Hierarchies of the 4 Texture Zero
Quark Mass Matrices and their equal
spacing rule

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Abstract

We show that the parameters of the quark mass matrices $M_u$ and $M_d$ of a 4 texture zero model, exhibit an interesting hierarchical regularity we have called equal spacing rule, and that this regularity leads us to a further observation that the quark mass matrices $(M_u, M_d)$ appear to exist in a plurality of states, each state labelled by two numbers $(N, k)$. We give interpretation to these observations.

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1 Introduction

The 4 texture zero hermitian quark mass matrix [1-12] continues to present interesting study, first with respect to the extent the model reproduces experimental data, and secondly the extent any such agreement with data is controlled by the hierarchies, including geometric hierarchy, of the various parameters of the texture zero quark mass matrices \( M_u \) and \( M_d \). We focus in the present article on the latter problem.

To reach the hierarchy problem we have first to state in some detail the formalism of the 4 texture zero hermitian mass matrix. This we do in section 2. In section 3 we adopt some approximations that simplify our formalism, but enable us to calculate analytically, the various parameters of our mass matrices \( M_u \) and \( M_d \). Armed with the parameter values, we proceed in section 4 to discuss the hierarchies of these parameters, and discover the equal spacing rule that underlines the good agreement the 4 texture zero model has with experimental data. In section 5 we consider the possible interpretation of the observed equal spacing rule of the 4 texture zero mass matrices. We state our final results and conclusions in section 6.

2 The 4 Texture zero quark mass matrix

In their most familiar form, the mass matrices of the 4 texture zero hermitian model are given by:

\[
M_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u^* & D_u & B_u \\ 0 & B_u^* & C_u \end{pmatrix}; \quad M_d = \begin{pmatrix} 0 & A_d & 0 \\ A_d^* & D_d & B_d \\ 0 & B_d^* & C_d \end{pmatrix}
\]

where \( D_u, D_d \) are new non-zero elements in the (2,2) position, not possessed by the original 6 texture zero quark mass matrix model of Fritzsch [13,14]. The claim in the literature [1-12] is that this 4 texture zero model endowed with the extra non-zero (2,2) element, gives better agreement with data than the 6 texture zero model of Fritzsch. Our interest is in seeing to what extent this acclaimed success of the 4 texture zero model is tied up with some intrinsic features, probably of hierarchical nature between mass matrix elements, of the 4 texture zero model.

A standard formalism of analysis of \( M_u \) and \( M_d \) is to first separate out the complex phases of their elements:
$$A_u = |A_u|e^{i\phi_{A_u}}, B_u = |B_u|e^{i\phi_{B_u}};$$

$$A_d = |A_d|e^{i\phi_{A_d}}, B_d = |B_d|e^{i\phi_{B_d}},$$

where the phases are however unknown and need to be determined later. Next one factors out these phases from $M^u$ and $M^d$, and leaves real mass matrices $\hat{M}_u, \hat{M}_d$ defined by: $M_u \rightarrow P_u \hat{M}_u P_u^T$ and $M_d \rightarrow P_d \hat{M}_d P_d^T$, where

$$\hat{M}^u = \begin{pmatrix}
0 & |A_u| & 0 \\
|A_u^*| & D_u & |B_u| \\
0 & |B_u^*| & C_u
\end{pmatrix}; \hat{M}^d = \begin{pmatrix}
0 & |A_d| & 0 \\
|A_d^*| & D_d & |B_d| \\
0 & |B_d^*| & C_d
\end{pmatrix}$$

(2)

and

$$P_u = \begin{pmatrix}
e^{i\alpha_u} & 0 & 0 \\
0 & e^{i\beta_u} & 0 \\
0 & 0 & 1
\end{pmatrix}; P_d = \begin{pmatrix}
e^{i\alpha_d} & 0 & 0 \\
0 & e^{i\beta_d} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

(3)

with the relations:

$\phi_{A_u} = (\alpha_u - \beta_u); \phi_{A_d} = (\alpha_d - \beta_d); \phi_{B_u} = \beta_u; \phi_{B_d} = \beta_d$ so that the product:

$$P = P_u^T P_d$$

(4)

$$= \begin{pmatrix}
e^{i(\alpha_d - \alpha_u)} & 0 & 0 \\
0 & e^{i(\beta_d - \beta_u)} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
e^{i((\phi_{A_d} - \phi_{A_u}) + (\phi_{B_d} - \phi_{B_u}))} & 0 & 0 \\
0 & e^{i(\phi_{A_d} - \phi_{A_u})} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
e^{i\psi} & 0 & 0 \\
0 & e^{i\phi} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

where

$$\psi = (\phi_1 + \phi_2); \phi_1 = (\phi_{A_d} - \phi_{A_u}); \phi_2 = (\phi_{B_d} - \phi_{B_u}) = \phi.$$  

(5)

We next diagonalize $\hat{M}_u$ and $\hat{M}_d$ through the equations: $\hat{M}_{u,d} \rightarrow O^T \hat{M}_{u,d} O = diag(\lambda_1, \lambda_2, \lambda_3)_{u,d}$, where $O$ is some orthogonal matrix whose elements we can calculate using the standard method of eigenvectors. The $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues we can identify with quark masses in the following way: $(\lambda_1, \lambda_2, \lambda_3)_u = (m_u, -m_c, m_t)$ and $(\lambda_1, \lambda_2, \lambda_3)_d = (m_d, -m_s, m_b)$. The reason for the negative sign for $m_c, m_s$ will be explained below.

We invoke similarity transformation invariants to enable us relate the unknown elements $|A|, |B|, D, C$, of the mass matrices to the mass eigenvalues $\lambda_i$. Because there are only three invariant conditions while there are four unknown parameters in this 4 texture zero model, we can remove at most only three of the four parameters, leaving one unknown parameter we choose
as $C$. Then the above parameters are determined for now as:

$$|A_u| = \sqrt{\frac{-\lambda_1 \lambda_2 \lambda_3}{C_u}} = \sqrt{\frac{m_u m_s m_t}{C_u}}$$

$$|B_u| = \sqrt{\frac{(C_u - \lambda_1)(C_u - \lambda_2)(\lambda_3 - C_u)}{C_u}} = \sqrt{\frac{(C_u - m_u)(C_u + m_c)(m_t - C_u)}{C_u}}$$

$$D_u = \lambda_1 + \lambda_2 + \lambda_3 - C_u = m_u - m_c + m_t - C_u$$

$$|A_d| = \sqrt{\frac{-\lambda_1 \lambda_2 \lambda_3}{C_d}} = \sqrt{\frac{m_d m_s m_b}{C_d}}$$

$$|B_d| = \sqrt{\frac{(C_d - \lambda_1)(C_d - \lambda_2)(\lambda_3 - C_d)}{C_d}} = \sqrt{\frac{(C_d - m_d)(C_d + m_s)(m_b - C_d)}{C_d}}$$

$$D_d = \lambda_1 + \lambda_2 + \lambda_3 - C_d = m_d - m_s + m_b - C_d$$

We note the negative sign under the square root $\lambda$ expression for $|A_u|$ and $|A_d|$. This requires that one of the $\lambda_1, \lambda_2, \lambda_3$ eigenvalues must be negative so that $|A|$ can be a real parameter like $C$. We can achieve this by putting $(\lambda_1, \lambda_2, \lambda_3) = (m_t, -m_2, m_3)$. But other choices are possible, a fact Fritzsch and Xing [1] have tried to deal with by introducing an $\eta = \pm 1$ sign convention. We have not pursued that line of elaboration here.

Upon setting up the eigenvector equations of our new orthogonal matrices, we determine their normalized eigenvectors to be:

$$\vec{X}_1^u = \begin{pmatrix} x_1^u \\ y_1^u \\ z_1^u \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_3(C_u - \lambda_1)}{C_u(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}} \\ \sqrt{\frac{\lambda_1(1-C_u)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}} \\ \sqrt{\frac{(C_u-C_2)(C_u-C_1)}{C_u(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m_u m_1(C_u-m_u)}{C_u(m_u-m_u)(m_t-m_u)}} \\ \sqrt{\frac{m_u(C_u-m_u)}{(m_u+m_u)(m_u-m_u)}} \\ \sqrt{\frac{m_u(C_u+m_u)(m_t-m_u)}{C_u(m_u+m_u)(m_t-m_u)}} \end{pmatrix}$$

(7)

$$\vec{X}_2^u = \begin{pmatrix} x_2^u \\ y_2^u \\ z_2^u \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_1 \lambda_2 (C_u - \lambda_2)}{C_u(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_2 (C_u - \lambda_2)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{(C_u-C_2)(C_u-C_1)}{C_u(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m_u m_1(C_u-m_u)}{C_u(m_u-m_u)(m_t-m_u)}} \\ \sqrt{\frac{m_u(C_u-m_u)}{(m_u+m_u)(m_u-m_u)}} \\ \sqrt{\frac{m_u(C_u+m_u)(m_t-m_u)}{C_u(m_u+m_u)(m_t-m_u)}} \end{pmatrix}$$

(8)
\[
\tilde{X}_3^u = \begin{pmatrix} x_1^u \\ x_2^u \\ x_3^u \\ y_1^u \\ y_2^u \\ y_3^u \\ z_1^u \\ z_2^u \\ z_3^u \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_1 \lambda_2 (C_u - \lambda_1)}{C_u (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_2 (C_u - \lambda_2)}{(\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_1 (C_u - \lambda_1)}{C_u (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m_u m_c (m_1 - C_u)}{m_1 (m_1 - m_u) (m_1 + m_c)}} \\ \sqrt{m_1 (m_1 - C_u)} \\ \sqrt{m_1 (C_u - m_u) (C_u + m_c)} \end{pmatrix}
\]
whence we determine \( O_u \) as:

\[
O_u = \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix} = \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix} \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix} \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix}
\]
or

\[
O_u = \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix} = \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix} \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix} \begin{pmatrix} \lambda_1 \lambda_3 (C_u - \lambda_1) \\ \lambda_1 \lambda_2 (C_u - \lambda_2) \\ \lambda_1 \lambda_3 (C_u - \lambda_3) \end{pmatrix}
\]

Similarly we have for the \( dsb \) sector:

\[
\tilde{X}_1^d = \begin{pmatrix} x_1^d \\ x_2^d \\ x_3^d \\ y_1^d \\ y_2^d \\ y_3^d \\ z_1^d \\ z_2^d \\ z_3^d \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_2 \lambda_3 (C_d - \lambda_1)}{C_d (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_1)}} \\ \sqrt{\frac{\lambda_1 (C_d - \lambda_1)}{(\lambda_2 - \lambda_1) (\lambda_3 - \lambda_1)}} \\ \sqrt{\frac{\lambda_3 (C_d - \lambda_3)}{C_d (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_1)}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m_s m_d (m_2 - C_d)}{m_2 (m_2 - m_s) (m_2 + m_d)}} \\ \sqrt{m_2 (m_2 - C_d)} \\ \sqrt{m_2 (C_d - m_s) (m_d - C_d)} \end{pmatrix}
\]
\[ \hat{X}^d_2 = \begin{pmatrix} x^d_2 \\ y^d_2 \\ z^d_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_1 \lambda_2 (\lambda_2 - \lambda_1)}{C_d (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_2 (\lambda_2 - \lambda_2)}{(\lambda_2 - \lambda_2) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\Lambda_3 (C_d - \lambda_1) (\lambda_3 - \lambda_2)}{C_d (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_2)}} \end{pmatrix} \]  

\[ \hat{X}^d_3 = \begin{pmatrix} x^d_3 \\ y^d_3 \\ z^d_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_1 \lambda_2 (C_d - \lambda_1)}{C_d (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_2 (\lambda_3 - C_d)}{(\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\Lambda_3 (C_d - \lambda_3)}{C_d (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \end{pmatrix} \]  

whence we determine \( O_d \) as:

\[ O_d = \begin{pmatrix} x^d_1 \\ x^d_2 \\ x^d_3 \\ y^d_1 \\ y^d_2 \\ y^d_3 \\ z^d_1 \\ z^d_2 \\ z^d_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\lambda_1 \lambda_2 (C_d - \lambda_1)}{C_d (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_1 \lambda_3 (\lambda_2 - C_d)}{C_d (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_1 \lambda_2 (C_d - \lambda_3)}{C_d (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_1 (\lambda_3 - C_d)}{(\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_2 (\lambda_2 - \lambda_1)}{(\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \\ \sqrt{\frac{\lambda_2 (C_d - \lambda_2)}{(\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}} \end{pmatrix} \]  

or

\[ O_d = \begin{pmatrix} x^d_1 \\ x^d_2 \\ x^d_3 \\ y^d_1 \\ y^d_2 \\ y^d_3 \\ z^d_1 \\ z^d_2 \\ z^d_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m_d m_s (C_d - m_d)}{C_d (m_s - m_d) (m_s + m_d)}} \\ \sqrt{\frac{m_d m_s (m_s + C_d)}{C_d (m_s + m_d) (m_s + m_s)}} \\ \sqrt{\frac{m_d m_s (m_s - C_d)}{C_d (m_s - m_d) (m_s + m_s)}} \\ \sqrt{\frac{m_s (C_d - m_d)}{(m_s + m_d) (m_s + m_s)}} \\ \sqrt{\frac{m_s (m_s + C_d)}{(m_s + m_d) (m_s + m_s)}} \\ \sqrt{\frac{m_s (m_s - C_d)}{(m_s - m_d) (m_s + m_s)}} \end{pmatrix} \]  

Given now \( O_u \) and \( O_d \), we compute elements of \( V_{CKM} \) predicted by the four texture zero model. We have:

\[ V_{CKM} = O_u^t O_d = O_u^t P_u P_d O_d = O_u^t P O_d \]  

(17)
where $P = P_d^d P_d$ was given earlier in equation (4).

The various elements of $V_{CKM}$ are thereafter given by:

\[
\begin{align*}
V_{ud} &= x_1^d x_1^e^i \phi + y_1^d y_1^e^i \phi + z_1^d z_1^e^i \\
V_{us} &= x_1^d x_2^e^i \phi + y_1^d y_2^e^i \phi + z_1^d z_2^e^i \\
V_{ub} &= x_1^d x_3^e^i \phi + y_1^d y_3^e^i \phi + z_1^d z_3^e^i \\
V_{cd} &= x_2^d x_1^e^i \phi + y_2^d y_1^e^i \phi + z_2^d z_1^e^i \\
V_{cs} &= x_2^d x_2^e^i \phi + y_2^d y_2^e^i \phi + z_2^d z_2^e^i \\
V_{cb} &= x_2^d x_3^e^i \phi + y_2^d y_3^e^i \phi + z_2^d z_3^e^i \\
V_{td} &= x_3^d x_1^e^i \phi + y_3^d y_1^e^i \phi + z_3^d z_1^e^i \\
V_{ts} &= x_3^d x_2^e^i \phi + y_3^d y_2^e^i \phi + z_3^d z_2^e^i \\
V_{tb} &= x_3^d x_3^e^i \phi + y_3^d y_3^e^i \phi + z_3^d z_3^e^i
\end{align*}
\]

Explicitly we obtain:

\[
\begin{align*}
V_{ud} &= e^{i \phi} \sqrt{\frac{m_c m_t (C_u - m_u)}{C_u (m_c - m_u) (m_t - m_u)}} - e^{i \phi} \sqrt{\frac{m_u (C_u - m_u)}{(m_c + m_u) (m_t - m_u)}} - \sqrt{\frac{m_u (C_u + m_c) (m_t - C_u)}{C_u (m_c + m_u) (m_t - m_u)}} \\
V_{us} &= e^{i \phi} \sqrt{\frac{m_c m_t (C_u - m_u)}{C_u (m_c - m_u) (m_t - m_u)}} - e^{i \phi} \sqrt{\frac{m_u (C_u - m_u)}{(m_c + m_u) (m_t - m_u)}} - \sqrt{\frac{m_u (C_u + m_c) (m_t - C_u)}{C_u (m_c + m_u) (m_t - m_u)}} \\
V_{ub} &= e^{i \phi} \sqrt{\frac{m_c m_t (C_u - m_u)}{C_u (m_c - m_u) (m_t - m_u)}} - e^{i \phi} \sqrt{\frac{m_u (C_u - m_u)}{(m_c + m_u) (m_t - m_u)}} - \sqrt{\frac{m_u (C_u + m_c) (m_t - C_u)}{C_u (m_c + m_u) (m_t - m_u)}}
\end{align*}
\]
\[ V_{cd} = e^{i\nu} \left[ \frac{m_u m_t (m_c + C_u)}{C_u (m_c + m_u) (m_t + m_c)} + \frac{m_d m_b (C_d - m_d)}{C_d (m_s - m_d) (m_b - m_d)} \right] \]
\[ + e^{i\phi} \left[ \frac{m_c (C_u + m_c)}{(m_c + m_u) (m_t + m_c)} + \frac{m_d (C_d - m_d)}{(m_s + m_d) (m_b - m_d)} \right] \]
\[ + \frac{m_c (m_c - m_u) (m_t - C_u)}{C_u (m_c + m_u) (m_t + m_c)} \sqrt{C_d} \frac{m_d (C_d + m_s) (m_b - C_d)}{C_d (m_s + m_d) (m_b - m_d)} \]
\[ (22) \]

\[ V_{cs} = e^{i\nu} \left[ \frac{m_u m_t (m_c + C_u)}{C_u (m_c + m_u) (m_t + m_c)} + \frac{m_d m_b (m_s + C_d)}{C_d (m_s + m_d) (m_b + m_s)} \right] \]
\[ + e^{i\phi} \left[ \frac{m_c (C_u + m_c)}{(m_c + m_u) (m_t + m_c)} + \frac{m_s (C_d + m_s)}{(m_s + m_d) (m_b + m_s)} \right] \]
\[ + \frac{m_c (C_u - m_u) (m_t - C_u)}{C_u (m_c + m_u) (m_t + m_c)} \sqrt{C_d} \frac{m_s (C_d - m_d) (m_b - C_d)}{C_d (m_s + m_d) (m_b - m_d)} \]
\[ (23) \]

\[ V_{cb} = e^{i\nu} \left[ \frac{m_u m_t (m_c + C_u)}{C_u (m_c + m_u) (m_t + m_c)} + \frac{m_d m_s (m_b - C_d)}{C_d (m_b - m_d) (m_b + m_s)} \right] \]
\[ + e^{i\phi} \left[ \frac{m_c (C_u + m_c)}{(m_c + m_u) (m_t + m_c)} + \frac{m_b (m_b - C_d)}{(m_b - m_d) (m_b + m_s)} \right] \]
\[ + \frac{m_c (C_u - m_u) (m_t - C_u)}{C_u (m_c + m_u) (m_t + m_c)} \sqrt{C_d} \frac{m_b (C_d - m_d) (C_d + m_s)}{C_d (m_b - m_d) (m_b + m_s)} \]
\[ (24) \]

\[ V_{td} = e^{i\nu} \left[ \frac{m_u m_c (m_t - C_u)}{C_u (m_t - m_u) (m_t + m_c)} + \frac{m_s m_b (C_d - m_d)}{C_d (m_s - m_d) (m_b - m_d)} \right] \]
\[ + e^{i\phi} \left[ \frac{m_t (m_t - C_u)}{(m_t - m_u) (m_t + m_c)} + \frac{m_d (C_d - m_d)}{(m_s + m_d) (m_b - m_d)} \right] \]
\[ + \frac{m_t (C_u - m_u) (C_u + m_c)}{C_u (m_t - m_u) (m_t + m_c)} \sqrt{C_d} \frac{m_d (C_d + m_s) (m_b - C_d)}{C_d (m_s + m_d) (m_b - m_d)} \]
\[ (25) \]

\[ V_{ts} = e^{i\nu} \left[ \frac{m_u m_c (m_t - C_u)}{C_u (m_t - m_u) (m_t + m_c)} + \frac{m_d m_b (m_s + C_d)}{C_d (m_s + m_d) (m_b + m_s)} \right] \]
We adopt at this stage, a minimal set of reasonable assumptions designed to simplify the various $|V_{ij}|$ expressions above. We invoke the known hierarchy of quark masses: $m_t \gg m_c \gg m_u; m_b \gg m_s \gg m_d$; also we assume $C_u \gg m_c \gg m_u; C_d \gg m_s \gg m_d$, but leave the ratios $C_u/m_t, C_d/m_b$ as quantities we need to specifically calculate and determine. With these approximations the above $V_{ij}$ quantities become:

$$V_{ud} \approx e^{i\psi} + e^{i\phi} \sqrt{\frac{m_um_d}{m_cm_s}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + \sqrt{\frac{m_um_d}{m_cm_s}} \sqrt{(1 - \frac{C_u}{m_t})} \sqrt{(1 - \frac{C_d}{m_b})} \tag{28}$$

### 3 Approximations and numerical Analysis of the 4 texture zero model

We note that unlike the 6 texture zero case where each element $|V_{ij}|$ is expressed entirely in terms of the known quark masses (and two phase angles), in the present case of 4 texture zero the unknown parameters $C_u$ and $C_d$ appear in every element $|V_{ij}|$, plus the same two phase angles $\psi$ and $\phi$. These parameters can play the role of adjustable parameters needed to accommodate a range of measured flavor mixing. More generally the 4 texture zero model with the above feature allows us to regard $C_u, C_d$ as well as all the other parameters $|A|, |B|, D$, linked to $C_u, C_d$ by equation (6), as adjustable parameters, a feature the 6 texture zero model did not have. Armed with this feature we can analyze the 4 texture zero model further and determine directly the magnitudes and hierarchies of these parameters, supported by experimental data.
or

\[ |V_{ud}| \approx \left| 1 + e^{-i\phi_1} \sqrt{\frac{m_u m_d}{m_c m_s}} \frac{C_u}{m_t} \frac{C_d}{m_b} + e^{-i\psi} \sqrt{\frac{m_u m_d}{m_c m_s}} \sqrt{(1 - \frac{C_u}{m_t}) (1 - \frac{C_d}{m_b})} \right| \]

\[ \approx 1 \]  \hspace{1cm} (29)\\

\[ V_{us} \approx e^{i\psi} \sqrt{\frac{m_d}{m_s}} + e^{i\phi} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{C_u}{m_t} \frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_u}{m_c}} \sqrt{(1 - \frac{C_u}{m_t}) (1 - \frac{C_d}{m_b})} \]

or

\[ |V_{us}| \approx \left| \sqrt{\frac{m_d}{m_s}} + e^{-i\phi_1} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_u}{m_c}} \sqrt{(1 - \frac{C_u}{m_t}) (1 - \frac{C_d}{m_b})} \right| \]

\[ \approx \frac{\sqrt{m_d m_s}}{m_b C_d} (1 - \frac{C_d}{m_b}) + e^{i\phi} \sqrt{\frac{m_c}{m_s}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_c}{m_s}} \sqrt{(1 - \frac{C_u}{m_t}) (1 - \frac{C_d}{m_b})} \]  \hspace{1cm} (30)\\

\[ V_{ab} \approx e^{i\psi} \frac{m_d m_s}{m_b C_d} (1 - \frac{C_d}{m_b}) + e^{i\phi} \sqrt{\frac{m_u}{m_s}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_u}{m_s}} \sqrt{(1 - \frac{C_u}{m_t}) (1 - \frac{C_d}{m_b})} \]

or

\[ |V_{ab}| \approx \left| \frac{m_d m_s}{m_b C_d} (1 - \frac{C_d}{m_b}) + e^{-i\phi_1} \sqrt{\frac{m_u}{m_s}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_u}{m_s}} \sqrt{(1 - \frac{C_u}{m_t}) (1 - \frac{C_d}{m_b})} \right| \]

\[ \approx \frac{\sqrt{m_u m_s}}{m_c} \sqrt{(1 - \frac{C_d}{m_b}) (1 - \frac{C_u}{m_t})} \]  \hspace{1cm} (31)\\

\[ V_{cd} \approx e^{i\psi} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_d}{m_s}} \sqrt{(1 - \frac{C_d}{m_b}) (1 - \frac{C_u}{m_t})} \]

or

\[ |V_{cd}| \approx \left| \sqrt{\frac{m_u}{m_c}} + e^{-i\phi_1} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{-i\psi} \sqrt{\frac{m_d}{m_s}} \sqrt{(1 - \frac{C_d}{m_b}) (1 - \frac{C_u}{m_t})} \right| \]

\[ \approx \frac{\sqrt{m_u m_c}}{m_s} \sqrt{(1 - \frac{C_d}{m_b}) (1 - \frac{C_u}{m_t})} \]  \hspace{1cm} (32)
\[ V_{cs} \approx e^{i\psi} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} + e^{i\phi} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + \sqrt{(1 - \frac{C_u}{m_t})} \sqrt{(1 - \frac{C_d}{m_b})} \]  

(36)

or

\[ |V_{cs}| \approx \left| \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} + e^{i\phi_1} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} + e^{-i\phi} \sqrt{(1 - \frac{C_u}{m_t})} \sqrt{(1 - \frac{C_d}{m_b})} \right| \]

\[ \approx \left| \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} \right| \]  

(37)

\[ V_{cb} \approx e^{i\psi} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d m_s}{m_b C_d}} + e^{i\phi} \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} + \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} \]

(38)

or

\[ |V_{cb}| \approx \left| \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d m_s}{m_b C_d}} + e^{-i\phi_1} \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} + e^{-i\psi} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} \right| \]

\[ \approx \left| \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} \right| \]  

(39)

\[ V_{td} \approx e^{i\psi} \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{i\phi} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{m_d}{m_s}} \sqrt{(1 - \frac{C_d}{m_b})} \]

(40)

or

\[ |V_{td}| \approx \left| \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{-i\phi_1} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{-i\psi} \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{m_d}{m_s}} \sqrt{(1 - \frac{C_d}{m_b})} \right| \]

\[ \approx \left| \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{-i\phi_2} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{m_d}{m_s}} \sqrt{(1 - \frac{C_d}{m_b})} \right| \]  

(41)

\[ V_{ts} \approx e^{i\psi} \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{i\phi} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} \]

(42)
or

\[ |V_{ts}| \approx \left| \frac{m_d}{m_s} \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{-i\phi_1} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{-i\psi} \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} \right| \]

\[ \approx \left| \frac{C_d}{m_b} \sqrt{(1 - \frac{C_u}{m_t})} + e^{-i\phi_2} \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} \right| \]

(43)

\[ V_{tb} \approx e^{i\psi} \sqrt{\frac{C_d}{m_b}} \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{i\phi} \sqrt{(1 - \frac{C_u}{m_t})} \sqrt{(1 - \frac{C_d}{m_b})} + \sqrt{\frac{C_u}{m_t}} \frac{C_d}{m_b} \]

or

\[ |V_{tb}| \approx \left| \sqrt{\frac{C_u}{m_t}} \frac{C_d}{m_b} + e^{i\psi} \sqrt{\frac{C_d}{m_b}} \sqrt{\frac{m_u m_c}{m_t C_u}} \sqrt{(1 - \frac{C_u}{m_t})} + e^{i\phi} \sqrt{(1 - \frac{C_u}{m_t})} \sqrt{(1 - \frac{C_d}{m_b})} \right| \]

\[ \approx \sqrt{\frac{C_u}{m_t}} \frac{C_d}{m_b} \]

(44)

Notably, we see in this 4 texture zero model, that many of the \(|V_{ij}|\) quantities owe any non-zero value they may have experimentally to the values we attach to the ratios \(C_u/m_t\) and \(C_d/m_b\), and by implication from equation (6), to the hierarchical values of the rest parameters of \(M_u\) and \(M_d\). Such critical dependence of the predictions of a texture zero model on hierarchies of the parameters of the model while present in the 4 texture zero case as shown above, are not present in the 6 texture zero model. We show this helps explain and illuminate some of the lack of consistency and agreement of the 6 texture zero model with experiment.

Thus we note from our equation (31) that the familiar result for the Cabibbo angle obtained in the 6 texture zero model [13, 14] namely:

\[ |V_{us}| = |\sin \theta_C| \approx \left| \sqrt{\frac{m_d}{m_s}} + e^{-i\phi_1} \sqrt{\frac{m_u}{m_c}} \right| \approx |V_{cd}| \]

(46)

is only reproduced here (in the 4 texture zero formalism), when we assume \(\sqrt{C_u/m_t} = \sqrt{C_d/m_b} = 1\) and \(\sqrt{1 - C_u/m_t} = \sqrt{1 - C_d/m_b} = 0\). But then from our equations (33), (39), (41), (43), the same assumption leads to zero values for \(|V_{ub}|, |V_{cb}|, |V_{td}|\) and \(V_{ts}|\). We see this as a good reason why the 6 texture zero model has difficulty with its other predicted relations, namely:

\[ \frac{|V_{ub}|}{|V_{cb}|} = \sqrt{\frac{m_u}{m_c}} \]

(47)
and
\[
\frac{|V_{td}|}{|V_{ts}|} = \sqrt{\frac{m_d}{m_s}}
\]  \hspace{1cm} (48)

Equations (47) and (48) would appear not intrinsically compatible with equation (46). The only way equations (47) and (48) can arise from our 4 texture zero model, is if we adopt a different approximation for \(V_{ub}\) and \(V_{cb}\) namely :

\[
V_{ub} \approx \frac{m_u}{m_c} \left[ e^{i\phi} \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} + \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} \right]
\]  \hspace{1cm} (49)

and

\[
V_{cb} \approx e^{i\phi} \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} + \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})}
\]  \hspace{1cm} (50)

whence

\[
\frac{|V_{ub}|}{|V_{cb}|} = \sqrt{\frac{m_u}{m_c}}
\]  \hspace{1cm} (51)

as in the 6 texture zero model.

By similarly dropping third order quantities and adopting a different approximation for \(V_{td}\) and \(V_{ts}\) in equations (41-43), namely:

\[
V_{td} \approx \sqrt{\frac{m_d}{m_s}} \left[ e^{i\phi} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})} \right]
\]  \hspace{1cm} (52)

and

\[
V_{ts} \approx e^{i\phi} \sqrt{\frac{C_d}{m_b}} \sqrt{(1 - \frac{C_u}{m_t})} + \sqrt{\frac{C_u}{m_t}} \sqrt{(1 - \frac{C_d}{m_b})}
\]  \hspace{1cm} (53)

we obtain :

\[
\frac{|V_{td}|}{|V_{ts}|} = \sqrt{\frac{m_d}{m_s}}
\]  \hspace{1cm} (54)

found in the 6 texture zero model. The fact remains however that these re-approximations (49)-(53) are not compatible with the approximation leading to equation (46).
3.1 Determining the values of $C_u/m_t, C_d/m_b$

We now take advantage of the intrinsic features of the 4 texture zero model, contained in the several relations above, to determine directly the quantities $C_u/m_t, C_d/m_b$. We use either of the equations:

$$|V_{cs}| = 0.97341 = \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}}$$

(55)

or

$$|V_{tb}| = 0.999135 = \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}}$$

(56)

as our basis to determine $C_u/m_t$ and $C_d/m_b$. We make the simple assumption that the relativity values $C_u/m_t$ and $C_d/m_b$ are the same in the $M_u$ sector as in the $M_d$ sector of texture zero mass matrices. This means we take $C_u/m_t = C_d/m_b$ whence using $|V_{cs}|$ equation (37), we obtain:

$$\sqrt{\frac{C_u}{m_t}} = \sqrt{\frac{C_d}{m_b}} = \sqrt{0.97341} = 0.9866154$$

(57)

Then we deduce that:

$$\sqrt{(1 - \frac{C_u}{m_t})} = \sqrt{(1 - \frac{C_d}{m_d})} = \sqrt{0.02659} = 0.1631.$$ 

(58)

As input data for measured $V_{CKM}$ elements we have taken values quoted in a recent paper by Lenz [15], namely:

$$
\begin{pmatrix}
|V_{ud}| & |V_{us}| & |V_{ub}| \\
|V_{cd}| & |V_{cs}| & |V_{cb}| \\
|V_{td}| & |V_{ts}| & |V_{tb}|
\end{pmatrix} = 
\begin{pmatrix}
0.97426 \pm 0.00030 & 0.22545 \pm 0.00095 & 0.00356 \pm 0.00020 \\
0.22529 \pm 0.00077 & 0.97341 \pm 0.00021 & 0.045085 \pm 0.000057 \\
0.00861 \pm 0.00021 & 0.04068 \pm 0.00138 & 0.999135 \pm 0.000018
\end{pmatrix}
$$

(59)

For quark masses, we shall use the following values, with the observation that the quark masses do in general run because of renormalization group effects which we do not consider here.

$$
m_u = 2.57\text{MeV} \\
m_d = 5.85\text{MeV} \\
m_s = 111.0\text{MeV} \\
m_c(m_c) = 1.25\text{GeV} \\
m_b(m_b) = 5.99\text{GeV} \\
m_t = 173.0\text{GeV}
$$

(60)
3.2 The phase angles $\phi_2$ and $\phi_1$

We can next plug these quantities into the equation

$$|V_{cb}| \approx |C_u| \sqrt{(1 - C_d/m_b)} + e^{-i\phi_2} |C_d/m_b| \sqrt{(1 - C_u/m_t)}$$  \hspace{1cm} (61)$$

or

$$|V_{ts}| \approx |C_d/m_b| \sqrt{(1 - C_u/m_t)} + e^{-i\phi_2} |C_u/m_t| \sqrt{(1 - C_d/m_b)}$$  \hspace{1cm} (62)$$

and determine phase angle $\phi_2$ therefrom. We use the $|V_{cb}|$ equation and insert its measured value of $|V_{cb}| = 0.04508$. We now find the choice of angle $\phi_2$ for which the right hand side yields the same numerical value as the data. We get $\phi_2 = 164^\circ$. If we use the $|V_{ts}|$ equation whose measured value is $|V_{ts}| = 0.04068$ we get $\phi_2 = 165.5^\circ$.

We can next determine the phase angle $\phi_1$ from the two equations:

$$|V_{us}| \approx \sqrt{\frac{m_d}{m_s}} \sqrt{\frac{m_u}{m_c}} \frac{C_u}{C_d}$$  \hspace{1cm} (63)$$

or

$$|V_{cd}| \approx \sqrt{\frac{m_u}{m_s}} \sqrt{\frac{m_d}{m_c}} \frac{C_u}{C_d}$$  \hspace{1cm} (64)$$

We obtain the following values: $\phi_1 = 100.6^\circ$ from $|V_{us}| = 0.22545$, or $\phi_1 = 100.7^\circ$ from $|V_{cd}| = 0.22529$. Phase angle $\psi = \phi_1 + \phi_2$, while angle $\phi = \phi_2$ from equation (5).

3.3 Other parameters of $M_u$ and $M_d$.

Finally we find other parameters of the texture zero mass matrices $M_u$ and $M_d$, by using the set of equations (6):

$$|A_u| = \sqrt{\frac{m_u m_c m_t}{C_u}} = \frac{m_u m_c}{C_u/m_t} = 0.05744$$  \hspace{1cm} (65)$$

$$|B_u| = \sqrt{\frac{(C_u - m_u)(C_u + m_c)(m_t - C_u)}{C_u}} \approx \sqrt{m_t C_u (1 + m_c/C_u)} \frac{C_u}{m_t} (1 - C_u/m_t) = 27.942$$  \hspace{1cm} (66)$$
\[ D_u = m_u - m_c + m_t - C_u = m_t \left( \frac{m_u}{m_t} - \frac{m_c}{m_t} + (1 - \frac{C_u}{m_t}) \right) \approx m_t(1 - \frac{C_u}{m_t}) = 4.60 \] (67)

\[ |A_d| = \sqrt{\frac{m_d m_s m_b}{C_d}} = \sqrt{\frac{m_d m_s}{C_d/m_b}} = 0.02583 \] (68)

\[ |B_d| = \sqrt{\frac{(C_d - m_d)(C_d + m_s)(m_b - C_d)}{C_d}} \approx \sqrt{m_b C_d(1 + \frac{m_s}{C_d}) \sqrt{1 - \frac{C_d}{m_b}}} = 0.9730 \] (69)

\[ D_d = m_d - m_s + m_b - C_d \approx m_b(1 - \frac{C_d}{m_b}) = 0.1592 \] (70)

Then in summary we have for the 4 texture zero mass matrices \( M_u, M_d \) that their parameters have the values (in GeV):

\[ M_u : C_u = 168.39993; |B_u| = 27.942; D_u = 4.60; |A_u| = 0.05744 \]

\[ M_d : C_d = 5.831; |B_d| = 0.9730; D_d = 0.1592; |A_d| = 0.02583 \] (71)

We see that these parameters have a definite hierarchy : \( C_u \gg |B_u| \gg D_u \gg |A_u| \). Also \( C_d \gg |B_d| \gg D_d \gg |A_d| \). Systematically too, the \( M_u \) parameters are larger in value than the corresponding \( M_d \) parameters. But superposed on these features, is a striking hierarchical relationship among the (2,2), (2,3) and (3,3) parameters in each of \( M_u \) and \( M_d \). The relationship is that for each of \( M_u, M_d \), we have : \(|B_q|^2 = C_q D_q\) where \( q = u, d \). This can be verified by our actual parameter values above: \(|B_u|^2 = 27.942^2 = 780.755\) against \( C_u D_u = (168.39993)(4.60) = 774.63\) for \( M_u \). Also \(|B_d|^2 = 0.9730^2 = 0.9467\) against \( C_d D_d = (5.831)(0.1592) = 0.9283\) for \( M_d \). This relationship has been called geometric hierarchy by Fritzsch and Xing [1] who first noticed it. Our model exhibits it closely.

But then, we notice other striking hierarchical feature of our model when we transform our above parameter values into ratio quantities, thus:

\[ |B_u|/C_u = 0.1659; |B_d|/C_d = 0.1668 \]

\[ D_u/|B_u| = 0.1646; D_d/|B_d| = 0.1636 \]

\[ |A_u|/D_u = 0.01248; |A_d|/D_d = 0.1622 \] (72)

We see the ratios appear to obey equal spacing rule that is particularly noticeable in the \( M_d \) case. The ratios of succeeding elements, including the (1,2) or (2,1) element, are the same across the mass matrix. This prompts further examination of the hierarchy issue which we now undertake.
4 The hierarchy question in the 4 texture zero Model

It is clear from the analysis above that the 4 texture zero model as presented is characterized by certain distinctive hierarchies among the parameters of the mass matrices $M_u$ and $M_d$. A logical deduction is that whatever success the model has in explaining experimental data on flavor mixing and flavor physics generally, must have as a factor, this dominant hierarchical state or phase of the model. Then a question arises whether the model has one unique hierarchical state that is in agreement with experiment, or a number of such experimentally allowed hierarchical states or phases. What in the end would such duplicity of allowed hierarchical states of $M_u, M_d$ mean from a symmetry point of view? We find we can provide below some answers and insights into these questions.

Our approach to the problem is to hold as central, the observed equal spacing rule equation (72) found for $M_u$ and $M_d$, and use it to characterize the 4 texture zero model. We assume as reasonably exhibited in equation (72) that the rule applies equally in the $M_u$ sector as in the $M_d$ sector. We denote the value of the equal spacing by a number $k$ where $k = 0.16$ for equation (72) that we now regard as just one of several possible allowed hierarchical states or modes of the 4 texture zero model. We show that similar states of $(M_u, M_d)$ exist that are also dictated by present experimental data, but that each such other state has a different $k$ value that distinguishes it. Thus $k$ is to be seen as playing the role of a quantum number that labels different hierarchical states of $(M_u, M_d)$, where each $k$-state describes experimental data in much the same way as other $k$ labelled quantum states of $M_u$ and $M_d$.

Before discussing $k$ further, we notice that our specification of the parameters of $M_u$ and $M_d$ in equations (65) - (72) needed a prior input data or specification for the quantities $C_u/m_t$ and $C_d/m_b$. This choice and specification we made in equation (55), and this alone enabled us to calculate and fix the parameters $C, |B|, D, |A|$ of the model for both $M_u$ and $M_d$. We can say then that our 4 texture zero model was characterized not only by $k$, but also by another quantum number we denote by $N$ and define as:

$$N = \frac{C_u}{m_t} = \frac{C_d}{m_b}$$

(73)

Given $N$ and $k$, we can specify completely the parameters of $M_u$ and $M_d$. It was a specific choice of $N = N_1 = |V_{cs}| = 0.97341$ in equation (55) that
defined our initial hierarchical state of \((M_u, M_d)\), with its \(k = k_1 = 0.16\) given in equation (72).

We can assert, on the above basis, that the 4 texture zero model through its mass matrices \(M_u\) and \(M_d\), exists in hierarchical quantum states that are described by just two mutually commuting quantum numbers \((N, k)\). The quantum number \(N\) specifies the mass scale of the system while the quantum number \(k\) specifies the rate of fall off of the mass scale with flavor, for given \(N\).

Our assertion is buttressed further by the fact that equation (56) which is another valid experimental data like equation (55), already shows that another choice of \(N\) value is possible given by \(N = N_2 = 0.999135 = |V_{tb}|\). We can calculate the \(k\) number associated with this \(N_2\) and show that \(k = k_2\) so obtained is different from \(k_1\) value of equation (72), thus defining another allowed \((N, k)\) quantum state of \((M_u, M_d)\). We give these details.

### 4.1 The \((N_2, k_2)\) quantum state of \(M_u\) and \(M_d\)

In place of equation (55) for \(N = N_1\) we choose \(N = N_2\) from equation (56) given by:

\[
N = N_2 = |V_{tb}| = 0.999135 = \sqrt{\frac{C_u}{m_t}} \sqrt{\frac{C_d}{m_b}} = \frac{C_u}{m_t} = \frac{C_d}{m_b}
\]  

(74)

Then working our way through equations (66)-(72) as before, we obtain the following values for the \(N_2\) state:

\[
M_u : \quad C_u = 172.850; \quad |B_u| = 5.104238; \quad D_u = 0.149645; \quad |A_u| = 0.05670
\]

\[
M_d : \quad C_d = 5.9848; \quad |B_d| = 0.177720; \quad D_d = 0.00518134; \quad |A_d| = 0.025493375
\]

(75)

whence:

\[
|B_u|/C_u = 0.02952981; \quad |B_d|/C_d = 0.029695216
\]

\[
D_u/|B_u| = 0.029327794; \quad D_d/|B_d| = 0.02915449
\]

\[
|A_u|/D_u = 0.378919777; \quad |A_d|/D_d = 4.9
\]

(76)

which defines a sustained equal spacing rule with a \(k_2 \approx 0.0294\), though the rule appears visibly broken by the low mass \(A_u, A_d\) sector as we saw
partially in equation (72) for \((N_1, k_1)\) state.

The phase angles computed for \(M_u\) and \(M_d\) in the \((N_2, k_2)\) state are:

\[
\begin{align*}
\phi_1 &= 100.88^\circ \text{from : } |V_{us}| \\
\phi_1 &= 100.84^\circ \text{from : } |V_{cd}| \\
\phi_2 &= 79.88^\circ \text{from : } |V_{cb}| \\
\phi_2 &= 92.44^\circ \text{from : } |V_{ts}| \\
\end{align*}
\]

(77)

These angles are comparable to the ones found earlier for the \((N_1, k_1)\) state, suggesting that the phase angles are not suitable parameters or numbers to use to characterize \(M_u\) and \(M_d\).

We can generate other \((N_i, k_i)\), \(i, j = 1, 2, 3, 4, 5, \ldots\) quantum states of the \(M_u, M_d\) system by stretching the quantities \(|V_{cs}|\) and \(|V_{tb}|\) used earlier to define \(N_1\) and \(N_2\), by their experimental error bars, in a manner suggested by Xing and Zhang [3] and also Verma et. al. [4]. For example in place of \(|V_{cs}| = 0.97341 = N_1\) used in equation (55) to define \(N_1\), we can choose from equation (59): \(|V_{cs}| = 0.97362 = N_3\) and calculate its corresponding \(k_3\) value thus generating a new and still experimentally allowed quantum state \((N_3, k_3)\) of our \(M_u, M_d\) system. Therefore our analysis leads us to the conclusion that the flavor systems \(M_u\) and \(M_d\) live in a plurality of allowed quantum states \((N_i, k_i)\).

Independent investigations carried out by Yu-Feng Zhou[12] and by Giraldo[17] indicate also that other variants of the 4 texture zero model, not necessarily of equation (1) pattern, exhibit also hierarchical features amenable to the \((N, k)\) quantum state description or labelling for \((M_u, M_d)\).

5 Possible symmetry import of the equal spacing rule

It remains for us to attach some meaning, possibly of symmetry nature, to our finding of the above plurality of states \((N_i, k_i)\) for \(M_u\) and \(M_d\), each state labelled by only two mutually commuting quantum numbers, and each state representing some observed or observable state of the multi-flavor quark mass system \(M_u\) of \((u, c, t)\) system, and \(M_d\) of \((d, s, b)\). We argue that one way to immediately understand the result is to recall the intimate connection between certain texture zero mass matrices and weak basis transformations(WBT). It is known through several authors [16-18] that hermitian
quark mass matrices \((M_u, M_d)\) possess in general the property known as Weak
Basis (WB) transformation property whereby given any one pair \((M_u, M_d)\),
one can generate several other pairs by merely applying some unitary operator \(U\) thus:

\[
(M_u, M_d) \rightarrow U^\dagger (M_u, M_d) U = (M'_u, M'_d)
\]
or

\[
M_u \rightarrow M'_u = U^\dagger M_u U \\
M_d \rightarrow M'_d = U^\dagger M_d U
\]

(78)

and that these different pairs \((M_u, M_d), (M'_u, M'_d), (M''_u, M''_d)\)...... are equivalent with respect to the \(V_{CKM}\) flavor physics content of the system. These
matrices \(U\) constitute a group we may call the WB group of transformations, and denote by \(G_{WB}\). In a general situation of arbitrary hermitian
quark mass matrices \(G_{WB}\) is the \(U(3)\) symmetry group. But in a situation
where we require or restrict all pairs \((M_u, M_d), (M'_u, M'_d), (M''_u, M''_d)\) ... to
have the same 4 texture zero patterns stated in our equations (1) and (2),
we expect the relevant WB group to be some subgroup of \(U(3)\). The most
appropriate subgroup choice is \(SU(3)\), which we now show ties up with our
\((N_i, k_i)\) observed quantum states of \(M_u, M_d\).

The fact is that the WB transformations of equation (78) imply that each
paired matrix \((M_u, M_d)\) is a representation of \(G_{WB}\). Any such representation
or pair, must carry some commuting quantum numbers of \(G_{WB}\) that label
the representation and distinguish one representation or pair from another.
Since the various pairs we actually computed from data have each two com-
muting quantum number labels, and since SU(3) is known to have two such
commuting number labelling for its representations, we conclude that our
finding can be explained as the operation of WB symmetry in which SU(3)
is specifically the WB transformation group. What this throws up imme-
diately is that in the ongoing efforts[10,11] to embed the 4 texture zero model
in a GUT scheme and its low energy effective theory, we have to reckon that
our 4 texture zero model is subject to a plurality of \((N_i, k_i)\) quantum number
labelling. We will pursue this line of thought elsewhere.

6 Summary and Conclusions

In conclusion, we have shown that the 4 texture zero quark mass matrix
model possesses strong hierarchical features which manifest in a full blown
equal spacing rule, or minimally a geometric hierarchy. The features lead us to an important observation that the quark mass matrices \( M_u, M_d \) exist in a series of quantum states each labelled by two commuting numbers \( (N, k) \). The two numbers select SU(3) as a Weak Basis (WB) symmetry group of the pair of mass matrices \( (M_u, M_d) \). The two quantum numbers \( (N,k) \) are expected to be relevant in ongoing efforts to embed the 4 texture zero model in SO(10) and other GUT theories of Particle physics.

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