Action functionals for relativistic perfect fluids

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ABSTRACT

Action functionals describing relativistic perfect fluids are presented. Two of these actions apply to fluids whose equations of state are specified by giving the fluid energy density as a function of particle number density and entropy per particle. Other actions apply to fluids whose equations of state are specified in terms of other choices of dependent and independent fluid variables. Particular cases include actions for isentropic fluids and pressureless dust. The canonical Hamiltonian forms of these actions are derived, symmetries and conserved charges are identified, and the boundary value and initial value problems are discussed. As in previous works on perfect fluid actions, the action functionals considered here depend on certain Lagrange multipliers and Lagrangian coordinate fields. Particular attention is paid to the interpretations of these variables and to their relationships to the physical properties of the fluid.
1. Introduction

This paper contains a description of several action functionals whose extrema are the classical histories of a relativistic perfect fluid. The possible benefits of these actions are difficult to judge, but experience has shown generally that action principles are a powerful and conceptually elegant means of specifying a dynamical system [1]. They concisely encode the classical equations of motion, provide a direct and simple way of relating symmetries and conserved charges, and serve as the starting point for a canonical Hamiltonian analysis and for Hamilton–Jacobi theory. For a quantum mechanical description of a system, the existence of an action functional appears to be a necessity [2].

Perfect fluids are described locally by various thermodynamical variables. In the notation of Misner, Thorne, and Wheeler [3], they are

\[ n = \text{particle number density}, \quad (1.1a) \]
\[ \rho = \text{energy density}, \quad (1.1b) \]
\[ p = \text{pressure}, \quad (1.1c) \]
\[ T = \text{temperature}, \quad (1.1d) \]
\[ s = \text{entropy per particle}. \quad (1.1e) \]

These variables are spacetime scalar fields whose values represent measurements made in the rest frame of the fluid. The fluid motion can be characterized by its unit four–velocity vector field \( U^a \).

Previous works on relativistic and nonrelativistic perfect fluid actions [4–13] have revealed a number of common features that any such action principle must possess. In particular, it is known that no perfect fluid action can be constructed solely from the variables (1.1) and \( U^a \) unless the variations among those variables are constrained [5]. Two of the required constraints are particle number conservation \((nU^a)_{,a} = 0\) and the absence of entropy exchange between neighboring flow lines \((nsU^a)_{,a} = 0\). The remaining constraint is that the fluid flow lines should be fixed on the boundaries of spacetime. One way of enforcing this constraint is to vary

* Spacetime tensor indices are denoted by \( a, b, \text{ etc.} \), the spacetime metric is \( g_{ab} \), and spacetime covariant derivatives are denoted by a semicolon. The sign conventions of reference [3] are used throughout.
the flow lines by Lie dragging along some vector field $\xi^a$, where $\xi^a$ vanishes on the boundaries. The flow line variation then induces variations in the four–velocity $U^a$ with the result

$$\delta U^a = (\delta b^a + U_b U^a) \mathcal{L}_\xi U^b,$$

where $\mathcal{L}_\xi$ is the Lie derivative along $\xi^a$.

There is another method for handling the constraint that the flow lines should be fixed on the spacetime boundaries, suggested by the work of Lin [9, 10]. Instead of the four–velocity $U^a$, the history of the fluid is characterized by a set of spacetime scalar fields $\alpha^A$, $A = 1, 2, 3$, that are interpreted as Lagrangian coordinates for the fluid. That is, $\alpha^A(x)$ serve as labels for the fluid, specifying which flow line passes through a given spacetime point $x$. A set of Lagrangian coordinates can be generated by choosing an arbitrary spacelike hypersurface and a coordinate system $\alpha^A$ on that surface. Then each flow line is labeled by the coordinate value of the point where it intersects the hypersurface. By building an action functional using the Lagrangian coordinates $\alpha^A$, the fluid flow lines are held fixed on the spacetime boundaries by simply fixing $\alpha^A$ on the boundaries.

The particle number and entropy exchange constraints can be incorporated directly into the action via Lagrange multipliers. An action $S$ is presented in section 2 that makes use of such Lagrange multipliers along with the Lagrangian coordinates $\alpha^A$ to enforce the constraints. This action describes a perfect fluid whose equation of state is specified by giving the energy density $\rho$ as a function of number density $n$ and entropy per particle $s$. The action $S$ incorporates various features found in the action functionals discussed in references [5, 8, 14–16], but to my knowledge it has not been written previously in precisely the form discussed here. In sections 2 and 3 the Lagrangian and Hamiltonian forms of the action $S$ are developed, the Lagrangian and Hamiltonian equations of motion are displayed explicitly, global symmetries and conserved charges are analyzed, and the initial and boundary value problems are discussed.

The use of Lagrangian coordinates $\alpha^A$ and Lagrange multipliers is sometimes criticized on the grounds that these extra variables are “unphysical”. One of the results of the present work is to show that the values of each of these variables are determined to within a global symmetry transformation by the physical properties of the fluid. The situation here is formally analogous to that of a free nonrelativistic point particle moving in flat space with cartesian coordinates. In that case, Newton’s law is $\ddot{\vec{v}} = 0$, where $\vec{v}$ is the particle’s velocity and the dot denotes a time derivative. An action functional that yields this equation of motion is

$$\int dt (\vec{v} \cdot \vec{v}/2),$$
where the variations in $\vec{v}$ are constrained to those having the form $\delta \vec{v} = \vec{\xi}$. Here, $\vec{\xi}$ is an infinitesimal vector that vanishes at the initial and final times. An unconstrained action is obtained by introducing a “Lagrangian” coordinate $\vec{x}$ and constructing the velocity from $\vec{v} = \vec{x}$. The new variable $\vec{x}$ is just the particle’s coordinate location. The value of $\vec{x}$ at any given time specifies the physical location of the particle to within a symmetry translation or rotation of the cartesian coordinates. Similarly, the values of the Lagrangian coordinates $\alpha^A$ specify the physical location of the fluid flow lines to within a symmetry transformation that amounts to a change of Lagrangian coordinates. The values of the Lagrange multipliers that appear in the fluid action $S$ are also determined to within a global symmetry transformation by certain physical properties of the fluid. (The action $S$ is the analogue of the free particle action $\int dt (\vec{v} \cdot \dot{\vec{x}} - \vec{v} \cdot \vec{v}/2)$ in the sense that it is a functional of both the four–velocity $U^a$ and the Lagrangian coordinates $\alpha^A$.)

One of the fluid equations of motion obtained from the action $S$ relates the fluid four–velocity $U_a$ to the Lagrange multipliers, Lagrangian coordinates $\alpha^A$, and their gradients. Expressions of this type are often called velocity–potential representations or Clebsch [29] representations of the four–velocity. There have been numerous discussions in the literature concerning both the number of scalar fields that are mathematically required for such a representation, and the number of scalar fields that are physically natural for such a representation [4, 5, 10, 12, 15]. Section 4 contains a further discussion of these issues.

The Hamiltonian form of the perfect fluid action $S$ is a functional of the fluid number density, the fluid entropy density, the Lagrangian coordinates $\alpha^A$, and their canonical conjugates. In this case the number and entropy densities are Eulerian, that is, their values correspond to measurements made by observers at rest in space. The Eulerian densities are related to the Lagrangian (or comoving) densities $n$ and $n_s$ by a kinematical “gamma” or boost factor that in turn is determined by the local spatial velocity of the fluid. The fluid contributions to the Hamiltonian and momentum constraints are just the appropriate projections of the perfect fluid stress–energy–momentum tensor [17], and involve the spatial components $U_i$ of the fluid four–velocity. These components $U_i$ are explicitly expressed in terms of the canonical fluid variables and the Lagrangian particle number density $n$. The Lagrangian number density $n$ is itself a function of the canonical variables as implicitly determined by the equation relating the Lagrangian and Eulerian number densities.
A Hamiltonian formulation of perfect fluids that makes use of only “physical” fluid variables has been developed (see [18] and references therein) based on the Lie–Poisson brackets. However, Lie–Poisson brackets are not canonical, so there is no underlying symplectic structure and the usual “\( \int (p\dot{q} - H) \)” form of the action does not apply. On the other hand, the Lie–Poisson structure can be derived by a so–called Lagrangian to Eulerian map starting from a certain canonical Hamiltonian structure that involves the Lagrangian coordinates [18]. This canonical Hamiltonian formulation of perfect fluids is derived in section 5, starting from an associated action \( \bar{S} \) that is a functional of the Lagrangian coordinates \( \alpha^A \) only. In this action, the particle number and entropy exchange constraints are enforced by restricting \( nU^a \) and \( s \) to depend on spacetime only through certain combinations of the Lagrangian coordinates. In particular, \( s \) is given by a function \( s(\alpha) \), and \( nU^a \) is given by 
\[-\eta_{123}\epsilon^{abcd}\alpha^1,\alpha^2,\alpha^3,\alpha^4, \]
where \( \eta_{123} \) is a function of \( \alpha^A \). \( (\epsilon^{abcd} \) is obtained by raising indices on the totally antisymmetric spacetime volume form \( \epsilon_{abcd} \).) The action \( \bar{S} \), like \( S \), describes a perfect fluid whose equation of state is specified by giving the energy density \( \rho \) as a function of number density \( n \) and entropy per particle \( s \).

In addition to the local thermodynamical variables (1.1), it is also convenient to define [3]
\[
\begin{align*}
\mu & := \frac{\rho + p}{n} = \text{chemical potential } , \quad (1.2a) \\
 a & := \frac{\rho}{n} - Ts = \text{physical free energy } , \quad (1.2b) \\
f & := \frac{\rho + p}{n} - Ts = \text{chemical free energy } . \quad (1.2c)
\end{align*}
\]
The chemical potential \( \mu \) is the energy per particle required to inject a small amount of fluid into a fluid sample, keeping the sample volume and the entropy per particle \( s \) constant [3]. Similarly, the physical free energy \( a \) is the injection energy at constant number density and constant total entropy, while the chemical free energy \( f \) is the injection energy at constant volume and constant total entropy. The thermodynamical variables (1.1–2) are related by the local expression of the first law of thermodynamics, namely [3]
\[
d\rho = \mu \, dn + nT \, ds . \quad (1.3)
\]
This relationship shows that an equation of state for a perfect fluid can be specified by giving the function \( \rho(n, s) \), the energy density as a function of number density
and entropy per particle. The first law also can be written as

\[ dp = n d\mu - nT ds \]  

(1.4)

which naturally suggests an equation of state of the form \( p(\mu, s) \). Another possibility is

\[ d(na) = f \, dn - ns \, dT \]  

(1.5)

which suggests an equation of state \( a(n, T) \). In section 6, various action functionals are presented for perfect fluids specified by the equations of state \( p(\mu, s) \) and \( a(n, T) \). As special cases, actions for isentropic fluids and pressureless dust are obtained.

The perfect fluid action functionals developed by Taub [6, 7] and Schutz [15, 19, 20] have been applied to the analysis of the stability of stellar models [7, 21–23], and it is hoped that the actions discussed here can be put to similar use. Schutz’s action also has been used to quantize the combined gravity–fluid system [24, 25], where one of the Lagrange multipliers is chosen as a time coordinate. Along these same lines, the action for a pressureless dust, discussed in section 6, has been used in an investigation of the problem of time in canonical quantum gravity [26]. Another application of perfect fluid actions is found in the work of Gibbons and Hawking [27] (also see [28] and references therein). There, the action is used to obtain a thermodynamical potential that characterizes the global thermal properties of a gravitating fluid system.

2. Action \( S \) with equation of state \( \rho(n, s) \)

2.1 Action and equations of motion

By definition, the stress–energy–momentum tensor for a perfect fluid has the form

\[ T^{\alpha\beta} = \rho U^a U^b + p(g^{ab} + U^a U^b) \]  

(2.1)

where \( U^a \) is the unit four–velocity of the fluid. The equations of motion for a perfect fluid consist of the stress tensor equation of motion, namely \( T^a_{\, ; a} = 0 \), and the equation \( (n U^a)_{; a} = 0 \) expressing conservation of particle number [3]. The action functional presented in this section incorporates the stress tensor (2.1), the required equations of motion, and the first law of thermodynamics (1.3) for a perfect fluid with equation of state \( \rho(n, s) \).
The perfect fluid action $S$ is a functional of a spacetime contravariant vector density $J^a$ that is interpreted as the densitized particle number flux vector $\sqrt{-g}nU^a$. That is, the fluid four–velocity is defined by

$$U^a := J^a / |J| \quad ,$$

where

$$|J| := \sqrt{-J^ag_{ab}J^b} \quad ,$$

is the magnitude of $J^a$, and the particle number density is given by

$$n := |J| / \sqrt{-g} \quad .$$

This action is also a functional of the spacetime metric $g_{ab}$, the entropy per particle $s$, the Lagrangian coordinates $\alpha^A$, and spacetime scalars denoted by $\varphi$, $\theta$, and $\beta_A$. (The indices $A, B$ take the values 1, 2, 3.) In terms of an arbitrary equation of state $\rho(n, s)$, the action reads

$$S[g_{ab}, J^a, \varphi, \theta, s, \alpha^A, \beta_A] = \int d^4x \left\{ - \sqrt{-g} \rho(|J|/\sqrt{-g}, s) \right.$$  

$$ + J^a(\varphi_{,a} + s\theta_{,a} + \beta_A\alpha^A_{,a}) \right\} \quad .$$

The stress–energy–momentum tensor derived from this action is

$$T^{ab} := \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{ab}} = \rho U^aU^b + \left( n \frac{\partial \rho}{\partial n} - \rho \right) \left( g^{ab} + U^aU^b \right) \quad ,$$

and has the perfect fluid form (2.1) with energy density $\rho$ and pressure defined by

$$p := n \frac{\partial \rho}{\partial n} - \rho \quad .$$

This expression for pressure agrees with the relationship implied by the first law of thermodynamics (1.3).

The fluid equations of motion derived from the action (2.5) are as follows:

$$0 = \frac{\delta S}{\delta J^a} = \mu U_a + \varphi_{,a} + s\theta_{,a} + \beta_A\alpha^A_{,a} \quad ,$$

$$0 = \frac{\delta S}{\delta \varphi} = - J^a_{,a} \quad ,$$

$$0 = \frac{\delta S}{\delta \theta} = -(sJ^a)_{,a} \quad ,$$

$$0 = \frac{\delta S}{\delta s} = -\sqrt{-g} \frac{\partial \rho}{\partial s} + \theta_{,a}J^a \quad ,$$

$$0 = \frac{\delta S}{\delta \alpha^A} = -(\beta_AJ^a)_{,a} \quad ,$$

$$0 = \frac{\delta S}{\delta \beta_A} = \alpha^A_{,a}J^a \quad .$$
Solutions to these equations extremize the action $S$ with respect to variations that leave $\varphi$, $\theta$, and $\alpha^A$ fixed on the spacetime boundaries. According to equations (2.8b,c), the fields $\varphi$ and $\theta$ serve as Lagrange multipliers for the particle number conservation constraint $J^a\,_{,a} = 0$ and the entropy exchange constraint $(sJ^a)\,_{,a} = 0$, respectively. Equation (2.8f) shows that $\beta_\lambda$ are Lagrange multipliers for the constraints $\alpha^A\,_{,a}J^a = 0$ that restrict the fluid four–velocity vector to be directed along the flow lines $\alpha^A = \text{constant}$.

As discussed in the introduction, the fields $\alpha^A(x)$ are interpreted as Lagrangian coordinates for the fluid and serve as labels that specify which flow line passes through a given spacetime point $x$. A set of Lagrangian coordinates can be generated by choosing an arbitrary spacelike hypersurface and specifying a coordinate system $\alpha^A$ on that surface. Then each flow line is labeled by the coordinate value of the point where it intersects the hypersurface. It is useful to view this arbitrary spacelike hypersurface as an abstract “fluid space” whose points represent the fluid flow lines, and to view the Lagrangian coordinates $\alpha^A$ as a coordinate system on the fluid space. (The concept of “fluid space” is also discussed in reference [14], where it is called the “matter space”.) So conceptually, the fluid space is the space of Lagrangian coordinate labels, and as a manifold it is isomorphic to any spacelike hypersurface. It may be impossible to cover the fluid space with a single coordinate chart, so the Lagrangian coordinates $\alpha^A$ generally must be defined in open subsets of the fluid space. With the above interpretation, it is assumed that the fields $\alpha^A$ constitute a good set of Lagrangian coordinates, in the sense that within the appropriate open subsets each flow line carries a unique label $\alpha^A$ and the gradient $\alpha^A\,_{,a}$ is nonvanishing.

The significance of the scalar field $\theta$ is revealed by comparing the equation of motion (2.8d) with the first law of thermodynamics (1.3). This leads to the identification

$$T = \theta\,_{,a}U^a = \frac{1}{n} \frac{\partial \rho}{\partial s}$$

(2.9)

for the fluid temperature $T$, and shows that $\theta$ is the “thermasy” discussed by van Dantzig [30]. Kijowski et al. [14] have proposed a physical interpretation for thermasy, relating it to the difference between proper time along the fluid flow lines and proper time along the history of a typical fluid particle that executes chaotic motion about the flow lines. The field $\varphi$ plays a role that is mathematically analogous to $\theta$, as seen by contracting equation (2.8a) with $U^a$ and using equations (2.8f) and
(2.9) to obtain

\[ f = \varphi_a U^a . \]  

Thus, \( \theta \) is a “potential” for the temperature \( T \) and \( \varphi \) is a “potential” for the chemical free energy \( f \). The interpretations of the fields \( \theta, \varphi, \) and \( \beta_A \) are discussed further in sections 3 and 4.

As stated above, the perfect fluid equations of motion (2.8) should imply particle number conservation and the stress tensor equation of motion \( T^{ab}_{;b} = 0 \). The first of these, particle number conservation, is expressed explicitly in equation (2.8b). For the stress tensor equation of motion, first consider its projection along the fluid flow lines. This yields [3]

\[ 0 = U_a T^{ab}_{;b} = -\frac{\partial \rho}{\partial s} s_{;a} U^a , \]  

where particle number conservation has been used. This equation combines particle number conservation (2.8b) and the entropy exchange constraint (2.8c), and is indeed implied by the equations of motion. This result shows that the fluid flow is locally adiabatic, that is, the entropy per particle along the fluid flow lines is conserved.

The projection of the stress tensor equation of motion orthogonal to the flow lines gives the Euler equation, relating the fluid acceleration to the gradient of pressure [3]:

\[ 0 = (g_{ab} + U_a U_b) T^{bc}_{;c} = (\rho + p) U_{a;b} U^b + (\delta^b_a + U_a U^b) p_{;b} . \]  

Using expression (2.7) for \( p \), the Euler equation can be written as

\[ 0 = \mu U_{a;b} U^b + (\delta^b_a + U_a U^b) \frac{1}{n} p_{;b} 
\]

\[ = 2(\mu U_{a;b}) U^b + (\mu U_b)_{;a} U^b - \left( \frac{\partial \rho}{\partial n} \right)_{;b} U_a U^b \\
+ (\delta^b_a + U_a U^b) \frac{1}{n} \left( n \frac{\partial \rho}{\partial n} - \rho \right)_{;b} \\
= 2(\mu U_{a;b}) U^b - (\delta^b_a + U_a U^b) \frac{1}{n} \frac{\partial \rho}{\partial s} s_{;b} , \]  

(2.13)
where the square brackets denote antisymmetrization. Now combine equation (2.13) with the entropy conservation equation (2.11) to obtain

\[ 2V_{[a:b]}U^b = Ts_{,a}, \tag{2.14} \]

where

\[ V_a := \mu U_a \tag{2.15} \]

and the definition (2.9) for temperature \( T \) has been used. The vector \( V_a \), sometimes called the Taub current, plays an important role in relativistic fluid dynamics, especially in the description of circulation and vorticity [31]. From the interpretation of \( \mu \) as an injection energy, \( V^a \) is identified with the four–momentum per particle of a small amount of fluid to be injected in a larger sample of fluid without changing the total fluid volume or the entropy per particle.

It remains to be verified that the equations of motion (2.8) imply the Euler equation (2.12). For this purpose, it suffices to show that the equations of motion imply equation (2.14), since that equation and the Euler equation are related by use of the equations of motion. Accordingly, compute \( 2V_{[a:b]}U^b \) using the expression for \( V_a \) from equation (2.8a) to obtain

\[ 2V_{[a:b]}U^b = -2(\varphi_{,a} + s\theta_{,a} + \beta_{A\alpha}^{\lambda\beta})\theta_{,b}U^b. \tag{2.16} \]

The equations of motion imply that the terms involving gradients of \( s, \alpha^A \) and \( \beta_A \) along the flow lines all vanish. Only the term \( s_{,a}\theta_{,b}U^b \) remains on the right–hand–side of equation (2.16). According to equation (2.9), this term equals \( Ts_{,a} \) so that the equations of motion indeed imply equation (2.14). The conclusion is that the equations of motion (2.8) derived from the action (2.5) imply the relevant perfect fluid equations of motion, including the Euler equation.

The “on shell” action, that is, the value of the action (2.5) when the equations of motion (2.8) hold, is just the proper volume integral of the pressure:

\[ S(\text{on shell}) = \int d^4x \sqrt{-g} p . \tag{2.17} \]

Of course, the addition of surface integrals to the action will change its on–shell value without affecting the equations of motion. In particular, the surface integral \(-\int d^4x(\varphi J_a)\) can be added to \( S \), which amounts to replacing the term \( \varphi_{,a}J^a \)
by the term $-\varphi J^a,a$. Likewise, adding the surface integral $-\int d^4x(\theta s J^a)_a$ to $S$ results in replacing $s\theta J^a$ by $-\theta (s J^a)_a$. These surface integrals have on–shell values $-\int d^4x \sqrt{-g} n_f$ and $-\int d^4x \sqrt{-g} n s$, respectively. Thus, by adding these surface integrals in various combinations, action functionals can be obtained whose on–shell values are, for example, the proper volume integrals of $-\rho$ or $-na$.

There are a number of alternative perfect fluid actions that differ from the action (2.5) in the placement of derivatives in each term. In some cases, such as changing $\varphi J^a$ to $-\varphi J^a,a$, this amounts to the simple addition to the action of a boundary term as discussed above. A valid action functional is also obtained by replacing $s\theta J^a$ with $-\theta s J^a$. This change is brought about by defining $\varphi := \varphi' - s\theta$ then dropping the prime on the new field ($\varphi' \rightarrow \varphi$). In place of equation (2.10), $\varphi$ then satisfies $\mu = \varphi' U^a$ and is a “potential” for the chemical potential. Nevertheless, the resulting action yields the required equations of motion and stress tensor. The action (2.5) has an advantage over its alternatives in that the momenta conjugate to $\varphi$ and $\theta$ that naturally follow from that action have clear physical interpretations as the particle number density and entropy density seen by (Eulerian) observers at rest in space.

2.2 Symmetries of the action $S$

The perfect fluid action (2.5) is invariant under the infinitesimal transformations*

$$\delta \phi = \epsilon f - \epsilon s(\partial F/\partial s) - \epsilon \beta A(\partial F/\partial \beta A) \quad (2.18a)$$
$$\delta \theta = \epsilon (\partial F/\partial s) \quad (2.18b)$$
$$\delta s = 0 \quad (2.18c)$$
$$\delta \alpha A = \epsilon (\partial F/\partial \beta A) \quad (2.18d)$$
$$\delta \beta A = -\epsilon (\partial F/\partial \alpha A) \quad (2.18e)$$

along with $\delta J^a = 0$ and $\delta g_{ab} = 0$, where $\epsilon$ is an infinitesimal parameter and $F$ is an arbitrary function of the Lagrangian coordinates $\alpha A$, the entropy per particle $s$, and $\beta A$. These invariances, one for each function $F(\alpha, \beta, s)$, constitute global symmetries of the theory, symmetries in which the field transformations are fixed.

* These symmetries are derived in the Appendix.
for all time. Such a symmetry is to be distinguished from a gauge symmetry, in which the transformation can vary from one instant of time to the next. Observe that the particular combination \( \varphi_{,a} + s \theta_{,a} + \beta_{A} \alpha^{A},a \) that appears in the action (2.5), and equals \(-\mu U_{a}\) when the equation of motion (2.8a) holds, is invariant under the transformations (2.18). Also note that the function \( F(\alpha, \beta, s) \) has vanishing gradient \( \dot{F}_{a}U^{a} = 0 \) along the fluid flow lines, according to the equations of motion (2.8). Thus, the symmetries (2.18) produce variations in \( \varphi, \theta, \alpha^{A} \), and \( \beta_{A} \) that are constant along the flow lines. Such variations do not affect the temperature \( T = \theta_{,a}U^{a} \), chemical free energy \( f = \varphi_{,a}U^{a} \), or the constancy of \( \alpha^{A} \) and \( \beta_{A} \) along the flow lines.

The physical meaning of the symmetries (2.18) can be explored by considering special choices for the function \( F(\alpha, \beta, s) \). If \( F \) is a function of \( \alpha^{A} \) only, the transformations (2.18) reduce to

\[
\begin{align*}
\delta \varphi &= \epsilon F \\
\delta \beta_{A} &= -\epsilon (\partial F/\partial \alpha^{A}) ,
\end{align*}
\]

with all other fields unchanged. Since the Lagrangian coordinates \( \alpha^{A} \) uniquely label the flow lines, at least within open subsets of the fluid space, the change (2.19a) in \( \varphi \) along one flow line is independent of the changes in \( \varphi \) along the other flow lines (subject to the requirement of continuity in \( \varphi \)). Therefore, the symmetries (2.19) can be understood as arising from the freedom to shift the value of \( \varphi \) along each flow line by a constant amount. In particular, any solution to the fluid equations of motion can be transformed via equations (2.19) into a solution with, say, \( \varphi = 0 \) on any given spacelike hypersurface.

If the function \( F \) has the form \( s \bar{F}(\alpha) \) for some function \( \bar{F} \) of \( \alpha^{A} \), the transformations (2.18) reduce to

\[
\begin{align*}
\delta \theta &= \epsilon \bar{F} \\
\delta \beta_{A} &= -\epsilon s(\partial \bar{F}/\partial \alpha^{A}) ,
\end{align*}
\]

with the other fields unchanged. These symmetries can be used to shift the value of the thermasy \( \theta \) along each flow line by a constant amount. Thus, any solution of the equations of motion can be transformed into a solution with \( \theta = 0 \) on any given spacelike hypersurface.
Now consider functions $F$ of the form $\beta_A\bar{F}^A(\alpha)$ where $\bar{F}^A$ is a set of functions of $\alpha^A$. In this case, the transformations (2.18) reduce to

\begin{align}
\delta \alpha^A &= \epsilon \bar{F}^A \\
\delta \beta_A &= -\epsilon \beta_B (\partial \bar{F}^B/\partial \alpha^A).
\end{align}

These symmetry transformations describe changes of coordinates $\alpha^A$ in the fluid space, where $\bar{F}^A$ is viewed as a vector in the fluid space. Moreover, observe that the equations of motion (2.8b,e) imply the constancy of $\beta_A$ along the fluid flow lines, so that $\beta_A$ can be expressed as a function of the Lagrangian coordinates $\alpha^A$. Then according to its transformation (2.21b) under changes of coordinates (2.21a), $\beta_A$ can be viewed as the components of the covariant vector (or one–form) $\beta := \beta_A d\alpha^A$ in the fluid space.

The conserved Noether currents associated with the symmetries (2.18) are obtained from the general variation of the action (2.5), which is

\[\delta S = \text{(terms giving the equations of motion)} + \int d^4x (J^a \delta \varphi + s J^a \delta \theta + \beta_A J^a \delta \alpha^A).\]  

The action $S$ is invariant under the symmetry transformations (2.18), so when the equations of motion hold, $0 = \delta S$ becomes

\[0 = \int d^4x (F J^a).\]  

Therefore $F J^a$ are the conserved currents that satisfy $(F J^a)_a = 0$ by virtue of the equations of motion (2.8). If space is closed, or if suitable boundary conditions are imposed at spatial infinity, the volume integral (2.23) can be written as a surface integral $Q[F]$ evaluated on a final spacelike hypersurface, minus the same surface integral evaluated on an initial spacelike hypersurface. That surface integral is the Noether charge

\[Q[F] = \int_{\Sigma} d^3x \sqrt{h} n_a (-n_a U^a) F(\alpha, \beta, s),\]  

and equation (2.23) just expresses the conservation of $Q[F]$, that is, the independence of $Q[F]$ on the choice of hypersurface $\Sigma$. In equation (2.24), $h$ is the determinant of the three–metric on $\Sigma$, and $n_a$ is the future pointing unit normal to $\Sigma$. 

13
Insight into the meaning of the conserved charges $Q[F]$ is obtained by considering again a special choice for $F(\alpha, \beta, s)$. Let $F$ be a function of $\alpha^a$ only, and in particular choose $F$ equal to unity for all $\alpha^a$ in some ball $B$ in the fluid space, and equal to zero outside $B$. Then the charge (2.24) becomes a proper volume integral over the subspace of the hypersurface $\Sigma$ that contains flow lines in $B$, with integrand $n(-n_a U^a)$. The factor $-n_a U^a$ is the relativistic “gamma factor” characterizing a boost from the Lagrangian observers with four–velocity $U^a$ to the Eulerian observers with four–velocity $n^a$ [32]. Thus, $n(-n_a U^a)$ is the particle number density as seen by the Eulerian observers. Then conservation of the charge (2.24) expresses the conservation of particle number within a flow tube defined by the bundle of flow lines contained in the ball $B$.

For general functions $F(\alpha, \beta, s)$, the conserved charges (2.24) can be given a physical interpretation in terms of the case discussed above. Since $\beta^A$ and $s$ are constant along the flow lines, and because $\alpha^A$ are unique flow line labels, the fields $\beta^A$ and $s$ can be expressed as functions of the spacetime point $x$ through the combination $\alpha^A(x)$. Thus, a general function $F$ can be viewed as having functional dependence $F(\alpha, \beta(\alpha), s(\alpha))$. Then the charge $Q[F]$ is the number of particles contained in a distribution of flow lines, with each flow line weighted by the distribution function $F(\alpha, \beta(\alpha), s(\alpha))$. For example, choose $F$ equal to $s\tilde{F}$, where $\tilde{F}$ equals unity for all $\alpha^A$ in a ball $B$, and zero outside $B$. In this case the conserved charge $Q[F]$ equals the number of particles contained in the flow lines included in $B$, weighted by the entropy per particle $s$. Conservation of $Q[F]$ expresses the conservation of total entropy within the flow tube defined by the bundle of flow lines contained in $B$.

The conserved charges (2.24) must be distinguished from quantities that are conserved in a more restricted sense, such as the fluid circulation $C$. Circulation is defined as the integral of the Taub current one–form $V_a = \mu U_a$ around a spacelike loop [31]. $C$ is conserved in the sense that its value is a constant for particular families of loops, namely those obtained by evolving an initial loop along the fluid flow lines by an amount proportional to the thermasy $\theta$ [33]. (If the loop is evolved along the flow lines by an amount proportional to proper time, changes in $C$ are determined by the gradient of the entropy per particle [31].) On the other hand, a conserved charge $Q[F]$ has the same value for any spacelike hypersurface, not just for a particular family of hypersurfaces. See reference [34] for a discussion of the
various types of conservation laws that arise in fluid mechanics.

3. Hamiltonian form of the action $S$

3.1 Action and Hamiltonian

The perfect fluid action (2.5) can be written in Hamiltonian form by introducing first a space–time decomposition of the fields. Accordingly, let $t$ denote a scalar function on spacetime, whose gradient is nonzero and (future pointing) timelike. The $t =$ constant surfaces foliate spacetime into spacelike hypersurfaces with unit normal $n_a = -N t_a$, where $N := (-t_{a} g^{ab} t_b)^{-1/2}$ defines the lapse function. Also introduce a time vector field $t^a$ such that $t_a t^a = 1$ and the projection tensor $h_{ab} := \delta_{ab} + n^a n_b$ onto the leaves of the foliation. With the shift vector defined by $N^a := h_{ab} t^b$, the time vector field can be written as $t^a = N n^a + N^a$.

Using the above relationships, the vector density $J^a$ is decomposed as

$$J^a = h_{ab} J^b - n^a n_b J^b$$

$$= (hJ)^a + (t^a - N^a) \Pi ,$$

where the definitions $(hJ)^a := h_{ab} J^b$ and $\Pi := t_a J^a$ are used. Because $J^a = \sqrt{-g} n U^a$ is the densitized particle number flux vector, this latter definition becomes

$$\Pi = \sqrt{h} n (-n_a U^a) ,$$

where $\sqrt{-g} = N \sqrt{h}$ has been used. Recall that $-n_a U^a$ is the relativistic “gamma factor” characterizing a boost from the Lagrangian observers with four–velocity $U^a$ to the Eulerian observers with four–velocity $n^a$. Then $\Pi$ is recognized as the spatially densitized Eulerian particle number density; that is, $\Pi/\sqrt{h}$ is the number density of fluid particles as seen by the Eulerian observers who are at rest in the $t =$ constant hypersurfaces.

With the decomposition (3.1), the action (2.5) becomes

$$S = \int d^4 x \left\{ -\sqrt{-g} \rho (|J|/\sqrt{-g}, s) + \Pi t^a (\varphi, a + s \theta, a + \beta \alpha^a, a) \\
+ ((hJ)^a - \Pi N^a)(\varphi, a + s \theta, a + \beta \alpha^a, a) \right\} .$$

(3.3)
Now tie the spacetime coordinates to the foliation by choosing \( t \) as the “timelike” coordinate, and choosing spatial coordinates such that \( (\partial/\partial t)^a = t^a \). In these coordinates, the spacetime metric has the ADM form \[35\]
\[
g_{ab}dx^a dx^b = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) , \tag{3.4}
\]
where \( x^i \) are the spatial coordinates on the \( t = \) constant hypersurfaces, and \( h_{ij} \) are the spatial components of the spacetime tensor \( h_{ab} \). Note that derivatives along \( t^a \) are just ordinary \( t \) derivatives, and will be denoted by a dot. Since the contravariant tensors \( N^a \) and \( (hJ)^a \) have vanishing \( t \)-components, the fluid action (3.3) can be written as
\[
S = \int dt d^3 x \left\{ \Pi(\dot{\varphi} + s \dot{\theta} + \beta_A \dot{\alpha}^A) - N\sqrt{h} \rho(|J|/N\sqrt{h}, s) \\
+ ((hJ)^i - \Pi N^i)(\varphi,i + s\theta,i + \beta_A \alpha^A,i) \right\} , \tag{3.5}
\]
where
\[
|J| = \left[ (N\Pi)^2 - (hJ)^i(hJ)_i \right]^{1/2} , \tag{3.6}
\]
is the magnitude of \( J^a \). (Spatial indices are raised and lowered by the spatial metric \( h_{ij} \) and its inverse \( h^{ij} \).)

Since the fields with a time derivative appear linearly in the action (3.5), it is clear that \( \Pi \) is canonically conjugate to \( \varphi \), \( \Pi \theta \) is canonically conjugate to \( \theta \), and \( \Pi \beta_A \) is canonically conjugate to \( \alpha^A \). Only the fields \( (hJ)^i \) have no canonical counterpart. The action can be cast into canonical form with respect to the variables \( (hJ)^i \) by following the standard analysis due to Dirac [36]. The momenta conjugate to \( (hJ)^i \) are constrained to vanish, and the preservation in time of these primary constraints is equivalent to the equation of motion produced by varying the action with respect to \( (hJ)^i \); this is just the projection of the equation of motion (2.8a) onto the spacelike hypersurfaces, namely
\[
U_i = -(\varphi,i + s\theta,i + \beta_A \alpha^A,i)/\mu . \tag{3.7}
\]
The complete set of constraints is second class, and their elimination is equivalent to replacing \( (hJ)^i \) in the action (3.5) by the solution to the equation of motion (3.7).

Equation (3.7) can be solved implicitly for \( (hJ)^i \) as a function of \( \Pi, \varphi, \theta, s, \beta_A, \alpha^A \), and \( h_{ij} \) in the following way. Observe that \( (hJ)^i \) appears on the left-hand-side of that equation through the combination \( U_i = (hJ)_i/|J| \), with \( |J| \) given by
equation (3.6). The equation \( U_i = (hJ)_i / |J| \) can be solved for \((hJ)^i\) as a function of \(U_i\), with the result
\[
(hJ)^i = \frac{N \Pi}{\sqrt{1 + U_j^2 U_j^i}} U^i .
\] (3.8)

Now consider \(U_i\) as shorthand notation for the right–hand–side of equation (3.7). This expression for \(U_i\) depends on the chemical potential \(\mu = \partial \rho / \partial n\), which is a function of \(n\) and \(s\) as determined by the equation of state \(\rho(n, s)\). In turn, \(n\) is determined implicitly as a function of \(\Pi, \varphi, \theta, s, \beta_A, \alpha^A\), and \(h_{ij}\) by the equation
\[
\Pi = \sqrt{h} n (1 + U_i h_{ij} U_j)^{1/2} .
\] (3.9)

This expression (3.9) relates the Lagrangian number density \(n\) to the Eulerian number density \(\Pi / \sqrt{h}\), and is obtained from equation (3.2) with the gamma factor written as \(-n a U^a = (1 + U_i h_{ij} U_j)^{1/2}\). The relationship \(-n a U^a = (1 + U_i h_{ij} U_j)^{1/2}\) is proved by inserting the space–time split of the inverse metric \(g^{ab}\) into the identity \(-1 = U_a g^{ab} U_b\) and solving for \((1 + U_i h_{ij} U_j)\). To summarize, \(U_i\) as it appears in equations (3.8) and (3.9) should be viewed as shorthand notation for the right–hand–side of equation (3.7). Then equations (3.8) and (3.9) implicitly determine \((hJ)^i\) and the Lagrangian number density \(n\) as functions of the Eulerian number density \(\Pi\), the fields \(\varphi, \theta, s, \beta_A, \alpha^A\), and the spatial metric \(h_{ij}\).

The Hamiltonian form of the action is now obtained by substituting expression (3.8) for \((hJ)^i\) into the action (3.5) and using equation (3.7). This yields
\[
S = \int dt d^3x \left\{ \Pi \dot{\varphi} + (\Pi s) \dot{\theta} + (\Pi \beta_A) \dot{\alpha}^A - N H_{\text{fluid}} - N_i H_{\text{fluid}}^i \right\} ,
\] (3.10)
where the fluid contributions to the Hamiltonian and momentum constraints are
\[
H_{\text{fluid}} = \sqrt{h} \left[ \rho (1 + U_i U_i) + p U^i U_i \right] ,
\] (3.11a)
\[
H_{\text{fluid}}^i = -\mu \Pi U_i .
\] (3.11b)

The corresponding perfect fluid Hamiltonian is
\[
H_{\text{fluid}} := \int d^3x \left( N H_{\text{fluid}}^\text{fluid} + N_i H_{\text{fluid}}^i \right) .
\] (3.12)

In equations (3.10–12), \(n\) and \(U_i\) are considered to be functions of the variables \(\Pi, \varphi, (\Pi s), \theta, (\Pi \beta_A), \alpha^A\), and \(h_{ij}\) as determined by equations (3.7) and (3.9).

17
The fundamental Poisson brackets among the fluid variables are
\[
\{ \varphi(x), \Pi(x') \} = \delta(x, x') , \tag{3.13a}
\]
\[
\{ \theta(x), (\Pi s)(x') \} = \delta(x, x') , \tag{3.13b}
\]
\[
\{ \alpha^A(x), (\Pi \beta_i)(x') \} = \delta^A_i \delta(x, x') . \tag{3.13c}
\]
Observe that the fluid contributions (3.11) to the Hamiltonian and momentum constraints are given by the projections
\[
H^\text{fluid} = \sqrt{h} n_a T^{ab} n_b \quad \text{and} \quad H^\text{fluid}_i = \sqrt{h} n_a T^a_i
\]
of the stress tensor (2.1), as expected of a matter action with nonderivative coupling to gravity [17]. Correspondingly, the gravitational field contributions to the constraints are the same as for vacuum general relativity. The fluid contribution (3.11a) to the Hamiltonian constraint can be rewritten in various useful forms by using expression (3.9) to relate the Eulerian and Lagrangian number densities:
\[
H^\text{fluid} = \left[ (\mu \Pi)^2 + H^\text{fluid}_i h^{ij} H^\text{fluid}_j \right]^{1/2} - \sqrt{h} p \tag{3.14a}
\]
\[
= \mu \Pi^2 / (\sqrt{h} n) - \sqrt{h} p . \tag{3.14b}
\]
Also observe that the momentum constraint (3.11b) equals
\[
H^\text{fluid}_i = \Pi \varphi_{,i} + (\Pi s) \theta_{,i} + (\Pi \beta_i) \alpha^A_{,i} , \tag{3.15}
\]
which is the form dictated by the role of \( H^\text{fluid}_i \) as the generator of spatial diffeomorphisms for the scalar fields \( \varphi, \theta, \) and \( \alpha^A \) and their conjugates.

3.2 Canonical equations of motion and symmetries

In order to compute the canonical fluid equations of motion, first vary expression (3.14b) for \( H^\text{fluid} \) with respect to \( n, s, \Pi, \) and \( h_{ij} \). This yields
\[
\delta H^\text{fluid} = \sqrt{h} \left[ -\mu(1 + U \cdot U) + n(U \cdot U) \frac{\partial^2 \rho}{\partial n^2} \right] \delta n + \sqrt{h} \left[ n(U \cdot U) \frac{\partial^2 \rho}{\partial n \partial s} + \frac{\partial \rho}{\partial s} \right] \delta s
\]
\[
+ \frac{2}{\sqrt{h}} \frac{\mu}{n} \Pi \delta \Pi + \frac{1}{2} \sqrt{h} \left[ -n\mu(1 + U \cdot U) - p \right] h^{ij} \delta h_{ij} , \tag{3.16}
\]
where \( U \cdot U := U^i U_i \) and \( U^i := h^{ij} U_j \), and expression (3.9) is used. By varying equation (3.9) for \( \Pi, \delta n \) is given in terms of variations in the canonical variables by
\[
0 = \sqrt{h} \left[ -\mu(1 + U \cdot U) + n(U \cdot U) \frac{\partial^2 \rho}{\partial n^2} \right] \delta n - \frac{1}{2} \sqrt{h} n \mu \left[ (1 + U \cdot U) h^{ij} - U^i U^j \right] \delta h_{ij}
\]
\[
+ \sqrt{h} n \left[ (1 + U \cdot U) \frac{\partial^2 \rho}{\partial n \partial s} \right] \delta s + \mu(1 + U \cdot U)^{1/2} \delta \Pi - \sqrt{h} n U^i \delta (\mu U_i) . \tag{3.17}
\]
Combining these results gives

$$\delta \mathcal{H}^\text{fluid} = -\frac{1}{2} \sqrt{h} \left[ p h_{ij} + (\rho + p) U^i U^j \right] \delta h_{ij}$$

$$+ (1 + U \cdot U)^{-1/2} \left[ f \delta \Pi + T \delta (\Pi s) - U^i \delta \mathcal{H}^\text{fluid}_i \right], \quad (3.18)$$

where $U_i$ and $n$ are functions of the canonical variables as determined by equations (3.7) and (3.9). In addition, the pressure $p$, temperature $T$, and chemical free energy $f$ are functions of the canonical variables defined through the relations $p = n(\partial \rho / \partial n) - \rho$, $T = (\partial \rho / \partial s)/n$, and $f = (\partial \rho / \partial n) - T s$.

The functional derivatives of the fluid Hamiltonian (3.12) follow from the variation (3.18) and the variation of $\mathcal{H}^\text{fluid}_i$ from equation (3.15). In canonical form the perfect fluid equations of motion are

$$\dot{\varphi} = \frac{\delta \mathcal{H}^\text{fluid}}{\delta \Pi} = N(f - U^i \varphi, _i)(1 + U \cdot U)^{-1/2} + N^i \varphi, _i, \quad (3.19a)$$

$$\dot{\theta} = \frac{\delta \mathcal{H}^\text{fluid}}{\delta (\Pi s)} = N(T - U^i \theta, _i)(1 + U \cdot U)^{-1/2} + N^i \theta, _i, \quad (3.19b)$$

$$\dot{\alpha}^A = \frac{\delta \mathcal{H}^\text{fluid}}{\delta (\Pi \beta, _A)} = N(-U^i \alpha^A, _i)(1 + U \cdot U)^{-1/2} + N^i \alpha^A, _i, \quad (3.19c)$$

$$\dot{\Pi} = -\frac{\delta \mathcal{H}^\text{fluid}}{\delta \varphi} = -(N\sqrt{h}nU^i), _i + (N^i \Pi), _i, \quad (3.19d)$$

$$(\Pi s)^{\prime} = -\frac{\delta \mathcal{H}^\text{fluid}}{\delta \theta} = -(N\sqrt{h}nsU^i), _i + (N^i \Pi s), _i, \quad (3.19e)$$

$$(\Pi \beta, _A)^{\prime} = -\frac{\delta \mathcal{H}^\text{fluid}}{\delta \alpha^A} = -(N\sqrt{h}n\beta, _A U^i), _i + (N^i \Pi \beta, _A), _i. \quad (3.19f)$$

These are precisely equations (2.10), (2.9), (2.8f), (2.8b), (2.8c), and (2.8e), respectively, written in terms of the 3+1 decomposition (3.1), (3.8) for $J^a$. Also observe that the variation (3.18) yields

$$\frac{\delta \mathcal{H}^\text{fluid}}{\delta h_{ij}} = -\frac{1}{2} N\sqrt{h} \left[ p h_{ij} + (\rho + p) U^i U^j \right], \quad (3.20)$$

which is the fluid contribution to the canonical equation of motion for the spatial metric $h_{ij}$.

The global symmetries (2.18) described in section 2.2 appear in the canonical formalism as transformations on the canonical variables generated through the Poisson brackets by

$$Q[F] = \int d^3 x \Pi F(\alpha, \beta, s). \quad (3.21)$$
This phase space functional is obtained from the charge (2.24) by using definition (3.2) for Π. Note that the combination µUi from equation (3.7) is invariant under these transformations, so from equation (3.9) n is invariant as well. It follows that Q[F] has vanishing Poisson brackets with the Hamiltonian and momentum constraints (3.11), confirming that Q[F] indeed generates a symmetry of the theory. Also observe that the symmetry generators (3.21) close under the Poisson brackets according to

\[ \{Q[F_1], Q[F_2]\} = Q[F], \tag{3.22} \]

where

\[ F = (\partial F_1/\partial \alpha^A)(\partial F_2/\partial \beta_A) - (\partial F_1/\partial \beta_A)(\partial F_2/\partial \alpha^A). \]

3.3 Initial and boundary value problems

A perfect fluid with equation of state ρ(n, s) coupled to the gravitational field is described by the canonical action (3.10) plus the canonical action for gravity. The Cauchy data for this system consist of the fluid variables ϕ, Π, θ, (Πs), αA, (ΠβA), and the canonical gravitational variables. These initial data cannot be specified independently, but must satisfy the Hamiltonian and momentum constraints. Any set of initial data that does satisfy these constraints can be transformed into an equivalent set by the symmetries (2.18), which are generated by the phase space functional Q[F] of equation (3.21).

According to the analysis of section 2.2, the fields ϕ and θ can be brought to zero on any spacelike hypersurface by a symmetry transformation. Thus, there is no loss of generality in choosing ϕ and θ to be zero on the initial hypersurface. Moreover, the Lagrangian coordinates αA can be chosen to coincide with the coordinates x^i on the initial surface, so that α^A,i = δ^A_i. With these choices, equation (3.7) shows that the spatial components U_i of the covariant fluid four–velocity on the initial hypersurface are

\[ \mu U_i = -\beta_i. \tag{3.23} \]

This reveals the geometrical significance of the fields β_A: with the choices ϕ = θ = 0, α^A,i = δ^A_i allowed by symmetry, the fluid space covector components β_A are just −μ times the spatial components of the fluid four–velocity.

Recall the definition V_a := μU_a for the Taub current vector from equation (2.15). The result (3.23) shows that in specifying initial data, −β_A can be identified with the spatial components V_i of the Taub vector. Thus, a complete set of
initial data for a perfect fluid consists of the (Eulerian) particle number density, the (Eulerian) entropy density, and the spatial part of the Taub vector, along with \( \varphi = \theta = 0 \) and \( \alpha^A = \) (spatial coordinates). These initial data are then evolved according to the Hamiltonian differential equations of motion (3.19).

Now consider the boundary value problem. Assume the spacetime manifold admits closed spacelike hypersurfaces so the boundary data is specified only on initial and final hypersurfaces. One possible set of boundary data consists in specifying the canonical coordinates \( \varphi, \theta, \) and \( \alpha^A \) on the initial and final hypersurfaces. These boundary data include \( 10 \times \infty^3 \) boundary values, \( 5 \times \infty^3 \) on the initial surface and \( 5 \times \infty^3 \) on the final surface, where \( \infty^3 \) is the number of space points. With these boundary data, the \( 10 \times \infty^3 \) Hamiltonian first order differential equations of motion (3.19) generically determine a solution for the ten canonical field variables.

Another possible set of boundary data consists in specifying the canonical momentum \( \Pi \) along with the coordinates \( \theta \) and \( \alpha^A \) on the initial and final hypersurfaces. In this case, the data on the initial and final surfaces are related through the conserved charges \( Q[F] \) of equation (3.21). In particular, if \( \alpha^A \) and \( \Pi \) are specified initially and \( \alpha^A \) is specified finally, then the conserved charges \( Q[F] \) can be used to compute \( \Pi \) on the final surface by considering a complete set of functions \( F(\alpha^A) \). This means that the independent boundary data consist of only \( 9 \times \infty^3 \) boundary values. With these boundary data, the \( 10 \times \infty^3 \) Hamiltonian first order differential equations of motion generically determine the ten canonical field variables to within the symmetry transformation (2.19). In particular, the field \( \varphi \) is obtained only to within an additive constant along each of the \( \infty^3 \) flow lines. Other sets of boundary data are restricted by the conserved charges \( Q[F] \) as well.

The distinctions among the various types of boundary data can be clarified by a simple example, namely, the free nonrelativistic particle. If the initial and final positions of the particle are given as boundary data, the equations of motion can be solved uniquely for the particle position as a function of time. But the initial and final momenta cannot be specified independently, because space translation invariance implies that the momentum is conserved. By specifying equal values for the initial and final momenta, the equations of motion can be solved to within a constant spatial translation of the particle.

4. Velocity–potential representation
In equation (2.8a), namely
\[ U_a = -\left( \varphi_{,a} + s\theta_{,a} + \beta_{\Lambda}\alpha^{\Lambda}_{,a} \right)/\mu, \] (4.1)
the fluid four-velocity is written in terms of various scalar fields and their gradients. Expressions of this type are common in the literature on fluid dynamics. They are often called velocity-potential representations or Clebsch [29] representations of \( U_a \), and the scalar fields themselves are called velocity potentials or Clebsch potentials.

Two related questions naturally arise: Is the velocity-potential representation (4.1) sufficiently general to allow for any four-velocity, or does it restrict the four-velocity in some way? If the representation (4.1) is sufficiently general, then is it overly general in the sense that fewer potentials would be adequate?

The first of these questions can be answered in the affirmative by explicitly constructing a set of velocity potentials \( \varphi, \theta, \alpha^A, \) and \( \beta_{\Lambda} \) that correspond to an arbitrary timelike, unit normalized vector field \( U_a \), along with arbitrary spacetime scalar fields \( s, T, \) and \( \mu \neq 0 \). Begin by defining the thermasy through the relationship \( T = \theta_{,a}U^a \). More precisely, assign arbitrary values for \( \theta \) on some spacelike hypersurface, then define the value of \( \theta \) at any other spacetime point \( x \) by integrating the scalar \( T \) along a flow line of \( U_a \) from the hypersurface to the point \( x \). Likewise, choose arbitrary values for \( \varphi \) on some spacelike hypersurface then extend \( \varphi \) off this hypersurface by integrating the relationship \( f := \mu - Ts = \varphi_{,a}U^a \) along the flow lines of \( U^a \). Now observe that the vector
\[ \beta_a := -\left( \mu U_a + \varphi_{,a} + s\theta_{,a} \right) \] (4.2)
is orthogonal to the flow lines of \( U^a \); that is, \( \beta_a U^a = 0 \). Thus, \( \beta_a \) can be expressed as a linear combination of three independent basis vector fields, where the basis vectors span the subspace of spacetime vector fields that are orthogonal to \( U^a \). Such basis vectors are the gradients of three scalar fields \( \alpha^A, \) where \( \alpha^A \) are a set of Lagrangian coordinates that (uniquely) label the flow lines of \( U^a \). Therefore \( \beta_a \) can be written as \( \beta_a = \beta_{\Lambda}\alpha^{\Lambda}_{,a} \) where \( \beta_{\Lambda} \) are scalar fields. Inserting this expression for \( \beta_a \) into equation (4.2) yields the velocity-potential representation (4.1).

The construction described above shows that the fluid four-velocity \( U_a \) always can be expressed in the velocity-potential representation (4.1) for any entropy per particle \( s \), any nonzero chemical potential \( \mu \), and any temperature \( T \) such that the
thermasy satisfies $T = \theta_a U^a$. This construction also shows the extent to which the velocity potentials are arbitrary. In particular, $\varphi$ and $\theta$ can be chosen arbitrarily on a single spacelike hypersurface, and the fields $\alpha^A$ can be chosen as any set of unique flow line labels. These ambiguities in the velocity–potential representation appear as the invariances (2.19–21) of the perfect fluid action.

Regarding the second question, it is indeed possible to reduce the number of potentials that appear in the velocity–potential representation by replacing the three pairs of fields $\alpha^A$, $\beta_A$, by a single pair (see references [4, 5, 15]). The demonstration invokes Pfaff’s theorem [37] and the observation from section 2.2 that $\beta_A$ can be viewed as a covector or one–form on the three–dimensional fluid space. According to Pfaff’s theorem, $\beta_A$ can be written in terms of three fluid space scalar fields as $\beta_A = \bar{\varphi},_A + \bar{\beta} \bar{\alpha},_A$. Then the velocity–potential representation (4.1) becomes

$$U_a = - (\bar{\varphi},_a + s\theta,_{a} + \bar{\beta} \bar{\alpha},_a)/\mu$$  \hspace{1cm} (4.3)

where the definition $\bar{\varphi} := \varphi + \tilde{\varphi}$ has been used. Comparing equations (4.1) and (4.3) shows that the seven potentials $\varphi$, $\alpha^A$, $\beta_A$, have been replaced by three potentials $\bar{\varphi}$, $\bar{\alpha}$, $\bar{\beta}$. In effect, this reduction amounts to restricting the fluid space indices on $\alpha^A$ and $\beta_A$ to a single value.

The perfect fluid action (2.5) admits the symmetry transformations (2.18) regardless of the range of values assumed by the fluid space indices. However, the identification of $\alpha^A$ as Lagrangian coordinates was used repeatedly in the interpretation of those symmetries and the corresponding conserved charges (2.24). It should be emphasized that the reduction in the number of velocity potentials precludes this possibility of identifying $\alpha^A$ as Lagrangian coordinates for the fluid. Thus, with just one pair of variables in place of $\alpha^A$ and $\beta_A$, the physical interpretation of the symmetries and conserved charges is lost. Also recall that with three pairs $\alpha^A$, $\beta_A$, the fields $\beta_A$ can be given a direct geometrical interpretation by relating them to the spatial components of $-V_a$ to within a symmetry transformation (see section 3.3). This allows for the specification of initial data in terms of simple physical quantities, namely the Eulerian number density, the Eulerian entropy density, and the spatial part of the Taub current.

Another disadvantage of the reduced representation (4.3) is that the velocity potentials $\bar{\varphi}$, $\bar{\alpha}$, $\bar{\beta}$ are not always single valued functions on spacelike hypersurfaces. (This has been recognized in the context of nonrelativistic perfect fluids in reference
For example, consider a fluid whose four-velocity is described by the representation (4.1) with \( \beta_1 = -\alpha^2, \beta_2 = \alpha, \beta_3 = 0 \). Such a fluid has nonzero vorticity with axis of rotation (in the isentropic case) in the direction \( \epsilon_{abcd} \phi_{a,b} \alpha^1, c \alpha^2, d \). Using the Pfaff reduction, \( \beta_A \) can be written as \( \beta_A = \tilde{\phi}_A + \bar{\alpha}_A \), where \( \tilde{\phi} = R^2 \phi, \bar{\alpha} = R^2 \), and \( \bar{\beta} = - \phi \). Here, \( R \) and \( \phi \) are polar coordinates in the \( \alpha^1-\alpha^2 \) plane of the fluid space, defined by \( R^2 = (\alpha^1)^2 + (\alpha^2)^2 \) and \( \tan \phi = \alpha^2/\alpha^1 \). Since the value of \( \phi \) jumps by \( 2\pi \) along a spacelike loop surrounding the vorticity axis, the velocity potentials \( \tilde{\phi} \) (and hence \( \bar{\phi} \)) and \( \bar{\beta} \) are not single valued functions on space.

This example shows that the representation (4.3), with potentials \( \tilde{\phi}, \bar{\alpha}, \bar{\beta} \) that are single valued, does not allow for an arbitrary four-velocity. As a particular consequence, it can be shown [38, 39] that the fluid helicity is restricted to vanish. (Helicity is defined, for example, in references [34] and [40].) On the other hand, the velocity-potential representation (4.1) can be used to represent any four-velocity with potentials that are single valued on all spacelike hypersurfaces. Since several coordinate charts may be needed to cover the fluid space with Lagrangian coordinates \( \alpha_A \), a more precise statement is that any four-velocity can be represented as in equation (4.1) by velocity potentials that are single valued within open subsets of spacelike hypersurfaces, where the open subsets contain flow lines \( \alpha^A \) from a single fluid space coordinate chart.

In order to verify this claim, recall the discussion at the beginning of this section in which an arbitrary four-velocity \( U_a \) was constructed from the velocity-potential representation (4.1). The goal now is to argue that the potentials used in that construction are single valued. Assuming \( \varphi \) and \( \theta \) are chosen to be single valued on one spacelike hypersurface, then integration of the equations \( f = \varphi_a U^a \) and \( T = \theta_a U^a \) yields spacetime fields \( \varphi \) and \( \theta \) that are single valued on any spacelike hypersurface. Of course, \( s \) is always single valued since it has a direct physical interpretation. The Lagrangian coordinates \( \alpha_A \) from a given coordinate chart on the fluid space are also single valued functions on any spacelike hypersurface. From equation (4.2), the fields \( \beta_A \) on a hypersurface with coordinates \( x^i \) are now defined by

\[
\beta_A = -\alpha_A^i (\mu U_i + \varphi_{,i} + s \theta_{,i}) ,
\]

(4.4)

where \( \alpha_A^i \) is the inverse of \( \alpha^A, i \). It follows that \( \beta_A \) are single valued functions on any hypersurface since each field appearing on the right-hand-side of equation (4.4), including the spatial components \( U_i \) of the fluid four-velocity, are single valued. The
conclusion is that within a subset of any spacelike hypersurface that contains flow lines from a given fluid space coordinate chart, the velocity–potentials appearing in the representation (4.1) can be assumed to be single valued without any restriction on $U_a$.

Another consequence of reducing the number of velocity potentials is that the counting of degrees of freedom (per space point) is changed. With all three pairs $\alpha^A$, $\beta_A$, the perfect fluid is described by five pairs of unconstrained canonical variables and therefore has the expected five degrees of freedom. Three of these correspond to the fluid’s freedom of motion in three–dimensional space, while the remaining two degrees of freedom correspond to the fluid number density and entropy density. With just one pair of potentials replacing $\alpha^A$ and $\beta_A$, the system apparently has just three degrees of freedom. Evidently, this difference in the number of degrees of freedom arises because in the steps leading to the reduced representation (6.3), the equations of motion $\beta_{A,a}U^a = 0$ and $\alpha^{A,a}U^a = 0$ were used in order to interpret $\beta_A$ as a fluid space covector. The significance of using the equations of motion can be understood by an analogy: Consider a nonrelativistic particle in three spatial dimensions moving in a central potential. The particle has three degrees of freedom, corresponding to motion in each of the three spatial directions. But the equations of motion show that the particle actually moves in a two–dimensional plane. Using this result, the problem can be reduced to that of a particle moving in its orbital plane, and the number of degrees of freedom is reduced to two.

Finally, observe that if the term $\varphi_a$ had been omitted from the representation (4.1), the Pfaff reduction would nevertheless lead to the representation (4.3) with an apparently trivial change of notation $\varphi \to \tilde{\varphi}$. This reasoning has incorrectly lead to the conclusion (see the appendix of reference [5]) that the potential $\varphi$ is unnecessary, and that a term $\varphi_aJ^a$ is not needed for a valid action principle if all three pairs of fields $\alpha^A$, $\beta_A$ are used. The conclusion is not valid because $\tilde{\varphi}$ is a function on the fluid space, and satisfies $\tilde{\varphi}_aU^a = 0$. This contradicts the result obtained by contracting equation (4.3) (with $\varphi \to \tilde{\varphi}$) with $U^a$, which shows that $\tilde{\varphi}_aU^a$ equals the chemical free energy $f$. Another (invalid) justification [5] for dropping $\varphi$ is that the variation of $\varphi$ in the action just yields $J^a,a = 0$, and particle number conservation is already implied by the constancy of the labels $\alpha^A$ along the fluid flow lines. The error in logic appears to lie in the interpretation of $\alpha^A$ as particle labels, in which case their existence would imply conservation of
particle number. However, true particle labels are not continuous spacetime fields, so \( \alpha^A \) do not label individual particles. Rather, \( \alpha^A \) label the fluid flow lines that are physically determined by the average particle motion. The particle number conservation equation \( J^a,\alpha = 0 \) must be imposed as a separate equation of motion to insure that the number of particles within a flow tube (defined by a bundle of flow lines) is conserved.

5. Action \( \bar{S} \) with equation of state \( \rho(n, s) \)

5.1 Action and equations of motion

Express the densitized fluid number flux vector as

\[
J^a = -\sqrt{-g}\epsilon^{abcd}\eta_{123}(\alpha)\alpha^1_{,a}\alpha^2_{,b}\alpha^3_{,c},
\]

where \( \eta_{123} \) is a function of \( \alpha \). The significance of \( \eta_{123}(\alpha) \) can be seen by using the number flux vector \( n U^a = J^a/\sqrt{-g} \) at a spacetime point \( x \) to construct a differential three–form on the spacelike hypersurface orthogonal to the fluid flow line at \( x \). This three–form is

\[
\frac{1}{3!} n U^a \epsilon^{abcd} dx^b \wedge dx^c \wedge dx^d = \eta_{123} d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3,
\]

and is interpreted as the number of particles in the infinitesimal three–volume \( (1/3!) U^a \epsilon^{abcd} dx^b \wedge dx^c \wedge dx^d \). Then equation (5.2) shows that \( \eta_{123}(\alpha) \) is the component of a three–form \( \eta = (1/3!) \eta_{ABC} d\alpha^A \wedge d\alpha^B \wedge d\alpha^C \) on the fluid space whose integral over a region \( B \) gives the number of fluid particles whose flow lines are included in \( B \):

\[
\int_B \eta = \text{(number of particles in } B) \,.
\]

Under changes of coordinates (2.21a) in the fluid space, \( \eta_{123} \) transforms as the component of a three–form. Correspondingly, expression (5.1) for \( J^a \) is independent of the choice of fluid space coordinates. Note that locally, \( \eta_{123} \) can be set to unity by an appropriate choice of coordinates.

The equations of motion (2.8b,f), expressing conservation of particle number and constancy of the Lagrangian coordinates \( \alpha^A \) along the fluid flow lines, are immediately satisfied by virtue of the ansatz (5.1) for \( J^a \). If the entropy per particle \( s \)
is given as a function of $\alpha^A$, the entropy exchange constraint (2.8c) is also automatically satisfied. Observe that $s\eta$ is the fluid–space three–form whose integral over a region $B$ gives the total entropy contained in the flow lines included in $B$.

Given a three–form $\eta$, a function $s$, and coordinates $\alpha^A$ on the fluid space, a perfect fluid action $\bar{S}$ can be constructed that is a functional of the Lagrangian coordinates $\alpha^A$ only. First consider the action $S' = S - \int d^4 x (\varphi J^a + s\theta J^a)_a$, that differs from $S$ of equation (2.5) by boundary terms. This action $S'$ yields the equations of motion (2.8) that correctly describe a relativistic perfect fluid with equation of state $\rho(n, s)$. From the action $S'$, construct the functional $\bar{S}$ by substituting expression (5.1) for $J^a$ and the function $s(\alpha)$ for $s$. The terms $-\varphi J^a, -\theta(sJ^a)_a + \beta_A\alpha^A, a J^a$ drop out, leaving the result

$$\bar{S} = -\int d^4 x \sqrt{-g} \rho(|J|/\sqrt{-g}, s), \quad (5.4)$$

where $s = s(\alpha)$ and $J^a$ is defined as a function of $\alpha^A$ through equation (5.1). The procedure used to obtain this functional (5.4) does not guarantee its validity as an action principle. In particular, the equations of motion that follow by varying $\bar{S}$ with respect to $\alpha^A$ are not necessarily equivalent to the original equations of motion from $S$ or $S'$ with $J^a$ expressed as a function of $\alpha^A$ through equation (5.1). Such a discrepancy would occur if the variations in $J^a$ induced by variations in $\alpha^A$ are not the most general possible variations consistent with the constraint equations $J^a, a = 0, (sJ^a)_a = 0$, and $\alpha^A, a J^a = 0$. However, in this case the representation (5.1) for $J^a$ is general, and the functional (5.4) does constitute a valid perfect fluid action. This can be confirmed explicitly by varying the functional (5.4) and showing that the correct equations of motion follow. (Alternatively, the validity of the functional (5.4) as an action would be assured if it could be demonstrated that expression (5.1) and $s = s(\alpha)$ arise from solving the equations of motion (2.8a–d,f) for $J^a$, $\varphi$, $\theta$, $s$ and $\beta_A$ in terms of $\alpha^A$.)

In varying the functional (5.4), observe first that variations with respect to the metric $g_{ab}$ are unchanged by the substitution (5.1) for $J^a$. So $\bar{S}$ yields the correct stress–energy tensor (2.1). Variation with respect to $\alpha^A$ gives

$$\delta \bar{S} = \int d^4 x [\mu U_\alpha \delta J^a(\alpha) - \sqrt{-gnT}\delta s(\alpha)], \quad (5.5)$$
where

\[ \delta J^a(\alpha) = -\frac{1}{3!} \sqrt{-g} \epsilon^{abcd} \left[ 3 \eta_{ACD} \delta \alpha^A, b \alpha^C, c \alpha^D, d + \left( \frac{\partial \eta_{BCD}}{\partial \alpha^A} \right) \alpha^B, b \alpha^C, c \alpha^D, d \delta \alpha^A \right] \]

\[ = -\frac{1}{3!} \sqrt{-g} \epsilon^{abcd} \left[ 3 (\eta_{ACD} \delta \alpha^A), b \alpha^C, c \alpha^D, d \right. \\
\left. + \left( \frac{\partial \eta_{BCD}}{\partial \alpha^A} - 3 \frac{\partial \eta_{ACD}}{\partial \alpha^B} \right) \alpha^B, b \alpha^C, c \alpha^D, d \delta \alpha^A \right] . \tag{5.6} \]

The last term above is proportional to the four–form \( 4 \eta_{[BCD,A]} = \eta_{BCD,A} - 3 \eta_{A[CD,B]}, \)
and must vanish because the fluid space is three–dimensional. Inserting \( \delta J^a(\alpha) \) into
the variation (5.5) and integrating by parts yields the equation of motion

\[ 0 = \frac{1}{\sqrt{-g}} \frac{\delta \bar{S}}{\delta \alpha^A} = \frac{1}{2} \epsilon^{abcd} V_{a;b} \eta_{ACD} \alpha^C, c \alpha^D, d - n T s_A , \tag{5.7} \]

where \( V_a = \mu U_a \) is the Taub current. Equation (5.2) implies \( \eta_{ACD} \alpha^A, a \alpha^C, c \alpha^D, d = n U^b \epsilon_{bacd}, \) so the equation of motion simplifies to

\[ 0 = \frac{1}{2} \epsilon^{efcd} V_{e;f} U^b \epsilon_{bacd} - T s_A \\
= 2 V_{[a;b]} U^b - T s_A . \tag{5.8} \]

This is the combination (2.14) of the Euler equation and the entropy conservation equation. This analysis shows that the functional \( \bar{S} \) is indeed a valid perfect fluid action: \( \bar{S} \) has the correct stress tensor, incorporates particle number conservation and entropy conservation by virtue of expressions (5.1) for \( J^a \) and \( s = s(\alpha), \) and yields the Euler equation from its equations of motion.

### 5.2 Hamiltonian form of the action \( \bar{S} \)

The momenta conjugate to \( \alpha^A \) are

\[ P_A = -\frac{1}{2} \sqrt{-g} \mu U_a \epsilon^{abcd} \eta_{ACD} \alpha^C, c \alpha^D, d \tag{5.9a} \]

\[ = -\mu \Pi U_i \alpha^i_A . \tag{5.9b} \]

Here, \( \alpha^i_A \) is the matrix inverse of \( \alpha^A, i \) and \( \Pi := J^i \) is the spatially densitized Eulerian particle number density (3.9):

\[ \Pi = \sqrt{n h (1 + U_i h^i U_j)^{1/2}} . \tag{5.10} \]
In principle, the next step in deriving the Hamiltonian is to solve equations (5.9) for \(\dot{\alpha}^A\) as functions of \(P_A\) and \(\alpha^A\). This is not possible for arbitrary equations of state \(\rho(n, s)\). The situation here is closely analogous to the expression of the Hamiltonian for the action \(S\), discussed in section 3. In that case the Hamiltonian (3.11–12) is given as a function of the Lagrangian number density \(n\) and the spatial components \(U_i\) of the fluid velocity. In turn, \(n\) and \(U_i\) are determined as functions of the canonical variables through equations (3.7) and (3.9). For the case at hand, the Hamiltonian is again expressed by equations (3.11) and (3.12), namely

\[
H_{\text{fluid}} = \int d^3 x \left\{ N \sqrt{h} \left( \rho (1 + U^i U_i) + p U^i U_i \right) + N^i (-\mu \Pi U_i) \right\} .
\] (5.11)

This result is dictated by the fact that the fluid contributions to the Hamiltonian and momentum constraints are just the energy and momentum densities of the fluid and are given by appropriate projections of the fluid stress–energy–momentum tensor [17]. The quantities \(n\) and \(U_i\) that appear in the Hamiltonian (5.11) are determined as functions of the canonical variables \(\alpha^A\), \(P_A\) through equations (5.9b) and (5.10). For this purpose observe that, according to expression (5.1),

\[
\Pi = \frac{1}{3!} \sqrt{h} \epsilon^{ijk} \eta_{ABC} \alpha^A_{,i} \alpha^B_{,j} \alpha^C_{,k} .
\] (5.12)

Thus, \(\Pi\) only depends on \(\alpha^A\) and their spatial derivatives, not on \(\dot{\alpha}^A\). Also note that the fluid contribution to the momentum constraint equals \(P_A \alpha^A_{,i}\) and is the canonical generator of spatial diffeomorphisms for the scalar fields \(\alpha^A\) and their conjugates \(P_A\).

The canonical equations of motion follow from the variation (3.18) of the fluid contribution to the momentum constraint, where \(\Pi\) is the function (5.12) of \(\alpha^A\). This calculation shows that once again \(\dot{\alpha}^A\) is given by equation (3.19c), and that \(\dot{P}_A\) is given by

\[
\dot{P}_A = -\frac{\delta H_{\text{fluid}}}{\delta \alpha^A} = - \left( \frac{NP_A U^i}{(1 + U \cdot U)^{1/2}} \right)_{,i} + (N^i P_A)_{,i} - N \sqrt{h} T_{ns.A} + \Pi_{,i} \alpha^A \left( \frac{N \mu}{(1 + U \cdot U)^{1/2}} \right)_{,i} .
\] (5.13)

Equation (5.13) is just the space–time decomposition of the Lagrangian equation of motion (5.7). Also note that the first two terms in \(\dot{P}_A\) above coincide with \((\Pi \beta_\lambda)\), from equation (3.19f).
As mentioned in the introduction, the canonical Hamiltonian formulation of perfect fluids derived here is related to the Lie–Poisson Hamiltonian formulation by a Lagrangian to Eulerian map (see [18] and references therein). The result of this mapping is a Hamiltonian description of perfect fluids in which \( \Pi, s, \) and the fluid momentum density are the fundamental variables. Since the Lie–Poisson brackets are not canonical, there is no corresponding action functional of the form “\( \int (p \dot{q} - H) \)”.

5.3 Symmetries of the action \( \bar{S} \)

For fixed fluid space tensors \( \eta \) and \( s \), the local coordinate expressions \( s(\alpha) \) and \( \eta_{123}(\alpha) \, d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3 \) can be constructed from any set of coordinates \( \alpha^A \) on the fluid space. Under a change of coordinates \( \delta \alpha^A = \xi^A(\alpha) \), the functions \( \eta_{123}(\alpha) \) and \( s(\alpha) \) at a given value of \( \alpha^A \) (at a given coordinate location) change according to 
\[
- (\mathcal{L}_\xi \eta)_{123} \quad \text{and} \quad - (\mathcal{L}_\xi s)(\alpha),
\]
respectively, and correspondingly the tensors \( \eta_{123}(\alpha) \, d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3 \) and \( s(\alpha) \) at a given point on the fluid space manifold remain unchanged. (Here, \( \mathcal{L}_\xi \) is the Lie derivative in the fluid space along the vector field \( \xi \).) In this sense the functional \( \bar{S} \), which depends on \( \alpha^A \) only through the combinations \( \eta_{123}(\alpha) \, d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3 \) and \( s(\alpha) \), is invariant under changes of fluid space coordinates. Note however that this invariance of \( \bar{S} \) involves a transformation of the functions \( \eta_{123}(\alpha) \) and \( s(\alpha) \) at a given value of \( \alpha^A \), as well as a transformation of the field variables \( \alpha^A \). Consequently this invariance does not in general correspond to a conserved Noether current. On the other hand, the subset of fluid space coordinate transformations \( \delta \alpha^A = \xi^A \) that satisfy \( \mathcal{L}_\xi \eta = 0 \) and \( \mathcal{L}_\xi s = 0 \) do not involve a transformation of the functions \( \eta_{123}(\alpha) \) and \( s(\alpha) \) at a given value of \( \alpha^A \) and do give rise to conserved Noether currents. This same conclusion can be reached by considering the changes in \( \eta_{123}(\alpha) \, d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3 \) and \( s(\alpha) \) induced by a general field transformation \( \delta \alpha^A = \xi^A \), where \( \eta_{123}(\alpha) \) and \( s(\alpha) \) are treated as fixed functions of \( \alpha^A \). Those changes are given by
\[
\begin{align}
\delta (\eta_{123}(\alpha) \, d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3) &= (\mathcal{L}_\xi \eta)_{123}(\alpha) \, d\alpha^1 \wedge d\alpha^2 \wedge d\alpha^3, \\
\delta (s(\alpha)) &= (\mathcal{L}_\xi s)(\alpha),
\end{align}
\]
and show that \( \bar{S} \) is invariant under transformations \( \delta \alpha^A = \xi^A \) that satisfy \( \mathcal{L}_\xi \eta = 0 \) and \( \mathcal{L}_\xi s = 0 \). The Noether current associated with such transformations is obtained
from a general variation of the action, which is
\[
\delta \bar{S} = \text{(terms giving the equations of motion)} + \frac{1}{2} \int d^4x \left[ \sqrt{-g} \mu U_b \epsilon^{abcd} \eta_{ACD} \delta \alpha^A \alpha^C, \epsilon \alpha^D, \alpha \right]_a.
\] (5.15)

For the transformations \( \delta \alpha^A = \xi^A \) that leave \( \bar{S} \) invariant, the last integral in expression (5.15) must vanish when the equations of motion hold and the conserved current is
\[
\frac{1}{2} \sqrt{-g} \mu U_b \epsilon^{abcd} \eta_{ACD} \xi^A(\alpha) \alpha^C, \epsilon \alpha^D, \alpha.
\] (5.16)

The Noether charge is the integral of the time component of this current over a spacelike hypersurface. Using the definition (5.9) for the conjugate momenta, this charge can be written as
\[
Q[\xi] = \int_{\Sigma} d^3x P_A \xi^A.
\] (5.17)

\( Q[\xi] \) is independent of the spacelike hypersurface \( \Sigma \) for any fluid space vector \( \xi \) that leaves the tensors \( \eta \) and \( s \) invariant under Lie transport.

6. Other action functionals

6.1 Equation of state \( p(\mu, s) \)

A relativistic perfect fluid with equation of state \( \rho(n, s) \) is described by histories that extremize the action (2.5), along with the gravitational action, under independent variations of \( J^a, \varphi, \theta, s, \alpha^A, \beta_A, \) and \( g_{ab} \). This choice of variables is convenient, but not unique. For example, replace \( J^a/\sqrt{-g} \) by \( nU^a \) with \( n \) and \( U^a \) varied separately, subject to the normalization condition \( U^a U_a = -1 \). This changes the individual equations of motion, but as a set the new equations are equivalent to the original equations (2.8). In particular, the stress tensor has the standard perfect fluid form (2.1) only when certain other equations of motion hold.

The action (2.5) with \( J^a/\sqrt{-g} \) replaced by \( nU^a \) is
\[
S = -\int d^4x \sqrt{-g} \left\{ \rho(n, s) - nU^a (\varphi, a - \theta s, a + \beta_A \alpha^A, a) \right\},
\] (6.1)

where \( n \) is an independent variable. Now, according to the first law (1.3), the chemical potential \( \mu \) is the function of \( n \) and \( s \) defined by \( \mu = \partial \rho/\partial n \). If this
equation could be inverted for \( n \) as a function of \( \mu \) and \( s \), then \( \mu \) could be used as an independent variable in the action (6.1) by substituting \( n(\mu, s) \) in place of \( n \). In effect, this inversion is accomplished by using the pressure \( p \) as a function of \( \mu \) and \( s \) to generate the functions \( n(\mu, s), \rho(\mu, s) \) through the first law in the form (1.4). That is, let \( p(\mu, s) \) specify the equation of state and determine \( n \) and \( \rho \) through

\[
  n := \frac{\partial p}{\partial \mu} ,
\]

\[
  \rho := \mu \frac{\partial p}{\partial \mu} - p .
\]

(6.2a, 6.2b)

Inserting these relationships into the action (6.1) yields an action \( S(p) \) for a perfect fluid with equation of state \( p(\mu, s) \).

For the action \( S(p) \), it is convenient to use the Taub vector \( V^a = \mu U^a \) as the independent variable in place of \( \mu \) and \( U^a \). With this definition, the action with equation of state \( p(\mu, s) \) reads

\[
  S(p) = \int d^4x \sqrt{-g} \left\{ p - \left( \frac{\partial p}{\partial \mu} \right) \left( |V| - V^a (\varphi, a + s \theta, a + \beta_A \alpha^A, a) / |V| \right) \right\} ,
\]

(6.3)

where \( p = p(\mu, s) \) and \( \mu := |V| = \sqrt{-V^a g_{ab} V^b} \). The equations of motion obtained by varying \( \varphi, \theta, s, \alpha^A, \) and \( \beta_A \) yield equations (2.8b–f) respectively. The equation of motion obtained by varying \( V^a \) is

\[
  0 = \frac{1}{\sqrt{-g}} \frac{\delta S(p)}{\delta V^a} = \frac{1}{|V|^2} \left( \frac{\partial^2 p}{\partial \mu^2} \right) \left( |V|^2 - V^b (\varphi, a + s \theta, a + \beta_A \alpha^A, a) \right) V_a
\]

\[
  - \frac{1}{|V|} \left( \frac{\partial p}{\partial \mu} \right) (\varphi, b + s \theta, b + \beta_A \alpha^A, b) \left( \delta^b_a + \frac{V^b V_a}{|V|^2} \right) .
\]

(6.4)

The projection of this equation orthogonal to the fluid flow lines shows that \( (\varphi, a + s \theta, a + \beta_A \alpha^A, a) \) is proportional to \( V_a \), and then projection along the flow lines gives

\[
  V_a = -(\varphi, a + s \theta, a + \beta_A \alpha^A, a) .
\]

(6.5)

This is the equation of motion (2.8a), so the action \( S(p) \) indeed yields the complete set of fluid matter equations (2.8). As shown in section 2, these equations of motion imply particle number conservation, conservation of entropy along the flow lines, and the Euler equation. The stress tensor derived from \( S(p) \) has the perfect fluid form (2.1) when equation (6.5) holds.
The equation of motion (6.5) can be used to eliminate $V^a$ from the action $S_{(p)}$ yielding an action functional that is essentially the one found by Schutz [15]:

$$\tilde{S}_{(p)} = \int d^4x \sqrt{-g} \rho(\mu, s). \quad (6.6)$$

Here, $\mu$ is treated as the function of $\varphi, \theta, s, \alpha^A, \beta_A$, and $g_{ab}$ determined by $\mu^2 = -V^aV_a$, where $V_a$ is given by equation (6.5). The only difference between the action (6.6) and Schutz’s action is a difference in the number of pairs of fields $\alpha^A$ and $\beta_A$. Schutz uses just one pair, in accordance with the Pfaff reduction discussed in section 4. The canonical form of Schutz’s action is constructed in references [19] and [20].

6.2 Equation of state $a(n, T)$

An action $S_{(a)}$, for a perfect fluid with equation of state specified by $a(n, T)$, the physical free energy as a function of number density and temperature, is obtained by eliminating the entropy per particle $s$ from the action (2.5). Thus, consider solving the equation of motion (2.8d) for $s$ as a function of $n$ and $T = \theta_a U^a$, then eliminating $s$ from the action (2.5) by substituting the result $s(n, T)$. The term $-\sqrt{-g} \rho + s \theta_a J^a$ becomes a function of $n = |J|/\sqrt{-g}$ and $T = \theta_a J^a / |J|$ that is identified with $na$, where $a$ is the physical free energy (1.2b). The resulting action for a perfect fluid with equation of state $a(n, T)$ is

$$S_{(a)} = -\int d^4x \left\{ |J| a(|J|/\sqrt{-g}, \theta_a J^a / |J|) - J^a (\varphi_a + \beta_a \alpha^A) \right\}. \quad (6.7)$$

$S_{(a)}$ is a functional of $J^a, \varphi, \theta, \alpha^A, \beta_A$, and $g_{ab}$.

It is straightforward to show that the equation of motion $\delta S_{(a)}/\delta J^a = 0$ is equivalent to equation (2.8a), and the equations of motion obtained by varying $\varphi, \theta, \alpha^A$, and $\beta_A$ are just equations (2.8b,c,e,f). Of course, the equation of motion (2.8d) is missing, since $S_{(a)}$ does not depend on $s$. But with the identification of thermasy as in (2.9), this equation simply reiterates the relationship dictated by the first law of thermodynamics and is, in this sense, superfluous. Therefore the equations of motion derived from the action $S_{(a)}$, along with the interpretation of the variables in a manner that is consistent with the first law (1.5), are complete in the sense that they imply particle number conservation, conservation of entropy along the flow lines, and the Euler equation. In addition, the stress tensor obtained from the action (6.7) has the perfect fluid form (2.1).
The action discussed by Kijowski et al. [14] can be obtained from the action \( S(a) - \int d^4 x (\varphi J^a) \), by substituting expression (5.1) for \( J^a \). The terms \(-\varphi J^a, \alpha^A, \beta_A \) drop out, leaving
\[
\bar{S}(a) = -\int d^4 x \left\{ |J| a(|J|/\sqrt{-g}, \theta, J^a/|J|) \right\} . \tag{6.8}
\]

Variations with respect to the metric \( g_{ab} \) and the thermasy \( \theta \) are unchanged by this substitution, so \( \bar{S}(a) \) yields the correct stress tensor (2.1) and the conservation of entropy (2.8c). A calculation similar to the one appropriate for \( \bar{S} \) shows that variations of \( \bar{S}(a) \) with respect to the Lagrangian coordinates \( \alpha^A \) yield equation (2.14), which implies the Euler equation. The Hamiltonian form of \( \bar{S}(a) \) and the symmetries and conserved charges can be found along the same lines as the analysis found in section 5 for \( \bar{S} \). The Hamiltonian form of this action also has been considered in reference [41].

6.3 Isentropic fluids and dust

Isentropic fluids are perfect fluids with a constant entropy per particle \( s \). The first law of thermodynamics in the form (1.5) indicates that isentropic fluids are described by an equation of state of the form
\[
a(n, T) = \frac{\rho(n)}{n} - sT , \tag{6.9}
\]
where \( s \) is the constant value of the entropy per particle. Inserting this equation of state into the action (6.7) yields a functional like \( S \) of equation (2.5), but with two differences: the function \( \rho \) only depends on \( n = |J|/\sqrt{-g} \), and \( s \) appears as a fixed constant, not a variable. Thus, \( s\theta \) can be absorbed into \( \varphi \) by the change of variables \( \varphi' := \varphi + s\theta \), which reduces the combination \( \varphi, a + s\theta, a \) that appears in the action to \( \varphi', a \). This leads to the isentropic fluid action
\[
S_{\text{isentropic}} = \int d^4 x \left\{ -\sqrt{-g} \rho(|J|/\sqrt{-g}) + \varphi', a J^a + \beta_A \alpha^A, a J^a \right\} , \tag{6.10}
\]
which is a functional of \( J^a, \varphi', \alpha^A, \beta_A \), and \( g_{ab} \). In terms of the new variable \( \varphi' \), the equation of motion (2.8a) obtained by varying \( J^a \) becomes \( 0 = \mu U_a + \varphi', a + \beta_A \alpha^A, a \). Contracting with \( U^a \) gives \( \mu = \varphi', a U^a \) and shows that \( \varphi' \) is a “potential” for the chemical potential \( \mu \).
Dust is a particular case of an isentropic fluid in which the energy density \( \rho \) is proportional to the number density \( n \) and the pressure (2.7) is zero. The proportionality constant is the rest mass–energy of a fluid particle, which equals the chemical potential \( \mu = \rho/n \). Inserting the equation of state \( \rho = \mu n \) into the isentropic fluid action (6.10) yields

\[
S_{\text{dust}} = \int d^4x \left\{ -\mu |J| + \varphi_a J^a + \beta_A \alpha^A, a J^a \right\},
\]

where the prime has been dropped from \( \varphi \). An alternative to this dust action is obtained by replacing \( |J|/\sqrt{-g} \) in the action (6.11) with the function \( [(|J|/\sqrt{-g})^2/n + n]/2 \) and treating \( n \) as a new dynamical variable. This is justified because the action (6.11) is recovered when \( n \) is eliminated from the new action by using the solution \( n = |J|/\sqrt{-g} \) to the equation of motion for \( n \). After introducing the variable \( n \) into the dust action, \( J^a \) can be eliminated by using the \( J^a \) equation of motion. This yields an action for dust that is a functional of \( n, \varphi, \alpha^A, \) and \( \beta_A \), namely

\[
S'_{\text{dust}} = -\frac{\mu}{2} \int d^4x \sqrt{-gn} (U_a g^{ab} U_b + 1),
\]

where \( U_a := (\varphi_a + \beta_A \alpha^A, a)/\mu \).

The Hamiltonian form of the dust action (6.11) or (6.12) is just the canonical action (3.10) but with the terms containing \( \theta \) and \( s \) omitted:

\[
S_{\text{dust}} = \int dt d^3x \left\{ \Pi \dot{\varphi} + \left( \Pi \beta_A \right) \dot{\alpha}^A - N H_{\text{dust}} - N^i H_{i, \text{dust}} \right\}.
\]

The dust contributions to the Hamiltonian and momentum constraints can be found explicitly, because the chemical potential appearing in equation (3.7) for \( U_i \) has no \( n \) dependence. Thus, from equations (3.14a) and (3.15),

\[
H_{i, \text{dust}} = \Pi \varphi,i + (\Pi \beta_A) \alpha^{A,i}, \quad (6.14a)
\]

\[
H_{\text{dust}} = \left[ (\mu \Pi)^2 + H_{i, \text{dust}} h^{ij} H_{j, \text{dust}} \right]^{1/2}. \quad (6.14b)
\]

In reference [26], the action (6.12) and its canonical form (6.13) are used to analyze the quantum theory of gravity coupled to dust.

The equation of state (6.9) also can be inserted into the action functional (6.8). This yields an isentropic fluid action

\[
S_{\text{isentropic}} = -\int d^4x \left\{ \sqrt{-g} \rho(|J|/\sqrt{-g}) \right\},
\]

35
where a total divergence (boundary term) has been discarded. Here, $J^\alpha$ is a function of the Lagrangian coordinates $\alpha^A$ as given by equation (5.1). Specialized to a pressureless dust equation of state, this action becomes $\tilde{S}_{\text{dust}} = -\int d^4x \mu |J|$. In the canonical form of this dust action, the dust contribution to the momentum constraint is $P_A \alpha^A, i$, and the dust contribution to the Hamiltonian constraint is given by equation (6.14b) where $\Pi$ is a function of $\alpha^A$ through equation (5.12).

**Acknowledgments**

I would like to thank C.R. Evans, K.V. Kuchař, V. Moncrief, T. Piran, and J.W. York for helpful discussions. I also thank the Center for Relativity at The University of Texas at Austin, the Aspen Center for Physics, and the Institute of Field Physics at the University of North Carolina for hospitality during various stages of this work. Research support was received from National Science Foundation grant number PHY–8908741 to the University of North Carolina.

**Appendix**

The symmetries of the action (2.5) are transformations among the potentials $\varphi$, $s$, $\theta$, $\beta_A$, and $\alpha^A$ that leave the one–form $-\mu U = d\varphi + s d\theta + \beta_A d\alpha^A$ and the entropy per particle $s$ invariant at each spacetime point. That is, the action satisfies $S[\varphi, s, \theta, \beta_A, \alpha^A] = S[\varphi', s', \theta', \beta'_A, \alpha'^A]$ if the primed and unprimed variables are related by

$$
d\varphi + s d\theta + \beta_A d\alpha^A = d\varphi' + s' d\theta' + \beta'_A d\alpha'^A,
$$

$$
s = s'.
$$

Equation (A.1a) can be viewed as the expression of a canonical transformation with generating function $\varphi - \varphi'$ for a fictitious phase space with coordinates $\theta, \alpha^A$. In general, the primed variables can be nonlocal functions of the unprimed variables. Thus, for example, $\varphi'(x)$ can depend on the unprimed variables at spacetime points other than $x$. The analysis that follows is restricted to the case of ultralocal transformations in which the primed variables at $x$ depend only on the unprimed variables at $x$. In particular, the primed variables are not allowed to depend on the derivatives of the unprimed variables.
and momenta \( s, \beta_A \). In terms of the true phase space variables discussed in section 3, the symmetry transformations (A.1) are given by

\[
\Pi d\varphi + (\Pi s) d\theta + (\Pi \beta_A) d\alpha^A = \Pi' d\varphi' + (\Pi s)' d\theta' + (\Pi \beta_A)' d\alpha^A',
\]
\[
\Pi = \Pi',
\]
\[
(\Pi s) = (\Pi s)' .
\]

Equation (A.2a) expresses a Mathieu transformation among the canonical fluid variables at each space point. (Mathieu transformations are canonical transformations with zero generating function; that is, they preserve the form \( p_i dq_i \) [1].) The extra restrictions (A.2b) and (A.2c) can be analyzed by computing Poisson brackets of the primed variables with one another using the definition of the Poisson brackets in terms of the unprimed variables. For example, condition (A.2b) along with the Poisson bracket relationship \( \{ \varphi', \Pi' \} = 1 \) lead to \( \partial \varphi' / \partial \varphi = 1 \); therefore,

\[
\varphi' = \varphi + f ,
\]

where \( f \) is independent of \( \varphi \). The Poisson bracket of \( \varphi' \) and \( (\Pi s)' \) along with condition (A.2c) show that \( f \) is independent of \( \theta \). A similar analysis gives

\[
\theta' = \theta + g ,
\]

where \( g \), like \( f \), is a function only of \( \Pi, (\Pi s), \alpha^A \), and \( (\Pi \beta_A) \). It also follows from conditions (A.2b) and (A.2c) that \( \alpha' \) and \( (\Pi \beta_A)' \) are independent of \( \varphi \) and \( \theta \).

Using the results (A.3) and (A.4), equation (A.2a) can be written as

\[
d(\Pi F) = fd\Pi + gd(\Pi s) + (\Pi \beta_A) d\alpha^A + \alpha^A' d(\Pi \beta_A)' ,
\]

where \( F \) is defined by \( \Pi F = \Pi f + (\Pi s) g + (\Pi \beta_A)' \alpha^A' \). This relationship can be investigated by treating various combinations of primed and unprimed variables as independent. For example, assume that \((\Pi \beta_A)\) as a function of \( \Pi, (\Pi s), \alpha^A \), and \((\Pi \beta_A)\) can be inverted for \((\Pi \beta_A)\) as a function of \( \Pi, (\Pi s), \alpha^A \), and \((\Pi \beta_A)\). Then \( \Pi, (\Pi s), \alpha^A \), and \((\Pi \beta_A)\)' can be chosen as independent variables and equation (A.5) immediately yields

\[
\varphi' - \varphi = f = F + \Pi \frac{\partial F}{\partial \Pi} ,
\]

37
\[ \theta' - \theta = g = \Pi \frac{\partial F}{\partial (\Pi s)} , \quad (A.6b) \]

\[ (\Pi \beta_A) = \Pi \frac{\partial F}{\partial \alpha^A} , \quad (A.6c) \]

\[ \alpha^{A'} = \Pi \frac{\partial F}{\partial (\Pi \beta_A)^'} . \quad (A.6d) \]

These equations imply

\[ 0 = \Pi \left( \frac{\partial F}{\partial \Pi} + (\Pi s) \left( \frac{\partial F}{\partial (\Pi s)} \right) + (\Pi \beta_A) \left( \frac{\partial F}{\partial (\Pi \beta_A)^'} \right) \right) , \quad (A.7) \]

and show that \( F \) is homogeneous of degree zero in the momenta \( \Pi, (\Pi s), \) and \( (\Pi \beta_A)^' \). As a consequence, the functional dependence of \( F \) can be expressed as

\[ F = F(\alpha^A, (\Pi s)/\Pi, (\Pi \beta_A)^'/\Pi) \]

and equation (A.6a) can be written as

\[ \varphi' = \varphi + F - (\Pi s) \frac{\partial F}{\partial (\Pi s)} - (\Pi \beta_A)^' \frac{\partial F}{\partial (\Pi \beta_A)^'} . \quad (A.8) \]

Other choices of independent variables lead to alternative sets of transformations, in the same way as canonical transformations are categorized into various types.

The infinitesimal symmetry transformations are obtained by setting

\[ F = \left( \alpha^A (\Pi \beta_A)^' \right)/\Pi + \epsilon F , \quad (A.9) \]

where \( \epsilon \) is an infinitesimal parameter. Equations (A.8) and (A.6b,c,d) then become

\[ \delta \varphi = \epsilon F - \epsilon (\Pi s) \frac{\partial F}{\partial (\Pi s)} - \epsilon (\Pi \beta_A) \frac{\partial F}{\partial (\Pi \beta_A)} , \quad (A.9a) \]

\[ \delta \theta = \epsilon \Pi \frac{\partial F}{\partial (\Pi s)} , \quad (A.9b) \]

\[ \delta (\Pi \beta_A) = -\epsilon \Pi \frac{\partial F}{\partial \alpha^A} , \quad (A.9c) \]

\[ \delta \alpha^A = \epsilon \Pi \frac{\partial F}{\partial (\Pi \beta_A)} , \quad (A.9d) \]

where \( F \) has functional dependence \( F(\alpha^A, (\Pi s)/\Pi, (\Pi \beta_A)/\Pi) \). The factors of \( \Pi \) can be eliminated from these relationships, leading directly to the expression (2.18) of the infinitesimal symmetry transformations in terms of the fluid potentials \( \varphi, s, \theta, \beta_A, \) and \( \alpha^A \).

38
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40