Approval Voting in Circular Societies: Piercing Numbers and Agreement

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Abstract

In the system of approval voting, individuals vote for all candidates they find acceptable. Many approval voting situations can be modeled geometrically and thus, geometric concepts such as the piercing number, can have a natural interpretation. In this paper we explore piercing numbers and agreement in the setting where preferences can be modeled by arcs on a circle – i.e., in circular societies. We give bounds on piercing and agreement in circular societies with fixed-length approval sets. We also present probabilistic results about the average piercing number for such circular societies.

Keywords: approval voting, piercing number, circular societies, discrete geometry, applied topology, probability

1 Introduction

Approval voting is a system in which each individual casts a vote for all candidates they approve of and the winning candidate is the one that receives the most votes. This system is of interest in that it has many advantages over the more common plurality voting systems and it can be applied outside of the political arena \cite{4}. For instance, approval voting is often the way that committees approach scheduling problems: each committee member indicates the times they are available and the meeting is held at the time most members are able to meet. This situation can be modeled mathematically by considering each person’s availability as an interval in a line or an arc on a circle, the latter corresponding to a 24-hour clock which may be especially useful for scheduling virtual meetings occurring across several time zones.

To expand upon the scheduling problem above, suppose the committee requires all of its members to attend a training session. They plan to hold the same training session at multiple times so that all members will be able to attend one of the sessions. After viewing each member’s availability the committee then asks, “what is the minimum number of training sessions needed so that each member can attend at least one of the sessions?” This question in approval voting corresponds to the notion of piercing in convex geometry. Namely, given a collection of sets, a piercing set is a collection of points such that each set contains at least one of the points. The size of the smallest possible piercing set is the piercing number of the collection of sets. Thus, if we model the above scenario geometrically, then the time intervals at which the members are available is a collection of sets and the piercing number of the collection is the minimum number of training sessions needed.

In 1957, Hadwiger and Debrunner \cite{6} developed an upper bound on the piercing number of convex sets in $\mathbb{R}^d$ based on a local intersection property. For example, they showed that given a collection of intervals on the real number line, if two out of every four intervals intersect then the piercing number is at most three. Su and Zerbib recently contextualized piercing numbers in terms of approval voting \cite{13}. They present a
collection of piercing results that best apply to approval voting scenarios that are modeled using intervals on a line. On the other hand, we model the training session scenario discussed above using arcs on a circle, to account for the periodic nature of the calendar year or the 24-hour clock (e.g., members are able to select an availability interval of December-January or 10:00pm-1:00am). In approval voting we call this model a circular society. The piercing number for circular societies has scarcely been explored (see Final Remark 1 in [13]).

In this paper, we develop a collection of piercing number results that apply specifically to circular societies. After reviewing mathematical background on approval voting in Section 2, we give an algorithm for finding a piercing set of a circular society in Section 3. We prove theorems about piercing in circular societies in which all arcs have the same fixed length in Section 4. In Section 5, we introduce a local counting function that allows us to develop further results on both the piercing number and on the maximum number of arcs that intersect at a single point. Lastly, in Section 6, we explore the probability of certain piercing numbers when arcs of a fixed length are randomly distributed around the circle.

2 Mathematical Background

We refer to the set of all possible candidates that voters may select from as a spectrum \( X \). The set of candidates acceptable to a voter is called an approval set. Thus, each approval set is a subset of the spectrum. We define a society \( S \) of \( n \) voters to be a pair \((X, A)\) in which \( X \) is a spectrum and \( A \) is a collection of \( n \) approval sets. While the term “candidate” may seem to imply that spectra and thus approval sets are necessarily discrete, we only use this language to remain consistent with previous literature, and we assume that each point in a continuous spectrum represents a possible candidate. In [3], Berg et al. study societies in which the spectrum \( X \) is a closed subset of the real number line, and approval sets are closed and bounded intervals. They call such societies linear societies. On the other hand, Hardin [7] considers circular societies in which the spectrum is a circle and approval sets are arcs on the circle. Furthermore, the authors of this paper previously studied products of linear and circular societies [10].

The main goal in each of [3], [7], and [10] is to understand “global” agreement in each type of society based on an idea of “local” agreement. The global agreement of a society \( S \) is the maximum number of voters in \( S \) who agree on a single candidate. We call this number the agreement number \( a(S) \). We examine local agreement by polling subcollections of voters instead of considering all voters at once. Specifically, the local agreement condition studied in the aforementioned papers is that given any collection of \( m \) voters, at least \( k \) of them agree on a single candidate. In other words, they define a society to be \((k, m)\)-agreeable if \( k \) out of every \( m \) approval sets intersect on the spectrum. We provide examples of \((2,3)\)-agreeable linear and circular societies in Figure 1. Berg et al. develop a lower bound on the agreement number in a linear society based on its \((k, m)\)-agreeability. Hardin develops an analogous result for circular societies. We state both theorems below.

**Theorem 1** (Agreement in Linear Societies [3]). If \( S \) is a \((k, m)\)-agreeable linear society with \( n \) voters then at least \( n(k - 1)/(m - 1) \) voters agree on a single candidate. Thus, \( a(S) \geq n(k - 1)/(m - 1) \).

**Theorem 2** (Agreement in Circular Societies [7]). If \( S \) is a \((k, m)\)-agreeable circular society with \( n \) voters then at least \( n(k - 1)/m \) voters agree on a single candidate. Thus, \( a(S) > n(k - 1)/m \).

A linear society and a circular society with the same local agreement may have different global agreements. For example, if \( S \) is a \((2, 3)\)-agreeable linear society with five voters, then at least three voters must agree on a single candidate. However, if \( S \) is a \((2, 3)\)-agreeable circular society with five voters, then only at least two voters must agree on a single candidate. See Figure 1. In general, the lower bound on agreement in circular societies given in Theorem 2 is smaller than the lower bound for linear societies given in Theorem 1.

The piercing number is another measure of global agreeability in a society. Given a society \( S = (X, A) \), Su and Zerbib [13] call a piercing set of \( A \) a representative candidate set of \( S \). Hence, a representative candidate set is a collection of candidates in the spectrum such that each voter approves of at least one candidate in the collection. We continue to call the size of the smallest possible representative candidate set the piercing number of the society \( S \), and denote it \( \tau(S) \). Moreover, if an approval set \( A \) contains a candidate \( x \) we say that \( x \) pierces \( A \). Su and Zerbib [13] give an upper bound on the piercing number of a \((k, m)\)-agreeable linear society.
Theorem 3 (Piercing Numbers in Linear Societies [13]). If $S$ is a $(k,m)$-agreeable linear society then we can find a representative candidate set with $m - k + 1$ candidates in it. Hence, $\tau(S) \leq m - k + 1$.

Theorem 3 is a corollary of a more general result from discrete geometry that assumes the $(p,q)$-property.

A family of convex sets satisfies the $(p,q)$-property if out of every $p$ sets, $q$ of them intersect. Thus, if a collection of approval sets are $(k,m)$-agreeable then they satisfy the $(m,k)$-property, and so Theorem 3 follows directly from the Hadwiger-Debrunner Theorem stated below.

Theorem 4 (Hadwiger-Debrunner, 1957 [6]). Let $F$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the $(p,q)$-property for $p \geq q > 1$. If $(d-1)p < d(q-1)$, then $\tau(F) \leq p - q + 1$.

In [13], Su and Zerbib indicate that we can use Theorem 3 to find an upper bound on the piercing number of a $(k,m)$-agreeable circular society. While it may seem that we can obtain this result from Theorem 4 by replacing each approval set with its convex hull, this requires working in $\mathbb{R}^2$ and thus only applies to $(k,m)$-agreeable circular societies for which $m < 2(k-1)$. In order to obtain a result for any $(k,m)$-agreeable circular society, Su and Zerbib recognize that some circular societies may be viewed linearly. Specifically, given a circular society $S = (X, A)$, if there exists a point $x$ in $X$ that is not contained in any approval set in $A$ then we can “cut” the spectrum at $x$ and “unroll” it to produce a linear society that maintains all of the agreement properties (agreement number, $(k,m)$-agreeability, piercing number, etc.) of $S$. So, this linear society is, in essence, equivalent to $S$. We call a circular society whose spectrum contains such a point $x$ a linear-equivalent circular society; see Figure 2 for an example. Hence, if $S$ is a $(k,m)$-agreeable linear-equivalent circular society with $n$ voters then by Theorem 3, $\tau(S) \leq m - k + 1$ and by Theorem 1, $a(S) \geq n(k-1)/(m-1)$. If a circular society $S$ is not linear-equivalent, then we can find a linear-equivalent sub-society by removing approval sets. More precisely, let $S = (X, A)$ be a circular society, let $x$ be a point in $X$, and let $A'$ be all approval sets that do not contain $x$. Then $S' = (X, A')$ is a sub-society of $S$ that is linear-equivalent.

We restate Su and Zerbib’s upper bound on the piercing number of a $(k,m)$-agreeable circular society in Theorem 6 and provide a formal proof, which relies on the following property of $(k,m)$-agreeability.

Lemma 5. If $S$ is a $(k,m)$-agreeable society then it is also a $(k-i, m-i)$-agreeable society for any $0 < i < k$.

Proof. Let $S$ be a $(k,m)$-agreeable society, and let $A$ be a collection of $m - i$ approval sets in $S$. Adding any $i$ additional approval sets to $A$ creates a collection of $m$ sets, and thus at least $k$ of them must intersect.
Since at most $i$ of these $k$ sets are not in $A$, it follows that at least $k - i$ approval sets from $A$ intersect. Therefore, $S$ is $(k - i, m - i)$-agreeable.

**Theorem 6.** Suppose $S$ is a $(k, m)$-agreeable circular society. Then $S$ has a representative candidate set of size $m - k + 2$. Hence, $\tau(S) \leq m - k + 2$.

**Proof.** Suppose that $S$ is a $(k, m)$-agreeable circular society, and choose an arbitrary point $x$ on the spectrum of $S$. Remove all approval sets that contain $x$ to create a linear-equivalent sub-society $S'$.

If $S'$ maintains at least $m$ approval sets, then it is $(k, m)$-agreeable, and thus by Theorem 3 it follows that $\tau(S') \leq m - k + 1$. If $S'$ has only $m - i$ sets for some $i > 0$, then by Lemma 5 it must be $(k - i, m - i)$-agreeable, in which case $\tau(S') \leq (k - i) - (m - i) + 1 = m - k + 1$. Regardless, there is a representative candidate set $W'$ of $S'$ containing $m - k + 1$ points. It then follows that $W' \cup \{x\}$ is a representative candidate set of the original circular society $S$. Therefore, $\tau(S) \leq k - m + 2$.

As in the results concerning the agreement number for linear and circular societies, Theorems 3 and 6 provide different bounds for the piercing number of $(k, m)$-agreeable linear and circular societies. In particular, the bound for the piercing number of a circular society is larger than that of a linear society. We can see this is necessary by examining Figure 1: the circular society requires 3 piercing points whereas the linear society requires only 2.

### 3 Piercing Number Algorithms

If $S$ is a linear society, then not only can we find an upper bound on the piercing number via Theorem 3, but we can actually produce a minimum representative candidate set using the algorithm described below. This algorithm is given without proof in [9, Section 1.4]. We restate the algorithm using approval voting language, provide a proof, and highlight its connection to a set packing problem. We then extend the algorithm to circular societies.

**Algorithm 7**

Given a linear society $S = (X, A)$, a minimum representative candidate set $W$ can be found as follows.

1: Find the approval set in $A$ with the leftmost right endpoint $x$, and add $x$ to $W$.

2: Remove from $A$ all approval sets that contain $x$.

3: Repeat lines 1 and 2 until $A$ is empty.

**Example 8.** In Figure 3 we apply Algorithm 7 to the linear society below. The algorithm produces a minimum representative candidate set containing the right endpoints of the approval sets $A_1$, $A_5$, $A_9$, and $A_{13}$. Hence, the piercing number of this society is four.

![Diagram of linear society with approval sets and points](image)

**Theorem 9.** Let $S$ be a linear society. Algorithm 7 produces a minimum representative candidate set of $S$.

**Proof.** Since $S$ contains a finite number of approval sets, Algorithm 7 terminates in a finite number of iterations and produces a valid representative candidate set. Let $W = \{x_1, \ldots, x_k\}$ be the representative candidate set produced by Algorithm 7. It remains to show that $W$ is a minimum representative candidate set. The points $x_1, x_2, \ldots, x_k$ are the right endpoints of $k$ approval sets, call them $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$. By construction of $W$, these approval sets are disjoint. Therefore, any representative candidate set must contain a point in each of $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ and as these sets are disjoint, the points must be distinct. Hence, any representative candidate set must contain at least $k$ points, so $W$ is a minimum representative candidate set.
The problem of finding a minimum representative candidate set in a society, or in other words finding a minimum piercing set of a collection of sets, is closely related to another geometric optimization problem called the set packing problem: given a collection of sets \( A \), find the largest subcollection of sets in \( A \) that are pairwise disjoint. The two problems are related since the maximum number of pairwise disjoint sets in \( A \) is less than or equal to the size of the minimum piercing set for \( A \) \[5\]. In some special cases, such as when \( A \) is a collection of intervals on a line (i.e., when \( S = (X, A) \) is a linear society) the two numbers are equal\(^1\) [8]. However, the two numbers are not necessarily equal for a collection of arcs on a circle (i.e., for a circular society).

Algorithm 7 not only produces a minimum representative candidate set for a linear society but also provides a solution to the set packing problem. Furthermore, in Proposition 10 we show that under a mild additional assumption on the approval sets of the society, Algorithm 7 produces a collection of pairwise-disjoint sets that have a special property that we use in Section 6.

**Proposition 10.** Let \( S \) be a linear society with the property that no approval set is contained in another, and let \( x_1 < x_2 < \ldots < x_k \) be the piercing points produced by Algorithm 7. Let \( x_i \) be the right endpoint of approval set \( A_i \), reindexing approval sets if necessary. Then \( \{A_1, \ldots, A_k\} \) is the unique collection of pairwise-disjoint approval sets whose union contains the left endpoints of all remaining approval sets.

**Proof.** As noted in the proof of Theorem 9, approval sets \( \{A_1, \ldots, A_k\} \) are pairwise disjoint.

For each \( i = 1, \ldots, k \), let \( S_i \) be the collection of approval sets that contain \( x_i \) but not \( x_j \) for any \( j < i \). Since the \( A_i \) are pairwise disjoint, \( A_i \in S_i \). Since no approval set is contained in another, any other approval set in \( S_i \) must have its left endpoint in \( A_i \).

Since the \( x_i \) form a piercing set, every approval set is in one of the \( S_i \). Thus, \( \{A_1, \ldots, A_k\} \) collectively contain the left endpoints of all approval sets in the society.

For uniqueness, \( A_1 \) must be the leftmost approval set in \( S \), since no other approval set contains its left endpoint. Likewise, \( A_2 \) must be the leftmost approval set that is disjoint from \( A_1 \). Similarily, \( A_i \) must be the leftmost approval set that is disjoint from \( A_1 \cup A_2 \cup \ldots \cup A_{i-1} \). This results in a unique collection of approval sets \( A_1, \ldots, A_k \) that contain the left endpoints of all remaining approval sets. \( \square \)

If \( S \) is a linear-equivalent circular society, then we can use Algorithm 7 to produce a minimum representative candidate set of \( S \). If \( S \) is not linear-equivalent, we can still adapt Algorithm 7 to construct a representative candidate set. While this set may not be a minimum representative candidate set, we show that the size of this set is at most one more than the piercing number of \( S \).

**Theorem 12.** Let \( S \) be a circular society. Then Algorithm 11 produces a representative candidate set \( W \) whose size is at most one more than the piercing number of \( S \).

\(^1\)Furthermore, even though both problems are known to be NP-hard in general (including when \( X = \mathbb{R}^2 \) and \( A \) is a collection of axis-parallel boxes in \( \mathbb{R}^2 \) \[5\]), they are both tractable in the context of linear societies (that is, when \( X = \mathbb{R} \) and \( A \) is a collection of closed intervals). Indeed, we saw in Theorem 9 that Algorithm 7 produces a minimum representative candidate set in a small number of iteration.
Algorithm 11 Given a circular society $S = (X, A)$, a representative candidate set $W$ can be found as follows.

1. Select an arbitrary point $x$ on the spectrum $X$. Let $x$ be the first point in $W$.
2. Remove from $A$ all approval sets that contain $x$ to produce a linear-equivalent sub-society $S'$.
3. Apply Algorithm 7 to $S'$ to complete $W$.

Proof. Let $W'$ be the minimum representative candidate set of $S'$ created in Step 3 of Algorithm 11. Thus, $W = W' \cup \{x\}$. Since all sets in $S'$ contain a point in $W'$ and $x$ is contained in all approval sets removed from $A$ in Step 2, the set $W$ is a representative candidate set of $S$.

Next, let $\tau(S)$ be the piercing number of $S$, and $|W|$ be the size of $W$. We will show that $|W|$ is at most $\tau(S) + 1$. Suppose for contradiction that $W$ contains at least $\tau(S) + 2$ points, equivalently $\tau(S) \leq |W| - 2$. Then since $W = W' \cup \{x\}$ and $W'$ is a minimum representative candidate set of $S'$, it follows that $|W| = \tau(S') + 1$. Hence, $\tau(S) \leq \tau(S') - 1$. Since $S'$ is a sub-society of $S$, any representative candidate set of $S$ is also a representative candidate set of $S'$. Therefore, $\tau(S') \leq \tau(S) \leq \tau(S') - 1$, which is a contradiction.

The choice of the point $x$ in Step 1 of Algorithm 11 affects whether the representative candidate set produced contains a minimum number of points. Unfortunately, it is not an easy task to determine which point to select as it depends on the overall structure of the society.

Example 13. Consider the following circular society $S$ with $\tau(S) = 3$. If the point $x_1$ (see below left) is selected in Step 1 of Algorithm 11, then the algorithm produces a representative candidate set $W$ containing four points. However, if a different $x_1$ is selected first (see below right) then $W$ contains three points.

![Example 13](image1.png)

Though Algorithm 11 is new in the context of approval voting, a similar algorithm appears in a computer science context [8]. That algorithm involves an exhaustive process in which all intersections of a given arc with all other arcs are used as locations for the first piercing point. This produces a potentially large number of piercing sets, the smallest of which is a minimum piercing set. In approval voting, however, one wishes to obtain a minimum representative candidate set without resorting to an exhaustive search. This leads us to the following question.

Question 14. Is there a non-exhaustive way to choose the initial point $x$ in Algorithm 11 so that the algorithm finds a minimum representative candidate set?

4 Piercing Numbers in Fixed-Length Circular Societies

In this section we examine the piercing number of circular societies in which all approval sets are the same length. We call these societies fixed-length circular societies. In order to work with these societies more clearly and concisely, we first lay out some useful conventions for circular societies that we use throughout the remaining sections of this paper.

Remark 15. We use the classic topological property that a circle is equivalent to $\mathbb{R}/\mathbb{Z}$, identifying the points $x$ and $x + n$ for any real number $x$ and integer $n$. In other words, we represent the circular spectrum $X$ as the interval $[0, 1)$ in which we identify 0 and 1. In doing so, we force the circumference of $X$ to be 1, rescaling all approval sets as needed. Identifying the spectrum with $\mathbb{R}/\mathbb{Z}$ allows us to freely consider either linear or circular coordinates for approval sets, and to refer to approval sets as either “arcs” or “intervals” without confusion. We also assume that approval sets are closed unless otherwise noted.
Definition 16. Let $S$ be a circular society. We say that $S$ is a fixed-length circular society if all approval sets have the same length $p < 1$.

The approval set length $p$ can be thought of as a proportion of the circumference of the spectrum. For example, suppose we are modeling time preferences on a 24-hour clock. If we require individual approval sets to be 3-hour time windows, then since the spectrum circumference is 1, the approval set length $p$ is $\frac{1}{8}$. By Remark 15 we can also depict fixed-length circular societies “linearly” with approval sets as intervals (or a union of two intervals if the corresponding arc crosses the point corresponding to 0) of length $p$. See Figure 4.

![Figure 4: Two depictions of the same fixed-length circular society with approval set length $p = \frac{1}{4}$. We cut the figure on the left at the dashed line and “unroll” to obtain the figure on the right. The arrows on two approval sets on the right represent that these approval set “wrap around”.

In this section we explore the relationship between the approval set length $p$ of a fixed-length circular society and the piercing number of the society. Intuitively one expects that for longer approval sets, fewer points are required to pierce all sets. Indeed, the following result demonstrates this inverse relationship between approval set length and piercing number.

Theorem 17. Let $q \in \mathbb{N}$ and $S$ be a fixed-length circular society with approval sets of length $p \geq \frac{1}{q}$. Then there exists a representative candidate set containing $q$ candidates. Therefore, $\tau(S) \leq q$.

Proof. Consider $q$ points on the spectrum that are evenly spaced so the distance between any two points is exactly $\frac{1}{q}$. As each approval set has length at least $\frac{1}{q}$, each approval set must contain at least one of the points. Thus, the $q$ points form a representative candidate set.

It is often prudent to consider possible “best-case” and “worst-case” scenarios in approval voting. The best-case minimum representative candidate set contains a single candidate, whereas the worse-case minimum representative candidate set contains as many candidates as voters. The following propositions give conditions in which these extreme scenarios could occur.

Proposition 18. Let $S$ be a fixed-length circular society with $n$ voters whose approval sets have length $p \geq \frac{n-1}{n}$. Then there is a representative candidate set containing only one candidate. Thus, $\tau(S) = 1$.

Proof. Suppose $S = (X, A)$ is a fixed-length circular society with approval sets $A_1, A_2, \ldots, A_n$, and suppose the length of each $A_i$ is at least $\frac{n-1}{n}$. Then, the complement of each $A_i$, $X \setminus A_i$, has length less than $1/n$.

So $\bigcup_{i=1}^{n}(X \setminus A_i)$ has total length less than 1. Thus, $X \setminus \left(\bigcup_{i=1}^{n}(X \setminus A_i)\right)$ must contain some candidate $x$. In other words, $x \in \bigcap_{i=1}^{n}A_i$, so $\{x\}$ is a representative candidate set.

While the condition in the statement of Proposition 18 looks very strong, it is necessary. There are fixed-length circular societies with $n$ voters and approval set length $p < \frac{n-1}{n}$ for which the piercing number is greater than 1. We provide an example below.

Example 19. Let $S$ be a fixed-length circular society with 4 voters and approval set length $p < \frac{3}{4}$ as shown below. Since the agreement number is 3, any representative candidate set must contain at least 2 points.
Proposition 20. In order for a fixed-length circular society with \( n \) voters to have a minimum representative candidate set of size \( n \), the length of each approval set must be less than \( 1/n \).

Proof. Suppose for contradiction that \( S \) is a fixed-length circular society with approval sets of length \( p \geq 1/n \). Thus, the total length of all \( n \) approval sets in \( S \) is greater than or equal to 1. As these sets are closed, at least two must intersect, making the size of the minimum representative candidate set at most \( n - 1 \). \qed

The upper bound on the piercing number in Theorem 17 gives a “worst-case” scenario for fixed-length circular societies with approval set length \( p \geq 1/q \). In Proposition 21 we construct such a society that realizes this worst-case scenario. Hence, we construct a society in which the piercing number is \( q \). On the other hand, in Proposition 22 we show that the piercing number is strictly less than \( q \) when the society consists of a small number of voters.

Proposition 21. Let \( q \in \mathbb{N}, q \geq 2 \). There exists a fixed-length circular society \( S \) with \( 2q - 1 \) voters and approval set length \( p = 1/q \) that has a minimum representative candidate set containing \( q \) candidates, hence, \( \tau(S) = q \).

Proof. Consider the following fixed-length circular society with \( n = 2q - 1 \) voters and approval set length \( 1/q \). As per Remark 15, we identify the spectrum with \([0,1)\), with 0 and 1 identified. We construct the approval sets recursively as follows. Define \( A_1 = [\ell_1, r_1] = [0, 1/q] \). For \( 2 \leq i \leq 2q - 1 \), define \( A_i = [\ell_i, r_i] \) where \( \ell_i = r_{i-1} + \frac{1}{2q^2} \) and \( r_i = \ell_i + \frac{1}{q} \). This implies \( \ell_i = (i - 1) \left( \frac{1}{q} + \frac{1}{2q^2} \right) \), reduced modulo 1. In essence, we wrap approval sets of length \( 1/q \) around the circle so that adjacent approval sets \( A_{i-1} \) and \( A_i \) are separated by a distance of \( \frac{1}{2q^2} \). Thus, the total length of the approval sets combined with the gaps between adjacent approval sets is less than 2, as in Figure 5.

Consider \( W = \{\ell_1 = 0, \ell_2, \ldots, \ell_q\} \). We will show \( W \) is a minimum representative candidate set. The points in \( W \) are clearly in \( A_1, A_2, \ldots, A_q \), respectively. The remaining sets \( A_{q+1}, \ldots, A_{2q-1} \) can be written as \( A_{q+i} \) where \( 1 \leq i \leq q - 1 \). We will show each \( A_{q+i} \) contains \( \ell_{i+1} \in W \) for \( 1 \leq i \leq q - 1 \). First,

\[
\ell_{q+i} = (q + i - 1) \left( \frac{1}{q} + \frac{1}{2q^2} \right) = \ell_i + q \left( \frac{1}{q} + \frac{1}{2q^2} \right) = \ell_i + \frac{1}{2q}.
\]

This implies

\[
A_{q+i} = [\ell_{q+i}, r_{q+i}] = \left[ \ell_i + \frac{1}{2q}, \ell_i + \frac{1}{2q} + \frac{1}{q} \right].
\]
Since \( \ell_{i+1} = \ell_i + \frac{1}{q} + \frac{1}{2q^2} \), it follows that \( \ell_{q+i} < \ell_{i+1} < r_{q+i} \), so \( \ell_{i+1} \in A_{q+i} \). Thus, \( W \) is a representative candidate set.

Further, by construction, \( a(S) = 2 \). If \( W' \) is a set containing \( q - 1 \) or fewer points on the spectrum, then since \( a(S) = 2 \), at most \( 2(q-1) = 2q - 2 \) approval sets can contain a point in \( W' \). Thus, \( W \) is a minimum representative candidate set and \( \tau(S) = q \).

**Proposition 22.** Let \( q \in \mathbb{N}, q \geq 2 \). Any fixed-length circular society \( S \) with fewer than \( 2q - 1 \) voters and approval set length \( p = \frac{1}{q} \) has a minimum representative candidate set containing fewer than \( q \) candidates. Hence, \( \tau(S) < q \).

**Proof.** Let \( S = (X, A) \) be a fixed-length circular society with approval set length \( p = \frac{1}{q} \). If \( S \) has fewer than \( q \) voters, then \( \tau(S) < q \). So, suppose \( S \) has at least \( q \) but fewer than \( 2q - 1 \) voters: since all approval sets have length \( \frac{1}{q} \), at most \( q - 1 \) approval sets can be pairwise disjoint. If there are fewer than \( q - 1 \) pairwise disjoint approval sets, then \( S \) is \((2, q - 1)\)-agreeable. Therefore, by Theorem 6, \( \tau(S) \leq (q - 1) - 2 + 2 = q - 1 \).

Now suppose there exists a subset \( D = \{A_1, A_2, \ldots, A_{q-1}\} \) consisting of \( q - 1 \) pairwise disjoint approval sets in \( A \). Let \( B = \{B_1, B_2, B_3, \ldots\} \) be the collection of remaining approval sets in \( A \) so that \( A = D \cup B \). As \( S \) has fewer than \( 2q - 1 \) voters, \( B \) contains at most \( q - 1 \) approval sets. Since there are no more than \( q - 1 \) pairwise disjoint sets, each set in \( B \) must intersect at least one set in \( D \). Further, since all sets in \( D \) are disjoint and all approval sets have the same length, it follows that every set in \( B \) intersects at most two sets in \( D \). Hence, every set in \( B \) intersects either one or two sets in \( D \).

We now consider two cases: either there is a set in \( D \) that intersects fewer than two sets in \( B \), or every set in \( D \) intersects exactly two sets in \( B \). To see that these two cases are sufficient, suppose that some set in \( D \) intersects more than two sets in \( B \): since \( |B| \leq |D| \), and each set of \( B \) intersects at most two sets of \( D \), there must be another set in \( D \) that intersects fewer than two sets in \( D \).

First, suppose there is a set \( A_i \) in \( D \) that intersects fewer than two sets in \( B \). We remove \( A_i \) from \( A \), and if \( A_i \) intersects some \( B_j \) in \( B \) then we remove \( B_j \) from \( A \) as well, to create a linear-equivalent sub-society of \( S \). We next reduce the spectrum of the sub-society by deleting all points in \( X \) contained in \( A_i \). Hence, we cut out a segment of the spectrum whose length is that of \( A_i \), effectively creating a linear society \( S' = (X', A') \) in which the length of \( X' \) is \( 1 - \frac{1}{q} = \frac{q - 1}{q} \), and \( A' = A - \{A_i\} \) or \( A' = A' - \{A_i, B_j\} \) if we remove \( B_j \) as well. If \( A' \) contains less than \( q - 1 \) approval sets then \( \tau(S') \leq q - 2 \). If \( A' \) contains at least \( q - 1 \) approval sets, then since the spectrum length is \( \frac{q - 1}{q} \) and all approval sets have length \( \frac{1}{q} \), it follows that at most \( q - 2 \) approval sets can be pairwise disjoint. Therefore, \( S' \) is \((2, q - 1)\)-agreeable, and by Theorem 3, \( \tau(S') \leq (q - 1) - 2 + 2 = q - 2 \). Thus, all sets in \( A' \) can be pierced by \( q - 2 \) points in \( X' \). These points along with a point in \( A_i \) (that is also in \( B_j \) if we removed some \( B_j \) as well) form a representative candidate set of the original society \( S \). Therefore, \( \tau(S) \leq q - 1 \).

Next suppose that every approval set in \( D \) intersects exactly two sets in \( B \). This can only occur if \( B \) contains \( q - 1 \) approval sets. We create a representative candidate set of \( S \) containing \( q - 1 \) points by examining a subgraph of the agreement graph of \( S \). Recall from [3] that the agreement graph of a society contains a vertex for each approval set and an edge between vertices if the corresponding approval sets intersect. Consider the subgraph \( G \) of the agreement graph of \( S \) that consists of all vertices but only edges with one vertex corresponding to a set in \( D \) and the other corresponding to a set in \( B \). Label the vertices by their corresponding approval sets. Hence, \( G \) is a bipartite graph whose two parts correspond to \( D \) and \( B \). Since each approval set in \( D \) intersects exactly two sets in \( B \), the degree of each \( A_i \) in \( D \) is two. So, \( \sum_{A_i \in D} \deg(A_i) = 2(q - 1) \), which implies that \( \sum_{B_i \in B} \deg(B_i) = 2(q - 1) \) as well. Thus, since all sets in \( B \) intersect at most two sets in \( D \) and there are \( q - 1 \) sets in \( B \), it follows that \( \deg(B_i) = 2 \) for all \( B_i \). Hence, since all vertices in \( G \) have degree 2, \( G \) is a 2-regular bipartite graph. By Hall’s Marriage Theorem, \( G \) has a perfect matching \( M \).

We create a representative candidate set of the circular society \( S \) by selecting a point in each intersection represented by an edge in \( M \). Because \( M \) is a perfect matching and both \( D \) and \( B \) contain \( q - 1 \) sets, this representative candidate set contains \( q - 1 \) points. Therefore, \( \tau(S) \leq q - 1 \).

While the term fixed-length society has not previously appeared in approval voting literature, the authors of this paper are not the first to investigate such societies. In [7], Hardin uses a specific type of fixed-length circular society to develop the upper bound on agreement stated in Theorem 2. He calls these societies uniform societies. We construct a minimum representative candidate set for any uniform society, and thus
compute its piercing number. Below we rephrase Hardin’s definition of uniform society using the conventions that the spectrum is identified with \( \mathbb{R}/\mathbb{Z} \) and the length of the approval sets is strictly less than 1. See Figure 6 for an example.

**Definition 23** (Definition 2.1 in [7]). For integers 1 \( \leq h < n \), we define the uniform circular society \( U(n,h) \) to be a fixed-length circular society with \( n \) voters and approval set length \( \frac{h}{n} \) in which \( A_i = \left[ \frac{i-1}{n}, \frac{i}{n}+\frac{h}{n} \right) \) for \( 1 \leq i \leq n \).

![Figure 6: The uniform society \( U(7,3) \). The red points on the spectrum form the minimum representative candidate set constructed in Theorem 25.](image)

**Remark 24.** The approval sets in Definition 23 are half-open intervals in order to be consistent with Hardin’s original definition of uniform societies. A similar society could be defined using closed approval sets that are a bit shorter. More precisely, we could consider the approval sets to be \( \left\lfloor \frac{i-1}{n} \right\rfloor \leq x < \left\lfloor \frac{i}{n}+\frac{h}{n} \right\rfloor \) for an example.

**Theorem 25.** For integers 1 \( \leq h \leq n \) let \( U(n,h) \) be a uniform circular society. Then the set \( W = \left\{ 0, \frac{h}{n}, \frac{2h}{n}, \ldots, \frac{(\left\lfloor \frac{n}{h} \right\rfloor - 1)h}{n} \right\} \) is a minimum representative candidate set of \( U(n,h) \). Hence, the piercing number of \( U(n,h) \) is \( \left\lfloor \frac{n}{h} \right\rfloor \).

**Proof.** First we show that \( W \) is a representative candidate set; then we show that any representative candidate must contain at least \( \left\lfloor \frac{n}{h} \right\rfloor \) points. In essence, \( W \) is a representative candidate set because it contains enough points that are spaced out along the spectrum as evenly as possible. We make these ideas more precise below.

Observe that \( W \) consists of \( \left\lfloor \frac{n}{h} \right\rfloor \) points in \([0,1)\), and the distance between neighboring points in \( W \) is \( \frac{h}{n} \). Since \([0,1) = \mathbb{R}/\mathbb{Z}\), the distance between the points 0 and \( \frac{(\left\lfloor \frac{n}{h} \right\rfloor - 1)h}{n} \) on the circular spectrum may be less than \( \frac{h}{n} \), as in Figure 6. Since each approval set is a half-open interval of length \( \frac{h}{n} \), each approval set must contain a point in \( W \). Thus, \( W \) is a representative candidate set.

To show that \( W \) is a minimum representative candidate set, suppose to the contrary that there is a smaller representative candidate set, call it \( W' \), where \( |W'| < \left\lfloor \frac{n}{h} \right\rfloor \). In [7], Hardin states that every point on the spectrum is contained in exactly \( h \) approval sets in a uniform society \( U(n,h) \). So, the points in \( W' \) can pierce at most \( |W'|h \) approval sets. We will show that \( |W'|h < n \), which implies that not all \( n \) sets are pierced by \( W' \) and contradicts our assumption that \( W' \) is a representative candidate set.

If \( h \) divides \( n \), then \( |W'|h < \left\lfloor \frac{n}{h} \right\rfloor h = n \). If \( h \) does not divide \( n \), then \( n = hm + r \) for some integer \( m \) and integer \( 0 < r \leq n - 1 \). So, \( \left\lfloor \frac{n}{h} \right\rfloor = m + 1 \). Let \( |W'| = \left\lfloor \frac{n}{h} \right\rfloor - i \) for some integer \( i \geq 1 \). Thus,

\[
|W'|h = \left( \frac{n}{h} \right) - i)h = (m + 1 - i)h = mh + h(1 - i) \leq mh < n.
\]

\[\square\]

5 A Functional Approach to Agreement in Circular Societies

We now introduce a new perspective on approval voting that allows us to explore local and global agreement of fixed-length circular societies. Our key insight for this is that we define a local counting function on
the spectrum $X$ that we can integrate to derive agreeability results. This draws on ideas from applied topology which, to our knowledge, have not appeared before in the approval voting literature. Many of the results in this section apply to general circular societies, without the requirement that approval sets have fixed length, though Theorems 34 and 35 require the fixed-length assumption. Furthermore, our functional perspective can be extended to other types of societies (e.g., linear or product societies) with slightly modified assumptions—we invite the reader to explore!

**Definition 26.** Let $S = (X, A)$ be a society. The local counting function for $S$ is $C(x): X \rightarrow \{0, 1, 2, 3, \ldots \}$, where $C(x)$ is the number of approval sets in $A$ that contain the point $x$ in $X$.

Figure 7 shows an example of a circular society and its local counting function. Recall from Section 2 that for a circular society $S$, the agreement number $a(S)$ is the maximum number of approval sets that have a common intersection. Thus, the agreement number of a circular society $S$ is the maximum value of the local counting function: $a(S) = \max_X C(x)$.

![Figure 7](image)

**Figure 7:** Illustration of the local counting function on a circular society. Four approval sets are shown below the spectrum. A plot of the local counting function $C(x)$ appears above the spectrum in purple.

We rely on two types of integrals that allow us to extract information from the local counting function. The familiar Riemann integral connects $C(x)$ to the length of approval sets in a circular society, while the lesser-known Euler integral provides results about the number of approval sets. Abstractly, these two integrals offer two different ways of quantifying the size of the collection of approval sets. We can consider size as the total length of the approval sets, or as the number of approval sets (that is, the number of voters in the society).

To state our integral results, we first define $1_I$ to be the indicator function on set $I$; that is, the function that takes value 1 on $I$ and 0 elsewhere. We let $\text{len}(A)$ denote the length of approval set $A$. We now begin with the Riemann integral.

**Proposition 27.** The sum of the lengths of the approval sets in a circular society $S$ is $\int_X C(x) \, dx$.

**Proof.** The local counting function is a sum of indicator functions of the approval sets: $C(x) = \sum_{i=1}^n 1_{A_i}$. Thus,

$$\int_X C(x) \, dx = \int_X \sum_{i=1}^n 1_{A_i} \, dx = \sum_{i=1}^n \int_X 1_i \, dx = \sum_{i=1}^n \text{len}(A_i).$$

Turning to the Euler integral, we first introduce the Euler characteristic $\chi$, which is an integer-valued function on geometric sets that can be viewed as a topological counting measure.\(^2\) The Euler characteristic is a homeomorphism invariant, meaning that $\chi(X) = \chi(Y)$ if there is a continuous function $f: X \rightarrow Y$ with a continuous inverse. Informally, $\chi(X) = \chi(Y)$ if $X$ can be transformed into $Y$ by continuous stretching and bending. This allows us to consider the Euler characteristic on both intervals in a line and arcs in a circle.

\(^2\)The Euler characteristic appears in many areas of geometry and topology; the interested reader should consult Richeson [11]. For more about the Euler characteristic as a topological counting measure, with applications, see Baryshnikov and Ghrist [1].
As in Remark 15, we may refer to approval sets in a circular spectrum as “intervals,” and thus “interval” in the following definition is interchangeable with “arc.”

**Definition 28.** The *Euler characteristic* $\chi$ of an interval is determined solely by whether or not the interval includes its endpoints. Let $I_o$ be an open interval, $I_h$ a half-open interval, and $I_c$ a closed interval. Then define

$$\chi(I_o) = -1, \quad \chi(I_h) = 0, \quad \text{and} \quad \chi(I_c) = 1.$$  

The Euler characteristic extends to finite unions and intersections of intervals via the additivity property,

$$\chi(A \cup B) = \chi(A) + \chi(B) + \chi(A \cap B),$$

with $\chi(\emptyset) = 0$.

The Euler integral appears in applied topology and has a simple definition for integer-valued functions whose level sets are finite unions of intervals [1]. This integral uses Euler characteristic as a “measure” (more properly, a *valuation*), and is denoted with $d\chi$.

**Definition 29.** For an interval $I$, the *Euler integral* of $1_I$ is the Euler characteristic of $I$:

$$\int_X 1_I \, d\chi = \chi(I).$$

For a function $f$ that is a finite sum of indicator functions $f = \sum_{i=1}^n 1_{I_i}$, the Euler integral is

$$\int_X f \, d\chi = \sum_{i=1}^n \chi(I_i).$$

As promised, the Euler integral connects the local counting function to the number of approval sets in a circular society. The following proposition makes this precise.

**Proposition 30.** Let $S$ be a circular society with closed approval sets. Then the number of approval sets in $S$ is given by the Euler integral of the local counting function $C(x)$:

$$\int_X C \, d\chi = n.$$

*Proof.~* $C(x)$ is a finite sum of indicator functions on closed intervals $A_1, \ldots, A_n$, each of which has Euler characteristic 1. Thus,

$$\int_X C \, d\chi = \sum_{i=1}^n \chi(A_i) = \sum_{i=1}^n 1 = n. \quad \Box$$

The number of approval sets of a circular society $S$ can be expressed in terms of the local minimum and maximum values of $C(x)$. These local minimum and maximum values are analogous to local extrema in calculus, adapted to our integer-valued step function $C(x){}^3$. A local maximum (resp. local minimum) interval of $C$ is an interval in $X$ on which $C$ attains a constant value that is greater than (resp. less than) the value of $C$ on neighboring intervals. We define these intervals formally below.

**Definition 31.** Let $C : X \to \{0, 1, 2, 3, \ldots\}$ be a local counting function. An interval $I = [x_1, x_2] \subset X$ is a *local maximum interval* of $C$ if $C(x) = m$ for $x \in I$ and for some $\epsilon > 0$, $C(x') < m$ whenever $x_1 - \epsilon < x' < x_1$ or $x_2 < x' < x_2 + \epsilon$. Similarly, an interval $I = (x_1, x_2) \subset X$ is a *local minimum interval* of $C$ if $C(x) = m$ for $x \in I$ and for some $\epsilon > 0$, $C(x') > m$ whenever $x_1 - \epsilon < x' < x_1$ and for $x_2 < x' < x_2 + \epsilon$.

Since approval sets are closed, then $C$ is a finite sum of indicator functions on closed intervals, all local maximum intervals are closed, and all local minimum intervals are open. (C is an upper-semicontinuous function.) The local counting function depicted in Figure 8 has two local maximum intervals and two local minimum intervals.

Let $l_{\text{max}}(C)$ and $l_{\text{min}}(C)$ denote the sets of local maximum and local minimum intervals, respectively. The following theorem expresses the number of approval sets in a circular society in terms of these intervals.

---

3 A reader familiar with Morse theory may be interested to explore connections between the Euler integral and critical points of continuous functions [2].
Theorem 32. Let $S$ be a circular society with closed approval sets. Then the number of approval sets in $S$ is

$$n = \sum_{I \in \text{Imax}(C)} C(I) - \sum_{I \in \text{Imin}(C)} C(I).$$

Proof. The Euler integral $\int_X C \, d\chi$ can be expressed in terms of level sets $C^{-1}(r)$ for $r$ in the image of $C$. To see this, first write $C$ as a sum of indicator functions on level sets $C^{-1}(r)$ for all $r$ in the image of $C$:

$$C = \sum_{r \in \text{Im}(C)} r \cdot 1_{C^{-1}(r)}.$$

Taking the Euler integral of both sides, we obtain

$$\int_X C \, d\chi = \sum_{r \in \text{Im}(C)} r \cdot \int_X 1_{C^{-1}(r)} \, d\chi = \sum_{r \in \text{Im}(C)} r \cdot \chi(C^{-1}(r)).$$

If $r \in \text{Im}(C)$, then $C^{-1}(r)$ has a finite number of connected components, each of which is an interval. That is, $C^{-1}(r)$ is a disjoint union $C^{-1}(r) = \bigsqcup I_{r,j}$, where each $I_{r,j}$ is an interval. Since Euler characteristic is additive,

$$\chi(C^{-1}(r)) = \sum_j \chi(I_{r,j}).$$

If $I_{r,j}$ is a local maximum interval, then $I_{r,j}$ is closed and $\chi(I_{r,j}) = 1$. If $I_{r,j}$ is a local minimum interval, then $I_{r,j}$ is open and $\chi(I_{r,j}) = -1$. If $I_{r,j}$ is neither a local maximum or local minimum interval, then $I_{r,j}$ is half-open and $\chi(I_{r,j}) = 0$. Thus, only local maximum and local minimum intervals contribute to the sum.

We can rearrange the sum to express the Euler integral as

$$\int_X C \, d\chi = \sum_{r \in \text{Im}(C)} r \cdot \sum_j \chi(I_{r,j}) = \sum_{I \in \text{Imax}(C)} C(I) - \sum_{I \in \text{Imin}(C)} C(I).$$

Since $n = \int_X C \, d\chi$, the proof is complete.

We now turn to our main results of this section, which use the local counting function to provide bounds on the agreement number of a fixed-length circular society. We require that the local counting function is not constant, which the following lemma shows is true as long as the society has at least one voter.

Lemma 33. If $S$ is a circular society with $n \geq 1$ closed approval sets, none of which is the entire circle $X$, then $C(x)$ is not constant.

Proof. The lemma is clearly true when $n = 1$.

Assume the lemma is true for $n > 1$ approval sets. Let $r = \max_X C(x)$. Since approval sets are closed, and there are finitely many approval sets, for any $r \in \mathbb{R}$, the superlevel set $\{C \geq r\} := \{x \in X \mid C(x) \geq r\}$ is closed. Its complement, the sublevel set $\{C < r\} := \{x \in X \mid C(x) < r\}$, is open. Since $C < r$ on an open set, the addition of one more (closed) approval set cannot make $C$ be constant, so the lemma is true for $n + 1$ approval sets. Thus, the lemma holds by induction on the number of approval sets.
We are now ready to prove the following lower bound on the agreement number of a fixed-length circular society.

**Theorem 34.** Let $S$ be a fixed-length circular society with $n$ voters and approval sets of length $p$. Then the agreement number $a(S) \geq \lceil np \rceil + 1$.

**Proof.** Let $S$ be a fixed-length circular society with $n$ voters and approval set length $p$. We will show $a(S) \geq \lceil np \rceil + 1$ by showing that there is a point $x$ in the spectrum $X$ for which $C(x) \geq \lceil np \rceil + 1$.

First, since $S$ is a circular society with $n$ voters and all approval sets of length $p$, Proposition 27 gives

$$\int_X C(x) \, dx = np.$$  

Now, for the purpose of contradiction, suppose $C(x) \leq \lceil np \rceil$ for all $x \in X$. If $np \notin \mathbb{N}$, this means that $C(x) < np$ for all $x \in X$. Thus, $\int_X C(x) \, dx < np$. On the other hand, if $np \in \mathbb{N}$, as $C$ is not constant by Lemma 33, there must exist an interval in $X$ on which $C(x) < np$. Thus, $\int_X C(x) \, dx < np$. In either case we obtain a contradiction.

Therefore, there must be some point $x$ in the spectrum $X$ such that $C(x) > \lceil np \rceil$, and so $a(S) \geq \lceil np \rceil + 1$. $\square$

Finally, we establish guaranteed $(k, m)$-agreeability in a fixed-length circular society $S = (X, A)$. For this, let $C_B(x)$ be the local counting function of some subcollection $B \subseteq A$ of approval sets: $C_B = \sum_{i \in B} 1_I$.

**Theorem 35.** Let $k \in \mathbb{N}$, $k \geq 2$. If $S$ is a fixed-length circular society with approval sets of length $p$, then $S$ is $(k, \lceil \frac{k-1}{p} \rceil)$-agreeable, but $S$ is not necessarily $(k, \lceil \frac{k-1}{p} \rceil - 1)$-agreeable.

**Proof.** We first prove that $S$ is $(k, \lceil \frac{k-1}{p} \rceil)$-agreeable by contradiction. Suppose the society is not $(k, \lceil \frac{k-1}{p} \rceil)$-agreeable. Then there is a collection $B \subseteq A$ containing $\lceil \frac{k-1}{p} \rceil$ approval sets, no $k$ of which have a common intersection. This means that every point on the circle is contained in at most $k-1$ of the sets in $B$. Thus, the local counting function $C_B(x) \leq k-1$ for all $x \in X$. Lemma 33 implies that $C_B(x)$ is not constant, and in fact $C_B(x) < k-1$ on some interval of positive length. This implies that

$$\int_X C_B(x) \, dx < k-1.$$  

However, the sum of the lengths of the sets in $B$ is

$$\lceil \frac{k-1}{p} \rceil \geq k-1.$$  

This contradicts Proposition 27, that the sum of the lengths of the approval sets in $B$ is $\int_X C_B(x) \, dx$. Thus, the society is $(k, \lceil \frac{k-1}{p} \rceil)$-agreeable.

Next, we show that $S$ is not necessarily $(k, \lceil \frac{k-1}{p} \rceil - 1)$-agreeable. Let $m = \lceil \frac{k-1}{p} \rceil - 1$. Since $m = \lceil \frac{k-1}{p} \rceil - 1 < \frac{k-1}{p}$, there exists $\epsilon > 0$ such that $m + \frac{\epsilon}{p} < \frac{k-1}{p}$, or equivalently $mp + \epsilon < k-1$. Consider $m$ approval sets of length $p$, arranged in sequence along a line, each approval set separated from its neighbors by a distance of $\frac{\epsilon}{m}$. The total length of this arrangement is

$$mp + (m - 1) \frac{\epsilon}{m} < mp + \epsilon < k-1.$$  

Thus, if we wrap this linear arrangement around a circle of circumference 1, each point of the circle will be in at most $k-1$ of the approval sets, completing the proof. $\square$

We leave it to the reader to explore the connection between the local counting function and the piercing number of a circular society. On first glance, it may seem that a minimum representative candidate set must include a point at which the local counting function attains its maximum value, but this is not true—see the society in Example 13.

**Question 36.** Given the local counting function of a circular society, can the piercing number of the society be determined? Furthermore, can a minimum representative candidate set be found?
6 Probability Distribution of Piercing Numbers of Fixed-Length Circular Societies

In Section 4, we focused on circular societies where the approval sets have equal lengths and explored best-case and worst-case scenarios in this setting. For instance, given an approval set length \( p = \frac{1}{q}, \) \( q \in \mathbb{N}, \) we can construct a circular society of \( n \geq 2q - 1 \) voters in which the minimum representative candidate set contains \( q \) points, the largest possible given \( p = \frac{1}{q} \) (Proposition 21). However, we might believe that most arbitrarily-constructed fixed-length circular societies with \( p = \frac{1}{q} \) would have a minimum representative candidate set containing fewer than \( q \) points. More generally, we might expect that most fixed-length circular societies have piercing numbers that are between the best- and worst-case scenarios. However, existing literature on piercing numbers has focused mostly on finding upper bounds ("worst cases") to the piercing number of a particular family of sets. The overall distribution of piercing numbers has not been thoroughly explored.

We propose a direction for understanding the overall distribution of piercing numbers and a notion of an "average-case scenario" for piercing numbers. We use a probabilistic model to explore the space of all fixed-length circular societies, by considering fixed-length circular societies whose approval sets are randomly-generated with a uniform distribution on the spectrum \( X. \) Recall from Remark 15 that we identify \( X \) with \( \mathbb{R}/\mathbb{Z}. \)

**Definition 37.** A random fixed-length circular society with \( n \) voters and approval set length \( p \) is a circular society \( S = (X,A) \) in which \( A \) consists of \( n \) approval sets of the form \( A_i = [\ell_i, r_i] \) for \( i = 1,\ldots,n \) where

- \( \ell_1, \ldots, \ell_n \) are independent random variables, uniformly distributed on the spectrum \( X, \) and
- \( r_i = \ell_i + p. \)

Our ultimate goal is to provide a complete probability distribution for the piercing number as a function of the number of voters, \( n, \) and the approval set length, \( p. \) In other words, we hope to answer the question below.

**Question 38.** Given a random fixed-length circular society \( S \) with \( n \) voters and approval sets of length \( p \in (0,1), \) and a positive integer \( k, \) what is the probability that \( \tau(S) = k? \)

While this question largely remains open, we determine the probability that the piercing number is \( k \) when the approval set length is sufficiently small. Specifically, we state the following theorem, which is the main result of this section.

**Theorem 39.** For a random fixed-length circular society \( S \) with \( n \) voters and approval set length \( p < \frac{1}{2k}, \) the probability that \( \tau(S) = k \) is

\[
\binom{n}{k} (1 - kp)^{k-1} (kp)^{n-k}. \tag{1}
\]

In Section 6.1, we prove Theorem 39 and answer Question 38 in the special case where \( n = 3 \) and \( k = 1. \) In Section 6.2, we discuss simulation results, including those that suggest that Theorem 39 might apply for a greater range of values of \( p. \) Throughout, we use the following observations.

**Remark 40.** Let \( S = (X,A) \) be a random fixed-length circular society with \( n \) voters, approval set length \( p, \) and approval sets \( A_i = [\ell_i, r_i] \) as defined in Definition 37. Since \( \ell_i \) is a random variable that is uniform on \( \mathbb{R}/\mathbb{Z} = [0,1), \) its probability density function is \( f_{\ell_i}(x) = 1_{[0,1)}(x). \) This means that

1. given any interval \( I \subseteq \mathbb{R}/\mathbb{Z}, \) the probability that \( \ell_i \in I \) is equal to the length of \( I, \) and
2. the approval sets \( A_1,\ldots,A_n \) are distinct with probability 1.

6.1 Proofs of Piercing Probabilities

We now prove our probabilistic results for the piercing numbers of random fixed-length circular societies. Specifically, Theorem 39 gives the probability that \( \tau(S) = k \) when the length of approval sets satisfies \( p < \frac{1}{2k}. \) The proof of Theorem 39 relies on a lemma that we prove first. Following the proof of Theorem 39, we give the probability that \( \tau(S) = 1 \) for a society of \( n = 3 \) voters with \( \frac{1}{2} \leq p < \frac{2}{3}. \)
Lemma 41. Let $S = (X, A)$ be a fixed-length circular society with $n$ distinct approval sets of length $p < \frac{1}{2k}$. Then $S$ has a minimum representative candidate set containing $k$ candidates (i.e., $\tau(S) = k$) if and only if there exists a unique collection of $k$ pairwise-disjoint approval sets in $A$ whose union contains the left endpoints of all other approval sets.

Proof. Consider a fixed-length circular society $S = (X, A)$ with distinct approval sets $A_1, \ldots, A_n$ of length $p < \frac{1}{2k}$. Denote $A_i = [\ell_i, r_i]$, where $r_i = \ell_i + p$.

First, suppose there exists a unique collection of $k$ pairwise-disjoint approval sets $A_1, \ldots, A_k$ whose union contains the left endpoints of all other approval sets. This means that each of the remaining $n - k$ approval sets contains at least one of the right endpoints of $A_1, \ldots, A_k$. Thus, the set of the right endpoints of $A_1, \ldots, A_k$ is a representative candidate set of size $k$. Since $A_1, \ldots, A_k$ are pairwise disjoint, a minimum representative candidate set of $S$ must contain at least $k$ points. Therefore, $\tau(S) = k$.

Conversely, suppose $\tau(S) = k$. Then, there is a minimum representative candidate set $W = \{x_1, \ldots, x_k\}$ of $S$. For each $i = 1, \ldots, k$, let $S_i$ denote the collection of all sets pierced by $x_i$. Let $L_i$ denote the leftmost left endpoint among sets in $S_i$ and let $R_i$ denote the rightmost right endpoint among sets in $S_i$. Since each set in $S_i$ is pierced by $x_i$, then $L_i$ and $R_i$ are each within $p$ from $x_i$. Therefore, $R_i - L_i \leq 2p < \frac{1}{k}$.

So, the union of all approval sets in each $S_i$ is a closed interval of length strictly less than $\frac{1}{k}$. This means that the union of all sets in $S$ is a union of closed intervals whose total length is strictly less than $\frac{1}{k}$, which implies that not all points on the circle are covered by the approval sets of $S$. Therefore, $S$ is a linear-equivalent circular society. Choosing a point $x$ not in any approval set and following the cut-and-unroll procedure illustrated in Figure 2, we treat $S$ as a linear society in the remainder of the proof.

Applying Algorithm 7—the piercing number algorithm for linear societies—to $S$ produces a minimum representative candidate set, call it $W'$, which might be different than $W$ but must also contain $k$ piercing points. Note that $W'$ consists of $k$ right endpoints, call them $r_1, \ldots, r_k$, the right endpoints of approval sets $A_1 = [\ell_1, r_1], \ldots, A_k = [\ell_k, r_k]$. Since $S$ is a fixed-length society with distinct approval sets, no approval set is contained in another. Therefore, by Proposition 10, $\{A_1, \ldots, A_k\}$ is the unique subcollection of $k$ pairwise-disjoint approval sets in $A$ whose union contains the left endpoints of the remaining $n - k$ approval sets.

The choice of the point $x$ for the cut-and-unroll procedure does not affect this result: choosing a different point $x$ may only have the effect of rearranging non-intersecting sub-societies of $S$, but does not affect the results of Algorithm 7 or Proposition 10.

We are now ready to prove Theorem 39.

Proof of Theorem 39. Let $S = (X, A)$ be a random fixed-length circular society with approval sets $A_1, \ldots, A_n$ of length $p < \frac{1}{2k}$. Denote $A_i = [\ell_i, r_i] = [\ell_i, \ell_i + p]$. We can assume that these approval sets are distinct, which occurs with probability 1 (Remark 40).

Let $E$ denote the event that $\tau(S) = k$. By Lemma 41, $E$ is the event that there exists a unique collection of $k$ pairwise-disjoint approval sets whose union contains the left endpoints of the remaining $n - k$ sets. Our goal is to compute $P(E)$, by expressing $E$ as a union of disjoint events.

Suppose $T$ is a subset of $\{1, \ldots, n\}$ of cardinality $k$. Let $D_T$ denote the event that the approval sets $\{A_i\}_{i \in T}$ are pairwise disjoint. Let $C_T$ be the event that $\bigcup_{i \in T} A_i$ contains the left endpoints of the remaining $n - k$ approval sets. Then, $C_T \cap D_T$ denotes the event that $\{A_i\}_{i \in T}$ is pairwise disjoint and the union $\bigcup_{i \in T} A_i$ contains left endpoints of the remaining sets. The uniqueness of the collection of $k$ pairwise disjoint sets specified in Lemma 41 implies that the events $\{C_T \cap D_T\}_{T \subseteq \{1, \ldots, n\}, |T| = k}$ are pairwise disjoint.

Thus, $E$ is the union of the disjoint events $\{C_T \cap D_T\}_{T \subseteq \{1, \ldots, n\}, |T| = k}$, which implies that

$$P(E) = P \left( \bigcup_{T \subseteq \{1, \ldots, n\}, |T| = k} (C_T \cap D_T) \right) = \sum_{T \subseteq \{1, \ldots, n\}, |T| = k} P(C_T \cap D_T).$$

It remains to compute $P(C_T \cap D_T) = P(D_T)P(C_T | D_T)$. Solomon [12, Equation (4.5)] shows that the probability that $k$ arcs of length $p$, randomly placed on the circle, are pairwise disjoint is $(1 - kp)^{k-1}$. Thus, $P(D_T) = (1 - kp)^{k-1}$. Next, we compute $P(C_T | D_T)$. Given that $k$ approval sets $\{A_i\}_{i \in T}$ are disjoint, the
length of $\bigcup_{i \in T} A_i$ is $kp$. Hence, the probability that the left endpoints of the remaining $n - k$ approval sets are in $\bigcup_{i \in T} A_i$ is $(kp)^{n-k}$. Therefore, $P(C_T \cap D_T) = P(D_T)P(C_T|D_T) = (1 - kp)^{k-1}(kp)^{n-k}$.

Since the number of possible combinations of $k$ disjoint approval sets is $\binom{n}{k}$, the probability that $\tau(S) = k$ is

\[ P(E) = \sum_{T \subseteq \{1, \ldots, n\}, |T| = k} P(C_T \cap D_T) = \binom{n}{k} (1 - kp)^{k-1}(kp)^{n-k}. \]

The proof of Lemma 41 relies on the fact that a fixed-length circular society $S$ with $p < \frac{1}{2k}$ and $\tau(S) = k$ is linear-equivalent. If such a society is not linear-equivalent, then it might not have a unique collection of $k$ disjoint approval sets that contain the left endpoints of all other approval sets (e.g., see Figure 1). This complicates the task of finding the probability distribution of $\tau(S)$.

If we restrict our attention to the probability that $\tau(S) = 1$, we are able to provide a more complete answer to Question 38. In this case, Theorem 39 says that if $p < \frac{1}{2}$, then $P(\tau(S) = 1) = np^{n-1}$. Proposition 18 states that $\tau(S) = 1$ for any fixed-length circular society $S$ with approval set length $p \geq \frac{n-1}{n}$. It remains to find $P(\tau(S) = 1)$ when the approval set length is $\frac{1}{2} \leq p < \frac{n-1}{n}$. The following proposition gives this probability when the number of voters is $n = 3$. Hence, we have a complete picture of the probability $P(\tau(S) = 1)$ for fixed-length circular societies with $n = 3$ voters. Figure 10 shows this probability as a function of the approval set length $p$.

**Proposition 42.** For a random fixed-length circular society $S$ with $n = 3$ voters and approval set length $\frac{1}{2} \leq p < \frac{2}{3}$, $P(\tau(S) = 1) = -9p^2 + 12p - 3$.

![Figure 9: The five cases in the proof of Proposition 42.](image)

Case 1  Case 2  Case 3  Case 4  Case 5

**Proof.** Without loss of generality, let $A_1 = [0, p]$. We consider five disjoint cases for the position of $A_2 = [\ell_2, \ell_2 + p]$ as illustrated in Figure 9.

Case 1: Let $E_1$ be the event that $0 < \ell_2 < 2p - 1$. In this case, $A_1 \cap A_2$ is an interval of length greater than $1 - p$, so $A_3$ must intersect $A_1 \cap A_2$. Thus, $P((\tau = 1) \cap E_1) = P(E_1) = 2p - 1$.

Case 2: Let $E_2$ be the event that $2p - 1 < \ell_2 < 1 - p$. In this case, $A_1 \cap A_2$ is an interval of length $p - \ell_2$, which is less than $1 - p$. The probability that $E_2$ occurs and $A_3$ intersects $A_1 \cap A_2$ is

\[ P((\tau = 1) \cap E_2) = \int_{2p-1}^{1-p} (2p - \ell_2) \, d\ell_2 = \frac{3}{2} (2p - 3p^2). \]

Case 3: Let $E_3$ be the event that $1 - p < \ell_2 < p$. In this case, $A_1 \cap A_2$ is a union of two disconnected intervals: $A_1 \cap A_2 = [0, \ell_2 + p - 1] \cup [\ell_2, p]$. Since one of these intervals contains 0 and the other contains $\frac{1}{2}$, $A_3$ must intersect $A_1 \cap A_2$. Thus, $P((\tau = 1) \cap E_3) = P(E_3) = 2p - 1$. 

\[ 17 \]
Case 4: Let $E_4$ be the event that $p < \ell_2 < 2 - 2p$. In this case, $A_1 \cap A_2$ is an interval of length $\ell_2 + p - 1$, which is less than $1 - p$. The probability that $E_4$ occurs and $A_3$ intersects $A_1 \cap A_2$ is
\[
P((\tau = 1) \cap E_4) = \int_p^{\ell_2 - 2} (2p - 1 + \ell_2) \, d\ell_2 = \frac{3}{2} (2p - 3p^2).
\]

Case 5: Let $E_5$ be the event that $2 - 2p < x < 1$. In this case, $A_1 \cap A_2$ is an interval of length greater than $1 - p$. As in Case 1, $(A_1 \cap A_2) \cap A_3 \neq \emptyset$, so $P((\tau = 1) \cap E_5) = P(E_5) = 2p - 1$.

By the law of total probability,
\[
P(\tau = 1) = \sum_{i=1}^{5} P((\tau = 1) \cap E_i) = 3(2p - 1) + 2 \cdot \frac{3}{2} (2p - 3p^2) = -9p^2 + 12p - 3.
\]

Figure 10: For fixed-length circular societies with $n = 3$ voters, this plot shows $P(\tau(S) = 1)$ as a function of the approval set length $p$. The purple curve for $0 < p < \frac{1}{2}$ is given by Theorem 39, the orange curve for $\frac{1}{2} \leq p < 2/3$ is given by Proposition 42, and the blue line for $2/3 \leq p < 1$ by Proposition 18.

It seems possible to extend Proposition 42 to $n > 3$, but the number of cases grows quickly with $n$, since we must consider many different possible intersections of approval sets. We state the following question, which is a special case of Question 38.

**Question 43.** For a random fixed-length circular society $S$ with $n > 3$ voters and $\frac{1}{2} \leq p < \frac{n-1}{n}$, what is $P(\tau(S) = 1)$?

### 6.2 Simulating Societies

Since Theorem 39 only applies for fixed-length circular societies with small approval set length $p$, we now use simulation to explore the probability of various piercing numbers and compute average piercing numbers of random fixed-length circular societies with larger $p$. For a particular number of approval sets $n$ of any length $p$, we can estimate the probability that the piercing number is $k$ by randomly generating $N$ fixed-length circular societies and computing the piercing number of each. The proportion of the $N$ circular societies whose piercing number is equal to $k$ is our estimate of $P(\tau(S) = k)$.

Theorem 39 states that the probability $P(\tau(S) = k)$ for a random fixed-length circular society with approval set length $p < \frac{1}{2}$ is given by Equation (1). Our simulations suggest that Equation (1) may give the correct probability even when $p$ is somewhat larger than $\frac{1}{2}$, at least for certain values of $k$. Figure 11 compares the simulated probability with the value given by Equation (1) for random fixed-length circular
societies with $n = 4$ voters, each $k \in \{1, 2, 3, 4\}$, and a range of approval set lengths $p \in (0, 0.8)$. We see that for each $\tau = k$ the simulated probability agrees with Equation (1) for $p < \frac{1}{2k}$, as it should. However, for $\tau > 2$, these values agree for $p$ beyond the bound stated in Theorem 39. For $\tau = 3$, the values seem to match well up to $p = \frac{1}{3}$, and for $\tau = 4$, they seem to match well up to $p = \frac{1}{4}$. This leads us to the following conjecture.

**Conjecture 44.** For a random fixed-length circular society $S$,

$$P(\tau(S) = k) = \binom{n}{k} (1 - kp)^{k-1}(kp)^{n-k}$$

for $p$ in an interval $(0, \frac{1}{2k} + \delta)$ for some $\delta > 0$.

**Figure 11:** Comparison of simulated probabilities $P(\tau(S) = k)$ with those given by Theorem 39 for each $k = 1, \ldots, 4$ in fixed-length circular societies with $n = 4$ voters and approval set lengths $0 \leq p \leq 0.8$. Simulated probabilities are plotted in color; polynomial curves given by Equation (1) are plotted in gray. The simulated probabilities suggest that Equation (1) gives the correct probability for $p$ somewhat larger than $\frac{1}{2k}$. Specifically, as highlighted by the dashed vertical lines, when $\tau = 3$ the formula seems to hold for $p \leq \frac{1}{3}$; and when $\tau = 4$ it seems to hold for $p \leq \frac{1}{4}$.

Furthermore, Table 1 displays simulated probabilities $P(\tau(S) = k)$ for certain values of $p$ greater than $\frac{1}{2k}$. For each combination of $n$, $k$, and $p$ listed in the table, we randomly generated $N = 100,000$ fixed-length circular societies and computed the piercing number for each. The table shows that the simulated probabilities $P(\tau(S) = k)$ are close to the values given by Equation (1) and Conjecture 44. Hence, there is evidence that the upper bound on $p$ should be greater than $\frac{1}{2k}$. However, we do not yet know the exact upper bound on $p$ at which Conjecture 44 fails to hold. We state this as a open question.

**Question 45.** What is the largest value of $\delta$ for which Conjecture 44 gives the probability that $\tau(S) = k$? What is the relationship between $\delta$, the number of voters $n$, and the piercing number $k$?

As previously mentioned, our probabilistic approach helps us understand the “average-case scenario” for piercing numbers of fixed-length circular societies. Thus, we conclude with a discussion of the average piercing number. If $p < \frac{1}{2n}$, then Theorem 39 holds for all $k \in \{1, \ldots, n\}$, and the expected value of $\tau(S)$ is:

$$E[\tau(S)] = \sum_{k=1}^{n} k \binom{n}{k} (1 - kp)^{k-1}(kp)^{n-k}. \quad (2)$$

However, given Conjecture 44, the formula above may hold for $p$ somewhat larger than $\frac{1}{2n}$. We simulated the average piercing number in random fixed-length circular societies such that $p \geq \frac{1}{2n}$; selected results
Table 1: Simulation results. For each pair of \( N \) and \( p \), we randomly generated \( N = 100,000 \) fixed-length circular societies and computed the proportion that have each possible piercing number \( \tau \). We compare these simulated averages with the probabilities given by Theorem 39. Results are rounded to four decimal places; simulated results of zero are not shown.

| \( n \) | \( p \) | simulated average | probability given by Conjecture 44 |
|---|---|---|---|
| 5  | 0.15 | 4   | 0.1903 | 0.1920 |
| 5  | 0.15 | 5   | 0.0038 | 0.0039 |
| 8  | 0.12 | 5   | 0.3102 | 0.3040 |
| 8  | 0.12 | 6   | 0.0254 | 0.0250 |
| 8  | 0.12 | 7   | 0.0001 | 0.0001 |
| 10 | 0.10 | 5   | 0.4896 | 0.4922 |
| 10 | 0.10 | 6   | 0.2808 | 0.2787 |
| 10 | 0.10 | 7   | 0.0309 | 0.0300 |
| 10 | 0.10 | 8   | 0.0004 | 0.0004 |

Table 2: For each combination of values of \( n \) and \( p \), we simulated the average piercing number by randomly generating \( N = 10,000 \) fixed-length circular societies. We compare the simulation averages with the conjectured expected values, computed from Equation (2).

| \( n \) | \( p \) | simulated average piercing number | expected value given by Equation (2) |
|---|---|---|---|
| 5  | 0.100 | 3.494 | 3.492 |
| 5  | 0.200 | 2.633 | 2.632 |
| 5  | 0.250 | 2.340 | 2.324 |
| 25 | 0.050 | 11.207 | 11.222 |
| 25 | 0.100 | 7.198 | 7.201 |
| 25 | 0.125 | 6.099 | 5.986 |
| 45 | 0.020 | 23.823 | 23.816 |
| 45 | 0.050 | 13.916 | 13.912 |
| 45 | 0.080 | 9.822  | 9.816  |

We invite readers to explore our simulations of piercing probabilities and average values using our circular society visualization and simulation tool, which is available at https://github.com/tiasondjaja/circular_societies.
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