Shifting and splitting of resonance lines due to dynamical friction in plasmas

V. N. Duarte,1, * J. B. Lestz,2, 3, * N. N. Gorelenkov,1 and R. B. White1

1Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ, 08543, USA
2General Atomics, San Diego, CA, 92121, USA
3Department of Physics and Astronomy, University of California, Irvine, CA, 92697, USA

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A quasilinear plasma transport theory that incorporates Fokker-Planck dynamical friction (drag) and pitch angle scattering is self-consistently derived from first principles for an isolated, marginally-unstable mode resonating with an energetic minority species. It is found that drag fundamentally changes the structure of the wave-particle resonance, breaking its symmetry and leading to the shifting and splitting of resonance lines. In contrast, scattering broadens the resonance in a symmetric fashion. Comparison with fully nonlinear simulations shows that the proposed quasilinear system preserves the exact instability saturation amplitude and the corresponding particle redistribution of the fully nonlinear theory. Even in situations in which drag leads to a relatively small resonance shift, it still underpins major changes in the redistribution of resonant particles. This novel influence of drag is equally important in plasmas and gravitational systems. In fusion plasmas, the effects are especially pronounced for fast-ion-driven instabilities in tokamaks with low aspect ratio or negative triangularity, as evidenced by past observations. The same theory directly maps to the resonant dynamics of the rotating galactic bar and massive bodies in its orbit, providing new techniques for analyzing galactic dynamics.

Introduction.—Resonances are channels for non-adiabatic energy exchange between particles and waves in kinetic systems. In linear theory, resonant interactions occur when a specific synchronization condition is satisfied exactly. In reality, however, this condition is smeared out due to effects such as finite mode amplitude, turbulence, and collisions. In plasmas, the effect of resonance broadening has been historically investigated in strong turbulence [1–5] and quasilinear [6–11] approaches, with important implications to the dynamics of plasma echoes [12] and Alfvén waves [13]. Previously unexplored aspects, but similarly important for influencing the resonant particle relaxation, are mechanisms responsible for the shifting and splitting of discrete resonances.

The competition between convective (dynamical friction, also known as drag) and diffusive (scattering) collisions plays a key role in determining the dynamical behavior of wave-particle resonant systems [14]. For instance, it is crucial to explaining the observation of Alfvén eigenmode frequency chirping in tokamaks [15]. Although global, non-resonant effects of drag are well known [16], the influence of drag on the structure of narrow resonant layers has not previously been determined.

In this work, we explore kinetic instabilities close to their threshold to analytically show that coherent Fokker-Planck drag breaks the symmetry of the resonances with respect to their original location. This occurs not only by shifting and splitting them but also by altering the relative strength of regions in the vicinity of a resonance. The skewed dependence of the resonance due to drag is in turn responsible for significant modifications to the particle distribution function. Moreover, it is demonstrated that the fully nonlinear collisional kinetic system naturally reduces to the form of a quasilinear (QL) theory in the limit of marginal stability and when stochastic processes dominate the relaxation.

An important application of these findings in fusion plasmas is the forecasting of deleterious fast ion transport by Alfvén eigenmodes. Fully nonlinear simulations are numerically costly and therefore it is of practical interest to develop reduced models, such as quasilinear theory, capable of reproducing essential features of more complete descriptions [17, 18]. The drag-induced modifications of the instability saturation level, resonance function, and fast ion redistribution are anticipated to be more significant in tokamaks with low aspect ratio or negative triangularity, as evidenced by past observations of a greater propensity for dynamical Alfvén Eigenmode behavior in these configurations [19–23], consistent with theoretical predictions [24–26].

A deep connection exists between kinetic processes in plasmas and those present in self-gravitating systems, as both are well-described by mean field theories governed by long-range, inverse square laws. In fact, the respective distributions obey the same evolution equation in phase space, giving rise to analogous phenomena such as Landau damping and resonant relaxation in both plasmas [27–29] and self-gravitating systems [30–33]. Consequently, understanding the structure of collisional wave-particle resonances is equally relevant to kinetic plasma instabilities and galactic dynamics, for instance, in determining the torque applied to the rotating galactic bar by orbiting heavy bodies [34].

Self-consistent transport theory with dynamical friction and scattering.— Krook or scattering collisions are known to lead to an anti-symmetric modification of the particle distribution around a resonance [9]. As will be demonstrated, drag introduces a distinct, asymmetric
response. Hence, we investigate the 1D nonlinear dynamics of an electrostatic wave resonating with a hot minority species, i.e. the canonical bump-on-shoulder, in the presence of both scattering and drag. The generalization to more complicated geometries and to waves of any polarization can be achieved with the methods of Refs. [9, 35]. The wave field is represented as

\[ E(x, t) = \text{Re} \left( \hat{E}(t) e^{i(kx - \omega t)} \right), \]

with complex amplitude \( \hat{E}(t) \). The hot minority species is described via a distribution function \( f(x, v, t) \) with an initial discrete resonance at \( kv_{\text{res}} = \omega \). For resonances narrow with respect to the resonance scale of the distribution, it is sufficient to evaluate the Fokker-Planck coefficients at the resonance center and the kinetic equation becomes [14, 35]

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{1}{k} \text{Re} \left( \omega_b^2 e^{i(kx - \omega t)} \right) \frac{\partial f}{\partial v} = C[f - f_0], \tag{1}
\]

where

\[
C[g] = \frac{\nu^3}{k^2} \frac{\partial^2 g}{\partial v^2} + \frac{\alpha^2}{k} \frac{\partial g}{\partial v}. \tag{2}
\]

Here \( f_0(v) \) is the equilibrium distribution function, which is assumed to have an approximately constant positive slope in the vicinity of the resonance. The rate of mode drive at \( t = 0 \) in the absence of damping is proportional to this slope: \( \gamma_{L,0} = (2\pi^2 q^2 \omega/mk^2) \partial f_0/\partial v \), while the mode damping rate due to interaction with the background thermal plasma is given by a constant \( \gamma_d \). The nonlinear bounce frequency for deeply-trapped resonant particles, \( \omega_b(t) \equiv \sqrt{gk\hat{E}(t)/m} \), is a convenient measure of the mode amplitude.

The coefficients \( \nu \) and \( \alpha \) are the effective scattering and drag collision frequencies, which are enhanced with respect to the 90° pitch angle scattering rate and the inverse slowing down time [7, 12, 25, 36]. (see [37] for heuristic arguments). Due to the narrowness of the resonances, the resonant particle dynamics can be strongly controlled by collisions even if their non-resonant dynamics hardly feel their effect.

Due to spatial periodicity, the distribution can be assumed of the form \( f(x, v, t) = f_0(v, t) + \sum_{i=1}^{\infty} f_i(v, t) e^{i(kx - \omega t)} \). In the collisional regime, an asymptotic expansion exists in orders of the small parameter \( |\omega_b^2|/\nu^2 \ll 1 \). Re-writing Eq. (1) in orders of such a small parameter naturally implies the ordering \( f_0^{(0)} \gg f_1^{(1)} \gg f_2^{(2)} \gg f_3^{(3)} \gg \ldots \) [38]. The prime denotes the derivative with respect to \( v \) while the superscript denotes the order in the wave amplitude (proportional to \( |\omega_b^2|/\nu^2 \)). Note that \( f_0^{(0)} \) corresponds to the equilibrium distribution \( f_0 \). This ordering of the perturbation theory is satisfied for the entire time evolution of the system so long as (i) steady asymptotic solutions exist (guaranteed when \( \alpha/\nu < 0.96 \) and \( \nu > \nu_{\text{crit}} \approx 2.05(\gamma_{L,0} - \gamma_d) \) [14]) and (ii) the system is sufficiently close to marginal stability, i.e. \( \gamma_{L,0} - \gamma_d \ll \gamma_{L,0} \). When conditions (i) and (ii) are not satisfied, the mode may grow to an amplitude that violates the \( |\omega_b^2|/\nu^2 \) assumption. From Eq. (1), the \( f_l^{(i)} \) satisfy

\[
\frac{\partial f_l^{(i)}}{\partial t} + il(kv - \omega) f_l^{(i)} + \frac{\omega_b^2}{k} f_{l-1}^{(i)} + \frac{\omega_b^2}{k} f_{l+1}^{(i)} = C[f]. \tag{3}
\]

In general, the solution of Eq. (3) is an integral over the time history of the system that involves delays in the argument of \( \omega_b(t) \). However, when stochastic collisions dominate the system’s dynamics, the system’s memory is poorly retained and the dynamics instead become essentially local in time [39, 40]. This occurs when \( \nu \gg \gamma_{L,0} - \gamma_d \). The constraints on these parameters become more restrictive as the ratio of drag to scattering is increased [41]. Then, to first order in \( |\omega_b^2|/\nu^2 \) one can disregard the time derivative in (3), yielding

\[
f_1^{(1)} = -\frac{F_0\omega_b^2(t)}{2k
\nu} \int_0^\infty ds e^{-i(kv - \omega)s} e^{-\alpha^2s^2/2 - s^3/3}. \tag{4}
\]

Using the reality rule \( f_{l-1}^{(1)} = f_{l-1}^{(1)*} \), Eq. (3) can be written to second order in \( |\omega_b^2|/\nu^2 \),

\[
\frac{\partial f_2^{(2)}}{\partial t} + \frac{1}{2k} \left( \omega_1^2 f_1^{(1)} * + \omega_2^2 f_1^{(1)} \right) = C[f_0]. \tag{5}
\]

Substitution of Eq. (4) into Eq. (5) shows that only the spatially averaged distribution \( f(v, t) = f_0(v) + f_2^{(2)}(v, t) \) explicitly appears in the wave-particle power exchange to second order in \( |\omega_b^2|/\nu^2 \). Since \( \partial F_0/\partial t = 0 \) and \( |F_0| \gg |f_0^{(2)}| \), one then obtains from Eq. (5) that the resonant particle response is regulated by a QL diffusion-advective transport equation

\[
\frac{\partial f_2^{(2)}(v, t)}{\partial t} - \frac{\partial}{\partial v} \left[ \frac{\pi}{2k^3} |\omega_b^2|^2 R(v) \frac{\partial f}{\partial v} \right] = C[f - f_0]. \tag{6}
\]

with the resonance function given by

\[
R(v) = \frac{k}{\pi
\nu} \int_0^\infty ds \cos \left( \frac{(kv - \omega)}{\nu} s + \frac{\alpha^2 s^2}{\nu^2/2} \right) e^{-s^3/3}. \tag{7}
\]

\( R(v) \) gives the velocity-dependent strength of the resonant wave-particle interaction. In the absence of collisions, \( R(v) = \delta(v - \omega/k) \) would describe an exact, unbroadened resonance. For all values of \( \alpha/\nu \), the property \( \int_{-\infty}^{\infty} R(v) dv = 1 \) is exactly satisfied. The complete set of equations describing the QL system is then given by Eqs. (6) and (7), coupled with the amplitude evolution equation

\[
\frac{d|\omega_b^2(t)|^2}{dt} = 2 (\gamma_{L,0} - \gamma_d) |\omega_b^2(t)|^2, \tag{8}
\]
Remarkably, a QL transport theory has emerged spontaneously from the nonlinear one under the assumption that stochasticity dominates over the relaxation timescale \( \nu \gg \gamma_{L,0} - \gamma_d \), even in the presence of coherent drag which acts to preserve phase space correlations. Furthermore, the QL system was obtained for an isolated, discrete resonance without invoking any overlapping condition.

Shift of the resonance function due to symmetry breaking.—Plots the resonance function \( R(v) \) for different values of \( \alpha/\nu \), as done in Fig. 1(a), demonstrates that drag shifts the location of the strongest wave-particle interaction in phase space. Moreover, drag acts to increase the strength of the interaction downstream of the resonance and diminish it upstream. Although the quantitative changes in the resonance function due to drag are seemingly small, this fundamental change in the character of the wave-particle resonance has substantial consequences for how the energetic particles are redistributed. This shift is not analogous to a simple Doppler shift, but rather the result of asymmetry present in the collisional dynamics due to drag, which has a preferred direction. Without drag, \( R(v) \) is perfectly symmetric about \( kv - \omega = 0 \).

The shift of the resonance function implies a previously unrecognized collisional modification to the resonance condition, which can be obtained by calculating the location of the peak of \( R(v) \), leading to

\[
\int_0^\infty \sin \left( s \left( \frac{kv - \omega}{\nu} \right)_{\text{peak}} + \frac{\gamma^2 s^2}{2} \right) s \exp \left( -\frac{s^3}{3} \right) ds = 0.
\]

Noting that only small \( s \) contributes due to the strongly decaying cubic exponential, the drag-modified resonance condition becomes \( (kv - \omega)_{\text{peak}} \approx -3^{1/3} \Gamma(4/3) \alpha^2/2\nu \), which is accurate to within 1.5% for \( \alpha/\nu < 1 \). Shifted resonance lines, captured here using drag in a considerably reduced theory, are also known to exist in the strong turbulence framework \cite{44} as a result of turbulence modification of ensemble average orbits.

Particle relaxation.—The derived QL system also reproduces key features of the complete nonlinear system, namely the perturbed distribution function \( \delta f \equiv f - F_0 \) and wave saturation amplitude. When stochasticity regulates the timescale for the mode growth, Eq. (6) can be further simplified to

\[
\frac{\delta f (v, t)}{F_0} = \left( \frac{\omega_b^2 (t)}{2 k \nu^3} \right) \left( \frac{\alpha}{\nu} \right) - \int_0^\infty ds e^{-s^3/3} \left\{ \gamma^2 s \cos \left( \frac{(kv - \omega) s}{\nu} + \frac{\alpha^2}{\nu^2} s^2 \right) + s \sin \left( \frac{(kv - \omega) s}{\nu} + \frac{\alpha^2}{\nu^2} s^2 \right) \right\}.
\]
The integration constant \( c(\alpha/\nu) \) is determined by enforcing particle conservation: 
\[
\int_{v_{\text{max}}}^{v_{\text{max}} - \delta f} \delta f dv = 0.
\]

The relaxed distribution is plotted in Fig. 1(b) for several values of \( \alpha/\nu < 1 \). Remarkably, a small quantitative asymmetry in the collisional dynamics due to drag can have a large qualitative effect on the saturated distribution, even though the corresponding changes in the resonance function are less dramatic. When no drag is present \( (\alpha = 0) \), \( \delta f \) is antisymmetric with constant plateaus outside of a narrow transition region near the peak of the resonance. As the ratio of drag to scattering is increased, the plateau upstream of the resonance \( (kv > \omega) \) instead decays in velocity space at a rate proportional to \( \alpha^2/\nu^2 \). Particle conservation shifts the entire distribution downward once the symmetry is broken, eliminating nearly all of the downstream particle redistribution, even for very small amounts of drag. The drag-induced modifications to \( \delta f \) were compared against simulations performed with the nonlinear 1D Vlasov code BOT, which solves the plasma kinetic equation (Eq. (1)) directly [24, 45]. The dashed black curves in Fig. 1(b) show the simulation results for each value of \( \alpha/\nu \), demonstrating excellent agreement.

**Instability saturation level.**—Drag has a destabilizing effect on the underlying instability, leading the wave to saturate at a larger amplitude with increasing \( \alpha/\nu \). An analytically tractable example is the first order correction to the saturation amplitude due to drag. Substituting Eq. (10) into Eq. (9), to lowest order in \( \alpha^2/\nu^2 \), one finds (the details of the derivation are given in the Supplemental Material [46])

\[
\gamma_L (t) \simeq \gamma_{L,0} \left\{ 1 - \frac{\omega_b^2 (t)}{2\nu^2} \left[ \Gamma \left( \frac{1}{3} \right) \left( \frac{3}{2} \right)^{1/3} \frac{1}{3} - \frac{\pi \alpha^2}{2 \nu^2} \right] \right\}.
\]

At saturation, i.e., when \( \gamma_L (t) = \gamma_d \), then the mode amplitude \( \hat{E} \) is given by

\[
|\omega_{b,\text{sat}}| = \sqrt{\frac{q k F_{\text{sat}}}{m}} \simeq \left[ \frac{2 (1 - \gamma_d/\gamma_{L,0})}{\Gamma \left( \frac{1}{3} \right) \left( \frac{3}{2} \right)^{1/3} \frac{1}{3} - \frac{\pi \alpha^2}{2 \nu^2}} \right]^{1/4} \nu,
\]

which is the same as calculated directly from nonlinear theory (Ref. [40]). In the absence of drag, the saturation in Eq. (12) recovers the level calculated in Ref. [35]. While not possible to express in terms of elementary functions, it can nonetheless be proven analytically that the QL saturation level also reproduces the fully nonlinear one whenever a steady state solution exists.

Fig 1(c) shows the time evolution of the mode amplitude in fully nonlinear simulations. The saturation levels are in excellent agreement with the unapproximated analytic expression [43], and demonstrate that even moderate amounts of drag can appreciably increase the saturated mode amplitude. For example, \( \alpha/\nu = 0.6 \) yields a 40% increase in the saturated amplitude relative to if drag were neglected. Consequently, this increase in mode amplitude enhances the resonant particle transport, as shown in Fig. 1(b). As \( \alpha/\nu \) is increased from 0.6 to 0.8, the magnitude of \( \delta f \) nearly doubles.

**Resonance splitting due to large drag.**—It is also interesting to consider the structure of the resonance when drag dominates over scattering, corresponding to instabilities which will eventually reach a strongly nonlinear regime, with the potential for substantial transport. Although for \( \alpha/\nu > 0.96 \) no steady state solution is allowed within the near-threshold perturbation theory [14], all of the derivations to this point nonetheless remain valid during the early growth phase, up until the mode amplitude exceeds the assumed \( |\omega_b^2|/\nu^2 \ll 1 \) ordering.

For \( \alpha \gg \nu \), a pronounced splitting of the resonance function occurs, as shown in Fig. 2(a). Simultaneously, the resonance function broadens beyond its original width, proportional to \( \nu \), instead becoming proportional to \( \alpha \). It can be shown from Eq. (7) that for \( kv < \omega \) in the limit of \( (kv - \omega)^2 \gg \alpha^2 \gg \nu^2 \gg \gamma^2 \), the resonance function has the following asymptotic behavior:
\[ R(v) = \frac{1}{\alpha} \sqrt{\frac{2}{\pi}} \exp \left[ \frac{1}{3} \left( \frac{\nu}{\alpha} \frac{k v - \omega}{\alpha} \right)^3 \right] \cos \left[ \frac{(k v - \omega)^2}{2\alpha^2} - \frac{\pi}{4} \right]. \]  

(13)

Hence the resonance function’s extrema are given by 
\[(k v - \omega)_{\text{crit}} = -\alpha \sqrt{2\pi} \sqrt{n + 1/4} \text{ for } n \geq 0.\]

In particular, 
\[n = 0\] gives the central shift in the resonance function, showing that it becomes proportional to \(\alpha\) when drag dominates instead of \(\alpha^2\) when scattering dominates. Strictly speaking, similar extrema are present even when \(\alpha \ll \nu\), however they are substantially smaller than the primary peak.

The splitting of the resonance function is a novel quasilinear effect not previously identified in plasmas. This behavior implies that for sufficiently large drag there can be multiple regions of phase space where particles are efficiently interacting with the wave, even for the idealized case of a monochromatic wave with a uniform background.

During the early growth phase in the drag-dominated regime, \(\delta f\) has identical asymptotic behavior as the resonance function: \(\delta f(v) \propto -R(v)\), inheriting the same pattern of decaying oscillations. Strong agreement is found between the analytic \(\delta f\) calculated with Eq. 10 and a nonlinear BOT simulation with \(\alpha \gg \nu\), as shown in Fig. 2(b) for the cases with clear splitting. Consequently, large amounts of drag act to extend the downstream region of phase space where significant transport occurs.

**Relevance to fusion plasmas.**—The effects of drag derived in this Letter are most prominent when drag approaches or exceeds scattering within a narrow resonance. A heuristic scaling useful for guiding intuition is given by \(\alpha/\nu \sim \xi_{\text{res}}^{1/2} n_e^{1/10} T_e^{3/4} \omega_{\text{e}}^{1/6}\) [40]. Quantitatively, \(\alpha/\nu\) can be numerically calculated with rigor in fusion plasmas using a kinetic equilibrium reconstruction, solving for the eigenmode structure with an MHD code, following realistic orbits, and averaging over resonant surfaces. This method is outlined in the Supplemental Material [47] and was previously applied to NSTX, DIII-D, and ITER [15, 25, 26], including the effect of enhanced diffusion due to microturbulence [48, 49].

An increase in \(\alpha/\nu\) due to the reduction of turbulence has been previously identified as a reliable indicator of the onset of chirping behavior for Alfvénic modes [15], which is ubiquitous in spherical tokamaks but rarely observed in conventional tokamaks. Turbulence is typically lower in spherical tokamaks such as NSTX due to favorable curvature and enhanced rotation shear [50], allowing \(\alpha \sim \nu\). Similar findings exist on other devices. Calculations by Lilley et al. found that \(\alpha/\nu = 0.6 - 5\) at the TAE resonance for beam-heated MAST plasmas [14], with the practical consequence of explaining why those discharges often feature bursting TAEs, in contrast to radio-frequency-heated discharges with larger scattering rates. In addition, Lesur et al. analyzed an uncommon case of chirping in JT-60U by fitting the observed frequency sweeping to a theoretical model, enabling the extraction of the collisional coefficients, finding \(\alpha/\nu\) in the range of 0.2 – 0.7 consistent with measured plasma parameters [51]. Furthermore, simulations of an ITPA tokamak benchmark case and the W7-X stellarator using a global hybrid MHD-kinetic code performed by Slaby et al. concluded that including realistic amounts of drag in the Fokker-Planck collision operator affected both the AE saturation level and its long term nonlinear behavior [52]. Lastly, negative triangularity experiments on DIII-D observed a greater tendency for chirping than in matched positive triangularity discharges [23]. Van Zeeland et al. attributed this difference to the lower level of turbulent scattering in negative triangularity, increasing \(\alpha/\nu\) and making non-steady behavior more likely according to theory. The relative propensity for chirping in several distinct scenarios empirically supports the relevance of drag to wave-particle dynamics in fusion plasmas, especially tokamaks with low aspect ratio or negative triangularity.

**Connection to galactic dynamics.**—Beyond fusion plasmas, the methods presented in this Letter are directly applicable to the resonant gravitational interaction of the rotating galactic bar and heavy bodies in its orbit such as black holes, massive astrophysical compact halo objects, and stars in a tepid disk [53]. As recently demonstrated by Hamilton et al. [34], the collisional bump-on-tail problem in plasmas (studied in this work) can be made isomorphic to this application. In this formalism, the role of the AE is played by the rigidly rotating galactic bar, the resonant fast ions are replaced by the orbiting bodies, and the electrostatic potential is replaced by a gravitational one. Diffusive scattering occurs due to stochastic potential fluctuations, while heavy bodies experience drag when passing through a background of lighter masses. The resonant interaction exerts a torque on the bar and leaves a collisionally-dependent imprint on the spatial distribution of surrounding bodies in the galaxy, which in tandem with observations can be used to constrain theoretical models of dark matter. Specifically, the collisions experienced by orbiting heavy compact objects are dominated by drag when their mass ratio to the background particles is sufficiently large [54, 55]. Hence, the novel influence of drag on wave-particle interactions derived in this Letter are relevant for an accurate description of the resonant galactic bar-heavy body system in the presence of collisions.

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* VND and JBL contributed equally to this work.

Correspondence may be addressed to vduarte@pppl.gov and lestzj@fusion.gat.com.

[1] T. H. Dupree, Phys. Fluids 9, 1773 (1966).
[2] J. Weinstock, Phys. Fluids 11, 1977 (1968).
[3] A. Salat, Phys. Fluids 31, 1499 (1988).
[4] O. Ishihara, H. Xia, and A. Hirose, Phys. Fluids B 4, 349 (1992).
[5] J. A. Krommes, J. Plasma Phys. 81, 205810601 (2015).
[6] H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).
[7] J. D. Callen, Phys. Plasmas 21, 052106 (2014).
[8] N. H. Bian, E. P. Kontar, and H. Ratcliffe, J. Geophys. Res.: Space Phys. 119, 4239 (2014).
[9] V. N. Duarte, N. N. Gorelenkov, R. B. White, and H. L. Berk, Phys. Plasmas 26, 120701 (2019).
[10] P. J. Catto, J. Plasma Phys. 86, 815860302 (2020).
[11] P. J. Catto and E. A. Tolman, J. Plasma Phys. 87, 905870606 (2021).
[12] C. H. Su and C. Oberman, Phys. Rev. Lett. 20, 427 (1968).
[13] R. B. White, V. N. Duarte, N. N. Gorelenkov, and G. Meng, Phys. Plasmas 26, 032508 (2019).
[14] M. K. Lilley, B. N. Breizman, and S. E. Sharapov, Phys. Rev. Lett. 102 (2009).
[15] V. N. Duarte, H. L. Berk, N. N. Gorelenkov, W. W. Heidbrink, G. J. Kramer, R. Nazikian, D. C. Pace, M. Podestā, B. J. Tobias, and M. A. Van Zeeland, Nucl. Fusion 57, 054001 (2017).
[16] J. D. Gaffey, J. Plasma Phys. 16, 149 (1976).
[17] N. Gorelenkov, V. Duarte, M. Podestā, and H. Berk, Nucl. Fusion 58, 082016 (2018).
[18] M. Podestā, M. Gorelenkova, and R. B. White, Plasma Phys. Control. Fusion 56, 055003 (2014).
[19] M. Gryaznevich and S. Sharapov, Nucl. Fusion 46, S942 (2006).
[20] M. Podestā, R. Bell, A. Bortoloni, N. Crocker, D. Darrow, A. Diao, E. Fredrickson, G.-Y. Fu, N. Gorelenkov, W. Heidbrink, G. Kramer, N. Kubota, B. LeBlanc, S. Medley, and H. Yuh, Nucl. Fusion 52, 094001 (2012).
[21] Y. V. Petrov, N. N. Bakharev, V. K. Gusev, V. B. Minaev, V. A. Kornev, G. S. Kurskiev, M. I. Patrov, N. V. Sakharov, S. Y. Tolstyakov, P. B. Shchegolev, and et al., J. Plasma Phys. 81, 515810601 (2015).
[22] K. G. McClements and E. D. Fredrickson, Plasma Phys. Control. Fusion 59, 055003 (2017).
[23] M. Van Zeeland, C. Collins, W. Heidbrink, M. Austin, X. Du, V. Duarte, A. Hyatt, G. Kramer, N. Gorelenkov, B. Grierson, D. Lin, A. Marinoni, G. McKee, C. Muscatello, C. Petty, C. Sung, K. Thome, M. Walker, and Y. Zhu, Nucl. Fusion 59, 086029 (2019).
[24] M. K. Lilley, B. N. Breizman, and S. E. Sharapov, Phys. Plasmas 17, 092305 (2010).
[25] V. N. Duarte, H. L. Berk, N. N. Gorelenkov, W. W. Heidbrink, G. J. Kramer, R. Nazikian, D. C. Pace, M. Podestā, and M. A. Van Zeeland, Phys. Plasmas 24, 122508 (2017).
[26] V. N. Duarte, N. N. Gorelenkov, M. Schneller, E. D. Fredrickson, M. Podestā, and H. L. Berk, Nucl. Fusion 58, 082013 (2018).
[27] L. D. Landau, Sov. Phys. JETP. 10, 25 (1946).
[28] T. Servers, Phys. Fluids 8, 2255 (1965).
[29] R. K. Mazitov, J. Appl. Mech. Tech. Phys. 6, 22 (1965).
[30] D. Lynden-Bell, Mon. Notices Royal Astron. Soc. 124, 279 (1962).
[31] P. A. Sweet, Mon. Notices Royal Astron. Soc. 125, 285 (1962).
[32] Mikhailovskii, A.A. and Fridman, A.M., Sov. Phys. JETP 34, 243 (1972).
[33] D. Ryutov, Plasma Phys. Control. Fusion 41, A1 (1999).
[34] C. Hamilton, E. Tolman, L. Arzamasskiy, and V. Duarte, Submitted to Astrophys. J. (2022).
[35] H. L. Berk, B. N. Breizman, and M. Pekker, Plasma Phys. Rep. 23, 778 (1997).
[36] H. L. Berk, B. N. Breizman, and H. Ye, Phys. Rev. Lett. 68, 3563 (1992).
[37] See Supplemental Material [url], Sec. 1, which includes [7, 12, 25, 36].
[38] H. L. Berk, B. N. Breizman, and M. Pekker, Phys. Rev. Lett. 76, 1256 (1996).
[39] V. N. Duarte and N. N. Gorelenkov, Nucl. Fusion 59, 044003 (2019).
[40] J. B. Lestz and V. N. Duarte, Phys. Plasmas 28, 062102 (2021).
[41] See Supplemental Material [url], Secs. 6 - 8, which includes [14, 40].
[42] See Supplemental Material [url], Secs. 1 and 3, which includes [38-40].
[43] See Supplemental Material [url], Sec. 6, which includes [14, 40].
[44] T. J. Birmingham and M. Bornatici, Phys. Fluids 15, 1785 (1972).
[45] The bot code repository, https://github.com/mklilley/BOT.
[46] See Supplemental Material [url], Sec. 2, which includes [9, 15, 40].
[47] See Supplemental Material [url], Secs. 4 and 5, which includes [15, 26, 35, 48-50, 56-60].
[48] J. Lang and G.-Y. Fu, Phys. Plasmas 18, 055002 (2011).
[49] W. Zhang, Z. Lin, and L. Chen, Phys. Rev. Lett. 101, 10.1103/physrevlett.101.095001 (2008).
[50] S. Kaye, F. Levinton, D. Stutman, K. Tritz, H. Yuh, M. Bell, R. Bell, C. Domier, D. Gates, W. Horton, J. Kim, B. LeBlanc, N. Luhmann, R. Maini, E. Mazzucato, J. Menard, D. Mikelsen, D. Mueller, H. Park, G. Rewoldt, S. Sabbagh, D. Smith, and W. Wang, Nucl. Fusion 47, 490 (2007).
[51] M. Lesur, Y. Idomura, K. Shinozaka, X. Garbet, and the JT-60 Team, Phys. Plasmas 17, 122311 (2010).
[52] C. Slaby, A. Koenies, R. Kleiber, and H. Leyh, Nuclear Fusion 59, 046006 (2019).
[53] Fouvry, J. B., Pichon, C., Magorrian, J., and Chavanis, P. H., A&A 584, A129 (2015).
[54] J. Binney and S. Tremaine, Galactic Dynamics (Princeton University Press, 2008).
[55] L. Ciotti, Introduction to Stellar Dynamics (Cambridge University Press, 2021).
[56] Y. Zhang, W. W. Heidbrink, H. Boehmer, R. McWilliams, G. Chen, B. N. Breizman, S. Vincena,
T. Carter, D. Leneman, W. Gekelman, P. Pribyl, and B. Brugman, Phys. Plasmas 15, 012103 (2008).

[57] W. W. Heidbrink, H. Boehmer, R. McWilliams, A. Preiwisch, Y. Zhang, L. Zhao, S. Zhou, A. Bovet, A. Fasoli, I. Furno, K. Gustafson, P. Ricci, T. Carter, D. Leneman, S. K. P. Tripathi, and S. Vincena, Plasma Phys. Control. Fusion 54, 124007 (2012).

[58] W. Gekelman, P. Pribyl, Z. Lucky, M. Drandell, D. Leneman, J. Maggs, S. Vincena, B. Van Compernolle, S. K. P. Tripathi, G. Morales, T. A. Carter, Y. Wang, and T. DeHaas, Rev. Sci. Instr. 87, 025105 (2016).

[59] M. Sarfaty, S. De Souza-Machado, and F. Skiff, Phys. Rev. Lett. 80, 3252 (1998).

[60] J. W. R. Schroeder, F. Skiff, G. G. Howes, C. A. Kletzing, T. A. Carter, and S. Dorfman, Phys. Plasmas 24, 032902 (2017).
Supplemental Material of the Letter

"Shifting and splitting of resonance lines due to dynamical friction in plasmas"

V. N. Duarte,1, * J. B. Lestz,2,3, * N. N. Gorelenkov,1 and R. B. White1

1Princeton Plasma Physics Laboratory, Princeton University, Princeton, N.J. 08543, USA
2General Atomics, San Diego, CA, 92121, USA
3Department of Physics and Astronomy, University of California, Irvine, CA, 92697, USA

I. DERIVATION OF THE EVOLUTION EQUATION, THE GROWTH RATE AND THE RELAXED DISTRIBUTION

The power transfer from the resonant particles satisfying \( kv \approx \omega \) to the wave is

\[
P = -\frac{q_0 \omega}{k} \int dv \int dx E^* (x,t) f(x,v,t).
\]

The power balance equation, for a constant background damping rate \( \gamma_d \), is

\[
\frac{d}{dt} \int dx \frac{|E(x,t)|^2}{8\pi} = P - \gamma_d \int dx \frac{|E(x,t)|^2}{8\pi}.
\]  

(1)

As in the Letter, the distribution is considered in the form

\[
f(x,v,t) = f_0(v,t) + \sum_{i=1}^{\infty} \left( f_i(v,t) e^{i(kv'-\omega t)} + \text{c.c.} \right),
\]

with the small parameter \( |\omega_0^2|/v^2 \ll 1 \) leading to the ordering \( |f_i^{(0)}| \gg |f_i^{(1)}| \gg |f_i^{(2)}| \), [1]. The prime denotes the derivative with respect to \( v \) while the superscript denotes the order in the wave amplitude (proportional to \( |\omega_0^2|/v^2 \)). Performing the spatial integration of Eq. (1) over a wavelength leads to

\[
\frac{d\hat{E}(t)}{dt} = -\frac{4\pi q_0 \omega}{k} \int dv f_1(v,t) - \gamma_d \hat{E}(t)
\]  

(2)

where \( q \) is the energetic particle electric charge. Consider the above equation up to cubic order in the electric field amplitude, i. e., \( f_1 \approx f_1^{(1)} + f_1^{(3)} \) \( f_1^{(2)} = 0 \) because the structure of the kinetic equation [Eq. (2) of the Letter] does not allow for a first-order Fourier \( f_1 \) to have even powers in the electric field amplitude, where \( f_1^{(1)} \) and \( f_1^{(3)} \) can be found by direct integration of the kinetic equation (the details are given in Sec. III of this Supplemental Material)

\[
f_1^{(1)} = -\frac{\omega_0^2}{2kv} \frac{\partial f_0}{\partial v} \int_0^\infty \! ds \, e^{-i\left(\frac{kv}{\nu} - s\right)} s^{-3/2 - 3/3}
\]

(3)

and

\[
f_1^{(3)} = -\frac{\omega_0^2}{2kv} \frac{\partial f_0^{(2)}}{\partial v} \int_0^\infty \! ds \, e^{-i\left(\frac{kv}{\nu} - s\right)} s^{-3/2 - 3/3}.
\]  

(4)

Using the definition \( \omega_0^2 \equiv qk\dot{E}/m \) and Eqs. (3) and (4), Eq. (2) becomes

\[
\frac{d\omega_0^2(t)}{dt} = -\gamma_d \omega_0^2(t) + \frac{2\pi q_0^2 \omega}{kv m} \omega_0^2(t) \times
\]

\[
\int_0^\infty \! dv \left( \frac{\partial F_0}{\partial v} + \frac{\partial f_0^{(2)}}{\partial v} \right) \int_0^\infty \! ds \, e^{-i\left(\frac{kv}{\nu} - s\right)} s^{-3/2 - 3/3}.
\]  

(5)

The equilibrium distribution can be approximated by a constant slope within narrow resonance layers. Since \( \partial F_0/\partial v \gg \partial f_0^{(2)}/\partial v \) and

\[
\int dv \int_0^\infty \! ds \, e^{-i\left(\frac{kv}{\nu} - s\right)} s^{-3/2 - 3/3} = 0,
\]

then the power balance equation [Eq. (2)] leads to the amplitude evolution equation

\[
\frac{d\omega_0^2(t)}{dt} = 2(\gamma_L(t) - \gamma_d) |\omega_0^2(t)|^2
\]  

(6)

with the growth rate \( \gamma_L(t) \) given by

\[
\gamma_L(t) = \frac{2\pi q_0^2 \omega}{mk^2} \int_{-\infty}^\infty \! dv \mathcal{R}(v) \frac{\partial f(v,t)}{\partial v}
\]

(7)

where the spatially-integrated distribution used in quasilinear theory is \( f(v,t) \approx f_0(v) + f_0^{(2)}(v,t) \). At \( t = 0 \), \( \gamma_L = \gamma_{L,0} = (2\pi q_0^2 \omega/mk^2) \partial F_0/\partial \nu \). Eqs. (6) and (7), together with the quasilinear equation for the distribution relaxation (Eq. (5) of the Letter, derived for a marginally unstable mode)

\[
\frac{\partial f(v,t)}{\partial t} - \frac{\partial}{\partial v} \left( \frac{\pi}{2k^3} |\omega_0^2|^2 \mathcal{R}(v) \frac{\partial f(v,t)}{\partial v} \right) =
\]

\[
= \nu^3 \frac{\partial^2 (f(v,t) - F_0)}{\partial v^2} + \frac{\alpha^2}{k} \frac{\partial (f(v,t) - F_0)}{\partial v}
\]

(8)

constitute the full set of transport equations. In Eq. (8), the resonance function is given by

\[
\mathcal{R}(v) = \frac{k}{\pi \nu} \int_0^\infty \! ds \, e^{-i\left(\frac{kv}{\nu} - s\right)} s^{-3/2 - 3/3}.
\]

(9)
\( R(v) \) gives the velocity-dependent strength of the resonant wave-particle interaction. The full dynamic solution of Eq. (8) for \( \delta f \equiv f(v, t) - F_0(v) \) is

\[
\delta f = \frac{F'_{\nu}}{2k^2 \nu} \int_0^t dt_1 \left[ \frac{\omega_0^2(t_1)}{v^2} \right]^2 \frac{\partial}{\partial v} \int_0^\infty dse^{-s^2/3} e^{-\nu s^2(t-t_1)} \times \\
\times \cos \left[ \frac{kv - \omega}{\nu} s + \frac{\alpha^2 s^2}{\nu} \left( \frac{s}{2\nu} + t - t_1 \right) \right] + \text{const.}
\]

Quasi-steady solutions can be obtained by neglecting the time derivative in Eq. (8). The solution for such system is

\[
\delta f(v, t) = \frac{[\omega_0^2(t)]^2}{2k^2 \nu^3} F'_{\nu} \left\{ c \left( \frac{\alpha}{\nu} \right) e - \\
\int_0^v dv' R(v') \exp \left[ -\frac{\alpha^2}{\nu^3} k(v - v') \right] \right\}.
\]

Using Eq. (9), the velocity integral in Eq. (10) can be performed, which leads to the expression shown in Eq. (10) of the Letter.

**II. DERIVATION OF THE GROWTH RATE TO LOWEST ORDER IN \( \alpha^2/\nu^2 \)**

The condition \( \alpha \ll \nu \), common in scenarios of fast ion resonances with Alfvén eigenmodes in tokamaks [6] further allows the resonance function [Eq. (9)] to be approximately written as

\[
R(v) \approx \left[ 1 - \frac{\alpha^2}{2\nu^2} \frac{k v - \omega}{\nu} \right] R_{\text{scatt}}(v),
\]

where

\[
R_{\text{scatt}}(v) = \frac{k}{\nu \pi} \int_0^\infty ds \cos \left( \frac{k v - \omega}{\nu} s \right) e^{-s^2/3}
\]

is the symmetric resonance function in the absence of drag [7]. Substituting Eq. (10), into Eq. (7), one obtains

\[
\gamma_L(t) = \gamma_{L0} \left\{ 1 - \frac{\pi}{2} \frac{[\omega_0(t)]^2}{\nu^2} \left[ k \int_{-\infty}^{\infty} dv R^2(v) - \\
- \frac{a^2}{\nu^2} k^2 \int_{-\infty}^{\infty} dv R(v) \int v' dv' R(v') \exp \left[ -\frac{a^2}{\nu^2} k(v - v') \right] \right] \right\}
\]

To first order in \( \alpha^2/\nu^2 \), the resonance function in Eq. (13) can be approximated by Eq. (11). The first term of Eq. (13) can be simplified by noting that because \( R_{\text{scatt}}(v) \) (given by Eq. (12)) is an even function with respect to \( kv = \omega \), then \( \int_{-\infty}^{\infty} (kv - \omega) R_{\text{scatt}}(dv) = 0 \). Therefore the first term only contributes via \( \int_{-\infty}^{\infty} R_{\text{scatt}}(dv) = \frac{2k}{\nu^2} \left[ \frac{\pi}{3} \Gamma \left( \frac{1}{3} \right) \right]^{1/3} \).

For the second term of Eq. (13), it should be noted that, because \( v' \leq v \) in the inner integral, then the exponential is always within the interval \((0, 1]\). The term involving the double integral in Eq. (13) is already multiplied by \( \alpha^2/\nu^2 \), so we seek to calculate the integrals to zeroth order in this small parameter.
Then, the term involving the double integral can be approximated by
\[
\int_{-\infty}^{\infty} d\nu R(\nu) \int_{-\infty}^{\nu} d\nu' R(\nu') = 1/2 \left[ \int_{-\infty}^{\infty} d\nu' R(\nu') \right] \left( \frac{3}{2} \right)^{1/3} \left( \frac{3}{2} - \frac{\pi \alpha^2}{2} \right)
\]
which is the same expression calculated directly from nonlinear theory (Ref. [8]).

### III. INTEGRATION OF THE KINETIC EQUATION

Since Eq. (2) of the Letter implies \( f_2^{(0)} = 0 \), the kinetic equation at first order in the amplitude parameter to be integrated is
\[
\frac{\partial f_1^{(1)}}{\partial t} + i (kv - \omega) f_1^{(1)} + \frac{\nu^2 F_0}{2k} f_1^{(1)} = \frac{\nu^3 (1)}{k^2} f_1^{(1)} + \frac{\alpha^2}{k} f_1^{(1)}
\]
Eq. (15) has the solution
\[
f_1^{(1)}(t) = \frac{F_0'}{2k} \int_0^t dt_1 \omega_0^2(t_1) \times e^{-i(kv - \omega)(t-t_1)} e^{i\alpha_0^2(t-t_1)^2/2-\nu^3(t-t_1)^3/3},
\]
Defining \( s \equiv \nu (t - t_1) \),
\[
f_1^{(1)} = \frac{F_0'}{2k} \int_0^{\nu t} ds \omega_0^2(t - s/\nu) e^{-i(kv - \omega)s - i\alpha_0^2 s^2/2-\nu^3 s^3/3}. \tag{16}
\]
Due to the cubic exponential, only small values of \( s \lesssim 2 \) contribute significantly to the integral. Physically, when stochastic collisions dominate the system’s dynamics over other processes, the system memory is poorly retained and the dynamics instead become essentially local in time \([8, 9]\), which occurs when \( \nu \gg \gamma_L_0 - \gamma_d \). Therefore, the amplitude can be evaluated at that peak of the exponential kernel and be moved out of the integral and the upper limit of integration can be set to infinity,
\[
f_1^{(1)} = \frac{F_0' \omega_0^2(t)}{2
u k} \int_0^{\infty} ds e^{-i(kv - \omega)s - i\alpha_0^2 s^2/2-\nu^3 s^3/3}. \tag{17}
\]
Eq. (18) is also a solution of Eq. (15) when the time derivative is neglected from the outset. That can be easily seen by substituting Eq. (18) into Eq. (15), which leads to
\[
\frac{F_0' \omega_0^2(t)}{2k} = \frac{F_0' \omega_0^2(t)}{2k} \int_0^{\infty} ds \times \left[ -i \left( \frac{kv - \omega}{\nu} \right) - i \frac{\alpha_0^2}{\nu^2} s - s^2 \right] e^{-i(kv - \omega)s - i\alpha_0^2 s^2/2-\nu^3 s^3/3}. \tag{19}
\]
Noticing that the integrand of Eq. (19) can be re-written as \( \frac{d}{ds} e^{-i(kv - \omega)s - i\alpha_0^2 s^2/2-\nu^3 s^3/3} \), it follows straightforwardly that the equality (Eq. (19)) is satisfied and Eq. (18) is a solution of Eq. (15) without the time derivative.

### IV. GENERALIZATION TO TOKAMAK GEOMETRY

Axisymmetric tokamaks admit the kinetic energy \( \mathcal{E} \), the canonical toroidal momentum \( P_\varphi \) and the magnetic moment \( \mu \) as invariants of the unperturbed particle motion. In the presence of low-frequency (compared to the cyclotron frequency) perturbations, the magnetic moment remains approximately preserved, while neither \( \mathcal{E} \) nor \( P_\varphi \) are conserved independently. Instead, a new invariant emerges, \( \mathcal{E}' = \mathcal{E} + \omega P_\varphi/n \), as long as the ansatz \( e^{-i(n \varphi \omega t)} \) holds for the wave perturbation. The existence of two invariants implies that the resulting resonant particle motion is, therefore, one dimensional in the vicinity of an isolated resonance determined by the condition \( \Omega = \omega + n \omega_p - p \omega_p = 0 \), where \( \omega_p \) and \( \omega_p \) are the mean poloidal and toroidal transit frequencies of the unperturbed orbit, \( p \) is an integer and \( \omega \) is the frequency of a mode with toroidal number \( n \). The fact that the dynamics near a single resonance are one-dimensional allows the idealized bump-on-tail system to be mapped into more realistic geometries, as shown in Ref. [10]. In particular, modeling the resonant interaction of fast ions with Alfvénic waves in toroidal geometry can be made more convenient by employing the relevant relaxation variable \( \Omega (P_\varphi, \mathcal{E}, \mu) \), which directly maps onto the bump-on-tail framework (by replacing \( kv - \omega \) with \( \Omega \)).

### V. CALCULATION OF \( \alpha/\nu \) IN REALISTIC TOKAMAK SCENARIOS

As discussed in the Letter, heuristic arguments describe the enhancement of the effective collisional frequencies \( \alpha \) and \( \nu \) within a narrow resonance, relative to the inverse slowing down time \( \tau_s^{-1} \) and 90-degree pitch angle scattering frequency \( \nu_\perp \), respectively. Combination of those scaling expressions with well-known formulas for the macroscopic collision frequencies together yield a simple expression for \( \alpha/\nu \) in a uniform plasma, appropriate for the bump-on-tail problem, which gives the relation analyzed in the letter. However, in realistic scenarios with nontrivial magnetic geometries and plasma profiles, such an expression is an incomplete accounting of the competing collisional effects within the resonance. In a nonuniform plasma, such as a tokamak, Alfvénic eigenstructures can be spatially localized, and its resonances with fast ions are spread out across multiple surfaces in the \( (P_\varphi, \mathcal{E}, \mu) \) constant-of-motion space. In this case, it is necessary to perform averages when calculating \( \alpha \) and \( \nu \) over both the bounce orbit motion and over the phase-space resonance surfaces.
weighted by their relative contribution to driving the mode.

In a tokamak, these averages depend on details of the equilibrium, eigenmode properties, and fast ion distribution, and thus must be performed numerically with a presently labor-intensive workflow. In Refs. [6] and [11], α and ν were evaluated in this way for several shots from NSTX, DIII-D, TFTR and ITER, yielding much more accurate values than could be obtained from the simple parametric scaling relation. There, it was found that different eigenmodes can have substantially different values of α and ν, even within the same time of a discharge. Turbulence enters the calculation via an enhanced diffusivity, utilizing scaling laws arising from gyrokinetic simulations [12, 13]. Whereas the inclusion of turbulence significantly decreases the value of α/ν in conventional tokamaks, to the point of suppressing non-steady behavior, it hardly affects the calculation of α/ν in NSTX, as shown in Fig. 3 of Ref. [5], due to the fact that ion transport is typically near neoclassical levels in spherical tokamaks [14]. Consequently, it can be concluded that configurations with reduced turbulence, such as spherical tokamaks and negative triangularity plasmas, are more susceptible to the influence of drag and more closely adhere to the scaling expression for α/ν derived without turbulent effects. Quantitatively reliable calculations of this ratio require case-by-case calculation, even for a given tokamak. One situation where values obtained from the heuristic expression alone can be considered more reliable is a linear device, such as LAPD [15–17], where the simpler geometry leads to less complicated resonance surfaces (in addition to better control over the level of turbulent transport), such that careful averaging is not as essential as in a tokamak. Due to much lower plasma temperatures, laboratory plasmas such as those studied in LAPD are a promising candidate for experimentally probing the influence of drag in a controlled setting, with α/ν ≥ 5. The shifting and splitting of the resonance function can in principle be detected by laser induced florescence measurements of the perturbed distribution in a set-up similar to that of Ref. [18] for the case of resonant ions or using whistler mode wave absorption for resonant electrons [19].

VI. NONLINEAR THEORY IN THE TIME-LOCAL REGIME

When the rate of scattering is much larger than the growth rate of the mode (ν ≫ γ), the evolution of the bump-on-tail problem becomes local in time, for any value of α/ν. In general, the evolution of the complex mode amplitude is given by the paradigmatic Berk-

Breizman cubic equation

\[
\frac{dA(\tau)}{d\tau} = A(\tau) - \frac{1}{2} \int_0^{\tau/2} dz z^2 A(\tau - z) \int_0^{\tau-2z} dx e^{-\nu z^2 (2z^3 + z + x)} A(\tau - z - x) A^*(\tau - 2z - x). \tag{20}
\]

For compactness, normalized variables were introduced, with A(\tau) = (|\omega^2_0(\tau)/|\gamma|^2)\sqrt{1 - \gamma_d/\gamma_{L,0}}, γ = γ_{L,0} - γ_d, τ = t_γ, \nu = \nu/γ, and \alpha = \alpha/γ. When \nu ≫ 1, the integrand in Eq. (20) is vanishingly small except near x = 0 and z \approx 1/\nu ≪ 1. In this case, the time delays present in the arguments of the mode amplitude become negligible, such that the product A(\tau - z)A(\tau - z - x)A^*(\tau - 2z - x) ≈ A(\tau)|A(\tau)|^2 can be pulled out of the integral. This approximation reduces the time-delayed integro-differential equation for the mode amplitude into an ordinary differential equation, dubbed the time-local cubic equation since it makes explicit the irrelevance of the system’s history in determining its future evolution:

\[
\frac{dA(\tau)}{d\tau} = A(\tau) - b(\hat{\alpha}, \hat{\nu}) A(\tau)|A(\tau)|^2, \tag{21}
\]

\[
b(\hat{\alpha}, \hat{\nu}) = \frac{1}{2\hat{\nu}^2} \int_0^\infty e^{-2\alpha^2/z + i\alpha^2 u/\nu}/(1 + i\alpha^2/\nu^2) \, du. \tag{22}
\]

A complete analytic solution for the time-local cubic equation was derived in Ref. [8]. Of relevance here is its asymptotic behavior in the limit of τ → ∞, which gives regular oscillations at a fixed amplitude

\[
\lim_{\tau \to \infty} A(\tau) = A_{\text{sat}} e^{-i \delta \omega_{\text{sat}} \tau/\gamma} = \frac{e^{-i(\text{Im}[b]/\text{Re}[b])\tau}}{\sqrt{\text{Re}[b]}} \tag{23}
\]

Hence the collisional constant b contains information both on the saturation amplitude of the wave and drag-induced frequency shift δω_{sat} (not discussed in the Letter). \text{A}_{\text{sat}} = 1/\sqrt{|\text{Re}[b]|} gives the wave saturation amplitude for all values of α/ν < 0.96. An equivalent expression for \text{A}_{\text{sat}} was first derived by Lilley [20]. Its expansion when α ≪ ν is given by Eq. 26 of Ref. [8], which matches the saturation of the quasilinear system (Eq. 12) derived in the Letter exactly. The quasilinear system in the Letter emerges from the nonlinear system with two key assumptions: 1) |\omega^2_0| ≪ \nu^2, used to expand the Vlasov equation in orders of the mode amplitude and 2) the “time-local” approximation, enabling the time delays to be ignored in deriving the perturbed distribution function. The constraints implied by these assumptions will be outlined here. Define a new function I^{-2}(α/ν) = \hat{\nu}^2\text{Re}[b], which isolates the dependence of the saturation level on the ratio \alpha/\nu. Then in order for |\omega^2_0| ≪ \nu^2 to be valid through mode saturation, one must require that

\[
1 - \gamma_d/\gamma_{L,0} \ll \frac{1}{I^2(\alpha/\nu)}. \tag{24}
\]
Since $A_{\text{sat}}$ (and therefore, $I^{-2}$) is an increasing function of $\alpha/\nu$, this shows that larger values of $\alpha/\nu$ require the system to be closer to marginal stability for the expansion to hold at all times. As points of reference, Eq. (24) requires $1 - \gamma_d/\gamma_{L,0} \approx 0.51$ in the absence of drag, $1 - \gamma_d/\gamma_{L,0} \approx 0.33$ for $\alpha/\nu = 0.5$, and $1 - \gamma_d/\gamma_{L,0} \approx 0.03$ for $\alpha/\nu = 0.9$. For any value of $\alpha/\nu < 0.96$, there exist sufficiently small values of $1 - \gamma_d/\gamma_{L,0} \ll 1$ to satisfy Eq. (24). Regarding the time-local assumption, neglecting the time delays is justified when $A$ does not change much between $\tau$ and $\tau - \delta$, where $\delta$ is a general time delay. Quantitatively, one must require $|A(\tau - \delta/\nu) - A(\tau)|/|A(\tau - \delta/\nu)| \ll 1$, which can be rearranged for the maximum relevant time delay as $|A'(\tau)|/|A(\tau)| \ll \dot{\nu}/3$. Writing $A(\tau) = |A(\tau)| e^{i\phi(\tau)}$, this condition becomes $\sqrt{(|A'|||A|^2) + (\phi')^2} \ll \dot{\nu}/3$. Rigorous arguments to bound $|A'|||A|$ and $|\phi'|$ can be found in Appendix A of Ref. [8], which is beyond the scope of this Supplemental Material. In a nutshell, $|A'|||A| < 1$ because the fastest growth of the mode occurs during the initial linear phase, and $|\phi'| < \text{Im}[b]/\text{Re}[b]$ because the fastest variation of the phase occurs in the saturation phase. Combination of these bounds gives

$$3\sqrt{1 + \frac{\text{Im}[b]^2}{\text{Re}[b]^2}} \ll \frac{\nu}{\gamma}. \tag{25}$$

Since $\text{Im}[b]/\text{Re}[b]$ is an increasing function of $\alpha/\nu$, Eq. (25) demonstrates that larger values of $\alpha/\nu$ require relatively larger $\nu/\gamma$ in order to justify the time-local assumption. With only scattering, the condition becomes $\dot{\nu} \gg 3$. When $\alpha/\nu \approx 0.6$, the two terms under the radical are equal, and Eq. (25) requires $\dot{\nu} \gg 4.2$. As $\alpha$ approaches $\nu$, the constraint becomes more restrictive, with $\alpha/\nu = 0.9$ requiring $\dot{\nu} \gg 22$.

VII. SENSITIVITY TO THE MARGINALITY PARAMETER $\gamma_d/\gamma_{L,0}$

In this section, we document a scan of the marginality parameter $1 - \gamma_d/\gamma_{L,0}$, which is assumed to be small in this work. This scan is shown in Fig. 1. There, simulations are performed using the BOT code varying the marginality parameter while keeping the collisional constants $\nu/(\gamma_{L,0} - \gamma_d) = 20$ and $\alpha/(\gamma_{L,0} - \gamma_d) = 10$ fixed. The theoretical prediction using a small parameter expansion in $1 - \gamma_d/\gamma_{L,0} \ll 1$ is overlayed as the dashed black curve. In this case, there is hardly any discrepancy between theory and simulations until $\gamma_d/\gamma_{L,0} \leq 0.84$. Even then, the shape of $\delta f$ is still very similar, with its magnitude requiring higher order corrections in the small parameter for accuracy. Note that this value of $\gamma_d/\gamma_{L,0} \approx 0.8$ is not a universal parameter where the theory is expected to break down. As described in Sec. VI, the quantitative requirement on $\gamma_d/\gamma_{L,0}$ for the theory to remain valid is itself a function of $\alpha/\nu$ and $\nu/\gamma$. Specifically, for $\nu/\gamma = 20$ and $\alpha/\nu = 0.5$, Eqs. (24) and (25) require $1 - \gamma_d/\gamma_{L,0} \ll 0.33$ and $\nu/\gamma \gg 3.7$, respectively, for the theory to be valid. Hence it is consistent that the simulations and theory begin to disagree when $1 - \gamma_d/\gamma_{L,0} \gtrsim 0.2$ for these parameters.

VIII. SENSITIVITY TO THE COLLISIONALITY PARAMETER $\nu/\gamma$

In this section, we document a scan of the collisionality parameter $\nu/\gamma$, which is assumed to be large in this work. BOT simulations are performed with fixed values of $\gamma_d/\gamma_{L,0} = 0.99$ and $\alpha/\nu = 0.5$, while $\nu/\gamma$ is varied from 1.25 to 20. The perturbed distribution $\delta f$ at saturation in the simulations is shown in Fig. 2(a) for $\nu/\gamma = 2.5 - 20$. The analytic expression for $\delta f$ in the limit of $\nu/\gamma \gg 1$ is overlayed as the dashed black curve. Note that the plotted quantity $\delta f$ is normalized by the squared mode amplitude (proportional to $|\omega_0^2|^2 \propto \nu^4$) in order to facilitate comparison on the same scale. Good quantitative agreement is found for $\nu/\gamma \gtrsim 10$, with differences appearing for $\nu/\gamma \lesssim 5$. This is consistent with the theoretical formalism, which involves the implicit quantitative assumption of $\nu/\gamma \gg 3.7$ for these parameters (this threshold value depends on $\alpha/\nu$, as prescribed by Eq. (25)).

The corresponding time evolution of the mode amplitude is shown in Fig. 2(b), extending down to $\nu/\gamma = 1.25$. Consequently, this parameter scan includes a transition from the collisional regime where orbits are stochastic ($|\omega_0^2| \ll \nu^2$) to the strongly nonlinear regime where the resonant island structure is preserved ($|\omega_0^2| \gg \nu^2$). To convert from the plotted quantity to the normalized collisional parameter $|\omega_0^2|/\nu^2$, one would multiply by $\gamma^2/\nu^2(1 - \gamma_d/\gamma_{L,0})^2$. For sufficiently large values of $\nu/\gamma$, the linear growth phase is followed almost immediately by saturation at a steady level, with the collisional parameter remaining small ($|\omega_0^2|/\nu^2 \approx 0.18$ at saturation). As $\nu/\gamma$ is lowered, the saturation level
\[ \left| \frac{\omega_b^2}{\gamma_{L,0}^2} \right| \text{ decreases since } |\omega_{0,\text{sat}}| \propto \nu \text{ at constant } \alpha/\nu, \text{ as in Eq. (12) of the Letter}, \text{ and decaying oscillations occur about the saturation level. At very small values of } \nu/\gamma, \text{ the initial growth phase transitions from the stochastic to strongly nonlinear regime, and consequently the quasi-steady saturation seen in simulations with larger } \nu/\gamma \text{ is replaced by pulsating solutions. Since no steady state is achieved in these simulations due to the pulsations, the } \delta f \text{ can not be compared in Fig. 2(a) against the other simulations. To reiterate, the focus of this work is on near threshold instabilities } (1 - \gamma_d/\gamma_{L,0} \ll 1) \text{ that are expected to be in the collisional regime since } \nu/\gamma \text{ is typically a large parameter when } \gamma \text{ is so small. However, the results are also valid in the early phase of the instabilities that will eventually enter the strongly nonlinear regime, up to the point where } |\omega_b^2| \ll \nu^2 \text{ is no longer satisfied, which is where the near threshold ordering breaks down.} \]

**IX. CONCLUSIONS AND OUTLOOK**

This work has identified several ways in which the presence of collisional drag fundamentally modifies the character of resonant wave-particle interactions in kinetic systems and the associated transport. It has been shown that near marginal stability and with sufficiently large effective scattering, nonlinear theory naturally reduces to QL theory even when coherence-preserving dynamical friction is included. This finding provides further physical justification for the use of QL models, such as the resonance broadened quasilinear code [21], for predicting fast ion transport due to Alfvén eigenmodes in otherwise computationally intensive burning plasma scenarios. Of equal importance, an isomorphism between the collisional bump-on-tail problem in plasmas, studied in this work, and the resonant interaction of the rotating galactic bar with orbit heavy bodies [22] allows these results and techniques to be applied to the distinct context of galactic dynamics. The strength of the wave-particle interaction is characterized by a collisionally-broadened resonance function. The inclusion of drag breaks a symmetry otherwise present in the system, shifting the resonance function from its usual location in phase space. When drag dominates, the resonance function splits into multiple peaks of similar magnitude. In particular, the particle redistribution is found to be very sensitive to even small amounts of drag, indicating the importance of including it in realistic models of resonant particle transport.
W. Wang, *Nucl. Fusion* **47**, 499 (2007).

[15] Y. Zhang, W. W. Heidbrink, H. Boehmer, R. McWilliams, G. Chen, B. N. Breizman, S. Vincena, T. Carter, D. Leneman, W. Gekelman, P. Pribyl, and B. Brugman, *Phys. Plasmas* **15**, 012103 (2008).

[16] W. W. Heidbrink, H. Boehmer, R. McWilliams, A. Preiwisch, Y. Zhang, L. Zhao, S. Zhou, A. Bovet, A. Fasoli, I. Furno, K. Gustafson, P. Ricci, T. Carter, D. Leneman, S. K. P. Tripathi, and S. Vincena, *Plasma Phys. Control. Fusion* **54**, 124007 (2012).

[17] W. Gekelman, P. Pribyl, Z. Lucky, M. Drandell, D. Leneman, J. Maggs, S. Vincena, B. Van Compernolle, S. K. P. Tripathi, G. Morales, T. A. Carter, Y. Wang, and T. DeHaas, *Rev. Sci. Instr.* **87**, 025105 (2016).

[18] M. Sarfaty, S. De Souza-Machado, and F. Skiff, *Phys. Rev. Lett.* **80**, 3252 (1998).

[19] J. W. R. Schroeder, F. Skiff, G. G. Howes, C. A. Kletzing, T. A. Carter, and S. Dorfman, *Phys. Plasmas* **24**, 032902 (2017).

[20] M. K. Lilley, B. N. Breizman, and S. E. Sharapov, *Phys. Rev. Lett.* **102** (2009).

[21] N. Gorelenkov, V. Duarte, M. Podestà, and H. Berk, *Nucl. Fusion* **58**, 082016 (2018).

[22] C. Hamilton, E. Tolman, L. Arzamasskiy, and V. Duarte, *Submitted to Astrophys. J.* (2022).