Repeatable quantum memory channels

Tomáš Rybár and Mário Ziman

1Research Center for Quantum Information, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia
2Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic

Within the framework of quantum memory channels we introduce the notion of repeatability of quantum channels. In particular, a quantum channel is called repeatable if there exist a memory device implementing the same channel on each individual input. We show that random unitary channels can be implemented in a repeatable fashion, whereas the nonunital channels cannot.

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I. INTRODUCTION

A quantum channel is any transformation taking a state \(\rho\) of a quantum system as an input and transforming it to some state \(\rho'\) on the output. Within the standard model \([1,2]\) of quantum dynamics the channels are represented by completely positive trace-preserving linear maps acting on the set of (trace-class) Hilbert space operators \(T(\mathcal{H})\). Let us note that quantum states are represented by so-called density operators, i.e. positive trace-class operators with unit trace. The physical picture of quantum channels as the correct description of the evolutions of open quantum systems follows from the Stinespring theorem \([3]\). According to this theorem each quantum channel can be understood as the unitary evolution of the isolated “supersystem” composed of the system and its environment, where the unitary evolution is governed by the Schrödinger equation (see Fig.1). In other words quantum channels are implemented by suitable quantum devices consisting of intrinsic degrees of freedom (associated with the environment) and acting on the system via particular interactions between the system and the environment.

The statistical nature of quantum physics requires that the experiments must be typically repeated large number of times in order to make some relevant conclusions about the properties of quantum devices. A typical example is the problem of quantum channel tomography, in which the goal is to identify which channel the given device is implementing. Any estimation procedure is based on repeated use of the device. Considering the above model of the quantum device implementing some quantum channel we encounter the following problem. Although the concept of the quantum channel itself does not need any particular specification of the environment properties, for the repeated use of the same device the details about the environment can play a significant role. In particular, let us consider the following (not entirely realistic) example. Consider an optical device “storing” a single photon in some polarization state \(\xi\). After inserting another photon (in the polarization state \(\xi_1\)) this device will output the photon that was originally stored in it and the new photon will remain stored in the device. From the theory point of view the device implements the transformation mapping the whole state space onto the state \(\xi\), i.e. \(S(\mathcal{H}) \mapsto \xi\), where by \(S(\mathcal{H})\) we denote the system’s state space. However, using the same fiber once more we get the transformation \(S(\mathcal{H}) \mapsto \xi_1\), i.e. the channel action is completely different (unless \(\xi_1 = \xi\)). The main aim of this paper is to analyze and characterize the situations, in which the device can be reused infinitely many times and still implementing the same quantum channel. As we shall see such reusable devices would be good for saving resources. Instead of infinite amount of resources, needed to provide the channel transformation forever, finite resources would be sufficient.

The paper is organized as follows: in the Section II we shall recall the basics of quantum memory channels, after that in Section III we shall define the problems of reusability of quantum channels and prove the main theorems. In the last section we shall discuss the derived results.

II. THE EFFECT OF MEMORY

In the case when the subsequent actions of the device are independent of the previous ones we say that the device implements a memoryless channel. If the output does not depend on future inputs we say that the channel is causal. Let us note that memoryless channels are automatically causal. In what follows we shall assume that all physically relevant channels are causal. Under such condition it was shown in the seminal paper by Kretschmann and Werner \([4]\) that each causal memory channel can be understood as a sequence of collisions between the system and its environment playing the role of the memory. In the last few years different aspects of quantum memory channels attracted researchers \([1,2,3,4,5,6,7,8,9]\) and many interesting results have been achieved concerning capacity, structure and physical implementations for memory channels.

A personification of the Stinespring’s theorem describing one usage of a device implementing a quantum channel is depicted in Fig.1. According to this picture the device is consisting of some internal degrees of freedom forming the effective environment affecting the system transferred through the channel. We shall refer to this internal degrees of freedom as to channel’s memory associated with the Hilbert space \(\mathcal{H}_M\). The interaction be-
We shall address more general question whether the perfect relaxation is possible, or needed, within the unitary model of quantum memory channel (see Fig. 3). In other words we are asking under which conditions the model depicted on Fig. 3 can embed the reset model depicted in Fig. 2.

Let us now define the model and formulate the problem in an intuitive way. According to Fig. 3 after the $n$th usage of the device the memory is described by the state
\[
\xi_{n+1} = \text{Tr}_S[U_n(\xi \otimes \varrho_1 \otimes \cdots \otimes \varrho_n)U_n^\dagger] = F_n \circ \cdots \circ F_1[\xi],
\]
where $U_n = U_{MS_n} \cdots U_{MS_1}$, $U_{MS_j}$ acts on the memory and $j$th input system, $F_j \equiv F_{\varrho_j}$ and $S^\otimes n$ denotes the composite system of $n$ input systems. In this model we assume that the input states $\varrho_1, \ldots, \varrho_n$ are uncorrelated.

On one hand this assumption is motivated by standard algorithms for channel testing. On the other hand if we allow correlations, then we are getting outside of the validity of the mathematical model for quantum channel as depicted in Fig. 1. In fact, after the first usage of the device the subsequent inputs become to be correlated with the memory system. These issues extending the standard quantum channel framework are studied in Refs. [10, 11, 12, 13, 14].

The induced channel transformation $E_j$ on the $j$th trial depends on the state of the memory $\xi_{j-1}$ which is dependent on the particular choice of the input states $\varrho_1, \ldots, \varrho_{j-1}$. We shall investigate in which cases the particular choice of the input states does not matter. But before that, let us consider the following example.

### A. Memory channel induced by SWAP interaction

Consider an experimentalist who would like to repeat his experiment aiming to describe the device schematically depicted on Fig. 1. If he is able to set the initial conditions of the experiment to some initial values and repeat the experiment he is fine. This refers to the model on Fig. 2. If not, then he is not repeating the same experiment (with the same channel) again. This has some severe consequences for channel tomography. As an illustrative example we will consider a quantum device with a two-dimensional memory implementing a single qubit channel. The interaction between the system (qubit) and the memory is described by the SWAP transformation

![FIG. 1: One use of quantum channel](image1)

![FIG. 2: Standard (memoryless) model.](image2)

![FIG. 3: Unitary memory model.](image3)
ORY_Effects are suppressed. We call the triple repetable quantum memory channels is that the memory after the unitary transformation $U$ is applied. We obtain

$$E[\rho] = \frac{1}{N} \sum_{j,k} (|j\rangle\langle j| \otimes \rho) (|k\rangle\langle k| \otimes U^j_k)$$

for the system’s transformation and

$$F[\xi] = \frac{1}{N} \sum_{j,k} (|j\rangle\langle j| \otimes \xi) (|k\rangle\langle k| \otimes U^j_k)$$

for the memory transformation.

In order to implement random unitary channel $E[\rho] = \sum_j p_j U^j_\rho$ it is sufficient to choose a state with diagonal elements $\xi_{jj} = \langle j|\xi|j\rangle = p_j$. Let us note that diagonal elements of density operator always form a probability distribution. From Eq. (3.2) it follows that diagonal elements of the memory state are preserved, because

$$\langle j|F[\xi]|j\rangle = \sum_{j,k} \xi_{jk} \langle j|U^j_k|k\rangle = \xi_{jj}.$$

The last equality holds because $\langle j|U^j_k|k\rangle = \delta_{jk}$. Since the diagonal elements of $\xi$ defines the random unitary channels and, moreover, they are preserved, it follows that random unitary channels are indeed repeatable. In particular, $\text{diag}[\xi_1] = \text{diag}[\xi_2] = \cdots = \text{diag}[\xi_n]$ and therefore $E_1 = E_2 = \cdots = E_n$ for all $n > 0$.
As a result we get that a particular Stinespring’s dilation of any random unitary transformation forms a reusable quantum device, meaning that random unitary transformations are repeatable. Could it be that all transformations have such dilation? The following theorem gives a negative answer saying that nonrepeatable channels do exist.

**Theorem 2.** If $\mathcal{E}$ is a nonunital channel, i.e. $\mathcal{E}[I] \neq I$, then it is not repeatable with finite memory.

**Proof.** We shall prove that repeatability of quantum memory channel (specified by $U$) implies unitality of the induced channels $\mathcal{E}$. Let us start with the entropy analysis of the memory channel. Becuse of the unitarity it follows that

$$S(\xi_1) + S(\xi_2) = S(U(\rho_1 \otimes \xi_1)U^\dagger), \quad (3.4)$$

where $S(\rho) = -\text{Tr} [\rho \log \rho]$ is the von Neumann entropy of state $\rho$. The entropy is subadditive, i.e. $S(\omega_{AB}) \leq S(\omega_A) + S(\omega_B)$, where $\omega_A = \text{Tr}_B \omega_{AB}$ and $\omega_B = \text{Tr}_A \omega_{AB}$ are the states of the subsystems $A, B$, respectively. Applying this inequality for our situation we obtain

$$S(\xi_1) + S(\xi_2) \leq S(\mathcal{E} [\rho_1]) + S(\xi_2). \quad (3.5)$$

Let us repeat the quantum memory channel $n$ times by using the same input state, i.e. $\rho_1 = \rho_2 = \ldots = \rho_n = \rho$. The repeatability of the channel $\mathcal{E}$ implies that

$$nS(\xi_1) \leq nS(\mathcal{E} [\rho_1]) + S(\xi_{n+1}). \quad (3.6)$$

From this immediately follows the inequality

$$n\Delta(\xi_1) \leq S(\xi_{n+1}) - S(\xi_1) \leq \log (\text{dim} \mathcal{H}_M). \quad (3.7)$$

where $\Delta(\xi_1) = S(\rho_1) - S(\mathcal{E} [\rho_1])$.

For unital channels the entropy cannot decrease, i.e. $\Delta(\xi_1) \leq 0$ for all states $\xi_1$ (see Appendix). Consequently, the above inequality is satisfied by all unital channels. For nonunital channels the complete mixture decreases its entropy, i.e. $\Delta(\xi_{n+1}) > 0$. The right hand side is bounded by the dimension of the memory system. However, since $n$ is arbitrarily large, the left hand side goes to infinity, hence necessarily also the dimension of the memory system must be infinite. Thus repeatability requires unitality as it is stated in the theorem.

IV. CONCLUSION

We investigated the problem of reusability of quantum devices implementing (in each single use) state transformations described by quantum channels. Due to interaction of the system with the device both, the system and the device, are affected by some noise, hence the original settings of the device have changed. Consequently the repeated usage of the same quantum device can result in a different noise, i.e. different quantum channel. This picture leads to an emergence of memory effects in the description of quantum channels. If the channel can be repeated infinitely many times without resetting the memory we say it is repeatable. For such type of channels the memory effects are suppressed although the memory itself undergoes a nontrivial dynamics. It was shown in this paper that any random unitary channel is repeatable with a finite memory, whereas the repeatable implementation of nonunital channels requires infinite resources. For qubit channels we can make even stronger statement that unitality is equivalent to repeatability, because each unital channel can be expressed as a random unitary channel. For general systems we leave the question of repeatable implementation of unital, but not random unitary channels open.

One possible way how to tackle the problem is to investigate the channels that can be implemented by a quantum device with the memory initialized in the total mixture. For such channels the reset operation can be implemented in a repeatable way, since the channel $A$ transforming the whole state space into the total mixture is random unitary and therefore is repeatable. That is, whatever is the output memory state, it can be reset to the total mixture by using only finite resources. Interestingly, since the entropy of the total system is preserved, it follows that if the memory is initially in the total mixture, then the implemented channel is necessarily unital. In fact, if the system is initially in the total mixture, then necessarily also output must be in the total mixture, because the entropy achieves its maximum for a unique state being the total mixture. But, this is nothing else as the unitality of the channel. It is an open
problem whether there are some unital but not random unitary channels that are implementable in the described repeatable way.

Let us note that the concept of repeatability is similar to the concept of quantum cloning [16] in a sense that the channels (just like copies in quantum cloning) are not completely independent if measurements are taken into account. In fact, the memory system may act as a mediator of correlations between the channel outputs although the inputs are factorized. For sure, the impact of measurements on repeatability of quantum memory channels deserves further investigation. The presented analysis of the repeatability of quantum channels is a part of the research program aiming to understand and develop realistic models of quantum dynamics of open systems including the memory effects.

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APPENDIX A: MONOTICITY OF VON NEUMANN ENTROPY UNDER UNITAL CHANNELS

Lemma 1. If $E$ is a unital channel, then $S(E[\rho]) \geq S(\rho)$ for all states $\rho$.

Proof. The proof of entropy monotonicity for unital channels is a consequence of the monoticity of the relative entropy [17]. In particular, for arbitrary quantum channel $E$

$$S(E[\rho] || E[\omega]) \leq S(\rho || \omega), \quad (A1)$$

where $S(\rho || \omega) = \text{Tr}[\rho (\log \rho - \log \omega)]$ is the quantum relative entropy. Setting $\omega = \frac{1}{d} I$ we get $S(\rho || I/d) = -S(\rho) + \log d$. Using this fact and assuming that $E$ is unital the above inequality can be rewritten as

$$S(E[\rho] || I/d) \leq S(\rho || I/d) - S(E[\rho]) \leq -S(\rho),$$

from which the lemma follows. □

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