2p}} \left( \int_0^\infty \eta^\alpha \| u(\eta) \|_{L^p_x}^q d\eta \right)^{\frac{2}{q}}.
\]

Hence, \( T_K \mathbb{P} \div \chi(x, t, s) ds \in C(0, \infty; \mathbb{L}^p_x) \). This completes the proof of Theorem 1.4.
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A. Proof of Lemma 3.2

Let

\[ T_f(x) = \nabla^2_x \int_{\mathbb{R}_+^n} N(x - y)f(y)dy. \]

We first recall the following lemma.

**Lemma A.1.** If \(1 < p < \infty\) and \(0 \leq s < \infty\), then

\[ \|T_f\|_{\dot{H}_p^s(\mathbb{R}_+^n)} \lesssim \|f\|_{\dot{H}_p^s(\mathbb{R}_+^n)}, \]

where \(\dot{H}_p^s(\mathbb{R}_+^n)\) is the space of all restriction of \(f \in \dot{H}_p^s(\mathbb{R}^n)\) on \(\mathbb{R}_+^n\), which is the standard homogenous fractional Sobolev space defined by the Fourier inversion. (In particular, \(\dot{H}_p^0(\mathbb{R}_+^n) = L^p(\mathbb{R}_+^n)\).)

**Proof.** See Lemma 3.3 in [25]. \(\square\)

To prove Lemma 3.2, we apply the real interpolation (see, e.g., Section 2 in [24]) to Lemma A.1 to obtain that

\[ \|T_f\|_{\dot{B}_{p,q}^{-s}(\mathbb{R}_+^n)} \lesssim \|f\|_{\dot{B}_{p,q}^{-s}(\mathbb{R}_+^n)}. \]

Since the operator \(T_f\) is symmetric, we use the duality argument to get

\[ \|T_f\|_{\dot{B}_{p,q}^{-s}(\mathbb{R}_+^n)} \lesssim \|f\|_{\dot{B}_{p,q}^{-s}(\mathbb{R}_+^n)}. \]

We note that if \(-1/p < s < 1/p'\), then

\[ \dot{B}_{p,q}^{-s}(\mathbb{R}_+^n) = \dot{B}_{p,q}^{-s}(\mathbb{R}_+^n). \]

This proves the first estimate. Since the proof of the second estimate is almost the same, we omit the detail.

B. Proof of Lemma 3.5

Fix \(1 \leq p \leq \infty\). If \(1 \leq q < \infty\), then by Lemma 3.4

\[ \|\Gamma_t * f\|_{L^q(0,\infty;L^p(\mathbb{R}^n))} = \int_0^\infty \sum_{j \in \mathbb{Z}} 2^{\beta j} \|\Delta_j(\Gamma_t * f)\|_{L^p}^q dt \]

\[ \|\Gamma_t * f\|_{L^q(0,\infty;L^p(\mathbb{R}^n))} = \int_0^\infty \sum_{j \in \mathbb{Z}} 2^{\beta j} \|\Delta_j(\Gamma_t * f)\|_{L^p}^q dt \]
\[ \leq \int_{0}^{\infty} \sum_{j \in \mathbb{Z}} 2^{q \beta j} \exp(-ct^{2^{j}q}) \| \Delta f \|_p^q dt \]
\[ = \sum_{j \in \mathbb{Z}} 2^{q \beta j} \| \Delta f \|_p^q \int_{0}^{\infty} \exp(-ct^{2^{j}q}) dt \]
\[ = \frac{1}{cq} \sum_{j \in \mathbb{Z}} 2^{(q \beta - 2)j} \| \Delta f \|_p^q \]
\[ = \frac{1}{cq} \left\| f \right\|_{\dot{B}^{\beta}_{pq} (\mathbb{R}^n)}. \]

Note that the operator norm does not depend on \( q \) since \( \lim_{q \to \infty} (cq)^{1/q} = 1 \). If \( q = \infty \), then by Lemma 3.4
\[ \| \Gamma_t * f \|_{L^\infty(0, \infty; \dot{B}^{\beta}_{pq} (\mathbb{R}^n))} = \sup_{t > 0} \sup_{j \in \mathbb{Z}} 2^{\beta j} \| \Delta_j (\Gamma_t * f) \|_p \]
\[ \leq \sup_{j \in \mathbb{Z}} 2^{\beta j} \| \Delta_j f \|_p \]
\[ = \sup_{j \in \mathbb{Z}} 2^{\beta j} \| \Delta_j f \|_p \]
\[ = \| f \|_{\dot{B}^{\beta}_{pq} (\mathbb{R}^n)}. \]

By the same method, we get the estimates for \( \Gamma_t^* * f \).

**C. Proof of Lemma 3.6**

We first consider the case \( q = \infty \). Using the identity \( f = \sum_{j \in \mathbb{Z}} \Delta_j f \) and applying Lemma 3.4, we get
\[ t^{\alpha} \| \Gamma_t * f \|_p \leq t^{\alpha} \sum_{j \in \mathbb{Z}} \| \Delta_j (\Gamma_t * f) \|_p \leq t^{\alpha} \sum_{j \in \mathbb{Z}} \exp(-ct^{2^{j}}) \| \Delta_j f \|_p. \]

Since \( 2^{-2\alpha j} \| \Delta_j f \|_p \leq \| f \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)} \) for all \( j \), we have
\[ t^{\alpha} \sum_{j \in \mathbb{Z}} \exp(-ct^{2^{j}}) \| \Delta_j f \|_p \lesssim \| f \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)} \sum_{j \in \mathbb{Z}} (t^{2^{j}})^{\alpha} \exp(-ct^{2^{j}}) \lesssim \| f \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)}. \]

Thus, we have for all \( t \),
\[ t^{\alpha} \| \Gamma_t * f \|_p \lesssim \| f \|_{\dot{B}^{-2\alpha}_{pq} (\mathbb{R}^n)}. \]

This proves the lemma for the case \( q = \infty \).

We now consider the case \( q = 1 \). By the same method,
\[ \int_{0}^{\infty} t^{\alpha} \| \Gamma_t * f \|_p dt \leq \int_{0}^{\infty} t^{\alpha} \sum_{j \in \mathbb{Z}} \| \Delta_j (\Gamma_t * f) \|_p dt \]
\[ \lesssim \sum_{j \in \mathbb{Z}} \| \Delta_j f \|_p \int_{0}^{\infty} t^{\alpha} \exp(-ct^{2^{j}}) dt \]
since the integral \( \int_0^\infty (t2^{2i})^\alpha \exp(-ct2^{i})2^{2i} dt \) is bounded uniformly on \( j \) by a change of variable. This proves the lemma for the case \( q = 1 \). We get the result by using complex interpolation (see 5.6.3. Theorem in [26])

\[
[L_{\alpha}^\infty (0, \infty; L^p (\mathbb{R}^n)), L_{\alpha}^1 (0, \infty; L^p (\mathbb{R}^n))]_{\theta} = L_{\alpha}^q (0, \infty; L^p (\mathbb{R}^n))
\]

and (see 6.4.5. Theorem in [26])

\[
[B_{p}^{2\alpha} (\mathbb{R}^n), B_{p1}^{2\alpha-2} (\mathbb{R}^n)]_{\theta} = \dot{B}_{pq}^{2\alpha-2/q} (\mathbb{R}^n)
\]

with \( 1/q = 1 - \theta \). By the same method, we get the estimates for \( \Gamma_{t}^* \ast f \).

### D. proof of Lemma 3.8

We prove the technical estimate (3.1). Since the estimate of \( D_x \mathcal{U}^* F \) can be derived in the same way as that of \( D_x \mathcal{U} F \), we focus on estimating the term \( D_x \mathcal{U} F \), which is done by duality argument. If \( \psi \in L^{q}_t (0, \infty; \dot{B}_{pq}^{2\alpha} (\mathbb{R}^n)) \), then by Fubini’s theorem

\[
\int_0^\infty \int_{\mathbb{R}^n} D_x \mathcal{U} F(x,t) \psi(x,t) dx dt = \int_0^\infty \int_{\mathbb{R}^n} D_x \mathcal{U} F(x,t) \tilde{\psi}(x,t) dx dt = \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^t \int_{\mathbb{R}^n} F(y,s) D_y \Gamma(x-y,t-s) dy ds \right) \tilde{\psi}(x,t) dx dt
\]

Integrating by parts and using Hölder’s inequality, we have

\[
\int_{\mathbb{R}^n} F(y,s) \int_{\mathbb{R}^n} D_y \Gamma(x-y,t-s) \tilde{\psi}(x,t) dx dy = \left| \int_{\mathbb{R}^n} F(y,s) \int_{\mathbb{R}^n} \Gamma(y-x,t-s) D_x \tilde{\psi}(x,t) dx dy \right|
\]

\[
= \left| \int_{\mathbb{R}^n} F(y,s) \int_{\mathbb{R}^n} \Gamma(t-s) \ast D_y \tilde{\psi}(y,t) dy \right|
\]

\[
\leq \| F(s) \|_{L^p (\mathbb{R}^n)} \| \Gamma(t-s) \ast D_y \tilde{\psi}(t) \|_{L^p_1 (\mathbb{R}^n)}
\]

Using the semigroup property of the heat kernel and Young’s convolution inequality, we get

\[
\| \Gamma(t-s) \ast D_x \tilde{\psi}(t) \|_{L^p_1 (\mathbb{R}^n)} \leq \| \Gamma(t-s) \ast D_x \tilde{\psi}(t) \|_{p'_1}
\]

\[
= \| (t-s)/2 \ast (t-s)/2 \ast D_x \tilde{\psi}(t) \|_{p'_1}
\]

\[
\leq \| \Gamma(t-s)/2 \|_{L^r (\mathbb{R}^n)} \| \Gamma(t-s)/2 \ast D_x \tilde{\psi}(t) \|_{p'_1}
\]
\[ \left\langle (t-s)^{-\frac{\alpha}{2}} \| \Gamma_{(t-s)/2} \ast D_x \tilde{\psi}(t) \|_{p'} \right\rangle = (t-s)^{\frac{\alpha}{2}} \frac{1}{p'} \| \Gamma_{(t-s)/2} \ast D_x \tilde{\psi}(t) \|_{p''}, \]

where \( 1 + 1/p'_1 = 1/r + 1/p' \). Since Lemma 3.6 yields

\[ \| \Gamma_{(t-s)/2} \ast D_x \tilde{\psi}(t) \|_{p'} \lesssim (t-s)^{-\frac{1}{2} + \alpha} \| D_x \tilde{\psi}(t) \|_{B^{1+2\alpha}_{p',q}(\mathbb{R}^n)} \lesssim (t-s)^{-\frac{1}{2} + \alpha} \| \tilde{\psi}(t) \|_{B^{2\alpha}_{p',q}(\mathbb{R}^n)}, \]

we obtain that

\[ \| \Gamma_{t-s} \ast D_x \psi(t) \|_{L^{p'(\mathbb{R}^n)}} \lesssim (t-s)^{\frac{1}{2} - \frac{1}{p'} - \frac{1}{2} + \alpha} \| \psi(t) \|_{B^{2\alpha}_{p',q}(\mathbb{R}^n)}. \]

Combining the estimate above and then using Hölder’s equality, we get

\[
\begin{align*}
&\left| \int_0^\infty \int_{\mathbb{R}^n} D_x \mathcal{U}F(x,t) \psi(x,t) dx dt \right| \\
&\leq \int_0^\infty \int_0^t |F(s)| \| \Gamma_{(t-s)/2} \ast D_x \tilde{\psi}(t) \|_{L^{p'}(\mathbb{R}^n)} (t-s)^{-\frac{1}{2} + \alpha} \| \tilde{\psi}(t) \|_{B^{2\alpha}_{p',q}(\mathbb{R}^n)} ds dt \\
&= \int_0^\infty I_s f(t) \| \psi(t) \|_{B^{2\alpha}_{p',q}(\mathbb{R}^n)} dt \\
&\leq \| I_s f \|_{L^q(0,\infty)} \| \psi \|_{L^{q'}(0,\infty;B^{2\alpha}_{p',q}(\mathbb{R}^n))},
\end{align*}
\]

where

\[ \lambda := -\frac{n}{2} \left( \frac{1}{p'_1} - \frac{1}{p'} \right) + \frac{1}{2} - \alpha \quad \text{and} \quad f(s) := \| F(s) \|_{L^{p'}(\mathbb{R}^n)}. \]

If \( 0 < \lambda < 1, 1 < q_1 \leq q < \infty, 0 \leq \alpha_1 < 1/q_1' \), and \( 1 + 1/q = 1/q_1 + \lambda + \alpha_1 \), then by Lemma 3.11

\[ \| I_s f \|_{L^{q_1}(0,\infty)} \lesssim \| f \|_{L^{q_1'}(0,\infty)} = \| F \|_{L^{q_1}_{q_1'}(0,\infty;L^1(\mathbb{R}^n))}. \]

Thus, we have

\[ \left| \int_0^\infty \int_{\mathbb{R}^n} D_x \mathcal{U}F(x,t) \psi(x,t) dx dt \right| \lesssim \| F \|_{L^{q_1}_{q_1'}(0,\infty;L^1(\mathbb{R}^n))} \| \psi \|_{L^{q'}(0,\infty;B^{2\alpha}_{p',q}(\mathbb{R}^n))} \]

and therefore by duality \( \| D_x \mathcal{U}F \|_{L^q(0,\infty;B^{-2\alpha}_{p,q}(\mathbb{R}^n))} \lesssim \| F \|_{L^{q_1}_{q_1'}(0,\infty;L^1(\mathbb{R}^n))} \). This completes the proof of the estimate (3.1).

**E. proof of Lemma 3.9**

We prove the estimate (3.2). Notice that \( p > n \geq 2 \). The Burkholder-Davis-Gundy inequality (Section 2.7 in [17]) yields

\[
\mathbb{E} \int_0^\infty \| \Delta_j \mathcal{V} \tilde{g}(t) \|_{p'}^q dt \\
= \mathbb{E} \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \int_0^t |\Delta_j \Gamma_{t-s} \ast \tilde{g}(x,s) dB_s| dx \right)^{p} ds \right)^{q/p} dt \\
\lesssim \mathbb{E} \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \int_0^t |\Delta_j \Gamma_{t-s} \ast \tilde{g}(x,s) |^2 ds \right)^{p/2} dx \right)^{q/p} dt.
\]
By Minkowski’s integral inequality and Lemma 3.4, we obtain

\[
\mathbb{E} \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \int_0^t |\Delta_j (\Gamma_{t-s} \ast \bar{g})(x,s)|^2 ds \right)^{p/2} dx \right)^{q/p} dt \\
\leq \mathbb{E} \int_0^\infty \left( \int_0^t \left( \int_{\mathbb{R}^n} \left| \Delta_j (\Gamma_{t-s} \ast \bar{g})(x,s) \right|^p dx \right)^{2/p} ds \right)^{q/2} dt \\
= \mathbb{E} \int_0^\infty \left( \int_0^t \| \Delta_j (\Gamma_{t-s} \ast \bar{g})(s) \|^2_p ds \right)^{q/2} dt \\
\lesssim \mathbb{E} \int_0^\infty \left( \int_0^t \exp(-c(t-s)2^{2j}) \| \Delta_j \bar{g}(s) \|^2_p ds \right)^{q/2} dt 
\]

for some positive constant \( c \). By Young’s convolution inequality, we have

\[
\mathbb{E} \int_0^\infty \left( \int_0^t \exp(-c(t-s)2^{2j}) \| \Delta_j \bar{g}(s) \|^2_p ds \right)^{q/2} dt \\
\leq \mathbb{E} \left( \int_0^\infty \exp(-ct2^{2j}) dt \right)^{q/2} \int_0^\infty \| \Delta_j \bar{g}(t) \|^q_p dt \\
= (c2^{2j})^{-q/2} \mathbb{E} \int_0^\infty \| \Delta_j \bar{g}(t) \|^q_p dt.
\]

Thus, we get (3.2).

**F. Proof of Lemma 4.3**

By the Burkholder–Davis–Gundy inequality and Minkowski’s integral inequality, we have

\[
\mathbb{E} \int_0^\infty t^{\alpha q} \| w_3 \|^q_{L^p(\mathbb{R}^n_+)} dt = \mathbb{E} \int_0^\infty t^{\alpha q} \left( \int_{\mathbb{R}^n_+} \left| \int_0^t T_K \Pi g(x,t,s) dB_s \right|^p dx \right)^{q/p} dt \\
\lesssim \mathbb{E} \int_0^\infty t^{\alpha q} \left( \int_{\mathbb{R}^n_+} \left| \int_0^t |T_K \Pi g(x,t,s)|^2 ds \right|^p dx \right)^{q/2} dt \\
\leq \mathbb{E} \int_0^\infty t^{\alpha q} \left( \int_{\mathbb{R}^n_+} \left| \int_0^t \left| T_K \Pi g(x,t,s) \right|^p dx \right| ds \right)^{q/2} dt.
\]

(See Section 2.7 in [17]) for the Burkholder–Davis–Gundy inequality.)

We now recall the following estimate

\[
\int_{\mathbb{R}^n_+} |T_K \Pi g(x,t,s)|^p dx \lesssim \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} \Gamma_{t-s}(x-y) \Pi g(y,s) dy \right|^p dx \\
+ \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} \Gamma_{t-s}(x-y^*) \Pi g(y,s) dy \right|^p dx \tag{F.1}
\]
in Section 3.1 in [22]. Then we obtain that, by Young’s inequality and Lemma 3.11,

\[
\mathbb{E} \int_0^\infty t^{\alpha q} \| w_3 \|_{L^p(\mathbb{R}^n_+)}^q \, dt \leq \mathbb{E} \int_0^\infty t^{\alpha q} \left( \int_0^t \left( \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} \Gamma_{t-s}(x-y) \mathbb{P} g(y, s) \, dy \right|^p \, dx \right)^{\frac{2}{p}} \, ds \right)^{\frac{q}{2}} \, dt \\
+ \mathbb{E} \int_0^\infty t^{\alpha q} \left( \int_0^t \left( \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} \Gamma_{t-s}(x-y) \mathbb{P} g(y, s) \, dy \right|^p \, dx \right)^{\frac{2}{p}} \, ds \right)^{\frac{q}{2}} \, dt \\
\leq \mathbb{E} \int_0^\infty t^{\alpha q} \left( \int_0^t \left( t-s \right)^{-\frac{n}{p} + \frac{2}{p} + \frac{n}{q}} \left\| \mathbb{P} g(s) \right\|_{L^p(\mathbb{R}^n_+)}^2 \, ds \right)^{\frac{q}{2}} \, dt \\
\leq \mathbb{E} \int_0^\infty t^{\alpha q} \| g \|_{L^p(\mathbb{R}^n_+)}^q \, dt,
\]

where \( \left( \frac{n}{p^2} - \frac{n}{p} \right) + 2(\alpha_2 - \alpha) = 1 \). Combining this with Lemma 4.2, we obtain that

\[
\| w_3 \|_{L^q(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))} \lesssim \| g \|_{L^q(\Omega \times (0,\infty), \mathcal{P}; L^p(\mathbb{R}^n_+))}.
\]

We complete the proof of Lemma 4.3.

**G. Proof of Lemma 5.1**

Note first that \( \chi_{v(\omega)}(t) = 1 \) if \( \| v(\omega) \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq R \) and that \( \chi_{v(\omega)}(t) = 0 \) if \( \| v(\omega) \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \geq 2R \). To prove the claim, we may consider the following six cases.

1. If \( \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq R \), then \( |x_u - x_v| = |1 - 1| = 0 \).
2. If \( \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq R \leq \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq 2R \), then

\[
|x_u - x_v| = |1 - (2 - R^{-1} \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))})| \\
= R^{-1} \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} - R| \\
\leq R^{-1} \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} - \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}| \\
\leq R^{-1} \| u - v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}.
\]

3. If \( \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq R \leq 2R \leq \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \), then

\[
|x_u - x_v| = |1 - 0| \leq \frac{\| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} - \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}}{2R - R} \\
\leq R^{-1} \| u - v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}.
\]

4. If \( R \leq \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} \leq 2R \), then

\[
|x_u - x_v| = |(2 - R^{-1} \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}) - (2 - R^{-1} \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))})| \\
= R^{-1} \| v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))} - \| u \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}| \\
\leq R^{-1} \| u - v \|_{L^q(0,t; L^p(\mathbb{R}^n_+))}.
\]
(5) If $R \leq \|u\|_{L^q_0(0, t; L^p(R^n_+))} \leq 2R \leq \|v\|_{L^q_0(0, t; L^p(R^n_+))}$, then
\[
|\chi_u - \chi_v| = |(2 - R^{-1})\|u\|_{L^q_0(0, t; L^p(R^n_+))} - 0|
\leq R^{-1}|2R - \|u\|_{L^q_0(0, t; L^p(R^n_+))}|
\leq R^{-1}\|v\|_{L^q_0(0, t; L^p(R^n_+))} - \|u\|_{L^q_0(0, t; L^p(R^n_+))}|
\leq R^{-1}\|u - v\|_{L^q_0(0, t; L^p(R^n_+))}.
\]

(6) If $2R \leq \|u\|_{L^q_0(0, t; L^p(R^n_+))} \leq \|v\|_{L^q_0(0, t; L^p(R^n_+))}$, then $|\chi_u - \chi_v| = |0 - 0| = 0$.

Thus, (5.5) always holds.

H. Proof of Lemma 7.1

Using Minkowski’s integral inequality, for $1 \leq r < \infty$, we have
\[
\int_{R^n_+} |\Gamma(x, t) - \Gamma(x, s)|^r \, dx \leq \int_{R^n_+} \left( \int_0^1 |D_t \Gamma(x, (1 - \theta)s + \theta t)| \, d\theta \right)^r \, dx \, dt
\leq c \left( \int_0^1 \left( (1 - \theta)s + \theta t \right)^{-\frac{n+2}{2}r} e^{-\frac{\theta^2 s^2 + \theta^2 t^2}{1-\theta^2}} \, d\theta \right)^r \, (t-s)^r
= c \left( \int_0^1 (1 - \theta)s + \theta t \right)^{-\frac{n+2}{2}r + \frac{n}{2}} \, (t-s)^r
= cs^{-\frac{n+2}{2}r + \frac{n}{2}} \left( \int_0^{\frac{t-s}{r}} (1 + \theta)^{-\frac{n+2}{2}} \, d\theta \right)^r
\leq c \left\{ \begin{array}{ll}
s^{-\frac{n+2}{2}r + \frac{n}{2}} (t-s)^r, & t < 2s, \quad r \geq 1, \\
s^{-\frac{n}{2}r + \frac{n}{2}}, & t \geq 2s, \quad r > 1.
\end{array} \right. \quad \text{(H.1)}
\]

Similarly, we have
\[
\int_{R^n_+} |D_x \Gamma(x, t) - D_x \Gamma(x, s)|^r \, dx \leq c \left\{ \begin{array}{ll}
s^{-\frac{n+3}{2}r + \frac{n}{2}} (t-s)^r, & t < 2s, \quad r \geq 1, \\
s^{-\frac{n+3}{2}r + \frac{n}{2}}, & t \geq 2s, \quad r > 1.
\end{array} \right. \quad \text{(H.2)}
\]

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