Cosmological perturbations in Palatini-modified gravity

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Received 10 May 2007, in final form 18 June 2007
Published 17 July 2007
Online at stacks.iop.org/CQG/24/3951

Abstract
Two approaches to the study of cosmological density perturbations in modified theories of Palatini gravity have recently been discussed. These utilize, respectively, a generalization of Birkhoff’s theorem and a direct linearization of the gravitational field equations. In this paper these approaches are compared and contrasted. The general form of the gravitational Lagrangian for which the two frameworks yield identical results in the long-wavelength limit is derived. This class of models includes the case where the Lagrangian is a power-law of the Ricci curvature scalar. The evolution of density perturbations in theories of the type \( f(R) = R - c/R^b \) is investigated numerically. It is found that the results obtained by the two methods are in good agreement on sufficiently large scales when the values of the parameters \((b, c)\) are consistent with current observational constraints. However, this agreement becomes progressively poorer for models that differ significantly from the standard concordance model and as smaller scales are considered.

PACS numbers: 98.80.-k, 95.36.+x

(Some figures in this article are in colour only in the electronic version)

1. Introduction

There is now overwhelming observational evidence that the universe is currently undergoing a phase of accelerated expansion. This evidence arises from high redshift supernovae surveys [1–5], observations of large-scale structure [6, 7], baryon acoustic oscillations [8] and high-precision data from the cosmic microwave background (CMB) [9, 10].

An accelerating universe poses a great challenge to modern cosmology. It is very difficult to explain such behaviour within a conventional general relativistic framework. The simplest way to generate a phase of acceleration is to introduce a cosmological constant into the field equations. Although this is entirely consistent with the available data, the microphysical origin of such a term remains a mystery.
In view of this, a large number of alternative models have been proposed (see [11] for a recent review). In most cases, these can be classified into two broad groups: those that invoke an exotic matter source for the dark energy [12] and those that modify the gravitational sector of the theory [13]. Examples of the latter include generalized theories of gravity based on nonlinear functions, $f(R)$, of the Ricci scalar $R$. Such modifications to the (linear) Einstein–Hilbert action typically arise in effective actions derived from string/M-theory [14–17].

Given the large number of candidates within both approaches, it is important to determine whether observations can be employed to discriminate between them. In general, the homogeneous dynamics encoded in the modified Friedmann equations of $f(R)$ theories can always be expressed in terms of a conventional relativistic cosmology sourced by an effective perfect fluid. Thus, it is necessary to include inhomogeneities in order to lift the degeneracy between the two frameworks. This can be achieved, for example, by considering the evolution of density perturbations, and a study of perturbation theory in $f(R)$ gravity is therefore of considerable interest. Two approaches have been developed recently in this context. One possibility is to follow the standard procedure employed in relativistic perturbation theory and derive the perturbation equations by linearizing the gravitational field equations [18, 19]. On the other hand, an alternative procedure has recently been put forward by Lue, Scoccimarro and Starkman (LuSS) [20] which employs a generalized version of Birkhoff’s theorem (see also [21]). This procedure has the benefit of greatly simplifying the analysis, but suffers from the drawback that the degree of its applicability in more general settings is presently not known in detail.

The purpose of the present work is to perform a detailed comparative study of the evolution of perturbations obtained by employing the LuSS procedure and the direct linearization of the field equations. Such a comparison can serve as a crucial step in clarifying the status of the LuSS approach in nonlinear gravity theories.

When studying generalized $f(R)$ gravity, a choice must be made as to which independent fields should be varied in the action. There are two possibilities, which are referred to as the ‘metric’ and ‘Palatini’ frameworks, respectively. In the former case, only variations with respect to the metric are considered, whereas the action is varied with respect to both the metric and the connection in the Palatini framework [22]. Both approaches result in identical field equations for the Einstein–Hilbert action, where $f(R) \propto R$. More generally, however, the metric approach results in fourth-order equations, whereas the Palatini variation generates a second-order system [23]. Moreover, the theories of the type $f(R) = R - c/R^b$ based on metric formalism have difficulty in passing the solar system tests [24] (see, however, [25]) and producing the correct Newtonian limit [26]. Such theories also suffer from gravitational instabilities as discussed in [27]. Along with these concerns, a recent study has found that these types of theories cannot produce a standard matter epoch [28]. In this paper, we consider theories based on the Palatini variational method which can avoid the problems outlined above. Such theories have received considerable attention in recent years as candidates for explaining the present-day acceleration of the universe [18, 22, 23, 29–34].

The outline of the paper is as follows. In section 2, we briefly review the Palatini variational method and present the perturbation evolution equations derived by using the LuSS procedure and the direct linearization of the field equations. In section 3, we derive the necessary conditions on the general form of the gravitational Lagrangian $f(R)$ for the two approaches to be compatible. We perform a numerical analysis in section 4 to quantitatively compare the two frameworks for the class of $f(R)$ theories where the action includes an inverse power of the Ricci curvature scalar. We conclude with a discussion in section 5.
2. \( f(R) \) gravity in the Palatini formalism

2.1. The field equations

We consider the class of nonlinear gravity theories defined by the action

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [f(R)] + S_m(g, \psi),
\]

where \( \kappa \equiv 8\pi G \) is a constant, \( f(R) \) is a differentiable function of the Ricci scalar \( R \equiv g^{\mu\nu} R_{\mu\nu}(g, \hat{\Gamma}) \) and \( R_{\mu\nu}(g, \hat{\Gamma}) \) is the Ricci tensor of the affine connection \( \hat{\Gamma}^{\alpha}_{\beta\gamma} \). The matter action \( S_m \) is a functional only of the metric tensor and matter fields, \( \psi \).

In the Palatini framework, the affine connection and the metric are treated as independent variables. Extremizing action (1) with respect to the metric tensor yields the gravitational field equations (see, e.g., [22]):

\[
F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu},
\]

where \( F(R) \equiv df/dR \) and

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta S_m /\delta g^{\mu\nu}
\]

defines the energy–momentum tensor. Contraction of equation (2) yields the algebraic constraint equation

\[
RF(R) - \frac{2}{2} f(R) = \kappa T,
\]

where \( T \equiv T_{\mu\mu} \).

On the other hand, varying action (1) with respect to the affine connection \( \hat{\Gamma}^{\alpha}_{\beta\gamma} \) and contracting implies that [22]

\[
\hat{\nabla}_{\rho}[F(R) \sqrt{-g} g^{\mu\nu}] = 0,
\]

where \( \hat{\nabla} \) is the covariant derivative defined by \( \hat{\Gamma}^{\alpha}_{\beta\gamma} \). The solution to equation (5) is given by writing \( \hat{\Gamma}^{\alpha}_{\beta\gamma} \) as the Levi-Civita connection for a new metric \( h_{\mu\nu} \equiv F(R) g_{\mu\nu} \), which is conformally equivalent to the spacetime metric \( g_{\mu\nu} \). As a result, we may write the Ricci tensor of the affine connection in the form

\[
R_{\mu\nu}(\hat{\Gamma}) = R_{\mu\nu}(g) + \frac{3}{2} \nabla_{(\mu} F \nabla_{\nu)} F - \frac{1}{F} \nabla_{\mu\nu} F - \frac{1}{2F} g_{\mu\nu} \nabla^\lambda \nabla^\lambda F,
\]

where \( R_{\mu\nu}(g) \) is the Ricci tensor of the Levi-Civita connection. It can then be shown that the field equations (2) and constraint equation (4) can be derived from a scalar-tensor action of the form [35]

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \phi R(g) + \frac{3}{2\phi} (\nabla\phi)^2 - V(\phi) \right] + S_m,
\]

where the scalar field \( \phi \equiv F \) and \( V(\phi) \equiv R(\phi) F - f[R(\phi)] \) represents the potential. Action (7) corresponds to the Brans–Dicke theory with a dilaton–graviton coupling \( \omega_0 = -3/2 \).

We assume throughout that the background spacetime is given by the spatially flat and isotropic Friedman–Robertson–Walker (FRW) metric

\[
dx^2 = a(\tau)^2 (-d\tau^2 + \delta_{ij} dx^i dx^j),
\]

where \( a(\tau) \) is the scale factor of the universe and \( \tau \equiv \int dt / a(t) \) defines the conformal time. We further assume that the matter content of the universe is represented by a pressureless
perfect fluid with energy–momentum tensor $T^\mu_\nu = \text{diag}(-\rho_m, 0, 0, 0)$. The Ricci tensor (6) may then be employed to derive the generalized Friedmann equation for this cosmology:

$$6F \left( H + \frac{\dot{F}}{2F} \right)^2 - f = \kappa \rho_m,$$

where a dot denotes differentiation with respect to coordinate time $t$ and $H \equiv \dot{a}/a$ defines the Hubble expansion parameter. After substituting the trace equation (4), the generalized Friedmann equation can be rewritten in the form

$$H^2 = \frac{3f - RF}{6F} \left( 1 - \frac{3}{2} \frac{F, R(2f - RF)}{F(F - RF, R)} \right)^{-2},$$

where a comma denotes differentiation.

2.2. The perturbation evolution equations

In conventional cosmology, there exists an interesting equivalence between the Newtonian and general relativistic frameworks. Both approaches result in identical background evolution equations (i.e., Friedmann equations) as well as evolution equations for the scalar perturbations. The former coincidence results from the fact that there is an analogue of Newton’s sphere theorem in general relativistic settings, i.e., Birkhoff’s theorem holds. The correspondence for the perturbation evolution arises in the absence of vector and tensor fluctuations.

Recently, a procedure has been put forward by Lue, Scoccimarro and Starkman [20] which relies on the assumption that this Newtonian analogy, including Birkhoff’s theorem, holds in the more general setting of modified gravity theories. According to this procedure, it is assumed that the growth of the large-scale structure can be modelled in terms of a uniform sphere of dust of constant mass, such that the evolution inside the sphere is determined by the FRW metric. Using Birkhoff’s theorem, the spacetime metric in the empty exterior is then taken to be Schwarzschild-like. The components of the exterior metric are then uniquely determined by smoothly matching the interior and exterior regions.

The overdensity $\delta(t)$ of the spherical distribution of pressureless matter with mass $M$ and radius $r$ is defined by

$$1 + \delta(t) \equiv \frac{3M}{4\pi \rho r^3},$$

where $\rho(t)$ represents the background energy density. The matching conditions imply that $\dot{r} = r(H^2 + \dot{H})$ and the evolution of the density perturbation is then given by [20, 36]

$$\ddot{\delta} + 2H \dot{\delta} - \left( 2H + \frac{\dot{H}}{H} \right) \delta = 0,$$

or, equivalently, by

$$\dddot{\delta} + \dot{H} \ddot{\delta} - \left( \frac{\dot{H}''}{H} - 2\dot{H}' \right) \delta = 0,$$

where a prime denotes differentiation with respect to conformal time and $\dot{H} \equiv aH = \dot{a}$. Equation (13) can also be derived by assuming that the continuity and Friedmann equations apply directly to the fluctuations [21].

Recently, the evolution of perturbations in $f(R)$ gravity was investigated using the LuSS procedure [20, 31]. The advantage of this approach is that the growth of the density contrast can be expressed in terms of a single quadrature involving the Hubble parameter and the scale factor [20, 37]:

$$\delta \propto H \int \frac{dt}{a^2 H^2}. $$
In principle, therefore, the evolution of the perturbations can be determined once the background dynamics has been specified. However, the validity of the LuSS procedure has yet to be established in generalized gravity. It is important, therefore, to compare this approach with the method that directly linearizes the gravitational field equations. The corresponding evolution equation for comoving, linear density perturbations in a pressureless universe was recently derived by Koivisto and Kurki-Suonio (KKS) using this direct method and found to have the form \[ \delta'' + \frac{2F'H(H^2 + F') - 2F^2'H + F'F(-2H' + H)^2}{3F'H^2(2F'H + F')} \delta' = \frac{6F^2'H(H'' - 2H'H) + 6F^2H(H^2 - H')}{3F'H^2(2F'H + F')} \delta = 0, \] where \( k \) is the comoving wavenumber that arises due to the Fourier decomposition.

We will refer to equations (13) and (15) as the LuSS and KKS perturbation equations, respectively. We will be interested in identifying the domain where the equation based on the LuSS procedure provides an accurate description for the evolution of the perturbations. In the following section, we adopt an analytical approach with the aim of identifying the general form of the gravitational Lagrangian, \( f(R) \), for this to be the case.

3. Analytical results

A direct comparison between the LuSS equation (13) and the KKS equation (15) suggests that the latter should be rewritten in the form
\[
\delta'' + \xi H \delta' - \zeta \left( \frac{H''}{H} - 2H' \right) \delta = 0,
\]
where the parameters \( \xi \) and \( \zeta \) are defined by
\[
\xi \equiv 1 + \frac{2FF''H - 2F^2H' - 2FF'H'}{F'H^2(2F'H + F')},
\]
and
\[
\zeta \equiv 1 + \frac{H^2 - H'}{H'' - 2H'H}(1 - \xi) - \frac{F'H}{3(2F'H + F')(H'' - 2H'H)}k^2,
\]
respectively. The form of equation (16) implies that the LuSS and KKS equations are equivalent when \( \xi = \zeta = 1 \), but it is clear that this occurs only for Einstein gravity where \( F' = 0 \). Indeed, the most striking difference is the presence of the gradient term in the KKS equation. Such a term also arises in the corresponding density perturbation equation derived in the metric variational approach [38]. The origin of this term can be understood from the dynamical equivalence between Palatini gravity and Brans–Dicke theory, as expressed in equation (7). Fluctuations in the pressureless matter induce perturbations in the scalar field \( \phi \) (i.e. the Ricci curvature), which in turn generate a pressure gradient in the fluid. In general, the sound speed of the fluctuations in the cold dark matter is given by
\[
c_s^2 = \frac{F'}{3(2F'H + F')}.
\]
The magnitude of \( \xi \) is independent of \( k \) and is therefore unaffected by the specific choice of scale. However, \( \zeta \) contains a gradient term which is proportional to \( k^2 \) and this may be significant on small scales. Consequently, the evolution of the perturbations will indeed be
different in the two approaches. However, the gradient term becomes negligible in the long-wavelength limit (which corresponds formally to $k^2 \to 0$). In this limit, a necessary and sufficient condition for equivalence between the LuSS and KKS equations is that $\xi = 1$ and this constraint is satisfied when

$$FF''\dot{\mathcal{H}} - F'^2\dot{\mathcal{H}} - FF'\dot{\mathcal{H}}' = 0. \tag{20}$$

Equation (20) may be viewed as a second-order, nonlinear differential equation for $F(\tau)$. One solution to this equation is that of general relativity with a cosmological constant, $f(R) = R - \Lambda$. More generally, if $F' \neq 0$ and $F'' \neq 0$, we may define a parameter $Y \equiv F'/F$. This reduces equation (20) to the remarkably simple form

$$\frac{Y'}{Y} = \frac{\dot{\mathcal{H}}'}{\dot{\mathcal{H}}}, \tag{21}$$

which admits the integral $Y = Y_0 \dot{\mathcal{H}}$, where $Y_0$ is an arbitrary integration constant. This in turn implies that

$$F = F_0 a^{Y_0}, \tag{22}$$

where $F_0$ is a second integration constant.

On the other hand, the trace equation (4) for a universe sourced by pressureless matter reduces to the condition \[31\]

$$a \propto \left(2f - R \frac{df}{dR}\right)^{-1/3}. \tag{23}$$

Hence, substitution of equation (23) into equation (22) yields a first-order, nonlinear differential equation in the gravitational Lagrangian $f(R)$:

$$\left(\frac{df}{dR}\right)^n \left(2f - R \frac{df}{dR}\right) = \text{constant}, \tag{24}$$

where $n \equiv 3/Y_0$.

Equation (24) is a particular example of d’Alembert’s equation and may be solved in full generality [39]. Since we are interested in the functional dependence of the Lagrangian on the Ricci scalar, we may rescale $f$ without loss of generality such that the constant on the right-hand side of equation (24) is unity. If we now define the functions

$$M \equiv \frac{1}{2} \frac{df}{dR}, \quad N \equiv \frac{1}{2} \left(\frac{df}{dR}\right)^{-n}, \tag{25}$$

and denote $p \equiv df/dR$, equation (24) can be expressed in the form $f(R) = RM(p) + N(p)$. Differentiating this expression with respect to $R$ then yields

$$p = M(p) + \frac{dp}{dR} \left[R \frac{dM(p)}{dp} + \frac{dN(p)}{dp}\right]. \tag{26}$$

However, equation (26) can be expressed as a linear differential equation in the dependent variable $R$ and independent variable $p$:

$$\frac{dR}{dp} - \frac{R}{p} = -\frac{n}{p^{2+n}}. \tag{27}$$

Hence, solving equation (27) by the method of integrating factors yields the general solution to equation (24) in a parametric form

$$R = C_0 P + \frac{n}{n + 2} \frac{1}{p^{1+n}}. \tag{28}$$
\[ f = \frac{1}{2} R P + \frac{1}{2} P^n, \]  
\[ \text{(29)} \]

where \( C_0 \) is an arbitrary integration constant and \( P \) is a free parameter.

Equations (28)–(29) represent the general form of the gravitational Lagrangian \( f(R) \) for the LuSS and KKS equations to be compatible in the long wavelength limit. It is interesting that for this class of theories the sound speed of the fluctuations is constant with a numerical value given by

\[ c_s^2 = \frac{1}{3 + 2n}. \]  
\[ \text{(30)} \]

When \( C_0 = 0 \), which is equivalent to the asymptotic limit where \( R \) is sufficiently small, the gravitational action depends on a simple power of the Ricci scalar:

\[ f(R) \propto R^{n/(1+n)}. \]  
\[ \text{(31)} \]

For this class of theories the Friedmann equation (10) reduces to

\[ H^2 = \frac{3 + 2n}{6n} \left( 1 + \frac{3}{2n} \right)^2 R, \]  
\[ \text{(32)} \]

which in turn implies that the background dynamics is given by a power-law solution for the scale factor, \( a \propto \mathcal{H}^{2/(3+n)} \propto \eta^{2/(3+n)} \) [40]. Consequently, the cosmic dynamics is equivalent to that of a conventional relativistic universe dominated by a perfect fluid with a constant equation of state. Finally, the parameter \( \zeta \) simplifies in this case to

\[ \zeta = 1 - \frac{2n^2}{3(1+n)(3+n)(3+2n)} \frac{k^2}{\mathcal{H}^2}. \]  
\[ \text{(33)} \]

In conclusion, therefore, the above analysis indicates that the LuSS equation should provide a good approximation to the full evolution equation for the linear density perturbation on sufficiently large scales in any modified gravity theory that asymptotes in the low-energy limit to a power-law in the Ricci curvature scalar. On the other hand, for fixed values of \( n \) and \( \mathcal{H} \), the LuSS equation becomes progressively less accurate as we move to smaller scales (i.e. as \( k \) increases). In the following section, we will quantify these conclusions further by performing numerical calculations for a specific class of modified gravity theories.

4. Numerical results

Motivated by the results of the previous section, we consider the class of gravity theories defined by

\[ f(R) = R - \frac{c}{R^b}, \]  
\[ \text{(34)} \]

where \( b \) and \( c \) are free parameters whose values are constrained by observations. Such theories have recently been considered as possible candidates for explaining the late-time acceleration of the universe [32–34]. In particular, a recent study found that data obtained from CMB, baryon oscillation and large-scale structure observations constrain the parameters \((b, c)\) to lie in the ranges \( b \in [-0.2, 1.2] \) and \( c \in [-3.5, 6.6] \) at the 68% confidence level [34]. The best-fit model corresponds to the values \((b, c) = (0.027, 4.63)\) and the \( \Lambda \)CDM concordance model is represented by \((b, c) = (0, 4.38)\). These values are consistent with the results of other studies that employ CMB and supernovae data [31].

For the above choice of parameters, we have made a detailed comparative study of the evolution of the density perturbations for both the LuSS equation (13) and the KKS
equation (15). The results of such a comparison can be quantified by defining a ‘fractional difference’ parameter

$$\Delta \equiv \frac{\delta_{\text{LuSS}} - \delta_{\text{KKS}}}{\delta_{\text{KKS}}}$$

where subscripts ‘LuSS’ and ‘KKS’ refer to the results obtained using the LuSS and KKS equations, respectively. Thus, the two approaches are completely compatible when $$\Delta = 0$$. This parameter is defined in such a way that the difference between the two approaches is of the same order as the KKS approach when $$\Delta \sim O(1)$$. To a first approximation, therefore, it is reasonable to suppose that the LuSS equation becomes unreliable when $$\Delta \approx 1$$.

There are three physical parameters in the field equations whose values need to be specified in the numerical integrations. These are $$\Omega_{\text{m}0}$$, $$R_0$$ and $$H_0$$, where a subscript zero indicates present-day values and $$\Omega_{\text{m}}$$ is the normalized matter energy density. However, only two constraint equations are available, corresponding to the Friedmann equation (10) and the trace equation (4). In order to be consistent, therefore, we specify the value of $$H_0$$ to be unity, as is the usual practice (see, e.g., [31]). We then use the constraint equations (4) and (10) to determine $$\Omega_{\text{m}0}$$ and $$R_0$$. The choice of equation (35) implies that the initial value of the perturbation $$\delta$$ is unimportant. Finally, we need to specify the scale of the perturbations. By fixing the wavenumber at a particular value, one focuses on perturbations that entered the horizon at a particular epoch.

For illustrative purposes we consider the values $$k = 5$$ and $$k = 20$$, corresponding to scales which remain within the horizon throughout our numerical evolution.

The left panel of figure 1 illustrates the evolution of $$\Delta$$ when $$c = 4.38$$ and $$k = 5$$, with $$b$$ taking values in the range $$b \in [0, 1]$$.

As expected, $$\Delta = 0$$ for the $$\Lambda$$CDM concordance model (given by $$b = 0$$), since it is known that the LuSS equation is exact in this case. On the other hand, increasing the value of $$b$$ causes the behaviour of the two approaches to deviate and the quantitative difference becomes more pronounced as $$b$$ is increased.

We have verified that our results remain qualitatively similar when the parameter values lie in the ranges $$b \in [-0.2, 1.2]$$ and $$c \in [-3.5, 6.6]$$, respectively. An important outcome of these

Note that in modified gravity theories of the type considered here, this parameter need not necessarily be unity in a spatially flat universe.
Figure 2. Illustrating the evolution of the parameter $\zeta$ defined by equation (18) in the text for the parameter values $k = 5$ (left panel) and $k = 20$ (right panel). The LuSS procedure for the evolution of the perturbations becomes progressively less accurate as the deviation of this quantity from unity becomes more pronounced.

results is that for values of the parameters consistent with recent observations, the agreement between the LuSS and KKS approaches is good in the sense that $\Delta < 0.1$ for $b < 0.2$. This implies that the LuSS equation provides a good approximation to the full (linear) perturbation theory (for this value of $k$). This can be understood by noting that observations constrain theoretical models to lie close to the $\Lambda$CDM point, where it is known that the LuSS equation is exact.

Further inspection of the left panel of figure 1 indicates that as the value of $b$ is increased, the models take longer to move away from the $\Lambda$CDM point $\Delta = 0$, but those with smaller values of $b$ subsequently find it easier to approach $\Delta = 0$ at later times. We may gain further insight into the origin of this behaviour by investigating the evolution of the quantity $Q \equiv 1 - F$. This vanishes at all times for Einstein gravity but is given by $Q = -bcR^{-(1+b)}$ for the class of models (34). This parameter therefore provides a measure of the deviation away from general relativity. Our numerical calculations indicate that initially $R \approx \mathcal{O}(10^3)$ and, consequently for larger values of $b$, the scale factor must grow to a larger value before the Ricci scalar has fallen sufficiently for the correction term $Q$ to become dynamically significant. In other words, the onset of acceleration occurs at later times for larger $b$. On the other hand, the correction term in $f(R)$ that is proportional to $R^{-b}$ will become more important as the universe expands. The analysis of section 3 then indicates that the accuracy of the LuSS equation will improve as $f(R)$ asymptotes to a power-law form. Consequently, $\Delta$ will begin to decrease back to zero at later times.

We find qualitatively similar behaviour at larger values of $k$. The right panel of figure 1 illustrates the corresponding evolution of $\Delta$ when $k = 20$. As expected, models with lower values of $b$ move away from the $\Lambda$CDM point $\Delta = 0$ at smaller values of the scale factor. The model with the lowest non-zero value of $b = 0.2$ crosses the solutions for $b = 0.4$ and $b = 0.6$. This can be understood from equation (33), which implies that the magnitude of $\zeta$ depends on the ratio $k^2/\mathcal{H}^2 = k^2/a^4$. At a formal level, therefore, increasing the value of $k$ is equivalent to ending the numerical calculation at a fixed $k$ but with a smaller value for the scale factor.

However, the quantitative agreement between the solutions of the LuSS and KKS equations is poor when $k = 20$ and $\Delta$ rapidly exceeds unity in this case. This discrepancy arises primarily because the deviation of the parameter $\zeta$ away from unity is more pronounced at larger $k$. Figure 2 illustrates the evolution of $\zeta$ for the different values of $k$. 
5. Conclusion

In this paper, we have studied the evolution of density perturbations in generalized theories of gravity where the field equations are derived via the Palatini variational approach. We focused on models where the energy–momentum tensor is sourced by a pressureless perfect fluid. Two approaches to the study of density perturbations have recently been developed in the literature [18, 20, 21]. These involve, respectively, an application of Birkhoff’s theorem to modified gravity (the LuSS method) and the linearization of the full field equations (the KKS approach). In the former case, the evolution of the perturbations is determined entirely by the background dynamics and no pressure gradients are present in the perturbation evolution equation. However, such terms do arise in the linearization approach, which takes into account the fact that perturbations in the fluid induce fluctuations in the Ricci curvature which in turn modify the sound speed of the fluctuations in the matter.

In the long-wavelength limit, these gradient terms are negligible. We have identified the most general $f(R)$ theory of gravity, as summarized in equations (28) and (29), for the LuSS and KKS approaches to be compatible in this limit. A particular case of this class of theories arises when $f(R)$ is a simple power law of the Ricci curvature scalar. This is interesting because such terms are expected to arise generically as corrections to the Einstein–Hilbert action at low energies. Furthermore, theories of this type result in a background scaling solution, in the sense that the homogeneous dynamics is equivalent to that of a conventional relativistic cosmology where the pressure and energy density of the perfect fluid redshift at the same rate. It would be interesting to explore whether this scaling behaviour is a necessary condition for compatibility between the LuSS and linearization methods in more general theories of modified gravity. For example, a power-law cosmology arises in the Palatini variation of Ricci-squared gravity, where $f \propto (R^{\mu\nu}R_{\mu\nu})^{n/2}$ [41].

We numerically investigated a specific class of power-law theories of the type (34) and compared the LuSS and KKS approaches on smaller scales where gradient terms become significant. We found that when the parameters of the underlying theory take values that are consistent with cosmological observations, the LuSS procedure provides a reasonably good approximation to the complete linearized theory if $k$ is not too large (i.e. of the order of a few or less). However, the agreement between the two approaches soon breaks down on smaller scales.

Acknowledgments

KU is supported by the Science and Technology Facilities Council (STFC). We thank S Fay, C Hidalgo and K Malik for helpful discussions.

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