Manifolds with positive curvature operator and strictly convex boundary

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Abstract

In [AMW], it is proved that if a compact 3-manifold has positive Ricci curvature and strictly convex boundary, then this manifold is diffeomorphic to the standard 3-dimensional Euclidean disk. In this paper, we prove its higher-dimensional generalization: if a compact n-manifold has positive curvature operator and strictly convex boundary, then this manifold is diffeomorphic to the standard n-dimensional Euclidean disk.

1 Introduction

The implications of curvature conditions on the topology have been interesting topics in the study of geometry, especially ever since the invention of the Ricci flow [H1]. To list a few results of this fashion, closed 3-manifolds with positive Ricci curvature (see [H1]), closed 4-manifolds with positive curvature operator (see [H2]), closed n-manifolds with 2-positive curvature operator (see [BW]), and closed n-manifolds with point-wise quarterly pinched sectional curvature (see [BS]) must all be diffeomorphic to round space forms. All the results listed above are proved by showing that the respective curvature conditions are preserved and improved under the Ricci flow, and hence the normalized Ricci flow deforms the Riemannian metric to one with constant positive sectional curvature.

By constructing a $C^2$ metric with positive Ricci curvature on the doubling space (this method is suggested by Perelman [P]), and subsequently deforming the metric with the Ricci flow [H1], Aché-Maximo-Wu [AMW] proved that a 3-manifold with strictly convex boundary and positive Ricci curvature is diffeomorphic to a standard 3-dimensional Euclidean disk. Naturally, this method should be generalized to higher dimensions, if the curvature condition assumed is also preserved and improved under the Ricci flow. In this paper, we prove a result of this fashion.

Theorem 1.1. Let $(M^n,\partial M)$ be an n-dimensional compact manifold with boundary. If $(M^n,\partial M)$ admits a smooth Riemannian metric with positive curvature operator, such
that \( \partial M \) is strictly convex in \((M^n, g)\), then \((M^n, \partial M)\) is diffeomorphic to the standard \(n\)-dimensional Euclidean disk \((D^n, S^{n-1})\).

Note that the Riemannian metric of the doubling space is only known to be \(C^0\), and not differentiable at the common \(\partial M\), unless \(\partial M\) is totally geodesic in \((M, g)\). Perelman [P] proved the following gluing and smoothing theorem: let \((M_1, \partial M_1)\) and \((M_2, \partial M_2)\) be two compact manifolds with positive Ricci curvature, if their boundaries are isometric and the summation of their second fundamental forms is positive definite, then the \(C^0\) metric on the connected sum can be smoothed to a \(C^2\) metric with positive Ricci curvature. A similar construction is the main technique for the proof of Theorem 1.1.

**Proposition 1.2.** Let \((M^n, \partial M)\) be a smooth Riemannian manifold with positive curvature operator and strictly convex boundary. Then the \(C^0\) Riemannian metric on its doubling space can be smoothed to a reflectionally symmetric \(C^2\) Riemannian metric with positive curvature operator.

Once Proposition 1.2 is established, we may apply a result of Böhm and Wilking [BW] to show that the doubling space is a round space form. Hence the bulk of this paper is the proof of Proposition 1.2.

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# 2 The construction of a \(C^1\) metric

As in [P], the proof of Proposition 1.2 is divided into two steps, and this section focuses on the first step: we will show that the \(C^0\) metric on the doubling manifold can be smoothed to a reflectionally symmetric \(C^1\) metric with positive curvature operator wherever it is \(C^2\). The idea is the same as Perelman [P]: we construct a new metric by a polynomial interpolation near the common boundary of the two copies of \(M\). Note that because of the reflectional symmetry, our construction also admits a simplification as in [AMW].

Let \(\bar{M} = M \# \partial M\) be the doubling manifold and \(g\) the \(C^0\) metric on \(\bar{M}\), which, when restricted to either copy of \(M\), is identical to the original metric. We shall henceforth use the notation \(\Sigma\) to denote the \((n - 1)\)-dimensional embedded submanifold \(\partial M \subset \bar{M}\), which is identified with the boundaries of the two copies of \(M\). By our construction, \(\mu\) is reflectionally symmetric with respect to \(\Sigma\) and \(Rm\mu > 0\) on \(\bar{M} \setminus \Sigma\). Let \(r(x)\) be the signed distance function from \(\Sigma\), then the two copies of \(M\) can be expressed as \(\{r(x) \leq 0\}\) and \(\{r(x) \geq 0\}\), respectively. Obviously, when \(|r|\) is small enough, the level set \(\Sigma_r := \{x : r(x) = r\}\) is smoothly embedded into \(\bar{M}\) and is isotopic to \(\Sigma = \Sigma_0\); let \(\rho > 0\) be a small number to be fixed, such that \(\Sigma_r\) is isotopic to \(\Sigma_0\) whenever \(r \in [-\rho, \rho]\). Then, in the
tubular neighborhood
\[ \Sigma \times [-\rho, \rho] = \bigcup_{r \in [-\rho, \rho]} \Sigma_r \times \{r\}, \]
the \( C^0 \) metric \( g \) can be written as
\[ g = dr^2 + g^r, \]
where \( g^r \) is a smooth Riemannian metric on \( \Sigma \), depending continuously on \( r \) for all \( r \in [-\rho, \rho] \), and depending smoothly on \( r \) for all \( r \in [-\rho, 0) \cup (0, \rho] \).

On the other hand, since \( \partial M \) is strictly convex in \( M \), we have
\[ (g^r)'|_{r=0} = -(g^r)'|_{r=0} > 0. \]
Hence, by the compactness of \( M \) and \( \partial M \), when \( \rho \) is small enough, we have
\[ (g^r)'|_{r=-\rho} = -(g^r)'|_{r=\rho} > c g^0, \tag{2.1} \]
where \( c \) is a constant independent of \( \rho \).

With these preparations, we are ready to construct a \( C^1 \) metric on \( \overline{M} \) by replacing the restriction of \( g \) on \( \Sigma \times [-\rho, \rho] \) with the following polynomial interpolation
\[ \bar{g} = dr^2 + \bar{g}^r, \]
where
\[ \bar{g}^r = br^2 + d, \tag{2.2} \]
and \( b, d \) are symmetric 2-tensors on \( \Sigma \). For the sake of convenience, we will let \( \bar{g} \) also denote the newly constructed metric on \( \overline{M} \). The boundary conditions of this interpolation is given by
\[ (\bar{g}^r)|_{r=\pm \rho} = (g^r)|_{r=\pm \rho}, \quad (\bar{g}^r)'|_{r=\pm \rho} = (g^r)'|_{r=\pm \rho}. \tag{2.3} \]
In order to match the boundary conditions, Perelman [P] uses a cubic polynomial interpolation. However, because of the reflectional symmetry with respect to \( \{r = 0\} \), we may omit the odd terms in this polynomial; see also [AMW].

**Lemma 2.1.** For \( \rho > 0 \) small enough, it holds that
\[ (\bar{g}^r)|_{r=\pm \rho} \leq -\frac{c}{\rho} \bar{g}^r \text{ for all } r \in (-\rho, \rho), \tag{2.4} \]
where \( c \) is a positive constant independent of \( \rho \).
Proof. To simplify our notations, let us denote
\[
(g^r)'|_{r=-\rho} = -(g^r)'|_{r=\rho} = a_0, \quad g^{-\rho} = g^\rho = b_0.
\]
Then, using (2.3), one may straightforwardly compute the coefficients in (2.2).
\[
b = -\frac{1}{2\rho}a_0, \quad d = b_0 + \frac{\rho}{2}a_0.
\]
Consequently, we have
\[
(g^r)'' = -\frac{1}{\rho}a_0,
\]
and (2.4) then follows from (2.1). Note that Lemma 2.2 shows that \( g^r \) and \( g^0 \) are very close in the \( C^0 \) sense, and the proof of Lemma 2.2 does not rely on the conclusion of the current lemma.

**Lemma 2.2.** For all \( r \in [-\rho, \rho] \), it holds that
\[
|\bar{g}^r - g^{\pm \rho}|_{C^2(\Sigma)} = o(1),
\]
\[
|{(\bar{g}^r)'}|_{C^0(\Sigma)} \leq C
\]
where \( o(1) \to 0 \) whenever \( \rho \to 0 \), \( C \) is a constant independent of \( \rho \), and the norm can be computed by using, say, \( g^0 \).

**Proof.** We continue using the notations defined in formula (2.5). By (2.6) we have
\[
\bar{g}^r - g^{\pm \rho} = \frac{r^2}{2\rho}a_0 + b_0 + \frac{\rho}{2}a_0 - g^{\pm \rho} = \frac{-r^2}{2\rho}a_0 + \frac{\rho}{2}a_0,
\]
\[
(\bar{g}^r)' = -\frac{r}{\rho}a_0.
\]
Since the original manifold \( M \) is compact and \( g \) is smooth on \( M \), we have
\[
|a_0|_{C^2(\Sigma)} \leq |g|_{C^3(M)} \leq C,
\]
where \( C \) is a constant independent of \( \rho \); (2.8) and (2.9) follow immediately.

**Lemma 2.3.** If \( \rho > 0 \) is taken small enough, then \( \bar{g} \) is a \( C^1 \) metric on \( \bar{M} \), and the eigenvalues of its curvature operator has positive lower bound wherever \( \bar{g} \) is smooth, that is, on \( \bar{M} \setminus (\Sigma \times \{-\rho, \rho\}) \).
Proof. We need only to verify the second statement on $\Sigma \times (-\rho, \rho)$. Throughout this proof we will use indices $i, j, k$, etc., to denote the tensor components along the tangential direction of $\Sigma_r$, and the index $r$ to denote the tensor components along the normal direction. Then, by direct computation and the Gauss-Codazzi equations we obtain the following formulas for the Riemann curvature tensor of $\bar{g}$. 

\[
R_{irjr} = Rm^\tilde{g}(\partial_i, \partial_r, \partial_j, \partial_r) = -\frac{1}{2}(\tilde{g}^r)_{ij} + \frac{1}{4}(\tilde{g}^r)^{pq}(\tilde{g}^r)^{ip}(\tilde{g}^r)^{jq}
\]  

(2.11)  

\[
R_{ijk\ell} = Rm^\tilde{g}(\partial_i, \partial_j, \partial_k, \partial_\ell) = \nabla_i(\tilde{g}^r)_{jk} - \nabla_j(\tilde{g}^r)_{ki},
\]  

(2.12)  

\[
R_{ijkl} = Rm^\tilde{g}(\partial_i, \partial_j, \partial_k, \partial_l) = Rm^\tilde{g}(\partial_i, \partial_j, \partial_k, \partial_l) + \frac{1}{4}\left((\tilde{g}^r)_{ik}(\tilde{g}^r)_{jk} - (\tilde{g}^r)_{ij}(\tilde{g}^r)_{jk}\right),
\]  

(2.13)  

for all $\rho$ small enough, where $c_1$ is a constant independent of $\rho$. Note that in the computation of (2.11) we have applied (2.4) and (2.9). Next, we estimate the right-hand-sides of (2.12) and (2.13), respectively.

First of all, we may estimate the covariant derivatives of the second fundamental form easily, that is,  

\[
|R_{ijk\ell}| \leq C|\nabla(\tilde{g}^r)'| = C\frac{|r|}{\rho}|\nabla a_0| \leq C|g|_{C^2(M)} \leq C,
\]  

(2.14)  

where $C$ is independent of $\rho$. Our main focus would be estimating the right-hand-side of (2.13). By (2.8), we have  

\[
|Rm^\tilde{g}(\partial_i, \partial_j, \partial_k, \partial_\ell) - Rm^{g^{\pm \rho}}(\partial_i, \partial_j, \partial_k, \partial_\ell)| = o(1).
\]  

(2.15)

We will again use the same notations as defined in (2.5). The following computation is straightforward.  

\[
(\tilde{g}^r)_{il}(\tilde{g}^r)_{jk}' - (\tilde{g}^r)_{ik}(\tilde{g}^r)_{jl}' = \frac{r^2}{\rho^2}(a_0)_{il}(a_0)_{jk} - (a_0)_{ik}(a_0)_{jl}
\]  

(2.16)  

\[
= (a_0)_{il}(a_0)_{jk} - (a_0)_{ik}(a_0)_{jl} - (1 - \frac{r^2}{\rho^2})(a_0)_{il}(a_0)_{jk} - (a_0)_{ik}(a_0)_{jl} \\
\geq (a_0)_{il}(a_0)_{jk} - (a_0)_{ik}(a_0)_{jl} \\
= (g^{\pm \rho})_{il}(g^{\pm \rho})_{jk}' - (g^{\pm \rho})_{ik}(g^{\pm \rho})_{jl}',
\]

where the inequality is in terms of the eigenvalues of a symmetric operator on 2-vectors. To see why this inequality is true, recall that $a_0 > 0$ and hence 

\[
(a_0)_{il}(a_0)_{jk} - (a_0)_{ik}(a_0)_{jl}
\]
is a negative definite linear operator on the space of 2-vectors. Combining (2.13), (2.15), and (2.16), we obtain
\[
R_{ijkl} \geq Rm^{\pm \rho}(\partial_i, \partial_j, \partial_k, \partial_l) + \frac{1}{4} \left((g^{\pm \rho})'_{ik}(g^{\pm \rho})'_{jk} - (g^{\pm \rho})'_{ik}(g^{\pm \rho})'_{jl}\right) + o(1) \quad (2.17)
\]
where \(c_0\) depends only on the lower eigenvalue bound for \(Rm^g\); here \(R_{ijkl}^g\) stands for the Riemann curvature tensor of \(g\), and the inequality is in terms of the eigenvalues of a symmetric operator on 2-vectors.

Finally, combining (2.11), (2.14), and (2.17), we may prove that, when \(\rho\) is small enough, the eigenvalues of \(Rm^\bar{g}\) have positive lower bound on \(\Sigma \times (-\rho, \rho)\). Let
\[
\phi = \sum_{i,j=1}^{n-1} \alpha^{ij} \partial_i \wedge \partial_j + \sum_{k=1}^{n-1} \beta^k \partial_k \wedge \partial_r
\]
be an arbitrary 2-vector, where \(\alpha^{ij} = -\alpha^{ji}\), then we have
\[
Rm^\bar{g}(\phi, \phi) = \sum_{i,j,k,l=1}^{n-1} R_{ijkl} \alpha^{ij} \alpha^{kl} + 2 \sum_{i,j,k=1}^{n-1} R_{ijkr} \alpha^{ij} \beta^k + \sum_{i,j=1}^{n-1} R_{irjr} \beta^i \beta^j
\]
\[
\geq c_0 |\alpha|^2 - 2C |\alpha||\beta| + \frac{c_1}{\rho} |\beta|^2
\]
\[
\geq c_0 |\alpha|^2 - \frac{1}{2} c_0 |\alpha|^2 - 2 \frac{C^2}{c_0} |\beta|^2 + \frac{c_1}{\rho} |\beta|^2
\]
\[
= \frac{1}{2} c_0 (|\alpha|^2 + |\beta|^2) + \left(\frac{c_1}{\rho} - 2 \frac{C^2}{c_0} - \frac{1}{2} c_0\right) |\beta|^2.
\]
Therefore, if we take \(\rho \leq c_1 \left(\frac{2 C^2}{c_0} + \frac{1}{2} c_0\right)^{-1}\), then
\[
Rm^\bar{g}(\phi, \phi) \geq \frac{1}{2} c_0 |\phi|^2
\]
for any 2-vector \(\phi\); this finishes the proof of the lemma.

\[\square\]

## 3 The construction of a \(C^2\) metric

Let \(\rho > 0\) and \(\tilde{g}\) be as constructed in the previous section. From its construction, we have that \(\tilde{g}\) is \(C^1\) on \(\bar{M}\), is smooth and has positive curvature operator everywhere except for \(\Sigma \times \{-\rho, \rho\}\). By implementing the interpolation with a polynomial of degree 5 near
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$\Sigma \times \{-\rho\}$ and $\Sigma \times \{\rho\}$, we will construct a reflectionally symmetric $C^2$ Riemannian metric with positive curvature operator everywhere. This process is sketched by Perelman [P], yet he did not supply any details. We consider only $\Sigma \times \{-\rho\}$, and the construction near $\Sigma \times \{\rho\}$ is symmetric to that near $\Sigma \times \{-\rho\}$.

In this section, we will let $t(x) := \rho + r(x)$ denote the signed distance from $\Sigma_{-\rho}$, and let $\Sigma_t := \{x : t(x) = t\}$ be the level set; $t$, rather than $r$, shall be the parameter used throughout this section. Let $\tau \in (0, \rho)$ be a small positive number to be fixed, then the new metric is defined by replacing the restriction of $\bar{g}$ on $\Sigma \times [-\tau, \tau] := \bigcup_{t \in [-\tau, \tau]} \Sigma_t \times \{t\}$

with

$$\bar{g} = dt^2 + \bar{g}^t;$$

where the construction of $\bar{g}^t$ is the chief end of this section. The boundary conditions for $\bar{g}^t$ are

$$(\bar{g}^t)|_{t=\pm \tau} = \frac{d}{dt}\bar{g}^t|_{t=\pm \tau}, \quad (\bar{g}^t)|_{t=\pm \tau} = \frac{d}{dt}\bar{g}^t|_{t=\pm \tau}, \quad (\bar{g}^t)|_{t=\pm \tau} = \frac{d}{dt}\bar{g}^t|_{t=\pm \tau}.$$

To simplify the notations, we denote

$$(\bar{g}^t)|_{t=\pm \tau} = a_1, \quad (\bar{g}^t)|_{t=-\tau} = a_0,$$

$$(\bar{g}^t)|_{t=\pm \tau} = b_1, \quad (\bar{g}^t)|_{t=-\tau} = b_0,$$

$$(\bar{g}^t)|_{t=\pm \tau} = c_1, \quad (\bar{g}^t)|_{t=-\tau} = c_0,$$

where the right-hand-sides are all symmetric 2-tensors on $\Sigma$. Note that these notations do not have the same meanings as in section 2: in order not to introduce too much notational complexity, we used the same letters. Next, we defined $\bar{g}^t$ to be the following modified polynomial interpolation

$$\bar{g}^t = at^5 + bt^4 + ct^3 + dt^2 + et + f + h_\varepsilon(t^2)\tau^2(a_1 - a_0),$$

where $a, b, c, d, e,$ and $f$ are symmetric 2-tensors on $\Sigma$, and $h_\varepsilon(s)$ is a $C^2$ function defined on $[-1, 1]$, satisfying

$$h_\varepsilon(\pm1) = h'_\varepsilon(\pm1) = h''_\varepsilon(\pm1) = 0,$$  

(3.1)

whose exact definition and necessity will be explained later.
By straightforward computation, one obtains the following interpolation coefficients; note that by (3.1), $h_\varepsilon$ does not affect this computation.

$$\begin{align*}
a &= \frac{1}{16\tau^5} (a_1 - a_0) - \frac{3}{16\tau^4} (b_1 + b_0) + \frac{3}{16\tau^3} (c_1 - c_0), \\
b &= \frac{1}{16\tau^2} (a_1 + a_0) - \frac{1}{16\tau^3} (b_1 - b_0), \\
c &= -\frac{1}{8\tau^2} (a_1 - a_0) + \frac{5}{8\tau^2} (b_1 + b_0) - \frac{5}{8\tau^3} (c_1 - c_0), \\
d &= -\frac{1}{8} (a_1 + a_0) + \frac{3}{8\tau^3} (b_1 - b_0), \\
e &= \frac{1}{16} \tau (a_1 - a_0) - \frac{7}{16} (b_1 + b_0) + \frac{15}{16\tau} (c_1 - c_0), \\
f &= \frac{1}{16} \tau^2 (a_1 + a_0) - \frac{5}{16} \tau (b_1 - b_0) + \frac{1}{2} (c_1 + c_0).
\end{align*}$$

To further simplify the interpolation coefficients, we define

$$c^0 := (\tilde{g}^t)|_{t=0} \text{ and } b^o := (\tilde{g}^t)'|_{t=0}. \tag{3.3}$$

Then, by the fact that $\tilde{g}$ is $C^1$ everywhere and $C^2$ outside $\Sigma \times \{0\}$, we have

$$\begin{align*}
b_0 &= b^o - a_0 \tau + o(\tau), \quad b_1 = b^o + a_1 \tau + o(\tau), \\
c_0 &= c^o - b^o \tau + \frac{1}{2} a_0 \tau^2 + o(\tau^2), \quad c_1 = c^o + b^o \tau + \frac{1}{2} a_1 \tau^2 + o(\tau^2),
\end{align*} \tag{3.4}$$

where $o(\tau)$ and $o(\tau)$ stand for small quantities satisfying

$$\frac{o(\tau)}{\tau} \to 0, \quad \frac{o(\tau^2)}{\tau^2} \to 0,$$

whenever $\tau \to 0$.

By (3.4) and straightforward computation, one obtains the following simple formula for $(\tilde{g}^t)''$.

$$\begin{align*}
(\tilde{g}^t)'' &= \left( \lambda \left( \frac{t}{\tau} \right) + h_\varepsilon'' \left( \frac{t}{\tau} \right) \right) (a_1 - a_0) + a_0 + o(1), \tag{3.5}
\end{align*}$$

where

$$\lambda(s) = -\frac{5}{8} s^3 + \frac{9}{8} s + \frac{1}{2}. \tag{3.6}$$

One may verify that $\lambda(s) \in [-0.081, 1.081] \not\subset [0, 1]$ for $s \in [-1, 1]$, and this is exactly why we have to introduce $h_\varepsilon$. 
Lemma 3.1. For every $\varepsilon > 0$, there exists a $C^2$ function $h_\varepsilon : [-1, 1] \to \mathbb{R}$ satisfying (3.1) and the following conditions

$$-\lambda(s) - \varepsilon \leq h''_\varepsilon(s) \leq 1 - \lambda(s) + \varepsilon, \quad |h_\varepsilon(s)| \leq C, \quad |h'_\varepsilon(s)| \leq C,$$

for all $s \in [-1, 1]$, where $C$ is a positive constant independent of $\varepsilon$. Here $\lambda(s)$ is as defined in (3.6).

Proof. Let us construct $h_\varepsilon$ through its second derivative function $\eta$. The boundary conditions (3.1) are covered by the following prescriptions on $\eta$

1. $\eta(\pm 1) = 0$;
2. $\eta$ is a continuous odd function;
3. $\int_{-1}^{0} \int_{-1}^{t} \eta(s)dsdt = 0$.

Let us construct $\eta$ by using $\lambda$. Since

$$-\lambda(-1) = 0, \quad 1 - \lambda(1) = 0, \quad \lambda(-s) = 1 - \lambda(s), \quad \int_{-1}^{0} \int_{-1}^{t} \lambda(s)dsdt = 0,$$

one may define $\eta_\delta$ as

$$\eta_\delta(s) = \begin{cases} 
-\lambda(s) & \text{ for } s \in [-1, -\delta], \\
1 - \lambda(s) & \text{ for } s \in [\delta, 1], \\
\text{linear interpolation} & \text{ for } s \in [-\delta, \delta],
\end{cases}$$

where $\delta > 0$ is a small number. Then $\eta_\delta$ satisfies (1)(2) above and

$$-\lambda(s) \leq \eta_\delta(s) \leq 1 - \lambda(s), \quad \int_{-1}^{0} \int_{-1}^{t} \eta_\delta(s)dsdt = \psi(\delta) > 0,$$

where $\psi(\delta) \to 0$ as $\delta \to 0$. Finally, by taking $\delta$ small enough and perturbing $\eta_\delta(s)$, one obtains a function $\eta$ satisfying all conditions (1)(2)(3) and

$$-\lambda(s) - \varepsilon \leq \eta(s) \leq 1 - \lambda(s) + \varepsilon;$$

the construction of $\eta$ is illustrated in Figure 1. Hence,

$$h_\varepsilon(s) = \int_{-1}^{s} \int_{-1}^{t} \eta(u)dudt,$$

satisfies all the conditions in (3.7). Note that the bounds of $h_\varepsilon$ and of $h'_\varepsilon$ follow from the bound of $\eta$. \qed
Henceforth, we will denote

$$\lambda_\varepsilon(t) := \lambda(t) + h''(t).$$

Then, by (3.3) we have

$$(\tilde{g}'')' = (1 - \lambda_\varepsilon(t))a_0 + \lambda_\varepsilon(t)a_1 + o(1),$$

(3.9)

where $\lambda_\varepsilon(t) \in [-\varepsilon, 1 - \varepsilon]$ for all $t \in [-\tau, \tau]$.

**Lemma 3.2.** For all $t \in [-\tau, \tau]$, it holds that

$$\left| \tilde{g}' - c_i \right|_{C^2(\Sigma)} = o(1),$$

(3.10)

$$\left| (\tilde{g}')' - b_i \right|_{C^1(\Sigma)} = o(1),$$

(3.11)

where $i = 0, 1$, the norm is computed by using, say, $\tilde{g}^0$, and $o(1) \to 0$ as $\tau \to 0$.

**Proof.** (3.10) and (3.11) are obtained by writing out the formulas of $\tilde{g}'$ and of $(\tilde{g}')'$ explicitly, and verifying that every term is small. In the computations, one needs to apply (3.4). Note that (3.4) is not only true for $a_i$, $b_i$, and $c_i$, where $i = 0, 1$, but also for their corresponding tangential derivatives along $\Sigma_t$. This is due to the fact that $\tilde{g}$ is smooth both on \( \{ t \leq 0 \} \) and on \( \{ 0 \leq t \leq 2\rho \} \). (In fact, the non-smoothness of $\tilde{g}$ is only resulted from the low regularity of its directional derivative along $\partial_t$ when $t = 0$.)

\[\square\]
Proof of Proposition 1.1. The Proposition is a consequence of the following claim.

Claim: There exists \( \varepsilon > 0 \), such that for all \( \tau > 0 \) small enough, the metric \( \tilde{g} \) is \( C^2 \) with positive curvature operator everywhere.

The rest of the proof mainly focuses on the Claim. It is obvious that \( \tilde{g} \) is \( C^2 \), and we need only to show that the curvature operator of \( \tilde{g} \) is positive on \( \Sigma \times [-\tau, \tau] \). We may write the Riemann curvature tensor of \( \tilde{g} \) on \( \Sigma \times [-\tau, \tau] \) as follows

\[
Rm\tilde{g} = L((\tilde{g}^t)^\prime, \partial_x(\tilde{g}^i)^\prime, \partial_x(\tilde{g}^i)) + Q(\tilde{g}^t, (\tilde{g}^i)^\prime, \partial_x(\tilde{g}^i)),
\]

where \( \partial_x \) stands for tangential derivatives along \( \Sigma_t \), \( L \) is linear in all its arguments, and \( Q \) is a quadratic operator. By using (3.9), (3.10), and (3.11), we obtain

\[
Rm\tilde{g} = (1 - \lambda_\varepsilon(t))L(a_0, \partial_x(\tilde{g}^t)^\prime) + \lambda_\varepsilon(t)Q(c_0, b_0, \partial_x c_0) + o(1)
\]

Recall that \( \{ Rm > 0 \} \) is a convex cone. By Lemma 2.3, the distance from \( Rm\tilde{g}(x) \) to the boundary of \( \{ Rm > 0 \} \) is positive and bounded from below for all \( x \in \tilde{M} \setminus (\Sigma \times \{-\rho, \rho\}) \). Hence, if we take \( \varepsilon \) small enough, depending only on

\[
\inf_{x, y \in \tilde{M} \setminus (\Sigma \times \{-\rho, \rho\})} \frac{\lambda(Rm\tilde{g}(x))}{\Lambda(Rm\tilde{g}(y))} > 0,
\]

where \( \lambda \) and \( \Lambda \) stand for the smallest and largest eigenvalues of the curvature operator, respectively, then, the distance from

\[
(1 - \lambda_\varepsilon(t))Rm\tilde{g}|_{t=-\tau} + \lambda_\varepsilon(t)Rm\tilde{g}|_{t=\tau}
\]

to the boundary of \( \{ Rm > 0 \} \) is also positive and bounded from below; here we used the fact \( \lambda_\varepsilon(t) \in [-\varepsilon, 1-\varepsilon] \) for all \( t \in [-\tau, \tau] \). Finally, taking \( \tau \) small enough, the claim follows.

Proof of Theorem 1.1. First of all, let us assume that the manifold \( M \) in question is simply connected. Let \( \tilde{g} \) be the \( C^2 \) metric on \( \tilde{M} = M \#_{\partial M} M \) constructed above. Since \( \tilde{g} \) has positive curvature operator, we may apply [BW] to assert that the normalized Ricci flow deforms \( (\tilde{M}, \tilde{g}) \) to a round space form. It then follows that \( \partial M \) is connected, for otherwise \( \tilde{M} \) has infinite fundamental group; this cannot happen on a round space form. Hence, by the Van Kampen Theorem, \( \tilde{M} \) is simply connected, and must be the standard sphere \( S^n \).
On the other hand, the Ricci flow preserves reflectional symmetry, hence $\partial M$ must be an embedded submanifold with respect to which $\mathbb{S}^n$ is reflectionally symmetric, that is, a tropical $\mathbb{S}^{n-1}$. This shows that $(M, \partial M)$ is diffeomorphic to the standard Euclidean disk $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

Next, if $M$ is not simply connected, we first observe that $M$ has finite fundamental group, because otherwise, by the Gromoll-Meyer Theorem, the doubling of its universal cover must be diffeomorphic to $\mathbb{R}^n$; a contradiction (Note that $\partial M$ must also be a round space form, hence $\partial \tilde{M}$ consists of infinitely many copies of round space forms). Hence $(M, \partial M)$ is a finite quotient of $(\mathbb{D}^n, \mathbb{S}^{n-1})$. However, a finite nontrivial group action on the Euclidean disk always has fixed points, and this cannot happen on a smooth manifold. Therefore $M$ is simply connected.

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\square
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