The Reeh-Schlieder Property for Thermal Field Theories

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Abstract. We show that the Reeh-Schlieder property w.r.t. KMS states is a direct consequence of locality,
additivity and the relativistic KMS condition. The latter characterizes the thermal equilibrium states of a
relativistic quantum field theory. The statement remains valid even if the given equilibrium state breaks spatial
translation invariance.

I. Introduction

In a relativistic quantum theory (even of massive particles), the vacuum state, which is
simply characterized by Poincaré invariance and the spectrum condition, has a very rich
intrinsic structure. The full content of the theory can be described in terms of vacuum
expectation values. And there are more surprises out there. While studying a relativistic
quantum theory and its vacuum state, one encounters a number of most peculiar prop-
erties, which require a drastic departure from ‘classical’ quantum mechanics and its in-
terpretation. The famous Reeh-Schlieder property may be seen as one of the origins of
these peculiarities; with astonishing consequences for the theory and its interpretation.

The Reeh-Schlieder property for thermal equilibrium states was first proven by Jung-
las. He assumed that the thermal equilibrium state is locally normal w.r.t. the vacuum and
his proof relied on a result of Borchers concerning timelike cylinders in the vacuum repre-
sentation. Only time translations were used and therefore the (standard) KMS condition,
which characterizes equilibrium states, was sufficient. Here we present a self-contained
derivation of the Reeh-Schlieder property, which does not rely on results concerning the
vacuum sector, but instead takes advantage of the relativistic KMS condition, recently
proposed by Bros and Buchholz. We will introduce it in the next section. We would like
to emphasize that we do not require that there exists a group of unitary operators which
implies spacelike translations in the thermal representation associated with a given
equilibrium state. In fact, our proof remains applicable even if the thermal state breaks translation or rotation symmetry. It uses Glaser’s Theorem and exploits the characteristic analyticity properties of an equilibrium state. The latter simply reflect the basic stability and passivity properties of an equilibrium state.

**Remark.** Recently the Cluster Theorem has been generalized to thermal states. By simply combining the KMS condition with the locality assumption the author was able to show that there is a tight relation between the infrared properties of the generator of time translations and the decay of spatial correlations in any extremal KMS state, in complete analogy to the well understood case of the vacuum state. To be more precise, since the spectrum of the generator of the time-evolution in the thermal sector does not have a mass gap, a new condition proposed by Buchholz, which may be interpreted as a type of Hölder continuity of the spectrum at the discrete eigenvalue zero, has been used to show that the correlations between two spacelike separated measurements decay like some inverse power of their spatial distance. The correlations of free massless bosons in two dimensions saturate these bounds.

To conclude this introduction we line out the content of this paper. In Section 2 basic properties of thermal quantum field theories and their representations are collected, including the relativistic KMS condition of Bros and Buchholz. Section 3 contains the derivation of the Reeh-Schlieder property for thermal equilibrium states. A brief outlook is given in the final section.

## II. Thermal Quantum Field Theory

In the algebraic formulation a QFT is casted into an inclusion preserving map

\[ \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \]  

which assigns to any open bounded region \( \mathcal{O} \) in Minkowski space \( \mathbb{R}^4 \) a unital \( C^* \)-algebra \( \mathcal{A}(\mathcal{O}) \). The Hermitian elements of the abstract \( C^* \)-algebra \( \mathcal{A}(\mathcal{O}) \) are interpreted as the observables which can be measured at times and locations in \( \mathcal{O} \). The net \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \) is isotonous, i.e., there exists a unital embedding

\[ \mathcal{A}(\mathcal{O}_1) \hookrightarrow \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2. \]  

For mathematical convenience the local algebras are embedded in the \( C^* \)-inductive limit algebra

\[ \mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{A}(\mathcal{O})}^{C^*}. \]  

The space–time symmetry of Minkowski space manifests itself in the existence of a representation

\[ \alpha: (\Lambda, x) \mapsto \alpha_{\Lambda, x} \in Aut(\mathcal{A}), \quad (\Lambda, x) \in \mathcal{P}^+_+, \]
of the (orthochronous) Poincaré group $\mathcal{P}_+$. Lorentz-transformations $\Lambda$ and space–time translations $x$ act geometrically:

$$\alpha_{\Lambda,x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda \mathcal{O} + x) \quad \forall (\Lambda, x) \in \mathcal{P}_+.$$  

(5)

Einstein causality is implemented by locality: observables localized in spacelike separated space–time regions commute, i.e.,

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}^c(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'.$$

(6)

Here $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathcal{A}^c(\mathcal{O})$ denotes the set of operators in $\mathcal{A}$ which commute with all operators in $\mathcal{A}(\mathcal{O})$.

Remark. Let $h \in L^1(\mathbb{R}^4, d^4x)$ such that the Fourier-transform $\tilde{h}$ of $h$ has compact support. Strong continuity of the group of automorphisms $x \mapsto \alpha_x$ implies that the Bochner integral

$$a_h = \int d^4x \ h(x) \alpha_x(a), \quad a \in \mathcal{A},$$

(7)

exists in $\mathcal{A}$ and defines an entire analytic element for the translations. (If the map $x \mapsto \alpha_x$ fails to be strongly continuous, then we may proceed by simply restricting the given net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ to the subnet consisting of those elements of $\mathcal{A}$ which comply with the continuity condition (see, e.g., Ref. 5, Proposition 1.18).) Recall that $b \in \mathcal{A}$ is called an entire analytic element for the group of automorphisms $x \mapsto \alpha_x$, if there exists a function $g: \mathbb{C}^4 \to \mathcal{A}$ such that

i) $g(x) = \alpha_x(b)$ for all $x \in \mathbb{R}^4$;

ii) $z \mapsto \omega(g(z))$ is entire analytic for all positive linear funtionals $\omega$ over $\mathcal{A}$.

The algebra $\mathcal{A}_\alpha$ of entire analytic elements is norm dense in $\mathcal{A}$.

States are, by definition, positive, linear and normalized functionals over $\mathcal{A}$. It is an advantage of the abstract setting that the thermal equilibrium states can be distinguished among the set of all (physical) states by first principles such as time invariance, stability against small perturbations or passivity properties (see Ref. 6 and 7). Adding a few technical assumptions one usually ends up (see Ref. 8 and 9) with a precise mathematical selection criterion: the KMS condition. Today the KMS condition is generally accepted as the appropriate mathematical criterion for equilibrium. But only recently, Buchholz and Junglas have shown that the characterization of equilibrium states by the KMS condition applies even to a large class of relativistic models.10

Lorentz invariance is always broken by a KMS state.11,12 A KMS state might also break spatial translation or rotation invariance, but the maximal propagation velocity of signals, which is characteristic for a relativistic theory, will not be affected by such a lack of symmetry. It was first recognized by Bros and Buchholz that a finite maximal propagation velocity of signals basically implies that the KMS states of a relativistic QFT have stronger analyticity properties in configuration space than those imposed by the traditional KMS condition.2 These properties are summarized in the following
**Definition II.1.** A state \( \omega_\beta \) satisfies the relativistic KMS condition at inverse temperature \( \beta > 0 \) if and only if there exists some positive timelike vector \( e \in V_+ \) such that for every pair of elements \( a, b \) of \( A \) there exists a function \( F_{a,b} \) which is analytic in the domain 

\[-T_{\beta e/2} \times T_{\beta e/2}, \]

(8) 

where \( T_{\beta e/2} = \{ z \in \mathbb{C} : \exists z \in V_+ \cap (\beta e/2 + V_-) \} \) is a tube, and continuous at the boundary sets \( \mathbb{R}^4 \times \mathbb{R}^4 \) and \( (\mathbb{R}^4 - \frac{1}{2} \beta e) \times (\mathbb{R}^4 + \frac{1}{2} \beta e) \) with boundary values given by 

\[ F_{a,b}(x_1, x_2) = \omega_\beta(\alpha_{x_1}(a)\alpha_{x_2}(b)) \]

\[ F_{a,b}(x_1 - \frac{i}{2} \beta e, x_2 + \frac{i}{2} \beta e) = \omega_\beta(\alpha_{x_2}(b)\alpha_{x_1}(a)) \quad \forall x_1, x_2 \in \mathbb{R}^4. \]

(9) 

The relativistic KMS condition can be understood as a remnant of the relativistic spectrum condition in the vacuum sector. It has been rigorously established\(^2\) for the KMS states constructed by Buchholz and Junglas.\(^10\) In this letter we will show that together with the condition of additivity (see (13) below) it implies that the KMS state has the Reeh-Schlieder property.

Once a relativistic KMS state \( \omega_\beta \) for some inverse temperature \( \beta \) is fixed, the well known GNS-construction provides a Hilbert space \( \mathcal{H}_\beta \), a cyclic vector \( \Omega_\beta \in \mathcal{H}_\beta \) and a ‘thermal representation’ \( \pi_\beta \) of \( A \) such that 

\[ \omega_\beta(a) = (\Omega_\beta, \pi_\beta(a)\Omega_\beta) \quad \forall a \in A. \]  

(10) 

Due to the KMS condition the vector \( \Omega_\beta \) is not only cyclic for \( \mathcal{R}_\beta := \pi_\beta(A)'' \) but also separating. (Note that a priori the relativistic KMS condition only applies to elements of \( A \) and in general it will not extend to \( \mathcal{R}_\beta \).) Thus any state, which is normal w.r.t. \( \pi_\beta \), is a vector state (see Ref. 9, 2.5.31).

**Remark.** Our main concern may be formulated as follows: Given a state \( \omega : A \rightarrow \mathcal{C} \), can we find an element \( a_\omega \) in \( A(\mathcal{O}) \) (representing a strictly local operation in \( \mathcal{O} \)) such that \( \| \omega - \hat{\omega} \| < \epsilon \), where \( \hat{\omega} \) is identified with the normal state induced by \( \Omega_{\hat{\omega}} := \pi_\beta(a_\omega)\Omega_\beta \)? Obviously, it is sufficient to prove that \( \Omega_{\hat{\omega}} \) can be chosen arbitrarily close to \( \Omega_\omega \) in the Hilbert space topology, where \( \Omega_\omega \) is the state vector associated with the state \( \omega \). This will be a direct consequence of Theorem III.9.

The representation \( \pi_\beta \) assigns to any \( \mathcal{O} \subset \mathbb{R}^4 \) a von Neumann algebra 

\[ \mathcal{R}_\beta(\mathcal{O}) = \pi_\beta(A(\mathcal{O}))'' . \]

(11) 

The weak closure respects the local structure, i.e., 

\[ \mathcal{R}_\beta(\mathcal{O}_1) \subset \mathcal{R}_\beta(\mathcal{O}_2)' \quad \text{for} \quad \mathcal{O}_1 \subset \mathcal{O}_2. \]

(12) 

Note that \( \mathcal{R}_\beta(\mathcal{O})' \) denotes the commutant of \( \mathcal{R}_\beta(\mathcal{O}) \) in the algebra \( \mathcal{B}(\mathcal{H}_\beta) \) of all bounded operators on \( \mathcal{H}_\beta \). We emphasize that \( \mathcal{R}_\beta(\mathcal{O})' \) includes both the algebra \( \mathcal{R}_\beta' \), which itself is isomorphic to \( \mathcal{R}_\beta \), and \( \mathcal{R}_\beta(\mathcal{O}') \) as subalgebras.
Definition II.2. The net $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ is called additive, if

$$
\cup_{i \in I} \mathcal{O}_i = \mathcal{O} \Rightarrow \vee_{i \in I} \mathcal{R}_\beta(\mathcal{O}_i) = \mathcal{R}_\beta(\mathcal{O}).
$$
(13)

Here $I$ is some index set and $\vee_{i \in I} \mathcal{R}_\beta(\mathcal{O}_i)$ denotes the von Neumann algebra generated by the algebras $\mathcal{R}_\beta(\mathcal{O}_i)$, $i \in I$.

If $\omega_\beta$ is locally normal w.r.t. the vacuum representation, then additivity in the vacuum sector and additivity in the thermal sector are equivalent. As is well known, additivity in the vacuum sector can be proven, if the net of local algebras is constructed from a Wightman field theory.

The Reeh-Schlieder Property

We start with the following adapted and simplified version of Glaser’s Theorem 1 (see Ref. 13, see also Ref. 14, 15):

Theorem III.1. (Glaser): Let $a \in \mathcal{A}$ and let $F_{a^*,a}$ denote the function introduced in (9). The following properties are equivalent:

i.) There exists an open neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^4$ and a point $z_1 \in \mathcal{T}_{\beta e/2}$ such that $z_1 + \mathcal{V} \subset \mathcal{T}_{\beta e/2}$ and such that for each complex-valued testfunction $f$ with support in $\mathcal{V}$

$$
\int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4y_1 d^4y_2 \ F_{a^*,a}(y_1 + \bar{z}_1, y_2 + z_1) \overline{f(y_1)} f(y_2) \geq 0.
$$
(14)

ii.) There exists a sequence $\{ f^{(n)}_a : \mathcal{T}_{\beta e/2} \to \mathbb{C} \}_{n \in \mathbb{N}}$ of functions holomorphic in $\mathcal{T}_{\beta e/2}$ such that for $(z_1, z_2) \in -\mathcal{T} \times \mathcal{T}$

$$
F_{a^*,a}(z_1, z_2) = \sum_{n \in \mathbb{N}} \overline{f^{(n)}_a(\bar{z}_1)} f^{(n)}_a(z_2)
$$
(15)

holds in the sense of uniform convergence on every compact subset of $-\mathcal{T}_{\beta e/2} \times \mathcal{T}_{\beta e/2}$.

The next step is to show that condition i.) is indeed satisfied, if $\omega_\beta$ is a relativistic KMS state:
Proposition III.2. Let $\omega_\beta$ be a state which satisfies the relativistic KMS condition at inverse temperature $\beta > 0$ and let $\mathcal{V}$ be an open neighborhood of 0 in $\mathbb{R}^4$.

It follows that for each complex-valued test function $f$ with support in $\mathcal{V}$

$$\int_{\mathbb{R}^4} d^4y_1 d^4y_2 F_{a^*, a}(y_1 - i\kappa, y_2 + i\kappa) f(y_1) f(y_2) \geq 0$$

for all $0 < \kappa < \beta/2$. Here $e$ denotes the unit vector in the time direction distinguished by the relativistic KMS condition.

Proof. Let $a \in A_\alpha$ be an entire analytic element for the translations. Put

$$\Psi_f := \int_\mathcal{V} d^4y_1 f(y_1) \alpha_{\kappa e}(\alpha_{i\kappa e}(a)) \Omega_\beta \in \mathcal{H}_\beta.$$ (17)

Exploring the definition (9) of $F_{a^*, a}$ one finds

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4y_1 d^4y_2 F_{a^*, a}(y_1 - i\kappa, y_2 + i\kappa) f(y_1) f(y_2) = \|\Psi_f\|^2 \geq 0.$$ (18)

For general $a \in A$, choose a sequence $\{a_n \in A_\alpha\}_{n \in \mathbb{N}}$ such that

$$\|a_n\| \leq \|a\| \quad \text{and} \quad \pi_\beta(a_n) \Omega_\beta \to \pi_\beta(a) \Omega_\beta \quad \text{as} \quad n \to \infty.$$ (19)

Now define, for $y_1, y_2 \in \mathbb{R}^4$ and $0 < \kappa < \beta/2$,

$$F_n(y_1 - i\kappa, y_2 + i\kappa) := F_{a_n^*, a_n}(y_1 - i\kappa, y_2 + i\kappa).$$ (20)

The Three-line Theorem (see Ref. 9, 5.3.5) implies that

$$\left| F_n(y_1 - i\kappa, y_2 + i\kappa) - F_m(y_1 - i\kappa, y_2 + i\kappa) \right|$$

assumes its maximum value on the boundary of its domain and for $\kappa = 0, \beta/2$, the boundary values, the relativistic KMS condition yields

$$\left| F_n(y_1 - i\kappa, y_2 + i\kappa) - F_m(y_1 - i\kappa, y_2 + i\kappa) \right|
\leq \max\left\{ \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\beta(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_n)) - \omega_\beta(\alpha_{y_1}(a_m^*) \alpha_{y_2}(a_m)) \right|, \right.$$

$$\sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\beta(\alpha_{y_2}(a_n) \alpha_{y_1}(a_n^*)) - \omega_\beta(\alpha_{y_2}(a_m) \alpha_{y_1}(a_m^*)) \right| \right\}
\leq \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\beta(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_n)) - \omega_\beta(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_m)) \right|
+ \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\beta(\alpha_{y_1}(a_n^*) \alpha_{y_2}(a_m)) - \omega_\beta(\alpha_{y_1}(a_n) \alpha_{y_2}(a_m)) \right|
+ \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\beta(\alpha_{y_2}(a_n) \alpha_{y_1}(a_n^*)) - \omega_\beta(\alpha_{y_2}(a_n) \alpha_{y_1}(a_m^*)) \right|
+ \sup_{y_1, y_2 \in \mathbb{R}^4} \left| \omega_\beta(\alpha_{y_2}(a_n^*) \alpha_{y_1}(a_m)) - \omega_\beta(\alpha_{y_2}(a_n^*) \alpha_{y_1}(a_n^*)) \right|
\leq 2 \|a\| \sup_{y \in \mathbb{R}^4} \|\pi_\beta(\alpha_y(a_n - a_m))\| + 2 \|a\| \sup_{y \in \mathbb{R}^4} \|\pi_\beta(\alpha_y(a_m^* - a_n^*))\|.$$ (22)
In the last inequality we have used \( \|a_n\| = \|a^*_n\| \leq \|a\| \) and \( \|\Omega_\beta\| = 1 \). Strong continuity of \( \alpha \) now implies that \( \{F_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence uniformly on \( \overline{\mathcal{U}} \), where

\[
\mathcal{U} := \{(y_1 - i\kappa, y_2 + i\kappa) : y_1, y_2 \in \mathbb{R}^4, 0 < \kappa < \beta/2\}.
\]  

(23)

The limit function \( F_\infty \) is therefore continuous and bounded on \( \overline{\mathcal{U}} \) and analytic in \( \mathcal{U} \). Moreover,

\[
F_\infty(y_1, y_2) = F_{a^*, a}(y_1, y_2) \quad \text{for} \quad y_1, y_2 \in \mathbb{R}^4.
\]  

(24)

Thus, due to their analyticity properties, the functions \( F_\infty \) and \( F_{a^*, a} \) must coincide on \( \mathcal{U} \).

(25)  

The crucial step in the proof of the Reeh-Schlieder property is now summarized in the following

**Proposition III.3.** For each \( a \in \mathcal{A} \) the vector valued function \( \Phi_a : \mathbb{R}^4 \to \mathcal{H}_\beta \),

\[
x \mapsto \pi_\beta(\alpha_x(a))\Omega_\beta
\]  

(26)

can be analytically continued from the real axis into the domain \( T_{\beta e/2} \) such that it is weakly continuous for \( \Im z \searrow 0 \).

**Proof.** Let \( a, b \in \mathcal{A} \) with \( \|a\| = 1 \). Because of

\[
\overline{\pi_\beta(\mathcal{A})\Omega_\beta} = \mathcal{H}_\beta
\]  

(27)

the set of vectors \( \mathcal{S} := \{\pi_\beta(b)\Omega_\beta : b \in \mathcal{A}\} \) is dense in \( \mathcal{H}_\beta \). Moreover, according to Theorem III.1.ii) there exists a sequence \( \{f_a^{(n)} : T_{\beta e/2} \to \mathbb{C}\}_{n \in \mathbb{N}} \) of functions holomorphic in \( T_{\beta e/2} \) which satisfies (15). This allows us to consider — for \( z \in T_{\beta e/2} \) and \( a \in \mathcal{A} \) fixed — the map \( \hat{\phi}_{a,z} : \mathcal{S} \to \mathbb{C} \)

\[
\pi_\beta(b)\Omega_\beta \mapsto \sum_{n \in \mathbb{N}} \overline{f_a^{(n)}(z)} f_b^{(n)}(0).
\]  

(28)

(Recall that \( \Omega_\beta \) is separating for \( \pi_\beta(\mathcal{A}) \). Hence the map \( b \mapsto \pi_\beta(b)\Omega_\beta \) is injective and consequently the map \( \pi_\beta(b)\Omega_\beta \mapsto f_b^{(n)}(0) \) is well-defined.) Using \( \|a\| \leq 1 \),

\[
\sum_{n \in \mathbb{N}} |f_a^{(n)}(z)|^2 = F_{a^*, a}(\bar{z}, z),
\]  

(29)
and the Schwarz inequality, we find
\[
\left| \sum_{n \in \mathbb{N}} \overline{f_a^{(n)}(\bar{z})} f_b^{(n)}(0) \right|^2 \leq F_{a^*,a}(\bar{z}, z) \cdot \|\pi_\beta(b) \Omega_\beta\|^2.
\] (30)

By the Hahn-Banach Theorem the map \( \hat{\phi}_{a,z} : S \to \mathbb{C} \) extends to a (bounded) continuous linear functional \( \phi_{a,z} \) on \( H_\beta \). The Riesz Lemma ensures that there exists a vector \( \Phi_a(z) \in H_\beta \) such that
\[
\phi_{a,z}(\Psi) = (\Phi_a(z), \Psi) \quad \forall \Psi \in H_\beta.
\] (31)

The map
\[
z \mapsto \Phi_a(z)
\] (32)
is analytic for \( z \in T_{\beta e/2} \). (This can be shown by an approximation argument similar to the one given in the proof of Proposition III.2.) As can be seen more easily, the map (32) is also weakly continuous at the boundary set \( \Im z = 0 \), where it satisfies
\[
\Phi_a(x) = \pi_\beta(\alpha_x(a)) \Omega_\beta \quad \forall x \in \mathbb{R}^4.
\] (33)

Although we will not directly use it, we believe that it is worthwhile to spell out the following

**Corollary III.4.** Let \( a, b \in A \) and let \( \Phi_a, \Phi_b \) denote the associated vector valued functions introduced in (32). It follows that
\[
F_{a^*,b}(\bar{z}_1, z_2) = (\Phi_a(z_1), \Phi_b(z_2))
\] (34)
for all \( z_1, z_2 \in T_{\beta e/2} \). Here \( F_{a^*,b} \) denotes the analytic function introduced in (9).

**Proof.** The l.h.s. as well as the r.h.s. defines a holomorphic function on
\[
-T_{\beta e/2} \times T_{\beta e/2} \subset \mathbb{C}^4 \times \mathbb{C}^4.
\] (35)

Moreover, for \( \Im z_1 \searrow 0 \) and \( \Im z_2 \searrow 0 \) we find
\[
(\Phi_a(x_1), \Phi_b(x_2)) = (\pi(\alpha_{x_1}(a)) \Omega_\beta, \pi(\alpha_{x_2}(b)) \Omega_\beta) = \omega_\beta(\alpha_{x_1}(a^*) \alpha_{x_2}(b)) = F_{a^*,b}(x_1, x_2)
\] (36)
for all \( x_1, x_2 \in \mathbb{R}^4 \). Applying the Edge-of-the-Wedge Theorem we conclude that the l.h.s. and the r.h.s. in (34) describe the same analytic function. \[\square\]
What remains to be proven in order to establish the Reeh-Schlieder property is fairly standard. Borchers and Buchholz \cite{Ref16} recently gave a nice and transparent formulation of this final part of the argument and therefore we will simply reproduce their formulation here, up to minor notational differences.

**Definition III.5.** Let \( \mathcal{O} \) be any open region. The \(*\)-algebra \( \mathcal{B}(\mathcal{O}) \) is defined as the set of operators \( b \in \mathcal{A}(\mathcal{O}) \) for which there exists some neighborhood \( \mathcal{N} \subset \mathbb{R}^4 \) of the origin such that

\[
\alpha_x(b) \in \mathcal{A}(\mathcal{O}) \quad \forall x \in \mathcal{N}, \tag{37}
\]

where the neighborhood \( \mathcal{N} \) may depend on \( b \).

\( \mathcal{B}(\mathcal{O}) \) is a \(*\)-algebra and

\[
\mathcal{A}(\mathcal{O}_\circ) \subset \mathcal{B}(\mathcal{O}) \tag{38}
\]

for any region \( \mathcal{O}_\circ \) whose closure lies in the interior of \( \mathcal{O} \).

**Lemma III.6.** Let \( \Psi \in \mathcal{H}_\beta \) be a vector with the property that

\[
(\Psi, \pi_\beta(b)\Omega_\beta) = 0 \quad \forall b \in \mathcal{B}(\mathcal{O}). \tag{39}
\]

It follows that for each \( b \in \mathcal{B}(\mathcal{O}) \) the function

\[
\mathbb{R}^4 \ni x \mapsto (\Psi, \pi_\beta(\alpha_x(b))\Omega_\beta) \tag{40}
\]

vanishes.

**Proof.** Let \( b \in \mathcal{B}(\mathcal{O}) \) and let \( \mathcal{N} \) as in (37). It follows from the definition of \( \mathcal{B}(\mathcal{O}) \) and the geometrical action (5) of the translations that there exists some \( \epsilon > 0 \), which may depend on \( b \), such that

\[
\alpha_x(b) \in \mathcal{B}(\mathcal{O}) \quad \text{for} \quad |x| < \epsilon. \tag{41}
\]

On the other hand the function

\[
\mathbb{R}^4 \ni x \mapsto \pi_\beta(\alpha_x(b))\Omega_\beta \tag{42}
\]

extends analytically to some vector-valued function in the domain \( \mathcal{T}_{\beta\epsilon/2} \) by Proposition III.3. Combining (42) with (41) and (39) we find that

\[
(\Psi, \pi_\beta(\alpha_x(b))\Omega_\beta) = 0 \tag{43}
\]

for all \( x \in \mathbb{R}^4 \). \( \square \)
Lemma III.7. Assume that the additivity assumption (13) holds. It follows that
\[ \forall x \in \mathbb{R}^4 \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right)'' = \mathcal{R}_\beta. \] \hspace{1cm} (44)
Once again, \( \forall x \in \mathbb{R}^4 \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right)'' \) denotes the von Neumann algebra generated by the algebras \( \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right)'' \), \( x \in \mathbb{R}^4 \).

Proof. Let \( \mathcal{O}_o \) be an open subset of \( \mathcal{O} \) such that its closure \( \overline{\mathcal{O}_o} \) is contained in the interior of \( \mathcal{O} \). It follows that
\[ \forall x \in \mathbb{R}^4 \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right)'' \supset \forall x \in \mathbb{R}^4 \pi_\beta \left( \alpha_x (\mathcal{A}(\mathcal{O}_o)) \right)'' \].
Combining (5) with (13) we conclude that the r.h.s. equals \( \mathcal{R}_\beta \). \( \square \)

Corollary III.8. Let \( \Psi \in \mathcal{H}_\beta \) be a vector with the property that
\[ (\Psi, \pi_\beta (b) \Omega_\beta) = 0 \quad \forall b \in \mathcal{B}(\mathcal{O}). \] \hspace{1cm} (46)
It follows that \( \Psi = 0 \).

Proof. First, we apply Lemma III.6 and conclude from (40) that
\[ \Psi \perp \forall x \in \mathbb{R}^4 \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right) \Omega_\beta. \] \hspace{1cm} (47)
Then we recall that the orthogonal complement of \( \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right) \Omega_\beta \) is closed, therefore it coincides with the orthogonal complement of \( \pi_\beta \left( \alpha_x (\mathcal{B}(\mathcal{O})) \right)'' \Omega_\beta \). Hence Lemma III.7 implies
\[ \Psi \perp \overline{\mathcal{R}_\beta \Omega_\beta}. \] \hspace{1cm} (48)
By construction \( \overline{\mathcal{R}_\beta \Omega_\beta} = \mathcal{H}_\beta \), thus \( \Psi = 0 \). \( \square \)

We will now show that for every vector \( \Phi \in \mathcal{H}_\beta \) there exists an operator in \( \pi_\beta (\mathcal{A}(\mathcal{O})) \), which, when applied to \( \Omega_\beta \), generates a vector which is arbitrarily close to \( \Phi \):

Theorem III.9. Consider a QFT as specified in Section 2 and let \( \omega_\beta \) be a state, which satisfies the relativistic KMS condition. If the additivity assumption (13) holds, then
\[ \mathcal{H}_\beta = \overline{\pi_\beta (\mathcal{A}(\mathcal{O})) \Omega_\beta}, \] \hspace{1cm} (49)
for any open space–time region \( \mathcal{O} \subset \mathbb{R}^4 \). Moreover, if the spacelike complement of \( \mathcal{O} \) is not empty, then \( \Omega_\beta \) is separating for \( \mathcal{R}_\beta (\mathcal{O}) \).
Proof. We have to show that the orthogonal complement of \( \pi_\beta(A(\mathcal{O}))\Omega_\beta \) vanishes. Assume that
\[
\Psi \perp \pi_\beta(A(\mathcal{O}))\Omega_\beta.
\] (50)
Obviously, this implies
\[
\Psi \perp \pi_\beta(B(\mathcal{O}))\Omega_\beta
\] (51)
and then Corollary III.8 yields \( \Psi = 0 \). On the other hand, if the spacelike complement \( \mathcal{O}' \) of \( \mathcal{O} \) is not empty, then \( \Omega_\beta \) is cyclic for \( R_\beta(\mathcal{O}') \supset \pi_\beta(A(\mathcal{O}')) \). Since \( R_\beta(\mathcal{O})' \supset R_\beta(\mathcal{O})' \), this implies that \( \Omega_\beta \) is cyclic for \( R_\beta(\mathcal{O})' \) and therefore separating for \( R_\beta(\mathcal{O}) \). \( \square \)

Similar to the situation where \( \beta = \infty \), in the so-called vacuum sector, \( \Omega_\beta \) shares the Reeh-Schlieder property with a large class of vectors in \( \mathcal{H}_\beta \).

**Theorem III.10.** There exists a dense set \( D_\alpha \subset \mathcal{H}_\beta \), such that for all \( \Psi \in D_\alpha \)
\[
\mathcal{H}_\beta = \overline{\pi_\beta(A(\mathcal{O}))\Psi},
\] (52)
where \( \mathcal{O} \subset \mathbb{R}^4 \) is again an arbitrary open space–time region.

Proof. A set \( D_\alpha \subset R_\beta\Omega_\beta \) of suitable entire analytic vectors in \( \mathcal{H}_\beta \) may be specified by putting
\[
D_\alpha = \left\{ \left( 1 - \frac{\pi_\beta(a)}{2\|a\|} \right) \Omega_\beta : a \in A_\alpha \right\}.
\] (53)
Note that \( D_\alpha \) is dense in \( \mathcal{H}_\beta \):
\[
\overline{D_\alpha} = \overline{\pi_\beta(A_\alpha)\Omega_\beta} = \overline{\pi_\beta(A)\Omega_\beta} = \mathcal{H}_\beta.
\] (54)
The essential step is to show that for arbitrary \( b \in A \) the function
\[
\mathbb{R}^4 \ni x \mapsto \pi_\beta(\alpha_x(b))\Psi
\] (55)
extends to some analytic vector-valued function in the domain \( T_{\beta e/2} \). The reader is invited to check that Theorem III.1 and Proposition III.2 can easily be adapted and that the proofs given remain valid if we replace \( \Omega_\beta \) by some vector \( \Psi \in D_\alpha \). Finally, we note that \( (1 - a/2\|a\|) \) is invertible in \( A \), thus
\[
\pi_\beta(A)\left( 1 - \frac{\pi_\beta(a)}{2\|a\|} \right)\Omega_\beta = \pi_\beta(A)\Omega_\beta.
\] (56)
We conclude that \( \overline{\pi_\beta(A)\Psi} = \mathcal{H}_\beta \), which ensures that the arguments given in the proof of Corollary III.8 apply also in this slightly more general case. \( \square \)
Outlook

Although both quantum statistical mechanics as well as quantum field theory can nicely be formulated in terms of operator algebras, little Of course, one should not forget to mention the beautiful progress that has recently been achieved in the Wightman approach to thermal field theory (see, e.g., Ref. 17). In fact, only recently the relativistic KMS condition, which provides the necessary substitute for the spectrum condition, was formulated. As we have demonstrated, it allows us to treat the thermal theory independently from the vacuum theory. In a series of forthcoming papers by the author basic results like the Cluster Theorem, the Schlieder property and the Borchers property have been derived. The nuclearity condition, which distinguishes theories with decent phase-space properties, was used to derive the split property, which expresses a strong form of statistical independence of spacelike separated measurements. A rather involved argument, based on a rigorous version of what is commonly called ‘doubling the degrees of freedom’ in thermal field theory, establishes the ‘convergence of local charges’ in the thermal sector. These results use the Reeh-Schlieder property as a crucial input; and without the present result one would have to recourse to the physically sound but unproven assumption that relativistic KMS states are locally normal w.r.t. the vacuum representation.

Many other bricks are still missing in the wall, for instance a thermal Jost-Lehmann-Dyson representation. Scattering theory in the thermal context is one of the major challenges; in fact, the author would like to emphasize that the severe problems encountered in perturbation theory open up a fair chance for the operator algebraic approach to attract some interest from outside, provided it can offer some progress on this topic in time.

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