Abstract. For any prism $(A, d)$ we construct a canonical map
$W_r(A/d) \to A/d\phi(d)\ldots\phi^{r-1}(d)$. This map is necessary for the
existence of a canonical base change comparison between prismatic cohomology and de Rham-Witt forms. We construct a
canonical map from de Rham-Witt forms to prismatic cohomology and prove that it is an isomorphism in the perfect case. Using
this we get an explicit description of the prismatic cohomology $H^i((S/A)_{\Delta}, \mathcal{O}/d\ldots\phi^{n-1}(d))$ when $S$ is the$p$-completion of a polynomial algebra over $A/d$.

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1. Introduction

This paper mainly deals with prismatic cohomology which was recently introduced
by B. Bhatt and P. Scholze in [5]. This theory works for varieties over $p$-adic rings, and
admits comparisons to various cohomology theories. When the base ring is $R = \mathcal{O}_C$, where
$C$ is a complete algebraically closed extension of $\mathbb{Q}_p$, this theory had been
previously constructed by B. Bhatt, M. Morrow and P. Scholze in [4]. In that case
they proved a comparison with continuous de Rham-Witt forms. This comparison
was not generalized to the general setting of prismatic cohomology so far.

The goal of our work was to obtain some results in that direction. We started with
the following question:

Question 1.1. Let $(A, d)$ be a prism and $S$ be a $p$-completely smooth $A/d$-algebra. Then for any $r \geq 1$ there is a functorial isomorphism of $A/d\ldots\phi^{r-1}(d)$-modules

$$W_r\Omega^i_{S/(A/d)} \otimes_{W_r(A/d)} A/d\ldots\phi^{r-1}(d) \to H^i((S/A)_{\Delta}, \mathcal{O}/d\ldots\phi^{r-1}(d)).$$

If the previous statement is true one requires that for any principal prism $(A, d)$
there is a canonical map $W_r(A/d) \to A/d\phi(d)\ldots\phi^{r-1}(d)$. The construction of these
maps is the first result of this paper. Previously, the existence of these maps was
known only in the perfect case. Our proof was obtained via reduction to the case
of perfect prisms. We also present another proof provided to us by the anonymous
referee in Appendix.
After constructing of these maps it is formal procedure to get a map
\[ W_r(S) \rightarrow H^0((S/A)_{\Delta}, \mathcal{O}_\Delta/d\phi(d) \ldots \phi^{r-1}(d)). \]

Moreover, similar to Hodge-Tate complex $\Delta_{S/A}$, the prismatic cohomology groups
\[ \bigoplus_{n \geq 0} H^n((S/A)_{\Delta}, \mathcal{O}_\Delta/d\phi(d) \ldots \phi^{r-1}(d)) \{n\} \]
formally form a commutative differential graded algebra via a Bockstein homomorphism as a differential. From this one can conclude that the desired maps from de Rham-Witt forms are unique if they exist. The next result of this paper is the existence of these maps under some mild condition. As with the previous result this was obtained by a complicated reduction to the case of perfect prisms. For them we prove the following theorem:

**Theorem 1.2.** Let $(A, d)$ be a perfect prism such that $A/d$ is $p$-torsion free and $S$ be a $p$-completely smooth $A/d$-algebra. Then we have the functorial isomorphism
\[ W_r^n \Omega^{i, \text{cont}}_{S/(A/d)} \rightarrow H^i(\Delta_{S/A} \otimes_{A/d} A/d \ldots \phi^{n-1}(d)). \]

The perfect case reduces to a result of B. Bhatt, M. Morrow and P. Scholze in [4] via André’s lemma (cf. [[5], Theorem 7.12]).

In general, Peter Scholze informed us that the statement in the Question 1.1 is false. Right now it is unclear how one needs to modify the left-hand side. Nevertheless, when $S = A/d(T_1, \ldots, T_n)$ is a $p$-completed polynomial algebra, both sides of the 1.1 admit a similar explicit description. Indeed, it is known from [11] that $W_r^n \Omega^{i, \text{cont}}_{S/(A/d)}$ is a certain infinite direct sum of copies of $W_i(A/d)$ for $1 \leq i \leq r$ and it turns out that $H^i(S/A)_{\Delta}, \mathcal{O}_\Delta/d\phi(d) \ldots \phi^{r-1}(d)$ is a $p$-completed infinite direct sum, with the same index set, of copies of $A/d\phi(d) \ldots \phi^{r-1}(d)$. This is the third main result of the paper.

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2. Prisms and distinguished elements

Fix a prime $p$. In this section we list basic definitions and results from [5], of which we will make a constant use in the paper. We also use [3] a lot. We start with some preliminary information about $\delta$-rings.

2.1. $\delta$-rings. The goal of this subsection is to record some facts about $\delta$-rings. They were introduced by Joyal in [10]. This notion provides a good language to deal with rings with a lift of Frobenius modulo $p$. The good reference for this theory is [6].

Definition 2.1. A $\delta$-ring is a pair $(A, \delta)$ where $A$ is a commutative ring and $\delta : A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$, satisfying the following two identities

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

and

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}.$$

In the literature, a $\delta$-structure is often called a $p$-derivation. The main feature of the $\delta$-structure is that it gives a Frobenius lift:

Lemma 2.2. ($\delta$-structures give Frobenius lifts)

1. Suppose $\delta : A \to A$ gives a $\delta$-structure on $A$ then the map $\phi : A \to A$ given by $\phi(x) = x^p + p\delta(x)$ defines an endomorphism of $A$ that is a lift of the Frobenius on $A/p$.

2. When $A$ is $p$-torsion free the construction in (1) gives a bijection between $\delta$-structures on $A$ and Frobenius lifts modulo $p$ on $A$.

3. If $A$ is a $\delta$-ring then $\phi : A \to A$ is a $\delta$-map, i.e. $\phi(\delta(x)) = \delta(\phi(x))$ for any $x \in A$.

The proof is standard and can be found in [3] and [5].

Example 2.3. The previous lemma gives us many simple examples of $\delta$-rings when $A$ is $p$-torsion free.

1. The ring $\mathbb{Z}$ with identity $\phi(x) = x$. Moreover, it is easy to see that this is an initial object in the category of $\delta$-rings.

2. The ring $\mathbb{Z}[x]$ with $\phi$ determined by $\phi(x) = x^p + pg(x)$ for any $g(x) \in \mathbb{Z}[x]$.

3. If $k$ is a perfect field of characteristic $p > 0$ then the ring of Witt vectors $W(k)$ admits a unique standard lift of Frobenius which also gives a unique $\delta$-structure on $W(k)$.

Definition 2.4. A $\delta$-ring $A$ is called perfect if $\phi$ is an isomorphism.

2.2. Prisms and distinguished elements. In this subsection we discuss definitions and some properties of prisms and distinguished elements. We assume that all rings that appear are $p$-local, i.e. $p \in \text{Rad}(A)$ where $\text{Rad}(A)$ is the Jacobson radical of $A$. We clearly have this condition if $A$ is $p$-adically complete.

Definition 2.5. Let $A$ be a $\delta$-ring. An element $d \in A$ is called distinguished or primitive if $\delta(d)$ is a unit in $A$.

Note that since $\delta$ commutes with $\phi$ by Lemma 2.2 and all our rings are $p$-local we see that $d$ is distinguished if and only if $\phi(d)$ is distinguished.

Example 2.6. The main "cohomology" examples are the following:
(1) Let \((A, d)\) be \((\mathbb{Z}_p, p)\). This will give crystalline cohomology theory.

(2) Let \((A, d)\) be \((\mathbb{Z}_p[[q - 1]], d = [p]_q = \frac{q^d - 1}{q - 1})\) with \(\delta\)-structure given by \(\phi(q) = q^p\). \([p]_q\) is distinguished since \(\delta([p]_q) \equiv \delta(p) \mod q - 1\). Indeed, one then uses part (1) and \((q - 1)\)-adic completeness. This will give \(q\)-de Rham cohomology theory.

(3) Let \(\mathcal{C}/\mathbb{Q}_p\) be a perfectoid field and \((A, d)\) be \((A_{\text{perf}}(\mathcal{O}_C), \xi)\), where \(\xi\) is any generator of the kernel of Fontaine’s map \(A \to \mathcal{O}_C\). Then \(A\) admits a unique \(\delta\)-structure given by the lift of Frobenius. The distinguishedness of \(\xi\) can be seen similarly to part (2). This will give \(A_{\text{perf}}\)-cohomology theory.

**Definition 2.7.** (Category of prisms).

1. A \(\delta\)-pair \((A, I)\) is a prism if \(I \subset A\) is an invertible ideal such that \(A\) is derived \((p, I)\)-complete and \(p \in IA + \phi(I)A\).
2. A map \((A, I) \to (B, J)\) is (faithfully) flat if the map \(A \to B\) is \((p, I)\)-completely (faithfully) flat, i.e. \(A/(p, I) \to B \otimes^L_A A/(p, I)\) is (faithfully) flat.
3. A prism \((A, I)\) is called
   - bounded if \(A/I\) has bounded \(p^\infty\)-torsion, i.e. \(A/I[p^\infty] = A/I[p^c]\) for some \(c \geq 0\).
   - crystalline if \(I = (p)\).
   - orientable if the ideal \(I\) is principal, the choice of a generator is called an orientation.
   - perfect if \(A\) is a perfect \(\delta\)-ring, i.e. \(\phi\) is an isomorphism.

**Remark 2.8.** By \([5]\), Lemma 3.1 the condition \(p \in IA + \phi(I)A\) is equivalent to the fact that \(I\) is pro-Zariski locally on \(\text{Spec}(A)\) generated by a distinguished element. Therefore, it is usually not much harm to assume that \(I = (d)\), i.e. \((A, I)\) is principal.

**Example 2.9.** Let \(A\) be a \(p\)-torsion free and \(p\)-complete \(\delta\)-ring. Then the pair \((A, (p))\) is a crystalline prism. Conversely, any crystalline prism is of this form.

**Example 2.10.** Let \(A_0 = \mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}\) be the displayed localization of the free \(\delta\)-ring on a variable \(d\). We denote by \(A\) the \((p, d)\)-completion of \(A_0\). Then \((A, (d))\) is a bounded prism. Moreover, it is the universal oriented prism.

Let us recall the relation between perfectoid rings and perfect prisms. We start we the notion of the perfection of a prism.

**Lemma 2.11.** Let \((A, I)\) be a prism. Let us denote the perfection \(\text{colim}_\phi A\) of \(A\) by \(A_{\text{perf}}\). Then \(IA_{\text{perf}} = (d)\) is generated by a distinguished element, both \(d\) and \(p\) are nonzerodivisors in \(A\) and \(A/d[p^\infty] = A/d[p]\). In particular, the derived \((p, I)\)-completion \(A_{\text{perf}}\) agrees with the classical one and \((A_{\text{perf}}, IA_{\text{perf}})\) is the universal perfect prism under \((A, I)\).

**Proof.** Cf. \([5]\), Lemma 3.9. \(\square\)

**Theorem 2.12.** The functor

\[
\{\text{perfect prisms } (A, I)\} \to \{\text{(integral) perfectoid rings } R\}, \quad (A, I) \mapsto A/I
\]

is an equivalence of categories with the inverse given by \(R \mapsto (A_{\text{perf}}(R), \ker(\theta))\), where \(A_{\text{perf}}(R) = W(R^\theta)\) and \(\theta = \phi \circ \phi^{-1}\), being Fontaine’s theta map.

**Proof.** Cf. \([5]\), Theorem 3.10. \(\square\)
2.3. The prismatic site. Now we are ready to define the prismatic cohomology of a smooth $A/I$-algebra $R$. In some sense the definition is a mixed-characteristic analogue of a crystalline site.

**Definition 2.13.** Let $(A, I)$ be a prism and $R$ – a formally smooth $A/I$-algebra. The prismatic site of $R$ relative to $A$ is the category whose objects are prisms $(B, IB)$ over $(A, I)$ with an $A/I$-algebra map $R \to B/IB$. Morphisms in this category are defined in the obvious way. We shall denote this site as $(R/A)_\Delta$, denote its typical object as $(R \to B/IB \leftarrow B)$ and write it as

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A/I & \longrightarrow & R & \longrightarrow & B/IB.
\end{array}
$$

We endow $(R/A)_\Delta$ with the indiscrete topology, so all presheaves are sheaves automatically. We also define a structure sheaf $\mathcal{O}_\Delta$ and a structure sheaf modulo $I$ denoted by $\mathcal{O}_\Delta$ as functors that send $(R \to B/IB \leftarrow B) \in (R/A)_\Delta$ to $B$ and $B/IB$ respectively. Note that we really have $\mathcal{O}_\Delta \cong \mathcal{O}_\Delta/I\mathcal{O}_\Delta$.

**Remark 2.14.** Strictly speaking the category defined above is the opposite of what should to be called the prismatic site. However, we think that there will not be any confusion because of this abuse of notation. We also refer to covariant functors $\mathcal{O}_\Delta$ and $\mathcal{O}_\Delta$ as sheaves on $(R/A)_\Delta$.

**Example 2.15.** Assume that $A/I = R$. In this case the category $(R/A)_\Delta$ identifies with the category of prisms over $(A, I)$ and its initial object is $(R \cong A/I \leftarrow A)$.

**Definition 2.16.** The prismatic complex $\Delta_{R/A}$ of $R$ is defined to be $R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta)$. This is a $(p, I)$-complete commutative algebra object of $D(A)$. Note that the Frobenius on $\mathcal{O}_\Delta$ induced a $\phi$-semi-linear map $\Delta_{R/A} \to \Delta_{R/A}$. Sometimes we write $\Delta_{R/A}$ as $R \Gamma(\Delta, \mathcal{O}_\Delta)$.

We also define the Hodge-Tate complex $\Delta_{R/A} := R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta)$. There is an obvious isomorphism $\Delta_{R/A} \otimes^L_{A/I} A/I \cong \Delta_{R/A}$.

**Example 2.17.** If $R = A/I$, then $\Delta_{R/A} \cong A$ and $\Delta_{R/A} \cong A/I$. This follows immediately as $(R \cong A/I \leftarrow A) \in (R/A)_\Delta$ is the initial object.

In the following we will need that the Hodge-Tate complex localizes for étale topology.

**Lemma 2.18.** (Étale localization). Let $R$ and $S$ be $p$-completely smooth $A/I$-algebras. Assume that we have a $p$-completely étale map $R \to S$. Then there is a natural isomorphism $\Delta_{R/A} \otimes^L_{R/S} S \cong \Delta_{S/A}$.

**Proof.** Cf. [5], Lemma 4.19. $\square$
3. De Rham - Witt forms

As we mentioned in the introduction if the statement of Question 1.1 is true one requires that for any prism \((A, d)\) there is a canonical map
\[
W_r(A/d) \to A/d\phi(d) \ldots \phi^{r-1}(d).
\]
The construction of these maps is contained in this section. The alternative construction is contained in Appendix A. We also give the necessary information on the de Rham-Witt complex following [8] and [11]. The section ends by the application of the higher Cartier isomorphism to some form of the de Rham-Witt comparison for crystalline prisms.

3.1. The universal map. Let \((A, d)\) be an oriented prism. We start with the following proposition.

**Proposition 3.1.** There is a functorial map
\[
W_n(A/d) \to A/d\phi(d) \ldots \phi^{n-1}(d)
\]
which is an isomorphism when \((A, d)\) is perfect. Moreover, in the perfect case the inverse map can be explicitly written as \(\hat{\theta}_r \circ \phi = \theta_r \circ \phi^{n-1}\), where \(\theta_r\) and \(\hat{\theta}_r\) are Fontaine’s theta maps.

**Proof.** First we assume that \(A/d\) is \(p\)-torsion free. Then it follows that the obvious map
\[
A/d\phi(d) \ldots \phi^{n-1}(d) \xhookrightarrow{} \prod_{i=0}^{n-1} A/\phi^i(d)
\]
is injective. Indeed, since \(A/d[p] = 0\) the prism \((A, d)\) is bounded. Therefore, the derived \((p, d)\)-completion is the same as the classical one. Then it easy to see that \((p, d)\) is a regular sequence in \(A\). Hence \((p, \phi^i(d))\) is also regular. By induction one sees that \(\phi^{k+\varepsilon}(d) \equiv p \cdot \text{unit} \mod \phi^i(d)\). This implies the required result.

Thus to define the required map we can define a map from Witt vectors \(W_n(A/d)\) to \(\prod_{i=0}^{n-1} A/\phi^i(d)\) and show that the image of this map actually lies in \(A/d\phi(d) \ldots \phi^{n-1}(d)\).

So we define a map \(r_n : W_n(A/d) \to \prod_{i=0}^{n-1} A/\phi^i(d)\) on generators by
\[
V^j([x]) \mapsto (p^j x p^{n-j-1}, p^j \phi(x) p^{n-j-2}, \ldots, p^j \phi^{n-j-1}(x), 0, \ldots, 0).
\]

Now we will prove the proposition in the universal case, i.e when \((A, d)\) is the universal oriented prism. It is proved in [5] that the map \(A \to A_{\text{perf}} \to A_{\text{f}}\) is faithfully flat, where \(A_{\text{perf}}\) is the perfection of \(A\) and \(A_{\text{f}}\) is the derived \((p, d)\)-completion of \(A_{\text{perf}}\). Using faithfully flat descent we can reduce to the case of \(A_{\text{f}}\). From now on let us denote this ring simply by \(A\).

Recall the following result from [4]:

**Lemma 3.2.** Let \(R\) be a perfectoid ring and \(\theta_r : A_{\text{inf}}(R) \to W_r(R)\) be a Fontaine’s map. Let \(w_r : W_r(R) \to R^*\) be a ghost map. Then
\[
w \circ \theta_r = (\theta, \theta \phi, \theta \phi^2, \ldots, \theta \phi^{r-1}).
\]

Observe that by Theorem 2.12 the construction \((A, I) \mapsto A/I\) defines an equivalence of categories between perfect prisms and perfectoid rings. Moreover, \(A\) can be recovered from \(A/I\) as \(A_{\text{inf}}(A/I)\) and \(I\) can be recovered as the kernel of Fontaine’s
map. Since in our case \((A, d)\) is a perfect prism the ring \(A/d\) is perfectoid and \(A\) can be recovered from \(A/d\) as \(A_{\text{int}}(A/d)\). Hence by Lemma 3.12 from [4] we have the following isomorphisms:

\[
A/d\phi(d) \ldots \phi^{n-1}(d)A \rightarrow A/d\phi^{-1}(d) \ldots \phi^{-n+1}(d)A \rightarrow W_n(A/d).
\]

We denote the inverse map by \(r'_n\). We will check that \(r_n\) coincides with \(r'_n\) on generators. First, for simplicity, we show that \(r'_n(V(1)) = (p, p, \ldots, p, 0) \in \prod A/\phi^i(d)\). In order to do this we observe that since \(A/d\) is \(p\)-torsion free it follows that the ghost map \(w_n : W_n(A/d) \rightarrow (A/d)^n\) is injective. Let \(t \in A/d\phi^{-1}(d) \ldots \phi^{-n+1}(d)A\) be such an element that \(\theta_n(t) = V(1)\) and \(y \in A/d(\phi(d) \ldots \phi^{-n+1}(d)A\) be such that \(y = \phi^{n-1}(t)\). Thus we know that \(w_n(V(1)) = w_n(\theta_n(t))\) and \(w_n(V(1)) = (0, p, \ldots, p)\). Also from Lemma 3.2 we see that

\[
w_n(\theta_n(t)) = (\theta(t), \theta(\phi(t)), \ldots, \theta(\phi^{n-1}(t))).
\]

Note that in our case \(\theta : A \rightarrow A/d\) is just a quotient map, so we can conclude that \(x \equiv 0 \mod d\) and \(\phi^i(x) \equiv p \mod d\) where \(i = 1, \ldots, n - 1\) or in terms of \(y\) we have \(\phi^{-n+1}(y) \equiv 0 \mod d\) and \(\phi^{-n+k}(y) \equiv p \mod d\) where \(k = 2, \ldots, n\). Applying \(\phi^{n-k}\) to both sides of the congruence we see that \(y \equiv p \mod \phi^i(d)\), where \(i = 0, \ldots, n - 2\) and \(y \equiv 0 \mod \phi^{n-1}(d)\). We are done with this case.

By similar computation one checks that

\[
r'_n : V^j([x]) \mapsto (p^jx^{p^{n-j-1}}, p^j\phi(x)^{p^{n-j-2}}, \ldots, p^j\phi^{n-j-1}(x), 0, \ldots, 0) \in \prod_{i=0}^{n-1} A/\phi^i(d)
\]

thus the last term actually lies in \(A/d\phi(d) \ldots \phi^{n-1}(d)\).

Indeed, take an element \(t\) which maps to \(V^j([x])\) under \(\theta_n\) and let \(y = \phi^{n-1}(t)\). We know that \(w_n(V^j([x])) = (0, 0, \ldots, 0, p^jx, p^jx^p, \ldots, p^jx^{p^{n-j-1}})\). Using this identity and that

\[
w_n(V^j([x])) = w_n(\theta_n(t)) = (\theta(t), \theta(\phi(t)), \ldots, \theta(\phi^{n-1}(t)))
\]

we see that \(\phi^i(t) \equiv 0 \mod d\) for \(i = 0, \ldots, j - 1\) and \(\phi^i(t) \equiv p^jx^{p^{i-j}} \mod d\) for \(i = j, \ldots, n - 1\). Using that \(y = \phi^{n-1}(t)\) and applying the corresponding power of \(\phi\) we conclude that

\[
y \equiv p^j\phi^i(x)^{p^{n-j-1}} \mod \phi^i(d)
\]

for \(i = 0, \ldots, n - j - 1\) and \(y \equiv 0 \mod \phi^i(d)\) for \(i = n - j, \ldots, n - 1\).

So the image of generators under the map \(r_n\) lies in \(A/d \ldots \phi^{n-1}(d)A\). Moreover, the compatibility with the ring structure for \(r_n\) follows since the map \(r_n\) coincides with \(r'_n = (\theta_n \circ \phi^{n-1})^{-1}\) on generators as we have shown above. We conclude that \(r_n\) coincides with \(r'_n\) and the result for the universal prism follows.

Note that from the faithfully flat base change the result and the formula follows automatically for free \(\delta\)-rings over the universal prism. Now we assume that \(A/d\) is \(p\)-torsion free. We can take a part of its simplicial resolution \(\tilde{A} \Rightarrow \hat{A} \rightarrow A\) where \(\tilde{A}\) and \(\hat{A}\) are free \(\delta\)-rings over the universal prism. Then one has the following commutative diagram:
where the right downward arrow exists by the universal property since $W_n(A/d)$ is a coequalizer of $W_n(\tilde{A}/d) \Rightarrow W_n(\tilde{A}/d)$. The explicit formula for a $p$-torsion case follows from the above and the surjectivity of right horizontal maps. Therefore we get the result and the explicit formula for arbitrary $p$-torsion free $A/d$.

To prove the result for a general prism $A$ one repeats the argument from the above paragraph: take a part of a simplicial resolution $\tilde{A} \Rightarrow \tilde{A} \rightarrow A$ where $\tilde{A}$ and $\tilde{A}$ are free $\delta$-rings over the universal prism. Note that in the general case there is no way to get the explicit formula since the map $A/d . . . \phi^{n-1}(d) \rightarrow \prod_{i=0}^{n-1} A/\phi^i(d)$ is not injective anymore. Again consider the following commutative diagram

\[
W_n(\tilde{A}/d) \xrightarrow{\lambda_{n+1}} W_n(\tilde{A}/d) \xrightarrow{r} W_n(A/d)
\]

\[
\tilde{A}/d . . . \phi^{n-1}(d) \xrightarrow{\lambda_{r}} \tilde{A}/d . . . \phi^{n-1}(d) \xrightarrow{r} A/d . . . \phi^{n-1}(d)
\]

and conclude because $W_n(A/d)$ is a coequalizer of $W_n(\tilde{A}/d) \Rightarrow W_n(\tilde{A}/d)$.

**Remark 3.3.** One of the key points of the above proof was the inclusion

$$A/d\phi(d) . . . \phi^{n-1}(d) \hookrightarrow \prod_{i=0}^{n-1} A/\phi^i(d)$$

for a special prism. Due to Anschütz and Le Bras (cf. [[2], Definition 3.2]) a prism $(A,d)$ is called transversal if $(p,d)$ is a regular sequence on $A$. They got several useful technical results for them. In particular in [2], Lemma 3.6 they got the desired inclusion for transversal prisms. By our assumptions the prism is transversal thus we can use their results. Indeed, in our case $(d,p)$ is a regular sequence and the prism is bounded. Hence the derived completion coincides with the usual one. Then it is well-known that for any ring $R$ with a regular sequence $(r,s)$ such that $R$ is $r$-adically complete the sequence $(s,r)$ is also regular. For additional information see [2], Section 3.

**Remark 3.4.** From the construction of the map above and Lemma 3.4 from [4] we see that the following diagram is commutative:

\[
W_{r+1}(A/d) \xrightarrow{\lambda_{r+1}} A/d . . . \phi^r(d)
\]

\[
W_r(A/d) \xrightarrow{\lambda_r} A/d . . . \phi^{r-1}(d)
\]

where $\lambda_r$ is the constructed map. We will use this later in Propositions 4.9, 4.10.
Remark 3.5. There is an étale localization property for $\Delta_{R/A}/d \ldots \phi^{n-1}(d)$. Indeed, with the notation of Lemma 2.18 one has the following isomorphism:

$$\Delta_{R/A}/d \ldots \phi^{n-1}(d) \overset{L}{\otimes}_{W_n(R)} W_n(S) \simeq \Delta_{S/A}/d \ldots \phi^{n-1}(d).$$

To see this by derived Nakayama argument it is enough to check the isomorphism after a base change $- \otimes_{A/d} \phi^{n-1}(d) A/d$. Hence it suffices to check

$$\Delta_{R/A} \otimes_{W_n(R)} W_n(S) \simeq \Delta_{S/A}.$$

Next, the left-hand side can be rewritten as $\Delta_{R/A} \otimes_R (R \otimes_{W_n(R)} W_n(S))$. Since $R \to S$ is $p$-completely étale, the term in the last brackets is exactly $S$ (follows from [4] or [11]). Hence we conclude the result by Lemma 2.18.

3.2. The de Rham-Witt complex. In this subsection we introduce basic definitions and necessary results for the de Rham-Witt complex following [11]. The other reference for this subsection is [4]. Assume $A$ is a $\mathbb{Z}_p$-algebra.

Definition 3.6. ($F$-$V$-procomplex). Let $B$ be an $A$-algebra. An $F$-$V$-procomplex consists of the following data $(W^\bullet_r, R, F, V, \lambda_r)$:

(i) a commutative differential graded $W_r(A)$-algebra $W^\bullet_r = \bigoplus_{n \geq 0} W^n_r$ for $r \geq 1$;
(ii) morphisms $R : W^\bullet_{r+1} \to R, W^\bullet_r$ of differential graded $W_{r+1}(A)$-algebras for $r \geq 1$;
(iii) morphisms $F : W^\bullet_{r+1} \to F, W^\bullet_r$ of graded $W_{r+1}(A)$-algebras for $r \geq 1$;
(iv) morphisms $V : F, W^\bullet_r \to W^\bullet_{r+1}$ of graded $W_{r+1}(A)$-modules for $r \geq 1$;
(v) morphisms $\lambda_r : W_r(B) \to W^0_r$ for each $r \geq 1$, commuting with $R, V, F$;

such that the following holds:
- $R$ commutes with both $F$ and $V$;
- $FV = d$;
- $V(F(x)y) = xV(y)$;
- (Teichmüller identity). $F d \lambda_{r+1}([b]) = \lambda_r([b])^{p-1} d \lambda_r([b])$ for $b \in B$ and $r \geq 1$.

These complexes are also called Witt complexes.

Theorem 3.7. (See [11], sketch of the proof is adopted from [8]). There is an initial object $\{W^\bullet_r, \Omega^\bullet_{nR/A}\}_r$ in the category of $F$-$V$-procomplexes, called the relative de Rham-Witt complex, i.e. if $(W_r, R, F, V, \lambda_r)$ is any $F$-$V$-procomplex for $B/A$, then there are unique maps of graded $W_r(A)$-algebras

$$\lambda^\bullet_r : W_r \Omega^\bullet_{nR/A} \to W^\bullet_r$$

which are compatible with $R, F, V$ in the obvious sense and such that

$$\lambda^0_r : W_r(B) \to W^0_r$$

is the structure map $\lambda_r$ of the Witt complex $W_r$ for $r \geq 1$.

Proof. (sketch). We need to check the following two key points:

(i) Assume $W^\bullet_r$ is a Witt complex. Then for all $n$ the map $d : W_n(B) \to W^1_n$ is a pd-derivation, in particular, $W_n$ is a pd-dga. This implies that for $x \in B$

$$d \gamma_p(V[x]) = \gamma_{p-1}(V[x]) dV[x].$$
Indeed, from the definition of pd-structure on $W(B)$ this holds if and only if $p^{p-2}dV[x]^p = p^{p-2}V[x]p^{-1}dV[x]$, but
\[dV[x]^p = d((x)V(1)) = V(1)d[x] = VF d[x] = V([x]p^{-1}d[x]) = V[x]p^{-1}dV[x].\]
(ii) If $D : W_n(A) \to M$ is a pd-derivation into $W_n(A)$-module $M$, then $FD : W_{n-1}A \to F_nM$ defined by
\[FDx = [a^{p-1}]D[a] + DV[b]\]
for $x = [a] + V[b]$ is a pd-derivation.
It follows from (ii) that the projective system $\tilde{\Omega}^\bullet_{W_n(B)/W_n(A)}$ (pd-de Rham complex)
acquires maps of graded algebras $F : \tilde{\Omega}^\bullet_{W_n(B)/W_n(A)} \to \Omega^\bullet_{W_{n-1}(B)/W_{n-1}(A)}$ satisfying some of the identities in the definition of $F-V$-procomplex
\[(FdVx = dx \text{ for } x \in W_n(B), \ Fd[x] = [x^{p-1}]d[x] \text{ for } x \in B, \ dFx = pFdx).\]
Then the projective system $W^\bullet \Omega^\bullet_{B/A}$ is constructed inductively as a quotient of $\tilde{\Omega}^\bullet_{W_n(B)/W_n(A)}$.

**Definition 3.8.** The continuous de Rham-Witt complex of a morphism $A \to B$ of $\mathbb{Z}_p$-algebras is given by:
\[W_i \Omega_{B/A}^{cont} = \lim_{\leftarrow s} W_i \Omega_{(B/p^s)/(A/p^s)} = \lim_{\leftarrow s} W_i \Omega^i_{B/A}/p^s.\]

**Theorem 3.9.** (*Higher Cartier isomorphism*). Let $X/k$ be a smooth scheme over a perfect field $k$ of char $p > 0$. For $n \geq 1$, $F^n : W_{2n}\Omega^n_X \to W_n\Omega^n_X$ induces an isomorphism
\[W_n\Omega^n_X \to \mathcal{H}^nW_n\Omega^\bullet_X,
\]
compatible with products and equal to $C^{-1}$ for $n = 1$.

**Proof.** We note that the main point of the proof is to show that $F^nW_{2n}\Omega^n_X = ZW_n\Omega^n_X$. The proof was given in [7] is insufficient, it was corrected in [9]. One uses the description of $W_n\Omega^n_X$ for $X = \text{Spec}(k[t_1, \ldots, t_n])$ in terms of the complex of integral forms and the Cartier isomorphism.

**Lemma 3.10.** (*Étale extensions*). Let $A \to B$ be a morphism of $\mathbb{Z}_p$-algebras, and let $S$ be an étale $B$-algebra. Then the natural map
\[W_r \Omega^n_{B/A} \otimes_{W_r(B)} W_r(S) \to W_r \Omega^n_{S/A}\]
is an isomorphism.

**Proof.** If $p$ is nilpotent in $B$ or $B$ is $F$-finite then the result follows from [11], Proposition 1.7. This assumption used in [11] only to ensure the étaleness of the map $W_r(B) \to W_r(S)$. However, by [4], Theorem 10.4 the above map is always étale. It follows that the argument of [11] works in general.

The main advantage of the de Rham-Witt complex is that there is a comparison theorem with crystalline cohomology.

**Theorem 3.11.** Let $S$ be a ring such that $p$ is nilpotent in $S$, $X/S$ be a smooth scheme over $S$. There exists a canonical isomorphism of projective systems of $D(X, W_n(S))$:
\[Ru_* \Omega_X/W_n(S) \simeq W_n\Omega_X/S.\]

**Proof.** For $S = \text{Spec}(k)$, where $k$ is a perfect field the proof was given in [7]. The general case was obtained in [11].
3.3. The de Rham-Witt complex of a polynomial algebra. Now we recall some
results of Langer-Zink on the relative de Rham-Witt complex of $A[\mathbb{T}] = A[T_1, \ldots, T_n]$ and
$A[\mathbb{T}^{\pm 1}] = A[T_{i \pm 1}, \ldots, T_d]$.

Let $A$ be a $\mathbb{Z}_p$-algebra. We will give an explicit form of the relative de Rham-Witt complex of a polynomial (Laurent) algebra $A[\mathbb{T}]$. Let $a : \{1, \ldots, d\} \to p^{-r}\mathbb{Z}$ be a weight. We define $\nu(a) := \min_i \nu(a(i))$, where $\nu(a(i)) = \nu_p(a(i)) \in \mathbb{Z} \cup \{\infty\}$ is the $p$-adic valuation of $a(i)$. We also set $\nu(a) := \min_{i \in I} \nu(a(i))$ for any subset $I \subset \{1, \ldots, d\}$.

Let $P_\alpha$ be the collection of disjoint partitions $I_0, \ldots, I_n$ of $\{1, \ldots, d\}$ such that:
1. all but $I_0$ are not empty, $I_0$ is possibly empty;
2. the $p$-adic valuation of all elements of $a(I_{j-1})$ is less or equal then of those elements of $a(I_j)$ for $j \in \{1, \ldots, n\}$
3. we fix a total ordering $\succeq_\alpha$ on $\{1, \ldots, d\}$ such that $\nu : \{1, \ldots, d\} \to \mathbb{Z}$ is weakly increasing. Then we assume that all elements $I_{j-1}$ are strictly $\succeq_\alpha$-less than all elements of $I_j$.

Let $(I_0, \ldots, I_d) \in P_\alpha$ be such a partition and denote by $\rho_1$ the greatest integer between 0 and $n$ such that $\nu(a|_{I_{\rho_1}}) < 0$. Also $\rho_2$ is the greatest integer between 0 and $n$ such that $\nu(a|_{I_{\rho_2}}) < \infty$.

We denote $\max\{0, -\nu(a)\}$ by $u(a)$. Now we are ready to define the basic elements $e(x, a, I_0, \ldots, I_n) \in W_7\Omega^n_{A[\mathbb{T}]/A}$ for $x \in W_{7-u(a)}(A)$ as follows:

1. $(I_0 \neq \emptyset)$ the product of elements

$$dV^{-\nu(a|_{I_0})} \prod_{i \in I_0} [T_i]^{a(i)/p^{\nu(a|_{I_0})}}$$

$$dV^{-\nu(a|_{I_j})} \prod_{i \in I_j} [T_i]^{a(i)/p^{\nu(a|_{I_j})}}, \text{ where } j = 1, \ldots, \rho_1,$$

$$F^{\nu(a|_{I_j})} d \prod_{i \in I_j} [T_i]^{a(i)/p^{\nu(a|_{I_j})}}, \text{ where } j = \rho_1 + 1, \ldots, \rho_2,$$

$$d \log \prod_{i \in I_j} [T_i], \text{ where } j = \rho_2 + 1, \ldots, n.$$ 

2. $(I_0 = \emptyset, \nu(a) < 0)$ the product of elements

$$dV^{-\nu(a|_{I_1})} \prod_{i \in I_1} [T_i]^{a(i)/p^{\nu(a|_{I_1})}}$$

$$dV^{-\nu(a|_{I_j})} \prod_{i \in I_j} [T_i]^{a(i)/p^{\nu(a|_{I_j})}}, \text{ where } j = 2, \ldots, \rho_1,$$

$$F^{\nu(a|_{I_j})} d \prod_{i \in I_j} [T_i]^{a(i)/p^{\nu(a|_{I_j})}}, \text{ where } j = \rho_1 + 1, \ldots, \rho_2,$$

$$d \log \prod_{i \in I_j} [T_i], \text{ where } j = \rho_2 + 1, \ldots, n.$$ 

3. $(I_0 = \emptyset, \nu(a) \geq 0)$ the product of $x \in W_7(A)$ with the elements

$$F^{\nu(a|_{I_j})} d \prod_{i \in I_j} [T_i]^{a(i)/p^{\nu(a|_{I_j})}}, \text{ where } j = 1, \ldots, \rho_2.$$

\[ d \log \prod_{i \in I_j} [T_i], \text{ where } j = \rho_2 + 1, \ldots, n. \]

**Theorem 3.12.** ([4], [11]). The map of \( W_r(A) \)-modules:
\[
e : \bigoplus_{a: \{1, \ldots, d\} \to p^{-r}\mathbb{Z}, (I_0, \ldots, I_n) \in P_n} V^{u(a)} W_{r-u(a)}(A) \to W_r \Omega^n_{A[\mathbb{T}^\pm]/A}
\]
which sends \( V^{u(a)}(x) \) to \( e(x, a, I_0, \ldots, I_n) \) is an isomorphism.

**Proof.** Cf. [[4], Theorem 10.12]. \[ \square \]

**Remark 3.13.** To describe the de Rham-Witt complex for a polynomial algebra \( A[\mathbb{T}] \) instead of \( A[\mathbb{T}^\pm] \) one replaces \( p^{-r}\mathbb{Z} \) with \( p^{-r}\mathbb{Z}_{\geq 0} \).

### 3.4. An application of the higher Cartier isomorphism to special perfect prisms

Our main goal is to prove Theorem 1.2. We give its proof in the next section. First we prove the special case when \( d = p \) and \( A/p \) is a perfect field.

**Theorem 3.14.** Let \( (A, p) \) be a crystalline prism such that \( A/p \) is a perfect field and \( S \) be a smooth \( A/p \)-algebra. Then
\[
W_r \Omega^i_{S/(A/p)} \simeq H^i((S/A)_{\Delta^i}, \mathcal{O}/p^r).
\]

**Proof.** Let us denote \( A/p \) by \( k \). Our reasoning are very similar to the proof of the Hodge-Tate comparison in characteristic \( p \). By étale localization we can assume that \( S = k[T_1, \ldots, T_n] \). By higher Cartier isomorphism (Theorem 3.9) we have \( W_r \Omega^i_{S/k} \simeq H^i(W_r \Omega^i_{S/k}) \). By Theorem 3.11 \( H^i(W_r \Omega^i_{S/k}) \simeq H^i(Ru_* \mathcal{O}_S/W_r(k)) \). Then by the crystalline comparison for prismatic cohomology (cf. [[5], Theorem 5.2]) we know that
\[
H^i(Ru_* \mathcal{O}_S/W_r(k)) \simeq H^i(Ru_* \mathcal{O}_S/W_r(k) \otimes^{W_r}_{W(k)} W_r(k)) \simeq H^i(\phi^* \Delta_S^{(1)} W(k) \otimes^{L}_{W(k)} W_r(k)),
\]
but by our assumptions on \( S \) and \( A \) it follows that \( \phi^* \Delta_S^{(1)} W(k) \simeq \Delta_S^{(1)} W(k) \). Hence, we see that
\[
W_r \Omega^i_{S/k} \simeq H^i(\Delta_S^{(1)} W(k) \otimes^{L}_{W(k)} W_r(k)) \simeq H^i((S/A)_{\Delta^i}, \mathcal{O}/p^r).
\]
\[ \square \]
4. The de Rham-Witt comparison

In this section we get the canonical map from Question 1.1 and prove Theorem 1.2. We also give an explicit description of the prismatic cohomology for a polynomial algebra. These are the main results of the paper.

We start with the comparison of prismatic cohomology with [4] which was proved in [5]. We fix a perfectoid field \( C \) of char 0 such that it contains \( \mu_{p^\infty} \). Let \( R \) be a \( p \)-completely smooth \( \mathcal{O}_C \)-algebra. In [4] the complex \( A\Omega \) was constructed. The connection with prismatic cohomology is given in the following theorem:

**Theorem 4.1.** There is an isomorphism \( A\Omega_{R/A} \simeq \Delta_{R^{(1)}/A} = \phi^*_AA\Omega_{R/A} \) of \( E_{\infty} \)-\( A \)-algebras compatible with the Frobenius.

**Proof.** Cf. [[5], Theorem 17.2].

This can be used to show Theorem 1.2 in particular case. First recall the following sequence of isomorphisms from [4]

\[
H^i(A\Omega_{R/\mathcal{O}_C} \otimes^L_{\mathcal{O}_C} \hat{\theta}_r, W_r(\mathcal{O}_C)) \simeq H^i(\hat{W}_r\Omega_{R/\mathcal{O}_C}) \simeq W_r\Omega^{\text{cont}}_{R/\mathcal{O}_C} \{ -i \}.
\]

But from Theorem 4.1 and Proposition 3.1 we know that

\[
H^i(\hat{\Delta}_{R/A} \otimes^L_{A/d} \hat{\theta}_r, W_r(\mathcal{O}_C)) \simeq H^i(A\Omega_{R/\mathcal{O}_C} \otimes^L_{\mathcal{O}_C} \hat{\theta}_r, W_r(\mathcal{O}_C)).
\]

(The Breul-Kisin twist disappears since the map constructed in Proposition 3.1 is inverse to \( \hat{\theta}_r \) up to \( \phi^{-1} \)). Hence in this particular case the statement of Question 1.1 holds true. From now on we omit Tate twist in the notation of de Rham-Witt forms.

The main ingredient in the proof of Theorem 1.2 is the André’s lemma which was first proved in [1] and reproved in [5].

**Lemma 4.2.** (André’s lemma). Let \( R \) be a perfectoid ring. There exists a \( p \)-completely faithfully flat map \( R \to S \) of perfectoid rings such that \( S \) is absolutely integrally closed. In particular, every element of \( S \) admits a compatible system of \( p \)-power roots.

**Proof.** For the proof see [5], Theorem 7.12.

Now we give a proof of Theorem 1.2.

**Theorem 4.3.** Let \( (A, d) \) be a perfect prism such that \( A/d \) is \( p \)-torsion free and \( S \) be a \( p \)-completely smooth \( A/d \)-algebra. Then we have the functorial isomorphism

\[
W_n\Omega^{\text{cont}}_{S/(A/d)} \to H^i(\hat{\Delta}_{S/A} \otimes^L_{A/d} A/d \ldots \phi^{-1}d).
\]

**Proof.** Let us denote \( \Delta_{S/A} \) as \( R\Delta(S/A) \). The proof will consist of several reduction steps.

By our assumptions \( (A, d) \) is a perfect prism. Hence \( R := A/d \) is a perfectoid ring. By André’s lemma one has a \( p \)-completely faithfully flat map \( R \to \hat{R} \) with \( \hat{R} \) absolutely integrally closed. In particular, \( \hat{R} \) has a compatible system of \( p \)-power roots of unity. Therefore, one has a map \( \mathbb{Z}_p \to \hat{R} \). First we explain how to deduce the comparison for \( \hat{R} \) from the comparison for \( \mathbb{Z}_p \) which was explained in the beginning of this section. Recall the following well-known homological algebra lemma:

**Lemma 4.4.** Let \( P \to Q \) be any map of rings and \( C \in D^{-}(P) \). Assume that \( H^i(C) \otimes^L_P Q \simeq H^i(C) \otimes_P Q \) for any \( i \). Then \( H^i(C \otimes^L_P Q) \simeq H^i(C) \otimes_P Q \).
Remark 4.5. The version of the above lemma with completed base change is also true.

By base change and étale localization arguments for the de Rham-Witt complex in the perfectoid case we may assume that we work with a polynomial algebra $\hat{R}(T)$ where $\hat{R}(T) = R(T_1, \ldots, T_k)$. So let $C$ be $R\Gamma_D(\hat{\Z}_p(T)/A_{\inf}(\hat{\Z}_p)) \otimes_{A_{\inf}(\hat{\Z}_p)} W_n(\hat{\Z}_p)$. By our assumption we know that $H^i(C) \simeq W_n\Omega^{i,\cont}_{\hat{\Z}_p(T)/\hat{\Z}_p}$ and since $\hat{R}$ is perfectoid from [4], Proposition 10.14 we know that

$$W_n\Omega^{i,\cont}_{\hat{\Z}_p(T)/\hat{\Z}_p} \otimes^{L}_{W_n(\hat{\Z}_p)} W_n(\hat{R}) \simeq W_n\Omega^{i,\cont}_{\hat{\Z}_p(T)/\hat{\Z}_p} \otimes^{L}_{W_n(\hat{\Z}_p)} W_n(\hat{R}) \simeq W_n\Omega^{i,\cont}_{\hat{R}(T)/\hat{R}}.$$  

Hence by Lemma 4.4 and Remark 4.5 we have the second isomorphism in

$$W_n\Omega^{i,\cont}_{\hat{R}(T)/\hat{R}} \simeq W_n\Omega^{i,\cont}_{\hat{\Z}_p(T)/\hat{\Z}_p} \otimes^{L}_{W_n(\hat{\Z}_p)} W_n(\hat{R}) \simeq H^i(C \otimes^{L}_{W_n(\hat{\Z}_p)} W_n(\hat{R})).$$  

But the last term is exactly $H^i(R\Gamma_D(\hat{R}(T)/A_{\inf}(\hat{R})) \otimes_{A_{\inf}(\hat{R})} W_n(\hat{R}))$ since

$$H^i(R\Gamma_D(\hat{\Z}_p(T)/A_{\inf}(\hat{\Z}_p)) \otimes^{L}_{A_{\inf}(\hat{\Z}_p)} W_n(\hat{\Z}_p) \otimes^{L}_{W_n(\hat{\Z}_p)} W_n(\hat{R}))$$

is exactly

$$H^i(R\Gamma_D(\hat{\Z}_p(T)/A_{\inf}(\hat{\Z}_p)) \otimes^{L}_{A_{\inf}(\hat{\Z}_p)} W_n(\hat{R}) \simeq H^i(R\Gamma_D(\hat{R}(T)/A_{\inf}(\hat{R})) \otimes^{L}_{A_{\inf}(\hat{R})} W_n(\hat{R})).$$

Hence the result for $\hat{R}$ follows.

Moreover, the argument above works with any perfectoid $Q$ instead of $\hat{\Z}_p$ and any perfectoid $Q'$ above $Q$ instead of $\hat{R}$. Indeed, the proof is the same as above and uses only the base change and étale localization for perfectoid rings. So finally, we conclude that if the result is true for some perfectoid $Q$ then it is true for any $Q'$ above $Q$.

Now we want to prove the comparison isomorphism for $R$ assuming the result for $\hat{R}$. Since $R \to \hat{R}$ is $p$-completely faithfully flat and both $A_{\inf}(R)$ and $A_{\inf}(\hat{R})$ are $d$-torsion free it follows that $A_{\inf}(R) \to A_{\inf}(\hat{R})$ is $(p, d)$-completely faithfully flat. Hence it follows that $W_n(R)/p^m \to W_n(\hat{R})/p^m$ should be faithfully flat. Recall that for perfectoid rings we have $W_n(R) \simeq A_{\inf}(R)/d \ldots \phi^{n-1}(d)$. Since both rings are perfectoid we know that

$$W_n\Omega^{i,\cont}_{S/R}/p^m \otimes_{W_n(R)/p^m} W_n(\hat{R})/p^m \simeq W_n\Omega^{i,\cont}_{S/\hat{R}/\hat{R}}/p^m$$

and that

$$H^i(R\Gamma_D(S/A_{\inf}(R)) \otimes^{L}_{A_{\inf}(R)} W_n(R))/p^m \otimes^{L}_{W_n(R)/p^m} W_n(\hat{R})/p^m$$

is isomorphic to

$$H^i(R\Gamma_D(S/A_{\inf}(R)) \otimes^{L}_{A_{\inf}(R)} W_n(R))/p^m \simeq H^i(R\Gamma_D(S/\hat{R}/A_{\inf}(\hat{R})) \otimes^{L}_{A_{\inf}(\hat{R})} W_n(\hat{R}))/p^m.$$  

Here $S_{\hat{R}}$ is the base change $S \otimes_R \hat{R}$.

Now we want to apply faithfully flat descent for $W_n(R)/p^m \to W_n(\hat{R})/p^m$. Let us denote $W_n(R)/p^m$ by $B$, $W_n(\hat{R})/p^m$ by $C$, $W_n\Omega^{i,\cont}_{S/\hat{R}/\hat{R}}/p^m$ by $M$ and prismatic cohomology $H^i(R\Gamma_D(S/A_{\inf}(R)) \otimes^{L}_{A_{\inf}(R)} W_n(R))/p^m$ by $N$. Then since $B \to C$ is faithfully flat we have two exact complexes $0 \to M \to M \otimes_B C \to M \otimes_B C \otimes_B C \to \ldots$ and $0 \to N \to N \otimes_B C \to N \otimes_B C \otimes_B C \to \ldots$. Moreover, from the above we see that
$M \otimes_B C$ and $N \otimes_B C$ are isomorphic to each other since they are terms of the comparison isomorphism for $\bar{R} \mod p^n$. Also since $\bar{R} \otimes_R \bar{R}$ is perfectoid over $\bar{R}$ by the first part of the proof we know the comparison isomorphism for it. The same argument as above shows that terms of that isomorphism $\mod p^n$ are precisely $M \otimes_A B \otimes_A B$ and $N \otimes_A B \otimes_A B$.

Using the same machinery for other terms we get the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \rightarrow & M & \rightarrow & M \otimes_A B & \rightarrow & M \otimes_A B \otimes_A B & \rightarrow & M \otimes_A B \otimes_A B \otimes_A B & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N & \rightarrow & N \otimes_A B & \rightarrow & N \otimes_A B \otimes_A B & \rightarrow & N \otimes_A B \otimes_A B \otimes_A B & \rightarrow & \ldots \\
\end{array}
$$

where all solid vertical arrows are isomorphisms. Therefore, we conclude that the dash vertical arrow gives an isomorphism between $M$ and $N$.

So we get the comparison isomorphism $\mod p^n$ for $R$. After taking the limit we obtain the result for $R$. So we are done for any perfect prism $(A,d)$ such that $A/d$ is $p$-torsion free.

\begin{remark}
Let us reformulate the result above explicitly for $S = A/d(T_1,\ldots,T_k)$ being a $p$-completion of a polynomial algebra over $A/d$. Let $S_0 = A/d[T_1,\ldots,T_k]$. Then $W_n^{\Omega^i_{S_0/\langle A/d \rangle}} \simeq W_n^{\Omega^i_{S_0/\langle A/d \rangle}}$. During next several propositions we will see the notion of $\bigoplus$ many times. By this we mean $\bigoplus_{n:\{1,\ldots,k\} \to \mathbb{Z}_{\geq 0}} \bigoplus_{(I_0,\ldots,I_t) \in P_n}$ with notations of Theorem 3.12. From this theorem we know that

$$
\bigoplus \bigoplus V^{u(a)} W_{n-u(a)}(A/d) \simeq W_{n}^{\Omega^i_{S_0/\langle A/d \rangle}}.
$$

Since $(A,d)$ is perfect we see that

$$
\bigoplus \bigoplus V^{u(a)} W_{n-u(a)}(A/d) \simeq \bigoplus \bigoplus A/d \ldots \phi^{n-1-u(a)}(d).
$$

Finally we rewrite $W_n^{\Omega^i_{S_0/\langle A/d \rangle}}$ as

$$
\lim_s W_n^{\Omega^i_{S_0/\langle A/d \rangle}}/((A/d)/p^s) \simeq \lim_s W_n^{\Omega^i_{S_0/\langle A/d \rangle}}/p^s \simeq \lim_s \bigoplus A/d \ldots \phi^{n-1-u(a)}(d)/p^s
$$

and get the following isomorphism:

$$
\left( \bigoplus \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{h_p} \simeq H^i(R\Gamma_{\hat{\Delta}}(S/A) \otimes^L_{\hat{A}} A/d \ldots \phi^{n-1}(d)).
$$

We want to generalize this result for more arbitrary settings.

\begin{lemma}
Let $(\tilde{A},d) \rightarrow (A,d)$ be a map of prisms and $S = A/d(T_1,\ldots,T_k)$. Suppose that $\phi^i(d)$ are nonzerodivisors in $A'$ and $A$. Also suppose that $\phi^i(d)$ are nonzerodivisors in $A'$ and $A$. Also suppose that

$$
\left( \bigoplus \bigoplus \tilde{A}/d \ldots \phi^{n-1-u(a)}(d) \right)^{h_p} \simeq H^i(R\Gamma_{\hat{\Delta}}(\tilde{S}/\tilde{A}) \otimes^L_{\tilde{A}} \tilde{A}/d \ldots \phi^{n-1}(d)),
$$

where $S = \tilde{A}/d(T_1,\ldots,T_k)$. Then there is the explicit functorial isomorphism

$$
\left( \bigoplus \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{h_p} \simeq H^i(R\Gamma_{\hat{\Delta}}(S/A) \otimes^L_{A} A/d \ldots \phi^{n-1}(d))
$$

of $A/d \ldots \phi^{n-1}(d)$-modules.
\end{lemma}
Proof. From our assumptions we know that
\[
\left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p} \cong H^i(R\Gamma_{\Delta}(\bar{S}/\bar{A}) \otimes_{\bar{A}} A/d \ldots \phi^{n-1}(d)),
\]
where \( \bar{S} = \bar{A}/d(T_1 \ldots , T_k) \). Note that from the zero divisor condition we have
\[
\left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p} \otimes_{\bar{A}} A \cong \left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p}
\]
and also
\[
\left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p} \otimes_{\bar{A}} A \cong \left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p}.
\]
Let us denote \( R\Gamma_{\Delta}(\bar{S}/\bar{A}) \otimes_{\bar{A}} A/d \ldots \phi^{n-1}(d) \) by \( T \). By the previous observation and Remark 4.5 we see that
\[
H^i(T \otimes_{\bar{A}} A) \cong H^i(T \hat{\otimes}_{\bar{A}} A).
\]
This exactly means that
\[
H^i(R\Gamma_{\Delta}(S/A) \otimes_{\bar{A}} A/d \ldots \phi^{n-1}(d)) \cong \left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p}
\]
which is the desired result.

The above result means that under zero divisor condition the explicit isomorphism for \( \bar{A} \) induces the result for \( A \).

**Proposition 4.8.** Let \((A, d)\) be the universal oriented prism or a free prism over the universal one \( S = A/d(T_1, \ldots , T_k) \). Then there is the explicit functorial isomorphism
\[
\left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p} \cong H^i(R\Gamma_{\Delta}(S/A) \otimes_{\bar{A}} A/d \ldots \phi^{n-1}(d))
\]
of \( A/d \ldots \phi^{n-1}(d) \)-modules. In particular, this holds true for the universal oriented prism and for free prisms over the universal one.

*Proof.* From Remark 4.6 we know the explicit isomorphism result for \( A_{\infty} \). Moreover, from [5] we know that \( A \to A_{\infty} \) is faithfully flat. Now we would like to apply faithfully flat descent argument as in the proof of Theorem 4.3. For that it is enough to get the explicit isomorphisms for all terms of the Čech nerve of \( A \to A_{\infty} \). But it follows from the explicit construction of the universal prism that all the terms of the Čech nerve satisfies non-zero-divisor assumptions of Lemma 4.7. Applying this Lemma we get the explicit isomorphism for all terms of the Čech nerve of \( A \to A_{\infty} \), thus we could apply descent along this nerve and obtain the explicit isomorphism for \( A \). \( \square \)

Now we can deduce the same explicit result for almost arbitrary prism \((A, d)\).

**Proposition 4.9.** Let \((A, d)\) be a prism and \( S = A/d(T_1, \ldots , T_k) \) be a polynomial \( A/d \)-algebra. Suppose that \( \phi^i(d) \) are nonzerodivisors in \( A \) for all \( i \). Then we have the functorial isomorphism
\[
\left( \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge_p} \cong H^i(R\Gamma_{\Delta}(S/A) \otimes_{\bar{A}} A/d \ldots \phi^{n-1}(d))
\]
of \( A/d \ldots \phi^{n-1}(d) \)-modules.

*Proof.* Choose a surjection \( \bar{A} \to A \) of prisms such that \( \bar{A} \) is a free prism over the universal one and apply Lemma 4.7 and Proposition 4.8 to get the result. \( \square \)

Now we are ready to construct the de Rham-Witt comparison map for almost arbitrary prism \((A, d)\).
**Proposition 4.10.** Let \((A, d)\) be a prism and \(S\) be a \(p\)-completely smooth \(A/d\)-algebra. Suppose that \(\phi^i(d)\) are nonzerodivisors for all \(i\). Then there is the de Rham-Witt comparison map

\[
W_n \Omega^{i, \text{cont}}_{S/(A/d)} \otimes_{W_r(A/d)} A/d \rightarrow H^i(R\Gamma_{A}(S/A) \otimes^L_A A/d \ldots \phi^{n-1}(d))
\]
of \(A/d \ldots \phi^{n-1}(d)\)-modules.

**Proof.** As usual we may work étale locally, so may assume that \(S = A/d(T_1, \ldots, T_k)\). By Proposition 4.9 we know that

\[
K := \left( \bigoplus \bigoplus A/d \ldots \phi^{n-1-u(a)}(d) \right)^{\wedge p} \simeq H^i(R\Gamma_{A}(S/A) \otimes^L_A A/d \ldots \phi^{n-1}(d)).
\]

But

\[
W_n \Omega^{i, \text{cont}}_{S/(A/d)} \simeq \left( \bigoplus \bigoplus V^{u(a)}W_{n-u(a)}(A/d) \right)^{\wedge p}
\]

and the right hand-side maps to \(K\) by Proposition 3.1. To get the desired map one tensors \(W_n \Omega^{i, \text{cont}}_{S/(A/d)}\) with \(A/d \ldots \phi^{n-1}(d)\) over \(W_n(A/d)\). \qed
Appendix A. Alternative Proof of Proposition 3.1

In this appendix we want to give an alternative construction of maps
\[ W_n(A/d) \to A/d \ldots \phi^{n-1}(d) \]
which was provided to us by the anonymous referee. In contrast with our proof, this
does not give an explicit formula for the image of generators in \( \prod_{i=0}^{n-1} A/\phi^i(d) \) but
it is much simpler.

Proposition A.1. For any prism \((A, d)\) there is a functorial map
\[ W_n(A/d) \to A/d \phi(d) \ldots \phi^{n-1}(d). \]

Proof. First, by faithfully flat descent it is enough to work with prisms \((A, d)\) such
that \(A/d\) is \(p\)-torsion free and such that the Frobenius \(\phi: A \to A\) is surjective. In this
case, by the universal property of Witt vectors we get a map of \(\delta\)-rings
\(A \to W_n(A/d)\) which has the property that the composite \(A \to W(A/d) \to W_n(A/d)\) is surjective.
Next, we consider the composition of \(\phi^{n-1}: A \to A\) with the natural projection
\(A \to A/d \ldots \phi^{n-1}(d). \) We claim that this composition factors through the surjection
\(A \to W_n(A/d).\) For that is is enough to show that if \(x \in A\) maps to zero in
\(W_n(A/d)\) then \(\phi^i(x) \equiv 0 \mod d\) for \(0 \leq i \leq n - 1.\) Indeed, if the last congruence holds, then
applying the corresponding power of \(\phi\) we get that \(y = \phi^{n-1}(x) \equiv 0 \mod \phi^i(d)\) for
\(0 \leq i \leq n - 1.\) Recall that the first argument of Proposition 3.1 gives us the inclusion
\(A/d \ldots \phi^{n-1}(d) \hookrightarrow \prod_{i=0}^{n-1} A/\phi^i(d) \) (also see Remark 3.3). From this inclusion we see
that \(y = \phi^{n-1}(x) \equiv 0 \mod d \ldots \phi^{n-1}(d)\) which is the desired assertion.

Thus we need to show that if \(x \in A\) maps to 0 in \(W_n(A/d)\) then \(\phi^i(x) \in (d)\). For
that we consider the Witt vector Frobenius \(F^i: W_n(A/d) \to W_{n-i}(A/d)\) composed
with the projection \(W_{n-i}(A/d) \to A/d.\) Finally, from the commutative diagram
\[
\begin{array}{ccc}
A & \longrightarrow & W_n(A/d) \\
\downarrow \phi^i & & \downarrow F^i \\
A & \longrightarrow & W_{n-i}(A/d)
\end{array}
\]
we see easily that \(\phi^i(x) \in (d)\) as desired.

Remark A.2. From the argument above it is clear that the constructed map
\[ W_n(A/d) \to A/d \ldots \phi^{n-1}(d) \]
is the unique map such that the composition \(A \to W_n(A/d) \to A/d \ldots \phi^{n-1}(d)\) is
given by \(A \xrightarrow{\phi^{n-1}} A \to A/d \ldots \phi^{n-1}(d).\)
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