Nilpotent residual of fixed points

Emerson de Melo, Aline de Souza Lima, and Pavel Shumyatsky

Abstract. Let $q$ be a prime and $A$ a finite $q$-group of exponent $q$ acting by automorphisms on a finite $q'$-group $G$. Assume that $A$ has order at least $q^3$. We show that if $\gamma_\infty(C_G(a))$ has order at most $m$ for any $a \in A^\#$, then the order of $\gamma_\infty(G)$ is bounded solely in terms of $m$ and $q$. If $\gamma_\infty(C_G(a))$ has rank at most $r$ for any $a \in A^\#$, then the rank of $\gamma_\infty(G)$ is bounded solely in terms of $r$ and $q$.

1. Introduction

Suppose that a finite group $A$ acts by automorphisms on a finite group $G$. The action is coprime if the groups $A$ and $G$ have coprime orders. We denote by $C_G(A)$ the set

$$C_G(A) = \{ g \in G \mid g^a = g \text{ for all } a \in A \},$$

the centralizer of $A$ in $G$ (the fixed-point subgroup). In what follows we denote by $A^\#$ the set of nontrivial elements of $A$. It has been known that centralizers of coprime automorphisms have strong influence on the structure of $G$.

Ward showed that if $A$ is an elementary abelian $q$-group of rank at least 3 and if $C_G(a)$ is nilpotent for any $a \in A^\#$, then the group $G$ is nilpotent [20]. Later the third author showed that if, under these hypotheses, $C_G(a)$ is nilpotent of class at most $c$ for any $a \in A^\#$, then the group $G$ is nilpotent with $(c, q)$-bounded nilpotency class [17].

Throughout the paper we use the expression “$(a, b, \ldots)$-bounded” to abbreviate “bounded from above in terms of $a, b, \ldots$ only”. In the recent article [3] the above result was extended to the case where $A$ is not necessarily abelian. Namely, it was shown that if $A$ is a finite group

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of prime exponent $q$ and order at least $q^3$ acting on a finite $q'$-group $G$ in such a manner that $C_G(a)$ is nilpotent of class at most $c$ for any $a \in A^\#$, then $G$ is nilpotent with class bounded solely in terms of $c$ and $q$. Many other results illustrating the influence of centralizers of automorphisms on the structure of $G$ can be found in [9].

In the present article we study finite groups $G$ acted on by a (possibly non-abelian) group $A$ of prime exponent $q$ and order at least $q^3$ such that $C_G(a)$ has “small” nilpotent residual for every $a \in A^\#$. Recall that the nilpotent residual $\gamma_\infty(K)$ of a group $K$ is the last term of the lower central series of $K$. It can also be defined as the intersection of all normal subgroups of $K$ whose quotients are nilpotent. The order of a finite group $K$ is denoted by $|K|$. The rank of (a finite group) $K$ is the minimal number $r$, denoted by $r(K)$, such that every subgroup of $K$ can be generated by at most $r$ elements. Guralnick [6] and, independently, Lucchini [15] proved that $r(K) \leq 1 + \max_p \{r(P) \mid P \text{ a Sylow } p\text{-subgroup of } K\}$.

We obtain the following results.

**Theorem 1.1.** Let $q$ be a prime and $A$ a finite $q$-group of exponent $q$ acting by automorphisms on a finite $q'$-group $G$. Assume that $A$ has order at least $q^3$ and $|\gamma_\infty(C_G(a))| \leq m$ for any $a \in A^\#$. Then $\gamma_\infty(G)$ has $(m, q)$-bounded order.

**Theorem 1.2.** Let $q$ be a prime and $A$ a finite $q$-group of exponent $q$ acting by automorphisms on a finite $q'$-group $G$. Assume that $A$ has order at least $q^3$ and $r(\gamma_\infty(C_G(a))) \leq r$ for any $a \in A^\#$. Then $r(\gamma_\infty(G))$ is $(r, q)$-bounded.

Unsurprisingly, these results depend on the classification of finite simple groups and it seems unlikely that one could find a classification-free proof.

**2. Preliminaries**

If $A$ is a group of automorphisms of a group $G$, the subgroup generated by elements of the form $g^{-1}g^\alpha$ with $g \in G$ and $\alpha \in A$ is denoted by $[G, A]$. It is well-known that the subgroup $[G, A]$ is an $A$-invariant normal subgroup in $G$. Our first lemma is a collection of well-known facts on coprime actions (see for example [4]). Throughout the paper we will use it without explicit references.

**Lemma 2.1.** Let $A$ be a group of automorphisms of a finite group $G$ such that $(|G|, |A|) = 1$. Then

i) $G = C_G(A)[G, A]$. 


ii) $[G, A, A] = [G, A]$.

iii) $A$ leaves invariant some Sylow $p$-subgroup of $G$ for each prime $p \in \pi(G)$.

iv) $C_{G/N}(A) = C_G(A)N/N$ for any $A$-invariant normal subgroup $N$ of $G$.

v) If $G$ is nilpotent and $A$ is a noncyclic abelian group, then $G = \prod_{a \in A^\#} C_G(a)$.

We will require the following well-known fact.

**Lemma 2.2.** Let $N, H_1, \ldots, H_l$ be subgroups of a group $G$ with $N$ being normal. If $K = \langle H_1, \ldots, H_l \rangle$, then $[N, K] = [N, H_1] \ldots [N, H_l]$.

Throughout the rest of this section we will assume the following hypothesis. As usual, $Z(K)$ stands for the center of a group $K$.

Let $q$ be a prime and $A$ a group of exponent $q$ and order $q^3$. Denote by $B$ a subgroup of order $q$ of $Z(A)$. Let $A$ act on a finite $q'$-group $G = PH$, where $P$ and $H$ are $A$-invariant subgroups such that $P$ is a normal $p$-subgroup for a prime $p$ and $H$ is a nilpotent $p'$-subgroup. It is clear that $A$ has precisely $q + 1$ subgroups of order $q^2$ containing $B$.

**Lemma 2.3.** Let $A_1, \ldots, A_{q+1}$ be the subgroups of order $q^2$ of $A$ containing $B$. Then $C_P(B) = \prod C_P(A_i)$ and $C_H(B) = \prod C_H(A_i)$.

**Proof.** Denote by $\overline{A}$ the quotient-group $A/B$. Since $\overline{A}$ is not cyclic and both centralizers $C_P(B)$ and $C_H(B)$ are $A$-invariants, it follows that $C_P(B) = \prod C_{P(B)}(\overline{a})$ and $C_H(B) = \prod C_{H(B)}(\overline{a})$ where $\overline{a} \in \overline{A}^\#$. An alternative way of expressing this is to write that $C_P(B) = \prod C_P(A_i)$ and $C_H(B) = \prod C_H(A_i)$. □

**Lemma 2.4.** Suppose that $P$ is abelian. Then $[P, C_H(B)]$ is contained in $\prod [C_P(a), C_H(a)]$, where the product is taken over all $a \in A^\#$.

**Proof.** In view of Lemma 2.3 we have $C_H(B) = \prod C_H(A_i)$ and $P = \prod_{a \in A^\#} C_P(a)$ for each $i$. By Lemma 2.2 $[P, C_H(B)] = \prod [P, C_H(A_i)]$. Since $P$ is abelian, for each $i$ we have $[P, C_H(A_i)] = \prod [C_P(a), C_H(A_i)]$ where the product is taken over all $a \in A^\#$. In particular, $[P, C_H(A_i)] = \prod [C_P(a), C_H(a)]$. □

**Lemma 2.5.** If $[C_P(a), C_H(a)] = 1$ for any $a \in A^\#$, then $[P, H] = 1$.

**Proof.** First we assume that $P$ is abelian. By Lemma 2.4 $[P, C_H(B)] = 1$. Let us prove that $[C_P(B), H] = 1$. Using the notation of Lemma 2.3 we have that $C_P(B) = \prod C_P(A_i)$ and $H = \prod_{a \in A^\#} C_H(a)$ for each $i$. Since $P$ is abelian, we conclude that $[C_P(B), H] = \prod [C_P(A_i), H]$. 

NILPOTENT RESIDUAL OF FIXED POINTS

3
Thus, $[C_P(A_i), H] = 1$ as $[C_P(A_i), C_H(a)] = 1$ for any $a \in A_i^\#$. Hence, $[C_P(B), H] = 1$.

The above shows that $C_P(B) \leq Z(G)$ and $C_H(B)$ centralizes $P$. If $H$ is abelian, then $C_G(B) \leq Z(G)$. Hence, $B$ acts fixed-point-freely on $G/Z(G)$ and so $G/Z(G)$ is nilpotent by Thompson’s theorem \cite{Thompson}. Consequently, $G$ is nilpotent and so in the case where $P$ and $H$ both are abelian we have $[P, H] = 1$.

Suppose that $H$ is not abelian. By the previous paragraph, $[P, Z(H)] = 1$. Considering the action of $H/Z(H)$ on $P$ and arguing by induction on the nilpotency class of $H$ we deduce that $[P, H] = 1$. Thus, in the case where $P$ is abelian the lemma holds.

Assume that $P$ is not abelian. Consider the action of $HA$ on $P/\Phi(P)$. By the above, $[P, H] \leq \Phi(P)$. We see that $P = C_P(H)[P, H] \leq C_P(H)\Phi(P)$, which implies that $P = C_P(H)$ and $[P, H] = 1$.

Lemma 2.6. Suppose that $P$ is abelian and $P = [P, H]$. Assume that the order of $[C_P(a), C_H(a)]$ is at most $m$ for any $a \in A^\#$. Then $\langle C_P(B)^H \rangle$ has $(m, q)$-bounded order.

Proof. Recall that by Lemma 2.3 we have $C_P(B) = \prod C_P(A_i)$, where $A_1, \ldots, A_{q+1}$ are the subgroups of order $q^2$ of $A$ containing $B$. First, we prove that the order of $C_P(B)$ is $(m, q)$-bounded. It suffices to bound the order of $C_P(A_i)$ for each $i$. For each $a \in A_i$ we denote by $P_a$ and $H_a$ the centralizers $C_P(a)$ and $C_H(a)$ respectively. It is clear that $P_a$ is normal in $C_G(a)$. Set $D_a = C_{P_a}(H_a)$ and $D_i = \cap_{a \in A_i} D_a$.

The index of $D_a$ in $P_a$ is $m$ since $[P_a, H_a] = m$ and so the index of $D_i$ in $C_P(A_i)$ is $(m, q)$-bounded. Now, let $x \in D_i$. Taking into account that $H = \prod_{a \in A_i^\#} H_a$ we deduce that $[x, H] = 1$. Thus, $x = 1$ since $P = [P, H]$. We conclude that $D_i$ is trivial for each $i$. Therefore $C_P(A_i)$ has $(m, q)$-bounded order for any $i$ as desired.

Now, let $E_a$ be the centralizer of $[P_a, H_a]$ in $H_a$. Note that $E_a$ has $m$-bounded index in $H_a$ since $H_a/E_a$ embeds in the automorphism group of $[P_a, H_a]$. Moreover, $[P_a, E_a] = 1$ because $P_a = C_{P_a}(H_a)[P_a, H_a]$. Set $E_i = \langle E_a \mid a \in A_i \rangle$. Note that $E_i$ has $(m, q)$-bounded index in $H$ since $H = \prod_{a \in A_i^\#} H_a$. Further, note that $[C_P(A_i), E_i] = 1$. It becomes clear that $E = \cap_i E_i$ has $(m, q)$-bounded index in $H$ and $[C_P(B), E] = 1$. Therefore, $\langle C_P(B)^H \rangle$ has $(m, q)$-bounded order.

We use $Z_i(K)$ to denote the $i$th term of the upper central series of $K$.

Lemma 2.7. Suppose that the order of $[C_P(a), C_H(a)]$ is at most $m$ for any $a \in A^\#$. Then the order of $[P, H]$ is $(m, q)$-bounded.
PROOF. We can assume that $P = [P, H]$. Note that if $N$ is a normal subgroup of $P$ such that $[N, H] = 1$, then $N \leq Z(P)$. Indeed, in this case we have $[P, N] \leq N$ and so $[P, [N, H]] = 1$ and $[H, [P, N]] = 1$. Consequently $[N, [P, H]] = 1$ by the three subgroup lemma.

For a normal $A$-invariant subgroup $M$ of $P$ and $a \in A^\#$, we write $j_a(P/M)$ for the order of $[C_{P/M}(a), C_H(a)]$. We write $j_a(P)$ when $M$ is trivial. Set $k(P) = \sum_{a \in A^\#} j_a(P)$. It is clear that $k(P)$ is $(m, q)$-bounded. By induction on $k(P)$ we will prove that the nilpotency class of $P$ is at most $t = 2k(P) + 1$. If $k(P) = q^3 - 1$ (the smallest possible value for $k(P)$ - it occurs if and only if $[C_P(a), C_H(a)] = 1$ for any $a \in A^\#$), then $P$ is trivial by Lemma 2.5, since $P = [P, H]$. Further, if $P$ is abelian there is nothing to prove. Suppose that $P$ is not abelian. Then $[Z_2(P), H] \neq 1$ and so, by Lemma 2.5, $[C_{Z_2(P)}(a), C_H(a)] \neq 1$ for some $a \in A^\#$. Therefore $k(P/Z_2(P)) < k(P)$. By induction, $P/Z_2(P)$ has nilpotency class at most $2(k(P) - 1) + 1$ and so the nilpotency class of $P$ is at most $2k(P) + 1$.

Clearly, $|P|$ is bounded in terms of $|P/P'|$ and the nilpotency class of $P$. Hence, by the previous paragraph in order to prove the lemma it is sufficient to prove that the order of $P/P'$ is $(m, q)$-bounded. In particular, without loss of generality we may assume that $P$ is abelian.

By Lemma 2.6 the subgroup $\langle C_P(B)^H \rangle$ has $(m, q)$-bounded order and since $P$ is abelian we conclude that it is normal in $G$. We can pass to the quotient $G/\langle C_P(B)^H \rangle$ and without loss of generality assume that $C_P(B) = 1$.

By Lemma 2.4 $[P, C_H(B)]$ has $(m, q)$-bounded order. Hence, it is sufficient to show that $[P, H] = [P, C_H(B)]$. First, suppose that $H$ is abelian. Then $[P, C_P(B)]$ is normal in $G$ and passing to the quotient we can assume that $[P, C_P(B)] = 1$. Thus, $C_G(B) = C_H(B)$ belongs to $Z(G)$ and so $G/Z(G)$ admits a fixed-point-free automorphism of prime order $q$. By Thompson’s theorem [18], $G/Z(G)$ is nilpotent. Therefore, $G$ is nilpotent and $[P, H] = 1$, as required.

Suppose that $H$ is not abelian. We have proved that $[P, Z(H)] = [P, C_{Z(H)}(B)]$. The subgroup $[P, Z(H)]$ is normal in $G$. Hence, passing to the quotient $G/[P, Z(H)]$ we can consider the action of $H/[Z(H)]$ on $P$ and arguing by induction on the nilpotency class of $H$ we deduce that $[P, H] = [P, C_P(B)]$. \hfill \Box

**Lemma 2.8.** Let $N$ be a normal $HA$-invariant subgroup of $P$. Assume that $P = [P, H]$ and $[[N, H]] = p^n$. Then $N \leq Z_{2n+1}(P)$.

**Proof.** If $[N, H] = 1$, then $[H, [P, N]] = [P, [N, H]] = 1$ and so $N \leq Z(P)$ by the three subgroup lemma. Let $M = N \cap Z_2(P)$ and suppose that $N$ is not in $Z(P)$. Thus, it is clear that $M \not\leq Z(P)$ and
\[ [M, H] \neq 1. \] Using induction on \( n \) with \( G \) and \( N \) replaced by \( G/M \) and \( N/M \) respectively we derive that \( N/M \leq Z_{2n-1}(P/M) \), whence \( N \leq Z_{2n+1}(P) \). \hfill \Box 

Recall that a \( p \)-group is powerful if \( G' \leq G^p \) (for \( p \) odd) or \( G' \leq G^4 \) (for \( p = 2 \)). The reader can consult [2] for information on such groups.

For odd prime the next lemma can be found for example in [16, Lemma 2.2].

**Lemma 2.9.** Let \( N \) be a group of rank \( r \) and prime exponent \( p \) if \( p \) is odd or exponent 4 if \( p = 2 \). Then \( |N| \leq p^s \) where \( s \) is an \( r \)-bounded number.

**Proof.** By Theorem 2.13 of [2] \( G \) has a powerful characteristic subgroup \( N \) of index at most \( p^{\mu(r)} \) where \( \mu(r) \) is a number depending only on \( r \). Corollary 2.8 in [2] shows that \( N \) is a product of at most \( r \) cyclic subgroups. Therefore, \( N \) is of order at most \( p^r \) if \( p \) is odd or 4\(^r \) if \( p = 2 \) and the lemma follows. \hfill \Box 

**Lemma 2.10.** Suppose that \( r([C_P(a), C_H(a)]) \leq r \) for any \( a \in A^\# \). Then \( r([P, H]) \) is \((r, q)\)-bounded.

**Proof.** Without loss of generality we assume that \( P = [P, H] \). Let \( M \) be any normal \( A \)-invariant subgroup of exponent \( p \) (or exponent 4 if \( p = 2 \)) in \( P \). By Lemma 2.9, the order of \([C_M(a), C_H(a)]\) is \( r \)-bounded for any \( a \in A \). Hence, applying Lemma 2.7 we deduce that \([M, H]\) is of \((r, q)\)-bounded order. In other words, there exists an \((r, q)\)-bounded number \( t \) such that \([M, H] \leq p^t \). Applying this argument to \( P/\Phi(P) \) we conclude that \( P \) can be generated with at most \( t \) elements since the minimal number of generators of \( P \) is equal to the rank of \( P/\Phi(P) \) by the Burnside Basis Theorem.

Let the symbol \( p \) denote \( p \) if \( p \) is odd and 4 if \( p = 2 \). Let \( N = \gamma_{2t+1}(P) \) where \( t \) is as above. We will show that \( N \) is powerful, that is \( \gamma'_i \leq N^p \). Pass to the quotient \( G/N^p \) and assume that \( N \) has exponent \( p \). By the first paragraph \([N, H] \leq p^t \) and so \( N \leq Z_{2t+1}(P) \) by Lemma 2.8. Note that \([\gamma_i(P), Z_i(P)] = 1 \) for any positive integer \( i \). Therefore, \( N \) is abelian modulo \( N^p \). This means that \( N \) is powerful.

We now wish to show that \( N \) can be generated with bounded number of elements. We can pass to the quotient \( P/\Phi(N) \) and assume that \( N \) is elementary abelian. Let \( d \) be the minimal number of generators of \( P \). For each \( n \) the section \( \gamma_n(P)/\gamma_{n+1}(P) \) is generated by \( d^n \) elements (see for example [9, Corollary 2.5.6]). Hence, it suffices to
bound the nilpotency class of $P$. Recall that we have already proved that under our assumptions $N \leq Z_{2t+1}(P)$. Therefore $P$ is nilpotent of class at most $4t + 2$ and so the minimal number of generators of $N$ is $(r, q)$-bounded.

Since $N$ is powerful we obtain that $r(N)$ is $(r, q)$-bounded. The lemma follows since obviously $r(P) \leq r(P/N) + r(N)$. □

We finish this section with a useful result on coprime action. In the next lemma we will use the fact that if $D$ is any coprime group of automorphisms of a finite simple group, then $D$ is cyclic (see for example [12]).

**Lemma 2.11.** Let $D$ be a non-cyclic $q$-group of order $q^{2}$ acting on a finite $q'$-group $N = S_{1} \times \cdots \times S_{l}$ which is a direct product of $t$ nonabelian simple groups. Suppose that $r(\gamma_{\infty}(C_{N}(d))) \leq r$ for any $d \in D^{\#}$. Then $t$ is an $(r, q)$-bounded number and each direct factor $S_{i}$ has rank at most $r$.

**Proof.** First, we prove that each direct factor $S_{i}$ has rank at most $r$. Indeed, if $S_{i}$ is $D$-invariant, then $S_{i}$ is contained in $C_{G}(d)$ for some $d \in D^{\#}$ and so $r(S_{i}) \leq r$ by the hypotheses. Suppose that $S_{i}$ is not $D$-invariant. Choose $d \in D^{\#}$ such that $S_{i}^{d} \neq S_{i}$. Write $S = S_{i} \times S_{i}^{d} \times S_{i}^{d_{i}} \cdots$. We see that $C_{S}(d)$ is exactly the diagonal subgroup of $S$ and so $C_{S}(d)$ is isomorphic to $S_{i}$. Thus, we conclude that $r(S_{i}) \leq r$.

Now we prove that $t$ is $(r, q)$-bounded. Write $G = K_{1} \times \cdots \times K_{s}$ where each $K_{i}$ is a minimal normal $D$-invariant subgroup. Then each $K_{i}$ is a product of at most $|D|$ simple factors and so $t \leq |D|s$. Therefore it is sufficient to bound $s$.

Let $S_{j}$ be a direct factor of $K_{i}$. If $S_{j}$ is $D$-invariant, then $S_{j}$ is contained in $C_{G}(d)$ for some $d \in D^{\#}$. If $S_{j}$ is not $D$-invariant, then we can choose $d \in D^{\#}$ such that $S_{j}^{d} \neq S_{j}$. Now, it is clear that $C_{S}(d)$ is exactly the diagonal subgroup of $S_{i} \times S_{i}^{d} \times S_{i}^{d_{i}} \cdots$ and so $C_{S}(d)$ is isomorphic to $S_{j}$. In other words, for every $i$ there exists $d \in D^{\#}$ such that $C_{K_{i}}(d)$ contains a subgroup isomorphic to some $S_{j}$. Therefore $\gamma_{\infty}(C_{K_{i}}(d))$ has even order. Since $r(\gamma_{\infty}(C_{G}(d))) \leq r$, it follows that $\gamma_{\infty}(C_{K_{i}}(d))$ can have even order for at most $r$ indexes $i$. Taking into account that there are only $|D| - 1$ nontrivial elements in $D$, we deduce that $s \leq (|D| - 1)r$. □

## 3. Main results

We will give a detailed proof only for Theorem 1.2. The proof of Theorem 1.1 is easier and can be obtained by just obvious modifications.
of the proof of Theorem \[1.2\] The following elementary lemma will be useful (for the proof see for example [1, Lemma 2.4]).

**Lemma 3.1.** Let $G$ be a finite group such that $\gamma_\infty(G) \leq F(G)$. Let $P$ be a Sylow $p$-subgroup of $\gamma_\infty(G)$ and $H$ be a Hall $p'$-subgroup of $G$. Then $P = [P, H]$.

Let $F(G)$ denote the Fitting subgroup of a group $G$. Write $F_0(G) = 1$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. If $G$ is soluble, the least number $h$ such that $F_h(G) = G$ is called the Fitting height $h(G)$ of $G$. Let $B$ be a coprime group of automorphisms of a finite soluble group $G$. It was proved in [19] that the Fitting height of $G$ is bounded in terms of $h(C_G(B))$ and the number of prime factors of $|B|$ counting multiplicities. The nonsoluble length $\lambda(G)$ of a finite group $G$ is defined as the minimum number of nonsoluble factors in a normal series of $G$ all of whose factors are either soluble or (non-empty) direct products of nonabelian simple groups. It was proved in [11] (see Corollary 1.2) that the nonsoluble length $\lambda(G)$ of a finite group $G$ does not exceed the maximum Fitting height of soluble subgroups of $G$. If $B$ is a coprime group of automorphisms of a finite group $G$, then the nonsoluble length $\lambda(G)$ of $G$ is bounded in terms of $\lambda(C_G(B))$ and the number of prime factors of $|B|$ counting multiplicities [10].

Let us now assume the hypothesis of Theorem \[1.2\] Thus, $A$ is a finite group of prime exponent $q$ and order at least $q^3$ acting on a finite $q'$-group $G$ in such a manner that $r(\gamma_\infty(C_G(a))) \leq r$ for any $a \in A^\#$. We wish to show that $r(\gamma_\infty(G))$ is $(r, q)$-bounded. It is clear that $A$ contains a subgroup of order $q^3$. Thus, replacing if necessary $A$ by such a subgroup we may assume that $A$ has order $q^3$.

Suppose that $G$ is soluble. In that case $C_G(a)$ has $r$-bounded Fitting height for any $a \in A^\#$ (see for example Lemma 1.4 of [12]). Hence, $G$ has $(r, q)$-bounded Fitting height and we can use induction on $h(G)$.

In the case where $h(G) = 2$ the proof is immediate from Lemma 2.10. Indeed, let $P$ be a Sylow $p$-subgroup of $\gamma_\infty(G)$ and $H$ a Hall $A$-invariant $p'$-subgroup of $G$. Then by Lemmas 2.10 and 3.1 the rank of $P = [P, H]$ is $(r, q)$-bounded. Therefore the rank of $\gamma_\infty(G)$ is $(r, q)$-bounded. Suppose that the Fitting height of $G$ is $h > 2$ and let $N = F_2(G)$ be the second term of the Fitting series of $G$. It is clear that the Fitting height of $G/\gamma_\infty(N)$ is $h - 1$ and $\gamma_\infty(N) \leq \gamma_\infty(G)$. Hence, by induction we have that $\gamma_\infty(G)/\gamma_\infty(N)$ has $(r, q)$-bounded rank. Now, the result follows since $r(\gamma_\infty(G)) \leq r(\gamma_\infty(G)/\gamma_\infty(N)) + r(\gamma_\infty(N))$. 
We now drop the assumption that $G$ is soluble. Remark that $\lambda(C G(a))$ is $r$-bounded for any $a \in A^\# \text{ by } [11]$ since its soluble subgroups have $r$-bounded Fitting height. Hence, $\lambda(G)$ is $(r, q)$-bounded and we can use induction on $\lambda(G)$.

First, assume that $G = G'$ and $\lambda(G) = 1$. Since $G = G'$, it follows that $G/R(G)$ is a product of nonabelian simple groups where $R(G)$ is the soluble radical of $G$. By the above $\gamma_\infty(R(G))$ has $(r, q)$-bounded rank. We can factor out $\gamma_\infty(R(G))$ and assume $R(G)$ is nilpotent, that is $R(G) = F(G)$.

We now wish to show that the rank of $[F(G), G]$ is $(r, q)$-bounded. Clearly, it is sufficient to consider the case when $F(G) = P$ where $P$ is a Sylow $p$-subgroup of $F(G)$. Note that if $s$ is a prime different from $p$ and $H$ is an $A$-invariant Sylow $s$-subgroup of $G$, then $r(\gamma_\infty(PH))$ is $(r, q)$-bounded because $PH$ is soluble. We will require the following observation about finite simple groups.

**Lemma 3.2.** Let $K$ be a nonabelian finite simple group and $p$ a prime. There exists a prime $s$ different from $p$ such that $K$ is generated by two Sylow $s$-subgroup.

**Proof.** If $p \neq 2$, then we can use Guralnick’s result [5], Theorem A] that $K$ is generated by an involution and a Sylow 2-subgroup. We therefore can take $s = 2$. If $p = 2$, we can use King’s results [8] that $K = \langle i, a \rangle$, where $|i| = 2$ and $|a|$ is an odd prime. We have $K = \langle a, a^i \rangle$ since this is an $a$-invariant and $i$-invariant subgroup, which is therefore normal. Hence, $K$ is generated by two elements of odd prime order and the lemma follows. \qed

By Lemma [2.11] the quotient $G/F(G)$ is a product of a $(r, q)$-bounded number of normal $A$-invariant subgroups $K_1 \times \cdots \times K_s$ where $K_i$ is a product of at most $|A|$ nonabelian simple groups. Hence, without loss of generality we can assume that $G/F(G)$ is a product of isomorphic nonabelian simple groups. In view of Lemma 3.2 we deduce that $G/P$ is generated by the image of two Sylow $s$-subgroup $H_1$ and $H_2$ where $s$ is a prime different from $p$. On the other hand, $H_1$ and $H_2$ are conjugate of an $A$-invariant Sylow $s$-subgroup of $G$. It follows that both $[P, H_1]$ and $[P, H_2]$ have $(r, q)$-bounded rank.

Let $H = \langle H_1, H_2 \rangle$. Thus $G = PH$. Since $G = G'$, it is clear that $G = [P, H]H$ and $[P, G] = [P, H]$. By Lemma [2.2] we have $[P, H] = [P, H_1][P, H_2]$ and therefore the rank of $[P, H]$ is $(r, q)$-bounded. Passing to the quotient $G/[P, G]$ we can assume that $P = Z(G)$. So we are in the situation where $G/Z(G)$ has $(r, q)$-bounded rank. By a theorem
of Lubotzky and Mann [14] (see also [13]) the rank of $G'$ is $(r, q)$-bounded as well. Taking into account that $G = G'$ we conclude that the rank of $G$ is $(r, q)$-bounded.

Let us now deal with the case where $G \neq G'$. Let $G^{(l)}$ be the last term of the derived series of $G$. The argument in the previous paragraph shows that $r(G^{(l)})$ is $(r, q)$-bounded. Consequently, $r(\gamma_\infty(G))$ is $(r, q)$-bounded since $G/G^{(l)}$ is soluble and $\gamma_\infty(G^{(l)}) \leq \gamma_\infty(G)$. This proves the theorem in the particular case where $\lambda(G) \leq 1$.

Assume that $\lambda(G) \geq 2$. Let $T$ be a characteristic subgroup of $G$ such that $\lambda(T) = \lambda(G) - 1$ and $\lambda(G/T) = 1$. By induction, the rank of $\gamma_\infty(T)$ is $(r, q)$-bounded. It is clear that $\lambda(G/\gamma_\infty(T)) = 1$. Therefore, the result follows since $r(\gamma_\infty(G)) \leq r(G/\gamma_\infty(T)) + r(\gamma_\infty(T))$.

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**Department of Mathematics, University of Brasília, Brasília-DF 70910-900, Brazil**

*E-mail address: emerson@mat.unb.br*

**Department of Mathematics and Statistics, Federal University of Goiás, Goiânia-GO, 74001-970, Brazil**

*E-mail address: alinelima@ufg.br*

**Department of Mathematics, University of Brasília, Brasília-DF, 70910-900, Brazil**

*E-mail address: pavel@unb.br*