SUBELLIPTICITY OF SOME COMPLEX VECTOR FIELDS RELATED TO THE WITTEN LAPLACIAN

WEIXI LI

School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science
Wuhan University, Wuhan 430072, China

CHAOJIANG XU*

Department of Mathematics, Nanjing University of Aeronautics and Astronautics
Nanjing 211106, China
Université de Rouen, CNRS UMR 6085, Laboratoire de Mathématiques
76801 Saint-Etienne du Rouvray, France

In honor of the 80th birthday of Professor Shuxing CHEN

ABSTRACT. We consider some system of complex vector fields related to the semi-classical Witten Laplacian, and establish the local subellipticity of this system basing on condition (Ψ).

1. Introduction and main results. Let \( \Omega \subset \mathbb{R}^n \) be a neighborhood of 0, and denote by \( i \) the square root of \(-1\). We consider in this paper the following system of complex vector fields:

\[
P_j = \partial_{x_j} - i \left( \partial_{x_j} \varphi(x) \right) \partial_t, \quad j = 1, \cdots, n, \quad (x,t) \in \Omega \times \mathbb{R}, \tag{1.1}
\]

where \( \varphi(x) \) is a real-valued function defined in a neighborhood \( \Omega \) of 0. This system was first studied by Treves [22], and considered therein is more general case for \( t \) varying in \( \mathbb{R}^m \) rather than \( \mathbb{R} \).

Denote by \( (\xi, \tau) \) the Fourier variables of \( (x,t) \). Then the principal symbol \( \sigma \) for the system \( \{P_j\}_{1 \leq j \leq n} \) is

\[
\sigma(x,t; \xi, \tau) = (i\xi_1 + (\partial_{x_1} \varphi) \tau, \cdots, i\xi_n + (\partial_{x_n} \varphi) \tau) \in \mathbb{C}^n
\]

with \( (x,t; \xi, \tau) \in T^* (\Omega \times \mathbb{R}_t) \setminus \{0\} \), and thus the characteristic set is

\[
\{(x,t; \xi, \tau) \in T^* (\Omega \times \mathbb{R}_t) \setminus \{0\} \mid \xi = 0, \tau \neq 0, \nabla \varphi(x) = 0\}.
\]

Since outside the characteristic set the system \( \{P_j\}_{1 \leq j \leq n} \) is elliptic, we only need to study the microlocal hypoellipticity in the two components \( \{\tau > 0\} \) and \( \{\tau < 0\} \) under the assumption that

\[
\nabla \varphi(0) = 0.
\]

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* Corresponding author.
Note we may assume \( \varphi(0) = 0 \) if replacing \( \varphi \) by \( \varphi - \varphi(0) \). By maximal hypoellipticity (in the sense of Helffer-Nourrigat [8]) it means the existence of a neighborhood \( \tilde{\Omega} \subset \Omega \) of 0 and a constant \( C \) such that for any \( u \in C^\infty_0(\tilde{\Omega} \times \mathbb{R}) \) we have
\[
\sum_{j=1}^n \left( \| \partial_{x_j} u \|_{L^2(\mathbb{R}^{n+1})} + \| (\partial_{x_j} \varphi) \partial_t u \|_{L^2(\mathbb{R}^{n+1})} \right) \leq C \left( \sum_{j=1}^n \| P_j u \|_{L^2(\mathbb{R}^{n+1})} + \| u \|_{L^2(\mathbb{R}^{n+1})} \right).
\]

Note the maximal hypoellipticity will yield the following subelliptic estimate
\[
\sum_{j=1}^n \| \partial_{x_j} u \|_{L^2(\mathbb{R}^{n+1})} + \| |D|^{1+} u \|_{L^2(\mathbb{R}^{n+1})} \leq C \sum_{j=1}^n \| P_j u \|_{L^2(\mathbb{R}^{n+1})} + C \| u \|_{L^2(\mathbb{R}^{n+1})},
\]
provided \( \varphi \) is finite type, i.e., for some integer \( k \geq 1 \),
\[
\sum_{|\alpha| \leq k+1} |\partial_\alpha \varphi(0)| \neq 0.
\]
Thus the subellipticity is in some sense intermediate between the maximal hypoellipticity and the local hypoellipticity.

We first recall the related works concerned with the subellipticity and hypoellipticity as well as the links between these properties. For the system of real vector fields, it is well-known ([9, 12, 20]) that the Hörmander’s bracket condition is sufficient to obtain the subellipticity in terms of the Lie algebra generated by the real vector fields. Moreover for real vector fields with analytic coefficients, Derridj’s work [1] implies that the Hörmander’s condition and thus subellipticity is also a necessary condition to obtain hypoellipticity. But the situation is quite different in the setting of complex vector fields; Kohn [13] constructed a counterexample, showing we may lose the subellipticity even though the Hörmander’s bracket condition is fulfilled. The hypoellipticity for the system (1.1) was initiated by Treves [22], and then continued by Maire [16]. The standard \( L^2 \) subellipticity was investigated in Derridj [3, 2] and a series of works [6, 5, 4] of Derridj-Helffer, where the crucial tool is to find suitable escaping curves. So far it remain unclear for the links between hypoellipticity and subellipticity for systems of complex vector fields. In fact we may ask the same question as in the case of real vector fields, whether hypoellipticity implies subellipticity for system (1.1) under the assumption that \( \varphi \) is analytic in \( \Omega \). This is true for \( n = 1 \), but may be false when \( n > 1 \) due to the work [11] of Journé and Trépreau, where they showed for some specific polynomial potential \( \varphi \) that the system is hypoelliptic without being subelliptic. For specific quasihomogeneous analytic potential functions Derridj-Helffer [5] proved hypoellipticity system (1.1) is indeed subelliptic for the dimension \( n = 2 \).

Observe for the system \( \{P_j\}_{1 \leq j \leq n} \), we may perform partial Fourier transform with respect to \( t \), and study the microhypoellipticity, in the two directions \( \tau > 0 \) and \( \tau < 0 \). Indeed we only need consider without loss of generality the microhypoeellipticity in the positive direction \( \tau > 0 \), since the other direction \( \tau < 0 \) can be treated similarly by replacing \( \varphi \) by \( -\varphi \). Consider the resulting system as follows after taking partial Fourier transform for \( t \in \mathbb{R} \):
\[
L_j = \partial_{x_j} + \tau (\partial_{x_j} \varphi), \quad j = 1, \cdots, n.
\] (1.2)
By maximal microhypoeellipticity at 0 in the positive direction in \( \tau > 0 \), it means the existence of a positive number \( \tau_0 > 0 \), a constant \( C > 0 \) and a neighborhood \( \tilde{\Omega} \subset \Omega \) of 0, such that
∀ τ ≥ τ₀, ∀ u ∈ C^∞₀(\tilde{Ω}),
\sum_{j=1}^{n} (\|\partial_{x_j} u\|_{L^2} + \|\tau (\partial_{x_j} \varphi) u\|_{L^2}) \leq C \left( \sum_{j=1}^{n} \|L_j u\|_{L^2} + \|u\|_{L^2} \right), \quad (1.3)
where and throughout the paper we denote \(\|\cdot\|_{L^2(\mathbb{R}^n)}\) by \(\|\cdot\|_{L^2}\) for short if no confusion occurs. We remark the operators defined in (1.2) are closely related to the semi-classical Witten Laplacian

\[ \Delta_{\tau}^{(0)} = -\Delta_x + \tau^2 |\partial_x \varphi|^2 - \tau \Delta_x \varphi \]

with \(\tau^{-1}\) the semi-classical parameter, by the relationship

\[ \sum_{j=1}^{n} \|L_j u\|_{L^2}^2 = \left( \Delta_{\tau}^{(0)} u, u \right)_{L^2}, \]
where \((\cdot, \cdot)_{L^2}\) stands for the inner product in \(L^2(\mathbb{R}^n)\). Helffer-Nier [7] conjectured \(\Delta_{\tau}^{(0)}\) is subelliptic near 0 if \(\varphi\) is analytic and has no local minimum near 0, and this still remains open so far. Note (1.3) is a local estimate concerning the sharp regularity near 0 ∈ \(\mathbb{R}^n\) for \(\tau > 0\), and we have also its global counterpart, which is of independent interest for analyzing the semi-classical lower bound of Witten Laplacian. We refer to Helffer-Nier’s work [7] for the detailed presentation on the topic of global maximal hypoellipticity and its application to the spectral analysis on Witten Laplacian; see also the first author’s work [15] for the extension to the non-polynomial potentials.

Considered in this article is a specific case of the overdetermined system of pseudo-differential operators. The general case was well investigated in the book of Hellfer-Nourrigat [8] and further developed by Nourrigat [18, 19], basing on the technique of nilpotent Lie group. The idea of nilpotent Lie group was initiated by Rothschild-Stein [20] when studying the hypoellipticity property of the Hörmander’s operators and Rothschild-Stein lifting theorem says that one can obtain the sharp local regularity by lifting the vector fields to nilpotent Lie groups and then using analysis for the corresponding left invariant operators defined on the groups. This kind of nilpotent Lie techniques were developed further by Helffer and Nourrigat [8] for systems of pseudo-differential operators, where the pseudo-differential operators are approximated by ones defined in Euclidean space with polynomial coefficients and the problem is then reduced to the analysis of the operators with polynomial coefficients. This text is motivated by Nourrigat’s work [19] where he introduced some geometrical conditions that are invariant by the above nilpotent representation to characterize the maximal hypoellipticity for general systems of one order pseudo-differential operators. One of these geometrical conditions involves the control of eigenvalues of the Lévi matrix, and the global counterpart of this kind of condition is applied recently to analyze the spectral property of the Witten Laplacian (see [15] and Helffer-Nier’s lecture [7]). Another one is the geometrical condition (Ψ). Recall condition (Ψ) was initially introduced to study the solvability of a pseudo-differential operator \(p^w\) with complex-valued symbol \(p_1 + ip_2\); we refer to [10, 14] for the comprehensive discussion on condition (Ψ). For a specific scalar operator

\[ \partial_x + \mu(s), \]
with $\mu$ real-valued function defined in an interval $I \subset \mathbb{R}$, condition $(\Psi)$ means the following change-of-sign condition:

$$s \to \mu(s)$$
does not change sign from $+ \to -$ as $s$ increases in $I$.

Moreover we say $\partial_s + \mu(s)$ satisfies condition $(\overline{\Psi})$ if $\partial_s - \mu(s)$ satisfies condition $(\Psi)$, that is,

$$s \to \mu(s)$$
does not change sign from $- \to +$ as $s$ increases in $I$.

For the scalar operator $\partial_s - i\mu(s)\partial_t$, the subellipticity is well explored by condition $(\overline{\Psi})$; for instance if $\mu$ is finite type and has no sign change from $- \to +$ as $s$ increases, then it is subelliptic (c.f. [14, 21, 10]). Moreover Nourrigat’s criterion [19, Theorem 2.1] says if $\mu$ is a polynomial and $s \to \mu(s)$ does not change sign from $- \to +$ as $s$ increases then the following maximal microhypoellipticity

$$\forall \tau > 0, \quad \|\partial_s u\|_{L^2(\mathbb{R})} + \|\tau \mu(s) u\|_{L^2(\mathbb{R})} \leq C \left( \|\partial_s + \tau \mu(s)\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})} \right)$$

holds for any smooth $u$ with compaction support in $\mathbb{R}$.

In this work we aim to make use of condition $(\overline{\Psi})$ to investigate the hypoellipticity of systems rather than scalar operators. Note there is no condition $(\Psi)$ for systems, in particular for the system $\mathcal{L}_j$, $1 \leq j \leq n$, defined by (1.2). For clear presentation we consider the case of $n = 2$ and our argument may be applied to more general case of $n \geq 2$. In what follows we use $(x, y)$ instead of $(x_1, x_2)$ as variables. Then we can rewrite $\mathcal{L}_j$, $j = 1, 2$, as

$$\begin{cases} 
\mathcal{L}_1 = \partial_x + \tau (\partial_x \varphi), \\
\mathcal{L}_2 = \partial_y + \tau (\partial_y \varphi).
\end{cases}$$

(1.4)

To state our main result we first introduce the following assumption which, roughly speaking, says that the set of the points $(x, y)$ at which both of the functions $s \to \partial_x \varphi(s, y)$ and $s \to \partial_y \varphi(x, s)$ change sign from $- \to +$ for increasing $s$, has measure 0.

**Definition 1.1 (Assumption $\mathcal{H}_1(\alpha)$ and $\mathcal{H}_2(\alpha)$).** Let $\alpha > 0$ and let $\varphi$ be defined in a neighborhood $\Omega$ of 0. Write $\Omega = \Omega_x \times \Omega_y$ with $\Omega_x, \Omega_y$ being the projections of $\Omega$ onto $x$ and $y$ axes, respectively. We say $\varphi$ satisfies Assumption $\mathcal{H}_1(\alpha)$ in $\Omega$ if there exists a family of disjoint open sets $\omega_j$ and $C^1$-functions $r_j$, $j = 1, 2, \cdots, N_1$ with $N_1$ an integer such that the following properties are fulfilled by $\omega_j$ and $r_j$.

(i) Denote by $\bar{\omega}_j$ the closure of $\omega_j$. Then

$$\bigcup_{1 \leq j \leq N_1} \bar{\omega}_j = \overline{\Omega}_y.$$

(ii) Given $y \in \omega_j$ for $1 \leq j \leq N_1$, as $s$ increases in $\Omega_x$ the function $\partial_x \varphi(s, y)$ either has no sign change from $- \to +$, or only changes sign from $- \to +$ at the point $r_j(y)$. In the latter we suppose additionally that

$$s \mapsto (\partial_y \varphi)(r_j(y), s)$$

doesn’t change sign from $- \to +$ as $s$ increases.

(iii) For any $1 \leq j \leq N_1$ we have $|\frac{\partial r_j}{\partial y}| \geq \alpha$ in $\omega_j$.

Moreover Assumption $\mathcal{H}_2(\alpha)$ can be defined symmetrically by a family of pairs $(U_j, \tau_j)$, $1 \leq j \leq N_2$, which satisfy similar properties as that of $(\omega_j, r_j)$ such that for each $x \in U_j$,

$$s \mapsto (\partial_x \varphi)(s, \tau_j(x)).$$
does not change sign from $-$ to $+$ as $s$ increases if $s \mapsto \partial_y \varphi(x, s)$ changes sign from $-$ to $+$ at point $\tau_j(x)$.

**Theorem 1.2.** Let $\alpha > 0$ be given and denote by $P_m$ the set of polynomials with degree $\leq m$. Then there exists a number $\delta$ depending only on the degree $m$, and two constants $\tau_0$ and $C$ depending only on $m$ and $\alpha$, such that the following estimate

$$\forall \ u \in C_0^\infty([-\delta, \delta[ \times ] - \delta, \delta]), \quad \|\partial_x u\|_{L^2} + \|\tau(\partial_x \varphi) u\|_{L^2} \leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}$$

holds for all $\tau \geq \tau_0$ and for all $\varphi \in P_m$ satisfying Assumption $H_1(\alpha)$ in the neighborhood $[-\delta, \delta[ \times ] - \delta, \delta]$ of 0. Symmetrically the estimate

$$\forall \ u \in C_0^\infty([-\delta, \delta[ \times ] - \delta, \delta]), \quad \|\partial_y u\|_{L^2} + \|\tau(\partial_y \varphi) u\|_{L^2} \leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}$$

holds for all $\tau > \tau_0$ and for all $\varphi \in P_m$ satisfying $H_2(\alpha)$ in $[-\delta, \delta[ \times ] - \delta, \delta$.

**Corollary 1.** If $\varphi \in P_m$ satisfies Assumption $H_1(\alpha)$ in a neighborhood of 0 and the condition that

$$\sum_{0 \leq j \leq k} |\partial_x^{k+1} \varphi(0)| \neq 0$$

for some $k$. Then system (1.4) is (micro)subelliptic at positive direction, that is, there exists a constant $C > 0$, a neighborhood $\tilde{\Omega}$ of 0 and a positive number $\tau_0$ such that

$$\forall \ \tau \geq \tau_0, \ \forall \ u \in C_0^\infty(\tilde{\Omega}), \quad \|\tau^{1/(k+1)} u\|_{L^2} \leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}.$$

If $\varphi \in P_m$ satisfies Assumptions $H_1(\alpha)$ and $H_2(\alpha)$ in a neighborhood of 0 then we have the following maximal (micro)hypoellipticity: for any $\tau \geq \tau_0$ and any $u \in C_0^\infty(\tilde{\Omega})$,

$$\|\partial_x u\|_{L^2} + \|\partial_y u\|_{L^2} + \|\tau(\partial_x \varphi) u\|_{L^2} + \|\tau(\partial_y \varphi) u\|_{L^2} \leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}.$$

**Remark 1.** The maximal estimate was established by Nourrigat (see Theorem 2.1 of [19]) for scalar pseudo-differential operators with complex-valued symbol which satisfies condition $(\tilde{\Psi})$. For the system considered here the maximal estimate is also possible even though the sign may change from $-$ to $+$ for some operator in this system, provided the set of points where condition $(\tilde{\Psi})$ is violated for every operator has Lebesgue measure 0 (see for instance the Maire’s example below); this gives a complement to the Nourrigat’s results.

We would end up the introduction by an example of Maire (cf. [16, Example 1.2]). These are the functions $\varphi$ defined by

$$\varphi(x, y) = x^{2\ell+1} - xy^2, \quad \ell \in \mathbb{Z}_+ \setminus \{0\}.$$  

Then the system (1.4) has the form of

$$\begin{cases}
\mathcal{L}_1 = \partial_x + \tau ((2\ell + 1)x^{2\ell} - y^2) \\
\mathcal{L}_2 = \partial_y - 2\tau xy.
\end{cases}$$

Note

$$x \mapsto ((2\ell + 1)x^{2\ell} - y^2)$$
changes sign from \(-\) to \(+\) at \(x = \left(\frac{y}{\sqrt{2\tau+1}}\right)^\frac{1}{\tau}\) with \(y > 0\), so that we can not apply Nourrigat’s result to conclude, for any \(\tau > 0\),

\[
\|\partial_x u\|_{L^2} + \|\tau(\partial_x \varphi)u\|_{L^2} \leq C\|L_1u\|_{L^2} + C\|u\|_{L^2}.
\]

On the other hand, direct verification shows that Assumption \(H_1(\alpha)\) in \(\mathbb{R}^2\) is fulfilled if we take \(\omega_1 = ]-\infty, 0]\) and \(r_1(y) = \left(-\frac{y}{\sqrt{2\tau+1}}\right)^\frac{1}{\tau}\), and \(\omega_2 = [0, \infty]\) and \(r_2(y) = \left(-\frac{y}{\sqrt{2\tau+1}}\right)^\frac{1}{\tau}\). Then it follows Theorem 1.2 that there exists two constants \(C > 0, \tau_0 \geq 0\) such that for any \(u \in C_0^\infty(\mathbb{R}^2)\) and for all \(\tau \geq \tau_0\),

\[
\|\partial_x u\|_{L^2} + \|\tau(\partial_x \varphi)u\|_{L^2} \leq C\|L_1u\|_{L^2} + C\|L_2u\|_{L^2}.
\]

As a result

\[
\|\tau \frac{1}{\sqrt{\tau+1}} u\|_{L^2} \leq C\|L_1u\|_{L^2} + C\|L_2u\|_{L^2}.
\]

And hence the system \(L_j, 1 \leq j \leq 2\), is subelliptic with exponent \(\frac{1}{\sqrt{\tau+1}}\). We give here a new proof for the Maire’s example basing on condition (Ψ). As proven by Helffer-Nier (see Chapter 11 of [7]), the exponent \(\frac{1}{\sqrt{\tau+1}}\) is optimal and can not be improved. This implies the system \((L_1, L_2)\) is not maximally hypoelliptic when \(\ell > 1\), otherwise a better exponent \(\sigma = \frac{1}{3}\) would be deduced.

2. Proof of Theorem 1.2. In this part we prove the main result. Firstly we introduce some notations to be used frequently later. Let \(\varphi(x, y)\) be a polynomial of degree \(m\). Then we can write \(\partial_x \varphi(x, y)\) and \(\partial_y \varphi(x, y)\) as the following forms

\[
\partial_x \varphi(x, y) = \sum_{j=0}^{\ell} a_j(y)x^j, \quad \text{and} \quad \partial_y \varphi(x, y) = \sum_{j=0}^{d} b_j(x)y^j,
\]

where \(\ell, d \leq m\) and \(a_j(y), b_j(x)\) are polynomials of \(y\) and \(x\) respectively, such that the leading coefficients \(a_{\ell} \neq 0\) and \(b_d \neq 0\). For each pair \((x, y)\) we recall the quantities \(M_j(x, y)\) used in Nourrigat’s work [19], that are defined by

\[
\begin{align*}
M_1(x, y) &= \sum_{j=0}^{\ell} \left|\tau^{\frac{1}{\tau+1}} \partial_x^{j+1} \varphi(x, y)\right|^{\frac{\tau+1}{\tau}}, \\
M_2(x, y) &= \sum_{j=0}^{d} \left|\tau^{\frac{1}{\tau+1}} \partial_y^{j+1} \varphi(x, y)\right|^{\frac{\tau+1}{\tau}},
\end{align*}
\]

and set

\[
G(x, y) = \sum_{1 \leq i+j \leq m} \left|\tau^{\frac{1}{\tau+1}} \partial_x^i \partial_y^j \varphi(x, y)\right|^{\frac{\tau+1}{\tau}}.
\]

Let \(\mathcal{N}_1, \mathcal{N}_2\) be defined by

\[
\mathcal{N}_1 = \{y \in \mathbb{R}; \ M_1(x, y) \neq 0 \text{ for all } x\} , \quad \mathcal{N}_2 = \{x \in \mathbb{R}; \ M_2(x, y) \neq 0 \text{ for all } y\}.
\]

We remark that \(\mathcal{N}_1\) and \(\mathcal{N}_2\) are dense in \(\mathbb{R}\), since \(\mathcal{N}_1\) and \(\mathcal{N}_2\) include, respectively, the sets \(\{y; \ a_{\ell}(y) \neq 0\}\) and \(\{x; \ b_d(x) \neq 0\}\) which are dense in \(\mathbb{R}\).

To simplify the notation we will use the capital letter \(C\) to denote some generic constant that may vary from line to line and depend only on the number \(\alpha\) in Definition 1.1 and the degree \(m\) of the polynomial \(\varphi\), but are independent of \(\tau\).
Lemma 2.1. Let $\delta > 0$ be a given number, and let $I$ be a subset of $\mathbb{R}$. Then there exists a constant $C$ such that for any $u \in C_0^\infty (I - \delta, \delta)$ we have

$$
\|G u\|_{L^2(I - \delta, \delta)} \leq C \sum_{1 \leq j \leq 2} \|L_j u\|_{L^2} + C \left( \|M_1 u\|_{L^2} + \|M_2 u\|_{L^2(I - \delta, \delta)} \right),
$$

recalling $G$ is defined by (2.2).

Proof. If $I = [a, b]$ the estimate (2.4) is a straightforward consequence of Baker-Campbell-Hausdorff formula (see e.g. [17, Lemma 4.14]). Next we treat the general case, following the argument presented in [17]. In this proof we always let $u \in C_0^\infty (I - \delta, \delta)$. Direct verification shows

$$
\|\partial_x u\|_{L^2}^2 + \|\tau (\partial_x \varphi) u\|_{L^2}^2 = \|L_1 u\|_{L^2}^2 + \left( \tau (\partial_x^2 \varphi) u, u \right)_{L^2}.
$$

Then

$$
\|\partial_x u\|_{L^2}^2 \leq \|L_1 u\|_{L^2}^2 + \|\tau \partial_y^2 \varphi\|_{L^2}.
$$

Furthermore since the function $y \mapsto u(x, y)$ with $x \in I$ fixed has compact support in the interval $[a, b]$, then

$$
\|\partial_y u\|_{L^2(I - \delta, \delta)} \leq \|L_2 u\|_{L^2(I - \delta, \delta)} + \|\tau \partial_y^2 \varphi\|_{L^2(I - \delta, \delta)}.
$$

It follows that

$$
\|X u\|_{L^2} + \|Y u\|_{L^2} \leq \sum_{1 \leq j \leq 2} \|L_j u\|_{L^2} + 2\|M_1 u\|_{L^2} + \|M_2 u\|_{L^2(I - \delta, \delta)},
$$

where the operators $X$ and $Y$ are defined by $X = \frac{\partial}{\partial x}$ and $Y = \chi_I(x) \frac{\partial}{\partial y}$ with $\chi_I(x)$ being the characteristic function of the set $I$. Thus the desired estimate will follow if there exists a constant $C$ such that for any integer $q \geq 1$ one has

$$
\sum_{p \geq 0} \|A_{p,q}\|_{L^2}^2 \leq C \left( \|X u\|_{L^2} + \|Y u\|_{L^2} + \|\tau (\partial_x \varphi) u\|_{L^2} \right),
$$

where $A_{p,q}$ are purely imaginary-valued functions defined iteratively by

$$
A_{0,q} = i^{q-1} \left[ A_{0}, \{ X, \ldots, X, \tau \partial_x \varphi \}, \ldots \right], \quad A_{p,q} = i^{q}[Y, A_{p-1,q}],
$$

for $p \geq 1$.

Here we denote by $[P, Q]$ the commutator of two operators $P$ and $Q$, which is defined by $[P, Q] = PQ - QP$. Firstly note that for any $q \geq 1$ we have

$$
\|A_{0,q}\|_{L^2} \leq C \left( \|\partial_x u\|_{L^2} + \|\tau (\partial_x \varphi) u\|_{L^2} + \|u\|_{L^2} \right),
$$

which is just the assertion of [17, Lemma 4.14]). Next we will treat the case when $p \geq 1$. Baker-Campbell-Hausdorff formula implies

$$
\exp(tY) \exp(t^{p+q} A_{p-1,q}) \exp(-tY) \exp(-t^{p+q} A_{p-1,q}) = \exp(t^{p+q+1} A_{p,q} + \sum_{j=1}^{m} c_j t^{p+q+j+1} A_{p+j,q}),
$$

where $c_j$ are a family of constants. As a result we use induction on $p$ to conclude that there exist two integers $n_1$ and $n_2$ depending only on $p$, and a sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n_1})$ with $\varepsilon_j = \pm 1$, such that for each $p \geq 1$ one has

$$
\exp(t^{p+q+1} A_{p,q}) = \prod_{j=1}^{n_2} Z_j \exp \left( \sum_{k=1}^{m} c_k t^{p+q+k+1} A_{p+k,q} \right) = \prod_{j=1}^{n_2} Z_j \prod_{k=1}^{m} \exp \left( c_k t^{p+q+k+1} A_{p+k,q} \right)
$$

for
with $Z_j \in \{ \exp(\varepsilon_1 tY), \ldots, \exp(\varepsilon_{n_1} tY), \exp\left( t^{q+1} A_{0,q}\right) \}$. The last equality holds because $A_{p,q}$ are functions and hence commutative. Then

$$\exp\left( t^{p+q+1} A_{p,q}\right) - \text{Id}$$

with $\text{Id}$ the identity operator, can be rewritten as

$$\prod_{k=1}^{n_2} Z_k \sum_{i=1}^{m} \prod_{j=1}^{i-1} \left( c_j t^{q+j+1} A_{p+j,q}\right) \left( \exp\left( c_i t^{q+i+1} A_{p+i,q}\right) - \text{Id}\right) + \sum_{i=1}^{n_2} \prod_{j=1}^{i-1} Z_j \left( Z_i - \text{Id}\right),$$

which, together with fact that the norms of operators $Z_j$ are smaller than 1, implies

$$\| \exp\left( t^{p+q+1} A_{p,q}\right) u - u \|_{L^2}^2 \leq C \sum_{1 \leq k \leq m} \| \exp\left( c_k t^{k+q+1} A_{k+k,q}\right) u - u \|_{L^2}^2$$

$$+ C \sum_{\varepsilon_i \in \{e_1, \ldots, e_{n_1}\}} \| \exp(\varepsilon_i tY) u - u \|_{L^2}^2 + C \| \exp(\varepsilon t\gamma) u - u \|_{L^2}^2. \quad (2.8)$$

Since the functions $iA_{p,q}$ are real-valued and hence self-adjoint as operators acting on $L^2$. So by the spectral theorem the norm $\| iA_{p,q} \|_{\frac{1}{p+q+1}} u \|_{L^2}$ is equivalent to

$$(p+q+1)^{-1} \int_0^\infty \| \exp\left( t^{p+q+1} A_{p,q}\right) u - u \|_{L^2}^2 \frac{dt}{t^3}. $$

As a result, integrating both sides of (2.8) with respect to $t$ with the measure $t^{-3}dt$ gives that

$$\| A_{p,q} \|_{\frac{1}{p+q+1}} u \|_{L^2} \leq C \sum_{k=1}^m \| A_{p+k,q} \|_{\frac{1}{p+q+k+1}} u \|_{L^2} + C \| Y u \|_{L^2} + C \| A_{0,q} \|_{\frac{1}{1+q}} u \|_{L^2}. $$

Using reverse induction starting from $p = m$ and combing (2.7), we get the desired estimate (2.6), completing the proof. \hfill \square

**Lemma 2.2.** There exist two constants $C_s \geq 1$ and $r_0 \leq 1$ depending only on $m$, such that for any $y \in \mathcal{N}_1$ and $x \in \mathcal{N}_2$ with $\mathcal{N}_j$ defined by (2.3), we have

$$\forall x_*, \tilde{x} \in \mathbb{R}, \ |x_* - \tilde{x}| M_1(x_*, y) < r_0 \Rightarrow C_s^{-1} \leq \frac{M_1(x_*, y)}{M_1(\tilde{x}, y)} \leq C_s, \quad (2.9)$$

$$\forall y_*, \tilde{y} \in \mathbb{R}, \ |y_* - \tilde{y}| M_2(x_*, y_*) < r_0 \Rightarrow C_s^{-1} \leq \frac{M_2(x_*, y_*)}{M_2(\tilde{x}, \tilde{y})} \leq C_s. \quad (2.10)$$

And for any $x \in \mathbb{R},$

$$\forall y_*, \tilde{y} \in \mathbb{R}, \ |y_* - \tilde{y}| G(x, y_*) < r_0 \Rightarrow C_s^{-1} \leq \frac{G(x, y_*)}{G(x, \tilde{y})} \leq C_s. \quad (2.11)$$

**Proof.** We firstly show (2.9), and claim that for any $y \in \mathcal{N}_1$ we have, with each $0 \leq j \leq \ell$ with $\ell$ the integer given in (2.1),

$$\forall |x_* - \tilde{x}| M_1(x_*, y) < 1, \ |\tau \partial_x^{j+1} \varphi(\tilde{x}, y)| \leq (\ell + 2) M_1(x_*, y)^{j+1}. \quad (2.12)$$

In fact, by Taylor’expansion

$$\partial_x^{j+1} \varphi(\tilde{x}, y) = \partial_x^{j+1} \varphi(x_*, y) + \sum_{i=0}^{\ell-j} \frac{\partial_x^{j+j+1} \varphi(x_*, y)}{i!} (\tilde{x} - x_*)^i,$$
Lemma 2.3. We have
\[ |\tau \partial_x^{i+1} \varphi(x, y)| \leq M_1(x_*, y)^{i+1} + \sum_{i=0}^{\ell-j} \frac{M_1(x_*, y)^{i+j+1} |\tilde{x} - x_*|^i}{\ell-j}, \]
which yields the assertion (2.12). As a result it follows from (2.12) that
\[ \forall x_*, \tilde{x} \in \mathbb{R} \text{ with } |x_* - \tilde{x}| M_1(x_*, y) < 1, \quad M_1(\tilde{x}, y) \leq (\ell + 2)^2 M_1(x_*, y). \]
(2.13)
Now for any \( x_* \) and \( \tilde{x} \) with \( |x_* - \tilde{x}| M_1(x_*, y) < r_0 \leq 1 \), we use (2.13) to get
\[ |x_* - \tilde{x}| M_1(\tilde{x}, y) \leq (\ell + 2)^2 |x_* - \tilde{x}| M_1(x_*, y) < (\ell + 2)^2 r_0. \]
As a result, if we choose \( r_0 \) sufficiently small such that \( (\ell + 2)^2 r_0 < 1 \) then we use (2.13) to conclude
\[ M_1(x_*, y) \leq (\ell + 2)^2 M_1(\tilde{x}, y). \]
We have proven (2.9). The inequalities (2.10) and (2.11) can be deduced similarly. This completes the proof. \( \square \)

Lemma 2.3. Let \( P_m \) be the set of all polynomials of degree \( \leq m \), and \( r \in C^1(\omega; \mathbb{R}) \) be a given function defined in \( \omega \). Then there exists a number \( \delta > 0 \) and a constant \( C > 0, \) both depending only on \( m \), such that the following estimate
\[ \|M_2u\|_{L^2(I \times |x - \delta, \delta|)} \leq C\|L_2u\|_{L^2} \]
holds for all \( u \in C_{00}^{\infty} [\delta - |x| - \delta, 0) \) and for all \( \varphi \in P_m \) satisfying that for each \( s \in \omega \) the function \( y \mapsto \partial_y \varphi(r(s), y) \) has no sign change from \( - \) to \( + \) for increasing \( y \), where \( I \) stands for the range of \( r \), i.e., \( I = r(\omega) \).

Remark 2. If \( r = I d \) then this is just a local version of the result of Nourrigat (see Theorem 2.1 of [19]).

To prove the above lemma we need the following

Lemma 2.4 (Lemmas 3.1.1-3.1.3 of [14]). Let \( [a, b] \subset \mathbb{R} \) be a given interval, and let \( \phi(s) \in C^0([a, b]; \mathbb{R}) \) such that \( \phi \) does not change sign from \( - \) to \(+ \) as \( s \) increases in \([a, b]. \) Then for any \( v \in C^1([a, b]) \) with \( v(a) = 0 \) we have
\[ \max_{s \in [a, b]} |v(s)|^2 + 2 \int_a^b |\tau \phi(s)||v(s)|^2 ds \leq |v(b)|^2 + 2 \int_a^b \left( \frac{d}{ds} + \tau \phi \right)v(s) |v(s)| ds, \]
where \( \tau > 0 \) is a given number. In particular, for any for any \( v \in C^1_0([a, b]) \) we have
\[ \max_{s \in [a, b]} |v(s)|^2 + 2 \int_a^b |\tau \phi(s)||v(s)|^2 ds \leq 2 \int_a^b \left( \frac{d}{ds} + \tau \phi \right)v(s) |v(s)| ds. \]
Moreover if
\[ \min_{s \in [a, b]} \left| \frac{d^k \phi}{ds^k} (s) \right| \geq c_* \]
for some \( k \in \mathbb{Z}_+ \) and some constant \( c_* > 0 \), then for any \( v \in C^1_0([a, b]), \) we have
\[ \tau \frac{\pi^2}{4} \int_a^b |v(s)|^2 ds \leq C \left( \max_{s \in [a, b]} |v(s)|^2 + \int_a^b |\tau \phi(s)||v(s)|^2 ds \right), \]
with \( C \) a constant depending only on \( k \) and \( c_* \) above.

Proof. The proof just follows from the argument for proving [14, Lemma 3.1.1] and the assertions in [14, Lemmas 3.1.2-3.1.3]. \( \square \)
Proof of Lemma 2.3. Recall $\mathcal{N}_2$ is defined by (2.3). Since $r(\omega) \setminus \mathcal{N}_2$ has measure 0 then we may assume without loss of generality that $r(\omega) \subset \mathcal{N}_2$. For $x_0 = r(s_0) \in r(\omega) \subset \mathcal{N}_2$, $y_0 \in \mathbb{R}$ and $\varphi \in P_m$ we define a new function $\zeta(x_0, \cdot)$ by setting

$$\zeta(x_0, y) = \frac{(\partial_y \varphi)(x_0, y_0 + \frac{y}{M_2(x_0, y_0)})}{M_2(x_0, y_0)}.$$  

Then we use Taylor’s expansion to write

$$\zeta(x_0, y) = \sum_{0 \leq j \leq d} \frac{\partial^{j+1}_y \varphi(x_0, y_0)}{M_2(x_0, y_0)^{j+1}} \frac{y^j}{j!},$$  

where the integer $d$ is given in (2.1). By direct verification,

$$\forall \; 0 \leq j \leq d, \quad \frac{|\partial^{j+1}_y \varphi(x_0, y_0)|}{M_2(x_0, y_0)^{j+1}} \leq 1,$$  

which yields

$$\forall \; 0 \leq j \leq d, \quad |\partial^j \zeta(x_0, 0)| \leq 1. \quad (2.15)$$

On the other hand, we can find an integer $k$ with $0 \leq k \leq d \leq m$ such that

$$|\partial^{k+1}_y \varphi(x_0, y_0)|^{1/(k+1)} = \max_{0 \leq j \leq d} |\partial^{j+1}_y \varphi(x_0, y_0)|^{1/(j+1)}.$$  

Thus

$$|(\partial^k_y \zeta)(x_0, 0)| = \frac{\partial^{k+1}_y \varphi(x_0, y_0)}{M_2(x_0, y_0)^{k+1}} \geq c_0$$  

for some constant $c_0 > 0$ depending only on $m$. This, with (2.15), implies

$$|(\partial^k_y \zeta)(x_0, y)| \geq c_0 - \sum_{1 \leq j \leq d-k} |y|^j.$$  

As a result one can find a number $\delta > 0$ depending only on $m$, such that

$$\inf_{y \in [-\delta, \delta]} |(\partial^k_y \zeta)(x_0, y)| \geq c_0/2. \quad (2.16)$$

Moreover by assumption $\partial_y \varphi(x_0, y)$ has no sign change from $- \to +$ for increasing $y$. Then the function $\zeta(x_0, y)$ doesn’t change sign from $- \to +$ as $y$ increases in $] - \delta, \delta[$. As a result we apply Lemma 2.4 with $\phi = \zeta$ to conclude for any $v \in C_0^\infty([-\delta, \delta])$ we have

$$\tau^{k+1} \int_{-\delta}^{\delta} |v(y)|^2 \, dy \leq C \int_{-\delta}^{\delta} |(\partial_y + \tau \zeta(x_0, y))v(y)|^2 \, dy + C \int_{-\delta}^{\delta} |v(y)|^2 \, dy.$$  

Inspired by [10], for any function $v \in C^\infty([-\delta, \delta])$ having no compact support in $[-\delta, \delta]$, we can introduce a cut-off function $\chi(y)$ supported in the interval $[-\delta, \delta]$ and equal to 1 in $]-\delta/2, \delta/2[$. Then applying the above inequality to the function $\chi(y)v(y)$ gives that, for any $v \in C^\infty([-\delta, \delta])$,

$$\tau^{k+1} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |v(y)|^2 \, dy \leq C \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |(\partial_y + \tau \zeta(x_0, y))v(y)|^2 \, dy + C \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |v(y)|^2 \, dy. \quad (2.17)$$

Now for any given $u \in C_0^\infty([-\delta, \delta])$ we define $v \in C^\infty([-\delta, \delta])$ by setting

$$v(y) = u \left(x_0, y_0 + \frac{y}{M_2(x_0, y_0)} \right).$$
For such a function \( v \) we can verify that

\[
(\partial_y + \tau \zeta(x_0, y_0)) v(y) = \frac{(L_2 u)(x_0, y_0 + y/M_2(x_0, y_0))}{M_2(x_0, y_0)}.
\]

As a result, applying (2.17) to the function \( v \) above and then using the change of variables, we get

\[
\tau \int_{I_{y_0}} M_2(x_0, y_0)^3 |u(x_0, y)|^2 \, dy \leq C \int_{J_{y_0}} M_2(x_0, y_0) |(L_2 u)(x_0, y_0)|^2 \, dy + C \int_{J_{y_0}} M_2(x_0, y_0)^3 |u(x_0, y)|^2 \, dy,
\]

where

\[
I_{y_0} = \{ y : |y - y_0| M_2(x_0, y_0) < \delta/2 \}, \quad J_{y_0} = \{ y : |y - y_0| M_2(x_0, y_0) < \delta \}.
\]

By (2.10) we see \( C^{-1}_* \leq M_2(x_0, y_0)/M_2(x_0, y) \leq C_* \) when \( y \in J_{y_0} \), shrinking \( \delta \) if necessary. Then we may replace \( M_2(x_0, y_0) \) by \( M_2(x_0, y) \) in the above integrands, that is,

\[
\tau \int_{I_{y_0}} M_2(x_0, y)^3 |u(x_0, y)|^2 \, dy \leq C \int_{J_{y_0}} M_2(x_0, y) |(L_2 u)(x_0, y)|^2 \, dy + C \int_{J_{y_0}} M_2(x_0, y)^3 |u(x_0, y)|^2 \, dy.
\]

Thus taking integration with respect to \( y_0 \in \mathbb{R} \) on the both sides of above inequality and observing

\[
\{ y : |y - y_0| M_2(x_0, y) < \delta/(2C_*) \} \subset I_{y_0}, \quad J_{y_0} \subset \{ y : |y - y_0| M_2(x_0, y) < C_* \delta \},
\]

we obtain, applying Fubini’s theorem,

\[
\tau \int_{J_{y_0}} \int_{-\delta}^{\delta} M_2(x_0, y) u(x_0, y) \, dy \, dx \leq C \int_{J_{y_0}} \int_{-\delta}^{\delta} |(L_2 u)(x_0, y)|^2 \, dy \, dx.
\]

Note that the constant \( C \) is independent of \( \tau \), and hence we can choose \( \tau \) large enough such that \( \tau^{-1} \geq 2C \). This implies

\[
\tau \int_{J_{y_0}} \int_{-\delta}^{\delta} M_2(x_0, y) u(x_0, y) \, dy \, dx \leq C \int_{J_{y_0}} \int_{-\delta}^{\delta} |(L_2 u)(x_0, y)|^2 \, dy \, dx.
\]

Now integrating both sides with respect to \( x_0 \in r(\omega) \) gives the desired estimate (2.14). The proof is hence completed.

**Lemma 2.5.** Let \( \delta, \alpha > 0 \) be two given numbers with \( \delta \) small sufficiently and let \( r(y) \in C^1(\omega; \mathbb{R}) \) such that \( \left| \frac{\partial r}{\partial y}(y) \right| \geq \alpha \) for any \( y \in \omega \). Suppose \( \varphi \in P_m \) satisfies that for each \( y_0 \in \omega \) the function \( y \mapsto (\partial_y \varphi)(r(y_0), y) \) has no sign change from \( -t_0 \) to \( +t_0 \) for increasing \( y \). Then the following estimate

\[
\int_{\omega} G(r(y), y) |u(r(y), y)|^2 \, dy \leq C \sum_{1 \leq j \leq 2} \| L_j u \|_{L^2}^2 + C \| M_1 u \|_{L^2}^2
\]

holds for all \( u \in C^\infty_0 \{ [-\delta, \delta[ \times \ldots ] - \delta, \delta[ \} \). Recall \( G = \sum_{i+j \geq 1} |\tau \partial_x^i \partial_y^j r|^\frac{i+j}{2} \).
Proof. This proof is quite similar as Lemma 2.3. Let \( y_0 \in \omega \) be given and we define a new function \( y \mapsto \eta(r(y_0), y) \) by setting
\[
\eta(r(y_0), y) = \frac{(\partial_y \varphi) \left( r(y_0), y_0 + y/G(r(y_0), y_0) \right)}{G(r(y_0), y_0)}.
\]
Then by the assumption, the function \( y \mapsto \eta(r(y_0), y) \) does not change sign from \(-\) to \(+\) as \( y \) increases in \([-\delta, \delta]\). Then we apply Lemma 2.4 to conclude, for any \( v \in C_0^\infty (\omega) \),
\[
\max_{y \in [-\delta, \delta]} |v(y)|^2 \leq 2 \int_{-\delta}^\delta \left| (\partial_y + \tau \eta(r(y_0), y))v(y) \right| |v(y)| \, dy.
\]
Thus applying the above inequality to \( \chi(y)v(y) \), with \( \chi \in C_0^\infty (\omega) \) and \( \chi \equiv 1 \) on \([-\delta/2, \delta/2]\), we obtain, for any \( v \in C_0^\infty (\omega) \)
\[
|v(0)|^2 \leq \max_{y \in [-\delta, \delta]} |v(y)|^2 \leq C \int_{-\delta}^\delta \left| (\partial_y + \tau \eta(r(y_0), y))v(y) \right|^2 \, dy + C \int_{-\delta}^\delta |v(y)|^2 \, dy.
\]
As a result for any \( u \in C_0^\infty (\omega) \), we apply the above estimate to
\[
v(y) := u \left( r(y_0), y_0 + \frac{y}{G(r(y_0), y_0)} \right);
\]
this yields, by virtue of the change of variables,
\[
G(r(y_0), y_0)^2 |u(r(y_0), y_0)|^2 \leq C \int_{K_{y_0}} |(\mathcal{L}_2 u)(r(y_0), y)|^2 \, dy + C \int_{K_{y_0}} G(r(y_0), y_0)^2 |u(r(y_0), y)|^2 \, dy,
\]
where \( K_{y_0} = \{ y : |y - y_0| < \delta \} \). It follows from (2.11) that
\[
\forall y \in K_{y_0}, \quad C_*^{-1} G(r(y_0), y) \leq G(r(y_0), y_0) \leq C_* G(r(y_0), y).
\]
Then we may replay \( G(r(y_0), y_0) \) by \( G(r(y_0), y) \) in the above integrands. This gives
\[
G(r(y_0), y_0)^2 |u(r(y_0), y_0)|^2 \leq C \int_{-\delta}^\delta |(\mathcal{L}_2 u)(r(y_0), y)|^2 \, dy + C \int_{-\delta}^\delta G(r(y_0), y)^2 |u(r(y_0), y)|^2 \, dy.
\]
Integrating both sides with respect to \( y_0 \in \omega \) gives
\[
\int_\omega G(r(y_0), y_0)^2 |u(r(y_0), y_0)|^2 \, dy_0 \leq C \int_{-\delta}^\delta \left( |(\mathcal{L}_2 u)(r(y_0), y)|^2 + G(r(y_0), y)^2 |u(r(y_0), y)|^2 \right) \, dy_0 \, dy_0 \leq C \| \mathcal{L}_2 u \|_{L^2}^2 + C \| Gu \|_{L^2(I \times [-\delta, \delta])}^2
\]
with \( I = r(\omega) \), the last inequality using the change of variables \( r(y_0) \to x \) and the fact that \(|r'(y)| \geq \alpha \) for all \( y \in \omega \). Moreover we apply Lemma 2.1 and Lemma 2.3 to conclude
\[
\| Gu \|_{L^2(I \times [-\delta, \delta])} \leq C \sum_{1 \leq j \leq 2} \| \mathcal{L}_j u \|_{L^2} + C \left( \| M_1 u \|_{L^2} + \| M_2 u \|_{L^2(I \times [-\delta, \delta])} \right) \leq C \sum_{1 \leq j \leq 2} \| \mathcal{L}_j u \|_{L^2} + C \| M_1 u \|_{L^2}.
\]
Combining the above inequalities gives the desired estimate (2.18). The proof is completed.

Proof of Theorem 1.2. Let \( \varphi \in P_m \) satisfy Assumption \( H_1(\alpha) \) and let \( \omega_j, r_j, 1 \leq j \leq N_1 \), be given therein. For any \( y_0 \in \omega_j \cap \mathcal{N}_1 \) with \( 1 \leq j \leq N_1 \) and \( x_0 \in \mathbb{R} \) we define a new function \( x \mapsto \xi(x, y_0) \) by setting

\[
\xi(x, y_0) = \frac{(\partial_x \varphi)(x_0 + \frac{x}{M_1(x_0, y_0)}, y_0)}{M_1(x_0, y_0)}.
\]

Similar to the proof of (2.16), we may find an integer \( k \) such that

\[
\inf_{x \in [-\delta, \delta]} |(\partial_x^k \xi)(x, y_0)| \geq c_0/2
\]

with \( c_0, \delta \) depending only on \( m \). Denote

\[
\mathcal{A}(y_0) = \{ x \in \mathbb{R}; |x - r(y_0)| M_1(x, y_0) < \delta \}.
\]

We have two cases that either \( x_0 \in \mathcal{A}(y_0) \) or \( x_0 \notin \mathcal{A}(y_0) \).

**Case (a)**. We firstly consider the case when \( x_0 \in \mathcal{A}(y_0) \), that is,

\[
|r(y_0) - x_0| M_1(x_0, y_0) \leq \delta.
\]

Since \( \varphi \in P_m \) satisfies Assumption \( H_1(\alpha) \) then the function \( x \mapsto \partial_x \varphi(x, y_0) \) only changes sign from \( - \) to \( + \) at \( r(y_0) \) for increasing \( x \) and the function \( y \mapsto \partial_y \varphi(r(y_0), y) \) has no sign change from \( - \) to \( + \) for increasing \( y \). This with (2.20) implies that \( x \mapsto \xi(x, y_0) \) only changes sign from \( - \) to \( + \) at the critical point \( (r(y_0) - x_0) M_1(x_0, y_0) \) for \( x \) increases in \( ] - \delta, \delta [ \). Then we apply the first assertion in Lemma 2.4 with

\[
[a, b] = [-\delta, (r(y_0) - x_0) M_1(x_0, y_0)] \quad \text{or} \quad [a, b] = [(r(y_0) - x_0) M_1(x_0, y_0), \delta],
\]

to conclude for any \( v \in C_0^\infty(] - \delta, \delta [) \),

\[
\int_{-\delta}^{\delta} |v(x)|^2 dx + 2 \int_{-\delta}^{\delta} |\tau \xi(x, y_0)| |v(x)|^2 dx \leq 2 |v((r(y_0) - x_0) M_1(x_0, y_0))| + 4 \int_{-\delta}^{\delta} |(\partial_x + \tau \xi(x, y_0))v(x)| |v(x)| dx.
\]

with (2.19) and the last assertion in Lemma 2.4, yields, for any \( v \in C_0^\infty(] - \delta, \delta [) \),

\[
\int_{-\delta}^{\delta} |v(x)|^2 dx \leq C |v((r(y_0) - x_0) M_1(x_0, y_0))| + C \int_{-\delta}^{\delta} |(\partial_x + \tau \xi(x, y_0))v(x)| |v(x)| dx.
\]

Then we follow the argument for proving Lemma 2.3, for any \( v \in C^\infty([-\delta, \delta]) \), we have

\[
\int_{-\delta/2}^{\delta/2} |v(x)|^2 dx \leq C |v((r(y_0) - x_0) M_1(x_0, y_0))| + C \int_{-\delta}^{\delta} |(\partial_x + \tau \xi(x, y_0))v(x)| |v(x)| dx.
\]

Now for any \( u \in C_0^\infty(] - \delta, \delta [ \times ] - \delta, \delta [) \) we apply the above estimate to

\[
v(x) := u \left( x_0 + \frac{x}{M_1(x_0, y_0)}, y_0 \right);
\]
this gives, for any \( u \in C_0^\infty([ - \delta, \delta) \times \mathbb{R}) \),
\[
\tau \int_{I_{x_0}} M_1(x_0, y_0)^3 |u(x, y_0)|^2 \, dx 
\leq CM_1(x_0, y_0)^3 |u(r(y_0), y_0)|^2 + C \int_{J_{x_0}} M_1(x_0, y_0)|\mathcal{L}_1 u(x, y_0)|^2 \, dx 
+ C \int_{J_{x_0}} M_1(x_0, y_0)^3 |u(x, y_0)|^2 \, dx, \tag{2.21}
\]
where
\[
\tilde{I}_{x_0} = \{ x; |x - x_0| M_1(x_0, y_0) < \frac{\delta}{2} \}, \quad \tilde{J}_{x_0} = \{ x; |x - x_0| M_1(x_0, y_0) < \delta \}.
\]
It follows from (2.9) that \( C_*^{-1} \leq M_1(x_0, y_0)/M_1(x, y) \leq C_* \) when \( x \in \tilde{J}_{x_0} \). Moreover in view of (2.20),
\[
C_*^{-1} M_1(r(y_0), y_0) \leq M_1(x_0, y_0) \leq C_* M_1(r(y_0), y_0).
\]
Consequently we replay \( M_1(x_0, y_0) \) by \( M_1(x, y) \) in the above integrands and and by \( M_1(r(y_0), y_0) \) in the first term on the right side, and then integrate both side with respect to \( x_0 \in \mathcal{A}(y_0) \); this gives
\[
\tau \int_{\mathcal{A}(y_0)} \int_{I_{x_0}} M_1(x, y_0)^3 |u(x, y_0)|^2 \, dx \, dx_0 
\leq CM_1(r(y_0), y_0)^2 |u(r(y_0), y_0)|^2 + C \int_{\mathcal{A}(y_0)} \int_{J_{x_0}} \left( M_1(x, y_0)|\mathcal{L}_1 u(x, y_0)|^2 
+ M_1(x, y_0)^3 |u(x, y_0)|^2 \right) dx \, dx_0. \tag{2.22}
\]
Here we have used the fact that \( \mathcal{A}(y_0) \subset \{ x; |x - r(y_0)| M_1(r(y_0), y_0) < C_* \delta \} \) and hence the measure of \( \mathcal{A}(y_0) \) is less than \( 2C_* \delta/M_1(r(y_0), y_0) \).

Case (b). Next we consider the case that \( x_0 \notin \mathcal{A}(y_0) \). Then
\[
|r(y_0) - x_0| M_1(x_0, y_0) \geq \delta.
\]
This implies \( \xi(x, y_0) \) has no sign change from \(- \) to \( + \) as \( x \) increases in \( [ - \delta, \delta) \). Then the estimate (2.21) still holds with the absence of the first term on the right side. Then repeating the above arguments, we can show that, with \( x_0 \) varying in \( \mathcal{A}^c(y_0) \seteq \mathbb{R} \setminus \mathcal{A}(y_0) \),
\[
\tau \int_{\mathcal{A}^c(y_0)} \int_{I_{x_0}} M_1(x, y_0)^3 |u(x, y_0)|^2 \, dx \, dx_0 
\leq C \int_{\mathcal{A}^c(y_0)} \int_{J_{x_0}} \left( M_1(x, y_0)|\mathcal{L}_1 u(x, y_0)|^2 + M_1(x, y_0)^3 |u(x, y_0)|^2 \right) dx \, dx_0. \tag{2.23}
\]
Completing of the proof of Theorem 1.2. In conclusion we take sum of (2.22) and (2.23), to get
\[
\tau \int_R \int_{I_{x_0}} M_1(x, y_0)^3 |u(x, y_0)|^2 \, dx \, dx_0 
\leq CM_1(r(y_0), y_0)^2 |u(r(y_0), y_0)|^2 
+ C \int_R \int_{J_{x_0}} \left( M_1(x, y_0)|\mathcal{L}_1 u(x, y_0)|^2 + M_1(x, y_0)^3 |u(x, y_0)|^2 \right) dx \, dx_0,
\]
which, with Fubini’s theorem as well as the facts that
\[
\begin{aligned}
\{x; \ |x - x_0| M_1(x, y_0) < \frac{\delta}{2C_*}\} \subset \tilde{I}_{x_0}, \quad \tilde{I}_{x_0} \subset \{x; \ |x - x_0| M_1(x, y_0) < C_* \delta\},
\end{aligned}
\]
implies
\[
\begin{aligned}
\tau \frac{1}{\pi} \int_{-\delta}^{\delta} M_1(x, y_0)^2 |u(x, y_0)|^2 \, dx
\leq C M_1(r(y_0), y_0)^2 |u(r(y_0), y_0)|^2 + C \int_{-\delta}^{\delta} \left( |\mathcal{L}_1 u(x, y_0)|^2 + M_1(x, y_0)^2 |u(x, y_0)|^2 \right) \, dx.
\end{aligned}
\]
Observe the above inequality holds for all \( y_0 \in \omega_j \cap N_1 \) with \( N_1 \) dense in \( \omega_j \). Then after integration with respect to \( y_0 \) we obtain
\[
\begin{aligned}
\tau \frac{1}{\pi} \|M_1 u\|_{L^2(-\delta, \delta \times \omega_j)} \leq C \int_{\omega_j} M_1(r(y), y) |u(r(y), y)|^2 \, dy + C \|\mathcal{L}_1 u\|_{L^2}^2 + C \|M_1 u\|_{L^2}^2
\leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}^2 + C \|M_1 u\|_{L^2}^2,
\end{aligned}
\]
the last inequality following from (2.18) since \( M_1 \leq G \). Observe \( -\delta, \delta \subset \bigcup_{1 \leq j \leq 2} \bar{\omega}_j \) disjoint, and thus
\[
\begin{aligned}
\tau \frac{1}{\pi} \|M_1 u\|_{L^2} \leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}^2 + C \|M_1 u\|_{L^2}^2.
\end{aligned}
\]
Now we choose \( \tau_0 \) such that \( \tau_0 \frac{1}{\pi} = 4C \); this gives
\[
\forall \, u \in C_0^\infty (-\delta, \delta), \quad \forall \, \tau \geq \tau_0, \quad \|M_1 u\|_{L^2} \leq C \sum_{1 \leq j \leq 2} \|\mathcal{L}_j u\|_{L^2}^2.
\]
In view of (2.5) we have for any \( u \in C_0^\infty (-\delta, \delta) \),
\[
\|\partial_x u\|_{L^2}^2 + \|\tau (\partial_x \varphi) u\|_{L^2}^2 \leq \|\mathcal{L}_1 u\|_{L^2}^2 + \|M_1 u\|_{L^2}^2,
\]
and thus the desired estimate follows. The proof of Theorem 1.2 is completed.

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*E-mail address: wei-xi.li@whu.edu.cn*
*E-mail address: xuchaojiang@nuaa.edu.cn*