The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach: Potential contributions

J. Blümleina,*, A. Maiera, P. Marquarda, G. Schäferb

a Deutsches Elektronen–Synchrotron, DESY, Platanenallee 6, D–15738 Zeuthen, Germany
b Theoretisch-Physikalisches Institut, Friedrich Schiller-Universität, Max Wien-Platz 1, D–07743 Jena, Germany

Received 27 October 2020; received in revised form 8 February 2021; accepted 18 February 2021
Available online 24 February 2021
Editor: Stephan Stieberger

Abstract

We calculate the potential contributions of the motion of binary mass systems in gravity to the fifth post–Newtonian order ab initio using coupling and velocity expansions within an effective field theory approach based on Feynman amplitudes starting with harmonic coordinates and using dimensional regularization. Furthermore, the singular and logarithmic tail contributions are calculated. We also consider the non–local tail contributions. Further steps towards the complete calculation are discussed. We calculate all but the rational $O(\nu^2)$ contributions to the bound state energy for circular motion and periastron advance $K(\dot{E}, j)$. Comparisons are given to results in the literature.

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1. Introduction

The measurement of gravitational wave signals from merging black holes and neutron stars [1] has been a recent milestone in astrophysics. The different gravitational wave detectors are reaching higher and higher sensitivity [2], which requests to provide more detailed predictions at the theoretical side. Currently in binary Hamiltonian dynamics the level of the 4th post–Newtonian...
(PN) order has been fully understood and agreeing results have been obtained using a variety of different computation techniques in quite a series of gauges which lead to identical predictions in all key observables [3–9]. Moreover, it has been shown by applying canonical transformations [9], that all descriptions are dynamically equivalent. The different approaches can only be compared either by using canonical transformations, which requires local representations, or by calculating observables.

At the level of the 5th post–Newtonian order, first two agreeing results on the static potential in the harmonic gauge were calculated [10,11]. Later first results were derived using different matching techniques for the Hamiltonian in the effective one body (EOB) approach in [12,13]. Here two parameters, $\tilde{d}_5$ and $a_6$, which are of $O(v^2)$, with $v = m_1m_2/(m_1 + m_2)^2$, remained yet undetermined and the $O(v)$ terms have been fixed using results from self–force calculations, see e.g. [24].

The conserved Hamiltonian of the motion of binary mass systems in gravity has the following expansion

$$H = \sum_{k=0}^{\infty} H_{kPN},$$

where $k$ labels the post–Newtonian order, with $H_{0PN} \equiv H_N$. From $k = 4$ onward $H_{kPN}$ consists out of the term due to potential interactions, $H_{pot}^{kPN}$, and the tail (far-zone) terms, $H_{tail}^{kPN}$,

$$H_{kPN} = H_{pot}^{kPN} + H_{tail}^{kPN}.$$  

In effective field theory approaches based on Feynman diagrams this is the most natural decomposition. In [12,13] another decomposition has been chosen into the so–called non–local terms $H_{kPN}^{nl}$ and the local terms $H_{kPN}^{loc}$,

$$H_{kPN} = H_{kPN}^{loc} + H_{kPN}^{nl}.$$  

The non–local terms are fully contained in the tail terms and the local contributions are given by the local parts of the tail terms and the potential contributions. Later on we will use both representations, since the EFT method accesses them both.

In the present paper we calculate the 5PN potential corrections and some first parts of the 5PN tail terms using an effective field theory (EFT) approach; for related reviews see [25–29]. Here we follow Ref. [30]. A series of technical details for the calculation of the potential terms have already been presented in Refs. [9,11] before. In the case of the tail terms one first applies the multi–pole expansion valid for the far zone [3,13,15,26,28,32–39,41] to the respective post–Newtonian order and then applies EFT methods to calculate their contribution, cf. [42].

Expansions of this type generally belong to the operator product expansions [43]. In the calculation one also applies the method of expansion by regions [44,45].

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1 First results at 6PN order have been given in [14–16] recently. There is also a lot of activity in calculating post–Minkowskian corrections, cf. [14] Ref. [12], and [17–23].
2 Here we do not deal with conserved half PN contributions occurring from 5.5 PN onward.
3 For the 4PN calculations see [3,4,11].
4 Following the ideas in [31].
5 The present authors notified the authors of [42] having obtained differing results, also by sending notes, which has led to version_4 of the said paper.
Table 1

Numbers of contributing diagrams at the different loop levels and master integrals.

| #Loops | QGRAF | Source irred. | No source loops | No tadpoles | Masters |
|--------|-------|---------------|-----------------|-------------|---------|
| 0      | 3     | 3             | 3               | 3           | 0       |
| 1      | 72    | 72            | 72              | 72          | 1       |
| 2      | 3286  | 3286          | 3286            | 2702        | 1       |
| 3      | 81526 | 62246         | 60998           | 41676       | 7       |
| 4      | 545812| 264354        | 234934          | 116498      | 4       |
| 5      | 332020| 128080        | 101570          | 27582       | 4       |

In the present paper observables at 5PN such as the energy and periastron advance at circular orbits could not yet be calculated in complete form, since a series of differences with the literature have still to be fully clarified. This concerns rational terms of \(O(\nu^2)\) contributing to the tail term. However, we obtain all other contributions, including the \(O(\nu)\) terms and the \(\pi^2\) contributions to the yet undetermined constants \(d_5\) and \(a_6\) at \(O(\nu^2)\), in [13]. Furthermore, quite a series of comparisons could be performed with the literature, where different methods have been used.

The paper is organized as follows. In Section 2 we describe the calculation of the 5PN potential terms and present the associated Hamiltonian \(H_{\text{pot}}^{\text{5PN}}\) in the harmonic gauge. We use dimensional regularization in \(D = 4 - 2\epsilon\) dimensions. It is this method which allows a particular elegant merging of the potential and tail contributions in the conservative Hamiltonian, as we will show below. Already at 3PN the contributions to \(H_{\text{pot}}^{\text{5PN}}\) have poles in \(1/\epsilon\), cf. [46]. From 4PN corresponding poles also appear in the tail terms. We will discuss the main aspects of the 5PN tail term in Section 3 and construct a pole–free Hamiltonian in Section 4. Here we show that the poles in the combined Hamiltonian can be transformed away by a canonical transformation. In Section 5 we compare to results given in the literature and discuss open questions. A canonical transformation from harmonic to EOB coordinates is performed. We derive the non–local tail contributions and calculate their contribution to the binding energy in the circular case and to the local contributions to periastron advance. We then determine the \(\pi^2\) contributions to the previously unknown constants \(d_5\) and \(a_6\). Section 6 contains the conclusions. In the appendices some technical aspects are presented on the merging of the potential and tail terms using the method of expansion by regions. As well we present longer formulae, which emerge from the present calculation.

2. The potential contributions to the Hamiltonian

The calculation of the 5PN corrections is performed in the same way as has been described in Refs. [9,11]. Starting from the Einstein–Hilbert Lagrangian, we parameterize the metric \(g_{\mu\nu}\) according to Ref. [30] in terms of scalar, vector and tensor fields, and work in the harmonic gauge.\(^6\) The Feynman diagrams are generated using QGRAF [47]. The Lorentz algebra is carried out using Form [48] and we perform the integration by parts (IBP) reduction to master integrals using the code Cruncher [49]. Table 1 gives an overview on the present calculation.

From the graphs generated by QGRAF one has to remove the source reducible graphs, graphs with source loops and tadpoles. In this way the 962719 initial diagrams reduce to 188533 diagrams. The computation time amounts to about one week, including the time for the IBP

\(^6\) At a later stage technical steps will require to move away from the harmonic gauge, which we will explain in detail.
reduction, on an Intel(R) Xeon(R) CPU E5-2643 v4 and it grows exponentially with the loop order. Most of the CPU time is needed to perform the time derivatives. Only one non–trivial master integral contributes, see \[8,11\].

One first obtains a Lagrange function of \(m\)th order still containing the accelerations \(a_i\) and time derivatives thereof. They are removed by using first double zero insertions \[50,51\] together with partial integration and the remaining linear accelerations by a shift \[26,51–53\], cf. \[9\]. By this operation we leave harmonic coordinates. A Legendre transformation leads then to the potential contributions of the Hamiltonian, which still contains pole terms in the dimensional parameter \(\epsilon\).\(^7\) The reduced Hamiltonian in the cms is given by

\[
\hat{H} = \frac{H - Mc^2}{\mu c^2},
\]

with \(c\) the velocity of light, \(M = m_1 + m_2\) the rest mass of the binary system and \(\mu = m_1 m_2 / M\),

\[
\hat{H}_{\text{pot}}^{\text{SPN}} = \frac{-21 p_1^{12}}{1024} + \frac{5}{16r^6} - \frac{125p_2^2}{16r^6} - \frac{499 p_4^2}{64r^4} + \frac{161p_6^4}{32r^3} - \frac{445 p_8^2}{256r^2} - \frac{77 p_{10}^2}{256r} + \frac{17(p.n)^2}{4r^5} - \frac{29 p_2^2(p.n)^2}{8r^4} + \frac{21 p_4^4(p.n)^2}{16r^3} + \frac{5 p_6^6(p.n)^2}{32r^2} - \frac{(p.n)^4}{8r^4} + \frac{1}{\epsilon} \left\{ v^3 \left[ -\frac{520900}{37800r^6} - \frac{698242p_2^2}{112r^3} + \frac{52957p_4^4}{28r^4} - \frac{13583p_6^6}{336r^3} + \frac{23569(p.n)^2}{540r^5} - \frac{1895597p_2^2(p.n)^2}{2520r^4} + \frac{162323(p.n)^4}{28r^4} - \frac{130p_2^2(p.n)^4}{r^3} - \frac{91(p.n)^6}{6r^3} \right] \right\} + v \left[ -\frac{272309}{12600r^6} + \frac{2439p_2^2}{12600r^5} - \frac{49023p_4^4}{560r^4} + \frac{1173p_6^6}{80r^3} - \frac{210947(p.n)^2}{2520r^5} + \frac{25169p_2^2(p.n)^2}{105r^4} - \frac{2271p_4^4(p.n)^2}{80r^3} - \frac{13059(p.n)^4}{70r^4} - \frac{81p_2^2(p.n)^4}{r^3} + \frac{77(p.n)^6}{r^3} \right] \right\} + v \left[ \frac{28811p_2^2}{210r^5} - \frac{29750p_4^4}{2520r^4} - \frac{6889p_6^6}{360r^3} - \frac{3068(p.n)^2}{7r^5} + \frac{352834p_2^2(p.n)^2}{315r^4} + \frac{16538p_4^4(p.n)^2}{105r^3} - \frac{304669(p.n)^4}{240r^4} - \frac{18979p_2^2(p.n)^4}{56r^3} + \frac{2891(p.n)^6}{12r^3} \right] \right\} + v \left[ \frac{231p_1^{12}}{1024} - \frac{253555919}{529200r^6} - \frac{145787251p_2^2}{2116800r^5} + \frac{2128837091p_4^4}{1411200r^4} + \frac{11206267p_6^6}{141120r^3} + \frac{937p_2^8}{32r^2} + \frac{805p_1^{10}}{256r} + \frac{70399}{1152r^6} + \frac{65291p_2^2}{1152r^5} - \frac{132847p_4^4}{12288r^4} - \frac{7719p_6^6}{4096r^3} + \frac{6649(p.n)^2}{576r^5} + \frac{5042575p_2^2(p.n)^2}{6144r^4} + \frac{58887p_4^4(p.n)^2}{4096r^3} - \frac{3293913(p.n)^4}{4096r^4} \right].
\]

\(^7\) As also the case in renormalizable quantum field theories, Langrangians and Hamiltonians are in general no observables and are generally singular.
\[-\frac{89625 p^2(p.n)^4}{4096 r^3} + \frac{42105 (p.n)^6}{4096 r^3} \right) + \ln \left( \frac{r}{r_0} \right) \left( -\frac{272309}{1050 r^6} + \frac{22439 p^2}{1260 r^5} \right. \\
\left. - \frac{49023 p^4}{70 r^4} + \frac{3519 p^6}{40 r^3} - \frac{210947 (p.n)^2}{252 r^5} + \frac{201352 p^2 (p.n)^2}{105 r^4} - \frac{6813 p^4 (p.n)^2}{40 r^3} \right. \\
\left. - \frac{52326 (p.n)^4}{35 r^4} - \frac{486 p^2 (p.n)^4}{r^3} + \frac{462 (p.n)^6}{r^3} \right) + \frac{467022407 (p.n)^2}{2116800 r^5} \\
\left. - \frac{2385014243 p^2 (p.n)^2}{282240 r^4} - \frac{162949463 p^4 (p.n)^2}{235200 r^3} - \frac{589 p^6 (p.n)^2}{16 r^2} \right. \\
\left. - \frac{35 p^8 (p.n)^2}{256 r} + \frac{1895797259 (p.n)^4}{235200 r^4} + \frac{31715507 p^2 (p.n)^4}{235200 r^3} + \frac{8951 p^4 (p.n)^4}{384 r^2} \right. \\
\left. - \frac{627281 (p.n)^6}{960 r^3} - \frac{5117 p^2 (p.n)^6}{320 r^2} + \frac{159 (p.n)^8}{28 r^2} \right) + \frac{v^2}{r^2} \left[ -\frac{231 p^{12}}{256} + \frac{295859}{1050 r^6} \right] \\
\left. + \frac{165238903 p^2}{529200 r^5} - \frac{420686323 p^4}{132300 r^4} + \frac{3605263 p^6}{29400 r^3} - \frac{11535 p^8}{128 r^2} - \frac{2865 p^{10}}{256 r} \right. \\
\left. + \ln \left( \frac{r}{r_0} \right) \left( -\frac{520909}{3150 r^6} - \frac{1396484 p^2}{945 r^5} + \frac{592957 p^4}{315 r^4} - \frac{13583 p^6}{56 r^3} + \frac{23569 (p.n)^2}{54 r^5} \right) \right. \\
\left. - \frac{1895597 p^2 (p.n)^2}{315 r^4} + \frac{69141 p^4 (p.n)^2}{56 r^3} + \frac{32446 (p.n)^4}{7 r^4} - \frac{780 p^2 (p.n)^4}{r^3} \right. \\
\left. - \frac{91 (p.n)^6}{r^3} + \pi^2 \left( \frac{11573}{768 r^6} - \frac{121315 p^2}{768 r^5} + \frac{2076041 p^4}{12288 r^4} + \frac{29987 p^6}{4096 r^3} \right. \right. \\
\left. \left. - \frac{200359 (p.n)^2}{768 r^5} - \frac{5962205 p^2 (p.n)^2}{6144 r^4} - \frac{172311 p^4 (p.n)^2}{4096 r^3} + \frac{2617363 (p.n)^4}{4096 r^4} \right) \right. \\
\left. + \frac{127125 p^2 (p.n)^4}{4096 r^3} + \frac{14175 (p.n)^6}{4096 r^3} \right) - \frac{944072707 (p.n)^2}{264600 r^5} \\
\left. + \frac{35606467999 p^2 (p.n)^2}{2116800 r^4} - \frac{1945067 p^4 (p.n)^2}{2450 r^3} + \frac{4969 p^6 (p.n)^2}{64 r^2} + \frac{275 p^8 (p.n)^2}{256 r} \right. \\
\left. - \frac{1742633989 (p.n)^4}{117600 r^4} + \frac{848889 p^2 (p.n)^4}{1568 r^3} + \frac{925 p^4 (p.n)^4}{24 r^2} \right. \\
\left. + \frac{15 p^6 (p.n)^4}{128 r} + \frac{18031 (p.n)^6}{3360 r^3} - \frac{8331 p^2 (p.n)^6}{160 r^2} + \frac{751 (p.n)^8}{28 r^2} \right) + \pi^2 \left[ -\frac{1617 p^{12}}{1024} \right. \\
\left. - \frac{298537367 p^2}{151200 r^5} + \frac{617770201 p^4}{423360 r^4} + \frac{108551131 p^6}{4233600 r^3} + \frac{16283 p^8}{256 r^2} + \frac{3995 p^{10}}{256 r} + \pi^2 \right. \\
\left. \times \left( -\frac{2339 p^2}{192 r^5} + \frac{98447 p^4}{3072 r^4} - \frac{20259 p^6}{192 r^5} + \frac{16111 (p.n)^2}{1536 r^4} + \frac{131231 p^2 (p.n)^2}{1536 r^4} \right) \right. \\
\left. + \frac{106947 p^4 (p.n)^2}{1024 r^3} - \frac{361499 (p.n)^4}{1024 r^3} - \frac{30075 p^2 (p.n)^4}{1024 r^3} - \frac{65625 (p.n)^6}{1024 r^3} \right) \right]
\[ + \ln \left( \frac{r}{r_0} \right) \left( \frac{28811p^2}{21r^5} - \frac{297509p^4}{315r^4} - \frac{6889p^6}{60r^3} - \frac{30680(p.n)^2}{7r^5} \right) \]
\[ + \frac{2822672p^2(p.n)^2}{315r^4} + \frac{33076p^4(p.n)^2}{35r^3} - \frac{304669(p.n)^4}{30r^4} - \frac{56937p^2(p.n)^4}{28r^3} \]
\[ + \frac{2891(p.n)^6}{2r^3} \left( \frac{966353501(p.n)^2}{151200r^5} - \frac{3656476457p^2(p.n)^2}{235200r^4} \right) \]
\[ - \frac{2369976949p^4(p.n)^2}{1411200r^3} + \frac{177p^6(p.n)^2}{256r^2} - \frac{221p^8(p.n)^2}{64r} + \frac{1403555739(p.n)^4}{705600r^4} \]
\[ + \frac{373945981p^2(p.n)^4}{94080r^3} - \frac{125225p^4(p.n)^4}{768r^2} - \frac{3p^6(p.n)^4}{128r} - \frac{14830647(p.n)^6}{4480r^3} \]
\[ + \frac{136977p^2(p.n)^6}{1280r^2} - \frac{15p^4(p.n)^6}{128r} - \frac{289839(p.n)^8}{4480r^2} - \frac{35p^2(p.n)^8}{256r} \]
\[ + \nu^4 \left[ - \frac{1155p^{12}}{1024} - \frac{593p^6}{32r^3} + \frac{6649p^8}{256r^2} - \frac{1615p^{10}}{256r} + \frac{549p^4(p.n)^2}{32r^3} \right] \]
\[ - \frac{62143p^6(p.n)^2}{256r^2} + \frac{867p^8(p.n)^2}{256r} - \frac{5749p^2(p.n)^4}{96r^3} - \frac{3p^6(p.n)^4}{64r} \]
\[ + \frac{652381p^4(p.n)^4}{768r^2} - \frac{17623p^6(p.n)^6}{240r^3} - \frac{1178329p^2(p.n)^6}{1280r^2} \]
\[ - \frac{45p^4(p.n)^6}{128r} + \frac{1443091p^2(p.n)^8}{4480r^2} + \frac{105p^2(p.n)^8}{128r} \]
\[ + \nu^5 \left[ \frac{231p^{12}}{1024} - \frac{63p^{10}}{256r} \right] \]
\[ + \frac{35p^8(p.n)^2}{256r} - \frac{15p^6(p.n)^4}{128r} - \frac{15p^4(p.n)^6}{128r} - \frac{35p^2(p.n)^8}{256r} - \frac{63(p.n)^10}{256r} \] \tag{5}

with

\[ r_0 = \frac{e^{-\gamma_E/2}}{2\sqrt{\pi} \mu_1}, \] \tag{6}

where \( \gamma_E \) is the Euler–Mascheroni constant and \( \mu_1 \) the mass scale accounting for Newton’s constant \( G_N \to G_N \mu_1^{-2e} \) in \( D \) dimensions. The corresponding contributions up to 4PN have been presented in [9] before. We rescale

\[ p = p_{\text{phys}}/(\mu c), \quad r = \left( \frac{G_NM}{c^2} \right)^{-1} r_{\text{phys}}, \] \tag{7}

where \( p \) and \( r \) are now the rescaled (dimensionless) cms momentum and the distance of the two masses, with \( \tilde{m} = \tilde{r}/r \). In the following we will as widely as possible work with dimensionless quantities.

Pole and logarithmic contributions appear at \( O(\nu) \), \( O(\nu^2) \) and \( O(\nu^3) \), in accordance with the lower PN orders, where also always one more order in \( \nu \) contributes from 3PN onward. We will see in Section 4 that the tail term is only singular for \( O(\nu) \) and \( O(\nu^2) \) at 5PN.

In the Schwarzschild limit, \( \nu \to 0 \), one obtains the following contributions,

\[ \hat{H}_{5\text{PN}}^{\text{Schw}} = -\frac{21p^{12}}{1024} - \frac{77p^{10}}{256r} - \frac{445p^8}{256r^2} - \frac{161p^6}{32r^3} - \frac{499p^4}{64r^4} - \frac{125p^2}{16r^5} + \frac{5p^6(p.n)^2}{32r^2} \]
\[ + \frac{21 p^4 (p.n)^2}{16 r^3} + \frac{29 p^2 (p.n)^2}{8 r^4} - \frac{(p.n)^4}{8 r^4} + \frac{17 (p.n)^2}{4 r^5} + \frac{5}{16 r^6}, \]  

in agreement with the expansion of Eq. (30), \cite{9}, to 5PN, cf. \cite{54,55}.

3. Remarks on the tail term

We will will derive a pole–free Hamiltonian at 5PN in Section 4. For this we will add the singular and logarithmic terms of the tail term, \( \hat{H}_{5\text{PN}}^{\text{tail}, \text{sing, log}} \), to the potential term \( \hat{H}_{5\text{PN}}^{\text{pot}} \). Since these contributions are calculated, by different methods, either in the far zone (FZ) or in the near zone (NZ), the question arises whether potential overlap contributions have to be considered. We remind that the calculation is performed in \( D \) dimensions, not using any other regularization.

One may apply the method of expansion by regions, which has been introduced for the asymptotic expansion of Feynman integrals for bound states in the non–relativistic limit in \cite{38,39}. Here each loop integral is split into four \textit{distinct} momentum regions, which are denoted as hard, soft, potential, and ultrasoft. Integrals over the hard and soft region correspond to quantum corrections and are not considered in the context of classical gravity.

The potential region, characterized by the momentum scaling

\[ |k_0| \sim \frac{v}{r_{\text{phys}}}, \quad |k_i| \sim \frac{1}{r_{\text{phys}}}, \]

with \( v \in [v_1, v_2] \) and \( v = v_{\text{phys}}/c \) the typical velocities, is also referred to as orbital region. Here \( k_i \) is not rescaled. However, we set the associated action variable to 1. \footnote{In the quantum field theoretic case this would correspond to \( \hbar = 1 \).} It can be identified with the near zone of the literature (i.e. the potential terms). In the ultrasoft (or radiation region), corresponding to the far zone (i.e. the tail terms), momenta exhibit the uniform four–momentum scaling

\[ |k_\mu| \sim \frac{v}{r_{\text{phys}}}. \]

For definiteness, let us introduce an auxiliary distance parameter \( R \) with \( r_{\text{phys}} \leq R \leq \frac{r_{\text{phys}}}{|v|} \) to separate the two regions. The ultrasoft spatial momentum region is then given by the \( D - 1 \) dimensional solid sphere with radius \( \frac{1}{R} \), viz.

\[ \vec{k} \in D_{\text{us}} = \left\{ \vec{p} \in \mathbb{R}^{D-1} : |\vec{p}| \leq \frac{1}{R} \right\}. \]

and the kinematic region of the potential term is

\[ \vec{k} \in D_{\text{pot}} = \mathbb{R}^{D-1} \setminus D_{\text{us}}. \]

In the former region the exchanged fields are ultrasoft gravitons and in the latter region potential gravitons. One performs a Taylor expansion of the integrands according to the respective momentum scaling in \( v \) up to the respective post–Newtonian order by observing that

\[ v^2 \sim \frac{1}{r}. \]
For the tail terms this expansion includes the multi–pole expansion, which we will discuss below.\textsuperscript{9} Let us introduce the operators $T_{\text{pot}}$ and $T_{\text{us}}$, which describe the Taylor expansions (with a few Laurent–terms in some cases) in the potential region and the ultrasoft region. In the post–Newtonian expansion they are given, more precisely, by

$$T_i^N I_0(v) := \theta(N) \sum_{k=0}^{\infty} T_{i,k} v^k,$$

(14)

with the quantifier $\theta(N)$ truncating the series at a maximal term $v^N$, $N \in \mathbb{N}$, which is idempotent $\theta^2(N) = \theta(N)$. Here the coefficients $T_{i,k}$ denote the expansion coefficients of the function $I_0(v)$. The corresponding integrals have the following form

$$I_1 = \int_{D_{\text{pot}}} d^{D-1}k T_{\text{pot}}^N I + \int_{D_{\text{us}}} d^{D-1}k T_{\text{us}}^N I,$$

(15)

where $I$ denotes the original integrand. One further obtains

$$I_1 = \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{pot}}^N I - \int_{D_{\text{us}}} d^{D-1}k T_{\text{pot}}^N I + \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{us}}^N I - \int_{D_{\text{pot}}} d^{D-1}k T_{\text{us}}^N I.$$

(16)

In the respective domains $D_{\text{pot}}$ and $D_{\text{us}}$ one may further apply the operators $T_{\text{pot}}$ and $T_{\text{us}}$ given the post–Newtonian accuracy one is working in. One then obtains

$$I_1 = \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{pot}}^N I - \int_{D_{\text{us}}} d^{D-1}k T_{\text{us}}^N I + \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{us}}^N I - \int_{D_{\text{pot}}} d^{D-1}k T_{\text{us}}^N I.$$

(17)

Here the 2nd and 4th term are the overlap integrals. Eq. (17) can be further arranged to

$$I_1 = \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{pot}}^N I + \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{us}}^N I - \int_{\mathbb{R}^{D-1}} d^{D-1}k T_{\text{us}}^N I,$$

(18)

provided that

$$T_{\text{us}} T_{\text{pot}}^N - T_{\text{pot}} T_{\text{us}}^N = 0$$

(19)

holds, which we prove in Appendix A. Furthermore, the operation $T_{\text{us}} T_{\text{pot}}$ leads to scaleless integrands, implying that the last term in Eq. (18) vanishes in $D$ dimensions, see Appendix A.

We finally would like to make some remarks on the relation on the multi–pole expansion [31] in the far zone to the ultrasoft region. One is starting from the full theory of general relativity in harmonic coordinates, i.e. the bulk action

$$S_{\text{GR.bulk}} = 2\Lambda^2 \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} \Gamma^\mu \Gamma_\mu \right).$$

(20)

\textsuperscript{9} As is often the case in EFT representations, the corresponding expansions are not just kinematic. An important example in this respect is the light-cone expansion [56]. In the most simple case of the twist–2 contributions its results are also obtained by the QCD improved parton model, resulting from a kinematic expansion. This is much more subtle at higher twist, where partonic pictures require further conditions to give the same result, cf. [57] for a survey.
Here \( \Lambda = c^2 \mu_1^e / \sqrt{32 \pi G_N} \), \( R \) is the Ricci scalar, \( \Gamma^\mu = \Gamma^\mu_{\alpha\beta} g^{\alpha\beta} \) with \( \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} (g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) \) is the Christoffel symbol. (20) is coupled to compact objects via the action

\[
S_{pp} = - \sum_{a=1}^{2} m_a \int d\tau_a,
\]

with proper times \( \tau_1, \tau_2 \), one decomposes the metric into

\[
g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{\Lambda} (H_{\mu\nu} + h_{\mu\nu}),
\]

where \( \eta_{\mu\nu} \) is the Minkowski metric. The momenta associated with \( H_{\mu\nu} \) are of the potential type and the momenta of the \( h_{\mu\nu} \) fields are ultrasoft, cf. Eqs. (12) and (11). The resulting loop integrals therefore have the same form as Eq. (15), as obtained from the asymptotic expansion.

The action of the full theory is matched to the non–relativistic general relativity (NRGR) action given by

\[
S_{NRGR} = S_{NRGR, bulk} + \left( S_{NRGR, h^0} + S_{NRGR, h^1} + \mathcal{O} \left( h^2 \right) \right),
\]

with

\[
S_{NRGR, h^0} = \int dt \left( T - V_{NZ} \right),
\]

\[
S_{NRGR, h^1} = \frac{1}{2\Lambda} \int d^D x \ T^{\mu\nu} h_{\mu\nu},
\]

where there are no potential modes anymore. As indicated by the \( \mathcal{O}(h^2) \) term in Eq. (23), we neglect contributions involving the effective coupling of two or more ultrasoft modes to matter or the classical potential. In the contribution to the action \( S_{NRGR, h^0} \) without any ultrasoft fields \( T \) denotes the kinetic term and \( V_{NZ} \) the near–zone potential. \( S_{NRGR, bulk} \) is the same as the general relativity bulk action \( S_{GR, bulk} \) from Eq. (20), but without the potential contributions to the metric. Both \( V_{NZ} \) and the effective stress–energy tensor \( T^{\mu\nu} \) are fixed by requiring that the NRGR action produces the same predictions as the asymptotic expansion of the full theory. In other words, the integrals over the potential region are absorbed into \( V_{NZ} \) and \( T^{\mu\nu} \).

We now elaborate on the relation to the multi–pole expansion. Consider

\[
h_{\mu\nu}(x) = \int \frac{d^D p}{(2\pi)^D} e^{i p x} h_{\mu\nu}(p),
\]

where the momentum is ultrasoft by definition, i.e.

\[
\vec{p} \cdot \vec{x} \sim v.
\]

We can Taylor expand the exponential in Eq. (26) to obtain

\[
h_{\mu\nu}(x) = \int \frac{d^D p}{(2\pi)^D} e^{-i p_0 x_0} \left( \sum_{n=0}^{N} \frac{i^n (\vec{p} \cdot \vec{x})^n}{n!} + \mathcal{O} \left( v^{N+1} \right) \right) h_{\mu\nu}(p).
\]

Rewriting

\[
i^n p_{i_1} \ldots p_{i_n} = \left[ \partial x_{i_1} \ldots \partial x_{i_n} e^{i \vec{p} \cdot \vec{x}} \right]_{\vec{x}=0}
\]

now yields
\begin{equation}
N(x) = \sum_{n=0}^{N} \frac{1}{n!} x_{i_1} \ldots x_{i_n} \left[ \partial x_{i_1} \ldots \partial x_{i_n} h_{\mu\nu}(x) \right]_{\tilde{\alpha}=0} + O\left(v^{N+1}\right).
\end{equation}

Inserting this expression back into the linear ultrasoft action Eq. (25) we retrieve the familiar starting point of the multi–pole expansion

\begin{equation}
S_{NRGR,h_1} = \frac{1}{2\Lambda} \int d^{D}x \ T^{\mu\nu}(x) \sum_{n=0}^{N} \frac{1}{n!} x_{i_1} \ldots x_{i_n} \left[ \partial x_{i_1} \ldots \partial x_{i_n} h_{\mu\nu}(x) \right]_{\tilde{\alpha}=0} + O\left(v^{N+1}\right),
\end{equation}

with an explicit velocity power counting. The remaining steps in the multi–pole expansion are standard. In short, one defines moments

\begin{equation}
M_{n}^{\mu\nu} = \int d^{D-1}\vec{x} \ T^{\mu\nu}(x) x_{i_1} \ldots x_{i_n},
\end{equation}

and decomposes them into the standard irreducible representations, choosing a symmetric trace–free (STF) basis, cf. e.g. [37]. In the more general scenario of \( D - 1 \) space dimensions the expressions for the multipole moments have to be adapted, see Ref. [39].\(^{10}\) We then follow Ref. [42] in order to extract the singular and local logarithmic tail terms from the multipole-expanded NRGR action.

4. A pole-free Hamiltonian at 5PN

While the potential and tail contributions are separately divergent, their sum has to yield finite predictions for all physical observables. In order to prove that our result fulfills this vital property, we now construct a manifestly pole-free Hamiltonian. For reference we also list the corresponding local contribution for the 4PN tail term, \( \hat{H}^{\text{tail,loc}}_{4\text{PN}} \), where we also included the rational term\(^{11}\)

\begin{equation}
\hat{H}^{\text{tail,loc}}_{4\text{PN}} = v \left[ \frac{1}{\epsilon} \left( \frac{4(p.n)^2}{5r^4} - \frac{8p^2}{3r^4} - \frac{8}{15r^5} \right) + \left( \frac{-376(p.n)^2}{75r^4} + \frac{48p^2}{5r^4} + \frac{184}{75r^5} \right) \right. \\
\left. + \left( \frac{16(p.n)^2}{5r^4} - \frac{32p^2}{3r^4} - \frac{16}{5r^5} \right) \ln \left( \frac{r}{r_0} \right) \right].
\end{equation}

We add the singular and local logarithmic pieces of the Hamiltonian of the tail term in 5PN, \( \hat{H}^{\text{tail,sing,log}}_{5\text{PN}} \),

\begin{equation}
\hat{H}^{\text{tail,sing,log}}_{5\text{PN}} = \frac{1}{\epsilon} \left[ \left( \frac{16\nu}{105} - \frac{332\nu^2}{105} \right) \frac{1}{r^6} + \left[ \left( \frac{236\nu}{35} - \frac{212\nu^2}{35} \right) p^2 - \left( \frac{684\nu}{35} + \frac{1264\nu^2}{105} \right) \right] \right].
\end{equation}

\(^{10}\) See also [40].
\(^{11}\) In [9] the corresponding term was obtained by directly adding it at the Hamilton level leading to a slightly different but equivalent representation.
\[
\times (p.n)^2 \frac{1}{r^5} + \left[ \left( \frac{533 \nu}{21} + \frac{706 \nu^2}{21} \right) p^4 - \left( \frac{7732 \nu}{35} + \frac{10936 \nu^2}{105} \right) p^2 (p.n)^2 \right. \\
+ \left( \frac{6197 \nu}{35} + \frac{2656 \nu^2}{35} \right) (p.n)^4 \right] \frac{1}{r^4} + \left[ \left( \frac{94 \nu}{15} - \frac{94 \nu^2}{5} \right) p^6 \\
+ \left( \frac{172 \nu}{5} + \frac{516 \nu^2}{5} \right) p^4 (p.n)^2 + \left( 26 \nu - 78 \nu^2 \right) p^2 (p.n)^4 \right] \frac{1}{r^3} \\
+ \left[ \left( \frac{128 \nu}{105} - \frac{2656 \nu^2}{105} \right) \nu \right] \frac{1}{r^3} + \left[ \left( \frac{1416 \nu}{35} - \frac{1272 \nu^2}{35} \right) p^2 \\
- \left( \frac{4104 \nu}{35} + \frac{2528 \nu^2}{35} \right) (p.n)^2 \right] \frac{1}{r^5} + \left[ \left( \frac{2132 \nu}{21} + \frac{2824 \nu^2}{21} \right) p^4 \\
- \left( \frac{30928 \nu}{35} + \frac{43744 \nu^2}{105} \right) p^2 (p.n)^2 \right] \frac{1}{r^4} \\
+ \left( \frac{24788 \nu}{35} + \frac{10624 \nu^2}{35} \right) (p.n)^4 \right] \frac{1}{r^4} + \left[ \left( \frac{188 \nu}{15} - \frac{188 \nu^2}{5} \right) p^6 + \left( - \frac{344 \nu}{5} \right) \\
+ \left( \frac{1032 \nu^2}{5} \right) p^4 (p.n)^2 + \left( 52 \nu - 156 \nu^2 \right) p^2 (p.n)^4 \right] \frac{1}{r^3} \right] \ln \left( \frac{r}{r_0} \right) 
\]

We have calculated the corresponding tail–term contributions due to the electric quadrupole, octupole, and magnetic quadrupole moments in a similar manner to [42]. We insert the explicit expressions for the multipole moments given in [39], eliminate higher time derivatives using multiple–zero subtraction and coordinate shifts, and perform a Legendre transformation to arrive at the singular and logarithmic tail contributions to the Hamiltonian. The logarithmic terms originate from \( \mathcal{O}(D - 4) \) contributions to the equations of motion.

The sum of the potential term and this contribution is not pole–free yet, as is the case from 3PN onward, cf. [9]. However, after performing the following canonical transformation a pole–free Hamiltonian is obtained, which is not the case for \( H_{\text{5PN}}^{\text{pot}} \) and \( H_{\text{5PN}}^{\text{tail,sing}} \) individually. By this transformation one further moves away from the harmonic coordinates, which were used at the starting point of the calculation. Still a prediction of all observables is possible. Moreover, the comparison with EOB results becomes simpler, since they are given in pole–free form [13].

Following the formalism described in Ref. [9], Eqs. (38)–(41), one obtains the corresponding generating function\(^\text{12}\)

\[
G(p^2, p.n, r; \epsilon) = p.n \left\{ \frac{1}{\epsilon} \left[ -t_3 \frac{17 \nu}{6r^2} + t_4 \left( \frac{1}{90} \nu (585 + 4 \nu) p^2 - \frac{1}{3} \nu (12 + 37 \nu) (p.n)^2 \right) \\
- \frac{\nu (65 + 264 \nu)}{30r^3} \right] + t_5 \left[ \frac{1}{r^3} \left( -\frac{1}{420} \nu (-43 - 69088 \nu + 52078 \nu^2) p^2 \right) \right] \right\}
\]

\(^\text{12}\) Note that this function is different from the one in [9] at 4PN. There we added the tail terms at the Hamiltonian level. Here we add the contributions at the Lagrangian level.
Furthermore, we transform the logarithmic part to explicitly match the structure of the non-local contribution from the tail term in harmonic coordinates,

\[
\delta H_{\log}^{4+5\text{PN}} = 2 \frac{G N E}{c^{10}} \left( \frac{1}{5} I^{(3)}(t)^2 + \frac{1}{189 c^2} O^{(4)}(t)^2 + \frac{16}{45 c^2} J^{(3)}(t)^2 \right) \ln \left( \frac{r}{r_0} \right),
\]

see also Section 5.2. Here the multi-pole moments \(I_{ab}, O_{abc}\) and \(J_{ab}\) are those of Eq. (2.4) in [13], with indices contracted, and \(E\) is the total energy. The higher-order derivatives have been eliminated by using the 1PN equation of motions. The corresponding transformation reads

\[
G(p^2, p.n; \ln(r/r_0)) = p.n \left\{ \ln \left( \frac{r}{r_0} \right) \left\{ -t_3 \frac{17 \nu}{r^2} + t_4 \left[ \frac{1}{r^2} \left( \frac{v(585 + 4\nu)}{15} p^2 - 2v(12 + 37\nu) \right. \right. \right. \\
- \left. \left. \left. \frac{4v(73 + 264\nu)}{15r^3} \right) \right] + t_5 \left[ \frac{1}{r^3} \left( -\frac{2}{105} v(7557 - 65496\nu) \right. \right. \right. \\
+ \left. \left. \left. 52078\nu p^2 + \frac{2}{315} v(21345 - 195484\nu + 264212\nu p^2) p^2 \nu(12 + 37\nu) \right) \right. \right. \right. \\
+ \frac{1}{r^2} \left( -\frac{1}{840} v(-106162 + 311132\nu + 70007\nu^2) p^4 + \frac{1}{140} v(1232 \right. \right. \right. \\
\left. \left. + 27888\nu + 43567\nu^2) p^2(p.n)^2 - \frac{1}{2} v(132 - 26\nu + 413\nu^2)(p.n)^4 \right) \right. \right. \right. \\
+ \frac{v(886461 + 331090\nu)}{3780r^4} \right\} \right\}.
\]

Here \(t_i, i = 1...5\) labels the \(i\)th post-Newtonian order.

The pole-free Hamiltonian based on the above contributions is then given by\(^\text{13}\)

\[
H_{5\text{PN}}^{\text{polefree}} = -\frac{21 p^{12}}{1024} + \frac{5}{16r^6} - \frac{125 p^2}{16r^5} - \frac{499 p^4}{64r^4} - \frac{161 p^6}{32r^3} - \frac{445 p^8}{256r^2} - \frac{77 p^{10}}{256r} \\
+ v \left[ \frac{231 p^{12}}{1024} - \frac{279775133}{529200r^6} - \frac{1450584679 p^2}{2116800r^5} + \frac{2010713771 p^4}{1411200r^4} \right].
\]

\(^{13}\) Note, that since finite local contributions from the far zone have not been included, \(H_{5\text{PN}}^{\text{polefree}}\) does not yet correspond to the complete local Hamiltonian.
\[ \pi^2 \left( \frac{70399}{1152 r^6} + \frac{65291 p^2}{1152 r^5} - \frac{1328147 p^4}{12288 r^4} - \frac{7719 p^6}{4096 r^3} + \frac{6649 (p.n)^2}{576 r^5} \right) \]

\[ + \frac{5042575 p^2 (p.n)^2}{6144 r^4} + \frac{58887 p^4 (p.n)^2}{4096 r^3} - \frac{3293913 (p.n)^4}{4096 r^3} + \frac{89625 p^2 (p.n)^4}{4096 r^3} \]

\[ + \frac{42105 (p.n)^6}{4096 r^3} - \frac{34514593 (p.n)^2}{2116800 r^5} - \frac{2395722563 p^2 (p.n)^2}{282240 r^4} \]

\[ - \frac{62196341 p^4 (p.n)^2}{78400 r^3} - \frac{589 p^6 (p.n)^2}{256 r} - \frac{35 p^8 (p.n)^2}{256 r} + \frac{631107353 (p.n)^4}{78400 r^4} \]

\[ + \frac{31226291 p^2 (p.n)^4}{23520 r^3} + \frac{8951 p^4 (p.n)^4}{384 r^2} - \frac{563921 (p.n)^6}{960 r^3} - \frac{5117 p^2 (p.n)^6}{320 r^2} \]

\[ + \frac{159 (p.n)^8}{28 r^2} + v^2 \left[ \frac{231 p^{12}}{256} + \frac{72454}{225 r^6} + \frac{1353196483 p^2}{529200 r^5} - \frac{787300061 p^4}{264600 r^4} \right] \]

\[ + \frac{3605263 p^6}{29400 r^3} - \frac{11535 p^8}{128 r^2} - \frac{2865 p^{10}}{256 r} - \ln \left( \frac{r}{r_0} \right) \left( \frac{256}{105} + \frac{3392 p^2}{105 r} \right) \]

\[ - \frac{432 p^4}{35 r^4} - \frac{2992 (p.n)^2}{105 r^5} - \frac{6824 p^2 (p.n)^2}{105 r^4} + \frac{496 (p.n)^4}{7 r^4} + \frac{5453}{768 r^6} \]

\[ - \frac{121315 p^2}{768 r^5} + \frac{2076041 p^4}{12288 r^4} + \frac{29987 p^6}{4096 r^3} + \frac{200359 (p.n)^2}{768 r^5} - \frac{172311 p^4 (p.n)^2}{4096 r^3} \]

\[ - \frac{5962205 p^2 (p.n)^2}{6144 r^4} + \frac{2617363 (p.n)^4}{4096 r^4} + \frac{127125 p^2 (p.n)^4}{4096 r^3} + \frac{14175 (p.n)^6}{4096 r^3} \]

\[ - \frac{857318207 (p.n)^2}{264600 r^5} + \frac{34200172759 p^2 (p.n)^2}{2116800 r^4} - \frac{5034763 p^4 (p.n)^2}{9800 r^3} \]

\[ + \frac{4969 p^6 (p.n)^2}{64 r^2} + \frac{275 p^8 (p.n)^2}{256 r} - \frac{498943687 (p.n)^4}{352800 r^4} + \frac{2674877 p^2 (p.n)^4}{7840 r^3} \]

\[ + \frac{925 p^4 (p.n)^4}{24 r^2} + \frac{15 p^6 (p.n)^4}{128 r} - \frac{25649 (p.n)^6}{3360 r^3} - \frac{8331 p^2 (p.n)^6}{160 r^2} \]

\[ + \frac{751 (p.n)^8}{28 r^2} + v^3 \left[ \frac{1617 p^{12}}{1024} - \frac{238966727 p^2}{151200 r^5} + \frac{127702733 p^4}{84672 r^4} \right] \]

\[ + \frac{108551131 p^6}{4233600 r^3} + \frac{16283 p^8}{256 r^2} + \frac{3995 p^{10}}{256 r} + \pi^2 \left( \frac{2339 p^2}{192 r^5} + \frac{98447 p^4}{3072 r^4} \right) \]

\[ - \frac{20259 p^6}{1024 r^3} - \frac{16111 (p.n)^2}{192 r^5} + \frac{131231 p^2 (p.n)^2}{1536 r^4} + \frac{106947 p^4 (p.n)^2}{1024 r^3} \]
\[
- \frac{361499(p.n)^4}{1024^4} - \frac{30075p^2(p.n)^4}{1024^3} - \frac{65625(p.n)^6}{1024^3} + \frac{758233181(p.n)^2}{151200r^5} \\
- \frac{1037428811p^2(p.n)^2}{705600r^3} - \frac{2207947669p^4(p.n)^2}{1411200r^3} + \frac{177p^6(p.n)^2}{256r^2} \\
- \frac{221p^8(p.n)^2}{64r} + \frac{12810612439(p.n)^4}{705600r^4} \\
+ \frac{355111837p^2(p.n)^4}{94080r^3} - \frac{125225p^4(p.n)^4}{768r^2} - \frac{3p^6(p.n)^4}{128r} - \frac{13905527(p.n)^6}{4480r^3} \\
+ \frac{136977p^2(p.n)^6}{1280r^2} - \frac{15p^4(p.n)^6}{128r} - \frac{289839(p.n)^8}{4480r^2} - \frac{35p^2(p.n)^8}{256r} \\
+ \nu^4 \left[ -\frac{1155p^{12}}{1024} - \frac{593p^6}{32r^3} + \frac{6649p^8}{256r^2} - \frac{1615p^{10}}{256r} + \frac{549p^4(p.n)^2}{32r^3} \\
- \frac{62143p^6(p.n)^2}{256r^2} + \frac{867p^8(p.n)^2}{256r} - \frac{5749p^2(p.n)^4}{96r^3} + \frac{652381p^4(p.n)^4}{768r^2} \\
- \frac{3p^6(p.n)^4}{64r} - \frac{17623(p.n)^6}{240r^3} - \frac{1178329p^2(p.n)^6}{1280r^2} - \frac{45p^4(p.n)^6}{128r} \\
+ \frac{1443091(p.n)^8}{4480r^2} + \frac{105p^2(p.n)^8}{128r} \right] + \nu^5 \left[ -\frac{231p^{12}}{1024} + \frac{63p^{10}}{256r} - \frac{35p^8(p.n)^2}{256r} \\
- \frac{15p^6(p.n)^4}{128r} - \frac{15p^4(p.n)^6}{128r} - \frac{35p^2(p.n)^8}{256r} - \frac{63(p.n)^{10}}{256r} \right] + \frac{17(p.n)^2}{4r^3} \\
+ \frac{29p^2(p.n)^2}{8r^4} + \frac{21p^4(p.n)^2}{16r^3} + \frac{5p^6(p.n)^2}{32r^2} - \frac{(p.n)^4}{8r^4} \right].
\] (38)

By this we have shown in explicit form the cancellation of the singularities originally occurring in harmonic coordinates, for reasons of regularization only. In the case of the binary point–mass problem up to 5PN order no singularities survive requiring another method to be removed. At 4PN this has also been shown in Ref. [9], see also [58]. Furthermore, logarithmic terms do now only occur at \(O(\nu)\) and \(O(\nu^2)\). From this expression the local Hamiltonian in the notion of [15] can be obtained by setting \(\nu = 0\),

\[
\hat{H}^{\text{loc}} = \lim_{r \to 0} \hat{H}^{\text{polefree}}
\] (39)

5. Comparison to the literature

In the following we perform a series of comparisons with results in the literature. Many results given in the literature use the distinction local vs. non-local instead of near zone vs. far zone.

Since we finally would like also to compare our results of the binding energy for circular orbits and that of the periastron advance, we have to check also the non-local terms derived first in [13], for completeness. We first will perform a canonical transformation from the pole–free Hamiltonian derived in Section 4 and those contributions of the EOB Hamiltonian [13], stemming from the potential terms only in Section 5.1. We recalculate the 5PN non–local contributions in Section 5.2. In Section 5.3 we calculate the 5PN local contributions to periastron advance \(K(\dot{E}, j)\)
for all contributions but those of $O(\nu^2)$ and derive the results for the 5PN binding energy at circular orbits and the periastron advance without the $O(\nu^2)$ terms in Section 5.4. The $O(\nu^2)$ terms contain local contributions to the tail terms which have not yet been calculated in complete form.

### 5.1. Canonical transformation to EOB

Let us first compare to the EOB results of Ref. [13], Eq. (11.8), for the contributions at $O(\nu^0)$ and $O(\nu^3)$ and higher given in EOB coordinates in complete form. These terms do not receive contributions due to tail terms and one can therefore just refer to the pole–free Hamiltonian of Section 4 to construct the canonical transformation.

It is given by

$$G(p^2, p.n, r) = p.n \left\{ t_1 \left\{ v \left( -\frac{5}{4} p^2 + \frac{1}{2} p^2 r \right) - 1 \right\} + t_2 \left\{ v \left( -\frac{5}{4} p^2 + \frac{5}{4} - \frac{1}{4} p^4 r \right) \right\} - \frac{1}{2} (p.n)^2 \right\} + v^2 \left\{ \frac{1}{4} p^2 - \frac{3}{16} - \frac{1}{8} (p.n)^2 \right\} + t_3 \left\{ v \left( \frac{29}{6} p^2 + \frac{4}{3} (p.n)^2 \right) \right\} + \frac{7}{16} p^4 + \frac{1795 - 63 \pi^2}{72 r^2} + \frac{1}{16} p^6 r + \frac{1}{4} p^2 (p.n)^2 \right\}$$

$$+ v^2 \left\{ -\frac{19}{24} p^2 - \frac{83}{24} (p.n)^2 \right\} - \frac{37}{96} p^4 - \frac{3}{16 r^2} - \frac{1}{16} p^6 r + \frac{73}{24} p^2 (p.n)^2$$

$$+ \frac{1}{4} (p.n)^4 \right\} + v^3 \left\{ \frac{8}{24} p^2 - \frac{1}{8} (p.n)^2 \right\} - \frac{1}{48} p^4 - \frac{3}{16 r^2} + \frac{1}{96} p^2 (p.n)^2 \right\}$$

$$+ t_4 \left\{ v^3 \left[ -\frac{143}{16} p^4 + \frac{493}{32} p^2 (p.n)^2 + \frac{31}{80} (p.n)^4 \right] - \frac{163}{64} \frac{p^2 - \frac{15}{64} (p.n)^2}{r^2} + \frac{229}{384} p^6 + \frac{7}{96 r^3} - \frac{1}{48} p^8 r$$

$$+ \frac{61}{384} p^4 (p.n)^2 - \frac{1}{16} p^2 (p.n)^4 \right\} + v^4 \left[ \frac{1}{r} \left( -\frac{1}{24} p^4 + \frac{1}{48} p^2 (p.n)^2 + \frac{1}{48} (p.n)^4 \right) + \frac{5}{128} p^2 - \frac{3}{8} (p.n)^2 \right\}$$

$$+ \frac{1}{96} p^4 (p.n)^2 - \frac{7}{128} p^2 (p.n)^4 + \frac{5}{128} (p.n)^6 \right\} + t_5 \left\{ v^3 \left[ \frac{1}{r^3} \left( -\frac{p^2}{100800} \right) \right] 100800$$

$$+ \left( -\frac{84723358 + 562625 \pi^2}{33600} (p.n)^2 \right)$$

---

14 For definiteness we use the minimal choice (8.24) of the flexibility parameters.
\[
+ \frac{1}{r^2} \left( -\frac{p^4(502480088 + 75854205\pi^2)}{6773760} \right) \\
+ \frac{p^2(-2251644296 + 120889125\pi^2)(p.n)^2}{5644800} \\
+ \left( \frac{17505304 + 16879275\pi^2(p.n)^4}{2257920} \right) \\
+ \frac{1}{r} \left( \frac{7447}{128} p^6 + \frac{235}{12} p^4(p.n)^2 - \frac{21323}{384} p^2(p.n)^4 + \frac{4937}{420} (p.n)^6 \right) \\
+ \frac{595}{1536} p^8 + \frac{-186973 + 12400\pi^2}{640r^4} - \frac{5}{48} p^{10}r - \frac{347}{384} (p.n)^6 \\
+ \frac{109}{256} p^4(p.n)^4 - \frac{155}{256} p^2(p.n)^6 \right] \\
+ v^4 \left[ \frac{1}{r^2} \left( -\frac{7121}{640} p^4 + \frac{105479}{5760} p^2(p.n)^2 - \frac{7073(p.n)^4}{1440} \right) \\
+ \frac{1}{r} \left( \frac{14675}{384} p^6 - \frac{100025}{1152} p^4(p.n)^2 + \frac{484729}{5760} p^2(p.n)^4 - \frac{63677(p.n)^6}{2688} \right) \\
+ \frac{-\frac{7}{40} p^2 + \frac{3829(p.n)^2}{2880}}{r^3} - \frac{1927}{768} p^8 + \frac{13}{96r^4} + \frac{145}{768} (p.n)^2 \\
+ \frac{29}{128} p^4(p.n)^4 - \frac{203}{768} p^2(p.n)^6 + \frac{7}{48} (p.n)^8 \right] \\
+ v^5 \left[ \frac{1}{r^2} \left( \frac{77p^4}{1280} + \frac{23}{720} p^2(p.n)^2 + \frac{697(p.n)^4}{11520} \right) \\
+ \frac{1}{r} \left( -\frac{1}{384} p^6 + \frac{13}{576} p^4(p.n)^2 - \frac{199p^2(p.n)^4}{1440} + \frac{13}{128} (p.n)^6 \right) \\
+ \frac{1}{r^3} \left( \frac{107}{480} p^2 - \frac{377(p.n)^2}{2880} \right) + \frac{1}{768} p^8 - \frac{31}{192r^4} + \frac{p^6(p.n)^2}{4608} \\
+ \frac{37p^4(p.n)^4}{7680} - \frac{7p^2(p.n)^6}{1536} \right] \right) \right].
\]

(40)

In this way we confirm all the contributions of \(O(v^6)\) and \(O(v^3)\) or higher given in [13] by an explicit Feynman diagram calculation ab initio.

5.2. The non-local terms

Next we turn to the non–local terms defined in [13], cf. Eq. (3). We perform the eccentricity expansion of the non–local contributions for \(\langle \delta H_{4+5PN}^{nl} \rangle\) with
\[
\frac{\langle \delta H_{4+5\text{PN}}^{\text{nl}} \rangle}{MC^2} = n \frac{2\pi}{2\pi MC^2} \int_0^{2\pi} dt \delta H(t) \equiv F_{4+5\text{PN}}(a_r, e_t)
\]
\[
= \frac{v^2}{a_r^2} [A_{4\text{PN}} + B_{4\text{PN}} \ln(a_r)] + \frac{v^2}{a_r^6} [A_{5\text{PN}} + B_{5\text{PN}} \ln(a_r)]
\]
\]

starting from harmonic coordinates. Here \(a_r\) is the semimajor axis of the orbit, which we rescaled by \(a_r = a_r,\text{phys}c^2/(GNM)\). It appears in the parameterization of the radial coordinate distance \(r\) in the form \(r = a_r[1 - e_r \cos(u)]\), where \(e_r\) denotes the “radial eccentricity” of the orbit and \(u\) the “eccentric anomaly”. The Kepler equation reads \(n \cdot t = 1 - e_t \sin(u)\), with \(n = 2\pi/P\). Here \(P\) is the orbital period and \(t\) the coordinate time defines the eccentricity \(e_t\) and one uses standard relations otherwise, cf. [59].

In the limit of vanishing eccentricity \(e_t\) we obtain the following contribution for \(\langle \delta H_{5\text{PN}}^{\text{nl}} \rangle\), Eq. (2.12), [13],

\[
\frac{\langle \delta H_{4+5\text{PN}}^{\text{nl}} \rangle}{MC^2} = \frac{v^2}{a_r^2} \left\{ -\frac{32}{5} \left( \ln(a_r) - 2\gamma_E \right) + \frac{128}{5} \ln(2) \right\}
\]
\[
+ \frac{v^2}{a_r^6} \left\{ \left( \frac{5854}{105} + \frac{56}{5} \right) \ln(a_r) - 2\gamma_E \right\}
\]
\[
- \left( \frac{25276}{105} - \frac{912}{35} \right) \ln(2) + \left( \frac{243}{14} - \frac{486}{7} \right) \ln(3) + \frac{32}{5} \nu - \frac{96}{5} \right\},
\]
which agrees with the results in [13], also being derived in harmonic coordinates. The terms up to \(O(e_t^{20})\) are given in Appendix B. They agree with the expansion coefficients of Table I of [13] up to \(O(e_t^{10})\) (in the harmonic gauge). The non–local contribution to the energy for circular orbits is then obtained by

\[
\frac{E_{\text{nl}}^{\text{circ}}}{\mu c^2} = v \left\{ -\frac{64}{5} \left( \ln(j) - \gamma_E \right) + \frac{128}{5} \ln(2) \right\} \eta_{j10}^8 + \left[ \frac{32}{5} + \frac{28484}{105} \ln(2) \right] \eta_{j10}^8
\]
\[
+ v \left[ \frac{32}{5} + \frac{112}{5} \left( \ln(j) - \gamma_E \right) + \frac{912}{35} \ln(2) - \frac{486}{7} \ln(3) \right] + \frac{243}{14} \ln(3)
\]
\[
- \frac{15172}{105} \left( \ln(j) - \gamma_E \right) \eta_{j12}^{10},
\]
with

\[
a_r = j^2 - 4\eta^2 + O \left( \eta^4 \right),
\]
where \(j = J_{\text{phys}}c/(GNM)\). Here we have also introduced the dimensionless quantity \(\eta^2\), accounting for \(1/c^2\).

The contributions up to \(O(e_t^{2})\) are needed below to derive periastron advance for circular motion. One may now further express the variables \(a_r\) and \(e_t\) in terms of the normalized Delaunay variables [60] \(i_r, i_\phi\) and \(i_{r\phi}\), cf. [16], Eq. (A11), with \(i_{r\phi} = i_r + i_\phi, i_\phi = j\). By this one obtains

\[\text{Note a difference to Eq. (8.27), [16] in the ln(2)v^2 term at 5PN.}\]
\[
\hat{H}(i_r, j) = F(a_r, e_r). 
\]

(46)

The variables \(\hat{H}, i_r,\) and \(i_\phi\) are related by Euler’s chain rule

\[
\frac{\partial \hat{H}}{\partial i_\phi} \bigg|_{i_r} \frac{\partial i_\phi}{\partial i_r} \frac{\partial i_r}{\partial \hat{H}} \bigg|_{i_\phi} = -1 
\]

(47)

since \(\hat{H}\) depends only on \(i_r\) and \(i_\phi\) and therefore a function \(f\) exists with \(f(\hat{H}, i_r, i_\phi) = 0\). By applying the chain rule one obtains the periastron advance, \(K\), defined in (48).\(^16\) One obtains

\[
K = \frac{1}{\Omega_R} \frac{\partial \hat{H}(i_r, j)}{\partial i_\phi} \bigg|_{i_r} 
\]

(48)

\[
\Omega_R = \frac{\partial \hat{H}(i_r, j)}{\partial i_r} \bigg|_{j} 
\]

(49)

Here \(\hat{H}\) denotes the complete Hamiltonian. One may express

\[
K = K_{\text{loc}} + K_{\text{nl}}, 
\]

(50)

\[
\hat{H} = \hat{H}_{\text{loc}} + \hat{H}_{\text{nl}}, 
\]

(51)

where \(K_{\text{nl}}\) starts at 4PN and \(\Omega_R\) receives non–local (nl) contributions from 4PN on, cf. (49). The contributions to \(K_{\text{loc}}\) are calculated in Section 5.3. For \(K_{\text{nl}}^{4+5PN}\) the 4PN non–local contributions to \(\Omega_R\) are necessary beyond the 1PN (local) correction

\[
\Omega_{R,1PN}^{\text{loc}} = i_{\phi}^{-3} \left[ 1 + \frac{(3 + \nu)i_\phi + 18i_r}{2i_\phi i_r^2} \eta^2 \right] + O(\eta^4), 
\]

(52)

[16] with \(\Omega_R = (GNM/c^3)\Omega_{R,\text{phys}}\). We first calculate \(\Omega_{R,4PN}^{\text{nl}}\), setting \(i_\phi = j\), for circular orbits

\[
\Omega_R^{\text{nl},4PN} = \frac{\partial \hat{H}_{\text{nl},4PN}(i_r, j)}{\partial i_r} \bigg|_{j; i_r = 0} = -\frac{64}{10} \frac{\eta^8}{j^{11}} \left[ 13 + \frac{37}{6} (\ln(j) - \gamma_E) + \frac{203}{6} \ln(2) - \frac{729}{16} \ln(3) \right]. 
\]

(53)

The Newtonian term of \(\Omega_R\) for circular orbits is \(1/j^3\). Since we are only considering the 4 and 5PN contributions, the post–Newtonian expansion of \(1/\Omega_R\) can be done separately for (52) and (53), keeping the Newtonian contribution. The second term hits the \(O(\eta^2)\) term of \(K_{\text{loc}}\) (66). In this way Eq. (8.21) in [16] needs a slight extension.

We obtain

\[
K_{\text{nl}}^{4PN}(j) = -\frac{64}{10} \frac{\eta^8}{j^8} \left[ -11 - \frac{157}{6} (\ln(j) - \gamma_E) + \frac{37}{6} \ln(2) + \frac{729}{16} \ln(3) \right], 
\]

(54)

\[
K_{\text{nl}}^{5PN}(j) = -\frac{64}{10} \frac{\eta^{10}}{j^{10}} \left[ -\frac{59723}{336} - \frac{9421}{28} (\ln(j) - \gamma_E) + \frac{7605}{28} \ln(2) + \frac{112995}{224} \ln(3) \right]. 
\]

\(^16\) Note that also a related quantity, \(k = K - 1\), is sometimes denoted by periastron advance.
which is calculated in a different way than the local contributions. The representations are, however, equivalent. Here one has

\[ a_r = i_r^2 - 2 \frac{3i_r + 2i_\phi}{i_\phi} \eta^2 + O(\eta^4), \]  
\[ e_i^2 = \frac{i_r}{i_r^2} \left[ i_r + 2i_\phi + 2\frac{i_r(v-1) + i_\phi(2v-5)}{i_r^2} \eta^2 \right] + O(\eta^4), \]  

(cf. [16]). Eq. (56) turns into (45) for circular orbits \( (i_r \to 0) \). Eq. (54) agrees with Eq. (5.7) in [6], see also the expression of the related function \( \rho(x) \) in [5] and Eqs. (54), (55) agree with Eq. (8.29) of [16].

5.3. Periastron advance: local terms

The local contribution to periastron advance is obtained by

\[ K_{\text{loc}} = \frac{1}{\pi \frac{\partial}{\partial j}} \int_{r_{\text{min}}}^{r_{\text{max}}} dr \sqrt{R(r, \hat{E}, j)}. \]  

Here \( \hat{E} \) results from (4) by \( H \to E \).

To obtain also the terms of \( O(\nu) \) for \( K_{\text{loc}} \) and the binding energy for circular orbits \( E_{\text{circ}} \) we add the contributions due to the electric quadrupole, octupole, and magnetic quadrupole, cf. [42], in complete form. These contributions are those, which exhibit poles in \( 1/\varepsilon \) and contribute to \( O(\nu) \). The further local contributions to the 5PN tail terms contribute only to the rational terms of \( O(\nu^2) \).

Since the magnetic quadrupole contribution contains a product of contracted 3–index Levi-Civita tensors, special care is required in dimensional regularization. This is reminiscent of the treatment of the axial anomaly in quantum field theory, which can be addressed consistently by introducing a finite renormalization [64]. Indeed, consistent expressions for all observables are obtained by introducing the following \( Z \)-factor for the contribution due to the magnetic quadrupole moment in the Hamiltonian,

\[ Z_J = 1 - c_J \varepsilon. \]  

Similar to QFT, the parameter \( c_J \) is fixed referring to an observable, for which we choose the \( O(\nu) \) contribution to the binding energy at circular orbits, cf. (70) yielding

\[ c_J = \frac{1}{3}. \]  

The finite contribution to the Hamiltonian reads

\[ \delta_J H = -\frac{4}{135r^4}(1 - 4\nu) \left[ 26p^2(p.n)^2 - 21(p.n)^4 - 5p^4 + \frac{4}{r}(p^2 - (p.n)^2) \right]. \]  

\[ ^{17} \text{Other prescriptions were given in [65].} \]
The function $R$ is given by

$$R(r, \hat{E}, j)|_{\text{5PN}} = A + \frac{2B}{r} + \frac{C}{r^2} + \eta^2 \frac{D_1}{r^3} + \sum_{k=1}^{4} \eta^{2(k+1)} \left[ \frac{D_{2k}}{r^{2k+2}} + \frac{D_{2k+1}}{r^{2k+3}} \right]. \tag{62}$$

It is convenient to refer to the local terms rather than to a separation of potential and tail terms. The former ones have no logarithmic terms and the corresponding integrals are therefore somewhat simpler. The logarithmic terms have already been dealt with in Section 5.2.

The integrand of (58) has the form

$$\sqrt{R(r, \hat{E}, j)} = \frac{1}{r} \sqrt{Ar^2 + 2Br + C + \eta^2 \frac{D_1}{2r^2 \sqrt{Ar^2 + 2Br + C}}} + O(\eta^4). \tag{63}$$

The relation

$$\hat{E} = \hat{H}(p^2, (p.n)^2, r) \tag{64}$$

is solved iteratively for $R(r, \hat{E}, j) = (p.n)^2$ by applying

$$p^2 = (p.n)^2 + \frac{j^2}{r^2}, \tag{65}$$

through which the functions $A, B, C$ and $D_k$ become polynomials in $\hat{E}$ and $j$. The integral (58) is usually solved by a mapping to a contour integral [61] applying the residue theorem, expanding in $\eta^2$ up to 5PN. Except of the integral for the Newtonian term, involving only $A, B$ and $C$, all other integrals have only one residue at $r = 0$, see Appendix C.

We calculate the local contribution to periastron advance starting from harmonic coordinates and compare to Eq. (F5) of [16] resulting from the local EOB Hamiltonian Eq. (11.8) [13]. This is necessary to fix the notion of the parameters $d_5$ and $a_6$ in $K(E, j)_{\text{loc,t}}$ to 5PN. We rather use $K(E, j)_{\text{loc,t}}$ than $K_{\text{circ,5PN}}$ to test three relations between the parameters $d_5$ and $a_6$, which is advantageous.

To 4PN one obtains

$$K(\hat{E}, j)_{\text{loc,t}}^{\leq 4\text{PN}} = 1 + \frac{3}{j^2} \eta^2 + \left[ \left( \frac{15}{2} - 3\nu \right) \hat{E} j^2 + \left( \frac{105}{4} - \frac{15\nu}{2} \right) j^4 \right] \eta^4 + \left[ \left( \frac{15}{4} (1 - \nu) + 3\nu^2 \right) \hat{E} j^2 + \left( \frac{315}{2} + \frac{123\pi^2}{64} - 218 \right) \nu + \frac{45}{2} \nu^2 \right] \hat{E} j^4$$

$$+ \left[ \frac{1155}{4} + \frac{615\pi^2}{128} - \frac{625}{2} \nu \frac{105}{8} \nu^2 \right] \frac{1}{j^6} \eta^4 + \left[ \frac{15}{4} - 3\nu \right] \nu^2 \hat{E}^3 j^2$$

$$+ \left[ \frac{4725}{16} + \frac{35569\pi^2}{2048} - \frac{20323}{24} \nu + \left( \frac{4045}{8} - \frac{615\pi^2}{128} \nu^2 - 45\nu^3 \right) j^4 \frac{1}{j^6} \right] \hat{E}^2 j^4$$

$$+ \left[ \frac{257195\pi^2}{2048} - \frac{293413}{48} \nu + \frac{45045}{16} \nu^2 \hat{E} j^6 + \frac{225225}{64} + \frac{2975735\pi^2}{24576} \right].$$
We also rederived the periastron advance starting with the ADM Hamiltonian [3] and confirm the result given in [6,16]. For the 5PN terms we obtain the result

\[
K(\hat{E}, j)_{\text{loc},i}^{5\text{PN}} = \left\{ -\frac{1736399}{288} + \left( \frac{132475}{96} - \frac{7175\pi^2}{256} \right) v^2 - \frac{315}{16} v^3 \right\} \frac{1}{j^8} \eta^8. \tag{66}
\]

leaving out the rational terms at \(O(\nu^2)\) since they receive also contributions from the 5PN tail term that have not been included here. The terms given in (67) agree with those of Ref. [16] considering our results in (68) and (69). Also the \(O(\nu)\) terms agree, which confirms the validity of (59), (60).

Eq. (67) allows to derive the \(\pi^2\) contributions to \(\tilde{d}_5\) and \(a_6\) to which we turn now.

5.4. The \(\pi^2\) contributions at \(O(\nu^2)\)

The \(\pi^2\)-contributions stem from the potential terms. They also contribute to the yet open constants \(\tilde{d}_5\) and \(a_6\) in [13] at \(O(\nu^2)\). They can be extracted from the corresponding contribution to the binding energy for circular motion (for \(a_6\)) and the circular periastron advance [6,62,63], respectively. One obtains

\[
\tilde{d}_5 = r_{\tilde{d}_5} + \frac{306545}{512} \pi^2 \tag{68}
\]

\[
a_6 = r_{a_6} + \frac{25911}{256} \pi^2 \tag{69}
\]

from Eqs. (67), (70), (71). Let us finally summarize the terms for the binding energy \(E^{\text{circ}}(j)\) and the periastron advance \(K^{\text{circ}}(j)\) at circular orbits obtained in the present calculation,
\[
\frac{E_{\text{circ}}(j)}{\mu c^2} = -\frac{1}{2j^2} + \left( -\frac{v}{8} - \frac{9}{8} \right) \frac{1}{j^4} \eta^2 + \left( -\frac{v^2}{16} + \frac{7v}{16} - \frac{81}{16} \right) \frac{1}{j^6} \eta^4 + \left[ -\frac{5v^3}{128} + \frac{5v^2}{64} \right.
\]
\[
+ \left( \frac{8833}{384} - \frac{41v^2}{64} \right) \frac{1}{j^8} \eta^6 + \left[ -\frac{7v^4}{256} + \frac{3v^3}{128} + \left( \frac{41v^2}{128} - \frac{8875}{768} \right) \right] \eta^2
\]
\[
+ \left( \frac{989911}{3840} - \frac{6581v^2}{1024} \right) \frac{1}{j^{10}} \eta^8 + \left[ \frac{132979v^2}{2048} \right] \eta^2 - \frac{21v^5}{1024}
\]
\[
+ \frac{5v^4}{1024} + \left( \frac{41v^2}{512} - \frac{3769}{3072} \right) \frac{1}{j^{12}} \eta^{10} + O\left( \eta^{12} \right),
\]
\[
(70)
\]

\[
K_{\text{circ}}(j) = 1 + 3 \frac{1}{j^2} \eta^2 + \left( \frac{45}{2} - 6v \right) \frac{1}{j^4} \eta^4 + \left[ \frac{405}{2} + \left( -202 + \frac{123}{32} \right) \right] \frac{1}{j^6} \eta^6
\]
\[
+ \left[ \frac{15795}{8} + \left( \frac{185767}{3072} - \frac{105991}{36} \right) \right] \frac{1}{j^8} \eta^8
\]
\[
+ \left[ \frac{161109}{8} + \left( -\frac{18144676}{525} + \frac{488373}{2048} \right) \right] \frac{1}{j^{10}} \eta^{10} + K_{4+5\text{PN}}^\text{nl}(j) + O\left( \eta^{12} \right),
\]
\[
(71)
\]

leaving out the terms of \( O(v^2) \). We agree in all other terms with [62], Eq. (5.25), and [6], Eq. (5.9).

To obtain the corresponding relations in terms of the variable \( x = (G_N M \Omega_\phi / c^3)^{2/3} \), with \( \Omega_\phi \) the angular frequency, one may apply \( j = j(x) \), Eq. (8.31) of [16]. In particular one has in the Schwarzschild limit of (70), (71)

\[
\frac{E_{\text{Schw.}}(x)}{\mu c^2} = \frac{1 - 2x}{\sqrt{1 - 3x}} - 1,
\]
\[
(72)
\]

\[
K_{\text{Schw.}}(x) = \frac{1}{\sqrt{1 - 6x}},
\]
\[
(73)
\]

where (72) has been given in [28]. The relation for \( K_{\text{Schw.}}(j) \) has been given in [63], Eq. (A8).

6. Conclusions

We have presented the 5PN potential contributions to the Hamiltonian of binary motion in gravity starting form the harmonic gauge and a part of the 5PN tail term. The calculation has thoroughly been performed in \( D \) dimensions, based on 188533 Feynman diagrams using effective field theory methods, as a calculation ab initio. The singular and logarithmic contributions to the 5PN tail terms have been calculated. We have shown the explicit cancellation of the singularities between both contributions, performing an additional canonical transformation to a
pole–free Hamiltonian. We have shown in an explicit calculation how to match the potential and the tail terms, using dimensional regularization. Here the overlap–terms are canceling.

Comparisons to the literature have been performed. Firstly, we have shown that all terms of $O(v^0)$ and $O(v^3)$ and higher agree with the results presented in the literature. Because of the emergence of contracted 3-index Levi-Civita pseudo–tensors in the tail contribution by the magnetic quadrupole moment [42], Eq. (20), working consequently in $D$ dimension, a finite renormalization is performed. Like in quantum field theory, this requires reference to an observable. This is provided here, for the $O(v)$ terms, by the self-force expressions of the respective observables.\footnote{It has also consequences for the $O(v^2)$ terms because of the corresponding contributions to this multipole moment.} We fixed this in the case of the pole–free Hamiltonian using the binding energy for circular motion [24]. This implied our result also for the local periastron advance $K^\text{loc}(E, j)$, which agrees with the result given in [16]. At $O(v^2)$ we determined the $\pi^2$ contributions to $d_5$ and $a_6$. Furthermore, we also agree in the effect of the non–local terms on $E_{\text{circ},5\text{PN}}$ and $K_{\text{circ},5\text{PN}}$. We still observe a few differences in the purely rational (local) contributions to the tail term at $O(v^2)$ comparing to the present literature, which have to be clarified to obtain the complete 5PN result.

We note that the results of an independent calculation of a subset of the four-loop diagrams contributing to the near-zone potential [69] were made public at the same time as the present work. We find full agreement for these diagrams. Furthermore, recent post–Minkowskian results at $O(G_N^4)$ confirm our results, cf. [70], see also [71].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

We thank Z. Bern, D. Bini, Th. Damour, S. Foffa, C. Kavanagh, K. Schönwald, V. Smirnov, and B. Wardell for discussions and L. Blanchet for a remark. This work has been funded in part by EU TMR network SAGEX agreement No. 764850 (Marie Sklodowska–Curie) and COST action CA16201: Unraveling new physics at the LHC through the precision frontier. G. Schäfer has been supported in part by Kolleg Mathematik Physik Berlin (KMPB) and DESY. Part of the text has been typesetted using SigmaToTeX of the package Sigma [67,68].

Appendix A. Joining the potential and the tail term and the method of expansion by regions

We now prove that the Taylor expansions in the potential and ultrasoft region commute and that the occurring overlap integrals are indeed scaleless and vanish in $D$ dimensions. Here we use arguments given in [45].

The general form of the contributing integrands $I$ is

$$I = \frac{\exp ik[x(t_1) - x(t_2)]}{k^2 \prod_{i=1}^{P_{\text{int}}}(k + p_i)^2 \prod_{j=1}^{P_{\text{pot}}}(k + q_j)^2} J(q_i, \{p_i\}) P(k, \{q_i\}, \{p_i\}, v_1, v_2).$$

(74)
Apart from the loop momentum \( k \), \( I \) depends on ultrasoft momenta \( p_i \), \( 1 \leq i \leq P_{us} \) and potential momenta \( q_i \), \( 1 \leq i \leq P_{pot} \). The loop is associated with one of the two worldlines, whose four–position at the time \( t \) is given by \( x(t) \). \( J \) is a function with the same structure as \( I \) itself, but independent of \( k \). Finally, \( \mathcal{P} \) denotes a polynomial in the momenta and worldline velocities \( v_1, v_2 \). Our aim is to show that

\[
T_{pot}^N T_{us}^N I = T_{us}^N T_{pot}^N I = \sum_{i=0}^{i_{\text{max}}(N)} \frac{J_i}{k^{2i}},
\]

(75)

according to the power counting in the respective region. The \( \mathcal{P}_i \) are polynomials in the components of the four–vectors appearing in Eq. (74) and the \( J_i \) are independent of \( k \). This structure implies

\[
\int_{\mathbb{R}^d} d^4 \vec{k} \ T_{pot}^N T_{us}^N I = 0.
\]

(76)

We first note that

\[
T_{pot}^N f g = (T_{pot}^N f)(T_{pot}^N g),
\]

(77)

and similar for \( T_{us}^N \). This allows us to expand each factor in Eq. (74) separately. \( J \) is independent of \( k \), and \( \mathcal{P} \) is a polynomial, so trivially

\[
T_{pot}^N T_{us}^N J = T_{us}^N T_{pot}^N J = J,
\]

(78)

\[
T_{pot}^N T_{us}^N \mathcal{P} = T_{us}^N T_{pot}^N \mathcal{P} = \mathcal{P}.
\]

(79)

For the propagators without additional momenta, we obtain

\[
T_{pot}^N T_{us}^N \frac{1}{k^2} = T_{pot}^N \frac{1}{k^2} = \sum_{n=0}^{N} \frac{k^{2n}}{(k^2)^{n+1}} = T_{us}^N \sum_{n=0}^{N} \frac{k^{2n}}{(k^2)^{n+1}} = T_{us}^N T_{pot}^N \frac{1}{k^2},
\]

(80)

and similar

\[
T_{pot}^N T_{us}^N \frac{1}{(k + p_i)^2} = T_{pot}^N \frac{1}{(k + p_i)^2} = \sum_{n=0}^{N} \frac{T_{pot}^N[(k_0 + p_{i0})^2 - 2 \vec{k} \vec{p}_i - \vec{p}_i^2]^n}{(k^2)^{n+1}},
\]

(81)

\[
T_{us}^N T_{pot}^N \frac{1}{(k + p_i)^2} = T_{us}^N \sum_{n=0}^{N} \frac{T_{pot}^N[(k_0 + p_{i0})^2 - 2 \vec{k} \vec{p}_i - \vec{p}_i^2]^n}{(k^2)^{n+1}}
\]

\[
= \sum_{n=0}^{N} \frac{T_{pot}^N[(k_0 + p_{i0})^2 - 2 \vec{k} \vec{p}_i - \vec{p}_i^2]^n}{(k^2)^{n+1}}.
\]

(82)

With respect to the remaining propagators, we first observe the absence of poles in \( v \), i.e.

\[
T_{pot}^Z \frac{1}{(k + q_i)^2} = T_{pot}^Z T_{us}^N \frac{1}{(k + q_i)^2} = T_{us}^Z \frac{1}{(k + q_i)^2} = T_{us}^Z T_{pot}^N \frac{1}{(k + q_i)^2} = 0
\]

for all \( Z < 0 \),

(83)

and note the following algebraic properties of the Taylor expansion operators.
\[ T_{\text{pot}}^N P = P T_{\text{pot}}^{N-1}, \quad P \in \{k_0, p_{i0}, q_{i0}, \tilde{p}_i\} \] (84)

\[ T_{\text{pot}}^P = P T_{\text{pot}}^N, \quad P \in \{\bar{k}, \bar{q}_i\} \] (85)

\[ T_{\text{us}}^N P = P T_{\text{us}}^{N-1}, \quad P \in \{k_0, p_{i0}, q_{i0}, \bar{k}, \bar{p}_i\} \] (86)

\[ T_{\text{us}}^N P = P T_{\text{us}}^N, \quad P \in \{\bar{q}_i\}. \] (87)

We now show that

\[ T_{\text{pot}}^N T_{\text{us}}^M \frac{1}{(k + q_i)^2} = T_{\text{us}}^M T_{\text{pot}}^N \frac{1}{(k + q_i)^2} \] (88)

by induction over \( N + M \). The case \( N + M = 0 \) is straightforward. For \( N + M \geq 1 \) we observe

\[ T_{\text{pot}}^N T_{\text{us}}^M \frac{1}{(k + q_i)^2} = \frac{T_{\text{pot}}^N}{q_i^2} \left( 1 + \frac{([k_0 + q_0]^2 - \bar{k}^2) T_{\text{us}}^{N-2} - 1}{(k + q_i)^2} \right) \]

\[ = \frac{1}{q_i^2} \left( 1 + (k_0 + q_0)^2 T_{\text{pot}}^{N-2} \frac{1}{(k + q_i)^2} \right) \]

\[ - 2 \bar{k} \bar{q}_i T_{\text{us}}^{M-1} \frac{1}{(k + q_i)^2} \]

\[ = T_{\text{us}}^M T_{\text{pot}}^N \frac{1}{(k + q_i)^2} \] (89)

and similar

\[ T_{\text{us}}^M T_{\text{pot}}^N \frac{1}{(k + q_i)^2} = \frac{T_{\text{us}}^M}{q_i^2} \left( 1 + (k_0 + q_0)^2 T_{\text{pot}}^{N-2} \frac{1}{(k + q_i)^2} \right) \]

\[ - 2 \bar{k} \bar{q}_i T_{\text{us}}^{M-1} \frac{1}{(k + q_i)^2} \]

\[ = T_{\text{us}}^M T_{\text{pot}}^N \frac{1}{(k + q_i)^2} \] (90)

where we have used the induction hypothesis

\[ T_{\text{pot}}^n T_{\text{us}}^m \frac{1}{(k + q_i)^2} = T_{\text{us}}^m T_{\text{pot}}^n \frac{1}{(k + q_i)^2} \quad \text{for } n + m < N + M \] (91)

in the last step.

Furthermore, we find

\[ T_{\text{pot}}^N T_{\text{us}}^N \frac{1}{(k + q_i)^2} = \sum_{n=0}^N \frac{T_{\text{pot}}^n T_{\text{us}}^n (k_0 + q_{i0})^2 - 2 \bar{q}_i \bar{k} - \bar{k}^2)^n}{(q_i^2)^{n+1}} \] (92)

where the numerators are simply polynomials in the components of \( k \) and \( q_i \).

Finally, the exponential (74) can be expanded by observing

\[ \tilde{x}(t) \sim R, \] (93)

i.e.
\[ \vec{k} \vec{x}(t) \sim 1 \quad (94) \]
in the potential region and
\[ \vec{k} \vec{x}(t) \sim v \quad (95) \]
in the ultrasoft region. This yields
\[
T_N^{pot} T_N^{us} \exp \left( i k[x(t_1) - x(t_2)] \right) = T_N^{pot} \exp \left( -ik_0c(t_1 - t_2) \right) \sum_{n=0}^{N} \frac{(i\vec{k}[\vec{x}(t_1) - \vec{x}(t_2)])^n}{n!} \]
\[
= \exp \left( -ik_0c(t_1 - t_2) \right) \sum_{n=0}^{N} \frac{(i\vec{k}[\vec{x}(t_1) - \vec{x}(t_2)])^n}{n!} \quad (96) \]
\[
T_N^{us} T_N^{pot} \exp \left( i k[x(t_1) - x(t_2)] \right) = T_N^{us} \exp \left( i k[x(t_1) - x(t_2)] \right) \]
\[
= \exp \left( -ik_0c(t_1 - t_2) \right) \sum_{n=0}^{N} \frac{(i\vec{k}[\vec{x}(t_1) - \vec{x}(t_2)])^n}{n!} \quad (97) \]

We have now shown that the Taylor expansions commute for each of the factors in Eq. (74) and that the product of all expanded factors Eqs. (78)–(81), (92), (96) has the required form Eq. (75).

**Appendix B. The eccentricity expansion of the non–local terms**

We have calculated the eccentricity expansion of the non–local terms using the orbital average of the corresponding part of the Hamiltonian and applying representations in terms of Bessel functions, cf. [66]. For the expansions to \( O(e_r^{20}) \) we obtain

\[
F_{4PN}(a_r, e_r) = \frac{v^2}{a_r^5} \left( \frac{-32}{5} (\ln(a_r) - 2\gamma_E) + \frac{128}{5} \ln(2) + e_r^2 \left[ -\frac{176}{5} - \frac{628}{15} (\ln(a_r) - 2\gamma_E) \right. \right.
\]
\[
+ \frac{296}{15} \ln(2) + \frac{729}{5} \ln(3) \right) + e_r^4 \left[ -\frac{2681}{15} - 121 (\ln(a_r) - 2\gamma_E) \right. \right.
\]
\[
+ \frac{29966}{15} (\ln(2) - \frac{13851}{20} \ln(3)) + e_r^6 \left[ -\frac{90017}{180} - \frac{763}{3} (\ln(a_r) - 2\gamma_E) \right. \right.
\]
\[
- \frac{116722}{15} \ln(2) + \frac{419661}{320} \ln(3) + \frac{1953125}{576} \ln(5) \right) + e_r^8 \left[ -\frac{306433}{288} \right. \right.
\]
\[
- \frac{3605}{8} (\ln(a_r) - 2\gamma_E) + \frac{5381201}{180} \ln(2) + \frac{26915409}{2560} \ln(3) \right. \right.
\]
\[
- \frac{83984375}{4608} \ln(5) \left] + e_r^{10} \left[ -\frac{18541327}{9600} - \frac{114807}{160} (\ln(a_r) - 2\gamma_E) \right. \right. \right.
\]
\[
- \frac{4697998651}{54000} \ln(2) - \frac{138733913079}{2048000} \ln(3) + \frac{18736328125}{442368} \ln(5) \right. \right.
\]
\[
+ \frac{678223072849}{18432000} \ln(7) \right) + e_r^{12} \left[ -\frac{364045577}{115200} - \frac{679679}{640} \right. \right.
\]
\[
+ \frac{26059216929}{138240000} \ln(9) \right) + e_r^{14} \left[ -\frac{1675932659}{77760000} - \frac{2191279}{192000} \right. \right.
\]
\[
+ \frac{141276754849}{6350400000} \ln(11) \right) + e_r^{16} \left[ -\frac{1305332073}{3932160000} - \frac{784531}{15360000} \right. \right.
\]
\[
+ \frac{1742231609621}{154880000000} \ln(13) \right) + e_r^{18} \left[ -\frac{121026919}{5184000000} - \frac{52121}{43200000} \right. \right.
\]
\[
+ \frac{2129508565227}{6755040000000} \ln(15) \right) + e_r^{20} \left[ -\frac{79499335}{235929600000} - \frac{37777}{226800000} \right. \right.
\]
\[
+ \frac{4548422425219}{94371840000000} \ln(17) \right) \right) \right).
\[
\times ( -2\gamma_E + \ln (a_r) ) + \frac{110301092701}{216000} \ln (2) + \frac{1437894581679}{8192000} \ln (3)
- \frac{100439453125}{1769472} \ln (5) - \frac{678223072849}{2949120} \ln (7) + e_t^{14} \left[ -\frac{775035553}{161280} \right] \\
- \frac{95381}{64} ( -2\gamma_E + \ln (a_r) ) - \frac{38217199661503}{15876000} \ln (2)
+ \frac{996383367472131}{3211264000} \ln (3) + \frac{155008544921875}{3121348608} \ln (5)
+ \frac{4122918059849071}{6370099200} \ln (7) + e_t^{16} \left[ -\frac{992166951}{143360} \right] \\
- \frac{1028313}{512} ( -2\gamma_E + \ln (a_r) ) + \frac{2885944821108703}{381024000} \ln (2)
- \frac{199179784215931689}{51380224000} \ln (3) + \frac{35973968505859375}{49941577728} \ln (5)
- \frac{55887819388980017}{509607936000} \ln (7) + e_t^{18} \left[ -\frac{13153280515}{1376256} \right] \\
- \frac{299426170973969703}{164602368000} \ln (2) + \frac{114141707388359168251}{822083584000} \ln (3)
- \frac{1113553185418308608323}{88060251340800} \ln (7) + \frac{8140274938683976111321}{43149523156992000} \ln (11)
+ e_t^{20} \left[ \frac{30115692673}{2359296} - \frac{1639570933}{491520} \right] \\
+ \frac{321972869963109745379}{7054387200000} \ln (2) - \frac{1923804568946611360809}{82208358400000} \ln (3)
+ \frac{8504215528481591796875}{4142354223071232} \ln (5) - \frac{3124529501361780187583}{29353417113600000} \ln (7)
- \frac{13268648150054881061471323}{862990463139840000} \ln (11) \right]
+ O ( e_t^{22} ).
\]

\( F_{\text{PN}} (a_r, e_t) = \)

\[
\frac{v^2}{a^2} \left\{ \frac{2}{105} (2927 + 588\nu)(\ln (a_r) - 2\gamma_E) + \frac{4}{105} (-6319 + 684\nu) \ln (2)
- \frac{243}{14} (-1 + 4\nu) \ln (3) - \frac{96}{5} + \frac{32\nu}{5} + e_t^{2} \left[ \frac{2}{105} (8004 + 11935\nu)(\ln (a_r) - 2\gamma_E)
- \frac{4}{105} (-14268 + 100247\nu) \ln (2) + \frac{729}{70} (-76 + 129\nu) \ln (3) - \frac{5441}{35} + \frac{4672\nu}{21} \right] \right\}
\]

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\[\begin{align*}
+e_t^4 \left[ \frac{1}{70} (-14003 + 78435 \nu)(\ln(a_r) - 2\gamma_E) + \frac{1}{105} (-599911 + 3476231 \nu) \ln(2) \\
- \frac{729(2085 + 34396 \nu)}{4480} \ln(3) - \frac{9765625(-1 + 4 \nu)}{2688} \ln(5) - \frac{1160639}{840} + \frac{59756 \nu}{35} \right] \\
+e_t^6 \left[ \frac{1}{30} (-53391 + 100135 \nu)(\ln(a_r) - 2\gamma_E) - \frac{73}{945} (-226057 + 3011889 \nu) \ln(2) \\
- \frac{243}{896} (-95283 + 292001 \nu) \ln(3) + \frac{78125(-5557 + 43575 \nu)}{24192} \ln(5) - \frac{15761437}{2520} \right] \\
+ \frac{474653 \nu}{72} + e_t^8 \left[ \frac{1}{64} (-356481 + 490280 \nu)(\ln(a_r) - 2\gamma_E) \\
+ \frac{(-1814239887 + 33331273432 \nu)}{30240} \ln(2) + \frac{729(-181288681 + 1304180292 \nu)}{1146880} \ln(2) \\
\times \ln(3) - \frac{78125(325441 + 43174300 \nu)}{6193152} \ln(5) - \frac{96889010407(-1 + 4 \nu)}{884736} \ln(7) \\
- \frac{168508293}{8960} + \frac{2591779 \nu}{144} \right] + e_t^{10} \left[ \frac{561}{320} (-7258 + 8547 \nu)(\ln(a_r) - 2\gamma_E) \\
+ \frac{(1135478771202 - 693434234023 \nu)}{756000} \ln(2) \\
- \frac{2187(-1924018874 + 43692700941 \nu)}{2867200} \ln(3) + \frac{1953125(69962 + 1244683 \nu)}{33600} \ln(7) \\
\times \ln(5) + \frac{282475249(-244698 + 1611757 \nu)}{11059200} \ln(7) - \frac{1492974817}{3600} + \frac{255777929 \nu}{6400} \ln(7) \\
+ \frac{409595282746879 \nu}{1814400} \ln(2) + \frac{(431854060307859)}{131072000} \ln(3) - \frac{37854312670341 \nu}{6553600} \ln(3) \\
- \frac{90299298142188709 \nu}{5308416000} \ln(7) - \frac{20691354791}{230400} + \frac{3566474429 \nu}{46080} \ln(7) \\
+ e_t^{12} \left[ \left( -\frac{3926923}{160} + \frac{6733727 \nu}{256} \right)(\ln(a_r) - 2\gamma_E) + \left( \frac{3014093225433 \nu}{3402000} \right) \ln(2) \\
+ \frac{109595282746879 \nu}{1814400} \ln(2) + \left( \frac{431854060307859}{131072000} - \frac{37854312670341 \nu}{6553600} \right) \ln(3) \\
- \frac{47376871796875}{891813888} + \frac{60736572265625 \nu}{37158912} \ln(5) + \left( \frac{30496744521746713}{21233664000} \right) \ln(5) \\
- \frac{90299298142188709 \nu}{5308416000} \ln(7) - \frac{20691354791}{230400} + \frac{3566474429 \nu}{46080} \ln(7) \\
+ e_t^{14} \left[ \left( \frac{190225893}{4480} + \frac{5466461 \nu}{128} \right)(\ln(a_r) - 2\gamma_E) + \left( \frac{4290237684292667}{13358400} \right) \ln(2) + \left( \frac{19381110496948853 \nu}{74088000} \right) \ln(2) + \left( -\frac{21715453037739003}{8991539200} \right) \ln(3) \\
+ \frac{5487909996792714903 \nu}{4495769600} \ln(3) + \left( \frac{248411703945390625}{43698880512} \right) \ln(3) \\
\right]
\end{align*}\]
\[
\begin{align*}
&\left( -\frac{34494852294921875\nu}{16184770056} \right) \ln(5) + \left( -\frac{5548114000135259}{2359296000} \right) \\
+ &\left( \frac{295225368493618129\nu}{7077880000} \right) \ln(7) - \left( \frac{122529921403}{752640} + \frac{6267828367\nu}{46080} \right) \\
+ &e^{\nu_1}\left[ \left( -\frac{39012948117}{573440} + \frac{267733323\nu}{4096} \right) \ln(a_r) - 2\gamma_E \right] \\
- &\frac{87487776901657}{32112640} + \frac{51000706143}{229376} \nu + \left( -\frac{569049681664742359}{5689958400} \right) \\
+ &\left( \frac{17916377627561875829\nu}{21337344000} \right) \ln(2) + \left( \frac{19205491762769990753}{2301834035200} \right) \\
- &\frac{1707020784493974596637}{2877292544000} \nu \ln(3) + \left( -\frac{8138674822699358046875}{178990614577152} \right) \\
+ &\left( \frac{214230939178466796875\nu}{913217421312} \right) \ln(5) + \left( \frac{28987110744247081153}{7247757312000} \right) \\
- &\frac{224384811477454209253}{3261490790400} \nu \ln(7) + \left( \frac{81402749386839761113321}{4474765364428800} \right) \\
+ &\frac{81402749386839761113321\nu}{1118691341107200} \ln(11) + e^{\nu_2}\left[ -\frac{30986035007243}{72253440} \right] \\
+ &\frac{945865602119}{2752512} \nu + \left( -\frac{4426707571}{43008} + \frac{1565114265}{16384} \nu \right) \ln(a_r) - 2\gamma_E \\
+ &\left( \frac{328393145609360398807}{864162432000} - \frac{862890935509435128773}{329204736000} \right) \ln(2) \\
+ &\left( -\frac{203719103345458519569}{1438646272000} + \frac{16662707951693189189427}{11509170176000} \right) \ln(3) \\
+ &\left( \frac{106107911818692516640625}{604093324197888} - \frac{772369312821197509765625}{690392370511872} \right) \ln(5) \\
+ &\left( -\frac{4145697650045386774577}{660451885056000} + \frac{14418620964621562673877}{1761205026816000} \right) \ln(7) \\
- &\left( \frac{35656966913567027996954261}{226534996574208000} - \frac{470263683207773299951655417}{60409332419788800} \right) \ln(11)
\right]
\end{align*}
\]
\[
\begin{align*}
&+ \left( -\frac{48797431397}{327680} + \frac{132236197093}{983040} \nu \right) \ln(\epsilon_E) \\
&+ \left( -\frac{1113890589789673281623}{691329945600000} + \frac{6591498923973012595615229}{691329945600000} \nu \right) \ln(2) \\
&+ \left( -\frac{2196423646340482773962163}{1473173872800000} - \frac{27978900275432534562921069}{36829344563200000} \nu \right) \ln(3) \\
&- \left( -\frac{420710016768367311936953125}{927887345967955968} - \frac{760603377767734527587890625}{231971836491988992} \nu \right) \ln(5) \\
&+ \left( -\frac{3170132399738511368405793}{4226892064358400000} - \frac{77737887155913025713859691}{1056723016089600000} \nu \right) \ln(7) \\
&+ \left( -\frac{7854405708884154695671648813}{11835297780203520000} - \frac{112327084059654478481943854653}{28996479561498624000} \nu \right) \ln(11) \\
&+ \left( -\frac{917333019326816658399616099}{57992959122972480000} - \frac{917333019326816658399616009}{144982397807493120000} \nu \right) \ln(13) \bigg]\bigg] \\
+ O(\epsilon_E^{22}).
\end{align*}
\]

(99)

Up to $\epsilon_E^{10}$ this agrees with the results given in [13]. Higher-order terms are given for future applications.

Appendix C. The contour integral for the Delaunay variable $i_r$

The integral $J_1$, describing effect of the Newton dynamics, is given by

\[
J_1 = \frac{1}{2\pi i} \oint dx \sqrt{A + \frac{2B}{x} + \frac{C}{x^2}} = \frac{B}{\sqrt{-A}} - \sqrt{-C}.
\]

(100)

All the remaining integrals are directly obtained from the residue at $x = 0$.

The integral in (58) reads to 5PN

\[
\begin{align*}
i_r &= \frac{B}{\sqrt{-A}} - \sqrt{-C} \left\{ \frac{1}{2} \eta_2 \frac{BD_1}{2C^2} + \eta^4 \left[ \frac{3D_1^2}{16C^4} \left( -\frac{5B^2}{2} + AC \right) - \frac{D_2}{4C^5} \left( -\frac{3B^2}{2} + AC \right) \right] + \frac{BD_3}{4C^4} \left( -\frac{5B^2}{2} + 3AC \right) \right\} + \eta^6 \left[ \frac{BD_5}{16C^6} \left( \frac{63B^4}{8} - 70AB^2C + 15A^2C^2 \right) \right] \\
&+ \frac{D_1}{8C^5} \left( \frac{5B^2}{2} - 3AC \right) - \frac{15D_3}{32C^6} \left( \frac{21B^4}{32} - 14AB^2C + A^2C^2 \right) \right\} \\
&+ \frac{35BD_3}{32C^6} \left( -\frac{3B^2}{2} + AC \right) + \frac{D_4}{16C^5} \left( \frac{35B^4}{32} - 30AB^2C + 3A^2C^2 \right) \right\}. 
\end{align*}
\]
\begin{align*}
+ \eta^8 \left[ \frac{7D_3^2}{128C^8} \left( -429B^6 + 495AB^4C - 135A^2B^2C^2 + 5A^3C^3 \right) \\
- \frac{D_6}{32C^7} \left( -231B^6 + 315AB^4C - 105A^2B^2C^2 + 5A^3C^3 \right) + \frac{BD_7}{32C^8} \left( -429B^6 \\
+ 693AB^4C - 315A^2B^2C^2 + 35A^3C^3 \right) + D_1 \left( \frac{7D_5}{64C^8} \left( -429B^6 + 495AB^4C \\
- 135A^2B^2C^2 + 5A^3C^3 \right) + \frac{21BD_4(33B^4 - 30AB^2C + 5A^2C^2)}{32C^7} \right) \right] \\
+ D_1^2 \left( \frac{105D_2(33B^4 - 18AB^2C + A^2C^2)}{128C^7} \\
- \frac{63BD_3(143B^4 - 110AB^2C + 15A^2C^2)}{128C^8} \\
- \frac{105D_4^2(143B^4 - 66AB^2C + 3A^2C^2)}{1024C^8} - \frac{15D_2^2(21B^4 - 14AB^2C + A^2C^2)}{64C^6} \\
+ \frac{21BD_2D_3(33B^4 - 30AB^2C + 5A^2C^2)}{32C^7} \right] + \eta^{10} \left[ - \frac{D_8}{256C^9} \left( 6435B^8 \\
- 12012AB^6C + 6930A^2B^4C^2 - 1260A^3B^2C^3 + 35A^4C^4 \right) + \frac{BD_9}{256C^{10}} \left( 12155B^8 \\
- 25740AB^6C + 18018A^2B^4C^2 - 4620A^3B^2C^3 + 315A^4C^4 \right) \right] \\
+ D_2 \left( - \frac{9BD_5}{64C^9} \left( 715B^6 - 1001AB^4C + 385A^2B^2C^2 - 35A^3C^3 \right) + \frac{7D_4}{64C^8} \left( 429B^6 \\
- 495AB^4C + 135A^2B^2C^2 - 5A^3C^3 \right) \right) + D_1^2 \left( \frac{495BD_5}{256C^{10}} \left( 221B^6 - 273AB^4C \\
+ 91A^2B^2C^2 - 7A^3C^3 \right) - \frac{315D_4}{256C^9} \left( 143B^6 - 143AB^4C + 33A^2B^2C^2 - A^3C^3 \right) \right) \\
+ D_3 \left( - \frac{9BD_4}{64C^9} \left( 715B^6 - 1001AB^4C + 385A^2B^2C^2 - 35A^3C^3 \right) \right) \\
+ \frac{45D_5}{512C^{10}} \left( 2431B^8 - 4004AB^6C + 2002A^2B^4C^2 - 308A^3B^2C^3 + 7A^4C^4 \right) \\
+ D_1^3 \left( \frac{1155D_3}{512C^{10}} \left( 221B^6 - 195AB^4C + 39A^2B^2C^2 - A^3C^3 \right) \\
- 1155B(39B^4 - 26AB^2C + 3A^2C^2)D_2 \right) \right] \\
+ D_1 \left( - \frac{9BD_6}{64C^9} \left( 715B^6 - 1001AB^4C + 385A^2B^2C^2 - 35A^3C^3 \right) + \frac{495BD_2}{256C^{10}} \left( 221B^6 \right) \right]
\end{align*}
\[-273AB^4C + 91A^2B^2C^2 - 7A^3C^3\] \[-\frac{315D_2D_3}{128C^9}(143B^6 - 143AB^4C)\]
\[+ 33A^2B^2C^2 - A^3C^3] + \[\frac{45D_7}{512C^{10}}(2431B^8 - 4004AB^6C + 2002A^2B^4C^2)\]
\[-308A^3B^2C^3 + 7A^4C^4\] \[+ \frac{63B(143B^4 - 110AB^2C + 15A^2C^2)D_2^2}{128C^8}\]
\[+ \frac{9009B(17B^4 - 10AB^2C + A^2C^2)D_1^5}{2048C^{10}}\]\]

The coefficients $A$ to $D_9$ are determined iteratively expanding (62) in powers of $\eta^2$. They depend on the respective Hamiltonian for which one may choose a pole- and log–free form.

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