Dynamic Hardy type inequalities via alpha-conformable derivatives on time scales

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Abstract

We prove new Hardy-type $\alpha$-conformable dynamic inequalities on time scales. Our results are proved by using Keller’s chain rule, the integration by parts formula, and the dynamic Hölder inequality on time scales. When $\alpha = 1$, then we obtain some well-known time-scale inequalities due to Hardy. As special cases, we obtain new continuous, discrete, and quantum inequalities.

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1 Introduction

Hardy (1877–1947) established in 1920 a now classical discrete inequality.

\textbf{Theorem 1.1} (See [26]). Consider the nonnegative sequence of real numbers $\{\varrho(i)\}_{i=1}^\infty$. For $p > 1$, one has

$$\sum_{i=1}^\infty \frac{1}{i^p} \left( \sum_{j=1}^i \varrho(j) \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{i=1}^\infty \varrho^p(i). \quad (1)$$

In 1928, by using the calculus of variations, Hardy introduced the continuous version of inequality (1).

\textbf{Theorem 1.2} (See [28]). Let $\eta$ be a nonnegative continuous function on $[0, \infty)$. If $p > 1$, then

$$\int_0^\infty \frac{1}{\pi^p} \left( \int_0^\pi \eta(s)ds \right)^p d\pi \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty \eta^p(\pi)d\pi. \quad (2)$$

Moreover, the constant $\left( \frac{p}{p-1} \right)^p$ in (2) is sharp.

In 1927, Hardy and Littlewood (1885–1977) proved a discrete inequality that is an extension of (1).

\textbf{Theorem 1.3} (See [43]). Consider the sequence of nonnegative real numbers $\{\varrho(i)\}_{i=1}^\infty$.

(i) For $p > 1$ and $\alpha > 1$, one has

$$\sum_{i=1}^\infty \frac{1}{i^\alpha} \left( \sum_{j=1}^i \varrho(j) \right)^p \leq \zeta(\alpha, p) \sum_{i=1}^\infty \frac{1}{i^{\alpha-p}} \varrho^p(i); \quad (3)$$
(ii) For \( p > 1 \) and \( \alpha < 1 \), one has
\[
\sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left( \sum_{j=1}^{\infty} \varrho(j) \right)^{p} \leq G(\alpha, p) \sum_{i=1}^{\infty} \frac{1}{i^{\alpha} - p} \varrho^{p}(i); \tag{4}
\]
where the \( G(\alpha, p) \geq 0 \) in the inequalities \((3)\) and \((4)\) depend on \( \alpha \) and \( p \).

In \([43]\), the authors also studied the continuous analogous of Theorem 1.3.

**Theorem 1.4** (See \([43]\)). Let \( \eta \) be a nonnegative continuous function on \([0, \infty)\). If \( p > 1 \), then
\[
\int_{0}^{\infty} \left( \frac{1}{\pi} \int_{\pi}^{\infty} \eta(s) ds \right)^{p} \, d\pi \leq p^{p} \int_{0}^{\infty} \eta^{p}(\pi) \, d\pi
\]
or, by a trivial transformation,
\[
\int_{0}^{\infty} \left( \int_{\pi}^{\infty} \eta(s) ds \right)^{p} \, d\pi \leq p^{p} \int_{0}^{\infty} \pi^{p} \eta^{p}(\pi) \, d\pi. \tag{5}
\]

Hardy studied the integral form of \((3)\) and \((4)\) as follows.

**Theorem 1.5** (See \([28]\)). Consider the continuous function \( \eta \geq 0 \) on \([0, \infty)\).

(i) For \( \alpha > 1 \) and \( p > 1 \), we have
\[
\int_{0}^{\infty} \frac{1}{\pi^\alpha} \left( \int_{0}^{\pi} \eta(s) ds \right)^{p} \, d\pi \leq \left( \frac{p}{\alpha - 1} \right)^{p} \int_{0}^{\infty} \frac{1}{\pi^{\alpha - p}} \eta^{p}(\pi) \, d\pi. \tag{6}
\]

(ii) For \( \alpha < 1 \) and \( p > 1 \), we have
\[
\int_{0}^{\infty} \frac{1}{\pi^\alpha} \left( \int_{\pi}^{\infty} \eta(s) ds \right)^{p} \, d\pi \leq \left( \frac{p}{1 - \alpha} \right)^{p} \int_{0}^{\infty} \frac{1}{\pi^{\alpha - p}} \eta^{p}(\pi) \, d\pi. \tag{7}
\]

In the same year of 1928, Copson also extended \([11]\).

**Theorem 1.6** (See \([16]\)). Consider the nonnegative sequences \( \{\varrho(i)\}_{i=1}^{\infty} \) and \( \{\varsigma(i)\}_{i=1}^{\infty} \) of real numbers. Then,
\[
\sum_{i=1}^{\infty} \frac{\varsigma(i)}{\left( \sum_{j=1}^{i} \varsigma(j) \varrho(j) \right)^{\alpha}} \leq \left( \frac{p}{\alpha - 1} \right)^{p} \sum_{i=1}^{\infty} \frac{\varsigma(i)}{\left( \sum_{j=1}^{i} \varrho(j) \right)^{\alpha}} \left( \sum_{j=1}^{i} \varsigma(j) \right)^{p - \alpha}, \quad \text{for} \quad p \geq \alpha > 1, \tag{8}
\]
and
\[
\sum_{i=1}^{\infty} \frac{\varsigma(i)}{\left( \sum_{j=1}^{i} \varsigma(j) \varrho(j) \right)^{\alpha}} \leq \left( \frac{p}{1 - \alpha} \right)^{p} \sum_{i=1}^{\infty} \frac{\varsigma(i)}{\left( \sum_{j=1}^{i} \varrho(j) \right)^{\alpha}} \left( \sum_{j=1}^{i} \varsigma(j) \right)^{p - \alpha}, \quad \text{for} \quad p > 1 > \alpha \geq 0. \tag{9}
\]

In 1970, Leindler discussed the result \([8]\) when the limit of summation \( \sum_{n=1}^{\infty} r(m) < \infty \) changed from \( i \) to \( \infty \).

**Theorem 1.7** (See \([42]\)). Consider the nonnegative real numbers sequences \( \{\varrho(i)\}_{i=1}^{\infty} \) and \( \{\varsigma(i)\}_{i=1}^{\infty} \) with \( \sum_{j=1}^{\infty} \varsigma(j) < \infty \). For \( p > 1 > \alpha \geq 0 \), one has
\[
\sum_{i=1}^{\infty} \frac{\varsigma(i)}{\left( \sum_{j=1}^{i} \varsigma(j) \varrho(j) \right)^{\alpha}} \leq \left( \frac{p}{1 - \alpha} \right)^{p} \sum_{i=1}^{\infty} \frac{\varsigma(i)}{\left( \sum_{j=1}^{i} \varrho(j) \right)^{\alpha}} \left( \sum_{j=1}^{i} \varsigma(j) \right)^{p - \alpha}. \tag{10}
\]
Copson investigated the continuous form of (8) and (11) in 1976.

Theorem 1.8 (See [17]). Consider the continuous function \( \eta \geq 0 \) and \( \xi \) on \([0, \infty)\). Then,

\[
\int_0^\infty \frac{\xi(\pi) \left( \int_0^\infty \xi(s) \eta(\pi) ds \right)^p}{\left( \int_0^\infty \xi(s) ds \right)^\alpha} d\pi \leq \left( \frac{p}{\alpha - 1} \right)^p \int_0^\infty \xi(\pi) \eta^p(\pi) \left( \int_0^\pi \xi(s) ds \right)^{p-\alpha} d\pi
\]

for \( 1 < \alpha \leq p \) and

\[
\int_0^\infty \frac{\xi(\pi) \left( \int_0^\infty \xi(s) \eta(\pi) ds \right)^p}{\left( \int_0^\infty \xi(s) ds \right)^\alpha} d\pi \leq \left( \frac{p}{1 - \alpha} \right)^p \int_0^\infty \xi(\pi) \eta^p(\pi) \left( \int_0^\pi \xi(s) ds \right)^{p-\alpha} d\pi
\]

for \( 0 < \alpha \leq 1 < p \).

In 1987 Bennett, similarly to what Leindler did in Theorem 1.7, proved the following result.

Theorem 1.9 (See [10]). Consider the nonnegative real numbers sequences \( \{s(i)\}_{i=1}^\infty \) and \( \{\varsigma(i)\}_{i=1}^\infty \) with \( \sum_{j=1}^\infty \varsigma(j) < \infty \). For \( 1 < \alpha \leq p \), then

\[
\sum_{i=1}^\infty s(i) \left( \sum_{j=1}^\infty \varsigma(j) s(j) \right)^p \leq \left( \frac{p}{\alpha - 1} \right)^p \sum_{i=1}^\infty \varsigma(i) s^p(i) \left( \sum_{j=1}^\infty \varsigma(j) \right)^{p-\alpha}.
\]

Over several decades, Hardy-type inequalities have attracted many researchers and several refinements and extensions have been done to the previous results. We refer the reader to the works [7, 10, 11, 12, 10, 27, 30, 33, 35, 39, 12, 41, 77], and the references cited therein. Here we are particularly interested in the following extensions proved by Renaud in 1986.

Theorem 1.10 (See [51]). Consider the nonnegative real numbers and nonincreasing sequence \( \{s(i)\}_{i=1}^\infty \). For \( p > 1 \), we have

\[
\sum_{i=1}^\infty \left( \sum_{j=1}^\infty s(j) \right)^p \geq \sum_{i=1}^\infty s^p(i).
\]

Theorem 1.11 (See [51]). Consider a nonnegative and nonincreasing function \( \eta \) on the interval \([0, \infty)\). For \( 1 < p \), we have

\[
\int_0^\infty \left( \int_s^\infty \eta(s) ds \right)^p d\pi \geq \int_0^\pi \pi^p \eta^p(\pi) d\pi.
\]

Theorem 1.12 (See [51]). Consider a nonnegative and nonincreasing function \( \eta \) on the interval \([0, \infty)\). For \( p > 1 \), we have

\[
\int_0^\infty \frac{1}{\pi^p} \left( \int_0^\pi \eta(s) ds \right)^p d\pi \geq \frac{p}{p - 1} \int_0^\pi \eta^p(\pi) d\pi.
\]

The theory of time scales has become a trend and is now part of the mathematics subject classification: see 26E70, for “Real analysis on time scales”; 34K42, for “Functional-differential equations on time scales”; 34N05, for “Dynamic equations on time scales”; and 35R07, for “PDEs on time scales”. The subject has begun with the PhD thesis of Hilger, in order to get continuous and discrete results together [31]. In books [13, 14], Bohner and Peterson introduce most basic concepts and definitions related with the theory of time scales. In [11, 3, 13, 20, 24, 21, 42, 50], several mathematicians investigate new forms of dynamic inequalities. Rehák seems to be the first mathematician to have introduced a time-scale version of Hardy’s inequality, by obtaining in 2005 a dynamic inequality that unifies inequalities (1) and (2).
Theorem 1.13 (See [52]). Let $\mathbb{T}$ be a time scale, and $f \in C_{rd}([a, \infty)_\mathbb{T}, [0, \infty))$. If $p > 1$, then
\begin{equation}
\int_a^\infty \left( \frac{\int_a^\infty \eta(s) \Delta s}{\sigma(t) - a} \right)^p \Delta t < \left( \frac{p}{p - 1} \right)^p \int_a^\infty \eta^p(t) \Delta t,
\end{equation}
unsaless $\eta \equiv 0$. Furthermore, if $\mu(t)/t \to 0$ as $t \to \infty$, then inequality (17) is sharp.

Many other dynamic inequalities followed. For instance, in 2014 Saker et al. established the following results on time scales.

Theorem 1.14 (See [52]). Let $\mathbb{T}$ be time scale and $1 \le c \le k$. Let
\begin{equation}
\chi(t) = \int_a^t \lambda(s) \Delta s, \quad \text{for any } t \in [a, \infty)_\mathbb{T},
\end{equation}
and define
\begin{equation}
\Theta(t) = \int_a^t \lambda(s) \xi(s) \Delta s \quad \text{for any } t \in [a, \infty)_\mathbb{T}.
\end{equation}
Then,
\begin{equation}
\int_a^\infty \frac{\lambda(t)}{\chi^c(t)} (\Theta^c(t))^k \Delta t \le \frac{k}{c - 1} \int_a^\infty \chi^{1-c}(t) \lambda(t) \xi(t) (\Theta(t))^{k-1} \Delta t.
\end{equation}
and
\begin{equation}
\int_a^\infty \frac{\lambda(t)}{(\chi^c(t))^c} (\Theta^c(t))^k \Delta t \le \left( \frac{k}{c - 1} \right)^k \int_a^\infty \left( \frac{\chi(t)}{\lambda(t)} \right)^{(k-1)c} \lambda(t) \xi^k(t) \Delta t.
\end{equation}

Theorem 1.15 (See [52]). Let $\mathbb{T}$ be a time scale and $k > 1$ and $0 \le c < 1$. Let $\chi$ be defined as in (18) and define
\begin{equation}
\overline{\Theta}(t) = \int_t^\infty \lambda(s) \xi(s) \Delta s \quad \text{for any } t \in [a, \infty)_\mathbb{T}.
\end{equation}
Then,
\begin{equation}
\int_a^\infty \frac{\lambda(t)}{\chi^c(t)} (\overline{\Theta}(t))^k \Delta t \le \frac{k}{1-c} \int_a^\infty \chi^{1-c}(t) \lambda(t) \xi(t) (\overline{\Theta}(t))^{k-1} \Delta t
\end{equation}
and
\begin{equation}
\int_a^\infty \frac{\lambda(t)}{(\chi^c(t))^c} (\overline{\Theta}(t))^k \Delta t \le \left( \frac{k}{1-c} \right)^k \int_a^\infty \left( \frac{\chi(t)}{\lambda(t)} \right)^{(k-1)c} \lambda(t) \xi^k(t) \Delta t.
\end{equation}

In 2015, Saker et al. [53] established the following forms of the Hardy-type inequality.

Theorem 1.16 (See [53]). Let $\eta$ and $\xi$ be nonnegative rd-continuous functions on $[a, \infty)_\mathbb{T}$ with $\mathbb{T}$ a time scale and $a \in [0, \infty)_\mathbb{T}$.

(i) For $1 < \alpha \le p$, one has
\begin{equation}
\int_a^\infty \frac{\xi(\varpi) \left( \frac{1}{\sigma(t)} \xi(\varpi) \eta(\varpi) \Delta \varpi \right)^p}{\left( \frac{1}{\sigma(t)} \xi(\varpi) \Delta \varpi \right)^{\alpha}} \Delta \varpi \le \left( \frac{p}{\alpha - 1} \right)^p \int_a^\infty \frac{\xi(\varpi) \eta^p(\varpi) \left( \int_a^\infty \xi(\varpi) \Delta \varpi \right)^{\alpha(p-1)}}{\left( \int_a^\infty \xi(\varpi) \Delta \varpi \right)^{\alpha(p-1)}} \Delta \varpi.
\end{equation}
(ii) If \( p > 1 > \alpha \geq 0 \), then
\[
\int_{a}^{\infty} \xi(\varpi) \left( \frac{\int_{a}^{\infty} \xi(\vartheta) \eta(\vartheta) \Delta \vartheta}{(\int_{a}^{\infty} \xi(\vartheta) \Delta \vartheta)^{\alpha}} \right)^{p} \Delta \varpi \leq \left( \frac{p}{1 - \alpha} \right)^{p} \int_{a}^{\infty} \xi(\varpi) \eta^{p}(\varpi) \left( \int_{a}^{\infty} \xi(\vartheta) \Delta \vartheta \right)^{p - \alpha} \Delta \varpi.
\] (21)

(iii) If \( p > 1 > \alpha \geq 0 \), then
\[
\int_{a}^{\infty} \xi(\varpi) \left( \frac{\int_{a}^{\infty} \xi(\vartheta) \eta(\vartheta) \Delta \vartheta}{(\int_{a}^{\infty} \xi(\vartheta) \Delta \vartheta)^{\alpha}} \right)^{p} \Delta \varpi \leq \left( \frac{p}{1 - \alpha} \right)^{p} \int_{a}^{\infty} \xi(\varpi) \eta^{p}(\varpi) \left( \int_{a}^{\infty} \xi(\vartheta) \Delta \vartheta \right)^{p - \alpha} \Delta \varpi.
\] (22)

(iv) If \( p \geq \alpha > 1 \), then
\[
\int_{a}^{\infty} \xi(\varpi) \left( \frac{\int_{a}^{\infty} \xi(\vartheta) \eta(\vartheta) \Delta \vartheta}{(\int_{a}^{\infty} \xi(\vartheta) \Delta \vartheta)^{\alpha}} \right)^{p} \Delta \varpi \leq \left( \frac{p}{\alpha - 1} \right)^{p} \int_{a}^{\infty} \xi(\varpi) \eta^{p}(\varpi) \left( \int_{a}^{\infty} \xi(\vartheta) \Delta \vartheta \right)^{p - \alpha} \Delta \varpi.
\] (23)

Agarwal et al. \[4\] generalized inequality (16) to time scales as follows: for \( p > 1 \),
\[
\int_{0}^{\infty} \frac{1}{t^{p}} \left( \int_{0}^{t} \eta(s) \Delta s \right)^{p} \Delta t \geq \frac{p}{p - 1} \int_{0}^{\infty} \eta^{p}(t) \Delta t.
\] (24)

Recently, in 2020, Saker \[55\] proved the following theorem.

**Theorem 1.17.** Assume that \( \mathbb{T} \) is a time scale with \( \omega \in (0, \infty)_{\mathbb{T}} \). If \( m > 0 < h < 1 \), \( \chi(t) = \int_{t}^{\infty} \lambda(s) \Delta s \) and \( \Theta(t) = \int_{t}^{\infty} \lambda(s) \xi(s) \Delta s \), then
\[
\int_{\omega}^{\infty} \frac{\lambda(t)}{\chi^{m}(t)} (\Theta^{\sigma}(t))^{h} \Delta t \geq \left( \frac{h}{1 - m} \right)^{h} \int_{\omega}^{\infty} \lambda(t) \xi^{h}(t) \chi^{h - m}(t) \Delta t.
\]
If \( 0 < h < 1 < m \), \( \chi(t) = \int_{t}^{\infty} \lambda(s) \Delta s \) and \( \overline{\Theta}(t) = \int_{t}^{\infty} \lambda(s) \xi(s) \Delta s \), then
\[
\int_{\omega}^{\infty} \frac{\lambda(t)}{\chi^{m}(t)} (\overline{\Theta}(t))^{h} \Delta t \geq \left( \frac{hM}{m - 1} \right)^{h} \int_{\omega}^{\infty} \lambda(t) \xi^{h}(t) \chi^{h - m}(t) \Delta t,
\]
where
\[
M := \inf_{t \in \mathbb{T}} \frac{\chi^{\sigma}(t)}{\chi(t)} > 0.
\]

Also in 2020, El-Deeb et al. \[22\] established a generalization of (24) that unifies (14) and (15): for \( p \geq 1 \) and \( \gamma > 1 \), the inequality
\[
\int_{a}^{\infty} \frac{\lambda(\zeta) \tilde{\Psi}(\zeta)}{\Lambda^{\gamma}(\zeta)} \Delta \zeta \geq \frac{p}{\gamma - 1} \int_{a}^{\infty} \lambda(\zeta) \tilde{\Lambda}^{\gamma - \gamma}(\zeta) \eta^{p}(\zeta) \Delta \zeta
\] (25)
holds where
\[
\tilde{\Psi}(\zeta) = \int_{a}^{\zeta} \tilde{\lambda}(\eta) \eta(s) \Delta s \quad \text{and} \quad \tilde{\Lambda}(\zeta) = \int_{a}^{\zeta} \tilde{\lambda}(\eta) \Delta \eta.
\]

Furthermore, El-Deeb et al. \[23\] established a generalization of inequalities \[20\], \[21\], \[22\] and \[23\] on time scales as follows.
Fractional calculus (FC), the theory of integrals and derivatives of noninteger order, is a field of research with a history dating back to Abel, Riemann and Liouville: see [44] for an historical account. Here we are interested in such inequalities in the fractional sense.

Recently, just based on the classical limit definition of derivative, Khalil et al. [35] proposed a much simpler definition of a fractional derivative of a function \( f: \mathbb{R}^+ \to \mathbb{R} \), called the conformable derivative \( T_\alpha f(t) \), \( \alpha \in (0, 1] \), defined by

\[
T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}
\]

for all \( t > 0 \). This definition found wide resonance in the scientific community interested in fractional calculus, due to the fact that calculating the derivative by this definition is trivial compared with the definitions that are based on integration. The researchers in [35] also suggested a definition for the \( \alpha \)-conformable integral of a function \( \eta \) as follows:

\[
\int_a^b \eta(t) d_{\alpha}t = \int_a^b \eta(t)t^{\alpha-1}dt.
\]

After the seminal paper [35], Abdeljawad [2] made an extensive research of the newly introduced conformable calculus. In his work, he generalizes the definition of conformable derivative \( T_\alpha^a f(t) \) of \( f: \mathbb{R}^+ \to \mathbb{R} \) for \( t > a \in \mathbb{R}^+ \) as

\[
T_\alpha^a f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t - a)^{1-\alpha}) - f(t)}{\epsilon}.
\]

Theorem 1.18 (See [24]). Let \( \mathbb{T} \) be a time scale with \( a \in [0, \infty)_\mathbb{T} \). In addition, let \( f, g, k, r, w \) and \( v \) be nonnegative rd-continuous functions on \( [a, \infty)_\mathbb{T} \) such that \( k \) is nonincreasing. Assume there exist \( \theta, \beta \geq 0 \) such that

\[
\frac{w(t)}{w(t)} \leq \theta \left( \frac{G^\alpha(t)}{G^\beta(t)} \right) \quad \text{and} \quad \frac{w(t)}{v(t)} \leq \beta \left( \frac{K^\alpha(t)}{K(t)} \right),
\]

where

\[
G(t) = \int_a^t g(s)\Delta s \quad \text{with} \quad G(\infty) = \infty \quad \text{and} \quad K(t) = \int_a^t r(s)\Delta s, \quad t \in [a, \infty)_\mathbb{T}.
\]

If \( p \geq 1 \) and \( \alpha > \theta + 1 \), then

\[
\int_a^\infty k^\sigma(t)v^\sigma(t)w(t)(G^\sigma(t))^{-\alpha}(K^\alpha(t))^p \Delta t \leq \left( \frac{p + \beta}{\alpha - \theta - 1} \right)^p \int_a^\infty k^\sigma(t)v^\sigma(t)w(t)r^p(t)(G^\sigma(t))^{\alpha(p-1)}g^{p-1}(t)G^{\sigma(p-1)}(t) \Delta t. \tag{26}
\]

For more results on Hardy-type inequalities on time scales we refer to [5, 19, 46, 48, 54] and references therein. Here we are interested in such inequalities in the fractional sense.
Benkhettou et al. \[9\] introduced a conformable calculus on an arbitrary time scale, which is a natural extension of the conformable calculus. Based on such results, in the last few years many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g. polymers. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This seems to be the main advantages of fractional derivatives in comparison with classical integer-order models.

By using the conformable fractional calculus, many inequalities have been investigated like Hardy’s \[56,64\], Hermite–Hadamard’s \[15,36,60\], Opial’s \[57,59\] and Steffensen’s inequalities \[58\]. For example, in 2020, Saker et al. \[56\] proved a \(\alpha\)-conformable version of Theorems 1.14 and 1.15 on time scales as follows.

**Theorem 1.19** (See \[56\]). Let \(\mathbb{T}\) be a time scale and \(1 < c \leq k\). Define

\[
\chi(x) = \int_a^x \lambda(s) \Delta_\alpha s, \quad \text{and} \quad \Theta(x) = \int_a^x \lambda(s) \xi(s) \Delta_\alpha s.
\]

If

\[
\Theta(\infty) < \infty, \quad \text{and} \quad \int_a^\infty \frac{\lambda(s)}{(\chi^\sigma(s))^{c-\alpha+1}} \Delta_\alpha s < \infty,
\]

then

\[
\int_a^\infty \frac{\lambda(x)}{(\chi^\sigma(x))^{c-\alpha+1}} (\Theta(x))^k \Delta_\alpha x \leq \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty \frac{\lambda(x)(\chi(x))^{K(c-\alpha)}}{(\chi^\sigma(x))^{(1-k)(c-\alpha+1)}} \xi^k(x) \Delta_\alpha x.
\]

**Theorem 1.20** (See \[56\]). Let \(\mathbb{T}\) be a time scale, \(0 < c < 1\) and \(k > 1\). Define

\[
\chi(x) = \int_a^x \lambda(s) \Delta_\alpha s \quad \text{and} \quad \Theta(x) = \int_a^\infty \lambda(s) \xi(s) \Delta_\alpha s.
\]

If

\[
\Theta(\infty) < \infty \quad \text{and} \quad \int_a^\infty \frac{\lambda(s)}{(\chi^\sigma(s))^{c-\alpha+1}} \Delta_\alpha s < \infty,
\]

then

\[
\int_a^\infty \frac{\lambda(x)}{(\chi^\sigma(x))^{c-\alpha+1}} (\Theta^\sigma(x))^k \Delta_\alpha x \leq \left(\frac{k}{c-\alpha}\right)^k \int_a^\infty (\chi^\sigma(x))^{k(c-\alpha+1)} \lambda(x) \xi^k(x) \Delta_\alpha x.
\]

In 2021, Zakarya et al. \[64\] obtained \(\alpha\)-conformable versions on time scales of Theorem 1.19.

**Theorem 1.21** (See \[64\]). Assume that \(\mathbb{T}\) is a time scale with \(\omega \in (0, \infty)_\mathbb{T}\). Let \(k \leq 0 < h < 1\), \(\alpha \in (0, 1]\), and define

\[
\chi(t) = \int_t^\infty \lambda(s) \Delta_\alpha s \quad \text{and} \quad \Theta(t) = \int_t^\omega \lambda(s) \xi(s) \Delta_\alpha s.
\]

Then,

\[
\int_\omega^\infty \frac{\lambda(t)}{\chi^{k-\alpha+1}(t)} (\Theta^\sigma(t))^h \Delta_\alpha t \geq \left(\frac{h}{\alpha-m}\right)^h \int_\omega^\infty \lambda(t) \xi^h(t) \chi^{h-m+\alpha-1}(t) \Delta_\alpha t.
\]

**Theorem 1.22** (See \[64\]). Assume that \(\mathbb{T}\) is a time scale with \(\omega \in (0, \infty)_\mathbb{T}\), \(0 < h < 1 < k\) and \(\alpha \in (0, 1]\). Define

\[
\chi(t) = \int_t^\infty \lambda(s) \Delta_\alpha s \quad \text{and} \quad \Theta(t) = \int_t^\infty \lambda(s) \xi(s) \Delta_\alpha s
\]
such that

\[ M := \inf_{t \in \mathbb{R}} \frac{\chi^\sigma(t)}{\chi(t)} > 0. \]

Then,

\[
\int_{\omega}^{\infty} \frac{\lambda(t)}{\chi^{k-\alpha+1}(t)} \xi(t) \Delta_{\alpha} t \geq \left( \frac{hM^{k-\alpha+1}}{k-\alpha} \right)^{h} \int_{\omega}^{\infty} \lambda(t) \xi(t) \chi^{h-k+\alpha-1}(t) \Delta_{\alpha} t. 
\]

Here, we prove new Hardy-type dynamic inequalities via the \(\alpha\)-conformable calculus on time scales. Our inequalities have a completely new form and may be considered as extensions of inequalities \([20], [21], [22] \) and \([23]\). As special cases, we obtain some new continuous, discrete and quantum inequalities of Hardy-type, generalizing those obtained in the literature.

The paper is organized as follows. In Section 2, we briefly recall necessary results and notions. The original results being then given and proved in Section 3. We end with Section 4 of conclusion.

## 2 Preliminaries

We recall the necessary definitions and concepts about the time-scale \(\alpha\)-conformable calculi, which are used in the next section. For more details we refer the readers to \([9], [13], [14], [15]\).

Every nonempty arbitrary closed subset of the real numbers is called a time scale, being denoted by \(\mathbb{T}\). One assumes that \(\mathbb{T}\) has the standard topology on the real numbers \(\mathbb{R}\). The forward jump operator \(\sigma : \mathbb{T} \rightarrow \mathbb{T}\) is defined by

\[ \sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}, \quad t \in \mathbb{T}, \quad (27) \]

and the backward jump operator \(\rho : \mathbb{T} \rightarrow \mathbb{T}\) by

\[ \rho(t) := \sup \{ s \in \mathbb{T} : s < t \}, \quad t \in \mathbb{T}. \quad (28) \]

In Definitions \([27]\) and \([28]\) we set \(\sup \mathbb{T} = \inf \emptyset \) (i.e., \(\sigma(t) = t\) if \(t\) is the minimum of \(\mathbb{T}\)) and \(\inf \mathbb{T} = \sup \emptyset \) (i.e., \(\rho(t) = t\) if \(t\) is the maximum), where \(\emptyset\) is the empty set.

**Definition 2.1** (See \([9]\)). Let \(\eta : \mathbb{T} \rightarrow \mathbb{R}, \ t \in \mathbb{T}^k\), and \(\alpha \in (0, 1]\). For \(t > 0\), we define \(T^\Delta_{\alpha}(\eta)(t)\) to be the number (provided it exists) with the property that, given any \(\varepsilon > 0\), there is a \(\delta\)-neighborhood \(U_t \subset \mathbb{T}\) of \(t, \ \delta > 0\), such that

\[
|\eta(\sigma(t)) - \eta(s)|t^{1-\alpha} - T^\Delta_{\alpha}(\eta)(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|
\]

for all \(s \in U_t\). We call \(T^\Delta_{\alpha}(\eta)(t)\) the conformable fractional derivative of \(\eta\) of order \(\alpha\) at \(t\), and the conformable fractional derivative on \(\mathbb{T}\) at \(0\) is defined as \(T^\Delta_{\alpha}(\eta)(0) = \lim_{t \rightarrow 0^+} T^\Delta_{\alpha}(\eta)(t)\).

**Lemma 2.2** (See \([9]\)). Let \(\alpha \in (0, 1]\). Suppose \(\alpha\)-conformable differentiable of order \(\alpha\) at \(\zeta \in \mathbb{T}^k\), and continuous function \(\xi : \mathbb{T} \rightarrow \mathbb{R}\) and the differentiable continuously function \(\eta : \mathbb{R} \rightarrow \mathbb{R}\). There exists a constant \(c \in [\zeta, \sigma(\zeta)]_\mathbb{R}\) such that

\[
T^\Delta_{\alpha}(\eta \circ \xi)(\zeta) = \eta'(\xi(c))T^\Delta_{\alpha}(\xi)(\zeta). \quad (29)
\]

**Lemma 2.3** (See \([9]\)). Let \(\eta : \mathbb{R} \rightarrow \mathbb{R}\) be continuously differentiable, \(\alpha \in (0, 1]\) and \(\xi : \mathbb{T} \rightarrow \mathbb{R}\) be a \(\alpha\)-conformable differentiable function. Then \((\eta \circ \xi) : \mathbb{T} \rightarrow \mathbb{R}\) is \(\alpha\)-conformable differentiable and we have

\[
T^\Delta_{\alpha}(\eta \circ \xi)(t) = \left\{ \int_{0}^{\eta'(\xi(t)) + h\mu(t) t^{\alpha-1} T^\Delta_{\alpha}(\xi(t))} \right\} T^\Delta_{\alpha}(\xi)(t). \quad (30)
\]

For the continuous functions \(\eta\) and \(\xi\), we have that the product \(\eta \xi : \mathbb{T} \rightarrow \mathbb{R}\) is conformable fractional differentiable with

\[
T^\Delta_{\alpha}(\eta \xi) = T^\Delta_{\alpha}(\eta) \xi + \eta T^\Delta_{\alpha}(\xi) = T^\Delta_{\alpha}(\eta) \xi^\sigma + \eta T^\Delta_{\alpha}(\xi). \quad (31)
\]

The \(\alpha\)-conformable integration by parts formula on time scales is given in the following Lemma.
Lemma 2.4 (See [9]). Suppose that \( a, b \in \mathbb{T} \) where \( b > a \). If \( \eta, \xi \) are conformable \( \alpha \)-fractional differentiable and \( \alpha \in (0, 1] \), then
\[
\int_a^b \eta(t)T_\alpha^\Delta \xi(t)\Delta \alpha t = \left[ \eta(t)\xi(t) \right]_a^b - \frac{1}{\alpha} \int_a^b T_\alpha^\Delta \eta(t)\xi^\sigma(t)\Delta \alpha t.
\] (32)

Lemma 2.5 (The \( \alpha \)-conformable Hölder inequality, see [33]). Let \( a, b \in \mathbb{T} \) with \( a < b \). If \( \alpha \in (0, 1] \) and \( \eta, \xi : \mathbb{T} \rightarrow \mathbb{R} \), then
\[
\int_a^b |\eta(t)\xi(t)|\Delta \alpha t \leq \left( \int_a^b \eta^p(t)\Delta \alpha t \right)^{1/p} \left( \int_a^b \xi^q(t)\Delta \alpha t \right)^{1/q}
\] (33)
where \( p, q > 1 \) and \( 1/p + 1/q = 1 \).

We need relations between different types of calculus on general time scales \( \mathbb{T} \) and for the particular cases of continuous, discrete, and quantum calculi. Such relations are found in [9, 13, 14, 45]:

(i) for any time scale \( \mathbb{T} \), we have
\[
(\eta)^{\Delta_{\alpha}}(t) = (\eta)^{\Delta}(t)^{1-\alpha},
\]
\[
\int_a^b \eta(t)\Delta_{\alpha} t = \int_a^b \eta(t)t^{\alpha-1}\Delta t.
\]
(ii) If \( \mathbb{T} = \mathbb{R} \), then
\[
t = \sigma(t),
\]
\[
0 = \mu(t),
\]
\[
\eta^{\Delta}(t) = \eta'(t),
\] (34)
\[
\int_a^b \eta(t)\Delta t = \int_a^b \eta(t)dt.
\]
(iii) If \( \mathbb{T} = \mathbb{Z} \), then
\[
\sigma(t) = t + 1,
\]
\[
\mu(t) = 1,
\]
\[
\eta^{\Delta}(t) = \Delta \eta(t),
\] (35)
\[
\int_a^b \eta(t)\Delta t = \sum_{t=a}^{b-1} \eta(t).
\]
(iv) If \( \mathbb{T} = h\mathbb{Z} \), then
\[
\sigma(t) = t + h,
\]
\[
\mu(t) = h,
\]
\[
\eta^{\Delta}(t) = \frac{\eta(t+h) - \eta(t)}{h},
\] (36)
\[
\int_a^b \eta(t)\Delta t = \sum_{t=a}^{b-1} h\eta(ht).
\]
(v) If \( \mathbb{T} = q^\mathbb{Z} \), then
\[
\sigma(t) = qt,
\]
\[
\mu(t) = (q-1)t,
\]
\[
\eta^{\Delta}(t) = \frac{\eta(qt) - \eta(t)}{(q-1)t},
\] (37)
\[
\int_a^b \eta(t)\Delta t = (q-1) \sum_{t=\log_q a}^{\log_q b-1} q^t \eta(q^t).
\]
3 Main Results

First, we enlist assumptions for the proofs of our main results.

$$(S_1) \ \mathbb{T} \text{ is a time scale, } p \geq 1, \ a \in \mathbb{R}, \ \text{and } \alpha \in (0, 1].$$

$$(S_2) \ f, \ g, \ k, \ r, \ w \ \text{and} \ v \geq 0 \text{ are rd-continuous functions on } [a, \infty) \ \text{with } k \text{ monotonous}.$$

$$(S_3) \ \theta \text{ and } \beta \text{ are nonnegative constants}.$$

$$(S_4) \ \gamma > \theta + 1.$$

$$(S_5) \ 0 \leq \gamma < \alpha.$$

$$(S_6) \ G(\varsigma) = \int_a^\varsigma g(\varsigma) \Delta \varsigma, \ G(\infty) = \infty, \ t \in [a, \infty).$$

$$(S_7) \ H(\varsigma) = \int_\varsigma^\infty g(\varsigma) \Delta \varsigma, \ t \in [a, \infty).$$

$$(S_8) \ K(\varsigma) = \int_a^\varsigma r(\varsigma) f(\varsigma) \Delta \varsigma, \ t \in [a, \infty).$$

$$(S_9) \ F(\varsigma) = \int_\varsigma^\infty r(\varsigma) f(\varsigma) \Delta \varsigma, \ t \in [a, \infty).$$

$$(S_{10}) \ \frac{T_\alpha^\varsigma w(\varsigma)}{w(\varsigma)} \leq \theta \left( \frac{T_\alpha^\varsigma G(\varsigma)}{G(\varsigma)} \right).$$

$$(S_{11}) \ \frac{T_\alpha^\varsigma w(\varsigma)}{w(\varsigma)} \leq \theta \left( \frac{T_\alpha^\varsigma H(\varsigma)}{H(\varsigma)} \right).$$

$$(S_{12}) \ \frac{T_\alpha^\varsigma w(\varsigma)}{w^\sigma(\varsigma)} \geq \theta \left( \frac{T_\alpha^\varsigma G(\varsigma)}{G(\varsigma)} \right).$$

$$(S_{13}) \ \frac{T_\alpha^\varsigma w(\varsigma)}{w^\sigma(\varsigma)} \geq \theta \left( \frac{T_\alpha^\varsigma H(\varsigma)}{H(\varsigma)} \right).$$

$$(S_{14}) \ \frac{T_\alpha^\varsigma v(\varsigma)}{v^\sigma(\varsigma)} \leq \beta \left( \frac{T_\alpha^\varsigma K(\varsigma)}{K(\varsigma)} \right).$$

$$(S_{15}) \ \frac{T_\alpha^\varsigma v(\varsigma)}{v^\sigma(\varsigma)} \geq \beta \left( \frac{T_\alpha^\varsigma F(\varsigma)}{F(\varsigma)} \right).$$

$$(S_{16}) \ k(t) = v(t) = w(t) = 1.$$

$$(S_{17}) \ r(t) = g(t).$$

$$(S_{18}) \ r(t) = g(t) = 1.$$

$$(S_{19}) \ \theta = \beta = 0.$$

$$(S_{20}) \ a = 0.$$

$$(S_{21}) \ a = 1.$$

$$(S_{22}) \ \gamma = p.$$

Now, we are ready to state and prove our original results, which extend several results in the literature.
Proof. Applying (32) with Theorem 3.1. 

\[
\int_a^\infty k^\alpha(t)\nu^\alpha(t)w(t)g(t)\left(G^\alpha(t)\right)^{\alpha-\gamma-1}(K^\alpha(t))^{p-\alpha+1} \Delta \alpha t \\
\leq \left(\frac{p + \beta - \alpha + 1}{\gamma - \theta - \alpha}\right) \int_a^\infty k^\alpha(t)\nu^\alpha(t)w(t)r^p(t)f^p(t)(G^\alpha(t))^{\alpha+\gamma-1}(K^\alpha(t))^{1-\alpha} \Delta \alpha t.
\]  

(38)

Using (29), (31), and (32), we have

\[
\int_a^\infty k^\alpha(t)\nu^\alpha(t)w(t)g(t)\left(G^\alpha(t)\right)^{\alpha-\gamma-1}(K^\alpha(t))^{p-\alpha+1} \Delta \alpha t \\
= \left[ u(t)k(t)v(t)(K^{p-\alpha+1}(t)) \right]_a^\infty + \int_a^\infty (-u(t))T^\alpha_\alpha(k(t)v(t)(K^{p-\alpha+1}(t)) \Delta \alpha t,
\]

(39)

where

\[
u(t) = -\int_t^\infty w(s)g(s)\left(G^\alpha(s)\right)^{\alpha-\gamma-1} \Delta \alpha s.
\]

Using (29), (31), and (32), we have

\[
T^\alpha_\alpha w(s)G^\alpha(s) = T^\alpha_\alpha w(s)G^\alpha(s) + w(s)T^\alpha_\alpha G^\alpha(s) \\
\leq \theta w(s)T^\alpha_\alpha G^\alpha(s)G^\alpha(s) + (\alpha - \gamma)w(s)G^\alpha(s)G^\alpha(s)G^\alpha(s).
\]

Since \( T^\alpha_\alpha G^\alpha(s) = g(s) \geq 0, c \leq \sigma(s) \) and \( \gamma > 1 \), we get

\[
\leq \theta w(s)g(s)(G^\alpha(s))^{\alpha-\gamma-1} + (\alpha - \gamma)w(s)g(s)\left(G^\alpha(s)\right)^{\alpha-\gamma-1} \\
= (\alpha - \gamma + \theta)w(s)g(s)\left(G^\alpha(s)\right)^{\alpha-\gamma-1}
\]

for \( c \in [s, \sigma(s)] \). This gives us that

\[
w(s)g(s)\left(G^\alpha(s)\right)^{\alpha-\gamma-1} \leq \frac{1}{\alpha - \gamma + \theta}T^\alpha_\alpha w(s)G^\alpha(s).
\]

Hence,

\[
-u(t) = \int_t^\infty w(s)g(s)\left(G^\alpha(s)\right)^{\alpha-\gamma-1} \Delta \alpha s \leq \frac{1}{\alpha - \gamma + \theta} \int_t^\infty T^\alpha_\alpha w(s)G^\alpha(s) \Delta \alpha s \\
= \frac{1}{\gamma - \theta - \alpha} w(t)G^\alpha(t).
\]

(40)

Using (31) and (29), we have

\[
T^\alpha_\alpha k(t)v(t)(K^{p-\alpha+1}(t)) = T^\alpha_\alpha (k(t)v(t))K^{p-\alpha+1}(t) + k^\alpha(t)v^\alpha(t)T^\alpha_\alpha (K^{p-\alpha+1}(t)) \\
= T^\alpha_\alpha k(t)v(t)(K^{p-\alpha+1}(t)) + k^\alpha(t)v^\alpha(t)K^{p-\alpha+1}(t) \\
+ (p - \alpha + 1)k^\alpha(t)v^\alpha(t)r(t)f(t)K^{p-\alpha}(t)
\]

for \( c \in [t, \sigma(t)] \). Since \( \sigma(t) \geq c, 1 \leq p, 0 \leq T^\alpha_\alpha k(t), r(t)f(t) = T^\alpha_\alpha K(t) \geq 0, \) and (31), we have

\[
T^\alpha_\alpha k(t)v(t)(K^{p}(t)) \leq \beta k^\alpha(t)v^\alpha(t)r(t)f(t)K^{p-\alpha}(t) + (p - \alpha + 1)k^\alpha(t)v^\alpha(t)r(t)f(t)(K^{p}(t))^{p-\alpha} \\
\leq (p + \beta - \alpha + 1)k^\alpha(t)v^\alpha(t)r(t)f(t)(K^{p}(t))^{p-\alpha}.
\]

(41)
Combining (39), (40) and (41), we get \((K(a) = 0 \text{ and } u(\infty) = 0)\) that
\[
\int_a^\infty k^\sigma(t)v^\sigma(t)w(t)g(t)(G^\sigma(t))^{\alpha-\gamma-1}(K^\sigma(t))^{\beta-p-\alpha+1}\Delta at
\leq \frac{p+\beta-\alpha+1}{\gamma-\theta-\alpha} \int_a^\infty k^\sigma(t)v^\sigma(t)w(t)r(t)f(t)G^\alpha\gamma(t)(K^\sigma(t))^{\beta-\alpha}\Delta at
\]
or, equivalently,
\[
\int_a^\infty k^\sigma(t)v^\sigma(t)w(t)g(t)(G^\sigma(t))^{\alpha-\gamma-1}(K^\sigma(t))^{\beta-p-\alpha+1}\Delta at
\leq \frac{p+\beta-\alpha+1}{\gamma-\theta-\alpha} \int_a^\infty \left( k^\sigma(t)v^\sigma(t)w(t)g(t) \right)^{(p-1)/p} \left( G^\sigma(t) \right)^{(\alpha-\gamma-1)(p-1)/p} \left( K^\sigma(t) \right)^{(p-1)(p-\alpha+1)/p}
\times \frac{\left( k^\sigma(t)v^\sigma(t)w(t) \right)^{(1/p)} f(t) \left( G^\sigma(t) \right)^{(1+\alpha)(p-1)/p} \left( K^\sigma(t) \right)^{(1-\alpha)/(p-1)}}{g^{1/(p-1)}(t)G^{\gamma-\alpha}(t)} \Delta at,
\]
Using (33) with \(p \text{ and } p/(p-1)\) indices, we have
\[
\int_a^\infty k^\sigma(t)v^\sigma(t)w(t)g(t)(G^\sigma(t))^{\alpha-\gamma-1}(K^\sigma(t))^{\beta-p-\alpha+1}\Delta at
\leq \frac{p+\beta-\alpha+1}{\gamma-\theta-\alpha} \int_a^\infty \left( k^\sigma(t)v^\sigma(t)w(t)g(t) \right)^{(p-1)/p} \left( G^\sigma(t) \right)^{(\alpha-\gamma-1)(p-1)/p} \left( K^\sigma(t) \right)^{(p-1)(p-\alpha+1)/p}
\times \left( \int_a^\infty k^\sigma(t)v^\sigma(t)w(t)r(t)f(t)G^\alpha\gamma(t) \left( G^\sigma(t) \right)^{(1+\gamma-\alpha)(p-1)/p} \left( K^\sigma(t) \right)^{(1-\alpha)/(p-1)}}{g^{1/(p-1)}(t)G^{\gamma-\alpha}(t)} \Delta at\right)^{1/p},
\]
which implies that
\[
\int_a^\infty k^\sigma(t)v^\sigma(t)w(t)g(t)(G^\sigma(t))^{\alpha-\gamma-1}(K^\sigma(t))^{\beta-p-\alpha+1}\Delta at
\leq \left( \frac{p+\beta-\alpha+1}{\gamma-\theta-\alpha} \right)^p \int_a^\infty k^\sigma(t)v^\sigma(t)w(t)r(t)f(t)G^\alpha\gamma(t) \left( G^\sigma(t) \right)^{(1+\gamma-\alpha)(p-1)/p} \left( K^\sigma(t) \right)^{(1-\alpha)/(p-1)}}{g^{1/(p-1)}(t)G^{\gamma-\alpha}(t)} \Delta at.
\]
The proof is complete.

Remark 3.2. If we take \(\alpha = 1\) in Theorem 3.1 then we recapture Theorem 1.18.

Corollary 3.3. Theorem 3.1 with \(S_{16}, S_{117} \text{ and } S_{18}\) give us that
\[
\int_a^\infty g(t)(G^\sigma(t))^{\alpha-\gamma-1}(K^\sigma(t))^{p-\alpha+1}\Delta at
\leq \left( \frac{p-\alpha+1}{\gamma-\alpha} \right)^p \int_a^\infty g(t)f^p(t)(G^\gamma(t))^{(1-\alpha)(p-1)/(p-\alpha)} \left( K^\sigma(t) \right)^{(1-\alpha)/(p-1)}}{G^{\gamma-\alpha}(t)} \Delta at. \quad (42)
\]

Remark 3.4. If we set \(\alpha = 1\) in Corollary 3.3 then (12) gives (20).

Now, as special cases of our results, we will obtain continuous, discrete and quantum \(\alpha\)-conformal inequalities. This is obtained by choosing, respectively, the time scales \(T = \mathbb{R}, T = h\mathbb{Z} \text{ and } T = \mathbb{Z}, \text{ and } T = q^T\).

Corollary 3.5. Putting \(T = \mathbb{R}\) in Theorem 3.1 one obtains from (34) and (35) that
\[
\int_a^\infty k(t)v(t)w(t)g(t)G^{\alpha-\gamma-1}(t)K^{p-\alpha+1}(t)^{\alpha-1}dt
\leq \left( \frac{p+\beta-\alpha+1}{\gamma-\theta-\alpha} \right)^p \int_a^\infty \frac{k(t)v(t)w(t)r(t)f(t)G^{\gamma-\alpha-1}(t)K^{1-\alpha}(t)}{g^{\alpha-1}(t)}t^{\alpha-1}dt, \quad (43)
\]
where
\[ G(t) = \int_a^t g(s)s^{\alpha-1}ds \quad \text{and} \quad K(t) = \int_a^t r(s)s^{\alpha-1}ds. \]

**Remark 3.6.** Corollary 3.5 with $S_{16}, S_{17}, S_{19}$ and $S_{20}$ gives
\[
\int_0^\infty g(t)G^{\alpha-1}(t)K^{\alpha-1}(t)t^{\alpha-1}dt 
\leq \left( \frac{p - \alpha + 1}{\gamma - \alpha} \right)^p \int_0^\infty \int_0^t f^p(s)G^{\alpha-1}(s)K^{\alpha-1}(s)t^{\alpha-1}dt. \quad (44)
\]

**Remark 3.7.** If we set $\alpha = 1$ in inequality (44), then (44) reduces to (11).

**Remark 3.8.** If we use assumptions $S_{16}, S_{18}, S_{19}$ and $S_{20}$ with Corollary 3.5, then (43) gives
\[
\int_0^\infty G^{\alpha-1}(t)f^p(t)\left( \int_0^t f(s)s^{\alpha-1}ds \right)^{1-\alpha}t^{\alpha-1}dt. \quad (45)
\]

**Remark 3.9.** If we set $\alpha = 1$ in inequality (45), then (45) gives (6).

**Remark 3.10.** Under assumptions $S_{16}, S_{18}, S_{19}$ and $S_{20}$, the inequality (43) of Corollary 3.5 asserts that
\[
\int_0^\infty G^{\alpha-1}(t)f^p(t)\left( \int_0^t f(s)s^{\alpha-1}ds \right)^{1-\alpha}t^{\alpha-1}dt. \quad (46)
\]

**Remark 3.11.** If we set $\alpha = 1$ in inequality (46), then (46) gives (2).

**Corollary 3.12.** Putting $T = hZ$ in Theorem 3.1, inequality (48) gives
\[
\sum_{t=1}^\infty \frac{1}{h^\alpha} \sum_{s=1}^t g(hs)s^{\alpha-1} = h^\alpha \sum_{s=1}^{t-1} r(hs)f(hs)s^{\alpha-1}.
\]

**Corollary 3.13.** If $T = Z \quad (h = 1)$ in Corollary 3.12, then it follows from inequality (48) that
\[
\sum_{t=1}^\infty \frac{1}{h^\alpha} \sum_{s=1}^{t-1} g(s)s^{\alpha-1} = h^\alpha \sum_{s=1}^{t-1} r(s)f(s)s^{\alpha-1}.
\]
Remark 3.14. Using assumptions $S_{16}, S_{17}, S_{19}$ and $S_{21}$, it follows from (38) of Corollary 3.13 that

$$\sum_{t=1}^{\infty} g(t) \left( \sum_{s=1}^{t} g(s) f(s) \right)^{p} \leq \left( \frac{p}{\alpha - 1} \right)^{p} \sum_{t=1}^{\infty} g(t) f^{p}(t) \left( \sum_{s=1}^{t} g(s) \right)^{\gamma(p-1)}$$

which is another form of the discrete inequality (35).

Corollary 3.15. Putting $T = q^{\sigma}$ in Theorem 3.7, inequality (38) with (37) gives

$$\sum_{t=\log_{a} \alpha}^{\infty} k(q^{t+1}) v(q^{t+1}) w(q^{t}) g(q^{t}) G^{-\gamma}(q^{t+1}) K^{p}(q^{t+1}) q^{s t} \leq \left( \frac{p + \beta - \alpha + 1}{\gamma - \theta - 1} \right)^{p} \sum_{t=\log_{a} \alpha}^{\infty} k(q^{t+1}) v(q^{t+1}) w(q^{t}) r^{p}(q^{t}) f^{p}(q^{t}) G^{\gamma(p-1)}(q^{t+1}) q^{s t}$$

where

$$G(t) = (q - 1) \sum_{s=\log_{a} \alpha}^{\infty} g(q^{s}) q^{s t} \quad \text{and} \quad K(t) = (q - 1) \sum_{s=\log_{a} \alpha}^{\infty} r(q^{s}) f(q^{s}) q^{s t}.$$

Theorem 3.16. Let $S_{1}, S_{2}, S_{3}, S_{5}, S_{6}, S_{9}, S_{12},$ and $S_{15}$ be satisfied. Then,

$$\int_{a}^{\infty} k(t) v(t) w^{\sigma}(t) g(t) \left( G^{\sigma}(t) \right)^{\alpha - \gamma - 1} F^{p - \alpha + 1}(t) \Delta_{\alpha} t \leq \left( \frac{p + \beta - \alpha + 1}{\alpha - \gamma + 1} \right) \int_{a}^{\infty} k(t) v(t) w^{\sigma}(t) r^{p}(t) f^{p}(t) \left( G^{\sigma}(t) \right)^{\beta - \gamma - 1} F^{1 - \alpha}(t) \Delta_{\alpha} t.$$ (51)

Proof. From (52), we get

$$\int_{a}^{\infty} k(t) v(t) w^{\sigma}(t) g(t) \left( G^{\sigma}(t) \right)^{\alpha - \gamma - 1} F^{p - \alpha + 1}(t) \Delta_{\alpha} t \leq \left[ u(t) k(t) v(t) F^{p - \alpha + 1}(t) \right]_{a}^{\infty} + \int_{a}^{\infty} w^{\sigma}(t) T_{\alpha}^{A} \left( - k(t) v(t) F^{p - \alpha + 1}(t) \right) \Delta_{\alpha} t,$$ (52)

since

$$u(t) = \int_{a}^{t} w^{\sigma}(s) g(s) \left( G^{\sigma}(s) \right)^{\alpha - \gamma - 1} \Delta_{\alpha} s.$$

Applying (29), (51), and $S_{12}$, one has

$$T_{\alpha}^{A} \left( w(s) G^{\alpha - \gamma}(s) \right) = T_{\alpha}^{A} w(s) G^{\alpha - \gamma}(s) + \Delta_{\alpha} w(s) T_{\alpha}^{A} \left( G^{\alpha - \gamma}(s) \right) \geq \theta w^{\sigma}(s) G^{\alpha - \gamma - 1}(s) T_{\alpha}^{A} G(s) + (\alpha - \gamma) w^{\sigma}(s) G^{\alpha - \gamma - 1}(s) T_{\alpha}^{A} G(s)$$

for $c \in [s, \sigma(s)]$. As $T_{\alpha}^{A} G(s) = g(s) \geq 0, c \leq \sigma(s)$ and $0 \leq \gamma < \alpha$, then we get

$$T_{\alpha}^{A} \left( w(s) G^{\alpha - \gamma}(s) \right) \geq \theta w^{\sigma}(s) g(s) (G^{\sigma}(s))^{\alpha - \gamma - 1} + (\alpha - \gamma) w^{\sigma}(s) g(s) (G^{\sigma}(s))^{\alpha - \gamma - 1}$$

so that

$$w^{\sigma}(s) g(s) (G^{\sigma}(s))^{\alpha - \gamma - 1} \leq \frac{1}{\alpha - \gamma + \theta} T_{\alpha}^{A} \left( w(s) G^{\alpha - \gamma}(s) \right).$$
Therefore,

\[ u^\gamma(t) = \int_0^{\sigma(t)} \alpha - \gamma - 1 \Delta a s \leq \frac{1}{\alpha - \gamma + \theta} \int_a^{\sigma(t)} T_{\alpha} \left( w(s) G^{\alpha - \gamma}(s) \right) \Delta a s \]

Using (33) and (20), let \( c \in [t, \sigma(t)] \). Then,

\[ T_{\alpha} \left( -k(t)v(t)F^{p-\alpha+1}(t) \right) = -\left( (k(t)v(t))^\Delta(F^\alpha(t))^{p-\alpha+1} + k(t)v(t)\left(F^{p-\alpha+1}(t)\right)^\Delta \right) \]

\[ = -k^\Delta(t)v^\sigma(t)\left(F^\alpha(t)\right)^{p-\alpha+1} + k(t)v^\Delta(t)\left(F^\alpha(t)\right)^{p-\alpha+1} + (p - \alpha + 1)k(t)v(t)F^{p-\alpha}(c)T_{\alpha}^S F(t). \]

From \( 0 \leq T_{\alpha}^S k(t), -r(t)f(t) = T_{\alpha}^S F(t) \leq 0, t \leq c, p \geq 1 \) and \( S_5 \), we have

\[ T_{\alpha}^S \left( -k(t)v(t)F^{p-\alpha+1}(t) \right) \leq \beta k(t)v(t)r(t)f(t)\left(F^\alpha(t)\right)^{p-\alpha} + (p - \alpha + 1)k(t)v(t)r(t)f(t)F^{p-\alpha}(t) \]

\[ \leq (p - \alpha + \beta + 1)k(t)v(t)r(t)f(t)F^{p-\alpha}(t). \]  

Using (22), (33) and (34), it follows that \((F(\infty) = 0 \text{ and } u(a) = 0)\)

\[ \int_a^\infty k(t)v(t)w^\sigma(t)g(t)\left(G^\alpha(t)\right)^{\alpha - \gamma - 1} \left(F^\alpha(t)\right)^{p-\alpha+1} \Delta a t \]

\[ \leq \frac{(p - \alpha + \beta + 1)}{\alpha - \gamma + \theta} \int_a^\infty k(t)v(t)w^\sigma(t)r(t)f(t)\left(G^\alpha(t)\right)^{\alpha - \gamma} \Delta a t. \]

Equivalently,

\[ \int_a^\infty k(t)v(t)w^\sigma(t)g(t)\left(G^\alpha(t)\right)^{\alpha - \gamma - 1} F^{p-\alpha+1}(t) \Delta a t \]

\[ \leq \frac{p + \beta - \alpha + 1}{\alpha - \gamma + \theta} \int_a^\infty \left( (k(t)v(t)w^\sigma(t)g(t))^{(p-1)/p} \left(G^\alpha(t)\right)^{p-\gamma+\alpha} F^{p-\alpha+1}(t) \right) \Delta a t. \]

Applying (33) with indexes \( p \) and \( p/(p-1) \), we obtain

\[ \int_a^\infty k(t)v(t)w^\sigma(t)g(t)\left(G^\alpha(t)\right)^{\alpha - \gamma + 1} F^{p-\alpha+1}(t) \Delta a t \]

\[ \leq \frac{p + \beta - \alpha + 1}{\alpha - \gamma + \theta} \left( \int_a^\infty k(t)v(t)w^\sigma(t)g(t)\left(G^\alpha(t)\right)^{\alpha - \gamma - 1} F^{p-\alpha+1}(t) \Delta a t \right)^{(p-1)/p} \]

\[ \times \left( \int_a^\infty k(t)v(t)w^\sigma(t)r(t)f(t)\left(G^\alpha(t)\right)^{p-\gamma+\alpha} F^{1-\alpha}(t) \right)^{1/p} \Delta a t. \]

This gives

\[ \int_a^\infty k(t)v(t)w^\sigma(t)g(t)\left(G^\alpha(t)\right)^{\alpha - \gamma + 1} F^{p-\alpha+1}(t) \Delta a t \]

\[ \leq \frac{(p + \beta - \alpha + 1)^p}{\alpha - \gamma + \theta} \int_a^\infty k(t)v(t)w^\sigma(t)r(t)f(t)\left(G^\alpha(t)\right)^{p-\gamma+\alpha} F^{1-\alpha}(t) \Delta a t, \]

which is our desired result.
Corollary 3.17. If we take $\alpha = 1$ in Theorem 3.16, then we get the following inequality:

$$\int_a^\infty k(t)v(t)u(t)g(t)(G^\sigma(t))^{-\gamma}F^p(t)\Delta t \leq \left(\frac{\beta + p}{\alpha - \gamma + \theta}\right)^p \int_a^\infty \frac{k(t)v(t)u(t)r^p(t)f^p(t)(G^\sigma(t))^{p-\gamma}}{g^{p-1}(t)} \Delta t,$$

where

$$G(t) = \int_a^t g(s)\Delta s \quad \text{with} \quad G(\infty) = \infty, \quad \text{and} \quad F(t) = \int_t^\infty r(s)f(s)\Delta s,$$

which is Theorem 3.11 of [23].

Remark 3.18. Under hypotheses $S_{16}, S_{17}$ and $S_{19}$, then [51] of Theorem 3.16 tell us that

$$\int_a^\infty g(t)(G^\sigma(t))^{\alpha-\gamma+1}F^{\gamma+1}(t)\Delta t = \left(\frac{p-\alpha + 1}{\alpha - \gamma}\right)^p \int_a^\infty g(t)f^p(t)(G^\sigma(t))^{p-\gamma+\alpha+1}\Delta t. \quad (55)$$

Remark 3.19. If we set $\alpha = 1$ in [55], then we obtain inequality [21].

As special cases of our results, now we obtain continuous, discrete and quantum $\alpha$-conformable inequalities. Precisely, we consider the special cases of time scales $T = \mathbb{R}, T = h\mathbb{Z}, T = \mathbb{Z}$ and $T = q^\mathbb{Z}$.

Corollary 3.20. Putting $T = \mathbb{R}$ in Theorem 3.16 we get from [51] and [51] that

$$\int_a^\infty k(t)v(t)u(t)g(t)G^{\alpha-\gamma-1}(t)F^{\gamma+1}(t)t^{\alpha-1}dt \leq \left(\frac{p+\beta - \alpha + 1}{\alpha - \gamma + \theta}\right)^p \int_a^\infty \frac{k(t)v(t)u(t)r^p(t)f^p(t)G^{\gamma+\gamma+1}(t)F^{1-\alpha}(t)}{g^{p-1}(t)}t^{\alpha-1}dt \quad (56)$$

with

$$G(t) = \int_a^t g(s)s^{\alpha-1}ds \quad \text{and} \quad F(t) = \int_t^\infty r(s)f(s)s^{\alpha-1}ds.$$

Remark 3.21. Under assumptions $S_{16}, S_{17}, S_{19}$ and $S_{20}$, inequality [50] of Corollary 3.20 asserts that

$$\int_0^\infty g(t)G^{\alpha-\gamma-1}(t)F^{\gamma+1}(t)t^{\alpha-1}dt \leq \left(\frac{p+\beta - \alpha + 1}{\alpha - \gamma + \theta}\right)^p \int_0^\infty g(t)f^p(t)G^{\gamma+\gamma+1}(t)F^{1-\alpha}(t)t^{\alpha-1}dt. \quad (57)$$

Remark 3.22. If $\alpha = 1$, then inequality [57] reduces to [12].

Remark 3.23. With hypotheses $S_{16}, S_{18}, S_{19}$ and $S_{20}$, inequality [50] of Corollary 3.20 gives us that

$$\int_0^\infty G^{\alpha-\gamma-1}(t)\left(\int_0^\infty f(s)s^{\alpha-1}\right)^{p-\gamma+1}t^{\alpha-1}dt \leq \left(\frac{p-\alpha + 1}{\alpha - \gamma}\right)^p \int_0^\infty f^p(t)G^{\gamma+\gamma+1}(t)\left(\int_0^\infty f(s)s^{\alpha-1}\right)^{1-\alpha}t^{\alpha-1}dt. \quad (58)$$

Remark 3.24. If we set $\alpha = 1$, then the inequality [58] simplifies to [7].
Remark 3.25. With $S_{16}$, $S_{18}$, $S_{19}$, $S_{20}$, and $S_{22}$, of Corollary 3.20 gives inequality

$$
\int_0^\infty G^{\alpha-p-1}(t) \left( \int_0^\infty f(s)s^{\alpha-1} \right)^{p-\alpha+1} t^{\alpha-1} dt 
\leq \left( \frac{p-\alpha+1}{\alpha-p} \right)^p \int_0^\infty f^p(t)G^{\alpha-1}(t) \left( \int_0^\infty f(s)s^{\alpha-1} \right)^{1-\alpha} t^{\alpha-1} dt.
$$

(59)

Remark 3.26. In the particular case $\alpha = 1$, inequality (59) gives us (5).

Corollary 3.27. Choosing $T = h\mathbb{Z}$ in Theorem 3.16, we obtain from inequality (51) that

$$\sum_{t=\frac{1}{T}}^\infty k(ht)v(ht)w(ht+h)g(ht)G^{\gamma}(ht+h)F^p(ht)t^{\alpha-1}$$

$$\leq \left( \frac{p+\beta}{1-\gamma+\theta} \right)^p \sum_{t=\frac{1}{T}}^\infty k(ht)v(ht)w(ht+h)G^{\alpha-1}(ht)F^p(ht)G^{\gamma}(ht+h)l^{\alpha-1},$$

(60)

where

$$G(t) = h^\alpha \sum_{s=\frac{1}{T}}^{\frac{t-1}{T}} g(hs)s^{\alpha-1} \quad \text{and} \quad F(t) = h^\alpha \sum_{s=\frac{1}{T}}^\infty r(hs)f(hs)s^{\alpha-1}.$$

Corollary 3.28. Putting $h = 1$ in Corollary 3.27, that is, for the discrete time-scale $T = \mathbb{Z}$, we obtain from (51) the inequality

$$\sum_{t=a}^\infty k(t)v(t)w(t+1)g(t)G^{\gamma}(t+1)F^p(t)t^{\alpha-1}$$

$$\leq \left( \frac{p+\beta}{1-\gamma+\theta} \right)^p \sum_{t=a}^\infty k(t)v(t)w(t+1)r^p(t)f^p(t)G^{\gamma}(t+1)l^{\alpha-1},$$

(61)

where

$$G(t) = \sum_{s=a}^{\frac{t-1}{T}} g(s)s^{\alpha-1} \quad \text{and} \quad F(t) = \sum_{s=t}^\infty r(s)f(s)s^{\alpha-1}.$$

Remark 3.29. With $S_{16}$, $S_{17}$, $S_{19}$ and $S_{21}$, then (61) of Corollary 3.28 reduces to (9).

Corollary 3.30. Putting $T = \mathbb{Q}^+$ in Theorem 3.16, it follows from (37) that (38) simplifies to

$$\sum_{t=\log_q a}^\infty k(q^t)v(q^t)w(q^{t+1})g(q^t)G^{\gamma}(q^{t+1})F^p(q^t)q^{\alpha t}$$

$$\leq \left( \frac{p+\beta}{1-\gamma+\theta} \right)^p \sum_{t=\log_q a}^{(\log_q t)-1} k(q^t)v(q^t)w(q^{t+1})r^p(q^t)f^p(q^t)G^{\gamma}(q^{t+1})q^{\alpha t}$$

(62)

with

$$G(t) = (q-1) \sum_{s=\log_q a} g(q^s)q^{\alpha s} \quad \text{and} \quad F(t) = (q-1) \sum_{s=\log_q t} r(q^s)f(q^s)q^{\alpha s}.$$

Theorem 3.31. Let $S_1$, $S_2$, $S_3$, $S_5$, $S_7$, $S_8$, $S_{11}$, $S_{15}$ be satisfied. Then,

$$\int_a^\infty k^{\sigma}(t)v^{\sigma}(t)w(t)g(t)H^{\alpha-\gamma-1}(t)(K^{\sigma}(t))^{p-\alpha+1} \Delta_{\alpha t}$$

$$\leq \left( \frac{p-\alpha+1}{\alpha-\gamma+\theta} \right)^p \int_a^\infty k^{\sigma}(t)v^{\sigma}(t)w(t)r^p(t)f^p(t)H^{\alpha+\alpha-1}(t)(K^{\sigma}(t))^{(1-\alpha)}\Delta_{\alpha t}.$$ 

(63)
Proof. Using the $\alpha$-conformable integration by parts formula on time scales with
\[
T_\alpha^\Delta u(t) = w(t)g(t)H^{\alpha-\gamma-1}(t) 
\text{ and } z_\sigma(t) = k_\sigma(t)v_\sigma(t)(K_\sigma(t))^{p-\alpha+1},
\]
we have
\[
\int_a^\infty k_\sigma(t)v_\sigma(t)w(t)g(t)H^{\alpha-\gamma-1}(t)(K_\sigma(t))^{p-\alpha+1}\Delta_\alpha t
= \left[ u(t)k(t)v(t)K^{p-\alpha+1}(t) \right]_a^\infty
+ \int_a^\infty (-u(t))T_\alpha^\Delta \left( k(t)v(t)K^{p-\alpha+1}(t) \right) \Delta_\alpha t, \tag{64}
\]
where
\[
u(t) = -\int_t^\infty w(s)g(s)H^{\alpha-\gamma-1}(s)\Delta_\alpha s.
\]
From (20), (31), and $S_{11}$, then
\[
T_\alpha^\Delta \left( -w(s)H^{\alpha-\gamma}(s) \right)
= -\left( T_\alpha^\Delta w(s)(H^\alpha(s))^{\alpha-\gamma} + w(s)T_\alpha^\Delta (H^{\alpha-\gamma}(s)) \right)
\geq -\left( \theta w(s)(H^\alpha(s))^{\alpha-\gamma-1}T_\alpha^\Delta H(s) + (\alpha-\gamma)w(s)H^{\alpha-\gamma-1}(c)T_\alpha^\Delta H(s) \right).
\]
Since $s \leq c$ , $0 \leq T_\alpha^\Delta H(s) = -g(s)$ and $\alpha > \gamma > 0$, we get
\[
T_\alpha^\Delta \left( -w(s)H^{\alpha-\gamma}(s) \right)
\geq \theta w(s)g(s)H^{\alpha-\gamma-1}(s) + (\alpha-\gamma)w(s)g(s)H^{\alpha-\gamma-1}(s)
= (\alpha-\gamma+\theta)w(s)g(s)H^{\alpha-\gamma-1}(s).
\]
Thus,
\[
w(s)g(s)H^{\alpha-\gamma-1}(s) \leq \frac{1}{\alpha-\gamma+\theta} T_\alpha^\Delta \left( -w(s)H^{\alpha-\gamma}(s) \right).
\]
Hence,
\[
-u(t) = \int_t^\infty w(s)g(s)H^{\alpha-\gamma-1}(s)\Delta_\alpha s \leq \frac{1}{\alpha-\gamma+\theta} \int_t^\infty T_\alpha^\Delta \left( -w(s)H^{\alpha-\gamma}(s) \right) \Delta_\alpha s
= \frac{1}{\alpha-\gamma+\theta} w(t)H^{\alpha-\gamma}(t). \tag{65}
\]
Using (31) and (29), one obtains
\[
T_\alpha^\Delta \left( k(t)v(t)K^{p-\alpha+1}(t) \right) = T_\alpha^\Delta \left( k(t)v(t)K^{p-\alpha+1}(t) + k^\sigma(t)v^\sigma(t)T_\alpha^\Delta \left( K^{p-\alpha+1}(t) \right) \right)
= T_\alpha^\Delta k(t)v(t)K^{p-\alpha+1}(t) + k^\sigma(t)T_\alpha^\Delta v(t)K^{p-\alpha+1}(t) + (p-\alpha+1)k^\sigma(t)v^\sigma(t)K^{p-\alpha}(c)T_\alpha^\Delta K(t)
\]
with $c \in [t, \sigma(t)]$. Considering $r(t)f(t) = T_\alpha^\Delta K(t) \geq 0$, $0 \leq T_\alpha^\Delta k(t)$, $c \leq \sigma(t)$, $1 \leq p \geq 1$ and $S_{14}$, we arrive to
\[
T_\alpha^\Delta \left( k(t)v(t)K^\sigma(t) \right) \leq \beta k^\sigma(t)v^\sigma(t)r(t)f(t)K^{p-\alpha}(t) + (p-\alpha+1)k^\sigma(t)v^\sigma(t)r(t)f(t)(K^\sigma(t))^{p-\alpha}
\leq (p-\alpha+\beta+1)k^\sigma(t)v^\sigma(t)r(t)f(t)(K^\sigma(t))^{p-\alpha}. \tag{66}
\]
Now, combining (31), (65) and (66), we get ($K(a) = 0$ and $u(\infty) = 0$)
\[
\int_a^\infty k_\sigma(t)v_\sigma(t)w(t)g(t)H^{\alpha-\gamma-1}(t)(K_\sigma(t))^{p-\alpha+1}\Delta_\alpha t
\leq \frac{p-\alpha+\beta+1}{\alpha-\gamma+\theta} \int_a^\infty k_\sigma(t)v_\sigma(t)w(t)f(t)H^{\alpha-\gamma}(t)(K_\sigma(t))^{p-\alpha}\Delta_\alpha t. \tag{67}
\]
Inequality (69) becomes
\[
\int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g(t) H^{\alpha-\gamma+1}(t) (K^\sigma(t))^{p-\alpha+1} \Delta_t
\]
\[
\leq \frac{p-\alpha+1}{\alpha-\gamma+1} \int_a^\infty \left( k^\sigma(t) v^\sigma(t) w(t) g(t) H^{\alpha-\gamma+1}(t) (K^\sigma(t))^{p-\alpha+1} \right)^{(p-1)/p} \Delta_t
\]
\[
\times \left( \frac{1}{g^{p-1}(t)} \right) \Delta_t.
\]
Using the \(\alpha\)-conformable Hölder inequality (33) with \(p\) and \(p/(p-1)\) indices, we obtain that
\[
\int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g(t) H^{\alpha-\gamma+1}(t) (K^\sigma(t))^{p-\alpha+1} \Delta_t
\]
\[
\leq \left( \frac{p-\alpha+1}{\alpha-\gamma+1} \right)^p \int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g^p(t) H^{p-\gamma+1}(t) (K^\sigma(t))^{(1-\alpha)} \Delta_t
\]
This implies that
\[
\int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g(t) H^{\alpha-\gamma-1}(t) (K^\sigma(t))^{p-\alpha+1} \Delta_t
\]
\[
\leq \left( \frac{p-\alpha+1}{\alpha-\gamma+1} \right)^p \int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g^p(t) H^{p-\gamma+1}(t) (K^\sigma(t))^{(1-\alpha)} \Delta_t
\]
which is the desired result.

Corollary 3.32. Putting \(\alpha = 1\) in Theorem 3.31, then inequality (69) reduces to
\[
\int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g(t) H^{-\gamma}(t) (K^\sigma(t))^P \Delta t
\]
\[
\leq \left( \frac{p+1}{p-1} \right)^p \int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g^p(t) H^{p-\gamma}(t) \Delta t
\]
where
\[
H(t) = \int_t^\infty g(s) \Delta s \quad \text{and} \quad K(t) = \int_a^t r(s)f(s) \Delta s, \quad t \in [a, \infty)_T,
\]
which is Theorem 3.21 of [23].

Remark 3.33. Under \(S_{16}, S_{17}\), and \(S_{19}\), of Theorem 3.31 tell us that
\[
\int_a^\infty g(t) H^{\alpha-\gamma-1}(t) (K^\sigma(t))^{p-\alpha+1} \Delta t
\]
\[
\leq \left( \frac{p-\alpha+1}{\alpha-\gamma+1} \right)^p \int_a^\infty g(t) f^p(t) H^{p-\gamma+1}(t) (K^\sigma(t))^{(1-\alpha)} \Delta t. \quad (68)
\]

Remark 3.34. For \(\alpha = 1\), inequality (68) gives us (22).

Now, as special cases of our results, we obtain continuous, discrete and quantum \(\alpha\)-conformable inequalities. For that, we fix the time scale as \(T = \mathbb{R}, T = h\mathbb{Z}, T = \mathbb{Z}, \) or \(T = \mathbb{Z}^d\).

Corollary 3.35. Putting \(T = \mathbb{R}\) in Theorem 3.31, it follows from (33) and (63) that
\[
\int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g(t) H^{-\gamma+1}(t) K^{p-\alpha+1}(t)^{\alpha-1} dt
\]
\[
\leq \left( \frac{p+\beta-\alpha+1}{\alpha-\gamma+1} \right)^p \int_a^\infty k^\sigma(t) v^\sigma(t) w(t) g^p(t) H^{p-\gamma+1}(t) K^{(1-\alpha)}(t) t^{\alpha-1} dt, \quad (69)
\]
where
\[ H(t) = \int_t^\infty g(s)s^{\alpha-1}ds \quad \text{and} \quad K(t) = \int_t^\infty r(s)f(s)s^{\alpha-1}ds. \]

**Remark 3.36.** Under \( S_{16}, S_{17}, S_{19}, \) and \( S_{20}, \) inequality (69) of Corollary 3.35 gives us
\[
\int_0^\infty g(t)H^{\alpha-\gamma-1}(t)K^{p-\alpha+1}(t)t^{\alpha-1}dt \leq \left( \frac{p - \alpha + 1}{\alpha - \gamma} \right)^p \int_0^\infty g(t)f^p(t)H^{p-\gamma+\alpha-1}(t)K^{1-\alpha}(t)t^{\alpha-1}dt. \quad (70)
\]

**Remark 3.37.** If we take \( \alpha = 1, \) then inequality (70) reduces to
\[
\int_0^\infty r(t)\left( \int_0^t g(s)f(s)ds \right)^p \left( \int_0^\infty g(s)ds \right)^{p-\gamma} dt \leq \left( \frac{p}{1 - \gamma} \right)^p \int_0^\infty g(t)f^p(t)\left( \int_0^t g(s)ds \right)^{p-\gamma} dt, \quad (71)
\]
that is, we get the continuous analog of (10).

**Corollary 3.38.** Putting \( \mathbb{T} = h\mathbb{Z} \) in Theorem 3.31, then inequality (63) gives
\[
\sum_{t=0}^\infty k(ht+h)v(ht+h)w(ht)g(ht)H^{-\gamma}(ht)K^p(ht+h)t^{\alpha-1}
\leq \left( \frac{p + \beta}{1 - \alpha + \theta} \right)^p \sum_{t=0}^\infty k(ht+h)v(ht+h)w(ht)r^p(ht)f^p(ht)H^{-\gamma}(ht)g^{p-1}(ht)t^{\alpha-1}, \quad (72)
\]
where
\[ H(t) = h^\alpha \sum_{s=0}^{t-1} g(hs)s^{\alpha-1} \quad \text{and} \quad K(t) = h^\alpha \sum_{s=0}^{t-1} r(hs)f(hs)s^{\alpha-1}. \]

**Corollary 3.39.** Putting \( h = 1 \) in Corollary 3.38, that is, fixing the time scale to be \( \mathbb{T} = \mathbb{Z}, \) then one obtains that
\[
\sum_{t=0}^\infty k(t+1)v(t+1)w(t)g(t)H^{-\gamma}(t)K^p(t+1)t^{\alpha-1}
\leq \left( \frac{p + \beta}{1 - \alpha + \theta} \right)^p \sum_{t=0}^\infty k(t+1)v(t+1)w(t)r^p(t)f^p(t)H^{-\gamma}(t)g^{p-1}(t)t^{\alpha-1}, \quad (73)
\]
where
\[ H(t) = \sum_{s=0}^{\infty} g(s)s^{\alpha-1} \quad \text{and} \quad K(t) = \sum_{s=0}^{t-1} r(s)f(s)s^{\alpha-1}. \]

**Remark 3.40.** Under assumptions \( S_{16}, S_{17}, S_{19}, \) and \( S_{20}, \) inequality (63) of Corollary 3.39 gives us (10).**Corollary 3.41.** Choosing \( \mathbb{T} = q^\mathbb{Z} \) in Theorem 3.31, it follows from (72) and (73) that
\[
\sum_{t=\log_q a}^\infty k(q^{t+1})v(q^{t+1})w(q^t)g(q^t)H^{-\gamma}(q^t)K^p(q^{t+1})q^{\alpha t}
\leq \left( \frac{p + \beta}{1 - \alpha + \theta} \right)^p \sum_{t=\log_q a}^\infty k(q^{t+1})v(q^{t+1})w(q^t)r^p(q^t)f^p(q^t)H^{-\gamma}(q^t)g^{p-1}(q^t)q^{\alpha t}, \quad (74)
\]
where

\[ H(t) = (q - 1) \sum_{s = \log_q t}^{\infty} g(q^s)q^{as} \quad \text{and} \quad K(t) = (q - 1) \sum_{s = \log_q a}^{(\log_q t) - 1} r(q^s)f(q^s)q^{as}. \]

**Theorem 3.42.** Let \( S_1, S_2, S_3, S_4, S_7, S_9, S_{13}, \) and \( S_{15} \) be satisfied. Then,

\[
\int_{a}^{\infty} k(t)v(t)w^{\sigma}(t)g(t)H^{\alpha - \gamma - 1}(t)F^{p - \alpha + 1}(t)\Delta \alpha t \\
\leq \left( \frac{p + \beta - \alpha + 1}{\gamma - \theta - \alpha} \right)^{p} \int_{a}^{\infty} k(t)v(t)w^{\sigma}(t)g^{p}(t)H^{(1 - \alpha + \gamma)(p - 1)}(t)F^{(1 - \alpha)/p}(t)g^{p-1}(t)(H^{\sigma}(t))^{p(\gamma - \alpha)} \Delta \alpha t. \tag{75}
\]

**Proof.** Using (32), we get

\[
\int_{a}^{\infty} k(t)v(t)w^{\sigma}(t)g(t)H^{\alpha - \gamma - 1}(t)F^{p - \alpha + 1}(t)\Delta \alpha t \\
= \left[ u(t)k(t)v(t)F^{p - \alpha + 1}(t) \right]_{a}^{\infty} + \int_{a}^{\infty} u^{\sigma}(t)T_{\alpha}^{\Delta}( - k(t)v(t)F^{p - \alpha + 1}(t))\Delta \alpha t \tag{76}
\]

with

\[ u(t) = \int_{a}^{t} w^{\sigma}(s)g(s)H^{\alpha - \gamma - 1}(s)\Delta \alpha s. \]

Now, using (29), (31), and \( S_{13} \), it follows that

\[
T_{\alpha}^{\Delta}(w(s)H^{\alpha - \gamma}(s)) = w_{\alpha}^{\Delta}(s)H^{\alpha - \gamma}(s) + w^{\sigma}(s)T_{\alpha}^{\Delta}(H^{\alpha - \gamma}(s)) \\
\geq \theta w^{\sigma}(s)T_{\alpha}^{\Delta}H(s)H^{\alpha - \gamma - 1}(s) + (\alpha - \gamma)w^{\sigma}(s)H^{\alpha - \gamma - 1}(c)T_{\alpha}^{\Delta}H(s). \\
\]

Since \( T_{\alpha}^{\Delta}H(s) = -g(s) \leq 0, c \geq s \) and \( \gamma > 1 \), we obtain that

\[
T_{\alpha}^{\Delta}(w(s)H^{\alpha - \gamma}(s)) \geq -\theta w^{\sigma}(s)g(s)H^{\alpha - \gamma - 1}(s) + (\gamma - \alpha)w^{\sigma}(s)g(s)H^{\alpha - \gamma - 1}(s) \\
= (\gamma - \theta - \alpha)w^{\sigma}(s)g(s)H^{\alpha - \gamma - 1}(s)
\]

This gives us that

\[ w^{\sigma}(s)g(s)H^{\alpha - \gamma - 1}(s) \leq \frac{1}{\gamma - \theta - \alpha} T_{\alpha}^{\Delta}(w(s)H^{\alpha - \gamma}(s)). \]

Therefore,

\[
u^{\sigma}(t) = \int_{a}^{\sigma(t)} w^{\sigma}(s)g(s)H^{\alpha - \gamma - 1}(s)\Delta \alpha s \leq \frac{1}{\gamma - \theta - \alpha} \int_{a}^{\sigma(t)} T_{\alpha}^{\Delta}(w(s)H^{\alpha - \gamma}(s))\Delta \alpha s \\
= \frac{1}{\gamma - \theta - \alpha} \left( w^{\sigma}(t)(H^{\sigma}(t))^{\alpha - \gamma} - w(a)H^{\alpha - \gamma}(a) \right) \tag{77}
\]

Let \( c \in [t, \sigma(t)] \). Then, using (31) and (29), one has

\[
T_{\alpha}^{\Delta}( - k(t)v(t)F^{p}(t)) = -T_{\alpha}^{\Delta}(k(t)v(t)(F^{\sigma}(t))^{p - \alpha + 1} + k(t)v(t)T_{\alpha}^{\Delta}(F^{p - \alpha + 1}(t))) \\
= -T_{\alpha}^{\Delta}(k(t)v^{\sigma}(t)(F^{\sigma}(t))^{p - \alpha + 1} + k(t)T_{\alpha}^{\Delta}v(t)(F^{\sigma}(t))^{p - \alpha + 1} \\
+ (p - \alpha + 1)k(t)v(t)F^{p - \alpha + 1}(c)T_{\alpha}^{\Delta}F(t)).
\]
Since \( t \leq c, p > 1, 0 \leq T_0^k(t)0, -r(t)f(t) = T_0^k F(t) \leq 0, \) and \( S_1, \) we get
\[
T_0^k \left( -k(t)v(t)F^{p-\alpha + 1}(t) \right) \leq \beta k(t)v(t)r(t)f(t) \left( F^\sigma(t) \right)^{p-\alpha} + (p - \alpha + 1)k(t)v(t)r(t)f(t) F^{p-\alpha}(t) \\
\leq (p + \beta + \alpha - 1)k(t)v(t)r(t)f(t) F^{p-\alpha}(t).
\]

From (76), (77) and (78), we obtain that \( u(a) = 0 \) and \( F(\infty) = 0 \)
\[
\int_a^\infty k(t)v(t)w^\sigma(t)g(t)H^{\alpha-\gamma-1}(t)F^{p-\alpha+1}(t) \Delta t \\
\leq \frac{p + \beta + \alpha - 1}{\gamma - \theta - \alpha} \int_a^\infty k(t)v(t)w^\sigma(t)r(t)f(t) \left( H^\sigma(t) \right)^{\alpha-\gamma}F^{p-\alpha}(t) \Delta t,
\]
or, equivalently,
\[
\int_a^\infty k(t)v(t)w^\sigma(t)g(t)H^{\alpha-\gamma-1}(t)F^{p-\alpha+1}(t) \Delta t \\
\leq \frac{p + \beta + \alpha - 1}{\gamma - \theta - \alpha} \left( \int_a^\infty k(t)v(t)w^\sigma(t)g(t)H^{\alpha-\gamma-1}(t)F^{p-\alpha+1}(t) \Delta t \right)^{(p-1)/p} \\
\times \left( \int_a^\infty k(t)v(t)w^\sigma(t)r^p(t)f^p(t)H^{(\alpha-\gamma)(p-1)}(t)F^{(1-\alpha)p}(t) \Delta t \right)^{1/p},
\]
Applying the dynamic Hölder inequality \( \Box \) with indices \( p \) and \( p/(p-1) \), we get
\[
\int_a^\infty k(t)v(t)w^\sigma(t)g(t)H^{\alpha-\gamma-1}(t)F^{p-\alpha+1}(t) \Delta t \\
\leq \left( \frac{p + \beta - \alpha + 1}{\gamma - \theta - \alpha} \right)^p \int_a^\infty k(t)v(t)w^\sigma(t)r^p(t)f^p(t)H^{(\alpha-\gamma)(p-1)}(t)F^{(1-\alpha)p}(t) \Delta t,
\]
which implies that
\[
\int_a^\infty k(t)v(t)w^\sigma(t)g(t)H^{\alpha-\gamma-1}(t)F^{p-\alpha+1}(t) \Delta t \\
\leq \left( \frac{p + \beta - \alpha + 1}{\gamma - \theta - \alpha} \right)^p \int_a^\infty \frac{k(t)v(t)w^\sigma(t)r^p(t)f^p(t)H^{(\alpha-\gamma)(p-1)}(t)F^{(1-\alpha)p}(t)}{g^{(\gamma-\alpha)}(H^\sigma(t))^{p(\gamma-\alpha)}} \Delta t.
\]
The proof is complete. \( \Box \)

**Corollary 3.43.** If one takes \( \alpha = 1 \) in Theorem 3.42, then inequality (78) reduces to
\[
\int_a^\infty v(t)k(t)v(t)w(t)g(t)H^{-\gamma}(t)F^p(t) \Delta t \\
\leq \left( \frac{p + \beta}{\gamma - \theta - 1} \right)^p \int_a^\infty \frac{k(t)v(t)w^\sigma(t)r^p(t)f^p(t)H^{(\alpha-\gamma)(p-1)}(t)}{g^{(\gamma-\alpha)}(H^\sigma(t))^{p(\gamma-1)}} \Delta t \tag{79}
\]
with
\[
H(t) = \int_t^\infty g(s) \Delta s \quad \text{and} \quad F(t) = \int_t^\infty r(s)f(s) \Delta s, \quad t \in [a, \infty),
\]
which is Theorem 3.29 of [22].
Remark 3.44. Under assumptions $S_{16}$, $S_{17}$, and $S_{19}$, (85) of Theorem 3.42 gives
\[
\int_a^\infty g(t)H^{\alpha-\gamma}(t)F^{p-1}(t)\Delta t
\leq \left( \frac{p-\alpha+1}{\gamma-\alpha} \right)^p \int_a^\infty \frac{g(t)f^p(t)H^{(1-\alpha+\gamma)(p-1)}(t)F^{(1-\alpha)}(t)}{(H^p(t))^{p(\gamma-\alpha)}} \Delta t. \quad (80)
\]

Remark 3.45. If we take $\alpha = 1$ in inequality (80), then we obtain (23).

Now, as special cases of our results, we give continuous, discrete, and quantum $\alpha$-conformable inequalities. Namely, the following results are obtained by choosing $T$ as time scales $T = \mathbb{R}$, $T = h\mathbb{Z}$, $T = \mathbb{Z}$, and $\varphi = H^p$.

Corollary 3.46. Putting $T = \mathbb{R}$ in Theorem 3.42, then inequality (85) becomes
\[
\int_a^\infty k(t)v(t)w(t)g(t)H^{\alpha-\gamma}(t)F^{p-1}(t)t^{\alpha-1}dt
\leq \left( \frac{p+\beta-\alpha+1}{\gamma-\theta-\alpha} \right)^p \int_a^\infty \frac{k(t)v(t)w(t)r^p(t)f^p(t)H^{\alpha-\gamma}(t)F^{1-\alpha}(t)}{g^{p-1}(t)} t^{\alpha-1}dt, \quad (81)
\]
where
\[
H(t) = \int_t^\infty g(s)s^{\alpha-1}ds \quad \text{and} \quad F(t) = \int_t^\infty r(s)f(s)s^{\alpha-1}ds.
\]

Remark 3.47. Under hypotheses $S_{16}$, $S_{17}$, $S_{19}$, and $S_{20}$, inequality (81) of Corollary 3.46 gives
\[
\int_0^\infty g(t)H^{\alpha-\gamma}(t)F^{p-1}(t)t^{\alpha-1}dt
\leq \left( \frac{p-\alpha+1}{\gamma-\alpha} \right)^p \int_0^\infty g(t)f^p(t)H^{\alpha-\gamma}(t)F^{1-\alpha}(t)t^{\alpha-1}dt. \quad (82)
\]

Remark 3.48. If we take $\alpha = 1$ in inequality (82), then one gets
\[
\int_0^\infty g(t)\left( \int_t^\infty g(s)f(s)ds \right)^p dt \leq \left( \frac{p}{\gamma-1} \right)^p \int_0^\infty g(t)f^p(t)\left( \int_t^\infty g(s)f(s)ds \right)^{p-\gamma} dt,
\]
which is the continuous analogous of Bennett’s inequality (13).

Corollary 3.49. Choosing $T = h\mathbb{Z}$ in Theorem 3.42, inequality (75) gives that
\[
\sum_{t=0}^\infty k(ht)v(ht)w(ht+h)g(ht)H^{-\gamma}(ht)F^p(ht)t^{\alpha-1}
\leq \left( \frac{p+\beta}{\gamma-\theta-1} \right)^p \sum_{t=0}^\infty \frac{k(ht)v(ht)w(ht+h)r^p(ht)f^p(ht)H^{(p-1)}(ht)}{g^{p-1}(ht)H^{(\gamma-1)}(ht)} t^{\alpha-1}, \quad (83)
\]
where
\[
H(t) = h^\alpha \sum_{s=h}^\infty g(hs)s^{\alpha-1} \quad \text{and} \quad F(t) = h^\alpha \sum_{s=h}^\infty r(hs)f(hs)s^{\alpha-1}.
\]

Corollary 3.50. In the particular case $h = 1$, that is, in the discrete time scale $T = \mathbb{Z}$, Corollary 3.49 reduces to
\[
\sum_{t=a}^\infty k(t)v(t)w(t+1)g(t)H^{-\gamma}(t)F^p(t)t^{\alpha-1}
\leq \left( \frac{p+\beta}{\gamma-\theta-1} \right)^p \sum_{t=a}^\infty \frac{k(t)v(t)w(t+1)r^p(t)f^p(t)H^{(p-1)}(t)}{g^{p-1}(t)H^{(\gamma-1)}(t+1)} t^{\alpha-1}, \quad (84)
\]
where

\[ H(t) = \sum_{s=t}^{\infty} g(s)s^{\alpha-1} \quad \text{and} \quad F(t) = \sum_{s=t}^{\infty} f(s)r(s)s^{\alpha-1}. \]

Remark 3.51. With \( S_{16}, S_{17}, S_{19} \) and \( S_{21} \), then (84) of Corollary 3.50 gives a different form of inequality (83).

Corollary 3.52. If \( T = q^t \) in Theorem 3.42, it follows from (37) and (75) that

\[
\sum_{t=\log_q a}^{\infty} \frac{k(q^t)v(q^t)w(q^{t+1})g(q^t)H^{-\gamma}(q^t)F^p(q^t)q^{\alpha t}}{q^{p-1}(q^t)H^{p\gamma}(q^t)F^p(q^t)q^{\alpha t}} \leq \left( \frac{p+\beta}{\gamma - \theta - 1} \right)^p \sum_{t=\log_q a}^{\infty} k(q^t)v(q^t)w(q^{t+1})g(q^t)H^{-\gamma}(q^t)F^p(q^t)q^{\alpha t}, \tag{85}
\]

where

\[ H(t) = (q-1) \sum_{s=\log_q t}^{\infty} g(q^s)q^{\alpha s} \quad \text{and} \quad F(t) = (q-1) \sum_{s=\log_q t}^{\infty} r(q^s)f(q^s)q^{\alpha s}. \]

4 Conclusion

Hardy-type inequalities have many applications and are subject to strong research: see the books [6, 40, 41, 63] and the recent publications [25, 34, 61]. In this manuscript, by employing the \( \alpha \)-conformable fractional calculus on time scales of Benkhettou et al. [9], several new Hardy-type inequalities were proved. The results extend several dynamic inequalities known in the literature, being new even in the discrete, continuous and quantum settings.

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Data Availability Statement

The authors declare that all data supporting the findings of this study are available within the article.

Conflict of Interest Statement

The Authors declare that there is no conflict of interest.

References

[1] A. Abdeldaim and A. A. El-Deeb. On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations. *Applied Mathematics and Computation*, 256:375–380, 2015.

[2] T. Abdeljawad. On conformable fractional calculus. *Journal of computational and Applied Mathematics*, 279:57–66, 2015.

[3] R. P. Agarwal, M. Bohner, and A. Peterson. Inequalities on time scales: a survey. *Math. Inequal. Appl.*, 4(4):535–557, 2001.

[4] R. P. Agarwal, R. R. Mahmoud, D. O’Regan, and S. H. Saker. Some reverse dynamic inequalities on time scales. *Bull. Aust. Math. Soc.*, 96(3):445–454, 2017.
[5] R. P. Agarwal, D. O'Regan, and S. H. Saker. *Hardy Type Inequalities on Time Scales*. Springer, Cham, 2016.

[6] R. P. Agarwal, D. O'Regan, and S. H. Saker. *Hardy type inequalities on time scales*. Springer, Cham, 2016.

[7] K. F. Andersen and H. P. Heinig. Weighted norm inequalities for certain integral operators. *SIAM J. Math. Anal.*, 14(4):834–844, 1983.

[8] K. F. Andersen and B. Muckenhoupt. Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions. *Studia Math.*, 72(1):9–26, 1982.

[9] N. Benkhettou, S. Hassani, and D. F. M. Torres. A conformable fractional calculus on arbitrary time scales. *Journal of King Saud University-Science*, 28(1):93–98, 2016.

[10] G. Bennett. Some elementary inequalities. *Quart. J. Math. Oxford Ser. (2)*, 38(152):401–425, 1987.

[11] G. Bennett. Some elementary inequalities. II. *Quart. J. Math. Oxford Ser. (2)*, 39(156):385–400, 1988.

[12] G. Bennett. Some elementary inequalities. III. *Quart. J. Math. Oxford Ser. (2)*, 42(166):149–174, 1991.

[13] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales*. Birkhäuser Boston, Inc., Boston, MA, 2001. An Introduction with Applications.

[14] M. Bohner and A. Peterson, editors. *Advances in Dynamic Equations on Time Scales*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[15] Y. Chu, M. A. Khan, T. Ali, and S. S. Dragomir. Inequalities for α-fractional differentiable functions. *Journal of Inequalities and Applications*, 2017(1):1–12, 2017.

[16] E. T. Copson. Note on Series of Positive Terms. *J. London Math. Soc.*, 3(1):49–51, 1928.

[17] E. T. Copson. Some integral inequalities. *Proc. Roy. Soc. Edinburgh Sect. A*, 75(2):157–164, 1976.

[18] V. Daftardar-Gejji and H. Jafari. Analysis of a system of nonautonomous fractional differential equations involving caputo derivatives. *Journal of Mathematical Analysis and Applications*, 328(2):1026–1033, 2007.

[19] T. Donchev, A. Nosheen, and J. Pe arić. Hardy-type inequalities on time scale via convexity in several variables. *ISRN Math. Anal.*, pages Art. ID 903196, 9, 2013.

[20] A. A. El-Deeb. Some Gronwall-Bellman type inequalities on time scales for Volterra-Fredholm dynamic integral equations. *J. Egypt. Math. Soc.*, 26(1):1–17, 2018.

[21] A. A. El-Deeb. A variety of nonlinear retarded integral inequalities of Gronwall type and their applications. In *Advances in Mathematical Inequalities and Applications*, pages 143–164. Springer, 2018.

[22] A. A. El-Deeb, H. A. El-Sennary, and Z. A. Khan. Some reverse inequalities of Hardy type on time scales. *Advances in Difference Equations*, 2020(1):1–18, 2020.

[23] A. A. El-Deeb, H. A. Elsennary, and D. Baleanu. Some new hardy-type inequalities on time scales. *Advances in Difference Equations*, 2020(1):1–21, 2020.

[24] A. A. El-Deeb, S. D. Makharesh, and D. Baleanu. Dynamic Hilbert-type inequalities with fenchel-legendre transform. *Symmetry*, 12(4):582, 2020.
[25] F. Gesztesy, I. Michael, and M. M. H. Pang. Optimality of constants in power-weighted Birman-Hardy-Rellich-type inequalities with logarithmic refinements. Cubo, 24(1):115–165, 2022.

[26] G. H. Hardy. Note on a theorem of Hilbert. Math. Z., 6(3-4):314–317, 1920.

[27] G. H. Hardy. Notes on some points in the integral calculus (lx). Messenger of Math, 54:150–156, 1925.

[28] G. H. Hardy. Notes on some points in the integral calculus (lxii). Messenger of Math., 57:12–16, 1928.

[29] G. H. Hardy, J. E. Littlewood, and G. Pólya. Inequalities. Cambridge, at the University Press, 1952. 2d ed.

[30] H. P. Heinig. Weighted norm inequalities for certain integral operators. II. Proc. Amer. Math. Soc., 95(3):387–395, 1985.

[31] S. Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math., 18(1-2):18–56, 1990.

[32] R. Hilscher. A time scales version of a Wirtinger-type inequality and applications. J. Comput. Appl. Math., 141(1-2):219–226, 2002.

[33] M. Jleli and B. Samet. Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. Math. Inequal. Appl, 18(2):443–451, 2015.

[34] Z. Kayar and B. Kaymakçalan. Applications of the novel diamond alpha Hardy–Copson type dynamic inequalities to half linear difference equations. J. Difference Equ. Appl., 28(4):457–484, 2022.

[35] R. Khalil, M. A. Horani, A. Yousef, and M. Sababheh. A new definition of fractional derivative. Journal of Computational and Applied Mathematics, 264:65–70, 2014.

[36] M. A. Khan, T. Ali, S. S. Dragomir, and M. Sarikaya. Hermite–Hadamard type inequalities for conformable fractional integrals. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 112(4):1033–1048, 2018.

[37] A. Kilbas, H. Srivastava, and J. Trujillo. Theory and Applications of Fractional Differential Equations, volume 204. elsevier, 2006.

[38] A. Kufner, L. Maligranda, and L.-E. Persson. The Hardy Inequality. Vydavatelský Servis, Plzeň, 2007. About its history and some related results.

[39] A. Kufner and L.-E. Persson. Weighted Inequalities of Hardy Type. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[40] A. Kufner and L.-E. Persson. Weighted inequalities of Hardy type. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[41] A. Kufner, L.-E. Persson, and N. Samko. Weighted inequalities of Hardy type. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2017.

[42] L. Leindler. Generalization of inequalities of Hardy and Littlewood. Acta Sci. Math. (Szeged), 31:279–285, 1970.

[43] J. E. Littlewood and G. H. Hardy. Elementary theorems concerning power series with positive coefficients and moment constants of positive functions. J. Reine Angew. Math., 157:141–158, 1927.
[44] K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. Wiley, 1993.

[45] E. R. Nwaeze and D. F. M. Torres. Chain rules and inequalities for the bht fractional calculus on arbitrary timescales. *Arabian Journal of Mathematics*, 6(1):13–20, 2017.

[46] J. A. Oguntuase and L.-E. Persson. Time scales Hardy-type inequalities via superquadracity. *Ann. Funct. Anal.*, 5(2):61–73, 2014.

[47] B. Opic and A. Kufner. *Hardy-Type Inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.

[48] U. M. Ozkan and H. Yildirim. Hardy-Knopp-type inequalities on time scales. *Dynam. Systems Appl.*, 17(3-4):477–486, 2008.

[49] I. Podlubny. *Fractional Differential Equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[50] P. Řehák. Hardy inequality on time scales and its application to half-linear dynamic equations. *J. Inequal. Appl.*, 5:495–507, 2005.

[51] P. F. Renaud. A reversed Hardy inequality. *Bull. Austral. Math. Soc.*, 34(2):225–232, 1986.

[52] S. H. Saker, D. O’Regan, and R. Agarwal. Generalized hardy, copson, leindler and bennett inequalities on time scales. *Mathematische Nachrichten*, 287(5-6):686–698, 2014.

[53] S. H. Saker, D. O’Regan, and R. Agarwal. Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales. *Math. Nachr.*, 287(5-6):686–698, 2014.

[54] S. H. Saker, D. O’Regan, and R. P. Agarwal. Dynamic inequalities of Hardy and Copson type on time scales. *Analysis (Berlin)*, 34(4):391–402, 2014.

[55] S. H. Saker, S. S. Rabie, G. AlNemer, and M. Zakarya. On structure of discrete muchenhoupt and discrete gehring classes. *Journal of Inequalities and Applications*, 2020(1):1–18, 2020.

[56] S. H. Saker, M. Kenawy, G. AlNemer, and M. Zakarya. Some fractional dynamic inequalities of hardy’s type via conformable calculus. *Mathematics*, 8(3):434, 2020.

[57] M. Sarikaya and H. Budak. New inequalities of opial type for conformable fractional integrals. *Turkish Journal of Mathematics*, 41(5):1164–1173, 2017.

[58] M. Sarikaya, H. Yaldız, and H. Budak. Steffensen’s integral inequality for conformable fractional integrals. *International Journal of Analysis and Applications*, 15(1):23–30, 2017.

[59] M. Z. Sarikaya and C. C. Billisik. Opial type inequalities for conformable fractional integrals via convexity. *Transylv. J. Math. Mech.*, 11(1-2):163–170, 2019.

[60] E. Set, A. Gözpınar, and A. Ekinci. Hermite-Hadamard type inequalities via confortable fractional integrals. *Acta Mathematica Universitatis Comenianae*, 86(2):309–320, 2017.

[61] L. Tang, H. Chen, S. Shen, and Y. Jin. Hardy-Rellich Type Inequalities Associated with Dunkl Operators. *Chinese Ann. Math. Ser. B*, 43(2):281–294, 2022.

[62] P. Řehák. Hardy inequality on time scales and its application to half-linear dynamic equations. *J. Inequal. Appl.*, 5:495–507, 2005.

[63] B. Yang and M. T. Rassias. *On Hilbert-type and Hardy-type integral inequalities and applications*. SpringerBriefs in Mathematics. Springer, Cham, 2019.

[64] M. Zakaryaed, M. Altanj, G. AlNemer, A. El-Hamid, A. Hoda, C. Cesarano, and H. M. Rezk. Fractional reverse coposn’s inequalities via conformable calculus on time scales. *Symmetry*, 13(4):542, 2021.