ON OPERATOR-VALUED FREE CONVOLUTION POWERS.

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Abstract. We give an explicit realization of the \( \eta \)-convolution power of an \( A \)-valued distribution, as defined earlier by Anshelevich, Belinschi, Fevrier and Nica. If \( \eta : A \to A \) is completely positive and \( \eta \geq \text{id} \), we give a short proof of positivity of the \( \eta \)-convolution power of a positive distribution. Conversely, if \( \eta \not\geq \text{id} \), and \( s \) is large enough, we construct an \( s \)-tuple whose \( A \)-valued distribution is positive, but has non-positive \( \eta \)-convolution power.

1. Introduction.

In this note, we investigate the question of positivity of \( \eta \)-free convolution powers of an \( A \)-valued distribution. Such \( \eta \)-convolution powers were introduced by Anshelevich, Belinschi, Fevrier and Nica in \cite{1}, following a question due to Bercovici. For \( A = \mathbb{C} \) these correspond to the free convolution powers considered by Nica and Speicher \cite{2}. The main theorem of \cite{1} is a generalization (with a rather complicated proof) of a result from \cite{2}: if \( \mu \) is a positive \( A \)-valued distribution and \( \eta : A \to A \) is a completely positive map so that \( \eta - \text{id} \) is completely positive, then the convolution power \( \mu \boxast \eta \) is also positive.

In the case that \( A = \mathbb{C} \), a simple proof of this theorem exists: for \( t > 1 \), the convolution powers \( \mu^{\boxast t} \) are realized (after some rescaling) in an explicit way by starting with some random variable \( X \) with distribution \( \mu \) and compressing \( X \) to a suitable projection which is free from \( X \) (see the appendix to \cite{2} by Voiculescu).

We construct an explicit realization of the distribution of \( \mu^{\boxast \eta} \) as the distribution of \( v^*Xv \), where \( X \) has distribution \( \mu \), and \( v \) is a certain specially constructed element free from \( X \) with amalgamation over \( A \) \((v \) is a multiple of isometry if \( \eta(1) \) is a multiple of \( 1 \)). Positivity of the distribution of \( v^*Xv \) is then immediate. The condition \( \eta \geq \text{id} \) appears naturally in the construction of \( v \). Our construction can be viewed as a version of the proof of an explicit realization of \( \mu^{\boxast t} \) using free compression and the Fock space model given in \cite{6}.

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In addition, we prove a converse to the theorem of Anshelevich et al: if \( \eta - \text{id} \) is not completely positive, for \( s \) large enough, there is an \( s \)-tuple which has a positive joint \( A \)-valued distribution, but so that the \( \eta \)-convolution power of this distribution is not positive.

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1.1. \( A \)-valued distributions and realizability in a \( C^* \)-probability space. We refer the reader to the book [3] for some background on operator-valued free probability theory. Let \( A \) be a unital \( C^* \)-algebra. Recall that an \( A \)-probability space \([8, 7]\) is a unital \( * \)-algebra \( B \supset A \) together with a conditional expectation (i.e, an \( A \)-linear map) \( E_B^A : B \to A \). For \( X \in B \) and a non-commutative monomial \( W = a_0Xa_1X \cdots Xa_m \), the value \( E_B^A(W) \) is called the non-commutative (\( A \)-valued) moment of \( X \); the map \( \mu : W \mapsto E_B^A(W) \) is called the (\( A \)-valued) distribution of \( X \). We say that an \( A \)-valued distribution \( \mu \) is positive if it is possible to find some \( C^* \)-algebra \( B \), a positive \( A \)-linear map \( E_B^A : B \to A \) and \( X \in D \) so that the \( A \)-valued distributions of \( X \) is exactly \( \mu \). We say that such an \( X \) realizes \( \mu \).

Positivity is an important property of an \( A \)-valued distribution; for \( A = \mathbb{C} \) positivity of a distribution corresponds to positivity of a probability measure.

1.2. Free cumulants \( \omega^X_k \). Associated to any \( A \)-valued distribution \( \mu \) one has a sequence of \( \mathbb{C} \)-multilinear maps \( \omega^\mu_k : A^{k-1} \to A \) called the free cumulants of \( X \) (here \( \omega^\mu_1 \) is simply an element of \( A \)) \([8, 7]\). Let \( Q \) be the universal algebra generated by elements \( L_1^\dagger, L_0, L_1, \ldots \) and \( A \) subject to the relations:

\[
L_0 = \omega^\mu_1 \in A \subset Q
\]

\[
L_1^\dagger a_1L_1^\dagger a_2L_1^\dagger a_3 \cdots L_1^\dagger a_kL_k = \omega^\mu_{k+1}(a_1, \ldots, a_k) \in A \subset Q.
\]

Finally, let \( E_A^Q : Q \to A \) be determined by requiring that \( E_A^Q|_A = \text{id} \) and that for any non-commutative monomial \( W \) in elements of \( A, L_1, L_2, \text{etc.} \), \( E_A^Q(W) = 0 \) unless \( W \) can be reduced to an element of \( A \) using the relations \((1.1)\). Then the sequence \( \{\omega^\mu_k\}_{k \geq 1} \) is uniquely determined by the requirement that if we set \( Y = L_1^\dagger + \sum_{k \geq 0} L_k \), the \( A \)-valued distribution of \( Y \) is \( \mu \).

1.3. \( \eta \)-free convolution powers. Let \( \mu \) be an \( A \)-valued distribution, and let \( \eta : A \to A \) be a linear map. Define a new distribution \( \mu^{\eta \circ \mu} \) by requiring that its free cumulants are given by \( \omega^{\mu^{\eta \circ \mu}}_k = \eta \circ \omega^\mu_k \). This
distribution is called, by definition, the $\eta$-convolution power of $\mu$ (see equation (1.4) in [1]).

2. An explicit realization of the $\eta$-convolution powers.

2.1. Construction of the operator $v \in (C, E^C_A)$. Let $A$ be a $C^*$-algebra, let $\psi : A \to A$ be a completely-positive map, and let $\eta = \psi + \text{id}$. Let $\mathcal{H}$ be an $A, A$ Hilbert bimodule and $\xi \in \mathcal{H}$ be such that $
abla \xi = \psi(a)$. Let $\mathcal{K} = \mathcal{H} \oplus A$ with the inner product $\langle h \oplus a, h' \oplus a' \rangle_{\mathcal{K}} = \langle h, h' \rangle_{\mathcal{K}} + a_1 a'$. Then $\mathcal{K}$ is an $A, A$ Hilbert bimodule with the diagonal left and right actions of $A$. Finally let $F^C_A = A \oplus \mathcal{K} \oplus \mathcal{K} \otimes_A \mathcal{K} \oplus \cdots \oplus \mathcal{K}^n \oplus \cdots$ be the full Fock space associated to $\mathcal{K}$ (see [3, 7]). We view $F^C_A$ as an $A, A$-bimodule using the diagonal left and right actions of $A$. We’ll denote the left action of $A$ on $F$ by $\lambda$.

Let us denote by $v$ the operator $v : F \to F^C_A$, $\zeta_1 \otimes \cdots \otimes \zeta_n \mapsto (\xi 1) \otimes \zeta_1 \otimes \cdots \otimes \zeta_n$. Then an easy computation shows that $v^* \lambda(a) v = \lambda(\eta(a))$.

Finally, for a bounded adjointable right $A$-linear operator $T : F \to F$, set $E^C_A(T) = \langle 0 1, T(0 1) \rangle_{\mathcal{F}}$, where we regard $0 1 \in \mathcal{K} \subset F$. Then 

$$E^C_A(v \lambda(a) v^*) = \langle 0 1, v \lambda(a) v^*(0 1) \rangle_{\mathcal{F}} = \langle 0 1, v 1 \lambda(a) \rangle_{\mathcal{F}} = \langle 0 1, \xi 1 \rangle_{\mathcal{F}} = a.$$

Letting $C = C^*(\lambda(A), v)$, we note that $(C, E^C_A)$ is an $A$-probability space. We’ll also identify $A$ with $\lambda(A)$.

Remark 2.1. (i) Note that $v^* v = \lambda(1)v = \eta(1)$. Thus if $\eta(1) = \alpha 1$ with $\alpha \in \mathbb{R}$, then $\alpha^{-1/2} v$ is an isometry. For general $\eta$, $v$ is not an isometry. (ii) In the case that $A = \mathbb{C}$ and $\eta(a) = \lambda a$, $\lambda \in [1, +\infty)$, the conditional expectation $E^C_A$ is non-tracial. Indeed, we have that $E^C_A(vv^*) = E^C_A(v 1 v^*) = 1$ but $E^C_A(v^* v) = E^C_A(\alpha) = \lambda$. 


2.2. **The main result.** Let \( X \in (B, E_A^B) \) and assume that \( X \) has \( A \)-valued distribution \( \mu \). We will now compute the \( A \)-valued distribution of \( \hat{X} = vXv^* \).

**Proposition 2.2.** Assume that \( \psi : A \to A \) is a completely-positive map, and let \( \eta = \psi + \text{id} \) and let \( v \in (C, E_C^A) \) be as in \( \text{[2.1]} \). Let \( B \) be a \( C^* \)-algebra and \( E_A^B : B \to A \) be a positive \( A \)-linear map. Let \( X = X^* \in B \) having distribution \( \mu \). Consider \((M, E_A^M) = (B, E_A^B) \ast_A (C, E_C^A)\), let \( \hat{X} = vXv \), and let \( \hat{\mu} \) be the distribution of \( \hat{X} \). Then the free cumulants \( \omega_k^\mu \) satisfy:

\[
\omega_k^\mu = \eta \circ \omega_k^\mu.
\]

In particular, the \( A \)-valued distribution of \( \hat{X} \) is positive.

**Proof.** Consider \((N, E_A^N) = (B, E_A^B) \ast_A (Q, E_Q^A)\), and let \( Y = L^1 + \sum_{k \geq 0} L_k \in Q \) be as in \( \text{[1.2]} \). Let \( \hat{Y} = vYv \).

The \( A \)-valued distribution of \( X' \) is the same as the \( A \)-valued distribution of \( Y' \).

Since \( Q \) is free from \( B \) with amalgamation over \( A \), we may thus assume \([3]\) that \( L^1 \) and \( L_k \) satisfy the relations

\[
L^1b_1L^1b_2L^1b_3 \cdots L^1b_kL_k = \omega_{k+1}^\mu(E_A^C(b_1), \ldots, E_A^C(b_k)), \quad b_j \in C
\]

and moreover for any monomial \( W \) in elements of \( C \) and \( L^1, L_1, L_2, \ldots, E_A^N(W) = 0 \),

Let \( \hat{L}^1 = vL^1v \) and \( \hat{L}_k = vL_kv \). Then we have:

\[
\hat{L}^1a_1\hat{L}^1a_2 \cdots \hat{L}^1a_k\hat{L}_k = v^*L^1va_1v^*L^1va_2 \cdots v^*L^1va_kv^*L_kv = v^*\omega_{k+1}^\mu(E_A^C(va_1v^*), \ldots, E_A^C(va_kv^*))v = v^*\omega_{k+1}^\mu(a_1, \ldots, a_k)v = \eta(\omega_{k+1}^\mu(a_1, \ldots, a_k)).
\]

Moreover, if \( W \) is a non-commutative monomial in elements of \( A \) and \( \hat{L}^1, \hat{L}_1, \hat{L}_2, \ldots \), then \( E_A^N(W) = 0 \) unless \( W \) can be reduced to an element of \( A \) using this relation. It then follows that if \( \hat{\mu} \) is the distribution of \( \hat{Y} \) (and is the same as the distribution of \( \hat{X} \)), then its free cumulants are given by

\[
\omega_k^\hat{\mu} = \eta \circ \omega_k^\mu.
\]

This completes the proof. \( \square \)

**Theorem 2.3.** Let \((B, E_A^B)\) be an \( A \)-probability space and let \( X \in B \) be a random variable whose \( A \)-valued distribution \( \mu \) is positive. Let \( \eta : A \to A \) be completely-positive map so that \( \eta - \text{id} \) is completely positive. Let \( \hat{X} = vXv^* \) be as in Proposition \( 2.2 \).
Then the distribution of \( \hat{X} \) is the same as that of the \( \eta \)-convolution power \( \boxplus_1 X \boxplus_1 \); in other words, \( v^* X v \) is an explicit realization of \( \mu^{\boxplus \eta} \).

In particular, the \( A \)-valued distribution of \( \mu^{\boxplus \eta} \) is also positive.

Proof. Let \( \psi = \eta - \text{id} \), so that \( \eta = \psi + \text{id} \). Let \( \hat{X} \) be as in Proposition 2.2. By (2.1) and (1) equation (1.4), the free cumulants of the distribution of \( \hat{X} \) and of \( \mu^{\boxplus \eta} \) are equal. Thus these \( A \)-valued distributions are also equal. But \( \hat{X} \) is explicitly realized in a \( C^* \)-probability space and so its distribution is positive. \( \square \)

3. A CONVERSE.

It is natural to ask whether the condition that \( \eta - \text{id} \) be completely-positive is necessary for \( \eta \)-convolution powers to always remain positive (no matter what the initial distribution is). We show that this is indeed the case if one considers joint distributions of all \( s \)-tuples.

**Theorem 3.1.** Assume that \( \eta : A \to A \) is a completely positive map. Then \( \eta - \text{id} \) is completely-positive iff for every \( s \geq 1 \) and every positive \( A \)-valued distribution \( \mu \) of an \( s \)-tuple, \( \mu^{\boxplus \eta} \) is also positive.

Proof. There is a natural equivalence between \( A \)-valued distributions \( \mu^{(X_{ij})} \) of \( m^2 \)-tuples of variables \( (X_{ij})_{i,j=1}^m \) and of the \( M_{m \times m}(A) \)-valued distribution \( \mu^X \) of the matrix \( X = (X_{ij}) \). In fact, one easily obtains that the \( \eta \)-convolution power of \( \mu^{(X_{ij})} \) (defined by the requirement that the joint cumulants are composed with \( \eta \)) correspond exactly to the \( \text{id}_m \otimes \eta \)-convolution powers of \( \mu^X \). Thus positivity of \( \mu^{\boxplus \eta} \) for every \( A \)-valued distribution of an \( s \)-tuple is equivalent to positivity of \( \nu^{\boxplus \eta} \text{id}_m \otimes \eta \) for every \( M_{m \times m}(A) \)-valued distribution of a single variable \( \nu \). This completes the proof of one direction of the theorem.

Assume now that there exists integer \( m \) and \( a \in M_{m \times m}(A) \), \( a > 0 \) so that \( \eta_m(a) - a \) is not positive (here \( \eta_m = \text{id} \otimes \eta \)). Let \( \phi \) be a state on \( M_{m \times m}(A) \) so that \( \phi(\eta_m(a) - a) < 0 \). Passing from \( A \) to the enveloping von Neumann algebra \( A^{**} \), and from \( a \) to a spectral projection of \( a \), we may assume that \( a \in M_{m \times m}(A^{**}) \) is projection and \( \phi \) still satisfies \( \phi(\eta_m(a) - a) < -2\kappa < 0 \) for some fixed \( \kappa > 0 \). By replacing \( \phi \) with a convex linear combination with a state that is strictly positive on \( a \) we may assume that \( \phi(a) > 0 \) and still \( \phi(\eta_m(a)) < \phi(a) - \kappa \).

Let \( \pi : M_{m \times m}(A^{**}) \to B(H) \) be the GNS construction for \( \phi \) and denote by \( \xi \) the associated cyclic vector in \( H \). Let \( P \in B(H) \) be the rank one projection onto \( \xi \). Denote by \( \hat{A} \) the \( C^* \)-algebra generated by \( M_{m \times m}(A^{**}) \) and \( P \) inside \( B(H) \).

Choose \( \delta > 0 \) so that \( \delta < \kappa \).
Note that $Tr(aPa) = \phi(a)$. Since $aPa$ is finite-rank, we can find $N$ orthonormal vectors $\xi_j \in H$, $j = 1, \ldots, N$ so that $\xi_1 = \xi$ and $|Tr(aPa) - \sum \langle aPa\xi_j, \xi_j \rangle| < \delta$. Thus

$$\left| \sum \langle aPa\xi_j, \xi_j \rangle - \phi(a) \right| < \delta.$$ 

Let $\vartheta(x) = \frac{1}{N} \sum \langle x\xi_j, \xi_j \rangle$ be a state on $\hat{A}$. Then $\vartheta(P) = \frac{1}{N}$ and so

$$\left| \frac{\vartheta(aPa)}{\vartheta(P)} - \phi(a) \right| < \delta. \tag{3.1}$$

Let $X \in (B, \psi)$ be a self-adjoint random variable in a $\mathbb{C}$-valued $C^*$-probability space $B$, and consider $(C, \theta) = (\hat{A}, \vartheta) * (B, \psi)$.

Denote by $E = E_C^{\hat{A}}$ the conditional expectation from $C$ onto $\hat{A}$. If $\omega_n$ denotes the $n$-th scalar-valued cumulant of $X$, then the $\hat{A}$-valued cumulants of $a^{1/2}Xa^{1/2}$ are given by

$$\omega'_n(h_1, \ldots, h_n) = \omega_n a\vartheta(ah_1a) \cdots \vartheta(ah_na)a$$

(see [4]) and thus (recalling that $a^2 = a$) the $\hat{A}$-valued cumulants of the $\eta$-amplification $Y$ of the distribution of $aXa$ are given by

$$w''_{n+1}(h_1, \ldots, h_n) = \eta_m(a) \omega_{n+1} \prod \vartheta(ah_ja).$$

This means that the $\hat{A}$-valued cumulants of $PYP$ are given by

$$w''_{n+1}(h_1, \ldots, h_n) = P\eta_m(a)P \cdot \omega_{n+1} \prod \vartheta(ah_ja)$$

=since $P\eta(a)P = \phi(\eta(a))P$. From this we see that the scalar-valued cumulants of $PYP$ with respect to $\theta$ are given by

$$\tilde{\omega}_{n+1} = \vartheta(P)\phi(\eta_m(a))\vartheta(aPa)^n \cdot \omega_{n+1}.$$ 

Let us finally set $Z = \vartheta(aPa)^{-1}PYP$. Then its scalar-valued cumulants are given by

$$\tilde{\omega}_{n+1} = \vartheta(P)\phi(\eta_m(a))\frac{\vartheta(aPa)^n}{\vartheta(aPa)^{n+1}}\omega_n = \frac{\vartheta(P)}{\vartheta(aPa)^n} \phi(\eta_m(a))\omega_{n+1}.$$ 

Thus $\tilde{\omega}_n = \lambda \omega_n$ with

$$\lambda = \frac{\phi(\eta_m(a))}{\vartheta(aPa)/\vartheta(P)} < \frac{\phi(a) - \kappa}{\vartheta(aPa)/\vartheta(P)}.$$
By (3.1) and our choice of $$\delta < \kappa$$, we conclude that

$$\lambda < \frac{\phi(a) - \kappa}{\phi(a) - \delta} < 1.$$  

In other words, the $$\mathbb{C}$$-valued distribution of $$Z$$ is the same as that of the $$\lambda$$-convolution power of the $$\mathbb{C}$$-valued distribution of $$X$$ for some $$\lambda < 1$$.

Assume now for contradiction that the laws of all $$\text{id} \otimes \eta$$-convolution powers of $$M_{m \times m}(A)$$ are positive. Choose $$a_k \in M_{m \times m}(A)$$ so that $$a_k \to a$$ weakly and $$\sup \|a_k\| < \infty$$. Then if we set $$Y_k$$ to be the $$\eta$$-convolution power of the distribution of $$a_kXa_k$$ (which is positive by our assumption), and $$Z_k = \vartheta(aPa)^{-1}PY_kP$$, then we see that $$Z_k \to Z$$ in moments. But then positivity of distributions of $$Z_k$$ implies positivity of the distribution of $$Z$$.

To summarize, assuming that the $$\eta$$-convolution power of distribution of every $$M_{m \times m}(A)$$-valued distribution is positive, we concluded that the law of the scalar-valued distribution of $$X$$ admits a positive $$\lambda$$-convolution power for some $$\lambda < 1$$. But this is not always possible: for example, we could start with $$X$$ having as distribution the sum of two equal point masses; it is known that this distribution admits no $$\lambda$$-convolution power if $$\lambda < 1$$. □

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