Instanton constraints
in supersymmetric gauge theories II.

$N = 2$ Yang-Mills theory

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Abstract
The analysis in previous publications of the instanton constraints required to produce a finite action of the theory is carried out also for $N = 2$ supersymmetric Yang-Mills theory.

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1 Introduction

Instantons play a prominent role in nonperturbative studies of supersymmetric gauge theories. In the case of $N = 2$ supersymmetric Yang-Mills theory, the connection from instantons to the celebrated Seiberg-Witten solution [1] was established by Finnell and Pouliot [2]. The instantons in question are constrained instantons [3] since the scalar field of the theory has a nonvanishing vacuum expectation value. The choice of constraint was shown in [4] to be restricted since not all constraints lead to a finite action instanton solution.

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In the present paper the investigation reported in [4] on permissible instanton constraints is extended to the case of $N = 2$ supersymmetric Yang-Mills theory. Here the scalar field is in the adjoint representation. As a consequence, two different prepotentials, obeying a complicated set of coupled differential equations, are necessary for the gauge field. Remarkably it is found, however, that the constraint can be chosen in a gauge covariant way, and the main feature of the result of [4] persists: Constraint terms in the gauge field equations are only necessary at second and fourth order of the scalar field vacuum expectation value. This result actually applies to all $SU(2)$ Yang-Mills field theories where the gauge field couples to a scalar field in the adjoint representation, and thus also to $N = 4$ supersymmetric Yang-Mills theory.

The fermion zero modes are also investigated, and it is found that the supersymmetric zero modes automatically have acceptable asymptotic behaviour to all orders in the scalar field vacuum expectation value, whereas the superconformal zero modes (the eigenvalue of which is lifted) have unpermitted large-distance behaviour at first order.

This paper parallels one on supersymmetric QCD [5], and many features of the analysis reported there persist in the present case. Especially the analysis of the fermion zero modes turns out to be very similar to, though simpler than, the analysis of [5]: for the sake of completeness it is nevertheless presented in some detail. The analysis of the instanton mass corrections, on the other hand, is considerably more complicated here, though the final result on constraints is the same as in [4], [5].

In sec.II the action of supersymmetric $N = 2$ Yang-Mills theory and its continuation to Euclidean space are presented, while the analysis of the instanton mass corrections is given in sec.III and of the fermion zero modes in sec.IV. The results are summarized in a brief conclusion, while an appendix contains supplementary material for sec.III.

2 The action

The Lagrangian of $N = 2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$ is:

$$L_{N=2} = -(D^\mu A^a)^\dagger D_\mu A^a - i(q_L^a)^\dagger (\sigma^\mu D_\mu)^{ab} q_L^b$$

$$+ (F^a)^* F^a - i(\lambda_R^a)^\dagger (\sigma^\mu D_\mu)^{ab} \lambda_R^b$$
\[ + \frac{g}{\sqrt{2}} \epsilon^{abc}(A^a)^\dagger (\lambda^b_R)^\dagger q^c_R - \frac{g}{\sqrt{2}} \epsilon^{abc}(q^a_R)^\dagger \lambda^b_R A^c \\
- \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} (D^a)^2 - ig \epsilon^{abc} D^a (A^b)^* A^c. \tag{2.1} \]

Here \( \lambda^a \) and \( q^a \) are the two spinor fields of the theory, \( A^a \) is the complex scalar field and \( A^a_\mu \) the gauge field with the corresponding field strength \( F^{a\mu\nu} \), while \( D^a \) and \( F^a \) are auxiliary fields. As in [5] the conventions of Wess and Bagger [6] are used, with \( \eta^{\mu\nu} = (-1, 1, 1, 1) \), while \( \sigma^\mu = (-1, \bar{\sigma}) \), \( \bar{\sigma}^\mu = (-1, -\bar{\sigma}) \), where \( \bar{\sigma} \) are the Pauli matrices. \( \epsilon^{abc} \) is the standard Levi-Civita symbol.

The flat directions, where gauge invariance is spontaneously broken, are now characterized by

\[ D^a = 0; \ A^a = (A^a)^* \neq 0. \tag{2.2} \]

Continuation of the fermionic part of the action to Euclidean space requires some care because the spinor fields, when expressed as Dirac spinors, obey the Majorana condition. It is carried out by the Vainshtein-Zakharov doubling trick [7]. We skip the detailed manipulations, which run as in [5]. The outcome is the fermionic Lagrangian:

\[
L_{N=2, \text{Fermi, Euclid}} = -(\lambda^3_B)^\dagger (\sigma \cdot D)^{ab} \lambda^b_B - (\lambda^3_B)^\dagger (\bar{\sigma} \cdot D)^{ab} \lambda^b_A \\
-(q^3_A)^\dagger (\sigma \cdot D^{ab} q^b_B) - (q^3_B)^\dagger (\bar{\sigma} \cdot D^{ab} q^b_A) \\
+ g \sqrt{2} \epsilon^{abc} (A^a)^* (\lambda^b_B)^\dagger q^c_B - g \sqrt{2} \epsilon^{abc} (q^a_B)^\dagger \lambda^b_A A^c \\
- g \sqrt{2} \epsilon^{abc} (A^a)^* q^b_B \lambda^c_B + g \sqrt{2} \epsilon^{abc} (\lambda^a_A)^\dagger q^b_B A^c \tag{2.3}
\]

where now \( \bar{\sigma}_\mu = (i\bar{\sigma}, 1) \), \( \sigma_\mu = (-i\bar{\sigma}, 1) \), and where two different Weyl spinors are present, labelled \( A \) and \( B \), for each Minkowski space Weyl spinor.

### 3 Instanton mass corrections

#### 3.1 General setting

Ignoring fermions and picking a particular flat direction, one fulfills (2.2) by having

\[ A^3 = (A^3)^* = A \neq 0 \tag{3.1} \]
and $A^1, A^2$ vanish. Then the gauge field equations are:

$$\partial_\mu F^a_{\mu\nu} + g\epsilon^{ab}(A^b_\mu F^3_{\mu\nu} - A^3_\mu F^b_{\mu\nu}) - 2g^2 A^2 A^a_\nu = 0; \ a = 1, 2 \quad (3.2)$$

and

$$\partial_\mu F^3_{\mu\nu} + g\epsilon^{ab} A^a_\mu F^b_{\mu\nu} = 0 \quad (3.3)$$

while the scalar field equations are:

$$\partial^2 A - g^2 A^a_\mu A^a_\mu = 0; \ a = 1, 2 \quad (3.4)$$

and

$$2g\epsilon^{ab} A^b_\mu \partial_\mu A + g^2 A^a_\mu A^3_\mu A = 0; \ a, b = 1, 2. \quad (3.5)$$

The gauge fields are assumed to have the following form in the singular gauge:

$$A^a_\mu = -\frac{1}{g}\tilde{\eta}^a_{\mu\nu}\partial_\nu \log \alpha_\perp; \ a = 1, 2, \ A^3_\mu = -\frac{1}{g}\tilde{\eta}^3_{\mu\nu}\partial_\nu \log \alpha_\parallel, \quad (3.6)$$

where $\tilde{\eta}^a_{\mu\nu}$ is the standard 't Hooft symbol. Here the prepotentials $\alpha_\perp$ and $\alpha_\parallel$, as well as the function $A$ introduced in (3.1), are real functions of the parameter $t = \rho^2 x^2$, with $\rho$ the scale of the instanton. Then (3.5) is trivially valid.

Expressed in terms of the prepotentials, the field strength components are

$$F^a_{\mu\nu} = -\frac{4}{g}\tilde{\eta}^a_{\mu\nu}\frac{t^2}{\rho^2} \frac{d \log \alpha_\perp}{dt} (1 - t \frac{d \log \alpha_\parallel}{dt})$$

$$-\frac{4}{g}(\tilde{\eta}^a_{\mu\lambda} x_\mu x_\lambda - \tilde{\eta}^a_{\mu\lambda} x_\nu x_\lambda)(\frac{t^2}{\rho^2} \frac{d \log \alpha_\perp}{dt}$$

$$- \frac{t^4}{\rho^2} \frac{d \log \alpha_\perp}{dt} \frac{d \log \alpha_\parallel}{dt}); \ a = 1, 2 \quad (3.7)$$

and

$$F^3_{\mu\nu} = -\frac{4}{g}\tilde{\eta}^3_{\mu\nu}\frac{t^2}{\rho^2} \frac{d \log \alpha_\parallel}{dt} - t(\frac{d \log \alpha_\perp}{dt})^2$$

$$-\frac{4}{g}(\tilde{\eta}^3_{\mu\lambda} x_\mu x_\lambda - \tilde{\eta}^3_{\mu\lambda} x_\nu x_\lambda)(\frac{t^2}{\rho^2} \frac{d \log \alpha_\parallel}{dt}$$

$$- \frac{t^4}{\rho^2} \frac{d \log \alpha_\perp}{dt} \frac{d \log \alpha_\parallel}{dt}) \quad (3.8)$$
where the following identity was used:

\[ \epsilon^{abc} \eta_{\mu \lambda} \eta_{\nu \rho} x_\lambda x_\rho = x^2 \eta^a_{\mu \nu} + \eta^a_{\nu \lambda} x_\mu x_\lambda - \eta^a_{\mu \lambda} x_\nu x_\lambda. \]  

(3.9)

The action density is then (for a flat direction):

\[
-(D_\mu A^a)^\dagger D_\mu A^a - \frac{1}{4} F^a_{\mu \nu} F^a_{\mu \nu} \\
= \frac{-4t^2}{\rho^2} \left( \frac{dA}{dt} \right)^2 - \frac{8t^3}{\rho^2} \left( \frac{d \log \alpha_\perp}{dt} \right)^2 A^2 \\
- \frac{16 t^3}{g^2 \rho^4} \left( \frac{d \log \alpha_\perp}{dt} \right) \left( \frac{d \log \alpha_\parallel}{dt} - 1 \right)^2 \\
- \frac{8 t^4}{g^2 \rho^4} \left( \frac{d \log \alpha_\perp}{dt} \right)^2 \left( \frac{d \log \alpha_\parallel}{dt} \right)^2 \\
- \frac{16 t^4}{g^2 \rho^4} \left( \frac{d}{dt} \frac{t}{\rho^2} \left( \frac{d \log \alpha_\perp}{dt} \right) \right)^2 - \frac{16 t^4}{g^2 \rho^4} \left( \frac{d}{dt} \frac{t}{\rho^2} \left( \frac{d \log \alpha_\parallel}{dt} \right) \right)^2. 
\]  

(3.10)

(3.10) is negative semidefinite, and hence each term has to give a finite contribution to the action. From this observation follows bounds on the functions \( \alpha_\perp, \alpha_\parallel \) and \( A \). The analysis runs as in [4], with the difference that only \( \alpha_\perp \) falls off exponentially at small \( t \) whereas \( \alpha_\parallel \) and \( A \) have a power law decrease. The outcome is that \( \alpha_\perp \) and \( \alpha_\parallel \) at large \( t \) at most should grow logarithmically and for small \( t \) the leading terms of \( \alpha_\perp \) should conspire to the modified Bessel function \( K_1 \):

\[
\alpha_\perp = \alpha_{\perp,0} + \alpha_{\perp,2} + \alpha_{\perp,4} + \cdots \simeq 1 + \sqrt{2t} \rho g v K_1(\sqrt{\frac{2}{t} \rho g v}), 
\]  

(3.11)

thus ensuring exponential falloff of the gauge field in the nonflat directions.

The field equations (3.2), (3.3) and (3.4) are in terms of the functions \( \alpha_\perp, \alpha_\parallel \) and \( A \):

\[
\alpha_\perp^2 \frac{d}{dt} \left( \frac{t^3}{\rho^2} \frac{d^2 \alpha_\perp}{dt^2} \right) - 3t^2 \alpha_\perp^{-1} \frac{d \alpha_\perp}{dt} \left( \alpha_\perp^{-1} \frac{d \alpha_\perp}{dt} - \alpha_\parallel^{-1} \frac{d \alpha_\parallel}{dt} \right) \\
- t^3 \alpha_\perp^{-1} \frac{d \alpha_\perp}{dt} \left( \frac{d \alpha_\parallel}{dt} \right)^2 - \alpha_\perp^{-2} \left( \frac{d \alpha_\parallel}{dt} \right)^2 \\
= \frac{\rho^2 g^2 A^2}{2} \alpha_\perp^{-1} \frac{d \alpha_\perp}{dt}. 
\]  

(3.12)
\[\begin{align*}
\alpha_{\parallel} \frac{d}{dt} \left( \alpha_{\parallel}^{-3} t^3 \frac{d^2 \alpha_{\parallel}}{dt^2} \right) &+ 3t^2 \left( \alpha_{\parallel}^{-2} \left( \frac{d\alpha_{\parallel}}{dt} \right)^2 - \alpha_{\parallel}^{-2} \left( \frac{d\alpha_{\parallel}}{dt} \right)^2 \right) \\
+ 2t^3 \alpha_{\parallel}^{-1} \frac{d\alpha_{\parallel}}{dt} \left( \alpha_{\parallel}^{-2} \left( \frac{d\alpha_{\parallel}}{dt} \right)^2 - \alpha_{\parallel}^{-2} \left( \frac{d\alpha_{\parallel}}{dt} \right)^2 \right) &+ 3t^2 \left( \alpha_{\perp}^{-2} \left( \frac{d\alpha_{\perp}}{dt} \right)^2 - \alpha_{\perp}^{-2} \left( \frac{d\alpha_{\perp}}{dt} \right)^2 \right) = 0 \quad (3.13) \\
\end{align*}\]

and

\[\begin{align*}
\frac{d^2 A}{dt^2} - 2 \alpha_{\perp}^2 \left( \frac{d\alpha_{\perp}}{dt} \right)^2 A & = 0. \quad (3.14)
\end{align*}\]

### 3.2 Iteration of the bosonic field equations up to second order

(3.12), (3.13) and (3.14) are solved by iteration, where in the two lowest orders:

\[\alpha_{\perp,0} = \alpha_{\parallel,0} = 1 + t; \quad A_1 = \frac{v}{1+t} \quad (3.15)\]

with \(v\) a free parameter, in terms of which the iteration is carried out, and the subscript indicates the order.

At second order we require, according to the discussion leading to (3.11):

\[\alpha_{\perp,2} \approx \alpha_{\perp,2,\text{min}} \quad (3.16)\]

near \(t \approx 0\), with

\[\alpha_{\perp,2,\text{min}} = \frac{\rho^2 g^2 v^2}{2} \left( \log \frac{\rho^2 g^2 v^2}{2t} + 2\gamma - 1 \right) \quad (3.17)\]

by the power series expansion of the modified Bessel function \(K_1\).

(3.12) and (3.13) are at second order:

\[\begin{align*}
(1 + t)^2 \frac{d}{dt} \left( \left( \frac{t}{1+t} \right)^3 \frac{d^3 \alpha_{\perp,2}}{dt^3} \right) &- t^2 \left( 3 - \frac{2t}{1+t} \right) \frac{d}{dt} \alpha_{\perp,2} \\
&= \frac{\rho^2 g^2 v^2}{2} \frac{1}{(1+t)^3} \quad (3.18)
\end{align*}\]

and

\[\begin{align*}
(1 + t)^2 \frac{d}{dt} \left( \left( \frac{t}{1+t} \right)^3 \frac{d^3 \alpha_{\parallel,2}}{dt^3} \right) &+ \frac{t^2}{1+t} \left( 6 - \frac{4t}{1+t} \right) \frac{d}{dt} \alpha_{\parallel,2} = 0 \quad (3.19)
\end{align*}\]
with

\[ \alpha_{-2} = \alpha_{\perp,2} - \alpha_{\parallel,2}. \]  

(3.20)

The terms involving \( \alpha_{-2} \) are eliminated by forming a linear combination of (3.18) and (3.19), leading to

\[
(1 + t)^2 \frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d^2(2\alpha_{\perp,2} + \alpha_{\parallel,2})}{dt^2} \right) = \frac{\rho^2 g^2 v^2}{2} \frac{2}{(1 + t)^3}
\]

(3.21)

whence

\[
\frac{d^2(2\alpha_{\perp,2} + \alpha_{\parallel,2})}{dt^2} = -\frac{\rho^2 g^2 v^2}{4} \left( \frac{1}{t^3(1 + t)} + c_{2\perp+\parallel,2} \left( \frac{1 + t}{t} \right)^3 \right)
\]

(3.22)

where \( c_{2\perp+\parallel,2} \) is an integration constant that should be chosen such that the leading singularity for \( t \to 0 \) is eliminated, i.e.

\[
c_{2\perp+\parallel,2} = \frac{\rho^2 g^2 v^2}{4}.
\]

(3.23)

However, a nonvanishing value of \( c_{2\perp+\parallel,2} \) introduces on the right hand side of (3.22) terms that for \( t \to \infty \) are nonvanishing or only fall off as \( \frac{1}{t} \). These terms must be eliminated by a constraint. Hence, (3.22) has to be replaced by:

\[
\frac{d^2(2\alpha_{\perp,2} + \alpha_{\parallel,2})}{dt^2} = \frac{\rho^2 g^2 v^2}{4} \left( -\frac{1}{t^3(1 + t)} + \frac{1}{t^3} + \frac{3}{t^2} \right)
\]

(3.24)

that serves as a guideline for the detailed determination of possible constraints.

From (3.18)-(3.19) also follows

\[
(1 + t)^2 \frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d\alpha_{-2}}{dt^2} \right) = \frac{\rho^2 g^2 v^2}{2} \frac{1}{(1 + t)^3}
\]

(3.25)

This equation can be solved by quadrature, but it is convenient instead to reformulate it slightly.

The function \( \Psi_2 \), defined by:

\[
\Psi_2 = \frac{1}{1 + t} \frac{d}{dt} \frac{\alpha_{-2}}{1 + t}.
\]

(3.26)
fulfils in terms of the variable \( u = \frac{t}{1+t} \) an inhomogeneous hypergeometric equation:

\[
u(1-u)\frac{d^2\Psi_2(u)}{du^2} + 3\frac{d\Psi_2(u)}{du} = \frac{\rho^2 g^2 v^2}{2} \frac{(1-u)^4}{u^2}.
\] (3.27)

The general version \( \{(A.5) \} \) of this equation is treated in detail in the appendix, and the general solution is given in \( \{(A.12) \} \). In second order the solution is, with the integration constant chosen to make \( \Psi_2(1) = 0 \):

\[
\Psi_2(u) = \frac{\rho^2 g^2 v^2}{2} \left( -\frac{1}{u} - \frac{3}{2} + 3u - \frac{1}{2}u^2 - 3 \log u \right).
\] (3.28)

Inserting \( \{(3.28) \} \) into \( \{(3.26) \} \) one gets the result

\[
\frac{\alpha_{-2}}{1+t} = \frac{\rho^2 g^2 v^2}{2} \left( \frac{1}{2} \log u - \frac{3}{2} \log u \right) \left. \right|_{u=\frac{t}{1+t}} + C_1
\] (3.29)

with \( C_1 \) an integration constant determined below.

The integrated versions of \( \{(3.18) \} \) and \( \{(3.19) \} \) are:

\[
\left( \frac{t}{1+t} \right)^3 \frac{d^2\alpha_{\perp,2}}{dt^2} = -\int_0^1 \left( (\frac{u'}{1-u'})^2 + \frac{\rho^2 g^2 v^2}{2} \right) d\Psi_2(u') \left. \right|_{u=\frac{1}{1+t}} + c_{\perp,2}
\] (3.30)

and

\[
\left( \frac{t}{1+t} \right)^3 \frac{d^2\alpha_{\parallel,2}}{dt^2} = \int_0^1 \left( (\frac{u'}{1-u'})^2 + \frac{\rho^2 g^2 v^2}{2} \right) d\Psi_2(u') \left. \right|_{u=\frac{1}{1+t}} + c_{\parallel,2}.
\] (3.31)

Here \( c_{\perp,2} \) and \( c_{\parallel,2} \) are integration constants. Since \( \Psi_2(u) = O((1 - u)^4) \) for \( u \to 1 \) it follows that the integrals in \( \{(3.30) \} \) and \( \{(3.31) \} \) are \( O((1 - u)^3) = O(t^{-3}) \) in this limit that after two integrations are \( O(t^{-1}) \). Nonzero values of the integration constants, however, lead to singular solutions for \( t \to \infty \).

Considering separately

\[
\left( \frac{t}{1+t} \right)^3 \frac{d^2\alpha_{\perp,2}}{dt^2} \simeq c_{\perp,2}
\] (3.32)

and

\[
\left( \frac{t}{1+t} \right)^3 \frac{d^2\alpha_{\parallel,2}}{dt^2} \simeq c_{\parallel,2}
\] (3.33)
one obtains the solutions
\[ \alpha_{\perp,2} \simeq c_{\perp,2}(\frac{1}{2t} - 3 \log t + 3(t \log t - t) + \frac{1}{2}t^2) \] (3.34)
and
\[ \alpha_{\parallel,2} \simeq c_{\parallel,2}(\frac{1}{2t} - 3 \log t + 3(t \log t - t) + \frac{1}{2}t^2) \] (3.35)
where terms involving \( t \log t \) and \( t^2 \) must be eliminated by a constraint. This is achieved by making in (3.32) and (3.33) the replacements:
\[
c_{\perp,2} \rightarrow c_{\perp,2} - c_{\perp,2}(\frac{t}{1 + t})^3(1 + \frac{3}{t});

c_{\parallel,2} \rightarrow c_{\parallel,2} - c_{\parallel,2}(\frac{t}{1 + t})^3(1 + \frac{3}{t}).
\] (3.36)

The integral occurring in (3.30) and (3.31) is by insertion of (3.28) explicitly:
\[
\int \frac{\left(\frac{u'}{1 - u'}\right)^2(3 - 2u')\Psi_2(u')}{2} du' = \frac{\rho^2 g^2 v^2}{2} \left( -\frac{1}{12} + u^3(3 \log u + \frac{3}{2}u + 1) - \frac{1}{4}u \right).
\] (3.37)
The requirement of acceptable behaviour of \( \alpha_{\perp,2} \) and \( \alpha_{\parallel,2} \) near \( t \simeq 0 \) fixes the two integration constants:
\[ c_{\perp,2} = c_{\parallel,2} = \frac{\rho^2 g^2 v^2}{12} \] (3.38)
thus making the right hand sides of (3.30) and (3.31), modified according to (3.36), vanish for \( u \to 0 \) \( (t \to 0) \). There is obviously agreement between (3.23) and (3.38), with \( c_{21+\parallel,2} = 2c_{\perp,2} + c_{\parallel,2} \).
(3.30) becomes after these transformations:
\[ \frac{d^2\alpha_{\perp,2}}{dt^2} = \frac{\rho^2 g^2 v^2}{2}(3(1 + t) \log \frac{1 + t}{t} - 3 - \frac{3}{2}t + \frac{1}{t^2}). \] (3.39)
\( \alpha_{\perp,2} \) is allowed to grow only logarithmically for \( t \to 0 \), where it has to agree with (3.16). This fixes the integration constants of (3.39) completely and the solution is:
\[ \alpha_{\perp,2} = \frac{\rho^2 g^2 v^2}{4}((1 + t)^3 \log \frac{1 + t}{t} + \log t - \frac{5}{2}t - t^2) + \alpha_{\perp,2,\min}. \] (3.40)
Likewise it follows from (3.31)

\[
\frac{d^2\alpha_{\parallel,2}}{dt^2} = \frac{\rho^2 g^2 v^2}{2} \left( -6(1 + t) \log \frac{1 + t}{t} + 6 + \frac{5}{2} \frac{1}{t} + \frac{1}{2} \frac{1}{1 + t} \right)
\]  

(3.41)

that with (3.39) is consistent with (3.24). From (3.41) follows by integration and taking the boundary conditions into account:

\[
\alpha_{\parallel,2} = \frac{\rho^2 g^2 v^2}{2} \left( -(1 + t)^3 \log \frac{1 + t}{t} + \frac{5}{2} t + t^2 \right) \\
+ \frac{\rho^2 g^2 v^2}{4} \left( (1 + t) \log \frac{1 + t}{t} + \log t \right) \\
+ \frac{\rho^2 g^2 v^2}{2} \left( \log \frac{\rho^2 g^2 v^2}{2t} + 2\gamma - 1 \right) + C_2
\]  

(3.42)

where the terms in the first line have logarithmic growth for \( t \to \infty \) and where \( C_2 \) is an integration constant. From (3.40) and (3.42) is formed \( \alpha_{-,2} = \alpha_{\perp,2} - \alpha_{\parallel,2} \), and the outcome is in agreement with (3.29) for

\[
C_1 = -\frac{\rho^2 g^2 v^2}{2} \frac{3}{4}, \quad C_2 = \frac{\rho^2 g^2 v^2}{2} \frac{9}{4}.
\]  

(3.43)

The modification of the field equations means that in (3.12) and (3.13) an extra term occurs on the right hand side:

\[-\frac{\rho^2 g^2 v^2}{2} \frac{t}{(1 + t)^2}.\]  

(3.44)

So far the modification of the field equations, apparent in the replacement 3.36 in 3.30 and 3.31, with the two integration constants given by (3.38), has been as small as possible in the sense that it is the modification that is necessary in order to obtain permitted asymptotic behaviour of the instanton solution. However, once this objective is achieved, additional constraint terms that do not upset the asymptotic behaviour can be added freely to the right hand sides of (3.30) and (3.31). One might thus modify (3.31) in such a way that \( \alpha_{\perp,2} \) reduces to \( \alpha_{\perp,2,\min} \). Then the right hand side of (3.30) should contain the additional term

\[-3(\frac{t}{1 + t})^3 \frac{\rho^2 g^2 v^2}{2} \left( (1 + t) \log \frac{1 + t}{t} - 1 - \frac{1}{2} \frac{1}{t} \right) .\]  

(3.45)
In order to keep the constraint gauge covariant one then should add this term also to the right hand side of (3.31).

It is preferable in general to restrict additional constraint terms by gauge covariance, having the same constraint terms in all modifications of (3.30) and (3.31) as we have by (3.38) in the simplest case considered here. Then \( \alpha_{-2} \) given in (3.29) (with (3.43)) is independent of the choice of constraint.

### 3.3 Third order and fourth order

The third order scalar field is according to (3.14) determined by:

\[
(1 + t)^2 \frac{d^2 A_3}{dt^2} - 2 A_3 = 4v \frac{d}{dt} \frac{\alpha_{\perp,2}}{1 + t}
\]  

whence

\[
A_3 = -\frac{v \alpha_{\perp,2}(t)}{(1 + t)^2} - (1 + t)^2 \int_t^\infty \frac{d \alpha_{\perp,2}(t')}{dt'} \frac{dt'}{(1 + t')^4}.
\]  

(3.46)

Here an integration constant is taken equal to zero to ensure that \( A_3 \) is \( O(t^{-2}) \) for \( t \to \infty \), since \( \frac{d \alpha_{\perp,2}}{dt} \) is \( O(t^{-1}) \) in this limit because of the constraint. This means that no additional constraint is necessary here. Using (3.40) one obtains the final result:

\[
A_3 = -\frac{v \alpha_{\perp,2}(t)}{(1 + t)^2} + (1 + t)^2 \rho^2 \frac{g^2 v^3}{4} \left( \frac{3t}{1 + t} + 2 \right) \log \frac{1 + t}{t} \\
- \frac{5}{1 + t} + \frac{1}{2(1 + t)^2} \left( \frac{1}{1 + t}^{\frac{1}{3}} \right)
\]  

(3.47)

Replacing \( \alpha_{\perp,2} \) with \( \alpha_{\perp,2,\min} \) in (3.17), which as mentioned above can be obtained by a modified constraint, we get instead of (3.48):

\[
A_3 = -\frac{v \alpha_{\perp,2}(t)}{(1 + t)^2} + (1 + t)^2 \rho^2 \frac{g^2 v^3}{2} \left( \log \frac{1 + t}{t} - \frac{1}{1 + t} \right) \\
- \frac{1}{2(1 + t)^2} \left( \frac{1}{3(1 + t)^3} \right)
\]  

(3.49)

The equations determining the fourth order gauge prepotentials are according to (3.12) and (3.13):

\[
(1 + t)^2 \frac{d}{dt} \left( \frac{(t}{1 + t})^3 \frac{d^2 \alpha_{\perp,4}}{dt^2} \right) - \frac{t^2}{1 + t} \left( 3 \frac{2t}{1 + t} \right) \frac{d}{dt} \frac{\alpha_{-4}}{1 + t}
\]
\[ (1 + t)^2 \frac{d}{dt} \left( \frac{t^2}{1 + t} \frac{d^2 \alpha_{\perp}}{dt^2} \right) + \frac{t^2}{1 + t} (6 - \frac{4t}{1 + t}) \frac{d \alpha_{\perp}}{dt} = 3 \frac{(1 + t)^2}{2} \frac{d}{dt} \left( \frac{\alpha_{\perp} t^3}{(1 + t)^4} \frac{d^2 \alpha_{\perp}}{dt^2} \right) \]

\[ + \frac{\rho^2 g^2 v^2}{2} (1 + t)^2 \frac{d}{dt} \alpha_{\perp} + \tilde{\chi}_{\perp,4} \quad (3.50) \]

and

\[ (1 + t)^2 \frac{d}{dt} \left( \frac{t}{1 + t} \frac{d^2 \alpha_{\parallel}}{dt^2} \right) + \frac{t^2}{1 + t} (6 - \frac{4t}{1 + t}) \frac{d \alpha_{\parallel}}{dt} = 3 (1 + t)^2 \frac{d}{dt} \left( \frac{\alpha_{||} t^3}{(1 + t)^4} \frac{d^2 \alpha_{||}}{dt^2} \right) + \tilde{\chi}_{\parallel,4} \quad (3.51) \]

with

\[ \tilde{\chi}_{\perp,4} = -\frac{3t^2}{(1 + t)^2} \left( \frac{d \alpha_{\perp}}{dt} + \frac{d \alpha_{\parallel}}{dt} \right) - \alpha_{\perp} \frac{d \alpha_{\perp}}{dt} \frac{d \alpha_{\perp}}{dt} + \frac{1}{2} \left( \frac{d \alpha_{\perp}}{dt} - \frac{2 \alpha_{\perp}}{(1 + t)^2} \right) \]

\[ + t^2 (3 + t) \left( \frac{d \alpha_{\perp}}{dt} - \frac{2 \alpha_{\perp}}{(1 + t)^2} \right) \Psi_2 \]

\[ + \frac{\rho^2 g^2 v^2}{2} \frac{2 (\alpha_{\perp} + \alpha_{\parallel})}{(1 + t)^2} \quad (3.52) \]

and

\[ \tilde{\chi}_{\parallel,4} = -t^2 (3 - \frac{2t}{1 + t}) \left( \frac{d \alpha_{\perp}}{dt} + \frac{d \alpha_{\parallel}}{dt} \right) \frac{d \alpha_{\perp}}{dt} + \frac{2 \alpha_{\perp}}{(1 + t)^2} \]

\[ - \frac{d \alpha_{\parallel}}{dt} \frac{d \alpha_{\parallel}}{dt} - 2 \frac{\alpha_{\parallel}}{(1 + t)^2} \right) \]

\[ + \left( \frac{2t^2}{1 + t} (6 - \frac{4t}{1 + t}) \right) \alpha_{\parallel} + 4t^3 \frac{d \alpha_{\parallel}}{dt} \Psi_2. \quad (3.53) \]

For \( t \to \infty \) the right-hand sides of (3.50) and (3.51) are \( O(t^{-2}) \). This should be compared to the right-hand side of the second order equation (3.18), which is \( O(t^{-3}) \) in this limit.
The function $\Psi_4 = \frac{1}{1+t} \frac{d^2}{dt^2} \frac{\alpha_\perp}{1+t}$ is by (3.50)-(3.51) a solution of the inhomogeneous hypergeometric equation (A.5) with $X = X_4$ given by:

$$
\frac{t^2}{1+t} X_4 = 3(1+t)^2 \frac{d}{dt}\left( \frac{t^3}{(1+t)^4} \alpha_{\perp,2} \frac{d^2 \alpha_{\perp,2}}{dt^2} \right) + \frac{\rho^2 g^2 v^2}{2} \frac{d}{dt} \left( \alpha_{\perp,2} \right)
$$

$$
+ \chi_{\perp,4} - \chi_{\|,4}.
$$

(3.54)

Most of the right-hand side of (3.54) is $O(t^0)$ in the limit $t \to 0$ where

$$
X_4 \approx -\left( \frac{\rho^2 g^2 v^2}{2} \right)^2 \frac{1}{t^3}
$$

(3.55)

while $X_4$ is $O(t^{-3})$ for $t \to \infty$.

With $\Psi_4$ given by (A.12) specialized to fourth order, its derivative has by (A.10) the following leading term for $u \to 0$ by an appropriate choice of the integration constant:

$$
\frac{d\Psi_4(u)}{du} \approx \frac{1}{u^3} \int_0^u (u')^2 (1-u')^{-4} X_4(u') du'.
$$

(3.56)

From (3.55) follows the splitting:

$$
X_4(u) = \frac{\rho^2 g^2 v^2}{2} \frac{(1-u)^4}{u^2} \frac{d \alpha_{\perp,2}}{du} + \tilde{X}_4(u)
$$

(3.57)

where $\tilde{X}_4(u) = O(u^{-2})$ for $u \to 0$, and hence:

$$
\frac{d\Psi_4(u)}{du} \approx \frac{1}{u^3} \frac{\rho^2 g^2 v^2}{2} \alpha_{\perp,2}(u)
$$

(3.58)

giving the leading singularity of $\Psi_4$ for $u \to 0$, while $\Psi_4 = O((1-u)^4)$ near $u = 1$ by the argument after (A.12).

By integration of (3.50) and (3.51) follows:

$$
\left( \frac{t}{1+t} \right)^3 \frac{d^2 \alpha_{\perp,4}}{dt^2} = 3\alpha_{\perp,2} \frac{t^3}{(1+t)^4} \frac{d^2 \alpha_{\perp,2}}{dt^2} + \frac{\rho^2 g^2 v^2}{2} \alpha_{\perp,2} \frac{1}{1+t}
$$

$$
- \int_u^1 \left( \frac{u'}{1-u'} \right)^2 (3-2u') \Psi_4(u') + \tilde{\chi}_{\perp,4}(u') \right) du' \bigg|_{u=\frac{t}{1+t}}
$$

$$
+ c_{\perp,4}
$$

(3.59)
\[
\frac{(t + t)^3}{t^3} \frac{d^2 \alpha_{\|,4}}{dt^2} = 3 \alpha_{\|,2} \frac{t^3}{(1 + t)^3} \frac{d^2 \alpha_{\|,2}}{dt^2} \\
- \int_u^1 \left( \frac{u'}{1 - u} \right)^2 \left( -6 + 4u' \right) \Psi_4(u') + \tilde{\chi}_{\|,4}(u') du' \mid_{u = \frac{t}{t + t}} + c_{\|,4} \tag{3.60}
\]

where the integrals as those of \((3.30)-(3.31)\) are \(O((1 - u)^3) = O(t^{-3})\) for \(t \to \infty\).

The integration constants \(c_{\perp;4}\) and \(c_{\|;4}\) are as at second order fixed by the requirement of acceptable asymptotic behaviour of the gauge prepotentials for \(t \to 0\), to

\[
c_{\perp;4} = \int_0^1 \left( \frac{u}{1 - u} \right)^2 (3 - 2u) \Psi_4(u) + \tilde{\chi}_{\perp,4}(u) du \tag{3.61}
\]

and

\[
c_{\|;4} = \int_0^1 \left( \frac{u}{1 - u} \right)^2 (-6 + 4u) \Psi_4(u) + \tilde{\chi}_{\|,4}(u) du. \tag{3.62}
\]

With this choice of integration constants it follows from \((3.59)\) and \((3.60)\) for \(t \to 0\):

\[
\frac{d^2 \alpha_{\perp,4}}{dt^2} \simeq \frac{\rho^2 g^2 v^2}{2t^3} \alpha_{\perp,2} \tag{3.63}
\]

and

\[
\frac{d^2 \alpha_{\|,4}}{dt^2} = O(t^{-2}) \tag{3.64}
\]

and from \((3.63)\):

\[
\alpha_{\perp,4} \simeq \frac{1}{2} \left( \frac{\rho^2 g^2 v^2}{2t} \right)^2 \left( \log \frac{\rho^2 g^2 v^2}{2t} + 2\gamma - \frac{5}{2} \right) \frac{1}{t} \tag{3.65}
\]

in agreement with \((3.11)\). No other terms singular as \(\frac{1}{t}\) for \(t \to 0\) occur in \(\alpha_{\perp,4}\) or \(\alpha_{\|,4}\); this is the reason why the integration constants \(c_{\perp;4}\) and \(c_{\|;4}\) have been chosen according to \((3.61)\) and \((3.62)\), respectively.

The integrals \((3.61)\) and \((3.62)\) converge, since \(\Psi_4(u)\) is \(O((1 - u)^4)\) for \(u \to 1\) and \(O(u^{-2})\) for \(u \to 0\). The integration constants are again equal (cf. \((3.38)\)). This is seen by use of \((3.31)\), \((3.58)\) and \((A.5)\):

\[
c_{\perp;4} - c_{\|;4}
\]

14
\[\begin{align*}
&= \int_0^1 \frac{d}{du}(u^3 \frac{d\Psi_4(u)}{du'}) + 3\Psi_4(u) \\
&\quad -3u^3(1-u)(\alpha_{\perp,2} \frac{d^2\alpha_{\perp,2}}{dt^2} - \alpha_{\parallel,2} \frac{d^2\alpha_{\parallel,2}}{dt^2})(u) \\
&\quad -\frac{\rho^2 g^2 v^2}{2}(1-u)^5\alpha_{\perp,2}(u)du \\
&= -\lim_{u \to 0}(u^3 \frac{d\Psi_4(u)}{du} - \frac{\rho^2 g^2 v^2}{2} \alpha_2(u)) = 0. \tag{3.66}
\end{align*}\]

This result can be used to get a different expression for \(c_{\perp;4} = c_{\parallel;4}\) from (3.61)-(3.62):

\[c_{\perp;4} = \frac{1}{3} \int_0^1 (2\tilde{\chi}_{\perp,4}(u') + \tilde{\chi}_{\parallel,4}(u'))du'. \tag{3.67}\]

where, according to (3.52) and (3.53):

\[2\tilde{\chi}_{\perp,4} + \tilde{\chi}_{\parallel,4} = -\frac{\Psi_2}{1-u}(\frac{3u^2}{(1-u)^3} \Psi_2 \\
+ \frac{\rho^2 g^2 v^2}{2} \frac{2u^2}{1-u}(3-2u)(\log u + \frac{1}{u} - 1)) \big|_{u = t+1} \\
+ \frac{\rho^2 g^2 v^2}{2} \frac{4(A_3 v + \alpha_{\perp,2})}{(1+t)^2} \tag{3.68}\]

where also (3.29) and (3.43) were used.

Using the result (3.48) for the scalar field \(A\) one obtains a term of (3.67):

\[\frac{1}{3} \frac{\rho^2 g^2 v^2}{2} \int_0^1 \frac{4(A_3 v + \alpha_{\perp,2})}{(1+t)^2} \big|_{t = \frac{u}{t+1}} du = \frac{1}{4} \left(\frac{\rho^2 g^2 v^2}{2}\right)^2. \tag{3.69}\]

Using instead (3.49) it is replaced by

\[\frac{1}{3} \left(\frac{\rho^2 g^2 v^2}{2}\right)^2. \tag{3.70}\]

The rest of (3.67) only depends on \(\alpha_{\perp,2}\) and is thus independent of the precise form of the second order constraint, provided the same constraint is used for \(\alpha_{\perp,2}\) and \(\alpha_{\parallel,2}\), i.e. the constraint is chosen gauge covariant. By insertion of (3.28) one obtains:

\[\frac{2}{3} \left(\frac{\rho^2 g^2 v^2}{2}\right)^2 \left(\frac{1}{6} + \frac{1}{18} - \frac{1}{16}\right). \tag{3.71}\]
and thus the value of the two fourth order integration constants is:

\[ c_{\perp;4} = c_{\parallel;4} = \left( \frac{\rho^2 g^2 v^2}{2} \right)^2 \left( \frac{1}{3} \right) + \frac{2}{3} \left( \frac{1}{6} + \frac{1}{18} - \frac{1}{16} \right). \] (3.72)

Nonzero integration constants upset, like in second order, the asymptotic behaviour for \( t \to \infty \) and necessitate a constraint that modifies (3.59) according to (cf. (3.36)):

\[ c_{\perp;4} \to c_{\perp;4} - c_{\perp;4} \left( \frac{t}{1 + t} \right)^3 \left( 1 + \frac{3}{t} \right) \] (3.73)

and the same for (3.60). This in its turn means that (3.12) and (3.13) must have an extra term on the right hand sides (cf. (3.44)):

\[ -6c_{\perp;4} \frac{t}{(1 + t)^2}. \] (3.74)

### 3.4 Higher orders; the short-distance limit

At fifth order one gets from (3.14):

\[ (1 + t)^2 \frac{d^2 A_5}{dt^2} - 2A_5 = 4v \frac{d}{dt} \left( \frac{\alpha_{\perp,4}}{1 + t} \right) + 2v(1 + t) \left( \frac{d}{dt} \left( \frac{\alpha_{\perp,2}}{1 + t} \right) \right)^2 - \frac{4v\alpha_{\perp,2}}{1 + t} \frac{d}{dt} \alpha_{\perp,2} + 4(1 + t)A_5 \frac{d}{dt} \alpha_{\perp,2}. \] (3.75)

Here the right-hand side is \( O(t^{-2}) \) for \( t \to \infty \), and hence \( A_5 \) is also \( O(t^{-2}) \) in this limit.

At sixth order (3.12) and (3.13) imply corresponding to (3.18) and (3.19), with \( \alpha_{-,6} = \alpha_{\perp,6} - \alpha_{\parallel,6} \):

\[ (1 + t)^2 \frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_{\perp,6}}{dt^2} \right) - \frac{t^2}{1 + t} \left( 3 - \frac{2t}{1 + t} \right) \frac{d}{dt} \alpha_{-,6} \]

\[ = \frac{\rho^2 g^2 v^2}{2} \left( 1 + t \right)^2 \frac{d}{dt} \frac{\alpha_{\perp,4}}{(1 + t)^3} + \tilde{\chi}_{\perp,6} = \chi_{\perp,6} \] (3.76)

and

\[ (1 + t)^2 \frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_{\parallel,6}}{dt^2} \right) + \frac{t^2}{1 + t} \left( 6 - \frac{4t}{1 + t} \right) \frac{d}{dt} \alpha_{-,6} \]

\[ = \chi_{\parallel,6} \] (3.77)
where $\tilde{\chi}_{\bot,6}$ and $\chi_{\|,6}$ are $O(t^{-1})$ for $t \to 0$ and $O(t^{-2})$ for $t \to \infty$. Here $\Psi_6 = \frac{1}{1+t} \frac{d\alpha_{\bot,6}}{dt}$ was introduced, which is a solution of (A.5), with $X_6$ given by:

$$\frac{t^2}{1+t} X_6 = \chi_{\bot,6} - \chi_{\|,6}. \quad (3.78)$$

For $t \to 0$ it follows from (A.14) that $\Psi_6 = O(t^{-3})$ while it is $O(t^{-4})$ for $t \to \infty$ by (A.12) as at second and fourth order.

By integration of (3.76) and (3.77) follows:

$$\left(\frac{t}{1+t}\right)^3 \frac{d^2 \alpha_{\bot,6}}{dt^2} = - \int_1^u \left(\frac{u}{1-u'}\right)^2 (3 - 2u') \Psi_6(u') \left. + \chi_{\bot,6}(u') \right|_{u=\frac{1}{1+t}} du' \quad (3.79)$$

and

$$\left(\frac{t}{1+t}\right)^3 \frac{d^2 \alpha_{\|,6}}{dt^2} = - \int_1^u \left(\frac{u}{1-u'}\right)^2 (-6 + 4u') \Psi_6(u') \left. + \chi_{\|,6}(u') \right|_{u=\frac{1}{1+t}} du' \quad (3.80)$$

respectively. The right-hand sides of (3.79) and (3.80) are $O(t^{-2})$ for $t \to \infty$, and the integration constants, which upset this asymptotic behaviour in second and fourth order, can here be chosen equal to zero; they are no longer fixed by requiring Bessel function behaviour of $\alpha_{\bot}$ because the Bessel function term now is $O(t^{-3})$. In the limit $t \to 0$ one gets from (3.79), using also (3.76) and (3.65):

$$\frac{d^2 \alpha_{\bot,6}}{dt^2} \simeq \frac{1}{2} \left(\rho^2 g^2 v^2\right)^3 (\log \frac{\rho^2 g^2 v^2}{2t} + 2\gamma - \frac{5}{2}) \frac{1}{t^4} \quad (3.81)$$

with the solution

$$\alpha_{\bot,6} \simeq \frac{1}{12} \left(\rho^2 g^2 v^2\right)^3 (\log \frac{\rho^2 g^2 v^2}{2t} + 2\gamma - \frac{10}{3}) \frac{1}{t^2} \quad (3.82)$$

in agreement with (3.11).

The iteration at higher orders proceeds in the same way. It is proved by induction that the $n$th order terms, $n \neq 0$, of $\alpha_{\bot}(t)$ and $\alpha_{\|}(t)$ as well as $t^2 A(t)$ in all orders at most have logarithmic growth for $t \to \infty$. At each order the complication arising from the coupling of the two equations of the gauge preponentials is handled in the same way as in the sixth order calculation, by means of (A.12) and the asymptotic estimates it implies.
3.5 Constraint terms in the vector field equation

It was found in (3.44) and (3.74) that the equations (3.12) and (3.13) both require the following additional term on the right-hand side

\[-6c \frac{t}{(1 + t)^2}\]  

with \(c = c_{\perp,2} + c_{\perp,4}\). Thus the gauge field equation has to be modified to:

\[
\partial_\mu F^a_{\mu\nu} + g e^{abc} A^b_{\mu} F^c_{\mu\nu} + \frac{c}{g} \tilde{\eta}^{a}_{\nu\lambda} x_{\lambda} \frac{48 \rho^2}{x^2 (\rho^2 + x^2)^2} 
\]

\[-2g^2 (A^b A^b A^a_{\nu} - A^a A^b A^b_{\nu}) = 0\]  

(3.84)

where the extra term can be obtained from a source term in the Lagrangian that arises from a constraint on the path integral. This modification of the gauge field equation has the same structure as that determined for the case where the scalar field is in the fundamental representation [4], [5].

Modifying the equations according to (3.45), one obtains correspondingly an additional source term in the gauge field equation.

3.6 Leading and subleading terms at large distances

The leading terms for \(t \to 0\) fulfil in each order of \(v\):

\[
\alpha_{\perp,||,n} \propto t^{1 - \frac{n}{2}}, \quad A_n \propto t^3 - \frac{n}{2}, \quad n > 1
\]  

(3.85)

and

\[
\alpha_{\perp,||,0} \simeq 1, \quad A_1 \simeq v.
\]  

(3.86)

The sums of the leading terms are denoted \(\alpha^{(2)}_{\perp}, \alpha^{(2)}_{||}\) and \(A^{(2)}\) where the superscript indicates that they contain a factor \(\rho^2\) when expressed in terms of \(\rho\) and \(x\) (when the functions are expressed in terms of \(\rho\) and \(\sqrt{t}\), the superscript indicates the combined power of \(\rho\) and \(\sqrt{t}\)). Then (3.12), (3.13) and (3.14) imply:

\[
\frac{d}{dt}(t^3 \frac{d^2 \alpha^{(2)}_{\perp}}{dt^2}) = \frac{\rho^2 g^2 v^2}{2} \frac{d \alpha^{(2)}_{\perp}}{dt},
\]  

(3.87)

\[
\frac{d}{dt}(t^3 \frac{d^2 \alpha^{(2)}_{||}}{dt^2}) = 0
\]  

(3.88)
and

\[ \frac{d^2 A^{(2)}}{dt^2} = 0. \]  

(3.89)

The sum of the leading terms of \( \alpha_{\perp 2} \) to all orders in \( v \), which is a solution of (3.87), is the second term of (3.11), i.e.:

\[ \alpha_{\perp}^{(2)} = \sqrt{2t\rho g v K_1(\sqrt{\frac{2}{t}\rho g v}). \]  

(3.90)

To lowest order this reduces to

\[ \alpha_{\perp}^{(2)} \simeq t \]  

(3.91)

in accordance with (3.15). The solutions of (3.88) and (3.89) that are compatible with (3.42) and (3.15) are

\[ \alpha_{\parallel}^{(2)} = t + \frac{\rho^2 g^2 v^2}{4} \left( \log \frac{\rho^2 g^2 v^2}{2} + 2\gamma + \frac{5}{4} \right), \quad A^{(2)}(t) = -vt. \]  

(3.92)

At next order (3.12), (3.13) and (3.14) are:

\[ \frac{d^2}{dt^2} \frac{d\alpha_{\perp}^{(4)}}{dt} + 3 \frac{d}{dt} \frac{d\alpha_{\perp}^{(4)}}{dt} - \frac{\rho^2 g^2 v^2}{2t^3} \frac{d\alpha_{\perp}^{(4)}}{dt} \]

\[ = \frac{3}{t} \left( \frac{d\alpha_{\perp}^{(2)}}{dt} + \frac{d\alpha_{\parallel}^{(2)}}{dt} \right) + \frac{\rho^2 g^2 v}{2t^3} \left( 3v\alpha_{\perp}^{(2)} + 2A^{(2)} \right) \frac{d\alpha_{\perp}^{(2)}}{dt}, \]  

(3.93)

\[ \frac{1}{t^3} \frac{d}{dt} \frac{d^2 \alpha_{\perp}^{(4)}}{dt^2} = -\frac{3}{t} \left( \frac{d\alpha_{\perp}^{(2)}}{dt} \right)^2 - \left( \frac{d\alpha_{\parallel}^{(2)}}{dt} \right)^2 \]  

(3.94)

and

\[ \frac{d^2 A^{(4)}}{dt^2} = 2 \left( \frac{d\alpha_{\perp}^{(2)}}{dt} \right)^2 v. \]  

(3.95)

Here (3.93) and (3.94) have an additional term on the right hand side from the constraint:

\[ -\frac{\rho^2 g^2 v^2}{2} \frac{1}{t^2}. \]  

(3.96)

Modifying the constraint with an exponential factor, one obtains exponential falloff for \( t \to 0 \) of (3.96) and consequently of \( \alpha_{\perp}^{(4)} \), and \( A^{(4)} \) has also
exponential falloff by (3.95). The detailed analysis is very similar to that carried out in [5], the only difference being that (3.96) does not cancel the $O(t^{-2})$ term on the right hand side of (3.93). In contrast, $\alpha^{(4)}_\parallel$ has power law falloff by the following term on the right hand side of (3.94):

$$\frac{3}{t} \left( \frac{d\alpha^{(2)}_\parallel}{dt} \right)^2$$

whence:

$$\alpha^{(4)}_\parallel = \frac{1}{2} t^2 + \cdots.$$  (3.98)

This summation procedure can clearly be iterated

4 Coupled equations for fermionic zero modes

4.1 General setup

From (2.3) one obtains the following coupled equations for the fermionic zero modes:

$$(\sigma \cdot D)^{ab} \lambda_B^b + g \epsilon^{abc} \sqrt{2} A^b q^c_A = 0 \quad (4.1)$$

and

$$(\bar{\sigma} \cdot D)^{ab} q_A^b - g \epsilon^{abc} \sqrt{2} A^b \lambda_B^c = 0 \quad (4.2)$$

and a second set of equations obtained by the substitution $(\lambda_B, q_A) \rightarrow (q_B, -\lambda_A)$.

4.2 The supersymmetric zero mode

For the supersymmetric zero mode the following Ansatz for the gluino field $\lambda_B$ is used:

$$\lambda_B^a = f_\perp(t) \bar{\sigma} \cdot x \sigma^a \sigma \cdot x u_\sigma, \ a = 1, 2 \quad (4.3)$$

and

$$\lambda_B^3 = f_\parallel(t) \bar{\sigma} \cdot x \sigma^3 \sigma \cdot x u_\sigma, \ a = 1, 2, \quad (4.4)$$

where $u_\sigma$ is a constant unity twospinor. The quark field $q_A$ only has transverse components with the Ansatz:

$$q_A^a = \phi(t) \epsilon^{ab} \sigma^b \sigma \cdot x u_\sigma.$$  (4.5)
From (4.1) and (4.2) the following set of coupled equations is then obtained:

\[ 6f_{\perp}(t) - 2t \frac{df_{\perp}(t)}{dt} - 2t \frac{d\log \alpha_{\parallel}(t)}{dt} f_{\perp}(t) - 2t \frac{d\log \alpha_{\perp}(t)}{dt} f_{\parallel}(t) + g\sqrt{2}A(t)\phi(t) = 0, \]

(4.6)

\[ 6f_{\parallel}(t) - 2t \frac{df_{\parallel}(t)}{dt} - 4t \frac{d\log \alpha_{\parallel}(t)}{dt} f_{\perp}(t) = 0 \]

(4.7)

and

\[-2t^2 \rho^2 \left( \frac{d\phi(t)}{dt} - \frac{d\log \alpha_{\parallel}(t)}{dt} \phi(t) \right) + g\sqrt{2}A(t)f_{\perp}(t) = 0.\]

(4.8)

These equations are solved by iteration in the parameter \( v \), with even orders for \( f_{\perp} \) and \( f_{\parallel} \) and odd orders for \( \phi \), and with the order of \( v \) indicated by a subscript.

At zeroth order (4.6) and (4.7) reduce to:

\[ 6f_{\perp,0}(t) - 2t \frac{df_{\perp,0}(t)}{dt} - \frac{2t}{1+t}(f_{\perp,0}(t) + f_{\parallel,0}(t)) = 0 \]

(4.9)

with the solution

\[ f_{\perp,0}(t) = f_{\parallel,0}(t) = \frac{t^3}{(1+t)^2}. \]

(4.10)

At first order (4.8) implies:

\[-2t^2 \rho^2 \left( \frac{d\phi_1(t)}{dt} - \frac{\phi_1(t)}{1+t} \right) = -g\sqrt{2}v \left( \frac{t^3}{(1+t)^3} \right). \]

(4.11)

By integration follows:

\[ \phi_1(t) = g\sqrt{2}v\rho^2 \frac{1}{4} \frac{t^2}{(1+t)^2} + \frac{1}{3} t^3 \]

(4.12)

where an integration constant was chosen to make \( \phi(t) \) vanish for \( t \to 0 \).

At second order the following equations arise from (4.6) and (4.7):

\[ 6f_{\perp,2}(t) - 2t \frac{df_{\perp,2}(t)}{dt} - \frac{2t}{1+t} f_{\perp,2}(t) - \frac{2t}{1+t} f_{\parallel,2}(t) = 0 \]

\[ = \frac{2t^4}{(1+t)^2} \left( \frac{d\alpha_{\perp,2}}{dt} \frac{1}{1+t} + \frac{d\alpha_{\parallel,2}}{dt} \frac{1}{1+t} \right) - \frac{\rho^2 g^2 v^2 t^2}{2} \frac{1}{(1+t)^3} \]

(4.13)
and
\[ 6f_{\parallel,2}(t) - 2t \frac{df_{\parallel,2}(t)}{dt} - \frac{4t}{1+t} f_{\perp,2}(t) = \frac{4t^4}{(1+t)^2} \frac{d}{dt} \frac{\alpha_{\perp,2}}{1+t} \quad (4.14) \]

whence
\[
\frac{d}{dt} \frac{(1+t)^2}{t^3} (2f_{\perp,2}(t) + f_{\parallel,2}(t)) = \frac{1}{3} \frac{\rho^2 g^2 v^2}{2t^2} (1 + \frac{2}{1+t}) - 4 \frac{d}{dt} \frac{2\alpha_{\perp,2} + \alpha_{\parallel,2}}{1+t} \quad (4.15)
\]

and
\[
\frac{d}{dt} \frac{f_{\perp,2}(t) - f_{\parallel,2}(t)}{t^3(1+t)} = \frac{\Psi_2(t)}{(1+t)^2} + \frac{\rho^2 g^2 v^2}{4} \frac{1 + \frac{1}{3} t}{t^2(1+t)^4} \quad (4.16)
\]

that are both solved by quadrature. For \( t \to \infty \) the right hand sides of (4.15) and (4.16) are \( O(t^{-2}) \) and \( O(t^{-5}) \), respectively. Thus it follows by integration of these equations that both \( f_{\perp,2}(t) \) and \( f_{\parallel,2}(t) \) are \( O(t^0) \) in this limit. In the limit \( t \to 0 \) it follows from the two equations that both \( f_{\perp,2}(t) \) and \( f_{\parallel,2}(t) \) are \( O(t^2) \). The supersymmetric zero mode is therefore square integrable also in second order.

### 4.3 Asymptotic behaviour

A proof by induction similar to that given in the bosonic case is carried out on the basis of (4.6), (4.7) and (4.8) that \( \phi(t) \), \( f_{\perp}(t) \) and \( f_{\parallel}(t) \) at most have logarithmic growth for \( t \to \infty \) at each order in \( v \), except \( f_{\perp,0} \) and \( f_{\parallel,0} \) which grow linearly. Assuming the above-mentioned estimates hold to orders less than \( n \), one sees immediately from (4.6) and (4.7) that \( f_{\perp,0} \) and \( f_{\parallel,0} \) are \( O(t^0) \) for \( t \to \infty \). From (4.8) we get:
\[
\frac{d\phi_n(t)}{dt} - \frac{\phi_n(t)}{1+t} \propto t^{-2} \quad (4.17)
\]

establishing the estimate for \( \phi \) also to order \( n \). Thus the estimate holds to all orders.

Then the limit \( t \to 0 \) is investigated. For the leading terms for \( t \to 0 \) one has the estimate
\[
f_{\perp,n} \propto t^{3-n/2}, \quad \phi_n(t) \propto t^{5-n/2}. \quad (4.18)
\]
The analysis of the leading, nextleading etc. terms of $f$ and $\phi$ is quite similar to that carried out on the corresponding quantities for supersymmetric QCD in [5], to which we refer for details. The new feature is the presence of $f_\parallel(t)$, which by (4.7) and (4.10) has the leading term in the double series expansion of $\rho$ and $\sqrt{t}$:

$$f_\parallel^{(6)}(t) = t^3$$

while the nextleading terms are determined by

$$-2t^4 \frac{d}{dt} \frac{f_\parallel^{(8)}(t)}{t^3} = 4t \frac{d\alpha_\parallel^{(2)}(t)}{dt} f_\perp^{(6)}(t)$$

with the solution

$$f_\parallel^{(8)}(t) = -2t^3 \int_0^t \frac{dt'}{(t')^3} \frac{d\alpha_\parallel^{(2)}(t')}{dt} f_\perp^{(6)}(t')$$

which in lowest order of the expansion in $\nu$ agrees with the $O(t^4)$ term of (4.10). The right-hand side of (4.20) shows exponential decrease for $t \to 0$ ($x \to \infty$) after summation over all orders of $\nu$, and so does therefore $f_\parallel^{(8)}(t)$ by (4.21), in contrast to the leading term that by (4.19) has a power law decrease for $x \to \infty$. A similar phenomenon was observed for the scalar field $A$.

### 4.4 The superconformal zero mode

To obtain the superconformal zero mode the equations (4.11) and (4.12) are solved by the following Ansatz for the gluino field:

$$\lambda_B^a = f_\perp(t) \sigma \cdot x \sigma^a u_\sigma, \quad a = 1, 2$$

and

$$\lambda_B^3 = f_\parallel(t) \sigma \cdot x \sigma^3 u_\sigma, \quad a = 1, 2$$

and for the quark field:

$$q_A^a = \phi(t) \epsilon^{ab} \sigma^b u_\sigma; \quad a, b = 1, 2$$

where the functions $f_\perp(t)$, $f_\parallel(t)$ and $\phi(t)$ obey the coupled equations:

$$4f_\perp(t) - 2t \frac{df_\perp(t)}{dt} - 2t \frac{d\log \alpha_\parallel(t)}{dt} f_\perp(t)$$

$$-2t \frac{d\log \alpha_\perp(t)}{dt} f_\parallel(t) + g \sqrt{2} A(t) \phi(t) = 0,$$
\[ 4f_\parallel(t) - 2t \frac{df_\parallel(t)}{dt} - 4t \frac{d\log \alpha_\perp(t)}{dt} f_\perp(t) = 0 \quad (4.26) \]

and

\[ - \frac{2t^2}{\rho^2} \left( \frac{d\phi(t)}{dt} - \frac{d\log \alpha_\parallel(t)}{dt} \phi(t) \right) + g\sqrt{2}A(t)f_\perp(t) = 0. \quad (4.27) \]

At zeroth order the solution of (4.25) and (4.26) is:

\[ f_\perp,0(t) = f_\parallel,0(t) = \frac{t^2}{(1+t)^2}. \quad (4.28) \]

At first order (4.27) is:

\[ - \frac{2t^2}{\rho^2} \left( \frac{d\phi_1(t)}{dt} - \frac{\phi_1(t)}{1+t} \right) + g\sqrt{2}v \frac{t^2}{(1+t)^3} = 0 \quad (4.29) \]

with the solution

\[ \phi_1(t) = -\frac{1}{6} g\sqrt{2}v\rho^2 \frac{1}{(1+t)^2} \quad (4.30) \]

where an integration constant has to vanish for the sake of the asymptotic behaviour for \( t \to \infty \).

Adding to \( \phi_1(t) \) a term

\[ \phi_{1,\text{add}} = \frac{1}{6} g\sqrt{2}v\rho^2 \quad (4.31) \]

in order to obtain a solution that vanishes for \( t \to 0 \), means having a nonzero right-hand side of (4.2):

\[ \frac{t^2}{1+t} \varepsilon^{ab\sigma} \cdot x\sigma^b u_\sigma \simeq \frac{1}{3} g\sqrt{2}v(1+t)\varepsilon^{ab}\lambda_B^b \quad (4.32) \]

where the last version of (4.32) is produced by an additional Yukawa coupling term in (2.3):

\[ \frac{1}{3} g\sqrt{2}v(1+t)(q_B^a)^\dagger \varepsilon^{ab}\lambda_B^b. \quad (4.33) \]

For \( t \to 0 \) the analysis of leading, nextleading etc. terms is very similar to the corresponding analysis of supersymmetric QCD carried out in [5]. In the other limit, \( t \to \infty \), it follows from (4.25), (4.26) and (4.27) that \( f_\perp,0 \to 1, f_\parallel,0 \to 1 \) while \( f_\perp,n \propto t^{-1}, f_\parallel,n \propto t^{-1}, n \neq 0 \) and \( \phi \propto t^{-2} \).
5 Conclusion

The main result of this paper is in a sense a negative one: Despite the considerable complications arising from the presence of two different gauge prepotentials, while \[4\], \[5\] had only one, the result of the analysis is exactly the same: Constraints are required that produce additional terms in the gauge field equations at second and fourth order of the gauge breaking parameter \(v\).

It is perhaps less surprising that also the outcome of the analysis of the fermionic zero modes is the same as in \[5\]: While the supersymmetric zero mode is perfectly well behaved, the superconformal zero mode has a nonpermissible large distance behaviour at first order in \(v\) that is eliminated by an additional Yukawa coupling.

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A Solution of the inhomogeneous hypergeometric equation

The starting point is the following two differential equations fulfilled according to (3.12)-(3.13) by the gauge field prepotentials \(\alpha_\perp\) and \(\alpha_\parallel\) in a general order of the expansion in the parameter \(v\):

\[
(1 + t)^2 \frac{d}{dt} \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_\perp}{dt^2} - \frac{t^2}{1 + t} (3 - \frac{2t}{1 + t}) \frac{d}{dt} \frac{\alpha_\perp}{1 + t} = \chi_\perp \tag{A.1}
\]

and

\[
(1 + t)^2 \frac{d}{dt} \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_\parallel}{dt^2} + \frac{t^2}{1 + t} (6 - \frac{4t}{1 + t}) \frac{d}{dt} \frac{\alpha_\parallel}{1 + t} = \chi_\parallel \tag{A.2}
\]
with $\alpha_\perp = \alpha_\perp - \alpha_\parallel$, and the function $\Psi$ is defined as:

$$\Psi = \frac{1}{1 + t} \frac{d}{dt} \frac{\alpha_\perp}{1 + t}. \quad (A.3)$$

By combination of (A.1) and (A.2) follows:

$$(1 + t)t^3 \frac{d^2 \Psi}{dt^2} + (3t^2(1 + t) + 2t^3) \frac{d\Psi}{dt} = \frac{1}{1 + t} \frac{d}{dt} t^3(1 + t)^2 \frac{d\Psi}{dt} = \chi_\perp - \chi_\parallel. \quad (A.4)$$

This equation can obviously be solved by quadrature, but it is convenient to rewrite it as an inhomogeneous hypergeometric equation. Switching to the variable $u = \frac{t}{1+t}$ one finds:

$$u(1 - u) \frac{d^2 \Psi}{du^2} + 3 \frac{d\Psi}{du} = X \quad (A.5)$$

with

$$X = \frac{1 + t}{t^2} (\chi_\perp - \chi_\parallel). \quad (A.6)$$

The homogeneous equation:

$$u(1 - u) \frac{d^2 \Psi}{du^2} + 3 \frac{d\Psi}{du} = 0 \quad (A.7)$$

has the independent solutions

$$F(u) = 1 \quad (A.8)$$

and

$$G(u) = -\frac{1}{2u^2} + \frac{3}{u} + 3 \log u - u - \frac{3}{2} \quad (A.9)$$

with

$$\frac{dG(u)}{du} = (\frac{1 - u}{u})^3; \quad G(1) = 0. \quad (A.10)$$

A particular solution of the inhomogeneous equation (A.5) is, with $0 < u_0 \leq 1$:

$$\Psi(u) = \int_{u_0}^{u} (G(u) - G(u'))(u')^2(1 - u')^{-4}X(u')du' \quad (A.11)$$
to which is added a linear combination of $F(u)$ and $G(u)$ in order to obtain the most general solution. $X(u)$ is $O((1 - u)^3)$ near $u = 1$ for all orders of $v$ and the first integral in (A.11) thus diverges at most logarithmically for $u \to 1$. The integration constants are chosen such that the final solution of (A.5) is

$$
\Psi(u) = \int_1^u G(u')(u')^2(1 - u')^{-4}X(u')du' \\
+ \left( \int_{u_0}^u (u')^2(1 - u')^{-4}X(u')du' + C_G \right)G(u).
$$

(A.12)

with $C_G$ an integration constant, and it follows from (A.10) that $\Psi(u)$ is $O((1 - u)^4)$ near $u = 1$.

In general the leading term at $n$th order for $t \to 0$ ($u \to 0$) is:

$$
X_n \propto t^{-1 - \frac{n}{2}}.
$$

(A.13)

From this estimate then follows near $u = 0$:

$$
\Psi_n(u) \propto u^{-\frac{n}{2}}.
$$

(A.14)

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