DEGENERATE RIEemann-HILBERT-BIRKHOFF PROBLEMS, SEMISIMPLICITY, AND CONVERGENCE OF WDVV-POTENTIALS

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Abstract. In the first part of this paper, we give a new analytical proof of a theorem of C. Sabbah on integrable deformations of meromorphic connections on $\mathbb{P}^1$. This theorem generalizes a previous result of B. Malgrange to the case of connections admitting irregular singularities of Poincaré rank 1 with coalescing eigenvalues. In the second part of this paper, as an application, we prove that any semisimple formal Frobenius manifold (over $\mathbb{C}$), with unit and Euler field, is the completion of an analytic pointed germ of a Dubrovin-Frobenius manifold. In other words, any formal power series, which provides a quasi-homogenous solution of WDVV equations and defines a semisimple Frobenius algebra at the origin, is actually convergent under no further tameness assumptions.

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1. Introduction

In this paper, we address the problem of convergence of formal solutions, in the ring of formal power series, of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) associativity equations. This is the overdetermined system of non-linear partial differential equations, in a single scalar function $F(t_1, \ldots, t_n)$, given by

$$
\sum_{\mu, \nu} \frac{\partial^3 F}{\partial t_\alpha \partial t_\beta \partial t_\mu} \eta^{\mu \nu} \frac{\partial^3 F}{\partial t_\gamma \partial t_\delta} = \sum_{\mu, \nu} \frac{\partial^3 F}{\partial t_\delta \partial t_\beta \partial t_\mu} \eta^{\mu \nu} \frac{\partial^3 F}{\partial t_\gamma \partial t_\alpha}, \quad \alpha, \beta, \gamma, \delta = 1, \ldots, n,
$$

$$
\frac{\partial^3 F}{\partial t_1 \partial t_\alpha \partial t_\beta} = \eta_{\alpha \beta} = \text{const.}, \quad \eta = (\eta_{\alpha \beta})_{\alpha, \beta}, \quad \eta^{-1} = (\eta^{\alpha \beta})_{\alpha, \beta} \quad \alpha, \beta = 1, \ldots, n.
$$

Introduced in the physics of topological field theories [Wit90, DVV91], the geometry of solutions $F$ of WDVV equations, satisfying a further quasi-homogeneity condition

$$
\sum_{\alpha} [(1 - q_\alpha)t_\alpha + r^\alpha] \frac{\partial F}{\partial t_\alpha} = (3 - d)F(t) + \text{quadratic terms in } t,
$$

for suitable complex numbers $q_\alpha, r^\alpha, d \in \mathbb{C}$, was firstly axiomatized by B. Dubrovin, with the notion of Frobenius manifolds [Dub92, Dub96, Dub98, Dub99].

It was soon realized that these quasi-homogeneous solutions of WDVV equations arise in areas of mathematics which are very apart from each other (singularity theory, algebraic and symplectic geometry, integrable systems, mirror symmetry, to name just a few), often leading to new and non-trivial relations between them, see [Dub96, Man99, Her02, Sab07].

Typically, the corresponding solutions $F(t)$ of WDVV equations are given as generating functions of numerical sequences of geometrical interest (e.g. Gromov-Witten invariants). Consequently, they can be handled just as formal power series in $k[[t]]$, where $k$ is a commutative $\mathbb{Q}$-algebra, defining a formal Frobenius manifold structure on the formal spectrum $\text{Spf} k[[t]]$, see [Man99, III.§1]. This defines a formal family of Frobenius algebras with structure constants given by $c_{\alpha \beta}^\gamma(t) := \eta^{\alpha \gamma} \partial_{\alpha \beta}^3 F(t)$.

The relevance of these formal structures is further highlighted by their deep relations with the cohomology of the Deligne-Mumford moduli stacks $\overline{M}_{g,n}$ of $n$-pointed stable curves of genus $g$, [KM94, Man99]. Remarkably enough, any formal Frobenius manifold is equivalent to a tree level\(^1\) Cohomological Field Theory (CohFT), i.e. the datum of a family of

\[^1\]A richer notion of complete CohFT on a given $(H, \eta)$ is also available, in which the datum is enriched to a family $(\Omega_{g,n})_{g,n}$ of $k$-linear tensors $\Omega_{g,n} \in (H^*)^\otimes n \otimes_k H^*(\overline{M}_{g,n}; k)$, satisfying further compatibility
\( \mathfrak{S}_n \)-covariant\(^2 \) tensors \( \Omega_{0,n} \in (H^*)^n \otimes_k H^*(\mathcal{M}_{0,n}; k) \), on a given free metric \( k \)-module \((H, \eta)\) of finite rank, satisfying some compatibility conditions with respect to the natural forgetful morphisms \( \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1} \), and gluing morphisms \( \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \to \mathcal{M}_{0,n_1+n_2} \). The corresponding WDVV-potential \( F(t) \) is a generating power series for integrals of the form \( \int_{\mathcal{M}_{0,n}} \Omega_{0,n}(\bigotimes_{j=1}^n \Delta_{\alpha_j}) \) for a \( k \)-basis \((\Delta_j)_j \) of \( H \). See [KM94, Man99, Pan18] for more details.

One of the main points of the current paper is to find sufficient conditions ensuring the convergence of quasi-homogeneous solutions \( F \in k[[t]] \) of WDVV equations, in the case \( k = \mathbb{C} \). The convergence condition allows to jump from the formal category to the complex analytic category: formal Frobenius manifolds can be promoted to analytic Frobenius manifolds, the class of geometrical objects originally conceived by Dubrovin, and for this reason also called Dubrovin-Frobenius manifolds.

The new main result of this paper, Theorem 5.1, claims that any formal semisimple Frobenius manifold over \( k = \mathbb{C} \) is actually the completion of a pointed germ of an analytic Dubrovin-Frobenius manifold. Alternatively stated, given a quasi-homogeneous formal solution \( F \in \mathbb{C}[t] \) of WDVV equations whose corresponding Frobenius \( \mathbb{C} \)-algebra at the origin \( t = 0 \) is semisimple, its domain of convergence is non-empty, and it thus carries a Dubrovin-Frobenius manifold structure. This statement is a refinement of a seemingly known result, referred to as a "general fact" in [Man99, III.§7.1, pag.135], and stated under stronger unnecessary tameness assumptions\(^3 \) (see the next paragraph).

At the core of our proof there is the local identification of semisimple points \( t \) of a Dubrovin-Frobenius manifold with the parameters of isomonodromic deformations of ordinary differential equations with rational coefficients, of the form

\[
\frac{d}{dz} Y(z, t) = \left( \mathcal{U}(t) + \frac{1}{z} \mu(t) \right) Y(z, t), \tag{1.1}
\]

where \( \mathcal{U}, \mu \) are (matrices representing) suitably defined tensors on the Dubrovin-Frobenius manifold. This identification – one of the main points of the theory of Dubrovin– was originally established in [Dub96, Dub98, Dub99] at tame semisimple points, i.e. points \( t \) at which the leading term \( \mathcal{U}(t) \) of the coefficient of (1.1) has simple spectrum. Subsequently, in [CG17, CG18, CDG19, CDG20] the isomonodromic approach to the Frobenius geometry was extended to all semisimple points, including points \( t \) at which some of the eigenvalues of \( \mathcal{U}(t) \) coalesce.

The proof of Theorem 5.1 consists of two parts. Firstly, given a formal Frobenius manifold \( F \in \mathbb{C}[t] \), it is constructed an analytic family (1.1) of ODEs specializing to the given one\(^4 \)

\(^2\)Here \( \mathfrak{S}_n \) denotes the symmetric group on a finite set with \( n \) elements.

\(^3\)I do not know any reference in literature where a complete proof is given. I thank Yu.I.Manin for a friendly e-mail correspondence on this point. The current paper both recovers a proof of this known fact, and it also removes the tameness assumption.

\(^4\)Given a formal Frobenius manifold, the system (1.1) has coefficients in \( M_n(\mathbb{C}[t]) \). Hence, for \( t = 0 \), we have a well defined differential system with coefficients in \( M_n(\mathbb{C}) \).
for $t = 0$, and defining a Dubrovin-Frobenius manifold. Secondly, it is proved that the underlying analytic\(^5\) WDVV-potential $F^{\text{an}} \in \mathbb{C}\{t\}$ coincides with the original formal one, i.e. $F = F^{\text{an}}$. It is thus clear that the first step of the proof of Theorem 5.1 relies on the existence of solutions of families of Riemann-Hilbert-Birkhoff boundary value problems. In the case $t = 0$ is a tame semisimple point of the given formal Frobenius manifold, a well-known result of B. Malgrange [Mal83a, Mal83b, Mal86], on the existence of universal integrable deformations of meromorphic connections on $\mathbb{P}^1$ with irregular singularities, can be applied. This leads to the already known result mentioned\(^6\) in [Man99, III.§7, pag.135].

In [Sab18], C. Sabbah obtained an extension of the theorem of Malgrange, in order to include the case of meromorphic connections on $\mathbb{P}^1$ which admit irregular singularities with coalescing eigenvalues. In the geometrical case attached to Frobenius manifolds, the assumptions of [Sab18, Th. 4.9] are satisfied. Sabbah Theorem can thus be applied in the first step of the proof of Theorem 5.1, in the case $t = 0$ is a coalescing semisimple point for the given formal Frobenius manifold. Remarkably enough, the assumptions of [Sab18, Th. 4.9] exactly coincide with the sharp conditions, found in [CG17, CDG19], under which the resulting analytic family (1.1) of ODEs has a well-behaved deformation theory of both formal and genuine solutions.

The original proof of [Sab18, Th. 4.9] is actually only one of the outcomes of a more general study, invoking a mix of techniques, including properties of good and very-good formal decompositions of flat meromorphic bundles [Sab93, Sab00], and recent results on meromorphic connections in dimension $\geq 2$ due to K. Kedlaya (in the complex analytic case) and T. Mochizuki (in the algebraic case), see [Ked10, Ked11, Moc09, Moc11a, Moc11b, Moc14]. In Section 3, we give an alternative proof of [Sab18, Th. 4.9], with a more analytical perspective, closer to the one of [CG17, CDG19]. Our proof is uniquely based on properties of Fredholm-operator-valued holomorphic functions. In particular, a result due to B. Gramsch [Gra70] – an analytical Fredholm alternative with respect to several parameters– will be invoked to prove that the solvability of a family of Riemann-Hilbert-Birkhoff boundary value problems is an open property, in the same spirit of [Zho89]. This is a well-known strategy for proving the Painlevé property of solutions of the isomonodromy deformation equations, see e.g. [FZ92, FIKN06].

Many of the results of this paper can be extended to the case of flat $F$-manifolds [HM99, Man05]. These are slightly weaker structures than the Frobenius one, but whose geometry encompasses even more areas of modern mathematics, such as special solutions of the oriented associativity equations [LM04], quantum $K$-theory [Lee04], all Painlevé transcendents [AL15], open WDVV equations [BB19], $F$-cohomological field theories [ABLR20], and even information geometry [CM20]. We plan to give more analytical details in a future publication.

**Structure of the paper.** In Section 2 we review necessary background material on the Riemann-Hilbert-Birkhoff problems with a geometrical perspective. The main results of

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\(^5\)Here $\mathbb{C}\{t\}$ denotes the algebra of convergent power series in $t$.

\(^6\)We warn the reader that in the exposition of [Man99], the isomonodromic system (1.1) is replaced by a Fuchsian one obtained by applying a (formal) Laplace transform, see [Man99, Ch. II.§1-3].
B. Malgrange on the existence of the universal integrable deformation of meromorphic connection, as well as their generalization to degenerate cases due to C. Sabbah, are presented and summarized.

Section 3 is devoted to an analytical proof of Malgrange-Sabbah Theorem. After introducing the notion of admissible data, we formulate a Riemann-Hilbert-Birkhoff boundary value problem \( P[u, \tau, M] \), depending on parameters \( u \in \mathbb{C}^n \). We factorize its solutions via two auxiliary RHB problems, and we analyze its solvability with respect to \( u \).

In Section 4 basic notions in the theory of both formal and analytic Frobenius manifolds are given. We explain how to pass from the analytic to the formal category, and vice-versa under convergence assumption of the WDVV potential.

In Section 5, we review necessary results on the extended deformed connection on both formal and analytic Frobenius manifolds, properties of solutions of the Darboux-Egoroff system of partial differential equations, and the reconstruction procedure of the Frobenius potential. Consequently, we prove the main new result, Theorem 5.1.

In the last Section 6, reformulations and applications of Theorem 5.1 to cohomological field theories and quantum cohomology are given.

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2. Degenerations of Riemann-Hilbert-Birkhoff inverse problems

2.1. Riemann-Hilbert-Birkhoff inverse problems. Let \( D \) be a disc centered at \( z = \infty \) in \( \mathbb{P}^1 \). Given a (trivial) vector bundle on \( D \) equipped with a meromorphic connection with a pole at \( z = \infty \), the Riemann-Hilbert-Birkhoff (RHB) problem is the following:

Problem 2.1. Does there exist a trivial vector bundle \( E^o \) on \( \mathbb{P}^1 \) equipped with a meromorphic connection \( \nabla^o \), restricting to the given data on \( D \), and with a further logarithmic pole only at \( z = 0 \)?

Assume that the pole at \( z = \infty \) is of Poincaré rank 1: in a basis of sections on \( D \), the meromorphic connection has matrix of connection 1-forms \( \Omega = -A(z)dz \), where the \( n \times n \) matrix \( A(z) \) equals

\[
A(z) = \sum_{k=0}^{\infty} A_k z^{-k}, \quad A_0 \neq 0.
\]

Denote by \( \mathcal{O}(\frac{1}{z}) \) the ring of convergent power series in \( \frac{1}{z} \). The RHB problem 2.1 is then equivalent to find a so-called Birkhoff normal form: does it exist a matrix \( G \in GL_n(\mathcal{O}(\frac{1}{z})) \)
such that \( B(z) = G^{-1}AG - G^{-1} \frac{d}{dz}G \) is of the form

\[
B(z) = B_0 + \frac{B_1}{z}, \quad B_0, B_1 \in M_n(\mathbb{C})?
\]

2.2. Universal integrable deformations of Birkhoff normal forms: Malgrange Theorems. In this paper we consider families of RHB problems parametrized by a parameter space \( X \), see [Mal83a, Mal83b, Mal86][Sab07, Ch.VI]. We assume \( X \) is a complex manifold.

**Definition 2.2.** Let \((E^o, \nabla^o)\) be a trivial vector bundle on a disc \( D \) equipped with a meromorphic connection with a pole of order 2 at \( z = \infty \). An integrable deformation of \((E^o, \nabla^o)\) parametrized by \( X \) is the datum \((E, \nabla)\) of

- a trivial vector bundle \( E \) on \( D \times X \),
- a flat connection \( \nabla \) on \( E \) with a pole of Poincaré rank 1 along \( \{\infty\} \times X \),

such that \((E, \nabla)\) restricts to \((E^o, \nabla^o)\) at a point \( x_o \in X \). The integrable deformation is called versal if any other deformation with base space \( X' \) is induced by the previous one via pull-back by a holomorphic map \( \varphi: (X', x'_o) \to (X, x_o) \). It is universal if the germ at \( x'_o \) of the base-change \( \varphi \) is uniquely determined.

Let \((E^o, \nabla^o)\) be a solution of a RHB problem 2.1, i.e. a trivial vector bundle on \( \mathbb{P}^1 \) with meromorphic connection with matrix (in a suitable basis of sections) of the form

\[
\Omega = - \left( \Lambda_o + \frac{B_o}{z} \right) dz. \tag{2.1}
\]

Recall that a matrix \( A \in M_n(\mathbb{C}) \) is said to be regular if any (and hence all) of the following equivalent conditions is satisfied:

1. the characteristic polynomial of \( A \) equals its minimal polynomial,
2. the commutator of \( A \) in \( M_n(\mathbb{C}) \) is of minimal dimension (i.e. it equals \( n \)),
3. the commutator of \( A \) in \( M_n(\mathbb{C}) \) is \( \mathbb{C}[A] \).

**Theorem 2.3 ([Mal83a, Mal86]).** Assume that the matrix \( \Lambda_o \) is regular. The connection \( \nabla^o \) with matrix (2.1) has a germ of universal deformation.

This result can be refined to a global statement, under the further semisimplicity assumption on \( \Lambda_o \). Let us then assume that \( \Lambda_o = \text{diag}(u^1_o, \ldots, u^n_o) \) with \( u^i_o \neq u^j_o \) for \( i \neq j \).

Let \( \Delta \) be the union of big diagonal hyperplanes in \( \mathbb{C}^n \), defined by the equations

\[
\Delta := \bigcup_{i<j} \{ u \in \mathbb{C}^n : u^i = u^j \},
\]

let \( X_n \) be the complement \( \mathbb{C}^n \setminus \Delta \), with base point \( u_o := (u^1_o, \ldots, u^n_o) \). Denote by \( \pi: (\tilde{X}_n, \tilde{u}_o) \to (X_n, u_o) \) the universal cover of \( X_n \), equipped with fixed base points \( \tilde{u}_o \) and \( u_o \), respectively. The space \( X_n \) is identified with the space of diagonal regular \( n \times n \) matrices.

**Theorem 2.4 ([JMU81, Mal83b]).** There exists on \( \mathbb{P}^1 \times \tilde{X}_n \) a vector bundle \( E \), equipped with a meromorphic connection \( \nabla \), such that

1. \( \nabla \) is flat, with a pole of Poincaré rank 1 along \( \{\infty\} \times \tilde{X}_n \), and a logarithmic pole along \( \{0\} \times \tilde{X}_n \),
2. \( \nabla \) is flat, with a pole of Poincaré rank 1 along \( \{\infty\} \times \tilde{X}_n \), and a logarithmic pole along \( \{0\} \times \tilde{X}_n \),
(2) $(E, \nabla)$ restricts to $(E^o, \nabla^o)$ at $\tilde{u}_o$.

(3) For any $\tilde{u} \in \tilde{X}_n$, the eigenvalues of the residue of $\nabla$ at the point $(\infty, \tilde{u})$ equal (up to permutation) the $n$-tuple $\pi(\tilde{u})$.

Let $\Theta \subseteq \tilde{X}_n$ be the hypersurface of points $\tilde{u} \in \tilde{X}_n$ such that $E|_{\mathbb{P}^1 \times \{\tilde{u}\}}$ is not trivial. The coefficients of $\nabla$ have poles along $\Theta$. Moreover, for any $\tilde{u} \in \tilde{X}_n \setminus \Theta$, the bundle with meromorphic connection $(E, \nabla)$ induces a universal deformation of its restriction $(E, \nabla)|_{\mathbb{P}^1 \times \{\tilde{u}\}}$.

It is possible to explicitly describe the matrix of the connection 1-forms of the universal deformation of Theorem 2.4.

For $u \in \mathbb{C}^n$, denote $\Lambda(u) := \text{diag}(u^1, \ldots, u^n)$, so that $\Lambda(u_o) = \Lambda_o$. Given a matrix $A$ denote by $A'$ its diagonal part, and by $A''$ its off-diagonal part.

For $u_o \notin \Delta$, there exists an off-diagonal matrix $F''(u)$, holomorphic near $u_o$, such that the flat connection $\nabla$ of Theorem 2.4 has matrix of connection 1-forms
\begin{equation}
-d(z\Lambda(u)) - ([\Lambda(u), F''(u)] + B'_o) \frac{dz}{z} - [d\Lambda(u), F''(u)],
\end{equation}
e.g. see [Sab07, VI.§3.f, eq. (3.12)]. Notice that the $dz$-component of (2.2) restricts to (2.1) at $u = u_o$. Moreover, there exists a $z^{-1}$-formal base change which transforms (2.2) into
\begin{equation}
-d(z\Lambda(u)) - B'_o \frac{dz}{z}.
\end{equation}

2.3. Integrable deformations of degenerate Birkhoff normal forms: Sabbah Theorem. In the notations of the previous section, assume $u_o \in \Delta$. Define the partition $\{1, \ldots, n\} = \bigsqcup_{r \in \mathbb{R}} I_r$ such that for any $r \in \mathbb{R}$ we have
\[ \{i, j\} \subseteq I_r \quad \text{if and only if} \quad u^i_o = u^j_o. \]

In [Sab18], C. Sabbah addressed the following problem.

**Question:** Is it possible to find an integrable deformation of the form (2.2) of the Birkhoff normal form (2.1) with $z^{-1}$-formal normal form (2.3)?

Remarkably, in [Sab18, Section 4] it is shown that the answer is positive, under (sharp) sufficient conditions on the coefficient $B_o$ of the normal form (2.1).

**Theorem 2.5** ([Sab18, Th. 4.9]). Let $u_o \in \Delta$, and $\mathcal{V}$ a neighborhood of $u_o$ in $\mathbb{C}^n$. Assume that
\begin{enumerate}
    \item $B''_o \in \text{Im } \text{ad}(\Lambda(u^o))$,
    \item $B'_o$ is partially non-resonant, i.e.
    \[ \forall r \in \mathbb{R}, \ \forall i, j \in I_r, \ (B'_o)_{ii} - (B'_o)_{jj} \notin \mathbb{Z} \setminus \{0\}. \]
\end{enumerate}
If $\mathcal{V}$ is sufficiently small, there exists a holomorphic hypersurface $\Theta \in \mathcal{V} \setminus \{u_o\}$ and a holomorphic off-diagonal matrix $F''(u)$ on $\mathcal{V} \setminus \Theta$, such that the meromorphic connection, on the trivial vector bundle on $\mathbb{P}^1 \times (\mathcal{V} \setminus \Theta)$, with matrix (2.2) is integrable, restricts to (2.1) at $u_o$, and it is formally equivalent at $z = \infty$ to the matrix connection (2.3).
3. An analytical proof of Sabbah Theorem

In this section we provide an analytical proof of Sabbah Theorem 2.5, based on properties of holomorphic Fredholm-operator-valued functions. We recast Sabbah Theorem in terms of suitable Riemann-Hilbert-Birkhoff boundary value problems \( \mathcal{P}[\mathbf{u}, \tau, \mathcal{M}] \) depending on parameters \((\mathbf{u}, \tau, \mathcal{M})\), the admissible data. Sabbah Theorem claims that the solvability of \( \mathcal{P}[\mathbf{u}, \tau, \mathcal{M}] \), for fixed \((\tau, \mathcal{M})\), is an open property with respect to \(\mathbf{u}\). We prove this statement by factorizing solutions of \( \mathcal{P}[\mathbf{u}, \tau, \mathcal{M}] \) in terms of two auxiliary RHB boundary value problems, \( \mathcal{P}_1[\mathbf{u}, \tau, \mathcal{M}] \) and \( \mathcal{P}_2[\mathbf{u}, \tau, \mathcal{M}] \). Their solvability is studied via results in the \( L^p \)-theory of RH boundary value problems. General references for this section are [AB94, Bot20, CG81, CG18, CDG19, DZ02a, DZ02b, FIKN06, Its03, Its11, MP80, TO16, Vek67, Zho89].

3.1. Admissible data and Riemann-Hilbert-Birkhoff boundary value problem. Denote by \( \operatorname{Arg}(z)\in ]-\pi, \pi]\) the principal branch of the argument of the complex number \(z\). Let \(\mathbf{u}\in \mathbb{C}^n\), and set

\[
\mathcal{I}(\mathbf{u}) := \left\{ \operatorname{Arg}\left[-\sqrt{-1}(\bar{u}^i - u^i)\right] + 2\pi k : k \in \mathbb{Z}, \ i, j \text{ s.t. } u^i \neq u^j \right\}.
\]

Any element \(\tau\in \mathbb{R} \setminus \mathcal{I}(\mathbf{u})\) will be said to be admissible at \(\mathbf{u}\).

**Definition 3.1.** Let \(\mathbf{u}\in \mathbb{C}^n\) and \(\tau\) be admissible at \(\mathbf{u}\). A \((\mathbf{u}, \tau)\)-admissible datum is a 6-tuple \(\mathcal{M} := (B, D, L, S_1, S_2, C)\) of matrices in \(M_n(\mathbb{C})\) such that:

1. the matrix \(B\) is diagonal, i.e. \(B = B'\),
2. \(D\) is a diagonal matrix of integers,
3. we have
\[
\operatorname{tr} B = \operatorname{tr} D + \operatorname{tr} L. \tag{3.1}
\]
4. the matrices \(S_1, S_2, C\) are invertible, with \(\det S_1 = \det S_2 = 1\),
5. \((S_1)_{ii} = (S_2)_{ii} = 1\),
6. if \(i \neq j\), then \((S_1^{-1})_{ij} = 0\) if \(\Re \left( e^{\sqrt{-1}(\tau - \pi)}(u^i - u^j) \right) > 0\),
7. if \(i \neq j\), then \((S_2)_{ij} = 0\) if \(\Re \left( e^{\sqrt{-1}(\tau)}(u^i - u^j) \right) > 0\),
8. we have
\[
S_1^{-1} e^{2\pi \sqrt{-1} B} S_2^{-1} = C^{-1} e^{2\pi \sqrt{-1} L} C. \tag{3.2}
\]

If \(\mathbf{u}\in \Delta\), define the partition \(\{1, \ldots, n\} = \bigsqcup_{r\in R} I_r\) such that for any \(r\in R\) we have \(\{i, j\} \subseteq I_r\) if and only if \(u^i = u^j\). We then require the further vanishing condition

9. \((S_1^{-1})_{ij} = (S_2)_{ij} = 0\) if \(i, j\in I_r\), with \(i \neq j\), for some \(r\in R\).

**Lemma 3.2.** Let \(\mathbf{u}_o\in \mathbb{C}^n\) and \(\tau\) admissible at \(\mathbf{u}_o\). If \(\mathcal{M}\) is \((\mathbf{u}_o, \tau)\)-admissible, then there exists a sufficiently small neighborhood \(\mathcal{V}\) of \(\mathbf{u}_o\) such that

1. \(\tau\) is admissible at \(\mathbf{u}\), for all \(\mathbf{u}\in \mathcal{V}\),
2. \(\mathcal{M}\) is \((\mathbf{u}, \tau)\)-admissible for all \(\mathbf{u}\in \mathcal{V}\). \hfill \Box

Let \(\mathbf{u}\in \mathbb{C}^n\) and \(\tau\) admissible at \(\mathbf{u}\). Consider the complex \(z\)-plane with a branch cut from 0 to \(\infty\):

\[
\tau - \pi < \operatorname{arg} z < \tau + \pi.
\]

Let \(r > 0\) and denote by \(\Gamma = \Gamma(\tau, r)\) the union of the following oriented paths, see Figure 1:
we use the symbol \( \beta \) by Problem 3.4.

A simple computation shows that (3.4) follows from (3.2). Equation (3.5) is easily proved.

\[ (3.5) \]

**Proposition 3.3.** The following identities hold true identically in \( z \in \mathbb{C}^* \):

\[ H_{-\infty}(z; u)H_2(z e^{2\pi i}; u)H_1(z; u)^{-1} = I, \]

\[ H_1(z; u)H_2(z; u)^{-1}H_{+\infty}(z; u)^{-1} = I. \]

**Proof.** A simple computation shows that (3.4) follows from (3.2). Equation (3.5) is easily checked. \( \square \)

**Problem 3.4.** (Problem \( P[u, \tau, \mathcal{M}] \)). Find an analytic function \( G : \mathbb{C} \setminus \Gamma \to M_n(\mathbb{C}) \) such that

1. \( G|_{\Pi_\nu} \) extends continuously to \( \bar{\Pi}_\nu \) for \( \nu = 0, L, R \);
2. the non-tangential limits \( G_\pm : \Gamma \setminus \{T_1, T_2\} \to M_n(\mathbb{C}) \) of \( G \), from the – and + sides of \( \Gamma \), are related by

\[ G_+(z) = G_-(z)H(z; u); \]
3. \( G(z) \) tends to the identity matrix \( I \) as \( z \to \infty \).
**Remark 3.5.** Let $G$ be a solution of $\mathcal{P}[\mathbf{u}, \tau, \mathcal{M}]$. Let $\tilde{\mathbb{C}}^*$ be the universal cover of $\mathbb{C}^*$. Define the functions $Y_i(\cdot; \mathbf{u}) : \tilde{\mathbb{C}}^* \to \mathbb{C}$, with $i = 0, 1, 2, 3$, by

\[
Y_0(z; \mathbf{u}) := G(z; \mathbf{u})z^D z_L, \quad z \in \Pi_0,
\]

\[
Y_1(z; \mathbf{u}) := G(e^{2\pi \sqrt{-1}} z; \mathbf{u})z^{B^o} e^\Lambda(z), \quad z \in e^{-2\pi \sqrt{-1}} \Pi_L,
\]

\[
Y_2(z; \mathbf{u}) := G(z; \mathbf{u})z^{B^o} e^\Lambda(z), \quad z \in \Pi_R,
\]

\[
Y_3(z; \mathbf{u}) := G(z; \mathbf{u})z^{B^o} e^\Lambda(z), \quad z \in \Pi_L.
\]

We have

\[
Y_2(z; \mathbf{u}) = Y_1(z; \mathbf{u}) S_1, \quad Y_3(z; \mathbf{u}) = Y_2(z; \mathbf{u}) S_2, \quad Y_2(z; \mathbf{u}) = Y_0(z; \mathbf{u}) C. \tag{3.6}
\]

It follows that

\[
\frac{\partial Y_0}{\partial z} Y_0^{-1} = \frac{\partial Y_1}{\partial z} Y_1^{-1} = \frac{\partial Y_2}{\partial z} Y_2^{-1} = \frac{\partial Y_3}{\partial z} Y_3^{-1},
\]

and the resulting function $\mathcal{A}(\cdot; \mathbf{u}) := \partial_y Y_i(\cdot; \mathbf{u}) \cdot Y_i(\cdot; \mathbf{u})^{-1}$ is analytic with respect to $z \in \mathbb{C}^*$. Hence, for any fixed $\mathbf{u}$, the matrices $Y_i(\cdot; \mathbf{u})$, with $i = 0, 1, 2, 3$, are fundamental systems of solutions of the differential system

\[
\frac{dY}{dz} = \mathcal{A}(z; \mathbf{u}) Y. \tag{3.7}
\]

Such a system will be studied in Section 3.4. The admissible data $\mathcal{M}$ determine the monodromy of the solutions $Y_i$. The matrices $S_1, S_2$ are the *Stokes matrices* of the system, the matrix $B$ is the *formal monodromy*, and the matrix $C$ is the *central connection matrix*. The independence of $\mathcal{M}$ with respect to $\mathbf{u}$ implies that the system (3.7) is *isomonodromic*. Equations (3.6) are the reason for the precise shape of the RHB problem $\mathcal{P}[\mathbf{u}, \tau, \mathcal{M}]$.

3.2. **Factorization of solutions.** We factorize solutions of the problem $\mathcal{P}[\mathbf{u}, \tau, \mathcal{M}]$ via two auxiliary RHB boundary value problems, $\mathcal{P}_1[\mathbf{u}, \tau, \mathcal{M}]$ and $\mathcal{P}_2[\mathbf{u}, \tau, \mathcal{M}]$. First, we describe the contours for both problems.

Let $P_1 \in \Gamma_{-\infty}$, and $P_2 \in \Gamma_{+\infty}$ such that $|P_1|, |P_2| > r$. Set

- $\ell_1 \subseteq \Gamma_{-\infty}$ to be the closed half-line from $\infty$ to $P_1$,
- $\ell_2 \subseteq \Gamma_{+\infty}$ to be the closed half-line from $P_2$ to $\infty$.

Define

- $\Gamma'$ to be the union $\ell_1 \cup \ell_2$,
- $\Gamma''$ to be a circle of radius $R > \max(|P_1|, |P_2|)$.

See Figure 2 for specification of the orientations of $\Gamma'$ and $\Gamma''$. We also denote by $\Omega_\pm$ the half-planes defined by

\[
\Omega_- := \{z : \tau - \pi < \arg z < \tau\}, \quad \Omega_+ := \{z : \tau < \arg z < \tau + \pi\},
\]

and by $\Gamma_0$ the oriented line $\Gamma_0 := \Gamma' \cup [P_1, P_2]$, where $[P_1, P_2]$ denotes the oriented segment from $P_1$ to $P_2$. We have $\partial \Omega_+ = -\partial \Omega_- = \Gamma_0$.

Second, following [Dur70, Ch. 11], we introduce some natural classes of analytic functions on which to work.
Definition 3.6. We define the Smirnov class $E^p(\Omega_{\pm})$, with $0 < p < \infty$, as the set of analytic functions $f$ on $\Omega_{\pm}$ such that
\[
\sup_{\ell \in \mathcal{R}_\pm} \int |f(\zeta)|^p |d\zeta| < +\infty,
\]
where $\mathcal{R}_\pm$ is the set of lines $\ell \subseteq \Omega_{\pm}$ parallel to $\Gamma_0$.

Remark 3.7. One can prove that if $f \in E^p(\Omega_+)$ (resp. $\Omega_-$), with $0 < p < \infty$, then $f(z) \to 0$ for $z \to \infty$ within each half-plane in $\Omega_+$ (resp. $\Omega_-$) with boundary $\ell \in \mathcal{R}_+$ (resp. $\ell \in \mathcal{R}_-$).

See [Dur70, Cor. 2 of Th. 11.3]. Essentially, the Smirnov classes $E^p(\Omega_{\pm})$ are the largest classes of analytic functions (vanishing at $z = \infty$) for which Cauchy integral formula holds on $\Omega_{\pm}$. More precisely, we have the following results.

Theorem 3.8 ([Dur70, Th. 11.1 and Th. 11.8]).

1. If $f \in E^p(\Omega_{\pm})$, with $0 < p < \infty$, then $f$ has non-tangential limit at the boundary $\Gamma_0$ of $\Omega_{\pm}$ for almost all $z_0 \in \Gamma_0$, and $f|_{\Gamma_0} \in L^p(\Gamma_0; |d\zeta|)$.

2. If $f \in E^p(\Omega_+)$ (resp. $\Omega_-)$), with $1 \leq p < \infty$, then
\[
f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega_+} \frac{f(\zeta)}{\bar{\zeta} - z} d\zeta, \quad z \in \Omega_+,
\]
(resp. $f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega_-} \frac{f(\zeta)}{\bar{\zeta} - z} d\zeta, \quad z \in \Omega_-$),

and the integral vanishes for $z \in \Omega_- \setminus \Omega_+$ (resp. $\Omega_+ \setminus \Omega_-$).

3. Conversely, if $h \in L^p(\Gamma_0; |d\zeta|)$, with $1 \leq p < \infty$, and
\[
\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega_+} \frac{h(\zeta)}{\bar{\zeta} - z} d\zeta \equiv 0, \quad z \in \Omega_-, \quad (\text{resp. } z \in \Omega_+),
\]

then for $z \in \Omega_+ \setminus \Omega_-$ this integral represents a function $f \in E^p(\Omega_+)$ (resp. $E^p(\Omega_-)$) whose boundary function $f = h \ a.e.$

Problem 3.9 (Problem $P_1[u, \tau, \mathfrak{M}]$). Find an analytic function $\Psi : \mathbb{C} \setminus \Gamma' \to M_n(\mathbb{C})$ such that
(1) \((\Psi - I)|_{\Omega_\pm} \in \mathcal{E}^2(\Omega_\pm)\), so that the non-tangential limits \(\Psi_\pm\) of \(\Psi\) from the \(-\) and \(+\) sides of \(\Gamma'\) exist a.e., and \(\Psi_\pm \in L^2(\Gamma'; |d\zeta|)\).

(2) the limits \(\Psi_\pm\) are related by

\[
\Psi_+(z) = \Psi_-(z)H(z; u). \tag{3.8}
\]

**Lemma 3.10.** We have \(H(\zeta; u) - I \to 0\) exponentially fast for \(\zeta \to \infty\) along \(\Gamma'\). In particular, \(H(-; u) - I \in L^2(\Gamma'; |d\zeta|)\).

**Proof.** The \((i, j)\)-entry of \(H(\zeta; u) - I\) equals

\[
c_{ij} \exp\{(u^i - u^j)\zeta + (B_{ii} - B_{jj}) \log \zeta\} \delta_{ij}, \tag{3.9}
\]

where

\[
c_{ij} = \begin{cases} 
(S_1^{-1})_{ij}, \text{ along } \ell_1, \\
(S_2)_{ij}, \text{ along } \ell_2.
\end{cases}
\]

By conditions (5), (6), (7) (and (9) if \(u \in \Delta\)) of Definition 3.1, we deduce that (3.9) goes to zero exponentially fast for \(\zeta \to \infty\) along \(\ell_1\) and \(\ell_2\).

**Theorem 3.11.** If \(\min\{|P_1|, |P_2|\}\) is sufficiently big, then there exists a unique solution \(\Psi\) of the problem \(\mathcal{P}_{\Gamma}[u, \tau, \mathcal{M}]\), holomorphically depending on \(u \in \mathcal{V}\). Moreover, \(\det \Psi \equiv 1\).

**Proof.** If \(\Psi\) is a solution of \(\mathcal{P}_{\Gamma}[u, \tau, \mathcal{M}]\), the condition \((\Psi - I)|_{\Omega_\pm} \in \mathcal{E}^2(\Omega_\pm)\) captures both the condition \(\Psi(\infty) = I\), and the validity of a Cauchy integral representation. Indeed, we have

\[
\Psi(z; u) = I + \int_{\Gamma'} \frac{\Psi_+(\zeta) - \Psi_-(\zeta)}{\zeta - z} \frac{d\zeta}{2\sqrt{-1}} - \int_{\Gamma'} \frac{\Psi_-(\zeta)(H(\zeta; u) - I)}{\zeta - z} \frac{d\zeta}{2\sqrt{-1}}, \quad z \notin \Gamma',
\]

by Theorem 3.8, and the jump condition (3.8). See also [FIKN06, Ch. 3][Its11, §5.1.3][TO16, Ch. 2]. Set \(\delta \Psi := \Psi - I\) and \(\delta H := H - I\). The previous equation can be written as

\[
\delta \Psi(z) = \int_{\Gamma'} \frac{\delta \Psi_-(\zeta) \delta H(\zeta; u)}{\zeta - z} \frac{d\zeta}{2\sqrt{-1}} + \int_{\Gamma'} \frac{\delta H(\zeta; u)}{\zeta - z} \frac{d\zeta}{2\sqrt{-1}}, \quad z \notin \Gamma'. \tag{3.10}
\]

Given a function \(f\) defined on \(\Gamma'\), introduce the functions \(C^\pm_{\Gamma'}[f]\) on \(\Gamma'\) defined by the Cauchy integrals

\[
C^\pm_{\Gamma'}[f](p) := \lim_{z \to p_\pm} \int_{\Gamma'} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\sqrt{-1}}, \quad p \in \Gamma',
\]

whenever the integral is finite. General results ensure that if \(f \in L^p(\Gamma'; |d\zeta|)\), with \(1 \leq p < \infty\), then \(C^\pm_{\Gamma'}[f]\) exists for \(p \in \Gamma'\) a.e.. Moreover, the Cauchy operators \(C^\pm_{\Gamma'}\) are bounded in \(L^p(\Gamma'; |d\zeta|)\), with \(1 < p < \infty\), i.e. there exists a constant \(k_p > 0\) such that

\[
\|C^\pm_{\Gamma'}[f]\|_{L^p(\Gamma')} \leq k_p \|f\|_{L^p(\Gamma')}, \quad \text{if } f \in L^p(\Gamma'; |d\zeta|).
\]

See [MP80, Zho89, DZ02a, DZ02b, TO16] for details and proofs. Taking the limit \(z \to z_-\) in (3.10), we obtain the following integral equation for \(\delta \Psi_-\):

\[
C[\delta H; \Gamma'] \delta \Psi_- = C_{\Gamma'}[\delta H], \tag{3.11}
\]
where
\[ C[\delta H; \Gamma']f := f - C^{-1}_T[f \cdot \delta H]. \]

Notice that \( C^{-1}_T[\delta H] \in L^2(\Gamma'; |d\zeta|) \) by Lemma 3.10. Moreover, if \( f \in L^2(\Gamma'; |d\zeta|) \), we have
\[ \|C^{-1}_T[f \cdot \delta H]\|_{L^2(\Gamma')} \lesssim k_2 \|f \cdot \delta H\|_{L^2(\Gamma')} \lesssim \sup_{\zeta \in \Gamma'} \|\delta H(\zeta; u)\| \cdot \|f\|_{L^2(\Gamma')} \]

By Lemma 3.10, we can assume that \( \min\{|P_1|, |P_2|\} \) is so big that
\[ \sup_{\zeta \in \Gamma'} \|\delta H(\zeta; u)\| < \frac{1}{1 + k_2}, \]

then the operator \( C[\delta H; \Gamma'] : L^2(\Gamma'; |d\zeta|) \to L^2(\Gamma'; |d\zeta|) \) is invertible with inverse
\[ C[\delta H; \Gamma']^{-1} = \sum_{m=0}^{\infty} C^{-m}_T[(\cdot) \cdot \delta H|^m], \]

that is, for any \( f \in L^2(\Gamma'; |d\zeta|) \), we have
\[ C[\delta H; \Gamma']^{-1} f = f + C^{-1}_T[f \cdot \delta H] + C^{-2}_T[f \cdot \delta H] \cdot \delta H] + \ldots. \]

Equation (3.11) can be uniquely solved in \( \delta \Psi_+ \), and the formula (3.10) gives the unique solution \( \Psi \) of the RHB boundary value problem. Notice that the Cauchy operator \( C^{-1}_T[(-) \cdot \delta H] \) depends holomorphically on \( u \), so that \( \Psi(z; u) \) is holomorphic in \( u \). Finally, notice that the jump condition (3.8) implies
\[ \det \Psi_+ = \det \Psi_- \det H = \det \Psi_-, \]

since \( \det H(\zeta; u) \equiv 1 \) along \( \Gamma' \). Hence, \( \det \Psi \) is an entire function, and from the asymptotic condition \( \Psi \to I \) for \( z \to \infty \), we deduce \( \det \Psi \equiv 1 \) by Liouville Theorem. \( \square \)

**Remark 3.12.** It follows from general results of [Mus72, Chapter 4] that the function \( \Psi \) has two singularities at \( P_1, P_2 \). One can also prove that \( \Psi_{\pm} \in L^2(\Gamma'; |d\zeta|) \) admit continuous representatives on \( \Gamma' \setminus \{P_1, P_2\} \).

Define the function \( \mathcal{S}(-; u) : C \setminus \Gamma \to GL(n, \mathbb{C}) \) by
\[ \mathcal{S}(z; u) := \begin{cases} I, & \text{for } z \in \Pi_0, \\ H_1(z; u)^{-1}, & \text{for } z \in \Pi_R, \\ H_2(z; u)^{-1}, & \text{for } z \in \Pi_L. \end{cases} \]

**Lemma 3.13.** The function \( \mathcal{S}(-; u) \) is a naive solution of \( \mathcal{P}[u, \tau, \mathcal{M}] \): it satisfies conditions (1), (2), but not (3).

**Proof.** This is easily checked, by invoking equations (3.4), (3.5). \( \square \)

Consider the function \( \widetilde{H}(-; u) : C \setminus \Gamma \to GL(n, \mathbb{C}) \) defined by
\[ \widetilde{H}(z; u) := \Psi(z, u) \mathcal{S}(z; u)^{-1}, \]

where \( \Psi \) is the unique piecewise analytic solution of \( \mathcal{P}_1[u, \tau, \mathcal{M}] \), as in Theorem 3.11.

**Lemma 3.14.**

(1) The function \( \widetilde{H}(-; u) \) is continuous along \( \Gamma'' \).
(2) The function \( \det \widetilde{H}(-; u) \) has zero index across \( \Gamma'' \), i.e.
\[
\text{ind}_{\Gamma''} \det \widetilde{H}(-; u) := \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma''} d \log \det \widetilde{H}(\zeta; u) = 0.
\]

Proof. Point (1) is obvious. For point (2) notice that, for \( \zeta \in \Gamma'' \), we have
\[
\log \det \widetilde{H}(\zeta; u) = \log \det S(\zeta; u) - 1
= \log \det H_1(\zeta; u)
= \text{tr} Q(\zeta; u) - \log \det C - (\text{tr} L + \text{tr} D) \log \zeta
= \zeta \sum_{i=1}^{n} u_i - \log \det C - \left( \text{tr} B - \text{tr} L - \text{tr} D \right) \log \zeta.
\]

For the third line, see equation (3.3). This completes the proof. \(\square\)

We can now introduce a second auxiliary RHB boundary value problem, with continuous coefficients on the simple closed contour \( \Gamma'' \).

**Problem 3.15** (Problem \( \mathcal{P}_2[u, \tau, \mathcal{M}] \)). Find an analytic function \( \Upsilon : \mathbb{C} \setminus \Gamma'' \to M_n(\mathbb{C}) \) such that

1. the non-tangential limits \( \Upsilon_{\pm} : \Gamma'' \to M_n(\mathbb{C}) \) of \( \Upsilon \) from the - and + sides of \( \Gamma'' \) exist,
2. they are related by
\[
\Upsilon_+(z) = \Upsilon_-(z) \widetilde{H}(z; u),
\]
and
3. \( \Upsilon(z) \) tends to the identity matrix \( I \) as \( z \to \infty \).

**Theorem 3.16.** The solvability of \( \mathcal{P}[u, \tau, \mathcal{M}] \) is equivalent to the solvability of \( \mathcal{P}_2[u, \tau, \mathcal{M}] \).

Proof. If \( G \) is the solution of \( \mathcal{P}[u, \tau, \mathcal{M}] \), then
\[
\Upsilon(z; u) := \begin{cases} 
G(z; u) \Psi(z; u)^{-1}, & \text{for } z \text{ outside } \Gamma'', \\
G(z; u) \mathcal{S}(z; u)^{-1}, & \text{for } z \text{ inside } \Gamma'', 
\end{cases}
\]
is the solution of \( \mathcal{P}_2[u, \tau, \mathcal{M}] \). Vice-versa, if \( \Upsilon \) is the solution of \( \mathcal{P}_2[u, \tau, \mathcal{M}] \), then the solution \( G \) of \( \mathcal{P}[u, \tau, \mathcal{M}] \) is obtained by inverting the equations above. \(\square\)

3.3. **Solvability as an open property.** If \( \Upsilon \) is a solution of \( \mathcal{P}_2[u, \tau, \mathcal{M}] \), then we have
\[
\Upsilon(z) = I + \int_{\Gamma''} \frac{\Upsilon_-(\zeta)(\widetilde{H}(\zeta; u) - I)}{\zeta - z} \frac{d\zeta}{2\pi \sqrt{-1}}.
\]
In the limit \( z \to z_- \), we obtain the integral equation
\[
\Upsilon_- = I + \mathcal{C}_{\Gamma''}^{-} \left[ \Upsilon_- \delta \widetilde{H} \right], \quad \delta \widetilde{H} := \widetilde{H} - I,
\]
where \( \mathcal{C}_{\Gamma''}^{\pm} \) denotes the Cauchy integrals with respect to the contour \( \Gamma'' \). Conversely, if \( \Upsilon_- \) is a solution of (3.14), then (3.13) gives the solution of \( \mathcal{P}_2[u, \tau, \mathcal{M}] \), see [FIKN06, Ch. 3][Its11, §5.1.3][TO16, Ch. 2].
Theorem 3.17. The operator
\[ T(u) : L^2(\Gamma''; |d\zeta|) \to L^2(\Gamma''; |d\zeta|), \quad f \mapsto f - C_{\Gamma''}^-[f \cdot \delta\tilde{H}(u)] \]
is a Fredholm operator with index\(^7\) 0.

Proof. Assume we are given a factorization \( \tilde{H}(\zeta; u) = (I - W^- (\zeta; u))^{-1} (I + W^+ (\zeta; u)) \) with \( W^+(z; u) \in L^\infty(\Gamma'') \cap L^2(\Gamma'') \), \( (I - W^- (\zeta; u))^{-1} - I \in L^\infty(\Gamma'') \cap L^2(\Gamma'') \).

Define the Cauchy type operator
\[ C_W : L^2(\Gamma'') \to L^2(\Gamma''), \quad f \mapsto C_{\Gamma''}^+[fW^-] + C_{\Gamma''}^-[fW^+]. \]
Standard results imply that the operator \( f \mapsto f - C_W[f] \) is Fredholm, and its index is given by
\[ \text{ind}(Id - C_W) = n \text{ind}_{\Gamma''} \det \tilde{H} = 0, \]
by Lemma 3.14 point (2), see [Zho89, TO16]. In our case, we can take \( W^- = 0 \) and \( W^+ = \delta\tilde{H} \) by Lemma 3.14 point (1). This completes the proof. \( \square \)

A set \( \Theta \subseteq \mathbb{C}^n \) is said to be analytic if it is the zero locus of a scalar analytic function.

Theorem 3.18. Let \( u_o \in \mathbb{C}^n \). Assume that the pair \( (\tau, \mathcal{M}) \) is admissible at each point of a sufficiently small open neighborhood \( \mathcal{V} \) of \( u_o \). If \( \mathcal{P}_{\tau\mathcal{M}} \) is solvable, there exists an analytic set \( \Theta \subseteq \mathcal{V} \setminus \{u_o\} \) such that \( \mathcal{P}_{\tau\mathcal{M}} \) is solvable for all \( u \in \mathcal{V} \setminus \Theta \). Moreover, the solution \( G(z; u) \) is holomorphic with respect to \( u \in \mathcal{V} \setminus \Theta \).

For the proof we firstly invoke the following Lemma.

Lemma 3.19 ([Gra70, Lemma 10]). Let \( X \) be a Banach space and \( \mathfrak{F}(X) \) be the set of its Fredholm operators. Let \( \Omega \subseteq \mathbb{C}^n \) be a connected domain, and \( T : \Omega \to \mathfrak{F}(X) \) a holomorphic function. If \( T(u_o)^{-1} \) exists for some \( u_o \in \Omega \), then \( T(u)^{-1} \) exists on the complement \( \Omega \setminus \Theta \) of an analytic set \( \Theta \), and \( T^{-1} \) is meromorphic on \( \Omega \). \( \square \)

Remark 3.20. Lemma 3.19 was originally due to I. Gohberg and E. Sigal in the case \( n = 1 \), [GS70]. The general case was proved by B. Gramsch, though special cases were previously obtained by several authors. For a sketch of a proof, based on arguments of [GS70] and [GGK90, XI.8], see [Kab12, Sec. 2].

Proof of Theorem 3.18. By assumption and Theorem 3.16, the \( \mathcal{P}_{\tau\mathcal{M}} \) is solvable. We claim that the solution \( \Upsilon \) is unique. The function \( \det \Upsilon_+(z; u_o) \) solves the scalar RH problem
\[ \det \Upsilon_+(z; u_o) = \det \Upsilon_-(z; u_o) \det \tilde{H}(z; u_o). \]
Since the function \( \det \tilde{H}(-; u_o) \) has zero index along \( \Gamma'' \), this scalar equation can be uniquely solved: the solution is given by
\[ \det \Upsilon(z; u_o) = \exp \int_{\Gamma''} \frac{\log \det \tilde{H}(\mu; u_o)}{\mu - z} \frac{d\mu}{2\pi\sqrt{-1}}. \]

\(^7\)Recall that the index of a Fredholm operator \( T \) is the integer \( \text{ind} T := \dim \ker T - \dim \text{coker} T \).
see e.g. [TO16, §2.3.1]. In particular, $\Upsilon(z; \mathbf{u}_o)$ is invertible. Assume that $\Upsilon(z; \mathbf{u}_o), \breve{\Upsilon}(z; \mathbf{u}_o)$ are two solutions of $\mathcal{P}_2[\mathbf{u}_o, \tau, \Omega]$. Put $X(z) := \Upsilon(z; \mathbf{u}_o)\breve{\Upsilon}(z; \mathbf{u}_o)^{-1}$. For $z \in \Gamma''$ we have

$$X_+(z) = \Upsilon_+(z; \mathbf{u}_o)\breve{\Upsilon}_+(z; \mathbf{u}_o)^{-1} = \Upsilon_-(z; \mathbf{u}_o)\breve{H}(z; \mathbf{u}_o)\breve{\Upsilon}(z; \mathbf{u}_o)^{-1}\breve{\Upsilon}_-(z; \mathbf{u}_o)^{-1} = X_-(z).$$

Hence $X(z)$ is analytic, and moreover $X(z) \to I$ for $z \to \infty$. By Liouville Theorem we have $X(z) \equiv I$, and $\Upsilon = \breve{\Upsilon}$. It follows that the Fredholm operator $T(\mathbf{u}_o)$ has both trivial kernel and index zero. Hence $T(\mathbf{u}_o)^{-1}$ exists, Lemma 3.19 applies, and the problem $\mathcal{P}_2[\mathbf{u}_o, \tau, \Omega]$ is solvable on the complement of an analytic set $\Theta \subseteq V \setminus \{\mathbf{u}_o\}$. By Theorem 3.16 one concludes.

3.4. Proof of Sabbah Theorem. Let $(E_o, \nabla)$ be in Birkhoff normal form (2.1) with $\Lambda_o = \text{diag}(u_o^1, \ldots, u_o^n)$ and $\mathbf{u}_o \in \Delta$. Consider the differential system defining $\nabla_o$-flat sections

$$\frac{dY}{dz} = \left(\Lambda_o + \frac{1}{z}B_o\right)Y,$$

where $Y$ is a matrix-valued function.

**Proposition 3.21** ([AB94],[CG18],[CDG19, Section 16]). The differential system (3.15) has a fundamental system of solutions in Birkhoff-Levelt normal form

$$Y_0(z) = G_0(z)z^Dz^{S+R}, \quad G_0(z) = K\left(I + \sum_{j=1}^{\infty} A_jz^j\right),$$

where

- $K$ puts $B_o$ in Jordan form $J = K^{-1}B_oK$,
- $\mathcal{D}$ is a diagonal matrix of integers (called valuations),
- $S$ is a Jordan matrix whose eigenvalues have real part in $[0,1[$,
- $R$ is a nilpotent matrix, with possibly non-vanishing entries only if some of the eigenvalues of the matrix $B_o$ differ by a non-zero integer.

Moreover, we have

$$J = \mathcal{D} + S.$$

**Proposition 3.22** ([CDG19, Prop. 4.2]). Assume that

1. $B''_o \in \text{Im} \text{ad}(\Lambda(\mathbf{u}_o))$,
2. $B''_o$ is partially non-resonant, i.e. $(B_o)_{ii} - (B_o)_{jj} \notin \mathbb{Z} \setminus \{0\}$ if $u_o^i = u_o^j$.

Then, the differential system (3.15) has a unique formal solution of the form

$$Y_F(z) = \left(I + \sum_{k=1}^{\infty} F_kz^{-k}\right)z^{B'_o}e^{\Lambda_0 z}.$$

If $\tau$ is admissible at $\mathbf{u}_o$, then there exist three fundamental systems of solutions $Y_1, Y_2, Y_3$ of (3.15), satisfying respectively

$$Y_h(z) \sim Y_F(z), \quad |z| \to +\infty, \quad \tau - (3-h)\pi < \arg z < \tau + (h-2)\pi, \quad h = 1, 2, 3. \quad (3.16)$$

and uniquely determined by these conditions.
Remarks 3.23. For \( h = 1, 2, 3 \), set
\[
G_h(z) := Y_h(z)e^{-A_o z}z^{-B'_o},
\]
\[
W_{\tau,h} := \left\{ z \in \widehat{\mathbb{C}}^*: \tau - (3 - h)\pi < \arg z < \tau + (h - 2)\pi \right\}.
\]
Denote by \( \overline{W} \) an arbitrary unbounded closed sector, in the universal cover \( \widehat{\mathbb{C}}^* \) of \( \mathbb{C}^* \), with vertex at \( \infty \). The precise meaning of the asymptotic relations (3.16) is the following:
\[
\forall h \in \{1, 2, 3\}, \; \forall \ell \in \mathbb{N}, \; \forall W \subsetneq \overline{W}, \; \exists C_{h,\ell,\overline{W}} > 0: \text{ if } z \in \overline{W} \setminus \{0\} \text{ then }
\left\| G_h(z) - \left( I + \sum_{m=1}^{\ell-1} \frac{F_m}{z^m} \right) \right\| < \frac{C_{h,\ell,\overline{W}}}{|z|^\ell}.
\]

In the notations of Propositions 3.21, 3.22, consider the 6-tuples \( \mathfrak{M} = (B, D, L, S_1, S_2, C) \) where
\[
B := B'_o, \quad D := D, \quad L := S + R,
\]
and the matrices \( S_1, S_2, C \) are defined by
\[
Y_2(z) = Y_1(z)S_1, \quad Y_3(z) = Y_2(z)S_2, \quad Y_2(z) = Y_0(z)C.
\] (3.17)

Proposition 3.24. The 6-tuple \( \mathfrak{M} \) is a \( (u_o, \tau) \)-admissible datum. The RHB boundary value problem \( \mathcal{P}[u_o, \tau, \mathfrak{M}] \) is solvable, with unique solution
\[
G(z; u_o) = \begin{cases} 
G_0(z), & z \in \Pi_0, \\
G_2(z), & z \in \Pi_R, \\
G_3(z), & z \in \Pi_L.
\end{cases}
\]

Proof. Conditions (1),(2),(3) of Definition 3.1 are trivially satisfied. The proof of conditions (4),(5),(6),(7) for the Stokes matrices \( S_1, S_2 \) are standard, see e.g. [CDG19, Section 6.3]. Denote by \( \widehat{\mathbb{C}}^* \) the universal cover of \( \mathbb{C}^* \). Notice that
\[
Y_3(ze^{2\sqrt{-1}\pi}) = Y_1(z)e^{2\sqrt{-1}\pi B'_o}, \quad z \in \widehat{\mathbb{C}}^*,
\]
both sides having the same asymptotic expansion \( Y_F(ze^{2\sqrt{-1}\pi}) \) for \( |z| \to +\infty \), and \( \tau - 2\pi < \arg z < \tau - \pi \). We deduce that
\[
Y_0(ze^{2\sqrt{-1}\pi})C S_2 = Y_0(z)C S_1^{-1}e^{2\sqrt{-1}\pi B'_o},
\]
so that
\[
e^{2\sqrt{-1}\pi L} = CS_1^{-1}e^{2\sqrt{-1}\pi B'_o}S_2^{-1}C^{-1}.
\]
This proves condition (8) of Definition 3.1. Finally, condition (9) follows from [CG18, Th. 2.1], [CDG19, Prop. 6.1]. The remaining part of the statement follows from equations (3.17), and the uniqueness stated in Proposition 3.22.
By Proposition 3.24 and Theorem 3.18, there exist an open neighborhood \( V \) of \( u_o \), an analytic set \( \Theta \subset V \setminus \{u_o\} \) on which the RHB problem \( P[u, \tau, \mathbb{M}] \) is solvable, with unique solution \( G(z; u) \) holomorphic with respect to \( u \in V \setminus \Theta \). Define the functions

\[
Y_2(z; u) := G(z; u)z^{B_0^1}e^{\Lambda(u)z}, \quad z \in \Pi_R,
\]
\[
Y_3(z; u) := G(z; u)z^{B_0^1}e^{\Lambda(u)z}, \quad z \in \Pi_L,
\]
\[
Y_0(z; u) := G(z; u)z^{D}z^L, \quad z \in \Pi_0.
\]

By Remark 3.5, we have

\[
Y_2(z; u) = Y_0(z; u) \cdot C, \quad Y_3(z; u) = Y_2(z; u) \cdot S_2.
\]

Moreover, we have \( G(z; u) = I + \frac{F_i(u)}{z} + O \left( \frac{1}{z^2} \right) \) in \( z \to \infty \) in \( \Pi_{L/R} \), so that

\[
\frac{\partial Y_{2/3}}{\partial z} \cdot Y_{2/3}^{-1} = \frac{\partial G}{\partial u} \cdot G^{-1} + \frac{1}{z} GB_0^1 G^{-1} + GAG^{-1}
\]
\[
\quad = \Lambda(u) + \frac{1}{z} \left( [F_1(u), \Lambda(u)] + B_0^1 \right) + O \left( \frac{1}{z^2} \right), \quad z \to \infty,
\]
\[
\frac{\partial Y_0}{\partial z} \cdot Y_0^{-1} = \frac{\partial G}{\partial u} \cdot G^{-1} + \frac{1}{z} \left( GDG^{-1} + Gz^D Lz^{-D} G^{-1} \right)
\]
\[
\quad = \frac{1}{z} K (D + L) K^{-1} + O(1), \quad z \to 0.
\]

The matrices \( S_1, S_2, C \) are constant with respect to both \( u \) and \( z \): we deduce that the r.h.s. of the two equalities above are equal. This implies that \( Y_2, Y_3, \) and \( Y_0 \) are solutions of the differential equation

\[
\frac{\partial}{\partial z} Y = \left[ \Lambda(u) + \frac{1}{z} V(u) \right] Y, \quad V(u) := [F_1(u), \Lambda(u)] + B_0^1.
\]  

Similarly, we have

\[
\frac{\partial Y_{2/3}}{\partial u^i} \cdot Y_{2/3}^{-1} = \frac{\partial G}{\partial u^i} \cdot G^{-1} + z G E_i G^{-1} = z E_i + [F_1, E_i] + O \left( \frac{1}{z} \right),
\]
\[
\frac{\partial Y_0}{\partial u^i} \cdot Y_0^{-1} = \frac{\partial G}{\partial u^i} \cdot G^{-1} = \frac{\partial G_0}{\partial u^i} \cdot G_0^{-1} + O(z),
\]

where \( (E_i)_{ab} = \delta_{ab} \delta_{ia} \) and \( G(z; u) = G_0(u) + O(z) \) for \( z \to 0 \) (and in particular \( G_0(u_o) = K \)). The matrices \( S_1, S_2, C \) being constant, we deduce that the r.h.s. of the two equalities above are equal. Hence \( Y_2, Y_3, \) and \( Y_0 \) are solutions of the differential systems

\[
\frac{\partial}{\partial u^i} Y = (z E_i + V_i(u)) Y, \quad V_i(u) := [F_1(u), E_i] = \frac{\partial G_0}{\partial u^i} \cdot G_0^{-1}, \quad i = 1, \ldots, n.
\]  

The datum of the compatible joint differential systems (3.18) and (3.19), for \( u \in V \setminus \Theta \), proves the statement of Sabbah Theorem 2.5.

**Remark 3.25.** Note that in equations (3.18) and (3.19) we can replace \( F_1 \) with its off-diagonal part \( F'' \), since both \( \Lambda(u) \) and \( E_i \) are diagonal.

**Remark 3.26.** Propositions 3.21 and 3.22 also hold true for \( u_o \in \mathbb{C}^n \setminus \Delta \), these are standard results. All the subsequent arguments can be applied, giving an analytical proof of Theorem 2.4.
4. Formal Frobenius and Dubrovin-Frobenius Manifolds

We briefly review basic notions of the theory of Frobenius manifolds, in both formal and analytic frameworks. General references are [Dub96, Dub98, Dub99, Man99, Her02, Sab07].

4.1. Formal Frobenius manifolds. Let
- $k$ be a commutative $\mathbb{Q}$-algebra,
- $H$ be a free $k$-module of finite rank,
- $\eta: H \otimes H \to k$ be a symmetric pairing, inducing an isomorphism $\eta': H \to H^T$, where $H^T$ is the dual module,
- $K := k[H^T]$ be the completed symmetric algebra of $H^T$.

Fix a basis $(\Delta_1, \ldots, \Delta_n)$ of $H$, and denote by $t = (t^1, \ldots, t^n)$ the dual coordinates. The algebra $K$ is then identified with the algebra of formal power series $k[t]$. Denote by $\text{Der}_k(K)$ the $K$-module of $k$-linear derivations of $K$. Put $\partial_{\alpha} = \frac{\partial}{\partial t^\alpha} : K \to K$. It is well known that $\text{Der}_k(K)$ is a free $K$-module with basis $(\partial_1, \ldots, \partial_n)$, see e.g. [Now86]. If $\Phi \in K$, we will write $\Phi_\alpha$ for $\partial_\alpha \Phi$.

Elements of $H_K := K \otimes_k H$ will be identified with derivations on $K$, by $\Delta_\alpha \mapsto \partial_\alpha$.

For $\alpha, \beta = 1, \ldots, n$, set $\eta_{\alpha \beta} := \eta(\Delta_\alpha, \Delta_\beta)$. The matrix $(\eta^{\alpha \beta})$ will denote the inverse of the Gram matrix $(\eta_{\alpha \beta})$ of $\eta$. Einstein summation rule will be used over repeated Greek indices.

**Definition 4.1.** A formal Frobenius manifold structure on $(H, \eta)$ is given by a formal power series $\Phi \in K$, called WDVV potential, such that

$$
\Phi_{\alpha \beta \gamma} \eta^{\gamma \delta} \Phi_{\delta \varepsilon \varphi} = \Phi_{\varphi \beta \gamma} \eta^{\gamma \delta} \Phi_{\delta \varepsilon \alpha}, \quad \alpha, \beta, \varepsilon, \varphi = 1, \ldots, n. \tag{4.1}
$$

Define the $K$-linear multiplication $\circ$ on $H_K$ by

$$
\Delta_\alpha \circ \Delta_\beta := c_{\alpha \beta}^{\gamma} \Delta_\gamma, \quad \alpha, \beta = 1, \ldots, n, \tag{4.2}
$$

where $c_{\alpha \beta}^{\gamma} := \Phi_{\alpha \beta \gamma} \eta^{\gamma \delta}$. The WDVV equations (4.1) are equivalent to the associativity of $\circ$.

An element $e \in H_K$ is called identity if it is the identity for $\circ$. It is called flat identity if $e \in H$. An element $E \in H_K$ is called Euler if

$$
\mathcal{L}_E \eta = D\eta, \quad D \in k, \tag{4.3}
$$

$$
\mathcal{L}_E c = c. \tag{4.4}
$$

Here $\eta$ is $K$-bilinearly extended to $H_K$. Moreover, $\mathcal{L}_E$ denotes the Lie derivative along $E$, and it is extended to the full tensor algebra of the $K$-module $H_K$, as follows. Set

$$
T_0^0 = K, \quad T_p^q := H_K^{\otimes p} \otimes_K \text{Hom}_K(H_K, K)^{\otimes q}, \quad p, q \geq 1, \quad T := \bigoplus_{p, q \geq 0} T_p^q.
$$

We extend $\mathcal{L}_E$ on $T$, by requiring that:

1. $\mathcal{L}_E$ acts on elements $f \in T_0^0 = K$ by $\mathcal{L}_E f = Ef$;
2. $\mathcal{L}_E$ acts on elements $X \in T_1^1 = H_K \cong \text{Der}_k(K)$ by $\mathcal{L}_E X = [E, X]$;
3. $\mathcal{L}_E$ is a tensorial derivation, i.e. it satisfies the Leibniz rule

$$
\mathcal{L}_E(S_1 \otimes S_2) = (\mathcal{L}_E S_1) \otimes S_2 + S_1 \otimes (\mathcal{L}_E S_2), \quad S_1 \in T_{q_1}^{p_1}, S_2 \in T_{q_2}^{p_2};
$$

---

8For this standard algebraic approach to Lie derivatives of tensors, see e.g. [AMR88]
(4) $\mathcal{L}_E$ commutes with contractions, i.e.
\[
\mathcal{L}_E[V(\alpha_1, \ldots, \alpha_r, X_1, \ldots, X_s)] = (\mathcal{L}_EV)(\alpha_1, \ldots, \alpha_r, X_1, \ldots, X_s)
\]
\[
+ \sum_{j=1}^r V(\alpha_1, \ldots, \mathcal{L}_E\alpha_j, \ldots, \alpha_r, X_1, \ldots, X_s) + \sum_{k=1}^s V(\alpha_1, \ldots, \alpha_r, X_1, \ldots, \mathcal{L}_EX_k, \ldots, X_s),
\]
for any $V \in T_s, \alpha_1, \ldots, \alpha_r \in T_0^0, X_1, \ldots, X_s \in T_0^1$.

In this paper we always consider Frobenius manifolds equipped with a flat identity $e = \Delta_1$, and an Euler element $E$. Equations (4.3) and (4.4) are then equivalent to the single equation
\[
E\Phi = (1 + D)\Phi + \text{quadratic terms in } t.
\]

**Lemma 4.2.** Let $E = \sum_{i=1}^r E^\alpha(t) \Delta_\alpha$ be an Euler element. The series $E^\alpha(t) \in K$ are linear affine expressions in $t$, i.e. $E^\alpha(t) = \sum_\beta A^\alpha_\beta t^\beta + r^\alpha$, for suitable coefficients $A^\alpha_\beta, r^\alpha \in k$.

**Proof.** The Killing-conformal condition (4.3) reads $\eta_{\alpha\beta} \partial_{\alpha} E^\alpha + \eta_{\gamma\alpha} \partial_{\beta} E^\alpha = D\eta_{\beta\gamma}$. Differentiating this equation by $\partial_\alpha$, and permuting indices, we obtain
\[
\partial_\alpha \partial_\beta E^\gamma + \partial_\beta \partial_\alpha E^\gamma + \partial_\gamma \partial_\alpha E^\beta = 0, \quad \partial_\beta \partial_\gamma E^\alpha + \partial_\gamma \partial_\beta E^\alpha = 0, \quad \partial_\gamma \partial_\alpha E^\beta + \partial_\beta \partial_\alpha E^\gamma = 0,
\]
where $E^\gamma := \sum_\nu E^{\nu\nu} \eta_{\nu\lambda}$. Since $\partial_\alpha \partial_\beta \partial_\gamma = \partial_\beta \partial_\gamma \partial_\alpha$, we conclude that $\partial_\alpha \partial_\beta E^\gamma = \partial_\beta \partial_\gamma E^\alpha = \partial_\gamma \partial_\alpha E^\beta = 0$. It follows that $(E^\gamma, \text{and consequently}) E^\gamma$ are linear functions in $t$. \hfill \Box

### 4.2. Dubrovin-Frobenius manifolds.

Given a complex analytic manifold $M$, we denote by $TM, T^*M$ its holomorphic tangent and cotangent bundles. If $E$ is a vector bundle on $M$, its $k$-th symmetric power is denoted by $\bigotimes^k E$.

A *Dubrovin-Frobenius manifold* structure on a complex manifold $M$ of dimension $n$ is defined by giving

(FM1) a symmetric non-degenerate $\mathcal{O}(M)$-bilinear form $\eta \in \Gamma(\bigotimes^2 T^*M)$, called *metric*,

(FM2) a $(1, 2)$-tensor $c \in \Gamma(TM \otimes \bigotimes^2 T^*M)$ such that

(a) the induced multiplication of vector fields $X \circ Y := c(-, X, Y)$, for $X, Y \in \Gamma(TM)$, is associative,

(b) $c^\alpha \in \Gamma(\bigotimes^3 T^*M),

(c) $\nabla c^\alpha \in \Gamma(\bigotimes^4 T^*M)$;

(FM3) a vector field $e \in \Gamma(TM)$, called the *unity vector field*, such that

(a) the bundle morphism $c(-, e, -) : TM \to TM$ is the identity morphism,

(b) $\nabla e = 0$;

(FM4) a vector field $E \in \Gamma(TM)$, called the *Euler vector field*, such that

(a) $\mathcal{L}_Ec = c$,

(b) $\mathcal{L}_E\eta = (2 - d) \cdot \eta$, where $d \in C$ is called the *charge* of the Frobenius manifold.

---

In what follows, the musical isomorphisms with respect to the metric $\eta$ will be denoted by $(-)^\flat$ and $(-)^\sharp$, respectively. If $\xi \in \Gamma(TM)$, the 1-form $\xi^\flat \in \Gamma(T^*M)$ is defined by $\xi^\flat(X) = \eta(\xi, X)$, where $X \in \Gamma(TM)$. Conversely, if $\xi \in \Gamma(T^*M)$, the vector field $\xi^\sharp \in \Gamma(TM)$ is uniquely defined by the identity $\xi(X) = \eta(\xi^\sharp, X)$, where $X \in \Gamma(TM)$. Thus $(-)^\flat : \Gamma(TM) \to \Gamma(T^*M)$ and $(-)^\sharp : \Gamma(T^*M) \to \Gamma(TM)$ are mutually inverse. In components, these operations are also known as “lowering” and “raising” indices, respectively. These operations naturally extend to mixed tensors.
Dubrovin-Frobenius manifolds will be also called analytic Frobenius manifolds.

By axiom (FM1), there exist systems of flat coordinates \( t = (t^1, \ldots, t^n) \), with respect to which the Levi-Civita connection \( \nabla \) coincides with partial derivatives \( \partial_\alpha := \frac{\partial}{\partial x^\alpha} \), for \( \alpha = 1, \ldots, n \). Without loss of generality, we assume that the coordinate \( t^1 \) is such that \( \partial_1 = e \).

A pointed Dubrovin-Frobenius manifold is a pair \( (M, p) \), where \( M \) is a Dubrovin-Frobenius manifold, and \( p \in M \) is a fixed base point. Given \( (M, p) \) we will always consider flat coordinates \( t = (t^1, \ldots, t^n) \) vanishing at \( p \).

### 4.3. From Dubrovin-Frobenius to formal Frobenius structures, and vice-versa.

Given a pointed Dubrovin-Frobenius manifold \( (M, p) \), we can associate to it a formal Frobenius structure \( (H, \eta, \Phi) \) over \( k = \mathbb{C} \). Choose flat coordinates \( t \) vanishing at \( p \), and set \( H := T_pM \) equipped with the metric \( \eta|_p \). By axiom (FM2-c), the tensor \( \partial_\alpha c_{\beta\gamma} \) is completely symmetric: hence we deduce the local existence of a function \( F \) such that \( \partial^3_{\alpha\beta\gamma} F = c_{\alpha\beta\gamma} \). By axioms (FM2-a), (FM2-b), we deduce that \( F \) is a solution of WDVV equations, i.e.

\[
\partial^3_{\alpha\beta\gamma} F \eta^{\delta} \partial^3_{\delta\epsilon\varphi} F = \partial^3_{\epsilon\beta\gamma} F \eta^{\delta} \partial^3_{\delta\alpha\varphi} F, \quad \alpha, \beta, \epsilon, \varphi = 1, \ldots, n.
\]

Moreover, axiom (FM4) can be rephrased as the single equation

\[
EF = (3 - d)F + \text{quadratic terms in } t.
\]

Let \( O_{M,p} \) be the local ring of germs at \( p \), and \( m \) be its maximal ideal. The formal potential \( \Phi \) is given by the image of \( F \) in the completion \( \widehat{O_{M,p}} := \lim_{\leftarrow} (O_{M,p}/m^t) \) of the local ring \( O_{M,p} \): this means that \( \Phi \) is defined by the Taylor series expansion of \( F \) at \( p \) in coordinates \( t \). Moreover, the formal Frobenius structure \( (H, \eta, \Phi) \) is also equipped with a flat unit \( e|_p \) and an Euler vector field \( E|_p \). We will say that the formal Frobenius structure constructed in this way, starting from a pointed Dubrovin-Frobenius manifold, is convergent.

Conversely, let us assume that \( (H, \eta, \Phi) \) is a formal Frobenius structure over \( k = \mathbb{C} \), with flat identity \( e = \Delta_1 \), and Euler vector \( E \). If the domain of convergence \( \Omega \subset H \) of the power series \( \Phi \in k[[t]] \) is non-empty, then \( \Omega \) is equipped with a Dubrovin-Frobenius manifold structure. The product \( \circ \) defined in equation (4.2), indeed, turns out to have analytic structure constants \( c^\gamma_{\alpha\beta} = \Phi_{\alpha\beta\lambda} \eta^{\lambda\gamma} \). These are the components of a well-defined holomorphic section \( c \) of the bundle \( T\Omega \otimes \bigotimes^2 T^*\Omega \) as in axioms (FM2). The metric \( \eta \) and the potential \( \Phi \) are related by \( \eta_{\alpha\beta} = \partial^3_{\alpha\beta\gamma} \Phi \). The unit element \( e \in H_K \) is identified with the unit vector field \( \partial_1 \in \Gamma(T\Omega) \). The Euler element \( E \in H_K \) has linear components, by Lemma 4.2, so it can be identified with a well-defined holomorphic vector field on \( \Omega \), satisfying axioms (FM4).

### 4.4. Semisimplicity of Frobenius structures.

In this Section we collect main results and properties which hold true for a wide class of Frobenius structures (both formal and analytic), namely semisimple Frobenius structures. We begin our exposition with the formal case.

Let \( (H, \eta, \Phi) \) be a formal Frobenius manifold, and denote by \( \circ_0 \) the product on \( H \) with structure constants \( \Phi_{\alpha\beta\gamma}^\nu(0) \). We say that \( (H, \eta, \Phi) \) is

- **semisimple at the origin** if the \( k \)-algebra \( (H, \circ_0) \) is isomorphic to \( k^n \);
- **formally semisimple** if the \( K \)-algebra \( (H_K, \circ) \) is isomorphic to \( K^n \).
In the first (resp. second) case there exist an idempotent basis \((\pi_1, \ldots, \pi_n)\) of \(H\) (resp. \(H_K\)) such that
\[
\pi_i \circ \pi_j = \pi_i \delta_{ij}, \quad \eta(\pi_i, \pi_j) = 0, \quad i \neq j.
\]
(4.5)
Notice that the idempotent vectors \(\pi_i\) are uniquely defined up to re-ordering.

**Lemma 4.3.** A formal Frobenius manifold \((H, \eta, \Phi)\) is formally semisimple if and only if it is semisimple at the origin.

**Proof.** Formal semisimplicity clearly implies semisimplicity at the origin. Let us prove the converse. Denote by \(m := (t^1, \ldots, t^n)\) the maximal ideal of \(K\). We will denote by \(O(m^p)\) an arbitrary sum of elements of \(m^p \cdot H_K\). For any fixed \(h \in \mathbb{N}\) we call an \(h\)-order idempotent basis of \(H_K\) a basis \((\pi^H_1, \ldots, \pi^H_n)\) such that
\[
\pi^H_i \circ \pi^H_i = \pi^H_i + O(m^{h+1}), \quad \pi^H_i \circ \pi^H_j = O(m^{h+1}),
\]
for \(i, j = 1, \ldots, n\) and \(i \neq j\). Assume that \((H, \eta, \Phi)\) is semisimple at the origin. We claim there exist a \(h\)-order idempotent basis of \(H_K\) for any \(h \in \mathbb{N}\). We prove it by induction on \(h\). For \(h = 0\), it is trivial: if \((\pi_0^H, \ldots, \pi_n^0)\) is an idempotent basis of \((H, \omega_0)\), then it is a 0-order idempotent basis of \(H_K\). Assume that \((\pi^H_1, \ldots, \pi^H_n)\) is an \(h\)-order idempotent basis of \(H_K\): we have
\[
\pi^H_i \circ \pi^H_i = \sum_k a_{ik} \pi^H_k, \quad a_{ij} \in m^{h+1}, \quad \pi^H_i \circ \pi^H_j = \sum_k b_{ijk} \pi^H_k, \quad b_{ijk} \in m^{h+1},
\]
for \(i, j = 1, \ldots, n\) and \(i \neq j\). By commutativity and associativity, one deduces the following constraints on \(a_{ij}, b_{ijk}\):
\[
\begin{align*}
b_{ijk} &= b_{ijk}, & i, j, k = 1, \ldots, n, \\
b_{ijk} &\in m^{2h+2}, & i, j, k = 1, \ldots, n, \text{ distinct}, \\
b_{ij} + a_{ij} &\in m^{2h+2}, & i, j = 1, \ldots, n, \quad i \neq j.
\end{align*}
\]
(4.6) (4.7) (4.8)
Set
\[
\pi'_i := \pi^H_i + \sum_j w_{ij} \pi^H_j, \quad i = 1, \ldots, n,
\]
with arbitrary coefficients \(w_{ij} \in m^{h+1}\). The \(n\)-tuple \((\pi'_1, \ldots, \pi'_n)\) is an \((h+1)\)-order idempotent basis\(^{10}\) of \(H_K\) if and only if
\[
w_{ii} = -a_{ii}, \quad w_{ij} = a_{ij},
\]
for \(i, j = 1, \ldots, n\) and \(i \neq j\). This easily follows from (4.6)-(4.8). \(\square\)

In the analytic case, we will say that a Dubrovin-Frobenius manifold \(M\) is *(generically) semisimple* if the set \(M_{ss} := \{p \in M : (T_p M, \omega_p) \cong \mathbb{C}^n\}\) is non-empty. In such a case, it can be proved that \(M_{ss}\) is an open dense subset of \(M\). At each point \(p \in M_{ss}\) there exists tangent vectors \(\pi^p_1|_p, \ldots, \pi^p_n|_p\) satisfying the relations
\[
\pi^p_i|_p \circ \pi^p_j|_p = \pi^p_i|_p \delta_{ij}, \quad \eta_p(\pi^p_i|_p, \pi^p_j|_p) = 0, \quad i \neq j.
\]

\(^{10}\)In fact, the resulting basis \((\pi'_1, \ldots, \pi'_n)\) is not just an \((h+1)\)-order idempotent, but even a \((2h+1)\)-order idempotent basis.
It can be proved that, on sufficiently small open subsets $M_{ss}$, a coherent labeling of the idempotent tangent vectors can be chosen so that the resulting local vector fields are holomorphic. For a detailed discussion and proofs, see [Her02, Chapter 2].

**Remark 4.4.** In both the formal and analytic case we have $e = \sum_i \pi_i$.

**Proposition 4.5 ([Dub92, Dub96, Man99]).** For both formal and analytic semisimple Frobenius manifolds, the idempotents vector fields $\pi_1, \ldots, \pi_n$ are pairwise commuting, i.e. $[\pi_i, \pi_j] = 0$. Equivalently, the dual differential forms $\pi_i^\flat$, defined by $\langle \pi_i^\flat, \pi_j \rangle = \delta_{ij}$, are closed. \hfill \square

In both the formal and analytic cases, this result implies the existence of a local system of coordinates $u := (u_1, \ldots, u_n)$ such that

$$du_i = \pi_i^\flat, \quad \frac{\partial}{\partial u_i} = \pi_i.$$  

We will refer to $u$ as the *formal/analytic canonical coordinates*. These functions are defined up to re-ordering and shifts by constants. In the formal case, the functions $u_i$’s are just formal functions, i.e. elements of $k[\![t]\!]$.

**Proposition 4.6 ([Dub92, Dub96, Man99]).** The formal/analytic canonical coordinates can be uniquely chosen (up to re-ordering) so that $E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i}$. \hfill \square

In all the subsequent part of the paper, we will reserve Latin indices for canonical coordinates $u_1, \ldots, u_n$ and their vector fields $\partial_i := \frac{\partial}{\partial u_i}$. Einstein summation rule will be used only for repeated Greek indices.

## 5. Convergence of semisimple formal Frobenius manifolds

In this Section we prove the main result of the second part of this paper.

**Theorem 5.1.** Let $(H, \Phi, \eta, e, E)$ be a semisimple formal Frobenius manifold over $\mathbb{C}$. Then the domain of convergence of $\Phi$ is non-empty.

It follows that with any semisimple formal Frobenius manifold there is an associated analytic Dubrovin-Frobenius manifold, as explained in Section 4.3.

For the proof, we require some preliminary material.

### 5.1. Extended deformed connection.

We introduce one of the main objects attached to Frobenius structures, namely an integrable connection. It can be introduced in both formal and analytic frameworks.

**Formal case.** Let $k$ be a commutative $\mathbb{Q}$-algebra and $(H, \eta, \Phi)$ a formal Frobenius manifold as in Section 4.1. Denote by $k((z))$ the $k$-algebra of formal Laurent series in an auxiliary indeterminate $z$. Set $K((z)) := k[\![t]\!]((z))$ to be the Laurent series with coefficients in $k[\![t]\!]$, and $H_{K((z))} := H \otimes_k K((z))$. In the following paragraphs we will define two connections on the modules $H_K$ and $H_{K((z))}$ respectively. We firstly recall some basic notions.
5.1.1. **Algebraic connections on modules.** Let $A$ be a commutative and unital $k$-algebra, and $P$ an $A$-module. Denote by $\text{Diff}_1(P,P)$ the set of first order differential operators on $P$, i.e. the $k$-linear morphisms $\mathcal{D} \in \text{Hom}_k(P,P)$ such that

$$ab\mathcal{D}(p) - b\mathcal{D}(ap) - a\mathcal{D}(bp) + \mathcal{D}(abp) = 0, \quad a, b \in A, \quad p \in P.$$ 

Both $\text{Der}_k(A)$ and $\text{Diff}_1(P,P)$ are naturally equipped with an $A$-module structure. A connection $\nabla$ on $P$ is defined by an $A$-linear morphism $\nabla : \text{Der}_k(A) \to \text{Diff}_1(P,P)$, $u \mapsto \nabla_u$ satisfying the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_up, \quad a \in A, \quad p \in P.$$ 

The curvature of $\nabla$ is the $A$-bilinear morphism $R : \text{Der}_k(A) \times \text{Der}_k(A) \to \text{Hom}_A(P,P)$ defined by

$$R(u, v) := [\nabla_u, \nabla_v] - \nabla_{[u,v]}, \quad u, v \in \text{Der}_k(A).$$ 

Given a connection on $P$ we can induce connections on all the tensor products (over $A$) $P^\otimes p \otimes \text{Hom}_A(P,A)^\otimes q$ by requiring that

1. $\nabla$ commutes with contractions,
2. on $A$ (i.e. $p = q = 0$) the morphism $\nabla : \text{Der}_k(A) \to \text{Diff}_1(A,A)$ is just the inclusion.

As a general reference, see e.g. [Sar12].

5.1.2. **Deformed connections on $H_K$.** Consider the case $(A, P) = (K, H_K)$. Define a one-parameter family of connections $\nabla^z : \text{Der}_k(K) \cong H_K \to \text{Diff}_1(H_K, H_K)$, with $z \in \mathbb{C}$, on the module $H_K$ by the formula

$$\nabla^z_{\Delta_\alpha} \Delta_\beta := z\Delta_\alpha \circ \Delta_\beta, \quad \alpha, \beta = 1, \ldots, n.$$ 

**Theorem 5.2** ([Dub92, Man99]). WDVV equations (4.1) are equivalent to the flatness of $\nabla^z$, for any $z \in \mathbb{C}$. □

**Remark 5.3.** The connection $\nabla := \nabla^0$ is the (formal) Levi-Civita connection for $\eta$, i.e. the unique torsion-free connection satisfying $\nabla \eta = 0$. If $(e_1, \ldots, e_n)$ is a basis of $H_K$, set $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$. One can show that

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell \eta^{\ell k} (e_i \eta_{jk} + e_j \eta_{ik} - e_k \eta_{ij}).$$

The standard differential-geometrical proof works verbatim in this formal framework.

**Remark 5.4.** The Euler vector field is an affine vector field, i.e. $\nabla \nabla E = 0$. This is an equivalent formulation of Lemma 4.2.

5.1.3. **Extended deformed connection on $H_K((z))$.** We consider now the case $(A, P) = (K((z)), H_K((z)))$. In what follows we assume that the $K$-linear operator $\nabla^0 E : \text{Der}_k(K) \cong H_K \to H_K$ is (diagonalizable and) in diagonal form in the basis $(\Delta_1, \ldots, \Delta_n)$. Define two new $K$-linear operators $U, \mu$ by the formulae

$$U : H_K \to H_K, \quad X \mapsto E \circ X,$$

$$\mu : \text{Der}_k(K) \cong H_K \to H_K, \quad X \mapsto \frac{D}{2} - \nabla^0_X E,$$
where $D \in k$ is as in (4.3). All the tensors $\eta, \circ, U, \mu$ can be $K((z))$-linearly extended to $H_{K((z))}$. We will denote such an extension by the same symbols.

The extended deformed connection $\hat{\nabla}: \text{Der}_k(K((z))) \to \text{Diff}_1(H_{K((z))}, H_{K((z))})$ is defined by the formulae

\[ \hat{\nabla}_{\frac{\partial}{\partial \alpha}} X = \nabla_{\frac{\partial}{\partial \alpha}} X = \nabla_{\frac{\partial}{\partial \alpha}} X = \frac{\partial}{\partial z} X + U(X) - \frac{1}{z} \mu(X), \]

where $X \in H_{K((z))}$.

**Theorem 5.5** ([Dub96, Dub98, Dub99]). The connection $\hat{\nabla}$ is flat.

**Proof.** The flatness of $\hat{\nabla}$ is equivalent to the following conditions: $\partial_\beta \Phi_{\alpha, \beta, \gamma}$ is completely symmetric in $(\alpha, \beta, \gamma, \delta)$, the product $\circ$ is associative, $\nabla \nabla E = 0$, and $\mathcal{L}_E c = c$. This can be checked by a straightforward computation. \qed

**Analytic case.** Let $M$ be a Dubrovin-Frobenius manifold. Introduce the $(1,1)$-tensors $U, \mu \in \Gamma(\text{End}(TM))$ by the formulae

\[ U(X) = E \circ X, \quad \mu(X) := 2 - d X - \nabla X E, \quad X \in \Gamma(TM), \]

where $d$ is the charge of the Dubrovin-Frobenius structure, and $\nabla$ is the Levi-Civita connection of $\eta$. We assume that $\mu$ is (diagonalizable and) in diagonal form in the frame $(\partial t_1, \ldots, \partial t_n)$.

Denote by $\pi: M \times \mathbb{C}^* \to M$ the canonical projection on the first factor. If $\mathcal{F}_M$ denotes the tangent sheaf of $M$, then $\pi^* \mathcal{F}_M$ is the sheaf of sections of $\pi^*TM$, and $\pi^{-1} \mathcal{F}_M$ is the sheaf of sections of $\pi^*TM$ constant along the fibers of $\pi$. All the tensors $\eta, c, c, E, U, \mu$ can be lifted to the pulled-back bundle $\pi^*TM$, and we denote these lifts with the same symbols. Consequently, also the Levi-Civita connection $\nabla$ can be uniquely lifted on $\pi^*TM$ in such a way that $\nabla_{\frac{\partial}{\partial \alpha}} Y = 0$ for $Y \in \pi^{-1} \mathcal{F}_M$.

The extended deformed connection $\hat{\nabla}$ is the connection on $\pi^*TM$ defined by the formulae

\[ \hat{\nabla}_{\frac{\partial}{\partial \alpha}} Y = \nabla_{\frac{\partial}{\partial \alpha}} Y + z \frac{\partial}{\partial t^\alpha} \circ Y, \quad \hat{\nabla}_{\frac{\partial}{\partial z}} Y = \nabla_{\frac{\partial}{\partial z}} Y + U(Y) - \frac{1}{z} \mu(Y), \quad (5.1) \]

where $Y \in \pi^* \mathcal{F}_M$.

**Remark 5.6.** If we consider a formal Frobenius manifold associated to a pointed Dubrovin-Frobenius manifold $(M, p)$ as in Section 4.3, the Christoffel symbols of the formal connection $\hat{\nabla}$ constructed in Section 5.1.3 are germs of the Christoffel symbols of (5.1) at the point $p$.

Theorem 5.5 and its proof hold verbatim for the connection $\hat{\nabla}$ defined by (5.1).

**Remark 5.7.** In both the formal and analytic case, the operator $U$ is $\eta$-self-adjoint, and $\mu$ is $\eta$-skew-symmetric: for arbitrary $X, Y \in H_K$ (resp. sections of $TM$), we have

\[ \eta(U(X), Y) = \eta(X, U(Y)), \quad \eta(\mu(X), Y) = -\eta(X, \mu(Y)). \quad (5.2) \]
5.2. Darboux-Egoroff equations. Given a formal (resp. analytic) semisimple Frobenius manifold with idempotent vectors \( \pi_1, \ldots, \pi_n \) define the formal (resp. analytic) functions \( \eta_{ii}, \gamma_{ij} \in k[[u]] \) (resp. \( \mathbb{C}\{u\} \)) by

\[
\eta_{ii}(u) := \eta(\pi_i(u), \pi_i(u)), \quad i = 1, \ldots, n,
\]

\[
\gamma_{ij}(u) := \frac{\partial \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}}, \quad i, j = 1, \ldots, n.
\]

Lemma 5.8. We have

\[
\gamma_{ij}(u) = \frac{1}{2} \frac{\partial_i \partial_j t_1(u)}{\sqrt{\eta_{ii}(u) \eta_{jj}(u)}}, \quad t_1 := \sum_\alpha \eta_{1\alpha} t^\alpha. \tag{5.3}
\]

In particular, \( \gamma_{ij} = \gamma_{ji} \).

Proof. Consider the co-unit 1-form \( \theta := \eta(e, -) \). We have \( \eta_{ii} = \eta(\pi_i, \pi_i) = \langle \theta, \partial_i \rangle = \partial_i t_1 \). Equation (5.3) follows by the definition of \( \gamma_{ij} \).

Theorem 5.9. The functions \( \gamma_{ij}(u) \) satisfy the Darboux-Egoroff equations, i.e.

\[
\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad i, j, k \text{ distinct}, \tag{5.4}
\]

\[
\sum_{k=1}^n \partial_k \gamma_{ij} = 0, \quad i \neq j \tag{5.5}
\]

\[
\sum_{k=1}^n u_k \partial_k \gamma_{ij} = -\gamma_{ij}, \quad i \neq j. \tag{5.6}
\]

Proof. The proofs of [Man99, Prop. 3.4.1, Th. 3.7.2] apply verbatim also to the formal case. Notice that (5.4) and (5.5) are equivalent to the flatness of \( \eta \).

Corollary 5.10. For \( i \neq j \), we have

\[
(u_i - u_j) \partial_i \gamma_{ij} = \sum_{k \neq i, j} (u_j - u_k) \gamma_{ik} \gamma_{kj} - \gamma_{ij}. \tag{5.7}
\]

Proof. An easy consequence of (5.4), (5.5), and (5.6).

5.3. \( \nabla \)-flatness in canonical coordinates. Given a formal (resp. analytic) semisimple Frobenius manifold with idempotent vectors \( \pi_1, \ldots, \pi_n \) define the vectors

\[
f_i(u) := \eta_{ii}(u)^{-\frac{1}{2}} \pi_i(u), \quad i = 1, \ldots, n,
\]

for some choices of the square roots, and introduce the matrix \( \Psi \in GL(n, k[[u]]) \) (resp. \( GL(n, k\{u\}) \)) defined by

\[
\Psi = (\Psi_{i\alpha})_{i,\alpha}, \quad \frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \Psi_{i\alpha} f_i, \quad i = 1, \ldots, n.
\]

Lemma 5.11 ([CDG20, Lemma 2.29]). We have

\[
\Psi^T \Psi = \eta, \quad \Psi_{i1} = \sqrt{\eta_{ii}}, \quad \partial_i = \sum_{\alpha, \beta=1}^n \Psi_{i\beta} \eta^{\alpha\beta} \partial_\alpha, \quad c_{\alpha\beta\gamma} = \sum_{i=1}^n \frac{\Psi_{i\alpha} \Psi_{i\beta} \Psi_{i\gamma}}{\Psi_{i1}}.
\]

Lemma 5.12. We have \( \mu(f_i) = \sum_{j \neq i} (u_j - u_i) \gamma_{ij} f_j \).
Proof. Set $\nabla_{\pi_i} \pi_j = \sum_k \Gamma_{ij}^k \pi_k$. The only nonzero Christoffel symbols are
\[
\Gamma_{ii}^i = \frac{1}{2} \eta_{ii}^{-1} \frac{\partial \eta_{ii}}{\partial u_i}, \quad \Gamma_{ij}^j = -\frac{1}{2} \eta_{ij}^{-1} \frac{\partial \eta_{ii}}{\partial u_j}, \quad \Gamma_{ij}^i = \frac{1}{2} \eta_{ii}^{-1} \frac{\partial \eta_{ii}}{\partial u_j}, \quad i \neq j,
\]
see Remark 5.3. The claim follows by straightforward computations. \hfill \Box

The connection $\hat{\nabla}$ can be extended to the full tensor algebra of $H_K((z))$ (resp. $\pi^* TM$).

Lemma 5.13. Let $\xi \in H^T_K((z))$ (resp. $\pi^* T^* M$) such that $\hat{\nabla} \xi = 0$. If $\xi^i$ is $\eta$-dual to $\xi$, then we have
\[
\hat{\nabla}_{\frac{\partial}{\partial t}} \xi^i = 2z \frac{\partial}{\partial t} \circ \xi^i, \quad \hat{\nabla}_{\frac{\partial}{\partial z_i}} \xi^i = 2\mathcal{U}(\xi^i).
\]

Proof. A simple computation shows that $\hat{\nabla}_{\frac{\partial}{\partial t}} \eta_{\alpha\beta} = -2zc_{\alpha\beta}$ for $\alpha, \beta, \gamma = 1, \ldots, n$, and $\hat{\nabla}_{\frac{\partial}{\partial z_i}} \eta_{\alpha\beta} = -2\eta_{\alpha\beta} \mathcal{U}_i$, for $\alpha, \beta = 1, \ldots, n$. The claim then follows. \hfill \Box

Set $\xi^i(t, z) = \sum_{i=1}^n y_i(u(t), z) f_i$ for some formal (resp. analytic) functions $y_i$ of the formal canonical coordinates $u$ and $z$. Then, equations (5.8) are equivalent to
\[
\frac{\partial}{\partial u_i} y = \left(zE_i + V_i(u)\right) y, \quad \frac{\partial}{\partial z} y = \left(U(u) + \frac{1}{z} V(u)\right) y,
\]
where $y = (y_1, \ldots, y_n)^T$,
\[
V_i := \frac{\partial \Psi}{\partial u_i} \Psi^{-1}, \quad (E_i)_{ab} = \delta_{ai} \delta_{bi},
\]
and
\[
U = \text{diag}(u_1, \ldots, u_n), \quad V := \Psi \mu \Psi^{-1},
\]
where $\mu$ is the matrix of the operator $\mu : H_K \rightarrow H_K$ associated with the basis $(\frac{\partial}{\partial t}, \ldots, \frac{\partial}{\partial z})$. The compatibility of the system (5.9),(5.10) is equivalent to the equations
\[
\frac{\partial V}{\partial u_i} = [V_i, V], \quad [U, V_i] = [E_i, V].
\]

Lemma 5.14. Set $\Gamma = (\gamma_{ab})$. We have
\[
V^T + V = 0, \quad V = [\Gamma, U], \quad V_i^T + V_i = 0, \quad V_i = [\Gamma, E_i], \quad i = 1, \ldots, n.
\]

Proof. The identity $V = [\Gamma, U]$ is Lemma 5.12. The identity $\Psi(u)^T \Psi(u) = \eta$ implies $\partial_i \Psi^T \Psi + \Psi^T \partial_i \Psi = 0$, so that $V_i^T + V_i = 0$. We have $[U, V_i] = [U, [\Gamma, E_i]]$, by (5.12) and Jacobi identity. The kernel of the operator $[U, -] : M_n(k[u]) \rightarrow M_n(k[u])$ consists of diagonal matrices: if $A \in M_n(k[u])$ is such that $[U, A] = 0$, then $(u_a - u_b) A_{ab}(u) = 0$ for any $a, b = 1, \ldots, n$ with $a \neq b$. We deduce that $A_{ab}(u) = 0$, with $a \neq b$, since $k[u]$ is an integral domain. Hence $V_i = D + [\Gamma, E_i]$, where $D$ is a diagonal matrix. The skew-symmetry of $V_i$ implies that $D = 0$. \hfill \Box
Corollary 5.15. The equations
\[ V_i = [\Gamma, E_i], \quad V = [\Gamma, U], \quad \partial_i V = [V_i, V], \]
are equivalent to the Darboux-Egoroff equations (5.4),(5.5),(5.6) on \( \Gamma \).

Proof. This can be checked by a direct computation. \( \square \)

5.4. Reconstruction of the Frobenius structure. In this section we recall how it is possible to reconstruct the WDVV potential \( \Phi \) of a formal Frobenius manifold starting from the system of \( \nabla^z \)-flatness for 1-forms. By Theorem 5.2 we can look for formal functions \( \tilde{t} := (\tilde{t}_1, \ldots, \tilde{t}_n) \) of the form
\[ \tilde{t}_\alpha(t, z) := \sum_{p=0}^{\infty} h_{\alpha,p}(t)z^p \in k[[t, z]], \quad h_{\alpha,0}(t) = t_\alpha \equiv t^\beta \eta_{\alpha \beta}, \]
such that \( \nabla^zd\tilde{t}_\alpha = 0 \) for \( \alpha = 1, \ldots, n \).

Lemma 5.16 ([Dub92]). The functions \( h_{\alpha,p} \) satisfy the recursive equations
\[ h_{\alpha,0}(t) = t_\alpha \equiv t^\beta \eta_{\alpha \beta}, \quad \partial_\beta \partial_\gamma h_{\alpha,p+1} = c^\varepsilon_{\beta \gamma} \partial_\varepsilon h_{\alpha,p}, \quad p \in \mathbb{N}. \] \( \square \)

For \( f, g \in K \) write \( f \approx g \) iff \( f - g \) is a (at most) quadratic polynomial in \( t \).

Lemma 5.17 ([Dub92]). We have
\[ h_{\alpha,1} \approx \partial_\alpha \Phi, \quad \alpha = 1, \ldots, n, \]
\[ h_{1,2} \approx t^\alpha \partial_\alpha \Phi - 2\Phi. \] \( \square \)

Proof. We have \( \partial_\beta h_{\alpha,0} = \eta_{\alpha \beta} \), so that \( \partial_\beta \partial_\gamma h_{\alpha,1} = c_{\alpha \beta \gamma} \). Equation (5.13) follows.
We have \( \partial_1 \Phi = \frac{1}{2} \eta_{\alpha \beta} t^\alpha t^\beta \), so that \( \partial_\gamma \partial_\beta h_{1,2} = c_{\alpha \beta \gamma} \partial_\gamma h_{1,1} = c_{\alpha \beta \gamma} \partial_\gamma \partial_1 \Phi = c_{\alpha \beta \gamma} \partial_\gamma h_{1,0} t^\nu \). We also have \( \partial_\alpha \partial_\beta (t^\lambda \partial_\lambda \Phi - 2\Phi) = c_{\alpha \beta \lambda} t^\lambda \), and (5.14) follows. \( \square \)

Given a function \( f \in K \), we denote by \( \text{gr} f \in H_K \) the \( \eta \)-gradient of \( f \), defined by \( \text{gr} f := \sum_\alpha \eta^{\alpha \beta} \partial_\beta f \Delta_\alpha \). The following result allows to reconstruct the potential \( \Phi \) (up to quadratic terms) from the first coefficients \( h_{\alpha,p} \), with \( p \leq 3 \).

Theorem 5.18 ([Dub96, Dub99]). We have
\[ \Phi \approx \frac{1}{2} \left[ \eta(\text{gr} h_{\alpha,1}, \text{gr} h_{1,1}) \eta^{\alpha \beta} \eta(\text{gr} h_{\beta,0}, \text{gr} h_{1,1}) - \eta(\text{gr} h_{1,1}, \text{gr} h_{1,2}) - \eta(\text{gr} h_{1,3}, \text{gr} h_{1,0}) \right]. \] (5.15)

Proof. The expression in square brackets in the r.h.s. of (5.15) equals
\[ \eta^{\mu \lambda} \partial_\mu h_{\alpha,0} \partial_\lambda h_{1,1} \eta^{\alpha \beta} \eta(\text{gr} h_{\beta,1,1}, \text{gr} h_{1,1}) - \eta^{\tau \varepsilon} \partial_\tau h_{1,2} \partial_\varepsilon h_{1,2} - \partial_1 h_{1,3} \]
\[ \approx \eta^{\mu \lambda} \eta^{\alpha \beta} \partial_\nu \Phi \partial_\mu \Phi \partial_\lambda \Phi - \eta^{\tau \varepsilon} \partial_\tau \Phi (t^\lambda \partial_\lambda \Phi - \partial_1 \Phi) - \partial_1 h_{1,3}. \] (5.16)

We have \( \partial_\alpha \Phi = \eta_{\alpha \varepsilon} t^\varepsilon \), and \( \partial_\alpha \partial_1 h_{1,3} = c_{\alpha \beta} \partial_\beta h_{1,2} = \partial_\alpha h_{1,2} \) so that \( \partial_1 h_{1,3} \approx h_{1,2}. \) Hence (5.16) equals \( 2\Phi \) up to quadratic terms. \( \square \)
5.5. **Proof of Theorem 5.1.** Let \((H, \eta, \Phi, e, E)\) be a formal Frobenius manifold. Fix one ordering \(u_0 \in \mathbb{C}^n\) of the eigenvalues of \(U(t)\) specialized at the origin \(t = 0\). We have \(n \times n\) matrix-valued (a priori) formal power series in \(u\)

\[
V(u) = V_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} V^{(k)}_{\ell_1, \ldots, \ell_k} \prod_{i=1}^{\ell_k} \bar{u}_i, \quad V_i(u) = V_{i,0} + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} V_i^{(k)}_{\ell_1, \ldots, \ell_k} \prod_{i=1}^{\ell_k} \bar{u}_i, \quad \Psi(u) = \Psi_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} \Psi^{(k)}_{\ell_1, \ldots, \ell_k} \prod_{i=1}^{\ell_k} \bar{u}_i, \quad \Gamma(u) = \Gamma_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} \Gamma^{(k)}_{\ell_1, \ldots, \ell_k} \prod_{i=1}^{\ell_k} \bar{u}_i,
\]

where \(\bar{u}_i := u_i - u_{o,i}\) for \(i = 1, \ldots, n\). These power series are well defined by the semisimplicity assumption, and they satisfy properties described in Theorem 5.9, and Lemmata 5.11, 5.12 and 5.14. We subdivide the proof in two parts. In the first part, we construct a pointed germ \((M, p)\) of a Dubrovin-Frobenius manifold \(M\) starting from the datum of \(u_0, V_o, \Psi_o, \Gamma_o\). In the second part, we prove that the formal regularity structure \((H, \eta, \Phi, e, E)\) is the completion of the pointed analytic germ \((M, p)\).

**Part I.** The system (5.10) specialized at \(u_0\), namely \(\frac{\partial Y}{\partial z} = (U_o + \frac{1}{z} V_o)Y\), can be identified with equation (3.15) (in the special case \(B'_o = 0\)). Notice that conditions (1) and (2) of Theorem 2.5 and Proposition 3.22 are satisfied: by Lemma 5.12 we have \(V = [\Gamma, U]\) and therefore \(B'_o = 0\) and \(B''_o = [\Gamma_o, U_o]\). The arguments of Section 3.4 can be applied, in both cases \(u_0 \in \mathbb{C}^n \setminus \Delta\) and \(u_0 \in \Delta\). We can fix an admissible \(\tau\) at \(u_0\), the \((u_0, \tau)\)-admissible datum \(\mathfrak{M}\) is well-defined, and we can set the RHB problem \([u, \tau, \mathfrak{M}]\). This problem is solvable with respect to \(u\) on an open neighborhood \(V \setminus \Theta\) of \(u_0\), by Theorem 3.18. The unique solution \(G(z; u)\) is holomorphic in \(u \in V \setminus \Theta\), and with expansion

\[
G(z; u) = I + \frac{1}{2} F_{1, an}^{an}(u) + O \left( \frac{1}{z^2} \right), \quad z \to \infty, \quad z \in \Pi_{L/R},
\]

\[
G(z; u) = G_0(u) + G_1(u)z + G_2(u)z^2 + G_3(u)z^3 + O(z^4), \quad z \to 0.
\]

Here the superscript “an” stands for \emph{analytic}. As output of Section 3.4, we also obtain a compatible joint system of differential equations (with analytic coefficients in \(u\), not just formal) of the form

\[
\frac{\partial Y}{\partial u_i} = (z E_i + V_{i, an}^{an}(u))Y, \quad \frac{\partial Y}{\partial z} = \left( U + \frac{1}{z} V^{an}(u) \right)Y, \quad (5.17)
\]

where \(V^{an}(u) := [F_{1, an}^{an}(u), U], \) and \(V_{i, an}^{an}(u) := [F_{1, an}^{an}(u), E_i]. \) Moreover, we have

\[
V^{an}(u_0) = V_o, \quad G_0(u_0) = \Psi_o, \quad \partial_i G_0 = V^{an}_{i, 0}, \quad i = 1, \ldots, n.
\]

From the datum of \(G_i(u)\), with \(i = 0, 1, 2, 3\), we can construct a Dubrovin-Frobenius manifold as follows: set

\[
t^\alpha(u) := \eta^{\alpha, \beta} \sum_{i=1}^{n} G_{0, i, \beta}(u) G_{1, i, \alpha}(u), \quad \alpha = 1, \ldots, n,
\]

\[
F(u) := \frac{1}{2} \left[ t^\alpha(u) t^\beta(u) \sum_{i=1}^{n} G_{0, i, \alpha}(u) G_{1, i, \beta}(u) - \sum_{i=1}^{n} (G_{1, i, \alpha}(u) G_{2, i, \alpha}(u) + G_{0, i, \alpha}(u) G_{3, i, \alpha}(u)) \right].
\]
Invert the first series expansions, to obtain \( u = \mathbf{u}(t) \). The function \( F(\mathbf{u}(t)) \) gives a solution of WDVV equations, and defines an analytic Dubrovin-Frobenius manifold on an open subset of \( H \). The formulae above are, in their essence, re-writing of formulae of Lemma 5.11 and formula (5.15). See [Dub99, Guz01].

**Part II.** We need to prove that the series expansion \( F(\mathbf{u}(t)) \) obtained in Part I equals (up to quadratic terms) the original potential \( \Phi(t) \). For that, it is sufficient to prove that \( F^{an}_1(\mathbf{u})'' = \Gamma(\mathbf{u})'' \). By Lemma 5.14, indeed, it follows that \( V^{an}(\mathbf{u}) = V(\mathbf{u}) \), and \( V^{an}_i(\mathbf{u}) = V_i(\mathbf{u}) \). Consequently, from the equations

\[
\partial_t G_0 = V^{an}_i G_0, \quad \partial_t \Psi = V_i \Psi, \quad G_0(\mathbf{u}_o) = \Psi_o,
\]

we deduce \( G_0(\mathbf{u}) = \Psi(\mathbf{u}) \). From this, the equations defining \( F(\mathbf{u}(t)) \) given in Part I, and the last two formulas of Lemma 5.11 (or equivalently the reconstruction formula (5.15) for \( \Phi(t) \)), we obtain \( \partial^3_{\alpha\beta\gamma} \Phi(t) = \partial^3_{\alpha\beta\gamma} F(\mathbf{u}(t)) \). This proves the thesis.

We now prove that \( F^{an}_1(\mathbf{u})'' = \Gamma(\mathbf{u})'' \).

**Lemma 5.19.** We have \( F^{an}_1(\mathbf{u})'' = \Gamma(\mathbf{u})'' \).

**Proof.** By Proposition 3.22, the system (5.10) specialized at \( \mathbf{u}_o \), namely \( \frac{\partial Y}{\partial z} = (U_o + \frac{1}{z} V_o) Y \), admits a unique formal solution \( Y_F(z) = (I + A_1 z^{-1} + A_2 z^{-2} + O(z^{-3})) e^{z U} \). Let us recall how to compute \( A_1 \). It is uniquely determined by the two equations

\[
[A_1, U_o] = V_o, \quad [A_2, U_o] = A_1 + V_o A_1.
\]

The first equation uniquely determines all the entries \((A_1)_{ab}\) for indices \(a \neq b\) such that \( u_{o,a} \neq u_{o,b} \):

\[
(A_1)_{ab} = \frac{V_{o,ab}}{u_{o,b} - u_{o,a}} = \Gamma_{o,ab},
\]

by Lemma 5.12. All the remaining entries \((A_1)_{ab}\), with \( a \neq b \) such that \( u_{o,a} = u_{o,b} \), are uniquely determined by the second equation:

\[
(A_1)_{ab} = -\sum_{\ell} V_{o,a\ell} (A_1)_{\ell b} = -\sum_{\ell} (u_{o,\ell} - u_{o,a}) \Gamma_{o,a\ell} \Gamma_{\ell b} = \Gamma_{o,ab}.
\]

The last equality follows by specializing equation (5.7) to \( u = \mathbf{u}_o \). This prove that \( A''_1 = \Gamma'' \). By uniqueness of the formal solution we clearly have \( F^{an}_1(\mathbf{u}_o) = A_1 \).

**Lemma 5.20.** The off-diagonal entries of \( F^{an}_1(\mathbf{u}) \) satisfy the Darboux-Egoroff system (5.4), (5.5), (5.6), (5.7).

**Proof.** From the compatibility conditions \( \partial_i \partial_j = \partial_j \partial_i \) of the system (5.17), we have

\[
[E_j, \partial_t F^{an}_1] = [E_i, \partial_j F^{an}_1] + [[E_i, F^{an}_1], [E_j, F^{an}_1]] = 0,
\]

This coincides with equations (5.4) and (5.5). Let \( \kappa \in \mathbb{R}_{>0} \). The piecewise analytic function \( \tilde{G} : (\Pi_0 \cup \Pi_L \cup \Pi_R) \times (V \setminus \kappa \Theta) \to \mathbb{C} \) defined by

\[
\tilde{G}(z; \mathbf{u}) := G(\kappa z; \kappa^{-1} \mathbf{u}) \kappa^D z^D \kappa^L z^{-D}, \quad z \in \Pi_0, \\
\tilde{G}(z; \mathbf{u}) := G(\kappa z; \kappa^{-1} \mathbf{u}), \quad z \in \Pi_{L/R},
\]

where
solves the same RHB problem $\mathcal{P}[\mathbf{u}, \tau, \mathcal{M}]$ as $G$. By uniqueness of solution we have $\tilde{G} = G$. This implies that $F^\text{an}_1(\kappa^{-1}\mathbf{u}) = \kappa F^\text{an}_1(\mathbf{u})$, and (5.6) follows by Euler’s homogeneous function theorem.

□

Lemma 5.21. Let

$$\Gamma(\mathbf{u}) = \Gamma_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k = 1}^{n} \frac{1}{k!} \Gamma^{(\ell)} \prod_{j=1}^{k} \mathcal{u}_{\ell_j}, \quad \mathcal{u}_i := u_i - u_{0,i},$$

be a matrix-valued formal power series, with $\Gamma(\mathbf{u})^T = \Gamma(\mathbf{u})$, and whose off-diagonal entries $\Gamma_{ij}$ are formal solutions of the Darboux-Egoroff system (5.4), (5.5), (5.6). The off-diagonal entries of the coefficients $\Gamma^{(\ell)}$ can be uniquely reconstructed from the off-diagonal entries of $\Gamma_o$.

Proof. We have to show that the derivatives $\partial_i \ldots \partial_i \Gamma_{ij}(\mathbf{u}_o)$ can be computed from the only knowledge of the numbers $\Gamma_{ij}(\mathbf{u}_o)$. We proceed by induction on $N$. Let us start with the case $N = 1$.

**Step 1.** For $i, j, k$ distinct, by expanding both sides of $\partial_k \Gamma_{ij} = \Gamma_{ik} \Gamma_{kj}$ in power series, and equating the coefficients, one reconstructs the coefficients of $\partial_k \Gamma_{ij}(\mathbf{u}_o)$. 

**Step 2.** From the identity (5.7) for $\Gamma_{ij}$, one can compute $\partial_i \Gamma_{ij}(\mathbf{u}_o)$ provided that $u_{0,i} \neq u_{0,j}$. 

**Step 3.** Assume that $u_{0,i} = u_{0,j}$. By taking the $\partial_i$-derivative of both sides of (5.7) we obtain

$$2 \partial_i \Gamma_{ij}(\mathbf{u}) + (u_i - u_j) \partial_i \partial_i \Gamma_{ij}(\mathbf{u}) = \sum_{k \neq i, j} (u_j - u_k) \left[ \partial_i \Gamma_{ik}(\mathbf{u}) \Gamma_{kj}(\mathbf{u}) + \Gamma_{ik}(\mathbf{u}) \partial_i \Gamma_{kj}(\mathbf{u}) \right].$$

(5.18)

By evaluating (5.18) at $\mathbf{u} = \mathbf{u}_o$ we can compute all the numbers $\partial_i \Gamma_{ij}(\mathbf{u}_o)$, namely

$$\partial_i \Gamma_{ij}(\mathbf{u}_o) = \frac{1}{2} \sum_{k \neq i, j} (u_{0,j} - u_{0,k}) \left[ \partial_i \Gamma_{ik}(\mathbf{u}_o) \Gamma_{kj}(\mathbf{u}_o) + \Gamma_{ik}(\mathbf{u}_o)^2 \Gamma_{kj}(\mathbf{u}_o) \right].$$

Notice that the only terms $\partial_i \Gamma_{ik}(\mathbf{u}_o)$ appearing in this sum are those computed in Step 2.

**Step 4.** By the symmetry condition $\Gamma(\mathbf{u})^T = \Gamma(\mathbf{u})$, we have $\partial_i \Gamma_{ij}(\mathbf{u}_o) = \partial_j \Gamma_{ji}(\mathbf{u}_o)$, and these numbers can be computed as in Steps 2 and 3.

This proves that all the first derivatives $\partial_k \Gamma_{ij}(\mathbf{u}_o)$ can be computed.

**Inductive step.** Assume to know all the $N$-th derivatives $\partial_i \ldots \partial_i \Gamma_{ij}(\mathbf{u}_o)$. We show how to compute the number $\partial_{h_1} \ldots \partial_{h_{N+1}} \Gamma_{ij}(\mathbf{u}_o)$ for any $(N + 1)$-tuple $(h_1, \ldots, h_{N+1})$.

**Step 1.** Assume that there exists $\ell \in \{1, \ldots, N + 1\}$ such that $h_\ell \neq i, j$. We have

$$\partial_{h_1} \ldots \partial_{h_{N+1}} \Gamma_{ij} = \partial_{h_1} \ldots \partial_{h_{\ell-1}} \partial_{h_{\ell+1}} \ldots \partial_{h_{N+1}} \Gamma_{ij} = \partial_{h_1} \ldots \partial_{h_{\ell-1}} \partial_{h_{\ell+1}} \ldots \partial_{h_{N+1}} \Gamma_{i h_\ell} \Gamma_{h_\ell j}.$$

By evaluation at $\mathbf{u} = \mathbf{u}_o$, we can compute all the numbers $\partial_{h_1} \ldots \partial_{h_{N+1}} \Gamma_{ij}(\mathbf{u}_o)$.

Now we need to compute the mixed derivatives $\partial_i^p \partial_j^{N+1-p} \Gamma_{ij}(\mathbf{u}_o)$, with $0 \leq p \leq N + 1$.

**Step 2.** Assume $p > 0$ and $u_{0,i} \neq u_{0,j}$. Take the $\partial_i^p \partial_j^{N+1-p}$-derivative of both sides of (5.7): by evaluation at $\mathbf{u} = \mathbf{u}_o$ we can reconstruct the numbers $\partial_i^p \partial_j^{N+1-p} \Gamma_{ij}(\mathbf{u}_o)$.

**Step 3.** Assume $p > 0$ and $u_{0,i} = u_{0,j}$. Take the $\partial_i^p \partial_j^{N+1-p}$-derivative of both sides of
Step 4. Repeating this procedure, by decreasing nonnegative integers identifying the last two punctures of a same curve. Specialize (5.19) for the morphism forgetting the last puncture, by Cohomological field theories. This proves that all the derivatives \( \partial_i N^1 \Gamma_{ij} (u_o) \).

Specialize (5.19) for \( p = N \): by evaluation at \( u = u_o \) of both sides, we can compute the derivative \( \partial_i N^1 \Gamma_{ij} (u_o) \).

Repeating this procedure, by decreasing \( p \mapsto p - 1 \) at each step, we can compute all the mixed derivatives \( \partial_i^p \partial_j^p \Gamma_{ij} (u_o) \).

**Step 4.** Assume \( p = 0 \). By symmetry of \( \Gamma (u) \), we have \( \partial_j N^1 \Gamma_{ij} (u_o) = \partial_j N^1 \Gamma_{ji} (u_o) \), and we can proceed as in Steps 2 and 3.

This proves that all the \((N + 1)\)-th derivatives \( \partial_{h_1} \ldots \partial_{h_{N+1}} \Gamma_{ij} (u_o) \) can be computed. \( \square \)

This proves that \( F_1^{\infty} (u)^\prime = \Gamma (u)^\prime \). The proof of Theorem 5.1 is complete.

6. Application to CohFT’s and Gromov-Witten theory

6.1. Cohomological field theories. Let \( k \) and \((H, \eta, e)\) be as in Section 4.1. For a pair of nonnegative integers \((g, n)\) in the stable range \( 2g - 2 + n > 0 \), denote by \( \overline{M}_{g,n} \) the Deligne-Mumford moduli space of stable \( n \)-pointed curves of genus \( g \). Denote by \( \pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n} \) the morphism forgetting the last puncture, by \( \sigma : \overline{M}_{g,1,n+1} \times \overline{M}_{g,2,n+1} \to \overline{M}_{g+1,2,n+1} \) the morphism which identifies the last markings, and by \( \tau : \overline{M}_{g,n+2} \to \overline{M}_{g+1,n} \) the morphism identifying the last two punctures of a same curve.

A Cohomological field theory (CohFT) on \((H, \eta, e)\) is the datum of a system \((\Omega_{g,n})_{2g-2+n>0}\) of \( k \)-multilinear maps \( \Omega_{g,n} : H^{2n} \to H^*(\overline{M}_{g,n}, k) \) satisfying the following axioms:

1. each tensor \( \Omega_{g,n} \) is \( \mathcal{G}_n \)-covariant with respect to the natural actions of the symmetric group \( \mathcal{G}_n \) on both \( H^{2n} \) and \( H^*(\overline{M}_{g,n}, k) \),
2. \( \Omega_{0,3}(e \otimes \Delta_\alpha \otimes \Delta_\beta) = \eta_{\alpha, \beta} \),
3. \( \pi^* \Omega_{g,n}(\bigotimes_{i=1}^n v_{\alpha_i}) = \Omega_{g,n}(\bigotimes_{i=1}^n v_{\alpha_i} \otimes e) \),
4. \( \sigma^* \Omega_{g_1+g_2,n_1+n_2}(\bigotimes_{i=1}^{n_1+n_2} v_{\alpha_i}) = \eta^{\mu \nu} \Omega_{g_1,n_1+1}(\bigotimes_{i=1}^{n_1} v_{\alpha_i} \otimes \Delta_\mu) \Omega_{g_2,n_2+1}(\bigotimes_{i=n_1+1}^{n_1+n_2} v_{\alpha_i} \otimes \Delta_\nu) \),
5. \( \tau^* \Omega_{g+1,n}(\bigotimes_{i=1}^n v_{\alpha_i}) = \eta^{\mu \nu} \Omega_{g,n+2}(\bigotimes_{i=1}^n v_{\alpha_i} \otimes \Delta_\mu \otimes \Delta_\nu) \).

Given a CohFT, we may introduce generating functions, in infinitely many variables \( t^\ast = (t^\ast_d)_{d \in \mathbb{N}} \) of intersection numbers with psi-classes,

\[
\mathcal{F} (t^\ast) := \sum_{\substack{n \geq 0 \atop 2g-2+n>0}} \frac{1}{n!} \sum_{\alpha_1, \ldots, \alpha_n=1, \ldots, n} \left( \prod_{i=1}^n \tau_{d_i, \Delta_\alpha_i} \right)_g \prod_{i=1}^n t_{d_i}^\alpha_i , \quad (6.1)
\]

\[
\left( \prod_{i=1}^n \tau_{d_i, \Delta_\alpha_i} \right)_g := \int_{\overline{M}_{g,n}} \Omega_{g,n} \left( \bigotimes_{i=1}^n \Delta_\alpha_i \right) \prod_{i=1}^n \psi_{\alpha_i}^{d_i} . \quad (6.2)
\]
In the genus zero sector and restricting to the small phase space, i.e. by setting $t_d^α = 0$ for $d > 0$ and $t_d^α = t^α$ for $α = 1, \ldots, n$, the expression above simplifies to

$$\mathcal{F}_0(t) = \sum_{n>2} \sum_{α_1,\ldots,α_n=1}^n \frac{t^{α_1} \cdots t^{α_n}}{n!} \int_{\mathcal{M}_{0,n}} \Omega_{0,n}(\Delta_{α_1} \otimes \cdots \otimes \Delta_{α_n}). \quad (6.3)$$

The power series $\mathcal{F}_0 \in k[t]$ is a solution of WDVV equations, and it defines a formal Frobenius manifold (over $k$) on $(H, η, e)$, see [KM94, Man99]. The CohFT will be said to be 

semisimple

if the corresponding formal Frobenius manifold is semisimple.

If $E = \sum_α (w_α t^α + y_α) \partial_α$ is a Killing-conformal vector field on $H$, i.e. $\mathcal{E}Eη = (2 - d)η$ for some $d \in k$, we have a natural action of $E$ on the CohFT $(\Omega_{g,n})_{g,n}$. Denote by $\deg: H^*(\overline{M}_{g,n}, k) \to H^*(\overline{M}_{g,n}, k)$ the operator which acts on $H^{2k}$ by multiplication by $k$. Then we set

$$(E\Omega)_{g,n} \left( \bigotimes_{j=1}^n \Delta_{α_j} \right) := \left( \deg + \sum_{ℓ=1}^n w_ℓ \right) \Omega_{g,n} \left( \bigotimes_{j=1}^n \Delta_{α_j} \right) + π∗Ω_{g,n+1} \left( \bigotimes_{j=1}^n \Delta_{α_j} \otimes \sum_{ℓ=1}^n y_ℓΔ_ℓ \right).$$

A CohFT is called homogeneous in genus $g$ if $(E\Omega)_{g,n} = [(g-1)d+n]Ω_{g,n}$ for all $n > 2 - 2g$. When a CohFT is homogeneous in genus zero, $E$ is an Euler vector field for the underlying formal Frobenius manifold.

**Remark 6.1.** Teleman Reconstruction Theorem [Tel12, Th. 1] asserts that a CohFT, semisimple and homogeneous in all genera, can be uniquely reconstructed from the underlying formal Frobenius manifold. The reconstruction is performed via the Givental group action [Giv01].

The following result immediately follows from Theorem 5.1.

**Theorem 6.2.** For any semisimple and homogeneous (at least in genus 0) CohFT over $k = \mathbb{C}$, the potential $\mathcal{F}_0(t)$ is convergent. In particular, there exist real positive constants $m, ρ_1, \ldots, ρ_n$ such that

$$\left| \int_{\mathcal{M}_{0,[α]}} \Omega_{0,[α]}(\Delta_1^α \otimes \cdots \otimes \Delta_n^α) \right| \leq m \alpha! \prod_{i=1}^n ρ_i^α_i, \quad α \in \mathbb{N}^n,$$

where we set $\alpha! := \prod_j α_j$, and $|α| := \sum_k α_k$. \hfill \Box

### 6.2. Gromov-Witten theory

Let $X$ be a smooth complex projective variety with vanishing odd cohomology $H^{odd}(X; \mathbb{C}) = 0$. Let $(\Delta_1, \ldots, \Delta_n)$ be a homogeneous basis of $H^*(X; \mathbb{C})$, with $\Delta_1 = 1$ and $(\Delta_2, \ldots, Δ_{r+1})$ a NEF$^{11}$ $\mathbb{Z}$-basis of $H^2(X; \mathbb{Z})/\text{torsion}$. Denote by $η$ the Poincaré metric $η(α, β) := \int_X α \cup β$. Introduce indeterminates $Q := (Q_1, \ldots, Q_r)$, and define the Novikov ring $Λ := \mathbb{Q}[Q]$.

Gromov-Witten theory naturally provides a CohFT over the $Λ$-module $H^*(X; Λ)$ with $Λ$-bilinearly extended Poincaré metric $η$. The maps $Ω_{g,n}$ are given by the counting of curves on $X$,

$$Ω_{g,n} \left( \bigotimes_{i=1}^n Δ_{α_i} \right) := \sum_β φ_*( \left[ \overline{M}_{g,n}(X, β) \right]_{\text{vir}} \cap \bigcap_{i=1}^n \text{ev}_i^∗Δ_{α_i} ) Q_β \in H^*(\overline{M}_{0,n}; Λ), \quad (6.4)$$

---

$^{11}$This means that each $Δ_2, \ldots, Δ_{r+1}$ intersect every effective curve class $β \in \text{Eff}(X)$ non-negatively.
where \( Q^g := \prod_{i=1}^r Q_i^{f_i \Delta_i + 1} \), \( \overline{M}_{g,n}(X, \beta) \) is the Deligne-Mumford moduli space of \( n \)-pointed stable maps with target \( X \), genus \( g \) and degree \( \beta \), \( \text{ev}_i : \overline{M}_{g,n}(X, \beta) \to X \) are the evaluation morphisms and \( \phi : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n} \) is the morphism forgetting the map.

Equation (6.3) defines then a formal power series \( F_0^X \in \Delta[t] \), called the genus 0 Gromov-Witten potential of \( X \). The corresponding formal Frobenius manifold over \( k = \Lambda \) is the quantum cohomology of \( X \). In order to work with formal Frobenius manifolds over \( \mathbb{C} \) we make the following assumption.

**Assumption A:** There exist a point \( q \in \mathbb{C}^r \) such that the series \( \int_{\overline{M}_{0,n}} \Omega_{0,n} \left( \bigotimes_{i=1}^n \Delta_{a_i} \right) |_{Q=q} \) are convergent for any \( n \geq 3 \).

If Assumption A holds true, then the specialization \( F_0^X |_{Q=q} \) is a formal power series in \( \mathbb{C}[t] \). We call big quantum cohomology of \( X \) (at \( Q = q \)) the corresponding formal Frobenius manifold over \( \mathbb{C} \). We call small quantum cohomology of \( X \) (at \( Q = q \)) the Frobenius \( \mathbb{C} \)-algebra structure defined on \( H^*(X; \mathbb{C}) \) with structure constants \( c^g_{\alpha \beta} := \eta^{\eta \mu} \int_{\overline{M}_{0,3}} \Omega_{0,3} (\Delta_{\alpha} \Delta_{\beta} \Delta_{\mu}) |_{Q=q} \).

**Remark 6.3.** Assumption A holds true for all Fano varieties. This is because any sum \( \sum_\beta \) in (6.4) reduces to a finite number of terms, so that \( \Omega_{0,n} \left( \bigotimes_{i=1}^n \Delta_{a_i} \right) \in \mathbb{Q}[Q] \). See e.g. [CK99, Prop. 8.1.3].

**Remark 6.4.** By the Divisor axiom of Gromov-Witten invariants, it follows that the potential \( F_0^X \), can be seen as a formal power series in \( \mathbb{Q}[t^1, Q_1 e^{t^1}, \ldots, Q_r e^{t^r+1}, t^{r+2}, \ldots, t^n] \), see [CK99, Man99]. If Assumption A holds true, without loss of generalities we can assume that \( q = (1,1,1,\ldots,1) \): this correspond to a shift of coordinates \( t^{i+1} \mapsto t^{i+1} - \log q_i \) for \( i = 1,\ldots,r \).

**Remark 6.5.** If \( X \) has generically semisimple quantum cohomology (as a formal Frobenius manifold over \( \Lambda \)), then \( X \) is of Hodge-Tate type, i.e. the Hodge numbers \( h_{p,q}(X) := \dim_\mathbb{C} H^q(X, \Omega^p) \) vanish for \( p \neq q \), see [HMT09].

Theorem 5.1 implies then the following result.

**Theorem 6.6.** Let Assumption A hold true. If the small quantum cohomology of \( X \) at \( q \) is semisimple, then the function \( F_0^X(t) |_{Q=q} \) has a non-empty domain of convergence \( M_q \subseteq H^*(X; \mathbb{C}) \), which is equipped with a Dubrovin-Frobenius manifold structure. \( \square \)

Theorem 6.6 should be compared with other results in literature, differing in techniques. In [Iri07], H. Iritani proved convergence of the big quantum cohomology of \( X \) under a different assumption, namely that \( H^*(X; \mathbb{C}) \) is generated by \( H^2(X; \mathbb{C}) \), see [Iri07, Corollary 5.9]. Subsequently, in [CI15] T. Coates and H. Iritani proved the convergence (suitably defined) of all potentials \( F_g^X \) given by (6.1), by assuming both convergence of \( F_0^X \) and semisimplicity.

Whenever the three-point Gromov-Witten correlators \( \int_{\overline{M}_{0,3}} \Omega_{0,3} (\Delta_{\alpha} \Delta_{\beta} \Delta_{\mu}) \) of \( X \) are explicitly known, and thus generators and relations for the small quantum cohomology ring are given, it is purely a problem in computational commutative algebra to check generic semisimplicity of the small quantum cohomology. Here, we limit ourselves to the following claim\(^\text{12}\), which follows from [BM04, BM19, CMP10, Cio04, Cio05, Iri07, Per14].

\(^{12}\)Surely enough, such a list does not cover all the known cases of semisimple small quantum cohomologies available in literature.
Corollary 6.7. We have $F^X_0 \in \mathbb{Q}\{Q, t\}$ in the following cases (not mutually excluding):

1. $X = G/P$ is a (co)minuscule homogeneous variety;
2. $X$ is a del Pezzo surface;
3. $X$ is a Fano toric variety;
4. $X$ is one of the following Fano threefolds:
   - $\mathbb{P}^3$, a quadric $Q_3, V_5, V_{22}$,
   - $M^3_k$ with $21 \leq k \leq 36$ and $k \neq 23, 25, 28$,
   - $M^3_k$ with $k = 10, 12, 15, 17, 18, 20, 24, 25, 27, 28, 30, 31$,
   - $\mathbb{P}^4 \times \mathbb{P}^2_k$ where $\mathbb{P}^2_k$ is the blow-up of $\mathbb{P}^2$ at $k$ points ($1 \leq k \leq 8$);
5. $X$ is a Fano general hyperplane section with index $i(X) > \frac{1}{2} \dim_{\mathbb{C}} X$ of a homogeneous space in the following list:
   - $\mathbb{P}^n$, the $n$-dimensional quadric $Q_n$, $LG(3, 6)$, $F_4/P_1$
   - $Gr(2, 2n + 1)$, $OG(5, 10)$, $OG(2, 2n + 1)$, $G_2/P_1$;
6. $X$ is the Cayley Grassmannian parametrizing four dimensional subalgebras of the complex octonions. 

Remark 6.8. It is known that there exist homogeneous spaces with non-semisimple small quantum cohomology, [CMP10, CP11]. Isotropic Grassmannians $IG(2, 2n)$ furnish an example. It is also known, however, that their big quantum cohomology is generically semisimple [GMS15, Per14, CMMPS19]. For these varieties, the results of the current paper do not allow to infer the convergence of the genus zero Gromov–Witten potential, a working assumption in [CMMPS19, Th. B].

There is an intriguing conjecture due to B. Dubrovin [Dub98, Conj. 4.2.2] stating the equivalence of the semisimplicity of the (big) quantum cohomology of a variety $X$ (originally assumed to be Fano) and the existence of full exceptional collections in the derived category of coherent sheaves $D^b(X)$. In its most updated formulation, under the assumption of convergence of the genus zero Gromov-Witten potential $F^X_0$, Dubrovin’s conjecture also predicts the monodromy data of the system (5.10) (in the terminology of the current paper, the admissible data $M$) in terms of characteristic classes of the objects of these exceptional collections, see [GGI16, CDG18, Cot20]. In [Dub98, §4.2, Problem 1] Dubrovin also briefly addressed the problem of convergence of the genus zero Gromov-Witten potential $F^X_0$. In this regard, Dubrovin adds: «Hopefully, in the semisimple case the convergence can be proved on the basis of the differential equations of n.3»\(^{13}\). Theorems 6.6 fulfills Dubrovin’s hope.

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\(^{13}\) These include the equations in Part I of our proof of Theorem 5.1.
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