A $C^0$ LINEAR FINITE ELEMENT METHOD FOR A SECOND ORDER ELLIPTIC EQUATION IN NON-DIVergence FORM WITH CORDES COEFFICIENTS

MINQIANG XU∗, RUNCHANG LIN†, AND QINGSong ZOU‡

Abstract. In this paper, we develop a gradient recovery based linear (GRBL) finite element method (FEM) and a Hessian recovery based linear (HRBL) FEM for second order elliptic equations in non-divergence form. The elliptic equation is casted into a symmetric non-divergence weak formulation, in which second order derivatives of the unknown function are involved. We use gradient and Hessian recovery operators to calculate the second order derivatives of linear finite element approximations. Although, thanks to low degrees of freedom (DOF) of linear elements, the implementation of the proposed schemes is easy and straightforward, the performances of the methods are competitive. The unique solvability and the $H^2$ seminorm error estimate of the GRBL scheme are rigorously proved. Optimal error estimates in both the $L^2$ norm and the $H^1$ seminorm have been proved when the coefficient is diagonal, which have been confirmed by numerical experiments. Superconvergence in errors has also been observed. Moreover, our methods can handle computational domains with curved boundaries without loss of accuracy from approximation of boundaries. Finally, the proposed numerical methods have been successfully applied to solve fully nonlinear Monge-Ampère equations.

AMS subject classifications. Primary 65N30; Secondary 45N08

Key words. non-divergence form; discontinuous coefficients; Cordes condition; gradient recovery; Hessian recovery; linear finite element; Monge-Ampère equations; superconvergence.

1. Introduction. In this paper, we develop and analyze a $C^0$ linear FEM for the following second-order linear elliptic partial differential equation (PDE) in non-divergence form:

$$\begin{cases}
Lu = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

(1.1)

where $\Omega \subset \mathbb{R}^d$ is a bounded open convex domain with boundary $\partial\Omega$, $f \in L^2(\Omega)$ is a given function, and the differential operator $L$ has a non-divergence form

$$Lv = A : D^2v = \sum_{j,k=1}^d a_{jk} \partial_{jk}^2 v, \quad \forall v \in V := H^1_0(\Omega) \cap H^2(\Omega).$$

(1.2)

Here and in the rest of this paper, standard definitions and notations of Sobolev spaces are used [1]. We suppose that the coefficient tensor $A = (a_{ij})_{d \times d}$ is symmetric and uniformly bounded. Assume further that $A$ is positive definite; namely, there exist positive constants $\alpha, \beta$ such that

$$\alpha \xi^T \xi \leq A(x) \xi^T \xi \leq \beta \xi^T \xi, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \Omega.$$  

(1.3)

In addition, we assume that the coefficient tensor satisfies the Cordes condition; i.e. there exists an $\epsilon \in [0,1]$ such that

$$|A|^2/(\text{tr}A)^2 \leq 1/(d - 1 + \epsilon),$$

(1.4)

∗College of Science, Zhejiang University of Technology, Hangzhou, 310023, P.R. China, and School of Data and Computer Science, Sun Yat-sen University, Guangzhou 510275, P.R. China. Email: xumq9@mail2.sysu.edu.cn. The research of this author was supported in part by NSFC Grant 12271049, the General Scientific Research Projects of Zhejiang Education Department (Y202147013) and the Opening Project of Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University (2021008).

†Department of Mathematics and Physics, Texas A&M International University, Laredo, Texas 78041, USA. Email: rlin@tamiu.edu.

‡Corresponding author. School of Computer Science and Engineering, and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen University, Guangzhou 510275, China. Email: mcszqs@mail.sysu.edu.cn. The research of this author was supported in part by NSFC Grant 12071496, Guangdong Provincial NSF Grant 2017B030311001, and Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University (2020B1212060032).
where $|A|^2 = \sum_{i,j=1}^{d} a_{ij}^2$. It has been proven in [45] that the condition (1.4) can be derived from the positive definiteness condition (1.3) for two dimensional problems. But, in three dimensional cases, the Cordes condition (1.4) is essential; the PDE may be ill-posed in absence of this condition.

Problem (1.1) arises in many fields, such as stochastic processes and game theory [19]. The non-divergence equations are also frequently found in linearizations of second order fully nonlinear differential equations, such as the Hamilton-Jacobi-Bellman equation and the Monge-Ampère equation (cf., e.g., [8, 41]). In many important applications, the coefficients are hardly smooth, or even discontinuous, so that the differential equations cannot be written in divergence forms. On the other hand, compared with studies for elliptic problems in divergence form, the literature on numerical analysis of differential equations in non-divergence form is limited. Therefore, it is crucial to develop efficient numerical methods for the problem (1.1) with rough coefficients to accommodate its wide application.

In this paper, we will consider the problem (1.1) in two dimensional cases. If the coefficient matrix $A \in [C^1(\Omega)]^{2 \times 2}$, then (1.1) can be recast into the following divergence form:

$$\nabla \cdot (A \nabla u) - (\nabla \cdot A) \cdot \nabla u = f.$$  (1.5)

A weak formulation of the problem (1.5) is to find $u \in H^1(\Omega)$ such that

$$- \int_{\Omega} (A \nabla u) \cdot \nabla v - \int_{\Omega} (\nabla \cdot A) \cdot (\nabla u) v = \int_{\Omega} f v, \quad \forall v \in H^1_0(\Omega).$$  (1.6)

Therefore, standard Lagrange finite elements can be applied to discrete formulation (1.6). But for a non-divergence form (1.1), the formulation (1.6) may fail to work. To circumvent this difficulty, many numerical approaches have been developed; see, e.g., [6, 9, 11, 16, 22, 29, 34, 35, 38, 40, 43, 45, 46, 47, 53] and the references therein for an incomplete list of references. Among these methods, we are interested in three approaches.

The first approach is based on an asymmetric form, which reads: finding $u \in H^2(\Omega)$ such that

$$\int_{\Omega} (A : D^2 u) v = \int_{\Omega} f v, \quad \forall v \in H^1_0(\Omega).$$  (1.7)

To discretize formulation (1.7), Wang et al. introduced and analyzed a primal-dual weak Galerkin (WG) method [47], which characterizes the numerical solution as a minimization of a nonnegative quadratic functional with constraints. This method involves second derivatives of test and trial functions, which means polynomials of degree at least two are required for the finite element space. Lakkis et al. provided a nonconforming FEM by introducing finite element Hessian [35]. Stability and convergence of the method were provided in the case of quadratic or higher degree elements.

The second approach involves a fourth-order variational form of the non-divergence equation (1.1). Smears and Süli [45] designed an $hp$-version discontinuous Galerkin (DG) FEM based on the formulation

$$(\gamma A : D^2 u, \Delta v)_{\Omega} = (\gamma f, \Delta v)_{\Omega}, \quad \forall v \in H^2(\Omega),$$  (1.8)

which was the first contribution to the non-divergence equations (1.1) with Cordes coefficients. The stability of the presented scheme was shown by applying a discrete Miranda-Talenti estimate. Feng et al. [18] utilized continuous Lagrange finite elements to discrete scheme (1.8) and proved the well-posedness of the proposed scheme using a discrete inf-sup condition under the assumption that coefficients are continuous. Neilan et al. [16] proposed and investigated a $C^0$ DG method. They used an interior penalty term from the jump of flux across interior element edges, which can be obtained by applying DG integration by parts formula to the first term of the formulation (1.6).

The third approach is based on a symmetric form from the least-squares technique, which has widely applications in scientific computing (see, e.g., [28, 49, 50]). It reads: seeking $u \in H^2(\Omega)$ such that

$$(A : D^2 u, A : D^2 v)_{\Omega} = (f, A : D^2 v)_{\Omega}, \quad \forall v \in H^2(\Omega).$$  (1.9)
The formulation \(1.9\) can be obtained from minimizing the functional \(\|A : D^2u - f\|_2^2\), for which \(H^2\) elements are usually required. Gallistl \([22]\) applied a conforming mixed FEM (MFEM) for the numerical approximation. Adaptive algorithms were also discussed. Ye et al. \([38]\) presented a nonconforming FEM with interior penalty term. There are many other approaches for solving this classic problem; e.g. the vanishing moment method \([17]\), the Alexandroff-Bakelman-Pucci (ABP) method \([43]\), the tailored nonconforming FEM \([7]\), etc. Recently, Kawecki \([29]\) extended the DG technique to curved domains.

Recently, some differential operator recovery based linear FEMs have been proposed to solve high order partial differential equations (see, e.g., \([27, 48]\)). The main purpose of this work is to design linear FEMs for problems \(1.1\). A challenge of applying low degree elements is in the calculation of second order derivatives of the linear finite element approximation. To overcome this difficulty, we adopt a gradient recovery operator \(G_h\) (see, e.g., \([2, 3, 4]\)) to lift the discontinuous piecewise constant \(\nabla v_h\) to a continuous piecewise linear function \(G_h v_h\), such that differentiation \(DG_h v_h\) is possible. Alternatively, we can apply Hessian recovery operators \(H_h\) (see, e.g., \([25]\)) to discretize the second order differential operator \(D^2\) directly. Once the recovery operators \(DG_h\) or \(H_h\) have been constructed, numerical schemes for \(1.1\) can be designed by applying least-squares weak formulation \(1.9\). We shall remark that the difference operator \(DG_h\) is asymmetric in general. Consequently, a direct application of \(DG_h\) to \(1.9\) may lead to an instable numerical scheme. To ensure stability, the rotation of the recovery gradient may be included as a penalty in the scheme.

Comparing to other techniques for the non-divergence form \(1.1\), the proposed methods have two advantages. First, linear elements induce fewer DOFs in comparison to \(C^1\) and/or mixed elements, which hence leads to more convenient implementation and less cost in computation. For example, in Table 1.1, we demonstrate local and global DOFs from different methods for a benchmark problem on a square domain and a uniform mesh with \(2N^2\) triangles. In particular, the DG method in \([14]\), the MFEM in \([22]\), the WG method in \([47]\), and the internal penalty FEM (IPFEM) in \([38]\) are included in the Table. The total DOFs of the GRBL and HRBL FEMs are both \((N + 1)^2\), which are the smallest in these methods. Second, the recovery operators \(G_h\) and \(H_h\) can be defined on a general unstructured grid. Thus the numerical algorithm can be applied for problems on domains with arbitrary geometries. Meanwhile, our proposed numerical schemes have nice convergence properties. Under the assumption of \(H^3(\Omega)\) regularity of the exact solution, it is observed that numerical errors measured in \(L^2\) and \(H^1\) norms converge optimally on unstructured grids, which are of second and first orders, respectively. These convergence rates are competitive to the rates of other methods in the literature. For example, for both the WG method in \([47]\) and the internal penalty method in \([38]\), the convergence rates of numerical errors in \(L^2\) norm are of second order when quadratic elements are used, which are not optimal. On the other hand, superconvergence phenomena are also observed in our numerical experiments. In particular, numerical errors of the HRBL have a convergence rate of 1.5 when they are measured in the \(H^2\) seminorm. The recovered gradient of both schemes converges in second-order. In addition, numerical experiments show that, when inexact approximations of curved boundaries are employed, the proposed methods capture optimal second order convergence rate as well. Even for less smooth solutions with only \(H^{2+r}(\tau > 0)\) regularity, the proposed schemes using linear elements can still achieve same convergence rates as those obtained from the WG method using quadratic elements. Finally, as an application, we have applied the recovery based linear FEM to solve fully nonlinear Monge-Ampère equations. A convex solution with optimal convergence rates is obtained.
In this paper, theoretical investigation for the GRBL scheme has been developed. Error estimation in the $H^2$ seminorm converges with linear convergence order has been given under the assumption of $H^3$ regularity of the exact solution and sufficient regularity of the grid. Moreover, in the special case that $A = \alpha I$, we can prove superconvergence of the recovered gradient by applying the Aubin-Nitsche technique. Consequently, an optimal $L^2$ error estimate can be obtained by the discrete Poincaré inequality.

This paper is organized as follows. In Section 2, notations and some preliminary results on the gradient and Hessian recovery operators are introduced. In Section 3, we first introduce the GRBL and HRBL FEMs for problem (1.1). In section 4, the stability of the GRBL FEM is proven. Moreover, some optimal error estimates, including estimations in $H^2$ seminorm, recovered gradient seminorm, and $L^2$ norm, are established. In Section 5, we introduce an application of the proposed method to fully nonlinear Monge-Ampère equations. In Section 6, some typical (including benchmark) numerical experiments are presented to demonstrate the effectiveness of the new numerical methods.

2. Preliminary results. In this paper, we use $C$ to denote a generic positive constant independent of data of the PDE and mesh size, which may be different at different occurrences. For convenience, we write $x \lesssim y$ provided $x \leq Cy$ for some constants $C$, and $x \sim y$ if both $x \lesssim y$ and $y \lesssim x$ hold. Standard definitions and notations for Sobolev spaces are used. In particular, $(\cdot, \cdot)$ is the $L^2$-inner product, and $\| \cdot \|_i$ and $| \cdot |_i$ are the norm and seminorm in $H^i(\Omega)$, respectively.

For simplicity of presentation, we focus our attention on the two-dimensional case. Let $T_h$ be a regular triangulation of the domain $\Omega$ with mesh-size $h$. We use $N_h$ to denote the set of vertices of $T_h$. We denote by $V_h$ the standard $C^0$ linear finite element space associated with $T_h$ and define $V_h^0 = \{ v_h \in V_h : v_h|_{\partial \Omega} = 0 \}$. For each vertex $z$ in the triangulation, we define the element patch and control volume of $z$ as $\omega_z = \bigcup \{ \tau \in T_h : z \in \bar{\tau} \}$ and $V_z$ (see Figure 2.1).

![Fig. 2.1. (a) local patch $\omega_z$, and (b) control volume $V_z$ on uniform mesh.](image)

2.1. Calculation of first derivatives. In this subsection, we introduce a reconstructed gradient $G_h v_h$ which is an improvement of a piecewise constant function $\nabla v_h$. We shall define a gradient reconstruction operator $G_h : V_h \to V_h \times V_h$. We first determine the value of $G_h v_h$ at all vertices, and then obtain the gradient function by interpolation over the whole domain, namely,

$$G_h v_h(x, y) = \sum_{z \in N_h} G_h v_h(z) \phi_z(x, y),$$

where $\phi_z$ is the linear nodal shape function of vertex $z$. There are three popular ways to determine $G_h v_h(z)$ at a vertex $z \in N_h$, which are specified in below.

(a) Weighted average: define

$$G_h v_h(z) = \frac{1}{|\omega_z|} \int_{\omega_z} \nabla v_h \, dx dy,$$  \hspace{1cm} (2.1)
(b) Recovery techniques: using a local discrete least-squares fitting operator to smooth the gradient. The ZZ approach proposed by Zienkiewicz and Zhu [51] and polynomial preserving recovery (PPR) proposed by Naga and Zhang [52] are frequently used operators in post-processing technology. Specifically, they are defined as follows.

**ZZ:** seeking two linear polynomials \( p_l \in P_1(\omega_z) \) satisfying

\[
\sum_{i=1}^{m} [p_l(x_i,y_i) - \partial_l v_h(x_i,y_i)] q(x_i,y_i) = 0, \quad \forall q \in P_1(\omega_z),
\]

where \( l = x \) or \( y \), and \((x_i,y_i), i = 1,2, \cdots, m,\) are \( m \) given points in \( \omega_z \). Then the nodal value of \( G_h v_h \) can be defined as

\[
G_h v_h(z) = (p_x(z), p_y(z)).
\]

**PPR:** seeking a quadratic function \( p \in P_2(\omega_z) \), such that

\[
\sum_{i=1}^{m} [p(x_i,y_i) - v_h(x_i,y_i)] q(x_i,y_i) = 0, \quad \forall q \in P_2(\omega_z).
\]

Then the nodal value of \( G_h v_h \) can be defined as

\[
G_h v_h(z) = (\partial_x p(z), \partial_y p(z)).
\]

(c) Green’s formula: the determination of \( G_h v_h(z) \) proceeds with the help of the Green’s formula,

\[
\int_{V_z} \partial_i v d\mathbf{x} = \int_{\partial V_z} v n_i d\mathbf{s}.
\]

Here \( n_i \) is the \( i \)th component of the unit outward-pointing normal \( \mathbf{n} \). Then the nodal value of \( G_h v_h \) can be defined as

\[
G_h v_h(z) = \frac{1}{|V_z|} \left( \int_{\partial V_z} v_h n_1 d\mathbf{s}, \int_{\partial V_z} v_h n_2 d\mathbf{s} \right).
\]

**Remark 2.1.** The three definitions above are equivalent on a uniform triangular mesh.

We next review some properties of the gradient recovery operator. In particular, properties (2.6) and (2.7) are always valid on general grids.

(a) Boundedness (cf., e.g., [39, 51, 52]):

\[
\|G_h v_h\|_0 \lesssim |v_h|_1, \quad \forall v_h \in V_h.
\]

(b) Consistency (cf., e.g., [24, 52]):

\[
\|\nabla u - G_h u_I\|_0 \lesssim h^2 \|u\|_3, \quad \forall u \in H^3(\Omega),
\]

where \( u_I \) is the linear interpolation of \( u \) in \( V_h \).

Throughout the rest of this paper, we assume that the mesh \( T_h \) is sufficiently regular such that the following discrete Poincaré inequality holds:

\[
\|v_h\|_i \lesssim \|G_h v_h\|_i, \quad \forall v_h \in V_h^0, \quad i = 0, 1.
\]

**Remark 2.2.** In [27], the authors proved that (2.8) is valid for some uniform meshes. Moreover, numerical results indicate that (2.8) holds for some weakly regular grids.

**Remark 2.3.** On the boundary of a domain \( \Omega \), modification in the gradient recovery operator \( G_h \) is necessary to maintain the superconvergence property (2.7), as long as numerical data nearby are available. For more details, we refer to [26].
2.2. Calculation of the second derivatives. It is impossible to calculate the second derivatives of a linear finite element function directly since its gradient is piecewise constant and discontinuous across element boundaries. To overcome this difficulty, we introduce some techniques for approximating the second derivatives of linear elements in this subsection.

The first technique for approximating Hessian is derived from gradient reconstruction techniques. As $G_h v_h$ is continuous piecewise linear, hence further differentiation $DG_h v_h$ is possible. Therefore, the Hessian matrix of a linear function can be approximated as follows:

$$DG_h v_h = \begin{pmatrix} \frac{\partial^2 G_h^x}{\partial x^2} v_h & \frac{\partial^2 G_h^y}{\partial x \partial y} v_h \\ \frac{\partial^2 G_h^x}{\partial y \partial x} v_h & \frac{\partial^2 G_h^y}{\partial y^2} v_h \end{pmatrix}. \quad (2.9)$$

Note that $DG_h v_h$ is piecewise constant. Therefore, $DG_h v_h$ is not well defined on the common side of two elements.

The second technique for approximating Hessian overcomes this difficulty. The basic idea of a Hessian reconstruction operator $H_h : V_h \rightarrow V_h^2 \times V_h^2$ is either applying the gradient recovery operator twice or directly computing the second derivative of quadratic polynomial $p$ in (2.3). That is, the nodal value of the reconstructed Hessian is determined by:

$$H_{h} v_h (z) = \begin{pmatrix} H^x_{h} v_h (z) \\ H^y_{h} v_h (z) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 p}{\partial x^2} (z) \\ \frac{\partial^2 p}{\partial x \partial y} (z) \\ \frac{\partial^2 p}{\partial y \partial x} (z) \\ \frac{\partial^2 p}{\partial y^2} (z) \end{pmatrix}. \quad (2.11)$$

By interpolating the whole region, we obtain

$$H_h v_h = \sum_{z \in N_h} H_{h} v_h (z) \phi_z.$$

The third approach for approximating Hessian is to apply Green’s formula, namely,

$$\int_{V_z} \partial_{ij} v d\mathbf{x} = \int_{\partial V_z} (\partial_i v) n_j ds.$$

Then the nodal value of $H_{h}^{ij} v_h$ can be defined by

$$H_{h}^{ij} v_h (z) = \frac{1}{|V_z|} \int_{\partial V_z} (\partial_i v) n_j ds, \ i,j = x,y. \quad (2.12)$$

**Remark 2.4.** We shall remark that the approximated Hessian derived from (2.11) satisfies the symmetric property $H_{h}^{xy} = H_{h}^{yx}$. Moreover, the discrete Laplace operator defined by $H_{h}^{xx} + H_{h}^{yy}$ on regular pattern uniform grids is the well-known five-point finite difference scheme.

3. The Recovery Based Linear Finite Element Methods.

3.1. Algorithm. Recall the symmetric weak formulation (1.9) of the problem (1.1). We introduce a bilinear form

$$a(v, w) = \int_{\Omega} (A : D^2 v) \cdot (A : D^2 w), \ \forall \ v, w \in H^2(\Omega). \quad (3.1)$$

The formulation (1.9) can be written as: Finding $u \in H^2(\Omega)$ such that

$$a(u, v) = (f, A : D^2 v), \ \forall \ v \in H^2(\Omega), \quad (3.2)$$
which allows naturally the application of $C^1$ finite elements. However, due to the inconvenience of construction and implementation of $C^1$ elements, alternative discretization approaches of \cite{19} are desirable. In \cite{22}, the author designed a mixed FEM in the framework of saddle-point problems.

In this paper, we shall discretize \eqref{1.9} in a recovered FEM. The essential issue is the calculation of second derivatives $D^2$. Define the function space

$$V_h^{0,0} = \{ v_h \in V_h^0 : \text{the tangential trace of } G_h v_h \text{ on } \partial \Omega \text{ vanishes} \}.$$ 

For any given function $v_h \in V_h^{0,0}$, there are two approaches to approximate the Hessian matrix of $v_h$. The first approach uses the gradient recovery operator $G_h$ to obtain a continuous piecewise linear function $G_h v_h = (G_h^x v_h, G_h^y v_h)$, then the Hessian matrix of $v_h$ can be approximated by differentiating $G_h v_h$. In this paper, we utilize the PPR technique \cite{23} to produce the $G_h$. Correspondingly, for all $v_h, w_h \in V_h$, we define a bilinear form as:

$$a_{1,h}(v_h, w_h) = \sum_{T \in T_h} \int_T \left( (A : DG_h v_h) : (A : DG_h w_h) + (\text{rot} G_h v_h) : (\text{rot} G_h w_h) \right) dx dy, \quad (3.3)$$

where the rotation for a two-dimensional $H^1$ vector $v = (v_1, v_2)$ is defined as

$$\text{rot} v = \partial_y v_1 - \partial_x v_2.$$ 

We propose the first numerical method in below. **Scheme 1**: The gradient recovery based linear (GRBL) FEM for approximation of \eqref{1.1} is to find $u_h \in V_h^{0,0}$ such that

$$a_{1,h}(u_h, v_h) = (f, A : DG_h v_h), \quad \forall v_h \in V_h. \quad (3.4)$$

Note that the second term of the bilinear form $a_{1,h}(\cdot, \cdot)$ is a penalty term to ensure the stability of the method. We accordingly define an energy seminorm as

$$\| v_h \|^2_{a,1} = a_{1,h}(v_h, v_h) = \| A : DG_h v_h \|^2_0 + \| \text{rot} G_h v_h \|^2_0, \quad \forall v_h \in V_h^0.$$ 

Another approach of approximating the Hessian matrix is the direct application of the Hessian recovery operator. Here, we use \cite{2.11} to obtain operator $H_h$. We can thus alternatively define a bilinear form as:

$$a_{2,h}(v_h, w_h) = \sum_{T \in T_h} \int_T \left( (A : H_h v_h) : (A : H_h w_h) \right) dx dy. \quad (3.5)$$

The second numerical method is proposed in below. **Scheme 2**: The Hessian recovery based linear (HRBL) FEM for approximation of \eqref{1.1} is to find $u_h \in V_h^{0,0}$ such that

$$a_{2,h}(u_h, v_h) = (f, A : H_h v_h), \quad \forall v_h \in V_h. \quad (3.6)$$

We shall remark that, since the finite element space $V_h$ is not in $C^1$, numerical schemes \eqref{3.4} and \eqref{3.6} are both nonconforming methods.

**3.2. Stability of scheme** (3.4). Let the space of $H^1$ vector fields with vanishing tangential trace be

$$W = \{ v \in H^1(\Omega, \mathbb{R}^2) : \text{the tangential trace of } v \text{ on } \partial \Omega \text{ vanishes} \}.$$ 

It is useful to note that, on convex domains, the following estimate holds \cite{12} \[3.3\]

$$\| D w \|^2_0 \leq \| \text{rot} w \|^2_0 + \| \text{div} w \|^2_0, \quad \forall w \in W. \quad (3.7)$$
Following the strategy proposed in [45], we define the function $\gamma$ by

$$\gamma = \text{tr}(A)/|A|^2.$$ 

We have the following lemma.

**Lemma 3.1.** There exists a positive constant $c$ independent of $h$, such that for any $v_h \in V_h^{0,0}$, it holds

$$\|\gamma A : DG_h v_h\|_0^2 + \|\text{rot } G_h v_h\|_0^2 \geq c\|DG_h v_h\|_0^2. \tag{3.8}$$

**Proof.** From [22, 45], it holds that for any $B \in \mathbb{R}^{2 \times 2}$

$$|\gamma A - I : B| = |\gamma A : B - \text{tr}(B)| \leq \sqrt{1-\epsilon}|B|.$$ \tag{3.9}

By triangle inequality, one gets

$$|\gamma A : B| \geq \|\text{tr}(B)| - |\gamma A - I : B||. \tag{3.10}$$

We take $B = DG_h v_h$, the combination of (3.8), (3.9), and the Young inequality results in

$$\|\gamma A : DG_h v_h\|_0^2 \geq \|\text{tr}(B)| - |\gamma A - I : B||_0^2 = \|\text{div}(G_h v_h)\|_0^2 + |\gamma A - I : DG_h v_h|_0^2 - 2|\text{div}(G_h v_h)\|_0^2 \|\gamma A - I : DG_h v_h\|_0^2$$

$$\geq (1 - \beta)|\text{div}(G_h v_h)\|_0^2 + \left(1 - \frac{1}{\beta}\right)(1 - \epsilon)|\text{tr}(BG_h)\|_0^2 \beta|\text{rot } G_h v_h\|_0^2$$

$$\geq (1 - \beta)|\text{div}(G_h v_h)\|_0^2 + \left(1 - \frac{1}{\beta}\right)(1 - \epsilon)\|DG_h v_h\|_0^2.$$ \tag{3.11}

In the last inequality, a constant $0 < \beta < 1$ is required.

Since $G_h v_h \in W$, by (3.7) and (3.11), a simple calculation shows that

$$\|\gamma A : DG_h v_h\|_0^2 + \|\text{rot } G_h v_h\|_0^2$$

$$\geq (1 - \beta)|DG_h v_h|_0^2 + \left(1 - \frac{1}{\beta}\right)(1 - \epsilon)|DG_h v_h|_0^2 + \beta|\text{rot } G_h v_h|_0^2$$

$$\geq \left(1 - \beta + \left(1 - \frac{1}{\beta}\right)(1 - \epsilon)\right)\|DG_h v_h\|_0^2$$

$$= c(\epsilon, \beta)\|DG_h v_h\|_0^2.$$ 

Here $\beta$ is chosen to satisfy $1 - \epsilon < \beta < 1$, so that $c(\epsilon, \beta) > 0$. This completes the proof. \[\]

The proposed algorithm is well-posed.

**Theorem 3.1.** If the mesh is sufficiently regular such that (2.8) holds, then scheme (3.4) is uniquely solvable.

**Proof.** The lower bound (3.8) yields that

$$\max\left\{\|\gamma\|_{L^\infty(\Omega)}, 1\right\}\|v_{h,1}\|_{a,1}^2 \geq \|\gamma A : DG_h v_h\|_0^2 + \|\text{rot } G_h v_h\|_0^2 \geq c(\epsilon, \beta)\|DG_h v_h\|_0^2.$$ \tag{3.12}

which implies that

$$\|v_{h,1}\|_{a,1} \geq d(\gamma, \epsilon, \beta)\|DG_h v_h\|_0.$$ \tag{3.13}

Here $d(\gamma, \epsilon, \beta) = c(\epsilon, \beta)^{1/2}/\max\left\{\|\gamma\|_{L^\infty(\Omega)}, 1\right\}$ is independent of mesh size $h$.

On the other hand, we have

$$\|v_{h,1}\|_{a,1} \leq (\|A\|_{\infty}^2 + 2)^{1/2}\|D(G_h u_h)\|_0.$$ \tag{3.14}

Therefore, the seminorm $\|v_{h,1}\|_{a,1}$ is equivalent to $\|D(G_h u_h)\|_0$ in $V_h^{0,0}$. In [27], it has been proven that $\|D(G_h u_h)\|_0$ is a norm in $V_h^{0,0}$, which implies $\|u_{h,1}\|_{a,1}$ is a norm in $V_h^{0,0}$. By the Lax-Milgram theorem, there exists a unique solution to (3.4) in $V_h^{0,0}$. \[\]
4. Error Estimates. In this section, we first develop error estimation in the $H^2$ norm. Then, we establish the $H^1$ and $L^2$ error bounds in a special case.

4.1. Error Estimate in $H^2$ Norm. We have an optimal error estimate.

**Lemma 4.1.** Let $u \in H^{2+\alpha}(\Omega)$ with $0 < \alpha < 1$, then the following inequality holds true for any general grids,

$$\|D^2 u - DG_h u_I\|_0 \lesssim h^\alpha \|u\|_{2+\alpha}. \tag{4.1}$$

Consequently, it follows that

$$\|\text{rot} \, G_h u_I\|_0 \lesssim h^\alpha \|u\|_{2+\alpha}. \tag{4.2}$$

Here $u_I$ is the linear interpolation of $u$.

**Proof.** By the triangle inequality and the inverse inequality, we have, $\forall \, w_h \in V_h \times V_h$,

$$\|D^2 u - DG_h u_I\|_0 \leq \|D^2 u - Dw_h\|_0 + \|Dw_h - DG_h u_I\|_0 \lesssim \|D^2 u - Dw_h\|_0 + h^{-1} \|w_h - G_h u_I\|_0.$$  

Setting $w_h = (\text{rot} \, u_I)$, then $\|D^2 u - Dw_h\|_0 \lesssim h^\alpha \|u\|_{2+\alpha}$. Using (2.7) and standard estimates of linear interpolation, we derive that

$$h^{-1} \|(\text{rot} \, u_I) - G_h u_I\|_0 \lesssim h^{-1} \|(\text{rot} \, u_I) - \text{rot} \, u\|_0 + \|\text{rot} \, u - G_h u_I\|_0 \lesssim h^\alpha \|u\|_{2+\alpha}. \tag{4.1}$$

Hence,

$$\|D^2 u - DG_h u_I\|_0 \lesssim h^\alpha \|u\|_{2+\alpha}.$$  

Notice that,

$$\|\text{rot} \, G_h u_I\|_0 = \|\text{rot} \, G_h u_I - \text{rot} \, u\|_0 \leq \|D^2 u - DG_h u_I\|_0,$$

we get (4.2) immediately. \Box

**Theorem 4.1.** Let $u$ and $u_h$ be solutions of (1.1) and (3.4), respectively. If $u \in H^{2+\alpha}(\Omega)$, then for any general grids, there holds

$$\|u_h - u_I\|_{a,1} \lesssim h^\alpha \|u\|_{2+\alpha}. \tag{4.3}$$

Consequently, we have

$$\|D^2 u - DG_h u_h\|_0 \lesssim h^\alpha \|u\|_{2+\alpha}. \tag{4.4}$$

**Proof.** As $u$ solves (1.1) strongly in $L^2(\Omega)$, it holds that

$$\|u_h - u_I\|_{a,1}^2 = a_{1,h}(u_h - u_I, u_h - u_I) \nonumber$$

$$= a_{1,h}(u_h, u_h - u_I) - a_{1,h}(u_I, u_h - u_I) \nonumber$$

$$= (f, A : DG_h(u_h - u_I)) - a_{1,h}(u_I, u_h - u_I) \nonumber$$

$$= (A : (D^2 u - DG_h u_I), A : DG_h(u_h - u_I)) - (\text{rot} \, G_h u_I, \text{rot} \, G_h(u_h - u_I)).$$

Using the Cauchy-Schwarz inequality and the triangle inequality, it yields that

$$\|u_h - u_I\|_{a,1}^2 \leq \|A\|_\infty \|D^2 u - DG_h u_I\|_0 \cdot \|A : DG_h(u_h - u_I)\|_0 \nonumber$$

$$+ \|\text{rot} \, G_h u_I\|_0 \cdot \|\text{rot} \, G_h(u_h - u_I)\| \nonumber$$

$$\lesssim \|u_h - u_I\|_{a,1} \cdot (\|D^2 u - DG_h u_I\|_0 + \|\text{rot} \, G_h u_I\|_0).$$
Dividing \( \| u_h - u_I \|_a \) on both sides, we have

\[
\| u_h - u_I \|_{a,1} \lesssim \| D^2 u - DG_h u_I \|_0 + \| \text{rot} G_h u_I \|_0.
\]  

(4.5)

Plugging (4.1) and (4.2) into (4.5), we obtain the estimate (4.3).

Combining (3.12) and (4.3), we derive that

\[
\| DG_h (u_h - u_I) \|_0 \lesssim h^\alpha \| u \|_{2+\alpha}.
\]  

(4.6)

The estimate (4.4) is a direct consequence of (4.1) and (4.6).

**Remark 4.1.** By (4.6), we have

\[
\| \text{rot} G_h (u_h - u_I) \|_0 \lesssim h^\alpha \| u \|_{2+\alpha}.
\]  

Combining the above estimate with (4.2), we obtain that

\[
\| \text{rot} G_h u_h \|_0 \lesssim h^\alpha \| u \|_{2+\alpha}.
\]  

(4.7)

**Remark 4.2.** We observe that the proof of convergence and stability analysis for the scheme (3.4) is not applicable to the scheme (3.6). Nonetheless, numerical results confirm the robustness and the optimal convergence of the scheme (3.6).

### 4.2. Error Estimates in \( H^1 \) and \( L^2 \) Norms.

In this subsection, we establish error bounds in a special case of \( A = \alpha I \), where \( \alpha \) is a constant. We first apply the Aubin-Nitsche technique to estimate the \( H^1 \) error. To this end, we introduce the following auxiliary problem. For \( \phi \in H^1_0(\Omega) \), let \( \partial \Omega \) be \( C^3 \) and \( U_\phi \in H^3_0(\Omega) \) be a weak solution of the following equation:

\[
\begin{align*}
-\alpha^2 \Delta u &= \phi & \text{in} & \quad \Omega, \\
 u &= 0 & \text{on} & \quad \partial \Omega.
\end{align*}
\]

From [15, 6.3, Theorem 5], we have \( U_\phi \in H^3(\Omega) \) and

\[
\| U_\phi \|_3 \lesssim \| \phi \|_1.
\]  

(4.8)

By applying Green's formula, we derive that

\[
(\alpha^2 \Delta U_\phi, \text{div} \, \mathbf{v}) = (\mathbf{v}, \nabla \phi), \quad \forall \ \mathbf{v} \in (H^1(\Omega))^2.
\]  

(4.9)

The following estimates hold.

**Theorem 4.2.** Let \( u \) and \( u_h \) be solutions of (1.1) and (3.4), respectively. If \( u \in H^3(\Omega) \), then there holds

\[
\| G_h u_h - G_h u_I \|_0 \lesssim h^2 \| u \|_3.
\]  

(4.10)

consequently,

\[
\| \nabla u - G_h u_h \|_0 \lesssim h^2 \| u \|_3.
\]  

(4.11)
Proof. For any $\phi \in H^3_0$, we have
\[
(G_h(u_h - u_I), \nabla \phi)
= (\text{div} G_h(u_h - u_I), \alpha^2 \Delta U_\phi)
= (\text{div} G_h u_h - \alpha \Delta u, \alpha \Delta U_\phi - \text{div} G_h(U_\phi)) + (\alpha^2 \Delta U_\phi, \Delta u - \text{div} G_h u_I) + (\text{div} G_h u_h - \alpha \Delta u, \text{div} G_h(U_\phi))
= (\text{div} G_h u_h, \text{div} G_h(U_\phi)) - (\alpha \Delta u, \text{div} G_h(U_\phi)) + (\text{div} G_h u_h, \text{div} G_h(U_\phi)) - (f, \text{div} G_h(U_\phi))
= (\alpha(\text{div} G_h u_h - \Delta u), \alpha(\Delta U_\phi - \text{div} G_h(U_\phi)) - (\nabla \phi, \nabla u - G_h u_I) + (\text{div} G_h u_h, \text{div} G_h(U_\phi)) - (\text{rot} G_h u_h, \text{rot} G_h(U_\phi))
= (\alpha(\text{div} G_h u_h - \Delta u), \alpha(\Delta U_\phi - \text{div} G_h(U_\phi)) - (\nabla \phi, \nabla u - G_h u_I) - (\text{rot} G_h u_h, \text{rot} G_h(U_\phi))
\leq h^2\|u\|^3(\|U_\phi\|^4 + \|\phi\|_1)
\] (4.12)
where we have used the estimates \cite{2.7, 4.1, 4.4, and 4.7} in the last inequality. Combining 4.8 and 4.12, we derive the desired estimation (4.10). The estimate 4.11 is a direct consequence of (2.7) and (4.10). \hfill \Box

Remark 4.3. For the case of $A = \text{diag}(\alpha_1, \alpha_2)$, where $\alpha_1$ and $\alpha_2$ are constants with same sign, we have
\[
A : \mathcal{D}^2 u = \alpha_1 u_{xx} + \alpha_2 u_{yy}.
\]
By using variable substitution $x = \frac{\alpha_2}{\alpha_1} s$, $y = t$, we obtain that
\[
\begin{align*}
  u_{xx} &= \frac{\alpha_2}{\alpha_1} u_{ss},
  u_{yy} &= u_{tt},
\end{align*}
\]
therefore $A : \mathcal{D}^2 u = \alpha_2(u_{xx} + u_{tt})$, indicating that the problem can be reduced to the situation $A = \alpha_2 I$.

Theorem 4.3. Let $u$ and $u_h$ be solutions of \cite{1.1} and \cite{3.4}, respectively. If the mesh is sufficiently regular such that \cite{2.8} holds and $u \in H^3(\Omega)$, then there holds
\[
\|u_h - u\|_0 \lesssim h^2\|u\|_3;
\] (4.13)

Proof. By discrete Poincaré inequality \cite{2.8}, we have
\[
\|u_h - u_I\|_0 \lesssim \|G_h(u_h - u_I)\|_0 \lesssim h^2\|u\|_3.
\]
Applying triangle inequality, we obtain
\[
\|u - u_h\|_0 \leq \|u - u_I\|_0 + \|u_I - u_h\|_0 \lesssim h^2\|u\|_3.
\]
This completes the proof. \hfill \Box

Remark 4.4. The $H^1$ and $L^2$ error estimates are difficult for non-divergence elliptic equations in general, for which we have not yet found a theoretical proof in this article. Nonetheless, the $O(h^2)$ order of convergence of GRBL and HRBL finite element schemes in $L^2$ norm can be confirmed by numerical experiments, even for problems with non-smooth and discontinuous coefficients.

5. Application to the Monge-Ampère Equation. In this section, we apply the recovered linear element method to solve the fully nonlinear Monge-Ampère equation
\[
\begin{align*}
  \det(D^2 u) &= f & \text{in} & \Omega, \\
  u &= g & \text{on} & \partial \Omega,
\end{align*}
\] (5.1)
where $\Omega \subseteq \mathbb{R}^2$, and $D^2 u$ is the Hessian of the function $u$. The Monge-Ampère equation arises naturally from differential geometry and has widely applications in applied science such as mass transportation meteorology and geostrophic fluid dynamics.

If $f > 0$, $\Omega$ and $u$ are convex, and $D^2 u$ is positive definite, then problem (5.1) admits a unique solution. Numerical approximation of the Monge-Ampère equation is very challenging. Some numerical schemes in finite difference methods and/or FEMs have been designed for fully nonlinear equations in recent years, see, e.g., [5],[8],[10],[20],[21],[30],[31],[32],[33] and the references therein.

In this paper, we first use the efficient Newton’s technology to linearize the Monge-Ampère equation. Given $u_0 \in V$, let $\{u_k\}_{k=1}^\infty \in V$ be a sequence, such that

$$
\begin{aligned}
\text{cof}(D^2 u^{k-1}) : D^2 u^k &= f + \det(D^2 u^{k-1}) \quad \text{in} \quad \Omega, \\
u^k &= g \quad \text{on} \quad \partial \Omega,
\end{aligned}
$$

(5.2)

where the cofactor matrix of the Hessian $D^2 u$ is defined as follows:

$$
\text{cof} (D^2 u) = \begin{pmatrix} u_{yy} & -u_{yx} \\ -u_{xy} & u_{yy} \end{pmatrix}.
$$

For more details about the Newton’s method, we refer to [35],[36],[37]. It has been proved in [36],[37],[44] that each iteration $u^k$ in the continuous Newton’s scheme (5.2) will be convex provided that the initial guess is strictly convex. From [13],[36], a reasonable initial guess data is the solution of

$$
\begin{aligned}
\Delta u_0^0 &= 2\sqrt{f} \quad \text{in} \quad \Omega, \\
u_0^0 &= g \quad \text{on} \quad \partial \Omega.
\end{aligned}
$$

(5.3)

Clearly, problem (5.2) is an elliptic equation in non-divergence form. Define

$$
V^g_h = \{ v_h \in V_h : v_h|_{\partial \Omega} = g \}.
$$

The gradient recovery linear element method for solving (5.2) is to find $\{u_h^k\}_{k=1}^\infty \in V^g_h$ such that, $\forall v_h \in V_h$,

$$
\begin{aligned}
(\text{cof}(DG_h u_h^{k-1}) : DG_h u_h^k, \text{cof}(DG_h u_h^{k-1}) : DG_h v_h) + \sigma(\text{rot}(G_h u_h^k), \text{rot}(G_h v_h)) \\
= (f + \det(DG_h u_h^{k-1}), \text{cof}(DG_h u_h^{k-1}) : DG_h v_h),
\end{aligned}
$$

(5.4)

where $D^2 u_h$ is the weak Hessian of the linear function $u_h$ and $\sigma > 0$ is a penalty parameter.

We will exhibit the robustness and convergence of the proposed algorithm in Section 6.2 by using numerical examples. We observe that the solution of the scheme (5.4) can converge to a convex solution.

6. Numerical Experiments. In this section, we present numerical results for some representative examples to confirm our theoretical findings. In all examples, uniform meshes are used. We apply PPR (2.4) and (2.11) to generate $G_h u_h$ and $H_h u_h$ for HRBL and GRBL FEMs, respectively. We shall examine several numerical errors, which will be denoted in the following notations:

- $L^2$ norm: $\|e\|_0 = \|u - u_h\|_{0, \Omega},$
- $H^1$ seminorm: $|e|_1 = |u - u_h|_{1, \Omega},$
- Recovered $H^1$ seminorm: $|e|_{1,r} = \|\nabla u - G_h u_h\|_{0, \Omega},$
- $H^2$ seminorm: $|e|_2 = \|D^2 u - D^2 u_h\|_{0, \Omega}.$

where $D^2 u_h = DG_h u_h$ for the scheme (3.4) and $D^2 u_h = H_h u_h$ for the scheme (3.6).
6.1. Numerical Experiments of Non-divergence Form Elliptic Equations. We first consider four examples of second-order linear elliptic PDEs, including examples with non-smooth and/or discontinuous coefficients over convex domains (cf. Figure 6.1) or $L$-shaped domain.

**Example 6.1. A problem with non-smooth coefficients.** We consider numerical approximation for the problem (1.1) with exact solution $u = \sin x \sin y$. The coefficient function is

$$A = \begin{pmatrix}
1 + |x| & 0.5|x||y|^{1/3} \\
0.5|x||y|^{1/3} & 1 + |y|
\end{pmatrix},$$

which is continuous. This elliptic problem cannot be written in a divergence form, because the two off diagonal entries have a singularity at the origin.

| $1/h$ | $\|e\|_0$ order | $|e|_1$ order | $|e|_{1,r}$ order | $|e|_2$ order | $\|e\|_1$ order | $|e|_{1,r}$ order | $|e|_2$ order |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 16    | 1.99E-3        | 1.09E-1        | 1.04E-2        | 1.15E-2        | 2.10E-3        | 2.01E-3        | 5.63E-2        |
| 32    | 5.57E-4        | 2.03E-2        | 2.56E-3        | 2.01E-3        | 5.63E-2        | 1.01E-1        |
| 64    | 1.27E-4        | 2.01E-3        | 6.35E-4        | 2.01E-3        | 2.79E-2        | 1.01E-1        |
| 128   | 3.75E-5        | 2.00E-3        | 1.58E-4        | 2.01E-3        | 1.39E-2        | 1.00E-1        |
| 256   | 8.27E-6        | 2.00E-3        | 3.94E-5        | 2.00E-3        | 6.96E-3        | 1.00E-1        |
| 512   | 2.06E-6        | 2.00E-3        | 3.82E-6        | 2.00E-3        | 3.48E-3        | 1.00E-1        |

**Table 6.1** Example 6.1 – Numerical results of the scheme (3.4) on uniform meshes

| $n$ | $\|e\|_0$ order | $|e|_1$ order | $|e|_{1,r}$ order | $|e|_2$ order |
|-----|----------------|----------------|----------------|----------------|
| 2   | 1.00E-3        | 6.17E-2        | 5.28E-2        | 6.80E-1        |
| 3   | 2.46E-4        | 3.08E-2        | 1.28E-3        | 3.29E-1        |
| 4   | 6.07E-5        | 1.54E-2        | 3.11E-4        | 1.61E-1        |
| 5   | 1.50E-5        | 7.69E-3        | 7.68E-5        | 8.00E-2        |

**Table 6.2** Example 6.1 – Numerical results of the scheme (3.4) on nonuniform meshes

Numerical results of the recovered linear finite element approximation for problem (1.1) on a square $\Omega = (-1,1)^2$, an $L$-shaped domain $\Omega = (-1,1)^2 \setminus (0,1)^2$ and a circular domain $\Omega = \{(x,y)|x^2 + y^2 \leq 1\}$ are collected in Tables 6.1-6.8. For the GRBL FEM (3.4) on both uniform and unstructured meshes (see Figure 6.1), the convergence orders of numerical errors in $L^2$- and $H^1$- norms are $O(h^2)$ and $O(h)$, respectively, which are both optimal. Superconvergence phenomenon is also observed. In particular, $DG_hu_h$ and $G_hu_h$ converge to the $D^2 u$
and $\nabla u$ with convergence rates $O(h)$ and $O(h^2)$, respectively. As for the HRBL FEM (3.6), the convergence rates of numerical errors in $L^2$, $H^1$, and recovered gradient norms are the same as those from the scheme (3.4); but the convergence rate in $H^2$-seminorm is $O(h^{1.5})$, which is half order higher than that of the first scheme 

| $1/h$ | $\|e\|_0$ | order | $\|e\|_1$ | order | $\|e\|_{1,r}$ | order | $\|e\|_2$ | order |
|-------|--------|--------|--------|--------|--------------|--------|--------|--------|
| 16    | 2.20E-3 | 1.09E-1 | 1.05E-2 | 6.56E-2 |
| 32    | 5.47E-4 | 2.03    | 5.43E-1 | 1.05E-2 | 2.44E-2 | 1.49 |
| 64    | 1.36E-4 | 2.01    | 2.72E-2 | 1.00    | 6.43E-4 | 2.01 | 8.33E-3 | 1.49 |
| 128   | 3.39E-5 | 2.00    | 1.36E-2 | 1.00    | 1.60E-4 | 2.01 | 2.95E-3 | 1.49 |
| 256   | 8.45E-6 | 2.00    | 6.79E-3 | 1.00    | 3.98E-5 | 2.00 | 1.05E-3 | 1.50 |
| 512   | 2.12E-6 | 2.00    | 3.39E-3 | 1.00    | 9.82E-6 | 2.00 | 3.74E-4 | 1.50 |

| $n$ | $\|e\|_0$ | order | $\|e\|_1$ | order | $\|e\|_{1,r}$ | order | $\|e\|_2$ | order |
|-----|--------|--------|--------|--------|--------------|--------|--------|--------|
| 2   | 9.63E-3 | 6.15E-2 | 5.37E-2 | 2.65E-2 |
| 3   | 2.42E-4 | 2.02    | 3.08E-2 | 1.00    | 1.29E-3 | 2.04 | 9.37E-2 | 1.50 |
| 4   | 6.04E-5 | 2.02    | 1.54E-2 | 1.00    | 3.15E-4 | 2.04 | 3.32E-2 | 1.50 |
| 5   | 1.50E-5 | 2.02    | 7.69E-3 | 1.00    | 7.71E-5 | 2.02 | 1.18E-2 | 1.50 |

| $1/h$ | $\|e\|_0$ | order | $\|e\|_1$ | order | $\|e\|_{1,r}$ | order | $\|e\|_2$ | order |
|-------|--------|--------|--------|--------|--------------|--------|--------|--------|
| 16    | 2.33E-3 | 9.79E-1 | 8.53E-3 | 9.37E-2 |
| 32    | 5.76E-4 | 2.02    | 4.88E-2 | 1.00    | 2.11E-3 | 2.01 | 4.55E-2 | 1.04 |
| 64    | 1.42E-4 | 2.01    | 2.44E-2 | 1.00    | 5.23E-4 | 2.01 | 2.25E-2 | 1.02 |
| 128   | 3.55E-5 | 2.00    | 1.22E-2 | 1.00    | 1.30E-4 | 2.01 | 1.12E-2 | 1.01 |
| 256   | 8.85E-6 | 2.00    | 6.99E-3 | 1.00    | 3.23E-5 | 2.01 | 5.59E-3 | 1.00 |
| 512   | 2.23E-6 | 2.00    | 3.05E-3 | 1.00    | 8.05E-6 | 2.00 | 2.79E-3 | 1.00 |

| $1/h$ | $\|e\|_0$ | order | $\|e\|_1$ | order | $\|e\|_{1,r}$ | order | $\|e\|_2$ | order |
|-------|--------|--------|--------|--------|--------------|--------|--------|--------|
| 16    | 2.40E-3 | 9.75E-1 | 1.38E-3 | 1.23E-1 |
| 32    | 5.84E-4 | 2.01    | 2.67E-2 | 1.00    | 8.88E-4 | 2.01 | 2.72E-2 | 1.07 |
| 64    | 1.50E-4 | 2.01    | 1.33E-2 | 1.00    | 6.60E-5 | 2.01 | 1.33E-2 | 1.03 |
| 128   | 4.02E-5 | 2.00    | 6.65E-3 | 1.00    | 5.51E-5 | 2.01 | 6.60E-3 | 1.01 |

| $1/h$ | $\|e\|_0$ | order | $\|e\|_1$ | order | $\|e\|_{1,r}$ | order | $\|e\|_2$ | order |
|-------|--------|--------|--------|--------|--------------|--------|--------|--------|
| 16    | 2.46E-3 | 1.07E-1 | 1.38E-3 | 1.23E-1 |
| 32    | 6.26E-4 | 2.02    | 5.34E-2 | 1.01    | 3.57E-3 | 2.01 | 5.70E-2 | 1.11 |
| 64    | 1.59E-4 | 2.01    | 2.67E-2 | 1.00    | 8.88E-4 | 2.01 | 2.72E-2 | 1.07 |
| 128   | 4.02E-5 | 2.00    | 1.33E-2 | 1.00    | 2.12E-4 | 2.01 | 1.33E-2 | 1.03 |
| 256   | 1.01E-5 | 2.00    | 6.65E-3 | 1.00    | 5.51E-5 | 2.01 | 6.60E-3 | 1.01 |
Table 6.8
Example 6.1 – Numerical results of the scheme (3.6) on a circular area

| 1/h | ||e||₀ | order | ||e||₁ | order | ||e||₁,r | order | ||e||₂ | order |
|-----|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 16  | 2.35E-3 | 1.07E-1 | 1.29E-3 | | 7.12E-1 | | |
| 32  | 5.89E-4 | 2.00 | 5.32E-2 | 1.01 | 3.42E-3 | 1.99 | 3.47E-2 | 1.48 |
| 64  | 1.51E-4 | 1.96 | 2.66E-2 | 1.00 | 8.71E-4 | 1.97 | 9.70E-2 | 1.46 |
| 128 | 3.84E-5 | 1.97 | 1.33E-2 | 1.00 | 8.71E-4 | 1.97 | 3.47E-2 | 1.48 |
| 256 | 9.71E-5 | 1.98 | 6.66E-3 | 1.00 | 5.50E-4 | 2.00 | 1.23E-3 | 1.49 |

(3.4). It can also be concluded from Tables 6.7-6.8 that our proposed methods maintain optimal convergence rates over a circular domain. Moreover, numerical we observe that the convergence order under $H^2$-seminorm error obtained by the scheme (3.6) is 1.5 for the convex domain, but it reduces to 1.0 for non-convex domains.

Example 6.2. A problem with discontinuous coefficients. In this example, we consider the test problem in [22, 45, 47]. The coefficient reads

$$A = \begin{pmatrix} 2 & xy/|xy| \\ xy/|xy| & 2 \end{pmatrix},$$

and the function $f$ is chosen such that the exact solution of (1.1) is

$$u(x, y) = xy(1 - \exp(1 - |x|))(1 - \exp(1 - |y|)).$$

Note that the coefficient is discontinuous across the x- and y-axis. It is straightforward to verify that example 6.2 satisfies Cordes condition with $\epsilon = 3/5$. Tables 6.9-6.11 demonstrate the performance of the GRBL FEM for the test problem over a square domain, a $L$-shaped domain, and a circular domain, respectively, as specified in Example 6.1. Numerical results illustrate that the convergence rates over these domains are all optimal for problems with discontinuous coefficients, which are $O(h)$ for the approximation of the Hessian, $O(h^2)$ for the approximation of the gradient in $H^1$ norm, and $O(h^2)$ for the approximation of exact solution in $L^2$ norm. It worths mentioning that the convergence order of the WG method in $L^2$ norm [47] and the internal penalty method in $L^2$ norm [38] are both $O(h^2)$ when quadratic elements are employed.

Table 6.9
Example 6.2 – Numerical results of the scheme (3.4) on a square domain

| 1/h | ||e||₀ | order | ||e||₁ | order | ||e||₁,r | order | ||e||₂ | order |
|-----|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 16  | 7.71E-3 | 1.94E-1 | 2.25E-2 | 6.41E-1 |
| 32  | 1.90E-3 | 2.02 | 9.33E-2 | 1.01 | 5.73E-3 | 1.99 | 3.83E-2 | 1.01 |
| 64  | 4.81E-4 | 1.98 | 4.66E-2 | 1.00 | 1.46E-3 | 1.97 | 1.55E-1 | 1.02 |
| 128 | 1.21E-4 | 1.99 | 2.32E-2 | 1.00 | 3.69E-4 | 1.98 | 7.72E-2 | 1.01 |
| 256 | 3.02E-5 | 2.00 | 1.15E-2 | 1.00 | 9.31E-5 | 1.99 | 3.83E-2 | 1.01 |
| 512 | 7.55E-6 | 2.00 | 5.07E-3 | 1.00 | 2.33E-5 | 1.99 | 1.90E-2 | 1.01 |

Example 6.3. A problem with a singular solution. In this example, we consider the problem suggested in [47, 45]. The test equation is given by

$$\sum_{i,j=1}^{2} (\delta_{i,j} + \frac{x_i x_j}{|x|^2}) \partial_{ij}^2 u = f \quad \text{in} \ \Omega, \quad (6.1)$$

where $\delta_{i,j}$ is the Kronecker delta and $x = (x_1, x_2)$. For $\alpha > 1$, it is straightforward to confirm that $u = |x|^\alpha \in H^2(\Omega)$ satisfies (6.1) with $f = (2\alpha^2 - \alpha)|x|^{\alpha-2}$. In fact, the solution $u \in H^{1+\alpha^-}(\Omega)$ for arbitrarily small $\tau > 0$. Moreover, the coefficient satisfies the Cordes condition with $\epsilon = 4/5$. In the numerical experiments, we take $\alpha = 1.6$ with problem (6.1) defined on two square domains $(0, 1)^2$ and $(-1, 1)^2$. 

15
Table 6.10
Example 6.2 – Numerical results of the scheme (3.4) on a L-Shaped domain

| 1/h  | ||e||₀ | order | ||e||₁ | order | ||e||₁,r | order | ||e||₂ | order |
|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 16   | 8.05E-3 | 1.65E-1 |       | 2.75E-2 |       | 5.50E-1 |       |
| 32   | 2.44E-3 | 1.72   | 4.79E-2 | 1.14   | 8.09E-3 | 1.77   | 2.63E-1 | 1.06   |
| 64   | 6.87E-4 | 1.83   | 4.69E-2 | 1.02   | 2.22E-3 | 1.86   | 1.29E-1 | 1.03   |
| 128  | 1.81E-4 | 1.93   | 1.82E-2 | 1.02   | 5.83E-4 | 1.94   | 6.38E-2 | 1.01   |
| 256  | 4.51E-5 | 2.00   | 9.01E-3 | 1.02   | 1.47E-4 | 1.99   | 3.19E-2 | 1.00   |
| 512  | 1.13E-5 | 2.00   | 4.45E-3 | 1.02   | 3.67E-5 | 2.00   | 1.60E-2 | 1.00   |

Table 6.11
Example 6.2 – Numerical results of the scheme (3.4) on a circular area

| 1/h  | ||e||₀ | order | ||e||₁ | order | ||e||₁,r | order | ||e||₂ | order |
|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 16   | 6.65E-3 | 1.74E-1 |       | 4.95E-2 |       | 8.32E-1 |       |
| 32   | 1.91E-3 | 1.80   | 9.17E-2 | 0.92   | 1.14E-2 | 2.11   | 4.13E-1 | 1.01   |
| 64   | 4.86E-4 | 1.98   | 4.58E-2 | 1.00   | 2.73E-3 | 2.07   | 2.06E-1 | 1.00   |
| 128  | 1.18E-4 | 2.04   | 2.23E-2 | 1.04   | 6.73E-4 | 2.02   | 1.04E-1 | 0.98   |
| 256  | 2.89E-5 | 2.03   | 1.09E-2 | 1.03   | 1.69E-4 | 1.99   | 5.35E-2 | 0.97   |

Results in Tables 6.12-6.15 are from the computational domain \( \Omega = (0,1)^2 \), for which the coefficient matrix is discontinuous at the origin. Numerical results suggest a convergence rate of \( \mathcal{O}(h^{0.6}) \) in the \( H^2 \) seminorm, which is consistent with the estimate [1,4]. The convergence rates in \( L^2 \) norm and \( H^1 \) seminorm are of \( \mathcal{O}(h^{2}) \) and \( \mathcal{O}(h) \), respectively. The recovered numerical gradient has a superconvergence order of \( \mathcal{O}(h^{1.6}) \). Tables 6.16 and 6.17 display the performance of the recovered linear finite element schemes for (6.1) on the domain \( \Omega = (-1,1)^2 \). Due to the discontinuity of the coefficient matrix at the origin, the convergence rates of numerical results are reduced. In particular, the numerical results suggest a convergence rate of \( \mathcal{O}(h^{0.6}) \) in the \( H^2 \) seminorm. The convergence rates in both recovered gradient norm and \( L^2 \) norm are of \( \mathcal{O}(h^{1.1}) \), which is consistent with the numerical results reported in [47].

Table 6.12
Example 6.3 – Numerical results of the scheme (3.4) on uniform meshes

| 1/h  | ||e||₀ | order | ||e||₁ | order | ||e||₁,r | order | ||e||₂ | order |
|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 32   | 3.15E-4 | 1.93   | 1.92E-2 | 1.07   | 1.45E-3 | 1.58   | 1.33E-1 | 0.60   |
| 64   | 8.02E-5 | 1.98   | 9.23E-3 | 1.05   | 4.81E-4 | 1.59   | 8.76E-2 | 0.60   |
| 128  | 1.98E-5 | 2.02   | 4.52E-3 | 1.03   | 1.59E-4 | 1.60   | 5.78E-2 | 0.60   |
| 256  | 4.69E-6 | 2.07   | 2.23E-3 | 1.02   | 5.26E-5 | 1.60   | 3.82E-2 | 0.60   |
| 512  | 1.06E-6 | 2.14   | 1.11E-3 | 1.01   | 1.74E-5 | 1.60   | 2.52E-2 | 0.60   |

Table 6.13
Example 6.3 – Numerical results of the scheme (3.4) on nonuniform meshes

| n    | ||e||₀ | order | ||e||₁ | order | ||e||₁,r | order | ||e||₂ | order |
|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2    | 6.29E-4 | 2.76E-2 |       | 2.67E-3 |       | 1.49E-1 |       |
| 3    | 1.70E-4 | 1.89   | 1.36E-2 | 1.02   | 9.22E-4 | 1.53   | 1.03E-1 | 0.53   |
| 4    | 4.50E-5 | 1.92   | 6.59E-3 | 1.05   | 2.89E-4 | 1.67   | 6.81E-2 | 0.59   |
| 5    | 1.20E-5 | 1.90   | 3.21E-3 | 1.04   | 9.43E-5 | 1.62   | 4.43E-2 | 0.62   |

Example 6.4. A problem with degenerate coefficients. In this example, we consider the problem (1.1)
Table 6.14
Example 6.3 – Numerical results of the scheme (3.6) on uniform meshes

| $1/h$ | $\|e\|_0$ order | $\|e\|_1$ order | $\|e\|_{1,r}$ order | $\|e\|_2$ order |
|-------|-----------------|-----------------|-----------------|-----------------|
| 32    | 2.63E-4         | 1.75E-2         | 1.41E-3         | 1.10E-1         |
| 64    | 6.61E-5         | 1.99E-2         | 8.74E-3         | 5.18E-2         |
| 128   | 1.58E-5         | 2.06E-2         | 1.56E-4         | 4.77E-2         |
| 256   | 3.75E-6         | 2.07E-2         | 5.18E-5         | 3.15E-2         |
| 512   | 8.83E-6         | 2.08E-2         | 1.72E-5         | 2.08E-2         |

Table 6.15
Example 6.3 – Numerical results of the scheme (3.6) on nonuniform meshes

| $n$  | $\|e\|_0$  | $\|e\|_1$  | $\|e\|_{1,r}$ | $\|e\|_2$  |
|------|-------------|-------------|----------------|-------------|
| 2    | 5.39E-4     | 2.43E-2     | 2.53E-3        | 1.27E-1     |
| 3    | 1.46E-4     | 1.90E-2     | 1.21E-2        | 8.42E-1     |
| 4    | 3.75E-5     | 1.96E-2     | 6.00E-3        | 5.55E-2     |
| 5    | 9.73E-5     | 1.94E-2     | 3.00E-3        | 3.66E-2     |

Example 6.3 – Numerical results of the scheme (3.6)

Table 6.16
Example 6.3 – Numerical results of the scheme (3.4)

| $1/h$ | $\|e\|_0$ order | $\|e\|_1$ order | $\|e\|_{1,r}$ order | $\|e\|_2$ order |
|-------|-----------------|-----------------|-----------------|-----------------|
| 32    | 8.63E-3         | 8.52E-2         | 2.21E-2         | 3.42E-1         |
| 64    | 3.99E-3         | 1.11E-2         | 4.27E-2         | 1.00E-1         |
| 128   | 1.85E-3         | 1.10E-2         | 2.14E-2         | 5.55E-2         |
| 256   | 8.65E-4         | 1.10E-2         | 1.07E-2         | 9.96E-2         |
| 512   | 4.03E-4         | 1.10E-2         | 5.04E-3         | 6.58E-2         |

Table 6.17
Example 6.3 – Numerical results of the scheme (3.6)

| $1/h$ | $\|e\|_0$ order | $\|e\|_1$ order | $\|e\|_{1,r}$ order | $\|e\|_2$ order |
|-------|-----------------|-----------------|-----------------|-----------------|
| 32    | 6.29E-3         | 8.44E-2         | 1.67E-2         | 2.05E-1         |
| 64    | 2.79E-3         | 1.17E-2         | 7.36E-3         | 1.35E-1         |
| 128   | 1.27E-3         | 1.14E-2         | 3.31E-3         | 8.90E-2         |
| 256   | 5.85E-4         | 1.12E-2         | 1.52E-3         | 5.87E-2         |
| 512   | 2.70E-4         | 1.11E-2         | 7.03E-4         | 3.87E-2         |

with degenerate coefficients suggested in [16]. The coefficient reads

$$A = \frac{16}{9} \begin{pmatrix} x^{2/3} & -x^{1/3} y^{1/3} \\ -x^{1/3} y^{1/3} & y^{2/3} \end{pmatrix}.$$  

The exact solution of this problem is set as $u = x^{4/3} - y^{4/3}$. We take $\Omega = (0, 1)^2$. Note that $A : D^2 u = 0$. Unlike the first three example problems, this problem is not uniformly elliptic as $\det(A) \equiv 0$ in $\Omega$. Therefore, the error estimates developed in this paper are not applicable. Nevertheless, numerical results by the GRBL FEM are presented in Table 6.18 and 6.19. The experiment illustrates that $u - u_h$ measured in the $L^2$ norm and the $H^1$ seminorm have convergence rates of $O(h^{1.27})$ and $O(h^{0.78})$, respectively, for the GRLEM; and are of $O(h^{1.36})$ and $O(h^{0.83})$, respectively, for the HRLEM. These rates are competitive to the numerical results reported in [16].

**Example 6.5. A 3D problem with non-smooth coefficients.** We extend our proposed method to solve a
3D problem on cuboid meshes with exact solution \( u = \sin \pi x \sin \pi y \sin \pi z \). The coefficient function is

\[
A = \begin{pmatrix}
1 + |x| & 0.5|xy|^{1/3} & 0.5|xz|^{1/3} \\
0.5|xy|^{1/3} & 1 + |y| & 0.5|yz|^{1/3} \\
0.5|xz|^{1/3} & 0.5|yz|^{1/3} & 1 + |z|
\end{pmatrix},
\]

which is continuous but has singularities at the origin. We take \( \Omega = (-1,1)^3 \).

Numerical results by the GRBL FEM and HRBL FEM are presented in Table 6.20 and 6.21. The experiment illustrates that \( u - u_h \) measured in the \( L^2 \) norm and the \( H^1 \) seminorm have convergence rates of \( O(h^2) \) and \( O(h^1) \) for both GRBL FEM and HRBL FEM, which are optimal. Superconvergence phenomenon is also observed. For GRBL FEM, \( DG_h u_h \) and \( G_h u_h \) converge to the \( D^2 u \) and \( \nabla u \) with convergence rates \( O(h) \) and \( O(h^2) \), respectively. As for the HRBL FEM, the convergence rate of numerical errors in recovered gradient norms are the same as that from the scheme (3.4), but the convergence rate in \( H^2 \)-seminorm is nearly \( O(h^2) \), which is one order higher than that of the first scheme (3.4).
6.2. Numerical Experiments of the fully nonlinear Monge–Ampère equations. We shall next test the performance of the proposed numerical scheme (5.4) for solving the fully nonlinear Monge–Ampère equation. We will study the impact of the penalty term on stability and accuracy in Examples 6.8 and 6.9. In all examples, we denote \(K\) the number of iterations and \(T\) the CPU time cost. We use \(\|u^{k+1} - u^k\|_2 \leq 10^{-8}\) as the stopping criteria of Newton iterations.

**Example 6.6. A problem with an exact radial solution.** In this test, we solve problem (5.1) on the unit square \(\Omega = (0,1)^2\) with the data

\[
 f = (1 + x^2 + y^2)e^{(x^2+y^2)/2}, \quad g = e^{(x^2+y^2)/2}.
\]

This example problem is found in [5, 21]. The exact solution of this problem is \(u = e^{(x^2+y^2)/2}\).

We take \(\sigma = 10\) in (5.4) and the numerical results are collected in the Table 6.22. It shows clearly that the proposed numerical method converges with optimal orders of \(O(h^2), O(h),\) and \(O(h)\) in the \(L^2\) norm, \(H^1\) seminorm, and \(H^2\) seminorm, respectively. The recovered numerical gradient converges with a superconvergence order of \(O(h^2)\) as expected.

We study also the effects of the presentation of high-frequency sinusoidal noise to the data (i.e. the source \(f\) and the boundary conditions \(g\)). Numerical results are shown in Table 6.23. The proposed method yields a solution that is convex except at the boundary although the noisy data is not convex; cf. Figure 6.2. The numerical results show that the noise does not have any effect on the rate of convergence for the presented method.

| 1/h | K | T(s) | \(\|\epsilon\|_0\) | order | \(\|e\|_1\) | order | \(\|e\|_1,r\) | order | \(\|e\|_2\) | order |
|-----|----|------|----------------|-------|-------------|-------|--------------|-------|-------------|-------|
| 8   | 5  | 0.08 | 6.21E-3        |       | 1.44E-1     |       | 3.08E-2      |       | 3.28E-1     |
| 16  | 5  | 0.22 | 1.68E-3        | 1.88  | 7.12E-2     | 1.02  | 8.10E-3      | 1.93  | 1.56E-1     | 1.08  |
| 32  | 5  | 0.74 | 4.40E-4        | 1.94  | 3.55E-2     | 1.01  | 2.04E-3      | 1.99  | 7.60E-2     | 1.04  |
| 64  | 5  | 2.96 | 1.12E-5        | 1.97  | 1.77E-2     | 1.00  | 5.11E-4      | 2.00  | 3.76E-2     | 1.01  |
| 128 | 5  | 12.4 | 2.84E-5        | 1.99  | 8.86E-3     | 1.00  | 1.27E-4      | 2.00  | 1.88E-2     | 1.01  |
| 256 | 6  | 64.1 | 7.12E-6        | 1.99  | 4.43E-3     | 1.00  | 3.18E-5      | 2.00  | 9.36E-3     | 1.00  |

**Example 6.7. A problem with blow-up at boundary.** In this test, we choose data such that the exact solution to problem (5.1) is \(u = (x^2 + y^2)^{5/3}\). As shown in Example 6.3 \(u \in H^{8/3-\tau}\) for arbitrarily small \(\tau > 0\). The source function \(f\) blows up at the boundary.

We take \(\Omega = (0,1)^2\) and set \(\sigma = 2\). The numerical results are illustrated in Table 6.24. Numerical results suggest convergence orders of \(O(h^2), O(h),\) \(O(h^{5/3})\) in the \(L^2\) norm, \(H^1\) seminorm, and \(H^2\) seminorm, respectively. While the recovered numerical gradient converges with a superconvergence order of \(5/3\). Figure 6.3 shows the profiles of the approximated solution with \(\sigma = 2\) on a 64×64 grid.
Example 6.6. Numerical results with noisy data on a 64×64 grid

Example 6.7. Numerical results of the scheme (5.4)

| 1/h | K | $\|e\|_0$ order | $|e|_1$ order | $|e|_{1,r}$ order | $|e|_2$ order |
|------|----|-----------------|----------------|-----------------|-------------|
| 8    | 6  | 4.15E-3         | 8.34E-2        | 1.13E-2         | 2.17E-1     |
| 16   | 6  | 1.10E-3         | 4.27E-2        | 0.97            | 3.62E-3     | 1.64        | 1.37E-2     | 0.68         |
| 32   | 7  | 2.78E-4         | 2.19E-2        | 0.97            | 1.16E-3     | 1.64        | 8.66E-2     | 0.66         |
| 64   | 7  | 6.52E-5         | 1.09E-2        | 1.00            | 3.74E-4     | 1.63        | 5.52E-2     | 0.65         |
| 128  | 7  | 1.32E-5         | 5.22E-3        | 1.06            | 1.25E-4     | 1.60        | 3.52E-2     | 0.65         |
| 256  | 8  | 3.14E-6         | 2.47E-3        | 1.08            | 4.12E-5     | 1.60        | 2.25E-2     | 0.65         |

Example 6.8. Problem in [10, Example 3]. In this test, we consider (5.1) on $\Omega = (-0.5, 0.5)^2$ with the data

$$f(x, y) = \max(1 - \frac{0.1}{\sqrt{x^2 + y^2}}, 0), \quad g(x, y) = \frac{1}{2}(\sqrt{x^2 + y^2} - 0.1)^2.$$ 

The exact solution is given by $u = \frac{1}{2}\max(\sqrt{x^2 + y^2} - 0.1, 0)^2$. The singularity appears along the circle $x^2 + y^2 = 0.1^2$.

We set $\sigma = 2$ and $\sigma = 0$ in the numerical scheme (5.4), which represent the cases with and without penalty, respectively. Numerical results are shown in Tables 6.25 and 6.26. It is observed that the absence of penalty leads to unsteady results. Compared with the penalty method, the penalty-free method requires more iterations to converge to the exact solution and has a worse convergence rate.

Example 6.9. Problem whose solution is a cone. In the last test, we choose the data such that the exact
solution is a cone.

\[ u = \sqrt{(x - 0.5)^2 + (y - 0.5)^2}, \quad f = \pi \delta_{(0.5,0.5)} \].

Following a similar strategy as in \[5, 10, 20, 40\], we replace \( f \) by its regularized discrete version:

\[
 f_h = \begin{cases} 
 \pi/(4h^2) & \text{if } |x - 0.5| < h \text{ and } |y - 0.5| < h, \\
 0 & \text{otherwise}. 
\end{cases}
\]

We find that the absence of penalty leads to divergent Newton iterations. For example, when we take \( \sigma = 0 \) and \( n = 64 \), the numerical solution does not converge after 500 iterations. But the numerical scheme with penalty reaches the stopping tolerance after 64 iterations. The surface plots of the numerical solution and absolute error are demonstrated in Figure 6.4.

Fig. 6.4. Example 6.9 – Numerical results with \( \sigma = 2 \) on a 64×64 grid

### 7. Concluding remarks.

In this paper, we present GRBL and HRBL FEMs for second order elliptic equations in non-divergence form. By utilizing the gradient and Hessian recovery operators, we discrete the least square variation in the simplest linear element space. For the GRBL FEM, we prove the stability by
adding a rotation. The optimal order of the convergence error is theoretically shown under a discrete $H^2$ seminorm. When coefficients are constants, optimal error estimates in $L^2$ norm and $H^1$ seminorm have also been proven. For the HRBL FEM, optimal convergence in $L^2$ and $H^1$ norms and stability are confirmed from a series of benchmark tests. Finally, the GRBL FEM has been applied to solve the fully nonlinear Monge-Ampère equation. Numerical results verify the robustness and the optimal-order convergence.

REFERENCES

[1] R. Adams, Sobolev Space. Academic Press, New York (1975).
[2] M. Ainsworth and J. T. Oden, A Posteriori Error Estimation in Finite Element Analysis. Wiley Interscience, New York (2000).
[3] R. E. Bank and A. Weiser, Some a posteriori error estimators for elliptic partial differential equations. Math. Comp. 44, 283-301 (1985).
[4] R. E. Bank and J. Xu, Asymptotically exact a posteriori error estimators, Part I: Grid with superconvergence. SIAM J. Numer. Anal., 41, 2294-2312 (2003).
[5] J. D. Benamou, B. D. Froese and A. M. Oberman, Two numerical methods for the elliptic Monge-Ampère equation. M2AN Math. Model. Numer. Anal. 44 (4) (2010) 737-758.
[6] J. Blechschmidt, R. Herzog, and M. Winkler. Error estimation for second-order pdes in nonvariational form. arXiv preprint arXiv:1909.12676, 2019.
[7] J. Blechschmidt, R. Herzog and M. Winkler, Error estimation for second-order partial differential equations in nonvariational form. Numer. Methods Partial Differential Equations. 2020. https://doi.org/10.1002/num.22678.
[8] S. C. Brenner, T. Gudi, M. Neilan and L. Y. Sung, A $C^0$ penalty method for the fully nonlinear Monge-Ampère equation. Math. Comp. 80 (2011) 1979-1995.
[9] S. C. Brenner and E. L. Kwak, Adaptive $C^0$ interior penalty methods for Hamilton-Jacobi-Bellman equations with Cordes coefficients. J. Comput. Appl. Math. to appear, 2020.
[10] Y. G. Chen, J. W. L. Wan and J. Lin, Monotone Mixed Finite Difference Scheme for Monge-Ampère equation. J. Sci. Comput. 76 (3) (2018) 1839-1867.
[11] H. O. Cordes, Uber die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen. Math. Ann., 131:278-312, 1956.
[12] M. Costabel and M. Dauge, Maxwell and Lamé eigenvalues on polyhedra. Math. Methods. Appl. Sci. 22 (1999), 243-258.
[13] E. J. Dean and R. Glowinski, Numerical solution of the two-dimensional elliptic Monge-Ampère equation with Dirichlet boundary conditions: An augmented Lagrangian approach, C. R. Math. Acad. Sci. Paris, 336(2003), 779-784.
[14] A. Dedner and T. Pryer, Discontinuous Galerkin methods for non variational problems. arXiv:1304.2265v1.
[15] L. Evans, Partial Differential Equations. American Mathematical Society, Providence, Rhode Island (1998).
[16] X. Feng, L. Hennings and M. Neilan, Finite element methods for second order linear elliptic partial differential equations in non-divergence form. Math. Comp. 86 (2017), 2025-2051.
[17] X. Feng and M. Neilan, Vanishing moment method and moment solutions for fully nonlinear second order partial differential equations. J. Sci. Comput. 38 (2008), 74-98.
[18] X. Feng, M. Neilan, and S. Schnake, Interior penalty discontinuous Galerkin methods for second order linear non-divergence form elliptic PDEs, J. Sci. Comput. 74 (2018), 1651-1676.
[19] W. H. Fleming and H. M. Soner, Controlled Markov Processes and Viscosity Solutions. 2nd ed., Stoch. Model. Appl. Probab. 25, Springer, New York, 2006.
[20] B. D. Froese and A.M. Oberman, Convergent difference schemes for viscosity solutions of the elliptic Monge-Ampère equation in dimensions two and higher. SIAM J. Numer. Anal. 49 (4) (2011) 1692-1714.
[21] B. D. Froese and A.M. Oberman, Convergent filtered schemes for the Monge–Ampère partial differential equation. SIAM J. Numer. Anal. 51 (1) (2013) 423-444.
[22] D. Gallistl, Variational Formulation and Numerical Analysis of Linear Elliptic Equations in Nondivergence form with Cordès Coefficients. SIAM J. Numer. Anal. 55 (2017), 737-757.
[23] D. Gallistl, Numerical approximation of planar oblique derivative problems in nondivergence form. Math. Comp. 2018.
[24] H. Guo and X. Yang, Polynomial preserving recovery for high frequency wave propagation. J. Sci. Comput. 71(2017), 594-614.
[25] H. Guo, Z. Zhang and R. Zhao, Hessian recovery for finite element methods, Math. Comput. 86 (2017) 1671-1692.
[26] H. Guo, Z. Zhang, R. Zhao, and Q. Zou, Polynomial preserving recovery on boundary. J. Comput. Allp. Math. 307 (2016), 119-133.
[27] H. Guo, Z. Zhang and Q. Zou, A $C^0$ Linear Finite Element Method for Biharmonic Problems. J. Sci. Comp. 74(3) (2018), 1397-1422.
[28] Y. Jia, M. Xu, Y. Lin, and D. Jiang. An efficient technique based on least-squares method for fractional integro-differential equations, Alex. Eng. J., (2022), https://doi.org/10.1016/j.aej.2022.08.035.
[29] E. Kawecki, A DGFEM for nondivergence form elliptic equations with Cordes coefficients on curved domains. Numer. Meth. Part. D. E. 35(5)(2019)1717-1744.
[30] E. Kawecki, A discontinuous Galerkin finite element method for uniformly elliptic two dimensional oblique boundary-value problems. SIAM J. Numer. Anal. 57(2)(2019):751-778.
[31] E. Kawecki, O. Lakkis, and T. Pryer. A finite element method for the Monge–Ampère equation with transport boundary conditions. arXiv preprint [arXiv:1807.03535], 2018.
[32] E. Kawecki and I. Smears. Convergence of adaptive discontinuous Galerkin and $C^0$-interior penalty finite element methods for Hamilton–Jacobi–Bellman and Isaacs equations. 2020. [arXiv:2006.07215]
[33] E. Kawecki and I. Smears. Unified analysis of discontinuous galerkin and $C^0$-interior penalty finite element methods for Hamilton-Jacobi-Bellman and Isaacs equations. 2020. [arXiv:2006.07202]
[34] O. Lakkis and A. Mousavi. A least-squares galerkin approach to the Monge-Ampère equation. C. R. Math. Acad. Sci. Paris, 340(1) (2010), 319-324.
[35] O. Lakkis and T. Pryer. A finite element method for second order nonvariational elliptic problems. SIAM J. Sci. Comput. 33(2) (2011), 786-801.
[36] O. Lakkis and T. Pryer. A finite element method for nonlinear elliptic problems. SIAM J. Sci. Comput. 35(4)(2013), 2025-2045.
[37] L. Mu and X. Ye. A simple finite element method for non-divergence form elliptic equations. Int. J. Numer. Anal. Mod. 14(2) (2017), 306-311.
[38] L. C. Zienkiewicz and J. Z. Zhu. The superconvergence patch recovery and a posteriori error estimates part 1: the recovery technique. Int. J. Numer. Methods. Eng. 33(1992), 1331-1364.
[39] P. Zhu and X. Wang. A least square based weak Galerkin finite element method for second order elliptic equations in non-Divegence form. Acta Math. Sci. Ser. B (Engl. Ed.), 40(5) (2020), 1553-1562.