GBDT version of the Darboux transformation for the matrix coupled dispersionless equations (local and non-local cases)

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We introduce matrix coupled (local and non-local) dispersionless equations, construct GBDT (generalized Bäcklund-Darboux transformation) for these equations, derive wide classes of explicit multipole solutions, give explicit expressions for the corresponding Darboux and wave matrix valued functions and study their asymptotics in some interesting cases. We consider the scalar cases of coupled, complex coupled and non-local dispersionless equations as well.

Keywords: matrix coupled dispersionless equation; matrix non-local dispersionless equation; complex dispersionless equation; Darboux matrix; generalized matrix eigenvalue; transfer matrix function.

1. Introduction

The coupled dispersionless equations are integrable systems, which are actively studied since the important works [1, 2] (see e.g. [3–8] and various references therein). In particular, the coupled dispersionless equations in the form

\[ q_{tx} + (rs)_t = 0, \quad r_{tx} = 2qt r, \quad s_{tx} = 2qt s, \tag{1.1} \]

which describe a current-fed string within an external magnetic field, were introduced and actively studied by H. Kakuhata and K. Konno (see, e.g., [1, 9]). Also see [10, 11] for further results and references. In [12], a special case of (1.1) with \( r = \overline{s} \), where \( \overline{s} \) is the complex conjugate of \( s \), was solved by the inverse scattering transform method and found to be equivalent to the Pohlmeyer–Lund–Regge equations. The so-called complex coupled dispersionless equations

\[ \rho_x + \frac{1}{2} \kappa |v|^2 t = 0, \quad v_{tx} = \rho v, \quad (\rho = \overline{\rho}, \quad \kappa = \pm 1) \tag{1.2} \]

play an essential role in the study of the short pulse equations (see [1, 5, 13] and references therein). Moreover, these equations are connected with the Kadomtsev–Petviashvili hierarchy and describe (via hodograph transformations) the motion of space curves [13].

Non-local non-linear integrable equations attracted essential attention during the last years (see the important papers [3, 4, 14–17] and numerous references therein), starting from the article [18] on the non-local non-linear Schrödinger equation. The non-local (scalar) dispersionless equations were considered in [3, 4, 19].

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We will construct GBDT (generalized Bäcklund-Darboux transformation) for the matrix coupled
dispersionless equations. In particular, we will apply the obtained results to the systems (1.1) and (1.2)
and will also obtain new explicit solutions for the non-local scalar case
\[
\rho_i(x,t) = \frac{(-1)^p}{2}(v(x,t)v_i(-x,t) + v_i(x,t)v(-x,t)),
\]
\[
\nu_i(x,t) = \rho(x,t)v(x,t) \quad (\rho(x,t) = -\rho(-x,t)).
\]

We consider first the matrix generalization of the coupled dispersionless equations (MCDE):
\[
R_x = \frac{(-1)^p}{2}(VV_t + V_tV), \quad V_t = \frac{1}{2}(VR + RV) \quad \left(R_x = \frac{\partial}{\partial x}R\right),
\]
\[
R(x,t) = \text{diag}(\rho_1(x,t), \rho_2(x,t)), \quad V(x,t) = \begin{bmatrix}
0 & v_1(x,t) \\
v_2(x,t) & 0
\end{bmatrix},
\]
where diag stands for the block diagonal matrix, the blocks \( \rho_k \) are \( m_k \times m_k \) matrix functions \( k = 1, 2; \ m_k > 0 \), \( v_1 \) is an \( m_1 \times m_2 \) matrix function, \( v_2 \) is an \( m_2 \times m_1 \) matrix function and \( R \) and \( V \) are \( m \times m \) matrix functions \( m := m_1 + m_2 \). Clearly, it suffices for \( p \) in (1.5) to take the values \( \{0, 1\} \), that is, \( p \) is either 0 or 1.

It is easy to see that system (1.5) is equivalent to the compatibility condition
\[
G_t - F_x + [G, F] = 0, \quad [G, F] := GF - FG,
\]
of the following auxiliary linear systems:
\[
w_x = G(x,t,\lambda)w; \quad G(x,t,\lambda) := \frac{i}{4}\lambda j - \frac{i}{2}j^{p+1}V(x,t), \quad j := \begin{bmatrix}
I_{m_1} & 0 \\
0 & -I_{m_2}
\end{bmatrix},
\]
where \( i \) stands for the imaginary unit \( (i^2 = -1) \), \( I_k \) is the \( k \times k \) identity matrix, and
\[
w_i = F(x,t,\lambda)w, \quad F(x,t,\lambda) := \left(-ijR(x,t) + j^pV_i(x,t)\right)/\lambda.
\]
The complex coupled dispersionless equations (1.2) appear, when we set in (1.5), (1.8) and (1.9):
\[
m_1 = m_2 = 1, \quad \rho_1 = \rho_2 = \rho = \bar{\rho}, \quad v_1 = v, \quad v_2 = \bar{v}, \quad p = (1 + \varepsilon)/2.
\]

We note that the generalized coupled dispersionless system in [1, 7] is more general than MCDE
(1.5). However, MCDE is more specific and it is also well known (see e.g. [20]) that the matrix
and multicomponent generalizations are of interest in applications.

Soliton and more general multipole solutions are of a special interest for integrable systems, and
various versions of Bäcklund–Darboux transformations provide important tools for their construction,
see (for instance) [21–24] and references therein. See also [25–30] for some other important methods
to construct explicit solutions. (Closely related results are also used in the study of linear Schrödinger,
generalized Schrödinger and Dirac systems, see, for instance, [31–33].) For the case of MCDE, we
introduce the GBDT version of the Bäcklund–Darboux transformation. We note that a constant matrix
A is used (instead of eigenvalues) as a set of parameters in GBDT, and Darboux matrices, wave functions
w and various multipole solutions are easily constructed in this way (see e.g. [33–35] and references therein). The Darboux matrices for the generalized coupled dispersionless systems given in [7] were constructed in [6] by an iterative procedure and for a special case of diagonal generalized eigenvalues. GBDT allows to achieve an essential progress in this respect since neither the diagonal structure of the generalized eigenvalues nor iterative procedure are required there.

We will also study the non-local case $R(-x) = -R(x)^*$, $V(-x) = V(x)^*$, that is,

$$R(-x) = -R(x)^*, \quad v_1(x) = v(x), \quad v_2(x) = v(-x)^*. \tag{1.11}$$

We will develop further the non-local results from [36], introduce GBDT for the non-local equations (1.5), (1.11) and construct the corresponding explicit solutions and wave functions. The explicit construction of the wave functions is new even for the local and non-local scalar dispersionless equations.

In Section 2.3, we also consider asymptotics of the Darboux matrix functions (Darboux matrices) and of the wave matrix functions (wave functions) in the case of explicit solutions.

In the article, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the real axis, $\mathbb{C}$ stands for the complex plane and $\mathbb{C}_+$ ($\mathbb{C}_-$) stands for the open upper (lower) half-plane. The spectrum of a square matrix $A$ is denoted by $\sigma(A)$. The notation diag for diagonal (or block diagonal) matrix was explained after formula (1.6).

2. GBDT for the matrix coupled dispersionless equations

2.1 Preliminaries

1. GBDT, which we consider here, is a particular case of the GBDT introduced in [35, Theorem 1.1]. After fixing some $n \in \mathbb{N}$, each GBDT for MCDE (1.5) is determined by the initial system (1.5) itself and by five parameter matrices with complex-valued entries: three $n \times n$ invertible parameter matrices $A_1$, $A_2$ and $S(0,0)$ (det $A_i \neq 0$, $i = 1, 2$; det $S(0,0) \neq 0$), and two $n \times m$ parameter matrices $\Pi_1(0,0)$ and $\Pi_2(0,0)$ such that

$$A_1S(0,0) - S(0,0)A_2 = \Pi_1(0,0)\Pi_2(0,0)^*. \tag{2.1}$$

2. In our next step, we introduce matrix functions $S(x,t)$, $\Pi_1(x,t)$ and $\Pi_2(x,t)$. Darboux matrix and transformed solutions will be expressed in terms of these matrix functions. Similar to [35], we use coefficients $q_1(x,t)$, $q_0(x,t)$ and $Q_{-1}(x,t)$ in $G$ and $F$:

$$G = -\lambda q_1 - q_0, \quad F = -\frac{1}{\lambda}Q_{-1}. \tag{2.2}$$

Hence, in view of (1.8) and (1.9) we have

$$q_1 = -\frac{i}{4}j, \quad q_0 = \frac{i}{2}j^{p+1}V, \quad Q_{-1} = ijR - j^pV. \tag{2.3}$$

If (1.5) holds (and $V_t$ is continuous with respect to both variables combined), then the following linear differential systems are compatible and (jointly with the initial values $S(0,0)$, $\Pi_1(0,0)$ and $\Pi_2(0,0)$) determine matrix functions $S(x,t)$, $\Pi_1(x,t)$ and $\Pi_2(x,t)$, respectively:

$$(\Pi_1)_x = \sum_{i=0}^1 A_i^t\Pi_1 q_i, \quad (\Pi_1)_t = A_1^{-1}\Pi_1 Q_{-1}; \tag{2.4}$$
(\Pi_2)_t = - \sum_{i=0}^{1} (A_x^i)_{\Pi_2} q_i^*, \quad (\Pi_2)_t = -(A_x^2)^{-1}_{\Pi_2} Q^{-1}_{x-1}; \quad (2.5)

S_x = \Pi_1 q_1, \quad S_t = -A^{-1}_{x-1} \Pi_1 Q^{-1}_{x-1} A_{x-1}^{-1}. \quad (2.6)

Although the point \( x = 0, t = 0 \) is chosen above as the initial point, it is easy to see that any other point may be chosen for this purpose as well. Consider \( S(x, t), \Pi_1(x, t) \) and \( \Pi_2(x, t) \) in some domain \( D \), for instance,

\[ D = \{ (x, t) : -\infty \leq a_1 < x < a_2 \leq \infty, \quad -\infty \leq b_1 < t < b_2 \leq \infty \}, \]

such that \( R(x, t) \) and \( V(x, t) \) of the form (1.6) are well defined in \( D \) and satisfy (1.5), and such that \( (0, 0) \in D \). Then \( S(x, t), \Pi_1(x, t) \) and \( \Pi_2(x, t) \) are well defined and the identity

\[ A_1 S(x, t) - S(x, t) A_2 = \Pi_1(x, t) \Pi_2(x, t)^* \quad (2.7) \]

follows from (2.1) and (2.3)–(2.6) [35].

3. When \( R \) and \( V \) satisfy MCDE (1.5), (1.6), GBDT transforms initial auxiliary systems (1.8) and (1.9) into transformed systems

\[ \tilde{w}_x = \tilde{G} \tilde{w}, \quad \tilde{w}_t = \tilde{F} \tilde{w}, \quad (2.8) \]

where \( \tilde{G} \) and \( \tilde{F} \) are obtained by the substitution of \( \tilde{V} \) and \( \tilde{R} \) (instead of \( V \) and \( R \)) into the expressions for \( G \) and \( F \) in (1.8) and (1.9), respectively. The formulas for \( \tilde{G} \) and \( \tilde{F} \), and for \( \tilde{V} \) and \( \tilde{R} \) are studied in the Appendix. Namely, we have:

\[
\tilde{V} = \begin{bmatrix} 0 & \tilde{v}_1 \\ \tilde{v}_2 & 0 \end{bmatrix} := V + \frac{1}{2} j^p (X_0 - jX_0),
\]

\[
\tilde{R} = \frac{1}{2} (Q_{x-1} j + jQ_{x-1}), \quad \tilde{Q}_{x-1} := (I_m - X_{x-1}) Q_{x-1} (I_m + Y_{x-1}),
\]

\[
X_0 := \Pi_2^* S^{-1} \Pi_1, \quad X_{x-1} := \Pi_2^* S^{-1} A_{x-1}^{-1} \Pi_1, \quad Y_{x-1} := \Pi_2^* A_{x-1}^{-1} S^{-1} \Pi_1. \quad (2.10)
\]

Introduce (in the points of invertibility of \( S(x, t) \) in \( D \)) the matrix functions

\[ w_A(x, t, \lambda) = I_m - \Pi_2(x, t)^* S(x, t)^{-1} (A_1 - \lambda I_s)^{-1} \Pi_1(x, t). \quad (2.12) \]

In view of (2.7), the matrix function \( w_A(x, t, \lambda) \) is the so-called transfer matrix function in Lev Sakhnovich’s form (see [33, 37, 38] and references therein) at each point \((x, t)\) of invertibility of \( S(x, t) \).

**Remark 2.1** The transfer matrix function \( w_A \) plays fundamental role in GBDT. It is shown in Theorem 2.2 below that the Darboux matrix, which transforms the wave function \( w \) of the initial auxiliary systems (1.8), (1.9) into the wave function \( \tilde{w} \) of the transformed auxiliary systems (2.8), is given by \( w_A(x, t, \lambda) \). That is, (2.8) holds for the matrix function \( \tilde{w} = w_A w \) (if \( w \) satisfies (1.8) and (1.9)).
2.2 Darboux matrix

**Theorem 2.2** Let $R$ and $V$ have the form (1.6), let $V_j(x, t)$ be a continuous function of $(x, t)$, and let $R$ and $V$ satisfy MCDE (1.5) in $D$. Assume that three $n \times n$ parameter matrices $A_1$, $A_2$ and $S(0, 0)$ (det $A_i \neq 0$, $i = 1, 2$; det $S(0, 0) \neq 0$), and two $n \times m$ parameter matrices $\Pi_1(0, 0)$ and $\Pi_2(0, 0)$ are given, and that the matrix identity (2.1) holds.

Then, $\Pi_1(x, t)$, $\Pi_2(x, t)$, $S(x, t)$ and $w(x, t, \lambda)$ (where $w$ is the wave function, i.e., $w(x, t, \lambda)$ satisfies (1.8), (1.9) and det $w(0, 0, \lambda) \neq 0$) are well defined in $D$. Moreover, in the points of invertibility of $S(x, t)$ in $D$, the matrix functions $R$ and $V$ given by (2.9) and (2.10), respectively, have the form (1.6) and satisfy (1.5), that is,

\[
\begin{align*}
\tilde{R}_i &= \frac{(-1)^p}{2}(\tilde{V}V_i + \tilde{V}_iV), \\
\tilde{R}(x, t) &= \text{diag}(\tilde{\rho}_1(x, t), \tilde{\rho}_2(x, t)), \\
\tilde{V}(x, t) &= \begin{bmatrix} 0 & \tilde{v}_1(x, t) \\ \tilde{v}_2(x, t) & 0 \end{bmatrix}. 
\end{align*}
\] (2.13)

The wave function $\tilde{w}$ (def $\tilde{w}(0, 0, \lambda) \neq 0$), which corresponds to the transformed MCDE (2.13), is given by the product $w_A w$:

\[
\begin{align*}
\tilde{w}(x, t, \lambda) &= w_A(x, t, \lambda)w(x, t, \lambda); \\
\tilde{w}_i &= \tilde{G}\tilde{w}, \\
\tilde{w}_t &= \tilde{G}\tilde{w}; \\
\tilde{G}(x, t, \lambda) &= \frac{i}{4} \lambda + \frac{i}{2} j^{-1} \tilde{V}(x, t), \\
\tilde{F}(x, t, \lambda) &= \left( -ij\tilde{R}(x, t) + j\tilde{v}(x, t) \right) / \lambda. 
\end{align*}
\] (2.14)

**Proof.** It follows from [39] that $\Pi_1$, $\Pi_2$, $S$ and $w$ are well defined. Then, according to (1.8), (1.9) and Proposition A.1, $\tilde{w}$ determined by the first equality in (2.15) satisfies the second and third equalities in (2.15), where $\tilde{G}$ and $\tilde{F}$ are given by (A.3). Moreover, the second and third equalities in (2.15) imply that the compatibility condition

\[
\tilde{G}_i - \tilde{F}_i + [\tilde{G}, \tilde{F}] = 0
\]

holds. Relations (A.3), (A.4) and (A.7) imply that (2.16) holds. Relations (A.3) and (A.11) yield (2.17). Moreover, according to the first equalities in (2.9) and (2.10), the matrix functions $R$ and $V$ have the form (2.14).

Taking into account (2.14), (2.16) and (2.17), one can see that $\tilde{G}$ and $\tilde{F}$ have the same structure as $G$ and $F$, respectively. Thus, the compatibility condition $\tilde{G}_i - \tilde{F}_i + [\tilde{G}, \tilde{F}] = 0$ yields (2.13) in the same way as (1.7) yields (1.5). \hfill $\square$

It is convenient to partition both $\Pi_1$ and $\Pi_2$ into $n \times m_1$ and $n \times m_2$ blocks:

\[
\Pi_1 = [\Phi_1 \Phi_2], \quad \Pi_2 = [\Psi_1 \Psi_2].
\] (2.18)

The simplest cases where explicit solutions appear are the cases $V = 0$ and $R = I_m$, $R = j$, $R = iI_m$ or $R = ij$. For instance, when $V = 0$ and $R = I_m$ we obtain (in view of (2.3)–(2.5)) that

\[
\Phi_1(x, t) = \exp \left\{ -i (x/4)A_i - tA_i^{-1} \right\} \Phi_1(0, 0),
\] (2.19)
Example 2.3 Consider the case of trivial $V$ and constant diagonal matrix $R$ ($R = D$) with the entries $d_i$ (or, written in the block form, blocks $D_1$ and $D_2$) on the main diagonal:

\[ V(x, t) = 0, \quad R(x, t) = D = \text{diag}(d_1, d_2, \ldots, d_m) = \text{diag}(D_1, D_2), \quad D_1 = \text{diag}(d_1, \ldots, d_m), \quad D_2 = \text{diag}(d_{m+1}, \ldots, d_m). \]

We set also

\[ n = 1, \quad A_1 = a_1 \in \mathbb{C} \setminus \{0\}, \quad A_2 = a_2 \in \mathbb{C} \setminus \{0\}, \quad a_1 \neq a_2. \]

Then relations (2.3)–(2.5), (2.23) and (2.25) yield

\[ \begin{align*}
\Phi_1(x, t) &= \Phi_1(0, 0) \exp \left\{ -i \left( (x/4)a_1 I_{m_1} - (t/a_1) D_1 \right) \right\}, \\
\Phi_2(x, t) &= \Phi_2(0, 0) \exp \left\{ i \left( (x/4)a_1 I_{m_2} - (t/a_1) D_2 \right) \right\}, \\
\Psi_1(x, t) &= \Psi_1(0, 0) \exp \left\{ -i \left( (x/4)a_2 I_{m_1} - (t/a_2) D_1^* \right) \right\}, \\
\Psi_2(x, t) &= \Psi_2(0, 0) \exp \left\{ i \left( (x/4)a_2 I_{m_2} - (t/a_2) D_2^* \right) \right\},
\end{align*} \]

where $\Phi_i$ and $\Psi_i$ are vector (row) functions. The function $S(x, t)$ may be found using (2.7) and (2.26)–(2.29):

\[ S(x, t) = (a_1 - a_2)^{-1} (\Phi_1(x, t) \Psi_1(x, t)^* + \Phi_2(x, t) \Psi_2(x, t)^*), \]

\[ \begin{align*}
\Phi_1(x, t) \Psi_1(x, t)^* \\
= \Phi_1(0, 0) \exp \left\{ -i \left( (x/4)(a_1 - a_2) I_{m_1} - (t/a_1) - (t/a_2) D_1 \right) \right\} \Psi_1(0, 0)^*, \\
\Phi_2(x, t) \Psi_2(x, t)^* \\
= \Phi_2(0, 0) \exp \left\{ i \left( (x/4)(a_1 - a_2) I_{m_2} - (t/a_1) - (t/a_2) D_2 \right) \right\} \Psi_2(0, 0)^*.
\end{align*} \]

Using (2.9) and (2.14), we derive

\[ \begin{align*}
\tilde{\nu}_1(x, t) &= \frac{1}{S(x, t)} \Psi_1(x, t)^* \Phi_2(x, t), \\
\tilde{\nu}_2(x, t) &= \frac{(-1)^p}{S(x, t)} \Psi_2(x, t)^* \Phi_1(x, t),
\end{align*} \]

where $\Phi_i$, $\Psi_i$ and $S$ are given explicitly in (2.26)–(2.32). Finally, from (2.11), (2.10) and (2.14) we obtain

\[ \begin{align*}
\tilde{\rho}_1(x, t) &= \left( \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{a_1 S(x, t)} \Psi_1(x, t)^* \begin{bmatrix} \Phi_1(x, t) & \Phi_2(x, t) \end{bmatrix} \right) fD \\
&\times \left( \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{a_2 S(x, t)} \begin{bmatrix} \Psi_1(x, t)^* & \Psi_2(x, t)^* \end{bmatrix} \Phi_1(x, t) \right).
\end{align*} \]
and we have a similar formula for $\tilde{\rho}_2$ as well. Clearly, taking into account (2.12), (2.18) and (2.26)–(2.32) we have also an explicit formula for the Darboux matrix $w_A$.

### 2.3 Local matrix dispersionless equations and asymptotics of the Darboux matrix

Let us set in (1.6)

$$v_1(x, t) = v(x, t), \quad v_2(x, t) = v(x, t)^*; \quad R(x, t) = R(x, t)^*,$$

i.e., $\rho_i(x, t) = \rho_i(x, t)^*$ $(i = 1, 2)$.

Then, MCDE (1.5) takes the form of the local matrix dispersionless equation

$$v_{tx} = (\rho_1 v + v \rho_2)/2, \quad (\rho_1)_x = (-1)^p((vv^*)_t)/2, \quad (\rho_2)_x = (-1)^p((v^*v)_t)/2. \quad (2.36)$$

Put in (2.4)–(2.7),

$$A_1 = A, \quad A_2 = A^*, \quad \Pi_1(x, t) = \Pi(x, t), \quad \Pi_2(0, 0) = -i\Pi(0, 0)j^{p+1},$$

$$S(0, 0) = S(0, 0)^*. \quad (2.37, 2.38)$$

We will show that GBDT of the initial solutions of (2.36) into the transformed solutions of (2.36) is determined by the triple of matrices $\{A, S(0, 0), \Pi(0, 0)\}$, where $\det A \neq 0$,

$$AS(0, 0) - S(0, 0)A^* = i\Pi(0, 0)j^{p+1}\Pi(0, 0)^*, \quad (2.39)$$

and (2.38) holds. According to (2.3) and (2.34), we have

$$(Q - j^{p+1})^* = -Q - j^{p+1}. \quad (2.40)$$

It follows from (2.4), (2.5) and from (2.37), (2.40) that

$$\Pi_2(x, t) = -i\Pi(x, t)j^{p+1}. \quad (2.41)$$

Equations (2.4)–(2.7) take the form

$$\Pi_x = A\Pi q_1 + \Pi q_0, \quad \Pi_t = A^{-1}\Pi Q^{-1}, \quad (2.42)$$

$$S_x = \Pi j^{p+1}\Pi^*/4, \quad S_t = A^{-1}\Pi(j^{p+1}jV_t)\Pi(A^*)^{-1}, \quad (2.43)$$

$$AS(x, t) - S(x, t)A^* = i\Pi(x, t)j^{p+1}\Pi(x, t)^*. \quad (2.44)$$

Relations (2.38) and (2.43) yield

$$S(x, t) = S(x, t)^*.$$

Next, we show that

$$\tilde{R}(x, t) = \tilde{R}(x, t)^*, \quad \tilde{V}(x, t) = \tilde{V}(x, t)^*$$
in the case considered in this subsection. In other words, if (2.34) holds for the MCDE solutions, then (2.34) holds for the GBDT-transformed solutions as well. Indeed, relations (A.5), (2.11), (2.37), (2.40), (2.41) and (2.45) imply that

\[
(\tilde{Q}_{-j}^{p+1})* = -\tilde{Q}_{-j}^{p+1}.
\]

(2.47)

From the first equalities in (1.6) and (2.10), and from (2.47) we derive the first equality in (2.46). Taking into account (2.41), we rewrite (2.9) in the form

\[
\tilde{V} = V + \frac{i}{2} (j\Pi S^{-1} \Pi - \Pi S^{-1} \Pi j) \quad \left( V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right),
\]

(2.48)

and the second equality in (2.46) follows. Now, Theorem 2.2 yields the following corollary.

**Corollary 2.4** Let the matrix functions \(v(x,t), \rho_1(x,t), \rho_2(x,t)\) of sizes \(m_1 \times m_2, m_1 \times m_1\) and \(m_2 \times m_2\), respectively, satisfy the local matrix dispersionless equation (2.36) and the equalities (2.35), and let \(v(x,t)\) be continuous in \(D\). Assume that the parameter matrices \(A (\det A \neq 0), S(0,0) = S(0,0)^*\) and \(\Pi(0,0)\) satisfy (2.39).

Then, the matrix functions \(\tilde{v}, \tilde{\rho}_1, \tilde{\rho}_2\) given by (2.48) and equalities

\[
\tilde{V} = \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \tilde{V} \begin{bmatrix} 0 & 0 \\ I_{m_2} & 0 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\rho}_1 & 0 \\ 0 & \tilde{\rho}_2 \end{bmatrix} = \frac{i}{24} (\tilde{Q}_{-j} + j\tilde{Q}_{-1}),
\]

\[
\tilde{Q}_{-1} = (I_{m} - ij^{p+1} \Pi S^{-1} A^{-1} \Pi)(ijR_t - j^p V_t)(I_{m} + ij^{p+1} \Pi^* (A^*)^{-1} S^{-1} \Pi),
\]

(2.49)

(2.50)

where \(S(x,t)\) and \(\Pi(x,t)\) are determined by (2.42) and (2.43), satisfy the local matrix dispersionless equation and the equalities \(\tilde{\rho}_i(x,t) = \tilde{\rho}_i(x,t)^* \quad (i = 1, 2)\).

The corresponding Darboux matrix \(w_A\) takes the form

\[
w_A(x,t,\lambda) = I_{m} - ij^{p+1} \Pi(x,t)^* S(x,t)^{-1} (A - \lambda I_n)^{-1} \Pi(x,t).
\]

(2.51)

Further in this subsection, we consider the case

\[
R(x,t) \equiv R(t), \quad v(x,t) \equiv 0
\]

(2.52)

and study the asymptotics of \(v(x,t)\) and \(w_A(x,t)\) as \(x \to \infty\). The asymptotics of \(v(x,t)\) and \(w_A(x,t)\) as \(x \to -\infty\) can be studied in the same way.

Formula (2.15) for the fundamental solution \(\tilde{w}\) of the auxiliary systems (for the wave function) takes in this case the form

\[
\tilde{w}(x,t,\lambda) = w_A(x,t,\lambda)e^{ix\lambda/4}w(t,\lambda); \quad w_j(t,\lambda) = \frac{1}{i\lambda} jR(t)w(t,\lambda),
\]

\[
(2.53)
\]

where \(w_A\) is given by (2.51). Hence, the asymptotics of the wave function with respect to \(x\) is described by the asymptotics of the Darboux matrix \(w_A\). Moreover, when we partition \(\Pi(0,t)\) into the \(n \times m_1\) and \(n \times m_2\) blocks \(\Phi_1\) and \(\Phi_2\), we have

\[
\Pi(x,t) = \begin{bmatrix} e^{xA/(4i)} \Phi_1(0,t) & e^{-xA/(4i)} \Phi_2(0,t) \end{bmatrix}.
\]

(2.54)
In view of (2.43), under the assumptions

\[ S(0, 0) > 0, \quad \text{sgn}(t)\eta R(t) \geq 0 \]  

(2.55)

we have

\[ S(0, t) > 0; \quad S(x, t) > 0 \quad \text{for} \quad p = 0, \quad x \geq 0. \]  

(2.56)

When \( p = 1 \) and (2.55) holds, relations (2.43) and (2.44) yield

\[ \left(e^{i\lambda t/4}S(x, t)e^{-i\lambda t/4}\right)' \leq 0, \quad \left(e^{-i\lambda t/4}S(x, t)e^{i\lambda t/4}\right)' \geq 0. \]  

(2.57)

Since \( S(0, t) > 0 \), inequalities (2.57) imply that \( \det S(x, t) \neq 0 \) and \( \tilde{\nu}(x, t) \) is well defined for all \( x \in \mathbb{R} \).

We note that \( w_A(4x, t, \lambda)e^{i\lambda t} \) (for each fixed value \( t \)) is the fundamental solution of the ‘normalized’ Dirac system

\[ y' = i(\lambda j + \eta^{p+1}\tilde{\nu}_N(x))y, \quad \tilde{\nu}_N = \begin{bmatrix} 0 & -2\tilde{\nu}(4x, t) \\ -2\tilde{\nu}(4x, t)^* & 0 \end{bmatrix}. \]  

(2.58)

Systems (2.58) as the systems generated by the triples \( \{A, S(0, t), \Pi(0, t)\} \) have been studied in a series of papers (see [33, 40, 41] and references therein). In particular, Weyl functions of the systems (2.58) are rational, and inverse problems to recover systems from the rational Weyl functions have unique and explicit solutions.

Consider the case \( p = 0 \). According to [42, (3.13)] we have

\[ \tilde{\nu}(x, t) \in L^2_{m_1 \times m_2}(\mathbb{R}^+), \quad \tilde{\nu}(x, t) \to 0 \quad \text{for} \quad x \to \infty. \]  

(2.59)

Without changing \( \tilde{\nu} \) and \( w_A \) we may choose \( n, A, S(0, t) > 0 \) and \( \Pi(0, t) \) (see [42]) such that

\[ \sigma(A) \in (\mathbb{C}^- \cup \mathbb{R}), \quad \text{span} \bigcup_{k=0}^{n-1} \text{im} (A^k \Phi_1(0, t)) = \mathbb{C}^n, \]  

(2.60)

where \( \text{im} \) stands for image and the second equality in (2.59) means that the pair \( \{A, \Phi_1(0, t)\} \) is controllable. Then, we have the following asymptotic relation [42, (3.28)]

\[ w_A(x, t, \lambda) = \begin{bmatrix} I_{m_1} & 0 \\ 0 & \chi(t, \lambda) \end{bmatrix} + o(1) \quad \text{as} \quad x \to \infty, \]  

(2.61)

\[ \chi(t, \lambda) := I_{m_2} + i\Phi_2(0, t)^* \varepsilon(t)(A - \lambda I_n)^{-1}\Phi_2(0, t), \]  

(2.62)

where \( \varepsilon(t) = \lim_{x \to \infty} \left(e^{-i\lambda t}S(x, t)e^{i\lambda t}\right)^{-1} \), and this limit always exists.

**Corollary 2.5** Let \( p = 0 \) and assume that (2.55) holds. Then \( \tilde{\nu}(x, t) \) does not have singularities when \( x \geq 0 \), relations (2.59) are valid and (under a proper choice of \( n, A, S(0, t) > 0 \) and \( \Pi(0, t) \)) equality (2.61) holds.
When (2.55) holds (and so \( S(0, t) > 0 \)), one can combine [41, Theorem 2.5] (see also references therein) and [42, Theorem 3.7] in order to obtain the existence of the Jost solutions
\[
\tilde{w}_L = (i/4)(\lambda j - 2j\tilde{V}(x, t))\tilde{w}_L; \quad \tilde{w}_L(x, t, \lambda) = e^{(i/4)xj}(I_m + o(1)) \quad (2.63)
\]
for \( \lambda \in \mathbb{R} \) and \( x \to \infty \). Moreover, from the above-mentioned theorems follows the expression for the reflection coefficient \( R_L \) of the form
\[
R_L(t, \lambda) := \begin{bmatrix} 0 & I_{m_2} \\ I_{m_1} \end{bmatrix} \tilde{w}_L(0, t, \lambda) \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{w}_L(0, t, \lambda) \\ 0 \end{bmatrix}^{-1}.
\]

**Corollary 2.6** Let \( p = 0 \) and assume that (2.55) holds. Then the reflection coefficient \( R_L \) of system (2.63) on the semi-axis \([0, \infty)\) is given by the formula
\[
R_L(t, \lambda) = -i\Phi_2(0, t)^\ast S(0, t)^{-1}(\lambda I_n - \theta)^{-1}\Phi_1(0, t), \quad (2.64)
\]
where \( \theta = A - i\Phi_1(0, t)\Phi_1(0, t)^\ast S(0, t)^{-1} \).

Consider the case \( p = 1 \). We have shown that \( \tilde{v}(x, t) \) does not have singularities if (2.55) holds. Moreover, the equality
\[
\lim_{x \to \infty} \tilde{v}(x, t) = 0
\]
is valid [40, Corollary 3.6]. The asymptotics of \( w_A \) can be studied in a way similar to the case \( p = 0 \) (see [42]) although the result is somewhat more complicated.

The complex coupled dispersionless equation, which we will consider in Section 4, is a scalar subcase of the local matrix dispersionless equation (2.36).

### 3. GBDT for the non-local matrix dispersionless equations

Recall that the non-local matrix dispersionless equations are characterized by the equalities (1.11). Equivalently, the non-local matrix dispersionless equations (NLDE) are equations (1.5), where \( R \) and \( V \) have the form
\[
R(x, t) = \text{diag}(\rho_1(x, t), \rho_2(x, t)) = -R(-x, t)^\ast, \quad (3.1)
\]
\[
V(x, t) = \begin{bmatrix} 0 & v(x, t) \\ v(-x, t)^\ast & 0 \end{bmatrix}, \quad \text{i.e.,} \quad V(x, t) = V(-x, t)^\ast. \quad (3.2)
\]

In the non-local case, we assume (3.1) and (3.2) instead of (1.6) and (similar to the Section 2.3) determine GBDT by three parameter matrices. However, these matrices satisfy somewhat different relations. Namely, we set
\[
A_1 = A \quad (\det A \neq 0), \quad A_2 = -A^\ast, \quad \Pi_1(x, t) = \Pi(x, t), \quad (3.3)
\]
\[
\Pi_2(0, 0) = -i\Pi(0, 0)^\ast, \quad S(0, 0) = -S(0, 0)^\ast \quad (\det S(0, 0) \neq 0). \quad (3.4)
\]
so that the identity (2.1) takes the form

$$A S(0,0) + S(0,0) A^* = i \Pi(0,0) j^p \Pi(0,0)^*.$$  \hfill (3.5)

It easily follows from (2.4) to (2.6) that (3.3) and (3.4) yield

$$\Pi_2(x,t) \equiv -i \Pi(-x,t) j^p, \quad S(x,t) \equiv -S(-x,t)^*.$$  \hfill (3.6)

Thus, the identity (2.7) takes the form

$$A S(x,t) + S(x,t) A^* = i \Pi(x,t) j^p \Pi(-x,t)^*.$$  \hfill (3.7)

In view of (3.6), we have $X_0 = j^p \Pi(-x,t)^* S(x,t)^{-1} \Pi(x,t)$ and formula (2.9) takes the form

$$\tilde{V}(x,t) = V + \frac{i}{2} \left( \Pi(-x,t)^* S(x,t)^{-1} \Pi(x,t) - j \Pi(-x,t)^* S(x,t)^{-1} \Pi(x,t) j^p \right).$$  \hfill (3.8)

Let us again partition $\Pi$ into two blocks: $\Pi = [\Phi_1 \quad \Phi_2]$, where $\Phi_1$ is an $m \times m_1$ matrix function. Now, (3.2) and (3.8) imply that

$$\tilde{V}(x,t) = \begin{bmatrix} 0 & \tilde{V}(x,t)^* \\ \tilde{V}(x,t) & 0 \end{bmatrix} = \tilde{V}(-x,t)^*, \quad (3.9)$$

$$\tilde{V}(x,t) = \nu(x,t) + i \Phi_1(-x,t)^* S(x,t)^{-1} \Phi_2(x,t). \quad (3.10)$$

Next, we show that $\tilde{R}$ given by (2.10) satisfies (under the assumptions of this section) the non-local requirement

$$\tilde{R}(x,t) = \text{diag}(\tilde{\rho}_1(x,t), \tilde{\rho}_2(x,t)) = -\tilde{R}(-x,t)^*.$$  \hfill (3.11)

Indeed, in view of the relations (2.11), (3.3) and (3.6), we have

$$X_{-1}(x,t) = j^p \Pi(-x,t)^* S(x,t)^{-1} A^{-1} \Pi(x,t), \quad (3.12)$$

$$Y_{-1}(x,t) = -j^p \Pi(-x,t)^* (A^{-1})^* S(x,t)^{-1} \Pi(x,t), \quad (3.13)$$

$$X_{-1}(-x,t)^* = -j^p Y_{-1}(x,t) j^p, \quad Y_{-1}(-x,t)^* = -j^p X_{-1}(x,t) j^p.$$  \hfill (3.14)

From the last equality in (2.3) and the formulas (3.1) and (3.2), we derive

$$Q_{-1}(-x,t)^* = j^p Q_{-1}(x,t) j^p.$$  \hfill (3.15)

Formulas (A.5), (3.14) and (3.15) imply that

$$\tilde{Q}_{-1}(-x,t)^* = j^p \tilde{Q}_{-1}(x,t) j^p.$$  \hfill (3.16)

Finally, the first equality in (2.10) and formula (3.16) yield (3.11).
Rewriting (2.4), (2.6) and (2.12) under assumptions of this section, we obtain

\[
\Pi_x = \sum_{i=0}^{1} A_i^* q_i, \quad \Pi_r = A^{-1} Q_{-1}; \quad S_x(x,t) = i \Pi(x,t) q_j^p \Pi(-x,t)^*, \quad (3.17)
\]

\[
S_r = i A^{-1} \Pi(x,t) Q_{-1}(x,t) q_j^p \Pi(-x,t)^*(A^*)^{-1}, \quad (3.18)
\]

\[
w_A(x,t,\lambda) = I_m - i j^p \Pi(-x,t)^* S(x,t)^{-1}(A - \lambda I_n)^{-1} \Pi(x,t). \quad (3.19)
\]

Recall that in this section, we assume that the relations (3.3) and (3.4) hold. In particular, GBDT is determined by the triple \( \{A, S(0,0), \Pi(0,0)\} \). Now, we can rewrite Theorem 2.2 for the NMDE case.

**Theorem 3.1** Let \( R \) and \( V \) have the forms (3.1) and (3.2), respectively, let \( V_t(x,t) \) be continuous in \( D \), and let \( R \) and \( V \) satisfy (1.5) in \( D \). Assume that two \( n \times n \) parameter matrices \( A \) and \( S(0,0) \) and one \( n \times m \) parameter matrix \( \Pi(0,0) \) are given, and that (3.5) holds.

Then, \( \Pi(x,t), S(x,t) \) and \( w(x,t,\lambda) \) (where \( w \) is the wave function, i.e., \( w(x,t,\lambda) \) satisfies (1.8), (1.9) and \( \det w(0,0,\lambda) \neq 0 \) are well defined in \( D \).

Moreover, in the points of invertibility of \( S(x,t) \) in \( D \), the matrix function \( \tilde{R} \) given by (2.10) (where \( X_{-1} \) and \( Y_{-1} \) have the forms (3.12) and (3.13)) and the matrix function \( \tilde{V} \) given by (3.8) satisfy NMDE, that is, the relations (3.11) and (3.9) are valid and the equations

\[
\tilde{R}_t = \frac{(-1)^p}{2} (\tilde{V} \tilde{V}_t + \tilde{V}_t \tilde{V}), \quad \tilde{V}_{tt} = \frac{1}{2} (\tilde{V} \tilde{R} + \tilde{R} \tilde{V}) \quad (3.20)
\]

are satisfied.

The wave function \( \tilde{w} \) (\( \det \tilde{w}(0,0,\lambda) \neq 0 \)), which corresponds to the transformed NMDE (3.20), is given by the product \( w_A w \), where the Darboux matrix \( w_A \) has the form (3.19). In other words, the relations (2.15)–(2.17) are valid.

### 4. GBDT for the complex coupled dispersionless equations

Recall that in order to obtain the complex coupled dispersionless equations (CCDE)

\[
\rho_x + \frac{1}{2} \kappa(|v|^2), = 0, \quad v_{tt} = \rho v \quad (\rho = \bar{\rho}, \quad \kappa = \pm 1), \quad (4.1)
\]

we consider MCDE (1.5), (1.6) satisfying (1.10). In particular, since \( m_1 = m_2 = 1 \), the functions \( \rho \) and \( v \) are scalar functions and we rewrite (2.3) and (1.6) in the form

\[
q_1 = -\frac{i}{4} j, \quad q_0 = \frac{i}{2} j^{p+1} V, \quad Q_{-1} = i \rho j - j^p V; \quad (4.2)
\]

\[
j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R(x,t) = \rho(x,t) I_2, \quad V(x,t) = \begin{bmatrix} 0 & v(x,t) \\ v(x,t) & 0 \end{bmatrix}. \quad (4.3)
\]

In view of (4.3), the coefficients \( q_1, q_0, Q_1 \) given by (4.2) have the property

\[
q_k^* = -j^{p+1} q_j j^{p+1} \quad (k = 1, 0), \quad Q_{-1}^* = -j^{p+1} Q_{-1} j^{p+1}. \quad (4.4)
\]
In order to construct GBDT for the CCDE equations (4.1), we set in the GBDT for MCDE in Section 2 the equalities

\[ A_1 = A^*_2 = A, \quad \Pi_1(x, t) = \Pi(x, t), \quad \Pi_2(0, 0) = -i\Pi(0, 0)^{j^{p+1}}, \]  

and \( S(0, 0) = S(0, 0)^* \). It means that GBDT is determined by three parameter matrices:

\[ A \quad (\det A \neq 0), \quad S(0, 0) = S(0, 0)^* \quad (\det S(0, 0) \neq 0) \quad \text{and} \quad \Pi(0, 0). \]

In view of (4.5), we rewrite (2.4) in the form

\[ A_{1/2} x = A_{1/2} q_1 + A_{1/2} q_0, \quad A_{1/2} t = A_{1/2} Q - 1. \]  

Taking into account (2.5) and (4.4)–(4.6), we see that

\[ S(0, 0) = S(0, 0)^*. \]

Hence, the identity (2.7) takes the form

\[ AS(x, t) - S(x, t)A^* = i\Pi(x, t)^{j^{p+1}} \Pi(x, t)^* \]  

The matrix function \( S(x, t) = S(x, t)^* \) is determined now by \( S(0, 0) \) and the equations

\[ S_x = i\Pi q_{1/2}^{j^{p+1}} \Pi^*, \quad S_t = -i\Pi^{-1} Q - 1^{j^{p+1}} \Pi^*(A^*)^{-1}, \]

which follow from (2.6).

The GBDT-transformed solution \( \tilde{R}, \tilde{V} \), Darboux matrix \( w_A \) and wave function \( \tilde{w} \) are expressed via \( A, \Pi(x, t) \) and \( S(x, t) \). Let us show that \( \tilde{R} \) and \( \tilde{V} \) have the form (4.3):

\[ \tilde{R}(x, t) = \tilde{\rho}(x, t) I_2 \quad (\tilde{\rho} = \tilde{\rho}), \quad \tilde{V}(x, t) = \begin{pmatrix} 0 & \tilde{v}(x, t) \\ \tilde{v}(x, t) & 0 \end{pmatrix}, \]

and so \( \tilde{\rho} \) and \( \tilde{v} \) satisfy CCDE. Indeed, \( X_0, X_{-1} \) and \( Y_{-1} \) (given by (2.11)) take now the form

\[ X_0 = ij^{p+1} \Pi^* S^{-1} \Pi, \]
\[ X_{-1} = ij^{p+1} \Pi^* S^{-1} A^{-1} \Pi, \quad Y_{-1} = ij^{p+1} \Pi^*(A^*)^{-1} S^{-1} \Pi. \]

Thus, we rewrite (2.9) as

\[ \tilde{V} = V + \frac{i}{2}(j\Pi^* S^{-1} \Pi - \Pi^* S^{-1} \Pi j). \]  

According to (4.12), \( \tilde{V} \) has the form (4.9), where

\[ \tilde{v} = v + i\Phi_1 S^{-1} \Phi_2 \quad ([\Phi_1, \Phi_2] := \Pi)_j. \]
According to (A.5), (A.7) and (4.2) we have
\[ \text{tr}(\tilde{Q}_{-1}) = \text{tr}(Q_{-1}) = 0, \] (4.14)
where tr stands for trace. In view of (2.10) and (4.14), the first equality in (4.9) holds.

It remains to prove that \( \tilde{\rho} = \bar{\rho} \). From (4.11) we see that
\[ X_1 = -j^{p+1}Y_{p+1}. \]

Hence, the last equality in (4.4) and the equalities in (2.10) imply that \( \tilde{Q}_{-1} = -j^{p+1}\tilde{Q}_{p+1} \) and so \( \tilde{R} = \bar{R} \). That is, we have
\[ \tilde{\rho} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{Q}_{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \bar{\rho} \]
\[ (\tilde{Q}_{-1} = (I_n - X_{-1})Q_{-1}(I_n + Y_{-1})), \] (4.15)
which finishes the proof of (4.9). We obtained the following corollary of Theorem 2.2.

**Corollary 4.1** Let \( v(x, t) \) be continuous in \( D \), and let the functions \( \rho \) and \( v \) satisfy CCDE (4.1) in \( D \). Assume that two \( n \times n \) parameter matrices \( A \) and \( S(0, 0) = S(0, 0)^* \) and one \( n \times 2 \) parameter matrix \( \Pi(0) \) are given, and that the relations
\[ AS(0, 0) - S(0, 0)A^* = i\Pi(0, 0)^p \Pi(0, 0)^* \quad (\det A \neq 0, \quad \det S(0, 0) \neq 0) \] (4.16)
hold. Introduce \( \Pi(x, t) \) and \( S(x, t) \) using (4.6), (4.8) (and (4.2), (4.3)), where \( p = (1 + \kappa)/2 \).

Then, in the points of invertibility of \( S(x, t) \) in \( D \), the functions \( \tilde{\rho} \) (given by (4.15) and (4.11)) and \( \tilde{v} \) (given by (4.13)) satisfy CCDE:
\[ \tilde{\rho}_x + \frac{1}{2}x(\tilde{v})^2 = 0, \quad \tilde{v}_x = \tilde{\rho}\tilde{v} \quad (\tilde{\rho} = \bar{\rho}). \] (4.17)

Moreover, a wave function \( w(x, t, \lambda) \) (where \( \det w(0, 0, \lambda) \neq 0 \)) is well defined in \( D \) via (4.3) and auxiliary systems
\[ \begin{align*}
 w_x &= G(x, t, \lambda)w, \quad G(x, t, \lambda) := \frac{i}{4}\lambda + \frac{i}{2}j^{p+1}V(x, t); \\
 w_t &= F(x, t, \lambda)w, \quad F(x, t, \lambda) := (-i\rho j + j^{p+1}V(x, t))/\lambda.
\end{align*} \] (4.18)
\[ \text{The wave function } \tilde{\omega} \text{ (det } \tilde{\omega}(0, 0, \lambda) \neq 0), \text{ which corresponds to the transformed CCDE (4.17), is given by the product } \tilde{\omega} = w_A w, \text{ where the Darboux matrix } w_A \text{ has the form}
\]
\[ w_A(x, t, \lambda) = I_2 - ij^{p+1}\Pi(x, t)^*S(x, t)^{-1}(A - \lambda I_n)^{-1}\Pi(x, t). \] (4.20)

**Example 4.2** In order to present an example of the solution of CCDE (4.1), we set in Example 2.3 (in accordance with (1.10) and (4.5))
\[ m_1 = m_2 = 1, \quad a_1 = a, \quad a_2 = a, \quad D_1 = D_2 = d = d, \]
\( \Psi_1(0, 0) = -i \Phi_1(0, 0) \) and \( \Psi_2(0, 0) = (-1)^p i \Phi_1(0, 0) \). For simplicity of notations, we put \( \Phi_i(0, 0) = c_i \).

In view of (2.30)–(2.32) we have

\[
S(x, t) = \frac{i}{a - \bar{a}} \left( |c_1|^2 \exp \left\{ -i((a - \bar{a})(x/4) - d((t/a) - (t/\bar{a}))) \right\} 
+ (-1)^p |c_2|^2 \exp \left\{ i((a - \bar{a})(x/4) - d((t/a) - (t/\bar{a}))) \right\} \right),
\]

and \( S(x, t) = \overline{S(x, t)} \). The formula for \( \tilde{\nu}_1 \) in Example 2.3 takes the form

\[
\tilde{\nu}(x, t) = \frac{ic_1c_2}{S(x, t)} \exp \left\{ i((a + \bar{a})(x/4) - d((t/a) + (t/\bar{a}))) \right\}.
\]

Finally, formula (2.33) takes the form

\[
\tilde{\rho}(x, t) = d \left( 1 - i |c_1|^2 \exp \left\{ -i((a - \bar{a})(x/4) - d((t/a) - (t/\bar{a}))) \right\} / (aS(x, t)) \right)
\times \left( 1 + i |c_1|^2 \exp \left\{ -i((a - \bar{a})(x/4) - d((t/a) - (t/\bar{a}))) \right\} / (\bar{a}S(x, t)) \right)
+ (-1)^p d |\tilde{\nu}(x, t)|^2 / |a|^2.
\]

5. Coupled dispersionless equations

Similarly to Section 4, we consider here the case of scalar function \( \rho \) (and scalar \( v_1 \) and \( v_2 \)). Setting

\[
m_1 = m_2 = 1, \quad p = 1, \quad \rho_1 = \rho_2 = \rho, \quad v_1 = r, \quad v_2 = s,
\]

we rewrite (1.5) in the form

\[
\rho_{s} + \frac{1}{2} (rs_{t} + r_{s} s) = 0, \quad r_{s t} = \rho r, \quad s_{s t} = \rho s,
\]

which is equivalent to (1.1) in introduction (see also [1, 9, 10], [11, (1.2)] and references therein). The following corollary of Theorem 2.2 is valid.

**Corollary 5.1** Let the conditions of Theorem 2.2 hold, and assume additionally that

\[
m_1 = m_2 = 1, \quad \rho_1 = \rho_2 = \rho.
\]

Then, \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \) given by (2.9), and \( \tilde{\rho} \) given by the formula

\[
\tilde{\rho} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{Q}_{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \tilde{Q}_{-1} = (I_2 - X_{-1})Q_{-1}(I_2 + Y_{-1})
\]

satisfy equations (5.2), that is,

\[
\tilde{\rho}_{s} = \frac{(-1)^p}{2} (\tilde{\nu}_1(\tilde{\nu}_2)_s + (\tilde{\nu}_1)\tilde{\nu}_2), \quad (\tilde{\nu}_1)_{s t} = \rho \tilde{\nu}_1, \quad (\tilde{\nu}_2)_{s t} = \rho \tilde{\nu}_2.
\]
Proof. Taking into account (1.5), (1.6) and (2.10), the only fact which we need to prove is that \( \tilde{R} \) has the form \( \tilde{I}_2 \), that is, that \( \tilde{\rho}_1 = \tilde{\rho}_2 \). Similar to the calculation in the previous section, relations (2.3) and (A.7) yield

\[
\text{tr}(\tilde{Q}_{-1}) = \text{tr}(Q_{-1}) = 0, \tag{5.6}
\]
and the equality \( \tilde{\rho}_1 = \tilde{\rho}_2 \) follows from (2.10) and (5.6).

For the non-local situation

\[
\hat{\rho}(x, t) = -\rho(-x, t), \quad \hat{v}(x, t) := v_1(x, t) = v_2(-x, t), \tag{5.7}
\]
equations (5.5) have the form

\[
\rho_x(x, t) = \frac{(-1)^\rho}{2}(v(x, t)v(-x, t) + v_1(x, t)v_2(-x, t)), \tag{5.8}
\]
\[
v_{tx}(x, t) = \rho(x, t)v(x, t). \tag{5.9}
\]
In other words, under conditions (5.3) and (5.7) system (1.5) is equivalent to the system (5.8), (5.9).

(Note that \( \rho \) and \( v \) are scalar functions.)

Assume further that the relations (3.3)–(3.5) hold. In particular, GBDT is determined by the triple \( \{A, S(0, 0), \Pi(0, 0)\} \). Below, we formulate a corollary of Theorem 3.1.

**Corollary 5.2** Let \( v \) be continuous in \( D \), assume that \( \rho(x, t) = -\rho(-x, t) \), and let \( v \) and \( \rho \) satisfy (5.8), (5.9) in \( D \).

Then, \( \tilde{v} \) given by (3.10) and \( \tilde{\rho} \) given by (5.4) satisfy (5.8), (5.9) (as well as \( v \) and \( \rho \)), that is, the following equalities hold:

\[
\tilde{\rho}_1(x, t) = \frac{(-1)^\rho}{2}(\tilde{v}(x, t)\tilde{v}_1(-x, t) + \tilde{v}_1(x, t)\tilde{v}(-x, t)), \tag{5.10}
\]
\[
\tilde{v}_{tx}(x, t) = \tilde{\rho}(x, t)\tilde{v}(x, t). \tag{5.11}
\]
We also have \( \overline{\rho(x, t)} = -\tilde{\rho}(-x, t) \).

**Proof.** Similar to the Corollary 5.1 we need only to prove that \( \tilde{\rho}_1 = \tilde{\rho}_2 \) (after which we may use Theorem 3.1). The equality \( \tilde{\rho}_1 = \tilde{\rho}_2 \) follows from (2.10) and (5.6).

**6. Examples and figures**

In these examples, we construct explicit solutions of the non-local equations (5.10), (5.11). We set

\[
v(x, t) \equiv 0, \quad R(x, t) \equiv iI_2, \text{ i.e. } \rho(x, t) \equiv 1. \tag{6.1}
\]

We put

\[
n = 2, \quad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad (a + \overline{a} \neq 0), \tag{6.2}
\]
which corresponds \[42\] to the simplest case of the Weyl function (reflection coefficient) with a pole of the order more than one (so-called multipole case). For the literature on the multipole cases see, for instance, \[43, 44\] and the references therein.

Remark 6.1 It is often convenient to recover \( S \) from the identities (2.7), (3.7) or (4.7). In particular, \( S \) satisfying (3.7) exists and is unique if only \( \sigma(A) \cap \sigma(-A^*) = \emptyset \). Therefore, we require in (6.2) that \( a + \overline{a} \neq 0 \).

Using the notation \( \Phi_i(0, 0) = C_i = \begin{bmatrix} c_{i1} \\ c_{i2} \end{bmatrix} \), we (similarly to the deduction of (2.19)) obtain

\[
\Phi_1(x, t) = \exp \left\{ - \left( i(x/4)A + tA^{-1} \right) \right\} C_1
\]

\[
\Phi_2(x, t) = \exp \left\{ - \left( (i/4)ax + (t/a) \right) \right\} (I_2 - (i/4)xA_0 + (t/a^2)A_0) C_1,
\]

where \( A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). In the same way as (6.3), we derive

\[
\Phi_2(x, t) = \exp \left\{ (i/4)ax + (t/a) \right\} (I_2 + (i/4)xA_0 - (t/a^2)A_0) C_2.
\]

Relations (6.3)–(6.4) provide an explicit expression for

\[
K(x, t) = \{ K_{ii}(x, t) \}_{i, j = 1}^2 = i \left( \Phi_1(x, t) \Phi_1(-x, t)^* + (-1)^p \Phi_2(x, t) \Phi_2(-x, t)^* \right).
\]

Next, using (3.7) we easily express \( S(x, t) = \{ S_{ii}(x, t) \}_{i, j = 1}^2 \) in terms of \( K(x, t) \):

\[
S_{22}(x, t) = K_{22}(x, t)/(a + \overline{a}),
\]

\[
S_{12}(x, t) = (K_{12}(x, t) - S_{22}(x, t))/(a + \overline{a}),
\]

\[
S_{21}(x, t) = (K_{21}(x, t) - S_{22}(x, t))/(a + \overline{a}),
\]

\[
S_{11}(x, t) = (K_{11}(x, t) - S_{12}(x, t) - S_{21}(x, t))/(a + \overline{a}).
\]

Finally, from (3.10), (5.4) and (6.1) it follows that

\[
\tilde{\nu}(x, t) = i \Phi_1(-x, t)^* S(x, t)^{-1} \Phi_2(x, t),
\]

\[
\tilde{\rho}(x, t) = i \left( 1 - i(-1)^p \Phi_2(-x, t)^* S(x, t)^{-1} A^{-1} \Phi_2(x, t) \right)
\]

\[
\times (1 - i(-1)^p \Phi_2(-x, t)^* (A^*)^{-1} S(x, t)^{-1} \Phi_2(x, t))
\]

\[
+ i(-1)^p \Phi_2(-x, t)^* S(x, t)^{-1} A^{-1} \Phi_1(x, t)
\]

\[
\times \Phi_1(-x, t)^* (A^*)^{-1} S(x, t)^{-1} \Phi_2(x, t).
\]

Recall that (according to Corollary 5.2) \( \tilde{\nu} \) and \( \tilde{\rho} \) satisfy (5.10), (5.11).

The fundamental solution (wave function) \( w(x, t, \lambda) \) of the initial auxiliary systems (1.8) and (1.9), where \( V = 0 \) and \( R = \lambda/2 \) is given by the formula

\[
w(x, t, \lambda) = \exp \left\{ [(i/4)\lambda x + (t/\lambda)]j \right\}.
\]
In view of (6.2)–(6.9) we have explicit formulas for the Darboux matrix \( w_A(x,t,\lambda) = I_2 - i p \Pi(-x,t)^* S(x,t)^{-1} (A - \lambda I_2)^{-1} \Pi(x,t) \). Thus the wave function \( w_A(x,t,\lambda) w(x,t,\lambda) \) of the transformed system with \( \tilde{v} \) and \( \tilde{\rho} \) given by (6.10) and (6.11), respectively, is also expressed explicitly.

Let us consider several explicit formulas in greater detail. In the following, we set \( a_1 = \Re(a) \) and \( a_2 = \Im(a) \).

**Case 1.** The simplest case is the case where \( c_{12} = c_{21} = 0 \), that is, \( C_1 = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix} \) and \( C_2 = \begin{bmatrix} 0 \\ c_{22} \end{bmatrix} \). Here, relations (6.3)–(6.11) after some calculations yield:

\[
\tilde{v} = \frac{4 a_1^2 c_{11} c_{22} e^{-a_2(x + 4i t/|a|^2)/2}}{4 a_1^2 |c_{11}|^2 e^{-i a_1(x - 4i t/|a|^2)/2} + (-1)^p |c_{22}|^2 e^{i a_1(x - 4i t/|a|^2)/2}},
\]

\[
\tilde{\rho} = i - \frac{32 i (-1)^p a_1^4 |c_{11}|^2 |c_{22}|^2 / |a|^2}{(4 a_1^2 |c_{11}|^2 e^{-i a_1(x - 4i t/|a|^2)/2} + (-1)^p |c_{22}|^2 e^{i a_1(x - 4i t/|a|^2)/2})^2}.
\]

In particular, for

\( p = 1, \quad a = \frac{1}{2} + \frac{1}{3} i, \quad c_{11} = 1 + 2 i, \quad c_{22} = 4 + 3 i, \)

the behaviour of \( |\tilde{v}| \) and \( \ln |\tilde{\rho}| \) is shown on Fig. 1.

**Case 2.** When \( p = 1 \), \( C_1 = \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} \) and \( C_2 = \begin{bmatrix} 0 \\ c_{22} \end{bmatrix} \) (\( c_{11}, c_{12}, c_{22} \in \mathbb{R} \)), our choice of non-diagonal \( A \) leads to polynomials

\( \gamma_1 := i c_{12} x - 2 c_{11} - 4 c_{12} t / \bar{a}^2 \) and \( \gamma_2 := i c_{12} x - 2 c_{11} - 4 c_{12} t / a^2 \).
(in addition to the exponents) in the formulas for \( \tilde{v} \) and \( \tilde{\rho} \). Namely, we have:

\[
\tilde{v} = 2a_1c_{22} \frac{1}{c_{22}^2 e^{i a_1 x + 4a_1 t/|a|^2} + c_{12}^4 e^{-i a_1 x - 4a_1 t/|a|^2} - 2c_{12}^2 c_{22}^2 - a_1^2 c_{22}^2} \times \left( a_1 \gamma_1 - 2c_{12} \right) e^{i a_1 x + 4a_1 t/|a|^2} + c_{12}^4 e^{-i a_1 x - 4a_1 t/|a|^2} - 2c_{12}^2 c_{22}^2 - a_1^2 c_{22}^2 \gamma_1 \gamma_2 \\
+ 2c_{12}^2 c_{22}^2 ( - a_1^2 (a_1 + a_2)^2 \gamma_1 \gamma_2 + 32 i c_{12} \left( - 2a_1 c_{12}^2 a_1^4 |a|^{-4} t - 4a_1^2 c_{12}^2 \right) )
\]

The behaviour of \( \tilde{v} \) and \( \tilde{\rho} \) is in this case more complicated, see Fig. 2, where

\[
p = 1, \quad a = \frac{1}{3} + \frac{1}{5} i, \quad c_{11} = 3, \quad c_{12} = 1, \quad c_{22} = \frac{1}{2}
\]

In some other cases, the formulas are more complicated and we restrict ourselves to figures only. See Fig. 3, where

\[
p = 1, \quad a = \frac{1}{2} + \frac{1}{3} i, \quad C_1 = \left[ \begin{array}{c} 1 \\ 2i \end{array} \right], \quad C_2 = \left[ \begin{array}{c} 3i \\ 4 \end{array} \right]
\]

see Fig. 4, where

\[
p = 0, \quad a = \frac{3}{2} + \frac{1}{2} i, \quad C_1 = \left[ \begin{array}{c} 1 + 3i \\ 3 + 2i \end{array} \right], \quad C_2 = \left[ \begin{array}{c} 6 + i \\ 2 - 4i \end{array} \right]
\]
and see Fig. 5, where

\[ p = 1, \quad a = 1 + i, \quad C_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}. \]

A. Two propositions on GBDT for MCDE

The first proposition refers to the particular case of [35, (1.34)].

PROPOSITION A.1 Let \( R \) and \( V \) have the form (1.6) and satisfy the MCDE (1.5). Assume that \( S(x, t) \), \( \Pi_1(x, t) \) and \( \Pi_2(x, t) \) satisfy (2.1) and (2.4)–(2.6), and that \( w_A \) is given by (2.12). Then, in the points of
invertibility of $S(x, t)$ we have

\[
\frac{\partial}{\partial x}w_A(x, t, z) = \tilde{G}(x, t, z)w_A(x, t, z) - w_A(x, t, z)G(x, t, z), \tag{A.1}
\]

\[
\frac{\partial}{\partial t}w_A(x, t, z) = \tilde{F}(x, t, z)w_A(x, t, z) - w_A(x, t, z)F(x, t, z), \tag{A.2}
\]

where $G$ and $F$ are given by (2.2) and (2.3),

\[
\tilde{G} = -\lambda \tilde{q}_1 - \tilde{q}_0, \quad \tilde{F} = -\frac{1}{\lambda} \tilde{Q}_{-1}, \tag{A.3}
\]

\[
\tilde{q}_1 = q_1 = -\frac{i}{4}, \quad \tilde{q}_0 = q_0 - (q_1X_0 - X_0q_1), \tag{A.4}
\]

\[
\tilde{Q}_{-1} = (I_m - X_{-1})Q_{-1}(I_m + Y_{-1}), \tag{A.5}
\]

$q_0$ and $Q_{-1}$ are defined by (2.3), and $X_0, X_{-1}$ and $Y_{-1}$ are given by the formulas

\[
X_0 := \Pi_2^*S^{-1}\Pi_1, \quad X_{-1} := \Pi_2^*S^{-1}A_{-1}^{-1}\Pi_1, \quad Y_{-1} := \Pi_2^*A_{-2}^{-1}S^{-1}\Pi_1. \tag{A.6}
\]

Remark A.2 According to (2.2) and (A.3), $G$ and $\tilde{G}$ (respectively $F$ and $\tilde{F}$) depend on $\lambda$ in a similar way (e.g. $G$ and $\tilde{G}$ are polynomials of the first degree with respect to $\lambda$ and their leading coefficients coincide).

Compare (2.12) and (A.6) in order to see that $w_A(x, t, 0) = I_m - X_{-1}(x, t)$. Hence, the equality [33, (1.76)] (at $z = \zeta = 0$) yields

\[
(I_m - X_{-1})(I_m + Y_{-1}) = I_m. \tag{A.7}
\]
Finally, according to [35, (1.17)] the following useful relations are valid:

\[
\left(\Pi_2 S^{-1}\right)_t = -\tilde{q}_1 \Pi_2 S^{-1} A_1 - \tilde{q}_0 \Pi_2 S^{-1}, \quad \left(\Pi_2 S^{-1}\right)_r = -\tilde{Q}_{-1} \Pi_2 S^{-1} A_1^{-1}.
\]  

(A.8)

It follows from (2.2), (2.3) and (A.4) that \(\tilde{q}_0\) has the same form as \(q_0\), and so \(\tilde{G}\) has the same form as \(G\). More precisely, in the expression for \(\tilde{q}_0\) we substitute only \(\tilde{V}\) instead of \(V\), \(\tilde{v}_1\) instead of \(v_1\) and \(\tilde{v}_2\) instead of \(v_2\), that is

\[
\tilde{q}_0 = \frac{i}{2} j^{p+1} \tilde{V},
\]  

(A.9)

where

\[
\tilde{V} = \begin{bmatrix} 0 & \tilde{v}_1 \\ \tilde{v}_2 & 0 \end{bmatrix} = V + \frac{1}{2} j^p (X_0 - jX_0j) \quad (X_0 = \Pi_2 S^{-1} \Pi_1). \tag{A.10}
\]

According to (2.2), (2.3) and (A.3), the proposition below means that \(\tilde{F}\) has the same form as \(F\) and \(\tilde{Q}_{-1}\) has the same form as \(Q_{-1}\).

**Proposition A.3** Let the conditions of Proposition A.1 hold. Then, in the points of invertibility of \(S\), we have

\[
\tilde{Q}_{-1} = ij\tilde{R} - j^p \tilde{V}_t,
\]  

(A.11)

where \(\tilde{V}\) is given by (A.10) and

\[
\tilde{R} = \frac{1}{2i} (\tilde{Q}_{-1} j + j\tilde{Q}_{-1}), \quad \tilde{Q}_{-1} = (I_m - X_{-1}) Q_{-1} (I_m + Y_{-1}). \tag{A.12}
\]

**Proof.** The second equality in (A.12) coincides with (A.5). In view of the first equality in (A.12), the block diagonal part of \(\tilde{Q}_{-1}\) equals \(ij\tilde{R}\). Thus, in order to prove (A.11) it remains to show that the block antidiagonal part of \(\tilde{Q}_{-1}\) equals \(-j^p \tilde{V}_t\), that is,

\[
\frac{1}{2} (\tilde{Q}_{-1} - j\tilde{Q}_{-1} j) = -j^p \tilde{V}_t.
\]  

(A.13)

First, let us find the derivative \((\Pi_2 S^{-1} \Pi_1)_r\). The second equalities in (2.4) and (A.8) and the definition of \(X_{-1}\) in (A.6) yield

\[
(\Pi_2 S^{-1} \Pi_1)_r = \Pi_2 S^{-1} A_1^{-1} \Pi_1 Q_{-1} - \tilde{Q}_{-1} \Pi_2 S^{-1} A_1^{-1} \Pi_1
\]

\[= X_{-1} Q_{-1} - \tilde{Q}_{-1} X_{-1}. \tag{A.14}
\]

Using the second equality in (A.12) and formula (A.14), we derive

\[
(\Pi_2 S^{-1} \Pi_1)_r = X_{-1} Q_{-1} - (I_m - X_{-1}) Q_{-1} (I_m + Y_{-1}) X_{-1}
\]

\[= X_{-1} Q_{-1} + (I_m - X_{-1}) Q_{-1} (I_m + Y_{-1}) (I_m - X_{-1}) - \tilde{Q}_{-1}. \tag{A.15}
\]
By virtue of (A.7), we rewrite (A.15) in the form

\[(\Pi_2^2 S^{-1} \Pi_1)_t = X_{-1} Q_{-1} + (I_m - X_{-1}) Q_{-1} - \tilde{Q}_{-1} = Q_{-1} - \tilde{Q}_{-1}.\]  

(A.16)

Now, formulas (A.10) and (A.16) yield

\[\tilde{V}_t = V_t + \frac{1}{2} j^p (Q_{-1} - j \tilde{Q}_{-1}) - \frac{1}{2} j^p (\tilde{Q}_{-1} - j \tilde{Q}_{-1}).\]  

(A.17)

Hence, in view of the second equality in (2.3) we have

\[\tilde{V}_t = -\frac{1}{2} j^p (\tilde{Q}_{-1} - j \tilde{Q}_{-1}),\]

which implies (A.13). □

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