Pretty Good State Transfer via Adaptive Quantum Error Correction

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We examine the role of quantum error correction (QEC) in achieving pretty good quantum state transfer over a class of 1-d spin Hamiltonians. Recasting the problem of state transfer as one of information transmission over an underlying quantum channel, we identify an adaptive QEC protocol that achieves pretty good state transfer. Using an adaptive recovery and approximate QEC code, we obtain explicit analytical and numerical results for the fidelity of transfer over ideal and disordered 1-d Heisenberg chains. In the case of a disordered chain, we study the distribution of the transition amplitude, which in turn quantifies the stochastic noise in the underlying quantum channel. Our analysis helps us to suitably modify the QEC protocol so as to ensure pretty good state transfer for small disorder strengths and indicates a threshold beyond which QEC does not help in improving the fidelity of state transfer.

I. INTRODUCTION

Quantum communication entails transmission of an arbitrary quantum state from one spatial location to another. Spin chains are a natural medium for quantum state transfer over short distances, with the dynamics of the transfer being governed by the Hamiltonian describing the spin-spin interactions along the chains. Starting with the original proposal by Bose [1] for state transfer via a 1-d Heisenberg chain, several protocols have been developed for perfect as well as pretty good quantum state transfer via spin chains. Perfect state transfer protocols typically involve engineering the coupling strengths between the spins in such a way as to ensure perfect fidelity between the state of sender’s spin and that of the receiver’s spin [2–5]. Alternately, there have been proposals to use multiple spin chains in parallel, and apply appropriate encoding and decoding operations at the sender and receiver’s spins so as to transmit the state perfectly [6, 7].

Protocols for pretty good state transfer use permanently coupled spin chains with minimal control over the resources and aim at transmitting the information with minimal dispersion. One approach is for example to encode the information as a Gaussian wave packet in multiple spins at the sender’s end [8, 9]. Moving away from ideal spin chains, quantum state transfer has also been studied over disordered chains [7, 10]. There have been experimental studies which demonstrate state transfer [11–17].

Here, we study the problem of pretty good state transfer from a quantum channel point of view. It is known [1] that state transfer over an ideal XXX chain (also called the Heisenberg chain) can be realized as the action of an amplitude damping channel [18] on the encoded state. Naturally, this leads to the question of whether quantum error correction (QEC) can improve the fidelity of quantum state transfer. QEC-based protocols that achieve pretty good transfer have been developed for ideal as well as disordered XX [19] and Heisenberg spin chains [20].

In our work we study the role of adaptive QEC in achieving pretty good transfer over a class of 1-d spin systems which preserve the total spin. This includes both the XX as well as the Heisenberg chains, and more generally, the XXZ chain. We use an approximate QEC (AQEC) code, which has been shown to achieve the same level of fidelity as perfect QEC codes for certain noise channels while making use of fewer physical resources [21–23]. We demonstrate the efficacy of approximate codes and adaptive recovery in achieving pretty good state transfer over such a class of spin Hamiltonians.

Finally, we present explicit results for the fidelity of state transfer obtained using our QEC scheme, for ideal as well as disordered XXX chains. The presence of such disorder in a 1-d spin chain is known to lead to the phenomenon of localization [24]. Here, we analyze the distribution of the transition amplitude for a disordered XXX chain with random coupling strengths drawn from a uniform distribution. We modify the QEC protocol suitably so as to ensure pretty good transfer when the disorder strength is small. As the disorder strength increases, our analysis points to a threshold beyond which QEC does not help in improving the fidelity of state transfer.

The rest of the paper is organized as follows. We discuss the basic state transfer protocol over a general class of spin-preserving Hamiltonians and the underlying quantum channel description in Sec. II. We discuss the adaptive QEC protocol and the resulting fidelity in Sec. III. We present results specific to the ideal XXX chain in Sec. IV and discuss the disordered chain in Sec. V.

II. PRELIMINARIES

We consider a general 1-d spin chain with nearest neighbour interactions described by the Hamiltonian,

\[
\mathcal{H} = - \sum_k J_k \left( \sigma_x^k \sigma_x^{k+1} + \sigma_y^k \sigma_y^{k+1} \right) - \sum_k \tilde{J}_k \sigma_z^k \sigma_z^{k+1} + \sum_k B_k \sigma_z^k,
\]

(1)
where, \( \{ J_k \} > 0 \) and \( \{ \tilde{J}_k \} > 0 \) are site-dependent exchange couplings of a ferromagnetic spin chain, \( \{ B_k \} \) denote the magnetic field strengths at each site, and, \( (\sigma_x^k, \sigma_y^k, \sigma_z^k) \) are the Pauli operators at the \( k \)th site. The spin sites are numbered as \( j = 1, 2, \ldots, N \). We assume that the sender’s site is the \( s \)th spin and receiver’s site is the \( r \)th spin.

We denote the ground state of the spin as \( |0\rangle = |000\ldots0\rangle \). Since we are interested in transmitting a qubit worth of information along the chain, we will work within the subspace spanned by the set of single particle excited states \( |j\rangle \), with \( |j\rangle \) denoting the state with the \( j \)th spin alone flipped to \( |1\rangle \). The Hamiltonian in Eq. (1) preserves the total number of excitations, that is, \( \{ \mathcal{H}, \sum_{i=1}^N \sigma_z^i \} = 0 \) and hence the resulting dynamics is restricted to the \((N + 1)\)-dimensional subspace spanned by the single particle excited states and the ground state.

The sender (Alice) encodes an arbitrary quantum state \( |\psi_{in}\rangle = a|0\rangle + b|1\rangle \) at the \( s \)th site so that the state of the spin chain is now given by,

\[
|\Psi(0)\rangle = a|0\rangle + b|s\rangle,
\]

where \(|s\rangle\) is the state of the spin chain with only the \( s \)th spin flipped to \( |1\rangle \) and all other spins set to \( |0\rangle \). Under the action of the Hamiltonian \( \mathcal{H} \) described in Eq. (1), after time \( t \), the spin chain evolves to the state,

\[
|\Psi(t)\rangle = e^{-i\mathcal{H}t}|\Psi(0)\rangle = a|0\rangle + b\sum_{j=1}^N |j\rangle e^{-i\mathcal{H}t}|s\rangle|j\rangle.
\]

The state of the receiver’s spin at \( r \)th site after time \( t \), denoted as \( \rho_{out}\), is obtained by tracing out all the other spins from \( \rho(t) = |\Psi(t)\rangle \langle \Psi(t)|\):

\[
\rho_{out}(t) = \text{tr}_{1,2,\ldots,r-1,r+1,N-1}\left[ \rho(t) \right] = \left[ |a|^2 + |b|^2 \left( 1 - f_{r,s}^N(t)^2 \right) \right] |0\rangle \langle 0| + ab^* f_{r,s}^N(t) |0\rangle \langle 1| + ba^* f_{r,s}^N(t) |1\rangle \langle 0| + |b|^2 f_{r,s}^N(t) |1\rangle \langle 1|,
\]

where,

\[
f_{r,s}^N(t) = \langle r| e^{-i\mathcal{H}t}|s\rangle
\]

is the transition amplitude, which gives the probability amplitude for the excitation to transition from the \( s \)th site to \( r \)th site. Here, and in what follows, we set \( \hbar = 1 \).

The function \( f_{r,s}^N(t) \) satisfies,

\[
\sum_{r=1}^N |f_{r,s}^N(t)|^2 = 1, \forall \ s = 1, 2, \ldots, N.
\]

\[
\sum_{k=1}^N f_{N,k}^N(t)(f_{N,k}^N(t))^* = \delta_{NI}, \forall \ l = 1, 2, \ldots, N.
\]

We thus obtain the reduced state in Eq. (2) at receiver’s end as the action of a quantum channel on the input state. Specifically,

\[
\rho_{out}(t) = \mathcal{E}(\rho_{in}) = \sum_k E_k \rho_{in} E_k^*,
\]

where \( E_0 \) and \( E_1 \) are the Kraus operators that describe the action of the channel. It is easy to see that the operators \( E_0, E_1 \) have the following form when written in the \( \{ |0\rangle, |1\rangle \} \) basis.

\[
E_0 = \begin{pmatrix} 1 & 0 \\ 0 & f_{r,s}^N(t) \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{1 - |f_{r,s}^N(t)|^2} \\ 0 & 0 \end{pmatrix}.
\] (4)

The Kraus operators in Eq. (4) lead to a channel that has the same structure as the structure as the amplitude damping channel, but is more general since the channel parameter \( f_{r,s}^N(t) \) is complex.

Recall that the standard amplitude damping channel is parameterized by a real noise parameter \( p \) and is described by a pair of Kraus operators, written in the \( \{ |0\rangle, |1\rangle \} \) basis as [18].

\[
E_{0 AD} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - p} \end{pmatrix}, \quad E_{1 AD} = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.
\] (5)

This is the quantum channel induced in the original state transfer protocol in [1] where the Hamiltonian considered is a Heisenberg chain with an external \( \vec{B} \) field:

\[
\vec{H} = -\frac{J}{2} \sum_{\langle i,j \rangle} \vec{\sigma}^i \cdot \vec{\sigma}^j - B \sum_{i} \sigma_z.
\] (6)

By choosing the \( B \) field appropriately, it is possible to adjust the phase of the complex amplitude \( f_{r,s}^N(t) \) to be a multiple of \( 2\pi \) and hence replace \( f_{r,s}^N(t) \) by \( |f_{r,s}^N(t)| \), thus obtaining the amplitude damping channel described in Eq. (5) above.

### III. STATE TRANSFER PROTOCOL BASED ON ADAPTIVE QEC

While much of the past work on state transfer has focused on the Heisenberg Hamiltonian in Eq. (6), here, we will focus on the more general Hamiltonian in Eq. (1). We study the problem of transmitting an arbitrary quantum state from the 1st site to the \( N \)th site of an \( N \)-spin chain. We quantify the performance of the protocol in terms of the fidelity between the final state \( \rho_{out} \equiv \mathcal{E}(|\psi_{in}\rangle \langle \psi_{in}|) \) and the input state \( |\psi_{in}\rangle \). Specifically, we use the worst-case fidelity, which is defined as [18].

\[
F_{min}^2(\mathcal{E}) = \min_{a,b} \langle \psi_{in}| \rho_{out} |\psi_{in}\rangle,
\]

where the minimization is over all possible input states \( a|0\rangle + b|1\rangle \). Using our knowledge of the underlying quantum channel described in Eq. (4), it is easy to show that
the worst-case fidelity is related to the transition amplitude as,
\[ F_{\text{min}}^2(\mathcal{E}) = |f_{N,1}^N(t)|^2. \]  
(7)

The function \(|f_{N,1}^N(t)|^2\) is thus a measure of the fidelity of state transfer over a single spin chain evolving under a spin-conserving Hamiltonian.

Here, we examine how the worst-case fidelity may be improved using techniques from Quantum Error Correction (QEC). While several QEC protocols have been proposed to correct for amplitude damping noise \([21, 23, 24]\), the channel corresponding to the Heisenberg chain is more general, as seen from Eq. (4). Taking inspiration from the structural similarity to the amplitude damping channel, we propose a QEC protocol using an approximate 4-qubit code \([21]\) along with the channel-adapted near-optimal recovery proposed in \([23]\). The 4-qubit code \(C\) is realized as the span of the following pair of orthogonal states,
\[ |0_L\rangle = \frac{1}{\sqrt{2}} \left( |0000\rangle + |1111\rangle \right), \]
\[ |1_L\rangle = \frac{1}{\sqrt{2}} \left( |1100\rangle + |0011\rangle \right), \]
and was shown to be approximately correctable for amplitude damping noise \([21]\). The recovery map we use is adapted to a given noise map \(\mathcal{E}\) and code \(C\), and can be described in terms of the Kraus operators of the noise and the projector \(P\) onto the codespace, as follows,
\[ \mathcal{R}(.) = \sum_i P E_i^\dagger \mathcal{E}(P)^{-1/2}(.\mathcal{E}(P)^{-1/2} E_i P, \]
(9)
where the inverse of \(\mathcal{E}(P)\) is taken on its support. Such a recovery map \(\mathcal{R}\) has been shown to achieve worst-case fidelity close to that of the optimal recovery map for channel \(\mathcal{E}\) \([23]\).

The quantum state transfer protocol with QEC is implemented using a set of 4 unmodulated, identical, spin chains. Fig. [1] depicts a schematic of our protocol. The initial, encoded state \(|\psi_{\text{enc}}\rangle\) is now an entangled state across the four chains.
\[ |\psi_{\text{enc}}\rangle = a|0\rangle_L + b|1\rangle_L. \]
(10)

Once the initial state is prepared, the four chains are allowed to evolve in an uncoupled fashion, according to the Hamiltonian in Eq. (1). After time \(t\), the state at the receiver’s site is a joint state of the \(N^\text{th}\) site of the four chains, and is described by action of the map \(\mathcal{E}^\otimes 4\) with the time-dependent noise parameter \(f_{r,s}(t)\). Thus,
\[ \rho_{\text{err}} = \mathcal{E}^\otimes 4(\rho_{\text{enc}}) = \sum_i E_i^{(4)} \rho_{\text{enc}} \left( E_i^{(4)} \right)^\dagger, \]
where \(E_i^{(4)}\) are the Kraus operators of the 4-qubit noise channel realized as four-fold tensor products of the operators \(E_0\) and \(E_1\) in Eq. (1). After evolving the chains for time \(t\), the recovery map \(\mathcal{R}^{(4)}\) is applied at the receiver’s site of the four spin chains. The final state at the receiver’s end at the end of the QEC protocol is obtained as,
\[ \rho_{\text{rec}} = \sum_{i,j} R_j^{(4)} E_i^{(4)} \rho_{\text{enc}} \left( E_i^{(4)} \right)^\dagger \left( R_j^{(4)} \right)^\dagger, \]
with the Kraus operators \(R_i^{(4)}\) are given by,
\[ R_i^{(4)} = P \left( E_i^{(4)} \right)^\dagger \mathcal{E}^\otimes 4(P)^{-1/2}, \]
(11)
where \(P \equiv |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|\) is the projector onto the 4-qubit space described in Eq. (8). The fidelity of the 4-chain quantum state transfer protocol is then given by,
\[ F_{\text{min}}^2 \left( \mathcal{R}^{(4)} \circ \mathcal{E}^\otimes 4, C \right) \equiv \min_{a,b} \langle \psi_{\text{enc}} | \rho_{\text{rec}} | \psi_{\text{enc}} \rangle, \]
where the minimization is over all states in the codespace \(C\).

**Theorem 1.** The fidelity of quantum state transfer from site \(s\) to site \(r\) under a spin-conserving Hamiltonian as in Eq. (1), using the 4-qubit code \(C\) and adaptive recovery \(\mathcal{R}^{(4)}\) at time \(t\), is given by,
\[ F_{\text{min}}^2 \left( \mathcal{R}^{(4)} \circ \mathcal{E}^\otimes 4, C \right) \approx 1 - \frac{7p^2}{4} + O(p^3), \]
(12)
where \(p = 1 - |f_{r,s}^N(t)|^2\).

**Proof.** We first rewrite the Kraus operators given in Eq. (1), as,
\[ E_0 = |0\rangle\langle 0| + |f| e^{i\Theta} |1\rangle\langle 1| \]
\[ E_1 = |0\rangle\langle 1| \sqrt{1 - |f|^2}, \]
where, \(|f|\) and \(\Theta\) are the absolute value and phase of the complex-valued transition amplitude \(|f_{r,s}^N(t)|\). The state after the 4-qubit recovery map is then given by,
\[ \rho_{\text{rec}} = \left( \mathcal{R}^{(4)} \circ \mathcal{E}^\otimes 4 \right) (\rho_{\text{enc}}). \]

The composite map \(\mathcal{R}^{(4)} \circ \mathcal{E}^\otimes 4\) comprising noise and recovery has Kraus operators of the form,
\[ P \left( E_j^{(4)} \right)^\dagger \mathcal{E}^\otimes 4(P)^{-1/2} E_i^{(4)} P. \]
(13)
The key step in obtaining the desired fidelity is to show that the Kraus operators of the composite map written above are independent of $\Theta$. First, we write out $\mathcal{E}^{\otimes 4}(P)^{−1/2}$ in the (standard) computational basis of the 4-qubit space.

$$
\mathcal{E}^{\otimes 4}(P)^{−1/2} = \sum_{i=1}^{16} G_i |i⟩⟨i| + e^{−4i\Theta} G_{17} |0000⟩⟨1111| + e^{4i\Theta} G_{18} |1111⟩⟨0000| + G_{16} (|1100⟩⟨0011| + |0011⟩⟨1100|),
$$

where $\{G_i\}$ are polynomial functions of the transition amplitude $|f|$. The $\Theta$-dependence in this pseudo-inverse operator occurs only in the span of $\{|0000⟩, |1111⟩\}$. Since $\mathcal{E}^{\otimes 4}(P)^{−1/2}$ is sandwiched between the Kraus operators of the 4-qubit channel and their adjoints, we also write down the Kraus operators $\{E^{(i)}\}$ in the computational basis. Then, an explicit computation reveals that the $\Theta$-dependence gets conjugated out for each of the Kraus operators $\{E^{(i)}\}$.

IV. RESULTS FOR THE 1-D HEISENBERG CHAIN

We consider a special case of the Hamiltonian in Eq. (4), namely, an $N$-length, ideal Heisenberg chain, with $J_k = \tilde{J}_k = J/2 (J > 0)$ and $B_k = 0$, for all $k$. This is also often referred to as the $XXX$-chain in the literature.

![FIG. 2: Worst-case fidelity as a function of chain length $N$.](image)

Fig. 2 compares the performance of state transfer protocols with and without QEC. In particular, it compares the performance of our 4-chain state transfer protocol with the single-chain (no QEC) protocol [1] and the 5-chain protocol proposed in [20]. For each $N$, we plot the fidelity of state transfer from the 1st site to the $N$th site on a $N$-length spin chain, after a time $t^*$ chosen such that $|f^N_{1,N}(t)|$ is maximum at $t = t^*$.

From the plot we see that with QEC we achieve pretty good state transfer over longer distances. Using approximate QEC (AQEC) it is possible to achieve as high as fidelity as with the standard 5-qubit code, using fewer spin chains. Specifically, in the regime of small noise parameter $p$, we can show that the worst-case fidelity obtained using the 5-qubit code is,

$$
F^2_{\text{min}} \approx 1 - \frac{15p^2}{8} + O(p^3).
$$

For the ideal Heisenberg chain, it was recently shown that [20], there always exists a time $t$ at which $|f^N_{1,N}(t)|^2 > 1 - \epsilon$ if and only if the length of the chain is a power of 2, that is, $N = 2^m$. In other words, pretty good state transfer is always possible between the ends of a Heisenberg spin chain whose length $N$ is of the form $N = 2^m (m > 1)$. We may therefore consider improving the performance of our QEC-based protocol by repeating the error correction procedure every $2^m$ sites. Specifically, we can achieve pretty good state transfer over a chain of arbitrary length $L$, by stitching together smaller chains whose lengths are of the form $N = 2^m$. At every stage of the repeated QEC protocol, there are exactly $2^m$ interacting spins and the rest of the spin-spin-interactions are turned off.
FIG. 3: Worst-case fidelity using repeated QEC

Fig. 3 shows an example of the resulting improvement in fidelity when the QEC protocol is repeated every 8 sites. We first implement our QEC protocol for an 8-spin chain, evolving for time $t^*$ at which $|f_{k,1}^8(t)|$ maximizes. We repeat this procedure some $k$ times, where $k$ is the largest integer such that $8k < N$ and finally perform QEC for the remaining $N-7k$ sites for the same waiting time $t^*$. Such a repeated QEC protocol indeed enables pretty good transfer up to very long lengths, as seen in the plot.

More generally, if $F_{\min}^2 \approx 1 - \alpha p^2$ is the fidelity of the single-shot QEC protocol, repeating the procedure $k$ times gives us a fidelity of $F_{\min}^2 = 1 - (p_{\text{new}})$, with,

$$p_{\text{new}} = (1 - (1 - \alpha p^2)^k)$$

where $p_{\text{new}}$ is the noise parameter obtained after repeating QEC $k$ times.

V. QUANTUM STATE TRANSFER ON A DISORDERED HEISENBERG CHAIN

Moving away from an ideal spin chain with a fixed, uniform coupling between successive spins, we now study state transfer over a disordered XX chain, where the spin-spin couplings are randomly drawn from some distribution. It is well known that the presence of disorder in a 1-d spin chain leads to the phenomenon of localization [24] of information close to one end of the chain. It is therefore a challenging task to identify protocols which achieve perfect or pretty good transfer over disordered spin chains, overcoming the effects of localization.

Past work on disordered chains has primarily focused on the $XX$ chain. Starting with a modulated chain that admits perfect state transfer, both random magnetic field and random couplings have been studied [27]. Alternatively, an unmodulated chain with random couplings at all except the sender and receiver sites has also been studied [19].

When viewed in the quantum channel picture, the presence of disorder becomes as an additional source of noise. The role of QEC in overcoming the effects of disorder have been studied both for the $XX$ [19] as well as the Heisenberg chains [20]. The QEC protocol for the $XX$ chain with random couplings involves encoding into multiple spins at the sender’s end [19] using modified CSS codes, and is reported to be universally applicable except for Heisenberg model. On the other hand, the QEC protocol for the disordered Heisenberg chain [20] considers encoding into multiple identical, uncoupled chains using the standard 5-qubit code. However, this protocol also requires access to multiple spins at the sender and receiver ends of each of the chains.

The CSS-based encoding and decoding procedure for the $XX$ chain [19] is universal in the sense that it does not rely on knowledge of the specific value of disorder realized at each instance. However, the protocol based on the 5-qubit code involves choosing an encoding basis based on the phase of the transition amplitude, which in turn is specific to the disorder realization. This makes the QEC procedure hard to implement in a practical sense.

Here, we show how the channel-adapted QEC procedure described in Sec. III can be used to achieve pretty good state transfer over an $XXX$ chain with random couplings. We consider a disordered Heisenberg chain with couplings $J_k = \mathcal{J}(1 + \Delta_k)$, where $\Delta_k$ are independent, identically distributed random variables drawn from a uniform distribution between $[-\delta, \delta]$ and $\mathcal{J}$ is the mean value of the coupling strength, which we may set to 1, without loss of generality. Note that such a Hamiltonian conserves the total spin and hence falls within the universality class discussed in Sec. II.

Consider a state transfer protocol, where the sender wishes to transmit the state $|\psi_{\text{in}}\rangle = a|0\rangle + b|1\rangle$ from the $s^{th}$ site to the $r^{th}$ site via the natural dynamics of the chain. As before, the final state at the receiver’s site, tracing out the other spins can be realized as the action of a quantum channel $\mathcal{E}$,

$$\rho_{\text{out}} = \mathcal{E}(\rho_{\text{out}}) = \sum_k E_k \rho_{\text{in}} E_k^\dagger,$$

with the same Kraus operators $\{E_0, E_1\}$ as in Eq. [4]. The key difference however is in the nature of the noise parameter $p \equiv 1 - |f_{r,s}^N(t)|^2$: in the case of the disordered chain, the transition amplitude $f_{r,s}^N(t, \{\Delta_k\})$ between site $s$ and $r$ for a chain of length $N$ allowed to evolve for a time $t$, is a random variable whose value depends on the specific realization of the disorder variables $\{\Delta_k\}$. The distribution of $f_{r,s}^N(t, \{\Delta_k\})$ for a given set of $r, s, N, t$ values depends on the distribution over which the disorder variables $\{\Delta_k\}$ are sampled.
To illustrate our point, we specifically consider the case where the coupling strengths \( \{ \Delta_k \} \) are independently sampled from a uniform distribution. The Heisenberg Hamiltonian \( \mathcal{H} \) with static disorder in the coupling strengths, has the form,

\[
\mathcal{H}_{\text{dis}} = - \sum_k \frac{J(1+\Delta_k)}{2} (\sigma^k_x \sigma^{k+1}_x + \sigma^k_y \sigma^{k+1}_y + \sigma^k_z \sigma^{k+1}_z).
\]

Here, the effect of disorder is introduced via the i.i.d. random variables \( \{ \Delta_i \} \) which take values over a uniform distribution between \([-\delta, \delta]\). \( \delta \) is called the disorder strength, and \( J \) is the mean value of coupling coefficient. We may rewrite the disordered Hamiltonian as \( \mathcal{H}_{\text{dis}} = \mathcal{H}_0 + \mathcal{H}_\delta \), where \( \mathcal{H}_0 \) denotes the ideal XXX Hamiltonian studied in the previous section, and \( \mathcal{H}_\delta \) which captures the effect of disorder can be treated as a perturbation of the ideal Hamiltonian \( \mathcal{H}_0 \).

The transition amplitude between the \( N \)th and 1st site for the disordered Hamiltonian \( \mathcal{H}_{\text{dis}} \) in Eq. \( \text{(15)} \) is given by,

\[
f^{N}_{N,1}(t, \delta) = \langle N | e^{-i(\mathcal{H}_0 + \mathcal{H}_\delta)t} | 1 \rangle \\
= \langle N | e^{-i\mathcal{T} T} \left[ \exp \left( -i \int_0^t e^{i\mathcal{H}_0 t'} \mathcal{H}_\delta e^{-i\mathcal{H}_0 t'} dt' \right) \right] | 1 \rangle,
\]

where, \( \mathcal{T} \) is the time-ordered product and \( \hbar = 1 \) as before. Correspondingly, the transition amplitude in the presence of disorder can be written as a perturbation around the ideal amplitude \( f^{N}_{N,1}(t) \), of the form,

\[
f^{N}_{N,1}(t, \{ \Delta_k \}) = f^{N}_{N,1}(t) + \sum_i c_i^N(t) \Delta_i + \sum_{i,j} q_{ij}^N \Delta_i \Delta_j + \ldots,
\]

where the explicit forms of the complex coefficients \( c_i^N(t) \) are obtained in Appendix \[13\].

Starting from Eq. \( \text{(16)} \), we show that up to first order in disorder strength \( \delta \), the real part of the transition amplitude \( f^{N}_{N,1}(t, \{ \Delta_k \}) \) is distributed as,

\[
P_{\mathcal{R}}^{\delta,N,t}(x) \propto \sum_{s_1=1}^{2^{N-1}} (-1)^{s_1} (q_i)^{N-2} \text{Sign}(q_i),
\]

where \( s_i \in [0,1] \), and \( q_i(x, \mathcal{R}[f^{N,1}(t)], \{ \mathcal{R}[c_i^N(t)] \}) \) are linear combinations of the form,

\[
q_i = x - \mathcal{R}[f^{N,1}(t)] + \delta \sum_{i=1}^{N-1} (-1)^{r_i} \mathcal{R}[c_i^N(t)],
\]

where \( r_i \in [0,1] \forall i = 1, \ldots, N-1 \). The form of the distribution is identical for the imaginary part of \( f^{N}_{N,1}(t, \{ \Delta_k \}) \), with the real parts of \( \{ c_i^N(t) \} \) and \( f^{N}_{N,1}(t) \) replaced by their imaginary parts.

The key salient feature we observe from calculating the distribution functions above is that the limiting distribution in the case of no disorder (\( \delta \to 0 \)), is indeed a Delta distribution peaked around \( f^{N}_{N,1}(t) \). Furthermore, we also explicitly evaluate the mean and variance of \( f^{N}_{N,1}(t, \{ \Delta_k \}) \) and show that the mean is equal to the zero-disorder value of \( f^{N}_{N,1}(t) \), up to \( O(\delta^2) \). The variance goes as \( O(\delta^2) \), making it vanishingly small in the limit of small \( \delta \).

Figs. \[4, 5\] show the distribution of the real and imaginary of the transition amplitude for an 8-spin chain, with disorder strengths \( \delta = 0.001 \) and \( \delta = 1 \), respectively. We see that when the disorder strength is small enough, the transition amplitude is indeed distributed like a delta function peaked around the zero-disorder value. For large values of \( \delta \) the distribution spreads out quite a bit and its mean also shifts closer to zero, giving rise to a very small transition amplitude. This observation leads us to propose a modified QEC protocol for state transfer over disordered XXX chains, that uses an adaptive recovery \( \mathcal{R}_{\text{avg}} \) based on the disorder-averaged transition amplitude \( \langle f^{N}_{N,1}(t, \{ \Delta_k \}) \rangle \).
The disorder-averaged transition amplitude has also been studied as an indicator of localization in disordered chains [10, 27]. In Fig. 6, we plot the disorder-averaged transition amplitude \( \langle f_{N,1}(t, \{ \Delta_k \}) \rangle_\delta \) for a fixed time \( t^* \) and different disorder strengths \( \delta \), for the Heisenberg chain in Eq. (15). Empirically, we see that this plot follows an exponential distribution. Those curves where the peak is towards the left take the form exp \(- (\alpha N + \beta) / \text{Loc} \), where \( \alpha, \beta \) are constants and Loc is the localization length. We see that with the increase in disorder strength \( \delta \), the localization effects become more pronounced.

A. Adaptive QEC for 1-d disordered chain

As discussed in Sec. V above, the presence of disorder in the coupling strengths implies that the underlying channel for state transfer has the same structure as the ideal chain. Since the transition amplitude \( f_{r,s}^{N}(t, \{ \Delta_k \}) \) is now a random variable whose value depends on the random couplings \( \{ \Delta_k \} \), the underlying quantum channel is stochastic. However, as shown above, for small enough disorder strengths, \( f_{r,s}^{N}(t, \{ \Delta_k \}) \) is peaked sharply around its mean value, and we may consider the disorder-averaged amplitude \( \langle f_{r,s}^{N}(t, \{ \Delta_k \}) \rangle_\delta \) as a good estimate of the noise.

We therefore propose an adaptive QEC procedure for a disordered XXX chain involving the 4-qubit code in Eq. (5) and a recovery map \( \mathcal{R}_{\text{avg}} \) with the same structure as that used in the case of the ideal chain, described in Eq. (11). However, unlike the ideal case, the value of the channel parameter used in the recovery is different from the one in actual noise channel: the recovery map uses the disorder-averaged amplitude \( \langle f_{r,s}^{N}(t, \{ \Delta_k \}) \rangle_\delta \), and is therefore independent of the specific disorder realization, whereas the noise channel has the parameter \( f_{r,s}^{N}(t, \{ \Delta_k \}) \) which changes with every realization.

Fig. 7 shows the disorder-averaged worst-case fidelity \( \langle F_{\text{min}}^{2} \rangle_\delta \) obtained using the adaptive recovery \( \mathcal{R}_{\text{avg}} \), for an 8-spin chain. For disorder strengths \( \delta < 0.05 \), our adaptive QEC protocol achieves pretty good transfer. Beyond \( \delta \geq 0.05 \), localization effects are too strong to be counteracted by QEC. This is borne out by our detailed analysis of the distribution of the transition amplitude in the presence of disorder (see Appendix B). In particular, our expressions for the mean and standard deviation of the transition amplitude indicate that until \( \delta \leq 0.01 \), the disorder-averaged value \( \langle f_{r,s}^{N}(t, \{ \Delta_k \}) \rangle_\delta \) is close to the value of the transition amplitude in the ideal (zero-disorder) case, and the standard deviation is insignificant compared to the mean. However, as the disorder strength increases beyond \( \delta = 0.05 \), the disorder-averaged value \( \langle f_{r,s}^{N}(t, \{ \Delta_k \}) \rangle_\delta \) starts dropping and the standard deviation becomes comparable to the average value. Thus for \( \delta > 0.05 \), the effective noise parameter of the underlying quantum channel becomes too strong for the QEC procedure to be effective.

VI. CONCLUSIONS

We develop a pretty good state transfer protocol based on adaptive quantum error correction (QEC), for a universal class of Hamiltonians which preserve the total spin excitations on a linear spin chain. Based on the structure the underlying quantum channel, we choose an approximate code and near-optimal, adaptive recovery map, to solve for the fidelity of state transfer explicitly. For the specific case of the ideal Heisenberg chain, our protocol performs as efficiently as perfect-QEC-based protocols. Using repeated QEC on the chain, we are able to achieve high enough fidelity over longer distances for an ideal spin chain.

In the case of disordered spin chains we modify the recovery map parameters based on the distribution of the transition amplitude, which is the effective noise parameter. By suitably adapting the recovery procedure, we demonstrate pretty good transfer, on average, for low disorder strengths.

It is an interesting question as to whether similar channel-adapted QEC techniques maybe used to achieve
pretty good state transfer other universal classes, such as the transverse-field Ising model, the XYZ-chain etc. It is also an open problem to obtain an efficient circuit implementation of the adaptive recovery map discussed here.

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Appendix A: Effect of noise channel $\mathcal{E}$ on 4-qubit code

We note the following structure for the Kraus operators of the 4-qubit channel, by expanding them in the 4-qubit computational basis. First, we note that the only Kraus operator diagonal in the computational basis is $E_0^{\otimes 4}$, with diagonal entry $e^{ij\Theta}|f|^j$, corresponding to those basis vectors with $j$ 1’s in them. All the other operators are off-diagonal matrices with support on some subset of computational basis states. For example, a three-qubit error operator (involving $E$ off-diagonal matrices with support on some subset of computational basis states. For example, a three-qubit error with diagonal entry $e$ $E$

\begin{equation}
E_0 \otimes E_1^{\otimes 3} = (1 - |f|^2)^{3/2} |0000\rangle \langle 0111| + e^{i\Theta} |f| (1 - |f|^2)^{3/2} |1000\rangle \langle 1111|. \tag{A1}
\end{equation}

The remaining three-qubit errors are of the same form, with the strings $\{0111, 1000\}$ replaced by their permutations. Similarly, an operator which has $E_1$ errors on two of the qubits is a linear combination of the form,

\begin{equation}
E_0^{\otimes 2} \otimes E_1^{\otimes 2} = (1 - |f|^2) |0000\rangle \langle 0011| + e^{2i\Theta} |f|^2 (1 - |f|^2) |1100\rangle \langle 1111| + e^{i\Theta} |f|(1 - |f|^2) (|0100\rangle \langle 0111| + |1000\rangle \langle 1011|). \tag{A2}
\end{equation}

Other two-qubit error operators are realized by replacing the strings $\{0011, 1100, 0100, 1000\}$ with permutations thereof. A single-qubit error operator, with $E_1$ error on only one of the qubits has the form,

\begin{equation}
E_0^{\otimes 3} \otimes E_1 = (1 - |f|^2) |0000\rangle \langle 0001| + e^{3i\Theta} |f|^3 \sqrt{1 - |f|^2} |1110\rangle \langle 1111| + e^{2i\Theta} |f|^2 (1 - |f|^2) |1100\rangle \langle 1101| + \ldots \tag{A3}
\end{equation}

Finally, the four-qubit error operator $E_1^{\otimes 4}$ is of the form,

\begin{equation}
E_1^{\otimes 4} = (1 - |f|^2) |0000\rangle \langle 1111|. \tag{A4}
\end{equation}
We next explicitly write out the operator $\mathcal{E}^{\otimes 4}(P)$ in the computational basis of the 4-qubit space.

\[
\mathcal{E}^{\otimes 4}(P) = \begin{bmatrix}
Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-4i\Theta} Q_{17} \\
0 & Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{14} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{16} \\
e^{-4i\Theta} Q_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

with $\{Q_i\}$ denoting polynomial functions of the transition amplitude $|f|$. In terms of the rank-1 projectors onto the computational basis states, we may write $\mathcal{E}^{\otimes 4}(P)$ as,

\[
\mathcal{E}^{\otimes 4}(P) = \sum_{i=1}^{16} Q_i |i\rangle \langle i| + e^{-4i\Theta} Q_{17} |0000\rangle \langle 1111| + e^{i4\Theta} Q_{17} |1111\rangle \langle 0000| + Q_{18} (|1100\rangle \langle 0011| + |0011\rangle \langle 1100|), \tag{A5}
\]

wherein $|i\rangle \in \{|0000\}, |0001\}, |0010\}, |0011\}, |0100\}, \ldots |1111\}$ denote the computational basis states of the 4-qubit space.

Similarly, we can also express the pseudo-inverse $\mathcal{E}^{\otimes 4}(P)^{-1/2}$ in the 4-qubit computational basis, as follows:

\[
\mathcal{E}^{\otimes 4}(P)^{-1/2} = \begin{bmatrix}
G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-4i\Theta} G_{17} \\
0 & G_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{14} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{15} & 0 \\
e^{-4i\Theta} G_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \tag{A6}
\]

with $\{G_i\}$ denoting a set of polynomials in $|f|$. In terms of the rank-1 projectors onto the computational basis states, we have,

\[
\mathcal{E}^{\otimes 4}(P)^{-1/2} = \sum_{i=1}^{16} G_i |i\rangle \langle i| + e^{-4i\Theta} G_{17} |0000\rangle \langle 1111| + e^{i4\Theta} G_{17} |1111\rangle \langle 0000| + G_{18} (|1100\rangle \langle 0011| + |0011\rangle \langle 1100|). \tag{A7}
\]

Upon sandwiching the operator in Eq. (A7) between the different error operators of the four-qubit noise channel (as described in Eqs. (A1), (A2), (A3), (A4)) and their adjoints, it is easy to see that the phases cancel out everywhere. In other words, the Kraus operators of the composite channel comprising noise and recovery are all independent of the phase $\Theta$ of the transition amplitude.

**Appendix B: Distribution of the transition amplitude for a disordered XXX chain**

Here we derive the distribution of the transition amplitude $f_{N,t}^{\delta}(t, \{\Delta_k\}$ for the disordered XXX chain described in Eq. (15), as function of time $t$ and disorder strength $\delta$. Recall that the transition amplitude between the $N^{th}$ and
where, \( f^N_{N,1}(t, \delta) = \langle N | e^{-i(\mathcal{H}_0 + \mathcal{H}_d) t} | 1 \rangle = \langle N | e^{-i\mathcal{H}_d t} \mathcal{T} \left[ \exp \left( -i \int_0^t e^{i\mathcal{H}_d t'} \mathcal{H}_d e^{-i\mathcal{H}_d t'} dt' \right) \right] | 1 \rangle, \) (B1)

where \( \mathcal{T} \) denotes the time-ordered product and \( h = 1 \). We first expand the time-ordered perturbation series in Eq. (B1) as follows,

\[
f^N_{N,1}(t, \{\Delta_k\}) = \sum_{k=1}^N \langle N | e^{-i\mathcal{H}_d t} | k \rangle \langle k | \mathcal{T} \left[ e^{-i f^N_{N,k} t} e^{i\mathcal{H}_d t} e^{-i\mathcal{H}_d t} \right] | 1 \rangle.
\]

(B2)

where, \( f^N_{N,k}(t) = \langle N | e^{-i\mathcal{H}_d t} | k \rangle \) is the transition amplitude in the absence of disorder. Expanding the first order term \((O(\mathcal{H}_d))\) in the time-ordered expansion, we have,

\[
\langle k | O(\mathcal{H}_d) | 1 \rangle = \int_0^t \langle k | e^{i\mathcal{H}_d t'} \mathcal{H}_d e^{-i\mathcal{H}_d t'} | 1 \rangle dt'
\]

(B3)

where,

\[
\langle l | \mathcal{H}_d | m \rangle = \frac{J}{2} \left( \sum_{i=1}^{N-1} (s^i_l \Delta_i) \delta_{im} - 2\Delta_i \delta_{m(t+1)} - 2\Delta_i - 2\Delta_i \delta_{m(t-1)} \right),
\]

(B4)

with the coefficients \( s^i_l \in \{\pm 1\} \). For example, \( \mathcal{H}_d \) for a 4-qubit spin chain is a tridiagonal matrix of the form,

\[
\mathcal{H}_d = \frac{J}{2} \begin{pmatrix}
-\Delta_1 - \Delta_2 + \Delta_3 & -2\Delta_3 & 0 & 0 \\
-2\Delta_3 & -\Delta_1 + \Delta_2 + \Delta_3 & -2\Delta_2 & 0 \\
0 & -2\Delta_2 & \Delta_1 + \Delta_2 - \Delta_3 & -2\Delta_1 \\
0 & 0 & -2\Delta_1 & -\Delta_1 - \Delta_2 - \Delta_3
\end{pmatrix}.
\]

Substituting the form of \( \mathcal{H}_d \) in Eq. (B4) to the first order term in Eq. (B2), and setting \( J = 1 \) throughout, we get,

\[
f^N_{N,1}(t, \{\Delta_k\}) = f_{N,1}(t) - \frac{i}{2} \int_0^t \sum_{l,k=1}^N f_{N,k}(t) f^*_{k,l}(t') f_{l,1}(t') \left( \sum_{i=1}^{N-1} s^i_l \Delta_i \right) dt' - \frac{i}{2} \int_0^t \sum_{l,k=1}^N f_{N,k}(t) f^*_{k,l+1}(t') f_{l+1,1}(t') \left( -2\Delta_l \right) dt'
\]

Thus, up to first order in perturbation, \( f^N_{N,1}(t, \{\Delta_k\}) \) is simply a linear combination of the random variables \( \{\Delta_k\} \), of the form,

\[
f^N_{N,1}(t, \{\Delta_k\}) = f^N_{N,1}(t) + \sum_i c^N_i(t) \Delta_i,
\]

(B5)

where \( c^N_i(t) \) are complex coefficients given by,

\[
c^N_i = \frac{i}{2} \sum_{k=1}^N f_{N,k}(t) \left[ \int_0^t s^i_l \sum_{i=1}^{N-1} f^*_{k,i}(t') f_{i,l}(t') dt' + \int_0^t \sum_{l=1}^{N-1} f^*_{k,l}(t') f_{l,1}(t') dt' + \int_0^t \sum_{l=1}^{N-1} f^*_{k,l+1}(t') f_{l,1}(t') dt' \right]. \quad \text{(B6)}
\]

We first note that in the limit of large \( N \), the distribution of \( f^N_{N,1}(t) \) tends towards a normal distribution. This is a direct consequence of the central limit theorem, since \( \{\Delta_i\} \) are i.i.d random variables. In what follows, we will obtain the exact form of the distribution of \( f^N_{N,1}(t) \), specifically, the real and imaginary parts of \( f^N_{N,1}(t) \) in terms of \( N, t, \) and \( \delta \).
Since the $\{\Delta_i\}$ are randomly drawn from a uniform distribution between $[-\delta, \delta]$, the joint probability density $P(\Delta_1, \Delta_2, \ldots, \Delta_{N-1})$ is given by,

$$P(\Delta_1, \Delta_2, \ldots, \Delta_{N-1}) = \left\{ \begin{array}{ll} \frac{1}{(2\delta)^{N-1}}, & -\delta \leq \Delta_i \leq \delta, \forall i = 1, 2, \ldots, N, \\ 0, & \text{otherwise.} \end{array} \right. \quad (B7)$$

Let $x \equiv \Re[f_{N,1}^N(t, \{\Delta_k\})]$ and $y \equiv \Im[f_{N,1}^N(t, \{\Delta_k\})]$ denote the real and imaginary parts of the transition amplitude in Eq. (B5). Then, we may obtain the distribution of $x$ and $y$ as follows:

$$P_{\Re}^{\delta,t,N}(x) = \int_{\Delta_1=-\delta}^{\delta} \cdots \int_{\Delta_{N-1}=-\delta}^{\delta} \left( \prod_{i=1}^{N-1} d\Delta_i \right) P(\Delta_1, \Delta_2, \ldots, \Delta_{N-1}) \delta \left( x - (\Re[f_{N,1}^N(t)] + \sum_{i=1}^{N-1} \Re[c_i^N(t)\Delta_i]) \right),$$

$$P_{\Im}^{\delta,t,N}(y) = \int_{\Delta_1=-\delta}^{\delta} \cdots \int_{\Delta_{N-1}=-\delta}^{\delta} \left( \prod_{i=1}^{N-1} d\Delta_i \right) P(\Delta_1, \Delta_2, \ldots, \Delta_{N-1}) \delta \left( y - (\Im[f_{N,1}^N(t)] + \sum_{i=1}^{N-1} \Im[c_i^N(t)\Delta_i]) \right).$$

Replacing the Dirac delta functions with their Fourier transforms, and then integrating out the $\{\Delta_k\}$ variables, we get,

$$P_{\Re}^{\delta,t,N}(x) = \frac{1}{(2\delta)^{N-1}2\pi} \int_{\Delta_1=-\delta}^{\delta} \cdots \int_{\Delta_{N-1}=-\delta}^{\delta} \int_k \prod_{i=1}^{N-1} d\Delta_i \exp \left( -ik \left( x - \left[ \Re[f_{N,1}^N(t)] + \sum_{i=1}^{N-1} \Re[c_i^N(t)\Delta_i] \right] \right) \right) \frac{2\sin(k\delta \Re[c_i^N(t)])}{k \Re[c_i^N(t)]}.$$ \hspace{1cm} (B8)

$$P_{\Re}^{\delta,t,N}(x) = \left( \frac{1}{(2\delta)^{N-1}2\pi} \int_{\Delta_1=-\delta}^{\delta} \cdots \int_{\Delta_{N-1}=-\delta}^{\delta} \int_k \prod_{i=1}^{N-1} d\Delta_i \exp \left( -ik \left( x - \left[ \Re[f_{N,1}^N(t)] + \sum_{i=1}^{N-1} \Re[c_i^N(t)\Delta_i] \right] \right) \right) \frac{2\sin(k\delta \Re[c_i^N(t)])}{k \Re[c_i^N(t)]} \right)^{2^{N-1}}.$$ \hspace{1cm} (B9)

where $s_i \in [0,1]$, and $q_i(x, \Re[f_{N,1}^N(t)], \{\Re[c_i^N(t)]\})$ are linear combinations of the form,

$$q_i \equiv x - \Re[f_{N,1}^N(t)] + \delta \sum_{i=1}^{N-1} (-1)^i \Re[c_i^N(t)], \; r_i \in [0,1], \; \forall i = 1, \ldots, N - 1.$$ \hspace{1cm} (B10)

We may evaluate the distribution of the imaginary part of the transition amplitude in a similar fashion, to get,

$$P_{\Im}^{\delta,t,N}(y) = \left( \frac{1}{(2\delta)^{N-1}2\pi} \int_{\Delta_1=-\delta}^{\delta} \cdots \int_{\Delta_{N-1}=-\delta}^{\delta} \int_k \prod_{i=1}^{N-1} d\Delta_i \exp \left( -ik \left( y - \left[ \Im[f_{N,1}^N(t)] + \sum_{i=1}^{N-1} \Im[c_i^N(t)\Delta_i] \right] \right) \right) \frac{2\sin(k\delta \Im[c_i^N(t)])}{k \Im[c_i^N(t)]} \right)^{2^{N-1}}.$$ \hspace{1cm} (B11)

where the $\tilde{q}_i(x, \Im[f_{N,1}^N(t)], \{\Im[c_i^N(t)]\})$ are linear combinations of the form,

$$\tilde{q}_i \equiv x - \Im[f_{N,1}^N(t)] + \delta \sum_{i=1}^{N-1} (-1)^i \Im[c_i^N(t)], \; r_i \in [0,1], \; \forall i = 1, \ldots, N - 1.$$ \hspace{1cm} (B12)

We see from Eq. (B8) that the limiting distribution in the case of no disorder ($\delta \to 0$), is indeed a Delta distribution peaked around $\Re[f_{N,1}^N(t)]$:

$$\lim_{\delta \to 0} P_{\Re}^{\delta,t,N}(x) = \frac{1}{\sqrt{2\pi}} \int_{k=-\infty}^{\infty} dk \exp \left( -ik(x - \Re[f_{N,1}^N(t)]) \right) = \delta(x - \Re[f_{N,1}^N(t)]).$$ \hspace{1cm} (B13)

Finally, we compute the disorder-averaged value of the transition amplitude up to $O(H_3^2)$. We first modify the expression in Eq. (B5) to include the second-order perturbation terms:

$$f_{N,1}^N(t, \{\Delta_k\}) = f_{N,1}^N(t) + \sum_i c_i^N(t)\Delta_i + \sum_{i,j} d_{ij}^N(t)\Delta_i\Delta_j + \ldots,$$ \hspace{1cm} (B14)

where $\{d_{ij}^N\}$ are complex coefficients which are convolutions of the zero-disorder transition amplitude, similar to $\{c_i^N(t)\}$. Next, using the fact that the random couplings $\{\Delta_i\}$ are drawn from a uniform distribution, we obtain,

$$\langle f_{N,1}^N(t, \{\Delta_k\}) \rangle_\delta = \frac{1}{(2\delta)^{N-1}} \int \left( f_{N,1}^N(t) + \sum_i c_i^N(t)\Delta_i + \sum_{i,j} d_{ij}^N(t)\Delta_i\Delta_j + \ldots \right) \prod_{i=1}^{N-1} d\Delta_i$$

$$= f_{N,1}^N(t) + \delta^2 \sum_{i,m} d_{im}^N(t) + O(\delta^4).$$ \hspace{1cm} (B15)
The second moment of $f_{N,1}^N(t,\{\Delta_k\})$ up to $O(H^2\delta)$ is similarly obtained as,

\[
\langle (f_{N,1}^N(t,\{\Delta_k\}))^2 \rangle_{\delta} = \frac{1}{(2\delta)^{N-1}} \int \left( f_{N,1}^N(t) + \sum_i c_i^N(t)\Delta_i + \sum_{l,m} d_{lm}^N(t)\Delta_l\Delta_m + \ldots \right)^2 \prod_{i=1}^{N-1} d\Delta_i
\]

\[
= (f_{N,1}^N(t))^2 + \frac{\delta^2}{3} \left( 2(f_{N,1}^N(t) \sum_i d_{ii}^N(t)) + \sum_j (c_j^N(t))^2 \right) + \frac{\delta^4}{5} \sum_{i} (d_{ii}^N(t))^2 + \frac{\delta^4}{9} \sum_{l \neq m} (d_{lm}^N(t))^2 \ldots \tag{B16}
\]

We can now calculate the variance from Eq [B15] and Eq [B16] as follows:

\[
\text{Var}[f_{N,1}^N(t,\{\Delta_k\})] = \langle (f_{N,1}^N(t,\{\Delta_k\}))^2 \rangle_{\delta} - \langle f_{N,1}^N(t,\{\Delta_k\}) \rangle_{\delta}^2
\]

\[
= \frac{\delta^2}{3} \sum_j (c_j^N(t))^2 + \frac{\delta^4}{5} \sum_{i} (d_{ii}^N(t))^2 + \frac{\delta^4}{9} \sum_{i \neq m} (d_{lm}^N(t))^2 - \frac{\delta^4}{9} \left( \sum_{i} d_{ii}^N(t) \right)^2 \tag{B17}
\]