CLASSIFICATION OF FOUR-REBIT STATES

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Abstract. We classify states of four rebits, that is, we classify the orbits of the group \( \hat{G}(\mathbb{R}) = \text{SL}(2, \mathbb{R})^4 \) in the space \((\mathbb{R}^2)^{\otimes 4}\). This is the real analogon of the well-known SLOCC operations in quantum information theory. By constructing the \( \hat{G}(\mathbb{R}) \)-module \((\mathbb{R}^2)^{\otimes 4}\) via a \( \mathbb{Z} / 2\mathbb{Z} \)-grading of the simple split real Lie algebra of type \( D_4 \), the orbits are divided into three groups: semisimple, nilpotent and mixed. The nilpotent orbits have been classified in Dietrich et al. (2017), yielding applications in theoretical physics (extremal black holes in the STU model of \( N = 2, D = 4 \) supergravity, see Ruggeri and Trigiante (2017)). Here we focus on the semisimple and mixed orbits which we classify with recently developed methods based on Galois cohomology, see Borovoi et al. (2021). These orbits are relevant to the classification of non-extremal (or extremal over-rotating) and two-center extremal black hole solutions in the STU model.

1. Introduction

In a recent paper [25], we obtained a complete and irredundant classification of the orbits of the group \( \hat{G} = \text{SL}(2, \mathbb{C})^4 \) acting on the space \((\mathbb{C}^2)^{\otimes 4}\). This is relevant to Quantum Information Theory because it amounts to the classification of the entanglement states of four pure multiparticle quantum bits (qubits) under the group \( \hat{G} \) of reversible Stochastic Local Quantum Operations assisted by Classical Communication (SLOCC). Here we obtain the classification of the orbits of the real group \( \hat{G}(\mathbb{R}) = \text{SL}(2, \mathbb{R})^4 \) on the space \((\mathbb{R}^2)^{\otimes 4}\). This is relevant to real quantum mechanics, where the elements of \((\mathbb{R}^3)^{\otimes 4}\) are called four-rebit states. Via the “black hole/qubit correspondence” our classification has also applications to high-energy theoretical physics. We refer to Section 2 for a short introduction into rebits and their relevance to extremal black holes in string theory.

The main idea behind the complex classification is to construct the representation of \( \hat{G} \) on \((\mathbb{C}^2)^{\otimes 4}\) using a \( \mathbb{Z} / 2\mathbb{Z} \)-grading \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) of the simple Lie algebra \( \mathfrak{g} \) of type \( D_4 \) (see Section 3 for more details). In this construction the spaces \((\mathbb{C}^2)^{\otimes 4}\) and \( \mathfrak{g}_1 \) are identified, yielding a Jordan decomposition of the elements of \((\mathbb{C}^2)^{\otimes 4}\). This way the elements of (and hence the orbits in) the space \((\mathbb{C}^2)^{\otimes 4}\) are divided into three groups: nilpotent, semisimple, and mixed. The main result of [25] is a classification of the semisimple and mixed elements; the classification of the corresponding nilpotent orbits was already completed a decade earlier by Borsten et al. [10].

Analogously to the complex case, the representation of \( \hat{G}(\mathbb{R}) \) on \((\mathbb{R}^2)^{\otimes 4}\) can be constructed using a \( \mathbb{Z} / 2\mathbb{Z} \)-grading \( \mathfrak{g}(\mathbb{R}) = \mathfrak{g}_0(\mathbb{R}) \oplus \mathfrak{g}_1(\mathbb{R}) \) of the split real form \( \mathfrak{g}(\mathbb{R}) \) of \( \mathfrak{g} \). Also here the nilpotent orbits have been classified in previous work, see Dietrich et al. [25]: There are 145 nilpotent orbits, and 101 of these turned out to be relevant to the study of (possibly multi-center) extremal black holes (BHs) in the STU model (see [5,30]), this application was discussed in full detail in a subsequent paper by Ruggeri and Trigiante [49]. While in various papers, such as Bossart et al. [14,15], the classification of extremal BH solutions had been essentially based on the complex nilpotent \( \hat{G} \)-orbits in \( \mathfrak{g}_1 \), a more intrinsic, accurate and detailed treatment was provided in [26,49].

The present paper deals with the real \( \hat{G}(\mathbb{R}) \)-orbits of semisimple and mixed elements in \((\mathbb{R}^2)^{\otimes 4}\). Such non-nilpotent orbits are relevant for the classification of non-extremal (or extremal over-rotating) as well as of two-center extremal BH solutions in the STU model of \( N = 2, D = 4 \) supergravity, see Section 2. A detailed discussion of the application of our classification to the study of BHs goes beyond the scope of the present investigation and we leave it for future work. From now on we focus on the mathematical side of this research.
The methods that we use to classify the \( \hat{G}(\mathbb{R}) \)-orbits in \((\mathbb{R}^2)^{\otimes 4}\) are based on [7,8] and employ the theory of Galois cohomology. One of the main implications of this theory is the following: Let \( v \in (\mathbb{R}^2)^{\otimes 4} \) and consider its complex orbit \( \hat{G} v \subset (\mathbb{C}^2)^{\otimes 4} \). Then the \( \hat{G}(\mathbb{R}) \)-orbits contained in \( \hat{G} v \cap (\mathbb{R}^2)^{\otimes 4} \) are in bijection with the Galois cohomology set \( H^1(\hat{Z}_G(v)) \), where \( \hat{Z}_G(v) = \{ g \in \hat{G} : gv = v \} \) is the stabiliser of \( v \) in \( \hat{G} \). So in principle the only thing one has to do is to compute \( H^1(\hat{Z}_G(v)) \) for each \( \hat{G} \)-orbit in \((\mathbb{C}^2)^{\otimes 4}\) that has a real representative \( v \). This works well for the nilpotent orbits because they are finite in number and all have real representatives (we do not discuss the nilpotent case here since a classification is already given in [26]). However, for the orbits of semisimple and mixed elements this is not straightforward: firstly, there is an infinite number of them and, secondly, it is a problem to decide whether a given complex orbit has a real representative or not.

Our approach to classifying semisimple elements is described in Section 5 and analogous to the method developed in [8]. However, the work in [8] relies on some specific preliminary results that do not apply to our case; as a first step, we therefore need to establish the corresponding results for the situation discussed here. Mixed elements are considered in Section 6 also here the methods are similar to those in [8]. The main difference is that in the case treated in [8], the stabilisers of semisimple elements all have trivial Galois cohomology. This is far from being the case in the situation discussed here, which requires significant amendments. For example, we will work with sets of 4-tuples \((p, h, e, f)\), which do not explicitly appear in [8].

In the course of our research we have made frequent use of the computer algebra system GAP4 [38]. This system makes it possible to compute with the simple Lie algebra \( g \) of type \( D_4 \). We have used additional GAP programs of our own, for example to compute defining equations of the stabilisers of elements in the group \( \hat{G} \).

Our main result is summarised by the following theorem, it is proved with Theorems 5.9 and 6.3.

**Theorem 1.1.** The following is established.

a) Up to \( \hat{G}(\mathbb{R}) \)-conjugacy, the nonzero semisimple elements in \((\mathbb{R}^2)^{\otimes 4}\) are the elements in Tables 6–11.

b) Up to \( \hat{G}(\mathbb{R}) \)-conjugacy, the mixed elements in \((\mathbb{R}^2)^{\otimes 4}\) are the elements in Tables 13–27.

c) Up to \( \hat{G}(\mathbb{R}) \)-conjugacy, the nilpotent elements in \((\mathbb{R}^2)^{\otimes 4}\) are given in [26] Table I.

The notation used in these tables is explained in Definition 1.

**Structure of this paper.** In Section 2 we briefly comment on applications to real quantum mechanics and non-extremal black holes. In Section 3 we introduce more notation and recall the known classifications over the complex field; these classifications are the starting point for the classifications over the real numbers. In Section 4 we discuss some results from Galois cohomology that will be useful for splitting a known complex orbit into real orbits. Section 5 presents our classification of real semisimple elements; we prove our first main result Theorem 5.9 in Section 6 we prove Theorem 6.3 which completes the classification of the real mixed elements. The Appendix contains tables listing our classifications.

2. **Rebits and the black hole/qubit correspondence**

2.1. **On rebits.** Real quantum mechanics (that is, quantum mechanics defined over real vector spaces) dates back to Stückelberg [51]. It provides an interesting theory whose study may help to discriminate among the aspects of quantum entanglement which are unique to standard quantum theory and those aspects which are more generic over other physical theories endowed with this phenomenon [16]. Real quantum mechanics is based on the rebit, a quantum bit with real coefficients for probability amplitudes of a two-state system, namely a two-state quantum state that may be expressed as a real linear combination of \(|0\rangle\) and \(|1\rangle\) (which can also be considered as restricted states that are known to lie on a longitudinal great circle of the Bloch sphere corresponding to only real state vectors). In other words, the density matrix of the processed quantum state \( \rho \) is real; that is, at each point in the quantum computation, it holds that \(|x|\rho|y\rangle \in \mathbb{R}\) for all \(|x\rangle\) and \(|y\rangle\) in the computational basis.

As discussed in [1, 41, and 9 Appendix B], quantum computation based on rebits is qualitatively different from the complex case. Following [16], some entanglement properties of two-rebit systems have been discussed in [34], also exploiting quaternionic quantum mechanics. Moreover, as recalled in [21], rebits were shown in [48] to be sufficient for universal quantum computation; in that scheme, a quantum state of \( n \) qubits

\[
|\psi\rangle = \sum_{\nu \in \mathbb{Z}_2^d} r_\nu e^{i\theta_\nu} |\nu\rangle \quad (r_\nu \in \mathbb{R}_+, \theta_\nu \in \mathbb{R})
\]

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can be encoded into a state of $n + 1$ rebits,

$$\overline{\psi} = \sum_{\mathbf{v} \in \mathbb{Z}_2^n} \left( r_{\mathbf{v}} \cos \theta_{\mathbf{v}} |\mathbf{v}\rangle \otimes |R\rangle + r_{\mathbf{v}} \sin \theta_{\mathbf{v}} |\mathbf{v}\rangle \otimes |I\rangle \right),$$

where the additional rebit (which has been also named *universal rebit* or *ubit* [2]), with basis states $|R\rangle = |0\rangle$ and $|I\rangle = |1\rangle$, allows one to keep track of the real and imaginary parts of the unencoded $n$-qubit state.

It should also be remarked that in [55] the three-tangle for three rebits has been defined and evaluated, resulting to be expressed by the same formula as in the complex case, but without an overall absolute value sign: thus, unlike the usual three-tangle, the rebit three-tangle can be negative. In other words, by denoting the pure three rebits state as

$$|\phi\rangle = \sum_{i,j,k \in \mathbb{Z}_2} a_{ijk} |ijk\rangle,$$

where the binary indices $i, j, k$ correspond to rebits A, B, C, respectively, the three-tangle is simply four times the *Cayley’s hyperdeterminant* [17] of the cubic $2 \times 2 \times 2$ matrix $a_{ijk}$, see [55].

### 2.2. Rebits and black holes

In recent years, the relevance of rebits in high-energy theoretical physics was highlighted by the determination of striking relations between the entanglement of pure states of two and three qubits and extremal BHs holes in string theory. In this framework, which has been subsequently dubbed as the “black hole / qubit correspondence” (see for example [11–13] for reviews and references), rebits acquire the physical meaning of the electric and magnetic charges of the extremal BH, and they linearly transform under the generalised electric-magnetic duality group (named U-duality group in string theory) $G(\mathbb{R})$ of the Maxwell-Einstein (super)gravity theory under consideration.[4] This development started with the seminal paper [27], in which Duff pointed out that the entropy of the so-called extremal BPS STU BHs can be expressed in a very compact way in terms of Cayley’s hyperdeterminant [17], which, as mentioned above, plays a prominent role as the three-tangle in studies of three-qubit entanglement [55]. Crucially, the electric and magnetic charges of the extremal BH, which are conserved due to the underlying Abelian gauge invariance, are forced to be real because they are nothing but the fluxes of the two-form field strengths of the Abelian potential one-forms, as well as of their dual forms, which are real. Later on, for example in [40, 42–43, 28–29] and subsequent developments, Duff’s observation was generalised and extended to non-BPS BHs (which thus break all supersymmetries), also in $(\mathcal{N} > 2)$-extended supergravity theories in four and five space-time dimensions. Further mathematical similarities were thoroughly investigated by Lévy, which for instance showed that the frozen values of the moduli in the calculation of the macroscopic, Bekenstein-Hawking BH entropy in the STU model are related to finding the canonical form for a pure three-qubit entangled state, whereas the extremisation of the BPS mass with respect to the moduli is connected to the problem of finding the so-called optimal local *distillation protocol* [45–46].

Another application of rebits concerns extremal BHs with two-centers. Multi-center BHs are a natural generalisation of single-center BHs. They occur as solutions to Maxwell-Einstein equations in $4D$, regardless of the presence of local supersymmetry, and they play a prominent role within the dynamics of high-energy theories whose ultra-violet completion aims at describing Quantum Gravity, such as $10D$ superstrings and $11D$ M-theory. In multi-center BHs the *attractor mechanism* [31–34, 52] is generalised by the so-called *split attractor flow* [22–24], concerning the existence of a co-dimension-one region - the *marginal stability wall* - in the target space of scalar fields, where a stable multi-center BH may decay into its various single-center constituents, whose scalar flows then separately evolve according to the corresponding attractor dynamics.

In this framework, the aforementioned real fluxes of the two-form Abelian field strengths and of their duals, which are usually referred to as *electric* and *magnetic* charges of the BH, fit into a representation $R$ of the $4D$ $U$-duality group $G(\mathbb{R})$. In the STU model of $\mathcal{N} = 2, D = 4$ supergravity, $G(\mathbb{R}) = SL(2, \mathbb{R})^3$ and $R = (\mathbb{R}^2)^{\otimes 3}$, and each $SL(2, \mathbb{R})^3$-orbit supports a *unique* class of single-center BH solutions. In general, in presence of a multi-center BH solution with $p$ centers, the dimension $I_p$ of the ring of $G(\mathbb{R})$-invariant homogeneous polynomials constructed with $p$ distinct copies of the $SL(2, \mathbb{R})^3$-representation charge $R$ is given by the general formula [35]

$$p \dim_R R = \dim_R \mathcal{O}_p + I_p,$$

where $\mathcal{O}_p = G(\mathbb{R}) / \mathcal{H}_p(\mathbb{R})$ is a generally non-symmetric coset describing the generic, open $G(\mathbb{R})$-orbit, spanned by the $p$ copies of the charge representation $R$, each pertaining to one center of the multi-center solution. A crucial feature of *multi-center* ($p > 1$) BHs is that the various ($I_p > 1$) $G(\mathbb{R})$-invariant polynomials arrange into

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[1] In supergravity, the approximation of real (rather than integer) electric and magnetic charges of the BH is often considered, thus disregarding the charge quantization.
multiplets of a global, “horizontal” symmetry group $SL_p(\mathbb{R})$ [35], encoding the combinatoric structure of the $p$-center solutions of the theory, and commuting with $G(\mathbb{R})$ itself. Thus, by considering two-center BHs (that is, $p = 2$ – an assumption which does not imply any loss of generality due to tree structure of split attractor flows in the STU model), it holds that $\dim \mathbb{R} = 8$ and the stabiliser of $O_{p=2}$ has trivial identity connected component. The two-center version of formula (2.1) in the STU model yields

$$STU: I_{p=2} = 2 \dim \mathbb{R} \left( (\mathbb{R}^2)^{\otimes 3} \right) - \dim \mathbb{R} (SL(2,\mathbb{R}))^3 = 2 \cdot 8 - 9 = 7,$$

implying that the ring of $SL(2,\mathbb{R})^3$-invariant homogeneous polynomials built out of two copies of the tri-fundamental representation $(\mathbb{R}^2)^{\otimes 3}$ has dimension 7. As firstly discussed in [35] and then investigated in [4183637], the seven $SL(2,\mathbb{R})^3$-invariant generators of the aforementioned polynomial ring arrange into one quintuplet (in the spin-2 irreducible representation 5) and two singlets $1 \oplus 1'$ of the “horizontal” symmetry group $SL_{\text{hor}}(2,\mathbb{R})$:

$$I_{p=2} = 7 = \deg 4 + \deg 2 \otimes \deg 4 \quad \text{under } SL_{\text{hor}}(2,\mathbb{R})$$

where the degrees of each term (corresponding to one or more homogeneous polynomials) has been reported. The overall semisimple global group providing the action of the $U$-duality as well as of the “horizontal” symmetry on two-center BHs is

$$SL_{\text{hor}}(2,\mathbb{R}) \otimes SL(2,\mathbb{R})^3 \cong SL(2,\mathbb{R})^4 = \hat{G}(\mathbb{R}),$$

acting on the $SL_{\text{hor}}(2,\mathbb{R})$-doublet of $G$-representations $R$'s, namely

$$R^2 \otimes R = \mathbb{R}^2 \otimes (\mathbb{R}^2)^{\otimes 3} \cong (\mathbb{R}^2)^{\otimes 4}.$$

Since the “horizontal” factor $SL_{\text{hor}}$ stands on a different footing than the $U$-duality group $SL(2,\mathbb{R})^3$, only the discrete group $\text{Sym}_3$ of permutations of the three tensor factors in $\mathbb{R} = (\mathbb{R}^2)^{\otimes 3}$ should be taken into account when considering two-center BH solutions in the STU model, to which a classification invariant under $\text{Sym}_3 \ltimes SL(2,\mathbb{R})^3$ thus pertains. Clearly, the two singlets in the right hand side of (2.3) are invariant under the whole $SL_{\text{hor}}(2,\mathbb{R}) \otimes SL(2,\mathbb{R})^3$; on the other hand, when enforcing the symmetry also under the “horizontal” $SL_{\text{hor}}(2,\mathbb{R})$, one must consider its non-transitive action on the quintuplet 5 occurring in the right hand side of (2.3). As explicitly computed (for example, in [35]) and as known within the classical theory of invariants (see for example [54] as well as the Tables of [39]), the spin-2 $SL_2$-representation 5 has a two-dimensional ring of invariants, finitely generated by a quadratic and a cubic homogeneous polynomial:

$$I_{\text{spin-2}} = \dim \mathbb{R} (5) - \dim \mathbb{R} SL_{\text{hor}}(2,\mathbb{R}) = 5 - 3 = 2 = \deg 2 \oplus \deg 3 \quad \text{under } SL_{\text{hor}}(2,\mathbb{R})$$

This results into a four-dimensional basis of $(\text{Sym}_3 \ltimes (SL_{\text{hor}}(2,\mathbb{R}) \otimes SL(2,\mathbb{R})^3))$-invariant homogeneous polynomials, respectively of degree 2, 4, 8 and 12 in the elements of the two-center BH charge representation space $(\mathbb{R}^2)^{\otimes 4}$. However, as discussed in [35], a lower degree invariant polynomial of degree 6 can be introduced and related to the degree-12 polynomial, giving rise to a 4-dimensional basis of $(\text{Sym}_3 \ltimes (SL_{\text{hor}}(2,\mathbb{R}) \otimes SL(2,\mathbb{R})^3))$-invariant homogeneous polynomials with degrees 2, 4, 6 and 8, respectively, see [35]. We recall that the enforcement of the whole discrete permutation symmetry $\text{Sym}_3$ (as done in Quantum Information Theory applications) allows for the degrees of the four $(\text{Sym}_4 \ltimes (SL_{\text{hor}}(2,\mathbb{R}) \otimes SL(2,\mathbb{R})^3))$-invariant polynomial generators to be further lowered down to 2, 4, 4 and 6; this is explicitly computed in [4753] and then discussed in [44] in relation to two-center extremal BHs in the STU model. In all cases, the lowest-order element of the invariant basis, namely the homogeneous polynomial quadratic in the BH charges, is nothing but the symplectic product of the two copies of the single-center charge representation $\mathbb{R} = (\mathbb{R}^2)^{\otimes 3}$; such a symplectic product is constrained to be non-vanishing in non-trivial and regular two-center BH solutions with mutually non-local centers [35]. This implies that regular two-center extremal BHs are related to non-nilpotent orbits of the whole symmetry $SL_{\text{hor}}(2,\mathbb{R}) \otimes SL(2,\mathbb{R})^3$ (with a discrete factor $\text{Sym}_3$ or $\text{Sym}_4$, as just specified) on $(\mathbb{R}^2)^{\otimes 4}$. The application of the classification of such orbits (which are the object of interest in this paper) to the study of two-center extremal BHs in the prototypical STU model goes beyond the scope of the present investigation, and we leave it for further future work. 

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2Actually, the “horizontal” symmetry group is $GL(p,\mathbb{R})$, where the additional scale symmetry with respect to $SL(p,\mathbb{R})$ is encoded by the homogeneity of the $G(\mathbb{R})$-invariant polynomials in the BH charges. The subscript “hor” stands for “horizontal” throughout.
3. Notation and classifications over the complex field

3.1. The grading. Let $\mathfrak{g}$ be the simple Lie algebra of type $D_4$ defined over the complex numbers. Let $\Psi$ denote its root system with respect to a fixed Cartan subalgebra $\mathfrak{t}$. Let $\gamma_1, \ldots, \gamma_4$ be a fixed choice of simple roots such that the Dynkin diagram of $\Psi$ is labelled as follows

![Dynkin diagram of D_4](image)

We now construct a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$: let $\mathfrak{g}_0$ be spanned by $\mathfrak{t}$ along with the root spaces $\mathfrak{g}_\gamma$, where $\gamma = \sum_i k_i \gamma_i$ has $k_2$ even, and let $\mathfrak{g}_1$ be spanned by those $\mathfrak{g}_\gamma$ where $\gamma = \sum_i k_i \gamma_i$ has $k_2$ odd. Let $\gamma_0 = \gamma_1 + 2 \gamma_2 + \gamma_3 + \gamma_4$ be the highest root of $\Psi$. The root system of $\mathfrak{g}_0$ is $\{\pm \gamma_0, \pm \gamma_1, \pm \gamma_3, \pm \gamma_4\}$, hence

$$\mathfrak{g}_0 \cong \mathfrak{sl}(2, \mathbb{C})^4 = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}).$$

Taking $-\gamma_0, \gamma_1, \gamma_3, \gamma_4$ as basis of simple roots of $\mathfrak{g}_0$ we have that $-\gamma_2$ is the highest weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$, which therefore is isomorphic to $(\mathbb{C}^2)^{\otimes 4}$. We fix a basis $\{e_1, e_4\}$ of $\mathbb{C}^2$ and denote the basis elements of $(\mathbb{C}^2)^{\otimes 4}$ by

$$|i_1 i_2 i_3 i_4\rangle = e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}.$$

Mapping any nonzero root vector in $\mathfrak{g}_{-\gamma_2}$ to $|0000\rangle$ extends uniquely to an isomorphism $\mathfrak{g}_1 \rightarrow (\mathbb{C}^2)^{\otimes 4}$ of $\mathfrak{sl}(2, \mathbb{C})^4$-modules. We denote by $G$ the adjoint group of $\mathfrak{g}$, and we write $G_0$ for the connected algebraic subgroup of $G$ with Lie algebra $\text{ad}\mathfrak{g}_0 \cong \mathfrak{sl}(2, \mathbb{C})^4$. The isomorphism $\mathfrak{sl}(2, \mathbb{C})^4 \rightarrow \mathfrak{g}_0$ lifts to a surjective morphism $\pi : \hat{G} \rightarrow G_0$ of algebraic groups, which makes $\mathfrak{g}_1$ into a $\hat{G}$-module isomorphic to $(\mathbb{C}^2)^{\otimes 4}$.

In order to define a similar grading over $\mathbb{R}$ we take a basis of $\mathfrak{g}$ consisting of root vectors and basis elements of $\mathfrak{t}$, whose real span is a real Lie algebra (for example, we can take a Chevalley basis of $\mathfrak{g}$). We denote this real Lie algebra by $\mathfrak{g}(\mathbb{R})$. We set $\mathfrak{g}_0(\mathbb{R}) = \mathfrak{g}_0 \cap \mathfrak{g}(\mathbb{R})$ and $\mathfrak{g}_1(\mathbb{R}) = \mathfrak{g}_1 \cap \mathfrak{g}(\mathbb{R})$, so that

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{g}_0(\mathbb{R}) \oplus \mathfrak{g}_1(\mathbb{R}).$$

If $G_0(\mathbb{R})$ denotes the group of real points of $G_0$, then $\pi$ restricts to a morphism $\pi : \hat{G}(\mathbb{R}) \rightarrow G_0(\mathbb{R})$ that makes $\mathfrak{g}_1(\mathbb{R})$ a $\hat{G}(\mathbb{R})$-module isomorphic to $(\mathbb{R}^2)^{\otimes 4}$.

A first consequence of these constructions is the existence of a Jordan decomposition of the elements of the modules $(\mathbb{C}^2)^{\otimes 4}$ and $(\mathbb{R}^2)^{\otimes 4}$. Indeed, the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(\mathbb{R})$ have such decompositions as every element $x$ can be written uniquely as $x = s + n$ where $s$ is semisimple, $n$ nilpotent, and $[s, n] = 0$. It is straightforward to see that if $x$ lies in $\mathfrak{g}_1$ or $\mathfrak{g}_1(\mathbb{R})$, then the same holds for its semisimple and nilpotent parts. Thus, the elements of $(\mathbb{C}^2)^{\otimes 4}$ and $(\mathbb{R}^2)^{\otimes 4}$ are divided into three groups: semisimple, nilpotent and mixed. Since the actions of $\hat{G}$ and $G(\mathbb{R})$ respect the Jordan decomposition, also the orbits of these groups in their respective modules are divided into the same three groups.

A second consequence is that we can consider $\mathfrak{sl}_2$-triples instead of nilpotent elements; we use these in Section 6 when considering mixed elements: the classification of the orbits of mixed elements with a fixed semisimple part $p$ reduces to the classification of the nilpotent orbits in the centraliser of $p$, which in turn reduces to the classification of orbits of certain $\mathfrak{sl}_2$-triples. We provide more details in Section 6.

3.2. Notation. We now recall the notation used in [25] to describe the classification of $\hat{G}$-orbits in $(\mathbb{C}^2)^{\otimes 4}$.

A Cartan subspace of $\mathfrak{g}_1$ is a maximal space consisting of commuting semisimple elements. A Cartan subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ (and in fact a Cartan subalgebra of $\mathfrak{g}$) is spanned by

$$u_1 = |0000\rangle + |1111\rangle, \quad u_2 = |0110\rangle + |1001\rangle, \quad u_3 = |0101\rangle + |1010\rangle, \quad u_4 = |0011\rangle + |1100\rangle.$$

We denote by $\Phi$ the corresponding root system with Weyl group $W$. This group acts on $\Phi$ and $\mathfrak{h}$ in the following way. For $\alpha \in \Phi$ let $s_\alpha \in W$ be the corresponding reflection. If $\beta \in \Phi$ and $h \in \mathfrak{h}$, then $s_\alpha(\beta) = \beta - \beta(h)\alpha$ and $s_\alpha(h) = h - \alpha(h)\alpha$ where $h_\alpha$ is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_-\alpha] \leq \mathfrak{h}$ with $\alpha(h_\alpha) = 2$. This defines a $W$-action on $\mathfrak{h}$ and we write $W_\mathfrak{h} = \{\alpha \in W : \alpha(p) = p\}$ for the stabiliser of $p \in \mathfrak{h}$ in $W$; the latter is generated by all $s_\alpha$ with $\alpha \in \Phi_\mathfrak{h}$, where $\Phi_\mathfrak{h} = \{\alpha \in \Phi : \alpha(p) = 0\}$, see [25] Lemma 2.4. For a root subsystem $\Pi \subseteq \Phi$ define

$$\mathfrak{h}_\Pi = \{p \in \mathfrak{h} : \alpha(p) = 0 \text{ for all } \alpha \in \Pi\}, \quad W_\Pi = \langle s_\alpha : \alpha \in \Pi \rangle,$$

$$\mathfrak{h}_\Pi^\perp = \{p \in \mathfrak{h}_\Pi : \alpha(p) \neq 0 \text{ for all } \alpha \in \Phi \setminus \Pi\}, \quad \Gamma_\Pi = N_W(W_\Pi)/W_\Pi.$$
Let $\zeta$ be a fixed primitive 8-th root of unity; for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $u \in \mathbb{C}^\times$ we write

$$A^# = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad D(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

(3.1) $F = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$

Throughout this paper, we freely identify the spaces $g_1 \cong (\mathbb{C}^2)^{\otimes 4}$ and $g_1(\mathbb{R}) \cong (\mathbb{R}^2)^{\otimes 4}$ and we write elements of $\hat{G}$ and $\hat{G}(\mathbb{R})$ as 4-tuples $(A, B, C, D)$, with $A, B, C, D$ in $SL(2, \mathbb{C})$, respectively in $SL(2, \mathbb{R})$.

### 3.3. Complex classifications.

In [25] Section 3.1] we have determined 11 subsystems $\Pi_1, \ldots, \Pi_{11}$ to classify the semisimple $\hat{G}$-orbits in $g_1$; these sets are also described in Table [1]. The following result summarises [25] Proposition 2.5, Lemma 2.9, Theorem 3.2, Lemma 3.5, Proposition 3.6].

**Theorem 3.1** (Complex classification of semisimple elements [25]).

a) Each semisimple $\hat{G}$-orbit in $(\mathbb{C}^2)^{\otimes 4}$ intersects exactly one of the sets $h_{\Pi_i}$ nontrivially. Two elements $h_{\Pi_i}$ and $h_{\Pi_j}$ are $\hat{G}$-conjugate if and only if they are $\Gamma_{\Pi_i}$-conjugate. Each $\Gamma_{\Pi_i}$ can be realised as complement subgroup to $W_{\Pi_i}$ in $N_{W_i}(W_{\Pi_i})$, so as a matrix group relative to the basis $u_1, \ldots, u_4$ of $h$. The group $\Gamma_{\Pi_4} \cong (\mathbb{Z}/2\mathbb{Z})^3$ is generated by all $4 \times 4$ diagonal matrices that have two 1s and two $-1$s on the diagonal; the groups $\Gamma_{\Pi_1}, \Gamma_{\Pi_3}, \Gamma_{\Pi_6} \cong Dih_4$ are isomorphic to the dihedral group of order 8 and defined as

$$\Gamma_{\Pi_4} = \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rangle, \quad \Gamma_{\Pi_5} = \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rangle, \quad \Gamma_{\Pi_6} = \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rangle.$$

Furthermore, $\Gamma_{\Pi_1} = W$ and the remaining $\Gamma_{\Pi_i}$ are equal to $\{ \pm 1 \}$.

b) If $x, y \in h_{\Pi_1}$, then $Z_G(x) = Z_G(y)$, and the group $Z_G(x)$ is given in Row i of Table [2].

See [25] Remark 3.3] for a comment on the $\Gamma_{\Pi_i}$-orbits in $h_{\Pi_0}$; this yields a complete and irredundant classification of the semisimple $\hat{G}$-orbits in $g_1$, see Table [1]. The next theorem is [25] Theorem 3.7].

**Theorem 3.2** (Complex classification of mixed elements [25]). For $i \in \{2, \ldots, 10\}$ let $\Sigma_i$ be a set of $\hat{G}$-conjugacy representatives of semisimple elements in $h_{\Pi_i}$ as specified in Table [1]. Up to $\hat{G}$-conjugacy, the mixed elements in $g_1$ are the elements $s + n_{ij}$ where $i \in \{2, \ldots, 10\}, s \in \Sigma_i$, and $n_{ij}$ as specified in Table [3].

The nilpotent $\hat{G}$-orbits in $g_1$ and the nilpotent $\hat{G}(\mathbb{R})$-orbits in $g_1(\mathbb{R})$ are determined in [10] and [26], respectively, see also [25] Table 7]; therefore we do not recall these classifications here. We conclude this section by mentioning [25] Remark 3.1]; the symmetries described in this remark allow us to simplify our classifications.

**Remark 3.3.** If $\sigma \in \text{Sym}_4$, then the linear map $\pi_\sigma: g_1 \to g_1$ that maps each $i_1i_2i_3i_4$ to $i_1i_2i_3i_4\sigma$ extends to a Lie algebra automorphism of $g_1$ that preserves $g_0$ and $g_1$. The group generated by all these $\pi_\sigma$ fixes $u_1$ and permutes $\{u_2, u_3, u_4\}$ as $\text{Sym}_3$. Specifically, $\pi_{(2,3)}$ swaps $u_3$ and $u_4$, and $\pi_{(2,4)}$ swaps $u_2$ and $u_4$.

### 4. Galois cohomology

We describe some results from Galois cohomology that we use for determining the real orbits within a complex orbit; see [7] Section 3] for a recent treatment of Galois cohomology in the context of orbit classifications.

In this section only we consider the following notation. Let $G$ be a group with conjugation $\sigma: G \to G$, that is, an automorphism of $G$ of order 2; often $\sigma$ is the complex conjugation of a complex group. An element $c \in G$ is a cocycle (with respect to $\sigma$) if $\sigma(c) = 1$; write $Z^1(G, \sigma)$ for the set of all cocycles. Two cocycles $c, c'$ are equivalent if $c' = a\sigma(a)^{-1}$ for some $a \in G$; the equivalence class of $c$ is denoted $[c]$, and the set of equivalence classes is denoted $H^1(G, \sigma)$. We also write $Z^1(G)$ and $H^1(G)$ if it is clear which conjugation is used; these definitions are an adaption of the definitions in [50] §5.1] to the special case of an acting group $\langle \sigma \rangle$ of size 2.

We now list a few results that help to determine $H^1(G, \sigma)$. In the following we write $G^\sigma = \{ g \in G : \sigma(g) = g \}$.

Let $X$ be a set on which $G$ acts. We suppose that $X$ has a conjugation, also denoted $\sigma$ (that is, a map $\sigma: X \to X$ with $\sigma^2 = 1d(X)$, such that $\sigma(\sigma(x)) = \sigma(x)$ for all $x \in X$ and $g \in G$). Let $O$ be a $G$-orbit in $X$ that has a real point, that is, there is $x_0 \in O$ with $\sigma(x_0) = x_0$. In this situation, $O$ is stable under $\sigma$, and we are interested in listing the $G^\sigma$-orbits in $O^\sigma = \{ y \in O : \sigma(y) = y \}$. For this we consider the stabiliser

$$Z_G(x_0) = \{ g \in G : gx_0 = x_0 \}$$
and the exact sequence

$$1 \to Z_G(x_0) \xrightarrow{i} G \xrightarrow{j} O \to 1$$

resulting from the orbit-stabiliser theorem; here $j$ maps $g$ to $gx_0$. This sequence gives rise to the exact sequence

$$1 \to (Z_G(x_0))^{\sigma} \xrightarrow{i^\sigma} G^{\sigma} \xrightarrow{j^\sigma} O^{\sigma} \xrightarrow{\delta} H^1(Z_G(x_0), \sigma) \xrightarrow{i^*} H^1(G, \sigma),$$

where $i^*$ sends the class defined by $g \in Z^1(Z_G(x_0), \sigma)$ to its class in $H^1(G, \sigma)$; moreover, $\delta(gx_0)$ is the class of the cocycle $g^{-1}\sigma(g)$. The following is one of the main theorems in Galois cohomology, see [50] Proposition 36.

**Theorem 4.1.** The map $\delta$ induces a bijection between the orbits of $G^\sigma$ in $O^{\sigma}$ and the set $\ker i^*.$

**Remark 4.2.** It is known for the usual complex conjugation $\bar{\cdot}$ that $H^1(Gl(n, \mathbb{C})) = 1 = H^1(SL(n, \mathbb{C}))$ for all $n \geq 1$, see for example [6] Proposition III.8.24 and Corollary III.8.26, in particular, $H^1(C_\times) = 1.$ Moreover, if $\bar{\cdot}$ acts entry-wise on a complex matrix group $G = X \times Y,$ then $H^1(X \times Y) \cong H^1(X) \times H^1(Y).$ Since a torus $T \leq G$ is a direct product of copies of $C_\times$, we have that $H^1(T, \bar{\cdot}) = 1.$

4.1. **Cartan subspaces.** Recall that $\mathfrak{h}$ is the fixed Cartan subspace spanned by $\{u_1, \ldots, u_4\}$. Semisimple elements that lie in $\mathfrak{h}$ are represented as a linear combination of these basis elements, however, most of our real orbit representatives lie in a Cartan subspace different to $\mathfrak{h}$. To simplify the notation in our classification tables, we classify all Cartan subspaces and then represent our semisimple orbit representatives with respect to fixed bases of these spaces.

It follows from Galois cohomology that the real Cartan subspaces in $q_1$ are, up to $\hat{G}$-conjugacy, in bijection with $H^1(N)$ where $N = N_{G}(\mathfrak{h})$, see [7] Theorem 4.4.9]. The group $N$ fits into an exact sequence

$$1 \to Z_{\hat{G}}(\mathfrak{h}) \to N \to W \to 1.$$ 

Since $Z_{\hat{G}}(\mathfrak{h})$ and $W$ are finite groups of orders 32 and 192, respectively, $N$ is a finite group of order $32 \cdot 192 = 6144$. Because we know $Z_{\hat{G}}(\mathfrak{h})$ and $W$ (for the former see the first line of Table 4), we can determine $N$. Since $N$ is finite, a brute force calculation determines $H^1(N)$, and we obtain $|H^1(N)| = 7.$

For a fixed $[n] \in H^1(N)$ define $\tau : \mathfrak{h} \to \mathfrak{h}$ by $\tau(u) = nui$. Since $\tau$ is an anti-involution of $\mathfrak{h}$, the $\mathbb{R}$-dimension of the fixed space $\mathfrak{h}^\tau = \{u \in \mathfrak{h} : \tau(u) = u\}$ equals the $C$-dimension of $\mathfrak{h}$. Let $g \in \hat{G}$ be such that $g^{-1}\bar{g} = n$; if $u \in \mathfrak{h}^\tau$, then $u = nri$, and the element $\eta u = gmn^{-1}u = gu$ is real. Thus, the real span of all $gu$ with $u \in \mathfrak{h}^\tau$ gives a real Cartan subspace. Iterating this procedure for all $[n] \in H^1(N)$ gives all real Cartan subspaces up to $G(\mathbb{R})$-conjugacy; we fix the notation in the following definition.

**Definition 1.** There are seven classes in $H^1(N)$ corresponding to cocycles $n^*_1, \ldots, n^*_7 \in Z^1(N)$; for each $i \in \{1, \ldots, 7\}$ choose $g^*_i \in \hat{G}$ such that $(g^*_i)^{-1}g^*_i = n^*_i$ and $\zeta_i = g^*_i(\mathfrak{h}^\tau)$. Specifically, using the notation introduced in (3.3), we choose

$$g^*_1 = (I, I, I, I), \quad g^*_2 = (L, I, I, I), \quad g^*_3 = (D(\eta^5), D(\eta^5), -D(\eta^3), -D(\eta^7)), \quad g^*_4 = (M, I, I, M)$$

$$g^*_5 = (I, M, I, M), \quad g^*_6 = (I, I, M, M), \quad g^*_7 = (D(\eta^5), D(\eta^5), D(\eta^5), D(\eta^5)).$$

where $\eta$ is a primitive 16-th root of unity with $\eta^2 = \zeta.$ Moreover, we fix the following bases for the seven Cartan subspaces $\zeta_1, \ldots, \zeta_7$ constructed above:

$$\{u_1 = [0000] + [1111], \quad u_2 = [0110] + [1001], \quad u_3 = [0101] + [1010], \quad u_4 = [0011] + [1100]\}$$

$$\{v_1 = [0000] - [1111], \quad v_2 = [0110] - [1001], \quad v_3 = [0101] - [1010], \quad v_4 = [0011] - [1100]\}$$

$$\{w_1 = [0000] - [1111], \quad w_2 = [0110] - [1001], \quad w_3 = [0101] - [1010], \quad w_4 = [0011] + [1100]\}$$

$$\{x_1 = [0000] - [1111], \quad x_2 = [0110] - [1001], \quad x_3 = [0101] + [1010], \quad x_4 = [0011] + [1100]\}$$

$$\{y_1 = [0000] - [1111], \quad y_2 = [0110] + [1001], \quad y_3 = [0101] - [1010], \quad y_4 = [0011] + [1100]\}$$

$$\{z_1 = [0000] - [1111], \quad z_2 = [0110] + [1001], \quad z_3 = [0101] + [1010], \quad z_4 = [0011] - [1100]\}$$

$$\{t_1 = [0000] - [1111], \quad t_2 = [0110] + [1001], \quad t_3 = [0101] + [1010], \quad t_4 = [0011] + [1100]\}$$

5. **Real semisimple elements**

Throughout this section, we fix one of the subsystems \( \Pi_\Pi \), of Theorem 3.1 and abbreviate \( C = \mathfrak{h}^0_\Pi \). We fix a complex $\hat{G}$-orbit $O = \hat{G}t$ for some nonzero $t \in C$. We now discuss the following problems related to the orbit $O$: 
1) Decide whether $O \cap g_1(\mathbb{R})$ is nonempty, that is, whether $O$ has a real point.
2) If $O$ has real points, how can we find one?
3) Determine representatives of the real $\hat{G}(\mathbb{R})$-orbits contained in $O$.

We prove a number of results that help to decide these questions. These results as well as the proofs are similar to material found in [3]. However, the results in [3] concern a specific $\mathbb{Z}/3\mathbb{Z}$-grading of the Lie algebra of type $E_8$. Since here we consider a different situation, we have included the new proofs.

In the following, the centraliser and normaliser of $C$ in $\hat{G}$ are denoted by

$$Z_{\hat{G}}(C) = \{g \in \hat{G} : gx = x \text{ for all } x \in C\}$$

$$N_{\hat{G}}(C) = \{g \in \hat{G} : gx \in C \text{ for all } x \in C\}$$

**Lemma 5.1.** Let $t_1, t_2 \in C$. If $gt_1 = t_2$ for some $g \in \hat{G}$, then $g \in N_{\hat{G}}(C)$.

**Proof.** Theorem 3.1 shows that $w(t_1) = t_2$ for some $w \in N_W(W_H)$. If $\hat{w} \in N_{\hat{G}}(h)$ is a preimage of $w$, then $g^{-1}\hat{w} \in Z_{\hat{G}}(t_2)$. Theorem 3.1 shows that $g^{-1}\hat{w} \in Z_{\hat{G}}(x)$ for every $x \in C$, so $gx = \hat{w}x = w(x)$ for all $x \in C$.

Now we define a map $\varphi : N_{\hat{G}}(C) \to \Gamma_H$. If $g \in N_{\hat{G}}(C)$, then $gg = x \in C$, hence $w(g) = x$ for some $w \in N_W(W_H)$. We define $w(g) = wW_H \in \Gamma_H$. This is well-defined: if $w' \in N_W(W_H)$ satisfies $w'(q) = x$, then $w'^{-1}w \in W_H$, so $w'W_H = wW_H$.

**Lemma 5.2.** The map $\varphi : N_{\hat{G}}(C) \to \Gamma_H$ is a surjective group homomorphism with kernel $Z_{\hat{G}}(C)$. Moreover, if $g \in N_{\hat{G}}(C)$ and $x \in C$, then $gx = \varphi(g)x$.

**Proof.** We start with a preliminary observation. If $g \in N_{\hat{G}}(C)$ and $w \in N_W(W_H)$ such that $wg = w(q)$, then $g^{-1}g = w(y)$ for all $y \in C$; indeed, if $\hat{w} \in N_{\hat{G}}(h)$ is a preimage of $w$, then $h^{-1}\hat{w}g = Z_{\hat{G}}(g)$ and $h^{-1}\hat{w}g \in Z_{\hat{G}}(y)$ by Theorem 3.1. Now let $g_1, g_2 \in N_{\hat{G}}(C)$ and let $w_1, w_2 \in N_W(W_H)$ be such that each $w_i(q) = g_iq$. By the made observation, $g_1g_2w = w_1(w_2q)$; this implies that that $\varphi$ is a group homomorphism. If $wZ_{\hat{G}}(g) \in \Gamma_H$ with preimage $w \in N_{\hat{G}}(h)$, then $\hat{w} \in N_{\hat{G}}(C)$ and $\varphi(\hat{w}) = wW_H$, which shows that $\varphi$ is surjective. If $g \in \ker(\varphi)$, then $g = q$ and the first part of the proof shows that $g \in Z_{\hat{G}}(C)$.

By abuse of notation, we also write

$$\varphi : N_{\hat{G}}(C)/Z_{\hat{G}}(C) \to \Gamma_H$$

for the induced isomorphism. The next theorem provides solutions to Problems 1) and 2); it is similar to [8] Proposition 5.2.4. Recall that we fixed $\Pi = \Pi_1$ and $O = \hat{G}t$ with $t \in C$.

**Theorem 5.3.** Write $H^1(\Gamma_H) = [\gamma_1, \ldots, \gamma_s]$. Suppose that for each $\gamma_i \in Z(\Gamma_H)$ there is $n_i \in Z^1(N_{\hat{G}}(C))$ with $\varphi(n_i) = \gamma_i$. Then $O$ has a real point if and only if there exist $q' \in O \cap C$ and $i \in \{1, \ldots, s\}$ with $q' = \gamma_i^{-1}q$.

If the latter holds, then $gg' = \gamma_i^{-1}q$ is a real point of $O$, where $g \in \hat{G}$ is such that $g^{-1}\hat{g} = n_i$.

**Proof.** The elements $n_i$ exist by Lemma 5.2. If $O$ has a real point, say $p = gt$ for some $g \in \hat{G}$, then $\overline{gt} = gt$, and so $\bar{t} = n^{-1}t$ for $n = g^{-1}\hat{g}$; note that $n$ is a cocycle since $n\overline{n} = 1$. Because $t, \bar{t} \in C$, Lemma 5.1 shows that $n \in N_{\hat{G}}(C)$, so we can define $\bar{\gamma} = \varphi(n)$. Since $\gamma \in Z(\Gamma_H)$, there is $i \in \{1, \ldots, s\}$ and $\beta \in \Gamma_H$ with $\gamma = \beta^{-1}\gamma_i\beta$. Now Lemma 5.2 shows that $\bar{\gamma} = \gamma_i^{-1}t = \beta^{-1}\gamma_i^{-1}\beta t$, so if we set $q' = \beta t$, then $q' = \gamma_i^{-1}q'$, as claimed. Conversely, let $q' \in O \cap C$ and $\gamma_i$ be such that $q' = \gamma_i^{-1}q'$. By hypothesis there is $n_i \in Z^1(N_{\hat{G}}(C))$ with $\varphi(n_i) = \gamma_i$, hence $\overline{n_i} = \gamma_i^{-1}q'$ by Lemma 5.2. Because $n_IN_{\hat{G}} = 1$ in $\hat{G}$ and $H^1(\hat{G}) = 1$, there is a $g \in \hat{G}$ with $n_i = g^{-1}\hat{g}$.

**Remark 5.4.** The real point $gg'$ mentioned in Theorem 5.3 might lie in a Cartan subspace different to $h$. The real points corresponding to the class $[\gamma_1] = [1]$ can be chosen to lie in the Cartan subspace $h$.

**Remark 5.5.** One of the hypotheses of the theorem is that for each $\gamma_i$ there is a cocycle $n_i \in Z^1(N_{\hat{G}}(C))$ such that $\varphi(n_i) = \gamma_i$. We cannot prove this a priori, but for the cases that are relevant to the classification given in this paper we have verified it.

Galois cohomology also comes in handy for a solution to Problem 3): the next theorem follows from Theorem 4.1 taking into account that $H^1(\hat{G}) = 1$, see Remark 4.2.
Theorem 5.6. Let $p \in \mathcal{O}$ be a real representative. There is a 1-to-1 correspondence between the elements of $H^1(Z_G(p))$ and the $G(\mathbb{R})$-orbits of semisimple elements in $\mathcal{O}$ that are $G$-conjugate to $p$: the real orbit corresponding to $[z] \in H^1(Z_G(p))$ has representative $b\gamma$ where $b \in \hat{G}$ is chosen with $z = b^{-1}\gamma$.

5.1. Classification approach. Now we explain the classification procedure in full detail. For $i \in \{1, \ldots, 10\}$ we compute some information related to Row $i$ of Table 1. First, we construct the cohomology sets $H^1(\Gamma_{\Pi_i})$. The $\Gamma_{\Pi_i}$ are finite groups with trivial conjugation, so the cohomology classes coincide with the conjugacy classes of elements of order dividing 2. These can be computed brute-force. If the complex orbit of a semisimple element $q$ has a real point, then Theorem 5.3 shows that there is some $\gamma_j$ and some $q'$ in the orbit of $q$ such that $q'^{-1}q = \gamma_j^{-1}q'$; now $gq'$ is a real point in the orbit of $q$, where $g$ is defined in Theorem 5.3. We therefore proceed by looking at each $[\gamma_j]$ in $H^1(\Gamma_{\Pi_i})$ and determining all $q'$ such that $q'^{-1}q = \gamma_j^{-1}q'$; this will eventually determine the orbits of elements in $\hat{b}_{\Pi_i}$ that have real points, along with a real point in each such orbit. The next lemma clarifies that the elements determined for different $j$ yield real orbit representatives of different orbits.

Lemma 5.7. With the above notation, if $j \neq k$, then the real orbit representatives obtained for $[\gamma_j]$ are not $G$-conjugate to those representatives obtained for $[\gamma_k]$.

Proof. Suppose in the above procedure we construct $q_j, q_k \in \hat{b}_{\Pi_i}$ such that $q_j = \gamma_j^{-1}q_j$ and $q_k = \gamma_k^{-1}q_k$. If $q_j$ and $q_k$ lie in the same $\hat{G}$-orbit, then Theorem 5.1 shows that $q_k = \beta q_j$ for some $\beta \in \Gamma_{\Pi_i}$, that is, $q_k = \gamma_k^{-1}\beta q_j$, and solving for $q_j$ yields $q_j = \beta^{-1}\gamma_k\beta q_j = \beta^{-1}\gamma_k\beta^{-1}\gamma_j^{-1}q_j$. Since $q_j, q_k \in \hat{b}_{\Pi_i}$, we have $W_{q_j} = W_{q_k}$, see Section 3.2, and so $\beta^{-1}\gamma_k\beta^{-1} \in W_{q_j}$. Since $\Gamma_{\Pi_i} = N_W(W_{q_k})/W_{q_k}$, it follows that $[\gamma_j] = [\gamma_k]$ in $H^1(\Gamma_{\Pi_i})$. □

Our algorithm now proceeds as follows; recall that each of our $\Gamma_{\Pi_i}$ is realised as a subgroup of $W$. (A) For each component $C \in \hat{b}_{\Pi_i}$ and each cohomology class $[\gamma_j] \in H^1(\Gamma_{\Pi_i})$ with $\gamma_j \in W$, we determine all $q' \in C$ that satisfy $q'^{-1}q = \gamma_j^{-1}q'$ (using Table 3 this condition on $q'$ is easily obtained). We then determine $q_j \in \hat{G}$ such that $n_j = g_j^{-1}q_j$ (using Table 5), and set $p = g_jq'$. Theorem 5.3 shows that $p$ is a real representative in the complex orbit $\hat{G}q'$ of $q'$. We note that in this approach we do not fix a complex orbit $\mathcal{O}$ and look for $q' \in \mathcal{O} \cap C$ as in Theorem 5.3 but we first look for suitable $q' \in C$ and then consider the reduction up to $\Gamma_{\Pi_i}$-conjugacy.

(B) Next, we determine the real orbits contained in $\hat{G}p = \hat{G}q'$. Using Theorem 5.6 we need to consider $Z = Z_{\hat{G}}(p)$ and determine $H^1(Z)$ with respect to the usual complex conjugation $\bar{\cdot}$. Note that $Z$ is one of the centralisers in Table 2 and $Z = g_jZ_{\hat{G}}(q)g_j^{-1}$. It will turn out that in most cases we can decompose $Z = \tilde{Z} \times H$ where $H$ is abelian of finite order; then $H^1(Z) = H^1(\tilde{Z}) \times H^1(H)$ by Remark 4.2 which is useful for determining $H^1(Z)$. We will see that the component group $\tilde{Z}/Z^o$ is of order at most 2; if it is nontrivial, then is generated by the class of $(J,J,J,J)$ where $J$ is as in (4.1). The connected component $Z^o$ is in most cases parametrised by a torus or by $SL(2, \mathbb{C})$, and we show that $H_1(Z^o)$ is trivial (see Remark 4.2). To compute $H^1(Z)$ it remains to consider the cohomology classes of elements of the form $u = w(J,J,J,J) \in \hat{Z} \setminus Z^o$ where $w \in Z^o$. We determine conditions on $w$ such that $u$ is a 1-cocycle, and then solve the equivalence problem. All these calculations can be done by hand, but we have also verified them computationally with the system GAP.

(C) Finally, given $\gamma_j, n_j, g_j$ and $H^1(Z)$, Theorem 5.3 shows that a real orbit representative corresponding to $[z_k] \in H^1(Z)$ is $g_kp$ where $g_k \in \hat{G}$ is chosen such that $b_k^{-1}g_k^{-1} = z_k$ (using the elements $\epsilon(M)$ given in Table 5); specifically, we obtain the following real orbits representatives, recall that $p = g_jq'$:

\[(\epsilon(A), \epsilon(B), \epsilon(C), \epsilon(D))(g_jq') : [(A, B, C, D)] \in H^1(Z) \text{ and } q' \in \hat{b}_{\Pi_i}, \text{ with } q' = \gamma_j^{-1}q'.\]

Remark 5.8. Some real points and some real orbit representatives computed in (A) and (C) lie outside our fixed Cartan subspaces $c_1, \ldots, c_7$ as defined in Definition 1. If this is the case, then we rewrite these elements by following steps (A') and (C') after (A) and (C), respectively:

(A') If the real point $p$ in (A) is not in one of our fixed Cartan spaces $c_1, \ldots, c_7$, then we search for $g \in g_k^*N_{\hat{G}}(b)$ for some $k \in \{1, \ldots, 7\}$ such that $n_j = g^{-1}g_k$; recall the definition of $g_k^*$ and $n_k^*$ from Definition 1 then we replace $g_j$ by $g$. This is indeed possible: by construction, there is $g_0 \in N_{\hat{G}}(b)$ and $k \in \{1, \ldots, 7\}$ with $g_0n_jg_0^{-1} = n_k = (g_k^*)^{-1}g_k^*$ so $g = g_0n_jg_0 \in g_k^*N_{\hat{G}}(b)$ is a suitable element, and $p = gg'$ lies in $c_k$.

(C') If one of the $b_kp$ in Step (C) is not in our Cartan spaces $c_1, \ldots, c_7$, then we proceed as follows (using Definition 1). Note that $b_kp$ is $\hat{G}(\mathbb{R})$-conjugate to one of our Cartan spaces, say $b_k'bp \in c_j$ for some $b_k' \in \hat{G}(\mathbb{R})$. Classification of four-dimensional rebits
and \( j \in \{1, \ldots, 7\} \). Since \( b'_k \) is real, \((b'_k b_k)^{-1}b'_k b_k = z_k\), and it follows that \( b_0 = b'_k b_k \in \hat{G} \) satisfies \( b_0^{-1} b_0 = z_k \) and \( b_0 p \in \xi_j \), as required. Since \((g_j^*)^{-1}b_0 p \in \hat{h}\) is \( \hat{G}\)-conjugate to \( q \in \hat{h}\), there exists \( w \in \mathcal{N}_{\hat{G}}(h) \) such that \((g_j^*)^{-1}b_0 p = w q = w g^{-1} p\), where \( p = g q \) as in \((A)\); we always succeed finding \( b_0 \) in \( g_j^* \mathcal{N}_{\hat{G}}(h) g^{-1}\).

To simplify the exposition, in our proof below we do not comment on the rewriting process \((A')\) and \((C')\), but only describe the results for \((A), (B), \) and \((C)\).

### 5.2. Classification results

The procedure detailed in Section 5.1 leads to the following result; we prove it in this section.

**Theorem 5.9.** Up to \( \hat{G}(\mathbb{R})\)-conjugacy, the nonzero semisimple elements in \( g_1(\mathbb{R}) \) are the elements in Tables 6-17 in Appendix A.2, see Definition 4 for the notation used in these tables.

It follows from Theorem 3.1 and Theorem 5.9 that there are many complex semisimple orbits that have no real points. For example, consider the Case \( i = 10 \) and let us fix \( q = \lambda_1 u_1 \) with \( \lambda = a + ib \) and \( a, b \neq 0 \). According to the description given in Case 10 of the proof of Theorem 5.9 there exists a real point in that orbit if and only if there is \( q' \) with \( \overline{q'} = \gamma_0^{-1} q' \); this is equivalent to requiring that \( \lambda \) is either a real or a purely imaginary number, which is a contradiction; thus the complex semisimple orbit determined by \( q \) has no real points.

We now prove Theorem 5.7 by considering each case \( i \in \{1, 2, 3, 4, 7, 10\} \) individually; due to Remark 5.3 the classifications for the cases \( i \in \{5, 6, 8, 9\} \) can be deduced from those for \( i \in \{4, 7\} \). For each \( i \) we comment on the classification steps \((A), (B), (C)\) as explained in the previous section; we do not comment on the rewriting process \((A')\) and \((C')\). In Case i below we write \( Z = Z_{\hat{G}}(p_i) \) as in Table 2. Throughout, we use the notation introduced in \( 3.1 \).

**Case i = 1.** There are 7 equivalence classes of cocycles in \( \Gamma_{\Pi_1} = W \), with representatives \( \gamma_1 = I, \gamma_2 = -I, \gamma_3 = \text{diag}((-1,-1,-1,1)), \gamma_4 = \text{diag}((-1,1,1,1)), \gamma_5 = \text{diag}((-1,1,1,1)), \gamma_6 = \text{diag}((-1,1,1,1)) \), and \( \gamma_7 = \text{diag}((-1,1,1,1)). \) The centraliser \( Z \) is finite so the cohomology can be easily computed, and \( H^1(Z) \) has 12 classes with representatives

\[
\begin{align*}
&z_1 = (1, 1, 1, 1), \quad z_2 = (1, 1, -1, 1), \quad z_3 = (1, -1, 1, 1), \quad z_4 = (1, -1, -1, 1), \\
&z_5 = (K, K, K, K), \quad z_6 = (K, K, -K, -K), \quad z_7 = (K, -K, K, -K), \quad z_8 = (K, -K, -K, K), \\
&z_9 = (L, L, L, L), \quad z_{10} = (L, L, -L, -L), \quad z_{11} = (L, -L, L, -L), \quad z_{12} = (L, -L, -L, L).
\end{align*}
\]

We follow the procedure outlined in Section 5.1 and consider the various \([c] \in H^1(\Gamma_{\Pi_1})\). We note that in all cases \((1, j)\) below we start with a complex semisimple element \( \lambda_1 u_1 + \cdots + \lambda_4 u_4 \) as in Table 4 with \( (\lambda_1, \ldots, \lambda_4) \) reduced up to \( \Gamma_{\Pi_1}\)-conjugacy, where \( \Gamma_{\Pi_1} = W \) is the Weyl group.

**\((i, j) = (1, 1)\):** Let \([c] = [\gamma_1] = [\text{diag}(1,1,1,1)]\). We first determine all \( q' \in h_0^{\Pi_1} \) with \( \overline{q'} = q' \); by Theorem 5.1 these are the elements \( q' = \lambda_1 u_1 + \cdots + \lambda_4 u_4 \in \Pi_{\Pi_1} \) with \( \lambda_1, \ldots, \lambda_4 \in \mathbb{R} \setminus \{0\} \) and \( \lambda_1 \neq \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \). Since \( \gamma_1 = I \), we can choose \( n_1 = g_1 = (1, 1, 1, 1) \), and obtain \( p = g_1 q' = q' \) as real point in the complex \( G\)-orbit of \( q' \).

Since the first cohomology group of \( Z_{\hat{G}}(q') = Z_{\hat{G}}(p) \) has 12 elements, it follows from Theorem 5.6 that \( \hat{G}\)-split into 12 real orbits with representatives determined as in (5.1). For \( z_1 = (1, 1, 1, 1) \) we have \( b_1 = (1, 1, 1, 1) \), with real orbit representative \( b_1 p = b_1 q' = q' \). For \( z_2 = (1, 1, -1, 1) \) we choose \( b_2 = (1, 1, L, L) \), with real orbit representative \( b_2 p = b_2 q' = -\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 - \lambda_4 u_4 \in \Pi_{\Pi_1} \). In the same way we obtain the orbit representatives for \( z_3, \ldots, z_{12} \), which we summarise in Table 11 in the block \( j = 1 \).

**\((i, j) = (1, 2)\):** Now let \([c] = [\gamma_2] = [\text{diag}(-1,-1,1,1)]\). We determine all \( q' \in h_0^{\Pi_1} \) with \( \overline{q'} = -q' \); these are the elements \( q' = \lambda_1 u_1 + \cdots + \lambda_4 u_4 \in \Pi_{\Pi_1} \) with \( \lambda_1, \ldots, \lambda_4 \in i\mathbb{R} \setminus \{0\} \) and \( \lambda_1 \neq \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \). Table 4 shows that \( \text{diag}(-1,-1,1,1) \in W \) is induced by \( n_2 = (-I, I, I, I) \). An element \( g_2 \) with \( g_2^{-1} q' = n_2 \) is \( g_2 = (L, I, I, I) \). Now \( p = g_2 q' \) is a real point in the complex orbit of \( q' \). Since \( Z_{\hat{G}}(p) = g_2 \mathcal{Z}_{\hat{G}}(q') \), we can directly compute \( H^1(Z_{\hat{G}}(p)) \) and obtain 12 classes with the following representatives

\[
\begin{align*}
&z_1 = (1, 1, 1, 1), \quad z_2 = (-1, -1, 1, 1), \quad z_3 = (1, -1, 1, 1), \quad z_4 = (-1, 1, 1, 1), \\
&z_5 = (K, K, K, -K), \quad z_6 = (-K, -K, K, K), \quad z_7 = (-K, -K, K, K), \quad z_8 = (-K, K, K, K), \\
&z_9 = (-L, -L, -L, -L), \quad z_{10} = (L, L, -L, -L), \quad z_{11} = (-L, L, -L, -L), \quad z_{12} = (L, -L, -L, L).
\end{align*}
\]

Real orbits representatives are now determined as in Equation (5.1), see Table 11. Note that here every \( \lambda_i \) is purely imaginary, but each product \( \varepsilon(A), \varepsilon(B), \varepsilon(C), \varepsilon(D) \) of \( \lambda_1 u_1 + \cdots + \lambda_4 u_4 \) is a real point.
We repeat the same procedure for \( \gamma_3, \ldots, \gamma_7 \); for each case we only summarise the important data, and we refer to Table 11 for the list of real orbits representatives.

(i, j) = (1, 3): If \( [c] = [\gamma_3] \), then \( q' \in b_{H_1} \) satisfies \( q' = \gamma_3^{-1} q' \) if and only if \( q' = \lambda_1 u_1 + \ldots + \lambda_4 u_4 \) with \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\} \) and \( \lambda_i \notin \{ \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \} \); we have \( n_3 = (M, M, -N, N) \) and \( g_3 = (D(\eta^3), D(\eta^3), -D(\eta^3), -D(\eta^3)) \); the first cohomology of \( Z_G(p) = g_3 z_G(q') g_3^{-1} \) consists of four classes defined by the representatives
\[
z_1 = (-I, -I, -I, -I), \quad z_2 = (-I, -I, I, I), \quad z_3 = (-I, -I, I, I), \quad z_4 = (-I, I, -I, I).
\]

(i, j) = (1, 4): If \( [c] = [\gamma_4] \), then \( q' \in b_{H_1} \) satisfies \( q' = \gamma_4^{-1} q' \) if and only if \( q' = \lambda_1 u_1 + \ldots + \lambda_4 u_4 \) with \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\} \) and \( \lambda_i \notin \{ \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \} \). We have \( n_4 = (L, I, I, L) \), \( g_4 = (M, I, I, M) \), and the first cohomology of \( Z_G(p) = g_4 z_G(q') g_4^{-1} \) consists of four classes defined by the representatives
\[
z_1 = (I, I, I, I), \quad z_2 = (I, I, -I, -I), \quad z_3 = (L, L, L, L), \quad z_4 = (L, L, -L, -L).
\]

(i, j) = (1, 5): If \( [c] = [\gamma_5] \), then \( q' \in b_{H_1} \) satisfies \( q' = \gamma_5^{-1} q' \) if and only if \( q' = \lambda_1 u_1 + \ldots + \lambda_4 u_4 \) with \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\} \) and \( \lambda_i \notin \{ \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \} \). We get \( n_5 = (I, I, I, L) \), \( g_5 = (I, M, I, M) \), and the first cohomology of \( Z_G(p) = g_5 z_G(q') g_5^{-1} \) consists of four classes defined by the representatives
\[
z_1 = (I, I, I, I), \quad z_2 = (I, I, -I, -I), \quad z_3 = (L, L, L, L), \quad z_4 = (L, L, -L, -L).
\]

(i, j) = (1, 6): If \( [c] = [\gamma_6] \), then \( q' \in b_{H_1} \) satisfies \( q' = \gamma_6^{-1} q' \) if and only if \( q' = \lambda_1 u_1 + \ldots + \lambda_4 u_4 \) with \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\} \) and \( \lambda_i \notin \{ \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \} \). Now \( n_6 = (I, I, L, L) \), \( g_6 = (I, I, M, M) \), and the first cohomology of \( Z_G(p) = g_6 z_G(q') g_6^{-1} \) consists of four classes defined by the representatives
\[
z_1 = (I, I, I, I), \quad z_2 = (I, I, -I, -I), \quad z_3 = (L, L, L, L), \quad z_4 = (L, L, -L, -L).
\]

(i, j) = (1, 7): If \( [c] = [\gamma_7] \), then \( q' \in b_{H_1} \) satisfies \( q' = \gamma_7^{-1} q' \) if and only if \( q' = \lambda_1 u_1 + \ldots + \lambda_4 u_4 \) with \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\} \) and \( \lambda_i \notin \{ \pm \lambda_2 \pm \lambda_3 \pm \lambda_4 \} \). We have \( n_7 = (M, M, M, M) \) and \( g_7 = (D(\eta^2), D(\eta^2), D(\eta^2), D(\eta^2)) \); the first cohomology of \( Z_G(p) = g_7 z_G(q') g_7^{-1} \) consists of four classes defined by the representatives
\[
z_1 = (I, I, I, I), \quad z_2 = (I, I, -I, -I), \quad z_3 = (I, -I, -I, -I), \quad z_4 = (I, -I, I, I).
\]

Case \( i = 2 \). Since \( \Gamma_{H_2} \) is elementary abelian of order 8, there are 8 equivalence classes of cocycles in \( H^1(\Gamma_{H_2}) \), with representatives
\[
\gamma_1 = \text{diag}(1, 1, 1, 1), \quad \gamma_2 = \text{diag}(1, -1, -1, 1), \quad \gamma_3 = \text{diag}(1, -1, 1, -1), \quad \gamma_4 = \text{diag}(-1, -1, 1, 1),
\]
\[
\gamma_5 = \text{diag}(1, 1, -1, -1), \quad \gamma_6 = \text{diag}(-1, 1, -1, 1), \quad \gamma_7 = \text{diag}(-1, -1, 1, -1), \quad \gamma_8 = \text{diag}(-1, -1, -1, -1).
\]
The centraliser decomposes as \( Z = \tilde{Z} \times H \) where \( H \) is abelian of order 4, generated by \( (I, -I, -I, -I) \) and \( (-I, -I, -I, -I) \). Furthermore \( \tilde{Z}/Z^0 \) has size 2, generated by the class of \( (J, J, J, J) \), and \( Z^0 \) is a 1-dimensional torus consisting of elements \( T_1(a) = (D(a^{-1}), D(a^{-1}), D(a)) \) with \( a \in \mathbb{C} \setminus \{0\} \). The main difference to the Case \( i = 1 \) is that here \( Z \) is not finite; we include some details to explain our computations. First, a direct calculation shows that \( H^1(H) \) consists of four classes defined by the representatives
\[
z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I), \quad z_3 = (-I, -I, -I, -I), \quad z_4 = (-I, I, -I, I).
\]

Next, we look at \( H^1(\tilde{Z}) \). Since \( Z^0 \) is a 1-dimensional torus, a direct computation (together with Remark 4.3) shows that \( H^1(Z^0) \) is trivial. It remains to consider the cohomology classes of elements in \( \tilde{Z}/Z^0 \). Let \( w = u j^* \) where \( w \in Z^0 \) and
\[
j^* = (J, J, J, J).
\]
A short calculation shows that \( u = 1 \) is a cocycle if and only if there is \( a \in \mathbb{R} \setminus \{0\} \) with
\[
u = \begin{pmatrix}
0 & -ia & 0 & 0 \\
0 & -ia & 0 & 0 \\
0 & 0 & ia & 0 \\
0 & 0 & 0 & ia
\end{pmatrix} \times \begin{pmatrix}
0 & 0 & ia & 0 \\
a & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & 0 & a
\end{pmatrix} = \begin{pmatrix}
0 & 0 & ia & 0 \\
a & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & 0 & a
\end{pmatrix}.
\]
Moreover, every such \( u \) is equivalent to \( k = (-K, -K, K, K) \), thus \( H^1(\tilde{Z}) = \{[1], [k]\} \). Indeed, we can verify (by a short calculation or with the help of GAP) that two 1-cocycles \( u, u' \) satisfying \( u' = g u g^{-1} \) where \( g = (D(c), D(c), D(c), D(c)) \in Z^0 \) for some \( c \in \mathbb{C}^\times \) if and only if \( au' = a|c|^2 \), thus we can assume \( a = \pm 1 \). Now \( u' = g j^* u (g j^* g)^{-1} \) for some \( g \in Z^0 \) if and only if \( au' = (c/|c|)^2 \), then the 1-cocycles corresponding to \( a = 1 \).
and \(a' = -1\) are equivalent. Since \(H^1(Z) = H^1(\tilde{Z}) \times H^1(H)\), representatives of the classes in \(H^1(Z_G(p))\) are \(z_1, \ldots, z_4\) and
\[
\begin{align*}
z_5 &= z_1 k = (-K, -K, K, K), \quad z_6 = z_2 k = (K, K, K, K), \\
z_7 &= z_3 k = (K, -K, -K, K), \quad z_8 = z_4 k = (-K, K, -K, K).
\end{align*}
\]
With the same approach we obtain \(H^1(g_2 Z g_2^{-1})\) for all \(\gamma_j\) for \(j = 1, \ldots, 8\). Representatives of the real orbits are listed in Table 10 below we only list the important data.

(i, j) = (2, 1): If \([c] = [\gamma_1]\), then \(q' \in h_{\Pi_2}\) satisfies \(\overline{q'} = q'\) if and only if \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\) are nonzero and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\). We have \(n_1 = g_1 = (I, I, I, I)\), and \(H^1(Z)\) was computed above.

(i, j) = (2, 2): If \([c] = [\gamma_2]\), then \(q' \in h_{\Pi_2}\) satisfies \(\overline{q'} = q^{-1}\) if and only if \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\) and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\). We have \(n_2 = (I, I, L, -L)\) and \(g_2 = (I, I, M, D(\zeta))\), with real point \(p = g_2 q'\). A direct calculation shows that \(u = g_2 w j^* g_2^{-1}\), \(w = (D(a)^{-1}, D(a)^{-1}, D(a), D(a)) \in Z^0\) for some \(a \in C^\infty\) is a 1-cocycle if and only if \(a = -\pi\) and \(a = \pi\), which is a contradiction. This shows that there is no 1-cocycle with representative \(u = g_2 w j^* g_2^{-1}\), so \(H^1(Z) = H^1(H)\).

For \(\gamma_3, \ldots, \gamma_8\) we also deduce that \(H^1(Z) = H^1(H)\); the case of \(\gamma_8\) is similar to \(\gamma_1\).

(i, j) = (2, 3): If \([c] = [\gamma_3]\), then the condition on \(q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in h_{\Pi_2}\) is \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\), and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\). In this case \(n_3 = (I, I, I, -L)\) and \(g_3 = (I, M, I, D(\zeta))\).

(i, j) = (2, 4): If \([c] = [\gamma_4]\), then the condition on \(q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in h_{\Pi_2}\) is \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\), and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\). We have \(n_4 = (I, I, I, L)\) and \(g_4 = (M, I, I, M)\).

(i, j) = (2, 5): If \([c] = [\gamma_5]\), then the condition on \(q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in h_{\Pi_2}\) is \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\), and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\); we have \(n_5 = (L, I, I, -L)\) and \(g_5 = (M, I, I, N)\).

(i, j) = (2, 6): If \([c] = [\gamma_6]\), then the condition on \(q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in h_{\Pi_2}\) is \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\), and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\); we have \(n_6 = (I, I, I, L)\) and \(g_6 = (I, M, I, M)\).

(i, j) = (2, 7): If \([c] = [\gamma_7]\), then the condition on \(q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in h_{\Pi_2}\) is \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\), and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\); we have \(n_7 = (I, I, L, L)\) and \(g_7 = (I, M, M, M)\).

(i, j) = (2, 8): If \([c] = [\gamma_8]\), then the condition on \(q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in h_{\Pi_2}\) is \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}\), and \(\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\); we have \(n_8 = (-I, I, I, I)\) and \(g_8 = (L, I, I, I)\). A short calculation shows that \(u = w j^*\) with \(w = (D(a)^{-1}, D(a)^{-1}, D(a), D(a)) \in Z^0\) is a 1-cocycle if and only if \(a\) is purely imaginary. Moreover, every such \(u\) is equivalent to \(k = (K, -K, K, K)\), thus \(|H^1(\tilde{Z})| = 2\). In conclusion, \(H^1(Z)\) has 8 classes with representatives \(z_1, \ldots, z_4\) and
\[
\begin{align*}
z_5 &= z_1 k = (K, -K, K, K), \quad z_6 = z_2 k = (-K, K, K, K), \\
z_7 &= z_3 k = (K, -K, -K, K), \quad z_8 = z_4 k = (-K, K, -K, K).
\end{align*}
\]

Case i = 3. Since \(H^1(\Pi_{\Pi_3}) = \{[I], [-I]\}\), there are 2 equivalence classes of cocycles with representatives \(\gamma_1 = \text{diag}(1, 1, 1, 1)\) and \(\gamma_2 = \text{diag}(-1, -1, -1, -1)\). We decompose \(Z = Z^0 \times H\), where \(H\) is the same as in the case \(i = 2\) and
\[
Z^0 = \{(A\# , A^\# , A, A) : A \in \text{SL}(2, \mathbb{C})\}.
\]
A short calculation and Remark 4.2 show that \(H^1(Z^0) = 1\), so \(H^1(Z) = H^1(H) = \{[z_1], [z_2], [z_3], [z_4]\}\) as determined for \(i = 2\). Representatives of the real orbits are listed in Table 9 below we give some details.

(i, j) = (3, 1): If \([c] = [\gamma_1]\), then \(q' = \lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3) \in h_{\Pi_3}\) satisfies \(\overline{q'} = q'\) if and only if \(\lambda_1, \lambda_2 \in \mathbb{R}\) and \(\lambda_1\lambda_2(\lambda_1 + \lambda_2) \neq 0\); we have \(n_1 = g_1 = (I, I, I, I)\).

(i, j) = (3, 2): If \([c] = [\gamma_2]\), then the condition on \(q' = \lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3) \in h_{\Pi_3}\) is \(\lambda_1, \lambda_2 \in \mathbb{R}\) and \(\lambda_1\lambda_2(\lambda_1 + \lambda_2) \neq 0\); we have \(n_2 = (-I, I, I, I)\) and \(g_2 = (L, I, I, I)\). A short calculation shows that \(H^1(g_2 Z^0 g_2^{-1})\) is in bijection with \(H^1(\text{SL}(2, \mathbb{C})) = 1\); in conclusion, \(H^1(Z_G(p)) = H^1(H)\).

Case i = 4. This case is similar to \(i = 2\). Here \(H^1(\Pi_{\Pi_3})\) consists of the classes of
\[
\begin{align*}
\gamma_1 &= \text{diag}(1, 1, 1, 1), \quad \gamma_2 = \text{diag}(-1, 1, 1, 1), \quad \gamma_3 = \text{diag}(-1, 1, 1, -1), \quad \gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]
We have $Z = \tilde{Z} \times H$ where $H$ is abelian of order 2 and generated by $(-I, I, -I, I)$. Furthermore $\tilde{Z}/Z^o$ has order 2, generated by the class of $(J, J, J, J)$, and $Z^o$ is a 2-dimensional torus consisting of elements $T_2(a, b) = \{(D(a)^{-1}, D(a), D(b)^{-1}, D(b)) : a, b \in \mathbb{C}^\times\}$. First, $H_1(H)$ consists of 2 classes defined by the representatives

$$z_1 = (I, I, I, I), \quad z_2 = (-I, I, -I, I).$$

Since $Z^o$ is parametrised by a 2-dimensional torus, a direct computation and Remark\[42\] shows that $H_1(Z^o)$ is trivial. Now consider the cohomology classes of elements in $\tilde{Z}/Z^o$, that is, $u = wj^\ast$ where $w \in T_2(a, b)$. Computations similar to the ones in Case $i = 2$ show that $u$ is a 1 cocycle if and only if $a, b \in \mathbb{R}$. Moreover, such a 1-cocycle $u$ is equivalent to either $(-K, K, -K, K)$ or $(-K, K, K, -K)$, thus $|H_1(\tilde{Z})| = 3$. In conclusion, $H_1(Z)$ has 6 classes with representatives $z_1, z_2$ and

$$z_3 = (-K, K, -K, K), \quad z_4 = (-K, K, K, -K), \quad z_5 = (K, K, K, K), \quad z_6 = (K, K, -K, K).$$

Real orbit representatives are listed in Table[8] we summarise the important data below.

(i, j) = (4, 1): If $[c] = [\gamma_1]$, then $q' = \lambda_1u_1 + \lambda_4u_4 \in \mathfrak{h}_{14}$, satisfies $q' = q'$ if and only if $\lambda_1, \lambda_4 \in \mathbb{R}$ and $\lambda_1\lambda_4(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_4) \neq 0$. We have $n_1 = g_1 = (I, I, I, I)$ and $p = g_1q' = q'$

(i, j) = (4, 2): If $[c] = [\gamma_2]$, then the condition on $q' = \lambda_1u_1 + \lambda_4u_4 \in \mathfrak{h}_{14}$ is $\lambda_1, \lambda_4 \in \mathbb{R} \setminus \{0\}$; we have $n_2 = (M, M, M, M)$ and $g_2 = (D(\eta_1^\ast), D(\eta_1^\ast), D(\eta_1^\ast), D(\eta_1^\ast))$. Let $u = gwj^\ast g_2^{-1}$ with $w \in Z^o$. As in Case $i = 2$, a short calculation shows that there is no 1-cocycle with representative $u$, so $H_1(Z) = H_1(H)$ has size 2.

(i, j) = (4, 3): If $[c] = [\gamma_3]$, then the condition on $q' = \lambda_1u_1 + \lambda_4u_4 \in \mathfrak{h}_{14}$ with $\lambda_1, \lambda_4 \in \mathbb{R}$ and $\lambda_1\lambda_4(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_4) \neq 0$; we have $n_3 = (I, I, L, L)$ and $g_3 = (I, I, M, M)$. A direct calculation shows that a 1-cocycle $u = gwj^\ast g_3^{-1}$ with $w \in Z^o$ is equivalent to $(-K, K, -K, K)$ or $(-K, K, K, K)$. In conclusion $H_1(g_3Zg_3^{-1})$ consists of 6 classes with representatives $z_1, z_2$ and

$$z_3 = (-K, K, -K, K), \quad z_4 = (K, K, K, K), \quad z_5 = (-K, K, -K, K), \quad z_6 = (K, K, -K, K).$$

(i, j) = (4, 4): Let $[c] = [\gamma_4]$, then the condition on $q' = \lambda_1u_1 + \lambda_4u_4 \in \mathfrak{h}_{14}$ is $\lambda_1 - \lambda_4 \neq 0$ and $\lambda_1 + \lambda_4 \in \mathbb{R}$. We have $g_4 = (M, M, F, L, F)$, and a direct calculation (assisted by GAP) shows that $H_1(g_4Zg_4^{-1})$ has size 4 with representatives $z_1, z_2$ and $z_3 = (-K, -K, -L, -L)$ and $z_4 = (K, -K, L, L)$.

Case $i = 7$. Since $H_1(\Gamma_{Pi_5}) = \{[I], [-I]\}$, we have the same $\gamma_1, \gamma_2$ as in the Case $i = 3$. We decompose $Z = Z^o \times H$ where $H$ has order 2, generated by $(-I, I, -I, I)$, and $Z^o$ is parametrised by SL$(2, \mathbb{C}) \times$ SL$(2, \mathbb{C})$. As before, $H_1(Z^o) = 1$, so $H_1(Z) = H_1(H)$ consists of the classes of $z_1 = (I, I, I, I)$ and $z_2 = (-I, I, -I, I)$. Table[7] lists the real orbit representatives.

(i, j) = (7, 1): If $[c] = [\gamma_1]$, then $q' \in \mathfrak{h}_{17}$ satisfies $q' = q'$ if and only if $q' = \lambda(u_1 - u_4) \in \mathbb{R}$ with $\lambda \neq 0$; we have $n_1 = g_1 = (I, I, I, I)$.

(i, j) = (7, 2): If $[c] = [\gamma_2]$, then the condition on $q' = \lambda(u_1 - u_4)$ is $\lambda \in \mathbb{R} \setminus \{0\}$; we have $n_2 = (-I, I, I, I)$ and $g_2 = (L, I, I, I)$. A direct computation shows that $H_1(g_2Zg_2^{-1}) = 1$, so $H_1(g_2Zg_2^{-1}) = H_1(H)$ determines 2 real orbits.

Case $i = 10$. Here we have $H_1(\Gamma_{Pi_{10}}) = \{[I], [-I]\}$ and $Z = \tilde{Z}$, where $Z^o$ is a 3-dimensional torus consisting of elements $T_3(a, b, c) = \{(D(abc)^{-1}, D(a), D(b), D(c)) : a, b, c \in \mathbb{C}^\times\}$ and $\tilde{Z}/Z^o$ is of order 2, generated by the class of $(J, J, J, J)$. As before, $H_1(Z^o) = 1$, and elements of the form $u = wj^\ast$ with $w \in T_3(a, b, c) \in Z^o$ are 1-cocycles if and only if $a, b, c \in \mathbb{R}$. Moreover, every such 1-cocycle $u$ is equivalent to $(K, K, K, K)$, $(-K, K, K, -K)$, $(-K, K, -K, K)$, or $(K, -K, -K, K)$, thus $H_1(Z)$ has 5 classes with representatives $z_1 = (I, I, I, I), z_2 = (K, K, K, K), z_3 = (-K, K, K, -K), z_4 = (-K, K, -K, K), z_5 = (K, K, -K, K)$.

Table[7] lists the real orbit representatives.

(i, j) = (10, 1): If $[c] = [\gamma_1]$, then the condition on $q' = \lambda u_1 \in \mathbb{R} \setminus \{0\}$; we have $n_1 = g_1 = (I, I, I, I)$.

(i, j) = (10, 2): If $[c] = [\gamma_2]$, then the condition on $q' = \lambda u_1 \in \mathfrak{h}_{10}$ is $\lambda \in \mathbb{R} \setminus \{0\}$; we have $n_2 = (-I, I, I, I)$ and $g_2 = (L, I, I, I)$. Since $L$ commutes with diagonal matrices, $H_1(g_2Zg_2^{-1}) = H_1(Z^o)$. On the other hand, every 1-cocycle $u = wj^\ast$ with $w \in Z^o$ is equivalent to $(K, K, K, K)$, $(-K, K, K, -K)$, $(-K, K, -K, K)$, or $(K, K, K, K)$; thus $H_1(Z)$ has 5 classes with representatives $z_1 = (I, I, I, I), z_2 = (-K, K, K, K), z_3 = (K, K, K, K), z_4 = (K, K, -K, K), z_5 = (-K, K, -K, K)$. 

Classification of four-dimensional rebits
6. Real elements of mixed type

An element of mixed type is of the form \( p + e \), where \( p \) is semisimple, \( e \) is nilpotent and \( [p, e] = 0 \). From the uniqueness of the Jordan decomposition it follows that two elements \( p + e \) and \( p' + e' \) of mixed type are \( \hat{G} \)-conjugate if and only if there is a \( g \in \hat{G} \) with \( gp = p' \) and \( ge = e' \). So if we want to classify orbits of mixed type then we may assume that the semisimple part is one of a fixed set of orbit representatives of semisimple elements. For \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \) we define

\[
\mathcal{M}(\mathbb{K}) = \{ p + e : p + e \text{ is of mixed type in } \mathfrak{g}_1(\mathbb{K}) \}.
\]

We know the \( \hat{G} \)-orbits in \( \mathcal{M}(\mathbb{C}) \) and we want to classify the \( \hat{G}(\mathbb{R}) \)-orbits in \( \mathcal{M}(\mathbb{R}) \). Applying the general Galois cohomology approach will lead to additional challenges. To avoid these, instead of working with \( \mathcal{M}(\mathbb{K}) \), we will consider 4-tuples \( (p, h, e, f) \), where \( p + e \) is a mixed element and \((h, e, f)\) is a suitable \( \mathfrak{sl}_2 \)-triple. This has the advantage that the stabiliser of such a 4-tuple in \( \hat{G} \) is smaller than the stabiliser of \( p + e \), and secondly it is reductive. This makes it easier to compute the Galois cohomology sets. We now explain the details of this approach.

Let \( p \in \mathfrak{g}_1(\mathbb{K}) \) be a semisimple element. The nilpotent parts of mixed elements with semisimple part \( p \) lie in the subalgebra

\[
\mathfrak{a} = \mathfrak{z}_\mathbb{K}(\mathfrak{p})(p) = \{ x \in \mathfrak{g}(\mathbb{K}) : [x, p] = 0 \}.
\]

This subalgebra inherits the grading from \( \mathfrak{g} \), that is, if we set \( \mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_1(\mathbb{K}) \) then \( \mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \). Moreover, the possible nilpotent parts of mixed elements with semisimple part \( p \) correspond, up to \( \hat{G}(\mathbb{K}) \)-conjugacy, to the \( Z_{\hat{G}(\mathbb{K})}(p) \)-orbits of nilpotent elements in \( \mathfrak{a}_1 \). The latter are classified using homogeneous \( \mathfrak{sl}_2 \)-triples, which are triples \( (h, e, f) \) with \( h \in \mathfrak{a}_0 \) and \( e, f \in \mathfrak{a}_1 \) such that

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.
\]

By the Jacobson-Morozov Theorem (see [7, Proposition 4.2.1]), every nonzero nilpotent \( e \in \mathfrak{a}_1 \) lies in some homogeneous \( \mathfrak{sl}_2 \)-triple. Moreover, if \( e, e' \in \mathfrak{a}_1 \) lie in homogeneous \( \mathfrak{sl}_2 \)-triples \((h, e, f)\) and \((h', e', f')\), then \( e \) and \( e' \) are \( Z_{\hat{G}(\mathbb{K})}(p) \)-conjugate if and only if the triples \((h, e, f)\) and \((h', e', f')\) are \( Z_{\hat{G}(\mathbb{K})}(p) \)-conjugate. For this reason we consider the set of quadruples

\[
\mathcal{Q}(\mathbb{K}) = \{(p, h, e, f) : p \in \mathfrak{g}_1(\mathbb{K}) \text{ is semisimple and } (h, e, f) \text{ is a homogeneous } \mathfrak{sl}_2 \text{-triple in } \mathfrak{z}_\mathbb{K}(\mathfrak{p})(p)\}.
\]

We have just shown that there is a surjective map \( \mathcal{Q}(\mathbb{K}) \to \mathcal{M}(\mathbb{K}), (p, h, e, f) \mapsto p + e \). By the next lemma, this map defines a bijection between the \( \hat{G}(\mathbb{K}) \)-orbits in the two sets.

**Lemma 6.1.** Let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \). Let \( p, \hat{p} \in \mathfrak{g}_1(\mathbb{K}) \) be semisimple and let \((h, e, f)\) and \((\hat{h}, \hat{e}, \hat{f})\) be homogeneous \( \mathfrak{sl}_2 \)-triples in \( \mathfrak{z}_\mathbb{K}(\mathfrak{p})(p) \) and \( \mathfrak{z}_\mathbb{K}(\mathfrak{p})(\hat{p}) \), respectively. Then \( p + e \) and \( \hat{p} + \hat{e} \) are \( \hat{G}(\mathbb{K}) \)-conjugate if and only if \((p, h, e, f)\) and \((\hat{p}, \hat{h}, \hat{e}, \hat{f})\) are \( \hat{G}(\mathbb{K}) \)-conjugate.

**Proof.** Only one direction needs proof. If \( g(p + e) = \hat{p} + \hat{e} \) with \( g \in \hat{G}(\mathbb{K}) \), then \( gp = \hat{p} \) and \( ge = \hat{e} \) by uniqueness of the Jordan decomposition. Now \((gh, \hat{e}, gf)\) is a homogeneous \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{z}_\mathbb{K}(\mathfrak{p})(\hat{p}) \). By [7, Proposition 4.2.1], there is \( g_1 \in Z_{\hat{G}(\mathbb{K})}(\hat{p}) \) such that \( g_1(gh, \hat{e}, gf) = (\hat{h}, \hat{e}, \hat{f}) \), so \((g_1g)(p, h, e, f) = (\hat{p}, \hat{h}, \hat{e}, \hat{f})\). \( \square \)

Our approach now is to classify the \( \hat{G}(\mathbb{R}) \)-orbits in \( \mathcal{Q}(\mathbb{R}) \); the main tool for this is the following theorem which follows directly from Theorem 4.3 and the fact that \( \hat{G} \) has trivial cohomology.

**Theorem 6.2.** Let \((p', h', e', f')\) be a real point in the \( \hat{G} \)-orbit of \((p, h, e, f)\) in \( \mathcal{Q}(\mathbb{C}) \). There is a 1-to-1 correspondence between \( H^1(Z_{\hat{G}}(p, h', e', f')) \) and the \( \hat{G}(\mathbb{R}) \)-orbits in \( \hat{G}(p, h, e, f) \): the orbit corresponding to the class \([z] \in H^1(Z_{\hat{G}}(p', h', e', f'))\) has representative \( b(p', h', e', f')\) where \( b \in \hat{G} \) satisfies \( z = b^{-1}z \).

The complex semisimple and mixed orbits are parameterised as follows. For each \( i \in \{1, \ldots, 10\} \) let \( \Sigma_i \) be a set of \( \hat{G} \)-orbit representatives of semisimple elements in \( \mathfrak{h}_L^0 \) as specified in Table 1. By Theorem 3.2 up to \( \hat{G} \)-conjugacy, the complex elements in \( \mathfrak{g}_1 \) of mixed type are \( s + n \) where \( s \in \Sigma_i \) for some \( i \) and \( n = n_{i,r} \) for some \( r \), as specified in Table 3. In the following we write

\[
\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_{10}.
\]
The first problem is to decide which orbits in $Q(\mathbb{C})$ have real representatives, but we already know which semisimple orbits have real representative. So let us consider $p \in \Sigma$ such that $p' \in g_1(\mathbb{R})$, is a real element in its $\hat{G}$-orbit. We define $\alpha = \delta_{\hat{G}(\mathbb{C})}(p')$ as above, with the induced grading $\alpha = \alpha_0 \oplus \alpha_1$. It remains to determine which nilpotent $Z_{\hat{G}(\mathbb{C})}(p')$-orbits in $\alpha_1$ have real representatives. In the case that the real point $p'$ also lies in $\Sigma$, this is straightforward; we discuss this case in Section 6.1. We treat the case $p' \notin \Sigma$ in Section 6.2.

In conclusion, our efforts lead to the following theorem.

**Theorem 6.3.** Up to $\hat{G}(\mathbb{R})$-conjugacy, the mixed elements in $g_1(\mathbb{R})$ are the elements in Tables A.3–A.7 in Appendices A.4 and A.5; see Definition 1 for the notation used in all tables.

### 6.1. Classification for the case $p' \in \Sigma$. Here we suppose that the $\hat{G}$-orbit of $p$ has a real point $p'$ in $\Sigma$. As before set $\alpha = \delta_{\hat{G}(\mathbb{C})}(p')$. If $p' \in \Sigma_1$, then we can assume that $p'$ corresponds to an element in the first row of block $j = 1$ in the table for Case $i$ (see Tables 6–10). With this assumption, it follows from Theorem 5.2 that every nilpotent $Z_{\hat{G}(\mathbb{C})}(p')$-orbit in $\alpha_1$ has a real representative; in particular, we can assume that $e' = n_{i,r}$ for some $r$, as specified in Table 3. This then yields a real 4-tuple $(p', h', e', f') \in Q(\mathbb{R})$.

We start by computing the centralisers $Z_{\hat{G}}(p', h', e', f')$, similarly to how we computed $Z_{\hat{G}}(p')$ before; the result is listed in Table 12. Due to Theorem 3.1b), we can always take one explicit element for our computations; for example, in Case $i = 2$, we can always take $p' = u_1 + u_2 + u_3$.

We now consider the different cases $i = 2, \ldots, 10$; for $i = 1$ and $i = 11$ there are no elements of mixed type. As before, cases $i \in \{5, 6\}$ and $i \in \{8, 9\}$ follow from $i = 4$ and $i = 7$, respectively. We compute the first cohomology of each centraliser by using the same approach as described in Section 5.2; all the centralisers in this section can be found in Table 12.

**Case $i = 2$.** We can assume $p'$ corresponds to the first element in block $j = 1$ in Table 10. By Theorem 5.2 there is only one nilpotent element $e' = (0011)$ such that $p' + e'$ has mixed type; this is the real point we use. First, we compute a real $\delta_{\hat{G}}$-triple associated to $e'$. A direct calculation shows that $H^1(Z)$ for $Z = Z_{\hat{G}}(p', h', e', f')$ has 8 classes with representatives

\[
\begin{array}{cccc}
z_1 = (I, I, I, I), & z_2 = (I, I, I, I), & z_3 = (I, I, I, I), & z_4 = (I, I, I, I), \\
z_5 = (L, L, L, L), & z_6 = (L, L, L, L), & z_7 = (L, L, L, L), & z_8 = (L, L, L, L).
\end{array}
\]

This shows that the complex orbit $\hat{G}(p', h, e, f)$ splits into 8 real orbits: each $[z] \in H^1(Z_1)$ determines some $b \in \hat{G}$ with $z = b^{-1}b_0$, and then $b(p' + e')$ is the real representative of the mixed type orbit corresponding to $[z]$; the resulting orbit representatives are listed in Table 13.

**Case $i = 3$.** We proceed as before and the resulting real orbit representatives are exhibited in Table 14. Here we have to consider the nilpotent elements $n_{3.1}$ and $n_{3.2}$. The case $e' = n_{3.1}$ yields the same centraliser $Z_1 = Z$ as in Case $i = 2$. The centraliser $Z_2$ for the case $e' = n_{3.2}$ leads to a first cohomology $H^1(Z_2)$ with 8 classes, given by representatives

\[
\begin{array}{cccc}
z_1 = (I, I, I, I), & z_2 = (I, I, I, I), & z_3 = (I, I, I, I), & z_4 = (I, I, I, I), \\
z_5 = (I, I, I, I), & z_6 = (I, I, I, I), & z_7 = (I, I, I, I), & z_8 = (I, I, I, I).
\end{array}
\]

**Case $i = 4$.** Here we have four nilpotent elements $n_{4,r}$, with $r \in \{1, 2, 3, 4\}$; the real orbit representatives for this case are given in Table 15. The centralisers for $r = 1$ and $r = 2$ coincide with $Z$ as in Case $i = 2$; if $r \in \{3, 4\}$, then the centraliser has 4 classes with representatives

\[
\begin{array}{cccc}
z_1 = (I, I, I, I), & z_2 = (I, I, I, I), & z_3 = (I, I, I, I), & z_4 = (I, I, I, I), \\
z_5 = (I, I, I, I), & z_6 = (I, I, I, I), & z_7 = (I, I, I, I), & z_8 = (I, I, I, I).
\end{array}
\]

**Case $i = 7$.** There are six nilpotent elements $n_{7,r}$, with $r \in \{1, \ldots, 6\}$; the real orbit representatives are exhibited in Table 16. The centraliser for $r = 1, 2, 3$ is the same as $Z_2$ as in Case $i = 3$; the centraliser for $r = 6$ is as in Case $i = 4$ and $n_{4,4}$. It therefore remains to consider $r \in \{4, 5\}$; we use the approach described in Section 5.1. For $r = 4$, a short calculation shows that $k = (L, L, K, K)$ is the only 1-cocycle arising from $(L, L, -J, J)$; it follows that the first cohomology has 8 classes with representatives $z_i$ as in (6.3) along with $z_i k$, that is

\[
\begin{array}{cccc}
z_1 = (I, I, I, I), & z_2 = (I, I, I, I), & z_3 = (I, I, I, I), & z_4 = (I, I, I, I), \\
z_5 = (L, L, K, K), & z_6 = (L, L, K, K), & z_7 = (L, L, K, K), & z_8 = (L, L, K, K).
\end{array}
\]
For $r = 5$, the first cohomology has representatives $z_1, \ldots, z_4$ as above along with $z_5(K, K, -L, L)$:

\begin{equation}
(6.5) \quad z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I), \quad z_3 = (-I, I, -I, I), \quad z_4 = (-I, I, I, -I), \quad z_5 = (K, K, -L, L), \quad z_6 = (-K, -K, -L, L), \quad z_7 = (-K, K, L, L), \quad z_8 = (-K, K, -L, -L).
\end{equation}

**Case $i = 10$.** There are 12 nilpotent elements; the real orbit representatives are exhibited in Tables 17 and 18. Cases $r = 1, 3, 7, 9$ lead to centralisers that have the same first cohomology as in Case $i = 2$; Cases $r = 2, 4, 6, 8, 10, 12$ yield the same first cohomology as Case $(i, r) = (4, 4)$, see (6.3). For $r \in \{5, 11\}$ we obtain the following cohomology representatives:

\begin{equation}
(6.6) \quad z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I).
\end{equation}

For $j = 13$ the first cohomology has 2 classes with representatives

\begin{equation}
(6.7) \quad z_1 = (I, I, I, I), \quad z_2 = (-I, I, I, -I).
\end{equation}

**Remark 6.4.** The semisimple parts of the real orbit representatives arising from cocycles involving $K$ or $-K$ are not in our fixed Cartan subspaces $c_1, \ldots, c_7$, and we use Remark 5.8 to replace these elements by elements in our spaces. For example, consider the cocycle $z = (-L, L, K, K)$ in Case $(i, r) = (7, 4)$. There is $b \in \hat{G}$ with $b^{-1}b = z$, and we compute $bp'$, so that $b(p' + e')$ is a real point; in this case, $bp' \notin c_1 \cup \ldots \cup c_7$. However, $bp'$ is $(\hat{G}(\mathbb{R}))$-conjugate to an element in some space $c_j$, so there is $b' \in \hat{G}(\mathbb{R})$ such that $(b'b)p' \in c_j$. Since $b'$ is real, $(b'b)^{-1}b' = z$. Thus, there is $b_0 \in \hat{G}$ with $b_0^{-1}b_0 = z$ and $b_0p' \in c_j$. Now $(b'b)e'$ is real and $b_0(p' + e')$ is a real point.

6.2. **Classification for the case $p' \notin \Sigma$.** As before, we consider a complex semisimple element $p$. By Theorem 3.2 we can assume that $p \in \Sigma$. Let $p'$ be a real point in $\hat{G}p$ as in Theorem 5.9. This time we consider the case $p' \notin \Sigma$, so if $a = \delta_{p(\mathbb{C})}(p') = a_0 \oplus a_1$, then we do not know which nilpotent $Z_{\hat{G}}(p')$-orbits in $a_1$ have real points. We now discuss how decide this question. The method that we describe is borrowed from [8] Section 5.3. However, some difficulties that occurred in the case considered in [3] do not appear here, see Remark 6.8.

Recall that our proof of Theorem 5.9 has exhibited an explicit $g \in \hat{G}$ such that $p' = gp$; this construction used Remark 5.8 and a 1-cocycle $n = g^{-1}g' \in N_{\hat{G}}(b_0')$. In the following write

\[ U_{p'} = \delta_{p}(p') \cap g_1 \quad \text{and} \quad U_{p} = \delta_{p}(p) \cap g_1. \]

Since $Z_{\hat{G}}(p') = gZ_{\hat{G}}(p)g^{-1}$, the next lemma allows us to determine the nilpotent $Z_{\hat{G}}(p')$-orbits in $U_{p'}$ from the known $Z_{\hat{G}}(p)$-orbits in $U_{p}$, cf. Theorem 3.2.

**Lemma 6.5.** The map $\varphi : U_{p} \to U_{p'}, x \mapsto gx$, is a bijection that maps $Z_{\hat{G}}(p)$-orbits to $Z_{\hat{G}}(p')$-orbits.

Having determined the $Z_{\hat{G}}(p')$-orbits in $U_{p'}$, it remains to decide when such a complex orbit has a real point. Note that if $e'$ is a real nilpotent element in $a_1$ then $e' = \varphi(x)$ for some $x \in U_{p}$ lying in the $Z_{\hat{G}}(p)$-orbit of some $e = n_{x,e}$. Motivated by this observation, we proceed as follows: we fix $p \in \Sigma$, and $p' = gp$, and for each $e = n_{x,e}$ we look for $x$ in the complex $Z_{\hat{G}}(p)$-orbit of $e$ such that $\varphi(x)$ is real. Note that the condition that $\varphi(x)$ is real is equivalent to $n_{x,e} = x$ where $n = g^{-1}g$ as above. Thus, we define

\begin{equation}
(6.8) \quad \mu : U_{p} \to U_{p}, \quad x \mapsto n_{x,e},
\end{equation}

note that $\mu^2 = 1$ since $n_{x,e} = 1$, and, by construction, $\varphi(x)$ is real if and only if $\mu(x) = x$. The following lemma is analogous to [7] Lemma 5.3.1.

**Lemma 6.6.** Let $Y = Z_{\hat{G}}(p)e$. Then $\mu(Y) = Y$ if and only if $\mu(y) \in Y$ for some $y \in Y$.

**Proof.** It follows from Theorem 5.1(b) that $Z_{\hat{G}}(p) = Z_{\hat{G}}(b_0'^{n_{x,e}})$, where $b_0'^{n_{x,e}}$ is the component containing $p$. Suppose there is $y \in Y$ such that $\mu(y) \in Y$; we have to show $\mu(Y) = Y$. Write $y = he$ with $h \in Z_{\hat{G}}(p)$. We know that $\mu(y) = n_{x,e} = n_{x,e} = ke$ for some $k \in Z_{\hat{G}}(p)$; note that $he = \overline{he}$ because $e$ is real. Since $\overline{e} \in Z_{\hat{G}}(p) = Z_{\hat{G}}(b_0'^{n_{x,e}})$ and the latter is normal in $N_{\hat{G}}(b_0'^{n_{x,e}})$, we have $n_{x,e} = 1$ for some $z \in Z_{\hat{G}}(p)$, and the previous equation yields $ne = e^{-1}ke$. Now let $w \in Y$, say $w = te$ with $t \in Z_{\hat{G}}(p)$. There is some $s \in Z_{\hat{G}}(p)$ such that $sn = sn = s^{-1}ke$ and, since $s^{-1}ke \in Z_{\hat{G}}(p)$, we deduce that $\mu(w) \in Y$, so $\mu(Y) = Y$. The other implication is trivial.

**Corollary 6.7.** If there is $x \in Z_{\hat{G}}(p)e$ such that $\varphi(x)$ is real, then $\mu(e)$ is $Z_{\hat{G}}(p)$-conjugate to $e$. 

If \( e = n_{i,r} \) does not satisfy \( \mu(e) \in Z_G(p)e \), then we can discard the pair \((p,e)\). If \( \mu(e) \) is \( Z_G(p) \)-conjugate to \( e \), then we attempt to compute \( x \in Z_G(p)e \) such that \( \mu(x) = x \); then \( e' = \varphi(x) \) is a real nilpotent element commuting with \( p' \). Once this real point is found, we construct a real 4-tuple \((p', h', e', f')\) and apply Theorem 6.2 to compute the \( \hat{G}(\mathbb{R}) \)-orbits in \( \hat{G}(p', h', e', f') \).

**Remark 6.8.** In our classification, using ad hoc methods, we always found suitable elements \( x \) as above. We note that if such ad hoc methods would not have worked, then we could have used a method described in [8 Section 5.3] for finding such elements (or for deciding that none exists). This method is based on computations with the second cohomology set \( H^2(Z_G(p)) \).

**Classification approach.** We summarise our approach for classifying the real mixed orbits in \( \hat{G}(p + e) \), where the real point \( p' \) in \( \hat{G} \) (as in Theorem 5.9) does not lie in \( \Sigma \).

1. Recall that Tables 6–11 list our real semisimple orbits; each table corresponds to a case \( i \in \{1, \ldots, 10\} \) and has subcases \( j = 1, 2, \ldots \), where \( j = 1 \) lists elements in \( \Sigma \). For each \( i \in \{2, \ldots, 10\} \) and each \( j > 1 \) listed in the corresponding table, choose the first real element \( p' \) in the block labelled \( j \). The proof of Theorem 5.9 shows that \( p' = gp \) where \( p \in \Sigma_i \) as given in Table 1 in particular, the element \( g \) is determined by our classification, and we define \( n = g^{-1} \).

2. For each \( p \in \Sigma_i \) as determined in (1) we consider each \( e = n_{i,r} (r = 1, 2, \ldots) \) such that the elements \( p + n_{i,r} \) are the mixed orbit representatives as determined in Theorem 5.2. Using (6.8), we then define \( \mu \) with respect to \( n \), and check whether \( \mu(e) \in Z_G(p)e \). If true, then we compute \( x \in Z_G(p)e \) with \( \mu(x) = x \) by computing the 1-eigenspace \( U_p^\mu = \{ u \in U_p : \mu(u) = u \} \) and looking for some \( x \in U_p^\mu \) that is \( Z_G(p) \)-conjugate to \( e \). Note that \( \dim \mathbb{R} U_p^\mu = \dim \mathbb{C} U_p^\mu \); this follows from the fact that \( \mu \) is an \( \mathbb{R} \)-linear map of order 2, so \( U_p \) is the direct sum of the \( \pm 1 \)-eigenspaces. Since multiplication by \( i \) is a bijective \( \mathbb{R} \)-linear map swapping these eigenspaces, they have equal dimension. In our classification, this search is always successful, and we set \( e' = \varphi(x) \).

3. We determine a real 4-tuple \((p', h', e', f')\) and apply Theorem 6.2 to find the real \( \hat{G}(\mathbb{R}) \)-orbits in the \( \hat{G} \)-orbit of this 4-tuple. If \((p'', h'', e'', f'')\) is a representative of such an orbit then \( p'' + e'' \) is the corresponding element of mixed type. This is a representative of a \( \hat{G}(\mathbb{R}) \)-orbit contained in \( \hat{G}(p + e) \).

We now discuss the individual cases in detail. The following case distinction determines the relevant real points \( n_{i,j,r} \) for Case \( i \) and the cohomology class \([\gamma_j] \) of \( H^1(\Gamma_p) \) with \( \gamma_j \) as determined in the proof of Theorem 5.9; note that we do not have to consider the trivial class \([\gamma_1]\) because this class produces elements in \( \Sigma \).

**Case \( i = 2 \).** We have to consider cocycles \( \gamma_2, \ldots, \gamma_8 \), with corresponding elements \( n_j \) given by (see Section 6)

\[
\begin{align*}
    n_2 &= (I, I, L, -L), \\
    n_3 &= (I, L, I, -L), \\
    n_4 &= (L, I, I, L), \\
    n_5 &= (I, L, L, L), \\
    n_6 &= (I, I, L, L), \\
    n_7 &= (I, I, I, L), \\
    n_8 &= (-I, I, I, I).
\end{align*}
\]

For this case there is only one nilpotent element \( e = n_{2,1} \). After a short calculation we can see that \( \mu(e) = n_{2,1}e \) is conjugate to \( e \) for all \( j = 2, \ldots, 8 \). For \( n_2 = (I, I, L, -L) \) we obtain \( g_2 = (I, I, M, D(\zeta)) \); following Remark 5.3 we replace \( g_2 \) by \( g_2 = (-I, -J, -M, -MJ) \). We then compute \( U_p^\mu \) and verify that \( e \in U_p^\mu \), thus \( x = e \) is the element we are looking for, and therefore we set \( e' = \varphi(x) = g_2e = -[\mathfrak{001}] \); we denote the latter by \( n_{2,2,1} \). For \( n_3 \) we obtain \( g_3 = (I, M, I, D(\zeta)) \). Again, to get elements in one of the seven Cartan subspaces, it is necessary to replace it by \( g_3 = (-I, -M, -J, -MJ) \). After computing \( U_p^\mu \) we have that \( x = ie \in U_p^\mu \), which is conjugate to \( e \) under the action of \( Z_G(p) \) via \( g = (D(a^2), D(a), D(a^{-1}), D(a^{-1})) \). Therefore we set \( e' = \varphi(x) = g_2e = [0000] \); the latter is denoted by \( n_{2,3,1} \). The other cases \( j > 3 \) are computed along the same way.

**Case \( i = 3 \).** We have to consider the cocycle \( \gamma_2 \) with \( n_2 = (-I, I, I, I) \). There are two nilpotent elements \( n_{3,1} \) and \( n_{3,2} \). We compute \( U_p^\mu \) and see that \( x = i[00111] \) is a nilpotent element in \( U_p^\mu \) conjugate to \( n_{3,1} \) via \( (D(a)^{-1}, D(a^{-1}), D(a), D(a)) \) with \( a^{-4} = 1 \). Thus, the real nilpotent element is \( e' = \varphi(x) = [-\mathfrak{001}] \), denoted \( n_{3,2,1} \). Similarly, we find \( x \in U_p^\mu \) such that \( x \) is conjugate to \( n_{3,2} \), see \( \varphi(x) = n_{3,3,2} \) in Table 19.

**Case \( i = 4 \).** We have to consider cocycles \( \gamma_2, \ldots, \gamma_4 \), with corresponding elements

\[
\begin{align*}
    n_2 &= (M, M, M, M), \\
    n_3 &= (I, I, L, L), \\
    n_4 &= (L, L, -K, K).
\end{align*}
\]

For each of them, we have to consider four nilpotent elements \( n_{4,1}, \ldots, n_{4,4} \). After computing \( U_p^\mu \) for \( n_2 \), we observe that each \( n_{4,\ell} \in U_p^\mu \) thus the real representatives are \( n_{4,1,1}, \ldots, n_{4,2,4} \), defined as \( n_{4,\ell} = g_2n_{4,\ell} = n_{4,\ell} \) with \( g_2 = (D(\eta^5), D(\eta^6), D(\eta^6), D(\eta^6)) \). The case \( n_3 \) is similar, so now we let us consider \( n_4 \). We obtain that \( \mu(n_{4,\ell}) \) is not conjugate to \( n_{4,\ell} \) for \( \ell \in \{3, 4\} \), thus we have to only consider \( n_{4,1} \) and \( n_{4,2} \). For \( n_{4,1} \) there is no
$x \in U^n_p$ such that $x$ is conjugate to $n_{4.1}$. This is seen by acting on $n_{4.1}$ by a general element $g$ of $Z_G(p)$ which is $g = h(D(a^{-1}), D(a), D(b^{-1}), D(b))$ where $a, b \in \mathbb{C}^*$ and $h$ lies in the component group (see Table 2). From the expressions obtained it is straightforward to see that the image can never lie in $U^n_p$.

On the other hand, we observe that $n_{4.2}$ lies in $U^n_p$ and therefore we set $\varepsilon = \varphi(n_{4.2}) = g_{4}n_{4.2} = \frac{1}{2}(-|1110| - |1101| + |1010| + |0101| + |0110| - |0010| - |0001|)$. In this case it is necessary to replace $g_{4}$ in order to get elements in one of the seven Cartan subspaces.

**Cases $i \in \{7, 10\}$.** The nilpotent elements for the remaining cases $i = 7$ and $i = 10$ are computed analogously.

The next step is to compute the centralisers $Z = Z_G(p', h', e', f')$ of the 4-tuples $(p', h', e', f')$, and then the first cohomology $H^1(Z)$ as we did in Table 12. The difference here is that $p' \notin \Sigma$ and $e'$ is a nilpotent element from Table 19 with corresponding $\mathfrak{sl}_2$-triple $(h', e', f')$ in $\mathfrak{z}_0(p')$. We exemplify the details with a few examples:

**Example 6.9.** For $(i, j) = (2, 2)$ let $p'$ be the first element in the second block of Table 10 and let $e' = n_{2,2,1} = -|0011|$. We compute an $\mathfrak{sl}_2$-triple with nilpotent element $e'$ in the whole Lie algebra and then check that it centralises $p'$. Since the centralisers of semisimple elements in the same component are equal, in order to compute $Z_G(p')$ we can assume that the parameters defining $p'$ are $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = \tau$. Using Groebner basis techniques, we determine that $Z_G(p', h', e', f')$ is the same as in the first row of Table 12 hence $H^1(Z)$ has 8 classes with representatives given by (6.1). The results are listed in the first block of Table 20.

In fact, in most of the cases, the centraliser $Z = Z_G(p', h', e', f')$ for Case $i$, cohomology class $[\gamma_i]$, and nilpotent element $e' = n_{i,j,r}$ is exactly the same given in Table 12 for parameters $i$ and $r$. For example, if $i = 2$ and $e' = n_{2,j,1}$ with $j \in \{2, 3, 4, 5, 6, 7, 8\}$, then $Z_G(p', h', e', f')$ is always given in the first row of Table 12.

The only exceptions where this behaviour was not observed are the following two cases.

**Example 6.10.** For $(i, j) = (4, 4)$ let $p'$ be the first element in the second block of Table 12 and let $e' = n_{4,4,1}$ as in Table 21. Here the centraliser $Z = Z_G(p', h', e', f')$ is generated by $(-I, -I, I, I), (-I, I, -I, I), (K, K, J, J)$, therefore different from the centraliser given in Table 12 for parameters $(i, r) = (4, 1)$. The first cohomology $H^1(Z)$ has 4 classes with representatives

\begin{equation}
(6.9) \quad z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I), \quad z_3 = (-I, I, -I, I), \quad z_4 = (-I, I, I, -I).
\end{equation}

For $(i, j) = (7, 2)$ and $e' = n_{7,2,5}$ the centraliser is generated by $(-I, I, -I, I), (-I, I, I, -I), (-J, J, L, L)$, and \{(D(a)^{-1}, D(a), I, I) : a \in \mathbb{C}^*\}, and therefore is different from the centraliser given in Table 12 for parameters $(i, r) = (7, 5)$; its first cohomology has 8 classes with representatives

\begin{equation}
(6.10) \quad z_1 = (I, I, I, I), \quad z_2 = (-I, I, -I, I), \quad z_3 = (-I, I, I, -I), \quad z_4 = (I, I, -I, -I),
\end{equation}
\begin{equation}
z_5 = (K, K, L, L), \quad z_6 = (-K, K, -L, L), \quad z_7 = (-K, K, L, -L), \quad z_8 = (K, K, -L, -L).
\end{equation}

Having found all real points and having determined the corresponding cohomology class representatives, the last step is to employ the usual Galois theory approach to obtain a real representative for each of the mixed orbits obtained above. We exhibit the results in Tables 13-27. This proves Theorem 6.3.
A.1. Complex classification.

| $i$ | type of $\Pi_i$ | roots of $\Pi_i$ | elements of $\mathfrak{h}_0^\Pi$ | condition for being in $\mathfrak{h}_0^\Pi$ | $\Gamma_i$ | $\mathfrak{g}_\Phi(p_i)$ | $\mathfrak{g}_\Phi(p_i)'$ |
|-----|-----------------|------------------|---------------------------------|---------------------------------|----------|-----------------|-----------------|
| 1   | $\emptyset$     |                  | $\lambda_1 u_1 + \cdots + \lambda_4 u_4$ | $\lambda_i \neq 0$ and $\lambda_i \notin \{\pm \lambda_2 \pm \lambda_3 \pm \lambda_4\}$ | $W$      | $0$             | $\mathfrak{g}(2,\mathbb{C})$ |
| 2   | $A_1$           | $\alpha_4$      | $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ | $\lambda_i \neq 0$ and $\lambda_i \notin \{\pm \lambda_2 \pm \lambda_3\}$ | $(\mathbb{Z}/2\mathbb{Z})$ | $\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{g}(2,\mathbb{C})$ |
| 3   | $A_2$           | $\alpha_2, \alpha_4$ | $\lambda_1 (u_1 - u_2) + \lambda_2 (u_1 - u_3)$ | $\lambda_i \neq 0$ and $\lambda_i \notin -\lambda_2$ | $(-I_4)$ | $\mathfrak{g}(3,\mathbb{C})$ | $\mathfrak{g}(2,\mathbb{C})$ |
| 4   | $2A_1$          | $\alpha_1, \alpha_3$ | $\lambda_1 u_1 + \lambda_2 u_4$ | $\lambda_i \neq 0$ and $\lambda_i \notin \{\pm \lambda_2\}$ | $\text{Dih}_4$ | $\mathfrak{g}(2,\mathbb{C})^2$ | $\mathfrak{g}(2,\mathbb{C})^2$ |
| 5   | $2A_1$          | $\alpha_1, \alpha_4$ | $\lambda_1 u_1 + \lambda_2 u_3$ | $\lambda_i \neq 0$ and $\lambda_i \notin \{\pm \lambda_2\}$ | $\text{Dih}_4$ | $\mathfrak{g}(2,\mathbb{C})^2$ | $\mathfrak{g}(2,\mathbb{C})^2$ |
| 6   | $2A_1$          | $\alpha_3, \alpha_4$ | $\lambda_1 u_1 + \lambda_2 u_2$ | $\lambda_i \neq 0$ and $\lambda_i \notin \{\pm \lambda_2\}$ | $\text{Dih}_4$ | $\mathfrak{g}(2,\mathbb{C})^2$ | $\mathfrak{g}(2,\mathbb{C})^2$ |
| 7   | $A_3$           | $\alpha_1, \alpha_2, \alpha_3$ | $\lambda_1 (u_1 - u_4)$ | $\lambda_1 \neq 0$ | $(-I_4)$ | $\mathfrak{g}(4,\mathbb{C})$ | $\mathfrak{g}(4,\mathbb{C})$ |
| 8   | $A_3$           | $\alpha_1, \alpha_2, \alpha_4$ | $\lambda_1 (u_1 - u_3)$ | $\lambda_1 \neq 0$ | $(-I_4)$ | $\mathfrak{g}(4,\mathbb{C})$ | $\mathfrak{g}(4,\mathbb{C})$ |
| 9   | $A_3$           | $\alpha_2, \alpha_3, \alpha_4$ | $\lambda_1 (u_1 - u_2)$ | $\lambda_1 \neq 0$ | $(-I_4)$ | $\mathfrak{g}(4,\mathbb{C})$ | $\mathfrak{g}(4,\mathbb{C})$ |
| 10  | $3A_1$          | $\alpha_1, \alpha_3, \alpha_4$ | $\lambda_1 u_1$ | $\lambda_1 \neq 0$ | $(-I_4)$ | $\mathfrak{g}(2,\mathbb{C})^3$ | $\mathfrak{g}(2,\mathbb{C})^3$ |
| 11  | $D_4$           | $\alpha_1, \ldots, \alpha_4$ | $0$ | $\lambda_1 \neq 0$ | $1$ | $\mathfrak{s}(4,\mathbb{C})$ | $\mathfrak{s}(4,\mathbb{C})$ |

Table 1. This is [25 Table 2]: Complete root subsystems $\Pi_i$ of $\Phi$ and related data, with parameters $\lambda_1, \ldots, \lambda_4 \in \mathbb{C}$; the last column displays the derived algebra of the centraliser $\mathfrak{g}_\Phi(p_i)$ for $p_i \in \mathfrak{h}_0^\Pi$. Elements in $\mathfrak{h}_0^\Pi$ are not $\tilde{G}$-conjugate to elements in $\mathfrak{h}_0^\Pi$ if $i \neq j$; two elements in the same component $\mathfrak{h}_0^\Pi$ are $\tilde{G}$-conjugate if and only if they are $\Gamma_i$-conjugate.

| $i$ | identity component $Z_\tilde{G}(s)^\circ$ | preimages of generators of $Z_\tilde{G}(s)/Z_\tilde{G}(s)^\circ$ |
|-----|-------------------------------------------|---------------------------------------------------------------|
| 1   | $1$                                       | $(J, J, J), (-I, -I, I), (-I, -I, I), (J, J, J)$               |
| 2   | $\{(D(a)^{-1}, D(a)^{-1}, D(a)), D(a)\}$ : $a \in \mathbb{C}^\times$ | $(-I, -I, I), (J, J, J), (J, J, J)$ |
| 3   | $\{(A^#, A^#, A, A) : A \in \text{SL}(2,\mathbb{C})\}$ | $(-I, -I, I), (-I, -I, I)$ |
| 4   | $\{(D(a)^{-1}, D(a), D(b)^{-1}, D(b)) : a, b \in \mathbb{C}^\times\}$ | $(-I, -I, I), (J, J, J)$ |
| 5   | $\{(D(a)^{-1}, D(b)^{-1}, D(a), D(b)) : a, b \in \mathbb{C}^\times\}$ | $(-I, -I, I), (J, J, J)$ |
| 6   | $\{(D(a)^{-1}, D(b), D(b)^{-1}, D(a)) : a, b \in \mathbb{C}^\times\}$ | $(-I, -I, I), (J, J, J)$ |
| 7   | $\{(A^#, A, B^#, B) : A, B \in \text{SL}(2,\mathbb{C})\}$ | $(-I, -I, I)$ |
| 8   | $\{(A^#, B^#, A, B) : A, B \in \text{SL}(2,\mathbb{C})\}$ | $(-I, -I, I)$ |
| 9   | $\{(A^#, B, B^#, A) : A, B \in \text{SL}(2,\mathbb{C})\}$ | $(-I, -I, I)$ |
| 10  | $\{(D(a b c)^{-1}, D(a), D(b), D(c)) : a, b, c \in \mathbb{C}^\times\}$ | $(J, J, J)$ |

Table 2. This is [25 Table 3]: the groups $Z_{\tilde{G}}(s)$: the entry $i$ is the label of the canonical semisimple set $\mathfrak{h}_0^\Pi$ that contains $s$, as in Table 1 the notation is explained in (3.1).
i nilpotent elements

2 \( n_{2,1} = (001) \)
3 \( n_{3,1} = (001), n_{3,2} = (0111) + (1011) + (0010) + (0001) \)
4 \( n_{4,1} = (0110) + (1010), n_{4,2} = (0110) + (0101), n_{4,3} = (0110), n_{4,4} = (0101) \)
5 \( n_{5,1} = (0110) + (1100), n_{5,2} = (0110) + (0101), n_{5,3} = (1010), n_{5,4} = (0111) \)
6 \( n_{6,1} = (0011) + (1010), n_{6,2} = (0011) + (0101), n_{6,3} = (0011), n_{6,4} = (0101) \)
7 \( n_{7,1} = (1101) + (1011) + (0000) + (0001), n_{7,2} = (1101) + (1010) + (0001), \)
\( n_{7,3} = (1101) + (1000) + (0101), n_{7,4} = (1101) + (1000), n_{7,5} = (1101) + (0001), n_{7,6} = (1001) \)
8 \( n_{8,1} = (1011) + (1101) + (0000) + (0100), n_{8,2} = (1101) + (1100) + (0111), \)
\( n_{8,3} = (1101) + (1000) + (0011), n_{8,4} = (1101) + (1000), n_{8,5} = (1101) + (0001), n_{8,6} = (1001) \)
9 \( n_{9,1} = (1101) + (1110) + (1000) + (0100), n_{9,2} = (1101) + (1010) + (0100), \)
\( n_{9,3} = (1110) + (1000) + (0101), n_{9,4} = (1110) + (1000), n_{9,5} = (1101) + (0100), n_{9,6} = (1100) \)
10 \( n_{10,1} = (1100) + (1010) + (0110), n_{10,2} = (1010) + (0101), n_{10,3} = (1010) + (0110) + (0011), \)
\( n_{10,4} = (1100) + (0110), n_{10,5} = (0110), n_{10,6} = (0110) + (0011), n_{10,7} = (1100) + (0110) + (0101), \)
\( n_{10,8} = (0110) + (0101), n_{10,9} = (0110) + (0101) + (0011), n_{10,10} = (1100) + (1010), \)
\( n_{10,11} = (1010), n_{10,12} = (1010) + (0011), n_{10,13} = (0011) \).

Table 3. This is from [25 Theorem 3.7]: The nilpotent elements \( n_{i,j} \) used in Theorem 3.2.

A.2. Semisimple elements.

| \( w \in W \) | cocycle in \( \hat{G} \) | \( w \in W \) | cocycle in \( \hat{G} \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \text{diag}(-1, -1, -1, -1) \) | \((-I, I, I, I)\) | \(0 0 0 -1\) | \((L, L, -K, K)\) |
| \( \text{diag}(-1, -1, -1, 1) \) | \((-I, I, I, I)\) | \(0 0 0 -1\) | \((L, L, -K, K)\) |
| \( \text{diag}(1, -1, -1, 1) \) | \((L, I, I, L)\) | \(0 0 0 -1\) | \((I, I, I, K)\) |
| \( \text{diag}(1, -1, 1, -1) \) | \((I, I, I, L)\) | \(0 0 0 -1\) | \((I, I, I, K)\) |
| \( \text{diag}(1, -1, 1, 1) \) | \((L, I, I, L)\) | \(0 0 0 -1\) | \((K, I, I, K)\) |
| \( \text{diag}(1, 1, -1, -1) \) | \((L, I, I, L)\) | \(0 0 0 -1\) | \((K, I, I, K)\) |
| \( \text{diag}(1, 1, -1, 1) \) | \((M, M, M, M)\) | \(-1 0 0 0\) | \((M, M, M, M)\) |
| \( \text{diag}(1, 1, 1, -1) \) | \((N, M, M, N)\) | \(-1 0 0 0\) | \((N, M, M, N)\) |
| \( \text{diag}(1, 1, 1, 1) \) | \((M, N, M, N)\) | \(-1 0 0 0\) | \((N, M, M, N)\) |
| \( \text{diag}(1, -1, -1, -1) \) | \((-N, N, N, N)\) | \(0 0 0 -1\) | \((-N, N, N, N)\) |

Table 4. Cocycles in \( \hat{G} \) that induce the cocycles in \( W \) whose equivalence classes form the various sets \( H^1(\Gamma_{\Pi_i}) \); these will be the elements \( n_i \) that map under \( \varphi \) to \( \gamma_i \) as described in Theorem 5.3. Matrices \( I, J, K, L, M, N \) are from (5.1) and elements in \( H^1(\Gamma_{\Pi_i}) \) are considered as elements in \( W \).

\[
\begin{array}{cccccccc}
\text{A} & -I & M & -M & N & -N & L & -L & K & -K \\
\varepsilon(A) & L & D(\eta^\alpha) & D(\eta) & D(\eta^\beta) & -D(\eta^\beta) & M & D(\zeta) & LF & F \\
\end{array}
\]

Table 5. Matrices \( F, I, J, K, L, M, N \) are from (6.1), and \( \eta \) is a primitive 16-th root of unity with \( \eta^2 = \zeta \); if \( A \in \text{SL}(2, \mathbb{C}) \), then \( \varepsilon(A) \in \text{SL}(2, \mathbb{C}) \) satisfies \( \varepsilon(A)^{-1} \varepsilon(A) = A \).
### Table 6. Case $i = 10$: The table lists real orbit representatives corresponding to $\gamma_j$ and $[z_k] \in H^1(Z)$.

| $j$ | $k$ | real orbit representatives (semisimple) | Coefficients: | Conditions: |
|-----|-----|----------------------------------------|---------------|------------|
| 1   | 1, 2| $\lambda_1(1, 0, 0, 0)$, $(-2, 2, 2, 2)/\lambda_1$ | with respect to the basis $\{u_1, u_2, u_3, u_4\}$ | $\lambda_1$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$ |
| 2   | 1, 2| $\nu \lambda_1(1, 0, 0, 0)$, $(-2, 2, 2, 2)/(\nu \lambda_1)$ | with respect to the basis $\{v_1, v_2, v_3, v_4\}$ | $\lambda_1$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$ |

### Table 7. Cases $i = 7, 8, 9$: The table lists real orbit representatives corresponding to $\gamma_j$ and $[z_k] \in H^1(Z)$ for $\mathfrak{h}_{\Pi_7}^0$; the real representatives for Cases $i \in \{8, 9\}$ are obtained via Remark 3.3

| $j$ | $k$ | real orbit representatives (semisimple) | Coefficients: | Conditions: |
|-----|-----|----------------------------------------|---------------|------------|
| 1   | 1, 2| $(\lambda_1, 0, 0, \lambda_4)$, $(-\lambda_1, 0, 0, \lambda_4)$ | with respect to the basis $\{u_1, u_2, u_3, u_4\}$ | $(\lambda_1, \lambda_4)$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\lambda_1, \lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_4\}$ |
| 2   | 1, 2| $(\nu \lambda_1, \lambda_4)$, $(-\nu \lambda_1, \lambda_4)$ | with respect to the basis $\{t_1, t_4\}$ | $(\lambda_1, \lambda_4)$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \in \mathbb{R}^\times$ |
| 3   | 1, 2| $(\nu (-\lambda_1, 0, 0, \lambda_4), (\nu \lambda_1, 0, 0, \lambda_4))$ | with respect to the basis $\{v_1, v_2, v_3, v_4\}$ | $(\lambda_1, \lambda_4)$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\lambda_1, \lambda_2 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_2\}$ |
| 4   | 1   | $(-\nu (-\lambda_1 + \lambda_4), -\lambda_1 - \lambda_4, -\lambda_1 + \lambda_4, -\nu (-\lambda_1 + \lambda_4))$ | with respect to the basis $\{z_1, z_2, z_3, z_4\}$ | $(\lambda_1, \lambda_4)$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\nu (\lambda_1 + \lambda_4), \lambda_1 - \lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_4\}$ |

### Table 8. Cases $i = 4, 5, 6$: The table lists real orbit representatives corresponding to $\gamma_j$ and $[z_k] \in H^1(Z)$ for $\mathfrak{h}_{\Pi_4}^0$; the real representatives for Cases $i \in \{5, 6\}$ are obtained via Remark 3.3

| $j$ | $k$ | real orbit representatives (semisimple) | Coefficients: | Conditions: |
|-----|-----|----------------------------------------|---------------|------------|
| 1   | 1   | $(\nu (\lambda_1 + \lambda_4), \lambda_1 - \lambda_4, -\lambda_1 + \lambda_4, \nu (-\lambda_1 + \lambda_4))$ | with respect to the basis $\{u_1, u_2, u_3, u_4\}$ | $(\lambda_1, \lambda_4)$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\lambda_1 - \lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_4\}$ |
| 2   | 1   | $\nu (\lambda_1 + \lambda_4), \lambda_1 - \lambda_4, -\lambda_1 + \lambda_4, -\nu (-\lambda_1 + \lambda_4)$ | with respect to the basis $\{v_1, v_2, v_3, v_4\}$ | $(\lambda_1, \lambda_4)$ up to $\Gamma_{\Pi_{\mathfrak{h}}}^0$-conjugacy, $\nu (\lambda_1 + \lambda_4), \lambda_1 - \lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_4\}$ |
**Table 9. Case \( i = 3 \):** The table lists real orbit representatives corresponding to \( \gamma_j \) and \([z_k] \in H^1(Z)\).

| \( j \) | \( k \) | real orbit representatives (semisimple) |
|---|---|---|
| 1 | 1, \ldots, 4 | \((\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2)\) \(-\lambda_1, -\lambda_2, -\lambda_1, -\lambda_2\) \((-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2)\) \((-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2)\) \((-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2)\) |
| Coefficients: | with respect to the basis \(\{v_1, u_2, u_3\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, each \( \lambda_i \in \mathbb{R}^\times, \lambda_1 \neq -\lambda_2 \) |

| 2 | 1, \ldots, 4 | \(i(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2)\) \(-i(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2)\) \(-i(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2)\) \(-i(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2)\) \(-i(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2)\) |
| Coefficients: | with respect to the basis \(\{v_1, v_2, v_3\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, each \( \lambda_i \in \mathbb{R}^\times, \lambda_1 \neq -\lambda_2 \) |

| \( j \) | \( k \) | real orbit representatives (semisimple) |
|---|---|---|
| 1 | 1, \ldots, 4 | \((\lambda_1, \lambda_2, \lambda_3, 0)\) \((\lambda_1, \lambda_2, \lambda_3, 0)\) \((\lambda_1, \lambda_2, -\lambda_3, 0)\) \((\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((\lambda_1, -\lambda_2, 3, 0)\) \((\lambda_1, 3, -\lambda_2, -\lambda_3)\) |
| Coefficients: | with respect to the basis \(\{v_1, v_2, v_3, z_2, z_3, z_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1 \in \mathbb{R}^\times, \lambda_2, \lambda_3 \in \mathbb{R}^\times \) |

| 2 | 1, \ldots, 4 | \(i(\lambda_1, -\lambda_2, \lambda_3, 0)\) \((\lambda_1, -\lambda_2, \lambda_3, 0)\) \((\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((\lambda_1, -\lambda_2, 3, 0)\) |
| Coefficients: | with respect to the basis \(\{x_1, x_2, x_3, x_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1 \in \mathbb{R}^\times, \lambda_2, \lambda_3 \in \mathbb{R}^\times, \lambda_3 \in \mathbb{R}^\times \) |

| 3 | 1, \ldots, 4 | \((0, -\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, -\lambda_3, 3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, 3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) |
| Coefficients: | with respect to the basis \(\{y_1, y_2, y_3, y_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1, \lambda_3 \in \mathbb{R}^\times, \lambda_2, \lambda_3 \in \mathbb{R}^\times \) |

| 4 | 1, \ldots, 4 | \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) \((0, -\lambda_3, -\lambda_2, -\lambda_1)\) |
| Coefficients: | with respect to the basis \(\{x_1, x_2, x_3, x_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1 \in \mathbb{R}^\times, \lambda_2, \lambda_3 \in \mathbb{R}^\times \) |

| 5 | 1, \ldots, 4 | \((0, i\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, i\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, i\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, i\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, i\lambda_3, -i\lambda_2, -\lambda_1)\) \((0, i\lambda_3, -i\lambda_2, -\lambda_1)\) |
| Coefficients: | with respect to the basis \(\{x_1, x_2, x_3, x_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1 \in \mathbb{R}^\times, \lambda_2 \in \mathbb{R}^\times \) |

| 6 | 1, \ldots, 4 | \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) |
| Coefficients: | with respect to the basis \(\{y_1, y_2, y_3, y_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1, \lambda_3 \in \mathbb{R}^\times, \lambda_2, \lambda_3 \in \mathbb{R}^\times \) |

| 7 | 1, \ldots, 4 | \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) |
| Coefficients: | with respect to the basis \(\{y_1, y_2, y_3, y_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1, \lambda_3 \in \mathbb{R}^\times, \lambda_2 \in \mathbb{R}^\times \) |

| 8 | 1, \ldots, 4 | \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) \((i\lambda_1, -\lambda_2, -\lambda_3, 0)\) |
| Coefficients: | with respect to the basis \(\{v_1, v_2, v_3, v_4\}\) |
| Conditions: | up to \( \Gamma_{\Pi_2}\)-conjugacy, \( \lambda_1 \in \mathbb{R}^\times, \lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\} \) |

**Table 10. Case \( i = 2 \):** The table lists real orbit representatives corresponding to \( \gamma_j \) and \([z_k] \in H^1(Z)\).
| j | k | real orbit representatives (semisimple) |
|---|---|----------------------------------|
| 1 | 1, ..., 4 | \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad (-\lambda_1, \lambda_2, \lambda_3, -\lambda_4) \quad (-\lambda_1, \lambda_2, -\lambda_3, \lambda_4) \quad (-\lambda_1, -\lambda_2, \lambda_3, \lambda_4)\) |
| 5 | | \((-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4, \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)/2\) |
| 6 | | \((-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4, -\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4, -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2\) |
| 7 | | \((-\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4, -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, -\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)/2\) |
| 8 | | \((-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4, -\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4, -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)/2\) |
| 9, ..., 12 | | \((-\lambda_1, \lambda_2, \lambda_3, -\lambda_4) \quad (\lambda_1, \lambda_2, \lambda_3, -\lambda_4) \quad (\lambda_1, -\lambda_2, \lambda_3, \lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, each } \lambda_i \in \mathbb{R}^\times \text{ and } \lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3 \pm \lambda_4\}\) |
| 2 | 1, ..., 4 | \(i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad i(-\lambda_1, \lambda_2, \lambda_3, -\lambda_4) \quad i(-\lambda_1, \lambda_2, -\lambda_3, \lambda_4) \quad i(-\lambda_1, -\lambda_2, \lambda_3, \lambda_4)\) |
| 5 | | \(i(-\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, -\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4, -\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)/2\) |
| 6 | | \(i(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, \lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, -\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2\) |
| 7 | | \(i(-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, -\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2\) |
| 8 | | \(i(-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4, -\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)/2\) |
| 9, ..., 12 | | \(i(-\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad i(\lambda_1, \lambda_2, \lambda_3, -\lambda_4) \quad i(\lambda_1, \lambda_2, -\lambda_3, \lambda_4) \quad i(\lambda_1, -\lambda_2, \lambda_3, \lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, each } \lambda_i \in i\mathbb{R}^\times \text{ and } \lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3 \pm \lambda_4\}\) |
| 3 | 1, ..., 4 | \((i\lambda_1, i\lambda_2, -i\lambda_3, \lambda_4) \quad (-i\lambda_1, i\lambda_2, -i\lambda_3, -\lambda_4) \quad (-i\lambda_1, -i\lambda_2, i\lambda_3, \lambda_4) \quad (-i\lambda_1, -i\lambda_2, -i\lambda_3, \lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, } \lambda_1, \lambda_2, i\lambda_3 \in \mathbb{R}^\times \text{ and } \lambda_4 \in i\mathbb{R}^\times \text{ and } \lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\) |
| 4 | 1, ..., 4 | \((-i\lambda_1, -i\lambda_2, \lambda_3, \lambda_4) \quad (i\lambda_1, -i\lambda_2, \lambda_3, -\lambda_4) \quad (i\lambda_1, -i\lambda_2, -\lambda_3, \lambda_4) \quad (-i\lambda_1, -i\lambda_2, -\lambda_3, -\lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{-i\lambda_1, -i\lambda_2, \lambda_3, \lambda_4\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, } \lambda_1, \lambda_2 \in i\mathbb{R}^\times \text{ and } \lambda_3, \lambda_4 \in \mathbb{R}^\times \text{ and } \lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}\) |
| 5 | 1, ..., 4 | \((-i\lambda_1, \lambda_2, i\lambda_3, \lambda_4) \quad (i\lambda_1, \lambda_2, i\lambda_3, -\lambda_4) \quad (i\lambda_1, \lambda_2, -i\lambda_3, \lambda_4) \quad (-i\lambda_1, \lambda_2, -i\lambda_3, -\lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, } \lambda_1, \lambda_2, i\lambda_3 \in i\mathbb{R}^\times \text{ and } \lambda_4 \in \mathbb{R}^\times \text{ and } \lambda_1 \notin \{\pm \lambda_3\}\) |
| 6 | 1, ..., 4 | \((-i\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad (i\lambda_1, \lambda_2, -\lambda_3, \lambda_4) \quad (i\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad (-i\lambda_1, \lambda_2, -\lambda_3, -\lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{-2, -2, \pm 2, \pm 2\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, } \lambda_1, \lambda_2, \lambda_3 \in i\mathbb{R}^\times \text{ and } \lambda_4 \in \mathbb{R}^\times \text{ and } \lambda_1 \notin \{\pm \lambda_3\}\) |
| 7 | 1, ..., 4 | \((i\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad (-i\lambda_1, \lambda_2, -\lambda_3, -\lambda_4) \quad (-i\lambda_1, \lambda_2, -\lambda_3, \lambda_4) \quad (-i\lambda_1, -\lambda_2, \lambda_3, \lambda_4)\) |
| **Coefficients:** | with respect to the basis \(\{t_1, t_2, t_3, t_4\}\) |
| **Conditions:** | \((\lambda_1, \ldots, \lambda_4) \text{ up to } W\text{-conjugacy, } \lambda_1 \in i\mathbb{R}^\times \text{ and } \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^\times \) |

**Table 11. Case i = 1:** The table lists real orbit representatives corresponding to \(\gamma_j\) and \([z_k] \in H^1(Z)\).
### A.3. Mixed elements: Centralisers.

| $i$ | $r$ | identity component $Z^\circ$ | preimages of generators of $Z/Z^\circ$ | $H^1$ ref. |
|-----|-----|-------------------------------|--------------------------------|-----------|
| 2   | 1   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-L, -L, L, L)$ | 6.1       |
| 3   | 1   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-L, -L, L, L)$ | 6.1       |
| 3   | 2   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-I, -I, -I, -I)$ | 6.2       |
| 4   | 1   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-I, -I, -I, -I)$ | 6.1       |
| 4   | 2   | $\{ (D(a)^{-1}, D(a), D(a)^{-1}, D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.1       |
| 4   | 3   | $\{ (D(a)^{-1}, D(a), D(a)^{-1}, D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 7   | 1   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-I, -I, -I, -I)$ | 6.2       |
| 7   | 3   | $\{ (I, I, D(a), D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.2       |
| 7   | 4   | $\{ (D(a)^{-1}, D(a), D(a)^{-1}, D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.1       |
| 10  | 1   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-L, -L, L, L)$ | 6.1       |
| 10  | 2   | $\{ (D(a)^{-1}, D(a)^{-1}, D(a), D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 10  | 3   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-L, -L, L, L)$ | 6.1       |
| 10  | 4   | $\{ (D(a)^{-1}, D(a), D(a)^{-1}, D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 10  | 5   | $\{ (D(a)^{-1}, D(b)^{-1}, D(b), D(a)) : a, b \in \mathbb{C}^\times \}$ | $(-I, -I, I, I)$ | 6.6       |
| 10  | 6   | $\{ (D(a)^{-1}, D(a), D(a)^{-1}, D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 10  | 7   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-L, -L, L, L)$ | 6.1       |
| 10  | 8   | $\{ (D(a)^{-1}, D(a)^{-1}, D(a), D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 10  | 9   | 1                             | $(-I, -I, I, I), (-I, I, -I, I), (-L, -L, L, L)$ | 6.1       |
| 10  | 10  | $\{ (D(a), D(a)^{-1}, D(a), D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 10  | 11  | $\{ (D(b)^{-1}, D(a)^{-1}, D(b), D(a)) : a, b \in \mathbb{C}^\times \}$ | $(-I, -I, I, I)$ | 6.6       |
| 10  | 12  | $\{ (D(a), D(a)^{-1}, D(a), D(a)) : a \in \mathbb{C}^\times \}$ | $(-I, -I, I, I), (-I, I, -I, I)$ | 6.3       |
| 10  | 13  | $\{ (D(b)^{-1}, D(b), D(a)^{-1}, D(a)) : a, b \in \mathbb{C}^\times \}$ | $(-I, -I, I, I)$ | 6.7       |

Table 12. Centralisers $Z = Z_G(p, h, e, f)$ for each $p \in \Sigma_i$ and $e = n_{i, r}$ as in Theorem 3.2. The last column lists reference labels for the equations describing the representatives of the classes in $H^1(Z)$; the notation used in the third and fourth columns is from 3.4.1.
A.4. Mixed elements, case $p \in \Sigma$.

| $i$ | $r$ | $k$ | semisimple part                      | nilpotent part |
|-----|-----|-----|--------------------------------------|----------------|
| 2   | 1   | 1   | $(\lambda_1, \lambda_2, \lambda_3, 0)$ | (0011)         |
|     | 2   |     | $(-\lambda_1, \lambda_2, \lambda_3, 0)$ | $-\langle 0011 \rangle$ |
|     | 3   |     | $(\lambda_1, \lambda_2, -\lambda_3, 0)$ | (0011)         |
|     | 4   |     | $(-\lambda_1, -\lambda_2, \lambda_3, 0)$ | (0011)         |
|     | 5   |     | $(\lambda_1, -\lambda_2, \lambda_3, 0)$ | (0011)         |
|     | 6   |     | $(\lambda_1, \lambda_2, -\lambda_3, 0)$ | $-\langle 0011 \rangle$ |
|     | 7   |     | $(\lambda_1, \lambda_2, -\lambda_3, 0)$ | (0011)         |
|     | 8   |     | $(\lambda_1, -\lambda_2, \lambda_3, 0)$ | (0011)         |

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, each $\lambda_i \in \mathbb{R}^\times$ and $\lambda_1 \not\in \{\pm \lambda_2 \pm \lambda_3\}$

Table 13. Case $i = 2$ and $\mu \in \Sigma$: Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

| $i$ | $r$ | $k$ | semisimple part                      | nilpotent part |
|-----|-----|-----|--------------------------------------|----------------|
| 3   | 1   | 1   | $(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$ | (0011)         |
|     | 2   |     | $(-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2, 0)$ | $-\langle 0011 \rangle$ |
|     | 3   |     | $(\lambda_1 - \lambda_2, -\lambda_1, \lambda_2, 0)$ | (0011)         |
|     | 4   |     | $(-\lambda_1 - \lambda_2, \lambda_1, -\lambda_2, 0)$ | (0011)         |
|     | 5   |     | $(\lambda_1 - \lambda_2, \lambda_1, -\lambda_2, 0)$ | (0011)         |
|     | 6   |     | $(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$ | $-\langle 0011 \rangle$ |
|     | 7   |     | $(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$ | (0011)         |
|     | 8   |     | $(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$ | (0011)         |

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: $(\lambda_1, \lambda_2)$ up to $\Gamma_{\Pi_2}$-conjugacy, each $\lambda_i \in \mathbb{R}^\times$, $\lambda_1 \neq -\lambda_2$

Table 14. Case $i = 3$ and $\mu \in \Sigma$: Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$. 
| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|----------------|
| 4   | 1   | 1   | $(\lambda_1,0,0,\lambda_4)$ | $(|0100\rangle + |0110\rangle)$ |
|     |     | 2   | $(-\lambda_1,0,0,-\lambda_4)$ | $(|0100\rangle + |0110\rangle)$ |
|     |     | 3   | $(-\lambda_1,0,0,\lambda_4)$ | $-|0100\rangle + |0110\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,\lambda_4)$ | $|0100\rangle - |0110\rangle$ |
|     |     | 5   | $(-\lambda_1,0,0,\lambda_4)$ | $|0100\rangle + |0110\rangle$ |
|     |     | 6   | $(\lambda_1,0,0,-\lambda_4)$ | $|0100\rangle + |0110\rangle$ |
|     |     | 7   | $(\lambda_1,0,0,\lambda_4)$ | $-|0100\rangle + |0110\rangle$ |
|     |     | 8   | $(\lambda_1,0,0,\lambda_4)$ | $|0100\rangle - |0110\rangle$ |
| 4   | 2   | 1   | $(\lambda_1,0,0,\lambda_4)$ | $|0110\rangle + |0101\rangle$ |
|     |     | 2   | $(-\lambda_1,0,0,-\lambda_4)$ | $|0110\rangle + |0101\rangle$ |
|     |     | 3   | $(-\lambda_1,0,0,\lambda_4)$ | $|0110\rangle - |0101\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,\lambda_4)$ | $-|0110\rangle + |0101\rangle$ |
|     |     | 5   | $(-\lambda_1,0,0,\lambda_4)$ | $|0110\rangle + |0101\rangle$ |
|     |     | 6   | $(\lambda_1,0,0,-\lambda_4)$ | $|0110\rangle + |0101\rangle$ |
|     |     | 7   | $(\lambda_1,0,0,\lambda_4)$ | $|0110\rangle - |0101\rangle$ |
|     |     | 8   | $(\lambda_1,0,0,\lambda_4)$ | $-|0110\rangle + |0101\rangle$ |
| 4   | 3   | 1   | $(\lambda_1,0,0,\lambda_4)$ | $|0110\rangle$ |
|     |     | 2   | $(-\lambda_1,0,0,-\lambda_4)$ | $|0110\rangle$ |
|     |     | 3   | $(-\lambda_1,0,0,\lambda_4)$ | $|0110\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,\lambda_4)$ | $-|0110\rangle$ |
| 4   | 4   | 1   | $(\lambda_1,0,0,\lambda_4)$ | $|0101\rangle$ |
|     |     | 2   | $(-\lambda_1,0,0,-\lambda_4)$ | $|0101\rangle$ |
|     |     | 3   | $(-\lambda_1,0,0,\lambda_4)$ | $-|0101\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,\lambda_4)$ | $|0101\rangle$ |

**Coefficients:** with respect to the basis \(\{u_1, u_2, u_3, u_4\}\)

**Conditions:** \((\lambda_1, \lambda_4)\) up to \(\Gamma_{H_4}\)-conjugacy. \(\lambda_1, \lambda_4 \in \mathbb{R}^\times, \lambda_1 \notin \{\pm \lambda_4\}\)

Table 15. **Case** \(i = 4\) **and** \(p \in \Sigma\): Real representatives corresponding to \([z_k] \in H^1(Z)\) and \(n_{i,r}\).
Classification of four-dimensional rebits

| $i$ | $1$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|----------------|---------------|
| $7$ | $1$ | $1$ | $\lambda_1(1,0,0,0,-1)$ | $|1101\rangle + |1011\rangle + |1000\rangle + |0001\rangle$ |
|     | $2$ |     | $\lambda_1(-1,0,0,1)$ | $-|1101\rangle + |1011\rangle + |1000\rangle - |0001\rangle$ |
|     | $3$ |     | $\lambda_1(-1,0,0,-1)$ | $|1101\rangle - |1011\rangle + |1000\rangle - |0001\rangle$ |
|     | $4$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1101\rangle - |1011\rangle + |1000\rangle + |0001\rangle$ |
|     | $5$ |     | $\lambda_1(-1,0,0,1)$ | $|1101\rangle + |1011\rangle - |1000\rangle - |0001\rangle$ |
|     | $6$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1101\rangle + |1011\rangle + |1000\rangle + |0001\rangle$ |
|     | $7$ |     | $\lambda_1(-1,0,0,1)$ | $|1101\rangle - |1011\rangle - |1000\rangle + |0001\rangle$ |
|     | $8$ |     | $\lambda_1(1,0,0,0,-1)$ | $-|1101\rangle + |1011\rangle - |1000\rangle - |0001\rangle$ |
| $7$ | $2$ | $1$ | $\lambda_1(1,0,0,-1)$ | $|1011\rangle + |1000\rangle + |0101\rangle$ |
|     | $2$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle + |1000\rangle + |0101\rangle$ |
|     | $3$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle + |1000\rangle - |0101\rangle$ |
|     | $4$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle + |1000\rangle + |0101\rangle$ |
|     | $5$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle - |1000\rangle + |0101\rangle$ |
|     | $6$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle - |1000\rangle - |0101\rangle$ |
|     | $7$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle - |1000\rangle + |0101\rangle$ |
|     | $8$ |     | $\lambda_1(1,0,0,-1)$ | $-|1011\rangle - |1000\rangle + |0101\rangle$ |
| $7$ | $3$ | $1$ | $\lambda_1(1,0,0,0,-1)$ | $|1011\rangle + |1000\rangle + |0001\rangle$ |
|     | $2$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle + |1000\rangle + |0001\rangle$ |
|     | $3$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle + |1000\rangle - |0001\rangle$ |
|     | $4$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle + |1000\rangle + |0001\rangle$ |
|     | $5$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle - |1000\rangle - |0001\rangle$ |
|     | $6$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle - |1000\rangle - |0001\rangle$ |
|     | $7$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle - |1000\rangle + |0001\rangle$ |
|     | $8$ |     | $\lambda_1(1,0,0,-1)$ | $-|1011\rangle - |1000\rangle + |0001\rangle$ |
| $7$ | $4$ | $1$ | $\lambda_1(1,0,0,0,-1)$ | $|1011\rangle + |1000\rangle$ |
|     | $2$ |     | $\lambda_1(-1,0,0,1)$ | $|1011\rangle + |1000\rangle$ |
|     | $3$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle + |1000\rangle$ |
|     | $4$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1011\rangle - |1000\rangle$ |
|     | $5$ |     | $\lambda_1(0,1,0,0)$ | $\frac{1}{2}(-|1111\rangle + |1100\rangle + |1011\rangle - |1000\rangle - |0111\rangle + |0100\rangle + |0011\rangle - |0000\rangle)$ |
|     | $6$ |     | $\lambda_1(0,1,0)$ | $\frac{1}{2}(|1111\rangle - |1100\rangle - |1011\rangle + |1000\rangle + |0111\rangle - |0100\rangle - |0011\rangle + |0000\rangle)$ |
|     | $7$ |     | $\lambda_1(0,1,-1,0)$ | $\frac{1}{2}(|1111\rangle + |1100\rangle + |1011\rangle + |1000\rangle + |0111\rangle + |0100\rangle + |0011\rangle + |0000\rangle)$ |
|     | $8$ |     | $\lambda_1(0,-1,1,0)$ | $\frac{1}{2}(|1111\rangle + |1100\rangle + |1011\rangle + |1000\rangle + |0111\rangle + |0100\rangle + |0011\rangle + |0000\rangle)$ |
| $7$ | $5$ | $1$ | $\lambda_1(1,0,0,-1)$ | $-|1101\rangle - |0001\rangle$ |
|     | $2$ |     | $\lambda_1(-1,0,0,1)$ | $-|1101\rangle - |0001\rangle$ |
|     | $3$ |     | $\lambda_1(-1,0,0,-1)$ | $|1101\rangle - |0001\rangle$ |
|     | $4$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1101\rangle + |0001\rangle$ |
|     | $5$ |     | $\lambda_1(0,-1,-1,0)$ | $\frac{1}{2}(-|1111\rangle - |1100\rangle + |1101\rangle + |1100\rangle + |0111\rangle + |0010\rangle - |0001\rangle - |0000\rangle)$ |
|     | $6$ |     | $\lambda_1(0,1,1,0)$ | $\frac{1}{2}(-|1111\rangle - |1100\rangle - |1101\rangle + |1100\rangle + |0111\rangle + |0010\rangle - |0001\rangle - |0000\rangle)$ |
|     | $7$ |     | $\lambda_1(0,-1,1,0)$ | $\frac{1}{2}(|1111\rangle - |1100\rangle - |1101\rangle + |1100\rangle + |0111\rangle - |0010\rangle - |0001\rangle + |0000\rangle)$ |
|     | $8$ |     | $\lambda_1(0,1,-1,0)$ | $\frac{1}{2}(|1111\rangle - |1100\rangle - |1101\rangle + |1100\rangle + |0111\rangle - |0010\rangle - |0001\rangle + |0000\rangle)$ |
| $7$ | $6$ | $1$ | $\lambda_1(1,0,0,-1)$ | $|1001\rangle$ |
|     | $2$ |     | $\lambda_1(-1,0,0,1)$ | $|1001\rangle$ |
|     | $3$ |     | $\lambda_1(-1,0,0,-1)$ | $|1001\rangle$ |
|     | $4$ |     | $\lambda_1(-1,0,0,-1)$ | $-|1001\rangle$ |

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: $\lambda_1$ up to $\Gamma_{17}$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$

Table 16. Case $i = 7$ and $p \in \Sigma$: Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$. 

| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|----------------|---------------|
| 10  | 1   | 1   | $(-\lambda_1,0,0,0)$ | $|1100\rangle + |1010\rangle + |0110\rangle$ |
|     |     | 2   | $(-\lambda_1,0,0,0)$ | $-|1100\rangle + |1010\rangle + |0110\rangle$ |
|     |     | 3   | $(-\lambda_1,0,0,0)$ | $|1100\rangle - |1010\rangle + |0110\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,0)$ | $|1100\rangle + |1010\rangle - |0110\rangle$ |
|     |     | 5   | $(-\lambda_1,0,0,0)$ | $|1100\rangle + |1010\rangle + |0110\rangle$ |
|     |     | 6   | $(\lambda_1,0,0,0)$  | $-|1100\rangle + |1010\rangle + |0110\rangle$ |
|     |     | 7   | $(\lambda_1,0,0,0)$  | $|1100\rangle - |1010\rangle + |0110\rangle$ |
|     |     | 8   | $(\lambda_1,0,0,0)$  | $|1100\rangle + |1010\rangle - |0110\rangle$ |
| 10  | 3   | 1   | $(-\lambda_1,0,0,0)$ | $|1010\rangle + |0110\rangle + |0011\rangle$ |
|     |     | 2   | $(-\lambda_1,0,0,0)$ | $|1010\rangle + |0110\rangle - |0011\rangle$ |
|     |     | 3   | $(-\lambda_1,0,0,0)$ | $-|1010\rangle + |0110\rangle + |0011\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,0)$ | $|1010\rangle - |0110\rangle + |0011\rangle$ |
|     |     | 5   | $(-\lambda_1,0,0,0)$ | $|1010\rangle + |0110\rangle + |0011\rangle$ |
|     |     | 6   | $(\lambda_1,0,0,0)$  | $|1010\rangle + |0110\rangle - |0011\rangle$ |
|     |     | 7   | $(\lambda_1,0,0,0)$  | $-|1010\rangle + |0110\rangle + |0011\rangle$ |
|     |     | 8   | $(\lambda_1,0,0,0)$  | $|1010\rangle - |0110\rangle + |0011\rangle$ |
| 10  | 7   | 1   | $(-\lambda_1,0,0,0)$ | $|1100\rangle + |0110\rangle + |0101\rangle$ |
|     |     | 2   | $(-\lambda_1,0,0,0)$ | $-|1100\rangle + |0110\rangle + |0101\rangle$ |
|     |     | 3   | $(-\lambda_1,0,0,0)$ | $|1100\rangle + |0110\rangle - |0101\rangle$ |
|     |     | 4   | $(-\lambda_1,0,0,0)$ | $|1100\rangle - |0110\rangle + |0101\rangle$ |
|     |     | 5   | $(-\lambda_1,0,0,0)$ | $|1100\rangle + |0110\rangle + |0101\rangle$ |
|     |     | 6   | $(\lambda_1,0,0,0)$  | $-|1100\rangle + |0110\rangle + |0101\rangle$ |
|     |     | 7   | $(\lambda_1,0,0,0)$  | $|1100\rangle + |0110\rangle - |0101\rangle$ |
|     |     | 8   | $(\lambda_1,0,0,0)$  | $|1100\rangle - |0110\rangle + |0101\rangle$ |
| 10  | 9   | 1   | $(\lambda_1,0,0,0)$  | $|0110\rangle + |0101\rangle + |0011\rangle$ |
|     |     | 2   | $(\lambda_1,0,0,0)$  | $|0110\rangle + |0101\rangle - |0011\rangle$ |
|     |     | 3   | $(\lambda_1,0,0,0)$  | $|0110\rangle - |0101\rangle + |0011\rangle$ |
|     |     | 4   | $(\lambda_1,0,0,0)$  | $-|0110\rangle + |0101\rangle + |0011\rangle$ |
|     |     | 5   | $(\lambda_1,0,0,0)$  | $|0110\rangle + |0101\rangle + |0011\rangle$ |
|     |     | 6   | $(\lambda_1,0,0,0)$  | $|0110\rangle + |0101\rangle - |0011\rangle$ |
|     |     | 7   | $(\lambda_1,0,0,0)$  | $|0110\rangle - |0101\rangle + |0011\rangle$ |
|     |     | 8   | $(\lambda_1,0,0,0)$  | $-|0110\rangle + |0101\rangle + |0011\rangle$ |
| 10  | 5   | 1   | $(\lambda_1,0,0,0)$  | $|0110\rangle$ |
|     |     | 2   | $(\lambda_1,0,0,0)$  | $|0110\rangle$ |
| 10  | 11  | 1   | $(\lambda_1,0,0,0)$  | $|1010\rangle$ |
|     |     | 2   | $(\lambda_1,0,0,0)$  | $|1010\rangle$ |
| 10  | 13  | 1   | $(\lambda_1,0,0,0)$  | $|0011\rangle$ |
|     |     | 2   | $(\lambda_1,0,0,0)$  | $|0011\rangle$ |

**Coefficients:** with respect to the basis $\{u_1, u_2, u_3, u_4\}$

**Conditions:** $\lambda_1$ up to $\Gamma_{110}$-conjugacy, $\lambda_1 \in \mathbb{R}^x$

**Table 17.** Case $i = 10$ and $p \in \Sigma$ (Part I): Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,p}$. 
| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|----------------|---------------|
| 10  | 2   | 1   | $(\lambda_1, 0, 0, 0)$ | $|1010\rangle + |0110\rangle$ |
|     |     | 2   | $(-\lambda_1, 0, 0, 0)$ | $|1010\rangle + |0110\rangle$ |
|     |     | 3   | $(-\lambda_1, 0, 0, 0)$ | $-|1010\rangle + |0110\rangle$ |
|     |     | 4   | $(-\lambda_1, 0, 0, 0)$ | $|1010\rangle - |0110\rangle$ |
| 10  | 4   | 1   | $(\lambda_1, 0, 0, 0)$ | $|1100\rangle + |0110\rangle$ |
|     |     | 2   | $(-\lambda_1, 0, 0, 0)$ | $-|1100\rangle + |0110\rangle$ |
|     |     | 3   | $(-\lambda_1, 0, 0, 0)$ | $|1100\rangle + |0110\rangle$ |
|     |     | 4   | $(-\lambda_1, 0, 0, 0)$ | $|1100\rangle - |0110\rangle$ |
| 10  | 6   | 1   | $(\lambda_1, 0, 0, 0)$ | $|0110\rangle + |0011\rangle$ |
|     |     | 2   | $(-\lambda_1, 0, 0, 0)$ | $|0110\rangle - |0011\rangle$ |
|     |     | 3   | $(-\lambda_1, 0, 0, 0)$ | $|0110\rangle + |0011\rangle$ |
|     |     | 4   | $(-\lambda_1, 0, 0, 0)$ | $-|0110\rangle + |0011\rangle$ |
| 10  | 8   | 1   | $(\lambda_1, 0, 0, 0)$ | $|0110\rangle + |0101\rangle$ |
|     |     | 2   | $(-\lambda_1, 0, 0, 0)$ | $|0110\rangle + |0101\rangle$ |
|     |     | 3   | $(-\lambda_1, 0, 0, 0)$ | $|0110\rangle - |0101\rangle$ |
|     |     | 4   | $(-\lambda_1, 0, 0, 0)$ | $-|0110\rangle + |0101\rangle$ |
| 10  | 10  | 1   | $(\lambda_1, 0, 0, 0)$ | $|1100\rangle + |1010\rangle$ |
|     |     | 2   | $(-\lambda_1, 0, 0, 0)$ | $-|1100\rangle + |1010\rangle$ |
|     |     | 3   | $(-\lambda_1, 0, 0, 0)$ | $|1100\rangle - |1010\rangle$ |
|     |     | 4   | $(-\lambda_1, 0, 0, 0)$ | $|1100\rangle + |1010\rangle$ |
| 10  | 12  | 1   | $(\lambda_1, 0, 0, 0)$ | $|1010\rangle + |0011\rangle$ |
|     |     | 2   | $(-\lambda_1, 0, 0, 0)$ | $|1010\rangle - |0011\rangle$ |
|     |     | 3   | $(-\lambda_1, 0, 0, 0)$ | $-|1010\rangle + |0011\rangle$ |
|     |     | 4   | $(-\lambda_1, 0, 0, 0)$ | $|1010\rangle + |0011\rangle$ |

**Coefficients:** with respect to the basis $\{u_1, u_2, u_3, u_4\}$

**Conditions:** $\lambda_1$ up to $\Gamma_{110}$-conjugacy, $\lambda_1 \in \mathbb{R}$.  

**Table 18.** Case $i = 10$ and $p \in \Sigma$ (Part II): Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$. 
A.5. **Mixed elements, case** $p \notin \Sigma$.

| $i$  | $j$  | nilpotent elements                                                                 |
|------|------|-----------------------------------------------------------------------------------|
| 2    | 2    | $n_{2,2,1} = -[0110]$                                                             |
| 4, 6, 7, 8 | $n_{2,j,1} = [0011]$                                                               |
| 3, 5 |      | $n_{2,2,1} = [0000]$                                                               |
| 3    | 2    | $n_{3,2,1} = -[0011], \ n_{3,2,2} = -[0111] + [1011] - [0010] - [0001]$           |
| 4    | 2, 3 | $n_{4,j,1} = [0110] + [1010], \ n_{4,j,2} = [0110] + [0101], \ n_{4,j,3} = [0110], \ n_{4,j,4} = [0101]$ | |
| 4    |      | $n_{4,4,1} = -\frac{1}{2}(-[1110] - [1101] + [1010] + [1001] + [0110] + [0101] - [0010] - [0001])$ |
| 7    | 2    | $n_{7,2,1} = -[1101] - [1011] - [1000] + [0001], \ n_{7,2,2} = -[1101] - [1010] + [0001], \ n_{7,2,3} = -[1011] - [1000] + [0101], \ n_{7,2,4} = -[1011] - [1000], \ n_{7,2,5} = -[1101] + [0001], \ n_{7,2,6} = -[1001]$ |
| 10   | 2    | $n_{10,2,1} = -[1100] - [1010] + [0110], \ n_{10,2,2} = -[1010] + [0110], \ n_{10,2,3} = -[1010] + [0110] + [0011], \ n_{10,2,4} = -[1100] + [0110], \ n_{10,2,5} = [0110], \ n_{10,2,6} = [0110] + [0011], \ n_{10,2,7} = -[1100] + [0110] + [0101], \ n_{10,2,8} = [0110] + [0101], \ n_{10,2,9} = [0110] - [0101] - [0011], \ n_{10,2,10} = -[1100] - [1010], \ n_{10,2,11} = -[1010], \ n_{10,2,12} = -[1010] + [0011], \ n_{10,2,13} = [0011].$ |

**Table 19.** The nilpotent elements $n_{i,j,r}$ used in the classification of mixed elements with semisimple part $p' \notin \Sigma$, sorted by Case $i$ and cohomology class $[\gamma_j]$. 
### Classification of four-dimensional rebits

| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 2   | 1   | $(\lambda_3, 0, \lambda_1, -i\lambda_2)$ | $-\{0110\}$ |
| 2   | 2   | 2   | $(i\lambda_3, 0, \lambda_1, i\lambda_2)$ | $-\{0110\}$ |
| 3   | 2   | 3   | $(\lambda_3, 0, -\lambda_1, -i\lambda_2)$ | $-\{0110\}$ |
| 4   | 2   | 4   | $(i\lambda_3, 0, \lambda_1, -i\lambda_2)$ | $\{0110\}$ |
| 5   | 2   | 5   | $(i\lambda_3, 0, \lambda_1, -i\lambda_2)$ | $-\{0110\}$ |
| 6   | 2   | 6   | $(-i\lambda_3, 0, \lambda_1, i\lambda_2)$ | $-\{0110\}$ |
| 7   | 2   | 7   | $(-i\lambda_3, 0, -\lambda_1, -i\lambda_2)$ | $-\{0110\}$ |
| 8   | 2   | 8   | $(-i\lambda_3, 0, \lambda_1, -i\lambda_2)$ | $\{0110\}$ |

**Coefficients:** with respect to the basis $\{z_1, z_2, z_3, z_4\}$

**Conditions:** $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$, $\lambda_2, \lambda_3 \in i\mathbb{R}^\times$

| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 3   | 1   | $(0, -\lambda_3, -i\lambda_2, \lambda_1)$ | $\{0000\}$ |
| 2   | 3   | 2   | $(0, -\lambda_3, -i\lambda_2, -\lambda_1)$ | $\{0000\}$ |
| 3   | 3   | 3   | $(0, -\lambda_3, i\lambda_2, \lambda_1)$ | $\{0000\}$ |
| 4   | 3   | 4   | $(0, \lambda_3, -i\lambda_2, \lambda_1)$ | $\{0000\}$ |
| 5   | 3   | 5   | $(0, -\lambda_3, -i\lambda_2, -\lambda_1)$ | $\{0000\}$ |
| 6   | 3   | 6   | $(0, -\lambda_3, -i\lambda_2, -\lambda_1)$ | $\{0000\}$ |
| 7   | 3   | 7   | $0, -\lambda_3, i\lambda_2, \lambda_1$ | $\{0000\}$ |
| 8   | 3   | 8   | $(0, \lambda_3, -i\lambda_2, \lambda_1)$ | $\{0000\}$ |

**Coefficients:** with respect to the basis $\{y_1, y_2, y_3, y_4\}$

**Conditions:** $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, $\lambda_1, \lambda_3 \in \mathbb{R}^\times$, $\lambda_2 \in i\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_3\}$

| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 4   | 1   | $(-i\lambda_1, -i\lambda_2, \lambda_3, 0)$ | $\{0011\}$ |
| 2   | 4   | 2   | $(i\lambda_1, -i\lambda_2, \lambda_3, 0)$ | $\{0011\}$ |
| 3   | 4   | 3   | $(i\lambda_1, -i\lambda_2, -\lambda_3, 0)$ | $\{0011\}$ |
| 4   | 4   | 4   | $(i\lambda_1, i\lambda_2, \lambda_3, 0)$ | $\{0011\}$ |
| 5   | 4   | 5   | $(i\lambda_1, -i\lambda_2, -\lambda_3, 0)$ | $\{0011\}$ |
| 6   | 4   | 6   | $(-i\lambda_1, -i\lambda_2, \lambda_3, 0)$ | $\{0011\}$ |
| 7   | 4   | 7   | $(-i\lambda_1, -i\lambda_2, -\lambda_3, 0)$ | $\{0011\}$ |
| 8   | 4   | 8   | $(-i\lambda_1, i\lambda_2, \lambda_3, 0)$ | $\{0011\}$ |

**Coefficients:** with respect to the basis $\{x_1, x_2, x_3, x_4\}$

**Conditions:** $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, $\lambda_3 \in \mathbb{R}^\times$, $\lambda_1, \lambda_2 \in i\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_3\}$

Table 20. **Cases $i = 2$ and $p \notin S$ (Part I):** Mixed real representatives corresponding to $\gamma_j$, $[z_k] \in H^1(Z)$, and $n_{2,j,1}$.
| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 5   | 1   | $(0, i\lambda_3, -\lambda_2, \lambda_1)$ |               |
| 2   | 5   | 2   | $(0, i\lambda_3, -\lambda_2, -\lambda_1)$ | $-\langle 0000 \rangle$ |
| 2   | 5   | 3   | $(0, i\lambda_3, \lambda_2, \lambda_1)$ |               |
| 4   | 5   | 4   | $(0, -i\lambda_3, -\lambda_2, \lambda_1)$ | $-\langle 0000 \rangle$ |
| 5   | 5   | 5   | $(0, i\lambda_3, -\lambda_2, \lambda_1)$ |               |
| 6   | 5   | 6   | $(0, i\lambda_3, -\lambda_2, -\lambda_1)$ | $\langle 0000 \rangle$ |
| 7   | 5   | 7   | $(0, i\lambda_3, \lambda_2, \lambda_1)$ | $\langle 0000 \rangle$ |
| 8   | 5   | 8   | $(0, -i\lambda_3, -\lambda_2, \lambda_1)$ | $\langle 0000 \rangle$ |

Coefficients: with respect to the basis $\{x_1, x_2, x_3, x_4\}$

Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, $\lambda_1, \lambda_2 \in \mathbb{R}^\times$, $\lambda_3 \in i\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm \lambda_2\}$

| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 6   | 1   | $(-i\lambda_1, \lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 2   | 6   | 2   | $(i\lambda_1, \lambda_2, i\lambda_3, 0)$ | $-\langle 0011 \rangle$ |
| 3   | 6   | 3   | $(i\lambda_1, \lambda_2, -i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 4   | 6   | 4   | $(i\lambda_1, -\lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 5   | 6   | 5   | $(i\lambda_1, \lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 6   | 6   | 6   | $(-i\lambda_1, \lambda_2, i\lambda_3, 0)$ | $-\langle 0011 \rangle$ |
| 7   | 6   | 7   | $(-i\lambda_1, \lambda_2, -i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 8   | 6   | 8   | $(-i\lambda_1, -\lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |

Coefficients: with respect to the basis $\{y_1, y_2, y_3, y_4\}$

Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, $\lambda_2 \in \mathbb{R}^\times$, $\lambda_1, \lambda_3 \in i\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm \lambda_3\}$

| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 7   | 1   | $(-i\lambda_1, \lambda_2, \lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 2   | 7   | 2   | $(i\lambda_1, \lambda_2, \lambda_3, 0)$ | $-\langle 0011 \rangle$ |
| 3   | 7   | 3   | $(i\lambda_1, \lambda_2, -\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 4   | 7   | 4   | $(i\lambda_1, -\lambda_2, \lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 5   | 7   | 5   | $(i\lambda_1, \lambda_2, \lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 6   | 7   | 6   | $(-i\lambda_1, \lambda_2, \lambda_3, 0)$ | $-\langle 0011 \rangle$ |
| 7   | 7   | 7   | $(-i\lambda_1, \lambda_2, -\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 8   | 7   | 8   | $(-i\lambda_1, -\lambda_2, \lambda_3, 0)$ | $-\langle 0011 \rangle$ |

Coefficients: with respect to the basis $\{y_1, y_2, y_3, y_4\}$

Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, $\lambda_2, \lambda_3 \in \mathbb{R}^\times$, $\lambda_1 \in i\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm \lambda_3\}$

| $i$ | $j$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|---------------|
| 2   | 8   | 1   | $(i\lambda_1, i\lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 2   | 8   | 2   | $(-i\lambda_1, i\lambda_2, i\lambda_3, 0)$ | $-\langle 0011 \rangle$ |
| 3   | 8   | 3   | $(-i\lambda_1, i\lambda_2, -i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 4   | 8   | 4   | $(i\lambda_1, -i\lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 5   | 8   | 5   | $(-i\lambda_1, i\lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 6   | 8   | 6   | $(i\lambda_1, i\lambda_2, i\lambda_3, 0)$ | $-\langle 0011 \rangle$ |
| 7   | 8   | 7   | $(i\lambda_1, i\lambda_2, -i\lambda_3, 0)$ | $\langle 0011 \rangle$ |
| 8   | 8   | 8   | $(i\lambda_1, -i\lambda_2, i\lambda_3, 0)$ | $\langle 0011 \rangle$ |

Coefficients: with respect to the basis $\{v_1, v_2, v_3, v_4\}$

Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to $\Gamma_{\Pi_2}$-conjugacy, each $\lambda_i \in i\mathbb{R}^\times$ and $\lambda_1 \notin \{\pm \lambda_2 \pm \lambda_3\}$

Table 21. Cases $i = 2$ and $S$ (Part II): Mixed real representatives corresponding to $\gamma_j$, $[z_k] \in H^1(Z)$, and $n_{2,j,1}$.
| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|----------------|
| 3   | 1   | 1   | $(i\lambda_1 + i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 2   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 3   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 4   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 5   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 6   |     |     | $(i\lambda_1 + i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 7   |     |     | $(i\lambda_1 + i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |
| 8   |     |     | $(i\lambda_1 + i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |0011| |

Coefficients: with respect to the basis $\{v_1, v_2, v_3\}$

Conditions: $(\lambda_1, \lambda_2)$ up to $\Gamma_{13}$-conjugacy, each $\lambda_i \in i\mathbb{R}^\times$, $\lambda_1 \neq -\lambda_2$

| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|----------------|
| 3   | 2   | 1   | $(i\lambda_1 + i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 2   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 3   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 4   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 5   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 6   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 7   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |
| 8   |     |     | $(-i\lambda_1 - i\lambda_2, -i\lambda_1, -i\lambda_2, 0)$ | |1011| |

Coefficients: with respect to the basis $\{v_1, v_2, v_3\}$

Conditions: $(\lambda_1, \lambda_2)$ up to $\Gamma_{13}$-conjugacy, each $\lambda_i \in i\mathbb{R}^\times$, $\lambda_1 \neq -\lambda_2$

Table 22. **Cases $i = 3$ and $p \notin S$:** Mixed real representatives corresponding to $\gamma_2$, $[z_k] \in H^1(Z)$, and $n_{3,2,r}$. 
| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|----------------|---------------|
| 7   | 1   | 1   | $(i\lambda, 0, 0, -i\lambda)$ | $-[1110] - [1011] - [1000] + [0001]$ |
|     | 2   | 0   | $(i\lambda, 0, 0, i\lambda)$ | $[1101] + [1011] - [1000] - [0001]$ |
|     | 3   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1101] + [1011] - [1000] - [0001]$ |
|     | 4   | $-i\lambda, 0, 0, -i\lambda)$ | $[1101] - [1011] + [1000] + [0001]$ |
|     | 5   | $-i\lambda, 0, 0, i\lambda)$ | $-[1101] - [1011] + [1000] - [0001]$ |
|     | 6   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1101] - [1011] + [1000] + [0001]$ |
|     | 7   | $i\lambda, 0, 0, i\lambda)$ | $[1101] + [1011] + [1000] + [0001]$ |
|     | 8   | $i\lambda, 0, 0, -i\lambda)$ | $[1101] + [1011] + [1000] - [0001]$ |
| 7   | 2   | 1   | same as $r = 1$ | $-[1101] - [1010] + [0001]$ |
|     | 2   | 0   | $(i\lambda, 0, 0, i\lambda)$ | $[1101] + [1010] - [0001]$ |
|     | 3   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1101] + [1010] - [0001]$ |
|     | 4   | same as $r = 1$ | $[1101] - [1010] + [0001]$ |
|     | 5   | $-i\lambda, 0, 0, i\lambda)$ | $-[1101] - [1010] - [0001]$ |
|     | 6   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1101] + [1010] + [0001]$ |
|     | 7   | same as $r = 1$ | $[1101] + [1010] + [0001]$ |
|     | 8   | $(i\lambda, 0, 0, i\lambda)$ | $[1101] + [1010] + [0001]$ |
| 7   | 3   | 1   | $(i\lambda, 0, 0, -i\lambda)$ | $[1011] + [1000]$ |
|     | 2   | 0   | $(i\lambda, 0, 0, i\lambda)$ | $[1011] + [1000]$ |
|     | 3   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1011] + [1000]$ |
|     | 4   | same as $r = 1$ | $[1011] - [1000] + [0001]$ |
|     | 5   | $-i\lambda, 0, 0, i\lambda)$ | $-[1011] - [1000] - [0001]$ |
|     | 6   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1011] + [1000] - [0001]$ |
|     | 7   | same as $r = 1$ | $[1011] + [1000] + [0001]$ |
|     | 8   | $(i\lambda, 0, 0, i\lambda)$ | $[1011] + [1000] + [0001]$ |
| 7   | 4   | 1   | $(i\lambda, 0, 0, -i\lambda)$ | $[1011] + [1000]$ |
|     | 2   | 0   | $(i\lambda, 0, 0, i\lambda)$ | $[1011] + [1000]$ |
|     | 3   | $-i\lambda, 0, 0, -i\lambda)$ | $-[1011] + [1000]$ |
|     | 4   | same as $r = 1$ | $[1011] - [1000]$ |
|     | 5   | $-i\lambda, 0, 0, i\lambda)$ | $[1011] - [1000] + [1001]$ |
|     | 6   | $(i\lambda, 0, 0, i\lambda)$ | $[1011] - [1000] - [0001]$ |
|     | 7   | $-i\lambda, 0, 0, -i\lambda)$ | $[1011] - [1000] + [0001]$ |
|     | 8   | $(i\lambda, 0, 0, -i\lambda)$ | $[1011] - [1000] + [0001]$ |
| 7   | 5   | 1   | $(i\lambda, 0, 0, -i\lambda)$ | $-[1101] + [0001]$ |
|     | 2   | 0   | $(i\lambda, 0, 0, i\lambda)$ | $-[1101] - [0001]$ |
|     | 3   | $-i\lambda, 0, 0, -i\lambda)$ | $+[1101] + [0001]$ |
|     | 4   | $(i\lambda, 0, 0, i\lambda)$ | $-[1101] + [0001]$ |
|     | 5   | $(i\lambda, 0, 0, -i\lambda)$ | $[1011] - [0001]$ |
|     | 6   | $(i\lambda, 0, 0, i\lambda)$ | $[1011] - [1000] + [0001]$ |
|     | 7   | $(i\lambda, 0, 0, i\lambda)$ | $[1011] - [0001]$ |
|     | 8   | $(i\lambda, 0, 0, -i\lambda)$ | $[1011] - [1000] + [0001]$ |
| 7   | 6   | 1   | $(i\lambda, 0, 0, -i\lambda)$ | $[1001]$ |
|     | 2   | 0   | $(i\lambda, 0, 0, i\lambda)$ | $[1001]$ |
|     | 3   | $-i\lambda, 0, 0, -i\lambda)$ | $[1001]$ |
|     | 4   | $(i\lambda, 0, 0, -i\lambda)$ | $-[1001]$ |

**Coefficients:** with respect to the basis $\{v_1, v_4\}$

**Conditions:** $\lambda$ up to $\Gamma_{17}$-conjugacy, $\lambda \in i\mathbb{R}^\times$

**Table 23.** Cases $i = 7$ and $p \notin S$: Mixed real representatives corresponding to $\gamma_2$, $[z_k] \in H^1(Z)$, and $n_{7,2,r}$. 
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| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|----------------|
| 10  | 1   | 1   | $(\alpha, 0, 0, 0)$ | $[1100] - [0101] + [0110]$ |
|     | 3   |     |                  | $[1100] + [0110] - [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] + [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |
| 1   | 2   |     | $(-\alpha, 0, 0, 0)$ | $[1100] - [0101] + [0110]$ |
|     | 3   |     |                  | $[1100] + [0110] - [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] + [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |
| 1   | 3   |     | $(-\alpha, 0, 0, 0)$ | $[1100] + [0110] - [0011]$ |
|     | 3   |     |                  | $[1100] + [0110] + [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] + [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |
| 1   | 4   |     | $(-\alpha, 0, 0, 0)$ | $[1100] - [0101] - [0011]$ |
|     | 3   |     |                  | $[1100] + [0110] + [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] - [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |
| 1   | 5   |     | $(-\alpha, 0, 0, 0)$ | $[1100] - [0101] + [0110]$ |
|     | 3   |     |                  | $[1100] + [0110] + [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] + [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |
| 1   | 6   |     | $(\alpha, 0, 0, 0)$ | $[1100] - [0101] + [0110]$ |
|     | 3   |     |                  | $[1100] + [0110] + [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] + [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |
| 1   | 7   |     | $(\alpha, 0, 0, 0)$ | $[1100] + [0110] + [0011]$ |
|     | 3   |     |                  | $[1100] + [0110] + [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] - [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] + [0011]$ |
| 1   | 8   |     | $(\alpha, 0, 0, 0)$ | $[1100] - [0101] - [0011]$ |
|     | 3   |     |                  | $[1100] - [0101] + [0011]$ |
|     | 7   |     |                  | $[0110] - [0101] - [0011]$ |
|     | 9   |     |                  | $[0110] - [0101] - [0011]$ |

**Coefficients:** with respect to the basis $\{v_1, v_2, v_3, v_4\}$

**Conditions:** $\lambda$ up to $\Gamma_{10}$-conjugacy, $\lambda \in \mathbb{R}^\times$

| Table 24. | Cases $i = 10$ and $p \notin S$ (Part 1): Mixed real representatives corresponding to $\gamma_2$, $[z_k] \in H^1(Z)$, and $n_{10,2,r}$. |
| $i$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----------------|----------------|
| 10  | 2, 4, 6, 8, 10, 12 | 1   | $(i\lambda, 0, 0, 0)$ | $-|1010\rangle + |0110\rangle$ |
|     |                  |     |                 | $-|1100\rangle + |0110\rangle$ |
|     |                  |     |                 | $|0110\rangle + |0011\rangle$ |
|     |                  |     |                 | $|0110\rangle + |0101\rangle$ |
|     |                  |     |                 | $-|1100\rangle - |1010\rangle$ |
|     |                  |     |                 | $-|1010\rangle + |0011\rangle$ |
| 2   |                  |     | $(-i\lambda, 0, 0, 0)$ | $-|1010\rangle + |0110\rangle$ |
|     |                  |     |                 | $|1100\rangle + |0110\rangle$ |
|     |                  |     |                 | $|0110\rangle - |0011\rangle$ |
|     |                  |     |                 | $|0110\rangle + |0101\rangle$ |
|     |                  |     |                 | $|1100\rangle - |1010\rangle$ |
|     |                  |     |                 | $-|1010\rangle - |0011\rangle$ |
| 3   |                  |     | $(-i\lambda, 0, 0, 0)$ | $|1010\rangle + |0110\rangle$ |
|     |                  |     |                 | $-|1100\rangle + |0110\rangle$ |
|     |                  |     |                 | $|0110\rangle + |0011\rangle$ |
|     |                  |     |                 | $|0110\rangle - |0101\rangle$ |
|     |                  |     |                 | $-|1100\rangle + |1010\rangle$ |
|     |                  |     |                 | $|1010\rangle + |0011\rangle$ |
| 4   |                  |     | $(-i\lambda, 0, 0, 0)$ | $-|1010\rangle - |0110\rangle$ |
|     |                  |     |                 | $-|1100\rangle - |0110\rangle$ |
|     |                  |     |                 | $-|0110\rangle + |0011\rangle$ |
|     |                  |     |                 | $-|0110\rangle + |0101\rangle$ |
|     |                  |     |                 | $-|1100\rangle - |1010\rangle$ |
|     |                  |     |                 | $-|1010\rangle + |0011\rangle$ |
| 10  | 5, 11            | 1   | $(i\lambda, 0, 0, 0)$ | $|0110\rangle$ |
|     |                  |     |                 | $-|1010\rangle$ |
|     |                  | 2   | $(-i\lambda, 0, 0, 0)$ | $|0110\rangle$ |
|     |                  |     |                 | $-|1010\rangle$ |
| 10  | 13               | 1   | $(i\lambda, 0, 0, 0)$ | $|0011\rangle$ |
|     |                  |     |                 | $|0011\rangle$ |
|     |                  | 2   | $(-i\lambda, 0, 0, 0)$ | $|0011\rangle$ |

**Coefficients:** with respect to the basis $\{v_1, v_2, v_3, v_4\}$

**Conditions:** $\lambda$ up to $\Gamma_{10}$-conjugacy, $\lambda \in i\mathbb{R}^\times$

**Table 25.** Cases $i = 10$ and $p \notin S$ (Part II): Mixed real representatives corresponding to $\gamma_2$, $[z_k] \in H^1(Z)$, and $n_{10,2,r}$.  

| $i$ | $j$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----|-----------------|---------------|
| 4   | 2   | 1   | 1   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 2   | $(-i\lambda_1, 0, 0, -\lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 3   | $(-i\lambda_1, 0, 0, \lambda_4)$ | $-|0110\rangle + |0110\rangle$ |
|     |     |     | 4   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle - |0110\rangle$ |
|     |     |     | 5   | $(i\lambda_1, 0, 0, -\lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 6   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 7   | $(i\lambda_1, 0, 0, \lambda_4)$ | $-|0110\rangle + |0110\rangle$ |
|     |     |     | 8   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle - |0110\rangle$ |
| 4   | 2   | 2   | 1   | same as $r = 1$ | $|0110\rangle + |0101\rangle$ |
|     |     |     | 2   | $|0110\rangle + |0101\rangle$ | $|0110\rangle - |0101\rangle$ |
|     |     |     | 3   | $-|0110\rangle + |0101\rangle$ | $|0110\rangle + |0101\rangle$ |
|     |     |     | 4   | same as $r = 1$ | $|0110\rangle + |0101\rangle$ |
|     |     |     | 5   | $|0110\rangle + |0101\rangle$ | $|0110\rangle - |0101\rangle$ |
|     |     |     | 6   | $|0110\rangle - |0101\rangle$ | $-|0110\rangle + |0101\rangle$ |
| 4   | 2   | 3   | 1   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle$ |
|     |     |     | 2   | $(i\lambda_1, 0, 0, -\lambda_4)$ | $|0110\rangle$ |
|     |     |     | 3   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle$ |
|     |     |     | 4   | $(i\lambda_1, 0, 0, \lambda_4)$ | $-|0110\rangle$ |
| 4   | 2   | 4   | 1   | same as $r = 3$ | $|0101\rangle$ |
|     |     |     | 2   | $|0101\rangle$ | $|0101\rangle$ |
|     |     |     | 3   | $-|0101\rangle$ | $|0101\rangle$ |
|     |     |     | 4   | $|0101\rangle$ | $|0101\rangle$ |

Coefficients: with respect to the basis $\{t_1, t_4\}$

Conditions: $(\lambda_1, \lambda_4)$ up to $\Gamma_{14}$-conjugacy, $\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \in i\mathbb{R}^\times$

| $i$ | $j$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----|-----------------|---------------|
| 4   | 2   | 1   | 1   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 2   | $(-i\lambda_1, 0, 0, -\lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 3   | $(-i\lambda_1, 0, 0, \lambda_4)$ | $-|0110\rangle + |0110\rangle$ |
|     |     |     | 4   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle - |0110\rangle$ |
|     |     |     | 5   | $(i\lambda_1, 0, 0, -\lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 6   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 7   | $(i\lambda_1, 0, 0, \lambda_4)$ | $-|0110\rangle + |0110\rangle$ |
|     |     |     | 8   | $(i\lambda_1, 0, 0, \lambda_4)$ | $|0110\rangle - |0110\rangle$ |

Cases $i = 4$ and $p \notin S$ (Part I): Mixed real representatives corresponding to $\gamma_j$, $[z_k] \in H^1(Z)$, and $n_{4,j,r}$.
| $i$ | $j$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----|-----------------|---------------|
| 4   | 3   | 1   | 1   | $(-i\lambda_1, 0, 0, i\lambda_4)$ | $|0110\rangle + |0110\rangle$ |
|     |     |     | 2   | $(i\lambda_1, 0, 0, -i\lambda_4)$ | $|1010\rangle + |0110\rangle$ |
|     |     |     | 3   | $(i\lambda_1, 0, 0, i\lambda_4)$ | $-|1010\rangle + |0110\rangle$ |
|     |     |     | 4   | $(i\lambda_1, 0, 0, -i\lambda_4)$ | $|1010\rangle + |0110\rangle$ |
|     |     |     | 5   | $(i\lambda_1, 0, 0, i\lambda_4)$ | $-|1010\rangle + |0110\rangle$ |
|     |     |     | 6   | $(-i\lambda_1, 0, 0, -i\lambda_4)$ | $|1010\rangle + |0110\rangle$ |
|     |     |     | 7   | $(-i\lambda_1, 0, 0, i\lambda_4)$ | $-|1010\rangle + |0110\rangle$ |
|     |     |     | 8   | $(-i\lambda_1, 0, 0, i\lambda_4)$ | $|1010\rangle - |0110\rangle$ |
| 4   | 3   | 2   | 1   | same as $r = 1$ | $|0110\rangle + |0101\rangle$ |
|     |     |     | 2   | | $|0110\rangle + |0101\rangle$ |
|     |     |     | 3   | | $|0110\rangle - |0101\rangle$ |
|     |     |     | 4   | | $-|0110\rangle + |0101\rangle$ |
|     |     |     | 5   | same as $r = 1$ | $|0110\rangle + |0101\rangle$ |
|     |     |     | 6   | | $|0110\rangle + |0101\rangle$ |
|     |     |     | 7   | | $|0110\rangle - |0101\rangle$ |
|     |     |     | 8   | | $-|0110\rangle + |0101\rangle$ |
| 4   | 3   | 3   | 1   | $(i\lambda_1, 0, 0, i\lambda_4)$ | $|0110\rangle$ |
|     |     |     | 2   | $(i\lambda_1, 0, 0, -i\lambda_4)$ | $|0110\rangle$ |
|     |     |     | 3   | $(i\lambda_1, 0, 0, i\lambda_4)$ | $|0110\rangle$ |
|     |     |     | 4   | $(i\lambda_1, 0, 0, -i\lambda_4)$ | $-|0110\rangle$ |
| 4   | 3   | 4   | 1   | same as $r = 3$ | $|0101\rangle$ |
|     |     |     | 2   | | $|0101\rangle$ |
|     |     |     | 3   | | $-|0101\rangle$ |
|     |     |     | 4   | | $|0101\rangle$ |

Coefficients: with respect to the basis \{$z_1, z_2, z_3, z_4\}$

Conditions: $(\lambda_1, \lambda_4)$ up to $\Gamma_{12}$-conjugacy, $\lambda_1, \lambda_2 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_2\}$

| $i$ | $j$ | $r$ | $k$ | semisimple part | nilpotent part |
|-----|-----|-----|-----|-----------------|---------------|
| 4   | 4   | 1   | 1   | $\frac{1}{2}(-i(\lambda_1 + \lambda_4), \lambda_1 - \lambda_4, -\lambda_1 + \lambda_4, -i(\lambda_1 + \lambda_4))$ | $\frac{1}{2}(|1110\rangle + |1101\rangle + |1010\rangle + |1001\rangle)$ |
|     |     |     |     | | $+|0110\rangle + |0101\rangle + |0010\rangle + |0001\rangle)$ |
|     |     |     | 2   | $\frac{1}{2}(i(\lambda_1 + \lambda_4), \lambda_1 - \lambda_4, -\lambda_1 + \lambda_4, i(\lambda_1 + \lambda_4))$ | $\frac{1}{2}(|1110\rangle + |1101\rangle + |1010\rangle + |1001\rangle)$ |
|     |     |     |     | | $+|0110\rangle + |0101\rangle + |0010\rangle + |0001\rangle)$ |
|     |     |     | 3   | $\frac{1}{2}(i(\lambda_1 + \lambda_4), \lambda_1 - \lambda_4, -\lambda_1 + \lambda_4, -i(\lambda_1 + \lambda_4))$ | $\frac{1}{2}(|1110\rangle - |1101\rangle - |1010\rangle + |1001\rangle)$ |
|     |     |     |     | | $-|0110\rangle - |0101\rangle + |0010\rangle + |0001\rangle)$ |
|     |     |     | 4   | $\frac{1}{2}(i(\lambda_1 + \lambda_4), -\lambda_1 + \lambda_4, -\lambda_1 + \lambda_4, i(\lambda_1 + \lambda_4))$ | $\frac{1}{2}(-|1110\rangle + |1101\rangle + |1010\rangle - |1001\rangle)$ |
|     |     |     |     | | $-|0110\rangle + |0101\rangle + |0010\rangle - |0001\rangle)$ |

Coefficients: with respect to the basis \{$z_1, z_2, z_3, z_4\}$

Conditions: $(\lambda_1, \lambda_4)$ up to $\Gamma_{12}$-conjugacy, $i(\lambda_1 + \lambda_4), \lambda_1 - \lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm \lambda_4\}$

Table 27. Cases $i = 4$ and $p \notin S$ (Part II): Mixed real representatives corresponding to $\gamma_j$, $[z_k] \in H^1(Z)$, and $n_{4,j,r}$. 
References

[1] A. Acín, A.Ándrianov, L. Costa, E. Jané, J.I. Latorre, R. Tarrach. Generalized Schmidt Decomposition and Classification of Three-Quantum-Bit States. Phys. Rev. Lett. 85, 1560 (2000).

[2] A. Aleksandрова, V. Borish, W.K. Wootters. Real-vector-space quantum theory with a universal quantum bit. Phys. Rev. A87, 052106 (2013).

[3] L. Andrianopolis, R. D’Auria, S. Ferrara, A. Marrani, M. Trigiante. Two-Centered Magical Charge Orbits. JHEP 2011, 41 (2011).

[4] J. Batle, A.R. Plastino, M. Casas, A. Plastino. Understanding Quantum Entanglement: Qubits, Rebits and the Quaternionic Approach. Optics and Spectroscopy 94 (5), 700-705 (2003).

[5] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, W. K. Wong, STU Black Holes and String Triality. Phys. Rev. D54, 6293 (1996).

[6] G. Berhuy. An Introduction to Galois Cohomology and its Applications. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press (2010).

[7] M. Borovoi, W.A. de Graaf, H. Ván Lé. Real graded Lie algebras, Galois cohomology and classification of trivectors in \( \mathbb{R}^9 \). arXiv:2106.00246v1

[8] M. Borovoi, W.A. de Graaf, H. Ván Lé. Classification of real trivectors in dimension nine arXiv:2108.00790v1.

[9] L. Borsten, D. Dahanayake, M.J. Duff, W. Rubens, H. Ebrahim. Freudenthal triple classification of three-qubit entanglement. Phys. Rev. A80 032326 (2009).

[10] L. Borsten, D. Dahanayake, M.J. Duff, A. Marrani, W. Rubens. Four-qubit entanglement classification from string theory. Phys. Rev. Lett. 105 100507, 4 (2010).

[11] L. Borsten, D. Dahanayake, M.J. Duff, H. Ebrahim, W. Rubens. Black Holes, Qubits and Octonions. Phys. Rept. 471, 113-219 (2009).

[12] L. Borsten, M.J. Duff, A. Marrani, W. Rubens. On the Black-Hole/Qubit Correspondence. Eur. Phys. J. Plus 126, 37 (2011).

[13] L. Borsten, M.J. Duff, P. Lévay. The black-hole/qubit correspondence: an up-to-date review. Class. Quant. Grav. 29 224008 (2012).

[14] G. Bossard. Octonionic black holes. JHEP 05 113 (2012).

[15] G. Bossard, C. Ruf. Interacting non-BPS black holes. Gen. Rel. Grav. 44, 21 (2012).

[16] C.M. Caves, C.A. Fuchs, P. Rungta. Entanglement of formation of an arbitrary state of two rebits. Found. Phys. Lett. 14, 199-212 (2001).

[17] A. Cayley. On the theory of linear transformations. Camb. Math. J. 4, 193-209 (1845).

[18] A. Ceresole, S. Ferrara, A. Marrani, A. Yeranyan. Small Black Hole Constituents and Horizontal Symmetry. JHEP 06 078 (2011).

[19] D.D.K. Chow, G. Compère. Black holes in \( \mathcal{N} = 8 \) supergravity from SO(4, 4) hidden symmetries. Phys. Rev. D90 2, 025029 (2014).

[20] D.D.K. Chow, G. Compère. Seed for general rotating non-extremal black holes of \( \mathcal{N} = 8 \) supergravity. Class. Quant. Grav. 31, 022001 (2014).

[21] N. Delfosse, P.A. Guerin, J. Bijn, R. Raussendorf. Wigner Function Negativity and Contextuality in Quantum Computation on Rebits. Phys. Rev. X5, 021003 (2015).

[22] F. Denef. Supergravity flows and \( D \)-brane stability. JHEP 0008, 050 (2000).

[23] F. Denef, B.R. Greene, M. Raugas. Split attractor flows and the spectrum of BPS \( D \)-branes on the quintic. JHEP 0105, 012 (2001).

[24] B. Bates, F. Denef. Exact solutions for supersymmetric stationary black hole composites. JHEP 11 127 (2011).

[25] H. Dietrich, W.A. de Graaf, A. Marrani, M. Origlia. Classification of four qubit states and their stabilisers under SLOCC operations. J. Phys. A: Math. Theor. (doi.org/10.1088/1751-8121/ac4b13 (2022).

[26] H. Dietrich, W.A. de Graaf, D. Ruggieri, M. Trigiante. Nilpotent orbits in real symmetric pairs and stationary black holes. Fortschr. Phys. 65, 2, 1600118 (2017).

[27] M.J. Duff. String triality, black hole entropy and Cayley’s hyperdeterminant. Phys. Rev. D76, 025017 (2007).

[28] M.J. Duff, S. Ferrara. \( E_8 \) and the tripartite entanglement of seven qubits. Phys. Rev. D76, 025018 (2007).

[29] M.J. Duff, S. Ferrara. \( E_6 \) and the bipartite entanglement of three qutrits. Phys. Rev. D76, 124023 (2007).

[30] M.J. Duff, J.T. Liu, J. Rahmfeld. Four-dimensional string/string/string triality. Nucl. Phys. B459, 125 (1996).

[31] S. Ferrara, G.W. Gibbons, R. Kallosh. Black Holes and Critical Points in Moduli Space. Nucl. Phys. B500, 75 (1997).

[32] S. Ferrara, R. Kallosh, A. Strominger, \( \mathcal{N} = 2 \) extremal black holes. Phys. Rev. D52, 5412 (1995).

[33] S. Ferrara, R. Kallosh. Supersymmetry and attractors. Phys. Rev. D54, 1514 (1996).

[34] S. Ferrara, R. Kallosh. Universality of supersymmetric attractors. Phys. Rev. D54, 1525 (1996).

[35] S. Ferrara, A. Marrani, E. Orazi, R. Stora, A. Yeranyan. Two-Center Black Holes Duality-Invariants for \( STU \) Model and its lower-rank Descendants. J. Math. Phys. 52, 062302 (2011).

[36] S. Ferrara, A. Marrani. On Symmetries of Extremal Black Holes with One and Two Centers. Springer Proc. Phys. 144, 345 (2013).

[37] S. Ferrara, A. Marrani, A. Yeranyan. On Invariant Structures of Black Hole Charges. JHEP 02, 071 (2012).

[38] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.11.1. [www.gap-system.org](http://www.gap-system.org) (2021).

[39] V.G. Kac. Some remarks on nilpotent orbits, J. Alg. 64, 190-213 (1980).

[40] R. Kallosh, A.D. Linde. Strings, black holes, and quantum information. Phys. Rev. D73, 104033 (2006).

[41] P. Lévay. Stringy black holes and the geometry of entanglement. Phys. Rev. D74, 024030 (2006).

[42] P. Lévay. Stringy black holes and the geometry of entanglement. Phys. Rev. D74, 024030 (2006).

[43] P. Lévay. A three-qubit interpretation of BPS and non-BPS STU black holes. Phys. Rev. D76, 106011 (2007).

[44] P. Levay. Two-Center Black Holes. Qubits and Elliptic Curves. Phys. Rev. D84, 025023 (2011).

[45] P. Levay. STU Black Holes as Four Qubit Systems. Phys. Rev. D82, 026003 (2010).

[46] P. Lévay, S. Szalay. The attractor mechanism as a distillation procedure. Phys. Rev. D82, 026002 (2010).

[47] J.-G. Luque, J.-Y. Thibon. The polynomial invariants of four qubits. Phys. Rev. A67, 042303, 1-5 (2003).

[48] T. Rudolph, L. Grover. A 2 Rebit Gate Universal for Quantum Computing. arXiv:quant-ph/0210187.
[49] D. Ruggeri, M. Trigiante. Stationary $D = 4$ Black Holes in Supergravity: The Issue of Real Nilpotent Orbits. Fortsch. Phys. 65, 5, 1700007 (2017).
[50] J.-P. Serre. Galois cohomology. Springer-Verlag (1997).
[51] E.C.G. Stueckelberg. Quantum Theory in Real Hilbert Space. Helv. Phys. Acta 33, 727 (1960).
[52] A. Strominger. Macroscopic entropy of $\mathcal{N} = 2$ extremal black holes. Phys. Lett. B383, 39 (1996).
[53] F. Verstraete, J. Dehaene, B. de Moor, H. Verschelde. Four qubits can be entangled in nine different ways. Phys. Rev. A36, 65, 051001 (2002).
[54] E.B. Vinberg. The Weyl group of a graded Lie algebra. Math. USSR-Izv. 10, 463–495 (1976).
[55] W.K. Wootters. The rebit three-tangle and its relation to two-qubit entanglement. J. Phys. A47, 424037 (2014).