Students’ Strategies to Solve Reversible Problems of Function: The Part of Reversible Thinking

S Maf’ulah¹, D Juniati²
¹Department of Mathematics Education, STKIP PGRI Jombang, Indonesia
²Department of Mathematics Education, Universitas Negeri Surabaya, Indonesia

E-mail: syarifatul.m@gmail.com

Abstract. This study aimed to reveal students’ strategies to solve reversible problems, particularly to function course. Individuals’ competence to solve reversible problems is the part of reversible thinking. This way of thinking refers to a reversible two-way relationship. Providing reversible problems is a way for teachers to develop their students’ reversible thinking. It was qualitative. 105 students who were prospective teachers of mathematics were selected as the subject of this study, and it took a test containing some reversible problems, such as function along with its graphic and between derivative and integral, as the instrument. The students’ results were further analyzed to reveal their strategies of solving those reversible problems. The finding showed that among 105 students, 62% of them were capable to draw the graph of an identified function through 4 different strategies, while 28% of them were capable to define the function of an identified graphic through 5 different strategies. The significant difference between those two percentages implied that only few students were capable to both drawing the graphic of an identified function and defining the function of an identified graphic. Overall, it indicated that the students were still less in constructing a reversible relationship between function and its graphic.

1. Introduction

This study was inspired by Jean Piaget’s thought on reversible thinking, and motivated by some references from published journals that described the importance of thinking reversibly for students in education field, especially those dealing with mathematics education field [1, 2, 3, 4, 5, 6, 7, 8]. Following Piaget [9], reversible thinking was an individual’s mental competence to reverse his/her way of thinking to the initial state of his/her thought. Hackenberg [10] argued that reversible thinking dealt with reversing a scheme. One may not only involve inversion—a reversing action into the initial state, but also involving a compensation to reverse a scheme—taking new action to reverse things into a condition equivalent to the initial state. Wong [11] argued that reversible thinking referred to mental flexibility—individual’s competence to anticipate and review in an analytical manner.

Thus, reversible thinking involved thinking for twice in the opposite aspect thus reducing the opportunity for someone to make a mistake in every decision making. Reversible thinking has a role when solving mathematical problems. When a person solves the mathematics problem with involving reversible thinking, he/she would use two solving processes from two opposite directions. In addition, he/she would also do looking back to check the result of problem solving was obtained by him/her. Thus, reversible thinking in mathematics learning should be considered and developed so that students’ abilities in problem solving are maximized.

This study aimed to reveal students’ strategies in solving reversible problems. The research focused on the subject matter of function. Function is one of mathematics courses that relates to other courses with higher level on their hierarchy, such as limits, continuity, derivative, integral, and the
others dealing with function. Hence, students should be able to comprehensively understand the course. Denbel [12] argued that function was useful in every branch of mathematics, as it is a unifying concept in all mathematics.

Four reversible problems related to function were given. The first two of them were between function and its graph, and the second ones were between derivative and integral. First, the students were asked to draw the graphic of an identified function. Second, they were asked to define the function of an identified graphic. Third, they were asked to find the derivative of an identified function. Fourth, they were asked to define the function of an identified derivative. The first two problems aimed to reveal the students’ competence in constructing a reversible relationship between function and graphic, while the second ones pointed to the reversible relationship between derivative and integral. Such competence was detected through the strategies they used, including the steps, to solving those problems.

2. Method

This study is qualitative. It took 105 students who were prospective mathematics teachers as the subject of the study. Furthermore, it took a test containing some reversible problems, such as function along with its graph and between derivative and integral, as the instrument. The students’ results were then analysed to reveal the strategies they used in solving those problems. This present study applied some procedures consisting of three primary phases as follow. (1) Preparation; in this stage, some theories of thinking reversible were viewed and examined. (2) Implementation; in this stage, the subject of this study was selected. Subsequently, the selected subject might have a test and interview respectively based on their work. (3) Analysis; in this stage, the data was analyzed and reported.

3. Result

The researchers gave a test containing some reversible problems to the students. The test was presented in Figure 1, as follow.

| TEST |
|------|

(Maximum time provided is 15 minutes)

**Instruction**: Complete the following problems clearly!

1. If $f(x) = 3x - 2$ with $x \in R$, draw the graphic of the function!
2. Look at the following graphic!

   ![Graph](image)

   Define the function $f(x)$ of the graphic!

3. If $f(x) = x^2 - 3x + 1$, define $f'(x)$!
4. If $f'(x) = 2x - 3$, define $f(x)$!

**Figure 1.** Test

The test as presented in Figure 1 was aimed to reveal the students’ strategies in solving the problems. They were usually asked to draw the graphic of an identified function. This test, however, had them define the function of an identified graphic. In addition, they might usually be asked to define the derivative or integral of an identified function. Otherwise, this test had them to find the integral of an identified derivative of a function. This test was designed to see how students construct a reversible two-way relationship as their competence. 105 students who were all prospective teachers participated. The result was presented as follow.
Based on Figure 2, it found the number of students (in percentage) who correctly solved the problems. The test result was then analyzed and classified based on the strategies they used for solving the problems. The following description showed the detail.

1. The students’ strategies for drawing the graphic of an identified function.
   The students’ strategies for drawing the graphic of an identified function were as follow.
   1) Two-point strategy without reviewing the final result
      The steps of this strategy were as follow.
      Step 1 : Defining two points of coordinate through the identified function; the coordinate that cut the axis X and the coordinate that cut the axis Y.
      Step 2 : Drawing a graphic based on the coordinates that had been found in step 1 by directly connecting those two points of coordinate using straight line, as it was a linear function.
      Step 3 : No review. This information was obtained from interview.

   The following figure (i.e., Figure 3) presented an example of the student’s work using problem solving through two-point strategy.

   ![Figure 3](image)

   **Figure 3.** An Example of Problem Solving through Two-Point Strategy.

2) Many-Point Strategy with a Review on the Final Result by Reversing It to the Initial Problem
   The steps of this strategy were as follow.
   Step 1 : Defining some coordinates (at least 3 coordinates) from a given identified function.
   Step 2 : Drawing the graphic based on the identified coordinates by connecting all of them through a straight line, as it was a linear function.
   Step 3 : Reversing the result to the initial problem by substituting those identified coordinates (on the graphic) into the function. This information was based on the students’ works and interview. A crossed graphic was found in their works. Furthermore, an interview was held to clarify the crossed graphic. The students reversed their work to its initial state by substituting those all points of coordinates on the graphic they have drawn.
The following figure (i.e., Figure 4) presented an example of the student’s work using many-point strategy with a review on the final result by reversing the result to the initial problem.

![Figure 4](image)

**Figure 4:** The Example of Many-Point Strategy with a Review on the Final Result by Reversing the Result to the Initial Problem

3) Many-Point Strategy with a Review on the Final Result by Recounting It

The steps of this strategy were as follow.

- **Step 1:** Defining some coordinates (at least 3 coordinates) from a given identified function.
- **Step 2:** Drawing a graphic based on the coordinates found in Step 1 by directly connecting those coordinates with a straight line, given that it was a linear function.
- **Step 3:** Recounting it. This information was obtained by interviewing the students.

The following figure (Figure 5) presented an example of students’ work using many-point strategy with a review on the final result by recounting it.

![Figure 5](image)

**Figure 5.** An Example of the Student’s Work Using Many-Point Strategy with a Review on the Final Result by Recounting It
4) Many-Point Strategy with no Review
The steps of this strategy were as follow.
Step 1 : Defining some coordinates (at least 3 coordinates) from a given identified function.
Step 2 : Drawing a graphic based on the coordinates found in Step 1 by directly connecting those coordinates with a straight line, given that it was a linear function.
Step 3 : No Review. This information was based on the result of interview.

The following figure (Figure 6) presented an example of the student’s work using many-point strategy with no review.

![Figure 6. An Example of the Student’s Work Using Many-Point Strategy with no Review](image)

2. The students’ strategies to define the function of an identified graphic
The students’ strategies to define the function of an identified graphic were as follow.
1) Trial-and-Error Strategy with a Review on the Final Result by Reversing It to the Initial Problem
The steps of this strategy were as follow.
   a. Defining some coordinates based on the identified graphic.
   b. Defining the function of the graphic by having trial and error based on the coordinates identified in step 1. This information was based on the result of interview. The steps of having trial-and-error strategy by students were as follow.
      • Taking \( f(x)=ax+b \) as the formula as they got that, in this category, the graphic was in the form of a straight line. Therefore, it must be linear with \( f(x)=ax+b \)
      In case that the coordinate cutting the axis Y was (0,1), the \( f(x)=ax+1 \), as \( x=0 \)
      • Defining the coefficient of X in such a way to get a function which graphic was equal to the identified one. Initially, the students wrote \( f(x)=-2x+1 \). However, as coordinate (-2,0) substituted to \( f(x)=-2x+1 \) resulted in \( f(-2)=-2(-2)+1=3\neq 0 \), \( f(x)=-2x+1 \) was found inappropriate. Then, they wrote \( f(x)=-2x+1 \) and the substitution of coordinate (-2,0) to (-2,0) resulted in \( f(-2)=-2(-2)+1=0 \), they found that \( f(x)=1/2 \ x+1 \) was appropriate as the function of the given graphic.
   c. Having a review. It was rechecking the correctness of the function by substituting the coordinates on the graphic into the function found. It was considered correct if the result was equal to the given graphic. this information was based on the result of interview.

The following figure (Figure 7) presented an example of the student’s work using trial-and-error strategy with a review on the final result by reversing the result to the initial problem.
2) Linear Equation Strategy with a Review on the Final Result by Reversing it to the Initial Problem

The steps of this strategy were as follow.

Step 1: Defining some coordinates based on the given graphic.

Step 2: Defining the function of the graphic using two identified coordinates and a formula of linear equation as follow.

\[ y - y_1 = \frac{x - x_1}{y_2 - y_1} x_2 - x_1 \]

It found \( f(x) = \frac{1}{2}x + 1 \)

Step 3: Having a review by substituting the coordinates on the graphic into the function. In case that the result was equal to the given graphic, the function was considered correct. This information was based on the result of interview.

The following figure (Figure 8) presented an example of the student’s work using linear equation strategy with a review on the final result by reversing the result to the initial problem.

3) Line Equation Strategy with no Review

The steps of this strategy were as follow.

Step 1: Defining the coordinates based on the given graphic.

Step 2: Defining the function of the given graphic using two coordinates and a formula of line equation as follow.

\[ y - y_1 = \frac{x - x_1}{y_2 - y_1} x_2 - x_1 \]

It found \( f(x) = \frac{1}{2}x + 1 \)

Step 3: No review. This information was based on the interview.

The following figure (Figure 9) presented an example of the student’s work using line equation strategy with no review.
4) Linear Equation Strategy with no Review

The steps of this strategy were as follow.

Step 1: Defining some coordinates based on the given graphic.

Step 2: Writing a formula $f(x) = ax + b$, given that the identified graphic was in the form of a straight line, which indicated a linear function.

Step 3: Defining the value of $a$ and $b$ by substituting the coordinates into function $f(x) = ax + b$, and it found $a = \frac{1}{2}$ and $b = 1$.

Step 4: Substituting $a = \frac{1}{2}$ and $b = 1$ into $f(x) = ax + b$, and thus it found $f(x) = \frac{1}{2}x + 1$.

Step 5: No review. This information was based on the interview.

The following figure (Figure 10) presented an example of the student’s work using linear equation strategy.

5) Gradient Strategy with a Review on the Final Result by Recounting It

The steps of this strategy were as follow.

Step 1: Defining some coordinates based on the given graphic.

Step 2: Defining a gradient (i.e., coded by $m$) using formula $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Step 3: Writing $f(x) = mx + c$, given that the identified graphic was in the form of a straight line, and thus, it must be linear.

Step 4: Defining the value of $c$ through the coordinates found, the gradient $m$, and $f(x) = mx + c$.

Step 5: Substituting gradient $m$ and value $c$ into $f(x) = mx + c$ in order to find a function of the given graphic.

Step 6: Having a review by recounting. This information was based on the interview.
The following figure (Figure 11) presented an example of the student’s work using gradient strategy with a review on the final result by recounting it.

Figure 11. An Example of the Student’s Work Using Gradient Strategy with a Review on the Final Result by Recounting It

3. The only strategy the students used to define the derivative of an identified function was through a derivative formula, “If \( f(x) = ax^n \), then \( f'(x) = nax^{n-1} \).” Using this formula made them able to define the derivative of \( f(x) \) without having any review on their final result. Therefore, it was called derivative formula strategy. The following figure (Figure 12) presented an example of the student’s work using this strategy.

Figure 12. An Example of the Student’s Work Using Derivative Formula Strategy

4. The only strategy the students used to define the function of an identified derivative was through an integral formula, “If \( f'(x) = ax^n \), then \( f(x) = \frac{a}{n+1}x^{n+1} + C \).” Using this formula made them able to define function \( f(x) \) without having any further review on their final result. Therefore, it was called integral formula strategy. The following figure (Figure 13) was an example of the student’s work using this strategy.

Figure 13. An Example of the Student’s Work Using Integral Formula Strategy.

In accordance to those all analyses, some information was collected as follow.

1. The students using two-point strategy with no review in drawing the graphic of an identified function were found using line equation formula with no review as well to define the function of the given graphic. However, some others decided to choose linear equation formula without having any review on their final result as their strategy.

2. In drawing the graphic of an identified function, the students who used many-point strategy with a review on their final result by reversing the result to the initial problem were found using line equation formula strategy as well without having any review on their result. In addition, some others decided to use linear equation with no review on their final result.

3. The students who used many-point strategy with a review on their final result by recounting it for drawing the graphic were found defining the function of an identified graphic using gradient strategy with a review on their final result by recounting it. Some others used line equation formula without having any review on their result.

4. The students who used many-point strategy with no review on their final result for drawing the graphic of an identified function were found defining the function of an identified graphic through
line equation formula with a review on their final result by reversing the result to the initial problem. Some others used linear equation formula with no review on their final result.

5. The students who used derivative formula in defining the derivative of an identified function were found defining the function of an identified derivative through integral formula.

Those five points were successively presented in Figure 14 and Figure 15, as follow.

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**Figure 14.** Scheme of the Students’ Competence to Construct a Reversible Two-Way Relationship between Function and Its Graphic

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**Figure 15.** Scheme of the Students’ Competence to Construct a Reversible Two-Way Relationship between Function and its Derivative
4. Discussion and Conclusion

This study aimed to reveal students’ strategies to solve reversible problems, particularly to function course. This competence was detected through some strategies the students used in solving the given problems. They took some steps to complete their strategies. Finally, the conclusion was made on Table 1 as follow.

| The students’ strategy to draw the graphic of an identified function | The students’ strategy to define the function of an identified graphic |
|---|---|
| Two-Point Strategy with no Review on the Final Result | Line equation formula with no review on the final result |
| Many-Point Strategy with a Review on the Final Result by Reversing the result to the Initial Problem | Trial-and-error strategy with a review on the final result by reversing the result to the initial problem |
| Many-Point Strategy with a Review on the Final Result by Recounting the Result | Gradient strategy with a review on the final result by recounting the result |
| Many-Point Strategy with no Review on the Final Result | Linear equation formula with a review on the final result by reversing the result to the initial problem |

Based on Figure 2 that presented the students’ test results, it found that 62% of 105 students were capable to draw a graphic of the given function through four different strategies, while 28% of them were capable to define a function of the given graphic through five different strategies. The significant difference between those two percentages implied that only few students were capable to both drawing the graphic of an identified function and defining the function of an identified graphic. Overall, it showed that students were still lacking in establishing reversible relationships between functions and its graphs. This was due to the fact that their reversible thinking development did not optimal yet on which can be seen from the competence to construct a reversible two-way relationship, as the opposite of one-way relationship which characterized on one single way [8]. Students with less competence in reversible thinking may not be able to define the function of the identified graph even though they are able to draw a graph. However, those who were able to define functions from identified graphs may have the ability to draw graphs from identified functions as well. This finding was consistent to Maf’ulah et al. [3] who took high school students as their research subject, especially their reversible thinking in solving function problems. That study found that among 123 high school students, only 5 students were capable to construct a reversible two-way relationship.

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