Universality in the dynamics of second-order phase transitions

G. Nikoghosyan,1,2 R. Nigmatullin*,1,3 and M.B. Plenio1,3

1Institut für Theoretische Physik, Albert-Einstein Allee 11, Universität Ulm, 89069 Ulm, Germany
2Institute of Physical Research, 378410, Ashtarak-2, Armenia
3Quantum Optics and Laser Science Group, The Blackett Laboratory, Imperial College London, London SW7 2BW, UK

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Introduction – The study of the non-equilibrium dynamics of systems undergoing phase transitions is an important problem of statistical physics [1, 2]. Of particular interest in this context is the analysis of the dynamics of systems undergoing slow quenches through a second-order phase transition from a symmetric to a symmetry broken phase [3]. When the quench is performed at finite rates the symmetry is broken locally, and spatially separated regions can select different states within the ground state manifold. The typical size of the correlated regions and hence the density of defects exhibit a power-law dependence on the quench rate. One approach for the prediction of the defect scaling, suggested in a theory which has become known as Kibble-Zurek mechanism [4–6], employs physically reasonable arguments involving the equilibrium concepts of divergence of correlation length, relaxation and freeze-out time near the critical point. Indeed, for homogeneous and large systems approaching the thermodynamic limit, Kibble-Zurek theory successfully predicts the scaling of defects with quench rate in terms of the critical exponents of the phase transition. This intuitive derivation may indeed be transparent for such systems but is much less so in more general cases involving for example those that exhibit spatial inhomogeneity [7, 8] which, in turn, poses the risk of incorrect conclusions to be drawn. Indeed, while some experiments aimed at measuring defect scaling laws were realized in systems that are well approximated as large and homogeneous systems, e.g. liquid crystals [9–11], liquid Helium [12, 13], superconducting films [14, 15] and multiferroics [16, 17], the consequences of inhomogeneity and finite-size effects cannot be ignored in recent experiments on Bose-Einstein condensates [18–20] and ion crystals [21, 22] which can exhibit non-standard scaling.

Hence it is important to develop an approach that does not appeal to physical intuition but exclusively relies on mathematical arguments in its derivation of scaling laws. This will be the main goal and result of the present Letter.

The basic idea – Due to the application of a quench, that is an external change of the system parameters, the parameters in the equations of motion adopt spatial and temporal dependencies whose rate of change is related to the quench time \( \tau_q \). The principal goal is the derivation of scaling laws for physical properties such as the rate of defect formation in terms of \( \tau_q \) without the explicit solution of the equations of motion and without resorting to physical arguments concerning equilibrium concepts such as the divergence of correlation length, relaxation and freeze-out time near the critical point. This goal is achieved by employing as the principal idea the rescaling of the parameters of the equations of motion in such a manner that the dependence on the quench rate is eliminated. The scaling of the spatial properties of the system is then contained within this transformation. As this method uses purely the structure of the equations of motion, it can be applied to a wide variety of systems. For homogeneous systems in the thermodynamic limit our technique predicts scaling laws that are consistent with the Kibble-Zurek theory in the mean-field regime. For the case of inhomogeneous and also finite size systems we show that the precise scaling can be derived, and its not only a function of quench rate but also of the characteristic system size. We illustrate the technique by applying it to power-law quenches in systems obeying the overdamped and underdamped time-dependent Ginzburg-Landau equations but note that this approach is applicable well beyond Landau theory of second-order phase transitions.

Ginzburg-Landau Theory – Let us start by introducing the free energy of the system in the vicinity of the phase transition. We assume that the state of the sys-
tem is governed by a vector field $\vec{\phi}(r, t)$. According to the Landau theory of phase transitions the free energy of our system near the critical point is given by

$$F_{\text{tot}} = \int dr \left[ F_{\text{grad}} + F_{\text{inh}} + F_L \right]. \quad (1)$$

The first term on the rhs of eq. (1) is the gradient

$$F_{\text{grad}} = \frac{1}{2} \left| \nabla \phi \right|^2, \quad (2)$$

the second term is the time-dependent Landau free energy

$$F_L = \varepsilon(t) \left| \phi^2 + \frac{g}{2} \phi^4 \right| \quad (3)$$

and the last term represents the inhomogeneous external potential

$$F_{\text{inh}} = V(r, L) \left| \phi^2 \right|. \quad (4)$$

Here, $\varepsilon(t)$ is the time dependent critical parameter and $L > 0$ is the characteristic system size. For $\varepsilon + V(r, L) > 0$ the minimum of $F_{\text{inh}} + F_L$ corresponds to $\phi = 0$, that is the field vanishes and the system is in the symmetric phase. For $\varepsilon + V(r, L) < 0$ there are several minima of $F_{\text{inh}} + F_L$ at $\left| \phi(r, t)^2 \right| = -\frac{t(t+V(r,L))}{g}$ and the system can adopt symmetry-broken states within the nontrivial ground-state manifold. According to Landau theory, the phase transition takes place when the critical parameter $\varepsilon(t) + V(r, L)$ changes its sign.

The spatially homogenous case – In a first example we illustrate our method by considering the equation of motion corresponding to phenomenological model A of [2]. The evolution of the field is described by the stochastical equation

$$\frac{\partial \phi_i}{\partial t} = -\Gamma \frac{\delta F_{\text{tot}}}{\delta \phi_i} + \theta_i(r, t) \sqrt{T} \quad (5)$$

where $\Gamma$ is the relaxation, $T$ is the temperature of the system, and $\theta_i(r, t)$ is the uncorrelated white-noise variable with $\langle \theta_i(r, t) \theta_i(r', t') \rangle = \Gamma k_B \delta(r - r') \delta(t - t')$. The use of an uncorrelated white-noise environment may be justified by taking the viewpoint that the physical environment couples to the microscopic degrees of freedom of the system whose evolution is faster than that of the macroscopic order parameter $\phi$. Hence, on the level of the macroscopic order parameter the environment is well approximated by a Markovian environment. Shortly it will become clear that the actual value of $\Gamma$ does not play any role in our analysis and in general $\Gamma$ can even be complex. First, we consider the dynamics of the system in a homogeneous setting, i.e. $F_{\text{inh}} \equiv 0$. By substituting eqs. (1), (2), and (4) into (5) we obtain the following equation of motion for the field

$$\frac{\partial \vec{\phi}}{\partial t} = -\Gamma \left[ \frac{1}{2} \nabla^2 \vec{\phi} + \varepsilon(t) \vec{\phi} + g \left| \vec{\phi}^2 \vec{\phi} \right| + \vec{\theta}(r, t) \sqrt{T} \right]. \quad (6)$$

In order to simplify the discussion we consider only polynomial dynamical quench functions. It should be noted here, that other functional dependencies of quench parameters can also be analyzed, but the analytical formulation of the result is considerably more involved. Without loss of generality we assume that the phase transition takes place at $t = 0$ and, furthermore, all quantities in the equations of motion are assumed to be in dimensionless natural units. Therefore, the critical parameter can be expressed as

$$\varepsilon(t) = -\left| \frac{t}{\tau_q} \right|^n \text{sign}(t) \quad (7)$$

where, $\tau_q$ defines the quench time scale. In general, the variation of the quench rate modifies the equation of motion (6) and thus affects the final state of the system. However, as we will show now, the dependence on $\tau_q$ can be eliminated and the equation of motion of the order parameter can be written in a universal form, if the parameters of eq. (6) are rescaled.

To this end we consider the following linear transformations

$$\eta = \alpha t, \quad \xi = \beta t, \quad \bar{T} = \sigma T, \quad \bar{\phi} = \frac{\phi}{\beta} \quad (7)$$

$$\alpha = \frac{\tau_q}{\tau_T}, \quad \beta = \frac{\tau_T \tau_q}{\tau_T + \tau_q}, \quad \sigma = \beta^{d-2}, \quad (8)$$

where $d$ is the dimensionality of the system. These transformations upon substitution into (6) yield

$$\frac{\partial \bar{\phi}}{\partial \eta} = -\Gamma \left[ \frac{1}{2} \nabla_\xi^2 \bar{\phi} - \text{sign}(\eta) |\eta|^n \bar{\phi} + g |\bar{\phi}|^2 \bar{\phi} \right] - \bar{\theta}(\xi, \eta) \sqrt{\bar{T}}. \quad (9)$$

Here, for the sake of simplicity we have assumed that the temperature $T$ is independent of $\xi$ and $\eta$. In general, time and space dependent temperatures can be considered, which results in more complicated dependence of $\sigma$ on $\alpha$ and $\beta$.

Equation (9) has been obtained after a rescaling of temporal and spatial variables as well as the magnitude of the order parameter by coefficients that are proportional to powers of $\tau_q$. Its kinetic and potential energy do not depend anymore on $\tau_q$. In the following we will explain that the $\tau_q$ dependence of the rescaled temperature is not relevant for defect formation which then completes the demonstration of the universality of the dynamics of the order parameter because in the rescaled coordinates the dynamics is now identical for different quench rates. This is the main idea of the present Letter as now the question of scaling in the quench time $\tau_q$ has been transferred from the analysis of a complicated equation to the scaling behavior of the parameter transformation that yields a quench rate independent equation of motion. The Kibble-Zurek theory of topological defect formation then follows directly from this result if we assume...
that topological defects are formed in the system upon traversal of the critical point.

In order to complete the argument, we now discuss the influence of the initial state and the temperature on the dynamics. To this end let us assume that the quench dynamics starts at a finite \( \eta_0 \) such that \( |\eta_0| \gg g\varphi_0|^{1/\alpha} \) as this implies that the nonlinear term in eq. (9) can be neglected. Therefore at sufficiently early times the equation of motion of the order parameter \( \bar{\varphi}_0 (\xi, \eta) \) can be reduced to

\[
\frac{\partial \bar{\varphi}_0}{\partial \eta} = -\Gamma \left[ \frac{1}{2} \nabla^2 \bar{\varphi}_0 - \sign (\eta) \bar{\varphi}_0^n + \theta (\xi, \eta) \sqrt{T} \right].
\]

(10)

Furthermore, we assume that before \( \eta_0 \) sufficient time has passed for the system to have reached thermal equilibrium which implies that any memory of the initial condition will have been lost. At this point the evolution becomes insensitive to the amplitude of the order parameter as it can be absorbed in the renormalized temperature \( \tilde{T} \).

Now eq. (10) implies that at \( \eta_0 \) the state of the system does not depend on the quench rate.

The quench dynamics can be terminated at a finite time \( \eta_m \) when the system is sufficiently deep in the non-linear regime such that thermal fluctuations cannot create defects. This is the case when the amplitude of the order parameter satisfies the condition

\[
F_L (\phi) \gg k_B T. \tag{11}
\]

Eq. (11) implies that thermal fluctuations cannot drive the system across the potential well which, in turn, prevents the spontaneous creation of topological defects. From this point onwards finite temperature effects will not influence the number of defects anymore in the subsequent evolution.

Therefore, the entire quench takes place across a finite time interval \([\eta_0, \eta_m]\) and we can always find a system-environment coupling \( \Gamma \) such that

\[
|\Gamma| \bar{T}(\eta_m - \eta_0) \ll 1. \tag{12}
\]

Under condition (12) the influence of the noise in eq. (9) is negligible across the entire quench. Then the formal solution of eq. (9) with initial condition \( \bar{\varphi} (\xi, \eta_0) = \bar{\varphi}_0 (\xi, \eta_0) \) is

\[
\bar{\varphi} (\xi, \eta) = U (\bar{\varphi}_0 (\xi, \eta_0), \xi, \eta), \tag{13}
\]

where \( U \) is a functional, which describes the dynamics of the system. The formal solution eq. (13) demonstrates that in the rescaled frame of reference the spatial properties of the system do not depend on the quench rate.

In \( d \)-dimensional space, the density of topological defects \( \rho (\mathbf{r}) \) and \( \rho (\bar{\xi}) \) in the original and the rescaled coordinates, respectively, satisfy

\[
\int \rho (\mathbf{r}) d^d \mathbf{r} = \int \rho (\bar{\xi}) d^d \bar{\xi}. \tag{14}
\]

It follows from the solution of eq. (13) that the density of defects in the rescaled frame of reference is independent of the quench rate. Making use of eq. (7) and (8) in eq. (14) we find that the defect density in real space scales as

\[
\rho_{def} \sim \beta^d \sim \tau_q^{-\frac{d}{d-2}}. \tag{15}
\]

Alternatively, this result can be formulated for characteristic domain size, which scales linearly with \( |\mathbf{r}| \). Thus in real space the characteristic domain size scales as \( \sim \beta^{-1} \). This analytical result is in agreement with predictions of the Kibble-Zurek theory in the so-called mean-field regime. For instance for linear quenches and one-dimensional systems, i.e. \( n = 1, d = 1 \), we find \( N_{def} \sim \tau_q^{-\frac{1}{2}} \) which agrees with the results presented in [22, 25].

It is not possible to have arbitrarily many defects in the system and therefore at high quench rates (high defect densities) one expects to observe a plateau in the defect scaling. This deviation from the simple power-law scaling can be anticipated by noting the scaling of the characteristic defect size and comparing it to the scaling of separation between defects. The characteristic size of the defects in rescaled coordinates is given by \( \bar{h} \sim 2/|\bar{\varphi}|^2 \), as according to eq. (14), a shorter defect has a kinetic energy that exceeds the potential barrier. By making use of the relations (14) one can see that in real space the typical defect size \( h \) has no dependence on \( \tau_q \). On the other hand the typical distance between defects, i.e. the domain size, scales as \( D \sim \beta^{-1} \). The simple power law scaling breaks down when the ratio \( h/D \) becomes of order of one, which happens for sufficiently small \( \tau_q \). We emphasize that this physical argumentation is not employed in our derivation of the power law-scaling, but is simply used to anticipate its breakdown at fast quench rates.

The spatially inhomogenous case – Now we will apply the same strategy to analyze the system in the presence of space-dependent external potential \( V (\mathbf{r}, L) \). In this case the dynamics of the system is governed by the equation

\[
\frac{\partial \tilde{\varphi}}{\partial t} = -\Gamma \left[ \frac{1}{2} \nabla^2 \tilde{\varphi} - \left| \frac{t}{\tau_q} \right|^n \sign (t) \tilde{\varphi} + g |\tilde{\varphi}|^2 \tilde{\varphi} - V (\mathbf{r}, L) \tilde{\varphi} \right] + \tilde{\theta} (\mathbf{r}, t) \sqrt{T}. \tag{16}
\]

which contains an additional spatially dependent term. Therefore the dependence on \( \tau_q \) can in general be eliminated only if the trapping potential \( V (\mathbf{r}, L) \) is also rescaled. To be specific let us consider here two such potentials:

(i) power law potential \( V (\mathbf{r}, L) = -|\mathbf{r}|^m \), with \( r = |\mathbf{r}| \). The dependence on \( \tau_q \) in eq. (16) is eliminated if in addition to rescalings eqs. (7) and (8) one assumes that the characteristic system size is rescaled according to \( \lambda = \gamma L \), where \( \gamma = \beta \alpha^{1/m} \). Then the equation of motion (16) is reduced to the following \( \tau_q \)-independent
universal equation

\[
\frac{\partial \varphi}{\partial \eta} = -\dot{\Gamma} \left[ \frac{1}{2} \nabla_\xi^2 \varphi - \text{sign} (\eta) |\eta|^n \varphi - \left| \frac{\xi}{\lambda} \right|^m \varphi + g |\varphi|^2 \varphi \right] \\
+ \dot{\theta} (\xi, \eta) \sqrt{T}.
\]

(17)

(ii) For an inverted Gaussian potential \( V(x, L) = \exp \left( -\frac{x^2}{\xi^2} \right) \) the equation of motion eq. (16) can be brought to universal form if the following nonlinear rescaling is applied \( L^2 = -\frac{\xi^2}{\beta^2 \ln (\lambda \alpha)} \). This allows us to write the equation of motion in the universal form

\[
\frac{\partial \varphi_0}{\partial \eta} = -\dot{\Gamma} \left[ \frac{1}{2} \nabla_\xi^2 \tilde{\varphi} - \text{sign} (\eta) |\eta|^n \tilde{\varphi} - \lambda \tilde{\varphi} + g |\tilde{\varphi}|^2 \tilde{\varphi} \right] \\
+ \dot{\theta} (\xi, \eta) \sqrt{T}.
\]

(18)

These examples illustrate that in inhomogeneous settings the number of topological defects depends not only on the quench rate but also on the characteristic system size \( L \), i.e. in the rescaled frame of reference we can write \( \rho (\xi, \lambda) \). Thus in original space for the density of defects we obtain the following relation \( \rho (r, \lambda (L, \tau_q)) \sim \beta^d \sim \tau_q^{-\frac{d}{2(\alpha + d)}} \) provided that \( \lambda \) is kept constant. This result differs from the standard inhomogeneous KZ theory based on intuitive arguments. In particular, it was predicted that the influence of external potential may change the power of dependence of \( \rho \) on \( \tau_q \) [2] [26]. However, as it was shown above the density of defects will follow a precise powerlaw scaling in \( \tau_q \) only if the characteristic system size \( L \) is varied in such a way that \( \lambda (L, \tau_q) \) does not depend on \( \tau_q \). The required transformation \( L (\tau_q) \) can be determined from the equations of motion, as it has been demonstrated in the two examples.

**Beyond Ginzburg-Landau** – Our method is based on the derivation of universal equations by means of a rescaling of spatial and temporal coordinates. It can also be applied to the systems whose dynamics is described by different equations of motion describing phase transition of systems in other dynamic universality classes [2]. Naturally, in this case the specific form of rescaling coefficients in (17) and (18) is modified. However, if the equation of motion is known, the method enables the simple derivation of number of defects scaling as a function of \( \tau_q \). In order to illustrate this let us analyze the dynamics of the system in the so-called underdamped regime [27]. The dynamics of the field in this case is governed by equation

\[
\frac{1}{2} \frac{\partial^2 \varphi}{\partial t^2} = \left[ \frac{1}{2} \nabla_\xi^2 \tilde{\varphi} - \frac{t}{\tau_q} \right]^n \text{sign} (t) \tilde{\varphi} + g |\tilde{\varphi}|^2 \tilde{\varphi}.
\]

(19)

Eq.(19) differs from eq. (6) in that the first time derivative is replaced by a second time derivative. Now the quench time in eq.(19) is eliminated if the following rescalings are applied

\[
\eta = \alpha t, \quad \xi = \beta r, \quad \alpha = \tau_q^{-\frac{1}{2}}, \quad \beta = \alpha.
\]

Thus in the linear quench regime \( n = 1 \) and for \( d = 1 \), the defect density scales according to \( \rho (r) \sim \beta \sim \tau_q^{-\frac{1}{2}} \).

This result is in perfect agreement with the analytical and numerical results presented in [24] [25].

It should be emphasized that the transformation approach that we have described in this Letter is not restricted to equations of motion of order parameters but does apply equally to the full microscopic equations of motion that are underlying the system. Therefore this approach is not limited to the Ginzburg-Landau theory of second order phase transitions but may also shed light on scaling laws in phase transitions with different order.

**Conclusions** – We have presented an analytical method for examining the dynamics of second-order phase transitions near the critical point that replaces physical arguments by mathematical reasoning based on transformations of the equations of motion of a system under consideration. The power of the method is demonstrated by considering two specific cases of the time dependent Landau-Ginzburg equation in the so-called overdamped and underdamped regimes. We have shown that by making use of linear transformations the equation of motion can be represented in a universal form with no dependence on quench rate. This has been used to derive the spatial scaling of the defect density with the quench rate of the transition. We have also applied our method to analyze the dynamics of inhomogeneous systems and have shown that in this case the number of defects also scales with the characteristic system size. The approach presented here goes well beyond these problems and can be applied to the microscopic equations of motion underlying arbitrary phase transitions.

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