Minimizing and Computing the Inverse Geodesic Length on Trees

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Abstract

The inverse geodesic length (IGL) of a graph $G = (V, E)$ is the sum of inverse distances between every two vertices: $\text{IGL}(G) = \sum_{\{u, v\} \subseteq V} \frac{1}{d_G(u, v)}$. In the MinIGL problem, the input is a graph $G$, an integer $k$, and a target inverse geodesic length $T$, and the question is whether there are $k$ vertices whose deletion decreases the IGL of $G$ to at most $T$. Aziz et al. (2018) proved that MinIGL is $W[1]$-hard for parameter treewidth, but the complexity status of the problem remains open in the case where $G$ is a tree. We show that MinIGL on trees is subexponential by giving a $2^{O\left((n \log n)^{5/6}\right)}$ time, polynomial space algorithm, where $n$ is the number of vertices of the input graph.

The distance distribution of a graph $G$ is a sequence $\{a_i\}$ describing the number of vertex pairs distance $i$ apart in $G$: $a_i = |\{\{u, v\} : d_G(u, v) = i\}|$. Given only the distance distribution, one can easily determine graph parameters such as diameter, Wiener index, and particularly, the IGL. We show that the distance distribution of a tree can be computed in $O(n \log^2 n)$ time by reduction to polynomial multiplication. We also extend the result to graphs of bounded treewidth by showing that the first $P$ values of the distance distribution can be computed in $2^{O(tw(G))} n^{1+\varepsilon \sqrt{P}}$ time.

1 Introduction

The Inverse Geodesic Length (IGL) is a widely-used metric for quantifying the connectedness and efficiency of a given graph or network. In mathematical chemistry, it is also known as the Harary Index [38], and in network science as the (global) efficiency [14].

To test the resilience of a graph to vertex failures, the problem of minimizing a particular metric by deleting a fixed number of vertices has been studied extensively [30, 26, 21]. In these cases, heuristics have been used to choose which vertices to delete, and their effect has been assessed using the chosen metric. In particular, Szczepański et al. [34] chose IGL as the metric to be minimized when examining this problem. Nonetheless, only recently has the exact optimization problem itself (MinIGL) been studied.

Veremyev et al. [36] formulated MinIGL as a special case of the Distance-Based Critical Node Detection Problem (DCNP), and gave an asymptotically inefficient solution by reduction to Integer Linear Programming. Aziz et al. [4] observed that MinIGL is NP-complete, since it corresponds to VERTEX COVER when $T = 0$, but it is also both NP-complete, and $W[1]$-hard for parameter $k$, on split and bipartite graphs. On the positive side, it was shown that MinIGL is Fixed-Parameter Tractable (FPT) for parameter twin (or vertex) cover number, and also for $\omega + k$, where $\omega$ is the neighbourhood diversity of the graph. In another paper, Aziz et al. [3] showed that MinIGL is $W[1]$-hard for parameter treewidth. The complexity status of MinIGL when the input graph is a
tree was stated as an open question in [4, 3], and in open problem sessions of IWOCA 2017 and the Sydney Algorithms Workshop 2017.

In Section 3, we examine this case of the problem and provide a subexponential time, polynomial space algorithm.

**Theorem 1.1.** There is a $2^{O((n \log n)^{5/6})}$ time, $O(n^3)$ space algorithm for MinIGL on a tree, on a real RAM.

Our approach is as follows. First, we give an algorithm that tries all subsets of $k$ vertices, so it is reasonably efficient when $k$ is relatively small. Next, we give a Dynamic Programming (DP) algorithm that solves MinIGL by matching ordered trees to the structure of the given tree, to give a forest with $n - k$ vertices and minimum IGL. The running time of this algorithm is exponential in $L$, but polynomial in $n$, where $L$ is the size of the largest tree in this forest. Finally, we prove that in any optimal solution, $L$ is $O\left(\left(\frac{n}{k}\right)^5\right)$, so the exponential factor in the DP algorithm is small when $k$ is relatively large with respect to $n$. Thus, we can pick an appropriate threshold (a function of $n$) and elect to use the first algorithm when $k$ is below the threshold, and the second algorithm when $k$ is above it. This gives Theorem 1.1.

IGL has also been used to identify key protein residues [9], compare the robustness of botnet structures [17] and assess the impact of attacks on power grids [39]. Thus, the ability to quickly compute the IGL of a graph efficiently serves practical purpose in identifying characteristics of real-world networks, and offers a potential speedup to algorithms for MinIGL.

Since the IGL of a graph can easily be computed from its distance distribution, we first examine the problem of computing the distance distribution of trees. By combining the relatively well-known techniques of centroid decomposition and fast polynomial multiplication, we have the following result on trees.

**Theorem 1.2.** The distance distribution of a tree with $n$ vertices can be computed in $O(n \log^2 n)$ time on a log-RAM.

We extend this result to graphs with bounded treewidth. This is of practical note, as real-world graphs for which IGL is an indicator of strength – such as electrical grids [2] and road transport networks [28] – have been found to have relatively small treewidth.

The distance distribution of a graph can be trivially computed from the All Pairs Shortest Paths (APSP). The output of APSP is of size $n^2$, so any APSP algorithm requires $\Omega(n^2)$ time. On graphs with treewidth $k$, APSP can be computed in $O(kn^2)$ time [31], so we seek algorithms that find the distance distribution with a subquadratic dependence on $n$.

Abboud et al. [1] proved that, under the Orthogonal Vectors Conjecture (OVC), there is no algorithm that distinguishes between graphs of diameter 2 and 3 in $2^{o(k)}n^{2-\varepsilon}$ time. Williams [37] showed that the OVC is implied by the Strong Exponential Time Hypothesis (SETH) of Impagliazzo, Paturi and Zane [23, 24]. Since the distance distribution of a graph immediately gives its diameter, this hardness result also applies to computing the distance distribution. We prove the following result.

**Theorem 1.3.** The prefix $a_1, \ldots, a_P$ of the distance distribution of a graph with $n$ vertices and treewidth $k$ can be computed in $2^{O(k)}n^{1+\varepsilon}\sqrt{P}$ time on a log-RAM, for any $\varepsilon > 0$.

In particular, when the diameter of the input graph is constant, the number of interesting values of $P$ is also constant, so we obtain an $2^{O(k)}n^{1+\varepsilon}$ time algorithm to compute the entire distance distribution. This matches the known hardness bounds above, in the sense that under the OVC (or the stronger SETH), the dependence on $k$ must be $2^{\Omega(k)}$ when the dependence on $n$ is subquadratic.
Cabello and Knauer [12] reduced the problem of computing the Wiener index (the sum of distances between every two vertices) to orthogonal range queries in $k - 1$ dimensions. They did so by applying a divide-and-conquer strategy that divides the graph with small separators that are found efficiently. Abboud et al. [1] adapted this approach to find radius and diameter. We take a similar approach, but reduce computing the distance distribution to the following problem rather than to orthogonal range queries.

If $v$ and $w$ are vectors in $\mathbb{R}^d$, write $v < w$ if each coordinate of $w$ is strictly greater than the corresponding coordinate in $v$. In this case, we say that $w$ (strictly) dominates $v$. We define the RedBluePolynomial problem as follows.

| RedBluePolynomial |
|-------------------|
| **Input:** $r$ red points $R_1, \ldots, R_r$, and $b$ blue points $B_1, \ldots, B_b$ in $\mathbb{R}^d$, along with corresponding non-negative integer values $r_1, \ldots, r_r$, and $b_1, \ldots, b_b$, respectively. |
| **Question:** Determine the non-zero coefficients of the polynomial $\sum_{R_p < B_q} x^{r_p + b_q}$, as a list of (exponent, coefficient) pairs. |

This problem can be solved naively in quadratic time, but we seek a more efficient solution in the case when the value of each point is bounded.

To our knowledge, this problem is new, and a variant of a well-known counting problem, which asks for the number of red points dominated by each blue point. Chan and Patrascu [13] showed that this counting problem can be solved in $O(n \sqrt{\log n})$ time on a Word RAM, using word operations to facilitate efficient counting. Bentley [6] gave a multidimensional divide-and-conquer approach for a similar problem, which Monier [29] showed had complexity $O(dn \cdot B(n, d))$ where $B(n, d) = \left(\frac{d + \lceil \log n \rceil}{d}\right)^d$.

Bringmann et al. [11] used this fact to show that the method employed by Cabello and Knauer [12], and Abboud et al. [1] can, in fact, be used to compute the Wiener index, radius, and diameter of graphs with treewidth $k$ in $2^{O(k)} n^{1+\varepsilon}$ time for any $\varepsilon > 0$, by proving that $B(n, k) = 2^{O(k)} n^\varepsilon$. Further, Husfeldt [22] gave an improved $2^{O(k)} n$ time algorithm for computing diameter and radius in the case where the graph also has constant diameter. However, it was noted that this result only pertains to the existence of pairs of vertices at certain distances, and not to counting the number of such pairs. Thus, the result does not directly give further insight to computing distance distributions.

We follow Bentley’s method, where it suffices to consider the one-dimensional case, $d = 1$. We resolve this case using square-root decomposition and fast polynomial multiplication. Applying the approach of Bringmann et al. to analyse the running time of this approach gives Theorem 1.3. A detailed discussion of this algorithm is given in Section 4.

## 2 Preliminaries

Let $G = (V, E)$ be a graph and suppose $u, v, w \in V$. We define the distance $d_G(u, v)$ between $u$ and $v$ to be the fewest number of edges in any path from $u$ to $v$, or $\infty$ if no such path exists.

The Inverse Geodesic Length or IGL of $G$ is $\text{IGL}(G) = \sum_{\{u, v\} \subseteq V} \frac{1}{d_G(u, v)}$, with the convention that $\frac{1}{\infty} = 0$. The MinIGL problem is defined as follows.

| MinIGL |
|--------|
| **Input:** a graph $G = (V, E)$, an integer $k$, and a rational number $Q$. |
| **Question:** Is there a vertex subset $S \subseteq V$ with $|S| \leq k$ such that $\text{IGL}(G - S) \leq Q$? |
In Section 3, we consider the problem when the provided graph is a tree $T$. In this case, precisely one simple path exists between every pair $\{u, v\} \subseteq V$. Define $\mathcal{P}_T(u, v)$ to be the set of vertices along the simple path from $u$ to $v$ in $T$, including the endpoints $u$ and $v$. Observe that $d_T(u, w) + d_T(w, v) = d_T(u, v)$ if and only if $w \in \mathcal{P}_T(u, v)$. For a vertex $w$, we also define $\mathcal{P}^{-1}_T(w)$ to be the set of all (unordered) pairs of vertices whose path in $T$ passes through $w$. Formally, $\mathcal{P}^{-1}_T(w) = \{\{u, v\} \subseteq V : w \in \mathcal{P}_T(u, v)\}$.

Let $u$ be a vertex in $T$. We call $u$ a centroid of $T$ if the maximum size of a connected component in $T - v$ is minimized. The following lemma characterises the existence of centroids in a tree.

**Lemma 2.1** (Jordan [25]). Every tree has either one centroid or two adjacent centroids. If a centroid is deleted from a tree, each tree in the remaining forest contains no more than $\frac{n}{2}$ vertices, where $n$ is the number of vertices in the original tree.

In Section 4.2, we consider the problem of computing the IGL on graphs with bounded treewidth. We define treewidth in terms of tree decompositions as follows. A tree decomposition of $G$ is a tree $T$ whose vertices (called nodes) are $\{1, \ldots, I\}$ and a sequence $V_1, \ldots, V_I$ of subsets of $V$ (called bags) such that

1. $V = \bigcup_{i=1}^I V_i$;
2. If $uv \in E$, then $\{u, v\} \subseteq V_i$ for some $i$;
3. $V_a \cap V_c \subseteq V_b$ whenever $b \in \mathcal{P}_T(a, c)$.

The width of such a tree decomposition is $\max_{i=1}^I |V_i| - 1$. The treewidth $tw(G)$ of $G$ is the minimum width among all tree decompositions of $G$. Further, if $T$ is a rooted binary tree, we say that $(T, \{V_i\})$ is a nice tree decomposition if each $i \in I$ also satisfies the following conditions:

1. If $i$ is a leaf of $T$, and not the root, then $|V_i| = 1$;
2. If $i$ has a single child $j$ in $T$, then either $|V_i| = |V_j| + 1$ and $V_j \subset V_i$, or, $|V_i| = |V_j| - 1$ and $V_i \subset V_j$;
3. If $i$ has two children $i_1$ and $i_2$ in $T$, then $V_i = V_{i_1} = V_{i_2}$.

The distance distribution of $G$ is the sequence $\{a_i\}$ which describes the number of pairs of vertices in $G$ that are distance $i$ apart. Formally, $a_i = |\{\{u, v\} : d_G(u, v) = i\}|$. This definition extends naturally to directed graphs (counting instead, the number of ordered pairs) and graphs with edges whose weights are non-negative integers. In Section 4, we examine the following problem on trees, and graphs with bounded treewidth.

| DistanceDistributionPrefix |
|----------------------------|
| **Input:** a graph $G = (V, E)$ and an integer $P$. |
| **Question:** What are the values $a_1, \ldots, a_P$ of the distance distribution of $G$? |

Particularly, computing the entire distance distribution of the graph corresponds to the case when $P = |V| - 1$. 

4
2.1 Model of computation

We establish our results on models of computation that closely reflect what is available to programmers of high-level languages on physical computing devices today.

In Section 3, we solve MinIGL by explicitly computing the minimum IGL that can be obtained by deleting \( k \) vertices from the given tree. We perform this on the real RAM formulated by Shamos [33], which allows addition, subtraction, multiplication, division and comparisons of real numbers in constant time, but does not support rounding a value to the nearest integer, or modulo as native operations. This allows us to efficiently add and compare contributions of distances between vertices to the IGL.

In Section 4, we reduce the problem of computing the IGL of a graph to finding its distance distribution. We solve this on a log-RAM introduced by Föurer [20], which is a Word RAM that also supports constant time arithmetic operations (including multiplication, integer modulo, and division) on words of length \( O(\log n) \). Föurer showed that on a log-RAM, multiplication of two \( n \)-bit integers can be done in \( O(n) \) time, using either the approach of Schönhage and Strassen [32] (performing a complex polynomial-based Fast Fourier Transform (FFT) and maintaining sufficient precision), or that of Föurer [19] (performing an FFT over a ring of polynomials).

We now extend this result to integer polynomials with bounded coefficients, as we would like to reduce computing the IGL on graphs with bounded treewidth to the multiplication of integer polynomials.

**Lemma 2.2.** Suppose \( P \) and \( Q \) are integer polynomials of degree \( n \) whose coefficients are non-negative integers, such that their product \( PQ \) has coefficients not exceeding some integer \( m \). Then, the coefficients of \( PQ \) can be computed from the coefficients of \( P \) and \( Q \) in \( O(n \log m) \) time on a log-RAM.

**Proof.** This can be done using a technique known as Kronecker substitution [27]. Let \( m' \) be the smallest power of two strictly greater than \( m \). Evaluate \( P(m') \) and \( Q(m') \) to give two integers, each containing \( O(n \log m) \) bits. This is easily done through bit manipulation because \( m' \) is a power of two. Next, multiply these together to obtain the value of \( (PQ)(m') \): this takes time linear in the number of bits in the multiplicands. Finally, one can unpack the coefficients of \( PQ \) from this value in \( O(n \log m) \) time by viewing it as a base \( m' \) integer: since \( m' \) is a power of two, this can be simply read off the bits of the result.

\[ \square \]

3 MinIGL on Trees

In this section we give a subexponential time, polynomial space algorithm for MinIGL on trees. Given a tree with \( n \) vertices, and a budget of deletions \( k \), we will solve MinIGL by directly computing the minimum IGL obtainable by deleting \( k \) vertices from the tree. To do this, we will propose two separate algorithms, each used depending on whether \( k \) is small or large, relative to \( n \).

**Lemma 3.1.** There is an \( O(n^{k+2}) \) time, \( O(n) \) space algorithm for MinIGL on a tree, on a real RAM.

**Proof.** Simply try all \( \binom{n}{k} = O(n^k) \) subsets of \( k \) vertices. The IGL of the forest that remains after each subset has been removed can be computed in \( O(n^2) \) time using a depth-first search from every vertex.

\[ \square \]

If \( k \) is small, this algorithm may be efficient. When \( k \) is large, one would expect that the vertices forming an optimal solution will leave a forest of relatively small trees after they are deleted.
Formally, call a forest $L$-trimmed if none of its trees contains more than $L$ vertices. Similarly, call a subset of vertices in a tree $L$-trimming if their deletion gives an $L$-trimmed forest. We first present an algorithm that minimizes IGL, considering only $L$-trimming subsets of $k$ vertices. The running time of this algorithm is parameterized by $L$. Then, we prove an upper bound on $L$, in terms of $n$ and $k$, such that any optimal solution to MinIGL on trees is $L$-trimming.

**Lemma 3.2.** Let $T = (V, E)$ be a tree with $n$ vertices. There is an $O\left(\frac{4L}{\sqrt{L}}n^2\right)$ time, $O(nkL)$ space algorithm on a real RAM, which finds the minimum value of $IGL(T - S)$ among all subsets $S$ of $k$ vertices such that $T - S$ is $L$-trimmed.

**Proof.** We root $T$ arbitrarily and will employ Dynamic Programming (DP) to compute this minimum value for every subtree and budget, in the case where the root of the subtree is deleted, and the case where it is not. Denote these minimum values by $f(u, b)$ and $g(u, b)$, respectively, for the subtree rooted at $u$ and budget $b$. The leaves of the tree form the base cases for this algorithm, and the final answer is derived from the minimum of $f(root, k)$ and $g(root, k)$. It remains to give recurrences for $f$ and $g$.

In the case where $u$ is deleted, we simply need to distribute the remaining $b - 1$ deletions among the subtrees rooted at each child of $u$. Let the children of $u$ be $v_1, \ldots, v_{ch_T(u)}$ in a fixed order. We run another DP algorithm to solve this: let $f'(u, i, b')$ be the minimum value of $f$ distributing a budget of $b'$ deletions among the subtrees rooted at the first $i$ children of $u$. Our recurrence is as follows:

$$f'(u, i, b') = \min_{0 \leq b'' \leq b'} (\min(f(v_i, b''), g(v_i, b'')) + f'(u, i - 1, b' - b''))$$

and we have that $f(u, b) = f'(u, ch_T(u), b - 1)$.

If $u$ is not deleted, it will be the root of some tree with no more than $L$ vertices after our chosen subset has been deleted. We fix the structure (an ordered tree) for this rooted tree, and attempt to match the vertices in this structure to vertices in the subtree rooted at $u$. Formally, let the structure be an ordered tree $T'$ over $L' \leq L$ vertices. Let its vertex set be $V' = \{1, \ldots, L'\}$ and, without loss of generality, suppose 1 is its root. We seek a total, injective mapping $m : V' \rightarrow V$ satisfying the following conditions.

1. $m(1) = u$;
2. Suppose $p$ and $p'$ are the parents of $q$ and $q'$ in $T$ and $T'$, respectively. If $m(q') = q$ then $m(p') = p$;
3. Let $p$ and $p'$ be vertices in $T$ and $T'$ such that their children are, in order, $q_1, \ldots, q_{ch_T(p)}$ and $q'_1, \ldots, q'_{ch_{T'}(p')}$, respectively. If $m(q'_{j_1}) = q_{i_1}$, $m(q'_{j_2}) = q_{i_2}$ and $j_1 \leq j_2$, then $i_1 \leq i_2$. That is, children are matched in order.

Figure 1 gives an example of a valid such mapping.

The structure of the ordered tree uniquely characterises the IGL of the component containing $u$, which can be found by considering the All Pairs Shortest Paths in the ordered tree. Let $v$ be some vertex in $T$. If $v$ is mapped to by $m$, then $v$ is a part of this component. Otherwise, if $v$ is not mapped to by $m$ but its parent is, then $v$ must be a vertex chosen for deletion, and so we should recursively consider each of its children’s subtrees.

This implies a DP approach to determine the optimal choice of $m$, similar to that of $f'$. We let $g'(u, i, b', u', j)$ be the minimum value (IGL) induced by a mapping which maps $u'$ to $u$ and maps the first $j$ children of $u'$ among the first $i$ children of $u$ with a total budget of $b'$ deletions in the
subtree rooted at \( u \). We have a choice to either delete the \( i \)th child \( v_i \) of \( u \), or map it to the \( j \)th child \( v'_j \) of \( u' \). In both cases, we allocate a budget of \( b'' \leq b' \) deletions to the subtree rooted at \( v_i \). This gives the following recurrence:

\[
g'(u, i, b', u', j) = \min_{0 \leq b'' \leq b'} \min \left\{ g'(u, i - 1, b' - b'', u', j) + f(v_i, b''), \right. \\
\left. g'(u, i - 1, b' - b'', u', j - 1) + g'(v_i, ch_T(v_i), b'', v'_j, ch_T(v'_j)) \right\}
\]

and \( g(u, b) = \min_{T'} g'(u, ch_T(u), b, 1, ch_T(1)) \). This concludes the description of the algorithm.

We now consider the complexity of computing \( f' \) (and thus that of computing \( f \)) over the entire tree. Each \( (u, i) \) pair corresponds to an edge of the tree, and \( b' \leq k \), so there are \( O(nk) \) states of \( f \) and \( f' \) to consider. Further, if \( b' \) is no less than the number of vertices in the subtree rooted at \( u \), we may delete every vertex in that subtree, giving a minimum IGL of 0. Hence, we need only consider values for \( b'' \) which do not exceed the number of vertices in the subtree rooted at \( v_i \), and also those where \( b' - b'' \) does not exceed the total number of vertices in the subtrees rooted at \( v_1, \ldots, v_{i-1} \). It can be shown (see, e.g., [15]) that there are only \( O(n^2) \) \( (u, i, b', b'') \) tuples satisfying these properties in the entire tree. Hence, all values of \( f \) can be computed in \( O(n^2) \) time and \( O(nk) \) space, assuming the required values of \( g \) are readily available.

Similarly, for a fixed \( T' \) we can examine the complexity of computing the values of \( g' \). Each \( (u', j) \) pair corresponds to an edge of \( T' \), and there are at most \( L - 1 \) of these. By a similar argument to that for \( f' \), we only need to consider \( O(n^2) \) tuples \( (u, i, b', b'') \). Hence, we can compute all values of \( g' \) in \( O(n^2L) \) time and \( O(nkL) \) space.

It is a well-known result that the number of ordered trees with \( l \) vertices is \( C_{l-1} \), the \( (l-1) \)-th Catalan Number (see, for example, [18]). The sum of the first \( l \) Catalan numbers was upper bounded by \( \frac{2^{l+1}}{3\sqrt{\pi l^3}} \) in [35], so there are \( O \left( \frac{4^L}{L^{\frac{3}{2}}} \right) \) different ordered trees we need to try as \( T' \). We generate these recursively. For each in turn, we compute its IGL in \( O(L^2) \) time. Then, we compute \( g' \), which dominates our running time and memory consumption. Thus, our algorithm runs in \( O \left( \frac{4^L}{\sqrt{L}}n^2 \right) \) time and uses \( O(nkL) \) space, as required. \( \square \)

We are also aware of an \( 3^L \cdot \text{poly}(n) \) time and space DP algorithm operating over the preorder traversal of the tree (once rooted). The state encodes a current vertex \( u \), and, for each ancestor \( a \) of the current vertex and for each distance \( l \), the number of vertices in the subtree rooted at \( a \) that are distance \( l \) from \( u \) in the final forest. We elect to use the result of Lemma 3.2 instead, as its exponent is larger only by a constant, whereas its memory consumption is strictly polynomial.
We now focus on proving the upper bound mentioned earlier. To do so, we choose to reason about the decrease in IGL caused by the removal of a subset of $k$ vertices, rather than the IGL itself. Maximizing this decrease (which we call utility) is equivalent to minimizing the IGL of the graph after removal.

**Definition 3.3.** Let $G = (V, E)$ be a graph. Then the utility $U_G(S)$ of some subset of vertices $S$ is defined as follows:

$$U_G(S) = IGL(G) - IGL(G - S).$$

If $S = \{v\}$ is a singleton, we write $U_G(v)$ instead of $U_G(\{v\})$, which we call the utility of $v$ in $G$.

Suppose $S = S' \cup \{v\}$ is the subset of $k$ vertices in a tree $T$ with maximum utility. Necessarily, $v$ must have maximum utility in $T - S'$. This means that $v$ has no less utility than any vertex in its component in $T - S'$, and that it also has no less utility than the optimal vertex in any other component. In this vein, we would like to consider the case when $k = 1$ so we can reason about the individual optimality of each vertex in an optimal solution. When $k = 1$, the MinIGL problem seeks the vertex with maximum utility, and so we derive the following results concerning the utility of individual vertices for this case.

**Lemma 3.4.** Let $G = (V, E)$ be a tree with $n$ vertices. Then $U_G(v) \leq IGL(G) \leq \frac{1}{2} n(n - 1)$ for any vertex $v \in V$.

**Proof.** From Definition 3.3, we have that $U_G(v) = IGL(G) - IGL(G - v)$. It is obvious that $IGL(T - v) \geq 0$, so $U_G(v) \leq IGL(G)$. The second inequality holds because there are $\binom{n}{2} = \frac{1}{2} n(n - 1)$ vertex pairs in the graph, and each contributes at most 1 to the IGL. 

Note that equality holds for the second inequality precisely when $G$ is the complete graph $K_n$, on $n$ vertices.

**Lemma 3.5.** Let $T = (V, E)$ be a tree with $n$ vertices. Then

$$\sum_{v \in V} U_T(v) = \binom{n}{2} + IGL(T).$$

**Proof.** Since $T$ is a tree, there is precisely one simple path with length $d_T(u, v)$ for every $\{u, v\} \subseteq V$. Now suppose $w \in V$ and compare $d_{T-w}(u, v)$ to $d_T(u, v)$. If $w \in P_T(u, v)$, then $u$ and $v$ are necessarily disconnected in $T - w$, so $d_{T-w}(u, v) = \infty$. Otherwise, the path remains, and $d_{T-w}(u, v) = d_T(u, v)$.

Hence, the utility of $w$ in $T$ can be written as follows.

$$U_T(w) = IGL(T) - IGL(T - w)$$

$$= \sum_{\{u, v\} \subseteq V} \frac{1}{d_T(u, v)} - \frac{1}{d_{T-w}(u, v)}$$

$$= \sum_{\{u, v\} \subseteq V} \frac{1}{d_T(u, v)}.$$
We can then sum this utility over all vertices in $T$.

$$\sum_{w \in V} U_T(w) = \sum_{w \in V} \sum_{\{u,v\} \subseteq V_{P_T(u,v)}} \frac{1}{d_T(u,v)}$$

$$= \sum_{\{u,v\} \subseteq V} \sum_{w \in P_T(u,v)} \frac{1}{d_T(u,v)}$$

$$= \sum_{\{u,v\} \subseteq V} |P_T(u,v)| \frac{1}{d_T(u,v)}$$

$$= \sum_{\{u,v\} \subseteq V} \frac{d_T(u,v) + 1}{d_T(u,v)}$$

$$= \left(\frac{n}{2}\right) + IGL(T),$$

as required.

\[\square\]

**Corollary 3.6.** Let $T = (V, E)$ be a tree with $n \geq 2$ vertices. Then, $\max_{v \in V} U_T(v) \geq n/2$.

*Proof.* From Lemma 3.5, we know the average utility of any vertex in $T$ is $\frac{1}{2} \left(\binom{n}{2} + IGL(T)\right) = \frac{n^2}{2} + \frac{1}{n} IGL(T)$. Since $T$ has $n - 1$ edges, each connecting a pair of vertices distance 1 apart, $IGL(T) \geq n - 1$ so $\frac{1}{n} IGL(T) \geq \frac{n-1}{n} \geq \frac{1}{2}$ whenever $n \geq 2$. Hence, the average utility is at least $n/2$ and the result follows. \[\square\]

If the diameter of the tree is bounded, we can improve the lower bound given in Corollary 3.6 by considering the utility of a centroid. First, we give a lower bound on the number of paths through a centroid.

**Lemma 3.7.** Let $T$ be a tree with $n \geq 2$ vertices, and suppose $u$ is a centroid of $T$. Then, $|P_T^{-1}(u)| \geq \frac{n^2}{4}$.

*Proof.* The statement is trivially true for the only tree with $n = 2$ vertices. Otherwise, root the tree at $u$, and let the children of $u$ be $v_1, \ldots, v_{c(u)}$. Note that $c(u) \geq 2$ when $n \geq 3$: if $c(u) = 1$ then $T - u$ has a single component with $n - 1 > \frac{n}{2}$ vertices, so $u$ would not be a centroid of $T$, according to Lemma 2.1.

Suppose the subtree rooted at $v_i$ has $a_i$ vertices, so $a_1 + \cdots + a_{c(u)} = n - 1$. Now $|P_T^{-1}(u)|$ is the number of paths in $T$ passing through $u$. We can view each path through $u$ as starting in some subtree $v_i$ and ending at either $u$ or some other subtree $v_j$ where $j \neq i$. Hence, the number of paths through $u$ can be determined only from these subtree sizes, so we need only consider the distributions of the number of vertices among the subtrees.

Without loss of generality, further assume that $a_1 \geq a_2 \geq \cdots \geq a_{c(u)} \geq 1$. We will show that $|P_T^{-1}(u)|$ is minimized when $c(u) = 2$.

Suppose, for a contradiction, that $c'(u) \geq 3$ and $a'_1 \geq a'_2 \geq \cdots \geq a'_{c'(u)} \geq 1$ is some sequence of subtree sizes that induces strictly fewer paths through $u$. We also have that $a'_{c'(u)} \geq 1$. Further, we must have that either $a'_1 < \lceil \frac{n-1}{2} \rceil$ or $a'_2 < \lceil \frac{n-1}{2} \rceil$. Now consider how $|P_T^{-1}(u)|$ changes if we move one vertex from the smallest subtree to the largest (or the second largest, if $a'_1 = \lceil \frac{n-1}{2} \rceil$) subtree: we have invalidated at least $a'_2$ paths and created only $a'_{c'(u)} - 1$ additional paths, so we have obtained a scenario with strictly fewer paths through $u$, which is a contradiction.
Hence, $|\mathcal{P}_T^{-1}(u)|$ is minimized when $c(u) = 2$, so we must have $a_1 = \lceil \frac{n-1}{2} \rceil$ and $a_2 = \lceil \frac{n-1}{2} \rceil$, because $a_1 \leq \frac{n}{2}$ by Lemma 2.1. Therefore, there are $(a_1 + 1)(a_2 + 1) - 1$ paths through $u$, accounting also for those paths with $u$ as an endpoint. We have

$$(a_1 + 1)(a_2 + 1) - 1 = \left\lceil \frac{n+1}{2} \right\rceil \left\lfloor \frac{n+1}{2} \right\rfloor - 1$$

$$\geq \frac{n}{2} \left( \frac{n}{2} + 1 \right) - 1$$

$$\geq \frac{n^2}{4}$$

whenever $n \geq 2$, as required. \hfill \square

By considering the utility of the centroid, we obtain the following lower bound for the maximum
utility of any vertex, in terms of the diameter of the tree.

**Corollary 3.8.** Let $T$ be a tree with $n \geq 2$ vertices and diameter no greater than some constant $D$ and suppose $u$ is a centroid of $T$. Then, $\max_{v \in V} U_T(v) \geq U_T(u) \geq \frac{n^2}{16D}$.

We use these results to show that the removal of a vertex with maximum utility leaves the remaining
forest somewhat balanced. Specifically, it is never the case that one tree in this forest is so large
that it contains all but $o(n^{1/4})$ vertices.

**Theorem 3.9.** Let $T = (V, E)$ be an unweighted tree, with $n \geq 3$ vertices and suppose $v \in V$ minimizes $IGL(T - v)$. Further, suppose $C$ is a connected component in $T - v$ containing $l$ vertices
and let $r = n - l - 1$ be the number of vertices in $T - v$ not in $C$. Then, there is a constant $0 < c < 1$
independent of $n$ such that $r \geq cn^{1/4}$.

**Proof.** We may assume $l \geq 1$, since the case when $l = 0$ is trivial. We may also assume that $r \geq 1$,
since if $r = 0$, $v$ is a leaf, which contradicts its optimality for any tree with at least three vertices.

Since $T$ is a tree, each of the neighbours of $v$ belong to a different component in $T - v$. Suppose $x_C$ is the neighbour of $v$ in $C$ and let $C'$ be the tree obtained by adding the vertex $v$ and the edge $vx_C$. Thus, $v$ is a leaf of $C'$. We use this structure (pictured in Figure 2) to give two different, but
related, upper bounds for the utility $U_T(v)$ of $v$ in $T$.

**Claim 3.9.1.** $U_T(v) \leq \frac{1}{2}r(r + 1) + (r + 1)U_{C'}(v)$.

**Proof.** Let us upper bound $U_T(v)$ by considering the utility of $v$ in $C'$ and also in $T - V(C)$. There
are $n - l$ vertices in $T - V(C)$, so by Lemma 3.4, we have that $U_{T - V(C)}(v) \leq \frac{1}{2}(n - l)(n - l - 1) = \frac{1}{2}r(r + 1)$. This accounts for the pairs of vertices disconnected by the deletion of $v$ in $T - V(C)$.

We still need to consider such pairs where one vertex is in $C$, and the other is in $T - V(C)$ (this
includes $v$). Since $v$ is a leaf in $C'$, the only pairs of vertices connected in $C'$ that are disconnected
in $C = C' - v$ are those of the form $\{v, v_C\}$, where $v_C$ ranges over $V(C)$. Now let $u$ be a vertex in $V \setminus V(C)$. The path from $u$ to $v_C$ must pass through $v$, and thus $d_T(u, v_C) \geq d_T(v, v_C)$. Hence, the
contribution of each disconnected $\{u, v_C\}$ pair is at most that of $\{v, v_C\}$ towards $U_T(v)$. Putting
Figure 2: Layout of the vertices of $T$, in Theorem 3.9. Shaded vertices are in $A$, and are no more than $D = 5$ away from $v$. The value of $D$ here has chosen for example’s sake, and is not the true value constructed in the proof.

these inequalities together gives us

$$U_T(v) = \sum_{(p,q) \in P_T(v)} \frac{1}{d_T(p, q)}$$

$$= U_{T-V(C)}(v) + \sum_{u \in V \setminus V(C) \atop v_C \in V(C)} \frac{1}{d_T(u, v_C)}$$

$$\leq \frac{1}{2} r(r+1) + |V \setminus V(C)| \sum_{v_C \in V(C)} \frac{1}{d_T(v, v_C)}$$

$$\leq \frac{1}{2} r(r+1) + (r+1) \sum_{v_C \in V(C)} \frac{1}{d_T(v, v_C)}$$

$$= \frac{1}{2} r(r+1) + (r+1)U_{C'}(v),$$

as required.

\begin{claim}
$U_T(v) \leq rn$.
\end{claim}

\begin{proof}
Since $v$ is a leaf of $C'$, it is distance 1 away from its sole neighbour, and only this neighbour, in $C'$. Also, the only pairs disconnected by $v$’s removal in $C'$ are those containing $v$ itself. Now there are $l-1$ other vertices in $C'$, each at least distance 2 away from $v$. Hence, $U_{C'}(v) \leq 1 + \frac{l-1}{2} = \frac{l+1}{2} = \frac{n-r}{2}$.

Since $r \geq 1$, we know that $r+1 \leq 2r$. Hence, by Claim 3.9.1

$$U_T(v) \leq r^2 + 2rU_{C'}(v)$$

$$\leq r^2 + r(n - r)$$

$$= rn,$$
as required.

Since the utility of deleting $v$ is maximal among all vertices, and $n \geq 2$, we know $U_T(v) \geq n/2$ from Corollary 3.6. Combining this with Claim 3.9.1 and rearranging gives

$$U_{C'}(v) \geq \frac{n - r(r + 1)}{2(r + 1)}. \quad (3.9.1)$$

Suppose, for a contradiction, that $r < \frac{1}{15} n^{1/4}$. Since $r$ is purported to be relatively small, $U_{C'}(v)$ must be rather large (note it is proportional to $n$). Intuitively, this implies that many vertices in $C'$ are close to $v$, and hints towards a more central choice of vertex to delete. We will formally show that such a vertex exists, and is a more optimal choice.

Fix some distance $D$. We can divide the vertices of $C'$ into two groups $A$ and $B$: those within distance $D$ of $v$ in $C'$ (and thus, also in $T$) and those that are not, respectively. Suppose that $|A| = t$ and that $|B| = |V(C)| - t$. We can then derive the following upper bound

$$U_{C'}(v) \leq t + \frac{|V(C)| - t}{D + 1} \leq t + \frac{n - t}{D + 1}, \quad (3.9.2)$$

because each vertex in $B$ is at least distance $D + 1$ away from $v$, and $|V(C)| \leq n$. Note that we do not have to account for $v$ itself, since the distance to itself does not contribute towards its utility.

Recall that $r < \frac{1}{15} n^{1/4}$. It is easy to see that $r(r + 1) \leq n/2$. Combining this with (3.9.1) and (3.9.2) gives us the following inequality.

$$\frac{n}{4(r + 1)} \leq U_{C'}(v) \leq t + \frac{n - t}{D + 1},$$

from which we can obtain

$$tD \geq \frac{n(D + 1)}{4(r + 1)} - n.$$

If we choose $D = 8(r + 1) - 1$, it holds that $t \geq \frac{n}{D} = \frac{n}{8(r + 1)} \geq \frac{n}{15r}$.

Consider the subgraph (a tree) $T_A$ induced by the vertex set $A \cup \{v\}$. $T_A$ contains at least two vertices as $v$ and $x_C$ both must be in $A$. Also, since $T_A$ is a tree, by Lemma 2.1 it must have a centroid. Let one of the centroids of $T_A$ be $v_A$. The diameter of $T_A$ is at most $2D$, since every vertex in $T_A$ is within distance $D$ of $v$. Then, by Corollary 3.8, we have

$$U_{T_A}(v_A) \geq \frac{t^2}{8D} \geq \frac{n^2}{8(15)^3 r^3}.$$

Now every pair in $T_A$ that is disconnected by the deletion of $v_A$ is also disconnected in $T$ by the deletion of $v_A$, so $U_{T_A}(v_A) \leq U_T(v_A)$. Also, by the optimality of $v$ in $T$, we have that $U_T(v_A) \leq U_T(v)$. Hence, using the result of Claim 3.9.2, we can conclude that

$$\frac{n^2}{8(15)^3 r^3} \leq U_{T_A}(v_A) \leq U_T(v) \leq rn.$$
Thus, we have that
\[
\begin{align*}
    r^4 &\geq \frac{n}{8(15)^3} \\
    &\geq \frac{n}{15^3},
\end{align*}
\]
so \( r \geq \frac{1}{15} n^{1/4} \), which is a contradiction. The result follows with a choice of \( c = \frac{1}{15} \).

We can use this result to finally upper bound the number of vertices in any component after an optimal set of vertices has been removed.

**Theorem 3.10.** Let \( T = (V, E) \) be a tree with \( n \) vertices, and let \( S \subseteq V \) be some subset of vertices such that \( |S| = k \geq 1 \). There exists a positive constant \( c' \), independent of \( T \) and \( k \), such that whenever \( S \) minimizes \( IGL(T - S) \), \( S \) is \( \left( c' \left( \frac{n}{k} \right)^5 \right) \)-trimming.

**Proof.** We will call the components of \( T - S \) remaining components and denote each of them by the set of vertices it contains. Suppose the remaining components are \( R = \{ R_1, R_2, \ldots, R_l \} \), where \( R_i \subseteq V \) and \( S \cup R_1 \cup \ldots \cup R_l = V \). We need to show that \( |R_i| \leq c' \left( \frac{n}{k} \right)^5 \) for each \( R_i \).

We first construct a new graph \( T' = (V', E') \) by collapsing each of the remaining components. Formally, \( V' = R \cup S \), and, for each \( R_i \in R \) and \( s \in S \), \( \{ R_i, s \} \in E' \) if and only if there exists some \( r \in R_i \) such that \( \{ r, s \} \in E \). It can be seen that \( T' \) is necessarily a tree, and that every \( R_i \) is only incident to elements in \( S \), and vice-versa.

Suppose \( |R_r| \) has no fewer vertices than any other remaining component, so it suffices to show the upper bound holds for \( R_r \). We root \( T' \) at \( R_r \). Since \( k > 0 \), there are strictly fewer than \( n \) vertices among the remaining components \( R \). Hence, by the Pigeonhole Principle, there must be some \( s \in S \) such that the children \( R_{s_1}, R_{s_2}, \ldots, R_{s_{ch(s)}} \) of \( s \) in \( T' \) together contain fewer than \( \frac{n}{k} \) vertices. Let the parent of \( s \) in \( T' \) be \( R_{p(s)} \). This is depicted in Figure 3.

Since \( S \) is optimal, \( s \) must be an optimal choice of vertex to delete in an instance of IGL with graph \( T - (S \setminus \{ s \}) \) and a budget of 1 deletion. In particular, it must also be the optimal choice of vertex to delete in the component containing \( s \) in \( T - (S \setminus \{ s \}) \). Hence, we may apply Theorem 3.9 to \( T - (S \setminus \{ s \}) \), in that component to give

\[
\frac{n}{k} \geq \sum_{i=1}^{ch(s)} |R_{s_i}| \geq c( |R_{p(s)}| + |R_{s_1}| + \cdots + |R_{s_{ch(s)}}| )^{1/4} \\
\geq c |R_{p(s)}|^{1/4},
\]

since \( c > 0 \). Thus, we have \( |R_{p(s)}| \leq c^{-4} \left( \frac{n}{k} \right)^4 \).

We now have two cases: if the parent \( R_{p(s)} \) of \( s \) in \( T' \) is the root, \( R_r \), or if it is not the root. If \( R_{p(s)} \) is the root, then \( s_p = r \), so \( |R_r| \leq c^{-4} \left( \frac{n}{k} \right)^4 \).

Otherwise, \( s \) is not a child of the root, and so \( s \) must have been a more optimal choice than the best choice in the component induced by \( R_r \) in \( T - (S \setminus \{ s \}) \). Since this component contains \( |R_r| \) vertices, the best choice had utility at least \( \frac{|R_r|}{k} \), by Corollary 3.6. Now the paths that pass through \( s \) in \( T - (S \setminus \{ s \}) \) must have one endpoint in some \( R_{s_j} \) and the other either in another \( R_{s_j} \) or in \( R_{p(s)} \). This is the case since no path can have both endpoints in \( R_{p(s)} \). Hence, there are at most \( \left( \frac{n}{k} + 1 \right) \left( \frac{n}{k} + |R_{p(s)}| \right) \) such pairs when we account for those paths starting at \( s \). Since each of these paths have length at least 1, we have that
Figure 3: Bounding the size of the largest remaining component in Theorem 3.10. In this case, the parent of $s$ is not the root and $s$ has $ch(s) = 3$ children in $T'$. The shaded vertices are those in $S$.

\[
\frac{|R_r|}{2} \leq \mathcal{U}_{T-(S\setminus\{s\})}(s) \leq \left(\frac{n}{k} + 1\right) \left(\frac{n}{k} + |R_p(s)|\right) \\
\leq 2\frac{n}{k} \left(\frac{n}{k} + |R_p(s)|\right),
\]

because $k \leq n$. Thus

\[
\frac{|R_r|}{2} \leq 2\frac{n}{k} \left(\frac{n}{k} + c^{-4} \left(\frac{n}{k}\right)^4\right) \\
\leq 4c^{-4} \left(\frac{n}{k}\right)^5,
\]

because $0 < c < 1$. Hence, $|R_r| \leq 8c^{-4} \left(\frac{n}{k}\right)^5$ and the result follows with a choice of $c' = 8c^{-4}$. 

Importantly, we now have an upper bound on the largest $L$ that we need to consider in Lemma 3.2. This gives us our second algorithm.

**Corollary 3.11.** There is an $O(4^{c(n/k)^5} n^2)$ time, $O(n^2k)$ space algorithm for MinIGL on a tree, on a real RAM.

With an appropriate threshold, we can combine this with the result of Lemma 3.1 to give the subexponential time, polynomial space algorithm for Theorem 1.1.

**Theorem 1.1.** There is a $2^{O((n\log n)^{5/6})}$ time, $O(n^3)$ space algorithm for MinIGL on a tree, on a real RAM.
Proof. Lemma 3.1 gives us an $O(n^{k+2}) = 2^O(k \log n)$ time algorithm for MinIGL on a tree. Corollary 3.11 gives us an alternate $O(4c(n/k)^s n^2) = 2^O((n/k)^3 + \log n)$ time algorithm for the same problem. Note that the memory consumption of both algorithms is bound by $O(n^3)$.

Let $k^* = n^{5/6} \log^{-1/6} n$. We select the former algorithm when $k \leq k^*$, and the latter algorithm otherwise. In both cases, our running time is bound by $2^O((n \log n)^{5/6})$, as required.

4 Computing the IGL

Computing the IGL of a graph is trivial once its distance distribution has been determined. In this section, we describe algorithms for efficiently computing the distance distribution of trees, and graphs with small treewidth. We first give an algorithm specific to trees that uses fast polynomial multiplication, and later generalise our approach to graphs with bounded treewidth.

4.1 Trees

We proceed using a divide-and-conquer method (commonly known as the centroid decomposition, as used in [7]) as follows. We pick a vertex and compute the contribution to the distance distribution of all paths passing through that vertex. Then, we delete the vertex from the tree, and recurse on the remaining connected subtrees. We first provide a method that efficiently computes this contribution.

Theorem 4.1. Let $T = (V, E)$ be an unweighted tree with $n$ vertices and suppose $r \in V$. Then, the contribution to the distance distribution of all pairs in $P^{-1}_T(r)$ can be found in $O(n \log n)$ time on a log-RAM.

Proof. We begin by rooting the tree at $r$. Suppose the children of $r$ are $s_1, \ldots, s_{\text{ch}(r)}$ and let $S_1, \ldots, S_{\text{ch}(r)}$ denote the set of vertices in the subtrees rooted at each child, respectively. With the addition of $S_0 = \{r\}$, the sets $S_i$ form a partition of $V$.

We perform a depth-first search from $r$, to find $d_T(r,u) = d_T(u,r)$ for each vertex $u$ and construct a sequence of distance polynomials $P_0, P_1, \ldots, P_{\text{ch}(r)}$, where $P_i(x) = \sum_{w \in S_i} x^{d_T(r,w)}$. This takes $O(n)$ time. Now let

$$P(x) = \left( \sum_{0 \leq i \leq \text{ch}(r)} P_i(x) \right)^2 - \left( \sum_{0 \leq i \leq \text{ch}(r)} P_i^2(x) \right) = \sum_{0 \leq j \leq n} b_j x^j.$$ 

We claim that

$$b_j = 2|\{\{u,v\} \in P^{-1}_T(r) : u \neq v \text{ and } d_T(u,v) = j\}|,$$

that is, $b_j$ is twice the number of pairs of distinct vertices which have a path of length $j$ passing through $r$.

Suppose $u, v \in V$ so that $u \in S_y$ and $v \in S_z$. We claim that $\{u,v\} \in P^{-1}_T(r)$ if and only if $y \neq z$. If $y \neq z$, $u$ contributes an $x^{d_T(u,r)}$ term to $P_y(x)$ and $v$ contributes an $x^{d_T(r,v)}$ term to $P_z(x)$. Hence, together they contribute $2x^{d_T(u,r) + d_T(r,v)} = 2x^{d_T(u,v)}$ term to $P_T(x)$. If $y = z$, the path from $u$ to $v$ in $T$ does not pass through $r$. However, this “erroneous” term will not appear in $P(x)$ as it is cancelled by an equal term in $P_y^2(x)$. Thus, $b_j$ satisfies (4.1.1).

Given the coefficients of $P(x)$, we produce the required contribution to the distance distribution by dividing each coefficient by 2. It remains to show that the coefficient form of $P(x)$ can be computed efficiently from the coefficients of each $P_i$. 

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Since any two vertices in $T$ are at most distance $n$ apart, the maximum degree of any distance polynomial is $n$. Hence, $\sum_{0 \leq i \leq \text{ch}(r)} \deg(P_i)x^i$ has degree at most $n$. As a result, we find the coefficients of the square of this polynomial in $O(n \log n)$ time by Lemma 2.2. In the same way, we also find the coefficient form of $P_i$ in $O(\deg(P_i)\log \deg(P_i))$ time. Now $\deg(P_i) \leq |S_i| \leq n$, so

$$\sum_{0 \leq i \leq \text{ch}(r)} \deg(P_i)\log \deg(P_i) \leq \sum_{0 \leq i \leq \text{ch}(r)} \deg(P_i) \log n \leq \sum_{0 \leq i \leq \text{ch}(r)} |S_i| \log n = n \log n,$$

because the sets $S_0, \ldots, S_{\text{ch}(r)}$ form a partition of $V$. All other steps in the algorithm can be performed in $O(n)$ time, and so the result follows.

If we always pick $r$ in Theorem 4.1 to be a centroid of the tree, Lemma 2.1 ensures that each vertex can appear in at most $\log n + 1$ trees throughout the execution of our divide-and-conquer algorithm. A centroid must always exist (also by Lemma 2.1), and we can find one in linear time by recursively computing, then examining, subtree sizes. This gives Theorem 1.2.

If we only wish to determine the first $P$ values of the distance distribution of $T$, we can modify Theorem 4.1 to run in $O(n + P\log n)$ time, by discarding all terms with degree $P + 1$ when constructing the polynomials. Thus, the expensive multiplication step costs $O(P\log n)$ time by Lemma 2.2, and we obtain the following corollary.

**Corollary 4.2.** The prefix $a_1, \ldots, a_P$ of the distance distribution of a tree with $n$ vertices can be computed in $O(n \log n + P \log^2 n)$ time on a log-RAM.

### 4.2 Graphs with bounded treewidth

In this section we prove Theorem 1.3. Let $G = (V, E)$ be an undirected graph with $n$ vertices, whose edges each have a non-negative weight. In time $2^{O(k)n}$, we can compute a tree decomposition of $G$ of width at most $k = 5 \cdot \text{tw}(G) + 4$ containing at most $O(kn)$ nodes [10]. Using a common technique, we can transform this decomposition into a *nice* tree decomposition with $N = O(kn)$ nodes (see, for example [16]).

A *portal* of a subset of vertices $A$ is a vertex in $A$ which has, as a neighbour, some vertex outside $A$. If these portals are contained in some set $S$, we can partition the vertices of the graph into three sets, $A, S$ and $V \setminus A$, such that every path from a vertex in $A$ to a vertex in $V \setminus A$ passes through some vertex in $S$.

Since the nice tree decomposition is a binary tree, there is some edge $ij$ in the decomposition whose removal splits the decomposition’s tree into two components $I$ and $J$ (containing $i$ and $j$, respectively), each containing between $\frac{N}{2}$ and $\frac{2N}{3}$ nodes. Let $A$ be the set of vertices that appear in component $I$, and let $S$ be the intersection $B_i \cap B_j$ of the bags corresponding to nodes $i$ and $j$. Necessarily, $S$ must contain all the portals of $A$ due to properties of the tree decomposition. Moreover, $|B_i \cap B_j| \leq \min(|B_i|, |B_j|) \leq k + 1$.

Given this fixed $A$, recursively find the distance distribution among all pairs of vertices in $A$ as follows. First, perform Dijkstra’s algorithm from all vertices in $S$. For every pair of vertices in $S$, add an edge whose weight equals the length of the shortest path between them. After these edges are added, the length of the shortest path in $G$ between any pair of vertices in $A$ can be found by only considering paths passing through the vertices of $A$. Hence, we remove all vertices in $V \setminus A$, and recurse on this smaller graph. Note that $I$ is a valid tree decomposition for this new graph.
Lemma 4.3. When \( d = 1 \), there is an algorithm that solves RedBluePolynomial in \( O(n\sqrt{M\log n} + n\log n) \) time on a log-RAM.

Proof. Sort the red and blue points together in non-decreasing order of the coordinate, placing blue points earlier in the order when there are ties. Let \( X \) be a positive integer no greater than \( n \). Assign points to groups of size no more than \( X \) by placing the first \( X \) points, in order, into a group \( G_1 \), followed by the next \( X \) points in order into a group \( G_2 \), and so on, so we create \( \lceil \frac{n}{X} \rceil \) groups in all. An example is given in Figure 4.
Figure 4: Square root decomposition in Lemma 4.3. The empty dots represent red points, and the shaded dots represent blue points. When processing $G_3$, we consider pairs of points within the group where the red point precedes the blue point. We then consider cross-group pairs whose blue point is in $G_3$ using fast polynomial multiplication.

We will separately consider pairs of points that both belong to the same group, and those that belong to different groups. In each group, consider every pair of points, and check if they contribute a term to the polynomial. This takes $O(nX)$ time over all groups.

It remains to consider pairs that belong to different groups: call these cross-group pairs. For each blue point in $G_i$, we must add an extra term for each red point among $G_1, \ldots, G_{i-1}$. Thus, the total cross-group contribution of all pairs with a blue point in $G_i$ can be written as the following product of two polynomials.

$$\sum_{B_q \in G_i} \sum_{R_p \in G_1 \cup \cdots \cup G_{i-1}} x_{R_p} b_q = \left( \sum_{R_p \in G_1 \cup \cdots \cup G_{i-1}} x_{R_p} \right) \left( \sum_{B_q \in G_i} x_{b_q} \right)$$

To compute these contributions, iterate over each group in order, maintaining the coefficient form of the polynomial representing all red points in groups processed thus far. This corresponds to the first multiplicand on the right hand side. We can quickly construct the second multiplicand directly from the elements in this group. Note that the degree of both multiplicands does not exceed $M$, and that the coefficients of the product do not exceed $n^2$. Hence, we can compute the product of these two polynomials in $O(M \log n)$ time by Lemma 2.2, so we can compute the cross-group contributions in $O(nX M \log n)$ time.

Combining these parts and accounting for our initial sort, we obtain a running time of $O(n \log n + nX + \frac{n}{X} M \log n)$. When $n \geq \sqrt{M \log n}$, a choice of $X = \sqrt{M \log n}$ gives the desired time complexity. Otherwise, $n < \sqrt{M \log n}$ and we can simply compare every pair of points in $O(n^2) = O(n \sqrt{M \log n})$ time, which also fits in the required time complexity.

**Theorem 4.4.** There is an algorithm that solves RedBluePolynomial in $2^{O(d)n^{1+\varepsilon}} \sqrt{M}$ time on a log-RAM, for every $\varepsilon > 0$.

**Proof.** When $d = 1$, we simply use Lemma 4.3 which has the required time complexity in this case. Otherwise, we will use the divide-and-conquer method of Bentley [6] to reduce the problem to smaller dimensions.

First, combine the red and blue points into one list and apply divide-and-conquer as follows. Let $x_m$ be the median value among the first coordinate of all points. This can found in $O(n)$ time [8]. We divide the list into two halves as follows. First assign those points with first coordinate less than $x_m$ into the first half, and those with first coordinate greater than $x_m$ into the second half. Among those with first coordinate precisely $x_m$, assign blue points to the first half until the first half has $\frac{n}{2}$ points. Assign the remaining points to the second half. This assignment can be done in
\(O(n)\) time and has the property that if \(R_p < B_q\), then either both points belong to the same half, or they belong to the first and second half, respectively.

Next, recursively compute the contribution of both groups to the final polynomial. The remaining pairs that may contribute terms to the result must have a red point in the first half, and a blue point in the second half. Since the ordering guarantees that all points in the first half have a first coordinate no greater than those in the second half, we project the red points in the first half together with the blue points in the second half onto a \((d-1)\)-dimensional space by simply ignoring the first coordinate of each point. We then solve \textbf{RedBluePolynomial} for this set of points in \(d-1\) dimensions recursively. This concludes the description of the algorithm.

The time complexity of our algorithm on \(n\) points and in \(k\) dimensions can be described by \(T(n, d)\), which satisfies the recurrence:

\[
T(n, d) = 2T(n/2, d) + T(n, d-1) + O(n)
\]
\[
T(n, 1) = O(n\sqrt{M\log n} + n \log n)
\]
\[
T(1, d) = O(1).
\]

Bringmann et al. [11], using the results of Monier [29], showed that the recurrence

\[
Q(n, d) = 2Q(n/2, d) + Q(n, d-1) + O(n)
\]
\[
Q(n, 2) = O(n\log n)
\]
\[
Q(1, d) = O(1)
\]

satisfies \(Q(n, d) = 2^{O(d)}n^{1+\varepsilon}\), for every \(\varepsilon > 0\). Observing that \(T(n, d) = O(Q(n, d+1)\sqrt{M\log n})\) and that \(\sqrt{\log n} = O(n^{\varepsilon^*})\) for any \(\varepsilon^* > 0\), completes the proof.

We are now able to evaluate the time complexity of our algorithm for computing the prefix of the distance distribution. To find the contribution of pairs in \((A \setminus S) \times (V \setminus A)\), we solve an instance of \textbf{RedBluePolynomial} in \(|S| - 1 \leq k\) dimensions, using the result of Theorem 4.4. As our algorithm performs divide-and-conquer over the nodes of the tree decomposition, each vertex induces the creation of a point in \(O(\log(\text{kn})) = O(\log n)\) instances of \textbf{RedBluePolynomial}. Hence, since the time complexity of Theorem 4.4 is superadditive with respect to \(n\), the total running time over all instances of \textbf{RedBluePolynomial} is \(2^{O(k)}n^{1+\varepsilon}\log n\sqrt{P} = 2^{O(k)}n^{1+\varepsilon'}\sqrt{P}\) for any \(\varepsilon' > 0\).

Since we are working on a (nice) tree decomposition with \(O(\text{kn})\) nodes, the running time of finding an appropriate dividing edge in the tree, and performing \(k\) Dijkstra’s per instance are negligible compared to that of solving our instances of \textbf{RedBluePolynomial}. By observing that \(k = O(\text{tw}(G))\), we obtain Theorem 1.3.

Theorem 1.3 can easily be extended to directed graphs, and weighted graphs with a suitable choice of \(P\). We omit the proof here, as the modifications are straightforward. On undirected graphs whose edges have unit weight, setting \(P = n-1\) determines the entire distance distribution.

**Corollary 4.5.** The distance distribution of an undirected graph with \(n\) vertices, edges of unit weight and treewidth \(k\) can be computed in \(2^{O(k)}n^{3/2+\varepsilon}\) time on a log-RAM, for any \(\varepsilon > 0\).

5 Conclusion

We have provided a method to solve \textbf{MinIGL} on trees in subexponential time and space. This method can be extended to problems that ask to minimize other metrics on a forest, by deleting \(k\) vertices from a tree with \(n\) vertices. Specifically, the method applies when:
1. the value of the metric on a forest can be computed in polynomial time, and is equal to the sum of the value of the metric on each of its trees; and

2. there exists some constant $c > 0$, such that, for any instance, there exists an optimal solution that is $O\left((\frac{n}{k})^c\right)$-trimming.

For graphs with treewidth $k$, we have shown that in $2^{O(k)}n^{3/2+\epsilon}$ time, one can compute the entire distance distribution of the input graph. Compared to the $O(kn^2)$ time algorithm for computing APSP [31], our dependence on $n$ is a factor of $O(\sqrt{n})$ less, though our dependence on $k$ is exponential. Our algorithm is a $O(\sqrt{n})$ factor slower than the current best-known $2^{O(k)}n^{1+\epsilon}$ time algorithm for diameter [1]. For graphs with constant diameter, the extra factor becomes $O(n^\epsilon)$ for any $\epsilon > 0$, when compared to the current best-known $2^{O(k)}n$ time algorithm for diameter in this setting. This might be expected, as the distance distribution implies the diameter, and is implied by the APSP, but we find it somewhat surprising that the distance distribution can be computed faster than APSP on graphs with bounded treewidth.

Our results can be immediately applied to compute any metric of a graph that can be expressed as a function of the distance distribution. However, they are difficult to adapt to metrics that compute properties of individual vertices in the graph, as we exploit properties exclusive to counting pairs that are certain distances apart, without expressly considering which vertices belong to such pairs. In particular, this means that our results are unlikely to directly provide further insight into the efficient computation of related metrics, such as the task of computing closeness centrality [5] of every vertex in a given graph.

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