Lorentz-covariant ultradistributions, hyperfunctions, and analytic functionals

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Dedicated to Professor V. Ya. Fainberg on the occasion of 75th birthday

Abstract

We generalize the theory of Lorentz-covariant distributions to broader classes of functionals including ultradistributions, hyperfunctions, and analytic functionals with a tempered growth. We prove that Lorentz-covariant functionals with essential singularities can be decomposed into polynomial covariants and establish the possibility of the invariant decomposition of their carrier cones. We describe the properties of odd highly singular generalized functions. These results are used to investigate the vacuum expectation values of nonlocal quantum fields with an arbitrary high-energy behavior and to extend the spin–statistics theorem to nonlocal field theory.
1 Introduction

The aim of this work is to extend the theory of Lorentz-covariant distributions to functionals with singularities of an infinite order. The theory of Lorentz-covariant distributions plays an important role in the axiomatic approach [1], [2] to quantum field theory (QFT); the main achievement of this approach is justly considered the derivation of the spin–statistics relation and the $\text{PCT}$ symmetry. My interest in highly singular quantum fields was aroused by Professor V. Ya. Fainberg more than 30 years ago, when I was his graduate student. The enthusiasm of those days, because of the works of Meiman [3] and Jaffe [4], was based on the hope to solve the nonrenormalizable interaction and zero-charge problems along this path and to construct a consistent nonlocal field theory. The subsequent development of gauge theory and superstring theory has shown that some of the ideas proposed then are still relevant.

Fainberg and Iofa [5], [6] first gave a formulation of nonlocal field theory at the rigor level of the axiomatic approach; the theorem on the global nature of local commutativity was shown to fail in the case of the exponential or faster growth of matrix elements of fields in momentum space. Exactly such a growth, with the exponent proportional to the Planck length, was later shown for the spectral density in a Källén–Lehmann-type representation of string propagators [7]. Restrictions on the scattering amplitudes of nonlocally interacting particles [8]–[10] are especially interesting because of the AdS/CFT duality (the correspondence between gravity theories in the anti-de Sitter space and conformal field theories on its boundary), which is currently in the focus of attention. In deriving the scattering matrix from conformal field theory correlators in the flat limit [11], these restrictions can indicate the presence of a nonlocality.

The use of propagators with nonlocal form factors suppressing ultraviolet divergences in the Euclidean momentum space has proved efficient in the Lagrangian formulation of nonlocal QFT [12] and in the phenomenological description of strong interactions [13]. Advanced schemes of this type involve quantum gravity and are proposed as a phenomenological alternative to string theory [14]. In [15], it was shown that the most general distributional framework for constructing local QFT is provided by the use of the Fourier-symmetric test function space $S_{1}$ and by the corresponding generalization of microcausality. Subsequently, such a formulation was developed in great detail in [16], [17] in terms of Fourier hyperfunctions. It was found to give a very symmetric
relation between QFT in Minkowski space and Euclidean field theory, which could not be achieved using the tempered distributions. The theory of the Fourier–Laplace transformation of functionals on the Gelfand–Shilov spaces $S^{\beta}_\alpha$ constructed in [18] proved useful in the operator realization of indefinite-metric gauge models [19]–[21], where singularities have the infrared origin.

A number of theorems on highly singular Lorentz-covariant generalized functions were established in [22]. However, the two most interesting, and also the most difficult, cases were not considered there: the case of tempered hyperfunctions defined on the space $S^1$, which is universal for local QFT, and the case of analytic functionals over the space $S^0$, on which nonlocal fields with an arbitrary high-energy behavior are defined. In this paper, we completely describe the theory for the functionals of class $S^{\beta}_\alpha$, $\beta \geq 0$. Although their use in field-theory constructions has a number of advantages, the test function spaces $S^{\beta}_\alpha$ are topologically more complicated than $S^{\beta}_\alpha$. While the latter belong to the well-studied class of DFS spaces (spaces that are duals of the Fréchet–Schwartz spaces), $S^{\beta}_\alpha$ are not in this class, which complicates the proof of certain structure theorems for the functionals defined there.

This complication is overcome using the acyclicity of a sequence of Fréchet spaces whose inductive limit is $S^\beta$; we establish the corresponding theorem in Sec. 2. In the three subsequent sections, we demonstrate the possibility of splitting the supports of functionals of class $S^{\beta}_\alpha$ for $\beta > 1$ and $\beta = 1$ and of their carrier cones for $\beta < 1$; proving this requires diverse arguments. In Sec. 6, these results are used to derive invariant decompositions of Lorentz-covariant generalized functions; we also describe the properties of odd invariant functionals with arbitrary singularities. In Sec. 7, we establish the density of covariant tempered distributions in the classes of covariant functionals under consideration, and in Sec. 8, we give the corresponding extension of the representation through polynomial covariants [4], [23]. In Sec. 9, these results are used to prove the spin–statistics theorem for nonlocal quantum fields; this proof is an alternative to the one in [24] using the notion of the analytic wave-front set.
2 The spaces $S^\beta(O)$ and their topology

By definition [25], the space of test functions $S^\beta(\mathbb{R}^n)$ with the index $\beta \geq 0$ consists of infinitely differentiable functions on $\mathbb{R}^n$ satisfying the inequalities

$$|\partial^{\kappa} f(x)| \leq C_N B^{k|k\beta_k|}(1 + |x|)^{-N},$$

where $\kappa$ ranges the set of multi-indices $\mathbb{Z}^n$, $N$ ranges the set $\mathbb{N}$ of nonnegative positive integers, and the positive constants $C_N$ and $B$ depend on the function $f$. If $\beta > 1$, the space $S^\beta$ contains functions with a compact support. The space $S^1$ consists of functions that are analytic in a complex neighborhood of $\mathbb{R}^n$. If $\beta < 1$, the space $S^\beta$ consists of functions that can be analytically continued to the whole of $\mathbb{C}^n$, i.e., are entire functions. The elements of the dual space $S^{\beta\prime}$ are called ultradistributions of class $\{\kappa \beta_k\}$ (and of tempered growth) in the first case, hyperfunctions in the second case, and analytic functionals (also of tempered growth) in the third case. Instead of specifying the topology, the convergence of sequences in $S^\beta$ was defined in [25]. Namely, the norms

$$\|f\|_{B,N} = \sup_{\kappa,x} (1 + |x|)^N \frac{|\partial^{\kappa} f(x)|}{B^{k|k\beta_k|}}$$

are associated with inequalities (1), and a sequence $f_\nu \in S^\beta$ is said to converge to zero if there exists $B$ such that $\|f_\nu\|_{B,N} \to 0$ for any $N$. We show that this definition is entirely consistent with the natural topologization of $S^\beta$ by taking the projective limit as $N \to \infty$ and the inductive limit as $B \to \infty$. Another addition to the theory in [25] that we need in what follows consists in using similar spaces over open sets in $\mathbb{R}^n$.

**Definition 1.** Let $O$ be a nonempty open set in $\mathbb{R}^n$. Then $S^{\beta,B,N}(O)$ denotes the normalized space of infinitely differentiable functions on $O$ with the norm $\|f\|_{O,B,N}$ defined similarly to Eq. (2), but with the sup operation taken over $x \in O$. Also, $S^{\beta,B}(O)$ denotes the intersection $\bigcap_N S^{\beta,B,N}(O)$ endowed with the projective topology, and $S^{\beta}(O)$ denotes the union $\bigcup_B S^{\beta,B}(O)$ endowed with the inductive topology.

It is easy to verify that the space $S^{\beta,B,N}(O)$ is complete and hence Banach. Because projective limits inherit the completeness property, $S^{\beta,B}(O)$ is a Fréchet space. To prove the completeness of $S^{\beta}(O)$, we use a sufficient condition given by Palamodov [26].

**Theorem 1.** The injective sequence of Fréchet spaces $S^{\beta,B}(O)$ is acyclic.
Proof. Let \( U_B \) be a neighborhood of the origin in \( S^{\beta,B}(O) \) specified by \( \|f\|_{O,B,0} < 1/2 \). Obviously, \( U_{B_0} \subset U_B \) for any \( B > B_0 \). In accordance with Theorem 6.1 in [26], it suffices to verify that the topology induced on \( U_{B_0} \) from \( S^{\beta,B}(O) \), \( B > B_0 \), is independent of \( B \). We assume that \( f_0 \in U_{B_0} \) and let \( V_{B,N,\epsilon} \) denote the trace on \( U_{B_0} \) of the neighborhood of the function \( f_0 \) in \( S^{\beta,B}(O) \) given by \( \|f - f_0\|_{O,B,N} < \epsilon \); we show that for any \( B > B_1 > B_0 \) and any \( N_1 \) and \( \epsilon_1 \), there exist numbers \( N \) and \( \epsilon \) such that \( V_{B,N,\epsilon} \subset V_{B_1,N_1,\epsilon_1} \). This then implies that the topology induced on \( U_{B_0} \) by that of \( S^{\beta,B}(O) \) is not weaker than the topology induced by that of \( S^{\beta,B_1}(O) \); the converse is obvious. In what follows, we set \( \beta = 0 \) for simplicity (formulas for the general case only differ by inessential factors). If \( f \in V_{B,N,\epsilon} \), we have two estimates for the function \( f_1 = f - f_0 \),
\[
|\partial^\kappa f_1(x)| < B_0^{[\kappa]} \quad \text{and} \quad |\partial^\kappa f_1(x)| < \epsilon B^{[\kappa]} (1 + |x|)^{-N}, \quad x \in O. \tag{3}
\]
We must show that for properly chosen \( N \) and \( \epsilon \), this implies that
\[
|\partial^\kappa f_1(x)| < \epsilon_1 B^{[\kappa]} (1 + |x|)^{-N_1}, \quad x \in O. \tag{4}
\]
We introduce the notation \( \epsilon = \epsilon_1 (1 + |x|)^{-N} \) and \( \epsilon_1 = \epsilon_1 (1 + |x|)^{-N_1} \) and define the number \( Q(x) \) by the equation \( B_0^Q = \epsilon_1 B_1^Q \). If \( x \) is fixed, the first inequality in (3) implies (4) for all \( |\kappa| \geq Q \). The second inequality in (3) implies (4) for \( |\kappa| < Q \) provided that \( \epsilon B^Q \leq \epsilon_1 B_1^Q \). Setting \( \epsilon B^Q = \epsilon_1 B_1^Q \), we obtain \( \epsilon = \epsilon_1 A \), where the number \( A = \log(B/B_0)/\log(B_1/B_0) \) is independent of \( x \). Therefore, the required implication follows if we take \( \epsilon \leq \epsilon_1 A \) and \( N \geq AN_1 \), which completes the proof.

In accordance with [26], the established acyclicity ensures the validity of the following statements.

Corollary. The space \( S^{\beta}(O) \) is Hausdorff and complete. The set \( B \subset S^{\beta}(O) \) is bounded if and only if it is entirely contained in some space \( S^{\beta,B}(O) \) and is bounded with respect to each of its norms.

It is certainly obvious that \( S^{\beta}(O) \) is a Hausdorff space because its topology majorizes the uniform convergence topology. We also note that the inductive limit of Fréchet spaces is a bornological space; therefore, the continuity of a linear mapping of \( S^{\beta}(O) \) into an arbitrary locally convex space is equivalent to its boundedness on all bounded sets, which in turn is equivalent to the sequential continuity [27]. The role of the \( S^{\beta}(O) \) spaces in localization problems can be seen from the following simple remark.
An ultradistribution \( v \in S^\beta \) has support in a compact set \( K \) if and only if for any of its neighborhoods \( O \), there exists a functional \( \hat{v} \in S^\beta(O) \) such that \( (v, f) = (\hat{v}, f|_O) \) for all \( f \in S^\beta \). For \( \beta \leq 1 \), when there are no functions of compact support among test functions, this can be taken as the basis for defining the notion of carrier, replacing the notion of support.

The nuclearity of the spaces \( S^\beta,B^+ = \bigcap_{\epsilon > 0} S^\beta,B^+\epsilon \) (more precisely, of the Fourier-isomorphic spaces \( S^\beta,B^+ \)) was proved in [28]. This implies the nuclearity of \( S^\beta \) because inductive limits of denumerable families of spaces inherit this important property (see Sec. 3.7.4 in [27]). In turn, the nuclearity of a space together with its completeness imply that the space is Montel and, in particular, reflexive (see Chap. 4, Exer. 19, in [27]). We use these properties of \( S^\beta(\mathbb{R}^n) \) in what follows. The spaces \( S^\beta(O) \) possess these properties only under certain restrictions on \( O \).

### 3 The decomposition of ultradistributions

Let \( v \) be a tempered distribution defined on the Schwartz space \( S \), and let the support of \( v \) be contained in the union of closed sets \( K_1 \) and \( K_2 \). It is known that a decomposition of the form \( v = v_1 + v_2 \) with distributions \( v_{1,2} \in S' \) supported by \( K_{1,2} \) is possible if these sets are sufficiently regular and are regularly positioned with respect to each other. The regularity conditions can be precisely formulated using the Whitney continuation theorem [29]. These conditions are trivially satisfied for sets represented as a union of finitely many closed convex subsets with a nonempty interior. For what follows, it is useful to recall the proof of the decomposition theorem in this simplest case (which is sufficient for most applications). We let \( K \) be a set in \( \mathbb{R}^n \) of the above form and let \( S(K) \) denote the space of infinitely differentiable functions on its interior \( \text{Int } K \) with the property that their derivatives extended by continuity to the boundaries of the convex constituents coincide with each other whenever the boundaries have common points; the functions are also required to be such that the norms

\[
\max_{|\kappa| \leq N} \sup_{x \in K} (1 + |x|)^N |\partial^\kappa f(x)|
\]

are finite. This space belongs to the class of FS spaces. By the Whitney theorem, \( f \) can be continued to a smooth function on \( \mathbb{R}^n \), which implies that the space of Schwartz distributions supported by \( K \) can be identified with the dual space \( S'(K) \) of \( S(K) \). The
canonical mapping

\[ S(K_1 \cup K_2) \rightarrow S(K_1) \oplus S(K_2) \tag{6} \]

is injective, is continuous, and has a closed image because the coincidence of convergent sequences \( f_{1\nu} \) and \( f_{2\nu} \) on the intersection \( K_1 \cap K_2 \) implies the coincidence of their limits, which thereby determine an element of the space \( S(K_1 \cup K_2) \). This space can therefore be considered a subspace of the sum \( S(K_1) \oplus S(K_2) \). This identification is valid not only algebraically but also topologically because a closed subspace of a sum of FS spaces is also a FS space, while in accordance with the open mapping theorem [27], a vector space cannot have two different comparable topologies such that it is a Fréchet space in each of them. Applying the Hahn–Banach theorem, we conclude that any functional \( v \in S'(K_1 \cup K_2) \) has a continuous extension to the sum. Letting \( \hat{v} \) denote it and writing \( v(f) = \hat{v}(f|_{K_1}, 0) + \hat{v}(0, f|_{K_2}) \), we obtain the desired decomposition because composing \( \hat{v} \) with the canonical embeddings \( S(K_i) \rightarrow S(K_1) \oplus S(K_2) \) gives elements of \( S'(K_i) \), \( i = 1, 2 \).

Conditions for the decomposability of ultradistributions were studied by Lambert [30], and his theorem covers the functionals of Gelfand–Shilov’s class \( S_0^\beta \). (We recall that the space \( S_0^\beta \) consists of those elements of \( S^\beta \) that have a compact support; this space is nontrivial for \( \beta > 1 \).) A similar theorem for the class \( S_\beta^\beta \) is difficult to prove, but in the particular case that is solely important for the relevant Lorentz-invariant decompositions considered in what follows, the corresponding statement is a direct consequence of Lambert’s results.

**Theorem 2.** Let \( K_1 \) and \( K_2 \) be closed convex cones in \( \mathbb{R}^n \) such that \( K_1 \cap K_2 = \{0\} \). Any functional \( v \in S_\beta^\beta \) with support in the cone \( K_1 \cup K_2 \) can be decomposed into a sum of functionals of the same class \( S_\beta^\beta \) supported by \( K_1 \) and \( K_2 \).

**Proof.** In accordance with Theorem 5.1.1 in [30], the restriction \( v|_{S_0^\beta} \) can be represented as a sum \( v_1 + v_2 \), where the respective functionals \( v_1 \) and \( v_2 \) are supported by \( K_1 \) and \( K_2 \). We need only verify that they have a continuous extension to \( S^\beta \), which is then necessarily unique, because \( S_0^\beta \) is dense in \( S^\beta \). Let \( \chi \) be an arbitrary function in \( S_0^\beta \) that is identically equal to 1 on the ball \( U = \{ x : |x| < 1 \} \), and let \( \chi_{1,2} \) be multipliers for \( S^\beta \) (and hence for \( S_0^\beta \)) that are equal to 1 in a neighborhood of \( K_{1,2} \setminus U \) and are equal to zero in a neighborhood of \( K_{2,1} \setminus U \). Because \( S_0^\beta \) is an algebra with respect to
multiplication, it follows that for all $f \in S_0^3$, we have

$$v_{1,2}(f) = v_{1,2}(\chi f) + v(\chi_{1,2}(1 - \chi)f).$$  \hspace{1cm} (7)

It remains to note that multiplying by $\chi$ continuously maps $S^3$ into $S_0^3$ and the right-hand side of (7) is therefore defined as a continuous functional on $S^3$.

4 The decomposition of hyperfunctions

The regularity conditions for sets established in [30], which guarantee the possibility of the splitting, become progressively weaker as the functionals become more singular, i.e., as the index $\beta$ of the test function space decreases. In the class $S^1$, the decomposition of a functional supported by $K_1 \cup K_2$ is already possible for any compact sets $K_1$ and $K_2$. However, the very definition of support is different in this case because the test functions are analytic. We let $A(K)$ denote the space of functions that are analytic in a complex neighborhood (depending on a chosen function) of a compact set $K \subset \mathbb{R}^n$ and recall that it is a DFS space when endowed with the natural topology [31]. Directly applying the Taylor formula shows that $S^1$ is continuously embedded in $A(K)$. In accordance with the standard definition [29], [31] of the carrier set of an analytic functional, the compact set $K$ is a carrier set of $v \in S^1$ if $v$ has a continuous extension to $A(K)$. This can be also expressed as $v \in A'(K)$ because $S^1$ is dense in $A(K)$. The decomposition theorem for $v \in A'(K_1 \cup K_2)$ can be easily proved using the same argument as in the previous section because a closed subspace of a sum of DFS spaces is a DFS space and a generalization of the open mapping theorem also applies to spaces of this class. A different proof, using a harmonic regularization of analytic functionals, is given in [29] (Sec. 9.1).

In formalizing the notion of support of elements $v \in S^1$, it must be taken into account that some of them can be naturally regarded as concentrated at infinity. The point is that the inductive limit $\lim_{R \to \infty} S^1(O_R)$, where $O_R = \{x : |x| > R\}$ and $R \to \infty$, is a Hausdorff space and $S^1$ is injectively and continuously embedded in it (see Proposition 1.19 in [22]). Therefore, in accordance with the Hahn–Banach theorem, there exist nonzero functionals in $S^1$ admitting a continuous extension to this space. The radial compactification $\mathbb{D}^n = \mathbb{R}^n \cup S^{{n-1}}_\infty$, where $S^{{n-1}}_\infty$ is an $(n-1)$-dimensional sphere at infinity, is used in the theory of Fourier hyperfunctions [32]. We also apply it in the
case of functionals of class $S'$1. We say that a compact set $K \in \mathbb{D}^n$ is a carrier of $v \in S'$1 if $v$ can be continuously extended to the space

$$S^1(K) = \lim_{\mathcal{O} \supset K} S^1(\mathcal{O} \cap \mathbb{R}^n),$$

(8)

where $\mathcal{O}$ ranges open neighborhoods of $K$ in $\mathbb{D}^n$. If $K \subset \mathbb{R}^n$, this definition reduces to the previous one because inductive limit (8) then coincides with $A(K)$. We recall that Fourier hyperfunctions compose a space that is the dual of the test function space $S^1$ defined by the inequalities

$$|\partial^\kappa f(x)| \leq CB^{|\kappa|} |\kappa|^\beta e^{-|x|/A},$$

(9)

where the constants $A$, $B$, and $C$ depend on $f$. The space $S^1_1(O)$ is a union of Banach spaces $S^1_{1,A}(O)$ related by compact embeddings with respect to both indices $A$ and $B$. Therefore, $S^1_1(O)$, as well as $S^1_1(K)$, is a DFS space. We next consider the canonical mapping

$$S^1_1(K_1 \cup K_2) \longrightarrow S^1_1(K_1) \oplus S^1_1(K_2)$$

(10)

and use the same simple argument as in the previous section to conclude that every Fourier hyperfunction with support in $K_1 \cup K_2$ admits a decomposition for any compact sets $K_1, K_2 \subset \mathbb{D}^n$.

The only obstruction to a similar proof of the decomposition theorem for the functionals of class $S'$1 is that the applicability of the open mapping theorem is no longer obvious. The most general formulation of this theorem assumes that the space of values of the mapping is ultrabornological, i.e., a Hausdorff space representable as the inductive limit of Fréchet spaces. It is by far nonobvious that $S^1(K)$ with $K = K_1 \cup K_2$ is in this class when it is endowed with the topology induced by the embedding in $S^1(K_1) \oplus S^1(K_2)$. For a deeper insight into this situation, we must consider the chain of mappings

$$0 \rightarrow S^1(K_1 \cup K_2) \overset{i}{\longrightarrow} S^1(K_1) \oplus S^1(K_2) \overset{s}{\longrightarrow} S^1(K_1 \cap K_2) \rightarrow 0,$$

(11)

where to a pair of functions $f_{1,2} \in S^1(K_{1,2})$, $s$ assigns the difference of their restrictions to $K_1 \cap K_2$. By Theorem 1.30 in [22], the sequence of vector spaces in (11) is exact, i.e., the kernel of each mapping involved coincides with the image of the preceding mapping.
The only nontrivial point is the exactness in the term \( S^1(K_1 \cap K_2) \), which means that any element of this space admits a decomposition into a sum of functions belonging to the spaces \( S^1(K_1) \) and \( S^1(K_2) \) that is established using the Hörmander \( L^2 \) estimates in complete analogy with the corresponding statement for Fourier hyperfunctions. This implies the existence of supports for the functionals of class \( S^0(\mathbb{R}^n) \).

**Theorem 3.** Each element of the space \( S^0(\mathbb{R}^n) \) has a unique minimal carrier in \( \mathbb{D}^n \).

**Proof.** We first note that if a functional \( v \in S^0(\mathbb{R}^n) \) has carriers \( K_1 \) and \( K_2 \) whose intersection is empty, then \( v = 0 \). This follows because \( S^1 \) is dense in \( S^1(K_1 \cup K_2) \) (which is asserted by Lemma 1.17 in [22]), and the latter space then contains a function that is identically equal to zero in a neighborhood of \( K_1 \) and is equal to any chosen element of \( S^1 \) in a neighborhood of \( K_2 \). We next let the carriers have a nonempty intersection \( K_1 \cap K_2 \). To prove the theorem, it suffices to show that this intersection is a carrier of \( v \). Supposing the converse, we let \( K \) denote the intersection of all carriers of \( v \). If there exists a finite subsystem of the carriers with an empty intersection, then \( v = 0 \). If there is no such subsystem, then \( K \neq \emptyset \). Let \( O \) be a neighborhood of \( K \) in \( \mathbb{D}^n \). The complements of the carriers constitute an open covering of the compact set \( \mathbb{D}^n \setminus O \). Choosing a finite subcovering, we conclude that \( K \) is a minimal carrier.

Going over to the orthogonal complements in the relation \( \text{Ima}i = \text{Ker}s \), we obtain \( \text{Ker}i' = \overline{\text{Ima}s'} \), where the primes indicate the conjugate mappings and the bar means the closure under the weak topology. Because \( s \) in (11) maps a \( \mathcal{UF} \) space \( \mathbb{H} \) to an ultrabornological space, Grothendieck’s formulation of the open mapping theorem \([24]\) applies to \( s \). In particular, \( s \) is a topological homomorphism. Therefore, the image of \( s' \) is weakly closed by Theorem 4.7.5 in \([27]\), and therefore \( \text{Ker}i' = \text{Ima}s' \). If \( v \in S^1 \) has continuous extensions \( v_{1,2} \) to \( S^1(K_{1,2}) \), then \( i'(v_1, v_2)|_{s^1} = 0 \). Because \( S^1 \) is dense in \( S^1(K_1 \cup K_2) \), it follows that \( i'(v_1, v_2) = 0 \). Therefore, the functionals \( v_{1,2} \) are restrictions to \( S^1(K_{1,2}) \) of elements belonging to \( S^1(K_1 \cap K_2) \), i.e., there exists a continuous extension of \( v \) to \( S^1(K_1 \cap K_2) \), which completes the proof.

**Theorem 4.** Let \( K_1 \) and \( K_2 \) be compact sets in \( \mathbb{D}^n \) that are closures of cones, and let \( K_1 \cap K_2 = \{0\} \). Any functional \( v \in S^1 \) with support in \( K_1 \cup K_2 \) admits a decomposition

\[^2\text{This class involves locally convex Hausdorff spaces that can be covered by a denumerable family of their Fréchet subspaces. Any } S^1(K) \text{ is in this class because } S^1,B(\mathcal{O} \cap \mathbb{R}^n), \mathcal{O} \supset K, \text{ are its Fréchet subspaces and } K \text{ has a denumerable fundamental system of neighborhoods in } \mathbb{D}^n.\]
into a sum of functionals of the same class with supports in the compact sets \( K_1 \) and \( K_2 \).

**Proof.** We let \( E \) denote the space \( S^1(K_1) \oplus S^1(K_2) \) and let \( \mathcal{O}^\nu_{1,2}, \nu = 1, 2, \ldots \), be denumerable fundamental systems of neighborhoods of \( K_{1,2} \) in \( \mathbb{D}^n \). The topology of \( E \) is identical to that of the inductive limit of the Fréchet spaces \( E^\nu = S^1(\mathcal{O}^\nu_1 \cap \mathbb{R}^n) \oplus S^1(\mathcal{O}^\nu_2 \cap \mathbb{R}^n) \). Let \( L \) be the image of \( S^1(K_1 \cup K_2) \) in \( E \) endowed with the topology induced by that of \( E \). It suffices to show that this topology coincides with the inductive limit topology for the family of subspaces \( L^\nu = E^\nu \cap L \). Indeed, the space \( L \) is then ultrabornological (because \( L \) being closed in \( E \) implies that \( L^\nu \) is closed in \( E^\nu \), i.e., each \( L^\nu \) is a Fréchet space), which allows applying the Grothendieck theorem \([34]\), thereby completing the proof using the previous argument. The condition for the coincidence of these two topologies of \( L \) follows from the Retakh theorem \([35]\) and amounts to the acyclicity of the sequence of quotient spaces \( F^\nu = E^\nu / L^\nu \), which are also Fréchet spaces and are related by canonical injections \( F^\nu \to F^\mu, \nu < \mu \). The subspace \( L^\nu \) is the kernel of the continuous mapping \( E^\nu \to E / L \); therefore, the corresponding mapping \( E^\nu / L^\nu \to E / L \) is also continuous. We let \( F \) denote the quotient space \( E / L \) endowed with the inductive topology relative to this family of mappings (which is in fact identical to its own topology). Because sequence (11) is exact, it follows that the continuous mapping \( F \to S^1(\{0\}) \) is bijective. The space \( S^1(\{0\}) \) coincides with the ring of germs of analytic functions at \( z = 0 \) and belongs to the DFS class; therefore, the above bijection is not only an algebraic but also a topological isomorphism in accordance with the cited theorem \([34]\). Writing \( a_\kappa = \partial^\kappa f(0) \), we identify the space \( S^1(\{0\}) \) with the space of strings \( \{a_\kappa\} \) of complex numbers such that \( |a_\kappa| \leq C B^{\kappa} |\kappa| K^\kappa \) for some \( C, B > 0 \). Let \( A^\nu \) be the Banach space of strings with the norm \( \sup_\kappa |a_\kappa| / \nu^{\kappa} K^\kappa \). The proof of Theorem 1 (where \( x = 0 \) must be set in this simplest case) shows that the injective sequence of spaces \( A^\nu \) is acyclic. Therefore, the sequence \( F^\nu \) is also acyclic because it is equivalent to the sequence \( A^\nu \) in view of the above isomorphism. Theorem 4 is thus proved.
5 The decomposition of functionals in the class $S^{\prime \beta}$, $\beta < 1$

It was shown in $[22], [36]$ that the analytic functionals of class $S^{\prime \beta}$ (and $S^{\prime \beta}_{\alpha}$), $0 < \beta < 1$, retain the angular localizability even though they do not possess supports. Namely, if arbitrary compact sets in $\mathbb{D}^n$ are replaced with only the closures of cones and $S^\beta(K)$ is defined similarly to Eq. (8), then the sequence

$$0 \rightarrow S^\beta(K_1 \cup K_2) \rightarrow S^\beta(K_1) \oplus S^\beta(K_2) \rightarrow S^\beta(K_1 \cap K_2) \rightarrow 0 \quad (12)$$

is exact, which implies that each element of $S^{\prime \beta}$ has the smallest closed carrier cone (we say that a closed cone in $\mathbb{R}^n$ is a carrier cone if the compact set obtained by adjoining to this cone the part of the compact covering $S^n_{\infty}^{-1}$ cut out by the cone is a carrier). As in the case of hyperfunctions, the derivation of this result is based on the complex-variable representation of the spaces $S^\beta(K)$ associated with the cones. Namely, let $O$ be the union of an open cone $U \subset \mathbb{R}^n$ and an $\epsilon$-neighborhood of the origin. As shown in $[36]$, the space $S^\beta(O)$ is isomorphic to the space of entire functions on $\mathbb{C}^n$ satisfying the inequalities

$$|f(z)| \leq C_N (1 + |x|)^{-N} \exp \left(|By|^{1/(1-\beta)} + d(Bx, U)^{1/(1-\beta)}\right), \quad z = x + iy, \quad (13)$$

where the constants $C_N$ and $B$ depend on $f$ and $d(\cdot, U)$ is the distance from the point to the cone $U$. If $K$ is the compact set in $\mathbb{D}^n$ corresponding to a closed cone $K$, then each element of $S^\beta(K)$ belongs to some space $S^\beta(O)$, where $O$ is of the above form and contains $K$. The space $S^\beta(\{0\})$ associated with the degenerate cone $\{0\}$ consists of entire functions with an order of growth at most $1/(1 - \beta)$ and of finite type, i.e., the entire functions satisfying the condition

$$|f(z)| \leq C \exp \left(|Bx|^{1/(1-\beta)} + |By|^{1/(1-\beta)}\right). \quad (14)$$

**Theorem 5.** Let $K_1$ and $K_2$ be closed cones in $\mathbb{R}^n$ such that $K_1 \cap K_2 = \{0\}$. Any functional $v \in S^{\prime \beta}$, $0 \leq \beta < 1$, with the carrier cone $K_1 \cup K_2$ admits a decomposition into a sum of functionals of the same class with the carrier cones $K_1$ and $K_2$.

**Proof.** The statement of the theorem follows from the exactness of sequence (12) via an argument completely similar to the proof of Theorem 4 because $S^\beta(\{0\})$ is the injective limit of an acyclic sequence of Banach spaces, as is $S_1(\{0\})$. This is based on
the possibility of decomposing every element of $S^\beta(K_1 \cap K_2)$ into functions belonging to $S^\beta(K_1)$ and $S^\beta(K_2)$. It is worth noting that for a nonzero $\beta$ and for the geometry in question, this possibility is almost obvious if we recall that the space $S^\beta_1$, $\beta > 0$, is nontrivial [25] and contains a function $\chi_0$ with the properties

$$|\chi_0(z)| \leq C_0 \exp \left( -\left| \frac{x}{A_0} \right|^{1/(1-\beta)} + |B_0y|^{1/(1-\beta)} \right)$$

(15)

and $\int \chi_0(\xi) \, d\xi = 1$. We let $W_1$ and $W_2$ be open cones such that $K_{1,2} \setminus \{0\} \subset W_{1,2}$ and $\overline{W}_1 \cap \overline{W}_2 = \{0\}$. For all $x \in W_1$ and $\xi \in W_2$, we then have the inequality $|x - \xi| \geq \theta|x|$, where $\theta > 0$. We set $\chi(z) = \int_{W_2} \chi_0(z - \xi) \, d\xi$. For any $A > A_0$, there is the obvious estimate

$$|\chi(z)| \leq C_A \exp \left( -\left| \frac{x}{A} \right|^{1/(1-\beta)} + |B_0y|^{1/(1-\beta)} \right), \quad x \in W_1.$$  

(16)

The desired decomposition of functions satisfying restrictions (14) is realized by $f = \chi f + (1 - \chi)f$ if we take $A_0 < \theta/B$, which is always possible. Indeed, an estimate of type (13) (with a sufficiently large $B_1$ instead of $B$) is then certainly satisfied for the function $\chi f$ when $x \in W_1$, and if the cone $U \supset K \setminus \{0\}$ is such that $\overline{U} \setminus \{0\} \subset W_1$, this estimate is also satisfied for $x \notin W_1$ because it then follows that $d(x, U) \geq \theta'|x|$ for some $\theta' > 0$. Therefore, $\chi f \in S^\beta(K_1)$. Similarly, $(1 - \chi)f \in S^\beta(K_2)$ because $1 - \chi = \int_{CW_2} \chi_0(z - \xi) \, d\xi$ and the cone $CW_2$ is separated from $K_2$ by a finite angular distance. For $\beta = 0$, this argument does not apply, and more sophisticated means must be used, as in the theory of hyperfunctions. It was proved in [37] that an analogue of sequence (12) for the spaces $S^0_\alpha$ with $\alpha > 1$ is exact. In particular, elements of $S^0_\alpha(\{0\}) = S^0_\alpha(\{0\})$ admit a decomposition even within this narrower class. Therefore, Theorem 5 is also valid for $\beta = 0$. We note that sequence (12) is itself exact for $\beta = 0$, which can be verified using Lemma 1.31 in [24], but this proof is considerably more involved.

**Lemma 1.** Any functional $v \in S^\ell_\beta$, $\beta \geq 0$, whose carrier is the origin is given by

$$v = \sum_\kappa c_\kappa \partial^\kappa \delta(x), \quad \lim_{|\kappa| \to \infty} |\kappa|^{\beta} |c_\kappa|^{1/|\kappa|} = 0.$$  

(17)

**Proof.** For $\beta > 1$, this statement is the simplest case of Theorem 4.1.1 in [30]. If $\beta \leq 1$, the functions $e_\kappa = x^\kappa/\kappa!$ obviously constitute an unconditional basis in $S^\beta(\{0\})$, and the functionals $e'_\kappa = (-1)^{|\kappa|} \partial^\kappa \delta(x)$ constitute the dual basis of functionals. Therefore,
$$(v, f) = \sum_\kappa (v, e_\kappa)(e'_\kappa, f),$$ which converts into representation (17) after the redefinition $c_\kappa = (-1)^{\kappa}(v, e_\kappa)$. By the continuity of $v$ in the topology of $S^\beta(\{0\})$, we have the inequality $|(v, e_\kappa)| \leq C_B\|e_\kappa\|_B$ for any $B$, where $\|e_\kappa\|_B = \sup_\ell |\partial^\ell e_\kappa(0)| B^{-|\ell|} e^{-\beta \ell} = B^{-|\kappa|} e^{-\beta \kappa}$, which implies the above restriction on the coefficients $c_\kappa$. Lemma 1 is proved.

6 The Lorentz-invariant decomposition

In considering Lorentz-invariant functionals on the Minkowski space $\mathbb{R}^4$, we use the notation $V = \{x \in \mathbb{R}^4: x^2 = x_0 y_0 - xy \geq 0\}$, $V_+ = \{x \in V: x_0 \geq 0\}$, and $V_- = \{x \in V: x_0 \leq 0\}$. We let $L^+_1$ denote the proper Lorentz group.

**Theorem 6.** Any Lorentz-invariant functional $v \in S^\beta$, $\beta \geq 0$, with the carrier cone $\bar{V}$ admits a decomposition into Lorentz-invariant functionals of the same class with the carrier cones $\bar{V}_+$ and $\bar{V}_-$.

**Proof.** Let $v = v_+ - v_-$ be a decomposition of $v$ into functionals with the forward and backward carrier cones; this decomposition exists in view of Theorems 2, 4, and 5 (the minus sign here is convenient in what follows). The invariance of the decomposition with respect to the subgroup of spatial rotations is ensured by the transition to the averaged functionals $\bar{v}_\pm$,

$$\left(\bar{v}_\pm, f\right) \overset{\text{def}}{=} \left(v_\pm, \int_{R \in SO(3)} f(Rx) \, dR\right).$$

The measure on the rotation group is taken as normalized to 1. This cannot be applied to pure Lorentzian transformations (boosts) because these transformations are noncompact. We let $N_j = x_j \partial_0 + x_0 \partial_j$ be the boost representation generators on the space of functionals. It is obvious that for $\beta > 1$, the functional

$$u = N_1 \bar{v}_+ = N_1 \bar{v}_-$$

is supported by the origin. For $\beta = 1$, a similar statement is valid in view of Theorem 3, and for $\beta < 1$, the degenerate cone $\{0\}$ is a carrier of $v$ because sequence (12) is exact (and $S^\beta(\bar{V}_\pm)$ is Lorentz invariant). To obtain the desired decomposition, it suffices to find a functional of form (17) that is an $SO(3)$-invariant solution of the equation

$$N_1 v = u.$$ 

14
Indeed, if such a solution \( v_0 \) exists, the functionals \( \bar{v}_\pm - v_0 \in S^0(\nabla_\pm) \) are invariant under the entire group \( L_+^\uparrow \) in view of the commutation relations

\[
[N_j, M_{ij}] = N_i, \quad i \neq j, \tag{21}
\]

where \( M_{ij} = x_j \partial_i - x_i \partial_j \) are the representation generators of the three-dimensional rotation group. Let \( C = - \sum_{i<j} M_{ij} \) be the Casimir operator of this group. It follows from Eqs. (21) that

\[
N_1 u = 0, \quad Cu = 2u. \tag{22}
\]

The finite-dimensional spaces \( E_n \) consisting of functionals of the form \( \sum_{|\kappa|=n} c_\kappa \partial^\kappa \delta(x) \) are invariant under \( L_+^\uparrow \). Therefore, the problem is reduced to verifying that inside each \( E_n \), the operator \( N_1 \) maps the subspace \( F_n = \bigcap_{i<j} \ker M_{ij} \) onto the subspace determined by Eqs. (22), which is denoted by \( G_n \). The Fourier transformation takes \( E_n \) into the space of homogeneous \( n \)th-order polynomials, which we decompose into the direct sum of \( SO(3) \)-invariant subspaces,

\[
E_n = \bigoplus_{l=0}^{n} p_0^{n-l} P_l, \tag{23}
\]

where \( P_l \) consists of homogeneous \( l \)th-order polynomials in the variables \( p_1, p_2, \) and \( p_3 \) (and where \( P_0 = \mathbb{C} \)). We recall \cite{38} that each subspace \( P_l \) is in turn a direct sum of the minimal invariant subspaces of the form \( (p^2)^k H_{l-2k}, k = 0, 1, \ldots, [l/2] \), where \( H_{l-2k} \) consists of harmonic homogeneous polynomials (i.e., those satisfying the Laplace equation). On \( H_{l-2k} \), the Casimir operator is a multiple of the unit operator, and the corresponding eigenvalue is equal to 2 only if the homogeneity degree \( l - 2k \) is equal to 1. Further, only those elements of \( H_1 \) that are multiples of \( p_1 \) satisfy the first condition in (22). Therefore, the polynomials

\[
p_0^{n-2k-1}(p^2)^k p_1, \quad k = 0, 1, \ldots, \left[ \frac{n-1}{2} \right], \tag{24}
\]

constitute a basis in \( G_n \). A basis in \( F_n \) is formed by the polynomials

\[
p_0^{n-2k}(p^2)^k, \quad k = 0, 1, \ldots, \left[ \frac{n}{2} \right]. \tag{25}
\]

Therefore, \( \dim F_n \geq \dim G_n \), and the mapping \( F_n \rightarrow G_n \) under consideration is indeed surjective. The occurrence of a one-dimensional kernel of this mapping for even \( n \)
corresponds to the obvious ambiguity of the sought decomposition due to the possibility of adding terms of the form $\sum_l c_l \square^l \delta(x)$. It remains to show that the solution of Eq. (20) can be chosen such that it satisfies the restriction on the coefficients in Eq. (17). For even $n$, we restrict the mapping $F_n \to G_n$ to the linear span of the first $(n/2-1)$ polynomials in (25), thereby specifying the choice of the solution $v_0$. Applying the generator $p_1 \partial_0 + p_0 \partial_1$ to these, we see that for any $n$, the matrix $(a_{kl})$ of the mapping in the above bases is quasi-diagonal with the only nonzero elements $a_{kk} = n - 2k$ and $a_{k,k+1} = 2(k+1)$, $0 \leq k \leq [(n-1)/2]$. The inverse matrix is also upper-triangular, the absolute value of its elements increases monotonically as both indices simultaneously increase and reaches its maximum at $l = [(n-1)/2]$, where

$$|a_{kl}^{(-1)}| = \frac{(n-1)!!}{(n-2k)!! (2k)!!} \tag{26}$$

for odd $n$. For even $n$, the numerator in (26) is replaced with $(n-2)!!$. The product of the left- and the right-hand sides of the inequality

$$\frac{(n-1)!!}{(n-2k)!! (2k)!!} \leq \frac{n!!}{(n-2k-1)!! (2k-1)!!}$$

is the binomial coefficient $\binom{n}{2k}$, which is majorized by $2^n$. Therefore, $|a_{kl}^{(-1)}| \leq 2^{n/2}$. The decomposition coefficients of the elements of $G_n$ with respect to basis (24) are a subset of the decomposition coefficients with respect to the monomials $p^\kappa$, and the decomposition coefficients of the elements of $F_n$ with respect to the monomials differ from the decomposition coefficients of the elements of $G_n$ with respect to basis (25) by factors that are not greater than $3^{n/2}$. Thus, if $u = \sum_\kappa c_\kappa p^\kappa \delta(x)$, then the coefficients $c_\kappa$ of the above solution $v_0$ to (20) satisfy the estimate $\max_{|\kappa|=n} |c_\kappa| \leq 6^{n/2} \max_{|\kappa|=n} |c_\kappa|$, which completes the proof of Theorem 6.

**Remark.** The proof presented can be directly extended to the space–time of an arbitrary dimension $d \geq 3$. However, for $d = 2$, where the rotation subgroup is absent, the proof loses its applicability, and a similar theorem is not true.

Indeed, we consider the distribution $\partial_+ \left[ \theta(x_+) \log x_+ \right] \delta(x_-) \in S'(\mathbb{R}^2)$, where $x_\pm = (x_0 \pm x_1)/\sqrt{2}$ are the light-cone coordinates. This distribution is not Lorentz-invariant in $\mathbb{R}^2$, is supported by a ray on the boundary surface of the cone $\mathbb{V}^{(2)}_+$, and satisfies the equation $N_1 v = \delta(x)$ (in these variables, $N_1 = x_+ \partial_+ - x_- \partial_-$). The same equation is satisfied by the distribution obtained from this one via the reflection $x \to -x$; however,
there is no solution to this equation among functionals (17). Therefore, the sum
\[ \partial_+ [\theta(x_+) \log x_+] \delta(x_-) + \partial_+ [\theta(-x_+) \log |x_+|] \delta(x_-) \]
(27)
is an odd Lorentz-invariant distribution, which does not admit an invariant decomposition in either the Schwartz class \( S' \) or the classes \( S'^\beta, \beta \geq 0 \). This example also shows that not every Lorentz-invariant distribution in \( \mathbb{R}^2 \setminus \{0\} \) admits an invariant extension to the whole of the \( \mathbb{R}^2 \) space, whereas for \( \mathbb{R}^4 \), such an extension always exists (Proposition 3.5 in \([2]\)). Because the distribution \( \theta(x_+)x_+^{-1}\delta(x_-) \) on \( \mathbb{R}^2 \setminus \{0\} \) cannot be continued to a Lorentz-invariant positive measure on \( \mathbb{R}^2 \), it is necessary to use an infrared indefinite metric in quantizing the massless scalar field in two-dimensional space–time \([39]\).

**Theorem 7.** Let \( v \) be a Lorentz-invariant functional of class \( S'^\beta, \beta \geq 0 \), with the carrier cone \( \mathbf{V} \). If its Fourier transform vanishes in a neighborhood of a spacelike point, then \( v \) is odd.

**Proof.** Let \( v = v_+ - v_- \) be the invariant decomposition of \( v \) into functionals with the carrier cones \( \mathbf{V}_+ \) and \( \mathbf{V}_- \). Theorem 4 in \([10]\) shows that there is a well-defined Laplace transform \( u_\pm(\zeta) = (v_\pm, e^{i(\cdot, \zeta)}) \) of \( v_\pm \). The functions \( u_\pm \) are analytic in the domain \( \mathbb{T}_\pm = \{ \zeta = p + i\eta: \eta \in \mathbb{V}_\pm \} \), and their boundary values as \( \eta \to 0, \eta \in \mathbb{V}_\pm' \) (where \( \mathbb{V}_\pm' \setminus \{0\} \subset \mathbb{V}_\pm \)), are just the Fourier transforms \( u_\pm = \mathcal{F}v_\pm \). In \([10]\), the classes \( S'^0_\alpha \supset S'^0 \) were considered, and the convergence to the boundary values was proved in the topology of \( S'^0_\alpha = \mathcal{F}(S'^0) \), but in the present case, certainly, this convergence also occurs in the topology \( S'_\beta = \mathcal{F}(S'^\beta) \), as can be verified using the same argument. In accordance with the Bargmann–Hall–Wightman theorem \([1, 2]\), each Lorentz-invariant function \( u_\pm \) can be analytically continued to the extended domain \( \mathbb{T}_{\text{ext}} \) that contains all spacelike points. This continuation is symmetric with respect to the complex Lorentz group \( L_+(\mathbb{C}) \) and, in particular, with respect to the full reflection \( \zeta \to -\zeta \). Taking the uniqueness property of analytic functions into account (Sec. 6 in \([11]\)), we conclude that the above assumption about the support of \( u = \mathcal{F}v \) implies the equality \( u_+(\zeta) = u_-(\zeta), \zeta \in \mathbb{T}_{\text{ext}} \). Therefore, for any test function \( g \in S_\beta = \mathcal{F}(S^\beta) \), we have
\[
(u, g) = \lim_{\eta \to 0, \eta \in \mathbb{V}_+'} \int_{\mathbb{V}_+'} (u_+(p + i\eta) - u_-(p - i\eta)) g(p) \, dp = - (u, g(- \cdot)),
\]
as was to be proved.
We note that Theorem 7 can be strengthened somewhat: in the condition of the theorem, it suffices to suppose that the functional \( v \) has a closed carrier cone \( K \supset V \) that is different from the entire space. Then \( V \) is also its carrier cone in view of the existence of the smallest closed carrier cone and in view of the Lorentz invariance of \( v \), because any spacelike direction can be rotated into the interior of the complement of \( K \) by an appropriate Lorentz transformation.

7 The denseness theorem

Let \( \mathcal{E} \) be a finite-dimensional complex vector space carrying a representation \( T \) of the group \( L^\uparrow_+ \). We let \( L(S^\beta, \mathcal{E}) \) denote the space of continuous linear mappings of \( S^\beta \) into \( \mathcal{E} \) endowed with the topology of uniform convergence on bounded sets. With a basis fixed in \( \mathcal{E} \), a mapping \( w \in L(S^\beta, \mathcal{E}) \) can be identified with the set of continuous linear functionals \( w^j \in S'^\beta \) (or, equivalently, with an element of the space \( (\mathcal{E}' \otimes S^\beta)' \)); the number of these functionals is equal to the dimension of the representation. Because the space \( S^\beta \) is Montel, the convergence of a sequence \( w_\nu \) in the above topology is equivalent to the condition that each sequence \( w^j_\nu \) is weakly convergent. A mapping \( w \in L(S^\beta, \mathcal{E}) \) is called a (vector-valued) Lorentz-covariant generalized function if it satisfies the condition

\[
w(f) = T(\Lambda)w(f_\Lambda), \quad \Lambda \in L^\uparrow_+, \quad f \in S^\beta,
\]

where the Lorentz group action in the space of test functions is defined in the standard way as \( f_\Lambda(x) = f(\Lambda^{-1}x) \). If \( T \) has an odd valence and realizes a representation of the \( SL(2, \mathbb{C}) \) group, which is the universal covering of \( L^\uparrow_+ \), then a condition similar to (28) implies the identical vanishing \( w \equiv 0 \), which therefore means that we are in fact dealing with single-valued Lorentz group representations. In the component notation, condition (28) becomes

\[
(w^j, f_{\Lambda^{-1}}) = \sum_k T^j_k(\Lambda)(w^k, f), \quad \Lambda \in L^\uparrow_+, \quad f \in S^\beta.
\]

Lorentz-covariant generalized functions constitute a closed subspace in \( L(S^\beta, \mathcal{E}) \), which we endow with the induced topology.

**Theorem 8.** For any \( \beta \geq 0 \), the space of tempered Lorentz-covariant generalized functions is dense in the space of covariant distributions of the class \( S'^\beta \) transforming
according to the same Lorentz group representation.

**Proof.** We regularize the ultraviolet behavior of a covariant generalized function \( w \) by multiplying its Fourier transform \( u = \mathcal{F} w \) by \( \chi(p/M) = \chi_0(p^2/M^2) \), where \( p^2 \) is the Lorentz square of the vector \( p \) and \( \chi_0(t) \) is an infinitely differentiable function with support in the interval \((-1, 1)\) and identically equal to 1 for \(|t| \leq 1/2\). The space \( S_\beta \) consists of smooth functions \( g \) such that

\[
\|g\|_{B,N} = \max_{\kappa \leq N} \sup_p |\partial^\kappa g(p)| \exp \left( \frac{|p|^{1/\beta}}{B} \right) < \infty
\]

for some \( B \) (depending on \( g \)) and any \( N = 0, 1, \ldots \). Therefore, it is obvious that \( \chi \) is a multiplier for \( S_\beta \), and the estimate

\[
|\partial^\kappa \chi \left( \frac{p}{M} \right)| \leq C_\kappa \left| \frac{p}{M^2} \right|^{\kappa}
\]

allows an easy verification that \( g_\chi(p/M) \to g \) in \( S_\beta \) as \( M \to \infty \). Therefore, \( u^j_M = u^j \chi(p/M) \to u^j \) in \( S' \). It remains to show that the functionals \( u^j_M \) admit a continuous extension to the Schwartz space \( S \). Because \( S'_\beta \subset S'_0 = \mathcal{D}' \), we can use the criterion noted in [12] (Sec. 2.10.7), in accordance with which a distribution in \( \mathcal{D}' \) can be continued to \( S \) if and only if its convolution with any test function \( g \in \mathcal{D} \) supported by the unit ball \(|p| < 1\) has no worse than a power growth at infinity. We first consider the case of Lorentz-invariant functionals. The value of the convolution \( (u_M * g) \) at a point \( q \) is the value taken by the distribution \( u_M \) on the shifted function \( g(p - q) \). We need only consider the shifts along the light-cone surface because for the other directions, \((u_M * g)(q)\) vanishes for sufficiently large \(|q|\). It can be additionally assumed that \( q_2 = q_3 = 0 \), because any vector \( q' \in \mathbb{R}^4 \) is taken into a certain point \( q \) in this plane by an appropriate spatial rotation \( R \) and \((u_M * g)(q') = (u_M * g_R)(q)\), where \( g_R(\cdot) = g(R^{-1}(\cdot)) \). Finally, we can set \( M = 1 \) without loss of generality because \( S' \) and \( S'_\beta \) are invariant under dilations. We now use the light-cone variables \( q_\pm = (q_0 \pm q_1)/\sqrt{2} \) and set \( q_- = 0 \) and \( q_+ \to +\infty \) for definiteness. We let \( \Lambda \) denote the Lorentz transformation \( p_+ \to p_+/q_+ \), \( p_- \to q_+ p_- \) in the plane \((p_0, p_1)\), which takes \( q \) into a vector with the unit Euclidean norm. In view of the Lorentz invariance of \( u \) and \( \chi \), we have

\[
(u \chi * g)(q) = (u, g_q), \quad \text{where } g_q(p) = \chi(p) g(q - \Lambda^{-1} p).
\]

The points of \( \text{supp } g_q \) satisfy the inequalities \(|p^2| < 1\) and \( p^2_2 + p^2_3 < 1 \) by construction, and hence \(|p_+ p_-| < 1\). In addition, \(|q_+ - q_+ p_+| < 1\), and consequently \(|p_-| < 1/(1-\)}
Therefore, if $q_+$ is sufficiently large, it follows that $\text{supp} \, g_q$ is contained in a ball of radius 2, and we have the estimate

$$|(u, g_q)| \leq \|u\|_{2,N} \|g_q\|_{2,N}, \quad (33)$$

where $\|g_q\|_{2,N} = \max_{|\kappa| \leq N} \sup_{|p| \leq 2} |\partial^\kappa g_q(p)|$ in accordance with Eq. (30) and $N$ has the meaning of the singularity order of the distribution $u$ in the ball $|p| < 2$. The transformation $\Lambda^{-1}$ results in contracting the graph of $g$ by $q_+$ times with respect to the variable $p_+$; therefore,

$$\sup_p |\partial^\kappa g(q - \Lambda^{-1}p)| = \sup_p |\partial^\kappa g(\Lambda^{-1}p)| \leq C_\kappa \|g\|_{1,N+1} |\kappa|, \quad |\kappa| \leq N. \quad (34)$$

Together with estimate (31), this gives

$$\|g_q\|_{2,N} \leq C_N \|g\|_{1,N} (1 + |q|)^N. \quad (35)$$

We therefore conclude that the behavior of $(u_M * g)(q)$ as $|q| \to \infty$ is indeed not worse than powerlike (with the power depending on $M$). For a Lorentz-covariant generalized function, the estimate $(u^j_M * g)(q)$ can be given similarly using transformation rule (29), which leads to the same conclusion, because matrix elements of the representation $T^j_k(\Lambda)$ are rational functions of the boost parameter $q_+$. Theorem 8 is proved.

8 The decomposition into polynomial covariants

We use the notation $(r, s)$, with nonnegative integer or half-integer $r$ and $s$, for irreducible finite-dimensional representations of the $SL(2, \mathbb{C})$ group and realize these representations in the ordinary way in the spaces of complex homogeneous polynomials of the respective degrees $2r$ and $2s$ in the spinor variables $\omega = (\omega_1, \omega_2)$ and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2)$. We recall that the standard polynomial covariant transforming according to the $(s, s)$ representation is given by $(\overline{\omega} \tilde{x} \omega)^{2s}$, where

$$\tilde{x} = \begin{pmatrix} x_0 - x_3 & -x_1 + ix_2 \\ x_1 + ix_2 & x_0 + x_3 \end{pmatrix}. \quad (36)$$

We now show that the representation $[4, 23]$ of Lorentz-covariant tempered distributions through polynomial covariants can be extended to functionals of the class $S^{\beta}_\alpha$. A vector-valued generalized function $w$ is now treated as a complex-valued generalized
function in the variable \( x \), which in addition polynomially depends on the variables \( \omega \) and \( \bar{\omega} \), and Lorentz-covariance condition (28) becomes

\[
w(x; \omega, \bar{\omega}) = w(\Lambda(A)x; A\omega, \bar{A}\bar{\omega}), \quad A \in SL(2, \mathbb{C}),
\]

(37)

where \( A \to \Lambda(A) \) is the canonical homomorphism of the \( SL(2, \mathbb{C}) \) group onto \( L^+ \).

**Theorem 9.** The Lorentz-covariant generalized function \( w \) that is defined on the space \( S^\beta, \beta \geq 0 \), and transforms according to the representation \((r,s)\) is different from zero only if \( r = s \); in this case, it can be represented as

\[
w(x; \omega, \bar{\omega}) = (\omega \xi \omega)^{2s} v(x),
\]

(38)

where \( v \in S^\beta \) is a Lorentz-invariant functional determined by \( w \) up to the term \( \sum_{l=0}^{2s-1} c_l \Delta^l \delta(x) \) involving arbitrary constants \( c_l \).

(We note that \((s, s)\) is a single-valued representation of the \( L^+ \) group.)

Proving this theorem requires two lemmas.

**Lemma 2.** Let \( f \in S^\beta(\mathbb{R}^n), \beta \geq 0 \). If \( f |_{x_1=0} = 0 \), then \( f(x) = x_1 f_1(x) \), where the function \( f_1 \) also belongs to \( S^\beta(\mathbb{R}^n) \).

**Proof.** We use the notation \( x' = (x_2, \ldots, x_n) \) and set

\[
f_1(x_1, x') = \int_0^1 (\partial_1 f)(tx_1, x') \, dt.
\]

(39)

For \( |x_1| \leq 1 \), we have the estimate

\[
|\partial^\kappa f_1(x)| \leq \|f\|_{B,N} B^{\kappa+1} \frac{(\kappa_1 + 1)\beta(\kappa_1 + 1)}{\kappa_1^{\beta_1}} \kappa^{\beta_\kappa} (1 + |x'|)^{-N} \leq C_{\epsilon,N} \|f\|_{B,N} (B + \epsilon)^{\kappa} \kappa^{\beta_\kappa} (1 + |x|)^{-N},
\]

(40)

where the norm is defined by Eq. (2) and \( \epsilon > 0 \) can be taken arbitrarily small. For \( |x_1| > 1 \), direct application of the Leibnitz formula to \( f_1 = f/x_1 \) gives

\[
|\partial^\kappa f_1(x)| \leq \|f\|_{B,N} \sum_{\ell \leq \kappa} \binom{\kappa}{\ell} B^{\kappa - \ell} \kappa^{\beta_\ell} (\kappa - \ell) \beta(\kappa - \ell) \ell! (1 + |x|)^{-N}.
\]

(41)

If \( \beta > 1 \), then \( \ell! \leq C_{\epsilon} \epsilon^{\ell} \ell^{\beta_\ell} \), and because of the inequality \( (\kappa - \ell) \beta(\kappa - \ell) \ell \leq \kappa^{\beta_\kappa} \), we conclude that an estimate of type (40) is also valid for the function \( f_1 \) in this domain, i.e., \( f_1 \in S^\beta \). We next let \( \beta = 1 \). The space \( S^1 \) can be represented as a union over \( B \) of
the spaces of functions that are analytic in the domains $T^B = \{ z = x + iy \in \mathbb{C}^n : |y_j| < 1/B \ \forall j \}$ and have the finite norms

$$
\|f\|_{B,N} = \sup_{z \in T^B} |f(z)| (1 + |x|)^N, \quad N = 1, 2, \ldots .
$$

(42)

The condition $f|_{z_1 = 0} = 0$ implies that $f(z) = z_1 f_1(z)$, where $f_1$ is an analytic function in the same tube as $f$ and is majorized by $f$ for $|z_1| \geq \delta > 0$. To estimate $f_1(z)$ for $|z_1| < \delta$, we use the Cauchy formula with respect to $z_1$, setting $\delta = 1/(3B)$ and taking a circle of radius $2\delta$ as the integration contour. This contour lies in $T^B$, and $|z_1| > \delta$ for any point $\zeta$ belonging to it. As a result, we obtain

$$
|f_1(z)| \leq C_{\delta,N} \|f\|_{B,N} (1 + |x|)^{-N}.
$$

(43)

For $\beta < 1$, the estimate can be obtained similarly. In this case, $f_1(z)$ is an entire function, the norms are given by

$$
\sup_z |f(z)| (1 + |x|)^N \exp\left(-|By|^{1/(1-\beta)}\right),
$$

(44)

and $\delta$ can be set equal to 1. Lemma 2 is thus proved.

We note that the occurrence of the norm of $f$ in right-hand sides of the estimates demonstrates the sequential continuity of the mapping that is inverse to the injective mapping $f \to x_1 f$ of $S^\beta$ into itself.

**Lemma 3.** Any function $f \in S^\beta(\mathbb{R}^n)$, $\beta \geq 0$, satisfying the condition $\partial^\kappa f(0) = 0$ for all $|\kappa| \leq mn$ admits a decomposition of the form

$$
f(x) = \sum_{i=1}^{n} x_i^{m+1} f_i(x), \quad \text{where } f_i \in S^\beta(\mathbb{R}^n).
$$

(45)

**Proof.** For $n = 1$, this representation directly follows from Lemma 2. We next use induction on $n$ with the notation

$$
g_j(x') = \frac{1}{j!} \partial^j f(x)|_{x_1 = 0}, \quad F(x) = f(x) - f_0(x_1) \sum_{j=0}^{m} x_i^j g_j(x'),
$$

(46)

where $x' = (x_2, \ldots, x_n)$ as before and the function $f_0 \in S^\beta(\mathbb{R})$ is subjected to the conditions $f_0^{(j)}(0) = 0$, $0 \leq j \leq m$. For all $|\kappa| \leq m(n-1)$, we have $\partial^\kappa g_j(0) = 0$, and by the induction hypothesis,

$$
g_j(x') = \sum_{i=2}^{n} x_i^{m+1} f_{ij}(x'), \quad \text{where } f_{ij} \in S^\beta(\mathbb{R}^{n-1}).
$$

(47)
Next, $\partial_j^1 F|_{x_1=0} = 0$ for all $j \leq m$. Therefore, $F(x) = x_1^{m+1} f_1(x')$ by Lemma 2, and Eq. (45) is satisfied with $f_i(x) = f_0(x_1) \sum_{j=0}^m x_1^j f_{ij}(x')$, $2 \leq i \leq n$.

**Proof of Theorem 9.** The identical vanishing of $w$ for $r \neq s$ follows from Theorem 8 and from Proposition 3.6 in [2] describing the structure of the Lorentz-covariant Schwartz distributions. The dimension of the representation $(s, s)$, i.e., the number of different monomials in $\omega$ and $\overline{\omega}$ of the degree $2s$ in each of these variables, is given by $(2s + 1)^2$. We enumerate these monomials and consider the mapping

$$h : \bigoplus_{i=1}^{(2s+1)^2} S^\beta \rightarrow S^\beta$$

that takes each set of test functions $\{f_i\}$, $1 \leq i \leq (2s + 1)^2$ to their linear combination obtained by replacing the monomials in the polynomial $(\overline{\omega} \bar{x} \omega)^{2s}$ with the test functions with the corresponding indices. Let $h'$ denote the dual mapping of $h$. To each functional $v \in S''^\beta$, it assigns the set of functionals obtained by multiplying $v$ by the coefficients of the polynomial, and the restriction of $h'$ to the subspace of invariant functionals takes them into covariant functionals of form (38). Because any covariant Schwartz distribution has this form and these distributions are dense in covariant generalized functions of the class under consideration, it suffices to show that the image of any closed subspace under $h'$ is closed in $\bigoplus_{i=1}^{(2s+1)^2} S''^\beta$.

We note that $\text{Ima} h$ contains a closed subspace of finite codimension in $S^\beta$, specified by the conditions $\partial^\kappa f(0) = 0$, $|\kappa| \leq 4s(4s - 1)$. Indeed, by Lemma 3, we have the decomposition $f(x) = \sum_{i=0}^3 x_1^{4s} f_i(x)$; with $x_1^{4s}$ expressed through the elements $x_{\rho\sigma}$ of matrix (36), each term of the resulting expression contains at least one of these elements raised to a power not less than $2s$. Therefore, the function $f$ can be written as the sum

$$\sum_{\rho, \sigma=1,2} x_1^{2s} f_{\rho\sigma}, \quad f_{\rho\sigma} \in S^\beta,$$

which is obviously in $\text{Ima} h$. The subspace $\text{Ima} h$ can therefore be represented as a sum of a finite-dimensional and a closed subspace and is therefore closed. Next, because $S^\beta$ is reflexive, the closedness of $\text{Ima} h$ implies that $h'$ is a homomorphism (an open mapping onto its image) with respect to the strong topology of the dual spaces [3] (Sec. 4.4.1). Let $L$ be a closed subspace of $S''^\beta$. Its sum with the finite-dimensional subspace $\text{Ker} h'$ (which is contained in the linear span of the functionals $\partial^\kappa \delta(x)$, $|\kappa| \leq 4s(4s - 1)$, is also closed. Therefore, for a point $v \notin L + \text{Ker} h'$, there exists a neighborhood $\mathcal{U}$ that
does not intersect \( L + \ker h' \). The set \( h'(U) \) is a neighborhood in \( \text{Im} h' \) and does not intersect the subspace \( h'(L) \). Therefore, \( h'(L) \) is closed, as was to be proved.

The last assertion in the theorem defines the kernels of the restriction of \( h' \) to the subspace of Lorentz-invariant functionals more exactly. Because the invariant combinations of the distributions \( \partial^s \delta(x) \) are of the form \( \sum c_l \Box^l \delta(x) \) and are converted into polynomials in \( p^2 \) by the Fourier transformation and because the Fourier transform of \( (\omega \tilde{\partial} \omega) \) is \( -i(\omega \tilde{\partial} \omega) \), the proof is completed by applying the identities \((\omega \tilde{\partial} \omega)(p^2) = 2(\omega \tilde{\partial} \omega)\) and \((\omega \tilde{\partial} \omega)(\omega \tilde{\partial} \omega) = 0\). This implies that \((\omega \tilde{\partial} \omega)^2(p^2)^l = 0\) only for \( l \leq 2s - 1 \). Theorem 9 is proved.

We can now extend Theorem 6 to covariant generalized functions, but this requires one more auxiliary statement, which is closely related to Lemma 2.

**Lemma 4.** Let \( v \in S^\beta, \beta \leq 1 \), and let \( K \) be a closed cone in \( \mathbb{R}^n \) that contains the plane \( x_1 = 0 \). If \( K \) is a carrier cone of \( x_1 v \), then it is also a carrier of \( v \).

**Proof.** Let \( O \) be the union of an open cone \( U \supset K \setminus \{0\} \) and an \( \epsilon \)-neighborhood of the origin, and let \( f \in S^\beta(O) \). We set \( g(z') = f(0, z') \). Because the points \((0, z') \) lie in \( O \), the function \( g \) belongs to \( S^\beta(\mathbb{R}^{n-1}) \), and

\[
\|g\|_{B,N} \leq \|f\|_{O,B,N}
\]

For \( \beta = 1 \), the norm \( \|f\|_{O,B,N} \) is defined differently than in (42); namely, sup is now taken over the complex \((1/B)\)-neighborhood of \( O \), while for \( \beta < 1 \), the norm involves the factor \( \exp\left\{-d(Bx, U)^{1/(1-\beta)}\right\} \) in addition to (44) in accordance with Eq. (13). Let \( f_0(z_1) \) be any function belonging to \( S^\beta(\mathbb{R}) \) that is equal to 1 at the origin. Then \( (f - f_0 g) \in S^\beta(O) \) and \( (f - f_0 g)|_{z_1=0} = 0 \). The same elementary argument as in the proof of Lemma 2 shows that \( (f - f_0 g)(z) = z_1 f_1(z) \), where \( f_1 \) belongs to \( S^\beta(O) \) and tends to zero in this space as \( f \) tends to zero in its topology. Therefore, the formula

\[
(\hat{v}, f) = (v, f_0 g) + (x_1 v, f_1)
\]

defines a continuous extension of the functional \( v \) to \( S^\beta(O) \), which proves the statement of the lemma. A similar statement is also true for \( \beta > 1 \), but it is trivial in that case. Lemma 4 is proved.

**Theorem 10.** Any Lorentz-covariant generalized function \( w \) over \( S^\beta, \beta \geq 0 \), with the carrier cone \( \nabla \) admits a decomposition into Lorentz-covariant generalized functions of the same class with the carrier cones \( \nabla_+ \) and \( \nabla_- \).
Proof. We assume that \( w \) transforms according to the \((s, s)\) representation. The cone \( \nabla \) is a carrier of the invariant functional \( v \) through which \( w \) is expressed by Eq. (38). Indeed, by the condition of the theorem, it is a carrier of \((x_0 - x_3)^{2s}v\). Viewing the difference \( x_0 - x_3 \) as the first coordinate and applying Lemma 4, we conclude that the complement of a conical neighborhood of the positive \( x_3 \) semiaxis is certainly a carrier cone of \( v \), and in view of the Lorentz invariance, the same is true for any other spacelike direction; the intersection of these complementary cones is exactly \( \nabla \). It remains to take the existence of the smallest carrier cone into account and apply Theorem 6. Theorem 10 is proved.

9 Application to the spin–statistics theorem

We consider a finite set of fields \( \{\phi_i\} \) that are operator-valued generalized functions over the space \( S^3(\mathbb{R}^4) \), \( \beta < 1 \), and satisfy the standard assumptions of the Wightman axioms [1], [2] except for the local commutativity, which is impossible to formulate using analytic test functions. A natural replacement for this axiom, with its meaning being closer to the physical requirement of macrocausality, is the condition that the closed light cone \( \nabla \) be a carrier cone of the matrix elements of the commutators of observable fields. (If the theory also involves nonobservable fields, then \( \nabla \) is a carrier cone of either commutators or anticommutators. For more details on the motivation and the exact formulation of this condition, which we call asymptotic commutativity, see [24].) We follow the standard assumption that the commutation relation type depends on only the type of the field and is the same for all of its Lorentz components; therefore, Lorentzian indices can be omitted in what follows. From the transformation properties of the fields under the Poincaré group and the invariance of the vacuum, it follows that the vacuum expectation values \( \langle \Psi_0, \phi(x_1)\psi(x_2)\Psi_0 \rangle \) are Lorentz-covariant generalized functions over \( S^3(\mathbb{R}^4) \) with respect to the difference variable \( \xi = x_1 - x_2 \). This allows applying Theorem 9 to generalize the derivation of the spin–statistics relation to nonlocal fields. In complete analogy with the standard theory of tempered quantized fields [1], [2], the weak cluster decomposition property of vacuum expectation values (which follows from the existence and uniqueness of the vacuum without using locality) implies that any pair of nonzero fields \( \phi, \psi \) defined on \( S^3 \) has commutation relations of the same type as the pair \( \phi, \psi^* \) (see Theorem 11 in [24]). Therefore, the problem reduces to the analysis
of asymptotic commutation relations between the field $\phi$ and its Hermitian conjugate field $\phi^*$.

**Theorem 11.** Let $\phi$ be a field defined on the space $S^\beta(\mathbb{R}^4)$, $0 \leq \beta < 1$, transforming according to the irreducible representation $(r, s)$ of the $SL(2, \mathbb{C})$ group. The anomalous asymptotic commutation relation between $\phi$ and $\phi^*$ (anticommutativity for an integer spin and commutativity for a half-integer spin) implies the equality $\phi(f)\Psi_0 = \phi^*(f)\Psi_0 = 0$ for all $f \in S^\beta(\mathbb{R}^4)$.

**Proof.** We use the notation

$$W(x_1 - x_2) = \langle \Psi_0, \phi(x_1)\phi^*(x_2)\Psi_0 \rangle, \quad W'(x_1 - x_2) = \langle \Psi_0, \phi^*(x_1)\phi(x_2)\Psi_0 \rangle$$

and first consider the scalar field case. The anomalous asymptotic commutation relation implies that the cone $\nabla$ is a carrier of the functional $W(\xi) + W'(-\xi)$. In accordance with Theorem 7, this functional is then odd because by the spectral condition, its Fourier transform vanishes for $p^2 < 0$. Therefore, the functional $W(\xi) + W'(-\xi)$ is also odd; in momentum space, this functional is supported by the cone $\nabla_+$ and must then be identically equal to zero. Taking its value on a test function of the form $\bar{f}(x_1)f(x_2)$, we obtain

$$\|\phi(f)\Psi_0\|^2 + \|\phi^*(f)\Psi_0\|^2 = 0, \quad f \in S^\beta.$$  

We now let the field $\phi$ transform according to the irreducible representation $(r, s)$. The anomalous commutation relation then implies that $\nabla$ is a carrier cone of $W(\xi) \pm W'(-\xi)$, where the plus sign corresponds to an integer spin case and the minus sign to a half-integer one. Lorentz-covariant generalized functions (50) transform according to the representation $(r, s) \otimes (s, r)$, whose decomposition into irreducible representations is given by

$$(r, s) \otimes (s, r) = \bigoplus_{|r - s| \leq r', s' \leq r + s, r', s' \in |r - s| + \mathbb{N}} (r', s').$$

Accordingly, the decomposition of $W(\xi) \pm W'(-\xi)$ into polynomial covariants then involves $2\min(r, s)+1$ terms. We note that in the integer spin case, where $r + s$ is an integer, the decomposition involves covariants of only even degrees, while for the half-integer spin, only odd. We apply Theorem 10 and perform the Laplace transformation.
In momentum space, the distribution $\tilde{W}(p) \pm \tilde{W}'(-p)$ is then represented as the difference of boundary values of Lorentz-covariant analytic functions that are holomorphic in the tubes $\mathbb{T}_\pm$ and can be analytically continued to the extended domain $\mathbb{T}^{\text{ext}}$ by the Bargmann–Hall–Wightman theorem [1], [2]. These analytic functions are symmetric with respect to the full reflection $p + i\eta \rightarrow -p + i\eta$ for an integer spin and are antisymmetric for a half-integer spin, and in view of the spectral condition and the uniqueness theorem, they coincide with each other in $\mathbb{T}^{\text{ext}}$. For the boundary values, this then implies the identity

$$\tilde{W}(p) \pm \tilde{W}'(-p) = \mp \tilde{W}(-p) - \tilde{W}'(p). \quad (53)$$

Again taking the spectral condition into account, we see that only the point $p = 0$ can be the support of $\tilde{W}(p) + \tilde{W}'(p)$. The singularity order of this distribution must be equal to zero by the positivity condition. In the case with a half-integer spin, there is no such term in its decomposition into covariants, while for an integer spin, it cannot transform in accordance with Eq. (53) under the reflection. Therefore, $\tilde{W}(p) + \tilde{W}'(p) \equiv 0$, which completes the proof.

### 10 Concluding remarks

The results obtained here allow treating highly singular Lorentz-covariant generalized functions as easily as the standard tempered distributions. An essential addition to Theorem 7 is given by Theorem 2.14 in [22] and Theorem 9 in [24], which show that the cone $\mathcal{V}$ is a carrier cone of odd Lorentz-invariant functionals with arbitrarily singular behavior. In Sec. 9, we considered two-point Wightman functions of a special form; however, there is a natural analogue of the covariant decomposition [2], [23] for the vacuum expectation values of any pair of fields over $S^\beta$ that transform according to finite-dimensional irreducible or simply reducible $SL(2, \mathbb{C})$ representations. Among open problems, we mention the proof of an analogue of the Methée representation for the even and odd invariant functionals of class $S^{\alpha\beta}$. We also note that the theorems proved above can be extended to functionals defined on the generalized Gelfand–Shilov spaces $S^b$ defined by an indicator function $b$ characterizing the growth of their Fourier transforms. The corresponding restrictions on the indicator function are established in [21].
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References

[1] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, New York, Benjamin (1964).

[2] N. N. Bogoliubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *General Principles of Quantum Field Theory*, Dordrecht, Kluwer (1990).

[3] N. N. Meiman, *JETP*, 20, 1320 (1965).

[4] A. Jaffe, *Phys. Rev.*, 158, 1454 (1967).

[5] M. Z. Iofa and V. Ya. Fainberg, *Teor. Mat. Fiz.*, 1, 187 (1969).

[6] M. Z. Iofa and V. Ya. Fainberg, *JETP*, 29, 880 (1969).

[7] V. Ya. Fainberg and A. V. Marshakov, *Phys. Lett. B*, 211, 82 (1988).

[8] M. Z. Iofa and V. Ya. Fainberg, *Nuovo Cimento A*, 5, 273 (1971).

[9] V. Ya. Fainberg, “On quantum theories with a nonpolynomial growth of matrix elements” [in Russian], in: *Problems in Theoretical Physics* (V. I. Ritus, ed.) Moscow, Nauka (1972), p. 119.

[10] V. Ya. Fainberg and M. A. Soloviev, *Ann. Phys.*, 113, 421 (1978).

[11] S. B. Giddings, *Phys. Rev. D*, 61, 106008 (2000).

[12] G. V. Efimov, *Nonlocal Interactions of Quantum Fields* [in Russian], Moscow, Nauka (1977).

[13] G. V. Efimov, *Problems in the Quantum Theory of Nonlocal Interactions Moscow* [in Russian], Moscow, Nauka (1985).

[14] J. W. Moffat, “Quantum field theory solution to the gauge hierarchy and cosmological constant problems”, hep-ph/0003171 (2000).
[15] M. A. Solov’ev, *Theor. Math. Phys.*, 7, 458 (1971).

[16] S. Nagamachi and N. Mugibayashi, *Commun. Math. Phys.*, 46, 119 (1976).

[17] S. Nagamachi and N. Mugibayashi, *Commun. Math. Phys.*, 49, 257 (1976).

[18] M. A. Solov’ev, *Theor. Math. Phys.*, 15, 317 (1973).

[19] U. Moschella and F. Strocchi, *Lett. Math. Phys.*, 24, 103 (1992).

[20] M. A. Soloviev, *Lett. Math. Phys.*, 41, 265 (1997).

[21] A. G. Smirnov and M. A. Solov’ev, *Theor. Math. Phys.*, 123, 709 (2000).

[22] M. A. Solov’ev, *Trudy Fiz. Inst. Lebedev*, 209, 121 (1993).

[23] A. I. Oksak and I. T. Todorov, *Commun. Math. Phys.*, 14, 271 (1969).

[24] M. A. Soloviev, *Theor. Math. Phys.*, 121, 1377 (1999).

[25] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. 2, New York, Academic Press (1964).

[26] V. P. Palamodov, *Russ. Math. Surv.*, 26, 1 (1971).

[27] H. H. Schaefer, *Topological Vector Spaces*, New York, MacMillan (1966).

[28] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions*, Vol. 4, New York, Academic Press (1964).

[29] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. 1, Berlin–Heidelberg–New York–Tokyo, Springer-Verlag (1983).

[30] A. Lambert, *Ann. Inst. Fourier*, 29, 57 (1979).

[31] P. Schapira, *Théorie des Hyperfonctions*, (Lect. Notes. Math., Vol. 126), Berlin–Heidelberg–New York, Springer-Verlag (1970).

[32] T. Kawai, *J. Fac. Sci. Univ. Tokyo. Sect. 1A. Math.*, 17, 467 (1970).

[33] D. A. Raikov, *Sib. Math. J.*, 7, 287 (1966).
[34] A. Grothendieck, *Mem. Amer. Math. Soc.*, 16, 1 (1955).

[35] V. S. Retakh, *Sov. Math. Dokl.*, 11, 1384 (1970).

[36] V. Ya. Fainberg and M. A. Soloviev, *Theor. Math. Phys.*, 93, 1438 (1992).

[37] M. A. Soloviev, *Lett. Math. Phys.*, 33, 49 (1995).

[38] D. P. Zhelobenko, *Compact Lie Groups and Their Representations* [in Russian], Moscow, Nauka (1970).

[39] A. S. Wightman, *Adv. Math. Suppl. Stud.*, 7B, 769 (1981).

[40] M. A. Soloviev, *Commun. Math. Phys.*, 184, 579 (1997).

[41] V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables* Moscow, Cambridge, Mass., MIT (1966).

[42] G. E. Shilov, *Mathematical Analysis: Second Special Course* [in Russian], Moscow, Nauka (1965).

[43] N. Bourbaki, *Espaces vectoriels topologiques*, Vol. 5, Paris, Hermann (1955).