HKT Geometry and de Sitter Supergravity

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Abstract

Solutions of five dimensional minimal de Sitter supergravity admitting Killing spinors are considered. It is shown that the “timelike” solutions are determined in terms of a four dimensional hyper-Kähler torsion (HKT) manifold. If the HKT manifold is conformally hyper-Kähler the most general solution can be obtained from a sub-class of supersymmetric solutions of minimal \( \mathcal{N} = 2 \) ungauged supergravity, by means of a simple transformation. Examples include a multi-BMPV de Sitter solution, describing multiple rotating black holes co-moving with the expansion of the universe. If the HKT manifold is not conformally hyper-Kähler, examples admitting a tri-holomorphic Killing vector field are constructed in terms of certain solutions of three dimensional Einstein-Weyl geometry.
1 Introduction

The connections between complex (in particular Kähler) geometry and supersymmetry have long been known. Well known examples arise in the context of string compactifications; \( \mathcal{N} = 1 \) supersymmetry in four dimensions requires the compact six dimensional manifold to be a Calabi-Yau 3-fold [1]. An even earlier example is the observation, by Zumino [2], that a two-dimensional non-linear sigma model admits an \( \mathcal{N} = 2 \) supersymmetric extension if and only if the target space metric is Kähler. An \( \mathcal{N} = 4 \) extension requires a hyper-Kähler target space geometry [3]; essentially each supersymmetry beyond the first requires the existence of a complex structure. Wess-Zumino-Witten couplings [4] in the sigma model can be interpreted as torsion potentials, from the target space viewpoint [5] [6]. Hence, the inclusion of such couplings leads naturally to Kähler and hyper-Kähler torsion (HKT) geometries [7] [8]. The latter are also called heterotic geometries [9], since they arise in the worldsheet description of soliton solutions of heterotic string theory [10]. HKT manifolds have also been found in the context of moduli space metrics of electrically charged five-dimensional black holes [11] [12].

Kähler and hyper-Kähler geometry also arise in connection with supersymmetric solutions in supergravity theories. As a particular example, all supersymmetric solutions with a timelike Killing vector field of minimal ungauged [13] and gauged five dimensional supergravity [14] are defined in terms of a four dimensional base space.
which is, respectively, hyper-Kähler and Kähler. These theories have, as the vacuum state, five dimensional Minkowski spacetime and AdS$_5$, respectively. The purpose of this paper is to show that HKT geometries also have a role to play in five dimensional supergravities: timelike solutions (in a sense to be defined below) of de Sitter supergravity [15, 16] are defined in terms of a base space which is an HKT geometry.

Even though de Sitter superalgebras have only non-trivial representations in a positive-definite Hilbert space in two dimensions [16], the perspective we wish to take here is that of fake supersymmetry, in analogy to the ‘Domain Wall/Cosmology’ correspondence [17]: that there is a special class of solutions in a gravitational theory with a positive cosmological constant admitting “pseudo Killing” spinors. Thus, we use supersymmetry as a solution generating technique. The theory we are considering has, nevertheless, an interpretation in terms of type IIB* theory [18], which is related to type IIA string theory via T-duality on a timelike circle [19].

This paper is organized as follows. In section 2 we integrate the Killing spinor equation of minimal five dimensional de Sitter supergravity. We obtain the general structure of the solutions of this theory that admit Killing spinors from which a timelike vector field is constructed. This structure is summarised in section 2.3. In section 3 we provide some examples and in section 4 we give some final remarks.

2 Integrating the Killing Spinor Equation

2.1 Basic Equations

The bosonic action is obtained from that in [14] by changing the signature and analytically continuing $\chi$. Thus

$$ S = \frac{1}{4\pi G} \int \left( \frac{1}{4} (5 R - \chi^2) \star 1 - \frac{1}{2} F \wedge \ast F - \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right) , $$

where $F = dA$ is a $U(1)$ field strength and $\chi \neq 0$ is a real constant. The equations of motion are

$$ 5 R_{\alpha\beta} - 2 F_{\alpha\gamma} F^\gamma_{\beta} + \frac{1}{3} g_{\alpha\beta} (F^2 - \chi^2) = 0 , $$

(2.1)

and

$$ d \ast F + \frac{2}{\sqrt{3}} F \wedge F = 0 , $$

(2.2)

where $F^2 \equiv F_{\alpha\beta} F^{\alpha\beta}$.
In the minimal theory, the gravitino Killing spinor equation acting on a Dirac spinor $\epsilon$ is given by

\[
\partial_M + \frac{1}{4} \Omega_{M, N_1 N_2} \Gamma_{N_1 N_2} - \frac{i}{4\sqrt{3}} F^{N_1 N_2} \Gamma_M \Gamma_{N_1 N_2} + \frac{3i}{2\sqrt{3}} F_M^N \Gamma_N \\
+ \chi \left( \frac{i}{4\sqrt{3}} \Gamma_M - \frac{1}{2} A_M \right) \epsilon = 0 ,
\]

where $\Omega$ denotes the spin connection. This Killing spinor equation is obtained from the standard Killing spinor equation of minimal gauged (AdS) D=5 supergravity (acting on Dirac spinors) by replacing $\chi \rightarrow i\chi$. Note that the metric has signature $(-, +, +, +, +)$.

We shall utilize this Killing spinor equation as a solution generating technique. In particular, if one has a non-vanishing Killing spinor satisfying (2.3), and if in addition the gauge field equations (2.2) are satisfied, then the integrability conditions of the Killing spinor equation place constraints on the Ricci tensor. For the solutions we consider here, in which the Killing spinor generates a timelike vector field, these constraints on the Ricci tensor are equivalent to the Einstein equations (2.1). This can be seen using exactly the same reasoning as in [14]. Hence we shall only solve the Killing spinor and the gauge equations, as the Einstein equations then follow automatically.

In order to analyse the Killing spinor equations, we make use of spinorial geometry techniques. These were initially used to analyse certain supersymmetric solutions in ten and eleven-dimensional supergravity theories. In the spinorial geometry method, one writes the spinors as differential forms. Then, by making use of appropriately chosen gauge transformations one can transform the spinors to simple canonical forms, which together with an appropriate choice of basis, simplifies the analysis. The result of this is a complete, and systematic, classification of the different types of spacetime geometry and fluxes of supersymmetric solutions.

This method has been particularly effective in classifying solutions preserving small and large amounts of supersymmetry in D=11 and type IIB supergravity [20], [21], [22], [23]. There has also been considerable progress in the analysis of generic solutions of type I supergravity using these methods [24]. Spinorial geometry techniques

\[1 \] However, it should be noted that the same result does not hold when the Killing spinor generates a null vector field. In particular, for these solutions, one component of the Einstein equations must be imposed in addition to the Killing spinor and gauge equations. We will not consider such solutions here.
have also been particularly effective in analysing solutions of lower-dimensional supergravity theories, for example in \cite{25}, \cite{26}, \cite{27}. Here we apply the same techniques to analyse the solutions of \eqref{2.3}.

For de Sitter supergravity in five-dimensions, one takes the space of Dirac spinors to be the space of complexified forms on $\mathbb{R}^2$, which are spanned over $\mathbb{C}$ by $\{1, e_1, e_2, e_{12}\}$ where $e_{12} = e_1 \wedge e_2$. The action of complexified $\Gamma$-matrices on these spinors is given by

\begin{align}
\Gamma_\alpha &= \sqrt{2} e_\alpha \wedge , \\
\Gamma_{\bar{\alpha}} &= \sqrt{2i} e^\alpha ,
\end{align}

for $\alpha = 1, 2$, and $\Gamma_0$ satisfies

\begin{align}
\Gamma_0 1 = -i1, & \quad \Gamma_0 e^{12} = -ie^{12}, \quad \Gamma_0 e^j = ie^j \quad j = 1, 2 ,
\end{align}

where we work with an oscillator basis in which the spacetime metric is

\begin{equation}
\ ds^2 = - (e^0)^2 + 2 \delta_{\alpha\bar{\beta}} e^\alpha e^{\bar{\beta}} .
\end{equation}

### 2.2 The timelike case

In this paper we will focus on integrating the Killing spinor equation \eqref{2.3} for the timelike case, i.e. when the vector field constructed from the Killing spinor is timelike. Note, however, that unlike the cases of the minimal ungauged \cite{13} and gauged theories \cite{14}, the timelike vector field obtained from the Killing spinor is not a Killing vector field. In this case, a generic spinor can be put in the form $\epsilon = f 1$ by making use of $Spin(4, 1)$ gauge transformations\footnote{When the vector field generated from the Killing spinor is null, then the Killing spinor can be reduced, using gauge transformations, to the simple canonical form $\epsilon = 1 + e_1$.}. Then, we find the following constraints from the Killing spinor equation

\begin{align}
\frac{\partial_0 f}{f} + \frac{1}{2} \Omega_{0,\mu}^\mu - \frac{1}{2\sqrt{3}} F_\mu^\mu + \frac{\chi}{4\sqrt{3}} - \frac{\chi}{2} A_0 &= 0 , \\
\Omega_{0,0\bar{\alpha}} - \frac{2}{\sqrt{3}} F_{0\bar{\alpha}} &= 0 , \\
(\Omega_{0,\alpha\bar{\beta}} - \frac{1}{\sqrt{3}} F_{\alpha\bar{\beta}}) e^{\alpha\bar{\beta}} &= 0 ,
\end{align}

\begin{equation}
\Omega_{0,\alpha\bar{\beta}} - \frac{1}{\sqrt{3}} F_{\alpha\bar{\beta}} e^{\alpha\bar{\beta}} = 0 ,
\end{equation}
\[ \frac{\partial \alpha f}{f} + \frac{1}{2} \Omega_{\alpha, \mu} + \frac{3}{2\sqrt{3}} F_{0\alpha} - \frac{\chi}{2} A_\alpha = 0, \quad (2.11) \]

\[ - \Omega_{\alpha,0\bar{\beta}} + \sqrt{3} F_{\alpha\bar{\beta}} - \frac{1}{\sqrt{3}} F_{\mu}^\alpha \delta_{\alpha\bar{\beta}} + \frac{\chi}{2\sqrt{3}} \delta_{\alpha\bar{\beta}} = 0, \quad (2.12) \]

\[ \Omega_{\alpha, \bar{\mu} \bar{\nu}} \epsilon^{\bar{\mu} \bar{\nu}} + \frac{2}{\sqrt{3}} F^{0\mu} \epsilon_{\alpha \mu} = 0, \quad (2.13) \]

\[ \frac{\partial \bar{\alpha} f}{f} + \frac{1}{2} \Omega_{\bar{\alpha}, \mu} + \frac{1}{2\sqrt{3}} F_{0\bar{\alpha}} - \frac{\chi}{2} A_{\bar{\alpha}} = 0, \quad (2.14) \]

\[ \Omega_{\bar{\alpha},0\bar{\beta}} - \frac{1}{\sqrt{3}} F_{\alpha\bar{\beta}} = 0, \quad (2.15) \]

\[ \Omega_{\alpha, \bar{\mu} \bar{\nu}} \epsilon^{\bar{\mu} \bar{\nu}} = 0. \quad (2.16) \]

However, observe that the Killing spinor equation is invariant under the \( \mathbb{R} \) transformation for which

\[ \epsilon \to e^g \epsilon, \quad A \to A + \frac{2}{\chi} dg. \quad (2.17) \]

Hence, without loss of generality, we can work in a gauge for which \( f = 1 \). This will simplify the analysis of the Killing spinor equation. In the gauge \( f = 1 \), we obtain the following constraints on the flux components

\[
A_0 = \frac{1}{2\sqrt{3}}, \\
A_\alpha = \frac{1}{\chi} \Omega_{0,0\alpha}, \\
F_{0\alpha} = \frac{\sqrt{3}}{2} \Omega_{0,0\alpha}, \\
F_{\alpha\beta} = \sqrt{3} \Omega_{[\alpha,0|\beta]}, \\
F_{\alpha\bar{\beta}} = \frac{1}{\sqrt{3}} \Omega\alpha,0\bar{\beta} + \left( -\frac{\chi}{2} + \frac{1}{\sqrt{3}} \Omega_{\mu,0\mu} \right) \delta_{\alpha\bar{\beta}},
\]

\[ (2.18) \]

together with the purely geometric constraints

\[
\Omega_{[0,\alpha]\beta} = 0, \quad \Omega_{\langle \alpha,|0|\beta \rangle} = \frac{\chi}{2\sqrt{3}} \delta_{\alpha\bar{\beta}}, \quad \Omega_{\mu,0\mu} - \Omega_{\alpha,0\mu} + \frac{\chi}{\sqrt{3}} = 0, \quad (2.19) \]

and
\[ \Omega_{\alpha,\mu\nu} = 0 , \]
\[ \Omega_{\alpha,\beta} + \frac{1}{2} \Omega_{0,0\alpha} = 0 , \]
\[ \Omega_{\alpha,\beta\bar{\rho}} - \frac{1}{2} \delta_{\alpha\beta} \Omega_{0,0\bar{\rho}} + \frac{1}{2} \delta_{\alpha\beta} \Omega_{0,0\bar{\rho}} = 0 . \]  

(2.20)

We begin by analysing the constraints (2.19). It is convenient to define the 1-form \( V = e^0 \), and introduce a \( t \) co-ordinate such that the dual vector field is \( V = -\frac{\partial}{\partial t} \). Let the remaining (real) co-ordinates be \( x^m \), for \( m = 1, 2, 3, 4 \). The vielbein is then given by

\[ e^0 = dt + \omega_m dx^m, \quad e^\alpha = e^\alpha_m dx^m . \]  

(2.21)

It is then straightforward to show that (2.19) is equivalent to

\[ (\mathcal{L}_V e^\alpha)_\beta = 0 , \]  

(2.22)

and

\[ (\mathcal{L}_V e^\alpha)_\beta = s^\alpha_\beta + \frac{\chi}{2\sqrt{3}} \delta^\alpha_\beta , \]  

(2.23)

where

\[ s^\alpha_\beta = \Omega_{0,\alpha\beta} + \frac{1}{2} \Omega_{\mu,\beta} \delta^\alpha_\beta - \Omega_{\beta,\alpha} + \frac{1}{2} \Omega_{\mu,\alpha} \delta^\alpha_\beta , \]  

(2.24)

and (2.19) implies that \( s \) is traceless and antihermitian (i.e. \( s \in su(2) \)). So, on defining \( \hat{e}^\alpha \) by

\[ e^\alpha = e^{-\frac{\chi}{2\sqrt{3}}} \hat{e}^\alpha , \]  

(2.25)

we find

\[ (\mathcal{L}_V \hat{e}^\alpha)_\beta = e^{-\frac{\chi}{2\sqrt{3}}} s^\alpha_\beta . \]  

(2.26)

However, one can without loss of generality apply a \( SU(2) \subset Spin(4,1) \) gauge transformation to the Killing spinors, but which leaves \( 1 \) invariant, and set \( s = 0 \) without loss of generality. In this gauge,

\[ \mathcal{L}_V \hat{e}^\alpha = 0 . \]  

(2.27)
It will be convenient to refer to the 4-manifold with $t$-independent metric
\[ ds^2_B = 2\delta_{\alpha\bar{\beta}} \hat{e}^\alpha \hat{e}^\bar{\beta} , \]  
(2.28)
as the base manifold $B$; the spin connection of this manifold is denoted by $\hat{\Omega}$ (whose components will always be taken with respect to the vielbein $\hat{e}^\alpha$).

Next consider the geometric constraints given in (2.20). The third constraint in (2.20) implies that
\[ \omega = \frac{2\sqrt{3}}{\chi} \mathcal{P} + e^{\sqrt{3}/\chi} \mathcal{Q} , \]  
(2.29)where
\[ \mathcal{P} \equiv \hat{\Omega}_{\bar{\beta},\alpha} \hat{\bar{e}}^\alpha + \hat{\Omega}_{\beta,\bar{\alpha}} \hat{\bar{e}}^\bar{\alpha} = \mathcal{P}_m dx^m , \]  
(2.30)and
\[ \mathcal{Q} = \mathcal{Q}_m dx^m , \]  
(2.31)are 1-forms on the base manifold $B$ with
\[ \mathcal{L}_V \mathcal{Q} = 0 . \]  
(2.32)
Note that, by construction,
\[ \mathcal{L}_V \mathcal{P} = 0 . \]  
(2.33)
The remaining geometric content of (2.20) can be expressed in terms of the spin connection of $B$ via
\[ \hat{\Omega}_{\alpha,\mu\nu} = 0, \quad \hat{\Omega}_{\alpha,\mu}^{\nu} - \hat{\Omega}_{\bar{\mu},\alpha}^{\bar{\nu}} = 0 , \]  
(2.34)where all components are with respect to the vielbein $\hat{e}^\alpha$.

Next, consider the constraints on the flux (2.18). The first two constraints fix the gauge potential to be
\[ A = \frac{1}{2\sqrt{3}} (dt + \omega) + \frac{1}{\sqrt{3}} e^{\sqrt{3}/\chi} \mathcal{Q} . \]  
(2.35)
We consider the consistency condition $F = dA$, using the last three constraints in (2.18) to compute $F$, and comparing the resulting expression with $dA$. There is no constraint in the “$0\alpha$” directions. However, from the $(2,0)$ component of $F = dA$ we obtain
\[ (d\mathcal{P})_{\alpha\beta} = 0 , \]  
(2.36)
and from the \((1, 1)\) component of \(F = dA\), we find

\[
(dQ)_{\alpha\dot{\beta}} - \frac{1}{2}(dQ)_\mu^{\mu} \delta_{\alpha\dot{\beta}} - 2(P_{\alpha} Q_{\dot{\beta}} - P_{\dot{\beta}} Q_{\alpha}) + \delta_{\alpha\dot{\beta}}(P_\mu Q^\mu - P_{\dot{\mu}} Q^{\dot{\mu}}) = 0 ,
\]

(2.37)

together with

\[
(dP)_\mu^{\mu} = 0 ,
\]

(2.38)

(again, all components are with respect to the vielbein \(\hat{e}^\alpha\)). Observe that (2.37) is traceless; in fact, it can be rewritten as

\[
(dQ - 2P \wedge Q)^+ = 0 ,
\]

(2.39)

where here \(+\) denotes the self-dual projection on the base-manifold \(B\), with positive orientation fixed with respect to the volume form \(\hat{e}^1 \wedge \hat{e}^\bot_1 \wedge \hat{e}^2 \wedge \hat{e}^\bot_2\). Similarly, the constraints (2.36) and (2.38) on \(dP\) can be rewritten as

\[
(dP)^- = 0 ,
\]

(2.40)

where \(-\) denotes the anti-self-dual projection. Finally, observe that the constraints (2.30), (2.34) and (2.40) are equivalent to

\[
dJ^i = -2P \wedge J^i , \quad i = 1, 2, 3 ,
\]

(2.41)

where

\[
J^1 = \hat{e}^1 \wedge \hat{e}^2 + \hat{e}^\bot_1 \wedge \hat{e}^\bot_2 , \\
J^2 = i\hat{e}^1 \wedge \hat{e}^\bot_1 + i\hat{e}^2 \wedge \hat{e}^\bot_2 , \\
J^3 = -i\hat{e}^1 \wedge \hat{e}^2 + i\hat{e}^\bot_1 \wedge \hat{e}^\bot_2 ,
\]

(2.42)

defines a triplet of anti-self-dual almost complex structures on \(B\) which satisfy the algebra of the imaginary unit quaternions.

It should be noted that the constraint (2.41) implies that the base \(B\) is hyper-Kähler with torsion (HKT), i.e.

\[
\nabla^+ J^i = 0 ,
\]

(2.43)

where the connection of the covariant derivative \(\nabla^+\) is given by

\[
\Gamma^{(+)}_{jk} = \{j_\dot{k}\} + H^i_{jk} ,
\]

(2.44)
and where \( H \) is the torsion 3-form on \( B \) given by
\[
H = \star_4 \mathcal{P} .
\] (2.45)

A HKT manifold is called *strong* HKT if \( H \) is closed. For the solutions under consideration here, one can without loss of generality take \( B \) to be a strong HKT manifold. This is shown in the Appendix.

This exhausts the content of the Killing spinor equation. Finally we evaluate the gauge field equations (2.2). The gauge potential is given by
\[
A = \frac{1}{2\sqrt{3}} dt + \frac{1}{\chi} \mathcal{P} + \frac{\sqrt{3}}{2} e^{\frac{\chi}{\sqrt{3}}} \mathcal{Q} ,
\] (2.46)
with gauge field strength
\[
F = dA = \frac{\chi}{2} e^{\frac{\chi}{\sqrt{3}}} e^0 \wedge \mathcal{Q} + \frac{1}{\chi} d\mathcal{P} + \frac{\sqrt{3}}{2} e^{\frac{\chi}{\sqrt{3}}} (d\mathcal{Q} - 2\mathcal{P} \wedge \mathcal{Q}) .
\] (2.47)
The Bianchi identity holds automatically (as the constraints obtained so far are sufficient to imply \( F = dA \)).

Note that
\[
\star F = \frac{\chi}{2} \star_4 \mathcal{Q} + \frac{1}{\chi} e^0 \wedge d\mathcal{P} - \frac{\sqrt{3}}{2} e^{\frac{\chi}{\sqrt{3}}} e^0 \wedge (d\mathcal{Q} - 2\mathcal{P} \wedge \mathcal{Q}) ,
\] (2.48)
where \( \star_4 \) denotes the Hodge dual on the 4-dimensional base space \( B \).

It is then straightforward to show that the gauge field equations are equivalent to
\[
d \star_4 \mathcal{Q} + \frac{16}{\sqrt{3} \chi^3} d\mathcal{P} \wedge d\mathcal{P} = 0 .
\] (2.49)

### 2.3 Summary

To summarize, the solutions of the five dimensional theory described in section 2.1 are constructed as follows:

1) Take the base space \( B \) to be a four dimensional HKT geometry with metric \( ds_B^2 \) and torsion tensor \( H \).

2) The 1-form \( \mathcal{P} \) is given by
\[
\mathcal{P} = - \star_4 H ,
\] (2.50)
where \( \star_4 \) denotes the Hodge dual on the base space.

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\(^3\)The five-dimensional volume form \( \epsilon^5 \) is related to the volume form of \( B \), \( \epsilon^5 = e^{-\frac{2\chi}{\sqrt{3}}} \epsilon^0 \wedge \epsilon^B \), and we use the convention \( (\star_4 \mathcal{Q})_{i_1 i_2 i_3} = (\epsilon^B)_{i_1 i_2 i_3} \mathcal{Q}_j \), where all indices are with respect to the \( \hat{e} \) vielbein, and are raised/lowered by \( ds_B^2 \).
3) Choose a 1-form $Q$ obeying the constraints (2.39) and (2.49). Note that one can always solve the gauge equation constraint (2.49); the general solution is given by

$$Q = \frac{16}{\sqrt{3\chi^3}} \star_4 (P \wedge dP) + \star_4 d\Phi,$$

where $\Phi$ is a 2-form on $B$. On substituting this expression back into (2.39), one finds an equation constraining $\Phi$, which must be solved.

4) The spacetime geometry is given by

$$ds^2 = - \left( dt + \frac{2\sqrt{3}}{\chi} P + e^{-\sqrt{3}t} Q \right)^2 + e^{-\sqrt{3}t} ds_B^2,$$

where the metric on the base manifold $ds_B^2$ does not depend on $t$, and $P$, $Q$ are two $t$-independent 1-forms on $B$. Note that, by construction, $B$ also admits three $t$-independent anti-self-dual almost complex structures $J^i$ which satisfy the algebra of the imaginary unit quaternions, and also satisfy (2.41). It follows that (2.40) is obeyed.

5) The gauge potential is given by (2.46).

Note that the Ricci scalar of the five dimensional metric is given by

$$R = \frac{F_{\mu\nu}F^{\mu\nu}}{3} + 5\chi^2,$$

using (2.47) and (2.48) we find that

$$R = \frac{5}{3} \chi^2 + \frac{e^{-\sqrt{3}t}}{3} \left[ \frac{(dP)^2}{\chi^2} - \frac{\chi^2}{2} e^{-\sqrt{3}t} Q^2 + \frac{3}{4} e^{-\sqrt{3}t} (dQ - 2P \wedge Q)^2 \right],$$

where the norms are computed with respect to the $t$-independent base space metric. Therefore, the $t$-dependence of the Ricci scalar can be read directly from the above expression. Thus, in particular, for the solution to be regular at both $t = \pm \infty$ we must require

$$Q = 0, \quad dP = 0.$$

In particular, this implies that the base space is conformally hyper-Kähler.

### 3 Examples

In this section, we present some examples of solutions.
3.1 Solutions with conformally hyper-Kähler base

Suppose that the base space $B$ is conformal to a hyper-Kähler manifold $HK$. We set $e^a = e^\phi E^a$, where $\phi$ is a real $t$-independent function, such that the manifold $HK$ with metric

$$ds^2_{HK} = 2\delta_{\alpha\beta}E^\alpha E^\beta,$$

is hyper-Kähler with closed Kähler forms $\tilde{J}^i$ related to $J^i$ by

$$J^i = e^{2\phi}\tilde{J}^i.$$  (3.2)

Then it is straightforward to show that (2.41) implies that

$$P = -d\phi,$$  (3.3)

so in particular, $dP = 0$. Also, (2.59) is equivalent to

$$d(e^{2\phi}Q)^+ = 0,$$  (3.4)

where here $+$ denotes the self-dual projection on $HK$, and the gauge equation constraint (2.49) is equivalent to

$$d(e^{2\phi}*_HK Q) = 0,$$  (3.5)

where $*_HK$ denotes the Hodge dual on $HK$.

The solution can be simplified further by making the co-ordinate transformation

$$t = t' + \frac{2\sqrt{3}}{\chi}\phi,$$  (3.6)

and setting

$$G = e^{2\phi}Q.$$  (3.7)

The metric is then given by

$$ds^2 = -(dt' + e^{-\sqrt{3}\phi'} G)^2 + e^{-\sqrt{3}\phi'} ds^2_{HK},$$  (3.8)

where

$$(dG)^+ = 0, \quad d*_HK G = 0.$$  (3.9)

In these co-ordinates, the gauge potential and gauge field strength are

$$A = \frac{1}{2\sqrt{3}}dt' + \frac{\sqrt{3}}{2}e^{-\sqrt{3}\phi'} G, \quad F = \frac{\sqrt{3}}{2}d(e^{-\sqrt{3}\phi'} G).$$  (3.10)
We can put this class of solutions in a more familiar form. Set

$$G = \sqrt{3}dV' + a'. \quad (3.11)$$

Making a coordinate transformation $t' \to t''$ given by

$$t'' = \frac{\sqrt{3}}{\chi} \left( V' - e^{-\chi t'} \right), \quad (3.12)$$

the solution can then be written in the form

$$ds^2 = -f^2(dt'' + a')^2 + f^{-1}ds_{HK}^2, \quad F = \frac{\sqrt{3}}{2}d\left( f(dt'' + a') \right), \quad (3.13)$$

where

$$f^{-1} = V' - \frac{\chi}{\sqrt{3}}t'', \quad (3.14)$$

and

$$\Delta_{HK}V' = -\frac{\chi}{\sqrt{3}}\nabla_{HK} \cdot a', \quad (da')^+ = 0, \quad (3.15)$$

where $\nabla_{HK} \cdot a'$ is the covariant divergence of $a'$ in $HK$. Finally we can choose a Lorentz-type gauge for $a'$: setting

$$a' = a + d\zeta, \quad t'' + \zeta = t, \quad V = V' + \frac{\chi}{\sqrt{3}}\zeta, \quad (3.16)$$

and choosing $\zeta$ such that $\nabla_{HK} \cdot a = 0$ we find the final form

$$ds^2 = -f^2(dt + a)^2 + f^{-1}ds_{HK}^2, \quad F = \frac{\sqrt{3}}{2}d\left( f(dt + a) \right), \quad (3.17)$$

where

$$f^{-1} = V - \frac{\chi}{\sqrt{3}}t, \quad (3.18)$$

and

$$\Delta_{HK}V = 0, \quad (da)^+ = 0. \quad (3.19)$$

$dS_5$ is obtained by taking $HK = \mathbb{R}^4$, $V = \text{const.}$ and $a = 0$.

Observe that in the limit of zero cosmological constant, the solutions (3.17)-(3.19) become a subset of the supersymmetric solutions of the ungauged theory [13], namely those with a timelike Killing vector field and $G^+ = 0$ (in the notation therein). Any
such solution - which we call seed solution - can be made into an asymptotically de Sitter solution, simply by adding a linear time dependence to the harmonic function of the seed solution. Such a procedure for building dS solutions was first observed in [28] and it underlies several asymptotically dS solutions constructed in the last few years. However, note that the result presented here is stronger than the result presented in [28]:

*Any solution of (2.1), (2.2) with a supercovariantly constant spinor and a base space which is conformal to a hyper-Kähler manifold is of the form (3.17)-(3.19). Thus it can be obtained from a seed solution of the $\mathcal{N} = 2$, $D = 5$ minimal ungauged supergravity theory with $G^+ = 0$ simply by adding a linear time dependence to the harmonic function, as in (3.18).*

For instance, the multi-centred, non-rotating, black hole solutions of [29] are obtained taking $HK = \mathbb{R}^4$ and $a = 0$; i.e. the five dimensional Majumdar-Papapetrou multi-black hole solution is the seed. Introducing rotation, one finds solutions whose seeds are the BMPV black hole [30], Gödel type universes [13] or black holes in Gödel type universes [13, 31]. To be concrete let us analyse a single-centred solution with rotation.

Set $HK = \mathbb{R}^4$ written in terms of left (or right) invariant forms on $SU(2)$:

$$ds^2(\mathbb{R}^4) = dr^2 + \frac{r^2}{4} \left( (\sigma_1^L)^2 + (\sigma_2^L)^2 + (\sigma_3^L)^2 \right)$$

$$= dr^2 + \frac{r^2}{4} \left( (\sigma_1^R)^2 + (\sigma_2^R)^2 + (\sigma_3^R)^2 \right).$$

An explicit expression for the 1-forms $\sigma_{L,R}$ in terms of Euler angles can be found, for instance, in [13]. Let $V$ be harmonic on $\mathbb{R}^4$. Set

$$a = g^L_i(r)\sigma^i_L + g^R_i(r)\sigma^i_R.$$  

(3.21)

Noting that

$$d\sigma_L = -\frac{1}{2} \epsilon^{ijk} \sigma^i_L \wedge \sigma^k_L, \quad d\sigma_R = \frac{1}{2} \epsilon^{ijk} \sigma^i_R \wedge \sigma^k_R,$$

(3.22)

it follows that the equations (3.19) are obeyed if

$$g^L_i(r) = C_i^L r^2, \quad g^R_i(r) = \frac{C_i^R}{r^2},$$

(3.23)

where $C_i^{L,R}$ are constants. For a particular choice of the constants, $C_i^R = 0 = C_3^L$, this is a solution dubbed ‘Gödel-de-Sitter Universe’ in [32]. If $V = 1$, the seed is the
maximal supersymmetric Gödel Universe found in [13]; but if $V = 1 + \mu/r^2$ the seed is actually the Gödel universe black hole found in [13] and discussed in [31]. On the other hand, taking $V = 1 + \mu/r^2$, $C_i^L = 0 = C_i^R = C_3^R = j$; then

$$f^{-1} = 1 + \frac{\mu}{r^2} - \frac{\chi}{\sqrt{3}}t, \quad a = \frac{j}{r^2} \sigma_R^3.$$  \hfill (3.24)

The seed is now the BMPV black hole [30]. Thus we dub this solution a BMPV-de Sitter black hole. It was first found, albeit not in this form, in [33]. This solution can be easily generalised to a multi-BMPV-de-Sitter solution by taking as seed the multi-centred BMPV solution [34]; using Cartesian coordinates on $\mathbb{R}^4$ we have

$$f^{-1} = 1 + \sum_i \frac{\mu_i}{|x - x^i|^2} - \frac{\chi}{\sqrt{3}}t, \quad a = dx^i J_i^k \partial_k \left( \sum_j \frac{j_i}{|x - x^j|^2} \right),$$ \hfill (3.25)

where $J$ is a complex structure on $\mathbb{R}^4$ and $\mu_i, j_i$ and $x^i$ are constants.

Let us note that the multi-black hole solutions displayed in this section are a five dimensional generalisation of the Kastor-Traschen solutions [35], but which can also carry rotation. Analogous solutions for branes have been studied in [36, 37].

Finally let us remark that the solutions of the ungauged supergravity theory with $G^+ \neq 0$ do not generalise straightforwardly to the de Sitter case. Most notably this includes the supersymmetric black ring of [38].

### 3.2 Solutions with a tri-holomorphic Killing vector

Suppose that the strong HKT base manifold $B$ has a tri-holomorphic Killing vector $X$, such that

$$\mathcal{L}_X h_B = 0, \quad \mathcal{L}_X J^i = 0, \quad i = 1, 2, 3,$$ \hfill (3.26)

where $h_B$ is the metric on $B$. Such manifolds have been classified in [39], [40], and their structure is specified in terms of a constrained 3-dimensional Einstein-Weyl geometry. This consists of a 3-dimensional manifold $E$ equipped with a metric $\gamma_{ij}$, together with a 1-form $u$ on $E$, and a scalar $u_0$ which satisfy the constraints

$$\star_E du = -du_0 - u_0 u,$$ \hfill (3.27)

and

$$(^E)R_{ij} + \nabla_i (u_j) + u_i u_j = \gamma_{ij} \left( \frac{1}{2} u_0^2 + u_k u^k \right).$$ \hfill (3.28)
where \((E)R_{ij}\) denotes the Ricci curvature of the Levi-Civita connection of \(E\), denoted here by \(\nabla_i\). Furthermore, \(*_E\) denotes the Hodge dual of \(E\). The 1-form \(u\) is also co-closed

\[
d *_E u = 0 .
\]

Then the 4-dimensional base geometry is obtained by introducing a local co-ordinate \(\tau\) such that \(X = \frac{\partial}{\partial \tau}\). We assume that \(X\) is a symmetry of the full five-dimensional solution. The metric on \(B\) is

\[
ds_B^2 = \frac{1}{W}(d\tau + \Psi)^2 + W ds_E^2 ,
\]

where \(W\) is a \(\tau\)-independent function, and \(\Psi\) is \(\tau\)-independent 1-form on the Einstein-Weyl manifold \(E\) which are related by the constraint

\[
*_E d\Psi = dW + Wu .
\]

The scalar \(u_0\), 1-form \(u\) and metric on \(\gamma_{ij}\) on \(E\) do not depend on \(\tau\).

Observe that the constraints \((3.27), (3.29), (3.31)\) imply that

\[
(\Delta_E + u^i \nabla_i)W = 0, \quad (\Delta_E + u^i \nabla_i)u_0 = 0 ,
\]

where \(\Delta_E\) is the Laplacian on \(E\). The volume form on \(B\) and the volume form on \(E\), \(dvol_E\) are related via

\[
\epsilon^B = W(d\tau + \Psi) \wedge dvol_E .
\]

The torsion is obtained using the identification

\[
P = -\frac{u_0}{2W}(d\tau + \Psi) - \frac{1}{2}u ,
\]

and it is straightforward to verify that \(P\) is co-closed on \(B\), so the geometry is indeed strong HKT.

To proceed, consider the constraint \((2.39)\), and write

\[
Q = Q_\tau(d\tau + \Psi) + \tilde{Q} ,
\]

where \(Q_\tau\) is a \(\tau\)-independent function, and \(\tilde{Q}\) is a \(\tau\)-independent 1-form on \(E\). It is then straightforward to show that \((2.39)\) is equivalent to

\[
d\tilde{Q} + u \wedge \tilde{Q} + *_E(Q_\tau dW - WdQ_\tau) + u_0 *_E \tilde{Q} = 0 .
\]
Next consider (2.49), this can be rewritten as
\[ d \star_E \tilde{Q} = \frac{8}{\sqrt{3}} \chi^3 d\left(\frac{u_0}{W}\right) \wedge \star_E d\left(\frac{u_0}{W}\right). \] (3.37)

By making use of (3.36), together with the other constraints this equation can be rewritten as
\[ (\Delta_E + u^i \nabla_i) \left( Q_\tau - \frac{4}{3\sqrt{3}} \frac{u_0^3}{W^2} \right) = 0, \] (3.38)
and hence
\[ Q_\tau = \frac{4}{3\sqrt{3}} \frac{u_0^3}{W^2} + M, \] (3.39)
where \( M \) satisfies \( (\Delta_E + u^i \nabla_i) M = 0. \) (3.40)

On substituting this expression back into (3.36) and defining \( \mathcal{K} \) by
\[ \tilde{Q} = \frac{4}{\sqrt{3}} \chi^3 d\left(\frac{u_0^2}{W}\right) + \frac{4}{\sqrt{3}} \frac{u_0^2}{W} u + \mathcal{K}, \] (3.41)
we then obtain the constraint
\[ d\mathcal{K} + u \wedge \mathcal{K} + \star_E (M dW - W dM) + u_0 \star_E \mathcal{K} = 0. \] (3.42)

Observe that this constraint implies that
\[ d \star_E \mathcal{K} = 0. \] (3.43)

To summarise, solutions with a tri-holomorphic Killing vector field are constructed in the following way:

1) Choose the 3-dimensional Einstein-Weyl data, \((\gamma_{ij}, u_i, u_0)\), which must satisfy (3.27), (3.28) and (3.29).

2) Choose a function \( W \) satisfying (3.32).

3) Solve (3.31) to obtain the 1-form \( \Psi; \mathcal{P} \) is then obtained from (3.34) and the base space from (3.30).

4) Choose a function \( M \) satisfying (3.40). \( Q_\tau \) is then obtained from (3.39).

5) Choose a 1-form \( \mathcal{K} \) obeying (3.42); \( \tilde{Q} \) is then obtained from (3.41) and \( Q \) from (3.35).

6) The five dimensional metric and gauge field are obtained from (2.52) and (2.46).
3.2.1 Example: The Round Sphere

A basic example for which the base space is not conformally hyper-Kähler ($d\mathcal{P} \neq 0$) is obtained by taking the Einstein-Weyl data to be

$$ds_E = b^2 \left( d\theta^2 + \sin^2 \theta (d\phi^2 + \sin^2 \phi d\psi^2) \right), \quad u = 0, \quad u_0 = -\frac{2}{b}, \quad (3.44)$$

which has been considered in [9] and [41]. Following the algorithm explained above we choose

$$W = \alpha \cot \theta, \quad (3.45)$$

where $\alpha$ is a constant, and obtain

$$\Psi = b\alpha \cos \phi d\psi, \quad \mathcal{P} = \frac{1}{b\alpha \cot \theta} \left( d\tau + b\alpha \cos \phi d\psi \right); \quad (3.46)$$

the base space is then

$$ds_B^2 = \frac{1}{\alpha \cot \theta} \left( d\tau + b\alpha \cos \phi d\psi \right)^2 + b^2 \alpha \cot \theta \left( d\theta^2 + \sin^2 \theta (d\phi^2 + \sin^2 \phi d\psi^2) \right), (3.47)$$

Now we choose

$$M = \beta_1 + \beta_2 \cot \theta, \quad (3.48)$$

then

$$Q_\tau = -\frac{32 \tan^2 \theta}{3\sqrt{3}\beta^3 \alpha^2 \chi^3} + \beta_1 + \beta_2 \cot \theta. \quad (3.49)$$

To solve (3.42) we take $\mathcal{K} = dV$, where $V$ is a function; then (3.42) is solved by taking

$$\mathcal{K} = \frac{b\alpha \beta_1}{2} d\cot \theta, \quad (3.50)$$

and hence

$$\tilde{Q} = d \left( \frac{16 \tan \theta}{\sqrt{3}\beta^3 \alpha \chi^3} + \frac{b\alpha \beta_1 \cot \theta}{2} \right). \quad (3.51)$$

Rescaling the coordinate $\tau \rightarrow b\alpha \tau$ and introducing a radial coordinate $R$ by\footnote{Note that $R$ has dimensions $[R] = L^2$.}

$$R\chi^2 = \tan \theta, \quad (3.52)$$
the five dimensional metric can be written
\[
ds_5^2 = - \left( dt' + 2\sqrt{3}\chi R (d\tau + \cos \phi d\psi) + e^{\frac{\chi}{\sqrt{3}}}Q \right)^2 + e^{-\frac{\chi}{\sqrt{3}}} ds_B^2 ,
\] (3.53)
where the base space is
\[
ds_B^2 = R (d\tau + \cos \phi d\psi)^2 + \frac{1}{R(1 + R^2\chi^4)} \left( \frac{dR^2}{1 + R^2\chi^4} + R^2(d\phi^2 + \sin^2 \phi d\psi^2) \right),
\] (3.54)
and \(Q\) is given by
\[
Q = \left( \frac{\sqrt{3}\mu\chi}{2} - \frac{32\chi^3}{3\sqrt{3}} R^2 + \frac{j}{4R} \right) (d\tau + \cos \phi d\psi) + d \left( \frac{16\chi}{\sqrt{3}} R + \frac{\sqrt{3}\mu}{4\chi R} \right).
\] (3.55)
We have introduced \(\mu \equiv 2b^3\alpha^2\beta_1\chi/\sqrt{3}\), \(j \equiv 4b^3\alpha^2\beta_2\). Note also that we have shifted the time coordinate, \(t = t' + \frac{\chi}{\sqrt{3}} \ln(b^2\alpha\chi^2)\). Observe that for \(\chi = 0\), the base space (3.54) is Euclidean 4-space written in a Gibbons-Hawking form \([42]\).

Finally, introducing a new radial coordinate \(r\) by
\[
R = \frac{r^2}{4},
\] (3.56)
and a new time coordinate \(t\) by
\[
t = \frac{4\chi}{\sqrt{3}} r^2 + \frac{\sqrt{3}\mu}{\chi r^2} - \frac{\sqrt{3}}{\chi} e^{-\frac{\chi}{\sqrt{3}}}t',
\] (3.57)
the metric is written
\[
ds_5^2 = - f^2 \left[ dt + \left( \frac{4\chi^3}{3\sqrt{3}} r^4 + \sqrt{3}\mu \frac{j}{r^2} - \frac{\chi^2}{2} tr^2 \right) \sigma_3 \right]^2 + f^{-1} ds_B^2 ,
\] (3.58)
where
\[
f^{-1} \equiv \frac{4\chi^2}{3} r^2 + \frac{\mu}{r^2} - \frac{\chi}{\sqrt{3}} t ,
\] (3.59)
and the base space is
\[
ds_B^2 = \frac{1}{1 + \left( \frac{\chi}{\sqrt{3}} \right)^4} \left( \frac{dr^2}{1 + \left( \frac{\chi}{\sqrt{3}} \right)^4} + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2) \right) + \frac{r^2}{4} \sigma_3^2 ,
\] (3.60)
where we have used the following (right) one forms on \(SU(2)\)
\[
\sigma_1 = \cos \tau \sin \phi d\psi - \sin \tau d\phi ,
\]
\[
\sigma_2 = \sin \tau \sin \phi d\psi + \cos \tau d\phi ,
\]
\[
\sigma_3 = \cos \phi d\psi + d\tau .
\] (3.61)
The gauge field strength for this solution is

\[ F = \frac{\sqrt{3}}{2} d \left( f \left[ dt + \left( \frac{2\mu \chi}{\sqrt{3}} + \frac{j}{r^2} - \frac{\chi^2}{6} tr^2 \right) \sigma_3 \right] \right). \] (3.62)

To consider two limits of the solution (3.58)-(3.62) it is convenient to shift

\[ t \rightarrow t - \frac{\sqrt{3}}{\chi} c. \] (3.63)

Then, observe that for \( \chi = 0 \) the solution is the BMPV black hole. For small \( r \), the dominating terms are also the ones of the BMPV black hole. Thus, this geometry contains a BMPV black hole. Note also that performing the rescalings

\[ (t, r, c) \rightarrow \left( \frac{t}{\eta^2}, r \eta, \frac{c}{\eta^2} \right), \] (3.64)

and taking the limit \( \eta \rightarrow 0 \) one recovers the solution (3.24).

Let us note that the solution just derived is singular. Using the \( t', R \) coordinates one verifies that, in accordance with the comments at the end of section (2.3), the solution has a curvature singularity at \( t' \rightarrow +\infty \) (and also at \( R \rightarrow +\infty \)). It remains to be seen if the singularities that will necessarily arise in the non conformally hyper-Kähler case can have interesting interpretations as Big Bang/Big Crunch singularities or black object singularities.

### 4 Final Remarks

In this paper we have shown that the timelike solutions of five dimensional, minimal de Sitter supergravity admitting Killing spinors are determined by a four dimensional HKT geometry, wherein two constraint equations have to be solved, as summarised in section (2.3). To give concrete examples we considered two distinct cases:

- When the HKT manifold is conformally hyper-Kähler, all solutions can be generated from supersymmetric solutions (with a timelike Killing vector field) of five dimensional, minimal, ungauged supergravity, in the following way: Take a solution with \( G^+ = 0 \) (in the notation of (13)) and add a linear time dependence with the appropriate coefficient (3.18) to the harmonic function of the solution. Our analysis shows that all solutions with a conformally hyper-Kähler base space can be put in this form, which is therefore a stronger statement than that of (28). Several examples were given, including a multi-BMPV de Sitter solution, describing multiple rotating black holes co-moving with the expansion of the universe.
• If one assumes that the HKT manifold is not conformally hyper-Kähler, but possesses a tri-holomorphic Killing vector field, solutions can be found in terms of certain constrained (special, in the terminology of [39]) three-dimensional Einstein-Weyl geometries. Taking the latter to be simply the round three-sphere an explicit example was constructed, describing a BMPV black hole inside a singular universe.

One immediate task that this work suggests is to look for more interesting solutions in the non-conformally hyper-Kähler case. As discussed in section 2.3, such solutions will always have curvature singularities at $t = \pm \infty$, but these might have a cosmological or black object interpretation.

It would be particularly interesting to determine whether there exist regular (pseudo) supersymmetric black ring solutions in de Sitter supergravity. One encouraging hint that such ring geometries may exist is the fact that certain solutions, such as multi-BMPV black holes, have been found in both the ungauged and the de Sitter supergravities. In contrast to this, there are no known supersymmetric, asymptotically $AdS_5$ multi-black hole solutions. However, it is not possible to straightforwardly construct a de-Sitter black ring using the asymptotically flat solution found in [38] as a seed, as one can do for the (multi) BMPV black hole. This is because the asymptotically flat ring solution has $G^+ \neq 0$. Therefore if a (pseudo) supersymmetric black ring exists in this theory it will be described by a base space which is not conformally hyper-Kähler.

One other issue which remains to be resolved is whether one can always construct a 5-dimensional solution given a generic HKT base space $B$. In particular, it is not a priori apparent that given $B$, one can always find a solution to both the constraints (2.39) and (2.49), although we have shown that one can always solve (2.49). Note that in the case of the $AdS$ supergravity theory, it was shown in [43] that not all Kähler bases give rise to a five dimensional solution.

Finally, an immediate continuation of this work consists of considering the null case. Other possible generalisations include going beyond the minimal theory by including vector multiplets, and by considering de Sitter supergravity in other dimensions.

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Appendix: Strong HKT manifolds

For the solutions under consideration here, one can without loss of generality take $B$ to be a strong HKT manifold. This can be achieved by making a conformal transformation, and defining a new vielbein $\hat{e}^\alpha$ by

$$\hat{e}^\alpha = e^\phi e^\alpha, \quad (A-1)$$

where $\phi$ is a $t$-independent real function. Then the manifold $B'$ with metric

$$ds^2_{B'} = 2\delta_{\alpha\beta}\hat{e}^\alpha\hat{e}^\beta, \quad (A-2)$$

admits forms $\tilde{J}^i$ related to $J^i$ by

$$J^i = e^{2\phi}\tilde{J}^i, \quad (A-3)$$

which satisfy the algebra of the imaginary unit quaternions, and

$$d\tilde{J}^i = -2\mathcal{P}' \wedge \tilde{J}^i, \quad (A-4)$$

where

$$\mathcal{P}' = \mathcal{P} + d\phi. \quad (A-5)$$

Hence $B'$ is also a HKT manifold, and by making an appropriate choice of $\phi$, one can ensure that

$$d\star_4'\mathcal{P}' = 0, \quad (A-6)$$

where $\star_4'$ denotes the Hodge dual on $B'$. This implies that the torsion $H' = \star_4'\mathcal{P}'$ is closed. With this choice, $B'$ is strong HKT. Furthermore, on defining

$$Q' = e^{2\phi}Q, \quad t' = t - \frac{2\sqrt{3}}{\chi}\phi, \quad (A-7)$$

one finds that the metric, gauge potential and the constraints $\mathcal{P}$ and $Q$ remain invariant, with $t, \star_4, \mathcal{P}$ and $Q$ replaced with $t', \star_4', \mathcal{P}'$ and $Q'$. Hence, one can without loss of generality drop the primes, and take the base manifold $B$ to be strong HKT.
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