OPTIMAL HÖLDER REGULARITY FOR NONAUTONOMOUS KOLMOGOROV EQUATIONS

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To Claude-Michel Brauner on the occasion of his 60th birthday

Abstract. We consider a class of nonautonomous elliptic operators \( A \) with unbounded coefficients defined in \([0, T] \times \mathbb{R}^N\) and we prove optimal Schauder estimates for the solution to the parabolic Cauchy problem \( Du = Au + f, \)

\( u(0, \cdot) = g. \)

1. Introduction

In this paper we deal with a class of nonautonomous elliptic operators \( A \) defined by

\[ A \varphi(t, x) = \sum_{i,j=1}^{N} q_{ij}(t, x) D_{ij} \varphi(x) + \sum_{j=1}^{N} b_{j}(t, x) D_{j} \varphi(x) + c(t, x) \varphi(x), \]
on smooth functions \( \varphi: \mathbb{R}^N \to \mathbb{R}. \) We consider possibly unbounded coefficients defined in \([0, T] \times \mathbb{R}^N\), smooth with respect to \( x \) and continuous or just measurable with respect to time. We are interested in optimal Schauder estimates for the solution to the Cauchy problem

\[
\begin{cases}
    D_t u(t, x) = A u(t, x) + g(t, x), & t \in [0, T], \ x \in \mathbb{R}^N, \\
    u(0, x) = f(x), & x \in \mathbb{R}^N.
\end{cases}
\]

In the case when the coefficients of the operator \( A \) are smooth enough in \([0, T] \times \mathbb{R}^N\) and satisfy suitable algebraic and growth conditions at infinity (see Hypotheses 2.1), we prove that, for any \( f \in C^{2+\theta}_{\mathbb{R}^N} \) and any continuous function \( g: [0, T] \times \mathbb{R}^N \to \mathbb{R} \) such that \( g(t, \cdot) \in C^\theta_{\mathbb{R}^N} \) for any \( t \in [0, T] \) and

\[ \sup_{t \in [0,T]} \| g(t, \cdot) \|_{C^\theta_{\mathbb{R}^N}} < +\infty, \]

there exists a unique classical bounded solution to problem (1.1), i.e., there exists a unique bounded function \( u: [0, T] \times \mathbb{R}^N \to \mathbb{R} \) which (i) is continuously differentiable in \([0, T] \times \mathbb{R}^N\), once with respect to the time variable and twice with respect to the spatial variables, (ii) solves the differential equation in (1.1), (iii) satisfies the condition \( u(0, \cdot) \equiv f. \) Further, we have a Schauder type regularity result, i.e., the

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function $u(t, \cdot)$ belongs to $C^{2+\theta}_{b}(\mathbb{R}^N)$ for any $t \in [0, T]$ and there exists a positive constant $C$ such that

$$\sup_{t \in [0,T]} \|u(t, \cdot)\|_{C^{2+\theta}_{b}(\mathbb{R}^N)} \leq C \left( \|f\|_{C^{2+\theta}_{b}(\mathbb{R}^N)} + \sup_{t \in [0,T]} \|g(t, \cdot)\|_{C^\theta_{b}(\mathbb{R}^N)} \right).$$

(1.3)

In the case of discontinuous coefficients, under suitable assumptions (see Hypotheses 3.2), we can still prove an existence and uniqueness theorem for problem (1.1) as well as optimal Schauder estimates. Here, we assume that the function $g$ is measurable in $[0, T] \times \mathbb{R}^N$ and it satisfies condition (1.2). The lack of regularity of the data with respect to the time variable prevents the solution from being continuously differentiable with respect to the time variable. Hence, we introduce an appropriate definition of solution to problem (1.1), adapted to the discontinuity of the data (see Definition 3.3). Then, we prove that there exists a unique solution to the problem (1.1) in the sense of Definition 3.3. This solution $u$ satisfies estimate (1.3). Roughly speaking, the main difference with the case when the coefficients are smooth, is that now the function $u(\cdot, x)$ is differentiable for any $x \in \mathbb{R}^N$ almost everywhere in $[0, T]$ and that the differential equation is satisfied almost everywhere in $[0, T] \times \mathbb{R}^N$. Both for smooth and nonsmooth coefficients, the uniqueness of the solution to problem (1.1) follows from a variant of the classical maximum principle (see Propositions 2.2 and 3.6), which can be proved assuming the existence of a suitable Lyapunov function (see Hypothesis 2.1(vii)).

For autonomous equations with unbounded smooth coefficients, Schauder theorems of this type were obtained in [1, 13] as a consequence of optimal estimates in the sup norm of the spatial derivatives of the solution to the homogeneous Cauchy problem

$$\begin{cases} 
D_t u(t, x) = \mathcal{A} u(t, x), & t \in [0, T], 
\quad x \in \mathbb{R}^N, 
\end{cases}$$

(1.4)

$$u(0, x) = f(x), 
\quad x \in \mathbb{R}^N,$$

when $f$ belongs to suitable spaces of Hölder continuous functions. See also [3].

To the best of our knowledge, the first papers dealing with optimal Schauder estimates of the type (1.3) are [7, 8, 9] where bounded and continuous coefficients satisfying (1.2) were considered. Recently, the results in [7, 8, 9] have been extended in [10] to discontinuous bounded coefficients still satisfying (1.2) and in [11] to operators $\mathcal{A}$ of the type

$$\mathcal{A} \varphi(t, x) = \sum_{i,j=1}^N q_{ij}(t) D_{ij} \varphi(x) + \sum_{i,j=1}^N b_{ij}(t) x_j D_i \varphi(x),$$

with bounded coefficients $q_{ij}$ and $b_{ij}$ $(i, j = 1, \ldots, N)$. When $\mathcal{A}$ is an Ornstein-Uhlenbeck operator things are easier than in the general case, since an explicit formula for the solution to (1.4) is known. Hence, one can obtain uniform estimates for the spatial derivatives of the solution to problem (1.4) just differentiating the formula which defines $u$. On the contrary, no explicit formulas are available for more general elliptic operators.

Very recently, Krylo and Priola (see [6]) have studied more general nonautonomous elliptic operators with unbounded and less regular coefficients, using different techniques. Their interesting results show global Schauder estimates for the solutions to the parabolic equation $D_t u + \mathcal{A} u = f$ in $(T, +\infty) \times \mathbb{R}^N$ when $T \in (-\infty, +\infty)$ and, as a byproduct, Schauder estimates for the solution to problem (1.1), thus extending the results of [10]. Roughly speaking, in [6] the diffusion
coefficients are supposed to be bounded, whereas the drift coefficients may grow at most linearly at infinity, with respect to the spatial variables.

In this paper we prove optimal Schauder estimates for nonautonomous elliptic operators whose coefficients may grow faster than linearly at infinity. In the first part of the paper (see Section 2), we consider the case when the coefficients are smooth in $[0, T] \times \mathbb{R}^N$, adapting the techniques in [1, 13]. First, in Subsection 2.1, we prove that problem (1.4) admits a unique bounded classical solution $u$ for any $f \in C_b(\mathbb{R}^N)$. Subsection 2.2 is then devoted to prove uniform estimates for the derivatives (up to third-order, and with respect to the sup-norm in $\mathbb{R}^N$) of the solution to problem (1.4) when $f$ belongs to suitable spaces of Hölder continuous functions. Finally, these uniform estimates and an abstract interpolation method (see [12]) yield optimal Schauder estimates (see Subsection 2.3). In the second part of the paper, we turn our attention to the case of discontinuous (in time) coefficients. We prove the optimal Schauder estimates approximating the operator $A$ by a sequence of elliptic operators $A^{(n)}$ which satisfy the assumptions of the first part of the paper, and using a compactness argument. The arguments in the proof of Theorem 3.7 can then be used to weaken a bit the assumptions of Section 2 and prove the Schauder estimates of Theorem 2.7 without any assumption of Hölder in time regularity of the coefficients of $A$. See Theorem 3.8. Finally, in Section 4, we exhibit a class of elliptic operators to which the optimal Schauder estimates may be applied.

Notations. $C_b(\mathbb{R}^N)$ denotes the set of all bounded and continuous functions $f : \mathbb{R}^N \to \mathbb{R}$. We endow it with the sup-norm $\|f\|_\infty$. For any $k > 0$ (possibly $k = +\infty$), $C^k_b(\mathbb{R}^N)$ denotes the subset of $C_b(\mathbb{R}^N)$ of all functions $f : \mathbb{R}^N \to \mathbb{R}$ that are continuously differentiable in $\mathbb{R}^N$ up to $[k]$th-order, with bounded derivatives and such that the $[k]$th-order derivatives are $([k] - [k])$-Hölder continuous in $\mathbb{R}^N$. $C^k_b(\mathbb{R}^N)$ is endowed with the norm $\|f\|_{C^k_b(\mathbb{R}^N)} := \sum_{j \leq [k]} \|D^j f\|_\infty + \sum_{|\alpha| = [k]} \|D^\alpha f\|_{C^k_b(\mathbb{R}^N)}$. $C^k_c(\mathbb{R}^N)$ ($k \in \mathbb{N} \cup \{+\infty\}$) denotes the subset of $C^k_b(\mathbb{R}^N)$ of all compactly supported functions.

For any domain $D \subset \mathbb{R} \times \mathbb{R}^N$ and any $\alpha \in (0, 1)$, $C^{\alpha/2, \alpha}(D)$ denotes the space of all Hölder-continuous functions with respect to the parabolic distance of $\mathbb{R}^{N+1}$. Similarly, for any $h, k \in \mathbb{N} \cup \{0\}$ and any $\alpha \in [0, 1)$, $C^{h+\alpha/2, k+\alpha}(D)$ denotes the set of all functions $f : D \to \mathbb{R}$ which (i) are continuously differentiable in $D$ up to the $h$th-order with respect to time variable, and up to the $k$th order with respect to the spatial variables, (ii) the derivatives of maximum order are in $C^{\alpha/2, \alpha}(D)$ (here, $C^{0, 0} := C$). Finally, we use the notation $C^{h+\alpha/2, k+\alpha}_{\text{loc}}(D)$ to denote the set of all functions $f : D \to \mathbb{R}$ which are in $C^{h+\alpha/2, k+\alpha}(D_0)$ for any compact set $D_0 \subset D$.

For any measurable set $E$, we denote by $1_E$ the characteristic function of $E$, i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ otherwise.

Given a $N \times N$ matrix we denote by $\text{Tr}(Q)$ its trace. Further, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product of $\mathbb{R}^N$.

2. The case of smooth coefficients

Throughout this section, we make the following assumptions on the coefficients $q_{ij}$, $b_j$ ($i, j = 1, \ldots, N$) and $c$ of the operator $A$. We denote by $Q(t, x)$ and $b(t, x)$ the matrix whose entries are the coefficients $q_{ij}(t, x)$, and the vector whose entries are the coefficients $b_j(t, x)$, respectively.
Hypotheses 2.1.

(i) the coefficients \( q_{ij}, b_j \ (i, j = 1, \ldots, N) \) and \( c \) are thrice continuously differentiable with respect to the spatial variables in \([0, T] \times \mathbb{R}^N\) and they belong to \( C^{3/2, \delta}([0, T] \times B(0, R))\) for some \( \delta \in (0, 1) \) and any \( R > 0 \), together with their first-, second- and third-order spatial derivatives;

(ii) \( q_{ij}(t, x) = q_{ji}(t, x) \) for any \( i, j = 1, \ldots, N \) and any \( (t, x) \in [0, T] \times \mathbb{R}^N \), and

\[
\langle Q(t, x)\xi, \xi \rangle \geq \nu(t, x)|\xi|^2, \quad t \in [0, T], \ \xi, x \in \mathbb{R}^N,
\]

for some function \( \nu : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) such that

\[
\inf_{(t, x) \in [0, T] \times \mathbb{R}^N} \nu(t, x) = \nu_0 > 0;
\]

(iii) there exist positive constants \( C_1, C_2, C_3 \) such that

\[
|Q(t, x)| \leq C_1(1 + |x|^2)\nu(t, x),
\]

\[
\text{Tr}(Q(t, x)) \leq C_2(1 + |x|^2)\nu(t, x),
\]

\[
|b(t, x), x| \leq C_3(1 + |x|^2)\nu(t, x),
\]

for any \( t \in [0, T] \) and any \( x \in \mathbb{R}^N \);

(iv) \( c(t, x) \leq c_0 \) for some real constant \( c_0 \) and any \( (t, x) \in [0, T] \times \mathbb{R}^N \);

(v) there exist three positive constants \( K_1, K_2 \) and \( K_3 \) such that

\[
|D^3 q_{ij}(t, x)| \leq K_1|\beta|\nu(t, x),
\]

\[
\sum_{i, j = 1}^{N} D_{ik} q_{jk}(t, x)\xi_k \xi_{i} \leq K_2\nu(t, x) \sum_{i, j = 1}^{N} \xi_{i} \xi_{j},
\]

for any \( i, j = 1, \ldots, N, \) any \( |\beta| = 1, 3 \), any \( N \times N \) symmetric matrix \( \Xi = (\xi_{jk}) \) and any \( (t, x) \in [0, T] \times \mathbb{R}^N \);

(vi) there exist three functions \( d, r, \varrho : [0, T] \times \mathbb{R}^N \to \mathbb{R} \), with \( \varrho \geq \varrho_0 \) for some positive constant \( \varrho_0 \), and positive constants \( L_1, L_2, L_3 \) such that

\[
\langle Db(t, x)\xi, \xi \rangle \leq d(t, x)|\xi|^2,
\]

\[
|D^3 b_j(t, x)| \leq r(t, x),
\]

\[
|D^2 c(t, x)| \leq \varrho(t, x),
\]

\[
d(t, x) + L_1 r(t, x) + L_2 \varrho^2(t, x) \leq L_3\nu(t, x),
\]

for any \( t \in [0, T] \), any \( |\beta| = 2, 3 \), any \( |\gamma| = 1, 2, 3 \), any \( j = 1, \ldots, N \) and any \( x, \xi \in \mathbb{R}^N \), where \( Db = (D_j b_i) \);

(vii) there exist a positive function \( \varphi : \mathbb{R}^N \to \mathbb{R} \) and \( \lambda > 0 \) such that \( \varphi \) tends to \( +\infty \) as \( |x| \to +\infty \) and

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^N} \{ \varphi(t, x) - \lambda \varphi(x) \} < +\infty.
\]

2.1. The homogeneous Cauchy problem associated with the operator \( \mathcal{A} \).

In this subsection, for any \( s \in [0, T) \), we consider the homogeneous Cauchy problem

\[
\left\{ \begin{array}{ll}
D_t u(t, x) = \mathcal{A} u(t, x), & t \in (s, T], \ x \in \mathbb{R}^N, \\
u(s, x) = f(x), & x \in \mathbb{R}^N.
\end{array} \right.
\]

We are going to prove that, for any \( f \in C_0(\mathbb{R}^N) \), problem (2.9) admits a unique bounded classical solution \( u \) (i.e., there exists a unique bounded and continuous function \( u : [s, T] \times \mathbb{R}^N \to \mathbb{R} \) which is continuously differentiable in \((s, T] \times \mathbb{R}^N\),
once with respect to time and twice with respect to the spatial variables, that satisfies (2.9)).

Uniqueness of the bounded classical solution to problem (2.9) follows from the following variant of the classical maximum principle.

**Proposition 2.2.** Fix $s \in [0, T)$. If $u \in C_b([s, T] \times \mathbb{R}^N) \cap C^{1,2}((s, T] \times \mathbb{R}^N)$ satisfies

\[
\begin{cases}
D_t u(t, x) - \mathcal{A} u(t, x) \leq 0, & t \in (s, T], \ x \in \mathbb{R}^N, \\
u(s, x) \leq 0, & x \in \mathbb{R}^N,
\end{cases}
\]

then $u \leq 0$.

**Proof.** The proof can be obtained repeating the arguments in the proof of the forthcoming Proposition 3.6, using the classical maximum principle instead of the Nazarov-Ural’tseva maximum principle.

**Theorem 2.3.** Under Hypotheses 2.1, for any $f \in C_b(\mathbb{R}^N)$, the Cauchy problem (2.9) admits a unique bounded classical solution $u$. Moreover,

\[
\|u(t, \cdot)\|_{\infty} \leq e^{c_0(t-s)} \|f\|_{\infty}, \quad t \in [s, T],
\]

where $c_0$ is as in Hypothesis 2.1(iv).

**Proof.** As it has been already remarked, the uniqueness part is a straightforward consequence of Proposition 2.2.

To prove the existence part of the statement, we first consider the case when $f$ is nonnegative. For any $n \in \mathbb{N}$, let us consider the Cauchy-Dirichlet problem

\[
\begin{cases}
D_t v(t, x) = \mathcal{A} v(t, x), & t \in (s, T], \ x \in B(0, n), \\
v(t, x) = 0, & t \in (s, T], \ x \in \partial B(0, n), \\
v(s, x) = f(x), & x \in B(0, n).
\end{cases}
\]

(2.11)

By classical results (see e.g., [4, Theorem 3.5]) this problem admits a unique bounded solution $u_n \in C^{1,2}_c((s, T] \times B(0, n))$ which is continuous in $[s, T] \times \overline{B(0, n)} \setminus \{(s, x) : x \in \partial B(0, n)\}$. Moreover, the classical maximum principle shows that

\[
|u_n(t, x)| \leq e^{c_0(t-s)} \|f\|_{\infty}, \quad (t, x) \in [s, T] \times B(0, n)
\]

(2.12)

and that the sequence $u_n(t, x)$ is increasing for any fixed $(t, x)$. Classical interior Schauder estimates imply that the sequence $(u_n)$ is bounded in $C^{1+\delta/2,2+\delta}(D)$ for any compact set $D \subset (s, T] \times \mathbb{R}^N$. Here, $\delta$ is the same number as in Hypothesis 2.1(i). The Ascoli-Arzelà theorem implies that $u_n$ converges in $C^{1,2}(D)$, for any $D$ as above, to a function $u$ which belongs to $C^{1+\delta/2,2+\delta}_{loc}((s, T] \times \mathbb{R}^N)$ and solves the differential equation in (2.11).

Showing that $u$ is continuous up to $t = s$ and $u(s, \cdot) = f$ is a bit more tricky. It is straightforward if $f \in C^{2+\delta}(\mathbb{R}^N)$ since, in this case, the classical Schauder estimates show that $(u_n)$ is bounded in $C^{1+\delta/2,2+\delta}(D)$ for any compact set $D \subset [s, T] \times \mathbb{R}^N$. Hence, $u_n$ converges to $u$ uniformly in $[s, T] \times K$ for any compact set $K \subset \mathbb{R}^N$, so that $u$ is continuous up to $t = s$ and therein equals the function $f$. Taking the limit as $n \to +\infty$ in (2.12), estimate (2.10) follows immediately. Using this estimate, it is then easy to show that $u$ is continuous up to $t = s$ and therein equals the function $f$ also in the case when $f \in C_c(\mathbb{R}^N)$.

In the general case when $f \in C_b(\mathbb{R}^N)$, continuity up to $t = s$ can be obtained by a localization argument. Let us fix $x_0 \in \mathbb{R}^N$ and let $\eta : \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $\mathbb{I}_{B(x_0, 1)} \leq \eta \leq \mathbb{I}_{B(x_0, 2)}$. Writing $f = f\eta + f(1 - \eta)$, we split $u_n$
into the sum of the functions $v_n$ and $w_n$ which are, respectively, the solutions to (2.11) with initial data $\eta f$ and $(1 - \eta)f$. Since $\eta f$ is compactly supported in $\mathbb{R}^N$, $v_n$ converges to the solution $v$ to problem (2.9) with $f$ being replaced by $\eta f$. On the other hand, the classical maximum principle shows that

$$|w_n(t,x)| \leq K(1 - z_n(t,x)), \quad (t,x) \in [s,T] \times \overline{B(0,n)},$$

for any $n \in \mathbb{N}$. Here, $K = \|f\|_{\infty}$ and $z_n$ denotes the solution to problem (2.11), with $f$ being replaced with the function $\eta$. Since the function $v_n + w_n$ converges to $u$ and $z_n$ converges to the solution to problem (2.9) as $n \to +\infty$, it follows that

$$|u(t,x) - f(x)| \leq |v(t,x) - f(x)| + K(1 - z(t,x)).$$

Letting $(t,x) \to (s,x_0)$, one easily obtains that $u(t,x) - f(x)$ tends to 0. Hence, $u$ is continuous at $(t,x) = (s,x_0)$, where it equals $f(x_0)$. This completes the proof for nonnegative data $f$.

For a general $f \in C_b(\mathbb{R}^N)$, the proof can be obtained splitting $f = f^+ - f^-$, where $f^+ = \max\{f,0\}$, $f^- = (-f)^+$. Clearly, the solution to problem (2.9) will be given by $u_+ - u_-$, where $u_+$ and $u_-$ are the solutions to problem (2.9), with $f$ being replaced, respectively, by $f^+$ and $f^-$. \hfill \Box

In the rest of the paper, for any $f \in C_b(\mathbb{R}^N)$, we denote by $G(t,s) f$ the value at time $t$ of the unique bounded classical solution $u$ to problem (2.9).

### 2.2. Uniform estimates

This subsection is devoted to the proof of the following theorem.

**Theorem 2.4.** Let Hypotheses 2.1 be satisfied. Then, for any $\alpha, \beta \in [0,3]$, with $\alpha \leq \beta$, there exists a positive constant $C = C(\alpha, \beta)$ such that

$$\|G(t,s)f\|_{C_b^\beta(\mathbb{R}^N)} \leq C(t-s)^{-\frac{\beta-\alpha}{2}} \|f\|_{C_b^\alpha(\mathbb{R}^N)}, \quad f \in C_b^\alpha(\mathbb{R}^N),$$

for any $t \in (s,T]$.

The proof will be obtained in two steps. In the first one we will prove (2.13) when $\alpha, \beta \in \mathbb{N}$. Then, using an interpolation argument we extend (2.13) to any $\alpha, \beta$ as in the statement of the theorem.

Let us introduce a few more notation. We denote by $\mathcal{L}$ and $\mathcal{B}$ the operators defined on functions $f,g \in C^{0,1}([0,T] \times \mathbb{R}^N)$ by

$$\mathcal{L}(f,g) = \langle Q\nabla_x f, \nabla_x g \rangle, \quad \mathcal{B}(f,g) = \langle D_b\nabla_x f, \nabla_x g \rangle.$$

#### 2.2.1. The case when $\alpha, \beta \in \mathbb{N}$

We first consider the case when $\alpha = 0, \beta = 3$. For any $n \in \mathbb{N}$, let $\eta : \mathbb{R}^N \to \mathbb{R}$ be the radial function defined by $\eta(x) = \psi(|x|/n)$ for any $x \in \mathbb{R}^N$, where $\psi$ is a smooth nonincreasing function such that $\mathbb{I}_{[0,1/2]} \leq \psi \leq \mathbb{I}_{[0,1]}$. We fix $s \in (0,T)$, and define the function

$$v_n(t,x) = |u_n(t,x)|^2 + a(t-s)\eta^2|\nabla_x u_n(t,x)|^2 + a^2(t-s)^2\eta^4|D_x^2 u_n(t,x)|^2 + a^3(t-s)^3\eta^6|D_x^3 u_n(t,x)|^2,$$

for any $t \in (s,T]$ and any $x \in B(0,n)$, where $u_n$ is the (unique) classical solution of the Dirichlet Cauchy problem

$$\begin{cases}
D_t u(t,x) = a^2 u(t,x), & t \in [s,T], \quad x \in B(0,n), \\
u(t,x) = 0, & t \in [s,T], \quad x \in \partial B(0,n), \\
u(s,x) = \eta(x)f(x), & x \in \overline{B(0,n)}.
\end{cases}$$

The proof of Theorem 2.4 when $\alpha, \beta \in \mathbb{N}$ follows from the following

**Proposition 2.5.** Let $\alpha, \beta \in \mathbb{N}$. Then, for any $t \in (s,T]$ and any $x \in B(0,n)$, we have

$$\|G(t,s) f\|_{C^\beta(\mathbb{R}^N)} \leq C(t-s)^{-\frac{\beta}{2}} \|f\|_{C^\alpha(\mathbb{R}^N)},$$

where $C = C(\alpha, \beta, n)$.
By classical results (see e.g., [4]), the function \( v_n \) belongs to \( C^{1,2}((s, T) \times B(0, n)) \). Moreover, it can be extended by continuity up to \( t = s \) setting \( v_n(s, \cdot) = |\eta f|^2 \).

A long but straightforward computation shows that the function \( v_n \) solves the Cauchy problem

\[
\begin{cases}
D_t v_n(t, x) = \mathcal{A} v_n(t, x) + g_n(t, x), & t \in [s, T], \ x \in B(0, n), \\
v_n(t, x) = 0, & t \in [s, T], \ x \in \partial B(0, n), \\
v_n(s, x) = (\eta f)^2(x),
\end{cases}
\]

where \( g_n = \sum_{i=1}^{9} g_{i,n} \), with

\[
g_{1,n} = -2 q(u_n, u_n) - 2a(t-s)\eta^2 \sum_{i=1}^{N} q(D_i u_n, D_i u_n) - 2a^2(t-s)^2 \eta^4 \sum_{i,j=1}^{N} q(D_{ij} u_n, D_{ij} u_n)
\]

\[
-2a^3(t-s)^3 \eta^6 \sum_{i,j,h=1}^{N} q(D_{ijh} u_n, D_{ijh} u_n),
\]

\[
g_{2,n} = 2a(t-s)\eta^2 q(u_n, u_n) + 4a^2(t-s)^2 \eta^4 \sum_{i=1}^{N} q(D_i u_n, D_i u_n)
\]

\[
+ 6a^3(t-s)^3 \eta^6 \sum_{i=1}^{N} q(D_{ij} u_n, D_{ij} u_n),
\]

\[
g_{3,n} = -2a(t-s)\eta \|\nabla_x u_n\|^2 + 6a(t-s)\eta^2 \|D_{ij} u_n\|^2 + 15a^2(t-s)^2 \eta^4 \|D_{ij}^3 u_n\|^2) q(\eta, \eta),
\]

\[
g_{4,n} = -2(q \eta - cn) \{ a(t-s)\eta \|\nabla_x u_n\|^2 + 2a^2(t-s)^2 \eta^3 \|D_{ij}^2 u_n\|^2
\]

\[
+ 3a^3(t-s)^3 \eta^5 \|D_{ij}^3 u_n\|^2 \}
\]

\[
- 8a(t-s)\eta \sum_{i=1}^{N} q(\eta, D_i u_n) D_i u_n - 16a^2(t-s)^2 \eta^3 \sum_{i,j=1}^{N} q(\eta, D_{ij} u_n) D_{ij} u_n
\]

\[
- 24a^3(t-s)^3 \eta^5 \sum_{i,j,h=1}^{N} q(\eta, D_{ijh} u_n) D_{ijh} u_n,
\]

\[
g_{5,n} = 2a(t-s)\eta^2 \sum_{i,j,h=1}^{N} D_{ij} q_{ij} D_{ij} u_n D_{ij} u_n
\]

\[
+ 4a^2(t-s)^2 \eta^4 \sum_{i,j,h,k=1}^{N} D_{ijk} q_{ijk} D_{ijk} u_n D_{ijk} u_n
\]

\[
+ 6a^3(t-s)^3 \eta^6 \sum_{i,j,h,k,l=1}^{N} D_{ijkl} q_{ijkl} D_{ijkl} u_n D_{ijkl} u_n,
\]

\[
g_{6,n} = 2a^2(t-s)^2 \eta^4 \sum_{i,j,h,k=1}^{N} D_{ijk} q_{ijk} D_{ijk} u_n D_{ijk} u_n
\]
Taking Hypothesis 2.1(ii) into account, we easily deduce that
\[ g_{8,n} = a \eta^2 |\nabla_x u_n|^2 + 2a^2(t-s)\eta^4 |D_x^2 u_n|^2 + 3a^2(t-s)^2 \eta^6 |D_x^3 u_n|^2, \]
\[ g_{9,n} = 2cv_n + 2a(t-s)\eta^2 u_n(\nabla_x c, \nabla_x u_n) + 4a^2(t-s)^2 \eta^4 (D_x^2 u_n \nabla_x c, \nabla_x u_n) \]
\[ + 2a^2(t-s)^2 \eta^4 u_n \text{Tr}(D_x^2 c D_x^2 u_n) + 2a^3(t-s)^3 \eta^6 u_n \sum_{i,j,h=1}^{N} D_{ijh} c D_{ijh} u_n \]
\[ + 6a^3(t-s)^3 \eta^6 \left( \sum_{i,j,h=1}^{N} D_{ijh} c D_{ijh} u_n + \sum_{i,j,h=1}^{N} D_{ijh} c D_{ijh} u_n \right). \]

As far as the function \( g_{2,n} \) is concerned, we observe that condition (2.4) implies that \( \mathcal{B}(\zeta, \zeta) \leq d|\nabla \zeta|^2 \) for any \( \zeta \in C^1(\mathbb{R}^N) \). Hence, we can estimate
\[ g_{2,n} \leq 2a(t-s)du^2 |\nabla_x u_n|^2 + 4a^2(t-s)^2 du^4 |D_x^2 u_n|^2 + 6a^2(t-s)^3 du^6 |D_x^3 u_n|^2. \]

The function \( g_{3,n} \) can be estimated trivially from above by zero. So, let us consider the function \( g_{4,n} \). Using conditions (2.1), (2.2) and (2.3) and recalling that \( \nabla \eta \) and \( D^2 \eta \) identically vanish in \( B(0, n/2) \) and in \( \mathbb{R}^N \setminus B(0, n) \), it is not difficult to check that
\[ |\text{Tr}(Q(t,x)D_x^2 \eta(x))| \leq \frac{1 + n^2}{n^3} (2C_1 \|\psi''\|_{\infty} + 2C_2 \|\psi'\|_{\infty} + 2C_1 \|\psi'\|_{\infty} + 2C_1 \|\psi''\|_{\infty}) \nu(t, x), \]
\[ \langle b(t, x), \nabla_x \eta \rangle \geq -2C_3 \|\psi'\|_{\infty} \frac{1 + n^2}{n^2} \nu(t, x), \]
for any \((t, x) \in [0, T] \times \mathbb{R}^N \). Hence, for \( n \) sufficiently large, it holds that
\[ \mathcal{A}_n - cn \geq -C' \nu := -4 ((C_1 + C_2 + C_3) \|\psi''\|_{\infty} + C_1 \|\psi'\|_{\infty}) \nu. \]

Arguing similarly, we can estimate
\[ |\mathcal{A}(\eta, \eta)| \leq 4C_1 \|\psi''\|_{\infty} \nu |\nabla_x \zeta| := C'' \nu |\nabla_x \zeta|, \]
for any function \( \zeta \in C^1(\mathbb{R}^N) \). Using (2.16) and (2.17), we now get easily that
\[ g_{4,n} \leq 2aC'(t-s)\nu |\nabla_x u_n|^2 + 4a^2C'(t-s)^2 \nu n^3 |D_x^3 u_n|^2 \]
\[ + 6a^3C'(t-s)^3\nu\eta^5|D_x^3u_n|^2 + 8aC'(t-s)\eta\nu|\nabla_x u_n||D_x^2u_n| \\
+ 16a^2C''(t-s)^2\eta^3\nu|D_x^3u_n||D_x^3u_n| + 24a^3C''(t-s)^3\eta^5\nu|D_x^3u_n||D_x^4u_n|. \]

Using Young inequality we can estimate
\[ \eta|\nabla_x u_n||D_x^2u_n| \leq \varepsilon \eta^2|D_x^2u_n|^2 + \frac{1}{4\varepsilon}|\nabla_x u_n|^2, \]
\[ \eta^3|D_x^3u_n||D_x^3u_n| \leq \varepsilon \eta^4|D_x^3u_n|^2 + \frac{1}{4\varepsilon}\eta^2|D_x^2u_n|^2, \]
\[ \eta^5|D_x^3u_n||D_x^4u_n| \leq \varepsilon \eta^6|D_x^4u_n|^2 + \frac{1}{4\varepsilon}\eta^4|D_x^3u_n|^2, \]
for any \( \varepsilon > 0 \). Hence,
\[ g_{4,n} \leq 2a \left( C' + \frac{C''}{\varepsilon} \right) T \nu|\nabla_x u_n|^2 \]
\[ + 4a \left( aC'T + 2C''\varepsilon + \frac{C''}{\varepsilon}aT \right) (t-s)\nu\eta^2|D_x^2u_n|^2 \]
\[ + 2a^2 \left( 3aC'T + 8\varepsilon C'' + \frac{3C''}{\varepsilon}aT \right) (t-s)^2\nu^2|D_x^3u_n|^2 \]
\[ + 24a^3C''\varepsilon(t-s)^3\nu\eta^6|D_x^4u_n|^2. \]

The terms \( g_{5,n}, g_{6,n} \) and \( g_{7,n} \) can be estimated in a similar way. Hypotheses \( 2.1(v) \) and \( 2.1(vi) \) imply that
\[ \left| \sum_{i,j,h=1}^N D_{ijkl}D_{ij}D_{h} \zeta \right| \leq NK_1 \nu|\nabla \zeta||D_x^2 \zeta|, \]
\[ \left| \sum_{i,j,h,k=1}^N D_{ijkl}D_{ij}D_{hk} \zeta \right| \leq K_2 \nu|D_x^2 \zeta|^2, \]
\[ \left| \sum_{i,j,h,k,l=1}^N D_{ijkl}D_{ij}D_{hk}D_{kl} \zeta \right| \leq NK_3 \nu|D_x^3 \zeta||D_x^3 \zeta|, \]
\[ \left| \sum_{i,j,h,k,l=1}^N D_{ijkl}D_{ij}D_{hk} \zeta \right| \leq rN^\frac{1}{2}|\nabla \zeta||D_x^2 \zeta|, \]
\[ \left| \sum_{i,j,h,k,l=1}^N D_{ijkl}D_{ij}D_{hk} \zeta \right| \leq rN^\frac{1}{2}|\nabla \zeta||D_x^3 \zeta|, \]
for any smooth function \( \zeta \). Hence,
\[ g_{5,n} \leq aTK_1 \frac{N^2}{2\varepsilon} \nu|\nabla_x u_n|^2 + aK_1 \left( \frac{2\varepsilon}{\varepsilon} + aT \frac{N^2}{2\varepsilon} \right) (t-s)\nu\eta^2|D_x^2u_n|^2 \]
\[ \left(2.19\right) + 2a^2K_1 \left( 4\varepsilon + 3aT \frac{N^2}{2\varepsilon} \right) (t-s)^2\nu^2\eta^4|D_x^3u_n|^2 + 6a^3(t-s)^3\varepsilon K_1 \nu\eta^6|D_x^4u_n|^2, \]
\[ g_{6,n} \leq a^2(t-s)\eta^2 \frac{N^3}{2\varepsilon} T|\nabla_x u_n|^2 + a^2(t-s)^2\eta^4 \left[ 2K_2 \nu + \left( \frac{2\varepsilon + 3aN}{2\varepsilon} \right) r \right] |D_x^3u_n|^2 \]
\[ \left(2.20\right) + 6a^3(t-s)^3\eta^6 \left( K_2 \nu + \varepsilon r \right) |D_x^3u_n|^2, \]
\[ g_{7,n} \leq a^3(t-s)T^2 \eta^2 r \frac{N}{2\varepsilon} |\nabla_x u_n|^2 + a^3(t-s)T^2 \eta^4 K_3 \nu \frac{N^2}{2\varepsilon} |D_x^2 u_n|^2 \]

(2.21) \quad + 2a^3(t-s)^3 \eta^6 \varepsilon (K_3 \nu + r) |D_x^2 u_n|^2.

The function \( g_{8,n} \) can be estimated as follows:

(2.22) \quad g_{8,n} \leq a|\nabla_x u_n|^2 + 2a^2(t-s)\eta^2 |D_x^2 u_n|^2 + 3a^3(t-s)^2 \eta^4 |D_x^2 u_n|^2.

Finally, taking Hypotheses 2.1(iv) and 2.1(vi) into account, we can estimate

\[
g_{9,n} \leq 2c_0 v_n + T(1+T+T^2)u_n^2 + a^2(t-s) \left( 2T \varrho_0^{-1} + \sqrt{a} + 3T^2 \varrho_0^{-1} a \right) \varrho^2 \eta^2 |\nabla_x u_n|^2 \]

\[
+ a^2(t-s)^2 \left( 2\varrho_0^{-1} + 3T \varrho_0^{-1} a \right) \eta^4 \varrho^2 |D_x^2 u_n|^2 \]

\[
+ a^2(t-s)^3 \left( 6\varrho_0^{-1} + a^2 \right) \eta^6 \varrho^2 |D_x^3 u_n|^2.
\]

From (2.14), (2.18)-(2.21) we obtain, for any \( t \in [0,T] \),

\[
g_n \leq \left\{ -\nu_0 + a + \nu \left[ -1 + aT \left( 2C' + 2\frac{C''}{\varepsilon} + K_1 \frac{N^2}{2\varepsilon} \right) \right] 
\quad + aT \left( 4C' + 4\frac{C''}{\varepsilon} + K_1 \frac{N^2}{2\varepsilon} + 2K_2 \right) 
\quad + a(t-s) \left[ 4d + \left( 2\varepsilon + 3a \frac{N}{2\varepsilon} T \right) r 
\quad + \sqrt{a} \left( 2\varrho_0^{-1} + 3T \varrho_0^{-1} a \right) \varrho^2 \right] \right\} |\nabla_x u_n|^2 \]

\[
+ a^2 \left\{ -\nu_0 + 3a + \nu \left[ -1 + 4\varepsilon(4C'' + K_1) \right] 
\quad + aT \left( 6C' + 6\frac{C''}{\varepsilon} + 3K_1 \frac{N^2}{2\varepsilon} + 2K_2(3+\varepsilon) \right) 
\quad + a(t-s) \left( 2(3d + 4\varepsilon r) + \sqrt{a}(6\varrho_0^{-1} + a^2 \varrho^2) \right) \eta^2 \right\} |D_x^2 u_n|^2 \]

\[
+ 2a^3 \left( -1 + 3\varepsilon(4C'' + K_1) \right) (t-s)^3 \nu_0^6 |D_x^3 u_n|^2 \]

(2.24) \quad + \left\{ 2c_0 + T(1+T+T^2) \right\} v_n.
\]

It is now easy to check that \( \varepsilon \) and \( a \) can be fixed sufficiently small such that all the terms in the right-hand side of (2.24), but the last one, are negative. We thus get

\[ g_n(t,x) \leq (2c_0 + T(1+T+T^2)) v_n(t,x) := c_1 v_n(t,x), \]

for any \( t \in [0,T] \) and any \( x \in B(0,n) \). The maximum principle now yields

(2.25) \quad |v_n(t,x)| \leq e^{c_1(t-s)} \|f\|_{\infty}^2 \leq e^{c_1T} \|f\|_{\infty}^2, \quad t \in (s,T], \ x \in B(0,n). \]

The proof of Theorem 2.3 shows that the function \( u_n \) converges to \( u \) in \( C^{1,2}(D) \) for any compact set \( D \subset (s,T] \times \mathbb{R}^N \). We claim that \( u \) is thrice continuously
differentiable with respect to the spatial variables in \((s, T] \times \mathbb{R}^N\) and \(D^2_{sT}u_n\) converges to \(D^2_s u\) as \(n \to +\infty\), locally uniformly in \((s, T] \times \mathbb{R}^N\). Indeed, since the coefficients of the operator \(A\) are smooth, the interior Schauder estimates imply that the first-order spatial derivatives of the function \(u_n\) are bounded in \(C^{1+\delta/2,2+\delta}(D)\) for any \(D\) as above. Ascoli-Arzelà theorem now yields the claim. Hence, taking the limit as \(n \to +\infty\) in \((2.25)\), estimate \((2.13)\) follows at once.

To prove \((2.13)\) in the other situations when \(\alpha, \beta \in \mathbb{N}\) and \(\alpha \leq \beta\), it suffices to apply the same arguments as above to the function
\[
w(t, x) = \sum_{j=0}^{\beta} a^j (t - s)^{(j-\alpha)^+} \eta(x)^{2j} |D^j u_n(t, x)|^2, \quad t \in (s, T], \ x \in B(0, n),
\]
where \((\cdot)^+\) denotes the positive part of the number in brackets.

2.2.2. **The case when \((\alpha, \beta) \notin \mathbb{N} \times \mathbb{N}\).** As it has been already claimed, to prove \((2.13)\) in the general case we use an interpolation argument. It is well known that, given four Banach spaces \(X_1, X_2, Y_1, Y_2\), with \(Y_i\) continuously embedded into \(X_i\) \((i = 1, 2)\), any linear operator \(T\), which is bounded from \(X_1\) into \(X_2\) and from \(Y_1\) into \(Y_2\), is bounded from the interpolation space \((X_1, Y_1)_{\theta, \infty}\) into the interpolation space \((X_2, Y_2)_{\theta, \infty}\) for any \(\theta \in (0, 1)\) and
\[
(2.26) \quad \|T\|_{L((X_1, Y_1)_{\theta, \infty}; (X_2, Y_2)_{\theta, \infty})} \leq \|T\|_{L((X_1, X_2)_{1-\theta}; (X_2, Y_2)_{1-\theta})}^{\theta} \|T\|_{L((Y_1, Y_2)_{1-\theta}; (X_2, Y_2)_{1-\theta})}^{1-\theta},
\]
see e.g., [15, pag. 25]. We apply estimate \((2.26)\) with \(X_1 = C_b(\mathbb{R}^N)\), \(X_2, Y_1, Y_2 = C^3_b(\mathbb{R}^N)\) and \(S = G(t, s)\). Since \((C_b(\mathbb{R}^N), C^3_b(\mathbb{R}^N))_{\theta, \infty} = C^3_{b, \theta}(\mathbb{R}^N)\) (for any \(\theta \in (0, 1)\) such that \(3\theta \notin \mathbb{N}\)) and \((C^3_{b, \theta}(\mathbb{R}^N), C^3_{b, \theta}(\mathbb{R}^N))_{\theta, \infty} = C^3_{b, \theta}(\mathbb{R}^N)\), with equivalence of the corresponding norms (see e.g., [15, Chapter 2, Section 7, Theorem 1]), from the results in Subsection 2.2.1, we obtain \((2.13)\) with \(\alpha \in (0, 3)\) and \(\beta = 3\). A similar argument allows us to prove \((2.13)\) also when \(\alpha < \beta = 1, 2\) and \(\alpha \notin \mathbb{N}\).

Now, we observe that the maximum principle yields
\[
\|G(t, s)\|_{L(C_b(\mathbb{R}^N))} \leq C, \quad s \leq t \leq T,
\]
for some positive constant \(C\). Hence, applying \((2.26)\) with \(X_1 = X_2 = C_b(\mathbb{R}^N)\) and \(Y_1 = Y_2 = C^3_b(\mathbb{R}^N)\), \((2.13)\) follows for any \(0 \leq \alpha = \beta \leq 3\) such that \(\alpha, \beta \notin \mathbb{N}\).

To prove \((2.13)\) in the general case, it now suffices to fix \(\alpha, \beta \in [0, 3]\), with \(\alpha < \beta\), \(\alpha, \beta \notin \mathbb{N}\), and apply \((2.26)\) with \(X_1 = X_2 = Y_1 = C_b(\mathbb{R}^N)\), \(Y_2 = C^3_b(\mathbb{R}^N)\) and \(\theta = (3 - \alpha)^{-1}(\beta - \alpha)\).

**Remark 2.5.** Let \(I\) be a right-halfline and assume that Hypotheses 2.1, but 2.1(i), are satisfied with \([0, T]\) being replaced by \(I\). Further assume that
\[
(\text{i})\text{ the coefficients } q_{ij}, b_j, c \ (i, j = 1, \ldots, N) \text{ are thrice continuously differentiable with respect to the spatial variables in } I \times \mathbb{R}^N \text{ and their first-, second- and third-order spatial derivatives are in } C^{3/2, \delta}(D) \text{ for some } \delta \in (0, 1) \text{ and any compact set } D \subset I \times \mathbb{R}^N.
\]
Then, for any \(\alpha, \beta \in [0, 3]\), with \(\alpha \leq \beta\), there exists a positive constant \(C = C(\alpha, \beta)\) such that
\[
(2.27) \quad \|G(t, s)f\|_{C^\alpha_b(\mathbb{R}^N)} \leq C(t - s)^{-\frac{\alpha}{\beta - \alpha}} e^{c_0(t - s)} \|f\|_{C^\alpha_b(\mathbb{R}^N)}, \quad f \in C^\alpha_b(\mathbb{R}^N),
\]
for any \(s, t \in I\) with \(s < t\). Here, \(c_0\) is the constant in Hypothesis 2.1(vii).

The proof of Theorem 2.4 shows that
\[
(2.28) \quad \|G(t, s)f\|_{C^\alpha_b(\mathbb{R}^N)} \leq C_1(t - s)^{-\frac{\alpha}{\beta - \alpha}} \|f\|_{C^\alpha_b(\mathbb{R}^N)}, \quad f \in C^\alpha_b(\mathbb{R}^N),
\]
for any \(s, t \in I\) with \(s < t\). Here, \(C_1\) is the constant in Theorem 2.4.
for any \( t, s \in I \) such that \( 0 < t - s \leq 1 \) and some positive constant \( C_1 \), independent of \( s \) and \( t \). On the other hand, if \( t > s + 1 \), by virtue of Proposition 2.2 we can split \( G(t, s)f = G(t, t-1)G(t-1, s)f \). Hence, from (2.10) and (2.13) with \( \alpha = 0, \beta = 3 \) we get

\[
\|G(t, s)f\|_{C^\alpha_b(\mathbb{R}^N)} = \|G(t, t-1)G(t-1, s)f\|_{C^\alpha_b(\mathbb{R}^N)} \\
\leq C_1 \|G(t-1, s)f\|_\infty \\
\leq C_1 e^{c_0 (t-s-1)} \|f\|_\infty \\
\leq C_1 e^{c_0 (t-s-1)} \|f\|_{C^\alpha_b(\mathbb{R}^N)}.
\]

Estimate (2.27) now follows immediately from (2.28) and (2.29).

2.3. Optimal Schauder estimates. Using the uniform estimates in Theorem 2.4, we will prove an existence and uniqueness result for the Cauchy problem

\[
\begin{align*}
D_t u(t, x) &= (Au)(t, x) + g(t, x), & t \in [0, T], \quad x \in \mathbb{R}^N, \\
u(0, x) &= f(x), & x \in \mathbb{R}^N.
\end{align*}
\]

(2.30)

as well as optimal Schauder estimates for its solution. For this purpose, we introduce the following definition.

Definition 2.6. For any \( \alpha \notin \mathbb{N} \), we denote by \( C^{0,\alpha}([0, T] \times \mathbb{R}^N) \) the set of all continuous functions \( f : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) such that \( f(t, \cdot) \in C^\alpha_b(\mathbb{R}^N) \) for any \( t \in [0, T] \) and

\[
\|f\|_{C^{0,\alpha}([0, T] \times \mathbb{R}^N)} := \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^\alpha_b(\mathbb{R}^N)} < +\infty.
\]

We state the main result of this first part of the paper.

Theorem 2.7. Suppose that Hypotheses 2.1 are satisfied. Fix \( \theta \in (0, 1) \), \( g \in C^{0,\theta}([0, T] \times \mathbb{R}^N) \) and \( f \in C^{2+\theta}_b(\mathbb{R}^N) \). Then, problem (2.30) admits a unique bounded classical solution. Moreover, \( u(t, \cdot) \in C^{2+\theta}_b(\mathbb{R}^N) \) for any \( t \in [0, T] \) and there exists a positive constant \( C \) such that

\[
\|u\|_{C^{0,2+\theta}([0, T] \times \mathbb{R}^N)} \leq C \left( \|f\|_{C^{2+\theta}_b(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0, T] \times \mathbb{R}^N)} \right).
\]

Proof. The proof can be obtained repeating almost verbatim the arguments in the proof of [13, Theorem 2] (see also [2, Chapter 5]). For the reader’s convenience we sketch it.

The uniqueness part of the assertion is an immediate consequence of the maximum principle in Proposition 2.2.

As far the existence part and the optimal Schauder estimates are concerned, we show that the solution to problem (2.30) is given by the variation-of-constants formula

\[
u(t, x) = (G(t, 0)f)(x) + \int_0^t (G(t, r)g(r, \cdot))(x)dr, \quad t \in [0, T], \quad x \in \mathbb{R}^N,
\]

(2.32)
as in the classical case of bounded coefficients. Of course, it is enough to consider the convolution term in (2.32) (which we denote by \( \nu \)). The main step of the proof consists in showing that \( \nu \) belongs to \( C^{0,2+\theta}([0, T] \times \mathbb{R}^N) \) and

\[
\|\nu\|_{C^{0,2+\theta}([0, T] \times \mathbb{R}^N)} \leq \tilde{C} \|g\|_{C^{0,\theta}([0, T] \times \mathbb{R}^N)},
\]

(2.33)
for some positive constant $\bar{C}$, independent of $g$. Estimate (2.33) follows from the interpolation argument in [12], based on the uniform estimates of Subsection 2.2.

For any $\xi \in (0,1)$, $v(t,\cdot)$ is split into the sum $v(t,\cdot) = a_\xi(t,\cdot) + b_\xi(t,\cdot)$ where

$$a_\xi(t,x) = \begin{cases} \int_{t-\xi}^{t} (G(t,r)g(r,\cdot))(x)dr, & \xi \in [0,t), \\ \int_{0}^{t} (G(t,r)g(r,\cdot))(x)dr, & \text{otherwise}, \end{cases}$$

and

$$b_\xi(t,x) = \begin{cases} \int_{0}^{t-\xi} (G(t,r)g(r,\cdot))(x)dr, & \xi \in [0,t), \\ 0, & \text{otherwise}. \end{cases}$$

The uniform estimates in Theorem 2.4 can be used to check that $a_\xi(t,\cdot)$ and $b_\xi(t,\cdot)$ belong respectively to $C_0^\alpha (\mathbb{R}^N)$ and $C_0^{2+\alpha} (\mathbb{R}^N)$ for any $\alpha \in (\theta,1)$. Moreover,

$$||a_\xi(t,\cdot)||_{C_0^\alpha (\mathbb{R}^N)} + ||b_\xi(t,\cdot)||_{C_0^{2+\alpha} (\mathbb{R}^N)} \leq C_1 \xi^{1-(\alpha-\theta)/2} ||g||_{C^{0,\theta}([0,T] \times \mathbb{R}^N)},$$

for some positive constant $C_1$, independent of $\xi$ and $g$. This estimate shows that $v(t,\cdot)$ belongs to the interpolation space $(C_0^\alpha (\mathbb{R}^N),C_0^{2+\alpha} (\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty}$ for any $t \in [0,T]$, and

$$||v(t,\cdot)||_{(C_0^\alpha (\mathbb{R}^N),C_0^{2+\alpha} (\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty}} \leq C_1 ||g||_{C^{0,\theta}([0,T] \times \mathbb{R}^N)},$$

for any $t \in [0,T]$. Since $(C_0^\alpha (\mathbb{R}^N),C_0^{2+\alpha} (\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty} = C_0^{2+\theta} (\mathbb{R}^N)$ with equivalence of the corresponding norms (see [15, Chapter 2, Section 7, Theorem 1]), estimate (2.33) follows.

Estimate (2.33) combined with the fact that $v$ is continuous in $[0,T] \times \mathbb{R}^N$, shows that the spatial derivatives of $u$, up to the second-order, are continuous in $[0,T] \times \mathbb{R}^N$.

To conclude the proof, one just needs to show that the function $v$ is differentiable with respect to time in $[0,T] \times \mathbb{R}^N$ and $D_tv = sA^*v + g$, but this is rather straightforward and the proof is omitted. □

**Remark 2.8.**  
(i) The proof of Theorem 2.7 shows that the constant $C$ in (2.31) only depends on the constants in (2.13) which, in turn, only depend on the constants $C_j$, $K_j$, $L_j$ ($j = 1,2,3$) as well as the ellipticity constant $\nu_0$ in Hypotheses 2.1.

(ii) Formula (2.32), Theorem 2.3 and the proof of Theorem 2.7 show that the assumption $f \in C_b(\mathbb{R}^d)$ is enough for problem (2.30) to have a bounded classical solution.

3. **The case when the diffusion coefficients are only measurable in the pair $(t,x)$**

In this section, we consider some situation in which the diffusion coefficients are bounded but not continuous in $[0,T] \times \mathbb{R}^N$.

To state our standing assumptions and, then, the main result of this section, let us give the following definition, which is the counterpart of Definition 2.6 in this new setting.

**Definition 3.1.** Fix $\alpha > 0$. 

(i) $M^{0,\alpha}([0,T] \times \mathbb{R}^N)$ denotes the space of all measurable functions $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that $f(t,\cdot) \in C^\alpha(B(0,R))$ for any $t \in [0,T]$ and any $R > 0$, and the supremum of the $C^\alpha(B(0,R))$-norms of $f(t,\cdot)$, when $t$ runs in $[0,T]$, is finite for any $R > 0$. Note that it may blow up as $R \to +\infty$.

(ii) $B^{0,\alpha}([0,T] \times \mathbb{R}^N)$ denotes the subset of $M^{0,\alpha}([0,T] \times \mathbb{R}^N)$ of all bounded functions $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that $f(t,\cdot) \in C^\alpha(B(0,R))$ for any $t \in [0,T]$ and the supremum of the $C^\alpha$-norms of $f(t,\cdot)$, when $t$ runs in $[0,T]$, is finite. We norm $B^{0,\alpha}([0,T] \times \mathbb{R}^N)$ by setting

$$
\|f\|_{B^{0,\alpha}([0,T] \times \mathbb{R}^N)} := \sup_{t \in [0,T]} \|f(t,\cdot)\|_{C^\alpha(B(0,R))}.
$$

**Hypotheses 3.2.**

(i) the coefficients $q_{ij} = q_{ji}$, $b_j$ $(i,j=1,\ldots,N)$ and $c$ belong to $M^{0,3+\delta}([0,T] \times \mathbb{R}^N)$ for some $\delta \in (0,1)$.

(ii) Hypotheses 2.1(ii) to 2.1(vii) are satisfied with $[0,T]$ being replaced by $\mathcal{D}$, where $[0,T] \setminus \mathcal{D}$ is a negligible set. Moreover, for any $x \in \mathbb{R}^N$, the functions $d(\cdot,x)$, $r(\cdot,x)$ and $\varrho(\cdot,x)$ are bounded and measurable in $(0,T)$.

Since the coefficients of the operator $\mathcal{A}$ are not continuous, we do not expect that the Cauchy problem

$$
\begin{align*}
D_t u(t,x) &= (\mathcal{A}u)(t,x) + g(t,x), \quad t \in [0,T], \quad x \in \mathbb{R}^N, \\
 u(0,x) &= f(x), \quad x \in \mathbb{R}^N.
\end{align*}
$$

has a solution $u$ with the smoothness properties in Theorem 2.7 even if the data $f$ and $g$ are smooth. In the spirit of [6, 10, 11], we give the following definition of solution to problem (3.1).

**Definition 3.3.** Let $f \in C^2_b(\mathbb{R}^N)$ and $g$ be a bounded and measurable function such that $g(t,\cdot)$ is continuous in $\mathbb{R}^N$ for any $t \in [0,T]$. A bounded function $u : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is called a solution to (3.1) if the following conditions are satisfied:

(i) the function $u$ is Lipschitz continuous in $[0,T] \times B(0,R)$ for any $R > 0$, its first- and second-order space derivatives are bounded and continuous functions in $[0,T] \times \mathbb{R}^N$;

(ii) $u(0,x) = f(x)$ for any $x \in \mathbb{R}^N$;

(iii) there exists a set $G \subset [0,T] \times \mathbb{R}^N$, with negligible complement, such that $D_t u(t,x) = \mathcal{A} u(t,x) + g(t,x)$ for any $(t,x) \in G$. Moreover, for any $x \in \mathbb{R}^N$, the set $G_x = \{t \in [0,T] : (t,x) \in G\}$ is measurable with measure $T$.

Let us now prove the following lemmas which play a fundamental role in the proof of the main result of this section.

**Lemma 3.4.** Let $f \in M^{0,\theta}([0,T] \times \mathbb{R}^N)$ for some $\theta \in (0,1)$. Then, the following properties hold.

(i) The function $f(\cdot,x)$ is measurable for any $x \in \mathbb{R}^N$.

(ii) There exists a measurable set $C \subset [0,T]$, whose complement is negligible in $[0,T]$, such that the function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$, defined by

$$
F(t,x) = \int_0^t f(s,x) ds, \quad (t,x) \in [0,T] \times \mathbb{R}^N,
$$

is differentiable with respect to $t$ in $C \times \mathbb{R}^N$ and, therein, $D_t F = f$. 

Proof. (i). Since \( f \) is measurable, the function \( f(\cdot, x) \) is measurable in \((0, T)\) for almost any \( x \in \mathbb{R}^N \). Let us denote by \( H \) the set of all \( x \in \mathbb{R}^N \) such that the function \( f(\cdot, x) \) is measurable, and prove that \( H = \mathbb{R}^N \). For this purpose, we observe that, since it has a negligible complement, \( H \) is a dense subset of \( \mathbb{R}^N \). Hence, for any \( x \in \mathbb{R}^N \), there exists a sequence \((x_n) \subset H\) converging to \( x \) in \( \mathbb{R}^N \). Since \( f \in \mathcal{M}^{\alpha, \theta}([0, T] \times \mathbb{R}^N) \), the function \( f(t, \cdot) \) is Hölder continuous in \( B(0, R) \) of exponent \( \theta \), uniformly with respect to \( t \in [0, T] \), where \( R := \max_{n \in \mathbb{N}} |x_n| + 1 \). Hence, there exists a positive constant \( C \), independent of \( n \) and \( t \), such that

\[
|f(t, x) - f(t, x_n)| \leq C|x - x_n|^{\theta}, \quad t \in [0, T].
\]

This shows that \( f(\cdot, x_n) \) converges to the function \( f(\cdot, x) \) uniformly in \([0, T]\). Consequently, \( f(\cdot, x) \) is measurable in \([0, T]\).

(ii). Since, by step (i), \( f(\cdot, x) \in L^\infty(0, T) \) for any fixed \( x \in \mathbb{R}^N \), the function \( F \) is well defined and, for any \( x \in \mathbb{R}^N \), there exists a measurable set \( C_x \), with negligible complement, such that \( F(\cdot, x) \) is differentiable in \( C_x \) and \( D_1F(t, x) = f(t, x) \) for any \( t \in C_x \).

Let us set \( C = \cap_{x \in \mathbb{Q}^N} C_x \) and fix \((t, x) \in C \times \mathbb{R}^N \). Further, let \((x_n) \subset \mathbb{Q}^N \) converge to \( x \) as \( n \to +\infty \). Then, for any \( h \in \mathbb{R} \setminus \{0\} \), we can estimate

\[
\left| \frac{F(t + h, x) - F(t, x)}{h} - f(t, x) \right| \\
\leq \frac{1}{|h|} \left| \int_t^{t+h} |f(s, x) - f(t, x)| \, ds \right| \\
\leq \frac{1}{|h|} \left| \int_t^{t+h} |f(s, x) - f(s, x_n)| \, ds \right| + \frac{1}{|h|} \left| \int_t^{t+h} |f(s, x_n) - f(t, x_n)| \, ds \right| \\
+ |f(t, x_n) - f(t, x)| \\
\leq 2 \sup_{t \in [0,T]} \|f(t, \cdot)\|_{C^\infty(B(0,R))} |x - x_n|^\theta + \frac{1}{|h|} \left| \int_t^{t+h} |f(s, x_n) - f(t, x_n)| \, ds \right|, 
\]

where \( R := \max_{n \in \mathbb{N}} |x_n| + 1 \). Letting \( h \to 0 \), we get

\[
\limsup_{h \to 0} \left| \frac{F(t + h, x) - F(t, x)}{h} - f(t, x) \right| \leq 2 \|f\|_{B^{\theta, \theta}([0,T] \times \mathbb{R}^N)} |x - x_n|^\theta, 
\]

for any \( n \in \mathbb{N} \), which implies that \( F \) is differentiable with respect to time at \((t, x)\) and \( D_1 F(t, x) = f(t, x) \). This completes the proof. \( \square \)

Lemma 3.5. Assume that Hypotheses 3.2 are satisfied. Then, there exist sequences \((q_{ij}^{(n)})\), \((b_{ij}^{(n)})\) and \((c^{(n)})\) with the following properties:

(i) \( q_{ij}^{(n)} \), \( b_{ij}^{(n)} \) \((i, j = 1, \ldots, N)\), \( c^{(n)} \) and their spatial derivatives, up to the third-order, belong to \( C^{3/2, \theta}([0, T] \times B(0, R)) \) for any \( R > 0 \);

(ii) there exists a measurable set \( E \subset [0, T] \), whose complement in \([0, T] \) is negligible, such that, for any \( i, j = 1, \ldots, N \), \( q_{ij}^{(n)} \), \( b_{ij}^{(n)} \) and \( c^{(n)} \) converge pointwise in \( E \times \mathbb{R}^N \), respectively to \( q_{ij} \), \( b_{ij} \) and \( c \), as \( n \to +\infty \);

(iii) for any \( n \in \mathbb{N} \), the functions \( q_{ij}^{(n)} \), \( b_{ij}^{(n)} \) \((i, j = 1, \ldots, N)\) and \( c^{(n)} \) satisfy Hypotheses 2.1(ii) to 2.1(vii) with the functions \( v, d, r, q \) being replaced by new functions \( v_n, d_n, r_n, q_n \) and the same constants \( C, L_1, L_2, L_3 \). Moreover, there
exist two positive constants \( \hat{\nu}_0 \) and \( c_0 \) such that \( \nu_n(t, x) \geq \hat{\nu}_0 \) and \( c_n(t, x) \leq \hat{c}_0 \) for any \( (t, x) \in [0, T] \times \mathbb{R}^N \) and any \( n \in \mathbb{N} \).

**Proof.** By Lemma 3.4, the functions \( q_{ij}(\cdot, x), b_{ij}(\cdot, x) \) \((i, j = 1, \ldots, N)\) and \( c(\cdot, x) \) are in \( L^\infty(0, T) \) for any \( x \in \mathbb{R}^N \). Thus, for any \( n \in \mathbb{N} \), we can define the functions \( q^{(n)}_{ij}, b^{(n)}_{ij} \) and \( c^{(n)} \) by setting

\[
q^{(n)}_{ij}(t, x) = \left(\frac{n}{4\pi}\right)^{\frac{1}{2}} \int_0^T q_{ij}(\tau, x) \exp\left(-\frac{n}{4} |t - \tau|^2\right) d\tau,
\]

\[
b^{(n)}_{ij}(t, x) = \left(\frac{n}{4\pi}\right)^{\frac{1}{2}} \int_0^T b_{ij}(\tau, x) \exp\left(-\frac{n}{4} |t - \tau|^2\right) d\tau,
\]

\[
c^{(n)}(t, x) = \left(\frac{n}{4\pi}\right)^{\frac{1}{2}} \int_0^T c(\tau, x) \exp\left(-\frac{n}{4} |t - \tau|^2\right) d\tau,
\]

for any \( (t, x) \in [0, T] \times \mathbb{R}^N \) and any \( i, j = 1, \ldots, N \).

Clearly, \( q^{(n)}_{ij}, b^{(n)}_{ij} \) and \( c^{(n)} \) and their spatial derivatives, up to the third-order, belong to \( C^{3/2,q}([0, T] \times B(0, R)) \) for any \( i, j = 1, \ldots, N \) and any \( R > 0 \).

Let us prove that, for any \( i, j = 1, \ldots, N \), \( q^{(n)}_{ij} \) converges pointwise in \( \mathcal{E} \times \mathbb{Q}^N \) to \( q_{ij} \), for some measurable set \( \mathcal{E} \subset [0, T] \) whose complement is negligible. Then, the same argument can be applied to prove the convergence of \( b^{(n)}_{ij} \) and \( c^{(n)} \) to \( b_{ij} \) and \( c \), respectively.

Since \( q^{(n)}_{ij}(\cdot, x) \to q_{ij}(\cdot, x) \) in \( L^p(0, T) \) for any \( p \in [1, +\infty) \), any \( x \in \mathbb{R}^N \) and any \( i, j = 1, \ldots, N \), we can find out an increasing sequence \( (n_k) \subset \mathbb{N} \) such that the subsequence \( q^{(n_k)}_{ij}(t, x) \) converges to \( q_{ij}(t, x) \) as \( n \) tends to \( +\infty \) almost everywhere in \((0, T)\). By a classical diagonal procedure, we can determine an increasing sequence \( (n_k) \subset \mathbb{N} \) and a measurable set \( \mathcal{E} \subset [0, T] \), whose complement is negligible in \([0, T] \), such that

\[
(3.2) \quad \lim_{k \to +\infty} q^{(n_k)}_{ij}(t, x) = q_{ij}(t, x), \quad (t, x) \in \mathcal{E} \times \mathbb{Q}^N, \ i, j = 1, \ldots, N.
\]

Let us now show that we can extend (3.2) to any \((t, x) \in \mathcal{E} \times \mathbb{R}^N\). For this purpose, we fix \((t, x) \in \mathcal{E} \times \mathbb{R}^N\) and a sequence \((x_m) \subset \mathbb{Q}^N\) converging to \(x\) as \(m \) tends to \(+\infty\). Since

\[
\sup_{t \in (0, T)} \|q^{(n_k)}_{ij}(t, \cdot)\|_{C^0(B(0, R))} \leq \|q_{ij}(t, \cdot)\|_{C^0(B(0, R))},
\]

for any \(i, j = 1, \ldots, N\), where \(R = 1 + \sup_{m \in \mathbb{N}} |x_m|\), we can write

\[
|q^{(n_k)}_{ij}(t, x) - q_{ij}(t, x)| \\
\leq |q^{(n_k)}_{ij}(t, x_m) - q^{(n_k)}_{ij}(t, x)| + |q^{(n_k)}_{ij}(t, x_m) - q_{ij}(t, x_m)| + |q_{ij}(t, x_m) - q_{ij}(t, x)|
\]

\[
\leq 2 \sup_{t \in (0, T)} \|q_{ij}(t, \cdot)\|_{C^0(B(0, R))} |x - x_m|^\theta + |q^{(n_k)}_{ij}(t, x_m) - q_{ij}(t, x_m)|,
\]

for any \(k, m \in \mathbb{N}\). Taking, first, the limsup as \(k \to +\infty\) in the first- and last-side of (3.3), and then letting \(m \to +\infty\), (3.2) follows, for any \(x \in \mathbb{R}^N\). Property (ii) is proved.

Let us now prove property (iii). Taking Hypotheses 2.1(i) into account, we get

\[
\langle Q^{(n)}(t, x) \xi, \xi \rangle \geq \langle \xi \rangle^2 \left(\frac{n}{4\pi}\right)^{\frac{1}{2}} \int_0^T \nu(\tau, x) e^{-\frac{1}{4} |t - \tau|^2} d\tau := \nu_n(t, x) |\xi|^2,
\]
for any $\xi \in \mathbb{R}^N$, any $(t, x) \in [0, T] \times \mathbb{R}^N$ and any $n \in \mathbb{N}$, where $Q^{(n)} = (q^{(n)}_{ij})$. Note that the function $\nu_n$ can be bounded from below in $[0, T] \times \mathbb{R}^N$ by a positive constant, independent of $n$. Indeed,

$$\nu_n(t, x) \geq \nu_0 \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \left\{ \int_0^t e^{-\frac{n}{4} r^2} \, dr + \int_0^{T-t} e^{-\frac{n}{4} r^2} \, dr \right\}$$

$$\geq \nu_0 \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T e^{-\frac{n}{4} s^2} \, ds$$

\[ (3.4) \]

for any $(t, x) \in [0, T] \times \mathbb{R}^N$.

Now, an easy computation shows that, for any $n \in \mathbb{N}$, the functions $a^{(n)}_{ij}, b^{(n)}_{ij}$ $(i, j = 1, \ldots, N)$ and $c^{(n)}$ satisfy Hypothesis 2.1(iii) and 2.1(v) as well as conditions (2.4)-(2.6), with the same constants $C_j$ and $K_j$ $(j = 1, 2, 3)$ and $\nu, d, r, g$ being replaced with the functions $\nu_n, d_n, r_n, g_n$, where

$$d_n(t, x) = \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T d(\tau, x) \exp \left( -\frac{n}{4} |t - \tau|^2 \right) \, d\tau,$$

$$r_n(t, x) = \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T r(\tau, x) \exp \left( -\frac{n}{4} |t - \tau|^2 \right) \, d\tau,$$

$$g_n(t, x) = \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T g(\tau, x) \exp \left( -\frac{n}{4} |t - \tau|^2 \right) \, d\tau,$$

for any $(t, x) \in [0, T] \times \mathbb{R}^N$. Arguing as in the proof of (3.4), we can easily show that

$$g_n(t, x) \geq \frac{g_0}{2\sqrt{\pi}} \int_0^T e^{-\frac{n}{4} s^2} \, ds.$$

Moreover, integrating condition (2.7) we get

$$d_n(t, x) + L_1 r_n(t, x) + L_2 \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T g^2(\tau, x) \exp \left( -\frac{n}{4} |t - \tau|^2 \right) \, d\tau \leq L_3 \nu_n(t, x),$$

for any $(t, x) \in [0, T] \times \mathbb{R}^N$ and any $n \in \mathbb{N}$. Hölder inequality yields

$$\left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T g^2(\tau, x) \exp \left( -\frac{n}{4} |t - \tau|^2 \right) \, d\tau \geq \frac{n}{4\pi} \left( \int_0^T g(\tau, x) \exp \left( -\frac{n}{4} |t - \tau|^2 \right) \, d\tau \right)^2$$

$$= (g^{(n)}(t, x))^2,$$

for any $(t, x)$ as above. Hence, condition (2.7) is satisfied with $d, r, g$ being replaced by $d_n, r_n, g_n$ and the same constants $L_1, L_2, L_3$.

Next, we observe that $c^{(n)}$ satisfies Hypothesis 2.1(iv) with $c_0$ being replaced by

$$\frac{c_0}{2\sqrt{\pi}} \int_0^T e^{-\frac{n}{4} s^2} \, ds.$$

Finally, integrating condition (2.8) with respect to time, we easily deduce that

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}^N} \{(c^{(n)})(t, x) - c^{(n)}(0, x)\} \leq \sup_{(t, x) \in (0, T) \times \mathbb{R}^N} \{(c^{(n)})(t, x) - c^{(n)}(0, x)\} < +\infty.$$
for any \( n \in \mathbb{N} \), where by \( \mathcal{A}^{(n)} \) we have denoted the elliptic operator whose coefficients are \( q_{ij}^{(n)}, b_j^{(n)} \) (\( i, j = 1, \ldots, N \)) and \( c^{(n)} \). This completes the proof. \( \Box \)

We now consider the following maximum principle, which generalizes Proposition 2.2 to the case of noncontinuous coefficients.

**Proposition 3.6.** Assume that Hypotheses 3.2 are satisfied and let \( u \) be a solution to problem (3.1), in the sense of Definition 3.3, corresponding to \( f \in C_0^2(\mathbb{R}^N) \) and a bounded and measurable function \( g : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) such that the function \( g(t, \cdot) \) is continuous for any \( t \in [0, T] \). If \( f \leq 0 \) and \( g \leq 0 \), then \( u \leq 0 \).

**Proof.** To begin with, we observe that there exists a positive function \( \hat{\phi} : \mathbb{R}^N \to \mathbb{R} \) which blows up as \( |x| \to +\infty \) and \( \mathcal{A} \hat{\phi} - \lambda \hat{\phi} < 0 \) in \([0, T] \times \mathbb{R}^N \) for some \( \lambda > 0 \). It suffices to replace in Hypothesis 2.1(vii) the function \( \varphi \) and the positive constant \( \lambda \), respectively, with the function \( \hat{\varphi} = \varphi + C \) and \( \hat{\lambda} = \max\{\lambda, 2c_0\} \), and to take \( C \) sufficiently large.

Let \( u \) be a solution to problem (3.1). For any \( n \in \mathbb{N} \), we introduce the function \( v_n : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) defined by

\[
v_n(t, x) = e^{-\lambda t} u(t, x) - \frac{1}{n} \hat{\varphi}(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^N.
\]

Clearly, \( v_n \in W^{1,1}_{0,1}((0, T) \times B(0, R)) \subset W^{1,2}_{N+1}((0, T) \times B(0, R)) \) for any \( R > 0 \), it satisfies the differential inequality \( D_t v_n - \mathcal{A} v_n + \hat{\lambda} v_n \leq e^{-\lambda} g \) in the sense of distributions, and \( v_n(0, \cdot) \leq f \). Hence, the Nazarov-Ural’tseva maximum principle (see [14, Theorem 1]) may be applied. It yields

\[
v_n(t, x) \leq \sup_{(t, x) \in (0, T) \times \partial B(0, R)} v_n^+(t, x),
\]

for any \((t, x) \in [0, T] \times \overline{B(0, R)}\) and any \( R > 0 \). Here, \( v_n^+ \) denotes the positive part of the function \( v_n \). Since, for any \( n \in \mathbb{N} \), \( v_n(t, x) \) tends to \(-\infty\) as \(|x| \to +\infty\), uniformly with respect to \( t \), \( v_n(t, x) \leq 0 \) for any \((t, x) \in [0, T] \times \mathbb{R}^N \). Letting \( n \to +\infty \), yields the assertion. \( \Box \)

We are now in a position to prove the main result of this section.

**Theorem 3.7.** Let Hypotheses 3.2 be satisfied. Fix \( \theta \in (0, 1) \) and suppose that \( f \in C_0^{2+\theta}(\mathbb{R}^N) \) and \( g \in B^{0,\theta}(0, T) \times \mathbb{R}^N \). Then, the Cauchy problem (3.1) admits a unique solution \( u \), in the sense of Definition 3.3. The function \( u \) belongs to \( B^{0,2+\theta}([0, T] \times \mathbb{R}^N) \) and there exists a positive constant \( C \), independent of \( f \) and \( g \), such that

\[
\|u\|_{B^{0,2+\theta}([0, T] \times \mathbb{R}^N)} \leq C \left( \|f\|_{C_0^{2+\theta}(\mathbb{R}^N)} + \|g\|_{B^{0,\theta}(0, T) \times \mathbb{R}^N} \right).
\]

**Proof.** The uniqueness of the solution is a straightforward consequence of the maximum principle in Proposition 3.6.

To prove that problem (3.1) actually admits a solution in the sense of Definition 3.3, we use an approximation argument. For any \( n \in \mathbb{N} \), we introduce the operator \( \mathcal{A}^{(n)} \) defined by

\[
\mathcal{A}^{(n)} = \sum_{i,j=1}^N q_{ij}^{(n)} D_{ij}^2 + \sum_{j=1}^N b_j^{(n)} D_j + c^{(n)} I,
\]
where the coefficients \( g_{ij}^{(n)}, b_{ij}^{(n)} \) \((i, j = 1, \ldots, N)\) and \( c^{(n)} \) are defined in Lemma 3.5. We further approximate the function \( g \) by a sequence of functions \( g^{(n)} \in C^{0, \theta}([0, T] \times \mathbb{R}^N) \), defined by

\[
g^{(n)}(t, x) = \left( \frac{n}{4\pi} \right)^{\frac{1}{2}} \int_0^T g(t, x) \exp \left( -\frac{n}{4} |\tau|^2 \right) d\tau, \quad (t, x) \in [0, T] \times \mathbb{R}^N.
\]

Clearly, \( \|g^{(n)}\|_{C^{0, \theta}([0, T] \times \mathbb{R}^N)} \leq \|g\|_{C^{0, \theta}([0, T] \times \mathbb{R}^N)} \) for any \( n \in \mathbb{N} \). Moreover, by the proof of Lemma 3.5, there exists a set \( \mathcal{D} \), whose complement is negligible in \([0, T]\), such that \( g^{(n)}(t, x) \) tends to \( g(t, x) \) as \( n \to +\infty \), for any \((t, x) \in \mathcal{D} \times \mathbb{R}^N \). Again, Lemma 3.5 implies that the coefficients of the operator \( \mathcal{A}^{(n)} \) satisfy Hypotheses 2.1, with constants independent of \( n \). Hence, the Cauchy problem

\[
\begin{aligned}
D_t u(t, x) &= \mathcal{A}^{(n)} u(t, x) + g^{(n)}(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}^N, \\
u(0, x) &= f(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\]

admits a unique classical solution \( u_n \) which belongs to \( C^{0,2+\theta}([0, T] \times \mathbb{R}^N) \) and

\[
\|u_n\|_{C^{0,2+\theta}([0, T] \times \mathbb{R}^N)} \leq C_1 \left( \|f\|_{C^{0,2+\theta}([0, T] \times \mathbb{R}^N)} + \|g^{(n)}\|_{C^{0, \theta}([0, T] \times \mathbb{R}^N)} \right)
\]

\[
\leq C_1 \left( \|f\|_{C^{0,2+\theta}([0, T] \times \mathbb{R}^N)} + \|g\|_{C^{0, \theta}([0, T] \times \mathbb{R}^N)} \right),
\]

for some positive constant \( C_1 \), independent of \( n \) (see Remark 2.8(i)).

From the differential equation in (3.5) and the estimate (3.6) it follows that, for any \( R > 0 \), the sequence \( (D_t u_n) \) is bounded in \([0, T] \times B(0, R)\). By a byproduct, \( \|u_n\|_{\text{Lip}([0, T] \times B(0, R))} \leq C \) for some positive constant, independent of \( n \). Using an interpolation argument, we can now show that the functions \( D_t u_n \) and \( D_{ij} u_n \) \((i, j = 1, \ldots, N, n \in \mathbb{N})\) are equibounded and equicontinuous in \([0, T] \times B(0, R)\). Indeed, it is well known that there exists a positive constant \( K \) such that

\[
\|\psi\|_{C^1(B(0, R))} \leq K \|\psi\|_{C^{1+\theta}(B(0, R))} \|\psi\|_{C^{2+\theta}(B(0, R))},
\]

\[
\|\psi\|_{C^2(B(0, R))} \leq K \|\psi\|_{C^{1+\theta}(B(0, R))} \|\psi\|_{C^{2+\theta}(B(0, R))},
\]

for any \( \psi \in C^{2+\theta}(B(0, R)) \). It follows that

\[
\|\nabla_x u_n(t, \cdot) - \nabla_x u_n(s, \cdot)\|_{C(B(0, R))} \leq K \|u_n(t, \cdot) - u_n(s, \cdot)\|_{C^{1+\theta}(B(0, R))} \|\nabla_x u_n(t, \cdot) - u_n(s, \cdot)\|_{C^{2+\theta}(B(0, R))}
\]

\[
\leq K_\theta \|u_n\|_{C^{1,2+\theta}([0, T] \times \mathbb{R}^N)} \|u_n\|_{\text{Lip}([0, T] \times \mathbb{R}^N)} \|\nabla_x u_n(t, s)\|_{C^{2+\theta}(B(0, R))} |t - s|^{\frac{1+\theta}{2+\theta}},
\]

\[
\leq K'_\theta \left( \|f\|_{C^{1,2+\theta}([0, T] \times \mathbb{R}^N)} + \|g\|_{C^{0, \theta}([0, T] \times \mathbb{R}^N)} \right) |t - s|^{\frac{1+\theta}{2+\theta}},
\]

and, similarly,

\[
\|D^2 u_n(t, \cdot) - D^2 u_n(s, \cdot)\|_{C(B(0, R))} \leq K''_\theta \left( \|f\|_{C^{1,2+\theta}([0, T] \times \mathbb{R}^N)} + \|g\|_{C^{0, \theta}([0, T] \times \mathbb{R}^N)} \right) |t - s|^{\frac{1+\theta}{2+\theta}},
\]

for any \( s, t \in [0, T] \). Here, \( K_\theta, K'_\theta \) and \( K''_\theta \) are positive constants which may blow up as \( R \to +\infty \). The previous estimates show that, for any \( i, j = 1, \ldots, N \), the sequences \( (D_t u_n) \) and \( (D_{ij} u_n) \) are equibounded and equicontinuous in \([0, T] \times B(0, R)\). Since \( R \) is arbitrary, by Ascoli-Arzel`a theorem there exists a function \( u \in C^{0,2}([0, T] \times \mathbb{R}^N) \) and a subsequence \( (u_{n_k}) \) converging to \( u \) in \( C^{0,2}([0, T] \times K) \).
for any compact set $K \subset \mathbb{R}^N$. Moreover, $u$ belongs to $\text{Lip}([0, T] \times \overline{B(0, R)})$ for any $R > 0$. Hence, for any $x \in \mathbb{R}^N$, the function $u(\cdot, x)$ is differentiable almost everywhere in $(0, T)$. Clearly, $u(0, \cdot) \equiv f$ since $u_{nk}(0, \cdot) \equiv f$ for any $k \in \mathbb{N}$.

To complete the proof, let us show that $u$ is differentiable with respect to $t$ in $G \times \mathbb{R}^N$, for some measurable set $G \subset [0, T]$, whose complement is negligible, and $D_t u(t, x) = \mathcal{A} u(t, x) + g(t, x)$ for such values of $t$. For this purpose, we observe that, for any $(t, x) \in [0, T] \times \mathbb{R}^N$, it holds that

$$u_{nk}(t, x) = f(x) + \int_0^t \left( \mathcal{A}^{(nk)} u_{nk}(s, x) + g^{(nk)}(s, x) \right) ds. \tag{3.7}$$

Taking Lemma 3.5 into account, we can let $k \to +\infty$ in both the sides of (3.7). This yields

$$u(t, x) = f(x) + \int_0^t (\mathcal{A} u(s, x) + g(s, x)) ds, \quad (t, x) \in [0, T] \times \mathbb{R}^N.$$ 

The assumptions on the coefficients of the operator $\mathcal{A}$ and the regularity properties of the function $u$, already proved, imply that the function $\mathcal{A} u + g$ satisfies the assumptions of Lemma 3.4(ii). Therefore, there exists a set $G \subset [0, T]$, whose complement is negligible in $[0, T]$, such that $u$ is differentiable in $G \times \mathbb{R}^N$ with respect to the time variable and $D_t u = \mathcal{A} u + g$ in $G \times \mathbb{R}^N$. This accomplishes the proof. \hfill \Box

Taking Remark 2.8(i) into account and using the same argument as in the proof of Theorem 3.7, one can show that Theorem 2.7 holds true also under a slightly weaker regularity assumption on the coefficients of the operator $\mathcal{A}$. More precisely,

**Theorem 3.8.** Suppose that Hypotheses 2.1 are satisfied, but 2.1(i), in which the space $C^{3/2, \delta}((0, T) \times B(0, R))$ is replaced with $C^{0, \delta}([0, T] \times B(0, R))$ (defined as in Definition 2.6, with $\mathbb{R}^N$ replaced by $B(0, R)$). Then, the assertion of Theorem 2.7 holds true.

4. **An example**

In this section, we exhibit a class of nonautonomous elliptic operators with unbounded coefficients that satisfy the assumptions of Theorems 3.7 and 3.8.

Let $\mathcal{A}$ be the elliptic operator defined by

$$(\mathcal{A} \psi)(t, x) = (1 + |x|^2)^p \sum_{i,j=1}^N q^{(0)}_{ij}(t, x) D_{ij} \psi(x) + b^{(0)}(t)(1 + |x|^2)^q \langle x, \nabla_x \psi(x) \rangle + (c^{(0)}(t, x) - |x|^{2r}) \psi(x),$$

for any $(t, x) \in [0, T] \times \mathbb{R}^N$, on smooth functions $\psi : \mathbb{R}^N \to \mathbb{R}$. We assume that the coefficients of the operator $\mathcal{A}$ satisfy either

**Hypotheses 4.1.**

(i) The functions $q^{(0)}_{ij}$ $(i, j = 1, \ldots, N)$ and $c^{(0)}$ are thrice continuously differentiable with respect to the spatial variables in $[0, T] \times \mathbb{R}^N$. Moreover, they are bounded together with their spatial derivatives up to the third-order. Further, there exists $\delta \in (0, 1)$ such that, for any $R > 0$, the third-order spatial derivatives of $q^{(0)}_{ij}$ $(i, j = 1, \ldots, N)$ and $c^{(0)}$ are $\delta$-Hölder continuous in $B(0, R)$, uniformly with respect to $t \in [0, T]$;
(ii) there exists a positive constant \( \nu_0 \) such that
\[
\langle Q^0(t, x) \rangle \geq \nu_0 |\xi|^2, \quad t \in [0, T], \ x, \xi \in \mathbb{R}^N;
\]
(iii) \( p, q, r \in \mathbb{N} \cup \{0\} \) satisfy \( p \leq q \);
(iv) the function \( b^{(0)} \) is continuous and \( b^{(0)}(t) < 0 \) for any \( t \in [0, T] \),
or
\begin{enumerate}
\item The functions \( q_{ij}^{(0)} \) \( (i, j = 1, \ldots, N) \) and \( c^{(0)} \) are \( B^{p, \bar{q}, \bar{r}}((0, T) \times \mathbb{R}^N) \) and the third-order spatial derivatives are in \( M^{p, \bar{q}, \bar{r}}((0, T) \times \mathbb{R}^N) \) for some \( \delta \in (0, 1) \);
(ii) there exists a positive constant \( \nu_0 \) such that
\[
\langle Q^0(t, x) \rangle \geq \nu_0 |\xi|^2, \quad t \in F, \ x, \xi \in \mathbb{R}^N,
\]
where \( F \) is a measurable set whose complement is negligible in \([0, T] \);
(iii) \( p, q, r \in \mathbb{N} \cup \{0\} \) satisfy \( p \leq q \);
(iv) the function \( b^{(0)} \) is bounded and measurable in \((0, T) \) and there exists a negative constant \( b_0 \) such that \( b^{(0)}(t) \leq b_0 \) for almost any \( t \in (0,T) \).
\end{enumerate}

Let us check that, under Hypotheses 4.1, the operator \( \mathcal{A} \) satisfies the assumptions of Theorem 3.8. The same arguments will show that, if \( \mathcal{A} \) satisfies Hypotheses 4.2, then it satisfies also Hypotheses 3.2, so that Theorem 3.7 holds true.

It is immediate to check that the coefficients \( q_{ij}, b_j \) and \( c \), where
\[
q_{ij}(t, x) = q_{ij}^{(0)}(t, x)(1 + |x|^2)^p,
b_j(t, x) = b^{(0)}(t)x_j(1 + |x|^2)^q,
c(t, x) = c^{(0)}(t, x) - |x|^{2r},
\]
for any \( (t, x) \in [0, T] \times \mathbb{R}^N \) and any \( i, j = 1, \ldots, N \), are thrice continuously differentiable with respect to \( x \) and, for any \( R > 0 \), the third-order derivatives are Hölder continuous of exponent \( \delta \) with respect to the variable \( x \in B(0, R) \), uniformly with respect to \( t \in [0, T] \). Similarly, checking Hypotheses 2.1(ii) to 2.1(v) is an easy task. As far as Hypothesis 2.1(vi) is concerned, we observe that
\[
\langle Db(t, x)\xi, \xi \rangle = b^{(0)}(t)(1 + |x|^2)^q - (1 + |x|^2)|\xi|^2 + 2q(\xi, x)^2,
\]
for any \( t \in [0, T] \) and any \( x, \xi \in \mathbb{R}^N \). Since \( b \) is negative in \([0, T] \), we can estimate
\[
\langle Db(t, x)\xi, \xi \rangle \leq b_0(1 + |x|^2)^q|\xi|^2 \equiv d(t, x)|\xi|^2,
\]
for any \( t, x, \xi \) as above, where \( b_0 := \sup_{t \in [0, T]} b^{(0)} \). It is then easy to check that
\[
|D^\beta b_j(t, x)| \leq \kappa_1(1 + |x|^2)^q, \quad |\beta| = 2, 3,
|D^\beta c(t, x)| \leq \kappa_2(1 + |x|^2)^r, \quad |\beta| = 1, 2, 3,
\]
for some positive constants \( \kappa_1 \) and \( \kappa_2 \), any \( (t, x) \in [0, T] \times \mathbb{R}^N \) and any \( j = 1, \ldots, N \). Hence, we can take \( r(t, x) := \kappa_1(1 + |x|^2)^q \) and \( g(t, x) := \kappa_2(1 + |x|^2)^r \) in (2.5) and (2.6). Condition (2.7) then reads as follows:
\[
b_0(1 + |x|^2)^q + L_1 \kappa_1(1 + |x|^2)^q + L_2 \kappa_2(1 + |x|^2)^2r \leq L_3 \nu_0(1 + |x|^2)^p,
\]
for any \( t \in [0, T] \) and any \( x \in \mathbb{R}^N \). This inequality is clearly satisfied by suitable constants \( L_1, L_2, L_3 \) by Hypothesis 4.1(iii). Finally, taking \( \varphi(x) = 1 + |x|^2 \) for any \( x \in \mathbb{R}^N \), we get
\[
(\mathcal{A}\varphi)(t, x) = 2\text{Tr}(Q(t, x)) + 2b^{(0)}(t)|x|^2(1 + |x|^2)^q + (c_0(t, x) - |x|^{2r})(1 + |x|^2)
\]
where $\|Q^{(0)}\|_{\infty} = \sup_{(t,x) \in [0,T] \times \mathbb{R}^N} |Q^{(0)}(t,x)|$. Due to Hypothesis 4.1(iii), we can estimate the last side of (4.8) from above by $\kappa_3 + \|c^{(0)}\|_{\infty}(1 + |x|^2)$ for any $(t,x) \in [0,T] \times \mathbb{R}^N$ and some positive constant $\kappa_3$. Hence, Hypothesis 2.1(vii) is satisfied with $\lambda = \|c^{(0)}\|_{\infty} + \kappa_3$.

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REFERENCES

[1] (2139521) M. Bertoldi and L. Lorenzi, Estimates of the derivatives for parabolic operators with unbounded coefficients, Trans. Amer. Math. Soc. 357 (2005), 2627–2664.
[2] (2313847) M. Bertoldi and L. Lorenzi, “Analytical Methods for Markov Semigroups”, Chapman and Hall/CRC Press, Boca Raton, FL, 2007.
[3] (1373775) S. Cerrai, Elliptic and parabolic equations in $\mathbb{R}^n$ with coefficients having polynomial growth, Comm. Partial Differential Equations 21 (1996), 281–317.
[4] (0181836) A. Friedman, “Partial Differential Equations of Parabolic Type,” Prentice-Hall Inc., 1964.
[5] M. Kunze, L. Lorenzi and A. Lunardi, Nonautonomous Kolmogorov parabolic equations with unbounded coefficients, Trans. Amer. Math. Soc. (to appear).
[6] N.V. Krylov and E. Priola, Elliptic and parabolic second-order PDEs with growing coefficients, preprint, arXiv:0806.3100.
[7] (0393848) S.N. Kružkov, A. Castro and M. Lopes, Schauder type estimates and theorems on the existence of the solutions of fundamental problems for linear and nonlinear parabolic equations, Dokl. Akad. Nauk. SSSR 220 (1975), 277–280 (in Russian); Soviet Math. Dokl. 16 (1975) 60–64 (in English).
[8] (0503903) H. Triebel, “Interpolation Theory, Function Spaces, Differential Operators,” North-Holland, Amsterdam, 1978.

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