CLASSIFICATION OF CODAZZI AND MINIMAL
HYPERSURFACES IN $Nil^4$

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Abstract. In this paper, we give a classification of Codazzi hyper-
surfaces in a Lie group $(Nil^4, \tilde{g})$. We also give a characterization of
a class of minimal hypersurfaces in $(Nil^4, \tilde{g})$ with an example, non
trivial, of a minimal surface in this class.

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1. Introduction

Let $(M^m, g)$ be a Riemannian manifold, $\nabla$, $R$, $S$ and $\tau$ denote the
Levi-Civita connection, the Riemannian curvature, the Ricci curvature,
and the scalar curvature of $(M, g)$, respectively. Thus

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

$$S(X, Y) = \sum_{i=1}^{m} g(R(X, e_i)e_i, Y),$$

$$\tau = \sum_{i,j=1}^{m} g(R(e_i, e_j)e_j, e_i),$$

where $\{e_i\}_{1 \leq i \leq m}$ is an orthonormal frame on $(M, g)$, and $X, Y, Z \in \mathfrak{X}(M^m)$. A symmetric $(0,2)$-tensor field $T$ on $(M^m, g)$ is said to be
a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M^m),$$

(see [6, 8]).

Let $(N^n, g)$ be a hypersurface in a Riemannian manifold $(M^{n+1}, \tilde{g})$,
where $g$ is the induced Riemannian metric by $\tilde{g}$. We denote by $\nabla$ (resp.
$\tilde{\nabla}$) the Levi-Civita connection of $(N^n, g)$ (resp. of $(M^{n+1}, \tilde{g})$), $R$ (resp.
$\tilde{R}$) the Riemannian curvature of $(N^n, g)$ (resp. of $(M^{n+1}, \tilde{g})$), $B(\cdot, \cdot) = h(\cdot, \cdot)\xi$ the second fundamental form of the hypersurface $(N^n, g)$, $A_\xi$
the shape operator with respect to the unit normal vector field \( \xi \), \( H = (1/n) \text{trace}_g B \) the mean curvature of \((N^n, g)\) (see [6]). Under the notation above, we have

\[ (5) \quad \widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi, \]

\[ (6) \quad A_\xi X = -\widetilde{\nabla}_X \xi, \quad \forall X, Y \in \mathfrak{X}(N^n), \]

Note that, the components of the second fundamental form \( B \) are given by

\[ (7) \quad h(X, Y) = g(A_\xi X, Y) = -g(\widetilde{\nabla}_X \xi, Y), \quad \forall X, Y \in \mathfrak{X}(N^n). \]

The equations of Gauss and Codazzi are given respectively by

\[ (8) \quad \bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + h(X, W)h(Y, Z) - h(X, Z)h(Y, W), \]

\[ (9) \quad \bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z), \]

where \((\nabla_X h)(Y, Z) = X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\) called the cubic form, and \(X, Y, Z, W \in \mathfrak{X}(N^n)\). The hypersurface \((N^n, g)\) is said to be parallel if the cubic form vanishes identically, i.e., \(\nabla h = 0\). A special case of parallel hypersurfaces are totally geodesic hypersurfaces, for which the second fundamental form \( B = 0 \). The hypersurface \((N^n, g)\) is called Codazzi (resp. minimal) if the symmetric \((0, 2)\)-tensor field \( h \) is a Codazzi tensor (resp. if \( H = 0 \)) (see [6]).

The 4-dimensional lie group \( Nil^4 \) is a well known nilpotent lie group. It is also one of the 4-dimensional thurston model geometries, [7]. As the \( Nil^4 \) space are well know and one of its left invariant Riemannian metric is well known and also used in many research works, we will begin with an explicit calculus of this metric and its geometric proprieties, (that one of the autors has do a part of it, in [3], based on the definition of \( Nil^4 \) and its most used left invariant metric), only to let the reader follow us easily.

The nilpotent Lie group \( Nil^4 = \mathbb{R}^3 \ltimes U \mathbb{R} \), where \( U(t) = \exp(tL) \), with

\[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \exp(tL) = I_3 + tL + \frac{t^2}{2}L^2 = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \]
The semidirect product in \( N_{il}^4 \) is given by

\[
(V, t)(V', t') = (V + \exp(tL)V', t + t')
\]

\[
= \left( \begin{array}{c} x \\ y \\ z \\ \end{array} \right) + \left( \begin{array}{ccc} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} x' \\ y' \\ z' \end{array} \right), t + t'
\]

\[
= \left( \begin{array}{c} x + x' + t y' + \frac{t^2}{2} z' \\ y + y' + t z' \\ z + z' \end{array} \right), t + t'
\]

(11)

for all \( V = \left( \begin{array}{c} x \\ y \\ z \end{array} \right), V' = \left( \begin{array}{c} x' \\ y' \\ z' \end{array} \right) \in \mathbb{R}^3 \), and \( t \in \mathbb{R} \). We have the parameterization

\[
\phi : N_{il}^4 \rightarrow \mathbb{R}^4.
\]

\[
\left( \begin{array}{c} x \\ y \\ z \end{array} \right), t \rightarrow (x, y, z, t)
\]

(12)

Taking the left-invariant frame fields

\[
e_1 = \frac{\partial}{\partial x},
\]

\[
e_2 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial y},
\]

\[
e_3 = \frac{t^2}{2} \frac{\partial}{\partial x} + t \frac{\partial}{\partial y} + \frac{\partial}{\partial z},
\]

\[
e_4 = \frac{\partial}{\partial t}.
\]

(13)

So that, the dual coframe fields are given by

\[
\theta_1 = dx - t dy + \frac{t^2}{2} dz,
\]

\[
\theta_2 = dy - tdz,
\]

\[
\theta_3 = dz,
\]

\[
\theta_4 = dt.
\]

(14)

The matrix of a Riemannian metric \( \tilde{g} = \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 \) is given by

\[
(\tilde{g}_{ij}) = \begin{pmatrix}
1 & -t & \frac{t^2}{2} & 0 \\
-t & 1 + t^2 & -t(1 + \frac{t^2}{2}) & 0 \\
\frac{t^2}{2} & -t(1 + \frac{t^2}{2}) & 1 + t^2 + \frac{t^4}{4} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
In this paper, we study some geometric properties of a Riemannian manifold \((\text{Nil}^4, \tilde{g})\). We also give a characterization of a minimal hypersurfaces in \((\text{Nil}^4, \tilde{g})\) which have a normal vector field depend only on, last coordinate, \(t\).

2. Geometric properties of \((\text{Nil}^4, \tilde{g})\)

Proposition 1. The non-zero of the Levi-Civita connection \(\tilde{\nabla}\) of \((\text{Nil}^4, \tilde{g})\) are given by

\[
\begin{align*}
\tilde{\nabla}_{e_1}e_2 &= \frac{1}{2}e_4, & \tilde{\nabla}_{e_1}e_4 &= -\frac{1}{2}e_2 \\
\tilde{\nabla}_{e_2}e_1 &= \frac{1}{2}e_4, & \tilde{\nabla}_{e_2}e_3 &= \frac{1}{2}e_4 \\
\tilde{\nabla}_{e_2}e_4 &= -\frac{1}{2}(e_1 + e_3), & \tilde{\nabla}_{e_3}e_2 &= \frac{1}{2}e_4 \\
\tilde{\nabla}_{e_3}e_4 &= -\frac{1}{2}e_2, & \tilde{\nabla}_{e_4}e_1 &= \frac{1}{2}e_2 \\
\tilde{\nabla}_{e_4}e_2 &= \frac{1}{2}(e_1 - e_3), & \tilde{\nabla}_{e_4}e_3 &= \frac{1}{2}e_2.
\end{align*}
\]

Proof. Note that, the non-zero of Christoffel symbols \(\tilde{\Gamma}^k_{ij}\) for \(i, j, k \in \{1, 2, 3, 4\}\) are given by

\[
\begin{align*}
\tilde{\Gamma}^4_{12} &= \frac{1}{2}, & \tilde{\Gamma}^4_{13} &= -\frac{t}{2} \\
\tilde{\Gamma}^4_{14} &= -\frac{t}{2}, & \tilde{\Gamma}^2_{14} &= -\frac{1}{2} \\
\tilde{\Gamma}^4_{22} &= -t, & \tilde{\Gamma}^4_{23} &= \frac{1}{2} + \frac{3t^2}{4} \\
\tilde{\Gamma}^1_{24} &= -\frac{1}{2} + \frac{t^2}{4}, & \tilde{\Gamma}^3_{24} &= -\frac{1}{2} \\
\tilde{\Gamma}^4_{33} &= -t(1 + \frac{t^2}{2}), & \tilde{\Gamma}^3_{34} &= \frac{1}{2} + \frac{3t^2}{4} \\
\tilde{\Gamma}^3_{34} &= \frac{t}{2}.
\end{align*}
\]

Proposition 1 follows from (13). \(\square\)

Corollary 2. The non-zero Lie brackets of the basis \(\{e_i\}_{1 \leq i \leq 4}\) are given by

\[
[e_4, e_2] = e_1, \quad [e_4, e_3] = e_2.
\]

Proof. Follows directly by Proposition 1 with \([e_i, e_j] = \tilde{\nabla}_{e_i}e_j - \tilde{\nabla}_{e_j}e_i\) for all \(i, j = 1, 2, 3, 4\). \(\square\)
Proposition 3. The only non-zero components of Riemannian curvature of $(\text{Nil}^4, \tilde{g})$ are given by

\[
\begin{align*}
\tilde{g}(\tilde{\mathcal{R}}(e_1, e_2)e_1, e_2) &= -\frac{1}{4}, & \tilde{g}(\tilde{\mathcal{R}}(e_1, e_2)e_2, e_3) &= \frac{1}{4} \\
\tilde{g}(\tilde{\mathcal{R}}(e_1, e_4)e_1, e_4) &= -\frac{1}{4}, & \tilde{g}(\tilde{\mathcal{R}}(e_1, e_4)e_3, e_4) &= \frac{1}{4} \\
\tilde{g}(\tilde{\mathcal{R}}(e_2, e_1)e_2, e_3) &= -\frac{1}{4}, & \tilde{g}(\tilde{\mathcal{R}}(e_2, e_3)e_2, e_4) &= \frac{3}{4} \\
\tilde{g}(\tilde{\mathcal{R}}(e_2, e_4)e_2, e_4) &= \frac{1}{2}, & \tilde{g}(\tilde{\mathcal{R}}(e_3, e_4)e_3, e_4) &= \frac{3}{4} 
\end{align*}
\]

Proof. Using the definition of Riemannian curvature (1), the Proposition 1, and the Corollary 2. □

According to Proposition 3, we have the following Corollary.

Corollary 4. The matrix of Ricci curvature of $(\text{Nil}^4, \tilde{g})$ is given by

\[
(S_{ij}) = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix},
\]

where $S_{ij} = \sum_{a=1}^{n} \tilde{g}(\tilde{\mathcal{R}}(e_i, e_a)e_a, e_j) \text{ for all } i, j = 1, 2, 3, 4$. Thus, the scalar curvature of $(\text{Nil}^4, \tilde{g})$ is $\tau = -1$.

3. CODAZZI HYPERSURFACES IN $\text{Nil}^4$

Let $(M^3, g)$ be a hypersurface in $(\text{Nil}^4, \tilde{g})$. We have, $\xi = ae_1 + be_2 + ce_3 + de_4$ the unit normal vector field on $(M^3, g)$, where $a, b, c, d$ are local functions on $M^3$. Thus

\[
\begin{align*}
X_1 &= be_1 - ae_2, & X_2 &= ce_1 - ae_3 \\
X_3 &= de_1 - ae_4, & X_4 &= ce_2 - be_3 \\
X_5 &= de_2 - be_4, & X_6 &= de_3 - ce_4 
\end{align*}
\]

are tangent vectors fields to the hypersurface $(M^3, g)$. Now, assume that the hypersurface $(M^3, g)$ is Codazzi, that is

\[
(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M^3).
\]

Then it follows from the equation of Codazzi (9) that

\[
\tilde{g}(\tilde{\mathcal{R}}(X_i, X_j)X_k, \xi) = 0, \quad \forall i, j, k \in \{1, ..., 6\}.
\]
By using the curvature components given in Proposition 3, we get the following
\[ \widetilde{g}(\widetilde{R}(X_1, X_2)X_3, \xi) = \frac{1}{4}abd(a - c) = 0, \]
from which we prove that \( a = 0 \) or \( b = 0 \) or \( d = 0 \) or \( a = c \).

- If \( a = 0 \), we have the following equations
  \[
  \begin{align*}
  \widetilde{g}(\widetilde{R}(X_1, X_4)X_1, \xi) &= -\frac{1}{4}b^3c = 0, \\
  \widetilde{g}(\widetilde{R}(X_2, X_4)X_4, \xi) &= \frac{1}{4}c^2b^2 + \frac{1}{4}c^2 = 0, \\
  \widetilde{g}(\widetilde{R}(X_1, X_5)X_4, \xi) &= \frac{1}{2}b^3d + \frac{1}{4}bc^2d = 0.
  \end{align*}
  \]

Thus \( c = 0 \) and \( bd = 0 \). So that, \( \xi = e_2 \) or \( \xi = e_4 \). Note that, in the case where \( \xi = e_2 \), the Lie bracket \( [e_4, e_3] = e_2 \) is not tangent vector field on \( M^3 \) despite \( e_2 \) and \( e_4 \) are tangent vector fields on \( M^3 \). So, by Frobenius Theorem (see [8]), this case is unacceptable. Then we have \( \xi = e_4 \).

- If \( b = 0 \), we obtain the equations
  \[
  \begin{align*}
  \widetilde{g}(\widetilde{R}(X_1, X_2)X_1, \xi) &= \frac{1}{4}a^2(a^2 - c^2) = 0, \\
  \widetilde{g}(\widetilde{R}(X_1, X_3)X_1, \xi) &= -\frac{1}{4}a^2d(3a + c) = 0.
  \end{align*}
  \]

For \( a = 0 \), we get \( c = 0 \). Thus \( \xi = e_4 \). For \( a = \pm c \), we find that \( ad = 0 \). Hence \( \xi = e_4 \) or \( \xi = \frac{1}{\sqrt{2}}(e_1 \pm e_3) \). Note that, in the case where \( \xi = \frac{1}{\sqrt{2}}(e_1 \pm e_3) \), the Lie bracket \([e_4, e_2]\) is tangent vector field on \( M^3 \) because \( e_2 \) and \( e_4 \) are tangent vector fields on \( M^3 \). But \( \widetilde{g}([e_4, e_2], \xi) = \widetilde{g}(e_1, \xi) = \frac{1}{\sqrt{2}} \neq 0 \), we obtain a contradiction with the fact that \( \xi \) in normal to \( M^3 \). Therefore, \( \xi = e_4 \).

- If \( d = 0 \), we have the equations
  \[
  \begin{align*}
  \widetilde{g}(\widetilde{R}(X_1, X_2)X_2, \xi) &= -\frac{1}{4}ab(a - c)^2 = 0, \\
  \widetilde{g}(\widetilde{R}(X_1, X_4)X_1, \xi) &= \frac{1}{4}b(a - c)(a^2 + b^2 + ac) = 0, \\
  \widetilde{g}(\widetilde{R}(X_2, X_4)X_4, \xi) &= -\frac{1}{4}c(a - c)(b^2 + c^2 + ac) = 0, \\
  \widetilde{g}(\widetilde{R}(X_1, X_2)X_1, \xi) &= \frac{1}{4}a(a - c)(a^2 + b^2 + ac) = 0, \\
  \widetilde{g}(\widetilde{R}(X_6, X_2)X_3, \xi) &= -a^2c^2 - \frac{1}{4}ac(a^2 - c^2) = 0.
  \end{align*}
  \]

Hence \( a = c = 0 \). Thus, \( \xi = e_2 \). It is unacceptable, because in this case \( e_3 \) and \( e_4 \) are tangent vector fields on \( M^3 \) but \( [e_4, e_3] = e_2 \) is not
tangent vector field on $M^3$.

- If $a = c$, we get the following equations
  \[
  \tilde{g}(\tilde{R}(X_6, X_2)X_3, \xi) = -c^2(c^2 + \frac{1}{2}d^2) = 0, \\
  \tilde{g}(\tilde{R}(X_1, X_5)X_6, \xi) = \frac{1}{2}b^2(c^2 - d^2) = 0.
  \]
  we obtain $c = 0$ and $bd = 0$. Hence, $\xi = e_4$. Here, $\xi = e_2$ is unacceptable.

**Theorem 5.** A hypersurface $(M^3, g)$ in the Lie group $(\text{Nil}^4, \tilde{g})$ is Codazzi if and only if the unit normal vector field to $(M^3, g)$ is $\xi = e_4$.

**Proof.** According to the previous calculations, it suffices to show that
\[
(17) \quad \tilde{g}(\tilde{R}(X, Y)Z, \xi) = (\nabla Y h)(X, Z) - (\nabla X h)(Y, Z) = 0,
\]
for all $X, Y, Z \in \mathfrak{X}(M^3)$ for $\xi = e_4$. You can easily check the equations
\[
\tilde{g}(\tilde{R}(e_i, e_j)e_k, e_4) = 0, \quad i, j, k = 1, ..., 3.
\]

\[\square\]

**Remark 6.**
1. According to (13), a Codazzi hypersurface $(M^3, g)$ in $(\text{Nil}^4, \tilde{g})$ which is given by
   \[
   f : (M^3, g) \rightarrow (\text{Nil}^4, \tilde{g}), \\
   (x, y, z) \mapsto (x, y, z, t_0)
   \]
   where $t_0 \in \mathbb{R}$.
2. Since $B(X, Y) = (\bar{\nabla} X Y)^\perp$ for all $X, Y \in \mathfrak{X}(M^3)$. By using the Proposition, the second fundamental form $h$ of the hypersurface $(M^3, g)$ in $(\text{Nil}^4, \tilde{g})$ is given by
   \[
   (h_{ij}) = \begin{pmatrix}
   0 & \frac{1}{2} & 0 & 0 \\
   \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
   0 & \frac{1}{2} & 0 & 0 \\
   \end{pmatrix}
   \]
   where $h_{ij} = h(e_i, e_j)$ for all $i, j = 1, ..., 3$. In this case, $(M^3, g)$ is not totally geodesic because $h \neq 0$, and minimal hypersurface in $(\text{Nil}^4, \tilde{g})$, that is $H = 0$. It is also parallel in $(\text{Nil}^4, \tilde{g})$ because $\nabla h = 0$.
3. The principal curvatures of $(M^3, g)$ in $(\text{Nil}^4, \tilde{g})$ are $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$.

From the previous remarks we can conclude the following corollary.

**Corollary 7.** A hypersurface $(M^3, g)$ in the Lie group $(\text{Nil}^4, \tilde{g})$ is Codazzi if and only if it is parallel. The Lie group $(\text{Nil}^4, \tilde{g})$ do not have any totally geodesic hypersurface.
4. Minimal hypersurfaces in $Nil^4$

Let $(M^3, g)$ be a hypersurface in $(Nil^4, \bar{g})$. In this section, we search the conditions for the hypersurface $(M^3, g)$ to be minimal in $(Nil^4, \bar{g})$, where the unit normal vector field on $(M^3, g)$ is given by $\xi = ae_1 + be_2 + ce_3 + de_4$, and assume that $\{X_i\}_{1 \leq i \leq 3}$ is a local orthonormal frame on $(M^3, g)$, where $X_i = a_ie_1 + b_ie_2 + c_ie_3 + d_ie_4$ for some local functions $\{a, b, c, d, a_i, b_i, c_i, d_i\}_{1 \leq i \leq 3}$ on $M^3$ depends only on the variable $t$.

**Theorem 8.** The hypersurface $(M^3, g)$ is minimal in $(Nil^4, \bar{g})$ if and only if

$$\sum_{i=1}^{3} \left[ a_i(b_id-bdi) + b_i(c_id-cdi) + d_i(aa'_i+bb'_i+cc'_i+dd'_i) \right] = 0.$$  

**Proof.** Let $i = 1, 2, 3$. We compute

$$\bar{\nabla}_{X_i}X_i = \bar{\nabla}_{a_ie_1+b_ie_2+c_ie_3+d_ie_4}(a_ie_1 + b_ie_2 + c_ie_3 + d_ie_4)$$

$$= a_i \left( a_i \bar{\nabla}_{e_1}e_1 + b_i \bar{\nabla}_{e_1}e_2 + c_i \bar{\nabla}_{e_1}e_3 + d_i \bar{\nabla}_{e_1}e_4 \right)$$

$$+ b_i \left( a_i \bar{\nabla}_{e_2}e_1 + b_i \bar{\nabla}_{e_2}e_2 + c_i \bar{\nabla}_{e_2}e_3 + d_i \bar{\nabla}_{e_2}e_4 \right)$$

$$+ c_i \left( a_i \bar{\nabla}_{e_3}e_1 + b_i \bar{\nabla}_{e_3}e_2 + c_i \bar{\nabla}_{e_3}e_3 + d_i \bar{\nabla}_{e_3}e_4 \right)$$

$$+ d_i \left( a'_ie_1 + a_i\bar{\nabla}_{e_4}e_1 + b'_ie_2 + b_i\bar{\nabla}_{e_4}e_2 + c'_ie_3 + c_i\bar{\nabla}_{e_4}e_3 \right. $$

$$\left. + d'_ie_4 + d_i\bar{\nabla}_{e_4}e_4 \right).$$

(18)

From Proposition [1] and equation (18), we obtain

$$\bar{\nabla}_{X_i}X_i = a_i \left( \frac{b_i}{2}e_4 - \frac{d_i}{2}e_2 \right) + b_i \left( \frac{a_i}{2}e_4 + \frac{c_i}{2}e_4 - \frac{d_i}{2}(e_1 + e_3) \right)$$

$$+ c_i \left( \frac{b_i}{2}e_4 - \frac{d_i}{2}e_2 \right) + d_i \left( a'_ie_1 - \frac{a_i}{2}e_2 + b'_ie_2 + \frac{b_i}{2}(e_1 - e_3) \right)$$

$$+ c'_ie_3 + \frac{c_i}{2}e_2 + d'_ie_4 \right),$$

it is equivalent to the following equation

(19) $\bar{\nabla}_{X_i}X_i = a'_id_ie_1 + d_i(b'_i-a_i)e_2 + d_i(c'_i-b_i)e_3 + [d_id'_i + b_i(a_i + c_i)]e_4$.

By equation (19), we have

$$\bar{g}(\bar{\nabla}_{X_i}X_i, \xi) = aa'_id_i + bd_i(b'_i-a_i) + cd_i(c'_i-b_i) + d[d_id'_i + b_i(a_i + c_i)].$$

(20)
Note that, $B(X_i, X_i) = (\tilde{\nabla} X_i, X_i)_{\perp}$, that is $h(X_i, X_i) = \tilde{g}(\tilde{\nabla} X_i, X_i, \xi)$.

Thus, the hypersurface $(M^3, g)$ is minimal if

\[(21) \quad H = \frac{1}{3} \sum_{i=1}^{3} \tilde{g}(\tilde{\nabla} X_i, X_i, \xi) = 0.\]

The Theorem 8 follows by equations (20) and (21).

\[\blacksquare\]

**Example 9.** We consider the following vector fields

\[
\begin{align*}
\xi &= \frac{2}{\sqrt{5}(2 + t^2)}e_1 + \frac{2t}{\sqrt{5}(2 + t^2)}e_2 + \frac{t^2}{\sqrt{5}(2 + t^2)}e_3 - \frac{2}{\sqrt{5}}e_4, \\
X_1 &= -\frac{t}{\sqrt{1 + t^2}}e_1 + \frac{1}{\sqrt{1 + t^2}}e_2, \\
X_2 &= -\frac{t^2}{(2 + t^2)\sqrt{1 + t^2}}e_1 - \frac{t^3}{(2 + t^2)\sqrt{1 + t^2}}e_2 + \frac{2\sqrt{1 + t^2}}{2 + t^2}e_3, \\
X_3 &= \frac{4}{\sqrt{5}(2 + t^2)}e_1 + \frac{4t}{\sqrt{5}(2 + t^2)}e_2 + \frac{2t^2}{\sqrt{5}(2 + t^2)}e_3 + \frac{1}{\sqrt{5}}e_4.
\end{align*}
\]

It is easy to verify that these vector fields satisfy

\[\tilde{g}(\xi, \xi) = 1, \quad \tilde{g}(\xi, X_i) = 0, \quad \tilde{g}(X_i, X_j) = \delta_{ij}, \quad \forall i, j = 1, 2, 3,
\]

and the condition of Theorem 8. Thus the hypersurface $(M^3, g)$ defined by these vector fields is minimal. According to (13), this hypersurface $(M^3, g)$ is given by

\[
\begin{align*}
f : (M^3, g) &\longrightarrow (Nil^4, \tilde{g}), \\
(y, z, t) &\longrightarrow (2t + \frac{t^3}{3}, y, z, t)
\end{align*}
\]

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