Algebraic structure of the anti-causal system

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Abstract. This paper presents a behavioral framework, as developed by J.C. Willems, of a discrete time anticausal pure predictor system. This study is deduced by identifying that the generalized state space of the system has a natural polynomial module structure over the formal series ring.

1. Introduction
The structure of the dynamical system was rapidly investigated since Rosenbrock published some results that related to it [1]. Willems introduced it as a behavioral framework by defining the dynamical system as triple $\Sigma = (T, W, B)$ where $T \subseteq \mathbb{R}$ is called the time axis, $W$ is called the signal space, $W^T$ is the set of all transformation from the time axis to the signal space and $B \subseteq W^T$ is called the behavior [2]. It is worth pointing out that the behavior of a dynamical system is representing the set of all trajectories of the system in Willems’ definition.

The dynamical system can be studied through the truncated Laurent series ring since its behavior form a module over it. One of which is the study the behavior of a linear, discrete and invariant system that developed by Fuhrmann [6]. The behavior of the system can be seen as a complete submodule over the polynomial ring. We are interested in finding the structure of the behavior of discrete time anticausal pure predictor systems. Accordingly, we review the notion of polynomial module over the formal series ring in Section 2. We also characterize its submodules within this section. In the third section, we examine the description of the behavior of discrete time anticausal pure predictor systems.

2. Polynomial Module over the Formal Series Ring
In this section, we review the notion of polynomial module over the formal series ring and characterize its submodules [4]. The ring action for this module is induced by a projection map that is defined based on a decomposition of the truncated Laurent series space. This action is known as the backward shift operator that associated with the behavior of a discrete time anticausal pure predictor systems. We start this section with introduce some set notations and its structure.

Let $F$ be an algebraically closed field, $F^m$ be the space of all $m$-tuples with components in $F$ and $F[z]$ be the polynomial ring with coefficients in $F$. We use the symbol $F((z^{-1}))$ to denote the truncated Laurent series space with coefficients in $F$ [6]. We see at once that the truncated Laurent series space $F((z^{-1}))$ can be decomposed into two parts as follows

$$F((z^{-1})) = F[z] \oplus z^{-1}F[[z^{-1}]]$$ (1)

where $F[[z^{-1}]]$ stands for the ring of all formal series with indefinite $z^{-1}$. The above notation can be naturally extended to the truncated Laurent series space with coefficients in the space $F^m$. 


Let $\pi \in F^m((z^{-1}))$. By generalized the decomposition in Equation (1), we can write $\pi = \gamma + z^{-1}h$ where $\gamma \in F^m[z]$ and $h \in F^m[[z^{-1}]]$. We also can define the projection map on $F^m[z]$ that perpendicular to $z^{-1}F^m[[z^{-1}]]$ by

$$\pi_+: F^m((z^{-1})) \to F^m[z]$$

for any $\pi \in F^m((z^{-1}))$ which is written as in the previous decomposition.

Now, we define a ring action $F[[z^{-1}]]$ on the abelian group $F^m[z]$ as follows

$$\ast: F[[z^{-1}]] \times F^m[z] \to F^m[z]$$

$$(\alpha, \pi) \mapsto \pi_+ (\alpha \pi)$$

where $\pi_+$ is the projection map that defined in Equation (2). By this action, we obtain that $F^m[z]$ is a torsion module over the formal series ring $F[[z^{-1}]]$. Meanwhile, it is evident that $F^m[[z^{-1}]]$ is a free module over $F[[z^{-1}]]$ and it follows immediately that $F^m((z^{-1}))$ is an $F[[z^{-1}]]$-module.

Particularly we have this lemma.

**Lemma 2.1** Consider $F^m[z]$ as an $F[[z^{-1}]]$-module. If $M \subseteq F^m[z]$ is a finitely generated submodule of $F^m[z]$, then $M = D F^m[[z^{-1}]]$ for some $D \in F[z]^{m \times m}$.

**Proof.** Let $M$ be a finitely generated submodule of $F[[z^{-1}]]$-module $F^m[z]$. The $F^m[z]$ as $F[[z^{-1}]]$-module is isomorphic to the quotient module

$$F^m((z^{-1}))/z^{-1}F^m[[z^{-1}]]$$

Any finitely generated submodule of $F^m((z^{-1}))/z^{-1}F^m[[z^{-1}]]$ is a submodule of finitely generated module $z^s F^m[[z^{-1}]]/z^{-1}F^m[[z^{-1}]]$ for some positive integer $s$. Since the quotient submodule $z^s F^m[[z^{-1}]]/z^{-1}F^m[[z^{-1}]]$ is $m$-generated, we have that $M$ is $n$-generated for some $n \leq m$. Choose $A = \{ \pi_1, \pi_2, \pi_3, \ldots, \pi_n \}$ as the generating set of $M$ where $\pi_i \in F^m[z]$ for $i = 1, 2, 3, \ldots, n$ and $n \leq m$. We next construct an $m \times m$ matrix

$$D = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_n & \bar{\pi} & \cdots & \bar{\pi} \end{pmatrix}$$

where $\pi_i \in F^m[z]$ is the generating element of $M$ for $i = 1, 2, 3, \ldots, n$.

We proceed to show that $M = D F^m[[z^{-1}]]$. Let $\pi \in M$. Write $\pi = \sum_{i=1}^n \alpha_i \pi_i$ for some $\alpha_i$ in $F^m([[z^{-1}]])$ where $i = 1, 2, 3, \ldots, n$. It follows that $\pi = D\gamma$ where $\gamma = (\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_m)^t$ is an element of $F^m([[z^{-1}])]$. Hence, $\pi \in D F^m[[z^{-1}]]$ and $M \subseteq D F^m[[z^{-1}]]$.

Now, let $\pi = D\gamma \in D F^m[[z^{-1}]]$ where $\gamma = (y_1 y_2 y_3 \cdots y_m)^t \in F^m[z]$. We check at once that

$$\pi = D\gamma = \sum_{i=1}^n y_i \pi_i$$

This gives $\pi \in M$ and $D F^m[[z^{-1}]] \subseteq M$. We deduce that $M = D F^m[[z^{-1}]]$ and the proof is complete.

From the above Lemma, we obtain a characterization of finitely generated submodule of $F[[z^{-1}]]$-module $F^m[z]$.

3. Behaviors of Discrete Time Anticausal Pure Predictor Systems

In this section, we identifies the structure of the behavior of a discrete time anticausal pure predictor systems. We consider the generalized state space of discrete time anticausal pure predictor systems of the form

$$\begin{align*}
Jx_{t+1} &= \bar{\pi}_t + B\bar{u}_t \\
\bar{y}_t &= C\bar{\pi}_t
\end{align*}$$

(3)
for some \( J, B, C \in \mathbb{F}^{m \times m} \) where \( J \) is a nilpotent matrix. If \( \mathbf{x}_0 \in \mathbb{F}^m \) is given, then we have the sequence states \( \{\mathbf{x}_t\}_{t=0}^{k-1} \). Using z-transform, the transfer matrix of such system is

\[
G(z) = -C \left( \sum_{i=0}^{k-1} J^i z^i \right) B.
\]

From [5] and [3], it may be conclude that the system in Equation (3) is a linear system around infinite poles.

The above system is also referred as a discrete and anticausal predictor system in [4] and it is part of descriptor systems. As we know, descriptor systems appears in a variety of fields such as electronic systems,

Our next goal is to give a description of a behavior of an ordinary discrete time anticausal pure predictor system.

### 3.1. Ordinary Discrete Time Anticausal Pure Predictor System

Consider the generalized state space of discrete time anticausal pure predictor systems of the form

\[
J \mathbf{x}_{t+1} = \mathbf{x}_t
\]

(4)

where \( J \in \mathbb{F}^{m \times m} \) is a nilpotent matrix. A matrix \( J \) is said to be nilpotent if \( J^k = 0 \) for some positive integer \( k \). If \( \mathbf{x}_0 \in \mathbb{F}^m \) is given, then we have \( \mathbf{x}_i = J^{-i} \mathbf{x}_0 \) for \( i = 0, -1, -2, \ldots, -(k-1) \). Taking the sequence \( \{\mathbf{x}_t\}_{t=0}^{k-1} \) in formal series form, we see that \( \sum_{i=0}^{k-1} \mathbf{x}_i z^i = \left( \sum_{i=0}^{k-1} J^i z^i \right) \mathbf{x}_0 \).

We can identified the behavior of linear system in Equation (4), based on Willems [2] definition, as follows

\[
\mathcal{B} = \left\{ \sum_{i=0}^{k-1} \mathbf{x}_i z^i \mid \mathbf{x}_i \text{ satisfies the system in Equation (4) for some } \mathbf{x}_0 \in \mathbb{F}^m \right\}.
\]

The structure of the behavior of linear system in Equation (4) can be seen on the following theorem.

**Theorem 3.1** The behavior \( \mathcal{B} = \left\{ \left( \sum_{i=0}^{k-1} J^i z^i \right) \mathbf{x}_0 \in \mathbb{F}[z] \mid \mathbf{x}_0 \in \mathbb{F}^m \right\} \) is a finitely generated submodule of \( \mathbb{F}[z^{-1}] \)-module \( \mathbb{F}^m[z] \).

**Proof.** We first observe that \( \mathcal{B} \) is a submodule of \( \mathbb{F}[z^{-1}] \)-module \( \mathbb{F}^m[z] \). It is easily seen that \( \mathcal{B} \) is a non-empty subset of \( \mathbb{F}^m[z] \) and \( \mathcal{B} \) is a submodule of \( \mathbb{F}[z^{-1}] \)-module \( \mathbb{F}^m[z] \).

We proceed to show that \( \mathcal{B} \) is finitely generated. Consider

\[
\sum_{i=0}^{k-1} J^i z^i = \left( \begin{array}{c} \overline{p}_1 \\ \vdots \\ \overline{p}_m \end{array} \right)
\]

where \( \overline{p}_j \in \mathbb{F}^m[z] \) is the vector column of the matrix \( \sum_{i=0}^{k-1} J^i z^i \) for \( j = 1, 2, 3, \ldots, m \). We will show that \( P = \{\overline{p}_1, \overline{p}_2, \overline{p}_3, \ldots, \overline{p}_m\} \) generate \( \mathcal{B} \). Let

\[
\overline{g} = \sum_{i=0}^{k-1} J^i z^i \overline{g}_0 \in \mathcal{B} \text{ where } \overline{g}_0 = \left( \begin{array}{c} g_{01} \\ g_{02} \\ \vdots \\ g_{0m} \end{array} \right) \in \mathbb{F}^m
\]

and \( g_{0j} \in \mathbb{F} \subseteq \mathbb{F}[z^{-1}] \) for \( j = 1, 2, 3, \ldots, m \). We see that \( \overline{g} = \sum_{i=0}^{k-1} J^i z^i \overline{g}_0 = \sum_{i=1}^{m} g_{0i} \overline{p}_i \). From the above it follows that \( P \) generate \( \mathcal{B} \), and the proof is complete.

The above theorem shows a description of a behavior of an ordinary discrete time anticausal pure predictor system.
Consider the generalized state space of discrete time anticausal pure predictor systems of the form

\[ J\mathbf{x}_{t+1} = \mathbf{x}_t + B\mathbf{u}_t \]  \hspace{1cm} (5)

for some \( B \in \mathbb{F}^{m \times m} \) and nilpotent matrix \( J \in \mathbb{F}^{m \times m} \). Let \( k \) be a positive integer such that \( J^k = 0 \). If \( \mathbf{x}_0 \in \mathbb{F}^m \) and \( \mathbf{u}_i \in \mathbb{F}^m \) for \( i \in \{-1, -2, -3, \ldots\} \) are given, then we have the sequence \( \{\mathbf{x}_i\}_{i=0}^{-k+1} \).

We can identify the behavior of linear system in Equation (5), based on Willems [2] definition, as follows

\[
C = \left\{ \sum_{i=0}^{k-1} \mathbf{w}_i z^i \right\} \mathbf{x}_i \text{ satisfied the linear system in Equation (5) for some } \mathbf{x}_0, \mathbf{w}_i \in \mathbb{F}^m \\
= \left\{ \sum_{i=0}^{k-1} (J\mathbf{x}_{i+1} - B\mathbf{w}_i) z^i \right\} \text{ for some } \mathbf{x}_0, \mathbf{w}_i \in \mathbb{F}^m 
\]

where \( \mathbf{w}_i \) is a submodule of \( \mathbb{F}^m \). If \( \mathbf{w}_i \) is a non-empty subset of \( \mathbb{F}^m \) and \( \mathbf{w}_i \) is a submodule of \( \mathbb{F}^m \) [2], then we have the sequence \( \{\mathbf{x}_i\}_{i=0}^{-k+1} \).

Theorem 3.2 The behavior

\[
C = \left\{ \sum_{i=0}^{k-1} (J\mathbf{x}_{i+1} - B\mathbf{w}_i) z^i \right\} \mathbf{x}_i \in \mathbb{F}^m, i = -1, -2, -3, \ldots 
\]

is a finitely generated submodule of \( \mathbb{F}[[z^{-1}]] \)-module \( \mathbb{F}^m[z] \).

Proof. It is immediate that \( C \) is a non-empty subset of \( \mathbb{F}^m[z] \) and \( C \) is a submodule of \( \mathbb{F}[[z^{-1}]] \)-module \( \mathbb{F}^m[z] \).

The task is now to show that \( C \) is finitely generated. Consider

\[
A = \sum_{j=0}^{k-1} J^j z^j = (\vec{p}_1 \vec{p}_2 \vec{p}_3 \cdots \vec{p}_m)
\]

where \( \vec{p}_i \in \mathbb{F}^m \) is the \( i \)-th column vector of \( A \). Meanwhile for \( j = 1, 2, \ldots, k - 1 \),

\[
A_j = -\sum_{i=j}^{k-1} J^{i-j} B z^i = (\vec{q}_{j1} \vec{q}_{j2} \vec{q}_{j3} \cdots \vec{q}_{jm})
\]

where \( \vec{q}_{ji} \in \mathbb{F}^m \) is the \( i \)-th column vector of \( A_j \). We will show that

\[
P = \{\vec{p}_1, \cdots, \vec{p}_m, \vec{q}_{11}, \cdots, \vec{q}_{1m}, \vec{q}_{21}, \cdots, \vec{q}_{2m}, \cdots, \vec{q}_{(k-1)m}\}
\]

generate \( C \).

Let \( \vec{x} = \sum_{i=0}^{k-1} (J\mathbf{x}_{i+1} - B\mathbf{w}_i) z^i \in C \) for some

\[
\mathbf{x}_0 = (x_{01} \ x_{02} \ x_{03} \cdots \ x_{0m})^t, \mathbf{u}_i = (u_{i1} \ u_{i2} \ u_{i3} \cdots \ u_{im})^t \in \mathbb{F}^m
\]

where \( x_{0j}, u_{ij} \in \mathbb{F} \) for \( l = 0, -1, -2, \ldots \) and \( j = 1, 2, \ldots, m \). It follows that

\[
\vec{x} = \sum_{i=0}^{k-1} (J\mathbf{x}_{i+1} - B\mathbf{w}_i) z^i = \left( \sum_{j=0}^{k-1} J^j z^j \right) \mathbf{x}_0 - \sum_{j=0}^{k-1} \left( \sum_{i=j}^{k-1} J^{i-j} B z^i \right) \mathbf{w}_i \\
= (\vec{p}_1 \vec{p}_2 \vec{p}_3 \cdots \vec{p}_m) (x_{01} \ x_{02} \ x_{03} \cdots \ x_{0m})^t + \sum_{j=1}^{k-1} (\vec{q}_{j1} \vec{q}_{j2} \vec{q}_{j3} \cdots \vec{q}_{jm}) (u_{j1} \ u_{j2} \ u_{j3} \cdots \ u_{jm})^t \\
= \sum_{i=1}^m x_{0i} \vec{p}_i + \sum_{j=1}^{k-1} (\sum_{i=1}^m u_{ji} \vec{q}_{ji})
\]

Hence, \( P \) generate \( C \).

From what has already been proved, we conclude that \( C \) is a finitely generated submodule of \( \mathbb{F}[[z^{-1}]] \)-module \( \mathbb{F}^m[z] \).

Theorem 3.2 shows a description of a behavior of a discrete time anticausal pure predictor system with input.
4. Conclusion
In this paper, we have identified the structure of the generalized state space of the discrete time anticausal pure predictor system. We also identified the description of the behavior of the system. For further research, we are planning to develop properties of the system in behavioral framework.

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