The Approach of Turbulence to the Locally Homogeneous Asymptote as Studied using Exact Structure-Function Equations

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ABSTRACT. An exact equation is obtained that relates the products of two-point differences of fluid velocity and those differences with the difference of pressure gradient and other quantities. The averages of such products are structure functions. Equations that follow from the Navier-Stokes equation and incompressibility but with no other approximations are called “exact” here. Exact equations for structure functions are obtained, as is an exact incompressibility condition on the second-order velocity structure function. Ensemble, temporal, and spatial averages are all considered because they produce different statistical equations and because they respectively apply to theoretical purposes, experiment, and numerical simulation of turbulence; those applications are addressed herein. The midpoint and the difference of the two points at which the hydrodynamic quantities are obtained are \( X \) and \( r \); \( t \) is time. The equations are organized in a revealing way by use of \( X \), \( r \), \( t \) as independent variables. Dependences on \( X \) and on the orientation of \( r \) and on \( t \) fade as the asymptotic statistical states of local homogeneity, local isotropy, and local stationarity, respectively, are approached. The exact equations are thus applicable to study of the approach toward those asymptotic states. Exact equations obtained by averaging over a sphere in \( r \)-space have a particularly simple form. The case of a simulation that has periodic boundary conditions leads to particularly simple equations. A new definition of local homogeneity is contrasted with previous definitions. The approach toward the asymptotic state of local homogeneity is studied by using scale analysis to determine the required approximations and the approximate equations pertaining to experiments and simulations of the small-scale structure of high-Reynolds-number turbulence, but without invoking local isotropy. Those equations differ from equations for homogeneous turbulence. The traces of both exact and approximate equations have particularly simple forms; in particular, the energy dissipation rate appears in the exact trace equation even without averaging, whereas in previous formulations the energy dissipation rate appears after averaging and use of local isotropy. The trace mitigates the effect of anisotropy in the equations, thereby revealing that the trace of the third-order structure function is expected to be superior for quantifying asymptotic scaling laws.

1. INTRODUCTION

The dynamic theory of the local structure of turbulence is so named by Monin and Yaglom (1975) (their Sec. 22) to mean the derivation and investigation of equations for structure functions by use of the Navier-Stokes equation. The structure functions are averages of differences of basic hydrodynamic quantities such as velocity and pressure gradient. Monin and Yaglom (1975) pointed out that the dynamic theory gives important relationships between structure functions, and that these relationships provide important extensions of predictions based on dimensional analysis and flow similarity. The dynamic theory is the basis for Kolmogorov’s (1941a) famous equation that relates second-order and third-order velocity structure functions, and is of fundamental importance in the theory of locally homogeneous and locally isotropic turbulence (Monin and Yaglom, 1975; Batchelor, 1947; Monin, 1959; Frisch, 1995). The dynamic theory does not uniquely determine the structure functions; this is known as the closure problem (Monin and Yaglom, 1975). Experimental data have been used to evaluate the balance of Kolmogorov’s equation and generalizations of it (Antonia, Chambers, and Browne, 1983; Chambers and Antonia, 1984; Lindborg, 1999; Danaila et al., 1999 a,b; Antonia et al., 2000). This report supports such experimental work, as well as more precise use of direct numerical simulation (DNS) by giving correct and complete equations to be used in such evaluations.

We derive exact equations for structure functions by use of differential operator identities. By “exact” we mean that the equations follow from the Navier-Stokes equation and the incompressibility condition with no additional approximations. This meaning is emphasized because turbulence researchers consistently use “exact” when they mean asymptotic. Exact equations satisfy the perceived need by Yaglom (1998) for careful derivation of dynamic-theory equations and the perceived value placed by Sreenivasan and Antonia (1997) on aspects of turbulence that can be understood precisely. In Sec. 2, the equations for products of differences is developed to the greatest extent possible before any average is performed. This mathematical method is similar to that used in the theory of wave propagation in random media where the equations for wave-field products are thoroughly developed before an average is performed.
The velocity field. Such an equation would be useful for interpreting the observed (Mydlarski and Warhaft, 1998; Sreenivasan, 1991) local anisotropy of scalar fields in the presence of a mean gradient of the scalar.

The exact equations retain all of the dependence of the structure functions on \( r, X \), and time \( t \), where \( r \) is the vector spacing between two points at which the measurements are obtained within the turbulent flow and \( X \) is the midpoint position of these points. Previous methods (Batchelor, 1956; Lindborg, 1996; Hill, 1997) of deriving dynamic-theory equations neglected the dependence of statistics on \( X \), and thereby limited the equations to the cases of homogeneous and locally homogeneous turbulence. To also study the approach toward local homogeneity, equations are needed that retain \( X \). Here, attention is given to the conditions that must be fulfilled for the \( X \)-dependence to be neglected. Previously (Hill, 1997), the approach toward local isotropy was examined, although exact equations were not then available. Consequently, the approach toward local isotropy is not considered here.

A scale analysis is performed to quantify terms that are to be neglected on the basis that \(|r|\) is much less than a length scale that will be called the outer scale, and to deduce all other required approximations. Such scale analysis is presented in detail in Sec. 7.4. Our analysis determines approximations that quantify the degree to which the small-scale structure of turbulence depends on its large-scale structure; such analysis was called for by Yaglom (1998). Our analysis sets the stage for DNS and experimental studies of the approximations.

The equations derived in Sections 2 and 3 are exact for every flow, whether laminar or turbulent. For example, the equations apply exactly to the edge of a jet, to a boundary layer, as well as to those experimental situations such as grid-generated wind-tunnel turbulence, for which local homogeneity is expected to be most accurate. The equations apply provided there are no forces on the fluid at the points of measurement. Forces can be applied near the point of measurement; for instance, the equations are exact for hot-wire anemometer supports just downstream of the measurement points. The equations apply for turbulence generated at places other than the points of measurement; examples are grid-generated turbulence measured downstream of the grid, and turbulence generated by rotating blades (Zocchi et al., 1994). The case of statistically homogeneous forces distributed throughout the fluid has been considered for the asymptotic case of isotropic turbulence by Novikov (1965) (see also Frisch, 1995). The case of forces at the points of measurement is considered in Appendix A.

The ensemble average is considered first (Sec. 3.1). It has the advantage for theoretical studies that temporal and spatial changes can be considered because the ensemble average does not eliminate dependence on \( X \) or \( t \). The temporal average is typically used with experimental data, and the spatial average is typically used for data from DNS. For this reason, exact equations for both temporal averaging (Sec. 3.2) and spatial averaging (Sec. 3.3) are also obtained. The connection between the derivations presented here and any experiment or DNS is important because the equations relate several statistics and therefore are most revealing when data are substituted into them.

A recently developed experimental method (Su and Dahm, 1996) has the potential to thoroughly evaluate terms in the equations derived here. As shown in Sec. 3.4, the exact equations have a particularly simple form for the case of DNS with periodic boundary conditions.

The equations can be evaluated with experimental or DNS data to determine the most significant terms in the equations for a given flow and thereby determine the effects that cause deviations from asymptotic laws. The ongoing interest in turbulence intermittency includes accurate evaluation of inertial-range exponents of structure functions, for which purpose precise definition of an observed inertial range is needed. The third-order structure function can serve this purpose because it has a well-known inertial-range power law and the 4/5 coefficient (Kolmogorov’s (1941a) 4/5 law) in the asymptotic limit of accurate local homogeneity and local isotropy. Deviations from the 4/5 coefficient are observed in experiments (Anselmet et al., 1984; Mydlarski and Warhaft, 1996, 1998; Lindborg, 1999); this casts doubt on the precision with which measured exponents apply to the intermittency phenomenon. The equations derived here, when evaluated with data, can reveal the effects contributing to the deviation from Kolmogorov’s 4/5 law. The usefulness of such evaluations is shown by Lindborg (1999); Danaila et al. (1999 a,b); and Antonia et al. (2000). They generalize Kolmogorov’s equation by the addition of a term describing streamwise inhomogeneity. To obtain this term from the present exact analysis, it is necessary to perform the Reynolds decomposition. The present analysis has the advantage that it reveals all terms that describe inhomogeneity. This is discussed in detail in Sec. 7. The equations derived here are obtained in the Eulerian framework, which is most useful for experimental evaluation.

Particular attention is given to the typical experimental case that is used to investigate universality of turbulence statistics at small scales and large Reynolds numbers. We derive the simplification of the exact equation that applies approximately to such experiments. Experimental data typically have the mean velocity subtracted before structure functions are calculated from the velocity fluctuation. For this reason, we derive the approximate equation obeyed by structure functions calculated from velocity fluctuations. The Reynolds decomposition (Sec. 5) is essential for
this purpose. The derivation is necessarily long in Sec. 7.4, but in this case, the journey is more significant than the destination because all required approximations are determined en route. Local homogeneity is the most important of the approximations. A necessary condition for local homogeneity is given in Sec. 7.3; it is not a sufficient condition.

The trace of the exact equation has a particularly simple form. When averaged over a sphere in r-space, and when the advective and time-derivative terms are neglected, this equation has the same form as Kolmogorov’s (1941a) equation (Sec. 4.3). This is true despite the fact that the r-space sphere-averaged equation is valid even for extreme violations of local isotropy.

1.1 Contrasting Definitions of Local Homogeneity

Local homogeneity has been given various definitions by different authors. Kolmogorov (1941b) introduced a space-time domain that is small compared to L and T=(L/U), where L and U are “typical length and velocity for the flow in the whole.” Kolmogorov considers the two-point differences of the velocities at spatial points in the domain; one point is common to all the differences. Kolmogorov (1941b) defines local homogeneity as follows: the joint probability distribution of the velocity differences is independent of the one common spatial point, and of the velocity at the one common point, and of time. Data of Praskovsky et al. (1993), Sreenivasan & Stolovitzky (1996), and Sreenivasan & Dhruva (1998) contradict the statistical independence of velocity difference and the velocity at either end point, as well as contradict the statistical independence of velocity difference and the velocity at the midpoint. The exception is isotropic turbulence (Sreenivasan & Dhruva, 1998) for which case local homogeneity is assured. An alternative possibility that is particularly relevant here is that the two-point velocity sum, \( u_n + u_n' \), might be statistically independent of velocity difference, but statements by Sreenivasan & Stolovitzky (1996) and Sreenivasan & Dhruva (1998) contradict that statistical independence as well; publication of supporting data would be useful. Kolmogorov’s definition should not be used because experimental data contradict that statistical independence (Praskovsky et al., 1993; Sreenivasan and Stolovitzky, 1996; Hill and Wilczak, 2001), as do theoretical considerations (Hill and Wilczak, 2001).

Monin and Yaglom (1975) define local homogeneity to mean that the joint probability distribution of the two-point velocity differences is unaffected by any translation of the spatial points. They do not impose a restriction on the translations to a spatial domain. It follows (Monin and Yaglom, 1975) that statistics composed entirely of the differences obey the same relationships that they do for homogeneous turbulence (namely, they are independent of where they are measured), and that the mean velocity depends linearly on position. In practice, statistics of differences and of derivatives do depend on where they are measured except in the ideal case of homogeneous turbulence. Frisch (1995) gives a definition that is equivalent to that of Monin and Yaglom (1975), except that the translations are restricted to a domain the size of the spatial scale characteristic of the production of turbulent energy (which he calls the integral scale). Two-point structure-function equations of all orders contain a statistic that is the product of not only factors of the difference of the two velocities but also one factor of the sum of the two velocities, i.e., \( u_n + u_n' \) (Hill, 2001). Because the definitions of local homogeneity by Monin and Yaglom (1975) and Frisch (1995) involve only the joint probability distribution of two-point differences, it follows that those definitions are not sufficient to simplify structure-function equations to the same level of simplification as does homogeneity.

The calculus of homogeneity by Batchelor (1956) is the commutation of spatial derivatives from within an average to outside the average where they become derivatives with respect to \( r \), and vice versa. The calculus of local homogeneity by Hill (1997) is a generalization of Batchelor’s calculus; specifically, local homogeneity was implemented by neglecting the derivative with respect to \( X \) relative to the derivative with respect to \( r \) when spatial derivatives were commuted with the averaging operation. That implementation is restricted to statistics that contain at least one difference or derivative of basic hydrodynamic quantities (such as velocity, pressure, temperature, etc.). This calculus differs from the aforementioned definitions of local homogeneity in that no translational invariance is required other than for the infinitesimal displacement in \( X \) implied by the derivative operation. In Appendix C, examples are given that show how this calculus produces the predictions of homogeneity for the homogeneous case. To simplify the structure-function equations, Hill (1997, 2001) found that it was necessary to apply that calculus to statistics of products containing not only at least one difference but also quantities that were not differences.

Consider grid-generated turbulence in a wind tunnel operated with constant mean velocity. For anemometers fixed relative to the position of the grid, the turbulence is stationary and streamwise inhomogeneous. For simplicity, ignore the cross-stream inhomogeneity. For anemometers moving relative to the grid in a direction parallel to the streamwise direction, the turbulence is both streamwise inhomogeneous and nonstationary. It is nonstationary because of downstream decay of the turbulence intensity. That example raises the question as to whether or not local stationarity and local homogeneity should be combined into a single definition that is independent of the motion of the coordinate system. In this author’s opinion such a combined definition is neither desirable nor practical. Thus, local homogeneity (or local stationarity) must be considered in a given coordinate system.
2. EXACT TWO-POINT EQUATIONS

Exact equations are given here that relate two-point quantities and that are obtained from the Navier-Stokes equations and incompressibility. The two spatial points are denoted \( x \) and \( x' \); they are independent variables: they have no relative motion; e.g., anemometers at \( x \) and \( x' \) are fixed relative to one another. To be concise, velocities are denoted \( u_i = u_i(x, t) \), \( u'_j = u_j(x', t) \), and the same notation is used for other quantities. \( p(x, t) \) is the pressure divided by the density (density is constant), \( \nu \) is kinematic viscosity, and \( \partial \) denotes partial differentiation with respect to its subscript variable. Summation is implied by repeated Roman indices; e.g., \( \partial_{x_n} \partial_{x_n} \) is the Laplacian operator. For brevity, define:

\[
\begin{align*}
d_{ij} &\equiv (u_i - u'_i) (u_j - u'_j) ; \\
d_{ijn} &\equiv (u_i - u'_i) (u_j - u'_j) (u_n - u'_n) ; \\
\tau_{ij} &\equiv (\partial_{x_i} p - \partial_{x'_i} p') (u_j - u'_j) + (\partial_{x_j} p - \partial_{x'_j} p') (u_i - u'_i) ; \\
ee_{ij} &\equiv (\partial_{x_n} u_i) (\partial_{x_n} u_j) + (\partial_{x'_n} u'_i) (\partial_{x'_n} u'_j) ; \\
F_{ijn} &\equiv (u_i - u'_i) (u_j - u'_j) u_n + u'_n / 2 .
\end{align*}
\]

We change independent variables from \( x \) and \( x' \) to the sum and difference independent variables:

\[
X \equiv (x + x') / 2 \quad \text{and} \quad r \equiv x - x' , \quad \text{and define} \quad r \equiv |r| .
\]

The derivatives \( \partial_X \) and \( \partial_r \) are related to \( \partial_x \) and \( \partial_{x'} \) by

\[
\partial_{x_i} = \partial_{r_i} + \frac{1}{2} \partial_X , \quad \partial_{x'_i} = \partial_{r_i} - \frac{1}{2} \partial_X , \quad \partial_X = \partial_x + \partial_{x'} , \quad \partial_r = \frac{1}{2} (\partial_x - \partial_{x'}) .
\]

It is essential to hold fixed the correct variables for each of the above partial derivative operations. The partial derivative \( \partial_{x'} \) is obtained with the following variables held fixed: \( x_j \), for \( j \neq i \), and \( x' \) and \( t \). Likewise for \( \partial_{x'} \), \( x'_j \), for \( j \neq i \), and \( x \) and \( t \) are held fixed. For \( \partial_X \), \( X_j \), for \( j \neq i \), and \( r \) and \( t \) are held fixed. For \( \partial_r \), \( r_j \), for \( j \neq i \), and \( X \) and \( t \) are held fixed. For any functions \( f(x, t) \) and \( g(x', t) \), (7) gives

\[
\partial_r [f(x, t) \pm g(x', t)] = \partial_X [f(x, t) \mp g(x', t)] / 2 .
\]

For example, \( \partial_r (u_j - u'_j) = \partial_{X_i} (u_j + u'_j) / 2 \), and \( \partial_r (u_j + u'_j) = \partial_{X_i} (u_j - u'_j) / 2 \).

Now, \( \tau_{ij} \) and the trace of (3) and (4) (i.e., \( \tau_{ii} \) and \( e_{ii} \)) can be expressed differently. Use of (6) in (3) as well as in \( e_{ii} \) and rearranging terms gives

\[
\begin{align*}
\tau_{ij} &= -2 (p - p') (s_{ij} - s'_{ij}) + \partial_{X_i} [(p - p') (u_j - u'_j)] + \partial_{X_j} [(p - p') (u_i - u'_i)] , \\
2 \nu e_{ii} &= 2 (\epsilon + \epsilon') + 2 \nu \partial_{X_n} \partial_{X_n} (p - p') ,
\end{align*}
\]

where

\[
s_{ij} \equiv (\partial_{x_i} u_j + \partial_{x_j} u_i) / 2 , \quad \epsilon \equiv 2 \nu s_{ij} s_{ij} ;
\]

to obtain (11) we used Poisson’s equation \( \partial_{x_n} \partial_{x_n} p = -\partial_{x_i} u_j \partial_{x_j} u_i \). Incompressibility requires that the trace of \( s_{ij} \) vanishes; thus, the trace of (6) is

\[
\tau_{ii} = 2 \partial_{X_i} [(p - p') (u_i - u'_i)] .
\]
2.1 Use of the Navier-Stokes equation

The Navier-Stokes equation for velocity component $u_i(x, t)$ and the incompressibility condition are

$$\partial_t u_i + \partial_x (u_i u_n) = -\partial_x p + \nu \partial_{xx} u_i, \text{ and } \partial_x u_n = 0.$$  \hspace{1cm} (13)

By multiplying the Navier-Stokes equation for $u_i$ by $u_i'$, we obtain an equation having $u_i' \partial_t u_i$ as its time-derivative term. We add and subtract eight such equations to obtain the equation having as its time-derivative term the expression $u_j \partial_t u_i - u_i' \partial_t u_j + u_i \partial_t u_j'$ + $u_i' \partial_t u_j - u_i \partial_t u_j' + u_j \partial_t u_i - u_j \partial_t u_i' = \partial_t [(u_i - u_j') (u_j - u_i')]$. Algebra is used to simplify the terms in the resultant equation, and zero is added to the equation (for convenience) in the form of $\partial_x (u_i' u_n u_n') + \partial_x' (u_i u_j u_n')$ (which vanishes by incompressibility). We thereby obtain

$$\partial_t d_{ij} + \partial_x (d_{ij} u_n) + \partial_x' (d_{ij} u_n') = -\tau_{ij} + \nu \left( \partial_x \partial_x d_{ij} + \partial_x' \partial_x' d_{ij} \right).$$  \hspace{1cm} (14)

Use of (7) in (14), and use of the identity $\partial_x \partial_x (fg) = f \partial_x \partial_x g + g \partial_x \partial_x f + 2 (\partial_x f) (\partial_x g)$ to simplify the terms proportional to $\nu$ gives

$$\partial_t d_{ij} + \partial_x \partial_x F_{ij} + \partial_x' d_{ij} = -\tau_{ij} + 2\nu \left( \partial_x \partial_x d_{ij} + \frac{1}{4} \partial_x \partial_x \partial_x d_{ij} - \epsilon_{ij} \right).$$  \hspace{1cm} (15)

As a check, one sees that (12) is the same as can be obtained by specializing, for the present case, equation (2.13) in Hill (2001). The trace of (13) and substitution of (11) and (12) give

$$\partial_t d_{ii} + \partial_x \partial_x F_{ii} + \partial_x' d_{ii} = 2\nu \partial_x \partial_x d_{ii} - 2(\epsilon + \epsilon') + w,$$

where

$$w \equiv -2\partial_x [(p - p') (u_n - u_n')] + \frac{\nu}{2} \partial_x \partial_x d_{ii} - 2\nu \partial_x \partial_x (p + p').$$  \hspace{1cm} (17)

The limit $r \to 0$ applied to (16) recovers the definition of $\epsilon$ in (11). It is significant that $\epsilon$ appears in the unaveraged exact equation (16) because $\epsilon$ will appear in the average of (13) only for the locally isotropic case.

2.2 Exact Incompressibility Relationships

Because $\mathbf{x}$ and $\mathbf{x}'$ are independent variables, $\partial_x u_i' = 0$, and $\partial_x u_j' = 0$. Then, incompressibility gives: $\partial_x u_n = 0$, $\partial_x u_n' = 0$, $\partial_x u_n = 0$, $\partial_x (u_n - u_n') = 0$, $\partial_x (u_n - u_n') = 0$. The combined use of incompressibility and (8) gives

$$\partial_x [(u_j - u_i') (u_n - u_n')] = \partial_x [(u_j + u_i') (u_n - u_n')] / 2,$$

$$\partial_{ij} \partial_x [(u_j - u_i') (u_n - u_n')] = \partial_{ij} \partial_x [(u_j + u_i') (u_n - u_n')] / 4.$$  \hspace{1cm} (18, 19)

3. EXACT AVERAGED EQUATIONS

3.1 Exact Equations: Ensemble Average

The ensemble is defined as a set of similar flows. An example is a set of mechanically identical wind tunnels operated with the same forcing. Points $\mathbf{x}$ and $\mathbf{x}'$ are defined in each flow relative to the mechanical structures or relative to the corresponding locations where the flow is (or was) forced. Time $t$ is defined for each flow from the start of the forcing. Thus, the space-time points $(\mathbf{x}, \mathbf{x}', t)$, or equivalently $(\mathbf{X}, \mathbf{r}, t)$, are in complete correspondence between flows in the ensemble. The ensemble average is defined at each point $(\mathbf{X}, \mathbf{r}, t)$ as the arithmetical average over the ensemble. We denote the ensemble average by angle brackets $\langle \cdot \rangle_E$, where the subscript $E$ is a mnemonic for ‘ensemble.’ Define the following statistics:
\[ D_{ij} (\mathbf{X}, \mathbf{r}, t) \equiv \langle d_{ij} \rangle_E, \quad D_{ijn} (\mathbf{X}, \mathbf{r}, t) \equiv \langle d_{ijn} \rangle_E, \quad T_{ij} (\mathbf{X}, \mathbf{r}, t) \equiv \langle \tau_{ij} \rangle_E, \]
\[ E_{ij} (\mathbf{X}, \mathbf{r}, t) \equiv \langle e_{ij} \rangle_E, \quad W (\mathbf{X}, \mathbf{r}, t) \equiv \langle w \rangle_E, \quad F_{ijn} (\mathbf{X}, \mathbf{r}, t) \equiv \langle f_{ijn} \rangle_E. \] (20)

The argument list \((\mathbf{X}, \mathbf{r}, t)\) is shown above to emphasize that the average applies to the general case of nonstationary, inhomogeneous turbulence, and that the ensemble average does not eliminate dependence on any independent variable. The argument list \((\mathbf{X}, \mathbf{r}, t)\) is suppressed where clarity does not suffer. Defining the symbols \(D_{ij}, D_{ijn}, T_{ij}, E_{ij}, W,\) and \(F_{ijn}\) causes brief notation in later sections. Because the ensemble average is a summation, it commutes with differential operators, and the average of (13) is therefore
\[ \partial_t D_{ij} + \partial_{X_n} F_{ijn} + \partial_{r_n} D_{ijn} = -T_{ij} + 2\nu \left[ \partial_{r_n} \partial_{r_n} D_{ij} + \frac{1}{4} \partial_{X_n} \partial_{X_n} D_{ij} - E_{ij} \right]. \] (21)

The average of (16) is
\[ \partial_t D_{ii} + \partial_{X_n} F_{iin} + \partial_{r_n} D_{iin} = 2\nu \partial_{r_n} \partial_{r_n} D_{ii} - 2 \langle \varepsilon + \varepsilon' \rangle_E + W, \] (22)
where
\[ W \equiv -2\partial_{X_n} \langle (p - p') (u_n - u'_n) \rangle_E + \frac{\nu}{2} \partial_{X_n} \partial_{X_n} D_{ii} - 2\nu \partial_{X_n} \partial_{X_n} \langle p + p' \rangle_E. \] (23)

Exact incompressibility conditions on the second-order velocity structure function are given by the average of (13) and (15) as
\[ \partial_{r_n} D_{jn} = \partial_{X_n} \langle (u_j + u'_j) (u_n - u'_n) \rangle_E / 2, \] (24)
\[ \partial_{r_j} \partial_{r_n} D_{jn} = \partial_{X_j} \partial_{X_n} \langle (u_j + u'_j) (u_n + u'_n) \rangle_E / 4. \] (25)

### 3.2 Exact Equations: Temporal Average

The ensemble average used above is important because it allows us to simultaneously investigate rapid temporal variation that a temporal average would smooth and to investigate sharp spatial variation that a spatial average would smooth. It is important to consider temporal and spatial averages because they are typical of experiments and DNS, respectively. Of course, an ensemble average can be approximated by widely separated temporal or spatial sampling for stationary or homogeneous turbulence, respectively. However, nearly continuous sampling is typical. Thus, we represent the temporal and spatial averages by integrals, but all results are valid for the sum of discrete points as well. The temporal average is most meaningful when the turbulence is nearly stationary, and the spatial average is most meaningful for nearly homogeneous turbulence.

Let \(t_0\) be the start time of the temporal average of duration \(T\). The operator effecting the temporal average of any quantity \(Q\) is denoted by \(\langle \cdot \rangle_T\), which has argument list \((\mathbf{X}, \mathbf{r}, t_0, T)\); that is,
\[ \langle Q \rangle_T \equiv \frac{1}{T} \int_{t_0}^{t_0 + T} Q (\mathbf{X}, \mathbf{r}, t) \, dt. \] (26)

For brevity the argument list \((\mathbf{X}, \mathbf{r}, t_0, T)\) is suppressed where clarity does not suffer. The temporal average of (15) is
\[ \langle \partial_t d_{ij} \rangle_T + \partial_{X_n} \langle F_{ijn} \rangle_T + \partial_{r_n} \langle d_{ijn} \rangle_T = -\langle \tau_{ij} \rangle_T 
+ 2\nu \left( \partial_{r_n} \partial_{r_n} \langle d_{ij} \rangle_T + \frac{1}{4} \partial_{X_n} \partial_{X_n} \langle d_{ij} \rangle_T - \langle e_{ij} \rangle_T \right). \] (27)

The temporal average of (16) is
\[ \langle \partial_t d_{ii} \rangle_T + \partial_{X_n} \langle F_{iin} \rangle_T + \partial_{r_n} \langle d_{iin} \rangle_T = 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ii} \rangle_T - 2 \langle \varepsilon + \varepsilon' \rangle_T + \langle w \rangle_T, \] (28)
where
\[ \langle w \rangle_T = -2\partial_{X_i} \langle (p - p') (u_i - u'_i) \rangle_T + \frac{\nu}{2} \partial_{X_n} \partial_{X_n} \langle d_{ij} \rangle_T - 2\nu \partial_{X_n} \partial_{X_n} \langle p + p' \rangle_T. \]
Now, (27) and (28) are exact because they are derived from (13) without approximations. They differ in form from (21) and (22) only in that the time derivative does not commute with the temporal average. Thus, (27) contains \( \langle \partial_t d_{ij} \rangle_T \), whereas (13) contains \( \partial_t D_{ij} \equiv \partial_t (d_{ij})_E \).

Because the data are taken in the rest frame of the anemometers and \( \partial_t \) is the time derivative for that reference frame, it follows that

\[
\langle \partial_t d_{ij} \rangle_T = \frac{1}{T} \int_{t_0}^{t_0+T} \partial_t d_{ij} dt = \left[ d_{ij} (X, r, t_0 + T) - d_{ij} (X, r, t_0) \right] / T.
\]

(29)

This shows that it is easy to evaluate \( \langle \partial_t d_{ij} \rangle_T \) using experimental data because only the first (at \( t = t_0 \)) and last (at \( t = t_0 + T \)) data in the time series are used. If \( [d_{ij}(X, r, t_0 + T) - d_{ij}(X, r, t_0)] \) is bounded and its ensemble mean varies less rapidly than \( T \), then we can make \( \langle \partial_t d_{ij} \rangle_T \) as small as we like by allowing \( T \) to be very large.

### 3.3 Exact Equations: Spatial Average

Let the spatial average be over a region \( \mathbb{R} \) in \( X \)-space. The spatial average of any quantity \( Q \) is denoted by \( \langle Q \rangle_{\mathbb{R}}(r, t, \mathbb{R}) \), and is defined by

\[
\langle Q \rangle_{\mathbb{R}} = \frac{1}{V} \int \int \int_{\mathbb{R}} Q(X, r, t) dX,
\]

(30)

where \( V \) is the volume of the space region \( \mathbb{R} \). For brevity, the argument list \( (r, t, \mathbb{R}) \) is suppressed where clarity does not suffer. The spatial average commutes with \( r \) and \( t \) differential and integral operations and with ensemble, time, and \( r \)-space averages. For the divergence in \( X \) of a vector \( q_n \), the divergence theorem relates the volume average to the surface average; that is,

\[
\langle \partial_{X_n} q_n \rangle_{\mathbb{R}} = \frac{1}{V} \int \int \int \partial_{X_n} q_n dX = \frac{S}{V} \left( \frac{1}{S} \int \int S_n q_n dS \right) \equiv \int_{X_n} q_n,
\]

(31)

where \( S \) is the surface area bounding the \( X \)-space region \( \mathbb{R} \), \( dS \) is the differential of surface area, and \( S_n \) is the unit vector oriented outward and normal to the surface. For brevity, the notation \( \int_{X_n} q_n \) is used for the \( X \)-space surface average in (31).

The spatial average of (15) is

\[
\partial_t \langle d_{ij} \rangle_{\mathbb{R}} + \frac{S}{V} \int_{X_n} F_{ij} + \partial_{r_n} \langle d_{ij} \rangle_{\mathbb{R}} = -\langle \tau_{ij} \rangle_{\mathbb{R}} + 2\nu \left( \partial_{r_n} \partial_{r_n} \langle d_{ij} \rangle_{\mathbb{R}} + \frac{1}{4} \frac{S}{V} \int_{X_n} \partial_{X_n} d_{ij} - \langle \epsilon_{ij} \rangle_{\mathbb{R}} \right).
\]

(32)

The spatial average of (16) is

\[
\partial_t \langle d_{ii} \rangle_{\mathbb{R}} + \frac{S}{V} \int_{X_n} F_{iin} + \partial_{r_n} \langle d_{iin} \rangle_{\mathbb{R}} = 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ii} \rangle_{\mathbb{R}} - 2 \langle \varepsilon + \varepsilon' \rangle_{\mathbb{R}} + \langle w \rangle_{\mathbb{R}},
\]

(33)

where \( \langle w \rangle_{\mathbb{R}} \equiv \frac{S}{V} \int_{X_n} \left[ -2 (p - p') (u_n - u'_n) + \frac{\nu}{2} \partial_{X_n} d_{ij} - 2\nu \partial_{X_n} (p + p') \right]. \)

The spatial average of the exact incompressibility condition (18) is

\[
\partial_{r_n} \langle d_{jn} \rangle_{\mathbb{R}} = \frac{S}{2V} \int_{X_n} (u_n - u'_n) (u_j + u'_j),
\]

(34)

which is, on the right-hand side, a surface flux of a quantity that depends on large-scale structures in the flow. Similarly, (19) gives

\[
\partial_{r_n} \langle d_{jn} \rangle_{\mathbb{R}} = \frac{S}{4V} \int_{X_n} [\partial_{X_j} (u_n + u'_n)] (u_j + u'_j).
\]

Of course, (32) and (34) are exact.
3.4 Spatial Average: DNS with Periodic Boundary Conditions

The spatial average is particularly relevant to DNS. DNS that is used to investigate turbulence at small scales often has periodic boundary conditions. For such DNS, consider the spatial average over the entire DNS domain. Contributions to $f_X q_n$ from opposite sides of the averaging volume cancel for that case such that $f_X q_n = 0$ and therefore $(\partial_X q_n)\rangle = 0$. In [12] we then have $f_X F_{ini} = 0$ and $f_X \partial_X d_{ij} = 0$. In [13] we have $f_X F_{ini} = 0$ and $\langle w\rangle_R = 0$. In [14], the right-hand side vanishes. Thus, in the important DNS case described above, we have the significant simplification that

$$\partial_t \langle d_{ij}\rangle_R + \partial_{r_n} \langle d_{ijn}\rangle_R = -\langle \tau_{ij}\rangle_R + 2\nu \left( \partial_{r_n} \partial_{r_n} \langle d_{ij}\rangle_R - \langle e_{ij}\rangle_R \right),$$  

(35)

$$\partial_t \langle di\rangle_R + \partial_{r_n} \langle d_{ijn}\rangle_R = 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ij}\rangle_R - 2 \langle \varepsilon + \varepsilon'\rangle_R,$$

(36)

and

$$\partial_{r_n} \langle d_{jn}\rangle_R = 0.$$  

(37)

Proof of $\partial_r \langle e_{ij}\rangle_R = 0$ follows: Using [8] in [4] we have $\partial_r e_{ij} = \partial_X \xi_{ij}$, where $\xi_{ij} \equiv \nu \left( \left[ (\partial_{r_n} u_i) (\partial_{r_n} u_j) - (\partial_{r_n} u'_i) (\partial_{r_n} u'_j) \right] \right)$ such that $\partial_r \langle e_{ij}\rangle_R = \langle \partial_r e_{ij}\rangle_R = \langle \partial_X \xi_{ij}\rangle_R = \frac{\partial}{\partial r} f_X \xi_{ij}$; this surface integral vanishes because of the DNS periodic boundary conditions and the selected averaging volume. Thus,

$$\partial_{r_n} \langle e_{jn}\rangle_R = 0.$$  

(38)

No approximations have been used to obtain these equations for the DNS case considered. It seems that [13]- [14] offer an ideal opportunity to evaluate the contribution of the time-derivative term $\partial_t \langle d_{ij}\rangle_R$ for freely decaying turbulence, as well as the contribution of the pressure term $\langle \tau_{ij}\rangle_R$ for anisotropic turbulence, as well as the balance of the off-diagonal components of [15].

Because we have not introduced a force generating the turbulence and because every point in the flow enters into the X-space average, the DNS must be freely decaying. As shown in Appendix A, it is straightforward to include forces in our equations.

Performing the r-space divergence of [13] and using [7]-[28], we have

$$\partial_r \partial_{r_n} \langle d_{ijn}\rangle_R = -\partial_{r_n} \langle \tau_{ij}\rangle_R.$$  

(39)

This exact result is analogous to the asymptotic result in Table 3 of Hill (1997).

We can further simplify the dissipation-rate term in [14]. Using Taylor’s series, we have $\varepsilon (x, t) = \varepsilon (X, t) + 1 \frac{r_n}{2} \partial_X X_n \varepsilon (X, t) + \ldots$. Clearly, a great number of terms will be needed when $|x - X|$ is outside of the viscous range, but the differentiability of hydrodynamic fields guarantees convergence of the Taylor series. The series for $\varepsilon (x', t)$ is the same as for $\varepsilon (x, t)$ in which $r/2$ is replaced by $-r/2$, such that

$$\varepsilon + \varepsilon' = 2\varepsilon (X, t) + 1 \frac{r_n}{4} \partial_X X_n \varepsilon (X, t) + \ldots = 2\varepsilon (X, t) + \partial_X \left[ 1 \frac{r_n}{4} \partial_X \varepsilon (X, t) + \ldots \right].$$  

(40)

Only terms having even-order derivatives appear in [14]. The right-most term in [14] has the form $\partial_X u_n$, and therefore vanishes when averaged in the X-space over the entire DNS domain for the periodic DNS case considered. Substituting [11] in [15] gives the term

$$-2 \langle \varepsilon + \varepsilon'\rangle_R = -4 \langle \varepsilon (X, t)\rangle_R = -4 \langle \varepsilon\rangle_R (t).$$  

(41)

The same method applied to the right-most term in [13] gives

$$\langle e_{ij}\rangle_R = -4\nu \langle \left[ (\partial_{x_n} u_i) \right] \rangle_{x=X} \langle e (X, t)\rangle_R (t),$$  

(42)

where the subscript $x = X$ means that the derivatives are evaluated at the point $X$. Of course, none of the quantities in [13]-[14] depends on $X$ because of the spatial average over $X$. An interesting feature of [11] and [14] is that their right-hand sides clearly do not depend on $r$, whereas this is not obvious in [13] and [15]. The only dependence of [11] and [14] is on $t$. Thus, (t) on the right-hand side of [11]-[14] is the entire argument list.

Of course, these results follow from the periodic boundary conditions and the fact that the averaging volume is over the whole periodic structure of the DNS domain. These results follow from the symmetry of that case.
4. AVERAGES OVER THE r-SPACE SPHERE

4.1 Definition of the r-Space Sphere Average and the Orientation Average

The energy dissipation rate averaged over a sphere in r-space has been a recurrent theme in small-scale similarity theories since its introduction by Obukhov (1962) and Kolmogorov (1962). By averaging our equations for the trace, we can, for the first time, produce an exact dynamical equation containing the sphere-averaged energy dissipation rate. The volume average over an r-space sphere of radius \( r_S \) of a quantity \( Q \) is denoted by

\[
\langle Q \rangle_{r\text{-sphere}} \equiv \left( 4\pi (r_S)^3 / 3 \right)^{-1} \int \int Q(X, r, t) \, dr.
\]

The orientation average over the surface of the r-space sphere of radius \( r_S \) of a vector \( q_n(X, r, t) \) is denoted as follows:

\[
\oint_{r_n} q_n \equiv \left( 4\pi r_S^2 \right)^{-1} \int \int r_n \, q_n(X, r, t) \, ds,
\]

where \( ds \) is the differential of surface area, and \( r_n/r \) is the unit vector oriented outward and normal to the surface of the r-space sphere. Both \( \langle Q \rangle_{r\text{-sphere}} \) and \( \oint_{r_n} q_n \) are functions of \( X, r_S, \) and \( t \), but the argument list \( (X, r_S, t) \) is suppressed. In this notation, the divergence theorem is

\[
\langle \partial_n q_n \rangle_{r\text{-sphere}} = (3/r_S) \oint_{r_n} q_n.
\]

Because \( r, X, \) and \( t \) are independent variables, the r-space volume and orientation averages commute with time and X-space averages and with \( X \)- and \( t \)-differential operators, and, of course, with the ensemble, temporal, and spatial averages as well. For instance, \( \langle \partial_t \langle d_{ii} \rangle_R \rangle_{r\text{-sphere}} = \partial_t \langle \langle d_{ii} \rangle_R \rangle_{r\text{-sphere}} = \langle \langle \partial_t d_{ii} \rangle_{r\text{-sphere}} \rangle_R = \partial_t \langle \langle d_{ii} \rangle_{r\text{-sphere}} \rangle_R \), etc.

4.2 Example of an Equation Operated upon by the r-Space Sphere Average

The r-space average \( \langle Q \rangle_{r\text{-sphere}} \) can operate on the structure-function equations \([24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44] \): indeed, it can operate on the unaveraged equations \([13], [14] \) as well. These equations have terms of the form \( \partial_n q_n \); examples are: \( q_n = \langle d_{ij} \rangle_R, \partial_n \langle d_{ij} \rangle_R, D_{ini}, \langle d_{ij} \rangle_T, \partial_n \langle d_{ij} \rangle_T \), etc. By means of \([34] \), the volume average in r-space of any term of the form \( \partial_n q_n \) produces the orientation average of \( q_n \) within the subject equation. After operating on \([22] \) with the volume average in r-space \([13] \), the right-most term in that equation contains \( \langle \langle \varepsilon + \varepsilon' \rangle_{r\text{-sphere}} \rangle_E \), which is the same as the sphere-averaged energy dissipation rate defined in the third equations of both Obukhov (1962) and Kolmogorov (1962) (after multiplication by 2).

The result of the r-space sphere average of any of our equations will be clear from operating on the simplest equation, namely, \([40] \) for the case of periodic DNS. The average of \([30] \) over a sphere in r-space of radius \( r_S \) and multiplication by \( r_S/3 \) and use of \([41] \) gives

\[
\frac{r_S}{3} \partial_t \langle \langle d_{ii} \rangle_{r\text{-sphere}} \rangle_R + \oint_{r_n} \langle \langle d_{ii} \rangle_{r\text{-sphere}} \rangle_R = 2\nu \oint_{r_n} \partial_n \langle \langle d_{ii} \rangle_{r\text{-sphere}} \rangle_R - \frac{4r_S}{3} \langle \langle \varepsilon \rangle_{r\text{-sphere}} \rangle_R.
\]

The terms have argument list \( (r_S, t) \), but \( \langle \langle \varepsilon \rangle_{r\text{-sphere}} \rangle_R \) depends only on \( t \). Of course, none of the quantities in \([40] \) depends on \( X \) because of the X-space average. Despite its simplicity, \([46] \) has been obtained without approximations for the freely decaying DNS case considered; \([46] \) applies to inhomogeneous and anisotropic DNS having periodic boundary conditions.
4.3 Kolmogorov’s Equation Derived from the Sphere-Averaged Equation

Most readers are familiar with Kolmogorov’s (1941a) famous equation that is valid for locally isotropic turbulence. A useful point of reference is to derive it from (46). An index 1 denotes projection in the direction of \( \mathbf{r} \) and indices 2 and 3 denote orthogonal directions perpendicular to \( \mathbf{r} \). For locally isotropic turbulence we recall that the only nonzero components of \( \langle d_{ij;n} \rangle \) are \( \langle d_{11} \rangle_{\mathbf{R}}, \langle d_{22} \rangle_{\mathbf{R}} = \langle d_{33} \rangle_{\mathbf{R}}, \) and of \( \langle d_{ij} \rangle_{\mathbf{R}} \) are \( \langle d_{11} \rangle_{\mathbf{R}}, \) and \( \langle d_{22} \rangle_{\mathbf{R}} = \langle d_{33} \rangle_{\mathbf{R}} \). These components depend only on \( r \) such that there is no distinction in an \( \mathbf{r} \)-space sphere average between \( r_S \) and \( r \); thus, we simplify the notation by replacing \( r_S \) with \( r \). The isotropic-tensor formula for \( \langle d_{ij;n} \rangle \) gives \( \langle d_{11} \rangle_{\mathbf{R}} = (r_n/r) \langle (d_{11})_{\mathbf{R}} \rangle_{\mathbf{R}} + 2 \langle d_{22} \rangle_{\mathbf{R}} = (r_n/r) \langle (d_{11})_{\mathbf{R}} \rangle_{\mathbf{R}}, \) substitution of which into (44) gives \( f_{r_n} \langle d_{11} \rangle_{\mathbf{R}} = (r_n/r) \langle (d_{11})_{\mathbf{R}} \rangle_{\mathbf{R}} = (r_n/r) (r_n/r) \langle (d_{11})_{\mathbf{R}} \rangle_{\mathbf{R}} = \langle d_{11} \rangle_{\mathbf{R}} \). Since \( \langle \partial_r d_{11} \rangle = (r_n/r) \), we have \( f_{r_n} \partial_r \langle d_{11} \rangle_{\mathbf{R}} = (r_n/r) \partial_r \langle d_{11} \rangle_{\mathbf{R}} = \partial_r \langle d_{11} \rangle_{\mathbf{R}} \). For locally stationary turbulence, which is the case considered by Kolmogorov (1941a), the time-derivative term in (46) is neglected; then (46) becomes

\[
\langle d_{11} \rangle_{\mathbf{R}} = 2 \nu \partial_r \langle d_{11} \rangle_{\mathbf{R}} - \frac{4}{3} \langle \varepsilon \rangle_{\mathbf{R}} r. \tag{47}
\]

Alternatively, we can time average (46); then the time derivative can be neglected with the weaker conditions noted with respect to the smallness of (29); then

\[
\langle \langle d_{11} \rangle_{\mathbf{R}} \rangle_T = 2 \nu \partial_r \langle \langle d_{11} \rangle_{\mathbf{R}} \rangle_T - \frac{4}{3} \langle \langle \varepsilon \rangle \rangle_{\mathbf{R}} T r. \tag{48}
\]

For simplicity of notation, continue with (17). To eliminate \( \langle d_{22} \rangle_{\mathbf{R}} \) and \( \langle d_{221} \rangle_{\mathbf{R}} \) from the expressions \( \langle d_{11} \rangle_{\mathbf{R}} = \langle d_{11} \rangle_{\mathbf{R}} + 2 \langle d_{22} \rangle_{\mathbf{R}} \) and \( \langle d_{111} \rangle_{\mathbf{R}} = \langle d_{111} \rangle_{\mathbf{R}} + 2 \langle d_{221} \rangle_{\mathbf{R}} \), we use the incompressibility conditions \( \frac{\nu}{r} \partial_r \langle d_{11} \rangle_{\mathbf{R}} + \langle d_{111} \rangle_{\mathbf{R}} - \langle d_{22} \rangle_{\mathbf{R}} = 0, \) and \( r \partial_r \langle d_{111} \rangle_{\mathbf{R}} + \langle d_{111} \rangle_{\mathbf{R}} - 6 \langle d_{221} \rangle_{\mathbf{R}} = 0 \), which are valid for local isotropy (Hill, 1997). Then (17) becomes, after multiplying by \( 3r^3 \partial_r \langle d_{111} \rangle_{\mathbf{R}} = 6 \nu r^3 \partial_r \langle (r^3 \langle d_{111} \rangle_{\mathbf{R}} \rangle) - 4 \langle \varepsilon \rangle_{\mathbf{R}} r^4, \) which is then integrated from 0 to \( r \). After the term proportional to \( \nu \) is integrated by parts and the resultant equation is divided by \( r^4 \) we have Kolmogorov’s equation

\[
\langle d_{111} \rangle_{\mathbf{R}} = 6 \nu \partial_r \langle d_{111} \rangle_{\mathbf{R}} - \frac{4}{5} \langle \varepsilon \rangle_{\mathbf{R}} r. \tag{49}
\]

Two integrations over \( r \) were required to obtain the equivalent of (19) in section 6 of Hill (1997), whereas one integration over \( r \) was required here to obtain (19); the reason is that the \( \mathbf{r} \)-space sphere average replaced the first integration. Kolmogorov’s 4/5 law, \( \langle d_{111} \rangle_{\mathbf{R}} = -\frac{4}{5} \langle \varepsilon \rangle_{\mathbf{R}} r, \) for the inertial range immediately follows from (19). For the viscous range, \( \langle d_{111} \rangle_{\mathbf{R}} \) can be neglected in (19) such that the known relation \( \langle \varepsilon \rangle_{\mathbf{R}} = (15 \nu/2r) \partial_r \langle d_{111} \rangle_{\mathbf{R}} = 15 \nu \langle \langle \partial_r u_1 \rangle^2 \rangle_{\mathbf{R}} \) is obtained, where the viscous-range asymptotic formula \( \langle d_{111} \rangle_{\mathbf{R}} = \langle \langle \partial_r u_1 \rangle^2 \rangle_{\mathbf{R}} r^2 \) was used.

5. Reynolds Decomposition

The Reynolds decomposition separates any hydrodynamic variable into its mean value and fluctuation and is essential when considering hot-wire anemometer data. In the next section, the Reynolds decomposition is used to elucidate the meaning of \( \partial_x U_{ij;n} \), and in Sec. 7.4 to perform the scale analysis.

For the ensemble average, the Reynolds decomposition of \( u_i(x, t) \) is defined by

\[
u_i(x, t) = U_i(x, t) + \tilde{u}_i(x, t) , \text{ where } U_i(x, t) = \langle u_i(x, t) \rangle_{E} , \text{ and } \langle \tilde{u}_i(x, t) \rangle_{E} = 0, \tag{50}
\]

and similarly at the point \( x' \). For brevity, \( U_i' = U_i(x', t) \), and \( \tilde{u}_i' = \tilde{u}_i(x', t) \), etc. Using (7), the incompressibility condition gives

\[
\partial_x u_n = 0, \quad \partial_x U_n = 0, \quad \partial_x \tilde{u}_n = 0, \quad \partial_r u_n = 0, \quad \partial_r U_n = 0, \quad \partial_r \tilde{u}_n = 0, \tag{51}
\]

and similarly for \( u'_n, U'_n, \tilde{u}'_n \).

For the time average (29), the mean velocity is \( U_i(x, t_0, T) = \langle u_i(x, t) \rangle_T \); as in (29) this notation emphasizes that the mean depends on the start, \( t_0 \), and duration, \( T \), of the time average, as well as on \( x \). The Reynolds decomposition is \( u_i(x, t) = U_i(x, t_0, T) + \tilde{u}_i(x, t, t_0, T) \), such that \( \langle \tilde{u}_i(x, t, t_0, T) \rangle_T = 0 \). Clearly the fluctuation,
rate of change of $D$ like with spatial derivatives. velocity is $\langle u_i(x,t) \rangle_{\mathcal{R}}$ as in (34) this average depends on the centroid and shape of the averaging volume, but this dependence is not denoted explicitly. The Reynolds decomposition is $u_i(x,t) \equiv U_i(t,\mathcal{R}) + \tilde{u}_i(x,t,\mathcal{R})$, which gives $\langle \tilde{u}_i(x,t,\mathcal{R}) \rangle_{\mathcal{R}} = 0$. Clearly, (21) is valid for the space average.

For brevity, the arguments of mean quantities are not shown in the following.

6. MEANING OF THE TERM $\partial_X F_{ijn}$

The Reynolds decomposition (30) used in the second term of (21) (i.e., $\partial_X F_{ijn}$) combined with (31) gives

$$\partial_X F_{ijn} = \frac{U_n + U'_n}{2} \partial_X D_{ij} + \partial_X \left( \Delta_i \tilde{\Gamma}_{jn} + \Delta_j \tilde{\Gamma}_{in} + \tilde{\Gamma}_{ijn} \right)$$

where, for brevity, we define

$$\Delta_i \equiv (U_i - U'_i), \quad \tilde{\Gamma}_{in} \equiv \left\langle \left( \tilde{u}_i - \tilde{u}'_i \right) \frac{\tilde{u}_n + \tilde{u}'_n}{2} \right\rangle_E, \quad \tilde{\Gamma}_{ijn} \equiv \left\langle \left( \tilde{u}_i - \tilde{u}'_i \right) \left( \tilde{u}_j - \tilde{u}'_j \right) \frac{\tilde{u}_n + \tilde{u}'_n}{2} \right\rangle_E.$$  

Note that $\hat{o}$ means that the quantity is a fluctuation, e.g. $\hat{u}_i$, and that a statistic is calculated from fluctuations, e.g., $\tilde{\Gamma}_{in}$ and $\tilde{\Gamma}_{ijn}$. Also, $D_{ij}$ appears in (32), not $\tilde{D}_{ij}$.

Consider the first term in (32), namely $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$. If the mean flow is spatially uniform to the extent that $U_n$ and $U'_n$ are equal, then $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$ becomes the same expression that Lindborg (1999) [his Eq.(8)] deduced as an addition to Kolmogorov’s equation. His deduction was based on Galilean invariance applied to a uniform mean flow. The combination of (32) and (21) shows that both $\partial_i D_{ij}$ and $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$ must appear in the dynamical equation as was correctly deduced by Lindborg (1999) on the basis of mean-flow Galilean invariance, but replacing $\partial_i D_{ij}$ with $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$, as was done by Danaila et al. (1999 a,b) on the basis of Taylor’s hypothesis, does not preserve that invariance. Now, $\partial_X D_{ij}$ is a measure of inhomogeneity because $\partial_X D_{ij}$ is the rate of change of $D_{ij} (\mathbf{X}, r, t)$ with respect to where the average is performed. Thus, $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$ describes the effect of the fluid moving relative to the anemometers in a direction in which $\tilde{D}_{ij} (\mathbf{X}, r, t)$ is inhomogeneous. Lindborg (1999) quantifies the contribution of $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$ to Kolmogorov’s (1941a) equation (Sec. 4.3) for several experiments and thereby shows that the contribution can be significant.

Now, $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$ is well illustrated by the case of turbulent flow in a pipe or wind tunnel. Perform the $\mathbf{X}$-space spatial average (30) of $\frac{1}{2} (U_n + U'_n) \partial_X D_{ij}$ over a cylinder having sides parallel to the mean velocity and having ends perpendicular to the mean velocity. For simplicity, assume that the mean velocity is uniform over the ends of the cylinder so that $U'_n = U_n = |U| \hat{s}_n$ where $\hat{s}_n$ is a unit vector in the streamwise direction, which is the 1-axis. Use of the divergence theorem (31) gives

$$\frac{1}{V} \int \int \int \partial_X \left[ \frac{1}{2} (U_n + U'_n) D_{ij} \right] d\mathbf{X} = \frac{1}{\mathcal{L}} \int \int \tilde{N}_n \hat{s}_n |U| D_{ij} dS$$

$$= \frac{|U|}{\mathcal{L}} \left[ \oint_{\mathcal{X}_n} \hat{s}_n D_{ij} \text{downstream} - \oint_{\mathcal{X}_n} \hat{s}_n D_{ij} \text{upstream} \right] ,$$

where $\oint_{\mathcal{X}_n} \hat{s}_n D_{ij} \text{downstream}$ and $\oint_{\mathcal{X}_n} \hat{s}_n D_{ij} \text{upstream}$ are the surface averages over just the downstream and upstream ends of the cylinder, respectively, and $\mathcal{L}$, $A$, and $V = \mathcal{L} A$ are the length, area of the ends, and volume of the cylinder, respectively. Now, $(|U|/\mathcal{L})^{-1}$ is the mean time for the flow to pass from the upstream end of the cylinder to the downstream end. Thus, (34) is the rate of downstream decay of $D_{ij}$ averaged over the cylinder cross section.

Now consider the term $\partial_X \left( \Delta_i \tilde{\Gamma}_{jn} + \Delta_j \tilde{\Gamma}_{in} + \tilde{\Gamma}_{ijn} \right)$ in (32). From (33) this term is important if there is strong correlation between velocity difference and velocity sum. One such case is when at least one anemometer is at the edge of a jet and is therefore sometimes immersed in quiescent entrained fluid and sometimes in turbulently agitated fluid. More generally, the second term in (32) is important for the case of large-scale structures. This term describes a contribution caused by inhomogeneity in the direction transverse to the mean flow direction as well as in the streamwise direction. Thus, this term is expected to contribute for pipe and jet flows when anemometers
are separated transverse to the flow. Experimental and/or numerical evaluation of these terms is needed to quantify their contribution to (21) for particular flows.

On the other hand, the second term in (52), i.e., \( \partial_X \left( \Delta_i \hat{\Gamma}_{jn} + \Delta_j \hat{\Gamma}_{in} + \hat{\Gamma}_{ijn} \right) \), does not grow if \( \frac{1}{2} (U_n + U_n') \) increases, as does the first term, i.e., \( \frac{U_n + U_n'}{\partial_X D_{ij}} \). Therefore, for a flow in which large-scale structures are minimized, such as grid-generated turbulence, and for a large enough Reynolds number such that \( r \) can be much less than the integral scale, the second term in (52) is expected to be negligible because it is \( \partial_X \) operating on an average. For such a flow, one expects that the two-point sum, \( \langle \hat{u}_n + \hat{u}_n' \rangle \), has a weak statistical relationship to the difference, \( \langle \hat{u}_i - \hat{u}_i' \rangle \). The negligibility of the second term in (52) when (52) is substituted in (21) will be considered further in Sec. 7.4.

7. APPROXIMATE EQUATIONS PERTAINING TO EXPERIMENTS ON THE SMALL-SCALE STRUCTURE OF HIGH-REYNOLDS-NUMBER TURBULENCE

We are now in a position to investigate three closely related objectives that will be considered simultaneously. One objective is to study the simplification of (21) on the basis of data for the small-scale structure of high-Reynolds-number turbulence; another is to determine the approximations required for that simplification. The third objective is to obtain from (21) an equation that is closer to the measurement process of extracting a mean velocity from anemometry data. We use the ensemble-averaged equations because they retain both temporal and spatial variability. Here, we consider the approach toward local homogeneity. For this purpose, our equations that depend on the location of measurements, i.e., \( \mathbf{X} \), are needed. We also consider the approach toward local stationarity, so the dependence on \( t \) is needed. The restrictions required by local isotropy are not used, so dependence on the orientation of measurement, i.e., \( \mathbf{r}/r \), is retained. On the other hand, assumptions about the order of magnitude of some quantities require that local isotropy is not greatly violated. The data used for this investigation are given in Appendix B, which includes the empirically verified (Monin and Yaglom, 1975) formulas for the inertial and viscous ranges for components of \( D_{ij} \) and \( D_{ijn} \).

7.1 Structure Functions of Fluctuations

An experimenter usually extracts \( U_i \) from the anemometer’s signal, then calculates statistics from \( \hat{u}_i \), e.g., \( \hat{D}_{ij} \equiv \langle (\hat{u}_i - \hat{u}_i') (\hat{u}_j - \hat{u}_j') \rangle_E \). Similarly define \( \hat{D}_{ijn}, \hat{T}_{ij}, \) and \( \hat{E}_{ij} \) in terms of the fluctuations of velocity and pressure. However, \( \langle \hat{u}_n + \hat{u}_n' \rangle /2 \) in (8) cannot be replaced by \( \langle \hat{u}_n + \hat{u}_n' \rangle /2 \) without destroying the meaning of \( F_{ijn} \); that replacement would result in \( \hat{F}_{ijn} \) being defined as \( \hat{\Gamma}_{ijn} \) in (53). A reasonable choice for the symbol \( \hat{F}_{ijn} \) is

\[
\hat{F}_{ijn} \equiv \frac{U_n + U_n'}{2} \langle \hat{u}_i - \hat{u}_i' \rangle \langle \hat{u}_j - \hat{u}_j' \rangle \frac{1}{E} = \frac{U_n + U_n'}{2} \hat{D}_{ij},
\]

Now (21) is not exactly satisfied by substitution of \( \hat{D}_{ij}, \hat{F}_{ijn}, \hat{D}_{ijn}, \hat{T}_{ij}, \) and \( \hat{E}_{ij} \), in place of \( D_{ij}, F_{ijn}, D_{ijn}, T_{ij}, \) and \( E_{ij} \), nor does that substitution satisfy any equations derived from (21). Kolmogorov’s equation being one such equation (see Hill, 1997). Substitution of the Reynolds decomposition of \( D_{ij}, F_{ijn}, D_{ijn}, T_{ij}, \) and \( E_{ij} \) (e.g., \( D_{ij} = \Delta_i \Delta_j + \hat{D}_{ij} \), etc.) in (21) gives a complicated equation. Below, simpler approximate equations are derived by scale analysis and are summarized in Sec. 8.

7.2 Experimentally Evaluable Exact Incompressibility Conditions

Because the approximations \( \partial_r D_{in} \approx 0 \) and \( \partial_r \hat{D}_{in} \approx 0 \) have an essential role in many theories, experimental evaluation of these approximations is desirable. However, the expressions \( \partial_r D_{in} \) and \( \partial_r \hat{D}_{in} \) are nearly impossible to evaluate experimentally. Use of (8) and (22) gives exact expressions for them that can be more readily evaluated; namely,

\[
\partial_r \hat{D}_{in} = \langle \partial_r (\hat{u}_n - \hat{u}_n') (\hat{u}_n - \hat{u}_n') \rangle_E = \partial_X \langle \hat{u}_n (\hat{u}_n - \hat{u}_n') \rangle_E /2, \tag{55}
\]

which is similar to (24). For the temporal average, the right-most expression in (55) requires, at most, measurements at four positions of the statistic \( \langle \hat{u}_i + \hat{u}_i' \rangle (\hat{u}_n - \hat{u}_n') \rangle_T \). If, as in the case of grid-generated turbulence,
inhomogeneity is streamwise, then only two positions displaced in the streamwise direction suffice to determine \( \partial \delta_x \langle (\hat{u}_i + \hat{u}_i') (\hat{u}_i - \hat{u}_i') \rangle_E / 2 \). The Reynolds decomposition gives \( \partial_r_n D_{in} = \Delta_x \partial_r \Delta_i + \partial_r_n D_{in} \), which shows that evaluation of \( \partial_r_n D_{in} \) only requires mean velocity measurements at several positions in addition to the previous evaluation of \( \partial_r_n D_{in} \).

### 7.3 A Necessary Condition for Local Homogeneity

We must define several scaling parameters determined by the flow. The integral scale, as traditionally defined, is strictly applicable only to homogeneous turbulence; see, for example, Tennekes and Lumley (1972). Here, however, we are studying inhomogeneous turbulence. As an example of the difficulty of defining integral scales in general inhomogeneous turbulence, consider the horizontally homogeneous atmospheric surface during daytime convective conditions. It is difficult to imagine a useful integral scale defined using data obtained along a line from the ground to the upper reaches of the surface layer. However, the horizontal homogeneity and Taylor’s hypothesis allow integral scales to be defined for all three velocity components measured at a point. Using surface-layer data, Kaimal et al. (1976) show that the horizontal velocity components scale with the depth of the entire boundary layer; that depth can be 1 to 2 km. Unlike the horizontal velocity component, the vertical velocity variance obeys Monin-Obukhov similarity such that its integral scale is proportional to the height above ground (Kaimal et al., 1976). For our study of the approach toward local homogeneity, it is necessary to define the large scale as the smallest of the integral scales or of the distance to boundaries. From the example of the atmospheric surface layer, that scale is the height above ground. Denote this chosen length scale by \( L \) and call it the outer scale. This name distinguishes it from the integral scale, which might not exist as traditionally defined in terms of the integral of a velocity correlation function. It is useful to define a velocity scale \( v \) by

\[
v \equiv \left( \langle \varepsilon \rangle_E / L \right)^{1/3}.
\]

Monin and Yaglom (1975) and Tennekes and Lumley (1972) determine that \( v \) is an estimate of the root-mean-square velocity, and that the mean shear is not greater than \( v / L \). If this is not so for our chosen outer scale \( L \), then \( L \) can be adjusted to make it so. From studies of nearly homogeneous turbulence, the right-hand side of (56) is proportional to velocity variance and the proportionality constant is independent of Reynolds number at high enough Reynolds numbers (Sreenivasan, 1998; Pearson, Krogstad, and van de Water, 2002). The proportionality constant is of order unity and depends somewhat on the large-scale structure of the flow (Sreenivasan, 1998; Pearson, Krogstad, and van de Water, 2002).

We define the scale \( \ell \) by

\[
\ell \equiv 10 \eta;
\]

\( \ell \) is a scale typical of the energy dissipation range (Appendix B.1). Here, Kolmogorov’s microscale \( \eta \), which is a scale typical of the viscous range, is defined by

\[
\eta \equiv \left( v^3 / \langle \varepsilon \rangle_E \right)^{1/4}.
\]

If the data have an inertial range, then \( \ell \) is closely related to the \( r \) at which asymptotic formulas for the inertial and viscous ranges are equal; this is demonstrated in Appendix B.1.

The basic tenet of local homogeneity is that as \( r \) is reduced relative to \( L \), nonlinear randomization causes statistics of differences of basic hydrodynamic quantities to decrease their dependence on the large-scale flow structure. For \( r < \ell \) and as \( r \) is further reduced, the nonlinear randomization is increasingly opposed by the smoothing effect of viscosity. Therefore, \( \ell \ll L \) is a necessary condition for local homogeneity. For \( r \geq \ell \), \( r \ll L \) is the necessary condition. That is, local homogeneity applies to the asymptotic case:

\[
\text{if } r < \ell, \text{ then } \ell \ll L; \text{ if } r > \ell, \text{ then } r \ll L; \text{ i.e., } \max (r, \ell) \ll L,
\]

where \( L \) is the outer scale. We study the approach toward local homogeneity by using (58) in scale analyses. We do so in Sec. 7.4, and find that some predictions of local homogeneity (such as \( \partial_{r,n} F_{ijn} = 0 \) and \( \partial_{r,n} D_{jn} = 0 \)) do not follow solely on the basis of the necessary condition (58). Thus, (58) is not a sufficient condition for local homogeneity.

Suppose for the moment that the turbulence under investigation is sufficiently homogeneous that an integral scale \( L \) can be defined in terms of an integral of the velocity correlation function. The microscale Reynolds number (Tennekes and Lumley, 1972) \( R_\lambda \) is well known to be related to integral scale \( L \) and \( \eta \) by \( L / \eta \propto R_\lambda^{3/2} \) (Tennekes and Lumley, 1972). Then, (57) gives \( L / \ell \propto R_\lambda^{3/2} \). Now, \( \ell \ll L \) is a necessary condition in (58); so \( R_\lambda \gg 1 \) is a necessary condition for local homogeneity, but it is not a sufficient condition. In a general inhomogeneous turbulence case, we assume that this is also true when \( L \) is the outer scale.
7.4 Scale Analysis

This section uses the data given in equations (B1) to (B13) of Appendix B.2. Those equations are distinguished by the prefix B.

Now, we consider the scale analysis of (21). First, consider the Reynolds decomposition of $D_{ij}$. Denote the local shear at point $X$ by

$$G_{i,n} = \partial_{X_a} U_i(X,t).$$

On the basis of (58) that max $(r, l) \ll L$, we retain only the first two terms of the Taylor series of $U_i$ and $U'_j$ around point $X$ to obtain that $\Delta_i \simeq r p G_{i,p}$ where the $1$-axis is parallel to $r$. Therefore, $\partial_{r_a} \Delta_i \simeq \partial_{r_a} G_{i,p}$, which is also follows from (8). Recall that the velocity scale $v$ is defined such that $\langle \epsilon \rangle_E$ is of order $v^3/L$ and a component of mean shear, i.e., $G_{i,p}$ is at most of order $v/L$. The Reynolds decomposition of $D_{ij}$ gives

$$D_{ij} = \Delta_i \Delta_j + \hat{D}_{ij} \simeq r^2 G_{i,1} G_{j,1} + \hat{D}_{ij}. \quad (59)$$

Now $G_{i,1}$ might be zero; if not, it is no greater than of order $v/L$. Use of (B1) gives $r^2 G_{\alpha,1} G_{\alpha,1}/D_{\alpha \alpha} \sim (r/L)^{4/3}$ in the inertial range, and use of (B3) gives $r^2 G_{\alpha,1} G_{\alpha,1}/D_{\alpha \alpha} \sim (\ell/L)^{4/3}$ in the viscous range. Thus, on the basis of (8), (72) gives $D_{\alpha \alpha} \simeq \hat{D}_{\alpha \alpha}$. Therefore, (B1) and (B3) are used below for $\hat{D}_{\alpha \alpha}$ as well as for $D_{\alpha \alpha}$.

Consider the Reynolds decomposition of the term $\partial_{r_a} \partial_{r_a} D_{ijn}$ in (21). Use of (58) and the assumption (see Appendix B) that the off-diagonal elements of $D_{ij}$ are no greater than the $D_{\alpha \alpha}$, the second, third, and fourth terms in (60) introduce off-diagonal elements of $\hat{D}_{ij}$ into the diagonal elements of (21). Using (B1) and the assumption above (Appendix B) that the off-diagonal elements of $D_{ij}$ are no greater than the $D_{\alpha \alpha}$, we can show that $\partial_{r_a} \partial_{r_a} D_{ijn}$ can be replaced by $\partial_{r_a} \partial_{r_a} \hat{D}_{ijn}$.

The stronger conclusion that $\partial_{r_a} \partial_{r_a} D_{ijn} \simeq \partial_{r_a} \partial_{r_a} \hat{D}_{ijn}$ can be obtained as follows. For an inertial range, the above comparison of terms with $\xi$ in (52) is of order $\xi$ relative to $\hat{\xi}_E$. The first term in (60) can be approximated by $\xi^2 (G_{\alpha,1} G_{\alpha,1} + G_{\alpha,1} G_{\alpha,1})$, which is at most of order $\xi^2 L^3$; this is of order $(r/L)^2$ relative to $\xi_E$. Hence, when (60) is substituted in (21), the first term in (60) can be neglected relative to the diagonal element $E_{\alpha \alpha}$ on the basis of (58). The second, third, and fourth terms in (60) introduce off-diagonal elements of $\hat{D}_{ij}$ into the diagonal elements of (21). Using (B1) and the assumption above (Appendix B) that the off-diagonal elements of $D_{ij}$ are no greater than the $D_{\alpha \alpha}$, we can show that $\partial_{r_a} \partial_{r_a} D_{ijn}$ can be replaced by $\partial_{r_a} \partial_{r_a} \hat{D}_{ijn}$.

The stronger conclusion that $\partial_{r_a} \partial_{r_a} D_{ijn} \simeq \partial_{r_a} \partial_{r_a} \hat{D}_{ijn}$ can be obtained as follows. For an inertial range, the above comparison of terms with $E_{\alpha \alpha}$ is equivalent to comparison with $\hat{\xi}_E$, and we therefore also neglected on the basis of (58). The same procedure can be used for the fifth and sixth terms in (60). On the other hand, substitution of the definition (24) of $\hat{\xi}_E$ in (55) and use of (B13) gives $\partial_{r_a} \hat{D}_{ijn} = \partial_{X_a} \hat{\Gamma}_{ijn} \leq v^2/L$ such that the fifth and sixth terms in (60) are much less than $(r/L)^2 (v^3/L)$ and are therefore negligible compared with $\langle \xi \rangle_E$ on the basis of both (B13) and (58). Therefore, for the projection of (21) in an arbitrary direction $\hat{\xi}$, incompressibility, and our data imply that $\partial_{r_a} \partial_{r_a} D_{ijn}$ can be replaced by $\partial_{r_a} \partial_{r_a} \hat{D}_{ijn}$.

The significance of there being a projection in an arbitrary direction $\hat{\xi}$ within (51), is that empirical evidence is lacking for the off-diagonal components of $D_{ijn}$. We are now ready to consider in more detail the second term in (52), namely $\partial_{X_a} \Delta_i \hat{\Gamma}_{ijn} + \Delta_j \hat{\Gamma}_{ijn}$. It is assumed that our data are chosen to mitigate large-scale structures such that (B13) is true. One part of the second term in (52) is $\partial_{X_a} \Delta_i \hat{\Gamma}_{ijn} = \partial_{X_a} \Delta_i \hat{\Gamma}_{ijn} + \Delta_i \partial_{X_a} \hat{\Gamma}_{ijn}$. Now, $\partial_{X_a} \Delta_i \simeq \partial_{X_a} G_{i,1}$; this is at most of order $(r/L)(v/L)$. Therefore, the ratio $\partial_{X_a} \Delta_i \hat{\Gamma}_{ijn} / E_{\alpha \alpha}$ is at most of order $(r/L) \hat{\Gamma}_{\alpha \alpha}/v^2$, which is very small compared to unity on the basis of (58) and (B13). Similarly, $\Delta_{ij} \hat{\Gamma}_{ijn} / E_{\alpha \alpha}$ is of order $(r/L) \hat{\Gamma}_{\alpha \alpha}/v^2$. Another part of the second term in (52) is $\partial_{X_a} \hat{\Gamma}_{ijn}$. The ratio $\partial_{X_a} \hat{\Gamma}_{\alpha \alpha} / E_{\alpha \alpha}$ is at most of order $\hat{\Gamma}_{\alpha \alpha}/v^3$, which
is very small because of (B13). Therefore, the entire second term in (52) is negligible compared to $E_{aa}$, and therefore it is negligible in diagonal components of (21). Neglecting the second term in (52) and using incompressibility, in the diagonal components of (21) we have

$$\partial_{X_n} F_{aan} (X,r,t) \simeq \frac{1}{2} (U_n + U'_n) \partial_{X_n} D_{aa}. \tag{62}$$

The Reynolds decomposition of (52) is

$$\partial_{X_n} F_{aan} \simeq M_{aa} + \frac{1}{2} (U_n + U'_n) \partial_{X_n} \hat{D}_{aa}, \tag{63}$$

where $M_{aa} = \frac{1}{2} (U_n + U'_n) r^2 \partial_{X_n} (G_{a1}G_{a1}).$

There are clearly flows for which we expect that $M_{aa}$ is negligible; an example is freely decaying grid-generated turbulence in a wind tunnel for which $G_{a1} = 0$. On the other hand, $M_{aa}$ might not be negligible in all cases. Consider that $M_{aa}$ is at most of order $(|U|/\nu) (r/L)^2$ relative to $E_{aa}$. Although $(r/L)^2 \ll 1$ follows from (58), $|U|/\nu$ can be much larger than unity. Thus, $M_{aa}$ cannot be neglected relative to $E_{aa}$ on the basis of (58); the same is true for $\frac{1}{2} (U_n + U'_n) \partial_{X_n} \hat{D}_{aa}$ because it is also proportional to $|U|/\nu$. We assume that the mean flow does not have an abrupt change near the positions of the anemometers. Then, use of (31) and (33) shows that $r^2 G_{a1}G_{a1}$ is of order $(r/L)^{4/3}$ and $(\ell/L)^{4/3}$ relative to $\hat{D}_{aa}$ in the inertial and viscous ranges, respectively. However, it is not clear on this basis that we can neglect $M_{aa}$ relative to $\frac{1}{2} (U_n + U'_n) \partial_{X_n} \hat{D}_{aa}$ because what is needed in (33) is the streamwise rate of change, i.e., $(U_n + U'_n) \partial_{X_n}$, operating on both $r^2 G_{a1}G_{a1}$ and $\hat{D}_{aa}$. Consequently, we will not further simplify (52).

Now consider the term $\partial_t D_{ij}$ in (21). Recall that the positions of the anemometers, namely $x$ and $x'$, are held fixed for the time-derivative operation $\partial_t$. Thus, the meaning of $\partial_t D_{ij}$ is the time rate of change of $D_{ij}$ in the anemometer’s rest frame. The sum of $\partial_t D_{aa}$ and $\frac{1}{2} (U_n + U'_n) \partial_{X_n} D_{aa}$ [see (52)] is the time rate of change of $\hat{D}_{aa}$ in the reference frame moving with velocity $(U + U')/2$; that is, moving with the fluid in the sense of moving with the local and momentary ensemble-averaged velocity. Now (61) is exact and therefore describes cases that include rapid changes of mean conditions in the rest frame of the anemometers. However, assume that the experimenter has chosen a case for which mean conditions are nearly constant in the anemometer’s rest frame; examples include fixed anemometer positions in a wind tunnel, pipe, or jet for constant mean flow, or freely decaying DNS. From the Reynolds decomposition (59) we have

$$\partial_t D_{aa} = r^2 \partial_t (G_{a1}G_{a1}) + \partial_t \hat{D}_{aa}. \tag{64}$$

For example, consider the case of turbulence that is freely decaying in the anemometer’s rest frame, or freely decaying DNS. In this case, $r^2 \partial_t (G_{a1}G_{a1})$ is at most of order $(r/L)^2$ relative to $E_{aa}$, whereas for the inertial and viscous ranges $\partial_t \hat{D}_{aa}$ is at most of orders $(r/L)^{2/3}$ and $(\ell/L)^{2/3}$ relative to $E_{aa}$. For this case, $\partial_t D_{aa}$ can be neglected in (64). More generally, $\partial_t D_{aa}$ is negligible because the experimenter chooses not to move the anemometers rapidly through positions where mean conditions differ greatly. Given the opposite choice, $\partial_t D_{aa}$ would not be negligible: it would be of order $(r/L)^2 (|V|/\nu)$ relative to $E_{aa}$, where $|V|$ is the speed of the anemometers relative to the large-scale inhomogeneous structures of the mean flow. Although $(r/L)^2$ is small compared with unity, $(|V|/\nu)$ can be made large by increasing the speed of the anemometers relative to the mean-flow structure. Thus, the term $\partial_t D_{aa}$ cannot be neglected from (21) solely on the basis of (58) for the same reason that applies to $\partial_{Xn} F_{ijn}$. We do neglect $\partial_t D_{aa}$ on the basis of the choice mentioned above.

Reconsider the term $M_{aa}$ in (63) together with $r^2 \partial_t (G_{a1}G_{a1})$, which appears in (64). Their sum, i.e.,

$$[\partial_t + \frac{1}{2} (U_n + U'_n) \partial_{X_n}] r^2 (G_{a1}G_{a1}),$$

is the temporal rate of change following the mean flow of $r^2 (G_{a1}G_{a1})$. This might not be negligible for some flows, such as a contraction in a wind tunnel or an expanding round jet, even though $\partial_t (G_{a1}G_{a1})$ might be zero. This helps illustrate that $M_{aa}$ might not be negligible.

Now consider the term proportional to $\nu$ in (21). The term $\frac{1}{2} \partial_{X_n} \partial_{X_n} D_{ij}$ is of order $(r/L)^2$ relative to $\partial_{r_n} \partial_{r_n} D_{ij}$, and is negligible. The Laplacian operating on (19) gives $\partial_{r_n} \partial_{r_n} D_{ij} \simeq 6 G_{ij1} G_{j1} + \partial_{r_n} \partial_{r_n} \hat{D}_{ij}$. Now, $2\nu (6 G_{ij1} G_{j1})$ is at most of order $2\nu (r/L)^2$, which is of orders $(r/L)^{4/3}$ and $(\ell/L)^{4/3}$ relative to (B3) and (B8), respectively. Therefore, (58) and (B3) and (B8) give $\hat{u}_i \hat{u}_j \partial_{r_n} \partial_{r_n} D_{ij} + \frac{1}{2} \partial_{X_n} \partial_{X_n} D_{ij} \simeq \hat{u}_i \hat{u}_j \partial_{r_n} \partial_{r_n} \hat{D}_{ij}$.

In the Reynolds decomposition of $E_{aa}$ the terms that depend on mean velocity are of the order of an inverse Reynolds number $(\nu L/\nu)^{-1} \ll 1$ relative to $\langle \varepsilon \rangle E$. Thus, (B3) gives $E_{aa} \simeq \tilde{E}_{aa}$. By the same method, use of the definition (14) of $\varepsilon$ gives $\langle \varepsilon \rangle E \simeq \langle \tilde{\varepsilon} \rangle E$. That is, the mean velocity produces negligible viscous dissipation.
In the average of (10), consider the term $2v \partial_X \partial_{X_n} (p + p')_E$, which also appears in (23). Excluding the case of nearby bodies in the flow that can cause sharp spatial variation of pressure, the mean pressure gradient scales with $v$ and $L$. Then, the term $2v \partial_X \partial_{X_n} (p + p')_E$ is of order $(vL/\nu)^{-1}$ relative to $\langle \varepsilon \rangle_E$, and is thus negligible. The Taylor series expansion (40) shows that $\langle \varepsilon + \varepsilon' \rangle_E \simeq 2 \langle \varepsilon \rangle_E$, where the neglected terms are at most of order $(r/L)^2$ relative to $\langle \varepsilon \rangle_E$ and are therefore negligible on the basis of (58). Then, the average of (10) gives the trace: $\langle \varepsilon \rangle_E \equiv E_{ii} \simeq 4 \langle \varepsilon \rangle_E \simeq 4 \langle \varepsilon \rangle_E$.

Finally, consider the Reynolds decomposition of $T_{ij}$. Denote the mean pressure gradient at point $X$ by $\Pi_n \equiv \partial_{X_n} \langle p(X,t) \rangle_E$. The Reynolds decomposition of the term $-2 \langle (p - p') (s_{ij} - s'_{ij}) \rangle$ in $T_{ij}$ [see (8)] gives a mean-gradients term that is approximated by $-r_q r_n \partial_{X_n} \langle G_{j,i} + G_{i,j} \rangle = -r^2 \Pi_1 \partial_{X_1} \langle G_{j,i} + G_{i,j} \rangle$. Recall that $\Pi_n$ scales with $v$ and $L$. Then, $-r^2 \Pi_1 \partial_{X_1} \langle G_{j,i} + G_{i,j} \rangle$ is of order $(r/L)^2$ relative to $E_{aa}$, such that this term is negligible in (21). Using (B11) and (B12) for the diagonal components of $-2 \langle (p - p') (s_{ij} - s'_{ij}) \rangle$, this term is seen to be negligible compared to $E_{aa}$ for $r$ within the inertial range through the viscous range. The Reynolds decomposition gives $\partial_{X_n} \langle (p - p') (u_j - u'_j) \rangle \simeq r_q r_n \partial_{X_n} \langle \Pi_1 G_{j,i} \rangle$. Since $\Pi_n$ scales with $v$ and $L$, the term $r_q r_n \partial_{X_n} \langle \Pi_1 G_{j,i} \rangle$ is at most of order $(r/L)^2$ relative to $E_{aa}$, and this term is therefore negligible in (21). On the basis of (B11) and the neglect of $-2 \langle \hat{p} - \hat{p}' \rangle \langle \hat{s}_{ij} - \hat{s}'_{ij} \rangle$, we also neglect $\partial_{X_n} \langle \hat{p} - \hat{p}' \rangle \langle \hat{u}_n - \hat{u}'_n \rangle$. Taken together, these approximations show that $T_{aa}$ is negligible in (21). On the other hand, mean pressure gradient can be large in the presence of bodies in the flow; a contraction of a wind tunnel is an example. Thus, like $\partial_{X_n} F_{ijn}$, terms containing the mean pressure gradient cannot be neglected on the basis of (58) alone. In effect, we have assumed that there are no bodies strongly affecting the local turbulent flow. For this case, $T_{aa}$ is negligible in the diagonal elements of (21).

The results of the above scale analysis are summarized in the following three sections.

8. APPROXIMATE EQUATIONS

8.1 Ensemble Average: Approximate Equations

Given the experimental case discussed above and quantified in Appendix B, the diagonal elements of (21) projected in arbitrary directions $\tilde{a}$ give the approximate equation

$$\tilde{a}_i \tilde{a}_j \left[ \frac{1}{2} (U_n + U'_n) \partial_{X_n} D_{ij} + \partial_{r_n} \tilde{D}_{ijn} = 2v \partial_{r_n} \partial_{r_n} \tilde{D}_{ij} - \tilde{E}_{ij} \right].$$

(65)

As examples, the direction $\tilde{a}$ can be chosen to be in the direction of some large-scale flow symmetry, such as streamwise or cross stream, etc., or in a direction defined by the separation of anemometers, such as $r$ or perpendicular to $r$. The appearance of $D_{ij}$, rather than $\tilde{D}_{ij}$, in the left-most term in (65) indicates that both terms in (53) are included. The trace of (21) becomes

$$\frac{1}{2} (U_n + U'_n) \partial_{X_n} D_{ii} + \partial_{r_n} \tilde{D}_{inn} = 2v \partial_{r_n} \partial_{r_n} \tilde{D}_{ii} - 4 \langle \varepsilon \rangle_E.$$

(66)

As shown above, derivation of (65) and (66) from the exact equation (21) requires more than just (58). A further requirement is that the experimenter avoids cases having large spatial and temporal variation of the mean flow. Of course, that choice improves the accuracy of local homogeneity for fixed values of $[\max (r, \ell)/L]$. Additional requirements are approximations (B113) and (B110), and that the inverse Reynolds number $(vL/\nu)^{-1}$ is very small. In general, those conditions are typical of an experimental situation that is sought for the study of the universality of turbulence statistics at small scales. Most experiments use Taylor’s hypothesis to estimate spatial statistics from temporal statistics, for which purpose $|U|/\nu$ must be large. For this reason, the left-most term is not neglected in (65), nor in (66).

Of course, (65) contains no information about the off-diagonal elements of (21). We cannot evaluate those off-diagonal elements because we lack the necessary data. Clearly, DNS or a very complete experiment (e.g., as in Su and Dahm, 1996) could be used to quantify those off-diagonal elements. The off-diagonal elements of (21) describe quantities that approach zero as local isotropy becomes accurate.
8.2 Temporal Average: Approximate Equations

Using \( \langle \partial_t d_{ij}, T \rangle \), we noted the case for which \( \langle \partial_t d_{ij}, T \rangle \) can be made as small as desired by use of a long averaging duration. This case is typical of experimental work for which the temporal average is also typical. Assume that this is the case such that in \( \langle \partial_t d_{ii}, T \rangle \) can be neglected, and, in the case of \( \langle \partial_t d_{ij}, T \rangle \) can be neglected. On the other hand, recall from \( \langle n \rangle \) that it is easy to evaluate \( \hat{a}_i \hat{a}_j \langle \partial_t d_{ij}, n \rangle \), from by use of experimental data. The Reynolds decomposition and the approximations that lead from \( \langle 21 \rangle \) and \( \langle 22 \rangle \) to \( \langle 65 \rangle \) and \( \langle 66 \rangle \) also apply to \( \langle 27 \rangle \) and \( \langle 28 \rangle \): we immediately obtain

\[
\hat{a}_i \hat{a}_j \left[ \frac{1}{2} (U_n + U'_n) \partial_XU_n \langle d_{ij}, T \rangle + \partial_{r_n} \left( \hat{d}_{ijn} \right)_T = 2 \nu \partial_{r_n} \partial_{r_n} \left( \hat{d}_{ij} \right)_T - \langle \hat{e}_{ij} \rangle_T \right],
\]

\[
\frac{1}{2} (U_n + U'_n) \partial_XU_n \langle d_{ii}, T \rangle + \partial_{r_n} \left( \hat{d}_{ii} \right)_T = 2 \nu \partial_{r_n} \partial_{r_n} \left( \hat{d}_{ii} \right)_T - 4 \langle \hat{e} \rangle_T,
\]

where, as before, the caret over the averaged quantity means that the quantity is calculated from fluctuations. These equations relate the statistics that experimenters (e.g., Antonia, Chambers, and Browne, 1983; Chambers and Antonia, 1984; Danaila et al., 1999 a,b) calculate from data. As shown in Sec. 5, the mean functions, i.e., \( U_n(x, t_0, T) \equiv \langle u_n(x, t) \rangle_T \), in the definition of the Reynolds decomposition \( \langle 50 \rangle \) are now time averages rather than ensemble averages such that \( \langle \hat{a}_i(x, t) \rangle_T = 0 \), etc. Except for replacing the ensemble average with the time average, \( \langle 27 \rangle \) and \( \langle 58 \rangle \) are the same as \( \langle 25 \rangle \) and \( \langle 65 \rangle \). However, the statistics in \( \langle 67 \rangle \) and \( \langle 68 \rangle \) can have dependence on \( t \), whereas the statistics in \( \langle 67 \rangle \) and \( \langle 68 \rangle \) depend on only the time of the start of the temporal average (i.e., \( t_0 \) and the duration of the average \( T \)), in addition to which the dependence on start time and duration must be slight because of the neglect of \( \langle \partial_t d_{ij}, T \rangle \).

8.3 Spatial Average: Approximate Equations

Now consider spatial averaging. Given the approximations that lead from \( \langle 21 \rangle \) and \( \langle 22 \rangle \) to \( \langle 65 \rangle \) and \( \langle 66 \rangle \), \( \langle 32 \rangle \) and \( \langle 33 \rangle \) become

\[
\hat{a}_i \hat{a}_j \left[ \partial_t \left( \hat{d}_{ij, r} \right)_R + \frac{S}{2V} \int_X(U_n + U'_n) d_{ij} + \partial_{r_n} \left( \hat{d}_{ijn, r} \right)_R = 2 \nu \partial_{r_n} \partial_{r_n} \left( \hat{d}_{ij, r} \right)_R - \langle \hat{e}_{ij, r} \rangle_R \right],
\]

\[
\partial_t \left( \hat{d}_{ii, r} \right)_R + \frac{S}{2V} \int_X(U_n + U'_n) d_{ii} + \partial_{r_n} \left( \hat{d}_{iin, r} \right)_R = 2 \nu \partial_{r_n} \partial_{r_n} \left( \hat{d}_{ii, r} \right)_R - 4 \langle \hat{e} \rangle_R.
\]

As shown in Sec. 5, the mean quantities, \( U_n(t, \langle \hat{r}_n(x, t) \rangle_R \), in the definition of the Reynolds decomposition \( \langle 34 \rangle \) are now space averages rather than ensemble averages such that \( \langle \hat{u}_i(x, t) \rangle_R = 0 \), etc. As in the previous case, the caret above a quantity designates that it is calculated from velocity fluctuations. The time-derivative terms \( \partial_t \left( \hat{d}_{ij, r} \right)_R \) and \( \partial_t \left( \hat{d}_{ij, r} \right)_R \) have been retained in \( \langle 33 \rangle \) and \( \langle 71 \rangle \) because they are more significant than the advective term for the case of freely decaying DNS. Another example is the forced DNS flow of Borue and Orszag (1996), because it exhibits temporal variation of total mean-squared vorticity by a factor of 2. It seems prudent to retain the time derivatives. For DNS data, the advective term in both \( \langle 32 \rangle \) and \( \langle 70 \rangle \) is seldom important. Consider the DNS flow of Borue and Orszag (1996), for which \( |U| / \nu \) was at most about 2. Then, on the basis of the scale analysis [see below \( \langle 63 \rangle \)], the advective term is negligible on the basis of \( \langle 58 \rangle \). In \( \langle 69 \rangle \) there is no information on the off-diagonal components because the approximations apply only to the diagonal components.

Also, \( \langle 34 \rangle \) and \( \langle 71 \rangle \) become

\[
\hat{a}_i \hat{a}_j \left[ \partial_t \left( \hat{d}_{ij, r} \right)_R + \partial_{r_n} \left( \hat{d}_{ijn, r} \right)_R = 2 \nu \partial_{r_n} \partial_{r_n} \left( \hat{d}_{ij, r} \right)_R - \langle \hat{e}_{ij, r} \rangle_R \right],
\]

\[
\partial_t \left( \hat{d}_{ii, r} \right)_R + \partial_{r_n} \left( \hat{d}_{iin, r} \right)_R = 2 \nu \partial_{r_n} \partial_{r_n} \left( \hat{d}_{ii, r} \right)_R - 4 \langle \hat{e} \rangle_R.
\]
9. DISCUSSION

Given data for which local homogeneity and/or local isotropy are approximate, it seems that \( D_{iin} \) is closer to that asymptotic case than is \( \hat{D}_{i11} \), and therefore, that data for the trace \( D_{iin} \) will more accurately show the asymptotic inertial-range power law than does \( D_{111} \). The reason is as follows. For the approach toward local isotropy in homogeneous turbulence, the anisotropy quantified by nonzero values of \( T_{ij} \) is balanced by that from the term \( \partial_r \), \( D_{ijn} \) in \( \hat{D}_{i11} \) (Hill, 1997). The trace of \( T_{ij} \) vanishes exactly for the homogeneous case because \( \partial_r \langle (p - p') (u_i - u'_i) \rangle_E = 0 \) for homogeneous turbulence and because \(-2 \langle (p - p') (s_{ii} - s'_{ii}) \rangle_E = 0 \) on the basis of incompressibility \( (s_{ii} = 0) \). Then, \( \partial_r D_{iin} \) must balance less anisotropy in \( \hat{D}_{i11} \) than does \( \partial_r D_{ijn} \) in \( \hat{D}_{i11} \). For inhomogeneous turbulence, the nonvanishing part of the trace, namely \( T_{ii} = 2 \partial_r \langle (p - p') (u_i - u'_i) \rangle_E \), is expected to approach zero rapidly as \( r \) decreases for two reasons. First, \( \langle (p - p') (u_i - u'_i) \rangle_E \) vanishes on the basis of local isotropy. Second, the operator \( \partial_r \), causes \( \partial_r \langle (p - p') (u_i - u'_i) \rangle_E \) to vanish on the basis of local homogeneity. The right-most two terms in \( \hat{D}_{i11} \) contain the operator \( \partial_X \partial_X \), which causes these terms in \( W \) to vanish rapidly on the basis of local homogeneity. Thus, all terms in \( W \) are negligible for locally homogeneous turbulence. By performing the trace it appears that anisotropy has been significantly reduced in \( \hat{D}_{i11} \) relative to \( D_{iin} \). It follows that the trace, \( \partial_r D_{iin} \), is affected less by anisotropy than is \( \partial_r D_{ijn} \), and therefore, that \( D_{iin} \) is less affected by anisotropy than is \( D_{ijn} \). This hypothesis should be checked by comparison with DNS. Evaluation of all terms in \( \hat{D}_{i11} \) and \( D_{iin} \) are the basis for such an investigation. We therefore expect that inertial-range power-law scaling would be more evident in \( D_{iin} \) than in \( D_{111} \). Of course, performing the trace requires that all three components of velocity be measured at both \( X \) and \( X' \).

To determine scaling properties of the third-order structure function, past theory has used the isotropic-tensor formula to produce a differential equation having the operator \( \partial_r \) and integration of that equation (as done in Sec. 4.3). However, one can use an equation like \( \hat{D}_{i11} \) without an assumption about the symmetry properties (e.g., isotropic) of the structure functions by means of the sphere average in \( r \)-space, as implemented in Sec. 4.2. Evaluating resultant terms in the \( r \)-space sphere-averaged equation implies a tedious experimental procedure if wire anemometers are used. On the other hand, both DNS and the experimental method of Su and Dahm (1996) are suited to such evaluation. In effect, the \( r \)-space sphere average solves the equation by producing the orientation-averaged third-order structure function. It would seem that the orientation average mitigates anisotropy effects. Thus, the orientation average of the trace of the third-order structure function, namely, \( \int d_r D_{i11} \), is expected to best exhibit properties of locally isotropic turbulence, such as the inertial-range power law with the 4/3 coefficient that appears in \( \hat{D}_{i11} \).

Lindborg (1999) estimates the contribution of \( \frac{1}{2} (U_n + U'_n) \partial_X D_{\alpha \beta} \) (for the case \( U_n = U'_n \)) to experimental measurements of \( \langle \hat{d}_{111} \rangle_T \) for grid, jet, and wake turbulence of moderate Reynolds number, and Danaila et al. (1999 a,b) do so for grid turbulence at \( R = 66, 99, \) and 448. They show that the term \( \frac{1}{2} (U_n + U'_n) \partial_X D_{\alpha \beta} \) accounts for much of the observed deviation of the data from Kolmogorov’s equation; Kolmogorov’s equation is \( \langle \hat{d}_{111} \rangle_T = 6\nu \frac{\partial_r \langle \hat{d}_{11} \rangle_T}{5} - \frac{4}{5} \langle \varepsilon \rangle_T r \). In the case of Danaila et al. (1999a), one must keep in mind that their estimation method reduces their equation to \( 2 \langle \hat{d}_{111} \rangle_T \) in the energy-containing range such that the balance of the equation is not tested in the energy-containing range.

10. CONCLUSION

The mathematical method of deriving exact structure-function equations from the Navier-Stokes equation is developed. The basic tools are the change of variables \( \Phi \) and the derivative identities \( \Phi \) and \( \Phi \) and algebra. Manipulations are performed to the greatest extent possible (in Sec. 2) before an average is performed. Then, exactly defined ensemble, time, and spatial averages are used. DNS makes study of exact structure-function equations practical. Also, experimental methods exist (Su and Dahm, 1995) that can completely evaluate terms in the exact structure-function equations. Exact incompressibility relationships, such as \( \hat{D}_{i11} \) and \( \hat{D}_{i11} \), are obtained. Following from the discussion in Sec. 9, the exact incompressibility relationship \( \hat{D}_{i11} \) will have a nonzero value at small \( r \) because of large-scale structures in the flow. At small \( r \), \( \hat{D}_{i11} \) is approximately the second derivative with respect to measurement location of the velocity covariance, and therefore clearly depends on flow inhomogeneity.

That the exact structure-function equations are an advance can be seen from previous work. It is no longer necessary to derive individual terms that describe effects of inhomogeneity that are missing from equations valid only for homogeneous turbulence, such as was done by Lindborg (1999). All such terms are now known. Sreenivasan and Dhruva (1998) note that one could determine scaling exponents with greater confidence if one has a theory that exhibits not only the asymptotic power law but also the trend toward the power law, and that without such a theory the method of computing local slopes is a “misplaced delusion.” The exact equations given here are the
required theory for the third-order structure function, given that data must be used to evaluate the equations in a manner analogous to previous evaluations in Antonia, Chambers, and Browne (1983), Chambers and Antonia (1984), Lindborg (1999), Danaila et al. (1999 a,b), and Antonia et al. (2000). The exact dynamical equations obtained here are useful for studies of the approach toward local homogeneity as well as to local isotropy. Toward that end, a scale analysis is given in Sec. 7.4, which leads to the approximate equations in Sec. 8. The exact equations provide insight into the time-derivative terms, as discussed in Sec. 6.

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**Appendix A: Forced Turbulence**

The Navier-Stokes equation (13) and the exact structure-function equations [e.g., (21)] apply to cases in which the turbulence is forced at places other than at the points of observation $x$ and $x'$, such as grid-generated turbulence, pipe flow, and boundary layers. Also, the Navier-Stokes equation (13) and (21) apply to freely decaying DNS such as that by Boratav and Pelz (1997) and the simulation of laboratory experiments as in de Bruyn Kops and J. J. Riley (1998). Some DNS employ spatially distributed forces to drive the turbulence to a steady state. The Navier-Stokes
equation (13), and the exact equations derived from it, do not apply to that case; instead, such forces must be introduced into (13) and the resultant additional terms derived for the exact structure-function equations.

If a force \( f_i \) is added to the right-hand side of the Navier-Stokes equation (13), then the term to be added to (15) is simply \(-\tau_{ij}\) defined in (3) with \(-\partial x_j p\) replaced by \( f_i \) and \(-\partial x_i p'\) by \( f_i'\). That is, the added term is

\[
(f_i - f_i') (u_j - u_j') + (f_j - f_j') (u_i - u_i') \equiv \Phi_{ij},
\]

and the average of this expression must appear in our subsequent structure-function equations. Consider the case of the deterministic force, \( f_i = \delta_{ij} F \cos(k_f x_i) \), that was used in the DNS in Borue and Orszag (1996), where we use subscripts 2 and 1 to denote their \( y \) and \( x \) directions, respectively, and \( \delta_{ij} \) is the Kronecker delta. Use the identity \( \cos(k_f x_1) - \cos(k_f x_2') = 2 \cos(k_f x_1/2) \cos(k_f x_1/2) \). The ensemble and temporal averages of \( \Phi_{ij} \) are \( 2 F \cos(k_f x_1/2) \cos(k_f x_1) \left[ \delta_{ij} (U_j - U_j') + \delta_{ij} (U_i - U_i') \right] \), the trace of which is \( 4 F \cos(k_f x_1/2) \cos(k_f x_1) (U_2 - U_2') \).

The \( X \)-space average of the first term in \( \Phi_{ij} \) is \( 2 F \delta_{ij} \cos(k_f x_1/2) \frac{1}{2} \oint \left[ \oint (u_j - u_j') \right] dx_2 dx_3 \cos(k_f x_1) dx_1 \); inter-change \( i \) and \( j \) to obtain the second term in \( \Phi_{ij} \). Whichever average is employed, this force introduces a term that has no small-scale spatial variation and is negligible in our scale analysis.

Forced turbulence is temporally intermittent such that a space average, e.g., \( \langle d_{ii} \rangle_R \), does not obey \( \partial_t \langle d_{ii} \rangle_R = 0 \). The temporal intermittency observed by Borue and Orszag (1996) illustrates this fact; of particular relevance is the observation of repeated events characterized by accumulation of space-averaged turbulent energy. Given the conditions mentioned below (29), one can time-average (71) such that

\[
\langle \partial_t \langle d_{ii} \rangle_R \rangle_T \approx -\frac{\langle \delta \rangle_R}{r_S} r_S S
\]

(recall that this is based on neglecting the time-derivative and viscous terms in (7) and the forcing because the forcing has no small-scale spatial variation and is therefore negligible in our scale analysis). A similar generalization of Kolmogorov’s 4/5 law, namely, \( \hat{f}_{r_n} D_{in} \approx -\frac{4}{3} \langle \delta \rangle_E r \), was obtained in Lindborg (1996), Frisch (1995), and Hill (1997) for the inertial range of homogeneous, anisotropic turbulence.

Appendix B: Data

B.1 Inner Scales

Inner scale was first defined by Obukhov (1949) as the \( r \) at which the asymptotic formulas for the inertial and viscous ranges are equal. Inner scales are more applicable in our scaling analysis by a factor of about 10 compared with \( \eta \). Inner scales for \( D_{111} \) and \( D_{11\beta} \), denoted \( \ell_{111} \) and \( \ell_{11\beta} \), can be related to \( \eta \) using \( \langle \delta \rangle_E = 15 \nu \left( \langle \partial_1 u_1 \rangle^2 \right)_E = (15/2) \nu \left( \langle \partial_1 u_1 \rangle^2 \right)_E \), which is valid on the basis of local isotropy and incompressibility. Then, \( \ell_{111} = (2/3)^{3/4} \ell_{11\beta} = 13 \eta \). Inner scales for \( D_{1111} \) and \( D_{11\beta\beta} \), denoted \( \ell_{1111} \) and \( \ell_{11\beta\beta} \), can be related to \( \eta \) on the additional empirical basis that the derivative skewness, \( \left( \langle (\partial_1 u_1)^3 \rangle_E \right)_E / \left( \langle (\partial_1 u_1)^2 \rangle_E \right)_E \), varies little from \(-0.5\) over observed values of Reynolds number (Sreenivasan and Antonia, 1997; Belin et al., 1997). Then \( \ell_{1111} = \sqrt{2} \ell_{11\beta\beta} = 9.6 \eta \). The average of these four inner scales is \( \langle \ell_{111} + \ell_{11\beta} + \ell_{1111} + \ell_{11\beta\beta} \rangle/4 = 10 \eta \). Even though the turbulence being studied need not be locally isotropic, we define \( \ell \equiv 10 \eta \). This is the background of definition (17).

B.2 Typical Data

Data are needed for the investigations in Sec. 7.4. Let \( r/r, \ell, \) and \( \hat{r} \) be orthogonal unit vectors. Let subscript 1 denote projection in the direction \( r/r \), e.g., \( D_{11} \equiv \langle r_i/r \rangle \langle r_j/r \rangle D_{ij} \). Let subscript \( \beta \) denote projection in either the \( \hat{r} \) or \( \hat{\ell} \) directions (we need not distinguish which direction), e.g., \( D_{11\beta} \) is either \( \ell_1 \ell_2 (r_k/r) D_{ijk} \) or \( \hat{r}_i \hat{r}_j (r_k/r) D_{ijk} \), but not \( \hat{\ell}_i \hat{\ell}_j (r_k/r) D_{ijk} \). If a distinction need not be made as to the direction of projection, then subscript \( \alpha \) is used; thus, \( D_{\alpha\alpha} \) is either \( D_{11} \) or \( D_{\beta\beta} \). No summation is implied by repeated Greek indices. A unit vector \( \hat{a} \) in an arbitrary direction is a linear combination of the unit vectors \( r/r, \ell, \) and \( \hat{r} \). Thus, if projections of a quantity have the same
order of magnitude and sign in all three directions \( r/r, \hat{i}, \) and \( \hat{e} \), then the projection in an arbitrary direction \( \hat{a} \) also has that order of magnitude.

For the inertial range we use the formulas

\[
D_{\alpha\alpha} = \langle \varepsilon \rangle_E^{2/3} r^{2/3} K_{\alpha\alpha}(X, r, t), \tag{B1}
\]
\[
D_{111} = - \langle \varepsilon \rangle_E r K_{111}(X, r, t), \quad D_{1\beta\beta} = - \langle \varepsilon \rangle_E r K_{1\beta\beta}(X, r, t). \tag{B2}
\]

The dimensionless coefficient functions, \( K_{\alpha\alpha}(X, r, t), K_{111}(X, r, t), \) and \( K_{1\beta\beta}(X, r, t) \), are included to emphasize that our inertial-range data, like real data, need not be precisely homogeneous, locally isotropic, or stationary. The coefficient functions are assumed to be of the order of unity, and when differentiating the structure functions with respect to \( r \), the derivatives of the coefficient functions are assumed to be negligible compared to the derivative of \( r^{2/3} \) in (\ref{B1}) and \( r \) in (\ref{B2}). As motivation for this assumption, consider that for the case of local isotropy the above coefficient functions are constants between 2.7 and 0.26. The choice to scale with \( \ell \equiv 10 \eta \), rather than with \( \eta \), causes the coefficient functions to be of the order of unity.

The slight effect of intermittency on the exponent 2/3 in (\ref{B1}) is not of concern here. Of more significance is the finding by Mydlarski and Warhaft (1996) of power-law ranges that are precursors to the inertial range. Their precursor power-law exponents are smaller than the 2/3 exponent of the inertial range, and the precursor exponents approach 2/3 as Reynolds number increases. Our scale analysis can be extended to apply for those weaker power-laws; the accuracy of the scaling condition would be correspondingly weakened.

Using \( \ell \) defined in (\ref{2}), the viscous-range formulas for the scale analysis are

\[
D_{\alpha\alpha} = \langle \varepsilon \rangle_E^{2/3} \ell^{2/3} r^{2/3} k_{\alpha\alpha}(X, r, t), \tag{B3}
\]
\[
D_{111} = - \langle \varepsilon \rangle_E \ell (r/\ell)^3 k_{111}(X, r, t), \quad D_{1\beta\beta} = - \langle \varepsilon \rangle_E \ell (r/\ell)^3 k_{1\beta\beta}(X, r, t), \tag{B4}
\]

where the dimensionless coefficient functions, \( k_{\alpha\alpha}(X, r, t), k_{111}(X, r, t), \) and \( k_{1\beta\beta}(X, r, t) \), are assumed to be of the order of unity, and when differentiating the structure functions with respect to \( r \), the derivatives of the coefficient functions are assumed to be negligible. In support of this assumption, note that for the case of local isotropy these coefficient functions are constants between 2.9 and 0.57. The choice to scale with \( \ell \equiv 10 \eta \), rather than with \( \eta \), causes the coefficient functions to be of the order of unity.

For \( r \) between the inertial and viscous ranges, the structure functions \( D_{\alpha\alpha}, D_{111}, \) and \( D_{1\beta\beta} \) have monotonic transitions between the asymptotic formulas (\ref{B1}), (\ref{B2}), and (\ref{B3}), (\ref{B4}). Therefore, if a quantity is negligible on the basis of both (\ref{B1}), (\ref{B2}), and (\ref{B3}), (\ref{B4}), then it is negligible for all \( r \) from within the inertial range to within the viscous range.

Consider the projection of \( \partial_r D_{ijn} \) in directions parallel to \( r \), i.e., \( (r_i/r) (r_j/r) \partial_r D_{ijn} \), and perpendicular to \( r \), e.g., \( \hat{e}_i \hat{e}_j \partial_r D_{ijn} \). Note that neither projection commutes with the derivatives \( \partial_r \), e.g., \( \partial_r (\hat{e}_i \hat{e}_j \partial_r D_{ijn}) \neq \hat{e}_i \hat{e}_j \partial_r \partial_r D_{ijn} \). For example, if \( D_{111} \) is locally isotropic, then differentiating the isotropic-tensor formula gives \( \hat{e}_i \hat{e}_j \partial_r D_{ijn} = \partial_r D_{\beta\beta} + \frac{3}{7} D_{33} \), whereas \( \partial_r (\hat{e}_i \hat{e}_j \partial_r D_{ijn}) = \partial_r D_{\beta\beta} + \frac{3}{7} D_{33} \); another example is: \( (r_i/r) (r_j/r) \partial_r D_{ijn} = \partial_r D_{111} + \frac{3}{7} D_{33} \), whereas \( \partial_r [(r_i/r) (r_j/r) \partial_r D_{ijn}] = \partial_r D_{111} = \partial_r D_{111} + \frac{3}{7} D_{33} \). By use of \( \hat{e}_i \hat{e}_j \partial_r D_{ijn} \) and \( (r_i/r) (r_j/r) \partial_r D_{ijn} \) are \(- \frac{3}{7} \langle \varepsilon \rangle_E \) for the locally isotropic case. Since the projections in all three directions are the same, the projection in the arbitrary direction \( \hat{a} \), i.e., \( \hat{a} \hat{a} \partial_r D_{ijn} \), also equals \(- \frac{3}{7} \langle \varepsilon \rangle_E \) for the locally isotropic case. Although our data are not locally isotropic, assume that for our data \( \hat{a} \hat{a} \partial_r D_{ijn} \) is of the order of \(- \langle \varepsilon \rangle_E \) in the inertial range. For the viscous range and for the locally isotropic case, \( \hat{e}_i \hat{e}_j \partial_r D_{ijn} = \partial_r D_{\beta\beta} + \frac{3}{7} D_{33} = \frac{3}{7} D_{111} \) and \( (r_i/r) (r_j/r) \partial_r D_{ijn} = \partial_r D_{111} + \frac{3}{7} D_{33} \); these formulas combined with (\ref{B4}) imply that \( \hat{a} \hat{a} \partial_r D_{ijn} \) is of the order of \(- \langle \varepsilon \rangle_E \) in the viscous range. Assume that this is true of our data as well.

The same manipulations apply to the projection of \( \partial_{\alpha} \partial_{\alpha} D_{ij} \). Differentiation of the isotropic-tensor formulas gives \( (r_i/r) (r_j/r) \partial_{\alpha} \partial_{\alpha} D_{ij} = \left( \partial_{\alpha} + \frac{\partial}{\partial r} \right) \partial_r D_{ij} + \frac{3}{7} (D_{\beta\beta} - D_{\alpha\alpha}) \) [in contrast, \( \partial_r \partial_{\alpha} D_{ij} = \left( \partial_{\alpha} + \frac{\partial}{\partial r} \right) \partial_r D_{ij} \)], and \( \hat{e}_i \hat{e}_j \partial_{\alpha} \partial_{\alpha} D_{ij} = \left( \partial_{\alpha} + \frac{\partial}{\partial r} \right) \partial_r D_{ij} - \frac{3}{7} (D_{\beta\beta} - D_{\alpha\alpha}) \). By use of (\ref{B1}) and (\ref{B3}) we find for the projection in an arbitrary direction \( \hat{a} \), that \( \hat{a} \hat{a} \partial_r D_{ijn} \) is about \( 2 \langle \varepsilon \rangle_E^{2/3} r^{-4/3} \) in an inertial range and is about \( 10 \langle \varepsilon \rangle_E^{2/3} r^{-4/3} \) \( \approx \langle \varepsilon \rangle_E / 2 \nu \) in the viscous range. Assume that this is true of our data as well.

In effect, our definition of the inertial range includes (\ref{B1}), (\ref{B2}), and that the projections in an arbitrary direction \( \hat{a} \) of the two terms in (\ref{2}) behave as

\[
\hat{a} \hat{a} \partial_r D_{ijn} \sim - \langle \varepsilon \rangle_E, \quad 2 \nu \hat{a} \hat{a} \partial_r D_{ijn} \sim 0.2 \langle \varepsilon \rangle_E (r/\ell)^{-4/3}, \tag{B5}
\]
where \( \sim \) means “is of the order of.” Our definition of the viscous range includes (B3) and (B4), and that the projections in an arbitrary direction behave as

\[
\hat{a}_i \hat{a}_j \partial_{r_n} D_{ijn} \sim -\langle \varepsilon \rangle_E (r/\ell)^2, \tag{B7}
\]

\[
2 \nu \hat{a}_i \hat{a}_j \partial_{r_n} D_{ij} \sim \langle \varepsilon \rangle_E. \tag{B8}
\]

For both ranges we include the additional assumption that the off-diagonal components \( D_{\alpha\beta} \) (for \( \alpha \neq \beta \)) are not greater in magnitude than \( D_{\alpha\alpha} \).

We assume that diagonal components of \( E_{ij} (X, r, t) \) are of order \( \langle \varepsilon \rangle_E \); that is, when projected on a arbitrary direction \( \hat{a} \),

\[
\hat{a}_i \hat{a}_j E_{ij} \sim \langle \varepsilon \rangle_E. \tag{B9}
\]

In support of this assumption, recall that \( E_{ii} = 4 \langle \varepsilon \rangle_E \) on the basis of homogeneity, and, in the case of local isotropy, \( E_{\alpha\alpha} \equiv 4 \langle \varepsilon \rangle_E / 3 \).

Data are needed for the diagonal elements \( \hat{a}_i \hat{a}_j T_{ij} \). Because \( \langle (p - p') (u_j - u'_j) \rangle_E \) vanishes on the basis of local isotropy and because \( \partial_{X_n} \) operating on any average vanishes on the basis of homogeneity, it is assumed that

\[
\left| \hat{a}_i \hat{a}_j \left( 2 (p - p') (s_{ij} - s'_{ij}) \right)_E \right| \geq \left| \hat{a}_i \hat{a}_j \partial_{X_n} \left( (p - p') (u_j - u'_j) \right)_E \right|. \tag{B10}
\]

Of course, this is not true for the sum of diagonal components because of \( B_{12} \). It is likely that the right side of (B10) is much smaller than the left side, but a more restrictive condition than (B10) is not needed. On the basis of DNS, Borue and Orszag (1996) show the cross spectrum of velocity and pressure gradient, where both velocity and pressure gradient are projected in their y-direction. For the inertial range, their data show that the corresponding diagonal component of \( T_{ij} \) is proportional to \( \langle \varepsilon \rangle_E r/L \). More details for other flows would be welcome because \( T_{ij} \) vanishes for locally isotropic turbulence (Hill, 1997). Therefore, its anisotropic behavior is of interest. However, based on the result by Borue and Orszag (1996), it is assumed that our data in the inertial range obeys

\[
\hat{a}_i \hat{a}_j T_{ij} \sim \langle \varepsilon \rangle_E (r/L). \tag{B11}
\]

The data by Alvelius and Johansson (2000) are consistent with (B11). Using data from nearly homogeneous turbulence, Lindborg (1996) found that the single-point pressure strain correlation has a longitudinal component that is approximately \(-4 \langle \varepsilon \rangle_E / 3\) and a transverse component that is approximately \( 2 \langle \varepsilon \rangle_E / 3\). For homogeneous turbulence in the limit \( r \to \infty \) Lindborg’s result corresponds to \( T_{11} \to -16 \langle \varepsilon \rangle_E / 3 \) and \( T_{33} \to -8 \langle \varepsilon \rangle_E / 3 \); this agrees in order of magnitude with (B11) evaluated at \( r = L \). The first nonvanishing term of the Taylor series expansion of \( \hat{a}_i \hat{a}_j \left( 2 (p - p') (s_{ij} - s'_{ij}) \right)_E \) is \( r^2 \) times the average of the product of pressure gradient and strain-rate gradient. This suggests that for the viscous range,

\[
\hat{a}_i \hat{a}_j T_{ij} \sim \langle \varepsilon \rangle_E (r/\ell)^2 (\ell/L). \tag{B12}
\]

The form of (B12) is chosen to equal (B11) at \( r = \ell \). In the absence of further information, (B12) is assumed to be valid.

Finally, data are needed for \( \hat{\Gamma}_{in} \) and \( \hat{\Gamma}_{ijn} \), which are defined in (B3) and appear in the second term of (B2). As described in Sec. 7.4, the second term in (B2) is important for the case of large-scale structures in the flow. Assume that the experimenter chooses a flow that mitigates against large-scale structure; grid-generated turbulence is an example. In this case, it is assumed that

\[
\hat{\Gamma}_{in} \leq v^2, \quad \text{and} \quad \hat{\Gamma}_{ijn} \ll v^3, \tag{B13}
\]

where \( v \) is defined in (66).

Relationships (B1)–(B13) serve as exemplary data in the scale analysis. The fact that we have data only for projections in an arbitrary direction \( \hat{a} \) means that we can investigate only the diagonal components of (B2). The off-diagonal components, which are obtained by projection in two orthogonal directions, cannot be studied here. Relationships like (B3)–(B9) are most often associated with the assumption of local isotropy. However, like anemometry data, these relationships can be fulfilled for coefficient functions [as defined in (B11)–(B14)] of the order of unity without the specific restrictions of local isotropy being precisely fulfilled. For instance, for \( D_{ij} \), the restrictions for local isotropy are that its off-diagonal elements are zero, and that \( \bar{i}_i \bar{i}_j D_{ij} = \bar{e}_i \bar{e}_j D_{ij} \), and that \( D_{11} \) is related to \( D_{\beta\beta} \) by an incompressibility condition. In the scale analysis, such restrictions are not used; therefore, local isotropy is not assumed.
Appendix C: Homogeneity Implemented Using the Calculus of Local Homogeneity

Although homogeneity is mentioned only briefly in this study, it is useful to introduce it and to show how the calculus of local homogeneity produces the predictions of homogeneity for the case of homogeneous turbulence. Homogeneity is the approximation that ensemble averages do not depend on the position at which the average is obtained (Monin and Yaglom, 1975). That position being $X$, we implement this approximation by neglecting the result of $\partial X_n$ operating on any average. For example, in (21) $\frac{1}{4} \partial X_n \partial X_n D_{ij} = 0$, $\partial X_n F_{ijn} = 0$, and for the average of (21) $\partial X_n \langle (p-p')(u_j - u'_j) \rangle_{E} = 0$ such that $T_{ij} = -2 \langle (p-p')(s_{ij} - s'_{ij}) \rangle_{E}$, from which we obtain $T_{ii} = 0$ because $s_{ii} = 0$ by incompressibility. The Taylor series of $p(x,t)$ around point $X$ is $p(x,t) = p(X,t) + (x_n - X_n) \partial X_n p(X,t) + \cdots$. Upon averaging, homogeneity requires that $\partial X_n \langle p(X,t) \rangle_{E} = 0$, etc., such that $\langle p(x,t) \rangle_{E} = \langle p(X,t) \rangle_{E}$, and similarly $\langle p(x',t) \rangle_{E} = \langle p(X,t) \rangle_{E}$; similarly, $\langle \varepsilon \rangle_{E} = \langle \varepsilon' \rangle_{E}$. Within the average of (10) we have $\partial X_n \partial X_n \langle p + p' \rangle_{E} = 2 \partial X_n \partial X_n \langle p(X,t) \rangle_{E} = 0$, etc., such that $E_{ii} = 4 \langle \varepsilon \rangle_{E}$. From (24) homogeneity gives the incompressibility condition, $\partial X_n D_{jn} = 0$.

In Sec. 3.3 the spatial average is a volume average in $X$-space such that the equations [e.g., (32) and (33)] do not contain $\partial X_n$ operating on an average. For those spatially averaged equations, homogeneity can be implemented by neglecting any average over the surface bounding the averaging volume of the surface-normal component of any vector. The basis for this implementation is that there are no net average fluxes in homogeneous turbulence.