Quantum Potential and Cosmological Singularities

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ABSTRACT

We apply the de Broglie-Bohm interpretation to the Wheeler-DeWitt equation for the quantum FRW cosmological model with a minimal massless scalar field. We find that the quantum FRW cosmological model has quantum potential dominated solutions that avoid the initial and the final cosmological singularities. It is suggested that the quantum potential and the back-reaction of geometry and matter fields may change the property of the cosmological singularities of the Universe.

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1 Introduction

One of the most prominent classical-mechanical problems that quantum mechanics can solve is the classical singularity. For example, an electron of an atom suffers from the instability that due to the infinitely negative Coulomb potential and the electron in a zero-momentum state should fall into the infinite potential. In quantum mechanics, however, the s-wavefunction (zero orbital angular momentum state) of the electron has zero probability at the singularity of potential.

Similarly, according to the classical singularity theorem by Hawking and Penrose, the Universe also suffers from the initial and the final singularities. One of the many prospects of canonical quantum gravity which is described by the Wheeler-DeWitt equation from the Hamiltonian formulation of the general relativity whose wavefunctions carry all the information of the gravity and matter system, has been to resolve the classical singularities, as quantum mechanics does. In spite of diverse approaches and attempts, the problem of singularities for gravity coupled to a general matter field has not been completely solved yet.

In this Letter we approach the problem of the initial and the final singularities of the Universe by applying the de Broglie-Bohm interpretation to the wavefunctions of the Wheeler-DeWitt equation for the quantum FRW cosmological model with a minimal massless scalar field. The de Broglie-Bohm interpretation is an alternative approach to quantum mechanics, in which the Schrödinger equation is replaced by two equations, the Hamilton-Jacobi equation with the quantum potential and the continuity equation for probability. The quantum trajectory described by these two equations are physically the same as the corresponding wavefunction of Schrödinger equation. By introducing a suitable cosmological time-parameter and solving the imaginary part of the Wheeler-DeWitt equation (the equation for the continuity of probability in quantum mechanics), we are able to find the semiclassical Einstein equation both with the quantum potential as a part of quantum fluctuation of geometry and with an operator ordering parameter. We find the solutions for the semiclassical Einstein. Depending on the relative magnitude of the operator ordering parameter and the energy density of the scalar field, there can be or cannot be particular solutions that avoid the cosmological singularities, either directly via the exact wavefunctions or via the de Broglie-Bohm interpretation.

The exact solvability of the de Broglie-Bohm trajectory for the quantum FRW model is rooted on the direct separability of the Wheeler-DeWitt equation into the gravitational field equation by quantum states of the massless scalar field. For a general scalar-field model, one may need the semiclassical quantum gravity. Several different ways have been introduced to derive the semiclassical quantum gravity. Firstly, one may use the Born-Oppenheimer idea of separating the different mass scales to get the semiclassical Einstein equation; secondly, one may apply the de Broglie-Bohm idea to the Wheeler-DeWitt equation; thirdly, one may use both the Born-Oppenheimer and the de Broglie-Bohm ideas.

\[\text{More rigorously, one should take into account the energy loss due to the radiation of the electron which renders it to fall finally into the nucleus}\]
2 Classical and Quantum FRW Model

We shall consider the FRW cosmological model minimally coupled to a massless scalar field. The action for the gravity coupled to the minimal massless scalar field is given by

\[ I = -\frac{m_P}{16\pi} \int d^4x \sqrt{-g}R + \frac{1}{2} \int d^4\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \]  

(1)

In the FRW geometry with the metric

\[ ds^2 = -N^2(t)dt^2 + a^2d\Omega^2, \]

(2)

the action takes the form

\[ I = \int dt \left[ -\frac{3m_P}{8\pi}a^3\left( \frac{\dot{a}^2}{Na^2} - \frac{k}{a^2} \right) + \frac{a^3\dot{\phi}^2}{2N} \right], \]

(3)

where \( k = 1, 0, -1 \) for a spatially closed, flat, open Universe, respectively.

From the action (3), one obtains the classical Einstein equation

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{4\pi}{3m_P} \ddot{\phi}, \]

(4)

and the classical equation for the scalar field

\[ \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = 0. \]

(5)

As a simple case, we shall consider the massless scalar field in the spatially flat Universe. Eq. (5) has the solution

\[ \dot{\phi} = \frac{p}{a^3}, \]

(6)

where \( p \) is a constant of integration (classical momentum), and the corresponding solutions of Eq. (4) are

\[ a(t) = a(t_0)\left[ 1 \pm \sqrt{\frac{4\pi p^2}{m_P}} \frac{3}{a^3(t_0)}(t - t_0) \right]^{1/3}. \]

(7)

The positive and the negative signs correspond to the expanding and the recollapsing Universes, respectively. Classically the Universe recollapses inevitably to the final singularity (big crunch) or has the initial singularity run backward in time.

The Wheeler-DeWitt equation for the quantum FRW cosmological model takes the form

\[ \left[ -\frac{\hbar^2}{2m_P} \frac{1}{a^\nu}\frac{\partial}{\partial a}\left( a^\nu\frac{\partial}{\partial a} \right) + \frac{9m_P}{32\pi^2}ka^2 + \frac{3\hbar^2}{8\pi a^2} \frac{\partial^2}{\partial \phi^2} \right] \Psi(a, \phi) = 0. \]

(8)

In the above equation, \( \nu \), which denotes an operator ordering parameter, may have a special meaning that one can choose a particular value in order to make the super-Hamiltonian operator hermitian in the range \([0, \infty)\) in which the wavefunctions are defined. In Refs. [3, 4, 5] an intensive investigation is done for the physical meaning of the parameter \( \nu \) in defining the Hilbert space of the wavefunctions.
3 de Broglie-Bohm Interpretation

The massless scalar-field model is a particularly simple case whose solutions can be found. The scalar field decouples from gravity, and the wavefunction has the form

$$\Psi(a, \phi) = \Psi(a) \Phi(\phi, a).$$ (9)

The scalar-field Hamiltonian has the eigenfunction

$$\Phi = \frac{1}{(2\pi)^{3/2}} e^{\pm i \phi}.$$ (10)

Then the Wheeler-DeWitt equation reduces to the gravitational field equation

$$\left[ -\frac{\hbar^2}{2m_P} \frac{\partial^2}{\partial a^2} \left( a^a \frac{\partial}{\partial a} \right) + \frac{9m_P}{32\pi^2} k a^2 - \frac{3h^2 p^2}{8\pi a^2} \right] \Psi(a) = 0.$$ (11)

One can either solve the Wheeler-DeWitt equation directly or follow the de Broglie-Bohm interpretation. We shall use first the semiclassical gravity and then the exact wavefunctions to test the validity of it. Following de Broglie and Bohm [7], we find the wavefunction of the form

$$\Psi(a) = F(a) \exp \left( \pm \frac{i}{\hbar} S(a) \right)$$ (12)

where the positive and the negatives signs correspond to the expanding and the recollapsing Universes, respectively. By equating the real part, we obtain the semiclassical Einstein (Hamilton-Jacobi) equation

$$\frac{1}{2m_P} \left( \frac{\partial S}{\partial a} \right)^2 + \frac{9m_P}{32\pi^2} k a^2 - \frac{3h^2 p^2}{8\pi a^2} + V_q(a) = 0,$$ (13)

where

$$V_q(a) = -\frac{\hbar^2}{2m_P} \left[ \frac{\partial^2 F}{\partial a^2} + \frac{\nu}{a} \frac{\partial F}{\partial a} \right]$$ (14)

is the quantum potential. The imaginary part leads to the continuity equation for probability

$$F \frac{\partial^2 S}{\partial a^2} + 2 \frac{\partial F}{\partial a} \frac{\partial S}{\partial a} + \frac{\nu}{a} F \frac{\partial S}{\partial a} = 0.$$ (15)

A cosmological time emerges into the classical world as parameterizing the de Broglie-Bohm trajectory along the tangential direction of the gravitational action:

$$\frac{\partial}{\partial t} = \mp \frac{4\pi}{3m_Pa} \frac{\partial S}{\partial a} \frac{\partial}{\partial a}.$$ (16)

It is not difficult to see that if one neglects the quantum potential and identifies the classical momentum by $p_c = \hbar p$, then the time parameter coincides with the cosmological time used to define the momentum in Hamiltonian formulation of the classical gravity:

$$\pi_a = m_P \dot{a} = \mp \frac{4\pi}{3a} \frac{\partial S}{\partial a}.$$ (17)
We rewrite the semiclassical Einstein equation (13) as
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{16\pi^2 \hbar^2}{9m_P a^6} \left( \frac{3}{4\pi} \rho^2 - \frac{2a^2}{\hbar^2} V_q(a) \right). \tag{18}
\]
Assuming a one-to-one mapping between \( t \) and \( a \) and using Eq. (17), we are able to invert \( S(a) \) and \( F(a) \) as functions of \( a(t) \) to find
\[
V_q = -\frac{\hbar^2}{2m_P} \left[ \left( \frac{(a\dot{a})^2}{2a^2} \right)^2 - \frac{1}{2\dot{a}} \left( \frac{(a\dot{a})^2}{a^2} + \nu \dot{a}^2 \right) - \frac{\nu (a^2 + \dot{a}^2)}{2a^2 \dot{a}^2} \right]. \tag{19}
\]
Finally, we obtain the semiclassical Einstein equation
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{16\pi^2 \hbar^2}{9m_P^2 a^6} \left[ \left( \frac{3m_P \rho^2}{4\pi} \right)^2 + a^2 \left( \frac{(a\dot{a})^2}{2a^2} \right)^2 - \frac{1}{2\dot{a}} \left( \frac{(a\dot{a})^2}{a^2} + \nu \dot{a}^2 \right) - \frac{\nu (a^2 + \dot{a}^2)}{2a^2 \dot{a}^2} \right]. \tag{20}
\]
Eq. (20) is highly nonlinear. In the flat space \((k = 0)\), however, the massless scalar-field model has a relatively simple solution. Trying an ansatz
\[
a(t) = a(t_0) \left[ 1 \pm \beta(t - t_0) \right]^{1/3}, \tag{21}
\]
simplifies the quantum potential as
\[
V_q(a) = \frac{\hbar^2}{2m_P} \frac{(1 - \nu)^2}{4a^2}, \tag{22}
\]
and leads to the semiclassical Einstein equation
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{16\pi^2 \hbar^2}{9m_P^2 a^6} \left[ \left( \frac{3m_P \rho^2}{4\pi} \right)^2 - \frac{(1 - \nu)^2}{4} \right]. \tag{23}
\]
Letting
\[
D_\nu = \frac{3m_P \rho^2}{4\pi} - \frac{(1 - \nu)^2}{4}, \tag{24}
\]
we see that depending on the signs of \( D_\nu \), there either can be or cannot be classically allowed solution for \( a(t) \). For a positive \( D_\nu \), we find the solutions
\[
a(t) = a(t_0) \left[ 1 \pm \frac{4\pi \hbar \sqrt{D_\nu}}{3m_P a^3(t_0)} (t - t_0) \right]^{1/3}. \tag{25}
\]
For a negative \( D_\nu \), there are no classically allowed solutions. It is found that the role of the quantum potential is to change the property of singularities. Since the wavefunctions do not oscillate but behave as a power-law, we may interpret the corresponding wavefunctions as quantum mechanical wormholes [13].

We now turn to the exact wavefunctions of the Wheeler-DeWitt equation. We treat separately the cases \( k = 0, 1, -1 \).
3.1 $k = 0$

For the case $k = 0$, we find the exact wavefunctions of the form

$$
\Psi(a) = C \exp(\gamma \ln(a)),
$$

(26)

where $\gamma$ satisfies

$$
\gamma^2 + (p - 1)\gamma + \frac{3m_P p^2}{4\pi} = 0.
$$

(27)

Again depending on the signs of $D_{\nu}$, the wavefunctions either oscillate or do not as the Universe recollapses. In the case $D_{\nu} > 0$, the wavefunctions are

$$
\Psi^\pm_I(a) = C_{\pm}a^{(1-p)/2}e^{\pm i\sqrt{D_{\nu}}\ln(a)},
$$

(28)

whereas in the case $D_{\nu} < 0$ they are

$$
\Psi^\pm_{II}(a) = C_{\pm}a^{(1-p)/2}\pm\sqrt{-D_{\nu}}.
$$

(29)

One sees that for the exact wavefunctions with the action

$$
S = \pm \hbar \sqrt{D_{\nu} \ln(a)}
$$

(30)

Eq. (15) and Eq. (13) are satisfied. It should be noted that the wavefunctions $\Psi_{\pm} \to 0$ as $a \to 0$.

The form of the wavefunctions also guarantees the Hermitivity of the Wheeler-DeWitt equation

$$
J_{+-}(0) = \left. i \left( \frac{\partial}{\partial a} \Psi_+ - \frac{\partial}{\partial a} \Psi_\ast_- \right) \right|_{a=0} = 0.
$$

(31)

From Eq. (17)

$$
m_P \dot{a} = \pm \frac{4\pi \hbar \sqrt{D_{\nu}}}{3} \frac{1}{a^2},
$$

(32)

we recover exactly the same solutions (25).

3.2 $k = 1$

We now turn to the cases of $k = \pm 1$. The exact wavefunctions of Eq. (11) are classified according to the signs of $D_{\nu}$, as in the $k = 0$ case. First, for $D_{\nu} > 0$ the wavefunctions are

$$
\Psi_{III}^{(\pm)} = C_{\pm}a^{(1-p)/2}H_{(1,2)}^{(1,2)}(i\beta a^2)
$$

(33)

whereas for $D_{\nu} < 0$ they are

$$
\Psi_{IV}^{(\pm)} = C_{\pm}a^{(1-p)/2}H_{(1,2)}^{(1,2)}(i\beta a^2)
$$

(34)
where $H^{(1,2)}$ are the Hankel functions, and $\beta = \frac{3m_p}{8\pi h}$. For $a \gg 1$, the wavefunctions are approximated as
\begin{align*}
\Psi^{(\pm)}_{III} &= C_{\pm} \sqrt{\frac{2}{\pi \beta a^2}} e^{\mp \left(\beta a^2 - \frac{\pi \nu}{2} + i\frac{\pi}{4}\right)}, \\
\Psi^{(\pm)}_{IV} &= C_{\pm} \sqrt{\frac{2}{\pi \beta a^2}} e^{\mp \left(\beta a^2 + i\frac{\pi \nu}{2} + i\frac{\pi}{4}\right)}. \tag{35}
\end{align*}

Regardless of the signs of $D_\nu$, the asymptotic wavefunctions $\Psi^{(\pm)}_{III}$ and $\Psi^{(\pm)}_{IV}$ describe the exponentially decreasing and the exponentially increasing wavefunctions dominated by the curvature, of which $\Psi^{(+)}_{III}$ and $\Psi^{(+)}_{IV}$ can be interpreted as the quantum wormholes \cite{[13]}. From the series expansion of the Hankel functions one can easily see that the approximate wavefunctions for $a \ll 1$ are the same as those of $k = 0$. We may use these wavefunctions to calculate the quantum potential and treat the semiclassical Einstein equation. Thus, we find that there is still a particular class of solutions, quantum mechanical or semiclassical, which may avoid classical singularities due to the quantum potential.

### 3.3 k = -1

We find the exact solutions for $k = -1$. For $D_\nu > 0$ the wavefunctions are
\begin{align*}
\Psi^{(\pm)}_{V} &= C_{\pm} a^{(1-n)/2} H^{(1,2)}_{i\sqrt{D_\nu}/2}(\beta a^2), \tag{36}
\end{align*}
whereas for $D_\nu < 0$ they are
\begin{align*}
\Psi^{(\pm)}_{VI} &= C_{\pm} a^{(1-n)/2} H^{(1,2)}_{-i\sqrt{-D_\nu}/2}(\beta a^2). \tag{37}
\end{align*}

For $a \gg 1$, we approximate the wavefunctions as
\begin{align*}
\Psi^{(\pm)}_{V} &= C_{\pm} \sqrt{\frac{2}{\pi \beta a^2}} e^{\pm i(\beta a^2 - \frac{\pi \nu}{2} - \frac{\pi}{4})}, \\
\Psi^{(\pm)}_{VI} &= C_{\pm} \sqrt{\frac{2}{\pi \beta a^2}} e^{\pm i(\beta a^2 + \frac{\pi \nu}{2} - \frac{\pi}{4})}, \tag{38}
\end{align*}

The asymptotic wavefunctions $\Psi^{(\pm)}_{V}$ and $\Psi^{(\pm)}_{VI}$ for all $D_\nu$, represent either the expanding or the recollapsing Universe. For $a \ll 1$ the wavefunctions have the same form as those for $k = 0, 1$. The quantum potential still avoids the cosmological singularities.

### 4 Conclusion

We applied the de Broglie-Bohm interpretation to the Wheeler-DeWitt equation for the FRW model with a massless scalar field. We not only found the exact wavefunctions but also solved
both the Hamilton-Jacob equation with the quantum potential (real part) and the continuity
equation for probability (imaginary part) at the same time. Both the exact wavefunctions and
the de Broglie-Bohm interpretation gave the identical result, as expected, that there can be or
cannot be quantum potential dominated particular solutions that avoid either the initial or the
final cosmological singularities, depending on the relative magnitude of the operator ordering
parameter and the momentum of massless scalar field. It was shown that the quantum potential
and the back-reaction of the massive scalar field changed significantly the effective energy density
in the semiclassical gravity [12]. These results suggest that the quantum potential and the back-
reaction of both geometry and matter fields might change the properties of and avoid the initial
and the final cosmological singularities.

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