On some $H$-Galois objects which are distinguished by their polynomial $H$-identities

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Abstract
When $k$ is an algebraically closed field of characteristic 0 and $H$ is a non-semisimple monomial Hopf algebra, we show that all Galois objects over $H$ are determined up to $H$-comodule algebra isomorphism by their polynomial $H$-identities, extending a previous result in Kassel [8].

Keywords: Hopf algebra, comodule algebra, Galois object, polynomial identity

Introduction
This paper contributes to the well-known question concerning whether the set of polynomials identities distinguishes PI-algebras (associative unital algebras over a field $k$ satisfying a nontrivial polynomial identity) up to isomorphism. For instance, it follows from the celebrated Amitsur-Levitsky theorem that the standard polynomial of degree $2n$ distinguishes the finite-dimensional central simple associative algebras over an algebraically closed field $k$ up to isomorphism. When $k$ is not algebraically closed, the situation can be quite different: the quaternions $\mathbb{H}$ are a central simple algebra of dimension 4 over $\mathbb{R}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C} \otimes_{\mathbb{R}} M_2(\mathbb{R})$, hence $\mathbb{H}$ and $M_2(\mathbb{R})$ have the same set of polynomial identities, but they are obviously not isomorphic as algebras.

When $k$ is algebraically closed and the $k$-algebras are “simple”, various results have settled the isomorphism question in the affirmative (here the meaning of simple depends on the full structure of the algebra). For example,

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Kushkulei and Razmyslov [11] on simple Lie algebras, Drensky and Racine [5] on simple Jordan algebras, Koshlukov and Zaicev [9] on simple associative algebras graded by an abelian group, and Aljadeff and Haile [1] extending this result to any group. Also, Shestakov and Zaicev [14] on arbitrary finite dimensional simple algebras and, more recently, Bahturin and Yasumura [2] on arbitrary (semigroup) simple graded algebras.

Extending the notion of graded polynomial identity, Kassel [8] introduced a notion of polynomial $H$-identity, $H$ being an arbitrary Hopf algebra, and considered the $H$-comodule algebras, namely $k$-algebras which are also right $H$-comodules and the right coaction is compatible with the multiplication. When $H = kG$ is the group Hopf algebra of a group $G$, the $H$-comodule algebras are essentially the same as the $G$-graded associative algebras and the polynomial $H$-identities become the usual $G$-graded polynomial identities. More specifically, Kassel studied certain $H$-comodule algebras which are cleft extensions of the ground field $k$, namely the $H$-Galois objects, when $H$ is a (generalized) Taft algebra or the Hopf algebra $E(n)$, and showed that these objects are distinguished by their polynomial $H$-identities.

Following Kassel [8], here we consider the case when $H$ is a non-semisimple monomial Hopf algebra and prove the $H$-Galois objects are distinguished by their sets of polynomial $H$-identities.

The non-semisimple monomial Hopf algebras were classified by Chen et al. [4] and include the Taft algebras. This classification associates to each such an algebra the notion of “group datum” which will be described below. In turn, Bichon [3] further classified the various group data into six different types and used this to determine, up to isomorphism, the $H$-Galois objects for the monomial Hopf algebras according to their associated group data. Our work draws from this classification.

As it turns out, the “generalized” Taft algebras are precisely the type I in Bichon’s classification of the non-semisimple monomial Hopf algebras, and the isomorphism question for the corresponding $H$-Galois objects has already been settled in [8]. Here we set out to investigate types II through VI.

Throughout this paper, $k$ denotes an algebraically closed field of characteristic 0.

1 Preliminaries

We shall follow closely the basic notations, definitions and results on Hopf algebras which are found for instance in [12] and [6]. In particular, we assume that the reader is familiar with concepts such as Hopf algebras, $H$-comodule algebras, cleft extensions, and the Heyneman-Sweedler-type notation for the
comultiplication $\Delta: H \to H \otimes H$ and the coaction $\delta: M \to M \otimes H$ for a (right) $H$-comodule $M$. As usual, the counit is $\varepsilon: H \to k$. Concerning the polynomial $H$-identities we keep the same notations and definitions from [8]. These will be briefly recalled in this section.

1.1 $H$-comodule algebras and $H$-Galois objects

We recall the definition of the twisted Hopf algebra $\alpha H$. If $H$ is a Hopf algebra and $\alpha \in \mathbb{Z}^2(H, k^\times)$ is a normalized convolution invertible (right) 2-cocycle then the algebra $\alpha H$ is defined as an algebra $V_H$ with the same vector space structure of $H$ and whose multiplication given by

$$v_x v_y = \alpha(x_1, y_1)v_{x_2 y_2}$$

where the $v$-symbols are the images of the elements of $H$ under the linear isomorphism $x \mapsto v_x$ onto $V_H$, and $x_1 \otimes x_2$ is the Heyneman-Sweedler notation for $\Delta(x) = \sum_i x_{i1} \otimes x_{i2}$. The cocycle condition

$$\alpha(x_1, y_1)\alpha(x_2 y_2, z) = \alpha(y_1, z_1)\alpha(x, y_2 z_2),$$

for all $x, y, z \in H$, is responsible for the associativity of the multiplication in $\alpha H$, and the normalization

$$\alpha(x, 1) = \alpha(1, x) = \varepsilon(x),$$

for all $x \in H$, implies that $1_{\alpha H} = u_1$. Moreover, $\alpha H$ is an $H$-comodule algebra with coaction $\delta: \alpha H \to \alpha H \otimes H$ given by

$$\delta(v_x) = v_{x_1} \otimes x_2.$$

Next we recall that an $H$-comodule algebra $A$ (with coaction $\rho: A \to A \otimes H$) is called a (right) $H$-Galois object if the subalgebra of coinvariants $A^{coH} = \{a \in A \mid \rho(a) = a \otimes 1_H\}$ is isomorphic to $k$ and the map

$$\beta: A \otimes A \to A \otimes H$$

$$a \otimes b \mapsto (a \otimes 1_H)\rho(b)$$

is a linear isomorphism.

The next two propositions show that the $H$-Galois objects over finite dimensional Hopf algebras are equivalent to the twisted $H$-comodule algebras $\alpha H$ for some 2-cocycle $\alpha$. The proofs are based on the discussions found in chapters 7 and 8 of [12].

**Proposition 1.1.** Let $\alpha$ be a normalized convolution invertible 2-cocycle. Then $\alpha H$ is an $H$-Galois object.
Proof. In fact, \( x \in H^{coH} \) if, and only if, \( \Delta(x) = x \otimes 1_H \), hence
\[
x = xS(1_H) = x_1S(x_2) = \varepsilon(x)1_H.
\]
It follows that \( u_x \in (\alpha H)^{coH} \) if, and only if, \( u_x = u_{\varepsilon(x)1_H} = \varepsilon(x)u_{1_H} \), which shows that \( (\alpha H)^{coH} \cong k \).

Let \( \varphi : H \to \alpha H \) be the linear map given by
\[
\varphi(x) = \sigma^{-1}(S(x_2), x_3)v_S(x_1).
\]
It is straightforward to check that \( \varphi \) is the convolution inverse to the isomorphism \( x \mapsto v_x \), that is \( v_x \varphi(x_2) = \varepsilon(x)v_{1_H} \). Using this map, define \( \gamma : \alpha H \otimes H \to \alpha H \otimes \alpha H \) by
\[
\gamma(v_x \otimes y) = v_x \varphi(y_1) \otimes v_{y_2}.
\]
A direct calculation shows that \( \beta \) and \( \gamma \) are mutual inverses:
\[
\begin{align*}
\beta \gamma(v_x \otimes y) &= \beta(v_x \varphi(y_1) \otimes y_2) \\
&= v_x \varphi(y_1)v_{y_2} \otimes y_3 \\
&= v_x \varepsilon(y_1)v_{1_H} \otimes y_2 = v_x \otimes y,
\end{align*}
\]
and
\[
\begin{align*}
\gamma \beta(v_x \otimes y) &= \gamma(v_x v_{y_1} \otimes y_2) \\
&= v_x u_{y_1} \varphi(y_2) \otimes v_{y_3} \\
&= v_x \varepsilon(y_1)v_{1_H} \otimes v_{y_2} = v_x \otimes v_y,
\end{align*}
\]
for all \( x, y \in H \). This completes the proof.

Proposition 1.2. Let \( H \) be a finite dimensional Hopf algebra and \( A \) an \( H \)-Galois object. Then there is a normalized convolution invertible 2-cocycle \( \alpha \) such that \( A \) and \( \alpha^H \) are isomorphic as \( H \)-comodule algebras.

Proof. In \cite{[10]} Proposition 2] it is proved that \( A \) is isomorphic to \( H^* \) as \( H^* \)-modules. In particular \( \dim_k A \) is finite, hence from \cite{[13]} Corollary 3.1.6] \( A \) is a rational \( H^* \)-module. It follows from \cite{[13]} Proposition 3.2.2(b)] that \( H \) is isomorphic to \( A \) as \( H \)-comodules. Let \( \phi : H \to A \) be this isomorphism. From the proof in \cite{[12]} Theorem 8.2.4], \( \phi \) has a convolution inverse \( \phi^{-1} : H \to A \). As a consequence, we can always assume that \( \phi(1_H) = 1_A \), for otherwhise we can simply replace \( \phi \) with \( \phi' = \phi(1_H)^{-1}\phi \).

Using these maps, we define
\[
\alpha(x, y) = \phi(x_1)\phi(y_1)\phi^{-1}(x_2y_2).
\]
To see that $\alpha$ is a cocycle, we first check that it’s image lies in $k1_A$. Since $A^{coh} \cong k$, it is enough to show that $\delta(\alpha(x, y)) = \alpha(x, y) \otimes 1_H$ (\delta being the coaction as in [13 Corollary 3.1.6]). Following the proof of [12 Proposition 7.2.3], observing that $\delta\phi = (\phi \otimes id_H)\Delta$ (since $\phi$ is $H$-comodule map), for all $x, y \in H$, we calculate

$$\delta(\alpha(x, y)) = \delta\phi(x_1)\delta\phi(y_1)\delta^{-1}(x_2y_2)$$

$$= (\phi(x_{11}) \otimes x_{12})(\phi(y_{11}) \otimes y_{12}) (\phi^{-1}(x_{22}y_{22}) \otimes S(x_{21}y_{21}))$$

$$= \phi(x_1)\phi(y_1)\phi^{-1}(x_4y_4) \otimes x_{2y_2}S(y_3)S(x_3)$$

$$= \phi(x_1)\phi(y_1)\phi^{-1}(x_{2y_2}) \otimes 1_H = \alpha(x, y) \otimes 1_H,$$

as required. Next, we have

$$\alpha(x, 1_H) = \phi(x_1)\phi(1_H)\phi^{-1}(x_21_H) = \phi(x_1)\phi^{-1}(x_2) = \varepsilon(x) = \alpha(1_H, x),$$

for all $\alpha \in H$, so $\alpha$ is normalized. It remains to check the cocycle condition:

$$\alpha(x_1, y_1)\alpha(x_2y_2, z) = \phi(x_1)\phi(y_1)\varepsilon(x_2y_2)\phi(z_1)\phi^{-1}(x_3z_3)$$

$$= \phi(x_1)\phi(y_1)\phi(z_1)\phi^{-1}(x_{2y_2}z_2)$$

$$= \alpha(y_1, z_1)\alpha(x, y_2z_2),$$

for $x, y, z \in H$.

To complete the proof, we consider the algebra $^\alpha H$ and the map $F : ^\alpha H \rightarrow A$ given by $F(v_x) = \phi(x)$. It is clear that $F$ is an $H$-comodule isomorphism, so it is enough to check that $F$ is an algebra map:

$$F(v_xv_y) = \alpha(x_1, y_1)\phi(x_2y_2)$$

$$= \phi(x_1)\phi(y_1)\phi^{-1}(x_{2y_2})\phi(x_3y_3)$$

$$= \phi(x_1)\phi(y_1)\varepsilon(x_2)\varepsilon(y_2)$$

$$= F(v_x)F(v_y),$$

and

$$F(v_{1_H}) = \phi(1_H) = 1_A,$$

as required. Therefore, $A$ is isomorphic to $^\alpha H$ as $H$-comodule algebras. □

### 1.2 Polynomial $H$-identities

Following [8], for each $i = 1, 2, \ldots$ let $X_i^H$ be a copy of a Hopf algebra $H$ and denote by $X_i^x$ ($x \in H$) the elements of $X_i^H$ (called $X$-symbols). Define
\[ X_H = \bigoplus_{i \geq 1} X_i^H \] and take the tensor algebra
\[ T = T(X_H) = T \left( \bigoplus_{i \geq 1} X_i^H \right). \]

\( T \) is an \( H \)-comodule algebra with coaction given by
\[ \delta(X_i^x) = X_i^{x_1} \otimes x_2. \]

We shall need a symmetric version \( S \) of \( T \) which is defined naturally as the symmetric algebra of \( X_H \). However, to avoid confusion, when referring to \( S \) we replace the \( X \)-symbols \( X_i^x \) by the \( t \)-symbols \( t_i^x \).

Given a linear basis \( \{x_1, \ldots, x_r\} \) for \( H \), it is easy to see that the tensor algebra \( T \) is isomorphic to free associative unital algebra defined by the indeterminates \( \bigcup_{i \geq 1} \{X_i^x \mid 1 \leq j \leq r\} \), while the symmetric algebra \( S \) is isomorphic to the algebra of commutative polynomials in the indeterminates \( \bigcup_{i \geq 1} \{t_i^x \mid 1 \leq j \leq r\} \). In view of this remark it should be clear that the following definition generalizes both the ordinary (case \( H = k \)) and the \( G \)-graded (case \( H = kG \)) polynomial identities:

**Definition 1.3.** [8, Definition 2.1] An element \( P \in T \) is a polynomial \( H \)-identity for the \( H \)-comodule algebra \( A \) if \( \mu(P) = 0 \) for all \( H \)-comodule algebra maps \( \mu: T \rightarrow A \).

We denote the set of the polynomial \( H \)-identities for an algebra \( A \) by \( I_H(A) \). Then
\[ I_H(A) = \bigcap_{\mu: T \rightarrow A} \ker \mu. \]

Central in PI-theory is the study of T-ideals, which can be very difficult. For \( A = \alpha H \), it happens that \( I_H(A) \) can be characterized as the kernel of a single \( H \)-comodule algebra map, to be described bellow.

Consider the algebra \( S \otimes \alpha H \) generated by the simple tensors \( t_i^x \otimes v_y \) (\( x, y \in H \)), hereby denoted simply by \( t_i^x v_y \). With the coaction
\[ \delta(t_i^x v_y) = t_i^x v_{y_1} \otimes y_2, \]
\( S \otimes \alpha H \) becomes an \( H \)-comodule algebra. Also consider the map
\[ \mu_\alpha: T \rightarrow S \otimes \alpha H, \]
defined by \( \mu_\alpha(X_i^x) = \sum t_i^{x_1} v_{x_2} \). A direct calculation shows that this is an \( H \)-comodule algebra map, and in [8, Proposition 2.7] it is shown that every
$H$-comodule algebra map $\mu: T \to {}^aH$ factors through $\mu_\alpha$, in the sense that given $\mu$ there is a unique algebra map $\xi: S \to k$ such that

$$\mu = (\xi \otimes \text{id}_H) \circ \mu_\alpha.$$  

The usefulness of $\mu_\alpha$ becomes clear in the following result.

**Theorem 1.4.** [8, Theorem 2.6] An element $P \in T$ is a polynomial $H$-identity for $^aH$ if and only if $\mu_\alpha(P) = 0$. Equivalently, $I_H(^aH) = \ker \mu_\alpha$.

## 2 Non-semisimple Monomial Hopf Algebras and their Galois objects

In this section, we recall the classification [4] of the non-semisimple monomial Hopf Algebras and the classification [3] of their Galois objects.

Following Chen, Huang, Ye and Zhang [4] we define the group datum:

**Definition 2.1.** A group datum is a quadruplet $G = (G, g, \chi, \mu)$, where:

(i) $G$ is a finite group, with an element $g$ in its center;

(ii) $\chi: G \to k^\times$ is a one-dimensional representation with $\chi(g) \neq 1$;

(iii) $\mu \in k$ is such that $\mu = 0$ if $o(g) = o(\chi(g))$ and, if $\mu \neq 0$, then $\chi^{o(\chi(g))} = 1$.

For each group datum $G = (G, g, \chi, \mu)$, they associate an associative unital algebra $A(G)$ with generators $x \in G$ and $y$ satisfying the relations

$$yx = \chi(x)xy$$

and

$$y^d = \mu(1 - g^d),$$

for all $x \in G$, where $d = o(\chi(g))$. It follows from the Diamond’s Lemma, that

$$\{xy^i | x \in G, 0 \leq i \leq d - 1\}$$

forms a basis for $A(G)$, therefore its dimension is $|G|d$. One can endow this algebra with a Hopf algebra structure with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$ by

$$\Delta(y) = 1 \otimes y + y \otimes g, \quad \varepsilon(y) = 0, \quad S(y) = -yg^{-1},$$

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1, \quad S(x) = x^{-1}, \quad \forall x \in G.$$  

Next, they show that this class of Hopf algebras is precisely that of the non-semisimple monomial Hopf algebras (the precise definition of a monomial Hopf algebra is given in [4]). Observe that $A(G)$ is indeed non-semisimple, since it has finite dimension and $S^2 \neq 1$.

For the $A(G)$-Galois objects, following Bichon [3], we define $n = o(g)$, $d = o(\chi(g))$, and $q = \chi(g)$ a primitive $d$-th root of the unity, and further divide the various group data $G = (G, g, \chi, \mu)$ into 6 different types:
• Type I: \( \mu = 0, \ d = n \) and \( \chi^d = 1 \);
• Type II: \( \mu = 0, \ d = n \) and \( \chi^d \neq 1 \);
• Type III: \( \mu = 0, \ d < n \) and \( \chi^d = 1 \);
• Type IV: \( \mu = 0, \ d < n \), \( \chi^d \neq 1 \) and no \( \sigma \in \mathbb{Z}_2(G, k) \times (G, k) \), with \( \sigma(g^d, x) = \chi^d(x) \sigma(x, g^d) \), exists for all \( x \in G \);
• Type V: \( \mu = 0, \ d < n \), \( \chi^d \neq 1 \) and there exists a \( \sigma \in \mathbb{Z}_2(G, k) \), with \( \sigma(g^d, x) = \chi^d(x) \sigma(x, g^d) \), for all \( x \in G \);
• Type VI: \( \mu \neq 0 \) (and hence \( d < n \) and \( \chi^d = 1 \)).

Here we observe that the Taft algebra \( H_n^2 \) occurs as a particular case of the type I \( A(G) \) when \( g \) is a generator of \( G = \mathbb{Z}/n\mathbb{Z} \).

To each \( A(G) \), 2-cocycle \( \sigma \), and \( a \in k \), Bichon constructs an associative unital algebra \( A_{\sigma,a}(G) \) as follows:

**Definition 2.2.** [3, Definition 2.2] Let \( \sigma \in \mathbb{Z}_2^2(G, k^*) \) and \( a \in k \). We define the algebra \( A_{\sigma,a}(G) \) to be the algebra with generators \( u_y, \{ u_x \}_{x \in G} \) and subject to the relations:

\[
u_x u_{x'} = \sigma(x, x') u_{xx'}, \ u_1 = 1, \ u_y u_x = \chi(x) u_x u_y \text{ and } u_y^d = a u_{g^d},\]

for all \( x, x' \in G \).

The following proposition shows that \( A_{\sigma,a}(G) \) is always an \( A(G) \)-comodule algebra, and gives a necessary and sufficient condition for this algebra to be an \( A(G) \)-Galois object.

**Proposition 2.3.** [3, Proposition 2.3] The algebra \( A_{\sigma,a}(G) \) has a right \( A(G) \)-comodule algebra structure with coaction \( \rho: A_{\sigma,a}(G) \rightarrow A_{\sigma,a}(G) \otimes A(G) \) defined by \( \rho(u_y) = u_1 \otimes y + u_y \otimes g \) and \( \rho(u_x) = u_x \otimes x \) for all \( x \in G \). Moreover, \( A_{\sigma,a}(G) \) is an \( A(G) \)-Galois object if and only if

\[
a \sigma(g^d, x) = a \chi(x)^d \sigma(x, g^d), \quad (2.1)
\]

for all \( x \in G \). In this case, the set \( \{ u_x u_y^i \mid x \in G \text{ and } 0 \leq i \leq d - 1 \} \) is a linear basis of \( A_{\sigma,a}(G) \) and the map \( \Psi: A(G) \rightarrow A_{\sigma,a}(G), \ xy^i \mapsto u_x u_y^i \) is an isomorphism of \( A(G) \)-comodules.

Next, Bichon shows that all \( A(G) \)-Galois objects are of the form \( A_{\sigma,a}(G) \) and gives a necessary and sufficient condition for two such objects to be isomorphic as \( A(G) \)-comodule algebras.
Proposition 2.4. [3, Proposition 2.9] Let $B$ be an $A(G)$-Galois object. Then there exists $\sigma \in Z^2(G, k^\times)$ and $a \in k$ such that $B \cong A_{\sigma,a}(G)$ as $A(G)$-comodule algebras.

Proposition 2.5. [3, Proposition 2.10] Let $\sigma, \tau \in Z^2(G, k^\times)$ and $a, b \in k$ such that $A_{\sigma,a}(G)$ and $A_{\tau,b}(G)$ are $A(G)$-Galois objects. Then the $A(G)$-comodule algebras $A_{\sigma,a}(G)$ and $A_{\tau,b}(G)$ are isomorphic if and only if exists $\nu: G \to k^\times$ with $\nu(1) = 1$ such that

$$\sigma = \partial(\nu)\tau \text{ and } b = a\nu(g^d).$$

It turns out that here we shall not need all of the details of Bichon’s classification, but only a few facts that we have tailored to our needs in the following propositions.

Proposition 2.6. Let $G$ be a group datum of types II or IV. Then any $A(G)$-Galois object is isomorphic to $A_{\sigma,0}(G)$ for some $\sigma \in Z^2(G, k^\times)$.

Proof. Let $B$ be an arbitrary $A(G)$-Galois object. By Proposition 2.4, $B \cong A_{\sigma,0}(G)$ for some $\sigma \in Z^2(G, k^\times)$ and $a \in k$.

In type II, $d = n$ and $x^d \neq 1$, while in type IV there is no $\sigma \in Z^2(G, k^\times)$ satisfying $\sigma(g^d, x) = \chi^d(x)\sigma(x, g^d)$, for all $x \in G$. In both cases, condition (2.1) is satisfied only if $a = 0$, hence $B \cong A_{\sigma,0}$. ■

Proposition 2.7. Let $G$ be a group datum of type III, V or VI. Then any $A(G)$-Galois object is isomorphic to either $A_{\sigma,0}(G)$ or $A_{\sigma,1}(G)$ for some $\sigma \in Z^2(G, k^\times)$.

Proof. Let $B$ be an arbitrary $A(G)$-Galois object. By Proposition 2.4, $B \cong A_{\tau,0}(G)$ for some $\tau \in Z^2(G, k^\times)$, $a \in k$ and we may assume that $a \neq 0$ (A_{\tau,0} is always a Galois object).

Let $\nu: G \to k^\times$ be such that $\nu(1) = 1$ and $\nu(g^d) = a$ (notice that $g^d \neq 1$). For $\sigma = \partial(\nu)\tau \in Z^2(G, k^\times)$,

$$\sigma(g^d, x) = \partial(\nu)\tau(g^d, x) = \partial(\nu)\chi(x)^d\tau(x, g^d) = \chi(x)^d\sigma(x, g^d),$$

for all $x \in G$, hence from Proposition 2.3, $A_{\tau,0}(G)$ is an $A(G)$-Galois object. It follows from Proposition 2.5 that $A_{\tau,0}(G) \cong A_{\sigma,1}(G)$. ■

3 The Main Theorem

In this section, $H$ always denotes the non-semisimple monomial Hopf algebra $A(G)$, for some group datum $G$, and our main objective is to prove the following result.
Theorem 3.1 (Main). Suppose that $A$ and $B$ are two $H$-Galois objects. If

$$I_H(A) = I_H(B),$$

then $A$ and $B$ are isomorphic as $H$-comodule algebras.

The first step is to check that the $H$-Galois objects $A_{\sigma,a}$ are indeed polynomial $H$-identity algebras. Here it is useful the fact that every $H$-Galois object is isomorphic to $^\alpha H$ for some normalized convolution invertible 2-cocycle $\alpha$ (see Proposition 1.2). It is easy to check that this isomorphism is the $H$-comodule algebra map $F: A_{\sigma,a} \to ^\alpha H$ given by $F(u_x u_y^i) = v_{xy^i}$. In particular, since

$$v_x v_{x'} = \sigma(x,x') v_{xx'}, \quad v_1 = 1, \quad v_y v_x = \chi(x) v_x v_y \text{ and } v_y^d = a v_y^d,$$

for all $x, x' \in G$, any valid relation in $A_{\sigma,a}(G)$ can be transferred to a same-format relation in $^\alpha H$.

To simplify the notation we set:

$$X = X_0^y, \quad t_y = t_1^y,$$
$$Y = X_0^y, \quad t_y = t_1^y,$$
$$E = X_1^1, \quad t_1 = t_1^1.$$

For use below, we first record an easy lemma.

Lemma 3.2. Fix $\zeta$ a primitive $m$-th root of unit and let $z$ and $w$ two variables such that $zw = \zeta wz$. Then

$$(z + w)^m = z^m + w^m.$$  

Proof. From [7, Proposition IV.2.2], we have the $\zeta$-analogue of the Binomial Theorem,

$$(z + w)^m = \sum_{i=0}^{m} \binom{m}{i}_\zeta z^i w^{m-i},$$

where

$$\binom{m}{i}_\zeta = \frac{(\zeta^m - 1) \cdots (\zeta^{m-i+1} - 1)}{(\zeta - 1) \cdots (\zeta - 1)}$$

is the $\zeta$-binomial. Obviously, $\binom{m}{0}_\zeta = \binom{m}{m}_\zeta = 1$ and, whenever $0 < i < m$, the factor $\zeta^m - 1 = 0$ remains in the numerator of $\binom{m}{i}_\zeta$ while no factor of the denominator is null. Therefore, $\binom{m}{i}_\zeta = 0$ in this case.  

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Proposition 3.3. Let $H$ be a non-semisimple monomial Hopf algebra and $A_{\sigma,a}(G)$ an $H$-Galois object. Then
\[ \mathcal{P} = \sum_{s \in S_3} \text{sgn}(s)E_{s(1)}X_{s(2)}Y_{s(3)}^{d} \]
is a polynomial $H$-identity for $A_{\sigma,a}(G)$.

Proof. Using the notation above, it is enough to calculate $\mu_\alpha(\mathcal{P})$ observing that
\[ \begin{align*}
\mu_\alpha(E) &= t_1v_1, \\
\mu_\alpha(X) &= t_yv_y, \\
\mu_\alpha(Y) &= t_1v_y + t_yv_g,
\end{align*} \]
and, since $(t_1v_y)(t_yv_g) = q(t_yv_g)(t_1v_y)$, by Lemma 3.2,
\[ \mu_\alpha(Y^d) = t_1^{d}v_y + t_y^{d}v_g. \]
Now, $\mathcal{P} = EXY^d - EY^dX + XY^dE - XY^dE + Y^dEX - Y^dXE$,
\[ \begin{align*}
\mu_\alpha(EXY^d) &= t_1v_1t_yv_y(t_1^{d}v_y + t_y^{d}v_g) \\
&= t_1^{d+1}t_yv_yv_g + t_1t_y^{d}v_gv^{d+1}_y,
\end{align*} \]
and the computation of $\mu_\alpha$ on the remaining terms of $\mathcal{P}$ is easy and left to the reader. After a few cancellations,
\[ \mu_\alpha(\mathcal{P}) = t_1^{d+1}t_yv_yv_g - v_y^{d}v_y = (1 - q^d)t_1^{d+1}t_yv_gv^{d} = 0. \]

The next theorem, a particular case of a result stated only for the type I monomial Hopf algebras in [8, Theorem 3.4], remains valid for types II-VI with virtually the same proof.

Theorem 3.4. Let $A_{\sigma,a}(G)$ and $A_{\tau,a}(G)$ be $H$-Galois objects for some $a \in k$ and $\sigma, \tau \in Z^2(G, k^\times)$. Then
\[ I_H(A_{\sigma,a}(G)) = I_H(A_{\tau,a}(G)) \]
if, and only if, $A_{\sigma,a}(G)$ and $A_{\tau,a}(G)$ are isomorphic as $H$-comodule algebras.
Proof. Denote by $k^\sigma G$ the subalgebra of $A_{\sigma,a}(G)$ generated the symbols $u_x$ ($x \in G$) and relations $u_x u_{x'} = \sigma(x, x') u_{x x'}$ and $u_1 = 1$, for all $x, x' \in G$. If we prove that $I_{kG}(k^\sigma G) = I_{kG}(k^\tau G)$ then by [1, Proposition 2.11] the cocycles $\sigma$ and $\tau$ are cohomologous. Therefore, by Proposition 2.5 $A_{\sigma,a}(G)$ and $A_{\tau,a}(G)$ are isomorphic as $H$-comodule algebras. It remains to prove that $I_{kG}(k^\sigma G) = I_{kG}(k^\tau G)$.

Consider the diagram:

$$
0 \longrightarrow I_{kG}(k^\sigma G) \longrightarrow T(X_{kG}) \longrightarrow S(t_{kG}) \otimes k^\sigma G \bigg|_{\mu_a}
$$

$$
0 \longrightarrow I_H(A_{\sigma,a}(G)) \longrightarrow T(X_H) \longrightarrow S(t_H) \otimes A_{\sigma,a}(G) \bigg|_{\mu_a}
$$

The vertical map $\iota_T$ is induced by the inclusion $kG \rightarrow H$. It is injective. The map $\iota_S$ is induced by the previous natural inclusion and the comodule algebra inclusion $k^\sigma G \subseteq A_{\sigma,a}(G)$. It sends its generators $t_x^\sigma u_x$ of $S(t_{kG}) \otimes k^\sigma G$ to itself viewed as an element of $S(t_H) \otimes A_{\sigma,a}(G)$. Note that the horizontal sequences are exacts by Theorem 1.4. It is straightforward to check that the diagram is commutative. Therefore the restriction $\iota$ of $\iota_T$ to $I_{kG}(k^\sigma G)$ send the latter to $I_H(A_{\sigma,a}(G))$ and is injective. By this injectivity we have

$$
I_{kG}(k^\sigma G) = T(X_{kG}) \cap I_H(A_{\sigma,a}(G)).
$$

Then, as $I_H(A_{\sigma,a}(G)) = I_H(A_{\tau,a}(G))$, we have that $I_{kG}(k^\sigma G) = I_{kG}(k^\tau G)$. This completes the proof. \hfill \blacksquare

For types III, V and VI we have shown that only isomorphism classes $[A_{\sigma,0}(G)]$ and $[A_{\tau,1}(G)]$ can occur (Proposition 2.7). Next, we show that these classes are disjoint. First we need a Lemma.

Lemma 3.5. Let $A_{\sigma,a}(G)$ be an $H$-Galois object. Then

$$
Q = (YX - qXY)^d - (1 - q)^d X^d Y^d
$$

is a polynomial $H$-identity for $A_{\sigma,a}(G)$ if, and only if, $a = 0$.

Proof. By Theorem 1.4 it is enough to check that $\mu_a(Q) = 0$ if, and only if, $a = 0$. Proceeding as in the proof of Proposition 3.3 we have

$$
\mu_a(YX - qXY) = (1 - q)t_y t_g v_g^2,
$$

$$
\mu_a(E^d) = t_1^d u_1,
$$

$$
\mu_a(X^d) = t_g^d v_g^d.
$$
\[\mu_\alpha((YX - qXY)^d) = (1 - q)^d t_g^d v_g^d.\]

Therefore,

\[\mu_\alpha(Q) = (1 - q)^d t_g^d v_g^d - (1 - q)^d t_g^d v_g^d(t_1^d v_g^d + t_2^d v_g^d)\]

\[= -a(1 - q)^d t_1^d v_g^d u_g^d.\]

Since \((1 - q)^d t_1^d v_g^d u_g^d \neq 0\), the result follows.

**Proposition 3.6.** Let \(\mathbb{G}\) be a group datum of type III, V or VI, \(A_{\sigma,a}(\mathbb{G})\) and \(A_{\tau,b}(\mathbb{G})\) be \(H\)-Galois objects. If \(A_{\sigma,a}(\mathbb{G}) \cong A_{\tau,b}(\mathbb{G})\) then \(a = b\).

**Proof.** Suppose that \(a \neq b\). By Proposition 2.7 we can assume that \(a = 0\) and \(b = 1\). Since \(A_{\sigma,a}(\mathbb{G}) \cong A_{\tau,b}(\mathbb{G})\) then \(I_H(A_{\sigma,a}(\mathbb{G})) = I_H(A_{\tau,b}(\mathbb{G}))\) hence, it follows from Lemma 3.5 that \(Q\) is a polynomial \(H\)-identity for \(A_{\sigma,0}(\mathbb{G})\) but it is not a polynomial \(H\)-identity for \(A_{\tau,1}(\mathbb{G})\), which is absurd.

With this setup, the proof of Theorem 3.1 is straightforward.

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