A Dynamic Approach to Complex Vector Reconstruction from Intensity Measurements

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Abstract

In this article we propose a dynamic approach to complex vector reconstruction in the context of quantum tomography. There are two underlying assumptions behind our reasoning. The first one claims that the evolution of a d-level pure quantum system is given by the Schrödinger equation with a time-independent Hamiltonian and the other states that the knowledge about the quantum state is provided from projective measurements, also called intensity measurements. The problem of quantum state reconstruction is connected with the notion known as phase retrieval – recovering a complex vector from modulus of inner product with frame vectors. Phase retrieval is widely studied in many areas of science but still there is a number of problems that remain to be answered. We believe that the dynamic approach can significantly improve the effectiveness of the vector reconstruction as it aims to decrease the number of distinct projectors by taking advantage of the knowledge about the evolution. General conditions and observations are applied to a specific unitary evolution model.

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1 Introduction

To clarify the problem, let us postulate that the achievable information about a d-level pure quantum system is encoded in a complex vector – called the state vector and denoted $|\psi\rangle$, which belongs to the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$ such that $\text{dim}\mathcal{H} = d < \infty$. Moreover, $|\psi\rangle$ is normalized, i.e. $\langle \psi | \psi \rangle = 1$, where $\langle . | . \rangle$ denotes the inner product in the Hilbert space. Then the goal of quantum tomography is to reconstruct the accurate representation of $|\psi\rangle$ on the basis of data accessible from an experiment. Naturally, multiplying the state vector by a scalar of unit modulus does not change the measurement results. Thus, the state vector can be determined up to a global phase factor.

Apparently, there are many approaches to quantum tomography which differ from one another when it comes to the kind of measurement(s) and the number of their repetitions. One can refer to papers [1, 2, 3, 4, 5] to get some insight about the methods used in quantum tomography. In case of pure states tomography, we are analyzing the problem of recovering a complex vector from intensity measurements – the very same kind of problem is considered in many other areas of science, from pure mathematics to speech recognition or signal processing. Thus, there is a vast
literature concerning phase retrieval. In recent years the attention was paid by many researchers to the connection between complex vector reconstruction and the theory of frames - out of many papers one can especially refer to [6, 7, 8, 9, 10, 11]. By an $N$-element complex frame in $\mathbb{C}^d$, denoted $\Phi = \{ |\theta_1 \rangle, \ldots, |\theta_N \rangle \}$ (where $|\theta_i \rangle \in \mathbb{C}^d$), one should understand a set of complex vectors that span $\mathbb{C}^d$. In articles not connected to quantum tomography authors usually consider in general the problem of reconstructing an unknown complex vector $x \in \mathbb{C}^d$ from its intensity measurements, i.e. it is discussed whether the knowledge about the non-linear map given by

$$J_\Phi : x \rightarrow (|\langle \theta_i | x \rangle|^2)_{i=1,\ldots,N},$$

is sufficient to determine the complex vector $x$. However, in papers not related to Physics authors do not usually discuss the problem of how the intensities can be obtained. Whereas in quantum tomography there is a tendency to look at the reconstruction problem from the point of view of 'economy of measurements'. Since each distinct kind of measurement requires, in general, preparing a new experimental setup, quantum physicists would prefer to perform quantum tomography by the lowest possible number of distinct kinds of measurements. It turns out that if one knows how the quantum state changes in time, one can obtain new data by performing the same kind of measurement at different time instants (apparently, each time one has to measure a distinct physical system). This observation gives the gist of the stroboscopic (dynamic) approach to complex vector reconstruction.

We believe that the stroboscopic approach seems the most advantageous as it focuses on determining the optimal criteria for quantum tomography. This approach originated in 1983 in the article [12], in which the author was considering the minimal number of distinct observables for quantum tomography of a system with evolution given by the von Neumann equation. Later, the stroboscopic tomography was developed in many papers, from which one can especially refer to [13, 14, 15]. Recently new results concerning the stroboscopic tomography have been proposed in [16].

Originally, the stroboscopic tomography was applied to mixed quantum states with evolution given by the Kossakowski-Lindblad equation of the form $\dot{\rho}_t = \mathbb{L}[\rho_t]$, where the operator $\mathbb{L}$ is called the generator of evolution [17, 18]. In order to reconstruct the initial density matrix $\rho_0$ one has to assume that it is possible to obtain from an experiment mean values of certain observables from a fixed set $\{Q_1, \ldots, Q_r\}$, which does not satisfy the condition for completeness. Provided one can measure the same observable more than once on the same system, one may exist a set of discrete time instants $\{t_1, \ldots, t_p\}$, such that the matrix of data $\langle Q_i | Q_j \rangle = Tr(Q_i \rho_t)$ where $i = 1, \ldots, r$ and $j = 1, \ldots, p$ accessible from an experiment enables one to reconstruct the initial density matrix. Moreover, knowledge about the evolution makes it possible to determine not only the initial density matrix, but also the complete trajectory of the quantum system. The minimal number of distinct observables needed for quantum tomography as well as the algebraic structure of those observables can be determined on the basis of the generator of evolution. Explicit theorems concerning density matrix reconstruction in this approach can be found in [15].

In this article we reformulate the assumptions of the stroboscopic tomography so that it will be applicable only for pure states with unitary evolution. One might think that it will be a simplification of the original stroboscopic approach as introduced in [12]. However, one should bear in mind that in case of density matrix reconstruction there are specific general criteria for quantum tomography (concerning the minimal number of observables, time instants etc.), whereas in case of complex vector reconstruction there are still many open questions – for instance it remains unknown what is the minimal number of intensity measurements for phase retrieval (see more in section 2). Therefore, this paper is not just an adaptation of the stroboscopic tomography to pure states, but it tackles in an original way current problems connected with phase retrieval, i.e. complex vector reconstruction.
Moreover, additional motivation behind this article is the observation that in quantum optics or quantum communication quite often one can be sure that the quantum state of a physical system is pure and, therefore, the achievable information is encoded in a complex vector. Thus, we believe that in such situations it is more efficient to employ a quantum tomography approach devised exclusively for pure states. Furthermore, we believe that the reasoning introduced here might be applicable to phase retrieval problems that arise in other areas of science provided one can access knowledge concerning how the vector changes in time.

In this article we consider the case when the evolution of a $d$-level quantum system is given by the Schrödinger equation of the form

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle,$$

where $|\psi(t)\rangle \in \mathcal{H}$ and $H$ is a self-adjoint operator such that $H: \mathcal{H} \rightarrow \mathcal{H}$. The symbol $\hbar$ denotes the Dirac’s constant, for simplicity, henceforth, it is assumed that $\hbar = 1$. Moreover, the vector space of all linear operators on $\mathcal{H}$ shall be denoted by $\mathcal{B}(\mathcal{H})$.

If the evolution of the state vector is given by (2), then $|\psi(t)\rangle$ for any $t \in \mathbb{R}^+$ can be computed from

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle,$$

where $U(t) \in \mathcal{B}(\mathcal{H})$ is a unitary operator such that $U(t) = \exp(-iHt)$.

Discussing the problem of quantum tomography, we will assume that the knowledge about the quantum system is provided by results of projective measurements. Thus, we assume to have a set of operators $\{M_1, \ldots, M_r\}$ for some $r \in \mathbb{N}$, and each $M_i \in \mathcal{B}(\mathcal{H})$. The operators are assumed to be projectors, i.e. $M_i = |i\rangle\langle i|$, where $|i\rangle \in \mathcal{H}$ for $i = 1, \ldots, r$. Naturally, the vectors $|i\rangle$ are normalized, i.e. $\langle i|i \rangle = 1$ for $i = 1, \ldots, r$. We assume that the set of projectors is incomplete and single measurement of each projector does not allow one to reconstruct the initial state vector $|\psi(0)\rangle$.

Thus, in order to determine the initial state vector one has to perform each projective measurement at some discrete time instants $\{t_1, \ldots, t_p\}$. Consequently, the data obtainable from an experiment can be expressed as $m_i(t_j) = \langle \psi(t_j) | M_i | \psi(t_j) \rangle$. In section 2 we consider how such data can be used for effective complex vector reconstruction. Obviously, we assume to have a large number of identically prepared quantum systems and, therefore, each individual system is measured only once, which allows us to skip the problem of how the measurement changes the state.

In this section we have formulated the problem of quantum tomography for pure states and enumerated the assumptions of the stroboscopic approach. In section 2 we shall propose general results and observations that are obtainable from employing the assumptions and combining them with current knowledge about stroboscopic tomography and phase retrieval. In section 3 we demonstrate how the stroboscopic tomography works for pure states by solving a specific example.

## 2 General Results of the Stroboscopic Tomography for Complex Vector

First, we shall expand the exponential formula for the unitary operator $U(t)$ introduced in (3). The most obvious way to expand this operator goes as follows

$$U(t) = \exp(-iHt) = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} H^k. \quad (4)$$

However, one can agree that it should be possible to expand $\exp(-iHt)$ by means of a finite sum of powers of $H$ due to the fact that for every matrix one can introduce the notion of the minimal
polynomial. If by $\mu$ one denotes the degree of the minimal polynomial of $H$, then the decomposition (4) can be rewritten in the form

$$U(t) = \exp(-iHt) = \sum_{k=0}^{\mu-1} \alpha_k(t)H^k,$$

(5)

where $\alpha_k(t)$ are certain time-dependent functions. It can be proved that they are mutually linearly independent and can be computed from a set of differential equations which depend on the coefficients of the minimal polynomial of $H$ [14].

The exponential form of the unitary operator is useful to analyze the experimental data. The formula for the measurement result can be written as

$$m_i(t_j) = \langle \psi(t_j) | M_i | \psi(t_j) \rangle = \langle \psi(t_j) | i \langle i | \psi(t_j) \rangle =$$

$$= |\langle i | \psi(t_j) \rangle|^2 = |\langle i | \exp(-iHt_j) | \psi(0) \rangle|^2$$

$$= \left| \langle i \sum_{k=0}^{\mu-1} \alpha_k(t_j)H^k | \psi(0) \rangle \right|^2 = \sum_{k=0}^{\mu-1} |\alpha_k(t_j)|^2 |\langle i | H^k | \psi(0) \rangle|^2 =$$

$$= \sum_{k=0}^{\mu-1} |\alpha_k(t_j)|^2 |\langle \phi_i^{(k)} | \psi(0) \rangle|^2,$$

(6)

where $|\phi_i^{(k)}\rangle = H^k|i\rangle$.

Now, if one assumes that projective measurement associated with the operator $M_i$ is performed at time instants $\{t_1, \ldots, t_p\}$, we obtain the set of $p$ equations concerning each operator $M_i$: 

$$m_i(t_1) = \sum_{k=0}^{\mu-1} |\alpha_k(t_1)|^2 |\langle \phi_i^{(k)} | \psi(0) \rangle|^2,$$

$$m_i(t_2) = \sum_{k=0}^{\mu-1} |\alpha_k(t_2)|^2 |\langle \phi_i^{(k)} | \psi(0) \rangle|^2,$$

$$\vdots$$

$$m_i(t_p) = \sum_{k=0}^{\mu-1} |\alpha_k(t_p)|^2 |\langle \phi_i^{(k)} | \psi(0) \rangle|^2.$$

(7)

One can notice that such a system of equations can be rewritten as a matrix equation

$$\begin{bmatrix}
m_i(t_1) \\
m_i(t_2) \\
\vdots \\
m_i(t_p)
\end{bmatrix} =
\begin{bmatrix}
|\alpha_0(t_1)|^2 & |\alpha_1(t_1)|^2 & \ldots & |\alpha_{\mu-1}(t_1)|^2 \\
|\alpha_0(t_2)|^2 & |\alpha_1(t_2)|^2 & \ldots & |\alpha_{\mu-1}(t_2)|^2 \\
\vdots & \vdots & \ddots & \vdots \\
|\alpha_0(t_p)|^2 & |\alpha_1(t_p)|^2 & \ldots & |\alpha_{\mu-1}(t_p)|^2
\end{bmatrix}
\begin{bmatrix}
|\langle \phi_1^{(0)} | \psi(0) \rangle|^2 \\
|\langle \phi_1^{(1)} | \psi(0) \rangle|^2 \\
\vdots \\
|\langle \phi_1^{(\mu-1)} | \psi(0) \rangle|^2
\end{bmatrix}.$$

(8)

On the left-hand side of the equation (8) we have a vector of data accessible from an experiment. The matrix $\Lambda = [|\alpha_k(t_j)|^2]$ where $k = 0, \ldots, \mu - 1$ and $j = 1, \ldots, p$ is computable on the basis of the minimal polynomial of $H$. Thus, if the matrix $\Lambda$ is invertible, one can calculate from the equation (8) the intensities (projections) $|\langle \phi_i^{(k)} | \psi(0) \rangle|^2$ for $k = 0, \ldots, \mu - 1$. The condition for computability of these projections can be stated as a theorem.
Theorem 2.1. From the matrix equation (8) one can calculate the projections $|\langle \phi_i^{(k)} | \psi(0) \rangle|^2$ for $k = 0, \ldots, \mu - 1$ if and only if

$\begin{align*}
p &= \mu, \\
det \Lambda &\neq 0.
\end{align*}$

(9a)

(9b)

One can easily notice that the condition (9a) ensures that the matrix $\Lambda$ is square and the condition (9b) ensures its invertibility.

Theorem 2.1 states that the number of time instants is equal to the degree of the minimal polynomial of $H$.

Now it should be clear that if the stroboscopic approach to quantum tomography is applied to each projective operator from the set $\{M_1, \ldots, M_r\}$, i.e. each projector is measured at time instants $\{t_1, \ldots, t_\mu\}$ such that the condition (9b) is fulfilled, then one can determine the correspondence $|\psi(0)\rangle \rightarrow |\langle \phi_i^{(k)} | \psi(0) \rangle|^2$ for $k = 0, \ldots, \mu - 1$ and $i = 1, \ldots, r$.

(10)

The set of vectors that we project $|\psi(0)\rangle$ onto shall be denoted as $\Phi = \{ |\phi_i^{(k)}\rangle \}_{i,k=(1,0)}^{(r,\mu-1)}$. Henceforth, the non-linear map that assigns to the unknown state vector $|\psi(0)\rangle$ the set of squares of absolute values of its inner product with the vectors from the set $\Phi$ as demonstrated in (10) shall be denoted by $J_\Phi$.

The natural question which arises here states: When can we reconstruct the initial state vector $|\psi(0)\rangle$ on the basis of intensities defined by the set $\Phi$?

Apparently, of one wants to reconstruct $|\psi(0)\rangle$ on the basis of the intensities given by the correspondence (10), the vectors that belong to $\Phi$ have to span the Hilbert space to which $|\psi(0)\rangle$ belongs. In other words, the necessary condition for complex vector reconstructability claims that the collection of vectors $\Phi$ has to constitute a frame in $\mathcal{H}$ such that $|\psi(0)\rangle \in \mathcal{H}$.

In order to formulate the theorem concerning the necessary (but not sufficient) condition for reconstructability of $|\psi(0)\rangle$ let us first introduce the notion of the Krylov subspace.

Definition 2.1. The Krylov subspace, which shall be denoted by $K_\mu(H, M_i)$, is defined as follows

$K_\mu(H, M_i) := \text{Span}\{ |\phi_i^{(0)}\rangle, |\phi_i^{(1)}\rangle, \ldots, |\phi_i^{(\mu-1)}\rangle \}$.

(11)

The stroboscopic approach to tomography is the more advantageous, the higher the dimension of the Krylov subspace is. One can instantly notice that the dimension of the Krylov subspace depends on algebraic properties of both $|i\rangle$ and $H$.

The theorem concerning the necessary condition for phase retrieval can be formally formulated as follows.

Theorem 2.2. The necessary condition for the initial state vector $|\psi(0)\rangle \in \mathcal{H}$ to be reconstructible on the basis of the intensity measurements $|\langle \phi_i^{(k)} | \psi(0) \rangle|^2$ for $k = 0, \ldots, \mu - 1$ and $i = 1, \ldots, r$ claims that

$\bigoplus_{i=1}^r K_\mu(H, M_i) = \mathcal{H}$,

(12)

where the symbol $\bigoplus$ denotes the Minkowski sum of subspaces.

To consider the sufficient condition for complex vector reconstruction let us revise the general considerations concerning the question when phase retrieval is possible. In a recent paper [10] the authors claim that phase retrieval is possible when any two vectors $|\psi(0)\rangle$ and $|\psi'(0)\rangle$ with identical
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The intensity measurements differ only by a scalar of norm one, i.e. $|\psi(0)\rangle = e^{i\theta} |\psi'(0)\rangle$. In other words, the same observation can be stated that it is possible to reconstruct the complex vector $|\psi(0)\rangle$ if and only if the non-linear map $J_\Phi$ is injective and $\Phi$ is a frame. Thus, henceforth in situations when phase retrieval is possible, we shall say that the frame $\Phi$ generates (or defines) injective measurements.

In [9] Bandeira, Cahill, Mixton and Nelson postulated a conjecture according to which if one wants to reconstruct a vector $x$ such that $x \in \mathbb{C}^d$, then a frame that contains less than $4d - 4$ vectors cannot generate injective intensity measurements, i.e. according to the authors fewer than $4d - 4$ modulus of inner product of $x$ with other vectors is not sufficient to obtain the structure of $x$. Furthermore, in the same paper the authors postulated the second part of the conjecture that a generic frame with $4d - 4$ vectors (or more) generates injective measurements on $\mathbb{C}^d$. The second part of the conjecture has recently been proved in [10], where the authors explained the notion of a generic frame and proved that for a generic frame $\Phi$ that contains at least $4d - 4$ elements the map $J_\Phi$ is injective.

Another recent paper [11] proves a result that contradicts the first part of the conjecture from [9]. Cynthia Vinzant proposes a frame in $\mathbb{C}^4$ which consists of 11 vectors and proves that it defines injective measurements on $\mathbb{C}^4$. Therefore, the current knowledge about the phase retrieval problem does not give an answer to the question what is the minimal number of elements of the frame $\Phi$ so that the map $J_\Phi$ is injective, i.e. so far it remains unknown in general how many intensity measurements at least are needed to reconstruct the unknown complex vector. However, in [9] the authors propose a relatively efficient way to check whether a frame $\Phi$ generates injective measurements. Their approach is presented below as a theorem.

**Theorem 2.3** (Bandeira et al. 2014). *A frame $\Phi = \{|\theta_1\rangle, \ldots, |\theta_N\rangle\}$ (where $|\theta_i\rangle \in \mathbb{C}^d$) defines injective measurements, i.e. one can reconstruct some unknown vector $x \in \mathbb{C}^d$ from intensity measurements $|\langle \theta_i|x \rangle|^2$ for $i = 1, \ldots, N$, if and only if the linear space

$$L_\Phi := \{Q \in \mathbb{C}^{d\times d} : \langle \theta_1|Q|\theta_1 \rangle = \cdots = \langle \theta_N|Q|\theta_N \rangle = 0\}$$

does not contain any non-zero Hermitian matrix of the rank $\leq 2$. 

The theorem 2.3 states clearly the sufficient condition that needs to be fulfilled so that a frame $\Phi$ defines injective measurements and, therefore, it is possible to reconstruct a complex vector on the basis of its intensity measurements. For a given frame one can relatively easy check whether the condition stated in theorem 2.3 is fulfilled or not. However, so far there have been no proposal concerning the procedure how to obtain such a sufficient frame.

### 3 An Example of the Stroboscopic Tomography Model for a Complex Vector from $H \cong \mathbb{C}^2$

In this section we demonstrate how to apply the general results and observations concerning complex vector reconstruction to a specific quantum tomography problem.

Let us analyze a quantum system associated with the Hilbert space $H$ such that $\text{dim} H = 2$. The knowledge about the system is encoded in a time-dependent state vector $|\psi(t)\rangle \in H$. The problem of reconstructing the initial state vector $|\psi(0)\rangle$ from experimentally accessible data has been widely studied. In [19] one can find the result according to which the initial state vector can be uniquely determined from projections onto 4 complex vectors which correspond to the vertices of a tetrahedron embedded within the Bloch sphere. The main goal of this section is to demonstrate an alternative approach to qubit reconstruction – by employing the stroboscopic tomography one
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can determine $|\psi(0)\rangle$ on the basis of two distinct projectors, but each one is measured at two different time instants.

As the stroboscopic approach to quantum tomography requires the knowledge about the evolution of the system, let us postulate that the state vector evolves in accordance with the equation

$$\frac{d|\psi(t)\rangle}{dt} = -iH|\psi(t)\rangle,$$

where the Hamiltonian, in this example, takes the form

$$H = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The goal of the quantum tomography is to determine the initial state vector $|\psi(0)\rangle$ from certain intensity measurements. It is commonly known that the structure of a two-level state vector can be represented by two real parameters $\theta$ and $\phi$. Thus, in order to reconstruct $|\psi(0)\rangle$ one needs to find $\theta, \phi$ such that

$$|\psi(0)\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix},$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ [20].

One can notice that for $H$ given by (15) the degree of the minimal polynomial is equal to the degree of the characteristic polynomial - both are equal 2. Bearing in mind representation (3) one can write

$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle,$$

where it is convenient to substitute $\tilde{H} = -iH$, which on the basis of (5) gives us

$$|\psi(t)\rangle = \exp(\tilde{H}t)|\psi(0)\rangle = \left(\alpha_0(t)I_2 + \alpha_1(t)\tilde{H}\right)|\psi(0)\rangle.$$

The functions $\alpha_0(t)$ and $\alpha_1(t)$ can be computed from the set of differential equations that they need to fulfill [14]. One can easily get

$$\alpha_0(t) = \cos(t) \text{ and } \alpha_1(t) = \sin(t).$$

Thus, $|\psi(t)\rangle$ can be expressed as

$$|\psi(t)\rangle = \cos(t)|\psi(0)\rangle + \sin(t)\tilde{H}|\psi(0)\rangle.$$

To determine the initial state vector $|\psi(0)\rangle$ we propose to perform the projective measurements onto two vectors $|1\rangle$ and $|2\rangle$ given by

$$|1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ and } |2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}.$$

In other words, we introduce projectors $M_1 = |1\rangle\langle 1|$ and $M_2 = |2\rangle\langle 2|$. Each distinct projective measurement is performed twice, as the degree of the minimal polynomial is equal 2. Let us denote the time instants by $t_1$ and $t_2$.

One can easily calculate $\tilde{H}|1\rangle$ and $\tilde{H}|2\rangle$ and obtain

$$\tilde{H}|1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \text{ and } \tilde{H}|2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} -i \\ 2 \end{pmatrix}.$$
Obviously, the four vectors introduced in (21)-(22) constitute a frame in $\mathbb{C}^2$.

As each projective measurement is performed two times, let us denote the results of the measurements by $m_i(t_j) = |\langle i|\psi(t_j)\rangle|^2$ where $i = 1, 2$ and $j = 1, 2$. For each projector we get two equations which can be combined into a matrix according to (28). For the projector $M_1$ we obtain

$$
\begin{bmatrix}
m_1(t_1) \\
m_1(t_2)
\end{bmatrix} =
\begin{bmatrix}
\cos^2 t_1 & \sin^2 t_1 \\
\cos^2 t_2 & \sin^2 t_2
\end{bmatrix}
\begin{bmatrix}
|\langle 1|\psi(0)\rangle|^2 \\
|\langle 1|\widetilde{H}|\psi(0)\rangle|^2
\end{bmatrix}.
$$

For simplicity we can assume that $t_1 = 0$. Then from (23) we can calculate the intensities if $\sin^2 t_2 \neq 0$, which means that

$$
t_2 \neq k\pi \text{ for } k \in \mathbb{N} \cup \{0\}.
$$

Assuming that the condition (24) concerning the time instant $t_2$ is fulfilled, we can calculate the intensities

$$
|\langle 1|\psi(0)\rangle|^2 = m_1(0),
$$

$$
|\langle 1|\widetilde{H}|\psi(0)\rangle|^2 = \frac{-\cos^2 t_2 m_1(0) + m_1(t_2)}{\sin^2 t_2}.
$$

In case of the projector $M_2$ one can write a matrix equation analogous to (23) which is solvable under the very same condition concerning the time instant $t_2$. Thus, one can get the projections

$$
|\langle 2|\psi(0)\rangle|^2 = m_2(0),
$$

$$
|\langle 2|\widetilde{H}|\psi(0)\rangle|^2 = \frac{-\cos^2 t_2 m_2(0) + m_2(t_2)}{\sin^2 t_2}.
$$

This analysis proved that by performing the two projective measurements given by $M_1$ and $M_2$ at two different time instants $t_1 = 0$ and $t_2$ satisfying (24), we can determine the intensity measurements defined by a complex frame $\Phi = \{|1\rangle, \widetilde{H}|1\rangle, |2\rangle, \widetilde{H}|2\rangle\}$. The question that remains to be answered states whether this frame defines injective measurements. To answer this question we propose to employ theorem 2.3. Thus, we shall analyze the linear space as introduced in (13). In this example it takes the form

$$
\mathcal{L}_\Phi = \{Q \in \mathbb{C}^{2 \times 2} : \langle 1|Q|1\rangle = \langle 1|\widetilde{H}Q\widetilde{H}|1\rangle = \langle 2|Q|2\rangle = \langle 2|\widetilde{H}Q\widetilde{H}|2\rangle = 0\}.
$$

According to the theorem 2.3 the frame $\Phi$ defines injective measurements if and only if the linear space $\mathcal{L}_\Phi$ does not contain any non-zero Hermitian matrix of rank $\leq 2$. In the analyzed case the proof of the injectivity can be done by a contradiction, i.e. we shall assume that there is a Hermitian matrix $Q'$ which belongs to $\mathcal{L}_\Phi$. Bearing in mind $\dim \mathcal{H} = 2$, $Q'$ can generally be presented in the form

$$
Q' = \begin{pmatrix}
a & c + id \\
b & c - id
\end{pmatrix}
$$

for some $a, b, c, d \in \mathbb{R}$. By employing the software Mathematica 10, one can get that $Q' \in \mathcal{L}_\Phi$ if and only if $a = b = c = d = 0$, which means that the frame $\Phi$ defines injective measurements. Therefore, from the intensity measurements $|\langle i|\widetilde{H}^k|\psi(0)\rangle|^2$ for $i = 1, 2$ and $k = 0, 1$ one can reconstruct the initial state vector $|\psi(0)\rangle$.

In order to observe that phase retrieval is possible let us calculate the intensities defined by $\Phi$ from the knowledge about the vectors (21)-(22) and the general representation of $|\psi(0)\rangle$ (16). One can easily get

$$
|\langle 1|\psi(0)\rangle|^2 = \frac{5 - 3\cos \theta - 4\cos \phi \sin \theta}{10},
$$

(29a)
Comparing the equations (25a)-(26b) with the equations (29a)-(29d) one can obtain

$$\cos \theta = \frac{10}{6} \frac{m_2(0) - m_2(t_2)}{\sin^2 t_2}. \quad (30)$$

Bearing in mind that $\theta \in [0, \pi]$, one can uniquely determine $\theta$ by

$$\theta = \arccos \left( \frac{10}{6} \frac{m_2(0) - m_2(t_2)}{\sin^2 t_2} \right). \quad (31)$$

To find the structure of the initial state vector $|\psi(0)\rangle$ parametrized as (16), we also need to determine $\phi$. Again by comparing the set of equations (25a)-(26b) with the equations (29a)-(29d) one can obtain

$$\sin \phi = \frac{5}{4} \frac{m_2(t_2) - m_2(0)\cos(2t_2) - \sin^2 t_2}{\sin^2 t_2 \sin \theta}, \quad (32a)$$

$$\cos \phi = \frac{5}{4} \frac{2(m_1(t_2) - \cos^2 t_2 m_1(0)) + m_2(t_2) - m_2(0) - \sin^2 t_2}{\sin \theta \sin^2 t_2}. \quad (32b)$$

One can agree that for $\phi \in [0, 2\pi]$ the knowledge about $\sin \phi$ and $\cos \phi$ is sufficient to uniquely determine $\phi$. Therefore, we have showed explicitly that for an isolated quantum system with evolution given by (14), projective measurement performed for two operators introduced in (21) in two distinct time instants $t_1 = 0$ and $t_2$ satisfying the condition (24) provides sufficient data for phase retrieval, i.e. the experimental data is sufficient to uniquely determine the structure of the initial state vector $|\psi(0)\rangle$.

## 4 Summary

The problem of complex vector reconstruction from intensity measurements appears in many areas of science. Thus, different aspects of phase retrieval have been deeply studied. However, there are still many open problems, concerning for example the optimal criteria for reconstruction. This article proposes in the context of quantum tomography a new approach to complex vector reconstruction, which allows to decrease the number of distinct projectors required for phase retrieval. The stroboscopic approach has been applied to one specific model of unitary evolution, but the number of problems that can be solved with this reasoning is unlimited. One can also apply the introduced analysis to phase retrieval problems not connected to quantum tomography provided one can access the knowledge concerning how the vector changes in time.

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References

[1] J. B. Altepeter, D. F. V. James, P. G. Kwiat, *Qubit Quantum State Tomography*, in: M. G. A. Paris, J. Řeháček (eds.), *Quantum State Estimation*, Springer, Berlin (2004), 111-145.

[2] G. Kimura, Phys. Lett. A 314 (2003) 339-349.

[3] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, C. Bamber, Nature 474 (2011) 188.

[4] O. Steuernagel, J. A. Vaccaro, Phys. Rev. Lett. 75 (1995) 3201-3205.

[5] S. Wu, Scientific reports 3 (2013) 1193.

[6] R. Balan, P. G. Casazza, D. Edidin, Appl. Comput. Harmon. Anal. 20 (2006) 345-356.

[7] J. Cahill, P. G. Casazza, J. Peterson, L. Woodland, arXiv:1305.6226 (2013).

[8] P. G. Casazza, L. M. Woodland, Contemp. Math. 626 (2014) 1-17.

[9] A. S. Bandeiraa, J. Cahillb, D. G. Mixone, A. A. Nelsonc, Appl. Comput. Harmon. Anal. 37 (2014) 106-125.

[10] A. Conca, D. Edidin, M. Hering, C. Vinzant, Appl. Comput. Harmon. Anal. 38 (2015) 346-356.

[11] C. Vinzant, arXiv:1502.04656 (2015).

[12] A. Jamiołkowski, Int. J. Theor. Phys. 22 (1983) 369-376.

[13] A. Jamiołkowski, Rep. Math. Phys. 46 (2000) 469-482.

[14] A. Jamiołkowski, Open Syst. Inf. Dyn. 11 (2004) 63-70.

[15] A. Jamiołkowski, *Fusion Frames and Dynamics of Open Quantum Systems*, in: S. Lyagushyn (eds.), *Quantum Optics and Laser Experiments* InTech, Rijeka (2012), 67-84.

[16] A. Czerwiński, Int. J. Theor. Phys. (2015). DOI: 10.1007/s10773-015-2703-2

[17] V. Gorini, A. Kossakowski, E. C. G. Sudarshan, J. Math. Phys. 17 (1976) 821-825.

[18] G. Lindblad, Commun. Math. Phys. 48 (1976) 119-130.

[19] J. Řeháček, B.-G. Englert, D. Kaszlikowski, Phys. Rev. A 70 (2004) 052321.

[20] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge (2000)