The Hubble Expansion is Isotropic in the Epoch of Dark Energy

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ABSTRACT

The isotropy of the universal Hubble expansion is a fundamental tenet of physical cosmology, but it has not been precisely tested during the current epoch, when dark energy is dominant. Anisotropic expansion will produce a shearing velocity field, causing objects to stream toward directions of faster expansion and away from directions of slower expansion. This work tests the basic cosmological assumption of isotropic expansion and thus the isotropy of dark energy. The simplest anisotropy will manifest as a quadrupolar curl-free proper motion vector field. We derive this theoretical signature using a tri-axial expanding metric with a flat geometry (Bianchi I model), generalizing and correcting previous work. We then employ the best current data, the Titov & Lambert (2013) proper motion catalog of 429 objects, to measure the isotropy of universal expansion. We demonstrate that the Hubble expansion is isotropic to 7% (1σ), corresponding to streaming motions of 1 microarcsecond yr\(^{-1}\), in the best-constrained directions (−19% and +17% in the least-constrained directions) and does not significantly deviate from isotropy in any direction. The Gaia mission, which is expected to obtain proper motions for 500,000 quasars, will likely constrain the anisotropy below 1%.

Key words: astrometry — cosmology: observations — cosmology: theory — cosmology: miscellaneous — dark energy — proper motions

1 INTRODUCTION

The isotropy of the cosmic expansion is well-constrained for the early universe, particularly by Cosmic Microwave Background observations, and is a basic tenet of physical cosmology. The change from a matter-dominated to a dark energy-dominated universe in recent times, however, raises the possibility of a dark energy-driven anisotropic expansion if dark energy is itself anisotropic. There is no obvious reason for such symmetry breaking, but observational tests of something as fundamental as the isotropy of the Hubble expansion should be made for late times (the current epoch).

One such test is possible via extragalactic proper motions: if the expansion is anisotropic, then quasars and galaxies will stream toward directions of faster expansion and away from directions of slower expansion. The signature of anisotropic expansion in a homogeneous universe is thus a curl-free proper motion vector field (to first order; Quercellini et al. 2009, Fontanini et al. 2009, Titov 2009).

The term “cosmic parallax” has been used by some to indicate a general relative angular motion of objects in the universe (e.g., Quercellini et al. 2009, Fontanini et al. 2009) and used by others in a more canonical sense to indicate apparent angular motion induced by the motion of the observer (e.g., Ding & Croft 2009). We favor the latter usage; this paper therefore treats the apparent proper motion induced by anisotropic cosmic expansion (Amendola et al. 2013), referenced to the International Celestial Reference Frame (ICRF) in the current epoch. The observed proper motions are therefore relative, but are not necessarily induced by the observer’s motion.

In this paper, we present a simple model of anisotropic expansion and fit the model to the Titov & Lambert (2013) proper motion catalog to place a new constraint on the isotropy of the Hubble expansion and thus on the isotropy of dark energy. We assume \( H_o = 72 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and a flat cosmology (this treatment is independent of specific assumptions about \( \Omega_\Lambda \) and \( \Omega_M \), provided \( \Omega_\Lambda + \Omega_M = 1 \)).

2 ANISOTROPIC EXPANSION MODEL

As described in Quercellini et al. (2009) and Fontanini et al. (2009), a homogeneous but anisotropic Bianchi I model with metric

\[
\text{ds}^2 = -dt^2 + a^2(t) \left( dx^2 + b^2(t) dy^2 + c^2(t) dz^2 \right)
\]
has three different expansion rates, $H_{x'} = \dot{a}/a$, $H_{y'} = b/b$, and $H_{z'} = c/c$, where the Hubble parameter as observed is $H = \frac{d}{dt}(abc)^{1/3}/(abc)^{1/3}$ and the Friedmann-Robertson-Walker metric is recovered for $a(t) = b(t) = c(t)$. This metric has a flat geometry and no global vorticity, but the anisotropic expansion will produce a shearing velocity field, causing objects to stream toward directions of faster expansion and away from directions of slower expansion. The shear can be characterized by the fractional deviation from the average Hubble expansion today ($t = t_0$),

$$\Sigma_{i'} = \frac{H_{i'} - H_0}{H_0} - 1,$$

where $i' = x'$, $y'$, or $z'$, isotropy corresponds to $\Sigma_{x'} = \Sigma_{y'} = \Sigma_{z'} = 0$ (no deviation from $H_0$ in any direction), and the expansion is “conserved”: $\Sigma_{x'} + \Sigma_{y'} + \Sigma_{z'} = 0$ (the directionless overall $H_0$ is preserved, despite anisotropy). Under the simplifying assumption of straight geodesics (incorrect, but a small error as demonstrated by Quercellini et al. 2009), the sky signal of an anisotropic expansion is a curl-free quadrupolar proper motion vector field that is independent of distance. Note that this is a “real-time” signal, meaning that the time derivatives are with respect to small coordinate time intervals today (decades) and therefore a constant $H_0$ is a good assumption.

The treatment here is therefore rotationally invariant (as desired), and it is therefore completely equivalent to fit the spherical $\ell = 2$ vector spherical harmonics to test the isotropy of expansion. It is not, however, correct to fit or select single or selected orders of a vector spherical harmonic model, as has been done previously. Table II lists the equivalent vector spherical harmonic coefficients (see Appendix A for the full equation), following the conventions:

$$V_{E2}(\alpha, \delta) = \sum_{m=0}^{\ell} \sum_{i=\alpha, \delta} \xi_{2m,i}^{Re,Im}(\alpha, \delta) \chi_{2m}^{Re,Im} \hat{e}_i.$$  (5)

Note that this equation has absorbed the factors of 2 for the $m > 0$ orders and the factors of $-1$ for the imaginary terms into the coefficients $\chi_{2m}^{Re,Im}$, in contrast to the definitions used by Mignard & Klioner (2012).

3 DATA ANALYSIS METHODS

We employ the Titov & Lambert (2013) proper motion measurements of 429 radio sources to examine the isotropy of the Hubble expansion. The data were obtained from sessions of the permanent geodetic and astrometric VLBI program, which includes the Very Long Baseline Array at 8.4 GHz in 1990–2013 using a relaxed per-session no-net rotation constraint and an iterative process to reject objects with large intrinsic or spurious proper motions (Titov & Lambert 2013). Objects in this catalog can show large intrinsic proper
Table 1. Anisotropy Model Coefficients

| $\epsilon_\alpha$ | $\epsilon_\delta$ | $\Sigma_{\psi'} + \frac{1}{2} \Sigma_{\psi''}$ | $\frac{1}{2} \Sigma_{\psi'}$ | $V_{E2}$ |
|----------------|----------------|-----------------|-----------------|---------|
| $\xi^{Re,Im}_{2m,\delta\alpha}(\alpha, \delta)$ | $\xi^{Re,Im}_{2m,\delta\delta}(\alpha, \delta)$ | $\xi^{Re,Im}_{2m,\psi'(\alpha, \delta', \psi')}$ | $\xi^{Re,Im}_{2m,\psi''(\alpha, \delta', \psi')}$ | $\xi^{Re,Im}_{2m,\psi''(\alpha, \delta', \psi')}$ |
| 0 | $\frac{3}{8} \sin 2\delta$ | $\cos 2\psi^*(1 + \cos 2\delta^*)$ | $-3 \cos \alpha^* \sin 2\delta^*$ | $s_{20}$ |
| $\frac{1}{2} \sin \alpha \sin \delta$ | $-\frac{1}{2} \cos \alpha \cos 2\delta^*$ | $2 \sin \alpha^* \cos \delta^* \sin 2\psi^*$ | $-3 \sin \alpha^* \cos 2\delta^*$ | $s_{21}$ |
| $\frac{1}{2} \cos \alpha \sin \delta$ | $\frac{1}{2} \sin \alpha \cos 2\delta^*$ | $2 \cos \alpha^* \cos \delta^* \sin 2\psi^*$ | $3 \sin \alpha^* \sin 2\delta^*$ | $s_{21}$ |
| $\frac{1}{4} \sin 2\alpha \cos \delta$ | $\frac{1}{4} \sin 2\alpha \sin 2\delta^*$ | $3 \cos 2\alpha^* \cos 2\psi^*$ | $-3 \cos 2\alpha^* (1 + \cos 2\delta^*)^*$ | $s_{22}$ |
| $\frac{1}{4} \cos 2\alpha \cos \delta$ | $-\frac{1}{4} \sin 2\alpha \sin 2\delta^*$ | $-3 \sin 2\alpha^* \cos 2\psi^*$ | $3 \sin 2\alpha^* (1 + \cos 2\delta^*)^*$ | $s_{22}$ |

Model coefficients and angular terms corresponding to the terms in Equations (4) and (5). See Equation (A1) for the full shear vector field and Equation (A2) for the full spheroidal quadrupolar vector field.

motions due to plasmoid ejection in jets or due to core shift effects, but these motions are uncorrelated between objects; they simply add intrinsic proper motion noise to any correlated global signals.

Titov et al. (2011) first detected the secular aberration drift quasar proper motion signature induced by the barycenter acceleration about the Galactic Center, which was later confirmed by Xu et al. (2012) and refined by Titov & Lambert (2013). The signature is an E-mode (curl-free) proper motion dipole with apex at the Galactic Center. In order to measure or constrain the E-mode quadrupolar anisotropy signal, we subtract the dipole proper motion pattern from the observed proper motion vector field, but employ the Reid et al. (2014) results obtained from trigonometric parallaxes and proper motions of masers associated with young massive stars: they obtain a Galactic Center distance of $R_0 = 8.34 \pm 0.16$ kpc and a rotation speed at $R_0$ of $\Theta_0 = 240 \pm 8$ km s$^{-1}$. Since the relevant quantity for aberration drift is the solar acceleration about the Galactic Center, we use the actual solar orbital motion that includes the solar motion in the direction of the Galactic rotation, $V_0 = 255.2 \pm 5.1$ km s$^{-1}$ (Reid et al. 2014), which yields an acceleration of $0.80 \pm 0.04$ cm s$^{-2}$ yr$^{-1}$ and a dipole amplitude of $5.5 \pm 0.2$ mas yr$^{-1}$. This dipole in the Titov & Lambert (2013) notation is $\tilde{d} = (d_1, d_2, d_3) = (-0.300 \pm 0.013, -4.80 \pm 0.21, -2.66 \pm 0.12)$ mas yr$^{-1}$; in the Mignard & Klienhon (2012) notation, it is $(s_{10}, s_{11}, s_{1m}) = (-7.71 \pm 0.34, +0.615 \pm 0.027, -9.82 \pm 0.44)$ mas yr$^{-1}$. We assume that the acceleration direction is exactly toward the Galactic Center and do not include the out-of-the-disk acceleration described by Xu et al. (2012) in our correction (following Darlin (2013)). Titov & Lambert (2013) do not confirm the acceleration detected by Xu et al. (2012).

The derived dipole has substantially smaller errors than the Titov & Lambert (2013) measurement, and when we subtract this dipole from the vector field, we introduce negligible statistical errors compared to the proper motion uncertainty in individual objects. Although dipole and quadrupole signals are in principle orthogonal, covariance between different-degree vector spherical harmonics does exist (e.g., Titov & Malkin 2009, Titov & Lambert 2013), so subtraction of the best-measured dipole — using completely independent observations — is appropriate before measuring the E-mode quadrupole.

After subtracting the dipole signal from the quasar proper motion catalog, we perform a least-squares minimization fit of the observed proper motions to the anisotropy model described by Equations (4) and (A1) and Table 1. The free parameters are the rotation angles between the equatorial reference frame and the anisotropy frame, $\alpha^*$, $\delta^*$, and $\psi^*$, and two of the shear parameters describing the anisotropy, $\Sigma_{\psi'}$ and $\Sigma_{\psi''}$. The third shear parameter, $\Sigma_{\psi''}$, is determined by the expansion conservation condition (Section 2). The Hubble constant is assumed. Unless genuine significant anisotropy is detected, this model does not constrain $H_0$, which acts as a scaling amplitude for the streaming proper motions ($H_0 = 15.2$ mas yr$^{-1}$). We also fit the spheroidal quadrupole vector spherical harmonics for comparison.

4 RESULTS

Table 2 shows the measured anisotropy based on the least-squares fitting of the proper motion catalog to the anisotropy model. Table 3 shows the more general vector spherical harmonic parameters for a spheroidal quadrupole proper motion vector field. None of the shear parameters nor the quadrupole vector spherical harmonic coefficients are significant; all are consistent with zero, indicating an isotropic Hubble expansion. The largest deviation from isotropy is $-19\% \pm 7\%$, and the largest positive deviation is $+17\% \pm 7\%$ (neither significant). The smallest deviation (and thus the best anisotropy constraint) is $+2\% \pm 7\%$. The anisotropy of the Hubble expansion in the epoch of dark energy is thus less than 7% ($1\sigma$) in the best-constrained direction.

Figure 1 shows the model fit to the proper motion vector field for the 429 objects in the Titov & Lambert (2013) catalog. Positive deviation from the Hubble expansion ($\Sigma_{\psi'} > 0$) appears as an antipodal pair of convergent points, and negative deviation appears as an antipodal pair of divergent points. The equatorial coordinates of the best-fit
Table 2. Measured Expansion Anisotropy

| $\Sigma_x$ | $\Sigma_y$ | $\Sigma_z$ | $\alpha^*(\circ)$ | $\delta^*(\circ)$ | $\psi^*(\circ)$ |
|------------|------------|------------|-------------------|-------------------|-----------------|
| 0.17(7)    | −0.19(7)   | 0.02(7)    | 193(15)           | 47(26)            | −2(21)          |

The expansion shear terms $\Sigma$ indicate the fractional departure from the average Hubble expansion (Equation 2). The $\hat{z}'$ axis lies in the ($\alpha^*,\delta^*$) direction, and the $y'$ and $z'$ axes are rotated about the $\hat{z}'$ axis by $\psi^*$; their coordinates are listed generally in Equations (21) and (22), and as measured in Section 4. Parenthetical values are 1$\sigma$ uncertainties on the final digit(s).

Table 3. Measured Spheroidal Quadrupole

| $s_{20}$ | $d_{21}^{Re}$ | $d_{21}^{Im}$ | $d_{22}^{Re}$ | $d_{22}^{Im}$ | $\sqrt{\chi^2}$ |
|----------|---------------|---------------|---------------|---------------|-----------------|
| 3.0(2.5) | 1.7(1.4)      | −0.5(1.7)     | 3.1(1.4)      | −1.4(1.3)     | 6.2(2.1)        |

The quadrupolar ($\ell = 2$) spheroidal vector spherical harmonic coefficients $s_{2m}$ and the total power in the $\ell = 2$ curl-free order $P_2$ follow the Mignard & Klioner (2012) conventions. Parenthetical values are 1$\sigma$ uncertainties. The units are $\mu$as yr$^{-1}$.

Figure 1 shows a remarkable (but non-significant) alignment of the fit anisotropy axes with the Galactic plane-equator intersection. This could be driven by a vertical spheroidal dipole component induced by vertical (out-of-the-plane) solar acceleration (a second aberration drift; Xu et al. 2012). A dipole fit to the proper motion vector field (after subtracting the galactocentric acceleration described in Section 3) is not significant and does not point toward the $\hat{x}'$ axis (the points of convergence). Since the anisotropy is neither significant in total power nor in individual parameters, this surprising alignment seems to be coincidental and should not be over-interpreted.

The main limitation to this technique is the proper motion precision and sample size. The overall proper motion precision of individual objects will improve with time, as will the sample size of ICRF “defining sources,” but these will be slow, secular improvements. The next order-of-magnitude improvement will be provided by the Gaia mission. Gaia is an optical astrometry mission that will measure 500,000 quasar proper motions with $\sim 80$ mas astrometry for $V = 18$ mag stars (de Bruijne et al. 2005). Unlike radio sources, compact optical extragalactic sources do not show significant internal intrinsic proper motions, so Gaia proper motion catalogs will not exhibit the uncorrelated proper motion signals that contaminate radio measurements. We estimate that the Gaia mission will constrain anisotropy below 1%.

6 CONCLUSIONS

We have demonstrated how anisotropic Hubble expansion can be measured or constrained using extragalactic proper motions, and we applied this technique to the best current proper motion catalog (Titov & Lambert 2013) to place a new constraint on the isotropy of the Hubble expansion and thus on the isotropy of dark energy. No significant anisotropy was detected; the Hubble expansion is isotropic to 7% (1$\sigma$), corresponding to streaming motions of 1 $\mu$as yr$^{-1}$, in the
best-constrained directions (−19% and +17% in the least-
constrained directions) and does not significantly deviate
from isotropy in any direction. The Gaia mission, which is
expected to obtain proper motions for 500,000 quasars, will
likely constrain the anisotropy below 1%.

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use of NASA’s Astrophysics Data System Bibliographic Ser-
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**APPENDIX A: ANISOTROPIC EXPANSION VECTOR FIELDS**

The complete vector field described in Section 2 Equation (4), and Table 1 is

\[
V_{\text{shear}}(\alpha, \delta) = H_0 \left( \frac{1}{2} \sin \alpha \sin \delta \left[ \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (2 \sin \alpha^* \cos \delta^* \sin 2 \psi^* - \cos \alpha^* \sin 2 \delta^* \cos 2 \psi^*) - \frac{3}{2} \Sigma_{y'} \cos \alpha^* \sin 2 \delta^* \right] \\
+ \frac{1}{2} \cos \alpha \sin \delta \left[ \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (2 \cos \alpha^* \cos \delta^* \sin 2 \psi^* + \sin \alpha^* \sin 2 \delta^* \cos 2 \psi^*) + \frac{3}{2} \Sigma_{y'} \sin \alpha^* \sin 2 \delta^* \right] \\
+ \frac{1}{4} \sin 2 \alpha \cos \delta \left[ \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (3 \cos 2 \alpha^* \cos 2 \psi^* - \cos 2 \alpha^* \cos 2 \delta^* \cos 2 \psi^* - 4 \sin 2 \alpha^* \sin \delta^* \sin 2 \psi^*) - \frac{3}{2} \Sigma_{x'} \cos 2 \alpha^* (1 + \cos 2 \delta^*) \right] \\
+ \frac{1}{4} \cos 2 \alpha \cos \delta \left[ \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (-3 \sin 2 \alpha^* \cos 2 \psi^* + \sin 2 \alpha^* \cos 2 \delta^* \cos 2 \psi^* - 4 \cos 2 \alpha^* \sin \delta^* \sin 2 \psi^*) + \frac{3}{2} \Sigma_{x'} \sin 2 \alpha^* (1 + \cos 2 \delta^*) \right] \right) \hat{e}_\alpha \\
+ H_0 \left( \frac{3}{8} \sin 2 \delta \left[ - \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) \cos 2 \psi^* (1 + \cos 2 \delta^*) \right] + \frac{1}{2} \Sigma_{x'} (1 - 3 \cos 2 \delta^*) \right) \\
+ \frac{1}{2} \cos \alpha \cos 2 \delta \left[ - \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (2 \sin \alpha^* \cos \delta^* \sin 2 \psi^* - \cos \alpha^* \sin 2 \delta^* \cos 2 \psi^*) + \frac{3}{2} \Sigma_{x'} \cos \alpha^* \sin 2 \delta^* \right] \\
+ \frac{1}{2} \sin \alpha \cos 2 \delta \left[ \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (2 \cos \alpha^* \cos \delta^* \sin 2 \psi^* + \sin \alpha^* \sin 2 \delta^* \cos 2 \psi^*) + \frac{3}{2} \Sigma_{y'} \sin \alpha^* \sin 2 \delta^* \right] \\
+ \frac{1}{8} \cos 2 \alpha \sin 2 \delta \left[ \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (3 \cos 2 \alpha^* \cos 2 \psi^* - \cos 2 \alpha^* \cos 2 \delta^* \cos 2 \psi^* - 4 \sin 2 \alpha^* \sin \delta^* \sin 2 \psi^*) - \frac{3}{2} \Sigma_{x'} \cos 2 \alpha^* (1 + \cos 2 \delta^*) \right] \\
+ \frac{1}{8} \sin 2 \alpha \sin 2 \delta \left[ - \left( \Sigma_{y'} + \frac{1}{2} \Sigma_{x'} \right) (-3 \sin 2 \alpha^* \cos 2 \psi^* + \sin 2 \alpha^* \cos 2 \delta^* \cos 2 \psi^* - 4 \cos 2 \alpha^* \sin \delta^* \sin 2 \psi^*) - \frac{3}{2} \Sigma_{x'} \sin 2 \alpha^* (1 + \cos 2 \delta^*) \right] \right) \hat{e}_\delta. \quad (A1)
\]

The E-mode (curl-free) quadrupole vector spherical harmonic is

\[
V_{E2}(\alpha, \delta) = \left( s_{21} \frac{1}{2} \sqrt{\frac{5}{2\pi}} \sin \alpha \sin \delta + s_{21}^f \frac{1}{2} \sqrt{\frac{5}{2\pi}} \cos \alpha \sin \delta - s_{21}^e \frac{1}{2} \sqrt{\frac{5}{2\pi}} \sin 2 \alpha \cos \delta - s_{22}^f \frac{1}{2} \sqrt{\frac{5}{2\pi}} \cos 2 \alpha \cos \delta \right) \hat{e}_\alpha \\
+ \left( s_{20} \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin 2 \delta - s_{21}^e \frac{1}{2} \sqrt{\frac{5}{2\pi}} \cos \alpha \cos 2 \delta + s_{21}^f \frac{1}{2} \sqrt{\frac{5}{2\pi}} \sin \alpha \cos 2 \delta - s_{22}^e \frac{1}{4} \sqrt{\frac{5}{2\pi}} \cos 2 \alpha \sin 2 \delta \\
+ s_{22}^f \frac{1}{4} \sqrt{\frac{5}{2\pi}} \sin 2 \alpha \sin 2 \delta \right) \hat{e}_\delta. \quad (A2)
\]