Walks: A Beginner’s Guide to Graphs and Matrices

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We provide an introduction to graph theory and linear algebra. The present article consists of two parts. In the first part, we review the transfer-matrix method. It is known that many enumeration problems can be reduced to counting walks in a graph. After recalling the basics of linear algebra, we count walks in a graph by using eigenvalues. In the second part, we introduce PageRank by using a random walk model. PageRank is a method to estimate the importance of web pages and is one of the most successful algorithms. This article is based on the author’s lectures at Tohoku University in 2018 and 2020.

KEYWORDS: Graphs, Walks, Eigenvalues, Transfer-Matrix Method, PageRank

1. Introduction

A relation in the real world can be modeled by using a graph consisting of vertices and edges. For example, in Fig. 1(a), vertices represent people and an edge connects two people if they know each other. In certain cases, directed graphs are more suitable than undirected graphs. In Fig. 1(b), species are represented by vertices and predator-prey relationships by directed edges. Figure 1(c) shows the structure of this article.

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We now give a formal definition of graphs. We first define undirected graphs. An undirected graph $\Gamma$ is defined as a pair $(V, E)$ such that $V$ is a set and $E$ is a set of two-element subsets of $V$. The set $V$ is called the vertex set of $\Gamma$ and $E$ is called the edge set of $\Gamma$.

**Example 1.1.** Let $\Gamma$ be the undirected graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set 
$$\{(1, 2), (2, 3), (1, 3), (2, 4)\}.$$ 

The graph $\Gamma$ can be drawn as shown in Fig. 2.

![Fig. 2. An undirected graph with five vertices and four edges.]

**Example 1.2.** Let $\Gamma$ be the undirected graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set 
$$\{(1, 2), (2, 3), (3, 4), (4, 5), (1, 5)\}.$$ 

The graph $\Gamma$ can be drawn as shown in Fig. 3.

![Fig. 3. An undirected graph with five vertices and five edges.]

There are many ways to draw a graph. For example, we can also draw $\Gamma$ as shown in Fig. 4.

![Fig. 4. Another way to draw $\Gamma$.]

Next, we define directed graphs. A directed graph (or digraph) $\Gamma$ is defined as a pair $(V, E)$ such that $V$ is a set and $E$ is a subset of $V \times V$. The set $V$ is called the vertex set of $\Gamma$ and $E$ is called the edge set of $\Gamma$. Throughout this article, we assume that the vertex set of a digraph is $\{1, 2, \ldots, n\}$ unless otherwise noted, where $n$ is a positive integer.

**Example 1.3.** Let $\Gamma$ be the digraph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{(1, 1), (1, 2), (1, 3), (2, 3), (3, 2), (1, 4)\}$. Note that $(2, 3) \neq (3, 2)$. The digraph $\Gamma$ can be drawn as shown in Fig. 5. An edge that connects a vertex to itself is called a loop. For example, $\Gamma$ has a loop $(1, 1)$.

![Fig. 5. A digraph with four vertices and six edges.]
**Remark 1.4.** In our definition, an undirected graph is not allowed to have a loop. A loop in an undirected graph can be considered as a multiset \([v, v]\). Thus to allow loops, we can define an *undirected graph with loops* as a pair \((V, E)\) such that \(V\) is a set and \(E\) is a set of two-element multisets.

**Example 1.5.** Let us consider the digraph shown in Fig. 6. In this digraph, each vertex represents a web page and each edge represents a link. For example, page 4 contains a link to page 1, so there is an edge from page 4 to page 1; however page 1 does not contain a link, so there is no edge from page 1 to page 4.

Our question is as follows. **Which page is the most important?** If there were a few web pages, for example, 100 or 1000 pages, then we could evaluate them by hand. There are, however, a huge number of web pages in the world, so it is impossible to rank them by hand. We want to find a good solution.

One of the most successful algorithms is PageRank invented by Page and Brin [9]. It is a method to estimate the importance of a page. The key ingredients of PageRank are random walks and eigenvalues. In Sect. 2, we review walks and eigenvalues. We then introduce PageRank by using a random walk model in Sect. 3.

**Exercise 1.6.** Find ten examples of (undirected or directed) graphs in the real world.

## 2. Counting Walks

In this section, we count walks by using eigenvalues. It is known that many enumeration problems can be reduced to counting walks in a graph. This technique is called the transfer-matrix method (see, for example, [32] for details).

### 2.1 Walks

A *walk from* \(v_0\) *to* \(v_l\) in a digraph \(\Gamma\) is a sequence of vertices \((v_0, v_1, \ldots, v_l)\) such that

\[
(v_0, v_1), (v_1, v_2), \ldots, (v_{l-1}, v_l)
\]

are edges of \(\Gamma\). The number \(l\) is called the *length* of this walk.

**Example 2.1.** Let \(\Gamma\) be the digraph with vertex set \([1, 2, 3]\) and edge set \([\{(1, 2), (1, 3), (2, 1), (3, 1)\}]\).

Then

- (1) is a walk from 1 to 1 of length 0.
- (1, 2) is a walk from 1 to 2 of length 1.
- (1, 1) is not a walk since (1, 1) is not an edge.
- (1, 2, 1) is a walk from 1 to 1 of length 2.
- (1, 2, 3) is not a walk since (2, 3) is not an edge.
- (1, 2, 1, 2, 1, 3) is a walk from 1 to 3 of length 5.

In this digraph, there are the following six walks of length 2:

\[(1, 2, 1), (1, 3, 1), (2, 1, 2), (2, 1, 3), (3, 1, 2), (3, 1, 3).\]

Fig. 6. A web graph.

Fig. 7. A digraph with three vertices and four edges.
Exercise 2.2. Let $\Gamma$ be the digraph with vertex set $\{1, 2, 3\}$ and edge set
\[ \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}. \]
Count walks from 1 to 1 in $\Gamma$ of length 2, 3, 4, and 5, respectively.

Remark 2.3. Exercise 2.2 can be immediately interpreted as a word counting problem as follows. Let $W_m$ be the set of $m$-letter words over $\{a, b, c\}$. For example,
\[ W_1 = \{a, b, c\}, \]
\[ W_2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}, \]
\[ W_3 = \{aaa, aab,aac, ..., ccc\}. \]
It is easy to see that the size of $W_m$ is $3^m$. We count the number of words $w \in W_m$ satisfies the following two conditions.
(w1) The first and the last letters of $w$ are $a$.
(w2) $w$ does not contain $aa$, $bb$, $cc$.
For example, $abcba$ satisfies (w1) and (w2), but $acbca$ does not satisfy (2) since it contains $cc$. We can see that this problem is equivalent to counting walks from 1 to 1 of length $m - 1$ in the digraph of Exercise 2.2.

2.2 Adjacency matrices
We have counted walks from 1 to 1 of length 5 in Exercise 2.2. Let us consider the following problem.

Problem 2.4. Count the number of walks from 1 to 1 of length 100 in the digraph $\Gamma$ shown in Fig. 8.

It seems impossible to count such walks because it may be complicated to even count them of length 5. However, we can count them by using the adjacency matrix of the digraph $\Gamma$. Our goal is to solve Problem 2.4.

An $n \times m$ matrix $A$ is an array of numbers with $n$ rows and $m$ columns. The number in the $i$th row and the $j$th column of $A$ is called the $(i, j)$ entry of $A$. Let $A_{ij}$ denote the $(i, j)$ entry of $A$. An $n \times n$ matrix is called a square matrix.

Example 2.5. The following is a $2 \times 2$ square matrix:
\[ A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}. \]
The $(1, 2)$ entry of $A$ is 3.

The identity matrix $I_n$ of size $n$ is the $n \times n$ square matrix such that
\[ (I_n)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \]
For example,
\[ I_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]
We will write $I$ instead of $I_n$ when no confusion can arise.

Matrices can be used to study graphs. For a digraph $\Gamma$ with $n$ vertices, its adjacency matrix $A$ is the $n \times n$ square matrix whose $(i, j)$ entry is defined by
\[ A_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is an edge of } \Gamma, \\ 0 & \text{if } (i, j) \text{ is not an edge of } \Gamma. \end{cases} \]

Example 2.6. Let $\Gamma$ be the digraph with vertex set $\{1, 2, 3\}$ and edge set $\{(1, 2), (1, 3), (3, 1)\}$.
Then its adjacency matrix $A$ is
\[
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]
For example, the $(1, 2)$ entry of $A$ is $1$ since $(1, 2)$ is an edge of $\Gamma$.

### 2.3 Matrix multiplication

We can count walks by using matrix multiplication. Let us recall this operation.

An $n \times 1$ matrix is called a **column vector of order $n$**. Let $A$ be a $2 \times 2$ matrix. We can consider that $A$ is a function that maps a column vector $v$ of order 2 to the following column vector $Av$:

\[
Av = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_{11}v_1 + A_{12}v_2 \\ A_{21}v_1 + A_{22}v_2 \end{bmatrix},
\]

where $v_i$ is the $(i, 1)$ entry of $v$.

**Example 2.7.** Let
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

Then
\[
Av = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} \quad \text{and} \quad Bv = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}.
\]

We can illustrate $A$ and $B$ by using the two-dimensional Euclidean space. The matrix $A$ corresponds to the reflection in the horizontal axis and $B$ corresponds to the rotation through angle $90^\circ$ (see Fig. 10). To define matrix multiplication, let us consider the composition $C$ of $A$ and $B$, that is $C$ maps a column vector $v$ to $A(Bv)$.

We can represent the composition $C$ as a matrix as follows:
\[
Cv = A(Bv) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)
\]
\[
= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}
\]
\[
= \begin{bmatrix} -v_2 \\ -v_1 \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]
Thus the composition $C$ is corresponding to the matrix
\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}.
\]

In general, if $A$ and $B$ are $2 \times 2$ matrices, then we can consider the composition $C$ of $A$ and $B$. We can calculate $Cv$ as follows:
\[
Cv = A(Bv) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11}v_1 + B_{12}v_2 \\ B_{21}v_1 + B_{22}v_2 \end{bmatrix} = \begin{bmatrix} A_{11}(B_{11}v_1 + B_{12}v_2) + A_{12}(B_{21}v_1 + B_{22}v_2) \\ A_{21}(B_{11}v_1 + B_{12}v_2) + A_{22}(B_{21}v_1 + B_{22}v_2) \end{bmatrix} = \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21})v_1 + (A_{11}B_{12} + A_{12}B_{22})v_2 \\ (A_{21}B_{11} + A_{22}B_{21})v_1 + (A_{21}B_{12} + A_{22}B_{22})v_2 \end{bmatrix}.
\]

From this, the composition $C$ corresponds to the matrix
\[
\begin{bmatrix}
A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\
A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22}
\end{bmatrix}.
\]

This matrix is called the product of $A$ and $B$ and is denoted by $AB$. Note that the first column of $AB$ equals $Ab_1$, where $b_1$ is the first column of $B$, since
\[
\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = Ab_1,
\]

Similarly, the second column of $AB$ equals $Ab_2$, where $b_2$ is the second column of $B$, since
\[
\begin{bmatrix} A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} = Ab_2.
\]

Thus we will write
\[
AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix}.
\]

More generally, let $A$ and $B$ be two $n \times n$ matrices. Then the matrix product $AB$ is defined to be an $n \times n$ matrix $C$ such that $Cv = A(Bv)$ for all column vectors $v$ of order $n$.

**Remark 2.8.** Although we can define the matrix product $AB$ for an $n \times m$ matrix $A$ and an $m \times k$ matrix $B$, we only discuss the case when $n = m = k$ for simplicity.

**Exercise 2.9.** Let $A$ and $B$ be two $n \times n$ matrices. Show that
\[
AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix},
\]

where $b_i$ is the $i$th column of $B$. In particular, the $(i, j)$ entry of $AB$ is equal to
\[
\sum_{k=1}^{n} A_{ik}B_{kj}.
\]

**Example 2.10.** The identity matrices play a role in matrix multiplication similar to the role of 1 in multiplication. For example, if $A$ is a $2 \times 2$ square matrix, then $AI_2 = I_2A$. Indeed,
\[
AI_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A \quad \text{and} \quad I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A.
\]

In general, if $A$ is an $n \times n$ square matrix, then
\[
AI_n = I_nA = A.
\]

**Example 2.11.** Let $A$ and $B$ be the same as in Example 2.7, that is,
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.

Then
\[AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.

Thus \(AB \neq BA\).

As we have seen in Example 2.11, matrix multiplication is not commutative, that is, \(AB\) may be different from \(BA\). However, matrix multiplication is associative, that is, if \(A, B,\) and \(C\) are \(n \times n\) matrices, then

\[A(BC) = (AB)C. \quad (2.1)\]

Indeed, the \((i, j)\) entries of \(A(BC)\) and \((AB)C\) are equal to

\[\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} A_{ik_1}B_{k_1k_2}C_{k_2j}.\]

Thus we will write

\[ABC = A(BC) = (AB)C.\]

We also see that \(A\) commutes with \(A^2\), that is, \(AA^2 = A^2A\). For a nonnegative integer \(l\), we define the \(l\)th power \(A^l\) of \(A\) by

\[A^l = A \cdots A,\]

where \(A^0 = I\).

**Example 2.12.** Let

\[A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}.

Let us calculate \(A^3\). We see that

\[A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 4 & 0 \\ 2 & 2 & 4 \end{bmatrix}.

Therefore

\[A^3 = A^2A = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 4 & 0 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 10 \\ 0 & 8 & 0 \\ 10 & 8 & 4 \end{bmatrix}.

Recall that \(A^3 = AA^2\). Indeed,
Exercise 2.13. Let \( \Gamma \) be the digraph of Exercise 2.2.
1. Calculate the adjacency matrix of \( \Gamma \).
2. Calculate \( A^2, A^3, A^4, \) and \( A^5 \).
3. Compare your answer with that of Exercise 2.2.

2.4 Counting walks using adjacency matrices

The key to solve Problem 2.4 is the following result.

Theorem 2.14. Let \( \Gamma \) be a digraph with adjacency matrix \( A \). Then the number of walks from \( i \) to \( j \) of length \( l \) in \( \Gamma \) is equal to the \((i,j)\) entry of \( A^l \).

We present an example to illustrate Theorem 2.14. Let us consider the digraph with vertex set \( \{1, 2, 3\} \) and edge set \( \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\} \). Its adjacency matrix \( A \) is

\[
A = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}.
\]

Theorem 2.14 asserts that the \((1,1)\) entry of \( A^2 \) is equal to the number of walks from 1 to 1 of length 2. Indeed, the \((1,1)\) entry of \( A^2 \) is equal to 2 since

\[
(A^2)_{11} = \sum_{k=1}^{3} A_{1k}A_{k1} = A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31} = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 2.
\]

and the number of walks from 1 to 1 of length 2 is 2. Why are they equal? This is because

\[
A_{1k}A_{k1} = 1 \iff A_{1k} = A_{k1} = 1
\]

\(\iff (1,k) \) and \((k,1)\) are edges

\(\iff (1,k,1)\) is a walk.

Indeed,
1. \( A_{11}A_{11} \) is equal to 0 and \((1,1,1)\) is not a walk.
2. \( A_{12}A_{21} \) is equal to 1 and \((1,2,1)\) is a walk.
3. \( A_{13}A_{31} \) is equal to 1 and \((1,3,1)\) is a walk.

Thus the \((1,1)\) entry of \( A^2 \) is equal to two and there are two walks from 1 to 1 of length 2.

In general, we can see that the number of walks from \( i \) to \( j \) of length 2 is equal to

\[
\sum_{k=1}^{3} A_{ik}A_{kj},
\]

that is, the \((i,j)\) entry of \( A^2 \).

Exercise 2.15. Complete the proof of Theorem 2.14.

Hint We can prove it by induction or using the fact that the \((i,j)\) entry of \( A^l \) is equal to

\[
\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_{l-1}=1}^{n} A_{ik_1}A_{k_1k_2} \cdots A_{k_{l-2}k_{l-1}}A_{k_{l-1}j}.
\]

2.5 Diagonalization

From Theorem 2.14, we can solve Problem 2.4 by computing \((A^{100})_{11}\). To this end, we introduce diagonalization.

A square matrix \( A \) is said to be diagonal if its \((i,j)\) entry is 0 whenever \( i \neq j \). For example, if

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}.
\]
then $A$ and $B$ are diagonal matrices, but $C$ is not since $C_{31} = 2$. Note that it is easy to calculate the $l$th power of a diagonal matrix. Indeed, we can see that

$$
\begin{bmatrix}
2 & 0 \\
0 & 5 
\end{bmatrix}^l = \begin{bmatrix} 2^l & 0 \\
0 & 5^l 
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 2 
\end{bmatrix}^l = \begin{bmatrix} (-1)^l & 0 & 0 \\
0 & 3^l & 0 \\
0 & 0 & 2^l 
\end{bmatrix}.
$$

Using eigenvalues and eigenvectors, we can reduce the calculation of $A^{100}$ to the calculation of the 100th power of a diagonal matrix as follows.

For simplicity, let $A$ be a $2 \times 2$ square matrix. Assume that there are two vectors $u_1$ and $u_2$ and two numbers $\lambda_1$ and $\lambda_2$ such that

$$
Au_1 = \lambda_1 u_1 \quad \text{and} \quad Au_2 = \lambda_2 u_2. \quad (2.2)
$$

Let $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$. Then

$$
AU = \begin{bmatrix} Au_1 & Au_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 \end{bmatrix}.
$$

On the other hand, let

$$
\Lambda = \begin{bmatrix} \lambda_1 & 0 \\
0 & \lambda_2 \end{bmatrix}.
$$

Then

$$
U\Lambda = \begin{bmatrix} U \begin{bmatrix} \lambda_1 \\
0 
\end{bmatrix} & U \begin{bmatrix} 0 \\
\lambda_2 
\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 \end{bmatrix}.
$$

Therefore

$$
AU = U\Lambda. \quad (2.3)
$$

Here, assume that $U$ is invertible, that is, there is the inverse matrix $U^{-1}$ such that $UU^{-1} = U^{-1}U = I$. By multiplying (2.3) from the right by $U^{-1}$, we see that

$$
(AU)U^{-1} = A(UU^{-1}) = AI = A
$$

and

$$
(U\Lambda)U^{-1} = U\Lambda U^{-1}.
$$

This implies that $A = U\Lambda U^{-1}$. Hence

$$
A^{100} = (U\Lambda U^{-1})^{100} = (U\Lambda U^{-1})(U\Lambda U^{-1}) \cdots (U\Lambda U^{-1})(U\Lambda U^{-1})
$$

$$
= U\Lambda(U^{-1}U)\Lambda(U^{-1}U) \cdots (U^{-1}U)\Lambda(U^{-1}U)\Lambda U^{-1} = U\Lambda^{100}U^{-1}.
$$

Since

$$
\Lambda^{100} = \begin{bmatrix} \lambda_1^{100} & 0 \\
0 & \lambda_2^{100} \end{bmatrix},
$$

we can calculate $A^{100}$ by using $A^{100} = U\Lambda^{100}U^{-1}$.

**Example 2.16.** Let

$$
A = \begin{bmatrix} 2 & 6 \\
9 & 5 
\end{bmatrix}, \quad u_1 = \begin{bmatrix} 2 \\
3 
\end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 1 \\
-1 
\end{bmatrix}.
$$

We see that

$$
Au_1 = \begin{bmatrix} 2 & 6 \\
9 & 5 
\end{bmatrix} \begin{bmatrix} 2 \\
3 
\end{bmatrix} = \begin{bmatrix} 22 \\
33 
\end{bmatrix} = 11 \begin{bmatrix} 2 \\
3 
\end{bmatrix} = 11u_1.
$$

and

$$
Au_2 = \begin{bmatrix} 2 & 6 \\
9 & 5 
\end{bmatrix} \begin{bmatrix} 1 \\
-1 
\end{bmatrix} = \begin{bmatrix} -4 \\
-4 
\end{bmatrix} = -4 \begin{bmatrix} 1 \\
-1 
\end{bmatrix} = -4u_2.
$$
Therefore, \( u_1 \) and \( u_2 \) satisfy (2.2). Let

\[
U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 11 & 0 \\ 0 & -4 \end{bmatrix}.
\]

We see that

\[
AU = \begin{bmatrix} 2 & 6 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 22 & -4 \\ 33 & 4 \end{bmatrix} = \begin{bmatrix} Au_1 & Au_2 \end{bmatrix}
\]

and

\[
U\Lambda = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 11 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 22 & -4 \\ 33 & 4 \end{bmatrix}.
\]

Therefore \( AU = U\Lambda \). Moreover, \( U \) is invertible. Indeed,

\[
\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.
\]

Therefore the inverse \( U^{-1} \) of \( U \) is

\[
\frac{1}{5} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}.
\]

Hence

\[
A = U\Lambda U^{-1}.
\]

It follows that

\[
A^i = U\Lambda^i U^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 11^i & 0 \\ 0 & (-4)^i \end{bmatrix} \left( \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} 2 \cdot 11^i & (-4)^i \\ 3 \cdot 11^i & -(4)^i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \cdot 11^i + 3 \cdot (-4)^i & 2 \cdot 11^i - 2 \cdot (-4)^i \\ 3 \cdot 11^i - 3 \cdot (-4)^i & 3 \cdot 11^i + 2 \cdot (-4)^i \end{bmatrix}.
\]

For example,

\[
\frac{1}{5} \begin{bmatrix} 2 \cdot 11 + 3 \cdot (-4) & 2 \cdot 11 - 2 \cdot (-4) \\ 3 \cdot 11 - 3 \cdot (-4) & 3 \cdot 11 + 2 \cdot (-4) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 & 30 \\ 45 & 25 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 9 & 5 \end{bmatrix} = A
\]

and

\[
\frac{1}{5} \begin{bmatrix} 2 \cdot 11^2 + 3 \cdot (-4)^2 & 2 \cdot 11^2 - 2 \cdot (-4)^2 \\ 3 \cdot 11^2 - 3 \cdot (-4)^2 & 3 \cdot 11^2 + 2 \cdot (-4)^2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 290 & 210 \\ 315 & 395 \end{bmatrix} = A^2.
\]

From the above example, we can see that \( u_1 \) and \( u_2 \) are significant. They are called eigenvectors of \( A \). If we can obtain them, then we can calculate \( A^{100} \). How can we find them?

### 2.6 Eigenvectors and eigenvalues

Let \( A \) be a square matrix and let \( \lambda \) be a complex number. If

\[
Au = \lambda u
\]

for some non-zero column vector \( u \) (that is, at least one entry of \( u \) is not zero), then \( \lambda \) is called an eigenvalue of \( A \) and \( u \) is called an eigenvector of \( A \) with eigenvalue \( \lambda \).

**Example 2.17.** Let \( A, u_1, \) and \( u_2 \) be the same as in Example 2.16. Since

\[
Au_1 = 11u_1 \quad \text{and} \quad Au_2 = -4u_2,
\]

it follows that \( u_1 \) and \( u_2 \) are eigenvectors of \( A \) with eigenvalues 11 and \(-4\), respectively.

We can find eigenvalues and eigenvectors by using determinants. Let \( A \) be an \( n \times n \) square matrix. If \( n = 1 \), then define
det(A) = A_{11}.

(2.5)

If \( n > 1 \), then define

\[
\det(A) = \sum_{i=1}^{n} (-1)^{i+1} A_{1i} \cdot \det(A_{(1i)}),
\]

(2.6)

where \( A_{(1i)} \) is the \((n - 1) \times (n - 1)\) square matrix obtained from \( A \) by removing the first row and the \( i \)th column. The number \( \det(A) \) is called the determinant of \( A \). The determinant \( \det(A) \) is also denoted by \(|A|\) or

\[
\begin{vmatrix}
  A_{11} & A_{12} & \cdots & A_{1n} \\
  A_{21} & A_{22} & \cdots & A_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  A_{n1} & A_{n2} & \cdots & A_{nn}
\end{vmatrix}.
\]

Example 2.18. If \( A \) is a \( 1 \times 1 \) matrix, then \( \det(A) = A_{11} \). Note that \( \det([2]) = |2| = -2 \).

Example 2.19. If \( A \) is a \( 2 \times 2 \) matrix, then

\[
\det(A) = A_{11} \det(A_{(11)}) - A_{12} \det(A_{(12)}) = A_{11} |A_{21}| - A_{12} |A_{21}|
\]

For example, if

\[
A = \begin{bmatrix}
  5 & 3 \\
  -2 & 4
\end{bmatrix},
\]

then

\[
\det(A) = \begin{vmatrix}
  5 & 3 \\
  -2 & 4
\end{vmatrix} = 5 \cdot 4 - (-2) = 26.
\]

Example 2.20. If \( A \) is a \( 3 \times 3 \) matrix, then

\[
\det(A) = A_{11} \det(A_{(11)}) - A_{12} \det(A_{(12)}) + A_{13} \det(A_{(13)}) = A_{11} |A_{21}A_{32}| - A_{12} |A_{21}A_{33}| + A_{13} |A_{21}A_{32}|
\]

Let \( A \) be an \( n \times n \) square matrix. Then eigenvalues of \( A \) are the solutions of the following equation:

\[\det(A - xI) = 0.\]

Example 2.21. Let \( A \) be the matrix of Example 2.16. We see that

\[
\det(A - xI) = \det\left(\begin{bmatrix}
  2 & 6 \\
  9 & 5
\end{bmatrix} - x \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix}\right)
\]

\[
= \begin{vmatrix}
  2 - x & 6 \\
  9 & 5 - x
\end{vmatrix}
\]

\[
= (2 - x)(5 - x) - 54
\]

\[
= x^2 - 7x - 44
\]

\[
= (x - 11)(x + 4).
\]
Therefore the eigenvalues of $A$ are $11$ and $-4$. Let $u$ be an eigenvector of $A$ with eigenvalue $-4$. Then

$$Au = \begin{bmatrix} 2 & 6 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 + 6u_2 \\ 9u_1 + 5u_2 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ -4u_2 \end{bmatrix}. $$

Therefore $u_1$ and $u_2$ satisfy the following system of linear equations:

$$\begin{cases} 6u_1 + 6u_2 = 0 \\ 9u_1 + 9u_2 = 0. \end{cases}$$

By solving this, we find a solution $u_1 = 1$ and $u_2 = -1$. Thus

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue $-4$. In general,

$$\begin{bmatrix} c \\ -c \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue $-4$ for $c \neq 0$.

**Exercise 2.22.** Let $A$ be the matrix of Example 2.21. Find an eigenvector of $A$ with eigenvalue $11$.

**Remark 2.23.** We can determine whether a square matrix is invertible by using its determinant. A square matrix $A$ is invertible if and only if $\det(A) \neq 0$. Moreover, if $A$ is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \begin{vmatrix} \det(A_{11}) & \cdots & \det(A_{1(n-1)}) \\ \vdots & \ddots & \vdots \\ \det(A_{(n-1)1}) & \cdots & \det(A_{(n-1)(n-1)}) \end{vmatrix},$$

where $A_{ij}$ is the $(n-1) \times (n-1)$ square matrix obtained from $A$ by removing the $i$th row and the $j$th column.

For example, if $A$ is a $2 \times 2$ matrix with $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}. $$

For example,

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}^{-1} = \frac{1}{-5} \begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}. $$

Indeed,

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \left( \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

### 2.7 Counting walks by using eigenvalues

Let $\Gamma$ be the digraph of Exercise 2.2. We can now count the number of walks from 1 to 1 of length 100 in $\Gamma$. Recall that the adjacency matrix $A$ of $\Gamma$ is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. $$

We first calculate the eigenvalues of $A$. Since

$$\det(A - xI) = \begin{vmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{vmatrix} = -x \begin{vmatrix} -x & 1 \\ 1 & -x \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -x \\ 1 & 1 \end{vmatrix} = -x(x^2 - 1) - (-x - 1) + (1 + x)$$

is the characteristic polynomial of $A$, we have the eigenvalues $1$ and $-1$ with multiplicity $2$. Therefore $A$ is a diagonalizable matrix, and we can write $A = X \Lambda X^{-1}$, where $X$ is the matrix of eigenvectors and $\Lambda$ is the diagonal matrix of eigenvalues. Since $1$ and $-1$ are the only eigenvalues, the characteristic polynomial of $A$ is

$$\det(A - xI) = (x - 1)^2 (x + 1).$$

Thus, the matrix $A$ has the eigenvalues $1$ and $-1$. We can now count the number of walks from 1 to 1 of length 100 in $\Gamma$. Let $u$ be an eigenvector of $A$ with eigenvalue $-1$. Then

$$Au = \begin{bmatrix} 2 & 6 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 + 6u_2 \\ 9u_1 + 5u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix},$$

Therefore $u_1$ and $u_2$ satisfy the following system of linear equations:

$$\begin{cases} 6u_1 + 6u_2 = 0 \\ 9u_1 + 9u_2 = 0. \end{cases}$$

By solving this, we find a solution $u_1 = 1$ and $u_2 = -1$. Thus

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue $-1$. In general,

$$\begin{bmatrix} c \\ -c \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue $-1$ for $c \neq 0$. Therefore, the number of walks from 1 to 1 of length 100 in $\Gamma$ is given by

$$\det(A^{100}) = (1)^{100} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$
it follows that the eigenvalues of $A$ are 2 and $-1$. Next, we calculate eigenvectors. Let $u$ be an eigenvector of $A$ with eigenvalue 2. Then

$$Au = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_2 + u_3 \\ u_1 + u_3 \\ u_1 + u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \\ 2u_3 \end{bmatrix}.$$  

By solving this, we can find a solution $u_1 = u_2 = u_3 = 1$. Thus

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue 2. Indeed,

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Next, let $u$ be an eigenvector of $A$ with eigenvalue $-1$. Then

$$Au = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_2 + u_3 \\ u_1 + u_3 \\ u_1 + u_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix}.$$  

By solving this, we find that $u_1 + u_2 + u_3 = 0$. Therefore, for example,

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

are eigenvectors of $A$ with eigenvalue $-1$. Let

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$  

Then $U$ is invertible because $\det(U) \neq 0$. Moreover,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

Therefore

$$A' = UA\Lambda U^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2^j & 0 & 0 \\ 0 & (-1)^j & 0 \\ 0 & 0 & (-1)^j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2^j & (-1)^j & 0 \\ 2^j & -(-1)^j & (-1)^j \\ 2^j & 0 & -(-1)^j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2^j - 2(-1)^j & 2^j - (-1)^j & 2^j + 2(-1)^j \\ 2^j - (-1)^j & 2^j + 2(-1)^j & 2^j - (-1)^j \\ 2^j - (-1)^j & 2^j + 2(-1)^j & 2^j + 2(-1)^j \end{bmatrix}.$$  

For example, the number of walks from 1 to 1 of length 5 is 10 since

$$\frac{2^5 + 2(-1)^5}{3} = 10.$$
and that of length 100 is 
\[ \frac{2^{100} + 2}{3} \].

Recall that the number of words with length \( m \) satisfying (w1) and (w2) defined in Remark 2.3 is equal to the number of walks from 1 to 1 of length \( m - 1 \). As this example shows, some problems can be reduced to counting walks. We could then solve it by using eigenvalues as we have seen. This technique is called the transfer-matrix method [32].

**Remark 2.24.** We can avoid computing \( U^{-1} \) in the above calculation by using the **trace** of \( A \) as follows. For an \( n \times n \) square matrix \( A \), define the **trace** \( \text{tr}(A) \) of \( A \) by

\[ \text{tr}(A) = \sum_{i=1}^{n} A_{ii}. \]

For example,

\[ \text{tr} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = 4 + 3 = 7. \]

We can show that

\[ \text{tr}(AB) = \text{tr}(BA). \quad (2.7) \]

Indeed,

\[ \text{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}B_{jk} = \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ik}A_{ki} = \sum_{k=1}^{n} (BA)_{kk} = \text{tr}(BA). \]

Let \( \Gamma \) be the digraph of Exercise 2.2 and let \( A \) be its adjacency matrix. We now calculate \((A^l)_{11}\) without computing \( U^{-1} \). Note that

\[ \text{tr}(A^l) = (A^l)_{11} + (A^l)_{22} + (A^l)_{33}. \]

By the symmetry of \( \Gamma \), we see that

\[ (A^l)_{11} = (A^l)_{22} = (A^l)_{33}. \]

Therefore

\[ (A^l)_{11} = \frac{\text{tr}(A^l)}{3}. \quad (2.8) \]

On the other hand, since \( A^l = U A^l U^{-1} \), it follows from (2.7) that

\[ \text{tr}(A^l) = \text{tr}((U A^l U^{-1}) = \text{tr}(U^{-1} U A^l)) = \text{tr}(A^l). \]

Recall that

\[ A^l = \begin{bmatrix} 2^l & 0 & 0 \\ 0 & (-1)^l & 0 \\ 0 & 0 & (-1)^l \end{bmatrix}. \]

Therefore

\[ \text{tr}(A^l) = \text{tr}(A^l) = 2^l + 2(-1)^l. \quad (2.9) \]

By combining (2.8) and (2.9), we see that

\[ (A^l)_{11} = \frac{2^l + 2(-1)^l}{3}. \]

**Exercise 2.25.** Let \( \Gamma \) be the digraph with vertex set \( \{1, 2\} \) and edge set

\[ \{(1, 1), (1, 2), (2, 1)\}. \]

Count walks from 1 to 1 of length \( l \) in \( \Gamma \).

\[ \begin{array}{c}
\circ \quad 1 \\\\
\end{array} \\
\begin{array}{c}
\text{Fig. 12. A digraph with two vertices and three edges.} \\
\end{array} \]
**Exercise 2.26.** Let \( W_m \) be the set of \( m \)-letter words over \( \{a, b, c, d\} \). Count the number of words \( w \in W_m \) satisfies the following two conditions.

- **(w1)** The first and the last letters of \( w \) are \( a \).
- **(w2)** \( w \) does not contain \( aa, bb, cc, dd \).

### 2.A One-dimensional Ising model

In this section, we have reviewed the basics of the transfer-matrix method. By using this method, we solved some enumeration problems. We can apply the transfer-matrix method also to other kinds of problems. Indeed, this method is successfully used in statistical mechanics and has a wide range of applications. As a typical example, we consider the one-dimensional Ising model of spins in this appendix. See [4] for details. The reader can skip this appendix without loss of continuity.

Let \((u_0, u_1, \ldots, u_l)\) be a sequence over \( \{\pm 1\} \). This sequence represents a state of \( N \) spins. The energy of \((u_0, \ldots, u_l)\) is defined by

\[
E(u_0, \ldots, u_l) = -J \sum_{i=0}^{l} u_i u_{i+1} - H \sum_{i=0}^{l} u_i,
\]

where \( u_{l+1} = u_0 \), and \( J \) and \( H \) are constants. For example,

\[
E(1, 1, -1) = -J(1 \cdot 1 + 1 \cdot (-1) + (-1) \cdot 1) - H(1 + 1 - 1) = J - H.
\]

The first term (2.10) represents the interaction between pairs of spins. We define the partition function by

\[
Z(\beta) = \sum_{(u_0, \ldots, u_l) \in \{\pm 1\}^{l+1}} \exp(-\beta E(u_0, \ldots, u_l)).
\]

For example, if \( l = 1 \),

\[
Z(\beta) = \sum_{(u_0, u_1) \in \{\pm 1\}^2} \exp(-\beta E(u_0, u_1))
\]

\[
= \exp(-\beta E(1, 1)) + \exp(-\beta E(1, -1)) + \exp(-\beta E(-1, 1)) + \exp(-\beta E(-1, -1)).
\]

Surprisingly, we can calculate \( Z(\beta) \) by the matrix-transfer method as follows. Note that

\[
E(u_0, \ldots, u_l) = -J \sum_{i=0}^{l} u_i u_{i+1} + H \frac{u_i + u_{i+1}}{2}
\]

\[
= \alpha_{u_0,u_1} + \cdots + \alpha_{u_l,u_0},
\]

where \( \alpha_{u_0,u_1} = -J \sum_{i=0}^{l} u_i u_{i+1} - H \frac{u_i + u_{i+1}}{2} \). It follows that

\[
Z(\beta) = \sum_{(u_0, \ldots, u_l) \in \{\pm 1\}^{l+1}} \exp(-\beta E(u_0, \ldots, u_l))
\]

\[
= \sum_{u_0 \in \{\pm 1\}} \cdots \sum_{u_l \in \{\pm 1\}} \exp(-\beta(\alpha_{u_0,u_1} + \cdots + \alpha_{u_l,u_0}))
\]

\[
= \sum_{u_0 \in \{\pm 1\}} \cdots \sum_{u_l \in \{\pm 1\}} \exp(-\beta \alpha_{u_0,u_1}) \cdots \exp(-\beta \alpha_{u_l,u_0}).
\]

Let

\[
A = \begin{bmatrix}
\exp(-\beta \alpha_{1,1}) & \exp(-\beta \alpha_{1,-1}) \\
\exp(-\beta \alpha_{-1,1}) & \exp(-\beta \alpha_{-1,-1})
\end{bmatrix} = \begin{bmatrix}
\exp(\beta (J + H)) & \exp(-\beta J) \\
\exp(-\beta J) & \exp(\beta (J - H))
\end{bmatrix}.
\]

We index the rows and the columns of \( A \) by \( \{\pm 1\} \), that is,

\[
A_{\pm 1} = \exp(\beta (J + H)), \quad A_{-1,1} = \exp(-\beta J), \quad A_{1,-1} = \exp(\beta (J - H)).
\]

Note that the matrix \( A \) can be considered as the adjacency matrix of the weighted digraph \( \Gamma \) shown in Fig. 13.

![Fig. 13. A weighted digraph with two vertices.](image)

Then \( \exp(-\beta \alpha_{u_0,u_1}) \cdots \exp(-\beta \alpha_{u_l,u_0}) \) corresponds to the walk \((u_0, u_1, \ldots, u_l, u_{l+1})\) in \( \Gamma \). Recall that \( u_{l+1} = u_0 \). This implies that
Let \( A \) be the digraph with vertex set \( \{1, 2, 3\} \) and edge set \( \{(1, 3), (2, 1), (3, 1), (3, 2)\} \). Then its transposed adjacency matrix is

\[
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]
Exercise 3.4. Let $\Gamma$ be the digraph with vertex set $\{1, 2, 3, 4\}$ and edge set
\[
\{(1, 4), (2, 1), (3, 1), (3, 2), (3, 4), (4, 1), (4, 3)\}.
\]
Calculate the transposed adjacency matrix of $\Gamma$.

3.1 Step A: In-degree rank

If page 1 is important, then it may have a lot of backlinks. In other words, many pages have links to page 1. For example, in Fig. 16, page 1 may be more important than page 2 since the former has more backlinks than the latter.

We now define our first rank function. Let $\Gamma$ be a digraph with $n$ vertices and transposed adjacency matrix $T$. For a vertex $i$ of $\Gamma$, define $r_d(i)$ to be the number of backlinks of $i$, that is,
\[
r_d(i) = \sum_{j=1}^{n} T_{ij}.
\]
(3.1)

In other words, $r_d(i)$ is the sum of the entries in the $i$th row of $T$. The number $r_d(i)$ is called the in-degree of $i$.

Example 3.5. Let $\Gamma$ be the digraph of Example 3.3. We see that $r_d(1) = 2$ since pages 2 and 3 have links to page 1 (see Fig. 17). Table 1 shows $r_d(i)$ for $i \in \{1, 2, 3\}$.

| $i$ | $r_d(i)$ |
|-----|----------|
| 1   | 2        |
| 2   | 1        |
| 3   | 1        |

Table 1. In-degrees.
From this, we estimate the importance of the three pages as follows: page 1 is the most important, and pages 2 and 3 are the second.

**Exercise 3.6.** Let $\Gamma$ be the digraph of Exercise 3.4. Estimate the importance of the four pages by using $r_a$.

### 3.2 Step B: Weighted in-degree rank

Suppose that some tourist attractions in Sendai are recommended on pages 1 and 2. Page 1 lists ten places and page 2 lists one hundred places. If pages 1 and 2 are equally reliable, we will consider that the recommendation of page 1 should be more worth than that of page 2. Based on this idea, we assign a weight to each edge by using out-degrees.

We first define out-degrees and weights. Let $\Gamma$ be a digraph with $n$ vertices and transposed adjacency matrix $T$. For a vertex $i$ of $\Gamma$, the out-degree $d_i$ of $i$ is defined to be the number of edges from $i$, that is,

$$d_i = \sum_{k=1}^{n} T_{ki}.$$ 

If $(i, j)$ is an edge of $\Gamma$, then we assign weight $1/d_i$ to this edge. Let $H$ denote the $n \times n$ square matrix whose $(i, j)$ entry is

$$T_{ij}/d_i.$$ 

The matrix $H$ is called the hyperlink matrix of $\Gamma$.

**Example 3.7.** Let $\Gamma$ be the digraph of Example 3.5. Since page 1 has one link, the out-degree $d_1$ of page 1 is one, so the edge $(1, 3)$ is weighted by 1. Similarly, the edge $(2, 1)$ is weighted by 1. Since page 3 has two links, the out-degree $d_3$ of page 3 is two, so the edges $(3, 1)$ and $(3, 2)$ are weighted by $1/2$. Figure 19 shows the weights and the hyperlink matrix of $\Gamma$.

We now define the second rank function. For a vertex $i$ of a digraph $\Gamma$, we define $r_b(i)$ as the weighted in-degree of $i$, that is,

$$r_b(i) = \sum_{j=1}^{n} \frac{T_{ij}}{d_j}. \quad (3.2)$$

In other words, $r_b(i)$ is the sum of the entries in the $i$th row of the hyperlink matrix of $\Gamma$.

**Example 3.8.** Let $\Gamma$ be the digraph of Example 3.7. By the definition of $r_b$, we see that

$$r_b(1) = 0 + 1 + \frac{1}{2} = \frac{3}{2},$$

$$r_b(2) = 0 + 0 + \frac{1}{2} = \frac{1}{2},$$

$$r_b(3) = 1 + 0 + 0 = 1.$$
Therefore we estimate the importance of the three pages as follows: page 1 is the most important, page 3 is the second, and page 2 is the third.

Exercise 3.9. Let \( \Gamma \) be the digraph of Exercise 3.6. Estimate the importance of the four pages by using \( r_b \).

3.3 Step C: Eigenvector rank

Suppose again that some tourist attractions are recommended on pages 1 and 2. This time, suppose that both pages list ten places, but page 1 is much more reliable than page 2. Thus we consider that the recommendation of page 1 should be more worth than that of page 2. How can we achieve this?

Let \( \Gamma \) be a digraph with \( n \) vertices and transposed adjacency matrix \( T \). For a vertex \( i \) of \( \Gamma \), we will define \( r_c(i) \) by

\[
\begin{align*}
    r_c(i) &= \sum_{j=1}^{T} \frac{T_{ij}}{d_j} r_c(j). \quad (3.3)
\end{align*}
\]

For example, suppose that

- \( r_c(1) = 500, d_1 = 10 \), page 1 has a link to page 3,
- \( r_c(2) = 3, d_2 = 10 \), page 2 has a link to page 4.

Then page 1 gives 50 scores to page 3, and page 2 gives 0.3 scores to page 4 as expected. Although this definition looks strange because \( r_c \) is defined by using \( r_c \), it works well as we will see in Example 3.10.

Example 3.10. Let \( \Gamma \) be the digraph of Example 3.7. By the definition of \( r_c \), we find that

\[
\begin{align*}
    r_c(1) &= 0 \cdot r_c(1) + \frac{1}{2} r_c(3) = r_c(2) + \frac{1}{2} r_c(3), \quad (3.4)
    r_c(2) &= 0 \cdot r_c(1) + 0 \cdot r_c(2) + \frac{1}{2} r_c(3) = \frac{1}{2} r_c(3), \quad (3.5)
    r_c(3) &= r_c(1) + 0 \cdot r_c(2) + 0 \cdot r_c(3) = r_c(1). \quad (3.6)
\end{align*}
\]

This can be seen as a system of linear equations of \( r_c(1), r_c(2), \) and \( r_c(3). \) Let us solve it. Since \( r_c(2) = \frac{1}{2} r_c(3) \) and \( r_c(3) = r_c(1) \), it follows that

\[
\begin{align*}
    \begin{bmatrix}
    r_c(1) \\
    r_c(2) \\
    r_c(3)
    \end{bmatrix}
    &= \begin{bmatrix}
    2 \\
    1 \\
    2
    \end{bmatrix}.
\end{align*}
\]

If \( c = 0 \), then \( r_c(i) = 0 \) for \( i \in \{1, 2, 3\} \), and hence \( c \) should be non-zero for our purpose. By using \( r_c \), we estimate the importance of the three pages as follows: pages 1 and 3 are the most important, and page 2 is the second.

Here, we can rewrite (3.4)–(3.6) as follows:

\[
\begin{align*}
    \begin{bmatrix}
    r_c(1) \\
    r_c(2) \\
    r_c(3)
    \end{bmatrix}
    &= \begin{bmatrix}
    0 & 1 & \frac{1}{2} \\
    0 & 0 & \frac{1}{2} \\
    1 & 0 & 0
    \end{bmatrix}
    \begin{bmatrix}
    r_c(1) \\
    r_c(2) \\
    r_c(3)
    \end{bmatrix}.
\end{align*}
\]
In other words,

$$r_c = Hr_c,$$

where $$r_c = [r_c(1) \ r_c(2) \ r_c(3)]^\top$$. Surprisingly, $$r_c$$ therefore is an eigenvector of $$H$$ with eigenvalue 1. This is the reason why we prefer transposed adjacency matrices to adjacency matrices. (We have to consider left eigenvectors if we use adjacency matrices. Since this article is for beginners, the author avoided using left eigenvectors to prevent confusion.)

From the above observation, we define $$r_c$$ as an eigenvector of $$H$$ with eigenvalue 1 such that $$\sum_{i=1}^n r_c(i) = 1$$. Here, the condition $$\sum r_c(i) = 1$$ is for normalization. Unfortunately, there is a problem with this definition because $$T$$ may not have such an eigenvector.

**Exercise 3.11.** Let $$\Gamma$$ be the digraph of Exercise 3.9. Estimate the importance of the four pages by using $$r_c$$.

**Exercise 3.12.** Let $$\Gamma$$ be the digraph with vertex set $$\{1, 2, 3\}$$ and edge set $$\{(1, 3), (3, 1), (3, 2)\}$$. Estimate the importance of the three pages by using $$r_c$$.

![Fig. 22. A digraph with three vertices and three edges.](image)

### 3.4 Random walks

A vertex said to be **dangling** or **sink** if its out-degree is 0. For example, if $$\Gamma$$ is the digraph of Exercise 3.12, then the out-degree $$d_2$$ of vertex 2 is 0, so this vertex is dangling. A digraph has no dangling vertices, then its hyperlink matrix has an eigenvector with eigenvalue 1 (see Remark 3.18), however, if it has a dangling vertex, then its hyperlink matrix may not have such an eigenvector as we have seen in Exercise 3.12.

To solve the above problem, Brin and Page adjusted the hyperlink matrix by using a random walk model. In this model, we randomly click a link to get another page. Suppose that page $$i$$ has a link to page $$j$$. If we are on page $$i$$, then we move to page $$j$$ in the next step with probability $$\frac{1}{d_i}$$.

**Example 3.13.** Let $$\Gamma$$ be the digraph with vertex set $$\{1, 2, 3, 4\}$$ and edge set $$\{(1, 2), (1, 3), (1, 4)\}$$. If we are now on page 1 in this digraph, then the probability that we next move to page 2 is $$\frac{1}{3}$$.

![Fig. 23. A random walk on $$\Gamma$$](image)

**Example 3.14.** Let $$\Gamma$$ be the digraph of Example 3.7. We are now on page 1. In the first step, we move to page 3. In the second step, we either move to page 1 with probability $$\frac{1}{2}$$ or move to page 2 with probability $$\frac{1}{2}$$.

![Fig. 24. A random walk on $$\Gamma$$](image)
Let \( p_t(i) \) be the probability of being on page \( i \) at step \( t \) and

\[
p_t = \begin{bmatrix}
p_t(1) \\
p_t(2) \\
p_t(3)
\end{bmatrix}.
\]

Then

\[
p_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad p_4 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.
\]

We see that

\[
\begin{bmatrix} p_t(1) \\ p_t(2) \\ p_t(3) \end{bmatrix}^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^2.
\]

Thus

\[
p_t = H p_{t-1} = H^t p_0.
\]

For example,

\[
p_{15} = H^{15} p_0 = \frac{1}{128} \begin{bmatrix} 51 \\ 26 \\ 51 \end{bmatrix} = \begin{bmatrix} 0.3984375 \\ 0.203125 \\ 0.3984375 \end{bmatrix} \approx \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}.
\]

Since

\[
r_c = \frac{1}{5} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix},
\]

we can observe that the limit of \( p_t \) as \( t \) approaches \( \infty \) may be equal to \( r_c \) (see Remark 3.15). From this observation, we will define PageRank by adjusting our random work model in the next two sections.

**Remark 3.15.** Let us illustrate the reason why the limit of \( p_t \) is equal to \( r_c \). The hyperlink matrix \( H \) has the following three eigenvalues:

\[
1, \quad \frac{-1 - i}{2}, \quad \frac{-1 + i}{2},
\]

where \( i \) is the imaginary unit. The corresponding eigenvectors are as follows:

\[
u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -i \\ -1 + i \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ i \\ -1 - i \end{bmatrix}.
\]

We can express \( p_0 \) as a linear combination of \( u_1, u_2, u_3 \). Indeed,

\[
p_0 = \frac{1}{5} u_1 + \frac{3 - i}{10} u_2 + \frac{3 + i}{10} u_3.
\]

Therefore

\[
p_t = H^t p_0 = H^t \left( \frac{1}{5} u_1 + \frac{3 - i}{10} u_2 + \frac{3 + i}{10} u_3 \right) = \frac{1}{5} u_1 + \frac{3 - i}{10} \left( \frac{-1 - i}{2} \right)^t u_2 + \frac{3 + i}{10} \left( \frac{-1 + i}{2} \right)^t u_3.
\]

Since the absolute values of \( \frac{-1 - i}{2} \) and \( \frac{-1 + i}{2} \) are less than 1, we see that the limit of \( p_t \) as \( t \) approaches \( \infty \) is equal to \( \frac{1}{5} u_1 \), that is, \( r_c \).

**3.5 Step D: First adjustment**

If we reach a dangling vertex, then we get stuck. For example, in the left digraph in Fig. 25, if we are on page 4, we cannot walk anymore. To keep going, we may enter an address into the URL bar. In other words, we jump to a random
page. This is called teleportation. By adding teleportation, we obtain the right graph in Fig. 25. For example, if we are on page 4, then we will move to page \(i\) with probability \(1/4\) for \(i \in \{1, 2, 3, 4\}\).

![Fig. 25. The right digraph is obtained from the left one by adding teleportation.](image)

We now define the fourth rank function. Let \(\Gamma\) be a digraph with hyperlink matrix \(H\). Let \(S\) be the matrix obtained from \(H\) by replacing the 0 columns with \(1\), where 0 and 1 are the column vectors whose all entries are 0 and 1, respectively. For example, if \(\Gamma\) is the left graph in Fig. 25, then

\[
H = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix},
\]

so the fourth column is 0. Hence

\[
S = \begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{3} \\
0 & 0 & 0 & \frac{1}{3} \\
0 & \frac{1}{2} & 0 & \frac{1}{3} \\
1 & 0 & 1 & \frac{1}{3}
\end{bmatrix}.
\]

The matrix \(S\) will be called the *stochastic matrix* of \(\Gamma\) in this article. We define \(rd\) to be an eigenvector of \(S\) with eigenvalue 1 satisfying \(\sum_{i=1}^{n} rd(i) = 1\).

**Exercise 3.16.** Let \(\Gamma\) be the digraph of Exercise 3.12. Estimate the importance of the three pages by using \(rd\).

**Exercise 3.17.** Let \(\Gamma\) be the digraph with vertex set \(\{1, 2, 3, 4\}\) and edge set \(\{(1, 2), (2, 1), (3, 4), (4, 3)\}\). Estimate the importance of the four pages by using \(rd\).

**Remark 3.18.** Although the hyperlink matrix \(H\) may not have an eigenvector with eigenvalue 1 as we have seen in Exercise 3.12, the matrix \(S\) always has such an eigenvector. Moreover, we can show that the first eigenvalue of \(S\) is 1, that is, if \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the eigenvalues of \(S\) with \(|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|\), then \(\lambda_1 = 1\).

We first show that 1 is an eigenvalue of \(S\). By the definition of \(S\), we see that the sum of the entries in the \(i\)th column of \(S\) is 1, that is,

\[
\sum_{i=1}^{n} S_{ii} = 1.
\]

Therefore the all-one vector \(1\) is an eigenvector of the matrix \(S^T\) with eigenvalue 1, since

\[
S^T 1 = \begin{bmatrix}
S_{11} & \cdots & S_{1n} \\
\vdots & \ddots & \vdots \\
S_{1n} & \cdots & S_{nn}
\end{bmatrix}
\begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} = \begin{bmatrix}
\sum_{k=1}^{n} S_{k1} \\
\vdots \\
\sum_{k=1}^{n} S_{kn}
\end{bmatrix} = \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix}.
\]

This implies that

\[
det(S^T - I) = 0.
\]

Since

\[
det(S - I) = det((S - I)^T) = det(S^T - I),
\]

it follows that 1 is also an eigenvalue of \(S\).

Next, we show that 1 is the first eigenvalue of \(S\). Suppose that \(S\) has an eigenvector \(u\) with eigenvalue \(\lambda\). Then
\[ \lambda u_i = \sum_{j=1}^{n} S_{ij} u_j \quad \text{for every } i \in \{1, 2, \ldots, n\}. \quad (3.7) \]

We choose \( i \) so that
\[ |u_i| = \max\{|u_1|, |u_2|, \ldots, |u_n|\}. \]

Note that \( |u_i| > 0 \) since \( u \) is an eigenvector. By (3.7), we see that
\[ |\lambda u_i| = \sum_{j=1}^{n} |S_{ij} u_j| = \sum_{j=1}^{n} |u_j|. \]

Therefore \( |\lambda| \leq 1 \).

### 3.6 Step E: PageRank

As we have seen in Exercise 3.17, \( r_d \) may not be uniquely determined. We make the second adjustment. Even when we are on a non-dangling page, we may enter an address into the URL bar. Fix a real number \( \alpha \) with \( 0 < \alpha < 1 \). On a non-dangling page, we either click a link with probability \( \alpha \) or teleport to a random page with probability \( 1 - \alpha \).

**Example 3.19.** Let \( \Gamma \) be the digraph of Exercise 3.17. Set \( \alpha = \frac{4}{5} \). Suppose that we are on page 1. We either click a link and move to page 2 with probability \( \frac{4}{5} \) or teleport to a random page with probability \( \frac{1}{5} \). Thus we move to page 2 with probability \( \frac{17}{20} = \left( \frac{4}{5} + \frac{1}{5} \right) \) or move to page \( i \) with probability \( \frac{1}{20} \) for \( i \in \{1, 3, 4\} \).

![Fig. 26. We either click a link or teleport to a random page.](image)

For a digraph \( \Gamma \) with \( n \) vertices, define
\[ G = \alpha S + (1 - \alpha) \frac{1}{n} J, \]
where \( S \) is the stochastic matrix of \( \Gamma \) and \( J \) is the all-one matrix, that is, \( J_{ij} = 1 \) for \( 1 \leq i, j \leq n \). Then the \((i, j)\) entry of \( G \) is equal to the probability of moving to page \( i \) from page \( j \). The matrix \( G \) is called the Google matrix (with parameter \( \alpha \)) of \( \Gamma \). We define \( r_e \) to be an eigenvector of \( G \) with eigenvalue 1 satisfying \( \sum r_e(i) = 1 \). The vector \( r_e \) is called the PageRank of \( \Gamma \).

**Example 3.20.** Let \( \Gamma \) be the digraph of Exercise 3.17. Set \( \alpha = \frac{4}{5} \). Then
\[ G = \frac{4}{5} S + \frac{1}{5} \cdot \frac{1}{4} J = \frac{4}{5} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{17}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{17}{20} & \frac{1}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} & \frac{17}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{17}{20} \end{bmatrix}. \]

Note that the probability moving page 1 to page 2 is \( \frac{17}{20} \), which is equal to the \((2, 1)\) entry of \( G \).

We now calculate \( r_e \). Let \( r_e = [x \ y \ z \ w]^\top \). By definition,
\[
\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 1 & 17 & 1 & 1 \\ 17 & 1 & 1 & 1 \\ 1 & 1 & 17 & 1 \\ 1 & 1 & 1 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.
\]

Hence
\[-19x + 17y + z + w = 0, \quad (3.8)\]
\[17x - 19y + z + w = 0, \quad (3.9)\]
\[x + y - 19z + 17w = 0, \quad (3.10)\]
\[x + y + 17z - 19w = 0. \quad (3.11)\]

By adding (3.8) and (3.9),
\[-2x - 2y + 2z + 2w = 0. \quad (3.12)\]

Hence
\[x + y = z + w. \quad (3.12)\]

It follows from (3.8), (3.10), and (3.12) that
\[-18x + 18y = 0 \quad \text{and} \quad -18z + 18w = 0. \]

Therefore \(x = y = z = w.\) Since \(x + y + z + w = 1,\) we see that
\[r_e = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \]

Note that \(r_e\) is uniquely determined.

In this section, we have considered the question of how we can estimate the importance of pages. As an answer to this question, we introduced PageRank \(r_e.\) In Steps A and B, we defined \(r_a\) and \(r_b\) by using degrees. We then provided \(r_c\) by using eigenvalues in Step C, and found that \(r_c\) sometimes does not exist. To solve this problem, by using the random walk model, we defined \(r_d\) and \(r_e\) in Steps D and E. Although \(r_d\) may not be uniquely determined, PageRank \(r_e\) is uniquely determined (see Remark 3.23).

**Exercise 3.21.** Let \(\Gamma\) be the digraph with vertex set \(\{1, 2, 3\}\) and edge set \(\{(1, 2), (1, 3), (3, 1), (3, 2)\}.\) Set \(\alpha = \frac{2}{3}.\) Estimate the importance of the three pages by using \(r_e.\)

![Fig. 27. A digraph with three vertices and four edges.](image)

**Exercise 3.22.** Let \(\Gamma\) be the digraph of Example 1.5. Set \(\alpha = \frac{1}{2}.\) Estimate the importance of the twelve pages by using \(r_e\) (we may need to use a computer program).

**Remark 3.23.** By using the Perron–Frobenius theorem [13–15, 28], we can show that \(r_e\) is positive and uniquely determined (see, for example, [22, 24] for details).

A matrix is said to be positive if all entries are positive. For example,
\[
\begin{bmatrix}
1 & \frac{1}{2} \\
\frac{1}{2} & 3
\end{bmatrix}
\]
is positive and
\[
\begin{bmatrix}
1 & 0 \\
4 & 3
\end{bmatrix}
\]
is not positive since the \((1, 2)\) entry is not positive. The Perron–Frobenius theorem asserts that if a square matrix \(A\) is positive then the following statements hold.

1. The first eigenvalue \(\lambda_1\) of \(A\) is positive and simple, that is, if \(u\) and \(u'\) are eigenvectors of \(A\) with eigenvalue \(\lambda_1,\) then \(u = cu'\) for some \(c \neq 0.\)
2. There is a positive eigenvector with eigenvalue \(\lambda_1.\)

As we have seen in Remark 3.18, the first eigenvalue of the matrix \(G\) is 1. Since \(G\) is positive, it follows from the Perron–Frobenius theorem that \(r_e\) is positive and uniquely determined.
A Solutions

A.1 Section 1

Exercise 1.6

We give some examples of graphs. See [3, 8, 10, 19, 33] for other examples.

(1) Transportation

The graph in Fig. A-1 shows the Tohoku University campus bus stops [35].

(2) Dressing

The digraph in Fig. A-2 shows the order of dressing in the morning. For example, we have to wear socks before wearing shoes.

(3) Matching

In the graph in Fig. A-3, vertices represent men and women, and there is an edge between a man and a woman if they prefer each other. We want to make as many couples as possible.

Let $\Gamma$ be an undirected graph without loops. A matching of $\Gamma$ is a set of edges such that no two of them share common vertices. A matching is said to be perfect if it contains all vertices. For example, Fig. A-4 shows a perfect matching.

(4) Scheduling

Suppose that there are five meetings. The members of each meeting are shown in Table A-1.

Each meeting uses an hour. We want to schedule them to avoid conflicts. For example, Meetings 1 and 2 cannot be scheduled for the same time slot because Alice has to attend both of them. If all members will gather at 1 pm, what time can they finish the five meetings in the shortest? We can model this problem by using the following graph.
In this graph, \(i, j\) is an edge if and only if there is a person who has to attend Meetings \(i\) and \(j\). For example, \(\{1, 2\}\) is an edge since Alice is a member of both Meetings 1 and 2. Our scheduling problem is equivalent to coloring the vertices of the graph so that two vertices \(i\) and \(j\) are assigned two different colors whenever \(\{i, j\}\) is an edge. For example, we can color this graph by using three colors as shown in Fig. A-6.

![Graph](image)

**Fig. A-6.** Vertices 1 and 4 are colored by black, 2 and 5 by gray, and 3 by white.

From this, we can schedule the meetings as follows:

| Time  | Meeting 1 | Meeting 2 | Meeting 3 | Meeting 4 | Meeting 5 |
|-------|-----------|-----------|-----------|-----------|-----------|
| 1 pm  | Meeting 1 | Meeting 4 |           |           |           |
| 2 pm  | Meeting 2 | Meeting 5 |           |           |           |
| 3 pm  | Meeting 3 |           |           |           |           |

(5) Rigidity

Consider a grid framework consists of rods and rotatable joints. Rods cannot stretch and compress. For example, the framework in Fig. A-7(a) is flexible. In contrast, the framework in Fig. A-7(b) is rigid because it cannot be deformed without changing the lengths of rods.

![Flexible and Rigid Frameworks](image)

**Fig. A-7.** Rigidity.

Let us consider the following two frameworks in Fig. A-8.

![Rigid and Flexible Grids](image)

**Fig. A-8.** Rigid or flexible?
The left framework is rigid and the right one is flexible. Indeed, the right framework can be deformed without changing the lengths of any rods as shown in Fig. A-9.

![Fig. A-9. We can deform the right framework.](image)

How can we determine whether a framework is rigid or flexible? We can solve this problem by using graphs. For an $n \times m$ grid framework, we consider the graph with vertex set $\{r_1, \ldots, r_n, c_1, \ldots, c_m\}$ and edge set $\{[r_i, c_j] : \text{there is a rod in the } i\text{th row and the } j\text{th column}\}$.

For example, the corresponding graphs of the two frameworks in Fig. A-8 are as shown in Fig. A-10.

![Fig. A-10. Corresponding graphs.](image)

We see that the left graph in Fig. A-10 is connected, that is, there is a walk from $i$ to $j$ for every pair of vertices $i$ and $j$. Here, a walk is a sequence $(v_0, v_1, \ldots, v_l)$ of vertices such that $(v_0, v_1), (v_1, v_2), \ldots, (v_l-1, v_l)$ are edges. See Sect. 2 for details. In contrast, the right graph is not connected. It is known that a grid framework is rigid if and only if the corresponding graph is connected (see, for example, [2, 7, 19]). For example, the framework in Fig. A-11 is flexible.

![Fig. A-11. A flexible framework.](image)

See [17] for details of rigidity.

(6) Gray codes

In the binary numeral system, a number is expressed by using 0 and 1. For example, $5 = 2^2 + 1 = 101_2$ and $11 = 2^3 + 2 + 1 = 1011_2$. The subscript 2 indicates the base. Let us consider $3 = 011_2$ and $4 = 100_2$. The two numbers 3 and 4 are near but their representations differ in three positions. In some cases, this is not convenient. For example, in some devices like rotary encoders, there is a risk of reading bits before the bit change is completed. Let us consider the transition from 011_2 (or 3) to 100_2 (or 4). Because bits do not change simultaneously, 011_2 changes to 100_2 via, for example, 001_2 and 101_2 as follows: $011_2 \rightarrow 001_2 \rightarrow 101_2 \rightarrow 100_2$. Thus there is a possibility to produce 001_2 or 101_2 accidentally.

We can solve this problem by using Gray codes [18]. In the following number system, two successive numbers differ only in one position.

|   | 000 | 001 | 101 | 100 | 110 | 111 | 011 | 010 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 0 | 000 | 001 | 101 | 100 | 110 | 111 | 011 | 010 |

This is called a Gray code. In general, an $n$-bit Gray code is a (cyclic) ordering of $n$-bit strings such that two successive strings differ only in one position. See [12, 29] for details of Gray codes.
Gray codes correspond to a special type of walks, called Hamiltonian cycle. Let $\Gamma$ be the graph shown in Fig. A-12. Then our Gray code corresponds to the following walk:

$$(000, 001, 101, 100, 110, 111, 011, 010).$$

A walk is called a Hamiltonian cycle if it satisfies the following two conditions:

1. It visits every vertex exactly once.
2. $v_f = v_0$.

(7) Games

We can represent certain games by using digraphs. Let us consider the following game called Treblecross or -007. This game is a Tic-Tac-Toe game played by two players with a strip of squares. Two players alternately write the symbol $X$ in a square. The player who creates a line of three crosses wins. Figure A-13 shows an example of Treblecross. Although this game is simple, it is not completely solved. We now represent Treblecross as a digraph. Consider the digraph whose vertex set is the set of positions in this game and there is an edge from a position $P$ to a position $Q$ if moving from $P$ to $Q$ is legal. Figure A-15 shows the digraph of Treblecross with four squares. In this way, many games can be represented by using digraphs. This allows us to analyze them in a unified manner. We will describe how to analyze a class of games, including Treblecross, called impartial games.

A (short) impartial game is defined as a digraph such that the maximum length of a walk from each vertex is finite. For example, Treblecross is an impartial game; however, the digraph with vertex set $\{1, 2\}$ with edge set $\{(1, 2), (2, 1)\}$ is not since the walk $(1, 2, 1, 2, \ldots)$ has infinite length. Let $\Gamma$ be an impartial game. We will consider the vertices of $\Gamma$ as the positions and the edges as the legal moves (see Remark A.1). If $P$ and $Q$ are positions in $\Gamma$, then $Q$ is called an option or out-neighbor of $P$ if $(P, Q)$ is an edge in $\Gamma$. We can show that every impartial game has a terminal position, that is, a position without options. For example, Treblecross with four squares has the three terminal positions shown in Fig. A-14.

![Fig. A-12. A Gray code corresponds to a Hamiltonian cycle.](image)

![Fig. A-13. An example of Treblecross.](image)

![Fig. A-14. Terminal positions.](image)

Remark A.1. We can consider $\Gamma$ as the following two-player game. Before the game, we pick an initial position and put a token on it. Two players alternately move the token to an option of the position where the token is currently placed. The winner is the player who moves the token to a terminal position. For example, in Treblecross, the initial position is the position where every square is empty, and moving to an option corresponds to writing the symbol $X$ in a square.

We can analyze impartial games in a unified manner by using Sprague–Grundy functions. For a vertex $P$ of an impartial game $\Gamma$, define the Sprague–Grundy value $sg(P)$ of $P$ by
\[ \text{sg}(P) = \text{mex}\{\text{sg}(Q) : Q \text{ is an option of } P\}, \]

where \( \text{mex} \) is the mex function. Recall that an impartial game has a terminal position. By definition, if \( P \) is terminal, then

\[ \text{sg}(P) = \text{mex} \emptyset = 0. \]

Sprague [31] and Grundy [20] independently proved that we can force a win by moving to an option with Sprague–Grundy value 0. In fact, they obtained a much stronger theorem, which states that every position in an impartial game can be characterized by its Sprague–Grundy value in some sense. See, for example, [1, 6, 11, 30] for details.

Let us calculate the Sprague–Grundy function of Treblecross with four squares. First, we introduce a notation for positions in Treblecross. We represent positions in Treblecross as strings over \( \{0, 1\} \). For example, the initial position is represented as 0000 and the position only marked the third square is 0010. We now calculate the Sprague–Grundy function of Treblecross. Since 1110, 0111, and 1111 are terminal, their Sprague–Grundy values are 0. Therefore

\[ \text{sg}(1101) = \text{sg}(1011) = \text{mex}\{\text{sg}(1111)\} = \text{mex}[0] = 1. \]

We also see that

\[ \text{sg}(1100) = \text{mex}\{\text{sg}(1110), \text{sg}(1101)\} = \text{mex}[0, 1] = 2 \quad \text{and} \quad \text{sg}(1010) = \text{mex}\{\text{sg}(1110), \text{sg}(1011)\} = \text{mex}[0, 1] = 2. \]

By symmetry, \( \text{sg}(0011) = \text{sg}(0101) = 2 \). In this way, we can calculate the Sprague–Grundy values of the other positions (see Fig. A·15). The Sprague–Grundy theorem asserts that the first player can force a win by moving to 0100 or 0010 since their Sprague–Grundy values are 0.

Let us consider the reason why we can force a win by moving to an option with Sprague–Grundy value 0. See Fig. A·16. Suppose that the Sprague–Grundy value of the current position \( P \) is greater than 0. Then \( P \) has an option \( Q \) with \( \text{sg}(Q) = 0 \), so we move to \( Q \). Since \( Q \) has no option with Sprague–Grundy value 0, the opponent cannot move anymore or has to move an option \( R \) with \( \text{sg}(R) > 0 \). If he cannot move, then we win. If he moves \( R \), then we again can move to an option \( S \) of \( R \) with \( \text{sg}(S) = 0 \). In this way, we can keep moving to an option with Sprague–Grundy value 0. Because the Sprague–Grundy value of a terminal position is 0, we can force a win in this way.

We have seen that games can be analyzed by using their Sprague–Grundy functions. Finding an explicit formula for the Sprague–Grundy function of a game is challenging. For example, the Sprague–Grundy function of Treblecross is not determined, which is one of the most famous open problems in combinatorial game theory [27].
A.2 Section 2

Exercise 2.2

(1) 2 walks of length 2: (1, 2, 1), (1, 3, 1).
(2) 2 walks of length 3: (1, 2, 3, 1), (1, 3, 2, 1).
(3) 6 walks of length 4: (1, 2, 1, 2, 1), (1, 2, 1, 3, 1), (1, 2, 3, 2, 1), (1, 3, 1, 2, 1), (1, 3, 1, 3, 1), (1, 3, 2, 3, 1).
(4) 10 walks of length 5: (1, 2, 1, 2, 3, 1), (1, 2, 1, 3, 2, 1), (1, 2, 3, 1, 2, 1), (1, 2, 3, 1, 3, 1), (1, 2, 3, 2, 3, 1), (1, 3, 1, 2, 3, 1), (1, 3, 1, 3, 2, 1), (1, 3, 2, 1, 3, 1), (1, 3, 2, 3, 2, 1).

We can easily count walks from 1 to 1 of length 5 by using a digraph as follows. There are two choices for v1. We first count the case when v1 is vertex 2. See the digraph in Fig. A.17. There are two choices for v2, vertex 1 or vertex 3. In this way, there are four choices for v3 and there are eight choices for v4. Note that there is no edge from vertex 1 to itself. Thus (1, 2, 1, 2, 1, 1) is not a walk, but (1, 2, 1, 2, 1, 3, 1) is a walk. We then find five walks with v1 = 2. By symmetry, we can see that there are five walks from 1 to 1 with v2 = 3. Therefore there are ten walks from 1 to 1 of length 5 in Π.

Fig. A.17. Ten walks from 1 to 1 of length 5.

Exercise 2.9

Let $C = AB$. By definition, $Cv = A(Bv)$ for every column vector $v$ of order $n$. Hence

$$
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1n} \\
C_{21} & C_{22} & \cdots & C_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1} & C_{n2} & \cdots & C_{nn}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1n} \\
B_{21} & B_{22} & \cdots & B_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
B_{n1} & B_{n2} & \cdots & B_{nn}
\end{bmatrix} =
\begin{bmatrix}
A_{11}B_{11} + A_{12}B_{21} + \cdots + A_{1n}B_{n1} \\
A_{21}B_{11} + A_{22}B_{21} + \cdots + A_{2n}B_{n1} \\
\vdots \vdots \vdots \vdots \\
A_{n1}B_{11} + A_{n2}B_{21} + \cdots + A_{nn}B_{n1}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}B_{v1}.
$$

Hence the first column of $C$ is equal to $A B v_1$. Similarly, we can show that its $i$th column is equal to $A B v_i$.

Exercise 2.13

The adjacency matrix $A$ is
Therefore
\[
A^2 = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2 \\
\end{bmatrix}, \quad A^3 = \begin{bmatrix}
2 & 3 & 3 \\
3 & 2 & 3 \\
3 & 3 & 2 \\
\end{bmatrix}, \quad A^4 = \begin{bmatrix}
6 & 5 & 5 \\
5 & 6 & 5 \\
5 & 5 & 6 \\
\end{bmatrix}, \quad A^5 = \begin{bmatrix}
10 & 11 & 11 \\
11 & 10 & 11 \\
11 & 11 & 10 \\
\end{bmatrix}.
\]

From this, we see that the \((1,1)\) entry of \(A^l\) is equal to the number of walks from 1 to 1 of length \(l\) for \(l \in \{1, 2, 3, 4, 5\}\).

**Exercise 2.15**

Note that the \((i,j)\) entry of \(A^l\) is
\[
\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_l=1}^{n} A_{k_1} A_{k_2} \cdots A_{k_{l-1}} A_{k_l}. \quad \text{for } \ell \leq 6.
\]

We see that
\[
A_{k_1} A_{k_2} \cdots A_{k_{l-1}} A_{k_l} = 1 \iff A_{k_1} = A_{k_2} = \cdots A_{k_{l-1}} = A_{k_l} = 1
\]
\(
\iff (i, k_1), (k_1, k_2), \ldots, (k_{l-2}, k_{l-1}), (k_{l-1}, j) \text{ are edges}
\]
\(\iff (i, k_1, k_2, \ldots, k_{l-1}, j) \text{ is a walk}.
\)

Therefore the \((i,j)\) entry of \(A^l\) is equal to the number of walks from \(i\) to \(j\) of length \(l\).

**Exercise 2.22**

Let \(u\) be an eigenvector of \(A\) with eigenvalue 11. Then
\[
Au = \begin{bmatrix}
2 & 6 \\
9 & 5 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix} = \begin{bmatrix} 2u_1 + 6u_2 \\
9u_1 + 5u_2 \\
\end{bmatrix} = \begin{bmatrix} 11u_1 \\
u_2 \\
\end{bmatrix}.
\]

Therefore \(u_1\) and \(u_2\) satisfy the following system of linear equations:
\[
\begin{align*}
-9u_1 + 6u_2 &= 0 \\
99u_1 - 6u_2 &= 0.
\end{align*}
\]

By solving this, we see that
\[
\begin{bmatrix}
2c \\
3c \\
\end{bmatrix}
\]

is an eigenvector of \(A\) with eigenvalue 11 for \(c \neq 0\). Indeed,
\[
\begin{bmatrix}
2 & 6 \\
9 & 5 \\
\end{bmatrix} \begin{bmatrix}
2 \\
3 \\
\end{bmatrix} = \begin{bmatrix}
22 \\
33 \\
\end{bmatrix}.
\]

**Exercise 2.25**

Let \(A\) be the adjacency matrix of \(\Gamma\). Then
\[
A = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}.
\]

We first calculate the eigenvalues of \(A\). Since
\[
\det(A - \lambda I) = -(1 - \lambda)x - 1 = x^2 - x - 1,
\]
it follows that the eigenvalues of \(A\) are \((1 \pm \sqrt{5})/2\). Let
\[
\tau = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \rho = \frac{1 - \sqrt{5}}{2}.
\]

Note that \(\tau^2 = \tau + 1\) and \(\rho^2 = \rho + 1\) since \(\tau\) and \(\rho\) are solutions of \(x^2 - x - 1 = 0\).

Next, we calculate eigenvectors. Let \(u\) be an eigenvector of \(A\) with eigenvalue \(\tau\). Then
\[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix} = \tau \begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix}.
\]

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Then 
\[
\begin{align*}
(1 - \tau)u_1 + u_2 &= 0 \\
u_1 - \tau u_2 &= 0.
\end{align*}
\]

From this, let 
\[
u_1 = \begin{bmatrix} \tau \\ 1 \end{bmatrix}.
\]

Then \(u_1\) is an eigenvector of \(A\) with eigenvalue \(\tau\). Indeed, since \(\tau^2 = \tau + 1\), we see that
\[
\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} \tau + 1 \\ \tau \end{bmatrix} = \tau \begin{bmatrix} \tau \\ 1 \end{bmatrix}.
\]

Similarly, let 
\[
u_2 = \begin{bmatrix} \rho \\ 1 \end{bmatrix}.
\]

Then \(u_2\) is an eigenvector of \(A\) with eigenvalue \(\rho\). Let 
\[
\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \tau & \rho \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \tau & 0 \\ 0 & \rho \end{bmatrix}.
\]

Then 
\[
U^{-1} = \frac{1}{\tau - \rho} \begin{bmatrix} 1 & -\rho \\ -1 & \tau \end{bmatrix}
\]

and
\[
AU = \begin{bmatrix} Au_1 & Au_2 \end{bmatrix} = \begin{bmatrix} \tau u_1 & \rho u_2 \end{bmatrix} = U\Lambda.
\]

Hence 
\[
A' = U\Lambda'U^{-1} = \frac{1}{\tau - \rho} \begin{bmatrix} \tau & \rho \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t' & 0 \\ 0 & \rho' \end{bmatrix} \begin{bmatrix} 1 & -\rho \\ -1 & \tau \end{bmatrix} = \frac{1}{\tau - \rho} \begin{bmatrix} \tau t' - \rho t' & \tau t' - \rho t' \\ \rho - \rho & -\rho - \rho' \end{bmatrix} = \frac{1}{\tau - \rho} \begin{bmatrix} \tau^{l+1} - \rho^{l+1} & \tau^l - \rho^l \\ \tau^l - \rho^l & \tau^l - \rho^l \end{bmatrix}.
\]

Therefore the number of walks from 1 to 1 of length \(l\) is equal to
\[
\frac{1}{\tau - \rho} (\tau^{l+1} - \rho^{l+1}).
\]

For example,
\[
\begin{align*}
\text{For } l = 0: & \quad \frac{1}{\tau - \rho} (\tau - \rho) = 1. \\
\text{For } l = 1: & \quad \frac{1}{\tau - \rho} (\tau^2 - \rho^2) = \frac{1}{\tau - \rho} (\tau + 1 - \rho - 1) = 1. \\
\text{For } l = 2: & \quad \frac{1}{\tau - \rho} (\tau^3 - \rho^3) = \frac{1}{\tau - \rho} \left( \tau^2 + \tau - \rho^2 - \rho \right) = \frac{1}{\tau - \rho} (2\tau + 1 - 2\rho - 1) = 2. \\
\text{For } l = 3: & \quad \frac{1}{\tau - \rho} (\tau^4 - \rho^4) = \frac{1}{\tau - \rho} \left( 2\tau^2 + \tau - 2\rho^2 - \rho \right) = \frac{1}{\tau - \rho} (3\tau + 2 - 3\rho - 2) = 3. \\
\text{For } l = 4: & \quad \frac{1}{\tau - \rho} (\tau^5 - \rho^5) = \frac{1}{\tau - \rho} \left( 3\tau^2 + 2\tau - 3\rho^2 - 2\rho \right) = \frac{1}{\tau - \rho} (5\tau + 3 - 5\rho - 3) = 5. \\
\text{For } l = 5: & \quad \frac{1}{\tau - \rho} (\tau^6 - \rho^6) = \frac{1}{\tau - \rho} \left( 5\tau^2 + 3\tau - 5\rho^2 - 3\rho \right) = \frac{1}{\tau - \rho} (8\tau + 5 - 8\rho - 5) = 8.
\end{align*}
\]

We see that the number of walks from 1 to 1 of length \(l\) is equal to the \(l\)th Fibonacci number.

**Exercise 2.26**

Let \(\Gamma\) be the digraph with vertex set \(\{1, 2, 3, 4\}\) and edge set
\[(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\].

It suffices to count the number of walks from 1 to 1 of length \(l\) in \(\Gamma\).

![A digraph with four vertices and twelve edges.](image)

Let \(\Gamma\) be the adjacency matrix of \(\Gamma\). Then

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}.
\]

We first calculate the eigenvalues of \(A\). Since

\[
\det(A - xI) = (x + 1)^3(x - 3),
\]

it follows that \(-1\) and 3 are the eigenvalues of \(A\). Next, we calculate eigenvectors. Let \(u\) be an eigenvector of \(A\) with eigenvalue \(-1\). Then

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}.
\]

Hence

\[u_1 + u_2 + u_3 + u_4 = 0.\]

From this, let

\[
u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.
\]

Then \(u_1\), \(u_2\), and \(u_3\) are eigenvectors of \(A\) with eigenvalue \(-1\). Let

\[
u_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\]

It is easy to see that \(u_4\) is an eigenvector of \(A\) with eigenvalue 3. Let \(U = [u_1, u_2, u_3, u_4]\). Then \(\det(U) \neq 0\), so \(U\) is invertible. Since \((A^3)_{11} = (A^3)_{22} = (A^3)_{33} = (A^3)_{44}\), it follows from Remark 2.24 that the number of walks from 1 to 1 of length \(l\) in \(\Gamma\) is equal to

\[
\frac{3^l + 3 \cdot (-1)^l}{4}.
\]

For example,

- \(l = 0\) : \(\frac{3^0 + 3 \cdot (-1)^0}{4} = 1.\)
- \(l = 1\) : \(\frac{3 - 3}{4} = 0.\)
- \(l = 2\) : \(\frac{3^2 + 3}{4} = 3.\)
- \(l = 3\) : \(\frac{3^3 - 3}{4} = 6.\)
- \(l = 4\) : \(\frac{3^4 + 3}{4} = 21.\)
- \(l = 5\) : \(\frac{3^5 - 3}{4} = 60.\)
Remark A.2. A square matrix $B$ is said to be symmetric if $B^\top = B$. Although we have checked that $U$ is invertible in the above, we do not have to check since it is known that a symmetric matrix is always diagonalizable. For your information, while a square matrix may not be diagonalizable, it is always triangularizable. From this, we can show that $\text{tr}(A^t) = \lambda_1^t + \cdots + \lambda_n^t$ even when $A$ is not diagonalizable, where $\{\lambda_1, \ldots, \lambda_n\}$ is the multiset of eigenvalues of $A$. See, for example, [26] for details.

A.3 Section 3

Exercise 3.4

The transposed adjacency matrix of $\Gamma$ is

$$
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}.
$$

Exercise 3.6

Table A-2 shows $r_a(i)$ for $i \in \{1, 2, 3, 4\}$. From this, we estimate the importance of the four pages as follows: page 1 is the most important, page 4 is the second, and pages 2 and 3 are the third.

| $i$ | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| $r_a(i)$ | 3 | 1 | 1 | 2 |

Exercise 3.9

Figure A-19 shows the weights and the hyperlink matrix of $\Gamma$.

It follows that

$$
\begin{align*}
  r_a(1) &= 0 + 1 + \frac{1}{3} + \frac{1}{2} = \frac{11}{6}, \\
  r_a(2) &= 0 + 0 + \frac{1}{3} + 0 = \frac{1}{3}, \\
  r_a(3) &= 0 + 0 + 0 + \frac{1}{2} = \frac{1}{2}, \\
  r_a(4) &= 1 + 0 + \frac{1}{3} + 0 = \frac{4}{3}.
\end{align*}
$$

Therefore we estimate the importance of the four pages as follows: page 1 is the most important, page 4 is the second, page 3 is the third, and page 2 is the fourth.

Exercise 3.11

Let $H$ be the hyperlink matrix of $\Gamma$ and let $u$ be an eigenvector of $H$ with eigenvalue 1. Then

$$
u_1 = 0 \cdot u_1 + 1 \cdot u_2 + \frac{1}{3} u_3 + \frac{1}{2} u_4 = u_2 + \frac{1}{3} u_3 + \frac{1}{2} u_4,$$

(A-1)
$$u_2 = 0 \cdot u_1 + 0 \cdot u_2 + \frac{1}{3} u_3 + 0 \cdot u_4 = \frac{1}{3} u_3,$$

(A-2)

$$u_3 = 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 + \frac{1}{2} u_4 = \frac{1}{2} u_4,$$

(A-3)

$$u_4 = 1 \cdot u_1 + 0 \cdot u_2 + \frac{1}{3} u_3 + 0 \cdot u_4 = u_1 + \frac{1}{3} u_3.$$

(A-4)

By (A-4) and (A-3),

$$u_4 = u_1 + \frac{1}{3} u_3 = u_1 + \frac{1}{6} u_4.$$

Hence

$$u_4 = \frac{6}{5} u_1$$

and

$$u_3 = \frac{3}{5} u_1.$$

From (A-2),

$$u_2 = \frac{1}{3} u_3 = \frac{1}{5} u_1.$$

This implies that

$$r_c = \frac{1}{15} \begin{bmatrix} 5 \\ 1 \\ 3 \\ 6 \end{bmatrix}.$$

Therefore we estimate the importance of the four pages as follows: page 4 is the most important, page 1 is the second, page 3 is the third, and page 2 is the fourth.

**Exercise 3.12**

Let $H$ be the hyperlink matrix of $\Gamma$ and let $u$ be an eigenvector of $H$ with eigenvalue 1. Then

$$\begin{bmatrix} 0 & 0 & \frac{1}{7} \\ 0 & 0 & \frac{1}{7} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Hence

$$\frac{u_3}{2} = u_1 = u_2$$

and

$$u_1 = u_3,$$

so $u_3 = u_1 / 2$. This implies that $u_3 = 0$ and $u_1 = u_2 = 0$. Therefore $H$ has no eigenvectors with eigenvalue 1. In other words, we cannot define $r_c$.

**Exercise 3.16**

The hyperlink matrix of $\Gamma$ is

$$\begin{bmatrix} 0 & 0 & \frac{1}{7} \\ 0 & 0 & \frac{1}{7} \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence

$$S = \begin{bmatrix} 0 & \frac{1}{7} & \frac{1}{7} \\ 0 & \frac{1}{7} & \frac{1}{7} \\ 1 & \frac{1}{7} & 0 \end{bmatrix}.$$

We now calculate $r_d$. Since $r_d$ is an eigenvector of $S$ with eigenvalue 1, it follows that

$$\begin{bmatrix} r_d(1) \\ r_d(2) \\ r_d(3) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{7} & \frac{1}{7} \\ 0 & \frac{1}{7} & \frac{1}{7} \\ 1 & \frac{1}{7} & 0 \end{bmatrix} \begin{bmatrix} r_d(1) \\ r_d(2) \\ r_d(3) \end{bmatrix}.$$

We see that $r_d(1) = r_d(2)$ since
\[ r_d(1) = \frac{1}{3} r_d(2) + \frac{1}{2} r_d(3) = r_d(2). \]

We also see that
\[ r_d(2) = \frac{3}{4} r_d(3). \]

Therefore
\[ r_d = \frac{1}{10} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}. \]

Thus we estimate the importance of the three pages as follows: page 3 is the most important, and pages 1 and 2 are the second.

**Exercise 3.17**

The hyperlink matrix \( H \) of \( \Gamma \) is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Since there are no dangling vertices, we see that
\[ S = H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \]

We now calculate \( r_d \). By definition,
\[
\begin{bmatrix} r_d(1) \\ r_d(2) \\ r_d(3) \\ r_d(4) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_d(1) \\ r_d(2) \\ r_d(3) \\ r_d(4) \end{bmatrix}.
\]

It follows that \( r_d(1) = r_d(2) \) and \( r_d(3) = r_d(4) \). Hence
\[ r_d = \begin{bmatrix} x \\ x \\ y \\ y \end{bmatrix}, \quad x + y = \frac{1}{2}. \]

This implies that \( r_d \) is not uniquely determined. For example,
\[
\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\]

are both eigenvectors of \( S \) with eigenvalue 1. If we use the former, then pages 1 and 2 will be estimated to be the most important; however, if we use the later, then pages 3 and 4 will be estimated to be the most important. In Sect. 3.6, we will solve this problem.

**Exercise 3.21**

Since page 2 is dangling, it follows that
\[
S = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}.
\]
Therefore the Google matrix of $\Gamma$ is as follows:

$$G = \frac{2}{3}S + \frac{1}{3} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 3 & 4 \\ 4 & 3 & 4 \\ 4 & 3 & 1 \end{bmatrix}.$$ 

Suppose that $u$ is an eigenvector of $G$ with eigenvalue 1. Then

$$u_1 + 3u_2 + 4u_3 = 9u_1, \quad (A-5)$$
$$4u_1 + 3u_2 + 4u_3 = 9u_2, \quad (A-6)$$
$$4u_1 + 3u_2 + u_3 = 9u_3. \quad (A-7)$$

By subtracting (A-6) from (A-5),

$$-3u_1 = 9u_1 - 9u_2,$$
so $u_1 = \frac{3}{4}u_2$. By subtracting (A-7) from (A-5),

$$-3u_1 + 3u_3 = 9u_1 - 9u_3,$$
and hence $u_1 = u_3$. This implies that

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$ 

Therefore

$$r = \frac{1}{10} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

because $r_1(1) + r_2(2) + r_3(3) = 1$. From this, we estimate the importance of the four pages as follows: page 2 is the most important, and pages 1 and 3 are the second.

**Exercise 3.22**

Since page 1 is dangling, it follows that

$$S = \begin{bmatrix} \frac{1}{12} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Using the following python program, we can obtain an eigenvector of $G$ with eigenvalue 1.
```python
import numpy as np
import numpy.linalg as LA
S = np.matrix(
    [[1/12, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 1, 1/2, 1/2, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 1/3, 0, 1/2],
     [1/12, 1/2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 1/3, 0, 0, 1/2, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
     [1/12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]]
)
J = np.ones((12, 12))
G = 4/5 * S + 1/5 * 1/12 * J
eigvals, eigvecs = LA.eig(G)
re = eigvecs[:, 0] / np.sum(eigvecs[:, 0])
print(re)
# [[0.02573529-0.j]
# [0.32979707-0.j]
# [0.28222001-0.j]
# [0.01838235-0.j]
# [0.11484421-0.j]
# [0.0433775 -0.j]
# [0.0937318 -0.j]
# [0.01838235-0.j]
# [0.01838235-0.j]
# [0.01838235-0.j]
# [0.01838235-0.j]
# [0.01838235-0.j]]
```

From this, we estimate the importance of the four pages as follows: page 2 > page 3 > page 5 > page 7 > page 6 > page 1 > page 4 = page 8 = page 9 = page 10 = page 11 = page 12.

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