Hypergeometric Functions and Feynman Diagrams

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Abstract The relationship between Feynman diagrams and hypergeometric functions is discussed. Special attention is devoted to existing techniques for the construction of the $\varepsilon$-expansion. As an example, we present a detailed discussion of the construction of the $\varepsilon$-expansion of the Appell function $F_3$ around rational values of parameters via an iterative solution of differential equations. As a by-product, we have found that the one-loop massless pentagon diagram in dimension $d = 3 - 2\varepsilon$ is not expressible in terms of multiple polylogarithms. Another interesting example is the Puiseux-type solution involving a differential operator generated by a hypergeometric function of three variables. The holonomic properties of the $F_N$ hypergeometric functions are briefly discussed.
1 Introduction

Recent interest in the analytical properties of Feynman diagrams has been motivated by processes at the LHC. The required precision demands the evaluation of a huge number of diagrams having many scales to a high order, so that a new branch of mathematics emerges, which we may call the Mathematical Structure of Feynman Diagrams\(^1\)[1,2], which includes elements of algebraic geometry, algebraic topology, the analytical theory of differential equations, multiple hypergeometric functions, elements of number theory, modular functions and elliptic curves, multidimensional residues, and graph theory. This mathematical structure has been extensively developed, studied and applied. For a more detailed discussion of the oldest results and their relation to modern techniques, see Refs. [3] and [4]). One of these approaches is based on the treatment of Feynman diagrams in terms of multiple hypergeometric functions [5]. For example, in the series of papers [6, 7, 8], the one-loop diagrams have been associated with the \( \zeta \) function (a particular case of the \( \zeta \) function \[9,10,11\]).

1.1 Mellin-Barnes representation, asymptotic expansion, NDIM

A universal technique based on the Mellin-Barnes representation of Feynman diagrams has been applied to one-loop diagrams in Ref. [12, 13] and to two-loop propagator diagrams in Ref. [14, 15, 16, 17, 18]. The multiple Mellin-Barnes representation for a Feynman diagram in covariant gauge can be written in the form

\[
\Phi(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}; \mathbf{z}) = \int_{-i\infty}^{+i\infty} \phi(t) dt = \int_{-i\infty}^{+i\infty} t_{a,b,c,d} \prod \frac{\Gamma(\sum_{i=1}^{m} A_{a_i t_i} + B_{a_i})}{\Gamma(\sum_{j=1}^{n} C_{b_j t_j} + D_{b_j})} dt \mathbf{z}^{\mathbf{a}} \mathbf{t}^{\mathbf{b}},
\]

where \( \mathbf{z} \) are ratios of Mandelstam variables and \( A, B, C, D \) are matrices and vectors depending linearly on the dimension of space-time \( n \) and powers of the propagators. Closing the contour of integration on the right (on the left), this integral can be presented around zero values of \( \mathbf{z} \) in the form

\[
\Phi(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}; \mathbf{z}) = \sum_{\mathbf{a}} f_{\mathbf{a}} H(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}; \mathbf{z}) \mathbf{z}^{\mathbf{a}},
\]

where the coefficients \( f_{\mathbf{a}} \) are ratios of \( \Gamma \)-functions and the functions \( H \) are Horn-type hypergeometric functions \[22\] (see Section 2 for details). The analytic continuation of the hypergeometric functions \( H(\mathbf{z}) \) into another region of the variables \( \mathbf{z} \) can be constructed via the integral representation (when available) \[23,24\]. \( H(\mathbf{z}) \rightarrow H(1 – \).

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\(^1\) Several programs are available for the automatic generation of the Mellin-Barnes representation of Feynman diagrams [19, 20, 21].
\( \vec{u} \). However, for more complicated cases of Horn-type hypergeometric functions, this type of analytic continuation is still under construction \[25, 26\].

A major set of mathematical results (see, for example, \[27, 28, 29\]) is devoted to the construction of the analytic continuation of a series around \( \vec{u} = 0 \) to a series of the form \( \vec{u} \vec{u} = \vec{u} \vec{v} = \vec{v} \vec{x} \). For example, the singular locus \( \vec{u} \vec{v} \) of the Appell function \( \vec{v} \vec{v} \) is \( \vec{u} \vec{v} = \vec{u} \vec{v} \vec{x} \vec{v} = 0 \) \( \cup \vec{u} \vec{v} \vec{x} \vec{v} \vec{w} = 0 \) \( \cup \vec{u} \vec{v} \vec{x} \vec{v} \vec{w} = 0 \) \( \cup \vec{u} \vec{v} \vec{x} \vec{v} \vec{w} = 0 \)

A similar problem, the construction of convergent series of multiple Mellin-Barnes integrals in different regions of parameters, has been analyzed in detail for the case of two variables \[30, 31, 32\]. However, to our knowledge, there are no systematic analyses of the relation between these series and the singularities of multiple Mellin-Barnes integrals.

It was understood long ago that there is a one-to-one correspondence between the construction of convergent series from Mellin-Barnes integrals and the asymptotic expansions; see Ref. \[33\] for example. The available software, e.g. Ref. \[34\], allows the construction of the analytical continuation of a Mellin-Barnes integral in the limit when some of the variables \( \vec{u} \) go to 0, or \( \infty \). These are quite useful in the evaluation of Feynman diagrams, but do not solve our problem. The current status of the asymptotic expansions is discussed in Ref. \[35, 36\].

Another technique for obtaining a hypergeometric representation is the so-called “Negative Dimensional Integration Method” (NDIM) \[37, 38, 39, 40, 41, 42\]. However, it is easy to show \[43\] that all available results follow directly from the Mellin-Barnes integrals \[12\].

For some Feynman diagrams, the hypergeometric representation follows from a direct integration of the parametric representation, see Ref. \[44, 45, 46, 47, 48, 49, 50, 51, 52\].

We also mention that the “Symmetries of Feynman Integrals” method \[53, 54, 55\] can also be used to obtain the hypergeometric representation for some types of diagrams.

### 1.2 About GKZ and Feynman Diagrams

There are a number of different though entirely equivalent ways to describe hypergeometric functions:

- as a multiple series;
- as a solution of a system of differential equations (hypergeometric D-module);
- as an integral of the Euler type;
- as a Mellin-Barnes integral.
In a series of papers, Gel’fand, Graev, Kapranov and Zelevinsky \cite{56, 57, 58} (to mention only a few of their series of papers devoted to the systematic development of this approach) have developed a uniform approach to the description of hypergeometric functions\cite{56}. The formal solution of the $A$-system is a so-called multiple $\Gamma$-series having the following form:

$$
\sum_{(l_1, \cdots, l_N) \in \mathbb{L}} \frac{z_1^{l_1+\gamma_1} \cdots z_N^{l_N+\gamma_N}}{\Gamma(l_1 + y_1 + 1) \cdots \Gamma(l_N + y_N + 1)},
$$

where $\Gamma$ is the Euler $\Gamma$-function and the lattice $\mathbb{L}$ has rank $d$. When this formal series has a non-zero radius of convergence, it coincides (up to a factor) with a Horn-type hypergeometric series \cite{58} (see Section 2). Any Horn-type hypergeometric function can be written in the form of a $\Gamma$-series by applying the reflection formula

$$
\Gamma(a + n) = (-1)^n \frac{\Gamma(a) \Gamma(1-a)}{\Gamma(1-a-n)}.
$$

Many examples of such a conversion – all Horn-hypergeometric functions of two variables – have been considered in Ref. \cite{62}.

The Mellin-Barnes representation was beyond Gelfand’s consideration. It was worked out later by Fritz Beukers \cite{63}; see also the recent paper \cite{64}. Beukers analyzed the Mellin-Barnes integral

$$
\int \prod_{i=1}^N \Gamma(-\gamma_i - \tilde{b}_i \tilde{s}) v^{\gamma_i + \tilde{b}_i \tilde{s}} d\tilde{s},
$$

and pointed out that, under the assumption that the Mellin-Barnes integral converges absolutely, it satisfies the set of $A$-hypergeometric equations. The domain of convergence for the $A$-hypergeometric series and the associated Mellin-Barnes integrals have been discussed recently in Ref. \cite{65}.

Following Beuker’s results, we conclude that any Feynman diagram with a generic set of parameters (to guarantee convergence, we should treat the powers of propagators as non-integer parameters) could be treated as an $A$-function. However our analysis has shown that, typically, a real Feynman diagram corresponds to an $A$-function with reducible monodromy.

Let us explain our point of view. By studying Feynman diagrams having a one-fold Mellin-Barnes representation \cite{66}, we have found that certain Feynman diagrams ($E_{1220}^q, B_{1220}^q, V_{1220}^q, J_{1220}^q$ in the notation of Ref. \cite{66}) with powers of propagator equal to one (the so-called master-integrals) have the following hypergeometric structure (we drop the normalization constant for simplicity):

$$
\Phi(n, \tilde{a}; z) = 3 \Phi_2(a_1, a_2, \tilde{a}; b_1, b_2; z) + z'' \Phi_4(1, c_1, \tilde{c}_2, c_3; p_1, p_2, p_3; z), \quad (3)
$$

where the dimension $n$ of space-time \cite{67} is not an integer and the difference between any two parameters of the hypergeometric function also are not integers. The holonomic rank of the hypergeometric function $\Phi_{p-1}$ is equal to $p$, so that the

\footnote{The detailed discussion of $A$-functions and their properties is beyond our current consideration. There are many interesting papers on that subject, including (to mention only a few) Refs. \cite{59, 60, 61}.}
Feynman diagram is a linear combination of two series having different holonomic rank. What could we say about the holonomic rank of a Feynman diagram $\Phi$? To answer that question, let us find the differential equation for the Feynman diagram $\Phi(n, \vec{1}; z)$ starting from the representation Eq. (3). This could be done by the Honomic Function Approach [68] or with the help of a programs developed by Frederic Chyzak [69] (it is a MAPLE package) or by Christoph Koutschan [70] (it is Mathematica package). We used a private realization of this approach based on ideas from the Gröbner basis technique. Finally, we obtained the result that the Feynman diagram $\Phi$ satisfies the homogeneous differential equation of the hypergeometric type of order 4 with a left-factorizable differential operator of order 1:

$$\left(\theta + A\right) \left[\left(\theta + B_1\right)\left(\theta + B_2\right)\left(\theta + B_3\right) + z\theta\left(\theta + C_1\right)\left(\theta + C_2\right)\right] \Phi(n, \vec{1}; z) = 0, \quad (4)$$

where none of the $B_j$ and $C_\alpha$ are integers and $\theta = \frac{d}{dz}$.

It follows from this differential equation that the holonomic rank of the Feynman diagram $\Phi$ is equal to 4, and factorization means that the space of solutions splits into a direct sum of two spaces of dimension one and three: $\Phi_{\text{dim}} = 4 = 1 \otimes 3$. As it follows from [71], the monodromy representation of Eq. (4) is reducible and there is a one-dimensional invariant subspace. Consequently, there are three non-trivial solutions (master-integrals) and the one-dimensional invariant subspace corresponds to an integral having a Puiseux-type solution (expressible in terms of $\Gamma$-functions).

We pointed out in Ref. [66] that a Feynman diagram can be classified by the dimension of its irreducible representation. This can be evaluated by the construction of differential equations or by using the dimension of the irreducible representation of the hypergeometric functions entering in the r.h.s. of Eq. (2). Indeed, in the example considered in Eq. (3), the dimension of the irreducible representation $4F3(1, \vec{c}; \vec{\rho}; z)$ is equal to 3 [71], so that the dimension of the irreducible space of the Feynman diagram $\Phi$ is equal to 3, see Eq. (4), and $\Phi$ is expressible via the sum of a series (see Eq. (3)) with an irreducible representation of dimension 3.

The results of the analysis performed in Ref. [66], are summarized in the following proposition:

**Proposition:** A Feynman diagram can be treated as a linear combination of Horn-type hypergeometric series where each term has equal irreducible holonomic rank.

Examining this new “quantum number,” irreducible holonomic rank, we discover, and can rigorously prove, an extra relation between master-integrals [72]. In many other examples we found a complete agreement between the results of differential reduction and and the results of a reduction based on the IBP relations [73, 74].

The Feynman diagram $J$ considered in Ref. [72] satisfies the differential equation

$$\left(\theta - \frac{n}{2} + I_1\right) \left(\theta - n + I_2\right) \left[\theta \left(\theta - n + \frac{1}{2} + I_3\right) + z \left(\theta - \frac{3n}{2} + I_4\right)\right] J = 0, \quad (5)$$

\[^3\text{See Christoph’s paper in the present volume.}\]
where $I_1, I_2, I_3, I_4$ are integers, $n$ is the dimension of space-time and $\theta = \frac{d}{d\tau}$.

The dimension of $J$ is 4 and there are two one-dimensional invariant subspaces, corresponding to two first-order differential operators: $J_{\text{dim}} = 4 = 1 \otimes 1 \otimes 2$.

Indeed, after integrating twice, we obtained

$$\left[ \theta \left( \theta - n + \frac{1}{2} + I_1 \right) + z \left( \theta - \frac{3n}{2} + I_2 \right) \right] J = C_1 z^{n/2 - I_1} + C_2 z^{n - I_2}.$$ 

Surprisingly, this simple relation has not been not reproduced (as of the end of 2016) by any of the powerful programs for the reduction of Feynman diagrams (see the discussion in Chapter 6 of Ref. [75]). In [76], it was shown that the extra relation [72] could be deduced from a diagram of more general topology by exploring a new relation derived by taking of the derivative with respect to the mass with a subsequent reduction with the help of IBP relations. However, it was not shown that the derivative with respect to mass can be deduced from derivatives with respect to momenta, so that the result of Ref. [76] could be considered as an alternative proof that, in the massive case, there may exist an extra relation between diagrams that does not follow from classical IBP relations.

Finally, we have obtained a very simple result [77]: Eq. (4) follows directly from the Mellin-Barnes representation for a Feynman diagram. (See Section 2 and Eq. (12) for details.) Based on this observation and on the results of our analysis performed in Ref. [66], and extending the idea of the algorithm of Ref. [78], we have constructed a simple and fast algorithm for the algebraic reduction of any Feynman diagram having a one-fold Mellin-Barnes integral representation to a set of master-integrals without using the IBP relations. In particular, our approach and our program cover some types of Feynman diagrams with arbitrary powers of propagators considered in Refs. [79, 80, 81].

In a similar manner, one can consider the multiple Mellin-Barnes representation of a Feynman diagram [75, 82]. In contrast to the one variable case, the factorization of the partial differential operator is much more complicated. The dimension of the Pfaff system related to the multiple Mellin-Barnes integral can be evaluated with the help of a prolongation procedure (see the discussion in Section 2). However, in this case, there may exist a Puiseux type solution even for a generic set of parameters (see for example Section 3.1.2).

Exploring the idea presented in Ref. [71], one possibility is to construct an explicit solution of the invariant subspace (see [75] and Section 3.1.2) and find the dimension of the irreducible representation. Our results presented in [75] were confirmed by another technique in Ref. [85].

\[\text{\footnotesize{\textsuperscript{4}}}\text{ Rigorously speaking, this system of equations is correct when there is a contour in } \mathbb{C}^n \text{ that is not changed under translations by an arbitrary unit vector, see Ref. [84], so that we treat the powers of propagator as parameters.}\]

\[\text{\footnotesize{\textsuperscript{5}}}\text{ The monodromy group is the group of linear transformations of solutions of a system of hypergeometric differential equations under rotations around its singular locus. In the case when the monodromy is reducible, there is a finite-dimensional subspace of holomorphic solutions of the hypergeometric system on which the monodromy acts trivially.}\]
Let us illustrate the notion of irreducible holonomic rank (or an irreducible representation) in an application to Feynman diagrams. As follows from our analysis of sunset diagrams \cite{75}, the dimension of the irreducible representation of two-loop sunset with three different masses is equal to 4. There is only one hypergeometric function of three variables having holonomic rank 4, the $F_D$ function. Then we expect that there is a linear combination of four two-loop sunsets and the product of one-loop tadpoles that are expressible in terms of a linear combination of the $F_D$ functions.

Another approach to the construction of a GKZ representation of Feynman Diagrams was done recently in the series of papers in Refs. \cite{86,87,88}. Based on the observation made in Ref. \cite{89} about the direct relation between A-functions and Mellin transforms of rational functions, and exploring the Lee-Pomeransky representation \cite{90}, the authors studied a different aspect of the GKZ representation mainly considering the examples of massless or one-loop diagrams. Two non-trivial examples have been presented in Ref. \cite{87}: the two-loop sunset with two different masses and one zero mass, which corresponds to a linear combination of two Appell functions $F_3$ (see Eq. (3.11) in \cite{94}) and a two-loop propagator with three different masses related to the functions $F_C$ of three variables \cite{15}.

A different idea on how to apply the GKZ technique to the analysis of Feynman Diagrams has been presented in \cite{91} and has received further development in \cite{92,93}.

\subsection{1.3 One-Loop Feynman Diagrams}

Let us give special attention to one-loop Feynman diagrams. In this case, two elegant approaches have been developed \cite{95,96} that allow us to obtain compact hypergeometric representations for the master-integrals. The authors of the first paper \cite{95} explored the internal symmetries of the Feynman parametric representation to get a one-fold integral representation for one-loop Feynman diagrams (see also \cite{97,98}). The second approach \cite{96} is based on the solution of difference equations with respect to the dimension of space-time \cite{99} for the one-loop integrals. In spite of different ideas on the analysis of Feynman diagrams, both approaches, \cite{95} and \cite{96}, produce the same results for one-loop propagator and vertex diagrams \cite{100,101,102}. However, beyond these examples, the situation is less complete: it was shown in Ref. \cite{96} that the off-shell one-loop massive box is expressible in terms of a linear combination of $F_5$ Horn-type hypergeometric functions of three variables (see also discussions in Refs. \cite{103,104,105,106}), or in terms of $F_N$ Horn-type hypergeometric functions of three variables \cite{107} (see Section 3.1.1).

Recently, it was observed \cite{108,109,110} that massive conformal Feynman diagrams are invariant under a Yangian symmetry that allows to get the hypergeometric representation for the conformal Feynman diagrams.

\footnote{It is interesting to note, that on-mass shell $z = 1$, this diagram has two Puiseux type solutions that do not have analytical continuations.}
1.4 Construction of $\varepsilon$-expansion

For physical applications, the construction of the analytical coefficients of the Laurent expansions of hypergeometric functions around particular values of parameters (integer, half-integer, rational) is necessary. Since the analytic continuations of hypergeometric functions is still an unsolved problem, the results are written in some region of variables in each order of the $\varepsilon$-expansion in terms of special functions like classical or multiple polylogarithms, and then these functions are analytically continued to another region. For this reason, the analytical properties of special functions were analyzed in detail. Also, tools for the numerical evaluation of the corresponding functions are important.

Each of the hypergeometric function representations (series, integral, Mellin-Barnes, differential equation) can be used for the construction of the $\varepsilon$-expansion, and each of them has some technical advantages or disadvantages in comparison with the other ones. The pioneering $\varepsilon$-expansion of the hypergeometric function around $\varepsilon = \pm 1$ was done by David Broadhurst. The expansion was based on the analysis of multiple series and it was interesting from a mathematical point of view as well as for its application to quantum field theory.

The integral representation was mainly developed by Andrei Davdychev and Bas Tausk, so that, finally, the all-order $\varepsilon$-expansion for the Gauss hypergeometric functions around a rational parameter, a case that covers an important class of diagrams, has been constructed in terms of generalized log-sine functions or in term of Nielsen polylogarithms.

The integral representation was also the starting point for the construction of the $\varepsilon$-expansion of hypergeometric functions, and also the $F_1$ and $F_2$ functions around integer values of parameters. Purely numerical approaches can be applied for arbitrary values of the parameters. However, this technique typically does not produce a stable numerical result in regions around singularities of the hypergeometric functions.

A universal technique which does not depend on the order of the differential equation is based on the algebra of multiple sums. For the hypergeometric function for which the nested-sum algorithms are applicable, the results of the $\varepsilon$-expansion are automatically obtained in terms of multiple polylogarithms.

The nested-sum algorithms have been implemented in a few packages and allow for the construction of the $\varepsilon$-expansion of hypergeometric functions $F_{p-1}$ and Appell functions $F_1$ and $F_2$ around integer values of parameters. However, the nested-sum approach fails for the $\varepsilon$-expansion of hypergeometric functions around rational values of parameters and it is not applicable.

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7 It was shown in that multiple Mellin-Barnes integrals related to Feynman diagrams could be evaluated analytically/numerically at each order in $\varepsilon$ via multiple sums, without requiring a closed expression in terms of Horn-type hypergeometric functions.

8 See also Refs. for an alternative realization.
to some specific classes of hypergeometric functions (for example, the $F_4$ function, see [166]).

In the series of papers [167][168], the generating function technique [169][170] has been developed for the analytical evaluation of multiple sums. Indeed, the series generated by the $\varepsilon$-expansion of hypergeometric functions has the form $\sum_k c(k)\varepsilon^k$, where the coefficients $c(k)$ include only products of the harmonic sums $\prod_{a,b} S_a(k-1)S_b(2k-1)$ and $S_a(k) = \sum_{j=1}^k \frac{1}{j^a}$. The harmonic sums satisfy the recurrence relations

$$S_a(k) = S_a(k-1) + \frac{1}{k^a}, \quad S_a(2k+1) = S_a(2k-1) + \frac{1}{(2k+1)^a} + \frac{1}{(2k)^a},$$

so that the coefficients $c(k)$ satisfy the first order difference equation

$$P(k+1)c(k+1) = Q(k)c(k) + R(k),$$

where $P$ and $Q$ are polynomial functions that can be defined from the original series. This equation could be converted into a first order differential equation for the generating function $F(z) = \sum_k c(k)z^k$.

$$\frac{1}{z}\left(\frac{d}{dz} \right) F(z) - P(1)C(1)z = Q \left(\frac{d}{dz}\right) F(z) + \sum_{k=1} R(k)z^k.$$

One of the remarkable properties of this technique, that the non-homogeneous part of the differential equation, the function $R(k)$, has one-unit less depth in contrast to the original sums, so that, step-by-step, all sums could be evaluated analytically. Based on this technique, all series arising from the $\varepsilon$-expansion of hypergeometric functions around half-integer values of parameters have been evaluated [167] up to weight 4. The limits considered were mainly motivated by physical reasons (at $O$(NNLO) only functions of weight 4 are generated) and, in this limit, only one new function $H_{1,0,1}(z)$ was necessary to introduce. These results [167] allow us to construct the $\varepsilon$-expansion of the hypergeometric functions $pF_{p-1}$ around half-integer values of parameters, see [171][172].

Other results and theorems relevant for the evaluation of Feynman diagrams are related with the appearance of a factor $1/\sqrt{3}$ in the $\varepsilon$-expansion of some diagrams [173] expressible in terms of hypergeometric functions [174] were derived in Refs. [147][174][175][176].

Let us consider typical problems arising in this program. We follow our analysis presented in Ref. [180], see also the closely related discussion in Ref. [181]. First, the construction of the difference equation for the coefficient functions $c(z)$ is not an easy task [182][183][184]. In the second step, the differential operator(s) coming

9 In general, it could be a more generic recurrence, $\sum_{k=0}^{k} p_{k+j} (k+j) c(k+j) = r(k)$.
10 Recent results on the analytical evaluation of inverse binomial sums for particular values of the arguments have been presented in [175][177][178].
11 The appearance of $1/\sqrt{3}$ in RG functions in seven loops was quite intriguing [179][183].
from the difference equation \( P \left(\frac{d}{dz}\right) - zQ \left(\frac{d}{dz}\right) \) should be factorized into a product of differential operators of the first order,

\[
P \left(\frac{d}{dz}\right) - zQ \left(\frac{d}{dz}\right) = \prod_{k=1}^{r} \left( p_k(z) \frac{d}{dz} - q_k(z) \right),
\]

where \( p_k(z) \) and \( q_k(z) \) are rational functions. Unfortunately, the factorization of differential operators into irreducible factors is not unique [185].

\[
\left( \frac{d^2}{dx^2} - \frac{2}{x} \frac{d}{dx} + \frac{2}{x^2} \right) = \left( \frac{d}{dx} - \frac{1}{x} \right) \left( \frac{d}{dx} - \frac{1}{x} \right) = \left( \frac{d}{dx} - \frac{1}{x(1 + ax)} \right) \left( \frac{d}{dx} - \frac{1 + 2ax}{1 + ax} \right),
\]

where \( a \) is a constant.

However, the following theorem is valid (see [186]): Any two decompositions of a linear differential operator \( L^{(p)} \) into a product (composition) of irreducible linear differential operators

\[
L^{(p)} = L_1^{(a_1)} L_2^{(a_2)} \cdots L_m^{(a_m)} = P_1^{(r_1)} P_2^{(r_2)} \cdots P_k^{(r_k)}
\]

have equal numbers of components \( m = k \) and the factors \( L_j \) and \( P_a \) have the same order of differential operators: \( L_a = P_j \) (up to commutation). In the application to the \( e \)-expansion of hypergeometric functions this problem has been discussed in Ref. [187].

After factorization, the iterated integral over rational functions (which is not uniquely defined, as seen in the previous example) would be generated that in general is not expressible in terms of hyperlogarithms. Indeed, the solution of the differential equation

\[
\left[ R_1(z) \frac{d}{dz} + Q_1(z) \right] \left[ R_2(z) \frac{d}{dz} + Q_2(z) \right] h(z) = F(z),
\]

has the form

\[
h(z) = \int \frac{dt_3}{R_2(t_3)} \left[ \exp - \int_{t_0}^{t_3} \frac{Q_2(t_3)}{R_2(t_3)} dt_4 \right] \int \frac{dt_1}{R_1(t_1)} \left[ \exp - \int_{t_0}^{t_1} \frac{Q_1(t_1)}{R_1(t_1)} dt_2 \right] F(t_1).
\]

From this solution it follows [188] that the following conditions are enough to convert the iterated integral into hyperlogarithms: there are new variables \( \xi \) and \( x \) so that

\[
\int \frac{Q_1(t)}{R_1(t)} dt = \ln \frac{M_1(\xi)}{N_1(\xi)} \Rightarrow \frac{dt}{R_1(t)} \bigg|_{\xi(t)} = \frac{N_1(\xi)}{M_1(\xi)} = dx \frac{K_1(x)}{L_1(x)},
\]

where \( M_1, N_1, K_1, L_1 \) are polynomial functions.

The last problem is related to the Abel-Ruffini theorem: the polynomial is factorizable into a product of its primitive roots, but there are not solutions in radicals for polynomial equations of degree five or more. The last problem got a very elegant solution by the introduction of cyclotomic polylogarithms [159], with the integration over irreducible cyclotomic polynomials \( \Phi_n(x) \). The first two irreducible polynomi-
als (see Eqs. (3.3) – (3.14) in [159]) are $\Phi_7$ and $\Phi_9$ (the polynomial of order 6). Two other polynomials of order 4, $\Phi_5$ and $\Phi_{10}$: $(x^4 \pm x^3 + x^2 \pm x + 1)$, have non-trivial primitive roots. But up to now, all these polynomials were not generated by Feynman diagrams. Surprisingly, by increasing the number of loops or number of scales, other mathematical structures are generated. [189, 190]. Detailed analyses of properties of the new functions have been presented in Refs. [191, 192] and automated by Jakob Ablinger [193, 194, 195]. The problem of integration over algebraic functions (typically square roots of polynomials) was solved by the introduction of a new type of functions, [196], intermediate between multiple and elliptic polylogarithms.

The series expansion is not very efficient for the construction of the $\epsilon$-expansion, since the number of series increases with the order of the $\epsilon$-expansion and increases the complexity of the individual sums. Let us recall that the Laurent expansion of a hypergeometric function contains a linear combination of multiple sums. From this point of view, the construction of the analytical coefficients of the $\epsilon$-expansion of a hypergeometric function can be carried out independently of existing analytical results for each individual multiple sum. The “internal” symmetry of a Horn-type hypergeometric function is uniquely defined by the corresponding system of differential equations. While exploring this idea, a new algorithm was presented in Refs. [197, 198], based on factorization, looking for a linear parametrization and direct iterative solution of the differential equation for a hypergeometric function. This approach allows the construction of the analytical coefficients of the $\epsilon$-expansion of a hypergeometric function, as well as obtaining analytical expressions for a large class of multiple series without referring to the algebra of nested sums.

Based on this approach, the all-order $\epsilon$-expansion of the Gauss hypergeometric function around half-integer and rational values of parameters has been constructed [197, 199], so that the first 20 coefficients around half-integer values of parameters, the 12 coefficients for $q = 4$ and 10 coefficients for $q = 6$ have been generated already in 2012 [2]. Another record is the generation of 24 coefficients for the Clausen hypergeometric function $3F_2$ around integer values of parameters, relevant for the analysis performed in [201]. To our knowledge, at the present moment, this remains the fastest and most universal algorithm.

Moreover, it was shown in Refs. [197, 198], that when the coefficients of the $\epsilon$-expansion of a hypergeometric function are expressible in terms of multiple polylogarithms, there is a set of parameters (not uniquely defined) such that, at each order of $\epsilon$, the coefficients of the $\epsilon$-expansion include multiple polylogarithms of a single uniform weight. A few years later, this property was established not only for hypergeometric functions, but for Feynman Diagrams [202].

A multivariable generalization [180] of the algorithm of Refs. [197, 198] has been described. The main difference with respect to the case of one variable is the construction of a system of differential equations of triangular form to avoid the appearance of elliptic functions. As a demonstration of the validity of the algorithm, the first few coefficients of the $\epsilon$-expansion of the Appell hypergeometric

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12 The results have been written in terms of hyperlogarithms of primitive $q$-roots of unity.

13 The results of [200] were relevant for the reduction of multiple zeta values to the minimal basis.
functions $F_1, F_2, F_3$ and $F_D$ around integer values of parameters have been evaluated analytically \[103\].

The $\varepsilon$-expansion of the hypergeometric functions $F_3$ and $F_D$ are not covered by the nested sums technique or its generalization. The differential equation technique can be applied to the construction of analytic coefficients of the $\varepsilon$-expansions of hypergeometric functions of several variables (which is equivalent to the multiple series of several variables) around any rational values of parameters via direct solution of the linear systems of differential equations.

The differential equation approach \[197, 198\] allows us to analyze arbitrary sets of parameters simultaneously and to construct the solution in terms of iterated integrals, but for any hypergeometric function the Pfaff system of differential equations should be constructed. That was the motivation for creation of the package(s) (the HYPERDIRE project) \[82\] for the manipulation of the parameters of Horn-type hypergeometric functions of several variables. For illustration, we describe in detail how it works in the application to the $F_3$ hypergeometric function in Section 3.2.

Recently, a new technique \[203\] for the construction of the $\varepsilon$-expansion of Feynman diagrams \[204\] as well as for hypergeometric functions has been presented \[205\]. It is based on the construction of a coaction \[14\] of certain hypergeometric functions. The structures of the $\varepsilon$-expansion of the Appell hypergeometric functions $F_1, F_2, F_3$ and $F_4$ as well as $F_D$ (for the last function $F_D$ see also the discussion in Ref. \[207\]) around integer values of parameters are in agreement with our analysis and partial results presented in Refs. \[180\] and \[103\]. However, the structure of the $\varepsilon$-expansion around rational values of parameters has not been discussed in \[205\], nor in \[207\].

2 Horn-type hypergeometric functions

2.1 Definition and system of differential equations

The study of solutions of linear partial differential equations (PDEs) of several variables in terms of multiple series, i.e. a multi-variable generalization of the Gauss hypergeometric function \[208\], began long ago \[209\].

Following the Horn definition \[22\], a multiple series is called a “Horn-type hypergeometric function,” if, about the point $\vec{z} = 0$, there is a series representation

$$H(\vec{z}) = \sum_{\vec{m}} C(\vec{m}) \vec{z}^{\vec{m}},$$

where $\vec{z}^{\vec{m}} = z_1^{m_1} \cdots z_r^{m_r}$ for any integer multi-index $\vec{m} = (m_1, \cdots, m_r)$, and the ratio of two coefficients can be represented as a ratio of two polynomials:

\[14\] An interesting construction of the coaction for the Feynman graph has been presented recently in Ref. \[206\].
The differential operator \( /u1D715 \) with polynomial coefficients is applied into Pfaff form (for simplicity, we assume that system is closed):

\[ C(\vec{m} + \vec{e}_j) = P_j(\vec{m}) \frac{Q_j(\vec{m})}{Q_j(\vec{m})} , \]

where \( \vec{e}_j \) denotes the unit vector with unity in its \( j \)th entry, \( \vec{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0) \).

The coefficients \( C(\vec{m}) \) of such a series can be expressed as products or ratios of Gamma-functions (up to some factors irrelevant for our consideration) \([210, 211]\):

\[ C(\vec{m}) = \frac{\prod_{j=1}^{p} \Gamma \left( \sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j \right)}{\prod_{k=1}^{q} \Gamma \left( \sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k \right)} , \]

where \( \mu_{ja}, \nu_{kb}, \sigma_j, \gamma_j \in \mathbb{Z} \) and \( m_a \) are elements of \( \vec{m} \).

The Horn-type hypergeometric function, Eq. (7), satisfies the following system of differential equations:

\[ 0 = L_j(\vec{z}) H(\vec{z}) = \left[ Q_j \left( \sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \frac{1}{z_j} - P_j \left( \sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \right] H(\vec{z}) , \]

where \( j = 1, \ldots, r \). Indeed,

\[ Q_j \left( \sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \frac{1}{z_j} \sum_{\vec{m}} C(\vec{m}) \vec{z} \vec{m} = \sum_{\vec{m}} Q_j(\vec{m}) C(\vec{m} + \vec{e}_j) \vec{z} \vec{m} = \sum_{\vec{m}} P_j(\vec{m}) C(\vec{m}) \vec{z} \vec{m} = P_j \left( \sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \sum_{\vec{m}} C(\vec{m}) \vec{z} \vec{m} . \]

The degrees of the polynomials \( P_i \) and \( Q_i \) are \( p_i \) and \( q_i \), respectively. The largest of these, \( r = \max \{ p_i, q_i \} \), is called the order of the hypergeometric series. To close the system of differential equations, the prolongation procedure should be applied: by applying the differential operator \( \partial_i \) to \( L_j \) we can convert the system of linear PDEs with polynomial coefficients into Pfaff form (for simplicity, we assume that system is closed):

\[ L_j H(\vec{z}) = 0 \implies \left\{ d\omega_j(\vec{z}) = \Omega^k_{ij}(\vec{z}) \omega_j(\vec{z}) d\omega_k(\vec{z}) , \quad d [ d\omega_j(\vec{z}) ] = 0 \right\} . \]

Instead of a series representation, one can use a Mellin-Barnes integral representation (see the discussion in [77]). Indeed, the multiple Mellin-Barnes representation for a Feynman diagram could be written in the form in Eq. (11). Let us define the polynomials \( P_i \) and \( Q_i \) as

\[ \frac{P_i(\vec{t})}{Q_i(\vec{t})} = \frac{\phi(\vec{t} + \vec{e}_i)}{\phi(\vec{t})} . \]
The integral then satisfies the system of linear differential equations

\[ Q_i(\vec{t}) \big|_{t_j \to \theta_j} \frac{1}{\zeta_i} \Phi(A, \vec{B}; C, \vec{D}; \vec{z}) = P_i(\vec{t}) \big|_{t_j \to \theta_j} \Phi(A, \vec{B}; C, \vec{D}; \vec{z}), \tag{12} \]

where \( \theta_i = z_i \frac{\partial}{\partial z_i} \). Systems of equations such as Eq. (12) are left ideals in the Weyl algebra of linear differential operators with polynomial coefficients.

### 2.2 Contiguous relations

Any Horn-type hypergeometric function is a function of two types of variables, continuous variables, \( z_1, z_2, \ldots, z_r \) and discrete variables: \( \{ J_a \} := \{ \gamma_k, \sigma_l \} \), where the latter can change by integer numbers and are often referred to as the parameters of the hypergeometric function.

For any Horn-hypergeometric function, there are linear differential operators changing the value of the discrete variables by one unit. Indeed, let us consider a multiple series defined by Eq. (6).

Two hypergeometric functions \( H \) with sets of parameters shifted by unity, \( H(\vec{\gamma} + \vec{\epsilon}; \vec{\sigma}; \vec{z}) \) and \( H(\vec{\gamma}; \vec{\sigma}; \vec{z}) \), are related by a linear differential operator:

\[ H(\vec{\gamma} + \vec{\epsilon}; \vec{\sigma}; \vec{z}) = \left( \sum_{a=1}^{r} \mu_{ca} z_a \frac{\partial}{\partial z_a} + \gamma_c \right) H(\vec{\gamma}; \vec{\sigma}; \vec{z}). \tag{13} \]

Similar relations also exist for the lower parameters:

\[ H(\vec{\gamma}; \vec{\sigma} - \vec{\epsilon}; \vec{z}) = \left( \sum_{b=1}^{r} \nu_{cb} z_b \frac{\partial}{\partial z_b} + \sigma_c - 1 \right) H(\vec{\gamma}; \vec{\sigma}; \vec{z}). \tag{14} \]

Let us rewrite these relations in a symbolic form:

\[ R_K(\vec{z}) \frac{\partial}{\partial z_K} H(\vec{J}; \vec{z}) = H(\vec{J} \pm e_K; \vec{z}), \tag{15} \]

where \( R_K(\vec{z}) \) are polynomial (rational) functions.

In Refs. [78][59] it was shown that there is an algorithmic construction of inverse linear differential operators:

\[ B_{L,N}(\vec{z}) \frac{\partial^L}{\partial z_N} \left( R_K(\vec{z}) \frac{\partial}{\partial z_K} \right) H(\vec{J}; \vec{z}) \equiv B_{L,N}(\vec{z}) \frac{\partial^L}{\partial z_N} H(\vec{J} \pm e_K; \vec{z}) = H(\vec{J}; \vec{z}) \tag{16} \]

Applying the direct or inverse differential operators to the hypergeometric function the values of the parameters can be changed by an arbitrary integer:
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\[ S(\vec{z}) H(\vec{J} + \vec{m}; \vec{z}) = \sum_{j=0}^{r} S_j(\vec{z}) \frac{\partial^j}{\partial \vec{z}^j} H(\vec{J}; \vec{z}) , \quad (17) \]

where \( \vec{m} \) is a set of integers, \( S \) and \( S_j \) are polynomials and \( r \) is the holonomic rank (the number of linearly independent solutions) of the system of differential equations Eq. (9). At the end of the reduction, the differential operators acting on the function \( H \) can be replaced by a linear combination of the function evaluated with shifted parameters.

We note that special considerations are necessary when the system of differential operators, Eq. (9), has a Puiseux-type solution (see Section 3.1.2). In this case, the prolongation procedure gives rise to the Pfaffian form, but this set of differential equations is not enough to construct the inverse operators [212], so that new differential equations should be introduced. In the application to the Feynman diagrams, this problem is closely related with obtaining new relations between master integrals, see Ref. [72] for details. In the Section 3.1.2 we present an example of the Horn-type hypergeometric equation of second order of three variables having a Puiseux-type solution.

Another approach to the reduction of hypergeometric functions is based on the explicit algebraic solution of the contiguous relations, see the discussion in Ref. [213]. This technique is applicable in many particular cases, including \( 2F_1, 3F_2 \), and the Appell functions \( F_1, F_2, F_3, F_4 \) (see the references in Ref. [214]), and there is a general expectation that it could be solved for any Horn-type hypergeometric function. However, to our knowledge, nobody has analyzed the algebraic reduction in the application to general hypergeometric functions having a Puiseux-type solution.

The multiple Mellin-Barnes integral \( \Phi \) defined by Eq. (1) satisfies similar differential contiguous relations:

\[ \Phi(A, \vec{B} + e_a; \vec{C}, \vec{D}; \vec{z}) = \left( \sum_{i=1}^{m} A_{ii} \theta_i + B_a \right) \Phi(A, \vec{B}; \vec{C}, \vec{D}; \vec{z}) , \]

\[ \Phi(A, \vec{B}; \vec{C}, \vec{D} - e_b; \vec{z}) = \left( \sum_{j=1}^{r} C_{bj} \theta_j + D_b \right) \Phi(A, \vec{B}; \vec{C}, \vec{D}; \vec{z}) , \quad (18) \]

so that the original diagram may be explicitly reduced to a set of basis functions without examining the IBP relations [73, 74]. A non-trivial example of this type of reduction beyond IBP relations has been presented in Ref. [72] (see also the discussion in Chapter 6 of Ref. [75]).
3 Examples

3.1 Holonomic rank & Puiseux-type solution

In addition to the examples presented previously in our series of publications, we present here a few new examples.

3.1.1 Evaluation of holonomic rank: the hypergeometric function $F_N$

The Lauricella-Saran hypergeometric function of three variables $F_N$ is defined about the point $z_1 = z_2 = z_3 = 0$ by

$$F_N(a_1, a_2, a_3; b_1, b_2; c_1, c_2; z_1, z_2, z_3) = \sum_{m_1, m_2, m_3 = 0}^{\infty} \left[ \Pi_{j=1}^{3} (a_j)_m \frac{z_j}{m_j!} \frac{(b_1)_m (b_2)_m}{(c_1)_m (c_2)_m (c_3)_m} \right]. \tag{19}$$

This function is related to one-loop box diagrams in an arbitrary dimension considered by Andrei Davydychev [107].

Following the general algorithm [103], the following result is easily derived:

**Theorem 1.** For generic values of the parameters, the holonomic rank of the function $F_N$ is equal 8.

In this way, for generic values of parameters, the result of differential reduction, Eq. (17), have the following form:

$$S(z) F_N(\vec{J} + \vec{m}; \vec{z}) = \left[ S_0 + S_i \sum_{j=1}^{3} \theta_j + \sum_{i,j=1}^{3} S_{i,j} \theta_i \theta_j + S_{123} \theta_1 \theta_2 \theta_3 \right] F_N(\vec{J}; \vec{z}),$$

where $\theta_i = \frac{d}{dz_i}$ and inverse differential operators can be easily constructed (Note that "easy" does not mean that these operators have a simple form, see Ref. [215]).

**Theorem 2.** The system of differential equations defined by the series (19) is reducible when the one of the following combinations of parameters is an integer:

$$\{a_1, a_2, a_3, b_1, b_2, \} \in \mathbb{Z},$$

$$\{a_2 - c_1 - c_2 + b_1, \ a_2 - c_2 + b_1, \ b_2 - c_1 - c_2 + b_1, \ b_2 - c_2 + b_1 \} \in \mathbb{Z} \quad (20)$$

When one or more conditions of the Theorem 2 are valid, the number of independent differential equations describing the function $F_N$ reduces and some additional analysis is necessary (see for example Ref. [75]) to evaluate the value of irreducible holonomic rank.
3.1.2 Puiseux-type solution: hypergeometric function $F_T$

The Lauricella-Saran hypergeometric function of three variables $F_T$ is defined about the point $z_1 = z_2 = z_3 = 0$ by

$$F_T(a_1; a_2; b_1, b_2; c; z_1, z_2, z_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{m_1} (a_2)_{m_2+m_3} (b_1)_{m_1+m_1} (b_2)_{m_2}}{(c)_{m_1+m_2+m_3}} z_1^{m_1} z_2^{m_2} z_3^{m_3}. \tag{22}$$

In this case, the differential operators, Eq. (9), are the following:

$$L_1 F_T : \quad \theta_1 \left(c-1+\sum_{j=1}^{3} \theta_j \right) F_T = z_1 \left( a_1+\theta_1 \right) (b_1+\theta_1+\theta_3) F_T, \tag{22a}$$

$$L_2 F_T : \quad \theta_2 \left(c-1+\sum_{j=1}^{3} \theta_j \right) F_T = z_2 \left( a_2+\theta_2+\theta_3 \right) (b_2+\theta_2) F_T, \tag{22b}$$

$$L_3 F_T : \quad \theta_3 \left(c-1+\sum_{j=1}^{3} \theta_j \right) F_T = z_3 \left( a_3+\theta_3+\theta_3 \right) (b_1+\theta_1+\theta_3) F_T, \tag{22c}$$

where $F_T = F_T(a_1; a_2; b_1, b_2; c; z_1, z_2, z_3)$.

Let us introduce the function

$$\Phi_T = \frac{z_1^{1-c+a_2} z_2^{1-c+b_1}}{z_3^{1-c+b_1+a_2}}. \tag{23}$$

It is easy to check that

$$L_1 \Phi_T = L_2 \Phi_T = L_3 \Phi_T = 0.$$

**Theorem 3.** The system of differential equations defined by Eq. (22) has a Puiseux-type solution:

$$\Phi_T = \frac{z_1^{1-c+a_1} z_2^{1-c+b_1}}{z_3^{1-c+b_1+a_2}}. \tag{24}$$

In particular, to construct the inverse contiguous relations for the function $F_T$, one extra differential equation should be added.

For completeness, we also note the following result:

**Theorem 4.** The monodromy group of the system of differential equations defined by Eq. (22) is reducible when the one of the following combinations of parameters is an integer:

$$\{a_1, a_2, b_1, b_2, c-b_1-b_2, c-a_1-a_2 \in \mathbb{Z} \}.$$
A Puiseux-type solution for the hypergeometric differential equation of two variables was established by Erdelyi [83] still in the 50’s and has been analyzed in detail in [216] in the framework of the GKZ approach.

3.2 Construction of the $\varepsilon$-expansion via differential equations: the Appell Function $F_3$:

To explain the technical details of our algorithm, let us analyze and construct the $\varepsilon$-expansion for the Appell hypergeometric function $F_3$. The preliminary results have been presented in [187] and [180].

3.2.1 Notations

Let us consider the Appell hypergeometric function $F_3$ defined about $x = y = 0$ as

$$\omega_0 \equiv F_3(a_1, a_2, b_1, b_2, c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n x^m y^n}{(c)_{m+n} m! n!}, \quad (25)$$

It is symmetric with respect to simultaneous exchange

$$a_1 \leftrightarrow a_2, \quad b_1 \leftrightarrow b_2, \quad x \leftrightarrow y. \quad (26)$$

so that $F_3(a_1, a_2, b_1, b_2, c; x, y) = F_3(a_2, a_1, b_1, b_2, c; y, x)$.

This function satisfies four differential equations (see Section 3.4 in [82]):

$$(1-x)\theta_{x,x} \omega_0 = -\theta_{x,y} \omega_0 + [(a_1+b_1)x-(c-1)] \theta_y \omega_0 + xa_1 b_1 \omega_0, \quad (27a)$$

$$(xy-x-y)\theta_{x,x,y} \omega_0 = [(1-y)(a_1+b_1)x-y(a_2+b_2+1-c)] \theta_{x,y} \omega_0$$

$$+(1-y)xa_1 b_1 \theta_y \omega_0 - ya_2 b_2 \theta_x \omega_0, \quad (27b)$$

and two other equations follow from Eqs. (27) and the symmetry relation (26). Eqs. (27) can be written in projective space with homogeneous coordinates on $\mathbb{P}^2(C)$ with $x = X/Z$ and $y = Y/Z$. We want also to remark that the system of differential equations for the Appell hypergeometric function $F_3(u, v)$ coincides with the system Eq. (27) by a redefinition $u = 1/x, v = 1/y$. This result follows also from analytic continuation of the Mellin-Barnes representation for Appell’s functions $F_2$ and $F_3$.

3.2.2 One-fold iterated solution

Assuming the most general form of the parameters
we obtain the full system of differential equations (under the conditions
\( p, r_i, p, q \) are integers, and writing the \( \varepsilon \)-expansion for the functions as

\[
\begin{align*}
\omega_j &= \sum_{k=0}^{\infty} \omega_j^{(k)} \varepsilon^k, \quad j = 0, 1, 2, 3, \\
\end{align*}
\]

we obtain the full system of differential equations (under the conditions \( p_j r_j = 0 \), \( j = 1, 2 \):

\[
\begin{align*}
&\left[ (1-x) \frac{d}{dx} - \frac{s_1}{q} + \frac{1}{x} \frac{p}{q} \right] \omega_1^{(r)} = -\frac{1}{x} \omega_3^{(r)} + \left[ (a_1 + b_1) - \frac{c}{x} \right] \omega_1^{(r-1)} + a_1 b_1 \omega_0^{(r-2)}, \\
&+ \left[ \frac{a_1 r_1 + b_1 p_1}{q} \right] \omega_0^{(r-1)} + a_1 b_1 \omega_0^{(r-2)}, \\
&\left[ (1-y) \frac{d}{dy} - \frac{s_2}{q} + \frac{1}{y} \frac{p}{q} \right] \omega_2^{(r)} = -\frac{1}{y} \omega_3^{(r)} + \left[ (a_2 + b_2) - \frac{c}{y} \right] \omega_2^{(r-1)} + a_2 b_2 \omega_0^{(r-2)}, \\
&+ \left[ \frac{a_2 r_2 + b_2 p_2}{q} \right] \omega_0^{(r-1)} + a_2 b_2 \omega_0^{(r-2)}, \\
&\left[ (xy-x-y) \frac{d}{dx} - (1-x) \frac{s_1}{q} + \frac{x}{y} \frac{s_2 + p}{q} \right] \omega_3^{(r)} = \left[ (1-y)(a_1 + b_1) - \frac{y}{x} (a_2 + b_2 - c) \right] \omega_3^{(r-1)} + \left[ a_1 r_1 + b_1 p_1 \right] \omega_2^{(r-1)} + a_1 b_1 \omega_1^{(r-2)} + \left[ \frac{a_2 r_2 + b_2 p_2}{q} \right] \omega_1^{(r-1)} + a_2 b_2 \omega_1^{(r-2)}, \\
&\left[ (xy-x-y) \frac{d}{dy} - (1-y) \frac{s_2}{q} + \frac{x}{y} \frac{s_1 + p}{q} \right] \omega_3^{(r)} = \left[ (1-x)(a_2 + b_2) - \frac{x}{y} (a_1 + b_1 - c) \right] \omega_3^{(r-1)} + \left[ a_2 r_2 + b_2 p_2 \right] \omega_1^{(r-1)} + a_2 b_2 \omega_1^{(r-2)} + \left[ \frac{a_1 r_1 + b_1 p_1}{q} \right] \omega_2^{(r-1)} + a_1 b_1 \omega_2^{(r-2)}. \\
\end{align*}
\]
where we have introduced new notations:

\[ s_1 = p_1 + r_1, \quad s_2 = p_2 + r_2. \]  

(30)

We want to mention that this system of differential equations does not have the $\varepsilon$-form *ala* Henn's form in [202, 217]. However, the system of equations Eq. (29) can be straightforwardly solved iteratively. Let us redefine functions $\omega_1, \omega_2, \omega_3$, as follows:

\[
\omega_1^{(r)} = \theta_x \omega_0 = h_1(x) \phi_1^{(r)}, \quad \omega_2^{(r)} = \theta_y \omega_0 = h_2(y) \phi_2^{(r)},
\]

(31)

where $h_{1,2}(x)$ are new functions, defined as

\[
h_1(x) = \sigma_1 \left[ \frac{x^p}{(x-1)^{\nu_1+p}} \right]^{\frac{1}{\nu_1}}, \quad h_2(y) = \sigma_2 \left[ \frac{y^p}{(y-1)^{\nu_2+p}} \right]^{\frac{1}{\nu_2}},
\]

(32)

and

\[
\omega_3^{(r)} = \theta_x \theta_y \omega_0^{(r)} = H(x, y) \phi_3^{(r)},
\]

(33)

with

\[
H(x, y) = \sigma_3 \left[ \frac{x^{\nu_2+p} y^{\nu_1+p}}{(xy-x-y)^{\nu_1+\nu_2+p}} \right]^{\frac{1}{\nu_1+\nu_2+p}},
\]

(34)

where $\sigma_j, j = 1, 2, 3$ are some normalization constants.

Substituting it to the original system, Eq. (29), we have (for completeness we show all four equations explicitly):

\[
(1-x) \frac{d}{dx} \phi_1^{(r)} = - \frac{1}{x} \frac{H(x, y)}{h_1(x)} \phi_3^{(r)} \]

\[
+ \left[ (a_1+b_1) - \frac{c}{x} \right] \phi_1^{(r-1)} + \left[ \frac{a_1 r_1 + b_1 p_1}{q} \right] \frac{1}{h_1(x)} \frac{\omega_0^{(r-1)}}{h_1(x)} + \frac{1}{h_1(x)} a_1 b_1 \omega_0^{(r-2)},
\]

(35a)

\[
(1-y) \frac{d}{dy} \phi_2^{(r)} = - \frac{1}{y} \frac{H(x, y)}{h_2(y)} \phi_3^{(r)} \]

\[
+ \left[ (a_2+b_2) - \frac{c}{y} \right] \phi_2^{(r-1)} + \left[ \frac{a_2 r_2 + b_2 p_2}{q} \right] \frac{1}{h_2(y)} \frac{\omega_0^{(r-1)}}{h_2(y)} + \frac{1}{h_2(y)} a_2 b_2 \omega_0^{(r-2)},
\]

(35b)
Keeping in mind that

Let us recall that the surface of singularities

completeness, it should be supplemented by the boundary condition. Boundary

can be understood as a smooth map

This is a system of linear differential equations with algebraic coefficients. For
completeness, it should be supplemented by the boundary condition. Boundary
conditions are discussed below.

Remark:

Let us recall that the surface of singularities \( L(x, y) \) of the \( F_3 \) hypergeometric
function is defined by the system of equations Eq. (27) and has the form

\[
L : L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \equiv x \cup y \cup (1 - x) \cup (1 - y) \cup (x + y - xy) .
\]  

(37)

The extra functions, Eqs. (32) and (33), can be understood as the ratio of elements \( L_a \)
of Eq. (37), and \( q \)-root is related with angle of rotations of curves \( L_j \) around zero.

Indeed, a multiple polylogarithm can be understood as a smooth map \( U \) from one
region of singularities, where the solution of differential equation exists, to another
one: \( \text{Li}_A(L_i) \xrightarrow{U} \text{Li}_A(L_j) \). Such map is nothing but an analytic continuation that
mixed the singularities of the differential system Eq. (27).

3.2.3 Boundary conditions

The boundary conditions for the system of equations, Eqs. (35), (36), are defined by
the series representation, so that

\[
\omega_0(z_1, z_2)|_{z_1=0} = 2F_1(a_2, b_2; c; z_2) , \quad \omega_0(z_1, z_2)|_{z_2=0} = 2F_1(a_1, b_1; c; z_1) .
\]

(38)

Keeping in mind that
\[
\omega_j(z_1, z_2) = z_j \frac{a_j b_j}{c} F_3(a_1, a_2, b_1, b_2, 1 + c; z_1, z_2) |_{a_j \to a_{j+1}} \,, \quad j = 1, 2, \\
\omega_3(z_1, z_2) = z_1 z_2 \frac{a_1 a_2 b_1 b_2}{c(1 + c)} F_3(1 + a_1, 1 + a_2, 1 + b_1, 1 + b_2; 2 + c; z_1, z_2) ,
\]
we have
\[
\omega_1(z_1, z_2)|_{z_1=0} = 0 \,, \quad \omega_1(z_1, z_2)|_{z_2=0} = z_1 \frac{a_1 b_1}{c} 2 F_1(1 + a_1, 1 + b_1; 1 + c; z_1) , \\
\omega_2(z_1, z_2)|_{z_2=0} = 0 \,, \quad \omega_2(z_1, z_2)|_{z_1=0} = z_2 \frac{a_2 b_2}{c} 2 F_1(1 + a_2, 1 + b_2; 1 + c; z_2) , \\
\omega_3(z_1, z_2)|_{z_1=0} = \omega_3(z_1, z_2)|_{z_2=0} = 0 .
\]

The construction of the all-order \( \epsilon \)-expansion of Gauss hypergeometric functions around rational values of parameters in terms of multiple polylogarithms has been constructed in Refs. [197], [199].

### 3.2.4 The rational parametrization: towards multiple polylogarithms

It is well-known that one-fold iterated integrals over algebraic functions are not, in general, expressible in terms of multiple polylogarithms but demand the introduction of a new class of functions [218, 219, 220].

The iterative solution of the system Eq. (35), (36), have the following form. In the first two orders of \( \epsilon \)-expansion, we have (these results follow from the series representation and a special choice of parameters):

\[
\omega_i^{(0)} = 1 \,, \quad \phi_1^{(0)} = \phi_2^{(0)} = \phi_3^{(0)} = 0 , \quad (40a) \\
\omega_i^{(1)} = \phi_1^{(1)} = \phi_2^{(1)} = \phi_3^{(1)} = 0 , \quad (40b)
\]

The second iteration produces:

\[
\phi_3^{(2)}(x, y) = 0 , \quad (41a) \\
\phi_1^{(2)}(x, y) = a_1 b_1 \int_0^x \frac{dt}{1 - t} h_1(t) = a_1 b_1 R_1(x) , \quad (41b) \\
\phi_2^{(2)}(x, y) = a_2 b_2 \int_0^y \frac{dt}{1 - t} h_2(t) = a_2 b_2 R_2(y) , \quad (41c) \\
\omega_0^{(2)}(x, y) = \int_0^x \frac{dt_1}{t_1} \phi_1^{(2)}(x) + \frac{dt_1}{t_1} \phi_3^{(2)}(x) + \frac{dt_1}{t_1} \phi_2^{(2)}(y) + 2 F_1^{(2)}(x) + 2 F_1^{(2)}(y) ,
\]

Originaly, such types of functions have been introduced in Ref. [196].
where $F_1^{(2)}(t)$ are the functions coming from boundary conditions. The finite part of the $F_3$ function is (in terms of $R$-functions):

$$
\phi_3^{(3)}(x, y) = -\left(\frac{a_1 r_1 + b_1 p_1}{q}\right) a_2 b_2 h_2(y) R_2(y) \int_0^x \frac{dt}{H(t, y) (t + \frac{y}{1-x})} \\
+ \left(\frac{a_2 r_2 + b_2 p_2}{q}\right) a_1 b_1 \int_0^x \frac{dt}{H(t, y)} \frac{h_1(t) R_1(t)}{H(t, y)} \left(1 - \frac{1}{t + \frac{y}{1-x}}\right) \\
- \left(\frac{a_2 r_2 + b_2 p_2}{q}\right) a_1 b_1 h_1(x) R_1(x) \int_0^y \frac{dt}{H(x, t)} \frac{1}{H(x, t) (t + \frac{x}{1-x})} \\
+ \left(\frac{a_1 r_1 + b_1 p_1}{q}\right) a_1 b_1 \int_0^y \frac{dt}{H(x, t)} \frac{h_2(t) R_2(t)}{H(x, t)} \left(1 - \frac{1}{t + \frac{x}{1-x}}\right). \quad (42)
$$

Up to some factor, the function $R(t)$ coincides with the Gauss hypergeometric function with a rational set of parameters:

$$
R(z) = \int_0^z \frac{dt}{h(t)} \frac{1}{(1-t)} \sim _2F_1 \left(\frac{r_j}{q}, \frac{p_j}{q}, 1 - \frac{p_j}{q}; z\right), \quad j = 1, 2,
$$

where $p_j r_j = 0$. The $\epsilon$-expansion of the Gauss hypergeometric function around rational values of parameters has been analyzed in detail in Ref. [199]. It was shown that only the following cases are relevant:

- Integer set: $r_j = p_j = p = 0$ ,
- Zero-balance type: $r_j + p_j = -p$ , $r_j p_j = 0$ .
- Binomial type: $p = p_j = 0$ , $r_j \neq 0$ , (and symmetric one: $p = r_j = 0$ , $p_j \neq 0$)
- Inverse binomial type: $r_j = p_j = 0$ , $p \neq 0$ .
- Full type: $r_j = p_j = -p$ .

For each particular set of parameters, the rational parametrization of Eqs. [35] should be cross-checked. For illustration, let us consider a few examples.

### 3.2.5 The rational parametrization: set 1

Let us consider the hypergeometric function

$$
_F^3 \left(a_1 \epsilon, a_2 \epsilon, b_1 \epsilon, b_2 \epsilon, 1 - \frac{p}{q} + c \epsilon; x, y\right),
$$

so that

$$
r_1 = p_1 = p_2 = s_2 = 0 , \quad p \neq 0 .
$$

For this set of parameters, we have $s_1 = 0 , s_2 = 0 , p \neq 0$ , and
There is a parametrization that converts the functions $h_1$ and $h_2$ into rational functions $P_1$ and $P_2$:

$$\frac{\varrho}{\varrho_1} = P_1 (\xi_1, \xi_2), \quad \frac{\varrho}{\varrho_2} = P_2 (\xi_1, \xi_2),$$

where the functions $P_1$ and $P_2$ have the form

$$P_m(x, y) = \prod_{i,j,k,l} \frac{(x-a_i)(y-b_j)}{(x-c_k)(y-d_l)}; \quad m = 1, 2,$$

for a set of algebraic numbers $\{a_i, b_j, c_k, d_l\}$. In terms of new variables, the function $H$ has the form

$$H(x, y) = \frac{xy}{x+y-xy} = \frac{1}{\frac{x}{\varrho_1} + \frac{1}{\varrho_2} + 1} = P_3^q,$$

where $P_3$ is again a rational function of two variables of the type (43).

After a redefinition $\frac{1}{\varrho_i} = Q_i^\varrho$, $i = 1, 2, 3$ we obtain the result that a statement about existence of a rational parametrization for the functions $h(x), h(y)$ and $H(x, y)$ is equivalent to the existence of three rational functions of two variables satisfying the equation

$$Q_1^q + Q_2^q + Q_3^q = 1.$$  

(44)

To our knowledge, for $q > 2$, a solution exists only in terms of elliptic functions. However, it may happen that such a parametrization exists for $q = 2$, but we are not able to find it.

This problem is closely related to the solution of a functional equation. For example, for the equation

$$f^n + g^n = 1,$$

the solution can be characterized as follows: [221] [222]:

- For $n = 2$, all solutions are the form

$$f = \frac{2\beta(z)}{1 + \beta^2(z)}, \quad g = \frac{1 - \beta^2(z)}{1 + \beta^2(z)},$$

where $\beta(z)$ is an arbitrary function.

- For $n = 3$, one solution is given by

$$f = \frac{1}{2\varphi} \left( 1 + \frac{1}{\sqrt{3}} \varphi^3 \right), \quad g = -\frac{1}{2\varphi} \left( 1 - \frac{1}{\sqrt{3}} \varphi^3 \right).$$  

(45)
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where \( \wp \) is the Weierstrass \( \wp \)-function satisfy \( (\wp')^2 = 4\wp^3 - 1 \). For \( n = 3 \), the original equation is of genus 1, so that uniformization theorem assures the existence of an elliptic solution.

One of the most natural sets of variables for the set of parameters under consideration is the following: \( P_1 = \xi_1, P_2 = \xi_2 \), so that

\[
H = \frac{1}{\xi_1^p \xi_2^q} (\xi_1^q \xi_2^q - \xi_1^q - \xi_2^q) \frac{\xi}{\pi}.
\]

### 3.2.6 The rational parametrization: set 2

In a similar manner, let us analyze another set of parameters:

\[
F_3 \left( \frac{P}{q} + a_1 \epsilon, a_2 \epsilon, b_1 \epsilon, -\frac{P}{q} + b_2 \epsilon, 1 - \frac{P}{q} + c \epsilon; x, y \right).
\]

In this case, \( s_1 = s_2 = -p \), and the functions \( h \) have the form \( h(x) = x^{\frac{\xi}{\pi}}, \quad h(y) = y^{\frac{\xi}{\pi}} \). Applying the same trick with the introduction of new functions \( P_1 \) and \( P_2 \), we would find that the existence of a rational parametrization corresponds in the present case to the validity of Eq. (44). In particular, by introducing new variables \( x^{\frac{\xi}{\pi}} = \xi_1, y^{\frac{\xi}{\pi}} = \xi_2 \), we obtain \( H = (\xi_1^q \xi_2^q - \xi_1^q - \xi_2^q)^{\frac{\xi}{\pi}} \).

### 3.2.7 The rational parametrization: set 3

Let us analyze the following set of parameters: \( p = 0, s_1, s_2 \neq 0 \), corresponding to

\[
F_3 \left( \frac{P_1}{q} + a_1 \epsilon, \frac{P_2}{q} + b_1 \epsilon, b_2 \epsilon, 1 + c \epsilon; x, y \right).
\]

In this case,

\[
h_1(x) = (1 - x)^{\frac{P_1}{q}}, \quad h_2(y) = (1 - y)^{\frac{P_2}{q}}, \quad H(x, y) = \left( \frac{x^{s_2} y^{s_1}}{(xy - x - y)^{s_1 + s_2}} \right)^{\frac{\xi}{\pi}}.
\]

For simplicity, we set \( s_1 = -s_2 = s \), and put \( 1 - x = P_1^q \) and \( 1 - y = P_2^q \), so that \( H = \left( \frac{1-P_1^q}{1-P_2^q} \right)^{\frac{\xi}{\pi}} \equiv P_3 \). In particular, for \( P_1 = \xi_1, P_2 = \xi_2, H = \left( \frac{1-\xi_1^q}{1-\xi_2^q} \right)^{\frac{\xi}{\pi}} \).

### 3.2.8 The rational parametrization: set 4

Let us analyze the following set of parameters: \( s_1 = -p, s_2 = 0 \), so that hypergeometric function is
In this case,

\[ h_1(x) = x_i^\alpha, \quad h_2(y) = \left(\frac{y}{y-1}\right)^{\alpha / \beta}, \quad H(x, y) = x_i^\alpha \equiv h(x). \quad (47) \]

Let us take a new set of variables \((x, y) \rightarrow (\xi_1, \xi_2)\):

\[ \xi_1 = x_i^\alpha, \quad \xi_2 = \left(\frac{y}{y-1}\right)^{\alpha / \beta}, \quad (48) \]

In terms of a new variables we have:

\[ H(x, y) \equiv h_1(x) = \xi_1^p, \quad h_2(y) = \xi_2^p, \quad x + y - xy = \frac{\xi_1^p - \xi_2^p}{1 - \xi_2^p}. \quad (49) \]

Thus, a rational parametrization exists.

### 3.2.9 The rational parametrization: set 5

Consider the set of parameters defined by \( s_2 = p = 0, s_1 \neq 0 \), that corresponds to the case \( F_3 \left( -\frac{P^1}{q} + a_1 e, a_2 e, b_1 e, b_2 e, 1 + ce; x, y \right) \). In this case,

\[ h_1(x) = (1 - x)^{\frac{\alpha}{\beta}}, \quad h_2(y) = 1, \quad H(x, y) = \left(\frac{y}{xy - x - y}\right)^{\frac{p}{q}}. \quad (50) \]

Let us suggest that

\[ 1 - x = P^q, \quad H(x, y) = \frac{y}{x + y - xy} = Q^q, \]

where \( P \) and \( Q \) are rational functions. Then, \( y = \frac{(1 - P^q)(Q^q)}{1 - P^q/Q^q} \). After the redefinition, \( PQ = R \), we get \( y = \frac{(1 - P^q)R^q}{1 - P^qR^q} \). The simplest version of \( P \) and \( R \) are polynomials: \( P = \xi_1 \), and \( R = \xi_2 \). In this parametrization,

\[ h(x) = \xi_1^{p_1}, \quad H(x, y) = \left(\frac{\xi_2}{\xi_1}\right)^{p_1}, \quad y = \frac{1 - \xi_1^q}{1 - \xi_2^q} \left(\frac{\xi_2}{\xi_1}\right)^q. \]

In this case, the rational parametrization exists.
3.2.10 Explicit construction of expansion: integer values of parameters

Let us consider the construction of the \( \varepsilon \)-expansion around integer values of parameters. If we put

\[
\omega_0 = F_3(a_1 \varepsilon, b_1 \varepsilon, a_2 \varepsilon, b_2 \varepsilon, 1 + c \varepsilon; x, y),
\]

then the system of differential equations can be presented in the form

\[
\begin{align*}
\frac{\partial}{\partial x} \omega_1 &= \left[ \frac{1}{x-1} - \frac{1}{x} \right] \omega_3 - \left[ \frac{c + (a_1 + b_1 - c)}{x - 1} \right] \varepsilon \omega_1 - a_1 b_1 \frac{1}{x-1} \varepsilon^2 \omega_0, \quad (51) \\
\frac{\partial}{\partial y} \omega_2 &= \left[ \frac{1}{y-1} - \frac{1}{y} \right] \omega_3 - \left[ \frac{c + (a_2 + b_2 - c)}{y - 1} \right] \varepsilon \omega_2 - a_2 b_2 \frac{1}{y-1} \varepsilon^2 \omega_0, \quad (52) \\
\frac{\partial}{\partial x} \omega_3 &= \left[ \frac{(a_2 + b_2 - c)}{x} - \frac{(a_1 + b_1 + a_2 + b_2 - c)}{x + \frac{1}{1-y}} \right] \varepsilon \omega_3 \\
&\quad - a_1 b_1 \frac{1}{x + \frac{1}{1-y}} \varepsilon^2 \omega_1 + \left[ \frac{1}{x + \frac{1}{1-y}} \right] a_2 b_2 \varepsilon^2 \omega_1, \quad (53) \\
\frac{\partial}{\partial y} \omega_3 &= \left[ \frac{(a_1 + b_1 - c)}{y} - \frac{(a_1 + b_1 + a_2 + b_2 - c)}{y + \frac{1}{1-x}} \right] \varepsilon \omega_3 \\
&\quad - a_2 b_2 \frac{1}{y + \frac{1}{1-x}} \varepsilon^2 \omega_1 + \left[ \frac{1}{y + \frac{1}{1-x}} \right] a_1 b_1 \varepsilon^2 \omega_2. \quad (54)
\end{align*}
\]

This system can be straightforwardly integrated in terms of multiple polylogarithms, defined via a one-fold iterated integral \( G \), where

\[
G(z; a_k, \vec{a}) = \int_0^z \frac{dt}{t - a_k} G(t; \vec{a}). \quad (55)
\]

In addition, the \( \varepsilon \)-expansion of a Gauss hypergeometric function around integer values of parameters is needed. It has the following form (see Eq. (34) in [172]):

\[
\begin{align*}
_{2}F_{1}(a \varepsilon, b \varepsilon; 1+c \varepsilon; z) &= 1 + ab \varepsilon^2 \text{Li}_2 (z) \\
&\quad + ab \varepsilon^3 \left[ (a+b-c) \text{Si}_2 (z) - c \text{Li}_3 (z) \right] + O(\varepsilon^4).
\end{align*}
\]

The first iteration gives rise to

\[
\begin{align*}
\omega_0^{(0)} &= 1, \quad \omega_1^{(0)} = \omega_2^{(0)} = \omega_3^{(0)} = 0, \quad \omega_0^{(1)} = \omega_1^{(1)} = \omega_2^{(1)} = \omega_3^{(1)} = 0.
\end{align*}
\]

The results of the second iteration are the following:

\[
\begin{align*}
\omega_3^{(2)} &= 0, \quad \omega_1^{(2)} = -a_1 b_1 \ln(1-x), \quad \omega_2^{(2)} = -a_2 b_2 \ln(1-y), \\
\omega_0^{(2)} &= a_1 b_1 \text{Li}_2 (x) + a_2 b_2 \text{Li}_2 (y).
\end{align*}
\]
where the classical polylogarithms \( \text{Li}_n (z) \) are defined as

\[
\text{Li}_1 (z) = - \ln (1 - z), \quad \text{Li}_{n+1} (z) = \int_0^z \frac{dt}{t} \text{Li}_n (t), \quad n \geq 1.
\] (58)

After the third iteration, we have

\[
\begin{align*}
\omega_3^{(3)} &= 0 \\
\omega_1^{(3)} &= \frac{1}{2} a_1 b_1 (a_1 + b_1 - c) \ln^2 (1 - x) - a_1 b_1 c \text{Li}_2 (x), \\
\omega_2^{(3)} &= \frac{1}{2} a_2 b_2 (a_2 + b_2 - c) \ln^2 (1 - y) - a_2 b_2 c \text{Li}_2 (y), \\
\omega_0^{(3)} &= -a_1 b_1 c \text{Li}_3 (x) - a_2 b_2 c \text{Li}_3 (y) \\
&\quad + a_1 b_1 (a_1 + b_1 - c) \text{S}_{1,2} (x) + a_2 b_2 (a_2 + b_2 - c) \text{S}_{1,2} (y),
\end{align*}
\]

where \( \text{S}_{a,b} (z) \) are the Nielsen polylogarithms:

\[
z \frac{d}{dz} \text{S}_{a,b} (z) = \text{S}_{a-1,b} (z), \quad \text{S}_{1,b} (z) = \frac{(-1)^b b!}{b} \int_0^1 \frac{dx}{x} \ln^b (1 - zx) .
\] (63)

The result of the next iteration, \( \omega_3^{(4)} (x, y) \), can be expressed in several equivalent forms:

\[
\begin{align*}
\frac{\omega_3^{(4)} (x, y)}{a_1 a_2 b_1 b_2} &= \ln (1 - y) G_1 \left( x; -\frac{y}{1-y} \right) - G_2 (x; 1) + G_{1,1} \left( x; -\frac{y}{1-y} , 1 \right), \\
\frac{\omega_2^{(4)} (x, y)}{a_1 a_2 b_1 b_2} &= \ln (1 - x) G_1 \left( y; -\frac{x}{1-x} \right) - G_2 (y; 1) + G_{1,1} \left( y; -\frac{x}{1-x} , 1 \right).
\end{align*}
\]

Keeping in mind that

\[
G_{1,1} \left( x; -\frac{y}{1-y} , 1 \right) = \int_0^x \frac{dt}{t + \frac{y}{1-y}} \ln (1 - t) \quad (64)
\]

\[
= -\ln (1 - y) \ln (x + y - xy) + \ln (1 - y) \ln y - \text{Li}_2 (x + y - xy) + \text{Li}_2 (y),
\]

the result can be written in a very simple form,

\[
\frac{\omega_1^{(4)} (x, y)}{a_1 a_2 b_1 b_2} = \text{Li}_2 (x) + \text{Li}_2 (y) - \text{Li}_2 (x + y - xy).
\] (65)

Taking into account that \( \frac{\omega_3^{(4)} (x, y)}{a_1 a_2 b_1 b_2} = \frac{1}{2} x y F_3 (1, 1, 1; 3; x, y) \), we obtain the well-known result [223]

\[
\frac{1}{2} x y F_3 (1, 1, 1; 3; x, y) = \text{Li}_2 (x) + \text{Li}_2 (y) - \text{Li}_2 (x + y - xy).
\]
There is also another form \[ \frac{1}{2} xy F_3 (1, 1, 1; 3; x, y) = \text{Li}_2 \left( \frac{x}{x + y - xy} \right) - \text{Li}_2 \left( \frac{x - xy}{x + y - xy} \right) - \ln (1 - y) \ln \left( \frac{y}{x + y - xy} \right). \]

One form can be converted to the other using the well-known dilogarithm identity

\[ \text{Li}_2 \left( \frac{1}{z} \right) = -\text{Li}_2 (z) - \frac{1}{2} \ln^2 (-z) - \zeta_2, \]

together with the attendant functional relations \[111, 225\].

The following expressions result from direct iterations in terms of $G$-functions:

\[
\begin{align*}
\frac{\omega_1^{(4)} (x, y)}{a_1 b_1} &= a_2 b_2 \left\{ \ln (1 - y) \left[ G_{1, 1} \left( x; 1, -\frac{y}{1 - y}, 1 \right) - G_2 \left( x; -\frac{y}{1 - y}, 1 \right) \right] \\
&- G_{1, 2} (x; 1, 1) + G_3 (x; 1) + G_{1, 1, 1} \left( x; 1, -\frac{y}{1 - y}, 1 \right) - G_{2, 1} \left( x; -\frac{y}{1 - y}, 1 \right) \right\} \\
&+ a_2 b_2 G_1 (x; 1) G_2 (y; 1) - c \Delta_1 G_{2, 1} (x; 1, 1) - \Delta_1^2 G_{1, 1, 1} (x; 1, 1, 1) - c^2 G_3 (x; 1) \\
&+ (a_1 b_1 - c \Delta_1) G_{1, 2} (x; 1, 1),
\end{align*}
\]

where

\[ \Delta_j = a_j + b_j - c. \]

The $G$-functions can be converted into classical or Nielsen polylogarithms with the help of the following relations:

\[
\begin{align*}
G_{1, 1} \left( x; 1, -\frac{y}{1 - y}, 1 \right) &= G_1 (x; 1) G_1 \left( x; -\frac{y}{1 - y}, 1 \right) - G_{1, 1} \left( x; 1, -\frac{y}{1 - y}, 1 \right), \\
G_1 (x; 1) G_{1, 1} \left( x, -\frac{y}{1 - y}, 1 \right) &= G_{1, 1, 1} \left( x; 1, -\frac{y}{1 - y}, 1 \right) \tag{67a} \\
&+ 2 G_{1, 1, 1} \left( x; -\frac{y}{1 - y}, 1, 1 \right), \\
G_{1, 1, 1} \left( x; -\frac{y}{1 - y}, 1, 1 \right) &= S_{1, 2} (x + y - xy) - S_{1, 2} (y) \tag{67c} \\
&+ \frac{1}{2} \ln^2 (1 - y) \left[ \ln (x + y - xy) - \ln y \right] + \ln (1 - y) \left[ \text{Li}_2 (x + y - xy) - \text{Li}_2 (y) \right], \\
G_{2, 1} \left( x; -\frac{y}{1 - y}, 1 \right) + \ln (1 - y) G_2 \left( x; -\frac{y}{1 - y}, 1 \right) &= G_{1, 2} \left( 1, 1; -\frac{1}{y}, x \right) + G_3 (x; 1), \\
&= \int_0^x \frac{du}{u} \left[ \text{Li}_2 (y) - \text{Li}_2 (u + y - uy) \right]. \tag{67d}
\end{align*}
\]
In a similar manner, 
\[
\frac{\omega^{(4)}_2(x, y)}{a_2b_2} = a_1 b_1 \left\{ \ln(1-x) \left[ G_{1,1} \left( y; 1, -\frac{x}{1-x} \right) - G_2 \left( y; -\frac{x}{1-x} \right) \right] \\
- G_{1,2}(y; 1, 1) + G_3(y; 1) + G_{1,1,1} \left( y; 1, -\frac{x}{1-x}, 1 \right) - \Delta G_{2,1} \left( y; -\frac{x}{1-x}, 1 \right) \right\}
\]
\[
- c \Delta_2 G_{2,1}(y; 1, 1) - \Delta_2^2 G_{1,1,1}(y; 1, 1, 1) - c^2 G_3(y; 1) - c \Delta_2 G_{1,2}(y; 1, 1)
\]
\[
+ a_1 b_1 G_1(y; 1) G_2(x; 1) + a_2 b_2 G_{1,2}(y; 1, 1),
\]
and the last term, expressible in terms of functions of order 3, is \(\omega_3^{(5)}\).

### 3.2.11 Construction of \(\varepsilon\)-expansion via integral representation

An alternative approach to construction of the higher order \(\varepsilon\)-expansion of generalized hypergeometric functions is based on their integral representation. We collect here some representations for the Appell hypergeometric function \(F_3\) extracted from Refs. [10] and [11].

For our purposes, the most useful expression is the following: (see Eq. (16) in Section 9.4. of [11], [223] [224]:

\[
\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(c_1 + c_2)} F_3(a_1, b_1, a_2, b_2, c_1 + c_2; x, y) = \int_0^1 du u^{c_1-1}(1-u)^{c_2-1} F_1(a_1, b_1; c_1; u) F_1(a_2, b_2; c_2; (1-u)y). 
\]

Indeed, expanding one of the hypergeometric functions as a power series leads to

\[
\int_0^1 u^{c_1-1}(1-u)^{j+c_2-1} F_1(a_1, b_1; c_1; u) \sum_{j=0}^{\infty} \frac{(a_2)_j (b_2)_j y^{j}}{(c_2)_j j!}. 
\]

The order of summation and integration can be interchanged in the domain of convergence of the series, and after integration we obtain Eq. (69).

The two-fold integral representation [10],

\[
\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(c - b_1 - b_2)}{\Gamma(c)} F_3(a_1, a_2; b_1, b_2; c; x, y) = \int \int_{0 \leq u, 0 \leq v, u + v \leq 1} du dv \left[ (1-y) u^{b_1-1} v^{b_2-1} (1-u-v)^{c-b_1-b_2-1} \right]^{-a_1} \left[ (1-y) \right]^{-a_2},
\]
can be reduced to the following integral:
can be reduced to the Appell function \( F_3 \) to prove the following relations:

\[
\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(c-b_1-b_2)}{\Gamma(c)} F_3(a_1, a_2; b_1, b_2; c; x, y) = \int_0^1 du \int_0^{1-u} dv u^{b_1-1}, b_2-1 (1-u-v)^{c-b_1-b_2} (1-ux)^{-a_1} (1-vy)^{-a_2}
\]

\[
= \frac{\Gamma(b_2)\Gamma(c-b_1-b_2)}{\Gamma(c-b_1)} \times \int_0^1 du u^{b_1-1} (1-ux)^{-a_1} (1-u)^{c-b_1-1} \binom{a_2, b_2}{c-b_1} (1-u)^y .
\]

For the particular values of parameters \((c_1 = a_1, c_2 = a_2)\), the integral Eq. \(69\) can be reduced to the Appell function \( F_1 \):

\[
F_3(a_1, a_2, b_1, b_2; a_1 + a_2; x, y) = \frac{1}{(1-y)^{b_2}} F_1 \left( a_1, b_1, b_2, a_1 + a_2; x, \frac{y}{1-y} \right) .
\]

Using the one-fold integral representation for the Appell function \( F_1 \), it is possible to prove the following relations:

\[
F_3(a, c-a, b, c-b, c, x, y) = (1-y)^{c+a-b} \binom{a, b}{c} \binom{x+y-xy} .
\]

Using Eqs. \(69\) and \(71\), the one-fold integral representation can be written for the coefficients of the \( \varepsilon \)-expansion of the hypergeometric function \( F_3 \) via the \( \varepsilon \)-expansion of the Gauss hypergeometric function, constructed in Refs. \[197, 199\].

The coefficients of the \( \varepsilon \)-expansion of the Gauss hypergeometric function can be expressed in terms of multiple polylogarithms of a \( q \)-root of unity with arguments \( \left( \frac{5}{6} \right)^{1/q}, \frac{1}{\sqrt{3}} \) or \( 1-\mathcal{Z} \) (see also \[158\]), so that the problem of finding a rational parametrization reduces to the problem of finding a rational parametrization of the integral kernel of Eqs. \(69\) and \(71\) in terms of variables generated by the \( \varepsilon \)-expansion of the Gauss hypergeometric function.

The construction of the higher-order \( \varepsilon \)-expansion of the Gauss hypergeometric function around rational values of parameters \[197, 199\], plays a crucial role in construction of the higher-order \( \varepsilon \)-expansion of many (but not all) Horn-type hypergeometric functions.

### 3.2.12 Relationship to Feynman Diagrams

Let us consider the one-loop pentagon with vanishing external legs. The higher-order \( \varepsilon \)-expansion for this diagram has been constructed \[166\] in terms of iterated one-fold integrals over algebraic functions. In Ref. \[226\], the hypergeometric representation
for the one-loop pentagon with vanishing external momenta has been constructed as a sum of Appell hypergeometric functions $F_3$.

In Ref. [180], where our differential equation approach is presented, it was pointed out that the one-loop pentagon can be expressed in terms of multiple polylogarithms. Ref. [155] verified the numerical agreement between the results of Refs. [166] and [226], and Ref. [227] constructed the iterative solution of the differential equation [228].

Let us recall the results of Ref. [226]. The one-loop massless pentagon is expressible in terms of the Appell function $F_3$ with the following set of parameters:

$$\Phi^{(d)}_5 \sim F_3 \left( 1, 1, \frac{7 - d}{2}, 1, \frac{10 - d}{2}; x, y \right), \quad (73)$$

where $d$ is dimension of space-time. Another representation presented in Ref. [226] has the structure

$$H^{(d)}_5 \sim F_3 \left( \frac{1}{2}, 1, 1, \frac{d - 2}{2}, \frac{d + 1}{2}; \frac{y}{x + y - xy}, \frac{1}{x} \right). \quad (74)$$

Let us consider the case of $d = 4 - 2\epsilon$. The first representation, Eq. (73), is

$$\Phi^{(4-2\epsilon)}_5 \sim F_3 \left( 1, 1, \frac{3}{2} - \epsilon, 1, 4 - \epsilon; x, y \right).$$

This case,

$$\{p_1 = p_2 = r_2 = p = 0\}, \{r_1 = 1, q = 2\} \implies s_1 \neq 0; \quad s_2 = 0, \quad p = 0,$$

corresponds to our set 5, so that the $\epsilon$-expansion is expressible in terms of multiple polylogarithms, defined by Eq. (55).

For the other representation, Eq. (74),

$$H^{(4-2\epsilon)}_5 \sim F_3 \left( \frac{1}{2}, 1, 1, \epsilon, \frac{5}{2} - \epsilon; \frac{y}{x + y - xy}, \frac{1}{x} \right),$$

so that it is reducible to the following set of parameters:

$$\{p_2 = r_1 = r_2 = 0\}, \quad \{p_1 = 1, q = 2, \ p = 1\} \implies s_1 = 1; \quad s_2 = 0, \quad p = -1.$$

This is our set 4, so that the $\epsilon$-expansion is expressible in terms of multiple polylogarithms, defined by Eq. (55).

The $\epsilon$-expansion of the one-loop pentagon about $d = 3 - 2\epsilon$ could be treated in a similar manner. In this case, the first representation corresponds to set 1

$$\Phi^{(3-2\epsilon)} \sim F_3 \left( 1, 1, 2 + \epsilon, 1, \frac{7}{2} + \epsilon; x, y \right) \implies p_1 = p_2 = r_1 = r_2 = 0, \quad q = 2, \ p = 1.$$
and there is no rational parametrization, so that the result of the $\varepsilon$-expansion is expressible in terms of a one-fold iterated integral over algebraic functions.

The other representation, Eq. (74), also cannot be expressed in terms of multiple polylogarithms:

$$H_5^{(3-2\varepsilon)} \sim F_3\left(\frac{1}{3}, 1, 1, \frac{1}{2} - \varepsilon, 2 - \varepsilon; \frac{y}{x + y - xy}, \frac{1}{x}\right) \Rightarrow p_2 = r_1 = p = 0, \quad q = 2, \quad p_1 = r_2 = 1, \quad s_1 = 1; \quad s_2 = 1.$$

This corresponds to set 3, and the $\varepsilon$-expansion is expressible in terms of one-fold iterated integral over algebraic functions.

In this way, the question of the all-order $\varepsilon$-expansion of a one-loop Feynman diagram in terms of multiple polylogarithms is reduced to the question of the existence of a rational parametrization for the (ratio) of singularities.

Remark: The dependence of the coefficients of the $\varepsilon$-expansion (multiple polylogarithms or elliptic function) on the dimension of space-time is not new. In particular, it is well known that the two-loop sunset in $3-2\varepsilon$ dimension is expressible in terms of polylogarithms \([229, 230]\) and demands introduction of new functions in $4-2\varepsilon$ dimension \([231, 232, 233, 234, 235, 236, 237]\).

## 4 Conclusion

The deep relationship between Feynman diagrams and hypergeometric functions has been reviewed, and we have tried to enumerate all approaches and recent results on that subject. Special attention was devoted to the discussion of different algorithms for constructing the analytical coefficients of the $\varepsilon$-expansion of multiple hypergeometric functions. We have restricted ourselves to multiple polylogarithms and functions related to integration over rational functions (the next step after multiple polylogarithms). The values of parameters related to elliptic polylogarithms was beyond our consideration.

We have presented our technique for the construction of coefficients of the higher order $\varepsilon$-expansion of multiple Horn-type hypergeometric functions, developed by the authors during the period 2006 – 2013. One of the main results of interest was the observation \([146, 147, 167, 197, 198, 199, 103]\) that for each Horn-type hypergeometric function, a set of parameters can be found so that the coefficients of the $\varepsilon$-expansion include only functions of weight one (so-called “pure functions,” in a modern terminology). As was understood in 2013 by Johannes Henn \([202, 217]\), this property is valid not only for hypergeometric functions but also for generic Feynman diagrams.

Our approach is based on the systematic analysis of the system of hypergeometric differential equations (linear differential operators of hypergeometric type with poly-

\[\text{Unpublished.}\]
nominal coefficients) and does not demand the existence of an integral representation, which is presently unknown for a large class of multiple Horn-type hypergeometric functions (they could be deduced, but are not presently available in the mathematical literature).

Our approach is based on the factorization of the system of differential equations into a product of differential operators, together with finding a rational parametrization and constructing iterative solutions. To construct such a system, an auxiliary manipulation with parameters (shifting by integer values) is required, which can be done with the help of the HYPERDIRE set of programs [82]. This technique is applicable not only to hypergeometric functions defined by series but also to multiple Mellin-Barnes integrals [77] – one of the representations of Feynman diagrams in a covariant gauge. We expect that the present technique is directly applicable (with some technical modifications) to the construction of the \( \varepsilon \)-expansion of hypergeometric functions beyond multiple polylogarithms, specially, that the two-loop sunset has a simple hypergeometric representation [232].

There are two of our considerations that have not been solved algorithmically: (a) the factorization of linear partial differential operators into irreducible factors is not unique, as has been illustrated by Landau [185] (see also Ref. [186]):

\[
(\partial_x + 1)(\partial_x + 1)(\partial_y + x\partial_y) = (\partial_x^2 + x\partial_x + \partial_x + (2 + \varepsilon)\partial_y)(\partial_y + 1)
\]

(b) The choice of parametrization is still an open problem, but there is essential progress in this direction [238, 239].

Our example has shown that such a parametrization is defined by the locus of singularities of a system of differential equations, so that the problem of finding a rational parametrization is reduced to the parametrization of solutions of the Diophantine equation for the singular locus of a Feynman diagram and/or hypergeometric function. It is well known that in the case of a positive solution of this problem (which has no complete algorithmic solution), the corresponding system of partial differential equations of a few variables takes the simplest structure. At the same time, there is a relationship between the type of solution of the Diophantine equation for the singular locus and the structure of the coefficients of the \( \varepsilon \)-expansion: a linear solution allows us to write the results of the \( \varepsilon \)-expansion in terms of multiple polylogarithms. An algebraic solution gives rise to functions different from multiple polylogarithms and elliptic functions, etc. It is natural to expect that, in the case of an elliptic solution of the Diophantine equation for the singular locus, the results for the \( \varepsilon \)-expansion are related to the elliptic generalization of multiple polylogarithms.

Another quite interesting and still algorithmically open problem is the transformation of multiple Horn-type hypergeometric functions with reducible monodromy to hypergeometric functions with irreducible monodromy. In the application to Feynman diagrams, such transformations correspond to functional relations, studied recently by Oleg Tarasov [240, 241, 242], and by Andrei Davydychev [107].

Further analysis of the symmetries of the hypergeometric differential equations related to the Mellin-Barnes representation of Feynman diagrams (for simplicity we will call it the hypergeometric representation) has revealed their deep connection to the holonomic properties of Feynman diagrams. In particular, a simple and fast algorithm was constructed for the reduction of any Feynman diagram having a one-fold Mellin-Barnes integral representation to a set of master integrals [77].
The importance of considering the dimension of the irreducible representation instead of generic holonomic rank has been pointed out in the application to Feynman diagrams [66]. In the framework of this approach, the set of irreducible, non-trivial master-integrals corresponds to the set of irreducible (with respect to analytical continuation of the variables, masses, and external momenta) solutions of the corresponding system of hypergeometric differential equations, whereas diagrams expressible in terms of Gamma-functions correspond to Puiseux-type solutions (monomials with respect to Mandelstam variables) of the original system of hypergeometric equations.

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