Abstract. We rebut Kowalenko’s claims in 2010 that he proved the irrationality of Euler’s constant \( \gamma \), and that his rational series for \( \gamma \) is new.

1. Introduction

The irrationality of Euler’s constant

\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215664901532860606512090082402431042 \ldots
\]

has long been conjectured. However, it remains an open problem.

In 2010 Kowalenko claimed that simple arguments suffice to settle this matter [4]. As he offered no general framework or new mathematical principle, we believe that the following illustrations are sufficient to describe the flaws in his very limited approach.

2. A faulty irrationality argument

Kowalenko derives the following formula for Euler’s constant in equation (65) of [4, p. 428]:

\[
\gamma = \sum_{k=1}^{\infty} \frac{A_0}{k(k+1)} - \sum_{k=2}^{\infty} \frac{A_1}{k^2} + \sum_{k=3}^{\infty} \frac{A_2}{k(k-1)} - \sum_{k=4}^{\infty} \frac{A_3}{k(k-2)} + \cdots.
\]

Here \( A_0, A_1, A_2, A_3, \ldots \) are certain rational numbers. He writes:

With the exception of the second series on the rhs, all the series in (65) can be easily evaluated by decomposing them into partial fractions. On the other hand, the second series on the rhs is virtually equal to \( \zeta(2) \).

He then transforms formula (65) into the following series in equation (69):

\[
\gamma = \frac{3}{2} - \frac{1}{2} \left( \frac{\pi^2}{6} + \frac{1}{12} + \frac{5}{144} + \frac{247}{12960} + \frac{77}{6400} + \frac{25027}{3024000} + \cdots \right).
\]

Kowalenko states:

For \( \gamma \) to be rational the term involving \( \pi^2/6 \), which arises solely from the summation over \( 1/k^2 \) or \( \zeta(2) \) in (65), has to be cancelled by the remaining sum. This means that we need to examine the methods for converting an irrational number into a rational number by the process of addition. There are only two possible methods for achieving this, which are best understood if we regard an irrational number as an infinite random distribution of decimal
digits. . . Therefore, for \( \gamma \) to be rational, we need to convert a random distribution into a non-random one.

But, for example, the distribution of digits in Liouville’s irrational number

\[
\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.11000100000000000000010\ldots
\]

is not random, as the sum formula shows. Thus Kowalenko’s understanding of irrationality is lacking.

He continues:

The first method by which an irrational number can be converted to a rational number is to add another number, which at some stage possesses the opposite random distribution to the original irrational number. This represents the situation whereby the second number can be expressed as \( C - \pi^2/6 \), with \( C \) a rational number. Such a situation, however, cannot occur with \( (69) \). First, we note that if we are to obtain \( C - \pi^2/6 \) from the remaining terms after the \( \pi^2/6 \) term in the parenthesis of \( (69) \), then these terms would have to yield a summation involving \( 1/k^2 \). This is simply not possible as all the \( 1/k^2 \) terms have already been removed as mentioned above.

Here Kowalenko apparently assumes that Euler’s series \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6 \) is the only way to represent \( \pi^2/6 \) as the sum of a series of rational numbers. Of course, that is not true; there are infinitely many such representations of \( \pi^2/6 \), and hence of \( C - \pi^2/6 \), for any rational number \( C > \pi^2/6 \). For instance, taking \( C = 2 \), an alternate series for \( 2 - \pi^2/6 \) is

\[
2 - \frac{\pi^2}{6} = \sum_{n=3}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k^n} = \frac{1}{8} + \frac{1}{16} + \frac{1}{27} + \frac{1}{32} + \frac{1}{64} + \frac{1}{81} + \frac{1}{125} + \cdots,
\]

from Goldbach’s theorem \( \sum_{n=2}^{\infty} (\zeta(n) - 1) = 1 \) (see, e.g., [5, p. 142, equation (13)]).

Kowalenko concludes the paragraph:

Furthermore, the summation would have to be negative. Yet all the terms in the parenthesis in \( (69) \) are positive definite. Therefore, it is simply impossible for all the terms in \( (69) \) to yield \( C - \pi^2/6 \).

Here he claims that the sum of a series of positive rational numbers cannot be equal to \( C - \pi^2/6 \). But, for example, decimal expansion does give such a series:

\[
C - \frac{\pi^2}{6} = n + 0.d_1d_2d_3\ldots = n + \sum_{k=1}^{\infty} \frac{d_k}{10^k}.
\]

In view of his misconceptions, Kowalenko has not proven that the “first method by which an irrational number can be converted to a rational number” does not lead to the rationality of Euler’s constant. Its irrationality therefore remains an open problem.

3. A KNOWN REPRESENTATION

Finally, we point out that Kowalenko’s claim to have found “a new representation for Euler’s constant” [4, p. 143] is also incorrect. Namely, what he calls “Hurst’s formula,” which is \( \gamma = \sum_{k=1}^{\infty} \frac{|A_k|}{k} \) in equation (60) of [4], is known. According to Gourdon and Sebah
The formula was discovered in 1924 by Kluyver \[2\] (see the translation \[3\]), who wrote it as

$$\gamma = \sum_{k=1}^{\infty} \frac{a_k}{k} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{868}{362880} + \frac{275}{169344} + \cdots.$$  

By comparing the values of Kluyver’s numbers \(a_k\) (see the end of Section 2.3 in \[1\]) with those of Kowalenko’s numbers \(A_k\) (see Table 1 in \[4\]) for \(1 \leq k \leq 7\), one readily observes that \(a_k = |A_k|\).

Indeed, letting

\[(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1) \cdots (z+n-1)\]

denote the Pochhammer symbol, with \(\Gamma\) the Gamma function, we have the following relations. From (18) of \[4\],

\[A_k = \frac{(-1)^k}{k!} \int_0^1 (-x) k dx,\]

while from \[2\ p. 150\] and \[3\ p. 143\],

\[a_k = \frac{1}{k!} \int_0^1 x(1-x)^{k-1} dx = -\frac{1}{k!} \int_0^1 (-x) k dx,\]

and we therefore conclude that \(a_k = (-1)^{k+1} A_k\).

4. Appendix: An anonymous referee’s report

The authors submitted a version of the first three sections of this paper to *Acta Applicandae Mathematicae*, along with a list of six experts in irrationality theory as potential referees. The editors accepted the paper, writing in part, “We have received the following clear and concise confirmation by one of the experts”:

I completely agree with arguments of J. Sondow and M. Coffey. The article of V. Kowalenko is baseless.

References

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