Device-Independent Certification of High-Dimensional Quantum Systems

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An important problem in quantum information processing is the certification of the dimension of quantum systems without making assumptions about the devices used to prepare and measure them, that is, in a device-independent manner. A crucial question is whether such certification is experimentally feasible for high-dimensional quantum systems. Here we experimentally witness in a device-independent manner the generation of six-dimensional quantum systems encoded in the orbital angular momentum of single photons and show that the same method can be scaled, at least, up to dimension 13.

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Introduction.—Dimensionality is a fundamental property of physical systems and a key resource in quantum information processing. Phenomena such as contextuality require systems of a certain minimum dimension to occur [1, 2]; applications such as quantum secure communication have different levels of security depending on the dimension of the systems [3, 4], and methods to characterize quantum states strongly depend on the assumed dimension of the systems [5]. It is therefore of crucial importance to develop methods to certify whether a source produces systems that have at least a certain dimension and to distinguish quantum systems from classical systems of the same dimension. The first theoretical tools for providing lower bounds on the dimension of quantum systems were based on Bell inequalities [6, 7] and random access codes [8].

Nevertheless, in cryptographic scenarios in which the preparation and measurements devices are not trustable, and also in scenarios in which the devices are imperfect or are not well characterized, it is desirable to assess the dimension of the physical systems in a “device-independent” (DI) manner [9–16]; that is, using only the correlations between preparations and the outcomes of different measurements and without making assumptions about the nature of the systems under observation or about the devices used to prepare and measure them. Theoretical tools achieving this goal where introduced by Gallego et al. [10] under the name of DI dimension witnesses (DWs). DI DWs for systems of arbitrary dimension have been recently proposed [14].

So far, DI DWs have only been used to experimentally certify the generation of classical and quantum systems of dimension 2 and 3 [17, 18]. However, realistic quantum information processing applications demand quantum systems of much higher dimensions. Moreover, the increasing technical complexity required for the generation of high-dimensional quantum systems turns DI DWs into fundamental testing tools.

The question is whether DI DWs are experimentally feasible to witness higher quantum dimensions or, on the contrary, the complexity of DI DWs may prevent their actual realization. For example, it can be shown (see below) that the number of parameters that have to be experimentally controlled to assess dimension $d$ in a DI manner increases as $2d^2$. This reveals that assessing high dimensions in a DI manner constitutes an experimental challenge, since it requires much higher control and accuracy than previous experiments assessing dimensions in a DI manner [17, 18]. This is the challenge that we address here.

The aim of this Letter is to experimentally witness the generation of six-dimensional quantum systems and show that the observed results cannot be simulated with classical systems of dimension 6, using for that theoretical and experimental techniques which, as we show, can be scaled up to higher dimensions.

Device-independent dimension witness for dimension 6.—To witness the generation of six-dimensional quantum systems and discriminate between classical and quantum systems of dimension 6, we implement the DI DW of the family $I_N$ [10] that requires the smallest number of preparations and measurements and is capable to achieve this goal. This DI DW is $I_7$ and is defined within the following scenario: physical systems prepared in states $\rho_x$ with $x \in \{1, \ldots, N\}$ are submitted to a measurement $y \in \{1, \ldots, N-1\}$ with possible outcomes $b \in \{-1,1\}$; see Fig. 1 (a). The point is to chose these preparations and measurements in such a way that an external observer does not need to know which are these states and measurements to reach conclusions about the dimension of the prepared systems. For that, $I_N$, with $N = 2, 3, \ldots$, are specific linear combinations of conditional probabilities $P(b|x,y)$ of obtaining the outcome $b$ when measure $y$ is performed upon the state $\rho_x$, and have different upper bounds depending on the dimension of the prepared systems and on whether the systems are
FIG. 1: (a) Schematic scenario for DI dimension witnessing: A physical system is prepared in a state \( \rho_x \) chosen among a set of states \( x \), and is sent to a measurement device. There, a dichotomic measurement, chosen among a set of observables \( y \), is performed and the outcome is recorded. Specifically, the DI DW tested in our experiment requires 7 states and 6 observables. (b) Experimental implementation of the previous scheme: The setup consists of two stages, labeled as “generation” and “analysis.” In the generation stage, heralded single photons are produced by spontaneous parametric down-conversion in beta barium borate nonlinear crystal. The heralding photon is directly sent to a detector which acts as a trigger (not shown in figure), the signal photon is projected on the fundamental TEM00 Gaussian state \( (l = 0) \) by means of a single mode fiber (Photon Source). The orbital angular momentum (OAM) state of signal photons is then manipulated with the spatial light modulator SLM1 in combination with a single mode fiber and a single photon detector. To avoid the Gouy phase shift effect, an imaging system (not shown in the figure) is implemented between the screens of the two spatial light modulators.

Classical or quantum. Specifically, \( I_N \) are of the form

\[
I_N = \sum_{j=1}^{N-1} E_{ij} + \sum_{j=2}^{N} \sum_{i=1}^{N+1-i} \alpha_{ij} E_{ij},
\]

where \( E_{xy} \equiv P(+1|x, y) - P(-1|x, y) \) and \( \alpha_{ij} = 1 \) if \( i + j \leq N \) and \( \alpha_{ij} = -1 \), otherwise. For classical systems of dimension \( d \leq N - 1 \), the maximum value of \( I_N \) is \( L_d = \frac{N(N-3)}{2} + 2d - 1 \). \( I_N \) is specially designed to certify systems of dimension \( d = N - 1 \) and discriminate between classical and quantum systems of dimension \( d = N - 1 \), since a quantum system of dimension \( d = N - 1 \) can give a value of \( I_N \) higher than \( L_d = N - 1 \).

Here, we use \( I_7 \) to certify the generation of six-dimensional qudit systems. \( I_7 \) is given by

\[
I_7 = |E_{11} + E_{12} + E_{13} + E_{14} + E_{15} + E_{16} + E_{21} + E_{22} + E_{23} + E_{24} + E_{25} + E_{26} + E_{31} + E_{32} + E_{33} + E_{34} - E_{35} + E_{41} + E_{42} + E_{43} - E_{44} + E_{51} + E_{52} - E_{53} + E_{61} - E_{62} - E_{71}|.
\]

Unlike for the classical limits, for the quantum limits of \( I_7 \) there are not known analytical expressions. Here we obtain these limits using the numerical technique of the conjugated gradient \[19\]. We focus on real \( d \)-dimensional states, hence, characterized by \( d - 1 \) parameters, and use the following parametrization based on \( d \)-dimensional spherical coordinates:

\[
\lambda_1 = \cos \phi_1, \\
\lambda_j = \cos \phi_{j-1} \prod_{k=1}^{j-1} \sin \phi_k, \\
\lambda_{d-1} = \cos \phi_{d-2} \prod_{k=1}^{d-2} \sin \phi_k, \\
\lambda_d = \sin \phi_{d-1} \prod_{k=1}^{d-2} \sin \phi_k,
\]

with \( j = 2, \ldots, d - 2 \), and where the quantum states to be prepared are \( \rho_x = |\Psi_x\rangle \langle \Psi_x| \), with

\[
|\Psi_x\rangle \equiv \sum_{i=0}^{d-1} \lambda_i^{(x)} |i\rangle.
\]

The measurements to be made are given by \( M_y^{+} = |\Psi_y\rangle \langle \Psi_y| \), where \( |\Psi_y\rangle = \sum_{i=0}^{d-1} \lambda_i^{(y)} |i\rangle \).

Applied to estimating bounds for \( I_7 \), the conjugated gradient method consists of calculating the local gradi-
TABLE I: Limits of $I_T$ for classical ($I_{Tc}$) and quantum ($I_{Tq}$) systems of dimension $d$.

| $d$ | $I_{Tc}$ (rad) | $I_{Tq}$ (rad) |
|-----|---------------|---------------|
| 2   | 4.5801        | 4.5888        |
| 3   | 4.5814        | 4.8358        |
| 4   | 4.8343        | 4.9946        |
| 5   | 4.6016        | 4.2510        |
| 6   | $\pi$         | 2.3258        |

TABLE II: Orientations of the states that maximize $I_T$ while considering six-dimensional qudit states.

| $x$ | $\phi_1^{(x)}$ (rad) | $\phi_2^{(x)}$ (rad) | $\phi_3^{(x)}$ (rad) | $\phi_4^{(x)}$ (rad) | $\phi_5^{(x)}$ (rad) |
|-----|----------------------|----------------------|----------------------|----------------------|----------------------|
| 1   | 4.5801               | 1.8679               | 5.1341               | 1.5056               | 4.493                |
| 2   | 1.0858               | 4.5814               | 4.8358               | 6.2086               |                      |
| 3   | 1.4347               | 1.3779               | 5.0763               | 4.5888               | -1.338               |
| 4   | 4.5814               | 1.4557               | 0.6297               | 4.5888               |                      |
| 5   | 4.8343               | 1.8268               | 4.9946               | 0.5796               | 1.6284               |
| 6   | 4.6016               | 1.3527               | 4.2510               | 0.5860               | 4.9691               |
| 7   | $\pi$                | 2.2358               | 5.2280               | 2.8465               | 0.5110               |

TABLE III: Orientations of the measurements that maximize $I_T$ while considering six-dimensional qudit states.

| $y$ | $\phi_1^{(y)}$ (rad) | $\phi_2^{(y)}$ (rad) | $\phi_3^{(y)}$ (rad) | $\phi_4^{(y)}$ (rad) | $\phi_5^{(y)}$ (rad) |
|-----|----------------------|----------------------|----------------------|----------------------|----------------------|
| 1   | 0                    | 6.0542               | 3.8912               | 1.8371               | 0.378                |
| 2   | 1.3491               | 1.3527               | 2.0322               | 2.5556               | 1.8275               |
| 3   | 1.7849               | 1.7926               | 1.3549               | 3.6187               | 1.6015               |
| 4   | 1.3718               | 1.5194               | -0.5647              | 4.8676               | 3.2279               |
| 5   | 1.3932               | 1.4595               | 5.0187               | 4.8270               | -0.1914              |
| 6   | 1.7268               | 6.4857               | 5.7731               | 1.2669               | 4.4619               |

The states and measurements that lead to the maximum value of $I_T$ when the prepared systems are quantum systems of dimension 6 are presented in Tables II and III, respectively. Notice that, with the parametrization (3), the number of parameters that have to be experimentally controlled in order to reach the closest maximum point of $I_T$ is obtained removing the contribution of the dark counts (DC).
investigate how far this technique can be scaled up. For dimensional systems with high fidelity. It is then worth to this work allowed us to generate and measure six-classical states of the same dimension.

Classical and quantum systems of dimension 6 were generated and measured experimentally. The experimental results obtained for the DW test with and without removing dark counts, respectively. Erasing dark counts is important because it improves the accuracy of the certification process. The accuracy required for dimension 6 is higher than 95%.

The experimental technique adopted in this work allowed us to generate and measure six-dimensional quantum states; see Table I. The experimental results are compared with the theoretical bounds in Fig. 2. Specifically, Fig. 2 shows the experimental results obtained for the DW test with and without removing dark counts, respectively. Erasing dark counts is important because it improves the accuracy of the certification process. The accuracy required for dimension 6 is higher than 95%.

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we have shown that, encoding high-dimensional quantum information in the transverse spatial mode of light, the accuracy needed to assess dimensions up to 13 (10 in a more adversarial scenario) is still within the experimental error of our experiment. These results demonstrate the feasibility of the device-independent approach for realistic high-dimensional quantum information processing.

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