LOG-PLURIGENERA IN STABLE FAMILIES OF SURFACES

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Abstract. We study the flatness of log-pluricanonical sheaves on stable families of surfaces.

The paper [Kol18] studies flatness of the pluricanonical sheaves $\omega_{X/S}^m([m\Delta])$ for locally stable morphisms over reduced base schemes. Positive results are obtained for families with normal generic fibers, provided every divisor appears in $\Delta$ with coefficient $\geq \frac{1}{2}$. While examples show that the bound $\geq \frac{1}{2}$ is sharp, presumably the normality condition is not necessary. The aim of this note is to prove this for families of surfaces. Unfortunately, the proof relies on the classification of slc surface pairs, thus it is unlikely to generalize to higher dimensions.

Theorem 1. Let $S$ be a reduced scheme over a field of characteristic 0 and $f : (X, \Delta) \to S$ a locally stable morphism of relative dimension 2. Assume that $\text{coeff } \Delta \subset [\frac{1}{2}, 1]$. Then, for every $m \in \mathbb{Z}$ and $B \subset \lfloor \Delta \rfloor$, the sheaves

$$\omega_{X/S}^m([m\Delta] - B)$$

are flat over $S$ and commute with base change.

Warning 1.2. The Theorem and its Corollary hold for every variant of local stability I know of if either $\text{coeff } \Delta \subset (\frac{1}{2}, 1]$ or if $S$ is unibranch. In general we need to assume also that $\lfloor \Delta_s \rfloor = \lfloor \Delta \rfloor_s$ holds for every $s \in S$. See [Kol18 Sec.6] for relevant examples and [Kol17 Chap.4] for a detailed discussion of the issues.

As a first consequence we obtain that, if $S$ is connected, then the Hilbert function of the fibers

$$\chi(X_s, \omega_{X_s}^m([m\Delta_s]))$$

is independent of $s \in S$. If $f : (X, \Delta) \to S$ a stable, that is, if $K_{X/S} + \Delta$ is also $f$-ample, then by Serre vanishing, the log plurigenera

$$p_m(X_s, \Delta_s) := h^0(X_s, \omega_{X_s}^m([m\Delta_s]))$$

are also independent of $s \in S$ for $m \gg 1$. We can be more precise if we restrict the coefficients further.

Corollary 2. Let $S$ be a reduced scheme over a field of characteristic 0 and $f : (X, \Delta) \to S$ a stable morphism of relative dimension 2 such that $\text{coeff } \Delta \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$. Then, for every $m \geq 2$,

1. $R^i f_* \omega_{X/S}^m([m\Delta]) = 0$ for $i > 0$ and
2. $f_* \omega_{X/S}^m([m\Delta])$ is locally free and commutes with base change.

Both the Therem and the Corollary should hold in higher dimensions as well, hence the surface case is rather special. Therefore the main interest of this note may be the observation that the gluing theory of log pluricanonical sheaves on slc
pairs seems much more complicated than the gluing of slc pairs themselves. The latter was introduced in [Kol16] and discussed in detail in [Kol13, Chap. 5].

In Section 1 we reduce the Theorem to a claim about slc threefolds, which is then proved in Section 2. The proof uses detailed information about certain non-normal slc surfaces. These include a partial classification of non-normal slc surfaces, given in Section 3, and the computation of the Poincaré residue map on their irreducible components, treated in Section 4. Corollary 2 is proved in Section 5.

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1. Non-normal slc threefolds

Using [Kol18, Prop. 16], it is sufficient to prove Theorem 1 when $S$ is regular and of dimension 1. Furthermore, the latter is equivalent to proving the following variant, which will be the focus of our attention from now on.

**Proposition 3.** Let $(x \in X, H + \Delta)$ be a 3-dimensional, local slc pair over a field of characteristic 0 where $H$ is Cartier and $\text{coeff} \Delta \subset \left[\frac{1}{2}, 1\right]$. Then

$$\text{depth}_x \omega_X^m \left(\left\lfloor m\Delta \right\rfloor - B\right) = 3 \text{ for every } m \in \mathbb{Z} \text{ and } B \subset \left\lfloor \Delta \right\rfloor.$$  

Equivalently, $\omega_X^m \left(\left\lfloor m\Delta \right\rfloor - B\right)$ satisfies Serre’s condition $S_3$.

For a coherent sheaf whose support has dimension $\leq 3$, being $S_3$ is equivalent to being Cohen-Macaulay. If $\dim X \geq 4$ then the sheaves $\omega_X^m \left(\left\lfloor m\Delta \right\rfloor - B\right)$ are frequently not Cohen-Macaulay, but a (slight modification of) the $S_3$ condition is expected to hold; see [Kol18, Prop. 5]. This is why we state Proposition 3 using the $S_3$ condition.

The method of [Kol18], which proves Proposition 3 in case $X$ is normal, has 3 steps. The first, going back to [Ale08, Kol11] establishes the case when $mK_X + \left\lfloor m\Delta \right\rfloor - B$ is $\mathbb{Q}$-Cartier. The second constructs a small modification $\pi : X' \to X$ such that $mK_{X'} + \left\lfloor m\Delta' \right\rfloor - B'$ is $\mathbb{Q}$-Cartier and the third uses $X'$ to obtain the conclusion. As observed in [Kol18, Exmp. 22], the second step usually does not hold if $X$ is not normal; there are obstructions in codimension 2 and also in higher codimensions. In this note we deal with the codimension 2 obstruction. In the theory of slc pairs, the higher codimension obstructions usually behave quite differently, and there are several instances when the higher codimension case is easier. So there is some reason to believe that handling the codimension 2 obstruction may be a useful step in general.

If $X$ is normal then the conclusion of Proposition 3 is known to hold in all dimensions by [Kol18, Prop. 5]. Thus it remains to understand what happens when $X$ is non-normal. The gluing method of [Kol16] suggests that one should be able to treat $X$ by first working on its normalization $(\bar{X}, \bar{D} + \bar{\Delta})$, then proving compatibility with the gluing involution $\tau$ and finally descending to $X$; see [Kol13, Chap. 5] for details. Compatibility with the gluing involution turns out to be quite subtle. There are 2 variants:

- divisor version, working with $mK_{\bar{X}} + m\bar{D} + \left\lfloor m\bar{\Delta} \right\rfloor$ and the different, and
- sheaf version, working with $\omega_{\bar{X}}^m \left(\left\lfloor m\bar{D} + \left\lfloor m\bar{\Delta} \right\rfloor \right\right)$ and the Poincaré residue maps $R_{\bar{X}, \bar{D}}^m$ as in [Kol13, Sec. 4.1].
Unexpectedly, the 2 variants are not equivalent, and compatibility fails for both of them. For the divisor version see Example 12 for the sheaf version see Examples 14 and 15. However, the sheaf version does hold in many instances and one can describe quite well all cases when it fails.

With this in mind, first we focus on $H$ and prove a rather complete étale-local classification of such non-normal surface pairs $(H, \text{Diff}_H \Delta)$ in Theorem 8. This in turn implies the following description of the pair $(X, H + \Delta)$. This classification also shows that $x \in \text{Supp } B$ can happen only in the simpler case (4.1), thus we can mostly ignore $B$ in the sequel.

Proposition 4. Let $(x \in X, H + \Delta)$ be a 3-dimensional, strictly Henselian, slc pair over a field of characteristic 0 where $H$ is Cartier and $\text{coeff } \Delta \subset [\frac{1}{2}, 1]$. Assume that $X$ is not normal. Then one of the following holds.

1. The point $x$ is an lc center and $2(K_X + H + \Delta)$ is Cartier at $x$.
2. The point $x$ is not an lc center and $X$ has 2 irreducible components $(x_i \in X_i, D_i + H_i + \Delta_i)$ where $D_i$ denotes the conductor. Furthermore, the $D_i$ are smooth and the Poincaré residue maps

$$R_{X/D_i}^m : \omega_X^m (mD_i + mH_i + \lfloor m\Delta_i \rfloor) \to \omega_{D_i}^m (\lfloor m \text{Diff}_{D_i} (H_i + \Delta) \rfloor)$$

are surjective for every $m$.
3. The point $x$ is not an lc center, $X$ is irreducible and it has a quasi-étale double cover as in (2).

2. Proof of the main results

5 (A reformulation of Proposition 3). The Poincaré residue map

$$R_{X/H}^m : \omega_X^m (mH + \lfloor m\Delta \rfloor - B) \to \omega_H^m (\lfloor m \text{Diff}_H \Delta \rfloor - B|_H)$$

can be factored through the injection

$$\omega_X^m (mH + \lfloor m\Delta \rfloor - B)|_H \to \omega_H^m (\lfloor m \text{Diff}_H \Delta \rfloor - B|_H),$$

which is an isomorphism on $H \setminus \{x\}$, where both sheaves are locally free. Thus we see that

$${\text{depth}}_x \omega_X^m (mH + \lfloor m\Delta \rfloor - B) = 3 \iff R_{X/H}^m \text{ is surjective.}$$

6 (Proof of Proposition 4). It is easy to establish that $\text{coeff } (\text{Diff}_H \Delta) \subset [\frac{1}{2}, 1]$, see for instance [Kol13] 3.45.

If $x$ is an lc center of $(X, H + \Delta)$ then it is also an lc center of $(H, \text{Diff}_H \Delta)$ by adjunction [Kol13] 4.9. Thus Theorem 8 shows that $2(K_H + \text{Diff}_H \Delta)$ is Cartier. Therefore $2(K_X + \Delta)$ is also Cartier by [Gro68 XIII] or [Kol17] 2.90), giving (4.1).

If $x$ is not an lc center of $(X, H + \Delta)$ then it is also not an lc center of $(H, \text{Diff}_H \Delta)$ by adjunction. Thus $(H, \text{Diff}_H \Delta)$ is as described in (8.2). In particular, $X$ has 1 or 2 irreducible components. If $X$ has only 1 irreducible component then by [Kol13] 5.23 it has a quasi-étale double cover with 2 irreducible components. This gives case (4.3).

It remains to consider the case when $X$ has 2 irreducible components. Then $C_i := D_i \cap H_i$ is smooth by (8.2) hence the $D_i$ are smooth. The various residue
maps sit in a diagram

\[
\begin{align*}
\omega_X^{[m]}(mD_i + mH_i + [m\Delta_i]) & \xrightarrow{R_{X_i/D_i}} \omega_D^{[m]}([m\text{Diff}_D(H_i + \Delta)]) \\
\omega_{H_i}^{[m]}([m(D_i + \Delta_i)]) & \xrightarrow{R_{H_i/C_i}} \omega_C^{[m]}([m\text{Diff}_C(H_i + \Delta)])
\end{align*}
\]

(Note that, as discussed in [Kol13, 4.18], this diagram commutes for even maps sit in a diagram

more, (11.1) shows that the bottom horizontal arrow

is even then

arguments apply in all dimensions.)

(Note that we could also have used the more general Corollary 18. Both of these

proof of Proposition 3.)

proving (4.2).

these into 3 cases, we consider them separately.

In case (4.1) coeff \( \Delta \subset \{ \frac{1}{2}, 1 \} \) and \( \omega_X^{[2]}(2\Delta) = \omega_X^{[2]}(2\Delta) \) is free. So, if \( m \) is even then \( \omega_X^{[m]}([m\Delta] - B) \cong \mathcal{O}_X(-B) \) and if \( m \) is odd then \( \omega_X^{[m]}([m\Delta] - B) \cong \omega_X([\Delta] - B) \). We can now use [Kol13, 7.20] first for \( B \leq \Delta \) and then for

\(-K_X - |\Delta| + B \sim \frac{1}{2} \{ \Delta \} + B \leq \Delta \) to conclude that \( \text{depth}_X \omega_X^{[m]}([m\Delta] - B) = 3 \). (Note that we could also have used the more general Corollary 15. Both of these arguments apply in all dimensions.)

In case (4.2) the \( H_i \) do not have any further role, so set \( \Theta_i := H_i + \Delta_i \). We know that \( X \) is obtained by gluing the 2 components \( (X_i, D_i + \Theta_i) \) using an isomorphism \( \tau : (D_1, \text{Diff}_D_1 \Theta_1) \cong (D_2, \text{Diff}_D_2 \Theta_2) \). Thus, by [Kol13, 5.8], a pair of sections \( \sigma_i \) of \( \omega_X^{[m]}([m\Theta_i] + mD_i) \) glues to a section of \( \omega_X^{[m]}([m\Theta]) \) iff

\[
\mathcal{R}_{X_i/D_i}(\sigma_1) = (-1)^m \cdot \tau^*(\mathcal{R}_{X_2/D_2}(\sigma_2)).
\]

Equivalently, we have an exact sequence

\[
0 \rightarrow \omega_X^{[m]}([m\Theta]) \rightarrow \bigoplus_{i=1,2} \omega_X^{[m]}([m\Theta_i] + mD_i) \xrightarrow{R_D} \omega_D^{[m]}([m\text{Diff}_D \Theta_1]) \rightarrow 0,
\]

where \( R_D = \mathcal{R}_{X_i/D_i} - (-1)^m \cdot \tau^* \mathcal{R}_{X_2/D_2} \). The sheaves in the middle and on the right are \( S_3 \) by [Kol18, Prop.5]. Thus the sheaf on the left is also \( S_3 \); cf. [Kol13, 2.60].

In case (4.3) let \( \pi : (\tilde{X}, \tilde{H} + \tilde{\Delta}) \rightarrow (X, H + \Delta) \) be the double cover. We already proved that \( \omega_{\tilde{X}}^{[m]}([m\tilde{\Delta}] - \tilde{D}) \) is \( S_3 \), hence so is \( \omega_X^{[m]}([m\Delta] - D) \), which is a direct summand of \( \pi_\ast \omega_{\tilde{X}}^{[m]}([m\tilde{\Delta}] - \tilde{D}) \).

\[
3. \text{ Non-normal surface pairs}
\]

Next we describe the pairs \( (H, \text{Diff}_H \Delta) \) that arise in Proposition 3. In order to emphasize that we work with a purely 2 dimensional question, we write \( (S, \Delta) \) for an slc, surface pair. It turns out that non-normal pairs such that coeff \( \Delta \subset \{ \frac{1}{2}, 1 \} \) have a rather simple structure.
Theorem 8. Let \((s \in S, \Delta)\) be a strictly Henselian, slc, surface pair over an algebraically closed field of characteristic 0 such that \(\text{coeff} \Delta \subset [\frac{1}{2}, 1]\) and \(s \in S\) a non-normal point. Then one of the following holds.

1. The point \(s\) is an lc center and \(2(K_S + \Delta)\) is Cartier.
2. The point \(s\) is not an lc center and \(S\) has 2 irreducible components \((s_i \in S_i, D_i + \Delta_i)\). For both of them the extended dual graph of the minimal resolution (over \(s \in S\)) is of the form

\[
\bullet \quad c_1 \quad \cdots \quad c_n \quad \circ
\]

where \(\text{coeff}(\bullet) = 1\), \(\text{coeff}(\circ) \in [\frac{1}{2}, 1)\), \(n \geq 0\) and \(c_i \geq 2\) for every \(i\).

Furthermore \(\text{Diff}_{D_i} \Delta_1 = \text{Diff}_{D_2} \Delta_2\). The local class group has rank 1.
3. The point \(s\) is not an lc center and \(S\) has a quasi-étale double cover as in (2). The local class group is torsion.

Note. The Theorem should hold more generally whenever the characteristic is not 2, but there may be a lack of references related to adjunction. In characteristic 2 there should be only one more case for which the normalization induces an inseparable map on the conductors.

Proof. Let \((s_i \in S_i, D_i + \Delta_i)\) be the irreducible components of the normalization of \((S, \Delta)\), where the \(D_i\) denote the conductors. The pairs \((s_i \in S_i, D_i + \Delta_i)\) are lc and \(D_i \neq 0\) for every \(i\) since \(S\) is not normal at \(s\).

The pairs \((s_i \in S_i, D_i + \Delta_i)\) are described in (9.1–3). Correspondingly, there are 2 cases.

Plt case. If \(s\) is not an lc center then none of the \(s_i\) is an lc center by [Kol13, 5.10.3]. Thus each \((s_i \in S_i, D_i + \Delta_i)\) is as in (9.1). In particular, \(D_i\) is irreducible and there are at most 2 irreducible components. If there are 2 irreducible components then the gluing is done by an isomorphism \(\tau : D_1 \to D_2\). These give case (2).

Any element of the local class group is given by a pair of divisors \(C_i \subset S_i\). Then \(\delta(C_1, C_2) := (C_1 \cdot D_1) - (C_2 \cdot D_2)\) is well defined and we get an exact sequence

\[
0 \to (\text{torsion subgroup}) \to \text{Cl}(s \in S) \xrightarrow{\delta} \mathbb{Q}.
\]

If there is 1 irreducible component \((s_i \in S_i, D_i + \Delta_i)\) then the gluing is done by an involution \(\tau : D_1 \to D_1\). Furthermore, it has a quasi-étale double cover \(\rho : (\tilde{S}, \tilde{\Delta}) \to (S, \Delta)\), with 2 irreducible components by [Kol13, 5.23], giving case (3). The covering involution \(\rho\) interchanges the 2 irreducible components \(\tilde{S}_i\), hence \(\delta \circ \rho = -\delta\). Thus only the torsion subgroup of \(\text{Cl}(\tilde{s} \in \tilde{S})\) descends to give divisors on \(S\).

Non-plt case. If \(s\) is an lc center then each \(s_i\) is an lc center by [Kol13, 5.10.3]. Thus the irreducible components \((S_i, D_i + \Delta_i)\) are as in (9.2–3), their number can be arbitrary. Let \(D_{ij}\) denote the irreducible components of the normalization of \(D_i\). Thus we have \(\rho_i : D_{i1} \amalg D_{i2} \to D_i\) in case (9.2) and \(\rho_i : D_{i1} \cong D_i\) in case (9.3).

Let \(D := \amalg_{ij}(s_{ij} \in D_{ij})\) be their disjoint union and let \(\tau : D \to D\) denote the gluing involution \(\tau\) acting on it. Furthermore, by (9.4), each \(\omega^2_{S_i}(2D_i + 2\Delta_i)\) is a line bundle on \(S_i\) and the Poicaré residue map gives canonical isomorphisms

\[
\mathcal{R}^2_{S_i/D_{ij}} : \rho^*_{ij}\omega^2_{S_i}(2 \text{Diff}_{D_{ij}} \Delta_i) \cong \omega^2_{D_{ij}}(2 \text{Diff}_{D_{ij}} \Delta) \cong \omega^2_{D_{ij}}(2[s_{ij}]).
\]

Applying the Poicaré residue map twice gives canonical isomorphisms

\[
\mathcal{R}^2_{S_i/s_{ij}} : \omega^2_{S_i}(2D_i + 2\Delta_i)|_{s_{ij}} \cong k(s_{ij}).
\]
We can thus pick $\tau$-invariant sections 

$$ (\sigma^D_{ij}) \in \oplus_{ij} H^0\left(D_{ij}, \omega^2_{D_{ij}} \left(2 \text{Diff}_{D_{ij}} \Delta_i\right)\right) $$

that have residue 1 at the points $s_{ij}$. Since they have the same residue, they descend to sections 

$$ (\sigma_i^D) \in \oplus_i H^0\left(D_i, \omega^2_{D_i} \left(2 \text{Diff}_{D_i} \Delta_i\right)\right). $$

We can lift these $\sigma_i^D$ back to sections 

$$ (\sigma_i) \in \oplus_i H^0\left(S_i, \omega^2_{S_i} \left(2 \text{Diff}_{S_i} \Delta_i\right)\right). $$

By [Kol13, 5.8] the $(\sigma_i)$ descend to a section of $\omega^2_{S}(2\Delta)$ that has residue 1 (hence nonzero) at the origin. Therefore $\omega^2_{S}(2\Delta)$ is locally free at $p$. This completes the proof of (1). 

**□**

**Reminder 9.** The list of all lc surface pairs $(S, \Theta)$ where $\text{coef} \Theta \subset [\frac{1}{2}, 1]$ is given in [Kol13, pp.125-128]. Here we are interested in those special cases when there is a divisor with coefficient 1. Since the list in [Kol13] is organized differently, the classification is summarised next where we use $\Theta := D + \Delta$.

Thus let $(S, \Theta)$ be an lc pair of dimension 2 over an algebraically closed field such that $\text{coef} \Theta \subset [\frac{1}{2}, 1]$ and $s \in \Theta$ a closed point. We describe $(s \in S, \Theta)$ using the extended dual graph of the minimal embedded resolution $\pi : S' \rightarrow S$. The exceptional divisors are denoted by the negative of their self-intersection: 2 or $c_i$ in the diagrams. The birational transforms of the local branches of $\Theta$ are denoted by

- if $\text{coef}(\bullet) = 1$ and by $\circ$ if $\text{coef}(\circ) \in [\frac{1}{2}, 1)$.

There are 3 distinct cases.

**Plt case.** If $s$ is not an lc center then the extended dual graph is one of the following, where $n \geq 0$ and $c_i \geq 2$ for every $i$.

1. $\bullet \quad c_1 \quad \cdots \quad c_n \quad \circ$ \hspace{1cm} (9.1)

**Cyclic non-plt case.** Here $s$ is an lc center, $n \geq 0$, $c_i \geq 2$ and the extended dual graph is

2. $\bullet \quad c_1 \quad \cdots \quad c_n \quad \circ$ \hspace{1cm} (9.2)

**Dihedral non-plt case.** Here $s$ is an lc center, $\text{coef}(\circ) = \frac{1}{2}$, $n \geq 0$ and $c_i \geq 2$ except that $c_n = 1$ is allowed in cases (9.3.2–3)

3. $\bullet \quad c_1 \quad \cdots \quad c_n \quad 2$ \hspace{1cm} (9.3.1)

4. $\bullet \quad c_1 \quad \cdots \quad c_n \quad 2$ \hspace{1cm} (9.3.2)

5. $\bullet \quad c_1 \quad \cdots \quad c_n \quad 2$ \hspace{1cm} (9.3.3)

Next let $\Sigma$ be the divisor on $S'$ that contains the curves marked by $\bullet$ or $c_i$ with coefficient 1 and the curves marked by 2 or $\circ$ with coefficient $\frac{1}{2}$. By inspection
we see that \(2(K_S + \Sigma)\) is a \(\mathbb{Z}\)-divisor that has 0 intersection with all \(\pi\)-exceptional divisors. Thus \(2(K_S + \Theta) = 2 \cdot \pi_*(K_S + \Sigma)\) is Cartier near \(s\) by [Kol13, 10.9.2].

**Conclusion 9.4.** If \(p\) is an lc center then \(2(K_S + \Theta)\) is Cartier near \(s\). \(\square\)

**Example 10.** The nice dichotomy of Theorem 8 does not seem to carry over to higher dimensions. As an example, pick \(\frac{1}{2} \leq c_1, c_2, c_3 \leq 1\) such that \(c_1 + c_2 + c_3 = 2\). Consider the pair \((x_3x_4 = 0, c_1(x_1 = 0) + c_2(x_2 = 0) + c_3(x_1 = x_2)) \subset \mathbb{A}^4\).

It is non-normal and the origin is an lc center.

### 4. The Poincaré Residue Map

We study the surjectivity of the Poincaré residue map for slc surface pairs. First we show surjectivity for the pairs listed in (9.1). We stress that this is a rather special property of such pairs. We see in Example 14 that it fails for some dihedral pairs, even when \(\Delta = 0\). Also, even on smooth surfaces, it fails for every other \(\Delta'\) for some \(m\); see Example 16.

**Proposition 11.** Let \((S, D + \Delta)\) be an lc surface pair as in (9.1), over a field of characteristic 0. Then the Poincaré residue map

\[
\mathcal{R}^m_{S/D} : \omega^m_S(mD + |m\Delta|) \to \omega^m_D(|m \text{ Diff}_D \Delta|) \tag{11.1}
\]

is surjective for every \(m\).

**Proof.** We use the (étale-local) representation of \((S, D + \Delta)\) as a quotient \((S, D + \Delta) := (\tilde{S}, \tilde{D} + \tilde{\Delta})/\mathbb{Z}^n(1, q), \tag{11.2}\)

where \((\tilde{S}, \tilde{D} + \tilde{\Delta}) := (A^2, (y = 0) + (1 - c)(x = 0)); \text{ cf. [Kol13, 3.32].} \]

We can write the sections of \(\omega^m_S(mD + |m\Delta|)\) in the form

\[
g(x, y) \left(\frac{dx}{x} \wedge \frac{dy}{y}\right)^{\otimes m}, \tag{11.3}
\]

where \(x^mc\) divides \(g(x, y)\) (in the ring of Puiseux series) and \(g(x, y)\) is \(\mu_n\)-invariant. Any monomial in \(g\) that contains \(y\) restricts to 0 on \(D\), thus in (11.2) only the sections of the form

\[
x^r \left(\frac{dx}{x} \wedge \frac{dy}{y}\right)^{\otimes m}
\]

have non-zero image. We need the \(\mu_n\)-invariant generator, which is

\[
\sigma := x^{n[mc/n]} \left(\frac{dx}{x} \wedge \frac{dy}{y}\right)^{\otimes m}. \tag{11.4}
\]

Setting \(\gamma := \frac{1}{n}\), the sign convention of [Kol13, 4.1] gives that

\[
\mathcal{R}^m_{S/D}(\sigma) = (-1)^m x^{n[m\gamma]} \left(\frac{dx}{x}\right)^{\otimes m}. \tag{11.5}
\]

On \(D\) the local coordinate is \(z = x^n\) and \(\frac{dz}{z} = n \frac{dx}{x}\), hence we get that

\[
\mathcal{R}^m_{S/D}(\sigma) = (-n)^m z^{m\gamma} \left(\frac{dz}{z}\right)^{\otimes m}. \tag{11.6}
\]

The different is computed by the formula (cf. [Kol13, 3.45])

\[
\text{Diff}_D \Delta = (1 - \frac{1}{n} + \frac{1}{n^2})[s] = (1 - \frac{1}{n})[s] = (1 - \gamma)[s], \tag{11.7}
\]
where $s \in D \subset S$ denotes the origin. Since $m - \lfloor m \gamma \rfloor = \lfloor m(1 - \gamma) \rfloor$, (11.6) shows the isomorphism (modulo torsion supported at $s$)
\[ R_{S/D}^m : \omega_S^m (mD + \lfloor m \Delta \rfloor)|_D \cong_{\text{tor}} \omega^m_D \left( \lfloor m \text{Diff}_D \Delta \rfloor \right). \quad \Box \quad (11.8) \]

Next we compute the $\mathbb{Q}$-divisor version of (11.1).

**Example 12.** Consider 2 surface pairs

\[ (S_i, D_i + (1 - c_i)C_i) := (\kappa^2, (y = 0) + (1 - c)(x = 0))/_{\mathbb{Q}} (1, 1) \quad (12.1) \]

and glue them using an isomorphism $\tau : D_1 \to D_2$ to get

\[ (S, \Delta) := (S_1, D_1 + (1 - c_1)C_1) \amalg (S_2, D_2 + (1 - c_2)C_2). \quad (12.2) \]

Note that $\text{Diff}_{D_i}(1 - c_i)C_i = 1 - \frac{\omega}{n_i}$, thus $(S, \Delta)$ is slc $\iff K_S + \Delta$ is $\mathbb{Q}$-Cartier $\iff \frac{\omega}{n_1} = \frac{\omega}{n_2}$.

Given any $n_1, n_2$, choose the $c_i$ such that $\frac{c_1}{n_1} = \frac{c_2}{n_2}$ and $c_i < \frac{1}{2}$. Then $2K_S + [2\Delta] = 2K_S + C_1 + C_2$. Note that $(2K_S + C_1 + C_2)|_{S_i} = 2K_{S_i} + 2D_i + C_1$ and

\[ (2K_{S_i} + 2D_i + C_i)|_{D_i} = 2(K_{D_i} + (1 - \frac{\omega}{n_i})[s]) + \frac{\omega}{n_i}[s] = 2K_{D_i} + (2 - \frac{\omega}{n_i})[s], \]

where $s \in D_i \subset S_i$ denotes the origin. Thus $2K_S + [2\Delta]$ is $\mathbb{Q}$-Cartier iff $n_1 = n_2$.

Formula (11.8) and Example 12 directly imply the following.

**Corollary 13.** Let $(S, \Delta) = (S_1, D_1 + \Delta_1) \amalg (S_2, D_2 + \Delta_2)$ be an slc surface as in (8.2). Then

\begin{enumerate}
  \item $\omega_{S_1}^m (mD_1 + \lfloor m \Delta_1 \rfloor)|_{D_1} \cong\text{tor} \omega_{S_2}^m (mD_2 + \lfloor m \Delta_2 \rfloor)|_{D_2}$, but in general
  \item $(mK_{S_1} + mD_1 + \lfloor m \Delta_1 \rfloor)|_{D_1} \neq (mK_{S_2} + mD_2 + \lfloor m \Delta_2 \rfloor)|_{D_2}. \quad \Box \quad (13.1)\]
\end{enumerate}

Next we compute the Poincaré residue map in the dihedral cases. Although these are not needed for the proof of Theorem 1, they show that the Poincaré residue map is not surjective in general. This suggests that it may not be easy to understand the pluricanonical sheaves for non-normal pairs using the normalization.

**Example 14.** Let $(S, B)$ be a pair as in the dihedral case (12.1). It can also be obtained as the quotient of the pair

\[ (\tilde{S}, \tilde{B}) := ((xy = z^{2n}), (z = 0)) \quad (14.1) \]

by the involution $\tau : (x, y, z) \mapsto (y, x, -z)$. Both the $x$ and $y$ axes map isomorphically to $B \subset S$.

A local generator of $\omega_{\tilde{S}}$ is $z^{-2n+1}dx \wedge dy$, thus a local generator of $\omega_{\tilde{S}}(\tilde{B})$ is $z^{-2n}dx \wedge dy$, which can be rewritten as

\[ \sigma := \frac{dx}{x} \wedge \frac{dy}{y}. \]

Thus we see that $\tau^* \sigma = -\sigma$ and so $\sigma$ does not descend to $S$. Hence $\omega_{\tilde{S}}(B)$ is not locally free. However $\tau^* \sigma \otimes \sigma = \sigma \otimes \sigma$, so $\sigma \otimes \sigma$ does descend to a local generator of $\omega_{\tilde{S}}^2(2B)$, which is thus locally free.

We can also find generators of $\omega_{\tilde{S}}(B)$, obtained from the $\tau$-invariant forms

\[ (x - y) \frac{dx}{x} \wedge \frac{dy}{y} \quad \text{and} \quad z \frac{dx}{x} \wedge \frac{dy}{y}. \quad (14.2) \]

By restricting these first to the $x$-axis and then using that the latter is isomorphic to $B$, we see that

\[ R_{S/B}((x - y) \frac{dx}{x} \wedge \frac{dy}{y}) = -dx \quad \text{and} \quad R_{S/B}(z \frac{dx}{x} \wedge \frac{dy}{y}) = 0. \quad (14.3) \]
Thus $R_{S/B} : \omega_S(B)|_B \to \omega_B$ is an isomorphism modulo torsion supported at the origin $s \in S$. By contrast, in degree 2 we have the Poincaré residue isomorphism

$$R^2_{S/B} : \omega^2_S(2B)|_B \cong \omega^2_B(2[s]).$$

This also shows that $\text{Diff}_B(K_S + B) = K_B + [s]$. Thus the Poincaré residue map in degree $m$ gives surjections

$$R^m_{S/B} : \omega^m_S(mB) \to (\omega_B([s]))^m \quad \text{if } m \text{ is even, but}$$

$$R^m_{S/B} : \omega^m_S(mB) \to (\omega_B([s]))^m (-[s]) \quad \text{if } m \text{ is odd.}$$

**Example 15.** Consider 2 pairs $(S_1, B_1)$ and $(S_2, B_2)$ as in the dihedral case (9.2.1) and $(T, C_1 + C_2) := (K^2, (u_1 = 0) + (u_2 = 0))$. Out of these we can assemble a reducible slc pair by gluing $S_i$ to $T$ using isomorphisms $\tau_i : C_i \to B_i$ such that $\tau^*_i dx_i = du_i$. We get an slc pair $(S, 0) \cong (S_1, B_1) \amalg \tau_1 (T, C_1 + C_2) \amalg \tau_2 (S_2, B_2)$. We claim that, modulo torsion supported at the origin $s \in S$,

1. $\omega_S|S \cong \omega_{S_1}(B_1)$ but
2. $\omega_S|T \cong \omega_T(C_1 + C_2)(-[s]).$

Thus, although $\omega_S$ is $S_2$, its restriction to $T$ is not $S_2$. This makes it hard to study the depth of pluricanonical sheaves on reducible slc pairs using the normalization.

In order to see (2) we need to show that every section of $\omega_{S_1}(B_1)$ extends to a section of $\omega_S$. It is enough to prove this for the generators in (14.2). Here $z \frac{dx_i}{x_i} \wedge \frac{du_i}{y_i}$ vanishes on $B_i$, so we can extend it by 0 to the other components. The other generator $(x_i - y_i) \frac{dx_i}{x_i} \wedge \frac{du_i}{y_i}$ restricts to $B_i$ as $-dx_i = -du_i$. This can be extended to $T$ as $u_i \frac{du_i}{y_i} \wedge \frac{du_{3-i}}{y_{3-i}}$ and then as zero to $S_{3-i}$.

There is a sign ambiguity in the definition of higher codimension restriction maps, but $\pm R_{S/C} = \pm R_{S/B} = \pm R_{S/S_1}$, hence $R_{S/C}$ gives a surjection $R_{S/C} : \omega_S \to \omega_C$. On the other hand, $R_{T/C}$ gives a surjection $R_{T/C} : \omega_T(C_1 + C_2) \to \omega_{C_1}(s)$. These show that the image of $\omega_S|T$ is contained in $\omega_T(C_1 + C_2)(-[s])$.

The latter is generated by the forms $u_i \frac{du_i}{y_i} \wedge \frac{du_{3-i}}{y_{3-i}}$ and we already showed that these extend to sections of $\omega_S$. This proves (3).

The next example shows that Proposition 11 does not hold if $\Delta$ has at least 2 irreducible components.

**Example 16.** Let $D, C_1, \ldots, C_r$ be distinct lines through the origin $s \in S := k^2$. For some positive rational numbers $c_i$ consider the pair $(S, D + \Delta)$ where $\Delta = \sum_i c_i C_i$. We claim that if $r \geq 2$ then there is an $m > 0$ such that

$$R^m_{S/D} : \omega^m_S([m\Delta])|_D \to \omega^m_D([m \text{Diff}_D \Delta])$$

is not surjective.

First note that $\text{Diff}_D \Delta = (\sum_i c_i)[s]$ and the 2 sides of (16.1) are

$$R^m_{S/D} : \omega^m_D(\sum_i [mc_i] \cdot [s]) \to \omega^m_D([m \sum_i c_i] \cdot [s]).$$

Thus our claim is equivalent to saying that

$$\sum_i [mc_i] < [m \sum_i c_i] \quad \text{for some } m > 0.$$
First choose the smallest $m > 0$ such that $m \sum_i c_i$ is an integer. Then $\sum_i \lceil mc_i \rceil \leq \sum_i mc_i = \lceil m \sum_i c_i \rceil$ and equality holds iff all the $mc_i$ are integers. Thus we are done unless $\sum_i c_i = a/m$ for some $(a, m) = 1$ and $c_i = a_i/m$ for some integers $a_i$.

Now choose $m' > 0$ such that $m'a \equiv 1 \mod m$. Note that $m'a_i/m$ is not an integer since $(m', m) = 1$ and $c_i < 1$. Thus

$$\sum_i \lceil m'c_i \rceil \leq \sum_i (m'c_i - \frac{1}{m'}) = m'(\sum_i c_i) - \frac{m'}{m} < m'(\sum_i c_i) - \frac{1}{m} = \lceil m'\sum_i c_i \rceil.$$  

5. Standard coefficients

In this section we prove Corollary 2. More generally, we study what happens in all dimensions if coeff $\Delta$ is contained in the standard coefficient set $T := \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$. The first observation is that then a strong form of Proposition $\text{Kol}^{11}$ holds in all dimensions. We write $(X, \Delta)$ for pairs of arbitrary dimension but change to $(S, \Delta)$ when the discussion is restricted to surfaces.

**Proposition 17.** Let $(X, \Delta)$ be an slc pair over a field of characteristic 0 where coeff $\Delta \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$. Assume that $x \in X$ is not an lc center. Then

$$\text{depth}_x \omega_X^{|m\Delta| - B|} \geq \min\{3, \text{codim}_x X\}. \quad (17.1)$$

for every $m \in \mathbb{Z}$ and $B \subset |\Delta|$. \hfill $\square$

Proof. Set $-D \sim mK_X + |m\Delta| - B$ and note that

$$D \sim_{\mathbb{Q}} -m(K_X + \Delta) + |m\Delta| + B \sim_{\mathbb{Q}} |m\Delta| + B \leq \Delta.$$  

Thus (17.1) follows from [Kol11]; see also [Kol13, 1.81]. \hfill $\square$

**Corollary 18.** Let $(X, H + \Delta)$ be an slc pair over a field of characteristic 0 where $H$ is Cartier and coeff $\Delta \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$. Then

$$\text{depth}_x \omega_X^{|m\Delta| - B|} \geq \min\{3, \text{codim}_x X\}.$$  

holds for every $m \in \mathbb{Z}$, $B \subset |\Delta|$ and $x \in H$. \hfill $\square$

Proof. By the monotonicity of discrepancies [KM98, 2.27], none of the lc centers of $(X, \Delta)$ is contained in $H$. Thus the Corollary follows from Proposition $17$. \hfill $\square$

As we noted at the beginning of Section 1, Corollary 18 and [Kol18, Prop.16] imply the following.

**Corollary 19.** Let $S$ be a reduced scheme over a field of characteristic 0 and $f : (X, \Delta) \to S$ a stable morphism such that coeff $\Delta \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$. Then, for every $m \geq 1$, $\omega_X^{|m\Delta|}_{f/S}(|m\Delta|)$ is flat over $S$ and commutes with base change. \hfill $\square$

By the Cohomology and Base Change Theorem, the $n$-dimensional version of Corollary 2 follows once we establish vanishing theorems for the fibers $\omega_X^{|m\Delta|}(|m\Delta|)$. More generally, the following should be true.

**Conjecture 20.** Let $(X, \Delta)$ be an proper, slc pair over a field of characteristic 0 such that $K_X + \Delta$ is ample. Fix $m \geq 2$ and assume that

$$\text{coeff } \Delta \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\} \cup \{1 - \frac{1}{m}, 1\}. \quad (20.1)$$

Then

$$H^i(X, \omega_X^{|m\Delta|}(|m\Delta|)) = 0 \quad \text{for } i > 0. \quad (20.2)$$
Proof attempt. Note that
\[ mK_X + [m\Delta] \sim_{q} K_X + (m-1)(K_X + \Delta) + [m\Delta] - (m-1)\Delta. \quad (20.3) \]

Our assumption (20.1) guarantees that \( 0 \leq [m\Delta] - (m-1)\Delta \leq \Delta \). If \( mK_X + [m\Delta] \) is \( \mathbb{R} \)-Cartier then we can apply the Ambro-Fujino form of Kodaira’s vanishing (see Theorem 22) and we are done. However, usually \( mK_X + [m\Delta] \) is not \( \mathbb{R} \)-Cartier. It is a natural idea to try to find a proper, birational morphism \( \pi : (X', \Delta') \to (X, \Delta) \) such that \( mK_{X'} + [m\Delta'] \) is \( \mathbb{R} \)-Cartier, establish vanishing on \( X' \) and then descend to \( X \). That is, we aim to find a proper, birational morphism \( \pi : (X', \Delta') \to (X, \Delta) \) with the following properties.

(4) \( \pi_*(\omega^{[m]}_{X'}(\lfloor m\Delta' \rfloor)) = \omega^{[m]}_X(\lfloor m\Delta \rfloor) \),
(5) \( H^i(X', \omega^{[m]}_{X'}(\lfloor m\Delta' \rfloor)) = 0 \) for \( i > 0 \) and
(6) \( R^i\pi_*(\omega^{[m]}_{X'}(\lfloor m\Delta' \rfloor)) = 0 \) for \( i > 0 \).

Then the Leray spectral sequence shows that \( H^i(X, \omega^{[m]}_X(\lfloor m\Delta \rfloor)) = 0 \) for \( i > 0 \).

If \( X \) is normal, then [Ko18 Prop.19] and Theorem 22 show that there is a small modification \( \pi : X' \to X \) with these properties. However, if \( X \) is not normal, then sometimes there is no such small modification. This is obvious for surfaces, since a demi-normal surface has no nontrivial small modifications. Therefore we have to use a birational morphism with exceptional divisors. This brings in 2 extra problems.

- In many cases the coefficient of an exceptional divisor in \( \Delta' \) should be the discrepancy, or a small perturbation of it. Thus it may not satisfy the numerical assumptions (20.1).
- If an exceptional divisor appears with coefficient 1, then the needed vanishing claims (20.5–6) usually do not hold.

For surfaces we can avoid these problems, but only with very special choices of \( \pi \).

21 (Proof of Conjecture 20 for surfaces). There are only finitely many points \( s \in S \) such that \( mK_S + [m\Delta] \) is not \( \mathbb{R} \)-Cartier at \( s \). At these points \( S \) is non-normal. We follow the classification of such points given in \([S1\text{–}3]\) and in each case give a local description of \( \pi : (S', \Delta') \to (S, \Delta) \). We describe \( \pi \) after passing to the strict Henselisation, but in each case this automatically descends to the original base field.

21.1 (Ã£â†‘ case with 2 components.) By \([S2]\) here \((S, \Delta)\) is glued together from 2 branches \( S_1, S_2 \), with resolution dual graphs

\[ \bullet \quad c_{i1} \quad \cdots \quad c_{in} \quad \circ \]

where \( \text{coeff}(\bullet) = 1, \text{coeff}(\circ) = 1 - d_i \). We choose the partial resolution \( S'_1 \to S_1 \) that extracts only the curve \( C_{i1} \). Thus \( S'_1 \) has 1 singular point (obtained by contracting the curves \( C_{i2}, \ldots, C_{in} \)) and \( D_1 \cap C_{i1} \) is a smooth point of \( S'_1 \). Note further that

\[ a(C_{i1}, S_1, \Delta_1) = -1 + \frac{1}{n_i} - \frac{1-d_i}{n_i} = -1 + \frac{d_i}{n_i}. \]

As we noted in Example 12 the quality \( \gamma := \frac{d_i}{n_i} \) is independent of \( i \). We add \( C_{i1} \) to \( \Delta'_1 \) with coefficient \( 1 - \gamma \), thus \( \Delta'_1 \) also satisfies the coefficient assumption (20.1).

We can now glue \( S'_1 \) and \( S'_2 \) to get \( S' \to S \). On \( S' \) we get 2 normal cyclic quotient singularities and 1 normal crossing point of the form \((xy = 0), (1 - \gamma)(z = 0)\).
21.2 (Plt case with 1 component.) By (8.3) in these cases the local class group is torsion, so \( mK_S + \lceil m\Delta \rceil \) is \( \mathbb{R} \)-Cartier at \( s \).

21.3 Log center case. If \( m\Delta \) is a \( \mathbb{Z} \)-divisor then \( mK_S + \lfloor m\Delta \rfloor = mK_S + m\Delta \) is \( \mathbb{Q} \)-Cartier by (8.1). Thus assume that \( \lfloor m\Delta \rfloor \not\equiv m\Delta \). For \( \pi : S' \to S \) we take the slc modification. That is, on the irreducible components listed in (4.1–3) we extract all curves marked \( c_i \), but we do not extract the curves marked 2 (these have discrepancy \(-\frac{1}{2}\)). Thus all the exceptional curves \( C_{ij} \) appear with coefficient 1. Set \( \Delta'' := \pi^{-1}\Delta + \sum C_{ij} \). As we noted, the problem is that we can not apply Theorem 22 to \((S', \Delta'')\).

Let \( \sigma \) be a section of \( \omega_S^{|m\Delta|}(\lfloor m\Delta \rfloor) \). We can then view \( \pi^{-1}\sigma \) as a section of \( \omega_S^{|m\Delta''|}(\lfloor m\Delta'' \rfloor) \). Since \( \lfloor m\Delta \rfloor \not\equiv m\Delta \), the section \( \pi^{-1}\sigma \) vanishes along all the exceptional curves \( C_{ij} \). Thus we can decrease the coefficients of the \( C_{ij} \) without violating (20.4).

To do this choose a \( \pi \)-exceptional, \( \mathbb{Q} \)-Cartier divisor \( E \) such that \(-E\) is \( \pi \)-ample and set \( \Delta' := \Delta'' - \varepsilon E \). Then \( K_{S'} + \Delta' \) is \( \pi \)-ample on \( S' \) for every \( \varepsilon \). Furthermore, once we patch the local modifications to a global \( S' \to S \), the divisor \( K_{S'} + \Delta' \) is still nef and log big on \( S' \) for \( 0 < \varepsilon \ll 1 \). (It has degree 0 only on the \( \pi \)-exceptional curves over the plt points.)

With these choices \( \pi : (S', \Delta') \to (S, \Delta) \) satisfies (20.4) and (20.5–6) follow from Theorem 22. Thus the Leray spectral sequence shows that \( H^i(S, \omega_S^{|m\Delta|}(\lfloor m\Delta \rfloor)) = 0 \) for \( i > 0 \).

The following is proved in [Amb03] and [Fuj14, 1.10], see also [Fuj17], where it is called a Reid-Fukuda–type vanishing theorem. The 2 dimensional case that we use is much easier.

**Theorem 22** (Ambro-Fujino vanishing theorem). Let \((X, \Delta)\) be an slc pair and \( D \) a Mumford \( \mathbb{Z} \)-divisor on \( X \) (that is, \( X \) is regular at all generic points of \( \text{Supp} \Delta \)). Let \( f : X \to S \) be a proper morphism. Assume that \( D \sim_R K_X + L + \Delta \), where \( L \) is \( \mathbb{R} \)-Cartier, \( f \)-nef and log \( f \)-big. Then \( R^i f_* \mathcal{O}_X(D) = 0 \) for \( i > 0 \). \( \square \)

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