FRACTIONAL DIFFUSION LIMITS OF NON-CLASSICAL TRANSPORT EQUATIONS

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Abstract. We establish asymptotic diffusion limits of the non-classical transport equation derived in [12]. By introducing appropriate scaling parameters, the limits will be either regular or fractional diffusion equations depending on the tail behaviour of the path-length distribution. Our analysis is based on a combination of the Fourier transform and a moment method. We put special focus on dealing with anisotropic scattering, which compared to the isotropic case makes the analysis significantly more involved.

1. Introduction. Anomalous diffusion, a diffusion process described by a fractional diffusion equation, has gained a lot of interest recently. Examples include Lévy glasses [25], plasma physics [5], spreading of diseases [22], chemical reactions [2], elementary particle physics [21], and flight patterns of birds [24]. Many more examples are contained in the aptly-titled review [19].

It is known that fractional diffusion can be rigorously derived from Continuous Time Random Walks (CTRWs) in the limit of many interactions by a Generalized Central Limit Theorem (cf. [19] and references therein). In addition, Marklof & Tóth [15] proved a superdiffusive Central Limit Theorem for the particle billiards of the so-called periodic Lorentz gas and showed that the periodic Lorentz gas is superdiffusive (but only logarithmically).

Our purpose in this paper is to prove a similar result as in [15], but by using techniques from kinetic theory instead of the techniques from CTRWs. To our knowledge, the first rigorous mathematical work to connect kinetic equations to fractional diffusion is by Mellet-Mischler-Mohout [18]. In that paper and in the two subsequent works [1, 16], the authors show that solutions to classical transport
equations can converge to solutions of fractional diffusion equations if their equilibrium states have algebraic decay in velocity or if their scattering coefficient is degenerate. Since then, fractional diffusion limits have been revealed in more applications [6, 17], and asymptotic-preserving numerical schemes have been designed [7, 26].

The main point for our work is to extend the theory developed for classical transport equations to non-classical transport equations. Roughly speaking, we will show that with slow decay of a certain path-length distribution, non-classical transport equations can also give rise to fractional diffusion equations. The particular model that we study is proposed by Larsen [12] (see (1) for the explicit equation). The original motivation for this model was from measurements of photon path-length in atmospheric clouds, which could not be explained by classical radiative transfer (cf. [20] or sections 5.1 and 8.3 in the review [8]). Classically, the amount of radiation, when it passes through a medium, is attenuated exponentially. This is the well-known law of Beer-Lambert. Recent measurements, however, have revealed that radiation through an atmospheric cloud is attenuated less, namely merely algebraically [20]. This has led Larsen to formulate a Boltzmann equation on an extended phase space [12], which he named non-classical transport equation. The equation models the transport of particles with a given path-length distribution $p(s)$, $s$ being the path-length, and $p$ its probability density function. In his original paper [12], Larsen has considered the formal diffusion limit of the non-classical transport equation, which is later made rigorous in [10]. However, the analysis in [12, 10] cannot capture the case when the second moment, i.e. the variance, of the path-length distribution does not exist. The purpose of our work is to extend the analysis to cover this case and make the limit process rigorous. As is mentioned before, in the case of an infinite variance of the path-length distribution, the limiting equation is a fractional diffusion equation. We therefore provide a connection between non-classical transport and anomalous diffusion.

Non-classical transport theory has now been extended [13] and has found applications for neutron transport in pebble bed reactors [23], and even computer graphics [9]. We also comment that recent results by Golse et al. (cf. [11] for a review), and by Marklof & Strömbergsson [14] show that an equation (periodic Lorenz gas equation) similar to the non-classical transport equation can be derived from particle transport in a regular lattice.

In this paper we study the equation in its time-independent version, because the steady case is the one that is relevant for the applications. The main content of our work is laid out as follows: The basic setting and the main result are stated mathematically in Section 2. In Section 3 we give a short proof of the well-posedness of the transport equation, which lays down the basic functional setting in this paper. The main part is in Section 4 where we establish various limits of the transport equation. We comment on the periodic Lorentz gas in Section 5.

2. Main result. The non-classical transport equation with a scaling parameter $\epsilon$ as considered in [12] has the form

\[
\frac{1}{\epsilon} \partial_s \psi_\epsilon(x, v, s) + v \cdot \nabla_x \psi_\epsilon(x, v, s) + \frac{\Sigma_t(s)}{\epsilon} \psi_\epsilon(x, v, s) = \delta(s) \int_{\mathbb{S}^{n-1}} \int_0^\infty (\sigma(v \cdot v') - \theta(\epsilon)(1 - c)) \frac{\Sigma_t(s')}{\epsilon} \psi_\epsilon(x, v', s') ds' dv' + \delta(s) \frac{\theta(\epsilon)}{\epsilon} Q.
\]
The unknown function $\psi_\epsilon$ is the angular flux of particles at position $x \in \mathbb{R}^n$, moving in the direction $v \in S^{n-1}$ (unit vector). The particles interact with a background medium. The interaction of the particles is described by the collision cross section $\Sigma_\epsilon$. What makes the equation non-classical is that $\Sigma_\epsilon = \Sigma(s)$ depends on the distance $s$ from the last collision. The angular scattering kernel $\sigma(v \cdot v')$ is independent of $s$. Moreover, the measure $dv$ is scaled to be the unit measure on $S^{n-1}$ and $\sigma$ satisfies that

$$\int_{S^{n-1}} \sigma(v \cdot v') \, dv = 1. \quad (2)$$

The equation is completed by the particle source $Q$, and the scattering ratio $c$ (when a particle interacts with the background, the probability that it is absorbed is $1-c$, the probability that it scatters is $c$). We will assume throughout the paper that $c < 1$, i.e. there is a small amount of absorption everywhere. The Dirac delta $\delta(s)$ on the right-hand side models that particles which scatter have their distance-to-previous-collision reset to zero. In the case of constant $\Sigma_\epsilon$, this equation reduces to the classical transport equation [12].

The parameter $\epsilon$ being small means that we have many collisions (small Knudsen number). Extending the scaling in [12] where $\theta(\epsilon) = \epsilon^2$, we have introduced a general function $\theta(\epsilon)$ to scale the absorption term and the source. We assume that $\theta(\epsilon)$ is monotonically increasing with $\epsilon$ and $\theta(\epsilon) \to 0$ as $\epsilon \to 0$. In most cases, we will use $\theta(\epsilon) = \epsilon^\alpha$ with $1 < \alpha < 2$. Some comments on this particular choice of the scaling are in order: First, as in [12], we have fixed the scale of $\Sigma_\epsilon$ to be $1/\epsilon$ and of $s$ to be $\epsilon$, which means the mean free path is small. Second, if we rearrange the equation as

$$\frac{1}{\epsilon} \partial_\tau \psi_\epsilon(x, v, s) + v \cdot \nabla_x \psi_\epsilon(x, v, s) + \frac{\Sigma_\epsilon(s)}{\epsilon} \psi_\epsilon(x, v, s)$$

$$- \delta(s) \int_0^\infty \sigma(v \cdot v')\frac{\Sigma_\epsilon(s')}{\epsilon} \psi_\epsilon(x, \Omega', s') \, ds' \, dv'$$

$$= \delta(s)\theta(\epsilon) \left( \frac{1}{\epsilon} Q - (1 - c) \int_0^\infty \frac{\Sigma_\epsilon(s')}{\epsilon} \psi_\epsilon(x, v', s') \, ds' \, dv' \right).$$

then it becomes clear that the scaling factor $\theta(\epsilon)$ controls the relative weakness of emission/absorption compared to scattering.

As in [12] we reformulate the equation in terms of the new unknown

$$\Psi_\epsilon(x, v) = \psi_\epsilon(s, x, v) e^{\int_0^s \Sigma_\epsilon(\tau) \, d\tau}. \quad (3)$$

Eq. (1) can be re-written as

$$\frac{1}{\epsilon} \partial_\tau \Psi_\epsilon + v \cdot \nabla_x \Psi_\epsilon = 0,$$

$$\Psi_\epsilon(0, x, v)$$

$$= \int_0^\infty \int_{S^{n-1}} (\sigma(v \cdot v') - \theta(\epsilon)(1 - c)) p(\tau) \psi_\epsilon(\tau, x, v') \, dv' \, d\tau + \theta(\epsilon) Q(x, v),$$

where $p(s)$ is the path-length distribution given by

$$p(s) = \Sigma_\epsilon(s) e^{-\int_0^s \Sigma_\epsilon(\tau) \, d\tau}.$$

The advantage of the new formulation is that we can work directly with the path-length distribution $p(s)$. 

The main purpose of this paper is to prove the following convergence result as \( \epsilon \to 0 \):

**Theorem 2.1.** Suppose the scattering constant \( c \) and the cross section \( \sigma \) satisfy the assumptions

\[
0 < c < 1, \quad \int_{S^{n-1}} \sigma(v \cdot v') \, dv' = 1, \quad \sigma(v \cdot v') \geq \sigma_0 > 0
\]  

for some constant \( \sigma_0 \). Suppose the path-length distribution function \( p \) satisfies

\[
\int_0^\infty p(s) \, ds = 1, \quad \int_0^\infty sp(s) \, ds < \infty, \quad p(s) = \frac{d_0}{s^{\alpha+1}} \quad \text{for } s > 1,
\]

where \( d_0 > 0 \) is a constant. Suppose the source term \( Q \in L^1 \cap L^2(\mathbb{R}^n \times S^{n-1}) \). Let

\[
\Psi_\epsilon(s, x, v) = \psi_\epsilon(s, x, v) e^{\int_0^s \Sigma(\tau) \, d\tau},
\]

where \( \psi_\epsilon \in L^\infty(0, \infty; L^2(\mathbb{R}^n \times S^{n-1})) \) is the solution to (1). Then there exists \( \Psi_0 \in L^2(\mathbb{R}^n) \) which only depends on \( x \) such that

\[
\Psi_\epsilon \to \Psi_0 \quad \text{in } w^* - L^\infty(0, \infty; L^2(\mathbb{R}^n \times S^{n-1})).
\]

Moreover, with the following choices of \( \theta(\epsilon) \), the limit \( \Psi_0 \) satisfies the (fractional) diffusion equation

(i) \( D_1(-\Delta)\Psi_0 + (1 - c)\Psi_0 = \int_{S^{n-1}} Q(x, v) \, dv \) if \( \alpha > 2 \) and \( \theta(\epsilon) = \epsilon^2 \);

(ii) \( D_2(-\Delta)^{\alpha/2}\Psi_0 + (1 - c)\Psi_0 = \int_{S^{n-1}} Q(x, v) \, dv \) if \( 1 < \alpha < 2 \) and \( \theta(\epsilon) = \epsilon^\alpha \);

(iii) \( D_3(-\Delta)\Psi_0 + (1 - c)\Psi_0 = \int_{S^{n-1}} Q(x, v) \, dv \) if \( \alpha = 2 \) and \( \theta(\epsilon) = \epsilon^2 |\ln \epsilon| \),

where the positive coefficients \( D_1, D_2, D_3 \) can be explicitly computed from \( p \) and \( \sigma \).

**Remark 1.** The complete main theorem will be stated in Theorem 4.5, where \( D_i \)'s are specified and additional convergences are obtained.

**Remark 2.** We only consider the case where \( \alpha > 1 \), so as to obtain a result in the original unknown \( \psi_\epsilon \) as well. Note that

\[
\int_0^\infty e^{\int_0^s \Sigma(\tau) \, d\tau} \, ds = \int_0^\infty sp(s) \, ds.
\]

Thus the transformation (3) is only sensible in the case where \( \int_0^\infty sp(s) \, ds < \infty \).

3. **Well-posedness.** In this section we establish the well-posedness of the non-classical transport equation in the spaces \( L^\infty(0, \infty; L^r(\mathbb{R}^n \times S^{n-1})) \) for any \( 1 \leq r \leq \infty \). This can be done either by applying the iteration method used in [10] or by using a fixed-point argument. Here we adopt the latter method.

First, we further re-formulate equation (4) using characteristics. This gives

\[
\Psi_\epsilon(s, x, v) = \int_0^\infty \int_{S^{n-1}} (\sigma(v \cdot v' - \theta(\epsilon)(1 - c)) \, p(\tau) \Psi_\epsilon(\tau, x - \epsilon vs, v') \, dv' \, d\tau
\]

\[
+ \theta(\epsilon)Q(x - \epsilon vs, v).
\]

It is this last formulation that we will use to carry out our analysis in this paper. The well-posedness result is

**Theorem 3.1.** Suppose the scattering coefficient \( c \) and the cross section \( \sigma \) satisfy the conditions

\[
0 < c < 1, \quad \int_{S^{n-1}} \sigma(v \cdot v') \, dv = 1, \quad \sigma(v \cdot v') \geq \sigma_0 > 0
\]
for some constant $\sigma_0 > 0$. Suppose the path-length distribution function $p$ and the source term satisfy

$$\int_0^\infty p(s) \, ds = 1, \quad Q \in L^r(\mathbb{R}^n \times S^{n-1}) \quad \text{for any } 1 \leq r \leq \infty. \quad (9)$$

Then for each fixed $\epsilon > 0$ small enough such that $\sigma - \theta(\epsilon)(1-c) \geq 0$, equation (4) has a unique solution $\Psi_\epsilon \in L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))$ in the sense of (7). Moreover, $\Psi_\epsilon$ satisfies the uniform-in-$\epsilon$ bound

$$\|\Psi_\epsilon\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))} \leq \frac{1}{1-c} \|Q\|_{L^r(\mathbb{R}^n \times S^{n-1})}, \quad 1 \leq r \leq \infty. \quad (10)$$

Furthermore, if $Q \geq 0$, then $\Psi_\epsilon \geq 0$.

Before proving Theorem 3.1, we state a simple lemma that will be used frequently in this paper:

**Lemma 3.2.** Suppose $c, \sigma, p$ satisfy (8)-(9) and $u \in L^\infty(0, \infty ; L^r(\mathbb{R}^n \times S^{n-1}))$ for any $1 \leq r \leq \infty$. Then

$$\left\| \int_0^\infty \int_{S^{n-1}} (\sigma(v \cdot v') - \theta(\epsilon)(1-c)) p(\tau) u(\tau, x - \epsilon v, v') \, dv' \, d\tau \right\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))} \leq (1 - \theta(\epsilon)(1-c)) \|u\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))} \quad (11)$$

for any $\epsilon > 0$. In the limit case $c = 1$ we have

$$\left\| \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(\tau) u(\tau, x - vs, v') \, dv' \, d\tau \right\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))} \leq \|u\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))}. \quad (12)$$

**Proof.** This result follows directly from the Minkowski and H"{o}lder inequalities. Denote

$$\tilde{\sigma} = \sigma - \theta(\epsilon)(1-c) \geq 0.$$

The case $r = \infty$ follows directly from the normalization conditions for $\sigma$ and $p$. If $1 \leq r < \infty$, then integrating in $x$ gives

$$\left\| \int_0^\infty \int_{S^{n-1}} \tilde{\sigma}(v \cdot v') p(\tau) u(\tau, x - \epsilon v, v') \, dv' \, d\tau \right\|_{L^r(\mathbb{R}^n)} \leq \int_0^\infty \int_{S^{n-1}} \tilde{\sigma}(v \cdot v') p(\tau) \|u(\tau, \cdot, v')\|_{L^r(\mathbb{R}^n)} \, dv' \, d\tau \leq \left[ \int_0^\infty \int_{S^{n-1}} \tilde{\sigma}(v \cdot v') p(\tau) \|u(\tau, \cdot, v')\|_{L^r(\mathbb{R}^n)}^r \, dv' \, d\tau \right]^{1/r} \left( \int_0^\infty \int_{S^{n-1}} \tilde{\sigma}(v \cdot v') p(\tau) \|u(\tau, \cdot, v')\|_{L^r(\mathbb{R}^n)}^r \, dv' \, d\tau \right)^{1/r}$$

where $\frac{1}{r'} + \frac{1}{r} = 1$. Then by integrating in $v$ we get

$$\left\| \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(\tau) u(\tau, x - \epsilon v, v') \, dv' \, d\tau \right\|_{L^r(\mathbb{R}^n \times S^{n-1})} \leq (1 - \theta(\epsilon)(1-c)) \|u\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))}, \quad (11)$$

$$\left\| \int_0^\infty \int_{S^{n-1}} \tilde{\sigma}(v \cdot v') p(\tau) u(\tau, x - \epsilon v, v') \, dv' \, d\tau \right\|_{L^r(\mathbb{R}^n)} \leq (1 - \theta(\epsilon)(1-c)) \|u\|_{L^\infty((0, \infty) ; L^r(\mathbb{R}^n \times S^{n-1}))}. \quad (12)$$
which gives the desired bound.

Now we proceed to prove Theorem 3.1.

Proof of Theorem 3.1. For each fixed $\epsilon > 0$ small enough such that $1 - \theta(\epsilon)(1 - c) \geq 0$, define the operator $T$ on $L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ as

$$
Tu = \int_0^\infty \int_{\mathbb{S}^{n-1}} (\sigma(v \cdot v') - \theta(\epsilon)(1 - c)) p(\tau) u(\tau, x - \epsilon vs, v') \, dv' \, d\tau \\
+ \theta(\epsilon)Q(x - \epsilon vs, v).
$$

We will show that $T$ maps $L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ into itself and $T$ is a contraction mapping.

First, the source term in $T$ satisfies

$$
\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} Q(x - \epsilon vs, v)^r \, dx \, dv = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} (Q(x, v))^r \, dx \, dv.
$$

(13)

Applying Lemma 3.2 to the integral term in $T$, we have

$$
\|Tu\|_{L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))} \leq (1 - \theta(\epsilon)(1 - c)) \|u\|_{L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))} \\
+ \theta(\epsilon) \|Q\|_{L^r(\mathbb{R}^n \times \mathbb{S}^{n-1})} < \infty.
$$

(14)

Hence $T$ maps $L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ into itself.

Next, we show that $T$ is a contraction mapping. To this end, let

$$
u, v \in L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1})).$$

Applying (3.2) to $u - v$, we get

$$
\|Tu - Tv\|_{L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))} \leq (1 - \theta(\epsilon)(1 - c)) \|u - v\|_{L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))},
$$

which shows that $T$ is a contraction mapping since the coefficient satisfies $1 - \theta(\epsilon)(1 - c) < 1$ for each fixed $\epsilon > 0$. We can now apply the fixed-point theorem to conclude that equation (7) has a unique solution $\Psi_\epsilon \in L^\infty(0, \infty; L^r(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ for any $1 \leq r \leq \infty$.

The uniform bound of $\Psi_\epsilon$ in (10) follows directly from (14) with $Tu$ and $u$ in the inequality being replaced by $\Psi_\epsilon$. The positivity of $\Psi_\epsilon$ can be obtained by noting that the mapping $T$ preserves positivity if $Q \geq 0$. Hence the unique solution obtained through iterations in the fixed-point argument must be non-negative. \qed

4. Passing to the Limit. In this section we show the limit of (7) as $\epsilon \to 0$. Roughly speaking, the main result is to recover a regular or fractional diffusion equation in the limit for the quantity $\int_{(0, \infty) \times \mathbb{S}^{n-1}} \psi_\epsilon(s, x, v) \, dv \, ds$. Throughout this section, we assume that $Q \in L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Then by Theorem 3.1, the solution $\Psi_\epsilon$ is uniformly bounded in $L^\infty(0, \infty; L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ with bounds satisfying (10) for $r = 1, 2$.

First, we show the convergence of $\Psi_\epsilon$ as $\epsilon \to 0$.

Theorem 4.1. Let $Q \in L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Then there exists a subsequence $\Psi_{\epsilon_k}$ and $\Psi_0 = \Psi_0(x)$ such that

$$
\Psi_{\epsilon_k} \to \Psi_0 \quad \text{in } w^* \to L^\infty(0, \infty; L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})).
$$

(15)
Our main goal is to prove that $\Psi_0$ holds. What remains to show is that the limiting function $\Psi$ of $\Psi$ is independent of $s,v$. Thus, we have

$$
\Psi_0(s,x,v) = \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(\tau) \Psi_0(\tau,x,v') \, dv' \, d\tau.
$$

Proof. The convergence of a subsequence of $\Psi_0$ is guaranteed by the uniform bound of $\Psi_0$ in (10). Therefore there exists $\Psi_0 \in L^\infty(0, \infty; L^2(\mathbb{R}^n \times S^{n-1}))$ such that (15) holds. What remains to show is that the limiting function $\Psi_0$ is independent of $s,v$. Hence $\Psi_0$ is independent of $s$ and equation (16) is reduced to

$$
\Psi_0(x,v) = \int_{S^{n-1}} \sigma(v \cdot v') \Psi_0(x,v') \, dv',
$$

where $\sigma \geq \sigma_0 > 0$ is again a probability measure. Since $\Psi_0 \in L^2(\mathbb{R}^n \times S^{n-1})$, we can take the $L^2$-norm on both sides and obtain

$$
\left\| \Psi_0 \right\|_{L^2(\mathbb{R}^n \times S^{n-1})}^2 = \int_{\mathbb{R}^n} \left| \int_{S^{n-1}} \sigma(v \cdot v') \Psi_0(x,v') \, dv' \right|^2 \, dx.
$$

Therefore the inequality in the above estimate must be an equality. This means the equality holds when using the H"older inequality for the integral of the product of $\sqrt{\sigma}$ and $\sqrt{\sigma'}$. This is only true when there exists $\lambda$ independent of $v'$ such that

$$
\sqrt{\sigma(v \cdot v')} = \lambda \sqrt{\sigma(v' \cdot v')} \Psi_0(x,v'),
$$

which implies that $\Psi_0(x,v')$ is independent of $v'$.

In order to show (16), we recall that $\Psi_0$ satisfies

$$
\Psi_0(s,x+\epsilon sv,v) = \int_0^\infty \int_{S^{n-1}} (\sigma(v \cdot v') - \theta(\epsilon)(1-c)) p(\tau) \Psi_0(\tau,x,v') \, dv' \, d\tau
$$

$$
+ \theta(\epsilon) Q(x,v).
$$

We will prove that (17) converges to (16) as $\epsilon \to 0$ in the sense of distributions. First we study the convergence of the right-hand side of (17). Note that the right-hand side of (17) is independent of $s$. Hence any convergence is uniform in $s$. The terms associated with $\theta(\epsilon)$ satisfy

$$
\theta(\epsilon) Q \to 0, \quad \theta(\epsilon)(1-c) \int_0^\infty \int_{S^{n-1}} p(\tau) \Psi_0(\tau,x,v') \, dv' \, d\tau \to 0.
$$
in \( L^\infty(0, \infty; L^2(\mathbb{R}^n \times S^{n-1})) \) by the uniform bound in (10) and (12). Next, for any \( h_1(x, v) \in L^2(\mathbb{R}^n \times S^{n-1}) \),
\[
\int_{\mathbb{R}^n} \int_{S^{n-1}} h_1(x, v) \left( \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(\tau, x, v') \, dv' \, d\tau \right) \, dv \, dx \\
= \int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h_1(x, v) \sigma(v \cdot v') \, dv \, p(\tau, x, v') \, dv' \, d\tau \, dx \\
\to \int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h_1(x, v) \sigma(v \cdot v') \, dv \, p(\tau, x, v') \, dv' \, d\tau \, dx 
\]
where the last step follows from (15) and Lemma 3.2, since by (12) (with \( s = 0 \)) we have
\[
\left( \int_{S^{n-1}} h_1(x, v) \sigma(v \cdot v') \, dv' \right) p(\tau) \in L^1(0, \infty; L^2(\,dx \,dv)) .
\]
Therefore as \( \epsilon \to 0 \),
\[
\text{RHS of (17)} \to \text{RHS of (16) weakly in } L^2(\mathbb{R}^n \times S^{n-1}). 
\]

Next we show the convergence of the left-hand side of (17). To this end, let \( h_2 \in C_\infty^0((0, \infty) \times \mathbb{R}^n \times S^{n-1}) \). Then the left-hand side term satisfies
\[
\int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h_2(s, x, v) \Psi_\epsilon(s, x + \epsilon \omega, v) \, dv \, dx \, ds \\
= \int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h_2(s, x, v) \Psi_\epsilon(s, x, v) \, dv \, dx \, ds \\
+ \int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h_2(s, x, v) \left( \Psi_\epsilon(s, x + \epsilon \omega, v) - \Psi_\epsilon(s, x, v) \right) \, dv \, dx \, ds \\
\triangleq I_{1, \epsilon} + I_{2, \epsilon} .
\]

By (15), the limit of \( I_{1, \epsilon} \) is
\[
I_{1, \epsilon} \to \int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h_2(s, x, v) \Psi_0(s, x, v) \, dv \, dx \, ds .
\]

Denote the compact support of \( h_2 \) as \( \Omega \). Then the limit of \( I_{2, \epsilon} \) is
\[
|I_{2, \epsilon}| = \left| \int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} (h_2(s, x - \epsilon \omega, v) - h_2(s, x, v)) \Psi_\epsilon(s, x, v) \, dv \, dx \, ds \right| \\
\leq \epsilon C_1(\Omega) \| h_2 \|_{C^1(\Omega)} \int_{\Omega} | \Psi_\epsilon(s, x, v) | \, dv \, dx \, ds \\
\leq \epsilon C_2(\Omega) \| h_2 \|_{C^1(\Omega)} \| \Psi_\epsilon \|_{L^\infty(0, \infty; L^2(\mathbb{R}^n \times S^{n-1}))} \to 0 \quad \text{as } \epsilon \to 0 .
\]

Combining (20) and (21), we have
\[
\text{LHS of (17)} \to \Psi_0 \quad \text{in } \mathcal{D}' .
\]
The limiting equation (16) hence follows from (19) and (22).

Next, we study the convergence of averages of \( \Psi_\epsilon \). To this end, we apply the Fourier transform in \( x \) to (7) for a.e. \( s, v \). This gives
\[
\tilde{\Psi}_\epsilon(s, \xi, v') = \left( \int_0^\infty \int_{S^{n-1}} (\sigma(v' \cdot \bar{v}) - \theta(\epsilon)(1 - c)) \, p(\tau) \tilde{\Psi}_\epsilon(\tau, \xi, \bar{v}) \, d\tau \right) e^{-iv' \cdot \xi s} \\
+ \theta(\epsilon) \tilde{Q}(\xi, v') e^{-iv' \cdot \xi s} ,
\]
(23)
where $\xi$ is the Fourier variable and $\hat{u}$ denotes the Fourier transform in $x$ of $u$. The free velocity variable is changed from $v$ to $v'$ for later notational convenience. Note that switching the order of integration on the right-hand side of (7) when applying the Fourier transform is valid. Indeed, denote

$$w_1(v') = \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{S^{n-1}} \sigma(v' \cdot \tilde{v}) \psi_\epsilon(\tau, x, \tilde{v}) d\tilde{v} d\tau \right) e^{-ix \cdot \xi} dx ,$$

$$w_2(v') = \int_0^\infty \int_{S^{n-1}} \sigma(v' \cdot \tilde{v}) \psi_\epsilon(\tau, x, \tilde{v}) e^{-ix \cdot \xi} d\tilde{v} d\tau .$$

By (12) with $s = 0$, we have $w_1, w_2 \in L^1(S^{n-1})$. Moreover, for any function $\phi_1 \in L^\infty(S^{n-1})$,

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} \int_0^\infty \int_{S^{n-1}} \sigma(v' \cdot \tilde{v}) \psi_\epsilon(\tau, x, \tilde{v}) \phi_1(v') d\tilde{v} d\tau dx d\bar{v} \leq \|\phi_1\|_{L^\infty(S^{n-1})} \|\psi_\epsilon\|_{L^\infty((0,\infty); L^1(\mathbb{R}^{n} \times S^{n-1}))} < \infty .$$

Hence, by Fubini’s Theorem, it holds that

$$\int_{S^{n-1}} w_1(v') \phi_1(v') d\tilde{v} = \int_{S^{n-1}} w_2(v') \phi_1(v') d\tilde{v}$$

for any $\phi_1 \in L^\infty(S^{n-1})$. This shows $w_1 = w_2$ and (23) is valid.

Hinted by (23), we consider the following averaged quantity of $\hat{\psi}_\epsilon$:

$$\hat{\phi}_\epsilon(\xi, v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \sigma(v \cdot \tilde{v}) p(s) \hat{\psi}_\epsilon(s, \xi, \tilde{v}) d\bar{v} ds . \tag{24}$$

We will first show that the limit of the velocity average of $\hat{\phi}_\epsilon$ satisfies a diffusion equation. The equation for $\hat{\phi}_\epsilon$ is derived by multiplying $\sigma(v \cdot v') p(s)$ to equation (23) and integrating in $s, v'$. It has the form

$$\hat{\phi}_\epsilon(\xi, v) = \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(s) \hat{\phi}_\epsilon(\xi, v') e^{-iv' \cdot \xi} d\tilde{v} ds$$

$$\quad - \theta(\epsilon)(1 - c) \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(s) \psi_\epsilon(\tau, \xi, \tilde{v}) e^{-iv' \cdot \xi} d\tilde{v} d\tau d\tilde{v} ds \tag{25}$$

$$\quad + \theta(\epsilon) \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(s) \hat{Q}(\xi, v') e^{-iv' \cdot \xi} d\tilde{v} ds .$$

For the ease of notation, we introduce the operator $\mathcal{K}_\epsilon$ and the terms $A_\epsilon \hat{\psi}_\epsilon, \hat{q}_\epsilon$ as

$$\mathcal{K}_\epsilon \hat{u} = \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(s) \hat{u}(\xi, v') e^{-iv' \cdot \xi} d\tilde{v} ds , \quad \hat{q}_\epsilon = \mathcal{K}_\epsilon \hat{Q} . \tag{26}$$

$$A_\epsilon \hat{\psi}_\epsilon = \int_0^\infty \int_{S^{n-1}} \int_0^\infty \int_{S^{n-1}} \sigma(v \cdot v') p(s) \psi_\epsilon(\tau, \xi, \tilde{v}) e^{-iv' \cdot \xi} d\tilde{v} d\tau d\tilde{v} ds . \tag{27}$$

Then (25) becomes

$$\hat{\phi}_\epsilon = \mathcal{K}_\epsilon \hat{\phi}_\epsilon - \theta(\epsilon)(1 - c) A_\epsilon \hat{\psi}_\epsilon + \theta(\epsilon) \hat{q}_\epsilon . \tag{28}$$

We can further write (28) as

$$\frac{1}{\theta(\epsilon)} (\hat{\phi}_\epsilon - \mathcal{K}_\epsilon \hat{\phi}_\epsilon) + (1 - c) A_\epsilon \hat{\psi}_\epsilon = \hat{q}_\epsilon . \tag{29}$$
To this end, let equation (7) hold.

Lemma 4.2. Suppose $\sigma, c, p$ satisfy the conditions in Theorem 3.1. Suppose $Q \in L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and $\Psi_\epsilon \in L^\infty(0, \infty; L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ is the solution to equation (7). Then $\hat{\phi}_\epsilon \in L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Moreover, there exists a constant $C_1 > 0$ which only depends on $Q, c$ such that

\[
\|\hat{\phi}_\epsilon\|_{L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})} \leq C_1, \quad \|\mathcal{K}_\epsilon \hat{\phi}_\epsilon\|_{L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})} \leq C_1,
\]

\[
\|\mathcal{A}_\epsilon \hat{\Psi}_\epsilon\|_{L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})} \leq C_1, \quad \|\hat{q}_\epsilon\|_{L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})} \leq C_1.
\]

Proof. These are all direct consequences of the inequality (12) in Lemma 3.2 and Parseval’s identity.

We can now derive the limit of $\hat{\phi}_\epsilon$ along a subsequence.

Lemma 4.3. Suppose $Q \in L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and $\Psi_\epsilon \in L^\infty(0, \infty; L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1}))$ is the solution to equation (7). Let $\Psi_0$ be the limit defined in Theorem 4.1. Then there exists a subsequence $\hat{\phi}_{\epsilon k}$ such that $\hat{\phi}_{\epsilon k} \rightarrow \hat{\Psi}_0$ weakly in $L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$.

As a consequence, $\langle \hat{\phi}_{\epsilon k} \rangle \rightarrow \hat{\Psi}_0$ weakly in $L^2(\mathbb{R}^n)$. Here $\langle \hat{\phi}_{\epsilon k} \rangle$ is defined as the velocity average

\[
\langle \hat{\phi}_{\epsilon k} \rangle = \int_{\mathbb{S}^{n-1}} \hat{\phi}_{\epsilon k}(\xi, v) \, dv.
\]

Proof. The convergence (along a subsequence) of $\hat{\phi}_\epsilon$ follows from the uniform bound of $\hat{\phi}_\epsilon$. Suppose $\hat{\phi}_{\epsilon k} \rightarrow \hat{\phi}_0$ in $L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$. We want to show that $\hat{\phi}_0 = \hat{\Psi}_0$. To this end, let $h_1(v)h_2(\xi)$ be any test function such that $h_1 \in C^\infty(\mathbb{S}^{n-1})$ and $h_2 \in C^\infty_c(\mathbb{R}^n)$. Then

\[
\int_{\mathbb{S}^{n-1}} \sigma(v \cdot \bar{v})h_1(v) \, dv \in L^2(\mathbb{S}^{n-1})
\]

and

\[
p(\cdot)h_2(\cdot) \left( \int_{\mathbb{S}^{n-1}} \sigma(v \cdot \bar{v})h_1(v) \, dv \right) \in L^1 \left( 0, \infty; L^2(\mathbb{R}^n \times \mathbb{S}^{n-1}) \right).
\]

Hence,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{S}^{n-1}} \sigma(v \cdot \bar{v})h_1(v) \, dv \right) h_2(\xi)p(s) \hat{\Psi}_\epsilon \\
\rightarrow \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{S}^{n-1}} \sigma(v \cdot \bar{v})h_1(v) \, dv \right) h_2(\xi)p(s) \hat{\Psi}_0,
\]

which implies that $\hat{\phi}_{\epsilon k} \rightarrow \hat{\Psi}_0$ in $\mathcal{D}'$. By the uniqueness of the limit, we have $\hat{\phi}_{\epsilon k} \rightarrow \hat{\Psi}_0$ in $L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$.

Define the operators $\mathcal{L}_\epsilon, \mathcal{K}, \mathcal{L}$ as

\[
\mathcal{K}\hat{u} = \int_{\mathbb{S}^{n-1}} \sigma(v \cdot v')\hat{u}(v') \, dv', \quad \mathcal{L}_\epsilon = \mathcal{I} - \mathcal{K}_\epsilon, \quad \mathcal{L} = \mathcal{I} - \mathcal{K}, \quad (30)
\]
where $I$ is the identity operator. Then (28) can be further reformulated as

$$
\frac{1}{\theta(\epsilon)} L \tilde{\phi}_\epsilon - \int_0^\infty \int_{\mathbb{S}^{n-1}} \sigma(v \cdot v') p(s) \tilde{\phi}_\epsilon(\xi, v') e^{-i ev' \cdot \xi s} - \frac{1}{\theta(\epsilon)} \, dv' \, ds = -(1 - e) \Lambda \hat{\Psi}_\epsilon + \tilde{q}_\epsilon. 
$$

(31)

We summarize some properties of $L$ in the following lemma:

**Lemma 4.4.** Let $L, \mathcal{K}$ be defined as in (30). Then

1. $\mathcal{K} : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ is compact and $L$ is Fredholm.
2. $\text{Null } \mathcal{L} = \text{span} \{1\}.$
3. Fix $\xi \in \mathbb{R}^n$. Then $v \cdot \xi \in (\text{Null } \mathcal{L})^\perp$ is an eigenfunction of $L$ with eigenvalue $1 - \bar{\mu}_0$. Here,

$$
\bar{\mu}_0 = \frac{1}{2} \int_{-1}^1 \mu \sigma(\mu) \, d\mu < 1
$$

is the average scattering cosine.

**Proof.** Part (1) are classical results regarding transport equations [3]. Part (2) is essentially shown in the proof of Theorem 4.1. Part (3) follows from a direct calculation. Indeed, by a symmetry argument

$$
\mathcal{K}(v \cdot \xi) = \int_{\mathbb{S}^{n-1}} \sigma(v \cdot v')(v' \cdot \xi) \, dv' = \left( \int_{\mathbb{S}^{n-1}} \sigma(v \cdot v')(v' \cdot v) \, dv' \right) (v \cdot \xi)
= \left( \frac{1}{2} \int_{-1}^1 \mu \sigma(\mu) \, d\mu \right) (v \cdot \xi). 
$$

(32)

Therefore $v \cdot \xi$ is an eigenfunction of $L$ with the associated eigenvalue $1 - \bar{\mu}_0$. \(\square\)

In the rest of the section we will find the macroscopic equation that $\hat{\Psi}_0$ satisfies by passing to the limit in (31). The main result builds upon the key estimates summarized in the following proposition:

**Proposition 1.** Suppose $p$ satisfies

$$
\int_0^\infty p(s) \, ds = 1, \quad \int_0^\infty sp(s) \, ds < \infty, \quad p(s) = \frac{d_0}{s^{\alpha+1}} \quad \text{for } s > 1, 
$$

(33)

where $d_0 > 0$ is a constant. For any given $v \in \mathbb{S}^{n-1}$ and $\xi \in \mathbb{R}^n$, define $\Lambda_\epsilon, \Gamma_\epsilon$ as

$$
\Lambda_\epsilon(\xi, v) = \int_0^\infty p(s) \frac{1 - \cos(\epsilon v \cdot \xi s)}{\theta(\epsilon)} \, ds, 
$$

$$
\Gamma_\epsilon(\xi, v) = \frac{1}{\theta(\epsilon)} \int_0^\infty p(s) \epsilon v \cdot \xi s \left( 1 - \frac{\sin(\epsilon v \cdot \xi s)}{\epsilon v \cdot \xi s} \right) \, ds, 
$$

Then there exists a generic constant $c_0 > 0$ which is independent of $\epsilon, v', \xi$ such that

(a1) if $\alpha > 2$ and we choose $\theta(\epsilon) = \epsilon^2$, then $|\Lambda_\epsilon| + |\Gamma_\epsilon| \leq c_0 |\xi|^2$;

(b1) if $1 < \alpha < 2$ and we choose $\theta(\epsilon) = \epsilon^\alpha$, then $|\Lambda_\epsilon| + |\Gamma_\epsilon| \leq c_0 |\xi|^\alpha$;

(c1) if $\alpha = 2$ and we choose $\theta(\epsilon) = \epsilon^2 |\ln\epsilon|$, then $|\Lambda_\epsilon| + |\Gamma_\epsilon| \leq c_0 (|\xi|^2 + 1)$.

Moreover, with $\bar{D}_1, \bar{D}_2, \bar{D}_3 > 0$ and $\bar{D}_1', \bar{D}_2'$ (that depend on $v \cdot \epsilon \xi$) defined below we have

(a2) if $\alpha > 2$ and we choose $\theta(\epsilon) = \epsilon^2$, then

$$
\lim_{\epsilon \to 0} \Lambda_\epsilon = \bar{D}_1 |\xi|^2 = \left( \frac{D_0}{2} |v \cdot \epsilon \xi|^2 \right) |\xi|^2, \quad \lim_{\epsilon \to 0} \Gamma_\epsilon = 0,
$$

where $D_0 = \int_0^\infty p(s)s^2 \, ds$ and $\epsilon \xi = \xi/|\xi|$. 

(b2) If $1 < \alpha < 2$ and we choose $\theta(\epsilon) = \epsilon^\alpha$, then
\[
\lim_{\epsilon \to 0} \Lambda_\epsilon = \tilde{D}_2|\xi|^\alpha = \left( \frac{d_0}{2} \int_0^\infty \frac{\sin^2 \tau}{\tau^{1+\alpha}} \, d\tau \right) |v \cdot e_\xi|^\alpha |\xi|^\alpha ,
\]
and
\[
\lim_{\epsilon \to 0} \Gamma_\epsilon = \tilde{D}_2'(|\xi|^\alpha = \left( \frac{d_0}{2} \int_0^\infty \frac{1}{\tau^\alpha} \left( 1 - \frac{\sin \tau}{\tau} \right) \, d\tau \right) |v \cdot \xi|^\alpha sgn(v \cdot \xi) .
\]
(c2) If $\alpha = 2$ and we choose $\theta(\epsilon) = -\epsilon^2 \ln \epsilon$, then
\[
\lim_{\epsilon \to 0} \Lambda_\epsilon = \tilde{D}_3|\xi|^2 = \frac{1}{2} |v \cdot e_\xi|^2 |\xi|^2 , \quad \lim_{\epsilon \to 0} \Gamma_\epsilon = 0 .
\]
The above convergences are all pointwise in $\xi, v$.

**Proof.** We will show these estimates case by case.

**Case (a1)-(a2)** First we consider $\Lambda_\epsilon$. Suppose $\alpha > 2$ and let $\theta(\epsilon) = \epsilon^2$. This is the small-tail case where
\[
D_0 \triangleq \int_0^\infty p(s)s^2 \, ds < \infty . \tag{34}
\]
In this case, for each fixed $v \in \mathbb{S}^{n-1}$ and $\xi \in \mathbb{R}^n$, the integrand of $\Lambda_\epsilon$ satisfies
\[
0 \leq p(s) \frac{1 - \cos(\epsilon v \cdot \xi s)}{\epsilon^2} = p(s) \frac{1 - \cos(\epsilon v \cdot \xi s)}{\epsilon^2} \leq 2|\xi|^2 s^2 p(s) \in L^1(0, \infty) . \tag{35}
\]
Therefore,
\[
|\Lambda_\epsilon| = \int_0^\infty p(s) \frac{1 - \cos(\epsilon v \cdot \xi s)}{\epsilon^2} \, ds \leq 2|\xi|^2 \int_0^\infty s^2 p(s) \, ds = 2|\xi|^2 D_0 .
\]
In addition, by Lebesgue’s Dominated Convergence Theorem, we have
\[
\lim_{\epsilon \to 0} \Lambda_\epsilon = 2 \int_0^\infty p(s) \lim_{\epsilon \to 0} \left( \frac{\sin^2 \left( \frac{\epsilon v \cdot \xi s}{\epsilon^2} \right)}{\theta(\epsilon)} \right) \, ds \, dv = \frac{|v \cdot \xi|^2}{2} D_0 = \tilde{D}_1|\xi|^2 , \tag{36}
\]
where $\tilde{D}_1$ is defined as
\[
\tilde{D}_1(v, e_\xi) = \frac{1}{2} |v \cdot e_\xi|^2 D_0 , \quad e_\xi = \xi/|\xi| . \tag{37}
\]
To show the bound of $\Gamma_\epsilon$, we separate the integration into two regions: $s > 1$ and $s < 1$. In the region $s < 1$, we further consider two cases: $\epsilon |v \cdot \xi| > 1$ and $\epsilon |v \cdot \xi| \leq 1$. In the first case, we have
\[
\frac{1}{\epsilon^2} \int_0^1 p(s) |v \cdot \xi s - \sin (\epsilon v \cdot \xi s)| \, ds \leq \frac{2}{\epsilon^2} \int_0^1 p(s) s \epsilon |v \cdot \xi| \, ds \\
\leq \frac{2}{\epsilon^2} (\epsilon |v \cdot \xi|)^2 \int_0^\infty p(s) s \, ds \leq c_0|\xi|^2 .
\]
In the latter case where $\epsilon |v \cdot \xi| \leq 1$, we have
\[
\frac{1}{\epsilon^2} \int_0^1 p(s) |v \cdot \xi s - \sin (\epsilon v \cdot \xi s)| \, ds \leq \frac{1}{6 \epsilon^2} \int_0^1 p(s) |v \cdot \xi|^3 \, ds \\
\leq \epsilon |v \cdot \xi|^3 \int_0^1 p(s) \, ds \leq \epsilon |v \cdot \xi|^3 \leq |\xi|^2 .
\]
Combining these two cases, we have
\[
\frac{1}{\epsilon^2} \int_0^1 p(s) |e v \cdot \xi s - \sin (e v \cdot \xi s)| \, ds \leq c_0 |\xi|^2. \tag{38}
\]

Note that if we do not constrain ourselves to the $|\xi|^2$-upper bound, then we can simply derive that
\[
\frac{1}{\epsilon^2} \int_0^1 p(s) |e v \cdot \xi s - \sin (e v \cdot \xi s)| \, ds \leq \frac{1}{6 \epsilon^2} \int_0^1 p(s) |e v \cdot \xi s|^3 \, ds \leq \epsilon |\xi|^3 \int_0^1 p(s) \, ds \leq \epsilon |\xi|^3. \tag{39}
\]

To estimate the integral over $s > 1$, we make the change of variable $\tau = \epsilon |v \cdot \xi| s$. Then,
\[
\left| \frac{1}{\epsilon^2} \int_1^\infty p(s) (e v \cdot \xi s) \left( 1 - \sin \left( \frac{e v \cdot \xi s}{e v \cdot \xi s} \right) \right) \, ds \right| \leq d_0 \frac{1}{\epsilon^2} \int_1^\infty \frac{1}{s^3} |e v \cdot \xi s| \left( 1 - \sin \left( \frac{|e v \cdot \xi s|}{|e v \cdot \xi s|} \right) \right) \, ds \leq d_0 |v \cdot \xi|^2 \int_1^{\epsilon |v \cdot \xi|} \frac{1}{\tau^2} \left( 1 - \sin \left( \frac{\tau}{\tau} \right) \right) \, d\tau. \tag{41}
\]

Note that
\[
\int_{\epsilon |v \cdot \xi|}^{\infty} \frac{1}{\tau^2} \left( 1 - \sin \left( \frac{\tau}{\tau} \right) \right) \, d\tau \leq \int_0^{\infty} \frac{1}{\tau^2} \left( 1 - \sin \left( \frac{\tau}{\tau} \right) \right) \, d\tau < \infty.
\]

We thereby obtain the bound for the integration over $s > 1$ as
\[
\frac{1}{\epsilon^2} \left| \int_1^\infty p(s) (e v \cdot \xi s) \left( 1 - \sin \left( \frac{e v \cdot \xi s}{e v \cdot \xi s} \right) \right) \, ds \right| \leq c_0 |v \cdot \xi|^2 \leq c_0 |\xi|^2. \tag{42}
\]

The bound for $\Gamma_\epsilon$ then follows from combining (38) with (42).

To show the limit of $\Gamma_\epsilon$ as $\epsilon \to 0$, we apply the same changes of variable $\tau = \epsilon |v \cdot \xi| s$. Then
\[
\left| \frac{1}{\epsilon^2} \int_1^\infty p(s) (e v \cdot \xi s) \left( 1 - \sin \left( \frac{e v \cdot \xi s}{e v \cdot \xi s} \right) \right) \, ds \right| \leq \frac{d_0}{\epsilon^2} \int_1^\infty \frac{1}{s^{\alpha+1}} |e v \cdot \xi s| \left( 1 - \sin \left( \frac{|e v \cdot \xi s|}{|e v \cdot \xi s|} \right) \right) \, ds = d_0 |v \cdot \xi|^{\alpha-1} \int_{\epsilon |v \cdot \xi|}^{\infty} \frac{1}{\tau^2} \left( 1 - \sin \left( \frac{\tau}{\tau} \right) \right) \, d\tau \leq c_0 |\xi|^{\alpha-2} \left( 1 + \int_{\epsilon |v \cdot \xi|}^{1} \frac{1}{\tau^{\alpha+2}} \, d\tau \right).
\]

This shows if $\alpha \neq 3$, then
\[
|\Gamma_\epsilon| \leq c_0 |\xi|^{\alpha-2} \left( 1 + \epsilon^{-\alpha+3} \right) \to 0, \quad \text{as } \epsilon \to 0.
\]

In the case when $\alpha = 3$, we have
\[
|\Gamma_\epsilon| \leq c_0 |\xi|^{\alpha-2} \left( 1 + |\ln \epsilon| + |\ln |\xi|| \right) \to 0, \quad \text{as } \epsilon \to 0.
\]

Overall, we have $\Gamma_\epsilon \to 0$ as $\epsilon \to 0$ when $\alpha > 2$. 
Case (b1)-(b2) Now suppose $1 < \alpha < 2$. In this case we take $\theta(\epsilon) = e^\alpha$. Similar as for Case (a1)-(a2), we break the integration domain in $\Lambda_{\epsilon}$ into $s > 1$ and $s < 1$ and let
\[
\Lambda_{\epsilon,1} = 2 \int_{s>1} p(s) \frac{\sin^2 \left( \frac{\epsilon |\xi| (v \cdot \xi s)}{2} \right)}{e^\alpha} \, ds, \quad \Lambda_{\epsilon,2} = 2 \int_{s<1} p(s) \frac{\sin^2 \left( \frac{\epsilon |\xi| (v \cdot \xi s)}{2} \right)}{e^\alpha} \, ds.
\]
The term $\Lambda_{\epsilon,2}$ is bounded as
\[
|\Lambda_{\epsilon,2}| \leq \frac{|\xi|^2 \epsilon^{2-\alpha}}{2} \int_{s<1} p(s) s^2 \, ds \, dv \leq \frac{|\xi|^2 \epsilon^{2-\alpha}}{2} \int_0^\infty p(s) \, ds = \frac{1}{2} |\xi|^2 \epsilon^{2-\alpha}. \tag{43}
\]
Since $1 < \alpha < 2$, we have
\[
\Lambda_{\epsilon,2} \to 0 \quad \text{as } \epsilon \to 0 \text{ for each fixed } v, \xi. \tag{44}
\]
To derive the bound of $\Lambda_{\epsilon,1}$, we apply the change of variable $\tau = \epsilon |\xi| s$. By the tail behaviour of $p(s)$ in (33),
\[
\Lambda_{\epsilon,1} = 2d_0 |\xi|^\alpha \int_{|\tau| > \epsilon |\xi|} \frac{\sin^2 \left( \frac{\tau (v \cdot \xi s)}{2} \right)}{\tau^{\alpha+1}} \, d\tau \, dv.
\]
Define
\[
\tilde{D}_2 \triangleq 2d_0 \int_0^\infty \sin^2 \left( \frac{\tau (v \cdot \xi s)}{2} \right) \frac{\tau^{\alpha}}{\tau^{\alpha+1}} \, d\tau. \tag{45}
\]
Then
\[
\tilde{D}_2 = \frac{d_0}{2} |v \cdot \xi|^\alpha \int_0^\infty \frac{\sin^2 \frac{\tau}{\tau^{1+\alpha}}} {\tau^{1+\alpha}} \, d\tau \leq \frac{d_0}{2} \int_0^\infty \frac{\sin^2 \frac{\tau}{\tau^{1+\alpha}}} {\tau^{1+\alpha}} \, d\tau < \infty.
\]
Therefore, we have
\[
0 \leq \Lambda_{\epsilon,1} \leq \tilde{D}_2 |\xi|^\alpha, \quad \Lambda_{\epsilon} \to \tilde{D}_2 |\xi|^\alpha \quad \text{as } \epsilon \to 0.
\]

The bound and limit for $\Gamma_{\epsilon}$ in Case (b1)-(b2) are derived in the same way as in Case (b1)-(b2) with $\theta(\epsilon) = e^\alpha$ changed to $\theta(\epsilon) = \epsilon^\alpha$. We thus omit the details. To derive the limit of $\Gamma_{\epsilon}$, we note that the integral over $0 < s < 1$ will approach zero as in (39). For the part where $s > 1$, we make the change of variable $\tau = \epsilon |v \cdot \xi| s$. Then,
\[
\frac{1}{\epsilon^\alpha} \int_1^\infty p(s) (ev \cdot \xi s) \left( 1 - \frac{\sin (ev \cdot \xi s)}{ev \cdot \xi s} \right) \, ds = \frac{d_0}{e^{\alpha}} \sgn(v \cdot \xi) \int_1^\infty \frac{1}{s^{1+\alpha}} |ev \cdot \xi s| \left( 1 - \frac{\sin (|ev \cdot \xi s|)}{|ev \cdot \xi s|} \right) \, ds = d_0 |v \cdot \xi|^\alpha \sgn(v \cdot \xi) \int_{|v \cdot \xi|}^\infty \frac{1}{\tau^{\alpha}} \left( 1 - \frac{\sin \frac{\tau}{\tau^{1+\alpha}}} {\tau^{1+\alpha}} \right) \, d\tau. \tag{46}
\]
Note that
\[
\int_{|v \cdot \xi|}^\infty \frac{1}{\tau^{\alpha}} \left( 1 - \frac{\sin \frac{\tau}{\tau^{1+\alpha}}} {\tau^{1+\alpha}} \right) \, d\tau \leq \int_0^\infty \frac{1}{\tau^{\alpha+2}} \left( 1 - \frac{\sin \frac{\tau}{\tau^{1+\alpha}}} {\tau^{1+\alpha}} \right) \, d\tau < \infty.
\]
We therefore have
\[
\Gamma_{\epsilon} \to \left( d_0 \int_0^\infty \frac{1}{\tau^{\alpha}} \left( 1 - \frac{\sin \frac{\tau}{\tau^{1+\alpha}}} {\tau^{1+\alpha}} \right) \, d\tau \right) |v \cdot \xi|^\alpha \sgn(v \cdot \xi), \quad \text{as } \epsilon \to 0.
\]
Case (c1)-(c2) In the borderline case where $\alpha = 2$, we choose $\theta(\epsilon) = \epsilon^2 |\ln \epsilon|$. The choice of $\theta$ is slightly less obvious than the previous two cases but it will be clear from the estimates below.

We again split the integration domain into $s < 1$ and $s > 1$ and apply the change of variable $\tau = \epsilon |s| \theta$ in the subdomain $s > 1$. Define

$$
\Lambda_{\epsilon,3} = \frac{2d_0 |\xi|^2}{|\ln \epsilon|} \int_{\tau > \epsilon |\xi|} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3}, \\
\Lambda_{\epsilon,4} = 2 \int_{s < 1} p(s) \frac{\sin^2 \left( \frac{\epsilon |s| (v \cdot \epsilon \xi)}{2} \right)}{\epsilon^2 |\ln \epsilon|} ds.
$$

First we bound $\Lambda_{\epsilon,4}$ as

$$
\Lambda_{\epsilon,4} = 2 \int_{s < 1} p(s) \frac{\sin^2 \left( \frac{\epsilon |s| (v \cdot \epsilon \xi)}{2} \right)}{\epsilon^2 |\ln \epsilon|} ds \leq \frac{2|\xi|^2}{|\ln \epsilon|} \int_{s < 1} p(s) ds dv \leq \frac{2|\xi|^2}{|\ln \epsilon|}.
$$

This shows

$$
\Lambda_{\epsilon,4} \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0 \quad \text{for each fixed} \quad \xi.
$$

To show the bound of $\Lambda_{\epsilon,3}$, we note that if $|\epsilon| |\xi| \geq 1$, then

$$
\Lambda_{\epsilon,3} \leq \frac{2d_0 |\xi|^2}{|\ln \epsilon|} \int_{1}^{\infty} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} \leq \frac{d_0 |\xi|^2}{|\ln \epsilon|},
$$

since

$$
\int_{1}^{\infty} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} \leq \frac{1}{2}.
$$

If $|\epsilon| |\xi| \leq 1$, then we separate $\Lambda_{\epsilon,3}$ as

$$
\Lambda_{\epsilon,3} = \frac{2d_0 |\xi|^2}{|\ln \epsilon|} \int_{|\xi|}^{1} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} + \frac{2|\xi|^2}{|\ln \epsilon|} \int_{1}^{\infty} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} \triangleq \Lambda_{\epsilon,3,1} + \Lambda_{\epsilon,3,2}.
$$

The second term $\Lambda_{\epsilon,3,2}$ is bounded as before which gives

$$
|\Lambda_{\epsilon,3,2}| \leq \frac{2|\xi|^2}{|\ln \epsilon|} \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0 \quad \text{for each fixed} \quad \xi.
$$

In the case where $|\xi| \geq 1$, the first term $\Lambda_{\epsilon,3,1}$ is bounded as

$$
0 \leq \Lambda_{\epsilon,3,1} \leq \frac{2|\xi|^2}{|\ln \epsilon|} \int_{|\epsilon|}^{1} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} \leq \frac{1}{2} |\xi|^2, \quad |\xi| \geq 1,
$$

while for $|\xi| \leq 1$, we have

$$
0 \leq \Lambda_{\epsilon,3,1} = \frac{2|\xi|^2}{|\ln \epsilon|} \int_{|\epsilon|}^{1} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} \leq \frac{1}{2} |\ln (|\xi|)| \leq |\xi|^2 (1 + |\ln |\xi||) \leq 2,
$$

Overall, we have

$$
0 \leq \Lambda_{\epsilon,3,1} \leq 2 \left( 1 + |\xi|^2 \right), \quad \text{for all} \quad \xi \in \mathbb{R}^n.
$$

The limit of $\Lambda_{\epsilon,3,1}$ are shown by L'Hôpital's rule, where

$$
\lim_{\epsilon \rightarrow 0} \frac{2}{|\ln \epsilon|} \int_{|\epsilon|}^{1} \sin^2 \left( \frac{\tau (v \cdot \epsilon \xi)}{2} \right) \frac{d\tau}{\tau^3} = 2 \lim_{\epsilon \rightarrow 0} \frac{\sin^2 \left( \frac{\epsilon |\xi| (v \cdot \epsilon \xi)}{2} \right)}{(\epsilon |\xi|)^2} = \frac{1}{2} |v \cdot \epsilon \xi|^2 \triangleq \bar{D}_3.
$$
Combining the bounds and limits for $\Lambda_{\epsilon,3.1}$, $\Lambda_{\epsilon,3.2}$, and $\Lambda_{\epsilon,4}$, we conclude that there exists a constant $c_0 > 0$ which is independent of $\epsilon, v', \xi$ such that

$$|\Lambda_{\epsilon}| \leq c_0 (|\xi|^2 + 1), \quad \Lambda_{\epsilon} \to \tilde{D}_3 |\xi|^2$$

for each $\xi \in \mathbb{R}^n$,

where $\tilde{D}_3$ is defined in (48).

To bound $\Gamma_{\epsilon}$, we apply the same division of ranges of $s$ and $\epsilon v \cdot \xi$ as in Case (a1)-(a2). If $s < 1$ and $\epsilon v \cdot \xi > 1$, then the integral is bounded by

$$\frac{2}{\epsilon^2 \ln \epsilon} \int_0^1 p(s) |\epsilon v \cdot \xi| \, ds \leq \frac{2\epsilon v \cdot \xi}{\epsilon^2 \ln \epsilon} \leq \frac{2 \epsilon^2 |v| \cdot |\xi|^2}{\epsilon^2 \ln \epsilon} \leq \frac{2 |\xi|^2}{\ln \epsilon}. \quad (49)$$

If $s < 1$ and $\epsilon v \cdot \xi \leq 1$, then the integral is bounded by

$$\frac{1}{6 \epsilon^2 \ln \epsilon} \int_0^1 p(s) |\epsilon v \cdot \xi|^3 \, ds \leq \frac{|\epsilon v \cdot \xi|^3}{6 \epsilon^2 \ln \epsilon} \leq \frac{\epsilon |\xi|^2}{\ln \epsilon}. \quad (50)$$

In the case where $s > 1$, we make the same change of variable $\tau = \epsilon v \cdot \xi s$. Then the integral satisfies

$$\frac{1}{\epsilon^2 \ln \epsilon} \int_1^\infty \frac{1}{s^3} (\epsilon v \cdot \xi s) \left( 1 - \frac{|\epsilon v \cdot \xi|}{|\epsilon v \cdot \xi|} \right) \, ds
= \frac{1}{\epsilon^2 \ln \epsilon} \int_0^\infty \frac{1}{\tau^2} \left( 1 - \frac{\sin \tau}{\tau} \right) (\epsilon v \cdot \xi) |\epsilon v \cdot \xi| \, d\tau
= \frac{(v \cdot \xi) |v| \cdot |\xi|}{\ln \epsilon} \int_0^\infty \frac{1}{\tau^2} \left( 1 - \frac{\sin \tau}{\tau} \right) \, d\tau.$$ 

Note that the difference between $\Gamma_{\epsilon}$ and $\Lambda_{\epsilon}$ when $\alpha = 2$ is that $\frac{1}{\tau^2} \left( 1 - \frac{\sin \tau}{\tau} \right)$ is integrable on $\mathbb{R}^+$ while the integrand for $\Lambda_{\epsilon}$ given by $\frac{1}{\tau^2} \sin^2 \left( \frac{\tau - \epsilon v \cdot \xi}{2} \right)$ in (48) is not.

Hence the integral for $\Gamma_{\epsilon}$ when $s > 1$ is bounded directly by

$$\left| \frac{1}{\epsilon^2 \ln \epsilon} \int_1^\infty \frac{1}{s^3} (\epsilon v \cdot \xi s) \left( 1 - \frac{|\epsilon v \cdot \xi|}{|\epsilon v \cdot \xi|} \right) \, ds \right| \leq \frac{c_0 |\xi|^2}{\ln \epsilon}. \quad (51)$$

Combining (49), (50), and (51), we have both the bound and the limit of $\Gamma_{\epsilon}$ as

$$|\Gamma_{\epsilon}| \leq \frac{c_0}{\ln \epsilon} |\xi|^2, \quad \lim_{\epsilon \to 0} \Gamma_{\epsilon}(\xi, v) = 0,$$

for each fixed $v, \xi$.

Using Proposition 1, we can now state our main theorem in more detail and show its proof.

**Theorem 4.5.** Suppose the scattering coefficient $c$ and the cross section $\sigma$ satisfy that

$$0 < c < 1, \quad \int_{\mathbb{S}^{n-1}} \sigma(v \cdot v') \, dv = 1, \quad \sigma(v \cdot v') \geq \sigma_0 > 0 \quad (52)$$

for some $\sigma_0 > 0$. Suppose the path-length distribution function $p$ satisfies (33) and $Q \in L^1 \cap L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Let $\psi_{\epsilon}$ be the solution to (1) and $\Psi_{\epsilon} = \psi_{\epsilon} e^{\int_0^t \Sigma_\epsilon(\tau) \, d\tau}$. Let $\phi_{\epsilon}, \varphi_{\epsilon}$ be the functions defined in (24) and (26). Then

(a) there exists $\Psi_0 \in L^2(\mathbb{R}^n)$ such that

$$\Psi_{\epsilon} \to \Psi_0 \quad \text{in} \ w^* - L^\infty(0, \infty; L^2(\mathbb{R}^n \times \mathbb{S}^{n-1})).$$
(b) there exists \( q \in L^2(\mathbb{R}^n \times S^{n-1}) \) such that \( \int_{\mathbb{R}^n} q(x, v) \, dv = \int_{\mathbb{R}^n} Q(x, v) \, dv \)

and

\[
\phi_\epsilon \to \Psi_0, \quad q_\epsilon \to q \quad \text{weakly in } L^2(\mathbb{R}^n \times S^{n-1}).
\]

Moreover, with the choices of \( \theta(\epsilon) \) in Proposition 1, the limit \( \Psi_0 \) satisfies

(b1) \( D_1(-\Delta)\Psi_0 + (1 - c)\Psi_0 = \int_{\mathbb{R}^n} Q(x, v) \, dv \) if \( \alpha > 2 \);

(b2) \( D_2(-\Delta)^{\alpha/2}\Psi_0 + (1 - c)\Psi_0 = \int_{\mathbb{R}^n} Q(x, v) \, dv \) if \( 1 < \alpha < 2 \);

(b3) \( D_3(-\Delta)\Psi_0 + (1 - c)\Psi_0 = \int_{\mathbb{R}^n} Q(x, v) \, dv \) if \( \alpha = 2 \),

where \( D_1, D_2, D_3 \) are positive constants defined as

\[
D_1 = \frac{1}{3} \left( \int_0^\infty p(s) s \, ds \right)^2 \bar{\mu}_0 + \frac{1}{6} \int_0^\infty p(s)^2 s \, ds,
\]

and

\[
D_2 = \frac{d_0}{2(\alpha + 1)} \left( \int_0^\infty \frac{\sin^2 \tau}{\tau^{1+\alpha}} \, d\tau \right), \quad D_3 = \frac{1}{2} \int_{\mathbb{R}^n} (v \cdot \epsilon \xi)^2 \, dv = \frac{1}{6},
\]

where \( d_0 \) is the constant in (33).

(c) Let \( \eta_\epsilon(x) = \int_0^\infty \int_{\mathbb{R}^n} \hat{\Psi}_\epsilon(s, x, v) \, dv \, ds \). Then there exists \( \eta_0 \in L^2(\mathbb{R}^n) \) such that

\[
\eta_\epsilon \to \eta_0 \quad \text{weakly in } L^2(\mathbb{R}^n).
\]

Moreover, \( \eta_0 = \beta_0 \Psi_0 \) where the positive constant \( \beta_0 \) is defined in (71). Therefore \( \eta_0 \) satisfies similar diffusion equations as in (b1)-(b3) with the source term replaced by \( \beta_0 \int_{\mathbb{R}^n} Q(x, v) \, dv \).

**Proof.** (a) The convergence along a subsequence is proved in Theorem 4.1. The convergence of the full sequence will be clear from the proof of Part (b) and (c).

(b) Integrating (31) in terms of \( v \) to annihilate the singular term \( \frac{1}{\theta(\epsilon)} \mathcal{L}\hat{\phi}_\epsilon \), we have

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(v \cdot v') \hat{\phi}_\epsilon(\xi, v') \hat{w}_\epsilon(\xi, v') \, dv' \, dv = -(1 - c) \int_{\mathbb{R}^n} \mathcal{A}_\epsilon \Psi_\epsilon \, dv + \int_{\mathbb{R}^n} \hat{q}_\epsilon(\xi, v) \, dv,
\]

where \( \hat{w}_\epsilon \) is defined as

\[
\hat{w}_\epsilon(\xi, v) = \int_0^\infty p(s) e^{-i\epsilon v \cdot \xi s} - \frac{1}{\theta(\epsilon)} \, ds,
\]  

(53)

By the assumption for \( \sigma \) in (52), the above equation can be written as

\[
- \int_{\mathbb{R}^n} \hat{\phi}_\epsilon(\xi, v') \hat{w}_\epsilon(\xi, v') \, dv' = -(1 - c) \int_{\mathbb{R}^n} \mathcal{A}_\epsilon \Psi_\epsilon \, dv + \int_{\mathbb{R}^n} \hat{q}_\epsilon(\xi, v) \, dv.
\]  

(54)

The eventual diffusion equation will be obtained by passing to the limit along the subsequence \( \hat{\phi}_\epsilon \) in (54). We study the limit of each term in (54) along the subsequence \( \hat{\phi}_\epsilon \) given in Lemma 4.3. Up to a further subsequence and an abuse of notation, suppose \( \hat{\phi}_\epsilon \to \hat{q} \) weakly in \( L^2(\mathbb{R}^n \times S^{n-1}) \). Then

\[
\int_{\mathbb{R}^n} \hat{q}_\epsilon(\xi, v) \, dv \to \int_{\mathbb{R}^n} \hat{q} \, dv \quad \text{weakly in } L^2(\mathbb{R}^n).
\]  

(55)
By the Lebesgue Dominated Convergence Theorem, we have \( \int_{\mathbb{R}^n} \tilde{q}(\xi, v) \, dv = \int_{\mathbb{R}^n} Q(\xi, v) \, dv \). Therefore,

\[
\int_{\mathbb{R}^n} \tilde{q}_e(\xi, v) \, dv \to \int_{\mathbb{R}^n} \tilde{Q} \, dv \text{ weakly in } L^2(\mathbb{R}^n). 
\]

(56)

Next, let \( \tilde{g} \in C_c(\mathbb{R}^n) \) be arbitrary. Then by Fubini's theorem,

\[
\int_{\mathbb{R}^n} \tilde{g}(\xi) \left( \int_{\mathbb{R}^n} A_{\epsilon_k} \tilde{\Psi}_{\epsilon_k} \, dv \right) \, d\xi \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty \tilde{g}(\xi) \sigma(v \cdot v') p(s) p(\tau) \\
\times \tilde{\Psi}_e(\tau, \xi, \tilde{\nu}) e^{-i\epsilon v' \cdot \xi s} \, d\tau \, dv' \, ds \, dv \, d\xi \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty \tilde{g}(\xi) \sigma(v \cdot v') p(s) p(\tau) \\
\times \tilde{\Psi}_e(\tau, \xi, \tilde{\nu}) \left( e^{-i\epsilon v' \cdot \xi s} - 1 \right) \, d\tau \, dv' \, ds \, dv \, d\xi \\
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \tilde{g}(\xi) \sigma(v \cdot v') p(s) p(\tau) \\
\times \tilde{\Psi}_e(\tau, \xi, \tilde{\nu}) \, d\tau \, dv' \, ds \, dv' \, d\xi \\
\rightarrow \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \tilde{g}(\xi) \sigma(v \cdot v') p(s) p(\tau) \tilde{\Psi}_0(\xi) \, ds \, dv' \, dv \, d\xi \\
= \int_{\mathbb{R}^n} \tilde{g}(\xi) \tilde{\Psi}_0(\xi) \, d\xi
\]

since \( \int_0^\infty s p(s) \, ds < \infty \). Hence,

\[
\int_{\mathbb{R}^n} A_{\epsilon_k} \tilde{\Psi}_{\epsilon_k} \, dv \to \tilde{\Psi}_0 \text{ in } \mathcal{D}'.
\]

(57)

Combining (56) with (57), we obtain that along the subsequence \( \epsilon_k \),

\[
\text{R.h.s. of (54)} \quad \rightarrow (1 - c) \tilde{\Psi}_0 + \int_{\mathbb{R}^n} \tilde{Q}(\xi, v) \, dv \quad \text{in the sense of distributions.}
\]

(58)

To find the limit of the left-hand side of (54) we rewrite it as

\[
\int_{\mathbb{R}^n} \hat{\phi}_e(\xi, v') \hat{\omega}_e(\xi, v') \, dv' \\
= \int_{\mathbb{R}^n} \left( \hat{\phi}_e(\xi, v') - \langle \hat{\phi}_e(\xi) \rangle \right) \hat{\omega}_e(\xi, v') \, dv' + \int_{\mathbb{R}^n} \langle \hat{\phi}_e(\xi) \rangle \hat{\omega}_e(\xi, v') \, dv' \\
\triangleq J_1' + J_2',
\]

where we have introduced the notation \( \langle \cdot \rangle = \int \cdot \, dv \) The limits of \( J_1' \) and \( J_2' \) are shown separately.
Limit of $J'_1$. Recall the definition of $w_\epsilon$ in (53) and rewrite $J'_1$ as

$$J'_1 = -i \int_{S^{n-1}} \int_0^\infty \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) p(s) \frac{\sin(\epsilon v' \cdot \xi s)}{\theta(\epsilon)} ds \, dv'$$

$$\quad - \int_{S^{n-1}} \int_0^\infty \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) p(s) \frac{1 - \cos(\epsilon v' \cdot \xi s)}{\theta(\epsilon)} ds \, dv'$$

$$\quad = \frac{-i \epsilon}{\theta(\epsilon)} \int_{S^{n-1}} \int_0^\infty \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) p(s) (v' \cdot \xi s) ds \, dv'$$

$$\quad + \frac{i \epsilon}{\theta(\epsilon)} \int_{S^{n-1}} \int_0^\infty \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) p(s) (v' \cdot \xi s) \left( 1 - \frac{\sin(\epsilon v' \cdot \xi s)}{\epsilon v' \cdot \xi s} \right) ds \, dv'$$

$$\quad - \int_{S^{n-1}} \int_0^\infty \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) p(s) \frac{1 - \cos(\epsilon v' \cdot \xi s)}{\theta(\epsilon)} ds \, dv'$$

$$\triangleq J'_{1,1} + iJ'_{1,2} - J'_{1,3}. \quad (59)$$

First we show that

$$J'_{1,2} \to 0 \quad \text{and} \quad J'_{1,3} \to 0 \quad \text{in} \quad D'(\mathbb{R}^n).$$

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a test function. Let $\Gamma$ be the pointwise limit of $\Gamma_\epsilon$ as in Proposition 1 in each case of $\alpha$. Then by the definition of $\Gamma_\epsilon$ in Proposition 1, we have

$$\int_{\mathbb{R}^n} J'_{1,2}(\xi) \chi(\xi) \, d\xi$$

$$= \int_{\mathbb{R}^n} \int_{S^{n-1}} \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) \Gamma_\epsilon(\xi, v') \chi(\xi) \, dv' \, d\xi.$$

$$= \int_{\mathbb{R}^n} \int_{S^{n-1}} \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) \Gamma(\xi, v') \chi(\xi) \, dv' \, d\xi$$

$$+ \int_{\mathbb{R}^n} \int_{S^{n-1}} \left( \hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi) \right) (\Gamma_\epsilon(\xi, v') - \Gamma(\xi, v')) \chi(\xi) \, dv' \, d\xi$$

The first term converges to zero since $\Gamma \chi \in L^2(\mathbb{R}^n \times S^{n-1})$ and $\hat{\phi}_\epsilon(\xi, v') - \left\langle \check{\phi}_\epsilon \right\rangle(\xi)$ converges to zero weakly in $L^2(\mathbb{R}^n \times S^{n-1})$. The second term is bounded by

$$2 \| \phi_\epsilon \|_{L^2(\mathbb{R}^n \times S^{n-1})} \| \Gamma_\epsilon - \Gamma \|_{L^\infty(supp(\chi) \times S^{n-1})}$$

$$\times \int_{\mathbb{R}^n} \int_{S^{n-1}} | \Gamma_\epsilon(\xi, v') - \Gamma(\xi, v') | \chi(\xi) \, dv' \, d\xi,$$

where the integral converges to zero by Lebesgue Dominated Convergence Theorem. We thereby verify that

$$J'_{1,2} \to 0 \quad \text{in} \quad D'(\mathbb{R}^n) \quad \text{as} \quad \epsilon \to 0.$$

By the bounds and limits of $\Lambda$, Proposition 1, the convergence of $J'_{1,3}$ to zero in $D'$ can be shown in exactly the same way as that of $\Gamma_\epsilon$. The details are thus omitted to avoid repetition.

The limit of $J'_{1,1}$ is more involved. First, by Lemma 4.4, we rewrite $v' \cdot \xi$ as

$$v' \cdot \xi = \nu_0 \mathcal{L}(v' \cdot \xi), \quad \nu_0 = \frac{1}{1 - \bar{\mu}_0}. \quad (60)$$
Then $J'_{1,1}$ becomes
\begin{align*}
J'_{1,1} &= -\nu_0 \epsilon \int_{S^{n-1}} \int_0^\infty \left( \hat{\phi}_\epsilon(\xi, v') - \left< \hat{\phi}_\epsilon \right> (\xi) \right) p(s) \mathcal{L}(v' \cdot \xi) s \, ds \, dv' \\
&= -\nu_0 \epsilon \int_{S^{n-1}} \int_0^\infty \mathcal{L} \left( \hat{\phi}_\epsilon(\xi, v') - \left< \hat{\phi}_\epsilon \right> (\xi) \right) p(s) (v' \cdot \xi) s \, ds \, dv' \\
&= -\nu_0 \epsilon \int_{S^{n-1}} \int_0^\infty \hat{\phi}_\epsilon(\xi, v')p(s) (v' \cdot \xi) s \, ds \, dv' .
\end{align*}

By the equation for $\mathcal{L} \hat{\phi}_\epsilon$ in (31), we have
\begin{align*}
J'_{1,1} &= -i\nu_0 \epsilon \int_{S^{n-1}} \left( \int_{S^{n-1}} \sigma(v \cdot v') \hat{\phi}_\epsilon(\xi, v)\hat{w}_\epsilon(\xi, v) \, dv \right) (v' \cdot \xi) \left( \int_0^\infty p(s) s \, ds \right) \, dv' \\
&\quad + i(1-c)\nu_0 \epsilon \int_{S^{n-1}} \mathcal{A}_v \hat{\Psi}_\epsilon(\xi, v')(v' \cdot \xi) \, dv' - i\nu_0 \epsilon \int_{S^{n-1}} \hat{q}_\epsilon(\xi, v')(v' \cdot \xi) \, dv' \\
&\quad \triangleq J'_{1,1,1} + J'_{1,1,2} + J'_{1,1,3} ,
\end{align*}
where $\hat{w}_\epsilon$ is defined in (53). By the uniform bounds of $\mathcal{A}_v \hat{\Psi}_\epsilon$ and $\hat{q}_\epsilon$ in Lemma 4.2, we have
\[ J'_{1,1,2}, J'_{1,1,3} \to 0 \quad \text{in the sense of distributions.} \quad (61) \]
To show the convergence of $J'_{1,1,1}$, we separate the cases where $\alpha > 2$ and $\alpha \leq 2$. First, if $\alpha \leq 2$, then
\[ |\hat{w}_\epsilon| \leq \frac{\epsilon}{\theta(\epsilon)} |v \cdot \xi| \left( \int_0^\infty sp(s) \, ds \right) \leq \frac{\epsilon}{\theta(\epsilon)} |\xi| \left( \int_0^\infty sp(s) \, ds \right) . \]
Therefore,
\begin{align*}
J'_{1,1,1} &\leq c_0 \left( 1 + |\xi|^2 \right) \frac{\epsilon^2}{\theta(\epsilon)} \left| \int_{S^{n-1}} \int_{S^{n-1}} \sigma(v \cdot v') \hat{\phi}_\epsilon(\xi, v') \, dv' \, dv \right| \\
&\leq c_0 \left( 1 + |\xi|^2 \right) \frac{\epsilon^2}{\theta(\epsilon)} \left| \hat{\phi}_\epsilon(\xi, \cdot) \right|_{L^2(S^{n-1})} .
\end{align*}
In the case where $\alpha \leq 2$, we have
\[ \frac{\epsilon^2}{\theta(\epsilon)} \to 0 \quad \text{as } \epsilon \to 0 . \]
Therefore, if $\alpha \leq 2$, then
\[ J'_{1,1,1} \to 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \text{ as } \epsilon \to 0 . \]
Together with (61), we have
\[ J'_{1,1} \to 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \text{ as } \epsilon \to 0 , \quad \alpha \leq 2 . \quad (62) \]
In the case where $\alpha > 2$, we have $\theta(\epsilon) = \epsilon^2$. Separate the real and imaginary parts of $\hat{w}$ in $J'_{1,1}$ and denote
\[ \text{Re}\left( J'_{1,1,1} \right) = -\nu_0 \left( \int_0^\infty p(s) s \, ds \right) \]
\[ \times \int_{S^{n-1}} \int_{S^{n-1}} \int_0^\infty \sigma(v \cdot v') \hat{\phi}_\epsilon(\xi, v) (v' \cdot \xi) p(s) \frac{\sin(\epsilon v \cdot \xi s)}{\epsilon} \, ds \, dv' . \]
and

\[ Im \left( J_{1,1,1}^{k} \right) = -\nu_0 \left( \int_0^\infty p(s) s \, ds \right) \int_{S^{n-1}} \int_{S^{n-1}} \sigma(v \cdot v') \times \hat{\phi}_\epsilon(\xi, v) (v' \cdot \xi) p(s) \frac{1 - \cos(\epsilon v \cdot \xi)}{\epsilon} \, ds \, dv \, dv'. \]

Since \( \hat{\phi}_\epsilon \to \tilde{\Psi}_0 \) in \( L^2(\mathbb{R}^n \times S^{n-1}) \), we have

\[ Re \left( J_{1,1,1}^{k} \right) \to -\nu_0 \left( \int_0^\infty p(s) s \, ds \right) \int_{S^{n-1}} \int_{S^{n-1}} \sigma(v \cdot v') (v \cdot \xi) (v' \cdot \xi) \, dv' \, dv \]

in \( D' \), and

\[ Im \left( J_{1,1,1}^{k} \right) \to 0 \quad \text{in} \ D'(\mathbb{R}^n). \] (64)

In the above convergences we have applied the bounds and limits

\[ \left| \frac{\sin(\epsilon v \cdot \xi)}{\epsilon} \right| \leq |\xi|, \quad \left| 1 - \cos(\epsilon v \cdot \xi) \right| \leq \frac{1}{2} |\xi|^2 \epsilon, \quad \frac{\sin(\epsilon v \cdot \xi)}{\epsilon} \to v \cdot \xi \]

pointwise as \( \epsilon \to 0 \). The limit of \( Re \left( J_{1,1,1}^{k} \right) \) can be simplified as

\[ -\nu_0 \left( \int_0^\infty p(s) s \, ds \right) \int_{S^{n-1}} \int_{S^{n-1}} \sigma(v \cdot v') (v \cdot \xi) (v' \cdot \xi) \, dv' \, dv \]

\[ = -\nu_0 \left( \int_0^\infty p(s) s \, ds \right) \int_{S^{n-1}} \left( \int_{S^{n-1}} \sigma(v \cdot v') (v \cdot \xi) \, dv' \right) (v' \cdot \xi) \, dv' \] (65)

\[ = -\nu_0 \left( \int_0^\infty p(s) s \, ds \right) \left( \frac{1}{2} \int_{-1}^1 \mu \sigma(\mu) \, d\mu \right) \tilde{\Psi}_0(\xi) \int_{S^{n-1}} (v' \cdot \xi)^2 \, dv' \]

\[ = -\frac{1}{3} \nu_1 |\xi|^2 \tilde{\Psi}_0(\xi), \]

where \( \nu_0 \) is defined in (60) and the constant \( \nu_1 \) is

\[ \nu_1 = \frac{\left( \int_0^\infty p(s) s \, ds \right)^2 \tilde{\mu}_0}{1 - \tilde{\mu}_0}. \] (66)

Combining (61), (63), (64), and (65), we have

\[ J_{1,1,1}^{k} \to -\frac{1}{3} \nu_1 |\xi|^2 \tilde{\Psi}_0(\xi) \quad \text{in} \ D', \quad \alpha > 2. \] (67)

As a summary, we have

\[ J_{1}^{k} \to \begin{cases} 0, & \alpha \leq 2, \\ -\frac{1}{3} \nu_1 |\xi|^2 \tilde{\Psi}_0(\xi), & \alpha > 2 \end{cases} \quad \text{in} \ D' \quad \text{as} \ \epsilon \to 0. \] (68)

**Limit of \( J_2^{k} \).** To find the limit of \( J_2^{k} \), we make use of the symmetry of the integral and obtain that

\[ J_2^{k} = \left( \int_{S^{n-1}} \tilde{\omega}_\epsilon(\xi, v) \, dv \right) \left( \hat{\phi}_\epsilon \right)(\xi) \]

\[ = \left( \int_{S^{n-1}} \int_{0}^{\infty} p(s) \frac{\cos(\epsilon v \cdot \xi s) - 1}{\theta(\epsilon)} \, ds \, dv \right) \left( \hat{\phi}_\epsilon \right)(\xi) \]

\[ = \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} \Lambda_v(\xi, v) \, dv \right) \left( \hat{\phi}_\epsilon \right)(\xi) \, d\xi. \] (69)
Applying Proposition 1, Lebesgue Dominated Convergence Theorem, and the weak convergence of \((\hat{\phi}_k)\) in Lemma 4.3, we have

\[
J^2_k \to \begin{cases} -\hat{D}_1 \xi^2 \hat{\Psi}_0, & \alpha > 2, \\ -\hat{D}_2 \xi^\alpha \hat{\Psi}_0, & 1 < \alpha < 2, \\ -\hat{D}_3 \xi^2 \hat{\Psi}_0, & \alpha = 2 \end{cases}
\]

weakly in \(L^2(\mathbb{R}^n)\) , \(\hat{D}_i\)'s are given by

\[
\hat{D}_1 = \int_{S^{n-1}} \tilde{D}_1 \, dv = \frac{D_0}{2} \int_{S^{n-1}} (v \cdot e_\xi)^2 \, dv = \frac{D_0}{6} = \frac{1}{6} \int_0^\infty p(s) s^2 \, ds,
\]

\[
\hat{D}_2 = \int_{S^{n-1}} \tilde{D}_2 \, dv = \left( \frac{d_0}{2} \int_0^\infty \sin^2 \frac{\tau}{\tau^1 + \alpha} d\tau \right) \int_{S^{n-1}} |v \cdot e_\xi|^{\alpha} \, dv
\]

\[
= \frac{d_0}{2(\alpha + 1)} \left( \int_0^\infty \sin^2 \frac{\tau}{\tau^1 + \alpha} d\tau \right),
\]

\[
\hat{D}_3 = \int_{S^{n-1}} \tilde{D}_3 \, dv = \frac{1}{2} \int_{S^{n-1}} (v \cdot e_\xi)^2 \, dv = \frac{1}{6}.
\]

Combining (58), (68), and (70) we obtain the desired diffusion equation for \(\Psi_0\). Moreover, since the solution to the diffusion equation in each case is unique in the space \(L^2(\mathbb{R}^n)\), the limit holds along the full sequence \(\{\hat{\phi}_k\}\).

(c) Note that by \(\int_0^\infty sp(s) \, ds < \infty\), we have

\[e^{-\int_0^\infty \Sigma_{s}(\tau) \, d\tau} \in L^1 \cap L^\infty(0, \infty).\]

Hence, for any \(h \in L^2(\mathbb{R}^n)\), we have

\[
\int_0^\infty \int_{\mathbb{R}^n} \int_{S^{n-1}} h(x) \Psi_\epsilon(s, x, v)e^{-\int_0^\infty \Sigma_{s}(\tau) \, d\tau} \, dv \, dx \, ds
\]

\[
\to \beta_0 \int_0^\infty h(x) \Psi_0(x) \, dx \text{ as } \epsilon \to 0.
\]

where

\[\beta_0 = \int_0^\infty e^{-\int_0^\infty \Sigma_{s}(\tau) \, d\tau} \, d\tau < \infty.\] (71)

Therefore, we have

\[\eta_\epsilon = \int_0^\infty \int_{S^{n-1}} \psi(s, x, v) \, dv \, ds = \int_0^\infty \int_{S^{n-1}} \Psi_\epsilon(s, x, v)e^{-\int_0^\infty \Sigma_{s}(\tau) \, d\tau} \, dv \, ds \to \beta_0 \Psi_0\]

weakly in \(L^2(\mathbb{R}^n)\). By Part (a), the limiting equations for \(\beta_0 \Psi_0\) are in the same format with the source term replaced by \(\beta_0 \int_{S^{n-1}} q(x, v) \, dv\).

**Remark 3.** Note that in the case where \(\alpha > 2\), there are two parts that contribute to the diffusion coefficient \(D_1\) such that

\[
D_1 = \frac{1}{3} \nu_1 + \tilde{D}_1 = \frac{1}{3} \nu_1 + \frac{1}{2} \left( \int_{S^{n-1}} (v \cdot e_\xi)^2 \, dv \right) D_0 = \frac{1}{3} \nu_1 + \frac{1}{6} D_0,
\]

where \(\nu_1\) and \(D_0\) are defined in (66) and (34) respectively. This coefficient is consistent with the one in [12] and captures anisotropic scattering. Interestingly, the anisotropy of the scattering vanishes from the limit equation in the heavy-tail case.
5. Concluding remarks and future work. In a series of papers, Golse et al. (for a review cf. [11]), and independently Marklof & Strömbergsson [14] show that an equation similar to the non-classical transport equation can be derived from particle transport in a regular lattice. A test particle moves between obstacles that are placed on a regular lattice, and undergoes specular reflections. In the Boltzmann-Grad limit of shrinking obstacles, while simultaneously increasing their number so that the collision frequency is fixed, one obtains a kinetic equation that contains two seemingly unphysical memory variables, namely the distance to the next collision (similar to the variable \( s \) in non-classical transport), as well as the impact factor for the next collision. This is the so-called periodic Lorentz gas equation.

In 2D, an explicit path-length distribution can be computed. Translated into our notation, it reads

\[
p(s) = \begin{cases} 
  \frac{24}{\pi^2} & \text{if } 0 \leq s < \frac{1}{2}, \\
  \frac{24}{\pi^2} \left( \frac{1}{2} s + 2 \left( 1 - \frac{1}{2} s \right)^2 \ln \left( 1 - \frac{1}{2} s \right) - \frac{1}{2} \left( 1 - \frac{1}{2} s \right)^2 \ln \left( 1 - \frac{1}{2} s \right) \right) & \text{if } s \geq \frac{1}{2}.
\end{cases}
\]

As \( s \to \infty \), this path-length distribution behaves like

\[
p(s) \sim \frac{2}{\pi^2} \frac{1}{s^3} + O\left( \frac{1}{s^4} \right).
\]

This means that the path-length distribution of the periodic Lorentz gas corresponds exactly to the borderline case between classical and anomalous diffusion, as its second moment diverges logarithmically. We thus expect a classical diffusion equation with a non-classical coefficient in the asymptotic limit. For this simplified equation, this reproduces the result of Marklof & Tóth [15] who proved a superdiffusive central limit theorem directly for the particle billiards underlying the periodic Lorentz gas equation. They showed that the periodic Lorentz gas is superdiffusive, but only logarithmically.

There are several open topics related to non-classical transport. Among them are the formulation of correct boundary and interface conditions for heterogeneous media. A first attempt has been made in [4]. In these media, it is also open how a fractional diffusion limit might look like. To study these questions, it will be necessary to generalize the Hilbert expansion technique to the fractional case along the lines of [1]. Furthermore, the asymptotic limit of the periodic Lorentz gas equation including impact factor should be studied, to see if the results of Marklof & Tóth [15] can be retrieved by kinetic theory techniques.

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