STUDY OF A NONLINEAR BOUNDARY-VALUE PROBLEM OF GEOPHYSICAL RELEVANCE

KATERYNA MARYNETS∗
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz 1
1090 Wien, Austria

Abstract. We present some results on the existence and uniqueness of solutions of a two-point nonlinear boundary value problem that arises in the modeling of the flow of the Antarctic Circumpolar Current.

1. Introduction. The two-point boundary-value problem
\[
\begin{cases}
  u''(t) = a(t) F(u(t)) - b(t), & t \in (0, 1), \\
  u(0) = u(1) = 0,
\end{cases}
\]
was recently derived as a model for the azimuthal horizontal jet flow components of the Antarctic Circumpolar Current (see the discussion in Section 2). The existence of nontrivial solutions is of considerable interest, since these correspond to azimuthal flows that feature variations in the meridional direction, being thus models that capture the essential geophysical features, confirmed by field data (some of these models were also studied in [12, 17]). In this context, the function $F$ represents the vorticity and it is desirable to identify large classes of vorticity distributions for which (1) has a unique (classical) solution. General studies of this type of two-point boundary-value problems have a long history, both with respect to special explicit solutions as well as with respect to abstract results of far-reaching generality. Our study is motivated by the need to go beyond the explicit solutions that were recently derived for the linear version of (1) in [13, 15] (see also the recent survey paper [10]) and to tackle cases of nonlinear functions $F$ for which one can not expect to provide explicit solutions. Among the most promising approaches that were devised are those that guarantee existence if the general initial-value problem does not present the finite-time blow-up phenomenon and if a somewhat more general associated boundary-value problem has at most one solution (uniqueness implies existence).

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∗ Corresponding author: Kateryna Marynets.
A quite useful result in this direction ensures the existence and uniqueness of a solution to (1) provided that for every $\varepsilon \in (0, 1)$ we have:

(H1) all solutions of the initial-value problem

\[
\begin{align*}
  u''(t) &= a(t) F(u(t)) - b(t), \quad t \in (0, 1), \\
  u(0) &= u_0, \\
  u'(0) &= u_1,
\end{align*}
\]

exist on $[0, 1 + \varepsilon)$ for all $u_0, u_1 \in \mathbb{R}$;

(H2) there do not exist two solutions on $[0, t^*]$ to the two-point boundary-value problem

\[
\begin{align*}
  u''(t) &= a(t) F(u(t)) - b(t), \quad t \in (0, 1), \\
  u(0) &= 0, \quad u(t^*) = u^*,
\end{align*}
\]

for any $t^* \in (1 - \varepsilon, 1 + \varepsilon)$ and $u^* \in \mathbb{R}$;

(see [18]). In particular, global existence for (3) and uniqueness for (4) ensure the solvability of (1). While this is by no means the only possible approach that can accommodate nonlinear functions $F$, it has some advantages over more classical methods (like the method of sub- and super-solutions, which are often very difficult to find for second-order nonlinear differential equations) in that the type of hypotheses that need to be verified seem to accommodate large classes of functions. For this reason, our aim will be to derive general conditions on $F$ that guarantee (H1) and (H2). We illustrate the usefulness of our results by means of some examples.

In Section 2 we provide an overview of the physical context in which (1) arises, Section 3 is devoted to finding general classes of functions $F$ for which all solutions to (3) exist for all times $t \geq 0$, while in Section 4 we provide conditions on the function $F$ that ensure uniqueness for (4). Section 5 combines the considerations made in Section 3 and Section 4, yielding thus the existence (and, due to (H2), also the uniqueness) of a solution to (1). Some examples of the practical applicability of our abstract considerations conclude Section 3 and the paper.

2. The modeling of the Antarctic Circumpolar Current. The Antarctic Circumpolar Current (ACC) dominates the Southern Ocean flow over a very large area (measured in thousands of km length and hundreds of km width), flowing from west to east around Antarctica. It is the largest ocean current and the only current that flows completely around the globe, unimpeded by continents due to the lack of any landmass connecting with the continent of Antarctica. The ACC flow connects the Atlantic, Pacific, and Indian oceans and, unlike other major ocean currents, it reaches from the surface to the bottom of the ocean (of average depth 4 km). However, the ACC has negligible vertical speeds, with the ratio of vertical speed to either of the horizontal speed components (with azimuthal speeds that can exceed 1 m/s) typically about $10^{-4}$, so that we may realistically model it as a shallow water flow on a rotating sphere. With this purpose, let us introduce spherical coordinates:

- $\theta \in [0, \pi]$ is the polar angle (with $\theta = 0$ corresponding to the North Pole and with $\theta = \pi/2$ along the Equator);
- $\varphi \in [0, 2\pi)$ is the angle of longitude (or azimuthal angle);

see Figure 1. We recall that the Earth is rotating eastwards around the polar axis with practically constant angular speed $\Omega' \approx 7.29 \times 10^{-5}$ radians per second, the radius of the spherical model of Earth being about 6378 km.
Let $(u', v', w')$ be the velocity field in physical variables. If $(e_r, e_\theta, e_\phi)$ are the unit vectors associated with a fixed point $P$ on the rotating sphere, where $e_r$ points upwards, $e_\phi$ points from west to east, and $e_\theta$ from north to south (see Figure 2), then the governing equations for inviscid flow are (see [4, 5] for a physical justification of
neglecting viscous effects for large-scale ocean flows) the Euler equation
\[
\left( \frac{\partial}{\partial t} + u' \frac{\partial}{\partial r'} + v' \frac{\partial}{\partial \theta'} + w' \frac{\partial}{\partial \varphi'} \right) (u', v', w') + \frac{1}{r'} (-v'^2 - w'^2, u'v' - w'^2 \cot \theta, u'w' + v'w' \cot \theta) + 2\Omega' (-u' \sin \theta, -u' \cos \theta, u' \sin \theta + v' \cos \theta) - r'\Omega^2 (\sin^2 \theta, \sin \theta \cos \theta, 0) = -\frac{1}{\rho'} \left( \frac{\partial p'}{\partial r'} \frac{1}{r' \sin \theta} + \frac{1}{r' \sin \theta} \frac{\partial p'}{\partial \varphi'} \right) + (-g', 0, 0),
\]

and the equation of mass conservation
\[
\frac{1}{r'^2} \frac{\partial}{\partial r'} (r'^2 u') + \frac{1}{r' \sin \theta} \frac{\partial}{\partial \theta'} (v' \sin \theta) + \frac{1}{r' \sin \theta} \frac{\partial w'}{\partial \varphi'} = 0,
\]

where \( p'(r', \theta, \varphi) \) is the pressure in the fluid, \( \rho' \) is the (constant) density and \( g' \approx 9.81 \text{ m/s}^2 \) is the (constant) gravitational acceleration of the Earth.

A suitable length scale for the ACC is the average depth of the Southern Ocean, \( H' = 4 \text{ km} \), and a suitable speed scale \( c' \) for ocean flows which are dominated by Coriolis effects (as it is the case for the ACC, see the discussion in [3]) is \( c' = 0.1 \text{ m/s} \). We non-dimensionalize the original physical variables by setting
\[
e' = H' z, \quad (u', v', w') = c'(ku, v, w), \quad p' = \rho' c' \frac{H'}{R'} g',
\]

where \( k \) is the scaling factor, associated with the vertical component of the velocity (typically, \( k \approx 10^{-4} \)).

Introducing the shallow-water parameter
\[
\varepsilon = \frac{H'}{R'},
\]

where \( R' \approx 6378 \text{ km} \) is the radius of the Earth, the governing equations (5)-(6) for a steady flow become
\[
\left( \frac{k}{\varepsilon} \frac{\partial}{\partial z} + \frac{v}{1 + \varepsilon \partial \theta} \frac{w}{(1 + \varepsilon z) \sin \theta} \frac{\partial}{\partial \varphi} \right) (ku, v, w) + \frac{1}{1 + \varepsilon z} (-v'^2 - w'^2, kw - w'^2 \cot \theta, ku + vw \cot \theta) + 2\Omega' \frac{R'}{c'} (-w \sin \theta, -w \cos \theta, ku \sin \theta \cos \theta, 0) - (1 + \varepsilon z) \left( \frac{\Omega' R'}{c'} \right)^2 (\sin^2 \theta, \sin \theta \cos \theta, 0) = -\left( \frac{1}{\varepsilon} \frac{\partial p}{\partial z} + \frac{1}{1 + \varepsilon \partial \theta} \frac{1}{z} \frac{\partial p}{\partial \varphi} \right) + \frac{R'}{c'^2} (-g', 0, 0),
\]

and
\[
\frac{k}{\varepsilon (1 + \varepsilon z)^2} \frac{\partial}{\partial z} [(1 + \varepsilon z)^2 u] + \frac{1}{(1 + \varepsilon z) \sin \theta} \left\{ \frac{\partial}{\partial \theta} (v \sin \theta) + \frac{\partial w}{\partial \varphi} \right\} = 0,
\]

respectively. In the context of large-scale ocean flows like the ACC, the scaling factor \( k \) is taken equal to \( \varepsilon^2 \) (see the discussion in [4]).

Defining
\[
P = p + \frac{H' g'}{c'^2} z,
\]
we obtain the following non-dimensional form of the governing equations:

\[
\frac{\varepsilon u}{1 + \varepsilon z} \frac{\partial}{\partial z} + \frac{v}{1 + \varepsilon z} \frac{\partial}{\partial \theta} + \frac{w}{(1 + \varepsilon z) \sin \theta} \frac{\partial}{\partial \varphi} \right) (\varepsilon^3 u, v, w)
\]

\[
+ \frac{1}{1 + \varepsilon z} (-\varepsilon u^2 - \varepsilon w^2 - \varepsilon^2 u v - w^2 \cot \theta + \varepsilon^2 u w + v w \cot \theta)
\]

\[
+ 2 \omega(-\varepsilon w \sin \theta, -w \cos \theta, \varepsilon^2 u \sin \theta + v \cos \theta)
\]

\[
- (1 + \varepsilon z) \omega^2 (\varepsilon \sin^2 \theta, \sin \theta \cos \theta, 0)
\]

\[
= - \left( \frac{\partial P}{\partial z}, \frac{1}{1 + \varepsilon z} \frac{\partial P}{\partial \theta}, \frac{1}{(1 + \varepsilon z) \sin \theta} \frac{\partial P}{\partial \varphi} \right),
\]

and

\[
\frac{\varepsilon}{(1 + \varepsilon z)^2} \frac{\partial}{\partial z} \left\{ (1 + \varepsilon z)^2 u \right\} + \frac{1}{(1 + \varepsilon z) \sin \theta} \left\{ \frac{\partial}{\partial \theta} (v \sin \theta) + \frac{\partial w}{\partial \varphi} \right\} = 0,
\]

where

\[
\omega = \frac{\Omega R'}{c'}
\]

is the non-dimensional Coriolis parameter.

The leading-order problem in \( \varepsilon \) is obtained in the limit \( \varepsilon \rightarrow 0 \), leading us to the shallow-water model

\[
\frac{\partial \Pi}{\partial z} = 0,
\]

\[
\left( \frac{v}{\sin \theta} + \frac{w}{\sin \theta} \frac{\partial}{\partial \varphi} \right) v - w^2 \cot \theta - 2 \omega w \cos \theta = - \frac{\partial \Pi}{\partial \theta},
\]

\[
\left( \frac{v}{\sin \theta} + \frac{w}{\sin \theta} \frac{\partial}{\partial \varphi} \right) w + v w \cot \theta + 2 \omega v \cos \theta = - \frac{1}{\sin \theta} \frac{\partial \Pi}{\partial \varphi},
\]

\[
\frac{\partial}{\partial \theta} (v \sin \theta) + \frac{\partial w}{\partial \varphi} = 0,
\]

where we did set

\[
\Pi = P + \frac{1}{4} \omega^2 \cos 2 \theta.
\]

Using (14) one can introduce the stream function \( \psi \) in spherical coordinates (see the discussion in [5] for the relevance of this concept for the particle paths of steady flows on the surface of the sphere) by requiring that

\[
\begin{cases}
  v = \frac{1}{\sin \theta} \psi_\varphi, \\
  w = -\psi_\theta.
\end{cases}
\]

The compatibility condition generated by the elimination of \( \Pi \) from (11)–(13) yields to the vorticity equation

\[
\psi_\varphi \left( \frac{1}{\sin^2 \theta} \psi_\varphi \varphi + \psi_\varphi \cot \theta + \psi_\theta \theta - 2 \omega \cos \theta \right)_\theta
\]

\[
- \psi_\theta \left( \frac{1}{\sin^2 \theta} \psi_\varphi \varphi + \psi_\varphi \cot \theta + \psi_\theta \theta - 2 \omega \cos \theta \right)_\varphi = 0,
\]

in which the vorticity in the flow, at leading order, is given in spherical coordinates by the expression

\[
\frac{1}{\sin^2 \theta} \psi_\varphi \varphi + \psi_\varphi \cot \theta + \psi_\theta \theta.
\]
In terms of the vorticity of the underlying motion of the ocean (relative to the Earth’s surface and not driven by the rotation of the Earth), given by 

\[ \Psi(\theta, \varphi) = \omega \cos \theta + \psi(\theta, \varphi), \]

equation (16) can be written equivalently as 

\[
(\Psi - \omega \cos \theta) \varphi \left( \frac{1}{\sin^2 \theta} \psi_{\varphi \varphi} + \psi_{\varphi} \cot \theta + \psi_{\theta \theta} \right)_\theta - (\Psi - \omega \cos \theta) \theta \left( \frac{1}{\sin^2 \theta} \psi_{\varphi \varphi} + \psi_{\varphi} \cot \theta + \psi_{\theta \theta} \right)_\varphi = 0. \quad (17)
\]

If \( \nabla (\Psi - \omega \cos \theta) \neq 0 \) throughout the relevant fluid-flow region, then the rank theorem ensures that (see the discussion in [5]) we can express (17) in the form

\[
\frac{1}{\sin^2 \theta} \psi_{\varphi \varphi} + \Psi_\theta \cot \theta + \Psi_{\theta \theta} = F(\Psi - \omega \cos \theta), \quad (18)
\]

where \( F(\Psi - \omega \cos \theta) \) is the oceanic vorticity, typically one order of magnitude larger than the planetary vorticity \( 2\omega \cos \theta \) due to the Earth’s rotation (see the data in [4]). The (total) vorticity of the flow is the sum of the oceanic vorticity, \( F(\Psi - \omega \cos \theta) \), and the planetary vorticity \( 2\omega \cos \theta \). The planetary vorticity is prescribed but the oceanic vorticity can change from location to location, being dependent on specific features (for example, the prevailing wind pattern, which induces near-surface currents) of the type of ocean flow that is under consideration. Irrotational flows (with zero oceanic vorticity) are rare occurrences in the real-world oceans, while flows with constant non-zero vorticity model tidal currents as well as wind-generated currents (see the discussions) in [2, 7, 9]. Furthermore, non-constant oceanic vorticities are often encountered. Let us also point out that while wave-current interactions are very important in the Southern Ocean (see [6] for recent theoretical considerations and [19] for field data), at the large scales that are relevant for our study, the waves play no role.

There are considerable complications associated with the use of spherical coordinates and, to avoid them, we rely on the stereographic projection of the unit sphere centred at origin from the North Pole to the equatorial plane (see Figure 3). The model (18) in spherical coordinates is thus transformed into an equivalent planar elliptic partial differential equation (see the discussion in [11]): in our chosen coordinates the stereographic projection is defined by 

\[
\xi = r e^{i \phi} \quad \text{with} \quad r = \cot \left( \frac{\theta}{2} \right) = \frac{\sin \theta}{1 - \cos \theta}, \quad (19)
\]

where \( (r, \phi) \) are the polar coordinates in the equatorial plane, and (18) becomes

\[
\psi_{\xi \xi} + 2\omega \frac{1 - \xi \xi}{(1 + \xi \xi)^3} - \frac{F(\psi)}{(1 + \xi \xi)^2} = 0.
\]

The above equation is equivalent, using the Cartesian coordinates \((x, y)\) in the complex \(\xi\)-plane, to the following semilinear elliptic partial differential equation

\[
\Delta \psi + 8\omega \frac{1 - (x^2 + y^2)}{(1 + x^2 + y^2)^3} - \frac{4F(\psi)}{(1 + x^2 + y^2)^2} = 0, \quad (20)
\]

where \( \Delta = \partial_x^2 + \partial_y^2 \) denotes the Laplace operator; see [4, 5].

Since the ACC presents a considerable uniformity in the azimuthal direction (see the discussion in [3, 19]), we can take advantage of this observed feature to simplify
Figure 3. Depiction of the stereographic projection $P \mapsto P'$ from the North Pole to the equatorial plane, illustrated for a location that corresponds to the region where the Antarctic Circumpolar Current is encountered.

the problem (20) further. Indeed, solutions with no variation in the azimuthal direction correspond to radially symmetric solutions $\psi = \psi(r)$ of the problem (20). The change of variables

$$
\psi(r) = U(s), \quad s_1 < s < s_2, \quad (21)
$$

with

$$
r = e^{-s/2} \quad \text{for} \quad 0 < s_1 = -2 \ln(r_+) < s_2 = -2 \ln(r_-), \quad (22)
$$

for $0 < r_- < r_+ < 1$, can now be used to transform the partial differential equation (20) to the second-order ordinary differential equation

$$
U''(s) - \frac{e^s}{(1+e^s)^2} F(U(s)) + \frac{2\omega e^s(1-e^s)}{(1+e^s)^3} = 0, \quad s_1 < s < s_2. \quad (23)
$$

Here the choice of $r_\pm \in (0,1)$ with $r_+/r_- \in (1,2)$ is guided by the flow in a jet component of the ACC, between parallels of latitude, and the appropriate boundary conditions for (23) are

$$
U(s_1) = U(s_2) = 0. \quad (24)
$$

These capture the feature that the boundary of the jet is a streamline: the flow being steady ensures that a particle there will be confined to the boundary at all times. We therefore derived (23)-(24) as a model for a jet component of the ACC. In this context the choice of the oceanic vorticity $F$, left unspecified, will drive different properties of the solution $U$ and will determine the entire flow pattern.

Note that, for $0 < s_1 < s_2$, the change of variables

$$
u(t) = U(s) \quad \text{with} \quad t = \frac{s - s_1}{s_2 - s_1}, \quad (25)$$
transforms the second-order differential equation (23) with the boundary conditions (24) to the equivalent two-point boundary-value problem
\[ u'' = a(t)F(u) - b(t), \quad 0 < t < 1, \quad (26) \]
\[ u(0) = u(1) = 0, \quad (27) \]
where
\[
\begin{align*}
  a(t) &= \frac{(s_2 - s_1)^2 e^{(s_2-s_1)t+s_1}}{(1 + e^{(s_2-s_1)t+s_1})^2} > 0, \\
b(t) &= \frac{2\omega(s_2 - s_1)^2 e^{(s_2-s_1)t+s_1}(1 - e^{(s_2-s_1)t+s_1})}{(1 + e^{(s_2-s_1)t+s_1})^3},
\end{align*}
\]
\[ t \in [0,1]. \]

The problem (26)-(27) is of type (1)-(2), with the functions \(a\) and \(b\) specified explicitly. We are interested in allowing large classes of functions for the nonlinearity \(F\). Note that explicit solutions for \(F\) constant and for \(F(u) = -2u\) were provided in [13] and some existence results for a special class of nonlinear functions \(F\) was proved in [14] (see also the recent survey [10]).

3. Global existence of solutions. The following result, inspired by the considerations made in [16], provides sufficient conditions for the global existence of the solutions to the initial-value problem (3).

**Theorem 3.1.** If the continuous function \(F : \mathbb{R} \to \mathbb{R}\) satisfies
\[
M + \int_0^u F(\xi)d\xi \geq W^{-1}(F^2(u)), \quad u \in \mathbb{R}, \quad (28)
\]
for some constant \(M > 0\) and some strictly increasing function \(W : [0, \infty) \to [0, \infty)\) with \(W(0) = 0, W(s) > 0\) for \(s > 0\) and satisfying
\[
\int_1^\infty \frac{du}{W(u)} = \infty, \quad (29)
\]
and if
\[
\lim_{|u| \to \infty} \int_0^u F(\xi)d\xi = \infty, \quad (30)
\]
then all solutions of (3) are global in time.

**Proof.** It is easy to see that the differential equation in (3) is equivalent to the first-order system
\[
\begin{align*}
x'(t) &= y(t), \\
y'(t) &= a(t)F(x(t)) - b(t).
\end{align*}
\]
Let us introduce the positive-definite function \(V : [0, \infty) \times \mathbb{R}^2 \to [0, \infty)\) by setting
\[
V(x, y) := M + \int_0^x F(\xi)d\xi + \frac{y^2}{2}, \quad x, y \in \mathbb{R}. \quad (32)
\]
Note that (30) ensures that \(V(x, y) \to \infty\) as \(|x| + |y| \to \infty\),
\[
M + \int_0^x F(\xi)d\xi \geq 0, \quad u \in \mathbb{R}.
\]
Assuming that a solution of (3) blows-up in finite time \( T > 0 \), set
\[
\alpha = 1 + \sup_{t \in [0, T]} \{a(t)\} + \sup_{t \in [0, T]} \{b(t)\}.
\]
Calculating the derivative of the functional (32) along this solution of the differential system (31), we get
\[
\frac{d}{dt}V(t, x(t), y(t)) = y(t) \{F(x(t))[1 + a(t)] - b(t)\} \\
\leq \alpha \|y(t)\| + \alpha |y(t)|F(x(t))| \\
\leq \alpha \left(1 + \frac{y^2}{2} + F^2(x(t))\right) \\
\leq \alpha \left(1 + V(t, x(t), y(t)) + W(V(t, x(t), y(t)))\right),
\]
in view of (28). The comparison method for scalar ordinary differential equations (see [8]) ensures that
\[
0 \leq V(t, x(t), y(t)) \leq m(t), \quad t \in [0, T),
\]
where \( m(t) \) is the maximal solution of the differential equation
\[
m'(t) = \alpha \left(1 + m(t) + W(m(t))\right), \quad t > 0,
\]
with initial data \( m(0) = V(x(0), y(0)) \). Note that the hypotheses made on the function \( W \) ensure (see [1]) that
\[
\int_1^\infty \frac{du}{1 + u + W(u)} = \infty,
\]
and this condition guarantees that all solutions of the differential equation (35) are global in time. But then (34) is incompatible with the assumed finite-time blow up, since \( V \) is a positive-definite radially unbounded function. The obtained contradiction proves that all solutions of (3) are global.

**Remark 1.**

(i) Note that the assumption of the boundedness of \( a \) and \( b \) on \([0, \infty)\) is not necessary in the proof of the above result.

(ii) Linear functions \( F(u) = \beta u \) with \( \beta > 0 \) clearly enter into the framework of the above result since (28)-(30) hold for \( W(u) = u/2\beta \). The main interest of the above result resides in the fact that it allows us to choose functions \( F \) with superlinear growth at infinity. For example, by choosing \( W(u) = \beta u \ln(1 + u) \) for \( u \geq 0 \), we can see that for \( \beta > 0 \) sufficiently large, the function
\[
F(u) = u \ln(1 + \|u\|), \quad u \in \mathbb{R},
\]
enters into the framework of validity of the above result.

4. **Uniqueness of solutions.** Since Theorem 3.1 proves the global existence of solutions of the initial-value problem (3), we now study the question of the uniqueness of solutions to the two-point boundary-value problem (4).

**Theorem 4.1.** If \( a(t) > 0 \) on \([0, \infty)\) and if the continuous function \( F : \mathbb{R} \rightarrow \mathbb{R} \) is monotone nondecreasing on \( \mathbb{R} \), then the solution of the (4) is unique.
Proof. Assume that there exist two solutions $u_1(t)$ and $u_2(t)$ with $u_1(t) \neq u_2(t)$ of (4), on some interval $[0, t^*]$ with $t^* \in (1 - \varepsilon, 1 + \varepsilon)$. Then the difference
\[ u(t) = u_1(t) - u_2(t) \] (37) satisfies the differential equation
\[ u''(t) = a(t) \left( F(u_1(t)) - F(u_2(t)) \right), \quad t \in (0, t^*) \] (38)with the boundary conditions
\[ u(0) = u(t^*) = 0. \] (39)

Multiplying (38) by $u(t)$, yields after integration by parts that
\[ 0 \geq - \int_0^1 [u'(t)]^2 \, dt = \int_0^1 a(t) \left( F(u_1(t)) - F(u_2(t)) \right) u(t) \, dt , \]
in view of (39). Since $a(t) > 0$ and the function $F$ is nondecreasing by assumption, we deduce that $u \equiv 0$, which means that the solution of the boundary-value problem (4) is unique. \(\square\)

5. The geophysical boundary-value problem. We can now combine the results of the previous two sections to derive an existence and uniqueness result for the two-point boundary-value problem (1).

**Theorem 5.1.** Assume that the continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$ is monotone nondecreasing and satisfies the conditions (28) and (30), for some nondecreasing continuous function $W : [0, \infty) \rightarrow [0, \infty)$ with $W(0) = 0$, $W(u) > 0$ for $u > 0$, and subject to the constraint (29). Then the problem (1) admits a unique solution.

Other than linear functions, the above main result applies also to nonlinear functions with superlinear growth at infinity. An example is provided in the Remark at the end of Section 3.

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*E-mail address: kateryna.marynets@univie.ac.at*