Abstract
In this paper we prove the Cheeger–Müller theorem for $L^2$-analytic torsion form under the assumption that there exists a fiberwise Morse function and the Novikov–Shubin invariants are positive.

1 Introduction
Let $F$ be a unitary flat vector bundle on a closed Riemannian manifold $M$. In [27], Ray and Singer defined an analytic torsion associated to $(M, F)$ and proved that it does not depend on the Riemannian metric on $M$. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on $M$ (cf. [21]). This conjecture was later proved in the celebrated papers of Cheeger [12] and Müller [22]. Later, Müller [23] generalized this result to the case where $F$ is a unimodular flat vector bundle on $M$. In [7], inspired by the considerations of Quillen [26], at the same time of [23], Bismut and Zhang reformulated the above Cheeger–Müller theorem as an equality between the Reidemeister and Ray–Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundle over $M$. In [7], they make use of the Witten deformation [31] of the de Rham complex.

Let $X \to M \xrightarrow{\pi} S$ be a fiber bundle with connected closed fibers $X_x = \pi^{-1}(x)$ and $F$ be a flat complex vector bundle on $M$ with a flat connection $\nabla^F$ and a Hermitian metric $h^F$. Let $T^HM$ be a horizontal distribution for the fiber bundle and $g^{TX}$ be a vertical Riemannian metric. Then in [6] Bismut and Lott introduced the torsion form $T(T^HM, g^{TX}, h^F) \in \Omega(S)$ (cf. [6, (3.118)]). Assuming that there exists a fiberwise $G$-invariant Morse function with a fibrewise Thom–Smale gradient field, Bismut and Goette [4] proved the Cheeger–Müller type theorem for the equivariant Bismut–Lott torsion form.

The $L^2$-analytic torsion was first introduced by Carey et al. [11,17,20], under the assumptions that the $L^2$-Betti numbers vanish and that certain technical “determinant class condition” is satisfied. The later condition is satisfied if the Novikov–Shubin invariants are positive. In [10], Carey et al. showed that the condition on the vanishing of the $L^2$-Betti numbers can be relaxed. In [8,9] proved the equality between the $L^2$-Reidemeister torsion and $L^2$-Ray-Singer...
torsion for unitary representations, under the “determinant class condition”. In [32], under the framework of [8], Zhang proved the equality between the $L^2$-Reidemeister torsion and $L^2$-Ray-Singer torsion for arbitrary flat vector bundle and without the “determinant class condition”.

In [14], Gong and Rothenberg defined the $L^2$-analytic torsion form and proved that the torsion form is smooth, under the condition that the Novikov–Shubin invariant is at least half of the dimension of the base manifold. In [2], Azzali, Goette and Schick proved that the integrand defining the $L^2$-analytic torsion form, as well as several other invariants related to the signature operator, converges provided the Novikov–Shubin invariants are positive (or of determinant class and $L^2$-acyclic). However, they did not prove the smoothness of the $L^2$-analytic torsion form. To consider transgression formula, they had to use weak derivatives. In [30], So and Su proved that under the condition that the Novikov-Shubin invariants are positive the $L^2$-analytic torsion form is a smooth form. Then it is natural to define the $L^2$-combinatorial torsion form and study the Cheeger-Müller theorem for the $L^2$-analytic torsion form and the $L^2$-combinatorial torsion form. In this paper, we will use the methods in [4,32] to prove the theorem.

The rest of the paper is organized as follows. In Sect. 2, we recall the definition of the $L^2$-analytic torsion form. In Sect. 3, we recall the family of Thom-Smale complexes and the $L^2$ torsion form of it. In Sect. 4, we prove the main theorem of this paper. In Sect. 5, we extend [4, Section 10] to the $L^2$-case and prove Theorem 4.3. In Sect. 6, we prove Theorem 4.2. In Appendixes, we generalize some results of [3,5] to the current case, which are needed in this paper.

2 $L^2$-analytic torsion form

In this section we will recall the definition of the $L^2$-analytic torsion form.

Let $X$ be a smooth oriented closed connected manifold. Let $\tilde{X}$ be a Galois covering of $X$ with covering group $\Gamma$. Let $\tilde{X} \rightarrow \tilde{M} \xrightarrow{\pi} S$ be a smooth fiber bundle. Then $\Gamma$ acts fiberwisely on $\tilde{M}$. We assume that $S$ is compact. Set $M = \tilde{M} / \Gamma$. Then $\pi : M \rightarrow S$ is a smooth fiber bundle with fiber $X$. Let $g^{TX}$ be a vertical Riemannian metric on the fiber bundle. We assume that there is a Riemannian metric $g^{TS}$ on $S$, although all final results will be independent of $g^{TS}$. Let $\nabla^{TS}$ be the Levi-Civita connection on $TS$. Choose a horizontal subbundle $T^H \tilde{M}$, so that

$$\tilde{T} \tilde{M} = T \tilde{X} \oplus T^H \tilde{M}. \quad (2.1)$$

Let $P^{T \tilde{M}} : T \tilde{M} \rightarrow T \tilde{X}$ be the projection. Then we have a Riemannian metric $g^{T \tilde{M}}$ on $\tilde{M}$ defined by

$$g^{T \tilde{M}} = g^{T \tilde{X}} \oplus \tilde{\pi}^* g^{TS}. \quad (2.2)$$

Let $\nabla^{T \tilde{M},L}$ be the Levi-Civita connection on $(T \tilde{M}, g^{T \tilde{M}})$. Let $\nabla^{T \tilde{X}} = P^{T \tilde{X}} \nabla^{T \tilde{M},L}$ be the connection on $T \tilde{X}$. The restriction of $\nabla^{T \tilde{X}}$ to a fiber coincides with the Levi-Civita connection of the fiber. Let $\nabla^{T \tilde{M}} = \tilde{\pi}^* \nabla^{TS} \oplus \nabla^{T \tilde{X}}$ be the connection on $T \tilde{M}$.

Put

$$\tilde{S} = \nabla^{T \tilde{M},L} - \nabla^{T \tilde{M}}.$$
Observe that
\[ T^H \tilde{M} \cong \tilde{\pi}^* TS. \] (2.3)

By (2.1) and (2.3), we have the identification of bundles of algebras
\[ \Lambda^\bullet(T^* \tilde{M}) \cong \tilde{\pi}^* \Lambda^\bullet(T^* S) \otimes \Lambda^\bullet(T^* \tilde{X}). \] (2.4)

Let \( F \to M \) be flat complex vector bundle on \( M \) with flat connection \( \nabla^F \) and Hermitian metric \( g^F \). Let \( \tilde{F} \to \tilde{M} \) be the lifted bundle of \( F \) with lifted flat connection \( \nabla^{\tilde{F}} \) and lifted metric \( g^{\tilde{F}} \). Set \( \omega(\nabla^{\tilde{F}}, g^{\tilde{F}}) = \left( g^{\tilde{F}} \right)^{-1} \nabla^{\tilde{F}} g^{\tilde{F}} \) and
\[ \nabla^{\tilde{F}, u} = \nabla^{\tilde{F}} + \frac{1}{2} \omega \left( \nabla^{\tilde{F}}, g^{\tilde{F}} \right). \]

Let \( \Omega^\bullet(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) be the fiberwise de Rham complex of \( (\tilde{X}, \tilde{F}|_{\tilde{X}}) \) and let \( \Omega^\bullet(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) be the subcomplex with compact support. We equip \( \Omega^\bullet(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) with the Hermitian product such that if \( s, s' \in \Omega^\bullet(\tilde{X}, \tilde{F}|_{\tilde{X}}) \),
\[ \langle s, s' \rangle = \int_{\tilde{X}} \langle s, s' \rangle_{\Lambda^\bullet(T^* \tilde{X})} \otimes \tilde{\pi}^* du_{\tilde{X}}. \] (2.5)

Let \( g^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) be the corresponding \( L^2 \) metric. Let \( (\Omega^\bullet(2))(\tilde{X}, \tilde{F}|_{\tilde{X}}), d\tilde{X} \) be the \( L^2 \)-de Rham complex along the fibers \( \tilde{X} \) with values in \( \tilde{F}|_{\tilde{X}} \), equipped with the fiberwise de Rham operator \( d\tilde{X} \). Let \( d\tilde{X}^* \) be the fiberwise adjoint of \( d\tilde{X} \) with respect to \( g^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \).

Let \( \nabla^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) be the connection on \( \Omega^\bullet(2)(\tilde{X}, \tilde{F}|_{\tilde{X}}) \), such that if \( U \in TS \) and \( s \in \Omega^\bullet(2)(\tilde{X}, \tilde{F}|_{\tilde{X}}) \), then
\[ \nabla^{\Omega^\bullet(2)}(U^\bullet, \tilde{F}|_{\tilde{X}}) s = L_{U^H} s, \] (2.7)
where \( U^H \in C^\infty(\tilde{M}, T^H \tilde{M}) \) such that \( \tilde{\pi}_* U^H = U \) and \( L_{U^H} \) is the Lie differentiation operator.

If \( U, V \) are smooth sections of \( TS \), set
\[ T^H(U, V) = -P_T \tilde{X} [U^H, V^H]. \] (2.8)

Let \( \Omega^\bullet(S, \Omega^\bullet(2)(\tilde{X}, \tilde{F}|_{\tilde{X}})) \) be the space of smooth sections of \( \Lambda^\bullet(T^* S) \otimes \Omega^\bullet(2)(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) on \( S \). Then \( dM \) naturally acts on \( \Omega^\bullet(S, \Omega^\bullet(2)(\tilde{X}, \tilde{F}|_{\tilde{X}})) \). Let \( i_{T^H} \) be the interior multiplication. Then (cf. [6, Proposition 3.4]) \( \tilde{A}' = d\tilde{M} \) can be decomposed as \( \tilde{A}' = d\tilde{X} + \nabla^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) + i_{T^H} \). Let \( \tilde{A}'' \) be the adjoint of the superconnection \( \tilde{A}' \) with respect to the metric \( g^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \). Let \( \nabla^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) be the connection on \( \Omega^\bullet(2)(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) which is adjoint to \( \nabla^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \) with respect to \( g^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \). Then (cf. [6, Proposition 3.7]) we have \( \tilde{A}'' = d\tilde{X} + \nabla^{\Omega^\bullet(2)}(\tilde{X}, \tilde{F}|_{\tilde{X}}) \).
Set
\[ D\tilde{X} = d\tilde{X} + d\tilde{X}^*, \quad D\tilde{X}, = \left( d\tilde{X} + d\tilde{X}^* \right)^2 = d\tilde{X} d\tilde{X}^* + d\tilde{X}^* d\tilde{X}. \]

Let \( D_i^{\tilde{X}} = D_{\tilde{X}} |_{\Omega^2(\tilde{X}, F|\tilde{X})} \) and let \( \tilde{P}_{\ker D_i^{\tilde{X}}} \) be the orthogonal projection onto \( \ker D_i^{\tilde{X}}, \)
\( i = 0, \ldots, \dim\tilde{X}. \)

Let \( H^{\bullet}(\tilde{X}, F|\tilde{X}) \) be the reduced \( L^2 \)-cohomology of \( (\Omega^{\bullet}(\tilde{X}, F|\tilde{X}), d\tilde{X}) \). Then by Hodge theory we have
\[ H^{\bullet}(\tilde{X}, F|\tilde{X}) \cong \ker D_i^{\tilde{X}}. \]

Let \( g^L_{\Omega^k(\tilde{X}, F|\tilde{X})} \) be the induced metric on \( H^{\bullet}(\tilde{X}, F|\tilde{X}) \). Let \( \nabla^{H^{\bullet}}(\tilde{X}, F|\tilde{X}) \) be the induced connection on \( H^{\bullet}(\tilde{X}, F|\tilde{X}) \).

**Definition 2.1** The analytic Novikov-Shubin invariants (\([18,24,25]\)) are defined by
\[ \alpha_i = \sup \left\{ \beta \geq 0 | \text{Tr}_\Gamma \left( \exp \left( -t D_i^{\tilde{X}} \right) \right) - \text{Tr}_\Gamma \left( \tilde{P}_{\ker D_i^{\tilde{X}}} \right) = O(t^{-\beta}) \right\}, \quad i = 0, \ldots, \dim\tilde{X}, \]
where \( \text{Tr}_\Gamma \) means the \( \Gamma \)-trace (cf. \([1]\)).

Set
\[ \tilde{A} = \frac{1}{2} (\tilde{A}' + \tilde{A}'), \quad \tilde{B} = \frac{1}{2} (\tilde{A}' - \tilde{A}). \] (2.9)

For \( t > 0 \), set
\[ g_t^{\tilde{X}} = g_t^{\tilde{X}} \Rightarrow \frac{g_t^{\tilde{X}}}{t}. \] (2.10)

Let \( N \) be the number operator of \( \Omega^{\bullet}(\tilde{X}, F|\tilde{X}) \), i.e., \( N \) acts by multiplication by \( k \) on \( \Omega^k(\tilde{X}, F|\tilde{X}) \). One verifies easily that
\[ g_t^{\Omega^k(\tilde{X}, F|\tilde{X})} = t^{N-k} g_{\Omega^k(\tilde{X}, F|\tilde{X})}. \] (2.11)

Let \( \tilde{A}_`` \) be the adjoint of \( \tilde{A} \) with respect to \( g_t^{\Omega^k(\tilde{X}, F|\tilde{X})} \), then we have
\[ \tilde{A}_`` = t^{-N} \tilde{A}'' t^N. \] (2.12)

Set
\[ \tilde{A}_t = \frac{1}{2} (\tilde{A}_t'' + \tilde{A}_t'), \quad \tilde{B}_t = \frac{1}{2} (\tilde{A}_t'' - \tilde{A}_t'). \] (2.13)

For \( t > 0 \), set
\[ \tilde{C}_t = t^{N/2} \tilde{A}_t - t^{-N/2}, \quad \tilde{C}_t'' = t^{-N/2} \tilde{A}_t t^{N/2}. \] (2.14)

Then \( \tilde{C}_t \) is a flat superconnection on \( \Omega^{\bullet}(\tilde{X}, F|\tilde{X}) \), and \( \tilde{C}_t'' \) is its adjoint with respect to \( g^{\Omega^k(\tilde{X}, F|\tilde{X})} \). Set
\[ \tilde{C}_t = \frac{1}{2} (\tilde{C}_t'' + \tilde{C}_t'), \quad \tilde{D}_t = \frac{1}{2} (\tilde{C}_t'' - \tilde{C}_t'). \] (2.15)
Then we have

$$\widetilde{\mathcal{C}}_t = it^{N/2}\tilde{A}_t t^{-N/2}, \quad \tilde{D}_t = it^{N/2}\tilde{B}_t t^{-N/2}. \quad (2.16)$$

For $t > 0$, let $\psi_t : \Lambda^\bullet(T^*S) \to \Lambda^\bullet(T^*S)$ be given by

$$\psi_t\omega = t^\deg\omega/2\omega. \quad (2.17)$$

We have

$$\widetilde{\mathcal{C}}_t = \psi_t^{-1}\sqrt{t}\tilde{A}_t\psi_t, \quad \tilde{D}_t = \psi_t^{-1}\sqrt{t}\tilde{B}_t\psi_t. \quad (2.18)$$

We fix a square root $i^{1/2}$ of $i$. Our formulas will not depend on the choice of the square root. Let $\varphi : \Lambda^\bullet(T^*S) \to \Lambda^\bullet(T^*S)$ be the linear map such that for all homogeneous $\omega \in \Lambda^\bullet(T^*S)$,

$$\varphi\omega = (2i\pi)^{-\deg(\omega)/2}\omega. \quad (2.19)$$

Set $h(x) = xe^{x^2}$. For $t > 0$, set

$$h^\wedge \left(\tilde{A}', g_t^{\Omega^2_2(\tilde{X}, \tilde{F}|\tilde{X})}\right) = \varphi\text{Tr}_{\Gamma,s} \left[\frac{N}{2} h'\left(\tilde{B}_t\right)\right], \quad (2.20)$$

where $s$ denotes the supertrace in the sense of [26].

Let $e(X) \in C^\infty(S)$ be the locally constant integer-valued function on $S$ which, to $s \in S$, assigns the Euler characteristic of the fiber $X_s$.

Put

$$\chi'(\tilde{F}) = \sum_{j=0}^{\dim\tilde{X}} (-1)^j j\dim\left(H^j_2(\tilde{X}, \tilde{F}|\tilde{X})\right), \quad \chi(\tilde{F}) = \text{rk}(\tilde{F})e(X). \quad (2.21)$$

Here $\dim\Gamma$ means the $\Gamma$ dimension. Then $\chi'(\tilde{F})$ and $\chi(\tilde{F})$ are locally constant functions on $S$.

In the following of this paper, we will assume that the analytic Novikov-Shubin invariants of $(X, F)$ are positive.

**Definition 2.2** The $L^2$-analytic torsion form is defined by

$$T_{L^2,h}(T^H\tilde{M}, g^{T\tilde{X}}, \nabla\tilde{F}, g\tilde{F}) = -\int_0^{+\infty} \left[ h^\wedge \left(\tilde{A}', g_t^{\Omega^2_2(\tilde{X}, \tilde{F}|\tilde{X})}\right) - \frac{1}{2} \chi'(\tilde{F})h'(0) \right. \right.$$ 

$$- \left. \left(\frac{n}{4} \chi(\tilde{F}) - \frac{1}{2} \chi'(\tilde{F})\right) h'\left(i\sqrt{t}/2\right) \right] \frac{dt}{t}. \quad (2.22)$$

By [30], the $L^2$-analytic torsion form $T_{L^2,h}(T^H\tilde{M}, g^{T\tilde{X}}, \nabla\tilde{F}, g\tilde{F}) \in \Omega^\bullet(S)$ is a smooth form.

### 3 A family of Thom-Smale gradient vector fields

Let $f : M \to \mathbb{R}$ be a smooth function. We assume that $f$ is Morse along every fiber $X$.

Set $\dim X = n$. Let $B$ be the set of critical points of $f$,

$$B = \{x \in X, df(x) = 0\}. \quad (3.1)$$

If $x \in B$, recall that the index $\text{ind}(x)$ is such that the quadratic form $d^2f(x)$ on $T_xX$ has signature $(n - \text{ind}(x), \text{ind}(x))$. 

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Let $h^{TX}$ be a metric on $TX$. Let $\nabla f \subset TX$ be the gradient field of $f$ along the fiber $X$ with respect to $h^{TX}$. Then $\nabla f \in TX$. We make the fundamental assumption that $Y = -\nabla f$ is Thom-Smale along every fiber $X$.

Let $B$ be the zero set of $Y$, i.e., the set of fiberwise critical points of $f$. Let $B'$ be the set of critical points of $f$ which have index $i$ along fibers $X$. Then $B$, $B'$ are finite covers of $S$. We denote by $B$, $B'$ the corresponding fibers.

Let $\tilde{f}$ be the lifted of $f$ to $\tilde{M}$, then the restriction of $\tilde{f}$ to the fiber $\tilde{X}$ is Morse. Let $\tilde{B}$ be the zero set of $\nabla \tilde{f}$ and $\tilde{B}$ be its fiber. Let $\tilde{B}'$ be the set of critical points of $\tilde{f}$ which have index $i$ along fibers of $\tilde{X}$. For $x \in \tilde{B}$, let $\tilde{W}^u(x)$ and $\tilde{W}^s(x)$ be the unstable and stable cells. If $x \in \tilde{B}$, set

$$T_x \tilde{X}^u = T_x \tilde{W}^u(x), \quad T_x \tilde{X}^s = T_x \tilde{W}^s(x).$$

Let $\tilde{T} \tilde{X}^u$ and $\tilde{T} \tilde{X}^s$ be the vector bundles on $\tilde{B}$ with fibers $T_x \tilde{X}^u$ and $T_x \tilde{X}^s$ respectively, and we have

$$\tilde{T} \tilde{X} \big|_B = T \tilde{X}^u \oplus T \tilde{X}^s. \quad (3.2)$$

Let $o^u$, $o^s$ be the $\mathbb{Z}_2$-lines on $\tilde{B}$, which are the orientation lines of $T \tilde{X}^u$, $T \tilde{X}^s$.

Let $(F, \nabla F)$ be a complex flat vector bundle over $X$ with flat connection $\nabla F$, and let $(F^*, \nabla F^*)$ be the corresponding dual flat vector bundle carrying the flat connection $\nabla F^*$ and the dual Hermitian metric $g_{F^*}$. Let $(\tilde{F}, \nabla \tilde{F})$ denote the lifted flat vector bundle over $\tilde{X}$ obtained as the pullback of $(F, \nabla F)$ through the covering map. Let $g_{F}$ be the naturally lifted Hermitian metric on $\tilde{F}$. Let $(\tilde{F}^*, \nabla F^*)$ and $g_{F^*}$ denote the corresponding lifted objects on $\tilde{X}$.

Set

$$C_*(W^u, F^*) = \bigoplus_{x \in \tilde{B}} \tilde{F}_x^* \otimes o_x^u,$$

$$C_1(W^u, F^*) = \bigoplus_{x \in \tilde{B}'} \tilde{F}_x^* \otimes o_x^u, \quad (3.3)$$

where $\bigoplus$ is meant in the $L^2$ sense. Then we have the $L^2$-Thom-Smale complex $(C_*(W^u, F^*), \partial)$ (cf. [32]).

Let $(C^*(W^u, \tilde{F}), \tilde{\partial})$ be the complex dual to the complex $(C_*(W^u, F^*), \partial)$. By (3.3), we get

$$C^*(W^u, \tilde{F}) = \bigoplus_{x \in \tilde{B}} \tilde{F}_x \otimes o_x^u,$$

$$C^1(W^u, \tilde{F}) = \bigoplus_{x \in \tilde{B}'} \tilde{F}_x \otimes o_x^u, \quad (3.4)$$

where $\bigoplus$ is meant in the $L^2$ sense.

Let $l^2(\Gamma)$ denote the Hilbert space obtained through the $L^2$-completion of the group algebra of $\Gamma$ with respect to the canonical trace on it. Then $C^1(W^u, \tilde{F})$ is a Hilbert space, which is isomorphic to the direct sum of $n_i$ copies of $l^2(\Gamma)$, where $n_i = \#B'$. We call $(C^*(W^u, \tilde{F}), \tilde{\partial})$ the $L^2$-Thom-Smale cochain complex.

The complex $C^*(W^u, \tilde{F})$ is a flat $\mathbb{Z}$-graded vector bundle on $S$. Let $\nabla C^*(W^u, \tilde{F})$ be the corresponding flat connection on $C^*(W^u, \tilde{F})$. Then

$$\tilde{\partial} C^*(W^u, \tilde{F})' = \tilde{\partial} + \nabla C^*(W^u, \tilde{F})$$
is a flat superconnection of total degree 1 on $C^\bullet(W^u, \tilde{F})$ and we have

$$H^\bullet_{(2)} (C^\bullet(W^u, \tilde{F}), \tilde{\partial}) \cong H^\bullet_{(2)} (\tilde{X}, \tilde{F}|\tilde{X}).$$  \hfill (3.5)

Let $g_{C^\bullet(W^u, \tilde{F})}$ be the metric on $H^\bullet_{(2)} (\tilde{X}, \tilde{F}|\tilde{X})$ induced by the isomorphism. Let $\tilde{\partial}^*$ be the adjoint of $\tilde{\partial}$ and $\nabla C^\bullet(W^u, \tilde{F}),*$ be the adjoint of $\nabla C^\bullet(W^u, \tilde{F}).$ For $t > 0,$ set

$$\tilde{A}_t^C(W^u, \tilde{F})'' = t\tilde{\partial}^* + \nabla C^\bullet(W^u, \tilde{F}),*,$$

$$\tilde{C}_t^C(W^u, \tilde{F})'' = \sqrt{t}\tilde{\partial} + \nabla C^\bullet(W^u, \tilde{F}),$$

$$\tilde{B}_t^C(W^u, \tilde{F})'' = \frac{1}{2} \left( \tilde{A}_t^C(W^u, \tilde{F})'' - \tilde{A}_t^C(W^u, \tilde{F})'' \right).$$  \hfill (3.6)

Then as [4, Definition 1.17], we define

$$h^\wedge \left( \tilde{A}_t^C(W^u, \tilde{F}), g_t C^\bullet(W^u, \tilde{F}) \right) = \varphi \Tr_{\Gamma, s} \left[ \frac{N(C^\bullet(W^u, \tilde{F})}{2} h' \left( B_t^C(W^u, \tilde{F}) \right) \right],$$

where $N(C^\bullet(W^u, \tilde{F})$ is the number operator of $C^\bullet(W^u, \tilde{F}).$

Set

$$\tilde{D} = \tilde{\partial} + \tilde{\partial}^*.$$  

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Let $\tilde{P}_{\ker \tilde{D}^\bullet}$ be the orthogonal projection onto $\ker \tilde{D}^\bullet,$ $i = 0, \ldots, \dim \tilde{X}.$

**Definition 3.1** The cellular Novikov-Shubin invariants (cf. [18]) are defined by

$$\alpha_i^c = \sup \left\{ \beta \geq 0 \left| \Tr_{\Gamma} \left( \exp \left( -it\tilde{D}^\bullet \right) \right) - \Tr_{\Gamma} \left( \tilde{P}_{\ker \tilde{D}^\bullet} \right) \right| = 0 \right\}, \quad i = 0, \ldots, \dim \tilde{X}.$$

**Theorem 3.2** ([18, Theorem 2.68]) Let $M$ be a cocompact free proper $G$-manifold without boundary and with $G$-invariant Riemannian metric. Then the cellular and the analytic spectral density functions are dilatationally equivalent and the cellular and analytic Novikov-Shubin invariants agree in each degree $i.$

Since we assume that $\alpha_i > 0, i = 0, \ldots, \dim \tilde{X},$ then by [13,15,18], we have $\alpha_i^c > 0, i = 0, \ldots, \dim \tilde{X}.$

Set

$$\chi^c(\tilde{F}) = \sum_{i=0}^{\dim \tilde{X}} (-1)^i i \dim \Gamma \left( C^i(W^u, \tilde{F}) \right).$$

As [4, Proposition 1.27], we have

**Proposition 3.3**

$$h^\wedge \left( \tilde{A}_t^C(W^u, \tilde{F}), g_t C^\bullet(W^u, \tilde{F}) \right) = \frac{1}{2} \chi^c(\tilde{F}) + O(t), \quad as \ t \to 0.$$

**Proof** The proof follows similarly as (4.18), Theorem 4.5 and (4.34) below.

Since $\alpha_i^c > 0, i = 0, \ldots, \dim \tilde{X},$ using Duhamel expansion developed in [2, Theorem 4.1], one has

**Theorem 3.4** There exists $\gamma > 0$ such that

$$h^\wedge \left( \tilde{A}_t^C(W^u, \tilde{F}), g_t C^\bullet(W^u, \tilde{F}) \right) = \frac{1}{2} \chi^c(\tilde{F}) + O(t^{-\gamma}), \quad as \ t \to \infty.$$
Proof As in [2], set
\[
\theta(t) = \text{Tr}_\Gamma \left[ \exp \left( -t \tilde{D}^{c,2}_{\kappa} \right) \right] - \text{Tr}_\Gamma \left[ \tilde{P}_{\kappa} \tilde{D}^{c,2} \right].
\]

Then by the same proof of [2, Lemma 4.5], there exists a monotone decaying function \( \tilde{s} = \tilde{s}(t) \) and \( \alpha > 0 \) such that
\[
\theta \left( t \tilde{s}(t) \right) \cdot \left( \frac{1}{\tilde{s}(t)} \right)^{\frac{\dim \Gamma}{2}} \leq t^{-\alpha}, \quad \text{as } t \to \infty.
\]

Choose \( T \) such that \( \tilde{s}(T) < \frac{1}{\dim \Delta + 1} \). Then as [2, Lemma 4.2], for \( k \geq 0 \), there exists a constant \( C \) such that for all \( s > 0, t > T \),
\[
\left\| \left( \sqrt{t} \tilde{D}^c \right)^k e^{-st \tilde{D}^{c,2}} \right\|_{\text{op}} \leq Cs^{-k}, \quad \text{for } k \geq 0,
\]
\[
\left\| e^{-st \tilde{D}^{c,2}} - \tilde{P} \tilde{D}^{c,2} \right\|_{\Gamma,1} = \theta(st),
\]
\[
\left\| \left( \sqrt{t} \tilde{D}^c \right)^k e^{-st \tilde{D}^{c,2}} \right\|_{\Gamma,1} \leq Cs^{-k} \theta \left( \frac{st}{2} \right), \quad \text{for } k \geq 1,
\]
where \( \left\| A \right\|_{\Gamma,1} = \text{Tr}_\Gamma(|A|) \) (cf. Definition A.3).

As (2.15) and (2.16), one has
\[
h^\wedge \left( \tilde{A}_{i}^{C^*(W^u, \bar{F})}, \tilde{g}_t^{C^*(W^u, \bar{F})} \right) = \varphi \text{Tr}_{\Gamma,s} \left[ \frac{N_{C^*(W^u, \bar{F})}}{2} h' \left( \frac{C_t^{C^*(W^u, \bar{F})} - C_i^{C^*(W^u, \bar{F})}}{2} \right) \right],
\]
where
\[
\frac{C_t^{C^*(W^u, \bar{F})} - C_i^{C^*(W^u, \bar{F})}}{2} = \frac{1}{2} \sqrt{t} \left( \tilde{\theta}^* - \tilde{\theta} \right) + \frac{1}{2} \left( \nabla C^*(W^u, \bar{F})^* - \nabla C^*(W^u, \bar{F}) \right).
\]

Set
\[
\chi_t^{C^*(W^u, \bar{F})} = \frac{C_t^{C^*(W^u, \bar{F})} - C_i^{C^*(W^u, \bar{F})}}{2}, \quad \Omega^{C^*(W^u, \bar{F})} = \frac{1}{2} \left( \nabla C^*(W^u, \bar{F})^* - \nabla C^*(W^u, \bar{F}) \right),
\]
\[
\omega^{C^*(W^u, \bar{F})} = \frac{1}{2} \left( \tilde{\theta}^* - \tilde{\theta} \right).
\]

We denote the standard \( m \)-simplex by
\[
\Delta^m = \left\{ (s_0, \ldots, s_m) \in [0,1]^{m+1} \middle| s_0 + \cdots + s_m = 1 \right\}
\]
and the standard volume form on \( \Delta^m \) by \( d^m(s_0, \ldots, s_m) \), so that \( \Delta^m \) has total volume \( \frac{1}{m!} \).

Since
\[
\chi_t^{C^*(W^u, \bar{F})} = \sqrt{t} \omega^{C^*(W^u, \bar{F})} + \Omega^{C^*(W^u, \bar{F})},
\]
we have
\[
\left( \chi_t^{C^*(W^u, \bar{F})} \right)^2 = -\frac{t}{4} \tilde{D}^{c,2} + \sqrt{t} \omega^{C^*(W^u, \bar{F})} \Omega^{C^*(W^u, \bar{F})} + \sqrt{t} \Omega^{C^*(W^u, \bar{F})} \omega^{C^*(W^u, \bar{F})} + \left( \Omega^{C^*(W^u, \bar{F})} \right)^2.
\]
Then by Duhamel’s principle, we have
\[
\exp \left( \left( \mathcal{X}_t C^*(W^u, \bar{F}) \right)^2 \right) = \sum_{m=0}^{\dim S} (\int_{\Delta^m} e^{-s_0 \frac{i}{2} \bar{D}^c} \left( \sqrt{\int_{\omega t} C^*(W^u, \bar{F})} \Omega C^*(W^u, \bar{F}) \right)^2) + \times \sqrt{t} \Omega C^*(W^u, \bar{F}) C^*(W^u, \bar{F}) + \Omega C^*(W^u, \bar{F}) e^{-s_1 \frac{i}{2} \bar{D}^c} \ldots \\
\times \left( \sqrt{t} \Omega C^*(W^u, \bar{F}) C^*(W^u, \bar{F}) + \sqrt{t} \Omega C^*(W^u, \bar{F}) C^*(W^u, \bar{F}) + \left( \Omega C^*(W^u, \bar{F}) \right)^2 \right) e^{-s_m \frac{i}{2} \bar{D}^c} \ldots \times d^m (s_0, \ldots, s_m).
\]
(3.10)

Then by (3.8), (3.9) and (3.10), proceeding as the proof of [2, Theorem 4.1], we have
\[
\lim_{t \to \infty} \left( \left( \mathcal{X}_t C^*(W^u, \bar{F}) \right)^k \right) = \tilde{P}_{\ker \bar{D}^c} \left( \Omega C^*(W^u, \bar{F}) \tilde{P}_{\ker \bar{D}^c} \right)^k \exp \left( \left( \Omega C^*(W^u, \bar{F}) \tilde{P}_{\ker \bar{D}^c} \right)^2 \right)
\]
in \| \cdot \|_{\Gamma, 1}, for k = 0, 1, 2.

Since
\[
\tilde{P}_{\ker \bar{D}^c} \Omega C^*(W^u, \bar{F}) \tilde{P}_{\ker \bar{D}^c} = \frac{1}{2} \left( \nabla h^{(2)} (\bar{x}, \bar{F}) \right) - \frac{1}{2} \left( \nabla h^{(2)} (\bar{x}, \bar{F}) \right) \left( \nabla h^{(2)} (\bar{x}, \bar{F}) \right) - \frac{1}{2} \chi' (\bar{F}) h' (0)
\]
(3.11)

then by (3.8)–(3.10), we get the theorem. □

**Definition 3.5** By Proposition 3.3 and Theorem 3.4, the \( L^2 \)-combinatorial torsion form is defined by
\[
T_{L^2, h} (\tilde{\Lambda} C^*(W^u, \bar{F}), g C^*(W^u, \bar{F})) = \int_0^{+\infty} \left[ \frac{1}{2} \chi' (\bar{F}) h' (0) - \left( \frac{1}{2} \chi' (\bar{F}) + \frac{1}{2} \chi' (\bar{F}) \right) h' (i / \sqrt{t}) + \frac{1}{2} \chi (\bar{F}) \right] \left( h \wedge (\tilde{\Lambda} C^*(W^u, \bar{F}), g C^*(W^u, \bar{F})) \right) dt.
\]
(3.13)

As in [30], one can also define the \( m \)-th Hilbert-Schmidt norm \( \| \cdot \|_{\text{HS} m} \) and \( m \)-th operator norm \( \| \cdot \|_{\text{op} m} \) for this case. Since \( \alpha_i^c > 0, i = 0, \ldots, \dim \bar{X} \), as in [30, Theorem 4.4], we have that (3.10) holds in \( \| \cdot \|_{\text{HS} m} \). Then under the condition of positive Novikov-Shubin invariants, \( T_{L^2, h} (\tilde{\Lambda} C^*(W^u, \bar{F}), g C^*(W^u, \bar{F})) \in \Omega (S) \) is a smooth form.

Let \( \Omega^* (S, C^*(W^u, \bar{F})) \) be the space of smooth sections of \( \Lambda^* (T^* S) \otimes C^*(W^u, \bar{F}) \) on \( S \).

**Definition 3.6** Let \( \tilde{P}^\infty : \Omega^* (S, \Omega^* (\tilde{X}, \bar{F} | \bar{X})) \to \Omega^* (S, C^*(W^u, \bar{F})) \) be given by
\[
\tilde{P}^\infty \alpha = \sum_{x \in B} \bar{W}^u (x)^* \int_{\bar{W}^u (x)} \alpha.
\]
(3.14)

Then the map \( \tilde{P}^\infty \) is a quasi-isomorphism of \( \mathbb{Z} \)-graded mapping \( (\Omega^* (S, \Omega^* (\tilde{X}, \bar{F} | \bar{X})), d^M) \) into the Thom-Smale complex \( (\Omega^* (S, C^*(W^u, \bar{F})), \tilde{\Lambda} C^*(W^u, \bar{F}')) \).

### 4 The main theorem

In this section, we will prove the main theorem of this paper.
Let $g_\ell$ be a path of metrics on $H^*_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}})$ connecting $g^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}})$ and $g^L_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}})$, let $\nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}})$ be adjoint of $\nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}})$ with respect to $g_\ell$, let

$$B_\ell = \frac{1}{2} \left( \nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}) - \nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}) \right).$$

As [4, Definition 1.10], we define

$$h^L_\ell \left( \nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}), g_\ell \right) = \int_0^1 \varphi Tr_{\Gamma, s} \left[ \frac{1}{2} (g_\ell)^{-1} \frac{\partial g_\ell}{\partial \ell} h'(B_\ell) \right] d\ell. \quad (4.1)$$

By the same argument in [4, Theorem 1.11], the class of $h^L_\ell(\nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}), g_\ell)$ in $\Omega^*(S)/d\Omega^*(S)$ only depends on $g^C_{\ell}(w, \tilde{F})$ and $g^L_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}})$. We denote the class by

$$\tilde{h}^L_\ell \left( \nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}), g^C_{\ell}(w, \tilde{F}), g^L_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}) \right) \in \Omega^*(S)/d\Omega^*(S).$$

Let $T^H M$, $g^{TX}$ and $g^{T\tilde{F}}$ be another triple of data, then the following anomaly formula holds ([30, Lemma 4.9]),

$$T^L_{\ell, h}(T^H M, g^{TX}, \nabla^F, g^{T\tilde{F}}) - T^L_{\ell, h}(T^H M, g^{TX}, \nabla^F, g^{T\tilde{F}}) = \int_X \tilde{e} \left( TX, \nabla^{TX}, \tilde{F} \tilde{F} \right) h \left( \nabla^F, g^{F} \right) + \int_X e \left( TX, \nabla^{TX} \right) \tilde{h} \left( \nabla^F, g^{F}, g^{F} \right)
$$

$$- \tilde{h}^L_\ell \left( \nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}), g^L(\tilde{X}, \tilde{F}|_{\tilde{X}}), \tilde{H}^L_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}) \right) \right) \in \Omega^*(S)/d\Omega^*(S). \quad (4.2)$$

Similarly we can define the class $\tilde{h}^L_\ell(\nabla^C(w, \tilde{F}), g^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F}))$ and the class $\tilde{h}^L_\ell(\nabla^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F}))$ in $\Omega^*(S)/d\Omega^*(S)$. By the same argument of [4, Theorem 1.31], we also have

$$T^L_{\ell, h}(\hat{A}^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F})) - T^L_{\ell, h}(\hat{A}^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F})) = h^L \left( \nabla^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F}), g^C_{\ell}(w, \tilde{F}) \right)
$$

$$- \tilde{h}^L_\ell \left( \nabla^H_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}), g^L(\tilde{X}, \tilde{F}|_{\tilde{X}}), \tilde{H}^L_\ell(\tilde{X}, \tilde{F}|_{\tilde{X}}) \right) \right) \in \Omega^*(S)/d\Omega^*(S). \quad (4.3)$$

Let $\psi(TX, \nabla^{TX})$ be the Mathai-Quillen current on $TX$, which is defined as in [4, Definition 6.7]. Let $h(\nabla^F, g^F)$ be defined in [6, (1.34)]. By [4, Section 7.1], we see that $\int_X h(\nabla^F, g^F)(\nabla f)^* \psi(TX, \nabla^{TX})$ is smooth on $S$.

Consider the $\mathbb{Z}_2$-graded vector bundle $TX|_B = TX^s|_B \oplus TX^u|_B$ over the manifold $B$, where $TX^s$, $TX^u$ are defined in analogue with $T\tilde{X}^s$, $T\tilde{X}^u$. To avoid any ambiguity, let us just say that $TX^s$ is the even part of $TX|_B$, and $TX^u|_B$ is the corresponding odd part. We define the form $0I(TX|_B, \nabla^{TX}|_B)$ as in [4, (4.64)–(4.65)]. Let $0I(TX|_B)$ be the corresponding cohomology class.

Now we state the main theorem of this paper.
Theorem 4.1 The following identity holds in $\Omega^*(S)/d\Omega^*(S)$,

$$
T_{L^2, h} \left( T^H \tilde{M}, g^T \tilde{\nabla}, \nabla \tilde{F}, \tilde{g} \right) - T_{L^2, h} \left( \tilde{A}^C(\tilde{W}, \tilde{F}'), g^C(\tilde{W}, \tilde{F}) \right)
$$

$$
= \int h(\nabla F, \tilde{g} F)(h F) \psi(TX, \nabla TX) + \sum_{x \in B} (-1)^{\text{ind}(x)} \text{rk}(F) 0 I(T_x X|B). \quad (4.4)
$$

Proof By the discussions in [4, Chapter 7], as in [4, Chapter 9], we will assume that $g^TX = h^TX$ and the metrics $g^TX, \tilde{g}$ satisfy the assumptions in [4, Section 9.1].

Following [4, (9.6)], set

$$
\tilde{\chi}'(F) = \sum_{x \in B} \text{rk}(F)(-1)^{\text{ind}(x)} \text{dim}(T_x X_u|B),
$$

$$
\tilde{\chi}^+(F) = \sum_{x \in B} \text{rk}(F)(-1)^{\text{ind}(x)} \text{dim}(T_x X^s|B),
$$

$$
\chi'(F) = \sum_{i=0} \text{dim}(X) (-1)^i \text{dim}(H^i(X, F|X)),
$$

$$
\text{Tr}_s[f] = \sum_{x \in B} (-1)^{\text{ind}(x)} \text{rk}(F) f(x).
$$

Recall that the Bismut-Lott torsion form is defined by (cf. [6, Definition 3.22])

$$
T_h(T^H M, g^TX, \nabla F, \tilde{g}) = - \int_0^{+\infty} \left[ h^\wedge \left( A', g_i^{\Omega^*(X,F|X)} \right) - \frac{1}{2} \chi'(F) h'(0) - \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(F) \right) h' \left( i \sqrt{t}/2 \right) \right] dt. \quad (4.5)
$$

In particular, $\chi(F) = \chi(\tilde{F})$ in the above formula.

By [4, Theorem 0.1], we only need to prove in $\Omega^*(S)/d\Omega^*(S)$,

$$
T_{L^2, h} \left( T^H \tilde{M}, g^T X, \nabla F, \tilde{g} \right) - T_{L^2, h} \left( \tilde{A}^C(\tilde{W}, \tilde{F}'), g^C(\tilde{W}, \tilde{F}) \right)
$$

$$
= \tilde{T}_h \left( T^H M, g^TX, \nabla F, g^F \right) - \tilde{T}_h \left( \tilde{A}^C(\tilde{W}, \tilde{F}'), g^C(\tilde{W}, \tilde{F}) \right)
$$

$$
= \tilde{T}_h \left( T^H M, g^TX, \nabla F, g^F \right) - \tilde{T}_h \left( \tilde{A}^C(\tilde{W}, \tilde{F}'), g^C(\tilde{W}, \tilde{F}) \right), \quad (4.6)
$$

where $T_h(\tilde{A}^C(\tilde{W}, \tilde{F}'), g^C(\tilde{W}, \tilde{F}))$ is the torsion form defined in [4, Chapter 5] and $\tilde{T}(\nabla H^*(X,F|X), H^*(X,F|X), H^*(X,F|X)) \in \Omega^*(S)/d\Omega^*(S)$ is the class defined in [6, Section 1].

For $T \geq 0$, let $g^F_T$ be the metric on $F$,

$$
g^F_T = e^{-2T f} g^F. \quad (4.7)
$$

Let $\tilde{g}^F_T$ be the metric on $\tilde{F}$,

$$
\tilde{g}^F_T = e^{-2T \tilde{f}} \tilde{g}^F.
$$

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By the anomaly formulas of Bismut-Lott analytic torsion form (cf. [6, Theorem 3.24]) and $L^2$-analytic torsion form (cf. [30, Lemma 4.9]), (4.6) is equivalent to the claim that for any $T \geq 0$, in $\Omega^\bullet(S)/d\Omega^\bullet(S)$, we have

$$
\begin{align*}
t_{L^2,h} \left( T^H M, gT^X, \nabla^F, g_T^F \right) - t_h \left( T^H M, gT^X, \nabla^F, g_T^F \right) & - \left( t_{L_0^2,h} \left( \tilde{A}^*(W^u,\bar{F})', g^C(W^u,\bar{F}) \right) - t_h \left( A^*(W^u,\bar{F})', g^C(W^u,\bar{F}) \right) \right) \\
& + \tilde{h}_{L_2^2} \left( \nabla^{H^\bullet(X,\bar{F}|\bar{x}), g^C(W^u,\bar{F}), g_{L_2,T}} \cdot \right) \\
& - \tilde{h} \left( \nabla^{H^\bullet(X,F|\bar{x}), g^C(W^u,\bar{F}), g_{L_2,T}} \cdot \right) = 0,
\end{align*}
$$

where we denote by $g_{L_2,T}$ and $g_{L_2,T}^F$ the corresponding metrics associated to $g_T^F$ and $g_T^F$ respectively.

So we need only to prove in $\Omega^\bullet(S)/d\Omega^\bullet(S)$,

$$
\begin{align*}
\lim_{T \to +\infty} \left\{ t_{L^2,h} \left( T^H M, gT^X, \nabla^F, g_T^F \right) - t_h \left( T^H M, gT^X, \nabla^F, g_T^F \right) \\
& - \left( t_{L_0^2,h} \left( \tilde{A}^*(W^u,\bar{F})', g^C(W^u,\bar{F}) \right) - t_h \left( A^*(W^u,\bar{F})', g^C(W^u,\bar{F}) \right) \right) \\
& + \tilde{h}_{L_2^2} \left( \nabla^{H^\bullet(X,\bar{F}|\bar{x}), g^C(W^u,\bar{F}), g_{L_2,T}} \cdot \right) \\
& - \tilde{h} \left( \nabla^{H^\bullet(X,F|\bar{x}), g^C(W^u,\bar{F}), g_{L_2,T}} \cdot \right) \right\} = 0.
\end{align*}
$$

We define $\tilde{C}_{t,T}$ and $\tilde{D}_{t,T}$ as in (2.15) with respect to $g_T^F$, then we need the following $L^2$-extension of [4, Theorems 9.7 and 9.8].

**Theorem 4.2** There exists $\delta \in (0, 1/2)$ such that if $\varepsilon$, $A$ are such that $0 < \varepsilon < A < +\infty$, there exists $C > 0$ such that if $t \in [\varepsilon, A]$, $T \geq 1$, then

$$
\left| \text{Tr}_{\Gamma,s} \left[ Nh'(\tilde{D}_{t,T}) \right] - \tilde{\chi}^-'(F) \right| \leq \frac{C}{T^\delta}.
$$

**Theorem 4.3** The following identity holds in $\Omega^\bullet(S)/d\Omega^\bullet(S)$,

$$
\begin{align*}
\lim_{T \to +\infty} \left\{ \int_1^{+\infty} \left( \text{Tr}_{\Gamma,s} \left[ Nh' \left( \tilde{D}_{t,T} \right) \right] - \chi'(F) \right) \frac{dt}{2t} \\
& \tilde{h}_{L_2^2} \left( \nabla^{H^\bullet(X,\bar{F}|\bar{x}), g^{B^\bullet(W^u,\bar{F}), g_{L_2,T}}} \cdot \right) - \text{Tr}_{\Gamma,s}[f]T \\
& - \frac{1}{4} \left( \tilde{\chi}^+(F) - \tilde{\chi}^-'(F) \right) \log(T) \right\} \\
& = \int_0^1 \left( \text{Tr}_{\Gamma,s} \left[ Nh^\bullet(W^u,\bar{F}) h' \left( B^\bullet(W^u,\bar{F}) \right) \right] - \tilde{\chi}^-'(F) \right) \frac{dt}{2t} \\
& + \int_1^{+\infty} \left( \text{Tr}_{\Gamma,s} \left[ Nh^\bullet(W^u,\bar{F}) h' \left( B^\bullet(W^u,\bar{F}) \right) \right] - \chi'(F) \right) \frac{dt}{2t} \\
& \tilde{h}_{L_2^2} \left( \nabla^{H^\bullet(X,\bar{F}|\bar{x}), g^{B^\bullet(W^u,\bar{F}), g_{L_2,T}}} \cdot \right) + \frac{1}{4} \left( \tilde{\chi}^-'(F) - \tilde{\chi}^+(F) \right) \log(\pi),
\end{align*}
$$

where $*$ means that the factors $2i\pi$ are omitted (cf. [4, Chapter 9]).
The proofs of Theorems 4.2 and 4.3 will be given in next sections.

Let $\alpha > 0$ be a positive constant which will be chosen later. Let $f : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$f(t) = 1 \text{ if } |t| \leq \frac{\alpha}{2}, \quad f(t) = 0 \text{ if } |t| \geq \alpha. \quad (4.12)$$

Set

$$g(t) = 1 - f(t). \quad (4.13)$$

**Definition 4.4** For $t \in (0, 1], a \in \mathbb{C},$ set

$$F_t(a) = \int_{-\infty}^{+\infty} \exp(iu\sqrt{2}a) \exp\left(-\frac{u^2}{2}\right) f(ut) \frac{du}{\sqrt{2\pi}},$$

$$G_t(a) = \int_{-\infty}^{+\infty} \exp(iu\sqrt{2}a) \exp\left(-\frac{u^2}{2}\right) g(ut) \frac{du}{\sqrt{2\pi}}. \quad (4.14)$$

Then

$$\exp(-a^2) = F_t(a) + G_t(a). \quad (4.15)$$

The functions $F_t(a), G_t(a)$ are even holomorphic functions. Therefore there exist holomorphic functions $\tilde{F}_t(a), \tilde{G}_t(a)$ such that

$$F_t(a) = \tilde{F}_t(a^2), \quad G_t(a) = \tilde{G}_t(a^2). \quad (4.16)$$

From (4.15), (4.16), we get

$$\exp(-a) = \tilde{F}_t(a) + \tilde{G}_t(a). \quad (4.17)$$

The restrictions of $F_t, G_t$ to $\mathbb{R}$ lie in the Schwartz space $S(\mathbb{R})$. Therefore the restrictions of $\tilde{F}_t, \tilde{G}_t$ to $\mathbb{R}$ also lie in $S(\mathbb{R})$.

From (4.17), we deduce that

$$\exp\left(-\tilde{C}_{i,T}^2\right) = \tilde{F}_{\sqrt{t}}(\tilde{C}_{i,T}^2) + \tilde{G}_{\sqrt{t}}(\tilde{C}_{i,T}^2). \quad (4.18)$$

We define $C_{i,T}$ and $D_{i,T}$ for $X \to M \to S$ in analogue with $\tilde{C}_{i,T}, \tilde{D}_{i,T}$ for $\tilde{X} \to \tilde{M} \to \tilde{S}$ as in (2.15) with respect to $g^F_T$.

We have the following analogue theorem of [3, Theorem 11.3] and [19, Theorem 5.3].

**Theorem 4.5** There exist $c > 0, C > 0$ such that for $t \in (0, 1], T \geq 0$, then

$$|\text{Tr}_{\Gamma,T} \left[ N \tilde{G}_{\sqrt{t}}(\tilde{C}_{i,T}^2) \right]| \leq c \exp\left(-\frac{C}{T}\right), \quad |\text{Tr}_{\Gamma} \left[ N \tilde{G}_{\sqrt{t}}(C_{i,T}^2) \right]| \leq c \exp\left(-\frac{C}{T}\right). \quad (4.19)$$

**Proof** Set

$$H_t(a) = \int_{-\infty}^{+\infty} \exp(iu\sqrt{2}a) \exp\left(-\frac{u^2}{2t}\right) g(u) \frac{du}{t^{\sqrt{2\pi}}} \quad (4.20)$$

Then

$$G_{\sqrt{t}}(a) = H_{\sqrt{t}}(\frac{a}{\sqrt{t}}). \quad (4.21)$$
By [5, eq. (13.23)], we find that for any $c \in \mathbb{R}_+, m \in \mathbb{N}$, there exist $c_m > 0$, $C_m > 0$ such that

$$\sup_{a \in C, |\text{Im}(a)| \leq c} |a|^m |H_{\sqrt{t}}(a)| \leq c_m \exp \left(-\frac{C_m}{t}\right).$$

Again there is a holomorphic function $\tilde{H}_t(a)$ such that

$$H_{\sqrt{t}}(a) = \tilde{H}_{\sqrt{t}}(a^2)$$

and so by (4.21), (4.23)

$$\tilde{G}_{\sqrt{t}}(a) = \tilde{H}_{\sqrt{t}} \left(\frac{a}{t}\right).$$

Let $\Delta'$ be the contour in $\mathbb{C}$ defined by

$$\Delta' = \{x + iy | +\infty > x \geq -1, y = 1\} \cup \{x + iy | x = -1, 1 \geq y \geq -1\} \cup \{x + iy | -1 \leq x < +\infty, y = -1\}.$$

From (4.22), we deduce that

$$\sup_{a \in \Delta'} |a|^m \left|\tilde{H}_{\sqrt{t}}(a)\right| \leq c \exp \left(-\frac{C}{t}\right).$$

Let $\tilde{H}_{t,p}(a)$ be a holomorphic function such that

$$\lim_{a \to +\infty} \tilde{H}_{\sqrt{t},p}(a) = 0,$$

$$\tilde{H}_{\sqrt{t},p}^{(p-1)}(a) = \tilde{H}_{\sqrt{t},p}(a).$$

By (4.22), we see that for any $m \in \mathbb{N}$,

$$\sup_{a \in \Delta'} |a|^m \left|\tilde{H}_{\sqrt{t},p}(a)\right| \leq c \exp \left(-\frac{C}{t}\right).$$

By (4.24) and (5.63), we have

$$\text{Tr}_{\Gamma,s} \left[N \tilde{G}_{\sqrt{t}} (\tilde{C}_{\sqrt{t}}^2)\right] = \psi_t^{-1} \text{Tr}_{\Gamma,s} \left[N \tilde{H}_{\sqrt{t}} (\tilde{A}_T^2)\right],$$

where $\tilde{A}_T$ is defined by (5.14).

Clearly

$$\tilde{H}_{\sqrt{t}} (\tilde{A}_T^2) = \frac{1}{2i\pi} \int_{\Delta'} \frac{\tilde{H}_{\sqrt{t}}(\lambda)}{\lambda - \tilde{A}_T^2} d\lambda.$$

Equivalently

$$\tilde{H}_{\sqrt{t}} (\tilde{A}_T^2) = \frac{1}{2i\pi} \int_{\Delta'} \frac{\tilde{H}_{\sqrt{t},p}(\lambda)}{(\lambda - \tilde{A}_T^2)^p} d\lambda.$$

Using (4.26), (4.31) and proceeding as in [3, Chapter 9], we find that for $t \in (0, 1]$, $T \geq 1$,

$$\left|\text{Tr}_{\Gamma,s} \left[N \tilde{H}_{\sqrt{t}} (\tilde{A}_T^2)\right]\right| \leq c \exp \left(-\frac{C}{t}\right).$$

Using (4.29)–(4.32), we get the first formula in (4.19). Similarly, one can get the second inequality in (4.19).
Proposition 4.6 The following formula holds,
\[
\lim_{T \to +\infty} \int_0^1 \left( \text{Tr}_{\Gamma,s} \left[ Nh'(\tilde{D}_{t,T}) \right] - \text{Tr}_s \left[ Nh'(D_{t,T}) \right] \right) \frac{dt}{2t} = 0. \tag{4.33}
\]

Proof Let \( \tilde{F}_{\sqrt{\gamma}}(\tilde{D}^2_{t,T}) \) be the smooth kernel of \( \tilde{F}_{\sqrt{\gamma}}(\tilde{D}^2_{t,T}) \) with respect to the volume \( d\sqrt{\chi}(x')/(2\pi)^{\dim \tilde{X}/2} \). As discussed in [4, p. 234], using the finite propagation speed property of hyperbolic equations, one can choose \( \alpha > 0 \) small enough so that for any \( x \in \tilde{X} \), \( \tilde{F}_{\sqrt{\gamma}}(\tilde{D}^2_{t,T})(x, x) \) only depends on the behavior of \( \tilde{D}_{t,T} \) in a sufficiently small neighborhood of \( x \in X \). In particular, one gets
\[
\text{Tr}_s \left[ N \tilde{F}_{\sqrt{\gamma}}(\tilde{D}^2_{t,T})(x, x) \right] = \text{Tr}_s \left[ N \tilde{F}_{\sqrt{\gamma}}(D^2_{t,T})(\pi_{\tilde{X}}(x), \pi_{\tilde{X}}(x)) \right],
\]
where \( \pi_{\tilde{X}} : \tilde{X} \to X \) is the covering map. Then by the definition of \( \Gamma \)-trace one has
\[
\text{Tr}_{\Gamma,s} \left[ N \tilde{F}_{\sqrt{\gamma}}(\tilde{D}^2_{t,T}) \right] = \text{Tr}_s \left[ N \tilde{F}_{\sqrt{\gamma}}(D^2_{t,T}) \right]. \tag{4.34}
\]

Then by (4.34) and Theorem 4.5, there exist \( c > 0, C > 0 \) such that for \( t \in (0, 1], T \geq 0, \)
\[
|\text{Tr}_{\Gamma,s} \left[ N h'(\tilde{D}_{t,T}) \right] - \text{Tr}_s \left[ N h'(D_{t,T}) \right]| \leq c \exp \left( -\frac{C}{t} \right). \tag{4.35}
\]

By (4.35), for any \( \varepsilon' > 0, \) there exists \( \varepsilon > 0 \) such that
\[
\left| \int_{\varepsilon}^1 \left( \text{Tr}_{\Gamma,s} \left[ N h'(\tilde{D}_{t,T}) \right] - \text{Tr}_s \left[ N h'(D_{t,T}) \right] \right) \frac{dt}{2t} \right| < \varepsilon'. \tag{4.36}
\]

By Theorem 4.2 and [4, Theorem 9.7], we have
\[
\left| \int_{\varepsilon}^1 \left( \text{Tr}_{\Gamma,s} \left[ N h'(\tilde{D}_{t,T}) \right] - \text{Tr}_s \left[ N h'(D_{t,T}) \right] \right) \frac{dt}{2t} \right| \leq \frac{C}{T^\delta} \frac{1}{\varepsilon}. \tag{4.37}
\]
Then by (4.36) and (4.37), we get (4.33). \( \square \)

Remark 4.7 Using the methods in Sect. 6, as the estimates (6.30) and (6.32), one can give another argument for (4.33).

By [4, (9.39) and (9.40)], we have
\[
\varphi \left( -\int_0^1 \left( \text{Tr}_s \left[ Nh'(D_{t,T}) \right] - \frac{1}{2} n \chi(F) h'(0) \right) \frac{dt}{2t} \right)
- \int_1^{+\infty} \left( \text{Tr}_s \left[ Nh'(D_{t,T}) \right] - \chi'(F) h'(0) \right) \frac{dt}{2t}
= T_h \left( T^H M, g^T X, \nabla^F, g^F_T \right)
- \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(F) \right) \left[ \int_0^1 \left( h'(i \sqrt{t}/2) - h'(0) \right) \frac{dt}{t} + \int_1^{+\infty} h'(i \sqrt{t}/2) \frac{dt}{t} \right]. \tag{4.38}
\]
For the $L^2$-analytic torsion form, by (2.16) and (2.22), we have

\[
T_{L^2,h} \left( T^H \tilde{M}, g^T \tilde{X}, \nabla \tilde{F}, g^F_T \right) = -\varphi \left( \int_0^1 \left[ \text{Tr}_{\Gamma,s} \left[ \frac{1}{2} Nh'(\tilde{D}_t, T) \right] - \frac{n}{4} \chi(F)h'(0) \right] \, dt \right)
\]

\[
- \int_0^1 \left[ \frac{n}{4} \chi(F)h'(0) - \frac{1}{2} \chi'(\tilde{F})h'(0) - \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(\tilde{F}) \right) h' \left( \frac{i\sqrt{t}}{2} \right) \right] \, dt
\]

\[
- \varphi \left( \int_1^{+\infty} \left[ \text{Tr}_{\Gamma,s} \left[ \frac{1}{2} Nh'(\tilde{D}_t, T) \right] - \frac{1}{2} \chi'(\tilde{F})h'(0) \right] \, dt \right)
\]

\[
+ \int_1^{+\infty} \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(\tilde{F}) \right) h' \left( \frac{i\sqrt{t}}{2} \right) \, dt.
\] (4.39)

Then we also have

\[
\varphi \left( - \int_0^1 \left( \text{Tr}_{\Gamma,s} \left[ Nh'(\tilde{D}_t, T) \right] - \frac{1}{2} n \chi(F)h'(0) \right) \, dt \right)
\]

\[
- \int_1^{+\infty} \left( \text{Tr}_{\Gamma,s} \left[ Nh'(\tilde{D}_t, T) \right] - \chi'(\tilde{F})h'(0) \right) \, dt
\]

\[
= T_{L^2,h} \left( T^H \tilde{M}, g^T \tilde{X}, \nabla \tilde{F}, g^F_T \right)
\]

\[
- \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(\tilde{F}) \right) \left[ \int_0^1 \left( h' \left( i\sqrt{t}/2 \right) - h'(0) \right) \, dt + \int_1^{+\infty} \left( h' \left( i\sqrt{t}/2 \right) \right) \, dt \right].
\] (4.40)

By [4, (9.71)],

\[
\int_0^1 \left( h' \left( i\sqrt{t}/2 \right) - h'(0) \right) \, dt + \int_1^{+\infty} \left( h' \left( i\sqrt{t}/2 \right) \right) \, dt = \Gamma'(1) + 2(\log(2) - 1).
\] (4.41)

Then by (4.38) and (4.40), we have

\[
T_{L^2,h} \left( T^H \tilde{M}, g^T \tilde{X}, \nabla \tilde{F}, g^F_T \right) - T_h \left( T^H M, g^T X, \nabla F, g^F_T \right)
\]

\[
= \varphi \left( - \int_0^1 \left( \text{Tr}_{\Gamma,s} \left[ Nh'(D_t, T) \right] - \frac{1}{2} n \chi(F)h'(0) \right) \, dt \right)
\]

\[
- \int_1^{+\infty} \left( \text{Tr}_{\Gamma,s} \left[ Nh'(D_t, T) \right] - \chi'(F)h'(0) \right) \, dt
\]

\[
- \varphi \left( - \int_0^1 \left( \text{Tr}_s \left[ Nh'(D_t, T) \right] - \frac{1}{2} n \chi(F)h'(0) \right) \, dt \right)
\]

\[
- \int_1^{+\infty} \left( \text{Tr}_s \left[ Nh'(D_t, T) \right] - \chi'(F)h'(0) \right) \, dt
\]

\[
- \frac{1}{2} \left( \chi'(\tilde{F}) - \chi'(F) \right) \left( \Gamma'(1) + 2(\log(2) - 1) \right).
\] (4.42)
By (4.33), (4.42), [4, Theorem 9.8] and Theorem 4.3, we have
\[
\lim_{T \to +\infty} \left( T L_2, h \left( T H M, g T \bar{X}, \nabla \bar{F}, g_T^F \right) - T h \left( T H M, g T X, \nabla F, g_T^F \right) \right)
\]
\[
+ \tilde{h} \left( \nabla^* H_02(\bar{X}, \bar{F}|\bar{\chi}) , g_{L_2,0} , g_{L_2,T} \right)
\]
\[
- \tilde{h} \left( \nabla^* H^*(X,F|\chi) , g_{L_2,0} , g_{L_2,T} \right)
\]
\[
+ h \left( \nabla^* H^*(X,F|\chi) , g_{L_2,0} , g_{C^\bullet(W^u,F)} \right)
\]
\[
- \tilde{h} \left( \nabla^* H^*(X,F|\chi) , g_{L_2,0} , g_{C^\bullet(W^u,F)} \right)
\]
\[
- \frac{1}{2} (\chi'(\bar{F}) - \chi'(F))(\Gamma'(1) + 2(\log(2) - 1))
\]
\[
= - \varphi \int_0^1 \left( \text{Tr}_t, s \left[ N C^\bullet(W^u,F) h^t \left( B_t C^\bullet(W^u,F) \right) \right] - \bar{\chi}^t_0(F) \right) \frac{dt}{2t}
\]
\[
- \varphi \int_0^1 \left( \text{Tr}_t, s \left[ N C^\bullet(W^u,F) h^t \left( B_t C^\bullet(W^u,F) \right) \right] - \chi'(\bar{F}) \right) \frac{dt}{2t}
\]
\[
+ \varphi \int_0^1 \left( \text{Tr}_t \left[ N C^\bullet(W^u,F) h^t \left( B_t C^\bullet(W^u,F) \right) \right] - \bar{\chi}^t_0(F) \right) \frac{dt}{2t}
\]
\[
+ \varphi \int_0^1 \left( \text{Tr}_t \left[ N C^\bullet(W^u,F) h^t \left( B_t C^\bullet(W^u,F) \right) \right] - \chi'(F) \right) \frac{dt}{2t}
\]
\[
- \frac{1}{2} (\chi'(\bar{F}) - \chi'(F))(\Gamma'(1) + 2(\log(2) - 1))
\]
\[
= T L_2, h \left( A C^\bullet(W^u,F)' , g C^\bullet(W^u,F) \right) - T h \left( A C^\bullet(W^u,F)' , g C^\bullet(W^u,F) \right). \tag{4.43}
\]

So the formula (4.9) holds, then we get Theorem 4.1.

\[ \square \]

5 A proof of Theorem 4.3

In this section, we will give a proof of Theorem 4.3. In [4], there are two proofs of [4, Theorem 9.8]. In this paper, we will use the method in [4, Chapter 10]. Almost the $L^2$ analogue results in [4, Chapter 10] hold, except that it should be modified for the convergence as $t \to +\infty$. Springer
such as [4, Theorem 10.37]. Also for the finite dimension of $F_T^{[0,1]}$ and etc., here it should be of finite $\Gamma$-dimension.

### 5.1 The harmonic oscillator near $\tilde{B}$

This subsection is the $L^2$-set up of [4, Section 10.1].

Recall that $\tilde{B}$ is the set of fiberwise critical points of $\tilde{f}$. Then we have a $\mathbb{Z}_2$-graded vector bundle $T\tilde{X}|_{\tilde{B}}$ (cf. 3.2). The metric $g^{T\tilde{X}}$ induces a metric $g^{T\tilde{X}|_{\tilde{B}}} = g^{T\tilde{X}}|_{\tilde{B}} \oplus g^{T\tilde{X}^u}|_{\tilde{B}}$. Recall that $T\tilde{X}^s|_{\tilde{B}}$ is the even part of $T\tilde{X}|_{\tilde{B}}$ and $T\tilde{X}^u|_{\tilde{B}}$ is the odd part of $T\tilde{X}|_{\tilde{B}}$. Let $\pi' : T\tilde{X}|_{\tilde{B}} \to \tilde{B}$ denote the $\mathbb{Z}_2$-graded real vector bundle.

**Definition 5.1** ([4, Definition 4.1]) For $x \in \tilde{B}$, let $\tilde{I}_x$ (resp. $\tilde{I}^0_x$) be the vector space of smooth (resp. square integrable) sections of $\pi'^* \Lambda ((T\tilde{X}|_{\tilde{B}})^*)$ along the fiber $T\tilde{X}_x$.

Let $dT\tilde{X}|_{\tilde{B}}$ be the de Rham operator acting along the fibers $\tilde{I}$.

The connection $\nabla^{T\tilde{X}|_{\tilde{B}}}$ induces a horizontal vector bundle $T^H(T\tilde{X}|_{\tilde{B}})$ on the total space $T\tilde{X}|_{\tilde{B}}$. Then one verifies easily that with the notation in (2.8), if $U, V \in T\tilde{B}$, $Z \in T\tilde{X}|_{\tilde{B}},$

$$T^H_Z(U, V) = R^{T\tilde{X}|_{\tilde{B}}}(U, V)Z.$$

Also, with the notation in [6, Definition 3.2] and in (2.7), if $U \in T\tilde{B}$, if $s$ is a smooth section of $\tilde{I}$ on $\tilde{B}$,

$$\nabla_U^s = \nabla_U^s((T\tilde{X}|_{\tilde{B}})^*)_s.$$

Let $q : T\tilde{X}|_{\tilde{B}} \to \mathbb{R}$ be the smooth function, such that if $Z = (Z_+, Z_-) \in T\tilde{X}|_{\tilde{B}} = T\tilde{X}^s|_{\tilde{B}} \oplus T\tilde{X}^u|_{\tilde{B}},$

$$q(Z) = \frac{1}{2} (|Z_+|^2 - |Z_-|^2),$$

where $Z_+ \in T\tilde{X}^s$, $Z_- \in T\tilde{X}^u$. Then $q$ is a fiberwise Morse function, whose only critical point 0 has index $\dim T\tilde{X}^u|_{\tilde{B}}$.

Clearly, $\tilde{F}|_{\tilde{B}}$ is equipped with a flat connection $\nabla^\tilde{F}|_{\tilde{B}}$. We denote by $\nabla^\tilde{I} \otimes \tilde{F}|_{\tilde{B}}$ the connection on $\tilde{I} \otimes \tilde{F}|_{\tilde{B}}$ induced by $\nabla^\tilde{I}$ and $\nabla^\tilde{F}|_{\tilde{B}}$. The metric $g^\tilde{F}$ induces a metric $g^\tilde{F}|_{\tilde{B}}$ on $\tilde{F}|_{\tilde{B}}$. Let $\mathcal{F}$ be the restriction of $\tilde{f}$ to $\tilde{B}$. We choose $\varepsilon > 0$ as in [4, Section 10.1]. Then by [4, (9.3)], using the notation in [4, (4.6)], if $x \in \tilde{B}$, $Z \in (T\tilde{X}|_{\tilde{B}})_x$, $|Z| \leq \varepsilon$, set

$$\tilde{f}(Z) = \mathcal{F}(x) + q(Z).$$

For $T \in \mathbb{R}$, the metric $g^\tilde{F}$ induces the metric $g^{\tilde{F}|_{\tilde{B}}}$, which is given by

$$g^\tilde{F}|_{\tilde{B}} = e^{-2T\mathcal{F}} g^\tilde{F}|_{\tilde{B}}.$$  

(5.2)

The exterior differentiation operator $C^{\tilde{I} \otimes \tilde{F}|_{\tilde{B}}'}$, acting on $\Omega^*(\tilde{M}|_{\tilde{B}}, \tilde{F}|_{\tilde{B}})$, has degree 1 and satisfies $(C^{\tilde{I} \otimes \tilde{F}|_{\tilde{B}}'})^2 = 0$. Thus $C^{\tilde{I} \otimes \tilde{F}|_{\tilde{B}}'}$ defines a flat superconnection of total degree 1 on $\tilde{I} \otimes \tilde{F}|_{\tilde{B}}$. As in [4, (4.5)], we have the identity

$$C^{\tilde{I} \otimes \tilde{F}|_{\tilde{B}}'} = dT\tilde{X}|_{\tilde{B}} + \nabla^\tilde{I} \otimes \tilde{F}|_{\tilde{B}}' + i_{R^{T\tilde{X}|_{\tilde{B}}}} Z.$$  

(5.3)
Given $T \in \mathbb{R}$, let $\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B}$ be the adjoint flat superconnection with respect to the metrics $g^{T \bar{X}}_\mathbb{B}, \tilde{F}_\mathbb{B}$. Then we have (cf. [4, (10.5)])

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = d^{T \bar{X}}_\mathbb{B}* + 2T i_{Z_+ - Z_-} + \nabla^{T \bar{X}}_\mathbb{B} + \omega \left( \tilde{F}_\mathbb{B}, g^{\tilde{F}_\mathbb{B}} \right) - 2T d\mathcal{F} - R^{T \bar{X}}_\mathbb{B} Z \wedge \mathcal{L},(5.4)$$

where

$$\omega \left( \tilde{F}_\mathbb{B}, g^{\tilde{F}_\mathbb{B}} \right) = \left( \nabla^{T \bar{X}}_\mathbb{B} \right)^* - \nabla^{T \bar{X}}_\mathbb{B}.$$ 

Set

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = \frac{1}{2} \left( \tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} + \tilde{C} \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} \right), \quad \mathcal{D}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = \frac{1}{2} \left( \tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} - \tilde{C} \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} \right). \quad (5.5)$$

Set

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = e^{-T \tilde{f}} \tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} e^{-T \tilde{f}}, \quad \tilde{C} \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = e^{T \tilde{f}} \tilde{C}_0 \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} e^{-T \tilde{f}},$$

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = \frac{1}{2} \left( \tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} + \tilde{C} \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} \right), \quad \mathcal{D}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = \frac{1}{2} \left( \tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} - \tilde{C} \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} \right). \quad (5.6)$$

By (5.1) and (5.6), we get

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = \frac{1}{2} \left( d^{T \bar{X}}_\mathbb{B}* + d^{T \bar{X}}_\mathbb{B} \right) + \frac{T}{2} (i_{Z_+ - Z_-} + (Z_+ - Z_-) \wedge ) + \nabla^{T \bar{X}}_\mathbb{B}$$

$$+ \frac{1}{2} \omega \left( \tilde{F}_\mathbb{B}, g^{\tilde{F}_\mathbb{B}} \right) + \frac{1}{2} \left( i_{R^T \bar{X}_\mathbb{B}} Z - R^{T \bar{X}_\mathbb{B} Z} \wedge \right),$$

$$\mathcal{D}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B} = \frac{1}{2} \left( d^{T \bar{X}}_\mathbb{B}* - d^{T \bar{X}}_\mathbb{B} \right) + \frac{T}{2} (i_{Z_+ - Z_-} - (Z_+ - Z_-) \wedge )$$

$$+ \frac{1}{2} \omega \left( \tilde{F}_\mathbb{B}, g^{\tilde{F}_\mathbb{B}} \right) - T d\mathcal{F} + \frac{1}{2} \left( i_{R^T \bar{X}_\mathbb{B}} Z + R^{T \bar{X}_\mathbb{B} Z} \wedge \right). \quad (5.7)$$

Let $e_1, \ldots, e_{n_+}$ be an orthonormal basis of $T \bar{X}_\mathbb{B}^s$, let $e_{n+1}, \ldots, e_n$ be an orthonormal basis of $T \bar{X}_\mathbb{B}^u$. Set (cf. [4, 3.33])

$$c(e_i) = e_i \wedge - i_{e_i}, \quad \tilde{c}(e_i) = e_i \wedge + i_{e_i},$$

where we identify $T \bar{X}_\mathbb{B}$ with its dual by the metric.

Then we have

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B}^2 = - \frac{1}{4} \left( \nabla_{e_i} + \left( R^{T \bar{X}_\mathbb{B} Z, e_i} \right) \right) + \frac{1}{4} \left( e_i, R^{T \bar{X}_\mathbb{B} e_j} \tilde{c}(e_i) \tilde{c}(e_j) \right)$$

$$- \frac{1}{4} \omega^2 \left( \tilde{F}_\mathbb{B}, g^{\tilde{F}_\mathbb{B}} \right) + \frac{T^2}{4} |Z|^2 + \frac{T}{4} \left( \sum_{1 \leq i \leq n_-} c(e_i) \tilde{c}(e_i) - \sum_{n_+ + 1 \leq i \leq n_+ + n_-} c(e_i) \tilde{c}(e_i) \right). \quad (5.8)$$

In particular, by (5.8), we get

$$\tilde{C}_T \tilde{\otimes} \tilde{F}^\nu_\mathbb{B}^2(0) = - \frac{1}{4} \Delta^{T \bar{X}_\mathbb{B}} + \frac{T^2}{4} |Z|^2 + \frac{T}{4} \left( \sum_{1 \leq i \leq n_-} c(e_i) \tilde{c}(e_i) - \sum_{n_+ + 1 \leq i \leq n_+ + n_-} c(e_i) \tilde{c}(e_i) \right). \quad (5.9)$$
The operator \( \mathcal{C}_T \) is a harmonic oscillator. Take \( \rho \) non zero in \( \Lambda^{\max}(T^* \tilde{X}^u |_{\tilde{B}}) \). Let \( \tilde{f}_T \) be the one dimensional vector space spanned by \( \exp(-T |Z|^2/2) \rho \). Then we have
\[
\ker \mathcal{C}_T \otimes \tilde{F}|_{\tilde{B}} = \tilde{f}_T \otimes \tilde{F}|_{\tilde{B}}.
\]

Let \( \tilde{p}_T \) be the orthogonal projection operator from \( \tilde{I} \) on \( \tilde{f}_T \). Then \( \tilde{p}_T \) extends to an endomorphism of \( \tilde{I} \otimes \tilde{F}|_{\tilde{B}} \). Note that when acting on \( \tilde{I} \otimes \tilde{F}|_{\tilde{B}} \), \( \tilde{p}_T \) is of the form \( \tilde{p}_T \otimes 1 \).

For \( T \in \mathbb{R}^+ \), using [4, (4.12)], we get
\[
\text{Sp} \left( \mathcal{C}_T \otimes \tilde{F}|_{\tilde{B}} \otimes \rho \right) = \frac{T}{2} \mathbb{N}.
\]

### 5.2 The eigenbundles associated to small eigenvalues

This section is the \( L^2 \)-set up of [4, Section 10.2]

For \( T \geq 0 \), recall that \( d^\ast_T \tilde{X} \) is the adjoint of \( d \tilde{X} \) with respect to the metrics \( g^T \tilde{X}, g^T \tilde{F} \). Let \( \nabla_{\tilde{T}^2} \) be the corresponding adjoint connection to \( \nabla_{\tilde{T}^2} \tilde{X}, \tilde{F} \) \( \tilde{X} \) and let \( \tilde{A}' \) be the superconnection on \( \tilde{A}'(\tilde{X}, \tilde{F}) \) adjoint to \( \tilde{A}' \) with respect to \( g^T \tilde{X}, g^T \tilde{F} \). Then
\[
\tilde{A}' = d\tilde{\bar{X}} + \nabla_{\tilde{T}^2} \tilde{X} + i_T \tilde{H}
\]
and
\[
\tilde{A}'' = d\tilde{\bar{X}} + \nabla_{\tilde{T}^2} \tilde{X} - T^H \wedge.
\]

Set
\[
\tilde{A}_T = \frac{1}{2} (\tilde{A}' + \tilde{A}''), \quad \tilde{B}_T = \frac{1}{2} (\tilde{A}' - \tilde{A}').
\]

Clearly,
\[
\tilde{A}_T^2 = \frac{1}{4} [\tilde{A}'', \tilde{A}'].
\]

In the sequel, we will also use the notations,
\[
\tilde{A}_T = e^{-T \tilde{f}_T} \tilde{A}_T e^{T \tilde{f}_T}, \quad \tilde{B}_T = e^{-T \tilde{f}_T} \tilde{B}_T e^{T \tilde{f}_T}.
\]

If \( H \in \Lambda^\ast(T^* \tilde{S}) \otimes \text{End}(\Omega_{(2)}^\ast(\tilde{X}, \tilde{F} | \tilde{X}^u)) \), let \( H^{(0)} \) be the component of \( H \) in \( \text{End}(\Omega_{(2)}^\ast(\tilde{X}, \tilde{F} | \tilde{X}^u)) \). Clearly,
\[
\text{Sp}(\tilde{A}_T) = \text{Sp}(\tilde{A}_T^2) = \text{Sp}(\tilde{A}_T^{(0)}) = \text{Sp}(\tilde{A}_T^{(0)}).
\]

**Definition 5.2** If \( s \in S \), let \( \tilde{F}_{T,s}^{(0,1)} \) (resp. \( \tilde{F}_{T,s}^{(0,1)} \)) be the eigenspaces of \( \tilde{A}_T^{(0)} \) (resp. \( \tilde{A}_T^{(0)} \)) associated to eigenvalues \( \lambda \in [0, 1] \), let \( \tilde{F}_{T,s}^{(0,1)} \) (resp. \( \tilde{F}_{T,s}^{(0,1)} \)) be the orthogonal projection operator
form \((\Omega^\bullet_2)(\tilde{X}, \tilde{F}_0), g_{T}\substack{\Omega^\bullet_2(\tilde{X}, \tilde{F}_0) & \Omega^\bullet_2(\tilde{X}, \tilde{F}_0) }\) on \(F_{T,s}^{[0,1]}\) (resp. from \((\Omega^\bullet_2)(\tilde{X}, \tilde{F}_0), g_{T}\substack{\Omega^\bullet_2(\tilde{X}, \tilde{F}_0) & \Omega^\bullet_2(\tilde{X}, \tilde{F}_0) }\) on \(F_{T,s}^{[0,1]}\)).

Clearly, we have the obvious orthogonal splittings,

\[
\tilde{F}_{T,s}^{[0,1]} = \bigoplus_{i=0}^{\dim X} \tilde{F}_{T,s}^{[0,1],i}, \quad \tilde{F}_{T,s}^{[0,1]} = \bigoplus_{i=0}^{\dim X} F_{T,s}^{[0,1],i} .
\]  

(5.18)

Also,

\[
\tilde{F}_{T,s}^{[0,1]} = e^{-T \tilde{f}} \tilde{F}_{T,s}^{[0,1]} e^{T \tilde{f}} .
\]  

(5.19)

Let \(M^i\) be the number of elements in \(B^i\). Equivalently, \(M^i\) is the number of critical points of \(f\) in a given fiber \(X\) whose index is equal to \(i\).

**Theorem 5.3** There exists \(T_0 \geq 0\) such that for \(T \geq T_0\),

\[
\text{Sp} \left( \tilde{A}^{2(0)}_T \right) \subset \left[ 0, \frac{1}{4} \right] \cup [4, \infty), \quad \dim_{\Gamma} \left( \tilde{F}_{T,s}^{[0,1],i} \right) = M^i, \quad 1 \leq i \leq \dim X .
\]  

(5.20)

**Proof** For a given \(s \in S\), this result was established in [9, Proposition 5.2 and Theorem 5.5]. Since \(S\) is compact, a trivial uniformity argument shows that we can find \(T_0 \in \mathbb{R}_+\) such that (5.20) holds for any \(s \in S, T \geq T_0\).

By the above, it follows that \(\tilde{F}_{T,s}^{[0,1]}, \tilde{F}_{T,s}^{[0,1]}\) are the fibers of smooth \(Z\)-graded vector bundles \(\tilde{F}_{T}^{[0,1]}, \tilde{F}_{T}^{[0,1]}\) on \(S\), which are subbundles of \(\Omega^\bullet_2(\tilde{X}, \tilde{F}_0)\).

Clearly,

\[
h'(x) = (1 + 2x^2)e^{x^2} .
\]  

(5.21)

Put

\[
r(x) = (1 - 2x)e^{-x} .
\]  

(5.22)

Let \(\delta_1\) be the unit circle in \(\mathbb{C}\). Let \(\Gamma = \Gamma_+ \cup \Gamma_-\) be the contour defined by

\[
\begin{align*}
\Gamma_+ &= \{ z = x + iy \mid x \geq 2, y = \pm 1 \} \cup \{ z = x + iy \mid x = 2, -1 \leq y \leq 1 \} , \\
\Gamma_- &= \{ z = x + iy \mid x \leq -2, y = \pm 1 \} \cup \{ z = x + iy \mid x = -2, -1 \leq y \leq 1 \} .
\end{align*}
\]  

(5.23)

**Definition 5.4** For \(t \in \mathbb{R}_+^\bullet, T \geq T_0\), put

\[
\tilde{K}_{t,T} = \psi_{t}^{-1} \frac{1}{2i\pi} \int_{\Gamma} \frac{r(t\lambda)}{\lambda - \tilde{A}_T^2} d\lambda \psi_{t}, \quad \tilde{L}_{t,T} = \frac{1}{2i\pi} \int_{\sqrt{\delta_1}} \frac{h'(\lambda)}{\lambda - \tilde{D}_{t,T}} d\lambda .
\]  

(5.24)

Then as [4, Proposition 10.4], we have

**Proposition 5.5** The following identity holds,

\[
h'(\tilde{D}_{t,T}) = \tilde{K}_{t,T} + \tilde{L}_{t,T} .
\]  

(5.25)

By (5.25),

\[
\text{Tr}_{\Gamma,s} \left[ Nh' \left( \tilde{D}_{t,T} \right) \right] = \text{Tr}_{\Gamma,s} \left[ N \tilde{K}_{t,T} \right] + \text{Tr}_{\Gamma,s} \left[ N \tilde{L}_{t,T} \right] .
\]  

(5.26)

By the same proof of [4, Theorem 10.5], we have the following analogue theorem in \(L^2\)-case.
Theorem 5.6 There exist $C > 0$, $c > 0$, $\delta \in (0, 1]$ such that for $t \geq 1$, $T \geq T_0$,
\[
|\text{Tr}_{\Gamma,s} \left[ N \tilde{K}_{t,T} \right] | \leq \frac{Ce^{-ct}}{T^\delta}.
\] (5.27)

Proof Set
\[
\tilde{M}_{t,T,a} = \psi_t^{-1} \frac{1}{2i\pi} \int_\Gamma \frac{\exp(-t\alpha \lambda)}{\lambda - A_T^2} d\lambda \psi_t.
\] (5.28)

Then by (5.22),
\[
\tilde{K}_{t,T} = \left( 1 + 2 \frac{\partial}{\partial a} \right) \tilde{M}_{t,T,a} |_{a=1}.
\] (5.29)

Put
\[
\bar{M}_{t,T,a} = e^{-T} \tilde{M}_{t,T,a} e^{T\bar{f}}.
\] (5.30)

By (5.16) and (5.28),
\[
\tilde{M}_{t,T,a} = \psi_t^{-1} \frac{1}{2i\pi} \int_\Gamma \frac{\exp(-t\alpha \lambda)}{\lambda - A_T^2} d\lambda \psi_t.
\] (5.31)

By (5.30),
\[
\text{Tr}_{\Gamma,s} \left[ N \tilde{M}_{t,T,a} \right] = \text{Tr}_{\Gamma,s} \left[ N \bar{M}_{t,T,a} \right].
\] (5.32)

Clearly
\[
d\tilde{X},* = d\tilde{X}, + 2T i_{\nabla \tilde{f}}.
\] (5.33)

Recall that $\nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),*$ is the connection adjoint to $\nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),*$ with respect to $g_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}})$. Let $(d \tilde{f})^H$ be the horizontal component of $d \tilde{f}$. Then
\[
\nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),* = \nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),* - 2T (d \tilde{f})^H.
\] (5.34)

From (5.13), (5.14), (5.33) and (5.34), we get
\[
\tilde{A}_T = \frac{1}{2} \left( D\tilde{X} + 2T i_{\nabla \tilde{f}} \right) + \nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),* - T (d \tilde{f})^H - \frac{1}{2} c \left( T^H \right).
\] (5.35)

If $\nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),* = \nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),*$, by (5.16), we obtain
\[
\tilde{A}_T = \frac{1}{2} \left( D\tilde{X} + T \tilde{c}(\nabla \tilde{f}) \right) + \nabla_{\Omega_0}^*(\tilde{X}, \tilde{F}|_{\tilde{X}}),* - \frac{1}{2} c \left( T^H \right).
\] (5.36)

The term $T (d \tilde{f})^H$ disappeared in (5.36).

Observe that fiberwise, $\tilde{B}$ is the zero set of $\nabla \tilde{f}$. Also $\tilde{c}(\nabla \tilde{f})$ anticommutes with the principal $c(t \xi)$ of $D\tilde{X}$. By the simplifying assumptions in [4, Section 9.1], near $\tilde{B}$,
\[
\tilde{A}_T = \tilde{c}_{T \otimes \tilde{F}|_{\tilde{B}}^\tau}.
\] (5.37)

By (5.11), the kernel of $\tilde{c}_{T \otimes \tilde{F}|_{\tilde{B}}^\tau}$ can be identified with $\tilde{F}|_{\tilde{B}} \otimes \delta |_{\tilde{B}}$. Using (5.7) and the fact, we get the easy formula,
\[
\tilde{p}_T \tilde{A}_T \tilde{p}_T = \nabla \tilde{F}|_{\tilde{B}}^\tau.
\] (5.38)
Therefore,
\[
\frac{1}{2i\pi} \int_{\gamma} \exp(-ta\lambda)\frac{d\lambda}{\lambda - \nabla \bar{F}_{\bar{B},u,2}} = 0.
\]

By (5.29), (5.32), (5.36) and (5.37)–(5.40), as in the proof of [4, Theorem 10.5], one needs a \(L^2\)-version of [3, (9.149)] which we prove in (B.76). Then the proof of the theorem is completed. \(\square\)

### 5.3 The projectors \(\widehat{P}^{[0,1]}_T\)

This section is the \(L^2\)-set up of [4, Sections 10.3 and 10.4].

Observe that \(\Lambda^\bullet(T^*S)\) acts on \(\Lambda^\bullet(T^*S) \otimes \Omega_{(2)}^\bullet (\widetilde{X}, \widetilde{F}|\widetilde{X})\). Let \(\delta_2 \subset \mathbb{C}\) be the circle of centre 0 and radius 1/4. Recall that for \(T \geq T_0\),
\[
\text{Sp}(\widehat{A}^\gamma_T) \cap \delta_2 = \emptyset.
\]

**Definition 5.7** For \(T \geq T_0\), put
\[
\widehat{P}^{[0,1]}_T = \frac{1}{2i\pi} \int_{\delta_2} \frac{d\lambda}{\lambda - \widehat{A}^\gamma_T}.
\]

Clearly, \(\widehat{P}^{[0,1]}_T \in \Lambda^\bullet(T^*S) \otimes \text{End}(\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X}))\). Then we write,
\[
\widehat{P}^{[0,1]}_T = \sum_{j=1}^{\text{dim}S} \widehat{P}^{[0,1],(j)}_T,
\]

where \(\widehat{P}^{[0,1],j}_T \in \Lambda^j(T^*S) \otimes \text{End}(\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X}))\).

Set \(\widehat{F}^{[0,1]}_T = \text{Im}(\widehat{P}^{[0,1]}_T)\), then \(\widehat{F}^{[0,1]}_T\) is a \(Z_2\)-graded subbundle of \(\Lambda^\bullet(T^*S) \otimes \Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X})\).

Also we have
\[
\widehat{P}^{[0,1],(0)}_T = \widehat{P}^{[0,1]}_T.
\]

In the sequel, the operator \(*\) acts on \(\Lambda^\bullet(T^*S) \otimes \text{End}(\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X}))\) as in [4, (1.8)], with respect to the metric \(g_T^{\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X})}\). We will often say that if \(k\) is such that \(k^* = k\), then it is self-adjoint.

If \(k \in \Lambda^\bullet(T^*S) \otimes \text{End}(\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X}))\), we can write \(k\) in the form,
\[
k = \sum_{j=0}^{\text{dim}S} k^{(j)}, \quad k^{(j)} \in \Lambda^j(T^*S) \otimes \text{End}(\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X})).
\]

Observe that \(\Lambda^\bullet(T^*S) \otimes \text{End}(\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X}))\) is a \(Z\)-graded bundle of algebras. Namely, if \(k\) is taken as in (5.45), \(\text{deg}k = p\) if for any \(j\),
\[
k^{(j)} \in \Lambda^{(j)}(T^*S) \otimes \text{Hom} (\Omega^\bullet_{(2)} (\widetilde{X}, \widetilde{F}|\widetilde{X}), \Omega^\bullet_{(2)}^{*p-j} (\widetilde{X}, \widetilde{F}|\widetilde{X})).
\]
Moreover $\Lambda^\bullet(T^*S) \otimes \text{End}(\Omega^\bullet_2(\widetilde{X}, \widetilde{F}|_X))$ inherits a filtration $F$ from the filtration of $\Lambda^\bullet(T^*S)$. We will say that $\deg(k) \geq 0$ if it is the sum of elements of non-negative degree. Also we will write that

$$\deg(k) \leq 2F(k)$$

if for any $j$, $\deg(k_{ij}) \leq 2j$.

**Theorem 5.8** For $T \geq T_0$, $\widetilde{P}_T^{[0,1]}$ is an even projection operator acting on $\Lambda^\bullet(T^*S) \otimes \Omega^\bullet_2(\widetilde{X}, \widetilde{F}|_X)$, which commutes with the action of $\Lambda^\bullet(T^*S)$ and with $\widetilde{X}'$, $\widetilde{A}'_T$, and is such that

$$\widetilde{P}_T^{[0,1],(0)} = \widetilde{F}_T^{[0,1]}.$$

Also

$$\widetilde{P}_T^{[0,1],*} = \widetilde{P}_T^{[0,1]}, \quad 0 \leq \deg \left( \widetilde{P}_T^{[0,1]} \right) \leq 2F(\widetilde{P}_T^{[0,1]}).$$

The linear map $\alpha \in \Lambda(T^*S) \otimes F_T^{[0,1]} \to \widetilde{P}_T^{[0,1]} \alpha \in \widetilde{F}_T^{[0,1]}$ is an isomorphism of $\mathbb{Z}_2$-graded filtered vector bundles.

**Proof** By definition, $\widetilde{P}_T^{[0,1]}$ commutes with $\widetilde{A}_T^2$. Since $\widetilde{X}'$, $\widetilde{A}'_T$ and the elements of $\Lambda^\bullet(T^*S)$ commute with $\widetilde{A}_T^2$, they also commute with $\widetilde{P}_T^{[0,1]}$. We write $\widetilde{A}_T^2$ in the form,

$$\widetilde{A}_T^2 = \widetilde{A}_T^2(0) + \widetilde{A}_T^2(>0).$$

Then if $\lambda \in \delta$,

$$\left( \lambda - \widetilde{A}_T^2 \right)^{-1} = \sum_{i=0}^{\dim X} \left( \lambda - \widetilde{A}_T^2(0) \right)^{-1} \widetilde{A}_T^2(>0) \ldots \widetilde{A}_T^2(>0) \left( \lambda - \widetilde{A}_T^2(0) \right)^{-1},$$

so that $\widetilde{A}_T^2(>0)$ appears $i$ times in the right-hand side of (5.49). The term corresponding to $i = 0$ is obviously equal to the projection operator $\widetilde{P}_T^{[0,1]}$, i.e. (5.46) holds. Also since $\widetilde{A}_T^2, \ast = \widetilde{P}_T^2$, the first identity in (5.47) also holds. Also $\widetilde{A}_T^2(0)$ is of degree 0. Using [4, Proposition 10.7], (5.42) and (5.49), we get the second identity in (5.47).

Obviously, $\widetilde{F}_T^{[0,1]}$ inherits a filtration from the filtration of $\Lambda^\bullet(T^*S) \otimes \Omega^\bullet_2(\widetilde{X}, \widetilde{F}|_X)$. If $\beta \in \widetilde{F}_T^{[0,1], \geq j}$, then

$$\beta = \beta^{(j)} + \beta^{(j+1)} + \ldots .$$

Since $\widetilde{F}_T^{[0,1]} \beta = \beta$, using (5.46), we get

$$\widetilde{P}_T^{[0,1]} \beta^{(j)} = \beta^{(j)}.$$ 

so that $\beta^{(j)} \in \Lambda^j(T^*S) \otimes \widetilde{F}_T^{[0,1]}$. This way, we defined an injective linear map $\text{Gr}^j(\widetilde{F}_T^{[0,1]}) \to \Lambda^j(T^*S) \otimes \widetilde{F}_T^{[0,1]}$. An obvious inverse for this map is just $\alpha \in \Lambda^j(T^*S) \otimes \widetilde{F}_T^{[0,1]} \to 

$$
\left[ \widetilde{F}_T^{[0,1]} \alpha \right] \in \text{Gr}^j(\widetilde{F}_T^{[0,1]}).$$

Therefore $\alpha \in \Lambda^\bullet(T^*S) \otimes \widetilde{F}_T^{[0,1]} \to \widetilde{F}_T^{[0,1]} \alpha \in \widetilde{F}_T^{[0,1]}$ is an isomorphism of $\mathbb{Z}_2$-graded filtered vector bundles. \hfill \Box
5.4 The projectors $\widetilde{P}_{t,T}^{[0,1]}$

This section is the $L^2$-set up of [4, Section 10.5].

Since $T \widetilde{M} = TH \widetilde{M} \oplus T \widetilde{X}$, we have a smooth identification $T \widetilde{M} \cong \pi^*TS \oplus T \widetilde{X}$. Therefore we have the identification,

$$\Lambda(T^*\widetilde{M}) \cong \pi^* \Lambda(T^*S) \otimes \Lambda(T^*\widetilde{X}).$$

(5.52)

Let $N^{\Lambda^*}(T^*\widetilde{M})$ be the operator defining the $Z_2$-grading of $\Lambda(T^*\widetilde{M})$. Then $N^{\Lambda^*}(T^*\widetilde{M})$ acts naturally on the space $\Lambda^*(T^*S) \otimes \Omega^*_2(\widetilde{X}, \widetilde{F}|\widetilde{X})$.

For $t > 0$, $T \geq 0$, let $\widetilde{A}'_{t,T}$ be the adjoint superconnection to $\widetilde{A}'$ with respect to the metrics $g_{T}\widetilde{X} / t$, $g_{\widetilde{F}}$. Then we have

$$t^{-N^{\Lambda^*}(T^*\widetilde{M})/2} \widetilde{A}'_{t,T}^{-N^{\Lambda^*}(T^*\widetilde{M})/2} = \frac{1}{\sqrt{t}} \widetilde{A}'_{t,T}, \quad t^{-N^{\Lambda^*}(T^*\widetilde{M})/2} \widetilde{A}'_{t,T}^{-N^{\Lambda^*}(T^*\widetilde{M})/2} = \frac{1}{\sqrt{t}} \widetilde{A}'_{t,T}. \quad (5.53)$$

We also have

$$\text{Sp} \left( \widetilde{A}'_{t,T} \right) \subset \left[ 0, \frac{t}{4} \right] \cup [2t, +\infty). \quad (5.54)$$

Definition 5.9 For $T \geq T_0$, $t > 0$, put

$$\widetilde{P}_{t,T}^{[0,1]} = \frac{1}{2i\pi} \int_{tb_2} \frac{d\lambda}{\lambda - A_{t,T}^2}. \quad (5.55)$$

Set $\widetilde{F}_{t,T}^{[0,1]} = \text{Im} \left( \widetilde{P}_{t,T}^{[0,1]} \right)$, then $\widetilde{P}_{t,T}^{[0,1]}$ is an even projection operator and $\widetilde{F}_{t,T}^{[0,1]}$ is a finite $\Gamma$-dimensional subbundle of $\Lambda^*(T^*S) \otimes \Omega^*_2(\widetilde{X}, \widetilde{F}|\widetilde{X})$.

Proposition 5.10 The following identity holds,

$$\widetilde{P}_{t,T}^{[0,1]} = t^{-N^{\Lambda^*}(T^*\widetilde{M})/2} \widetilde{P}_{T}^{[0,1]} t^{-N^{\Lambda^*}(T^*\widetilde{M})/2}, \quad \widetilde{F}_{t,T}^{[0,1]} = t^{-N^{\Lambda^*}(T^*\widetilde{M})/2} \widetilde{F}_{T}^{[0,1]} \quad (5.56)$$

Proof This is a consequence of (5.53). \qed

Definition 5.11 For $T \geq T_0$, put

$$\widetilde{P}_{\infty,T}^{[0,1]} = \frac{1}{2i\pi} \int_{b_2} \frac{d\lambda}{\lambda - \frac{1}{4} \left[ \widetilde{A}'_{t}, d_{t}^*\widetilde{A}'_{t} \right]} \quad (5.57)$$

Then by the same argument of [4, Proposition 10.14], we have the following $L^2$-analogue of it.

Proposition 5.12 Given $\alpha \in \Lambda^*(T^*S) \otimes \Omega^*_2(\widetilde{X}, \widetilde{F}|\widetilde{X})$, as $t \rightarrow +\infty$,

$$\widetilde{P}_{t,T}^{[0,1]} \alpha = \widetilde{P}_{\infty,T}^{[0,1]} \alpha + O(1/t) \alpha \quad (5.58)$$

For $t > 0$, $T \geq 0$,

$$\tilde{C}_{t,T} = t^{N/2} \tilde{A}'_{t,T} t^{-N/2}. \quad (5.59)$$
Definition 5.13 Put
\[ \hat{P}_{t,T}^{[0,1]} = \frac{1}{2i\pi} \int_{\delta_2} \frac{d\lambda}{\lambda - C_{t,T}^2}. \] (5.60)

Set \( \hat{F}_{t,T}^{[0,1]} = \text{Im} \left( \hat{P}_{t,T}^{[0,1]} \right) \). Then \( \hat{P}_{t,T}^{[0,1]} \) is a projection operator and \( \hat{F}_{t,T}^{[0,1]} \) is a finite \( \Gamma \)-dimensional vector bundle.

Proposition 5.14 The following identities hold,
\[ \hat{P}_{t,T}^{[0,1]} = \psi_t^{-1} \hat{P}_{T}^{[0,1]} \psi_t, \]
\[ \hat{P}_{t,T}^{[0,1]} = t^{N/2} \hat{P}_{t,T}^{[0,1]} t^{-N/2}, \]
\[ \hat{F}_{t,T}^{[0,1]} = \psi_t^{-1} \hat{F}_{T}^{[0,1]} = t^{N/2} \hat{F}_{t,T}^{[0,1]} \]. (5.61)

Also, as \( t \to +\infty \),
\[ \hat{P}_{t,T}^{[0,1]} = P_T^{[0,1]} s + O \left( 1/\sqrt{t} \right). \] (5.62)

Proof By (2.18),
\[ \tilde{C}_{t,T} = \psi_t^{-1} \sqrt{t} \hat{A}_T \psi_t. \] (5.63)
From (5.42), (5.56) and (5.63), we get (5.61). By (5.46) and (5.61), we get (5.62). \( \square \)

5.5 The maps \( \tilde{P}_T^\infty \)

This section is the \( L^2 \)-set up of [4, Section 10.6].

Definition 5.15 Let \( \tilde{P}_T^\infty \) be the map
\[ \alpha \in \Omega_{(2)}^*(\tilde{X}, \tilde{F}|\tilde{X}) \to \tilde{P}_T^\infty \alpha = \sum_{x \in B} \tilde{W}^u(x)^* \int_{\tilde{W}^u(x)} \alpha \in C^*(W^u, \tilde{F}). \]
Similarly as [4, Definitions 5.2, 5.7], we have \( \tilde{P}_T^\infty : \Omega_{(2)}^*(\tilde{X}, \tilde{F}|\tilde{X}) \to C^*(W^u, \tilde{F}) \) and \( \tilde{P}_T^\infty : \Lambda^*(T^*S) \otimes \Omega_{(2)}(\tilde{X}, \tilde{F}|\tilde{X}) \to \Lambda^*(T^*S) \otimes C^*(W^u, \tilde{F}) \). Then \( \tilde{P}_T^\infty \) and \( \tilde{P}_T^\infty \) are chain maps which preserves the \( \mathbb{Z} \)-grading. Also \( \tilde{P}_T^\infty \) preserves the filtrations associated to \( \Lambda^*(T^*S) \).

Definition 5.16 For \( T \geq T_0 \), let \( \tilde{P}_T^\infty : \hat{F}_{t,T}^{[0,1]} \to \Lambda^*(T^*S) \otimes C^*(W^u, \tilde{F}) \), \( \tilde{P}_T^\infty : \hat{F}_{t,T}^{[0,1]} \to C^*(W^u, \tilde{F}) \) be the restrictions of \( \tilde{P}_T^\infty, \tilde{P}_T^\infty \) to \( \hat{F}_{t,T}^{[0,1]} \).

By the same proof of [4, Theorem 10.18], there exists \( T'_0 \geq T_0 \) such that for \( T \geq T'_0 \), \( \tilde{P}_T^\infty \) is an isomorphism of \( \mathbb{Z}_2 \)-graded filtered vector bundles, which commutes with the action of \( \Lambda^*(T^*S) \). Moreover, \( \tilde{P}_T^\infty \tilde{A}' = \tilde{A}^\infty (W^u, \tilde{F})/\tilde{P}_T^\infty \).

Observe that \( \tilde{P}_{t,T}^{[0,1]} - \tilde{P}_{t,T}^{[0,1]} \) contains only terms of positive degree in the Grassmann variables in \( \Lambda^*(T^*S) \).

Proposition 5.17 For \( T \geq T'_0 \), the following \( L^2 \)-analogue of [4, (10.78)] holds,
\[ \left( \tilde{P}_T^\infty \right)^{-1} = \tilde{P}_{t,T}^{[0,1]} \left( \tilde{P}_T^\infty \right)^{-1} \left( \tilde{P}_{t,T}^{[0,1]} \left( \tilde{P}_T^\infty \right)^{-1} \right)^{-1}, \left( \tilde{P}_T^\infty \right)^{-1,0} = \left( \tilde{P}_T^\infty \right)^{-1}. \] (5.64)
Proof First by (5.46), we get
\[
(\tilde{P}_T^\infty \tilde{P}_T^{0,[1]} (\tilde{P}_T^\infty)^{-1})^{(0)} = 1. \tag{5.65}
\]

By (5.65), $\tilde{P}_T^\infty \tilde{P}_T^{0,[1]} (\tilde{P}_T^\infty)^{-1}$ is one to one. Recall that $(\tilde{P}_T^\infty)^{-1}$ and $\tilde{P}_T^\infty$ preserve the total degree. By Theorem 5.8, $\tilde{P}_T^\infty$ increases the total degree. Therefore $\tilde{P}_T^\infty \tilde{P}_T^{0,[1]} (\tilde{P}_T^\infty)^{-1}$ increases the total degree. Using the invertibility of $\tilde{P}_T^\infty \tilde{P}_T^{0,[1]} (\tilde{P}_T^\infty)^{-1}$, the first equation in (5.64) follows. From (5.46) and (5.65), we get the second equation in (5.64). \(\square\)

5.6 The generalized metric $g^C(W^u, \bar{F})$

This section is the $L^2$-set up of [4, Section 10.7].

The map $(\tilde{P}_T^\infty)^{-1}$ identifies $\Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$ and $\tilde{P}_T^{0,[1]} \subset \Lambda^\bullet(T^*S) \otimes \Omega^\bullet_2(\tilde{X}, \bar{F}|\tilde{X})$. In the sequel, we will consider $(\tilde{P}_T^\infty)^{-1}$ as a map from $\Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$ into $\Lambda^\bullet(T^*S) \otimes \Omega^\bullet_2(\tilde{X}, \bar{F}|\tilde{X})$.

Let $(\tilde{P}_T^\infty)^{-1,*}$ be the adjoint of $(\tilde{P}_T^\infty)^{-1}$ with respect to the metrics $g^C(W^u, \bar{F})$, $g^\Omega_2(\tilde{X}, \bar{F}|\tilde{X})$. Then $(\tilde{P}_T^\infty)^{-1,*}$ maps $\Lambda^\bullet(T^*S) \otimes \Omega^\bullet_2(\tilde{X}, \bar{F}|\tilde{X})$ into $\Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$.

Definition 5.18 For $T \geq T_0'$, put
\[
g^C_T(W^u, \bar{F}) := (\tilde{P}_T^\infty)^{-1,*}(\tilde{P}_T^\infty)^{-1}, \quad g^\Omega_2(W^u, \bar{F}) := (\tilde{P}_T^\infty)^{-1,*} (\tilde{P}_T^\infty)^{-1}. \tag{5.66}
\]

Observe that $g^C_T(W^u, \bar{F})$ is a generalized metric on $C^\bullet(W^u, \bar{F})$ in the sense of [4, Section 2.9]. Also $g^\Omega_2(W^u, \bar{F})$ is a standard metric on $C^\bullet(W^u, \bar{F})$, which is such that the $C^i(W^u, \bar{F})$'s are orthogonal in $C^\bullet(W^u, \bar{F})$ with respect to $g^C_T(W^u, \bar{F})$.

The following $L^2$-analogue of [4, Theorem 10.21] obviously holds.

Theorem 5.19 For $T \geq T_0'$,
\[
g^C_T(W^u, \bar{F}), (0) = g^C_T(W^u, \bar{F}). \tag{5.67}
\]

Moreover,
\[
\tilde{P}_T^\infty \tilde{A}^\bullet \tilde{P}_T^\infty = \tilde{A}^\bullet, \quad \tilde{P}_T^\infty \tilde{U}^\bullet \tilde{P}_T^\infty = \left( g^C_T(W^u, \bar{F}) \right)^{-1} \tilde{A}^\bullet \left( g^C_T(W^u, \bar{F}) \right)^{-1} \tag{5.68}
\]

5.7 The maps $\tilde{P}_T^\infty$ and the generalized metrics $g^C_{t,T}(W^u, \bar{F})$

This section is the $L^2$-set up of [4, Section 10.8].

Let $\Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$ be the number operator of $\Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$.

Definition 5.20 Given $T \geq T_0$, $t > 0$, let $\tilde{P}_t^\infty : \tilde{F}_t^\infty \rightarrow \Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$ be the restriction of $\tilde{P}_t^\infty$ to $\tilde{F}_t^\infty$.

Set $\tilde{G}_t^\infty = \text{Im}(\tilde{P}_t^\infty)$. If $\tilde{P}_\infty : \tilde{F}_\infty \rightarrow \Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \bar{F})$ is the restriction of $\tilde{P}_t^\infty$ to $\tilde{F}_\infty$, then $\tilde{P}_\infty$ is an isomorphism. Also we have the following $L^2$-analogue of [4, Proposition 10.24].
Proposition 5.21 The map $\tilde{P}^T_{t,T}$ is invertible. Moreover,
\[
(\tilde{P}^T_{t,T})^{-1} = t^{-N\Lambda^*(T^*S)/2}(\tilde{P}^\infty_T)^{-1} t^{-N\Lambda^*(T^*S)\otimes C(W_u,\tilde{F})/2}.
\] (5.69)
As $t \to +\infty$,
\[
(\tilde{P}^\infty_{t,T})^{-1} = (\tilde{P}^\infty_{\infty,T})^{-1} + O(1/t).
\] (5.70)

Let $g_{t,T}^{\Omega^*_2(\tilde{X},\tilde{F}|x)}$ be the metric on $\Omega^*_2(\tilde{X},\tilde{F}|x)$ which is associated to the metrics $g^{T\tilde{X}}/t$, $g_{T\tilde{X}}$ on $T\tilde{X}$, $\tilde{F}$. Let $(\tilde{P}^\infty_{t,T})^{-1,*}$ be the adjoint of $(\tilde{P}^\infty_{t,T})^{-1}$ with respect to the metrics $g_{t,T}^{\Omega^*_2(\tilde{X},\tilde{F}|x)}$, $g^{C^*(W_u,\tilde{F})}$.

Let $(\tilde{P}^\infty_T)^{-1,*}$ be the adjoint of $(\tilde{P}^\infty_T)^{-1}$ with respect to the metrics $g_{T}^{\Omega^*_2(\tilde{X},\tilde{F}|x)}$, $g^{C^*(W_u,\tilde{F})}$.

Proposition 5.22 The following identity holds,
\[
t^{N/2}(\tilde{P}^\infty_{t,T})^{-1} t^{-N\Lambda^*(W_u,\tilde{F})/2} = \psi^{-1}_t (\tilde{P}^\infty_T) \psi_t.
\] (5.71)

Proof This follows from Proposition 5.21. □

Let $(\tilde{P}^\infty_{t,T})^{-1,*}_0$ be the adjoint of $(\tilde{P}^\infty_{t,T})^{-1}$ with respect to the metrics $g_{T}^{\Omega^*_2(\tilde{X},\tilde{F}|x)}$, $g^{C^*(W_u,\tilde{F})}$.

Proposition 5.23 There is a smooth section $\tilde{J}$ of
\[
(\Lambda^*(T^*S)\otimes \text{Hom}(C^*(W_u,\tilde{F}), \Omega^*_2(\tilde{X},\tilde{F}|x)))^{\text{even}},
\]
such that as $t \to +\infty$,
\[
t^{N/2}(\tilde{P}^\infty_{t,T})^{-1} t^{-N\Lambda^*(W_u,\tilde{F})/2} = (\tilde{P}^\infty_T)^{-1} \frac{\tilde{J}}{\sqrt{t}} + O(1/t).
\] (5.72)

Proof Using (5.44), (5.64) and (5.71), we get the first identity in (5.72). By taking adjoints, we obtain the second identity. □

Definition 5.24 Put
\[
\varrho_{t,T}^{C^*(W_u,\tilde{F})} = (\tilde{P}^\infty_{t,T})^{-1,*}_0 (\tilde{P}^\infty_{t,T})^{-1}.
\] (5.73)

Then $\varrho_{t,T}^{C^*(W_u,\tilde{F})}$ is a generalized metric on $C^*(W_u,\tilde{F})$.

Theorem 5.25 There is a smooth section $\tilde{H}$ of
\[
(\Lambda^*(T^*S)\otimes \text{End}(C^*(W_u,\tilde{F}))^{\text{even}},
\]
such that as $t \to +\infty$,
\[
t^{-N\Lambda^*(W_u,\tilde{F})/2 + 2n/2}\varrho_{t,T}^{C^*(W_u,\tilde{F})} t^{-N\Lambda^*(W_u,\tilde{F})/2} = \tilde{G}_T^{C^*(W_u,\tilde{F})} + \frac{\tilde{H}}{\sqrt{t}} + O(1/t),
\] (5.74)
\[
t^{N\Lambda^*(W_u,\tilde{F})/2} \left[ \varrho_{t,T}^{C^*(W_u,\tilde{F})} \right]^{-1} \frac{\partial}{\partial t} \left[ \varrho_{t,T}^{C^*(W_u,\tilde{F})} \right] t^{-N\Lambda^*(W_u,\tilde{F})/2}
\]
\[= \left( N^{C^*(W_u,\tilde{F})} - \frac{n}{2} \right) \frac{1}{t} + O(1/t^{3/2}).
\] (5.75)
By Proposition 5.23 and by (5.77), we get (5.74).

\[ \text{Then we have} \]

\[ I^{-N}C^*(w, \bar{f})/2, g_{t,T} = t^{-N}C^*(w, \bar{f})/2 \]

\[ = \left( I^{-N}C^*(w, \bar{f})/2, \bar{t} - n/2 \left( \text{Proposition 5.23} \right) - 1, N/2 \right) \left( t^{N/2} \left( \text{Proposition 5.23} \right) - 1, I^{-N}C^*(w, \bar{f})/2 \right). \]  

(5.77)

By Proposition 5.23 and by (5.77), we get (5.74).

Also

\[ g_{t,T}^2 C^*(w, \bar{f}) = t^{-N}C^*(w, \bar{f})/2 - n/2 \left[ I^{n/2} I^{-N}C^*(w, \bar{f})/2, g_{t,T}^2 C^*(w, \bar{f}) I^{-N}C^*(w, \bar{f})/2 \right] t^{N}C^*(w, \bar{f}). \]  

(5.78)

Therefore,

\[ \frac{\partial}{\partial t} C^*(w, \bar{f}) = \frac{N C^*(w, \bar{f}) - n}{2t} C^*(w, \bar{f}) + \frac{g_{t,T} C^*(w, \bar{f})}{2t} \]

\[ + t^{-N}C^*(w, \bar{f})/2 - n/2 \left( \frac{\partial}{\partial t} \left( I^{n/2} I^{-N}C^*(w, \bar{f})/2, g_{t,T}^2 C^*(w, \bar{f}) I^{-N}C^*(w, \bar{f})/2 \right) t^{N}C^*(w, \bar{f}). \]  

(5.79)

By (5.69), (5.72) and (5.77), one verifies easily that

\[ t^{n/2} I^{-N}C^*(w, \bar{f})/2, g_{t,T}^2 C^*(w, \bar{f}) I^{-N}C^*(w, \bar{f})/2 \]

is a polynomial in $1/\sqrt{t}$. Therefore, as $t \to +\infty$,

\[ \frac{\partial}{\partial t} \left( t^{n/2} I^{-N}C^*(w, \bar{f})/2, g_{t,T}^2 C^*(w, \bar{f}) I^{-N}C^*(w, \bar{f})/2 \right) = O(1/t^{3/2}). \]  

(5.80)

From (5.74), (5.79) and (5.80), we get (5.75). \hfill \Box

**Proposition 5.26** The following identity holds,

\[ t^{-N}C^*(w, \bar{f})/2, \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 = \sqrt{t} \tilde{\partial} + \nabla C^*(w, \bar{f}). \]  

(5.81)

Also as $t \to +\infty$,

\[ t^{-N}C^*(w, \bar{f})/2, \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 = \tilde{P}_{T}^\infty \sqrt{t} dX, * (\tilde{P}_{T}^\infty)^{-1} + O(1). \]  

(5.82)

**Proof** Identity (5.81) follows from (5.68). Also by (5.68), we have

\[ \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} = \left[ t^{-N}C^*(w, \bar{f})/2, \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 \right] \]

\[ = \left[ t^{-N}C^*(w, \bar{f})/2, \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 \right] \]

\[ \times \left( t^{-N}C^*(w, \bar{f})/2, A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 \right) \]

(5.83)

Then we have

\[ t^{-N}C^*(w, \bar{f})/2, \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 \]

\[ = \left[ t^{-N}C^*(w, \bar{f})/2, \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2 \right] \]

\[ \times \tilde{P}_{t,T}^\infty A^* (\tilde{P}_{t,T}^\infty)^{-1} I^{-N}C^*(w, \bar{f})/2. \]  

(5.84)
Using Theorem 5.25 and (5.84), we find that as \( t \to +\infty \),
\[
\begin{align*}
&\int_{L} \frac{d\lambda}{\sqrt{\lambda - B_t, T}}
\end{align*}
\]

which is equivalent to (5.82). The proof is completed.

**5.8 The superconnection forms for \( \bar{F}_T \)**

This section is the \( L^2 \)-set up of [4, Sections 10.10, 10.13 and 10.14].

Observe that
\[
\bar{F}_T \in \mathbb{H}.
\]

Using the holomorphic functional calculus, we find that
\[
\begin{align*}
&h(\bar{B}_t, T)\bar{F}_T = \int_{L} \frac{h(\lambda)}{\sqrt{\lambda - B_t, T}} d\lambda,
&h'(\bar{B}_t, T)\bar{F}_T = \int_{L} \frac{h'(\lambda)}{\sqrt{\lambda - B_t, T}} d\lambda.
\end{align*}
\]

Similarly,
\[
\begin{align*}
&h(\bar{D}_t, T)\bar{F}_T = \int_{L} \frac{h(\lambda)}{\sqrt{\lambda - D_t, T}} d\lambda,
&h'(\bar{D}_t, T)\bar{F}_T = \int_{L} \frac{h'(\lambda)}{\sqrt{\lambda - D_t, T}} d\lambda,
\end{align*}
\]

**Definition 5.27** For \( t \in \mathbb{R}_+^*, T \geq T_0^t \), put
\[
\begin{align*}
\bar{a}_t, T &= \sqrt{2i\pi} \varphi \text{Tr}_{\Gamma,s} \left[ h \left( \bar{B}_t, T \right) \bar{F}_T \right],
\bar{b}_t, T &= \frac{1}{2} \varphi \text{Tr}_{\Gamma,s} \left[ \left( N - \frac{n}{2} \right) h' \left( \bar{B}_t, T \right) \bar{F}_T \right].
\end{align*}
\]

In (5.89), we may replace \( \bar{B}_t, T \) by \( \bar{D}_t, T \) and \( \bar{F}_T \) by \( \bar{F}_T \). Then \( \bar{a}_t, T, \bar{b}_t, T \) are forms on \( S \).

Let \( \bar{F}_T \) be the vector bundle of \( L_2 \) sections of \( \wedge^* (T^* X) \otimes \tilde{F} \) along the fibers \( \tilde{X} \), and let \( \| \cdot \|_0 \) be the norm of \( \bar{F}_T \) associated to the Hermitian product (2.5). If \( L \in \mathcal{L}(\bar{F}_T) \), for \( p \geq 1 \), put
\[
\| L \|_{\Gamma,p} = \text{Tr}_{\Gamma} \left[ (L^* L)^{p/2} \right]^{1/p}.
\]

Then (5.90) defines a norm on a vector subspace of \( \mathcal{L}(\bar{F}_T) \). For \( p = 1 \), we get the \( \Gamma \)-trace class operators.
As in [4, (10.159)], given $t' > 0$, if $t$ is close enough to $t'$, instead of (5.87), we can write
\[
 h(B_t, T) \overline{h}_{t, T}^{0, 1} = \frac{1}{2i\pi} \int_{\sqrt{T}\delta_2} \frac{h(\lambda)}{\lambda - B_t, T} d\lambda, \tag{5.91}
\]
the key point being that the contour of integration in (5.91) does not depend on $t$.

For $T \geq T'_0$, there exists $d_1 > 0$ such that
\[
 |\text{Sp}(B_T)| \subset \delta_2 \cup (2d_1, +\infty). \tag{5.92}
\]

Proposition 5.28 For $T \geq T'_0$, there exist $C > 0$, $c > 0$ such that for $t \geq 1$,
\[
 \|\tilde{G}_{t, T}\|_{\Gamma, 1} \leq Ce^{-ct}. \tag{5.93}
\]

Proof Take $p \in \mathbb{N}$, $p > \dim \tilde{X}$. Let $h_p(\lambda)$ be the unique holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ such that
1. As $\lambda \rightarrow \pm i\infty$, $h_p(\lambda) \rightarrow 0$.
2. The following identity holds,
\[
 h_p^{(p-1)}(\lambda) = h(\lambda). \tag{5.95}
\]

Clearly, if $\lambda \in \Delta''$,
\[
 |\text{Re}(\lambda)| \leq \frac{1}{2} |\text{Im}(\lambda)|. \tag{5.96}
\]

Using (5.96), we find that there exist $C > 0$, $C' > 0$ such that if $\lambda \in \Delta''$,
\[
 |h_p(\sqrt{t}\lambda)| \leq C \exp(-C' t |\lambda|^2). \tag{5.97}
\]
Theorem 5.29

For \( T \), such that if \( C \) and the expansion only contains a finite number of terms. Then we find that there is \( t \to +\infty \), and as \( t \to +\infty \),

\[
\left( \lambda - \tilde{B}_T \right)^{-1} = \left( \lambda - \tilde{B}^{(0)}_T \right)^{-1} + \left( \lambda - \tilde{B}^{(1)}_T \right)^{-1} + \cdots
\]  

(5.99)

and the expansion only contains a finite number of terms. Then we find that there is \( C > 0 \) such that if \( \lambda \in \Delta'' \),

\[
\left\| \left( \lambda - \tilde{B}_T \right)^{-1} \right\|_\infty \leq C.
\]  

(5.100)

Fix \( \lambda_0 \in \Delta'' \). Since \( p > \dim \tilde{X} \), and \( \tilde{B}_T \) is a fiberwise elliptic operator of order 1,

\[
\left\| \left( \lambda - \tilde{B}_T \right)^{-1} \right\|_{\Gamma, p} < +\infty.
\]  

(5.101)

If \( \lambda \in \Delta'' \),

\[
\left( \lambda - \tilde{B}_T \right)^{-1} = \left( \lambda - \tilde{B}^{(0)}_T \right)^{-1} + \left( \lambda - \lambda_0 \right) \left( \lambda_0 - \tilde{B}_T \right)^{-1} \left( \lambda - \tilde{B}_T \right)^{-1}.
\]  

(5.102)

From (5.100)–(5.102), we find that if \( \lambda \in \Delta'' \),

\[
\left\| \left( \lambda - \tilde{B}_T \right)^{-1} \right\|_{\Gamma, p} \leq C (1 + |\lambda|) \left\| \left( \lambda_0 - \tilde{B}_T \right)^{-1} \right\|_{\Gamma, p} \leq C' (1 + |\lambda|).
\]  

(5.103)

Using (5.103), we find that if \( \lambda \in \Delta'' \),

\[
\left\| \left( \lambda - \tilde{B}_T \right)^{-p} \right\|_{\Gamma, 1} \leq C (1 + |\lambda|)^p.
\]  

(5.104)

From (5.92), (5.97), (5.98) and (5.104), we get (5.94). The proof is completed. \( \square \)

Let \( \nabla^*_{T} \left( \tilde{X}, \tilde{F}|\tilde{X} \right) \) be the adjoint connection of \( \nabla^* \left( \tilde{X}, \tilde{F}|\tilde{X} \right) \) with respect to the metric \( g_{L_2, T} \). As in [4, Definition 1.7], we define

\[
\begin{align*}
&h_{L_2} \left( \nabla^*_{T} \left( \tilde{X}, \tilde{F}|\tilde{X} \right) \right) \\
&= (2\pi)^{1/2} \varphi_{T, \Gamma} \left[ h \left( \frac{1}{2} \left( \nabla^*_{T} \left( \tilde{X}, \tilde{F}|\tilde{X} \right) \right) \right) \right].
\end{align*}
\]

Theorem 5.29

For \( T \geq T_0' \), the form \( \tilde{a}_{r, T} \) is odd and closed, and the form \( \tilde{b}_{r, T} \) is even. Moreover,

\[
\frac{\partial}{\partial t} \tilde{a}_{r, T} = d \frac{\tilde{b}_{r, T}}{t}.
\]  

(5.105)

Also as \( t \to +\infty \), there exists \( \gamma' > 0 \) such that

\[
\tilde{a}_{r, T} = h_{L_2} \left( \nabla^*_{T} \left( \tilde{X}, \tilde{F}|\tilde{X} \right) \right) + O(t^{-\gamma'}),
\]

\[
\tilde{b}_{r, T} = \frac{1}{2} \chi' (\tilde{F}) - \frac{n}{4} \chi (F) + O(t^{-\gamma'}).
\]  

(5.106)
**Proof** By the same argument of [4, Theorem 10.37], we have that the form \( \widetilde{\eta}_T \) is odd and closed, the form \( \widehat{\eta}_T \) is even and (5.105) holds.

By (5.93), Proposition 5.28, [2, Theorem 4.1] and [30, Theorem 4.4], we get the first formula in (5.106). One can get the second formula in (5.106) similarly.

Set
\[
\widetilde{c} = \widetilde{\alpha}_t + \frac{dt}{t} \widetilde{b}_t ,
\]
then \( \widetilde{c} \) is closed.

**Definition 5.30** For \( T \geq T_0' \), put
\[
\widetilde{S}_h^{[0,1]}(T) = - \int_1^{+\infty} (\widetilde{b}_t - \widetilde{b}_t, 0) \frac{dt}{t} .
\]

The following \( L^2 \)-analogue of [4, Proposition 10.41] holds.

**Proposition 5.31** The following identity holds,
\[
\widetilde{S}_h^{[0,1]}(T) = - \int_1^{+\infty} \left( \varphi \text{Tr}_{\Gamma_s} [N \widetilde{L}_t, T] - \chi' (\widetilde{F}) \right) \frac{dt}{2t} .
\]

Let \( \widetilde{C}^*(W_u, \widetilde{F})' \) be the adjoint of \( \widetilde{C}^*(W_u, \widetilde{F}) \) with respect to \( \varphi_{t, T} \). Set (cf. [4, (2.134)])
\[
b_{t, T} = t \varphi \text{Tr}_{\Gamma_s} \left[ 1 \left( \varphi_{t, T} \right) \frac{1}{2} \frac{\partial}{\partial t} \varphi_{t, T} \right] .
\]

By the same proof of Theorem 5.29, as in [4, Definition 2.49], for \( T \geq T_0' \) we can define
\[
U_{L^2, h} (\widetilde{C}^*(W_u, \widetilde{F})', \varphi_{t, T}) = - \int_1^{+\infty} (b_{t, T} - b_{t, T}) \frac{dt}{t} .
\]

Then for \( T \geq T_0' \), in \( \Omega^*(S)/d\Omega^*(S) \), we have
\[
\widetilde{S}_h^{[0,1]}(T) = U_{L^2, h} (\widetilde{C}^*(W_u, \widetilde{F})', \varphi_{t, T}) .
\]

Let \( \widetilde{A}^*(W_u, \widetilde{F})' \) be the adjoint of \( \widetilde{A}^*(W_u, \widetilde{F}) \) with respect to \( G_{t, T} \). Let \( \eta_T \) be the even form associated to \( \widetilde{A}^*(W_u, \widetilde{F}) \) defined by (cf. [4, (2.143)])
\[
\eta_T = \varphi \left\{ \frac{1}{2} \text{Tr}_{\Gamma_s} \left[ \frac{1}{2} \left( N \widetilde{C}^*(W_u, \widetilde{F}) + \left( G_{t, T} \right)^{-1} N \widetilde{C}^*(W_u, \widetilde{F}) G_{t, T} \right) \right] \right\} .
\]

**Definition 5.32** For \( u \in \mathbb{R}^* \), let \( \eta_{t, u}^C(W_u, \widetilde{F}) \) be the generalized metric on \( C^*(W_u, \widetilde{F}) \),
\[
\eta_{t, u}^C(W_u, \widetilde{F}) = u N \widetilde{C}^*(W_u, \widetilde{F}) G_{t, T} \frac{N \widetilde{C}^*(W_u, \widetilde{F})}{2} .
\]
Let \( \widetilde{AC}^* (W^u, \widetilde{F}) \) be the adjoint of \( \widetilde{AC}^* (W^u, F) \) with respect to \( b T,u \). Put

\[
\widetilde{B}^*_{T,u} = \frac{1}{2} \left( \widetilde{AC}^* (W^u, \widetilde{F}) - \widetilde{AC}^* (W^u, F) \right)
\]  

(5.112)

Let \( N^{H^*_2} (\bar{X}, \bar{F} | \bar{Y}) \) be the number operator of \( H^*_2 (\bar{X}, \bar{F} | \bar{Y}) \).

By the same argument in [4, Theorem 10.49], the following identity holds in \( \Omega^* (S) / d\Omega^* (S) \),

\[
-U_{L^2, \lambda} \left( \widetilde{AC}^* (W^u, \widetilde{F}), \mathcal{G}_{T,u} \right) - \tilde{h}_{L^2} \left( \nabla^{H^*_2} (\bar{X}, \bar{F} | \bar{Y}), g_{L^2,0}, g_{L^2,T} \right)
\]

\[
= \varphi \left\{ \int_0^1 \left( \nabla^{C^* (W^u, \widetilde{F})} \right) \left( \nabla^{C^* (W^u, F)} \right) - \tilde{h}_{L^2} \right\} \left( \nabla^{H^*_2} (\bar{X}, \bar{F} | \bar{Y}), g_{L^2,0}, g_{C^* (W^u, \widetilde{F})} \right)
\]

(5.113)

**Theorem 5.33** As \( T \to +\infty \),

\[
\int_0^1 \left( \nabla^{C^* (W^u, \widetilde{F})} \right) \left( \nabla^{C^* (W^u, F)} \right) - \tilde{h}_{L^2} \left( \nabla^{C^* (W^u, \widetilde{F})}, g_{C^* (W^u, \widetilde{F})} \right) - \text{Tr}_s [f] T
\]

\[
- \frac{1}{4} \left( \tilde{\chi}'^+ (F) - \tilde{\chi}'^- (F) \right) \log (T) - \frac{1}{4} \left( \tilde{\chi}'^- (F) - \tilde{\chi}'^+ (F) \right) \log (\pi).
\]  

(5.114)

The rest part of this section is devoted to prove Theorem 5.33.

Recall that the operator \( \overline{C}^*_T \) was constructed in Sect. 5.1.

Given \( n' \in \mathbb{N} \), let \( Q^{n'} (T \bar{X}_u | B) \) be the algebra of polynomials of degree \( n' \) on \( T \bar{X}_u | B \).

**Definition 5.34** For \( T \geq 1 \), put

\[
\overline{F}_T = \frac{1}{2 \pi} \int_{\delta_2} \frac{d\lambda}{\lambda - \overline{C}^*_T}.
\]  

(5.115)

Then \( \overline{F}_T \) is a projection acting on \( \Lambda^* (T^* S) \otimes \bar{F} | B \) with finite \( \Gamma \)-dimensional range \( \overline{F}_T \). Then by [28, Theorem XII.5], for \( k \in \mathbb{N} \) large enough,

\[
\overline{F}_T = \ker D^*_T \otimes \bar{F} | B^k.
\]  

(5.116)

As [4, (10.191)], for \( n' \) large enough, we also have

\[
\overline{F}_T \subset \exp \left( -T |Z|^2 / 2 \right) Q^{n'} (T \bar{X}_u | B) \otimes \Lambda^* (T^* S) \otimes \Lambda^* (T^* \bar{X} | B) \otimes \bar{F} | B.
\]  

(5.117)
5.9 The maps $\tilde{J}_T$ and $\bar{e}_T$

This section is the $L^2$-set up of [4, Section 10.15].

Put

$$P_T^{(0.1)} = \frac{1}{2i\pi} \int_{\delta_2} \frac{d\lambda}{\lambda - \lambda_T^2}. \quad (5.118)$$

Then

$$P_T^{(0.1)} = e^{-T \tilde{J}_T} P_T^{(0.1)} e^{T \tilde{J}_T}. \quad (5.119)$$

So $P_T^{(0.1)}$ is a projector acting on $\Lambda^*(T^*S) \hat{\otimes} \Omega^*_{(2)}(\tilde{X}, \tilde{F}|\tilde{X})$.

Let $\gamma: \mathbb{R} \to [0, 1]$ be a smooth function such that

$$\gamma(a) = 1 \text{ for } a < \frac{1}{2}, \quad = 0 \text{ for } a > 1. \quad (5.120)$$

Let $\epsilon_0$ be chosen as in [4, Section 9.1]. If $Z \in \mathbb{R}^n$, set

$$\mu(Z) = \gamma(|Z|/\epsilon_0). \quad (5.121)$$

Then

$$\mu(Z) = 1 \text{ if } |Z| \leq \epsilon_0/2, \quad = 0 \text{ if } |Z| \geq \epsilon_0. \quad (5.122)$$

If $T > 0$, set

$$\alpha_T = \int_{\mathbb{R}^n} \mu^2(Z) \exp(-T|Z|^2) dZ. \quad (5.123)$$

Then there is $c > 0$ such that as $T \to +\infty$,

$$\alpha_T = \left( \frac{\pi}{T} \right)^{n/2} + O(e^{-cT}). \quad (5.124)$$

Take $x \in \tilde{B}$. Let $\rho_x \in \Lambda^{\max}(T^u_x, \tilde{X}|\tilde{B}) \otimes \mathbb{C}$ be of norm 1. Then $\rho_x$ is determined up to sign. It defines a section of $o(x) \otimes \Lambda^{\max}(T^u_x, \tilde{X}|\tilde{B})$.x.

**Definition 5.35** Let $\tilde{J}_T: \Lambda^*(T^*S) \hat{\otimes} C^*(W^u, \tilde{F}) \to \Lambda^*(T^*S) \hat{\otimes} \Omega^*_{(2)}(\tilde{X}, \tilde{F}|\tilde{X})$ be such that if $h \in o(x) \otimes \tilde{F}_x$, then

$$\tilde{J}_T h = \frac{\mu(Z)}{\alpha_T^{1/2}} \frac{1}{P_T^{(0.1)}} \left[ \exp(-T|Z|^2/2) \rho_x \right] h. \quad (5.125)$$

The induced map $\tilde{J}_T: \Lambda^*(W^u, \tilde{F}) \to \Omega^*_{(2)}(\tilde{X}, \tilde{F}|\tilde{X})$ is given by

$$\tilde{J}_T h = \frac{\mu(Z)}{\alpha_T^{1/2}} \exp(-T|Z|^2/2) \rho_x h. \quad (5.126)$$

**Definition 5.36** Let $\bar{e}_T: \Lambda^*(T^*S) \hat{\otimes} C^*(W^u, \tilde{F}) \to \bar{P}_T^{(0.1)}$ be given by

$$\bar{e}_T = P_T^{(0.1)} \tilde{J}_T. \quad (5.127)$$
The induced map \( \overline{\tau}_T : C^\bullet(W^u, \tilde{F}) \to \Omega^2_{(2)}(\tilde{X}, \tilde{F}|\tilde{X}) \) is given by
\[
\overline{\tau}_T = \overline{P}_{T}^{[0,1]} \tilde{\tau}_T.
\] (5.128)

In the sequel, we write that as \( T \to +\infty \), a family of smooth sections on \( \tilde{M} \) is \( O(e^{-cT}) \) if the sup norm of the derivatives is \( O(e^{-cT}) \). By the same argument of [4, Theorem 10.56], we have the following \( L^2 \)-extension.

**Theorem 5.37** There is \( c > 0 \) such that as \( T \to +\infty \), for any \( s \in C^\bullet(W^u, \tilde{F}) \),
\[
(\overline{\tau}_T - \tilde{\tau}_T) s = O(e^{-cT}) \text{ uniformly on } \tilde{M}.
\] (5.129)

**Proof** By (5.118) and holomorphic functional calculus, we know that for any \( k \in \mathbb{N}^* \),
\[
\overline{P}_{T}^{[0,1]} = \frac{1}{2i\pi} \int_{\delta_2} \frac{d\lambda}{\lambda - \overline{A}_T^2}.
\] (5.130)

In the sequel, we choose \( k \in \mathbb{N}^* \) large enough so that (5.116) holds.

Take \( x \in \tilde{B}, h \in o(x) \otimes \tilde{F}_x \). If \( \lambda \in \delta_2 \),
\[
\left( \lambda - \overline{A}_T^2 \right) \frac{\tilde{J}_T h}{\lambda} - \tilde{J}_T h = -\overline{A}_T^2 \frac{\tilde{J}_T h}{\lambda},
\] (5.131)
and so,
\[
\frac{\tilde{J}_T h}{\lambda} - \left( \lambda - \overline{A}_T^2 \right)^{-1} \tilde{J}_T h = -\left( \lambda - \overline{A}_T^2 \right)^{-1} \overline{A}_T^2 \frac{\tilde{J}_T h}{\lambda}.
\] (5.132)

Now, by the fundamental assumptions in [4, Section 9.1], with the required identifications, on \( \{x' \in \tilde{X}, d\tilde{x}(x, x') \leq \varepsilon \} \), the operator \( \overline{A}_T^2 \) coincides with \( \overline{C}_T \otimes \tilde{F}|\tilde{X} \). Since \( \mu(Z) = 1 \) for \( |Z| \leq \varepsilon/2 \), using (5.116), we get
\[
\overline{A}_T^2 \frac{\tilde{J}_T h(Z)}{\lambda} = 0 \text{ for } |Z| \leq \varepsilon/2.
\] (5.133)

By (5.118), we deduce from (5.133) that there exists \( c > 0 \) such that as \( T \to +\infty \),
\[
|\overline{A}_T^2 \tilde{J}_T h| = O\left(e^{-cT}\right)|h|.
\] (5.134)

Since
\[
\tilde{A} = \frac{1}{2}D\tilde{x} + \frac{1}{2} \left( \nabla^{\Lambda^2}_{(2)}(\tilde{x}, \tilde{F}|\tilde{x}) + \nabla^{\Lambda^2}_{(2)}(\tilde{x}, \tilde{F}|\tilde{x}) \right) - \frac{1}{2}c(T^H),
\]
we see that
\[
\overline{A}_T^2 \frac{\tilde{J}_T h}{\lambda} = \left( \frac{D\tilde{x}}{2} \right)^{2k} \tilde{K}_T,
\] (5.135)
where \( \tilde{K}_T \) is a differential operator of order \( 2k - 1 \), whose coefficients depend polynomially on \( T \), the polynomial being of degree \( 2k \).

If \( q \in \mathbb{N} \), let \( \| \cdot \|_q \) be the norm on the fiberwise \( q \)-th Sobolev space of sections of \( \Lambda^\bullet(T^s\tilde{X}) \otimes \tilde{F} \) defined in [9, Section 2].

Since \( D\tilde{x}^m \) is elliptic of degree \( 2k \), given \( q \in \mathbb{N} \), by [29, Lemma 1.4] (see Lemma A.1), there exists \( C > 0 \) such that for \( s \in \Lambda^\bullet(T^s\tilde{X}) \otimes \tilde{F}|\tilde{X} \),
\[
\|s\|_{q+2k} \leq C \left( \left\| D\tilde{x}^m s \right\|_q + \|s\|_0 \right).
\] (5.136)
By the considerations which follow (5.135) and (5.136), we see that given \( q \in \mathbb{N} \), there exists \( C > 0 \) such that for \( \lambda \in \delta_2 \), \( T \geq 1 \),
\[
\|s\|_{q+2k} \leq C \left( \left\| \left( \lambda - \bar{A}_T^{2k} \right) s \right\|_q + T^{2k} \|s\|_{q+2k-1} \right). \tag{5.137}
\]

Also given \( q \in \mathbb{N} \), there exists \( C > 0 \) such that for \( A > 0 \), \( s \in \Lambda^*(T^*S) \otimes \Omega^*_\varepsilon(\tilde{X}, \tilde{F}_{|\tilde{X}}) \),
\[
\|s\|_{q+2k-1} \leq C \left( \left\| \frac{s}{A} \right\|_q + A^{q+2k-1} \|s\|_0 \right). \tag{5.138}
\]

From (5.137), (5.138), we deduce that there exists \( C > 0 \), \( k' \in \mathbb{N} \) such that for \( \lambda \in \delta_2 \), \( s \in \Lambda^*(T^*S) \otimes \Omega^*_\varepsilon(\tilde{X}, \tilde{F}_{|\tilde{X}}) \),
\[
\|s\|_{q+2k} \leq C \left( \left\| \left( \lambda - \bar{A}_T^{2k} \right) s \right\|_q + T^{2k} \|s\|_0 \right). \tag{5.139}
\]

Also by (5.20), for \( T \geq T_0 \), we know that \( \text{Sp}(\bar{A}_T^{2k, (0)}) \cap \delta_2 = \emptyset \). More precisely, since \( \bar{A}_T^{2k, (0)} \) is self-adjoint, by (5.20), there exists \( C' > 0 \) such that for \( T \geq T_0 \), \( \lambda \in \delta_2 \), \( s \in \Omega^*_\varepsilon(\tilde{X}, \tilde{F}_{|\tilde{X}}) \),
\[
\left\| \left( \lambda - \bar{A}_T^{2k, (0)} \right)^{-1} s \right\|_0 \leq C' \|s\|_0. \tag{5.140}
\]

By (5.139), (5.140), we get
\[
\left\| \left( \lambda - \bar{A}_T^{2k, (0)} \right)^{-1} s \right\|_{q+2k} \leq C'T^{2k'} \|s\|_{q}. \tag{5.141}
\]

Moreover,
\[
\left( \lambda - \bar{A}_T^{2k} \right)^{-1} = \left( \lambda - \bar{A}_T^{2k, (0)} \right)^{-1} + \left( \lambda - \bar{A}_T^{2k, (0)} \right)^{-1} \bar{A}_T^{2k, (0)} \left( \lambda - \bar{A}_T^{2k, (0)} \right)^{-1} + \cdots \tag{5.142}
\]

and the expansion in (5.142) contains a finite number of terms. Also by (5.36), \( \bar{A}_T^{2k, (0)} \) is a differential operator of order \( 2k - 1 \), which depends polynomially on \( T \). By (5.141), (5.142), we find that there exists \( C'' > 0 \), \( k'' \in \mathbb{N} \) such that for \( \lambda \in \delta_2 \), \( T \geq T_0 \), \( s \in \Omega^*_\varepsilon(\tilde{X}, \tilde{F}_{|\tilde{X}}) \),
\[
\left\| \left( \lambda - \bar{A}_T^{2k} \right)^{-1} s \right\|_{q+2k} \leq C''T^{2k'} \|s\|_{q}. \tag{5.143}
\]

From (5.134), (5.143), we deduce that there exists \( c > 0 \) such that for \( T \geq T_0 \),
\[
\left\| \left( \lambda - \bar{A}_T^{2k} \right) \bar{A}_T^{2k} \mathbf{J}_T h \right\|_{q+2k} = O \left( e^{-cT} \right) \|h\|. \tag{5.144}
\]

Using (5.144) and Sobolev’s inequalities (cf. [16]), we see that there exists \( c > 0 \) such that for \( \lambda \in \delta_2 \), \( T \geq T_0 \), \( x \in \tilde{B} \), \( h \in \tilde{F}_x \),
\[
\left\| \left( \lambda - \bar{A}_T^{2k} \right)^{-1} \bar{A}_T^{2k} \mathbf{J}_T h \right\|_{q+2k} = O \left( e^{-cT} \right) \|h\|. \tag{5.145}
\]

From (5.130), (5.132), (5.145), we get (5.129). The proof is completed. \( \Box \)
Definition 5.38  For \( T \geq T_0 \), let \( \tilde{e}_T : \Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \tilde{F}) \to \tilde{F}_T^{[0,1]} \) be the linear map,

\[
\tilde{e}_T = e^T \tilde{f} \tilde{e}_T .
\]  

(5.146)

By (5.119), (5.127), we get

\[
\tilde{e}_T = \tilde{F}_T^{[0,1]} e^T \tilde{f} \tilde{e}_T .
\]  

(5.147)

Then \( \tilde{e}_T \) commutes with \( \Lambda^\bullet(T^*S) \). The induced map \( \tilde{e}_T : C^\bullet(W^u, \tilde{F}) \to \Omega^\bullet(\tilde{X}, \tilde{F}|\tilde{X}) \) is given by,

\[
\tilde{e}_T = \tilde{F}_T^{[0,1]} e^T \tilde{f} \tilde{e}_T .
\]  

(5.148)

In the sequel, we consider \( \tilde{e}_T \) as a linear map from \( \Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \tilde{F}) \) into \( \Lambda^\bullet(T^*S) \otimes \Omega^\bullet(\tilde{X}, \tilde{F}|\tilde{X}) \). Let \( \tilde{e}_T : \Lambda^\bullet(T^*S) \otimes \Omega^\bullet(\tilde{X}, \tilde{F}|\tilde{X}) \to \Lambda^\bullet(T^*S) \otimes C^\bullet(W^u, \tilde{F}) \) be the adjoint of \( \tilde{e}_T \) with respect to the metrics \( g_{\Lambda^\bullet(T^*S)} \), \( g_{\Omega^\bullet(\tilde{X}, \tilde{F}|\tilde{X})} \).

Recall that \( C^\bullet(W^u, \mathcal{R}) = \oplus_{x \in \tilde{B}} o(x) \). We will denote by \( \mathcal{O}_D(1/T) \) an element of \( \Lambda^\bullet(T^*S) \otimes \text{End}(C^\bullet(W^u, \mathcal{R})) \) which commutes with \( \Lambda^\bullet(T^*S) \), which is of positive degree in \( \Lambda^\bullet(T^*S) \), which preserves the \( \Lambda^\bullet(T^*S) \otimes o(x) \), which is \( \mathcal{O}(1/T) \) as \( T \to +\infty \). Of course \( \mathcal{O}_D(1/T) \) then acts on \( C^\bullet(W^u, \tilde{F}) \), and preserves the \( o(x) \otimes \tilde{F}_x \), \( x \in \tilde{B} \). Also the various \( \mathcal{O}_D(1/T) \) commute with each other.

Using Theorem 5.37, the following \( L^2 \)-extension of [4, Proposition 10.58] can be proved using the same argument of [4, Proposition 10.58].

Proposition 5.39  As \( T \to +\infty \),

\[
\tilde{e}_T^* \tilde{e}_T = 1 + \mathcal{O}_D(1/T) + \mathcal{O}\left(e^{-cT}\right). \]

(5.149)

Clearly, \( \tilde{F}_T^{\infty} \tilde{e}_T \in \Lambda^\bullet(T^*S) \otimes \text{End}(C^\bullet(W^u, \tilde{F})) \). Let \( \mathcal{F} : C^\bullet(W^u, \tilde{F}) \to C^\bullet(W^u, \tilde{F}) \) be acting on \( \tilde{F}_x \otimes o(x) \), \( x \in \tilde{B} \), by multiplication by \( \tilde{f}(x) \). Also we still denote by \( N : C^\bullet(W^u, \tilde{F}) \to C^\bullet(W^u, \tilde{F}) \) the operator acting on \( C^i(W^u, \tilde{F}) \) by multiplication by \( i \). The following \( L^2 \)-extension of [4, Theorem 10.59] can be proved by the same argument of [4, Theorem 10.59].

Theorem 5.40  There exists \( c > 0 \) such that as \( T \to +\infty \),

\[
\tilde{P}_T^{\infty} \tilde{e}_T = e^{T \mathcal{F}} \left( \frac{\pi}{T} \right)^{N_{C^i(W^u, \tilde{F})}/2-\eta/4} \left( 1 + \mathcal{O}_D(1/T) + \mathcal{O}\left(e^{-cT}\right) \right). \]

(5.150)

In particular for \( T \geq T_0 \), \( \tilde{P}_T^{\infty} \tilde{e}_T \in (\Lambda^\bullet(T^*S) \otimes \text{End}(C^\bullet(W^u, \tilde{F}))) \) is invertible.

5.10 A proof of the first part of Theorem 5.33

This section is the \( L^2 \)-set up of [4, Section 10.16].

By Theorem 5.40, for \( T \geq T_0 \) large enough, the map \( \tilde{P}_T^{\infty} \tilde{e}_T \) is invertible. Therefore, for \( T \) large enough,

\[
g_T^{C^\bullet(W^u, \tilde{F})} = (\tilde{P}_T^{\infty} \tilde{e}_T)^{-1} (\tilde{e}_T^* \tilde{e}_T) (\tilde{P}_T^{\infty} \tilde{e}_T)^{-1}. \]

(5.151)
By (5.111),
\[
\begin{align*}
& e^{-T\mathcal{F}} \sum C^*(W^u,\bar{F}) A e^{T\mathcal{F}} = \sqrt{\det e^{-T\mathcal{F}}} e^{-T\mathcal{F}} + \nabla C^*(W^u,\bar{F}) + T d\mathcal{F}, \\
& e^{-T\mathcal{F}} \sum C^*(W^u,\bar{F}) A_{T,t} e^{T\mathcal{F}} = \left( \sum_{i} e^{-T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}} \right)^{-1} \left( \sqrt{T} \partial^* + \nabla C^*(W^u,\bar{F}) \right) g^{T}(W^u,\bar{F}) e^{T\mathcal{F}}.
\end{align*}
\]
(5.152)

Set
\[
\tilde{k}_T = e^{-T\mathcal{F}} \tilde{p}_T e^{T\mathcal{F}}.
\]
(5.153)

Clearly, by (5.151) and (5.153), we get
\[
e^{T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}} = (\tilde{k}_T)^{-1} \left( \sum_{i} e^{T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}} \right) (\tilde{k}_T)^{-1}.
\]
(5.154)

From (5.151)–(5.154), we obtain,
\[
e^{-T\mathcal{F}} \sum C^*(W^u,\bar{F}) A e^{T\mathcal{F}} = \left( \sum_{i} e^{-T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}} \right)^{-1} \left( \sqrt{T} \partial^* e^{-T\mathcal{F}} + \nabla C^*(W^u,\bar{F}) - T d\mathcal{F} \left( \tilde{k}_T \left( \sum_{i} e^{T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}} \right) \right)^{-1} \tilde{k}_T \right)^{-1}.
\]
(5.155)

Also by Proposition 5.39 and Theorem 5.40, as \( T \to +\infty \),
\[
\tilde{k}_T (\sum_{i} e^{T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}})^{-1} \tilde{k}_T^* = \left( \frac{\pi}{T} \right)^{N C^*(W^u,\bar{F}) - n/2} (1 + O_D(1/T)) + O \left( e^{-cT} \right).
\]
(5.156)

By (5.156), we deduce that
\[
\tilde{k}_T (\sum_{i} e^{T\mathcal{F}} g^{T}(W^u,\bar{F}) e^{T\mathcal{F}})^{-1} \tilde{k}_T^* = \left( \sqrt{1 + O_D(1/T)} + O \left( e^{-cT} \right) \right).
\]
(5.157)

Similar as \( [4, (10.251)] \), there exists \( c > 0 \) such that as \( T \to +\infty \),
\[
e^{-T\mathcal{F}} \partial e^{-T\mathcal{F}} = O \left( e^{-cT} \right), \quad e^{T\mathcal{F}} \partial^* e^{-T\mathcal{F}} = O \left( e^{-cT} \right).
\]
(5.158)

From (5.152)–(5.158), we deduce that given \( t \in (0, 1] \), as \( T \to +\infty \),
\[
e^{-T\mathcal{F}} \sum C^*(W^u,\bar{F}) A e^{T\mathcal{F}} \sum C^*(W^u,\bar{F}) A_{T,t} e^{T\mathcal{F}} = \nabla C^*(W^u,\bar{F}),
\]
(5.159)

By \([4, (2.143)]\) and (2.150),
\[
\int_{0}^{t} \left( \phi \text{Tr}_{T,s} \left[ \frac{1}{2} \left( g^{T}(W^u,\bar{F}) \right)^{-1} \left( \partial T \right) \left[ \nabla C^*(W^u,\bar{F}) \right] \right] - \frac{T}{t} \right) dt
\]
\[
= \int_{0}^{t} \phi \text{Tr}_{T,s} \left[ \left( g^{T}(W^u,\bar{F}) \right)^{-1} \left( \nabla C^*(W^u,\bar{F}) \right) \right] dt - \frac{T}{t}.
\]
(5.160)
Also by (5.154),
\[ e^{-T\mathcal{F}} \left( N^* C^*(W^u, \tilde{F}) + \left( \mathcal{G}^*_T C^*(W^u, \tilde{F}) \right)^{-1} N^* C^*(W^u, \tilde{F}) \right) e^{T\mathcal{F}} = N^* C^*(W^u, \tilde{F}) + \tilde{k}_T (e_T e_T) N^* C^*(W^u, \tilde{F}) (k_T (e_T e_T)^{-1} k_T^{-1})^{-1}. \] (5.161)

By (5.156) and (5.161), it is clear that
\[ \tilde{k}_T (e_T e_T) N^* C^*(W^u, \tilde{F}) (k_T (e_T e_T)^{-1} k_T^{-1})^{-1} \]
remains uniformly bounded as \( T \to +\infty \).

From (5.159), and from the above boundedness result, we see that as \( T \to +\infty \), the integrand in the right-hand side of (5.160) tends to 0. Moreover, by (5.159), we can use dominated convergence in this integral, which tends to 0 as \( T \to +\infty \). We have thus established the first convergence result in Theorem 5.33.

5.11 A proof of the second part of Theorem 5.33

This section is the \( L^2 \)-set up of [4, Section 10.17].

Clearly,
\[ \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}), \mathcal{G}^*_T C^*(W^u, \tilde{F}), g C^*(W^u, \tilde{F}) \right) \]
\[ = \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}), g C^*(W^u, \tilde{F}), e^{-2T\mathcal{F}} g C^*(W^u, \tilde{F}) \right) \]
\[ + \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}), e^{-2T\mathcal{F}} g C^*(W^u, \tilde{F}), g C^*(W^u, \tilde{F}) \right). \] (5.162)

By (4.1), we have
\[ \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}), e^{-2T\mathcal{F}} g C^*(W^u, \tilde{F}), g C^*(W^u, \tilde{F}) \right) = T \text{Tr}_3 [f]. \] (5.163)

Also,
\[ \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}), g C^*(W^u, \tilde{F}), e^{-2T\mathcal{F}} g C^*(W^u, \tilde{F}) \right) \]
\[ = \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}) + T \mathcal{F}, e^{-2T\mathcal{F}} g C^*(W^u, \tilde{F}) e^{T\mathcal{F}}, g C^*(W^u, \tilde{F}) \right). \] (5.164)

Using (5.154), (5.156), (5.157) and (5.164), we find that as \( T \to +\infty \),
\[ \tilde{h}_{L^2} \left( \nabla^* C^*(W^u, \tilde{F}), g C^*(W^u, \tilde{F}), e^{-2T\mathcal{F}} g C^*(W^u, \tilde{F}) \right) \]
\[ + \frac{1}{4} (\bar{\varphi}^-(F) - \bar{\varphi}^+(F)) \log(T) \to \frac{1}{4} (\bar{\varphi}^- (F) - \bar{\varphi}^+(F)) \log(\pi). \] (5.165)

From (5.162)–(5.165), we get the second equation in (5.114). The proof of Theorem 5.33 is completed. Then by Theorem 5.6, Proposition 5.31, (5.110), (5.113) and Theorem 5.33, we get Theorem 4.3.

6 A proof of Theorem 4.2

By [4, Section 5.5], we may assume that \( f \) is fiberwise nice, also we may assume that the conditions in [4, Section 9.1] hold. Then the conditions in [4, Section 11.1] hold.
Let $d^\nabla X$ be the adjoint of $dX$ with respect to the metrics $g_T^nX, g_T^nX$. Let $\nabla^\Omega_{(X^*)}(X,F|\nabla X)$ be the corresponding adjoint connection to $F$ on $\nabla_2^{\Omega_{(X^*)}(X,F|\nabla X)}$, and let $\nabla^\Omega_{(2)}(X,F|\nabla X)$ to $\nabla$ with respect $g_T^nX, g_T^nX$.

Recall that we have defined $\nabla T, \nabla T, \nabla T, \nabla T$ in (5.14) and (5.16).

Then we have the following basis of $TX$.

Put (cf. [4, (3.44)])

$$\nabla T = \left\{ \lambda \in \mathbb{R}, \frac{1}{4} |\lambda| \leq c_1 \sqrt{T}, |\lambda| \geq \frac{1}{8} \right\}. \quad (6.1)$$

Then we have the following $L^2$-analogue of [4, Theorems 11.16 and 11.17]. The proofs are the same as [4, Theorems 11.16 and 11.17].

**Theorem 6.1** For $c_1 \in (0, 1]$ small enough, for any integer $p \geq \dim X + 2$, there exists $C > 0$ such that for $T \geq 1$ and $\lambda \in J T$,

$$\left| \nabla T \left[ \nabla T (\lambda - T) - P \right] - \nabla T \left[ N \nabla T (W_u, F) \left( \lambda - T \right) \right] \right| \leq \frac{C}{\sqrt{T}} (1 + |\lambda|) p+1. \quad (6.2)$$

**Proof** Let $(f_a)$ be an orthonormal basis of $T S$ and $(f_a^*)$ be its dual. Let $(e_i)$ be an orthonormal basis of $T X$ and $\tilde{e}_i$ be its lift to $T X$. Let $\nabla^\Lambda (T S) \otimes \Lambda^*(T^* X)$ be the connection along the fibers $\nabla X$ on $\Lambda^*(T S) \otimes \Lambda^*(T^* X)$ defined by (cf. [4, (3.41)])

$$1 \nabla^\Lambda (T S) \otimes \Lambda^*(T^* X) = \nabla^\Lambda (T S) \otimes \Lambda^*(T^* X) + 2 \left( \sqrt{T} f_a^* \right) \nabla T (e_i) f_a + \frac{1}{2} \left( \sqrt{T} f_a^* \right) f_a f_b^*.$$  

Put (cf. [4, (3.44)])

$$1 \nabla^\Lambda (T S) \otimes \Lambda^*(T^* X) \otimes F, u = \psi_1^{-1} \circ 1 \nabla^\Lambda (T S) \otimes \Lambda^*(T^* X) \otimes F, u \circ \psi_t.$$  

Then as [4, (11.53)], we have

$$\nabla T = -\frac{1}{2} \tilde{e}_i \left( \nabla T (e_i) \left( -\frac{1}{2} \nabla T (e_i) - T \nabla T (e_i) \right) - \frac{1}{2} \nabla T (e_i) \right) + \frac{1}{2} \left( \nabla T (e_i) \right) (f_a^*) \left( f_a \right). \quad (6.3)$$

From (6.3), we get

$$\nabla T = -\frac{1}{2} \tilde{e}_i \left( \nabla T (e_i) \left( -\frac{1}{2} \nabla T (e_i) - T \nabla T (e_i) \right) + \frac{1}{2} \left( \nabla T (e_i) \right) \left( f_a^* \right) \left( f_a \right). \quad (6.4)$$

By (6.4), we find that the $\nabla T$ does not depend on $T$. We will write $\nabla T$ instead of $\nabla T$.

Clearly, in (6.2), we can replace $\nabla T$ by $\nabla T$. By the simplifying assumptions made in [4, Section 9.1], on the support of $\mu$,

$$\nabla T = \nabla T \otimes F|_\mathbb{B}. \quad (6.5)$$

As in (5.11),

$$\ker \nabla T \otimes F|_\mathbb{B} = \nabla T \otimes F|_\mathbb{B}. \quad (6.6)$$
Moreover,
\[(\lambda - B_T)^{-1} = (\lambda - B_T^{(0)})^{-1} + (\lambda - B_T^{(0)})^{-1} \left( B_T^{>0} \right) (\lambda - B_T^{(0)})^{-1} + \cdots , \tag{6.7} \]
and the expansion in (6.7) only contains a finite number of terms. Set
\[\bar{P}_T^{(1, +\infty)} = 1 - \bar{P}_T^{[0, 1]} . \tag{6.8} \]

If \(\lambda \in U_T\), put
\[
L_{T, 1} = \bar{P}_T^{[0, 1]} (\lambda - B_T^{(0)})^{-1} \bar{P}_T^{[0, 1]} , \quad L_{T, 2} = \bar{P}_T^{[0, 1]} (\lambda - B_T^{(0)})^{-1} \bar{P}_T^{(1, +\infty)} , \quad L_{T, 3} = \bar{P}_T^{(1, +\infty)} (\lambda - B_T^{(0)})^{-1} \bar{P}_T^{[0, 1]} , \quad L_{T, 4} = \bar{P}_T^{(1, +\infty)} (\lambda - B_T^{(0)})^{-1} \bar{P}_T^{(1, +\infty)} . \tag{6.9} \]

We still use the notation in (5.90) to define the norms \(\|\cdot\|_{\Gamma, p}\). By proceeding as in Theorem A.6, we can define \(m_T(\lambda)\) \(\in\text{End}(C^*(W^u, \tilde{F}))\) such that for \(T \geq 0\) large enough,
\[L_{T, 1} = (\lambda m_T(\lambda))^{-1} , \tag{6.10} \]
and moreover by (A.22), if \(c_1 > 0\) is small enough, \(\lambda \in U_T\),
\[\left\| m_T^{-1}(\lambda) - 1 \right\|_{\infty} \leq \frac{C}{\sqrt{T}}(1 + |\lambda|) . \tag{6.11} \]

By using (6.5), (6.6) and by proceeding as in the proof of Theorem A.8, we find that for \(2 \leq j \leq 4\),
\[\left\| L_{T, j} \right\|_{\Gamma, p-1} \leq C , \quad \left\| L_{T, j} \right\|_{\infty} \leq \frac{C}{\sqrt{T}} . \tag{6.12} \]

From (6.7)–(6.12) we find that to establish (6.2), in (6.7), we may as well replace \((\lambda - B_T^{(0)})^{-1}\) by \(\bar{P}_T^{[0, 1]} / \lambda\).

Let \(\bar{P}_T\) be the orthogonal projection operator form \(\Omega^*_{(2)}(\tilde{X}, \tilde{F}|\tilde{X})\) on \(\text{Im}(\tilde{J}_T) \subset \Omega^*_{(2)}(\tilde{X}, \tilde{F}|\tilde{X})\). We claim that
\[\left\| \left(\bar{P}_T^{[0, 1]} - 1\right) \bar{P}_T \right\|_{\infty} = O(e^{-cT}) . \tag{6.13} \]
In fact, since \(\tilde{J}_T : C^*(W^u, \tilde{F}) \rightarrow \Omega^*_{(2)}(\tilde{X}, \tilde{F}|\tilde{X})\) is an isometric embedding, (6.13) follows from that \(\bar{P}_T - \tilde{J}_T = O(e^{-cT})\) as \(T \rightarrow +\infty\).

Recall that \(\|\cdot\|_{\Gamma, 2}\) is the \(\Gamma\)-Hilbert-Schmidt norm. Then
\[\left\| \bar{P}_T^{[0, 1]} - \bar{P}_T \right\|_{\Gamma, 2}^2 = \left\| \bar{P}_T^{[0, 1]} \right\|_{\Gamma, 2}^2 + \left\| \bar{P}_T \right\|_{\Gamma, 2}^2 - 2\text{ReTr}_{\Gamma} \left[ \bar{P}_T^{[0, 1]} \bar{P}_T \right] . \tag{6.14} \]

Since the \(\Gamma\)-dimension of the images of \(\bar{P}_T^{[0, 1]}\) and \(\bar{P}_T\) are both equal to \(\text{rk}(F|_B)\), using (6.13), (6.14), we get
\[\left\| \bar{P}_T^{[0, 1]} - \bar{P}_T \right\|_{\Gamma, 2}^2 = O(e^{-cT}) . \tag{6.15} \]
Let $Q_T$ be the orthogonal projection operator from $\Omega^*_2(\tilde{X}, \tilde{F}|_{\tilde{X}})$ on $\tilde{F}^{[0,1]} + \text{Im}(\tilde{J}_T)$. Then

$$
\left\| \overline{P}_T^{[0,1]} - \overline{P}_T \right\|_{\Gamma,1} = \left\| \left( \overline{P}_T^{[0,1]} - \overline{P}_T \right) Q_T \right\|_{\Gamma,1} \\
\leq \left\| \overline{P}_T^{[0,1]} - \overline{P}_T \right\|_{\Gamma,2} \| Q_T \|_{\Gamma,1} \leq 2r_k (F|_{\overline{B}}) \left\| \overline{P}_T^{[0,1]} - \overline{P}_T \right\|_{\Gamma,2}.
$$

(6.16)

From (6.15), (6.16), we get

$$
\left\| \overline{P}_T^{[0,1]} - \overline{P}_T \right\|_{\Gamma,1} = O \left( e^{-cT} \right).
$$

(6.17)

By the above, it follows that to establish (6.2), we may as well replace in (6.7) $(\lambda - \overline{B}_T^{(-)})^{-1}$ by $\overline{P}_T / \lambda$.

Let $r : \Lambda^* (T^* \tilde{X}|_{\overline{B}}) \to \Lambda^{\text{max}} (T^* \tilde{X}|_{\overline{B}})$ be the obvious orthogonal projection operator. Then using (5.11), (5.126), we find easily that if $s \in \Omega^*_2(\tilde{X}, \tilde{F}|_{\tilde{X}})$,

$$
\overline{P}_T s(Z) = \frac{\mu(Z)}{\alpha_T} \exp \left( -T |Z|^2 / 2 \right) r \int_{T \tilde{X}|_{\overline{B}}} \mu(Z') \exp \left( -T |Z'|^2 / 2 \right) s(Z') dv_{T \tilde{X}}(Z').
$$

(6.18)

Using (6.5), (6.18), we get

$$
\overline{P}_T B_T \overline{P}_T = \overline{P}_T D_T \overline{F}|_{\overline{B}} \overline{P}_T.
$$

(6.19)

By (5.126), since here $\partial \tilde{F} = 0$, we get

$$
\overline{D}_T^{\overline{F}|_{\overline{B}}}^{(-)} = \frac{1}{2} \left( \omega \left( \overline{F}|_{\overline{B}}, \overline{S}|_{\overline{B}} \right) - \overline{c} \left( R^T \tilde{X}|_{\overline{B}} Z \right) \right).
$$

(6.20)

Since $\overline{c}(R^T \tilde{X}|_{\overline{B}} Z)$ is an odd operator, we get

$$
r \overline{c} \left( R^T \tilde{X}|_{\overline{B}} Z \right) = 0.
$$

(6.21)

By (6.18), (6.19), (6.21), we obtain,

$$
\overline{P}_T B^{(-)} \overline{P}_T = \frac{1}{2} \omega \left( \overline{F}|_{\overline{B}}, \overline{S}|_{\overline{B}} \right).
$$

(6.22)

Using the above, we get (6.2). The proof is completed.

\[\square\]

**Theorem 6.2** For $c_1 \in (0, 1]$ small enough, given any integer $p \geq \text{dim}X + 1$, there exists $C > 0$ such that for $T \geq 1$ large enough, and $\lambda \in U_T$,

$$
\left\| (\lambda - \overline{B}_T)^{-1} \right\|_{\Gamma,p} \leq C (1 + |\lambda|)^p.
$$

(6.23)

**Proof** Again, we can replace in (6.23) $\tilde{B}_T$ by $\overline{B}_T$. First we claim that (6.23) holds for $\overline{B}_T^{(0)}$. Using (6.5), the proof is the same as the proofs of Theorems A.6 and A.8. To get (6.23), we use (6.7) and the fact that $\overline{B}_T^{(0)}$ is of order 0. The proof is completed.

Now we prove Theorem 4.2.

Take $p \in \mathbb{N}$. Let $k_p(\lambda)$ be the unique holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ such that

$$-A \lambda \to \pm i \infty, k_p(\lambda) \to 0.$$

[\square Springer]
The following identity holds,

\[
\frac{k_p^{(p-1)}(\lambda)}{(p-1)!} = h'(\lambda).
\] (6.24)

Let \( \Delta = \Delta_+ \cup \Delta_- \) be a contour defined by

\[
\Delta_+ = \left\{ z = x + iy | x = \pm \frac{1}{4}, \frac{1}{2} \leq y < +\infty \right\} \cup \left\{ z = x + iy | y = \frac{1}{2}, -\frac{1}{4} \leq x \leq \frac{1}{4} \right\},
\]

\[
\Delta_- = \left\{ z = x + iy | x = \pm \frac{1}{4}, -\infty < y \leq -\frac{1}{2} \right\} \cup \left\{ z = x + iy | y = -\frac{1}{2}, -\frac{1}{4} \leq x \leq \frac{1}{4} \right\}.
\] (6.25)

Clearly, if \( \lambda \in \Delta \),

\[
|\text{Re}(\lambda)| \leq \frac{1}{2} |\text{Im}(\lambda)|.
\] (6.26)

Using (6.26), we find that there exist \( C > 0, C' > 0 \) such that if \( \lambda \in \Delta \),

\[
\left| k_p(\sqrt{t}\lambda) \right| \leq C \exp \left( -C' t |\lambda|^2 \right).
\] (6.27)

By the argument of [4, Theorem 11.7], we find that for \( T \geq 0 \) large enough,

\[
\left| \text{Sp} \left( \overline{B}_T^{(0)} \right) \right| \subset \left( \frac{T}{\pi} \right)^{1/2} e^{-T} [0, d] \cup [1, +\infty).
\] (6.28)

Then for \( T \geq 0 \) large enough and \( t > 0 \), we have (cf. [4, (12.47)])

\[
h' \left( \widetilde{D}_{t,T} \right) = \psi_t^{-1} \frac{1}{2i\pi} \int_{\Delta} \frac{h' \left( \sqrt{t}\lambda \right)}{\lambda - \overline{B}_T} d\lambda \psi_t + \frac{1}{2i\pi} \int_{\delta_2} \frac{h' \left( \lambda \right)}{\lambda - \widetilde{D}_{t,T}} d\lambda.
\] (6.29)

Using Theorems 6.1 and 6.2, (6.27) and (6.29), there exists \( C > 0 \) such that for \( t \in [\varepsilon, A] \), \( T \geq 1 \), we have

\[
\left| \text{Tr}_{\Gamma,s} \left[ N \psi_t^{-1} \frac{1}{2i\pi} \int_{\Delta} \frac{h' \left( \sqrt{t}\lambda \right)}{\lambda - \overline{B}_T} d\lambda \psi_t \right] \right| \leq \frac{C}{\sqrt{T}}.
\] (6.30)

For \( T > 0 \) large enough, proceeding as in the proof of [4, Theorem 11.19], one has

\[
\lim_{t \to 0^+} \text{Tr}_{\Gamma,s} \left[ N \frac{1}{2i\pi} \int_{\delta_2} \frac{h' \left( \lambda \right)}{\lambda - \widetilde{D}_{t,T}} d\lambda \right] = \widetilde{\chi}^{-}(F).
\] (6.31)

Then by (6.28) and (6.31), for \( T > 0 \) large enough, \( t \in [\varepsilon, A] \), one has

\[
\left| \text{Tr}_{\Gamma,s} \left[ N \int_{\delta_2} \frac{h' \left( \lambda \right)}{\lambda - \widetilde{D}_{t,T}} d\lambda \right] - \widetilde{\chi}^{-}(F) \right| \leq \frac{C}{\sqrt{T}}.
\] (6.32)

Then by (6.30) and (6.32), we get the \( L^2 \) case of [4, (12.48)], i.e., given \( \varepsilon, A \) with \( 0 \leq \varepsilon \leq A \leq +\infty \), there exists \( C > 0 \) such that if \( t \in [\varepsilon, A] \), \( T \geq 1 \),

\[
\left| \text{Tr}_{\Gamma,s} \left[ N h' \left( \widetilde{D}_{t,T} \right) \right] - \widetilde{\chi}^{-}(F) \right| \leq \frac{C}{\sqrt{T}}.
\] (6.33)

Then we get Theorem 4.2.
If $f$ is not fiberwise nice, as in [4, (12.49)], (6.5) is replaced by

$$B_T^2 = \overline{\mathcal{D}_T} \otimes \overline{\mathcal{F}|B|^2}. \quad (6.34)$$

Let $\Gamma''$ be a contour defined by

$$\Gamma'' = \left\{ z = x + iy | x = - \frac{1}{4}, - \frac{1}{2} \leq y \leq \frac{1}{2} \right\} \cup \left\{ z = x + iy | y = \pm \frac{1}{2}, - \infty < x \leq - \frac{1}{4} \right\}.$$

Then the operator in (6.30) is replaced by

$$\psi_t^{-1} \frac{1}{2 \pi i} \int_{\Gamma''} \frac{(1 + 2 t \lambda) e^{i \lambda}}{\lambda - B_T^2} \, d \lambda \psi_t. \quad (6.35)$$

As in [4], proceedings as the proof of Theorem B.9, we get the corresponding estimate for (6.35). The proof is completed.

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Appendix A: $L^2$ analogue of some results of [5] in the current case

In this appendix, we generalize some results in [5] to the current case.

For $h \in o(x) \otimes \overline{\mathcal{F}|x}$, recall that $\overline{\mathcal{J}}_T(h)$ has been defined in (5.126). Let

$$\overline{\mathcal{E}}_T = \overline{\mathcal{J}}_T \left( L^2 \left( \overline{\mathcal{F}|B} \right) \right)$$

be the image of $\overline{\mathcal{J}}_T$. Since $\overline{\mathcal{J}}_T$ is an isometry, $\overline{\mathcal{E}}_T \subset \Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}})$ is closed.

Let $\overline{\mathcal{E}}_T^\perp$ denote the orthogonal complement of $\overline{\mathcal{E}}_T$ in $\Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}})$, that is

$$\Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}}) = \overline{\mathcal{E}}_T \oplus \overline{\mathcal{E}}_T^\perp.$$

Let $\overline{\mathcal{P}}_T$ (resp. $\overline{\mathcal{P}}_T^\perp$) denote the orthogonal projection from $\Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}})$ onto $\overline{\mathcal{E}}_T$ (resp. $\overline{\mathcal{E}}_T^\perp$).

Recall that we have the decompositions

$$\overline{\mathcal{A}}' = d \overline{\mathcal{X}} + \nabla \Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}}) + i_{T H}, \quad \overline{\mathcal{A}}'' = d \overline{\mathcal{X}}^* + \nabla \Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}})^* - T^H \wedge.$$

For any $T \geq 0$, following [31], set

$$d_T \overline{\mathcal{X}} = e^{-T \overline{\mathcal{F}}} d \overline{\mathcal{X}} e^{T \overline{\mathcal{F}}}, \quad \delta_T \overline{\mathcal{X}} = e^{T \overline{\mathcal{F}}} d \overline{\mathcal{X}}^* e^{-T \overline{\mathcal{F}}}.$$

Then $\delta_T \overline{\mathcal{X}}^*$ is the formal adjoint of $d_T \overline{\mathcal{X}}$ with respect to the usual inner product on $\Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}})$.

Set

$$\overline{\mathcal{D}}_T = d_T \overline{\mathcal{X}} + \delta_T \overline{\mathcal{X}}^*, \quad \overline{\mathcal{D}}_T^2 = \left( d_T \overline{\mathcal{X}} + \delta_T \overline{\mathcal{X}}^* \right)^2 = d_T \delta_T \overline{\mathcal{X}}^* + \delta_T \overline{\mathcal{X}}^* d_T \overline{\mathcal{X}}.$$

Then $\overline{\mathcal{D}}_T^2$ preserves the $\mathbb{Z}$-grading of $\Omega^* (\overline{\mathcal{X}}, \overline{\mathcal{F}|\overline{\mathcal{X}}})$.

Following Bismut-Lebeau [5, Section 9], we define

$$\overline{\mathcal{D}}_{T,1} = \overline{\mathcal{P}}_T \overline{\mathcal{D}}_T' \overline{\mathcal{P}}_T, \quad \overline{\mathcal{D}}_{T,2} = \overline{\mathcal{P}}_T \overline{\mathcal{D}}'_T \overline{\mathcal{P}}_T^\perp, \quad \overline{\mathcal{D}}_{T,3} = \overline{\mathcal{P}}_T^\perp \overline{\mathcal{D}}'_T \overline{\mathcal{P}}_T, \quad \overline{\mathcal{D}}_{T,4} = \overline{\mathcal{P}}_T^\perp \overline{\mathcal{D}}'_T \overline{\mathcal{P}}_T^\perp. \quad (A.1)$$
We then write the operator $\tilde{D}_T'$ in matrix form

$$\tilde{D}_T' = \begin{pmatrix} \tilde{D}_{T,1}' & \tilde{D}_{T,2}' \\ \tilde{D}_{T,3}' & \tilde{D}_{T,4}' \end{pmatrix}. \quad \text{(A.2)}$$

We recall the following elliptic estimates needed in this paper.

**Lemma A.1** [29, Lemma 1.4] Let $A : C^\infty(\tilde{X}, \tilde{F}|_{\tilde{X}}) \to C^\infty(\tilde{X}, \tilde{F}|_{\tilde{X}})$ be a $C^\infty$-bounded uniformly elliptic differential operator of order $m$. For any $i, j \geq 0$, there exists a constant $C$ such that for any $s \in C_0^\infty(\tilde{X}, \tilde{F}|_{\tilde{X}})$,

$$\|s\|_{i+m} \leq C \left( \|As\|_i + \|s\|_j \right).$$

Let $\mathbb{H}^1(\tilde{X}, \tilde{F}|_{\tilde{X}})$ denote the first Sobolev space with respect to a (fixed, $\Gamma$-invariant) first Sobolev norm on $\Omega^*(\tilde{X}, \tilde{F}|_{\tilde{X}})$.

The following analogue of [5, Proposition 9.12] holds.

**Proposition A.2** There exist $\epsilon \in (0, \epsilon_0/4]$, $C > 0$, $T_0 > 0$ such that for any $T \geq T_0$, any $s \in \tilde{E}_T \cap \mathbb{H}^1(\tilde{X}, \tilde{F}|_{\tilde{X}})$ whose support is included in $U_{2\epsilon}$, then

$$\|\tilde{D}_T s\|_0^2 \geq C \left( \|s\|^2_1 + T \|s\|_0^2 \right). \quad \text{(A.3)}$$

**Proof** Since $[D^\tilde{X}, \tilde{c}(d \tilde{f})]$ is an operator of zero order and the support of $s$ is included in $U_{2\epsilon}$, then there exists $\kappa > 0$ such that for any $\alpha \in (0, 1]$,

$$(1 - \alpha) \left\| \left( D^\tilde{X} + T \tilde{c}(d \tilde{f}) \right) s \right\|_0^2 + \alpha T \left\{ \left[ D^\tilde{X}, \tilde{c}(d \tilde{f}) \right] s, s \right\}_0 + \alpha \kappa T \|s\|_0^2 \geq 0. \quad \text{(A.4)}$$

Then we have

$$\left\| \left( D^\tilde{X} + T \tilde{c}(d \tilde{f}) \right) s \right\|_0^2 \geq \alpha \left\| \left( D^\tilde{X} + T \tilde{c}(d \tilde{f}) \right) s \right\|_0^2 - \alpha T \left\{ \left[ D^\tilde{X}, \tilde{c}(d \tilde{f}) \right] s, s \right\}_0 - \alpha \kappa T \|s\|_0^2 \geq \alpha \left\| D^\tilde{X} s \right\|_0^2 - \alpha \kappa T \|s\|_0^2. \quad \text{(A.5)}$$

By Lemma A.1, there exists $C > 0$ such that

$$\left\| D^\tilde{X} s \right\|_0^2 \geq \frac{1}{2C^2} \|s\|^2_1 - \|s\|^2_0. \quad \text{(A.6)}$$

On the other hand, by [32, (6.16)–(6.17)], there exist $C' > 0$, $T_0' > 0$ such that for $T \geq T_0'$, $s \in \tilde{E}_T \cap \mathbb{H}^1(\tilde{X}, \tilde{F}|_{\tilde{X}})$,

$$\|\tilde{D}'_T s\|_0^2 \geq C' T \|s\|_0^2. \quad \text{(A.7)}$$

By (A.5)–(A.7), one has

$$\|\tilde{D}'_T s\|_0^2 \geq \frac{\alpha}{4C^2} \|s\|^2_1 - \frac{\alpha}{2} \|s\|^2_0 - \frac{\alpha \kappa T}{2} \|s\|^2_0 + \frac{1}{2} C' T \|s\|_0^2. \quad \text{(A.8)}$$

Take $\alpha \in (0, 1]$ such that $\frac{1}{2} C' - \frac{\alpha}{2T} - \frac{\alpha \kappa}{2} \geq \frac{1}{4} C'$, then we get the proposition. \qed

Using Lemma A.1 and Proposition A.2, by proceeding as in [5, Section 9 b)], one deduces that
(i) The following identity holds (cf. [32, (6.15)]):
\[ \tilde{D}_{T,1} = 0. \]  
(A.9)

(ii) There exists \( C > 0 \) such that for any \( T \geq 1, \) any \( s \in \tilde{E}_T^1 \cap \mathbb{H}^1(\tilde{X}, \tilde{F}|\tilde{X}), \) set
\[ \| \tilde{D}_{T,2}s \|_0 \leq C \left( \frac{\|s\|_1}{\sqrt{T}} + \|s\|_0 \right) \] and
\[ \| \tilde{D}_{T,3}^s' \|_0 \leq C \left( \frac{\|s\|_1}{\sqrt{T}} + \|s\|_0 \right). \] (A.10)

(iii) There exist \( T_0 > 0, c > 0 \) such that for any \( T \geq T_0, s \in \tilde{E}_T^1 \cap \mathbb{H}^1(\tilde{X}, \tilde{F}|\tilde{X}), \) then
\[ \| \tilde{D}_{T,4}s \|_0 \geq c \left( \|s\|_1 + \sqrt{T}\|s\|_0 \right). \] (A.11)

(iv) There exists \( C > 0 \) such that for any \( T \geq T_0, \lambda \in \mathbb{C}, |\lambda| \leq \frac{c\sqrt{T}}{4}, s \in \tilde{E}_T^1, \) then
\[ \left\| (\lambda - \tilde{D}_{T,4})^{-1}s \right\|_0 \leq C \sqrt{T}\|s\|_0, \]
\[ \left\| (\lambda - \tilde{D}_{T,4}')^{-1}s \right\|_1 \leq C\|s\|_0. \] (A.12)

Following [5, (9.113)], for \( T \geq 1, \) set
\[ U_T' = \left\{ \lambda \in \mathbb{C} : \frac{1}{8} \leq |\lambda| \leq \frac{c\sqrt{T}}{4} \right\}. \] (A.13)

Note that one can also extend [5, Definitions 9.17 and 9.22] to the current case, and they still have the similar properties.

**Definition A.3** (Compare with [5, Definition 9.17]) If \( H, H' \) are \( \Gamma \)-Hilbert modules, set
\[ L_{\Gamma, p} = \left\{ A \in L(H, H') : \text{Tr}_{\Gamma} \left[ (A^*A)^{p/2} \right] < +\infty \right\}. \]

If \( A \in L_{\Gamma, p}(H, H'), \) set
\[ \|A\|_{\Gamma, p} = \left\{ \text{Tr}_{\Gamma} \left[ (A^*A)^{p/2} \right] \right\}^{1/p}. \]

From (A.9) to (A.13), one can show that there exists \( T_0 \geq 1 \) such that for any \( T \geq T_0, \lambda \in U_T', \lambda - \tilde{D}_{T} \) is invertible.

**Proposition A.4** If \( p \geq \text{dim}X + 1, \) there exists \( C > 0 \) such that for \( T \geq T_0, \lambda \in \mathbb{C}, |\lambda| \leq \frac{c\sqrt{T}}{4}, \) then
\[ \left\| (\lambda - \tilde{D}_{T,4})^{-1} \right\|_\infty \leq \frac{C}{\sqrt{T}}, \]
\[ \left\| (\lambda - \tilde{D}_{T,4}')^{-1} \right\|_{\Gamma, p} \leq C, \]
\[ \left\| \tilde{D}_{T,2}(\lambda - \tilde{D}_{T,4})^{-1} \right\|_\infty \leq \frac{C}{\sqrt{T}}. \] (A.14)

**Proof** The first line of (A.14) is from (A.12). Also
\[ \left\| (\lambda - \tilde{D}_{T,4})^{-1} \right\|_{\Gamma, p} \leq \left\| \left( D_{\tilde{X}} + \sqrt{-1} \right)^{-1} \right\|_{\Gamma, p} \left\| \left( D_{\tilde{X}} + \sqrt{-1} \right)(\lambda - \tilde{D}_{T,4})^{-1} \right\|_\infty. \]
Since $D\tilde{X}$ is elliptic of order one, when $p \geq \dim X + 1$, \(\| (D\tilde{X} + \sqrt{-1})^{-1}\|_{\Gamma, p} < +\infty\). Also by (A.12), for $T \geq T_0$,

\[
\left\| \left( D\tilde{X} + \sqrt{-1} \right) (\lambda - \tilde{D}_{T,4})^{-1} \right\|_{\infty} \leq C. \tag{A.15}
\]

The second line in (A.14) follows. Using (A.10) and (A.12), we get the third line in (A.14).

Since $\tilde{D}_{T,1}' = 0$ (cf. [32, (6.15)]), following [5, Definition 9.20], we make the following definition.

**Definition A.5** For $T \geq T_0$, $\lambda \in U_T'$, let $M_T(\lambda)$ be the linear map from $\tilde{E}_T \cap H^1(\tilde{X}, \tilde{F}|\tilde{X})$ into $\tilde{E}_T$

\[
M_T(\lambda) = \lambda - \tilde{D}_{T,2}' (\lambda - \tilde{D}_{T,4}')^{-1} \tilde{D}_{T,3}'. \tag{A.16}
\]

**Theorem A.6** $M_T(\lambda)$ is invertible and for any integer $p \geq \dim X + 1$, there exists $C > 0$ such that for $T \geq T_0$, $\lambda \in U_T'$,

\[
\left\| M_T^{-1}(\lambda) \right\|_{\infty} \leq C, \quad \left\| \tilde{D}_{T,3}' M_T^{-1}(\lambda) \right\|_{\infty} \leq C, \quad \left\| M_T^{-1}(\lambda) \right\|_{\Gamma, p} \leq C(1 + |\lambda|), \quad \left\| \tilde{J}_T^{-1} \left( M_T^{-1}(\lambda) \right)^p \tilde{J}_T - \lambda^{-p} \right\|_{\Gamma, 1} \leq \frac{C}{\sqrt{T}} (1 + |\lambda|)^{p+1}. \tag{A.17}
\]

**Proof** For $\lambda \in U_T'$, set

\[
m_T(\lambda) = 1 - \lambda^{-1} \tilde{D}_{T,2}' (\lambda - \tilde{D}_{T,4}')^{-1} \tilde{D}_{T,3}'. \tag{A.18}
\]

Clearly

\[
M_T(\lambda) = \lambda m_T(\lambda). \tag{A.19}
\]

For $\lambda \in U_T'$, by (A.10) and (A.12), there exits $C_1 > 0$ such that for $T \geq T_0$,

\[
\left\| \tilde{D}_{T,2}' (\lambda - \tilde{D}_{T,4}')^{-1} \right\|_{\infty} \leq \frac{C_1}{\sqrt{T}}. \tag{A.20}
\]

By [32, (6.16)], there exists $C_2 > 0$ such that

\[
\left\| \lambda^{-1} \tilde{D}_{T,3}' \right\|_{\infty} \leq C_2. \tag{A.20}
\]

Then there exists $C_3 > 0$ such that

\[
\left\| \lambda^{-1} \tilde{D}_{T,2}' (\lambda - \tilde{D}_{T,4}')^{-1} \tilde{D}_{T,3}' \right\|_{\infty} \leq \frac{C_3}{\sqrt{T}}. \tag{A.21}
\]

From (A.18)–(A.21), it is clear that if $1/\sqrt{T}$ and $|\lambda|/\sqrt{T}$ are small enough, the operator $m_T(\lambda)$ is invertible, and moreover for $T \geq 1$

\[
\left\| M_T^{-1}(\lambda) - 1 \right\|_{\infty} \leq \frac{C_3}{\sqrt{T}} (1 + |\lambda|). \tag{A.22}
\]

In particular

\[
\left\| M_T^{-1}(\lambda) \right\|_{\infty} \leq C_3. \tag{A.23}
\]
By (A.19), we get
\[ M_T^{-1}(\lambda) = \lambda^{-1}m_T^{-1}(\lambda). \]  
(A.24)

By (A.20), (A.23) and (A.24), we get the second inequality in (A.17).
Since \( \| \sqrt{-1}\tilde{\mathbf{T}} \|_{\Gamma'p} < +\infty \), by (A.23) and (A.24), for \( \lambda \in U_T' \), we have
\[
\| M_T^{-1}(\lambda) \|_{\Gamma'p} \leq C_3 \| \lambda^{-1} \|_{\Gamma'p} \left\| \left( -\sqrt{-1} - (\sqrt{-1} - \lambda)^{-1} \sqrt{-1} \right) \tilde{\mathbf{T}} \right\|_{\Gamma'p} \\
\leq C_4(1 + |\lambda|). 
\]  
(A.25)

Then we get the third inequality in (A.17).
Finally, using (A.22)–(A.25), we get the last inequality in (A.17).

If \( B \in \mathcal{L}(\Omega^\bullet(2)(\tilde{F}|\tilde{X})) \), for any \( T \geq 1 \), we write \( B \) as a matrix with respect to the splitting \( \Omega^\bullet(2)(\tilde{X}, \tilde{F}|\tilde{X}) = \tilde{E}_T \oplus \tilde{E}_T^\perp \) in the form
\[
B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}. 
\]

**Definition A.7** (Compare with [5, Definition 9.22]) If \( B \in \mathcal{L}(\Omega^\bullet(2)(\tilde{F}|\tilde{X})) \), \( C \in \mathcal{L}(L^2(\tilde{F}|\tilde{b})) \), set
\[
d_T(B, C) = \sum_{j=2}^4 \| B_j \|_{\Gamma,1} + \| \tilde{T}_j^{-1}B_1 \tilde{T}_j - C \|_{\Gamma,1}.
\]  
(A.26)

Clearly if \( B \in \mathcal{L}_{\Gamma,1}(\Omega^\bullet(2)(\tilde{X}, \tilde{F}|\tilde{X})) \), \( C \in \mathcal{L}_{\Gamma,1}(L^2(\tilde{F}|\tilde{b})) \),
\[
|\text{Tr}_T(B) - \text{Tr}_T(C)| \leq d_T(B, C).
\]  
(A.27)

From the proof of [5, Theorem 9.23], one can easily get the following analogue of [5, Theorem 9.23] in our case.

**Theorem A.8** There exists \( T_0 \geq 1 \) such that for any \( T \geq T_0 \), \( \lambda \in U_T' \), \( \lambda - \tilde{D}_T' \) is invertible. For any integer \( p \geq \dim X + 2 \), there exists \( C > 0 \) such that if \( T \geq T_0 \), \( \lambda \in U_T' \), then
\[
d_T \left( \left( \lambda - \tilde{D}_T' \right)^{-p}, \lambda^{-p}|_{L^2(\tilde{F}|\tilde{b})} \right) \leq \frac{C}{\sqrt{T}} \left( 1 + |\lambda| \right)^{p+1}.
\]

**Proof** Set
\[
\tilde{B}_T = (\lambda - \tilde{D}_T')^{-1}.
\]  
(A.28)

Then we find that
\[
\tilde{B}_{T,1} = M_T^{-1}(\lambda), \quad \tilde{B}_{T,2} = M_T^{-1}(\lambda) \tilde{D}_{T,2}(\lambda - \tilde{D}_{T,A}^{-1}), \\
\tilde{B}_{T,3} = (\lambda - \tilde{D}_{T,A}^{-1}) \tilde{D}_{T,3}M_T^{-1}(\lambda), \quad \tilde{B}_{T,4} = (\lambda - \tilde{D}_{T,A}^{-1})^{-1}(1 + \tilde{D}_{T,3} \tilde{B}_{T,2}).
\]  
(A.29)

Using Proposition A.4 and Theorem A.6, we find that if \( p \geq \dim X + 2 \), \( T \geq T_0 \), \( \lambda \in U_T' \), for \( 2 \leq j \leq 4 \), then
\[
\| \tilde{B}_{T,j} \|_{\Gamma,p-1} \leq C; \quad \| \tilde{B}_{T,j} \|_{\infty} \leq \frac{C}{\sqrt{T}}. 
\]  
(A.30)
From Theorem A.6 and from (A.30), we deduce that if \( j_1, \ldots, j_p \in \{1, 2, 3, 4\} \), if one of the \( j'_k \)'s is not equal to 1, then

\[
\| \tilde{B}_{T,j_1} \ldots \tilde{B}_{T,j_p} \|_{p,1} \leq \frac{C}{\sqrt{T}} (1 + |\lambda|)^{p-1}.
\]  
(A.31)

Then follows from the fourth inequality in (A.17) and (A.31), we get the theorem. \( \square \)

**Appendix B: \( L^2 \) analogue of [3, Theorem 9.5] in the current case**

In this appendix, we prove the \( L^2 \) analogue of [3, (9.149)] (cf. (B.76)).

Although [3, Theorem 9.1] does not hold in \( L^2 \)-case, but when \( T \) large enough, the spectral gap also holds. In this paper, we only need the spectral gap. This is the key point that we can get our theorem. So the proofs of the \( L^2 \)-case are almost the same as in [3].

We also denote by \( \tilde{F} \) the canonical projection \( T \tilde{X}|_{\tilde{B}} \to \tilde{B} \).

**Definition B.1** (Compare with [3, Definition 9.9]) For \( s \in S, \mu \in \mathbb{R} \), let \( \tilde{E}^\mu_{T,s} \) (resp. \( \tilde{E}^\mu_{T} \), resp. \( \tilde{E}^\mu_{T,s} \)) be the set of sections of \( \Lambda^*(T^*\tilde{X}) \otimes \tilde{F} \) over \( \tilde{X}_T \) (resp. of \( \tilde{F}^*(\Lambda^*(T^*\tilde{X}) \otimes \tilde{F})|_{\tilde{B}} \) over \( T \tilde{X}|_{\tilde{B}} \), resp. \( \tilde{F} \) over \( \tilde{B}_s \)) which lie in the \( \mu^{th} \) Sobolev space, and let \( \| \| \tilde{E}^\mu_{T} \) (resp. \( \| \| \tilde{E}^\mu_{T,s} \), resp. \( \| \| \tilde{F}^\mu_{T} \) be the corresponding Sobolev norm.

For \( s \in S \), let \( \tilde{E}_s \) be the set of smooth sections with compact support of \( \Lambda^*(T^*\tilde{X}) \otimes \tilde{F} \) over \( \tilde{X}_T \). Let \( \tilde{E} \) be a smooth \( Z \)-graded infinite dimensional vector bundle over \( S \) whose fiber at \( s \) is \( \tilde{E}_s \).

For \( 0 < a < +\infty \), set

\[
B_a = \left\{ Z \in T \tilde{X}|_{\tilde{B}}, |Z| < a \right\}.
\]

Fix \( \varepsilon_0 > 0 \) small enough such that the map \((y, Z) \in T \tilde{X}|_{\tilde{B}} \to \exp_{\tilde{X}}^\dagger Z \in \tilde{X} \) is a diffeomorphism from \( B_{2\varepsilon_0} \) to a neighborhood \( U_{\varepsilon_0} \) of \( \tilde{B} \). We take \( \varepsilon \in (0, \varepsilon_0/4] \).

For \( \mu \geq 0, T > 0 \), let \( \tilde{J}_T : \tilde{E}^\mu \to \tilde{E}^\mu \) be the linear map defined by (5.126). Let \( \tilde{E}^\mu_{\tilde{T}} \) be the image of \( \tilde{F}^\mu \) in \( \tilde{E}^{\tilde{T}} \) by \( \tilde{J}_T \). Then \( \tilde{J}_T \) is an isometric embedding of \( \tilde{E}^0 \) into \( \tilde{E}^0 \).

Let \( \tilde{E}^{\mu,\perp}_{\tilde{T}} \) be the orthogonal space to \( \tilde{E}^0_{\tilde{T}} \) in \( \tilde{E}^0_{\tilde{T}} \), let \( \tilde{p}_T, \tilde{p}^\perp_T \) be the orthogonal projection operators from \( \tilde{E}^0 \) on \( \tilde{E}^0_{\tilde{T}}, \tilde{E}^{\mu,\perp}_{\tilde{T}} \) respectively.

If \( \sigma \in \tilde{E}^\mu \), we consider \( \tilde{T} \sigma \) as an element of \( \tilde{E}^\mu \). Then \( \tilde{J}_T \) is an isometric embedding from \( \tilde{E}^0 \) into \( \tilde{E}^0 \). Let \( \tilde{E}^\mu_{\tilde{T}} \) be the image of \( \tilde{F}^\mu \) in \( \tilde{E}^{\tilde{T}} \). Let \( \tilde{E}^{\mu,\perp}_{\tilde{T}} \) be the orthogonal bundle to \( \tilde{E}^0_{\tilde{T}} \) in \( \tilde{E}^0 \).

For \( \mu \geq 0 \), set

\[
\tilde{E}^{\mu,\perp}_{\tilde{T}} = \tilde{E}^\mu \cap \tilde{E}^{0,\perp}_{\tilde{T}}.
\]  
(B.1)

Let \( \tilde{p}_T, \tilde{p}^\perp_T \) be the orthogonal projection operators from \( \tilde{E}^0 \) on \( \tilde{E}^0_{\tilde{T}}, \tilde{E}^{0,\perp}_{\tilde{T}} \).

Let \( e_1, \ldots, e_n \) be a locally defined smooth orthogonal basis of \( TX \). Denote by \( \tilde{e}_1, \ldots, \tilde{e}_n \) the lifting to \( T \tilde{X} \).

If \( s \in \tilde{E}^0 \), put

\[
|s|_0 = \|s\|_{\tilde{E}^0}, \quad \langle s, s' \rangle_0 = \langle s, s' \rangle_{\tilde{E}^0}.
\]  
(B.2)

**Definition B.2** For \( T \geq 1, s \in \tilde{E} \), set

\[
|s|_{T,1}^2 = \| \tilde{p}_T s \|_0^2 + T \left| \tilde{p}_T s \right|_0^2 + T^2 \left| \tilde{c}(\nabla \tilde{f}) \tilde{p}_T s \right|_0^2 + \sum_1^n \left| \nabla_{\tilde{e}_i} (T^*\tilde{X}) \otimes \tilde{F} \tilde{p}_T s \right|_0^2.
\]  
(B.3)
Then (B.3) defines a Hilbert norm on $\tilde{E}^1$. We identify $\tilde{E}^0$ to its antidual by $\langle \cdot , \cdot \rangle_0$. Then we can identify $\tilde{E}^{-1}$ to the antidual of $\tilde{E}^1$. Let $| \cdot |_{T, -1}$ be the norm on $\tilde{E}^{-1}$ associated to $| \cdot |_{T, 1}$. Then we have the continuous dense embeddings with norms smaller than 1,

$$\tilde{E}^1 \rightarrow \tilde{E}^0 \rightarrow \tilde{E}^{-1}.$$  \hspace{1cm} (B.4)

Then the definition of $|s|_0, |s|_{T, 1}$ obviously extends to $\Lambda(T^*S)\tilde{\otimes}E$.

Recall that $\tilde{A}_T$ was defined in (5.16). Let $\tilde{A}_T^{(0)}$ (resp. $\tilde{A}_T^{(>0)}$) be the piece of $\tilde{A}_T$ which has degree 0 (resp. positive degree) in $\Lambda^*(T^*S)$. Then

$$\tilde{A}_T = \tilde{A}_T^{(0)} + \tilde{A}_T^{(>0)}.$$  \hspace{1cm} (B.5)

As [3, (9.41)], we have

$$\tilde{A}_T^{(0)} = \frac{1}{2} D\tilde{x} + \frac{1}{2} T\tilde{c}(\nabla \tilde{f}).$$  \hspace{1cm} (B.6)

Set

$$\tilde{R}_T = \left[\tilde{A}_T^{(0)}, \tilde{A}_T^{(>0)}\right] + \tilde{A}_T^{(>0), 2}.$$  \hspace{1cm} (B.7)

Then $\tilde{R}_T$ is a first order differential operator and moreover

$$\tilde{A}_T^{(0), 2} = \tilde{A}_T^{(0), 2} + \tilde{R}_T.$$  \hspace{1cm} (B.8)

Then by Lemma A.1, proceeding as the proof of [3, Theorem 9.14], we have

**Theorem B.3** If $\varepsilon \in (0, \varepsilon_0/4]$ is small enough, there exist constants $C_1 > 0, C_2 > 0, C_3 > 0$ such that for $T \geq 1, s, s' \in \Lambda^*(T^*S)\tilde{\otimes}E$,

$$\left|\tilde{A}_T^{(0)} s\right|_0^2 \geq C_1 |s|_{T, 1}^2 - C_2 |s|_0^2,$$

$$\left|\tilde{A}_T^{(0)} s, \tilde{A}_T^{(0)} s'\right| \leq C |s|_{T, 1} |s'|_{T, 1},$$

$$\left|\tilde{R}_T s, s'\right|_0 \leq C_3 \left(|s|_{T, 1} |s'|_0 + |s|_0 |s'|_{T, 1}\right).$$  \hspace{1cm} (B.9)

**Proof** In the whole proof, $C, C', \ldots$ are positive constants, which may vary from line to line.

To establish the first inequality in (B.9), we may as well assume that $s \in \tilde{E}$. If $s \in \tilde{E}$, then

$$\left|\tilde{A}_T^{(0)} s\right|_0^2 = \left|\tilde{p}_T \tilde{A}_T^{(0)} s\right|_0^2 + \left|\tilde{p}_T^\perp \tilde{A}_T^{(0)} s\right|_0^2 \geq \frac{1}{2} \left|\tilde{p}_T \tilde{A}_T^{(0)} s\right|_0^2 - \left|\tilde{p}_T \tilde{A}_T^{(0)} \tilde{p}_T^\perp s\right|_0^2,$$

$$\hspace{1cm} - \left|\tilde{p}_T^\perp \tilde{A}_T^{(0)} \tilde{p}_T^\perp s\right|_0^2.$$  \hspace{1cm} (B.10)

since $\tilde{p}_T \tilde{A}_T^{(0)} \tilde{p}_T = 0$.

If $s \in \tilde{E}$ is supported in $\mathcal{U}_{\varepsilon_0}$,

$$C |Z| |s| \leq |\tilde{c}(\nabla \tilde{f}) s| \leq C' |Z| |s|.$$  \hspace{1cm} (B.11)

By [5, (9.87)] and (B.11), if $\varepsilon \in (0, \varepsilon_0/4]$ is small enough, if $s \in \tilde{E}_T^{1, \perp}$ is supported in $\mathcal{U}_{2\varepsilon}$, then

$$\left|\tilde{A}_T^{(0)} s\right|_0^2 \geq C |s|_{\tilde{E}_T^{1, \perp}}^2 + C' T^2 |\tilde{c}(\nabla \tilde{f}) s|_0^2 + C'' T |s|_0^2 - C''' |s|_0^2,$$  \hspace{1cm} (B.12)
By (B.6), we have
\[ A_T^{(0)} = \frac{1}{4} (D\bar{X})^2 + \frac{1}{4} T \left[ D\bar{X}, \hat{c}(\nabla \bar{f}) \right] + \frac{1}{4} T^2 |d\bar{f}|^2. \] (B.13)

If \( s \) vanishes on \( U_\varepsilon \), then
\[ \|(d\bar{f})s\|_0^2 \geq C\|s\|_0^2. \] (B.14)

Since \([D\bar{X}, \hat{c}(\nabla \bar{f})]\) is an operator of order zero, we have
\[ \|\left[D\bar{X}, \hat{c}(\nabla \bar{f})\right]_{s,s}\| \leq C\|s\|_0^2. \] (B.15)

By Lemma A.1, we have
\[ \|D\bar{X}s\|_0^2 \geq \frac{1}{2C_2} \|s\|_{\tilde{E}_1}^2 \cdot \|s\|_0^2. \] (B.16)

By (B.13)–(B.16), if \( s \in \tilde{E} \) vanishes on \( U_\varepsilon \), then
\[ \|A_T^{(0)}T s\|_0^2 \geq C\|s\|_{\tilde{E}_1}^2 + C'T^2|s|_0^2 - C''|s|_0^2. \] (B.17)

Using (B.12), (B.17), Lemma A.1 and proceeding as in [5, p. 115, 116], we find that if \( s \in \tilde{E}_1\perp T \),
\[ \|A_T^{(0)}T s\|_0^2 \geq C\|s\|_{\tilde{E}_1}^2 + C'T^2\|\hat{c}(\nabla \bar{f})_{s} s\|_0^2 + C''T|s|_0^2 - C'''|s|_0^2. \] (B.18)

By (A.10), one has
\[ \left|\overline{p_T}\right|_0 \leq C \left( \frac{\|p_T s\|_{\tilde{E}_1}}{|T|} + \|p_T s\|_0 \right), \]
\[ \left|\overline{p_T} A_T^{(0)} \overline{p_T} s\right|_0 \leq C \left( \frac{\|\overline{p_T} s\|_{\tilde{E}_1}}{|T|} + \|\overline{p_T} s\|_0 \right). \] (B.19)

Using (B.18) and the second inequality in (B.19), for \( T \geq 1 \) large enough,
\[ \left|\overline{p_T} A_T^{(0)} \overline{p_T} s\right|_0^2 \geq C \left( \frac{\|\overline{p_T} s\|_{\tilde{E}_1}}{|T|} \right)^2 + C'T^2\|\hat{c}(\nabla \bar{f})_{\overline{p_T} s}\|_0^2 + C''T \left|\overline{p_T} s\right|_0^2 - C'''\left|\overline{p_T} s\right|_0^2. \] (B.20)

By definition of \( \overline{p_T} \), we have
\[ \|\overline{p_T} s\|_{\tilde{E}_1} \leq C\sqrt{|T|} \|\overline{p_T} s\|_0. \] (B.21)

Using (B.10), (B.19)–(B.21), for \( T \geq 1 \) large enough, we obtain the first inequality in (B.9).

Now we prove the second inequality in (B.9). Clearly
\[ \left|A_T^{(0)} s\right|_0 \leq \left|A_T^{(0)} \overline{p_T} s\right|_0 + \left|A_T^{(0)} \overline{p_T} s\right|_0. \] (B.22)

By (B.6),
\[ \left|A_T^{(0)} \overline{p_T} s\right|_0 \leq C \left( \|s\|_{\tilde{E}_1} + T \|\hat{c}(\nabla \bar{f})_{\overline{p_T} s}\|_0 \right). \] (B.23)

From (B.19), (B.21)–(B.23), we get the second inequality in (B.9).
Now we prove the third inequality in (B.9). Put
\[ \overline{H} = \frac{1}{2} \left[ D^\overline{X}, \overline{A}^{(>0)} \right] + \overline{A}^{(>0), 2}. \] (B.24)

Then \( \overline{H} \) is a fiberwise first order differential operator, and moreover
\[ \overline{R}_T = \overline{H} + \frac{T}{2} \left[ \overline{\nabla}(\nabla \overline{f}), \overline{A}^{(>0)} \right]. \] (B.25)

Clearly, by Lemma A.1, if \( s, s' \in \Lambda^\bullet(T^s S) \otimes \overline{E} \),
\[ |\overline{H}s, s'_0| \leq C \left( |s|_{T, 1} |s'|_0 + |s|_0 |s'|_{T, 1} \right) + \left[ |\overline{H}_{pT} s, \overline{p}_T s'|_0 \right]. \] (B.26)

Moreover, if \( U \) is a smooth section of \( T \overline{X} \), as \([3, (9.66)]\), we have
\[ \left| Z|\nabla U(T^s \overline{X}) \otimes \overline{p}_{T} s'_0 \right| \leq C \left| \overline{p}_T s'_0 \right|. \] (B.27)

Then by \([3, (9.65)]\), we have
\[ \left| \overline{H}_{pT} s, \overline{p}_T s'_0 \right| \leq C \left( |s|_{T, 1} |s'|_0 + |s|_0 |s'|_{T, 1} \right). \] (B.28)

Also as \([3, (9.68)-(9.72)]\), we have
\[ T \left| \left[ \overline{\nabla}(\nabla \overline{f}), \overline{A}^{(>0)} \right] s, s' \right| \leq \left( |s|_{T, 1} |s'|_0 + |s|_0 |s'|_{T, 1} \right). \] (B.29)

Then from the above we get the third inequality in (B.9). \( \square \)

Now we fix \( \varepsilon > 0 \) as in Theorem B.3.
Take \( c_2 \in (0, 1] \), let \( D = \delta_3 \cup \Delta_3 \) be the contour in \( \mathbb{C} \) defined by
\[ \delta_3 = \left\{ \lambda \in \mathbb{C} ||\lambda|| = \frac{c_2}{2} \right\} \]
and
\[ \Delta_3 = \left\{ z = x + iy|x = \frac{3}{4} c_2, -1 \leq y \leq 1 \right\} \cup \left\{ z = x + iy|y = \pm 1, x \geq \frac{3}{4} c_2 \right\}. \] (B.30)

If \( A \in \mathcal{L}(\overline{E}^0, \overline{E}^0) \) (resp. \( A \in \mathcal{L}(\overline{E}^{-1}, \overline{E}^1) \)), let \( \|A\|^{0, 0}_T \) (resp. \( \|A\|^{-1, 1}_T \)) be the norm of \( A \) with respect to the norm \( \| \cdot \|_0 \) (resp. the norms \( \| \cdot \|_{T, -1}, \| \cdot \|_{T, 1} \)). The following \( L^2 \) analogue of \([3, \text{Theorem } 9.15]\) holds.

**Theorem B.4** There exist \( T_0 \geq 1, C > 0, p \in \mathbb{N} \), such that for \( T > T_0 \), \( \lambda \in \delta_3 \cup \Delta_3 \), the resolvent \( \left( \lambda - \overline{A}_T^2 \right)^{-1} \) exists and
\[ \left\| \left( \lambda - \overline{A}_T^2 \right)^{-1} \right\|_{T}^{0, 0, 1} \leq C(1 + |\lambda|)^p. \] (B.31)

**Proof** Recall that \( \overline{A}_T^{(0)} = \frac{1}{2} D \overline{X} + \frac{1}{2} T \overline{\nabla}(\nabla \overline{f}) \). For \( \delta' > 0 \), \( A > 0 \) set
\[ U = \{ \lambda \in \mathbb{C}, \text{Re}(\lambda) \leq \delta' \text{Im}^2(\lambda) - A \}. \] (B.32)

Using Lemma A.1, we have the following analogue of \([5, (11.83)]\),
\[ \|s\|_2 \leq C' \left( \left\| \left( \lambda - \overline{A}_T^{(0), 2} \right) s \right\|_0 + \|s\|_0 \right). \] (B.33)
Then using the first two inequalities in (B.9), proceeding as in [5, Theorem 11.27], we find that if \( \delta' \) is small enough, and \( A \) is large enough, for \( T \geq 1, \lambda \in U \),

\[
\left\| \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} \right\|_{0,0}^{0,0} \leq C, \\
\left\| \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} \right\|_{-1,1}^{-1,1} \leq C(1 + |\lambda|)^2. \tag{B.34}
\]

Since \( S \) is compact, then \( \overline{A}_T^{(0,2)} \) has a spectral gap for \( T \geq T_0 \) (cf. [9, Proposition 5.2]). Take \( \lambda \in \delta_3 \cup \Delta_3 \), then \( (\lambda - \overline{A}_T^{(0,2)})^{-1} \) exists for \( T \geq T_0 \) and moreover

\[
\left\| \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} \right\|_{0,0}^{0,0} \leq C. \tag{B.35}
\]

If \( \lambda_0 \in U, \lambda \in \delta_3 \cup \Delta_3, T \geq T_0 \), then

\[
\left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} = \left( \lambda_0 - \overline{A}_T^{(0,2)} \right)^{-1} + \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} (\lambda_0 - \lambda) \left( \lambda_0 - \overline{A}_T^{(0,2)} \right)^{-1}. \tag{B.36}
\]

From (B.34)–(B.36), we get

\[
\left\| \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} \right\|_{-1,0}^{-1,0} \leq C(1 + |\lambda|). \tag{B.37}
\]

Also

\[
\left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} = \left( \lambda_0 - \overline{A}_T^{(0,2)} \right)^{-1} + \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} (\lambda_0 - \lambda) \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1}. \tag{B.38}
\]

By (B.34), (B.37) and (B.38), we obtain

\[
\left\| \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} \right\|_{-1,1}^{-1,1} \leq C(1 + |\lambda|)^2. \tag{B.39}
\]

Moreover, if \( \lambda \in \delta_3 \cup \Delta_3 \), then

\[
\left( \lambda - \overline{A}_T^{2} \right)^{-1} = \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} + \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} \overline{R}_T \left( \lambda - \overline{A}_T^{(0,2)} \right)^{-1} + \cdots \tag{B.40}
\]

and the expansion terminates after a finite numbers of terms. By Theorem B.3,

\[
\left\| \overline{R}_T \right\|_{1,-1}^{1,-1} \leq C. \tag{B.41}
\]

Using (B.39)–(B.41), we get (B.31). The proof is completed. \( \square \)

Since \( B \) is compact, there exists a finite family of smooth functions \( f_1, \ldots, f_q \) on \( M \) with values in \([0, 1]\), such that

\[
B = \bigcap_{j=1}^{q} \left\{ x \in M, f_j(x) = 0 \right\}, \tag{B.42}
\]

and that on \( B \), \( df_1, \ldots, df_q \) span \( T^*X \). We lift \( f_1, \ldots, f_q \) to \( \tilde{M} \) and denote them by \( \tilde{f}_1, \ldots, \tilde{f}_q \).

Similarly, there exists a finite family of smooth sections \( U_1, \ldots, U_r \) of \( TX \) such that for any \( x \in M, U_1(x), \ldots, U_r(x) \) spans \( (TX)_x \). Let \( \tilde{U}_1, \ldots, \tilde{U}_r \) be the liftings of \( U_1, \ldots, U_r \).
Let $\gamma : \mathbb{R} \to [0, 1]$ be a smooth function such that

$$\gamma(a) = 1 \text{ for } a \leq 1/2,$$
$$0 \text{ for } a \geq 1.$$ 

If $Z \in T\tilde{X}$, set

$$\rho(Z) = \gamma\left(\frac{|Z|}{\varepsilon}\right).$$

**Definition B.5** For $T \geq 1$, let $\tilde{L}_T$ be the family of operators acting on $\tilde{E}$

$$\tilde{L}_T = \left\{ \nabla_\Lambda^\Lambda(T^*\tilde{X}) \otimes \tilde{F}, \frac{1}{\sqrt{T}}p_T^{-1/2}p_T^{-1}p_T^{-1}, \sqrt{T}p_T^{-1}f_jp_T^{-1} \right\}.$$  (B.43)

For $k \in \mathbb{N}$, let $Q^k_T$ be the family of operators $Q$ acting on $\tilde{E}$ which can be written in the form

$$Q = Q_1 \ldots Q_k, \quad Q \in \tilde{L}_T.$$  (B.44)

If $k \in \mathbb{N}$, we equip the Sobolev fibers $\tilde{E}^k$ with the Hilbert norm $\| s \|_{T,k}$ such that if $s \in \tilde{E}$,

$$\| s \|_{T,k}^2 = \sum_{l=0}^{k} \sum_{Q \in Q^l_T} |Qs|_{T,1}^2.$$  (B.45)

By the same proof of [3, Theorem 9.17], we have

**Theorem B.6** Take $k \in \mathbb{N}$. There exists $C_k > 0$ such that for $T \geq T_0$, $Q_1, \ldots, Q_k \in \tilde{L}_T$, $s, s' \in \Lambda^\Lambda(T^*S) \otimes \tilde{E}$,

$$\left\| \left[ Q_1, \left[ Q_2, \ldots, [Q_k, A^2_T] \right] \right] s, s' \right\|_0 \leq C_k |s|_{T,1}|s'|_{T,1}. $$  (B.46)

If $A \in \mathcal{L}(\tilde{E}^m, \tilde{E}^{m'})$, we denote by $\| A \|_{T,m,m'}$ the norm of $A$ with respect to the norms $\| \cdot \|_{T,m}$, $\| \cdot \|_{T,m'}$. Then by the same proof of [3, Theorem 9.18], we have

**Theorem B.7** For any $m \in \mathbb{N}$, there exist $p_m \in \mathbb{N}$, $C_m > 0$ such that for $T \geq T_0$, $\lambda \in \Delta_3$,

$$\left\| \lambda - A^2_T \right\|_{T,m,m+1}^{m,m+1} \leq C_m (1 + |\lambda|)^{p_m}. $$  (B.47)

**Proof** Clearly for $T \geq 1$,

$$\| s \|_{T,1} \leq C|s|_{T,1}. $$  (B.48)

When $m = 0$, our Theorem follows from Theorem B.4 and from (B.48).

Using Theorems B.4 and B.6, the proof of our Theorem proceeds as the proof of [5, Theorem 11.30].

Let $\Delta_3$ be defined as in (B.30). If $a \notin \Delta_3$, put

$$F_u(a) = \frac{1}{2\pi i} \int_{\Delta_3} \frac{\exp(-u^2\lambda)}{\lambda - a} d\lambda.$$  (B.49)

Then

$$F_u(a) = \begin{cases} \exp(-u^2a) & \text{if } a \text{ lies inside the contour } \Delta_3, \\ 0 & \text{if } a \text{ lies outside } \Delta_3. \end{cases}$$
Putting
\[ F_u \left( \overline{A_T^2} \right) = \frac{1}{2\pi i} \int_{\Delta_3} \frac{\exp(-u^2\lambda)}{\lambda - \overline{A_T^2}} \, d\lambda. \quad (B.50) \]

**Definition B.8** Let \( F_u(\overline{A_T^2})(x, x') (x, x' \in \mathcal{X}) \) be the smooth kernel associated to the operator \( F_u(\overline{A_T^2}) \) with respect to
\[ d\nu_u(x')/(2\pi)^{\dim\mathcal{X}/2}. \]

**Theorem B.9** For any \( \alpha > 0, m \in \mathbb{N} \), there exist \( C > 0, C' > 0 \) such that if \( x \in \mathcal{M}, d\mathcal{X}(x, \mathcal{B}) \geq \alpha \), for \( u \geq u_0, T \geq T_0 \),
\[ |F_u(\overline{A_T^2})(x, x')| \leq \frac{C \exp(-C'u^2)}{T^m}. \quad (B.51) \]

For any \( m \in \mathbb{N} \), there exist \( C > 0, C' > 0 \) such that for \( y \in \mathcal{B}, u \geq u_0, T \geq T_0 \),
\[ \sup_{|Z| \leq \frac{T}{2} \sqrt{T}} (1 + |Z|)^m \frac{1}{T^{\dim\mathcal{X}/2}} \left| F_u \left( \overline{A_T^2} \right) \left( \left( y, \frac{Z}{\sqrt{T}} \right), \left( y', \frac{Z'}{\sqrt{T}} \right) \right) \right| \leq C \exp(-C'u^2). \quad (B.52) \]

For any \( m \in \mathbb{N} \), there exist \( C > 0, C' > 0 \) such that for \( y \in \mathcal{B}, u \geq u_0, T \geq T_0 \),
\[ \sup_{|\alpha| \leq m', |\alpha'| \leq m'} |\alpha|^{|\alpha'|} \frac{1}{T^{\dim\mathcal{X}/2}} \left| F_u \left( \overline{A_T^2} \right) \left( \left( y, \frac{Z}{\sqrt{T}} \right), \left( y', \frac{Z'}{\sqrt{T}} \right) \right) \right| \leq C \exp(-C'u^2). \quad (B.53) \]

**Proof** Clearly for \( p \in \mathbb{N} \),
\[ \frac{1}{2\pi i} \int_{\Delta_3} \frac{\exp(-u^2\lambda)}{\lambda - \overline{A_T^2}} \, d\lambda = (-1)^{2p-1} \frac{(2p - 1)!}{2\pi i (u^2)^{2p-1}} \int_{\Delta_3} \frac{\exp(-u^2\lambda)}{(\lambda - \overline{A_T^2})^{2p}} \, d\lambda. \quad (B.54) \]

By Theorem B.7, we know that there exist \( C > 0, q \in \mathbb{N} \) such that if \( \lambda \in \Delta_3, Q \in Q_T^l \), \( l \leq p \),
\[ \left\| Q \left( \lambda - \overline{A_T^2} \right)^{-p} \right\|^{0,0}_T \leq C (1 + |\lambda|)^q. \quad (B.55) \]

By introducing the obvious adjoint operator with respect to \( \langle \cdot, \cdot \rangle_0 \), we also find that if \( \lambda \in \Delta_3, Q' \in Q_T^{l'}, l' \leq p \),
\[ \left\| \left( \lambda - \overline{A_T^2} \right)^{-p} Q' \right\|^{0,0}_T \leq C (1 + |\lambda|)^q. \quad (B.56) \]

From (B.55), (B.56), we see that if \( \lambda \in \Delta_3, Q \in Q_T^l, Q' \in Q_T^{l'}, l, l' \leq p \),
\[ \left\| Q \left( \lambda - \overline{A_T^2} \right)^{-2p} Q' \right\|^{0,0}_T \leq C (1 + |\lambda|)^{2q}. \quad (B.57) \]

From (B.54), (B.57), we find that if \( Q \in Q_T^l, Q' \in Q_T^{l'}, \) there exist \( C > 0, C' > 0 \) such that
\[ \left\| QF_u \left( \overline{A_T^2} \right) Q' \right\|^{0,0}_T \leq C \exp(-C'u^2). \quad (B.58) \]
By (B.58) and by Sobolev inequalities (cf. [16]), we get (B.51). Using (B.57) and proceeding as in [5, proof of Theorem 13.32], we get (B.52), (B.53). The proof is completed. □

By the same proof of [3, Proposition 9.21], we have

**Proposition B.10** There exist $C > 0$, $p \in \mathbb{N}$ such that for $T \geq T_0$, $\lambda \in \delta_3 \cup \Delta_3$, then

$$
\left\| \bar{\rho}_T^\perp (\lambda - \bar{A}_T)^{-1} \right\|_{0,0} \leq \frac{C}{\sqrt{T}} (1 + |\lambda|)^p.
$$

(B.59)

**Proof** This follows from Definition B.2 and Theorem B.4. □

If $A$ is an bounded operator acting on $\tilde{\mathcal{E}}^0$, we write $A$ in matrix form with respect to the splitting $\tilde{\mathcal{E}}^0 = \tilde{\mathcal{E}}^{0,0}_T \oplus \tilde{\mathcal{E}}^{0,\perp}_T$,

$$
A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.
$$

(B.60)

By the same proof of [3, Proposition 9.22], we have

**Proposition B.11** There exist $C > 0$, $p \in \mathbb{N}$, $T_0 \geq 1$ such that if $T \geq T_0$, $\lambda \in \delta_3 \cup \Delta_3$, the resolvent $(\lambda - \bar{A}_{T,4})^{-1}$ exists and moreover

$$
\left\| (\lambda - \bar{A}_{T,4})^{-1} \right\|^{-1,1}_T \leq C (1 + |\lambda|)^2.
$$

(B.61)

**Proof** By Theorem B.3, it is clearly that $\bar{A}_{T,4}$ verifies inequalities similar to (B.9). Therefore by using the notations in (B.32), for $\delta' > 0$ small enough, and $A > 0$ large enough, if $\lambda \in U$, then

$$
\left\| (\lambda - \bar{A}_{T,4})^{-1} \right\|^{-1,1}_T \leq C (1 + |\lambda|)^2.
$$

(B.62)

By Theorem B.3, for $T \geq 1$ large enough, if $s \in \tilde{\mathcal{E}}^{1,\perp}_T$,

$$
\left\langle \bar{A}^{(0),2}_T s, s \right\rangle_0 \geq CT \left\| \bar{\rho}_T^\perp s \right\|_0^2.
$$

(B.63)

By (B.63), we find that there is $C > 0$, $T_0 \geq 1$ such that for $T \geq T_0$, $\lambda \in \Delta_3 \cup \delta_3$,

$$
\left\| (\lambda - \bar{A}^{(0),2}_{T,4})^{-1} \right\|^{-0,0}_T \leq C.
$$

(B.64)

Using (B.62), (B.64), and proceeding as in (B.34)–(B.39), we get for $T \geq T_0$, $\lambda \in \Delta_3 \cup \delta_3$,

$$
\left\| (\lambda - \bar{A}_{T,4})^{-1} \right\|^{-1,1}_T \leq C (1 + |\lambda|)^2.
$$

(B.65)

Thus if $\lambda \in \Delta_3 \cup \delta_3$,

$$
(\lambda - \bar{A}_{T,4})^{-1} = (\lambda - \bar{A}^{(0),2}_{T,4})^{-1} + (\lambda - \bar{A}_{T,4})^{-1} \bar{R}_{T,4} (\lambda - \bar{A}^{(0),2}_{T,4})^{-1} + \cdots.
$$

(B.66)
By Theorem B.3, we get
\[ \| R_{T,4} \|_{T}^{-1} \leq C. \]  
(B.67)

By (B.66) and (B.67), we get (B.61). \( \square \)

By Proposition B.10, proceeding as [3, Theorem 9.23], we have

**Theorem B.12** There exist \( C > 0, C' > 0 \) such that for \( u \geq 1, \ T \geq T_0, \)
\[ \| \hat{p}_T F_u \left( \bar{A}_T^2 \right) \|_{0.0} \leq \frac{C}{\sqrt{T}}, \ \| \hat{p}_T F_u \left( \bar{A}_T^2 \right) \|_{0.0} \leq \frac{C}{\sqrt{T}}. \]  
(B.68)

**Proof** In view of Proposition B.10, we can proceed exactly as in \([5, p. 264-267] \).

By the same proof of [3, Theorem 9.27], we have

**Theorem B.13** There exist \( c > 0, C > 0 \) such that for \( u \geq u_0, \ T \geq 1, \)
\[ \| F_u \left( \bar{A}_T^2 \right) \|_{0.0} \leq \frac{c \exp(-Cu^2)}{T^{1/4}}. \]  
(B.69)

**Proof** The proof is the same as the proof of \([5, Theorem 13.42] \).

Using above results, we now prove the \( L^2 \)-case of \([3, (9.149)] \).

Let \( Y' \subset \tilde{B} \) be a fundamental domain and \( X' \subset \tilde{X} \) be a fundamental domain and such that \( Y' \subset X' \). Then
\[ \text{Tr}_{\Gamma,T} \left[ \frac{1}{\Delta} \int \frac{\exp(-u^2 \lambda)}{\lambda - \bar{A}_T^2} \right] = \int_{X'} \text{Tr}_s \left[ N F_u \left( \bar{A}_T^2 \right) (x, x) \right] \frac{dv_{\tilde{X}}(x)}{(2\pi)^{\dim\tilde{X}/2}}. \]  
(B.70)

Let \( d_{T,\tilde{X}} \) be the volume form on the fibers of \( T \tilde{X} \). For \( y \in Y', \ Z \in B_{\epsilon_0}, \) let \( k(y, Z) \) be the smooth positive function defined on \( B_{\epsilon_0} \) by the equation
\[ dv_{\tilde{X}}(y, Z) = k(y, Z)dv_{T,\tilde{X}}(Z). \]

By Theorem B.9, for any \( m \in \mathbb{N} \)
\[ \left| \int_{X' \cap \{ x, d_{T,\tilde{X}}(x, y') \geq \xi/4 \}} \text{Tr}_s \left[ N F_u \left( \bar{A}_T^2 \right) (x, x) \right] \frac{dv_{\tilde{X}}(x)}{(2\pi)^{\dim\tilde{X}/2}} \right| \leq \frac{C}{T^m} \exp \left(-C'u^2 \right). \]  
(B.71)

Also
\[ \int_{X' \cap \{ x, d_{T,\tilde{X}}(x, y') \leq \xi/4 \}} \text{Tr}_s \left[ N F_u \left( \bar{A}_T^2 \right) (x, x) \right] \frac{dv_{\tilde{X}}(x)}{(2\pi)^{\dim\tilde{X}/2}} \]
\[ = \sum_{y \in Y'} \int_{T X', |Z| \leq \xi/\sqrt{T}} \text{Tr}_s \left[ \frac{F_u \left( \bar{A}_T^2 \right)}{T^{\dim\tilde{X}/2}} \left( \left( y, \frac{Z}{\sqrt{T}} \right), \left( y, \frac{Z}{\sqrt{T}} \right) \right) \right] k \]
\[ \times \left( y, \frac{Z}{\sqrt{T}} \right) \frac{dv_{T,\tilde{X}}(Z)}{(2\pi)^{\dim\tilde{X}/2}}. \]  
(B.72)
By Theorems B.9 and B.13 and proceeding as in [5, Section 13 q], we find that there exist
$c > 0, C > 0, \delta \in (0, 1/2)$ such that if $y \in \tilde{B}, \, Z \in T\tilde{X}, \, |Z| \leq \frac{\epsilon \sqrt{T}}{4},$
\[
\left| \frac{1}{2\pi} \dim \frac{\chi}{2} F_\mu \left( \frac{\omega^2}{T \dim \chi/2} \right) \left( \left( y, \frac{Z}{\sqrt{T}} \right), \left( y, \frac{Z}{\sqrt{T}} \right) \right) \right| \leq \frac{c \exp(-Cu^2)}{T^\delta}.
\]  
(B.73)

By (B.52), (B.73), we find that for any $p \in N$, there exist $c > 0, C > 0$ such that if $y \in \tilde{B}, \, Z \in T_y\tilde{X}, \, |Z| \leq \frac{\epsilon \sqrt{T}}{4},$
\[
\left| \frac{1}{2\pi} \dim \frac{\chi}{2} F_\mu \left( \frac{\omega^2}{T \dim \chi/2} \right) \left( \left( y, \frac{Z}{\sqrt{T}} \right), \left( y, \frac{Z}{\sqrt{T}} \right) \right) \right| \leq \frac{c \exp(-Cu^2)}{(1 + |Z|)^\rho T^{3/2}}.
\]  
(B.74)

From (B.74), we deduce that there exist $c > 0, C > 0, \delta \in (0, 1/4)$ such that
\[
\sum_{y \in Y} \int_{|Z| \leq \frac{\epsilon \sqrt{T}}{4}} \operatorname{Tr}_{\gamma, s} \left[ N F_\mu \left( \frac{\omega^2}{T} \right) \left( \left( y, \frac{Z}{\sqrt{T}} \right), \left( y, \frac{Z}{\sqrt{T}} \right) \right) \right] \leq \frac{c \exp(-Cu^2)}{T^\delta}.
\]  
(B.75)

So by (B.70)–(B.72), (B.75), we obtain
\[
\left| \operatorname{Tr}_{\gamma, s} \left[ N F_\mu \left( \frac{\omega^2}{T} \right) \right] \right| \leq \frac{c \exp(-Cu^2)}{T^\delta}.
\]  
(B.76)

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