Talenti’s Comparison Theorem for Poisson Equation and Applications on Riemannian Manifold with Nonnegative Ricci Curvature

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Abstract
In this article, we prove Talenti’s comparison theorem for Poisson equation on complete noncompact Riemannian manifold with nonnegative Ricci curvature. Furthermore, we obtain the Faber–Krahn inequality for the first eigenvalue of Dirichlet Laplacian, $L^1$- and $L^\infty$-moment spectrum, especially Saint-Venant theorem for torsional rigidity and a reverse Hölder inequality for eigenfunctions of Dirichlet Laplacian.

Keywords Faber–Krahn · Isoperimetric inequality · Talenti’s comparison · Reverse Hölder inequality

Mathematics Subject Classification 53C44 · 53C42

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in Euclidean space, $f \in L^2(\Omega)$ be nonnegative and $u$ be the solution to

$$
\begin{cases}
-\Delta u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

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and \( v \) be the solution to
\[
\begin{align*}
-\Delta v &= f^\# , \quad \text{in } \Omega^\sharp, \\
v &= 0 , \quad \text{on } \partial \Omega^\sharp,
\end{align*}
\]

where \( \Omega^\sharp \) denotes the ball centered at the origin satisfying \(|\Omega^\sharp| = |\Omega|\) and \( f^\# \) is the Schwarz rearrangement of \( f \). In 1976, Talenti [29] proved

\[
u^\sharp(x) \leq v(x), \quad \text{for } x \in \Omega^\sharp.
\]

Talenti’s comparison results were generalized to semilinear and nonlinear elliptic equations, for instance, in [3, 30], and parabolic equation, for instance, in [3, 7]. In 2018, Colladay et al. [14] generalized Talenti’s result to compact Riemannian manifolds whose Ricci curvature has positive lower bound. Recently, Talenti’s comparison results were extended to RCD\((K, N)\) spaces in [26]. We also refer the reader to excellent books [5, 19, 21] for related topics.

The main aim in the present work is to generalize Talenti’s comparison result to complete noncompact Riemannian manifold with nonnegative Ricci curvature. Let \((M, g)\) be a noncompact, complete \(n\) \((n \geq 2)\) dimensional Riemannian manifold with nonnegative Ricci curvature and positive asymptotic volume ratio, i.e.

\[\text{AVR}(g) = \lim_{r \to \infty} \frac{|B_x(r)|}{\omega_n r^n} > 0.\]

where \(B_x(r)\) stands for the open metric ball centered at \(x \in M\) with radius \(r > 0\), \(\omega_n\) denotes the volume of the unit ball in Euclidean space \(\mathbb{R}^n\). For a given nonnegative (not identically zero) \(f \in L^2(\Omega)\), we consider the following problem
\[
\begin{align*}
-\Delta_g u &= f , \quad \text{in } \Omega , \\
u &= 0 , \quad \text{on } \partial \Omega.
\end{align*}
\]

We will establish a comparison principle with the solution to the following problem
\[
\begin{align*}
-\Delta v &= f^\# , \quad \text{in } \Omega^\sharp , \\
v &= 0 , \quad \text{on } \partial \Omega^\sharp ,
\end{align*}
\]

where \( \Omega^\sharp \) denotes Euclidean ball centered at the origin satisfying \(\text{AVR}(g) \mid \Omega^\sharp \mid = |\Omega|\) and \( f^\# \) is the Schwarz rearrangement of \( f \). Based on the isoperimetric inequality in [10], we have

**Theorem 1.1** Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, i.e. \(\text{AVR}(g) > 0\). Assume that \(\Omega\) is a bounded domain in \(M\), \(f \in L^2(\Omega)\) is nonnegative and \(u\) is the weak solution to Problem (1.1). Let \(\Omega^\sharp\) be an Euclidean ball satisfying \(\text{AVR}(g) \mid \Omega^\sharp \mid = |\Omega|\) and \(v\) be the solution to Problem (1.2). Then we have

\[
u^\sharp(x) \leq v(x), \quad x \in \Omega^\sharp.
\]
Moreover, the equality holds in (1.3) if and only if \((M, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, g_0)\) and \(\Omega\) is isoperimetric to the Euclidean ball \(\Omega^\sharp\), where \(g_0\) is the canonical metric of Euclidean space.

As an application of Theorem 1.1, we obtain the Faber–Krahn inequality for the first eigenvalue of Dirichlet Laplacian.

**Corollary 1.2** Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold with nonnegative Ricci curvature and \(\text{AVR}(g) > 0\). Assume that \(\Omega\) is a bounded domain in \(M\), \(\Omega^\sharp\) be an Euclidean ball satisfying \(\text{AVR}(g) \frac{|\Omega|^\frac{n}{2}}{|\Omega^\sharp|^\frac{n}{2}} = 1\) and \(\lambda_1(\Omega)\) denotes the first eigenvalue of Dirichlet Laplacian

\[
\begin{aligned}
-\Delta_g u &= \lambda_1(\Omega) u, \quad \text{in } \Omega, \\
\quad u &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then we have

\[
\lambda_1(\Omega^\sharp) \leq \lambda_1(\Omega),
\]

where \(\lambda_1(\Omega^\sharp)\) is the first eigenvalue of \(\Omega^\sharp\). Moreover, the equality holds if and only if \((M, g)\) is isometric to Euclidean space \((\mathbb{R}^n, g_0)\) and \(\Omega\) is isometric to Euclidean ball \(\Omega^\sharp\).

**Remark 1.3** The Faber–Krahn inequality (1.5) has been proved in [15] for \(3 \leq n \leq 7\) and in [6] for all dimension \(n\), both in terms of Pólya-Szegő principle. However, the proof we given here differs from theirs and the equality case follows Talenti’s comparison result in Theorem 1.1 directly. The inequality (1.5) can also be written as

\[
\frac{j_{\frac{n}{2}-1,1}^2(\omega_n \text{AVR}(g))}{\frac{n}{2}} \leq \lambda_1(\Omega) \frac{|\Omega|^\frac{n}{2}}{|\Omega^\sharp|^\frac{n}{2}},
\]

where \(j_{\frac{n}{2}-1,1}\) denotes the first positive zero of the Bessel function of the first kind with order \(\frac{n}{2} - 1\).

From the Faber–Krahn inequality (1.5), we can deduce estimates for the second eigenvalue of the Laplacian.

**Corollary 1.4** Under the same assumptions as in Corollary (1.2), \(\lambda_2(\Omega)\) denotes the second eigenvalue of Dirichlet laplacian

\[
\begin{aligned}
-\Delta_g u &= \lambda_2(\Omega) u, \quad \text{in } \Omega, \\
\quad u &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then we have

\[
\lambda_2(\Omega) > 2\frac{n}{2} \frac{j_{\frac{n}{2}-1,1}^2(\omega_n \text{AVR}(g))}{|\Omega|^\frac{n}{2}}. \tag{1.7}
\]
Remark 1.5 When $M^n$ is Euclidean space $\mathbb{R}^n$, the result is known as Krahn-Hong-Szegö inequality.

As another application of Theorem 1.1, we will establish the comparison theorem for so called $L^1$ and $L^\infty$ moment spectrum between complete Riemannian manifold and Euclidean space. Let $(M, g)$ be a complete Riemannian manifold and $\Omega \subset M$ be a smoothly bounded domain. Assume that $u$ solves the equation

\[
\begin{cases}
-\Delta_g u = 1, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\] (1.8)

The torsional rigidity $T(\Omega)$ of $\Omega$ is defined by

\[
T(\Omega) = \int_{\Omega} u(x) \, d\mu_g(x).
\] (1.9)

For the background on torsional rigidity, one can refer to [5, 21, 28]. In general, let $u_k$ be solution of a hierarchy of Poisson equation

\[
\begin{cases}
-\Delta_g u_k = ku_{k-1} & \text{in } \Omega, \\
u_k = 0 & \text{on } \partial \Omega, \quad k = 1, \ldots
\end{cases}
\] (1.10)

where $u_0 = 1$ by convention. For a positive integer $k$, we define

\[
T_k(\Omega) = \int_{\Omega} u_k \, d\mu_g, \quad \text{for } k = 1, \ldots
\] (1.11)

and

\[
J_k(\Omega) = \sup_{x \in \Omega} u_k, \quad \text{for } k = 1, \ldots
\] (1.12)

The collection $\{T_k(\Omega)\}_{k=1}^{\infty}$ and $\{J_k(\Omega)\}_{k=1}^{\infty}$ are called the $L^1$-moment spectrum and $L^\infty$-moment spectrum of $\Omega$, respectively. For interpretation for $L^1$ and $L^\infty$-moment spectrum in probability theory, we refer the reader to [11, 24, 25]. For related results for torsional rigidity, $L^1$ and $L^\infty$-moment spectrum, one can consult [2, 9, 11, 12, 14, 16, 18, 22, 24, 25] and references therein.

Corollary 1.6 Let $(M, g)$ be a noncompact, complete $n$-dimensional Riemannian manifold with nonnegative Ricci curvature and $\text{AVR}(g) > 0$. Assume that $\Omega$ is a bounded domain in $M$ and $\Omega^2$ is an Euclidean ball satisfying $\text{AVR}(g) |\Omega^2| = |\Omega|$. For the $L^1$-moment spectrum of $\Omega$, we have

\[
T_k(\Omega) \leq \text{AVR}(g) T_k(\Omega^2), \quad k = 1, \ldots
\] (1.13)

In particular, for $k = 1$, the Saint-Venant inequality holds

\[
T(\Omega) \leq \text{AVR}(g) T(\Omega^2).
\] (1.14)
For $L^\infty$-moment spectrum of $\Omega$, we have

$$J_k(\Omega) \leq J_k(\Omega^\sharp). \quad (1.15)$$

Moreover, equality holds for any $k$ in (1.14) or (1.15) if and only if $(M, g)$ is isometric to Euclidean space $(\mathbb{R}^n, g_0)$ and $\Omega$ is isometric to Euclidean ball $\Omega^\sharp$.

**Remark 1.7** The inequality (1.12) can also be written by

$$T_k(\Omega) \left| \Omega \right|^{-\frac{n+2k}{n}} \leq \left( \operatorname{AVR}(g)\omega_n \right)^{-\frac{2k}{n}} T_k(B_1)\omega_n^{-1}, \quad k = 1, \ldots, \quad (1.16)$$

where $B_1$ is the unit ball in Euclidean space $\mathbb{R}^n$.

The quantity $T_k(B_1)\omega_n^{-1}$ can be computed explicitly, for instance, $k = 1, 2$,

$$\frac{T_1(B_1)}{\omega_n} = \frac{1}{n(n+2)}, \quad \frac{T_2(B_1)}{\omega_n} = \frac{4}{n^2(n+2)(n+4)}.$$

In particular, the Saint-Venant inequality (1.14) can also be written as

$$(\operatorname{AVR}(g)\omega_n)^{\frac{2}{n}} T(\Omega) \left| \Omega \right|^{-\frac{n+2}{n}} \leq \frac{1}{n(n+2)}, \quad (1.17)$$

Our another aim here is to obtain a reverse Hölder inequality for eigenfunctions of Dirichlet eigenvalue problem. In 1972, Payne and Rayner [27] proved that the eigenfunction $u$ of the Dirichlet Laplacian corresponding to the first eigenvalue $\lambda_1(\Omega)$ for a bounded planar domain $\Omega \subset \mathbb{R}^2$ satisfies a reverse Hölder inequality

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^1(\Omega)}} \leq \frac{\sqrt{\lambda_1(\Omega)}}{2\sqrt{\pi}}.$$ 

The equality occurs if and only if $\Omega$ is a disk. In 1981, Kohler-Jobin [23] obtained an isoperimetric comparison between the $L^2$ and $L^1$ norms of the eigenfunction for Dirichlet Laplacian for bounded domain in $\mathbb{R}^n (n \geq 3)$. In 1982, Chiti [13] generalized the reverse Hölder inequality for the norms $L_q$ and $L_p$, $q \geq p > 0$, for bounded domains of $\mathbb{R}^n (n \geq 2)$. Chiti’s comparison results was generalized to bounded domain in hemisphere by Ashbaugh and Benguria [4] and in hyperbolic space by Benguria and Linde [8], respectively. Using isoperimetric comparison the result in [13] was extended to compact Riemannian manifolds whose Ricci curvature has positive lower bound and the integral Ricci curvature condition [12]. In this paper, we extend Chiti’s result to complete Riemannian manifolds with nonnegative Ricci curvature.

**Theorem 1.8** Let $(M, g)$ be a noncompact, complete $n$-dimensional Riemannian manifold with nonnegative Ricci curvature and $\operatorname{AVR}(g) > 0$. Let $\Omega$ be a bounded domain in $(M, g)$ and $u$ be one of solution corresponding to Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_g u = \lambda \cdot u, & \text{in } \Omega, \\
0 & \text{on } \partial \Omega. \end{cases} \quad (1.18)$$
For real numbers $p$ and $q$ satisfying $q \geq p > 0$, then we have

\[
\frac{\|u\|_{L^q(\Omega)}}{\|u\|_{L^p(\Omega)}} \leq K(p, q, \lambda, n, \text{AVR}(g)),
\]

where

\[
K(p, q, \lambda, n, \text{AVR}(g)) = \left( n\omega_n \text{AVR}(g) \left( j_{\frac{n}{2}-1,1}^{-\frac{1}{2}} \right)^n \right)^{\frac{1}{q}} \frac{\left( \int_0^1 r^{n-1+q(1-\frac{q}{2})} j_{\frac{n}{2}-1}^q (j_{\frac{n}{2}-1,1} r) dr \right)^{\frac{1}{q}}}{\left( \int_0^1 r^{n-1+p(1-\frac{q}{2})} j_{\frac{n}{2}-1}^p (j_{\frac{n}{2}-1,1} r) dr \right)^{\frac{1}{p}}}.
\]

Furthermore, equality holds in (1.19) if and only if $(M, g)$ is isometric to Euclidean space $(\mathbb{R}^n, g_0)$, $\Omega$ is isometric to Euclidean ball with radius $j_{\frac{n}{2}-1,1}^{\frac{1}{2}}$ and $\lambda$ is the first eigenvalue of Dirichlet eigenvalue problem (1.18).

The paper is organized as follows. In Sect. 2, we recall the isoperimetric inequality of complete Riemannian manifold with nonnegative Ricci curvature in [1, 10], Schwarz rearrangement and its properties for measurable functions; In Sect. 3, we establish Talenti’s comparison theorem 1.1 using the isoperimetric inequality and Schwarz rearrangement; In Sect. 4, we prove Faber–Krahn inequality for the first eigenvalue of Dirichlet Laplacian and comparison results for $L^1$ and $L^\infty$-moment spectrum. In the last Sect. 5, we establish Chiti’s comparison theorem 1.8 for eigenfunctions of Dirichlet Laplacian.

2 Preliminaries

2.1 Isoperimetric Inequalities

Let $(M, g)$ be a noncompact, complete $n$-dimensional Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, which means the asymptotic volume ratio positive, i.e.

\[
\text{AVR}(g) = \lim_{r \to \infty} \frac{|B_x(r)|_g}{\omega_n r^n} > 0,
\]

where $B_x(r)$ stands for open metric ball centered at $x \in M$ with radius $r > 0$ and $\omega_n$ denotes by the volume of the unit ball in $\mathbb{R}^n$. According to Brendle [10, Corollary 1.3], for every bounded domain $\Omega \subset M$ with smooth boundary, the isoperimetric inequality holds

\[
|\partial \Omega| \geq n\omega_n^\frac{1}{n} \text{AVR}(g)^\frac{1}{n} |\Omega|^{\frac{n-1}{n}}.
\]
The equality holds in (2.1) if and only if \((M, g)\) is isometric to \((\mathbb{R}^n, g_0)\) and \(\Omega\) is isometric to an Euclidean ball. Letting \(\Omega^\sharp \subset \mathbb{R}^n\) be an Euclidean ball centered at origin satisfying \(\text{AVR}(g) |\Omega^\sharp| = |\Omega|\), the inequality (2.1) can be equivalently rewritten as

\[
|\partial \Omega| \geq \text{AVR}(g) |\partial \Omega^\sharp|.
\] (2.2)

We also notice that the inequality (2.1) is proved by Agostiniani et al. [1, Theorem 1.8] for \(n = 3\) and then extended to \(3 \leq n \leq 7\) by Fogagnolo and Mazzieri [15]. The equality case in (2.1) is also characterized and the isoperimetric inequality still holds in \(CD(0, N)\) metric measure spaces based on the method of optimal mass transport by Balogh and Kristály in [6].

### 2.2 Schwarz Rearrangement

**Definition 2.1** Let \(h : \Omega \to \mathbb{R}\) be a measurable function. The distribution function of \(h\) is the function \(\mu : [0, +\infty) \to [0, +\infty)\) defined by

\[
\mu_h(t) = |\{x \in \Omega : |h(x)| > t\}|.
\]

Here, and in the whole paper, \(|A|\) stands for the \(n\)-dimensional measure of the set \(A\).

**Definition 2.2** Let \(\Omega\) be a bounded domain in complete manifold \((M, g)\) and \(h : \Omega \to \mathbb{R}\) be a measurable function. The decreasing rearrangement \(h^* : [0, |\Omega|_g] \to \mathbb{R}\) is defined using the distribution function,

\[
h^*(s) = \begin{cases} 
\text{ess sup}_{\Omega} h & \text{if } s = 0, \\
\inf \{t : \mu_h(t) \leq s\} & \text{if } t > 0.
\end{cases}
\] (2.3)

The Schwarz rearrangement \(h^\sharp : \Omega^* \to \mathbb{R}\) of \(h\) is defined by

\[
h^\sharp(x) = h^*(\text{AVR}(g) |B(r)|_{g_0}), \quad \text{for } x \in \Omega^\sharp,
\] (2.4)

where \(B(r)\) denotes the Euclidean ball with radius \(r\) centered at origin in \(\mathbb{R}^n\).

The Schwarz rearrangement \(h^\sharp\) and \(h\) satisfies that

\[
\mu_h(t) = \text{AVR}(g) \mu_{h^\sharp}(t).
\] (2.5)

It is easily checked that \(h, h^*\) and \(h^\sharp\) are equi-distributed in the sense that

\[
\|h\|_{L^p(\Omega)} = \|h^*\|_{L^p(\Omega^*_{g_0})} = \text{AVR}(g)^{\frac{1}{p}} \|h^\sharp\|_{L^p(\Omega^\sharp)}.
\] (2.6)

An important property of the decreasing rearrangement is the Hardy-Littlewood inequality

\[
\int_{\Omega} |h(x)w(x)| \, dx \leq \int_0^{\frac{\text{AVR}(g)}{\|h^\sharp\|_{L^p(\Omega^\sharp)}}} h^*(s)w^*(s) \, ds,
\] (2.7)
where $h, w$ are measurable functions defined on $\Omega$. By choosing $w = \chi_{\{|u|>t\}}$ in (2.7), one has

$$\int_{|x\in\Omega; |u|>t|} |h(x)| \, dx \leq \int_0^{\mu(t)} h^*(s) \, ds. \quad (2.8)$$

**Lemma 2.3** [17] Let $R, p, q$ be real numbers such that $0 < p \leq q$, $R > 0$, and $f, g$ real functions in $L^q([0, R])$. If the decreasing rearrangements of $f$ and $g$ satisfy the following inequality

$$\int_0^s (f^*)^p \, dt \leq \int_0^s (g^*)^p \, dt, \quad \text{for all } s \in [0, R],$$

then

$$\int_0^R f^q \, dt \leq \int_0^R g^q \, dt.$$

### 3 Proof of Theorem 1.1

In this section, we shall prove the Talenti’s comparison Theorem 1.1 using Schwarz rearrangement and the isoperimetric inequality in [10].

**Proof** Assuming that $v$ is the solution of (1.2), we claim that $v^\sharp = v$. Since $f^\sharp$ is radial function and $\Omega^\sharp$ is the Euclidean ball, $v$ solves the ordinary differential equation

$$\begin{cases} -\frac{1}{r^{n-1}} (r^{n-1} v')' = f^\sharp, \\ v'(0) = 0 = v(R), \end{cases}$$

where $R$ is the radius of the ball $\Omega^\sharp$. Integration by part we have

$$v(r) = \gamma_n^{-2} \int_{AVR(g)w_n r^n}^{[\Omega]} \xi^{-2+\frac{2}{n}} F(\xi) \, d\xi, \quad (3.1)$$

where $\gamma_n = n (AVR(g)w_n)^{\frac{1}{n}}$ and

$$F(\xi) = \int_0^\xi f^\sharp(\eta) \, d\eta.$$

From (3.1), it follows that $v(r)$ is nonincreasing since $f$ is nonnegative, and so $v = v^\sharp$. Therefore, we have

$$v^\sharp(s) = \gamma_n^{-2} \int_s^{[\Omega]} \xi^{-2+\frac{2}{n}} F(\xi) \, d\xi. \quad (3.2)$$

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Let $u$ be the solution to (1.1). For $t \geq 0$, we denote by

$$\Omega_t = \{x \in \Omega : u(x) > t\}, \quad \Gamma_t = \{x \in \Omega : u(x) = t\},$$

and by $\mu_u(t) = |\Omega_t|$, the volume of $\Omega_t$ in $(M^n, g)$. The Sard’s theorem implies that

$$\partial \Omega_t = \Gamma_t,$$

for almost every $t$.

A function $u \in H^1_0(\Omega)$ is a weak solution to (1.1) if

$$\int_{\Omega} \nabla_g u \cdot \nabla_g \phi \, d\mu_g = \int_{\Omega} f \, \phi \, d\mu_g, \quad \forall \phi \in H^1_0(\Omega). \tag{3.3}$$

For $t > 0$ and $h > 0$, define

$$\varphi_h(x) = \begin{cases} 0, & \text{if } 0 < u < t, \\ \frac{u-t}{h}, & \text{if } t < u < t + h, \\ 1, & \text{if } t + h < u. \end{cases} \tag{3.4}$$

Putting the test function (3.4) in (3.3) and letting $h$ go to 0 yields

$$- \frac{d}{dt} \int_{\Omega_t} |\nabla_g u|^2 \, d\mu_g = \int_{\Omega_t} f \, d\mu_g, \tag{3.5}$$

From the co-area formula, (3.5) and (2.8), we have

$$\int_{\Gamma_t} |\nabla_g u| \, d\sigma = \int_{\Omega_t} f \, d\mu_g \leq \int_0^{\mu_u(t)} f^*(\eta) \, d\eta. \tag{3.6}$$

From the isoperimetric inequality (2.1), the Cauchy–Schwarz inequality and (3.6), we infer

$$\gamma_n^2 \mu_u(t)^{2 - \frac{2}{n}} \leq |\Gamma_t|^2 \leq \int_{\Gamma_t} |\nabla_g u| \, d\sigma \int_{\Gamma_t} \frac{1}{|\nabla_g u|} \, d\sigma \leq (-\mu'_u(t)) \int_0^{\mu_u(t)} f^*(\eta) \, d\eta \leq (-\mu'_u(t)) F(\mu_u(t)), \tag{3.7}$$

i.e.

$$1 \leq \gamma_n^{-2} \mu_u(t)^{-2 + \frac{2}{n}} (-\mu'_u(t)) F(\mu_u(t))$$
Integrating both sides in (3.7) from 0 to $t$ yields

$$t \leq \gamma_n^{-2} \int_{\mu_u(t)}^{\mathcal{V}} \xi^{-2+\frac{2}{n}} F(\xi) \, d\xi. \quad (3.8)$$

From the definition of $u^*$ and the right continuous property for decreasing rearrangement, we can infer

$$u^*(s) \leq \gamma_n^{-2} \int_{s}^{\mathcal{V}} \xi^{-2+\frac{2}{n}} F(\xi) \, d\xi$$

which gives the proof of (1.3).

For the case of equality in (1.3), we adapt the technique of Kesavan [20]. If $u^\sharp = v$, we have

$$\mu_u(t) = \text{AVR}(g) \mu_v(t), \quad \text{for all} \quad t \geq 0. \quad (3.9)$$

From the explicit expression (3.1) for $v$, we get

$$t = \gamma_n^{-2} \int_{\text{AVR}(g)\mu_v(t)}^{\mathcal{V}} \xi^{-2+\frac{2}{n}} F(\xi) \, d\xi.$$  

Differentiating with respect to $t$, we have

$$1 = \gamma_n^{-2} (\text{AVR}(g)\mu_v(t))^{-2+\frac{2}{n}} F(\text{AVR}(g)\mu_v(t)) \left( -\text{AVR}(g)\mu'_v(t) \right). \quad (3.10)$$

From (3.7), (3.9) and (3.10), we deduce

$$\gamma_n^2 \mu_u(t)^{2-\frac{2}{n}} \leq |\Gamma_t|^2 = \left( -\mu'_u(t) \right) F(\mu_u(t)) = \gamma_n^2 \mu_u(t)^{2-\frac{2}{n}}, \quad (3.11)$$

which implies that

$$|\Gamma_t| = \gamma_n \mu_u(t)^{1-\frac{1}{n}}.$$  

The equality appears in isoperimetric equality (2.1) if and only if $\text{AVR}(g) = 1$, $(M, g)$ is isometric to $(\mathbb{R}^n, g_0)$ and $\Omega_t$ is isometric to an Euclidean ball centered at origin for almost every $t \geq 0$. Let $t_n$ denote a strictly decreasing sequence satisfying $t_n \to 0$ as $n \to \infty$. Then

$$\Omega = \{ x \in \mathbb{R}^n : u(x) > 0 \} = \bigcup_{n=1}^{n} \Omega_{t_n},$$

which implies that $\Omega$ is a nested union of Euclidean ball centered at origin in $\mathbb{R}^n$, and then we deduce that $\Omega$ is exactly an Euclidean ball $\Omega^\sharp$. This completes the proof of Theorem 1.1.  

\[ \square \]
4 Faber–Krahn Inequality, Comparisons for $L^1$ and $L^\infty$ Moment Spectrum

In this section, as applications of Theorem 1.1 we will give the alternative proof of that Faber–Krahn inequality by following an idea contained in [20]. We will obtain comparison results for $L^1$, $L^\infty$-moment spectrum and the estimate for the second eigenvalue of the Dirichlet Laplacian.

**Proof of Corollary 1.2** Assume that $v$ solve the equation

$$
\begin{cases}
-\Delta_{g_0} v = \lambda_1(\Omega) u^\sharp, & \text{in } \Omega^\sharp, \\
v|_{\partial\Omega^\sharp} = 0,
\end{cases}
$$

(4.1)

where $u^\sharp$ is the Schwarz rearrangement of the first eigenfunction $u$ corresponding to Eq. (1.4). From Talenti’s comparison result in Theorem 1.1, we have

$$u^\sharp(x) \leq v(x), \quad x \in \Omega^\sharp,$$

which implies that

$$\int_{\Omega^\sharp} v u^\sharp d\mu_{g_0} \leq \int_{\Omega^\sharp} v^2 d\mu_{g_0}.$$

Multiplying $v$ both sides in (4.1) and integrating by parts, we get

$$\lambda_1(\Omega) = \frac{\int_{\Omega^\sharp} |\nabla_{g_0} v|^2 d\mu_{g_0}}{\int_{\Omega^\sharp} v u^\sharp d\mu_{g_0}} \geq \frac{\int_{\Omega^\sharp} |\nabla_{g_0} v|^2 d\mu_{g_0}}{\int_{\Omega^\sharp} v^2 d\mu_{g_0}} \geq \lambda_1(\Omega^\sharp),$$

where we use the Rayleigh-quotient characterization of the first eigenvalue of $\Omega^\sharp$. This gives the proof of (1.5). If the equality holds in (1.5), all above inequalities become equalities, especially, $u^\sharp = v$. Hence the equality case follows directly from Theorem 1.1. $\square$

**Proof of Corollary 1.4** Assume that $u_2$ is the second eigenfunction associated to $\lambda_2(\Omega)$. According to Courant’s nodal theorem, $u_2$ must change its sign. Defining

$$\Omega_+ = \{x \in \Omega : u_2(x) > 0\}, \quad \Omega_- = \{x \in \Omega : u_2(x) < 0\},$$

we have

$$\begin{cases}
-\Delta u_2 = \lambda_2(\Omega) u_2 & \text{in } \Omega_+, \\
u_2 = 0 & \text{on } \partial\Omega_+,
\end{cases} \quad \begin{cases}
-\Delta u_2 = \lambda_2(\Omega) u_2 & \text{in } \Omega_- \\
u_2 = 0 & \text{on } \partial\Omega_-.
\end{cases}$$
Let $\mathbb{B}_+$ and $\mathbb{B}_-$ be two disjoint Euclidean balls with $\text{AVR}(g)|\mathbb{B}_+| = \text{AVR}(g)|\mathbb{B}_-| = |\Omega|/2$. From the Faber–Krahn inequality (1.5), we deduce that
\[
\lambda_2(\Omega) \geq \lambda_1(\Omega \pm) \geq j_{\frac{n}{2} - 1,1}^2 \left( \frac{\omega_n \text{AVR}(g)}{|\Omega \pm|} \right)^{\frac{2}{n}},
\]
which implies that
\[
2\lambda_2(\Omega) \geq j_{\frac{n}{2} - 1,1}^2 \left( \frac{2\omega_n \text{AVR}(g)}{|\Omega|} \right)^{\frac{2}{n}}, \quad (4.2)
\]
where we use the convexity of $t^{-\frac{2}{n}}$ and $|\Omega_+| + |\Omega_-| \leq |\Omega|$ in the second inequality. If the equality holds in (4.2), then $\Omega_+$ and $\Omega_-$ must be Euclidean balls by Faber–Krahn inequality (1.5). Since $\Omega_+$ and $\Omega_-$ have the same volume $\frac{1}{2} |\Omega|$, $\Omega$ would not be connected. This completes the proof of Corollary 1.4.

**Proof of Corollary 1.6** Let $\Omega^\sharp$ be the rearrangement of $\Omega$ in $\mathbb{R}^n$. Assume that $v_0 = 1$ and $v_k$ satisfy
\[
\begin{cases}
-\Delta g_0 v_k = k v_{k-1}, & \text{in } \Omega^\sharp, \\
v_k|_{\partial \Omega^\sharp} = 0, & k = 1, \ldots,
\end{cases} \quad (4.3)
\]
and $w_k$ solve
\[
\begin{cases}
-\Delta g_0 w_k = k u_{k-1}^\sharp, & \text{in } \Omega^\sharp, \\
w_k|_{\partial \Omega^\sharp} = 0, & k = 1, \ldots,
\end{cases} \quad (4.4)
\]
where $u_k$ are the solutions of the Eq. (1.12).

For $k = 1$, from the definition of $T_1(\Omega)$ and Theorem 1.1, we have
\[
T_1(\Omega) = \int_\Omega u_1 d\mu_g = \text{AVR}(g) \int_\Omega u_1^\sharp d\mu_{g_0} \leq \text{AVR}(g) \int_{\Omega^\sharp} u_1 d\mu_{g_0} = \text{AVR}(g) T(\Omega^\sharp).
\]
For $k = 2$, since $v_1 \geq u_1^\sharp$, the comparison theorem of elliptic equation implies $v_2 \geq w_2$. By Theorem 1.1,
\[
w_2 \geq u_2^\sharp.
\]
Therefore, we deduce
\[
T_2(\Omega) = \int_{\Omega} u_2 \, d\mu_g = \text{AVR}(g) \int_{\Omega^2} u_2^g \, d\mu_{g_0} \\
\leq \text{AVR}(g) \int_{\Omega^2} w_2 \, d\mu_{g_0} \\
\leq \text{AVR}(g) \int_{\Omega^2} v_2 \, d\mu_{g_0} \\
= \text{AVR}(g) T_2(\Omega^2).
\]

By induction for \( k \), we can infer, for \( k \geq 2 \),
\[
u_k^g \leq w_k \leq v_k, \quad \text{in } \Omega^2.
\] (4.5)

Finally, we deduce
\[
T_k(\Omega) = \text{AVR}(g) \int_{\Omega^2} u_k^g \, d\mu_{g_0} \leq \text{AVR}(g) \int_{\Omega^2} w_k \, d\mu_{g_0} \\
\leq \text{AVR}(g) \int_{\Omega^2} v_k \, d\mu_{g_0} = \text{AVR}(g) T_k(\Omega^2),
\]

which give the proof of (1.13). If the equality occurs in (1.13), the conclusion follows Theorem 1.1.

For \( k = 1 \), by (1.3) we have
\[
J_1(\Omega) = \sup_{x \in \Omega} u_1 = \sup_{x \in \Omega^2} u_1^g \leq \sup_{x \in \Omega^2} v_1 = J_1(\Omega^2) = \frac{|\Omega|}{2n \text{AVR}(g) \omega_n}.
\]

For \( k \geq 2 \), by (4.5) we can infer
\[
J_k(\Omega) = \sup_{x \in \Omega} u_k = \sup_{x \in \Omega^2} u_k^g \leq \sup_{x \in \Omega^2} w_k \leq \sup_{x \in \Omega^2} v_k = J_k(\Omega^2).
\]

In the following, we discuss the equality case in (1.15). If \( J_k(\Omega) = J_k(\Omega^2) \) for \( k \geq 1 \), denote
\[
\mu_k(t) = \mu_{u_k}(t) = |\{x \in \Omega : u_k(x) > t\}|
\]
and
\[
v_k(t) = v_{u_k}(t) = |\{x \in \Omega^2 : v_k(x) > t\}|,
\]
where \( u_k \) and \( v_k \) are the solution of (1.12) and (4.3), respectively. From (4.5), \( u_k^g \leq v_k \), we have
\[
\mu_k(t) \leq \text{AVR}(g) v_k(t).
\]
Define
\[ U_k(\eta) = \int_0^{\eta} u_k(s) \, ds, \quad V_k(\eta) = \int_0^{\eta} v_k(s) \, ds, \]
and
\[ \tilde{H}_k(\eta) = \gamma_n^{-2} \int_0^{\eta} \xi^{-2 + \frac{2}{n}} U_{k-1}(\xi) \, d\xi, \quad \tilde{G}_k(\eta) = \gamma_n^{-2} \int_0^{\eta} \xi^{-2 + \frac{2}{n}} V_{k-1}(\xi) \, d\xi. \]

Similar to (3.1),
\[ v_k(r) = \gamma_n^{-2} \int_{\text{AVR}(g)_{0n} r^n}^{\text{AVR}(g)_{\nu_k} n} \xi^{-2 + \frac{2}{n}} V_{k-1}(\xi) \, d\xi. \]

Letting \( v_k(r) = t \),
\[ t = \gamma_n^{-2} \int_{\text{AVR}(g)_{\nu_k} t}^{\text{AVR}(g)_{\nu_k} t} \xi^{-2 + \frac{2}{n}} V_{k-1}(\xi) \, d\xi. \]

Taking the derivative with respect to \( t \) both sides, we have
\[ 1 = \gamma_n^{-2} (\text{AVR}(g)_{\nu_k} t)^{-2 + \frac{2}{n}} V_{k-1}(\text{AVR}(g)_{\nu_k} t) \left( - \text{AVR}(g)_{\nu_k}' t \right). \]

Now recalling the proof of Theorem 1.1, similar arguments to (3.7), we obtain
\[ 1 \leq \gamma_n^{-2} (\mu_k(t))^{-2 + \frac{2}{n}} U_{k-1}(\mu_k(t)) \left( - \mu_k'(t) \right), \]
which gives
\[ \frac{d}{dt} \tilde{G}_k(\text{AVR}(g)_{\nu_k} t) \leq \frac{d}{dt} \tilde{H}_k(\mu_k(t)). \]

For \( t = 0 \),
\[ \tilde{G}_k(\text{AVR}(g)_{\nu_k} 0) = \tilde{G}_k(\text{AVR}(g)_{|\Omega|^2}) = \tilde{G}_k(|\Omega|) = 0, \]
\[ \tilde{H}_k(\mu_k(0)) = \tilde{H}_k(|\Omega|) = 0, \]

If \( J_k(\Omega) = J_k(\Omega^2) \), we have
\[ \tilde{G}_k(0) = \gamma_n^{-2} \int_0^{\text{AVR}(g)_{\nu_k} 0} \xi^{-2 + \frac{2}{n}} V_{k-1}(\xi) \, d\xi \geq \gamma_n^{-2} \int_0^{\text{AVR}(g)_{\nu_k} 0} \xi^{-2 + \frac{2}{n}} U_{k-1}(\xi) \, d\xi = \tilde{H}_k(0). \]

Set
\[ \zeta(t) = \tilde{G}_k(\text{AVR}(g)_{\nu_k} t) - \tilde{H}_k(\mu_k(t)). \]
Since
\[ \zeta(0) = 0, \quad \zeta(J_k(\Omega)) \geq 0, \quad \zeta'(t) \leq 0, \]
we get
\[ \zeta(t) \equiv 0, \]
which implies that
\[ \mu_k(t) = \text{AVR}(g)v_k(t), \quad \text{for all} \quad t > 0. \]
This completes the proof of Corollary 1.6.

\[ \square \]

5 Proof of Chiti’s Reverse Hölder Inequality

In this section, we will prove Chiti’s reverse Hölder inequality based on the isoperimetric inequality and Faber–Krahn inequality for the first eigenvalue.

**Lemma 5.1** Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold with nonnegative Ricci curvature and \(\text{AVR}(g) > 0\). Let \(\Omega\) be a bounded domain in \((M, g)\) and \(u\) be one of solution to

\[
\begin{aligned}
-\Delta gu &= \lambda u, \quad \text{in} \quad \Omega, \\
\left. u \right|_{\partial\Omega} &= 0, \quad \text{on} \quad \partial\Omega.
\end{aligned}
\]

(5.1)

Let \(B_\lambda\) be an Euclidean ball with the first eigenvalue \(\lambda\) and \(v\) be the solution to Dirichlet eigenvalue problem

\[
\begin{aligned}
-\Delta_{g_0} v &= \lambda v, \quad \text{in} \quad B_\lambda, \\
v &= 0, \quad \text{on} \quad \partial B_\lambda.
\end{aligned}
\]

(5.2)

If \(\bar{u} := \max_{x \in \Omega} u = v(0)\), then we have

\[ v^*(s) \leq u^*(s), \quad s \in [0, \text{AVR}(g) |B_\lambda|]. \]

(5.3)

Moreover, the equality holds in (5.3) if and only if \((M, g)\) is isometric to Euclidean space \((\mathbb{R}^n, g_0)\) and \(\Omega\) is isoperimetric to Euclidean ball \(B_\lambda\).

**Proof** For any \(s \in [0, \text{AVR}(g) |B_\lambda|]\), there exists a positive constant \(c\) such that

\[ v^*(s) \leq cu^*(s). \]

Define

\[ \zeta = \inf \left\{ c|v^*(s) \leq cu^*(s), s \in [0, \text{AVR}(g) |B_\lambda|] \right\}. \]
Obviously, \( c \geq 1 \) since \( \bar{u} = z(0) \). From Faber–Krahn inequality in Corollary 1.5 and the domain monotonicity of the first eigenvalue, we can infer \( B_\lambda \subseteq \Omega^{\sharp} \). If \( B_\lambda = \Omega^{\sharp} \), it is nothing to prove. Now assuming that \( B_\lambda \subset \Omega^{\sharp} \), we get

\[
v^*(|B_\lambda|) = 0 \quad \text{and} \quad u^*(|B_\lambda|) > 0.
\]

Since \( u^*(0) = \bar{u} = v(0) = v^*(0) \), there exists \( s_0 \in [0, \text{AVR}(g)|B_\lambda|] \) such that

\[
c u^*(s_0) = v^*(s_0).
\]

Setting

\[
w(s) = \begin{cases} c u^*(s), & s \in [0, s_0], \\ v^*(s), & s \in (s_1, |B_\lambda|], \end{cases}
\]

we have

\[
- \frac{d}{ds} w(s) \leq \lambda \gamma_n^{-2} s^{-2} s^{2+\frac{2}{n}} \int_0^s w(\xi) d\xi. \tag{5.4}
\]

Let

\[
w^\sharp(r) = w \left( \text{AVR}(g) \omega_n r^n \right), \quad \text{for} \quad r \in \left[0, j_{\frac{2}{2}-1}^{-1} \lambda^{-\frac{2}{2}} \right].
\]

By directly calculations and (5.4), we have

\[
\int_{B_\lambda} \left| \nabla g_0 w^\sharp \right|^2 d\mu_{g_0} = \text{AVR}(g)^{-1} \int_0^{\text{AVR}(g)|B_\lambda|} \left(- \frac{d w(s)}{ds}\right)^2 \gamma_n^2 s^{2-\frac{2}{n}} ds \\
\leq \text{AVR}(g)^{-1} \lambda \int_0^{\text{AVR}(g)|B_\lambda|} \left(- \frac{d w(s)}{ds}\right) \int_0^s w(\xi) d\xi ds \\
= \text{AVR}(g)^{-1} \lambda \int_0^{\text{AVR}(g)|B_\lambda|} w^2(s) ds,
\]

and

\[
\int_{B_\lambda} \left( w^\sharp(r) \right)^2 d\mu_{g_0} = \text{AVR}(g)^{-1} \int_0^{\text{AVR}(g)|B_\lambda|} w^2(s) ds.
\]

From the Rayleigh quotient on \( B_\lambda \), we have

\[
\lambda \leq \frac{\int_{B_\lambda} \left| \nabla g_0 w^\sharp \right|^2 (r) d\mu_{g_0}}{\int_{B_\lambda} \left( w^\sharp(r) \right)^2 d\mu_{g_0}} \leq \lambda.
\]

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By using the simplicity of the first eigenvalue and $u^*(0) = v(0)$, we get $c = 1$, which implies

$$v^*(s) \leq u^*(s), \quad \text{for all } s \in [0, \text{AVR}(g) |B_\lambda|].$$

This completes the proof of Lemma (5.1).

**Proof of Theorem 1.8** We normalize $u$ such that

$$\|u\|_{L^p(\Omega)} = \text{AVR}(g)^{\frac{1}{p}} \|v\|_{L^p(B_\lambda)},$$

which is equivalent to

$$\int_0^{\|\Omega\|} (u^*(s))^p \, ds = \int_0^{\text{AVR}(g)|B_\lambda|} (v^*(s))^p \, ds.$$  \hspace{1cm} (5.6)

From Lemma 5.1, we can infer that $v^*(0) \geq u^*(0)$.

Now we prove the theorem by dividing two cases.

Suppose that $v^*(0) = u^*(0)$. Form Lemma 5.1 and (5.6), we have

$$\int_0^{\|\Omega\|} (u^*(s))^p \, ds = \int_0^{\text{AVR}(g)|B_\lambda|} (v^*(s))^p \, ds \leq \int_0^{\text{AVR}(g)|B_\lambda|} (u^*(s))^p \, ds.$$  \hspace{1cm} (5.7)

This implies that $|\Omega|_g = \text{AVR}(g)|B_\lambda|$. Furthermore, $\Omega^\sharp = B_\lambda$ and $u^\sharp = v$, which implies equality cases in (1.19).

Now suppose that $v^*(0) > u^*(0)$. In this case, we get

$$u^*(|\Omega|_g) = 0, \quad v^*(\text{AVR}(g)|B_\lambda|) = 0, \quad \text{and } |B_\lambda| < |\Omega^\sharp|.$$

We claim that there exists only one $s_1 \in [0, \text{AVR}(g)|B_\lambda|]$ such that $v^*(s_1) = u^*(s_1)$. In fact, if one can find $s_2 \in (0, \text{AVR}(g)|B_\lambda|)$ and $s_2 > s_1$, such that

$$u^*(s_2) = v^*(s_2), \quad u^*(s) > v^*(s), \quad \text{for } s \in (s_1, s_2).$$  \hspace{1cm} (5.7)

Defining

$$h(s) = \begin{cases} 
  v^*(s), & s \in [0, s_1] \cup [s_2, \text{AVR}(g)|B_\lambda|], \\
  u^*(s), & s \in (s_1, s_2),
\end{cases}$$

we have

$$-\frac{d}{ds} h(s) \leq \lambda \gamma_n^{-2} s^{-2+\frac{2}{n}} \int_0^s h(\xi)d\xi.$$  \hspace{1cm} (5.8)
Setting
\[ h^\sharp(r) = h^*(\text{AVR}(g)\omega_n r^n), \quad r \in \left[0, \sqrt{\frac{g}{2}} - 1, \lambda^{-\frac{1}{2}} \right], \]
by (5.7) and (5.8), we get
\[ \int_{B_\lambda} |\nabla g^0 h^\sharp| d\mu_g \leq \int_{B_\lambda} (h^\sharp)^2 d\mu_g, \]
which is a contradiction with the minimum property of the first eigenvalue \( \lambda \) of \( \Omega^\sharp \). Therefore, there exists \( s_1 \) such that
\[ u^*(s) \begin{cases} \leq v^*(s), & s \in [0, s_1]; \\ \geq v^*(s), & s \in (s_1, \text{AVR}(g) |B_\lambda|]. \end{cases} \tag{5.9} \]
The function \( v^*(s) \) can be extended to 0 as \( s \in (\text{AVR}(g) |B_\lambda|, |\Omega|_g] \). We claim that
\[ \int_0^s (u^*)^p (\xi) d\xi \leq \int_0^s (v^*)^p (\xi) d\xi, \quad s \in [0, \text{AVR}(g) |B_\lambda|]. \tag{5.10} \]
In fact, if \( s \in [0, s_1] \), (5.10) holds obviously by (5.9). For \( s \in (s_1, \text{AVR}(g) |B_\lambda|] \), we deduce, from (5.6) and (5.9),
\[ \int_0^s (u^*)^p (\xi) d\xi = \int_0^{[\Omega]} (u^*)^p (\xi) d\xi - \int_0^s (u^*)^p (\xi) d\xi \\
= \int_0^{\text{AVR}(g) |B_\lambda|} (u^*)^p (\xi) d\xi - \int_0^s (u^*)^p (\xi) d\xi \\
\leq \int_0^{\text{AVR}(g) |B_\lambda|} (v^*)^p (\xi) d\xi - \int_0^s (v^*)^p (\xi) d\xi \\
= \int_0^s (v^*(\xi))^p d\xi. \]
From Lemma 2.3, we can infer
\[ \int_0^{[\Omega]} (u^*)^q (s) ds \leq \int_0^{[\Omega]} (v^*)^q (s) ds = \int_0^{\text{AVR}(g) |B_\lambda|} (v^*)^q (s) ds, \]
which is equivalent to
\[ \|u\|_{L^q(\Omega)} \leq \text{AVR}(g)^{\frac{1}{q'}} \|v\|_{L^q(B_\lambda)}. \]
Now we assume that the equality holds in (1.19), from (5.5), one can infer that
\[ |\Omega| = \text{AVR}(g) |B_\lambda|. \]
Since $B_\lambda$ and $\Omega^{\sharp}$ the ball center at origin, it yields

$\Omega^{\sharp} = B_\lambda$.

By assumption of Lemma 5.1, $\lambda$ is the first eigenvalue of $B_\lambda$, hence of $\Omega^{\sharp}$. Therefore, we deduce that

$\lambda_1(\Omega) = \lambda_1(\Omega^{\sharp}) = \lambda$.

From Corollary 1.2, we infer that $(M, g)$ is isometric to Euclidean space $(\mathbb{R}^n, g_0)$, $\Omega$ is isometric to Euclidean ball with radius $j_n^{-1} \lambda^{-1/2}$ and $\lambda$ is the first eigenvalue of problem (1.18). This completes the proof of Theorem 1.8. \qed

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