ON THE MODULI SPACE OF PAIRS CONSISTING OF A CUBIC
THREEFOLD AND A HYPERPLANE

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Abstract. We study the moduli space of pairs \((X, H)\) consisting of a cubic
threefold \(X\) and a hyperplane \(H\) in \(\mathbb{P}^4\). The interest in this moduli comes
from two sources: the study of certain weighted hypersurfaces whose middle
cohomology admit Hodge structures of \(K3\) type and, on the other hand, the
study of the singularity \(O_{16}\) (the cone over a cubic surface). In this paper,
we give a Hodge theoretic construction of the moduli space of cubic pairs by
relating \((X, H)\) to certain “lattice polarized” cubic fourfolds \(Y\). A period map
for the pairs \((X, H)\) is then defined using the periods of the cubic fourfolds
\(Y\). The main result is that the period map induces an isomorphism between a
GIT model for the pairs \((X, H)\) and the Baily-Borel compactification of some
locally symmetric domain of type IV.

Introduction

It is an interesting problem to study (weighted) hypersurfaces whose middle
cohomology admit Hodge structures of \(K3\) type (a Hodge structure is of \(K3\) type
if it is an effective weight 2 Hodge structure with \(h^{2,0} = 1\)) up to Tate twist. In
1979, Reid listed all 95 families of \(K3\) weighted hypersurfaces (see for example
[Rei80]). The next case is to consider quasi-smooth hypersurfaces of degree \(d\) in a
weighted projective space \(\mathbb{P}(w_0, w_1, w_2, w_3, w_4, w_5)\). By Griffiths residue calculus,
if \(d = \frac{1}{2}(w_0 + \cdots + w_5)\) then the middle cohomology of the weighted fourfolds are of
\(K3\) type (see Appendix A for more discussions on \(K3\) type varieties). In particular,
when \(w_0 = \cdots = w_5 = 1\) one obtains cubic fourfolds. Families of such weighted
fourfolds which contain a Fermat member can be easily classified (cf. Theorem A.8
and Table 1). There are 17 cases most of which have essentially appeared in Reid’s
list. In this paper we consider one of the new cases, namely, weighted degree 6
hypersurfaces in \(\mathbb{P}(1, 2, 2, 2, 2, 3)\).

A quasi-smooth hypersurface \(Z\) of degree 6 in \(\mathbb{P}(1, 2, 2, 2, 2, 3)\) is a double cover
of \(\mathbb{P}^4\) branched along a smooth cubic threefold \(X\) and a hyperplane \(H\). The isomor-
phism class of \(Z\) is determined by the projective equivalence class of the branched
data \(X\) and \(H\). Thus, we are interested in the moduli space of pairs \((X, H)\) where
\(X\) is a cubic threefold and \(H\) is a hyperplane in \(\mathbb{P}^4\). This moduli space is also
interesting from the prospective of singularity theory. By the theory of Pinkham
[Pin74] the study of deformations of the singularity \(O_{16}\) (which comes from the
affine cone over a cubic surface) is essentially reduced to the study of the moduli of
cubic pairs \((X, H)\). The strategy has been successfully applied to unimodal singu-
larities by Pinkham, Looijenga and Brieskorn, and to the minimally elliptic surface

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singularity $N_{16}$ by the first author (see [Laz09a] and references therein). Note that there is a close relation between the deformations of $N_{16}$ and $O_{16}$. We will discuss more about $O_{16}$ elsewhere.

We study the moduli of pairs $(X, H)$ consisting of a cubic threefold $X$ and a hyperplane $H$ via a period map. To construct periods for pairs $(X, H)$, we use a variation of the construction by Allcock, Carlson and Toledo [ACT02, ACT11] which allows us to encode a pair $(X, H)$ as a cubic fourfold $Y$. A cubic fourfold $Y$ coming from a cubic pair $(X, H)$ is characterized by the geometric property that it admits an involution fixing a hyperplane section, or equivalently, it has an Eckardt point (i.e. $Y$ contains a cone over a cubic surface as a hyperplane section and we call the vertex an Eckardt point). Note that our construction works for cubic pairs of any dimension. A smooth cubic fourfold $Y$ admitting an Eckardt point contains at least 27 planes (generated by the 27 lines on the cubic surface and the Eckardt point). The Hodge classes corresponding to these planes generate a saturated sublattice $M \subset H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$. In addition to the hyperplane class, the lattice $M$ contains a (scaled) $E_6$ lattice induced from the primitive cohomology of the cubic surface $X \cap H$. The cubic fourfolds $Y$ are characterized Hodge theoretically as “$M$-polarized” cubic fourfolds (analogous to lattice polarized $K3$ surfaces defined by Dolgachev [Dol96]).

Using periods of the cubic fourfolds $Y$ we get a period map $\mathcal{P}_0$ for cubic pairs $(X, H)$. It is well known that the periods of cubic fourfolds behave similarly to the periods of $K3$ surfaces. The Torelli theorem for cubic fourfolds was proven by Voisin [Voi86], “lattice polarized” cubic fourfolds were used by Hassett [Has00], and the image of the period map was described by Looijenga [Loo09] and the first author [Laz09b, Laz10]. Using Voisin’s Torelli theorem and lattice theory, we first prove that the moduli space of cubic pairs $(X, H)$ is birational to a certain locally symmetric domain of type IV. More precisely, we let $T = M_{\mathcal{H}_4(Y, \mathbb{Z})}^1$. Denote a connected component of the period domain for weight 4 Hodge structures on $T$ with Hodge numbers $[0, 1, 14, 1, 0]$ by $\mathcal{D}_M$. Also set $O^+(T) \subset O(T)$ to be the group of isometries of $T$ stabilizing $\mathcal{D}_M$.

**Theorem 0.1** (= Theorem 3.5). Let $\mathcal{M}_0$ be the moduli of pairs $(X, H)$ consisting of a cubic threefold $X$ and a hyperplane $H$ in $\mathbb{P}^4$ such that the associated cubic fourfold $Y$ is smooth. The period map $\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}_M/O^+(T)$ defined via periods of the cubic fourfolds $Y$ is an isomorphism onto the image.

Next we consider the problem of compactifying the period map $\mathcal{P}_0$. There is a natural compactification for the moduli of cubic pairs $(X, H)$ using geometric invariant theory (GIT). Specifically, we take the GIT quotient of $PH^0(\mathbb{P}^4, \mathcal{O}(3)) \times PH^0(\mathbb{P}^4, \mathcal{O}(1))$ with respect to the action of $\text{SL}(5, \mathbb{C})$. Note that our GIT construction depends on a parameter $t$ which corresponds to the choice of a linearization (see for example [Laz09a]). Write $\mathcal{M}(t) := PH^0(\mathbb{P}^4, \mathcal{O}(3)) \times PH^0(\mathbb{P}^4, \mathcal{O}(1)) \sslash SL(5, \mathbb{C})$. The natural question for us is how does $\mathcal{M}(t)$ compare to the Baily-Borel compactification of $\mathcal{D}_M/O^+(T)$ (for some related examples see [Sha80], [LS07], [Loo09], [Laz10] and [ACT11]). Note that $\mathcal{M}(t)$ is not empty when $0 \leq t \leq \frac{1}{4}$ (as $t$ increases, $X$ becomes more singular while transversality for $X \cap H$ becomes better). Also, $\mathcal{M}(t)$ and $\mathcal{M}(t')$ only differs in codimension 2 for $t \neq t'$. Our second result is that the GIT moduli $\mathcal{M}(_{\frac{1}{4}})$ at $t = \frac{1}{4}$ is isomorphic to the Baily-Borel compactification of $\mathcal{D}_M/O^+(T)$.
Theorem 0.2 (= Theorem 5.5). The period map \( P_0 : \mathcal{M}_0 \to \mathcal{D}_M/O^+(T) \) extends to an isomorphism \( \mathcal{M}(\frac{1}{t}) \cong (\mathcal{D}_M/O^+(T))^* \) where \((\mathcal{D}_M/O^+(T))^*\) denotes the Baily-Borel compactification of \( \mathcal{D}_M/O^+(T) \).

To prove this theorem, we apply a general framework developed by Looijenga [Loo03a, Loo03b] (see also [Loo86] and [LS08]) for comparing GIT compactifications to appropriate compactifications of period spaces. Specifically, we consider cubic pairs \((X, H)\) with \(X\) at worst nodal and \(H\) at worst simply tangent to \(X\). Such pairs are stable for any \(0 < t < \frac{4}{7}\). Denote the corresponding moduli by \( \mathcal{M} \) (N.B. \( \mathcal{M}_0 \subset \mathcal{M} \) and \( \mathcal{M} \subset \mathcal{M}(t) \) has codimension higher than 1). The period map \( P_0 \) extends to \( P : \mathcal{M} \to \mathcal{D}_M/O^+(T) \). Using the work of Looijenga [Loo09] and the first author [Laz09b, Laz10], we are able to show that the image \( \mathcal{P}(\mathcal{M}) \) of the period map has boundary of codimension at least 2 in \( \mathcal{D}_M/O^+(T) \). Indeed, the image of the period map for smooth cubic fourfolds is the complement of a union of the hyperplane arrangements \( \mathcal{H}_\Delta \) (which corresponds to the limiting mixed Hodge structures of cubic fourfolds with simple singularities, see [Has00, §4.2]) and \( \mathcal{H}_\infty \) (which parameterizes the limiting mixed Hodge structures of certain determinantal cubic fourfolds, see [Has00, §4.4]). The hyperplanes in \( \mathcal{H}_\infty \) do not intersect with the subdomain \( \mathcal{D}_M \). The intersection of the hyperplane arrangement \( \mathcal{H}_\Delta \) with \( \mathcal{D}_M \) gives two Heegner divisors \( H_n \) (the nodal Heegner divisor) and \( H_t \) (the tangential Heegner divisor) in \( \mathcal{D}_M/O^+(T) \). Geometrically, they correspond respectively to the divisors in \( \mathcal{M} \) parameterizing pairs \((X, H)\) with \(X\) nodal or \(H\) simply tangent to \(X\). To prove the theorem, we need to select \( t \in (0, \frac{4}{7}) \) such that the natural polarization on the GIT quotient \( \mathcal{M}(t) \) matches with the polarization on the Baily-Borel compactification \((\mathcal{D}_M/O^+(T))^*\) (recall that \( \mathcal{M} \) is an open subset with boundary of codimension \( \geq 2 \) in both).

Remark 0.3. Consider 2-dimensional cubic pairs \((S, D)\) consisting of a cubic surface \( S \) and a hyperplane \( D \). Following [ACT02] we associate a cubic threefold \( X \) (which is the triple cyclic cover of \( \mathbb{P}^3 \) branched along \( S \) and hence admits an automorphism of order 3) to \( S \). We can also view \( D \) as a hyperplane in \( \mathbb{P}^4 \). Our construction in this paper for \((X, D)\) seems to give a complex ball uniformization for the moduli space of \((S, D)\). Specifically, the associated cubic fourfold \( Y \) is not only “lattice polarized” but also admits an automorphism of order 3; the corresponding Mumford-Tate subdomain is a 7-dimensional ball.

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1. Associate a cubic fourfold to a cubic threefold and a hyperplane

In this section we construct a smooth cubic fourfold \( Y \) from a pair \((X, H)\) consisting of a smooth cubic threefold \( X \) and a hyperplane \( H \) intersecting \( X \) transversely.
Moreover, we characterize the cubic fourfold $Y$ geometrically by the fact that it contains a cone over a smooth cubic surface (more precisely, $Y$ contains an Eckardt point) and admits a certain involution. The construction is inspired by the work of Allcock, Carlson and Toledo [ACT11]. Given a smooth cubic threefold in $\mathbb{P}^3$ they consider the triple cyclic cover of $\mathbb{P}^4$ branched along the cubic threefold. In our case we take the double cover $Z$ of $\mathbb{P}^4$ branched over $X + H$. Note that $Z$ is isomorphic to a quasi-smooth hypersurface of degree 6 in $\mathbb{P}(1, 2, 2, 2, 3)$. The cubic fourfold $Y$ is obtained as a birational modification of the double cover $Z$ (blow up the ramification locus over $X \cap H$ and then blow down the strict transform of the preimage of $H$ in $Z$). Because our construction works for cubic pairs of any dimension, we shall present it in that generality.

Let $X \subset \mathbb{P}^n$ ($n \geq 3$) be a smooth cubic $(n - 1)$-fold cut out by a cubic homogeneous polynomial $f(y_0, \ldots, y_n) = 0$. Let $H$ be a hyperplane intersecting $X$ transversely. Assume the equation of $H$ is $l(y_0, \ldots, y_n) = 0$. Note that the intersection $X \cap H$ is a smooth cubic $(n - 2)$-fold in $H \cong \mathbb{P}^{n - 1}$. We consider the cubic $n$-fold $Y \subset \mathbb{P}^{n+1}$ defined by

$$f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2 = 0$$

where $y_{n+1}$ is a new variable. (In what follows we view $X$ and $H$ as subvarieties of the hyperplane $(y_{n+1} = 0) \cong \mathbb{P}^n$.)

**Remark 1.2.** Geometrically, the cubic $n$-fold $Y$ can be constructed as follows (see Proposition 1.9). Take the double cover $Z$ of $\mathbb{P}^n$ ramified over the union $X + H$. After blowing up $Z$ along the reduced inverse image of $X \cap H$ and then blowing down the strict transform of the inverse image of $H$ in $Z$, we obtain $Y$.

**Lemma 1.3.** Notations as above. The cubic $n$-fold $Y$ defined by $f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2 = 0$ is smooth.

**Proof.** Set $F(y_0, \ldots, y_{n+1}) = f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2$. A singular point is a common zero point for both $F$ and the vector $(\frac{\partial F}{\partial y_0}, \ldots, \frac{\partial F}{\partial y_n}, \frac{\partial F}{\partial y_{n+1}})$ which is $(\frac{\partial l}{\partial y_0} + \frac{\partial f}{\partial y_0}y_{n+1}^2, \ldots, \frac{\partial l}{\partial y_n} + \frac{\partial f}{\partial y_n}y_{n+1}^2, \frac{\partial l}{\partial y_{n+1}} + \frac{\partial f}{\partial y_{n+1}}y_{n+1}^2, 2y_{n+1})$. Suppose $l_{y_{n+1}} = 0$. There are two possibilities: either $y_{n+1} = 0$ or $l(y_0, \ldots, y_n) = 0$. If $y_{n+1} = 0$ (and hence $f(y_0, \ldots, y_n) = 0$) then the vector must be nonzero because $X$ is smooth. Now suppose $l(y_0, \ldots, y_n) = 0$ (which implies that $f(y_0, \ldots, y_n) = 0$). Notice that $[0, \ldots, 0, 1]$ is a smooth point of $Y$. Since $H$ intersects $X$ transversely, the following matrix has rank 2 for every point on $X \cap H$ (in particular, $X \cap H$ is smooth):

$$\begin{pmatrix}
\frac{\partial f}{\partial y_0} & \cdots & \frac{\partial f}{\partial y_n} & \frac{\partial f}{\partial y_{n+1}} \\
\frac{\partial l}{\partial y_0} & \cdots & \frac{\partial l}{\partial y_n} & \frac{\partial l}{\partial y_{n+1}}
\end{pmatrix}.$$

As a result, the vector $(\frac{\partial F}{\partial y_0}, \ldots, \frac{\partial F}{\partial y_n}, \frac{\partial F}{\partial y_{n+1}})$ is nonzero. \hfill $\Box$

In a similar way one can prove the converse of Lemma 1.3.

**Lemma 1.4.** Consider a cubic $n$-fold $Y \subset \mathbb{P}^{n+1}$ with equation

$$f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2 = 0.$$

Let $X$ (resp. $H$) be the variety cut out by the equations $f(y_0, \ldots, y_n) = y_{n+1} = 0$ (resp. $l(y_0, \ldots, y_n) = y_{n+1} = 0$). If $Y$ is smooth, then $X$ is smooth. Moreover, the linear subspace $H$ intersects $X$ transversely.
Now we study the geometry of the smooth cubic $n$-fold $Y$. From the equation it is easy to see that $Y$ contains the point $p = [0, \ldots, 0, 1]$. Furthermore, the hyperplane section of $Y$ defined by $l(y_0, \ldots, y_n) = 0$ is a cone over the smooth cubic $(n - 2)$-fold $X \cap H$ with vertex $p$. As a generalization of an Eckardt point for a cubic surface (see for example [Dol12, Chap. 9]), we call $p$ an Eckardt point of $Y$. The cubic $n$-fold $Y$ coming from a cubic $(n - 1)$-fold $X$ and a hyperplane $H$ is characterized geometrically by the fact that it contains an Eckardt point.

**Definition 1.5.** A point $p$ on a cubic hypersurface $V \subset \mathbb{P}^{n+1}$ is called an **Eckardt point** if the following two conditions hold:

1. $p$ is a smooth point of $V$;
2. $p$ is a point of multiplicity $3$ for the $(n - 1)$-dimensional cubic $V \cap T_pV$ (where $T_pY$ denotes the projective tangent space at $p \in Y$).

Eckardt points are also called star points by other authors (cf. [CC10]). If $V$ is smooth, then the second condition is equivalent to saying that $(V \cap T_pV) \subset T_pY$ is a cone with vertex $p$ over a $(n - 2)$-dimensional cubic hypersurface. It is also easy to prove the following lemma.

**Lemma 1.6.** If a cubic $n$-fold $V$ contains an Eckardt point, then we can choose coordinates such that the equation defining $V$ is $G(y_0, \ldots, y_n) = g(y_0, \ldots, y_n) + y_0y_5^2$ where $g$ is a homogenous cubic polynomial.

We give a second geometric characterization of $Y$. There is a natural involution

\[ \sigma : [y_0, \ldots, y_n, y_{n+1}] \mapsto [y_0, \ldots, y_n, -y_{n+1}] \]

acting on the smooth cubic $n$-fold $Y = (f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2 = 0)$. Clearly $\sigma$ fixes the hyperplane $(y_{n+1} = 0)$ pointwise and also the point $p = [0, \ldots, 0, 1]$. Conversely, we have the following lemma. Note that the fixed locus of an involution of $\mathbb{P}^{n+1}$ is the union of two subspaces of dimensions $k$ and $n - k$.

**Lemma 1.8.** Let $V \subset \mathbb{P}^{n+1}$ be a smooth cubic $n$-fold. Suppose $\tau$ is an involution of $\mathbb{P}^{n+1}$ preserving $V$ and the fixed locus of $\tau$ consists of a hyperplane and a point $p$ which belongs to $V$. Then there exist coordinates such that $V$ is cut out by the equation $f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2 = 0$ where $f$ (resp. $l$) is a cubic (resp. linear) homogenous polynomial.

**Proof.** Suppose $V$ admits a projective isomorphism of order 2 with one isolated fixed point $p$ on $V$. Choose coordinates on $\mathbb{P}^5$ such that $\tau$ is given by $[y_0, \ldots, y_n, y_{n+1}] \mapsto [y_0, \ldots, y_n, -y_{n+1}]$. The fixed point is $p = [0, \ldots, 0, 1]$. Then $V$ has equation $y_{n+1}^2l(y_0, \ldots, y_n) + y_{n+1}q(y_1, \ldots, y_n) + f(y_0, \ldots, y_n) = 0$. Because $\tau$ preserves $V$ we have $q = 0$. \qed

Now let us consider the projection of the smooth cubic $n$-fold $Y$ from the Eckardt point $p = [0, \ldots, 0, 1]$ to the hyperplane $(y_{n+1} = 0)$. Specifically we have a rational map

\[ \pi : Y \dashrightarrow \mathbb{P}^n, \quad [y_0, \ldots, y_n, y_{n+1}] \mapsto [y_0, \ldots, y_n, 0] \]

which has degree $2$ generically. Clearly $\sigma$ is the involution relative to $\pi$. To resolve the indeterminacy locus let us blow up $Y$ at $p$. Note that we view $X$ and $H$ as subvarieties of the linear subspace $(y_{n+1} = 0) \cong \mathbb{P}^n$. 
Proposition 1.9. Notations as above. After blowing up the point \( p = [0, \ldots, 0, 1] \) we obtain a morphism \( \pi : \text{Bl}_p Y \to \mathbb{P}^n \) which is also the blow-up \( \text{Bl}_{X \cap H} Z \to Z \) of the double cover \( Z \) of \( \mathbb{P}^n \) ramified over \( X + H \) along the reduced inverse image of \( X \cap H \).

\[
\pi \quad \text{Bl}_p Y \quad \xrightarrow{\sigma} \quad \text{Bl}_{X \cap H} Z \quad \xrightarrow{\sigma} \quad \mathbb{P}^n
\]

Proof. The equation of \( Y \) is \( f(y_0, \ldots, y_n) + l(y_0, \ldots, y_n)y_{n+1}^2 = 0 \). After choosing the open affine \( (y_{n+1} \neq 0) \) containing the point \( p = [0, \ldots, 0, 1] \) we assume that \( y_{n+1} = 1 \). The blow-up of the open affine at the point \( (0, \ldots, 0) \in \mathbb{C}^{n+1} \) is the morphism \( \text{Proj} \mathbb{C}[y_0, \ldots, y_n][A_0, \ldots, A_n]/(y_iA_j = y_jA_i) \to \text{Spec} \mathbb{C}[y_0, \ldots, y_n] \). The projection morphism \( \pi \) is defined by \( (y_0, \ldots, y_n)[A_0, \ldots, A_n] \mapsto [A_0, \ldots, A_n] \). Let us describe the exceptional divisor. Choose an open subset \( (A_0 \neq 0) \). Write \( a_i = \frac{1}{y_i} \) for \( 1 \leq i \leq n \). Now pull back the defining equation of \( Y \) using \( y_i = y_0a_i \). We get \( y_0g_0f(1, a_1, \ldots, a_n) + l(1, a_1, \ldots, a_n) = 0 \). The blow-up \( \text{Bl}_p Y \) is given locally by \( g_0f(1, a_1, \ldots, a_n) + l(1, a_1, \ldots, a_n) = 0 \). Over the point \( p \) we have \( y_0 = 0 \) and hence \( l(1, a_1, \ldots, a_n) = 0 \). One has similar descriptions when choosing different open subsets. It follows that globally the exceptional divisor is defined by \( l(A_0, \ldots, A_n) = 0 \) which is mapped onto the hyperplane \( H \) by \( \pi \). Over the complement \( \mathbb{P}^n \setminus (X \cap H) \) of \( X \cap H \) the morphism \( \pi \) is a double cover branched along \( (X + H) \setminus (X \cap H) \). The fibers of \( \pi \) over the cubic surface \( X \cap H \) are \( \mathbb{P}^1 \)'s (N.B. the involution \( \sigma \) acts on these fibers). Because \( \pi^{-1}(X \cap H) \) is a divisor in \( \text{Bl}_p Y \), there exists a unique morphism \( \text{Bl}_p Y \to \text{Bl}_{(X \cap H)} \mathbb{P}^n \) factoring \( \pi \). It is not difficult to see that the morphism \( \text{Bl}_p Y \to \text{Bl}_{(X \cap H)} \mathbb{P}^n \) is a double cover ramified over the strict transforms \( X' + H' \) of \( X + H \). Let \( Z \) be the double cover of \( \mathbb{P}^n \) branched along \( X + H \). By [CvS06, §3] \( \text{Bl}_p Y \) is the blow-up of \( Z \) along the locus over the cubic surface \( X \cap H \). \( \square \)

Remark 1.10. The cubic \( n \)-fold \( Y \) and the double cover \( Z \) are birational. Let us describe the birational maps. To get \( Z \) one blows up \( Y \) at the Eckardt point \( p \) and then blow down the ruling of the associated cone. Conversely, by the proof of Proposition 1.9 the cubic \( n \)-fold \( Y \) can be obtained by blowing down the strict transform of the inverse image of \( H \) in \( Z \) (to the point \( p \)) for \( \text{Bl}_{X \cap H} Z \).

Remark 1.11. Let \( Z \) be an \( n \)-dimensional quasi-smooth hypersurface of degree 6 in \( \mathbb{P}(1, 2, \ldots, 2, 3) \). Let \( z_0, z_1, \ldots, z_n, z_{n+1} \) be the weighted homogeneous coordinates. After a change of coordinates one can assume that \( Z \) has the equation

\[
z_{n+1}^2 + g(z_0^2, z_1, \ldots, z_n) = 0
\]

where \( g \) is a cubic homogeneous polynomial. It is not difficult to see that \( Z \) is a double cover of \( \mathbb{P}^n \) along a cubic \((n - 1)\)-fold \( X \) union a hyperplane \( H \). Moreover, both \( X \) and \( X \cap H \) are smooth. The cubic \( n \)-fold \( Y \) constructed from the pair \((X, H)\) is obtained as the closure of \( Z \) under the Veronese map of degree 3 from \( \mathbb{P}(1, 2, \ldots, 2, 3) \) to \( \mathbb{P}^{n+1} \).

In what follows we shall focus on the case when \( n = 4 \), that is, \( X \) is a cubic threefold and \( H \) is a hyperplane in \( \mathbb{P}^4 \). The cubic fourfold \( Y \) coming from \((X, H)\) contains an Eckardt point \( p = [0, 0, 0, 0, 0, 1] \) and admits an involution \( \sigma \) fixing
p and a hyperplane (see (1.7)). It is also birational to the double cover $Z$ of $\mathbb{P}^4$ branched along $X + H$ as described in Proposition 1.9.

To conclude this section, let us mention another geometric property of the smooth cubic fourfold $Y$. Still, $X$ and $H$ are considered as subvarieties of the hyperplane $(y_5 = 0) \cong \mathbb{P}^4$. Choose a line $m$ on the cubic surface $X \cap H$ and let $P \subseteq Y$ be the plane generated by $m$ and the Eckardt point $p$. Choose a plane $P'$ which is complementary to $m$ in the hyperplane $(y_5 = 0) \cong \mathbb{P}^4$ and project $Y$ from $P$ to $P'$. After blowing up $P$ inside $Y$ we obtain a quadratic bundle $\text{Bl}_P Y \to P'$ whose discriminant locus is a degree 6 curve (cf. [Voi86]). The sextic discriminant curve consists of two irreducible components: a quintic curve $C$ and a line $L$. Indeed, the quintic curve $C$ is the discriminant locus of the conic bundle obtained by projecting $X$ from the line $m$ to $P'$ in $(y_5 = 0) \cong \mathbb{P}^4$ and the line $L$ is the intersection of $H$ and $P'$. The proof is left to the reader.

Remark 1.12. More generally, there exists a finite correspondence between the moduli of pairs $(X, H)$ and $(C, L)$ (obtained by projecting $X$ from a line on $X \cap H$, see for example [Bea00, §4.2]). As discussed in [Laz09a], the moduli of $(C, L)$ is associated to the singularity $N_{16}$, and is birational to a locally symmetric variety of type IV. In conclusion, there is a close relation between the deformations of $(X, H)$ and $(C, L)$, and we expect the moduli of $(X, H)$ is also birational to a type IV locally symmetric domain.

2. A CYCLE THEORETICAL CHARACTERIZATION OF CUBIC FOURFOLDS WITH ECKARDT POINTS

We now study the smooth cubic fourfold $Y$ constructed in Section 1 from the prospective of Hodge theory. The cubic fourfold $Y$ is a special cubic fourfold (meaning it contains an algebraic surface which is not homologous to a complete intersection, see [Has00]). In fact, $Y$ contains more algebraic cycles. There exists a positive definite lattice $M$ of rank 7 and a primitive embedding $M \subset H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y) \subset H^4(Y, \mathbb{Z})$. Moreover, the sublattice $M$ is invariant with respect to the natural involution $\sigma$ (see (1.7)). We shall determine $M$ and its orthogonal complement $T = M_{H^4(Y, \mathbb{Z})}^\perp$ using lattice theory (in particular, we use the terminologies and notations of [Nik79]) and show that $Y$ can be characterized by the condition that it is “$M$-polarized”.

Let $Y$ be the smooth cubic fourfold (see (1.1)) coming from a pair $(X, H)$ where $X$ is a smooth cubic threefold and $H$ is a transverse hyperplane. Let us first describe how the involution $\sigma$ in (1.7) decomposes the Hodge structure on the primitive cohomology $H^4_{\text{prim}}(Y, \mathbb{Q})$.

Lemma 2.1. The Hodge structure $H^4_{\text{prim}}(Y, \mathbb{Q})$ splits as

$$H^4_{\text{prim}}(Y, \mathbb{Q}) = H^4_{\text{prim}}(Y, \mathbb{Q})_{+1} \oplus H^4_{\text{prim}}(Y, \mathbb{Q})_{-1}$$

where $H^4_{\text{prim}}(Y, \mathbb{Q})_{+1}$ (resp. $H^4_{\text{prim}}(Y, \mathbb{Q})_{-1}$) is the eigenspace of the induced action $\sigma^*$ with eigenvalue $+1$ (resp. $-1$). The Hodge numbers of $H^4_{\text{prim}}(Y, \mathbb{Q})_{+1}$ (resp. $H^4_{\text{prim}}(Y, \mathbb{Q})_{-1}$) are $[0, 0, 6, 0, 0]$ (resp. $[0, 1, 14, 1, 0]$).

Proof. It suffices to check the claim on Hodge numbers for a particular cubic fourfold $Y$. We assume that $Y$ is given by $F(y_0, \ldots, y_4) = y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_0 y_5^2$. The Jacobian ideal is $J(F) = \langle 3y_0^2 + y_5^2, 3y_1^2, 3y_2^2, 3y_3^2, 3y_4^2, 2y_0 y_5 \rangle$. Write $\Omega =
Lemma 2.1 \( \phi \)

\( C \) to prove \( f \) and Hodge-Riemann

The equations of the cones

The surface classes

\( g \) where

\( \Omega \) is a positive definite lattice of rank 7. It is also easy to see that

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The class of a hyperplane section of \( Y \) is clearly fixed by the induced involution \( \sigma^* \) (e.g. consider the hyperplane \( (y_5 = 0) \)). Let \( M \subset H^4(Y, Z) \cap H^{2,2}(Y) \) be the \( \sigma^* \)-invariant sublattice of \( H^4(Y, Z) \cap H^{2,2}(Y) \). By Lemma 2.1 and Hodge-Riemann relations, \( M \) is a positive definite lattice of rank 7. It is also easy to see that \( M \) is saturated (i.e. \( M = (M \otimes \mathbb{Q}) \cap (H^4(Y, Z) \cap H^{2,2}(Y)) \)) and hence a primitive sublattice (i.e. \( (H^4(Y, Z) \cap H^{2,2}(Y))/M \) is torsion free).

We construct some algebraic surfaces in \( Y \) whose cohomology classes belong to \( M \). Let \( p = [0, 0, 0, 0, 0, 1] \) be the Eckardt point of \( Y \). Recall that \( Y \) contains a cone over the cubic surface \( X \cap H \) with vertex \( p \). It is classically known that the smooth cubic surface \( X \cap H \) is isomorphic to \( \mathbb{P}^3 \) with 6 points blown up. Let \( F_0 \) be the pullback of a (general) line on \( \mathbb{P}^2 \). Denote the exceptional curves by \( F_1, \ldots, F_6 \). (The embedding of the blow-up of \( \mathbb{P}^2 \) at 6 points into \( \mathbb{P}^3 \) as a cubic surface is given by the linear system \( 3F_0 - F_1 - \cdots - F_6 \).

We shall use \( F_0, F_1, \ldots, F_6 \) to denote the cones over the corresponding curves on \( X \cap H \) with vertex \( p \). In particular, \( F_1, \ldots, F_6 \) are planes in \( Y \). Also let the corresponding surface classes be \( [F_0], [F_1], \ldots, [F_6] \).

**Lemma 2.2.** The surface classes \( [F_0], [F_1], \ldots, [F_6] \) are linearly independent in \( H^4(Y, Z) \cap H^{2,2}(Y) \). Moreover, they are left invariant by \( \sigma^* \).

**Proof.** The equations of the cones \( F_0, F_1, \ldots, F_6 \) only involves \( y_0, \ldots, y_4 \). Therefore, the involution \( \sigma : [y_0, \ldots, y_4, y_5] \mapsto [y_0, \ldots, y_4, -y_5] \) fixes them. Now we use Proposition 1.9 to prove \( [F_0], [F_1], \ldots, [F_6] \) are linearly independent. Let \( f : Z \to \mathbb{P}^4 \) be the double cover of \( \mathbb{P}^4 \) branched along \( X + H \). Set \( \varphi : \text{Bl}_{X \cap H} Z \to Z \) (resp. \( \psi : \text{Bl}_{X \cap H} \mathbb{P}^4 \to \mathbb{P}^4 \)) to be the blow-up of \( Z \) (resp. \( \mathbb{P}^4 \)) along the locus ramified over the cubic surface \( X \cap H \) (resp. the surface \( X \cap H \)). Let \( j : F \to \text{Bl}_{X \cap H} \mathbb{P}^4 \) be the exceptional divisor of \( \text{Bl}_{X \cap H} \mathbb{P}^4 \). Note that \( \psi^*(X + H) \sim X' + H' + 2F \) where \( X' \) (resp. \( H' \)) is the strict transform of \( X \) (resp. \( H \)). By [CvS06, §3] we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Bl}_{X \cap H} Z & \xrightarrow{\varphi} & Z \\
g \downarrow & & \downarrow f \\
\text{Bl}_{X \cap H} \mathbb{P}^4 & \xrightarrow{\psi} & \mathbb{P}^4
\end{array}
\]

(2.3)

where \( g : \text{Bl}_{X \cap H} Z \to \text{Bl}_{X \cap H} \mathbb{P}^4 \) is the double cover of \( \text{Bl}_{X \cap H} \mathbb{P}^4 \) along \( X' + H' \). By [Voi02, Thm. 7.31] there exists an isomorphism of Hodge structures

\[
H^4(\mathbb{P}^4, \mathbb{Z}) \oplus H^2(X \cap H, \mathbb{Z})(-1) \xrightarrow{\psi^* + j_* \circ \varphi^* \circ \psi^*} H^4(\text{Bl}_{X \cap H} \mathbb{P}^4, \mathbb{Z}).
\]

The covering map \( g \) induces an injective homomorphism \( g^* : H^4(\text{Bl}_{X \cap H} \mathbb{P}^4, \mathbb{Z}) \to H^4(\text{Bl}_{X \cap H} Z, \mathbb{Z}) \).

By Proposition 1.9 \( \text{Bl}_{X \cap H} Z \) is also the blow-up \( \text{Bl}_p Y \) of \( Y \) at the Eckardt point \( p \) (the exceptional divisor is \( g^{-1}(H') \)). Thus we have

\[
H^4(\text{Bl}_{X \cap H} Z, \mathbb{Z}) \cong H^4(Y, \mathbb{Z}) \oplus \]
$H^0(p,\mathbb{Z})(-2)$. By our construction, the classes represented by $F_0, F_1, \ldots, F_6$ are linearly independent in $H^2(X \cap H, \mathbb{Z})$. The strict transform of $F_i$ ($0 \leq i \leq 6$) in $Bl_p Y = Bl_{X \cap H} Z$ is $g^{-1}(\psi^{-1}(F_i))$. It follows that $[F_0], [F_1], \ldots, [F_6]$ are linearly independent in $H^4(Y, \mathbb{Z})$. \hfill $\square$

Let us consider the rank 7 sublattice $N \subset M$ generated by $[F_0], [F_1], \ldots, [F_6]$. Later we will show that $N$ is a saturated sublattice of $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ and hence $N = M$.

**Lemma 2.4.** The Gram matrix of the lattice $N$ is given as follows.

1. $[F_i] \cdot [F_i] = 3$ for $1 \leq i \leq 6$.
2. $[F_0] \cdot [F_0] = 7$.
3. $[F_0] \cdot [F_i] = 3$ and $[F_i] \cdot [F_j] = 1$ for $1 \leq i \neq j \leq 6$.

In particular, the lattice $N$ is positive definite and $\text{discr}(N) = 64$. Furthermore, the square $h^2$ of the hyperplane class $h$ can be expressed as $3[F_0] - [F_1] - \cdots - [F_6]$ in $H^4(Y, \mathbb{Z})$.

**Proof.** The classes $[F_0], [F_1], \ldots, [F_6]$ are represented by planes in $Y$. The class $[F_0]$ corresponds to a union of planes $P_0 + P'_0 + P''_0$ where $P_0$ share a line with $P'_0$ (resp. $P''_0$) and the intersection $P_0' \cap P''_0$ is a point (see for example [Rei97]). If two planes contained in a cubic fourfold intersect along a line then their intersection number is $-1$ (cf. [Voi86]). Also, the self intersection of a plane in a cubic fourfold equals 3. The Gram matrix of $N$ can be computed easily using these observations.

Alternatively, we compute the intersection numbers using Proposition 1.9. Denote the blow-up of the point $p \in Y$ by $q : Bl_p Y \rightarrow Y$. Let $i : E \hookrightarrow Bl_p Y$ be the exceptional divisor. By the blow-up formula (cf. [Ful98, Thm. 6.7.1]) we have $q^*[F_j] = [F_j] + i_*[V]$ ($1 \leq j \leq 6$) and $q^*[F_0] = [F_0] + 3i_*[V]$ where $[F_j] and [F_0]$ denote the corresponding proper transforms and $V$ is a 2-dimensional linear space of $E \cong \mathbb{P}^3$. Using Proposition 1.9 and Diagram (2.3) one computes that $[F'_0] \cdot [F'_0] = -2$, $[F'_j] \cdot [F'_j] = 2$ and $[F'_0] \cdot [F'_j] = 0$. Now we get $[F_j] \cdot [F_j] = q^*([F_j] \cdot [F'_j] = [F_j] \cdot [F_j] + 2[F_j] \cdot i_*[V] + i_*[V] \cdot i_*[V] = 2 + 2 \times 1 - 1 = 3$ (see [Ful98] Example 8.3.9). Similarly, one can verify that $[F_0] \cdot [F_0] = 7$ and $[F_0] \cdot [F_j] = 3$. It is clear that $[F_j] \cdot [F'_j] = 1$ for $1 \leq j \neq j' \leq 6$.

Take another general linear form $l'(y_0, \ldots, y_4)$ and consider the surface $D$ in $Y$ cut out by $l = l' = 0$. Clearly, the surface $D$ represents $h^2$ where $h$ is the hyperplane class of $Y$. Because $D$ is a cone (with vertex $p$) over the hyperplane section of the cubic surface $X \cap H$, we have $h^2 = [D] = 3[F_0] - [F_1] - \cdots - [F_6]$. \hfill $\square$

**Remark 2.5.** A direct computation shows the following:

- $h^2 \cdot [F_0] = (3[F_0] - [F_1] - \cdots - [F_6]) \cdot [F_0] = 3$;
- $h^2 \cdot [F_i] = (3[F_0] - [F_1] - \cdots - [F_6]) \cdot [F_i] = 1$ for $1 \leq i \leq 6$;
- $h^2 \cdot h^2 = (3[F_0] - [F_1] - \cdots - [F_6]) \cdot (3[F_0] - [F_1] - \cdots - [F_6]) = 3$.

In particular, the cubic fourfold $Y$ is special (cf. [Has00]) with discriminant 8 or 12.

To show that $N$ is a saturated sublattice of $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ we need to compute its discriminant group. By Lemma 2.4 the Gram matrix $G_N$ of $N$ (with
Lemma 2.6. Notations as above. The discriminant group $A_N$ of $N$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$. The discriminant bilinear form
\[ b_N : A_N \times A_N \to \mathbb{Q}/\mathbb{Z} \]
is given by the following matrix (with respect to the basis $\{[F_1]^*, \ldots, [F_6]^*\}$ of $A_N$).
\[
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
\]

Now we show that $N \subset H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ is saturated. As a result, we have $N = M$ and the natural embedding of $N$ into $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ is primitive.

Proposition 2.7. The lattice $N$ generated by $[F_0], [F_1], \ldots, [F_6]$ is saturated in $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$.

Proof: Assume the natural embedding $N \subset H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ is not saturated. Then it factors as $N \subset \text{Sat}(N) \subset H^4(Y, \mathbb{Z})$ where $\text{Sat}(N)$ denotes the saturation of $N$ in $H^4(Y, \mathbb{Z})$. Thus, $\text{Sat}(N)$ is a nontrivial overlattice of $N$ and corresponds to a nontrivial isotropic subgroup of $(A_N, b_N : A_N \times A_N \to \mathbb{Q}/\mathbb{Z})$ (cf. [Nik79, Prop. 1.4.1]). By Lemma 2.6 all the elements of $A_N$ are isotropic. These elements are
linear combinations of $[F_1]^*, \ldots, [F_6]^*$ with coefficients either 0 or 1. There are 6 cases depending on the number of nonzero coefficients. For example let us suppose $[F_i]^*$ is contained in the isotropic subgroup. From the expression of $G_N^{-1}$ it is easy to see $[F_i]^* \equiv \frac{1}{2}(h^2 - [F_i]) \mod N$. It follows that the element $\frac{1}{2}(h^2 - [F_i])$ belongs to $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$ (i.e. $h^2 - [F_i]$ is divisible by 2). The other cases are similar. In conclusion, $N \neq \text{Sat}(N)$ if and only if some of the following elements are 2-divisible in $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$:

1. $h^2 - [F_i]$;
2. $[F_i] - [F_j]$;
3. $[F_0] - [F_i] - [F_j] - [F_k]$;
4. $[F_i] + [F_j] - [F_k] - [F_l]$;
5. $[F_0] - [F_i]$;
6. $h^2 - [F_0]

for $1 \leq i, j, k, l \leq 6$. Let us do a case by case analysis.

1. Write $h^2 - [F_i] = 2x$ for some $x \in H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$. It is easy to see that $x \cdot h^2 = x^2 = 1$. Then $Y$ is a special cubic fourfold (cf. [Has00, Def. 3.1.2]) labeled by the rank 2 lattice generated by $h^2$ and $x$. The corresponding discriminant $d = 2$ (see op. cit. Definition 3.2.1). By op. cit. Section 4.4 this can not happen for the smooth cubic fourfold $Y$.

2. Write $[F_i] - [F_j] = 2x$ for some $x \in H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$. One computes that $x \cdot h^2 = 0$ (which implies that $x^2$ is even) and $x^2 = 1$. Clearly, this is a contradiction.

3. This case can not happen. The argument is similar to that for Case (2).

4. Write $[F_i] + [F_j] - [F_k] - [F_l] = 2x$ for some $x \in H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y)$. A direct computation shows that $h^2 \cdot x = 0$ and $x^2 = 2$. Then $Y$ is a special cubic fourfold with discriminant 6. This is impossible because $Y$ is smooth (cf. [Has00, §4.2]).

5. This case can not happen. The argument is similar to that for Case (1).

6. This case can not happen. The argument is similar to that for Case (2).

□

We give a Hodge theoretic condition for smooth cubic fourfolds to admit Eckardt points.

**Proposition 2.8.** Let $V \subset \mathbb{P}^5$ be smooth cubic fourfold. Suppose there exists a primitive embedding of $M \hookrightarrow H^4(V, \mathbb{Z}) \cap H^{2,2}(V)$ sending $h^2 \in M$ to the square of the hyperplane class of $V$. Then $V$ contains an Eckardt point (and hence can be constructed from a cubic threefold and a hyperplane as in Section 3).

**Proof.** We first show that $V$ contains two planes which intersect at one point. Write the class of a hyperplane section of $V$ by $h_V$. Denote the class corresponding to $[F_i] \in M$ by $[F_i]_V$ ($0 \leq i \leq 6$). Consider (the saturation of) the rank two lattice generated by $h^2_V$ and $[F_i]_V$ ($1 \leq i \leq 6$). Clearly, the discriminant is 8. By [Vois80, §3] (see also [Has00, §4.1]) $V$ contains a plane $P_i$ whose class is $[F_i]_V$. Choose $i, j$ with $1 \leq i \neq j \leq 6$. Because $[F_i]_V \cdot [F_j]_V = 1$, the corresponding planes $P_i$ and $P_j$ intersect at one point. (In a similar way one can show that $V$ contains at least 27 planes.)

Choose two such planes $P$ and $P'$. Suppose $P \cap P' = \{p\}$. Consider the linear subspace $\Pi \cong \mathbb{P}^4$ generated by $P$ and $P'$ in $\mathbb{P}^5$. (In fact, the 27 planes in $V$ are all
Lemma 2.4

Proposition 2.9

Lemma 2.6

Dol12

Has00

Nik79

and

Using

V

surface (N.B. \( \cap \)) The cubic threefold \( V \) contained in the hyperplane \( \Pi \) is the cubic threefold \( V \cap \Pi \) can contain \( V \cap \Pi \subset \Pi \) can be written as \( x_3^2l_3 + x_3q_3 + x_4q_4 + x_5q_5 + c = 0 \) where \( l_i, q_i \) and \( c \) are homogeneous polynomials in \( x_1, x_2 \) of degree 1, 2 and 3 respectively. But \( P' \) is also contained in \( V \cap \Pi \). It follows that \( x_3^2l_3 + x_5q_5 + c \) is the zero polynomial. The equation of \( V \cap \Pi \subset \Pi \) does not contain \( x_5 \), so \( V \cap \Pi \) is a cone with vertex \( p \) over a cubic surface (N.B. \( V \cap \Pi \) can not be a cone with vertex \( p \) because \( V \cap \Pi \) is smooth). Using [CC10, Lem. 2.12] it is easy to show that \( p \) is an Eckardt point of \( V \).

Now let us focus on the lattice \( M (= N) \) which is a positive definite lattice of rank 7 generated by \( [F_0], [F_1], \ldots, [F_6] \). The natural embedding \( M \subset H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y) \) is primitive. The Gram matrix \( G_M (= G_N) \) and the discriminant group \( (A_M, b_M) (= (A_N, b_N)) \) have been computed in Lemma 2.4 and Lemma 2.6. Note that \( M \) is an odd lattice. However, the primitive part \( M \cap H^4_{prim}(Y, \mathbb{Z}) = (h^2)_{M}^{+} \) is even and can be described explicitly.

**Proposition 2.9.** The primitive part \( M \cap H^4_{prim}(Y, \mathbb{Z}) \) of \( M \) is an even lattice. More precisely, we have \( M \cap H^4_{prim}(Y, \mathbb{Z}) \cong E_6(2) \).

**Proof.** Because \( (h^2)_{H^4(Y, \mathbb{Z})}^{+} \) is even (see for example [Has00, Prop. 2.1.2]), the sublattice \( (h^2)_{M}^{+} \) is even. We compute \( M \cap H^4_{prim}(Y, \mathbb{Z}) = (h^2)_{M}^{+} \) explicitly. Let \( x \) be an element of \( M \) satisfying \( x \cdot h^2 = 0 \). Write \( x = a_0[F_0] + a_1[F_1] + \cdots + a_6[F_6] \) where \( a_0, \ldots, a_6 \) are integers. Then \( 3a_0 + a_1 + \cdots + a_6 = 0 \). It is not difficult to verify that \( (h^2)_{M}^{+} \) is generated by \( [F_1] - [F_6] \) and \( [F_0] - [F_1] - [F_2] - [F_6] \) with \( 1 \leq i, j, k \leq 6 \). Consider the \( \mathbb{Z} \)-basis of \( (h^2)_{M}^{+} \) consisting of \( [F_1] - [F_6], [F_0] - [F_1] - [F_2] - [F_6], [F_4] - [F_3], [F_5] - [F_4], [F_6] - [F_5] \) and \( -[F_0] + [F_1] + [F_2] + [F_3] \) (compare [Dol12, Thm. 8.2.12]). The Gram matrix can be easily computed which coincides with the Gram matrix for \( E_6(2) \).

**Remark 2.10.** Proposition 2.9 shows that \( M \) is closely related to the lattice \( E_6 \). This is not surprising because \( M \) contains the classes of the planes generated by the Eckardt point \( p \) and the 27 lines on the cubic surface \( X \cap H \). The configuration of these planes is determined by that of the 27 lines. The relation between lines on a cubic surface and the \( E_6 \) lattice is classically known, see for example [Dol12, §9.1].

Next we study the transcendental lattice \( T := M^1_{H^4(Y, \mathbb{Z})} \) which is the orthogonal complement of \( M \) in \( H^4(Y, \mathbb{Z}) \). By [Has00, Prop. 2.1.2] the middle integral cohomology lattice \( H^4(Y, \mathbb{Z}) \) is the odd unimodular lattice \( I_{21,2} = (+1)^{\otimes 21} \oplus (-1)^{\otimes 2} \). We determine \( T \) using analogues of the results in [Nik79, §1] for odd lattices.

**Lemma 2.11.** The invariants of the transcendental lattice \( T \) are computed as follows.

- \( T \) is an even lattice of rank 16. The signature is \( (14, 2) \).
- \( A_T \cong (\mathbb{Z}/2\mathbb{Z})^6 \). In particular, \( T \) is 2-elementary and \( \text{discr}(M) = 64 \).
- The discriminant quadratic form \( q_T : A_T \to \mathbb{Q}/2\mathbb{Z} \) is isomorphic to the orthogonal direct sum \( v \oplus v \oplus v \) where \( v : (\mathbb{Z}/2\mathbb{Z})^\otimes 2 \to \mathbb{Q}/2\mathbb{Z} \) is the discriminant quadratic form of the \( D_4 \) lattice.
Proof. The lattice $T$ is contained in $(h^2)^1_{H^4(Y,\mathbb{Z})}$ and hence must be even. Clearly the signature is $(14,2)$. By [Nik79, Prop. 1.6.1] we have $(A_T, b_T) \cong (A_M, -b_M)$ (as pointed out on Page 110 of op. cit. the results in Sections 14.1.6 hold for odd lattices after replacing discriminant quadratic forms by discriminant bilinear forms). By op. cit. Theorem 1.11.3 the signature $14 - 2 \pmod{8}$ and $b_T$ determine $q_T$. Let $q$ be the finite quadratic form on $A_T \cong A_M$ given by the matrix in Lemma 2.6 (with respect to the basis $\{[F_1]^* \ldots, [F_6]^*\}$). It is straightforward to check that $b_T$ is the bilinear form of $q$ (cf. op. cit. Section 1.2). Also, $q$ equals 0 for 28 elements and equals 1 for 36 elements. Denote the quadratic form $u^{(2)}_T$ (resp. $v^{(2)}_T$) in op. cit. Section 1.8 by $u$ (resp. $v$). Note that $v$ is isomorphic to the discriminant quadratic form of $D_4$. Since $q$ only takes values in integers, the only possibilities are $q \equiv u \oplus u \oplus u \equiv u \oplus v \oplus v$ or $q \equiv v \oplus v \oplus v \equiv u \oplus u \oplus v$ (cf. op. cit. Propositions 1.8.1 and 1.8.2). Because both $q$ and $v \oplus v$ have Arf-invariant equal to 1, we deduce that $q \equiv v \oplus v \oplus v$. It follows that $q$ has signature 4 \pmod{8} (see also op. cit. Theorem 1.3.3) and hence $q_T = q$. \hfill \Box

**Proposition 2.12.** The lattice $T = M^\perp_{H^4(Y,\mathbb{Z})}$ is isomorphic to the orthogonal direct sum $U^{\oplus 2} \oplus D_4^{\oplus 3}$ and the natural map $O(T) \to O(q_T)$ is surjective.

**Proof.** This follows from Lemma 2.11, [Nik79] Theorem 1.13.2 and Theorem 1.14.2 (see also Theorem 3.6.2). \hfill \Box

**Lemma 2.13.** The automorphism group $O(q_T)$ is isomorphic to the Weyl group $W(E_6)$ of the $E_6$ lattice.

**Proof.** By [Bou02] Exercise VI.4.2 (see also [Dol12, §9.1.1]) the action of $W(E_6)$ on $E_6/2E_6 \cong (\mathbb{Z}/2\mathbb{Z})^6$ (equipped with the quadratic form $q(-) \equiv \frac{1}{2}(-, -)_{E_6}$ (mod 2\mathbb{Z})) induces an isomorphism $W(E_6) \cong O(E_6/2E_6, q)$. The quadratic form $q$ has Arf-invariant 1 and hence vanishes on 28 vectors (cf. [Dol12, §9.1]). As a result we get $q \cong q_T$ (see also the proof of Lemma 2.11). \hfill \Box

Let us identify $O(b_M)$ and $O(b_T)$ using $(A_M, b_M) \cong (A_T, -b_T)$ and view $O(q_T) \subset O(b_T)$ as a subgroup of $O(b_M)$. The following lemma allows us to extend an automorphism of $T$ to an automorphism of $H^4(Y, \mathbb{Z})$ fixing the square of the hyperplane class $h^2$.

**Lemma 2.14.** The subgroup $O(q_T) \subset O(b_T) \cong O(b_M)$ is the image of the stabilizer subgroup $O(M)_{h^2} \subset O(M)$ of $h^2$ under the natural map $O(M) \to O(b_M)$. Moreover, we have $O(M)_{h^2} \cong W(E_6)$.

**Proof.** The idea is to consider the action of $W(E_6)$ on the 27 lines of the cubic surface $X \cap H$ (and hence on the corresponding planes in $Y$ which generate the lattice $M$). Let us first set up some notations. The smooth cubic surface $X \cap H$ is isomorphic to $\mathbb{P}^2$ with 6 points blown up. Let $\tilde{F}_0$ be the pull-back of a line on $\mathbb{P}^2$. Denote the exceptional curves by $\tilde{F}_1, \ldots, \tilde{F}_6$. (We shall use the same notations $\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_6$ to denote the corresponding curve classes.) The classes $\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_6$ form a orthonormal basis of $\text{Pic}(X \cap H)$ which is isomorphic to $I^*_6 \cong (1, 1)^{\oplus 6} \oplus (1, -1)^{\oplus 6}$. By [Dol12, Thm. 8.2.12] the vectors $\beta_1 = -\tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3$ and $\beta_j = \tilde{F}_j - \tilde{F}_{j-1}$ for $2 \leq j \leq 6$ form a canonical basis (op. cit. Definition 8.2.11) of the lattice $E_6(-1) = (3\tilde{F}_0 - \tilde{F}_1 - \cdots - \tilde{F}_6)^\perp_{\text{Pic}(X \cap H)}$. The Weyl group $W(E_6)$ is generated by the reflections $r_{\beta_i}$ in the $(-2)$-vectors $\beta_i$. Let $p = [0, 0, 0, 0, 0, 1]$ be the Eckardt
point of $Y$. Set $[F_i]$ to be the surface class in $Y$ represented by the cone over $\bar{F}_i$ with vertex $p$ ($0 \leq i \leq 6$). Note that $\{[F_0],[F_1],\ldots,[F_6]\}$ forms a basis of $M$. Every element $\bar{C}$ of $\text{Pic}(X \cap H)$ is a linear combination of $\bar{F}_0, \bar{F}_1, \ldots, \bar{F}_6$. We use $[C]$ to denote the linear combination of $[F_0],[F_1],\ldots,[F_6]$ in $M$ with the same coefficients. In particular, the notation $[\beta_i]$ makes sense.

We define a homomorphism $W(E_6) \to O(M)$ by

$$r_{\beta_i} \to s_{\beta_i} : [C] \mapsto [C] + (\bar{C} \cdot \beta_i)[\beta_i].$$

(For example, because $\beta_1 \cdot \bar{F}_0 = -1$ on the cubic surface $X \cap H$ we have $s_{\beta_1}([F_0]) = [F_0] - [\beta_1] = 2[F_0] - [F_1] - [F_2] - [F_3].$) With respect to the basis $\{[F_0],[F_1],\ldots,[F_6]\}$ the transform $s_{\beta_i}$ is given by

$$\begin{pmatrix}
2 & 1 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and $s_{\beta_j}$ is the transposition between $[F_{j-1}]$ and $[F_j]$ for $2 \leq j \leq 6$. It is straightforward to verify that $s_{\beta_i} \in O(M)$. It is also clear that the homomorphism $W(E_6) \to O(M)$ is injective.

The reflections $s_{\beta_i}$ ($1 \leq i \leq 6$) fix $h^2 = 3[F_0] - [F_1] - \cdots - [F_6]$. In other words, the image of $W(E_6)$ is contained in the stabilizer subgroup $O(M)_{h^2} \subset O(M)$ of $h^2$. We claim the image is $O(M)_{h^2}$. Let $g$ be an element of $O(M)_{h^2}$. Fix the basis $\{[F_0],[F_1],\ldots,[F_6]\}$ for $M$ and the basis $\{\bar{F}_0, \bar{F}_1, \ldots, \bar{F}_6\}$ for $\text{Pic}(X \cap H)$. Let $\bar{g} : \text{Pic}(X \cap H) \to \text{Pic}(X \cap H)$ be the automorphism which has the same matrix as $g$ (with respect to the bases we choose). It suffices to show that $\bar{g}$ belongs to $W(E_6)$. The isometry $g$ fixes $h^2$. As a result, $\bar{g}$ fixes the anticanonical class $3\bar{F}_0 - \bar{F}_1 - \cdots - \bar{F}_6$ of $X \cap H$. Recall that $\{\beta_1,\ldots,\beta_6\}$ is a canonical basis in $E_6(-1)$. Because $(h^2)_{E_6}^+ \cong E_6(2)$ (cf. Proposition 2.9) and $(3\bar{F}_0 - \bar{F}_1 - \cdots - \bar{F}_6)_{\text{Pic}(X \cap H)} \cong E_6(-1)$, $\{\bar{g}(\beta_1),\ldots,\bar{g}(\beta_6)\}$ is also a canonical basis. By [Dol12] Theorem 8.2.12 and Corollary 8.2.15 there exists a unique $\bar{g}' \in W(E_6)$ so that $\bar{g}'$ fixes $3\bar{F}_0 - \bar{F}_1 - \cdots - \bar{F}_6$ and $\bar{g}' = \bar{g}$ when restricting to the orthogonal complement $\langle 3\bar{F}_0 - \bar{F}_1 - \cdots - \bar{F}_6 \rangle_{\text{Pic}(X \cap H)}^\perp$. It follows that $\bar{g} = \bar{g}'$ for a sublattice of finite index. Since $\text{Pic}(X \cap H)$ is torsion free we get $\bar{g} = \bar{g}'$.

The automorphisms $s_{\beta_i}$ induces $s_{\beta_i}^* \in O(b_M) \cong O(b_T)$. Specifically, $s_{\beta_i}^*$ corresponds to the matrix (with respect to the basis $\{[F_1]^*,\ldots,[F_6]^*\}$)

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}$$

and $s_{\beta_j}^*$ ($2 \leq j \leq 6$) swaps $[F_{j-1}]^*$ with $[F_j]^*$. The quadratic form $q_T$ has been computed in the proof of Lemma 2.11. One easily verifies that $s_{\beta_i}^* \in O(q_T)$. 


Lemma 2.13}) one gets $AGZV88$). To prove this we use $Ebe84$ and $Nik79$ Lemma 2.14 and $\Box$

Proof. The proof is similar to that of $\Box$ (which acts by permuting the transversely), and $\Box$ Proposition 2.15. O the subgroup of isometries of $T$ $\quad$ Because $\quad$ the singularity $O_{16}$ $\quad$ and $\quad$ lines on the cubic surfaces $\quad$ Another property of $\quad$ Consider the natural exact sequence $\quad$ where $\quad$ is right split and thus $\quad$ \quad$\Box$

Theorem 2.16. The lattice $T$ is isometric to the Milnor lattice associated to the singularity $O_{16}$ and satisfy $O^*(T) = W(T)$ (the Weyl group). Geometrically, $O^*(T)$ is the monodromy associated to $O_{16}$, $O^+(T)$ is the monodromy for the pairs $(X, H)$ (consisting of a smooth cubic threefold $X$ and a hyperplane $H$ intersecting $X$ transversely), and $W(E_6) \cong O^+(T)/O^*(T)$ represents the monodromy at infinity (which acts by permuting the 27 lines on the cubic surfaces $X \cap H$).

Proof. The proof is similar to that of $\Box$ [Laz09a, Prop. 4.22]. A singularity of type $O_{16}$ is the germ given by $f(x_1, x_2, x_3, x_4) = 0, 0) \subset (C^4, 0)$ where $f$ is a homogeneous polynomial of degree 3 (i.e. it comes from the cone over a cubic surface). The singularity $O_{16}$ is quasihomogeneous and has corank 4 (see $[AGZV85, \S11.1]$) and Milnor number $\mu = 16$. Any singularity of class $O_{16}$ has a $\mu$-constant deformation to $(x_1^3 + x_2^3 + x_3^3 + x_4^3, 0)$. The Milnor lattice can be computed using the theorems by Thom and Sebastiani and by Gabrielov (cf. $[AGZV88, \S2.7]$). By $[Ebe84, Thm. 5.5]$ $O^*(T)$ is the local monodromy group of $O_{16}$. Because the monodromy group is generated by Picard-Lefschetz transformations in the vanishing cycles (see for example $[AGZV88, \S2.3]$) we get $O^*(T) = W(T)$.

Consider the natural exact sequence $1 \to \overline{O}(T) \to O(T) \to O(q_T) \to 1$.

Because $\Box$ $O(q_T) \cong W(E_6)$ (see Lemma 2.13) one gets $O(T)/\overline{O}(T) \cong W(E_6)$. Moreover, by $\Box$ Proposition 2.12, Lemma 2.14 and $[Nik79, Cor. 1.5.2]$ the exact sequence is right split and thus $O(T) = \overline{O}(T) \times W(E_6)$. Similarly, we have $O^+(T)/O^*(T) \cong W(E_6)$ and $O^+(T) = O^*(T) \times W(E_6)$. 

\[ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \]

The matrix of $s^*_i$ under this basis coincides with the matrix corresponding to the action of $r_i$ on $E_6/2E_6$.
Let $\mathcal{U} \subset \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}(3)) \times \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}(1))$ be the open subset parameterizing pairs $(X, H)$ with $X$ a smooth cubic threefold and $H$ a hyperplane intersecting $X$ transversely. Fix a base point which corresponds to a pair $(X, H)$. Let $Y$ be the associated cubic fourfold. There exists a monodromy action of $\pi_1(\mathcal{U})$ on $H^4(Y, \mathbb{Z})$ and on $T$. The monodromy group $\Pi := \text{Im}(\pi_1(\mathcal{U}) \to \text{Aut}(H^4(Y, \mathbb{Z})))$ is contained in $O^+(T)$ (cf. [Ben86]). Now we show $O^+(T) \subset \Pi$. One has $O^+(T) \subset \Pi$ because $O^+(T)$ is generated by the reflections in vanishing cycles corresponding to the degenerations of $X$. Meanwhile, $\Pi$ acts as $W(E_6)$ on the set of lines on the cubic surface $X \cap H$ (and on the corresponding planes in $Y$). It follows that $\Pi = O^+(T)$.

3. A period map for cubic pairs $(X, H)$

We have associated a smooth cubic fourfold $Y$ to a pair $(X, H)$ consisting of a smooth cubic threefold $X$ and a transverse hyperplane $H$. In this section, we shall define a period map for cubic pairs $(X, H)$ using the period map for the cubic fourfolds $Y$ and investigate the local and global Torelli problems.

Let us first review Voisin's Torelli theorem [Voi86] for smooth cubic fourfolds (see also [Has00]). Let $\mathcal{C}_0$ be the moduli space for smooth cubic fourfolds (constructed using GIT). Denote by $\Lambda = I_{21,2} = (+1)^{\mathbb{Z}21} \oplus (-1)^{\mathbb{Z}2}$ the abstract lattice isomorphic to the integral middle cohomology of a cubic fourfold, by $h$ the class of a hyperplane section, and by $A_0 = (h^2)_{\Lambda} \cong A_2 \oplus (E_8^2 \oplus U^{\oplus 2}$ the primitive cohomology. Write

$$D := \{ \omega \in \mathcal{P}(A_0 \otimes \mathbb{C}) \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} < 0 \}_{\beta}$$

(where the subscript $0$ indicates the choice of a connected component). Set $\Gamma := \{ \gamma \in O(\Lambda) \mid \gamma(h^2) = h^2 \}$ and let $\Gamma^+ \subset \Gamma$ be the index 2 subgroup stabilizing $D$ (i.e. $\Gamma^+$ consists of automorphisms of $\Lambda$ that preserve $h^2$ and the orientation of a negative definite 2-plane in $\Lambda$).

**Theorem 3.1** (Torelli theorem for cubic fourfolds [Voi86]). The period map for cubic fourfolds $\mathcal{C}_0 \to D/\Gamma^+$ is an open immersion of analytic spaces.

In our situation, we view $Y$ as “lattice polarized” cubic fourfolds which are analogues of lattice polarized K3 surfaces (cf. [Dol96]) and have been used by Hassett [Has00]. Fix a sufficiently general pair $(X_b, H_b)$. Let $Y_b$ be the associated cubic fourfold which admits an involution $\sigma$ (see (1.7)). We have described the $\sigma^*$-invariant primitive sublattice $M \subset H^4(Y_b, \mathbb{Z})$ (which are generated by the classes $[F_0], [F_1], \ldots, [F_6]$) in Section 2. Let us fix a primitive embedding $M \subset H^4(Y_b, \mathbb{Z}) \cong \Lambda$. Write the image of $[F_i]$ in $\Lambda$ by $f_i$. In what follows, we identify $M$ with its image in $\Lambda$. In other words, $M \subset \Lambda$ will be considered as an abstract lattice spanned by $f_0, \ldots, f_6$ (together with a primitive embedding into $\Lambda$). The intersection form of $M$ is given by the Gram matrix in Lemma 2.4. In particular, $M$ contains $h^2$. Again $T = M^*_\Lambda$. Let $O^+(T) \subset O(T)$ (resp. $\tilde{O}(T) \subset O(T)$) be the subgroup of isometries of $T$ preserving the orientation of a negative definite 2-plane in $T$ (resp. the subgroup of isometries of $T$ that induce the identity on $A_T$). Set $O^+(T) = O^+(T) \cap \tilde{O}(T)$.

**Remark 3.2.** The embedding $M \subset \Lambda$ depends on the choice of a marking $H^4(Y_b, \mathbb{Z}) \xrightarrow{\sim} \Lambda$. It also depends on the choice of an orthonormal basis $\{\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_6\}$ of Pic$(X_b \cap H_b) \cong I_{1,6}$ (i.e. the choices of the classes $[F_0], [F_1], \ldots, [F_6]$). By Proposition 2.12,
Lemma 2.14 and [Nik79, Cor. 1.5.2] isometries of $M$ fixing $h^2$ extend to automorphisms of $\Lambda$. As a result, our choice of the embedding $M \subset \Lambda$ is unique up to the action of $O(\Lambda)$.

**Notation 3.3.** Let us introduce the following notations.

- $\mathcal{M}_0$: the moduli space of pairs $(X, H)$ consisting of a smooth cubic threefold $X$ and a transverse hyperplane $H$ (constructed as a Zariski open subset of the GIT quotient $\mathbb{F}H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(3)) \times \mathbb{F}H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(1)) \sslash \text{SL}(5, \mathbb{C})$). Because such pairs $(X, H)$ are GIT stable, $\mathcal{M}_0$ is a geometric quotient.
- $\mathcal{D}_M := \{ \omega \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) | \omega \cdot \omega = 0, \omega \cdot \bar{\omega} < 0, \omega \cdot M = 0 \}_0$. It is convenient to identify $\mathcal{D}_M$ to the domain $\{ \omega \in \mathbb{P}(T \otimes \mathbb{C}) | \omega \cdot \omega = 0, \omega \cdot \bar{\omega} < 0 \}_0$. Note that $O^+(T)$ acts on $\mathcal{D}_M$ naturally.
- $\Gamma_M^+ := \{ \gamma \in \Gamma^+ | \gamma(h^2) = h^2, \gamma(M) \subset M \}$. By Proposition 3.4, Lemma 2.14 and [Nik79, Cor.1.5.2] $\Gamma_M^+$ can be identified with $O^+(T)$.

We define a period map for cubic pairs $(X, H) \in \mathcal{M}_0$. Let $Y$ be the associated smooth cubic fourfold. After choosing an orthonormal basis of $\text{Pic}(X \cap H) \cong I_{1,6}$ we get a primitive embedding $j : M \rightarrow H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y) \subset H^4(Y, \mathbb{Z})$ (N.B. $j$ sends $h^2$ to the square of the hyperplane class). To associate a period point on $\mathcal{D}_M$ to $Y$, one needs to choose a marking $\phi : \Lambda \rightarrow H^4(Y, \mathbb{Z})$ satisfying $\phi_M = j$. Now we consider $\phi_C^{-1}(H^{3,1}(Y))$ (where $\phi_C := \phi \otimes \mathbb{C}$) which belongs to the subdomain $\mathcal{D}_M$. One also has to take the action of $\Gamma_M^+$ (which can be identified with $O^+(T)$) on the markings $\phi$ into consideration. Specifically, suppose we choose a different orthonormal basis of $\text{Pic}(X \cap H)$ which gives a different $j'$. Then $j$ and $j'$ are related by an element $g_M$ of the stabilizer group $O(M)_{h^2}$. By Proposition 2.12, Lemma 2.14 and [Nik79, Cor.1.5.2] $g_M$ can be extended to an automorphism $g \in O(\Lambda)$ such that $g|T \in O^+(T)$. The marking $\phi \circ g$ satisfies $(\phi \circ g)|_M = j'$. Moreover, it is not difficult to verify that the $O^+(T)$-orbit of the period point does not depend on the projective equivalence class of $(X, H)$. To sum up, we get the following period map (the choice of the monodromy group is also explained in Proposition 2.15)

$$\mathcal{P}_0 : \mathcal{M}_0 \rightarrow \mathcal{D}_M/O^+(T).$$

Now let us prove the local Torelli and the global Torelli theorems for $\mathcal{P}_0$.

**Proposition 3.4.** The period map $\mathcal{P}_0 : \mathcal{M}_0 \rightarrow \mathcal{D}_M/O^+(T)$ is a local isomorphism at every point of $\mathcal{M}_0$.

**Proof.** It is enough to show the differential $d\mathcal{P}_0$ is injective (N.B. dim$\mathcal{M}_0 = \dim \mathcal{D}_M/O^+(T)$ = 14). The infinitesimal deformations of hypersurfaces in projective spaces have been well studied (see for example [CMSP03, §5.4]). Let $Y$ be the smooth cubic fourfold associated to a smooth cubic threefold $X$ and a transverse hyperplane $H$. Write the equation of $Y$ as $F = f(y_0, \ldots, y_4) + l(y_0, \ldots, y_4)\tilde{y}_5$. In particular, $X$ and $H$ are subvarieties of $(y_5 = 0) \cong \mathbb{P}^4$ and are cut out in $\mathbb{P}^4$ by $f = 0$ and $l = 0$ respectively. Let $X'$ (resp. $H'$) be a deformation of the hypersurface $X$ (resp. $H$) in $\mathbb{P}^4$. Assume the equations for $X'$ and $H'$ are $f + \epsilon f' = 0$ and $l + \epsilon l' = 0$ respectively. We view $f'$ and $l'$ as elements of $\mathbb{C}[y_0, \ldots, y_4]/(C : f)$ and $\mathbb{C}[y_0, \ldots, y_4]/(C : l)$ (where $\mathbb{C}[y_0, \ldots, y_4]$ denotes the vector space of degree $d$ homogeneous polynomials) respectively. Notice that $f + \epsilon f' + (l + \epsilon l')\tilde{y}_5 = (f + ly_5) + \epsilon(f' + l'y_5)$. Suppose $d\mathcal{P}_0(X', H') = 0$. From the local Torelli for cubic fourfolds (see for example [Voi86, §0]) we deduce that
$f + l'y_0^2$ is contained in the Jacobian ideal $J(F)$ of $F$. A direct computation shows that

$$f' = \frac{\partial f}{\partial y_0} l_0 + \cdots + \frac{\partial f}{\partial y_4} l_4 \quad \text{and} \quad l' = \frac{\partial l}{\partial y_0} l_0 + \cdots + \frac{\partial l}{\partial y_4} l_4 + 2c \cdot l$$

where $l_k$ ($0 \leq k \leq 4$) are linear polynomials in $y_0, \ldots, y_4$ and $c$ is a constant number. Thus $f' \in J(f)/(\mathbb{C} \cdot l)$ and $l' \in J(l)/(\mathbb{C} \cdot l)$. Moreover, the pencil spanned by $(X', H')$ and $(X, H)$ is tangent to the $\text{GL}(5, \mathbb{C})$-orbit through $(X, H)$ (cf. [ACT11, Lem. 1.8]).

**Theorem 3.5.** The period map $\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}_M/O^+(T)$ is an isomorphism onto its image.

**Proof.** Suppose the pairs $(X, H)$ and $(X', H')$ (denote the associated smooth cubic fourfolds by $Y$ and $Y'$ respectively) have the same image in $\mathcal{D}_M/O^+(T)$. That is to say, there exist markings $\phi : \Lambda \to H^4(Y, \mathbb{Z})$ and $\phi' : \Lambda \to H^4(Y', \mathbb{Z})$ and an isometry $g_T \in O^+(T)$ such that $g_T(\phi^{-1}_C(H^3(Y))) = \phi'^{-1}_C(H^3(Y'))$ (N.B. $\phi^{-1}_C(H^3(Y))$ and $\phi'^{-1}_C(H^3(Y'))$ both belong to $T$). Proposition 2.12 and Lemma 2.14 (see also [Nik79, Cor.1.5.2]) allow one to extend $g_T$ to an automorphism $g \in O(\Lambda)$. In particular, $g$ fixes $h^2$. Now we apply the global Torelli theorem for cubic fourfolds (see Theorem 3.1). It follows that the associated cubic fourfolds $Y$ and $Y'$ (which admit Eckardt points) are projectively equivalent. In other words, there exists a projective transform $\beta$ sending $Y$ to $Y'$. Generically, $Y$ (resp. $Y'$) admits one Eckardt point $p$ (resp. $p'$) (cf. [CC10, Thm. 2.10, Thm. 4.2 and §6]). (Using Lemma 1.6 it is not difficult to see that the locus of smooth cubic fourfolds containing Eckardt points is irreducible, so we can speak of a generic such fourfold.) Consider the extended involution $\sigma$ (resp. $\sigma'$) (see (1.7)) on the blow-up $\text{Bl}_p Y$ (resp. $\text{Bl}_p Y'$). Since $\beta$ carries the fixed locus of $\sigma$ (which is $X + H$ by Proposition 1.9) to the fixed locus of $\sigma'$ (which is $X' + H'$), $(X, H)$ and $(X', H')$ are projectively equivalent. This proves that $\mathcal{P}_0$ is generically injective. From Proposition 3.4 we know $\mathcal{P}_0$ is an local isomorphism. As a result, $\mathcal{P}_0$ is an isomorphism onto its image. \hfill \Box

**Remark 3.6.** By [Has00, Prop. 2.2.3] the period map for cubic fourfolds $\mathcal{C}_0 \to \mathcal{D}/\Gamma^+$ is an algebraic map. A similar argument shows that $\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}_M/O^+(T)$ is also algebraically defined.

**Remark 3.7.** Let $Z$ be a quasi-smooth hypersurface of degree 6 in $\mathbb{P}(1, 2, 2, 2, 2, 3)$. Note that $Z$ is a double cover of $\mathbb{P}^4$ branched over a smooth cubic threefold $X$ and a hyperplane $H$ intersecting $X$ transversely (cf. Remark 1.11). The isomorphism class of $Z$ is determined by the isomorphism class of the pair $(X, H)$ consisting of the branched data $X$ and $H$. Let $Z$ be the smooth cubic fourfold associated to $(X, H)$. The morphism $\text{Bl}_p Y \cong \text{Bl}_X H Z \to Z$ (see Proposition 1.9) allows one to identify the Hodge structures on $H^4_{\text{prim}}(Z, \mathbb{Z})$ and $T = M^+_4(Y, \mathbb{Z})$. A global Torelli theorem for weighted degree 6 hypersurfaces in $\mathbb{P}(1, 2, 2, 2, 2, 3)$ can be derived from Theorem 3.5 (compare [Sai86] and [DT87]).

4. **Special Heegner divisors in the period domain**

We introduce two Heegner divisors in the locally symmetric domain $\mathcal{D}_M/O^+(T)$, namely, the nodal Heegner divisor $H_n$ and the tangential Heegner divisor $H_t$. We
also establish a Borcherds’ relation between the Hodge line bundle on $\mathcal{D}_M/O^+(T)$ and these Heegner divisors.

The locally symmetric domain $\mathcal{D}_M/O^+(T)$ is associated to the transcendental lattice $T \cong U^{\oplus 2} \oplus D_4^{\oplus 3}$. The lattice theoretical invariants (e.g. the discriminant group $A_T$ and the discriminant quadratic form $q_T$) for $T$ have been computed in Lemma 2.11, Proposition 2.12 and Lemma 2.13. We also need the following notations.

**Notation 4.1.** Let $v$ be an element of $T$.

- $\text{div}(v)$: the positive generator of the ideal $v \cdot T \subset \mathbb{Z}$ (called the *divisibility* of $v$).
- $\tilde{v} := v/\text{div}(v)$ (viewed as an element in the discriminant group $A_T$).
- $r_v : T \otimes \mathbb{Q} \to T \otimes \mathbb{Q}$: the reflection in the hyperplane $v^\perp$ (provided that $v$ is non isotropic) defined by $r_v(x) = x - \frac{2\langle x, v \rangle}{\langle v, v \rangle} v$.

For later use, let us choose a set of generators for $T \cong U^{\oplus 2} \oplus D_4^{\oplus 3}$. Denote a standard basis for the first copy of $U$ (resp. the second copy of $U$) by $e_1$ and $f_1$ (resp. $e_2$ and $f_2$). In particular, $e_i^2 = f_i^2 = 0$ and $e_i \cdot f_i = 1$ ($i = 1, 2$). The lattice $T$ contains 3 copies of $D_4$. For the first copy of $D_4$, we let $\alpha_1, \ldots, \alpha_4$ be the simple roots as in [Bou02, §VI.4.8]. Note that $\alpha_1^2 = \cdots = \alpha_4^2 = 2$, $\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = \alpha_2 \cdot \alpha_4 = \alpha_1 = -1$, and all the other intersection numbers are zero. The discriminant group $A_{D_4}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ which is generated by $\frac{1}{2}(\alpha_1 + \alpha_3)$ and $\frac{1}{2}(\alpha_1 + \alpha_4)$ (N.B. $\frac{1}{2}(\alpha_1 + \alpha_3) + \frac{1}{2}(\alpha_1 + \alpha_4) = \frac{1}{2}(\alpha_1 + \alpha + 4)$ in $A_T$). Also, $q_{D_4}(\frac{1}{2}(\alpha_1 + \alpha_3)) = q_{D_4}(\frac{1}{2}(\alpha_1 + \alpha_4)) = q_{D_4}(\frac{1}{2}(\alpha_3 + \alpha_4)) = 1$ (mod $2\mathbb{Z}$). Similarly, we choose a basis $\beta_1, \ldots, \beta_4$ (resp. $\gamma_1, \ldots, \gamma_4$) for the second copy (resp. the third copy) of $D_4$.

Before defining the Heegner divisors, we note the following lemmas.

**Lemma 4.2.** Let $v \in T$ be a primitive vector. Suppose that the reflection $r_v$ preserves $T$ (i.e. $r_v(T) \subset T$) and $\mathcal{D}_M \cap v^\perp \neq \emptyset$. Then there are two possibilities.

1. $v^2 = 2$ and $\text{div}(v) = 1$. In this case, $\tilde{v} = 0$ in $A_T$.
2. $v^2 = 4$ and $\text{div}(v) = 2$. In this case, $q_T(\tilde{v}) \equiv 1$ (mod $2\mathbb{Z}$).

**Proof.** The assumption $r_v(T) \subset T$ is equivalent to $v^2|2\text{div}(v)$. The intersection $\mathcal{D}_M \cap v^\perp$ is empty if $v^2 < 0$ (Hodge-Riemann bilinear relations). By the proof of Lemma 2.11 the discriminant quadratic form $q_T$ only takes values in integers. As a result, we get $\text{div}(v^2)|v^2$ (consider $q_T(\tilde{v})$). The lemma then easily follows. □

**Lemma 4.3.** Let $v$ be a primitive vector of $T$ with $v^2 = 2$ and $\text{div}(v) = 1$. Then $v$ is unique up to the action of $O^+(T)$ (and hence also unique up to the action of $O^+(T)$).

**Proof.** We use Eichler’s criterion (cf. [GHS09, Prop. 3.3] or [LO16, Prop. 1.3.2]). More precisely, let $u$ and $v$ be nonzero primitive vectors of an even lattice $L$ which contains $U \oplus U$. There exists an element $q \in O^+(L)$ such that $q(u) = v$ if and only if $u^2 = v^2$ and $\tilde{u} = \tilde{v}$. The lemma is a direct application of Eichler’s criterion (notice that $\tilde{v} = 0$ in $A_T$). □

**Lemma 4.4.** Consider primitive vectors $v \in T$ with $v^2 = 4$ and $\text{div}(v) = 2$. There are 36 equivalence classes of $v$ for the action of $O^+(T)$. Moreover, these vectors form a single $O^+(T)$-orbit.
Proof. We apply Eichler’s criterion to prove the first assertion. The discriminant quadratic form \( q_T : A_T \to \mathbb{Q}/2\mathbb{Z} \) vanishes on 28 elements and equals 1 (mod 2\( \mathbb{Z} \)) for the rest 36 elements (see the proof of Lemma 2.11). Notice that \( q_T(\tilde{v}) \equiv 1 \) (mod 2\( \mathbb{Z} \)) if \( v^2 = 4 \) and \( \text{div}(v) = 2 \). By Eichler’s criterion, there are at most 36 \( O^*(T) \)-equivalence classes for \( v \). Now we construct a representative for every class. Let \( \{e_1, e_2, f_1, f_2, a_1, \ldots, a_4, b_1, \ldots, b_4, \gamma_1, \ldots, \gamma_4 \} \) be a basis for \( T \cong U^\oplus 2 \oplus D_4^\oplus 3 \) (see the beginning of this section). Let \( i, j \in \{1, 3, 4\} \) be two distinct integers. Similarly for \( k \neq l \) and \( s \neq t \). Consider the 9 primitive vectors of the form \( \alpha_i + \alpha_j, \beta_k + \beta_l, \gamma_s + \gamma_t \) respectively. Also consider the following 27 primitive vectors: \( 2e_i - 2f_i + \alpha_i + \alpha_j + \beta_k + \beta_l + \gamma_s + \gamma_t \). All these 36 vectors satisfy \( v^2 = 4 \) and \( \text{div}(v) = 2 \). It is also easy to show that the corresponding \( \tilde{v} \in A_T \) are all different. In fact, these \( \tilde{v} \) are exactly the 36 elements for which \( q_T \) does not vanish.

Let us prove the second assertion. Let \( v_1 \) and \( v_2 \) be primitive vectors of \( T \) with \( v_1^2 = v_2^2 = 4 \) and \( \text{div}(v_1) = \text{div}(v_2) = 2 \). If \( \tilde{v}_1 = \tilde{v}_2 \) then \( v_1 \) and \( v_2 \) are \( O^*(T) \)-equivalent by Eichler’s criterion. Suppose that \( \tilde{v}_1 \neq \tilde{v}_2 \) in \( A_T \). By Lemma 2.13 and its proof, we have \( (E_6/2E_6, q) \cong (A_T, q_T) \) (recall that \( g(-) \equiv \frac{1}{6}(-, -)E_6 \) (mod 2\( \mathbb{Z} \)) which induces \( W(E_6) \cong O(q_T) \). Note that the 36 elements of \( E_6/2E_6 \) for which \( q \) does not equal to 0 corresponds to the 36 pairs of opposite roots of \( E_6 \). Because \( W(E_6) \) acts transitively on the set of roots (see for example [Dol12, §8.2]), these 36 elements form a single orbit. It follows that there exists \( \tilde{g} \in O(q_T) \) such that \( \tilde{g}(\tilde{v}_1) = \tilde{v}_2 \). Using the natural short exact sequence \( 1 \to O^*(T) \to O^+(T) \to O(q_T) \to 1 \) we lift \( \tilde{g} \) to \( g \in O^+(T) \). By Eichler’s criterion, \( g(v_1) \) and \( v_2 \) are in the same orbit of \( O^*(T) \). In other words, there is \( f \in O^+(T) \) such that \( f(g(v_1)) = v_2 \).

We introduce some Heegner divisors (cf. [Has00, §3] and [LO16, §1.3]) for \( D_M/O^*(T) \) and \( D_M/O^+(T) \). Let \( \Pi \subset O^+(T) \) be a finite-index subgroup (in our situation, \( \Pi \) is either \( O^*(T) \) or \( O^+(T) \)). Let \( \pi : D_M \to D_M/\Pi \) be the quotient map. For a nonzero \( v \in T \), we write

\[
H_v(\Pi) := \bigcup_{g \in \Pi} g(v)^\perp \cap D_M, \quad H_v(\Pi) := \pi(H_v(\Pi)).
\]

If \( v^2 > 0 \) then \( H_v(\Pi) \) is not empty and gives a hyperplane arrangement. Also, \( H_v(\Pi) \) is a prime divisor (which is also \( \mathbb{Q} \)-Cartier as discussed in [LO16, §1.3.1]) in the locally symmetric variety \( D_M/\Pi \). We call it the Heegner divisor associated to \( v \). Both \( H_v(\Pi) \) and \( H_v(\Pi) \) depend only on the \( \Pi \)-class of \( v \). Following [LO16] we say \( H_v(\Pi) \) is reflective if the reflection \( r_v \) belongs to \( \Pi \).

Consider primitive vectors \( v \in T \) of either Type (1) (i.e. \( v^2 = 2 \) and \( \text{div}(v) = 1 \)) or Type (2) (i.e. \( v^2 = 4 \) and \( \text{div}(v) = 2 \)) as in Lemma 4.2. By Lemma 4.3 there is a single \( O^*(T) \)-orbit for Type (1) vectors. We denote the corresponding Heegner divisor \( H_v(O^*(T)) \) in \( D_M/O^*(T) \) by \( H_0 \). Note that \( H_0 \) is reflective. There are 36 \( O^*(T) \)-equivalence classes of Type (2) vectors (see Lemma 4.4). Let us denote the corresponding Heegner divisors by \( H_1, \ldots, H_{36} \).

Definition 4.5.

(1) Let \( v \) be a vector of Type (1) as in Lemma 4.2. The nodal Heegner divisor \( H_0 \) in \( D_M/O^+(T) \) is \( H_v(O^+(T)) \).

(2) Let \( v \) be a vector of Type (2) as in Lemma 4.2. The tangential Heegner divisor \( H_0 \) in \( D_M/O^+(T) \) is \( H_v(O^+(T)) \).
(Thanks to Lemma 4.3 and Lemma 4.4 the definitions make sense. Moreover, both $H_n$ and $H_t$ are reflective Heegner divisors.)

**Remark 4.6.** Later we shall see that a generic point of $H_n$ (resp. $H_t$) corresponds to the a cubic pair $(X,H)$ with $X$ nodal (resp. $H$ simply tangent to $X$).

**Remark 4.7.** The work of Borcherds, Bruinier (see [Bru02] and references therein) and the refinement given in [BLMM17] allows us to compute the rank of the Picard group of $\mathcal{D}_M/O^+(T)$ (more generally, the Picard rank of certain modular varieties of type IV). More specifically, let $\mathcal{S}_{k,T}$ denote the space of (vector-valued) cusp forms of weight $k$ with values in $T$. Borcherds has defined a homomorphism $\mathcal{S}_{k,T} \to \text{Pic}(\mathcal{D}_M/O^+(T)) \otimes \mathbb{C}/\langle \lambda(O^+(T)) \rangle$ (where $\lambda(O^+(T))$ denotes the Hodge bundle, see the discussions below). Because $T$ contains two copies of $U$, the homomorphism is injective (cf. [Bru02]). By [BLMM17] this is in fact an isomorphism. A formula for computing the dimension of $\mathcal{S}_{k,T}$ is given by Bruinier [Bru02] (see also [LO16, §3.2]). Using Lemma 2.11 we get

$$\dim(\mathcal{S}_{k,T}) = 64 + \frac{64 \cdot 8}{12} - \frac{65}{3} - 18 - 28 = 21$$

and hence the Picard rank of $\mathcal{D}_M/O^+(T)$ is 22.

What will be important for us is a relation between the Hodge bundle on $\mathcal{D}_M/O^+(T)$ (resp. the Hodge bundle on $\mathcal{D}_M/O^+(T)$) and the Heegner divisors $H_n$ and $H_t$ (resp. $H_0, H_1, \ldots, H_{36}$) in $\text{Pic}(\mathcal{D}_M/O^+(T)) \otimes \mathbb{C}$ (resp. in $\text{Pic}(\mathcal{D}_M/O^+(T)) \otimes \mathbb{C}$). We call this type of relation a Borcherds’ relation. Let $\Pi$ be either $O^+(T)$ or $O^+(T)$. Recall that the Hodge bundle $\lambda(\Pi)$ is the fractional line bundle on $\mathcal{D}_M/\Pi$ defined as the quotient of $\lambda = O_{\mathcal{D}_M}(-1)$ (the restriction of the tautological line bundle on $\mathbb{P}(T \otimes \mathbb{C})$ to $\mathcal{D}_M$) by $\Pi$. By abuse of notation, we also denote the divisor class of the Hodge bundle by $\lambda(\Pi)$ (which is $\mathbb{Q}$-Cartier). The Baily-Borel compactification $(\mathcal{D}_M/\Pi)^* = \text{Proj} \bigoplus_{m \geq 0} H^0(\mathcal{D}_M/\Pi, \lambda^\otimes m)^\Pi$. The line bundle $\lambda(\Pi)$ extends to an ample fractional line bundle $\lambda(\Pi)$ on $(\mathcal{D}_M/\Pi)^*$ and the sections of $m\lambda(\Pi)$ are exactly the weight $m$ $\Pi$-automorphic forms.

We briefly describe the strategy for computing Borcherds’ relations which has been used for example in [CML09], [CMJL12] and [LO16] (see also [Kon99], [GHS07] and [TVA15]). The idea is to choose a primitive embedding of $T$ into the even unimodular lattice $\mathcal{H}_{26,2} \cong U^6 \oplus E_8^3$ of signature $(26,2)$ and study the quasi-pullback $\Phi_T$ (see [LO16, §3.3.1]) of Borcherds’ automorphic form $\Phi_{12}$ defined in [Bor95]. Let $T^\perp = T_{\mathcal{H}_{26,2}}^\perp$ be the orthogonal complement of $T$ in $\mathcal{H}_{26,2}$. For a lattice $L$ let us denote the set of roots by $R(L)$. $\Phi_T$ is an automorphism form on $\mathcal{D}_M$ for $O^+(T)$ of weight 12 + $|R(T^\perp)|/2$. The divisor of $\Phi_T$ is supported on the union of the hyperplanes $\delta^\perp \cap \mathcal{D}_M$ where $\delta \in R(\mathcal{H}_{26,2}) \setminus R(T^\perp)$:

$$\text{div}(\Phi_T) = \sum_{\pm \delta \in R(\mathcal{H}_{26,2}) \setminus R(T^\perp)} (\delta^\perp \cap \mathcal{D}_M).$$

Note that $\delta^\perp \cap \mathcal{D}_M \neq \emptyset$ if and only if the lattice $(\delta, T^\perp)$ spanned by $\delta$ and $T^\perp$ is positive definite. Given such a $\delta \in R(\mathcal{H}_{26,2}) \setminus R(T^\perp)$, we let $\nu(\delta)$ be a generator of $(\mathbb{Q}\delta \oplus \mathbb{Q}T^\perp) \cap T$ (which is unique up to a choice of the sign). As discussed in [LO16, Rmk 3.3.3], the right hand side of Equation (4.8) is a finite sum of the hyperplane arrangements $\mathcal{H}_{\nu(\delta)}(O^+(T))$. The coefficient of $\mathcal{H}_{\nu(\delta)}(O^+(T))$ is $\frac{1}{2}(|R(\text{Sat}(\delta, T^\perp))| - |R(T^\perp)|)$ where $\text{Sat}$ means taking the saturation in $\mathcal{H}_{26,2}$. 


Equation (4.8) descends to a relation between the Hodge bundle $\lambda(O^*(T))$ on $\mathcal{D}_M/O^*(T)$ and certain Heegner divisors. By pushing forward via the natural morphism $\mathcal{D}_M/O^*(T) \to \mathcal{D}_M/O^+(T)$, we obtain a Borcherds’ relation on $\mathcal{D}_M/O^+(T)$.

The proof is analogous to that of Lemma 4.9. We first choose a primitive embedding of $T \cong U^{\boxplus 2} \oplus D_4^{\boxplus 3}$ into $H_{26,2} \cong U^{\boxplus 2} \oplus E_8^{\boxplus 3}$.

**Lemma 4.9.** There exists a primitive embedding $T \hookrightarrow H_{26,2}$ with orthogonal complement $T^\perp$ isomorphic to $D_4^{\boxplus 3}$.

**Proof.** Because $T$ is isomorphic to $U^{\boxplus 2} \oplus D_4^{\boxplus 3}$ and $H_{26,2}$ is isomorphic to $U^{\boxplus 2} \oplus E_8^{\boxplus 3}$, it suffices to construct a primitive embedding of $D_4$ into $E_8$. Using Borel-de Siebenthal procedure (see for example [Dol12, §8.2]) it is easy to obtain a primitive embedding $D_4 \hookrightarrow D_8 \hookrightarrow E_8$. More explicitly, let $\delta_1, \ldots, \delta_8$ be a set of simple roots of $E_8$ (we label them as in [Bou02, §VI.4.10]). The sublattice generated by $\delta_2, \delta_3, \delta_4, \delta_5$ is isomorphic to $D_4$. The orthogonal complement $\langle \delta_2, \delta_3, \delta_4, \delta_5 \rangle_{E_8}$ is generated by $\delta_7, \delta_8, \delta = -2\delta_1 - 3\delta_2 - 4\delta_3 - 6\delta_4 - 5\delta_5 - 4\delta_6 - 3\delta_7 - 2\delta_8$ (the highest root for $E_8$) and $\delta' = 2\delta_1 + 3\delta_2 + 3\delta_3 + 4\delta_4 + 3\delta_5 + 2\delta_6 + \delta_7$ (coming from the highest root for $D_4$) which is isomorphic to $D_4$. (For any primitive embedding $D_4 \hookrightarrow E_8$, the orthogonal complement $(D_4)_{E_8}$ is in the genus of $D_4$ and hence must be isomorphic to $D_4$.) The claim then follows. $\square$

Next we show that the Heegner divisors that show up in the expression of $\text{div}(\Phi_T)$ (more precisely, its descent to $\mathcal{D}_M/O^*(T)$) are exactly the Heegner divisors $H_0, H_1, \ldots, H_36$ we introduce before.

**Lemma 4.10.** Choose an embedding of $T$ into the unimodular lattice $H_{26,2}$ as in Lemma 4.9. Suppose $\delta \in R(H_{26,2}) \setminus R(T^\perp)$ is such a root that $(\delta, T^\perp)$ is positive definite (in other words, $\delta^\perp \cap \mathcal{D}_M \neq \emptyset$). Let $\nu(\delta)$ be a generator of $(Q\delta \oplus QT^\perp) \cap T$. Then one of the following holds (compare Lemma 4.2):

1. $\nu(\delta)^2 = 2$ and $\text{div}(\nu(\delta)) = 1$;
2. $\nu(\delta)^2 = 4$ and $\text{div}(\nu(\delta)) = 2$.

**Proof.** The proof is analogous to that of [LO16] Proposition 3.3.10. Let $m$ be the minimal positive integer such that $m\delta \in T \setminus \nu(\delta)$. Write

$$m\delta = v + w$$

where $0 \neq v \in T$ and $w \in T^\perp$. Because the embedding $T \hookrightarrow H_{26,2}$ is defined for every piece of $D_4$ contained in $T \cong U^{\boxplus 2} \oplus D_4^{\boxplus 3}$, we have $m \in \{1, 2, 4\}$ (N.B. $[E_8 : D_4 \oplus D_4] = 4$). Note also that $v \in \langle \nu(\delta) \rangle$ and $\nu(\delta) = \pm v$ if and only if $v$ is primitive. Because $(\delta, T^\perp)$ is positive definite, we get $v^2 > 0$ and $w^2 \geq 0$. If $w^2 = 0$, then $\nu(\delta) = \pm \delta$ and Case (1) holds. Since $2m^2 = v^2 + w^2$, one of the following holds:

(i) $m = 1$, $v^2 = 2$, and $\nu(\delta) = \pm v$;
(ii) $m = 2$, $v^2 \in \{2, 4, 6\}$, $\nu(\delta) = \pm v$, and $\text{div}(v)$ is either 2 or 4 in $T$;
(iii) $m = 4$, $v^2 \in \{2, 4, 6, \ldots, 30\}$, $\nu(\delta) = \pm v$, and $\text{div}(v)$ equals 4 in $T$;
(iv) $m = 4$, $v^2 \in \{8, 16, 18, 24\}$, and $v$ is not primitive.

Let us do a case by case analysis.

(i) Suppose that (i) holds, then we have Case (1).
(ii) Suppose that (ii) holds. Because the discriminant quadratic form $q_T$ takes values in integers, the only possibility is $v^2 = 4$, $\nu(\delta) = \pm v$, and $\text{div}(v)$ equals 2 in $T$. This is Case (2).
(iii) We claim (iii) can not happen. It contradicts the fact that $A_T \cong (\mathbb{Z}/2\mathbb{Z})^6$.

(iv) We claim (iv) can not happen. Suppose (iv) holds. Let us observe that $w$ must be primitive. When $v^2 = 18$, this is clear. When $v^2 \in \{8, 16, 24\}$, $w^2 \in \{8, 16, 24\}$. If $w$ is not primitive, then one has $v = 2u$ and $w = 2z$ for $u \in T$ and $z \in T^\perp$. Thus, $2\nu(\delta) = u + z$ which contradicts our assumption that $m$ is minimal. But then $\text{div}(w) = 4$ in $T^\perp$ which is impossible ($A_{T^\perp} \cong A_T \cong (\mathbb{Z}/2\mathbb{Z})^6$ does not have 4-torsion elements).

□

Now let us compute Borcherds’ relations.

**Proposition 4.11.** In the $\mathbb{Q}$-Picard group $\text{Pic}(D_M/O^+(T))_\mathbb{Q}$ we have

$$\lambda(O^+(T)) \sim H_n + 2H_t$$

where $D \sim D'$ means $D = cD'$ for some $c \in \mathbb{Q}$.

*Proof.* Let us embed $T$ into $II_{26,2}$ as in Lemma 4.9. Let $\Phi_T$ be the quasi-pullback of Borcherds’ automorphism form $\Phi_{12}$ (see [LO16, §3.3.1]). Then $\Phi_T$ is automorphic form on $D_M$ for $O^+(T)$. The weight of $\Phi_T$ is $12 + |R(T^\perp)/2 = 12 + |R(D_3^{23})|/2 = 12 + 72/2 = 48$ (recall that $|R(D_3^k)| = 2k(k - 1)$). Set $\delta \in R(II_{26,2}) \setminus R(T^\perp)$ to be a root with $\langle \delta, T^\perp \rangle$ positive definite and denote a generator of $(\mathbb{Q}\delta \oplus \mathbb{Q}T^\perp) \cap T$ by $\nu(\delta)$. Then $\Phi_T$ vanishes on a union of hyperplane arrangements $\mathcal{H}_{\nu(\delta)}(O^+(T))$ (cf. [LO16, Rmk. 3.3.3]). By Lemma 4.10, the vectors $\nu(\delta) \in T$ defining these hyperplane arrangements are exactly the vectors we consider in Lemma 4.2. Type (1) vectors in Lemma 4.2 form a single $O^+(T)$-orbit and there are 36 orbits for Type (2) vectors. For every orbit it is easy to find a vector which can be realized as $\nu(\delta)$ for some $\delta \in R(II_{26,2}) \setminus R(T^\perp)$. The vanishing order of $\Phi_T$ of $\nu(\delta)$ is equals $\frac{1}{2}(|R(Sat(\delta, T^\perp))| - |R(T^\perp)|).$ If $\nu(\delta)$ is of Type (1), then the lattice $\langle \delta, T^\perp \rangle$ is saturated and isomorphic to $A_1 \oplus D_3^{23}$. If $\nu(\delta)$ is of Type (2), then the saturation of the lattice $\langle \delta, T^\perp \rangle$ is $D_5 \oplus D_3^{22}$. Note also that $H_6$ is reflexive. Putting everything together, we get an expression (in $\text{Pic}(D_M/O^+(T))_\mathbb{Q}$) for the descent of $\lambda(\Phi_T)$ from $D_M$ to $D_M/O^+(T)$:

$$48\lambda(O^+(T)) = \frac{1}{2}H_0 + 8(H_1 + \cdots + H_{36})$$

Now let us push forward this relation via $D_M/O^+(T) \to D_M/O^+(T)$. This means one needs to divide the Borcherds’ relation by the ramification order for the Heegner divisors $H_t$ which is 8 (by Proposition 5.4 a generic point of $H_t$ corresponds to a cubic fourfold with a pair of conjugate $A_1$ singularities and hence the local monodromy group has order 8: the product of Weyl groups for the two $A_1$ singularities and an involution interchanging the two singularities). Thus we obtain $\lambda(O^+(T)) \sim H_n + 2H_t$. □

5. Extending the period map

We compactify the period map $P_0 : \mathcal{M}_0 \to D_M/O^+(T)$ defined in Section 3. Specifically, we show that a certain GIT compactification of the moduli $\mathcal{M}_0$ of the pairs $(X, H)$ is isomorphic to the Baily-Borel compactification of the locally symmetric domain $D_M/O^+(T)$.

The natural parameter space for cubic pairs $(X, H)$ consisting of a cubic threefold $X$ and a hyperplane $H$ is

$$P := \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}(3)) \times \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}(1)) \cong \mathbb{P}^{34} \times \mathbb{P}^4$$
on which the group $G := \text{SL}(5, \mathbb{C})$ acts diagonally. Thus we consider the GIT quotient

$$\mathbf{P} // G = \text{Proj} \bigoplus_{m \geq 0} H^0(\mathbf{P}, \mathcal{L}^{\otimes m})^G$$

where $\mathcal{L}$ is an ample $G$-linearized line bundle. The dependence of the GIT quotient on the choice of an ample $G$-linearized line bundle was studied by Thaddeus [Tha96] and Dolgachev and Hu [DH98]. Note that $\text{Pic}^G(\mathbf{P}) \cong \text{Pic}(\mathbf{P}) \cong \mathbb{Z} \times \mathbb{Z}$. Following [Laz09a, Def. 2.2], an ample $G$-linearized line bundle $\mathcal{L}$ is said to be of slope $t \in \mathbb{Q}_{\geq 0}$ if $\mathcal{L} \cong \pi_1^* \mathcal{O}_{\mathbb{P}^3}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^4}(b)$ with $t = \frac{b}{a}$. By [Tha96] and [DH98] the quotient $\mathbf{P} // G$ only depends on the slope $t$ of $\mathcal{L}$. We denote the corresponding GIT by $\mathcal{M}(t)$ or $\mathbf{P} // G$. The lower and upper bounds for $t$ are $0$ and $\frac{4}{5}$ respectively (i.e. $\mathcal{M}(t) = \emptyset$ if $t < 0$ or $t > \frac{4}{5}$). Let $(X, H)$ be a $t$-semistable pair. As $t$ increases, $X$ may become more singular but we require better transversality for $X \cap H$. More precisely, we have the following proposition.

**Proposition 5.1.** Let $(X, H) \in \mathcal{P}$. Then there exists an interval (possibly empty) $[a, b] \subset [0, \frac{4}{5}]$ such that $(X, H)$ is $t$-semistable if and only if $t \in [a, b] \cap \mathbb{Q}_{\geq 0}$. Also, if $(X, H)$ is $t$-stable for some $t$ then it is $t$-stable for all $t \in (a, b) \cap \mathbb{Q}_{\geq 0}$. Furthermore,

1. $a = 0$ if and only if $X$ is a GIT semistable cubic threefold (which has been described in [All03]);
2. $b = \frac{4}{5}$ if and only if $X \cap H$ is a GIT semistable cubic surface (see for example [Muk03, §7.2]).

**Proof.** See [Laz09a, Thm. 2.4] or [GMG16, Cor. 3.3, Lem. 4.1].

By Proposition 5.1 the cubic pairs $(X, H)$ with $X$ at worst nodal (i.e. admitting nodes, or equivalently, $A_1$ hypersurface singularities, cf. [AGZV85, §15]) and $H$ at worst simply tangent to $X$ (that is, $X \cap H$ admits one $A_1$ singularity; in particular, $H$ does not pass through any singular point of $X$) are stable for any $t \in (0, \frac{4}{5})$. Let $\mathcal{M} \subset \mathcal{M}(t)$ ($t \in (0, \frac{4}{5})$) be the moduli space of cubic pairs $(X, H)$ consisting of a cubic threefold $X$ with at worst nodal singularities and a hyperplane $H$ which is at worst simply tangent to $X$. Note that $\mathcal{M}$ is a geometric quotient and a quasi-projective variety.

For $(X, H) \in \mathcal{M}$ we also consider the cubic fourfold $Y$ defined in (1.1). For projectively equivalent pairs $(X, H)$ the corresponding cubic fourfolds $Y$ are also projectively equivalent. When $X$ has a node and $H$ is in general position (i.e. $H$ avoids the node and intersects $X$ transversely), the cubic fourfold $Y$ also has a node. When $X$ is smooth and $H$ is simply tangent to $X$, the cubic fourfold $Y$ admits a pair of $A_1$ singularities which are conjugate to each other with respect to the involution $\sigma : [y_0, \ldots, y_4, y_5] \mapsto [y_0, \ldots, y_4, -y_5]$.

We recall the results of Looijenga [Loo09] and the first author [Laz09b, Laz10] on the image of the period map $\mathcal{C}_0 \to \mathcal{D}/\Gamma^+$ for smooth cubic fourfolds. Notations as in Section 3. For a saturated rank 2 sublattice $K \subset \Lambda$, we consider the hyperplane $\mathcal{D}_K := \{ \omega \in \mathcal{D} | \omega \perp K \}$. Geometrically, $\mathcal{D}_K$ corresponds to certain special cubic fourfolds (see [Has00, §3.1, §4]). The hyperplane $\mathcal{D}_K$ is said to be of discriminant $d = \text{det}(K)$. The hyperplanes $\mathcal{D}_K$ of a given discriminant $d$ form an arithmetic arrangement with respect to $\Gamma^+$ (cf. [Has00, §3.2]). Let $\mathcal{H}_\infty \subset \mathcal{D}$ (resp. $\mathcal{H}_\Delta \subset \mathcal{D}$) be the arrangement of hyperplanes of discriminant 2 (resp. 6). Note that $\mathcal{H}_\infty/\Gamma^+$ and $\mathcal{H}_\Delta/\Gamma^+$ are irreducible hypersurfaces in $\mathcal{D}/\Gamma^+$. According to [Laz10, Thm.
Lemma 1.3. Let \( \Lambda \) be a saturated rank 2 sublattice with discriminant 2. Denote the corresponding hyperplane in \( D \) by \( D_K \) which is a member of \( \mathcal{H}_\infty \). By [Has00, §4.4] \( K \) is generated by \( h^2 \) and an element \( x \) satisfying \( x \cdot h^2 = 1 \) and \( x^2 = 1 \). Suppose that \( D_M \cap D_K \neq \emptyset \) in \( D \). Then the lattice generated by \( M \) and \( x \) is positive definite (Hodge-Riemann bilinear relations). Recall that \( M \subset \Lambda \) is generated by \( f_0, \ldots, f_6 \) with the intersection form given by the Gram matrix in Lemma 2.4. A direct computation shows that \( x \cdot f_0 = 0, 1 \) or 2 and \( x \cdot f_i = 0 \) or 1 (1 \( \leq i \leq 6 \)). (For example, the Gram matrix of the lattice generated by \( h^2, x \) and \( f_i \) is as follows.

\[
\begin{pmatrix}
3 & 1 & 1 \\
1 & 1 & x \cdot f_i \\
1 & x \cdot f_i & 3
\end{pmatrix}
\]

Because the matrix is positive definite, one has \( x \cdot f_i = 0 \) or 1.) Note that \( h^2 = 3f_0 - f_1 - \cdots - f_6 \) and \( h^2 \cdot x = 1 \). There are two possibilities: either \( x \cdot f_0 = 1 \) (and \( x \cdot f_i = 1 \) for two \( f_i \)'s) or \( x \cdot f_0 = 2 \) (and \( x \cdot f_i = 1 \) for five \( f_i \)'s). But in both cases the Gram matrix of the lattice generated by \( x \) and \( M \) has determinant 0 which is a contradiction.  

Denote the moduli of cubic fourfolds with at worst simple singularities by \( C \) (cf. [Laz09b, Thm. 1.1]). By [Laz10, Thm. 1.1] the period map extends to an isomorphism \( C \longrightarrow (D \setminus \mathcal{H}_\infty)/\Gamma^+ \) with image the complement of \( \mathcal{H}_\infty \). In particular, \( C \setminus C_0 \) is mapped to \( \mathcal{H}_\Delta/\Gamma^+ \). A generic point of \( \mathcal{H}_\Delta/\Gamma^+ \) corresponds to a cubic fourfold with \( A_1 \) singularities (see also [Voi86, §4] and [Has00, §4.2]).

We analyze how \( \mathcal{H}_\Delta \) intersects with \( D_M \). By the previous paragraph, generically \( \mathcal{H}_\Delta \) parameterizes cubic fourfolds admitting \( A_1 \) singularities. Let \( Y \) be a cubic fourfold coming from a cubic pair \((X,H)\). If \( Y \) is singular then either \( X \) is singular or \( H \) is tangent to \( X \) (see the proof of Lemma 1.3). Recall that \( \mathcal{M} \) parameterizes cubic pairs \((X,H)\) with \( X \) at worst nodal and \( H \) at worst simply tangent to \( X \). There are two natural geometric divisors of \( \mathcal{M} \): cubic pairs with \( X \) admitting at least one node and \( H \) in general position (call the closure \( \Sigma \)) and cubic pairs with \( X \) smooth and \( H \) simply tangent to \( X \) (call the closure \( V \)). Now we extend the period map \( P_0 \) from \( \mathcal{M}_0 \) to \( \mathcal{M} \) and match \( \Sigma \) (resp. \( V \)) with the nodal Heegner divisor \( H_n \) (resp. the tangential Heegner divisor \( H_t \)) introduced in Section 4. As a result, the intersection of \( \mathcal{H}_\Delta \) and \( D_M \) produces the Heegner divisors \( H_n \) and \( H_t \).

Proposition 5.3. The period map \( P_0 : \mathcal{M}_0 \to D_M/O^+(T) \) extends to a morphism \( P : \mathcal{M} \to D_M/O^+(T) \).

Proof. We apply the removability singularity theorem (see for example [Gri84, p. 41]) and [Laz10, Proposition 3.2]. Let \( o \in \mathcal{M} \setminus \mathcal{M}_0 \) correspond to a pair \((X_0,H_0)\). The corresponding cubic fourfold \( Y_o \) has at worst \( A_1 \) singularities. The statement is analytically local at \( o \), and stable by finite base change. Since \( \mathcal{M} \) is a geometric quotient
we can assume (after shrinking and a possible finite cover) that a neighborhood of \( o \) in \( M \) is a 14-dimensional ball \( B \). We can further assume that there exists a family of cubic pairs over \( B \) (and hence there is a family of cubic fourfolds \( Y \to B \) with at worst \( A_1 \) singularities and fiber \( Y_0 \) over \( o \)). Let \( \Omega \) be the discriminant hypersurface. Over \( B \setminus \Omega \) the family \( Y \) gives a variation of Hodge structure on \( \Lambda_0 \cong H^4_{prim}(Y, \mathbb{Z}) \) (which defines the period map \( B \setminus \Omega \to D/\Gamma^+ \)) and a variation of Hodge structure on the transcendental part \( T \) (which corresponds to \( P_0 : B \setminus \Omega \to D_{\mathbb{P}^4}/O^+(T) \)). By the removable singularity theorem, the proposition is equivalent to the monodromy representation \( \pi_1(B \setminus \Omega, t) \to \text{Aut}(T) \) (where \( t \in B \setminus \Omega \)) having finite image. By [Laz10, Proposition 3.2] the period map \( B \setminus \Omega \to D/\Gamma^+ \) extends across the point \( o \). It follows that the local monodromy for the variation of Hodge structure on \( \Lambda_0 \) around \( Y_0 \) is finite. Consider the restriction of the monodromy group to \( T \). We conclude that \( P_0 : M_0 \to D_{\mathbb{P}^4}/O^+(T) \) has finite local monodromy.

We compare the geometric divisor \( \Sigma \) (resp. \( V \)) with the nodal Heegner divisor \( H_n \) (resp. the tangential Heegner divisor \( H_t \)).

**Lemma 5.4.** The generic point of the Heegner divisor \( H_n \) (resp. \( H_t \)) corresponds to a cubic pair \((X, H)\) with \( X \) admitting an \( A_1 \) singularity and \( H \) in general position (resp. a cubic pair \((X, H)\) with \( X \) smooth and \( H \) simply tangent to \( X \)) via the extended period map \( P \).

**Proof.** If \( X \) has an \( A_1 \) singularity and \( H \) is a general hyperplane, then the associated cubic fourfold \( Y \) has a single \( A_1 \) singularity. The limiting mixed Hodge structures of nodal cubic fourfolds have been studied in [Voi86, §4] and [Has00, §4.2]. Specifically, we project \( Y \) from the node. The surface parameterizing the lines of \( Y \) through the node is a \( K3 \) surface \( S \) (which is the complete intersection of a quadratic and a cubic in \( \mathbb{P}^4 \)). The desingularization of \( Y \) is isomorphic to the blow-up of \( \mathbb{P}^4 \) along \( S \). This (together with the Clemens-Schmid sequence) induces an embedding of \( H^2(S)(-1) \) into the limiting mixed Hodge structure \( H^4_{\text{lim}}(Y) \). The orthogonal complement is generated by a vector \( v_n \in T \) with \( v_n^2 = 2 \). Thus, the corresponding period point belongs to \( H_n \).

Suppose \( X \) is smooth and \( H \) is simply tangent to \( X \). The associated cubic fourfold \( Y \) has a pair of \( A_1 \) singularities \( p_1 \) and \( p_2 \). Note that \( p_1 \) and \( p_2 \) are conjugate to each other under the involution \( \sigma : [y_0, y_1, y_2, y_3, y_4, y_5] \mapsto [y_0, y_1, y_2, -y_3, y_4, -y_5] \). Let us project \( Y \) from \( p_1 \) and from \( p_2 \) to a common complementary hyperplane \( (y_5 = 0) \) in \( \mathbb{P}^5 \). As in the previous paragraph (see also [Laz10, Prop. 3.8]), we get a nodal \( K3 \) surface for \( p_1 \) and a nodal \( K3 \) surface for \( p_2 \). It is not difficult to prove that these \( K3 \) surfaces coincide. We blow up the node of the \( K3 \) surface (call the smooth \( K3 \) surface \( S \) and the class of exceptional curve \( e \)), embed \( H^2(S)(-1) \) into \( H^4_{\text{lim}}(Y) \), and take the orthogonal complement (call the generator \( v \)). Then \( v_t := e + v \) is a vector of \( T \) with \( v_t^2 = 4 \) and \( \text{div}(v_t) = 2 \). The lemma then follows.

(We also observe the following connection between \((X, H)\) and \( v_t \). Note that the cubic surface \( X \cap H \) has one \( A_1 \) singularity. It is classically known that \( X \cap H \) contains 21 lines (there are 6 lines through the node which are limits of 6 pairs of lines on a smooth cubic surface). In other words, a (marked) cubic surface containing one node determines a double-six (cf. [Dol12, §9.1]) of lines. By [Dol12, Lem. 9.1.2] a double-six corresponds to a pair of opposite roots \( \pm \alpha \) of \( E_6 \). Define \( q : E_6/2E_6 \to \mathbb{Z}/2\mathbb{Z} \) by \( q(-) \equiv \frac{1}{2}((-,-)_{E_6}) \pmod{2\mathbb{Z}} \) (cf. Lemma 2.13). The roots \( \pm \alpha \) gives an element \( \bar{\alpha} \in E_6/2E_6 \) with \( q(\bar{\alpha}) \equiv 1 \pmod{2\mathbb{Z}} \). Recall that we have
Now let us show that the GIT compactification \( \mathcal{M}(\mathcal{T}) \) is isomorphic to the Baily-Borel compactification of \( D_M/O^+(T) \). We follow the general framework developed by Looijenga [Loo03a, Loo03b]. Note that we get the Baily-Borel compactification because there is no hyperplane arrangement missing in the image of the extended period map \( \mathcal{P} \).

**Theorem 5.5.** The period map \( \mathcal{P}_0 : \mathcal{M}_0 \to D_M/O^+(T) \) extends to an isomorphism \( \mathcal{M}(\mathcal{T}) \cong (D_M/O^+(T))^* \) where \( (D_M/O^+(T))^* \) denotes the Baily-Borel compactification of \( D_M/O^+(T) \).

**Proof.** Consider the open subset \( \mathcal{U}' \) of \( \mathcal{P} = \mathbb{P} H^0(\mathbb{P}^4, O(3)) \times \mathbb{P} H^0(\mathbb{P}^4, O(1)) \) parameterizing cubic pairs \((X, H)\) with \( X \) at worst nodal and \( H \) at worst simply tangent to \( X \). Clearly, \( \mathcal{U}' \subset \mathcal{P} \) is invariant under the action of \( G = \text{SL}(5, \mathbb{C}) \) and \( \mathcal{M} = \mathcal{U}'/G \). Moreover, the complement \( \mathcal{P} \setminus \mathcal{U}' \) has codimension higher than 1 (N.B. \( \mathcal{M}(t) \) and \( \mathcal{M}(t') \) only differs in codimension 2 for \( t \neq t' \)). Using the extended period map \( \mathcal{P} : \mathcal{M} \to D_M/O^+(T) \) we identify a \( G \)-invariant open subset \( \mathcal{U} \subset \mathcal{U}' \) with an \( O^+(T) \)-invariant open subset \( D_M^0 \subset D_M \). Specifically, a cubic pairs \((X, H) \in \mathcal{U}' \) belongs to \( \mathcal{U} \) if the associated cubic fourfold \( Y \) has exactly one Eckardt point and its \( G \)-orbit corresponds to a smooth point of \( \mathcal{U}'/G \). Again \( \mathcal{P} \setminus \mathcal{U} \) has codimension bigger than 1 (cf. [CC10, Thm. 2.10 and §6]). The restriction of the extended period map \( \mathcal{P} \) to \( \mathcal{U}/G \) is an isomorphism onto its image \( D_M^0/O^+(T) \) (\( \mathcal{P}|_{\mathcal{U}/G} \) is injective by \textbf{Theorem 3.5} and [Laz10, Thm. 1.1], and it is a local isomorphism by the same argument in the proof of \textbf{Proposition 3.4}). Next we argue that the codimension of \( D_M^0 \) in \( D_M \) is larger than 1. According to [Laz10, Thm. 1.1] (see also \textbf{Proposition 2.8}), a period point \( \omega \in D_M \) which is not contained in the arrangement of hyperplanes \( \mathcal{H}_\Delta \) or \( \mathcal{H}_\infty \) corresponds to a cubic pair \((X, H)\) where \( X \) is smooth and \( H \) is transverse to \( X \). \textbf{Lemma 5.2} tells us that \( \mathcal{H}_\infty \) does not meet \( D_M \). A generic period point \( \omega_\Delta \in \mathcal{H}_\Delta \) corresponds to a cubic fourfold \( Y \) with \( A_1 \) singularities (cf. [Voi86, §4], [Has00, §4.2] and [Laz10, Thm. 1.1]). Suppose \( Y \) comes from a cubic pair \((X, H) \in \mathcal{U} \). Then either \( X \) is nodal or \( H \) is simply tangent to \( X \). Thus, \( \omega_\Delta \) corresponds to a generic point of the geometric divisor \( \Sigma \) or \( V \). Now it suffices to show that the extended period map \( \mathcal{P} \) preserves the polarizations. We have computed the Borchers' relation between the Hodge bundle \( \lambda(O^+(T)) \) on \( D_M/O^+(T) \) and the Heegner divisors \( H_n \) and \( H_t \) in Section 4. From \textbf{Proposition 4.11} we get \( \lambda(O^+(T)) \sim H_n + 2H_t \). We have also matched the geometric divisor \( \Sigma \) (resp. \( V \)) with the Heegner divisor \( H_n \) (resp. \( H_t \)) in \textbf{Proposition 5.4}. Write \( O(a, b) := p_1^* O(a) \otimes p_2^* O(b) \) (which is a \( G \)-linearized line bundle on \( \mathcal{P} \)). From [Ben12, Rmk. 1.2, Thm. 1.3] we deduce that \( \Sigma = O(80, 0) \) and \( V = O(32, 24) \). The line bundle \( \mathcal{L} \) corresponding to \( \lambda(O^+(T)) \) is hence \( O(144, 48) \) which has slope \( \frac{1}{3} \). This essentially completes the proof of the theorem. Indeed, recall that \( \mathcal{M}(\mathcal{T}) = \text{Proj} \bigoplus_{m \geq 0} H^0(\mathcal{P}, L^\otimes m)^G \) and \( (D_M/O^+(T))^* = \text{Proj} \bigoplus_{m \geq 0} H^0(D_M, \lambda^\otimes m)|O^+(T) \) (where \( \lambda = O_{D_M}(-1) \) is the natural automorphic bundle over \( D_M \)). The polarized isomorphism \( \mathcal{P}|_{\mathcal{U}/G} \) induces an isomorphism between \( H^0(\mathcal{U}, \mathcal{L}^\otimes m)^G \) and \( H^0(D_M^0, \lambda^\otimes m)|O^+(T) \) (see also the proof of [Loo03b, Thm. 7.6]). Because \( \mathcal{U} \subset \mathcal{P} \) has codimension higher than 1, we have \( H^0(\mathcal{U}, \mathcal{L}^\otimes m)^G \cong H^0(\mathcal{P}, \mathcal{L}^\otimes m)^G \). Similarly, \( H^0(D_M^0, \lambda^\otimes m)|O^+(T) \cong H^0(D_M, \lambda^\otimes m)|O^+(T) \). \qed
**Remark 5.6.** The stability of pairs \((X, H)\) for slope \(t = \frac{1}{3}\) is closely related to the stability of the associated cubic fourfold \(Y\). In particular, if \(Y\) has at worst simple singularities then \((X, H)\) is stable for \(t = \frac{1}{3}\).

**Appendix A. Algebraic varieties of K3 type**

The study of families of algebraic varieties whose period map takes values in the quotient of a Hermitian symmetric domain of type IV has a long and rich history. In this appendix we discuss two such classes of varieties. The difference is akin to the difference between Kunev surfaces (which have correct Hodge numbers \(h^{2,0} = 1\) but the holomorphic 2-forms vanish along curves) v.s. K3 surfaces (which have non-degenerate holomorphic 2-forms). The emphasis will be on weighted hypersurfaces (especially weighted fourfolds). In particular, we classify in Theorem A.8 families of weighted fourfolds satisfying:

1. A general member is a quasi-K3;
2. The family contains a Fermat hypersurface.

Let \(\mathbb{W}\) be a well formed weighted projective space and \(s(\mathbb{W})\) be the sum of the weights of \(\mathbb{W}\). Let \(Z\) be a quasi-smooth closed subvariety of \(\mathbb{W}\). Then, \(H^k(Z, \mathbb{Q})\) admits a pure Hodge structure of weight \(k\). Suppose \(Z\) has complex dimension \(2n\).

We say that \(Z\) is a numerical K3 if \(H^{2n}(Z, \mathbb{Q})\) is a Hodge structure of level 2 with \(h^{n+1,n-1} = 1\). In the case where \(Z = (f = 0)\) is a quasi-smooth hypersurface, we say that \(Z\) is a quasi-K3 if

\[(A.1) \quad \frac{1}{2}(\dim_{\mathbb{C}} Z) \deg(f) = s(\mathbb{W}).\]

**Lemma A.2.** Every quasi-K3 is a numerical K3.

**Proof.** We begin by recalling the following general description of the cohomology of a quasi-smooth hypersurface in a weighted projective space [Dol82]:

Let \(R = \mathbb{C}[z_0, \ldots, z_m]\) where \(z_j\) has weight \(w_j\), and \(U\) be the associated weighted projective space. Let \(d\Vol = dz_0 \wedge \cdots \wedge dz_m\) be the projective volume form and \(E = \sum_{j=0}^m z_j \frac{\partial}{\partial z_j}\) be the Euler vector field. Then, \(\Omega = i(E)d\Vol\) is a homogeneous differential form of degree \(s(U) = \sum_{j=0}^m w_j\). If \(V = (g = 0) \subset U\) is a quasi-smooth hypersurface, then \(F^m-qH^{m-1}_\text{prim}(V)\) is spanned by the Poincare residues of the algebraic differential forms \(\Omega(A) = (A\Omega)/(g^q)\) of homogeneous degree 0 (where \(A\) is a homogeneous polynomial on \(U\)). The residue of \(\Omega(A)\) belongs to \(F^{m+1-q}H^{m-1}_\text{prim}(V)\) if and only if \(A\) belongs to the Jacobian ideal \(J(g)\) of \(g\). In particular, this description of the Hodge filtration of the primitive cohomology of \(V\) implies that:

\[(A.3) \quad h^{m-1-j,j}_\text{prim}(V) = \dim(R/J(g))_{(j+1)\deg(g) - s(U)}\]

where \((R/J(g))_r\) is the subspace of homogeneous forms of degree \(\ell\) in \(R/J(g)\).

If \(Z \subset \mathbb{W}\) is a quasi-smooth hypersurface of dimension \(2n\) defined by the vanishing of a polynomial \(f\) of degree \(d\) then application of \((A.3)\) with \(m = 2n + 1\) and \(s(\mathbb{W}) = nd\) by \((A.1)\) implies that \(h^{2n-j,j}_\text{prim}(Z) = \dim(R/J(f))_{(j+1-n)d}\) which is zero for \(j < n - 1\), and equals to \(\dim(R/J(f))_0 = 1\) for \(j = n - 1\). \(\square\)

**Remark A.4.** The Fermat surface \(Z\) of degree 15 in \(\mathbb{P}(1,3,5,5)\) gives an example of a quasi-smooth hypersurface which is numerically of K3 type but not a quasi-K3. Indeed, by Equation \((A.3)\), \(h^{2,0}(Z) = \dim(R/J)_{15-14} = 1\).
Remark A.5. Each surface appearing on Reid’s list (see for instance [Rei80] or [Yon90]) of 95 weighted $K3$ hypersurfaces is a quasi-$K3$.

If $Z = (\sum z^n_j = 0)$ is a quasi-$K3$ Fermat (all $n_j > 1$) then dividing both sides of (A.1) by the degree of $f$ implies that

\[
\sum_j \frac{1}{n_j} = \frac{1}{2} \dim \mathbb{C} Z.
\]

(A.6)

In the case where $Z$ is a surface, this leads to the following 14 partitions of 1 into a sum of 4 unit fractions:

\[
\begin{align*}
1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15} \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{20} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{12} = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}.
\end{align*}
\]

(A.7)

Theorem A.8. Let $Z = (z_0^{n_0} + z_1^{n_1} + z_2^{n_2} + z_3^{n_3} + z_4^{n_4} + z_5^{n_5} = 0)$ be a quasi-$K3$ Fermat fourfold (all $n_j > 1$). Then, $2 = \frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5}$ is either a sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6}$ and hence classified by (A.7) or a sum $\alpha + \beta$ where $\alpha$ and $\beta$ are one of the following partitions:

\[
\begin{align*}
1 &= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}.
\end{align*}
\]

(A.9)

Proof. By reordering the variables as necessary, we assume that $1 > \frac{1}{n_0} \geq \cdots \geq \frac{1}{n_5}$. Then, either $\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} = 1$ or $\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} > 1$. In the first case, we have a partition $2 = \alpha + \beta$ with $\alpha$, $\beta$ from (A.9). In the second case, either $\frac{1}{n_0} = \frac{1}{n_1} = \frac{1}{2}$ and $\frac{1}{n_i} < 1$. If $\frac{1}{n_0} = \frac{1}{n_1} = \frac{1}{2}$ the remaining fractions give a partition from (A.7).

If $\frac{1}{n_0} + \frac{1}{n_1} < 1$ then $\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2}$ equals one of the following sums: $\frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{7}{6}$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{3}{3}$, or $\frac{1}{2} + \frac{1}{5} + \frac{1}{5} = \frac{7}{6}$. If $\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} = \frac{7}{6}$ then $\frac{1}{n_0} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{5}{6}$. This forces $\frac{1}{n_3} = \frac{1}{3}$. But then $\frac{1}{n_0} + \frac{1}{n_4} + \frac{1}{n_5} = 1$. Similar arguments work for the other two cases.

\[\square\]

Altogether, Theorem A.8 produces 17 families of quasi-smooth weighted fourfolds which are summarized in Table 1. For every family, a general hypersurface is a quasi-$K3$ and there exists a Fermat member.

Cases N1 (cubic fourfolds), N2 (studied in this paper) and N3 are new to dimension 4. The remaining cases N4-N17 have essentially appeared in Reid’s list of 95 weighted K3 hypersurfaces (by dropping the last two weights).

Remark A.10. The degree 12 Fermat hypersurface in $\mathbb{P}(2,3,3,4,4,6)$ is an example of numerical $K3$ type fourfold which is not a quasi-$K3$. A general hypersurface of
degree 20 in $\mathbb{P}(1, 4, 5, 5, 10, 15)$ is a quasi-K3 but there does not exist such a Fermat fourfold.

Remark A.11. More generally, if $Z$ is an almost Kähler $V$-manifold then $H^k(Z, \mathbb{Q})$ carries a pure Hodge structure of weight $k$. Accordingly, we say that $Z$ is a numerical K3 if the rational cohomology of $Z$ coincides with projective space except in the middle dimension (say $\dim_{\mathbb{C}} Z = 2n$), which is of Hodge level 2 with $h^{n+1,n-1} = 1$.

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