The inverse conductivity problem via the calculus of functions of bounded variation

Antonios Charalambopoulos1 | Vanessa Markaki1 | Drosos Kourounis2

1Department of Mathematics, School of Applied Mathematics and Physical Sciences, National Technical University of Athens, Athens, Greece
2NEPLAN AG, Küsnacht, Switzerland

Correspondence
Antonios Charalambopoulos, Department of Mathematics, School of Applied Mathematics and Physical Sciences, National Technical University of Athens, Athens 15780, Greece. Email: acharala@math.ntua.gr

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In this work, a novel approach for the solution of the inverse conductivity problem from one and multiple boundary measurements has been developed on the basis of the implication of the framework of BV functions. The space of the functions of bounded variation is recommended here as the most appropriate functional space hosting the conductivity profile under reconstruction. For the numerical investigation of the inversion of the inclusion problem, we propose and implement a suitable minimization scheme of an enriched—constructed herein—functional, by exploiting the inner structure of BV space. Finally, we validate and illustrate our theoretical results with numerical experiments.

KEYWORDS
boundary value problems for second-order elliptic equations, inverse problems

MSC CLASSIFICATION
35J25; 35R30

1 | INTRODUCTION

The problem of electrical impedance tomography is of great importance from the theoretical and application point of view. It consists in the inverse problem aiming at the determination of the conductivity of an electrically conductive region when sufficient data are given on the surface of this region. The data of this inverse problem include the knowledge of the voltage on the surface of the region—which is usually incorporated in the physical assumptions of the problem—along with the knowledge of the current density on the same surface (the role of these two surface fields may be interchanged), which is the outcome of a measurement process. The connection between surface voltage and current is linear and constitutes a well-known and important operator, the so called voltage-to-current map, which is of the type of Dirichlet-to-Neumann operators, encountered in elliptic boundary value problems of second order.

One of the most interesting cornerstone questions has been whether the knowledge of the Dirichlet-to-Neumann map on the known surface \( \partial \Omega \) of the conductive region is sufficient to determine the conductivity throughout the region \( \Omega \). In case where the conductivity is isotropic, it was proposed by Calderón in \(^1\) and developed later in the representative overviews,\(^2,3\) that any bounded conductivity might be determined solely from the boundary measurements, ie, from the Dirichlet-to-Neumann operator. In Astala and Paivarinta,\(^4\) this has been confirmed for the two-dimensional case. When the isotropic conductivity is smoother than just a \( L^1 \) function, the same conclusion is known to hold also in higher dimensions.

The first global uniqueness result was obtained for a \( C^\infty \) smooth conductivity in dimension \( d \geq 3 \) by Sylvester and Uhlmann in 1987.\(^5\) For the two-dimensional case, Nachman\(^6\) produced in 1996 a uniqueness result for two times differentiable conductivities. The developed therein algorithm has been successfully implemented and proven to work efficiently even with real data.\(^7,8\) The reduction of regularity assumptions has been since subject of active investigation. In two dimensions, the optimal \( L^{\infty} \) regularity was obtained in Astala and Paivarinta.\(^4\) The advantage of the reduction of the
smoothness assumptions up to $L^\infty$ does not lie solely on the fact that many conductivities have jump-type singularities, but it also allows us to consider much more complicated singular structures such as porous media. Uniqueness results for less smooth conductivities in dimensions three and higher were obtained in Brown and Torres\textsuperscript{9} and Krupchyk and Uhlmann.\textsuperscript{10} Further, works concerning uniqueness in 3D can be found in Haberman and Tataru\textsuperscript{11} and Caro and Rogers\textsuperscript{12} combined with the recent work of Haberman\textsuperscript{13} for $W^{1,p}(\Omega)$ ($n = 3, 4$) conductivities.

The stability of the inversion is also very important and has been extensively investigated. The main argumentation can be encountered in previous studies,\textsuperscript{14-17} which are very representative references. The majority of the approaches establishing stability require some uniform control on the oscillations of the conductivity function and so deal with some kind of conditional stability. This is expected, since extreme oscillations of a sequence of conductivities create an instability of the Calderón problem. This kind of asymptotics is well described via the implication of $H$ or $G$ convergence analysis,\textsuperscript{18-20} where the homogenization theory assigns the suitable convergence regime.\textsuperscript{21,22}

As already mentioned, the investigation method depends strongly on the assumptions made for the regularity of the sought conductivity profile. Clearly, it is essential to assure the potentially discontinuous character of the conductivity function. This priority introduces the space of functions of bounded variation ($BV$ functions), as the most appropriate functional setting for the conductivity. In fact, modeling and formulation of a large number of problems in physics and technology require the introduction of new functional spaces, permitting discontinuities of the solution. In phase transitions, image segmentation, plasticity theory and the study of cracks and fissures, in the study of the wake in fluid dynamics, and the shock theory in mechanics, the solution of the problem presents discontinuities along one-dimensional manifolds. Its first distributional derivatives are now measured, which may charge zero Lebesgue measure sets and its solutions cannot be considered as an element of Sobolev spaces throughout the entire domain of the problem.

In the conductivity regime, the solution of the direct problem is an ordinary Sobolev function, but the conductivity profile is not. There exist cases where the conductivity coefficients are just $L^\infty$ functions and the implication of the $BV$ structure is introduced in the constructed minimization scheme via a suitable regularization term whose particular influence can be modulated via the relevant regularization parameter. In other cases, as the one encountered in the inclusion problem which the present work mainly focuses on, it seems reasonable to select the space $BV$ as the appropriate hosting space for the conductivity functions. A function (with just $L^1$ integrability behavior) belongs to $BV(\Omega)$ iff its first distributional derivatives are bounded measures. In particular, the space $SBV(\Omega)$ is a subspace of $BV(\Omega)$, which is more adequate for our purposes, since it contains functions whose distributional derivatives are free of the peculiar Cantor part.\textsuperscript{23} A well-known result (see, for example, Attouch et al\textsuperscript{23}, p409) holds, according to which when a $BV$ function $\alpha$ belongs additionally to $L^\infty(\Omega)$ and there exists a closed subset $K$ of $\Omega$ (with finite Hausdorff measure, i.e., $H^{d-1}(K) < \infty$) such that $\alpha \in W^{1,1}(\Omega \setminus K)$, then $\alpha$ is necessarily a function of $SBV(\Omega)$, whose distributional derivative disposes no Cantor part, while its jump part is a subset of $K$ (i.e., $S_\alpha \subset K$). This situation is frequently encountered when dealing with the inclusion problem.

In the present work, the principal aim is the investigation of the inverse conductivity problem on the basis of a methodology developed inside the framework of $BV$ space. Our approach is motivated by the fundamental properties of the $BV$ functions, as these are well introduced in Attouch et al\textsuperscript{23} and Aubert and Kornprobst.\textsuperscript{24} Every $BV$ function (more accurately, every member of the subspace $SBV(\Omega)$) is a function in $L^1(\Omega)$, throughout the domain $\Omega$, which potentially disposes discontinuity surfaces. These surfaces consist the “jump” set, which represents the interfaces of the conductivity profile and is the support of the singular part (Young measures) of the distributional derivatives. Consequently, in case of reconstructing discontinuous conductivities, it is preferable to work within the $BV$ regime, since it is not necessary to define a priori a partition of the domain, every subdomain of which supports one of the continuous components of the conductivity function but just work with specific $BV$ functions, whose one of the main intrinsic characteristics is the possibly multi-connected “jump” set. Therefore, a function in $BV$ represents simultaneously its discontinuity surfaces in the formation of its domain of definition. In Appendix A, we present, for completeness, the necessary properties of $BV$ functions.

The implication of the regime of functions of bounded variation in the regularization of ill-posed problems has been already attempted and two of the primitive approaches can be encountered in previous studies.\textsuperscript{25,26} Some later indicative works referring to regularization by total variation in electrical impedance tomography can be found in previous studies\textsuperscript{27-29} and further analysed in Rondi.\textsuperscript{30,31} In the present work, a specific minimization scheme has been designed in order to solve the inverse conductivity problem, exploiting the advantages stemmed by the $BV$ calculus. The constructed functional deals with an “energy” over the whole domain $\Omega$, but when a $BV$ function is involved, a part of the energy is concentrated on the interface (“jump” set) of this function.\textsuperscript{23} The designed herein functional incorporates efficiently both the characteristics of the direct and inverse conductivity problem. Our motivation has been to detour, if necessary, solving
directly the full conductivity problem at every step of the minimization process, but just perform the optimization descent, implementing simultaneously minimization on the conductivity profile and the $H^1(\Omega)$ potential fields participating in the direct Dirichlet and Neumann conductivity problems. The concept has been inspired by Richins\textsuperscript{32} and Milton et al\textsuperscript{33} and is based on the simple idea that the Dirichlet and Neumann solutions, corresponding to the surface data, should have a degree of compatibility (theoretically, they are identical given exact and noiseless data). Nevertheless, the aforementioned methodology is not restrictive, in the sense that it is always possible to solve the intermediate, consecutive, direct problems and transform the initial functional to a new one depending only on the unknown conductivity function. Then, this functional is very reminiscent (in case that a specific intrinsic weight parameter increases sufficiently) of the commonly used optimization functional, forcing the Dirichlet and Neumann surface data to compromise on the region boundary $\partial \Omega$ or the functional introduced in Kohn and Vogelius.\textsuperscript{34} Besides the alternative to detour the solution of a sequence of direct problems, the main motive to construct such an enriched functional is the anticipation that strong advantages will emerge when the method will be generalized in problems where the involving physical fields are also discontinuous functions and thus should by themselves be represented as $BV$ functions (cracks, fissures in mechanics, etc). In that case, it might be preferable to develop a concurrent $BV$ minimization over profiles and fields.

In Section 2, the inverse conductivity problem is defined. It is clearly stated that this paper focuses on the investigation of the inclusion problem, though the suggested methodology has some characteristics potentially applicable to more general cases where the $BV$ behavior is introduced exclusively on the basis of regularization issues. Section 3 includes the construction and the properties of the optimization functional. In Section 4, we establish the necessary framework for the numerical implementation of the minimization scheme. Section 5 involves the numerical investigation of the reconstruction method applied to some indicative, characteristic conductivity profiles. It is noticed that both connected and disconnected inclusions are considered, and especially in the second more demanding case, a comparison of the applicability of the current minimization technique with older approaches is presented. Finally, Appendices A and B give the necessary background for $BV$ functions, as well as the convergence analysis of the minimizing conductivity boundary value problems connected with the minimization process under examination.

## 2 | THE INVERSE CONDUCTIVITY PROBLEM

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a Lipschitz domain, denoting a conductive medium, whose conductivity $\alpha(x)$, $x \in \Omega$, is the target of reconstruction. Usually, there exists a probably disconnected object $D$, which is an open subset of $\Omega$, being the target of detection and diversified from the remaining conductive material (ie, the matrix) via the possibly discontinuous change of the conductivity parameter on the interface $\partial D$.

The potential $u$ inside the structure $\Omega$ satisfies the direct boundary value problem\textsuperscript{*}

\begin{equation}
\nabla \cdot (\alpha(x)\nabla u(x)) = 0 \quad x \in \Omega, \tag{1}
\end{equation}

\begin{equation}
\alpha(x)\frac{\partial u}{\partial n}(x) = g(x) \quad x \in \partial \Omega, \tag{2}
\end{equation}

with the coefficient of conductivity having the form

\begin{equation}
\alpha(x) = \tilde{b}(x)\chi_{\Omega \setminus D}(x) + \tilde{c}(x)\chi_D(x), \tag{3}
\end{equation}

where we meet the characteristic function $\chi_A$ of a set $A$. The functions $\tilde{b}(x), \tilde{c}(x)$ characterize the background (matrix) and the inclusion region, respectively (Figure 1). In several applications, these functions are constant and constitute representations of two-phase materials. Here, we allow them to be variable functions with discontinuous behavior on the interfaces of the inclusions. The only quantitative restriction is that they take values in the interval $[b, c]$, where $b, c$ are real threshold values with $0 < b < c < \infty$. So it holds that $\alpha(x) \in L^\infty(\Omega, [b, c])$. The additional assumption in this work is that $\alpha(x) \in BV(\Omega)$. We mention here the simple pilot case of a two-valued conductivity profile. In that case, the inverse conductivity problem consists in the determination of the measurable function $\alpha(x) \in L^\infty(\Omega, [b, c]) \cap BV(\Omega)$. The boundary condition of the problem imposes a specific current $g \in H^{-\frac{1}{2}}(\partial \Omega)$ on the whole boundary $\partial \Omega$ of the body. It is well known that such a boundary value problem is solvable only when the surface data obey to compatibility condition

\textsuperscript{*}The symbol of the normal derivative $\frac{\partial}{\partial n}$ should not be confused with the index $n$ of the several sequences appeared in this work.
\( \langle g, 1 \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = 0 \), and there emerges an equivalence class of solutions \( u \in H^1(\Omega) \), whose members differ each other by a real constant. To unify the members of the equivalence classes above, we introduce the quotient space \( H^1(\Omega)/\mathbb{R} \) - consisted of all the cosets \( u + \mathbb{R} = \{ u + \lambda \mid \lambda \in \mathbb{R} \} \) when \( u \) runs over \( H^1(\Omega) \) — with the appropriate norm \( \| u \|_{H^1(\Omega)/\mathbb{R}} = \inf_{\lambda \in \mathbb{R}} \| u + \lambda \|_{H^1(\Omega)} \). Notice that every solution \( u \), along with the electric current flux \( \frac{\partial u}{\partial n} \), is continuous across the interface \( \partial D \) (Figure 1).

The inverse conductivity problem consists in the determination of \( \alpha(x) \), in case that we have at hand as supplementary data, in the realm of measurements, the voltage \( u|_{\partial \Omega} \) on the boundary of the body. This is equivalent to say that we have the knowledge of the Neumann to Dirichlet map\(^\dagger\) denoted by \( \Lambda_{N \rightarrow D} : H^{-\frac{1}{2}}(\partial \Omega) \to H^1(\partial \Omega) \) or equivalently its inverse \( \Lambda_{D \rightarrow N} : H^1(\partial \Omega) \to H^1(\partial \Omega) \), fact corresponding to the situation, in which we initially have knowledge about the Dirichlet values of the field, and we acquire a posteriori additional information about the current field on the surface \( \partial \Omega \). We will next denote \( \Lambda_{\alpha} = \Lambda_{D \rightarrow N} \), in order to make clear the dependence of this surface operator on the conductivity parameter function \( \alpha(x) \), and then, we have immediately that \( \Lambda_{\alpha}^{-1} = \Lambda_{N \rightarrow D} \).

Summarizing, the following inverse problem is formulated:

\[
\begin{align*}
\nabla \cdot (\alpha(x) \nabla u(x)) &= 0 & \text{in } \Omega, \\
u(x) &= f(x) & \text{on } \partial \Omega, \\
\alpha(x) \frac{\partial u}{\partial n}(x) &= g(x) & \text{on } \partial \Omega,
\end{align*}
\]

with known data \( f \in H^\frac{1}{2}(\partial \Omega) \) and \( g = \Lambda_{\alpha}(f) \in H^{-\frac{1}{2}}(\partial \Omega) \).

A thorough analysis, incorporating the necessary compatibility condition of the Neumann data, leads to adopting\(^\ddagger\) the more appropriate functional setting \( \Lambda_{\alpha} : B := (H^\frac{1}{2}(\partial \Omega)/\mathbb{R}) \to (H^1(\partial \Omega)/\mathbb{R})^* \), where an isometric isomorphism of the dual space \( B^* = (H^\frac{1}{2}(\partial \Omega)/\mathbb{R})^* \) is the space of distributions \( A := \{ g \in H^{-\frac{1}{2}}(\partial \Omega) : \langle g, 1 \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)} = 0 \} \), in which the data \( g \) should belong.

### 3 ON THE INVESTIGATION OF THE APPROPRIATE MINIMIZATION FUNCTIONAL FOR THE INVERSE CONDUCTIVITY PROBLEM

Building the appropriate functional under minimization is based on connecting efficiently the conductivity function \( \alpha \), along with the solution of the Dirichlet and the Neumann boundary value problem. More precisely, the functional under examination should involve the term

\[
J_1(\alpha, w) = \frac{1}{2} \int_{\Omega} \alpha(x) |\nabla w(x)|^2 dx - \langle g, w \rangle_{A \times B},
\]

where \( \gamma : H^1(\Omega) \to H^\frac{1}{2}(\partial \Omega) \) stands for the well-known trace operator, which is well established for Lipschitz domains.

The minimization of this term—even under the assumption of a specific function \( \alpha \)—over all possible \( w \in H^1(\Omega)/\mathbb{R} \), would lead to the solution of the Neumann boundary value problem \( u^\alpha \), corresponding to the conductivity coefficient \( \alpha(x) \). However, this is not the ultimate settlement, since the function \( \alpha \) is not known at all—as a matter of fact, it is the target of our investigation—and more data should be incorporated. The functional under construction should also contain the term

\[
J_2(\alpha, u) = \frac{1}{2} \int_{\Omega} \alpha(x) |\nabla (u(x) + (\eta f)(x))|^2 dx.
\]

Here, we meet the continuous right inverse \( \eta \) of the trace operator \( \gamma \), which is helpful to acquire a specific realization of the extension of surface fields to volume functions. A constructive characterization of this operator is encountered in McLean.\(^\ddagger\), p101 The term (8) would independently be minimized—over all functions \( u \in H^1_0(\Omega) \)—by the unique solution \( u^\eta(x) + (\eta f)(x) \) of the Dirichlet problem, in case that the conductivity function was considered as a fixed function. The third term of the functional should be a forcing term, imposing a theoretical vanishing of the difference of the solutions

\(^\dagger\) Or the current to voltage map.

\(^\ddagger\) \( H^\frac{1}{2}(\partial \Omega)/\mathbb{R} \) is defined similarly to the space \( H^1(\Omega)/\mathbb{R} \).
of the implication of the parameter $\Omega$ operator $R$ where gradients not to be extremely penalized.

is demanded for every conductivity problem, we need to incorporate, in the construction of $J_3$, the term $\int_{\Omega} a \nabla(u + \eta f - w) \cdot \nabla w dx + \kappa \langle g, \gamma w - f \rangle_{AXB}$. The specific manner this term leads the minimization tracing will be clarified in the proof of Theorem 1. Selecting a regularization parameter $\kappa$, we schedule then the penalty term $J_3$ as follows:

$$ J_3(a, u, w) = \frac{\kappa}{2} \int_{\Omega} a \nabla(u + \eta f - w)^2 dx + \kappa \int_{\Omega} a \nabla(u + \eta f - w) \cdot \nabla w dx + \kappa \langle g, \gamma w - f \rangle_{AXB}. \quad (9) $$

It is clear that all terms of (9) are identically zero in case of exact conductivity and data. Collecting the previous terms, we construct the following functional:

$$ E(u, w, a) = \frac{\kappa + 1}{2} \int_{\Omega} a |\nabla(u + \eta f)|^2 dx + \left(1 - \kappa\right) \frac{1}{2} \int_{\Omega} a |\nabla w|^2 dx - \langle g, \gamma w \rangle_{AXB} \right] - \kappa \langle g, f \rangle_{AXB}. \quad (10) $$

The above expression can be considered as a family of operators parametrized by the parameter $\kappa$, being restricted in the interval $(-1, 1)$, in order to preserve the coercivity of the first two terms of (10) while its selection gives obviously weighted preference to Dirichlet or Neumann data. Before facing regularization matters, we notice that (10) constitutes the primitive form of the functional to be minimized.

The coefficients $a$, in the general setting, are considered as elements of $L^\infty(\Omega, [b, c])$, while the interesting but more restrictive pilot case of the two-valued conductivity profiles $L^\infty(\Omega, [b, c])$ could also be included in this framework. One additional, essential property of the $a$ coefficient stems from the requirement that there exists a very thin region away the surface $\partial \Omega$, in which the conductivity does not change its value. We introduce the space $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) < \delta\}$ and demand that $a \in L^\infty_\delta(\Omega, [b, c])$, in case of the pilot two constant-valued profile, where $L^\infty_\delta(\Omega, [b, c]) = \{\beta \in L^\infty(\Omega, [b, c]) : \|\beta - b\|_{L^\infty(\Omega_\delta)} = 0\}$. Similarly, in case of a variable profile, we state that $a \in L^\infty_\delta(\Omega, [b, c])$, where now the values of the conductivity range inside the interval $[b, c]$. We mention here that the necessity of the implication of the parameter $\delta$ and the zone $\Omega_\delta$ is due to the desire to have the ability to take strong limits of the Calderón’s operator $A_\alpha$, when $\alpha$ follows specific limit processes.\(^{19}\)

The fourth term of the functional, offering the necessary regularization, is a term properly referring to the selection over conductivity profiles obeying to restriction rules concerning their total masses and some reference compatibility. This last term of the functional involves two regularization parameters $\lambda, \mu$ and obtains the following form:

$$ J_4 = \frac{\lambda}{2} \|Ra - a_0\|^2_{L^2(\Omega)} + \mu \int_{\Omega} \Phi(|Da|), \quad (11) $$

where $R$ is a bounded operator $R : L^2(\Omega) \rightarrow L^2(\Omega)$ and $a_0$ a reference value for the conductivity coefficient. The function $\Phi$ is a strictly convex function of its argument, which is selected to be a non decreasing function from $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, with $\Phi(0) = 0$ and $\lim_{s \rightarrow \infty} \Phi(s) = \infty$. In addition, the condition $ts - \tau \leq \Phi(s) \leq ts + \tau$ with specific constants $t > 0$ and $\tau \geq 0$, is demanded for every $s \geq 0$. The last requirement serves at imposing a “mild” growth condition in order for the strong gradients not to be extremely penalized.

Adding the previous terms (7) to (11), we construct the total functional

$$ E(u, w, a) = E(u, w, a) + \frac{\lambda}{2} \|Ra - a_0\|^2_{L^2(\Omega)} + \mu \int_{\Omega} \Phi(|Da|). \quad (12) $$

We will clarify now the admissible set for the conductivity functions $a$, and as a byproduct, the special role of the operator $R$ is going to be revealed. As explained previously, the abrupt discontinuities in the physical properties of the domain $\Omega$ are very well represented via the functions of bounded variation.

As far as the reference operator $R$ is concerned, we could select $Ra$ as the restriction $a|_{\Omega_0}$, and in consequence, $\|Ra - a_0\|_{L^2(\Omega)}$ is defined to be the norm $\|a - b\|_{L^2(\Omega)}$, expressing the $L^2(\Omega_0)$ difference of the conductivity coefficient from the
background level value $b$. This is a quantitative manner to force the conductivity to be close to the value $b$ in the vicinity of the boundary, throughout all the steps of the minimization process. Alternatively, the requirement $a|_{Ω_b} = b$ could be exactly imposed on the members of the minimization conductivity scheme. In any case, there exist certain realizations of $∥Ra − a_0∥_{L^2(Ω)}$ exploiting the function $a_0$ as an initial guess for the unknown conductivity in the whole or some part of $Ω$.

The fundamental result of this section is the following theorem:

**Theorem 1.** The minimization problem $\inf_{(u, w, a')} E(u, w, a')$ over elements $(u, w, a')$ in $H^1_0(Ω) × H^1(Ω)/\mathbb{R} × BV(Ω) \cap L_ δ^∞(Ω, {b, c})$ admits a solution belonging to the space $H^1_0(Ω) × H^1(Ω)/\mathbb{R} × BV(Ω) \cap L_ δ^∞(Ω, [b, c])$.

**Proof.** The proof will be accomplished in several steps. Let us consider a minimizing sequence $(u_n, w_n, a_n)$, $n \in \mathbb{N}$ such that the sequence $E(u_n, w_n, a_n)$ converges to the finite real number $\inf_{(u, w, a') \in K} E(u, w, a')$ (this necessity stems from the coercivity of the main structural parts of the functional\(^4\)). Clearly, there exists a positive constant $C$ such that

$$E(u_n, w_n, a_n) \leq C \quad \text{for all} \quad n \in \mathbb{N}. \quad (13)$$

On the other hand, given that $a_n \in L_ δ^∞(Ω, [b, c])$, we deduce that $∥a_n∥_{L^∞(Ω)} \leq C$, and so there exists a subsequence, still denoted by $a_n$, and an element $a \in L^∞(Ω)$ such that

$$a_n \rightharpoonup a \quad \text{weakly* in} \quad L^∞(Ω) \quad (as \quad n \rightarrow \infty). \quad (14)$$

Thanks to a very interesting result,\(^3\) the weak* closure of the space $L^∞(Ω, K)$, where $K$ is a subspace of $\mathbb{R}^d$, is $L^∞(Ω, Κ)$, where $Κ$ is the closed convex hull of $K$. In our case, we deduce that $a \in L^∞(Ω, [b, c])$. Furthermore, it is obvious that $\int_ Ω a_n \varphi \rightharpoonup \int_ Ω a \varphi$ for every $\varphi \in C_ c^∞(Ω_δ) \subset L^1(Ω)$ and then $\int_ Ω (b − a(x)) \varphi(x)dx = 0, \quad ∀ \varphi \in C_ c^∞(Ω_δ)$. Consequently, we deduce that $a(x) = b$, almost everywhere in $Ω_δ$, and so finally, $a \in L_ δ^∞(Ω, [b, c])$.

Returning to (13), we start taking advantage from this uniform boundedness by claiming that there exists a uniform bound $M$ (independent of $n$) such that

$$C_n := ⟨g, \gamma w_n⟩_{Α×B} − \frac{1}{2} \int_ Ω a_n |\nabla w_n|^2 dx \leq M. \quad (15)$$

Indeed, due to the boundedness of $g$ and the trace operator $γ$, we have that

$$⟨g, \gamma w_n⟩_{Α×B} − \frac{1}{2} \int_ Ω a_n |\nabla w_n|^2 dx \leq C'∥g∥_{L^∞(Ω)/\mathbb{R}} − \frac{b}{2}∥\nabla w_n∥_{L^2(Ω)}$$

$$= C'∥g∥_{L^∞(Ω)/\mathbb{R}} \inf_{v \in \mathbb{R}} ∥w_n + v∥_{H^1(Ω)} − \frac{b}{2}∥\nabla w_n∥_{L^2(Ω)}$$

$$\leq C'∥g∥_{L^∞(Ω)/\mathbb{R}} \inf_{v \in \mathbb{R}} ∥w_n∥_{H^1(Ω)} − \frac{1}{|Ω|} \int_ Ω w_n(x)dx∥_{H^1(Ω)} − \frac{b}{2}∥\nabla w_n∥_{L^2(Ω)}$$

$$\leq C''∥g∥_{L^∞(Ω)/\mathbb{R}} ∥\nabla w_n∥_{L^2(Ω)} − \frac{b}{2}∥\nabla w_n∥_{L^2(Ω)}^2, \quad (16)$$

where we have used the Poincaré-Wirtinger inequality for the field $w_n − \frac{1}{|Ω|} \int_ Ω w_n(x)dx$, with zero mean value over the finite region $Ω$. The last right term of the inequality (16) is necessarily bounded, since it is the negative of a coercive function. So (15) has been proved and along with the special form of $E(u_n, w_n, a_n)$ leads easily to the uniform boundedness of the integrals $\int_ Ω a_n |\nabla w_n|^2 dx$ are also uniformly bounded. We deduce from these results and the fact that $a_n$ is bounded below by the value $b$ that the norms $∥\nabla (u_n + η f)∥_{L^2(Ω)}$ and $∥\nabla w_n∥_{L^2(Ω)}$ are again uniformly bounded. This is inherited of course to the norms $∥u_n∥_{L^2(Ω)}$ and using the Poincaré inequality for the functions $u_n \in H^1_0(Ω)$, we infer that

$$∥u_n∥_{H^1_0(Ω)} \leq C, \quad n \in \mathbb{N}. \quad (17)$$

\(^4\)Invoking the forthcoming result (15) is clearly enough.
In addition, thanks again to the Poincaré-Wirtinger inequality, we find that

$$\inf_{v \in \mathbb{R}} \| w_n + v \|_{H^1(\Omega)} \leq \| w_n - \frac{1}{|\Omega|} \int_{\Omega} w_n(x) dx \|_{H^1(\Omega)} \leq \| \nabla w_n \|_{L^2(\Omega)} \leq C, \quad n \in \mathbb{N}. \quad (18)$$

We keep from the last two bounds that there exist $u \in H^1_0(\Omega)$, $w \in H^1(\Omega)/\mathbb{R}$, such that

$$u_n n \to \omega \quad \in H^1_0(\Omega), \quad (19)$$

$$\nabla w_n n \to \nabla w \quad \in (L^2(\Omega))^d. \quad (20)$$

Up to this point, the only convergence result valid for (a subsequence of) the minimizing sequence $a_n \in L^\infty_0(\Omega, \{b, c\})$ is given by (14). This is not of course enough for handling the convergence of the fluxes participating in the functional. In the framework of the homogenization theory and on the basis of the property $\alpha_n \in L^\infty_0(\Omega, \{b, c\})$ only, the best that can be done is the possibility to select a new subsequence such that $\alpha_n \rightharpoonup \alpha$ G-converges to a symmetric\(^4\) matrix $\Lambda^*(x) \in L^\infty(\Omega, M^c_{b,c})$ (see Allaire\(^2\)), fact giving prospects to the convergence of the flux terms. However, the regularization terms of the functional give the definite final character to the type of this convergence offering more regularity to the elements of the converging sequence of the conductivity coefficients. More precisely, using again the boundedness (15) and the special form of $E(u_n, w_n, a_n)$, we deduce the boundedness of the remaining terms participating in the functional

$$\int_\Omega \Phi(|Da_n|) \quad \text{and} \quad \| Ra_n - a_0 \|_{L^2(\Omega)} < \infty. \quad (21)$$

Using the specific convexity of the function $\Phi$, imposed by the assumptions made just below (11), it is easily deduced that $|D\alpha_n(\Omega)| \leq C$. Moreover, $\alpha_n \in L^\infty_0(\Omega, \{b, c\})$ and so $\alpha_n \in L^1(\Omega)$ with $\| a_n \|_{L^1(\Omega)} \leq c|\Omega|$. We infer that the sequence $\alpha_n$ is uniformly bounded in $BV(\Omega)$ and so thanks to the compactness theorem (Aubert and Kornprobst\(^2\)), we extract a subsequence, with the same symbolism, and an element $\tilde{\alpha} \in BV(\Omega)$ such that

$$\alpha_n \rightharpoonup \tilde{\alpha} \quad \text{in} \quad BV. \quad (22)$$

On the basis of a simple argument,\(^1\) it can be shown that $\tilde{\alpha} = a$, a.e. in $\Omega$. So the convergence of the sequence $a_n$ has been driven to obey to the rule

$$a_n \rightharpoonup a \quad \text{in} \quad L^1(\Omega) \cap L^\infty_0(\Omega, \{b, c\}), \quad (23)$$

plus the obvious subsequent relation $\alpha_n \to a$. Furthermore, given that the sequence $\alpha_n \in L^\infty_0(\Omega, \{b, c\})$ satisfies $\| a_n \|_{L^2(\Omega)} \leq c^2|\Omega|$, the sequence $a_n$—actually, a subsequence of it with which we keep on working—converges weakly in $L^2(\Omega)$ and so

$$Ra_n \rightharpoonup Ra. \quad (24)$$

The ultimate goal of the theorem is to take the limit in (13) as $n \to \infty$ and prove some kind of lower semicontinuity for the functional, in order to obtain the same inequality for the limiting triple $(u, w, a)$. First, we remark that due to the convexity of $\Phi$ and the lower semicontinuity of the $BV - w^*$ convergence (see Appendix A), it holds that

$$\mu \int_\Omega \Phi(|Da|) \leq \mu \liminf \int_\Omega \Phi(|Da_n|). \quad (24)$$

In addition, from the lower semicontinuity of the $L^2$ norm in the weak topology, we obtain

$$\frac{1}{2} \| Ra - a_0 \|_{L^2(\Omega)}^2 \leq \frac{1}{2} n \liminf \| Ra_n - a_0 \|_{L^2(\Omega)}^2. \quad (25)$$

\(^4\) $M^c_{b,c}$ represents the linear space of symmetric real matrices $M$ of order $d$, which along with their inverses are bounded below (ie, $M^c_{b,c} \cdot \xi \geq b|\xi|^2$ and $M^{-1}c_{b,c} \cdot \xi \geq c|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \quad b, c > 0$).
To step further, we consider the quadratic and the linear terms of the portion \( \tilde{E}(u, w, \alpha_n) \) of the functional.

We insert here in the scene the fields \( u_n^{\alpha_n} \) and \( w_n^{\alpha_n} \), which solve the exact Dirichlet and Neumann problems correspondingly, in case that the conductivity coefficient coincides with \( \alpha_n(x) \). These fields satisfy the boundary value problems I and II, respectively, defined and investigated in Appendix B. Then, we write in condensed form

\[
\frac{k + 1}{2} \int_\Omega a_n |\nabla (u_n + \eta f)|^2 = \frac{k + 1}{2} \int_\Omega a_n |\nabla (u_n^{\alpha_n} + \eta f)|^2 + \frac{k + 1}{2} \int_\Omega a_n |\nabla (u_n - u_n^{\alpha_n})|^2 \\
+ (k + 1) \int_\Omega a_n \nabla (u_n^{\alpha_n} + \eta f) \cdot \nabla (u_n - u_n^{\alpha_n}) \\
= \frac{k + 1}{2} \int_\Omega a_n |\nabla (u_n^{\alpha_n} + \eta f)|^2 + \frac{k + 1}{2} \int_\Omega a_n |\nabla (u_n - u_n^{\alpha_n})|^2.
\]

(26)

where we have used that \( u_n - u_n^{\alpha_n} \in H_0^1(\Omega) \) and the generalized Green’s formula to prove that

\[
(k + 1) \int_\Omega a_n \nabla (u_n^{\alpha_n} + \eta f) \cdot \nabla (u_n - u_n^{\alpha_n}) = 0.
\]

In addition

\[
\frac{1 - \kappa}{2} \int_\Omega a_n |\nabla w_n|^2 = \frac{1 - \kappa}{2} \int_\Omega a_n |\nabla w_n^{\alpha_n}|^2 + \frac{1 - \kappa}{2} \int_\Omega a_n |\nabla (w_n - w_n^{\alpha_n})|^2 \\
+ (1 - \kappa) \int_\Omega a_n \nabla w_n \cdot \nabla (w_n - w_n^{\alpha_n}) \\
= \frac{1 - \kappa}{2} \int_\Omega a_n |\nabla w_n^{\alpha_n}|^2 + \frac{1 - \kappa}{2} \int_\Omega a_n |\nabla (w_n - w_n^{\alpha_n})|^2 + (1 - \kappa) \langle g, \gamma w_n - \Lambda_n^{-1} g \rangle_{A \times B},
\]

(27)

where we have used again the Green’s formula and the Calderón operator to handle the surface term. Finally, the linear term of \( \tilde{E} \) becomes

\[
-(1 - \kappa) \langle g, \gamma w_n \rangle_{A \times B} - (1 - \kappa) \langle g, \gamma w_n^{\alpha_n} \rangle_{A \times B} - (1 - \kappa) \langle g, \gamma w_n - \Lambda_n^{-1} g \rangle_{A \times B}.
\]

(28)

The decomposition made in (26) to (28) allows to write

\[
\tilde{E}(u_n, w_n, \alpha_n) = \tilde{E}(u_n^{\alpha_n}, w_n^{\alpha_n}, \alpha_n) + \frac{k + 1}{2} \int_\Omega a_n |\nabla (u_n - u_n^{\alpha_n})|^2 + \frac{1 - \kappa}{2} \int_\Omega a_n |\nabla (w_n - w_n^{\alpha_n})|^2.
\]

(29)

The form of (29) is very helpful for monitoring the minimization descent. The structure of the subfunctional \( J_3 \) has been responsible for intermediate eliminations of linear terms. That is why it has been selected in this particular form. The outcome of these cancellations consists in expressing the difference of \( \tilde{E}(u_n, w_n, \alpha_n) \) and \( \tilde{E}(u_n^{\alpha_n}, w_n^{\alpha_n}, \alpha_n) \) as a positive flux—difference between the members of the whole minimizing sequence and those of the sequence solutions of the intermediate direct problems.

We exploit now the convergence results presented in Propositions 5 and 6 (see Appendix B). Then, clearly, we have

\[
\lim_{n \to \infty} \tilde{E}(u_n^{\alpha_n}, w_n^{\alpha_n}, \alpha_n) = \tilde{E}(u^\alpha, w^\alpha, \alpha).
\]

(30)

Taking the limit \( n \to \infty \) in (29) and exploiting (24), (25), and (30), we obtain that

\[
\inf_{(u, w, \alpha')} E(u, w, \alpha') = \lim_{n \to \infty} E(u_n, w_n, \alpha_n) \geq \lim_{n \to \infty} \tilde{E}(u_n^{\alpha_n}, w_n^{\alpha_n}, \alpha_n) + \frac{\lambda}{2} \lim_{n \to \infty} \| Ra_n - \alpha_n \|^2_{L^2(\Omega)} \\
+ \mu \lim_{n \to \infty} \int_\Omega \Phi(|D\alpha_n|) + \frac{k + 1}{2} \lim_{n \to \infty} \int_\Omega a_n |\nabla (u_n - u_n^{\alpha_n})|^2 + \frac{1 - \kappa}{2} \lim_{n \to \infty} \int_\Omega a_n |\nabla (w_n - w_n^{\alpha_n})|^2 \\
\geq \tilde{E}(u^\alpha, w^\alpha, \alpha) + \frac{\lambda}{2} \| Ra - \alpha \|^2_{L^2(\Omega)} + \mu \int_\Omega \Phi(|D\alpha|) + \frac{k + 1}{2} \lim_{n \to \infty} \int_\Omega a_n |\nabla (u_n - u_n^{\alpha_n})|^2 + \frac{1 - \kappa}{2} \lim_{n \to \infty} \int_\Omega a_n |\nabla (w_n - w_n^{\alpha_n})|^2 \\
\Rightarrow \inf_{(u, w, \alpha')} E(u, w, \alpha') \geq E(u^\alpha, w^\alpha, \alpha) + \frac{k + 1}{2} \lim_{n \to \infty} \int_\Omega a_n |\nabla (u_n - u_n^{\alpha_n})|^2 + \frac{1 - \kappa}{2} \lim_{n \to \infty} \int_\Omega a_n |\nabla (w_n - w_n^{\alpha_n})|^2.
\]

(31)
On the basis of the convergences (19) and (20), the limit processes declared by Propositions 5 and 6, and the lower semicontinuity of the $L^2(\Omega)$ norm in the weak topology, we obtain
\begin{align}
\lim_{n \to \infty} \int_\Omega \alpha_n |\nabla (u_n - u_n^a)|^2 & \geq b \lim_{n \to \infty} \int_\Omega |\nabla (u_n - u^a)|^2 \geq b \int_\Omega |\nabla (u - u^a)|^2, \\
\lim_{n \to \infty} \int_\Omega \alpha_n |\nabla (w_n - w_n^a)|^2 & \geq b \lim_{n \to \infty} \int_\Omega |\nabla (w_n - w^a)|^2 \geq b \int_\Omega |\nabla (w - w^a)|^2.
\end{align}

Consequently, the relation (31) becomes
\begin{equation}
\inf_{(u, w, \alpha)} E(u, w, \alpha') \geq E(u^a, w^a, \alpha) + b \frac{\kappa + 1}{2} \int_\Omega |\nabla (u - u^a)|^2 + b \frac{1 - \kappa}{2} \int_\Omega |\nabla (w - w^a)|^2. \tag{34}
\end{equation}

The last inequality can be satisfied only if $\nabla u = \nabla u^a$, $\nabla w = \nabla w^a$, a.e. in $\Omega$ or equivalently
\begin{equation}
u = u^a \quad \text{in } H^1_0(\Omega), \tag{35}
w = w^a \quad \text{in } H^1(\Omega)/\mathbb{R}. \tag{36}
\end{equation}

Necessarily, the inequality (34) becomes the equation
\begin{equation}
\inf_{(u, w, \alpha') \in (u, w, \alpha)} E(u, w, \alpha') = \lim_{n \to \infty} E(u_n, w_n, \alpha_n) = E(\lim_{n \to \infty} u_n, \lim_{n \to \infty} w_n, \lim_{n \to \infty} \alpha_n) = E(u^a, w^a, \alpha). \tag{37}
\end{equation}

So the minimization problem admits a solution, which necessarily is the limit of the minimizing sequence.

**Remark 1.** It is important to notice that one of the fundamental outcomes of the Theorem 1 is that it is not necessary to solve a direct conductivity problem in every step of the process (with a specific conductivity profile). This would of course accelerate the convergence of the minimization process, but the descent toward the minimum could be implemented independently for the three variables $u, w, \alpha$ of the problem.

**Remark 2.** Without loss of generality, the results of Theorem 1 hold, even if the members of the minimizing sequence belong themselves to $BV(\Omega) \cap L^2(\Omega, [b, c])$ (this space is closed with respect to the particular kind of activity the sequence elements participate in). The specific form of the theorem has been adopted, since it is common to use step-wise constant functions as approximation profiles, and so as to reveal that even starting with constant profiles the minimization process is well accomplished. This gives some extra degrees of freedom to the arsenal of the numerical implementation.

Theorem 1 states that the infima of the functional occur at triples $(u^a, w^a, \alpha)$, where the functions $u^a, w^a$ solve the Dirichlet and Neumann boundary value problems corresponding to the conductivity profile $\alpha$ and the data $f, g$. Actually, this would emerge naturally if we solved repeatedly the direct problem for every member $\alpha_n$ of the convergent sequence of profiles, as stated in Remark 1. If this was the plan, then the functional part $E(u_n, w_n, \alpha_n)$ would be replaced immediately by the term $E(u_n^a, w_n^a, \alpha_n)$ and the total functional would depend only on the conductivity function $\alpha$, i.e.,
\begin{align}
E_1(\alpha) &= E(u^a, w^a, \alpha) + \frac{\lambda}{2} \|Ra - a_0\|_{L^2(\Omega)}^2 + \mu \int_\Omega \Phi(|Da|) \\
&= \frac{\kappa + 1}{2} \langle \Lambda \alpha f, f \rangle_{\mathcal{A} \times B} - \frac{(1 - \kappa)}{2} \langle g, \Lambda^{-1} \alpha g \rangle_{\mathcal{A} \times B} - \kappa \langle g, f \rangle_{\mathcal{A} \times B} + \frac{\lambda}{2} \|Ra - a_0\|_{L^2(\Omega)}^2 + \mu \int_\Omega \Phi(|Da|) \tag{38}
\end{align}

In Theorem 1, the parameter $\kappa$ has been taken to belong strictly in the interval $(-1, 1)$, so that the concurrent minimization over fields and profiles shares coercive behavior. However, in the scheme (38), where only the conductivity profile appears, the parameter $\kappa$ can be released and detour the critical value 1. Thus, taking a sufficiently large parameter $\kappa$ would render the term $\langle g - \Lambda \alpha f, \Lambda^{-1} \alpha g - f \rangle_{\mathcal{A} \times B}$ primitively significant. It is worthwhile to notice that if it was additionally known that $g, \Lambda \alpha f \in L^2(\partial \Omega)$ (some slight additional regularity), then the second term
of $E_1$ would be responsible for the suppression of the difference $\|g - \Lambda_\alpha f\|^2_{L^2(\partial\Omega)}$ between data, over the measurement surface $\partial\Omega$. This is the commonly used minimization functional forcing the surface data to be in accordance. But even if this regularity is absent, the term $\langle g - \Lambda_\alpha f, \Lambda_\alpha^{-1} g - f \rangle_{AXB}$ is just the volume integral $\int_\Omega \eta^3 (w_a - (u_a + \eta f))^2$, which is identical with the functional proposed by Kohn and Vogelius$^{34}$ and analysed in Kohn and McKenney$^{38}$ in the single data case.

Theorem 1 establishes the existence of a minimum but does not offer any argument concerning some kind of uniqueness. In fact, it is not expected to have a unique solution working with just a pair of data $(f, g)$. As far as convexity is concerned, a more attentive examination of the terms participating in the formation of the functional $E_1(\alpha)$ leads to the result that there exist opposing terms. More precisely, the last two terms of (38), controlling purely the functional structure of the conductivity profiles, are convex.$^{24}$ The first two terms, emerging from the treatment of the involved direct problems, contain necessarily a concave term. Indeed, working rather with the decomposition

$$E(u^\alpha, w^\alpha, \alpha) = \frac{\kappa + 1}{2} \langle \Lambda_\alpha f, f \rangle_{AXB} - \frac{(1 - \kappa)}{2} \langle g, \Lambda_\alpha^{-1} g \rangle_{AXB} - \kappa \langle g, f \rangle_{AXB},$$

we see that the first term is concave. This can be shown by considering two discrete conductivity profiles $a_i, \quad i = 1, 2$, along with the accompanying fields $u_i, w_i, \quad i = 1, 2$, which solve the corresponding direct problems as described in Theorem 1. Let us select an arbitrary convex combination $\alpha_i = sa_i + (1 - s) a_2 (0 < s < 1)$ of the profiles $a_i, \quad i = 1, 2$ (with corresponding fields $u_i, w_i$). It turns out obviously that this function is an admissible conductivity function. We remark that

$$\langle \Lambda_\alpha f, f \rangle_{AXB} = \int_\Omega a_i |\nabla (u_i + \eta f)|^2 \geq s \int_\Omega a_1 |\nabla (u_1 + \eta f)|^2 + (1 - s) \int_\Omega a_2 |\nabla (u_2 + \eta f)|^2$$

$$= s \langle \Lambda_\alpha f, f \rangle_{AXB} + (1 - s) \langle \Lambda_\alpha f, f \rangle_{AXB},$$

which proves that (when of course $\kappa + 1 > 0$) the term $\frac{\kappa + 1}{2} \langle \Lambda_\alpha f, f \rangle_{AXB}$ is concave. In addition

$$-\langle g, \Lambda_\alpha^{-1} g \rangle_{AXB} = \int_\Omega a_i |\nabla w_i|^2 - 2 \langle g, \gamma w_i \rangle_{AXB} = s \int_\Omega a_1 |\nabla w_1|^2 + (1 - s) \int_\Omega a_2 |\nabla w_2|^2 - 2 \langle g, \gamma w_2 \rangle_{AXB}$$

$$\geq s \left[ \int_\Omega a_1 |\nabla w_1|^2 - 2 \langle g, \gamma w_1 \rangle_{AXB} \right] + (1 - s) \left[ \int_\Omega a_2 |\nabla w_2|^2 - 2 \langle g, \gamma w_2 \rangle_{AXB} \right]$$

$$(41)$$

Then, even the second term $-\frac{(1 - \kappa)}{2} \langle g, \Lambda_\alpha^{-1} g \rangle_{AXB}$ of $E(u^\alpha, w^\alpha, \alpha)$ is concave when $\kappa < 1$. In case that $\kappa > 1$, this term becomes convex and acts in collaboration with the term $\frac{\kappa + 1}{2} ||\nabla a_0||_{L^2(\partial\Omega)} + \mu \int_\Omega \Phi(||D a||)$. Indeed, when we perform a mixed minimization over $(u, w, a)$ (and then respect the coercivity condition $\kappa < 1$), we have more local minima than in the case of minimizing $E_1$ over $a$, when the relaxed selection $\kappa > 1$ can be adopted. Nevertheless, even the commonly used functional $\langle g - \Lambda_\alpha f, \Lambda_\alpha^{-1} g - f \rangle_{AXB}$ written as $\langle g, \Lambda_\alpha^{-1} g \rangle_{AXB} + \langle \Lambda_\alpha f, f \rangle_{AXB} - 2 \langle g, f \rangle_{AXB}$ disposes adversarial terms.

### 4 | TOWARDS THE NUMERICAL IMPLEMENTATION OF THE INVERSE CONDUCTIVITY PROBLEM

The following theorem is the first stage of the investigation stepping to the establishment of the suitable numerical scheme.

**Theorem 2.** The functional $E(u, w, a)$ expressed by (12) and (10) is lower semicontinuous with respect to the strong $L^1(\Omega)$ convergence on $a$ and the weak $L^1(\Omega) \times L^1(\Omega)$ convergence on $(\nabla u, \nabla w)$ (this combined convergence defines a topology, which will next be denoted $\tau$ topology).

**Proof.** We evoke herein mainly theorems 13.1.1 and 16.4.1 of Attouch et al.$^{23}$ to establish the sought semicontinuity. More precisely, we consider the function $h(\alpha, v)$, where $v = (v_1, v_2) \in \mathbb{R}^d \times \mathbb{R}^d$, defined as

$$h(\alpha, v) = \frac{\kappa + 1}{2} a |(v_1 + \nabla (\eta f))|^2 + \frac{(1 - \kappa)}{2} a |v_2|^2.$$  

(42)
Since the function \( h(a, v) \) is convex in \( v \) and lower semicontinuous in \( a \), we infer, on the basis of the aforementioned theorems of Aubert and Kornprobst, under the title of Half-Quadratic Minimization Approach, it is suggested to construct a

\[
\begin{align*}
\int_{\Omega} h(a(x), v(x)) \, dx
\end{align*}
\]

(43)
is lower semicontinuous with respect to the strong \( L^1 \) convergence on \( a \) and the weak \( L^1 \) convergence on \( v \). We take \( v = (v_1, v_2) = (\nabla u, \nabla w) \) and consider sequences \((u_n, w_n)\) with the properties implied by Theorem 1, according to which \( \nabla u_n \rightarrow \nabla u \) in \( L^2(\Omega) \) and \( \nabla w_n \rightarrow \nabla w \) in \( L^2(\Omega) \). Then, the pair \((\nabla u_n, \nabla w_n)\) converges \( L^1 \) weakly to \((\nabla u, \nabla w)\), while \( a_n \rightarrow a \) strongly. Incorporating the term \(-x'(g, f)_{A \times B}\), we construct the functional \( \tilde{E}(u, w, a) \) given by (10), which obeys then to the semicontinuity result:

\[
\tilde{E}(u, w, a) \leq \lim_{n \to \infty} \tilde{E}(u_n, w_n, a_n).
\]

(44)

Taking into consideration (24) and (25), it is finally proved that the total functional (12) satisfies

\[
E(u, w, a) \leq \lim_{n \to \infty} E(u_n, w_n, a_n).
\]

(45)

It is more informative to give the following extended form of the lower semicontinuous functional \( E(u, w, a) \)

\[
E(u, w, a) = E(u, w, a) + \frac{\lambda}{2} \|Ra - a_0\|_{L^2(\Omega)}^2 + \mu \int_{\Omega} |a^+ - a^-| dH^{d-1} + \mu \int_{\Omega} |C_a|,
\]

(46)

where we recognize the decomposition of the total mass into three parts: the absolutely continuous part with respect to the Lebesgue measure, the jump part, and the Cantor measure. The parameter \( \lambda \) stands for the limit \( \lim_{s \to \infty} \frac{\Phi(s)}{s} \). This makes sense, since the strictly convex function \( \Phi \), as stated just after (11), is selected to be nondecreasing function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \), with \( \Phi(0) = 0 \), while the condition \( ts - \tau \leq \Phi(s) \leq ts + \tau \) with specific constants \( t > 0 \) and \( \tau \geq 0 \), is satisfied for every \( s \geq 0 \).

The functional (46) could be exactly the target of the minimization process. On the basis of the methodology followed in Aubert and Kornprobst, under the title of Half-Quadratic Minimization Approach, it is suggested to construct a sequence of energies \( E_\epsilon \) labeled by the parameter \( \epsilon > 0 \), which constitute quadratic approximations of the functional (46).

The associated minimization problem to every functional \( E_\epsilon \) admits a minimum \((u_\epsilon, w_\epsilon, a_\epsilon)\) sharing increased regularity, i.e., belonging to \( H^1_0(\Omega) \times H^1(\Omega) \times (H^1(\Omega) \cap L^\infty(\Omega, [b, c])) \). All this is accomplished by the construction of the following sequence of auxiliary functions:

\[
\Phi_\epsilon(s) = \begin{cases} 
\frac{s^2 \Phi'(\epsilon s)}{2} + \Phi(s) - \epsilon \frac{\Phi'(\epsilon s)}{2}, & \text{if } 0 \leq s \leq \epsilon, \\
\Phi(s), & \text{if } \epsilon \leq s \leq \epsilon, \\
\frac{s^2 \Phi'(\epsilon s)}{2} + \Phi(s) - \epsilon \frac{\Phi'(\epsilon s)}{2}, & \text{if } s \geq \epsilon.
\end{cases}
\]

(47)
mollifying suitably the convex function \( \Phi \). Indeed, for every \( \epsilon \), it holds that \( \Phi_\epsilon(s) \geq 0 \), \( \forall s \) and \( \lim_{\epsilon \to 0} \Phi_\epsilon(s) = \Phi(s) \). We set \( \mathcal{E} = H^1_0(\Omega) \times H^1(\Omega)/\mathbb{R} \) and define now the functional \( E_\epsilon \) as follows:

\[
E_\epsilon(u, w, a) = \begin{cases} 
E(u, w, a) + \frac{\lambda}{2} \|Ra - a_0\|_{L^2(\Omega)}^2 + \mu \int_{\Omega} |a^+ - a^-| dH^{d-1} + \mu \int_{\Omega} |C_a|, & \text{if } (u, w) \in \mathcal{E} \quad \text{and} \quad a \in H^1(\Omega) \cap L^\infty(\Omega, [b, c]), \\
\infty, & \text{if } (u, w) \in \mathcal{E} \quad \text{and} \quad a \in \text{(BV}(\Omega) - H^1(\Omega)) \cap L^\infty(\Omega, [b, c]).
\end{cases}
\]

(48)

Evoking classical arguments, simpler but reminiscent of the steps followed in Theorem 1, we find that

**Proposition 1.** For each \( \epsilon > 0 \), the functional \( E_\epsilon \) admits a minimum \((u_\epsilon, w_\epsilon, a_\epsilon)\) in \( \mathcal{E} \times (H^1(\Omega) \cap L^\infty(\Omega, [b, c])) \).

This sequence of minimizers converges suitably to the minimizer of the initial functional, and this can be proved via the implication of \( \Gamma \) convergence as the next two propositions show. More precisely, we define

\[
E(u, w, a) = \begin{cases} 
E(u, w, a) - \infty, & \text{if } (u, w) \in \mathcal{E} \quad \text{and} \quad a \in H^1(\Omega) \cap L^\infty(\Omega, [b, c]), \\
\infty, & \text{if } (u, w) \in \mathcal{E} \quad \text{and} \quad a \in \text{(BV}(\Omega) - H^1(\Omega)) \cap L^\infty(\Omega, [b, c]).
\end{cases}
\]
Proposition 2. The lower semicontinuous envelope \( \mathcal{R}_\tau \bar{E} \) of \( \bar{E} \), with respect to the strong \( L^1(\Omega) \) convergence on \( \alpha \) and the weak \( L^1(\Omega) \times L^1(\Omega) \) convergence on \( (\nabla u, \nabla v) \), coincides with \( E \).

Proof. It is proved in Theorem 2 that \( E \) is lower semicontinuous with respect to the mixed type \( \tau \) topology stated there. Given that clearly \( \bar{E} \geq E \), what remains to be proved in order that \( \mathcal{R}_\tau \bar{E} = E \) is that there exists a sequence \( (u_h, w_h, a_h) \) in \( E \times (H^1(\Omega) \cap L^\infty(\Omega, [b, c])) \) converging in the \( \tau \) topology to \( (u, w, a) \) and \( E(u, w, a) = \lim_{h \to 0} \bar{E}(u_h, w_h, a_h) \). Such a sequence can be constructed using classical approximation theorems.\(^{39,40}\) We just pay attention on the exploitation of the intermediate convergence of measures, assuring the appropriate convergence of the involved measures. \( \square \)

Proposition 3. Every cluster point \( (u, w, a) \)—with respect to the strong \( L^1(\Omega) \) convergence on \( \alpha \) and the weak \( L^1(\Omega) \times L^1(\Omega) \) convergence on \( (\nabla u, \nabla v) \)—of the sequence \( \{(u_\epsilon, w_\epsilon, a_\epsilon); \epsilon > 0\} \), introduced in Proposition 1, is a minimizer of \( E \) and \( E_\epsilon(u_\epsilon, w_\epsilon, a_\epsilon) \) converges to \( E(u, w, a) \).

Proof. By construction, it is obvious that \( E_\epsilon(u, w, a) \) is a decreasing sequence converging pointwise to \( E(u, w, a) \). According to a well-known result of the theory of \( \Gamma \) convergence (theorem 2.1.8 in Aubert and Kornprobst\(^{24}\)), \( \Gamma \) converges to the lower semicontinuous envelope \( \mathcal{R}_\tau E \) of \( E \), which according to Proposition 2, coincides with the functional \( E \). In addition, thanks to the uniform bounds \( b, c \) of the conductivity coefficients the functionals \( E_\epsilon \) are equicoercive and so the set \( \{(u_\epsilon, w_\epsilon, a_\epsilon); \epsilon > 0\} \) is relatively compact in the aforementioned topologies. According to theorem 12.1.1 of Attouch et al.\(^{23}\) every cluster point \( (u, w, a) \) of the set \( \{(u_\epsilon, w_\epsilon, a_\epsilon); \epsilon > 0\} \) is a minimizer of \( E \) and \( E_\epsilon(u_\epsilon, w_\epsilon, a_\epsilon) \) converges to \( E(u, w, a) \) as \( \epsilon \) goes to zero.

Dealing with the quadratic approximations \( E_\epsilon \), instead of the initial functional \( E \), keeps the minimization searching inside the \( H^1(\Omega) \) framework, as far as the conductivity profiles are concerned, although in the limit \( \epsilon \to 0 \), large gradients of \( a \) simulate accordingly discontinuities of the coefficients in a mollified manner. In order to have an efficient algorithm not suppressing drastically the aforementioned gradients but in contrast revealing the subset of the discontinuities, we need to introduce in our analysis an indicator function \( \omega \), standing for a mollification of the characteristic function of these discontinuity curves. This is just the concept suggested in Aubert and Kornprobst,\(^{24}\) complementing the framework of the Half-Quadratic Minimization Approach, in the realm of image processing. This auxiliary function \( \omega \) has an additional role: Its constructive implication in the method offers a robust numerical algorithm for the minimization of the sequence of the quadratic functionals \( E_\epsilon \) and the initial functional itself. In the next section, these concepts will be exploited by extending them to the current minimization process concerning the inclusion problem kHz

5 THE NUMERICAL IMPLEMENTATION

Following the arguments presented in section 3.2.4 of Aubert and Kornprobst,\(^{24}\) we are interested in constructing a semiquadratic algorithm implementing numerically the optimization scheme described in the previous section. More precisely, we restrict ourselves to the case that the auxiliary function \( \Phi \), introduced above and participating in the part of the functional controlling the masses of the conductivity profiles, is selected to satisfy the requirements of proposition 3.2.4 (encountered in Aubert and Kornprobst\(^{24}\)). The main condition is that the function \( \Phi(\sqrt{s}) \) is concave in \((0, \infty)\), while we recall that \( \Phi(s) \) is a nondecreasing function. Consequently, the nonnegative parameters \( L = \lim_{s \to \infty} \Phi'(s) \) and \( M = \lim_{s \to 0} \frac{\Phi'(s)}{2s} \) are well defined and obey to the ordering \( L < M \). According to the main results of the proposition 3.2.4, there exists a convex and decreasing function \( \psi : [L, M] \to [\beta_1, \beta_2] \) such that

\[
\Phi(s) = \inf_{L \leq \omega \leq M} (\omega^2 + \psi(\omega)),
\]

(50)

where \( \beta_1 = \lim_{s \to 0^+} \Phi(s) \) and \( \beta_2 = \lim_{s \to +\infty} (\Phi(s) - \frac{\Phi'(s)}{2}) \). Moreover, for every \( s \geq 0 \), the value \( \omega \) at which the minimum is reached is exactly \( \omega = \frac{\Phi'(s)}{2s} \). This approach has the characteristics of a duality process, and the variable \( \omega \) is called the dual variable.
Applying this duality approach to the—quadratic near the end points—auxiliary function $\Phi$, given by (47), we have the possibility$^{24}$ to express the $\inf_{(u, w, \omega)} E_c(u, w, \omega, \omega)$ in the form

$$\inf_{(u, w, \omega)} E_c(u, w, \omega) = \inf_{\omega} \inf_{(u, w, \omega)} J_c(u, w, \omega),$$

(51)

where

$$J_c(u, w, \omega) = \bar{E}(u, w, \omega) + \frac{\lambda}{2 \epsilon}||Ra - a_0||^2_{L^2(\Omega)} + \mu \int_\Omega (\omega |\nabla a|^2 + \psi_c(\omega)) dx,$$

(52)

with $(u, w) \in \mathcal{E}$ and $\omega \in H^1(\Omega)$. The importance of the new type of minimization stems from the fact that the regularization part of $J_c$ (ie, the part $\frac{\lambda}{2 \epsilon}||Ra - a_0||^2_{L^2(\Omega)} + \mu \int_\Omega (\omega |\nabla a|^2 + \psi_c(\omega)) dx$) is convex in $\omega$, and for fixed $\omega \in H^1(\Omega)$, it is convex in $\omega$ (see Aubert and Kornprobst$^{2,4}$). This offers an extra robustness to the minimization process and in addition renders the role of $\omega$ very crucial for the determination of the jump set of the inclusions, as revealed in next paragraph.

### 5.1 BV regularized inversion algorithm

The main property of the semiquadratic algorithm we apply on (52) is to detect progressively the discontinuities of the conductivity profile $\alpha$. Our computational process will be started from an initial estimation of $(u, w, \omega)$. Following Aubert and Kornprobst$^{24}$ we have further taken advantage of the role of the dual variable, as attributed in Remark 3. The initial guess $\omega^0$ allows us to introduce into the formulation of the minimization problem (51) prior information about the location of the inhomogeneity’s boundary. The BV reconstruction algorithm is then read as follows: given a step size control $h \in (0, 1)$, an initial approximation $(u^0, w^0, \omega^0)$, a smoothing parameter $\epsilon$, a maximum number of iterations (max_iters = 10), and setting the regularization parameter $\lambda = 0$.

**Algorithm 1 BV Regularized Inversion**

Require: $(u^0, w^0, \omega^0)$

Ensure: $u_{\text{min}}, w_{\text{min}}, \alpha_{\text{min}}, \omega_{\text{min}}$ > initial guess for $(u^0, w^0, \omega^0, \omega^0)$

1. function BVRECONSTRUCTION($u^0, w^0, \omega^0, \omega^0$)
2. for $n \leftarrow 0$: max_iters do
3. $(u_c^{n+1}, w_c^{n+1}, \alpha_c^{n+1}) \leftarrow \text{argmin}_{(u, w, \omega)} J_c(u, w, \omega, \omega^n)$
4. $\omega^{n+1} \leftarrow \frac{\Phi'(\omega^{n+1})}{2||\nabla \omega^{n+1}||}$
5. end for
6. return $u_{\text{min}}, \omega_{\text{min}}, \alpha_{\text{min}}, \omega_{\text{min}}$
7. end function

As mentioned previously, the functions $\omega^n(x)$ can be seen as mollified characteristic functions of the discontinuity surfaces. Indeed, when the aforementioned limits $L = \lim_{r \to 0} \frac{\Phi_c(s)}{2s}$ and $M = \lim_{r \to 0} \frac{\Phi_s(s)}{2s}$ are selected close to 0 and 1, respectively, then the following remark holds.

**Remark 3.** The sequence of functions $\omega^n(x)$ can be seen as an indicator of contours.

1. If $\omega^n(x) \approx 0$, then $x$ belongs to a discontinuity surface.
2. If $\omega^n(x) \approx 1$, then $x$ belongs to a homogeneous region (ie, $x$ is an interior point of an inclusion).

According to this remark, near the sought interfaces of the inclusions, the function $\omega$ participating in the last regularization term of (52) takes very small values allowing for large values of the accompanying gradients of the conductivity coefficients. In contrast to that, inside the inclusions, the function $\omega$ is almost equal to 1 and the corresponding regularization term imposes locally smooth $H^1$ structure to the physical coefficients.

At the nth iteration of the algorithm, the variables $u, w, \alpha$ are computed for given $\omega$ as the argmin$_{(u, w, \omega)} J_c(u, w, \omega, \omega)$ (step 3).

Since we would like to exploit knowledge about lower and upper bounds on the conductivity, we adopt the interior point method implemented in the software package IPOPT$^{41,42}$ which accepts lower and upper bounds on the control parameters, can exploit gradients, and approximates the Hessian using BFGS updates$^{43}$ Gradient-based optimizers, such as IPOPT, typically terminate as soon as the dual infeasibility has been decreased below a specified threshold. Then, $\omega^{n+1}$ is computed as $\omega^{n+1} = \frac{\Phi'_c(|\nabla \omega^{n+1}|)}{2||\nabla \omega^{n+1}||} = \text{argmin}_{\omega} J_c(u_c^{n+1}, w_c^{n+1}, \alpha_c^{n+1}, \omega)$ (step 4).
TABLE 1  Delaunay triangulations

| Mesh Case         | $h$  | Elements | Nodes |
|-------------------|------|----------|-------|
| Concentric        | 0.27 | 1850     | 956   |
| Strong eccentric  | 0.24 | 1834     | 948   |
| Mild eccentric    | 0.26 | 1854     | 958   |
| Concentric        | 0.15 | 2024     | 3926  |
| Strong eccentric  | 0.14 | 2036     | 3950  |
| Mild eccentric    | 0.15 | 2036     | 3950  |

5.2  Numerical tests

Sufficiently general structures will support numerically our theoretical results. As an example, Figure 2 represents the interesting case of disconnected inclusions. We illustrate the numerical applications starting from simpler cases. For this purpose, we consider the boundary value problem

\[
\nabla \cdot (\alpha(x) \nabla u(x)) = 0 \quad x \in \Omega, \\
u(x) = f(x) \quad x \in \partial \Omega, \\
\alpha(x) \frac{\partial u}{\partial n}(x) = g(x) \quad x \in \partial \Omega,
\]

with known data $(f, g)$ and the conductive region $\Omega$ be the disc

\[
\Omega := \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < \rho_\Omega \right\}.
\]

(53)

The threshold values $b$ and $c$ in the definition domain of $\alpha(x)$ can vary accordingly.

We examine the efficiency of the conductivity reconstruction method via a single measurement for one concentric and two different eccentric inhomogeneities, each represented by the inclusion $D$ defined as

\[
D := \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{(x-x_D)^2 + (y-y_D)^2} < \rho_D \right\}
\]

with $(x_D, y_D)$ its center coordinates. We assume that the—known a priori though target of reconstruction—conductivity profile has the following simple profile:

\[
\alpha = \alpha(x, y) = \begin{cases} 2, & (x, y) \in D, \\ 1, & (x, y) \in \Omega \setminus D. \end{cases}
\]

(55)

Without loss of generality, the parameter $b$ has been selected—in the most part of our experiments—equal to 1 in order to comply, in a simple manner, with the requirements on the protective region $\Omega_\delta$. The threshold parameter $c$ has been chosen to be not less than the critical value 2 (Figure 2).

We discretize the computational domain, defined by the outer circle $(\rho_\Omega = 2)$, using Triangle with characteristic mesh size ($h$), reported in Table 1 below along with the number of elements and the number of nodes for each mesh (similar structure has been adopted later on for Table 2). We adopt the piecewise linear, continuous family of finite elements $(P1)$ for the discretization of the operators $J_e(u, w, a, \omega)$.

We emphasize here the sensitivity of the solutions with respect to the two fundamental parameters of the algorithm, $\omega$ and $\mu$. With regard to the first, we introduce the parameter $\ell$ for representing the average width of the ring-shaped $\omega^0$ profile. So the initial guess for $\omega^0$ is selected by

\[
\omega^0 = \begin{cases} 0, & \text{if } |r_i - \rho_D| \leq \frac{\ell}{2}, \\ 1, & \text{otherwise}, \end{cases}
\]

(56)

where $|r_i - \rho_D|$ stands for the Euclidean distance of the $i$ node from $\partial D$, $(\rho_D = 1)$. We consider three different configurations for $\omega^0$ constructed via (56), with $\ell = \{0.2, 0.3, 0.4\}$. Higher values of $\ell$ express higher uncertainty with respect to the location of the inclusion’s boundary. The region colored blue defines the area where $\omega^0(x) = 0$, representing so the
The inverse conductivity problem consists in the determination of $\alpha(x)$ [Colour figure can be viewed at wileyonlinelibrary.com].

**FIGURE 2** Profile of the target conductivity $\alpha$ [Colour figure can be viewed at wileyonlinelibrary.com]

discontinuity surface (Remark 3) (see, for example, the first column of Figures 3 to 5). We also consider the case where the initial $\omega$ is a constant function, $\omega^0 \equiv 1$.

The regularization parameter $\mu$ represents the influence of the $BV$ structure on the problem (although slightly regularized by the parameter $\epsilon$). It is surprising that only with one measurement, we are able to reconstruct both the geometrical configuration and the exact values of the conductivity profile, even though the initial guess (expressed via the configuration $\omega^0$) protects the minimization process from rambling. Besides, our aim is to examine numerically the robustness of the proposed methodology with respect to the choice of $\omega^0$.

The minimization reveals that when $\mu$ increases, the $BV$ norm of the conductivity dominates and highlights the discontinuities of the conductivity function. It is interesting to mention that when we impose constant $\omega$, (for example, $\omega^0 \equiv 1$), the associated term of the functional becomes $\mu \int_{\Omega} |\nabla \alpha|^2$, and we recover the commonly employed Tikhonov regularization, forcing the conductivity profiles to be $H^1$ functions, preventing the abruptness of the variations to be revealed across inclusion boundaries. The smoothing of the inclusion boundary, observed, for example, at the last row (p) of Figure 3, is indicative for the inexpediency of the common Tikhonov regularization ($\omega \equiv 1$), revealing convergence to false local minima (Table 2).

We proceed with the implementation of the algorithm for the numerical solution of the problem (51) setting the smoothing parameter $\epsilon = 0.1$ and the parameter $\kappa = 10$. The initial approximation for the conductivity is chosen to be the constant function $\alpha^0 = 2.5$. Namely, in the concentric (Section 5.2.1) and mild eccentric inhomogeneity (Section 5.2.0), we set

$$
\alpha^0 = \begin{cases} 
1, & r_1 > \rho_D + \frac{\ell}{2}, \\
2.5, & \text{otherwise}, 
\end{cases}
$$

(57)

(second column of Figures 3 and 5), while in the strong eccentricity inhomogeneity, we establish $a_0 = 2.5$ throughout the domain $\Omega \setminus \Omega_s$ (second column of Figure 4). The inversion is attempted for each aforementioned value of $\epsilon$ as well.
FIGURE 3  
BV regularized inversion solution pair \((\omega, \alpha)\) at the final iteration for the \(\omega^0\) profiles: \((\ell = 0.2, 0.3, 0.4), \quad \omega \equiv 1\) and \(\mu = \{1, 0.1, 0.1, 1\}\), respectively, for the concentric inhomogeneity problem [Colour figure can be viewed at wileyonlinelibrary.com]

as the case \(\omega^0 \equiv 1\) for \(n = 10\) iterations. The optimal solutions \(\alpha^n\) are depicted for the best—for each case—value of the regularization parameter \(\mu\) selected from the value range \(\mu = \{0.1, 0.5, 1, 5\}\). We clarify here that the “best” choice of the BV regularization term consists in that value of \(\mu\), which improves the solution \(\alpha^n\) to yield a better result both in the geometrical and physical sense (Tables 3 to 5). We display the solution pair \((\alpha, \omega)\) of the final iteration \((n = 10)\) for \(\ell = \{0.2, 0.3, 0.4\}\), and we choose only one value of \(\ell\) to present the \((\alpha^n, \omega^n)\) series in each geometry (Figures 6 to 8). In the latter display, we provide also the conductivity contour lines (black lines) along with their values, in order to highlight the uniformity of the physics in the conductivity reconstruction. Finally, we point out that the Tikhonov regularization corresponds to the case \(\omega^0 \equiv 1\) (consequently \(\omega \equiv 1\)). Hereafter, we use the notation \(\omega \equiv 1\) accounting for the \(\omega^0\) constant profile of the Tikhonov reconstruction.
FIGURE 4  *BV regularized inversion* solution pair \((\omega, \alpha)\) at the final iteration for the \(\omega^0\) profiles: \((\iota = 0.2, 0.3, 0.4), \ \mu \equiv 1\) and \(\mu = \{0.1, 0.5, 0.5, 0.1\}\), respectively, for the strong eccentricity problem [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3  Reconstructed values \((\alpha_{in}, \alpha_{out})\) of the target conductivity \(\alpha\) for the concentric inhomogeneity problem

| Number of Iterations \((n = 10)\) | \(\iota = 0.2\) | \(\mu = 1\) | \(\iota = 0.3\) | \(\mu = 0.1\) | \(\omega \equiv 1\) | \(\mu = 1\) |
|----------------------------------|-----------------|--------------|-----------------|-----------------|-----------------|--------------|
| \(n\)                           | \(\alpha_{in}\) | \(\alpha_{out}\) | \(\alpha_{in}\) | \(\alpha_{out}\) | \(\alpha_{in}\) | \(\alpha_{out}\) |
| 1                               | 1.92            | 1.00         | 1.84            | 1.00            | 1.75            | 1.00         |
| 2                               | 1.94            | 1.00         | 1.86            | 1.00            | 1.76            | 1.00         |
| 3                               | 1.93            | 1.00         | 1.87            | 1.00            | 1.74            | 1.00         |
| 4                               | 1.94            | 1.00         | 1.86            | 1.00            | 1.76            | 1.00         |
| \(\geq 4\)                      |                 |              |                 |                 |                 |              |

5.2.1  **Concentric inhomogeneity**

We consider the simple case where the inclusion \(D\) is located at \((x_D, y_D) = (0, 0)\). We additionally consider the synthetic data

\[
\begin{align*}
\mathcal{f} & = 1 + \frac{11}{4} \cos(\Phi), & \Phi \in [0, 2\pi), \\
\mathcal{g} & = \frac{13}{8} \cos(\Phi), & \Phi \in [0, 2\pi),
\end{align*}
\]  (58)
FIGURE 5  BV regularized inversion solution pair \((\omega, \alpha)\) at the final iteration for the \(\omega^0\) profiles: \((\epsilon = \{0.2, 0.3, 0.4\}, \omega \equiv 1)\) and \(\mu = \{0.1, 1, 0.1, 0.1\}\), respectively, for the mild eccentricity problem [Colour figure can be viewed at wileyonlinelibrary.com]

| Number of Iterations \((n = 10)\) | \(\epsilon = 0.2\) | \(\mu = 0.1\) | \(\alpha_{in}\) | \(\alpha_{out}\) | \(\epsilon = 0.3\) | \(\mu = 0.5\) | \(\alpha_{in}\) | \(\alpha_{out}\) | \(\epsilon = 0.4\) | \(\mu = 0.5\) | \(\alpha_{in}\) | \(\alpha_{out}\) | \(\omega \equiv 1\) | \(\mu = 0.1\) | \(\alpha_{in}\) | \(\alpha_{out}\) |
|---------------------------------|-------------------|----------------|--------------|----------------|-------------------|----------------|--------------|--------------|-------------------|----------------|--------------|--------------|----------------|--------------|--------------|--------------|
| \(\epsilon = 0.2\) \(\mu = 0.1\) | \(n\) \(\geq 1\) | \(\alpha_{in}\) | 1.99 | \(\alpha_{out}\) | 1.00 | \(\epsilon = 0.3\) | \(n\) \(\geq 1\) | 1.99 | \(\alpha_{out}\) | 1.00 | \(\epsilon = 0.4\) | \(n\) \(\geq 1\) | 1.99 | \(\alpha_{out}\) | 1.00 | \(\omega \equiv 1\) | \(\mu = 0.1\) | 1.46 | \(\alpha_{out}\) | 1.00 |

Using the computational process that was described in Section 5.1, in Figure 3, from left to right, we display the initial pair \((\omega^0, \alpha^0)\) and the final corresponding numerical solution \((\omega, \alpha)\) for the \(\omega^0\) configurations: \(\epsilon = \{0.2, 0.3, 0.4\}\) and \(\omega^0 \equiv 1\)

\[
u(\rho, \Phi) = \begin{cases} 
1 + \rho \cos(\Phi), & 0 \leq \rho \leq 1, \\
1 + \left(\frac{1}{2} \rho - \frac{1}{2\rho}\right) \cos(\Phi), & 1 < \rho \leq 2.
\end{cases}
\]
TABLE 5  Reconstructed values \((\alpha_{in}, \alpha_{out})\) of the target conductivity \(\alpha\) for the mild eccentricity problem

| Number of Iterations \((n = 10)\) | \(\ell\) | \(n\) | \(\omega\) | \(\alpha_{in}\) | \(\alpha_{out}\) |
|-----------------------------------|--------|-----|------|-------------|-------------|
| \(\ell = 0.2\) | 1 | \(\geq 2\) | \(\mu = 0.1\) | 1.92 | 1.91 |
| \(\ell = 0.3\) | 1 | 2 | 3-4 | \(\mu = 1\) | 1.84 | 1.83 |
| \(\ell = 0.4\) | 1 | 2 | 3-4 | \(\mu = 0.1\) | 1.79 | 1.78 |
| \(\omega \equiv 1\) | 1 | \(\geq 2\) | \(\mu = 0.1\) | 1.02 | 1.04 |

FIGURE 6  BV regularized inversion solutions and \(\omega^n\) profiles at the \(n\)th iteration for \(\ell = 0.2\) of the concentric inhomogeneity problem

[Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 6  BV physical reconstructed values \((\alpha_{in}, \alpha_{out})\) of the target conductivity \(\alpha\) for every inhomogeneity problem

| Number of Iterations \((n = 1)\) | Concentric | \(\ell = 0.02\) | \(\omega \equiv 1\) | \(\mu = 1\) | \(\alpha_{in}\) | \(\alpha_{out}\) |
|-----------------------------------|------------|----------------|----------------|-------------|-------------|-------------|
| Strong eccentric | \(\ell = 0.02\) | \(\omega \equiv 1\) | \(\mu = 1\) | 1.93 | 1.10 |
| Mild eccentric | \(\ell = 0.02\) | \(\omega \equiv 1\) | \(\mu = 1\) | 1.98 | 1.00 |

(last row). The inversion is performed for the different values of \(\mu = \{1, 0.1, 0.1, 1\}\) for \(\ell = \{0.2, 0.3, 0.4\}\), \(\omega^0 \equiv 1\), respectively, and mesh size \(h = 0.27\). In Figure 6, we present extensively the series of the optimal solutions \(\alpha^n\) (first row) corresponding to the profiles \(\omega^n\) (second row) for the chosen \(\omega^0\) configuration \(\ell = 0.2\). We summarize the numerical results for all iterations in Table 3.

All tables hereafter show the \(\omega^0\) refinement level \((\ell')\) along with the selected regularization value \((\mu)\), the value of the computed conductivity in the inclusion \(D_1(\alpha_{in})\), and that of the outer domain \(\Omega \backslash D_1(\alpha_{out})\), in every iteration \(n\). The variables \((\alpha_{in, out})\) denote the uniform values of the reconstructed conductivity \(\alpha\). We point out here for readers' convenience that the color bar in all figures pertains to the conductivity range \((\alpha \in [1, 2])\).
5.2.2 Eccentric inhomogeneity

Strong eccentricity problem

The center of an indicative strongly eccentric inclusion is chosen to be at \((x_D, y_D) = \left(\frac{\sqrt{5} - \sqrt{17}}{2}, 0\right)\). We assume that synthetic data are available in the form

\[
\begin{align*}
    f &= s_f(\alpha_2, \tau_1, \tau_2) \frac{\sin \Phi}{\sqrt{17} + 4 \cos \Phi}, \quad \Phi \in [0, 2\pi), \\
    g &= s_g(\alpha_2, \tau_1, \tau_2) \frac{1}{2} \frac{\sin \Phi}{(\sqrt{17} + 4 \cos \Phi)^2}, \quad \Phi \in [0, 2\pi),
\end{align*}
\]

(60)

where

\[
\begin{align*}
    s_f(\alpha_2, \tau_1, \tau_2) &= \frac{1}{2} (1 + \alpha_2)e^{-\tau_1} + \frac{1}{2} (1 - \alpha_2)e^{-2\tau_1}e^{\tau_1}, \\
    s_g(\alpha_2, \tau_1, \tau_2) &= \frac{1}{2} (1 + \alpha_2)e^{-\tau_1} - \frac{1}{2} (1 - \alpha_2)e^{-2\tau_1}e^{\tau_1},
\end{align*}
\]

(61)
FIGURE 9  Single-iteration BV regularized inversion solutions for the $\omega^0$ profiles: ($l = 0.02$, $\omega \equiv 1$) of every inhomogeneity problem and $\mu = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

generated by the exact solution (obtained by an analytical method in bipolar coordinates)

$$u_{\text{exact}} = \begin{cases} 
  (\rho^4 + c_1(\Phi)\rho^3 + \rho^2 c_2(\Phi) + c_3(\Phi)\rho + 16)^{-\frac{1}{2}} e^{-\tau} \rho \sin \Phi, & \text{if } \tau > \tau_2, \\
  (\rho^4 + c_1(\Phi)\rho^3 + c_2(\Phi)\rho^2 + c_3(\Phi)\rho + 16)^{-\frac{3}{2}} \tilde{c} \rho \sin \Phi, & \text{if } \tau_1 < \tau < \tau_2,
\end{cases} \tag{62}$$

with

$$\begin{align*}
  a &= \frac{1}{2}, \quad a_2 = 2, \\
  \tau_1 &= \ln(\sqrt{17} + 1)\frac{1}{4}, \\
  \tau_2 &= \ln(\sqrt{5} + 1)\frac{1}{2}, \\
  c_1(\Phi) &= 2\sqrt{17} \cos \Phi, \\
  c_2(\Phi) &= 17 + 8 \cos(2\Phi), \\
  c_3(\Phi) &= 8\sqrt{17} \cos \Phi, \\
  \tilde{c} &= \frac{1}{2}(1 + a_2)e^{-\tau} + \frac{1}{2}(1 - a_2)e^{-2\tau}e^{\tau}, \\
  \tau &= \frac{1}{2} \ln \left[ \frac{\rho^2 + \frac{4a^2}{(1-e^{-2\tau})^2} + \frac{4a\rho \cos \Phi}{1-e^{-2\tau}}}{\rho^2 + \frac{4\rho^2 e^{-4\tau}}{(1-e^{-2\tau})^2} + \frac{4\rho \cos \Phi e^{-2\tau}}{1-e^{-2\tau}}} \right].
\end{align*} \tag{63}$$

In our computational process, we have observed that the regularization values $\mu = \{0.1, 0.5, 0.5, 0.1\}$, corresponding to $\ell = \{0.2, 0.3, 0.4\}$ and $\omega^0 \equiv 1$, reveal appropriately the geometrical features of that case. We present the numerical results with respect to these values of $\mu$ and mesh size $h = 0.24$.

Figure 4 from left to right displays the chosen initial guess ($\omega^0, a^0$) along with the numerical solution pair ($\omega, a$) of the final iteration. Next, in Figure 7, we show the sequence of ($a^n, \omega^n$) for the most disturbed $\omega^0$ configuration $\ell = 0.4$. We perform the numerical results in Table 4.

**Mild eccentricity problem**

In this example, the coordinates of the center of $D$ are taken as $(x_D, y_D) = (-\frac{1}{3}, 0)$. We handle the following pair of synthetic data

$$\begin{align*}
  f &= m_f(a_2, \tau_1, \tau_2) \frac{\sqrt{10} \sin \Phi}{\frac{3}{2} + 4 \cos \Phi}, & \Phi \in [0, 2\pi), \\
  g &= m_g(a_2, \tau_1, \tau_2) \frac{320}{9} \frac{\sin \Phi}{(\frac{28}{3} + 4 \cos \Phi)^2}, & \Phi \in [0, 2\pi),
\end{align*} \tag{64}$$
where
\[
m_f(\alpha_2, \tau_1, \tau_2) = \frac{1}{2}(1 + \alpha_2)e^{-\tau_1} + \frac{1}{2}(1 - \alpha_2)e^{-2\tau_2}e^{\tau_1},
\]
\[
m_g(\alpha_2, \tau_1, \tau_2) = \frac{1}{2}(1 + \alpha_2)e^{-\tau_1} - \frac{1}{2}(1 - \alpha_2)e^{-2\tau_2}e^{\tau_1},
\]  
stemmed from the exact solution
\[
u_{exact} = \begin{cases} 
e^{-\tau} \frac{\sin \Phi \sinh \tau}{\cosh \tau \cos \Phi}, & \text{if } \tau > \tau_2, \\ \left[\frac{1}{2}(1 + \alpha_2)e^{-\tau} + \frac{1}{2}(1 - \alpha_2)e^{-2\tau_2}e^{\tau}\right] \frac{\sin \Phi \sinh \tau}{\cosh \tau \cos \Phi}, & \text{if } \tau_1 < \tau < \tau_2, \end{cases}
\]  
with
\[
a = \frac{4}{3} \sqrt{10}, \quad \alpha_2 = 2, \quad \tau_1 = \ln[\frac{2(\sqrt{10} + 7)}{3}], \quad \tau_2 = \ln[\frac{(4\sqrt{10} + 13)}{3}], \quad \tau = \frac{1}{2} \ln \left[\frac{\rho^2 + \frac{4a^2}{(1-e^{-2\tau_1})^2} + \frac{4\rho \cos \Phi e^{-2\tau_1}}{1-e^{-2\tau_1}}}{\rho^2 + \frac{4\rho e^{-4\tau_1}}{1-e^{-2\tau_1}} + \frac{4\rho \cos \Phi e^{-2\tau_1}}{1-e^{-2\tau_1}}}\right].
\]
In Figure 5, from left to right, we plot the initial pair \((\omega^0, \alpha^0)\) and its computed solution pair \((\omega, \alpha)\) at the 10th iteration for the \(\omega^0\) profiles: \(\ell' = \{0.2, 0.3, 0.4\}\), \(\omega^0 \equiv 1\), and the optimal regularization values—observed in computations—\(\mu = \{0.1, 1, 0.1, 0.1\}\) apiece, with respect to the mesh size \(h = 0.26\). Figure 8 presents the \((\omega^n, \alpha^n)\) reconstruction sequence generated by the value \(\ell' = 0.3\) for the finer mesh size \(h = 0.15\). The reconstruction parameters are given in Table 5.

5.3 \(BV\) physical reconstruction

In this paragraph, we would like to present the efficiency of the method in case we know a priori the correct geometry and we are aiming at the values of the conductivity function throughout the well-defined components of the whole structure. There exist two reasons why we present this analysis. First of all, it is really amazing to remark the efficiency as well as the abrupt convergence of the methodology to the desired exact solution with high accuracy. Secondly, the a priori information of the correct geometry of the inhomogeneity is introduced in the reconstruction technique via the specific form of the field \(\omega\), and this implication is strongly related to the efficiency of the numerical treatment of the method, on the basis of the hidden \(BV\) structure carried by \(\omega\) itself.

In what follows, we assume that \(\omega\) has indeed the expected behavior in the vicinity of the inhomogeneity, as depicted in second column of Figure 9, generated by the value \(\ell' = 0.02\) in formula (56). For all geometries (Concentric inhomogeneity, strong eccentricity problem, and mild eccentricity problem), we set \(\varepsilon = 0.1\), \(\kappa = 10\), the regularization parameter \(\mu = 1\).
and as initial approximation for the conductivity the three-valued constant function:

\[
\alpha^0 = \begin{cases} 
5, & (x, y) \in D, \\
0.5, & (x, y) \in \Omega \setminus (\Omega_2 \cup D), \\
1, & (x, y) \in \Omega_2, 
\end{cases} \tag{68}
\]

(Figure 9, first column). Here, we have selected a very small lower and a very large upper threshold value to show that even under an extended deviation range, the conductivity is recovered immediately and exactly.

The BV regularized inversion is performed for one iteration with respect to the mesh sizes \((h)\) displayed in Table 2. We also consider for comparison the case \(\omega^0 \equiv 1\), accounting for the Tikhonov reconstruction.

In Figure 9, from left to right, we display the initial guess \((\alpha^0)\), the “right” \((\omega)\) and the computed numerical solution \((\alpha)\) for the concentric, the strong eccentric and the mild eccentric inhomogeneity. Next to each (dashed line), we present the \(\omega \equiv 1\) profile along with the solution \((\alpha)\) provided by the Tikhonov reconstruction. We display the numerical results in Table 6.

### 5.4 Multidata reconstruction of the concentric inhomogeneity

#### 5.4.1 Exact observations

In this section, we investigate a simple case of multiple measurements. Thoroughly, we consider a specific family of exact synthetic data for the concentric inhomogeneity problem (Section 5.2.1). More precisely, we assume that we know a sufficient number \(m\) (\(m \in \mathbb{N}\)) of boundary data pairs of the form

\[
(f_m, g_m) = \left(1 + \frac{13}{8} \left(3 \cdot 2^{m+1} - 2\right) m(3 \cdot 2^m + 1) \cos(m\Phi), \frac{13}{8} \cos(m\Phi)\right), \quad \Phi \in [0, 2\pi), \quad m = 1, 2, \ldots, N, \tag{69}
\]

each one supporting the common exact solution

\[
u^{(m)}(\rho, \Phi) = 1 + \frac{13}{8} \begin{cases} 
\frac{2^{m+2}}{m(3 \cdot 2^m + 1)} \rho^m \cos(m\Phi), & 0 \leq \rho \leq 1, \\
\frac{2^{m+1}}{m(3 \cdot 2^m + 1)} (3 \rho^m - \rho^{-m}) \cos(m\Phi), & 1 < \rho \leq 2.
\end{cases} \tag{70}
\]
We perform the algorithm of Section 5.1 (setting $\lambda = 0$) on the problem (51), where now the functional (52) can be rewritten as

$$J_c(u, w, \alpha, \omega, N) = \sum_{m=1}^{N} |E_m(u, w, \alpha)| + \frac{\lambda}{2} \|Ra - a_0\|_{L^2(\Omega)}^2 + \mu \int_{\Omega} (\omega|\nabla\alpha|^2 + \psi_c(\omega))dx,$$

(71)
where

$$E_m(u, w, a) = \frac{k + 1}{2} \int_\Omega |\nabla(u + \eta f_m)|^2 dx + (1 - \kappa) \left[ \frac{1}{2} \int_\Omega |\nabla w|^2 dx - \langle g_m, \gamma w \rangle_{A \times B} \right] - \kappa \langle g_m, f_m \rangle_{A \times B}. \quad (72)$$

The problem parameters $\varepsilon$, $\kappa$ and the mesh size $h$ are selected as in Section 5.2.1, whereas we choose $a_0 = 2.5$ through the whole domain $\Omega \setminus \Omega_d$. We repeat the inversion for the $\omega^0$ configurations: $\varepsilon = \{0.2, 0.4, 0.6\}$ and $\omega^0 \equiv 1$ with regulariza-
tion values $\mu = \{1, 0.1, 1\}$ apiece, for $N = \{2, 5\}$ data pairs of the form (69), and we compare the results with the single data pair case $N = 1$ (Section 5.2.1). We notice here that the boundary data (69) for $N = 1$ are normalized to coincide with those in (58). Figure 10bA,B from left to the right displays the initial conductivity ($\alpha_0$) and the final reconstructed solutions ($\alpha$) for $N = \{1, 2, 5\}$ data pairs along with their corresponding solution pair ($\omega$) (beneath each), for the aforementioned $\omega^0$ configurations. Table 7 presents the numerical results for every selection of the number of the boundary data pairs ($N$) and every choice of the iteration level $n$ ($n = 10$).
We observe that the use of multiple measurements accelerates the computational process to an acceptable convergence both in the geometrical and physical sense. It is remarkable that for this mesh case, the reconstruction with just $N = 2$ data pairs is exact, coinciding with the inversion based on several data pairs ($N > 2$).

### 5.4.2 Noisy observations

In this example, we assume that noisy observations are available. More precisely, the range of the $\Lambda_{N,t,D}$ operator is subject to noise additions in the form

$$
(f_m, g_m) = (f_m + |f_m| \cdot R_{f_m} \cdot \theta, g_m),
$$

where $(f_m, g_m)$ are given in (69), $R_{f_m}$ are $\partial \Omega^k \times 1$ matrices of random numbers on the interval $(-1, 1)$, which are generated by the MATLAB function “rand” with $\partial \Omega^k$ the number of boundary nodes of the Delaunay triangulation and $\theta$ some noise level.

The minimization process on (71) will be started with $\lambda = 0$, $\epsilon = 0.1$, $\kappa = 10$, and $\alpha_0 = 2.5$ as the initial approximation for the conductivity function throughout the domain $\Omega \setminus \Omega_0$. We perform $n = 10$ iterations of the $BV$ Regularized Inversion algorithm for the $\omega^0$-configurations: $\ell = \{0.2, 0.4, 0.6\}$ with respect to the mesh size $h = 0.27$. At each $\omega^0$ refinement level ($\ell$), we recur the implementation for each value of the noise level $\theta = \{0.005, 0.01, 0.05\}$ and every choice of the number of the boundary data pairs $N = \{1, 2, 5\}$.

Figure 11a-C performs from left to the right the reconstructed solution pair $(\alpha, \omega)$ at the final iteration for each number of boundary data pairs ($N$) as the measurement noise $\theta$ becomes relatively larger. At each $\ell$ test case, the initial guess $(\alpha_0^0, \omega_0^0)$ is depicted on the right-hand side of the color bar. We start from the finest $\ell = 0.2$. The numerical results are summarized in Tables 8a to 8c, where we present the uniform values $(\alpha_{in, out})$ of the computed solution $\alpha$ at every iterate $n$ from the finest ($\ell = 0.2$) to the coarsest refinement level ($\ell = 0.6$).

| $\ell = 0.2$ | Number of Iterations ($n = 10$) |
|-------------|---------------------------------|
| $\theta = 0.005$ | $n$ | 1 | 2 | $\geq 3$ |
| $\mu = 1$ | $N = 1$ | 1.91 | 1.92 | 1.92 |
| $\alpha_{in}$ | $N = 2$ | 1.98 | 1.99 | 2.00 |
| $\alpha_{out}$ | $N = 5$ | 1.98 | 2.00 | 2.00 |
| $\theta = 0.01$ | $n$ | 1 | 2 | $\geq 3$ |
| $\mu = 1$ | $N = 1$ | 1.88 | 1.90 | 1.94 |
| $\alpha_{in}$ | $N = 2$ | 1.94 | 1.96 | 1.99 |
| $\alpha_{out}$ | $N = 5$ | 1.96 | 1.99 | 2.00 |
| $\theta = 0.05$ | $n$ | 1 | 2 | $\geq 4$ |
| $\mu = 1$ | $N = 1$ | 1.87 | 1.88 | 1.90 |
| $\alpha_{in}$ | $N = 2$ | 2.00 | 1.98 | 1.99 |
| $\alpha_{out}$ | $N = 5$ | 2.00 | 1.99 | 2.00 |

| $\ell = 0.4$ | Number of Iterations ($n = 10$) |
|-------------|---------------------------------|
| $\theta = 0.005$ | $n$ | 1 | 2 | $\geq 4$ |
| $\mu = 1$ | $N = 1$ | 1.60 | 1.62 | 1.67 |
| $\alpha_{in}$ | $N = 2$ | 1.99 | 1.99 | 1.99 |
| $\alpha_{out}$ | $N = 5$ | 1.99 | 1.99 | 2.00 |
| $\theta = 0.01$ | $n$ | 1 | 2 | $\geq 4$ |
| $\mu = 1$ | $N = 1$ | 1.60 | 1.65 | 1.70 |
| $\alpha_{in}$ | $N = 2$ | 1.99 | 1.98 | 1.98 |
| $\alpha_{out}$ | $N = 5$ | 1.99 | 2.00 | 1.99 |
| $\theta = 0.05$ | $n$ | 1 | 2 | $\geq 4$ |
| $\mu = 1$ | $N = 1$ | 1.56 | 1.71 | 1.76 | 1.78 |
| $\alpha_{in}$ | $N = 2$ | 1.98 | 1.99 | 2.00 | 1.99 |
| $\alpha_{out}$ | $N = 5$ | 1.99 | 2.00 | 1.99 | 2.00 |
| $\alpha_{out}$ | $N = 1, 2, 5$ | 1.00 | 1.00 | 1.00 | 1.00 |

**TABLE 8a** Reconstructed values $(\alpha_{in}, \alpha_{out})$ of the target conductivity $\alpha$ for $\ell = 0.2$ of the concentric inhomogeneity problem with $N$ “noisy” measurements

**TABLE 8b** Reconstructed values $(\alpha_{in}, \alpha_{out})$ of the target conductivity $\alpha$ for $\ell = 0.4$ of the concentric inhomogeneity problem with $N$ “noisy” measurements
TABLE 8c Reconstructed values ($\alpha_{\text{in}}, \alpha_{\text{out}}$) of the target conductivity $\sigma$ for $\ell' = 0.6$ of the concentric inhomogeneity problem with $N$ "noisy" measurements

| $\ell = 0.6$ | Number of Iterations ($n = 10$) |
|--------------|---------------------------------|
| $\theta = 0.005$ | $n$ | 1-6 | 7-8 | 9-10 |
| $\mu = 1$ | $N = 1$ | 1.67 | 1.90 | 1.99 |
| $\alpha_{\text{in}}$ | $N = 2$ | 1.97 | 1.99 | 2.00 |
| $\alpha_{\text{out}}$ | $N = 5$ | 1.99 | 2.00 | 1.99 |
| $\theta = 0.01$ | $n$ | 1-3 | 4-5 | 6-7 | 8-10 |
| $\mu = 1$ | $N = 1$ | 1.67 | 1.94 | 1.96 | 2.00 |
| $\alpha_{\text{in}}$ | $N = 2$ | 1.99 | 1.98 | 1.99 | 1.98 |
| $\alpha_{\text{out}}$ | $N = 5$ | 1.99 | 2.00 | 2.00 | 2.00 |
| $\theta = 0.05$ | $n$ | 1 | 2-3 | $\geq$4 |
| $\mu = 1$ | $N = 1$ | 1.71 | 1.90 | 1.96 |
| $\alpha_{\text{in}}$ | $N = 2$ | 1.97 | 1.98 | 1.99 |
| $\alpha_{\text{out}}$ | $N = 5$ | 1.98 | 2.00 | 1.99 |

FIGURE 12 $N$ data $BV$ regularized inversion solutions and $\sigma^0$ profiles at the first and final iteration for $\ell' = 0.3$ of the double inhomogeneity problem and $\mu = 0.06$ [Colour figure can be viewed at wileyonlinelibrary.com]

In the realm of the noisy observations, the Tikhonov approach ($\omega^0 \equiv 1$) totally fails, as expected, and therefore is omitted.
To conclude, it is revealing how the reconstruction with multiple measurements forcefully contributes to the geometrical and physical performance of the computed solution even in the presence of both relatively large noise and higher uncertainty with respect to the location of the inclusion’s boundary.

5.5 | Single and multidata reconstruction of disconnected inclusions

In this section, we would like to investigate further the applicability of the conductivity reconstruction method in case of disconnected inclusions. More precisely, we consider the conductive region Ω be a rectangular box of edge length $L = 6$, centered at the origin, with two different coaxial inhomogeneities, represented by the inclusions $D_2, D_3$ defined as follows:

$$D_2 := \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{(x - x_{D_2})^2 + (y - y_{D_2})^2} < \rho_{D_2} \right\}$$  \hspace{1cm} (74)

with center coordinates $(x_{D_2}, y_{D_2}) = (-1.34, 0)$ and $\rho_{D_2} = 0.9$, 

$$D_3 := \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{(x - x_{D_3})^2 + (y - y_{D_3})^2} < \rho_{D_3} \right\}$$  \hspace{1cm} (75)

with $(x_{D_3}, y_{D_3}) = (1.41, 0)$ and $\rho_{D_3} = 1$. 

**FIGURE 13** N data BV regularized inversion solutions and $\omega^n$ profiles at the first and final iteration for $\alpha = 0.5$ of the double inhomogeneity problem and $\mu = 0.06$ [Colour figure can be viewed at wileyonlinelibrary.com]
TABLE 9a Reconstructed values ($\alpha_{D_2}, \alpha_{D_3}, \alpha_{out}$) of the target conductivity $\alpha$ for $\ell' = \{0.3, 0.5\}$ of the double inhomogeneity problem with $N$ measurements

| Number of Iterations ($n = 10$) |
|-----------------|-----------------|-----------------|
| $\ell' = 0.3$   | $n$             | 1 | 2 | 3 | $\geq 4$ |
| $\mu = 0.06$    | $N = 1$         | $\alpha_{D_2}$ | 1.64 | 1.53 | 1.51 | 1.50 |
|                 |                 | $\alpha_{D_3}$ | 1.94 | 1.95 | 1.96 | 1.97 |
| $\mu = 0.06$    | $N = 2$         | $\alpha_{D_2}$ | 1.75 | 1.58 | 1.50 |
|                 |                 | $\alpha_{D_3}$ | 1.91 | 1.95 | 1.98 |
| $\ell' = 0.5$   | $n$             | 1 | 2 | $\geq 3$ |
| $\mu = 0.06$    | $N = 1$         | $\alpha_{D_2}$ | 1.53 | 1.58 | 1.60 | 1.61 |
|                 |                 | $\alpha_{D_3}$ | 1.95 | 1.95 | 1.95 | 1.96 |
| $\mu = 0.06$    | $N = 2$         | $\alpha_{D_2}$ | 1.45 | 1.72 | 1.76 | 1.58 | 1.50 |
|                 |                 | $\alpha_{D_3}$ | 1.96 | 1.93 | 1.95 | 1.95 | 1.97 |
| $\ell' = 0.5$   | $n$             | 1 | 2 | 3-7 | 8 |
| $\mu = 0.06$    | $N = 1$         | $\alpha_{D_2}$ | 1.00 | 1.00 | 1.00 | 1.00 |
|                 |                 | $\alpha_{D_3}$ | 1.00 | 1.00 | 1.00 | 1.00 |

FIGURE 14a A, $BV$ regularized inversion solutions and $\omega^n$ profiles at the first and final iteration for $\ell' = 0.4$ of the double inhomogeneity problem with $\kappa = \{0.5, 2\}$ and $\mu = 0.06$. B, $BV$ regularized inversion solutions and $\omega^n$ profiles at the first and final iteration for $\ell' = 0.4$ of the double inhomogeneity problem with $\kappa = \{5.11\}$ and $\mu = 0.06$. [Colour figure can be viewed at wileyonlinelibrary.com]

The geometry of the inclusions is selected to be adapted with the coordinate surfaces of a specific bispherical coordinate system. Then, all the underlying analytical tools are available in the service of the solution of the direct problem, giving flexibility to the acquisition of synthetic data. We assume that the target conductivity has the following profile (see
Figure 2):\[
\alpha = \alpha(x, y) = \begin{cases} 
1.5, & (x, y) \in D_2, \\
2, & (x, y) \in D_3, \\
1, & (x, y) \in \Omega \setminus (D_2 \cup D_3).
\end{cases}
\] (76)

The computational mesh, defined by the rectangular boundary forming the box’s vertices located at \{(-3, 3), (3, 3), (3, -3), (-3, -3)\}, is provided here as well by Triangle\textsuperscript{44,45} and consists of 3266 triangles and 1694 nodes. The characteristic mesh size is \(h = 0.28\).

The inversion is performed with the aid of the BV regularized inversion algorithm on the problem (51). The operators (71) are discretized using P1 elements for either \(N = 1\) or \(N = 2\) boundary data pairs. These pairs have alternative origin. The aforementioned analytical framework in conjunction with FEM techniques provide with a rich mixture of data of different features and characteristics.\textsuperscript{9}

We set \(\lambda = 0, \epsilon = 0.1, \kappa = 0.5,\) and \(a_0 = 2.5\) in \(\Omega \setminus \Omega_\delta\) as the initial approximation for the conductivity. Note that the \(\Omega_\delta\) zone along the vertical and horizontal sides of the box domain has been slightly shifted towards the center using \(\delta = 0.5\) for the inversion.

\* We constructed analytical data and synthetic data obtained on a much finer mesh using P1 discretization. The analytical ones constitute infinite series over bispherical eigensolutions, which carry naturally intrinsic truncation errors that are comparable with the errors involving in the alternatively produced data via FEM techniques. It is out of scope of the work to present the underlying analytical burden, the subsequent convergence analysis, and the involved error analysis.

\*\textsuperscript{Correction added on 8 March 2020 after initial online publication. Figure 14b has been corrected in this version.}
6 CONCLUSIONS

We presented a methodology for the solution of the inverse conductivity problem, honoring the space of functions of bounded variation as the most appropriate space hosting the conductivity profile. We demonstrated through representative examples, in the framework of the inclusion problem, that the BV regularization assists the gradient-based interior point algorithm to achieve accurate reconstructions of both the geometry of the inhomogeneity and the values of the conductivity, unlike the commonly employed for this purpose Tikhonov regularization, which enforces $H^1$ regularity. The BV regularization reconstruction of the geometry seems to be insensitive to the regularization parameter $\mu$, but this is
not the same as far as the estimation of the values of conductivity is concerned. Our investigation revealed that accurate reconstructions of both the geometry of the concentric and eccentric inclusions were possible with rather mild assumptions on their locations. In addition, we incorporated indicative cases of multidata and noisy measurements clarifying the sensitivity trend of the current reconstruction technique. Finally, an indicative disconnected inclusion case has been investigated, revealing the efficiency of the current minimization process in comparison with older approaches, whose minimization scheme can be actually provided by the present setting via the suitable selection of the intrinsic parameter $\kappa$. It is of interest to investigate further the applicability of the proposed methodology for multilayered or composite materials, in which cases the role of the $H$ convergence, which has been peripheral and indirect in the present work, is expected to obtain a parallel pace with $BV$ regularity monitoring. This project is under current investigation.

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ORCID

Antonios Charalambopoulos https://orcid.org/0000-0002-9505-0830
Vanessa Markaki https://orcid.org/0000-0002-7342-3050

REFERENCES

1. Calderón AP. On an inverse boundary value problem. Sem. on Numerical Analysis and its Applications to Continuum Physics (Soc. Brasil: Mat. Rio de Janeiro). 65-73; 1980.
2. Salo M. Calderón problem. Lecture notes, Department of mathematics and statistics university of helsinki; 2008.
3. Uhlmann G.. 30 Years of Calderón's Problem. Séminaire Laurent Schwartz EDP Appl.. 2012-2013, 1-25; 2012.
4. Astala K, Paivarinta L. Calderón’s inverse conductivity problem in plane. Ann of Math. 2006;163(1):265-299.
5. Sylvester J, Uhlmann G. A global uniqueness theorem for an inverse boundary value problem. Ann of Math. 1987;125(1):153-169.
6. Nachman A. Global uniqueness for a two-dimensional inverse boundary value problem. Ann of Math. 1996;143(2):71-96.
7. Siltanen S, Mueller J, Isaacson D.. An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem. Inverse Probl. 2000;16:681-699.
8. Mueller J, Siltanen S, Isaacson D. A direct reconstruction algorithm for electrical impedance tomography. IEEE Trans on Med Imaging. 2002;21(6):555-559.
9. Brown RM, Torres RH. Uniqueness in the inverse conductivity problem for conductivities with $3/2$ derivatives in $L^p$, $p > 2n$. J Fourier Anal Appl. 2003;9(6):563-574.
10. Krupchyk K, Uhlmann G. The Calderón problem with partial data for conductivities with $3/2$ derivatives. Commun Math Phys. 2016;348(1):185-219.
11. Haberman B, Tataru D. Uniqueness in Calderón’s problem with Lipschitz conductivities. Duke Math J. 2013;162(3):497-516.
12. Caro P, Rogers KM. Global uniqueness for the Calderón problem with Lipschitz conductivities. Forum Math Pi. 2015;3:e2. https://doi.org/10.1017/fmp.2015.9
13. Haberman B. Uniqueness in Calderón’s problem for conductivities with unbounded gradient. Commun Math Phys. 2015;340(2):639-659.
14. Alessandrini G. Stable determination of conductivity by boundary measurements. Appl Anal. 1988;27(1-3):153-172.
15. Barceló JA, Barceló T, Ruiz A. Stability of the inverse conductivity problem in the plane for less regular conductivities. J Differential Equations. 2001;173(2):231-270.
16. Clop A, Faraco D, Ruiz A. Stability of Calderón inverse conductivity problem in the plane for discontinuous conductivities. Inverse Probl Imaging. 2010;4:49-91.
17. Caro P, García A, Reyes JM. Stability of the Calderón problem for less regular conductivities. J Diff Equat. 2013;254:469-492.
18. Alessandrini G, Cabib E. EIT and the average conductivity. J Inverse Ill-posed Probl. 2008;16(8):727-736.
19. Faraco D, Kurylyev Y, Ruiz A. G-convergence, Dirichlet to Neumann maps and invisibility. J Funct Anal. 2013;267(7):2478-2506.
20. Rondi L. Continuity properties of Neumann-to-Dirichlet maps with respect to the $H$-convergence of the coefficient matrices. Inverse Probl. 2015;31(4). https://doi.org/10.1088/0266-5611/31/4/045002
21. Jikov VV, Kozlov SM, Oleinik OA. Homogenization of Differential Equations and Integral Functionals. Berlin: Springer-Verlag; 1994.
22. Allaire G. Shape Optimization by the Homogenization Method: Springer Applied Mathematical Sciences; 2002.
23. Attouch H, Buttazzo G, Michaille G. Variational Analysis in Sobolev and BV Functions. Applications to PDEs and Optimization. Society for Industrial and Applied Mathematics and the Mathematical Programming Society; 2006.
24. Aubert G, Kornprobst P. Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variations. New York, NY: Springer Verlag; 2002.
APPENDIX A: THE FRAMEWORK OF FUNCTIONS OF BOUNDED VARIATION

Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) and \( B(\Omega) \) its Borel field. \( M(\Omega, \mathbb{R}^d) \) denotes the space of all \( \mathbb{R}^d \)-valued Borel measures, which is also according to the Riesz theory the dual of the space \( C_0(\Omega, \mathbb{R}^d) \) of all continuous functions \( \Phi \) vanishing at infinity, equipped with the uniform norm \( \|\Phi\|_\infty = (\sum_{i=1}^d \sup_{x \in \Omega} |\Phi_i(x)|^2)^{1/2} \). We note that \( M(\Omega, \mathbb{R}^d) \) is isomorphic to the product space \( M^d(\Omega) \) and that

\[ \mu = (\mu_1, \ldots, \mu_d) \in M(\Omega, \mathbb{R}^d) \iff \mu_i \in C_0^d(\Omega), \ i = 1, \ldots, d. \]
Definition 1. We say that a function \( u : \Omega \to \mathbb{R} \) is a function of bounded variation if it belongs to \( L^1(\Omega) \) and its gradient \( Du \) in the distributional sense belongs to \( \mathbf{M}(\Omega, \mathbb{R}^d) \). We denote the set of all functions of bounded variation by \( \BV(\Omega) \). The four following assertions are then equivalent:\(^2\)

(i) \( u \in \BV(\Omega) \);
(ii) \( u \in L^1(\Omega) \) and \( \forall i = 1, \ldots, d, \frac{\partial u}{\partial x_i} \in \mathbf{M}(\Omega) \);
(iii) \( u \in L^1(\Omega) \) and \( \|Du\| := \sup\{\langle Du, \Phi \rangle : \Phi \in C_c(\Omega, \mathbb{R}^d), \|\Phi\|_{\infty} \leq 1 \} < +\infty \);
(iv) \( u \in L^1(\Omega) \) and \( \|Du\| = \sup\{\int_\Omega u \frac{d\Phi}{dx} \cdot dx : \Phi \in C^1_c(\Omega, \mathbb{R}^d), \|\Phi\|_{\infty} \leq 1 \} < +\infty \),

where \( C_c(\Omega, \mathbb{R}^d) \) denotes the space of all continuous functions with compact support in \( \Omega \) and the bracket \( \langle, \rangle \) in (iii) is defined by

\[
\langle Du, \Phi \rangle := \sum_{i=1}^d \int_\Omega \Phi_i \frac{\partial u}{\partial x_i} dx
\]

Theorem 3 (Riesz-Alexandroff representation). The topological dual of \( C_0(\Omega) \) can be isometrically identified with the space of bounded Borel measures. More precisely, to each bounded linear functional \( \Phi \) on \( C_0(\Omega) \), there is a unique Borel measure \( \mu \) on \( \Omega \) such that for all \( f \in C_0(\Omega) \),

\[
\Phi(f) = \int_\Omega f(x) d\mu(x).
\]

Moreover, \( \|\Phi\| = |\mu|(\Omega) \).

Remark 4. According to the vectorial version of the Riesz-Alexandroff representation theorem (Theorem 3), the dual norm \( \|Du\| \) is also the total mass \( |Du|(\Omega) = \int_\Omega |Du| \) of the total variation \( |Du| \) of the measure \( Du \).

Theorem 4 (Lebesgue decomposition). Let \( \mu \) a positive bounded measure on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) and \( \nu \) a vector-valued measure on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \). Then, there exists a unique pair of measures \( \nu_{ac} \) and \( \nu_s \) such that

\[
\nu = \nu_{ac} + \nu_s, \quad \nu_{ac} \ll \mu, \quad \nu_s \perp \mu.
\]

Moreover,

\[
\frac{d\nu}{d\mu} = \frac{d\nu_{ac}}{d\mu}, \quad \frac{d\nu_s}{d\mu} = 0, \quad \mu - a.e., \quad \text{and} \quad \nu(A) = \int_A \frac{d\nu}{d\mu} d\mu + \nu_s(A) \quad \forall A \in \mathcal{B}(\mathbb{R}^d),
\]

where \( \nu_{ac} \) and \( \nu_s \) are the absolutely continuous part and the singular part of \( \nu \).

If \( u \) belongs to \( \BV(\Omega) \) and in Theorem 4, we choose \( \mu = dx \), the \( d \)-dimensional Lebesgue measure, and \( \nu = Du \), we get

\[
Du = \nabla u \quad dx + \quad D_s u , \quad \text{(A1)}
\]

where \( \nabla u(x) = \frac{d(Du)}{dx}(x) \in L^1(\Omega) \) and \( D_s u \perp dx \). \( \nabla u(x) \) is also called the approximate derivative of \( u \). In fact, we can say more for \( \BV(\Omega) \) functions. The singular part \( D_s u \) of \( Du \) can be decomposed into a “jump” part \( J_u \) and a “Cantor” part \( C_u \).

In order to specify the \( J_u \) part, we need first the following definition.

Definition 2 (notion of approximate limit). Let \( B(x, r) \) be the ball of center \( x \) and radius \( r \) and let \( u \in \BV(\Omega) \). The approximate upper limit \( u^+(x) \) and the approximate lower limit \( u^-(x) \) are defined by

\[
u^+(x) = \inf \{ t \in [-\infty, +\infty] : \lim_{r \to 0} \frac{dx(\{u > t\} \cap B(x, r))}{rd^d} = 0 \},
\]

\[
u^-(x) = \sup \{ t \in [-\infty, +\infty] : \lim_{r \to 0} \frac{dx(\{u < t\} \cap B(x, r))}{rd^d} = 0 \}.
\]
If $u \in L^1(\Omega)$, then

$$\lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |u(x) - u(y)| dy = 0 \quad a.e. \ x.$$  \hspace{1cm} (A2)

A point $x$ for which (A2) holds is called a Lebesgue point of $u$, and we have

$$\lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy,$$  \hspace{1cm} (A3)

and $u(x) = u^+(x) = u^-(x)$. We denote by $S_u$ the jump set, that is, the complement, up to a set of $H^{d-1}$ measure** zero, of the set of Lebesgue points

$$S_u = \{ x \in \Omega; u^-(x) < u^+(x) \}.$$  

Then, $S_u$ is countably rectifiable, and for $H^{d-1}$-a.e. $x \in \Omega$, we can define a normal $n_u(x)$. Therefore, the measure $Du$ admits the following representation:

$$Du = \nabla u dx + (u^+ - u^-) n_u H^{d-1} + C_u.$$  \hspace{1cm} (A4)

From (A4), we can deduce the total variation of $Du$:

$$|Du|(\Omega) = \int_{\Omega} |Du| = \int_{\Omega} |\nabla u|(x) dx + \int_{S_u} |u^+ - u^-| dH^{d-1} + \int_{\Omega - S_u} |C_u|.$$  \hspace{1cm} (A5)

From (A1), we conclude that $W^{1,1}(\Omega)$ is a subspace of the vectorial space $BV(\Omega)$ and $u \in W^{1,1}(\Omega)$ iff $Du = \nabla u \mathcal{L}^d|\Omega$. The space $BV(\Omega)$ is equipped with the following norm, which extends the classical norm in $W^{1,1}(\Omega)$:

$$\|u\|_{BV(\Omega)} := |u|_{L^1(\Omega)} + \|Du\|.$$  

Equipped with its norm, $BV(\Omega)$ is a Banach space.

We will define two weak convergence processes in $BV(\Omega)$. The first provides compactness of bounded sequences and the second is an intermediate convergence between the weak and the strong convergence associated with the norm.

**Definition 3.** A sequence $(u_n)_{n \in \mathbb{N}}$ in $BV(\Omega)$ weakly (Mueller et al8 or weakly-*) converges to some $u$ in $BV(\Omega)$, and we write $u_n \rightharpoonup u$ (or $u_n \overset{\ast}{\rightharpoonup} u$) iff the following convergences hold:**

$$\begin{cases}
    u_n \rightharpoonup u \text{ strongly in } L^1(\Omega); \\
    Du_n \rightharpoonup Du \text{ weakly in } M(\Omega, \mathbb{R}^d).
\end{cases}$$

In the proposition below, we establish a compactness result related to this convergence, together with the lower semicontinuity of the total mass.

**Proposition 4.** Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $BV(\Omega)$ strongly converging to some $u$ in $L^1(\Omega)$ and satisfying $\sup_{n \in \mathbb{N}} \int_{\Omega} |Du_n| < +\infty$. Then,

(i) $u \in BV(\Omega)$ and $\int_{\Omega} |Du| \leq \lim_{n \to +\infty} \int_{\Omega} |Du_n|$;

(ii) $u_n$ weakly converges to $u$ in $BV(\Omega)$.

**Definition 4.** Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $BV(\Omega)$ and $u \in BV(\Omega)$. We say that $u_n$ converges to $u$ in the sense of the intermediate convergence iff

$$\begin{cases}
    u_n \rightharpoonup u \text{ strongly in } L^1(\Omega); \\
    \int_{\Omega} |Du_n| \rightharpoonup \int_{\Omega} |Du|.
\end{cases}$$

Finally, we state some fundamental theorems.**

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**Hausdorff measure.**

**In BV(Ω), one can refer to the weak convergence by denoting \( u_n \overset{\ast}{\rightharpoonup} u \) equivalently.
Theorem 5. The space $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$ equipped with the intermediate convergence. Consequently, $C^\infty(\bar{\Omega})$ is also dense in $BV(\Omega)$ for the intermediate convergence.

Theorem 6. Let $\Omega$ be a 1-regular open bounded subset of $\mathbb{R}^d$. For all $p$, $1 \leq p \leq \frac{d}{d-1}$, the embedding

$$BV(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous. More precisely, there exists a constant $C$ that depends only on $\Omega$, $p$, and $d$, such that for all $u \in BV(\Omega)$,

$$\left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}} \leq C|u|_{BV(\Omega)}.$$

Theorem 7. Let $\Omega$ be a 1-regular open bounded subset of $\mathbb{R}^d$. Then, for all $p$, $1 \leq p < \frac{d}{d-1}$, the embedding

$$BV(\Omega) \hookrightarrow L^p(\Omega)$$

is compact.

Theorem 8. Let $\Omega$ be a domain of $\mathbb{R}^d$ with a Lipschitz boundary $\Gamma$. There exists a linear continuous map $\gamma_0$ from $BV(\Omega)$ onto $L^1(\mathbb{R}^{d-1})$ satisfying

(i) for all $u$ in $C(\bar{\Omega}) \cap BV(\Omega)$, $\gamma_0(u) = u|\Gamma$;

(ii) the generalized Green’s formula holds: $\forall \Phi \in C^1(\bar{\Omega}, \mathbb{R}^d)$,

$$\int_{\Omega} \Phi Du = -\int_{\Omega} \text{div} \Phi \, dx + \int_{\Gamma} \gamma_0(u) \Phi \cdot n_\omega \, dH^{d-1},$$

where $n_\omega(x)$ is the outer unit normal at $H^{d-1}$ almost all $x$ in $\Gamma$.

APPENDIX B: RESULTS CONCERNING THE HOMOGENIZATION PROCESS

In the regime of 2D-homogenization theory, we consider a countable family of boundary value problems, each of them identified with the natural number $n \in \mathbb{N}$. In each problem of this sequence, the scalar conductivity profile of the type $A^{n}(x) = a_n(x)I_{2 \times 2}$ is involved. The convergence $a_n \rightharpoonup a$ weakly in $L^\infty(\Omega)$ is required in accordance with the supplementary assumption of $H$ convergence: $a_n(x)I_{2 \times 2} \rightharpoonup a(x)I_{2 \times 2}$, or equivalently of the corresponding $G$ convergence (given the scalar status). In the minimization process of the current investigation encountered in Section 3, a multiple nested subsequence $a_n$ of the minimizing sequence converges to $a$ strongly in $L^1$, and this implies naturally the $G$ convergence above for this particular subsequence. So $\forall f \in H^{-1}(\Omega)$, the sequence $\hat{u}^{a_n}_n(x)$ of the solutions of the problems

$$-\nabla \cdot (a_n(x)\nabla \hat{u}^{a_n}_n(x)) = h(x) \quad \text{in} \quad \Omega,$$

$$\hat{u}^{a_n}_n(x) = 0 \quad \text{on} \quad \partial \Omega,$$

obeys to the convergence rule:

$$\hat{u}^{a_n}_n \rightharpoonup u^a \quad \text{weakly in} \quad H^1_0(\Omega),$$

(and necessarily) $a_n(x)\nabla \hat{u}^{a_n}_n(x) \rightharpoonup a(x)\nabla u^a(x)$ weakly in $(L^2(\Omega))^2$ as $n \to \infty$, where $u^a$ is the unique solution of the problem

$$-\nabla \cdot (a(x)\nabla u^a(x)) = h(x) \quad \text{in} \quad \Omega,$$

$$u^a(x) = 0 \quad \text{on} \quad \partial \Omega.$$

The following two kinds of problems have been emerged in Section 3.
Problem I:

\[-\nabla \cdot (\alpha_n(x)\nabla u_n^a(x)) = h_n(x) \quad \text{in} \quad \Omega, \quad \tag{B7}\]

\[u_n^a(x) = 0 \quad \text{on} \quad \partial \Omega. \quad \tag{B8}\]

Problem II:

\[\nabla \cdot (\alpha_n(x)\nabla w_n^a(x)) = 0 \quad \text{in} \quad \Omega, \quad \tag{B9}\]

\[\alpha_n(x)\frac{\partial w_n^a}{\partial n}(x) = g(x) \quad \text{on} \quad \partial \Omega, \quad \tag{B10}\]

where \(h_n \in H^{-1}(\Omega)\) and \(g \in H^{-1}(\partial \Omega)\).

The convergence analysis would be very simple if the source term in (B7) was a constant function independent of the index \(n\). Given that this source is variable (as \(n\) runs over integers), some argumentation very reminiscent of the arguments presented in propositions 1.2.19 of Allaire\(^{22}\) (facilitated in case we focus on the aforementioned subsequence \(\alpha_n\) converging to \(\alpha\) in \(L^1\)) can be followed to settle the validity of the next two lemmas:

**Lemma 1.** In case that the solution of problem I obeys to the convergence \(u_n^a \rightharpoonup v\) (weakly in \(H^1_0(\Omega)\)) and the sequence \(h_n\) converges strongly in \(H^{-1}(\Omega)\), then \(\alpha_n(x)\nabla u_n^a \rightharpoonup \alpha \nabla v\) (weakly in \((L^2(\Omega))^2\)).

**Lemma 2.** In case that the stimulus \(h_n\) of problem I obeys to the convergence \(h_n \rightarrow h\) (strongly in \(H^{-1}(\Omega)\)), then the sequence \(u_n^a\) obeys to the energy convergence

\[\int_{\Omega} \alpha_n|\nabla u_n^a|^2 \rightarrow \int_{\Omega} \alpha|\nabla u^a|^2. \quad \tag{B11}\]

The trace operator \(\gamma : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)\) and its right inverse extension operator \(\eta : H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^1(\Omega)\) are used here to represent an interesting class of stimuli \(h_n\) as follows:

\[h_n(x) = \nabla \cdot (\alpha_n(x)\nabla (\eta f)(x)) \in H^{-1}(\Omega), \quad \tag{B12}\]

with \(f \in H^{\frac{1}{2}}(\partial \Omega)\).

Next, we consider problem I furnished with this particular type of nonhomogeneous terms (B12). In this framework, we state the following two propositions.

**Proposition 5.** The sequence \(u_n^a\), \(n \in \mathbb{N}\) converges to \(u^a\) weakly in \(H^1(\Omega)\) as \(n \rightarrow \infty\) and furthermore \(\alpha_n(x)\nabla u_n^a(x) \rightharpoonup \alpha(x)\nabla u^a(x)\) weakly in \((L^2(\Omega))^2\), where \(u^a\) is the unique solution of problems (B5) and (B6). In addition, it holds that

\[\int_{\Omega} \alpha_n|\nabla (u_n^a + \eta f)|^2 \rightarrow \int_{\Omega} \alpha|\nabla (u^a + \eta f)|^2. \quad \tag{B13}\]

**Proposition 6.** The sequence of solutions of problem II converges to \(w^a\) weakly in \(H^1_0(\Omega)\) as \(n \rightarrow \infty\), where \(w^a\) is the solution of the problem

\[-\nabla \cdot (\alpha(x)\nabla w^a(x)) = 0 \quad \text{in} \quad \Omega, \quad \tag{B14}\]

\[\alpha(x)\frac{\partial w^a}{\partial n}(x) = g(x) \quad \text{on} \quad \partial \Omega. \quad \tag{B15}\]

In addition,

\[\int_{\Omega} \alpha_n|\nabla w_n^a|^2 \rightarrow \int_{\Omega} \alpha|\nabla w^a|^2. \quad \tag{B16}\]

The proof of last two propositions is also based on the proof arguments presented in section 1.3 of Allaire.\(^{22}\) Here, only the proof of Proposition 6 merits to be presented thanks to the implication of the Dirichlet-to-Neumann operators and their convergence analysis:
Proof of Proposition 6. We make the decomposition

$$w^n_a = v^n_a + \eta \Lambda^{-1}_a g,$$  \hspace{1cm} (B17)

where we evoke the family of Neumann-to-Dirichlet operators $\Lambda^{-1}_a$ (or $\Lambda_{\alpha, L,D}$), which are well defined, uniformly bounded operators (over the integer $n$), since the members of the sequence of conductivities $a_n$ are uniformly bounded below, given that $a_n, a \in L^\infty(\Omega, [b, c])$. It is straightforward to show that the field $v^n_a$ satisfies the boundary value problem

$$-\nabla \cdot (a_n(x) \nabla v^n_a(x)) = \tilde{h}_n(x) \quad \text{in} \quad \Omega,$$  \hspace{1cm} (B18)

$$v^n_a(x) = 0 \quad \text{on} \quad \partial \Omega,$$  \hspace{1cm} (B19)

where $\tilde{h}_n = \nabla \cdot (a_n \nabla (\eta \Lambda^{-1}_a g)) \in H^{-1}(\Omega)$. For readers’ convenience, we briefly clarify here the following notations (see extensively in Section 2): $\Lambda_\alpha = \Lambda_{D,L,N}$ with $\Lambda_\alpha : B \colon \Omega \to B^*$ and $\Lambda^{-1}_\alpha = \Lambda_{\alpha, L,D}$ with $\Lambda^{-1}_\alpha : B^* \to B$.

At this point, we exploit the main result of Faraco et al.\textsuperscript{19} according to which when a very weak assumption is made for the conductivity profiles $a_n$, then the strong convergence $\|\Lambda_a - \Lambda_{\alpha}\|_H \rightarrow 0$ is valid. Actually, the mentioned weak assumption requires a specific convergence of the profiles $a_n$ to $\alpha$ in a very thin region around the boundary $\partial \Omega$. In this work, this assumption is a fortiori trivially satisfied, given that $a_n, a \in L^\infty(\Omega, [b, c])$ (Section 3) and then all the profiles totally agree on a very small region $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) < \delta\}$, with $\delta << 1$. Given that $\Lambda^{-1}_\alpha - \Lambda^{-1}_a = \Lambda^{-1}_\alpha (\Lambda_a - \Lambda_{\alpha}) \Lambda^{-1}_a$, it is easily inferred—thanks to the uniform boundedness of $\Lambda^{-1}_\alpha, \Lambda^{-1}_a$—that the strong convergence $\|\Lambda^{-1}_\alpha - \Lambda^{-1}_a\|_B \rightarrow 0$ is also valid.

The last result allows us to take the limit in $\tilde{h}_n$ to prove that $\tilde{h}_n \rightarrow \tilde{h} := \nabla \cdot (a \nabla (\eta \Lambda^{-1}_a g)) \in H^{-1}(\Omega)$ strongly in $H^{-1}(\Omega)$.

Indeed, given that $\Lambda^{-1}_\alpha g \leftarrow a \rightarrow \Lambda^{-1}_a g$ (strongly in $B^*$), we infer that $\nabla (\eta \Lambda^{-1}_a g) \cdot \nabla z \rightarrow \nabla (\eta \Lambda^{-1}_a g) \cdot \nabla z$, for every $z \in H^1(\Omega)$. We already know that $a_n \rightarrow a$ and then

$$\langle \tilde{h}_n, z \rangle_{H^{-1}(\Omega) \times H^1(\Omega)} = \int_\Omega a_n \nabla (\eta \Lambda^{-1}_a g) \cdot \nabla z - \int_\Omega a \nabla (\eta \Lambda^{-1}_a g) \cdot \nabla z = \langle \tilde{h}, z \rangle_{H^{-1}(\Omega) \times H^1(\Omega)}.$$  \hspace{1cm} (B20)

According to Lemma 2 and Proposition 5, we obtain that

$$v^n_a \rightarrow v^a \quad \text{weakly in} \quad H^1_0(\Omega),$$  \hspace{1cm} (B21)

$$a_n \nabla v^n_a \rightarrow a \nabla v^a \quad \text{weakly in} \quad L^2(\Omega)^2,$$  \hspace{1cm} (B22)

$$\int_\Omega a_n |\nabla v^n_a|^2 \rightarrow \int_\Omega a |\nabla v^a|^2,$$  \hspace{1cm} (B23)

where $v^a$ is the solution of the limiting problem

$$-\nabla \cdot (a(x) \nabla v^a(x)) = \tilde{h}(x) \quad \text{in} \quad \Omega,$$  \hspace{1cm} (B24)

$$v^a(x) = 0 \quad \text{on} \quad \partial \Omega.$$  \hspace{1cm} (B25)

In addition,

$$\int_\Omega a_n |\nabla (\eta \Lambda^{-1}_a g)|^2 \rightarrow \int_\Omega a |\nabla (\eta \Lambda^{-1}_a g)|^2,$$  \hspace{1cm} (B26)

due again to the limits $a_n \rightarrow a$ and $|\nabla (\eta \Lambda^{-1}_a g)|^2 \rightarrow |\nabla (\eta \Lambda^{-1}_a g)|^2$.

Finally,

$$\int_\Omega a_n \nabla v^n_a \cdot \nabla (\eta \Lambda^{-1}_a g) \rightarrow \int_\Omega a \nabla v^a \cdot \nabla (\eta \Lambda^{-1}_a g),$$  \hspace{1cm} (B27)

since $a_n \nabla v^n_a \rightarrow a \nabla v^a$, weakly in $(L^2(\Omega))^2$ and $\nabla (\eta \Lambda^{-1}_a g) \rightarrow \nabla (\eta \Lambda^{-1}_a g)$, strongly in $(L^2(\Omega))^2$.
Consequently, the sequence $w_n^{\alpha_n}$ converges (weakly in $H^1_0(\Omega)$) to $w^{\alpha} = v^{\alpha} + \eta \Lambda^{-1}_a g$, which readily solves problem (B14) and (B15), and furthermore, on the basis of limit processes (B23), (B26), and (B27), it comes out that

\[
\int_\Omega \alpha_n |\nabla w_n^{\alpha_n}|^2 = \int_\Omega \alpha_n |\nabla (\eta \Lambda^{-1}_n g)|^2 + 2 \int_\Omega \alpha_n |\nabla v_n^{\alpha_n}| \cdot |\nabla (\eta \Lambda^{-1}_n g)| + \int_\Omega \alpha_n |\nabla v_n^{\alpha_n}|^2
\]

\[-\rightarrow \int_{n \to \infty} \alpha |\nabla (\eta \Lambda^{-1}_a g)|^2 + 2 \int_\Omega \alpha |\nabla v^{\alpha}| \cdot |\nabla (\eta \Lambda^{-1}_a g)| + \int_\Omega \alpha |\nabla v^{\alpha}|^2 = \int_\Omega \alpha |\nabla w^{\alpha}|^2.
\]