INFINITE DIMENSIONAL SYMPLECTIC CAPACITY AND NONSQUEEZING PROPERTY FOR THE ZAKHAROV SYSTEM ON THE 1-DIMENSIONAL TORUS

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Abstract. We prove the invariant of the symplectic capacity for the Zakharov system on a torus. If the Zakharov solution map is well-defined, then it can be regarded as a symplectomorphism. Thus, we first show the global well-posedness via the local well-posedness and the conservation law. The invariant of the symplectic capacity can be obtained using an approximation method. Many authors use an approximation method to obtain the nonsqueezing theorem, instead of an invariant of the symplectic capacity. However, the conditions of the Hamiltonian system introduced by Kuksin can be relaxed by a new modified infinite dimensional Hamiltonian system. Thus we can back to the symplectic capacity which contains the nonsqueezing property. Heuristically, we obtain the invariant by using the Hamiltonian system which has linear flow at high frequencies and nonlinear flow at low frequencies.

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1. INTRODUCTION

In this paper, we consider the Zakharov system

\[
\begin{align*}
    i\partial_t u + \alpha \partial_x^2 u &= un, & (t, x) &\in \mathbb{R} \times \mathbb{T}, \\
    \beta^{-2} \partial_t^2 n - \partial_x^2 n &= \partial_x^2 (|u|^2), & (t, x) &\in \mathbb{R} \times \mathbb{T}, \\
    (u, n, \partial_t n)|_{t=0} &= (u_0(x), n_0(x), n_1(x)) \in L^2_x \times H^{-1/2}_x \times H^{-3/2}_x, & x &\in \mathbb{T} := \mathbb{R}/2\pi \mathbb{Z},
\end{align*}
\]

(1.1)

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\[ M \left[ u \right] (t) = \int_{T} \left| u(t) \right|^2 dx = \int_{T} \left| u(0) \right|^2 dx = M \left[ u_0 \right], \quad (1.2) \]

and
\[
H \left[ u, n, \partial_t n \right] (t) := H \left[ u, n, \dot{n} \right] (t) = \int_{T} \left( \alpha \left| \partial_x u(t) \right|^2 + \frac{|n(t)|^2}{2} + \frac{\beta^2 |i\partial_x^{-1}n|^2}{2} + n(t) |u(t)|^2 \right) dx = H \left[ u_0, n_0, n_1 \right]. \quad (1.3)
\]

The first (1.2) is called the mass conservation law and the second (1.3) is called the Hamiltonian. They are important tools for showing global well-posedness and to define the symplectic capacity, respectively. From (1.1), we have
\[
\frac{d^2}{dt^2} \int_{T} n(t, x) \, dx = \beta^2 \int_{T} \partial_x^2 \left( n + |u|^2 \right) dx = 0,
\]
and so
\[
\int_{T} n(t, x) \, dx = c_1 t + c_0,
\]
where
\[
c_0 = \int_{T} n_0(x) \, dx \quad \text{and} \quad c_1 = \int_{T} n_1(x) \, dx.
\]

Hence, we denote
\[
u'(t, x) = \exp \left[ i \left( \frac{1}{4\pi} c_1 t^2 + c_0 t \right) \right] u(t, x) \quad \text{and} \quad n'(t, x) = n(t, x) - \frac{1}{2\pi} (c_1 t + c_0), \quad (1.4)
\]
then \( u' \) and \( n' \) are also the solutions to (1.1), and
\[
\int_{T} n'(t, x) \, dx = 0 \quad \text{and} \quad \int_{T} \partial_t n'(t, x) \, dx = 0.
\]

If initial data \( n_0(x), n_1(x) \) have general mean, then one can easily change the data into the mean zero data by (1.4). Therefore, it will be convenient to work in the case when initial data \( n_0(x), n_1(x) \) have mean zero.

The system (1.1) was introduced by Zakharov [26]. It represents the propagation of Langmuir turbulence waves in unmagnetized ionized plasma [26]. In the system, \( u(t, x) \) expresses the slowly varying envelope of the electric field and \( n(t, x) \) describes the deviation in ion density from its mean. The constant \( \alpha \) is a dispersion coefficient and the constant \( \beta \) is the speed of an ion acoustic wave in plasma.

There are many results for the symplectic capacity and the nonsqueezing theorem for the infinite dimensional Hamiltonian system. The symplectic capacity was introduced by Ekeland and Hofer [8,9] for \( \mathbb{R}^{2n} \), and by Hofer and Zehnder [11,12] for \( 2n \)-dimensional general symplectic manifolds. It was developed from the Darboux width, which was discovered by Gromov [10]. Kuksin [19] was the first contributor of the infinite dimensional symplectic capacity for Hamiltonian Partial Differential Equations (PDEs). Kuksin’s concept, of course, is based on the finite dimensional symplectic capacity which was developed by Hofer and Zehnder. Indeed, Kuksin proved an invariance in the symplectic capacity for particular Hamiltonian flow,
and so he also captured its nonsqueezing property. Furthermore, he introduced an abstract method in which the Hamiltonian flow on the appropriate function space can be regarded as a symplectic map. Although there are results which have applied this condition \cite{19,22}, Kuksin’s condition for solution flow is somewhat strong. Thus, many contributors to this issue have turned to the nonsqueezing theorem for specific equations.

To prove the nonsqueezing results for Hamiltonian PDEs, one of the main steps is to find a ‘good’ truncation. Besides, the given Hamiltonian system turns out to be well-behaved with ‘good’ frequency truncations. There are two techniques for the truncation, the methods of \cite{4} and \cite{7}. In \cite{4}, Bourgain proved the nonsqueezing theorem of the cubic nonlinear Schrödinger equation (NLS) in its phase space $L^2_x(\mathbb{T})$ space. A sharp frequency truncation and the $X^{s,b}$ space were used to approximate the original solution. Later, this argument was extended by Colliander et al. \cite{7} for the KdV equation in its phase space $H^{-1/2}_x(\mathbb{T})$. The argument in \cite{7} is more complex than the one in \cite{4}. They used a smooth truncation, and also used the Miura transform which changes the KdV flow to a mKdV flow. Indeed, they showed an approximation using truncated mKdV flow and used Miura transform and its inverse. In this way, they obtained the estimate for the KdV flow. We use the methods of Bourgain \cite{4} instead of the method of Colliander et al. \cite{7}, because the modulation effects from the non-resonant interaction of (1.1) is better than that of the KdV equation. In Section \ref{sec:main} we show the bilinear estimates produced by these modulation effects and a similar calculation in \cite{6,23}. Specifically, bilinear estimates are needed to approximate the truncated solution flow, and this is stronger than the estimates of \cite{23} to prove the local well-posedness. Hong and Kwak \cite{13} extended the result to the higher-order KdV equation, and Mendelson \cite{21} also showed the nonsqueezing of the Klein-Gordon flow on $\mathbb{T}^3$ via a probabilistic approach. Moreover, Kwak \cite{20} proved the nonsqueezing and the local well-posedness for the fourth-order cubic nonlinear Schrödinger equation on a torus. Recently, Killip et al. \cite{17,18} proved the nonsqueezing theorem of the cubic NLS equation on a real line and a plane, respectively. These results are the first nonsqueezing study for an unbounded domain.

Nevertheless, we want to go back to the ‘capacity’ beyond ‘nonsqueezing.’ There are some results that are independent of the nonsqueezing theorem. For example, Abbondandolo and Majer \cite{1} constructed the symplectic capacity on a convex set in the Hilbert space without the approximation approach. However, we focus on the relaxation of Kuksin’s condition. As a result, we obtain a symplectic capacity for the Zakharov system flow which does not satisfy Kuksin’s condition. In particular, there is no nonsqueezing result associated with the Zakharov flow. Moreover, we do not even know the global well-posedness for the symplectic Hilbert space $L^2_x(\mathbb{T}) \times H^{-\frac{1}{2}}_x(\mathbb{T}) \times H^{-\frac{3}{2}}_x(\mathbb{T})$. We use the appropriate frequency truncation and approximate the finite dimensional solution to the original infinite dimensional solution, preserving the symplectic form. These are nontrivial facts, because the nonlinear terms in the Zakharov system does not satisfy Kuksin’s results. To overcome these obstacles, we need to prove that the frequency truncated solution flow well-approximates to the original solution flow. In addition, the truncated flow should be a Hamiltonian flow. We now introduce the main result.
Theorem 1.1. Assume that $\frac{\beta}{\alpha}$ is not an integer. Let $Z(T)$ be the Zakharov flow map at time $T$. For any bounded domain $O$ in $L^2_x \times H^{-\frac{1}{2}}_x \times H^{-\frac{3}{2}}_x$, we have
\[
\text{cap}(O) = \text{cap}(Z(T)(O))
\]
where $\text{cap}(\cdot)$ is the infinite dimensional symplectic capacity.

For the solution map to exist as the symplectic map for any $T > 0$, we should have the global well-posedness in the phase space as follows.

Theorem 1.2. Assume that $\frac{\beta}{\alpha}$ is not an integer. The initial value problem (1.1) is globally well-posed for any $(u_0, n_0, n_1) \in L^2_x(T) \times H^{\frac{1}{2}}_x(T) \times H^{-\frac{3}{2}}_x(T)$.

Theorem 1.2 can be proved by combining the local well-posedness with the mass conservation of $u$ (1.2). The details are in Section 5. It is the Duhamel’s formula for (1.1) which can be written as follows,
\[
u(t) := S(t)(u_0, n_0, n_1) = U(t)u_0 - i \int_0^t U(t-s)[un](s)ds, \quad (1.5)
\]
\[
n(t) := W(t)(u_0, n_0, n_1) = \partial_t V(t)n_0 + V(t)n_1 + \beta^2 \int_0^t V(t-s)\partial_x^2 |u|^2(s)ds, \quad (1.6)
\]
where $U(t) = e^{i\omega t}\partial_x^2$ and $V(t) = \frac{\sin(\beta(-\partial_x^2)^{1/2})}{\beta(-\partial_x^2)^{1/2}} = \frac{\sin(\beta\sqrt{-\Delta})}{\beta\sqrt{-\Delta}}$. We denote the solution to (1.1) by
\[
\mathbf{z}(t,x) = (u(t,x), n(t,x), \partial_t n(t,x)) = S(t)(u_0(x), n_0(x), n_1(x)) \times W(t)(u_0, n_0, n_1) = \partial_t W(t)(u_0, n_0, n_1).
\]
Thus, we also have $Z(t)$ as the solution flow to (1.1). In the same way as here, we will use bold fonts to present vectors in the appropriate space. The spatial Sobolev space is given by
\[
\|u\|_{H^s_x} = \|\langle k \rangle^s\hat{u}\|_{L^2_k} := \frac{1}{(2\pi)^{1/2}} \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{u}|^2 \right)^{1/2}
\]
for $s \in \mathbb{R}$, where $\langle k \rangle = \left(1 + |k|^2\right)^{1/2}$. Let $\mathcal{H}$ be the symplectic Hilbert space $L^2_x(T) \times H^{-\frac{1}{2}}_x(T) \times H^{-\frac{3}{2}}_x(T)$, and
\[
\|(u, v, w)\|_{\mathcal{H}} = \|u\|_{L^2_x} + \|v\|_{H^{-\frac{1}{2}}_x} + \|w\|_{H^{-\frac{3}{2}}_x}.
\]
We also define the absolute value in $\mathcal{H}$ by
\[
|\langle \tilde{u}_{k_0}, \tilde{v}_{k_0}, \tilde{w}_{k_0}\rangle|_{\mathcal{H}} = |\tilde{u}_{k_0}| + |k_0|^{-\frac{1}{2}}|\tilde{v}_{k_0}| + |k_0|^{-\frac{3}{2}}|\tilde{w}_{k_0}|
\]
for fixed frequency component $k_0$.

From Theorem 1.1 we can consider the nonsqueezing theorem of the Zakharov system as well. We first define a ball and a cylinder in the function space $\mathcal{H}$. 
**Definition 1.3.** Let $B_R^\infty (v_*)$ be an infinite dimensional ball in $\mathcal{H}$ which has the radius $R$ and is centered at $v_* \in \mathcal{H}$. That is,

$$B_R^\infty (v_*) := \{ v \in \mathcal{H} : \| v - v_* \|_{\mathcal{H}} \leq R \}.$$  

For any $k \in \mathbb{Z} \setminus \{ 0 \} := \mathbb{Z}^*$, $C_{k,r}^\infty (\eta)$ is defined an infinite dimensional $k$-th cylinder in $\mathcal{H}$ which has the radius $r$ and is centered at $\eta \in \mathbb{C}^3$. That is,

$$C_{k,r}^\infty (\eta) := \{ v \in \mathcal{H} : |\hat{v}_k - \eta|_{\mathcal{H}} \leq r \}$$

where $\hat{v}_k = (\hat{u}_k, \hat{v}_k, \hat{w}_k) \in \mathbb{C}^3$.

The nonsqueezing property of the Zakharov system is as follows,

**Corollary 1.4.** Let $0 < r < R$, $(u^*, n^*, n^{**}) \in L_x^2 \times H_x^{-\frac{1}{2}} \times H_x^{-\frac{3}{2}}$, $k \in \mathbb{Z}^*$, $(z, w_0, w_1) \in \mathbb{C}^3$ and $T > 0$. Then

$$Z (T) (B_R^\infty (u^*, n^*, n^{**})) \not\subseteq C_{k,r}^\infty (z, w_0, w_1).$$

In other words, there is a solution $Z (T) (u_0, n_0, n_1) \in C^t L_x^2 \times C^t H_x^{-\frac{1}{2}} \times C^t H_x^{-\frac{3}{2}}$ to (1.1) and $k_0 \in \mathbb{Z}^*$ such that

$$\| (u_0, n_0, n_1) - (u^*, n^*, n^{**}) \|_{\mathcal{H}} \leq R,$$

and

$$|F_x [(Z (T) (u_0, n_0, n_1))] (k_0) - (z, w_0, w_1) |_{\mathcal{H}} > r$$

where $F_x [\cdot]$ is the spatial Fourier transform.

**Remark 1.5.** There are no smallness conditions imposed on $(u^*, n^*, n^{**})$, $(z, w_0, w_1)$, $R$ or $T$ in Corollary 1.4.

Corollary 1.4, the symplectic nonsqueezing theorem, tells us the Zakharov flow cannot squash a large ball into a narrow cylinder, despite the fact that the cylinder has infinite volume.

### 2. Notations and Function spaces

In this section, we introduce notations to discuss our argument. The spatial Fourier transform, the space-time Fourier transform and the inverse Fourier transform are defined as follows,

$$F (u) = \hat{u} (k, \tau) = \int_{\mathbb{R} \times \mathbb{T}} e^{-ikx} e^{-i\tau \tau} u (t, x) dx dt,$$

$$F (u, v, w) = (\hat{u}, \hat{v}, \hat{w}),$$

$$F_x (u) = \hat{u}_k = \int_{\mathbb{T}} e^{-ikx} u (x) dx,$$

$$F_x (u, v, w) = (\hat{u}_k, \hat{v}_k, \hat{w}_k),$$

$$u (x) = \int e^{ikx} \hat{u} (k) dk := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \hat{u}_k e^{ikx}.$$
Moreover, respectively. Using the note that the first and the second are associated with Schrödinger flow and wave flow, and observe that
\[ T \in \text{a compact time interval } [0, T]. \]

We give some embeddings for the \( X \) spaces, we define \( Y^a_s, Z^a_s, Y^a_W \) and \( Z^a_W \) spaces for the solution and the nonlinear terms,
\[
\| f \|_{Y^a_S} = \| f \|_{X^{a, 1/2}_S} + \left\| \left\langle k \right\rangle^a \tilde{f}(k, \tau) \right\|_{I^0_k L^1_t}, \\
\| f \|_{Z^a_S} = \| f \|_{X^{a, -1/2}_S} + \left\| \left\langle k \right\rangle^a \tilde{f}(k, \tau) \right\|_{I^0_k L^1_t}, \\
\| f \|_{Y^a_W} = \| f \|_{X^{a, 1/2}_W} + \left\| \left\langle k \right\rangle^a \tilde{f}(k, \tau) \right\|_{I^0_k L^1_t}, \\
\| f \|_{Z^a_W} = \| f \|_{X^{a, -1/2}_W} + \left\| \left\langle k \right\rangle^a \tilde{f}(k, \tau) \right\|_{I^0_k L^1_t}.
\]

we give some embeddings for the \( Y \) and \( Z \) spaces
\[
Y^a_{S,W} \subset C_t H^a_x \subset L^\infty_t H^a_x, \\
L^2_t H^a \subset Z^a_{S,W}.
\]
in a compact time interval \([0, T]\) by the Hölder inequality. To simplify notations, \( \mathcal{Y} \) spaces would be
\[
\|(u, v)\|_{\mathcal{Y}} = \|u\|_{Y^0_S} + \|v\|_{Y^{-1/2}_W}.
\]

For each dyadic number \( N \), we denote the Littlewood-Paley projection by
\[
\hat{P}_N u(k) := 1_{N \leq |k| < 2N} (k) \widehat{u_k}, \\
\hat{P}_{\leq N} u(k) := 1_{|k| \leq N} (k) \widehat{u_k}, \\
\hat{P}_{\geq N} u(k) := 1_{|k| \geq N} (k) \widehat{u_k},
\]
where \( 1_\Omega \) is a characteristic function on \( \Omega \).

We call a connected open set a domain. For an nonempty domain \( O \subset \mathcal{H} \) and \( n \geq 1 \), we denote
\[
O_N = O \cap \bigcap_{n \leq N} \mathcal{H} \left( = \cap_{n \leq N} L^2_t x \times \cap_{n \leq N} H^{1/2}_x \times \cap_{n \leq N} H^{-1/2}_x \right), \\
\partial O_N = O \cap \left( 1 - P_{\leq N} \right) \mathcal{H},
\]
and observe that
\[
\partial O_N \subset \partial O \cap P_{\leq N} \mathcal{H}.
\]

For \( x, y \in \mathbb{R}_+ \), \( x \lesssim y \) denotes \( x \leq Cy \) for some \( C > 0 \) and \( x \sim y \) means \( x \lesssim y \) and \( y \lesssim x \). Using this, we denote \( f = O(g) \) by \( f \lesssim g \) for positive real-valued functions \( f \) and \( g \). Moreover, \( x \ll y \) denotes \( x \leq cy \) for small positive constant \( c \). Let \( a_1, a_2, a_3 \in \mathbb{R} \) and the
quantities $a_{\max} \geq a_{\text{med}} \geq a_{\min}$ can be defined to be the maximum, median and minimum values of $a_1, a_2, a_3$, respectively.

3. Symplectic capacity for Hilbert space

We begin with the definition of the symplectic Hilbert space $\mathcal{H}$. Let $\omega$ be a symplectic form $\mathcal{H}$ as follows,

$$\omega((u,v,w),(\dot{u},\dot{v},\dot{w})) = \omega_1(u,\dot{u}) + \omega_{-1/2}(v,\dot{v}) + \omega_{-3/2}(w,\dot{w})$$

where $\omega_1(f,g) = \text{Im} \int f \bar{g} \, dx$, $\omega_{-1/2}(f,g) = \int f \partial_x^{-1} g \, dx$, and $\omega_{-3/2}(f,g) = \int f \partial_x^{-3} g \, dx$. Let $J$ be an almost complex structure on $\mathcal{H}$ which is compatible with the Hilbert space inner product $\langle \cdot, \cdot \rangle$. In other words, a bounded self adjoint operator with $J^2 = -I$ such that $\omega(u,v) = \langle u, Jv \rangle$ for all $u, v \in \mathcal{H}$. One easily checks that the Zakharov solution can be written in the form

$$\partial_t u(t) := \dot{u}(t) = J \nabla H[u(t)]$$

where $u \in \mathcal{H}$. The notation $\nabla$ in (3.2) denotes the usual gradient with respect to the Hilbert space inner product. Hence, we have

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \nabla H[u(t), \cdot, \cdot] \\ \nabla H[\cdot, n(t), \cdot] \\ \nabla H[\cdot, \cdot, \dot{n}(t)] \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} dH[u(t), \cdot, \cdot] (v_1, \cdot, \cdot) \\ dH[\cdot, n(t), \cdot] (v_1, v_2, \cdot) \\ dH[\cdot, \cdot, \dot{n}(t)] (\cdot, v_1, v_2) \end{bmatrix}.$$

Definition 3.1. Consider a pair $(\mathcal{H}, \omega)$, where $\omega$ is a symplectic form on the Hilbert space $\mathcal{H} = L^2_x(\mathbb{T}) \times H^{1/2}_x(\mathbb{T}) \times H^{3/2}_x(\mathbb{T})$. We say that the pair $(\mathcal{H}, \omega)$ is the symplectic phase space for the Zakharov system.

One easily check that an equivalent way to write the Zakharov system corresponding to the Hamiltonian in (3.3) in $(\mathcal{H}, \omega)$ is

$$\partial_t \begin{bmatrix} u \\ n \\ \dot{n} \end{bmatrix} = \begin{bmatrix} \nabla \omega_1 H[u(t), \cdot, \cdot] \\ \nabla_{\omega_{-1/2}} H[\cdot, n(t), \cdot] \\ \nabla_{\omega_{-3/2}} H[\cdot, \cdot, \dot{n}(t)] \end{bmatrix}$$

where the symplectic gradient is defined an analogy with (3.3),

$$\omega \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \nabla \omega_1 H[u(t), \cdot, \cdot] \\ \nabla_{\omega_{-1/2}} H[\cdot, n(t), \cdot] \\ \nabla_{\omega_{-3/2}} H[\cdot, \cdot, \dot{n}(t)] \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} dH[u(t), \cdot, \cdot] (v_1, \cdot, \cdot) \\ dH[\cdot, n(t), \cdot] (v_1, v_2, \cdot) \\ dH[\cdot, \cdot, \dot{n}(t)] (\cdot, v_1, v_2) \end{bmatrix}.$$

Therefore, we can consider $L^2_x(\mathbb{T}) \times H^{1/2}_x(\mathbb{T}) \times H^{3/2}_x(\mathbb{T})$ as the phase space. In the phase space, we can consider an invariance of the symplectic capacity by the Zakharov solution flow.
In the following, we introduce the infinite dimensional symplectic capacity which was introduced by Kuksin [19]. The symplectic capacity was first discovered by Ekeland and Hofer [8, 9] in $\mathbb{R}^{2n}$ and was developed by Hofer and Zehnder [12]. Specifically, it is a symplectic invariant and the proof of existence is based on the variational principle.

**Definition 3.2 (Symplectic capacity).** A symplectic capacity on the phase space $(\mathcal{H}, \omega)$ with respect to 1.1 is a function $\text{cap} (\cdot)$ defined open subset $O \subset \mathcal{H}$ which takes values in $[0, \infty]$ and has the following properties:

i) **Translational invariant:**

$$\text{cap} (O) = \text{cap} (O + \xi) \quad \text{for } \xi \in \mathcal{H}.$$  

ii) **Monotonicity:** Let $O_1$ and $O_2$ be open sets in $\mathcal{H}$.

$$\text{cap} (O_1) \leq \text{cap} (O_2) \quad \text{if } O_1 \subseteq O_2.$$  

iii) **2-homogeneity:** For $\tau \in \mathbb{R}$,

$$\text{cap} (\tau O) = \tau^2 \text{cap} (O).$$  

iv) **Nontriviality:** For bounded nonempty set $O$,

$$0 < \text{cap} (O) < \infty.$$   

v) **For an $r$-ball $B_\infty^r$ in $\mathcal{H}$ and a $k$-th cylinder $C^\infty_{k,r}$ which has radius $r$ in $\mathcal{H}$,**

$$\text{cap} (B_\infty^r) = \text{cap} (C^\infty_{k,r}) = \pi r^2.$$  

We point out two notable remarks. Combining ii), v) and the invariant of the symplectic capacity, we can get the nonsqueezing theorem for a Hamiltonian flow. Moreover, we note that the symplectic capacity does not determine a unique capacity function. We, thus, have many ways to construct capacity functions, but we follow [19]. To construct a capacity function, we first introduce few definitions.

**Definition 3.3 (Admissible function).** Let $O$ be a simply connected open set in the phase space $\mathcal{H}$ (we call it domain in the sequel). Assume that a function $f$ is a smooth function in $O$, and $m > 0$. The function $f$ is called $m$-admissible if

i) $0 \leq f \leq m$ in $O$,

ii) $f \equiv 0$ in a nonempty subdomain of $O$,

iii) $f|_{\partial O} \equiv m$,

iv) The set $\{f < m\}$ is bounded, and the distance from this set to $\partial O$ is strictly positive, i.e., $\text{dist}_O (f) := \text{dist} (\{f < m\}, \partial O) > 0$.

For each $m$-admissible function, we denote

$$\text{Supp} f = \{u : 0 < f (u) < m\},$$

and so we have

$$\text{dist} (f^{-1} (0), \partial O) \geq \text{dist} (f), \quad (3.5)$$

$$\text{dist} (\text{Supp} f, \partial O) \geq \text{dist} (f). \quad (3.6)$$

Let $N \geq 1$ be integer. By using the Fourier basis and restricting the form $\omega$, we see that the truncated phase space $(P_{\leq N} \mathcal{H}, \omega)$ is a $3 \times 2N$-dimensional real symplectic space, and so is symplectomorphic to the standard phase space $(\mathbb{R}^{2N \times 3}, \omega_0)$ by Darboux theorem.
**Definition 3.4** (Fast function). Let \( f_N = f|_{O_N} \) be a frequency truncated function. We consider the corresponding symplectic Hamiltonian vector field \( V_{f_N} \) in \( O_N \). In other words, for \( u, v \in O_N \),

\[
\omega(v, V_{f_N}(u)) = df_N(u)(v).
\]

We call a periodic trajectory of \( V_{f_N} \) a ‘fast trajectory’ if it does not pass through a stationary point and the period \( T \leq 1 \). Furthermore, a \( m \)-admissible function \( f \) is called ‘fast’ if there exists \( N_0 = N_0(f) \) such that for all \( N \geq N_0 \) the vector field \( V_{f_N} \) has a fast trajectory.

A fast \( m \)-admissible function has a notable property which comes from its definition,

**Lemma 3.5** (Kuksin [19]). All fast periodic trajectories in \( V_{f_N} \) are contained in \( \text{Supp} f \cap P_{\leq N} \). This lemma can be proved by the definition of the fast trajectory and the fact that the derivative of \( f \) is zero in the complementary set of \( \text{Supp} f \). We next introduce a definition of the (infinite dimensional) symplectic capacity.

**Definition 3.6** (Infinite dimensional symplectic capacity, Kuksin [19]). For a nonempty domain \( O \in H \), its symplectic capacity \( \text{cap}(O) \) equals

\[
\inf \{ m_* : \text{each } m \text{-admissible function with } m > m_* \text{ is fast} \}
\]

From [19], we already have that the symplectic capacity \( \text{cap}(\cdot) \) satisfies Definition 3.2.

**4. Basic estimates**

In this section, we show estimates to prove Theorem 1.1 and 1.2. First of all, we recall the lemma regarding the linear estimates.

**Lemma 4.1.** Let \( \Psi(t) \in C_0^\infty(\mathbb{R}) \) such that \( \Psi(t) = 1 \) on \([-1, 1]\) and \( \Psi(t) = 0 \) outside of \([-2, 2]\). We have

\[
\left\| \Psi\left(\frac{t}{T}\right)U(t)u_0 \right\|_{Y_{\beta}^0} \lesssim \|u_0\|_{L_x^2},
\]

\[
\left\| \Psi\left(\frac{t}{T}\right) \int_0^t U(t-s)F(s)ds \right\|_{Y_{\beta}^0} \lesssim \|F(t)\|_{Z_{\beta}^0},
\]

\[
\left\| \Psi\left(\frac{t}{T}\right) \partial_x V(t)n_0 \right\|_{Y_{-\frac{1}{2}}^H} \lesssim \|n_0\|_{H_x^{-\frac{3}{2}}},
\]

\[
\left\| \Psi\left(\frac{t}{T}\right) V(t)n_1 \right\|_{Y_{-\frac{1}{2}}^H} \lesssim \|n_1\|_{H_x^{-\frac{3}{2}}},
\]

\[
\left\| \Psi\left(\frac{t}{T}\right) \int_0^t V(t-s)\partial_x F(s)ds \right\|_{Y_{-\frac{1}{2}}^H} \lesssim \|F(t)\|_{Z_{\beta}^{-1/2}}.
\]

where \( U(t) = e^{i\alpha t\partial_x^2} \) and \( V(t) = \frac{\sin(\beta t\partial_x^{1/2})}{\beta(\partial_x^{1/2})^{1/2}} \).
Lemma 4.1 will be used for the contraction mapping principle, and so it is well-known estimates. For that reason, we omit the proof of Lemma 4.1 but it can be found in [23, 35, 6, 15, 16]. From now on, we discuss bilinear estimates with respect to the Schrödinger and the wave flow. Define the resonant set and nonresonant set as follows,

\[ N_R = \{(k_0, k_1, k_2): k_0 \sim k_1 \sim k_2 \} \]

\[ N_{NR} = (N_R)^C. \]

**Proposition 4.2.** Let \( N_i \) be dyadic numbers, \( N_i \sim |k_i| \) and \( 0 < T < 1 \). Assume that \( \frac{\beta}{\alpha} \) is not an integer,

\[ \| P_{N_0} (P_{N_1} u P_{N_2} v) \|_{Z^0_S([0,T] \times \mathbb{T})} \lesssim T^\gamma \| u \|_{Y^0_S} \| v \|_{Y^{-1/2}_W}, \]

(4.1)

and

\[ \| P_{N_0} \partial_x (P_{N_1} u \overline{P_{N_2} v}) \|_{Z^{-1/2}_W([0,T] \times \mathbb{T})} \lesssim T^\gamma \| u \|_{Y^0_S} \| v \|_{Y^0_S}. \]

(4.2)

for sufficiently small \( \gamma > 0 \).

**Proposition 4.3.** Let \( N_i \) be dyadic numbers and \( N_i \sim |k_i| \). Assume that \( \frac{\beta}{\alpha} \) is not an integer,

\[ \| P_{N_0} (P_{N_1} u P_{N_2} v) \|_{Z^0_S} \lesssim N^{-\delta}_{max} \| u \|_{Y^0_S} \| v \|_{Y^{-1/2}_W}, \]

(4.3)

and

\[ \| P_{N_0} \partial_x (P_{N_1} u \overline{P_{N_2} v}) \|_{Z^{-1/2}_W} \lesssim N^{-\delta}_{max} \| u \|_{Y^0_S} \| v \|_{Y^0_S}. \]

(4.4)

for \( (k_0, k_1, k_2) \in N_{NR} \) and sufficiently small \( \delta > 0 \).

Takaoka [23] proved these type estimates (without dyadic decompositions) using the arguments in [14, 15], but we have to show slightly stronger estimates for the approximation step in the proof of Theorem 1.1. In addition, we prove the propositions using a little simpler argument in [24] than in [14, 15, 23]. Roughly, Proposition 4.3 can be deduced by observation of the intersection of hyperspace \( \tau = \frac{\beta}{\alpha} (k) \) (a function \( h \) be chosen by each equation). We first have the following algebraic results for observation of resonant relations.

**Lemma 4.4.** Let \( \text{sgn}(x) \) be the sign function. Then

i) Let \( \tau_0 = \tau_1 + \tau_2, k_0 = k_1 + k_2 \neq 0 \),

\[ \max \left\{ |\tau_0 - \alpha k_0^2|, |\tau_1 - \alpha k_1^2|, |\tau_2| - \beta |k_2| \right\} \gtrsim |\alpha| \left| k_2 \right| \left| k_0 + k_1 \right| - \frac{\beta}{\alpha} S_1, \]

(4.5)

for \( S_1 = \text{sgn}(\tau_2 k_0) \).

ii) Let \( \tau_0 = \tau_1 - \tau_2, k_0 = k_1 - k_2 \neq 0 \),

\[ \max \left\{ |\tau_0| - \beta |k_0|, |\tau_1 - \alpha k_1^2|, |\tau_2 - \alpha k_2^2| \right\} \gtrsim |\alpha| \left| k_0 \right| \left| k_1 + k_2 \right| - \frac{\beta}{\alpha} S_2, \]

(4.6)

for \( S_2 = \text{sgn}(\tau_0 k_0) \).
Proof. We first prove i). From the support of time and spatial frequencies,

$$\tau_0 - ak_0^2 - \tau_1 + ak_1^2 - \tau_2 + \beta S_1 k_2 = \tau_0 - \tau_1 - \tau_2 + \alpha \left(k_1^2 - k_0^2 - \frac{\beta}{\alpha}S_1k_2\right)$$

$$= -\alpha k_2 \left(k_0 + k_1 - \frac{\beta}{\alpha}S_1\right).$$

Similarly,

$$\tau_0 - \beta S_2 k_0 - \tau_1 + ak_1^2 + \tau_2 - ak_2^2 = \tau_0 - \tau_1 - \tau_2 + \alpha \left(k_1^2 - k_2^2 - \frac{\beta}{\alpha}S_2k_0\right)$$

$$= \alpha k_0 \left(k_1 + k_2 - \frac{\beta}{\alpha}S_1\right).$$

Therefore, we are done. \(\square\)

We can obtain Proposition 4.2 by the similar argument of Proposition 4.3 so we first prove the latter.

Proof of Proposition 4.3. Let \(L_i := \langle \tau_i - ak_i^2 \rangle\) and \(M_i = \langle |\tau_i| - |k_i| \rangle.\) To prove (4.3) and (4.4), we are going to show the \(X^{s,b}\) part and \(t_0^2 L_t^2\) part respectively. However, essential ideas are similar. From [2], we have the Strichartz estimate for the Schrödinger equation,

$$X^{0,3/8}_S \subset L^4_{t,x},$$

which will be used many times in the following calculation. We first consider the \(X^{s,b}\) part of (4.3). The left hand side of (4.3) can be rewritten by the Plancherel theorem, so our claim is

$$\left\| \sum_{k_0 = k_1 + k_2} \int_{t_0 = \tau_1 + \tau_2} \frac{|k_2|^{1/2}}{L_0^{1/2} L_1^{1/2} M_2^{1/2}} \tilde{u} \tilde{v} d\tau_1 \right\| \lesssim N_{\max}^{-\delta} \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}}.$$\hspace{1cm} (4.8)

i) max \(\{L_0, L_1, M_2\} = M_2\)

By (4.5), the left hand side of (4.8) is bounded by

$$\left\| \sum_{k_0 = k_1 + k_2} \int_{t_0 = \tau_1 + \tau_2} \frac{|k_2|^{1/2}}{L_0^{1/2} L_1^{1/2} |\alpha|^{1/2} |k_2|^{1/2} |k_0 + k_1 - \frac{\beta}{\alpha}S_1|^{1/2}} \tilde{u} \tilde{v} d\tau_1 \right\|_{L^2_{t_0}}.$$\hspace{1cm} .

Hence, it is reduced to show that

$$\left\| \sum_{k_0 = k_1 + k_2} \int_{t_0 = \tau_1 + \tau_2} \frac{N_{\max}^{\delta}}{L_0^{1/2} L_1^{1/2} |k_0 + k_1 - \frac{\beta}{\alpha}S_1|^{1/2}} \tilde{u} \tilde{v} d\tau_1 \right\|_{L^2_{t_0}} \lesssim \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}}.$$\hspace{1cm} .


Since \( k_0 = k_1 + k_2 \) and \( (k_0, k_1, k_2) \subset N_{\mathcal{R}} \), we have \( k_{\max} \) and \( k_{\text{med}} \) such that \( N_{\max} \sim k_{\max} \sim k_{\text{med}} \), and they are same sign. Thus, we have

\[
\frac{N_{\max}^\delta}{|k_0 + k_1 - \frac{\beta}{\alpha} S_1|^{1/2}} \lesssim 1
\]

for sufficiently small \( \delta \). By duality, it is enough to show that

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dx dt \right| \lesssim \| u_0 \|_{X_{\alpha}^{0,1/2}} \| u_1 \|_{X_{\alpha}^{0,1/2}} \| u_2 \|_{L_{t,x}^2},
\]

and then by the Hölder inequality and (4.7), we have

\[
\text{and (4.7), we have}
\]

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dx dt \right| \lesssim \| u_0 \|_{X_{\alpha}^{0,1/2}} \| u_1 \|_{X_{\alpha}^{0,1/2}} \| u_2 \|_{L_{t,x}^2}.
\]

ii) \( \max \{ L_0, L_1, M_2 \} = L_i \)

Without loss of generality, we may assume that \( \max \{ L_0, L_1, M_2 \} = L_0 \). From (4.3) again, the left hand side of (4.8) is bounded by

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dx dt \right| \lesssim \| u_0 \|_{L_{t,x}^4} \| u_1 \|_{L_{t,x}^4} \| u_2 \|_{L_{t,x}^2}.
\]

so the claim is

\[
\left| \sum_{k_0 = k_1 + k_2} \int_{\tau_0 = \tau_1 + \tau_2} \frac{N_{\max}^\delta}{\max \{ L_0, L_1, M_2 \}^{1/2}} \tilde{u}\tilde{v} \tilde{d} \tilde{r}_1 \right| \lesssim \| u \|_{L_{t,x}^2} \| v \|_{L_{t,x}^2} \quad (4.9)
\]

Again, we have \( k_0 \) and \( k_1 \) in the former case, but we prove in a different way from i). There are several subcases. We first consider \( |k_0| \sim |k_1| \gg |k_2| \sim N_{\min} \). Since \( k_0 = k_1 + k_2 \), we have that \( k_0 + k_1 - \frac{\beta}{\alpha} S_1 \) is similar to \( N_{\max} \). Thus, the left hand side of (4.9) is bounded by

\[
\left| \sum_{k_0 = k_1 + k_2} \int_{\tau_0 = \tau_1 + \tau_2} \frac{1}{\max \{ L_0, L_1, M_2 \}^{1/2}} \tilde{u}\tilde{v} \tilde{d} \tilde{r}_1 \right| \lesssim \| u \|_{L_{t,x}^2} \| v \|_{L_{t,x}^2}.
\]

By duality, the claim equivalent to

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dx dt \right| \lesssim \| u_0 \|_{L_{t,x}^2} \| u_1 \|_{X_{\alpha}^{0,1/2}} \| u_2 \|_{X_{\alpha}^{1/2-\delta,1/2}}.
\]
We now consider

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dxdt \right| \leq \|u_0\|_{L^2_{t,x}} \|u_1\|_{L^4_{t,x}} \|u_2\|_{L^4_{t,x}} \\
\lesssim \|u_0\|_{X^{0,0}_S} \|u_1\|_{X^{0,2}_S} \|u_2\|_{X^{1/2-\delta}_W},
\]

for sufficiently small \(\delta\). The remaining cases are \(|k_1| \sim |k_2| \gg |k_0|\) or \(|k_0| \sim |k_2| \gg |k_1|\).

Then we have

\[
\frac{N_{\max}^\delta}{|k_0 + k_1 - \frac{\beta}{\alpha} S_1|^{1/2}} \sim \frac{N_{\max}^\delta}{|k_1|^{1/2}} \sim \frac{1}{|k_1|^{1/2-\delta}} \ll \frac{1}{|k_0|^{1/2-\delta}} \quad (4.10)
\]

or

\[
\frac{N_{\max}^\delta}{|k_0 + k_1 - \frac{\beta}{\alpha} S_1|^{1/2}} \sim \frac{N_{\max}^\delta}{|k_0|^{1/2}} \sim \frac{1}{|k_0|^{1/2-\delta}} \quad (4.11)
\]

Thus, the left hand side of (4.9) is bounded by

\[
\left\| \sum_{k_0 = k_1 + k_2} \int_{\tau_0 = \tau_1 + \tau_2} \frac{1}{|k_0|^{1/2-\delta} L_1^{1/2} M_2^{1/2}} \bar{u} \bar{v} d\tau_1 \right\|_{L^2_{k_0} L^2_{\tau_0}} \quad (4.12)
\]

for both cases. By a similar calculation of the former case, we get the goal. More precisely, we need to show that

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dxdt \right| \lesssim \|u_0\|_{L^2_t H^{1/2-\gamma}_x} \|u_1\|_{X^{0,1/2}_S} \|u_2\|_{X^{0,1/2}_W}.
\]

By the Hölder inequality, (4.7), and the Sobolev embedding,

\[
\left| \int_{\mathbb{R} \times T} u_0 u_1 u_2 dxdt \right| \leq \|u_0\|_{L^2_t L^4_x} \|u_1\|_{L^4_{t,x}} \|u_2\|_{L^4_x L^2_t} \\
\lesssim \|u_0\|_{L^2_t H^{1/2-\gamma}_x} \|u_1\|_{X^{0,1/2}_S} \|u_2\|_{X^{0,1/2}_W}.
\]

We now consider \(L^1_{k_t} L^1_{\tau_t}\) part,

\[
\left\| \sum_{k_0 = k_1 + k_2} \int_{\tau_0 = \tau_1 + \tau_2} \frac{|k_2|^{1/2}}{L_0 L_1^{1/2} M_2^{1/2}} \bar{u} \bar{v} d\tau_1 \right\|_{L^2_{k_0} L^2_{\tau_0}} \lesssim N_{\max}^- \|u\|_{Y^0_S} \|v\|_{Y^{-1/2}_W} \quad (4.13)
\]
First of all, from the Cauchy-Schwarz inequality and the fact that $L_0 = \langle \tau_0 - \alpha k_0^2 \rangle$,

\[
\left\| \sum_{k_0 = k_1 + k_2} \int_{\tau_0 = \tau_1 + \tau_2} \frac{|k_2|^{1/2}}{L_0 L_1^{1/2} M_2^{1/2}} \tilde{\nu} \tilde{d} \tau_1 \right\|_{L_0^{1/2} L_0^{1/2}} \lesssim N_{\max}^{-\delta} \left\| u \right\|_{L_{t,x}^2} \left\| v \right\|_{L_{t,x}^2}.
\]

Next, we consider $|k_0| = |k_1| + |k_2|$, but it can be proved by the similar calculation and using (4.6) instead of (4.5). In other words, we can show that

\[
\left\| \sum_{k_0 = k_1 - k_2} \int_{\tau_0 = \tau_1 - \tau_2} \frac{|k_0|^{1/2}}{M_0^{1/2} L_1^{1/2} L_2^{1/2}} \tilde{\nu} \tilde{d} \tau_1 \right\|_{L_0^{1/2} L_0^{1/2}} \lesssim N_{\max}^{-\delta} \left\| u \right\|_{L_{t,x}^2} \left\| v \right\|_{L_{t,x}^2},
\]

\[
\left\| \sum_{k_0 = k_1 - k_2} \int_{\tau_0 = \tau_1 - \tau_2} \frac{|k_0|^{1/2}}{L_0 M_0 L_1^{1/2} L_2^{1/2}} \tilde{\nu} \tilde{d} \tau_1 \right\|_{L_0^{1/2} L_0^{1/2}} \lesssim N_{\max}^{-\delta} \left\| u \right\|_{Y_{s}^g} \left\| v \right\|_{Y_{s}^g}.
\]

As shown in (4.15), it is similar to (4.8) and (4.13) except for indices. Hence, we can obtain (4.4) by (4.6) and a similar argument for (4.4).
Proof of Proposition 4.2. Note that Proposition 4.2 has the time growth instead of the frequency decay. Hence, we need the following lemma.

Lemma 4.5 (Lemma 2.11, [25]). Let $\eta$ be a Schwartz function in time and $I$ be a time interval. If $-1/2 < b' \leq b < 1/2$, then for any interval $[0, T] \subset I$ such that $0 < T < 1$, we have
\[
\|\eta(t/T)u\|_{X^{s,b'}(I \times T)} \lesssim T^{b-b'}\|u\|_{X^{s,b}(I \times T)}.
\]

Proof. By duality, we may assume that $0 < b' \leq b < 1/2$. From the Christ-Kiselev lemma, we suffice to show that
\[
\|\eta(t/T)u\|_{X^{s,b'}(\mathbb{R} \times T)} \lesssim T^{b-b'}\|u\|_{X^{s,b}(\mathbb{R} \times T)}.
\]

From the following estimate,
\[
\langle \tau - \tau_0 - k^2 \rangle^b \lesssim \langle \tau_0 \rangle^b \langle \tau - k^2 \rangle^b,
\]
we have
\[
\|e^{it\tau_0}u\|_{X^{s,b}(\mathbb{R} \times T)} \lesssim \langle \tau_0 \rangle^b\|u\|_{X^{s,b}(\mathbb{R} \times T)}.
\]

Since $\eta(t)$ is a Schwartz function,
\[
\|\eta(t)u\|_{X^{s,b}(\mathbb{R} \times T)} \lesssim \left(\int_R |\hat{\eta} (\tau_0)| \langle \tau_0 \rangle^b d\tau_0 \right)\|u\|_{X^{s,b}(\mathbb{R} \times T)} \lesssim \|u\|_{X^{s,b}(\mathbb{R} \times T)}.
\]

To simplify our argument, we first assume that $s = 0$. Moreover, we may assume $b' = 0$ by the duality with $b' = b$ case. Thus, our claim is
\[
\|\eta(t/T)u\|_{L^2_tL^2_x(\mathbb{R} \times T)} \lesssim T^b\|u\|_{X^{0,b}(\mathbb{R} \times T)}
\]
for $0 < b < 1/2$. We now consider two cases separately, $\langle \tau - k^2 \rangle \geq 1/T$ and $\langle \tau - k^2 \rangle \leq 1/T$. In the former case, we have
\[
\|u\|_{X^{0,0}(\mathbb{R} \times T)} \leq T^b\|u\|_{X^{0,b}(\mathbb{R} \times T)}
\]
by the fact that $\eta$ is a Schwartz function and $0 < T < 1$. In the latter case, we have
\[
\|\eta(t/T)u\|_{L^2_tL^2_x} \lesssim T^{1/2} \|\hat{\eta} (\tau)\|_{L^2_x} \|\mathcal{F}_t u(t)(k)\|_{L^2_x}\|
\]
by the fact that $\eta$ is a Schwartz function.

Since the above argument does not depend $s$, we can get the same result in $s \in \mathbb{R}$. \(\square\)
From Lemma 4.5, we have a modified embedding,
\[ \| \eta(t^\gamma) u \|_{L^4_{1,x}} \lesssim \| \eta(t^\gamma) u \|_{X^{0,\frac{1}{8}}_S} \lesssim T^\gamma \| u \|_{X^{0,3/8+\gamma}_S} \]  
for sufficiently small \( \gamma > 0 \). We now get Proposition 4.2 by the similar argument in the proof of Proposition 4.3 and (4.16) instead of (4.7).

Remark 4.6. In fact, we can get the Proposition 4.2 without the time growth. Hence from Proposition 4.2 and summation with respect to each of dyadic frequency supports and the orthogonality of dyadic decomposition, we can obtain the full frequency estimate as follows,
\[ \|uv\|_{Z^S} \lesssim \|u\|_{Y^S} \|v\|_{Y^S}^{-1/2}, \]
and
\[ \|\partial_x (uv)\|_{Z^{-1/2}W} \lesssim \|u\|_{Y^S} \|v\|_{Y^S}^{-1}. \]  

5. Global well-posedness

In this section, we prove Theorem 1.2, the global well-posedness for (1.1). It can be easily proved by combining the local well-posedness and the conservation law. More precisely, after splitting the time interval as a finite union of intervals (obviously, the length of each interval depends on (1.2)), we get the solution for each interval by the local well-posedness (see [23]) and glue each solution by using the mass conservation law of \( u \). In particular, the nonlinear term of the wave part only consists of \( u \), so the mass conservation of \( u \) is sufficient to glue each solution.

We briefly explain a sketch of the proof of Theorem 1.2. It is sufficient to prove that
\[ \sup_{|t| \leq T} \| W(t)(u_0, n_0, n_1) \|_{H^{-1/2}W} \lesssim C \left( T, \| u_0 \|_{L^2_x}, \| n_0 \|_{H^{-1/2}_x}, \| n_1 \|_{H^{-3/2}_x} \right) \]  
and
\[ \sup_{|t| \leq T} \| S(t)(u_0, n_0, n_1) \|_{L^2_x} \lesssim C \left( T, \| u_0 \|_{L^2_x}, \| n_0 \|_{H^{-1/2}_x}, \| n_1 \|_{H^{-3/2}_x} \right). \]
for any \( T > 0 \). To prove (5.1) and (5.2), we define the norm as follows,
\[ \| W(t)(u_0, n_0, n_1) \|_W = \|(n, \partial_t n)\|_W := \left( \| n \|_{H^{-1/2}}^2 + \| \partial_t n \|_{H^{-3/2}}^2 \right)^{1/2}. \]

Let 0 < \( \alpha < 1 \) be a constant, we can choose a sufficiently small time \( T' \) such that
\[ \|u\|_{Y^S_{0,T'}} \lesssim \|u_0\|_{L^2_x}, \]  
\[ \|u_0\|_{L^2_x} \ll (T')^{-\alpha} \| (n_0, n_1) \|_W, \]  
and
\[ \|(n_0, n_1)\|_W \lesssim (T')^{-1} \]

\[ \text{\footnote{The details of this argument are in [6]. In fact, Colliander et al. [6] proved the global well-posedness of the Zakharov system on } \mathbb{R}, \text{ but we can apply a similar argument to a torus.}} \]
by (1.2). From (5.3), (2.1), (1.6), the triangular inequality, Lemma 4.1 (4.17), and (5.4), we estimate
\[
\sup_{|t| \leq T'} \|W(t)(u_0, n_0, n_1)\|_W \lesssim \left\| \frac{t}{T'} \right\| W(t)(u_0, n_0, n_1) \left\|_{Y_{W}^{-1/2}} \right. \\
\lesssim \left\| \frac{t}{T'} \right\| \partial_t V(t) n_0 \left\|_{Y_{W}^{-1/2}} \right. + \left\| \frac{t}{T'} \right\| V(t) n_1 \left\|_{Y_{W}^{-1/2}} \right. \\
+ \left\| \frac{t}{T'} \right\| \beta^2 \int_0^t V(t-s) \left[ \partial_x^2 \left( |u|^2 \right) \right] (s) ds \left\|_{Y_{W}^{-1/2}} \right. \\
\lesssim \|n_0\|_{H_x^{-1/2}} + \|n_1\|_{H_x^{-3/2}} + \left\| \partial_x \left( |u|^2 \right) \right\|_{L_{W}^{-1/2}} \\
\lesssim \|n_0\|_{H_x^{-1/2}} + \|n_1\|_{H_x^{-3/2}} + \|u\|_{Y_{W}^3}^2 \\
\lesssim \|(n_0, n_1)\|_W + \|u\|_{L_{x}^2}^2.
\]
From (5.5), we have
\[
\sup_{|t| \leq T'} \|(W(t)(u_0, n_0, n_1)\|_W \lesssim C \left( T', \|u_0\|_{L_{x}^2}, \|n_0\|_{H_x^{-1/2}}, \|n_1\|_{H_x^{-3/2}} \right)
\]  
and by the similar calculation with (1.5),
\[
\sup_{|t| \leq T'} \|S(t)(u_0, n_0, n_1)\|_{L_{x}^2} \lesssim C \left( T', \|u_0\|_{L_{x}^2}, \|n_0\|_{H_x^{-1/2}}, \|n_1\|_{H_x^{-3/2}} \right)
\]
for sufficiently small $T'$. \footnote{In (5.9), the time $T'$ is the same as in (5.8) because it is obtained by the local well-posedness, the fact that the nonlinear term of the Schrödinger part has $n(t, x)$, (5.3) and (5.8).}

We now consider the gluing step. For any time $T$, we divide the total time interval $[-T, T]$ into time intervals such that each interval satisfies (5.8) and (5.9). In the first such interval $[0, T']$, we directly obtain (5.1) and (5.2) and in the next interval, we let $T'$ be the initial time and then obtain the claim by (1.2). Hence, we can use the same iteration up to $\|W(T)(u_0, n_0, n_1)\|_W \gg \|u_0\|_{L_{x}^2}^2$. By taking this time as the initial time $\bar{T} = 0$, we can repeat the entire procedure again. To reach the given time $T$, we need to show that time $\bar{T}$ is independent of $W(t)(u_0, n_0, n_1)$. From the final term in (5.7) and (5.5), we can iterate $m$-times such that
\[
m \sim \frac{\|(n_0, n_1)\|_W}{\|u_0\|_{L_{x}^2}^2},
\]
and from (5.6), we have
\[
\bar{T} = mT \lesssim \frac{1}{\|u_0\|_{L_{x}^2}^2}
\]
which is independent of $W(t)(n_0, n_1)$. Therefore, we are done.
6. Proof of Theorem 1.1

In this section, we prove Theorem 1.1, the invariant of symplectic capacity with respect to the Zakharov flow.

6.1. Local approximation. We introduce a new system as follows,

\[
\begin{align*}
&i\partial_t (P_{\leq N}u) + \alpha\partial_x^2 (P_{\leq N}u) = P_{\leq N} \left[ (P_{\leq N}u)(P_{\leq N}n) \right], \\
&\beta^2 \partial_t^2 (P_{\leq N}n) - \partial_x^2 (P_{\leq N}n) = P_{\leq N} \left[ \partial_x^2 \left( |P_{\leq N}u|^2 \right) \right], \\
&i\partial_t ((1 - P_{\leq N})u) + \alpha\partial_x^2 ((1 - P_{\leq N})u) = 0, \\
&\beta^2 \partial_t^2 ((1 - P_{\leq N})n) - \partial_x^2 ((1 - P_{\leq N})n) = 0
\end{align*}
\]

(6.1)

for the initial data \((u_0, n_0, n_1)\) \(\in \mathcal{H}\). Let \(Z^N(t)\) be a solution flow with respect to (6.1), and its Hamiltonian is

\[
H^N [u, n, \dot{n}] = \int_T \left( \alpha |\partial_x u|^2 + \frac{|n|^2}{2} + \beta^2 \frac{|\partial_x^{-1}\dot{n}|^2}{2} + P_{\leq N}n |P_{\leq N}u|^2 \right) dx.
\]

Note that the new system (6.1) has the same nonlinear operator in the low frequencies, and a linear operator in the high frequencies. Hence, the solution map \(Z^N(t)\) is a smooth symplectomorphism with the symplectic form (5.1) on the Hilbert space \(\mathcal{H}\), and the new system has the global well-posedness as well.

**Proposition 6.1.** For a global-in-time \(T > 0\) and any large integer \(N\). The initial data \((u_0, n_0, n_1)\) is in \(\mathcal{H}\). Assume that \(Z(t)\) and \(Z^N(t)\) be the Zakharov flow and the solution flow for (6.1), respectively. Then we have

\[
\sup_{|s| \leq t} \| (Z(s)(u_0, n_0, n_1) - Z^N(s)(u_0, n_0, n_1)) \|_\mathcal{H} \leq C \left( T, \| (u_0, n_0, n_1) \|_\mathcal{H} \right) N^{-\delta}
\]

for a local-in-time \(0 < t \ll 1\) and \(\delta > 0\).

**Proof.** We denote that \(z_0 := (u_0, n_0, n_1)\), \(z(t) = (u(t), n(t), \partial_t u(t)) := Z(t)z_0\) and \(z^N(t) = (u^N(t), n^N(t), \partial_t n^N(t)) := Z^N(t)z_0\). From the global well-posedness, there exists constant \(C(T, \| z_0 \|_\mathcal{H})\) such that

\[
\| z(t) \|_Y + \| z^N(t) \|_Y \leq C \left( T, \| z_0 \|_\mathcal{H} \right) := \mathcal{R}.
\]

(6.2)

We split the solution into two portions as follows,

\[
z(t) = z_{lo} + z_{hi} := P_{\leq N}z(t) + (1 - P_{\leq N})z(t)
\]

\[
= P_{\leq N}u(t) + P_{\leq N}n(t) + (1 - P_{\leq N})u(t) + (1 - P_{\leq N})n(t)
\]

\[
= u_{lo} + n_{lo} + u_{hi} + n_{hi}.
\]

By (6.2), we also have

\[
\| z_{lo} \|_Y \leq \mathcal{R} \quad \text{and} \quad \| z_{hi} \|_Y \leq \mathcal{R}.
\]

Likewise, \(z^N\) is also split, and is bounded by \(\mathcal{R}\) for each flow. Especially, \(u_{hi}^N\) and \(n_{hi}^N\) are linear flow by the definition of the new Hamiltonian system flow. By the structure of the wave part and (2.1),

\[
\sup_{|s| \leq t} \| Z(t)(u_0, n_0, n_1) - Z^N(t)(u_0, n_0, n_1) \|_H \lesssim \| Z(t)(u_0, n_0, n_1) - Z^N(t)(u_0, n_0, n_1) \|_Y
\]
The right hand side is bounded by
\[ \left\| \int_0^t U (t-s) [u_n - P_{\leq N} (u_{l_0} n_{l_0})] (s) \right\|_{Y^0_S} + \left\| \int_0^t V (t-s) \partial^2_x \left[ |u|^2 - P_{\leq N} |u_{l_0}|^2 \right] (s) \right\|_{Y^{-1/2}_S} \]
by the Duhamel's formula and the fact that the initial data is same. We first estimate the Schrödinger part. By Lemma 4.1 and the Minkowski inequality, we have
\[ \left\| \int_0^t U (t-s) [u_n - P_{\leq N} (u_{l_0} n_{l_0})] ds \right\|_{Y^0_S} \lesssim \|u_n - P_{\leq N} (u_{l_0} n_{l_0})\| \]
by the global well-posedness (6.2), we have the estimate for the Schrödinger part as follows,
\[ \left\| \int_0^t V (t-s) \partial^2_x \left[ |u|^2 - P_{\leq N} |u_{l_0}|^2 \right] (s) ds \right\|_{Y^{-1/2}_S} \]
by the Duhamel's formula and the fact that the initial data is same. We first estimate the Schrödinger part. By Lemma 4.1 and the Minkowski inequality, we have
\[ \left\| \int_0^t U (t-s) [u_n - P_{\leq N} (u_{l_0} n_{l_0})] ds \right\|_{Y^0_S} \lesssim \|u_n - P_{\leq N} (u_{l_0} n_{l_0})\| \]
by the global well-posedness (6.2), we have the estimate for the Schrödinger part as follows,
\[ \left\| \int_0^t U (t-s) [u_n - P_{\leq N} (u_{l_0} n_{l_0})] ds \right\|_{Y^0_S} \lesssim \mathcal{R}^{-\delta} N^{-\delta} + t^\gamma \mathcal{R} \|z(t) - z^N(t)\|_Y. \]
By the similar calculation with (4.4), (4.2), and the global well-posedness, the wave part is bounded as well. Indeed,
\[ \left\| \int_0^t V (t-s) \partial^2_x \left[ |u|^2 - P_{\leq N} |u_{l_0}|^2 \right] (s) ds \right\|_{Y^{-1/2}_S} \lesssim N^{-\delta} \|u\|_{Y^0_S}^2 + t^\gamma \|u\|_{Y^0_S} \|u_n - u_{l_0}\|_{Y^0_S} \]
\[ \lesssim \mathcal{R}^{-\delta} N^{-\delta} + t^\gamma \mathcal{R} \|z(t) - z^N(t)\|_Y. \]
Therefore, we have
\[ \|Z(t) (u_0, n_0, n_1) - Z^N(t) (u_0, n_0, n_1)\|_Y \]
\[ \lesssim \left\| \int_0^t U (t-s) [u_n - P_{\leq N} (u_{l_0} n_{l_0})] (s) ds \right\|_{Y^0_S} + \left\| \int_0^t V (t-s) \partial^2_x \left[ |u|^2 - P_{\leq N} |u_{l_0}|^2 \right] (s) ds \right\|_{Y^{-1/2}_S} \]
\[ \lesssim \mathcal{R}^{-\delta} N^{-\delta} + t^\gamma \mathcal{R} \|Z(t) (u_0, n_0, n_1) - Z^N(t) (u_0, n_0, n_1)\|_Y. \]
Thus, choosing local-in-time \( t \) such that \( t < (\frac{1}{\mathcal{R}})^{\frac{1}{\delta}} \), we have
\[ \|Z(t) (u_0, n_0, n_1) - Z^N(t) (u_0, n_0, n_1)\|_Y \]
\[ \lesssim R N^{-\delta} = C(T, \|(u_0, n_0, n_1)\|_{H^1}) N^{-\delta}. \]
□

Remark 6.2. The local-in-time \( t \) in Proposition 6.1 does not depend on frequency \( N \). We thus conclude that the map \([Z - Z^N](t)\) is regarded as a small perturbation in a sufficiently short time interval.
6.2. **Proof of symplectic invariant.** We separate the solution flow, and use an iteration argument. There exists a local time length $\tau (T, \|(u_0, n_0, n_1)\|_H) > 0$ such that the Zakharov flow satisfies Proposition 6.1. The global time interval $[0, T]$ is split to $[0 = t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_n-1, t_n = T]$, and length of each interval is the constant $\tau = |t_{i+1} - t_i|$ that depends only on the implicit constant in Proposition 6.1. Let $\Omega_0$ be a initial domain which contains the initial data $u(x, 0)$. Likewise, we denote that $\Omega_i (:= Z(t_i)(\Omega_0))$ is a domain which has the solution $u(x, t_i)$.

**FIRST STEP** (Local-time symplectic invariant)

We first prove that

$$
cap(\Omega_1) = \cap(Z(t_1)(\Omega_0)) \leq \cap(\Omega_0). \tag{6.4}
$$

Let $f_1$ be a $m$-admissible function in $\Omega_1$ such that $m > \cap(\Omega_0)$. From the Definition 3.3, it suffices to show that the function $f_1$ is a fast function in $\Omega_1$. Since the fact that the initial domain $\Omega_0$ is bounded and the Zakharov system has the global well-posedness, the domain $\Omega_1$ is a bounded domain as well. Denoting $\tilde{\Omega}_0 = \Omega_0 \cap (Z(t_1))^{-1}(\Omega_1)$, we have $\cap(\tilde{\Omega}_0) \leq \cap(\Omega_0)$ by Definition 6.2. Define $\varepsilon_1 = \text{dist}_{\tilde{\Omega}_1}(f_1)$, we can get a sufficiently large integer $N$ such that

$$
\frac{\varepsilon_1}{2} > N_1^{-\delta}
$$

where $\delta$ is the implicit constant in Proposition 6.1. The Zakharov flow is decomposed to

$$
Z(t_1) = Z(t_1) - Z^{N_1}(t_1) + Z^{N_1}(t_1)
$$

$$
= \left[ I + (Z(t_1) - Z^{N_1}(t_1)) \circ (Z^{N_1}(t_1))^{-1} \right] \circ Z^{N_1}(t_1)
$$

$$
=: (I + Z_{\varepsilon_1}(t_1)) \circ Z^{N_1}(t_1),
$$

where $I$ is an identity map from $\mathcal{H}$ to $\mathcal{H}$, and the map $Z^{N_1}(t_1)$ is the solution map for (6.1). Note that $I + Z_{\varepsilon_1}(t_1)$ and $Z^{N_1}(t_1)$ are smooth symplectomorphisms. In the low frequencies, the solution map $Z^{N_1}(t_1)$ is composite operator with linear and nonlinear solution operators which is a finite dimensional symplectomorphism. In the high frequencies, the map are linear solution operator only, and they are isometries on the symplectic Hilbert space $\mathcal{H} = L^2 \times H^{-1/2} \times H^{-3/2})$. Hence, the classes of $m$-admissible functions are preserved by $Z^{N_1}(t_1)$. We thus show that

$$
\cap \left( (I + Z_{\varepsilon_1}(t_1))(\tilde{\Omega}_0) \right) \leq \cap(\tilde{\Omega}_0). \tag{6.5}
$$

where a domain $\tilde{\Omega}_0 = Z^{N_1}(t_1)(\tilde{\Omega}_0)$. By the decomposition of the Zakharov flow, we have

$$
(I + Z_{\varepsilon_1}(t_1))(\tilde{\Omega}_0) = \Omega_1.
$$

Since an inverse operator $(Z^{N_1}(t_1))^{-1}$ is also bounded, the operator $(Z(t_1) - Z^{N_1}(t_1)) \circ (Z^{N_1}(t_1))^{-1}$ has an estimate

$$
\left\| (Z(t_1) - Z^{N_1}(t_1)) \circ (Z^{N_1}(t_1))^{-1} \right\|_{\tilde{\Omega}_0 \rightarrow \Omega_1} \leq C(T, \tilde{\Omega}_0) N(\varepsilon_1)^{-\delta} =: N_1(T, \tilde{\Omega}_0, \varepsilon_1)^{-\delta} \tag{6.6}
$$

for the constant $\delta > 0$ by Proposition 6.1.

Let $V_{f_1}$ be a vector fields of the function $f_1$. It suffices to show that the vector field $V_{f_1}$
have a fast trajectory in the domain $\Omega_1$, for large integer $j$. The function $f_1$ is extended as $m$ outside $\Omega_1$, and provides an extended smooth function $g$ in $H$. Moreover, let $h$ be a function which is restriction $g$ to $\hat{\Omega}_0$. Since the operator $Z_{\varepsilon_1} (t_1)$ has a estimate (6.6), the $\varepsilon_1$-neighborhood of $\partial \Omega_1$ is enclosed in the $\varepsilon_2$-neighborhood of $\hat{\Omega}_0$, where $h \equiv m$. Furthermore, we have $h^{-1}(0) = f_1^{-1}(0) \subset \hat{\Omega}_0 \cap \Omega_1$ by (3.5). In other words, $\text{Supp} h$ is equal to $\text{Supp} f_1$.

Hence, the function $h$ is an $m$-admissible function in $\hat{\Omega}_0$. Since $m > \text{cap} \left( \hat{\Omega}_0 \right)$, the vector field $V_{h_j}$ has a fast trajectory in $\hat{\Omega}_0$ for all $j \gg 1$. By Lemma 3.5, this trajectory lies in $\text{Supp} h$, which equals $\text{Supp} f_1$ by (3.5). Therefore, the vector field $V_{f_1}$ has a fast trajectory in $\Omega_1$ for all $j \gg 1$. That is, the function $f_1$ is fast in $\Omega_1$.

The opposite case can be shown by the same argument for the inverse operator. Therefore, we have

$$\text{cap} (\Omega_1) = \text{cap} (Z(t_1) \Omega_0) = \text{cap} (\Omega_0)$$

for the local time $t_1$.

SECOND STEP (Iteration step)

Fix a domain $\Omega_{i-1}$ for any time $t_i$, we can get an appropriate constant $\varepsilon_i$ which is depended on $\text{dist}_{\Omega_i} (f_1)$. Thus we have $N_i \left( T, \hat{\Omega}_{i-1}, \varepsilon_i \right)$, and so we show that the symplectic capacity is preserved for $[t_i, t_{i+1}]$ by the similar argument of the first step, since the constant $N_i \left( T, \hat{\Omega}_{i-1}, \varepsilon_i \right)$ is independent of the local time length $\tau$. Repeating the process, we have that the Zakharov flow preserves the symplectic capacity is its the phase space for the given global-time $T$.

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