Proof of Lundow and Markström’s conjecture on chromatic polynomials via novel inequalities

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Abstract

It is well known that for a graph $G = (V, E)$ of order $n$, its chromatic polynomial $P(G, x)$ can be expressed as $\sum_{i=1}^{n} (-1)^{n-i} a_i x^i$, where $a_i$’s are non-negative integers. The number $\epsilon(G) = \frac{n \sum_{i=1}^{n} (n-i) a_i}{\sum_{i=1}^{n} a_i}$ is the mean size of a broken-cycle-free spanning subgraph of $G$. Lundow and Markström conjectured that $\epsilon(T_n) < \epsilon(G) < \epsilon(K_n)$ holds for any connected graph $G$ of order $n$ which is neither a tree $T_n$ of order $n$ nor the complete graph $K_n$. This conjecture is equivalent to the inequality $\epsilon(T_n, -1) < \epsilon(G, -1) < \epsilon(K_n, -1)$, where $\epsilon(G, x) = P'(G, x)/P(G, x)$. In this article, we prove this inequality and extend it to some new inequalities on chromatic polynomials. We first show that for any chordal and proper spanning subgraph $Q$ of $G$, $\epsilon(G, x) > \epsilon(Q, x)$ holds for all real $x < 0$. We then prove that $\epsilon(G, x) < \epsilon(K_n, x)$ holds for all non-complete graphs $G$ of order $n$ and all real $x < 0$, by applying the result that $(-1)^n (x-n+1) \sum_{u \in V} P(G-u, x) + (-1)^{n-1} n P(G, x) > 0$ holds for all non-complete graphs $G$ of order $n$ and all real $x < 0$. The last inequality is obtained by applying Whitney’s broken-cycle theorem and Greene and Zaslavsky’s interpretation on $a_1$ by special acyclic orientations.

Keywords: chromatic polynomial; broken cycle; acyclic orientation; combinatorial interpretation

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1 Introduction

For any loopless graph $G = (V, E)$ and any positive integer $k$, a proper $k$-coloring $f$ of $G$ is a mapping $f : V \rightarrow \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ holds whenever $uv \in E$. The

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chromatic polynomial of $G$ is the function $P(G, x)$ such that $P(G, k)$ counts the number of proper $k$-colorings of $G$ for any positive integer $k$. In this article, the variable $x$ in $P(G, x)$ is a real number and “$x < 0$” always means that $x$ is a negative real number.

The study of chromatic polynomials is one of the most active areas in graph theory. For basic concepts and properties on chromatic polynomials, we refer the reader to the monograph [2] and surveys [12, 7]. Many celebrated results on this topic have been obtained (for example, see [6, 16, 18, 17, 13, 9, 14]).

It is known that $P(G, x)$ can be expressed as $\sum_{i=1}^{n} (-1)^{n-i}a_i(G)x^i$, where $n = |V|$ is the order of $G$ and each $a_i(G)$ is a non-negative integer (for example, see [2, 10]). When there is no confusion, $a_i(G)$ is written as $a_i$ for short. Clearly $(-1)^n P(G, x) > 0$ holds for all $x < 0$.

Let $\eta : E \to \{1, 2, \ldots, |E|\}$ be a bijection. For any cycle $C$ in $G$, the path $C - e$ is called a broken cycle of $G$ with respect to $\eta$, where $e$ is the edge on $C$ with $\eta(e) \leq \eta(e')$ for every edge $e'$ on $C$. When there is no confusion, a broken cycle of $G$ is always assumed to be with respect to a bijection $\eta : E \to \{1, 2, \ldots, |E|\}$. In the celebrated Whitney’s Broken-cycle Theorem, we shall see that each coefficient $a_i$ has a combinatorial interpretation in terms of spanning forests of $G$ which contain no broken cycles.

**Theorem 1.1** ([19]). Let $G = (V, E)$ be a simple graph of order $n$. Then $a_i(G)$ is the number of spanning subgraphs of $G$ with $n - i$ edges and $i$ components which do not contain broken cycles.

As in [8], for $i = 0, 1, 2, \ldots, n-1$, we define $b_i(G)$ (or simply $b_i$) as the probability that a randomly chosen broken-cycle-free spanning subgraph of $G$ has size $i$. Then

$$b_i = \frac{a_{n-i}}{a_1 + a_2 + \cdots + a_n}, \quad \forall i = 0, 1, \ldots, n-1. \quad (1)$$

Let $\epsilon(G)$ denote the mean size of a broken-cycle-free spanning subgraph of $G$. Then

$$\epsilon(G) = \sum_{i=1}^{n} ib_i = \frac{(n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1}}{a_1 + a_2 + \cdots + a_n}. \quad (2)$$

An elementary property of $\epsilon(G)$ is given below.

**Proposition 1** ([8]). $\epsilon(G) = n + \frac{P'(G, -1)}{P(G, -1)}$ holds for every graph $G$ of order $n$.

By Proposition 1 $\epsilon(T_n) = \frac{n-1}{2}$ for any tree $T_n$ of order $n$, while

$$\epsilon(K_n) = n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \sim n - \log n - \gamma$$

as $n \to \infty$, where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant.

Lundow and Markström [8] proposed the following conjecture on $\epsilon(G)$.
Conjecture 1 (8). \(\epsilon(T_n) < \epsilon(G) < \epsilon(K_n)\) holds for any connected graph \(G\) of order \(n\) which is neither a \(T_n\) nor \(K_n\), where \(T_n\) is a tree of order \(n\) and \(K_n\) is the complete graph of order \(n\).

In this paper, we aim to extend Conjecture 1. For any graph \(G\), define the function \(\epsilon(G, x)\) as follows:

\[
\epsilon(G, x) = \frac{P'(G, x)}{P(G, x)}.
\]  

By Proposition 1, \(\epsilon(G, -1) = n + \epsilon(G, -1)\) holds for every graph \(G\) of order \(n\). Thus, for any graphs \(G\) and \(H\) of the same order, \(\epsilon(G) < \epsilon(H)\) if and only if \(\epsilon(G, -1) < \epsilon(H, -1)\). Conjecture 1 is equivalent to the statement that \(\epsilon(T_n, -1) < \epsilon(G, -1) < \epsilon(K_n, -1)\) holds for any connected graph \(G\) of order \(n\) which is neither \(K_n\) nor a tree \(T_n\).

A graph \(Q\) is said to be chordal if \(Q[V(C)] \not\sim C\) for every cycle \(C\) of \(Q\) with \(|V(C)| \geq 4\), where \(Q[V']\) is the subgraph of \(Q\) induced by \(V'\) for \(V' \subseteq V(G)\). In Section 2, we will establish the following result.

**Theorem 1.2.** For any graph \(G\), if \(Q\) is a chordal and proper spanning subgraph of \(G\), then \(\epsilon(G, x) > \epsilon(Q, x)\) holds for all \(x < 0\).

Note that any tree is a chordal graph and any connected graph contains a spanning tree. Thus, we have the following corollary which obviously implies the first part of Conjecture 1.

**Corollary 1.** For any connected graph \(G\) of order \(n\) which is not a tree, \(\epsilon(G, x) > \epsilon(T_n, x)\) holds for all \(x < 0\).

The second part of Conjecture 1 is extended to the following result.

**Theorem 1.3.** For any non-complete graph \(G\) of order \(n\), \(\epsilon(G, x) < \epsilon(K_n, x)\) holds for all \(x < 0\).

In order to prove Theorem 1.3, we will show in Section 3 that it suffices to establish the following result.

**Theorem 1.4.** For any non-complete graph \(G = (V, E)\) of order \(n\),

\[
(-1)^n(x - n + 1) \sum_{u \in V} P(G - u, x) + (-1)^{n+1}nP(G, x) > 0
\]  

holds for all \(x < 0\).

Note that the left-hand side of (4) vanishes when \(G \cong K_n\). The proof of Theorem 1.4 will be completed in Section 5 based on the interpretations for all coefficients \(a_i(G)\)'s of \(P(G, x)\) by acyclic orientations established in Section 4.
2 Proof of Theorem 1.2

A vertex \( u \) in a graph \( G \) is called a simplicial vertex if \( \{u\} \cup N_G(u) \) is a clique of \( G \), where \( N_G(u) \) is the set of vertices in \( G \) which are adjacent to \( u \). For a simplicial vertex \( u \) of \( G \), \( P(G, x) \) has the following property (see [2, 10, 11]):

\[
P(G, x) = (x - d(u))P(G - u, x),
\]

where \( G - u \) is the subgraph of \( G \) induced by \( V - \{u\} \) and \( d(u) \) is the degree of \( u \) in \( G \).

By (5), it is not difficult to show the following.

Proposition 2. If \( u \) is a simplicial vertex of a graph \( G \), then

\[
\epsilon(G, x) = \frac{1}{x - d(u)} + \epsilon(G - u, x).
\]

It has been shown that a graph \( Q \) of order \( n \) is chordal if and only if \( Q \) has an ordering \( u_1, u_2, \ldots, u_n \) of its vertices such that \( u_i \) is a simplicial vertex in \( Q[\{u_1, u_2, \ldots, u_i\}] \) for all \( i = 1, 2, \ldots, n \) (see [1, 3]). Such an ordering of vertices in \( Q \) is called a perfect elimination ordering of \( Q \). For any perfect elimination ordering \( u_1, u_2, \ldots, u_n \) of a chordal graph \( Q \), by Proposition 2

\[
\epsilon(Q, x) = \sum_{i=1}^{n} \frac{1}{x - d_Q(u_i)},
\]

where \( Q_i \) is the subgraph \( Q[\{u_1, u_2, \ldots, u_i\}] \).

Now we are ready to prove Theorem 1.2

Proof of Theorem 1.2 Let \( G \) be any simple graph of order \( n \) and \( Q \) be any chordal and proper spanning subgraph of \( G \). When \( n \leq 3 \), it is not difficult to verify that \( \epsilon(G, x) > \epsilon(Q, x) \) holds for all \( x < 0 \).

Suppose that Theorem 1.2 fails and \( G = (V, E) \) is a counter-example to this result such that \( |V| + |E| \) has the minimum value among all counter-examples. Thus the result holds for any graph \( H \) with \( |V(H)| + |E(H)| < |V| + |E| \) and any chordal and proper spanning subgraph \( Q' \) of \( H \), but \( G \) has a chordal and proper spanning subgraph \( Q \) such that \( \epsilon(G, x) \leq \epsilon(Q, x) \) holds for some \( x < 0 \).

We will establish the following claims. Let \( u_1, u_2, \ldots, u_n \) be a perfect elimination ordering of \( Q \) and \( Q_i = Q[\{u_1, \ldots, u_i\}] \) for all \( i = 1, 2, \ldots, n \). So \( u_i \) is a simplicial vertex of \( Q_i \) for all \( i = 1, 2, \ldots, n \).

Claim 1: \( u_n \) is not a simplicial vertex of \( G \).
Note that $Q - u_n$ is chordal and a spanning subgraph of $G - u_n$. By the assumption on the minimality of $|V| + |E|$, $\epsilon(G - u_n, x) \geq \epsilon(Q - u_n, x)$ holds for all $x < 0$, where the inequality is strict whenever $Q - u_n \not\cong G - u_n$.

Clearly $d_G(u_n) \geq d_Q(u_n)$. As $Q$ is a proper subgraph of $G$, $d_G(u_n) > d_Q(u_n)$ in the case that $G - u_n \cong Q - u_n$. If $u_n$ is also a simplicial vertex of $G$, then by Proposition 2

$$\epsilon(G, x) = \frac{1}{x - d_G(u_n)} + \epsilon(G - u_n, x), \quad \epsilon(Q, x) = \frac{1}{x - d_Q(u_n)} + \epsilon(Q - u_n, x),$$

implying that $\epsilon(G, x) > \epsilon(Q, x)$ holds for all $x < 0$, a contradiction. Hence Claim 1 holds.

Claim 2: $d_G(u_n) > d_Q(u_n)$.

Clearly $d_G(u_n) \geq d_Q(u_n)$. Since $u_n$ is a simplicial vertex of $Q$ and $Q$ is a subgraph of $G$, $d_G(u_n) = d_Q(u_n)$ implies that $u_n$ is a simplicial vertex of $G$, contradicting Claim 1. Thus Claim 2 holds.

For any edge $e$ in $G$, let $G - e$ and $G/e$ be the graphs obtained from $G$ by deleting $e$ and contracting $e$ respectively.

Claim 3: For any $e = u_nv \in E - E(Q)$, both $\epsilon(G - e, x) \geq \epsilon(Q, x)$ and $\epsilon(G/e, x) \geq \epsilon(Q - u_n, x)$ hold for all $x < 0$.

As $e = u_nv \in E - E(Q)$, $Q$ is a spanning subgraph of $G - e$ and $Q - u_n$ is a spanning subgraph of $G/e$. As both $Q$ and $Q - u_n$ are chordal, by the assumption on the minimality of $|V| + |E|$, the theorem holds for both $G - e$ and $G/e$. Thus this claim holds.

Claim 4: $\epsilon(G, x) > \epsilon(Q, x)$ holds for all $x < 0$. By Claim 2, there exists $e = u_nv \in E - E(Q)$. By Claim 3, $\epsilon(G - e, x) \geq \epsilon(Q, x)$ and $\epsilon(G/e, x) \geq \epsilon(Q - u_n, x)$ holds for all $x < 0$. By (7),

$$(\epsilon(G - e, x) - \epsilon(Q, x)) \times (-1)^nP(G - e, x) = (-1)^nP'(G - e, x) + (-1)^{n+1}P(G - e, x) \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)}.$$ (9)

As $(-1)^nP(G - e, x) > 0$ for all $x < 0$, $\epsilon(G - e, x) \geq \epsilon(Q, x)$ for all $x < 0$ implies that

$$(-1)^nP'(G - e, x) + (-1)^{n+1}P(G - e, x) \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)} \geq 0, \quad \forall x < 0.$$ (10)

As $u_1, \ldots, u_{n-1}$ is a perfect elimination ordering of $G - u_n$ and $\epsilon(G/e, x) \geq \epsilon(Q - u_n, x)$ holds for all $x < 0$, similarly we have:

$$(-1)^{n-1}P'(G/e, x) + (-1)^nP(G/e, x) \sum_{i=1}^{n-1} \frac{1}{x - d_{Q_i}(u_i)} \geq 0, \quad \forall x < 0.$$ (11)
As \((-1)^{n-1}P(G/e, x) > 0\) holds for all \(x < 0\), (11) implies that
\[
(-1)^nP(G/e, x) + (-1)^{n+1}P(G/e, x) \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)} = (-1)^{n+1}P(G/e, x) + \frac{1}{x - d_{Q_n}(u_n)} < 0, \quad \forall x < 0.
\] (12)

By the deletion-contraction formula for chromatic polynomials,
\[
P(G, x) = P(G - e, x) - P(G/e, x),
\]
and
\[
P'(G, x) = P'(G - e, x) - P'(G/e, x).
\]

Then (10) and (12) imply that
\[
(-1)^nP'(G, x) + (-1)^{n+1}P(G, x) \sum_{i=1}^{n} \frac{1}{x - d_{Q_i}(u_i)} > 0, \quad \forall x < 0.
\] (13)

By (7), \((\epsilon(G, x) - \epsilon(Q, x))(-1)^nP(G, x)\) is equal to the left-hand side of (13). Since \((-1)^nP(G, x) > 0\) holds for all \(x < 0\), inequality (13) implies that \(\epsilon(G, x) > \epsilon(Q, x)\) holds for all \(x < 0\).

Thus Claim 4 holds. As Claim 4 contradicts the assumption of \(G\), there are no counterexamples to this result and the theorem is proved. \(\square\)

3 An approach for proving Theorem 1.3

In this section, we will mainly show that, in order to prove Theorem 1.3, it suffices to prove Theorem 1.4. By (7), we have
\[
\epsilon(K_n, x) = \sum_{i=0}^{n-1} \frac{1}{x - i}.
\] (14)

Thus,
\[
\epsilon(K_n, x) - \epsilon(G, x) = \frac{(-1)^n}{P(G, x)} \left( (-1)^nP(G, x) \sum_{i=0}^{n-1} \frac{1}{x - i} + (-1)^{n+1}P'(G, x) \right). \quad (15)
\]

For any graph \(G\) of order \(n\), define
\[
\xi(G, x) = (-1)^nP(G, x) \sum_{i=0}^{n-1} \frac{1}{x - i} + (-1)^{n+1}P'(G, x).
\] (16)

Note that \(\xi(G, x) \equiv 0\) if \(G\) is a complete graph. For any non-complete graph \(G\) and any \(x < 0\), we have \((-1)^nP(G, x) > 0\) and so (15) implies that \(\epsilon(K_n, x) - \epsilon(G, x) > 0\) if and only if \(\xi(G, x) > 0\).
Proposition 3. Theorem 1.3 holds if and only if \( \xi(G, x) > 0 \) holds for every non-complete graph \( G \) and all \( x < 0 \).

It can be easily verified that \( \xi(G, x) > 0 \) holds for all non-complete graphs \( G \) of order at most 3 and all \( x < 0 \). For the general case, we will prove it by induction. In the rest of this section, we will give a relation between \( \xi(G, x) \) and \( \xi(G - u, x) \) in the two cases that \( d(u) = 0 \) or \( d(u) \geq 1 \), where \( u \) is a vertex of \( G \), and explain why Theorem 1.4 implies that \( \xi(G, x) > 0 \) holds for all non-complete graphs \( G \) and all \( x < 0 \).

**Lemma 1.** If \( u \) is an isolated vertex in \( G \), then

\[
\xi(G, x) = (-x)\xi(G - u, x) + \frac{(-1)^{n-1}(n-1)P(G - u, x)}{n - 1 - x}.
\]

(17)

**Proof.** As \( u \) is an isolated vertex, \( P(G, x) = xP(G - u, x) \). Thus

\[
P'(G, x) = P(G - u, x) + xP'(G - u, x).
\]

By (16),

\[
\xi(G, x) = (-1)^n xP(G - u, x) \sum_{i=0}^{n-1} \frac{1}{x - i} + (-1)^{n+1}(P(G - u, x) + xP'(G - u, x))
\]

\[
= (-x)\xi(G - u, x) + \frac{(-1)^{n}xP(G - u, x)}{x - n + 1} + (-1)^{n+1}P(G - u, x)
\]

\[
= (-x)\xi(G - u, x) + \frac{(-1)^{n-1}(n-1)P(G - u, x)}{n - 1 - x}.
\]

(18)

\[\square\]

Note that \( (-1)^{n-1}P(G - u, x) > 0 \) holds for all \( x < 0 \). Thus, if \( u \) is an isolated vertex of \( G \), then for any \( x < 0 \), by Lemma 1, \( \xi(G - u, x) > 0 \) implies that \( \xi(G, x) > 0 \). Also note that Lemma 1 can be extended to the case that \( u \) is any simplicial vertex of \( G \).

Now consider the case that \( u \) is a non-isolated vertex in \( G \). Assume that \( N(u) = \{u_1, u_2, \ldots, u_d\} \), where \( d \geq 1 \). For any \( i = 1, 2, \ldots, d - 1 \), let \( G_i \) denote the graph obtained from \( G - u \) by adding edges joining \( u_i \) to \( u_{i+1}, \ldots, u_d \). By applying the deletion-contraction formula for chromatic polynomials (see [2, 10]), it is not difficult to prove that

\[
P(G, x) = (x - 1)P(G - u, x) - \sum_{i=1}^{d-1} P(G_i, x).
\]

(19)

**Lemma 2.** Let \( G \) be a graph of order \( n \) and let \( u \) be a vertex of \( G \) with \( d(u) = d > 0 \). Then

\[
\xi(G, x) = (1 - x)\xi(G - u, x) + \sum_{i=1}^{d-1} \xi(G_i, x) + \frac{(-1)^{n} [(x - n + 1)P(G - u, x) - P(G, x)]}{n - x - 1},
\]

(20)

where \( G_1, \ldots, G_{d-1} \) are graphs defined above.
Proof. By (19), we have
\[ P'(G, x) = P(G - u, x) + (x - 1)P'(G - u, x) - \sum_{i=1}^{d-1} P'(G_i, x). \]

Thus
\[
\xi(G, x) = (-1)^n P(G, x) \sum_{i=0}^{n-1} \frac{1}{x - i} + (-1)^{n+1} P'(G, x)
\]
\[
+ (-1)^{n+1} \left[ P(G - u, x) + (x - 1)P'(G - u, x) - \sum_{i=1}^{d-1} P'(G_i, x) \right]
\]
\[
= (1 - x) \left[ (-1)^{n-1} P(G - u, x) \sum_{i=0}^{n-2} \frac{1}{x - i} + (-1)^n P'(G - u, x) \right]
\]
\[
+ \sum_{i=1}^{d-1} \left[ (-1)^{n-1} P(G_i, x) \sum_{i=0}^{n-2} \frac{1}{x - i} + (-1)^n P'(G_i, x) \right] + (-1)^{n+1} P(G - u, x)
\]
\[
+ (-1)^n \left[ \frac{(x - 1)P(G - u, x)}{x - (n - 1)} - \frac{1}{x - (n - 1)} \sum_{i=1}^{d-1} P(G_i, x) \right]
\]
\[
= (1 - x)\xi(G - u, x) + \sum_{i=1}^{d-1} \xi(G_i, x)
\]
\[
+ (-1)^{n} \frac{[(x - n + 1)P(G - u, x) - P(G, x)]}{x - (n - 1)}, \tag{21}
\]
where the last expression follows from (19) and the definitions of \( \xi(G - u, x) \) and \( \xi(G_i, x) \).

The result then follows. \( \square \)

It is known that \( \xi(G, x) > 0 \) holds for all non-complete graphs \( G \) of order at most 3 and all \( x < 0 \). To prove that \( \xi(G, x) > 0 \) holds for all non-complete graphs \( G \) of order larger than 3 and all \( x < 0 \), by Lemmas 1 and 2, it suffices to prove the following inequality for any non-complete graph \( G \) of order \( n \), some vertex \( u \) in \( G \) and all \( x < 0 \):
\[
(-1)^n ((x - n + 1)P(G - u, x) - P(G, x)) > 0. \tag{22}
\]

Note that the left-hand side of (22) vanishes when \( G \) is \( K_n \). Inequality (22) may also fail for some selections of vertex \( u \) in a non-regular graph. For example, it may fail when \( u \) is a vertex in the complete bipartite graph \( K_{p,q} \) with \( d(u) = q \), where \( p < q \). However, inequality (22) follows from the next inequality:
\[
(-1)^n (x - n + 1) \sum_{u \in V} P(G - u, x) + (-1)^{n+1} nP(G, x) > 0 \tag{23}
\]
for any non-complete graph $G = (V, E)$ of order $n$ and all $x < 0$.

By Proposition 3 and inequality (22), to prove Theorem 1.3, we can now just focus on proving inequality (23) (i.e., Theorem 1.4). The proof of Theorem 1.4 will be given in Section 5 based on the interpretations for the coefficients of chromatic polynomials established in Section 4.

4 Combinatorial interpretations for coefficients of $P(G, x)$

Let $G = (V, E)$ be any simple graph. In this section, we will construct combinatorial interpretations for the coefficients of $P(G, x)$ in terms of acyclic orientations. The result will be applied in the next section to prove inequality (23) (i.e., Theorem 1.4).

An orientation $D$ of $G$ is called acyclic if $D$ does not contain any directed cycle. Let $\alpha(G)$ be the number of acyclic orientations of a graph $G$. In [15], Stanley gave a nice combinatorial interpretation of $(-1)^n P(G, -1)$ for any positive integer $k$ in terms of acyclic orientations of $G$. In particular, he proved:

**Theorem 4.1** ([15]). For any graph $G$ of order $n$, $(-1)^n P(G, -1) = \alpha(G)$, i.e.,

\[
\sum_{i=1}^{n} a_i(G) = \alpha(G).
\]

In a digraph $D$, any vertex of $D$ with in-degree (resp. out-degree) zero is called a source (resp. sink) of $D$. It is well known that any acyclic digraph has at least one source and at least one sink. If $v$ is an isolated vertex of $G$, then $v$ is a source and also a sink in any orientation of $G$.

For any $v \in V$, let $\alpha(G, v)$ be the number of acyclic orientations of $G$ with $v$ as its unique source. Clearly $\alpha(G, v) = 0$ if and only if $G$ is not connected. In 1983, Greene and Zaslavsky[5] interpreted $a_1(G)$ by $\alpha(G, v)$.

**Theorem 4.2** ([5]). For any graph $G = (V, E)$, $a_1(G) = \alpha(G, v)$ holds for every $v \in V$.

This theorem was proved originally by using the theory of hyperplane arrangements. See [4] for three other nice proofs.

By Whitney’s Broken-cycle Theorem (i.e., Theorem 1.1), $a_i(G)$ equals the number of spanning subgraphs of $G$ with $i$ components and $n - i$ edges, containing no broken cycles of $G$. In particular, $a_1(G)$ is the number of spanning trees of $G$ containing no broken cycles of $G$. Now we have two different combinatorial interpretations for $a_1$. In the rest of this section, we will apply these two different combinatorial interpretations for $a_1$ to construct combinatorial interpretations for each $a_i(G)$, $1 \leq i \leq n$.  

9
Let \( \mathcal{P}_i(V) \) be the set of partitions \( \{V_1, V_2, \ldots, V_i\} \) of \( V \) such that \( G[V_j] \) is connected for all \( j = 1, 2, \ldots, i \) and let \( \beta_i(G) \) be the number of pairs \( (P, F) \), where

(a) \( P = \{V_1, V_2, \ldots, V_i\} \in \mathcal{P}_i(V) \);

(b) \( F \) is a spanning forest of \( G \) with exactly \( i \) components \( T_1, T_2, \ldots, T_i \) and each \( T_j \) is a spanning tree of \( G[V_j] \) containing no broken cycles of \( G \).

For any subgraph \( H \) of \( G \), let \( \tilde{\tau}(H) \) be the number of spanning trees of \( H \) containing no broken cycles of \( G \). By Theorem 1.1, \( \tilde{\tau}(H) = a_1(H) \) holds and the next result follows.

**Theorem 4.3.** For any graph \( G \) and any \( 1 \leq i \leq n \),

\[
a_i(G) = \beta_i(G) = \sum_{\{V_1, \ldots, V_i\} \in \mathcal{P}_i(V)} \prod_{j=1}^{i} \tilde{\tau}(G[V_j]). \tag{24}
\]

Now let \( V = \{1, 2, \ldots, n\} \). For any \( i : 1 \leq i \leq n \) and any vertex \( v \in V \), let \( \Omega\mathcal{P}_{i,v}(V) \) be the family of ordered partitions \( \{V_1, V_2, \ldots, V_i\} \) of \( V \) such that

(a) \( \{V_1, V_2, \ldots, V_i\} \in \mathcal{P}_i(V) \), where \( v \in V_1 \);

(b) for \( j = 2, \ldots, i \), the minimum number in the set \( \bigcup_{j-s \leq i \leq s} V_s \) is within \( V_j \).

Clearly, for any \( v \in V \) and any \( \{V_1, V_2, \ldots, V_i\} \in \mathcal{P}_i(V) \), there is exactly one permutation \( (\pi_1, \pi_2, \ldots, \pi_i) \) of \( 1, 2, \ldots, i \) such that \( (V_{\pi_1}, V_{\pi_2}, \ldots, V_{\pi_i}) \in \Omega\mathcal{P}_{i,v}(V) \).

**Example 1.** If \( |V| = 8 \) and \( \{V_1, V_2, V_3, V_4\} \in \mathcal{P}_4(V) \), where \( V_1 = \{3\} \), \( V_2 = \{2, 5, 8\} \), \( V_3 = \{4, 7\} \) and \( V_4 = \{1, 6\} \), then \( (3, 4, 2, 1) \) is the only permutation of \( 1, 2, 3, 4 \) such that \( (V_3, V_4, V_2, V_1) \in \Omega\mathcal{P}_{4,7}(V) \).

By Theorem 4.2, \( \tilde{\tau}(G[V_j]) = \alpha(G[V_j], u) \) holds for any vertex \( u \) in \( G[V_j] \) and Theorem 4.3 is equivalent to the following result.

**Theorem 4.4.** For any \( v \in V \) and any \( 1 \leq i \leq n \),

\[
a_i(G) = \sum_{(V_1, \ldots, V_i) \in \Omega\mathcal{P}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j), \tag{25}
\]

where \( m_j \) is the minimum number in \( V_j \) for \( j = 2, \ldots, i \).

Note that the theorem above indicates that the right hand side of (25) is independent of the choice of \( v \). Thus, for any \( 1 \leq i \leq n \),

\[
na_i(G) = \sum_{v \in V} \sum_{(V_1, \ldots, V_i) \in \Omega\mathcal{P}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j). \tag{26}
\]
5 Proof of Theorem \[1.4\]

From the discussion in Section 3, Theorem \[1.4\] implies Theorem \[1.3\] In this section, we will prove Theorem \[1.4\] by showing that the term \(x^i\) in the expansion of the left-hand side of \[1.1\] is in the form \((-1)^i d_i x^i\) with \(d_i \geq 0\) for all \(i = 1, 2, \ldots, n\). Furthermore, \(d_i > 0\) holds for some \(i\) when \(G\) is not complete.

We first establish the following result.

Lemma 3. Let \(G = (V, E)\) be a non-complete simple graph of order \(n \geq 3\) and component number \(c\).

(i). If \(c = 1\) and \(G\) is not the \(n\)-cycle \(C_n\), then there exist non-adjacent vertices \(u_1, u_2\) of \(G\) such that \(G - \{u_1, u_2\}\) is connected.

(ii). If \(2 \leq c \leq n - 1\), then for any integer \(i\) with \(c \leq i \leq n - 1\), there exists a partition \(V_1, V_2, \ldots, V_i\) such that \(G[V_j]\) is connected for all \(j = 2, \ldots, i\) and \(G[V_1]\) has exactly two components one of which is an isolated vertex.

Proof. (i). As \(c = 1\), \(G\) is connected. As \(G\) is non-complete, the result is trivial when \(G\) is 3-connected.

If \(G\) is not 2-connected, choose vertices \(u_1\) and \(u_2\) from distinct blocks \(B_1\) and \(B_2\) of \(G\) such that both \(u_1\) and \(u_2\) are not cut-vertices of \(G\). Then \(u_1u_2 \notin E(G)\) and \(G - \{u_1, u_2\}\) is connected.

Now consider the case that \(G\) is 2-connected but not 3-connected. Since \(G\) is not \(C_n\), there exists a vertex \(w\) such that \(d(w) \geq 3\). If \(d(w) = n - 1\), then \(G - \{u_1, u_2\}\) is connected for any two non-adjacent vertices \(u_1\) and \(u_2\) in \(G\). If \(G - w\) is 2-connected and \(d(w) \leq n - 2\), then \(G - \{w, u\}\) is connected for any \(u \in V - N_G(w)\). If \(G - w\) is not 2-connected, then \(G - w\) contains two non-adjacent vertices \(u_1, u_2\) such that \(G - \{w, u_1, u_2\}\) is connected, implying that \(G - \{u_1, u_2\}\) is connected as \(d(w) \geq 3\).

(ii). Let \(G_1, G_2, \ldots, G_c\) be components of \(G\) with \(|V(G_1)| \geq |V(G_j)|\) for all \(j = 1, 2, \ldots, c\). As \(c \leq n - 1\), \(|V(G_1)| \geq 2\). Choose \(u \in V(G_1)\) such that \(G_1 - u\) is connected. Then \(V(G_2) \cup \{u\}, V(G_1) - \{u\}, V(G_3), \ldots, V(G_c)\) is a partition of \(V\) satisfying the condition in (ii) for \(i = c\).

It remains to show that whenever (ii) holds for \(i = k\), where \(c \leq k < n - 1\), it also holds for \(i = k + 1\). Assume that \(V\) has a partition \(V_1, V_2, \ldots, V_k\) satisfying the condition in (ii). Then \(G[V_1]\) has an isolated vertex \(u\) and \(G[V'_j]\) is connected, where \(V'_i = V_i - \{u\}\). Since \(k \leq n - 2\), either \(|V'_i| \geq 2\) or \(|V'_j| \geq 2\) for some \(j \geq 2\).
If \(|V'| \geq 2\), then \(V'\) has a partition \(V'_{1,1}, V'_{1,2}\) such that both \(G[V'_{1,1}]\) and \(G[V'_{1,2}]\) are connected, implying that \(V'_{1,1} \cup \{u\}, V'_{1,2}, V_2, V_3, \ldots, V_k\) is a partition of \(V\) satisfying the condition in (ii) for \(i = k + 1\).

Similarly, if \(|V_j| \geq 2\) for some \(j \geq 2\) (say \(j = 2\)), then \(V_2\) has a partition \(V_{2,1}, V_{2,2}\) such that both \(G[V_{2,1}]\) and \(G[V_{2,2}]\) are connected, implying that \(V_1, V_{2,1}, V_{2,2}, V_3, \ldots, V_k\) is a partition of \(V\) satisfying the condition in (ii) for \(i = k + 1\).

\[\square\]

For any simple graph \(G = (V, E)\) of order \(n\), write

\[(-1)^n \left( x - n + 1 \right) \sum_{u \in V(G)} P(G - u, x) - nP(G, x) = \sum_{i=1}^{n} (-1)^i d_i x^i. \tag{27}\]

By comparing coefficients, it can be shown that

\[d_i = \sum_{u \in V(G)} [a_{i-1}(G-u) + (n-1)a_i(G-u)] - na_i(G), \quad \forall i = 1, 2, \ldots, n. \tag{28}\]

It is obvious that when \(G\) is the complete graph \(K_n\), the left-hand side of (27) vanishes and thus \(d_i = 0\) for all \(i = 1, 2, \ldots, n\). Now we consider the case that \(G\) is not complete.

**Proposition 4.** Let \(G = (V, E)\) be a non-complete graph of order \(n\) and component number \(c\). Then \(d_i \geq 0\) holds for any \(i = 1, 2, \ldots, n\), where the equality holds if and only if one of the following cases happens:

(i). \(i = n\);

(ii). \(1 \leq i \leq c - 2\);

(iii). \(i = c - 1\) and \(G\) does not have isolated vertices;

(iv). \(i = c = 1\) and \(G\) is \(C_n\).

**Proof.** We first show that \(d_i = 0\) in any one of the four cases above.

By (28), \(d_n = \sum_{u \in V} [1 + (n-1) \cdot 0] - n \cdot 1 = 0\).

It is known that for \(1 \leq i \leq n\), \(a_i(G) = 0\) if and only if \(i < c\) (see [2][10][11]). Similarly, \(a_i(G-u) = 0\) for all \(i\) with \(1 \leq i < c - 1\) and all \(u \in V\), and \(a_{c-1}(G-u) = 0\) if \(u\) is not an isolated vertex of \(G\). By (28), \(d_i = 0\) for all \(i\) with \(1 \leq i \leq c - 2\), and \(d_{c-1} = 0\) when \(G\) does not have isolated vertices.

If \(G\) is \(C_n\), then \(a_1(G) = n - 1\), \(a_0(G-u) = 0\) and \(a_1(G-u) = 1\) for each \(u \in V\), implying that \(d_1 = 0\) by (28).

In the following, we will show that \(d_i > 0\) when \(i\) does not belong to any one of the four cases.
If $G$ has isolated vertices, then $a_{c-1}(G - u) > 0$ for any isolated vertex $u$ of $G$ and

$$\sum_{u \in V} a_{c-1}(G - u) = \sum_{u \in V \text{ isolated}} a_{c-1}(G - u) > 0.$$ 

As $a_{c-1}(G) = 0$, by [25], we have $d_{c-1} > 0$ in this case. Now it remains to show that $d_i > 0$ holds for all $i$ with $c \leq i \leq n - 1$, except that $i = c = 1$ and $G$ is $C_n$.

For any $v \in V$, let $\mathcal{OP}'_{i,v}(V)$ be the set of ordered partitions $(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)$ with $V_1 = \{v\}$. As $\alpha(G[V_1], v) = 1$, for any $i$ with $c \leq i \leq n$, by Theorem 4.4,

$$a_{i-1}(G - v) = \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j),$$

(29)

where $m_j$ is the minimum number in $V_j$ for all $j = 2, \ldots, i$.

Let $s$ and $v$ be distinct members in $V$. For any $V_1 \subseteq V - \{s\}$ with $v \in V_1$, let $\alpha(G[V_1 \cup \{s\}], v, s)$ be the number of those acyclic orientations of $G[V_1 \cup \{s\}]$ with $v$ as the unique source and $s$ as one sink. Then $\alpha(G[V_1 \cup \{s\}], v, s) \leq \alpha(G[V_1], v)$ holds, where the inequality is strict if and only if $G[V_1]$ is connected but $G[V_1 \cup \{s\}]$ is not. Observe that

$$a_i(G - s) = \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V - \{s\})} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j)$$

$$\geq \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V - \{s\})} \alpha(G[V_1 \cup \{s\}], v, s) \prod_{j=2}^{i} \alpha(G[V_j], m_j)$$

(30)

$$= \sum_{(V_1', \ldots, V_i') \in \mathcal{OP}_{i,v,s}(V)} \alpha(G[V_1'], v, s) \prod_{j=2}^{i} \alpha(G[V_j'], m_j),$$

(31)

where $\mathcal{OP}_{i,v,s}(V)$ is the set of ordered partitions $(V_1', \ldots, V_i') \in \mathcal{OP}_{i,v}(V)$ with $s, v \in V_1'$. By the explanation above, inequality (30) is strict whenever $V - \{s\}$ has a partition $V_1, V_2, \ldots, V_i$ with $v \in V_1$ such that each $G[V_j]$ is connected for all $j = 1, 2, \ldots, i$ but $G[V_1 \cup \{s\}]$ is not connected.

By (28), we have

$$na_i(G) = \sum_{v \in V} \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j)$$

$$= \sum_{v \in V \{V_1, \ldots, V_i\} \in \mathcal{OP}'_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j)$$

$$+ \sum_{v \in V \{V_1, \ldots, V_i\} \in \mathcal{OP}'_{i,v}(V - \{v\}) \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j).$$

(32)
By (29),
\[ \sum_{v \in V} \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j) = \sum_{v \in V} a_{i-1}(G - v), \] (33)
and by (31),
\[ \sum_{v \in V} \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)} \alpha(G[V_1], v) \prod_{j=2}^{i} \alpha(G[V_j], m_j) \]
\[ \leq \sum_{v \in V} \sum_{s \in V - \{v\}} \sum_{(V_1, \ldots, V_i) \in \mathcal{OP}_{i,v,s}(V)} \alpha(G[V_1], v, s) \prod_{j=2}^{i} \alpha(G[V_j], m_j) \] (34)
\[ \leq \sum_{v \in V} \sum_{s \in V - \{v\}} a_i(G - s) \] (35)
\[ = (n - 1) \sum_{v \in V} a_i(G - v), \] (36)
where inequality (34) is strict if there exists \((V_1, \ldots, V_i) \in \mathcal{OP}_{i,v}(V)\) for some \(v \in V\) such that \(G[V_j]\) is connected for all \(j = 1, \ldots, i\) and \(G[V_1]\) has acyclic orientations with \(v\) as the unique source but with at least two sinks, and by (30) and (31), inequality (35) is strict if \(V\) can be partitioned into \(V_1, \ldots, V_i\) such that \(G[V_j]\) is connected for all \(j = 2, \ldots, i\) but \(G[V_1]\) has exactly two components, one of which is an isolated vertex in \(G[V_1]\).

As \(G\) is not complete, by Lemma 3 and the above explanation, the inequality of (36) is strict for all \(i\) with \(c \leq i \leq n - 1\), except that \(i = c = 1\) and \(G\) is \(C_n\). Then, by (22), (33) and (36), we conclude that
\[ d_i = \sum_{v \in V} [a_{i-1}(G - u) + (n - 1)a_i(G - u)] - na_i(G) > 0, \quad \forall c \leq i \leq n - 1, \] (37)
except that \(i = c = 1\) and \(G\) is \(C_n\). Hence the proof is complete.

Now everything is ready for proving Theorems 1.4 and 1.3.

**Proof of Theorem 1.4** Let \(G\) be a non-complete graph of order \(n\). Recall (27) that
\[ (-1)^n \left[ (x - n + 1) \sum_{u \in V(G)} P(G - u, x) - nP(G, x) \right] = \sum_{i=1}^{n} (-1)^i d_i x^i. \]
By Proposition 4 we know that \(d_i \geq 0\) for all \(i\) with \(1 \leq i \leq n\) and \(d_{n-1} > 0\). Thus \(\sum_{i=1}^{n} (-1)^i d_i x^i > 0\) holds for all \(x < 0\), which completes the proof of Theorem 1.4.

**Proof of Theorem 1.3** By the discussion in Section 3, Theorem 1.4 implies \(\xi(G, x) > 0\) holds for all non-complete graphs \(G\) and all \(x < 0\). By Proposition 5, Theorem 1.3 holds.
We end this section with the following remarks.

Remarks:

(i). Theorem 1.3 implies that for any non-complete graph $G$ of order $n$, $\frac{P(G,x)}{P(K_n,x)}$ is strictly decreasing when $x < 0$.

(ii). Let $G$ be a non-complete graph of order $n$ and $P(G,x) = \sum_{i=1}^{n} (-1)^{n-i}a_i x^i$. Then $\epsilon(G) < \epsilon(K_n)$ implies that

\[
\frac{a_1 + 2a_2 + \cdots + na_n}{a_1 + a_2 + \cdots + a_n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n}.
\]

(iii). When $x = -1$, Theorem 1.4 implies that for any graph $G$ of order $n$,

\[
(-1)^{n-1} \sum_{u \in V} P(G - u, -1) \geq (-1)^{n} P(G, -1),
\]

where the inequality holds if and only if $G$ is complete. By Stanley’s interpretation on $(-1)^{n} P(G, -1)$ in [15], the above inequality implies that for any graph $G = (V, E)$, the number of acyclic orientations of $G$ is at most the total number of acyclic orientations of $G - u$ for all $u \in V$, where the equality holds if and only if $G$ is complete.

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