PATH INTEGRAL QUANTIZATION FOR A TOROIDAL PHASE SPACE

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Abstract

A Wiener-regularized path integral is presented as an alternative way to formulate Berezin-Toeplitz quantization on a toroidal phase space. Essential to the result is that this quantization prescription for the torus can be constructed as an induced representation from anti-Wick quantization on its covering space, the plane. When this construction is expressed in the form of a Wiener-regularized path integral, symmetrization prescriptions for the propagator emerge similar to earlier path-integral formulas on multiply-connected configuration spaces.

1. INTRODUCTION

With the notion of “quantization” we associate the construction of quantum systems that are in correspondence with a given classical system. The various quantization prescriptions usually specify some Hilbert space and a mapping of suitable classical observables to operators on this Hilbert space. In Schrödinger’s prescription for canonical quantization, the vectors in the Hilbert space are square-integrable functions on classical configuration space, and their dynamics is derived from a partial differential equation commonly known as Schrödinger’s equation. There are other quantization schemes in which the Hilbert space consists of functions on the classical phase space, and among these, coherent states often play an important role [8, 2, 5].

An alternative way to express quantization is by path integration. Thanks to the Feynman-Kac formula [17], Schrödinger’s approach can be associated with a Wiener integral over paths in configuration-space. Similarly, a formula of Daubechies and Klauder [3] relates anti-Wick quantization to a so-called Wiener-regularized path integral, that

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is, a limit of certain Wiener-integrals over paths in phase space. In both approaches, only continuous paths contribute, which raises the question how the quantization of classical systems with topologically nontrivial phase spaces appears in these formulations. The torus as a simple example has been useful to develop such concepts, and we will restrict ourselves to that case.

The main goal of this paper is to derive a Wiener-regularized path integral formula associated to Berezin-Toeplitz quantization on a toroidal phase space. The result exhibits similarities to earlier investigations of Feynman path integrals on multiply connected configuration spaces \([16, 10, 7]\). The similar nature of both results is traced back to the appearance of induced representations which is a natural concept for the quantization on quotient spaces \([12, 13]\).

2. ANTI-WICK QUANTIZATION

First we will quickly review how Berezin-Toeplitz quantization applies to a system with a canonical degree of freedom, that is, when the phase space can be identified with the Euclidean plane \(\mathbb{R}^2\) equipped with the standard symplectic structure. In this special case Berezin-Toeplitz quantization is also known as anti-Wick quantization.

2.1. Operator Formulation

The Hilbert space for the description of quantum mechanical states is chosen as a closed subspace \(\mathcal{H}\) of \(L^2(\mathbb{R}^2)\) consisting of functions \(\psi\) which are square integrable with respect to the two-dimensional Lebesgue measure \(dpdq\) on \(\mathbb{R}^2\) and in addition to that satisfy the integral equation

\[
\psi(p',q') = (K\psi)(p',q') := \int_{\mathbb{R}^2} \frac{dpdq}{2\pi\hbar} K(p',q';p,q) \psi(p,q) \tag{2.1}
\]

with Planck’s constant \(2\pi\hbar\) and the so-called reproducing kernel

\[
K(p',q';p,q) := e^{-\frac{1}{\hbar} [(p' - p)^2 + (q' - q)^2] + \frac{i}{\hbar} (pq' - q'p)} \tag{2.2}
\]

This specifies the kinematical background of the quantum theory. The transition from classical to quantum dynamics is achieved by a mapping taking classical observables, represented by sufficiently nice real-valued functions on the phase space, to self-adjoint operators on \(\mathcal{H}\). Given the classical Hamilton function \(h: \mathbb{R}^2 \to \mathbb{R}\), anti-Wick quantization defines the quantum Hamiltonian by

\[
H\psi := K(h\psi) \tag{2.3}
\]

on functions \(\psi\) from a domain that is a suitable extension of the domain of \(h\) interpreted as multiplication operator on \(\mathcal{H}\).
Three remarks apply:

• It is well-known that this construction satisfies a correspondence principle [2].

• The above prescription can also be obtained from a group theoretical approach. In fact, the reproducing kernel can be identified with the inner product of so-called coherent vectors in \( \mathcal{H} \),

\[
K(p', q'; p, q) = (\eta_{p', q'}, \eta_{p, q}),
\]

which are given as the image \( \eta_{p, q} := D(p, q)\eta \) of a reference vector \( \eta(p, q) := K(p, q; 0, 0) \) under the unitary operators \( D(p, q) \) acting as a ray representation of the Heisenberg-Weyl group,

\[
(D(p, q)\psi)(p', q') := e^{ipq' - p'q}/2\hbar\psi(p + p', q + q')
\]

on vectors \( \psi \in \mathcal{H} \). This implies that the self-adjoint operators for position and momentum defined as the generators \( Q := (-i\hbar\frac{\partial}{\partial p}D)(0, 0) \) and \( P := (i\hbar\frac{\partial}{\partial q}D)(0, 0) \) satisfy the canonical commutation relation \( PQ - QP = -i\hbar \).

• For the convenience of relating the reproducing kernel to the unitary ray representation \( D(p, q) \) we chose the space \( \mathcal{H} \) and not the so-called Fock-Bargmann space of holomorphic functions which is usually associated with Berezin-Toeplitz quantization [2] for the planar phase space \( \mathbb{R}^2 \). The difference simply amounts to redistributing a Gaussian weight from the functions to the inner product [1].

2.2. Path Integral Formulation

The stochastic process underlying the path integrals considered hereafter is the Brownian bridge [17, 14]. Its realizations are continuous paths \( b : [0, r] \mapsto \mathbb{R}^2 \) in the plane, starting at \( b(0) = (p, q) \) and stopping at \( b(r) = (p', q') \) after a time \( r > 0 \). As a Gaussian process it is fully characterized by its mean and covariance,

\[
\langle b(s) \rangle = (p, q) + [(p', q') - (p, q)]s/r
\]

\[
\langle b_j(s)b_k(u) \rangle - \langle b_j(s) \rangle\langle b_k(u) \rangle = \delta_{jk}\hbar^2(\min\{s, u\} - su/r),
\]

respectively, with \( s, u \in [0, r] \). Kronecker’s delta \( \delta_{jk} \) shows that the path components denumerated with \( j, k \in \{1, 2\} \) are independent of each other. For notational convenience we also introduce the conditional Wiener measure \( \mu_{p', q', r;p,q} \), which relates to the expectation with respect to the Brownian bridge according to

\[
\int d\mu_{p', q', r;p,q}(\bullet) := \frac{1}{2\pi\hbar^2 r}e^{-((p' - p)^2 + (q' - q)^2)/2\hbar^2 r}\langle(\bullet) \rangle.
\]

It was demonstrated by Daubechies and Klauder [8] that anti-Wick quantization can be expressed in a so-called Wiener-regularized path integral. More precisely, the
continuous integral kernel of the time evolution operator $e^{-itH/\hbar}$ constructed from the Hamiltonian (2.3) is expressed as a limit of conditional Wiener-integrals
\[
\left(e^{-itH/\hbar}\right)(p', q'; p, q) = 2\pi\hbar \lim_{r \to \infty} e^{r\hbar/2} \int d\mu_{p', q'; r, p, q} e^{iS_r/\hbar} \tag{2.9}
\]
with the action functional
\[
S_r(b) := \frac{1}{2} \int_0^r \left[ db_2(s)b_1(s) - db_1(s)b_2(s) \right] - \frac{t}{r} \int_0^r ds \ h(b(s)) \tag{2.10}
\]
The stochastic integral appearing in $S_r$ is understood in the sense of Stratonovich [15].
Two remarks apply:
- The above path integral expression is valid for all Hamilton functions $h$ that are bounded from below and polynomially bounded from above [6].
- Setting $t = 0$ we obtain the reproducing kernel and thus all the information about the Hilbert space $\mathcal{H}$. Clearly, the path integral comprises information about quantization for both kinematics and dynamics in a nutshell!

3. INDUCED QUANTIZATION FOR A TOROIDAL PHASE SPACE

Let us regard the torus as the quotient space of the Euclidean plane $\mathbb{R}^2$ where points are identified which can be mapped into each other by adding vectors $g$ from a grid $\mathbb{G} := (a\mathbb{Z}, b\mathbb{Z})$ of spacings $a, b > 0$. For convenience we will denote the points on the torus with their representatives from the half-open rectangle $\mathbb{T}^2 := [0, a) \times [0, b)$.

Classically, Hamiltonian dynamics on the torus $\mathbb{T}^2$ can be lifted to the covering space $\mathbb{R}^2$ by periodically extending $h$ on $\mathbb{R}^2$ to make it compatible with the quotient construction. In the following section we will implement this concept within the framework of quantum mechanics.

3.1. Operator Formulation

It is well-known that Berezin-Toeplitz quantization only gives rise to Hilbert spaces of nonzero dimension if the volume of the torus is integral in units of Planck’s constant [5, 3], that is, $ab = 2\pi\hbar N$ with a positive integer $N$.

These Hilbert spaces, denoted as $\mathcal{H}_k$, are realized here (with an analogous identification as for $\mathcal{H}$ and Fock-Bargmann space) as closed subspaces of the space $L^2(\mathbb{T}^2)$ of square-integrable functions on the torus. Each function $\psi$ in $\mathcal{H}_k$ satisfies
\[
\psi(p', q') = \int_{\mathbb{T}^2} \frac{dp dq}{2\pi\hbar} K_k(p', q'; p, q) \psi(p, q) \tag{3.1}
\]
analogous to (2.4), where the reproducing kernel
\[
K_k(p', q'; p, q) := \sum_{g \in \mathbb{G}} (\eta_{p', q'}, D_g \eta_{p, q}) \tag{3.2}
\]
is defined in terms of the inner product and the coherent vectors of the preceding section and the unitary operators of the form

\[ D_{g,k} := D(g_1, g_2) e^{i(g_1k_2 - g_2k_1 + \frac{1}{2}g_1g_2)/\hbar} \]  

with a parameter \( k \) chosen from the “inverse” torus \([0, \frac{2\pi \hbar}{a}) \times [0, \frac{2\pi \hbar}{b})\).

A few remarks apply:

- With the integrality of \( ab/2\pi \hbar \) and the definition \((2.5)\) it can be confirmed that the set of operators \( \{D_{g,k}\}_{g \in G} \) is commutative and forms a genuine unitary group representation of the discrete phase-space translations by grid vectors \( g \in G \).

Thanks to the abelian group composition law for this set and the reproducing kernel property of \( K \) on \( \mathcal{H} \) it follows that \( K_k \) is also a reproducing kernel.

The additional freedom in the parameter \( k \) amounts to choosing a character of \( G \), which reflects in the kernel and thus in the boundary conditions that the functions in \( \mathcal{H}_k \) obey.

- The Hilbert space \( \mathcal{H}_k \) is \( N \)-dimensional. This can be read off from an orthogonal decomposition of the kernel

\[
K_k(p', q'; p, q) = \sum_{j=0}^{N-1} \phi_j^*(p', q') \phi_j(p, q),
\]

which also shows the connection to the (holomorphic) theta functions spanning the usual Hilbert space \([5, 11, 3]\) obtained from Berezin-Toeplitz quantization for the torus. The decomposition \((3.4)\) is derived with the help of Poisson’s sum formula, the use of the identity \( D_{-g,k}QD_{g,k} = Q - g_21 \) and the fact \( ab = 2\pi \hbar N \) by rewriting the summation in \((3.2)\) as

\[
\sum_{g \in G} D_{g,k} = \frac{b}{N} \sum_{m,n \in \mathbb{Z}} \sum_{j=0}^{N-1} e^{-ibm(P + k_1)/\hbar} \delta(Q + k_2 + \frac{b}{N}j) e^{ibn(P + k_1)/\hbar}. \tag{3.6}
\]

- The definition of the reproducing kernel \((3.2)\) can be interpreted as an induced representation. We briefly sketch this idea following the presentation in \([9]\). By the preceding remark, \( p_k : \psi \mapsto (\sum_g (\psi, D_{g,k}\psi))^{1/2} \) defines a semi-norm on \( \mathcal{H} \). Passing to the quotient space of \( \mathcal{H} \) with respect to the set \( \mathcal{M}_k := \{ \psi : p_k(\psi) = 0 \} \) of vectors having zero length and discarding vectors of infinite length yields a pre-Hilbert space equipped with the inner product derived from \( p_k \). The completion of this space is isomorphic to \( \mathcal{H}_k \). In the above setting, the symmetrization \((\psi + \mathcal{M}_k)(p, q) := \sum_g (D_{g,k}\psi)(p, q) \) realizes the quotient mapping, and the inner product derived from \( p_k \) coincides with that of \( L^2(T^2) \).
Once the reproducing kernel is given, it is straightforward to mimick the same procedure as for anti-Wick quantization and define the Hamiltonian as

\[ H_k\psi := K_k(h\psi) \]

for suitable \( \psi \in \mathcal{H}_k \). Again it is known that a correspondence principle holds \([3, 4]\) for this quantization prescription.

Alternatively, the continuous integral kernel for the time evolution operator constructed from \( H_k \) can be directly expressed by a symmetrization analogous to (3.2),

\[ (e^{-itH_k/\hbar})(p', q'; p, q) = \sum_{g \in G}(\eta_{p'q'}, D_{g,k}e^{-itH/\hbar}\eta_{pq}) \].

Hereby \( H \) is the operator obtained from anti-Wick quantization of the periodic extension of \( h \) on \( \mathbb{R}^2 \).

We will establish (3.8) under the assumption that \( h \) is bounded. It is sufficient to show that the right-hand side defines an integral kernel of the one-parameter group of unitary operators \( e^{-itH_k/\hbar} \) and that it is continuous in the parameters \( p', q' \) and \( p, q \). Unitarity and the group composition law are straightforward to verify. By the definition (3.7), the generator of this group can then be identified as \( H_k \). To see the claimed continuity, one shows that the expansion

\[ \sum_{g \in G}(\eta_{p'q'}, D_{g,k}e^{-itH/\hbar}\eta_{pq}) = \sum_{n=0}^{\infty}(\frac{-it}{\hbar^n n!})\sum_{g \in G}(\eta_{p'q'}, D_{g,k}H^n\eta_{pq}) \]

provides a uniformly convergent series. This follows from Hölder’s inequality, a bound on the functions \( \eta_{p'q'} + \mathcal{M}_k \), and an estimate on the norms of \( H^n\eta_{pq} \) in the Banach space \( L^1(\mathbb{R}^2) \) of Lebesgue-integrable functions on the plane. The last estimate is obtained from the boundedness of \( h \) interpreted as a multiplication operator on \( L^1(\mathbb{R}^2) \) and the boundedness of the operator on \( L^1(\mathbb{R}^2) \) associated with the kernel \( K(p', q'; p, q) \).

### 3.2. Path Integral Formulation

The quantization prescription in the preceding section can be expressed in the path integral formula

\[ (e^{-itH_k/\hbar})(p', q'; p, q) = 2\pi\hbar \lim_{r \to \infty} e^{r\hbar/2} \sum_{g \in G} e^{if_{g,k}(p', q')/\hbar} \int d\mu_{p'q'p+q+g_1q+g_2r} e^{iS_r/\hbar} \]

with the function \( f_{g,k}(p', q') := g_1k_2 - g_2k_1 + \frac{1}{2}(g_1q' - g_2p' + g_1g_2) \) and the previous action functional \( S_r(b) \) containing the periodic extension of \( h \). This result is derived from (3.8) in combination with the Daubechies-Klauder formula \([23]\) under the assumption that \( h \) is bounded.
We conclude with two remarks:

- The general structure of the path integral formula (3.10) is analogous to the construction in [16]. However, the extra phase factors independent of \( k \) cannot be anticipated by the argument given there, which is only concerned with multiply-connected configuration spaces.

- With hindsight one can interpret the result as an integral over the actual paths on the torus. This shows that each path receives a phase factor depending on its homotopy class and the transition functions of the complex line bundle underlying Berezin-Toeplitz quantization. From this viewpoint one may hope to deduce generalizations for more general phase-space manifolds.

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