MOLINO'S DESCRIPTION AND FOLIATED HOMOGENEITY

JESÚS A. ÁLVAREZ LÓPEZ AND RAMÓN BARRAL LIJÓ

Abstract. The topological Molino's description of equicontinuous foliated spaces, studied by the first author and Moreira Galicia, is sharpened by introducing a foliated action of a compact topological group on the resulting $G$-foliated space, like in the case of Riemannian foliations. Moreover a $C^\infty$ version is also studied. The triviality of this compact group characterizes compact minimal $G$-foliated spaces, which are also characterized by their foliated homogeneity in the $C^\infty$ case. Examples are also given, where the projection of the Molino's description is not a principal bundle, and the foliated homogeneity cannot be only checked by comparing pairs of leaves.

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1. INTRODUCTION

A description of certain compact minimal equicontinuous foliated spaces was given by the first author and Moreira Galicia [10]. It can be considered as a topological version of the Molino's description of Riemannian foliations.
on compact manifolds [34], in the minimal case. This gave another confirmation that equicontinuous foliated spaces should be considered as the topological Riemannian foliations, as asserted by Ghys [34, Appendix E]. That description reduces the study of such foliated spaces to the case of foliated spaces whose transverse dynamics is modeled by local left translations of a local group \( G \), called \( G \)-foliated spaces. According to the role played by Molino’s theory in the study of Riemannian foliations, its topological version should have interesting applications; for instance, it was already used in [10] to study the growth of the leaves. Dyer, Hurder and Lukina also gave an analogue of Molino’s description for equicontinuous actions on the Cantor set [18], and for equicontinuous matchbox manifolds [19] (the case of compact connected minimal foliated spaces of codimension zero).

Our first goal is to show the following slight sharpening of the main result of the topological Molino’s theory (Section 3). (The terminology and notation used here are recalled in Section 2.)

**Theorem A** (Cf. [10, Theorem A]). Suppose that a foliated space \( X \equiv (X, \mathcal{F}) \) is compact, minimal, equicontinuous and strongly quasi-analytic, and the closure of its holonomy pseudogroup is also strongly quasi-analytic. Then there is a local group \( G \), a compact topological group \( H \), a compact minimal \( G \)-foliated space \( \hat{X}_0 \equiv (\hat{X}_0, \hat{\mathcal{F}}_0) \), a foliated map \( \hat{\pi}_0 : \hat{X}_0 \to X \), and a free foliated right \( H \)-action on \( \hat{X}_0 \) such that the restrictions of \( \hat{\pi}_0 \) to the leaves of \( \hat{X}_0 \) are the holonomy coverings of the leaves of \( X \), and \( \hat{\pi}_0 \) induces a homeomorphism \( \hat{X}_0/\text{slash} \cdot H \to X \).

Precisely, our new contribution in Theorem A is the existence of \( H \) satisfying the stated properties. If \( H \) is the representative of the holonomy pseudogroup of \( X \) on a space \( T \) induced by the choice of a good foliated atlas, and we fix some \( u_0 \in T \), then \( H \) is the group of germs at \( u_0 \) of the maps \( g \) in the closure \( \overline{H} \) with \( u_0 \in \text{dom} \cdot g \) and \( g(u_0) = u_0 \). Following the construction of \( \hat{X}_0 \) in [10], we get a compatible compact topology on \( H \) and a right foliated \( H \)-action on \( \hat{X}_0 \) satisfying the statement of Theorem A.

We also show that the construction of \((G, H, \hat{X}_0, \hat{\pi}_0)\) is independent of the choices involved up to the obvious equivalences (Proposition 3.1), and therefore \((G, H, \hat{X}_0, \hat{\pi}_0)\) is called the Molino’s description of \( X \); in particular, \( G \) is called the structural local group according to [34, 10], and \( H \) is called the discriminant group according to [18]. Under the hypothesis of Theorem A we also prove the following additional properties:

- \( X \) is a \( G \)-foliated space for some local group \( G \) if and only if its discriminant group is trivial (Proposition 3.2).
- \( H \) contains the holonomy group of every leaf (Proposition 3.4).
- If \( X \) is \( C^\infty \), then its Molino’s description becomes \( C^\infty \) in a unique obvious sense (Proposition 5.1).
- The map \( \hat{\pi}_0 \) may not be a fiber bundle (an example is given in Section 5.2). This is the only missing property when comparing with the Riemannian foliation case.
Our second goal is to characterize $G$-foliated spaces using a property called foliated homogeneity. A foliated space $X \equiv (X, F)$ is called foliated homogeneous if the group Homeo$(X, F)$ of its foliated transformations acts transitively on itself (a foliated version of homogeneity). If $X$ is $C^\infty$, then $C^\infty$ foliated homogeneity can be similarly defined using $C^\infty$ foliated diffeomorphisms. This notion was studied by Clark and Hurder in the case of matchbox manifolds [15], where homogeneity and foliated homogeneity are equivalent conditions because any map between matchbox manifolds is foliated. Clark and Hurder have shown that a matchbox manifold is homogeneous if and only if it is a McCord solenoid (an inverse limits of towers of regular covers of compact connected manifolds). Since McCord solenoids are transversely modeled by left translations on profinite groups, they are particular cases of $G$-foliated spaces. Thus it makes sense to ask whether any compact minimal foliated space is foliated homogeneous if and only if it is a $G$-foliated space. We give the following answers to this question.

**Theorem B.** If a foliated space $X$ is compact, minimal and foliated homogeneous, then it satisfies hypotheses of Theorem A and is a $G$-foliated space for some local group $G$.

**Theorem C.** Suppose that a foliated space $X$ is compact, minimal and $C^\infty$. Then the following conditions are equivalent:

(i) $X$ is $C^\infty$ foliated homogeneous.

(ii) $X$ is foliated homogeneous.

(iii) $X$ satisfies the hypotheses of Theorem A and is a $G$-foliated space for some local group $G$.

Theorem B follows with an adaptation of an argument of Clark and Hurder [15, Theorem 5.2], using that the canonical left action of Homeo$(X, F)$ on $X$ is micro-transitive by a theorem of Effros [20, 39].

To prove Theorem C, it is enough to show “(iii) $\Rightarrow$ (i)” by Theorem B. Assuming (iii), we get the so-called structural right local transverse action, which has its own interest; for instance, it was introduced and used in [7] for Lie foliations. It is the unique “foliated right local action up to leafwise homotopies” of $G$ on $X$, which corresponds to the local right translations on $G$ via foliated charts (Proposition 6.6 and Section 6.3). Its construction uses a partition of unity subordinated to a foliated atlas and the leafwise center of mass to merge the obvious right local transverse actions on the domains of foliated charts. The structural right local transverse action gives (i) because we always have leafwise homogeneity (Proposition 7.1).

Since there exist leaves without holonomy, and since the quasi-isometry type of the leaves is independent of the choice of a (leafwise) Riemannian metric on $X$, it follows that $X$ is not foliated homogeneous if there is a leaf with holonomy, or if there is a pair of non-quasi-isometric leaves. The reciprocal statement is not true in general (Section 8.3). Precisely, we give an example of a (leafwise) Riemannian $C^\infty$ compact minimal equicontinuous
foliated space \( X \), with a Molino’s description, such that \( X \) is not foliated homogeneous, whereas \( X \) has no holonomy and all of its leaves are isometric to each other.

Hurder suggested that, generalizing McCord solenoids, an interesting example of compact minimal foliated homogeneous foliated space is defined by the inverse limit of any tower of foliated regular coverings between Lie foliations on compact connected manifolds (Section 8.4). We may ask whether all compact minimal homogeneous foliated spaces of finite topological codimension have such a description, like in the case of matchbox manifolds (zero topological codimension).

2. Preliminaries

See [35, Chapter II], [23] and [13, Chapter 11] for the needed preliminaries on foliated spaces and interesting examples, and [25, 26, 27] for the preliminaries on pseudogroups. We mainly follow [10, Sections 2 and 4A], which in turn follows [4, 5, 6]. Some ideas are also taken from [15, 9, 8]. The needed basic concepts and tools are recalled here for the reader’s convenience, and a few new observations are also made.

In the whole paper, unless otherwise stated, spaces are assumed to be locally compact and Polish, and maps are assumed to be continuous. In particular, this applies to foliated spaces, topological groups, local groups and partial maps.

2.1. Pseudogroups. For spaces \( T \) and \( T' \), recall that a \emph{paro map} \( \phi : T \rightarrow T' \) is a partial map whose domain is open in \( T \). The germ of \( \phi \) at any \( u \in \text{dom} \phi \) will be denoted by \( \gamma(\phi, u) \). If \( \phi \) is an open embedding, we may identify \( \phi \) with the homeomorphism \( \phi : \text{dom} \phi \rightarrow \text{im} \phi \) of an open subset of \( T \) to an open subset of \( T' \), whose inverse can be considered as a paro map \( \phi^{-1} : T' \rightarrow T \); in particular, when \( T = T' \), such a \( \phi \) is called a \emph{local transformation} of \( T \).

Let \( \Phi \) and \( \Psi \) be families of paro maps \( T \rightarrow T' \) and \( T' \rightarrow T'' \), respectively, for another space \( T'' \). We use the notation \( \Psi \Phi = \{ \psi \phi \mid \phi \in \Phi, \psi \in \Psi \} \); in particular, \( \Phi^n = \Phi \cdots \Phi \) (\( n \) times) if \( T = T' \) and \( n \in \mathbb{Z}^+ \). If \( \Phi \) consists of open embeddings, let \( \Phi^{-1} = \{ \phi^{-1} \mid \phi \in \Phi \} \).

Recall that a \emph{pseudogroup} \( \mathcal{H} \) on \( T \) is a family of local transformations of \( T \) that contains \( \text{id}_T \), and is closed by the operations of composite, inversion, restriction to open sets and union. It is said that \( \mathcal{H} \) is \emph{generated} by \( S \subset \mathcal{H} \) if \( \mathcal{H} \) can be obtained from \( S \) using the above operations. By considering a pseudogroup as a direct generalization of a group of transformations, the basic dynamical concepts have obvious generalizations to pseudogroups, like \emph{orbits}, \emph{saturation}, \( (\text{topological}) \) \emph{transitivity} and \emph{minimality}. The orbit space is denoted by \( T/\mathcal{H} \). The \( \mathcal{H} \)-saturation of any \( A \subset T \) is denoted by \( \mathcal{H}(A) \), and the orbit of any \( u \in T \) by \( \mathcal{H}(u) \). For any open \( V \subset T \), the \emph{restriction} \( \mathcal{H}|_V := \{ h \in \mathcal{H} \mid \text{dom} h, \text{im} h \subset V \} \) is a pseudogroup.

Given another pseudogroup \( \mathcal{H}' \) on \( T' \), a \emph{morphism} \( \Phi : \mathcal{H} \rightarrow \mathcal{H}' \) is a maximal collection of paro maps \( T \rightarrow T' \) such that \( \mathcal{H}'\Phi \mathcal{H} \subset \Phi \), \( T = \bigcup_{\phi \in \Phi} \text{dom} \phi \),
there is a unique morphism \( \Phi : H \to H \)

Let \( \Phi_0 \) be a family of paro maps \( T \to T' \) such that \( T = H(\bigcup_{\phi \in \Phi} \text{dom } \phi) \), and there is a subset \( S \) of generators of \( H \) such that, if \( \phi, \psi \in \Phi_0 \), \( h \in S \) and \( u \in \text{dom } \phi \cap \text{dom } \psi \), then there is some \( h' \in H' \) so that \( \phi(u) \in \text{dom } h' \) and \( \gamma(h'\phi, u) = \gamma(\psi h, u) \). Then there is a unique morphism \( \Phi : H \to H' \) containing \( \Phi_0 \), which is said to be generated by \( \Phi_0 \). For instance, \( \text{id}_T \) generates a morphism \( \text{id}_H : H \to H \)

Of course, \( \text{id}_H \) is the pseudogroup of all possible unions of maps in \( H \); in particular, \( H \subseteq \text{id}_H \). For another pseudogroup \( H'' \) on \( T'' \) and a morphism \( \Psi : H' \to H'' \), the family \( \Psi \Phi \) generates a morphism \( H \to H'' \), which may be also denoted by \( \Psi \Phi \) with some abuse of notation.

In this way, the morphisms of pseudogroups form a category \( \text{PsGr} \). There is a canonical functor \( \text{Top} \to \text{PsGr} \), assigning the pseudogroup generated by \( \text{id}_T \), also denoted by \( T \), to every topological space \( T \), and assigning the morphism generated by \( \phi \), also denoted by \( \phi \), to every map \( \phi : T \to T' \). A morphism \( \Phi : H \to H' \) is an isomorphism of \( \text{PsGr} \) if and only if it is generated by a family \( \Phi_0 \) of open embeddings such that \( \Phi_0^{-1} \) generates a morphism \( H' \to H \), which is the inverse \( \Phi^{-1} \) in \( \text{PsGr} \).

With the terminology of Haefliger [25, 26, 27], an étalé morphism \( \Phi : H \to H' \) is a maximal family of homeomorphisms of open subsets of \( T \) to open subsets of \( T' \) such that \( H' \Phi H \subseteq \Phi, T = \bigcup_{\phi \in \Phi} \text{dom } \phi \) and \( \Phi \Phi^{-1} \subseteq H' \).

If moreover \( \Phi^{-1} \) is an étalé morphism, then \( \Phi \) is called an equivalence, and the pseudogroups \( H \) and \( H' \) are said to be equivalent. If \( \Phi_0 \) is a family of homeomorphisms of open subsets of \( T \) to open subsets of \( T' \) such that \( T = H(\bigcup_{\phi \in \Phi} \text{dom } \phi) \) and \( \Phi_0 H \Phi_0^{-1} \subseteq H' \), then there is a unique étalé morphism \( \Phi : H \to H' \) containing \( \Phi_0 \), which is said to be generated by \( \Phi_0 \). Any equivalence generates an isomorphism in \( \text{PsGr} \), and, vice versa, any isomorphism in \( \text{PsGr} \)

is generated by a unique equivalence. Hence isomorphisms and equivalences are equivalent concepts. Equivalent pseudogroups are considered to have the same dynamics. For instance, \( H \) is equivalent to \( H|_V \) for any open \( V \subseteq T \) that meets all \( H \)-orbits. In fact, \( \Phi : H \to H' \) is an equivalence if and only if \( G = H \cup H' \cup \Phi \cup \Phi^{-1} \) is a pseudogroup on \( T \cup T' \) such that \( T \) and \( T' \) meet all \( G \)-orbits, \( G|_T = H \) and \( G|_T' = H' \).

Let \( \tilde{G} \) and \( \tilde{G}' \) be other pseudogroups on respective spaces \( Z \) and \( Z' \), and let \( \Psi : \tilde{G} \to \tilde{G}' \) be another morphism. The product \( H \times \tilde{G} \) is the pseudogroup on \( T \times Z \) generated by the maps \( h \times g \), for \( h \in H \) and \( g \in \tilde{G} \). The product \( \Phi \times \psi \) is the morphism \( H \times \tilde{G} \to H' \times \tilde{G}' \) generated by the maps \( \phi \times \psi \) for \( \phi \in \Phi \)

and \( \psi \in \Psi \).

The germs \( \gamma(h, u) \), \( h \in H \) and \( u \in \text{dom } h \), form a topological groupoid \( \mathcal{H} \), equipped with the sheaf topology and the operation induced by composite. Its unit subspace can be identified to \( T \). In fact, \( \mathcal{H} \) is an étalé groupoid (the source and target maps, \( s, t : \mathcal{H} \to T \), are local homeomorphisms).

Let us recall the following definitions of properties that \( H \) may have:

**Compact generation:** This means that there is a relatively compact open \( U \subseteq T \), which meets all orbits, such that \( H|_U \) is generated.
by a finite set, $E = \{h_1, \ldots, h_k\}$, and every $h_i$ has an extension $\tilde{h}_i \in \mathcal{H}$ with $\text{dom}\tilde{h}_i \subset \text{dom}h_i$. This $E$ is called a system of compact generation of $\mathcal{H}$ on $U$.

(Strong) equicontinuity: This means that there are an open cover $\{T_i\}$ of $T$ and a metric $d_i$ inducing the topology of every $T_i$, and $\mathcal{H}$ is generated by some subset $S \subset \mathcal{H}$, with $S^2 \subset S = S^{-1}$ ($S$ is symmetric and closed by composite), such that, for every $\epsilon > 0$, there is some $\delta > 0$ so that

$$d_i(x, y) < \delta \implies d_j(h(x), h(y)) < \epsilon$$

for all $h \in S$, indices $i, j$, and $x, y \in T_i \cap h^{-1}(T_j \cap \text{im } h)$.

Strong quasi-analyticity: This means that $\mathcal{H}$ is generated by some subset $S \subset \mathcal{H}$, with $S^2 \subset S = S^{-1}$, such that, if any $h \in S$ is the identity on some non-empty open subset of its domain, then $h = \text{id}_{\text{dom}h}$.

Strong local freeness: This means that $\mathcal{H}$ is generated by some subset $S \subset \mathcal{H}$, with $S^2 \subset S = S^{-1}$, such that, if any $h \in S$ fixes some point in its domain, then $h = \text{id}_{\text{dom}h}$.

These properties are invariant by equivalences. If compact generation holds with some $U$, then it also holds with any other relatively compact open subset of $T$ that meets all orbits. Let $\mathcal{P}$ denote any of the above last three properties. If $\mathcal{P}$ holds with $S$, then it also holds with its localization,

$$S_{\text{loc}} = \{h|_O \mid h \in S, O \text{ is open in dom } h\}.$$

Moreover we can add $\text{id}_T$ to $S$ if desired (obtaining $S^2 = S$). If $\mathcal{H}$ is compactly generated and satisfies $\mathcal{P}$, then, for every relatively compact open $U \subset T$ that meets all orbits, we can choose a system of compact generation $E$ of $\mathcal{H}$ on $U$ such that $\mathcal{H}|_U$ also satisfies $\mathcal{P}$ with $S = \bigcup_{n=1}^{\infty} E^n$. The following result lists some needed non-elementary properties.

**Proposition 2.1** ([8 Proposition 8.9, and Theorems 11.1 and 12.1], [38 and 2 Theorems 3.3 and 5.2]). Suppose that $\mathcal{H}$ is compactly generated, equicontinuous and strongly quasi-analytic. Then the following holds:

(i) Assume that $\mathcal{H}$ satisfies the condition of compact generation with $U$, $E = \{h_1, \ldots, h_k\}$ and $\tilde{h}_1, \ldots, \tilde{h}_k$. For every $h = h_{i_n \ldots i_1} \in \bigcup_{n=1}^{\infty} E^n$, let $\tilde{h} = \tilde{h}_{i_n \ldots i_1}$. Then there is a finite family $\mathcal{V}$ of open subsets of $T$ covering $U$ such that, for any $h \in \bigcup_{n=1}^{\infty} E^n$ and $V \in \mathcal{V}$, we have $V \subset \text{dom } \tilde{h}$ if $V \cap \text{dom } h \neq \emptyset$.

(ii) Suppose that $\mathcal{H}$ satisfies the equicontinuity condition with a set $S$. Then $C(O, T) \cap S_{\text{loc}}$ consists of local transformations for all small enough open subsets $O \subset T$, where the closure is taken in the compact-open topology, and the pseudogroup $\overline{\mathcal{H}}$ generated by such

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1The term pseudo-group was used in [11] when these conditions are satisfied. This term was introduced in [38] for a family that moreover contains $\text{id}_T$ and is also closed by restrictions to open subsets.
2.2. Relation of pseudogroups with local groups and local actions.

The general definition of local group is rather involved [31], but, in the locally compact case, a local group \( G \) can be considered as neighborhood of the identity element \( e \) in some topological group [16 17]. Two such neighborhoods in the same topological group define equivalent local groups; thus it can be said that, up to equivalences, a local group is the “germ” of a topological group at the identity element. For the sake of simplicity, the family of open neighborhoods of \( e \) in \( G \) will be denoted by \( \mathcal{N}(G,e) \). Given another local group \( G' \) with identity element \( e' \), a local homomorphism of \( G \) to \( G' \) is a paro map \( \sigma : G \rightarrow G' \) such that \( \sigma(e) = e' \), and \( \sigma(gh) = \sigma(g)\sigma(h) \) for all \( g,h \in \text{dom}\sigma \) such that the products \( gh \) and \( \sigma(g)\sigma(h) \) are defined, with \( gh \in \text{dom}\sigma \). Two local homomorphisms of \( G \) to \( G' \) are equivalent when they have the same germ at \( e \). If there is a local isomorphism \( \tau : G' \rightarrow G \) such that \( \tau \sigma \) and \( \sigma \tau \) are equivalent to \( \text{id}_G \) and \( \text{id}_{G'} \), then \( \sigma \) is called a local isomorphism. A local anti-homomorphism of \( G \) to \( G' \) is similarly defined, requiring \( \sigma(gh) = \sigma(h)\sigma(g) \) for all \( g,h \in \text{dom}\sigma \) such that the products \( gh \) and \( \sigma(h)\sigma(g) \) are defined, with \( gh \in \text{dom}\sigma \), and the corresponding equivalence relation has the same meaning. A right local action of \( G \) on \( T \) is a paro map \( \chi : T \times G \rightarrow T \), with \( T \times \{e\} \subset \text{dom}\chi \) and \( \chi(u,e) = u \) for all \( u \in T \), and such that, for all \( g,h \in G \) and \( u \in T \), if the product \( gh \) is defined and \( (u,g), (u,gh), (\chi(u,g),h) \in \text{dom}\chi \), then \( \chi(\chi(u,g),h) = \chi(u,gh) \). Two right local actions of \( G \) on \( T \) are equivalent when they agree around \( T \times \{e\} \). If \( T \) is compact, we can assume \( \text{dom}\chi = T \times O \) for some \( O \in \mathcal{N}(G,e) \). For any open \( V \subset T \), the restriction \( \chi : \chi^{-1}(V) \cap (V \times G) \rightarrow V \) is a right local action of \( G \) on \( V \), called the restriction of \( \chi \) to \( V \). Given an open cover \( \{T_i\} \) of \( T \) and a right local action \( \chi_i \) of \( G \) on every \( T_i \) such that the restrictions of \( \chi_i \) and \( \chi_j \) to \( T_i \cap T_j \) are equivalent, it is easy to check that there is a unique right local action of \( G \) on \( T \), up to equivalences, whose restriction to every \( T_i \) is equivalent to \( \chi_i \).

Consider another right local action \( \chi' \) of \( G' \) on \( T' \). A paro map \( \phi : T \rightarrow T' \) is called locally equivariant if there is some open neighborhood \( \Sigma \) of \( \text{dom}\phi \times \{e\} \) in \( \text{dom}\chi \cap (\phi \times \text{id}_G)^{-1}(\text{dom}\chi') \) such that \( \phi(\Sigma) \subset \text{dom}\phi \) and \( \phi \chi(u,g) = \chi'(\phi(u),g) \) for all \( (u,g) \in \Sigma \). Note that composites, restrictions to open sets and unions of locally equivariant paro maps are locally equivariant, as well as their inverses whenever defined. A family of paro maps \( T \rightarrow T' \) is called locally equivariant when all of its elements are locally equivariant.

Left local actions, their equivalences and corresponding locally equivariant maps are similarly defined.
Proposition 2.2 ([5, Theorems 3.3 and 5.2], [10, Lemma 2.36, Theorem 2.38 and Remark 21]). The following holds:

(i) Suppose that $\mathcal{H}$ is minimal, compactly generated, equicontinuous and strongly quasi-analytic. Then $\overline{\mathcal{H}}$ is strongly locally free if and only if $\mathcal{H}$ is equivalent to a pseudogroup on some local group $G$ generated by the left local action by local left translations of a finitely generated dense sub-local group $\Gamma \subset G$.

(ii) Let $\mathcal{G}$ and $\mathcal{G}'$ be the pseudogroups on local groups $G$ and $G'$ generated by the left local actions by local left translations of respective finitely generated dense sub-local groups $\Gamma$ and $\Gamma'$. Let $\Phi : \mathcal{G} \to \mathcal{G}'$ be a morphism such that $\mathcal{G}(e) \to \mathcal{G}'(e')$ by the induced map $G/\Gamma \to G/\Gamma'$. Then $\Phi$ is generated by a local homomorphism $G \to G'$ that restricts to a local homomorphism $\Gamma \to \Gamma'$.

Proposition 2.3. Let $\Phi : \mathcal{H} \to \mathcal{H}'$ be an equivalence between compactly generated pseudogroups. Let $\chi$ be a right local action of $G$ on $T$ such that $\mathcal{H}$ is locally equivariant. Then there is a unique right local action $\chi'$ of $G$ on $T'$, up to equivalences, such that $\Phi$ and $\mathcal{H}'$ are locally equivariant.

Proof. Let $E$ be a system of compact generation of $\mathcal{H}$ on a relatively compact open $U \subset T$, and let $h$ be an extension of every $h \in E$ with $\text{dom} \, h \subset \text{dom} \, h$. There is a subset $\Phi_0 \subset \Phi$ such that $\{ \text{dom} \phi \times \text{im} \phi \mid \phi \in \Phi_0 \}$ covers $U \times T'$.

Let $\Omega = \text{dom} \, \chi$, and let $\Sigma_{ij}$ be an open neighborhood of $U_{ij} \times \{e\}$ in $\Omega \cap (\tilde{\phi}_i^{-1} \phi_j \times \text{id}_G)^{-1}(\Omega)$ such that $\chi(\Sigma_{ij}) \subset \tilde{U}_{ij}$ and $\tilde{\phi}_i^{-1} \phi_j \chi(u, g) = \chi(\tilde{\phi}_i^{-1} \phi_j (u, g))$ for all $(u, g) \in \Sigma_{ij}$. Let $\Sigma_{ij}$.

$$\Omega'_i = \{ (u', g) \in T' \times G \mid u' \in \tilde{U}'_i \cap \tilde{U}'_j \Rightarrow (\tilde{\phi}_i^{-1}(u'), g) \in \Sigma_{ij}, \forall i, j \} .$$

Claim 2. $\Omega'_i$ is open in $T' \times G$.

Take some $(u', g) \in \Omega'_i$. Let $\mathcal{J}$ be the set of indices $i$ such that $u' \in \tilde{U}'_i$, and let $\mathcal{J}'$ be the set of pairs of indices, $(i, j)$, such that $u' \in \tilde{U}'_i \cap \tilde{U}'_j$, which are finite sets because $\{U'_i\}$ is locally finite in $T'$. Then, using that $\tilde{U}'_i \subset \tilde{U}'_i$, every $\tilde{\phi}_i$ is a homeomorphism, and $\Sigma_{ij}$ is an open neighborhood of $(\tilde{\phi}_j^{-1}(u'), g)$ in $\tilde{U}_{ij} \times G$ for all $(i, j) \in \mathcal{J}'$, it follows that there are open neighborhoods, $V$ of $u'$ in $T'$ and $P$ of $g$ in $G$, such that $V \cap \tilde{U}'_i = \emptyset$ if $i \notin \mathcal{J}$, and $\tilde{\phi}_j^{-1}(V) \times P \subset \Sigma_{ij}$ for all $(i, j) \in \mathcal{J}'$. Thus $V \times P \subset \Omega'_i$. 


Claim 3. A map $\chi' : \Omega'_0 \to T'$ is defined by $\chi'_0(u',g) = \tilde{\phi}_i \chi(\phi^{-1}_i(u'),g)$ if $u' \in U'_i$.

Let $(u',g) \in \Omega'_0$ such that $u' = \phi_i(u_i) = \phi_j(u_j)$ for some $u_i \in U_i$ and $u_j \in U_j$. We have $(u_j,g) \in \Sigma_{ij}$ because $(u',g) \in \Omega'_0$. Hence $\chi(u_j,g) \in \tilde{U}_{ij}$ and $\tilde{\phi}^{-1}_i \tilde{\phi}_j \chi(u_j,g) = \chi(u_i,g)$, obtaining $\tilde{\phi}_j \chi(u_j,g) = \tilde{\phi}_i \chi(u_i,g)$. This shows that $\chi'_0$ is well defined. Its continuity follows from the continuity of $\chi$ since the maps $\tilde{\phi}_i$ are homeomorphisms.

Let $\{V'_i\}$ be an open covering of $T'$ with $\overline{V'_i} \subset U'_i$, and define $V_i = \phi_i^{-1}(V'_i)$. Let

$$\Omega' = \{ (u',g) \in \Omega' \mid u' \in \overline{V'_i} \Rightarrow \chi'_0(u',g) \in U'_i, \forall i \}, \quad \chi' = \chi'_0|_{\Omega'}.$$  

Claim 4. $\chi'$ is a local action of $G$ on $T'$.

First, it is easy to check that $\Omega'$ is open in $\Omega'_0$, and therefore $\Omega'$ is also open in $T' \times G$ by Claim 2.

Second, $T' \times \{e\} \subset \Omega'_0$ because $\tilde{U}_{ij} \times \{e\} \subset \Sigma_{ij}$ for all $i,j$. Moreover, for all $u' \in T'$,

$$\chi'(u',e) = \tilde{\phi}_i \chi(\phi_i^{-1}(u'),e) = \tilde{\phi}_i \phi_i^{-1}(u') = u'.$$

Hence $T' \times \{e\} \subset \Omega'$ and $\chi'(u',e) = u'$ for all $u' \in T'$.

Third, assume that $gh$ is defined and $(u',g),(u',gh),(\chi'(u',g),h) \in \Omega'$ for some $g,h \in G$ and $u' \in T'$. Then, for $i$ with $u' \in V'_i$, we have $\chi'(u',g) \in U'_i$ and $(\tilde{\phi}^{-1}_i(u'),\phi_i^{-1}(u'),gh),(\tilde{\phi}^{-1}_i \chi'(u',g),h) \in \Omega$, obtaining

$$\chi'(\chi'(u',g),h) = \tilde{\phi}_i \chi(\phi_i^{-1} \chi'(u',g),h) = \tilde{\phi}_i \chi(\phi_i^{-1}(u'),gh)$$

because $\chi$ is a right local action. This completes the proof of Claim 4.

Obviously, all maps $\phi_i$ become locally equivariant by the definition of $\chi'_0$ and $\chi'$; indeed, up to equivalences, $\chi'$ is the unique local action satisfying this property because $\{U'_i\}$ covers $T'$. Therefore the maps $\phi_i \phi_j^{-1} : U_{ij} \to U_{ji}'$ are also locally equivariant. So $\Phi$ and $\mathcal{H}$ are locally equivariant by Claim 1 and because $\mathcal{H}$ is locally equivariant.

Let $\chi$ be a right local action of $G$ on $T$ such that $\mathcal{H}$ is locally equivariant. Consider the following property that $(T,\mathcal{H},\chi)$ may have:

$$\mathcal{H}(\chi(\{u\} \times P)) = T \quad \forall u \in T, \forall P \in \mathcal{N}(G,e) \mid \{u\} \times P \subset \text{dom} \chi. \quad (1)$$

Lemma 2.4. Property (1) is preserved by locally equivariant pseudogroup equivalences.

Proof. Elementary. \qed

2.3. Foliated spaces. The notation introduced here will be used in the remaining sections.

Let $X$ be a space and $n \in \mathbb{Z}^{\geq 0}$. The main results of the paper will require $X$ to be compact, but this condition is avoided for the basic concepts. Let $\mathcal{U}$ be a family consisting of pairs $(U_i,\xi_i)$, called foliated charts, where $\{U_i\}$ is
an open cover of $X$, and every $\xi_i$ is a homeomorphism $U_i \to B_i \times T_i$ for some contractible open subset $B_i \subset \mathbb{R}^n$ and a space $T_i$. Every $(U_i, \xi_i)$ induces a projection $p_i : U_i \to T_i$ whose fibers are called plaques. Assume that finite intersections of plaques are open in the plaques. Then the open subsets of the plaques form a base of a finer topology in $X$, becoming an $n$-manifold whose connected components are called leaves. In this case, it is said that $U$ defines a foliated structure $\mathcal{F}$ of dimension $n$ on $X$, $X \equiv (X, \mathcal{F})$ is called a foliated space (or lamination), and $U$ is called a foliated atlas. Two foliated atlases define the same foliated structure if their union is a foliated atlas. The subspaces $\xi_i^{-1}((v) \times T_i) \subset X$, $v \in B_i$, are called local transversals defined by the foliated chart $(U_i, \xi_i)$. A transversal is a subspace $\Sigma \subset X$ where any point has a neighborhood that is a local transversal of some foliated chart. A transversal is called global if it meets all leaves.

A foliated space can be considered as a weak version of a regular dynamical system where the the leaves play the role of the orbits. In this way, several basic dynamical concepts have obvious versions for foliated spaces, like saturation, (topological) transitivity and minimality. The partition of $X$ into leaves is enough to describe $\mathcal{F}$. The leaf through a point $x$ may be denoted by $L_x$, and the leaf space by $X/\mathcal{F}$. The saturation of a subset $A \subset X$ is denoted by $\mathcal{F}(A)$.

We can assume that the foliated atlas $U$ is regular in the sense that it satisfies the following properties [6, Definition 5.1] (see also [19, 13, 24]):

- there is another foliated atlas $\widetilde{U} = \{\widetilde{U}_i, \widetilde{\xi}_i\}$ of $X$, with $\widetilde{\xi}_i : \widetilde{U}_i \to \overline{B}_i \times \overline{T}_i$, such that $\overline{U}_i \subset \widetilde{U}_i$, $\overline{B}_i \subset \overline{B}_i$, $\overline{T}_i$ is an open subspace of $\overline{T}_i$, and $\xi_i = \widetilde{\xi}_i|_{\overline{U}_i}$ (thus $p_i = \widetilde{p}_i|_{\overline{U}_i}$);
- $\{\overline{U}_i\}$ is locally finite; and
- every plaque of $(U_i, \xi_i)$ meets at most one plaque of $(U_j, \xi_j)$.

By the last condition, there are homeomorphisms $h_{ij} : p_j(U_i \cap U_j) \to p_i(U_i \cap U_j)$, the elementary holonomy transformations, such that $h_{ij}p_j = p_i$ on $U_i \cap U_j$, obtaining the defining cocycle $\{U_i, p_i, h_{ij}\}$; it describes $\mathcal{F}$ and satisfies the cocycle condition $h_{ik} = h_{ij}h_{jk}$ on $p_k(U_i \cap U_j \cap U_k)$. So the changes of coordinates $\xi_i\xi_j^{-1} : \xi_j(U_i \cap U_j) \to \xi_i(U_i \cap U_j)$ are of the form

$$\xi_i\xi_j^{-1}(v, u) = (g_{ij}(v, u), h_{ij}(u)),$$

for some continuous maps $g_{ij} : \xi_j(U_i \cap U_j) \to B_i$.

The “transverse dynamics” of $X$ is described by its holonomy pseudogroup, which is (the equivalence class of) the pseudogroup $\mathcal{H}$ generated by the maps $h_{ij}$ on $T := \bigsqcup_i T_i$. Its elements are called holonomy transformations. There is a canonical identity $X/\mathcal{F} \equiv T/\mathcal{H}$, where the $\mathcal{H}$-orbit that corresponds to a leaf $L$ is $\bigsqcup_i p_i(L \cap U_i)$. Via this identity, $\mathcal{F}$-leaves and $\mathcal{H}$-orbits have corresponding dynamical concepts.

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2Regularity of the foliated atlas is used with another meaning in [19].

3This convention for the order of these subindices agrees with [15] and differs from [10]. The same kind of convention will be used in the local representations of foliated maps.
We can assume that \( \widetilde{U} \) is also regular, obtaining elementary holonomy transformations \( h_{ij} : \tilde{p}_j(U_i \cap \tilde{U}_j) \to \tilde{p}_i(U_i \cap \tilde{U}_j) \), extending the maps \( h_{ij} \), which generate another representative of the holonomy pseudogroup, \( \tilde{H} \) on \( \tilde{T} := \bigcup \tilde{T}_i \); \( \tilde{T} \) is an open subspace of \( \tilde{T} \) that meets all \( \tilde{H} \)-orbits, and \( \mathcal{H} = \tilde{H}_{|\tilde{T}} \).

Let \( \sigma_i : T_i \to U_i \) and \( \tilde{\sigma}_i : \tilde{T}_i \to \tilde{U}_i \) be the sections of every \( p_i \) and \( \tilde{p}_i \) defined by fixing an element of \( B_i \) (thus \( \sigma_i = \tilde{\sigma}_i|T_i \)). We can assume that the sets \( \tilde{\sigma}_i(\tilde{T}_i) \) are separated by open sets in \( X \), and therefore \( \bigcup_i \tilde{\sigma}_i : \tilde{T} \to \bigcup_i \tilde{\sigma}_i(\tilde{T}_i) \) and \( \bigcup_i \sigma_i : T \to \bigcup_i \sigma_i(T_i) \) are homeomorphisms to complete transversals.

Given a finite sequence of indices, \( I = (i_0, \ldots, i_\alpha) \), let \( h_I = h_{i_\alpha i_{\alpha-1}} \cdots h_{i_0 i_0} \) if \( \alpha > 0 \), and \( h_I = \text{id}_{T_{i_0}} \) if \( \alpha = 0 \). If \( \text{dom} h_I \neq \emptyset \), then \( I \) is called admissible. Let \( c : I := [0, 1] \to X \) be a path from \( x \) to \( y \), which is leafwise in the sense that \( c(I) \) is contained in some leaf \( L \). Let us say that \( c \) is (\( T \)-) covered by \( I \) if there is a partition of \( I \), \( 0 = t_0 < t_1 < \cdots < t_{\alpha+1} = 1 \), such that \( c([t_k, t_{k+1}]) \subseteq U_{i_k} \) for all \( k = 0, \ldots, \alpha \). In this case, \( u := p_{i_0}(x) \in \text{dom} h_I \) and \( h_I(u) = p_{i_\alpha}(y) \).

If \( I = (i_0, \ldots, i_\alpha) \) and \( J = (j_0, \ldots, j_\beta) \) cover \( c \) and \( c' \), respectively, with \( j_0 = i_\alpha \), then \( IJ := (i_0, \ldots, i_\alpha = j_0, \ldots, j_\beta) \) and \( I^{-1} := (i_\alpha, \ldots, i_0) \) cover \( c \) and \( c^{-1} \), respectively, and we have \( h_{IJ} = h_I h_J \) and \( h_{I^{-1}} = h_I^{-1} \). By using \( \tilde{U} \), we can similarly define \( h_{\tilde{I}} \), which is an extension of \( h_I \). Recall that, for another admissible sequence \( J = (j_0, \ldots, j_\beta) \) with \( j_0 = i_0 \) and \( j_\beta = i_\alpha \), covering another path \( c' \) from \( x \) to \( y \) in \( L \), if \( c \) and \( c' \) are endpoint-homotopic in \( L \), then \( u \in \text{dom} h_J \) and \( \gamma(h_I u) = \gamma(h_J u) \). Any leafwise path is covered by some admissible sequence, and, vice versa, for all \( I = (i_0, \ldots, i_\alpha) \), \( x \in U_{i_0} \) and \( y \in U_{i_\alpha} \), with \( p_{i_0}(x) \in \text{dom} h_I \) and \( h_I p_{i_0}(x) = p_{i_\alpha}(y) \), there is some leafwise path from \( x \) to \( y \) covered by \( I \).

The holonomy group of a leaf \( L \) at a point \( x \in L \cap U_i \) is the germ group,

\[ \text{Hol}(L, x) = \{ \gamma(h, u) \mid h \in \mathcal{H}, \ u \in \text{dom} h, \ h(u) = u \} , \]

which depends only on \( L \) up to conjugation by germs of holonomy transformations. The holonomy homomorphism, \( \text{hol} : \pi_1(L, x) \to \text{Hol}(L, x) \), is given by \( \text{hol}(c) = \gamma(h^{-1}, u) \) if \( c \) is covered by \( I = (i_0, \ldots, i_\alpha) \) with \( i_0 = i_\alpha = i \). This homomorphism is well defined and onto according to the previous observations, and it defines a regular covering \( \tilde{L}^{\text{hol}} \) of \( L \), the holonomy covering.

We will consider the canonical right action of \( \text{Hol}(L, x) \) on \( \tilde{L}^{\text{hol}} \) by covering transformations. A leaf is said to be without holonomy if its holonomy group is trivial, and \( X \) is called without holonomy when all leaves have no holonomy. The union of leaves without holonomy is a dense \( G_\delta \) in \( X \), and therefore Borel and residual \cite{23, 22}. A path connected subset of a leaf, \( D \subset L \), is said to be without holonomy if the composite

\[ \pi_1(D, x) \longrightarrow \pi_1(L, x) \overset{\text{hol}}{\longrightarrow} \text{Hol}(L, x) \]

is trivial for some (and therefore all) \( x \in D \).

It is said that \( X \) is (strongly) equicontinuous, strongly quasi-analytic or strongly locally free if \( \mathcal{H} \) satisfies these properties. In the definition of these

\footnote{This is a simplified version of standard terminology.}
conditions for $\mathcal{H}$, by refining $\mathcal{U}$ if necessary, we can assume that the metrics $d_i$ are defined on the sets $T_i$, and we can take

$$S = \{ h_T \mid T \text{ is an admissible sequence} \}$$

if desired. For a local group $G$, we say that $X$ is a $G$-foliated space if $H$ is equivalent to a pseudogroup generated by some local left translations on $G$.

If $X$ is compact, then $\mathcal{U}$ is finite and $\mathcal{T}$ is relatively compact in $\mathcal{T}$, obtaining that $\tilde{\mathcal{H}}$ satisfies the definition of compact generation with the generators $h_{ij}$ of $\mathcal{H}|_\mathcal{T} = \mathcal{H}$ and their extensions $\tilde{h}_{ij}$. So $\mathcal{H}$ is also compactly generated. If moreover $\mathcal{F}$ is equicontinuous, then the properties of Propositions 2.1 and 2.2 apply to $\mathcal{H}$; in particular, the leaf closures are minimal sets, and therefore $X$ is transitive if and only if it is minimal.

Foliated spaces with boundary can be defined in a similar way, adapting the definition of manifold with boundary: every $B_i$ would be a contractible open set in the half space $H^n \equiv \mathbb{R}^{n-1} \times [0, \infty)$. The boundary of $X$, $\partial X = \bigcup_i \xi_i^{-1}(\partial B_i \times T_i)$, becomes a foliated space without boundary, where $\partial B_i = B_i \cap \partial H^n$ for $\partial H^n \equiv \mathbb{R}^{n-1} \times \{0\} \equiv \mathbb{R}^{n-1}$. The basic concepts recalled here about foliated spaces have direct extensions to foliated spaces with boundary.

Any open $U \subset X$ becomes a foliated space with the restriction $\mathcal{F}|_U$, defined by all possible foliated charts of $\mathcal{F}$ with domain in $U$. Let $X' = (X', \mathcal{F}')$ be another foliated space with dim $\mathcal{F}' = n'$. Then $X \times X'$ has a product foliated structure of dimension $n + n'$, $\mathcal{F} \times \mathcal{F}'$, defined by the foliated charts that are products of foliated charts of $\mathcal{F}$ and $\mathcal{F}'$. Thus the leaves of $X \times X'$ are products of leaves of $X$ and $X'$. Any connected (second countable) manifold $M$ is a foliated space with one leaf. On the other hand, any space $Y'$ can be considered as a foliated space, denoted by $Y_{pt}$, whose leaves are the points. Thus we can consider the foliated spaces, $X \times M$ with leaves $L \times M$ and $X \times Y_{pt}$ with leaves $L \times \{y\}$, for leaves $L$ of $X$ and points $y \in Y$. Like in the case of foliations, a typical example of foliated space can be obtained by suspension of an action of the fundamental group of a manifold on a space (see Section 8.1). The concept of subfoliated structure has the obvious meaning, like the concept of subfoliation.

Let $X' \equiv (X', \mathcal{F}')$ be another foliated space, let $\mathcal{U}' = \{ U'_a, \xi'_a \}$ be a regular foliated atlas of $X'$, where $\xi'_a : U'_a \to B'_a \times T'_a$, giving rise to a defining cocycle $\{ U'_a, p'_a, h'_{ab} \}$, and the corresponding representative of the holonomy pseudogroup, $\mathcal{H}'$ on $T' := \bigcup_a T_a$ generated by $\{ h'_{ab} \}$. A map $\phi : X \to X'$ is called foliated when it maps leaves to leaves. Then every local representation $\xi'_a \phi^{-1} : \xi_i(U_i \cap \phi^{-1}(U'_a)) \to B'_a \times T'_a$ is of the form

$$\xi'_a \phi^{-1}(v, u) = (\phi_{ai}(v, u), \phi_{ai}(u)) \quad (3)$$

for some maps $\phi_{ai}^1 : \xi_i(U_i \cap \phi^{-1}(U'_a)) \to B'_a$ and $\phi_{ai}^2 : p_i(U_i \cap \phi^{-1}(U'_a)) \to T'_a$. The maps $\phi_{ai}^2$ generate a morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ [8, 9], which is said to be induced by $\phi$.

An action of a group on $X$ is called foliated when it is given by foliated homeomorphisms. A homotopy $H$ between foliated maps $\phi, \psi : X \to X'$ is
said to be leafwise (or integrable) if it is a foliated map $X \times I \to X'$; i.e., $H(L \times I)$ is contained in some leaf of $X'$ for all leaf $L$ of $X$, and therefore the path $H(x, \cdot) : I \to X'$ is leafwise for all $x \in X$. In this case, $\phi$ and $\psi$ induce the same morphism $\mathcal{H} \to \mathcal{H}'$ \cite{M} Proposition 6.1. This condition is stronger than being a foliated map $X \times I_{pt} \to X'$. A leafwise isotopy has a similar definition.

Let $V \subset \mathbb{R}^n \times Y$ be an open subset, and let $r \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. A map $g : V \to \mathbb{R}^n'$ is called (differentiable of class) $C^r$ when, for any integer $0 \leq k \leq r$ (it is enough to take $k = r$ if $r < \infty$), all partial derivatives of $g$ up to order $k$ with respect to the coordinates of $\mathbb{R}^n$ are defined and continuous on $V$. A change of coordinates $\xi_j \xi^{-1}_i$ is called $C^r$ when the map $g_{ij}$ in (2) is $C^r$. If all changes of coordinates are $C^r$, then $U$ defines a $C^r$ structure on $X$, which becomes a $C^r$ foliated space. In this case, $U$ and its foliated charts are called $C^r$. Two such foliated atlases of $X$ define the same $C^r$ structure if their union also defines a $C^r$ structure. The leaves of $C^r$ foliated spaces canonically become $C^r$ manifolds. Many concepts of $C^r$ manifolds have straightforward generalizations to $C^r$ foliated spaces, like $C^r$ foliated maps, $C^r$ foliated diffeomorphisms, $C^r$ foliated embeddings, $C^r$ foliated actions, $C^r$ leavewise homotopies/diffeotopies, $C^r$ vector bundles, $C^r$ sections, the (leafwise) tangent bundle $TX$ (or $TF$), the tangent map $T\phi : TX \to TX'$ of a $C^r$ foliated map $\phi : X \to X'$, (leafwise) Riemannian metrics, etc. For instance, a foliated map $\phi : X \to X'$ is $C^r$ when, for all local representations $\xi'_{ai} \xi^{-1}_i$, the maps $\phi^1_{ai}$ of \cite{M} are $C^r$.

Any $C^r$ foliated space has a $C^r$ partition of unity subordinated to any open cover \cite{2} Proposition 2.8. A version of the Reeb’s stability theorem holds for $C^2$ foliated spaces \cite{4} Proposition 1.7.

Recall that a subset $A$ in a Riemannian manifold $M$ is called convex when, for all $x, y \in A$, there is a unique minimizing geodesic segment from $x$ to $y$ in $M$ that lies in $A$ (see e.g. \cite{14} Section IX.6). For example, sufficiently small balls are convex. If $X$ is $C^\infty$, given any $C^\infty$ Riemannian metric on $X$, we can choose $U$ and $\tilde{U}$ so that the plaques of their charts are convex balls in the leaves. This follows from the relation between the convexity and injectivity radii \cite{14} Theorem IX.6.1, and the continuity of the injectivity radius on closed manifolds \cite{21, 37}—the case of closed manifolds easily yields local lower bounds of the injectivity radius on arbitrary manifolds, valid for all metrics that are close enough to a given metric in the weak $C^\infty$ topology.

2.4. Spaces of foliated maps. Suppose that $X$ and $X'$ are $C^r$ for some $r \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. We use the following notation\footnote{The foliated structures are added to this notation to avoid ambiguity.} for sets of maps $X \to X'$:

- $C^r(X, \mathcal{F}; X', \mathcal{F}')$; the set of $C^r$ foliated maps,
- Diffeo$^r(X, \mathcal{F}; X', \mathcal{F}')$ (or Diffeo$^r(X, \mathcal{F})$ if $X = X'$): the set of $C^r$ diffeomorphisms,
- Emb$^r(X, \mathcal{F}; X', \mathcal{F}')$: the set of $C^r$ foliated embeddings,
• Prop"(X, F; X', F') : the set of proper C everyone foliated maps, and
• Homeo(X, F; X', F') (or Homeo(X, F) if X = X') : the set of foliated homeomorphisms.

If r = 0 or it is clear that r = ∞, then r is removed from the above notation. Homeo(X, F) is a subgroup of the group of homeomorphisms, Homeo(X).

Let us define two foliated versions of the weak/strong C r topology. In the first version, consider any φ ∈ C r(X, F; X', F'), locally finite families of foliated charts, U = {U i, ξ i} of X and U' = {U' i, ξ' i} of X', a family of compact subsets of X, K = {K i}, so that K i ⊂ U i and f(K i) ⊂ U' i for all i and corresponding indices a i, a family E = {ε i} of positive numbers, and any integer 0 ≤ k ≤ r (it is enough to take k = r if r < ∞). Then let N F, K, E be the set of foliated maps ψ : X → X' such that ψ(K i) ⊂ U' a i and

\[ \left| \frac{\partial \alpha(\phi_{a,i} - \psi_{a,i})}{\partial \nu^\alpha}(v, u) \right| < \epsilon_i, \]

for all i, (v, u) ∈ ξ i(K i) and multi-indices α with |α| ≤ k, where φ_{a,i} and ψ_{a,i} are given by [3]. All possible sets N F, K, E form a base of open sets in a topology on C r(X, F; X', F'), called the strong foliated C r topology. The weak foliated C r topology is similarly defined by using finite families of indices i. The subindex “WF/SF” will be added to the notation to indicate that the weak/strong foliated C r topology in a family of C r foliated maps.

Note that C WF(X, F; X', F') has the compact-open topology. Of course both topologies coincide when X is compact, and only the subindex “F” will be added in this case.

If X is compact, then the group of homeomorphisms, Homeo(X), is a Polish topological group with the compact-open topology [11, Theorem 3]. Moreover Homeo(X, F) is a closed subgroup of Homeo(X), and therefore it is also a Polish topological group.

Some important results on spaces of C r maps between manifolds have straightforward generalizations to C r foliated spaces, like the following.

**Proposition 2.5.** The following properties hold:

(i) The injectivity/surjectivity of the restrictions of the tangent map to the fibers defines an open subset of C SF r(X, F; X', F') for 1 ≤ r ≤ ∞.
(ii) Prop"(X, F; X', F') is open in C SF r(X, F; X', F') for 0 ≤ r ≤ ∞.

**Proof.** Adapt the proofs of [30] Theorems 2.1.1 and 2.1.2. □

For general C r foliated maps X → X', r ≥ 1, the injectivity/surjectivity of the restrictions of their tangent maps to the fibers does not have any consequence on their transverse behavior, given by the induced morphisms H → H'. Thus the foliated immersions/submersions or foliated local homeomorphisms cannot be described using only the tangent map. So conditions on the induced morphisms H → H' must be added to extend some deeper results. For this reason, we use a second version of weak/strong C r topology.
introduced in [8], which is finer than the weak/strong foliated $C^r$ topology. The strong plaquewise $C^r$ topology has a base of open sets $N_F^k(\phi, \mathcal{U}, \mathcal{U}', \mathcal{K}, \mathcal{E})$, defined by adding the condition $p_i^* \phi = p_i^* \psi$ on every $K_i$ to the above definition of $N_F^k(\phi, \mathcal{U}, \mathcal{U}', \mathcal{K}, \mathcal{E})$; using [3], this extra condition can be also written as $q_{a,i}^* \phi = q_{a,i}^* \psi$ on $p_i(K_i)$ for all $i$. The weak plaquewise $C^r$ topology is similarly defined by requiring the conditions only for finite families of indices $i$. The subindex “WP/SP” will be added to the notation indicate that the weak/strong plaquewise foliated maps are close enough in $C^r$ as above where the plaques of the foliated charts in $\mathcal{U}$ are convex balls in the leaves for a given Riemannian metric on $X'$, and then using geodesic segments to define homotopies.

With the strong plaquewise $C^r$ topology, we can continue the direct extensions of results about spaces of $C^s$ maps between manifolds.

**Proposition 2.6.** The following properties hold:

(i) $\text{Emb}^r(X, \mathcal{F}; X', \mathcal{F}')$ is open in $C^r_{SP}(X, \mathcal{F}; X', \mathcal{F}')$ for $1 \leq r \leq \infty$.

(ii) For $1 \leq r \leq \infty$, the set of closed $C^r$ foliated embeddings is open in $C^r_{SP}(X, \mathcal{F}; X', \mathcal{F}')$.

(iii) $\text{Diffeo}^r(X, \mathcal{F}; X', \mathcal{F}')$ is open in $C^r_{SP}(X, \mathcal{F}; X', \mathcal{F}')$ for $1 \leq r \leq \infty$.

(iv) $C^s(X, \mathcal{F}; X', \mathcal{F}')$ is dense in $C^r_{SP}(X, \mathcal{F}; X', \mathcal{F}')$ for $0 \leq r < s \leq \infty$.

(v) $\text{Diffeo}^s(X, \mathcal{F}; X', \mathcal{F}')$ is dense in $\text{Diffeo}^r_{SP}(X, \mathcal{F}; X', \mathcal{F}')$ for $1 \leq r < s \leq \infty$.

(vi) If $1 \leq r < \infty$, any $C^r$ foliated space is $C^r$ diffeomorphic to a $C^\infty$ foliated space.

(vii) If $1 \leq r < s \leq \infty$, two $C^s$ foliated spaces are $C^s$ diffeomorphic if and only if they are $C^r$ diffeomorphic.

**Proof.** Adapt the proofs of [31] Theorems 2.1.4, 2.1.6, 2.2.7, 2.2.9 and 2.2.10, and Corollary 2.1.6. \qed

Like in the case of manifolds, it easily follows from Proposition 2.6 (iv) that, for $0 \leq r < s \leq \infty$, if there is a $C^r$ leafwise homotopy between $C^s$ foliated maps, then there is a $C^s$ leafwise homotopy between them.

The above openness statements are stronger with the strong foliated $C^r$ topology, whereas the denseness statements are stronger for the strong plaquewise $C^r$ topology. There is no version of Proposition 2.6 (ii) with the strong foliated $C^r$ topology (for instance, consider the case of compact spaces foliated by points). However we can prove a weaker form of that statement by using certain subspaces $C^r_{SP}(X, \mathcal{F}; X', \mathcal{F}')$ defined as follows. A foliated map $\phi : X \to X'$ is called a transverse embedding (respectively, transverse equivalence) if the induced morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is generated by embeddings (respectively, $\Phi$ is an isomorphism). Observe that $\mathcal{F}'(\phi(X)) = X'$ if $\phi$ is a transverse equivalence. A subset $\mathcal{M} \subset C(X, \mathcal{F}; X', \mathcal{F}')$ of transverse
embeddings (respectively, transverse equivalences) is called uniform if there are some foliated atlases, $U$ of $X$ and $U'$ of $X'$ like in Section 2.3 such that, for all $\phi \in M$, the maps $\phi_2^2$ in (33) are embeddings (respectively, open embeddings). Note that, if these properties hold with $U$ and $U'$, then they hold with all finer atlases. For example, $\text{Emb}(X, F; X', F')$ consists of uniform transverse embeddings, and $\text{Homeo}(X, F; X', F')$ consists of uniform transverse equivalences.

**Proposition 2.7.** For $1 \leq r \leq \infty$, let $M \subset C^r_sp(X, F; X', F')$ be a uniform subspace of transverse embeddings. Then $\text{Emb}^r(X, F; X', F') \cap M$ is open in $M$.

**Proof.** It is enough to prove the case $r = 1$. For any $\phi \in \text{Emb}^1(X, F; X', F') \cap M$, consider a basic open set $N_1 := N^1_F(\phi, U, U', K, E)$ in $C^1_sp(X, F; X', F')$ as above. We can assume that $K$ (and therefor $U$) covers $X$, and $U'$ covers $X'$. After refinements, we can choose $U, U'$ and $K$ such that the maps $\psi_2^2$ are embeddings for all $\psi \in M$, and the interiors $V_i := K_i$ cover $X$. Take an open cover $\{W_i\}$ of $X$ with $W_i \subset V_i$ for all $i$. By [30, Lemma 1.3], we can choose $E$ such that the maps $\psi : \psi_i^{-1}(u) \cap V_i \to \psi_2^{-1}(\psi_2^{-1}(u))$ are $C^1$ embeddings for $u \in p_i(V_i)$ and $\psi \in N_1$. Hence $\psi : V_i \to X'$ is a $C^1$ foliated embedding for all $\psi \in N_1 \cap M$.

Now, we adapt the final part of the proof of [30, Theorem 1.4] as follows. Since $\phi$ is an embedding, we get disjoint open subsets $V_i, W_i \subset X'$ for every $i$ such that $\phi(W_i) \subset W_i' \subset X'$ and $\phi(X \setminus V_i) \subset V_i'$. Then it is easy to find a neighborhood $N_0$ of $\phi$ in $C_sp(X, F; X', F')$ so that $\psi(W_i) \subset W_i'$ and $\psi(X \setminus V_i) \subset V_i'$ for all $\psi \in N_0$. We finally obtain $N_0 \cap N_1 \cap M \subset \text{Emb}^1(X, F; X', F')$.

**Proposition 2.8.** For $1 \leq r \leq \infty$, let $M \subset C^r_sp(X, F; X', F')$ be a uniform subspace of transverse equivalences. Then $\text{Diffeo}^r(X, F; X', F') \cap M$ is open in $M$.

**Proof.** We adapt the proofs of [30, Corollary 1.6 and Theorem 1.6]. The set $M' = \{ \phi \in \text{Prop}^r(X, F; X', F') \mid T_x \phi \text{ is surjective } \forall x \in X \}$ is closed in $C^r_sp(X, F; X', F')$ by Proposition 2.5 (i), (ii). On the other hand, $\text{Emb}^r(X, F; X', F') \cap M$ is open in $M$ by Proposition 2.7. Thus the result follows because $\text{Emb}^r(X, F; X', F') \cap M' = \text{Diffeo}^r(X, F; X', F')$.

According to Proposition 2.4 (vii), we will only consider either ($C^0$) foliated spaces or $C^\infty$ foliated spaces from now on.

**Proposition 2.9.** Let $\phi : X \to X'$ be a foliated map. Suppose that $X'$ is equipped with a $C^\infty$ structure. Then there is at most one $C^\infty$ structure on $X$ such that $\phi$ is $C^\infty$ and $T_x \phi$ is an isomorphism for all $x \in X$.

**Proof.** Consider two $C^\infty$ structures on $X$, and take $C^\infty$ foliated charts, $\xi_1 : U_1 \to B_1 \times T_1$ of the first $C^\infty$ structure on $X$, $\xi_2 : U_2 \to B_2 \times T_2$ of the second
We assume that \( \mathcal{C}(T) \) (of arbitrary order with respect to \( T \))

Proposition 2.11 (H. Karcher [32, Theorem 1.2])

Therefore, by the inverse function theorem, \( g \) is an isomorphism at any point. Therefore, by the inverse function theorem, we can assume that \( g'_1(\cdot, u_1) : B_1 \rightarrow g'_1(B_1 \times \{u_1\}) \) is a \( C^\infty \) diffeomorphism for all \( u_1 \in T_1 \). Its inverse function is denoted by \( \bar{g}'_1(\cdot, u_1) : g'_1(B_1 \times \{u_1\}) \rightarrow B_1 \).

Any \( f \) defining a function \( \bar{g}'_1(\cdot, u_1) : B_1 \rightarrow B_1 \) depends continuously on \( \bar{g}'_1(\cdot, u_1) \) with respect to \( u_1 \). Since

\[
g_{12}(v_2, u_2) = g_1(g_2(v_2, u_2), h_{12}(u_2))
\]
on \( B_2 \times h_{21}(T_{10}) \), the function \( g_{12} : B_2 \times h_{21}(T_{10}) \rightarrow B_1 \) has partial derivatives of arbitrary order with respect to \( v_2 \), continuous on \( B_2 \times h_{21}(T_{10}) \).

2.5. Center of mass. We will use the center of mass of a mass distribution on a Riemannian manifold \( M \) [32, Section IX.7].

Let \( \Omega \subset M \) be a compact submanifold with boundary with \( \dim \Omega = \dim M \).

For \( 0 \leq r \leq \infty \), let \( \mathcal{C}(\Omega) \) be the set of functions \( f \in C^{r+2}(\Omega) \) such that \( \operatorname{grad} f \) is an outward pointing vector field on \( \partial \Omega \) and \( \operatorname{Hess} f \) is positive definite on the interior \( \Omega \) of \( \Omega \). Note that \( \mathcal{C}(\Omega) \) is open in the Banach space \( C^{r+2}(\Omega) \) with the norm \( \| \cdot \|_{C^{r+2}(\Omega), g} \), and therefore it is a \( C^\infty \) Banach manifold. Moreover \( \mathcal{C}(\Omega) \) is preserved by the operations of sum and product by positive numbers. Any \( f \in \mathcal{C}(\Omega) \) attains its minimum value at a unique point \( m_\Omega(f) \in \Omega \), defining a function \( m_\Omega : \mathcal{C}(\Omega) \rightarrow \Omega \).

Lemma 2.10 (H. Karcher [32, Lemma 10.1 and Remark 11-(ii)]). \( m_\Omega \) is \( C^r \).

Suppose that \( M \) is connected and complete. Let \((A, \mu)\) be a probability space, \( B \) a convex open ball of radius \( r > 0 \) in \( M \), and \( f : A \rightarrow B \) a measurable map, which is called a mass distribution on \( B \). Consider the \( C^\infty \) function \( P_{f, \mu} : B \rightarrow \mathbb{R} \) defined by

\[
P_{f, \mu}(x) = \frac{1}{2} \int_A d(x, f(a))^2 \mu(a).
\]

Proposition 2.11 (H. Karcher [32, Theorem 1.2]). We have the following:

(i) \( \operatorname{grad} P_{f, \mu} \) is an outward pointing vector field on the boundary \( \partial B \).

(ii) If \( \delta > 0 \) is an upper bound for the sectional curvatures of \( M \) in \( B \), and \( 2r < \pi/2\sqrt{\delta} \), then \( \operatorname{Hess} P_{f, \mu} \) is positive definite on \( B \).
If the hypotheses of Proposition 2.11 are satisfied, then \( P_{f,\mu} \in C(B) \), and therefore \( C_{f,\mu} := m_B(P_{f,\mu}) \in B \) is defined and called the center of mass of \( f \) (with respect to \( \mu \)). The following is a consequence of Lemma 2.10.

**Corollary 2.12** ([3, Corollary 10.3]; cf. [32, Corollary 1.6]). The following properties hold:

(i) \( C_{f,\mu} \) depends continuously on \( f \) and the metric tensor of \( M \).

(ii) If \( A \) is the Borel \( \sigma \)-algebra of a metric space, then \( C_{f,\mu} \) depends continuously on \( \mu \) in the weak-* topology.

Consider the following particular case. Take \( A = B \) and \( f = \text{id}_B \). Fix a finite \( C^\infty \) partition of unity \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( B \). For every \( \bar{x} = (x_1, \ldots, x_k) \in B^k \), consider the probability measure \( \mu_{\lambda,\bar{x}} = \sum_{i=1}^k \lambda_i(x_i) \delta_{x_i} \), where \( \delta_y \) denotes the Dirac mass at every \( y \in M \), and let \( C_{\lambda,\bar{x}} = C_{\text{id}_B,\mu_{\lambda,\bar{x}}} \). The following sharpening of Corollary 2.12 also follows from Lemma 2.10.

**Corollary 2.13.** The map \( B^k \to B, \bar{x} \mapsto C_{\lambda,\bar{x}} \), is \( C^\infty \).

3. Molino’s description

Consider the notation of Section 2.3 in the rest of the paper.

**Proof of Theorem 2.11.** Most of the properties stated in this theorem were already proved in [10, Theorem A]. It only remains to prove the part concerning \( H \). For this purpose, we have to recall the construction of \( G, \tilde{X}_0, \tilde{\pi}_0 \), and \( \tilde{G} \) and \( \tilde{\pi}_0 \).

Some auxiliary definitions were given first: a space \( \tilde{T}_0 \), a proper open continuous surjection \( \tilde{\pi}_0 : \tilde{T}_0 \to T \) whose fibers are homeomorphic to each other, and a minimal compactly generated pseudogroup \( \tilde{H}_0 \) on \( \tilde{T}_0 \), which is equivalent to a pseudogroup generated by some local left translations on a local group \( G \), and such that \( \tilde{\pi}_0 \) generates a morphism \( \tilde{H}_0 \to H \). For this construction, we can assume that \( X \) satisfies the conditions of equicontinuity and strong quasi-analyticity with the same set \( S \), and that \( H \) satisfies the conditions of equicontinuity and strong quasi-analyticity with the induced set \( S \). Let \( \overline{S} \cap \cup \) be the space \( S \) with the restriction of the compact-open topology on the set of paro maps \( T \to T \). Consider the subspace

\[
\overline{S} \cap \cup T = \{ (g, u) \in \overline{S} \times T \mid u \in \text{dom } g \} \in \overline{S} \cap \cup T,
\]

and equip the set \( \overline{T} \) of all germs of maps in \( \overline{S} \) (or \( \overline{H} \)) with the final topology induced by the germ map \( \gamma : \overline{S} \cap \cup T \to \overline{T} \) (this is not the restriction of the sheaf topology). Consider the restrictions \( s, t : \overline{T} \to T \) of the source and target maps. The space \( \overline{T} \) is locally compact and Polish, and \( \overline{\pi} := (s, t) : \overline{T} \to T \times T \) is continuous and proper.

Fix some point \( u_0 \in T_{t^{-1}} \subset T \). Then the subspace \( T_0 := s^{-1}(u_0) \subset \overline{T} \) is locally compact and Polish. This definition is different from the one given in [10, Section 3D], where \( \overline{T}_0 = t^{-1}(u_0) \) was considered. This change can be
made because the inversion of local transformations defines a homeomorphism of $\mathcal{F}_{c.o} [10]$ Proposition 3.1], and therefore the germ inversion defines a homeomorphism of $\hat{T}$, which becomes a topological groupoid by [11 Proposition 10]. The rest of definitions and arguments of [10 Sections 3D–3F] must be changed accordingly. For instance, take $\hat{\pi}_0 = t : \hat{T}_0 \to T$ (instead of $\hat{\pi}_0 = s$, used in [10]), which is continuous and proper, and its fibers are homeomorphic to each other [10 Section 3D]. We have $\hat{T}_0 = \bigcup_i \hat{T}_{i,0}$, where $\hat{T}_{i,0} = \hat{\pi}_0^{-1}(T_i)$.

Note that $H := \hat{\pi}_0^{-1}(u_0) = \hat{\pi}^{-1}(u_0, u_0)$ becomes a compact Polish group since $\hat{T}$ is a topological groupoid. Moreover the germ product defines a continuous free right action of $H$ on $\hat{T}_0$ whose orbits are clearly equal to the fibers of $\hat{\pi}_0 : \hat{T}_0 \to T$. Thus this map induces a continuous bijection $\hat{T}_0/H \to T$. In fact this bijection is a homeomorphism, as easily follows by using also that $H$ is compact, $\hat{T}_0$ is locally compact, and $T$ is Hausdorff.

For any $h \in \mathcal{H}$, define $\hat{h} : \hat{\pi}_0^{-1}(\text{dom } h) \to \hat{\pi}_0^{-1}(\text{im } h)$ by $\hat{h}(\gamma(g, u_0)) = \gamma(hg, u_0)$ for $g \in \mathcal{S}$ with $u_0 \in \text{dom } h$ and $g(u_0) \in \text{dom } h$ (instead of $\hat{h}(\gamma(g, u)) = \gamma(gh^{-1}, h(u))$ for $u \in \text{dom } g \cap \text{dom } h$ with $g(u) = u_0$, used in [10].) The maps $\hat{h}$ are local transformations of $\hat{T}_0$ satisfying $h\hat{\pi}_0 = \hat{\pi}_0 h$, $\overline{\text{id}_T} = \text{id}_{\hat{T}_0}$, $\overline{h\hat{h}'} = \overline{hh'}$ and $\hat{h}^{-1} = \overline{h}^{-1}$ [10 Sections 3E]. Moreover it is easy to see that every $\hat{h}$ is $H$-equivariant (note that $\text{dom } \hat{h}$ and $\text{im } \hat{h}$ are $H$-invariant). Let $\hat{\mathcal{H}}_0$ be the pseudogroup on $\hat{T}_0$ generated by $\hat{S}_0 = \{ \hat{h} \mid h \in \mathcal{S} \}$. There is a local group $G$ and some dense finitely generated sub-local group $\Gamma \subset G$ such that $\hat{\mathcal{H}}_0$ is equivalent to the pseudogroup generated by the local action of $\Gamma$ on $G$ by local left translations [10 Proposition 3.41]—this was proved by checking that $\hat{\mathcal{H}}_0$ is compactly generated, equicontinuous and strongly locally free, and its closure is also strongly locally free, and then applying Proposition 2.2 [11].

Let $\hat{U}_i = U_i \times \hat{T}_{i,0} \times \{ i \} \equiv U_i \times \hat{T}_{i,0}$, equipped with the product topology, and consider the topological sum

$$X_0 := \bigcup_i U_i \times \hat{T}_{i,0} = \bigcup_i \hat{U}_{i,0},$$

and the closed subspaces

$$\hat{U}_{i,0} := \{ (x, \gamma, i) \in \hat{U}_{i,0} \mid p_i(x) = \hat{\pi}_0(\gamma) \} \subset \hat{U}_{i,0}, \quad \overline{X}_0 := \bigcup_i \hat{U}_{i,0} \subset X_0.$$

Note that $\overline{X}_0$ is the topological sum of the spaces $\hat{U}_{i,0}$. Consider the equivalence relation “~” on $\overline{X}_0$ defined by $(x, \gamma, i) \sim (y, \delta, j)$ if $x = y$ and $\gamma = \overline{h}_{ij}(\delta)$. Let $\overline{X}_0$ be the corresponding quotient space, let $q : \overline{X}_0 \to \overline{X}_0$ be the quotient map, let $[x, \gamma, i] = q(x, \gamma, i)$, let $\hat{U}_{i,0} = q(U_{i,0})$, and let $\hat{p}_{i,0} : \hat{U}_{i,0} \to \overline{T}_{i,0}$ denote the restriction of $\hat{p}_{i,0} : \hat{U}_{i,0} \equiv U_i \times \hat{T}_{i,0} \to \hat{T}_{i,0}$, which induces a map $\hat{p}_{i,0} : \hat{U}_{i,0} \to \overline{T}_{i,0}$. Moreover a map $\hat{\pi}_0 : \overline{X}_0 \to X$ is defined by $\hat{\pi}_0([x, \gamma, i]) = x$. Observe that $\hat{U}_{i,0} = \hat{\pi}_0^{-1}(U_i)$. Then $\overline{X}_0$ is compact and Polish, $\{ \hat{U}_{i,0}, \hat{p}_{i,0}, \overline{h}_{ij} \}$ is a defining cocycle of a minimal foliated structure $\mathcal{F}_0$ on $\overline{X}_0$, $\hat{\pi}_0$ is continuous and open, the fibers of $\hat{\pi}_0$ are homeomorphic to each other, and the
restiction of \( \hat{\pi}_0 \) to the leaves of \( \tilde{X}_0 \) are the holonomy coverings of the leaves of \( X \) \[10\] Section 4B]. In the proof of these properties, it was used that every restriction \( q : \tilde{U}_{i,0} \to \tilde{U}_{i,0} \) is a homeomorphism.

Since every \( \tilde{T}_{i,0} \) is \( H \)-invariant, we get an induced free right action of \( H \) on every \( \tilde{U}_{i,0} = U_i \times \tilde{T}_{i,0} \), acting as the identity on the factor \( U_i \), yielding a right \( H \)-action on \( \tilde{X}_0 \) by union. This restricts to a free right action of \( H \) on \( \tilde{X}_0 \), preserving every \( \tilde{U}_{i,0} \), because the \( H \)-orbits in \( \tilde{T}_0 \) are equal to the fibers \( \hat{\pi}_0 : \tilde{T}_0 \to T \). Since moreover every \( \tilde{T}_{ij} \) is \( H \)-equivariant, we get an induced right action on \( \tilde{X}_0 \), given by \([x, \gamma, i] \cdot \sigma = [x, \gamma \sigma, i] \) for \([x, \gamma, i] \in \tilde{X}_0 \) and \( \sigma \in H \). This action is also free because every restriction \( q : \tilde{U}_{i,0} \to \tilde{U}_{i,0} \) is a homeomorphism, and it is easy to see that its orbits equal the fibers of \( \hat{\pi}_0 : \tilde{X}_0 \to X \). Finally note that every map \( \bar{\pi}_{i,0} : \tilde{U}_{i,0} \to \tilde{T}_{ij,0} \) is \( H \)-equivariant, and therefore \( H \) acts on \( \tilde{X}_0 \) by foliated transformations. \( \square \)

In the rest of this section, assume that \( X \) satisfies the hypotheses of Theorem \( A \). Consider structures \((G, H, \tilde{X}_0, \hat{\pi}_0)\) satisfying the conditions of its statement, where \( \tilde{X}_0 \) is considered as a foliated space and \( H \)-space. If desired, we may also add a finitely generated dense sub-local group \( \Gamma \subset G \) to the notation, \((G, \Gamma, H, \tilde{X}_0, \hat{\pi}_0)\), so that the holonomy pseudogroup of \( \tilde{X}_0 \) is represented by the pseudogroup generated by the left local action of \( \Gamma \) on \( G \) by local left translations. It will be said that two such structures, \((G, \Gamma, H, \tilde{X}_0, \hat{\pi}_0)\) and \((G', \Gamma', H', \tilde{X}_0', \hat{\pi}_0')\), are equivalent if there are a local isomorphism \( \psi : G \to G' \) that restricts to a local isomorphism \( \Gamma \to \Gamma' \), an isomorphism \( \chi : H \to H' \), and a foliated \( \chi \)-equivariant homeomorphism \( \phi : \tilde{X}_0 \to \tilde{X}_0' \) such that \( \hat{\pi}_0 = \hat{\pi}_0' \phi \) (the condition on \( \Gamma \) and \( \Gamma' \) is omitted if \( \Gamma \) and \( \Gamma' \) are not considered). In this case, \((\psi, \chi, \phi)\) is called an equivalence.

**Proposition 3.1** (Cf. \[10\] Propositions 3.43, 4.12 and 4.13). All structures \((G, \Gamma, H, \tilde{X}_0, \hat{\pi}_0)\) constructed in the proof of Theorem \( A \) are equivalent.

**Proof.** We have to prove that the equivalence class of \((G, \Gamma, H, \tilde{X}_0, \hat{\pi}_0)\) is independent of the choices of \( u_0 \) and \( \{U_i, p_i, h_{ij}\} \). Most of this is already proved in \[10\] Propositions 3.43, 4.12 and 4.13. We only have to check what concerns \( H \).

To begin with, take another point of \( u_1 \in T_{i_1} \subset T \), and let \( \tilde{T}_1, \hat{\pi}_1, \tilde{S}_1, \bar{H}_1, G_1, \Gamma_1 \) and \( H_1 \) be constructed like \( \tilde{T}_0, \hat{\pi}_0, \tilde{S}_0, \bar{H}_0, G_0 := G, \Gamma_0 := \Gamma \) and \( H_0 := H \) by using \( u_1 \) instead of \( u_0 \). Now, for each \( h \in \bar{H}_1 \), let us use the notation \( \tilde{h}_0 := \tilde{h} \in \tilde{H}_0 \), and let \( \tilde{h}_1 : \hat{\pi}_1^{-1}(\text{dom } \tilde{h}) \to \hat{\pi}_1^{-1}(\text{im } \tilde{h}) \) be the map in \( \bar{H}_1 \) defined like \( h \). In particular, the maps \((\tilde{h}_{ij})_1 \) are defined like the maps \((\tilde{h}_{ij})_0 \) := \( \tilde{h}_{ij} \). There is some \( f_0 \in \tilde{S} \) such that \( u_0 \in \text{dom } f_0 \) and \( f_0(u_0) = u_1 \). Let \( \theta : \tilde{T}_0 \to \tilde{T}_1 \) be defined by \( \theta(\gamma(f, u_0)) = \gamma(f^{-1} f_0, u_1) \) (instead of \( \theta(\gamma(f, x)) = \gamma(f_0, f) \)) like in \[10\]). This map is a homeomorphism, and satisfies \( \hat{\pi}_0 = \hat{\pi}_1 \theta \), dom \( \tilde{h}_1 \) = \( \theta(\text{dom } \tilde{h}_0) \) and \( \tilde{h}_1 \theta = \theta \tilde{h}_0 \) for all \( h \in \bar{S} \), obtaining that \( \theta \) generates an equivalence \( \Theta : \tilde{H}_0 \to \tilde{H}_1 \) \[10\] Proposition 3.42. For \( k = 0, 1 \), let \( G_k \) be the pseudogroup on \( G_k \) generated by local left translations by
elements of $\Gamma_k$. Via equivalences $\widehat{\mathcal{H}}_k \to G_k$, $\Theta$ corresponds to an equivalence $\Theta' : G_0 \to G_1$. Since the local right translations of $G_1$ generate equivalences of $G_1$, we can assume that the orbits of the identity elements correspond by the induced map $G_0/G_0 \to G_1/G_1$. By Proposition 4.12, it follows that $\Theta'$ is generated by a local isomorphism $\psi : G_0 \to G_1$ that restricts to a local isomorphism $\Gamma \to \Gamma'$. On the other hand, the conjugation mapping, $\gamma(f, u_0) \mapsto \gamma(f_0 f_0^{-1}, u_1)$, defines an isomorphism $\chi : H_0 \to H_1$ so that $\theta$ is $\chi$-equivariant.

Now, define $\widehat{X}_1 \equiv (\widehat{X}_1, \widehat{\mathcal{F}}_1), [x, \gamma, i]_1$ and $\hat{\pi}_1 : \widehat{X}_1 \to X$ like $\widehat{X}_0 \equiv (\widehat{X}_0, \widehat{\mathcal{F}}_0), [x, \gamma, i]_0 \equiv [x, \gamma, i]$ and $\pi_0 : \widehat{X}_0 \to X$, using $\widehat{T}_1, \hat{\pi}_1 : \widehat{T}_1 \to T$ and the maps $(\widehat{h}_{ij})_1$ instead of $\widehat{T}_0, \pi_0 : \widehat{T}_0 \to T$ and the maps $(\widehat{h}_{ij})_0$. According to Proposition 4.12, a foliated homeomorphism $\phi : \widehat{X}_0 \to \widehat{X}_1$ is defined by $\phi([x, \gamma, i]_0) = [x, \theta(\gamma), i]_1$, which satisfies $\pi_0 = \hat{\pi}_1 \phi$ and induces the equivalence $\Theta : \widehat{H}_0 \to \widehat{H}_1$. Moreover, $\phi$ is $\chi$-equivariant: for all $[x, \gamma, i]_0 \in \widehat{X}_0$ and $\sigma \in H_0$,

$$\phi([x, \gamma, i]_0 \cdot \sigma) = \phi([x, \gamma \sigma, i]_0) = [x, \theta(\gamma \sigma), i]_1 = [x, \theta(\gamma), i]_1 \cdot \chi(\sigma).$$

All choices of $S$ define the same space $\widehat{T}_0$ by Proposition 3.43, giving rise to the same Molino's description.

To prove the independence of $\{U_i, p_i, h_{ij}\}$, it is enough to consider the case where $\{U_i, p_i, h_{ij}\}$ refines another defining cocycle $\{U_{a_i}, p_{a_i}, h'_{ab}\}$. Let $\mathcal{H}'$ be the corresponding representative of the holonomy pseudogroup on $T' = \bigcup_a T'_{a_i}$. If $U_i \subset U_{a_i}$, there is an induced open embedding $\phi_i : T_i \to T'_{a_i}$. These maps generate an equivalence $\Phi : \mathcal{H} \to \mathcal{H}'$. In fact, $\hat{h}'_{a_i a_j} \phi_j = \hat{h}_{ij} \phi_i$. Let $u'_0 = \phi_{i0}(u_0) \in T'_{a_i a_j} \subset T'$, and let $S' \subset \mathcal{H}'$ be a generating subset such that $S' \subset S'' \subset S''^{-1}$. We can also use $\{U_{a'_i}, p_{a'_i}, h'_{ab}\}$, $u'_0$ and $S'$ to define $\widehat{T}'_0, \hat{\pi}'_0 : \widehat{T}'_0 \to T'$ and $\widehat{\mathcal{H}}'_0$ like $\widehat{T}_0, \pi_0 : \widehat{T}_0 \to T$ and $\widehat{\mathcal{H}}_0$; in particular, the generators $\widehat{h}'_{a'b'}$ of $\widehat{\mathcal{H}}'_0$ are defined like the generators $\widehat{h}_{ij}$ of $\widehat{\mathcal{H}}_0$. We get open embeddings $\hat{\phi}_{i0} : \widehat{T}_{i0} \to \widehat{T}'_{a_{i0}}$ defined by $\hat{\phi}_{i0}(\gamma(g, u_0)) = \gamma(\phi_i g \phi_{i0}^{-1}, u'_0)$, which generate an equivalence $\widehat{\Phi}_0 : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}'_0$ (this is a corrected version of Proposition 3.44). Let $(G', \Gamma', \mathcal{H}', \widehat{X}'_0, \hat{\pi}'_0)$ be the Molino's description defined with $\widehat{T}'_0, \hat{\pi}'_0 : \widehat{T}'_0 \to T'$ and the maps $\widehat{h}'_{ab}$. Let us use the notation $[x, \gamma', a]_{\prime}$ for the element of $\widehat{X}'_0$ represented by a tern $(x, \gamma', a)$. Let $\mathcal{G}$ and $\mathcal{G}'$ be the pseudogroups on $G$ and $G'$ generated by the local left translations by elements of $\Gamma$ and $\Gamma'$. Via equivalences $\mathcal{H} \to \mathcal{G}$ and $\mathcal{H}' \to \mathcal{G}'$, $\widehat{\Phi}_0$ corresponds to an equivalence $\widehat{\Phi}_0' : \mathcal{G} \to \mathcal{G}'$. As above, we can assume that the orbits of the identity elements correspond by the induced map $G/G \to G'/G'$, and therefore, according to Proposition 2.2, $\widehat{\Phi}_0'$ is generated by a local isomorphism $\psi : G \to G'$ that restricts to a local isomorphism $\Gamma \to \Gamma'$. Moreover, $\hat{\phi}_{i0,0}$ restricts to an isomorphism $\chi : H \to H'$ so that any map in $\widehat{\Phi}_0$ is $\chi$-equivariant. Finally, a canonical foliated homeomorphism $\phi : \widehat{X}_0 \to \widehat{X}'_0$ is
well defined by \( \phi([x, \gamma, i]) = [x, \hat{\phi}_{i,0}(\gamma), a_i] \) [10] Proposition 4.13. It is easy to check that \( \phi \) is \( H \)-equivariant.

By Proposition 3.1, the equivalence class of any structure \((G, \Gamma, H, \tilde{X}_0, \tilde{\pi}_0)\) constructed in the proof of Theorem A can be called the Molino’s description of \( X \). By analogy with the original Molino’s theory, the local isomorphism class of \( G \) is called the structural local group \([10]\). On the other hand, with the terminology of [18], \( \tilde{X}_0 \) will be called the Molino space and \( H \) the discriminant group.

**Proposition 3.2.** \( X \) is a \( G \)-foliated space for some local group \( G \) if and only if its discriminant group is trivial.

**Proof.** The “if” part of the statement is directly given by Theorem A. To prove the “only if” part, assume \( X \) is a \( G \)-foliated space for some local group \( G \). Thus \( \mathcal{H} \) is strongly locally free, obtaining that \( H = \{ e \} \) according to the definition of \( H \) given in the proof of Theorem A.

For every \( \hat{x} \in \tilde{X}_0 \), let \( \tilde{L}_{\hat{x}} \) denote the leaf of \( \tilde{X}_0 \) through \( \hat{x} \), and consider the identity \( \tilde{L}_x^{\text{hol}} \equiv \tilde{L}_{\hat{x}} \) given by Theorem A.

**Lemma 3.3.** For \( x \in X \) and \( \hat{x} \equiv [x, \gamma, i] \in \tilde{\pi}_0^{-1}(x) \), let \( c : I \to X \) be a leafwise path from \( x \) to some point \( y \), and let \( \hat{c} \) be the unique lift of \( c \) to \( \tilde{L}_x^{\text{hol}} \equiv \tilde{L}_{\hat{x}} \) beginning at \( \hat{x} \). Then \( \hat{c}(1) = [y, \delta \gamma, j \beta] \), where \( \delta = \gamma(h, c, p_i(x)) \) for any \( J = (j_0, \ldots, j_\beta) \) covering \( c \) with \( j_0 = i \).

**Proof.** Take a partition \( 0 = t_0 < t_1 < \cdots < t_{\beta+1} = 1 \) of \( I \) such that \( c([t_k, t_{k+1}]) \subset U_{j_k} \) for \( k = 0, \ldots, \beta \). For \( s \in I \), the path \( c_s(t) := c(st) \) in \( L \) is covered by \( J_s := (j_0, \ldots, j_{\beta_s}) \), where \( \beta_s = \min\{ k \in \{0, \ldots, \beta \} \mid t_{k+1} \geq s \} \), and let \( \delta_s = \gamma(h, c_s, p_i(x)) \). Then it is easy to see that \( \hat{c}(s) = [c(s), \delta_s \gamma, j_{\beta_s}] \).

Fix some point \( x_0 \in p^{-1}_i(u_0) \subset U_{i_0} \subset X \).

**Proposition 3.4.** For \( \hat{x}_0 \in \tilde{\pi}_0^{-1}(x_0) \), we have

\[
\text{Hol}(L_{x_0}, x_0) = \{ \gamma \in H \mid \tilde{L}_{\hat{x}_0} \cdot \gamma = \tilde{L}_{\hat{x}_0} \},
\]

and the map \( \tilde{L}_x^{\text{hol}} \equiv \tilde{L}_{\hat{x}_0} \to \tilde{X}_0 \) becomes equivariant with respect to the homomorphism \( \text{Hol}(L_{x_0}, x_0) \to H \).

**Proof.** Consider the notation of the proof of Theorem A. Observe that \( \text{Hol}(L_{x_0}, x_0) \) is a subgroup of \( H \):

\[
\text{Hol}(L_{x_0}, x_0) = \{ \gamma(h, u_0) \mid h \in \mathcal{H}, \ u_0 \in \text{dom} \ h, \ h(u_0) = u_0 \}
\subset H = \{ \gamma(g, u_0) \mid g \in \overline{\mathcal{H}}, \ u_0 \in \text{dom} \ g, \ g(u_0) = u_0 \}.
\]

If \( \gamma = \text{hol}([c]) \in \text{Hol}(L_{x_0}, x_0) \) for some \([c] \in \pi_1(L_{x_0}, x_0)\), then \( \gamma = \gamma(h_{\mathcal{I}_0}^{-1}, u_0) \) for some \( \mathcal{I} = (i_0, i_1, \ldots, i_\beta) \) covering \( c \) with \( i_0 = i_\alpha \). For any \( y \in L_{x_0} \) and \( \hat{y} \in \tilde{L}_{\hat{x}_0} \tilde{\pi}_0^{-1}(y) \), we have \( \hat{y} \equiv [y, \delta, i] \), where \( \delta = \gamma(h, \mathcal{J}, u_0) \) for some admissible sequence \( J = (j_0, \ldots, j_\beta) \), with \( j_0 = i_0 \) and \( j_\beta = i \), which covers a leafwise
Lemma 3.3, applied to \( \hat{y} \) and \( \hat{c}_y \), because \( I^{-1}J \) is defined and covers \( c_y^{-1}cc_y \), and \( h_{J^{-1}J} = h_J h_T h_J^T h_J \). This proves the inclusion “\( c \)” in (4) and the equivariance of \( \tilde{L}_{x_0} \equiv \tilde{L}_{x_0} \hookrightarrow \tilde{X}_0 \). On the other hand, since the right \( H \)-action on \( \tilde{X}_0 \) is free, foliated and preserves every \( \pi_0 \)-fiber, any element of the right hand side of (4) defines a covering transformation of the restriction \( \pi_0 : \tilde{L}_{x_0} \to L_{x_0} \), showing the inclusion “\( c \)” in (4).

According to the proof of Proposition 3.4 it follows from Proposition 3.3 that, for all \( x \in X \) and \( \hat{x} \in \hat{\pi}_0^{-1}(x) \), there is an isomorphism

\[
\operatorname{Hol}(L_x, x) \cong \{ \gamma \in H \mid \hat{L}_x \cdot \gamma = \hat{L}_x \}
\]

so that the map \( \hat{L}_x^{\text{hol}} \equiv \hat{L}_x \hookrightarrow \hat{X}_0 \) becomes equivariant with respect to the induced injective homomorphism \( \operatorname{Hol}(L_x, x) \to H \). Nevertheless this isomorphism is not canonical in general.

4. Foliated homogeneous foliated spaces

The foliated space \( X \) is called \textit{foliated homogeneous} when the canonical left action of \( \operatorname{Homeo}(X, \mathcal{F}) \) on \( X \) is transitive. Similarly, if \( X \) is \( C^\infty \), it is called \textit{\( C^\infty \) foliated homogeneous} when the canonical left action of \( \operatorname{Diff}(X, \mathcal{F}) \) on \( X \) is transitive. A priori, \( C^\infty \) foliated homogeneity is stronger than foliated homogeneity, but we will see that indeed they are equivalent conditions for compact minimal \( C^\infty \) foliated spaces (Corollary ??).

Take any complete metric \( d \) inducing the topology of \( X \), and let \( D \) be the induced complete metric on \( \operatorname{Homeo}(X) \) defined by

\[
D(\phi, \psi) = \sup_{x \in X} d(\phi(x), \psi(x)) + \sup_{x \in X} d(\phi^{-1}(x), \psi^{-1}(x)).
\]

In this way, \( \operatorname{Homeo}(X) \) becomes a completely metrizable topological group, and its canonical left action on \( X \) is continuous. Moreover it is easy to check that \( \operatorname{Homeo}(X, \mathcal{F}) \) is closed in \( \operatorname{Homeo}(X) \), and therefore \( \operatorname{Homeo}(X, \mathcal{F}) \) is also a completely metrizable topological group.

Suppose that \( X \) is compact. Then \( D \) induces the compact-open topology on \( \operatorname{Homeo}(X) \), as follows from [11, Theorem 3], obtaining that \( \operatorname{Homeo}(X) \) is also second countable. So \( \operatorname{Homeo}(X) \) is a Polish group, and \( \operatorname{Homeo}(X, \mathcal{F}) \) a Polish subgroup. Therefore, by a theorem of Effros [20, 39], if \( X \) is foliated homogeneous, then the canonical left action of \( \operatorname{Homeo}(X, \mathcal{F}) \) on \( X \) is \textit{micro-transitive}; i.e., for all \( x \in X \) and any neighbourhood \( N \cdot x \) in \( \operatorname{Homeo}(X, \mathcal{F}) \), the set \( N \cdot x \) is a neighborhood of \( x \) in \( X \).
Proof of Theorem [A]. Clark and Hurder have proved that any \( C^\infty \) homogeneous matchbox manifold is equicontinuous [15, Theorem 5.2]. But the argument of their proof really applies to any compact minimal foliated homogeneous foliated space. Moreover the \( C^\infty \) structure is not used in that result. Thus the conditions of our statement are enough to get that \((X, \mathcal{F})\) is equicontinuous.

The rest of the proof uses the same main tool as in [15, Theorem 5.2], the indicated theorem of Effros.

Let us prove that \( \mathcal{H} \) is strongly locally free. Since \( \{U_i\} \) is finite, there is some \( \epsilon > 0 \) such that \( d(U_i, X \setminus \bar{U}_i) < \epsilon \) for all \( i \). Since the action of \( \text{Homeo}(X, \mathcal{F}) \) on \( X \) is micro-transitive, there is some \( \delta > 0 \) such that, for all \( x, y \in X \) with \( d(x, y) < \delta \), there exists some \( \phi \in \text{Homeo}(X, \mathcal{F}) \) so that \( D(\phi, \text{id}_X) < \epsilon \) and \( \phi(x) = y \).

Since every \( T_i \) has compact closure in \( \bar{T}_i \), we easily get a finite open cover \( \{T_{ia}\} \) of \( T_i \) such that the \( \nu \)-diameter of every \( \sigma_i(T_{ia}) \) is smaller than \( \delta \). Let \( U_{ia} = \xi_i^{-1}(B_i \times T_{ia}) \), \( \xi_{ia} = \xi_i|_{U_{ia}} \), \( \bar{U}_{ia} = \bar{U}_i \) and \( \bar{\xi}_{ia} = \bar{\xi}_i \). By using \( \{U_{ia}, \bar{U}_{ia}\} \) and \( \{\bar{U}_{ia}, \bar{\xi}_{ia}\} \), varying \( i \) and \( a \), instead of \( \{U_i, \xi_i\} \) and \( \{\bar{U}_i, \bar{\xi}_i\} \), it follows that we can assume that the \( \nu \)-diameter of every \( \sigma_i(T_i) \) is smaller than \( \delta \).

Take \( S \) equal to the family of the maps \( h_{\mathcal{I}} \) for admissible sequences \( \mathcal{I} \). Suppose that some \( h_{\mathcal{I}} \in S \) fixes a point \( u \in \text{dom} h_{\mathcal{I}} \). Thus \( \mathcal{I} = (i_0, \ldots, i_\alpha) \) with \( i_\alpha = i_0 \). Let \( x = \sigma_{i_0}(u) \in U_{i_0} \) and let \( c : I \to X \) be a leafwise loop in \( L_x \) based at \( x \) and \( \mathcal{U} \)-covered by \( \mathcal{I} \). Take any point \( v \in \text{dom} h_{\mathcal{I}} \), and let \( y = \sigma_{i_0}(v) \in U_{i_0} \). Since the \( \nu \)-diameter of \( \sigma_{i_0}(T_{i_0}) \) is smaller than \( \delta \), according to our application of the Effros theorem, there is some \( \phi \in \text{Homeo}(X, \mathcal{F}) \) with \( \phi(x) = y \) and \( d(c(t), \phi(c(t))) < \epsilon \) for all \( t \in I \). Hence the leafwise path \( \phi c : I \to X \) is \( \mathcal{U} \)-covered by \( \mathcal{I} \). It follows that \( \tilde{h}_\mathcal{I}(v) = p_{i_0} \phi c(1) = p_{i_0} \phi(x) = p_{i_0}(y) = v \), obtaining \( h_{\mathcal{I}}(v) = v \). This shows that \( h_{\mathcal{I}} = \text{id}_{\text{dom} h_{\mathcal{I}}} \), and therefore \( \mathcal{H} \) satisfies the condition of being strongly locally free with this \( S \).

Note that \( \mathcal{H} \) is strongly quasi-analytic since it is strongly locally free, and therefore the hypotheses of Theorem [A] are satisfied. In particular, the closure \( \overline{\mathcal{H}} \) is defined and generated by the set \( \overline{S} \) induced by the above \( S \).

Now, let us sharpen the above argument to prove that \( \overline{\mathcal{H}} \) is also strongly locally free, and therefore \((X, \mathcal{F})\) is a \( G \)-foliated space for some local group \( G \) by Proposition [22, 4]. For any \( g \in \overline{S} \) with \( O = \text{dom} g \), there is a sequence of admissible sequences, \( \mathcal{I}_k = (i_{k,0}, \ldots, i_{k,\alpha_k}) \), such that \( O \subset \text{dom} h_{\mathcal{I}_k} \) for all \( k \) and \( g = \lim_k h_{\mathcal{I}_k}|_O \) in the compact-open topology. Thus \( i_0 := i_{k,0} \) is independent of \( k \). Suppose that \( g(u) = u \) for some \( u \in O \), which means that \( u'_k := h_{\mathcal{I}_k}(u) \to u \) as \( k \to \infty \). So we can assume that \( i_{k,\alpha_k} = i_0 \) for all \( k \). Let \( x = \sigma_{i_0}(u) \in U_{i_0} \) and \( x'_k = \sigma_{i_0}(u'_k) \in U_{i_0} \). We get \( x'_k = \sigma_{i_0}(u'_k) \to \sigma_{i_0}(u) = x \) because \( u'_k \to u \) \( \nu \)-converges to \( u \). For every \( k \), there exits a leafwise path \( c_k, \mathcal{U} \)-covered by \( \mathcal{I}_k \), with \( c_k(0) = x \) and \( c_k(1) = x'_k \). For any \( v \in O \), we have \( u'_k := h_{\mathcal{I}_k}(v) \to g(v) \), and let \( y = \sigma_{i_0}(v) \in U_{i_0} \). As before, there is some \( \phi \in \text{Homeo}(X, \mathcal{F}) \) such that \( \phi(x) = y \) and \( d(c_k(t), \phi(c_k(t))) < \epsilon \) for all \( t \in I \), and let \( y'_k := \phi c_k(1) = \phi(x'_k) \). Hence the leafwise path \( \phi c_k \) is \( \mathcal{U} \)-covered by \( \mathcal{I}_k \), obtaining \( p_{i_0}(y'_k) = h_{\mathcal{I}_k}(v) = g(v) \).
So topological group Theorem A, and let \( \hat{\phi} \) where the second map is the restriction of \( \xi \) because \( \hat{\phi} \) changes of coordinates are of the form \( h \) for \( \pi \) by the condition that \( \hat{\phi} \) is equipped with the unique structure of \( U \) is determined by \( \hat{\phi} \) and satisfies the hypotheses of \( U \) is defined by \( \hat{\phi} \). Then \( \hat{\phi} = \pi_0(\gamma) \) for every \( i \), let \( \hat{\xi}_{i,0} : \hat{U}_{i,0} \to B_i \times \hat{T}_{i,0} \) be the composite of homeomorphisms:

\[
\hat{U}_{i,0} \xrightarrow{q^{-1}} \hat{U}_{i,0} \equiv \{(x,\gamma) \in U_i \times \hat{T}_{i,0} | p_i(x) = \hat{\pi}_0(\gamma) \} \quad \rightarrow \quad \{(v,\gamma) \in B_i \times \hat{T}_{i,0} | u = \hat{\pi}_0(\gamma) \} = B_i \times \hat{T}_{i,0},
\]

where the second map is the restriction of \( \hat{\xi}_i \times \text{id} : \hat{U}_{i,0} \to B_i \times \hat{T}_{i,0} \). Thus \( \hat{\xi}_{i,0}([x,\gamma,i]) = (v,\gamma) \) for \( [x,\gamma,i] \in \hat{U}_{i,0} \), where \( v \in B_i \) is determined by \( \hat{\xi}_i(x) = (v,\hat{\pi}_0(\gamma)) \). Then \( \hat{U} = \{\hat{U}_{i,0},\hat{\xi}_{i,0}\} \) is a foliated atlas of \( \hat{X}_0 \), whose changes of coordinates are of the form

\[
\hat{\xi}_{j,0}\hat{\xi}_{i,0}^{-1}(v,\gamma) = (g_{ij}(v,\hat{\pi}_0(\gamma)),\hat{h}_{ij}(\gamma)),
\]

for \( (v,\gamma) \in \hat{\xi}_{i,0}(\hat{U}_{i,0} \cap \hat{U}_{j,0}) \). Hence this atlas defines a \( C^\infty \) structure of \( \hat{X}_0 \).

The foliated map \( \hat{\pi}_0 : \hat{X}_0 \to X \) and the foliated \( H \)-action on \( \hat{X}_0 \) are \( C^\infty \) because \( \hat{U}_{i,0} = \hat{\pi}_0^{-1}(U_i) \) is \( H \)-invariant, and

\[
\hat{\xi}_i\hat{\pi}_0\hat{\xi}_{i,0}^{-1}(v,\gamma) = (v,\hat{\pi}_0(\gamma)), \quad \hat{\xi}_{i,0}(\hat{\xi}_{i,0}^{-1}(v,\gamma) \cdot \sigma) = (v,\gamma \cdot \sigma),
\]

for \( (v,\gamma) \in B_i \times \hat{T}_{i,0} \) and \( \sigma \in H \). It also follows that \( T\hat{\pi}_0 \) restricts to isomorphisms between the fibers.

By Proposition 5.1 applied to \( \hat{\pi}_0 \), the \( C^\infty \) structure on \( \hat{X}_0 \) is determined by the condition that \( \hat{\pi}_0 \) is \( C^\infty \) and \( T\hat{\pi}_0 \) restricts to isomorphisms between the fibers.

If \( \hat{X}_0 \) is equipped with the unique \( C^\infty \) structure given by Proposition 5.1, then \( (G,H,\hat{X}_0,\hat{\pi}_0) \) is called the \( C^\infty \) Molino’s description of \( X \).

6. Right local transverse actions

6.1. Topological right local transverse actions. The foliated homeomorphisms leafwisely homotopic to the identity form a normal subgroup \( \text{Homeo}_0(X,F) \) of \( \text{Homeo}(X,F) \), obtaining the (possibly non-Hausdorff) topological group

\[
\text{Homeo}(X,F) = \text{Homeo}(X,F)/\text{Homeo}_0(X,F).
\]
Suppose that $X$ is compact for the sake of simplicity. Then a right local transverse action of a local group $G$ on $X$ can be defined as a foliated map $\phi : X \times O_{pt} \to X$, for some $O \in \mathcal{N}(G,e)$, such that $\phi^g := \phi(\cdot,g) \in \text{Homeo}(X,F)$ for all $g \in O$, and $O \to \text{Homeo}(X,F)$, $g \mapsto [\phi^g]$, is a local anti-homomorphism of $G$ to $\text{Homeo}(X,F)$. Two right local transverse actions, $\phi : X \times O_{pt} \to X$ and $\psi : X \times P_{pt} \to X$, are declared to be equivalent if there is some $Q \in \mathcal{N}(G,e)$ such that $Q \subset O \cap P$ and the restrictions $\phi, \psi : X \times Q_{pt} \to X$ are leafwise homotopic.

**Lemma 6.1.** If $G$ is locally contractible, then the equivalence class of $\phi$ is determined by the induced local anti-homomorphism of $G$ to $\text{Homeo}(X,F)$.

**Proof.** Let $\psi : X \times P_{pt} \to X$ be another right local transverse action inducing the same local anti-homomorphism of $G$ to $\text{Homeo}(X,F)$ as $\phi$. Thus there is some $Q \in \mathcal{N}(G,e)$ such that $Q \subset O \cap P$ and $\phi^g$ is leafwisely homotopic to $\psi^g$ for all $g \in Q$. Since $G$ is locally contractible, we can suppose that there is a homotopy $E : Q \times I \to Q$ of const$_{g_0}$ to id$_Q$ for some point $g_0 \in Q$. By choosing $Q$ small enough and using $g_0^{-1}Q$ instead of $Q$, we can also assume that $g_0 = e$. Let $g_t = E(g,t)$ for $g \in Q$ and $t \in I$. Given any leafwise homotopy $H : X \times I \to X$ of $\phi^e$ to $\psi^e$, the map $F : X \times Q_{pt} \times I \to X$, defined by

$$F(x,g,t) = \psi^{gt}(\psi^e)^{-1}\phi^{gt^{-1}g}(\phi^e)^{-1}H(x,t),$$

is a leafwise homotopy between the restrictions $\phi, \psi : X \times Q_{pt} \to X$, as follows by using that $(\psi^e)^{-1}$, $(\phi^e)^{-1}$ and $H(\cdot,t)$ are leafwise homotopic to id$_X$, and $\psi^{gt}$ and $\phi^{gt}\phi^{gt^{-1}g}$ are leafwise homotopic to $\phi^{gt}$ and $\phi^g$, respectively. $\square$

According to Lemma [6.1] when $G$ is locally contractible, a right local transverse action of $G$ on $X$ could be defined as a local anti-homomorphism $G$ to $\text{Homeo}(X,F)$, given by a map $O \to \text{Homeo}(X,F)$, $g \mapsto [\phi^g]$, for some $O \in \mathcal{N}(G,e)$ and some foliated map $\phi : X \times O_{pt} \to X$ with $\phi^g \in \text{Homeo}(X,F)$ for all $g \in O$. This corresponds to the definition of right transverse action of Lie groups on foliated manifolds given in [7]. But it seems impossible to extend Lemma [6.1] to arbitrary local groups, which motivates our more involved definition.

**Lemma 6.2.** We can assume $\phi^e = \text{id}_X$.

**Proof.** The foliated map $\psi : X \times O_{pt} \to X$, defined by $\psi^g := \phi^g(\phi^e)^{-1}$, satisfies the stated conditions. In fact, if $H : X \times I \to X$ is a leafwise homotopy of $(\phi^e)^{-1}$ to id$_X$, then $F : X \times O_{pt} \times I \to X$, defined by $F(\cdot,g,t) = \phi^gH(\cdot,t)$, is a leafwise homotopy of $\phi$ to $\psi$. $\square$

From now on, suppose that $\phi^e = \text{id}_X$ according to Lemma [6.2]. Then, since $X$ is compact, there is some $O' \in \mathcal{N}(G,e)$ such that $O' \subset O$ and $\phi(U_i \times O') \subset \bar{U}_i$ for all $i$. The foliated restrictions $\phi : U_i \times O'_{pt} \to \bar{U}_i$ induce maps $\tilde{\phi} : T_i \times O' \to \bar{T}_i$, and let $\tilde{\phi} : T \times O' \to \bar{T}$ denote their union. Then the restriction $\phi : \Omega := \phi^{-1}(T) \to T$ is a right local action of $G$ on $T$, which will be said to be induced by $\phi$. 
Lemma 6.3. \( \mathcal{H} \) is locally equivariant (with respect to \( \tilde{\phi} : \Omega \to T \)).

Proof. It is enough to prove that the maps \( h_{ij} \) are locally equivariant. Let \( u \in p_j(U_i \cap U_j) \) and \( g \in O' \), and take any \( x \in U_i \cap U_j \) such that \( p_j(x) = u \). We have \( h_{ij}(u) = p_i(x) \), \( \phi(x,g) \in \overline{U}_i \cap \overline{U}_j \) and \( \tilde{\phi}(u,g) = p_j\phi(x,g) \), yielding

\[
\tilde{h}_{ij}\tilde{\phi}(u,g) = p_j\phi(x,g) = \tilde{\phi}(p_i(x),g) = \tilde{\phi}(h_{ij}(u),g).
\]

So \( h_{ij}\tilde{\phi}(u,g) = \tilde{\phi}(h_{ij}(u),g) \) for all \((u,g)\) in \((p_j(U_i \cap U_j) \times O') \cap \tilde{\phi}^{-1}(T_i)\), which is an open neighborhood of \( p_j(U_i \cap U_j) \times \{e\} \) in \( \Omega \). □

Lemma 6.4. If \( X \) has no holonomy, then the equivalence class of \( \phi \) determines the equivalence class of \( \tilde{\phi} : \Omega \to T \).

Proof. Suppose that \( \phi \) is equivalent to another right transverse local action \( \psi : X \times P \to X \) with \( \psi^e = \text{id}_X \). Take some \( P' \in \mathcal{N}(G,e) \) such that \( P' \subseteq P \) and \( \psi(U_i \times P') \subseteq \overline{U}_i \) for all \( i \). As above, consider the map \( \tilde{\psi} : T \times P' \to \overline{T} \) induced by the foliated restrictions \( \psi : U_i \times P_{pt} \to \overline{U}_i \), whose restriction \( \tilde{\psi} : \Sigma := \tilde{\phi}^{-1}(T) \to T \) is a right local action of \( G \) on \( T \). For some \( Q \in \mathcal{N}(G,e) \) with \( Q \cap O' \cap P' \), there is a leafwise homotopy \( H : X \times Q_{pt} \times I \to X \) between the foliated restrictions \( \phi, \psi : X \times Q_{pt} \to X \).

Claim 5. \( \tilde{\phi} = \tilde{\psi} \) on \( T \times Q' \) for some \( Q' \in \mathcal{N}(G,e) \) with \( Q' \subseteq Q \).

By absurdity, suppose that this assertion is not true. Then \( \tilde{p}_{ik}\phi^{g_k}(x_k) \neq \tilde{p}_{ik}\psi^{g_k}(x_k) \) for some sequences, of indices \( i_k \), of points \( x_k \in U_{i_k} \), and \( g_k \to e \) in \( G \). Since \( X \) is compact, we can assume that \( i_k = i \) for all \( k \), and \( x_k \to x \) in \( X \); thus \( x \in \overline{U}_i \). Consider the leafwise paths \( c_k = H(x_k,g_k) \) and \( c = H(x,e,.) \). Note that \( g_k \to c \) in the compact-open topology, and \( c \) is a loop in \( L_x \) based at \( x \) because \( \phi^e = \psi^e = \text{id}_X \). Let \( J = (j_0, \ldots, j_\beta) \) be a admissible sequence \( \tilde{U} \)-covering \( c \) with \( j_0 = j_\beta = i \). Hence \( J \) also \( \tilde{U} \)-covers \( x_k \) for \( k \) large enough, obtaining that \( \tilde{p}_i\phi^{g_k}(x_k) \in \text{dom} \tilde{h}_J \) and \( \tilde{h}_J\tilde{p}_i\phi^{g_k}(x_k) = \tilde{p}_i\psi^{g_k}(x_k) \) for \( k \) large enough. Since \( \tilde{p}_i\phi^{g_k}(x_k) \to \tilde{p}_i(x) \) in \( T_i \) and \( \tilde{h}_J \) is the identity on some neighborhood of \( p_i(x) \) because \( X \) has no holonomy, it follows that \( \tilde{h}_J\tilde{p}_i\phi^{g_k}(x_k) = \tilde{p}_i\phi^{g_k}(x_k) \) for \( k \) large enough, yielding \( \tilde{p}_i\phi^{g_k}(x_k) \to \tilde{p}_i\psi^{g_k}(x_k) \) for \( k \) large enough, a contradiction.

By Claim 5, we get \( \tilde{\phi} = \tilde{\psi} \) on \( \Omega \cap \Sigma \cap (T \times Q') \), showing that the right local actions \( \tilde{\phi} : \Omega \to T \) and \( \tilde{\psi} : \Sigma \to T \) are equivalent. □

6.2. \( C^\infty \) right local transverse actions. From now on, assume that \( X \) is \( C^\infty \), and consider also the (possibly non-Hausdorff) topological group

\[
\text{Diffeo}(X,\mathcal{F}) = \text{Diffeo}(X,\mathcal{F})/\text{Diffeo}_0(X,\mathcal{F})
\]

where \( \text{Diffeo}_0(X,\mathcal{F}) \) is the normal subgroup of \( \text{Diffeo}(X,\mathcal{F}) \) consisting of the foliated diffeomorphisms that are leafwisely homotopic to \( \text{id}_X \); i.e., \( \text{Diffeo}_0(X,\mathcal{F}) = \text{Diffeo}(X,\mathcal{F}) \cap \text{Homeo}_0(X,\mathcal{F}) \). It is said that the right local transverse action \( \phi \) is \( C^\infty \) if it is \( C^\infty \) as foliated map, \( \phi^g \in \text{Diffeo}(X,\mathcal{F}) \) for all \( g \in O \), and \( O \to \text{Diffeo}(X,\mathcal{F}) \), \( g \to [\phi^g] \), is a local anti-homomorphism of \( G \) to \( \text{Diffeo}(X,\mathcal{F}) \). A \( C^\infty \) equivalence between two \( C^\infty \) right local transverse
actions is defined like in the case of right local transverse actions. Suppose also that $\phi$ is $C^\infty$ from now on.

**Lemma 6.5.** The $C^\infty$ equivalence class of $\phi$ is determined by the equivalence class of $\phi : \Omega \to T$.

**Proof.** Let $\psi : X \times P_{pt} \to X$ be another $C^\infty$ right local transverse action of $G$ on $X$ with $\psi^e = \text{id}_X$. Take some $P' \in \mathcal{N}(G,e)$ such that $P' \subset P$ and $\phi(U_i \times P') \subset \tilde{U}_i$ for all $i$. Like in the proof of Lemma 6.2 let $\tilde{\psi} : T \times P' \to \tilde{T}$ be induced by the foliated restrictions $\psi : U_i \times P' \to \tilde{U}_i$, and consider the right local action $\tilde{\psi} : \Sigma := \tilde{\psi}^{-1}(T) \to T$. Suppose that $\tilde{\psi} = \tilde{\phi}$ on some open neighborhood $\Theta$ of $T \times \{e\}$ in $\Omega \cap \Sigma$. So $\tilde{p}_i \phi(x,g) = \tilde{p}_i \psi(x,g)$ for all $i$ and $(x,g) \in U_i \times (O \cap P)$ with $(p_i(x),g) \in \Theta$. Since $T$ is compact, the open neighborhood of $X \times \{e\}$ in $X \times (O \cap P)$,

$$
\bigcup_i \{ (x,g) \in U_i \times (O \cap P) \mid (p_i(x),g) \in \Theta \},
$$

contains $X \times Q$ for some $Q \in \mathcal{N}(G,e)$. Hence $\phi(x,g)$ and $\psi(x,g)$ lie in the same plaque of some $\tilde{U}_i$ for all $(x,g) \in X \times Q$. We can further assume that the plaques of the foliated charts in $\tilde{U}$ are convex for some choice of a Riemannian metric on $X$, obtaining a $C^\infty$ leafwise homotopy between the foliated restrictions $\phi, \psi : X \times Q_{pt} \to X$ by using geodesic segments. Therefore $\phi$ and $\psi$ are $C^\infty$ equivalent.

**Proposition 6.6.** If $X$ is without holonomy, then the assignment of the induced right local action defines a bijection of the set of $C^\infty$ equivalence classes of $C^\infty$ right local transverse actions of $G$ on $X$ to the set of equivalence classes of right local actions of $G$ on $T$ satisfying that $\mathcal{H}$ is locally equivariant.

**Proof.** By Lemmas 6.3, 6.4 and 6.5, it only remains to prove that, if $\mathcal{H}$ is locally equivariant with respect to a right local action $\chi : \Sigma \to T$ of $G$ on $T$, then $\chi$ is induced by some $C^\infty$ right local transverse action of $G$ on $X$.

By Proposition 2.4, $\mathcal{H}$ is locally equivariant with respect to some right local action $\tilde{\chi} : \tilde{\Sigma} \to \tilde{T}$ of $G$ on $\tilde{T}$, whose restriction to $T$ is equivalent to $\chi$. Since $T$ is relatively compact in $\tilde{T}$, there is some $P \in \mathcal{N}(G,e)$ such that $P \subset O$, $\tilde{T} \times P \subset \tilde{\Sigma}$ and $\tilde{\chi}(\tilde{T}_i \times P) \subset \tilde{T}_i$ for all $i$. Then, for $x \in U_i \subset \tilde{U}_i$ with $\tilde{\chi}(x) = (v,u)$ and $g \in P$, the point $\phi_i(x,g) := \tilde{\chi}_i^{-1}(v,\tilde{\chi}(u,g)) \in \tilde{U}_i$ is well defined because $u = \tilde{p}_i(x) \in \tilde{p}_i(U_i) = \tilde{T}_i$.

**Claim 6.** There is some $Q \in \mathcal{N}(G,e)$ such that $Q \subset P$ and, if $x \in U_i \cap U_j$ and $g \in Q$, then $\phi_i(x,g), \phi_j(x,g) \in \tilde{U}_i$ and $\tilde{p}_i \phi_i(x,g) = \tilde{p}_j \phi_j(x,g)$.

By absurdity, suppose that this assertion is not true. Hence $\tilde{p}_i \phi_i(x_k,g_k) \neq \tilde{p}_j \phi_j(x_k,g_k)$ for some sequences, of indices $i_k, j_k$, of points $x_k \in U_{i_k} \cap U_{j_k}$, and $g_k \to e$ in $P$. Since $X$ is compact, we can assume that $i_k = i$ and $j_k = j$ for all $k$, and $x_k \to x$ in $X$. Thus $x \in \bigcup_i \cap \bigcup_j \subset \tilde{U}_i \cap \tilde{U}_j$, $\phi_i(x_k,g_k) \in \tilde{U}_i$ and $\phi_j(x_k,g_k) \in \tilde{U}_j$. Let $u_k = \tilde{p}_j(x_k)$ and $u = \tilde{p}_j(x)$. Since $\mathcal{H}$ is locally equivariant,
there are some open neighborhood $W$ of $u$ in $\text{dom} \, \tilde{h}_{ij}$ and $Q \in \mathcal{N}(G, e)$ such that $Q \subset P$, $W \times Q, \tilde{h}_{ij}(W) \times Q \subset \tilde{\Sigma}$, $\tilde{\chi}(W \times Q) \subset \text{dom} \, \tilde{h}_{ij}$ and $\tilde{\chi}(\tilde{h}_{ij}(w), g) = \tilde{h}_{ij}\tilde{\chi}(w, g)$ for all $(w, g) \in W \times Q$. Take some open neighborhood $N$ of $x$ in $X$ so that $N \subset \tilde{U}_i \cap \tilde{U}_j$ and $\tilde{p}_j(N) \subset W$. Moreover we can choose $Q$ such that $\phi_j(N \times Q) \subset \tilde{U}_i \cap \tilde{U}_j$, and therefore

$$\tilde{p}_j\phi_j(N \times Q) = \tilde{\chi}(\tilde{p}_j(N) \times Q) \subset \text{dom} \, \tilde{h}_{ij}.$$  

For $k$ large enough, we have $(x_k, g_k) \in N \times Q$, obtaining

$$\tilde{p}_i\phi_i(x_k, g_k) = \tilde{\chi}(\tilde{h}_{ij}(u_k), g_k) = \tilde{h}_{ij}\tilde{\chi}(u_k, g_k) = \tilde{p}_i\phi_j(x_k, g_k),$$

a contradiction that proves Claim 6.

Given any Riemannian metric on $X$, we can assume that the plaques of every $(U_i, \xi_i)$ and $(\tilde{U}_i, \xi_i)$ are convex balls of diameter $< \pi/2\sqrt{\delta}$, where $\delta > 0$ is an upper bound for the sectional curvatures of the leaves.

Consider the open neighborhood $Q$ of $e$ in $P$ given by Claim 6 and let $\{\lambda_i\}$ be a $C^\infty$ partition of unity of $X$ subordinated to $\{U_i\}$. For all $(x, g) \in X \times Q$, a probability measure on $X$ is well defined by $\mu_{x, g} = \sum_i \lambda_i(x) \delta_{\phi_i(x, g)}$. By Claim 6, if $x \in \text{supp} \, \lambda_i$, then $\mu_{x, g}$ is supported in the plaque $\tilde{p}_i^{-1}(\chi(p_i(x), g))$ of $(\tilde{U}_i, \xi_i)$. Then, by Corollary 2.13 a $C^\infty$ foliated map $\phi : X \times Q_{pt} \to X$ is defined by taking $\phi(x, g)$ equal to the center of mass of $\mu_{x, g}$. Let $\phi^g = \phi(\cdot, g) : X \to X$ for $g \in Q$. Note that $\phi^g(U_i) \subset \tilde{U}_i$, and $\phi^e = \text{id}_X$ because $\phi_i(x, e) = x$ for $x \in \overline{U_i}$.

Claim 7. There exists some $Q' \in \mathcal{N}(G, e)$ such that $Q'^2 \subset Q$ and there is a $C^\infty$ leafwise homotopy of $\phi^{gh}$ to $\phi^h \phi^g$ for all $g, h \in Q'$.

Since $X$ is compact, there is $Q' \in \mathcal{N}(G, e)$ such that $Q'^2 \subset Q$ and

$$\phi_j((\text{supp} \, f_i \cap \text{supp} \, f_j) \times Q') \subset U_i$$

for all $i, j$. Then, for all $x \in \text{supp} \, f_i \cap \text{supp} \, f_j$ and $g \in Q'$, the points $\phi_i(x, g)$ and $\phi_j(x, g)$ are in the plaque $p_i^{-1}(\chi(p_i(x), g))$ of $(U_i, \xi_i)$ by Claim 6. Therefore $\phi(x, g) \in p_i^{-1}(\chi(p_i(x), g))$ according to Corollary 2.13. Applying again Claim 6 in a similar way, we get that $\phi(\phi(x, g), h)$ is in the plaque of $(\tilde{U}_i, \xi_i)$ over $\tilde{\chi}(\chi(p_i(x), g), h)$ for all $h \in Q'$. On the other hand, since $gh \in Q'^2 \subset Q$, the same kind of argument shows that $\phi(x, gh)$ is in the plaque of $(\tilde{U}_i, \xi_i)$ over $\tilde{\chi}(p_i(x), gh)$. Thus $\phi(\phi(x, g), h)$ and $\phi(x, gh)$ are in the same plaque of $(\tilde{U}_i, \xi_i)$. Since these plaques are convex, we can use geodesic segments to construct a $C^\infty$ leafwise homotopy between the foliated maps $\phi^h \phi^g$ and $\phi^{gh}$ for all $g, h \in Q'$.

Claim 8. There is some $Q'' \in \mathcal{N}(G, e)$ such that $Q'' \subset Q'$ and $\phi^g \in \text{Diffeo}(X, \mathcal{F})$ for all $g \in Q''$.

For all $g \in Q'$, every restricted foliated map $\phi^g : U_i \to \tilde{U}_i$ induces the open embedding $\tilde{\chi}^g : T_i \to \tilde{T}_i$; i.e., $\{\phi^g \mid g \in Q'\}$ is a uniform family of transverse equivalences. Hence, since $\phi^e = \text{id}_X$ and $g \mapsto \phi^g$ is continuous in
the $C^\infty$ foliated topology, it follows from Proposition 2.8 that there is some $Q'' \in \mathcal{N}(G,e)$ such that $\phi^g \in \text{Diff}_o(X,\mathcal{F})$ for all $g \in Q''$.

From Claims 7 and 8 and since $\phi^e = \text{id}_X$, we get that $\phi : X \times Q_{pt} \to X$ is a $C^\infty$ right transverse local action of $G$ on $X$. The induced right local action of $G$ on $T$ is equivalent to $\chi$ because every $\phi^g : U_i \to \overline{U_i}$ induces $\bar{\chi}^g : T_i \to T_i$. □

Consider the following property that $(X,\mathcal{F},\phi)$ may have:

$$\mathcal{F}(\phi(\{x\} \times P)) = X \quad \forall x \in X, \forall P \in \mathcal{N}(G,e) \mid P \in O.$$  \hspace{1cm} (5)

**Lemma 6.7.** Property (5) is invariant by equivalences of right transverse local actions.

**Proof.** Elementary. □

**Lemma 6.8.** $(X,\mathcal{F},\phi)$ satisfies (5) if and only if $(T,\mathcal{H},\tilde{\phi})$ satisfies (1).

**Proof.** Elementary. □

### 6.3. Structural right transverse local action.

Now, suppose that $X$ is a $C^\infty$ compact minimal $G$-foliated space. Fix any equivalence $\Psi$ of $\mathcal{H}$ to the pseudogroup $\mathcal{G}$ on $G$ generated by local left translations with respect to some finitely generated dense local subgroup $\Gamma \subset G$. The local multiplication $\mu : G \times G \to G$ is a right local action of $G$ on $G$ so that $\mathcal{G}$ becomes locally equivariant. By Proposition 2.3 there is a unique right local action $\chi : T \times G \to T$, up to equivalences, such that $\mathcal{H}$ and $\Psi$ become locally equivariant. According to Proposition 6.6 there is a unique right local transverse action $\phi : X \times O \to X$ of $G$ on $X$ inducing $\chi$, up to equivalences, (whose equivalence class is) called the *structural right transverse local action*.

### 7. $C^\infty$ $G$-foliated spaces are $C^\infty$ foliated homogeneous

Suppose that $X$ is compact and $C^\infty$. Then the following result states guarantees certain leafwise homogeneity.

**Proposition 7.1.** Let $L$ be the leaf of $X$, let $D$ be a relatively compact regular domain without holonomy in $L$, and let $c : I \to D$ be any $C^\infty$ path. Then, for any open neighborhood $U$ of $c(I)$ in $X$, there is some $C^\infty$ leafwise diffeotopy $\phi : X \times I \to X$ supported in $U$ with $\phi(c(0),\cdot) = c$.

**Proof.** Let $E$ be a relatively compact open subset of $L$ such that $c(I) \subset E$ and $\overline{E} \subset D \cap U$. By the homogeneity of $L$, there is a diffeotopy $\psi : L \times I \to L$ supported in $E$ so that $\psi(\cdot,0) = \text{id}_X$ and $\psi(c(0),\cdot) = c$. Let $\Sigma$ be a local transversal of $X$ through $x$. By the Reeb’s stability theorem for $C^\infty$ foliated spaces [4] Proposition 1.7, there is a $C^\infty$ foliated embedding $h : D \times \Sigma_{pt} \to X$ that can be identified to the identity on $D \times \{x\} \equiv D$ and $\{x\} \times \Sigma \equiv \Sigma$. Write $h^{-1} = (h',h'') : \text{im} h \to D \times \Sigma$. Take a compactly supported continuous
function $f: \Sigma \to I$ such that $h(E \times \text{supp } f) \subset U$. Then the statement is satisfied with the $C^\infty$ foliated diffeotopy $\phi: X \times I \to X$ defined by

$$
\phi(x,t) = \begin{cases} 
  h(\psi(x), fh''(x)), h'''(x)) & \text{if } x \in \text{im } h \\
  x & \text{otherwise}
\end{cases}
$$

**Corollary 7.2.** If there is a $C^\infty$ right transverse local action of $G$ on $X$ satisfying [5], then $X$ is $C^\infty$ foliated homogeneous.

**Proof.** Apply [5] and Proposition [7].

**Proof Theorem [5].** By Theorem [5] it is enough to prove “(iii) $\Rightarrow$ (i).” With the notation of Section [5.3], $(G, \mathcal{G}, \mu)$ satisfies (i) because

$$
\mu((\Gamma \times \mu((g) \times Q)) \cap \text{dom } \mu) = G
$$

for all $g \in G$ and $Q \in \mathcal{N}(G,e)$ with $(g) \times Q \subset \text{dom } \mu$. So $(T, \mathcal{H}, \chi)$ also satisfies (i) by Lemma [2.4] and therefore $(X, \mathcal{F}, \phi)$ satisfies (i) by Lemma [6.8]. Thus $X$ is $C^\infty$ foliated homogeneous by Corollary [7.2].

8. Examples and open problems

8.1. Molino’s description of equicontinuous suspensions. Let $T$ be a compact space with a transitive left action of a compact topological group $G$, which is quasi-analytic in the sense that any $g \in G$ is the identity element $e \in G$ if it acts as the identity on some non-empty open set, and let $H \subset G$ be the isotropy group at some fixed point $u_0 \in T$. Moreover let $\Gamma \subset G$ be a dense subgroup isomorphic to $\pi_1(M)/\pi_1(L)$ for a connected covering $L$ of some closed manifold $M$. Thus we have a right $\Gamma$-action on $L$ by covering transformations, and a left $\Gamma$-action on $T$ defined by the $G$-action. The induced diagonal $\Gamma$-action on $L \times T$, given by $(y,u) \cdot \gamma = (y \cdot \gamma, \gamma^{-1} \cdot u)$, is properly discontinuous and foliated. The corresponding foliated quotient space, $L \rtimes_\Gamma T$, is called the suspension of the $\Gamma$-action on $T$, and the quotient projection is a foliated covering map $L \times T \to L \times_\Gamma T$. The element in $L \times_\Gamma T$ defined by any $(y,u) \in L \times T$ will be denoted by $[y,u]$. Moreover the covering projection $\theta: L \to M$ induces a fiber bundle projection $\rho: L \times_\Gamma T \to M$, $\rho([y,u]) = \theta(y)$, with typical fiber $T$; in particular, $L \times_\Gamma T$ is compact. Note that the fibers of $\rho$ are transverse to the leaves; i.e., $\rho: L \times_\Gamma T \to M$ is a flat bundle. Recall that any flat bundle whose total space is compact is given by a suspension.

Let us use our notation $X = (X, \mathcal{F})$ for $L \times_\Gamma T$. Let $\mathcal{V} = \{V_i, \zeta_i\}$ be an atlas of $M$, with $\zeta_i: V_i \to B_i$ for some contractible open subset $B_i \subset \mathbb{R}^n$. Thus the flat bundle $\rho: X \to M$ is trivial over every $V_i$; i.e., there are homeomorphisms $\psi_i: U_i = \rho^{-1}(V_i) \to V_i \times T$ such that $\rho: U_i \to V_i$ corresponds to the first factor projection $V_i \times T \to V_i$ and the leaves of the $\mathcal{F}|_{U_i}$ correspond to the fibers of the second factor projection $V_i \times T \to T$. We get an induced foliated atlas $\mathcal{U} = \{U_i, \xi_i\}$ of $X$, where $\xi_i = (\zeta_i \times \text{id}_T)\psi_i: U_i \to B_i \times T_i$ with $T_i \equiv T$. Assuming obvious conditions on $\mathcal{V}$, we get that $\mathcal{U}$ is regular. Then $\mathcal{U}$ induces a representative $\mathcal{H}'$ of the holonomy pseudogroup of $X$ on $T' = \bigsqcup_i T_i$. For
any fixed index $i_0$, since $T_{i_0}' \equiv T$ meets all $H'$-orbits, by restricting $H'$ to $T_{i_0}'$, we get a pseudogroup $\mathcal{H}$ on $T$ equivalent to $\mathcal{H}'$, which is generated by the $\Gamma$-action on $T$. Thus $X$ is minimal, equicontinuous and strongly quasi-analytic (take $S = \Gamma$ to check the last two properties for $\mathcal{H}$). Moreover $\mathcal{H}$ is generated by the $G$-action on $T$, and therefore $\mathcal{H}$ also is strongly quasi-analytic. So $X$ satisfies the conditions of Theorem A.

Fix some $u_0 \in T \equiv T_{i_0}'$, and consider the associated space $\widetilde{T}_{\mathcal{H}}'$ with the pseudogroup $\widetilde{\mathcal{H}}'$, and the associated representative of the Molino’s description, $(G', H', \tilde{X}_0') = (\tilde{X}_0', \tilde{F}_0')$, constructed like in the proof of Theorem A. Then $\tilde{T}_0 := \tilde{T}_{i_0,0}$ meets all $\tilde{H}_0'$-orbits, obtaining that $\tilde{H}_0'$ is equivalent to its restriction $\tilde{H}_0 := \tilde{F}_0|\tilde{T}_0$. Thus $\tilde{T}_0 = \{ \gamma(g, u_0) \mid g \in G \}$ has the final topology induced by the map $G \to \tilde{T}_0$, $g \mapsto \gamma(g, u_0)$. This map is a continuous bijection, and therefore it is a homeomorphism because $G$ is compact and $\tilde{T}_0$ is Hausdorff. So $\tilde{T}_0 \equiv G$, $\tilde{H}$ is generated by the action of $G$ on itself by left translations, $G'$ is locally isomorphic to $G$, and $\tilde{\pi}_0 : \tilde{T}_0 \equiv G \to T$ is the orbit map $g \mapsto g \cdot u_0$. The composite $\rho \tilde{\pi}_0' : \tilde{X}_0' \to M$ is a fiber bundle with typical fiber $\tilde{T}_0 \equiv G$, and $(\tilde{X}_0', \rho \tilde{\pi}_0', \tilde{F}_0')$ is also a flat bundle. Thus there is a foliated homeomorphism of $\tilde{X}_0'$ to $\tilde{X}_0 \equiv (\tilde{X}_0, \tilde{F}_0') := L \times G_{pt}$. Moreover

$$H' \equiv H := \{ h \in G \mid h \cdot u_0 = \tilde{u}_0 \},$$

the right $H'$-action on $\tilde{X}_0'$ corresponds to the right $H$-action on $\tilde{X}_0$ given by $[y, g] \cdot h = [y, gh]$, and the map $\tilde{\pi}_0' : \tilde{X}_0' \to X$ corresponds to the map $\tilde{\pi}_0 : \tilde{X}_0 \to X$ defined by $\tilde{\pi}_0([y, g]) = [y, g \cdot u_0]$, which is induced by the foliated map $\text{id}_{L \times \tilde{T}_0} : L \times G_{pt} \to L \times T_{pt}$. Thus $(G, H, \tilde{X}_0, \tilde{\pi}_0)$ is another representative of the Molino’s description, which will be used in the next examples.

If $M$ is $C^\infty$, its $C^\infty$ structure can be lifted to a $C^\infty$ structure on $L$, which in turn can be lifted to $L \times T_{pt}$, which finally give rise to a $C^\infty$ structure on $X$ so that the projection $\rho : X \to M$ is $C^\infty$ and $T \rho$ has isomorphic restrictions to the fibers. This can be similarly applied to $\tilde{X}_0$, obtaining the $C^\infty$ structure given by Proposition 5.7. The same procedure can be applied to any Riemannian metric on $M$, obtaining induced Riemannian metrics on $X$ and $\tilde{X}_0$ so that the projections $\rho : X \to M$ and $\tilde{\pi}_0 : \tilde{X}_0 \to X$ have locally isometric restrictions to the leaves.

The following result is well known. A proof is included for completeness.

**Proposition 8.1.** The following properties are equivalent:

1. The $\Gamma$-action on $T$ has no fixed points.
2. $\Gamma \cap gHg^{-1} = \{ e \}$ for all $g \in G$.
3. The canonical foliated projection $L \times T_{pt} \to X$ restricts to homeomorphisms between the leaves.
Proof. Let us prove “(i) ⇔ (ii)”. Given any $\gamma \in \Gamma$ and $u \in T$, take some $g \in G$ such that $u = g \cdot u_0$. Then

$\gamma u = u \iff \gamma g \cdot u_0 = g \cdot u_0 \iff g^{-1} \gamma g \cdot u_0 = u_0 \iff g^{-1} \gamma g \in H$

$\iff \gamma \in \Gamma \cap gHg^{-1} = \{e\} \iff \gamma = e$.

Let us prove “(ii) ⇔ (iii)”. For all $y, y' \in L$ and $u \in T$, we have $[y, u] = [y', u]$ if and only if there is some $\gamma \in \Gamma$ such that $(y', u) = (y \cdot \gamma, \gamma^{-1} \cdot u)$, which means $\gamma = e$ and $y' = y$. □

When the conditions of Proposition 8.1 are satisfied, $X$ is strongly locally free (in particular, it has no holonomy), and all leaves are homeomorphic to $L$. If moreover $M$ is $C^\infty$/Riemannian, then $L \times T_{pt} \to X$ restricts to diffeomorphisms/isometries between the leaves, obtaining that all leaves are diffeomorphic/isometric to $L$.

8.2. The map $\hat{\pi}_0 : \widehat{X}_0 \to X$ may not be a principal bundle. Consider the canonical inclusion $SO(2) \subset SO(3)$, and the canonical transitive analytic action of $SO(3)$ on the sphere $S^2 \equiv SO(3)/SO(2)$. We get an induced transitive quasi-analytic left action of the compact topological group $G := SO(3)^N$ on the compact space $T := (S^2)^N$. Fix $u_0 \in S^2$ whose isotropy group is $SO(2)$, and let $\bar{u}_0 = (u_0, u_0, \ldots) \in T$. The orbit map $SO(3) \to S^2$, $g \mapsto g \cdot u_0$, is a non-trivial principal $SO(2)$-bundle, and therefore it has no global sections. Then, using the arguments of the first and second examples of [36] Section 1, it easily follows that the orbit map $G \to T$, $(g_i) \mapsto (g_i) \cdot \bar{u}_0 = (g_i \cdot u_0)$, has no local sections. Since $G$ is second countable, connected, compact and non-abelian, it contains a dense subgroup $\Gamma$ isomorphic to the fundamental group of the closed oriented surface $\Sigma_2$ of genus 2 [12, Corollary 8.3]. Let $L$ be the universal covering of $\Sigma_2$, which is diffeomorphic to the plane. Consider the corresponding suspension foliated space, $X = L \times_{\Gamma} T_{pt}$, which satisfies the conditions of Theorem A and the corresponding Molino’s description $(G, H, \widehat{X}_0, \hat{\pi}_0)$ constructed in Section 8.1 where $\widehat{X}_0 = L \times_G T_{pt}$, $H = SO(2)^N$, the right $H$-action on $\widehat{X}_0$ is given by $[y, g] \cdot h = [y, gh]$, and the map $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by $\hat{\pi}_0([y, g]) = [y, g \cdot u_0]$.

Proposition 8.2. The map $\hat{\pi}_0 : \widehat{X}_0 \to X$ has no local sections, and therefore it cannot be a principal $H$-bundle.

Proof. Since $\hat{\pi}_0 : \widehat{X}_0 \to X$ is induced by $id_L \times \hat{\pi}_0 : L \times G \to L \times T$, any local section of $\hat{\pi}_0$ with small enough domain defines a local section of $\hat{\pi}_0 : G \to T$. But this map has no local sections. □

8.3. Foliated homogeneity may not be told by the leaves.

Proposition 8.3. If $X$ is foliated homogeneous, then it is without holonomy, and all of its leaves are homeomorphic one another. If moreover $X$ is $C^\infty$ (respectively, Riemannian and compact), then all of its leaves are diffeomorphic (respectively, quasi-isometrically diffeomorphic) to each other.
Proof. Elementary, using that there always exist leaves without holonomy in the first assertion.

Let us exhibit an example where the reciprocal of Proposition 8.3 does not hold. To begin with, let $G_1$ and $G_2$ be second countable, connected compact topological groups, and let $G = G_1 \times G_2$. Assume that $G_1$ is non-abelian. Let us use the notation $g = (g_1, g_2)$ for the elements of $G$; in particular, we use $e = (e_1, e_2)$ for the identity element.

**Proposition 8.4.** There exists a subset $\mathcal{P} \subset G \times G$, which is both residual and of full Haar measure, such that, for all $(g, h) \in \mathcal{P}$, the subgroup $\langle g, h \rangle$ is dense in $G$ and freely generated by $g$ and $h$, and $(g, h) \cap (\{e_1\} \times G_2) = \{e\}$. 

Proof. By [12, Proposition 8.2], there are subsets, $O \subset G \times G$ and $O_1 \subset G_1 \times G_1$, which are residual and of full Haar measure, such that, for all $(g, h) \in O$ and $(a, b) \in O_1$, the subgroup $\langle g, h \rangle$ (respectively, $\langle a, b \rangle$) is dense in $G$ (respectively, $G_1$) and freely generated by $g$ and $h$ (respectively, $a$ and $b$). Then the statement is satisfied with

$$\mathcal{P} = O \cap \{ (g, h) \in G \times G \mid (g_1, h_1) \in O_1 \}. \quad \Box$$

Take $G_2 = \text{SO}(3)$, and consider $\text{SO}(2) \subset \text{SO}(3)$ and $S^2 \equiv \text{SO}(3)/\text{SO}(2)$ like in Section 8.2. By Proposition 8.4, $G$ has a dense subgroup $\Gamma$ freely generated by two elements such that $\Gamma \cap (\{e_1\} \times \text{SO}(3)) = \{e\}$. Hence the first factor projection $G_1 \times \text{SO}(3) \to G_1$ restricts to an injection $\Gamma \to G_1$, and $\Gamma$ does not meet any conjugate of $\langle e_1 \rangle \times \text{SO}(2)$ in $G$ (all of them are contained in $\langle e_1 \rangle \times \text{SO}(3)$). Consider the canonical left action of $G$ and $\Gamma$ on $T := G_1 \times S^2 \equiv G/\langle \{e_1\} \times \text{SO}(2) \rangle$. There is a regular covering $L$ of the closed oriented surface of genus two, $\Sigma_2$, whose group of covering transformations is isomorphic to $\Gamma$. Consider the corresponding suspension foliated space, $X = L \times_T \text{pt}$, which satisfies the conditions of Theorem A and the corresponding Molino’s description $(G, H, \tilde{X}_0, \tilde{\pi}_0)$ constructed in Section 8.1, where $\tilde{X}_0 = L \times_T G_{\text{pt}}, H = \text{SO}(2)$, the right $H$-action on $\tilde{X}_0$ is given by $[y, g] \cdot h = [y, gh]$, and the map $\tilde{\pi}_0 : \tilde{X}_0 \to X$ is defined by $\tilde{\pi}_0([y, g]) = [y, g \cdot u_0]$. We can equip $\Sigma_2$ with $C^\infty$ and Riemannian structures, and consider the induced $C^\infty$ and Riemannian structures on $X$ and $\tilde{X}_0$.

Since $H \neq \{e\}$, $X$ is not foliated homogeneous by Theorem C (or Theorem F and Proposition 8.2). However this cannot be seen by comparing any pair of leaves since all of them are isometric to $L$, and $X$ has no holonomy by “(i) $\leftrightarrow$ (iii)” in Proposition 8.1.

### 8.4. Inverse limits of minimal Lie foliations

This example was suggested by S. Hurder. Let $(X, \mathcal{G})$ be the McCord solenoid defined as the projective limit of a tower of non-trivial regular coverings between closed connected manifolds,

$$\cdots \to M_k \xrightarrow{\phi_k} M_{k-1} \to \cdots \to M_0.$$
Let $\Gamma_k = \pi_1(M_k)$, and consider the induced tower of homomorphisms between finite groups,

$$\cdots \to \Gamma_0/\Gamma_k \to \Gamma_0/\Gamma_{k-1} \to \cdots \to \Gamma_0/\Gamma_1,$$

whose inverse limit $K$ contains a canonical dense copy of $\Gamma_0$. Then $(X, \mathcal{G})$ can be also described as the suspension foliated space $\tilde{M}_0 \times_{\Gamma_0} K_{pt}$, where $\tilde{M}_0$ is the universal covering of $M_0$. We get induced maps $\psi_k : X \to M_k$, whose restrictions to the leaves are covering maps. Suppose that $M_0$ is equipped with a minimal Lie $G_0$-foliation $\mathcal{F}_0$, for some simply connected Lie group $G_0$. Then every $M_k$ can be endowed with the minimal Lie $G_0$-foliation $\mathcal{F}_k := (\phi_1 \cdots \phi_k)^* \mathcal{F}_0$, and $\mathcal{G}$ can be equipped with the subfoliated structure $\mathcal{F}$, whose restriction to every $G$-leaf $M$ is the pull-back of $\mathcal{F}_0$ by $\psi_0 : M \to M_0$. We can write $\mathcal{F} = \psi_0^* \mathcal{F}_0$, which equals $\psi_k^* \mathcal{F}_k$ for all $k$. We can also write $(X, \mathcal{F}) = (\tilde{M}_0, \tilde{\mathcal{F}}_0) \times_{\Gamma_0} K_{pt}$, where $\tilde{\mathcal{F}}_0$ is the lift of $\mathcal{F}_0$. This $\mathcal{F}$ is a “Lie $G_0$-subfoliated structure” of $\mathcal{G}$ in an obvious sense, and it easily follows that $(X, \mathcal{F})$ is a minimal $G$-foliated space for $G = G_0 \times K$.

8.5. Open problems.

8.5.1. Strong quasi-analyticity of $\overline{\mathcal{H}}$. This problem was proposed in [10]. It is really unknown to the authors if the strong quasi-analyticity of $\overline{\mathcal{H}}$ is needed in Theorem $\mathcal{A}$. If $\mathcal{H}$ is a compactly generated equicontinuous strongly quasi-analytic pseudogroup, then $\overline{\mathcal{H}}$ may not be strongly quasi-analytic (non-minimal counterexamples can be easily given). But the study of the strong quasi-analyticity of $\overline{\mathcal{H}}$ when $\mathcal{H}$ is minimal seems to be an interesting open problem.

8.5.2. Functoriality, universality and uniqueness of the Molino’s description. It would be desirable to have a uniqueness of the Molino’s description stronger than Proposition $\mathcal{A}$ stating that not only the structures $(G, \Gamma, H, X_0, \tilde{\pi}_0)$ constructed in the proof of Theorem $\mathcal{A}$ but also all possible structures $(G, \Gamma, H, X_0, \tilde{\pi}_0)$ satisfying the conditions of its statement are equivalent. This would follow by showing a universality property, which in turn would follow by exhibiting its functoriality with respect to some kind of foliated maps. Since the definition of $X_0$ uses germs of maps in $\overline{\mathcal{H}}$, the functoriality of Molino’s description could be achieved by showing that foliated maps between equicontinuous foliated spaces induce morphisms between the closures of their holonomy pseudogroups. This would be an extension of the case of Riemannian foliations, solved in [7, 8]. Such functoriality, universality and uniqueness of the Molino’s description is not even proved in the Riemannian foliation case. A direct consequence would be that $H$ is finite if and only if $X$ is a virtually foliated homogeneous foliated space (a finite fold covering of $X$ is foliated homogeneous as foliated space).
8.5.3. For matchbox manifolds, can foliated homogeneity be told by the leaves? S. Hurder and O. Lukina tells us that it is unknown if the reciprocal of Proposition 8.3 holds for matchbox manifolds. The argument given in Section 8.3 cannot produce matchbox manifolds because Proposition 8.4 requires $G$ to be connected (to apply [12, Proposition 8.2] in the proof).

8.5.4. How large is the class of inverse limits of minimal Lie foliations? Since any metrizable locally compact local group of finite topological dimension is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group [31, Theorem 107], it was asked by S. Hurder whether any compact minimal foliated homogeneous foliated space of finite “topological codimension” can be realized as inverse limit of minimal Lie foliations, like in Section 8.4. This would generalize the results of [15] (see also [2]), where an affirmative answer is given for homogeneous matchbox manifolds (the case of codimension zero). If this is true, using also the Molino’s description, it could be possible to prove that any equicontinuous foliated space satisfying the conditions of Theorem A is an inverse limit of Riemannian foliations.

8.5.5. Molino’s descriptions without assuming strong quasi-analyticity. This problem arises from the Molino spaces constructed by Dyer, Hurder and Lukina in [19] for equicontinuous matchbox manifolds, where strong quasi-analyticity is not needed. Their Molino spaces are also foliated homogeneous, but they may not be unique, and their leaves cover the leaves of the original matchbox manifold in an arbitrary way (they may not be the holonomy covers). Thus the following question makes sense. Does there exist this kind of Molino spaces for arbitrary compact minimal equicontinuous foliated spaces?

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E-mail address: jesús.alvarez@usc.es
E-mail address: ramonbarrallijo@gmail.com

Departamento de Matemáticas, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain