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The determinant of a hypergeometric period matrix

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1 Introduction

Let $f_1, \ldots, f_p$ be polynomials with real coefficients of degree one which define an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $\alpha_1, \ldots, \alpha_p$ complex numbers. Let $U_\alpha = f_1^{\alpha_1} \cdots f_p^{\alpha_p}$. Varchenko ([V1, Theorem 1.1], [V2]) calculated, for arrangements of hyperplanes in general position, the determinant of a (period) matrix $\text{PM}(\mathcal{A}, \alpha)$ whose entries are (hypergeometric) integrals $\int_\Delta U_\alpha \phi$, where $\Delta$ runs over the set $\text{Ch}(\mathcal{A})$ of bounded connected components of $\mathbb{R}^n - \bigcup_{i=1}^p \{f_i = 0\}$ and the $n$-form $\phi$ runs over the set $\Phi_\alpha(\mathcal{A}) = \{\alpha_{i_1} \cdots \alpha_{i_n} d\phi_{i_1} / f_{i_1} \wedge \cdots \wedge d\phi_{i_n} / f_{i_n} \mid 1 < i_1 < \cdots < i_n \leq p\}$. Since $|\text{Ch}(\mathcal{A})| = |\Phi_\alpha(\mathcal{A})| = \binom{p-1}{n}$, the period matrix $\text{PM}(\mathcal{A}, \alpha)$ is of size $\binom{p-1}{n} \times \binom{p-1}{n}$. The formula by Varchenko expresses the determinant of $\text{PM}(\mathcal{A}, \alpha)$ by the product of critical values and a certain function, called the Beta function $B(\mathcal{A}, \alpha)$ of the arrangement. The Beta function $B(\mathcal{A}, \alpha)$ is explicitly given as an alternating product of Gamma functions whose arguments are appropriate linear combinations of the parameters $\alpha_i$. In the general case, Varchenko conjectured an analogous explicit formula [V2, 6.3 Fundamental conjecture] and proved it for normal-crossing arrangements [V1, Theorem 1.4] and arrangements in general position at infinity [V2, Theorem 6.1]. Note that this determinant can also regarded as "Wronskian" of a certain system of partial differential equations (cf. for example [Kil]).

More recently, F. Loeser and C. Sabbah ([LS]), gave a general formula for such a determinant. In this formula enters the characteristic polynomial of some monodromies associated with the family $f_1, \cdots, f_p$ of polynomials. Since the bases of cycles and forms are not specified, there are some unknowns in their result and our purpose is to remove them. In order to do that, we use the $\beta$ nbc-bases of the cohomology defined in [FT, Theorem 3.6]. The calculation of the determinant of the period matrix in a $\beta$ nbc-basis can be done by using recurrence (deletion-restriction) and by studying zeros and poles of the determinant in terms

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of the non-resonance set. In the end, we show that the conjecture announced by Varchenko is true for any arrangement for an arbitrary \( \beta \) nbc-basis.

# 2 Arrangements

In this section we review results from [FT].

## 2.1

Let \( f_1, \ldots, f_p \) be linear polynomials defined on the \( n \)-dimensional complex affine space \( V \). Let \( I \) denote \( \{1, \ldots, p\} \) and \( \mathcal{A} \) be the arrangement \( \{H_i\}_{i \in I} \) where \( H_i = \ker f_i \) is the hyperplane defined by \( f_i \).

**Definition 2.1.1** An edge of \( \mathcal{A} \) is a nonempty intersection of some of its hyperplanes.

Let \( L(\mathcal{A}) \) denote the set of all these edges. \( L(\mathcal{A}) \) is partially ordered by reverse inclusion. The maximal elements of \( L(\mathcal{A}) \) all have the same codimension, which we denote by \( r \). Agree that \( V \in L(\mathcal{A}) \) which is the unique minimum element.

**Definition 2.1.2** (i) \( \mathcal{A} \) is said to be **essential** if \( r = n \) (in particular \( |I| \geq n \)). (ii) \( \mathcal{A} \) is said to be **real** if the polynomials \( f_i \) all have real coefficients.

**Definition 2.1.3** An arrangement \( \mathcal{A} \) is said to be in **general position** if, for all subarrangement \( \{H_{i_1}, \ldots, H_{i_k}\} \) of \( \mathcal{A} \), codim \( (H_{i_1} \cap \cdots \cap H_{i_k}) = k \) if \( 1 \leq k \leq n \) and \( H_{i_1} \cap \cdots \cap H_{i_k} = \emptyset \) if \( k > n \). An arrangement \( \mathcal{A} \) is said to be in **general position at infinity** if, for all subarrangement \( \{H_{i_1}, \ldots, H_{i_k}\} \) of \( \mathcal{A} \), \( H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset \) for \( k \leq n \). It is said to be **normal** if \( \bigcup H \) is a normal crossing divisor in \( V \).

**Notation 2.1.4** (i) Let

\[
M(\mathcal{A}) = V - \bigcup_{i \in I} H_i
\]

and, if \( \mathcal{A} \) is real,

\[
M_\mathbb{R}(\mathcal{A}) = M(\mathcal{A}) \cap V_\mathbb{R}
\]

where \( V_\mathbb{R} \) denote the real part of \( V \).

(ii) If \( \mathcal{A} \) is real, let \( \text{Ch}(\mathcal{A}) \) denote the set of all \( n \)-dimensional bounded components of \( M_\mathbb{R}(\mathcal{A}) \) and \( \beta(\mathcal{A}) \) its cardinality.

Until the end of this paper we suppose \( \mathcal{A} \) real and essential.
2.2 Linear orders

Let \( i_0 \in I \). We define a linear order \( <_{i_0} \) in \( \mathcal{A} \) putting \( H_i <_{i_0} H_j \) if \( i < j, i, j \neq i_0 \), and \( H_i <_{i_0} H_{i_0} \) for all \( i \in I - \{i_0\} \).

Remark 2.2.1 (i) \( <_p \) is the standard order defined in [OT, page 67].
(ii) If \( B \subseteq \mathcal{A} \) is a subarrangement which do not contains \( H_{i_0} \), it inherits the standard order.

2.3 Non-resonant weights

Let \( F \in L(\mathcal{A}) - \{V\} \) be an edge. Define

\[ I(F) = \{i \in I \mid F \subseteq H_i\}. \]

Definition 2.3.1 The edge \( F \) is called dense if the arrangement

\[ \{H_i \mid i \in I(F)\} \]

is not decomposable [STV, Section 2], that is, the arrangement is not a product of two nonempty arrangements.

Let \( \mathbb{P}^n \) be complex projective space, which is a compactification of \( V = \mathbb{C}^n \). Define the arrangement \( \mathcal{A}_\infty \) of projective hyperplanes by

\[ \mathcal{A}_\infty = \{\overline{H_1}, \overline{H_2}, \ldots, \overline{H_p}, \overline{H_\infty}\}, \]

where \( \overline{H_i} \) is the projective closure of \( H_i \) (\( 1 \leq i \leq p \)) and \( \overline{H_\infty} = \mathbb{P}^n - \mathbb{C}^n \). Let \( L(\mathcal{A}_\infty) \) be the collection of nonempty intersections of projective hyperplanes in \( \mathcal{A}_\infty \). Define \( L_-(\mathcal{A}_\infty) \) (resp. \( L_+(\mathcal{A}_\infty) \)) to be the set of edges of \( L(\mathcal{A}_\infty) \) contained (resp. not contained) in \( \overline{H_\infty} \). Then \( L(\mathcal{A}_\infty) = L_-(\mathcal{A}_\infty) \cup L_+(\mathcal{A}_\infty) \) (disjoint). Cover \( \mathbb{P}^n \) by the standard affine opens \( U_0, U_1, \ldots, U_n \), each of which is isomorphic to \( \mathbb{C}^n \). Let \( \mathcal{A}_i \) (\( 0 \leq i \leq n \)) be the arrangement in \( U_i \simeq \mathbb{C}^n \) obtained by restricting each projective hyperplane in \( \mathcal{A}_\infty \) to \( U_i \). Let \( F \in L(\mathcal{A}_\infty) - \{\mathbb{P}^n\} \). We say that \( F \) is dense if \( F \cap U_i \) is dense in \( \mathcal{A}_i \) for \( 0 \leq i \leq n \) with \( F \cap U_i \neq \emptyset \). (cf. [STV, Section 3].)

Let \( \mathcal{T} = \{1, \ldots, p, \infty\} \). For \( F \in L(\mathcal{A}_\infty) \) define

\[ \mathcal{T}(F) = \{i \in \mathcal{T} \mid F \subseteq \overline{H_i}\}. \]

To each polynomial \( f_i \) (and therefore to each hyperplane \( H_i \)), one associates a complex number \( \alpha_i \). These numbers are called weights. Define \( \alpha_\infty = -\sum_{i=1}^p \alpha_i \). For \( F \in L(\mathcal{A}_\infty) - \{\mathbb{P}^n\} \), let \( \alpha(F) \) be the sum of \( \alpha_i \) with \( i \in \mathcal{T}(F) \). In other words, for \( F \in L_+(\mathcal{A}_\infty) \),

\[ \alpha(F) = \sum_{i \in \mathcal{T}(F)} \alpha_i, \]
and, for $F \in L_\infty(A_\infty)$,

\[ \alpha(F) = \alpha_\infty + \sum_{j \in \mathcal{I}(F)} \alpha_i = -\sum_{i \in I} \alpha_i + \sum_{j \in \mathcal{I}(F)} \alpha_i = -\sum_{j \in I-\mathcal{I}(F)} \alpha_i. \]

Define the resonance set of $A$ by

\[ \text{Rsn}(A) = \{ \alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p \mid \alpha(F) \in \mathbb{Z} \text{ for some dense edge } F \in L(A_\infty) \}. \]

It is the union of a locally finite infinite family of hyperplanes.

**Definition 2.3.2** We say that the weights $\alpha = (\alpha_1, \ldots, \alpha_p)$ are non-resonant if $\alpha \not\in \text{Rsn}(A)$. In other words, the weights $\alpha$ are non-resonant if $\alpha(F) \not\in \mathbb{Z}$ for each dense edge $F \in L(A_\infty)$.

### 2.4 $\beta$ NBC-bases

Let $i_0 \in I$. We define the linear order $<_{i_0}$ in $A$.

A subset $\{H_i\}_{i \in J}$ of $A$ is dependent if $\cap_{i \in J} H_i \neq \emptyset$ and codim($\cap_{i \in J} H_i$) $< |J|$. A subset of $A$ which has nonempty intersection and is not dependent is called independent. Maximal independent sets are called bases. Every base has cardinality $n$.

A $k$-tuple $S = (H_1, \ldots, H_k)$ is a circuit if $(H_1, \ldots, H_k)$ is dependent and if, for each $l$, $1 \leq l \leq k$, the $(k-1)$-tuple $(H_1, \ldots, \widehat{H}_l, \ldots, H_k)$ is independent. A $k$-tuple $S$ is a broken circuit if there exists $H <_{i_0} \min(S)$ such that $\{H\} \cup S$ is a circuit, where $\min(S)$ denotes the minimal element of $S$ for $<_{i_0}$.

The collection of subsets of $A$ having nonempty intersection and containing no broken circuits is denoted by $\text{BC}$. $\text{BC}$ consists of independent sets. Maximal (with respect to inclusion) elements of $\text{BC}$ are bases of $A$ called NBC-bases.

A NBC-basis $B = (H_{i_1}, \ldots, H_{i_n})$ is ordered if $H_{i_1} <_{i_0} H_{i_2} <_{i_0} \cdots <_{i_0} H_{i_n}$.

We denote by $\text{NBC}_{i_0}(A)$ the set of all ordered NBC-bases of $A$. We introduce a linear order in $\text{NBC}_{i_0}(A)$ using the lexicographic order on the hyperplanes read from right to left.

**Definition 2.4.1** A basis $B$ is called $\beta$ NBC-basis if $B$ is a NBC-basis and if, for every $H \in B$, there exists $H' <_{i_0} H$ such that $(B - \{H\}) \cup \{H'\}$ is a base.

**Notation 2.4.2** Denote by $\beta \text{NBC}_{i_0}(A)$ the set of all $\beta$ NBC-bases, ordered by $<_{i_0}$.

**Remark 2.4.3** (i) $\beta \text{NBC}_{i_0}(A)$ inherits the order defined on $\text{NBC}_{i_0}(A)$.

(ii) In what follows, we omit the index $i_0$ when the linear order on $A$ is the standard order.

**Definition 2.4.4** If $B = (H_{i_1}, \ldots, H_{i_n}) \in \beta \text{NBC}_{i_0}(A)$, let

\[ F_j = \bigcap_{k=j+1}^{n} H_{i_k} \]
for $0 \leq j \leq n - 1$ and $F_n = V$. Define

$$\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n),$$

which is a flag of affine subspaces of $V$ with $\dim F_j = j$ ($0 \leq j \leq n$). This flag $\xi(B)$ is called the $\beta$ $\text{nbc-flag}$ associated with $B$.

**Notation 2.4.5** For $(i_1, \cdots, i_k) \subseteq I$, $\omega_{i_1} \omega_{i_2} \cdots \omega_{i_k} = df_{i_1} / f_{i_1} \wedge \cdots \wedge df_{i_k} / f_{i_k}$. If $F \in L(A) - \{V\}$, define

$$I(F) = \{i \in I \mid F \subseteq H_i\},$$

and

$$\omega_\alpha(F, A) = \sum_{i \in I(F)} \alpha_i \omega_i.$$  

For $B = (H_{i_1}, \cdots, H_{i_n}) \in \beta \text{nbc}_{i_0}(A)$, let $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ be the associated flag. Define

$$\Xi(B, A) = \omega_\alpha(F_0, A) \wedge \cdots \wedge \omega_\alpha(F_{n-1}, A).$$

**Definition 2.4.6** If $\beta \text{nbc}_{i_0}(A) = \{B_1, \cdots, B_{\beta(A)}\}$ and $\phi^i_0 = \phi^i_0(A) = \Xi(B_i, A)$, define

$$\Phi^i_0(A) = \{\phi^i_0, \cdots, \phi^i_{\beta(A)}\}.$$  

Define

$$I'_{i_0} = I - \{i_0\}$$

and

$$A'_{i_0} = \{H'_i\}_{i \in I'_{i_0}} \text{ with } H'_i = H_i.$$  

Let

$$A''_{i_0} = \{H_i \cap H_{i_0} \mid i \in I'_{i_0}, H_i \cap H_{i_0} \neq \emptyset\}.$$  

Then $A''_{i_0}$ is an arrangement of hyperplanes in $H_{i_0}$. The linear order on $A'_{i_0}$ is inherited from $A$. If $H'' \in A''_{i_0}$, let $\nu(H'')$ be the smallest hyperplane of $A'_{i_0}$ containing $H''$. We order $A''_{i_0}$ setting $H'' < K''$ if and only if $\nu(H'') < \nu(K'')$.

**Remark 2.4.7** $A'_{i_0}$ and $A''_{i_0}$ are equipped with standard orders.

Let

$$I''_{i_0} = \{i \in I'_{i_0} \mid H_i = \nu(H'') \text{ for some } H'' \in A''_{i_0}\}.$$  

To each $i \in I''_{i_0}$ (or $i \in I'''_{i_0}$), we associate the weights $\alpha'_i := \alpha_i$ (resp. $\alpha''_i := \sum \alpha_k$ where the sum runs over all the $H'_k \in A'_{i_0}$ such that $H'' \subseteq H'_k$).

There is an inductive definition (deletion-restriction) of $\beta \text{nbc}_{i_0}(A)$:
Proposition 2.4.8 (Ziegler[Z, Theorem 1.5] [FT, Theorem 2.4]) Let \((\mathcal{A}, \mathcal{A}_i', \mathcal{A}_i'')\) be the triple defined above. Suppose that \(\mathcal{A}_i'\) is essential. Then

\[
\beta \ nbc_\omega(A) = \beta \ nbc(\mathcal{A}_i') \cup \{(\nu(B''), H_i) \mid B'' \in \beta \ nbc(\mathcal{A}_i'')\}
\]

where \(\nu(B'') = (\nu(H''_1), \ldots, \nu(H''_k))\) if \(B'' = (H''_1, \ldots, H''_k)\).

Remark 2.4.9 Write \(\overline{\beta \ nbc(\mathcal{A}_i')} = \{(\nu(B''), H_i) \mid B'' \in \beta \ nbc(\mathcal{A}_i'')\}\). Elements of \(\beta \ nbc(\mathcal{A}_i')\) are always less than elements of \(\overline{\beta \ nbc(\mathcal{A}_i')}\).

Lemma 2.4.10 If \(B \in \beta \ nbc(\mathcal{A}_i')\), then \(\Xi(B, A)|_{|_{i_0} = 0} = \Xi'(B, \mathcal{A}_i')\).

Proof. Let \(F \in L(A)\). If \(i_0 \notin I(F)\), then \(\omega_\alpha(F, A) = \omega_\alpha(F, \mathcal{A}_i')\) and if \(i_0 \in I(F)\), then \(\omega_\alpha(F, A) = \omega_\alpha(F, \mathcal{A}_i') + \psi \alpha_i \omega_{i_0}\) for some form \(\psi\). \(\Box\)

Lemma 2.4.11 Let \(B'' \in \beta \ nbc(\mathcal{A}_i'')\) and \(B = (\nu(B''), H_i) \in \overline{\beta \ nbc(\mathcal{A}_i'')}\). Then the residue of \(\Xi(B, A)\) along \(H_{i_0}\) is equal to \(\Xi(B'', \mathcal{A}_i'')\).

Proof. The last factor of \(\Xi(B, A)\) is \(\alpha_{i_0} \omega_{i_0}\). Since the product is the exterior one, it follows that \(\alpha_{i_0} \omega_{i_0}\) may be removed as a summand from all the other factors of the product without changing its value. Taking residue of this rewritten product removes the factor \(\alpha_{i_0} \omega_{i_0}\) and restricts the remaining terms to \(H_{i_0}\). The residue is now just \(\Xi(B'', \mathcal{A}_i'')\). \(\Box\)

Example 2.4.12 Let \(A\) be an arrangement in general position. Consider the standard order \(<_p\). Then \(\beta(A) = \binom{p-1}{n}\).

\[
\beta \ nbc_\omega(A) = \{(H_{i_1}, \ldots, H_{i_n}) \mid 1 < i_1 < \cdots < i_n \leq p\},
\]

and

\[
\Phi^p(A) = \{\alpha_{i_1} \cdots \alpha_{i_n} \omega_{i_1} \cdots \omega_{i_n} \mid 1 < i_1 < \cdots < i_n \leq p\}.
\]

Example 2.4.13 Let \(A = \{H_i\}_{1 \leq i \leq 4}\) with

\[
f_1(x, y) = x - y, f_2(x, y) = 1 - x, f_3(x, y) = y, f_4(x, y) = 1 - y, f_5(x, y) = x.
\]

(i) Consider the order \(<_1\):

\[
H_2 <_1 H_3 <_1 H_4 <_1 H_5 <_1 H_1.
\]

• The circuits are \((H_2, H_4, H_1)\) and \((H_3, H_5, H_1)\) and the broken circuits are \((H_4, H_1)\) and \((H_5, H_1)\). Thus

\[
\text{nbc}_1(A) = \{(H_2, H_1), (H_3, H_1), (H_2, H_3), (H_2, H_4), (H_3, H_5), (H_4, H_5)\}
\]

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and
\[ \beta \text{nbc}_1(\mathcal{A}) = \{(H_4, H_5), (H_3, H_1)\}. \]

- We have
  \[ \mathcal{A}'_1 = \{H_2 < H_3 < H_4 < H_5\}, \]
  \[ \mathcal{A}''_1 = \{H_2 \cap H_1, H_3 \cap H_1\}, \]
  \[ \beta \text{nbc}(\mathcal{A}'_1) = \{(H_4, H_5)\}, \beta \text{nbc}(\mathcal{A}''_1) = \{H_{351}\} \]
  \[ (H_{351} := H_3 \cap H_5 = H_3 \cap H_5 \cap H_1, \nu(H_{351}) = \min(H_3, H_5, H_1) = H_2) \]
  and
  \[ \beta \text{nbc}_1(\mathcal{A}) = \beta \text{nbc}(\mathcal{A}'_1) \cup \beta \text{nbc}(\mathcal{A}''_1). \]

- \[ \Phi^1 = \{\alpha_4\alpha_5\omega_{45}, \alpha_3\alpha_1\omega_{31} + \alpha_5\alpha_1\omega_{51}\}. \]

(ii) In the same way,
  \[ \Phi^2 = \{\alpha_4\alpha_5\omega_{45}, \alpha_3\alpha_2\omega_{32}\} \]
  \[ \Phi^3 = \{\alpha_4\alpha_5\omega_{45}, \alpha_2\alpha_3\omega_{23}\} \]
  \[ \Phi^4 = \{\alpha_2\alpha_3\omega_{23}, \alpha_5\alpha_4\omega_{54}\} \]
  \[ \Phi^5 = \{\alpha_2\alpha_3\omega_{23}, \alpha_4\alpha_5\omega_{45}\}. \]

3 Hypergeometric period matrix

3.1 The \( \beta \text{nbc} \)-ordered homology basis

Recall that \( \text{Ch}(\mathcal{A}) \) is the set of real bounded chambers of \( \mathcal{A} \). Let \( \beta = \beta((\mathcal{A}) = |\text{Ch}(\mathcal{A})| \). In order to define the period matrix, we label \( \text{Ch}(\mathcal{A}) \) by \( \beta \text{nbc}_0(\mathcal{A}) \).

Definition 3.1.1 Let \( \xi = (F_0 \subset F_1 \subset \cdots \subset F_n) \) be a flag of affine subspaces \( F_i \in L(\mathcal{A}) \) \( \) with \( \dim F_i = i \) \( (i = 0, \ldots, n) \). Let \( \Delta \in \text{Ch}(\mathcal{A}) \) and \( \overline{\Delta} \) be its closure in \( \mathbb{R}^n \). We say that \( \xi \) is adjacent to \( \Delta \) if \( \dim(F_i \cap \overline{\Delta}) = i \) for \( i = 0, \ldots, n \).

For \( B = (H_{i_1}, \ldots, H_{i_n}) \in \beta \text{nbc}_0(\mathcal{A}) \), recall the associated \( \beta \text{nbc} \)-flag \( \xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n) \) from Definition 2.4.4.

Proposition 3.1.2 There exists a unique bijection
\[ C : \beta \text{nbc}_0(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{A}) \]
with the property that \( \xi(B) \) is adjacent to the bounded chamber \( C(B) \) for any \( B \in \beta \text{nbc}_0(\mathcal{A}) \).
Proof. If $\beta \, \text{nbc}_{t_0}(A) = \emptyset$, then $\text{Ch}(A) = \emptyset$. Suppose $\beta \, \text{nbc}_{t_0}(A) \neq \emptyset$. We will prove by induction on $|A|$. Assume that the maps $C'$ and $C''$ already exist for $A'_{t_0}$ and $A''_{t_0}$. There are the following four kinds of bounded chambers of $A$:

(i) $\Delta \in \text{Ch}(A)$ is called undivided if $\Delta \in \text{Ch}(A'_{t_0})$, i.e., $\Delta$ does not intersect $H_{t_0}$.

(ii) $\Delta \in \text{Ch}(A)$ is called newborn if there exists an unabounded chamber of $A'_{t_0}$ which contains $\Delta$.

(iii) Suppose that a bounded chamber $\Delta'$ of $A'_{t_0}$ is divided in two by $H_{t_0}$. (In this case, a bounded chamber $\Delta'$ of $A'_{t_0}$ is called divided.) Let $B' \in \beta \, \text{nbc}(A'_{t_0})$ with $\Delta' = C'(B')$. Then $\xi' := \xi(B')$ is adjacent to $\Delta'$. The two new chambers of $A$ inside $\Delta'$ are denoted by $\Delta^+$ and $\Delta^-$. We can easily observe that $\xi'$ is adjacent to exactly one of the two chambers $\Delta^+$ and $\Delta^-$, say, $\Delta^+$. The chamber $\Delta^+ \in \text{Ch}(A)$ is called the heir of $\Delta'$. The other chamber $\Delta^- \in \text{Ch}(A)$ is called the cutoff of $\Delta'$.

Define $C : \beta \, \text{nbc}_{t_0}(A) \rightarrow \text{Ch}(A)$ as follows:

(a) If $B \in \beta \, \text{nbc}(A'_{t_0})$ and $C''(B) \in \text{Ch}(A)$, then let $C(B) = C''(B)$.

(b) If $B \in \beta \, \text{nbc}(A'_{t_0})$ and $C'(B)$ is divided in two by $H_{t_0}$, then define $C(B) = \Delta^+$, where $\Delta^+$ is the heir of $C'(B)$.

(c) If $B = (\nu B'', H_{t_0})$ with $B'' \in \beta \, \text{nbc}(A''_{t_0})$ and $C''(B'')$ is inside an unabounded chamber of $A'_{t_0}$, then $C''(B'')$ is a wall of a unique newborn chamber $\Delta$. Define $C(B) = \Delta$.

(d) If $B = (\nu B'', H_{t_0})$ with $B'' \in \beta \, \text{nbc}(A''_{t_0})$ and $C''(B'')$ is inside a bounded chamber $\Delta'$ of $A'_{t_0}$, then $C''(B'')$ is a wall of a unique cutoff chamber $\Delta^-$ of $\Delta'$. Define $C(B) = \Delta^-$.

By construction, $C$ is bijective and satisfies the condition. The uniqueness is obvious from the construction. □

Let $\beta \, \text{nbc}_{t_0}(A) = \{B_1, \ldots, B_\beta\}$ be linearly ordered as in section 2.4. Define $\Delta_i := C(B_i) \in \text{Ch}(A)$ for $i = 1, \ldots, \beta$. We call $\Delta_1, \ldots, \Delta_\beta$ the $\beta \, \text{nbc}_{t_0}$-ordered chambers of $A$. If $\Delta_i$ is either undivided or an heir and $\Delta_j$ is either newborn or a cutoff, then $i < j$.

We give an orientation to each $\Delta \in \text{Ch}(A)$ as follows: Let $\Delta = C(B)$ with $B \in \beta \, \text{nbc}_{t_0}(A)$. Let $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ be the associated $\beta \, \text{nbc}$-flag. Choose the intrinsic orientation $[V2, 6.2]$ of $\Delta$ obtained from $\xi(B)$. In other words, an orthonormal frame $\{e_1, \ldots, e_n\}$ is chosen so that each $e_i$ is a unit vector originating from the point $F_0$ in the direction of $F_i \cap \Delta$. This orientation is called the $\beta \, \text{nbc}$-orientation of $\Delta$.

Example 3.1.3 Recall Example 2.4.13 with the order $<_1$:

$$H_2 <_1 H_3 <_1 H_4 <_1 H_5 <_1 H_1.$$ 

Let $B_1 = (H_4, H_5)$ and $B_2 = (H_3, H_1)$. Then $\beta \, \text{nbc}_1(A) = \{B_1, B_2\}$.
The bijection map

\[ C : \beta \text{ nbc}_1(A) \to \text{Ch}(A) \]

described in Proposition 3.1.2 is given by \( C(B_i) = \Delta_i \) for \( i = 1, 2 \). The \( \beta \text{ nbc-flags} \) are

\[ \xi(B_1) = (H_4 \cap H_5 \subset H_5 \subset V) \]

and

\[ \xi(B_2) = (H_3 \cap H_1 \subset H_1 \subset V). \]

The \( \beta \text{ nbc-orientations} \) of \( \Delta_1 \) and \( \Delta_2 \) are given by orthonormal frames \( \{e_1, e_2\} \) and \( \{e'_1, e'_2\} \) respectively.

Let \( L_\alpha \) be the rank one local system on \( M = M(A) \) (cf. 2.1.4) defined by the kernel of an integral connection

\[ \nabla_\alpha : \mathcal{O}_M \to \Omega^1_M \]

where \( \nabla_\alpha(f) = df + \sum_{i \in I} \alpha_i \omega_i \). Then its monodromy around \( H_k \) is \( e^{-2i\pi \alpha_k} \).

Proposition 3.1.4 (cf. [Ko], [AK, 4.1.1]) Suppose that the weights \( \alpha = (\alpha_1, \ldots, \alpha_p) \) are non-resonant (cf. Definition 2.3.2). We have
(i) \( H_j(M(A), L_\alpha) = H^j_{\text{lf}}(M(A), L_\alpha) = 0 \) pour \( j \neq n \). Here \( H^j_{\text{lf}} \) stands for the locally finite homology.
(ii) The natural map

\[ H_n(M(A), L_\alpha) \to H^j_{\text{lf}}(M(A), L_\alpha) \]

is an isomorphism.
(iii) \( \{[\Delta_j] \mid \Delta_j \in \text{Ch}(A)\} \) forms a basis for \( H^j_{\text{lf}}(M(A), L_\alpha) \).

Proof. The following proof is similar to Kohno's proof [Ko] in the case that the arrangement is generic to infinity.
For (i) and (ii), we use the exactly same argument as [Ko, Theorem 1] except that we blow up \( \mathbb{P}^n \) along all the dense edges of codimension of codimension > 1.
(iii): Write $M = M(A)$ and $\Delta = \bigcup_{j=1}^{g} \Delta_j$. In order to apply the argument in [Ko], it is enough to show $H^q(M - \Delta, L_{\alpha}) = 0$ for all $q$. Let $W$ be a small tubular neighborhood in $\mathbb{P}^n$ of the hyperplane at infinity $\mathbb{H}_{\infty}$. Note that the inclusion map $W \cap M \hookrightarrow M - \Delta$ is homotopy equivalent.

Let $j : W \cap M \hookrightarrow W$. Let $x \in W$. If $x \in W \cap M$, since there is no hyperplane in $A_{\infty}$ going through $x$ we have $(R^q j_* L_{\alpha})_x = 0$ for $q \neq 0$ and $(R^0 j_* L_{\alpha})_x \simeq \mathbb{C}$. If $x \in W - M$, then $W - M$ is locally a central arrangement near $x$. In this case, since the (local) Euler characteristic of $M$ (intersected with a small open ball centered at $x$) is zero, we have $(R^q j_* L_{\alpha})_x = 0$ for all $q$ by (i). Therefore we have $H^q(W \cap M, L_{\alpha}) \simeq H^q(W, j_* L_{\alpha})$. For each $x \in \mathbb{H}_{\infty}$, there exists a small neighborhood $W_x$ of $x$ in $W$ such that

$$H^q(W_{z_1} \cap \cdots \cap W_{z_k}, j_* L_{\alpha}) = 0$$

as long as $W_{z_1} \cap \cdots \cap W_{z_k} \neq \emptyset$. Since $\mathbb{H}_{\infty} \simeq \mathbb{P}^{n-1}$ is compact, we may choose $W_{z_1}, \ldots, W_{z_m}$ which cover $\mathbb{H}_{\infty}$. Let $W_0 = W_{z_1} \cup \cdots \cup W_{z_m}$. By applying the Mayer-Vietoris theorem repeatedly, we have $H^q(W_0, j_* L_{\alpha}) = 0$ for all $q$. By the Poincaré duality, we have

$$H^q(M - \Delta, L_{\alpha}) \simeq H^q(W_0 \cap M, L_{\alpha}) \simeq H^q(W_0, j_* L_{\alpha}) = 0. \Box$$

3.2 The $\beta$ nbc-ordered cohomology basis

**Proposition 3.2.1** Suppose that the weights $\alpha = (\alpha_1, \ldots, \alpha_p)$ are non-resonant. We have

(i) $H^j(M(A), L_{\alpha}) = H^j_c(M(A), L_{\alpha}) = 0$ pour $j \neq n$. Here $H_c$ stands for the compact support cohomology.

(ii) The natural map

$$H^n_c(M(A), L_{\alpha}) \longrightarrow H^n(M(A), L_{\alpha})$$

is an isomorphism.

(iii) The set $\Phi^{i_0}(A)$ (cf. Definition 2.4.6) forms a basis for $H^n(M(A), L_{\alpha})$.

**Proof.** Let $L^{\vee}_{\alpha}$ be the dual local system of $L_{\alpha}$. (i) and (ii) are obtained from the Poincaré dualities

$$H^q(M(A), L_{\alpha}) \simeq H^{1-n-q}_c(M(A), L^{\vee}_{\alpha}), \quad H^q_c(M(A), L_{\alpha}) \simeq H_{2n-q}(M(A), L^{\vee}_{\alpha}),$$

and Proposition 3.1.4. (iii) is [FT, Theorem 3.7]. \Box

If $j_0 \in I$ such that $j_0 \neq i_0$, it may happens that $\Phi^{j_0} \neq \Phi^{i_0}$ (cf. Example 2.4.13). However we have the following

**Proposition 3.2.2** [FT, Proposition 3.10] For all $j_0 \in I$ the transition matrix between the bases $\Phi^{i_0}$ and $\Phi^{j_0}$ is an integral unimodular matrix independent of $\alpha$.  

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3.3 The definition of hypergeometric period matrix

Definition 3.3.1 Let $\beta = \beta(\mathcal{A})$. Assume that $\text{Ch}(\mathcal{A}) = \{\Delta_1, \ldots, \Delta_\beta\}$ is the $\beta$ nbc-ordered chambers in 3.1.4 (iii) and $\Phi^0(\mathcal{A}) = \{\phi^0_1, \ldots, \phi^0_\beta\}$ is the $\beta$ nbc-ordered basis of $H^n(M, \mathcal{L}_\alpha)$ in 3.2.1 (iii). Choose a branch of $f_\alpha^{\circ}$ on each chamber $\Delta_i$. Let

$$U_\alpha := f_\alpha^{\circ_1} \cdots f_\alpha^{\circ_\beta}. $$

Also choose the $\beta$ nbc-orientation of each chamber $\Delta_i$. Define the hypergeometric period matrix $\text{PM}_{i_0}(\mathcal{A}, \alpha)$ by

$$\text{PM}_{i_0}(\mathcal{A}, \alpha) = \left[ \int_{\Delta_i} U_\alpha \phi^0_{i_0} \right].$$

If $\Re \alpha_j > 0, \forall j \in I$, then each entry of $\text{PM}_{i_0}(\mathcal{A}, \alpha)$ can be regarded as an holomorphic function of $\alpha = (\alpha_1, \ldots, \alpha_p)$ and it can be analytically continued to be a meromorphic function on the entire $\mathbb{C}^{\ast}$. In order to precise this point, first introduce the following

Definition 3.3.2 Let $\Delta_\beta^\ast$ be a twisted cycle representing the homology class in $H_n(M(\mathcal{A}), \mathcal{L}_\alpha^\ast)$ which is sent to the homology class of $\Delta_j$ in $H^n(\mathcal{L}(\mathcal{A}), \mathcal{L}_\alpha^\ast)$ via the isomorphism in Proposition 3.1.4 (ii). Define

$$\text{PM}^\ast_{i_0}(\mathcal{A}, \alpha) = \left[ \int_{\Delta_\beta^\ast} U_\alpha \phi^0_{i_0} \right].$$

Remark 3.3.3 It is known (e.g., see [LS, 4.2]) that each entry of $\text{PM}^\ast_{i_0}(\mathcal{A}, \alpha)$ can be regarded as a meromorphic function on $\mathbb{C}^{\ast}$ whose poles lie on some hypersurfaces defined by equations $\phi^0_{i_0} L(\alpha) - \lambda = 0$, where $L$ is a linear form of $\alpha_1, \ldots, \alpha_p$ and $\lambda$ is a nonzero complex number. Since the twisted de Rham pairing

$$H_n(M(\mathcal{A}), \mathcal{L}_\alpha^\ast) \times H^n(M(\mathcal{A}), \mathcal{L}_\alpha) \rightarrow \mathbb{C},$$

which is given by the hypergeometric integrals $(\Delta_\beta^\ast, \phi) \mapsto \int_{\Delta_\beta^\ast} U_\alpha \phi$, is a nondegenerate pairing (e.g., see [Ki, 1.4]), we may write $L(\alpha) = \alpha(F)$ for a dense edge $F$ and $\lambda = 1$ by Propositions 3.1.4 (iii) and 3.2.1 (iii). In other words, $\det \text{PM}^\ast_{i_0}(\mathcal{A}, \alpha)$ takes a finite nonzero value at each $\alpha \not\in \text{Rsn}(\mathcal{A})$. Moreover, if $\Re \alpha_j > 0, \forall j \in I$, then

$$\int_{\Delta_\beta^\ast} U_\alpha \phi^0_{i_0} = \int_{\Delta_i} U_\alpha \phi^0_{i_0}. $$

In particular, if $\Re \alpha_j > 0$ for all $j \in I$,

$$\text{PM}^\ast_{i_0}(\mathcal{A}, \alpha) = \text{PM}_{i_0}(\mathcal{A}, \alpha)$$

and the analytic continuation of the determinant of $\text{PM}_{i_0}(\mathcal{A}, \alpha)$ is equal to the determinant of $\text{PM}^\ast_{i_0}(\mathcal{A}, \alpha)$.

Remark 3.3.4 We formally define $\det \text{PM}_{i_0}(\mathcal{A}, \alpha) = 1$ if $\beta(\mathcal{A}) = 0$.

Proposition 3.3.5 The determinant of $\text{PM}_{i_0}(\mathcal{A}, \alpha)$ is independent (up to sign) of the choice of $i_0$.

Proof. Obvious from Proposition 3.2.2. ⊓⊔
4 Beta function of an arrangement

We keep the notations of Section 2.

4.1

Let $\mathcal{A}$ be an affine arrangement and $\mathcal{A}_\infty$ the arrangement of projective hyperplanes defined in 2.3. Recall $\overline{I} = I \cup \{\infty\} = \{1, \ldots, p, \infty\}$ and $\alpha_\infty = -(\alpha_1 + \cdots + \alpha_p)$. We note that $L_-(\mathcal{A}_\infty)$ (or $L_+(\mathcal{A}_\infty)$) is the set of edges of $L(\mathcal{A}_\infty)$ contained (resp. not contained) in $\overline{H}_\infty$. For $F \in L(\mathcal{A}_\infty)$, let

$$I(F) = \{i \in I \mid F \subset H_i\},$$

$$\overline{I}(F) = \{i \in \overline{I} \mid F \subset \overline{H}_i\},$$

$$\mathcal{A}_\infty^F = \{\overline{H}_i \mid i \in \overline{I}(F)\},$$

$$\mathcal{A}_\infty^F = \{\overline{H}_i \cap F \mid F \not\subset \overline{H}_i\},$$

$$\alpha(F) = \sum_{i \in \overline{I}(F)} \alpha_i = \begin{cases} \sum_{i \notin I(F)} \alpha_i & \text{if } F \in L_+(\mathcal{A}_\infty), \\ -\sum_{i \notin I(F)} \alpha_i & \text{if } F \in L_-(\mathcal{A}_\infty). \end{cases}$$

Following Varchenko [V1, 1.5], we associate to each edge $F$ of codimension $r$ a projective arrangement $\mathbb{P}\mathcal{A}_\infty^F$ in the $r-1$ dimensional projective space. If $\mathcal{A}_\infty$ is a projective arrangement, let $\chi(\mathcal{A}_\infty)$ denote the Euler characteristic of $\mathbb{P}^n - \cup_{i \in I} \overline{H}_i$. If $F$ is an edge of $\mathcal{A}_\infty$ defined above, we put

$$\mu(F, \mathcal{A}_\infty) = |\chi(\mathcal{A}_\infty^F)\chi(\mathbb{P}\mathcal{A}_\infty^F)|.$$

Proposition 4.1.1 If an edge $F \in L(\mathcal{A}_\infty)$ is not dense, then $\mu(F, \mathcal{A}_\infty) = 0$.

Proof. Recall that $F$ is dense if and only if $\chi(\mathbb{P}\mathcal{A}_\infty^F) \neq 0$ [STV, Proposition 7]. □

Definition 4.1.2 Let $j \in I$ and $i \in \{1, \ldots, \beta(A)\}$. Choose a branch of $f^*_j$ on each $\overline{\Delta}_i$. Define

$$R(f_j, \Delta_i)^{\alpha_j} = \{f_j^{\alpha_j}(x) \mid |f_j^{\alpha_j}(x)| \geq |f_j^{\alpha_j}(y)|, \forall y \in \Delta_i\},$$

$$R(A)^{\alpha} = \prod_{j=1}^p \prod_{i=1}^{\beta} R(f_j, \Delta_i)^{\alpha_j},$$

and

$$B(A, \alpha) = \prod_{F \in L_+(\mathcal{A}_\infty)} \Gamma(\alpha(F) + 1)^{\mu(F, \mathcal{A}_\infty)} \prod_{F \in L_-(\mathcal{A}_\infty)} \Gamma(-\alpha(F) + 1)^{-\mu(F, \mathcal{A}_\infty)}.$$
4.2 Recursion formulas for $B(A, \alpha)$ and $R(A)^\alpha$

Let $i_0 \in I$. Define the order $<_{i_0}$ in $A$, and orders in $A'_{i_0}$ and $A''_{i_0}$ as in 2.4.7. Define

$$I^*_{i_0} = \{ i \in I'_{i_0} \mid H_i \cap H_{i_0} = \emptyset \}.$$ 

**Theorem 4.2.1** We have

$$B(A, \alpha) = B(A'_{i_0}, \alpha')B(A''_{i_0}, \alpha'') \times \prod_{F \in L_+ (A_{i_0}), F \subseteq H_{i_0}} \Gamma(\alpha(F) + 1)/\Gamma(\alpha(F) - \alpha_{i_0} + 1) \mu(F, A_{i_0}) \times \prod_{F \in L_- (A_{i_0}), F \not\subseteq H_{i_0}} \Gamma(\alpha(F) - \alpha_{i_0} + 1)/\Gamma(-\alpha(F) + 1) \mu(F, A_{i_0})$$

and

$$R(A)^\alpha = R(A'_{i_0})^{\alpha'} R(A''_{i_0})^{\alpha''} \prod_{i=1}^\mu R(f_{i_0}, \Delta_i)^{\alpha_{i_0}} \prod_{i \in I^*_{i_0}} f_{i_0}^{\alpha_{i_0}} |H_{i_0}|^{\beta(A''_{i_0})}.$$ 

**Proof.** Analogous to the proof of Proposition 6.3 of [L]. \qed

**Corollary 4.2.2** Assume the arrangement $A$ normal. Then

$$B(A, \alpha) = B(A'_{i_0}, \alpha')B(A''_{i_0}, \alpha''\Gamma(\alpha+1))^{\beta(A''_{i_0})} \times \prod_{F \in L_- (A_{i_0}), F \not\subseteq H_{i_0}} \Gamma(\alpha(F) - \alpha_{i_0} + 1)/\Gamma(-\alpha(F) + 1) \mu(F, A_{i_0}).$$

This result was proved first by Varchenko ([V1, Theorem 2.5]).

**Corollary 4.2.3** We have

$$B(A, \alpha) |_{\alpha_{i_0}=0} = B(A'_{i_0}, \alpha')B(A''_{i_0}, \alpha'')$$

and

$$R(A)^\alpha |_{\alpha_{i_0}=0} = R(A'_{i_0})^{\alpha'} R(A''_{i_0})^{\alpha''} \prod_{j \in I^*_{i_0}} f_{j_0}^{\alpha_{j_0}} |H_{i_0}|^{\beta(A''_{i_0})}.$$ 

**Example 4.2.4** 1. If $A$ is in general position

$$B(A, \alpha) = \left[ \prod_{i=1}^p \Gamma(\alpha_i + 1)/\Gamma\left(\sum_{i=1}^p \alpha_i + 1\right) \right]^{\binom{p-2}{2}}.$$ 

2. If $A$ is the arrangement defined in Example 2.4.13,

$$B(A, \alpha) = \prod_{i=1}^5 \Gamma(\alpha_i + 1) \Gamma(\alpha_1 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_4 + 1) \times \left[ \Gamma\left(\sum_{i=1}^5 \alpha_i + 1\right) \Gamma(\alpha_1 + \alpha_3 + \alpha_4 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_5 + 1) \right]^{-1}.$$ 

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5 The main theorem and its proof

5.1 The main theorem

The main result of this paper is the following

Theorem 5.1.1 Suppose $\Re \alpha_i > 0$ for all $i \in I$. Then, for all $i_0 \in I$, we have

$$\det \Pi_{i_0}(A, \alpha) = R(A)^\circ B(A, \alpha),$$

$A$ being equipped with the order $<_{i_0}$ (cf. 2.2.1).

Remark 5.1.2 $R(A)^\circ B(A, \alpha)$ is independent of $i_0$. Thus so is the determinant.

Let $B = (H_{i_1}, \ldots, H_{i_n}) \in \beta$ nbc$_{i_0}(A)$. Recall the associated $\beta$ nbc-flag

$$\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n).$$

Let $\beta$ nbc$_{i_0}(A) = \{B_1, B_2, \ldots, B_\beta\}$. It is shown [BV] that the set $\{\xi(B_1), \ldots, \xi(B_\beta)\}$ of the $\beta$ nbc-flags gives a $\mathbb{Z}$-basis for the flag complex cohomology $H^n(F)$ which is studied by Schechtman and Varchenko in [SV, sections 2, 3] [V3, 10.1]. (It is also known that $H^n(F)$ is naturally isomorphic to the reduced cohomology $\tilde{H}^{n-1}(K(\tilde{L}), \mathbb{Z})$ where $\tilde{L} = L(A) - \{V\}$ and $K(\tilde{L})$ is the order complex of $\tilde{L}$ [FT, Remark 3.8].) For an arbitrary flag $\xi = (F_0 \subset F_1 \subset \cdots \subset F_n)$, $F_i \in L(A)$, $\dim F_i = i$ ($i = 0, \ldots, n$), associate a differential $n$-form

$$\Xi(\xi) = \omega_A(F_0, A) \wedge \cdots \wedge \omega_A(F_{n-1}, A)$$

(cf. Notation 2.4.5). Consider the homomorphism

$$\pi_\alpha : H^n(F) \otimes \mathbb{C} \longrightarrow H^n(M(A), L_\alpha)$$

such that $\pi_\alpha([\xi] \otimes 1) = [\Xi(\xi)]$. Then $\pi_\alpha([\xi(B_i)] \otimes 1) = [\Phi_i^\alpha(A)]$ for $1 \leq i \leq \beta$. The map $\pi_\alpha$ is an isomorphism when $\alpha$ is non-resonant.

Corollary 5.1.3 Suppose $\Re \alpha_i > 0$ for all $i \in I$. Choose a branch of $f_j^{\alpha_i}$ on each $\Delta_i \subset \text{Ch}(A)$. Let $\xi_1, \ldots, \xi_\beta$ be flags of length $n + 1$ such that their cohomology classes in the flag complex cohomology $H^n(F)$ form a $\mathbb{Z}$-basis. Let $\psi_i(\alpha) = \Xi(\xi_i)$ for $1 \leq i \leq \beta$. Then we have

$$\det \left[ \int_{\Delta_i} U_\alpha \psi_i(\alpha) \right] = \pm R(A)^\circ B(A, \alpha).$$

Proof. Since $\{\xi(B_1), \ldots, \xi(B_\beta)\}$ and $\{\xi_1, \ldots, \xi_\beta\}$ are connected by a unimodular integral constant matrix in $H^n(F)$, so are $\Phi^\alpha(A)$ and $\{\psi_1(\alpha), \ldots, \psi_\beta(\alpha)\}$ in $H^n(M(A), L_\alpha)$. Apply Theorem 5.1.1. □
Remark 5.1.4 Theorem 5.1.1 shows that the conjecture by Varchenko ([V2, 6.3 Fundamental conjecture]) is true for the β nbc-bases and the β nbc-orientations. If Varchenko’s flags $F_\Delta_j$ (1 ≤ j ≤ β) in [V2, 6.2] form a Z-basis for $H^n(\mathcal{F})$, then the affirmative answer (up to sign) to the original conjecture follows from Corollary 5.1.3. (In general, the original conjecture by Varchenko is always true up to a constant integral multiple by Theorem 5.1.1. Especially, for 2-dimensional arrangements, M. Neergaard and the second author have verified the original conjecture by studying the relationship between the β nbc-flags and Varchenko’s flags.) Note that the basis $\Phi^p$ coincides (up to sign) with Varchenko’s basis $\{\Xi(F_\Delta_j)\}$ when $\Lambda$ is normal or in general position at infinity.

5.2 A theorem by Loeser-Sabbah

Recall, at first, one of the main results of [LS].

Theorem 5.2.1 We have

$$\det \text{PM}_{(e)}(\mathcal{A}, \alpha) = c_1^{\alpha_1} \cdots c_p^{\alpha_p} B(\mathcal{A}, \alpha) h_{i_0}(\alpha),$$

where $c_1, \cdots, c_p$ are nonzero constants and $h_{i_0} \in C(\alpha_1, \cdots, \alpha_p)^*$. 

Proof. By [LS, 4.2.10], we have

$$\det \text{PM}^*(\mathcal{A}, \alpha) = \varphi_{i_0}(e^{2i\pi\alpha_1}, \cdots, e^{2i\pi\alpha_p}) c_1^{\alpha_1} \cdots c_p^{\alpha_p} B(\mathcal{A}, \alpha) h_{i_0}(\alpha),$$

where $\varphi_{i_0}$ is a periodic function of $\alpha = (\alpha_1, \cdots, \alpha_p)$, $c_1, \cdots, c_p$ are nonzero constants and $h_{i_0} \in C(\alpha_1, \cdots, \alpha_p)^*$. Since the polynomials $f_i$ take real values on each $\Delta_i$ and $\det \text{PM}^*(\mathcal{A}, \alpha)$ is holomorphic if $\Re \alpha_i > 0$ for all $i \in I$ by Remark 3.3.3, $\varphi_{i_0}(e^{2i\pi\alpha_1}, \cdots, e^{2i\pi\alpha_p})$ is constant by [LS, final remark] (see also Remark 5.4.3). Denote $\varphi_{i_0} h_{i_0}(\alpha)$ by $h_{i_0}(\alpha)$. □

Remark 5.2.2 The constants $c_k$ are defined as critical values (counted with multiplicities) of the polynomials $f_k$.

Recall that $\det \text{PM}^*(\mathcal{A}, \alpha) = \det \text{PM}_{(e)}(\mathcal{A}, \alpha)$ if $\Re \alpha_i > 0$ for all $i \in I$. We shall show that $h_{i_0}$ equals +1 and $c_k^{\alpha_k} = \prod_{j=1}^\infty R(f_k, \Delta_j)^{\alpha_k}$.

5.3 A recursion formula for hypergeometric period determinants

Let $i_0 \in I$. Give $\mathcal{A}$ the linear order $<_{i_0}$. Recall $\mathcal{A}_{i_0}', \mathcal{A}_{i_0}''$, $I_{i_0}'$, and $I_{i_0}''$ from 2.4. To each $i \in I_{i_0}'$ (or $i \in I_{i_0}''$), we associate the weights $\alpha_i := \alpha_i$ (resp. $\alpha_i' := \sum \alpha_k$ where the sum runs over all the $H_i' \in \mathcal{A}_{i_0}'$ such that $H_i'' \subset H_i'$). Let $\alpha' = (\alpha_i)_{i \in I_{i_0}'}$ and $\alpha'' = (\alpha_i')_{i \in I_{i_0}''}$. Denote the restriction of $f_j$ to $H_{i_0}$ by $\tilde{f}_j$. Define

$$U'_\alpha = \prod_{j \in I_{i_0}'} f_j^{\alpha'_j}, U''_{\alpha'} = \prod_{j \in I_{i_0}''} \tilde{f}_j^{\alpha''_j}.$$
Recall \( I_{i_0} = \{ i \in I_{i_0} \mid H_i \cap H_{i_0} = \emptyset \} \). Fix a branch of \( \prod_{i \in I_{i_0}} (f_{i_0}^j)_{i \in I_{i_0}} \) and call it \( c_{i_0} \). Then \( c_{i_0} \) is a constant number. Choose a branch of \( U' \) and a branch of \( U'' \) on each bounded chamber of \( \text{Ch}(\mathcal{A}_{i_0}') \) and on each bounded chamber of \( \text{Ch}(\mathcal{A}_{i_0}'') \) respectively. Also choose a branch of \( f_{i_0}^{j_{i_0}} \) on each bounded chamber \( \Delta \in \text{Ch}(\mathcal{A}) \). Define a branch \( U_{\Delta} \) of \( U \) on \( \Delta \in \text{Ch}(\mathcal{A}) \) as follows (we use the terminology from the proof of Proposition 3.1.2):

(i) if \( \Delta \) is divided, then \( \Delta \in \text{Ch}(\mathcal{A}_{i_0}') \). Define \( U_{\Delta} = (f_{i_0}^{j_{i_0}})_{\Delta}(U'_{\Delta} \circ \Delta) \).

(ii) if \( \Delta \) is the heir of \( \Delta' \in \text{Ch}(\mathcal{A}_{i_0}') \), then define \( U_{\Delta} = (f_{i_0}^{j_{i_0}})_{\Delta}(U'_{\Delta} \circ \Delta') \).

(iii) if \( \Delta \) is either a cutoff or newborn, then let \( \Delta'' \in \text{Ch}(\mathcal{A}_{i_0}'') \) be the wall of \( \Delta \). Choose a unique branch \( U''_{\Delta} \) of \( U'' \) on \( \Delta \) such that \( U''_{\Delta} \circ \Delta'' = \Delta (U''_{\Delta} \circ \Delta'') \). Let \( U_{\Delta} = (f_{i_0}^{j_{i_0}})_{\Delta}(U''_{\Delta} \circ \Delta') \).

Recall that we are using the \( \beta \) nbc-orientation for every chamber of \( \text{Ch}(\mathcal{A}_{i_0}') \), \( \text{Ch}(\mathcal{A}_{i_0}'') \), and \( \text{Ch}(\mathcal{A}) \). Then

(i) if \( \Delta \in \text{Ch}(\mathcal{A}) \) is the heir of \( \Delta' \in \text{Ch}(\mathcal{A}_{i_0}') \), then the corresponding \( \beta \) nbc-flags are equal. So the orientation of \( \Delta \) is the induced one from the orientation of \( \Delta' \), and

(ii) if \( \Delta \in \text{Ch}(\mathcal{A}) \) is either a cutoff or newborn, then the orthonormal frame for \( \Delta \) is given by the orthonormal frame for \( \Delta'' := \Delta \cap H_{i_0} \in \text{Ch}(\mathcal{A}_{i_0}'') \) together with the unit vector in the direction of \( \Delta \) as the last vector of the frame.

Define \( \text{PM}_{i_0}(\mathcal{A}, \alpha) \), \( \text{PM}(\mathcal{A}_{i_0}', \alpha') \) and \( \text{PM}(\mathcal{A}_{i_0}'', \alpha'') \) using these branches and orientations.

We analytically continue the determinant \( \text{det PM}_{i_0}(\mathcal{A}, \alpha) \) onto the hyperplane \( \alpha_{i_0} = 0 \).

**Proposition 5.3.1** Suppose that the real part of \( \alpha_i \) is positive for all \( i = 1, \ldots, p \), \( i \neq i_0 \). Then

\[
\text{det PM}_{i_0}(\mathcal{A}, \alpha)_{\alpha_{i_0} = 0} = \text{det PM}(\mathcal{A}_{i_0}', \alpha') \text{det PM}(\mathcal{A}_{i_0}'', \alpha'')^{\beta(\mathcal{A}_{i_0}'')}.
\]

**Proof.** Let \( \Delta' \in \text{Ch}(\mathcal{A}_{i_0}') \) be a divided chamber. Let \( \Delta^+ \) and \( \Delta^- \) be its heir and cutoff respectively. Let \( U^+ \) and \( U^- \) be the branches of \( U_{\Delta} \) on \( \Delta^+ \) and \( \Delta^- \). Choose a constant number \( c_{\Delta'} \) such that \( U^+ = c_{\Delta'} U^- \) on \( \Delta' \cap H_{i_0} \). Let \( M \) be the matrix obtained from \( \text{PM}_{i_0}(\mathcal{A}, \alpha) \) by adding for each divided chamber \( \Delta' \in \text{Ch}(\mathcal{A}_{i_0}') \) the column corresponding to the cutoff \( \Delta^- \) of \( \Delta' \) multiplied by \( c_{\Delta'} \) to the column corresponding to the heir \( \Delta^+ \) of \( \Delta' \) and setting \( \alpha_{i_0} = 0 \). Then \( \text{det} M = \text{det} \text{PM}_{i_0}(\mathcal{A}, \alpha)_{\alpha_{i_0} = 0} \). Note that the column of \( M \) corresponding to \( \Delta^+ \) is

\[
\left[ \int_{\Delta^+} U'_{\Delta} f_{i_0}^{j_{i_0}} \phi_i, \ldots, \int_{\Delta^+} U'_{\Delta} f_{i_0}^{j_{i_0}} \phi_\beta \right]_{i_{0}\alpha_{i_0} = 0}.
\]

Write

\[
M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}
\]

where \( P \) is a square matrix of size \( \beta(\mathcal{A}_{i_0}') \) and \( S \) is a square matrix of size \( \beta(\mathcal{A}_{i_0}'') \). Since the first \( \beta(\mathcal{A}_{i_0}') \) columns of \( M \) are labelled by \( \text{Ch}(\mathcal{A}_{i_0}') \), it follows from Lemma 2.4.10 that \( P = \text{PM}(\mathcal{A}_{i_0}', \alpha') \).

When computing \( R \) and \( S \) we may take \( \phi = \Xi(\mathcal{B}) \) where \( \mathcal{B} = (\nu B'', H_{i_0}) \in \beta \text{ nbc}(\mathcal{A}_{i_0}'') \) with \( B'' \in \beta \text{ nbc}(\mathcal{A}_{i_0}'') \). Write \( \phi = \psi(\alpha_{i_0} \omega_{i_0}) \). Let \( \Delta' \in \text{Ch}(\mathcal{A}_{i_0}') \). Set \( \Delta' = \Delta' \cap \{ f_{i_0} = t \} \).
and \( F(t) = \int_{\Delta_t} U'_t \psi \). Define real numbers \( a < b \) such that \( \Delta'_t \neq \emptyset \) if and only if \( a \leq t \leq b \). Using the variable \( t = t_{io} \), Fubini's theorem and integration by parts give

\[
\pm \int_{\Delta_t} U'_t \int_0^{\alpha_{io}} \phi = \int_a^b \alpha_{io} \int_0^{\alpha_{io}} F(t) dt = \left[ t^{\alpha_{io}} F(t) \right]_a^b - \int_a^b t^{\alpha_{io}} F'(t) dt.
\]

Taking the limit as \( \alpha_{io} \to 0 \), \( \Re \alpha_{io} > 0 \), we get

\[
\lim \left[ \left[ t^{\alpha_{io}} F(t) \right]_a^b - \int_a^b t^{\alpha_{io}} F'(t) dt \right] = \begin{cases} 0 & 0 \notin \{a, b\} \\ F(0) & 0 = a < b \\ -F(0) & a < b = 0. \end{cases}
\]

If \( \Delta' \) is divided, then we apply the first part to get zero. If \( H_{io} \) intersects \( \Delta' \) in a face of codimension \( > 1 \), then \( F(0) = 0 \). If \( H_{io} \) does not intersect \( \Delta' \), then the integral is again zero. Thus \( M(\Delta', \phi) = 0 \). This shows that \( R = 0 \).

It remains to compute the entries of \( S \). Let \( \Delta \in \text{Ch}(A) \) be either a cutoff or newborn. In this case \( H_{io} \) is a wall of \( \Delta \) so \( \Delta_0 = C''(B'') \). Let \( \Delta'' = C''(B'') \). It follows from Lemma 2.4.11 that \( \phi'' := \psi|_{\Delta''} = \Xi(B'', A'_{io}) \). Set \( \Delta_t = \Delta \cap \{f_{io} = t\} \) and \( G(t) = \int_{\Delta_t} U'_t \psi \), where \( U'_t \) is a unique branch of \( U'_{io} \) on \( \Delta \) such that \( U'_{\Delta|\Delta''} = \xi_{io}(U''_{io} \text{ on } \Delta'') \). Define real numbers \( a < b \) such that \( \Delta_t \neq \emptyset \) if and only if \( a \leq t \leq b \). Then \( 0 \notin \{a, b\} \). Recall the choice of branch of \( U'_{io} \) on \( \Delta \) and orientation of \( \Delta \). By the same calculation as above, using the variable \( t = f_{io} \), Fubini's theorem and integration by parts, we get

\[
M(\Delta, \phi) = \lim \int_{\Delta} U' \phi = G(0) = \xi_{io} \int_{\Delta''} U'' \phi'' = \xi_{io} M(\Delta'', \phi'').
\]

as \( \alpha_{io} \to 0 \), \( \Re \alpha_{io} > 0 \). So \( S = \xi_{io} \text{PM}(A''_{io}, \alpha'') \). Thus we have

\[
\det \text{PM}_{io}(A, \alpha)_{|\alpha_{io}=0} = \det M = (\det P)(\det S) = \det \text{PM}(A'_{io}, \alpha') \det \text{PM}(A''_{io}, \alpha'') \xi_{io}^{\beta(A''_{io})}. \]

Corollary 5.3.2

\[
\det \text{PM}^{*}(A, \alpha)_{|\alpha_{io}=0} = \det \text{PM}^{*}(A'_{io}, \alpha') \det \text{PM}^{*}(A''_{io}, \alpha'') \xi_{io}^{\beta(A''_{io})}.
\]

**Proof.** When the real part of \( \alpha_i \) is positive for all \( i = 1, \ldots, p \), \( i \neq i_0 \), this functional equality has been proved in Proposition 5.3.1. Therefore this equality holds true everywhere. □

**Remark 5.3.3** The recursion formula 5.3.1, in the case of arrangements in general position, is found in [V1, p.546].
5.4 Proof of the main theorem

We prove the theorem by induction on \((n, p)\) (equipped with lexicographical order). If \(n = 1\) the theorem is well-known. If \(\beta(A) = 0\), then the theorem asserts \(1 = 1\). Note that \(\beta(A) = 0\) whenever \(p \leq n\). Let \(i_0 \in I\). From the induction hypothesis we have

\[
\det PM^*(A_{i_0}', \alpha') = R(A_{i_0}')^{\alpha'}B(A_{i_0}', \alpha')
\]

and

\[
\det PM^*(A_{i_0}''', \alpha''') = R(A_{i_0}''')^{\alpha'''}B(A_{i_0}''', \alpha''').
\]

First step: we determine the product of critical values.

The induction hypothesis gives, together with Corollary 5.3.2 and Corollary 4.2.3,

\[
c_1^{\alpha_1} \cdots c_p^{\alpha_p} \mid_{\alpha_i = 0} = R(A)^{\alpha} \mid_{\alpha_i = 0}.
\]

Thus we have \(c_k^{\alpha_k} = \prod_{j=1}^g R(f_k, \Delta_j)^{\alpha_j} \) for \(k \neq i_0\). By considering another linear order \(\prec_k\) \((k \neq i_0)\), we have

\[
c_1^{\alpha_1} \cdots c_p^{\alpha_p} \mid_{\alpha_k = 0} = R(A)^{\alpha} \mid_{\alpha_k = 0}
\]

so \(c_k^{\alpha_k} = \prod_{j=1}^g R(f_k, \Delta_j)^{\alpha_j}\).

Second step: we determine the rational function.

We have

\[
\det PM^*_{i_0} (A, \alpha) = R(A)^{\alpha} B(A, \alpha) h_{i_0}(\alpha).
\]

First summarize what we know about the rational function \(h_{i_0}\). Let

\[
\mathcal{L} = \{\alpha(F) + m \mid F \text{ is a dense edge in } L(A_\infty), m \in \mathbb{Z}\}.
\]

Lemma 5.4.1 (i) \(h_{i_0}\) is independent (up to sign) of \(i_0\).

(ii) The numerator and the denominator of \(h\) are (up to sign) products of linear forms belonging to \(\mathcal{L}\).

(iii) For all \(i_0 \in I\), \(h(\alpha_1, \cdots, \alpha_p) \mid_{\alpha_i = 0}\) is equal to either 1 or \(-1\).

Proof. (i) follows from Proposition 3.3.5 and the fact that both \(B(A, \alpha)\) and \(R(A)^{\alpha}\) are independent of \(i_0\).

As for (ii), recall that the determinant of \(PM_{i_0}(A, \alpha)\) takes a finite nonzero value at each \(\alpha \notin \text{Rsn}(A)\) by Remark 3.3.3. Neither of \(B(A, \alpha)\) or \(R(A)^{\alpha}\) has a zero or pole at \(\alpha \notin \text{Rsn}(A)\) (cf. Remark 4.1.4). Therefore \(h\) is a rational function which takes a finite nonzero value at every \(\alpha \notin \text{Rsn}(A)\). Since \(\text{Rsn}(A)\) is the union of a locally finite infinite family of hyperplanes, we have (ii).

Lastly, (iii) is a consequence of the induction assumption, Corollary 5.3.2 and Corollary 4.2.3. \(\Box\)
Lemma 5.4.2 \( h \) is equal to a constant function which is either 1 or \(-1\).

Proof. Suppose that \( h \) is not constant. By Lemma 5.4.1(ii), we may write \( h \) as a fraction whose denominator and numerator are both products of finitely many elements of

\[ \mathcal{L} = \{ \alpha(F) + m \mid F \text{ is a dense edge in } L(A_{\infty}), m \in \mathbb{Z} \} \]

Suppose \( \alpha(F) + m \) appears in the expression. By Lemma 5.4.1(iii), \( \alpha(F) - \alpha_j + m \) also appears in the expression for each \( j \) such that \( \alpha_j \) appears in \( \alpha(F) \). Also, \( \alpha(F) + \alpha_j + m \) also appears in the expression for each \( j \) such that \( \alpha_j \) does not appear in \( \alpha(F) \). Therefore, by repeatedly using these observations, we finally can conclude that \( \sum_{i \in \mathcal{I}} \alpha_i + m \) appears for every subset \( J \) of \( I \). In particular, \( \alpha_1 + \cdots + \alpha_{p-1} + m \) appears in the expression. This implies either (i) \( F := H_1 \cap \cdots \cap H_{p-1} \) is dense and \( I(F) = \{1, 2, \ldots, p-1\} \), or (ii) \( F_\infty := H_p \cap H_\infty \) is dense and \( \overline{I(F_\infty)} = \{p, \infty\} \). Since (ii) is a contradiction, (i) always occurs. In particular, \( H_1, \ldots, H_{p-1} \) are dependent and there exists \( j_0 \in \{1, \ldots, p-1\} \) such that \( F = H_1 \cap \cdots \cap H_{j_0-1} \cap H_{j_0+1} \cap \cdots \cap H_{p-1} \). If \( A \) is central, there is nothing to prove. So we may assume \( \emptyset = H_1 \cap \cdots \cap H_p \). Thus

\[ \emptyset = H_1 \cap \cdots \cap H_p = F \cap H_p = H_1 \cap \cdots \cap H_{j_0-1} \cap H_{j_0+1} \cap \cdots \cap H_p. \]

This implies that \( \alpha_1 + \cdots + \alpha_{j_0-1} + \alpha_{j_0+1} + \cdots + \alpha_p + m \) does not appear in the expression of \( h \), which is a contradiction. This shows that \( h \) is a constant. By Lemma 5.4.1(iii), the constant is equal to either 1 or \(-1\). □

It follows from Lemma 5.4.2 that

\[ \det \text{PM}_\mathcal{I}_0(A, \alpha) = \pm R(A)^o B(A, \alpha). \]

Let us determine the sign. It is known that the sign is plus when \( n = 1 \) or \( \beta(A) = 0 \). By Corollaries 5.3.2 and 4.2.3, we can inductively show that the sign is always plus:

\[ \det \text{PM}_\mathcal{I}_0(A, \alpha) = R(A)^o B(A, \alpha). \]

This, together with Remark 3.3.3, proves the main theorem.

Remark 5.4.3 We could show in the same way that the periodic function \( \varphi_\mathcal{I}_0 \) appearing in the proof of theorem 5.2.1 is constant.

Remark 5.4.4 It should be interesting to study the connection between the roots of a Bernstein polynomial of \( f = (f_1, \cdots, f_p) \) ([S]) and the poles of \( \det \text{PM}_\mathcal{I}_0(A, \alpha) \).

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