From Spectrum Wavelet to Vertex Propagation: Graph Convolutional Networks Based on Taylor Approximation

Songyang Zhang, Member, IEEE, Han Zhang, Shuguang Cui, Fellow, IEEE, and Zhi Ding, Fellow, IEEE

Abstract—Graph convolutional networks (GCN) have been recently applied to semi-supervised classification problems with fewer labeled data and higher-dimensional features. Existing GCNs mostly rely on a first-order Chebyshev approximation of the graph wavelet-kernels. Such a generic propagation model may not always be well suited for the datasets. This work revisits the fundamentals of graph wavelet and explores the utility of spectral wavelet-kernels to signal propagation in the vertex domain. We first derive the conditions for representing the graph wavelet-kernels via vertex propagation. We next propose alternative propagation models for GCN layers based on Taylor expansions. We further analyze the choices of detailed propagation models. We test the proposed Taylor-based GCN (TGCN) in citation networks and 3D point clouds to demonstrate its advantages over traditional GCN methods.

Index Terms—graph convolutional network, graph spectral wavelet, Taylor approximation

1 INTRODUCTION

Learning and signal processing over graph models have gained significant traction owing to their demonstrated ability to capture the underlying data interactions. Modeling each data point as a node and their interactions as edges in a graph, graph-based methods have been adopted in various signal processing and analysis tasks, such as the semi-supervised classification [1]–[3], spectral clustering [4], [5], link prediction [6], [7] and graph classification [8], [9].

Among various graph-based tools, graph signal processing (GSP) has emerged as an efficient analytical tool for processing graph-modeled signals [10], [11]. Based on a graph Fourier space defined by the eigenspace of the representing adjacency or Laplacian matrix, different kinds of GSP filters have found applications in practice, including bridge health monitoring [12], point cloud denoising [13], and image classification [14]. Leveraging graph Fourier transform [15], graph wavelet [16] and graph spectral convolution [17] can extract additional features from graph signals. For example, graph convolutional filters have been proposed for edge detection and biomedical video segmentation [18]. Graph spectral analysis has been an important tool in data processing.

Although GSP-based spectral filtering has demonstrated successes in a variety of applications, it still suffers from the high-complexity of spectrum computation and the suitable choice of propagation models. To efficiently extract signal features and integrate traditional GSP within the machine learning framework, graph convolutional networks (GCN) [1] have been developed for semi-supervised classification problems. Approximating graph spectral convolution with first-order Chebyshev expansions, GCN has been effective in such learning problems. Furthermore, different GCN-related graph learning machines, such as personalized propagation of neural predictions (PPNP) [19] and N-GCN [20], have also been developed to process graph-represented datasets. However, traditional GCN based on Chebyshev expansions still has certain limitations. For example, it requires strong assumptions on maximum eigenvalues and Chebyshev coefficients in approximating spectral convolution, at the cost of possible information loss when compared against basic convolutional filters. Furthermore, systematic choice and design of propagation models for GCN remain elusive.

Our goal is to explore the relationship between GSP and GCN to improve GCNs. Specifically in this work, we explore the process from spectrum wavelet to vertex propagation, and investigate alternative designs for graph convolutional networks. Our contributions can be summarized as follows:

- We revisit the graph spectral convolution in GSP and determine the conditions for approximating spectrum wavelet via propagation in the vertex domain. These conditions could provide insights to design GCN layers.
- We propose alternative propagation models for the GCN layers and develop a Taylor-based graph convolutional networks (TGCN) based on the derived approximation conditions.

In our numerical tests, we illustrate the effectiveness of the proposed frameworks over several well-known datasets compared to other GCN-type and graph-based methods. We...
also provide guidelines on the choice of suitable propagation models for GCN layers.

2 Related Work

In this section, we provide an overview on state-of-the-art graph signal processing (GSP) and graph convolutional networks (GCN).

Graph Signal Processing: Graph signal processing (GSP) has emerged as an exciting and promising new tool for processing large datasets with complex structures, owing to its power to extract underlying relationships among signals [10], [11]. GSP has achieved significant success in generalizing traditional digital signal processing (DSP) and processing datasets with complex underlying structures. Modeling data points and their interactions as a graph, a graph Fourier space could be defined according to the eigenspace of a graph representing matrix, such as the Laplacian or adjacency matrix, to facilitate data processing operations including denoising [21], filter banks [22], and compression [23]. The framework of GSP is further generalized with fundamentals based on the graph Fourier space, including sampling theory [24], graph Fourier transform [15], frequency analysis [25], graph filters [14], graph wavelet [16], and graph stationary process [26], [27]. In addition, GSP has also been considered for high-dimensional geometric signal processing, such as hypergraph signal processing [28] and topological signal processing [29].

Graph Convolutional Networks: Graph-based learning machines have become important tools in data analysis. Developed from graph wavelet processing [16], graph convolutional networks (GCN) approximate the spectral wavelet convolution via first-order Chebyshev expansions [1] and has shown evident success in semi-supervised learning tasks. In addition, the authors of [19] have developed personalized propagation of neural predictions (PPNP) to integrate PageRank [30] with GCNs. Other typical graph-based learning machines include GatedGCN [31], GraphSAGE [32], Gaussian Mixture Model Network (MoNet) [33], Graph Attention Networks (GAT) [34], Differential Pooling (DiffPool) [35], and Graph Isomorphism Network (GIN) [36]. For additional information, interested readers are referred to a comprehensive literature review [37] and two extensive surveys [38], [39].

3 Graph Wavelet and Graph Convolutional Networks

In this section, we first review the fundamentals of graph spectral convolution and wavelets, necessary for the development of propagation models of the GCN layers. We will then briefly introduce the structures of traditional GCN [1].

3.1 Graph Spectral Convolution and Wavelet-Kernels

An undirected graph $G = (\mathcal{V}, \mathcal{E})$ with $N = |\mathcal{V}|$ nodes can be represented by a representing matrix (adjacency/Laplacian) decomposed as $A = \mathbf{V}\Sigma\mathbf{V}^T \in \mathbb{R}^{N \times N}$, where the eigenvectors $\mathbf{V} = \{\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_N\}$ form the graph Fourier basis and the eigenvalues $\lambda_i$’s represent graph frequency [15].

In GSP [17], [40], graph Fourier transform of convolution between two signals is a product between their respective Fourier transforms denoted by $\circ$, i.e.,

$$x \circ y = F_C^{-1}(F_C(x) \circ F_C(y)),$$

where $F_C(x) = C^T x$ refers to the graph Fourier transform (GFT) of signals $x$, $F_C^{-1}(x) = Cx$ is the inverse GFT and $\circ$ is the Hadamard product. This definition generalizes the property that convolution in the vertex domain is equivalent to multiplication in the corresponding graph spectral domain.

In [15], the graph wavelet transform is defined according to graph spectral convolution. Given a spectral graph wavelet-kernel $\hat{g} = [g(\lambda_1), g(\lambda_2), \cdots, g(\lambda_N)]^T$ with kernel function $g(\cdot)$, the graph wavelet operator is defined as

$$T_g x = V(\hat{g} \circ (V^T x))$$

$$= V \begin{bmatrix} g(\lambda_1) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & g(\lambda_N) \end{bmatrix} V^T x. \quad (3)$$

Note that graph wavelet can be interpreted as a graph convolutional filter with a spectrum wavelet-kernel $\hat{g}$. Depending on the datasets and applications, different kernel functions may be utilized in $\hat{g}$.

3.2 Graph Convolutional Networks and Their Limitations

To overcome the complexity of calculating the spectrum matrix $V$ and the difficulty of seeking suitable wavelet-kernel functions, the framework of GCN was developed in [1] via a first-order Chebyshev expansion. Consider Chebyshev polynomials $T_K(x)$ up to $K^{th}$ orders. Convolutional filter with wavelet-kernel $\hat{g}$ is approximated by

$$T_g(x) \approx \sum_k \theta_k T_k(\hat{L}) x, \quad (4)$$

where $\hat{L} = 2I/\lambda_{\text{max}} - I_N$. With careful choice of $\lambda_{\text{max}}$ and parameters $\theta_k$, the graph convolutional filter can be further approximated by the 1st-order Chebyshev expansion

$$T_g(x) \approx \theta(I_N + D^{-\frac{1}{2}} \hat{A} D^{-\frac{1}{2}}) x, \quad (5)$$

where $D$ is the diagonal matrix of node degree. Then, by generalizing the approximated graph convolutional filter to a signal $X \in \mathbb{X}^{N \times C}$ with $C$ features for each node, the filtered signals can be written as

$$Z = \hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} X \Theta, \quad (6)$$

where $\Theta \in \mathbb{R}^{C \times F}$ is the parameter matrix. Furthermore, by integrating the nonlinear functions with the approximated convolutional filters, a two-layer GCN can be designed with message propagation as

$$Z_{\text{GCN}} = \text{softmax} \left( \text{ReLU}(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} X W^{(0)}) W^{(1)} \right), \quad (7)$$

where $W^{(0)} \in \mathbb{R}^{N \times H}$ and $W^{(1)} \in \mathbb{R}^{H \times C}$ are the parameters for the $H$ hidden units.
Although GCN has achieved success in real applications, it still suffers from some drawbacks. First, several strong assumptions are taken to approximate the original convolutional filters. For example, \( \lambda_{\text{max}} \) are set as 2 to approximate while the Laplacian matrix is not normalized with \( \lambda_{\text{max}} = 2 \) in implementation, and the Chebyshev coefficients are set to \( \theta_1 = -\theta_0 = -\theta \). These assumptions may compromise the properties of the original spectral convolution. Second, the graph propagation model \( D^2 A D^2 \) may not always be the optimal choice. It remains unclear as to how to design a suitable kernel-function \( g \) and how to approximate it appropriately. Moreover, an insightful interpretation is needed from the spectral wavelet convolution to the vertex propagation.

To explore the alternative design of the propagation model for GCNs, we focus on the steps between graph spectral wavelet-kernels and propagation in the vertex domain. We will also propose some alternative propagation models for GCNs.

### 4 Taylor-based Graph Convolutional Networks

In this section, we investigate the conditions for approximating the spectral convolution via vertex propagation. Next, we propose alternative propagation models for graph convolution layers based on Taylor expansion, where the general convolutional filter can be written as

\[
Z = G_{\alpha}(P) X \Theta,
\]

where \( G_{\alpha}(P) \) is a polynomial function with parameter \( \alpha \), \( P \) is the representing matrix of the graph, and \( \Theta \) are parameters of feature projection.

#### 4.1 Approximation of Spectral Convolution

We first show the theoretical motivation for the design of polynomial propagation model, and its relationship to the graph spectral wavelets. For a polynomial filter in GSP, let \( P \) be the representing (adjacency/Laplacian) matrix. We can easily obtain the following property.

**Lemma 1.** Given a GSP polynomial filter \( H = h(P) = \sum_k \alpha_k P^k \), the filtered signals are calculated by

\[
H_s = h(P) s = \sum_{r=1}^{N} h(\lambda_r) f_r(f_r^T s),
\]

where \( f_r \)'s are the graph spectrum and \( \lambda_r \)'s are the eigenvalues of \( P \) related to graph frequency.

**Proof.** Let \( V = [f_1, \cdots, f_N] \) and \( \Sigma = \text{diag}(\lambda_1, \cdots, \lambda_N) \). Since \( V^T V = I \), we have

\[
P^k s = \underbrace{V \Sigma^T V \Sigma^T V \cdots V \Sigma^T V}_{\text{k times}} s
\]

\[
= V \Sigma^k V^T s
\]

\[
= \sum_{r=1}^{N} \lambda_r^k (f_r^T s) f_r.
\]

Since \( H = h(P) = \sum_k \alpha_k P^k \) is a polynomial graph filter, we can easily obtain

\[
H s = \sum_{k} \sum_{r=1}^{N} \alpha_k \lambda_r^k (f_r^T s) f_r = \sum_{r=1}^{N} h(\lambda_r)(f_r^T s) f_r.
\]

This lemma shows that the response of the filter to an exponential is the same exponential amplified by a gain that is the frequency response of the filter at the frequency of the exponential \([10]\). It works as the invariance property of exponentials as eigenfunctions/eigenvectors, similar to the linear systems in digital signal processing (DSP).

Looking into the graph wavelet convolutional filter in Eq. 3, the wavelet-kernel function \( g(\cdot) \) operates to modify frequency coefficients \( \lambda_r \)'s. Thus, we have the following property of transferring spectrum wavelet to vertex propagation.

**Theorem 1.** Given a polynomial wavelet kernel function \( g(\cdot) \), the GSP convolutional filter on signal \( x \) is calculated as

\[
\sum_{k} \alpha_k P^k = \sum_{r=1}^{N} h(\lambda_r)(f_r^T s) f_r = \sum_{r=1}^{N} h(\lambda_r)(f_r^T s) f_r.
\]

**Proof.** Since the convolution filter \( T_g(x) \) can be written in \( T_g(x) = \sum_{k} g(\lambda_r)(f_r^T x) \), it is easy to prove this theorem with Lemma 1.

This theorem indicates that we can avoid the computation of the spectrum by implementing the convolution directly in vertex domain if the wavelet kernel \( g(\cdot) \) is polynomial or could be approximated by a polynomial expansion. We can see that the Chebyshev expansion is a special case of Theorem 1. In addition to Chebyshev expansion, Legendre \([41]\) and Taylor \([42]\) expansions can be also considered to approximate the spectral convolution. In addition, other polynomial design on the wavelet-function \( g(\cdot) \) can be also applicable.

#### 4.2 Taylor-based Propagation Model

We now provide alternative propagation models for the GCN layers based on Taylor expansions, with which the wavelet-kernel function \( g(x) \) can be approximated via

\[
g(x) \approx \sum a(x - a)^k.
\]

Different from the Chebyshev expansion, the Taylor polynomials do not have limitations on the interval of the variable \( x \) (without using \( \lambda_{\text{max}} \)) and take the same form of exponentials regardless of \( a \). The only problem is that we may need prior knowledge on the derivatives of \( g(\cdot) \). However, since we aim to optimize the function \( g(\cdot) \) by using neural network, \( g^{(k)}(a) \) can be reparametrized as the parameters \( \theta_k \) of the convolution filter. In short, the graph spectral convolutional filter can be approximated as

\[
T_g(x) \approx \sum_{k} \theta_k (P - \text{diag}(\Phi))^k x.
\]

Different models can be proposed based on Eq. 16 to develop the Taylor-based GCN (TGNC).

The \( P \) matrix here can be any practical graph representing matrix used to capture the overall information of the
is a special case of single-layer higher-order polynomials. which is one term in the single-layer $k$ and the first-order polynomial is

$$
\Phi(x) = \theta'(P + \alpha I_N)x,
$$

where $\theta' = \theta_1$ and $\alpha = \frac{\theta_0 - \theta_1}{\theta_1}$ are the new parameters for the convolutional filter. Then, the GCN layer with generalized signal $X \in \mathbb{R}^{N \times C}$ can be designed as

$$
X^{(l+1)} = (P + \alpha_l I_N)X^{(l)}\Theta_l,
$$

where $\alpha_l$ and $\Theta_l$ are the trainable variables for the $l$th layer. Compared to GCN, the assumptions on the $\lambda_{\text{max}}$ and $\theta_k$ are not needed for approximation.

**Type-2 First-Order TGCN:** We also consider more complex but general diagonal matrix to replace the $\alpha I_N$, i.e.,

$$
X^{(l+1)} = (P + \text{diag}(\beta_l))X^{(l)}\Theta_l,
$$

where $\beta_l$ and $\Theta_l$ are the parameters of the $l$th layer. Here, the self-influence for each node varies from node to node, whereas each node affects itself equivalently in the type-1 first-order TGCN model.

**Type-3 $k^{th}$-Order TGCN:** We also consider the higher-order propagation models for each layer. To avoid overfitting and reduce the complexity, we force all the $\theta_k = \theta$ and the diagonal matrix as $\phi I_N$ in Eq. (16). Then, the TGCN layer can be designed as

$$
X^{(l+1)} = \sum_k (P + \alpha_l I_N)^k X^{(l)}\Theta_l.
$$

Note that the higher-order polynomial design is different from multiple layers of the first-order polynomials. Suppose that the single-layer $k^{th}$-order polynomial convolutional filter is written as

$$
Z_k = \sum_k \alpha_k (P + \text{diag}(\beta))X\Theta_k
$$

and the first-order polynomial is

$$
Z_1 = \alpha (P + \text{diag}(\beta))X\Theta.
$$

For a $k$-layer first-order polynomial convolutional filter, the filtered result can be written as

$$
Z^{(k)} = \alpha_1 \cdots \alpha_k (P + \text{diag}(\beta))^k X\Theta_1 \cdots \Theta_k
$$

$$
= \alpha' (P + \text{diag}(\beta))^k X\Theta',
$$

which is one term in the single-layer $k^{th}$-order polynomial convolutional filter. Thus, the multi-layer first-order TGCN is a special case of single-layer higher-order polynomials.

**Type-4 $k^{th}$-Order TGCN:** More general TGCN layers can be designed without requiring $\theta_k = \theta$ for all $k$ as follows:

$$
X^{(l+1)} = \sum_k [(P + \alpha_l I_N)^k X^{(l)}\Theta_{l,k}].
$$

Here, we only consider simple diagonal with one parameter $\alpha$ in the higher-order polynomials to avoid overfitting and high complexity. We will provide some insight into the choice of different propagation models in Section 5.

5 EXPERIMENTS

We now test the proposed TGCN models in comparison with other GCN-like models on citation datasets and point cloud datasets. We also experiment with different representing matrix and propagation model to explore the choice of suitable layer design and graph representations for GCNs.

5.1 Experimental Setup

**Convolution Layer:** For the first-order TGCN, we consider a two-layer structure designed as follows.

$$
Z = \text{softmax}(G_{\phi_1}(P)\text{RELU}(G_{\phi_1}(P)XW^{(0)})W^{(1)}),
$$

where $G_{\phi_1}(P)$ is the specific type of TGCN propagation model, and $W^{(l)} \in \mathbb{R}^{N \times H}$ together with $W^{(1)} \in \mathbb{R}^{H \times C}$ are the parameters of the $H$ hidden units. For the higher-order TGCN, we consider a single-layer structure, i.e.,

$$
Z = \text{softmax}(G_{\phi}(P)XW).
$$

When training the parameters, we let the neural networks learn the diagonal parameters $\beta$ for type-2 first-order TGCN, and $\alpha$ for higher-order TGCN. We applied Adam optimizer $[43]$ for network training. For type-1 first-order TGCN, we apply both manual and automatic adjustments on the diagonal parameters $\alpha$. We train the projection parameter $W$ for a variety of TGCNs. We also test different representing matrices $P$ including adjacency matrix and normalized $D^{-\frac{1}{2}}A D^{-\frac{1}{2}}$.

**Implementation:** Let $V_l$ be the set of labeled examples and $Y_l$ denote the labels. We evaluate the cross-entropy error over all labeled examples to train parameters, i.e.,

$$
L = -\sum_{i \in V_l} \sum_{j=1}^L Y_{lj} \ln Z_{lj}.
$$

**Baseline Method:** We compare traditional GCN in [1] with the propagation model in Eq. (7). We also consider the graph-based methods reported in [1], i.e., label propagation [2] and graph embeddings (DeepWalk) [44]. Since we are interested in the performance of different propagation models, we also compare with the personalized propagation of neural predictions (PPNP) [19], [45], whose propagation model is

$$
h(X) = \alpha (I_N - (1 - \alpha)A)^{-1}XW.
$$

**Hyperparameter:** For fair comparison of different propagation models, we use the similar hyperparameters for the first-order TGCNs, GCN, and PPNP, with dropout rate $d = 0.5$, learning rate $r = 0.01$ and weight decay $w = 5 \times 10^{-4}$. For the two-layer TGCNs, we let the number of hidden units be $H = 40$. For the higher-order TGCNs, we use fewer hidden units to reduce the complexity.
5.2 Results in Citation Networks

We first use three citation networks datasets for validation, i.e., Cora-ML [46], [47], Citeseer [48], and Pubmed [49]. In these citation networks, papers are represented by nodes and the citation relationship are represented by edges. The data statistics are shown in Table 1.

**Overall Accuracy:** We first measure the performances of first-order TGCNs by comparing different propagation models to the baseline methods. Note that, the type-1 first-order TGCN degenerates into the traditional GCN if the diagonal parameter $\alpha = 0$. To explore the difference between GCN and type-1 first-order TGCN, we adjust the parameter $\alpha$ both manually and automatically. In addition, we also test on different representing matrix for TGCNs. In our experiment, we randomly initialize the parameters both manually and automatically. The results are reported in $^*$ for manual and $^\dagger$ for auto.

From the results of Table 2, we can see that $\tilde{D}^{-\frac{1}{2}} \tilde{A} \tilde{D}^{-\frac{1}{2}}$ provides a better representing matrix for both type-1 and type-2 first-order TGCNs. For the type-1 first-order TGCN, our manual adjustment results in higher accuracy than automatic adjustment. This indicates the network may converge to a local optimum if allowed to learn $\alpha$ by itself. Usually, the optimal $\alpha$ for type-1 TGCN would be in $[0.15, 0.35]$ as shown in Fig. 1 while the TGCN degrades to traditional GCN if $\alpha = 0$. Generally, the type-2 first-order TGCN has a clear improvement in accuracy for all datasets, whereas the type-1 first-order TGCN shows only marginal improvement with suitable choice of the diagonal parameters.

**Performances of Different TGCN Propagation Models:** We compare different propagation models with different orders of polynomials under the experiment setup as aforementioned. We run 100 Monte Carlo random initializations and report the average accuracy of each model in Table 2. The first-order TGCNs achieve superior overall accuracy than higher-order TGCNs, although the higher-order methods may sometimes be better for some datasets. Recall that the multi-layer first-order TGCN is a special case of single-layer higher-order polynomials as illustrated in Section 4.2.

The reason why higher-order case may perform slightly worse than the first-order network may be attributed to its large number of parameters that may leads to overfitting and more likely local convergence. Generally, results in Table 3 shows that $\tilde{D}^{-\frac{1}{2}} \tilde{A} \tilde{D}^{-\frac{1}{2}}$ is a more efficient representing matrix than the adjacency matrix for TGCNs. It would be interesting for future works to explore additional designs of suitable representing matrices to improve both GCN and TGCN.

**Depth and Polynomials Orders:** We also test the effects of different polynomial orders and layer numbers. The accuracy and training time (200 epochs) for different polynomial orders (Type-3 as an example) are shown in the first sub-figures of Fig. 2. We note that performance improvement stagnates beyond sufficient order. Since the results also illustrate growing training time for higher order polynomials,
### Table 3
Accuracy for Higher-Order Propagation Model (Percent)

| Num of Layers | Polynomial Order | TGCN Type | Cora | Citeseer |
|---------------|------------------|-----------|------|----------|
| 2-Layer       | $1^{st}$-order   | Type-1    | 81.4/77.2 | 70.1/67.8 |
| 1-Layer       | $1^{st}$-order   | Type-1    | 95.7/79.2 | 70.1/67.8 |
| 2-Layer       | $2^{nd}$-order   | Type-3    | 76.4/61.2 | 68.7/62.5 |
| 1-Layer       | $2^{nd}$-order   | Type-1    | 78.9/74.7 | 70.1/67.8 |
| 2-Layer       | $3^{rd}$-order   | Type-3    | 74.9/69.2 | 68.7/62.5 |
| 1-Layer       | $3^{rd}$-order   | Type-1    | 79.2/75.8 | 70.1/67.8 |

* For each method, we test both on representing matrices $D^{-1/2} \tilde{A} D^{-1/2}$ and $\tilde{A}$, whose results are reported as ($D^{-1/2} \tilde{A} D^{-1/2}$ / $\tilde{A}$) in the table.

![Figure 2: Results of Different Polynomial Orders and Network Depth.](image)

...it would be more efficient to limit the polynomial order between two to three. We also test the performance of first-order TGCNs (Type-2 as an example) with different layers in the last two figures in Fig. 2, which also shows that a 2-layer or 3-layer TGCN would suffice.

**Convergence:** We evaluate the convergence of different TGCN models in Fig. 3. Here, we report the accuracy of training data and validation data for the cora dataset. From the results, we can see that TGCN models can converge well in the citation network datasets.

**Training Time:** We compare the training efficiency for different methods based on the average training time for each epoch over 200 epochs in total. We use the same number of hidden units for multi-layer graph convolutional networks to be fair. From the results of Table 4, 2-layer TGCN is nearly 10% slower than traditional 2-layer GCN because of larger number of parameters and matrix operations. Moreover, larger layer depth and higher polynomial order also increase TGCN training time.

**Discussions:** In terms of formulation, type-1 TGCN is an extension of GCN, which allows flexible self-influence for each node. Type-2 TGCN is an extension of type-1 TGCN, where different self-influence parameters are assigned for different nodes. In real applications, such self-influence does exist and may be less obvious within the data. For example, in the citation networks, the work from highly-cited authors may have greater impact and trigger the appearance of a series of related new works on its own, which indicates larger self-influence as well as higher impact on other works. Type-2 TGCN allows different self-influence parameters to be learned while training, which may lead to better performance in the citation networks. Higher-order TGCNs, as discussed in Section 4.2, is different from multi-layer TGCN and different orders may lead to different performances. However, to mitigate complexity increase, lower-order TGCNs are more efficient in applications. With the steady improvement of practical computation speed, higher-order TGCNs may play increasingly important roles in future data analysis.

### 5.3 Results in Point Cloud Datasets
We next test the performance of TGCN in the point cloud segmentation. The goal of point cloud segmentation is to identify and cluster points in a point cloud that share similar features into their respective regions [49]. The segmentation problem can be formulated as a semi-supervised classification problem if the labels of several samples are known [51].

In this work, we use the ShapeNet datasets [52, 53] as examples. In this dataset, there are 16 object categories, each of which may contain 2-6 classes. To implement TGCN efficiently, we randomly pick 20 point cloud objects from each category, and randomly set 70% points as training data with labels and use the remaining as the test data for each point cloud. We use $k$-nearest neighbor method to construct an adjacency matrix $A$ with elements $a_{ij} \in \{0, 1\}$, such that $a_{ij} = 1$, 0 indicates the presence or the absence of connection between two nodes $i,j$, respectively. More specifically, we set $k = 20$ in graph construction for all point clouds.

**Results and Discussions:** We compare the type-1 and type-2 TGCNs with traditional GCN in all categories. The mean accuracy is shown in Table 5. From the results, we can see that each method exhibits unique advantages in different categories. However, TGCN has a higher accuracy in general and provides the best performance in more categories. Its better performance can be attributed to better graph construction.

### 6 Conclusion
This work explores the inherent connection between GSP convolutional spectrum wavelet and the GCN vertex propa-
We show that spectral wavelet-kernel can be approximated in vertex domain if it admits a polynomial approximation. In addition, our work presents alternative design of GCN layers based on the simple Taylor expansion (TGCN), which exhibits computation efficiency and outperforms the state-of-art GCN-like methods. We further present insights on the choice of the representing matrix and the propagation model for TGCN layer design.

**Future Works:** In existing works, the design of GCN centers on performance while ignoring the correspondence of the original graph spectrum convolution. Evaluating GCN from the GSP-perspective may provide better insights for layer design and basis for performance analysis in the future. It is equally important for future works to explore the choice of representing matrix and propagation models to future enhance the performance of GCNs.

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