Pointwise strong and very strong approximation by Fourier series of integrable functions

Włodzimierz Łenski
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a
P O L A N D
W.Lenski@wmie.uz.zgora.pl

Abstract

We will present an estimation of the $H^q_{k_0,k_r}f$ and $H^{\lambda,\phi}_{\alpha}f$ means as a approximation versions of the Totik type generalization (see [8, 9]) of the results of J. Marcinkiewicz and A. Zygmund in [7, 10]. As a measure of such approximations we will use the function constructed on the base of definition of the Gabisonia points [1]. Some results on the norm approximation will also given.

Key words: Pointwise approximation, Strong and very strong approximation

2000 Mathematics Subject Classification: 42A24.
1 Introduction

Let \( L^p (1 < p < \infty) \) [resp.\( C \)] be the class of all \( 2\pi \)-periodic real-valued functions integrable in the Lebesgue sense with \( p \)-th power [continuous] over \( Q = [-\pi, \pi] \) and let \( X = X^p \) where \( X^p = L^p \) when \( 1 < p < \infty \) or \( X^p = C \) when \( p = \infty \). Let us define the norm of \( f \in X^p \) as

\[
\|f\|_{X^p} = \|f(x)\|_{X^p} = \begin{cases} 
\left(\int_{Q} |f(x)|^p \, dx\right)^{1/p} & \text{when } 1 < p < \infty, \\
\sup_{x \in Q} |f(x)| & \text{when } p = \infty.
\end{cases}
\]

Consider the trigonometric Fourier series

\[
S_f(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)
\]

and denote by \( S_kf \) the partial sums of \( S_f \). Then,

\[
H_{k_0, k_r}^q(x) := \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} |S_{k_\nu}f(x) - f(x)|^q \right\}^{1/q}, \quad (q > 0),
\]

where \( 0 \leq k_0 < k_1 < k_2 < \ldots < k_r \) (\( \geq r \)), and

\[
H_{u}^{\lambda_p}f(x) := \left\{ \sum_{\nu=0}^{\infty} \lambda_{\nu}(u) \varphi(|S_{\nu}f(x) - f(x)|) \right\}.
\]

where \( (\lambda_{\nu}) \) is a sequence of positive functions defined on the set having at least one limit point and a function \( \varphi : [0, \infty) \to \mathbb{R} \).

As a measure of the above deviations we will use the pointwise characteristic, constructed on the base of definition of the Gabisonia points (\( G_{p,s} - \text{points} \)), introduced in [1] as follows

\[
G_x f(\delta)_{p,s} := \left\{ \sum_{k=1}^{[\pi/\delta]} \left( 1/k \delta \int_{(k-1)\delta}^{k\delta} |\varphi_x(t)|^p \, dt \right)^s \right\}^{1/s},
\]

\[
G^2_x f(\gamma)_{p,s} := \sup_{0 < \delta \leq \gamma} \left\{ \sum_{k=1}^{[\pi/\delta]} \left( 1/k \delta \int_{(k-1)\delta}^{k\delta} |\varphi_x(t)|^p \, dt \right)^s \right\}^{1/s},
\]

and, constructed on the base of definition of the Lebesgue points (\( L^p - \text{points} \), defined as usually

\[
w_x f(\delta)_{p} := \left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x(t)|^p \, dt \right\}^{1/p},
\]

where \( \varphi_x(t) := f(x+t) + f(x-t) - 2f(x) \).
We can observe that, for any $s > 0$ and $p \in [1, \infty)$

$$w_x f(\delta) \leq G_x f(\delta)_{p,s},$$

for $p \in [1, \infty)$ and $f \in C$

$$w_x f(\delta) \leq \omega_C f(\delta).$$

By the Minkowski inequality, with $\tilde{p} \geq s > p \geq 1$ for $f \in X^{\tilde{p}}$,

$$\|G f(\delta)_{p,s}\|_{X^{\tilde{p}}} \leq \omega_{X^{\tilde{p}}} f(\delta)$$

(cf. [1]) and

$$\|w_x f(\delta)\|_{X^{\tilde{p}}} \leq \omega_{X^{\tilde{p}}} f(\delta),$$

where $\omega_X f$ is the modulus of continuity of $f$ in the space $X = X^{\tilde{p}}$ defined by the formula

$$\omega_X f(\delta) := \sup_{0 < |h| \leq \delta} \|\varphi(h)\|_X.$$

It is well-known that $H_{0,r}^{q} f(x)$ - means tend to 0 (as $r \to \infty$) at the $L^p - points x$ of $f \in L^p$ ($1 < p \leq \infty$) and at the $G_{1,s} - points x$ of $f \in L^1$ ($s > 1$). These facts were proved as a generalization of the Fejér classical result on the convergence of the $(C,1)$ -means of Fourier series by G. H. Hardy, J. E. Littlewood in [3] and by O. D. Gabisonia in [1]. In case $L^1$ and convergence almost everywhere the first results on this area belong to J. Marcinkiewicz [7] and A. Zygmund [10]. The estimates of $H_{0,r}^{q} f(x) - mean$ were obtained in [?, ?]. Here we present estimations of the $H_{0,r}^{q} f(x)$ and $H_{0,r}^{q} f(x)$ means as approximation versions of the Totik type (see [8, 9]) generalization of the result of O. D. Gabisonia [1]. We also give some corollaries on norm approximation.

By $K$ we shall designate either an absolute constant or a constant depending on the some parameters, not necessarily the same of each occurrence. We shall write $I_1 \ll I_2$ if there exists a positive constant $K$, sometimes depends on the some parameters, such that $I_1 \leq KI_2$.

2 Statement of the results

Let us consider a function $w_x$ of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2$ and let

$$L^p(w_x)_s = \left\{ f \in L^p : G_x f(\delta)_{p,s} \leq w_x(\delta), \text{ where } \delta > 0, \ s > p \geq 1 \right\}.$$

In the same way let

$$X^p(\omega)_s = \left\{ f \in X^p : \|G^o f(\delta)_{1,s}\|_{X^p} \leq \omega(\delta), \text{ with a modulus of continuity } \omega \right\}.$$

We start with theorems:
Theorem 1 If \( f \in L^1(w_x) \) and \( 0 \leq k_0 < k_1 < k_2 < \ldots < k_r (\geq r) \), then
\[
H_{k_0,k_r}^q(x) \ll w_x \left( \frac{\pi}{k_0 + 1} \right) \log \frac{k_r + 1}{r + 1/2},
\]
where \( 0 < q' \leq q \ (\geq 2) \) such that \( \frac{1}{s} + \frac{1}{q} = 1 \).

Theorem 2 If \( f \in X^p \) and \( 0 \leq k_0 < k_1 < k_2 < \ldots < k_r (\geq r) \), then
\[
\left\| H_{k_0,k_r}^{q'} f (\cdot) \right\|_{X^p} \ll \omega \left( \frac{\pi}{k_0 + 1} \right) \log \frac{k_r + 1}{r + 1/2},
\]
where \( 0 < q' \leq q \ (\geq 2) \) such that \( \frac{1}{s} + \frac{1}{q} = 1 \).

Denoting
\[
\Phi = \left\{ \varphi : \varphi(0) = 0, \ \varphi \not\rightarrow, \ \varphi(2u) \ll \varphi(u) \text{ for } u \in (0,1) \right\}
\]
and \( \log \varphi(u) = O(u) \) as \( u \to \infty \)

we can formulate the next theorems on the base of the before two.

Theorem 3 If \( f \in L^1 \), \( \varphi \in \Phi \) and \( \lambda_\nu (m) = \frac{1}{N_{m+1}} \) for \( \nu = N_{m-2} + 1, N_{m-2} + 2, \ldots, N_m \) and \( \lambda_\nu (m) = 0 \) otherwise, then
\[
H_{m}^{\lambda \nu} f (x) \ll \varphi \left( w_x \left( \frac{\pi}{N_{m-2} + 1} \right) \right),
\]
where \( m = 1,2,.. \) and \( s \geq 1 \).

Theorem 4 If \( f \in X^p \), \( \varphi \in \Phi \) and \( \lambda_\nu (m) = \frac{1}{N_{m+1}} \) for \( \nu = N_{m-2} + 1, N_{m-2} + 2, \ldots, N_m \) and \( \lambda_\nu (m) = 0 \) otherwise, then
\[
\left\| H_{m}^{\lambda \nu} f (\cdot) \right\|_{X^p} \ll \varphi \left( \omega \left( \frac{\pi}{N_{m-2} + 1} \right) \right),
\]
where \( m = 1,2,.. \) and \( s \geq 1 \).

Let, as in the Leindler monograph \[4\] p.15,
\[
\Lambda_\tau (N_m) = \left\{ (\lambda_\nu) : \left( \frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} (\lambda_\nu)^\gamma \right)^{1/\gamma} \ll \left( \frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu \right) \right\}
\]
for \( s \geq 1 \) and \( N_m < N_{m+1} \), \( N_0 = 0 \), \( N_{-1} = -1 \).

Finally, we present very general results deduced from the above theorems.

Theorem 5 If \( f \in L^1 \) then for \( (\lambda_\nu) \in \Lambda_\tau (N_m) \) with \( \tau > 1 \) and for \( \varphi \in \Phi \), we have
\[
H_{m}^{\lambda \nu} f (x) \ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_\nu (u) \varphi \left( w_x \left( \frac{\pi}{N_{m-2} + 1} \right) \right),
\]
for any real \( u \) and \( s \geq 1 \).
Theorem 6 If $f \in \mathcal{X}^p$ then, for $(\lambda, \nu) \in \Lambda_T(N_m)$ with $\tau > 1$ and for $\varphi \in \Phi$, we have

$$
\|H_u^\lambda f(\cdot)\|_{\mathcal{X}^p} \ll \sum_{m=1}^{\infty} \sum_{\nu=N_m-1}^{N_m} \lambda \varphi(u) \varphi\left(\omega\left(\frac{\pi}{N_m-1}\right)\right),
$$

for any real $u$ and $s > 1$.

From these theorems we can derive the following corollary.

Corollary 1 If we additionally suppose that $\lim_{u \to u_0} \lambda(\cdot) = 0$ for all $\nu$ and that $\sum_{\nu} \lambda(\cdot)$ converges, then we have

$$
\lim_{u \to u_0} H_u^\lambda f(x) = 0
$$

at every $G_{1,s}$-points $x$ of the function $f$, and

$$
\lim_{u \to u_0} \|H_u^\lambda f(\cdot)\|_{\mathcal{L}^p} = 0.
$$

for any real $s > 1$.

Remark 1 We can observe that in the light of the Gabisonia result \cite{2} our pointwise results remain true for $f \in \mathcal{L}^p$ ($p > 1$), since every $\mathcal{L}^p$-point of the function $f$ is its $G_{p,s}$-point.

3 Auxiliary results

At the begin we present some lemmas on pointwise characteristics.

Lemma 1 (Property 1 \cite{2}) If $f \in \mathcal{L}^p$ ($p \geq 1$) and $\lambda, \beta > 0$, then

$$
\left\{ \lambda^\beta \int_0^\pi t^{-(\beta+1)} |\varphi_x(t)|^p \, dt \right\}^{1/\beta} \ll G_x f(\lambda),
$$

with $s > p$ such that $s(1 - \beta) < p$.

Lemma 2 (Property 2 \cite{2}) If $f \in \mathcal{L}^p$ ($p \geq 1$) and $\lambda, \beta > 0$, then

$$
G_x f(2\lambda) \leq 2^{1/p-1/s} G_x f(\lambda),
$$

with $s > p$.

Lemma 3 If $f \in \mathcal{L}^p$ ($p \geq 1$), then

$$
\left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t + \gamma) - \varphi_x(t)|^p \, dt \right\}^{1/p} \leq \left( 2^{1/p} + 4^{1/p} \right) G_x f(\delta),
$$

for any positive $\gamma \leq \delta$ and $1 \leq p < s$. 

5
Proof. Since $\gamma \leq \delta$ we have

$$\left\{ \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} |\varphi_x (t+\gamma) - \varphi_x (t)|^p dt \right\}^{1/p} \leq \left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x (t+\gamma) - \varphi_x (t)|^p dt \right\}^{1/p} + \left\{ \frac{1}{\delta} \int_{-\delta}^{0} |\varphi_x (t+\gamma) - \varphi_x (t)|^p dt \right\}^{1/p}$$

$$\leq \left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x (t)|^p dt \right\}^{1/p} + \left\{ \frac{1}{\delta} \int_{-\delta}^{0} |\varphi_x (t)|^p dt \right\}^{1/p}$$

$$\leq \left\{ \frac{2}{2\delta} \int_{0}^{2\delta} |\varphi_x (t)|^p dt \right\}^{1/p} + \left\{ \frac{2}{\delta} \int_{-2\delta}^{-\delta} |\varphi_x (t)|^p dt \right\}^{1/p}$$

$$= \left( \frac{2^{1/p} + 4^{1/p}}{2} \right) \left\{ \frac{1}{2\delta} \int_{0}^{2\delta} |\varphi_x (t)|^p dt \right\}^{1/p}$$

and our inequalities are evident. 

Under the notation

$$\Phi_{x,f} (\delta, \gamma) := \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} \varphi_x (t) dt, \quad W_{x,f} (\delta, \gamma)_p := \left[ \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} |\varphi_x (t)|^p dt \right]^{1/p}$$

we can formulate a lemma.

**Lemma 4** If $f \in L^p$ ($p \geq 1$), then

$$|\Phi_{x,f} (\delta, \gamma)| \leq W_{x,f} (\delta, \gamma)_p \ll w_{x,f} (2\delta)$$

for any positive $\gamma \leq \delta$.

Proof. The first inequality is evidence, then we prove the second one only.

If $f \in L^p$, then

$$\left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x (t+\gamma) - \varphi_x (t)|^p dt \right\}^{1/p} \leq \left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x (t)|^p dt \right\}^{1/p}$$

whence

$$\left\{ \frac{1}{\delta} \int_{\gamma}^{\gamma+\delta} |\varphi_x (t)|^p dt \right\}^{1/p}$$

$$\leq \left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x (t+\gamma) - \varphi_x (t)|^p dt \right\}^{1/p} + \left\{ \frac{1}{\delta} \int_{0}^{\delta} |\varphi_x (t)|^p dt \right\}^{1/p}$$

$$= \left( \frac{2^{1/p} + 4^{1/p}}{2} \right) \left\{ \frac{1}{2\delta} \int_{0}^{2\delta} |\varphi_x (t)|^p dt \right\}^{1/p}$$
and by the previous lemma our second relation follows.

We will also need the inequalities for norms.

**Lemma 5** If \( f \in L^p \ (p \geq 1) \), then

\[
\| \Phi f (\delta, \gamma) \|_{L^p} \leq \left\| W f (\delta, \gamma) \right\|_{L^p} \leq 2 \omega_{L^p} f (\delta + \gamma)
\]

and

\[
\left\| \frac{1}{\delta} \int_0^\delta |\varphi_0 (t) - \varphi_0 (t \pm \gamma)|^p dt \right\|^{1/p} \leq 2 \omega_{L^p} f (\gamma),
\]

for any positive \( \gamma \) and \( \delta \).

**Proof.** If \( f \in L^p \), then, by monotonicity of the norm as a functional and by the above Lemma,

\[
\| \Phi f (\delta, \gamma) \|_{L^p} \leq \left\| W f (\delta, \gamma) \right\|_{L^p}
\]

and consequently

\[
\left\| \frac{1}{\delta} \int_0^\delta |\varphi_0 (t) - \varphi_0 (t \pm \gamma)|^p dt \right\|^{1/p} \leq 2 \omega_{L^p} f (\gamma),
\]

whence our first result follows.

In the next one we will change order of integration, whence

\[
\left\| \frac{1}{\delta} \int_0^\delta |\varphi_0 (t) - \varphi_0 (t \pm \gamma)|^p dt \right\|^{1/p} \leq 2 \omega_{L^p} f (\gamma).
\]
and thus our proof is complete. ■

In the sequel we will also need some another lemmas with the next notions.

Let
\[ \Psi_x f(\delta, \gamma)_p := \left\{ \frac{1}{\gamma} \int_{\gamma}^{\gamma+\delta} |\varphi_x(t)|^p \, dt \right\}^{1/p}, \]
then we have

**Lemma 6** If \( f \in L^p \) (\( p \geq 1 \)), then

\[ \Psi_x f(\delta, \gamma)_p \ll G_x f(\delta)_{p,s} \]
for any positive \( \delta \leq \gamma \) such that \( \gamma + \delta \leq \pi \) and \( 1 \leq p < s \).

**Proof.** There exists a natural \( k' \) such that \( (k' - 1) \delta \leq \gamma + \delta \leq k' \delta \). Then

\[
\begin{align*}
\Psi_x f(\delta, \gamma)_p & \ll \left( \frac{1}{k'\delta} \int_{(k'-2)\delta}^{k'\delta} |\varphi_x(t)|^p \, dt \right)^{1/p} \\
& \ll \left( \frac{1}{k'\delta} \int_{(k'-1)\delta}^{k'\delta} |\varphi_x(t)|^p \, dt \right)^{1/p} + \left( \frac{1}{(k'-1)\delta} \int_{(k'-2)\delta}^{(k'-1)\delta} |\varphi_x(t)|^p \, dt \right)^{1/p} \\
& \ll \left\{ \sum_{k=1}^{\lceil \pi/\delta \rceil} \left( \frac{1}{k\delta} \int_{(k-1)\delta}^{k\delta} |\varphi_x(t)|^p \, dt \right) \right\}^{s/p} = G_x f(\delta)_{p,s}
\end{align*}
\]
and our estimate is proved.

**Lemma 7** If \( f \in L^p \) (\( p \geq 1 \)), then

\[ \|\Psi . f(\delta, \gamma)_1\|_{L^p} \ll \omega L . f(\delta) \]
for any positive \( \delta \leq \gamma \) such that \( \gamma + \delta \leq \pi \).

**Proof.** Easy calculation gives

\[
\begin{align*}
\|\Psi . f(\delta, \gamma)_1\|_{L^p} & \ll \frac{1}{\gamma} \int_{\gamma}^{\gamma+\delta} \omega L . f(\gamma + \delta) \, dt \ll \frac{1}{\gamma} \int_{\gamma}^{\gamma+\delta} \omega L . f(\gamma) \, dt \\
& \ll \frac{\omega L . f(\gamma)}{\gamma} \int_{\gamma}^{\gamma+\delta} dt = \delta \frac{\omega L . f(\gamma)}{\gamma} \ll \delta \frac{\omega L . f(\delta)}{\delta}
\end{align*}
\]
and our Lemma is proved. ■ ■

### 4 Proofs of the results

We only prove Theorems 1, 3 and 5 because in the remain proofs we have to use Lemma 5 and Lemma 7 instead of Lemmas 3, 4 and 6.
4.1 Proof of Theorem 1

Let

\[
H^q_{k_0,k_\gamma}(x) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q}
\leq A_r + B_r + C_r,
\]

where

\[
A_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^{2\delta} \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q},
\]

\[
B_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_{2\delta}^{2\gamma} \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q},
\]

\[
C_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_{2\gamma}^{\pi} \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q},
\]

with \( D_{k_\nu}(t) = \frac{\sin((k_\nu+\frac{1}{2})t)}{2\sin \frac{\pi}{2}} \), \( \delta = \delta_\nu \) and \( \gamma = \gamma_\nu \), putting \( \delta_\nu = \frac{\pi}{k_\nu+1/2} \), \( \gamma_\nu = \frac{\pi}{k_\nu+1/2} \). In the case \( \gamma \geq \pi/2 \) we will have \( C_r \equiv 0 \). At the begin

\[
A_r \leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[ \frac{k_\nu + 1}{\pi} \int_0^{2\delta} |\varphi_x(t)| \, dt \right]^q \right\}^{1/q}
\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[ \frac{4k_\nu + 1/2}{2\pi} \int_0^{2\delta} |\varphi_x(t)| \, dt \right]^q \right\}^{1/q}
\leq \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[ 4w_x f(2\delta_\nu) \right]^q \right\}^{1/q}
\leq 4w_x(2\delta_0) \leq 8w_x(\delta_0).
\]

The terms \( B_r \) and \( C_r \), we estimate by the Totik method \([9]\) and its modification from \([6]\). We divide the term \( B_r \) into the two parts

\[
B_r = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^{2\gamma} \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q}
\leq \left\{ \frac{1}{r+1} \left( \sum_{\nu=0}^{r_0} + \sum_{\nu=r_0}^{r} \right) \left| \frac{1}{\pi} \int_{2\delta}^{2\gamma} \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q}
\]
Next the term $B_{r,v_0}$, we divide into the three parts.

$$B_{r,v_0} = \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^{r} \left[ \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} \varphi_x(t) D_{k_\nu}(t) \, dt \right]^q \right\}^{1/q}$$

$$= \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^{r} \frac{1}{\pi} \left( \int_{2\delta_\nu}^{2\gamma} + \int_{2\gamma-\delta_\nu}^{2\gamma} - \int_{2\gamma-\delta_\nu}^{2\delta_\nu} \right) \varphi_x(t) D_{k_\nu}(t) \, dt \right\}^q$$

where the first term

$$B_{r,v_0} = \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^{r} \frac{1}{\pi} \left( \int_{2\delta_\nu}^{2\gamma} + \int_{2\gamma-\delta_\nu}^{2\gamma} - \int_{2\gamma-\delta_\nu}^{2\delta_\nu} \right) \varphi_x(t) D_{k_\nu}(t) \, dt \right\}^q$$

$$= \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^{r} \frac{1}{\pi} \left( \int_{2\delta_\nu}^{2\gamma} + \int_{2\gamma-\delta_\nu}^{2\gamma} - \int_{2\gamma-\delta_\nu}^{2\delta_\nu} \right) \varphi_x(t) D_{k_\nu}(t) \, dt \right\}^q$$

$$\leq \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^{r} \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} (\varphi_x(t) - \varphi_x(t - \delta_\nu)) D_{k_\nu}(t) \, dt \right\}^q$$

$$+ \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^{r} \frac{1}{\pi} \int_{2\delta_\nu}^{2\gamma} \varphi_x(t - \delta_\nu) (D_{k_\nu}(t) + D_{k_\nu}(t - \delta_\nu)) \, dt \right\}^q$$

where the index $v_0$ is such that $k_{v_0-1} < r \leq k_{v_0}$ ($\delta_{v_0} \leq \gamma_r < \delta_{v_0-1}$ with $k_{-1} = 0$).
Using the partial integration we obtain

\[ B_{r,v_0}^1 \leq \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \| \frac{1}{\pi} \int_{2\delta_{v}}^{2\gamma} \frac{d}{dt} \left( \int_0^t (\varphi_x (u) - \varphi_x (u - \delta_{\nu})) \sin \left( \frac{2k_{\nu} + 1}{2} u \right) du \right) \frac{1}{2 \sin \frac{t}{2}} dt \right\}^{1/q} \]

\[ + \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \| \frac{1}{\pi} \int_{2\delta_{v}}^{2\gamma} \varphi_x (t - \delta_{\nu}) \left( \frac{1}{2 \sin \frac{t}{2}} - \frac{1}{2 \sin \frac{t-\delta_{\nu}}{2}} \right) \sin \left( \frac{2k_{\nu} + 1}{2} t \right) dt \right\}^{1/q} \]

\[ \ll \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \left\| \frac{1}{\pi} \int_0^t (\varphi_x (u) - \varphi_x (u - \delta_{\nu})) \sin \left( \frac{2k_{\nu} + 1}{2} u \right) du \frac{1}{2 \sin \frac{2\gamma}{2}} \right\|^{2\gamma} \right\}^{1/q} \]

\[ + \frac{1}{\pi} \int_{2\delta_{v}}^{2\gamma} \left( \int_0^t (\varphi_x (u) - \varphi_x (u - \delta_{\nu})) \sin \left( \frac{2k_{\nu} + 1}{2} u \right) du \frac{1}{2 \sin \frac{2\gamma}{2}} \right) \cos \left( \frac{\pi}{2 \sin \frac{t}{2}} \right) \|^{q} \right\}^{1/q} \]

\[ + \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \left\| \frac{1}{\pi} \int_{2\delta_{v}}^{2\gamma} \frac{\varphi_x (t - \delta_{\nu})}{t^2} dt \right\|^{q} \right\}^{1/q} \]

\[ \ll \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \left[ \frac{1}{8\gamma} \int_0^{2\gamma} |\varphi_x (u) - \varphi_x (u - \delta_{\nu})| du \right. \right. \]

\[ + \left. \frac{1}{4\delta_{\nu}} \int_0^{2\delta_{\nu}} |\varphi_x (u) - \varphi_x (u - \delta_{\nu})| du \right. \]

\[ + \left. \frac{\pi}{8} \int_{2\delta_{\nu}}^{2\gamma} \left( \frac{1}{t^2} \int_0^t |\varphi_x (u) - \varphi_x (u - \delta_{\nu})| du \right) dt \right\}^{q} \right\}^{1/q} \]

\[ + \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \left[ \delta_{\nu} \int_{\delta_{\nu}}^{\pi/t^2} |\varphi_x (t)| dt \right\}^{q} \right\}^{1/q} \]

and applying Lemmas 1, 2, 3 we have

\[ B_{r,v_0}^1 \ll \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \left[ \frac{1}{8\gamma} \int_0^{2\gamma} |\varphi_x (u) - \varphi_x (u - \delta_{\nu})| du \right. \right. \]

\[ + \left. \frac{1}{4\delta_{\nu}} \int_0^{2\delta_{\nu}} |\varphi_x (u) - \varphi_x (u - \delta_{\nu})| du \right. \]

\[ + \left. \frac{\pi}{8} \int_{2\delta_{\nu}}^{2\gamma} \left( \frac{1}{t^2} \int_0^t |\varphi_x (u) - \varphi_x (u - \delta_{\nu})| du \right) dt \right\}^{q} \right\}^{1/q} \]

\[ + \left\{ \frac{1}{r+1} \sum_{\nu=v_0}^r \left[ \delta_{\nu} \int_{\delta_{\nu}}^{\pi/t^2} |\varphi_x (t)| dt \right\}^{q} \right\}^{1/q} \]

\[ 11\]
\[ \ll w_x(\delta_0) \]
\[ + \frac{\pi}{8} \left\{ \frac{1}{r + 1} \sum_{\nu = \nu_0}^r \left[ \int_{2\delta \nu}^{2\gamma} \frac{1}{t} w_x(\delta \nu) \ dt \right]^q \right\}^{1/q} \]
\[ + \frac{1}{r + 1} \sum_{\nu = \nu_0}^r \left[ \frac{k_r}{\mu + 1} \sum_{\mu = 0}^r w_x \left( \frac{\pi}{\mu + 1} \right) \right]^q \]
\[ \ll w_x(\delta_0) + K w_x(\delta_0) \log \frac{\gamma}{\delta r} + K \delta_0 \sum_{\mu = 0}^{k_0} w_x \left( \frac{\pi}{\mu + 1} \right) \]
\[ \leq K w_x(\delta_0) \left( 1 + \log \frac{k_r + 1/2}{r + 1/2} \right). \]

Consequently, by Lemma 4,

\[ B_{r,\nu_0}^2 = \frac{1}{2} \left\{ \frac{1}{r + 1} \sum_{\nu = \nu_0}^r \left[ \frac{1}{\pi} \int_{2\gamma - \delta \nu}^{2\gamma} |\varphi_x(t)| D_{K_r}(t) \ dt \right]^q \right\}^{1/q} \]
\[ \leq \frac{1}{2} \left\{ \frac{1}{r + 1} \sum_{\nu = \nu_0}^r \left[ \frac{1}{\pi} \int_{2\gamma - \delta \nu}^{2\gamma} |\varphi_x(t)| \frac{\pi}{2t} \ dt \right]^q \right\}^{1/q} \]
\[ \leq \frac{1}{4} \left\{ \frac{1}{r + 1} \sum_{\nu = \nu_0}^r \left[ \int_{2\gamma - \delta \nu}^{2\gamma} \frac{d}{dt} \left( \int_0^t |\varphi_x(u)| \ du \right) \ dt \right]^q \right\}^{1/q} \]
\[ \ll \frac{1}{4} \left\{ \frac{1}{r + 1} \sum_{\nu = \nu_0}^r \left[ \int_0^t |\varphi_x(u)| \ du \right]^{t=2\gamma} + \int_{2\gamma - \delta \nu}^{2\gamma} w_x(t) \ t \ dt \right\}^q \]
\[ \ll \left\{ \frac{1}{2} \sum_{\nu = \nu_0}^r \left[ \frac{1}{2\gamma} \int_0^{2\gamma} |\varphi_x(u)| \ du - \frac{1}{2\gamma - \delta \nu} \int_{2\gamma - \delta \nu}^{2\gamma} |\varphi_x(u)| \ du \right. \]
\[ + \frac{w_x(2\gamma - \delta \nu)}{2\gamma - \delta \nu} \int_{2\gamma - \delta \nu}^{2\gamma} \ dt \right\}^{1/q} \]
\[ \ll \left\{ \frac{1}{2} \sum_{\nu = \nu_0}^r \left[ \frac{1}{2\gamma - \delta \nu} \int_0^{2\gamma} \left| |\varphi_x(u)| - |\varphi_x(u - \delta \nu)| \right| \ du \right. \]
\[ + \frac{1}{2\gamma - \delta \nu} \int_{2\gamma - \delta \nu}^{2\gamma} |\varphi_x(u - \delta \nu)| \ du + \frac{w_x(\delta \nu)}{\delta \nu} \right\}^{1/q} \]
\[ \ll \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^r \left[ \frac{1}{\gamma} \int_0^{2\gamma} \left| \varphi_x(u) - \varphi_x(u - \delta) \right| du \right. \right. \]
\[ \left. \left. + \frac{1}{\delta} \int_0^{\delta} \left| \varphi_x(u) \right| du + w_x(\delta) \right] \right\}^{1/q} \]
\[ \ll w_x(\delta_0) \]

and

\[ B_{r,\nu_0}^3 = \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^r \left[ \frac{1}{\pi} \int_{\delta}^{2\delta} \varphi_x(t) D_{\nu \nu} (t) \right. \right. \]
\[ \left. \left. \left| \frac{\varphi_x(t)}{2t} \right| \right]^{1/q} \right\} \]
\[ \leq \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^r \left[ \frac{1}{\pi} \int_{\delta}^{2\delta} \left| \varphi_x(t) \right| \right. \right. \]
\[ \left. \left. \left| \frac{\varphi_x(t)}{2t} \right| \right]^{1/q} \right\} \]
\[ \leq \frac{1}{2} \left\{ \frac{1}{r+1} \sum_{\nu=\nu_0}^r \left[ w_x(2\delta) \right]^{1/q} \right\} \ll w_x(\delta_0). \]

Thus

\[ B_r \ll w_x(\delta_0) \left( 1 + \log \frac{k_r + 1}{r+1/2} \right). \]

Finally we estimate the term \( C_r \) dividing it into the two parts.

\[ C_r^1 \leq \frac{1}{\delta_0} \int_0^{\delta_0} \left[ \int_{2\gamma}^{\gamma} \frac{1}{t} \left| \varphi_x(u + t) - \varphi_x(t) \right| dt \right] \]
\[ \leq \frac{1}{\delta_0} \int_0^{\delta_0} \left[ \int_{2\gamma}^{\gamma} \frac{1}{t} \left| \varphi_x(u + v) - \varphi_x(v) \right| dv \right] \]
\[ \leq \frac{1}{\delta_0} \int_0^{\delta_0} \left[ \int_{2\gamma}^{\gamma} \frac{1}{t} \left| \varphi_x(u + v) - \varphi_x(v) \right| dv \right] \]
\[ \leq \frac{1}{\delta_0} \int_0^{\delta_0} \left[ \int_{2\gamma}^{\gamma} \frac{1}{t} \left| \varphi_x(u + v) - \varphi_x(v) \right| dv \right] \]
\[ = C_r^1 + C_r^2. \]

Integrating by parts and applying Lemma 4 we obtain

\[ C_r^1 \leq \frac{1}{\delta_0} \int_0^{\delta_0} \left[ \int_{2\gamma}^{\gamma} \frac{1}{t} \left| \varphi_x(u + v) - \varphi_x(v) \right| dv \right] \]
\[ \leq \frac{1}{\delta_0} \int_0^{\delta_0} \left[ \int_{2\gamma}^{\gamma} \frac{1}{t} \left| \varphi_x(u + v) - \varphi_x(v) \right| dv \right] \]
\[ = C_r^1 + C_r^2. \]
\[
C_r^2
= \frac{1}{2 (r + 1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_{2\gamma}^{\pi} \frac{\Phi_x f (\delta_0, t) \ d (\cos ((k_{\nu} + \frac{1}{2}) t))}{2 \sin \frac{t}{2} \ k_{\nu} + \frac{1}{2}} \right|^{q/2} \right\}^{1/q}
\]

\[
= \frac{1}{2\pi (r + 1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{\Phi_x f (\delta_0, t) \cos ((k_{\nu} + \frac{1}{2}) t)}{2 \sin \frac{\gamma}{2} \ k_{\nu} + \frac{1}{2}} \right|^{q/2} \right\}^{1/q}
\]

\[
\leq \frac{1}{2\pi (r + 1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{\Phi_x f (\delta_0, 2\gamma) \cos ((k_{\nu} + \frac{1}{2}) 2\gamma)}{2 \sin \gamma \ k_{\nu} + \frac{1}{2}} \right|^{q/2} \right\}^{1/q}
\]

\[
\leq \frac{\left| \Phi_x f (\delta_0, 2\gamma) \right|}{\gamma (k_0 + 1)} + \frac{1}{k_0 + 1} \int_{2\gamma}^{\pi} \left| \frac{\Phi_x f (\delta_0, t)}{2 \sin \frac{t}{2}} \right| \ dt
\]

\[
\leq \frac{1}{\gamma (k_0 + 1) \delta_0} \int_0^{\delta_0} \left| \int_0^{\delta_0} \left| \frac{\Phi_x f (\delta_0, t + \varphi_x (u + 2\gamma) \right| }{\delta_0 t} \ dt \right| + \frac{1}{\delta_0} \int_0^{\delta_0} \left( \frac{1}{2\gamma} \left| \frac{\varphi_x (\delta_0 + t) - \varphi_x (t)}{t^2} \right| \ dt \right) \right.
\]

and additionally by Lemma 6

\[
\leq \frac{1}{\delta_0} \int_0^{\delta_0} \left( \frac{1}{2\gamma} \left| \frac{\varphi_x (\delta_0 + t) - \varphi_x (t)}{t^2} \right| \ dt \right) \ du
\]
\[ \leq |\Psi_x f(\delta_0, 2\gamma)| + \int_{2\gamma}^{\pi} \frac{1}{t^2} \left( \int_0^t |\varphi_x(\delta_0 + u) - \varphi_x(u)| \, du \right) \, dt \\
+ \frac{1}{\delta_0} \int_0^{\delta_0} \left( \delta_0 \int_{2\gamma}^{\pi} |\varphi_x(u + t)| \, dt \right) \, du \\
\leq G_x f(\delta_0)_{1,s} + \left[ \frac{1}{t} \int_0^t |\varphi_x(\delta_0 + u) - \varphi_x(u)| \, du \right]_{t=2\gamma}^{\pi} \\
+ \int_{2\gamma}^{\pi} \frac{w_x(\delta_0)}{t} \, dt + \frac{1}{\delta_0} \int_0^{\delta_0} \left( \delta_0 \int_{2\gamma}^{\pi} |\varphi_x(u + t)| \, dt \right) \, du \\
\leq w_x(\delta_0) \left( 1 + \int_{2\gamma}^{\pi} \frac{1}{t} \, dt \right) \leq w_x(\delta_0) \left( 1 + \log \frac{kr + 1}{r + 1} \right). \\
\]

Collecting our estimates we obtain desired estimate. ■

4.2 Proof of Theorem 3

If \( w_x(\delta) \equiv 0 \) then \( f \) is constant and our inequality is true. Thus we can suppose that \( w_x(\delta) > 0 \) for \( \delta > 0 \).

Let denote by

\[
\Delta_\mu = \{ \nu : |S_\nu f(x) - f(x)| \geq \mu w_x(u) \} \\
\Gamma_\mu = \{ \nu : (\mu - 1) G_x f(u)_{1,s} \leq |S_\nu f(x) - f(x)| \leq \mu w_x(u) \} \\
\Theta = \{ \mu : \Gamma_\mu \neq \emptyset \}
\]

the sets of integers \( \nu \in [N_{m-2} + 1, N_m] \) and \( \mu \), where \( u = \frac{\pi}{N_{m-2} + 1} \), then

\[
H_{m}^{\lambda, \varphi} f(x) \leq \frac{1}{N_m + 1} \sum_{\mu \in \Theta} \sum_{\nu \in \Gamma_\mu} \varphi(|S_\nu f(x) - f(x)|) \\
\leq \frac{1}{N_m + 1} \sum_{\mu \in \Theta} \sum_{\nu \in \Gamma_\mu} \varphi(\mu w_x(u)) \\
= \frac{1}{N_m + 1} \sum_{\mu \in \Theta} |\Gamma_\mu| \varphi(\mu w_x(u)) \\
\leq \frac{1}{N_m + 1} \sum_{\mu \in \Theta} |\Delta_{\mu-1}| \varphi(\mu w_x(u)).
\]

Using Theorem 1 we can compute that \( |\Delta_{\mu-1}| \leq N_m \exp \left( -\frac{\mu-1}{K} \right) \), whence

\[
H_{m}^{\lambda, \varphi} f(x) \leq \frac{1}{N_m + 1} \sum_{\mu \in \Theta} N_m \exp \left( -\frac{\mu-1}{K} \right) \varphi(\mu w_x(u)) \\
\ll \sum_{\mu \in \Theta} \exp \left( -\frac{\mu}{K} \right) \varphi(\mu w_x(u)).
\]
Since $\varphi \in \Phi$, we have

$$H^{\lambda \varphi} f(x) \ll \varphi(w_x(u))$$

$$+ \left( \sum_{n=0}^{n_0} + \sum_{n=n_0+1}^{\infty} \right) \sum_{\mu=2^n+1}^{2^{n+1}} \exp \left( -\frac{\mu}{K} \right) \varphi(\mu w_x(u))$$

$$\ll \varphi(w_x(u)) + \sum_{n=0}^{\infty} \sum_{\mu=2^n+1}^{2^{n+1}} \exp \left( -\frac{2^n}{K} \right) \varphi(2^{n+1} w_x(u))$$

$$\ll \varphi(w_x(u)) + \sum_{n=0}^{n_0} 2^n \exp \left( -\frac{2^n}{K} \right) \varphi(2^n w_x(u))$$

$$+ \sum_{n=n_0+1}^{\infty} 2^n \exp \left( -\frac{2^n}{K} \right) \varphi(2^n w_x(u))$$

$$\ll \varphi(w_x(u))$$

with some $n_0$, analogously as in [9] p.108, and therefore our proof is complete.

\[ \blacksquare \]

### 4.3 Proof of Theorem 5

We start with the obvious inequality

$$H^{\lambda \varphi} f(x) \ll \sum_{m=2}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_{\nu} \varphi \left( |S_\nu f(x) - f(x)| \right).$$

Using the Hölder inequality we obtain

$$H^{\lambda \varphi} f(x) \ll \sum_{m=1}^{\infty} \left\{ \sum_{\nu=N_{m-2}+1}^{N_m} (\lambda_{\nu})^s \right\}^{1/s} \left\{ \sum_{\nu=N_{m-2}+1}^{N_m} \varphi^q \left( |S_\nu f(x) - f(x)| \right) \right\}^{1/q}$$

with $\frac{1}{s} + \frac{1}{q} = 1$ ($s > 1$), and by the assumption $(\lambda_{\nu}) \in \Lambda_s(N_m)$, we have

$$H^{\lambda \varphi} f(x) \ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda_{\nu} \left\{ \frac{1}{N_m} \sum_{\nu=N_{m-2}+1}^{N_m} \varphi^q \left( |S_\nu f(x) - f(x)| \right) \right\}^{1/q}.$$
The second assumption $\varphi \in \Phi$ also implies that $\varphi^q \in \Phi$, and therefore, by the Theorem 3,

$$H_{\lambda \varphi} f(x) \ll \sum_{m=1}^{\infty} \sum_{\nu=N_{m-2}+1}^{N_m} \lambda \nu \left\{ \varphi^q \left( w_x \left( \frac{\pi}{N_{m-2} + 2} \right) \right) \right\}^{1/q}$$

Thus our result is proved. ■

References

[1] O. D. Gabisonia, On the point of strong summability of Fourier series, Mat. Zam. 14, No 5 (1973), 615-626 (in Russian).

[2] O. D. Gabisonia, Points of strong summability of Fourier series, Ukrainskij Matematicheskij Zhurnal, Vol. 44, No. 8, 1020-1031 (in Russian).

[3] G. H. Hardy, J. E. Littlewood, Sur la série de Fourier d’une fonction a carré sommable, Comptes Rendus, Vol. 28, (1913), 1307-1309.

[4] L. Leindler, Strong approximation by Fourier series, Akadémiai Kiadó, Budapest, 1985.

[5] W. Leinski, On the strong approximation by $(C, \alpha)$-means of Fourier series, Math Nachrichten 146 (1990), 207-220.

[6] W. Leinski, Pointwise strong and very strong approximation of Fourier series, Acta Math. Hungar. 115 (2007), no.3, 215-233.

[7] J. Marcinkiewicz, Sur la sommabilité forte de series de Fourier, J. London Math. Soc. 14 (1939), pp.162-168.

[8] V. Totik, On Generalization of Fejér summation theorem, Coll. Math. Soc. J. Bolyai 35 Functions series, operators, Budapest (Hungary) (1980), 1185-1199.

[9] V. Totik, Notes on Fourier series: Strong approximation, J. Approx. Th. 43 (1985), 105-111.

[10] A. Zygmund, On the convergence and summability of power series on the circle of convergence, P.L.M.S. 47 (1941), 326-50.