COMMUTING DIFFERENTIAL AND DIFFERENCE OPERATORS ASSOCIATED TO COMPLEX CURVES, II

B. ENRIQUEZ AND G. FELDER

Introduction. This paper is a sequel to [4]. Our main aim is to construct a commuting family of difference-evaluation operators \((T^\mathit{diff}_z)\), deforming the difference-evaluation operators \(T^\mathit{class}_z\) of [4], and to interpret them as the action of the center of a quantum algebra in the space of intertwiners of a “regular” subalgebra.

Let us recall first some points of [4]. In that paper, we proposed a functional approach to the Knizhnik-Zamolodchikov-Bernard (KZB) connection, relying on the functional picture for conformal blocks of [9]. Recall that conformal blocks are associated to a complex curve \(X\) with a marked point \(P_0\), a simple Lie algebra \(\bar{g}\) and representations \(V\) and \(V^{out}\) of \(g\) and \(g^{out}\) out, where \(g\) is the Kac-Moody algebra \((\bar{g} \otimes K) \oplus C K\), and \(g^{out}\) is the Lie subalgebra of \(g\) formed of the currents regular outside \(P_0\) (we denote by \(K\) is the local field of \(X\) at \(P_0\), and defined as the space of \(g^{out}\)-intertwiners \(\psi\) from \(V\) to \(V\). Twisted conformal blocks are defined in the same way, replacing \(g^{out}\) by the Lie subalgebra \(g^{out}_{\lambda_0}\) of \(g\) formed of the maps \(x\) from the universal cover of \(X\), regular outside the preimage of \(P_0\), with transformation properties \(x(\gamma_{A_z}) = x(z)\) and \(x(\gamma_{B_z}) = e^{\lambda_0(z)} x(z) e^{-\lambda_0(z)}\), where \(\gamma_{A_z}\) and \(\gamma_{B_z}\) are deck transformations corresponding to \(a\)- and \(b\)-cycles; \(\lambda_0 = (\lambda_a^{(0)})\) belongs to \(\bar{h}^\mathit{t}\), where \(h\) is the Cartan subalgebra of \(\bar{g}\) and \(g^{out}_{\lambda_0}\) is the Lie subalgebra of \(g\). We parametrize the space of twisted conformal blocks by associating to \(\psi_{\lambda_0}\) the twisted correlation functions of currents of \(g\) associated to the simple root generators of its nilpotent subalgebra \(\bar{n}_+\). Denote by \(e_i, f_i, h_i\) the Chevalley generators of \(\bar{g}\), so that the \(e_i\) generate \(\bar{n}_+\), and set \(x[f] = x \otimes f\), for \(x\) in \(\bar{g}\), \(f\) in \(K\). To a vector \(v\) of \(V\), annihilated by the \(h_i[z^k], f_i[z^{-1} \cdot g + k], k' > 0, k \geq 0\) (this property is shared by the extremal vectors in integrable modules), and to an intertwiner \(\psi_{\lambda_0}\), we associate the generating series

\[ f(\lambda_a^{(i)} | z_j^{(i)}) = \langle \psi_{\lambda} | \prod_{i} \prod_{j=1}^{n_j} e_i(z_j^{(i)}) v | \rangle, \]

where \(\psi_{\lambda} = \psi_{\lambda_0} \circ e^{\sum \lambda_a^{(0)} h [r_a]}\), the \(r_a\) are multivalued functions on \(X\), constant along \(a\)-cycles and with additive constants along \(b\)-cycles, and \(\xi\) is a lowest weight form on \(V\). The \(\lambda_a\) are formal parameters near \((\lambda_a^{(0)})\) and the \(z_i\) are formal parameters near \(P_0\).

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Theorem 0.1. (see [4]) Let $T_z^{\text{class}}$ be the differential-evaluation operator acting on functions $f(\lambda_1, \ldots, \lambda_g|z_1, \ldots, z_n)$ as

$$T_z^{\text{class}} = \frac{1}{2} \sum_{a} \omega_a(z) \partial_{\lambda_a} + 2 \sum_{i} G^{(I)}(z, z_i) - \sum_{j} \Lambda_j G^{(I)}(z, P_j)^2$$

$$+ \sum_{a} D_z^{(2\lambda)} \omega_a(z) \partial_{\lambda_a} + 2 \sum_{i} D_z^{(2\lambda)} G^{(I)}(z, z_i) - \sum_{j} \Lambda_j D_z^{(2\lambda)} G^{(I)}(z, P_j) + k \omega_{2\lambda}(z)$$

$$+ \sum_{i=1}^{n} \left( -2 G^{(I)}_{2\lambda}(z, z_i) \left[ \sum_{a} \omega_a(z_i) \partial_{\lambda_a} + 2 \sum_{j \neq i} G^{(I)}(z_i, z_j) - \sum_{k} \Lambda_k G^{(I)}(z_i, P_k) \right] - 4 G^{(I)}_{2\lambda}(z, z_i) G^{(I)}(z_i, z) + 2kd_z G^{(I)}_{2\lambda}(z, z_i) \right) \circ \text{ev}_z^{(i)},$$

where

$$(\text{ev}_z^{(i)} f)(\lambda|z_1, \ldots, z_n) = f(\lambda|z_1, \ldots, z \setminus z_i, z_n)$$

(z in $i$th position), we set $\lambda = (\lambda_1, \ldots, \lambda_g)$, $D_z^{(2\lambda)}$ is a connection on the bundle $K$ of differentials on $X$ and has simple pole at $P_0$, the $\omega_a$ are the holomorphic one-forms associated with the $a$-cycles, $\omega_{2\lambda}$ is a quadratic differential with double poles at $P_0$, $G^{(I)}$ and $G^{(I)}_{2\lambda}$ are (twisted) Green functions, $P_j$ are some point of $X - \{P_0\}$ and $\Lambda_i$ are some numbers.

If $V$ is the product $\otimes_i V_{-\Lambda_i}(P_i)$ of evaluation modules ($V_{-\Lambda_i}$ is the $\mathfrak{sl}_2$-module with lowest weight $-\Lambda_i$), we have the equality

$$\langle \psi_\lambda [T(z) \prod_i e_i(z_i^{(i)}) v], \xi \rangle = (T_z^{\text{class}} f)(\lambda|z_1, \ldots, z_n),$$

if $k$ is the level of $\mathcal{V}$. The operators $T_z^{\text{class}}$ commute when $k = -2$.

When $X$ is $\mathbb{CP}^1$, the expression of $T_z^{\text{class}}$ is similar to the expression for the action of the Hamiltonians on Bethe vectors obtained in the Bethe ansatz approach to the Gaudin system (see [10]).

In the present paper, we repeat these steps of [4] in the quantum case, at the critical level. We replace the Kac-Moody algebra $\mathfrak{g}$ by the quantum group $U_{h,\omega} \mathfrak{g}$ associated to a pair $(X, \omega)$ of a curve $X$ and a rational differential $\omega$ ([3]). The relations for this algebra depend on the choice of a Lagrangian subspace of $K$, that we construct in sect. [4]. We recall the presentation of $U_{h,\omega} \mathfrak{g}$ in terms of generating fields $e(z), f(z), k^\pm(z)$ (sect. [4]). The algebra $U_{h,\omega} \mathfrak{g}$ contains a subalgebra $U_h \mathfrak{g}^{\text{out}}$, which is a flat deformation of the enveloping algebra of $\mathfrak{g}^{\text{out}}$ ([4]).

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{m}$ be the maximal ideal at $P_0$ and $\mathfrak{b}_{in}$ be the subalgebra of $\mathfrak{g}$ defined as $\mathfrak{b}_{in} = (\mathfrak{h} \otimes \mathfrak{m}) \oplus (\mathfrak{n}_+ \otimes \mathcal{K})$. 

In [4], we expressed the KZB connection in terms of these correlation functions. Let $T(z)$ denote the Sugawara tensor; it is a series in $(U \mathfrak{g})_{\text{loc}}[[z, z^{-1}]]$, where $(U \mathfrak{g})_{\text{loc}}$ is the local completion of the universal enveloping algebra $U \mathfrak{g}$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, we have

$$T_z = \frac{1}{2} \sum_{a} \omega_a(z) \partial_{\lambda_a} + 2 \sum_{i} \Lambda_i G^{(I)}(z, z_i)$$

$$+ \sum_{a} D_z^{(2\lambda)} \omega_a(z) \partial_{\lambda_a} + 2 \sum_{i} D_z^{(2\lambda)} G^{(I)}(z, z_i) - \sum_{j} \Lambda_j D_z^{(2\lambda)} G^{(I)}(z, P_j) + k \omega_{2\lambda}(z)$$

$$+ \sum_{i=1}^{n} \left( -2 G^{(I)}_{2\lambda}(z, z_i) \left[ \sum_{a} \omega_a(z_i) \partial_{\lambda_a} + 2 \sum_{j \neq i} G^{(I)}(z_i, z_j) - \sum_{k} \Lambda_k G^{(I)}(z_i, P_k) \right] - 4 G^{(I)}_{2\lambda}(z, z_i) G^{(I)}(z_i, z) + 2kd_z G^{(I)}_{2\lambda}(z, z_i) \right) \circ \text{ev}_z^{(i)},$$

where

$$(\text{ev}_z^{(i)} f)(\lambda|z_1, \ldots, z_n) = f(\lambda|z_1, \ldots, z \setminus z_i, z_n)$$

(z in $i$th position), we set $\lambda = (\lambda_1, \ldots, \lambda_g)$, $D_z^{(2\lambda)}$ is a connection on the bundle $K$ of differentials on $X$ and has simple pole at $P_0$, the $\omega_a$ are the holomorphic one-forms associated with the $a$-cycles, $\omega_{2\lambda}$ is a quadratic differential with double poles at $P_0$, $G^{(I)}$ and $G^{(I)}_{2\lambda}$ are (twisted) Green functions, $P_j$ are some point of $X - \{P_0\}$ and $\Lambda_i$ are some numbers.
We construct, in $U_{h,\omega}\mathfrak{g}$, a subalgebra isomorphic to $(U_{\mathfrak{h}}[[[h]])$ (sect. 3). This subalgebra is expressed in terms of “new” generating fields $\tilde{e}(z)$, $\tilde{f}(z)$ and $k_{\text{tot}}^\pm(z)$; we study their relations in sect. 4. In the rational case, such generating fields appeared in [15]. We express a generating function for central elements $T(z)$ deforming the Sugawara tensor by the formula (see Thm. 5.1)

$$T(z) = e(z)\tilde{f}(z) + \lambda a_\lambda(z) k_{\text{tot}}^+(z) + \lambda b_\lambda(z) k_{\text{tot}}^-(z),$$

where $a(z)b(z) \lambda$ denotes a normal ordered product, depending on $\lambda$ and $a_\lambda(z)$ and $b_\lambda(z)$ are formal series of $K[[\lambda_a - \lambda_a^{(0)}]][[h]]$ defined by (18) and (19). We also obtain another expression for $T(z)$ of the type obtained in [12, 14], see (3).

We construct a subalgebra $U_{h,\omega}^{\text{out}}$ of $U_{h,\omega}\mathfrak{g}$ in sect. 3 deforming the enveloping algebra of $\mathfrak{g}_{\text{tot}}^{\text{out}}$ and study a class of its representations (sect. 7). We show that such representations have a lowest weight form $\xi$, such that

$$\xi \circ f[\tau_{-2\lambda}] = 0, \quad \xi \circ k^+(z) = \pi(z)\xi,$$

for $\tau_{-2\lambda}$ in $R_{\tau_{-2\lambda}}$ and $\pi(z)$ a formal series, which is an analogue of the Drinfeld polynomial.

To a module $V$ over $U_{h,\omega}\mathfrak{g}$, and to a morphism $\psi_{\lambda_0} : V \to V$ of $U_{h,\omega}^{\text{out}}$-modules, where $V$ is a product of evaluation modules, we associate the correlation function

$$f(\lambda_1, ..., \lambda_y, z_1, ..., z_n) = \langle \psi_{\lambda_0} \tilde{e}(u_1) \cdots \tilde{e}(u_n) \rangle,$$

where $\xi$ is a lowest weight form on $V$ and $\psi_{\lambda} = \psi_{\lambda_0} \circ e^{\sum(\lambda_a - \lambda_a^{(0)})\hbar[r_a]}$. We study the functional properties of $f(\lambda_1, ..., \lambda_y, z_1, ..., z_y)$ in sect. 8.

Our main result is then

**Theorem 0.2.** (see Thm. 10.1) Let for any formal series $\Pi$, $(T^\Pi_z)_z$ be the family of operators acting on functions $f(\lambda|u_1, ..., u_n)$, defined as

$$T^\Pi_z = \Pi(z) a'_{\lambda}(z|u_1, ..., u_n) \circ e^{\sum \omega^\Pi(z) \partial / \partial \lambda_a} + \Pi(q^{-\partial} z)^{-1} \Pi(u_1, ..., u_n) \circ e^{\sum \omega^\Pi(z) \partial / \partial \lambda_a} + \sum \Pi(u_i) c_{\lambda}^{\Pi(i)}(z|u_1, ..., u_n) \circ e^{\sum \omega^\Pi(z) \partial / \partial \lambda_a} \circ \text{ev}_z^{(i)}$$

where the multiplication operators are denoted as functions, and we set

$$a'_{\lambda}(z|u_1, ..., u_n) = a_{\lambda}(z) q \prod q_m(z, u_i), \quad a''_{\lambda}(z|u_1, ..., u_n) = b'_{\lambda}(z) \kappa(z) q \prod q_m(q^{-\partial} z, u_i)^{-1},$$

$$c_{\lambda}^{\Pi(i)}(z|u_1, ..., u_n) = -\frac{G_2}{h} G_{2\lambda}(z, u_i) q_m(u_i, z) \prod q_m(u_i, u_j),$$

$$c_{\lambda}^{\Pi(i)}(z|u_1, ..., u_n) = \frac{G_2}{h} G_{2\lambda}(z, q^{-\partial} u_i) \kappa(u_i) q_m(q^{-\partial} u_i, z)^{-1} \prod q_m(q^{-\partial} u_i, u_j)^{-1},$$
\[ \omega'_a = \hbar \frac{1}{1 + q^{-2}} (\omega_a / \omega)(z), \quad \omega''_a = -\hbar \frac{1}{1 + q^{-2}} (\omega_a / \omega)(z), \quad G_{2\lambda}(z, w) = G^{(f)}_{2\lambda}(z, w) / \omega(z), \]

where \( \partial \) is the derivation associated with \( \omega \), so that \( \partial f = df / \omega(z) \), \( a, b, c, q \) are defined in (48), (50), (59) and (60), and \( q = e^h \). The operators \( \hat{T}^{(\Pi)}_2 \) commute and normalize first order difference operators \( \hat{f}[^{\rho}] \) defined by (63). Moreover, we have, if the subalgebra \( U_h \mathfrak{b}^{\geq 1 - \gamma} \) of \( U_h \mathfrak{b}_m \) acts on \( \psi \) by the character \( \chi_n \) (see sect. [7]).

\[ \langle \psi_{\lambda}(T(z)\bar{c}(u_1) \cdots \bar{c}(u_n)\psi), \xi \rangle = T_z \{ \langle \psi_{\lambda}[\bar{c}(u_1) \cdots \bar{c}(u_n)\psi], \xi \rangle \}, \]

where \( \Pi \) can be expressed in terms of \( \pi \). We also set \( \Pi(q^\beta z) = (q^\beta \Pi)(z) \).

The \( T^{(\Pi)}_2 \) are difference deformations of the \( T^{class}_2 \). In the rational case, we identify the operators \( T^{(\Pi)}_2 \) with the commuting family of operators provided by the Yangian action on the hypergeometric spaces of [8] (see sect. [9]). In the elliptic case, we identify \( T^{(\Pi = 1)}_2 \) with the first \( q \)-Lamé operator (rem. [10]).

Let us say some words about possible prolongations of the present work:

1) noncritical level. One could try to prove analogues of the theta-behavior results of [6] for the twisted correlation functions of integrable modules over \( U_{h, \omega} \mathfrak{g} \).

Another problem is to find analogues of the KZB flows for noncritical level, by extending the approach of [4] for the twisted correlation functions of integrable modules over \( \mathfrak{g} \).

2) versions where \( \hbar \) takes complex values. When \( \omega \) is the pull-back of the form \( dz \) or \( dz/z \) from a morphism \( X \to \mathbb{C}P^1 \) or \( X \to E \), \( E \) some elliptic curve, \( q^\beta \) at least makes sense as some correspondence on \( X \). It could then be possible to find a presentation of \( U_{h, \omega} \mathfrak{g} \) allowing for complex values of \( \hbar \). This was done in [8] in the case \( X = \mathbb{C}P^1 \), \( \omega = z^N dz \).

3) Bethe ansatz for the operators \( T_z \). In [11], Bethe equations for the Gaudin system are shown to be equivalent to the existence of intertwining operators at critical level, and in turn to a trivial monodromy condition for some connection. The similar study should be possible for the systems constructed here, so that they could be viewed as \( q \)-deformations of the differential systems arising in [4].

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1. Geometric setting

1.1. Isotropic supplementaries. Let \( X \) be a smooth compact complex curve of genus \( g \), endowed with a nonzero holomorphic form \( \omega \). Let \( \sum_{i=1}^p n_i P_i \) be the divisor of \( \omega \) (we have \( n_i > 0 \), \( \sum_i n_i = 2(g - 1) \)). Let for each \( i \), \( \mathcal{K}_i \) be the local field at \( P_i \), \( \mathcal{O}_i \) the local ring at this point and \( \mathfrak{m}_i \) the maximal ideal of \( \mathcal{O}_i \). Define \( \mathcal{K} \) as \( \bigoplus_{i=1}^p \mathcal{K}_i \) and \( R \) as the space of rational functions on \( X \), regular outside \( \{ P_i \} \); we view it as a subring of \( \mathcal{K} \). For each \( i \), let \( z_i \) be a local coordinate at \( P_i \). Then \( \mathcal{O}_i = \mathbb{C}[[z_i]], \mathfrak{m}_i = z_i \mathcal{O}_i \) and \( \mathcal{K}_i = \mathbb{C}((z_i)) \).
\( \mathcal{K} \) is endowed with a scalar product \( \langle \cdot, \cdot \rangle_{\mathcal{K}} \) defined by
\[
\langle f, g \rangle_{\mathcal{K}} = \sum_{i=1}^{p} \text{res}_{P_i}(fg\omega).
\]

Let us fix on \( X \) a choice of \( a \)- and \( b \)-cycles \((A_a)_{1 \leq a \leq g}\) and \((B_a)_{1 \leq a \leq g}\). Let \( \tilde{X} \) be the universal cover of \( X \) and \( \pi: \tilde{X} \to X \) be the cover map. Denote by \( \gamma_{A_a} \) and \( \gamma_{B_a} \) the deck transformations associated with the cycles \( A_a \) and \( B_a \).

**Lemma 1.1.** There exists a linearly independent family of \( R \) formed by elements \( f_{(m_i)} \), where \( m_i, i = 1, \ldots, p \) are integers such that \( m_i \geq n_i \) for each \( i \) and \( \sum_i m_i \geq \sum_i n_i + 2 \), with \( \text{val}_{P_i}(f_{(m_i)}) = -m_i \), for each \( i = 1, \ldots, p \).

**Proof.** Let us first construct the \( f_{(m_i)} \). Assume \( m_j \geq n_j + 1 \), then by the Riemann-Roch theorem,
\[
h^0(O(\sum_i m_i P_i)) - h^0(O(\sum_i m_i P_i - P_j)) = 1 + h^1(O(\sum_i m_i P_i)) - h^1(O(\sum_i m_i P_i - P_j));
\]
by Serre duality this is equal to
\[
1 + h^0(O(\sum_i (n_i - m_i) P_i)) - h^0(O(\sum_i (n_i - m_i) P_i + P_j)).
\]
All the \( n_i - m_i + \delta_{ij} \) are \( \leq 0 \) and their sum is \( < 0 \), so not all of them are zero. Therefore both \( h^0 \) vanish. This proves the existence of the \( f_{(m_i)} \). \( \square \)

**Lemma 1.2.** We have \( g \) functions \( r_a \) defined on \( \tilde{X} \), regular outside \( \pi^{-1}(\{P_i\}) \), such that
i) \( r_a \circ \gamma_{B_b} = r_a - \delta_{ab} \),
ii) \( \text{val}_{P_i}(r_a) \leq -n_i - \delta_{i1} \) and
iii) \( \int_{A_a} r_a \omega = \frac{1}{2} \int_{A_a} \omega \delta_{ab} \).

**Proof.** The existence of rational functions \( \tilde{r}_a \) defined on \( \tilde{X} \), regular outside \( \pi^{-1}(\{P_i\}) \) and satisfying i) is a consequence of Cor. 1.1. Adding to them suitable combinations of the \( f_{(m_i)} \), one gets functions \( \tilde{r}_a \) satisfying both i) and ii). Let \( \omega_a \) be a basis of the space of holomorphic one-forms on \( X \). The ratios \( \omega_a/\omega \) are elements of \( R \), with valuation at each \( P_i \) less or equal to \(-n_i \). Adding to the \( \tilde{r}_a \) suitable combinations of the \( \omega_a/\omega \), one obtains elements \( r_a \) satisfying i), ii) and iii). \( \square \)

**Proposition 1.1.** Set \( \Lambda = (\oplus_a C r_a) \oplus (m_1 \oplus O_2 \oplus \cdots \oplus O_p) \). We have a direct sum decomposition
\[
\mathcal{K} = R \oplus \Lambda;
\]
moreover, \( R \) and \( \Lambda \) are both maximal isotropic subspaces of \( \mathcal{K} \).
Proof. The fact that $K = R \oplus \Lambda$ follows from [3], Prop. 1.1. That $\langle r_a, r_b \rangle = 0$ follows from [3], 4.1.1. Let us show that $\langle r_a, m_i \rangle$ vanishes: for $n > 0$, $\text{val}_{P_i}(r_a z_i^n \omega) > (-n_i - 1) + n_i = -1$ so $\text{res}_{P_i}(r_a z_i^n \omega)$ is zero; and for $i > 1$, $\langle r_a, O_i \rangle$ vanishes because for $n \geq 0$, $\text{val}_{P_i}(r_a z_i^n \omega) \geq -n_i + n_i = 0$ so $\text{res}_{P_i}(r_a z_i^n \omega)$ is zero.

Remark 1. In the case where $\omega$ has a unique zero of order 2($g - 1$) at some point $P_0$, $R$ is spanned by $f_0, f_{-a_1}, f_{-a_2}, ..., f_{-a_g}, f_{-g-1}, f_{-g-2}, ...$ with $\text{val}_{P_0}(f_i) = i$: if $\omega_1, ..., \omega_g$ be a basis of the space of holomorphic one-forms $H^0(X, \Omega_X)$, with $\text{val}_{P_0}(\omega_i) = b_i$, so that $0 \leq b_1 < b_2 < ... < b_g = 2(g - 1)$, then $f_0 = 1$, $f_{-a_1} = \omega_g^{-1}/\omega_g$, $f_{-a_2} = \omega_{g-1}/\omega_g$, etc. On the other hand, the $r_a$ may be chosen to have poles of order $b_1, ..., b_g$ at $P_0$, with $\{a_1, ..., a_g\} \cup \{b_1, ..., b_g\} = \{1, ..., 2g - 1\}$.

If $X$ is a hyperelliptic curve $y^2 = P_{2g+1}(x)$, $P_{2g+1}$ a polynomial of degree $2g + 1$, and $\omega = dx/y$; more generally, if $X$ is a plane curve of equation $P(z) = Q(y)$, and $\omega = dx/Q'(y) = -dy/P'(x)$, with $P$ and $Q$ generic polynomials of coprime degrees $p$ and $q$, (in that case, $g = \frac{(p-1)(q-1)}{2}$), $\omega$ has a zero of order $2(g - 1)$ at the point at infinity.

1.2. (Twisted) Green functions. We will denote by $z$ the $n$-uple $(z_i)$ of $K$. We will denote by $\mathbb{C}[[z, z^{-1}]]$ the set of series $\sum_{i=1}^{p} \sum_{n \in \mathbb{Z}} a_{in} z_i^n$, and by $\mathbb{C}[[z, w]]$ the space $\prod_{1 \leq i, j \leq p} \mathbb{C}[[z_i, w_j]]$.

We define $\delta(z, w)$ as the sum $\sum_i e^i(z) e_i(w)$, where $(e^i)$ and $(e_i)$ are dual bases of $K$ for $\langle \cdot, \cdot \rangle_K$.

The space of functions in two variables $z$ and $w$ will be identified with the tensor square of the space of functions in one variable, via the identification $a(z)b(w) \mapsto a \otimes b$.

1.2.1. Green function. Let $(e^i), (e_i)$ be dual bases of $R$ and $\Lambda$. We will assume that $(e_i)$ is the union of $(r_a)$ and a basis of $m = m_1 \oplus O_2 \oplus \cdots \oplus O_p$. We set

$$G = \sum_i e^i \otimes e_i.$$  \hfill (1)

We have then $\delta(z, w) = G(z, w) + G(w, z)$. $G(z, w)$ is the collection of expansions, for $w$ near each $P_i$, of a rational function defined on $X^2$, antisymmetric in $z$ and $w$, regular except for poles when $z$ of $w$ meets some $P_i$ and a simple pole at the diagonal.

1.2.2. Twisted Green functions. To $\lambda_0 = (\lambda_a^{(0)})_{1 \leq a \leq g}$ a vector of $\mathbb{C}^g$ is associated the line bundle $L_{2\lambda_0}$ over $X$. The space $H^0(X - \{P_0\}, L_{2\lambda_0})$ may be identified with the space of functions on $X$, regular outside $\pi^{-1}(P_0)$, with transformation properties

$$f(\gamma_{B_0} z) = f(z) \quad \text{and} \quad f(\gamma_{B_0} z) = e^{-2\lambda_0} f(z).$$  \hfill (2)

This space of functions will be denoted $R_{-2\lambda_0}$. For $\lambda_0$ generic, a complement in $K$ of this space is $z_i^{1-g} O_1 \oplus O_2 \oplus \cdots \oplus O_p$. 

Let $\lambda = (\lambda_a)_{1 \leq a \leq g}$ be $g$ formal parameters at the vicinity of $\lambda_0$, and define $R_{-2\lambda}$ as the $\mathbb{C}[\lambda_0 - \lambda_0^{(0)}]$-submodule of $\mathcal{K}[[\lambda_0 - \lambda_0^{(0)}]]$ generated by the $e^{2\sum_a (\lambda_a - \lambda_0^{(0)}) r_a \phi}$, $\phi \in R_{-2\lambda_0}$. Define also $\Lambda' = (z^{-q} \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \cdots \oplus \mathcal{O}_p)[[\lambda_0 - \lambda_0^{(0)}]]$.

Then we have a direct sum decomposition

$$\mathcal{K}[[\lambda_0 - \lambda_0^{(0)}]] = R_{-2\lambda} \oplus \Lambda'.$$

For $\phi$ in $\mathcal{K}[[\lambda_0 - \lambda_0^{(0)}]]$, we denote by $\phi_{\Lambda'}$ and $\phi_{R_{2\lambda}}$ the projections of $\phi$ on $\Lambda'$ parallel to $R_{2\lambda}$, resp. on $R_{2\lambda}$ parallel to $\Lambda'$. For $\phi(z)$ a series $\sum_i \phi_i(z)$, we define $\phi(z)_{z \to \Lambda'}$ as $\sum_i \phi_i(\epsilon_i)_{\Lambda'}(z)$ and $\phi(z)_{z \to R_{2\lambda}}$ as $\sum_i \phi_i(\epsilon_i)_{R_{2\lambda}}(z)$.

We have then $f(z) = f(z)_{z \to \Lambda'} + f(z)_{z \to R_{2\lambda}}$.

Let $(e_i^{\Lambda'}), (e_i^{R_{2\lambda}})$ be dual bases of $R_{2\lambda}$ and $\Lambda'$, and let us set

$$G_{2\lambda}(z, w) = \sum_i e_i^{\Lambda'}(z) e_i^{R_{2\lambda}}(w). \quad (3)$$

We have $\delta(z, w) = G_{2\lambda}(z, w) + G_{-2\lambda}(w, z)$. $G_{-2\lambda}(z, w) - G(z, w)$ belongs to $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$. The functions

$$g_+^\lambda(z) = (G_{-2\lambda} - G)(q^\theta z, z), \quad g_-^\lambda(z) = (G_{-2\lambda} - G)(q^{-\theta} z, z)$$

then belong to $\mathcal{K}[[h]]$.

**Remark 2. Relation with the Green functions of [3].** In [3], we introduced Green function $G(z, w)$ and a twisted Green function $G_{2\lambda}(z, w)$, that we denote here by $G^{(1)}(z, w)$ and $G_{2\lambda}^{(1)}(z, w)$. Let $\omega_a$ be the basis of one-forms on $X$, associated to $(A_a)_{1 \leq a \leq g}$. The relation of these Green functions with $G(z, w)$ and $G_{2\lambda}(z, w)$ defined by (3) (under the assumptions of Prop. [3,1]) and (3) is

$$G(z, w) = \left( G^{(1)}(z, w) + \sum_a \omega_a(z) r_a(w) \right) / \omega(z)$$

and

$$G_{2\lambda}(z, w) = G_{2\lambda}^{(1)}(z, w) / \omega(z).$$

Set

$$\bar{g}_\lambda(z) = \lim_{z \to w} (G_{\lambda}(z, w) - G(z, w)).$$

One can show that

$$\bar{g}_\lambda(z) = \sum_a \partial_{\epsilon_a} \ln \Theta(-\lambda + (g - 1)P_0 - \Delta) \omega_a(z) / \omega(z).$$

where $(\epsilon_a)_{1 \leq a \leq g}$ is the canonical basis of $\mathbb{C}^g$, $\Theta$ is the Riemann theta-function on the Jacobian of $X$ and points of $X$ are identified with their images by the Abel-Jacobi map ([3, 4.4]).
2. THE ALGEBRA $U_{h,\omega}g$

**Notation.** For $E$ a vector space and $E', E''$ two subspaces, such that $E$ is the direct sum $E' \oplus E''$, and for $\phi$ in $E$, we denote by $\phi_{E'|E''}$ the projection of $\phi$ on $E'$ parallel to $E''$. In the case of the decompositions $\mathcal{K} = R \oplus \Lambda$, $\mathcal{K}[[\lambda_a - \lambda_0]] = R_{2\lambda} \oplus \Lambda'$ and $\mathcal{K} = R(a) \oplus m$ below, we will simply denote $\phi_{E'|E''}$ and $\phi_{E''|E'}$ by $\phi_{E'}$ and $\phi_{E''}$ respectively.

For $(\epsilon_i)$ a basis of $\mathcal{K}$ and $f(z)$ a series $\sum_i \epsilon_i(z) \otimes v_i$ in some completion of $\mathcal{K} \otimes V$, $V$ some vector space, we define $f(z)_{z \to \Lambda}$ as $\sum_i (\epsilon_i)_\Lambda(z) \otimes v_i$, and $f(z)_{z \to R}$ as $\sum_i (\epsilon_i)_R(z) \otimes v_i$. One define in the same way $f(z)_{z \to R_{2\lambda}}$ and $f(z)_{z \to \Lambda'}$. If $f(z, w)$ is a series $\sum_{i,j} \epsilon_i(z) \epsilon_j(w) v_{ij}$, $f(z, w)_{z \to \Lambda}$ is $\sum_{i,j} (\epsilon_i)_\Lambda(z) \epsilon_j(w) v_{ij}$, etc.

If $f(z, w)$ belongs to $R_z((w))$ (the space of series $\sum_{i \geq 0} r_i(z) w^i$, with $r_i$ in $R$) and there exists $g(z, w)$ in $R_w((z))$, such that $f + g = (\pi \otimes \text{id}) \delta(z, w)$, where $\pi$ is some differential operator, then $f$ is the expansion of a rational function on $X - \{P_i\}^2$ with only poles at the diagonal, and $g$ may be viewed as the analytic prolongation of $-f$ in the region $z << w$. We write $g(z, w) = -f(z, w)_{z << w}$.

We write $\partial_{z}$ for $\partial \otimes \text{id}$, $\partial_{w}$ for $\text{id} \otimes \partial$. We set $\phi^{(21)}(z, w) = \phi(w, z)$.

### 2.1. Results on kernels. (see [6])

We have

$$\partial_{z} G(z, w) = -G(z, w)^2 - \gamma,$$

for some $\gamma \in R \otimes R$.

Let $\phi, \psi$ belong to $h\mathbb{C}[\gamma_0, \gamma_1, \ldots][[h]]$ such that

$$\partial_h \psi = D\psi - 1 - \gamma_0 \psi^2, \quad \partial_h \phi = D\phi - \gamma_0 \psi.$$

Here $D = \sum_{i \geq 0} \gamma_{i+1} \partial_{h_i}$. We have

$$\psi(h, \partial^i_{h\gamma}) = -h + o(h), \quad \phi(h, \partial^i_{h\gamma}) = \frac{1}{2} h^2 \gamma_0 + o(h^2).$$

Set $G^{(21)}(z, w) = G(w, z)$. From identity (3.11) of [6] (with $\partial$ transformed to $-\partial$) and by (3.8) of [6], we have

$$\sum_i \frac{1 - q^{-\partial}}{\partial} e_i(z) \otimes e^i(w) = -\phi(-h, \partial^i_{h\gamma}) + \ln(1 + G^{(21)}(\psi(-h, \partial^i_{h\gamma}))).$$

### 4

Set $T = \frac{\sinh h \partial}{h \partial}$. Let $\tau$ in $(R \otimes R)[[h]]$ satisfy

$$\tau + \tau^{(21)} = -\sum_i e^i \otimes (Te_i)_R.$$

Let $U$ be the linear map from $\Lambda$ to $R[[h]]$ such that $\tau = \sum_i U e_i \otimes e^i$. We have

$$\sum_i (T + U) e_i \otimes e^i + \sum_i e^i \otimes (T + U)e_i = (T \otimes \text{id}) \delta(z, w),$$

which means that after analytic prolongation the sum $\sum_i (T + U) e_i \otimes e^i$ is antisymmetric in $z$ and $w$. 

Set $T_+ = \frac{1 - q^{-\theta}}{2h\theta}$ and define $U_+ : \Lambda \to R[[h]]$ by the formula $U_+ = (1 + q^\theta)^{-1} \circ U$; we have

$$(T_+ + U_+)(\lambda) = \frac{1}{1 + q^\theta((T + U)(\lambda)).}$$

Define $q_+$ by

$$q_+(z, w) = q^2 \sum_i (T_+ + U_+) e_i(z) \otimes e^i(w),$$

it then follows from (4) that

$$q_+(z, w) = q^2 \sum_i (U_+ e_i)(z) \otimes e^i(w) e^{-\phi(-h, \partial_\gamma)} (1 + G^{(21)}(\psi(-h, \partial_\gamma))). \tag{6}$$

Remark 3. Formulas of this section correct a sign mistake in [3]: in sect. 3 of that paper, $\partial$ should be changed to $-\partial$.

2.2. The algebra $U_{h, \omega}$ has generators $h^+ [r], h^- [\lambda], e[\epsilon], f[\epsilon]$ and $K$, with $r$ in $R$, $\lambda$ in $\Lambda$ and $\epsilon$ in $K$; generating series

$$x(z) = \sum_i x[e^i] e_i(z), \quad h^+(z) = \sum_i h^+[e^i] e_i(z), \quad h^-(z) = \sum_i h^-[e_i] e^i(z),$$

$x = e, f$, and relations

$$x[\alpha \epsilon + \epsilon'] = \alpha x[\epsilon] + x[\epsilon'],$$

for $\alpha$ scalar, $x = h^+, \epsilon, \epsilon'$ in $R$; $x = h^-, \epsilon, \epsilon'$ in $\Lambda$; or $x = e, f, \epsilon, \epsilon'$ in $K$;

$$[h^+[r], h^+[r']] = 0, \tag{7}$$

$$[K, \text{anything}] = 0, \quad [h^+[r], h^-[\lambda]] = \frac{2}{h} \langle (1 - q^{-K}) r, \lambda \rangle, \tag{8}$$

$$[h^-[\lambda], h^-[\lambda']] = \frac{2}{h} \langle (T((q^{K\phi} \lambda)_R), q^{K\phi} \lambda') + \langle U \lambda, \lambda' \rangle - \langle U ((q^{K\phi} \lambda)_\Lambda), q^{K\phi} \lambda' \rangle \rangle \tag{9}$$

$$[h^+[r], e(w)] = 2r(w) e(w), \quad [h^-[\lambda], e(w)] = 2[(T + U)(q^{K\phi} \lambda)_\Lambda](w) e(w), \tag{10}$$

$$[h^+[r], f(w)] = -2r(w) f(w), \quad [h^-[\lambda], f(w)] = -2[(T + U) \lambda](w) f(w), \tag{11}$$

$$(\alpha(z) - \alpha(q^{-\theta} w)) e(z) e(w) = (\alpha(z) - \alpha(q^{-\theta} w)) q^2 \sum_i (T + U) e_i(z) \otimes e^i(w) e(w) e(z) \tag{12}$$

and

$$(\alpha(z) - \alpha(q^\theta w)) f(z) f(w) = (\alpha(z) - \alpha(q^\theta w)) q^{-2} \sum_i (T + U) e_i(z) \otimes e^i(w) f(w) f(z) \tag{13}$$

for any $\alpha$ in $K$,

$$[e(z), f(w)] = \frac{1}{h} [\delta(z, w) q^{(T + U) h^+}(z) - (q^{-K\phi} \delta(z, w) q^{-h^-}(w)]. \tag{14}$$
\[ \phi(z, w) = \phi(h, \partial_v \gamma), r, r' in R, \lambda, \lambda' in \Lambda (see \[3\]). \]

\[ U_{h,\omega}g \] is completed with respect to the topology defined by the left ideals generated by the \( x[\epsilon] \), \( \epsilon in \oplus_{i} z_{i} N \mathcal{O} \). The critical case corresponds to \( K = -2 \).

**Remark 4.** Relations (12) and (13) can be written

2.3. **Cartan currents.** In case we have a relation

\[ a(z)b(w)a(z)^{-1} = \mu(z, w)b(w), \]

with \( a(z), b(w) \) currents of \( U_{h,\omega}g \) and \( \mu(z, w) in \mathbb{C}(\{z\})(\{w\})[[\hbar]] \) or \( \mathbb{C}(\{w\})(\{z\})[[\hbar]] \), we will define \( (a(z), b(w)) \) as \( \mu(z, w) \).

Set \( K^{+}(z) = q^{(T+U)h^{+}(z)}, K^{-}(z) = q^{-h^{-}(z)} \). Let us also set

\[ q(z, w) = q^{2 \sum_{i}(T+U)e_{i}(z) \otimes e^{i}(w)}, \]

we have \( q(z, w) = (q(w, z)^{-1})_{w < z} \). Then the relations involving Cartan generators can be expressed as

\[ (K^{+}(z), K^{+}(w)) = 1, \quad (K^{+}(z), K^{-}(w)) = \frac{q(z, q^{-K\theta}(w))}{q(z, w)}, \quad (15) \]

\[ (K^{-}(z), K^{-}(w)) = \frac{q(q^{-K\theta}(z), q^{-K\theta}(w))}{q(z, w)} \]

\[ (K^{+}(z), e(w)) = q(z, w), \quad (K^{-}(z), e(w)) = q(w, q^{-K\theta}(z))^{-1}, \quad (17) \]

\[ (K^{+}(z), f(w)) = q(z, w)^{-1}, \quad (K^{-}(z), f(w)) = q(w, z). \quad (18) \]

Set

\[ k^{+}(z) = q^{(T+U)h^{+}(z)}, \quad k^{-}(z) = \lambda(z)q^{\frac{1}{1+q^{\theta}}} h^{-}(z), \]

with \( \lambda(z) \) the function such that

\[ \lambda(z)\lambda(q^{-\theta} z)q^{\left[ \frac{1}{1+q^{\theta}} h^{-}(z), \frac{1}{1+q^{\theta}} h^{-}(z') \right]} = 1, \]

that is

\[ \lambda(z) = \exp \left[ -\frac{1}{1 + q^{-\theta}} \left( \frac{1}{1 + q^{-\theta}} \otimes q^{-\theta} \right) [h^{-}(z), h^{-}(z')] \right]_{z'=z}. \]

We have

\[ K^{+}(z) = k^{+}(z)k^{+}(q^{\theta} z), \quad K^{-}(z) = k^{-}(z)^{-1}k^{-}(q^{-\theta} z)^{-1}. \]

Set

\[ q^{+}(z, w) = q^{2 \sum_{i}(T+U)e_{i}(z) \otimes e^{i}(w)}, q^{-}(z, w) = q^{-2 \sum_{i} \frac{1}{1+q^{\theta}} e^{i}(z) \otimes (T+U)e_{i}(w)}, \]

then we have \( q^{+}(z, w) = q^{-}(z, w) \) (up to analytic continuation). We have

\[ q^{+}(z, w)q^{+}(q^{\theta} z, w) = q(z, w), \]
and
\[ (k^+(z), e(w)) = q_+(z, w), \quad (k^-(z)^{-1}, e(w)) = q_-(q^{-(K+1)\partial} z, w), \]
\[ (k^+(z), f(w)) = q_+(z, w)^{-1}, \quad (k^-(z)^{-1}, f(w)) = q_-(q^\partial z, w)^{-1}. \]
Also when \( K = -2 \), we have
\[ (k^+(z), k^+(w)) = 1, \quad (k^+(z), k^-(w)) = \frac{q_+(z, q^\partial w)}{q_+(z, q^{2\partial} w)}, \]
and
\[ (k^-(z), k^-(w)) = \frac{q_+(q^{2\partial} z, q^\partial w) q_-(q^\partial w, q^2 z)}{q_+(q^\partial z, q^2 w) q_-(q^2 z, q^\partial w)}. \]

3. **Subalgebra \( \mathbb{U}_h \mathfrak{b}_m \)**

The quantity
\[ (q^{\partial_z + \partial_w} - 1) \sum_i ((T + U)e_i)(z)e^i(w) \]
belongs to \( (R \otimes R)[[h]] \), since \( Ue_i \) belongs to \( R \), \( T \) commutes with \( \partial_z + \partial_w \) and \((\partial_z + \partial_w)G(w, z) = - \sum_i e^i(z)(\partial e_i)_R(w)\) belongs to \( R \otimes R \). Moreover
\[ F(z, w) = 2h \frac{q^{\partial_z + \partial_w} - 1}{(1 + q^{-\partial_z})(1 + q^{-\partial_w})} \sum_i (T + U)e_i(z)e^i(w) \]
is symmetric in \( z \) and \( w \). Let \( \alpha(z, w) \) be an element of \( \mathbb{h}(R \otimes R)[[h]] \) such that
\[ \frac{\exp(2\alpha(q^{\partial w} z, z))}{\exp(2\alpha(q^{\partial z} w, w))} = \exp[2h \frac{q^{\partial_z + \partial_w} - 1}{(1 + q^{-\partial_z})(1 + q^{-\partial_w})} \sum_i (T + U)e_i(z)e^i(w)]; \]
we may choose
\[ \alpha(z, w) = \frac{1}{2} F(w, q^{-\partial} z). \]

Let us set
\[ \alpha(z, w) = \sum_{i,j} a_{ij} e^i(z)e^j(w), \]
and
\[ k_R(z) = \exp(\sum_{i,j} a_{ij} h[e^i(z)]e^j). \]

Define \( R_{(a)} \) as \( \oplus_a \mathfrak{c} r_a \oplus R \). Recall that we defined \( \mathfrak{m} \) as \( \mathfrak{m}_1 \oplus \mathfrak{O}_2 \oplus \cdots \oplus \mathfrak{O}_p \), so that \( \mathcal{K} = R_{(a)} \oplus \mathfrak{m} \). Let \( \mathcal{A} \) be the \( \mathbb{C}[[h]] \)-module automorphism of \( R_{(a)}[[h]] \) defined by
\[ \mathcal{A}(r) = r \text{ for } r \text{ in } R \text{, and } \mathcal{A}(r_a) = (T + U)r_a \text{ for } a = 1, \ldots, g. \]
Define $\beta(z, w)$ in $(R \otimes R_{(a)})[[h]]$ by

$$-2 (1 \otimes q^0 A) \beta(z, w)$$

$$= 2(\alpha(q^{2\beta} z, w) - \alpha(q^0 z, w)) - 2\hbar \sum_i \frac{1}{1 + q^\beta} e^i(z) \otimes (\beta(z + U) e_i)_{R_{(a)}}(w)$$

$$+ 2\hbar (q^{3\beta} \otimes q^0 - q^0 \otimes q^{-\beta}) (\frac{1}{1 + q^\beta} \otimes \frac{1}{1 + q^{-\beta}}) \sum_i (\beta(z + U) e_i)(z) \otimes e^i(w).$$

Set

$$\beta(z, w) = \sum_{a,i} b_{ai} e^i(z) r_a(w) + \sum_{i,j} c_{ij} e^i(z) e^j(w)$$

and

$$k_a(z) = \exp(\sum_{a,i} b_{ai} h[r_a] e^i(z) + \sum_{i,j} c_{ij} h[e^i] e^j(z)). \quad (23)$$

Set finally

$$k_m(z) = k_a(z)^{-1} k^-(z).$$

The currents $k_R(z), k_a(z)$ and $k_m(z)$ all belong to $U_h h \otimes R[[h]].$

**Proposition 3.1.** i) Set $\tilde{f}(z) = f(z) k_R(z) k^-(q^{-\beta} z)$. we have $\tilde{f}(z) \tilde{f}(w) = \tilde{f}(w) \tilde{f}(z)$.

ii) We have

$$(k_m(z), \tilde{f}(w)) \in \exp(h(R \otimes m)[[h]]).$$

iii) We have $k_m(z) k_m(w) = k_m(w) k_m(z)$.

**Proof.** Let us show that $\tilde{f}(z)$ commutes with itself. We have

$$[h[r], f(z) k^-(q^{-\beta} z)] = -2(q^0 r)(z) f(z) k^-(q^{-\beta} z),$$

therefore

$$\frac{(k_R(w), f(z) k^-(q^{-\beta} z))}{(k_R(z), f(w) k^-(q^{-\beta} w))} = \frac{\exp(2\alpha(q^0 z, w))}{\exp(2\alpha(q^0 w, z))}.$$
We have then
\[(w - z)f(z)k^-(q^{-\partial}z)f(w)k^-(q^{-\partial}w)\]
\[= i_-(z, w)(w - q^{-\partial}z)f(z)f(w)k^-(q^{-\partial}z)k^-(q^{-\partial}w)\]
\[= (k^-(q^{-\partial}z), k^-(q^{-\partial}w))i_-(q^\partial z, w)^{-1}(w - q^\partial z)f(z)f(z)k^-(q^{-\partial}w)k^-(q^{-\partial}z)\]
\[= (k^-(q^{-\partial}z), k^-(q^{-\partial}w))i_-(q^\partial z, w)^{-1}(w - z)\frac{w - q^\partial z}{q^{-\partial}w - z}f(z)k^-(q^{-\partial}w)f(z)k^-(q^{-\partial}z),\]
therefore
\[(w - z) \left[ (f(z)k^-(q^{-\partial}z)) (f(w)k^-(q^{-\partial}w)) \right.\]
\[= \frac{(k^-(q^{-\partial}z), k^-(q^{-\partial}w))}{j(z, w)} \left( f(w)k^-(q^{-\partial}w) \right) f(z)k^-(q^{-\partial}z) \right] = 0;\]

let \(B(z, w)\) be the term in brackets. It is equal to \(A(z)\delta(z, w)\), for some generating series \(A(z)\). Since \(B(z, w)\) also satisfies \(B(w, z) = -\frac{(k^-(q^{-\partial}w), k^-(q^{-\partial}z))}{j(w, z)} B(z, w)\), and \(\frac{(k^-(q^{-\partial}w), k^-(q^{-\partial}z))}{j(w, z)} = 1 + o(h)\), we obtain that \(B(z, w)\) vanishes so
\[\frac{(f(z)k^-(q^{-\partial}z)) (f(w)k^-(q^{-\partial}w))}{j(z, w)} \left( f(w)k^-(q^{-\partial}w) \right) f(z)k^-(q^{-\partial}z) \right) = 0;\]

We have
\[(k^-(z), k^-(w)) = \exp[2h(q^{2(\partial_z + \partial_w)} - 1)] \frac{1}{1 + q^{-\partial_z}} \frac{1}{1 + q^{-\partial_w}} \sum_i (T + U) e_i(z) e^i(w)],\]
\[j(z, w) = \exp[2h(1 - q^{-\partial_z - \partial_w})] \frac{1}{1 + q^{-\partial_z}} \frac{1}{1 + q^{-\partial_w}} \sum_i (T + U) e_i(z) e^i(w)],\]
so
\[\frac{(k^-(q^{-\partial}z), k^-(q^{-\partial}w))}{j(z, w)} = \exp[2h] \frac{q^{\partial_z + \partial_w} - 1}{(1 + q^{-\partial_z})(1 + q^{-\partial_w})} \sum_i (T + U) e_i(z) e^i(w)];\]
since \(k_R(z)\) commutes with \(k_R(w)\), and by (25), \(i)\) follows.

Let us prove \(ii)\). We have
\[(k^-(z), \tilde{f}(w)) = \exp[2(\alpha(q^{2\partial} z, w) - \alpha(q^\partial z, w))] q^-(q^\partial z, w)(k^-(z), k^-(q^{-\partial}w));\]
moreover,
\[q^-(q^\partial z, w) = \exp[-2h] \sum_i \frac{1}{1 + q^{-\theta}} e^i(z) \otimes (T + U) e_i(w)],\]
From (23) follows that
\[ (k_a(z), \tilde{f}(w)) = \exp\left[2(\alpha(q^\alpha z, w) - \alpha(q^\alpha z, w))\right] \cdot \exp[-2\hbar \sum_i \frac{1}{1+q^{-\theta_i}} e^i(z) \otimes ((T + U)e_i)_{R(a)}(w)](k^{-}(z), k^{-}(q^{-\theta} w)). \]

Therefore,
\[ (k_m(z), \tilde{f}(w)) = \exp[-2\hbar \sum_i \frac{1}{1+q^{-\theta_i}} e^i(z) \otimes ((T + U)e_i)m(w)], \quad (27) \]

which implies \(ii\).

Set for \(\phi = \sum \lambda_a r_a + r\), with \(\lambda_a\) in \(\mathbb{C}\) and \(r\) in \(R\), \(h[\phi] = \sum \lambda_a h^{-}[r_a] + h^{+}[r]\).

Then we have for \(\phi\) in \(R(a)\),
\[ [h[\phi], f(z)] = -2(\mathcal{A}\phi)(z)f(z) \]
and
\[ [h[\phi], k^{-}(z)] = 2[q^\theta (1 - q^\theta)\mathcal{A}\phi](z)k^{-}(z), \]
where \(\mathcal{A}\) is defined by (22), so that
\[ [h[\phi], \tilde{f}(z)] = -2[q^\theta \mathcal{A}\phi](z)\tilde{f}(z). \]

Therefore we get
\[ (k_a(z), k^{-}(w)) = \frac{(k_a(z), \tilde{f}(q^\theta w))}{(k_a(z), \tilde{f}(w))} = \exp[2(q^{2\theta_2 + \theta_w} - q^{\theta_2 + \theta_w} - q^{2\theta_2} + q^{\theta_2})\alpha(z, w)] \cdot \exp[-2\hbar \sum_i \frac{1}{1+q^{-\theta_i}} e^i(z) \otimes (q^\theta - 1)((T + U)e_i)_{R(a)}(w)] \cdot \frac{(k^{-}(z), k^{-}(w))}{(k^{-}(z), k^{-}(q^{-\theta} w))}. \]

On the other hand, \(iii\) is translated as
\[ (k_a(z)^{-1}, k^{-}(w))(k^{-}(z), k^{-}(w)) = 1, \]
that is
\[ \exp[(q^{\theta_2} - 1)(q^{\theta_w} - 1) (2\alpha(q^\theta z, w) - 2\alpha(q^\theta w, z))] \]
\[ \exp[-2\hbar \sum_i \frac{1}{1+q^{-\theta_i}} e^i(z) \otimes (q^\theta - 1)((T + U)e_i)_{R(a)}(w)]: (z \leftrightarrow w) \]
\[ \frac{(k^{-}(z), k^{-}(w))^2}{(k^{-}(z), k^{-}(q^{-\theta} w))(k^{-}(q^{-\theta} z), k^{-}(w))} = (k^{-}(z), k^{-}(w)), \]
in other terms

$$\exp[(q^\partial z - 1)(q^\partial w - 1) \log \left( \frac{(k^-(q^\partial z), k^-(q^\partial w))}{j(z, w)} \right)^{-1}]$$

$$\exp[-2\hbar \sum_i \frac{1}{1 + q^{-\partial}} e^i(z) \otimes (q^\partial - 1)((T + U)e_i)_{R(a)}(w)] : (z \leftrightarrow w)$$

$$\frac{(k^-(z), k^-(w))}{(k^-(z), k^-(q^\partial w))(k^-(q^\partial z), k^-(w))} = 1,$$

or

$$\exp[-2\hbar \sum_i \frac{1}{1 + q^{-\partial}} e^i(z) \otimes (q^\partial - 1)((T + U)e_i)_{R(a)}] : (z \leftrightarrow w) \tag{28}$$

$$\exp[(q^\partial z - 1)(q^\partial w - 1) \log j(z, w)]$$

$$= (k^-(q^\partial z), k^-(q^\partial w)).$$

The terms containing $U$ in the logarithm of (28) are

$$- 2\hbar \left( \frac{1}{1 + q^{-\partial}} \otimes (q^\partial - 1) \right) \sum_i e^i \otimes U e_i$$

$$+ 2\hbar ((q^\partial - 1) \otimes \frac{1}{1 + q^{-\partial}}) \sum_i U e_i \otimes e^i + 2\hbar (\frac{q^\partial - 1}{1 + q^{-\partial}} \otimes (q^\partial - 1)) \sum_i U e_i \otimes e^i$$

$$- 2\hbar ((q^\partial - 1) \otimes \frac{q^\partial - 1}{q^\partial + 1}) \sum_i U e_i \otimes e^i$$

$$- 2\hbar (q^\partial \otimes q^\partial - q^{-\partial} \otimes q^{-\partial})(\frac{1}{1 + q^{-\partial}} \otimes \frac{1}{1 + q^{-\partial}}) \sum_i U e_i \otimes e^i$$

which is equal to

$$2\hbar \left( \frac{1}{1 + q^{-\partial}} \otimes (q^\partial - 1) \right) \sum_i e^i \otimes (T e_i)_{R},$$

in view of (3).
Therefore \( iii) \) is written as

\[
\exp[-2\hbar\left(\frac{1}{1+q^{-\partial}} \otimes (q^\partial - 1)\right) \sum_i e^i \otimes (Te_i)_{R(a)}] : (z \leftrightarrow w)
\]

\[
\exp[2\hbar\left(\frac{1}{1+q^{-\partial}} \otimes (q^\partial - 1)\right) \sum_i e^i \otimes (Te_i)_{R(a)}] \sum_i e^i \otimes (Te_i)_{R(a)}] \exp[-2\hbar((q^\partial - 1) \otimes \frac{1}{1+q^\partial}) \sum_i Te_i \otimes e^i]
\]

\[
= \exp[2\hbar(q^\partial - 1 \otimes q^{-\partial})\left(\frac{1}{1+q^{-\partial}} \otimes \frac{1}{1+q^\partial}\right) \sum_i Te_i \otimes e^i]
\]

or

\[
\exp[2\hbar\left(\frac{1}{1+q^{-\partial}} \otimes (1-q^\partial)\right)\left(\sum_i Te_i \otimes e^i + e^i \otimes (Te_i)_{R(a)} - e^i \otimes (Te_i)_{R}\right)]
\]

\[
\exp[2\hbar((1-q^\partial) \otimes \frac{1}{1+q^{-\partial}})\left(\sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i\right)] = 1. \quad (29)
\]

Since

\[
\sum_i Te_i \otimes e^i + e^i \otimes (Te_i)_{R(a)} - e^i \otimes (Te_i)_{R}
\]

is

\[
(T \otimes id)\delta(z, w) - \left(\sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i\right) \quad (21), \quad (30)
\]

(29) is equivalent to the statement that

\[
((1-q^\partial) \otimes \frac{1}{1+q^{-\partial}})\left(\sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i\right) - (z \leftrightarrow w) \quad (31)
\]

\[
+ (q^\partial - 1)(1-q^{-\partial}) \otimes id)\delta(z, w) = 0
\]

(whose interpretation is that after analytic prolongation,

\[
((1-q^\partial) \otimes \frac{1}{1+q^{-\partial}})\left(\sum_i Te_i \otimes e^i - (Te_i)_{R(a)} \otimes e^i\right)
\]

is symmetric in \( z \) and \( w \).

To prove this, we first show that

**Lemma 3.1.** One can choose the dual bases \((e_i)_{i \geq 0}, (e^i)_{i \geq 0}\) as \((r_a; e'_i)_{a=1,\ldots,g,i \geq 0}\)
and \((\omega_a; \omega; e^i)_{a=1,\ldots,g,i \geq 0}\) with \((e'_i)_{i \geq 0}\) be a basis of \( m \) and \((e^i)_{i \geq 0}\) the dual basis of
the subspace $K_a$ of $R$ defined as \( \{ r \in R | \int_{A_i} r \omega = 0 \} \). We have
\[
\sum_{i \geq 0} (Te_i)_m \otimes e^i = \sum_{i \geq 0} e'_i \otimes Te^i. \tag{32}
\]

**Proof of Lemma.** The first statement follows from the fact that $K_a$ is the annihilator of $\oplus_a Cr_a$ in $R$ for \( \langle , \rangle_K \). Let us show the second statement. Both sides of the equality belong to $m \otimes K$. On the other hand, the annihilator of $K_a$ for \( \langle , \rangle_K \) is $R(a)$ and has therefore zero intersection with $m$. It follows that to show (32), it is enough to show that the pairing of both sides with $\rho \otimes \text{id}$ coincide, for $\rho$ in $K_a$. But \( \langle \text{left side}, \rho \otimes \text{id} \rangle = \sum_i (Te_i)_m, \rho \rangle e^i = \sum_i (Te_i, \rho) e^i \), because $K_a$ and $R(a)$ are orthogonal; this is equal to \( \sum_i \langle e^i, T\rho \rangle e^i \) because $T$ is self-adjoint and therefore to $T\rho$. On the other hand, \( \langle \text{right side}, \rho \otimes \text{id} \rangle = \sum_i \langle e'_i, \rho \rangle Te^i = T\rho \).

This proves (32).

(31) is equal to
\[
((1 - q^\partial) \otimes \frac{1}{1 + q^{-\partial}}) \sum_i (Te_i)_m \otimes e^i;
\]
by Lemma 3.1, this is
\[
((1 - q^\partial) \otimes \frac{1}{1 + q^{-\partial}}) \sum_i e'_i \otimes Te^i;
\]
which is
\[
(1 - q^\partial) \otimes \frac{q^\partial - 1}{2\partial} \sum_i e'_i \otimes e^i
\]
or
\[
- \frac{1}{2\partial} \left( \frac{q^\partial - 1}{\partial} \otimes \frac{q^\partial - 1}{\partial} \right) \sum_i \partial e'_i \otimes e^i.
\]

(31) now follows from
\[
\sum_i \partial e'_i \otimes e^i - \sum_i e''_i \otimes \partial e'_i = (\partial \otimes \text{id})\delta(z, w)
\]
(which means that after analytic continuation, $\sum_i \partial e'_i \otimes e^i$ is symmetric). This equality can be proved either by expressing $\sum_i e'_i \otimes e^i$ explicitly using theta-functions (see [4]), or as follows: $\sum_i \partial e'_i \otimes e^i - e''_i \otimes \partial e'_i - (\partial \otimes \text{id})\delta(z, w)$ is equal to
\[
\sum_i \partial e'_i \otimes e^i + e'_i \otimes \partial e^i + \sum_a \omega_a / \omega \otimes \partial r_a, \tag{33}
\]
which belongs to $K \otimes R$. To show that (33) is zero, let us pair in with $\text{id} \otimes \lambda$, $\lambda$ in $m \oplus \oplus_a Cr_a$. For $o$ in $m$, (33), $\text{id} \otimes o$ is equal to
\[
(\partial o)_{R(a)}|_m - \sum_a \omega_a / \omega \langle o, \partial r_a \rangle;
but \( \partial o \) belongs to \( m \oplus (\oplus a Cr_a) \oplus (\oplus a C \omega_a/\omega) \), therefore this vanishes. On the other hand, \( \langle [33], id \otimes r_a \rangle \) is equal to zero, because \( \langle \partial r_a, r_b \rangle = 0 \) for any \( a, b \). \( \square \)

From the proof of Prop. 3.1 follows that \( (k_m(z), \tilde{f}(w)) \) is of the form \( \exp(-2\hbar \sum_{i \geq 0} \varphi^i \otimes e'_i) \), so that we have

\[
(k_m(z), \tilde{f}(w)) = \exp(-2\hbar \sum_{i \geq 0} \varphi^i \otimes e'_i),
\]

with \( (\varphi^i)_{i \geq 0} \) a free family of \( R[[\hbar]] \). Set \( h_m(z) = \frac{1}{\hbar} \ln k_m(z) \); (34) implies that

\[
h_m(z) = \sum_{i \geq 0} \tilde{h}[e'_i] \varphi^i(z),
\]

with \( \tilde{h}[e'_i] \) linear combinations of the \( h^+[r] \) and \( h^-[\lambda] \). Define \( \tilde{h}[o] \) for \( o \) in \( m \) by linear extension.

**Corollary 3.1.** Define in \( U_{h,\omega} g \), \( \tilde{f}[\epsilon] = \sum_i \text{res}_P(\tilde{f}(z)\epsilon(z)\omega(z)) \). Let \( b_{in} \) be the Lie algebra

\[
b_{in} = (\hat{h} \otimes m) \oplus (\hat{n}_+ \otimes \mathcal{K}).
\]

Then there is an algebra injection \( U b_{in}[[\hbar]] \to U_{h,\omega} g \) defined by \( h \otimes o \to \tilde{h}[o], f \otimes \epsilon \mapsto \tilde{f}[\epsilon] \). We define \( U_{h} b_{in} \) as the image of this injection.

**Proof.** From the construction of \( \tilde{h}[o] \) follows that we have

\[
[\tilde{h}[o], \tilde{f}(z)] = -2o(z) \tilde{f}(z);
\]

Prop. 3.1, i) and iii) then imply the statement. \( \square \)

Define \( U_{h} b_- \) as the subalgebra of \( U_{h,\omega} g \) generated by the \( h^+[r], h^-[\lambda] \) and the \( f[\epsilon], r \) in \( R, \lambda \) in \( \Lambda, \epsilon \) in \( \mathcal{K} \); \( U_{h} b_{in} \) is then a subalgebra of \( U_{h} b_- \). Define \( U_{h} b_{out}^{out} \) as the subalgebra of \( U_{h} b_- \) generated by the \( h^+[r] \) and the \( \tilde{f}[r_{-2\lambda_0}], r \) in \( R, r_{-2\lambda_0} \) in \( R_{-2\lambda_0} \).

We have:

**Proposition 3.2.** \( U_{h} b_- \) is the direct sum of \( \mathbb{C}[h[r_a]] U_{h} b_{in} \) and of its right ideal generated by its right ideal \( \sum_{r \in R} h^+[r] U_{h} b_{in}^{out} + \sum_{r_{-2\lambda_0} \in R_{-2\lambda_0}} \tilde{f}[r_{-2\lambda_0}] U_{h} b_{out}^{out} \).

**Proof.** For \( \rho \) in \( R_{(a)} \), set \( \tilde{h}[\rho] = h[(q^a \mathcal{A})^{-1} \rho] \). Extend \( \tilde{h} \) to \( \mathcal{K} \) by linearity. A system of relations for \( U_{h} b_- \) is then

\[
[\tilde{f}[\epsilon], \tilde{f}[\epsilon']] = 0, \quad [\tilde{h}[\epsilon], \tilde{f}[\eta]] = -2\tilde{f}[\epsilon \eta],
\]

\[
[\tilde{h}[\epsilon], \tilde{h}[\epsilon']] = f(\epsilon, \epsilon'),
\]

with \( f(\epsilon, \epsilon') \) scalar, for \( \epsilon, \epsilon', \eta \) in \( \mathcal{K} \). Denote by \( \mathbb{C}(\phi, \phi \in F) \) the subalgebra of \( U_{h} b_- \) generated by the family \( F \) of elements of \( U_{h} b_- \). The product map from \( \mathbb{C}(\tilde{h}[o], \tilde{f}[\lambda], o \in m, \lambda' \in \Lambda') \otimes \mathbb{C}[h[r_a]] \otimes \mathbb{C}(\tilde{h}[r], \tilde{f}[r_{-2\lambda_0}], r \in R, r_{-2\lambda_0} \in R_{-2\lambda_0}) \)
to $U_{h\mathfrak{b}}$ then defines an isomorphism. Therefore $U_{h\mathfrak{b}}$ is the direct sum of $\mathbb{C}[h][r_a]U_{h\mathfrak{b}}^{\text{out}}$ and the left ideal $I$ generated by the $\tilde{h}[r], \tilde{f}[r_{-2\lambda_0}], r$ in $R$, $r_{-2\lambda_0}$ in $R_{-2\lambda_0}$. $\mathbb{C}[h][r_a]U_{h\mathfrak{b}}^{\text{out}}$ is equal to $\mathbb{C}[h][r_a]U_{h\mathfrak{b}}^{\text{out}}$, on the other hand, $\tilde{f}[r_{-2\lambda_0}] = \sum_i \text{res}_P(f(z)k^-(q^{-\theta}z)r_{-2\lambda_0}(z)\omega_z)$, and $k^-(q^{-\theta}z)$ belongs to $U_{h\omega}\mathfrak{g} \otimes R_z$, so that since $R_{-2\lambda_0}$ is a $R$-module, $\tilde{f}[r_{-2\lambda_0}]$ belongs to $f[r_{-2\lambda_0}] + h$(right ideal generated by the $f[\rho_{-2\lambda_0}], \rho_{-2\lambda_0}$ in $R_{-2\lambda_0}$). Therefore, $I$ coincides with the left ideal generated by the $h[r], f[r_{-2\lambda_0}], r$ in $R$, $r_{-2\lambda_0}$ in $R_{-2\lambda_0}$, which is the augmentation ideal of $U_{h\mathfrak{b}}^{\text{out}}$. The Lemma follows.

\[\text{Proposition 4.1. The following relations}

\[\tilde{e}(z) = k^+(q^\theta z)^{-1}k_R(z)^{-1}e(q^\theta z),\]

an

\[k^{+\text{tot}}(z) = k^+(q^{2\theta}z)k_R(q^\theta z)k_R(z)^{-1}k^-(z),\] \hspace{1cm} (35)

\[k^{-\text{tot}}(z) = k^+(q^\theta z)^{-1}k_R(z)^{-1}k_R(q^{-\theta}z)k^-(q^{-\theta}z)^{-1}.\] \hspace{1cm} (36)

are satisfied in $U_{h\omega}\mathfrak{g}$.

\[\text{Proof. The proof of } [\tilde{e}(z), \tilde{e}(w)] = [\tilde{f}(z), \tilde{f}(w)] = 0, \hspace{1cm} (37)\]

\[\tilde{e}(z), \tilde{f}(w) = \frac{1}{h} \delta(q^\theta z, w)k^{+\text{tot}}(z) - \frac{1}{h} \delta(z, q^\theta w)k^{-\text{tot}}(z) \frac{\exp(2\alpha(q^{-\theta}z, q^{-\theta}w))}{\exp(2\alpha(q^\theta z, q^{-\theta}w))}, \hspace{1cm} (38)\]

are satisfied in $U_{h\omega}\mathfrak{g}$.

Then we have

\[\tilde{f}(w)\tilde{e}(z) = (k_R(w)k^-(q^{-\theta}w), k^+(q^\theta z)^{-1}k_R(z)^{-1})

\[(f(w), k^+(q^\theta z)^{-1}k_R(z)^{-1})(k_R(w)k^-(q^{-\theta}w), e(z))

\[k^+(q^\theta z)^{-1}k_R(z)^{-1}f(w)e(q^\theta z)k_R(w)k^-(q^{-\theta}w);\]

this equation may be written as

\[\tilde{f}_n\tilde{e}_m = \sum_{p \geq 0} h^p \sum_{i \geq N(p), j \geq M(p)} A_{ij}^{(p)} k_{-i}^+ f_{n-j} e_{m+i} k_{-j}^-;\]
where we set \( x(z) = \sum_n x_n z^{-n} \), \( x = e, f, \vec{c}, \vec{f}, k^\pm \); the right side belongs to the completion \( U_{h,\omega}\mathfrak{g} \). Equation \( [38] \) then follows from the identity
\[
(K^-(q^{-\theta} z), k_R(q^{-\theta} z)) = \frac{\exp(2\alpha(q^{-\theta} z, q^{-\theta} \bar{z}))}{\exp(2\alpha(q^\theta z, q^{-\theta} z))},
\]

\( \square \)

**Theorem 4.1.** \( U_{h,\omega}\mathfrak{g}/(K+2) \) has a presentation with generating series \( \tilde{c}(z), \tilde{f}(z), k^\pm(z) \) and relations (19), (20), (35), (36), (37), (38), and

\[
(k^+(z), \tilde{c}(w)) = q_+(z, q^\theta w),
\]

(40)

\[
(k^-(z), \tilde{c}(w)) = \exp[2\alpha(q^\theta z, w) - 2\alpha(q^{2\theta} z, w)] \frac{q_+(q^\theta w, q^\theta z)}{q_+(q^\theta w, q^{2\theta} z)q_-(q^{3\theta} z, q^\theta w)}
\]

(41)

and

\[
(k^+(z), \tilde{f}(w)) = q_+(z, q^\theta w)^{-1}, \quad (k^-(z), \tilde{f}(w)) = (k^-(z), \tilde{c}(w))^{-1}.
\]

(42)

### 5. Central current \( T(z) \)

Recall that

\[
g^+_\lambda(z) = (G_{-\lambda} - G)(q^\theta z, z), \quad g^-_\lambda(z) = (G_{-\lambda} - G)(q^{-\theta} z, z).
\]

Define \( \sigma, \alpha, \beta \) in \( R[[h]] \) and \( A_\lambda, B_\lambda \) in \( \mathcal{K}[[h]] \) by

\[
\sigma(q^\theta z) = \left[ -e^{-2\sum_i (q^\theta u_{+} e_i)(z) \otimes e^i(w) e^{-\phi(h, \partial^i_z) \psi(h, \partial^i_z \gamma)} \right]_{z=w};
\]

(43)

\[
\alpha(q^\theta z) = \left[ -e^{-2\sum_i (q^\theta u_{+} e_i)(z) \otimes e^i(w) \partial_h \{ e^{-\phi(h, \partial^i_z) \psi(h, \partial^i_z \gamma)} \} \right]_{w=z},
\]

(44)

\[
\beta(q^\theta z) = \alpha(q^\theta z) - 2\partial_h [\tau_{w=z}] \sigma(q^\theta z),
\]

(45)

\[
A_\lambda(z) = \alpha(q^{2\theta} z) + \sigma(q^{2\theta} z) [g^+_\lambda(z) - \sum_i e^i(z)(q^{2\theta}(q^{-\theta} e_i)_R)(z)],
\]

(46)

\[
B_\lambda(z) = \beta(q^{2\theta} z) - \sigma(q^{2\theta} z) [g^-_\lambda(z) - \sum_i e^i(z)((q^{-\theta} e_i)_R)(z)],
\]

(47)

we have \( \sigma(z) = h + O(h^2), \alpha(z) = 1 + O(h), \beta(z) = 1 + O(h), A_\lambda = 1 + O(h), B_\lambda = 1 + O(h). \)

Let us set

\[
T(z) = \tilde{c}(z) \tilde{f}(z) z \rightarrow_{R_{2\lambda}} + \tilde{f}(z) z \rightarrow_{\Lambda} \tilde{c}(z) + a_\lambda(z) k^{+\alpha}_\text{tot}(z) + b_\lambda(z) k^{-\alpha}_\text{tot}(z),
\]

where

\[
a_\lambda(z) = \frac{1}{h} \frac{A_\lambda(z)}{\sigma(q^{2\theta} z)}.
\]

(48)
and

\[ b_\lambda(z) = \frac{1}{\hbar} \exp(2\alpha(q^{-\alpha}z, q^{-\alpha}z)) \frac{B_\lambda(z)}{\sigma(q^{2\alpha}z)}, \]

we will also set

\[ b'_\lambda(z) = \frac{1}{\hbar} \frac{B_\lambda(z)}{\sigma(q^{2\alpha}z)}. \tag{50} \]

**Theorem 5.1.** The Laurent coefficients of \( T(z) \) are central elements of \( U_{\hbar,\omega}g \).

The proof is contained in the next sections.

**5.1. Commutation of \( T(z) \) with \( \bar{e}(w) \).** Set

\[ k_{\text{tot}}^+(z)\bar{e}(w) := k^+(q^{2\alpha}z)k^+(q^{\alpha}w)^{-1}k_R(q^{\alpha}z)k_R(z)^{-1}k_R(w)^{-1}\bar{e}(q^{\alpha}w)k^-(z), \]

and

\[ k_{\text{tot}}^+(z)\bar{e}(w) := k^+(q^{\alpha}z)^{-1}k_R(z)^{-1}k_R(q^{-\alpha}z)k^+(q^{\alpha}w)^{-1}k_R(w)^{-1}\bar{e}(q^{\alpha}w)k^-(q^{-\alpha}z)^{-1}. \]

**Lemma 5.1.** We have

\[ k_{\text{tot}}^+(z)\bar{e}(w) = \exp(2\alpha(q^{\alpha}w, z) - 2\alpha(q^{\alpha}w, q^{\alpha}z)q_-(q^{2\alpha}z, q^{\alpha}w)^{-1}: k_{\text{tot}}^+(z)\bar{e}(w) :. \]

and

\[ \bar{e}(w)k_{\text{tot}}^+(z) = \exp(2\alpha(q^{\alpha}w, z) - 2\alpha(q^{\alpha}w, q^{\alpha}z))q_+(q^{2\alpha}z, q^{\alpha}w)^{-1} : k_{\text{tot}}^+(z)\bar{e}(w) :. \]

**Proof.** Let us prove the first identity. The factor in the right side is

\[ (k^-(z), k^+(q^{\alpha}w)^{-1})(k^-(z), k_R(w)^{-1})(k^-(z), e(q^{\alpha}w)). \tag{51} \]

we have

\[ (k^-(z), k_R(w)^{-1}) = \exp(2\alpha(q^{\alpha}w, z) - 2\alpha(q^{2\alpha}z, w)), \]

therefore (51) is equal to

\[ \frac{q_+(q^{\alpha}w, q^{\alpha}z)}{q_+(q^{\alpha}w, q^{2\alpha}z)} \exp(2\alpha(q^{\alpha}w, z) - 2\alpha(q^{2\alpha}z, w))q_-(q^{2\alpha}z, q^{\alpha}w)^{-1}. \tag{52} \]

Identity (51) can be formulated as

\[ \frac{\exp(2\alpha(q^{\alpha}w, z))}{\exp(2\alpha(q^{2\alpha}z, w))} = q_+(q^{\alpha}w, q^{\alpha}z)q_-(q^{2\alpha}z, q^{\alpha}w), \]

because the right side is \( j(q^{\alpha}z, q^{\alpha}w) \) (see (24)). Applying to this identity \( \exp \circ (1 - q^{\alpha}t) \circ \log \), we transform (52) into

\[ \exp(2\alpha(q^{\alpha}w, z) - 2\alpha(q^{\alpha}w, q^{\alpha}z))q_-(q^{2\alpha}z, q^{\alpha}w)^{-1}; \]

this implies the first equality.

The factor in the right side of the second identity is

\[ (e(q^{\alpha}w), k^+(q^{2\alpha}z)k_R(q^{\alpha}z)k_R(z)^{-1}). \]
Proposition 5.1.

which is equal to

\[ \exp(2\alpha(q^\theta w, z) - 2\alpha(q^\theta w, q^\theta z))q_+(q^{2\theta} z, q^\theta w)^{-1}. \]

\[ \] 

In the same way, one proves

Lemma 5.2. We have

\[ k_{i\text{tot}}^-(z)\tilde{e}(w) = \exp(2\alpha(q^\theta w, z) - 2\alpha(q^\theta w, q^{-\theta} z))q_- (q^\theta z, q^\theta w) : k_{i\text{tot}}^-(z)\tilde{e}(w) ; \]

and

\[ \tilde{e}(w)k_{i\text{tot}}^-(z) = \exp(2\alpha(q^\theta w, z) - 2\alpha(q^\theta w, q^{-\theta} z))q_+ (q^\theta z, q^\theta w) : k_{i\text{tot}}^-(z)\tilde{e}(w) ; \]

Then we have

Proposition 5.1. \( T(z) \) commutes with \( \tilde{e}(w) \).

Proof. We have

\[ [T(z), \tilde{e}(w)] = \tilde{e}(z)[\tilde{f}(z), \tilde{e}(w)]_{z \rightarrow R_{2\lambda}} + [\tilde{f}(z), \tilde{e}(w)]_{z \rightarrow \Lambda'}\tilde{e}(z) + a_{\lambda}(z)[k_{i\text{tot}}^+(z), \tilde{e}(w)] + b_{\lambda}(z)[k_{i\text{tot}}^-(z), \tilde{e}(w)] \]

\[ = -\tilde{e}(z) \left( \frac{1}{\hbar} \delta(w, q^\theta z)k_{i\text{tot}}^+(w) - \frac{1}{\hbar} \delta(w, q^\theta z)k_{i\text{tot}}^-(w) \exp(2\alpha(q^\theta z, q^{-\theta} w)) \exp(2\alpha(q^\theta w, q^{-\theta} w)) \right)_{z \rightarrow R_{2\lambda}} \]

\[ - \left( \frac{1}{\hbar} \delta(w, q^\theta z)k_{i\text{tot}}^+(w) - \frac{1}{\hbar} \delta(w, q^\theta z)k_{i\text{tot}}^-(w) \exp(2\alpha(q^\theta w, q^{-\theta} w)) \exp(2\alpha(q^\theta w, q^{-\theta} w)) \right)_{z \rightarrow \Lambda'} \tilde{e}(z) \]

\[ + a_{\lambda}(z)[k_{i\text{tot}}^+(z), \tilde{e}(w)] + b_{\lambda}(z)[k_{i\text{tot}}^-(z), \tilde{e}(w)] \]

\[ = -\frac{1}{\hbar} \left( (G_{-2\lambda}(q^\theta w, z)k_{i\text{tot}}^+(w)\tilde{e}(z) + G_{2\lambda}(z, q^\theta w)\tilde{e}(z)k_{i\text{tot}}^+(w)) \exp(2\alpha(q^{-\theta} z, q^{-\theta} w)) \exp(2\alpha(q^\theta w, q^{-\theta} w)) \right) \]

\[ + \frac{1}{\hbar} \left( (G_{-2\lambda}(q^{-\theta} w, z)k_{i\text{tot}}^-(w)\tilde{e}(z) + G_{2\lambda}(z, q^{-\theta} w)\tilde{e}(z)k_{i\text{tot}}^-(w)) \exp(2\alpha(q^{-\theta} z, q^{-\theta} w)) \exp(2\alpha(q^\theta w, q^{-\theta} w)) \right) \]

\[ + a_{\lambda}(z)[k_{i\text{tot}}^+(z), \tilde{e}(w)] + b_{\lambda}(z)[k_{i\text{tot}}^-(z), \tilde{e}(w)]; \]

the last equality follows from the identities

\[ \delta(w, z)_{z \rightarrow R_{2\lambda}} = G_{2\lambda}(z, w), \quad \delta(w, z)_{z \rightarrow \Lambda'} = G_{-2\lambda}(w, z). \]

We have

\[ G_{-2\lambda}(q^\theta w, z)k_{i\text{tot}}^+(w)\tilde{e}(z) + G_{2\lambda}(z, q^\theta w)\tilde{e}(z)k_{i\text{tot}}^+(w) \]

\[ = \exp(2\alpha(q^\theta z, w) - 2\alpha(q^\theta z, q^\theta w)) \]

\[ (G_{-2\lambda}(q^\theta w, z)q_- (q^{2\theta} w, q^\theta z)^{-1} + G_{2\lambda}(z, q^\theta w)q_+ (q^{2\theta} w, q^\theta z)^{-1}) : k_{i\text{tot}}^+(w)\tilde{e}(z) : \]

\[ = \exp(2\alpha(q^\theta z, w) - 2\alpha(q^\theta z, q^\theta w))A_{\lambda}(z)\delta(z, w) : k_{i\text{tot}}^+(w)\tilde{e}(z) ; \]

where the first equality follows from Lemma 5.1, and the second from Lemma A.3.
In the same way, we have

\[ G_{-2\lambda}(q^{-\vartheta}w, z)k_{tot}^-(w)\tilde{e}(z) + G_{2\lambda}(z, q^{-\vartheta}w)\tilde{e}(z)k_{tot}^-(w) \]
\[ = \exp(2\alpha(q^{\vartheta}z, w) - 2\alpha(q^{\vartheta}z, q^{-\vartheta}w)) \]
\[ (G_{-2\lambda}(q^{-\vartheta}w, z)q_-(q^{\vartheta}w, q^{\vartheta}z) + G_{2\lambda}(z, q^{-\vartheta}w)q_+(q^{\vartheta}w, q^{\vartheta}z)) : k_{tot}^-(z)\tilde{e}(w) : \]
\[ = \exp(2\alpha(q^{\vartheta}z, w) - 2\alpha(q^{\vartheta}z, q^{-\vartheta}w))B_\lambda(z, w) : k_{tot}^-(z)\tilde{e}(w) : \]

where the first equality follows from Lemma 5.2, and the second from Lemma A.3.

On the other hand, we have

\[ [k_{tot}^-(z), \tilde{e}(w)] \]
\[ = \exp(2\alpha(q^{\vartheta}w, z) - 2\alpha(q^{\vartheta}w, q^{\vartheta}z))[q_-(q^{2\vartheta}z, q^{\vartheta}w)^{-1} - q_+(q^{2\vartheta}z, q^{\vartheta}w)^{-1}] \]
\[ : k_{tot}^+(z)\tilde{e}(w) : \]
\[ = \exp(2\alpha(q^{\vartheta}w, z) - 2\alpha(q^{\vartheta}w, q^{\vartheta}z))\sigma(q^{2\vartheta}z)\tilde{e}(z, w) : k_{tot}^+(z)\tilde{e}(w) : \]

and

\[ [k_{tot}^-(z), \tilde{e}(w)] \]
\[ = \exp(2\alpha(q^{\vartheta}w, z) - 2\alpha(q^{\vartheta}w, q^{-\vartheta}z))[q_-(q^{\vartheta}w, q^{\vartheta}z) - q_+(q^{\vartheta}w, q^{\vartheta}z)] \]
\[ : k_{tot}^-(z)\tilde{e}(w) : \]
\[ = \exp(2\alpha(q^{\vartheta}w, z) - 2\alpha(q^{\vartheta}w, q^{-\vartheta}z))[-\sigma(q^{2\vartheta}w)\tilde{e}(z, w)] : k_{tot}^-(z)\tilde{e}(w) : .\]

The equalities \( \delta(z, w) : k_{tot}^\pm(z)\tilde{e}(w) := \delta(z, w) : k_{tot}^\pm(w)\tilde{e}(z) : \) then imply that (53) vanishes.

5.2. **Commutation of** \( T(z) \) **with** \( k^\pm(w) \). Let us denote by \( U_h^+ \) and \( U_h^- \) the subalgebras of \( U_{h,\omega}g \) generated respectively by the \( e[e] \), by the \( h^+[r] \), \( h^-[\lambda] \) and \( K \), and by the \( f[e] \). If we assign degree 1 to the \( e[e] \) and \( f[e] \), \( U_h^\pm \) are graded algebras. We denote by \( U_h^{\pm[i]} \) their homogeneous components of degree \( i \).

We will prove

**Lemma 5.3.** \( k^+(w)T(z)k^+(w)^{-1} - T(z) \) and \( k^-(w)T(z)k^-(w)^{-1} - T(z) \) both belong to \( U_h^\pm \).

**Proof.** It suffices to prove the same statements with \( T(z) \) replaced by \( T_0(z) \) defined by

\[ T_0(z) = \tilde{e}(z)\tilde{f}(z)_{z \to R_{2\lambda}} + \tilde{f}(z)_{z \to \lambda'}\tilde{e}(z). \]
Then from (44) and (12) follows that
\[ q_+(z, q^0 w)^{-1}k^+(w)T_0(z)k^+(w)^{-1} - T_0(z) \]
\[ = \tilde{c}(w)[q_+(z, q^0 w)^{-1}\tilde{f}(w)]_{w \to R_{2\lambda}} + [q_+(z, q^0 w)^{-1}\tilde{f}(w)]_{w \to \Lambda}\tilde{c}(w) \]
\[ - q_+(z, q^0 w)^{-1}[\tilde{c}(w)\tilde{f}(w)]_{w \to R_{2\lambda}} + \tilde{f}(w)_{w \to \Lambda}\tilde{c}(w) \]
\[ = [\tilde{c}(w), [q_+(z, q^0 w)^{-1}\tilde{f}(w)]_{w \to \Lambda'} - q_+(z, q^0 w)^{-1}\tilde{f}(w)_{w \to \Lambda'}] \]
because of the identity
\[ [q_+(z, q^0 w)^{-1}\tilde{f}(w)]_{w \to \Lambda'} - q_+(z, q^0 w)^{-1}\tilde{f}(w)_{w \to \Lambda'} \]
\[ = q_+(z, q^0 w)^{-1}\tilde{f}(w)_{w \to R_{2\lambda}} - [q_+(z, q^0 w)^{-1}\tilde{f}(w)]_{w \to R_{2\lambda}}. \]

For any \( \epsilon \) in \( \mathcal{K} \), \([\tilde{c}[\epsilon], \tilde{f}(z)]\) belongs to \( U_h\mathfrak{h} \), which proves the first part of the statement. The second part is proved in the same way, using (44) and (12). \( \square \)

Let us now prove

**Proposition 5.2.** \( T(z) \) commutes with \( U_h\mathfrak{h} \).

**Proof.** Set for \( r \) in \( R \) and \( \lambda \) in \( \Lambda \),
\[ x^+_\eta(r) = [h^+[r], T[\eta]], \quad x^-_\eta(\lambda) = [h^-[\lambda], T[\eta]]. \]

From Lemma 5.3 follows that \( x^+_\eta \) are linear maps from \( R \) and \( \Lambda \) to \( U_h\mathfrak{h} \). Moreover, we have \([x^+_\eta(r), \tilde{f}[\epsilon]] = [[h^+[r], T[\eta]], \tilde{f}[\epsilon]] = -[[T[\eta], \tilde{f}[\epsilon]], h^+[r]] - [[\tilde{f}[\epsilon], h^+[r]], T[\eta]]; \)
both terms are zero by Prop. 5.3, so that we have
\[ [x^+_\eta(r), \tilde{f}[\epsilon]] = 0; \]
in the same way, one shows that
\[ [x^-_\eta(\lambda), \tilde{f}[\epsilon]] = 0. \]

But any element \( x \) of \( U_h\mathfrak{h} \), such that \([x, \tilde{f}[\epsilon]] = 0 \) for any \( \epsilon \), is zero. To show this, one may divide \( x \) by the greatest possible power of \( \hbar \) and check that the same statement is true in the classical affine Kac-Moody algebra. \( \square \)

### 5.3. Commutation of \( T(z) \) with \( f(w) \).

**Lemma 5.4.** \( T(z) \) may be written
\[ T(z) = \tilde{f}(z)\tilde{c}(z)_{z \to R} + \tilde{c}(z)_{z \to \Lambda}\tilde{f}(z) + \kappa(z), \]
where \( \kappa(z) \) belongs to \( U_h\mathfrak{h}[[z, z^{-1}]] \).

**Proof.** We have
\[ \tilde{f}(z)\tilde{c}(z)_{z \to R} + \tilde{c}(z)_{z \to \Lambda}\tilde{f}(z) - T_0(z) \]
is equal to
\[ [\tilde{c}(z)_{z \to \Lambda'}, \tilde{f}(z)_{z \to \Lambda'}] - [\tilde{c}(z)_{z \to R_{2\lambda}}, \tilde{f}(z)_{z \to R_{2\lambda}}] \]
and therefore belongs to \( U_h\mathfrak{h}[[z, z^{-1}]]. \) \( \square \)
We first show:

**Lemma 5.5.** The commutator \([T(z), \tilde{f}(w)]\) belongs to \(U_h \mathfrak{h} U_h \mathfrak{n}_w^{[1]}\); in other words, there are formal series \(K_i(z, w)\) in \(U_h \mathfrak{h}[[z, z^{-1}, w, w^{-1}]]\), such that
\[
[T(z), \tilde{f}(w)] = \sum_i K_i(z, w) \tilde{f}[\epsilon_i].
\] (54)

**Proof.** It suffices to show this with \(T_0(z)\) instead of \(T(z)\). This follows from a reasoning analogous to the first part of the proof of Prop. 5.1.

From there follows:

**Proposition 5.3.** \(T(z)\) commutes with \(\tilde{f}(w)\).

**Proof.** Let \(\epsilon\) belong to \(\mathcal{K}\). \(\tilde{e}[\epsilon]\) commutes with the left side of (54), by Props. 5.3 and 5.1. Let us write that it commutes with the right side of this equality. We get \(\sum_i [\tilde{e}[\epsilon], K_i(z, w)] \tilde{f}[\epsilon_i] + \text{element of } U_h \mathfrak{h} = 0\). From there follows that \([\tilde{e}[\epsilon], K_i(z, w)] = 0\). The reasoning of the end of the proof of Prop. 5.2 applies to show that \(K_i(z, w)\) vanishes.

Props. 5.1, 5.2 and 5.3 imply Thm. 5.1.

**Remark 5.** Classical limit. Let us show that \(T(z)\) is, up to a scalar, a deformation of the Sugawara tensor. Let us denote by \(e_{cl}(z), h_{cl}(z)\) and \(f_{cl}(z)\) the generating currents of \(\mathfrak{g}\). Then we have
\[
e(z) = e_{cl}(z) + O(h), \quad f(z) = f_{cl}(z) + O(h),
\]
\[
k^+(z) = 1 + \frac{h}{2} h_{cl}(z)_{z \to \Lambda} + o(h), \quad k^-(z) = 1 + \frac{h}{2} h_{cl}(z)_{z \to R} + o(h),
\]
\[
k_R(z) = 1 + O(h^2), \text{ so that}
\]
\[
k^+_{tot}(z) = [1 + \frac{h}{2} q^\partial_{\mathfrak{s}}(h_{cl}(z)_{z \to \Lambda}) + h^2 s(z)] [1 + \frac{h}{2} h_{cl}(z)_{z \to R} + h^2 t(z)] + O(h^3)
\]
\[
k^-_{tot}(z) = [1 - \frac{h}{2} q^{-\partial_{\mathfrak{s}}}(h_{cl}(z)_{z \to \Lambda}) - h^2 s(z) + \frac{h^2}{4} (h_{cl}(z)_{z \to \Lambda})^2]
\]
\[
[1 - \frac{h}{2} q^{-\partial_{\mathfrak{s}}}(h_{cl}(z)_{z \to \Lambda}) - h^2 t(z) + \frac{h^2}{4} (h_{cl}(z)_{z \to R})^2] + O(h^3),
\]
where \(s(z)\) and \(t(z)\) are some currents. Then
\[
T(z) = e_{cl}(z)_{z \to \Lambda} f_{cl}(z) + f_{cl}(z) e_{cl}(z)_{z \to R} + \frac{1}{h^2} (k^+_{tot}(z) + k^-_{tot}(z)) + O(h)
\]
\[
= \frac{1}{h^2} + e_{cl}(z)_{z \to \Lambda} f_{cl}(z) + f_{cl}(z) e_{cl}(z)_{z \to R} + \frac{1}{2} \partial h_{cl}(z)
\]
\[
+ \frac{1}{4} (h_{cl}(z)_{z \to \Lambda} h_{cl}(z) + h_{cl}(z) h_{cl}(z)_{z \to R}) + O(h);
\]
so \(T(z) - h^{-2}\) coincides with the classical Sugawara tensor to order \(h\).  \(\square\)
Remark 6. Other expressions of $T(z)$. One may show that up to an additive scalar constant, $T(z)$ coincides with

$$T'(z) = k^+(q^\vartheta z)^{-1} \left( f(z)_{z \rightarrow \Lambda} e(q^\vartheta z) + e(q^\vartheta z) f(z)_{z \rightarrow R_{2\alpha}} \right) k^-(q^{-\vartheta} z)$$

$$+ \frac{\gamma'(z)}{\hbar \sigma(z)} k^+(q^\vartheta z)^{-1} k^-(q^{-\vartheta} z)^{-1} + \frac{\delta_2 \lambda(z)}{\hbar \sigma(q^{2\vartheta} z)} k^+(q^{2\vartheta} z) k^-(q^{-\vartheta} z),$$

with

$$\gamma'(z) = \alpha(q^{2\vartheta} z) - \sigma(q^{2\vartheta} z) g^{-}(z) + \sum_i e^i(z)(q^\vartheta(q^{-\vartheta} e_i)_R(z))$$

and

$$\delta_\lambda(z) = \beta(q^{2\vartheta} z) + \sigma(q^{2\vartheta} z) g^+(z).$$

It also coincides with $T''(z)$ defined by

$$T''(z) = k^+(z)^{-1} \left( e(z)_{z \rightarrow \Lambda} f(q^{-\vartheta} z) + f(q^{-\vartheta} z) e(z)_{z \rightarrow R} \right) k^-(q^{-2\vartheta} z)$$

$$+ \frac{1}{\hbar} \frac{\alpha(z)}{\sigma(z)} k^+(q^{-\vartheta} z) k^-(q^{-2\vartheta} z) + \frac{1}{\hbar} \frac{\beta'(z)}{\sigma(z)} (q^{-1} k^+ - (q^{-2} k^+))^{-1} k^-(q^{-\vartheta} z)^{-1},$$

up to an additive constant, with $\beta'(z) = \beta(q^\vartheta z) - \sigma(q^\vartheta z) \sum_i q^\vartheta((q^{-2\vartheta} e_i)_R(z)) e^i(z)$.

This formula is a generalization of the formula given in [12], which uses [14] and the new realizations isomorphism. To see the correspondence between this formula and ours, let us modify the notation in [12], so that the quantum parameter of that paper is denoted by $q$. The level in [12] is denoted by $k$ and the currents generating the algebra $U_q \mathfrak{g}$ are $k^\pm(z)$, $E(z)$ and $F(z)$.

Set $X = \mathbb{C}P^1$ and $\partial = z \frac{d}{dz}$. The algebra $U_{h, \omega} \mathfrak{g}$ is isomorphic to $U_q \mathfrak{g}$, the isomorphism $i$ being given by the formulas

$$i(K) = k, \quad i(k^+(z)) = k^+(z q^{\frac{\vartheta}{2} + 2}), \quad i(k^-(z)) = k^-(z q^{\frac{\vartheta}{2}}),$$

$$i(e(z)) = -\frac{1}{\hbar(q - q^{-1})} E(z), \quad i(f(z)) = F(q^k z),$$

with

$$q = q^{-2}, \quad q(z, w) = \frac{q^{-1} z - w}{z - q^{-1} w}, \quad q^+ (z, w) = \frac{q^{-1/2} z - q^{1/2} w}{z - w}.$$  

Formula (6.10) of [12] then gives

$$i^{-1}(\ell(z)) = \frac{1}{\hbar(q - q^{-1})} k^+(z)^{-1} : e(z) f(q^{-1} z) : k^-(z q^{-2})$$

$$+ q^{-1/2} k^+(z)^{-1} k^-(z q^{-2}) + q^{1/2} k^+(z)^{-1} k^-(z q^{-1})^{-1},$$

so $i^{-1}(\ell(z))$ is equal to $T(z)$ given by (56).
Remark 7. Genus 1 case. Assume $X$ is an elliptic curve $\mathbb{C}/L$, $L = \mathbb{Z} + \tau \mathbb{Z}$, and $\omega = dz$. Let $\theta$ be the Jacobi theta-function, equal to

$$\theta(z) = \frac{\sin(\pi z)}{\pi} \prod_{j=1}^{\infty} \frac{(1 - e^{2i\pi(j\tau+z)})(1 - e^{2i\pi(j\tau-z)})}{(1 - e^{2i\pi j\tau})^2}$$

The Weierstrass function is $\wp = -(d/dz)^2 \ln \theta(z)$. According to Prop. [14], we have $R = \mathbb{C}1 \oplus (\oplus_{i \geq 0} \mathbb{C}(d/dz)^i \wp)$ and $\Lambda = \mathbb{C}^\theta / \theta \oplus z \mathbb{C}[[z]]$. We have also $R_\lambda = \oplus_{i \geq 0} \mathbb{C}(d/dz)^i (\wp(z - \lambda))$ and $\Lambda' = \mathbb{C}[[z]]$. We have

$$G(z, w) = d/dz \ln \theta(z - w) - d/dz \ln \theta(z) + d/dz \ln \theta(w),$$

$$G_{2\lambda}(z, w) = \frac{\theta(-2\lambda + z - w)}{\theta(z - w)\theta(-2\lambda)},$$

$$q_-(z, w) = \frac{\theta(z - w - \hbar)}{\theta(z - w)},$$

viewed as a series in $\mathbb{C}((z))((w))[[\hbar]],$

$$\sigma(z) = \theta(h), \quad \gamma_\lambda(z) = \frac{\theta(2\lambda - \hbar)}{\theta(2\lambda)}, \quad \delta_\lambda(z) = \frac{\theta(2\lambda + \hbar)}{\theta(2\lambda)}.$$

The expression of $T'(z)$ is then

$$T'(z) = k^+(z + \hbar)^{-1} (f(z)_{z \to \lambda'} e(z + \hbar) + e(z + \hbar) f(z)_{z \to R_{2\lambda}}) k^-(z - \hbar)$$

$$+ \frac{\theta(2\lambda - \hbar)}{\hbar \theta(h) \theta(2\lambda)} k^+ (z + \hbar)^{-1} k^- (z - 2\hbar)^{-1} + \frac{\theta(2\lambda + \hbar)}{\hbar \theta(2\lambda) \theta(h)} k^+(z + 2\hbar) k^-(z - \hbar);$$

we have also

$$T(z) = \tilde{c}(z)_{z \to R_{2\lambda}} + \tilde{f}(z)_{z \to \lambda'} \tilde{c}(z) + \frac{1}{\hbar} \frac{\theta(2\lambda + \hbar)}{\hbar \theta(2\lambda) \theta(h)} k^+_\text{tot}(z) + \frac{1}{\hbar} \frac{\theta(2\lambda - \hbar)}{\hbar \theta(2\lambda) \theta(h)} k^-_{\text{tot}}(z).$$

□

6. Subalgebras $U_h\mathfrak{g}_{\lambda_0}^{\text{out}}$ and $U_h\mathfrak{g}_{\lambda_0}^{\text{out}}$ of $U_{h,\omega}\mathfrak{g}$ and coproducts

6.1. Subalgebras $U_h\mathfrak{g}^{\text{out}}$ and $U_h\mathfrak{g}_{\lambda_0}^{\text{out}}$. In [14], we showed that $U_{h,\omega}\mathfrak{g}$ contains a “regular” subalgebra $U_h\mathfrak{g}^{\text{out}}$, generated by the $h^+[r], e[r]$ and $f[r]$, for $r$ in $R$. The inclusion $U_h\mathfrak{g}^{\text{out}} \subset U_{h,\omega}\mathfrak{g}$ is a deformation of the inclusion of the classical enveloping algebra of $\mathfrak{g} \otimes R$ in that of $\mathfrak{g} = (\mathfrak{g} \otimes \mathcal{K}) \oplus \mathbb{C}K$.

For any $\lambda_0$ in $\mathbb{C}^g$, define $U_h\mathfrak{g}_{\lambda_0}^{\text{out}}$ as the subalgebra of $U_{h,\omega}\mathfrak{g}^{\text{out}}$, generated by the $h^+[r], e[r_{2\lambda_0}]$ and $f[r_{-2\lambda_0}]$, for $r$ in $R$, $r_{\pm 2\lambda_0}$ in $R_{\pm 2\lambda_0}$.

Proposition 6.1. Define $\mathfrak{g}_{\lambda_0}^{\text{out}}$ to be the Lie algebra $(\mathfrak{n}_+ \otimes R_{2\lambda_0}) \oplus (\mathfrak{k} \otimes R) \oplus (\mathfrak{n}_- \otimes R_{-2\lambda_0})$. The inclusion $U_h\mathfrak{g}_{\lambda_0}^{\text{out}} \subset U_{h,\omega}\mathfrak{g}$ is a deformation of the inclusion of the classical enveloping algebra of $\mathfrak{g}_{\lambda_0}^{\text{out}}$ in that of $\mathfrak{g}$.

Proof. [12] implies that the $e[\epsilon]$ satisfy the relations given by the pairing of

$$(1 + \psi(-\hbar, \partial_z \gamma)G(z, w) e(z) e(w) = e^{2(r-\phi)} (1 + \psi(\hbar, \partial_z \gamma)G(z, w)) e(w) e(z)$$

with any $\nu$ in $\mathcal{K} \otimes \mathcal{K}$, such that $m(\nu) = 0$, where $m$ is the multiplication map.
Taking for \( v \) any \( \alpha \otimes \beta - \beta \otimes \alpha \), with \( \alpha, \beta \) in \( R_{2\lambda_0} \), and using the fact that \( R_{2\lambda_0} \) is an \( R \)-module, we get relations of the form

\[
[e[\alpha], e[\beta]] = \sum_{i \geq 1, j} h^i e[\alpha_j^{(i)}] e[\beta_j^{(i)}],
\]

with \( \alpha_j^{(i)}, \beta_j^{(i)} \) in \( R_{2\lambda_0} \). Therefore, if \( e_{2\lambda_0;i} \) is a basis of \( R_{2\lambda_0} \), the family

\[
(e[e_{2\lambda_0;1}; \cdots ; e_{2\lambda_0;ip}])_{i_1 \leq \cdots \leq i_p}
\]

spans the subalgebra of \( U_{h,\omega}\mathfrak{g} \) generated by the \( e[r] \), \( r \) in \( R_{2\lambda_0} \). Since by [7], Lemma 3.3, this is also a free family, it forms a basis of this subalgebra. To finish the proof, one proves the similar basis result for the subalgebra generated by the \( f[r] \), \( r \) in \( R_{-2\lambda_0} \) and a triangular decomposition result (see [7], Prop. 3.2 and Prop. 3.5).

6.2. **Coproducts.** Set \( A = U_{h,\omega}\mathfrak{g}, B = U_{h}\mathfrak{g}^{\text{out}}, B_{\lambda_0} = U_{h}\mathfrak{g}^{\text{out}}_{\lambda_0} \).

Define for \( \mathfrak{n} = (n_i)_{1 \leq i \leq p}, I_n \) as the left ideal of \( A \) generated by the \( x[\epsilon], \epsilon \in \prod_i z_i^i \mathbb{C}[z_i] \). Define \( A \otimes_{\succ} A, A \otimes_{\prec} A \) and \( A \otimes A \) as the completions of \( A \otimes A \) with respect to the topologies defined by \( A \otimes I_n, I_n \otimes A \) and \( I_n \otimes A + A \otimes I_n \) (\( \otimes \) denotes the \( h \)-adically completed tensor product). We have the inclusions \( A \otimes_{\succ} A \subset A \otimes A, A \otimes_{\prec} A \subset A \otimes A \) and \( A \otimes A = (A \otimes_{\succ} A) \cap (A \otimes_{\prec} A) \).

We define also for any space \( V, V \otimes_{\succ} A \) as the completion of \( V \otimes A \) w.r.t. the topology defined by the \( V \otimes I_n, A^{\otimes>n} \) as \( A^{\otimes>n-1} \otimes_{\succ} A \), and \( A^{\otimes<n} \) in the same way.

In [7], we defined Drinfeld-type coproducts \( \Delta \) and \( \Delta \) on \( U_{h,\omega}\mathfrak{g} \) by formulas similar to those of [7]. \( \Delta \) and \( \Delta \) map \( A \) to \( A \otimes_{\prec} A \) and to \( A \otimes_{\succ} A \). Moreover, \( \Delta \) and \( \Delta \) are conjugated by an element \( F \) of \( A \otimes A \). \( F \) is decomposed as a product \( F_2 F_1 \), with \( F_1 \) in \( A \otimes_{\prec} B \) and \( F_2 \) in \( B \otimes_{\succ} A \), which are defined as \( \lim_{\leftarrow} A \otimes B/I_n \otimes B \) and \( \lim_{\rightarrow} B \otimes A/B \otimes I_n \).

\( \Delta_R \) is defined as \( \text{Ad}(F_1) \circ \Delta \). It maps therefore \( A \) to \( A \otimes_{\prec} A \). Since \( \Delta_R \) is equal to \( \text{Ad}(F_2^{-1}) \circ \Delta \), it also maps \( A \) to \( A \otimes_{\succ} A \) and therefore to \( A \otimes A \). Also we have \( \Delta_R(B) \subset B \otimes B \).

**Theorem 6.1.** We have \( \Delta(B_{\lambda_0}) \subset A \otimes_{\prec} B_{\lambda_0} \) and \( \Delta(B_{\lambda_0}) \subset B_{\lambda_0} \otimes_{\succ} A \). We have a decomposition

\[
F = F_{2;\lambda_0} F_{1;\lambda_0}, \text{ with } F_{1;\lambda_0} \in A \otimes_{\prec} B_{\lambda_0} \text{ and } F_{2;\lambda_0} \in B_{\lambda_0} \otimes_{\succ} A.
\]

Set \( \Delta_{\lambda_0} = \text{Ad}(F_{1;\lambda_0}) \circ \Delta \), then \( \Delta_{\lambda_0} \) defines a quasi-Hopf algebra structure on \( A \), for which \( B \) is a sub-quasi-Hopf algebra.

**Sketch of proof.** The first statement is proved like Prop. 4.4 of [7], using the fact that \( R_{\pm 2\lambda_0} \) are \( R \)-modules. The decomposition of \( F \) is proved using the same duality arguments, e.g. the annihilator of \( U_h \mathfrak{n}_+ \cap B_{2\lambda_0} \) in \( U_h \mathfrak{n}_- \) is equal to \( \sum_{r \in R_{-2\lambda_0}} U_h \mathfrak{n}_- f[r] \). The proof of the next statements follows [7].

\[\square\]
7. Finite dimensional representations of $U_{\hbar, \omega} g$.

In [3], we constructed a family $\pi_\zeta$ of 2-dimensional representations of $U_{\hbar, \omega} g$ at level zero, indexed by $\zeta$ in the infinitesimal neighborhood $\text{Spec}(K)$ of the $P_i$. We have

$$\pi_\zeta(K^+(z)) = \begin{pmatrix} q_-(z, \zeta) & 0 \\ 0 & q_-(q^\theta z, \zeta)^{-1} \end{pmatrix}, \quad \pi_\zeta(K^-(z)) = \begin{pmatrix} q_+(z, \zeta) & 0 \\ 0 & q_+(q^\theta z, \zeta)^{-1} \end{pmatrix},$$

$$\pi_\zeta(e(z)) = \begin{pmatrix} 0 & -h\sigma(z)\delta(z, \zeta) \\ 0 & 0 \end{pmatrix}, \quad \pi_\zeta(f(z)) = \begin{pmatrix} 0 & 0 \\ \delta(z, \zeta) & 0 \end{pmatrix}.$$  

This family extends to a family of representations of $U_{\hbar} g^{\text{out}}$, indexed by $\zeta$ in $X - \{P_i\}$. Formulas are

$$\pi_\zeta(K^+(z)) = \begin{pmatrix} q_-(z, \zeta) & 0 \\ 0 & q_-(q^\theta z, \zeta)^{-1} \end{pmatrix},$$

$$\pi_\zeta(e[r]) = \begin{pmatrix} 0 & -h\sigma(\zeta)\sigma(\zeta) \xi \zeta \\ 0 & 0 \end{pmatrix}, \quad \pi_\zeta(f[r]) = \begin{pmatrix} 0 & 0 \\ \delta(\zeta) & 0 \end{pmatrix}.$$  

It also extends to a family of representations of $U_{\hbar} g^{\text{out}}$ by the same formulas, where we fix a preimage of $\zeta$ in $\tilde{X} - \pi^{-1}(P_0)$. Changing the preimage of $\zeta$ amounts to conjugating the representation by a diagonal matrix.

Define a parenthesis order on $n$ objects as a binary tree with extremal vertices labelled $1, \ldots, n$. To each such order, and to $n$ points $\zeta_i$ of $X - \{P_i\}$, we associate some $B_{\lambda_0}$-module. In the case of the representation $V = (V(\zeta_1) \otimes V(\zeta_2)) \otimes (V(\zeta_3) \otimes V(\zeta_4))$, the space of the representation is $V = \bigotimes_{i=1}^{n} V(\zeta_i)$ and the morphism from $B_{\lambda_0}$ to $\text{End}(V)$ is $(\bigotimes_{i=1}^{n} \pi_{\zeta_i}) \circ (\Delta \otimes \Delta) \circ \Delta$.

In case the $\zeta_i$ are formal and $\zeta_1 << \zeta_2 << \cdots << \zeta_n$, the morphism $\rho_V^{(P)}$ is the restriction of a morphism from $A$ to $\text{End}(V)$, which is $\bigotimes_{i=1}^{n} \pi_{\zeta_i} \circ \text{Ad}(\Delta^{(P)}(F_{1;\lambda_0})$. For example for $V = (V(\zeta_1) \otimes V(\zeta_2)) \otimes (V(\zeta_3) \otimes V(\zeta_4)))$, $\Delta^{(P)}(F_1)$ is equal to $\Delta^{(P)}(F_{1;\lambda_0}) = F_{1;\lambda_0}^{(12)} F_{1;\lambda_0}^{(34)} (\Delta \otimes \Delta)(F_{1;\lambda_0})$.

Let $(z_1, z_2)$ be the canonical basis of $\mathbb{C}^2$, $\xi_1, \xi_2$ its dual basis.

**Proposition 7.1.** Let $\zeta_1, \ldots, \zeta_n$ be points of $X - \{P_i\}$; let $P$ be a parenthesis order, and define $V$ as the $B_{\lambda_0}$-module $\bigotimes_{i=1}^{n} V(\zeta_i)$ is then a $B_{\lambda_0}$-module; we denote by $\rho_V^{(P)}$ the corresponding morphism from $B_{\lambda_0}$ to $\text{End}(V)$. It has the following properties:

1) $e[r]$ and $f[r]$ act on $V$ as $\sum_i A_i(\zeta_1, \ldots, \zeta_n) r(\zeta_i)$, $A_i$ in $\text{End}(V) \otimes R^{\otimes n}$;

2) define the linear form $\xi$ on $V$ to be $\bigotimes_{i=1}^{n} \xi_i$. Then we have $\xi, \rho_V^{(P)}(f[r]) v = 0$ for any $r$ of $R_{2;\lambda_0}$ and $\langle \xi, \rho_V^{(P)}(K^+(z)) v \rangle = \prod \frac{1}{q_+(z, \zeta_i) \xi(\zeta_i, \zeta_i_1, \ldots, \zeta_n)}$.

**Proof.** Let us first show 1) when $\zeta_i$ are formal and $\zeta_1 << \zeta_2 << \cdots$. In that case, $\rho_V^{(P)}(e(z))$ is conjugate to $\Delta^{(n)}(e(z))$, which has the form

$$\sum_i A_i(\zeta, \zeta_1, \ldots, \zeta_n).$$
with $A'_i$ some endomorphisms of $V$ and $q_i$ in $\mathbb{C}((\zeta_1)) \cdots ((\zeta_n))$. Therefore $\Delta^{(n)}(e[r])$ is equal some $\sum_i A'_i r(\zeta_i) q_i(\zeta_1, \ldots, \zeta_n)$. On the other hand, $\rho^{(P)}(e[r])$ is equal to the conjugation of $\Delta^{(n)}(e[r])$ by $(\otimes_{i=1}^n \pi_{\zeta_i})(\Delta^{(P)}(F_{1;\lambda_0}))$. $\Delta^{(n)}(e[r])$ belongs to

$$\text{End}(V)((\zeta_1)) \cdots ((\zeta_n)),$$

so that $\rho^{(P)}(e[r])$ has the form

$$\sum_i B_i(\zeta_1, \ldots, \zeta_n) r(\zeta_i),$$

where $B_i(\zeta_1, \ldots, \zeta_n)$ belongs to $\text{End}(V)((\zeta_1)) \cdots ((\zeta_n))$. $r$ being fixed, $\rho^{(P)}(e[r])$ is an algebraic function in the $\zeta_i$, so the $B_i(\zeta_1, \ldots, \zeta_n)$ are algebraic functions and $\rho^{(P)}(e[r])$ has the form (57) for any $\zeta_i$ in $X - \{P_i\}$. This proves 1).

Let us prove 2). Since $\Delta^{(P)} F_{1;\lambda_0}$ has total weight zero (i.e. it commutes with $\Delta h[1] = \sum_i h[1]^{(i)}$), $\rho^{(P)}(f[r])$ has weight $-1$, which implies the first statement.

Let us prove the second statement. We can show by induction that

$$\langle \xi, \Delta^{(P)}(F_{1;\lambda_0}) \rangle = \langle \xi, v \rangle,$$

for any $v$ in $V$. For example, in the case of a representation $V = ((V(\zeta_1) \otimes V(\zeta_2)) \otimes (V(\zeta_3) \otimes V(\zeta_4)))$, we have

$$\langle \xi, (\otimes_{i=1}^4 \pi_{\zeta_i})(F_{1;\lambda_0}^{(12)} \otimes (\Delta \otimes \Delta)(F_{1;\lambda_0})) \rangle = \langle \xi, (\otimes_{i=1}^4 \pi_{\zeta_i})((\Delta \otimes \Delta)(F_{1;\lambda_0})) \rangle$$

because $F_{1;\lambda_0}$ belongs to $1 + U h_n^{[\geq 1]} \otimes U h_n^{[\geq 1]}$; as $(\Delta \otimes \Delta)(F_{1;\lambda_0})$ belongs to

$$1 + (U h_n^{[\geq 1]} \otimes A^{\otimes 3})[0] + (A \otimes U h_n^{[\geq 1]} \otimes A^{\otimes 2})[0]$$

(where [0] means the zero weight component w.r.t. the adjoint action of $\sum_i h[1]^{(i)}$), we have

$$\langle \xi, (\Delta \otimes \Delta)(F_{1;\lambda_0}) v \rangle = \langle \xi, v \rangle.$$

On the other hand, we have

$$\langle \xi, (\otimes_{i=1}^4 \pi_{\zeta_i})(\Delta^{(P)}(\Delta^+(z)))(v) \rangle = \prod_i q_+(z, \zeta_i) \langle \xi, v \rangle.$$

Together with (58), this shows the statement for $\Delta^+(z)$.

Remark 8. Prop. [7, 1, 2] means that the “Drinfeld polynomial” of $\otimes_i^{(P)} V(\zeta_i)$ is

$$\prod_i q_+(q^\partial z, \zeta_i)$$

(see [3]).

Define $k_{a \rightarrow R}(z)$ as $\exp(\sum_{i,j} c_{ij} h[e^i] e^j(z))$, where $c_{ij}$ are as in (23).

Corollary 7.1. There are formal series $\pi_{\alpha,\zeta}(z)$ such that

$$\langle \rho^{(P)}(k^+(q^\partial z)k_R(q^\partial z)k_R(z)^{-1}k_{a \rightarrow R}(z)v), \xi \rangle = \prod_i \pi_{\alpha,\zeta_i}(z) \langle v, \xi \rangle,$$

for any $v$ in $V$. 
8. Twisted correlation functions

Let $\mathcal{V}$ be a module over $U_{\mathbf{h},\omega}\mathfrak{g}/(K+2)$. Let $\psi_{\lambda_0}$ be a $U_{\mathbf{h},\omega}\mathfrak{g}_{\lambda_0}^\text{out}$-module map from $\mathcal{V}$ to $\mathcal{V}$ and set $\psi_\lambda = \psi_{\lambda_0} \circ e^{\sum(a_\lambda - \lambda_0^{(0)})^{h[r_\lambda]}}$. Fix $v$ in $\mathcal{V}$ and let us set

$$f_\lambda(u_1, \ldots, u_n) = \langle \psi_\lambda [\tilde{e}(u_1) \cdots \tilde{e}(u_n)]v, \xi \rangle,$$

where $\xi$ is the linear form defined in Prop. 7.1.

**Proposition 8.1.** $f_\lambda(u_1, \ldots, u_n)$ is a symmetric function in $(u_i)$, such that

$$\prod_{i=1}^n \prod_j \pi_{\zeta_j}(q^\alpha u_i)f_\lambda(u_1, \ldots, u_n)$$

is regular on $\tilde{X}^n$ except for poles for $u_i$ at $\pi^{-1}(P_i)$, and simple poles for $u_i$ at $\pi^{-1}(q^{-\theta}\zeta_j)$, and satisfy transformation properties (2), with $\lambda_0^{(0)}$ replaced by $\lambda_n$.

When $\mathcal{V}$ is the trivial representation, $f_\lambda(u_1, \ldots, u_n)$ is regular on $(\tilde{X}-\pi^{-1}(P_0))^n$.

**Proof.** From the commutation relations of $\tilde{e}(z)$ follows that $f_\lambda(u_1, \ldots, u_n)$ is symmetric in the $u_i$. We have

$$\prod_j \pi_{\zeta_j}(q^\alpha u_1)f_\lambda(u_1, \ldots, u_n) = \langle \psi_\lambda [e(q^\alpha u_1)w(u_1, \ldots, u_n)], \xi \rangle.$$

The fact that $\langle \psi_\lambda [e[r]w], \xi \rangle = 0$ for $r$ in $R_{2\lambda}$ vanishing at $\zeta_i$ implies that

$$\prod_j \pi_{\zeta_j}(q^\alpha u_1)f_\lambda(u_1, \ldots, u_n)$$

belongs to $[(\text{annihilator for } \langle , \rangle_\mathcal{K} \text{ of } \{r \in R_{2\lambda}|r(\zeta_i) = 0\}) \otimes \mathcal{K}^{n-1}[[\mathbf{h}]]$. This annihilator is the space of functions on $\tilde{X}$ with simple poles at $\zeta_i$ and a pole at the $P_i$, satisfying (2). \qed

9. Action of $T(z)$ on correlation functions

Let us set

$$q_m(z, w) = (k_m(z), \tilde{e}(w)), \quad \kappa(z) = \frac{\exp(2\alpha(q^{-\theta}z, q^{-\theta}z))}{\exp(2\alpha(q^\theta z, q^\theta z))}(k_\alpha(q^{-\theta}z), k^{-}(q^{-\theta}z))^{-1}.$$

**Lemma 9.1.** We have

$$q_m(z, w) = \exp[2\mathbf{h} \sum_i \left( \frac{1}{1 + q^{-\theta}e^i(z)}((T + U)e_i)m(w) \right)], \quad (59)$$

$$\kappa(z) = \exp[2\alpha(z, z) + 2\alpha(q^{-\theta}z, q^{-\theta}z) - 2\alpha(q^\theta z, z) - 2\alpha(q^{-\theta}z, z)] - \exp[2\mathbf{h} \sum_i \left( \frac{1}{1 + q^{-\theta}e^i(z)}((q^\theta - 1)(T + U)e_i)_{R(\alpha)}(z))(k^{-}(z), k^{-}(q^{-\theta}z))].$$

$$\cdots$$
$q_m(z, w)$ has the expansion

$$q_m(z, w) = i_m(z, w) \frac{q^d z - w}{z - w},$$

with $i_m(z, w)$ in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[h]] \times$.

**Proof.** $(k_m(z), \tilde{e}(w))$ is equal to $(k_a(z), \tilde{e}(w))^{-1}(k^-(z), \tilde{e}(w))$. We have already seen that $(k_a(z), \tilde{e}(w)) = (k_a(z), \tilde{f}(w))^{-1}$. Then

$$(k_a(z), \tilde{e}(w)) = (k_a(z), e(q^d w)) = (k_a(z), f(q^d w))^{-1}$$

$$= (k_a(z), \tilde{f}(q^d w)k^-(w)^{-1})^{-1} = (k_a(z), \tilde{f}(w))^{-1}.$$

Therefore $q_m(z, w) = (k_m(z), \tilde{f}(w))^{-1}$ and by (27), we get the statement on $q_m(z, w)$. \(\square\)

Fix $\Pi$ in $K[[h]]$. Let $U$ be an open subset of $\mathbb{C}^g$ and define $\mathcal{F}_U$ as the space of functions $f(\lambda u_1, \ldots, u_n)$ on $U \times (\bar{X} - \pi^{-1}(P))$, symmetric in $(u_1, \ldots, u_n)$ and with transformation properties (2), with $(\lambda^{(0)})$ replaced by $\lambda_a$. For $f$ in $\mathcal{F}_U$, set

$$(T_z^{(\Pi)}f)(\lambda u_1, \ldots, u_n)$$

$$= \Pi(z)a_\lambda(z) \prod_{i=1}^n q_m(z, u_i) f(\lambda_a + h(\frac{1}{1 + q^d \omega_a/\omega}))(z)|u_1, \ldots, u_n)$$

$$+ \Pi(q^{-\theta} z)^{-1} b'_a(z) \kappa(z) \prod_{i=1}^n q_m(q^{-\theta} z, u_i)^{-1} f(\lambda_a + h(\frac{1}{1 + q^d \omega_a/\omega}))(z)|u_1, \ldots, u_n)$$

$$+ \sum_i -\frac{1}{h}\Pi(u_i)G_{2\lambda}(z, q^\theta u_i)q_m(u_i, z) \prod_{j \neq i} q_m(u_i, u_j)$$

$$f(\lambda_a + h(1 + q^\theta \omega_a/\omega))(u_j)|u_1, \ldots, z, \ldots, u_n)$$

$$+ \sum_i \frac{1}{h}\Pi(q^{-\theta} u_i)^{-1} G_{2\lambda}(z, q^{-\theta} u_i) \kappa(u_i)q_m(q^{-\theta} u_i, z)^{-1} \prod_{j \neq i} q_m(q^{-\theta} u_i, u_j)^{-1} \cdot f(\lambda_a + h(1 + q^\theta \omega_a/\omega))(u_i)|u_1, \ldots, z, \ldots, u_n)$$

where $b'_a(z)$ is defined by (30) and in the two last sums, $q_m(u_i, z)$, $q_m(q^{-\theta} u_i, z)$, $q_m(u_i, u_j)$ and $q_m(q^{-\theta} u_i, u_j)$, $j < i$ are continued to the domains $u_i \ll z$ and $u_i \ll u_j$.

**Proposition 9.1.** Assume that $K$ acts by $-2$ on $\mathbb{V}$ and $v$ is such that $h[1]v = -2nv$, $h[\varepsilon]v = 0$ for $\varepsilon$ in $\mathfrak{m}$, and $\tilde{f}^{(1-g+k)}v = 0$ for $k \geq 0$. We have

$$\langle \psi_\lambda[z] \tilde{e}(u_1) \cdots \tilde{e}(u_n)v, \xi \rangle = T_z^{(\Pi)}(\langle \psi_\lambda[\tilde{e}(u_1) \cdots \tilde{e}(u_n)v, \xi \rangle),$$

with $T_z^{(\Pi)}$ defined by (61), and $\Pi(z) = \prod_i \pi_{\alpha_i}(z)$.
Proof. We have
\[
\langle \psi_\lambda | T(z) \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \sum_i \langle \psi_\lambda | \bar{e}(u_1) \cdots [\bar{f}(z), \bar{e}(u_i)] | z \rightarrow R_{2\lambda} \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
+ a_\lambda(z) \langle \psi_\lambda | k_\text{tot}^+(z) \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle + b_\lambda(z) \langle \psi_\lambda | k_\text{tot}^-(z) \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle,
\]
by the invariance of \( \psi_\lambda \).
The sum is equal to
\[
\sum_i - \frac{1}{\hbar} G_{2\lambda}(z, q^\beta u_i) \langle \psi_\lambda | \bar{e}(u_1) \cdots k_\text{tot}^+(u_i) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
+ \frac{1}{\hbar} G_{2\lambda}(z, q^{-\beta} u_i) \exp(2\alpha(q^{-\beta} u_i, q^{-\beta} u_i)) \exp(2\alpha(q^\beta u_i, q^{-\beta} u_i)) \langle \psi_\lambda | \bar{e}(u_1) \cdots k_\text{tot}^-(u_i) \cdots \bar{e}(u_n) v \rangle, \xi \rangle.
\]
Then
\[
\langle \psi_\lambda | k_\text{tot}^+(z) \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \Pi(z) \langle \psi_\lambda + \sum_i b_i e^i(z) | k_m(z) \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \Pi(z) \prod_{i=1}^n (k_m(z), \bar{e}(u_i)) \langle \psi_\lambda + \sum_i b_i e^i(z) | \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \Pi(z) \prod_{i=1}^n q_m(z, u_i) \langle \psi_\lambda + \sum_i b_i e^i(z) | \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle;
\]
the second equality follows from the covariance of \( \psi_\lambda \), the next follows from the fact that \( v \) is \( U_\beta b^{2-\beta} \)-invariant; in the same way
\[
\langle \psi_\lambda | k_\text{tot}^-(z) \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \Pi(q^{-\beta} z)^{-1}(k_a(q^{-\beta} z), k_m(q^{-\beta} z))^{-1} \langle \psi_\lambda - \sum_i b_i e^i(z) q^{-\beta} z | k_m(q^{-\beta} z)^{-1} \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \Pi(q^{-\beta} z)^{-1}(k_a(q^{-\beta} z), k^- q^{-\beta} z)^{-1} \prod_{i=1}^n (k_m(q^{-\beta} z)^{-1} \bar{e}(u_i))^{-1}
\]
\[
\langle \psi_\lambda - \sum_i b_i e^i(z) q^{-\beta} z | \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle
\]
\[
= \Pi(q^{-\beta} z)^{-1}(k_a(q^{-\beta} z), k^- q^{-\beta} z)^{-1} \prod_{i=1}^n q_m(q^{-\beta} z, u_i)^{-1}
\]
\[
\langle \psi_\lambda - \sum_i b_i e^i(z) q^{-\beta} z | \bar{e}(u_1) \cdots \bar{e}(u_n) v \rangle, \xi \rangle.
\]
We have
\[
-2(1 \otimes q^\beta \mathcal{A}) \beta(z, w) \in (R \otimes R)[[\hbar]] - 2\hbar \sum_a \left( \frac{1}{1 + q^{-\beta} \omega_a / \omega} \right) (z) r_a(w)
\]
so
\[
\beta(z, w) \in \hbar \sum_a \left( \frac{1}{1 + q^{-\beta} \omega_a / \omega} \right) (z) r_a(w) + (R \otimes R)[[\hbar]],
\]
therefore
\[ \sum_i b_{ii} e^i(z) = h\left(\frac{1}{1 + q^{-\ell}(\omega_a/\omega)}\right)(z) \]
and the Proposition follows. \hfill \Box

Remark 9. Dependence on \( \alpha \). The operators \( T_z \) depend on the choice of \( \alpha \) through their coefficients \( \kappa(z) \) and \( q_m(z, w) \). Operators \( T_z \) corresponding to different choices \( \alpha \) and \( \alpha' \) are conjugated. When \( \Pi(z) = 1 \), the conjugation is
\[ T_z^{(\alpha)} = M_{\alpha\alpha'} T_z^{(\alpha')} M_{\alpha\alpha}^{-1}, \]
where
\[ (M_{\alpha\alpha'}f)(\lambda_a|u_1, \ldots, u_n) = \prod_{i<j} \exp\{2(\alpha - \alpha')(q^0 u_i, u_j)\} f(\lambda_a|u_1, \ldots, u_n). \]

10. Commuting difference operators

Define \( U_{h,\omega} \mathfrak{g}_{\mathfrak{m}}^{>1-g} \) as the subalgebra of \( U_{h,\omega} \mathfrak{g} \) generated by \( h[1] \), the \( \tilde{h} \), \( \epsilon \in \mathfrak{m} \), and the \( \tilde{f}[z^{1-g+k}], k \geq 0 \).

Let \( \chi_n \) be the character of \( U_{h,\omega} \mathfrak{g}_{\mathfrak{m}}^{>1-g} \) defined by \( \chi_n(h[1]) = -2n \), \( \chi_n(\tilde{h}) = \chi_n(\tilde{f}[z^{1+g+k}]) = 0 \), \( \epsilon \in \mathfrak{m} \), \( k \geq 0 \).

Define \( \mathcal{V}_n \) as the \( U_{h,\omega} \mathfrak{g} \otimes U_{\mathfrak{g}_{\mathfrak{m}}^{>1-g}} \mathbb{C}_{\chi_n} \).

Proposition 10.1. The map \( \iota \) from \( (\mathcal{V}_n)_{U_{h,\omega} \mathfrak{g}^{\text{out}}_{\mathfrak{g}}}_{n} \) to the subspace \( \mathcal{F} \) of \( S^n \mathbb{K}[[\lambda_a - \lambda_a^{(0)}]][[h]] \) formed of the formal functions near \( \lambda_0 \), which can be continued in variables \( u_i \) to functions on \( \tilde{X} - \pi^{-1}(\{P_i\}) \) with transformation properties \( \mathcal{F} \) (with \( \lambda_a^{(0)} \) replaced by \( \lambda_a \)), defined by
\[ \psi_{\lambda_0} \mapsto \langle \psi_{\lambda}, \tilde{e}(u_1) \cdots \tilde{e}(u_n) \rangle, \]
is an isomorphism.

Proof. \( (\mathcal{V}_n)_{U_{h,\omega} \mathfrak{g}^{\text{out}}_{\mathfrak{g}}}_{n} \) is isomorphic to the space of forms \( \phi \) on \( U_{h,\omega} \mathfrak{g} \) such that
\[ \phi(x^{\text{out}} x) = \varepsilon x^{\text{out}} x \phi(x), \]
x \( U_{h,\omega} \mathfrak{g}^{\text{out}}_{\mathfrak{g}} \) and \( \phi(x x^{\text{in}}) = \phi(x) \chi_n(x^{\text{in}}), x^{\text{in}} \in U_{\mathfrak{g}_{\mathfrak{m}}^{>1-g}} \).

From Prop. \( \mathfrak{F} \) follows that the kernel of the product map
\[ \tilde{\pi} : U_{h,\omega} \mathfrak{g}_{\mathfrak{m}}^{\text{out}} \otimes \mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle \otimes U_{\mathfrak{g}_{\mathfrak{m}}^{>1-g}} \rightarrow U_{h,\omega} \mathfrak{g} \]
is spanned by the \( x \tilde{e}[r_{-2\lambda_0}] \otimes y \otimes z - x \otimes \tilde{e}[r_{-2\lambda_0}] y \otimes z, r_{-2\lambda_0} \in R_{-2\lambda_0}, x \) in \( U_{h,\omega} \mathfrak{g}_{\mathfrak{m}}^{\text{out}} \), \( y \) in \( \mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle \), \( z \) in \( U_{\mathfrak{g}_{\mathfrak{m}}^{>1-g}} \). An element of \( \mathcal{F} \) induces a form \( \phi \) on \( \mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle \), that we extend to the left side of \( \mathfrak{F} \) by the rule
\[ \phi(x \otimes y \otimes z) = \varepsilon(x) \phi(y) \chi_n(z). \]
The properties of the elements of \( \mathcal{F} \) imply that \( \phi \) maps \( \ker \tilde{\pi} \) to zero. It follows that \( \iota \) is surjective.

In the same way, if \( \iota(\psi_{\lambda_0}) = 0 \), then the restriction of \( \iota(\psi_{\lambda_0}) \) to \( \mathbb{C} \langle h[r_a], \tilde{e}[\epsilon], \epsilon \in \mathcal{K} \rangle \) is zero, so that \( \psi_{\lambda_0} \) is zero. \hfill \Box
Theorem 10.1. For any $\Pi(z)$ in $\mathcal{K}[[\hbar]]$ and $z$ in $\Spec(\mathcal{K})$, the operators $T_\pi^{(\Pi)}$ defined by (61) form a commuting family of evaluation-difference operators, acting on $S^n(\mathcal{K})[[\lambda_n - \lambda_n^{(0)}]][[\hbar]]$.

When $\Pi(z) = 1$, they form a commuting family of endomorphisms of $\mathcal{F}_U[[\hbar]]$, where $\mathcal{F}_U$ is defined in sect. 9, for $z$ in $X - \{P_0\}$. Set for $\rho = (\rho_\lambda)_{\lambda \in \Spec \mathbb{C}[[\lambda_n - \lambda_n^{(0)}]]}$ a family of elements of $R_{-2\lambda} \cap z^{-N} \mathcal{O}$,

$$ (\hat{f}[\rho]) (\lambda_a | u_1, \ldots, u_{n+1}) $$

$$ = \sum_{i=1}^{n+1} \frac{1}{h} \rho_\lambda (u_i) \Pi(u_i) \prod_{j \neq i} q_m(u_i, u_j) f(\lambda_a + \frac{h}{1 + q^{-\delta} / \omega_a / \omega(u_i)} | u_1, \ldots, u_{n+1}) $$

$$ - \sum_{i=1}^{n+1} \frac{1}{h} \rho_\lambda (q^{-\delta} u_i) \Pi(q^{-\delta} u_i)^{-1} \kappa(u_i) \prod_{j \neq i} q_m(q^{-\delta} u_i, u_j)^{-1} f(\lambda_a - \frac{h}{1 + q^\delta / \omega_a / \omega(u_i)} | u_1, \ldots, u_{n+1}). $$

Then the operators $T_\pi^{(\Pi=1)}$ normalize the $\hat{f}[\rho]$, which means that they preserve the intersection $\cap_{\rho_\lambda \in R_{-2\lambda} \cap z^{-N} \mathcal{O}} \Ker \hat{f}[\rho]$ for any integer $N$.

Proof. When $\Pi = 1$, the operators $T_\pi^{(\Pi)}$ can be identified with the action of $T(z)$ on the space of invariant forms $(\mathcal{V}_n^*) \otimes \mathcal{O}_{\mathbb{C}^n}^\ast$, by Prop. 10.1. Therefore, they preserve this space and commute with each other.

It follows that we have the cancellations of poles

$$ \text{res}_{z=w}[\alpha_\lambda(z)q_m(z, w)dz] = \text{res}_{z=w}[\frac{1}{h} G_{2\lambda}(z, q^\delta w)q_m(w, z)dz], $$

and

$$ \text{res}_{z=w}[\beta_\lambda(z)\kappa(z)q_m(q^{-\delta} z, w)^{-1}dz] + \text{res}_{z=w}[\frac{1}{h} G_{2\lambda}(z, q^{-\delta} w)q_m(q^{-\delta} w, z)^{-1}dz] = 0. $$

These relations imply that when $\Pi$ is arbitrary, $T_\pi^{(\Pi)}$ is a well-defined endomorphism of $S^n(\mathcal{K})[[\hbar]]$.

Set then $\Pi^+(z) = \Pi(z)$, $\Pi^-(z) = \Pi(q^{-\delta} z)^{-1}$, and

$$ [m(\varphi(u_i))f](\lambda_a | u_1, \ldots, u_n) = \varphi(u_i) f(\lambda_a | u_1, \ldots, u_n), $$

for any $\varphi$ in $\mathcal{K}[[\hbar]]$, and

$$ T_\pi^{(\Pi)} = \sum_{\epsilon=+, -} \Pi^\epsilon(z) A_\epsilon^\pi + \sum_{\epsilon=+, -} \sum_{i=1}^{n} m(\Pi^\epsilon(u_i)) \circ C_z^\epsilon(i). $$

Comparison of arguments in $(\lambda_a)$ in the relation $[T_\pi^{(\Pi=1)}, T_\pi^{(\Pi=1)}] = 0$ yields

$$ [A_\epsilon^\pi, A_\epsilon^\pi] = 0, \quad [C_z^\epsilon(i), C_z^\epsilon(j)] = 0 $$

for any $\epsilon, \epsilon'$ and if $i \neq j$ and

$$ [A_\epsilon^\pi, C_w^{\epsilon',(j)}] + C_w^{\epsilon',(j)} C_w^{\epsilon,(j)} = 0. $$
On the other hand, we have
\[ [m(\Pi^\epsilon(u_i)) \circ C_{w}^{\epsilon,(i)}, m(\Pi^{\epsilon'}(u_j)) \circ C_{w}^{\epsilon',(j)}] = m(\Pi^\epsilon(u_i)) \circ m(\Pi^{\epsilon'}(u_j)) \circ [C_{w}^{\epsilon,(i)}, C_{w}^{\epsilon',(j)}] = 0 \]
for any \( \epsilon, \epsilon', i \neq j \), and
\[
[\Pi^\epsilon(z)A_{z}, m(\Pi^{\epsilon'}(u_j)) \circ C_{w}^{\epsilon,(j)}] + m(\Pi^{\epsilon'}(u_j)) \circ C_{w}^{\epsilon',(j)} \circ m(\Pi^\epsilon(u_i)) \circ C_{w}^{\epsilon,(j)} = 0.
\]

Therefore \( [T_{z}^{(\Pi)}, T_{w}^{(\Pi)}] = 0 \).

The statement on \( \hat{f}[\rho] \) follows from the fact that \( \cap_{\rho \lambda \in R_{\lambda \in \mathbb{R}^+}} \mathrm{Ker} \hat{f}[\rho] \) is equal to \( (\mathbb{V}_{n,N})^{u_{\mathfrak{g}}} \), where \( \mathbb{V}_{n,N} \) is the \( U_{h_{\mathfrak{g}}\mathfrak{g}} \)-module \( U_{h_{\mathfrak{g}}\mathfrak{g}} \otimes U_{\mathfrak{g}_{\mathfrak{m}}}^{\geq N} \mathbb{C}_{\chi_{n}} \), \( U_{\mathfrak{g}_{\mathfrak{m}}}^{\geq N} \) is the subalgebra of \( U_{h_{\mathfrak{g}}\mathfrak{g}} \) generated by \( \hat{h}[\epsilon], \epsilon \in \mathfrak{m}, h[1] \) and the \( \hat{f}[z^{k}], k \geq -N \), and \( \chi_{n} \) is the character of this algebra defined by \( \chi_{n}(h[1]) = -2n \), \( \chi_{n}(\hat{h}[\epsilon]) = \chi_{n}(\hat{f}[z^{k}]) = 0 \) for \( k \geq -N \) and \( \epsilon \) in \( \mathfrak{m} \).

**Remark 10.** Write \( k^{+}(q_{20}z)k_{R}(q_{0}z)k_{R}(z)^{-1}k_{a \rightarrow R}(z) = \exp(\sum_{i} h[e_{i}] \rho_{i}(z)) \), with \( \rho_{i}(z) \) in \( \mathcal{K}[[h]] \). If \( \Pi(z) \) has the form \( \exp(\sum_{i} \lambda_{i} \rho_{i}(z)) \), for some \( \lambda_{i} \in \mathbb{C}[[h]] \), then \( T_{z}^{(\Pi)} \) may be interpreted as the action of \( T(z) \) on some space of intertwiners.

### 11. Connection with hypergeometric spaces

In \cite{16}, V. Tarasov and A. Varchenko proved the following result. Let \( W \) be a representation of the Yangian \( Y(\mathfrak{g}_{2}) \) and let \( \xi \) be a vector of \( W \) such that \( l_{21}^{+}(z)\xi = 0 \), and \( l_{ii}^{+}(z)\xi = \pi_{i}(z)\xi, i = 1, 2 \), for \( \pi_{i}(z) \) some formal series.

**Proposition 11.1.** (see \cite{10}). We can express \( (l_{11}^{+}(z) + l_{22}^{+}(z))l_{12}^{+}(u_{1}) \cdots l_{12}^{+}(u_{n})\xi \) in the form
\[
A(z|u_{1}, ..., u_{n})l_{12}^{+}(u_{1}) \cdots l_{12}^{+}(u_{n})\xi + \sum_{i=1}^{n} C^{(i)}(z|u_{1}, ..., u_{n})l_{12}^{+}(u_{1}) \cdots l_{12}^{+}(u_{n})l_{12}^{+}(z) \cdots l_{12}^{+}(u_{n})\xi;
\]
the family of operators acting on symmetric functions of \( (u_{1}, ..., u_{n}) \) defined by
\[
\hat{T}_{z} = A(z|u_{1}, ..., u_{n}) + \sum_{i=1}^{n} C^{(i)}(z|u_{1}, ..., u_{n}) \circ \text{ev}_{z}^{(i)}
\]
is commutative.

In this section, we will show that the operators \( \hat{T}_{z} \) are examples of the operators \( T_{z}^{(\Pi)} \) constructed above.

Let us consider now the case \( X = \mathbb{CP}^{1}, \omega = d\omega \). We have \( \sum_{i} n_{i}P_{i} = 2(\infty) \).
\( U_{h_{\mathfrak{g}}\mathfrak{g}} \) is then a completion of the central extension \( \overline{DY}(\mathfrak{g}_{2}) \) of the double of the Yangian \( Y(\mathfrak{g}_{2}) \) of \( \mathfrak{g}_{2} \). Let \( x[t^{n}], x \in \{e, f, h\}, n \in \mathbb{Z} \) be the “new realizations”
generators of $\mathcal{D}Y(\mathfrak{sl}_2)$ and $l_{ij}[n]$, $1 \leq i, j \leq 2$ and $n \in \mathbb{Z}$ its “matrix elements” generators.

Generators $x[t^n]$ are organized in generating series $e(z), f(z)$ and $k^\pm(z)$, as above; we further split $x(z)$ as the sum $x^+(z) + x^-(z)$, with $x^+(z) = \sum_{n \geq 0} x[t^n] z^{-n-1}$, $x^-(z) = \sum_{n < 0} x[t^n] z^{-n-1}$. Generating series for the $l_{ij}[n]$ are $l^+(z) = \sum_{n \geq 0} l_{ij}[n] z^{-n-1}$, $l^-(z) = \sum_{n < 0} l_{ij}[n] z^{-n-1}$.

We have the relations

$$(z - w + h)e(z)e(w) = (z - w - h)e(w)e(z),$$

$k^+(z)e(w)k^+(z)^{-1} = \frac{z - w + h}{z - w}e(w), \quad k^-(z)e(w)k^-(z)^{-1} = \frac{w - z + hK}{w - z + h(K + 1)}e(w)$,

and

$l^+_{12}(z) = -hk^+(z)^{-1}e^+(z), \quad l^-_{12}(z) = -he^-(z - hK)k^-(z - h)$

(see e.g. [3]). Moreover, we have

$$\bar{e}(z) = k^+(z + h)^{-1}e(z + h). \quad (64)$$

Define $Y^{\geq 0}$ and $Y^{< 0}$ as the subalgebras of $\overline{\mathcal{D}Y}(\mathfrak{sl}_2)$ generated the $x[t^n], n \geq 0$ (resp. by the $x[t^n], n < 0$). Let $\hat{Y}^{< 0}$ be the subalgebra generated by $K$ and $Y^{< 0}$. Then $U_h \mathfrak{g}^{aut}$ is equal to $Y^{\geq 0}$. Define $\mathcal{V}$ as the Weyl module $\overline{\mathcal{D}Y}(\mathfrak{sl}_2) \otimes_{\hat{Y}^{< 0}} \mathbb{C}_{-2}$, where $\mathbb{C}_{-2}$ is one-dimensional module over $\hat{Y}^{< 0}$ where all the generators act by zero, except for $K$, which acts by $-2$.

Let $\zeta_i$ be points of $\mathbb{C}$ and $V_i(\zeta_i)$ be evaluation modules over $Y^{\geq 0}$ associated with these points; $V_i$ is $(2\Lambda_i + 1)$-dimensional. Define $V$ as the tensor product (for the usual comultiplication of $Y^{\geq 0}$) of the $V_i(\zeta_i)$. Let $\psi$ be some $Y^{\geq 0}$-module map from $\mathcal{V}$ to $V$. We will view $V^*$ as a $Y^{\geq 0}$-module by the rule

$$\langle a\alpha, v \rangle = \langle \alpha, S(a)v \rangle$$

for $a$ in $Y^{\geq 0}$, $v$ in $V$ and $\alpha$ in $V^*$, where $S$ is the antipode of $Y^{\geq 0}$.

Let $\xi$ be a highest weight linear form as in Prop. 7.1 and let $\Omega$ be any vector of $\mathcal{V}$ annihilated by the $e^-(z)$ (for example, $\Omega$ could be the vector $1 \otimes 1$ of $\mathcal{V}$). We have

$$\langle \xi, k^+(z)v \rangle = \pi_V(z)\langle \xi, v \rangle,$$

with

$$\pi_V(z)\pi_V(z + h) = \prod_i \frac{\zeta_i - z + h(2\Lambda_i + 1)}{\zeta_i - z}$$

(see [3]).

**Lemma 11.1.** Let $\tilde{\zeta}$ be any linear form of $\mathcal{V}$ such that

$$\langle \tilde{\zeta}, k^+(z)v \rangle = \pi(z)\langle \tilde{\zeta}, v \rangle, \quad (65)$$
for any \(v\) in \(\mathbb{V}\) and some \(\pi(z)\) in \(\mathbb{C}[[z^{-1}]]\). Then we have

\[
\langle \tilde{\xi}, e(z_1) \cdots e(z_n) \rangle = \frac{1}{(-h)^n} \prod_{i<j} \frac{z_j - z_i}{z_j - z_i - h} \pi(z_1) \cdots \pi(z_n) \langle \tilde{\xi}, l_{12}^+(z_1) \cdots l_{12}^+(z_n) \rangle \tag{66}
\]

(identity in \(\mathbb{C}((z_1)) \cdots ((z_n))\)). In particular, we have

\[
\langle \psi(e(z_1) \cdots e(z_n)) \rangle = \frac{1}{h^n} \prod_{i<j} \frac{z_j - z_i}{z_j - z_i - h} \pi_V(z_1) \cdots \pi_V(z_n) \langle \psi(\Omega), l_{12}^+(z_1 - h) \cdots l_{12}^+(z_n - h) \rangle \tag{67}
\]

Proof. We proceed by induction. For \(n = 0\), the statement is trivial. Assume we have proved it at step \(n\) and let us try to prove it at step \(n + 1\). Apply the statement of step \(n\) for \(\tilde{\xi}' = \tilde{\xi} \circ e(z_0)\). \(\tilde{\xi}'\) satisfies (65) with \(\pi(z)\) replaced by \(\pi(z) = \frac{z - z_0}{z - z_0 - h}\). Therefore, we have

\[
\langle \tilde{\xi}, e(z_0) \cdots e(z_n) \rangle = \langle \tilde{\xi}', e(z_1) \cdots e(z_n) \rangle
\]

\[
= \frac{1}{(-h)^n} \prod_{0 \leq i < j \leq n} \frac{z_j - z_i}{z_j - z_i - h} \pi(z_1) \cdots \pi(z_n) \langle \tilde{\xi}', l_{12}^+(z_1) \cdots l_{12}^+(z_n) \rangle
\]

\[
= \frac{1}{h^n} \prod_{0 \leq i < j \leq n} \frac{z_j - z_i}{z_j - z_i - h} \langle \tilde{\xi}, e(z_0) l_{12}^+(z_1) \cdots l_{12}^+(z_n) \rangle.
\]

Now

\[
\langle \tilde{\xi}, e(z_0) l_{12}^+(z_1) \cdots l_{12}^+(z_n) \rangle
\]

\[
= -\frac{1}{h} \langle \tilde{\xi}, (k^+(z_0) l_{12}^+(z_0)) l_{12}^+(z_1) \cdots l_{12}^+(z_n) \rangle
\]

\[
= -\frac{1}{h} \pi(z_0) \langle \tilde{\xi}, l_{12}^+(z_0) \cdots l_{12}^+(z_n) \rangle + \langle \tilde{\xi}, k^-(z_0) l_{12}^+(z_1) \cdots l_{12}^+(z_n) l_{12}^+(z_0) \rangle.
\]

because \(l_{12}^+(z_0)\) commutes with the \(l_{12}^+(z_i)\). Since \(e^-(z_0) \Omega = 0\), we have \(l_{12}^+(z_0) \Omega = 0\), which proves (66) at step \(n + 1\). This shows (67).

Let us now show how (67) can be derived from (66). Let us set \(\tilde{\xi}(v) = \langle \xi, \psi(v) \rangle\). Then we have (66) with \(\pi(z) = \pi_V(z)\). Then

\[
\langle \psi(e(z_1) \cdots e(z_n)) \rangle = \frac{1}{(-h)^n} \prod_{i<j} \frac{z_j - z_i}{z_j - z_i - h} \pi_V(z_1) \cdots \pi_V(z_n) \langle \psi(l_{12}^+(z_1) \cdots l_{12}^+(z_n) \rangle, \xi
\]

\[
= \frac{1}{(-h)^n} \prod_{i<j} \frac{z_j - z_i}{z_j - z_i - h} \pi_V(z_1) \cdots \pi_V(z_n) \langle l_{12}^+(z_1) \cdots l_{12}^+(z_n) \psi(\Omega) \rangle, \xi
\]

\[
= \frac{1}{h^n} \prod_{i<j} \frac{z_j - z_i}{z_j - z_i - h} \pi_V(z_1) \cdots \pi_V(z_n) \langle \psi(\Omega), l_{12}^+(z_1 - h) \cdots l_{12}^+(z_n - h) \rangle, \xi.
\]
Corollary 11.1. We have

\[ \langle \psi[\bar{e}(z_1) \cdots \bar{e}(z_n)\Omega], \xi \rangle = \frac{1}{\hbar^n} \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle. \]

Proof. We have

\[
\begin{align*}
\langle \psi[\bar{e}(z_1) \cdots \bar{e}(z_n)\Omega], \xi \rangle & = \langle \psi[k^+(z_1 + \hbar)^{-1}e(z_1 + \hbar) \cdots k^+(z_n + \hbar)^{-1}e(z_n + \hbar)\Omega], \xi \rangle \\
& = \prod_{i<j} \langle e(z_i), k^+(z_j)^{-1} \rangle \prod_i \pi_1(\hbar(z_i + \hbar)^{-1} e(z_i + \hbar)\Omega), \xi \rangle \\
& = \frac{1}{\hbar^n} \prod_{i<j} (\hbar(z_i), k^+(z_j)^{-1}) \prod_{i<j} \frac{z_j - z_i}{z_j - z_i - \hbar} \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \\
& = \frac{1}{\hbar^n} \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle,
\end{align*}
\]

where the first equality follows from (64), the second from the commutation rules, and because of (68) is equal to

\[ \langle \psi[\bar{e}(z_1) \cdots \bar{e}(z_n)\Omega], \xi \rangle \]

(see [14].)

On the other hand, we have

\[ \langle \psi(T(z)\bar{e}(z_1) \cdots \bar{e}(z_n)v), \xi \rangle = \langle \psi(\bar{e}(z_1) \cdots \bar{e}(z_n)T(z)v), \xi \rangle \] (by centrality of \( T(z) \))

\[ = \frac{1}{\hbar^n} \langle \psi(T(z)\Omega), l_{12}^+(z_1 + \hbar) \cdots l_{12}^+(z_n + \hbar)\xi \rangle \]

(by Lemma [11.1] above and because \( T(z)\Omega \) is killed by \( e^{-i} \)).

Set \( L^\pm(z) = (l_{12}^\pm(z))_{1 \leq i,j \leq 2} \), then we have \( T(z) = \text{tr} L^+(z) L^-(z - 2\hbar)(\text{see [14]}) \). But since \( l_{12}^\pm(z)v = \delta_{ij}v \), we get \( T(z)v = (l_{11}^+(z) + l_{22}^+(z))v \); therefore the right side of (68) is equal to

\[ \frac{1}{\hbar^n} \langle \psi[(l_{11}^+(z) + l_{22}^+(z))\Omega], l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \]

\[ = \frac{1}{\hbar^n} \langle \psi(\Omega), (l_{11}^+(z) + l_{22}^+(z))l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \]

\[ = \frac{1}{\hbar^n} \hat{T}_z \{ \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \} \]

by Prop. [11.1] On the other hand, \( \langle \psi[T(z)\bar{e}(z_1) \cdots \bar{e}(z_n)\Omega], \xi \rangle \) is equal to

\[ T_z^{(H)} \{ \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \} = \frac{1}{\hbar^n} T_z^{(H)} \{ \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \} \]

by Thm. [10.1]. Since any symmetric polynomial can be realized as a correlation function \( \langle \psi(\Omega), l_{12}^+(z_1) \cdots l_{12}^+(z_n)\xi \rangle \), we have shown:
Proposition 11.2. The operators $T_z^{\Pi}$ and $\hat{T}_z$ are equal.

This fact can also be verified by direct computation.

Remark 11. Elliptic case. In the elliptic case, and when there is no $z_i$, $T_z^{\Pi=1}$ is independent on $z$ and coincides with the $q$-Lamé operator:

$$h\theta(h)(T_z \varphi)(\lambda) = \frac{\theta(2\lambda - h)}{\theta(2\lambda)} \varphi(\lambda - \frac{h}{2}) + \frac{\theta(2\lambda + h)}{\theta(2\lambda)} \varphi(\lambda + \frac{h}{2}).$$

It should be possible to obtain the $q$-Lamé operator for $m > 1$ with other $\Pi$.

Appendix A. Delta-function identities

Lemma A.1. We have

$$q_-(z, w)^{-1} - q_+(z, w)^{-1} = \sigma(z) \delta(q^{-\partial} z, w)$$

and

$$q_-(q^{-\partial} z, w) - q_+(q^{-\partial} z, w) = -\sigma(z) \delta(q^{-\partial} z, w),$$

with $\sigma$ defined by (43).

$\sigma$ has also the expression

$$\sigma(q^\partial z) = \left[ e^{2 \sum_i (U_i e_i)(z) \otimes e^i(w)} e^{-\phi(-h, \partial^i_\lambda \gamma)} \delta(-h, \partial^i_\lambda \gamma) \right]_{w=z},$$

Proof. From (3) follows that

$$q_-(q^\partial z, w)^{-1} - q_+(q^\partial z, w)^{-1} = -e^{-2 \sum_i (q^\partial U_i e_i)(z) \otimes e^i(w)} e^{-\phi(h, \partial^i_\lambda \gamma)} \delta(h, \partial^i_\lambda \gamma) \delta(z, w),$$

so that (69) follows, with $\sigma$ given by (43).

Recall that we have

$$q(z, w) = i(z, w) \frac{q^{-\partial} z - w}{z - q^{-\partial} w},$$

with $i(z, w)$ in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[h]]^\times$ such that $i(z, w) i(w, z) = 1$ (8, Prop. 3.1). We have seen that

$$q_-(z, w) = i_+(z, w) \frac{q^{-\partial} z - w}{z - q^{-\partial} w},$$

with $i_+(z, w)$ in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}][[h]]^\times$. Moreover, $i_+(z, w)$ satisfies

$$i_+(z, w) i_+(q^\partial z, w) = \frac{q^\partial z - w}{z - q^{-\partial} w} i(z, w).$$

On the other hand, we have

$$q_+(z, w) = i_+(z, w) \frac{w - q^{-\partial} z}{w - z},$$

so that

$$q_-(q^\partial z, w)^{-1} - q_+(q^\partial z, w)^{-1} = i_+(q^\partial z, z)(q^\partial z - z) \delta(z - w),$$
and
\[ q_-(z, w) - q_+(z, w) = i_+(z, z)(q^{-\partial}z - z)\delta(z - w), \]
with \( \delta(z - w) = \sum_{i \in \mathbb{Z}} z^i w^{-i-1} \). From \( i(z, z) = 1 \) and (72) follows that the prefactors of \( \delta(z - w) \) in both equations are opposite to each other. (70) follows.

On the other hand, we have
\[ q_+(z, w) = q^2 \sum_i (U_i e_i(z) \otimes e^i(w)) e^{-\phi(h, \partial_z^i \gamma)} (1 + G^{(21)}(z, w)\psi(-h, \partial_z^i \gamma)), \]
so that
\[ q_-(z, w) - q_+(z, w) = \rho(q^\partial z)\delta(z, w). \]
with \( \rho \) given by
\[ \rho(q^\partial z) = \left[ -e^{2 \sum_i (U_i e_i(z) \otimes e^i(w)) e^{-\phi(h, \partial_z^i \gamma)} \psi(-h, \partial_z^i \gamma)} \right]_{w=z}; \]
since \( \rho(q^\partial z) \) is equal to \(-\sigma(q^\partial z)\), we get expression (71) for \( \sigma \).

**Lemma A.2.** We have
\[ q_+(z, w)^{-1}G(w, z) + q_-(z, w)^{-1}G(z, w) = \alpha(z)\delta(q^{-\partial}z, w), \]
(73)
\[ q_+(z, w)G(w, q^{-\partial}z) + q_-(z, w)G(q^{-\partial}z, w) = \beta(q^\partial z)\delta(z, w), \]
(74)
with \( \alpha \) and \( \beta \) defined by (72) and (73). \( \beta \) has also the expression
\[ \beta(q^\partial z) = \left[ \partial_h [e^{-\phi(-h, \partial_z^i \gamma)} \psi(-h, \partial_z^i \gamma)] e^{2 \sum_i (U_i e_i(z) \otimes e^i(w))} \right]_{w=z}. \]
(75)

**Proof.** Let us prove (73). Applying \( q^\partial \otimes 1 \) to this equation, we write it as
\[ q_+(q^\partial z, w)^{-1}G(w, q^\partial z) + q_-(q^\partial z, w)^{-1}G(q^\partial z, w) = \alpha(q^\partial z)\delta(z, w). \]

We have
\[ q_+(q^\partial z, w)^{-1} = e^{-q^\partial U_i e_i(z) \otimes e^i(w)} e^{\sum_i \frac{1 - e^{q^\partial e_i(z) \otimes e^i(w)}}{q}}. \]
From sect. 2.1 follows that
\[ e^{\sum_i \frac{1 - e^{q^\partial e_i(z) \otimes e^i(w)}}{q}} = e^{-\phi(-h, (-\partial_z^i \gamma)) (1 - G^{(21)} \psi(-h, (-\partial_z^i \gamma))) (1 + G^{(21)} \psi(h, \partial_z^i \gamma))}. \]
(76)
\[ q_+(q^\partial z, w)^{-1}G^{(21)}(q^\partial z, w) \] is equal to
\[ q^{2q^\partial (T_r + U_i e_i(z) \otimes e^i(w))} G^{(21)}(q^\partial z, w) \]
\[ = e^{\sum_i \frac{1 - e^{q^\partial U_i e_i(z) \otimes e^i(w)}}{q}} G^{(21)}(q^\partial z, w) \]
\[ = -e^{-2 \sum_i e^{q^\partial U_i e_i(z) \otimes e^i(w)} \partial_h (e^{\sum_i \frac{1 - e^{q^\partial e_i(z) \otimes e^i(w)}}{q}}) \]
\[ = -e^{-2 \sum_i e^{q^\partial U_i e_i(z) \otimes e^i(w)} \partial_h [e^{-\phi(h, \partial_z^i \gamma)} (1 + G^{(21)} \psi(h, \partial_z^i \gamma))]}. \]
Differentiating \((77)\) with respect to \(\hbar\)

We have

\[
q_+(q^\theta z, w)^{-1} G^{(21)}(q^\theta z, w) - q_-(q^{-\theta} z, w)^{-1} G(q^\theta z, w) = -e^{-2 \sum_i q^\theta U_+ e_i(z) \otimes e^i(w)} \partial_\hbar [e^{-\phi(h, \partial^i z \gamma)} \psi(h, \partial^i z \gamma)] \delta(z, w).
\]

Therefore

\[
\alpha(q^\theta z) = \left[ -e^{-\sum_i q^\theta U_+ e_i(z) \otimes e^i(w)} \partial_\hbar [e^{-\phi(h, \partial^i z \gamma)} \psi(h, \partial^i z \gamma)] \right] \bigg|_{z=w}.
\]

Let us now prove \((74)\). From sect. \(2.1\) follows that

Using Lemmas \(A.2\) and \(A.1\), we find

\[
\text{Proof.}
\]

Differentiating \((77)\) with respect to \(\hbar\), we find

\[
G^{(21)}(q^{-\theta} z, w) e^{\sum_i \left( \frac{1 - e^{-\theta}}{\theta} - e^{-\theta} \right) e_i(z) \otimes e^i(w)} = \partial_\hbar [e^{-\phi(h, (-\partial_z)^i \gamma)} (1 - G^{(21)}(\psi(h, (-\partial_z)^i \gamma))].
\]

Therefore, we have also

\[
G(q^{-\theta} z, w) [e^{\sum_i \left( \frac{1 - e^{-\theta}}{\theta} - e^{-\theta} \right) e_i(z) \otimes e^i(w)}]_{w=\gamma} = \partial_\hbar [e^{-\phi(h, (-\partial_z)^i \gamma)} (1 - G^{(21)}(\psi(h, (-\partial_z)^i \gamma))].
\]

so that

\[
G^{(21)}(q^{-\theta} z, w) e^{\sum_i \left( \frac{1 - e^{-\theta}}{\theta} - e^{-\theta} \right) e_i(z) \otimes e^i(w)} = \partial_\hbar [e^{-\phi(h, (-\partial_z)^i \gamma)} (1 - G^{(21)}(\psi(h, (-\partial_z)^i \gamma))].
\]

and

\[
G^{(21)}(q^{-\theta} z, w) q_+(z, w) + G(q^{-\theta} z, w) q_-(z, w) = -\partial_\hbar [e^{-\phi(h, (-\partial_z)^i \gamma)} \psi(h, (-\partial_z)^i \gamma)] e^{2 \sum_i (U_+ e_i(z) \otimes e^i(w))} \delta(z, w),
\]

that is \((74)\) with \(\beta\) given by \((74)\). Identity \((74)\) allows then to write \(\beta\) in the form \((74)\). \(\square\)

Lemma A.3. We have

\[
G_{-2\lambda}(q^\theta w, z) q_-(q^{2\theta} w, q^\theta z)^{-1} + G_{2\lambda}(z, q^\theta w) q_+(q^{2\theta} w, q^\theta z)^{-1} = A_\lambda(z) \delta(z, w),
\]

and

\[
G_{-2\lambda}(q^{-\theta} w, z) q_-(q^{\theta} w, q^\theta z)^{-1} + G_{2\lambda}(z, q^{\theta} w) q_+(q^{\theta} w, q^\theta z)^{-1} = B_\lambda(z) \delta(z, w),
\]

where \(A_\lambda(z)\) and \(B_\lambda(z)\) are defined by \((44)\) and \((44)\).

Proof. Using Lemmas \(A.2\) and \(A.1\), we find

\[
A_\lambda(z) = \alpha(q^{2\lambda} z) + [G_{-2\lambda}(q^\theta w, z) - G(q^{2\theta} w, q^\theta z)]_{w=z} \sigma(q^{2\theta} z).
\]

Then \(G(q^\theta w, z) - G(q^{2\theta} w, q^\theta z)\) is equal to

\[
\sum_i e_i \otimes q^i e^i - \sum_i q^\theta e_i \otimes q^{2\theta} e^i;
\]

\[
(78)
\]
the pairing of $R$ with the first component of this tensor gives zero, so that it belongs to $(R \otimes R)[[h]]$ and its pairing with $\lambda$ in $\Lambda$ gives $-q^{2\partial}(q^{-\partial}\lambda)_R$; therefore (78) is equal to

$$-\sum_i e^i \otimes q^{2\partial}(q^{-\partial}e_i)_R.$$ 

so that $A_\lambda(z)$ is given by (16).

In the same way, we find

$$B_\lambda(z) = \beta(q^{2\partial}z) - \sigma(q^{2\partial}z)[G_{-2\lambda}(q^{-\partial}w,z) - G(w,q^\partial z)]_{w=z}.$$ 

Since

$$G(q^{-\partial}w,z) - G(w,q^\partial z)$$

is equal to $-\sum_i e^i(z)((q^{-\partial}e_i)_R)(z)$, it follows that $B_\lambda(z)$ is given by (17). \qed

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B.E.: Centre de Mathématiques, Ecole Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France
FIM, ETH-Zentrum, HG G46, CH-8092 Zurich, Switzerland
G.F.: D-Math, ETH-Zentrum, HG G44, CH-8092 Zurich, Switzerland