Holographic Superconductors in Hořava-Lifshitz Gravity

Kai Liu and E. Abdalla

Instituto de Física, Universidade de São Paulo, CP 66318, 05315-970, São Paulo, Brazil

Anzhong Wang

GCAP-CASPER, Physics Department, Baylor University, Waco, TX 76798-7316, U.S.A.

(Dated: September 8, 2018)

We consider holographic superconductors related to the Schwarzschild black hole in the low energy limit of Hořava-Lifshitz spacetime. The non-relativistic electromagnetic and scalar fields are introduced to construct a holographic superconductor model in Hořava-Lifshitz gravity and the results show that the $\alpha_2$ term plays an important role, modifying the conductivity curve line by means of an attenuation the conductivity.

PACS numbers: 11.25.Tq, 04.70.Bw, 74.20.-z

I. INTRODUCTION

Quantization of gravity is a key issue in modern theoretical gravitational theory, since as a quantum field theory Einstein's general relativity with Lorentz symmetry is unrenormalizable. A renormalizable candidate of quantum gravity has been recently proposed by Hořava [1], who assumed that the Lorentz symmetry is broken in the ultraviolet, so that the anisotropic scalings between space and time are given by

$$x \rightarrow \ell x, \quad t \rightarrow \ell^\gamma t.$$  \hfill (1.1)

A power-counting renormalizable gravity theory must satisfy $z \geq 3$ in 3+1 dimensional spacetime. Such a theory is called the Hořava-Lifshitz gravity.

Hořava-Lifshitz theory has attracted the attention of many theoretical physicists. However, it also faces several problems, in particular arising from the spin-0 graviton. In order to solve such shortcomings, a local U(1) gauge field $A$ is introduced [2]. Meanwhile some recent works prove that the problems from spin-0 graviton can be cancelled in Hořava-Lifshitz gravity by means of the U(1) gauge field $A$ [3], and the post-newtonian approximation is also satisfied [4].

On the other hand, recently, Hartnoll, Herzog and Horowitz considered the AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence principle to study the strongly correlated condensed matter physics with the gravitational duality. They found a correspondence between the instability of black string and the second-order phase transition from normal to superconductor state.

Subsequently, several authors generalized the idea to investigate holographic superconductors of various black hole solutions [5]. However, the holographic superconductor models are under the framework of spacetime with Lorentz symmetry. In this paper, we build a holographic superconductor model with non-relativistic matter in static Hořava-Lifshitz spacetime. The present research could help understanding the stability properties in Hořava-Lifshitz spacetime.

We plan this paper as follows. In section II, we generalize the model in general relativity to build a holographic superconductor action in Hořava-Lifshitz spacetime, and focus on discussing the correction from non-relativistic terms in section III. Then, in section IV, the conductivity will be calculated, and section V includes some conclusions and a summary.

II. SUPERCONDUCTOR ACTION IN HOŘAVA-LIFSHITZ GRAVITY

In this section, we build a holographic superconductor model in Hořava-Lifshitz gravity. We first analyze the holographic superconductor in general relativity. In [5], Hartnoll, Herzog and Horowitz proposed the model with the Lagrangian density

$$\mathcal{L}_G = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - |\nabla \bar{\Psi} - iq A \bar{\Psi}|^2 - V(|\bar{\Psi}|).$$ \hfill (2.1)

where $\bar{\Psi}$ is the scalar field, $A_{\mu}$ is electromagnetic four-potential in general relativity, and $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. The holographic superconductor model also requests

$$V(|\bar{\Psi}|) = m^2 |\bar{\Psi}|^2.$$ \hfill (2.2)

Rewriting the scalar field $\bar{\Psi}$ as $\Psi e^{ip}$ [9], with real $\Psi$ and $p$, Eq. (2.1) becomes

$$\mathcal{L}_G = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \nabla_{\mu} \Psi \nabla^{\mu} \Psi - V(|\Psi|)$$
$$- (q A_{\mu} - \partial_{\mu} p) (q A^{\mu} - \partial^{\mu} p) \Psi^2.$$  \hfill (2.3)

We choose the gauge $p = 0$ and get the action of the simplest models in general relativity,

$$S_G = \int d^4 x \sqrt{|g|} \left( \frac{1}{4} \mathcal{L}_G^E + 2 \mathcal{L}_G^S - \mathcal{L}_G^C \right),$$ \hfill (2.4)
where $g^{(4)}$ is the determinant of the 4 dimensional metric $g^{(5)}$, while the electromagnetic, scalar and coupling parts of the Lagrangian are, respectively, given by

\[ \mathcal{L}_G^E = F_{\mu
u}F^{\mu\nu}, \]
\[ \mathcal{L}_G^S = -\frac{1}{2}\nabla_{\mu}\nabla_{\nu}\Psi - \frac{1}{2}V(|\Psi|), \]
\[ \mathcal{L}_G^C = q^2 g^{(4)}_{\mu\nu} A_\mu A_\nu |\Psi|^2. \] (2.5)

Next, we generalize the action in Hořava-Lifshitz theory, with the Arnowitt-Deser-Misner metric

\[ ds^2 = -N^2 dt^2 + g_{ij} (dx^i - N^i dt) (dx^j - N^j dt). \] (2.6)

The non-relativistic matter in Hořava-Lifshitz gravity was proposed in [8] and the Lagrangians of complex scalar and electromagnetic fields are

\[ \mathcal{L}_E^H = \frac{2}{N^2} g^{ij} \left(F_{0i} - F_{ki}N^k\right) \left(F_{0j} - F_{lj}N^l\right) - F_{ij}F^{ij} - \beta_0 \Psi - \beta_1 A_i B^i + \beta_2 B_i B^i - G_E, \]
\[ \mathcal{L}_S^H = \frac{1}{2N^2} \left|\partial_\mu \Phi - N^\mu \partial_\mu \Psi\right|^2 + \frac{1}{2} |\partial \Psi|^2 - H_S, \] (2.7)

where $F_{ij} = \partial_i A_j - \partial_j A_i$. The Hořava-Lifshitz higher order corrections $G_E$ and $H_S$ are given by

\[ G_E = \beta_3 \left(B_i B^i\right)^2 + \beta_4 \left(B_i B^i\right)^3 + \beta_5 \left(\nabla_i B_j\right) \left(\nabla^i B^j\right) + \beta_6 \left(B_i B^i\right) \left(\nabla_k B_j\right) \left(\nabla^k B^j\right) + \beta_7 \left(\nabla_i B_j\right) \left(\nabla^i B^j\right) \left(\nabla^i B^j\right) + \beta_8 \left(\nabla_i \nabla_j B_k\right) \left(\nabla^i \nabla^j B^k\right), \]
\[ H_S = \alpha_3 \left(\Psi \Delta \Psi\right)^2 + \alpha_4 \left(\Psi \Delta \Psi\right)^3 + \alpha_5 \Psi^2 \Delta \Psi + \alpha_6 \left(\Psi \Delta \Psi\right) \left(\Psi \Delta \Psi\right) + \alpha_7 \Psi^2 \Delta \Psi, \] (2.8)

where $\alpha_i$ are arbitrary functions of $\Psi$ and $\beta_i$ arbitrary functions of $A_i A^i$. Nonetheless, we consider $\alpha_i$ and $\beta_i$ as constants in this paper, what can be seen as a weak field approximation. Moreover, $B^i = \frac{1}{\sqrt{g}} \epsilon^{ijk} F_{jk}$ with the Levi-Civita symbol $\epsilon^{ijk}$. What we want to consider is just the lower order terms of above equations: the higher order terms $G_E$ and $H_S$ are ignored in this paper.

Now, let’s construct the coupling between electromagnetic field and scalar field. The simplest transformations are given by

\[ p_i \to p_i - q A_i, \]
\[ p_0 \to p_0 - q A_0, \] (2.9)

where $A_i$ and $A_0$ satisfy the gauge invariant

\[ A_i \to A_i + \nabla_i \chi, \]
\[ A_0 \to A_0 - \partial_0 \chi. \] (2.10)

Therefore, we make the replacement

\[ \nabla_i \to \nabla_i - i q A_i, \]
\[ \partial_0 \to \partial_0 - i q A_0, \] (2.11)

and the complex scalar field is rewritten as

\[ \hat{\mathcal{L}}_H^S = \frac{1}{2N^2} \left|\partial_\mu \Psi - i q A_0 \Psi - N^i \left(\partial_i \Psi - i q A_i \Psi\right)\right|^2 - \left(\frac{1}{2} - \alpha_2\right) \left|\partial \Psi - i q A_1 \Psi\right|^2 - \frac{1}{2} V(|\Psi|) - \mathcal{H}_S, \] (2.12)

where $\mathcal{H}_S$ is replaced by $\mathcal{H}_S$ with $\partial_0 \to \partial_0 - i q A_1$.

Therefore, we build a holographic superconductor model in Hořava-Lifshitz gravity,

\[ S_H = \int dt d^3 x \sqrt{g} \left(\frac{1}{4} \mathcal{L}_H^E + 2 \hat{\mathcal{L}}_H^S \right). \] (2.13)

Considering the relationship [8]

\[ g^{(4)00} = - \frac{1}{N^2}, \quad g^{(4)ij} = \frac{N^i N^j}{N^2}, \]
\[ g^{(4)ij} = \epsilon^{ij} - \frac{N^i N^j}{N^2}, \] (2.14)

eq. (2.13) reduces into Eq. (2.1) when $\alpha_i = \beta_i = 0$.

### III. NUMERICAL RESULTS

In this paper, we aim at the holographic superconductor in the low energy limit of Hořava-Lifshitz gravity, while Schwarzschild spacetime is one of the solutions of the low energy limit in Hořava-Lifshitz gravity [9]. Thus, let us consider the Schwarzschild spacetime

\[ ds^2 = -N^2(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (dx^2 + dy^2), \] (3.1)

with

\[ N^2(r) = f(r) = r^2 - \frac{r_0^3}{r}, \] (3.2)

where we have chosen the AdS radius $L = 1$. In the low energy case, we consider the temperature of black hole as given by $T_h = \frac{4\pi}{4\pi}$. We also set the simplest form for the mass parameter $m^2 = -2 + 4\alpha_2$. Therefore, we derive the field equations in this Hořava-Lifshitz spacetime,

\[ \Psi'' + \left(\frac{2}{r} + \frac{f'}{f}\right) \Psi' + \left[\frac{2}{f} + \frac{\Phi^2}{(1 - 2\alpha_2 f^2)}\right] \Psi = 0, \]
\[ \Phi'' + \frac{2}{r} \Phi' - \frac{2\Psi^2}{f} \Phi = 0. \] (3.3)

At infinity, because $f(r) \to r^2$, we can get the boundary condition for $\Psi$ and $\Phi$

\[ \Psi = \frac{\sqrt{2} \langle O_1 \rangle}{r} + \frac{\sqrt{2} \langle O_2 \rangle}{r^2} + \cdots, \]
\[ \Phi = \mu - \frac{\rho}{r} + \cdots. \] (3.4)
Substituting the boundary condition into the main equations of the holographic superconductor, we can use the shooting method to calculate Eq. (3.3) numerically. Note that the correction from Hořava-Lifshitz gravity in Eq. (3.3) is $\alpha_2$, so we focus on studying the effect of $\alpha_2$.

We set $r_0 = 1$ in the definition of $f(r)$ and the charge of test particle as unit, $q = 1$. The critical temperature $T_c$ in $\mathcal{O}_1$ and $\mathcal{O}_2$ is given in Table I.

| $\alpha_2$ | $\langle \mathcal{O}_1 \rangle$ | $\langle \mathcal{O}_2 \rangle$ |
|------------|-------------------------------|-------------------------------|
| 0          | 0.2255$p^{1/2}$               | 0.1184$p^{1/2}$               |
| 0.1        | 0.2385$p^{1/2}$               | 0.1252$p^{1/2}$               |
| 0.2        | 0.2563$p^{1/2}$               | 0.1346$p^{1/2}$               |
| 0.3        | 0.2836$p^{1/2}$               | 0.1489$p^{1/2}$               |
| 0.4        | 0.3373$p^{1/2}$               | 0.1771$p^{1/2}$               |

The results show that the effect of $\alpha_2$ is to increase the critical temperature, while the effect of $\langle \mathcal{O}_2 \rangle$ is more obvious than the effect of $\langle \mathcal{O}_1 \rangle$.

Then, we draw the transition curve in FIG. and we find that, as the $\alpha_2$ increases, the curved line gets higher.

### IV. CONDUCTIVITY

Here, we discuss the conductivity. Considering the perturbed Maxwell field $A_i = \delta_i^x e^{-i\omega t}A_x(r)$ while $A_0 = 0$, we obtain the equation

$$A''_x(r) + \frac{8r^3}{2r^3} + f'(r)A'_x(r) + \frac{\omega^2}{(2r^3 + f(r)) f(r)} A_x(r) = 0 \quad ,$$

but what we are interested in is the case $\beta_2 = 0$, in which case the above equation is rewritten as

$$A''_x + \frac{f'(r)}{f(r)} A'_x + \left[ \frac{\omega^2}{f(r)^2} - \frac{2(1 - 2\alpha_2)}{f(r)} \right] A_x = 0 \quad (4.2)$$

We consider the low energy scale case, when the boundary condition at $r = r_0$ requires

$$A_x(r) \sim f(r) - \frac{\tilde{\Phi}}{\sigma_0} \quad ,$$

while the behavior of $A_x$ in the asymptotic AdS region is given by

$$A_x(r) = A_x^0 + \frac{A_x^{1}}{r} + \cdots \quad ,$$

and the definition of conductivity is

$$\sigma = -\frac{i A^{(1)}_x}{\omega A^{(0)}_x} \quad .$$

Using the above formulas, we draw the relation between $\text{Re}(\sigma)$ and $\omega$. We find that the real part of conductivity curves are lower as $\alpha_2$ increases. On the other hand, from the relations between $-\text{Im}(\sigma)$ and $\omega$, we find that the imaginary part of the conductivity lines are higher as $\alpha_2$ increases.

Finally, we plot $\Theta(0)$ with small $\omega$ (where $\Theta \equiv -\text{Im}(\sigma)$), and FIG. shows that the $\Theta(0)$ goes to a constant as $\omega$ is enough small, but the curves are lower as $\alpha_2$ increases. The values of $\Theta(0)$ are given in Table II.

| $\alpha_2$ | $\Theta_1(0)$ | $\Theta_2(0)$ |
|------------|--------------|--------------|
| 0          | 11.6798      | 6.6225       |
| 0.1        | 10.4294      | 6.1340       |
| 0.2        | 9.0088       | 5.5426       |
| 0.3        | 7.3211       | 4.7786       |
| 0.4        | 5.1131       | 3.6468       |

### V. CONCLUSION

We built the simplest holographic superconductor model in the low energy limit of Hořava-Lifshitz gravity and studied the property of transition near the critical temperature $T_c$ in static Hořava-Lifshitz spacetime. We found that the correction comes from the $\alpha_2 \Psi \Delta \Psi$ term in 3+1 dimensional static spacetime. The study also shows that the correction of $\langle \mathcal{O}_2 \rangle$ is more obvious than the effect on $\langle \mathcal{O}_1 \rangle$. From Eq. (3.3), if we make the transformation $\Phi \rightarrow \tilde{\Phi}$, we find the equations of superconductors will give the same results for any constant $\alpha_2$, but $\alpha_2$ can modify the conductivity in Section IV.

What we considered is just the simplest case, but it is possible that the correction comes from $\mathcal{H}_S$ and $\mathcal{G}_E$ as the black hole is charged or rotated. On the other hand, our work proves that it should introduce a $U(1)$ symmetrical field to avoid the difficulties from spin-0 graviton in Hořava-Lifshitz theory [3, 4], so it is more meaningful to research the holographic superconductor in general case with a $U(1)$ symmetric field.

The effect of the constant $\alpha_2$ is to decrease the value of conductivity and lower the superconductor effect (see table II). However, we did not see and insulator effect, a case in which we have to consider large values of $\alpha_2$, which then might be field dependent.

**Acknowledgements**

This work is supported in part by FAPESP No. 2012/08934-0 (EA, KL); CNPq (EA, KL); DOE Grant, DE-FG02-10ER41692 (AW); Ciência Sem Fronteiras, No.
FIG. 1: The condensate as a function of the temperature for the operators $O_1$ and $O_2$.

FIG. 2: The real part of conductivity for the two operators $O_1$ and $O_2$.

004/2013 - DRI/CAPES (AW); and NSFC No. 11375153 (AW).

[1] P. Hořava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775]. A. M. da Silva, Classical Quantum Gravity 28, 055011 (2011).
[2] P. Hořava and C. M. Melby-Thompson, Phys. Rev. D 82, 064027 (2010).
[3] A. Wang, Phys. Rev. D 82, 124063 (2010) [arXiv:1008.3637]. A. Borzou, K. Lin, and A. Wang, J. Cosmol. Astropart. Phys., 05, 006 (2011) [arXiv:1103.4390]. S. Mukohyama, J. Cosmol. Astropart. Phys., 06, 001 (2009) [arXiv:0904.2190]. T. Zhu, Q. Wu, A. Wang, and F.-W. Shu, Phys. Rev. D 84, 101502 (R) (2011) [arXiv:1108.1237]. T. Zhu, F.-W. Shu, Q. Wu, and A. Wang, Phys. Rev. D 85, 044053 (2012) [arXiv:1110.5106]. A. Wang and Y. Wu, Phys. Rev. D 83, 044031 (2011) arXiv:1009.2089; K. Lin, A. Wang, Q. Wu, and T. Zhu, Phys. Rev. D 84, 044051 (2011) [arXiv:1106.1480]. Elio Abdalla, Alan M. da Silva Phys.Lett. B707 (2012) 311-314.
[4] K. Lin, S. Mukohyama and A. Wang, Phys. Rev. D 86, 104024 (2012) [arXiv:1206.1338]. K. Lin and A. Wang, Phys. Rev. D87, 084041 (2013) [arXiv:1212.6794]. K. Lin, S. Mukohyama, A. Wang and T. Zhu, Phys. Rev. D (accept) [arXiv:1310.6666].
[5] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, Phys.Rev.Lett. 101, 031601 (2008); J. High Energy Phys. 12, 015 (2008).
[6] C.P. Herzog, J. Phys. A 42, 343001 (2009); S.A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009); G.T. Horowitz, Lect. Notes Phys. 828 313, 2011; [arXiv:1002.1722 [hep-th]]; G.T. Horowitz and R. C. Myers, Phys. Rev. D 59, 026005 (1999); T. Nishioka, S. Ryu, and T. Takayanagi, J. High Energy Phys. 03, 131 (2010); G.T. Horowitz and B. Way, J. High Energy Phys. 11, 011 (2010); Y. Peng, Q.Y. Pan, and B. Wang, Phys. Lett. B 699, 383 (2011); R.G. Cai, S. He, L. Li, and Y.L. Zhang, J. High Energy Phys. 07, 088 (2012); Y. Peng, X.M. Kuang, Y.Q. Liu, and B. Wang, arXiv:1204.2855 [hep-th]; R.G. Cai, H.F. Li, and H.Q. Zhang, Phys. Rev. D 83, 126007 (2011); [arXiv:1103.5568 [hep-th]]; Q.Y. Pan, J.L. Jing, and B. Wang, Phys. Rev. D 84, 126020 (2011); Chong Oh Lee, Eur. Phys. J. C 72, 2092 (2012); J.P. Wu, Y.Cao, X.M.Kuang and W.J. Li, Phys.Lett. B 697, 153 (2011); G. Siopsis and J. Therrien, J. High Energy Phys. 05, 013 (2010) [arXiv:1003.4275]. Q.Y. Pan, J.L. Jing, B. Wang and S.B. Chen, J. High Energy Phys. 06, 087 (2012) [arXiv:1205.3545]. E. Abdalla, O. P. F. Piedra, F. S. Nuez, Jeferson de Oliveira, Phys.Rev. D88 (2013) 6, 064035; E. Abdalla, Jeferson de Oliveira, A. Lima-Santos, A.B. Pavan Phys.Lett. B709 (2012) 276-279.
FIG. 3: The imaginary part of conductivity for the two operators $O_1$ and $O_2$.

FIG. 4: $\frac{\Theta(\omega)}{\Theta(0)}$ for the two operators $O_1$ and $O_2$ in rotating spacetime, where $\Theta(\omega) \equiv -\omega\text{Im}\sigma$

[7] S. Franco, A. García-García and D. Rodríguez-Gómez, J. High Energy Phys. 04, 092 (2010) arXiv:0906.1214.
[8] E. Kiritsis and G. Kofinas, Nucl. Phys. B821 467 (2009); I. Kimpton and A. Padilla, J. High Energy Phys. 04, 133 (2013) arXiv:1301.6950
[9] R. Cai, L. Cao and N. Ohta, Phys. Rev. D 80, 024003 (2009); R. Cai and H. Zhang, Phys. Rev. D 81, 066003 (2010); H. Lu, J. Mei, and C. N. Pope, Phys. Rev. Lett. 103, 091301 (2009) arXiv:0904.1595