Supersymmetric Matrix Quantum Mechanics with Non-Singlet Sector

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We consider a supersymmetric matrix model which is related to the non-critical superstring theory. We find new non-singlet terms in the supersymmetric matrix quantum mechanics. The new non-singlet terms from fermions, can eliminate other non-singlet terms from generators of $U(N)$ subalgebra and from time periodicity. The non-singlet terms from the generators violate the T-duality on the target space which is a circle. Therefore, we can retain the T-duality with a process of the elimination.

I. INTRODUCTION

The matrix models have been used widely in many mathematical and physical applications, such as combinatorics of graphs, topology, integrable systems, string theory, theory of mesoscopic systems and statistical mechanics on random surfaces [1, 2, 3]. In this paper we will focus on the supersymmetric matrix model which is related to the non-critical 2-dimensional string theory. The worldsheet of the 2-dimensional string theory is represented mathematically by random surfaces in matrix description [4, 5, 6]. The matrix description helps us to understand the non-perturbative effects of the string theory.

The supersymmetric matrix quantum mechanics [8] and the bosonic matrix model with a time periodicity [9] have been studied previously. In this paper we will focus on the supersymmetric matrix quantum mechanics with non-singlet sector. Such models are related to the 2-dimensional black hole [10].

In the matrix quantum mechanics, only the singlet sector has been considered because the non-singlet terms are difficult to handle. The non-singlet terms violate the T-duality on the target space. These non-singlet terms are from generators of $U(N)$ subalgebra. In this paper we construct new non-singlet terms from fermions. Our new non-singlet terms prevent the old non-singlet terms from violating the T-duality.

II. A BRIEF REVIEW

In this section, firstly, we review the quantum mechanics of the supersymmetric matrix model [8]. Secondly, we review the bosonic matrix model with periodic time condition [9]. Lastly, we review the non-singlet terms from generators of $U(N)$ subalgebra and the violation of the T-duality on the target space [4, 5, 15, 16, 17, 18].

A. Supersymmetric matrix quantum mechanics

1. Lagrangian and Hamiltonian

We will use a time-dependent, $N \times N$, $d = 1$, $\mathcal{N} = 2$ Hermitian matrix superfield as follows:

$$\Phi_{ij}(t) = M_{ij}(t) + i\theta_1 \Psi_{1ij}(t) + i\theta_2 \Psi_{2ij}(t) + i\theta_1 \theta_2 F_{ij}(t),$$

where $\theta_1$ and $\theta_2$ are real anticommuting parameters, $M_{ij}$ and $F_{ij}$ are $N \times N$ bosonic Hermitian matrices and $\Psi_{1ij}$ and $\Psi_{2ij}$ are $N \times N$ fermionic Hermitian matrices. The Lagrangian is

$$L = \int dt \, d\theta_1 d\theta_2 \left\{ \frac{1}{2} Tr D_1 \Phi D_2 \Phi + i W(\Phi) \right\},$$

where the potential, $W$, is a polynomial in $\Phi$,

$$W(\Phi) = \sum_n b_n Tr \Phi^n.$$  

In the above $b_n$ are real coupling parameters, and $D_I$ are superspace derivatives,

$$D_I = \frac{\partial}{\partial \theta_I} + i \theta_I \frac{\partial}{\partial t} \quad (I = 1, 2).$$

In component fields, the Lagrangian reads

$$L = \sum_{ij} \left\{ \frac{1}{2} (\dot{M}_{ij} M_{ji} + F_{ij} F_{ji}) + \frac{\partial W(M)}{\partial M_{ij}} F_{ij} \right\}$$

$$- \frac{i}{2} \sum_{ij} (\dot{\Psi}_{1ij} \Psi_{1ji} + \Psi_{2ij} \dot{\Psi}_{2ji})$$

$$- i \sum_{ijkl} \Psi_{1ij} \frac{\partial W(M)}{\partial M_{ij} \partial M_{kl}} \Psi_{2kl}.$$ 

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The equation of motion, \( F_{ij} = -\frac{\partial W(M)}{\partial M_{ij}} \), for the auxiliary matrix \( F_{ij} \) makes the Lagrangian as follows:

\[
L = \sum_{ij} \left\{ \frac{1}{2} \dot{M}_{ij} \dot{M}_{ji} - \frac{1}{2} \frac{\partial W(M)}{\partial M_{ij}} \frac{\partial W(M)}{\partial M_{ji}} \right\} - \frac{i}{2} \sum_{ij} (\Psi_{1ij} \dot{\Psi}_{1ji} + \Psi_{2ij} \dot{\Psi}_{2ji})
\]

\[
\text{(6)}
\]

If we turn off the fermionic terms in Eq.(6) then we get a pure bosonic model. The conjugate momenta to the matrices \( M, \Psi_1 \) and \( \Psi_2 \) are

\[
\Pi_{M_{ij}} = \dot{M}_{ij},
\]

\[
\Pi_{\Psi_{1ij}} = -\frac{i}{2} \Psi_{1ij},
\]

\[
\Pi_{\Psi_{2ij}} = -\frac{i}{2} \Psi_{2ij}.
\]

The Legendre transformation for the Lagrangian with Eq.(7) gives us following Hamiltonian,

\[
H = \sum_{ij} \left\{ \frac{1}{2} \Pi_{M_{ij}} \Pi_{M_{ji}} + \frac{1}{2} \frac{\partial W(M)}{\partial M_{ij}} \frac{\partial W(M)}{\partial M_{ji}} \right\} + \frac{i}{2} \sum_{ijkl} \Psi_{1ij} \Psi_{kl} \partial^2 W(M) \partial M_{ij} \partial M_{kl}.
\]

\[
\text{(8)}
\]

Let us introduce the following relations for the canonical quantization of the Hamiltonian,

\[
\{ \hat{\Pi}_{M_{ij}}, \hat{M}_{kl} \} = -i\delta_{ik}\delta_{jl},
\]

\[
\{ \hat{\Psi}_{ij}, \hat{\Psi}_{kl} \} = \delta_{ik}\delta_{jl},
\]

and the complex formation for the fermions,

\[
\hat{\Psi} = \frac{1}{\sqrt{2}}(\hat{\Psi}_1 + i\hat{\Psi}_2),
\]

\[
\hat{\bar{\Psi}} = \frac{1}{\sqrt{2}}(\hat{\Psi}_1 - i\hat{\Psi}_2).
\]

The anti-commutator for \( \hat{\Psi} \) and \( \hat{\bar{\Psi}} \) is \( \{ \hat{\Psi}_{ij}, \hat{\bar{\Psi}}_{kl} \} = \delta_{ik}\delta_{jl} \). Therefore, the Hamiltonian has a final form,

\[
\hat{H} = \sum_{ij} \left\{ \frac{1}{2} \hat{\Pi}_{M_{ij}} \hat{\Pi}_{M_{ji}} + \frac{1}{2} \frac{\partial W(M)}{\partial M_{ij}} \frac{\partial W(M)}{\partial M_{ji}} \right\} + \frac{1}{2} \sum_{ijkl} [\hat{\Psi}_{ij}, \hat{\bar{\Psi}}_{kl}] \partial^2 W(M) \partial M_{ij} \partial M_{kl}.
\]

\[
\text{(11)}
\]

2. Unitary transformation and the singlet sector

Now, let us take the unitary transformation

\[
\Phi(t) \rightarrow U^\dagger(t)\Phi(t)U(t)
\]

\[
\text{(12)}
\]

for the matrix superfield \( \Phi_j(t) \) in Eq.(1). This unitary transformation makes the bosonic matrix \( M \) which is diagonal as

\[
M_{ij} = \sum_k U_{ik}^\dagger \lambda_k U_{kj}.
\]

\[
\text{(13)}
\]

However, in general, the fermionic matrix \( \Psi \) is not be diagonalized simultaneously. Although the off-diagonal result of \( \chi_{kl} \) for \( k \neq l \), the only diagonal elements \( \chi_{kk} \equiv \chi_k \) for \( k = l \) have been mostly used in the supersymmetric matrix quantum mechanics up to date. It has diagonal formation such as

\[
\Psi_{ij} = \sum_{k} U_{ik}^\dagger \lambda_k U_{kj},
\]

\[
\text{(14)}
\]

by the unitary transformation.

In general case, the supersymmetric matrix quantum mechanics has so-called non-singlet terms related to the off-diagonal elements in the fermionic matrices. This makes us define a “rotated” fermion matrix

\[
\chi = U\Psi U^\dagger.
\]

\[
\text{(15)}
\]

We emphasize again that the unitary operator \( U \) diagonalizes the \( M \), but the \( \chi \) is not diagonalized simultaneously in general. However, the states \( |s\rangle \) on the \( U(N) \)-singlet sector in the Hilbert space, are annihilated by \( \partial U_j \) and are also annihilated by \( \bar{\chi}_{ij} \) where \( i \neq j \). Thus, if we concentrate those states to the singlet sector then we are able to take only diagonal terms of the fermionic matrices. In this case, the superfield, Eq.(1), is transformed into as follows:

\[
(U^\dagger \Phi(t)U)_i \equiv \lambda_i(t) + i\theta_1 \chi_{1i}(t) + i\theta_2 \chi_{2i}(t) + i\theta_1 \theta_2 f_i(t),
\]

\[
\text{(16)}
\]

where \( f_i \equiv (U^\dagger FU)_{ii} \). Another example for Eq.(16) is in Ref.[13].

3. Effective Lagrangian

Now, we can construct a Hamiltonian for the singlet sector, \( \hat{H}_s \), such that \( \hat{H}_s|s\rangle = \hat{H}_s|s\rangle \). The ordinary Hamiltonian, \( \hat{H} \), is given by Eq.(11) and the singlet-Hamiltonian is

\[
\hat{H}_s = \sum_i \left\{ \frac{1}{2} \hat{\Pi}_{\chi_i}^2 + i \frac{\partial W}{\partial \lambda_i} \hat{\Pi}_{\lambda_i} + \frac{1}{2} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 \right\} + \frac{1}{2} \sum_{ij} [\chi_i, \chi_j] \partial^2 W \partial \lambda_i \partial \lambda_j
\]

\[
+ \frac{1}{2} \sum_{ijkl} [\chi_{ij}, \chi_{kl}] \partial^2 W \partial \lambda_i \partial \lambda_j
\]

\[
\text{(17)}
\]

where \( \chi_i \equiv \chi_{ii} \) and

\[
w = -\sum_i \sum_{j \neq i} \ln |\lambda_i - \lambda_j|.
\]

\[
\text{(18)}
\]
The effective Lagrangian, \( L_s \), for \( \hat{H}_s \) is given by the Legendre transform and the gaussian integration for the \([d\Pi_x] \) in following partition function \( \mathcal{S} \),

\[
Z_N(b_n) = \int [d\Pi_x][d\lambda][d\chi][d\bar{\chi}] \exp \left[ i \int dt \sum_{ij} \bar{\chi}_i \chi_j \frac{\partial^2 w}{\partial \lambda_i \partial \lambda_j} \right]
\times \exp \left[ i \int dt \left\{ \Pi_\chi \dot{\chi}_i - i \bar{\chi}_i \chi_i - H_s \right\} \right]
= \int [d\lambda][d\chi][d\bar{\chi}] \exp \left( i \int dt L_s \right),
\]

(19)

Thus the effective Lagrangian for the singlet sector in the supersymmetric matrix case is

\[
L_s = \sum_i \left\{ \frac{1}{2} \dot{\chi}_i^2 - \frac{1}{2} \left( \frac{\partial W_{\text{eff}}}{\partial \lambda_i} \right)^2 - i \left( \bar{\chi}_i \dot{\chi}_i - \dot{\bar{\chi}}_i \chi_i \right) \right\}
- \sum_{ij} \bar{\chi}_i \chi_j \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial \lambda_j},
\]

(20)

where the effective potential is

\[
W_{\text{eff}}(\lambda_i) = W(\lambda_i) + w(\lambda_i).
\]

(21)

After inserting Eq. (21) into Eq. (20) we can rearrange terms to obtain the final effective Lagrangian,

\[
L_s = \sum_i \left\{ \frac{1}{2} \dot{\chi}_i^2 - \frac{1}{2} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 - \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} \right\}
- \frac{1}{2} \left( \frac{\partial w}{\partial \lambda_i} \right)^2 - i \left( \bar{\chi}_i \dot{\chi}_i - \dot{\bar{\chi}}_i \chi_i \right)
- \sum_{ij} \left\{ \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j} \bar{\chi}_i \chi_j + \frac{\partial^2 w}{\partial \lambda_i \partial \lambda_j} \bar{\chi}_i \chi_j \right\}.
\]

(22)

A detailed derivation for the effective Lagrangian is given in Appendix B of Ref. \( \mathcal{S} \).

\section*{B. The bosonic matrix quantum mechanics with time periodicity}

\subsection*{1. Constraint from time periodicity}

We take a partition function for a time dependent and periodic bosonic matrix,

\[
M_{ij}(t) = M_{ij}(t + \beta),
\]

(23)

as follows:

\[
Z_N = \int_{M(0)=M(\beta)} \mathcal{D}M \times \exp \left[ -N \text{Tr} \int_0^\beta dt \left\{ \frac{1}{2} \dot{M}^2 + V(M) \right\} \right].
\]

(24)

The unitary transformation \( M_{ij} = \sum_k N_{ik} U_k^j \), makes \( \text{Tr} \dot{M}^2 \) in Eq.(24) as follows:

\[
\text{Tr} \dot{M}^2 = \sum_i \lambda_i^2 + \sum_{i \neq j} (\lambda_i - \lambda_j)^2 |A_{ij}(t)|^2,
\]

(25)

where

\[
A_{ij}(t) = (U(t)U(t)^\dagger)_{ij}.
\]

(26)

The measure \( \mathcal{D}M \) is transformed into

\[
\mathcal{D}M_{ij} = [dA_{ij}] \prod_i (d\lambda_i) \prod_{i < j} (\lambda_i - \lambda_j)^2.
\]

(27)

In general, the measure of \([dA]\) can be dropped because the result of the integral for \([dA]\) becomes trivial gauge volume factor. However, in the case with the periodic time condition, we must keep the measure of \([dA]\) because the integral for \([dA]\) gives us non-trivial and important terms. Now, let us look into the non-trivial result from the time periodicity.

The periodic time condition in Eq.(23) gets the diagonal elements \( \lambda_i \) of \( M_{ij}(t) \) to be

\[
\lambda_k(t + \beta) = \sum_j P_{kj} \lambda_j(t) P_{jk}^{-1}
\]

(28)

where \( P \) is an operator which makes the unitary operator, \( U(t) \),

\[
U(t + \beta) = P U(t).
\]

(29)

The connection \( A_{ij}(t) \) is an independent variable in Eq.(27). However, in the case of time periodicity, \( A_{ij}(t) \) is not an independent variable but is constrained the constraint with \( P \) such that

\[
\hat{T} \exp i \int_0^\beta dt A(t) = P^{-1},
\]

(30)

where \( \hat{T} \) is a time-ordering operator. This constraint contributes a delta-function to the measure Eq.(27):

\[
\mathcal{D}M_{ij}(t) = dA_{ij}(t) \prod_i d\lambda_i(t) \prod_{t \in [0, \beta]} \Delta^2 (\lambda(t)) \times \delta (\hat{T} \exp i \int_0^\beta dt A(t), P^{-1}),
\]

(31)

where \( \Delta(\lambda) = \prod_{k < m} (\lambda_k - \lambda_m) \) is the Vandermond determinant. The delta-function can relate to the irreducible representation for the operator. The relation is

\[
\delta(U, P^{-1}) = \sum_R d_R \chi_R (PU),
\]

(32)
Now, let us write the final partition function by the gaus- 
boundary conditions, one should be more careful and use 
symmetry of wave functions. In the case of the periodic 
tion over all angular variables, only two Vandermonde 
tion.

$R_{symmetrizator}$, the skew-symmetry of the Vandermond

where $R_{\lambda}$ is the character $R_{\lambda}(\hat{R} \exp \left( i \int_0^\beta A(t)dt \right))$

$$ = \text{Tr}_R \left[ \hat{T} \exp \left( i \int_0^\beta dt \sum_{i,j} A_{ij} \tilde{R}_{ij}^R \right) \right] ,$$

where $\tilde{R}_{ij}^R$ is a generator of $U(N)$ in the $R$'th representation.

2. Non-singlet terms from the constraint

In the case of free boundary conditions (i.e. when $M(0)$ is independent of $M(\beta)$) the singlet partition function describes non-interacting fermions. After integration over all angular variables, only two Vandermonde determinants at the ends of the interval remain. They are $\Delta(\lambda(0)), \Delta(\lambda(\beta))$ and these terms assure the anti-
symmetry of wave functions. In the case of the periodic boundary conditions, one should be more careful and use delta functions to match the eigenvalues as follows:

$$ f(\lambda(\beta)) = \int \prod_k d\lambda_k(0) \Delta^2(\lambda(0)) f(\lambda(0)) \delta(\lambda(0) - \lambda(\beta)). \quad (34) $$

Now, let us write the final partition function by the gaussian integral for $dA_{ij}$ with preceding equations and relations. The partition function is

$$ Z_N = \frac{1}{N!} \sum_{\{P\}} (-1)^P \times \int \prod_i d\lambda_i \exp \left\{ -N \sum_i \int_0^\beta dt \left( \frac{1}{2} \dot{\lambda}_i^2 + V(\lambda_i) \right) \right\} \times \sum_R d_R \text{Tr}_R \left[ \hat{T} \exp \left( \frac{1}{4N} \int_0^\beta dt \sum_{i,j} \frac{\tilde{R}_{ij} R_{ij}^R}{(\lambda_i - \lambda_j)^2} \right) \right] \mathcal{P} , \quad (35) $$

where $\frac{1}{N!} \sum_{\{P\}} (-1)^P$ is the standard anti-
symmetrizer, the skew-symmetry of the Vandermonde determinant,

$$ \Delta(\mathcal{P}\lambda(0)|\mathcal{P}^{-1}) = (-1)^P \Delta(\lambda(0)). \quad (36) $$

Thus we have the Hamiltonian as follows:

$$ H_R = P_R \sum_i \left\{ -\frac{1}{2N} \frac{\partial^2}{\partial \lambda_i^2} + NV(\lambda_i) \right\} + \frac{1}{4N} \sum_{i,j} \frac{\tilde{R}_{ij} R_{ij}^R}{(\lambda_i - \lambda_j)^2} , \quad (37) $$

C. Non-singlet sector and T-duality

1. non-singlet terms from $U(N)$ generators

As the previous case, the effective Lagrangian, $L_{eff}$, for $H$ is given by the Legendre transformation, and by the gaussian integration for the $[d\Pi_{\lambda}]$ in following partition function :

$$ Z_N(b_n) = \int [d\Pi_{\lambda}] [d\chi_{ij}] [d\tilde{\chi}_{ij}] \times \exp \left[ i \int dt \left( \sum_i \Pi_{\lambda_i} \dot{\lambda}_i + \sum_{ij} (\Pi_{\chi_{ij}} \dot{\chi}_{ij} + \Pi_{\tilde{\chi}_{ij}} \dot{\tilde{\chi}}_{ij}) - H \right) \right] = \int [d\lambda] [d\chi_1] [d\chi_2] \exp \left( i \int dt L_{eff} \right) . \quad (38) $$

However, above equation is not a correct form because the Hamiltonian does not contain the non-singlet terms from the fermionic matrices involved in the Lagrangian.

Earlier works on the adjoint state(non-singlet state) of the supersymmetric matrix model can be found in Refs. [11] [12].

Now, we will consider the degrees of freedoms in the non-singlet terms and the angular terms. Let us decompose $A(t)$ into generators belonging to $U(N)$ as follows:

$$ A = \hat{U} U^\dagger = \sum_{i=1}^{N-1} \hat{\alpha}_i C_i + \frac{i}{\sqrt{2}} \sum_{i \neq j} \left( \tilde{\beta}_{ij} T_{ij} + \hat{\beta}_{ij} \hat{T}_{ij} \right) , \quad (39) $$

where

$$ (T_{ij})_{kl} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} , \quad (\hat{T}_{ij})_{kl} = -i(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \quad (40) $$

and $C_i$ is the Cartan subalgebra. If we insert Eq.(39) to Eq.(25), we get

$$ L = \sum_{i=1}^N \left( \frac{1}{2} \dot{\chi}_{ii}^2 - V \right) + \frac{1}{2} \sum_{i \neq j} \left( \lambda_i - \lambda_j \right)^2 (\hat{\beta}_{ij}^2 + \hat{\beta}_{ij}^2) . \quad (41) $$

The canonical relations,

$$ \Pi_{\lambda_i} = \frac{\partial L}{\partial \dot{\lambda}_i} , \quad \Pi_{ij} = \frac{\partial L}{\partial \dot{\beta}_{ij}}, \quad \tilde{\Pi}_{ij} = \frac{\partial L}{\partial \dot{\beta}_{ij}} \quad (42) $$

and the constraint,

$$ \Pi_{\alpha_i} = \frac{\partial L}{\partial \dot{\alpha}_i} = 0 \quad (43) $$

with the Legendre transformation,

$$ H(q_{ij}, \Pi_{ij}, \Pi_{i\alpha}) = \sum_{I} \Pi_{Iij} \dot{q}_{ij} - L(q_{ij}, \dot{q}_{ij}, t) \quad (44) $$
give us the following Hamiltonian,
\[ H = \sum_{i} \left( \frac{1}{2} \Pi_{i}^{2} + V(\lambda) \right) + \frac{1}{2} \sum_{i \neq j} \frac{\Pi_{ij}^{2} + \Pi_{ij}^{2}}{(\lambda_{i} - \lambda_{j})^2}. \] (45)

The second part in Eq. (45) is the interaction terms from the non-singlet sector. In the coordinate representation the momentum is realized as the following operator
\[ \Pi_{\lambda_{i}} = -i\hbar \frac{\partial}{\Delta(\lambda)} \Delta(\lambda). \] (46)

Therefore, Eq. (45) becomes
\[ H = \sum_{i} \left[ -\frac{\hbar^2}{2\Delta(\lambda)} \frac{\partial^2}{\partial \lambda_{i}^2} \Delta(\lambda) + V(\lambda) \right] + \frac{1}{2} \sum_{i \neq j} \frac{\Pi_{ij}^{2} + \Pi_{ij}^{2}}{(\lambda_{i} - \lambda_{j})^2}. \] (47)

Using the Hamiltonian derived in the previous paragraph, the partition function becomes
\[ Z_{N} = \text{Tr} e^{-\frac{\pi}{\hbar} H}, \] (48)
where \( t \) is the time interval we are interested in.

2. T-duality and neglecting of non-singlet sector

In this section, we will consider the relation of the non-singlet terms and the T-duality. Details can be found in Refs. 4, 15, 16. The 2-dimensional target space has a time direction and a spatial one. This spatial dimension is from the Liouville modes so the spatial direction can not be compactified. Thus we consider the time direction of the target space for the compactification.

Firstly, let us consider a non-compactified target space. In this case, we have an infinite real line for the time direction. The free energy of this target space is sufficient to show that we need only the ground state energy, \( E_{0} \), as follows:
\[ F = \lim_{t \to \infty} \frac{\log Z_{N}}{t} = -\frac{E_{0}}{\hbar}. \] (49)

Above minimum value of \( F \) is caused by fact that the angular modes from the non-singlet terms can be decoupled. This decoupling arises from fact that the non-singlet terms are positive definite operators. So, we can consider following Hamiltonian of the singlet sector which is independent of the angular degrees of freedom which is related to the non-singlet terms,
\[ H_{\text{singlet}} = \sum_{i} \left( \frac{1}{2} \Pi_{i}^{2} + V(\lambda) \right). \] (50)

Now, let us consider a compactified time direction of the target space as follows:
\[ t \sim t + \beta, \quad \beta = 2\pi R, \] (51)

where \( R \) is the radius of the compactified target space. In this case, we can not decouple the non-singlet terms because the compactification gives rise to new winding modes which are represented by the non-singlet terms mathematically. These winding modes are related to the vortices and the anti-vortices. The vortices and the anti-vortices bring about the vortex-anti-vortex condensation which is related to the Kosterlitz-Thouless phase transition at critical radius, \( R_{c} \). However, this phase transition violates the T-duality of the target space.

When the radius of the target space is big enough to be \( R > R_{c} \), we can suppress the phase transition. This means that we can decouple and truncate the non-singlet terms under restriction of \( R > R_{c} \). The decoupling is caused by following relation,
\[ E_{\text{non-singlet}} - E_{\text{singlet}} \sim \frac{\beta}{2\pi} |\ln \delta|, \] (52)
where \( \delta \to 0 \) as \( R \to \infty \). Thus we can truncate the non-singlet sector because the energy gap between \( E_{\text{non-singlet}} \) and \( E_{\text{singlet}} \) diverges to the infinity.

When the radius is small such that \( R < R_{c} \), one can resort to the lattice gauge theory. In this case, we can also suppress the vortices and anti-vortices. However, the lattice gauge theory transforms the circle of radius \( R \) into the 1-dimensional lattice. This modification of the target space is a restrictive condition.

III. A NEW METHOD FOR NON-SINGLET SECTOR IN SUPERSYMMETRIC MATRIX QUANTUM MECHANICS

Till now, we reviewed the supersymmetric matrix quantum mechanics and the bosonic matrix with time periodicity. We also reviewed the compactification of the target space and reviewed the non-singlet sector corresponding to the vortices which violate the T-duality. The compactification gives us the time periodicity.

The non-singlet terms from the angular operators and from the time periodicity violate the T-duality of the target space. However, when we want to maintain the T-duality, we must eliminate, or ignore at least, the non-singlet terms. We have seen, however, that we can only ignore the non-singlet terms under some restrictions. The first method is to use truncation with large difference of energy gap between singlet states and non-singlet states. The second method is to use the lattice gauge theory. However, this lattice gauge theory transforms the circle of the target space into the 1-dimensional lattice.

We will introduce a new method for the non-singlet terms. Our new suggestion is not to ignore the non-singlet terms but to eliminate of the non-singlet terms.

Firstly, let us look into the trivial singlet case of the fermionic sector. We consider the diagonalized fermionic matrices by the unitary transformation such as
\[ \Psi_{ij}(t) = \sum_{k} v_{ik}^{\dagger}(t) \chi_{k}(t) U_{kj}(t). \] (53)
The kinetic energy parts of Eq. (53) become as follows:

\[ \sum_{ij} \Psi_{ij} \dot{\Psi}_{ji} = \sum_{ij} \sum_{kl} U_{ik}^\dagger \chi_k U_{kj} \cdot \frac{\partial}{\partial \chi} \left( U_{ij}^\dagger \chi_l U_{li} \right) = \sum_{ij} \chi_k \dot{\chi}_k + 2 \sum_{ik} \chi_k \dot{U}_{ki} U_{ik}^\dagger = \sum_{ij} \chi_k \dot{\chi}_k \]

where \( A_k(t) = (\dot{U}(t)U^\dagger(t))_{kk} \) and \( \chi_k^2 = \chi_k \dot{\chi}_k = 0 \) because of the Pauli exclusion principle. Thus, in this case we have no non-trivial interaction terms for the fermionic parts. Next, we will investigate the non-singlet terms of the fermionic parts in following subsection.

A. Fermions in non-singlet sector

Now, we consider the off-diagonal elements of the fermions in non-singlet sector as follows:

\[ \Psi_{ij}(t) = \sum_{kl} U_{ik}^\dagger(t) \chi_{kl}(t) U_{lj}(t). \]  \hspace{1cm} (55)

The kinetic energy parts of Eq. (55) become such as

\[ \sum_{ij} \Psi_{ij} \dot{\Psi}_{ji} = \sum_{ij} \sum_{kl,mn} U_{ik}^\dagger \chi_{k} U_{lj} \cdot \frac{\partial}{\partial \chi} \left( U_{jm}^\dagger \chi_{mn} U_{ni} \right) = \sum_{kl} \chi_{kl} \dot{\chi}_{kl} + 2 \sum_{k \neq l} \chi_{kl} \chi_{ln} \dot{U}_{ki} U_{ik}^\dagger = \sum_{kl} \chi_{kl} \dot{\chi}_{kl} + 2 \sum_{k \neq l} \chi_{kl} \chi_{ln} A_{nk}. \]  \hspace{1cm} (56)

Thus, we have the non-trivial interaction terms from \( \chi_{kl} \chi_{ln} A_{nk} \) for \( k \neq l \neq n \). Note that the non-trivial terms have \( A_{ij} \). We will use this fact in the gaussian integral for \( A_{ij} \) later. Let us write the terms which have \( A_{ij} \), as follows:

\[ \frac{1}{2} \text{Tr} M^2 = \frac{1}{2} \sum_{i} \lambda_i^2 + \frac{1}{2} \sum_{ij} (\lambda_i - \lambda_j)^2 |A_{ij}|^2, \] \hspace{1cm} (57)

and

\[ -i \sum_{l} \sum_{ij} \Psi_{lj} \dot{\Psi}_{ji} \]

\[ = -i \sum_{l} \sum_{ij} \chi_{lj} \dot{\chi}_{ji} - i \sum_{ij} \sum_{k \neq j} \chi_{ik} \chi_{kj} A_{ij}. \] \hspace{1cm} (58)

If we use Eq. (57) and Eq. (58), the previous Lagrangian of Eq. (6) becomes

\[ L = \frac{1}{2} \left\{ \sum_{i} \lambda_i^2 + \sum_{ij} (\lambda_i - \lambda_j)^2 |A_{ij}|^2 \right\} - i \sum_{ij} \sum_{k \neq j} \chi_{ik} \chi_{kj} A_{ij} \]

\[ - \sum_{ij} \sum_{k \neq j} (\chi_{1ij} \chi_{1ji} + \chi_{2ij} \chi_{2ji}) + \frac{1}{2} \sum_{i \neq j} \left( \sum_{k} \lambda_k \chi_{ik} \chi_{kj} + \chi_{2ik} \chi_{2kj} \right)^2 \]  \hspace{1cm} (59)

where those potential terms remain off the unitary transformation.

B. Non-singlet terms from time periodicity

In the section II.2 considered the periodic time condition on the bosonic matrix case. Here we extend this time periodicity to the supersymmetric matrix model. However, in the supersymmetric case, using the gaussian integral for the \( A_{ij} \) with the periodic time condition, we have the same result of bosonic model. The same process in the section II.2 gives us following Lagrangian, instead of Eq. (59),

\[ L_{eff} = \frac{1}{2} \sum_{i} \lambda_i^2 - i \sum_{ij} \left( \chi_{1ij} \dot{\chi}_{1ji} + \chi_{2ij} \dot{\chi}_{2ji} \right) \]

\[ - \frac{1}{2} \sum_{i} \sum_{mn=1}^{K} mnb_n \lambda_i^{m+n-2} \]

\[ - i \sum_{ij} \sum_{n=2}^{K} n(n-1)b_n \chi_{1ij} \lambda_i^{n-2}\chi_{2ji} \] \hspace{1cm} (56)

\[ - i \sum_{m=2}^{K} \sum_{i \neq j} \sum_{n=2}^{m} mnb_m \chi_{1ij} \lambda_i^{m-n} \lambda_j^{n-2} \chi_{2ji} \]

\[ + \frac{1}{2} \sum_{i \neq j} \left( \sum_{R} \chi_{1ij} \chi_{1kj} + \chi_{2ik} \chi_{2kj} \right)^2 \]  \hspace{1cm} (60)

where the potential part is rewritten in forms of the matrix elements after unitary transformation and differentiation.

Consequently, we conclude that the non-singlet terms from the time periodicity and the non-singlet terms for fermions give us same effects. If we use the complex formation for the fermions like Eq. (10),

\[ \chi_{ij} = \frac{1}{\sqrt{2}} \left( \chi_{1ij} + i \chi_{2ij} \right) \]

\[ \dot{\chi}_{ij} = \frac{1}{\sqrt{2}} \left( \chi_{1ij} - i \chi_{2ij} \right) \] \hspace{1cm} (61)
then we have

\[ L_{\text{eff}} = \frac{1}{2} \sum_{i}^{N} \dot{\lambda}_{i}^{2} - \frac{i}{2} \sum_{ij}^{N} (\chi_{ij} \dot{\bar{x}}_{ji} + \bar{x}_{ij} \dot{\chi}_{ji}) \]

\[ - \frac{1}{2} \sum_{i}^{N} \sum_{m=1}^{K} mnb_{m} b_{n} \lambda_{i}^{m+n-2} \]

\[ - \frac{1}{2} \sum_{ij}^{N} n(n-1) b_{n} \lambda_{j}^{n-2} (\bar{x}_{ij} \chi_{ji} - \chi_{ij} \bar{x}_{ji}) \]

\[ - \frac{1}{2} \sum_{m=2}^{K} \sum_{n=2}^{N} mnb_{m} \lambda_{i}^{m-n} \lambda_{j}^{n-2} (\bar{x}_{ij} \chi_{ji} - \chi_{ij} \bar{x}_{ji}) \]

\[ + \frac{1}{2} \sum_{i \neq k \neq j} (\sum_{\mathcal{R}} \dot{\gamma}_{ij}^{\mathcal{R}} + \chi_{ik} \bar{x}_{kj} + \bar{x}_{ik} \chi_{kj}) \quad (\lambda_{i} - \lambda_{j})^{2} \]

\[ (62) \]

Now, let us investigate a Hamiltonian for above the Lagrangian.

C. Hamiltonian with fermionic non-singlet terms

The two ways which ignore the non-singlet terms, have the restricted conditions respectively. How do we get a more constructive and complete method for a eliminating the non-singlet terms which are related to the vortices? A possible answer is a using of the off-diagonal elements of the fermionic matrices in the supersymmetric matrix model.

Now, we have three types of the non-singlet terms. The first type is \( \Pi_{ij} \), the second is \( \bar{\tau}_{ij} \) and the third is \( \chi_{ik} \bar{x}_{kj} \) as follows:

1. The \( \Pi_{ij} \) is from the angular variable such as the \( A_{ij} = (\hat{U} U^\dagger)_{ij} \).

2. The \( \bar{\tau}_{ij} \) is from the time periodicity.

3. The \( \chi_{ik} \bar{x}_{kj} \) is from the non-diagonal elements of the fermions.

Let us consider following Lagrangian for a final Hamiltonian.

\[ L = \frac{1}{2} \sum_{i}^{N} \dot{\lambda}_{i}^{2} + \frac{1}{2} \sum_{i \neq j} (\lambda_{i} - \lambda_{j})^{2} |A_{ij}|^{2} - V(\lambda, \chi, \bar{\chi}) \]

\[ - \frac{i}{2} \sum_{ij}^{N} (\chi_{ij} \dot{\bar{x}}_{ji} + \bar{x}_{ij} \dot{\chi}_{ji}) \]

\[ - i \sum_{i \neq k \neq j} (\sum_{\mathcal{R}} \dot{\gamma}_{ij}^{\mathcal{R}} + \chi_{ik} \bar{x}_{kj} + \bar{x}_{ik} \chi_{kj}) A_{ji}, \]

\[ (63) \]

where the potential, \( V(\lambda, \chi, \bar{\chi}) \), is

\[ V(\lambda, \chi, \bar{\chi}) = \frac{1}{2} \sum_{i}^{N} \sum_{m=1}^{K} mnb_{m} b_{n} \lambda_{i}^{m+n-2} \]

\[ + \frac{1}{2} \sum_{ij}^{N} n(n-1) b_{n} \lambda_{j}^{n-2} (\bar{x}_{ij} \chi_{ji} - \chi_{ij} \bar{x}_{ji}) \]

\[ + \frac{1}{2} \sum_{m=2}^{K} \sum_{n=2}^{N} mnb_{m} \lambda_{i}^{m-n} \lambda_{j}^{n-2} (\bar{x}_{ij} \chi_{ji} - \chi_{ij} \bar{x}_{ji}). \]

\[ (64) \]

As previous case, let us decompose \( A(t) \) into generators belonging to \( U(N) \) as follows:

\[ A = \hat{U} U^\dagger = \sum_{i=1}^{N-1} \dot{\alpha} C_{i} + i \sqrt{2} \sum_{i \neq j} (\dot{\beta}_{ij} T_{ij} + \dot{\bar{T}}_{ij} \bar{T}_{ij}), \]

\[ (65) \]

where

\[ (T_{ij})_{kl} = \delta_{i k} \delta_{j l} + \delta_{i l} \delta_{j k}, \quad (\bar{T}_{ij})_{kl} = -i (\delta_{i k} \delta_{j l} - \delta_{i l} \delta_{j k}) \]

and \( C_{i} \) is Cartan subalgebra. In Eq.(63), the summation conditions, \( \sum_{i \neq j} \) and \( \sum_{i \neq k \neq j} \), make following constraint,

\[ \Pi_{n_{i}} = \frac{\partial L}{\partial \dot{\alpha}_{i}} = 0, \]

\[ (67) \]

thus we have

\[ A_{ij} = (\hat{U} U^\dagger)_{ij} = i \sqrt{2} \sum_{i \neq j} (\beta_{ij} T_{ij} + \bar{T}_{ij} \bar{T}_{ij}). \]

\[ (68) \]

Now, the description of Eq.(68) is comparatively complicated. Since the term \( A_{ij} \) which definite form is \( A_{ij}(t) = (\hat{U}(t) U^\dagger(t))_{ij} \), has time derivative part, we can redefine the \( A_{ij} \) as follows:

\[ A_{ij} \equiv \dot{\gamma}_{ij}, \]

\[ (69) \]

then we have following canonical relations instead of Eq.(42),

\[ \Pi_{\lambda_{i}} = \frac{\partial L}{\partial \dot{\lambda}_{i}}, \quad \Pi_{\gamma_{ij}} = \frac{\partial L}{\partial \dot{\gamma}_{ji}}. \]

\[ (70) \]

Comparing Eq.(42) to Eq.(70) we have following relation,

\[ \Pi_{\gamma_{ij}}^{2} = \Pi_{ij}^{2} + \Pi_{ij}^{2}. \]

\[ (71) \]

From now on, we relabel \( \Pi_{\gamma_{ij}} \) as follows:

\[ \Pi_{\gamma_{ij}} = \Pi_{ij}. \]

\[ (72) \]
Now, we have a following Lagrangian,

\[
L = \frac{1}{2} \sum_{i} \dot{\lambda}_{i}^{2} + \frac{1}{2} \sum_{i \neq j} (\lambda_{i} - \lambda_{j})^{2} |\dot{\gamma}_{ij}|^{2} - V(\lambda, \chi, \bar{\chi}) - \frac{i}{2} \sum_{i} \dot{\chi}_{ij} \hat{\chi}_{ji} - i \sum_{i \neq j} (\hat{\tau}_{ij} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj}) \dot{\gamma}_{ij} \tag{73}
\]

and the canonical relations as follows:

\[
\Pi_{\lambda_{i}} = \frac{\partial L}{\partial \dot{\lambda}_{i}} = \dot{\lambda}_{i},
\]

\[
\Pi_{\chi_{ij}} = \frac{\partial L}{\partial \dot{\chi}_{ij}} = -\frac{i}{2} \hat{\chi}_{ij},
\]

\[
\hat{\Pi}_{ij} = \frac{\partial L}{\partial \dot{\chi}_{ij}} = (\lambda_{i} - \lambda_{j})^{2} \dot{\gamma}_{ij} - i \sum_{k \neq j} (\hat{\tau}_{ij} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj}).
\]

Thus, we have following description for \( \dot{\gamma} \),

\[
\dot{\gamma}_{ij} = \frac{\hat{\Pi}_{ij} + i(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj})}{(\lambda_{i} - \lambda_{j})^{2}}. \tag{75}
\]

Inserting Eq.(75) into Eq.(73) then we have following Lagrangian

\[
L = \sum_{i} \frac{1}{2} \dot{\lambda}_{i}^{2} - V(\lambda, \chi, \bar{\chi}) - \frac{i}{2} \sum_{i} \dot{\chi}_{ij} \hat{\chi}_{ji} + \frac{1}{2} \sum_{i \neq j} \frac{\hat{\Pi}_{ij} \hat{\Pi}_{ji}}{(\lambda_{i} - \lambda_{j})^{2}} + \frac{1}{2} \sum_{i \neq j} \frac{(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj})^{2}}{(\lambda_{i} - \lambda_{j})^{2}} + \frac{i}{2} \sum_{i \neq j} \frac{\hat{\Pi}_{ij}(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj})}{(\lambda_{i} - \lambda_{j})^{2}} - \frac{i}{2} \sum_{i \neq j} \frac{(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj}) \hat{\Pi}_{ji}}{(\lambda_{i} - \lambda_{j})^{2}} \tag{76}
\]

With above Lagrangian, the canonical relations of Eq.(74), and the Legendre transformation,

\[
H(q_{\lambda_{i}}, \Pi_{\chi_{ij}}, t) = \sum_{i} \Pi_{\lambda_{i}} \dot{\lambda}_{i} - L(q_{\lambda_{i}}, \dot{q}_{\lambda_{i}}, t), \tag{77}
\]

where

\[
\sum_{i} \Pi_{\lambda_{i}} \dot{\lambda}_{i} = \Pi_{\lambda_{i}} \dot{\lambda}_{i} + \Pi_{\chi_{ij}} \dot{\chi}_{ji} + \Pi_{\bar{\chi}_{ij}} \dot{\bar{\chi}}_{ji} + \hat{\Pi}_{ij} \dot{\gamma}_{ji}. \tag{78}
\]

we have a extended Hamiltonian in the supersymmetric and periodic time (compactified target space) case,

\[
H = \sum_{i} \left( \Pi_{\lambda_{i}} \dot{\lambda}_{i} + \Pi_{\chi_{ij}} \dot{\chi}_{ji} + \Pi_{\bar{\chi}_{ij}} \dot{\bar{\chi}}_{ji} + \hat{\Pi}_{ij} \dot{\gamma}_{ji} \right) - L \tag{79}
\]

Inserting the Lagrangian of Eq.(76), into above Eq.(79), we have following result,

\[
H = \frac{1}{2} \sum_{i} \Pi_{\lambda_{i}}^{2} + V(\lambda, \chi, \bar{\chi}) + \frac{1}{2} \sum_{i \neq j} \frac{\hat{\Pi}_{ij} \hat{\Pi}_{ji}}{(\lambda_{i} - \lambda_{j})^{2}} + \frac{1}{2} \sum_{i \neq j} \frac{(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj})^{2}}{(\lambda_{i} - \lambda_{j})^{2}} + \frac{i}{2} \sum_{i \neq j} \frac{\hat{\Pi}_{ij}(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj})}{(\lambda_{i} - \lambda_{j})^{2}} \tag{80}
\]

At last, with some calculation and rearrangement of the terms in above equation, we arrive at this final Hamiltonian form,

\[
H = \frac{1}{2} \sum_{i} \Pi_{\lambda_{i}}^{2} + V(\lambda, \chi, \bar{\chi}) + \frac{1}{2} \sum_{i \neq j} \frac{\hat{\Pi}_{ij} \hat{\Pi}_{ji}}{(\lambda_{i} - \lambda_{j})^{2}} + \frac{i}{2} \sum_{i \neq j} \frac{(\sum_{k \neq j} \hat{\tau}_{ik} + \chi_{ik} \bar{\chi}_{kj} + \bar{\chi}_{ik} \chi_{kj})^{2}}{(\lambda_{i} - \lambda_{j})^{2}} \tag{81}
\]

Notice that the last four terms of Eq.(80) have been collected into a perfect square and the Hamiltonian is simplified. It is rather remarkable that the fermionic non-singlet terms and the non-singlet terms from the time periodicity and the angular variable conspire to give a simple form.

Here, the terms of \( \hat{\Pi}_{ij} \) and \( \hat{\tau}_{ij} \) are not controlled by us since they are given from the structure of the matrix model and some mathematical conditions. For example, the \( \hat{\Pi}_{ij} \) is from the angular variable of the \( A_{ij} \) and the unitary transformation. Similarly, the \( \hat{\tau}_{ij} \) is from the time periodicity. We can’t change these restricted conditions arbitrarily. However, we can control and vary the terms of \( \chi \) and \( \bar{\chi} \) since they are from the superfields of Eq.(1), which are introduced by us.

By the way, the non-singlet terms which are made of the \( \hat{\Pi}_{ij} \) and/or \( \hat{\tau}_{ij} \), violate the T-duality of the target space\[4][13][14][16]. But, if we want to maintain the T-duality then we must suppress the non-singlet terms. Really, we would like to retain the T-duality because that
Therefore we can always retain the T-duality on the target space without any constraint.

The target space of the 1-dimensional (time dimension) matrix model related to the non-critical 2-dimensional string theory, has three structures such as the infinite real line, the infinite 1-dimensional lattice and the circle of radius $R$. The target space can be represented by discretised random surfaces and have a dual structure of the fat Feynman graphs. If we consider the non-singlet sector in the matrix model then we can read the non-singlet terms into vortex or anti-vortex terms. Also these non-singlet terms correspond to the winding modes of the strings.

However, in the continuum limit and the double scaling limit on the random surfaces, the Kosterlitz-Thouless phase transition through the vortex-anti-vortex condensation, which violate the T-duality of the target space. In the case such that the target space is infinite line or infinite 1-dimensional lattice, we need not to consider the non-singlet terms since we need only ground state energy. Since the non-singlet terms are positive definite operators, corresponding excitation state energy is always above the ground state one. This excitation states occur where the target space is a circle of radius $R$. In general, the compact target space with radius $R$ has T-duality. But the target space in our case is composed of discretised random surfaces. Therefore we have vortex or anti-vortex terms on the surfaces. The non-singlet terms corresponding to the vortices and anti-vortices which violate the T-duality of the target space. Up to now, we have two old ways to exclude the violation of the T-duality. Firstly, in the continuum limit of discretised surfaces, we are able to truncate the vortex terms since the energy gap between ground state and excitation state diverges. This method corresponds to the infinite limit of the radius $R$ of the target space. Secondly, using the lattice gauge theory, we are able to read the circle into the 1-dimensional lattice. However these two ways are in restrictive conditions respectively.

In this paper we showed new non-singlet terms from the non-diagonal elements of the fermionic matrices in the Lagrangian. These new non-singlet terms can completely eliminate the old non-singlet terms which violate the T-duality so that we can retain the T-duality on the target space which is composed of the discretised random surfaces. What is more, we are also able to control the phase transition effect with these new non-singlet terms which are composed of $\chi$ and $\bar{\chi}$ instead of the elimination.

### IV. DISCUSSION

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