Yet another criterion for the total positivity of Riordan arrays

Jianxi Mao\textsuperscript{a}, Lili Mu\textsuperscript{b,*}, Yi Wang\textsuperscript{a},

\textsuperscript{a}School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P.R. China

\textsuperscript{b}School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, P.R. China

Abstract

Let $R = R(d(t), h(t))$ be a Riordan array, where $d(t) = \sum_{n \geq 0} d_n t^n$ and $h(t) = \sum_{n \geq 0} h_n t^n$. We show that if the matrix

\[
\begin{bmatrix}
    d_0 & h_0 & 0 & 0 & \cdots \\
    d_1 & h_1 & h_0 & 0 & \cdots \\
    d_2 & h_2 & h_1 & h_0 & \ddots \\
    \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

is totally positive, then so is the Riordan array $R$.

Keywords: Riordan array, Totally positive matrix, Pólya frequency sequence

2010 MSC: 15B48, 15B36, 15B05

Following Karlin \cite{karlin}, an infinite matrix is called \textit{totally positive} (or shortly, TP), if its minors of all orders are nonnegative. An infinite nonnegative sequence $(a_n)_{n \geq 0}$ is called a \textit{Pólya frequency sequence} (or shortly, PF), if its Toeplitz matrix

\[
[a_{i-j}]_{i,j \geq 0} = \begin{bmatrix}
    a_0 & a_0 & a_0 & \cdots \\
    a_1 & a_1 & a_0 & \cdots \\
    a_2 & a_2 & a_1 & \cdots \\
    a_3 & a_2 & a_1 & a_0 & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

is TP. We say that a finite sequence $a_0, a_1, \ldots, a_n$ is PF if the corresponding infinite sequence $a_0, a_1, \ldots, a_n, 0, \ldots$ is PF. A fundamental characterization for PF sequences is due to Schoenberg and Edrei, which states that a sequence $(a_n)_{n \geq 0}$ is PF if and only if its generating function

\[
\sum_{n \geq 0} a_n t^n = Ct^k e^{\gamma t} \prod_{j \geq 0} (1 + \alpha_j t) / \prod_{j \geq 0} (1 - \beta_j t),
\]

*Corresponding author.

Email addresses: maojianxi@hotmail.com (Jianxi Mao), lilimu@jsnu.edu.cn (Lili Mu), wangyi@dlut.edu.cn (Yi Wang)
where $C > 0$, $k \in \mathbb{N}$, $\alpha_j, \beta_j, \gamma \geq 0$, and $\sum_{j \geq 0} (\alpha_j + \beta_j) < +\infty$ (see [8, p. 412] for instance). In this case, the generating function is called a Pólya frequency formal power series. We refer the reader to [1, 3, 6, 11, 13] for the total positivity of matrices. Our concern in this note is the total positivity of Riordan arrays.

Riordan arrays play an important unifying role in enumerative combinatorics [10, 12].

Let $d(t) = \sum_{n \geq 0} d_n t^n$ and $h(t) = \sum_{n \geq 0} h_n t^n$ be two formal power series. A Riordan array, denoted by $\mathcal{R}(d(t), h(t))$, is an infinite matrix whose generating function of the $k$th column is $d(t)h^k(t)$ for $k \geq 0$. Chen and Wang [4, Theorem 2.1] gave the following criterion for the total positivity of Riordan arrays.

**Theorem 1** ([4, Theorem 2.1]). Let $R = (d(t), h(t))$ be a Riordan array. If both $d(t)$ and $h(t)$ are Pólya frequency formal power series, then $R$ is totally positive.

We say that $\mathcal{R}(d(t), h(t))$ is proper if $d_0 \neq 0, h_0 = 0$ and $h_1 \neq 0$. In this case, $\mathcal{R}(d(t), h(t))$ is an infinite lower triangular matrix. It is well known that a proper Riordan array $R = [r_{n,k}]_{n,k \geq 0}$ can be characterized by two sequences $(a_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ such that

$$r_{0,0} = 1, \quad r_{n+1,0} = \sum_{j \geq 0} z_j r_{n,j}, \quad r_{n+1,k+1} = \sum_{j \geq 0} a_j r_{n,k+j}$$

for $n, k \geq 0$ (see [6, 7] for instance). Call $(a_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ the $A$- and $Z$-sequences of $R$ respectively. Chen et al. [2, Theorem 2.1 (i)] gave the following criterion for the total positivity of Riordan arrays.

**Theorem 2** ([2, Theorem 2.1 (i)]). Let $R$ be the proper Riordan array with the $A$- and $Z$-sequences $(a_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$. If the product matrix

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 \\ z_2 & a_2 & a_1 & a_0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is totally positive, then so is $R$.

In this note we establish a new criterion for the total positivity of Riordan arrays, which can be viewed as a dual version of Theorem 2 in a certain sense.

**Theorem 3.** Let $R = (d(t), h(t))$ be a Riordan array, where $d(t) = \sum_{n \geq 0} d_n t^n$ and $h(t) = \sum_{n \geq 0} h_n t^n$. If the Hessenberg matrix

$$H = \begin{bmatrix} d_0 & h_0 & 0 & 0 & \cdots \\ d_1 & h_1 & h_0 & 0 \\ d_2 & h_2 & h_1 & h_0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is totally positive, then so is $R$. 

2
Proof. Let \( R[n] \) be the submatrix consisting of the first \( n + 1 \) columns of \( R \). Clearly, \( R \) is TP if and only if all submatrices \( R[n] \) are TP for \( n \geq 0 \). So it suffices to show that \( R[n] \) is TP for all \( n \geq 0 \). We proceed by induction on \( n \).

Let \( R = [r_{n,k}]_{n,k \geq 0} \). Since the generating function of the \( k \)th column of \( R \) is \( d(x)h^k(x) \), we have

\[
\begin{bmatrix}
  r_{0,0} \\
  r_{1,0} \\
  r_{2,0} \\
  \vdots
\end{bmatrix}
= \begin{bmatrix}
  d_0 \\
  d_1 \\
  d_2 \\
  \vdots
\end{bmatrix}
\quad \text{for } k \geq 1.
\]

It follows that

\[
\begin{bmatrix}
  r_{0,0} & r_{0,1} & \cdots & r_{0,n+1} \\
  r_{1,0} & r_{1,1} & \cdots & r_{1,n+1} \\
  r_{2,0} & r_{2,1} & \cdots & r_{2,n+1} \\
  \vdots & \vdots & \ddots & \vdots
\end{bmatrix}
= \begin{bmatrix}
  d_0 & h_0 & 0 & 0 & \cdots \\
  d_1 & h_1 & h_0 & 0 & \cdots \\
  d_2 & h_2 & h_1 & h_0 & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & \cdots & 0 \\
  0 & r_{0,0} & r_{0,1} & \cdots & r_{0,n} \\
  0 & r_{1,0} & r_{1,1} & \cdots & r_{1,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix},
\]

or equivalently,

\[
R[n + 1] = H \begin{bmatrix}
  1 & 0 \\
  0 & R[n]
\end{bmatrix}.
\tag{2}
\]

The first matrix \( H \) on the right-hand side of (2) is TP by assumption, which implies that all \( d_n \) are nonnegative, and the matrix \( R[0] \) is therefore TP. Assume now that the matrix \( R[n] \) is TP for \( n \geq 0 \). Then the second matrix \( \begin{bmatrix}
  1 & 0 \\
  0 & R[n]
\end{bmatrix} \) on the right-hand side of (2) is also TP. It is well known that the product of TP matrices is still TP by the classic Cauchy-Binet formula. Thus the matrix \( R[n + 1] \) on the left-hand side of (2) is TP. The matrix \( R \) is therefore TP by induction, and the proof is complete.

\[ \Box \]

Example 4. Consider Lucas polynomials \( L_n(x) = \sum_k L_{n,k}x^k \) defined by

\[
L_{n+1}(x) = L_n(x) + xL_{n-1}(x)
\tag{3}
\]

with \( L_0(x) = 2 \) and \( L_1(x) = 1 \). Lucas matrix is the lower triangular infinite matrix

\[
L = [L_{n,k}]_{n,k \geq 0} =
\begin{bmatrix}
  2 \\
  1 \\
  1 & 2 \\
  1 & 3 \\
  1 & 4 & 2 \\
  1 & 5 & 5 \\
  1 & 6 & 9 & 2 \\
  1 & 7 & 14 & 7 \\
  \vdots & \ddots
\end{bmatrix}.
\]
Let \( \mathcal{L}_k(t) = \sum_{n \geq 0} L_{n,k} t^n \) denote the generating function of the \( k \)th column of \( L \) for \( k \geq 0 \). Clearly, \( \mathcal{L}_0(t) = (2 - t)/(1 - t) \). On the other hand, we have \( L_{n,k} = L_{n-1,k} + L_{n-2,k-1} \) for \( n > k > 0 \) by \( \[3\] \). It follows that \( \mathcal{L}_n(t) = \frac{t^2}{1-t} \mathcal{L}_{n-1}(t) \) for \( n \geq 1 \). Thus \( L \) is a Riordan array:

\[
L = \mathcal{R}\left( \frac{2-t}{1-t}, \frac{t^2}{1-t} \right).
\]

The corresponding Hessenberg matrix is

\[
H = \begin{bmatrix}
2 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \\
1 & 1 & 0 & 0 & \\
1 & 1 & 1 & 0 & \\
1 & 1 & 1 & 1 & \\
\vdots & \vdots & \vdots & \ddots & \\
\end{bmatrix},
\]

which is clearly TP, and so is \( L \) by Theorem \( \[3\] \).

However, the total positivity of \( L \) can be followed neither from Theorem \( \[1\] \) since \( d(t) = (2 - t)/(1 - t) \) is not PF, nor from Theorem \( \[2\] \) since \( L \) is improper.

**Remark 5.** We can show that Theorem \( \[3\] \) implies Theorem \( \[1\] \). Consider first several important classes of proper Riordan arrays \( R = \mathcal{R}(d(t), h(t)) \).

(i) Let \( h(t) = t \). Then \( R \) is a Toeplitz-type Riordan array, which is precisely the Toeplitz matrix of the sequence \( (d_n)_{n \geq 0} \). If \( d(t) \) is PF, then \( \mathcal{R}(d(t),t) \) is TP.

(ii) Let \( h(t) = td(t) \). Then \( R \) is a Bell-type Riordan array. In this case, the corresponding Hessenberg matrix \( \[1\] \) is the Toeplitz matrix of \( (d_n)_{n \geq 0} \). If \( d(t) \) is PF, i.e., \( h(t) \) is PF, then \( \mathcal{R}(h(t)/t, h(t)) \) is TP by Theorem \( \[3\] \).

(iii) Let \( d(t) = 1 \). Then \( R \) is a Lagrange-type Riordan array. Note that

\[
\mathcal{R}(1,h(t)) = \begin{bmatrix} \ 1 & \ 0 \\ 0 & \mathcal{R}(h(t)/t, h(t)) \end{bmatrix}. 
\]

If \( h(t) \) is PF, then \( \mathcal{R}(h(t)/t, h(t)) \) is TP, and so is \( \mathcal{R}(1,h(t)) \).

It is well known \( \[10\] \) that every proper Riordan array can be decomposed into the product of a Toeplitz-type Riordan array and a Lagrange-type Riordan array:

\[
\mathcal{R}(d(t),h(t)) = \mathcal{R}(d(t),t) \cdot \mathcal{R}(1,h(t)).
\]

We conclude that if both \( d(t) \) and \( h(t) \) are PF, then \( R \) is TP. In other words, Theorem \( \[1\] \) follows from Theorem \( \[3\] \).
Acknowledgement

This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11771065, 11701249).

References

[1] T. Ando, Totally positive matrices, Linear Algebra Appl. 90 (1987) 165–219.

[2] X. Chen, H. Liang and Y. Wang, Total positivity of Riordan arrays, European J. Combin. 46 (2015) 68–74.

[3] X. Chen, H. Liang and Y. Wang, Total positivity of recursive matrices, Linear Algebra Appl. 471 (2015) 383–393.

[4] X. Chen and Y. Wang, Notes on the total positivity of Riordan arrays, Linear Algebra Appl. 569 (2019) 156–161.

[5] S.M. Fallat and C.R. Johnson, Totally Nonnegative Matrices, Princeton University Press, Princeton, NJ, 2011.

[6] T.-X. He, Matrix characterizations of Riordan arrays, Linear Algebra Appl. 465 (2015) 15–42.

[7] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, Discrete Math. 309 (2009) 3962–3974.

[8] S. Karlin, Total Positivity, Volume 1, Stanford University Press, 1968.

[9] A. Pinkus, Totally Positive Matrices, Cambridge University Press, Cambridge, 2010.

[10] L.W. Shapiro, S. Getu, W.-J. Woan and L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.

[11] R. Slowik, Some (counter) examples on totally positive Riordan arrays, Linear Algebra Appl. 594 (2020) 117–123.

[12] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1994) 267–290.

[13] Y. Wang and A.L.B. Yang, Total positivity of Narayana matrices, Discrete Math. 341 (2018) 1264–1269.