ON A GENERALIZED CLASS OF BOUNDARY VALUE PROBLEMS WITH DELAYED ARGUMENT

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ABSTRACT. In this work, spectrum and asymptotics of eigenfunctions of a generalized class of boundary value problems with a delay are obtained.

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1. Formulation of the problem

In this study we shall investigate discontinuous eigenvalue problems which consist of Sturm-Liouville equation

\[(1) \quad (-p(x)u'(x))' + q(x)u(x - \Delta(x)) = \lambda^2 u(x) = 0\]

on \(\Omega = \bigcup \Omega_\pm\) with boundary conditions

\[(2) \quad \delta_{10} u(a) - \delta_{11} u'(a) - \lambda^2 \left( \tilde{\delta}_{10} u(a) - \tilde{\delta}_{11} u'(a) \right) = 0,\]
\[(3) \quad \delta_{20} u(b) - \delta_{21} u'(b) + \lambda^2 \left( \tilde{\delta}_{20} u(b) - \tilde{\delta}_{21} u'(b) \right) = 0\]

and transmission conditions

\[(4) \quad \gamma_{10}^+ u(c+) + \gamma_{10}^- u(c-) = 0,\]
\[\gamma_{20}^+ u(c+) + \gamma_{21}^+ u'(c+) + \gamma_{20}^- u(c-) + \gamma_{21}^- u'(c-) = 0,\]
\[(5) \quad -\lambda^2 \left( \tilde{\gamma}_{20}^+ u(c+) + \tilde{\gamma}_{21}^+ u'(c+) + \tilde{\gamma}_{20}^- u(c-) + \tilde{\gamma}_{21}^- u'(c-) \right) = 0,\]

where \(p(x) = p_1^+\) for \(x \in \Omega^- = [a, c)\) and \(p(x) = p_2^+\) for \(x \in \Omega^+ = (c, b]\); the real-valued function \(q(x)\) is continuous in \(\Omega\) and has a finite limit \(q(c\pm) = \lim_{x \to c\pm} q(x)\), the real valued function \(\Delta(x) \geq 0\) continuous in \(\Omega\) and has a finite limit \(\Delta(c\pm) = \lim_{x \to c\pm} \Delta(x), x - \Delta(x) \geq a, \text{if } x \in \Omega^-; x - \Delta(x) \geq c, \text{if } x \in \Omega^+; \lambda\) is a real spectral parameter; \(p_i, \delta_{ij}, \tilde{\delta}_{ij}, \gamma_{ij}^\pm, \tilde{\gamma}_{ij}^\pm\) \((i = 1, 2; j = 0, 1)\) are arbitrary real numbers such that \(\gamma_{10}^\pm \left( \tilde{\gamma}_{21}^\pm - \lambda^2 \gamma_{21}^\pm \right) \neq 0\).

Sturm-Liouville problems with transmission conditions (also known as interface conditions, discontinuity conditions, impulse effects) arise in many applications of mathematical physics. Amongst the applications are thermal conduction in a thin laminated plate made up of layers of different materials and diffraction problems [11].

Sturm-Liouville problems with delayed argument is an active area of research and arise in many realistic models of problems in science, engineering, and medicine, where there is a time lag or after-effect (see [3]) and find applications in combustion.
in a liquid propellant rocket engine [5,10] and in systems of the type of an electromagnetic circuit-breaker [8,20]. The articles [1,4,6,9,15,17-19,22] are devoted to investigation of the spectral properties of eigenvalues and eigenfunctions of the Sturm-Liouville problems with delayed argument.

The main goal of this paper is to study the spectrum and asymptotics of eigenfunctions of the problem (1)-(5). Spectral properties of differential equations with delayed argument which contain such a generalized boundary and transmission conditions have not been studied yet. So, the results obtained in this work are extension and generalization of previous works in the literature. For example, if we take \( \Delta(x) \equiv 0 \) and/or \( \tilde{\delta}_{ij} = 0 \) (\( i = 1, 2; \ j = 0, 1 \)) and/or \( \tilde{\gamma}_{2j} = 0 \) (\( j = 0, 1 \)) and/or \( p(x) \equiv 1 \) then the asymptotic formulas for eigenvalues and eigenfunctions correspond to those for the classical Sturm-Liouville problem [2,7,12,13,21]. Moreover, results and methods of these kind of problems can be useful for investigating the inverse problems for partial differential equations.

Let \( \vartheta^-(x, \lambda) \) be a solution of Eq. (1) on \( \Omega^- = \Omega^- \cup \{c\} \), satisfying the initial conditions
\[
\vartheta^-(a, \lambda) = \delta_{11} - \lambda^2 \delta_{11}, \quad \frac{\partial \vartheta^-(a, \lambda)}{\partial x} = \delta_{10} - \lambda^2 \delta_{10}.
\]
The conditions (6) define a unique solution of Eq. (1) on \( \Omega^- \) [16].

After defining the above solution we shall define the solution \( \vartheta^+(x, \lambda) \) of Eq. (1) on \( \Omega^+ = \Omega^+ \cup \{c\} \) by means of the solution \( \vartheta^-(x, \lambda) \) using the initial conditions
\[
\vartheta^+(c+, \lambda) = -\frac{\gamma_{10}}{\gamma_{10}^+} \vartheta^-(c-, \lambda),
\]
\[
\frac{\partial \vartheta^+(c+, \lambda)}{\partial x} = \frac{1}{\gamma_{10}^+ (\gamma_{21}^+ - \lambda^2 \gamma_{21}^+)} 
\]
\[
\times \left[ \frac{\gamma_{10}^+ (\lambda^2 \gamma_{21}^+ - \gamma_{21}^+)}{\partial x} \right] + \left( \frac{\gamma_{10}^+ (\lambda^2 \gamma_{20}^+ - \gamma_{20}^+)}{\partial x} \right) \vartheta^-(c-, \lambda)
\]
The conditions (7)-(8) are defined as a unique solution of Eq. (1) on \( \Omega^+ \).

Consequently, the function \( \vartheta(x, \lambda) \) is defined on \( \Omega \) by the equality
\[
\vartheta(x, \lambda) = \begin{cases} 
\vartheta^-(x, \lambda), & x \in \Omega^-, \\
\vartheta^+(x, \lambda), & x \in \Omega^+
\end{cases}
\]
is a solution of the Eq. (1) on \( \Omega \); which satisfies one of the boundary conditions and both transmission conditions.

2. Spectrum and Asymptotics of Eigenfunctions

We begin by writing the problem (1)-(5) in terms of the following equivalent integral equations.
Lemma 1. Let \( \vartheta(x, \lambda) \) be a solution of Eq. (1) and \( \lambda > 0 \). Then the following integral equations hold:

\[
\vartheta^-(x, \lambda) = \left( \delta_{11} - \lambda^2 \tilde{\delta}_{11} \right) \cos \frac{\lambda(x-a)}{p_1} + \frac{p_1 \delta_{10} - \lambda^2 \tilde{\delta}_{10}}{\lambda} \sin \frac{\lambda(x-a)}{p_1} \\
+ \frac{1}{p_1 \lambda} \int_a^x q(\tau) \sin \frac{\lambda(x-\tau)}{p_1} \vartheta^- (\tau - \Delta (\tau), \lambda) \, d\tau,
\]

(9)

\[
\vartheta^+(x, \lambda) = -\frac{\gamma_{10} \vartheta^-(c-, \lambda)}{\gamma_{10}^+} \cos \frac{\lambda(x-c)}{p_2} + \frac{p_2 \lambda \gamma_{10}^+ (\gamma_{21} - \lambda^2 \tilde{\gamma}_{21})}{\gamma_{10}^+ (\gamma_{21} - \lambda^2 \tilde{\gamma}_{21})} \\
\times \left[ \gamma_{10}^+ (\lambda^2 \tilde{\gamma}_{21} - \gamma_{21}) \frac{\partial \vartheta^-(c-, \lambda)}{\partial x} + (\gamma_{10}^+ (\lambda^2 \tilde{\gamma}_{20} - \gamma_{20}) - \gamma_{10}^+ (\lambda^2 \tilde{\gamma}_{21} - \gamma_{21})) \right] \vartheta^- (c-, \lambda)
\]

(10)

\[
\times \sin \frac{\lambda(x-c)}{p_2} + \frac{1}{p_2 \lambda} \int_{c^+}^x q(\tau) \sin \frac{\lambda(x-\tau)}{p_2} \vartheta^+ (\tau - \Delta (\tau), \lambda) \, d\tau
\]

Proof. To prove this, it is enough to substitute \( \lambda^2 \vartheta^\pm (\tau, \lambda) + \frac{\partial^2 \vartheta^\pm (\tau, \lambda)}{\partial \tau^2} \) instead of \( q(\tau) \vartheta^\pm (\tau - \Delta (\tau), \lambda) \) in (9) and (10) respectively and integrate by parts twice. \( \Box \)

From Lemma 1, using the well-known successive approximation method, it is easy to obtain the following asymptotic expressions of fundamental solutions.

Lemma 2. The following asymptotic estimates

\[
\vartheta^-(x, \lambda) = -\lambda^2 \tilde{\delta}_{11} \cos \frac{\lambda(x-a)}{p_1} + O(\lambda),
\]

\[
\frac{\partial \vartheta^- (x, \lambda)}{\partial x} = \lambda^3 \tilde{\delta}_{11} \sin \frac{\lambda(x-a)}{p_1} + O(\lambda^2),
\]

\[
\vartheta^+(x, \lambda) = \frac{\lambda^4 p_2 \tilde{\delta}_{11} \tilde{\gamma}_{21}}{p_1 \gamma_{21}} \sin \frac{\lambda(c-a)}{p_1} \sin \frac{\lambda(x-c)}{p_2} + O(\lambda^3),
\]

\[
\frac{\partial \vartheta^+ (x, \lambda)}{\partial x} = \frac{\lambda^5 \tilde{\delta}_{11} \tilde{\gamma}_{21}}{p_1 \gamma_{21}^+} \sin \frac{\lambda(c-a)}{p_1} \cos \frac{\lambda(x-c)}{p_2} + O(\lambda^4)
\]

are valid as \( \lambda \to \infty \).

The function \( \vartheta(x, \lambda) \) defined in introduction is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting \( \vartheta(x, \lambda) \) into (3), we get the characteristic equation

\[
\Xi(\lambda) = \delta_{20} \vartheta^+(b, \lambda) - \delta_{21} \frac{\partial \vartheta^+(b, \lambda)}{\partial x} + \lambda^2 \left( \delta_{20} \vartheta^+(b, \lambda) - \delta_{21} \frac{\partial \vartheta^+(b, \lambda)}{\partial x} \right) = 0.
\]

Thus the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Eq. (11).

Theorem 1. The problem (1) – (5) has an infinite set of positive eigenvalues.
Putting the expressions (9), (10), (12) and (13) into (11), we get

\[
\frac{\partial \vartheta^- (x, \lambda)}{\partial x} = \frac{\lambda (\delta_{11} - \lambda^2 \delta_{11})}{p_1} \sin \frac{\lambda (x - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \cos \frac{\lambda (x - a)}{p_1} + \frac{1}{p_1^2} \int_a^x q (\tau) \cos \frac{\lambda (x - \tau)}{p_1} \vartheta^- (\tau - \Delta (\tau), \lambda) \, d\tau.
\]

(12)

Differentiating (10) with respect to \( x \), we get

\[
\frac{\partial \vartheta^+ (x, \lambda)}{\partial x} = \frac{\lambda \gamma_{10}}{p_2} \vartheta^- (c - \lambda, \lambda, \lambda) \sin \frac{\lambda (x - c)}{p_2} + \frac{1}{\gamma_{10} (\gamma_{21} - \lambda^2 \gamma_{21})} \left( \gamma_{10} (\lambda^2 \gamma_{21} - \gamma_{21}) \frac{\partial \vartheta^- (c - \lambda, \lambda)}{\partial x} + \left( \gamma_{10} (\lambda^2 \gamma_{20} - \gamma_{21}) - \gamma_{10} (\lambda^2 \gamma_{20} - \gamma_{20}) \right) \vartheta^- (c - \lambda, \lambda) \right)
\]

(13)

\[
\times \cos \frac{\lambda (x - c)}{p_2} + \frac{1}{p_2^2} \int_a^c q (\tau) \cos \frac{\lambda (x - \tau)}{p_2} \vartheta^+ (\tau - \Delta (\tau), \lambda) \, d\tau.
\]

Putting the expressions (9), (10), (12) and (13) into (11), we get

\[
\Xi (\lambda) \equiv \delta_{20} \left[ \frac{\gamma_{10}}{\gamma_{10}^2} \left( \frac{\delta_{11} - \lambda^2 \delta_{11}}{p_1} \right) \cos \frac{\lambda (c - a)}{p_1} + \frac{p_1 (\delta_{10} - \lambda^2 \delta_{10})}{p_1} \sin \frac{\lambda (c - a)}{p_1} \right.
\]

\[
+ \frac{1}{p_1 \lambda} \int_a^c q (\tau) \sin \frac{\lambda (c - \tau)}{p_1} \vartheta^- (\tau - \Delta (\tau), \lambda) \, d\tau \right] \cos \frac{\lambda (b - c)}{p_2} + \frac{p_2}{\gamma_{10} (\gamma_{21} - \lambda^2 \gamma_{21})} \left( \frac{\lambda (c - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \cos \frac{\lambda (c - a)}{p_1} \right)
\]

\[
\times \left( \gamma_{10} (\lambda^2 \gamma_{21} - \gamma_{21}) \left( \frac{\lambda (\delta_{11} - \lambda^2 \delta_{11})}{p_1} \right) \sin \frac{\lambda (c - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \cos \frac{\lambda (c - a)}{p_1} \right)
\]

\[
+ \frac{1}{p_2^2} \int_a^c q (\tau) \cos \frac{\lambda (c - \tau)}{p_1} \vartheta^- (\tau - \Delta (\tau), \lambda) \, d\tau + \left( \gamma_{10}^+ (\lambda^2 \gamma_{20} - \gamma_{21}) - \gamma_{10}^- (\lambda^2 \gamma_{20} - \gamma_{20}) \right)
\]

\[
\times \left( -\frac{\gamma_{10}^-}{\gamma_{10}^+} \left( \frac{\delta_{11} - \lambda^2 \delta_{11}}{p_1} \right) \cos \frac{\lambda (c - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \sin \frac{\lambda (c - a)}{p_1} \right)
\]

\[
+ \frac{1}{p_1 \lambda} \int_a^c q (\tau) \sin \frac{\lambda (c - \tau)}{p_1} \vartheta^- (\tau - \Delta (\tau), \lambda) \, d\tau \right) \right] \sin \frac{\lambda (b - c)}{p_2} + \frac{p_2}{\gamma_{10} (\gamma_{21} - \lambda^2 \gamma_{21})} \left( \frac{\lambda (c - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \cos \frac{\lambda (c - a)}{p_1} \right)
\]

\[
- \delta_{21} \left[ \frac{\lambda \gamma_{10}^-}{p_2 \gamma_{10}^+} \left( \frac{\lambda (c - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \sin \frac{\lambda (c - a)}{p_1} \right) \right]
\]

\[
- \delta_{21} \left[ \frac{\lambda \gamma_{10}^-}{p_2 \gamma_{10}^+} \left( \frac{\lambda (c - a)}{p_1} + \left( \frac{\delta_{10} - \lambda^2 \delta_{10}}{p_1} \right) \sin \frac{\lambda (c - a)}{p_1} \right) \right]
\]

Proof.
\[ + \frac{1}{p_1 \lambda} \int_a^c q(\tau) \sin \frac{\lambda(c-\tau)}{p_1} \vartheta^- (\tau - \Delta(\tau), \lambda) \, d\tau \left( \sin \frac{\lambda(b-c)}{p_2} \right) + \frac{1}{\gamma_{10}^+ (\gamma_{21}^- - \gamma_{21}^+)} \]

\[ \times \left( \gamma_{10}^+ (\lambda^2 \gamma_{21}^- - \gamma_{21}^-) \right) \left( \frac{\lambda(\delta_{11} - \lambda^2 \delta_{11})}{p_1} \sin \frac{\lambda(c-a)}{p_1} + \left( \delta_{10} - \lambda^2 \delta_{10} \right) \cos \frac{\lambda(c-a)}{p_1} \right) \]

\[ + \frac{1}{p_1 \lambda} \int_a^c q(\tau) \cos \frac{\lambda(c-\tau)}{p_1} \vartheta^- (\tau - \Delta(\tau), \lambda) \, d\tau \left( \cos \frac{\lambda(b-c)}{p_2} \right) + \frac{1}{\gamma_{10}^+ (\gamma_{21}^- - \gamma_{21}^+)} \]

\[ \times \left( \gamma_{10}^+ (\lambda^2 \gamma_{21}^- - \gamma_{21}^-) \right) \left( \frac{\lambda(\delta_{11} - \lambda^2 \delta_{11})}{p_1} \sin \frac{\lambda(c-a)}{p_1} + \left( \delta_{10} - \lambda^2 \delta_{10} \right) \cos \frac{\lambda(c-a)}{p_1} \right) \]
From Lemma 2, the following equalities

\[ \lambda (14) \Xi(\lambda) = \frac{\lambda^7}{\gamma_{21} p_1} \sin \frac{\lambda (c-a)}{p_1} \cos \frac{\lambda (b-c)}{p_2} + O(\lambda^6) = 0. \]

Let \( \lambda \) be sufficiently large. Obviously, for large \( \lambda \) Eq. (14) has, evidently, an infinite set of roots. The proof is complete. \( \square \)

By Theorem 2 we conclude that the problem (1)-(5) has infinitely many nontrivial solutions.

Solving the Eq. (14), we have

\[ \Gamma = \left\{ \lambda_n : \lambda_n = \frac{p_2 \pi (n + \frac{1}{2})}{b - c} + O\left(\frac{1}{n}\right) \text{ or } \lambda_n = \frac{p_1 \pi n}{c - a} + O\left(\frac{1}{n}\right), \ n = 1, 2, \ldots \right\} \]

for the spectrum of (1)-(5).

Now we are ready to present asymptotic expressions of eigenfunctions. Using Lemma 2 and replacing \( \lambda \) by \( \lambda_n \in \Gamma \) we obtain the next theorem. We see that there correspond two eigenfunctions for each \( n \).
Theorem 2. The following asymptotic formulas hold for eigenfunctions of boundary-value-transmission problem (1)-(5) for each \( x \in \Omega \) and \( \lambda \in \Gamma \) \((n = 1, 2, \ldots)\):

\[
\vartheta_{(1)}(x, \lambda) = -\frac{n^2 \pi^2 p_0^2 \delta_{11}}{(c-a)^2} \cos \frac{n \pi (x-a)}{c-a} + O(n),
\]

\[
\vartheta_{(2)}(x, \lambda) = -\frac{(n+\frac{1}{2})^2 \pi^2 p_0^2 \delta_{11}}{(b-c)^2} \cos \frac{(n+\frac{1}{2}) \pi p_2 (x-a)}{p_1 (b-c)} + O(n),
\]

\[
\vartheta^+_{(1)}(x, \lambda) = \frac{n^3 \pi^4 p_0^3 p_2 \delta_{11} \gamma_{21}^+}{\gamma_{21}^+ (c-a)^3} \sin \frac{n \pi p_1 (x-c)}{p_2 (c-a)} + O\left(n^2\right),
\]

\[
\vartheta^+_{(2)}(x, \lambda) = \frac{(n+\frac{1}{2})^4 \pi^4 p_0^3 p_2 \delta_{11} \gamma_{21}^+}{p_1 \gamma_{21}^+ (b-c)^3} \left\{ \sin \frac{(n+\frac{1}{2}) \pi p_2 (c-a)}{p_1 (b-c)} \right. \\
- \frac{\cos \left( (n+\frac{1}{2}) \pi (x-c) \right)}{b-c} - \frac{x-c}{np_2} \sin \left( \frac{(n+\frac{1}{2}) \pi (x-c)}{b-c} \right) \\
\left. - \frac{(n+\frac{1}{2}) \pi p_2 (c-a)}{p_1 (b-c)} \cos \left( \frac{(n+\frac{1}{2}) \pi (x-c)}{b-c} \right) \right\} + O\left(n^2\right).
\]

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