A note on the 3-rainbow index of $K_{2,t}$

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Abstract

A tree $T$, in an edge-colored graph $G$, is called a rainbow tree if no two edges of $T$ are assigned the same color. For a vertex subset $S \in V(G)$, a tree that connects $S$ in $G$ is called an $S$-tree. A $k$-rainbow coloring of $G$ is an edge coloring of $G$ having the property that for every set $S$ of $k$ vertices of $G$, there exists a rainbow $S$-tree $T$ in $G$. The minimum number of colors needed in a $k$-rainbow coloring of $G$ is the $k$-rainbow index of $G$, denoted by $rx_k(G)$. In this paper, we obtain the exact values of $rx_3(K_{2,t})$ for any $t \geq 1$.

Keywords: edge-coloring, $k$-rainbow index, rainbow tree, complete bipartite graph.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. Let $G$ be a nontrivial connected graph of order $n$ on which is defined an edge coloring, where adjacent edges may be the same color. A path $P$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is rainbow connected if $G$ contains a $u-v$ rainbow path for every pair $u, v$ of distinct vertices of $G$. The minimum number of colors that results in a rainbow connected graph $G$ is the rainbow connection number $rc(G)$ of $G$. These concepts were introduced by Chartrand et al. in [2].

Another generalization of rainbow connection number was also introduced by Chartrand et al. [3]. A tree $T$ is a rainbow tree if no two edges of $T$ are colored the same. For a vertex subset $S \in V(G)$, a tree that connects $S$ in $G$ is called an $S$-tree. Let $k$ be a fixed integer with $2 \leq k \leq n$. An edge coloring of $G$ is called a $k$-rainbow coloring if for every set $S$ of $k$ vertices of $G$, there exists a rainbow $S$-tree. The $k$-rainbow index $rx_k(G)$ of $G$ is the minimum number of colors needed in a $k$-rainbow coloring of $G$. It is obvious that $rc(G) = rx_2(G)$. It follows, for every nontrivial connected graph $G$ of order $n$, that

$$rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_k(G).$$

Chakraborty et al. [4] showed that computing the rainbow connection number of a graph is NP-hard. Thus, it is more difficult to compute $k$-rainbow index of general graphs.

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For complete bipartite graph, Chartrand et al. [2] obtained \(rc(K_{s,t}) = \min\{\sqrt{t}, 4\}\), for integers \(s\) and \(t\) with \(2 \leq s \leq t\). More results on the rainbow connection number can be found in the survey [5]. For 3-rainbow index, Li et al. [6] obtained the exact value of regular complete bipartite \(K_{r,r}\), \(rx_3(K_{r,r}) = 3\), with \(r \geq 3\).

In [7], we showed, for any integers \(s\) and \(t\) with \(3 \leq s \leq t\), \(rx_3(K_{s,t}) \leq \min\{6, s + t - 3\}\), and the bound is tight. But this bound can not be generalized to the graph \(K_{2,t}\). So in the paper, we derive the exact value of \(rx_3(K_{2,t})\) for different \(t(t \geq 1)\). We get the following theorem.

**Theorem 1.** For any integer \(t \geq 1\),

\[
rx_3(K_{2,t}) = \begin{cases} 
2, & \text{if } t = 1, 2; \\
3, & \text{if } t = 3, 4; \\
4, & \text{if } 5 \leq t \leq 8; \\
5, & \text{if } 9 \leq t \leq 20; \\
k, & \text{if } (k - 1)(k - 2) + 1 \leq t \leq k(k - 1), (k \geq 6);
\end{cases}
\]

### 2 Proof of Theorem [1]

In this section, we determine the 3-rainbow index of complete bipartite graphs \(K_{2,t}\). First of all, we need some new techniques and notions.

Let \(U\) and \(W\) be the two partite sets of \(K_{2,t}\), where \(U = \{u_1, u_2\}, W = \{w_1, w_2, \ldots, w_t\}\). Suppose that there exists a 3-rainbow coloring \(c : E(K_{2,t}) \rightarrow \{1, 2, \ldots, k\}\). Corresponding to the 3-rainbow coloring, there is a color code\((w)\) assigned to every vertex \(w \in W\), consisting of an ordered 2-tuple \((a_1, a_2)\), where \(a_i = c(u_iw) \in \{1, 2, \ldots, k\}\) for \(i = 1, 2\). In turn, for a subset \(Y\) of \(W\), given color codes of vertices in \(Y\) are acceptable if the corresponding coloring is 3-rainbow coloring of the graph induced by \(Y \cup U\). Let \(B\) be a set of colors. Color codes are \(B\)-limited if both colors in every color code, but not necessarily distinct, are from \(B\). The maximum number of color codes which are not only \(B\)-limited but also acceptable is denoted by \(\beta_B\).

Note that we adopt the following thought in the proof: we first give a certain \(B\) with \(k\) colors, then we consider the the maximum number of color codes which are not only \(B\)-limited but also acceptable, the number is the tight upper bound of \(t\) with \(rx_3(K_{2,t}) = k\).

The following claims are easy to verify and will be used later.

**Claim 1.** If \(|B| = 1\), then \(\beta_B \leq 1\).

**Claim 2.** If \(|B| = 2\), then \(\beta_B \leq 2\).

**Proof.** By contradiction. We may assume \(\beta_B \geq 3\). For three vertices in \(W\), we can find a rainbow tree containing them. We know the rainbow tree containing them uses at least an edge adjacent with every vertex of them, thus the tree uses at least three edges whose coloring are from \(B\). Since the color codes are \(B\)-limited and \(|B| = 2\), the tree is not a rainbow tree, a contradiction.

\(\Box\)
Lemma 2.1. For $t = 1, 2$, $\text{rx}_3(K_{2,t}) = 2$, and $\text{rx}_3(K_{2,t}) \geq 3$ for $t \geq 3$.

Proof. Since $K_{2,1}$ is a tree, $\text{rx}_3(K_{2,1}) = 2$. For $t = 2$, $\text{rx}_3(K_{2,2}) = \text{rx}_3(C_4) = 2$. From the Claim 2 we get if $t \geq 3$, then $\text{rx}_3(K_{2,t}) \geq 3$.

The following lemma reminds us how to construct color codes to some extent and is useful to show that an edge coloring is 3-rainbow coloring by character of color codes.

Lemma 2.2. Let $c$ be an edge coloring of $K_{2,t}$ with $\text{rx}_3(K_{2,t}) = k$ and $S = \{v_1, v_2, v_3\}$ be any a set of three vertices in $K_{2,t}$. We have the following.

1. $|S \cap W| = 3$. When $k = 3$, there is a rainbow $S$-tree if and only if there exists $i \in \{1, 2\}$ such that $c(u_i v_j)$ are distinct $(j = 1, 2, 3)$; when $k \geq 4$, if there are at least 4 colors used by the color codes of three vertices, then there is a rainbow $S$-tree.

2. $|S \cap W| = 2$. If both $i$-th $(i = 1, 2)$ elements of two color codes are distinct or at least three colors are used, then there is a rainbow $S$-tree.

3. $|S \cap W| = 1$. If $a_1 \neq a_2$ for any color code $(a_1, a_2)$, then there is a rainbow $S$-tree.

Proof. For (1), firstly, when $k = 3$, if there exists $i \in \{1, 2\}$ such that $c(u_i v_j)$ are distinct $(j = 1, 2, 3)$, then we find a rainbow $S$-tree $T = \{u_i v_1, u_i v_2, u_i v_3\}$. And if there exists no $i \in \{1, 2\}$ satisfying above condition, then we need to add at least two other vertices to obtain the rainbow $S$-tree, which implies there are at least four edges in rainbow $S$-tree. It contradicts that $k = 3$. Secondly, for $k \geq 4$, let $\text{code}(v_1) = (c(v_1 u_1), c(v_1 u_2)), \text{code}(v_2) = (c(v_2 u_1), c(v_2 u_2)), \text{code}(v_3) = (c(v_3 u_1), c(v_3 u_2))$. If there exists $i \in \{1, 2\}$ such that $c(u_i v_j)$ are distinct $(j = 1, 2, 3)$, the conclusion clearly holds. And if not, without loss of generality, assume $c(v_1 u_1) = c(v_2 u_1) \neq c(v_3 u_1)$, then we can find a rainbow $S$-tree $T = \{v_1 u_2, v_2 u_1, v_3 u_1, v_3 u_2\}$ or $T = \{v_1 u_1, v_2 u_2, v_3 u_1, v_3 u_2\}$.

For (2), suppose that $v_1 = u_1 \in U$, $v_2 = w_1 \in W$, $v_3 = w_2 \in W$. we can easily find a rainbow $S$-tree $T = \{u_1 w_1, u_1 w_2\}$ with length 2 or $T = \{u_1 w_1, w_1 u_2, w_2 u_2\}$ with length 3.

For (3), suppose that $v_1 = u_1 \in U$, $v_2 = w_2 \in U$, $v_3 = w_1 \in W$. Then the tree $T = \{u_1 w_1, w_1 u_2\}$ is a rainbow tree containing $S$.

Lemma 2.3. For $t = 3, 4$, $\text{rx}_3(K_{2,t}) = 3$, and $\text{rx}_3(K_{2,t}) \geq 4$ for $t \geq 5$.

Proof. First, we show the latter of conclusion that $\text{rx}_3(k_{2,t}) \geq 4$ for $t \geq 5$. By contradiction. We assume there exists $t \geq 5$ such that $\text{rx}_3(k_{2,t}) = 3$ by Lemma 2.1.

From Lemma 2.2 (1) and (2), if $\text{rx}_3(K_{2,t}) = 3$, then for any three color codes: $\text{code}(w_1)$, $\text{code}(w_2)$, $\text{code}(w_3)$, there exists $i \in \{1, 2\}$ such that $c(w_1 u_i)$, $c(w_2 u_i)$, $c(w_3 u_i)$ are different. Moreover, there is no same color code in this case.

Now we try to connect the problem to the game of chess. The only fact needed about the game is that rooks are isolate if and only if any three of them lie in the different rows or the different columns of the chessboard. We give each square on the board a pair $(i, j)$ of coordinates.
The integer $i$ designates the row number of the square and the integer $j$ designates the column number of the square, where $i$ and $j$ are integers between 1 and 3. Our concern is the maximum number of rooks which are isolate on the chess since it is the upper bound of $t$ with $rx_3(K_{2, t}) = 3$. We consider the condition from two factors:

(a) if the rooks lie in different rows and columns.

(b) if two of them lie in the same rows or columns.

It is easy to verify that at most 4 rooks are isolate, such as $(1, 2)$, $(2, 1)$, $(1, 3)$, $(3, 1)$ shown in Figure 1, a contradiction. Thus, the conclusion holds.

Second, we give the vertices of $K_{2,3}$ any three color codes shown in Figure 1 (b) and give the vertices of $K_{2,4}$ all color codes shown in Figure 1 (b). It is easy to check that corresponding coloring is a 3-rainbow coloring. So $rx_3(K_{2,3}) = rx_3(K_{2,4}) = 3$. □

![Figure 1: An example of (a) and (b) used in Lemma 2.3.](image)

From the proof of the Lemma 2.3, the following claim is easily obtained.

**Claim 3.** If $|B| = 3$, then $\beta_B = 4$.

**Lemma 2.4.** For $5 \leq t \leq 8$, $rx_3(K_{2, t}) = 4$, and $rx_3(K_{2, t}) \geq 5$ for $t \geq 9$.

**Proof.** Similarly, we first prove the latter of the lemma. By contradiction, we may assume that there exists $t \geq 9$ such that $rx_3(k_{2, t}) = 4$. It follows that $\beta_B \geq 9$. Let $B = \{1, 2, 3, 4\}$ be a set of 4 colors. Let $B_1 = \{1, 2, 3\}$, $B_2 = \{1, 2, 4\}$, $B_3 = \{1, 3, 4\}$, $B_4 = \{2, 3, 4\}$. Then $|B_i| = 3$, so $\beta_{B_i} = 4$ ($i = 1, 2, 3, 4$). Since $B$ is the union of four $B_i$ ($i = 1, 2, 3, 4$), thus $\beta_B \leq 16$. And we find that a color code is limited in at least two $B_i$ ($i = 1, 2, 3, 4$). So we get $\beta_B \leq 8$, a contradiction.

Then, we will get eight color codes such that the corresponding coloring is 3-rainbow coloring. We can seek eight rooks on the 4-by-4 board, shown in Figure 2. By the Lemma 2.2 for any
t (5 ≤ t ≤ 8) rooks in Figure 2, we can find a 3-rainbow coloring of $K_{2,t}$. Thus $rx_3(K_{2,t}) = 4, (5 ≤ t ≤ 8)$.

**Lemma 2.5.** For $9 ≤ t ≤ 20$, $rx_3(K_{2,t}) = 5$, and $rx_3(K_{2,t}) ≥ 6$ for $t ≥ 21$.

**Proof.** From the Claim 2, we know $t ≤ C_5^2 × 2 = 5 × 4 = 20$, if $rx_3(k_{2,t}) = 5$. That is, $rx_3(k_{2,t}) ≥ 6$ for $t ≥ 21$.

Next, we give $t$ vertices $t$ color codes $(9 ≤ t ≤ 20)$ and the corresponding coloring is 3-rainbow coloring. When $9 ≤ t ≤ 10$, we just give $t$ vertices the first $t$ codes successively: $(1,2), (2,3), (3,4), (4,5), (3,1), (4,2), (5,3), (1,4), (2,5), (5,1)$ (see Figure 3). When $11 ≤ t ≤ 20$, we choose randomly $t − 10$ color codes from the remaining color codes in Figure 4(a) to give the $t − 10$ vertices.

Then, it remains to show the coloring is 3-rainbow coloring. Let $S$ be a set of three vertices. By the Lemma 2.2, we can find a rainbow $S$-tree with the exception of the case: $|S ∩ W| = 3$ and the only 3 different colors used by the color codes of $S$. Note that 3 colors used by the color codes of $S$ may be allowed in this case. But there must exist a color code consisted of other two distinct colors. Thus we will find a rainbow $S$-tree with length 5, for example see Figure 3. When $t ≥ 10$, if 3 different colors used by the color codes of $S$, there must be a color code consisted of other two distinct colors appearing in the first 10 color codes of $K_{2,t}$ by the strategy of coloring. When $t = 9$, we only to check the subcase that color codes of $S$ are limited in {2,3,4}. It is easy to verify the fact there is a rainbow tree connecting $S$, which correspond to the color codes (2,3), (3,4), (4,2), respectively. So the coloring is 3-rainbow coloring. That is, for $9 ≤ t ≤ 20$, $rx_3(k_{2,t}) ≤ 5$. With the aid of Lemma 2.4 we get $rx_3(k_{2,t}) = 5, 9 ≤ t ≤ 20$.

**Lemma 2.6.** For $(k − 1)(k − 2) + 1 ≤ t ≤ k(k − 1)$, $rx_3(K_{2,t}) = k, k ≥ 6$.

**Proof.** By the Claim 2, if $rx_3(K_{2,t}) = k$, then it has at most $C_k^2 × 2 = k(k − 1)$ acceptable color codes. Thus when $t ≥ k(k − 1) + 1$, $rx_3(K_{2,t}) ≥ k + 1$. Similarly, when $t ≥ (k − 1)(k − 2) + 1$, $rx_3(K_{2,t}) ≥ k$.

Now we give a 3-rainbow coloring of $K_{2,t} ((k − 1)(k − 2) + 1 ≤ t ≤ k(k − 1))$ with $k$ colors. When $k ≥ 6$, $(k − 1)(k − 2) + 1 > \frac{1}{k}k(k − 1)$, thus $t > \frac{1}{k}k(k − 1)$. We randomly give the first $\frac{1}{k}k(k − 1)$ vertices $\frac{1}{k}k(k − 1)$ color codes in upper triangle of the chessboard, see Figure 4(b). Then the other $t − \frac{1}{k}k(k − 1)$ vertices are received any $t − \frac{1}{k}k(k − 1)$ remaining color codes in Figure 4(b).
Next we show this kind of coloring is 3-rainbow coloring. Let $S$ be the set of three vertices. By Lemma 2.2, we only to check the case: $|S \cap W| = 3$ and only 3 different colors used by the color codes of three vertices. Similar to the proof of Lemma 2.5, we need to find a color code consisted of other two distinct colors to construct a rainbow $S$-tree. Since the combinations of any two colors have appeared in first $k(k-1)/2$ color codes, we can easily find such a color code. Hence, in any case, there is a rainbow $S$-tree with length at most 5. That is, $(k-1)(k-2)+1 \leq t \leq k(k-1)$, $rx_3(K_{2,t}) \leq k$. So $rx_3(K_{2,t}) = k$ for $(k-1)(k-2)+1 \leq t \leq k(k-1)$. \hfill \Box

Now we complete the proof of Theorem 1.

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