Covariant Quantization of the Green-Schwarz Superstring
in a Calabi-Yau Background

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Abstract

After adding a scalar chiral boson to the usual superspace variables, the four-dimensional Green-Schwarz superstring is quantized in a manifestly SO(3,1) super-Poincaré covariant manner. The constraints are all first-class and form an N=2 superconformal algebra with \( c = -3 \). Since the Calabi-Yau degrees of freedom are described by an N=2 superconformal field theory with \( c = 9 \), the combined Green-Schwarz and Calabi-Yau systems form the \( c = 6 \) matter sector of a critical N=2 string.

Using the standard N=2 super-Virasoro ghosts, a nilpotent BRST charge is defined and vertex operators for the massless supermultiplets are constructed. Four-dimensional superstring amplitudes can be calculated with manifest SO(3,1) super-Poincaré invariance by evaluating correlation functions of these BRST-invariant vertex operators on N=2 super-Riemann surfaces.
I. Introduction

In previous papers by this author,\textsuperscript{1−4} it was shown that the ten-dimensional Green-Schwarz superstring can be quantized with free fields by constructing a nilpotent BRST operator out of an N=2 stress-energy tensor with critical central charge $c = 6$. Scattering amplitudes are calculated by evaluating correlation functions of BRST-invariant vertex operators on N=2 super-Riemann surfaces, where the integration over fermionic super-moduli is performed by inserting N=2 picture-changing operators. Because this N=2 GS formalism is conformally invariant, it has the advantage over the light-cone GS formalism\textsuperscript{5−7} that the picture-changing operators can be inserted anywhere on the surface. Furthermore, because some of the spacetime-supersymmetries are manifest, there is no need to perform a GSO projection or to sum over spin structures as in the RNS formalism.

A disadvantage of the N=2 GS formalism is that the free fields do not transform linearly under the full set of SO(9,1) super-Poincaré transformations. This prevents the amplitude calculations from being manifestly Lorentz-covariant in ten dimensions. However under an SO(3,1) subgroup of the super-Poincaré transformations, the free GS fields do transform covariantly.

Under this SO(3,1) subgroup, the ten-dimensional GS fields split naturally into two groups. The first group describes a four-dimensional GS superstring which contributes $c = -3$ to the conformal anomaly, while the second group describes a flat six-dimensional background with $c = 9$. Since the contribution of the flat six-dimensional background to the N=2 stress-energy tensor is decoupled from the contribution of the four-dimensional superstring, the flat background can be replaced by any Calabi-Yau background characterized by an N=2 superconformal field theory with $c = 9$ (in this paper, the term “Calabi-Yau” will mean Ricci-flat Kahler in the large-radius limit). One therefore has a manifestly SO(3,1) super-Poincaré covariant quantization of the GS superstring in a Calabi-Yau background.

In Section II of this paper, the worldsheet variables of the four-dimensional GS superstring are described. In addition to the usual four-dimensional superspace variables, there is a scalar chiral boson which is related both to $R$-transformations in superspace and to projective transformations in twistor-space. Using these worldsheet variables, an N=2 stress-energy tensor with central charge $c = -3$ is constructed (except for the chiral boson, all of the elements used to construct this N=2 tensor were described by Siegel in reference 8). After adding a $c = 9$ N=2 superconformal field theory representing the Calabi-Yau background, one obtains the $c = 6$ matter sector of a critical N=2 string.

In Section III, the N=2 super-Virasoro ghosts are introduced and a nilpotent BRST charge is defined. Manifestly SO(3,1) super-Poincaré covariant vertex operators are then constructed for the massless states of the heterotic superstring in a Calabi-Yau background. Unlike in the RNS formalism, all physical states are represented by vertex operators involving purely matter fields. In their lowest picture, the vertex operators are constructed out of prepotentials for the supermultiplets. Tree-level scattering amplitudes are calculated.
The free-field OPE’s for these worldsheet variables are $N=2$ string, and the scattering amplitudes coincide using either the $N=1$ or $N=2$ prescriptions. It was recently shown that any critical $N=1$ string can be embedded in a critical $N=2$ string in a manifestly SO(3,1) super-Poincaré invariant manner by evaluating correlation functions of these BRST-\(β\)-superstring amplitudes, calculating \(\beta\)-functions of the GS sigma model, constructing a GS superstring field theory, and quantizing with manifest SO(9,1) super-Poincaré invariance the superstring in a flat background.

In Section IV of the paper, the relation between this GS superstring formalism and the conventional RNS formalism is discussed. It was recently shown that any critical $N=1$ string can be embedded in a critical $N=2$ string in a Calabi-Yau background by first embedding the $N=1$ RNS string in an $N=2$ string, and then performing a field-redefinition of the worldsheet variables. This field-redefinition is a conformally invariant version of the light-cone gauge triality transformation and was described in reference 10 for the GS superstring in a flat background.

In the final section, possible applications of this paper are discussed. These include computing multiloop superstring amplitudes, calculating $\beta$-functions of the GS sigma model, constructing a GS superstring field theory, and quantizing with manifest SO(9,1) super-Poincaré invariance the superstring in a flat background.

II. The Green-Schwarz Superstring in a Calabi-Yau Background

A. The Four-Dimensional Green-Schwarz Superstring

The worldsheet variables of the four-dimensional GS superstring will consist of the spacetime variables, \(x^m\) \((m = 0 \text{ to } 3)\), the right-moving fermionic variables, \(\theta^\alpha\) and \(\bar{\theta}^\dot{\alpha}\) \((\alpha, \dot{\alpha} = 1 \text{ to } 2)\), the conjugate right-moving fermionic variables, \(p_\alpha\) and \(\bar{p}_\alpha\), and one right-moving boson \(\rho\). The chiral boson will be defined so that \(\rho\) is identified with \(\rho + 2\pi\) (in two-dimensional Minkowski space, \(\rho\) is imaginary valued and \(i\rho\) takes values on a circle of radius 1) and will be shown in Section IIIIC to be related to R-transformations of four-dimensional superspace. For the heterotic GS superstring, one also needs the 32 left-moving chiral fermions, \(\zeta_q\) \((q = 1 \text{ to } 32)\), which describe the SO(32) or \(E_8\times E_8\) lattice. For the four-dimensional Type II GS superstring, the left-moving fermionic fields, \(\hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\alpha}}, \hat{p}_\alpha, \hat{\bar{p}}_{\dot{\alpha}}\), and one left-moving boson, \(\hat{\rho}\), are needed.

In conformal gauge, the worldsheet action for these fields is:

\[
H\text{eterotic} : \quad \int dz^+dz^- \left[ \frac{1}{2} \partial_+ x^m \partial_- x_m + p_\alpha \partial_+ \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_+ \bar{\theta}^{\dot{\alpha}} + \frac{1}{2} \partial_+ \rho \partial_- \rho + \zeta_q \partial_- \zeta_q \right] \tag{1}
\]

\[
\text{Type II} : \quad \int dz^+dz^- \left[ \frac{1}{2} \partial_+ x^m \partial_- x_m + p_\alpha \partial_+ \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_+ \bar{\theta}^{\dot{\alpha}} + \frac{1}{2} \partial_+ \rho \partial_- \rho + \hat{p}_\alpha \partial_+ \hat{\theta}^\alpha + \hat{\bar{p}}_{\dot{\alpha}} \partial_+ \hat{\bar{\theta}}^{\dot{\alpha}} + \frac{1}{2} \partial_+ \hat{\rho} \partial_- \hat{\rho} \right].
\]

The free-field OPE’s for these worldsheet variables are

\[
x^m(y)x^n(z) \to -\eta^{mn} \log |y - z|, \quad \rho(y)\rho(z) \to -\log(y^- - z^-),
\]

\[
p_\alpha(y)\theta^\beta(z) \to \frac{\delta^\beta_\alpha}{y^+ - z^-}, \quad \bar{p}_{\dot{\alpha}}(y)\bar{\theta}^{\dot{\beta}}(z) \to \frac{\delta^{\dot{\beta}}_{\dot{\alpha}}}{y^+ - z^-}, \quad \zeta_q(y)\zeta_{q'}(z) \to \frac{\delta_{qq'}}{y^+ - z^+},
\]

\[
\hat{p}_\alpha(y)\hat{\theta}^\alpha(z) \to \frac{\delta^\alpha_\beta}{y^+ - z^+}, \quad \hat{\bar{p}}_{\dot{\alpha}}(y)\hat{\bar{\theta}}^{\dot{\alpha}}(z) \to \frac{\delta^{\dot{\alpha}}_{\dot{\beta}}}{y^+ - z^+}, \quad \hat{\rho}(y)\hat{\rho}(z) \to -\log(y^+ - z^+).
\]
Note that the chiral boson $\rho$ can not be fermionized since $e^{\rho(y)} e^{\rho(z)} \rightarrow e^{2\rho(z)}/(y^- - z^-)$ while $e^{\rho(y)} e^{-\rho(z)} \rightarrow (y^- - z^-)$. It has the same behavior as the negative-energy field $\phi$ that appears when bosonizing the RNS ghosts $\gamma = \eta e^{\phi}$ and $\beta = \partial \bar{\phi} e^{-\phi}$.11

These GS worldsheet variables are constrained by the $c = -3$ N=2 right-moving stress-energy tensor:

$$ L_{GS} = \frac{1}{2} \partial_x \partial_y \partial_z + p_\alpha \partial_\alpha \partial_\alpha + \frac{1}{2} \partial_\rho \partial_\rho, \quad G_{GS} = e^\rho(d)^2, \quad \bar{G}_{GS} = e^{-\rho}(\bar{d})^2, \quad J_{GS} = \partial_- \rho, \quad (2) $$

where

$$ d_\alpha = p_\alpha + i \theta^\alpha \partial_- x_{\alpha \dot{\alpha}} - \frac{1}{2} (\bar{\theta})^2 \partial_- \theta_{\alpha \dot{\alpha}} + \frac{1}{4} \theta_{\alpha \dot{\alpha}} \partial_- (\bar{\theta})^2, \quad \bar{d}_{\dot{\alpha}} = \bar{p}_{\dot{\alpha}} + i \theta^{\dot{\alpha}} \partial_- x_{\alpha \dot{\alpha}} - \frac{1}{2} \theta^{\dot{\alpha}} \partial_- \bar{\theta}_{\alpha \dot{\alpha}} + \frac{1}{4} \bar{\theta}_{\alpha \dot{\alpha}} \partial_- (\theta)^2, \quad (3) $$

and $(d)^2$ means $e^{\alpha \dot{\beta}} d_\alpha d_\beta$. As was shown by Siegel,8 $d_\alpha$ and $\bar{d}_{\dot{\alpha}}$ satisfy the OPE that $d_\alpha(y) \bar{d}_{\dot{\alpha}}(z) \rightarrow 2i \Pi_{\alpha \dot{\alpha}}/(y^- - z^-)$ where $\Pi_{\alpha \dot{\alpha}} = \sigma^{\alpha \dot{\alpha}} \partial_- x_m - i \theta_{\alpha \dot{\alpha}} \partial_- \bar{\theta}_{\alpha \dot{\alpha}} - i \bar{\theta}_{\alpha \dot{\alpha}} \partial_- \theta_{\alpha \dot{\alpha}}$, and $d_\alpha(y) \Pi^m(z) \rightarrow -2i \sigma^{m \alpha \dot{\alpha}} \partial_- \bar{\theta}_{\alpha \dot{\alpha}}/(y^- - z^-)$.

It is interesting to note that these four-dimensional GS variables are closely related to the twistor variables of Penrose12 (in fact, this N=2 description of the GS superstring grew out of the twistor descriptions of references 13-19). The relation is:

$$ \lambda_\alpha = e^{\rho} d_\alpha, \quad \bar{\lambda}_{\dot{\alpha}} = e^{-\rho} \bar{d}_{\dot{\alpha}}, \quad \omega^\alpha = (x^{\alpha \dot{\alpha}} + i \theta^\alpha \bar{\theta}^{\dot{\alpha}}) e^\rho d_\alpha, \quad \bar{\omega}^{\dot{\alpha}} = (x^{\alpha \dot{\alpha}} - i \theta^\alpha \bar{\theta}^{\dot{\alpha}}) e^{-\rho} \bar{d}_{\dot{\alpha}}, \quad (4) $$

which satisfy the twistor OPE’s,

$$ \lambda_\alpha(y) \bar{\lambda}_{\dot{\alpha}}(z) \rightarrow 2i \Pi_{\alpha \dot{\alpha}}, \quad \lambda_\alpha(y) \omega^{\dot{\alpha}}(z) \rightarrow \frac{2i \delta^{\dot{\alpha}}}{y^- - z^-}, \quad \bar{\lambda}_{\dot{\alpha}}(y) \omega^{\alpha}(z) \rightarrow \frac{2i \delta_{\dot{\alpha}}}{y^- - z^-}. \quad (5) $$

The U(1) current, $J = \partial_- \rho$, generates projective transformations in the twistor-space $CP^3$.

The advantage of working with the variables $d_\alpha$ and $\Pi^m$ is that they commute with the spacetime supersymmetry generators,

$$ q_\alpha = \int dz^- [p_\alpha - i \theta^{\alpha \dot{\alpha}} \partial_- x_{\alpha \dot{\alpha}} - \frac{1}{4} (\bar{\theta})^2 \partial_- \theta_{\alpha \dot{\alpha}}], \quad \bar{q}_{\dot{\alpha}} = \int dz^- [\bar{p}_{\dot{\alpha}} - i \theta^{\dot{\alpha} \alpha} \partial_- x_{\alpha \dot{\alpha}} - \frac{1}{4} \theta^2 \partial_- \bar{\theta}_{\alpha \dot{\alpha}}]. \quad (6) $$

Note that the N=2 tensor of equation (2) is spacetime supersymmetric since $L_{GS}$ can be written in the form

$$ L_{GS} = \frac{1}{2} \Pi^m \Pi_m + d_\alpha \partial_- \theta^\alpha + \bar{d}_{\dot{\alpha}} \partial_- \bar{\theta}_{\dot{\alpha}} + \frac{1}{2} \partial_\rho \partial_- \rho. $$

For the heterotic string, the left-moving stress-energy tensor is:

$$ \hat{L}_{GS} = \frac{1}{2} \partial_+ x^m \partial_+ x_m + \zeta_+ \partial_+ \zeta_+, \quad (7) $$

while for the Type II string, the left-moving N=2 stress-energy tensor is obtained from equation (2) by using hatted variables and replacing $\partial_- \rho$ with $\partial_+$. 

To check that this system correctly describes the four-dimensional GS superstring, one can use the N=2 constraints to gauge-fix to light-cone gauge. First, use $L_{GS}$ to gauge $\partial_- (x^0 + x^3)$ to 1, and use $J_{GS}$ to gauge $\rho = i \sigma$ where $\theta^1 = e^{i \sigma}$ and $p_1 = e^{-i \sigma}$ (note that $\rho - i \sigma$ has no singularities with itself). Since $G_{GS}$ in
describe curved backgrounds which correspond in the large-radius limit to Calabi-Yau manifolds.

Four dimensions. As was first shown by Gepner, the energy tensor of the four-dimensional GS superstring, the flat background can be replaced by any tensor to vanish fixes all variables except for \( x^1, x^2, \theta^1 \) and \( \bar{p}_1 \), which are just the light-cone GS variables in four dimensions.

B. The Calabi-Yau Background

The four-dimensional heterotic GS variables described in Section IIA are related in the following way to the ten-dimensional GS variables introduced in references 2-4 (for the rest of this paper, only the heterotic GS superstring will be discussed):

\[
x^m = x^m, \quad \theta^1 = \Gamma^4 e^{h^-}, \quad \theta^2 = \theta^-, \quad \bar{\theta}^1 = \Gamma^4 e^{h^+}, \quad \bar{\theta}^2 = \theta^+,
\]

\[
p_1 = \Gamma^4 e^{-h^-}, \quad p_2 = \varepsilon^+, \quad \bar{p}_1 = \Gamma^4 e^{-h^+}, \quad \bar{p}_2 = \varepsilon^-, \quad \partial_- \rho = \partial_-(h^+ - h^-) + \Gamma^4 \Gamma^4.
\]

The remaining ten-dimensional GS variables, \( x^\mu (\mu = 4 \text{ to } 9) \), \( \Gamma^l \) and \( \bar{\Gamma}^\l (l, \l = 1 \text{ to } 3) \), describe a flat six-dimensional background. These background variables can be combined in the following way into N=(2,0) chiral and anti-chiral superfields \( Y^j \) and \( \bar{Y}^j \) for \( j, \bar{j} = 1 \text{ to } 3 \):

\[
Y^j (z + \kappa \bar{\kappa}, \kappa) = y^j + \kappa \Gamma^j + \kappa \bar{\kappa} \partial_- y^j, \quad \bar{Y}^j (z - \kappa \bar{\kappa}, \bar{\kappa}) = \bar{y}^j + \bar{\kappa} \bar{\Gamma}^j - \kappa \bar{\kappa} \partial_- \bar{y}^j,
\]

where \( y^j = \frac{1}{\sqrt{2}} (x^{j+3} + ix^{j+6}) \) and \( \bar{y}^j = \frac{1}{\sqrt{2}} (x^{j+3} - ix^{j+6}) \). The worldsheet action for these variables is \( \int dz^+ dz^- dkd\bar{k}[\partial_+ Y^j \partial_+ \bar{Y}^\l - 3 \partial_+ \bar{Y}^j \partial_+ Y^\l + \Psi_a (e^V)^{\bar{a} \bar{b}} \bar{\Psi}_b] \) (10)

where \( K \) is the Kahler potential satisfying \( g_{jk} = \partial_j \partial_k K \) and \( V^{ab} \) is the gauge prepotential satisfying \( \bar{A}_j^{ab} \) (11) have been gauge-fixed to zero). \( \Psi_a \) and \( \bar{\Psi}_a \) for \( a, \bar{a} = 1 \text{ to } 3 \) are chiral and anti-chiral N=(2,0) superfields whose component expansions are

\[
\Psi_a (z - \kappa \bar{\kappa}, \bar{\kappa}) = \psi_a + \kappa \bar{\kappa} \partial_- \psi_a, \quad \bar{\Psi}_a (z - \kappa \bar{\kappa}, \bar{\kappa}) = \bar{\psi}_a + \bar{\kappa} \partial_- \bar{\psi}_a.
\]
where \( f_a \) and \( \bar{f}_a \) are auxiliary bosons, \( \psi_a = \zeta_a + i \zeta_{a+3} \) and \( \bar{\psi}_a = \zeta_a - i \zeta_{a+3} \) for \( a = 1 \) to \( 3 \), and \( \zeta_q \) for \( q = 1 \) to \( 6 \) are chiral fermions which are “borrowed” from the 32 fermions that describe the \( E_8 \times E_8 \) lattice.

In the large-radius limit, the \( c = 9 \) N=2 right-moving stress-energy tensor is given by

\[
T_{CY} = g_{j\bar{k}} \, D_{\kappa} Y^j \bar{D}_{\bar{\kappa}} \bar{Y}^k,
\]

while the \( c = 9 \) N=0 left-moving stress-energy tensor is given by

\[
\hat{L}_{CY} = g_{j\bar{k}} \, \partial_+ Y^j \partial_+ \bar{Y}^k - \partial_+ Y^j \, \partial_j (\Psi e^V)^\dagger \bar{\Psi} + \partial_+ \bar{Y}^k \, \bar{\Psi} \partial_j (e^V \bar{\Psi})^a \quad \text{at} \quad \kappa = \bar{\kappa} = 0.
\]

Note that the left-moving tensor from the four-dimensional GS string now contributes \( c = 17 \) to the central charge since six chiral fermions have been removed from the 32 \( \zeta_q \)’s.

The action and stress-tensors for the Calabi-Yau sector are invariant under the gauge transformations

\[
K \rightarrow K + \Lambda(Y) + \bar{\Lambda}(\bar{Y}), \quad e^V \rightarrow e^{\lambda(Y)} e^V e^{\bar{\lambda}(\bar{Y})}, \quad \Psi_a \rightarrow (\Psi e^{-\lambda})_a, \quad \bar{\Psi}_a \rightarrow (e^{-\bar{\lambda}} \bar{\Psi})_a,
\]

where \( \bar{\partial}_j \Lambda = \bar{\partial}_j \lambda^{ab} = \bar{\partial}_j \bar{\Lambda} = \partial_j \Lambda = \partial_j \tilde{\lambda}^{ab} = 0 \).

### III. Covariant Quantization

**A. N=2 Ghosts and the BRST Charge**

By adding together the right-moving stress-energy tensors of the four-dimensional GS superstring and of the Calabi-Yau background, one obtains a \( c = 6 \) N=2 stress-energy tensor which can be coupled to N=2 worldsheet supergravity. The right-moving N=2 super-Virasoro ghosts that gauge-fix this coupling consist of the N=(2,0) superfields

\[
C = c + \kappa \gamma + \bar{\kappa} \bar{\gamma} + \kappa \bar{\kappa} u, \quad B = v + \kappa \beta - \bar{\kappa} \bar{\beta} + \kappa \bar{\kappa} b,
\]

where \( (b, c) \) are the usual spin \( (2, -1) \) fermionic Virasoro ghosts, \( (\beta, \gamma) \) and \( (\bar{\beta}, \bar{\gamma}) \) are the two sets of spin \( (3/2, -1/2) \) bosonic superconformal ghosts, and \( (v, u) \) are the spin \( (1, 0) \) fermionic U(1) ghosts. In terms of these ghost superfields, the worldsheet action is \( \int d\zeta d\bar{\zeta} d\kappa d\bar{\kappa} [C \partial_+ B] \) and the \( c = -6 \) N=2 stress-energy tensor is \( T = \partial_- (CB) - D_a C \bar{D}_a \bar{B} - \bar{D}_a C D_a B \). It will be convenient to bosonize the \( (\beta, \gamma) \) and \( (\bar{\beta}, \bar{\gamma}) \) ghosts in the usual way as

\[
\beta = \partial_- \zeta e^{-\phi}, \quad \gamma = \eta e^{\phi}, \quad \bar{\beta} = \partial_- \bar{\zeta} e^{-\bar{\phi}}, \quad \bar{\gamma} = \bar{\eta} e^{\bar{\phi}}.
\]

Since the central charge contribution of the ghost fields cancels the contribution of the matter fields, it is easy to construct a nilpotent BRST charge as:

\[
Q = \int d\zeta d\bar{\zeta} d\kappa d\bar{\kappa} [C(T_{GS} + T_{CY}) + B(D_a C \bar{D}_a \bar{C} - C \partial_- C)]
\]

where \( T_{GS} \) is the \( c = -3 \) N=2 stress-energy tensor of the four-dimensional GS superstring and \( T_{CY} \) is the \( c = 9 \) N=2 stress-energy tensor of the Calabi-Yau background.
The left-moving BRST charge, $\hat{Q}$, is constructed in the usual way out of the left-moving Virasoro ghosts $(\hat{b}, \hat{c})$ as $\hat{Q} = \int dz^+ [\hat{c}(\hat{L}_{GS} + \hat{L}_{CY} - \partial_+ \hat{b})]$ where $L_{GS}$ is the left-moving $c = 17$ stress-energy tensor of the four-dimensional heterotic GS string and $\hat{L}_{CY}$ is the left-moving $c = 9$ stress-energy tensor of the Calabi-Yau background.

For any critical N=2 string, it is useful to define the N=2 picture-changing operators,$^2$

$$Z = \{Q, \xi\} = e^\phi [G_{GS} + G_{CY} + (b - \frac{1}{2} \partial_ - v) \gamma - v \partial_- \gamma] + c \partial_- \xi,$$

$$\hat{Z} = \{Q, \bar{\xi}\} = e^\phi [\bar{G}_{GS} + \bar{G}_{CY} + (b + \frac{1}{2} \partial_- v) \gamma + v \partial_+ \gamma] + \bar{c} \partial_+ \bar{\xi},$$

and the instanton-number-changing operators,

$$I = e^{\int \gamma J_{total}} = e^{\rho + \phi - \hat{\phi} + \int z J_{CY}}, \quad \bar{I}^{-1} = e^{-\int z J_{total}} = e^{-\rho - \phi - \bar{\phi} - \int \gamma J_{CY}}. \tag{19}$$

Note that these operators have the property that they are BRST-invariant, but their derivatives are BRST-trivial. Unlike in the N=1 string,$^1$ there is no BRST-invariant inverse picture-changing operator, $Y$, satisfying $Y(y) Z(z) = 1$.

### B. Vertex Operators

All physical states of the heterotic GS superstring can be represented by vertex operators $W$ of the form $W = c c e^{-\phi - \bar{\phi}} V$ where $V$ is an N=2 primary field which is constructed entirely out of matter fields and is dimension (0,1). In other words, $L$, $G$, and $\bar{G}$ have only $(y^- - z^-)^{-1}$ singularities with $V$, while $J$ with $V$ has no singularities. To obtain vertex operators in other pictures, one can attach arbitrary combinations of $Z$, $\hat{Z}$, $I$, and $\bar{I}$ onto $W$.

For example, if $V$ depends only on the four-dimensional GS fields and is independent of the Calabi-Yau manifold,

$$Z \hat{Z} W = c \bar{c} (\partial \bar{\theta} \bar{\nabla}_a V + \partial_- \bar{\theta} \bar{\nabla}_a V + \Pi^a \nabla_a \bar{\nabla}_a V) + \bar{c} \gamma e^{-\phi} d \bar{a} \bar{\nabla}_a V \tag{20}$$

where $\nabla_a V = \int dz^- d a , V$ and $\bar{\nabla}_a V = \int dz^- \bar{d} a , V$. Since $\bar{c} \bar{c}$ can be replaced with $\int dz^+ dz^-$, the vertex operator can be written in integrated form as:$$^24$$

$$\int dz^+ dz^- [\bar{d} \bar{a} \bar{\nabla}_a V + \partial_- \bar{\theta} \bar{\nabla}_a V + \Pi^a \nabla_a \bar{\nabla}_a V]. \tag{21}$$

Note that this vertex operator is invariant under the gauge transformation $\delta V = \Lambda + \bar{\Lambda}$ where $\bar{\nabla}_a \Lambda = \nabla_a \bar{\Lambda} = 0$.

The physical massless states of the heterotic superstring in a Calabi-Yau background consist of the four-dimensional graviton, gravitino, axion, dilaton, dilatino, gluon, and gluino, as well as the modulons and modulinos coming from the marginal deformations of the Calabi-Yau manifold. These massless states combine into supermultiplets which can be represented by BRST-invariant vertex operators in the following way:
The fields of the supergravity and dilaton multiplets can be combined into a superfield, \( E_m(x, \theta, \bar{\theta}) \), with the gauge invariances \( \delta E_m = \Lambda_m + \dot{\Lambda}_m + \partial_m F \) where \( \nabla_\alpha \Lambda_m = \nabla_\alpha \dot{\Lambda}_m = 0 \). The vertex operator for this superfield is

\[
W = c \hat{e} e^{-\phi - \bar{\phi}} E_m(x, \theta, \bar{\theta}) \partial_+ x^m, \tag{22}
\]

which is BRST-invariant if the following equations of motion and gauge-fixing conditions are imposed:

\[
\nabla^2 E_m = \nabla^2 \bar{E}_m = \eta^{mn} \partial_m E_n = \eta^{mn} \partial_m \partial_n E_p = 0. \tag{23}
\]

The component fields described by this vertex operator are contained in the \((\theta, \bar{\theta})\) components of \( E_m \). At \( \theta^\alpha = \bar{\theta}^\dot{\alpha} = 0 \), the traceless graviton is \( h_{mn} = \sigma_m^{\alpha\dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} E_m + \sigma_n^{\alpha\dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} E_n - \frac{1}{2} \eta_{mn} \eta^{rs} \sigma_\alpha^{\dot{r}\dot{s}} \nabla_\alpha \nabla_{\dot{r}} E_s \), the dilaton is \( D = \eta^{mn} \sigma_m^{\alpha\dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} E_n \), the axion is \( b_{mn} = \sigma_m^{\alpha\dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} E_n - \sigma_n^{\alpha\dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} E_m \), the gravitinos are \( \lambda_{mn} = \nabla_2 (\nabla_\alpha E_m - \frac{1}{2} \delta_{mn} \sigma^\alpha_{\dot{\alpha}\dot{\beta}} \nabla_{\dot{\alpha}} E_\beta) \), \( \bar{\lambda}_{mn} = \nabla_2 (\nabla_\alpha E_m - \frac{1}{2} \delta_{mn} \sigma^\alpha_{\dot{\alpha}\dot{\beta}} \nabla_{\dot{\alpha}} E_\beta) \), and the dilatinos are \( \delta^\alpha = \nabla^2 (\sigma^{mn\alpha} \nabla_m E_n) \), \( \bar{\delta}^\dot{\alpha} = \nabla^2 (\sigma^{mn\dot{\alpha}} \nabla_m E_n) \). It is easy to check that in Wess-Zumino gauge, the restrictions on \( E_m \) from equation (23) impose the usual polarization and mass-shell conditions on these component fields (e.g. \( \partial_m h_{mn} = \partial_n h_{mn} = \eta^{mn} \partial_m \partial_n h^{rs} = 0 \)).

The super-Yang-Mills fields can be combined into a superfield, \( V^I(x, \theta, \bar{\theta}) \), with the gauge invariance \( \delta V^I = \Lambda^I + \Lambda^I \) where \( \nabla_\alpha \Lambda^I = \nabla_\dot{\alpha} \Lambda^I = 0 \) and \( I \) takes values in the adjoint representation of the unbroken \( E_6 \times E_8 \). The vertex operator for this superfield is

\[
W = c \hat{e} e^{-\phi - \bar{\phi}} V^I(x, \theta, \bar{\theta}) j_I \tag{24}
\]

where \( j_I \) is the current for \( E_6 \times E_8 \) that is constructed out of the 26 chiral fermions \( j_I \) is quadratic in the chiral fermions for an \( SO(10) \times SO(16) \) subgroup of \( E_6 \times E_8 \), but for the other elements of \( E_6 \times E_8 \), \( j_I \) is realized non-linearly. This vertex operator is BRST-invariant after imposing the equations of motion and gauge-fixing conditions:

\[
\nabla^2 V^I = \nabla^2 \bar{V}^I = \eta^{mn} \partial_m \partial_n V^I = 0. \tag{25}
\]

At \( \theta^\alpha = \bar{\theta}^\dot{\alpha} = 0 \), the gluon is \( A^I_m = \sigma_m^{\alpha\dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} V^I \), and the gluinos are \( \chi^I_\alpha = \nabla^2 \nabla_\alpha V^I, \bar{\chi}^I_\dot{\alpha} = \nabla^2 \nabla_{\dot{\alpha}} V^I \).

Each marginal deformation of the Calabi-Yau manifold gives rise to a massless complex scalar and Weyl spinor in four dimensions. These combine into chiral and anti-chiral superfields, \( \Omega(x - i\theta\sigma\bar{\theta}, \theta) \) and \( \bar{\Omega}(x + i\theta\sigma\bar{\theta}, \bar{\theta}) \), satisfying \( \nabla_\alpha \Omega = \nabla_{\dot{\alpha}} \bar{\Omega} = 0 \). On-shell, these superfields satisfy the equations of motion \( \nabla^2 \Omega = \nabla^2 \bar{\Omega} = 0 \). In order to write the exact vertex operators for \( \Omega \) and \( \bar{\Omega} \), one needs to know the primary fields of the \( c = 9 \) \( N=2 \) superconformal field theory. However in the large-radius limit, these vertex operators can be approximated by expressions involving the Calabi-Yau fields.

For the massless states that come from deforming the Kahler structure of the Calabi-Yau manifold, the vertex operators are

\[
W = c \hat{e} e^{-\phi - \bar{\phi}} \Omega \partial_j K \partial_+ y^j, \quad \bar{W} = c \hat{e} e^{-\phi - \bar{\phi}} \bar{\Omega} \bar{\partial}_j K \partial_+ \bar{y}^j, \tag{26}
\]
where $K$ is the Kahler potential satisfying $\partial_j \bar{\partial}_k K = g_{jk}$. Note that these vertex operators change by a BRST-trivial quantity under the gauge transformation $\delta K = \Lambda + \bar{\Lambda}$ where $\partial_j \Lambda = \partial_j \bar{\Lambda} = 0$, and that in the picture $\bar{Z}W$ and $ZW$, this gauge invariance is manifest since

$$\bar{Z}W = c \bar{c} e^{-\phi} \Omega g_{jk} \bar{\Gamma}^k \partial_+ y^j, \quad Z\bar{W} = c \bar{c} e^{-\bar{\phi}} \bar{\Omega} g_{kj} \Gamma^j \partial_+ \bar{y}^k. \quad (27)$$

By introducing potentials for the one-forms on the Calabi-Yau manifold, it is also possible to write the vertex operators for the other massless states in the form $W = c \bar{c} e^{-\phi} \bar{\Omega} V$ and $\bar{W} = c \bar{c} e^{-\bar{\phi}} \bar{\Omega} \bar{V}$. However, it is more convenient to write them in the picture $\bar{Z}W$ and $ZW$ so that the one-forms appear directly.

For the massless states that come from deforming the complex structure of the manifold, the vertex operators are

$$\bar{Z}W = c \bar{c} e^{-\phi} \Omega g_{jk} \bar{K}^j_\ell \partial_+ \bar{y}^k, \quad Z\bar{W} = c \bar{c} e^{-\bar{\phi}} \bar{\Omega} g_{kj} \bar{K}^j_\ell \Gamma^\ell \partial_+ y^k, \quad (28)$$

where $\bar{K}^j_\ell$ and $K^j_\ell$ are the 101 complex elements in the Dolbeault cohomology $H^1(T)$ and $\bar{H}^1(\bar{T})$ ($T$ and $\bar{T}$ are the holomorphic and anti-holomorphic tangent bundles of the manifold).

Marginal deformations of the Calabi-Yau gauge field give rise to massless states which are singlets, 27's and 27's of the $E_6 \times E_8$. The vertex operators for the singlets are

$$\bar{Z}W = c \bar{c} e^{-\phi} \Omega M^{ab}_j \bar{\Gamma}^j \psi_u \bar{\psi}_b, \quad Z\bar{W} = c \bar{c} e^{-\bar{\phi}} \bar{\Omega} M^{ab}_j \Gamma^j \psi_u \bar{\psi}_b \quad (29)$$

where $M^{ab}_j$ and $M^{j\bar{a}}_j$ are the 224 complex elements in the Dolbeault cohomology $H^1(End T)$ and $\bar{H}^1(End \bar{T})$. The vertex operators for the 27's are

$$\bar{Z}W = c \bar{c} e^{-\phi} \Omega^u \bar{M}^a_k \bar{\Gamma}^k \bar{\psi}_{(u,a)}, \quad Z\bar{W} = c \bar{c} e^{-\bar{\phi}} \bar{\Omega}^u M^a_k \Gamma^k \psi_{(u,a)} \quad (30)$$

where $\bar{M}^a_k$ are the 101 elements in $H^1(T)$, $M^a_k$ is the unique element in $\bar{H}^1(T)$, $u$ is the label for the 27 representation, and $\bar{\psi}_{(u,a)}$ are the 81 currents constructed out of the $\zeta_q$'s which form the $(27,3)$ in the decomposition of the adjoint representation of $E_8$. The vertex operators for the 27's are the complex conjugates of the vertex operators for the 27's.

C. Scattering Amplitudes

Scattering amplitudes for this N=2 GS formalism are calculated in the same way as for any critical N=2 string. Correlations of BRST-invariant vertex operators are evaluated on an N=2 super-Riemann surface, and the moduli and super-moduli of the N=2 surface are integrated over. Integration over the fermionic super-moduli is straightforward and gives rise to insertions of the picture-changing operators, $Z$ and $\bar{Z}$, of equation (18). Integration over the U(1) moduli, however, is more subtle due to complications in defining the path integral over the negative-energy $\rho$ field. To avoid these subtleties, only tree-level scattering amplitudes will be discussed here, although another paper will hopefully be written soon discussing the loop amplitude calculations.
On a sphere with \( N \) punctures, there are no \( U(1) \) moduli, however there is still an instanton number, \( n_I \), coming from the integral of the field-strength of the \( U(1) \) gauge field. For the heterotic string, the number of bosonic moduli is \( 2N - 6 \), the number of fermionic moduli coming from \( G \) is \( N - 2 + n_I \), and the number of fermionic moduli coming from \( \tilde{G} \) is \( N - 2 - n_I \) (for \( |n_I| > N - 2 \), the amplitude vanishes).\(^2\) Evaluating a correlation function on a surface with instanton number \( n_I \) is equivalent to evaluating it on a surface with instanton number zero and inserting \( n_I \) instanton-number-changing operators \( I \) from eqn. (19). After choosing the bosonic moduli to be located at \( N - 3 \) of the punctures and after integrating over the fermionic moduli, the tree-level scattering amplitude for \( N \) external states \( V_1, ..., V_N \) is given by:

\[
A_{V_1, ..., V_N} = \sum_{n_I=2-N}^{N-2} <\hat{c}\hat{c}e^{-\phi-\phi}V_1(z_1)\hat{c}\hat{c}e^{-\phi-\phi}V_2(z_2)\hat{c}\hat{c}e^{-\phi-\phi}V_3(z_3) > \tag{31}
\]

\[
\int d^2z_4 e^{-\phi-\phi}V_4(z_4) ... \int d^2z_N e^{-\phi-\phi}V_N(z_N) I^{n_I} Z^{N-2+n_I} \tilde{Z}^{N-2-n_I} >
\]

where the locations of the \( I \)'s, \( Z \)'s and \( \tilde{Z} \)'s are arbitrary. Because of the background charges of the various fields, the correlation function

\[
< (\theta)^2(\bar{\theta})^2 c\partial_- c\partial_- c \bar{c}\partial_+ \bar{c}\partial_+ \bar{c} e^{-2\phi-2\bar{\phi}} > = 1. \tag{32}
\]

The presence of the \( (\theta)^2(\bar{\theta})^2 \) zero modes guarantees that the amplitude is \( SO(3,1) \) super-Poincaré invariant.

It is interesting to note that for tree amplitudes involving external states which are independent of the Calabi-Yau fields, only surfaces with \( n_I = 0 \) contribute. This can be seen by observing that \( I \) explicitly depends on the Calabi-Yau fields through the term \( e^{-\int J_{CY}} \). Although the \( Z \)'s and \( \tilde{Z} \)'s also depend on the Calabi-Yau fields, there are not enough \( Z \)'s and \( \tilde{Z} \)'s to cancel the dependence of the \( I \)'s (this result is not valid for loop amplitudes since there are more \( Z \)'s and \( \tilde{Z} \)'s present). As will now be shown, this property of tree amplitudes is caused by \( R \)-invariance of the classical four-dimensional action.

In four-dimensional superspace, the \( R \)-transformation scales \( \theta^a \rightarrow e^{ir}\theta^a, \bar{\theta}^a \rightarrow e^{-ir}\bar{\theta}^a, \) and \( \Omega \rightarrow e^{iw}\Omega \) where \( w \) is the \( R \)-weight of the four-dimensional superfield \( \Omega \). In the GS superstring, this transformation is generated by the BRST-invariant operator, \( R = \int dz^- (2\partial_- \rho - \theta^a p_a + \bar{\theta}^a \bar{p}_a) \), where \( [\partial_- \rho, \Omega] = -\frac{1}{2}w\Omega \) (the \( R \)-weights of the super-Yang-Mills prepotential, \( V \), and of the supergravity/dilaton potential, \( E_m \), are zero since the vertex operators have no \( e^\phi \) factors). Note that \([R, Z] = [R, \tilde{Z} ] = 0 \), but \([R, I] = -2I \).

If \( R \)-invariance is a symmetry of the classical four-dimensional action, the tree amplitude \( A_{[R,V_1,...,V_N]} = A_{[R,V_1]V_2...V_N} + ... + A_{V_1...V_{N-1}[R,V_N]} = 0 \). But by pulling the contour of \( \int dz^- (2\partial_- \rho - \theta^a p_a + \bar{\theta}^a \bar{p}_a) \) around the various operators in expression (31) for the scattering amplitude, this implies that \( \sum_{n_I=2-N}^{N-2} n_I A_{n_I} = 0 \) where

\[
A_{n_I} = <\hat{c}\hat{c}e^{-\phi-\phi}V_1(z_1)\hat{c}\hat{c}e^{-\phi-\phi}V_2(z_2)\hat{c}\hat{c}e^{-\phi-\phi}V_3(z_3) > \tag{33}
\]

\[
\int d^2z_4 e^{-\phi-\phi}V_4(z_4) ... \int d^2z_N e^{-\phi-\phi}V_N(z_N) I^{n_I} Z^{N-2+n_I} \tilde{Z}^{N-2-n_I} >.
\]
Similarly, $A_{[R,R_{i1}...V_{N}]} = 0$ implies that $\sum_{n_{j}=2-N}^{N-2} n_{j}^{2}A_{n_{j}} = 0$. By repeatedly commuting with $R$, one therefore proves that $A_{n_{l}} = 0$ for $n_{l} \neq 0$.

**IV. Relation with the RNS Formalism**

It was recently shown that any critical N=1 string can be embedded in a critical N=2 string and the scattering amplitudes coincide using the N=1 and N=2 prescriptions.\(^9\) It was also shown that the BRST operators in the N=1 and N=2 formalisms are related by a similarity transformation.\(^26\) If one chooses the critical N=1 string to be an RNS string in a Calabi-Yau background, the right-moving $c = 6$ N=2 stress-energy tensor is:

$$L = \frac{1}{2} \partial_{-} x^{m} \partial_{-} x_{m} + \psi^{m} \partial_{-} \psi_{m} + L_{\text{CY}}^{RNS} - 3 \frac{1}{2} \partial_{-} \gamma - \frac{1}{2} \gamma \partial_{-} \beta - \frac{3}{2} b \partial_{-} c + \frac{1}{2} c \partial_{-} b + \frac{1}{2} \partial_{-} (\xi \eta),$$

$$G = b, \quad \tilde{G} = \gamma (\psi^{m} \partial_{-} x_{m} + G_{\text{CY}}^{RNS} + \tilde{G}_{\text{CY}}^{RNS}) +$$

$$e (\frac{1}{2} \partial_{-} x^{m} \partial_{-} x_{m} + \psi^{m} \partial_{-} \psi_{m} + L_{\text{CY}}^{RNS} - 3 \frac{1}{2} \partial_{-} \gamma - \frac{1}{2} \gamma \partial_{-} \beta - b \partial_{-} c) - \gamma^{2} b + \partial^{2} c + \partial_{-} (\xi \eta),$$

$$J = cb + \eta \xi,$$

where $x^{m}$ and $\psi^{m}$ for $m = 0$ to 3 are the four-dimensional RNS matter fields, $[b,c]$ and $[\beta = \partial_{-} \xi e^{-\phi}, \gamma = \eta e^{\phi}]$ are the twisted RNS ghost fields, and $[L_{\text{CY}}^{RNS}, G_{\text{CY}}^{RNS}, \tilde{G}_{\text{CY}}^{RNS}, J_{\text{CY}}^{RNS}]$ is the $c = 9$ N=2 stress-energy tensor $T_{\text{CY}}^{RNS}$ for the RNS Calabi-Yau background (as will be shown in equation (36)), $T_{\text{CY}}^{RNS}$ is related by a field redefinition to the $c = 9$ N=2 stress-energy tensor of the GS Calabi-Yau background, $T_{\text{CY}}$ of equation (12).

In reference 10, it was shown that for a flat background, the N=2 tensor constructed out of the ten-dimensional RNS matter and ghost fields is mapped by a field redefinition onto the N=2 tensor for the ten-dimensional GS fields (this was how the N=2 RNS tensor was originally found). It is easy to modify this field redefinition for a curved Calabi-Yau background so that it maps the N=2 RNS tensor of equation (34) onto the N=2 GS tensor $T_{GS} + T_{CY}$ of equations (2) and (12).

The first step is to define a "chiral" set of GS variables by performing the unitary transformation,

$$\tilde{\phi} = e^{-\int d\tau [i \partial_{-} x_{\alpha} \theta^{\alpha} \tilde{\alpha}^{\alpha} + e^{-\phi}(\theta)^{2} G_{\text{CY}}]} \Phi e^{\int d\tau [i \partial_{-} x_{\alpha} \theta^{\alpha} \tilde{\alpha}^{\alpha} + e^{-\phi}(\theta)^{2} G_{\text{CY}}]},$$

where $\Phi$ includes all GS fields defined in Section II. In terms of these chiral GS variables, $G_{GS} + G_{\text{CY}}$ is simply $e^{\tilde{\phi}}(\tilde{\phi})^{2}$. The field redefinition from these chiral GS variables to the RNS variables is:

$$\tilde{x}_{GS}^{m} = x_{RNS}^{m}, \quad \partial_{-} \tilde{\phi} = -3 \partial_{-} \phi + cb + 2 \xi \eta - J_{\text{CY}}^{RNS},$$

$$\tilde{\beta}^{1} = c \xi e^{\frac{1}{3} - \phi + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} - J_{\text{CY}}^{RNS}]} \tilde{\beta}^{1} = c e^{\frac{1}{3} - \phi + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} - J_{\text{CY}}^{RNS}]},$$

$$\tilde{\beta}^{2} = e^{\frac{1}{3} + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} + J_{\text{CY}}^{RNS}]}, \quad \tilde{\beta}^{3} = e^{\frac{1}{3} + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} + J_{\text{CY}}^{RNS}]},$$

$$\tilde{\phi}^{1} = b \eta e^{\frac{1}{4} - \phi + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} + J_{\text{CY}}^{RNS}]}, \quad \tilde{\phi}^{2} = b \eta e^{\frac{1}{4} - \phi + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} + J_{\text{CY}}^{RNS}]},$$

$$\tilde{\phi}^{3} = b \eta e^{\frac{1}{4} - \phi + \int [-\psi_{0} \psi^{4} + \psi^{2} \psi^{3} + J_{\text{CY}}^{RNS}]}.$$
\[ \tilde{p}_1 = e^{\frac{1}{2}(\phi + \int^z [\psi^0 \phi^1 + \psi^2 \phi^3 - J_{GV}^{RNS}] )}, \quad \tilde{p}_2 = e^{\frac{1}{2}(\phi + \int^z [\psi^0 \phi^1 - \psi^2 \phi^3 - J_{GV}^{RNS}] )}, \]

\[ \tilde{L}_{GV} = L_{GV}^{RNS} + \frac{3}{2}(\partial_\phi + \eta \xi)^2 - (\partial_\phi + \eta \xi)J_{GV}^{RNS}, \quad \tilde{G}_{GV} = e^\phi \eta G_{GV}^{RNS} \]

Note that any GS field can be transformed to an RNS field, but only GS O-projected RNS fields (i.e., fields which have no square-root cuts with the spacetime-supersymmetry generator) can be transformed into single-valued GS fields. In addition to mapping the N=2 GS tensor \( T_{GS} + T_{GV} \) onto the N=2 RNS tensor of equation (34), this field redefinition maps the integrated GS vertex operators (in the picture of equation (34), this field redefinition maps the integrated GS vertex operators (in the picture of RNS operator. In other words, the GS operators for chiral fermions in the GS formalism get mapped onto Ramond states in the \( +\frac{1}{2} \) picture, while the vertex operators for anti-chiral fermions get mapped onto Ramond states in the \( -\frac{1}{2} \) picture. In the N=1 RNS formalism, this identification of chirality and ghost number would be inconsistent since it would force amplitudes to vanish unless they had a fixed number of chiral minus anti-chiral external states. In the N=2 formalism, however, there is no inconsistency because of the sum over instanton number which picks up contributions from different RNS pictures.

V. Possible Applications

There are various longstanding problems in superstring theory for which the results of this paper might be useful. One such problem is to calculate multiloop superstring amplitudes in a manifestly SO(3,1) super-Poincaré invariant manner. In previous papers by this author,\(^2\) multiloop GS amplitudes were calculated in a flat background using variables which were not manifestly SO(3,1) covariant. The advantage of using non-covariant variables is that it is easy to compare with light-cone gauge calculations to check that the
amplitudes are unitary. Because the rules for integrating over the U(1) moduli are not straightforward, this check of unitarity is especially important.

Of course, the advantage of using covariant variables is that the amplitude calculations would be manifestly SO(3,1) super-Poincaré invariant. However, proving equivalence with light-cone calculations may be tricky since, as was described at the end of Section IIA, light-cone gauge fixing using the covariant variables involves bosonization.

Another problem which can be re-evaluated using the techniques of this paper is to derive the superstring corrections to the classical equations of motion for the supergravity and super-Yang-Mills fields. This is done by coupling the GS superstring to a curved four-dimensional background and requiring that the resulting non-linear sigma model is conformally invariant. Previous attempts to derive the superstring corrections used semi-light-cone gauge fixing of the fermionic symmetries, which led to incomplete results in four dimensions. Since an N=(2,0) non-linear sigma model for the superstring in a curved background has already been constructed, it should be straightforward to use the N=(2,0) techniques described in this paper to derive the superstring corrections to the massless equations of motion.

A third possible application of this paper is to develop a manifestly SO(3,1) super-Poincaré invariant superstring field theory. Although the role of the chiral boson, \( \rho \), needs to be better understood, it should be possible to construct a field theory action of the form \( \langle \Phi Q \Phi + Z \bar{Z} \Phi^3 + \ldots \rangle \). As in the RNS string field theory, contact terms will be necessary to cancel the divergences of colliding picture-changing operators, \( Z \) and \( \bar{Z} \).

An obvious question is if it is possible to quantize the superstring in a flat background and preserve all SO(9,1) super-Poincaré invariance. A major obstacle to this goal is that the sixteen \( \theta \) variables of SO(9,1) superspace are not free fields. The easiest way to see this is that in the RNS formalism, \( \theta^\alpha = e^{\frac{2i}{\lambda}} S^\alpha \) where \( S^\alpha \) is the spin field constructed by bosonizing the ten \( \psi^\mu \)'s. But \( \theta^\alpha (y) \theta^\beta (z) \rightarrow (y - z)^{-1} \epsilon_{\alpha \beta} \psi^\mu \) which is not a free-field operator product. Note that the maximum number of \( \theta \)'s which have no singularities with each other is five, which transform as \( 1_1 \) and \( 4_2 \) representations under the subgroup SU(4) × U(1) of SO(9,1). So unless some new description of ten-dimensional superspace is constructed in which the \( \theta \)'s are not fundamental fields, it seems unlikely that the superstring will be quantizable in a manifestly SO(9,1) super-Poincaré invariant manner.

It is interesting to note that the most natural description of the GS superstring in ten flat dimensions contains N=8 worldsheet supersymmetry, rather than N=2. Since the superstring description of this paper can be obtained by embedding the N=1 RNS string into an N=2 string, perhaps one should try to embed the RNS string into an N=8 string. It is possible that a new picture of ten-dimensional superspace could emerge from the resulting N=8 description of the superstring.
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