Quantum algorithm for structured problems

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Abstract

Quantum algorithm can achieve exponential speedup over its classical counterparts in solving some problems by considering their structures, e.g. the Abelian hidden subgroup problems (HSP). In this work, we apply a quantum algorithm of finding the ground state of a Hamiltonian via a multi-step quantum computation process (arxiv: 1912.06959) for solving some structured problems, where the solution to a problem is encoded in the ground state of the corresponding problem Hamiltonian. By decomposing the problem based on its structure, we construct a sequence of intermediate Hamiltonians to approach the problem Hamiltonian, and evolve the system through the ground states of the intermediate Hamiltonians via quantum resonant transitions sequentially, finally obtain the ground state of the problem Hamiltonian. We find that this algorithm achieves exponential speedup over classical algorithms in solving some structured problems, including problems that can be reduced to both the Abelian and the non-Abelian HSP. The HSP and the unstructured search problem can be casted in the same framework in this algorithm.
I. INTRODUCTION

Solving a problem on a quantum computer can be transformed to finding the ground state of the corresponding problem Hamiltonian that encodes the solution to the problem. In Ref. [1], we proposed a quantum algorithm for finding the ground state of a problem Hamiltonian through a multi-step quantum computation process. In this algorithm, we construct a sequence of intermediate Hamiltonians \( \{H_i, i = 0, \cdots, m - 1\} \) by decomposing the problem based on its structure to form a Hamiltonian evolution path approaching the problem Hamiltonian \( H_P = H_m \). We start from the ground state of the initial Hamiltonian \( H_0 \), and evolve it through ground states of the intermediate Hamiltonians sequentially, finally reach the ground state of the problem Hamiltonian in \( m \) steps. In the \( i \)-th step of the algorithm, the ground state of \( H_{i-1} \) is protected in an entangled state and can be used repeatedly without making copies of the state, and the ground state of \( H_i \) is obtained deterministically through quantum resonant transitions (QRT) \([2, 3]\). This algorithm can be run efficiently if: \( i \) the energy gap between the ground and the first excited states of each Hamiltonian is not exponentially small and, \( ii \) the overlaps between the ground states of any two adjacent Hamiltonians are not exponentially small. We demonstrate that the conditions for efficiently running our algorithm are not equivalent to that of the common quantum adiabatic algorithm where the adiabatic evolution Hamiltonian is constructed as a linear interpolation of the initial and the problem Hamiltonians \([1]\).

The structure of a problem is considered as the key for whether it can be solved efficiently or not on a quantum computer \([4]\). We have applied this algorithm for solving a type of structured search problems, where the search space of the problem can be reduced in polynomial rate by using \( m = O(\log N) \) different oracles, and \( N \) is the dimension of the search space of the problem. Our algorithm achieves exponential speedup over Grover’s algorithm \([5]\) in solving the structured search problem in \( m \) steps, while it has the same efficiency as that of Grover’s algorithm in solving the unstructured search problem \([1]\).

It has been shown in \([6, 7]\) that quantum algorithms can be constructed for unstructured quantum search, quantum phase estimation, Hamiltonian simulation, and linear systems in the framework of quantum singular value transformation \([6]\). Here in this work, we analyze the structures of some structured problems including the hidden subgroup problems (HSP) and problems that can be reduced to both the Abelian and the non-Abelian HSP, a minimum-
search problem with distinct values, a minimum-search problem with a special structure, and a problem of minimizing a symmetrized cost function. We find that these problems can be reduced to the structured search problems, therefore can be solved efficiently by using our algorithm. Most of the problems for which quantum algorithms achieve exponential speedup over its classical counterparts can be reduced to the Abelian HSP, and solved by using the quantum Fourier transform, while there is no efficient quantum algorithm for solving the non-Abelian HSP. Here we show that in our algorithm, both the Abelian and the non-Abelian HSP can be reduced to the structured search problems and solved efficiently. The structured search problems and the unstructured search problem can be casted in the same framework in our algorithm.

This paper is organized as follows: in Sec. II, we briefly review a special case of the algorithm for finding the ground state of a Hamiltonian through a multi-step quantum computation process, and its application for solving a type of structured search problems; in Sec. III, we apply the algorithm for solving the HSP and problems that can be reduced to both the Abelian and non-Abelian HSP; in Sec. IV, we apply the algorithm for some other problems that have different structures than that of the HSP, while can be reduced to the structured search problems, including a minimum-search problem with a special structure and a minimization of a symmetrized cost function problem. We close with a discussion section.

II. THE ALGORITHM

In the following, we briefly describe a special case of the algorithm for finding the ground state of a problem Hamiltonian via multi-step quantum computation [1], where the ground state eigenvalues of the intermediate Hamiltonians and the overlaps between the ground states of any two adjacent Hamiltonians are known in advance. The algorithm needs $n + 1$ qubits with one probe qubit and an $n$-qubit register $R$ representing the problem of dimension $N = 2^n$. We use a step of the algorithm as an example to illustrate the working procedures of the algorithm. In the $i$-th step of the algorithm, given the Hamiltonian $H_{i-1}$, its ground state eigenvalue $E_0^{(i-1)}$ and the ground state $|\varphi_0^{(i-1)}\rangle$ obtained from the previous step, and the Hamiltonian $H_i$ and its ground state eigenvalue $E_0^{(i)}$, we are to prepare the ground state $|\varphi_0^{(i)}\rangle$ of $H_i$ deterministically by using the QRT method.
The algorithm Hamiltonian of the $i$-th step is constructed as

$$H^{(i)} = -\frac{1}{2} \omega \sigma_z \otimes I_N + H_R^{(i)} + c \sigma_x \otimes I_N,$$

where

$$H_R^{(i)} = \alpha_i |1\rangle\langle 1| \otimes H_{i-1} + |0\rangle\langle 0| \otimes H_i, \ i = 1, 2, \cdots, m,$$

$I_N$ is the $N$-dimensional identity operator, and $\sigma_{x,z}$ are the Pauli matrices. The first term in Eq. (1) is the Hamiltonian of the probe qubit, the second term contains the Hamiltonian of the register $R$ and describes the interaction between the probe qubit and $R$, and the third term is a perturbation. The parameter $\alpha_i$ is used to rescale the energy levels of $H_{i-1}$, and the ground state energy of $\alpha_i H_{i-1}$ is used as a reference energy level to the ground state eigenvalue of $H_i$, and $c \ll 1$.

The initial state of the $n+1$ qubits is set as $|1\rangle|\varphi_0^{(i-1)}\rangle$, which is an eigenstate of $H_R^{(i)}$ with eigenvalue $\alpha_i E_0^{(i-1)}$. By setting $\alpha_i = \left( E_0^{(i)} - \omega \right) / E_0^{(i-1)}$, the condition $E_0^{(i)} - \alpha_i E_0^{(i-1)} = \omega$ for a resonant transition between the probe qubit and the transition between states $|\varphi_0^{(i-1)}\rangle$ and $|\varphi_0^{(i)}\rangle$ driven by the unitary evolution $\exp\left(-i H^{(i)} t\right)$ is satisfied. The probe qubit decays to its ground state $|0\rangle$ and the register $R$ evolves to the ground state $|\varphi_0^{(i)}\rangle$ of $H_i$. In the case where the overlap $d_0^{(i)} = \langle \varphi_0^{(i-1)} | \varphi_0^{(i)} \rangle$ between the ground states of $H_{i-1}$ and $H_i$ are known in advance, we can set the optimal runtime $t_i = \pi / (2cd_0^{(i)})$ at which the probability for the system to be evolved to the state $|0\rangle|\varphi_0^{(i)}\rangle$ reaches its maximum.

The procedures of the $i$-th step of the algorithm are as follows:

1) Initialize the probe qubit in its excited state $|1\rangle$ and the register $R$ in state $|\varphi_0^{(i-1)}\rangle$.

2) Set $\alpha_i = \left( E_0^{(i)} - \omega \right) / E_0^{(i-1)}$, and implement the time evolution operator $U(t_i) = \exp\left(-i H^{(i)} t_i\right)$.

3) Read out the state of the probe qubit.

As the resonant transition occurs, the system is approximately in state $\sqrt{1 - p_0^{(i)}} |1\rangle|\varphi_0^{(i-1)}\rangle + \sqrt{p_0^{(i)}} |0\rangle|\varphi_0^{(i)}\rangle$, where $p_0^{(i)} = \sin^2 \left( ct_i d_0^{(i)} \right)$ is the decay probability of the probe qubit of the $i$-th step. The state $|\varphi_0^{(i-1)}\rangle$ from the previous step is protected in this entangled state. By performing a measurement on the probe qubit, if the probe decays to its ground state $|0\rangle$, it indicates that a resonant transition occurs and the system evolves from the state $|1\rangle|\varphi_0^{(i-1)}\rangle$ to the state $|0\rangle|\varphi_0^{(i)}\rangle$; otherwise if the probe qubit stays in state $|1\rangle$, it means the register $R$ remains in state $|\varphi_0^{(i-1)}\rangle$, then we repeat procedures ii)-iii) until the probe qubit decays to its ground state $|0\rangle$. Thus we obtain the ground state $|\varphi_0^{(i)}\rangle$ of $H_i$ deterministically. In
this way of protecting the state that encodes the calculation results of an intermediate step through entanglement, the state can be used repeatedly without making its copies, we do not need to run the algorithm from beginning once it fails to obtain the desired state in a middle step, we just repeat procedures of the step until the desired state is obtained, which is indicated by the outcome of the measurement on the probe qubit. Therefore, this algorithm circumvents the no-cloning theorem \[8, 9\] without making copies of a state to realize a multi-step computation process that is easy to implement in classical computation.

Considering errors accumulated in each step of the algorithm, the success probability of the algorithm satisfies \[1\]

\[
P_{\text{succ}} \geq \prod_{i=1}^{m} \left[ 1 - a_i^2 c^2 \right] \geq \left[ 1 - (a_{\text{max}} c)^2 \right]^m,
\]

where \(a_i^2 = \frac{4 \left[ 1 - \left( d_0^{(i)} \right)^2 \right]}{[E_1^{(i)} - E_0^{(i)}]^2 d_0^{(i)}^2}\), and \(a_{\text{max}}\) is the maximum value of \(a_i\). The coefficient \(c\) can be set such that \(a_{\text{max}} c < 1/\sqrt{m}\), then \(P_{\text{succ}} > 1/e\) in the asymptotic limit of \(m\). The runtime of each step is proportional to \(\pi/\left( 2 c d_0^{(i)} \right)\), therefore the total runtime of the algorithm scales as \(O(\sum_{i=1}^{m} \pi / (2 c d_0^{(i)}))\). The algorithm can be run efficiently if the overlap \(d_0^{(i)}\) and the energy gap \(E_1^{(i)} - E_0^{(i)}\) are not exponentially small. Then \(a_i\) is finite and the error in the \(i\)-th step is bounded by \(a_i^2 c^2\). The runtime of the algorithm is proportional to the number of steps of the algorithm.

This algorithm has been applied for solving a type of structured search problems \[1\]. The problem contains \(N\) items and can be decomposed by using \(m\) \((O (\log N))\) oracles to construct \(m\) Hamiltonians \(H_{P_i}\), respectively, as

\[
H_{P_i} = - \sum_{q_i \in \Pi_i} |q_i\rangle \langle q_i|,
\]

and

\[
H_{P_m} = H_m = H_P = -|q\rangle \langle q|,
\]

where the set \(\Pi_i\) contains \(N_i\) marked items in all the \(N\) items and \(|q_i\rangle\) are the marked states associated with the marked items, and \(|q\rangle\) is the marked state that defines the problem Hamiltonian of the structured search problem, the set \(\Pi_m\) only contains the target state \(|q\rangle\) of the problem. These sets satisfy the relation \(\Pi_1 \supset \cdot \cdot \cdot \supset \Pi_{m-1} \supset \Pi_m\) with sizes \(N_1, \cdot \cdot \cdot, N_{m-1}, N_m = 1\), respectively. We define \(H_0 = -|\psi_0\rangle \langle \psi_0|\) where \(|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle\), and
construct a sequence of intermediate Hamiltonians as

$$H_i = \frac{N_i}{N} H_0 + \left(1 - \frac{N_i}{N}\right) H_{P_i}, \quad i = 1, 2, \cdots, m - 1. \quad (5)$$

If the ratio $N_i/N_{i-1}$ ($i = 1, 2, \cdots, m$) are not exponentially small, where $N_0 = N$, then the conditions of our algorithm are satisfied and, the structured search problem can be solved efficiently in $m$ steps. In the case where $N_i$ are already known, the ground state eigenvalues of the intermediate Hamiltonians $H_i$ and the overlaps between the ground states of two adjacent Hamiltonians $H_{i-1}$ and $H_i$ can be calculated analytically [1]. The structured search problem can be solved with optimal runtime through the evolution path in Eq. (5) step by step.

In the following, we apply our algorithm for some structured problems that can be reduced to the structured search problem, including the HSP and other problems.

III. APPLICATION OF THE ALGORITHM FOR THE HIDDEN SUBGROUP PROBLEMS

The HSP encompass most of the quantum algorithms that are exponentially faster than their classical counterparts. Here, we analyze the structures of the HSP, and apply our algorithm for solving problems that can be reduced to both the Abelian and the non-Abelian HSP, and a minimum-search problem with distinct values, respectively. The Deutsch-Jozsa problem, the Simon’s problem, the factoring problem, the discrete logarithm problem, the period-finding problem and the order-finding problem can be reduced to the Abelian HSP, and solved efficiently by using the quantum Fourier transform. The graph isomorphism problem (GIP) and the poly($n$)-unique shortest vector problem (SVP) in a lattice can be reduced to the non-Abelian HSP of the symmetric group and the dihedral group, respectively. For the non-Abelian HSP, there is no known efficient quantum algorithm so far. We show that these problems can be reduced to the structured search problem, and can be solved efficiently by using our algorithm.

The HSP can be described as follows [4]: let $G$ be a finite group, $X$ be a finite set, and $f : G \rightarrow X$ be a function such that there exists a subgroup $K < G$ for which $f$ separates cosets of $K$, that is, for all $g_1, g_2 \in G$, $f(g_1) = f(g_2)$ only if $g_1K = g_2K$. Given a quantum black box for performing the unitary transform $U_f|g\rangle|h\rangle = |g\rangle|h\oplus f(g)\rangle$, for $g \in G$, $h \in X$,
and $\oplus$ a binary operation on $X$, using information gained from evaluations of $f$, find a generating set for $K$. The function $f$ hides the subgroup $K$ and has distinct values on different cosets of $K$.

For a finite group $G$, a classical algorithm calls a routine evaluating $f(g)$ once for each group element $g$, and determine $K$ with $|G|$ function calls. Quantum algorithms can reduce the time cost to $O(poly(\log |G|))$ for the finite Abelian groups. Suppose the function $f$ maps the subgroup $K = S_1$ and its cosets $S_2, \cdots, S_M$ to distinct constants $h_1, \cdots, h_M \in X$, respectively, where $M = |G|/|K|$. The standard quantum algorithm for solving the finite Abelian HSP is first querying the function $f$ to prepare the coset state: $\rho_K = \frac{1}{|G|} \sum_{i=1}^{M} |S_i\rangle\langle S_i|$, where $|S_i\rangle = \frac{1}{\sqrt{|K|}} \sum_{k_i \in S_i} |k_i\rangle$; then performing the quantum Fourier transform over $G$ to the coset state, measuring the resulting state, and repeat the whole process to output a generating set of the hidden subgroup $K$ from the measurement results [10]. The non-Abelian HSP is more difficult than the Abelian case, and there is no known quantum algorithm to solve the non-Abelian HSP efficiently.

The HSP can be solved if we can obtain the state of the subgroup that is labeled by the number $h_1 \in X$, and measure the state. In the following, we show that the HSP can be reduced to the structured search problem. In particular, both the Abelian and the non-Abelian HSP have the same structure in our algorithm, therefore can be solved efficiently in the same way.

The difficult case of HSP is when each coset contains polynomial number of group el-
elements (i.e. $|K| \ll |G|$). If we can obtain the state of the hidden subgroup $|S_1⟩ = \frac{1}{\sqrt{|K|}} \sum_{k_i \in K} |k_i⟩$, then we can obtain the group elements of the subgroup by measuring the state. By using the oracle $O_f : |j⟩|0⟩ \rightarrow |j⟩|f(j)⟩$, where $|j⟩$ is a state associated with a group element $g_j \in G$, the problem Hamiltonian for the HSP is:

$$H_P' |j⟩ = h_i |j⟩, \quad j \in S_i, \quad i = 1, 2, \cdots, M. \quad (6)$$

In Fig. 1, we plot the spectrum of the HSP, i.e., the dimension of the cosets v.s. the function $f(g_j)$. To apply our algorithm for the HSP, we use $m = \log_2 M$ oracles to divide the group elements of $G$ according to their corresponding eigenvalues $h_i$ by using the method of bisection to construct Hamiltonians $H_{P_i}$ as shown in Eq. (3), then construct the intermediate Hamiltonians $H_i$ as shown in Eq. (5). We prepare a division set $\{v_1 = h_{M/2}, v_2 = h_{M/2^2}, \cdots, v_m = h_1\}$, and construct the Hamiltonians $H_{P_i}$ by using $m$ different oracles as:

$$H_{P_i} |k⟩ = \begin{cases} -1 \cdot |k⟩, & \text{if } h_k \leq v_i \\ 0 \cdot |k⟩, & \text{if } h_k > v_i \end{cases}, \quad i = 1, \cdots, m. \quad (7)$$

This can be achieved by using an oracle that recognizes whether the integer $h_k$ is larger or less than $v_i$. It is a comparison logic circuitry and can be implemented efficiently on a quantum computer [4, 7, 11, 12].

The number of computational basis states associated with integers that are less than or equal to $v_i$ in each step as shown in Eq. (7) is $N_i$, by using the method of bisection, the ratio $N_i/N_{i-1} = 1/2$, therefore by constructing the intermediate Hamiltonians as shown in Eq. (5), the state of the hidden subgroup $|S_1⟩ = \frac{1}{\sqrt{|K|}} \sum_{k_i \in K} |k_i⟩$ can be obtained efficiently in $m$ steps with runtime $t = \pi/(\sqrt{2}c)$ in each step. Therefore the total runtime of the algorithm for solving the HSP scales as $O(\log |G|)$ by using $m$ oracles.

We have applied this algorithm for solving the Deutsch-Jozsa problem [1]. In the following, we apply the algorithm for solving problems that can be reduced to the Abelian HSP. We also apply the algorithm for the GIP and the poly$(n)$-unique SVP in a lattice that can be reduced to the non-Abelian HSP for the symmetric group and the dihedral group, respectively.
A. Simon’s problem

The Simon’s algorithm \[13\] is the first quantum algorithm that is demonstrated to be exponentially faster than any probabilistic classical algorithm in solving a black-box problem. In this problem \[13, 14\], there is an \(n\)-bit integer \(a\) such that for any two \(n\)-bit inputs \(i, j\), a black-box function \(f : \{0,1\}^n \rightarrow \{0,1\}^{n-1}\) outputs the integers \(f(i) = f(j)\) if and only if \(i \oplus j = a\), here \(\oplus\) denotes the bitwise XOR operation. The task is to find \(a\) by querying the function \(f\). It requires \(O(2^{n/2})\) queries of \(f\) for classical algorithms to solve this problem, while the Simon’s algorithm solves this problem with \(O(n)\) queries of \(f\) on a quantum computer.

In this problem, the black-box function represents a two-to-one mapping, two states \(|i\rangle, |j\rangle\) with \(i \oplus j = a\) form a set corresponds to an integer in \(\{0,1\}^{n-1}\). The period \(a\) can be calculated if we can obtain such a set by using our algorithm. The problem Hamiltonian for the Simon’s problem can be constructed by using the black-box function \(f\). Let \(h_k\) be the integer associated with the state \(|k\rangle (k = 0, \cdots, N - 1), h_k \in [0, N/2 - 1]\), then \(h_i = h_j\) if and only if \(j = i \oplus a\). To apply our algorithm for finding the set that contains two states associated with the integer \(h_k = 0\), we prepare a division set \(\{v_1 = N/4 - 1, v_2 = N/8 - 1, \cdots, v_{n-2} = 1, v_{n-1} = 0\}\), and use \(m = \log (N/2) = n - 1\) oracles to construct Hamiltonians \(H_{P_i}\) as

\[
H_{P_i}|k\rangle = \begin{cases} -1 \cdot |k\rangle, & \text{if } h_k \leq v_i, \\ 0 \cdot |k\rangle, & \text{if } h_k > v_i. \end{cases} \quad (8)
\]

Then we construct a sequence of intermediate Hamiltonians \(H_i\) as shown in Eq. (5). Our algorithm can be performed efficiently for obtaining the state of the set associated with integer 0 in \(n - 1\) steps to find the period \(a\) directly in runtime scales as \(O(n)\) by querying \(n - 1\) different oracles.

B. The factoring problem

Factoring an integer \(Z = x \times y\) on a quantum computer can achieve exponential speedup over the best known classical algorithm by using Shor’s algorithm \[15\]. An integer \(a < Z\) and co-prime with \(Z\) is used in the algorithm, the order of \(a\) is defined as the smallest integer \(r\) that satisfies \(a^r = 1 \pmod{Z}\), which can be found efficiently through quantum Fourier
transform. The factors of $Z$ can be calculated as $\gcd(a^{r/2} \pm 1, Z)$. The cost of the algorithm scales as $O(L^3)$, where $L = \lceil \log_Z Z \rceil$.

Our algorithm can also be applied for solving the factoring problem. The problem Hamiltonian for factoring an integer $Z$ is defined as

$$H_P|k\rangle = h_k|k\rangle = a^k (\text{mod } Z)|k\rangle, \ k = 0, 1, \ldots, N - 1,$$

where $N = 2^n \geq Z$ contains a few periods of the integer $a$. The ground state eigenvalue of $H_P$ is 1, and the corresponding eigenstate are in form of $|p \cdot r\rangle$, where $p = 0, 1, \ldots, [(N - 1)/r]$. The order $r$ of $a$ can be determined if the ground state of the problem Hamiltonian is obtained, which is in form of $\frac{1}{\sqrt{N/r}} \sum_{p=0}^{N/r} |p \cdot r\rangle$. The eigenvalues of the problem Hamiltonian are integers and distribute uniformly in the period of $a$, since $k$ ranges from 0 to $N - 1$ uniformly.

The ratio between the degeneracies of any two energy levels of the problem Hamiltonian is about one. We construct a division set $\{v_1 = \lceil \frac{Z}{2} \rceil, v_2 = \lceil \frac{v_1}{2} \rceil, \ldots, v_L = \lceil \frac{v_{L-1}}{2} \rceil = 1\}$ (it could be $v_{L-1} = \lceil \frac{v_{L-2}}{2} \rceil = 1$, then the division set contains $L - 1$ elements, for convenience, we assume there are $L$ elements). The Hamiltonians $H_{P_i}$ can be constructed by using $L$ oracles as

$$H_{P_i}|k\rangle = \begin{cases} -1 \cdot |k\rangle, & \text{if } h_k \leq v_i, \\ 0 \cdot |k\rangle, & \text{if } h_k > v_i, \end{cases}, \ i = 1, \ldots, L.\tag{10}$$

Then we construct a sequence of intermediate Hamiltonians as shown in Eq. (5) and let $H_P = H_{P_L}$. The ratio $N_i/N_{i-1}$ for the factoring problem Hamiltonian is about $1/2$ in each step of the algorithm, therefore the total runtime of the algorithm for obtaining the ground state of the problem Hamiltonian $H_P$ is proportional to the number of steps $L$ of the algorithm, and scales as $O(\log N)$. The order $r$ of $a$ can be obtained by measuring the ground state of the problem Hamiltonian, thus solving the factoring problem efficiently in $L$ steps.

### C. The discrete logarithm problem

The discrete logarithm problem is as follows [10]: for a cyclic group $G$ generated by an element $g$, given an element $x \in G$, find the discrete logarithm of $x$ with respect to $g$, $\log_g x$, which is the smallest nonnegative integer $j$ such that $g^j = x$.

The discrete logarithm problem can be reduced to a period-finding problem on the additive group of pairs of integers modulo $N$. Consider a function $f(x_1, x_2) = a^{x_1 + x_2} (\text{mod} N)$. The ratio between the degeneracies of any two energy levels of the problem Hamiltonian is about one. We construct a division set $\{v_1 = \lceil \frac{Z}{2} \rceil, v_2 = \lceil \frac{v_1}{2} \rceil, \ldots, v_L = \lceil \frac{v_{L-1}}{2} \rceil = 1\}$ (it could be $v_{L-1} = \lceil \frac{v_{L-2}}{2} \rceil = 1$, then the division set contains $L - 1$ elements, for convenience, we assume there are $L$ elements). The Hamiltonians $H_{P_i}$ can be constructed by using $L$ oracles as

$$H_{P_i}|k\rangle = \begin{cases} -1 \cdot |k\rangle, & \text{if } h_k \leq v_i, \\ 0 \cdot |k\rangle, & \text{if } h_k > v_i, \end{cases}, \ i = 1, \ldots, L.\tag{10}$$

Then we construct a sequence of intermediate Hamiltonians as shown in Eq. (5) and let $H_P = H_{P_L}$. The ratio $N_i/N_{i-1}$ for the factoring problem Hamiltonian is about $1/2$ in each step of the algorithm, therefore the total runtime of the algorithm for obtaining the ground state of the problem Hamiltonian $H_P$ is proportional to the number of steps $L$ of the algorithm, and scales as $O(\log N)$. The order $r$ of $a$ can be obtained by measuring the ground state of the problem Hamiltonian, thus solving the factoring problem efficiently in $L$ steps.
where all the variables are integers, and \( r \) is the smallest positive integer for which \( a^r \mod N = 1 \). This function is 2-tuple periodic, since \( f(x_1 + l, x_2 - ls) = f(x_1, x_2) \). Given \( a \) and \( b = a^s \), the discrete logarithm problem is to determine the integer \( s \). By applying the quantum order-finding algorithm [16], one can obtain \( r \) with one query of a quantum black box \( U \) that performs the unitary transform
\[
U|x_1⟩|y⟩ \rightarrow |x_1⟩|y \oplus f(x_1, x_2)⟩,
\]
and \( O\left(\lceil \log r \rceil^2 \right) \) other operations. Then the integer \( s \) can be obtained from \( r \), thus solving the discrete logarithm problem.

Our algorithm can also be applied for solving the discrete logarithm problem. For the periodic function \( f(x_1, x_2) \), we define the problem Hamiltonian as
\[
H'_P|x_1⟩|x_2⟩ = h_{x_1x_2}|x_1⟩|x_2⟩ = b^{x_1}a^{x_2}(\mod N)|x_1⟩|x_2⟩ = a^{sx_1+x_2}(\mod N)|x_1⟩|x_2⟩, \tag{11}
\]
where \( x_1, x_2 = 0, \ldots, N-1 \). The ground state eigenvalue of \( H'_P \) is 1, and the corresponding ground state is in form of
\[
\frac{1}{\sqrt{N/r}} \sum_{sx_1+x_2=kr} |x_1⟩|x_2⟩, \quad k = 0, \ldots, \lfloor (N - 1) / r \rfloor.
\]
The eigenvalues of \( H'_P \) distribute uniformly in the period of \( a \), since \( x_1 \) and \( x_2 \) range from 0 to \( N - 1 \) uniformly. We construct a division set as shown in solving the factoring problem by using the method of bisection, and construct the Hamiltonians \( H_i \) and a sequence of intermediate Hamiltonians \( H_i \). The ground state of \( H'_P \) can be obtained by using our algorithm in \( \lceil \log_2 N \rceil \) steps. Then by measuring the qubits to obtain states \( |x_1⟩ \) and \( |x_2⟩ \), one can calculate the integer \( s \), thus solving the discrete logarithm problem.

D. The period-finding problem and the order-finding problem

The period-finding problem can be described as follows [4]: suppose \( f \) is a periodic function producing a single bit as output and such that \( f(x + r) = f(x) \), for some unknown \( 0 < r < 2^L \), where \( x, r \in \{0, 1, \ldots\} \). Given a quantum black box \( U \) which performs the unitary transform \( U|x⟩|y⟩ \rightarrow |x⟩|y \oplus f(x)⟩ \), how many black box queries and other operations are required to determine \( r \)? Quantum algorithm solves this problem using one query and \( O(L^2) \) other operations [4]. In this problem, the states \( |x + kr⟩, k = 0, \ldots, 2^L/r \) have the same value, its spectrum is the same as shown in Fig. 1. Therefore it can be solved by obtaining the states \( |x + kr⟩ \) that correspond to the ground state eigenvalue of the function.
Our algorithm can be applied to solve this problem with $O(\log N)$ queries of $f$.

The order-finding problem is: for positive integers $y$ and $N$, $y < N$, that have no common factors, the order of $y$ modulo $N$ is defined to be the least positive integer, $r$, such that $y^r = 1 \pmod N$. The order-finding problem is to determine the order for some specified $y$ and $N$. Order-finding is believed to be a hard problem on a classical computer, and is used as the basis for the RSA crypto system. With $L \equiv \log_2 N$ being the number of bits needed to specify $N$, the order-finding problem can be solved efficiently by using our algorithm in $L$ steps, as shown in solving the factoring problem.

E. The graph isomorphism problem

The GIP is considered as one of the few natural problems in the complexity class of NP, and could be classified as neither NP-complete nor P. It is an important theoretical problem in mathematics and computer science since 1970s, for a review on the GIP, see Ref. [17] and references therein. Classically, it has been proven that the GIP is solvable in quasi-polynomial time in a recent work [18].

The GIP can be described as follows: given two undirected graphs $A_1(V_1, E_1)$ and $A_2(V_2, E_2)$ with vertex sets $V_1$, $V_2$, and edge sets $E_1$, $E_2$, respectively, and $|V_1| = |V_2| = n$, the graph $A_1$ is isomorphic to $A_2$, if there exists a bijection $\rho : V_1 \rightarrow V_2$ such that for all $x, y \in V_1$, $(x, y) \in E_1$ if and only if $(\rho x, \rho y) \in E_2$. The GIP can be reduced to the graph automorphism problem, which is a relabeling of vertices of a graph that preserves the graph edges. The graph automorphism problem for a graph $A(V, E)$ is to find a set of generators for the group of automorphisms $Aut(A)$ of $A$. We construct the graph $A = A_1 \cup A_2$ by putting two graphs $A_1$ and $A_2$ together. The graphs $A_1$ and $A_2$ are isomorphic if and only if there exists an automorphism of the graph $A$ that maps a vertex from $A_1$ to a vertex from $A_2$.

It has been found that solving the HSP on the symmetric group leads to a solution to the GIP [19]. The GIP can be casted to a HSP over a symmetric group $S_{2n}$ of all permutations of the graph $A$. The hidden function is $f : S_{2n} \rightarrow \{\text{permutations of graph } A\} \text{ by } f(\pi) = \pi(A)$. Therefore the function $f$ applies a permutation to the vertices of $A$, it separates cosets of $K = Aut(A)$, and is efficiently computable. Therefore the GIP can be reduced to a non-Abelian HSP that is to find a hidden subgroup $K = Aut(A)$ of the group $S_{2n}$. If $K$ contains
only an identity group element \( \{ e \} \), then we know the graphs \( A_1 \) and \( A_2 \) are not isomorphic; otherwise if \( K \) contains more than one group element, it indicates that the graph \( A \) is automorphic, and the two graphs \( A_1 \) and \( A_2 \) are isomorphic.

The symmetric group \( S_{2n} \) has \(|S_{2n}| = (2n)!\) group elements. By applying the hidden function \( f \), the group elements of \( S_{2n} \) are mapped to at most \((2n)!\) distinct integers from 1 to \((2n)!\). In this case, it means that the two graphs are not isomorphic. Otherwise, if the group elements of \( S_{2n} \) are mapped to less than \((2n)!\) distinct integers which start from 1 to at most \((2n)!/2\), it means that the two graphs are isomorphic. The decision version of the GIP can be solved by our algorithm in one step.

By using the oracle \( O_f : \ket{j} \ket{0} \rightarrow \ket{j} \ket{f(j)} \), where the state \( \ket{j} \) is associated with a permutation \( \pi_j \in S_{2n} \), the problem Hamiltonian of the GIP is:

\[
H_P' \ket{j} = h_j \ket{j}, \quad j = 1, \cdots, (2n)!.
\]  

We use a division threshold value \( v_1 = \frac{(2n)!}{2} \) to construct the Hamiltonian \( H_P \) as:

\[
H_P \ket{k} = \begin{cases} 
-1 \cdot \ket{k}, & \text{if } h_k \leq v_1 \\
0 \cdot \ket{k}, & \text{if } h_k > v_1
\end{cases}
\]  

If the group elements of \( S_{2n} \) are mapped to \((2n)!\) distinct integers, the ratio of \( N_1/N \) will be \( N_1/N = 1/2 \), here \( N = (2n)! \) and \( N_1 \) is the number of computational basis states with the eigenvalue \(-1\). To apply our algorithm, we set the initial state of the problem to be \( \ket{\psi_0} = \frac{1}{\sqrt{(2n)!}} \sum_{j=0}^{(2n)!-1} \ket{j} \), the overlap between the initial state and the target state \( \ket{\psi_1} = \frac{1}{\sqrt{(2n)!/2}} \sum_{x, f(x) = -1} \ket{x} \) is \( 1/\sqrt{2} \). In this case, by setting the runtime \( t = \pi/(\sqrt{2}c) \) and run the algorithm, a resonant transition occurs and can be identified by observing a decay of the probe qubit with probability one. It means the subgroup \( K \) contains only an identity group element \( \{ e \} \), therefore the two graphs \( A_1 \) and \( A_2 \) are not isomorphic. On the other hand, if the subgroup \( K \) contains more than one group element, the ratio \( N_1/N \) will be \( N_1/N = 1 \). By applying our algorithm, the resonant transition probability reaches one at runtime \( t = \pi/(2c) \). In this case, the two graphs are isomorphic. These two cases can be distinguished in our algorithm, therefore the decision version of the GIP can be solved in one step. To find out the exact group elements for the isomorphism of two graphs, we can use the division set \( \{ v_1 = \frac{(2n)!}{2}, v_2 = \frac{v_1}{2}, \cdots, v_m = \frac{v_{m-1}}{2} = 1 \} \) to run the algorithm step by step to find the state of the hidden subgroup.
The GIP can be solved in a simpler way if we are only interested in determining whether two graphs are isomorphic or not, instead of finding the permutations that keep the isomorphism of the graphs in addition. We can construct the problem Hamiltonian as:

$$\hat{H}_P|k\rangle = \begin{cases} -1 \cdot |k\rangle, & \text{if } h_k > v_1 \\ 0 \cdot |k\rangle, & \text{if } h_k \leq v_1 \end{cases},$$

where $v_1 = \frac{(2n)!}{2}$. In this construction, in the case where the group elements of $S_{2n}$ are mapped to $(2n)!$ distinct integers, the ratio $N_1/N = 1/2$, where $N_1$ represents the number of computational basis states with eigenvalue $-1$. Setting the initial state of the problem to be $|\psi_0\rangle = \frac{1}{\sqrt{(2n)!}} \sum_{j=0}^{(2n)!-1} |j\rangle$, its overlap with the state $|\psi_1\rangle = \frac{1}{\sqrt{(2n)!/2}} \sum_{x,f(x)=-1} |x\rangle$ is still $1/\sqrt{2}$. By applying our algorithm, a resonant transition occurs at time $t = \pi/((\sqrt{2}c)$, it can be identified by observing a decay of the probe qubit with probability one. In this case, we can determine that the hidden subgroup $K$ contains only an identity group element $\{e\}$, and the two graphs are not isomorphic. In the other case where there are more than one group element in either the subgroup or its cosets, the ground state eigenvalue of the Hamiltonian $\hat{H}_P$ is 0 in the above mapping. By applying our algorithm, no resonant transition will occur in this case. Therefore by setting the evolution time $t = \pi/(\sqrt{2}c)$ and run our algorithm, the GIP can be solved in one step by observing if a resonant transition occurs which is indicated by a decay on the probe qubit.

F. The poly($n$)-unique shortest vector problem

Another application of the non-Abelian HSP is the lattice problems. An $n$-dimensional point lattice is a discrete subset of $\mathbb{R}^n$ closed under addition and substraction. It is generated by a set of integer linear combination of $n$ linearly independent vectors $b_1, \ldots, b_n$ as

$$B = \left\{ \sum_{i=1}^{n} x_i b_i : x_i \in \mathbb{Z} \right\},$$

where $\mathbb{Z}$ denotes the integer set. The shortest vector problem is: given a lattice described by the basis vectors, find the shortest nonzero vector in the lattice.

The SVP is extremely difficult in $n$ dimensions, all known algorithms for the SVP requires exponential time in $n$. Approximate solutions have been developed, i.e., finding some vector that is not too much larger than the shortest one. A vector is $f(n)$-unique if it is a factor of
$f(n)$ shorter than all other nonparallel vectors. Approximating the shortest vector to within a constant, that is, the $O(1)$-unique SVP is known to be NP-hard under a randomized reduction \cite{20}. The $O(2^n)$-unique SVP is solvable in polynomial time by applying the so-called LLL algorithm \cite{21}. And the decision version of poly($n$)-unique SVP is known to be in NP and coNP \cite{22,23}.

It has been found \cite{24} that the poly($n$)-unique SVP can be reduced to the two-point problem on lattice, and the two-point problem can be reduced to the dihedral coset problem, which can be reduced to the dihedral HSP. Therefore the solution to the poly($n$)-unique SVP can be obtained if the dihedral HSP is solved. An efficient quantum algorithm for the HSP of the dihedral group $D_N$ will provide an efficient quantum algorithm for solving the poly($n$)-unique SVP.

The dihedral group $D_N$ has $2N$ elements, it is a group of symmetries of an $N$-sided regular polygon. There are $2N$ ways one can apply rotations or reflections in a distinct way. Every element of $D_N$ can be represented as a tuple $(r_1, r_2)$ where $r_1 \in \{0, 1\}$ representing the number of reflections and $0 \leq r_2 < N$ representing the number of rotations. The dihedral coset problem is to find a constant integer $l$, given a collection of states in the form

$$\frac{1}{\sqrt{2}}(|0\rangle|x\rangle + |1\rangle(x+l \mod N))$$ for random $x$. \hspace{1cm} (16)

As shown in Ref. \cite{24}, the dihedral HSP can be reduced to the case where the subgroup is of the form $K = \{(0, 0), (1, l)\}$. We define a function $f$ hides the subgroup $K$ by mapping $K$ to a constant (say 0), and each of the other $N-1$ cosets of $K$ to a distinct integer. The dihedral group $D_N$ has exponential number (in log $N$) of subgroups of order two, and are infeasible to solve by classical algorithms. While the dihedral HSP has the same structure as that of the Simon’s problem. And the state of the hidden subgroup $K$ can be obtained by applying our algorithm in time $O(\log N)$ in the same way as that of in solving the Simon’s problem. The constant $l$ can be obtained by measuring the state of the hidden subgroup $K$, and solving the dihedral coset problem, thus obtaining the solution to the poly($n$)-unique SVP.
G. A minimum-search problem with distinct values

Let $W[0, \cdots, N-1]$ be an unsorted table of $N$ items, each holding a distinct integer in the range of $[1, \cdots, N]$. The minimum-search problem is to find the index $y$ such that $W[y]$ is the minimum. This problem has the same spectrum as that of the Simon’s problem and the dihedral HSP, except that the hidden function is one-to-one mapping in this problem. Classically, the cost for solving this problem scales as $O(N)$, while it can be solved efficiently by using our algorithm.

The problem Hamiltonian $H_P'$ of the minimum-search problem is diagonal in the computational basis states of qubits with distinct eigenvalues, and can be constructed as:

$$H_P'|j\rangle = h_j|j\rangle, j = 0, 1, \cdots, N-1,$$

where $h_j = 1, 2, \cdots, N$. We prepare a division set $\{v_1 = \lfloor \frac{N}{2} \rfloor, v_2 = \lfloor \frac{v_1}{2} \rfloor, \cdots, v_m = \lfloor \frac{v_{m-1}}{2} \rfloor\}$ where $m = \lfloor \log_2^N \rfloor$, and construct the Hamiltonians $H_{P_k}$ by using $m$ different oracles as

$$H_{P_k}|j\rangle = h_{j_k}|j\rangle, k = 1, \cdots, m,$$

where

$$h_{j_k} = \begin{cases} -1, & \text{if } h_j \leq v_k \\ 0, & \text{if } h_j > v_k \end{cases}.$$

The state space of the problem is reduced in a rate of $1/2$ in each step. Then we can construct a sequence of intermediate Hamiltonians as shown in Eq. (5) and let $H_P = H_{P_m}$, and run our algorithm. The ground state of the problem Hamiltonian, thus the solution to the minimum-search problem, can be obtained in $m$ steps with runtime $t = \frac{\pi}{\sqrt{2c}}$ in each step.

IV. APPLICATION OF THE ALGORITHM FOR OTHER STRUCTURED PROBLEMS

In the above applications of our algorithm for solving the HSP and the minimum-search problem with distinct values, the problems have the structure that they can be divided to a number of sets with the same dimension and each set corresponds to a distinct value. These problems can be reduced to the structured search problem, and solved efficiently using our algorithm. In the following, we study two problems with different structures, while can also be reduced to the structured search problem, including a minimum search problem with special structure and a problem of minimizing a symmetrized cost function.
A. A minimum-search problem with a special structure

We consider another minimum-search problem: let $W[0, \cdots, N-1]$ be an unsorted table of $N$ items, each is associated with an integer in the range $[0, \cdots, n-1]$, where $N = 2^n$. There are $N/2 + 1$ items correspond to the integer $n-1$, $N/2$ items correspond to $n-2$, $\cdots$, 2 items correspond to the integer 1, and one item corresponds to 0. The problem is to find the index $y$ such that $W[y]$ is the minimum. Let $h_k$ be the integer associated with the state $|k\rangle$, the problem Hamiltonian $H_P$ of this problem is constructed as

$$H_P|k\rangle = \begin{cases} -1 \cdot |k\rangle, & \text{if } h_k < 1 \\ 0 \cdot |k\rangle, & \text{if } h_k \geq 1 \end{cases}$$

where $k = 0, 1, \cdots, N-1$. The problem Hamiltonian can be written as $H_P = -|y\rangle\langle y|$, where $|y\rangle$ is the state associated with the eigenvalue 0. To solve this problem, we prepare a division set $\{v_1 = n-1, v_2 = n-2, \cdots, v_{n-1} = 1\}$, and construct Hamiltonians $H_P$ by using $n-1$ different oracles as

$$H_{P_i}|k\rangle = \begin{cases} -1 \cdot |k\rangle, & \text{if } h_k < v_i \\ 0 \cdot |k\rangle, & \text{if } h_k \geq v_i \end{cases}, \quad i = 1, \cdots, n-1$$

which can also be written as $H_{P_i} = -\sum_{q_i \in \Pi_i} |q_i\rangle\langle q_i|$, and $\Pi_1 \supset \cdots \supset \Pi_{n-1}$. The size of $\Pi_1, \cdots, \Pi_{n-1}$ are $N_1 = N/2 + 1, N_2 = N/2^2, \cdots, N_{n-1} = 1$, respectively. The set $\Pi_i$ contains all the basis states associated with integers that are less than $v_i$. The ratio $N_i/N_{i-1} \approx 1/2$, therefore by constructing intermediate Hamiltonians as shown in Eq. (5), the minimum-search problem with a special structure can be solved efficiently by using our algorithm in $n-1$ steps.

B. Minimization of a symmetrized cost function

The quantum approximate optimization algorithm (QAOA) \cite{25} performs a hybrid quantum-classic multi-step computation and is applied for solving a problem of minimization of a symmetrized function. The problem is defined as follows \cite{26}: let

$$h_3(z, z', z'') = \begin{cases} 0 & z + z' + z'' = 0 \\ x & z + z' + z'' = 1 \\ 1 & z + z' + z'' = 2 \\ 1 & z + z' + z'' = 3 \end{cases}$$

$$h_3(z, z', z'') = \begin{cases} 0 & z + z' + z'' = 0 \\ x & z + z' + z'' = 1 \\ 1 & z + z' + z'' = 2 \\ 1 & z + z' + z'' = 3 \end{cases}$$

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where $h_3$ depends on three bits $z, z', z''$ with values either 0 or 1, and $x$ is an integer greater than or equal to 3, and

$$h(z_1, \ldots, z_n) = \sum_{i<j<k} h_3(z_i, z_j, z_k).$$  \hfill (23)

Then a cost function is defined as

$$h(w) \equiv h(z_1, \ldots, z_n) = \frac{x}{2} w(n - w)(n - w - 1) + \frac{1}{2} w(w - 1)(n - w) + \frac{1}{6} w(w - 1)(w - 2),$$  \hfill (24)

where $w = z_1 + z_2 + \cdots + z_n$. The goal is to find the minimum of this symmetrized function of the Hamming weight $w$ as a result of all bits being treated symmetrically. It has been shown that the minimum of the cost function (where $w = 0$) cannot be solved efficiently by using the usual quantum adiabatic algorithm with linear interpolation between an initial Hamiltonian and the problem Hamiltonian [25].

Our algorithm can be applied for solving this problem. The number of basis states $|z_1, \ldots, z_n\rangle$ that have Hamming weight $w$ is $N_w = \binom{n}{w}$, therefore as $w \leq \lfloor n/2 \rfloor$, the ratio $N_w/N_{w-1} \propto n$ is polynomially large. We can prepare a division set: \{\begin{align*} v_1 &= h(w_1), v_2 &= h(w_2), \cdots, v_{n/2} &= 1 \end{align*}\} where $w_1 = \lfloor n/2 \rfloor$, $w_2 = \lfloor n/2 \rfloor - 1$, $\cdots$, $w_{n/2} = 1$ ($n$ is even), or $w_{(n+1)/2} = 1$ ($n$ is odd). And construct the Hamiltonians $H_{P_i}$ as shown in Eq. (3), and a sequence of intermediate Hamiltonians $H_i$ as shown in Eq. (5) in the same way as the above problems. The minimum of the function can be obtained efficiently in $n/2$ steps by using our algorithm.

V. DISCUSSION

In this work, we study the structures of a number of problems including the hidden subgroup problems and problems that can be reduced to both the Abelian and non-Abelian HSP, a minimum-search problem with distinct values, a minimum-search problem with a special structure and a problem of minimizing a symmetrized cost function. We find that these problems can be reduced to a type of structured search problems [1] that can be solved efficiently by using our algorithm, therefore these problems can also be solved efficiently by using our algorithm via multi-step quantum computation.
In our algorithm, by decomposing the problem based on its structure using polynomial number \( O(\log N) \) of different oracles to construct the intermediate Hamiltonians considering the structure of the problem to approach the problem Hamiltonian, the problem is solved efficiently in polynomial number of steps. Each step is an independent process and an oracle is used to reduce the search space of the problem. Therefore the power of the oracles is multiplied in each step, making the multi-step quantum computation process to be exponentially (with respect to the number of oracles) more powerful than the computation process of using just one oracle. We have shown that our algorithm achieves exponential speedup over Grover’s algorithm in solving the structured search problem, and has the same efficiency as that of Grover’s algorithm in solving the unstructured search problem where only one oracle can be used to decompose the problem \[1\]. The method of constructing a Hamiltonian evolution path by considering the structure of the problem can also be used in adiabatic quantum computing for constructing an efficient adiabatic evolution path \[1, 27\].

Quantum algorithms achieve exponential speedup over classical algorithms in solving problems that can be reduced to the Abelian HSP by using the quantum Fourier transform, and quadratic speedup over classical algorithm in solving problems that can be reduced to the unstructured search problem by using the amplitude amplification technique. In this work, we find that both the Abelian and the non-Abelian HSP, and some other structured problems can be reduced to the structured search problems, and our algorithm achieves exponential speedup over classical algorithms in solving these problems. The structured search problems and the unstructured search problem can be casted in the same framework in our algorithm.

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[1] H. Wang, S. Yu and H. Xiang, Efficient quantum algorithm for solving structured problems via multi-step quantum computation, Arxiv: 1912.06959 (2019).
[2] H. Wang, Quantum algorithm for obtaining the eigenstates of a physical system, Phys. Rev. A 93, 052334 (2016).
[3] Z. Li, et al., Quantum simulation of resonant transitions for solving the Eigenproblem of an effective water Hamiltonian, Phys. Rev. Lett. 122, 090504 (2019).
[4] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge Univ. Press, Cambridge, England, 2000).
[5] L. K. Grover, Quantum mechanics helps in searching for a needle in a haystack, Phys. Rev. Lett. 79, 325 (1997).
[6] A. Gilyen, Y. Su, G. H. Low, and N. Wiebe, Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing (2019), arXiv:1806.01838.
[7] J. M. Martyn, Z. M. Rossi, A. K. Tan and I. L. Chuang, Grand unification of quantum algorithms, PRX Quantum, 2, 040203 (2021).
[8] W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature 299, 802 (1982).
[9] D. Dieks, Communication by EPR devices, Phys. Lett. A 92, 271 (1982).
[10] A. M. Childs and W. van Dam, Quantum algorithms for algebraic problems, Rev. Mod. Phys. 82, 1 (2010).
[11] C. Dürr and P. Høier, A quantum algorithm for finding the minimum, e-print quant-ph/9607014 (1996).
[12] W. P. Baritompa, D. W. Bulger and G. R. Wood, Grover’s quantum algorithm applied to global optimization, SIAM J. Optim. 15, 1170 (2005).
[13] D. R. Simon, On the power of quantum computation, in Proceedings of the 35th Annual Symposium on Foundations of Computer Science, edited by S. Goldwasser (IEEE) 1994, pp. 116–123.
[14] I. Hen, Period finding with adiabatic quantum computation, EPL, 105 50005 (2014).
[15] P. Shor, Algorithms for quantum computation: discrete logarithms and factoring. Proc. 35th Ann. Symp. on Found. of Comp. Sci., 124-134 (IEEE Comp. Soc. Press, Los Alamitos, CA, 1994).
[16] P. Shor, Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer, SIAM J. Comput. 26, 1484 (1997), preliminary version in FOCS 1994.
[17] M. Grohe and P. Schweitzer, The graph isomorphism, Communications of the ACM, 63,
128 (2020).

[18] L. Babai, Graph isomorphism in quasipolynomial time. In Proceedings of the 48th Annual ACM Symposium on Theory of Computing (STOC’16, 2016), 684–697.

[19] K. Johannes, S. Uwe, and T. Jacobo, The graph isomorphism problem: Its structural complexity, Birkhäuser Boston, Boston, 1993.

[20] D. Micciancio, The shortest vector in a lattice is hard to approximate to within some constant, SIAM J. Comput., 30:2008 (2001).

[21] A. K. Lenstra, H. W. Lenstra, and L. Lovász, Factoring polynomials with rational coefficients, Mathematische Annalen, 261:515-534 (1982).

[22] D. Aharonov and O. Regev, Lattice problems in NP ∩ coNP. J. ACM, 52:749 (2005).

[23] V. Lyubashevsky and D. Micciancio, On bounded distance decoding, unique shortest vectors, and the minimum distance problem. In Proceedings of the 29th Annual International Cryptology Conference on Advances in Cryptology, pages 577-594, Berlin, Heidelberg, 2009. Springer-Verlag.

[24] O. Regev, Quantum computation and lattice problems, Proceedings of the 43rd Annual Symposium on Foundations of Computer Science (FOCS), 2002.

[25] E. Farhi, J. Goldstone, S. Gutmann, A quantum approximate optimization algorithm e-print arXiv: 1411.4028 (2014).

[26] E. Farhi, J. Goldstone, S. Gutmann, Quantum adiabatic evolution algorithms versus Simulated Annealing, e-print arXiv: quant-ph/0201031 (2002).

[27] D. Aharonov and A. Ta-Shma, Adiabatic quantum state generation and statistical zero knowledge, e-print arXiv: quant-ph/0301023v2 (2003); Adiabatic quantum state generation, SIAM J. Comput. 37, 47 (2007).