ELLIPITIC QUANTUM MANY-BODY PROBLEM
AND DOUBLE AFFINE KNIZHNIK-ZAMOLODCHIKOV EQUATION

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ABSTRACT

The elliptic-matrix quantum Olshanetsky-Perelomov problem is introduced for arbitrary root systems by means of an elliptic generalization of the Dunkl operators. Its equivalence with the double affine generalization of the Knizhnik-Zamolodchikov equation (in the induced representations) is established.

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0. Introduction

We generalize the affine the Knizhnik-Zamolodchikov equation from [Ch1,2,3] replacing the corresponding root systems by their affine counterparts. To explain the construction in the case of the root system of $\mathfrak{gl}_n$, let us first introduce the affine Weyl group $S_n^a$. It is the semi-direct product of the symmetric group $S_n$ and the lattice $A = \oplus_{i=1}^{n-1} \mathbb{Z} \epsilon_{i+1}$, where the first acts on the second permuting $\{\epsilon_i, \epsilon_{ij} = \epsilon_i - \epsilon_j\}$ naturally. This group is generated by the adjacent transpositions $s_i = (ii + 1), 1 \leq i < n$, and $s_0 = s_n^{[1]}$, where $s_{ij}^{[k]} = (ij)(k\epsilon_{ij}) \in S_n^a$.

Setting

$$s_{ij}^{[k]}(b) = b - (\epsilon_{ij}, b)(\epsilon_{ij} + kc), \quad s_{ij}^{[k]}(c) = c, \quad b \in B = \oplus_{i=1}^{n} \mathbb{Z} \epsilon_i, \quad (0.1)$$

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we obtain an action of $S_n^a$ in $\tilde{B} = B \oplus \mathbb{Z}c$. In particular,

$$s_0(b) = b + (b, \epsilon_{1n})(c - \epsilon_{1n}), \quad a(\epsilon_i) = \epsilon_i - (a, \epsilon_i)c, \quad 1 \leq i \leq n.$$ 

Put

$$x_b = kx_c + \sum_{i=1}^{n} k_i x_i, \quad z_b = k\xi + \sum_{i=1}^{n} k_i z_i \quad \text{for} \quad \tilde{b} = kc + \sum_{i=1}^{n} k_i \epsilon_i.$$ 

The double affine degenerate (graded) algebra $\mathfrak{S}'$ is generated by the group algebra $\mathbb{C}[S_n^a]$, pairwise commutative elements $\{x_b, \ b \in B\}$, and central $x_c$, satisfying the relations (depending on degeneration of the double affine Hecke algebra from [Ch8] (for $\mathfrak{g}l_n$).

Let us fix $\mu \in \mathbb{C}$ and set

$$\text{ct}_{ij}^{[k]} = \text{ct}(z_{ij} + k\xi) \quad \text{for} \quad \text{ct}(t) = (\exp(t) - 1)^{-1}, \quad z_{ij} = z_i - z_j.$$ 

We introduce the differential operators of the first order:

$$\mathcal{D}_{\epsilon_i} = \mathcal{D}_i = \partial/\partial z_i - \eta \sum_{n \geq j > i} \text{ct}_{ij}^{[0]} (s_{ij}^{[0]} - \mu) + \eta \sum_{1 \leq j < i} \text{ct}_{ji}^{[0]} (s_{ji}^{[0]} - \mu)$$

$$- \eta \sum_{j \neq i} \sum_{k > 0} \left( \text{ct}_{ij}^{[k]} (s_{ij}^{[k]} - \mu) - \text{ct}_{ji}^{[k]} (s_{ji}^{[k]} - \mu) \right) + \mu \eta (n/2 - i + 1),$$

$$\mathcal{D}_c = \partial/\partial \zeta + \eta \mu n, \quad 1 \leq i, j \leq n.$$ 

We consider the sums formally as infinite linear combinations of the elements $\hat{w} \in S_n^a$ with the coefficients depending on $\{z, \xi\}$ and one more complex variable $\zeta$. Assuming that $\Re(\zeta) > 0$, we can introduce a norm in this space to make all series convergent.

The family of operators $\{\mathcal{D}'_i = \mathcal{D}_i - x_i, \ \mathcal{D}'_c = \mathcal{D}_c - x_c\}$ is commutative and $S_n^a$-invariant with respect to the following simultaneous action of this group on the coefficients (that are from $\mathfrak{S}'$) and the arguments $\{z_b, \zeta\}$:

$$\hat{w}(\hat{h}) = \hat{w} \hat{h} \hat{w}^{-1}, \quad \hat{h} \in \mathfrak{S}', \quad \hat{w}(z_b) = z_{\{\hat{w}(b)\}}, \quad \hat{b} \in \tilde{B},$$

$$s_i(\zeta) = \zeta \quad \text{for} \quad 1 \leq i < n, \quad s_0(\zeta) = \zeta - \xi + z_{1n}.$$ 

The invariance means that $\hat{w}(\mathcal{D}'_{\hat{u}}) = \mathcal{D}'_{\hat{w}(\hat{u})}$, where $\mathcal{D}'_{\alpha \hat{u} + \beta \hat{v}} = \alpha \mathcal{D}'_{\hat{u}} + \beta \mathcal{D}'_{\hat{v}}$ for $\alpha, \beta \in \mathbb{Z}, \ \hat{u}, \hat{v} \in \tilde{B}$, and $\hat{w} \in S_n^a$. Actually this family is invariant even with respect to the action of the bigger group generated by $W$ and $B$ (instead of $A$).
It leads to a natural extension of the above $\mathcal{S}'$. The precise choice of constants in (0.3) is necessary to ensure the $B$-invariance.

The double affine KZ is the system \( \{ D'_u \Phi = 0, \ u \in B \} \) for a function \( \Phi(z) \) with the values in \( \mathcal{S}' \) or its representations. Here \( \xi \) is considered as a parameter.

Let us factorize \( \mathcal{S}' \) by the ideal \( (x_c) \). The symmetric polynomials in \( x_1, \ldots, x_n \) constitute the center of the resulting algebra \( \mathcal{S}'_0 \). Given a fixed central character and a finite dimensional \( \mathbb{C}[S^n_a] \)-modules \( V \), the corresponding induced (universal) \( \mathcal{S}'_0 \)-module is finite dimensional as well. When considered in this representation, the series in (0.3) become convergent (at least for rather big \( \Re(\xi) \)) and turn into functions of elliptic type. The corresponding double KZ is equivalent to a \( V \)-valued version of the elliptic quantum many-body problem from [OP] (which also generalizes the matrix QMB from [Ch5]).

To introduce the latter let us first consider the same formulas (0.3) assuming now that \( s_{ij}^{[k]} \) act on the arguments \( \{ z_b, \zeta \} \) as in (0.4). We will write \( \sigma(\tilde{w}) \) and \( \sigma_{ij}^{[k]} \) instead of \( \tilde{w} \) and \( s_{ij}^{[k]} \) to emphasize this. The corresponding elliptic Dunkl operators (which are scalar but not pure differential anymore) will be denoted by \( \{ \Delta_i, \Delta_c, \Delta_b \} \). The map

\[ \tilde{w} \to \sigma(\tilde{w}), \quad x_b \to \Delta_b, \quad \tilde{w} \in W^a, \ b \in B, \]

gives a homomorphism from the algebra \( \mathcal{S}' \) into the algebra of operators acting on the space of (scalar) functions of \( \{ z, \zeta \} \). Imposing the relation \( \Delta_c = 0 \) we obtain an embedding of \( \mathcal{S}'_0 \).

Second, given an arbitrary symmetric polynomial \( p = p(x_1, \ldots, x_n) \), we use (0.4) to represent

\[ p(\Delta_1, \ldots, \Delta_n) = \sum_{\tilde{w} \in W^a} D_{\tilde{w}} \sigma(\tilde{w}), \] where \( D_{\tilde{w}} \) are differential.

Then we replace every \( \sigma(\tilde{w}) \) by the image of \( \tilde{w}^{-1} \) in Aut \( \mathbb{C} \) \( V \) setting \( \partial/\partial \zeta = -\eta \mu n \) afterwards. The resulting operators \( \{ L_p \} \) are \( S^n_a \)-invariant and pairwise commutative. If \( V \) is one-dimensional, they coincide with the OP operators for \( \mu = 0 \) and are conjugated to them (by proper remarkable scalar functions) when \( \mu = \pm 1 \) (with \( V \) of the same ”sign”).

The element \( p_2 = \sum_{i=1}^n x_i^2 \) leads up to a constant to the Schrödinger operator

\[ H = \sum_{i=1}^n \partial^2/\partial z_i^2 + \text{const} \sum_{i<j} \wp(z_i - z_j) \] (0.5)

in terms of the Weierstrass elliptic function with the periods \( \{(2\pi i), \xi \} \).

In this paper we consider arbitrary root systems and any initial representations \( V \) of the corresponding affine Weil groups. We note that the commutative families
of scalar $H$-operators for $A,B,D$ types (with certain uniqueness theorems) were obtained recently by direct methods (due to Heckman-Opdam) in [OOS].

It is worth mentioning that for $\mu = 1$ (and special $\eta$) the operators $L_p$ are expected to be the radial parts of Laplace operators for Kac-Moody symmetric spaces at the critical level $c + n = 0$. The latter condition gives the existence of the "big" center of the corresponding universal enveloping algebra (which is necessary to start the Harish-Chandra, Helgason theory of radial part). It is directly connected with the substitution $\partial/\partial \zeta = -\eta \mu n$.

Something can be done when $\partial/\partial \zeta = \eta \mu m$ for arbitrary $m$. Let us introduce one more operator

$$\Delta_d = \partial/\partial \xi - \eta \sum \sum k \left( \text{ct}_{ij}^{[k]} (\sigma_{ij}^{[k]} - \mu) + \text{ct}_{ji}^{[k]} (\sigma_{ji}^{[k]} - \mu) \right). \quad (0.6)$$

It does not commute with $\{\Delta_i\}$, but the operators $\Delta_b = \Delta_b + k \Delta_c + l \Delta_d$ for $\hat{b} = b + kc + ld$ still satisfy the cross-relations:

$$\sigma_i \Delta_b = \Delta s_i(b) \sigma_i + \eta (\epsilon_{ii+1}, \hat{b}), \quad 0 \leq i < n, \quad \epsilon_{01} = c - \epsilon_{1n},$$

$$s_i(c) = c, \quad s_j(d) = d \quad \text{for} \quad 1 \leq j < n, \quad s_0(d) = d - \epsilon_{01}, \quad (0.7)$$

where the form $(\ , \ )$ is extended to $\mathbb{R}^{n+2}$ in the following way:

$$(c, c) = (c, \epsilon_i) = 0 = (d, \epsilon_i) = (d, d), \quad 1 \leq i \leq n, \quad (c, d) = 1.$$ 

It gives that the operator $2\Delta_d \Delta_c + \sum_{i=1}^n \Delta_i^2$ is $S_n$-invariant. Its reduction in the above sense is also invariant and is conjugated to $\hat{H} = H + 2\eta \mu (n + m) \partial/\partial \xi$ in the setup of (0.5).

When $\mu = 1$ and $V$ is the corresponding one-dimensional representation, $\hat{H}$ was introduced in [EK]. Presumably it is related to the elliptic $r$-matrix KZ from [Ch1] (with the additional equation from [E]) and the so-called Bernard KZ equation (see [FW,EK]). Certain remarkable properties of this parabolic operator seem to be important for the affine harmonic analysis at arbitrary level.

In conclusion we would like to mention that all above constructions have difference counterparts and hopefully ensure a basis for the elliptic Macdonald theory (see e.g. [M,O,Ch6]).

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1. Double Hecke algebras

We follow [Ch3] (see also [Ch5,6,7]). Reduced root systems only will be discussed here. All the definitions and statements can be extended to the general case. Minor changes in formulas are necessary for divisible roots.
Given a Euclidean form \((\ ,\ )\) on \(\mathbb{R}^n\) and a root system \(R = \{\alpha\} \subset \mathbb{R}^n\) of type \(A_n, B_n, \cdots, G_2\), let \(s_\alpha\) be the orthogonal reflections in the hyperplanes \((\alpha, u) = 0,\ u \in \mathbb{R}^n\). Further, \(\{\alpha_1, \cdots, \alpha_n\}\) are the simple roots relative to some fixed Weyl chamber, \(R_+\) the set of all positive (written \(\alpha > 0\)) roots, \(W\) the Weyl group generated by \(s_\alpha\) (or by \(s_i = s_{\alpha_i}, 1 \leq i \leq n\)), \(C[W] = \oplus_w Cw\) the group algebra of \(W \ni w\).

We introduce \(a_i = \alpha_i^\vee\), where \(\alpha_i^\vee = 2\alpha/(\alpha, \alpha)\), the dual fundamental weights \(b_1, \cdots, b_n\) satisfying the relations \((b_i, \alpha_j) = \delta^i_j\) for the Kronecker delta, and the lattices

\[
A = \bigoplus_{i=1}^n \mathbb{Z}a_i \subset B = \bigoplus_{i=1}^n \mathbb{Z}b_i.
\]

Let us fix a \(W\)-invariant set \(\eta = \{\eta_\alpha \in C, \alpha \in R\}\). The \(W\)-invariance \(\langle w\eta_\alpha = \eta_{w(\alpha)}, w \in W \rangle\) gives that \(\eta_\alpha = \eta'_\alpha\) or \(\eta''\) respectively for the short and long roots \((R\) is supposed to be reduced). We put \(\eta_i = \eta_{\alpha_i}\) and define the \(\eta\)- generalization of the \(\rho\) and the Coxeter number \(\rho\) and the Coxeter number \(h:\)

\[
2\rho_\eta = \sum_{\alpha > 0} \eta_\alpha \alpha = \sum_{i=1}^n \eta_i (\alpha_i, \alpha_i) b_i \in \mathbb{R}^n, \quad h_\eta = \eta_\theta + (\rho_\eta, \theta) \quad \text{for the maximal root} \quad \theta \in R_+.
\]

We will use the same notations for other \(W\)-invariant sets instead of \(\eta\).

The following affine completion is common in the theory of the Kac-Moody algebras (see e.g. [Ka], [Ch6]). Let us extend the above pairing to \(\mathbb{R}^{n+1} = \mathbb{R}^n \oplus Rc\) setting \((c, c) = 0 = (c, u)\).

The vectors (affine roots) \(\tilde{\alpha} = \alpha + kc\) for \(\alpha \in R, k \in \mathbb{Z}\), form the affine root system \(R^a \supset R\). We add \(\alpha_0 \stackrel{\text{def}}{=} c - \theta\) to the set of simple roots and put \(\eta_\tilde{\alpha} = \eta_\alpha, \eta_0 = \eta_\theta = \eta''\). The corresponding set \(R^a_+\) of positive roots coincides with \(R_+ \cup \{\alpha + kc, \alpha \in R, k > 0\}\). Let \(\tilde{B} = B \oplus \mathbb{Z}c\). Given \(\tilde{\alpha} = \alpha + kc \in R^a, a \in A, \tilde{u} = u + \kappa c \in \mathbb{R}^{n+1},\)

\[
s_{\tilde{\alpha}}(\tilde{u}) = \tilde{u} - (u, \alpha^\vee)(\alpha + kc), \quad a'(\tilde{u}) = \tilde{u} - (u, a)c.
\]

The affine Weyl group \(W^a\) is generated by all \(s_{\tilde{\alpha}}\). One can take the simple reflections \(s_j = s_{\alpha_j}, 0 \leq j \leq n\), as its generators. This group is the semi-direct product \(W \rtimes A'\) of its subgroups \(W\) and \(A' = \{a', a \in A\}\), where

\[
a' = s_{\alpha} s_{\{\alpha + c\}} = s_{\{-\alpha + c\}} s_{\alpha} \quad \text{for} \quad a = \alpha^\vee, \alpha \in R.
\]

**Definition 1.1.**  The degenerate (graded) double affine Hecke algebra \(\mathcal{H}'\) is algebraically generated by the group algebra \(C[W^a]\) and the pairwise commutative

\[
x_{\tilde{u}} = \sum_{i=1}^n (u, \alpha_i) x_i + \kappa x_c \quad \text{for} \quad \tilde{u} = u + \kappa c \in \mathbb{R}^{n+1},
\]
satisfying the following relations:

\[ s_i x_{\tilde{u}} - x_{\{s_i(\tilde{u})\}} s_i = \eta_i(u, \alpha_i), \quad 0 \leq i \leq n. \] (1.4)

The restricted algebra \( \mathcal{H}'_0 \) is the factor-algebra \( \mathcal{H}'/(x_c) \) (the quotient by the central ideal \( (x_c) \)).

Without \( i = 0 \) we arrive at the defining relations

\[ s_i x_i - (x_i - x_{a_i}) s_i = \eta_i, \quad s_i x_j = x_j s_i, \quad \text{where} \quad 1 \leq i \neq j \leq n, \quad a_i = \alpha_i', \]

of the graded affine Hecke algebra from [L] (see also [Ch3,5]). We mention that \( \mathcal{H}' \) is a degeneration of the double affine Hecke algebras introduced in [Ch6,7].

Let \( C[x] \) be the algebra of polynomials in terms of \( \{x_{\tilde{u}}\} \). We denote the subalgebra of \( W \)-invariant polynomials (with respect to the action of \( W \) on \( \{\tilde{u}\} \)) by \( C[x]^W \). Later the same notations will be used for other letters instead of \( x \).

**Theorem 1.2.** An arbitrary element \( \hat{h} \in \mathcal{H}' \) can be uniquely represented in the (left) form \( \hat{h} = \sum_{\tilde{w} \in W^a} f_{\tilde{w}} \tilde{w} \) and the (right) form \( \hat{h} = \sum_{\tilde{w} \in W^a} \tilde{w} g_{\tilde{w}}, \) where \( f_{\tilde{w}}, g_{\tilde{w}} \in C[x] \). The center of \( \mathcal{H}'_0 \) coincides with \( C[x]^W \).

**Proof.** The first statement results from Theorem 2.3, [Ch7] established in the non-degenerate case (see also [Ch6]). Following [Ch3] one can check that the center of \( \mathcal{H}'_0 \) contains \( C[x]^W \) and belongs to \( C[x] \) (that is a maximal commutative subalgebra). The subalgebra generated by \( C[W] \) and \( C[x] \) is the degenerate (graded) affine Hecke algebra in the sense of [L,Ch3]. Hence its center coincides with \( C[x]^W \) (due to Bernstein).

**Induced representations.** Let \( V \) be a \( C[W^a] \)-module, \( V^o = \text{Hom}_C (V, C) \) its dual with the natural action \( (\tilde{w}(l(v)) = l(\tilde{w}^{-1}v), \quad l \in \text{Hom}_C (V, C)) \), \( \tau \) and \( \tau^o \) the corresponding homomorphisms from \( C[W^a] \) to \( \text{End}_C V \) and \( \text{End}_C V^o \). We will use the diagonal action:

\[ \delta(\tilde{w})(v \otimes x_{\tilde{u}}) = \tau^o(\tilde{w})(v) \otimes \tilde{w}(x_{\tilde{u}}), \quad \tilde{w}(x_{\tilde{u}}) = x_{\{\tilde{w}(\tilde{u})\}}, \]

for \( v \otimes x_{\tilde{u}} \in V \overset{\text{def}}{=} V^o \otimes_C C[x], \quad \tilde{w} \in W^a, \quad \tilde{u} \in \mathbb{R}^n. \) (1.5)

The next proposition holds good for the entire \( \mathcal{H}' \), but the latter has the trivial center = \( Cx_{\tilde{u}} \) (we need a "big" center to construct finite dimensional representations). Till the end of the section, \( x_c = 0 \) and \( x_{\tilde{u}} \) are identified with the corresponding \( x_u \).
Proposition 1.3. The universal (free) $\mathcal{S}'_0$-module generated by the $\mathbb{C}[W^a]$-module $V^\alpha$ is isomorphic to $\mathcal{V}$ with the natural action of $\mathbb{C}[x]$ by multiplications and the following action of $s_i$:

$$\hat{s}_i = \delta(s_i) + \eta_i x^{-1}_a (1 - s_i), \quad 0 \leq i \leq n, \quad a_0 = c - \theta^\vee,$$

where $x^{-1}_a (1 - s_i)(f) = x^{-1}_a (f - s_i(f))$ for $f \in \mathcal{V}$ ($s_i$ acts only on $x$).

\textbf{Proof}. follows [Ch3,5].

We fix a set $\lambda = \{\lambda_1, ..., \lambda_n\} \subset \mathbb{C}$ and consider the quotient $V^\alpha$ of $V$ by the (central) relations $p(x_1, ..., x_n) = p(\lambda_1, ..., \lambda_n)$ for all $p \in \mathbb{C}[x]^W$.

Finally, we introduce:

$$V^\alpha(\lambda) \overset{\text{def}}{=} (V^\alpha)^\circ = \text{Hom}_\mathbb{C}(V^\alpha, \mathbb{C}), \quad \hat{h}(l(u)) = l(\hat{h}^\alpha(u)), \quad u \in V^\alpha, l \in (V^\alpha)^\circ,$$

$$s_i^\circ = s_i, \quad x_i^\circ = x_i, \quad (\hat{h}_1 \hat{h}_2)^\circ = \hat{h}_2^\circ \hat{h}_1^\circ, \quad \hat{h}_{1,2} \in \mathcal{S}'_0.$$

(1.7)

The anti-involution $\hat{h} \to \hat{h}^\circ$ is well-defined because relations (1.4) are self-dual.

The above construction gives two canonical $W^a$-homomorphisms:

$$\text{id} : V^\circ \rightarrow \mathcal{V} \rightarrow V^\alpha, \quad \text{tr} : V^\alpha(\lambda) \rightarrow V.$$

Proposition 1.4. If a $\mathcal{S}'$-submodule $\mathcal{U} \subset V^\alpha(\lambda)$ is non-zero then its image $\text{tr}(\mathcal{U})$ is non-zero too.

\textbf{Proof}. It is clear, since $V^\alpha$ is generated by $V^\circ$ as an $\mathcal{S}'$-module. □

If $V$ is finite-dimensional then $\dim_{\mathbb{C}} V(\lambda) = |W| \dim_{\mathbb{C}} V$, where $|W|$ is the number of elements of $W$. The main examples will be for one-dimensional representations of $W^a$ which are described by $W$-invariant sets $\varepsilon \subset \{\pm 1\}$:

$$\tau_\varepsilon(s_i) = \varepsilon_i, \quad \tau_\varepsilon(a') = 1, \quad 0 \leq i \leq n, \quad a \in A.$$

(1.8)

Let us denote the corresponding $V$, $V^\alpha(\lambda)$ by $C_\varepsilon$, $C_\varepsilon(\lambda)$ for the latter reference.

2. Affine $r$-matrices

Following [Ch 1,3,5] we introduce abstract classical $r$-matrices with the values in an arbitrary $\mathbb{C}$-algebra $\mathcal{F}$ and show how to extend non-affine $r$-matrices to affine ones. The notations are from Section 1. Let us denote $\mathbb{R} \tilde{\alpha} + \mathbb{R} \tilde{\beta} \subset \mathbb{R}^a$ by $\mathbb{R} \langle \tilde{\alpha}, \tilde{\beta} \rangle$ for $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^a$.

\textbf{Definition 2.1.} a) A set $r = \{r_{\tilde{\alpha}} \in \mathcal{F}, \tilde{\alpha} \in R^a_+\}$ is an affine $r$-matrix if

$$[r_{\tilde{\alpha}}, r_{\tilde{\beta}}] = 0,$$

(2.1)
\[ [r_{\tilde{\alpha}}, r_{\tilde{\alpha} + \tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\beta}}] + [r_{\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\beta}}] = 0, \quad (2.2) \]

\[ [r_{\tilde{\alpha}}, r_{\tilde{\alpha} + \tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\beta}}] + [r_{\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\alpha} + 2\tilde{\beta}}] + [r_{\tilde{\alpha} + 2\tilde{\beta}}, r_{\tilde{\beta}}] = 0, \quad (2.3) \]

\[ [r_{\tilde{\alpha}}, r_{3\tilde{\alpha} + \tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{2\tilde{\alpha} + \tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\alpha} + \tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\beta}}] + [r_{3\tilde{\alpha} + \tilde{\beta}}, r_{2\tilde{\alpha} + \tilde{\beta}}] + [r_{2\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\alpha} + \tilde{\beta}}] + [r_{3\tilde{\alpha} + 2\tilde{\beta}}, r_{\tilde{\alpha} + \tilde{\beta}}] + [r_{\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\beta}}] = 0, \]

\[ [r_{3\tilde{\alpha} + \tilde{\beta}}, r_{3\tilde{\alpha} + 2\tilde{\beta}}] + [r_{3\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\beta}}] + [r_{3\tilde{\alpha} + 2\tilde{\beta}}, r_{\tilde{\beta}}] = 0 = [r_{3\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\alpha} + \tilde{\beta}}] = [r_{2\tilde{\alpha} + \tilde{\beta}}, r_{\tilde{\beta}}], \quad (2.4) \]

under the assumption that \(\tilde{\alpha}, \tilde{\beta} \in R^a_+\) and

\[ \mathbf{R}(\tilde{\alpha}, \tilde{\beta}) \cap R^a = \{ \pm \tilde{\gamma} \}, \tilde{\gamma} \text{ runs over all the indices} \quad (2.5) \]

in the corresponding identities.

b) A closed \( r \)-matrix (or a closure of the above \( r \)) is a set \( \{ r_{\tilde{\alpha}} \in \mathcal{F}, \tilde{\alpha} \in R^a \} \) (extending \( r \) and) satisfying relations (2.1) - (2.4) for arbitrary (positive, negative) \( \tilde{\alpha}, \tilde{\beta} \in R^a \) such that the corresponding condition (2.5) is fulfilled. If the indices are from \( R_+ \) (or \( R \)) we call \( r \) non-affine.

\[ \square \]

We note that (2.5) for identity (2.1) means that

\[ (\tilde{\alpha}, \tilde{\beta}) = 0 \text{ and } \mathbf{R}(\tilde{\alpha}, \tilde{\beta}) \cap R^a = \{ \pm \tilde{\alpha}, \pm \tilde{\beta} \}. \quad (2.6) \]

It is equivalent to the existence of \( \tilde{w} \in W^a \) such that \( \tilde{\alpha} = \tilde{w}(\alpha_i), \tilde{\beta} = \tilde{w}(\alpha_j) \) for simple \( \alpha_i \neq \alpha_j \) \((0 \leq i, j \leq n)\) disconnected in the affine Dynkin diagram of \( R^a \). In the most interesting examples, (2.1) holds true for arbitrary orthogonal roots.

The corresponding assumptions for (2.2) - (2.4) give that \( \tilde{\alpha}, \tilde{\beta} \) are simple roots of a two-dimensional root subsystem in \( R^a \) of type \( A_2, B_2, G_2 \). Here \( \tilde{\alpha}, \tilde{\beta} \) stay for \( \alpha_1, \alpha_2 \) in the notations from the figure of the systems of rank 2 from \([B]\). One can represent them as follows: \( \tilde{\alpha} = \tilde{w}(\alpha_i), \tilde{\beta} = \tilde{w}(\alpha_j) \) for a proper \( \tilde{w} \) from \( W^a \) and joined (neighbouring) \( \alpha_i, \alpha_j \).

Given an arbitrary \( r \), we always have the following closures (the standard one and the extension by zero):

\[ r_{-\tilde{\alpha}} = -r_{\tilde{\alpha}}, \quad r_{-\tilde{\alpha}} = 0, \quad \tilde{\alpha} \in R^a_+. \quad (2.7) \]

If there exists an action of \( W^a \triangleright \tilde{w} \) on \( \mathcal{F} \) such that

\[ \tilde{w}(r_{\tilde{\alpha}}) = r_{\tilde{w}(\tilde{\alpha})} \text{ for } \tilde{\alpha}, \tilde{w}(\tilde{\alpha}) \in R^a_+, \]

then the extension of \( r \) satisfying these relations for all \( \tilde{w} \) is well-defined and closed (the invariant closure).
Theorem 2.2. Let us assume that $r$ is a closed non-affine $r$-matrix and the group $A \ni a$ (see (1.2)) acts on the algebra $\mathcal{F} \ni f$ (written $f \to a(f)$) obeying the following condition

$$a(r_\alpha) = r_\alpha \quad \text{whenever} \quad (a, \alpha) = 0, a \in A, \alpha \in R. \quad (2.8)$$

Then the elements

$$r_\tilde{\alpha} \overset{\text{def}}{=} a(r_\alpha) \quad \text{for} \quad a \quad \text{such that} \quad \tilde{\alpha} = a'(\alpha) = \alpha - (a, \alpha)c \quad (2.9)$$

are well defined (do not depend on the choice of $a$ satisfying (2.9) for a given $\tilde{\alpha} \in R^a$) and form a closed (affine) $r$-matrix.

Proof is the same as that of Theorem 2.3 from [Ch4] in the case of quantum $R$-matrices.

Theorem 2.3. a) Given an affine $r$-matrix, let us suppose that the algebra $\mathcal{F}$ is supplied (as a $\mathbb{C}$-linear space) with a norm $||f||$ and the following series are absolutely convergent:

$$\tilde{r}_\alpha \overset{\text{def}}{=} r_\alpha + \sum_{k>0} (r_{kc+\alpha} - r_{kc-\alpha}), \alpha \in R_+,$$

$$y_u \overset{\text{def}}{=} \sum_{\tilde{\alpha} \in R^a} (u, \tilde{\alpha})r_{\tilde{\alpha}} = \sum_{\alpha \in R_+} (u, \alpha)\tilde{r}_\alpha, \quad u \in \mathbb{R}^n. \quad (2.10)$$

If any pairwise products of these series are also absolutely convergent, then $\tilde{r}$ is a non-affine $r$-matrix and $[y_u, y_v] = 0$ for any $u, v \in \mathbb{R}^n$.

b) Let the group $W^a$ act in $\mathcal{F}$ by continuous automorphisms relative to the norm and $r$ be $W^a$-invariant:

$$\tilde{w}(r_{\tilde{\alpha}}) = r_{\tilde{w}(\tilde{\alpha})} \quad \text{for all} \quad \tilde{w} \in W^a, \quad \tilde{\alpha} \in R^a, \quad (2.11)$$

for a proper closure of $r$. Then $\tilde{r}$ is $W$-invariant and

$$s_i(y_u) - y_{s_i(u)} = (u, \alpha_i)(r_{\alpha_i} + s_i(r_{\alpha_i})), \quad 0 \leq i \leq n, \quad u \in \mathbb{R}^n. \quad (2.12)$$

Proof. The commutativity in the non-affine case is proved in [Ch3], Proposition 3.2. As to (2.12), see [Ch3], Corollary 3.6 and the end of Section 1 from [Ch5]. The considerations in the affine case are the same. We calculate separately the sums of the pairwise commutators for any subspaces $R\langle \tilde{\alpha}, \tilde{\beta} \rangle \cap R^a$. □

Let us fix one more $W^a$-invariant set $\mu = \{\mu_{\tilde{\alpha}}, \tilde{\alpha} \in R^a\}$. 
Theorem 2.4.  a) Using the variables \{x\} from (1.3), (1.5), let \(\mathcal{F}\) be the algebra \(\mathcal{F}^b\) generated by \(C[W^a]\) and

\[
C\{x\} = C[\text{ct}(x_{\alpha'} + kx_c), \, \tilde{\alpha} = \alpha + kc \in R_+^a]
\]

with the cross-relations \(\tilde{w}x_u = x_{\tilde{w}(u)}\tilde{w}\), where \(\text{ct}(t) = (\exp(t) - 1)^{-1}\). Then

\[
r^b_{\tilde{\alpha}} = \eta_{\tilde{\alpha}}\text{ct}(x_a + kx_c)(\mu_{\tilde{\alpha}} - s_{\tilde{\alpha}}), \, \tilde{\alpha} = \alpha + kc \in R^a, \, a = \alpha', \quad (2.13)
\]

is a \(W^a\)-invariant closed \(r\)-matrix and \(s_i r^b_{\alpha_i} + r^b_{\alpha_i}s_i = \eta_i(s_i - \mu)\) for \(0 \leq i \leq n\).

b) Now \(\mathcal{F} = \mathcal{F}^\#\) is the algebra generated by \(\mathcal{F}^a\) and \(C\{z\} = C[\text{ct}(z_{\alpha})]\), where

\[
z_{u+k\xi} = \sum_{i>0} (u, b_i)z_i + \xi, \, u \in R^a, \text{ for complex } \tilde{z} = \{z_1, \ldots, z_n, z_c = \xi\}
\]

commuting with \(C[W^a]\). The following functions of \(\tilde{z}\)

\[
r^\#_{\tilde{\alpha}} = \eta_{\tilde{\alpha}}\text{ct}(z_\alpha + k\xi)(s_{\tilde{\alpha}} - \mu_{\tilde{\alpha}}) + r^b_{\tilde{\alpha}}, \, \tilde{\alpha} = \alpha + kc \in R^a, \quad (2.14)
\]

also form an \(r\)-matrix which is invariant relative to the diagonal (simultaneous) action \(\delta\) of \(W^a\) on \(\{x\}\) and the analogous action \(\sigma\) on \(\{z\}:: \sigma(\tilde{w})(z_u + \xi) = z_{\tilde{w}(u)} + \xi\). Moreover \(\delta(s_i)(r^\#_{\alpha_i}) + r^\#_{\alpha_i} = 0\) for \(0 \leq i \leq n\).

Proof. The theorem for \(\mu = 1\) is a straightforward affine extension of Corollary 3.6 from [Ch3] (see also the end of Section 2, [Ch5]). These \(r\)-matrices are quasi-classical limits of the quantum \(R\)-matrices from [Ch4], Propositions 3.5, 3.8 (cf. (1.6) that is a rational counterpart of one of them). Calculating the corresponding commutators (2.1-4) we obtain a set of relations that are the coefficients of \(s_{\tilde{\alpha}}\) and \(s_{\tilde{\alpha}}s_{\tilde{\beta}}\) (the latter never coincide with the first). If \(s_{\tilde{\alpha}}s_{\tilde{\beta}} = 1\) then \(\tilde{\alpha} = \tilde{\beta}\) and the corresponding commutator equals zero. Hence if the \(r\)-matrix relations are checked for one non-zero \(\mu\) they are valid for all of them. \(\square\)

Proposition 2.5. Let \(\alpha \geq \epsilon > 0, \, M > 0, \, m \in Z_+\),

\[
\Xi_{\alpha}(M) = \{ x, x_c, z, \xi, \mathcal{R}(\xi), \mathcal{R}(x_c) \geq \alpha, \text{ ct}(x_a + kx_c), \text{ ct}(z_\alpha + k\xi) < M \}
\]

for all \(k \in Z_+, \, \alpha \in R, \, a = \alpha'\). Setting \(f(z, \xi) = \sum_{\tilde{w}} f_{\tilde{w}}(z, x, \xi, x_c)\tilde{w}\) for scalar \(f_{\tilde{w}}, \, \tilde{w} \in W^a\), we introduce the norm as follows:

\[
||f|| \overset{\text{def}}{=} \sum_{\tilde{w}} \max\{||f_{\tilde{w}}|| \text{ in } \Xi_{\alpha}(M)\}||\tilde{w}||, \quad (2.15)
\]

where \(||\tilde{w}|| = \exp((\alpha - \epsilon)l(\tilde{w})(2h - 2)^{-1}4^{1-m})\).
$l(\hat{w})$ is the length of $\hat{w} \in W^a$ with respect to the generators $\{s_i, \ 0 \leq i \leq n\}$, $h$ the Coxeter number, $||\ |$ the absolute value. Then products of any $m$ series from (2.10) are absolutely convergent (for both $r^b$ and $r^\#$). The action of $W^a$ is continuous.

**Proof.** Let us start with $\mu = 0$. Without $r^3$, (2.15) follows from the estimate
\[
l(\alpha + kc) \leq kl(a^\prime) + \text{const} \leq k(2h - 2) + \text{const}, \quad \text{for} \quad k \geq 0, \ a = a^\vee, \ (2.16)
\]
(see e.g. [Ch4],Proposition 1.6 and [Ch7],(1.15)). Here the factor $4^{1-m}$ is not necessary. Given $\alpha(1), \ldots, \alpha(m) \in R_+$, let us consider the product $\tilde{r}^b_{\alpha(1)} \cdots \tilde{r}^b_{\alpha(m)}$ that is the sum of
\[
\Pi_k = \tilde{r}^b_{\alpha(1)} \cdots \tilde{r}^b_{\alpha(m)}, \ k = \{k(1), \ldots, k(n)\} \subset \mathbb{Z}_+, \ \tilde{\alpha}(i) = k(i) + \alpha(i) \in R^a_+.
\]
We should fix $C > 0$ and calculate the number of the terms such that $||\Pi_k|| > C$. A certain problem is that $\{s_{\tilde{\alpha}}\}$ from $\{r^3\}$ act on the arguments moving them from $\Xi_{\tilde{\alpha}}(M)$:
\[
\Pi_k = \prod_i \eta_{\tilde{\alpha}(i)}(\exp x_{\tilde{\alpha}(i)} - 1)^{-1} \hat{w}, \quad \text{where} \quad \hat{w} = \prod_i (\mu_{\tilde{\alpha}(i)} - s_{\tilde{\alpha}(i)}),
\]
\[
\tilde{\alpha}^1 = \tilde{\alpha}(1), \ \tilde{\alpha}^2 = s_{\tilde{\alpha}(1)}(\tilde{\alpha}(2)), \ldots, \ \tilde{\alpha}^2 = (s_{\tilde{\alpha}(1)} \cdots s_{\tilde{\alpha}(m-1)})(\tilde{\alpha}(m)). \ (2.17)
\]

**Lemma 2.6.** Let $\tilde{\alpha}^i = k^i + \alpha^i, \ \alpha^i \in R, \ k_\pm = \max\{0, \pm k^i, 1 \leq i \leq m\}, \ K > 0$. Then
\[
c_m k_+ \geq k_- \quad \text{for} \quad c_m = (\nu + 1)^{m-1} - 1, \ (2.18)
\]
where $\nu$ is 1 for $A, D, E, 3$ for $G_2$, and 2 for the other root systems. The number of the terms $\Pi_k$ such that $k_+ < K$ is less than $(c_m + 1)^m K^m$. The length of the corresponding element $\hat{w}$ is less than $(2h - 2)(c_m + 1)K$.

**Proof.** We argue by induction on $m$. The inequality for $k_\pm$ is clear for $m = 1$ since $k^1 = k(1)$ is always non-negative. Supposing that (2.18) is valid for $m$, let us add one more factor $\tilde{r}^b_{\tilde{\alpha}(0)}$ on the left and denote the new pair of extreme values of $\{\pm k^i, 0 \leq i \leq n\}$ by $k'_\pm$. Then
\[
k_+ - \nu k^0 \leq k'_+ \geq k^0, \quad k'_- \leq k_- + \nu k^0,
\]
\[
c_m(1 + \nu) k'_+ \geq c_m(k'_+ + \nu k^0) \geq k_- \geq k'_- - \nu k^0 \geq k'_- - \nu k'_+.
\]
Hence $(c_m(1 + \nu) + \nu)k'_+ \geq k'_-$, which provides the necessary estimate. As to the length, $l(\hat{w}) = l(\hat{w}^{-1}), \ \hat{w}^{-1} = s_{\tilde{\alpha}^m} \cdots s_{\tilde{\alpha}^1}$, and we can use (2.16).

The lemma gives that $||\Pi_k|| < \text{const} \exp(-K\epsilon)$ for a rather big $K$ if $k_+ > K$. The number of such terms grows polynomially in $K$. If we have a "mixed" product (2.17) where some of $x$ are replaced by $z$, then the reasoning is quite similar. We apply again the induction taking into consideration mostly the first term (with $k(1) = k^1$). The changes of the arguments of the others can be controlled in the same way. When $\mu \neq 0$, we can use the estimates without $\mu$ for smaller $m$. 

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Corollary 2.7. Let us denote the operators \( y \) from (2.10) considered for \( r^b \) by \( \{y_u^b, u \in \mathbb{R}^n\} \) and introduce
\[
x_u^b = y_u^b + (\rho_{\eta \mu}, u), \ x_c^b = h_{\eta \mu}' \overset{\text{def}}{=} h_{\eta \mu}(\theta, \theta)/2, \ x_{u+\kappa c}^b = x_u^b + \kappa x_c^b. \tag{2.19}
\]
Then the group algebra \( C[W^a] \) and \( \{x_u^b\} \) satisfy relations from Definition 1.1 and form a representation of \( \mathcal{S}_f' \) (which is faithful in \( \mathcal{S}_f/(x_c - h_{\eta \mu}') \)).

\[\square\]

3. Dunkl operators and KZ

Let us denote the operators \( \{\), \( \} \) to \( \mathbb{C}^n \) and then to \( \mathbb{C}^{n+2} = C^n \oplus Cc \oplus Cd \) setting \((c,d) = 1, (c,u) = 0 = (d,u) = (d,d) \) for \( u \in \mathbb{C}^n \) (see e.g. [Ka], Chapter 6). Given \( \tilde{\alpha} = \alpha + kc \in R^a, \ a \in A, \) the formulas
\[
s_{\tilde{\alpha}}(\tilde{u}) = \tilde{u} - \{(u,\alpha) + \nu k\}^\circ - \left\{ \nu k^2(\alpha^\circ, \alpha^\circ)/2 + (u, \alpha^\circ)k \right\} c, \quad a'(\tilde{u}) = \tilde{u} + \nu a - \left\{ \nu (a,a)/2 + (u,a) \right\} c, \quad \tilde{u} = u + \kappa c + \nu d \in \mathbb{C}^{n+2},
\]
\[
z_{\tilde{\alpha}} = \sum_{i=1}^n (u, b_i)z_i + \kappa \xi + \nu \zeta, \quad \sigma(\tilde{w})(z_{\tilde{\alpha}}) \overset{\text{def}}{=} z_{\tilde{w}(\tilde{u})}, \quad \sigma_{\tilde{\alpha}} = \sigma(s_{\tilde{\alpha}}). \tag{3.1}
\]
define an action of \( \tilde{w} \in W^a \) on \( \tilde{u} \in \mathbb{C}^{n+2} \) and \( W^a_{\sigma} \overset{\text{def}}{=} \sigma(W^a) \) on \( z_{\tilde{\alpha}} \).

The linear functions \( z_i = z_{\alpha_i}, 1 \leq i \leq n, \ \xi, \zeta \) will be regarded as coordinates of \( \mathbb{C}^{n+2} \). For instance, \( \partial z_{\tilde{\alpha}}/\partial z_i \) is the multiplicity of \( \alpha_i \) in \( \tilde{\alpha} = \alpha + kc \in R^a, \ \partial z_{\tilde{\alpha}}/\partial \xi = k, \ \partial z_{\tilde{\alpha}}/\partial \zeta = 0 \). We will also use the derivatives
\[
\partial \tilde{u} = \partial_u + \kappa \partial \zeta, \quad \partial_u(z_{\tilde{u}}) = (v,u), \quad \tilde{u} = u + \kappa c \in \mathbb{C}^{n+1}, \quad \tilde{v} \in \mathbb{C}^{n+2},
\]
with the following evident properties:
\[
\partial_{r\tilde{u} + t\tilde{v}} = r\partial \tilde{u} + t\partial \tilde{v}, \quad \sigma(\tilde{w})(\partial \tilde{u}) = \partial \tilde{w}(\tilde{u}), \quad r, t \in \mathbb{C}, \quad \tilde{w} \in W^a, \\
\partial_{\tilde{v}} = \partial / \partial z_i, \ 1 \leq i \leq n, \quad \partial_c = \partial / \partial \zeta. \tag{3.2}
\]

We extend \((\rho_{\eta \mu}, \cdot)\) to a linear function on \( \tilde{u} = u + kc \in \mathbb{C}^{n+1} \) by the formulas (see (1.1))
\[
\rho_{\eta \mu}(\tilde{u}) = (\rho_{\eta \mu} + h_{\eta \mu}(\theta, \theta)d/2, \ \tilde{u}) = (\rho_{\eta \mu}, u) + \kappa h_{\eta \mu}', \quad h_{\eta \mu}' \overset{\text{def}}{=} h_{\eta \mu}(\theta, \theta)/2. \tag{3.3}
\]
to ensure the relations \( \tilde{\rho}_{\eta \mu}(\alpha_i) = \eta_i \mu_i(\alpha_i, \alpha_i)/2 \) for all \( 0 \leq i \leq n \).

Following Theorem 2.4, let us introduce the algebra \( \mathcal{F}_\sigma^\# \) generated by \( C[W^a_{\sigma}] \) and \( C\{z\} = C[ct(z_{\tilde{\alpha}}), \ \tilde{\alpha} \in R^a_+\]. We will need another \( W^a \) (without \( \sigma \)) commuting with \( z \) and the corresponding algebra \( \mathcal{F}_\sigma^\# \) generated by \( W^a \) instead of \( W^a_{\sigma} \). Excluding \( \{\} \), the definition of the sequence of norms (depending on \( m, M \)) from Proposition 2.5 remains the same.

The algebra of differential operators in \( \partial_1, \ldots, \partial_n, \partial_c \) with the coefficients in \( \mathcal{F}_\sigma^\# \) will be denoted by \( \mathcal{F}_\sigma^\# [\partial] \). We will also use \( \mathcal{F}_\sigma^\# [\partial] \) (the derivatives are always with respect to \( z, \zeta \)).
Theorem 3.1. The following family of differential-difference Dunkl operators defined for \( \tilde{u} = u + \kappa c \in \mathbb{R}^{n+1} \)

\[
\Delta_{\tilde{u}} \overset{\text{def}}{=} \partial u + \kappa \partial / \partial \zeta - \sum_{\tilde{\alpha} > 0} \eta_{\tilde{\alpha}}(u, \alpha) \operatorname{ct}(z_{\tilde{\alpha}})(\sigma_{\tilde{\alpha} - \mu_{\tilde{\alpha}}}) + \tilde{\rho}_{\eta_{\mu}}(\tilde{u}), \tag{3.4}
\]

is commutative. Moreover, \( \{ \sigma_i = \sigma(s_i) \}, \ 0 \leq i \leq n, \) and \( \{ \Delta_{\tilde{u}} \} \) satisfy relations (1.4) and the map

\[
\Delta : \ s_{\tilde{\alpha}} \mapsto \sigma_{\tilde{\alpha}}, \ x_{\tilde{u}} \mapsto \Delta_{\tilde{u}} \tag{3.5}
\]

gives an injective homomorphism from \( \mathfrak{H}' \) into the algebra of convergent series from \( \mathcal{F}_{\sigma}'[\partial] \). The convergence of differential operators is coefficient-wise with respect to the norms for sufficiently big \( m, M \). If \( \Delta_c = \partial / \partial \zeta + h'_{\eta_{\mu}} \) is replaced by zero, then \( \Delta \) maps via \( \mathfrak{H}_0 \).

Proof. Without \( \{ \partial_{\tilde{\alpha}} \} \), it follows from Corollary 2.7. The contribution of the derivatives is trivial since \( [\partial_{\tilde{\alpha}}, r_{\tilde{\alpha}}] = 0 \) if \( (\tilde{u}, \tilde{\alpha}) = 0 \) and

\[
[\partial_{\tilde{u}}, (\tilde{v}, \tilde{\alpha}) r_{\tilde{\alpha}}] = [\partial_{\tilde{v}}, (\tilde{u}, \tilde{\alpha}) r_{\tilde{\alpha}}] = [\partial_{(\tilde{u}, \tilde{\alpha})\tilde{u} - (\tilde{u}, \tilde{\alpha})\tilde{v}}, r_{\tilde{\alpha}}] = 0 \text{ for all } \tilde{u}, \tilde{v}. \]

The theorem is valid even when the map \( \sigma \) satisfies the following weaker properties:

\[
\sigma_{\tilde{\alpha}} z_{\tilde{u}} = z_{\tilde{u}'}, \sigma_{\tilde{\alpha}} \partial_{\tilde{u}} = \partial_{\tilde{u}'}, \sigma_{\tilde{\alpha}}, \text{ for } \tilde{u}' = s_{\tilde{\alpha}}(\tilde{u}), \tilde{u} \in \mathbb{C}^{n+1}, \tag{3.6}
\]

\[
\sigma_{\tilde{\alpha}_1} \sigma_{\tilde{\alpha}_2} = \sigma_{\tilde{\beta}_1} \sigma_{\tilde{\beta}_2} \text{ if } s_{\tilde{\alpha}_1} s_{\tilde{\alpha}_2} = s_{\tilde{\beta}_1} s_{\tilde{\beta}_2}, \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^a. \tag{3.7}
\]

Indeed, the necessary relations are written in terms of commutators (cf. [Ch5], Section 2).

Definition 3.2. Let us take \( \mathbb{C}[W^a] \) which commutes with \( z, \xi, \zeta \) (we omit \( \sigma \) to differ it from \( \mathbb{C}[W^a_\sigma] \)). Given \( \Delta \in \mathcal{F}_{\sigma}'[\partial] \), we represent it in the form

\[
\Delta = \sum_{\tilde{w} \in W^a} D_{\tilde{w}} \sigma(\tilde{w}), \text{ where } D_{\tilde{w}} \text{ are differential}, \tag{3.8}
\]

and introduce the operator from \( \mathcal{F}'[\partial] \)

\[
\text{Red}(\Delta) \overset{\text{def}}{=} \sum_{\tilde{w} \in W^a} D_{\tilde{w}} \tilde{w}^{-1} \tag{3.9}
\]

with the coefficients in the completion of the group algebra \( \mathcal{F}' = \mathbb{C}\{z\} \otimes \mathbb{C}[W^a] \). Replacing \( \partial_c = \partial / \partial \zeta \) by \( -h'_{\eta_{\mu}} \) in \( \text{Red}(\Delta) \) we obtain \( \text{Red}_0(\Delta) \in \mathcal{F}'[\partial_1, \cdots, \partial_n] \). Both operations are continuous.

\(\square\)
**Theorem 3.3.** Given arbitrary $\Delta$ and $W^a_\sigma$-invariant $\Delta'$ from $F^\sigma_a[\partial]$,

$$\text{Red}(\Delta \Delta') = \text{Red}(\Delta)\text{Red}(\Delta').$$

If $p \in C[x_1, \ldots, x_n]^W$, then the (differential) OP operators

$$L_p \overset{\text{def}}{=} \text{Red}_0(p(\Delta b_1, \ldots, \Delta b_n)) \in F^\#_1[\partial_1, \ldots, \partial_n]$$

are pairwise commutative and $W^a_\sigma$-invariant with respect to the diagonal action $\delta(\tilde w) = \sigma(\tilde w) \otimes \tilde w$, where $\tilde w$ act in $C[W^a]$ by conjugations (cf. (1.5)).

**Proof.** We completely follow [Ch5], Theorem 2.4.

**Theorem 3.4.** Let us introduce the KZ operators that are differential operators of the first order with convergent coefficients from $C\{z\} \otimes C \mathcal{S}'$:

$$D_u = \partial u - \sum_{\tilde \alpha > 0} \eta_{\tilde \alpha}(\tilde u, \tilde \alpha)\text{ct}(z_{\tilde \alpha})(s_{\tilde \alpha} - \mu_{\tilde \alpha}) + \tilde \rho_{\mu \eta}(\tilde u) - x_{\tilde u}, \quad \tilde u \in C^{n+1}. \quad (3.10)$$

They are pairwise commutative and satisfy the following invariance property with respect to the above diagonal action $\delta$ extended to $\mathcal{S}' \supset C[W^a]$:

$$\delta(\tilde w)(D_u) = D_{\tilde w(u)}, \quad \tilde w \in W^a, \quad \tilde u \in C^{n+1}. \quad (3.11)$$

**Proof.** First of all, the contribution of the derivations is zero (see the proof of Theorem 3.1). Then the commutators $[D_u, D_v]$ and the differences $\tilde w(D_{\tilde u}) - D_{\tilde w(\tilde u)}$ for all $\tilde u, \tilde v, \tilde w$ belong to $C[W^a]$. We have to check that they vanish. Theorem 2.4 gives that they really equal zero in the representation of $\mathcal{S}'$ from Corollary 2.7. But the latter is faithful when restricted to $C[W^a]$.

**The isomorphism.** We will show that KZ considered in certain induced representations of $\mathcal{S}'$ is equivalent to the proper eigenvalue problem for the above Dunkl operators. It generalizes the constructions from [Ma] and [Ch5]. Let us start with the following general remark. If $\tilde w \in W^a$ are boundary operators in a certain algebra with a norm $\mathcal{N}$, then the series for $D$ and $\Delta$ and the products of any $m$ among them are convergent for rather big $\Re(\xi)$. Indeed, (2.15) leads to the estimate

$$\exp(a(2h - 2)^{-1}4^{1-m}) > \max\{\mathcal{N}(s_i), \ 0 \leq i \leq n\}.$$

It is always so if $W^a$ and $\{x\}$ act in finite dimensional representations.

Thus the KZ equation, which is the system

$$D_u \varphi(z_1, \ldots, z_n) = 0, \quad u \in C^n, \quad (3.12)$$
is well-defined when the values of $\varphi$ are taken in any finite dimensional representations of $\mathcal{S}^t$. The extended KZ is obtained for $\tilde{u}$ instead of $u$:

$$
\mathcal{D}_u \phi(z_1, \ldots, z_n, \zeta) = 0, \quad \partial \phi / \partial \zeta + h_{\eta \mu}' = 0, \quad h_{\eta \mu}' = h_{\eta \mu}(\theta, \theta)/2. \quad (3.12a)
$$

If $\varphi$ satisfies (3.12) then $\phi = \varphi \exp(-h_{\eta \mu}' \zeta)$ is a solution of (3.12a). But this trivial extension is important for the main theorem below.

We may use standard results about the solutions of differential equations (assuming that $\Re(\xi)$ is rather big). Here and further $\xi$ is considered as a parameter ($\partial / \partial \xi$ does not appear in $\mathcal{D}$).

Following Section 1, let $V$ be a finite dimensional $\mathbb{C}[W^a]$-module, $\tau$ the corresponding homomorphisms from $\mathbb{C}[W^a]$ to $\text{End}_\mathbb{C} V$. We fix a set $\lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ and consider the $\mathcal{S}_{0}'$-module $V(\lambda)$ introduced in (1.17) with the $\mathbb{C}[W^a]$-homomorphism $\text{tr} : V(\lambda) \mapsto V$. The homomorphism $\mathcal{S}_{0}' \mapsto \text{End}_\mathbb{C} V(\lambda)$ will be denoted by $\hat{\tau}$.

**Main Theorem 3.5.** Let $\mathcal{K}$ be the space of solutions $\varphi(z)$ of (3.12) in $V(\lambda)$ defined in a neighbourhood of a given point (its dimension coincides with $\dim_\mathbb{C} V(\lambda) = |W| \dim_\mathbb{C} V$). Then the map $\text{tr} : \varphi \mapsto \psi = \text{tr}(\varphi)$ is an isomorphism onto the space $\mathcal{M}$ of solutions of the quantum many-body problem

$$
L_\rho \psi(z) = p(\lambda_1, \cdots, \lambda_n) \psi(z), \quad p(x_1, \cdots, x_n) \in \mathbb{C}[x]^W, \quad (3.13)
$$

for the operators $\{L\}$ introduced in Theorem 3.3.

**Proof.** The statement is a direct generalization of Theorem 4.6 from [Ch5]. We will remind the main steps of the proof (adopted to the affine case).

In the set up of Theorem 2.5, let us pick a set $Z \subset \mathbb{C}^{n+1}$ obtained from $\Xi_{\mathcal{A}}(M)$ by certain cuts off and obeying the following conditions. It is connected and simply connected. The image of the intersection $\bigcap_{\tilde{w}} \tilde{w}(Z)$ in the quotien $\Xi_{\mathcal{A}}(M)/W^a$ is connected. Assuming that $\Re(\xi)$ is rather big, we can fix an invertible analitical solution $\Phi(z, \zeta)$ of (3.12a) for $z \in Z$ and arbitrary $\zeta$ with the values in $\text{End}_\mathbb{C} V(\lambda)$.

The functions $\sigma(\tilde{w})\Phi, \tilde{w} \in W^a$, are well-defined in open subsets of $Z$, we may introduce the "monodromy matrices" $T$:

$$
\hat{\tau}(\tilde{w}) \Phi(z, \zeta) = \sigma(\tilde{w}^{-1})(\Phi(z, \zeta)) T_{\tilde{w}}(z, \zeta), \quad \tilde{w} \in W^a, \quad (3.14)
$$

which are well-defined for almost all $z \in Z$ and locally constant (use the invariance of $\mathcal{D}_{\tilde{w}}$). They satisfy the one-cocycle relation

$$
T_{\tilde{w}_1 \tilde{w}_2} = \sigma(\tilde{w}_2^{-1})(T_{\tilde{w}_1}) T_{\tilde{w}_2}, \quad \tilde{w}_1, \tilde{w}_2 \in W^a,
$$

which results in the following action $\hat{\sigma}$ of $W^a$:

$$
\hat{\sigma}(\tilde{w})(F(z, \zeta)) \overset{\text{def}}{=} \sigma(\tilde{w})(F(z, \zeta)) T_{\tilde{w}^{-1}}(z, \zeta), \quad \tilde{w} \in W^a, \quad (3.15)
$$
on End $\mathbf{C}V(\lambda)$-valued functions $F$ defined for almost all $z \in Z$.

Substituting $\bar{\sigma}_\alpha = \bar{\sigma}(s_\alpha)$ for $\hat{\tau}(s_\alpha)$ we rewrite KZ for $\Phi$ as the system
\[
\bar{\Delta}_{\bar{u}}(\Phi) = \hat{\tau}(x_{\bar{u}})\Phi, \quad \bar{u} \in \mathbb{C}^{n+1},
\]
where the operators $\bar{\Delta}_{\bar{u}}$ are introduced by formulas (3.4) with $\sigma$ replaced by $\bar{\sigma}$. The latter obeys relations (3.6), (3.7), which ensure the validity of Theorems 3.1,3.3. The operators $\bar{L}_p$ constructed for $\bar{\sigma}$ (by replacing $\bar{\sigma}(\bar{w})$ on the right with $\bar{w}^{-1}$) coincide with $L_p$ for $\sigma$. Hence,
\[
\begin{align*}
 p(\bar{\Delta}_{\bar{u}})(\Phi) &= (\hat{\tau}(p(x_{\bar{u}}))(\Phi) = p(\lambda)\Phi, \\
 L_p(\Phi) &= p(\lambda)\Phi \text{ for } p(x_1, \cdots, x_n) \in \mathbb{C}[x]^W.
\end{align*}
\]

The last formula contains no $\{x\}$ and therefore commutes with $tr$. More precisely, given $e \in V(\lambda),
\[
tr(L_p(\Phi e)) = p(\lambda)tr(\Phi e).
\]

Since an arbitrary solution $\varphi \in K$ can be represented in the form $\Phi e$ for a proper $e$, the image of $K$ belongs to $M$. The dimension of the latter is not more than $\dim \mathbf{C}K$. However $tr$ has no kernel due to Proposition 1.4 (as it was checked in [Ch5]).

It is worth mentioning that one can introduce the monodromy of KZ more traditionally. It is necessary to fix a point $z^0$ and to replace $\Phi$ in right-hand side of (3.14) by its analitical continuation along a certain path from $z^0$ to $\bar{w}(z^0)$ (see [Ch2,5]). This approach gives a representaion of the ”elliptic” braid group which is directly connected with the induced representations of the double affine Hecke algebras from [Ch7,8].

4. Examples

We will calculate the first (quadratic) $L$-operators for the simplest $\mu \subset \{\pm 1\}$, $\mu = 0$ and discuss their basic properties. More complete analysis will be continued in the next paper(s).

The following elliptic functions $\varsigma, \vartheta$ ”almost” coincide (but do not coincide) with the classical $\zeta, \vartheta_1$. To avoid confusions we changed a little the standard notations. Let
\[
\begin{align*}
\varsigma(t) &= \sum_{k=0}^{\infty} \text{ct}(k\xi + t) - \sum_{k=1}^{\infty} \text{ct}(k\xi - t), \\
\vartheta(t) &= (\exp(t/2) - \exp(-t/2)) \prod_{k=1}^{\infty} (1 - \exp(-k\xi + t))(1 - \exp(-k\xi - t)), \\
\varrho(t) &= \sum_{k=1}^{\infty} k(\text{ct}(k\xi + t) + \text{ct}(k\xi - t)).
\end{align*}
\]
Here $t, \xi \in \mathbb{C}$, $\Re(\xi) > 0$. All these functions are $2\pi i\mathbb{Z}$-invariant. One has the following relations (which can be deduced from the corresponding properties of $\zeta$ and $\vartheta_1$ or proved directly):

\[
\begin{align*}
\zeta(t + m\xi) &= \zeta(t) + m, \quad \vartheta(t + m\xi) = -\exp(t + \xi/2)\vartheta(t), \\
\zeta(t) + \zeta(-t) &= -1, \quad \vartheta(-t) = -\vartheta(t), \quad \vartheta(t) = \vartheta(t), \\
\partial(\log \vartheta(t))/\partial t &= \zeta(t) \overset{\text{def}}{=} \zeta(t) + 1/2, \quad \partial(\log \vartheta(t))/\partial \xi = \vartheta(t), \\
\vartheta(t - \xi) &= \vartheta(t) + \zeta(t), \quad \zeta' \overset{\text{def}}{=} \partial \zeta/\partial t = \varpi - \zeta(t)^2 - 2\vartheta(t). \quad (4.2)
\end{align*}
\]

As to the latter (up to a constant $\varpi$), check that the difference of the two functions has no poles and is periodic with respect to the shifts by $\xi$ (everything is periodic relative to $2\pi i\mathbb{Z}$).

Let us take $\mu = \pm 1$ and the corresponding one-dimensional $V = \mathbb{C}_\mu$ (see (1.8)). Our first aim is to determine $L_2 = L_{p_2}$ (Theorem 3.3) for

\[
p_2(x_1, \ldots, x_n) = \sum_{i=1}^n x_i x_{\alpha_i}, \quad x_{\alpha_i} = \sum_j (b_j, b_i) x_j.
\]

The calculations are rather simple because $\text{Red}_0(\sigma_\alpha - \mu_\alpha) = 0$:

\[
\begin{align*}
\text{Red}_0(\Delta_{b_i}\Delta_{\alpha_i}) &= \partial_i \partial_{\alpha_i} + (\rho_{\eta\mu}, \alpha_i) \partial_i + (\rho_{\eta\mu}, b_i) \partial_{\alpha_i} + \\
(\rho_{\eta\mu}, \alpha_i)(\rho_{\eta\mu}, b_i) + \text{Red}_0\{- \sum_{\alpha > 0} \eta_{\alpha i}(b_i, \alpha) \text{ct}(z_{\alpha})(\sigma_\alpha - \mu_\alpha) \partial_{\alpha_i}\}, \quad (4.3)
\end{align*}
\]

where the last term equals

\[
+ \sum_{\alpha + k\mathbb{Z} > 0} \eta_{\alpha i}(b_i, \alpha) (\alpha_i, \alpha^\vee) \text{ct}(z_{\alpha} + k\xi)(\partial_{\alpha_i} - kh'_{\eta\mu}).
\]

Here we applied (3.1), (3.2) and replaced $\partial/\partial \zeta$ by $-h'_{\eta\mu}$. To sum up the terms (4.3) with respect to $i$, we use the definition of $\rho_{\eta\mu}$ and the relations

\[
b = \sum_{i=1}^n (b_i, b_i) \alpha_i = \sum_{i=1}^n (b, \alpha_i) b_i.
\]

Finally, $L_2 = \text{Red}_0(\sum_{i=1}^n \Delta_{b_i}\Delta_{\alpha_i}) =$

\[
\begin{align*}
\sum_{i=1}^n \partial_i \partial_{\alpha_i} + 2\partial_{\rho_{\eta\mu}} + (\rho_{\eta\mu}, \rho_{\eta\mu}) + 2 \sum_{\alpha \in R_+} \eta_{\alpha i}(z_{\alpha}) \partial_{\alpha} - h'_{\eta\mu} \vartheta(z_{\alpha}) &= \\
\sum_{i=1}^n \partial_i \partial_{\alpha_i} + (\rho_{\eta\mu}, \rho_{\eta\mu}) + 2 \sum_{\alpha \in R_+} \eta_{\alpha i}(z_{\alpha}) \partial_{\alpha} - h'_{\eta\mu} \vartheta(z_{\alpha}). \quad (4.4)
\end{align*}
\]
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The next calculation will be a reduction of $L^2$ to the Schrödinger operator (without linear differentiations). We will introduce the following elliptic generalization of the "standard product" playing the main role in the Macdonald theory, Heckman-Opdam theory, and the theory of integral solutions of KZ:

$$\omega(z) = \omega(-z) \overset{\text{def}}{=} \prod_{\alpha \in \mathbb{R}} \vartheta(z_\alpha)^{\eta_\alpha \mu_\alpha / 2}. $$

Actually we will need in this paper only the formulas (see (4.2)):

$$\partial_u(\omega) = \omega \sum_{\alpha \in \mathbb{R}_+} \eta_\alpha \mu_\alpha (u, \alpha) \xi(z_\alpha), \quad \text{for } u \in \mathbb{C}^n, $$

$$\partial \omega / \partial \xi = \omega \sum_{\alpha \in \mathbb{R}_+} \eta_\alpha \mu_\alpha q(z_\alpha). \quad (4.5)$$

The first gives that $H_2 \overset{\text{def}}{=} \omega L_2 \omega^{-1}$ is free of linear differential operators. More precisely,

$$U(z) = 2 \sum_{\alpha \in \mathbb{R}_+} \eta_\alpha \mu_\alpha (\xi(z_\alpha) \partial_\alpha (\omega) \omega^{-1} + h'_{\eta \mu} q(z_\alpha)) +
$$

$$\sum_{i=1}^{n} \left( (\partial_\alpha (\omega) \omega^{-1})(\partial_i (\omega) \omega^{-1}) - \partial_i \{\partial_\alpha (\omega) \omega^{-1}\} \right) =
$$

$$\sum_{\alpha > 0} \eta_\alpha \mu_\alpha \xi'(z_\alpha) + 2 h'_{\eta \mu} q(z_\alpha) +
$$

$$\sum_{\alpha, \beta > 0} \eta_\alpha \mu_\alpha \eta_\beta \mu_\beta (\alpha, \beta) \xi(z_\alpha) \xi(z_\beta). \quad (4.6)$$

**Lemma 4.1.**

$$\sum_{\alpha, \beta > 0} \eta_\alpha \eta_\beta (\alpha, \beta) \xi(z_\alpha) \xi(z_\beta) = h'_{\eta} \sum_{\alpha > 0} \eta_\alpha \xi^2(z_\alpha) + C(\eta). \quad (4.7)$$

**Proof.** Let us fix $b \in B$ and replace $z_u$ by $z_{b'(u)} = z_u - (b, u) \xi$ for $u = \alpha, \beta$ in (4.7). The change of the left-hand side is

$$\sum_{\alpha, \beta > 0} \eta_\alpha \eta_\beta (\alpha, \beta) ((b, \alpha) \xi(z_\alpha) + (b, \beta) \xi(z_\beta) + (b, \alpha)(b, \beta)) =
$$

$$h'_{\eta} \sum_{\alpha > 0} 2 \eta_\alpha (b, \alpha) \xi(z_\alpha) + (h'_{\eta})^2(b, b). \quad (4.8)$$

Here we used the main property of $h'_{\eta}$:

$$\sum_{\alpha > 0} \eta_\alpha (u, \alpha)(v, \alpha) = h'_{\eta}(u, v) \quad \text{for } u, v \in \mathbb{C}^n.$$
The same holds for the right-hand side. Hence, their difference is $B$-periodic and has no singularities. The latter can be checked directly or deduced from (4.8) with $t^{-1}$ instead of $\tilde{\xi}(t)$ (use the $r$-matrix relations). Thus the difference is a constant $C$ depending on $\eta$.

Finally, applying the lemma and replacing $2\varrho(z_\alpha) + \tilde{\varrho}(z_\alpha)^2$ by $\varpi - \varpi'(z_\alpha)$ (see (4.2)), we arrived at the formula for $U$ and the following

**Theorem 4.2.** a) If $\mu \subset \{\pm 1\}$ and $V = C_\mu$ is the corresponding one-dimensional representation of $W^a$, then the reduction procedure for $p_2 = \sum_i x_i x_{\alpha_i}$ gives the operator $L_2$ conjugated (by $\omega$) with $H_2 = \sum_{i=1}^{n} \partial_i \partial_{\alpha_i} + \sum_{\alpha > 0} \eta_{\alpha} \mu_{\alpha} \{h'_{\eta_{\mu}} - (\alpha, \alpha)\} \tilde{\xi}(z_\alpha) +$ $$(\rho_{\eta_{\mu}}, \rho_{\eta_{\mu}}) - \varpi h'_{\eta_{\mu}} \sum_{\alpha > 0} \eta_{\alpha} \mu_{\alpha} - C(\eta_{\mu}).$$ (4.9)

b) The operator $H_2$ can be included into the family of pairwise commutative differential operators $H_p \overset{\text{def}}{=} \omega L_p \omega^{-1}, \ p \in C[x]^W$, which are $W$-invariant. Their coefficients are $B$-periodic with respect to the action $z_u = z_u - (u, b)\xi$, $b \in B$. They are self-adjoint relative to the complex involution taking $z_u$ to $-z_u$ and leaving $\partial_u$ invariant.

c) Operators $\{L_p\}$ are $W$-invariant as well. Moreover, they are $B$-invariant for the action:
$$z_u \to z_u - (u, b)\xi, \ \partial_u \to \partial_u + (u, b)h_{\eta_{\mu}}, \ b \in B, \ u \in C^n,$$
and formally self-adjoint with respect to the following paring:
$$\langle f(z), g(z) \rangle = \int \omega^2 f(z)g(-z)dz_1 \ldots dz_n.$$

**Proof.** The previous calculation gives a). Beginning with c), the invariance relative to $W^a$ (generated by $W$ and $A$) is due to Theorem 3.3. It can be naturally extended to the bigger group with $B$ instead of $A$. We will not discuss this extension in this paper. The self-adjointness results from the same property of $\Delta_{\tilde{\omega}}$, which can be checked directly using the definition of $\omega$. It gives the analogous properties of $H$. For instance, let us check the periodicity:

$$\tilde{\omega}^{-1} \partial_u \tilde{\omega} = \partial_u + \sum_{\alpha > 0} \eta_{\alpha} \mu_{\alpha}(b, \alpha)(\alpha, u) = \partial_u + h'_{\eta_{\mu}}(u, b),$$
where $\tilde{\omega} = \omega(z_\alpha \to z_\alpha - (b, \alpha)\xi) = \omega \exp(-\sum_{\alpha \in R} \eta_{\alpha} \mu_{\alpha}(b, \alpha)\xi/2).$
Without going into detail we mention that one can generalize the construction of the shift operators from $[\text{Op},\text{He}]$ to the elliptic case. It is directly connected with Theorem 3.5 for $C_\mu$ (see [FV]). The most interesting applications of these operators are expected when $\mu = 1$ because in this case the operators $L_p$ preserve certain subspaces of $W$-invariant elliptic functions.

To define these spaces let us fix $m \in \mathbb{Z}_+$ and introduce the set

$$\tilde{P}_m^+ = \{ \tilde{\beta} = k_1 \omega_1 + \ldots + k_n \omega_n + kc, (\tilde{\beta}, \theta) \leq m', m' \stackrel{\text{def}}{=} m(\theta, \theta)/2, \}$$

where $\omega_i = (\alpha_i, \alpha_i) b_i/2$, $k_1, \ldots, k_n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. (4.10)

The linear space generated by the orbit sums

$$\Upsilon_{\tilde{\beta}^+} = \sum_{\tilde{\beta} \in W^+(\tilde{\beta}^+)} \exp(z_{\tilde{\beta}} + m'\zeta) \text{ for } \tilde{\beta} \in \tilde{P}_m^+$$

(4.11)

over the algebra of formal series $\sum_{l<\ell_0} c_l \exp(l\xi)$, $c_l \in \mathbb{C}$, will be denoted by $L_m$. This construction is due to Looijenga and closely related to the characters of Kac-Moody algebras. The operators $\{L_p\}$ for $\mu = 1$ leave $L_m$ invariant if $m' = -h'_\eta$. Moreover they preserve subspaces $L_m(\tilde{\beta}^+) \text{ for } \tilde{\beta}^+ \in \tilde{P}_m^+$ generated by

$$\Upsilon_{\tilde{\gamma}^+} \text{ such that } \tilde{\gamma}^+ = \tilde{\beta}^+ - \sum_{i=0}^n k_i \alpha_i \in \tilde{P}_m^+, \{k_i\} \subset \mathbb{Z}_+.$$

It results directly from the corresponding properties of the elliptic Dunkl operators and allows us to introduce the elliptic Jacobi-Jack-Macdonald polynomials $J_{\tilde{\beta}^+}$ as eigenfunctions of $\{L_p\}$ in $L_m(\tilde{\beta}^+)$ with leading terms $\Upsilon_{\tilde{\beta}^+}$. A further discussion will be continued in the next papers.

**Parabolic operator.** A demerit of the above constructions is the constraint $\partial/\partial \zeta + h'_\eta \mu = 0$ corresponding to the condition $x_c = 0$ in the Hecke algebra $S'$. We will show that something can be done even without this restriction.

Let $\Delta_{\hat{u}} = \Delta_{\tilde{u}} + \nu \Delta_d$ for $\hat{u} = \tilde{u} + \nu d \in \mathbb{C}^{n+2}$,

$$\Delta_d = \partial/\partial \zeta - \sum_{\alpha \in R} \sum_{k \in \mathbb{Z}_+} \eta_\alpha kct(z_{\alpha} + k\zeta)(\sigma_{\alpha+k\zeta} - \mu_{\alpha}).$$

(4.12)

The operators $\Delta_{\hat{u}}$ are not pairwise commutative but still satisfy the following cross-relations (see (1.4)):

$$\sigma_i \Delta_{\hat{u}} - \Delta_{\{s_i(\hat{u})\}} \sigma_i = \eta_i(\hat{u}, \alpha_i), \quad 0 \leq i \leq n, \ \hat{u} \in \mathbb{C}^{n+2},$$

(4.13)

relative to the action from (3.1). It gives (together with the previous considerations) the following theorem.
Theorem 4.3. The operator \( \mathcal{M} = 2 \Delta_d \Delta_c + \sum_{i=0}^n \Delta_{b_i} \Delta_{\alpha_i} \) and its reduction \( M = \text{Red}(\mathcal{M}) \) are \( W^a \)-invariant. If \( V = \mathbb{C}_\mu, \mu = 1, \partial/\partial \zeta = m', m \in \mathbb{Z}_+ \), then

\[
M = 2(m' + h'_{\eta})\partial/\partial \xi + \sum_{i=1}^n \partial_i \partial_{\alpha_i} + (\rho_{\eta \mu}, \rho_{\eta \mu}) + 2 \sum_{\alpha \in \check{R}_+} \eta_{\alpha} \mu_{\alpha} (\tilde{s}(z_{\alpha}) \partial_{\alpha} + m' \varrho(z_{\alpha})),
\]

\[
N \overset{\text{def}}{=} \omega M \omega^{-1} = 2(m' + h'_{\eta})\partial/\partial \xi + H_2 \text{ (see (4.9)).} \tag{4.14}
\]

The operator \( M \) preserves the spaces \( \mathcal{L}_m \) and \( \mathcal{L}_m(\tilde{\beta}^+) \) for arbitrary \( \tilde{\beta}^+ \in \tilde{P}_m^+ \).

The operator \( N \) was introduced by Etingof and Kirillov in [EK] for \( \mathfrak{sl}_n \) together with its certain eigenfunctions (the generalized characters that are the traces of proper vertex operators of \( \mathfrak{sl}_n \)). In a recent work, they extended the definition of \( M, N \) to arbitrary root systems and proved directly the properties mentioned in the theorem. To be more precise, their formulas are different but with certain minor changes seem to be equivalent to (4.14) (e.g. they use more special parameters). If it is so, then our approach (based on the Dunkl operators) gives another proof of their result. The construction of the generalized characters is still known for \( \mathfrak{sl}_n \) only.

Matrix Schrödinger operator. The next application (which is a straightforward extension of Corollary 2.8 from [Ch5]) will be for arbitrary representations and \( \mu = 0 \). Let us calculate \( L_2^0 = L_{\mu=0}^2 \) for \( p = p_2 \) (see above). Applying Red and imposing the condition \( \partial/\partial \zeta = 0 \), one has:

\[
L_2^0 = \sum_{i=1}^n \partial_{\alpha_i} - \sum_{\check{\alpha} = \alpha + \lambda \in R^+} \eta_{\check{\alpha}} (\alpha, \alpha) c t(\check{\alpha}) s_{\check{\alpha}} + \sum_{\check{\alpha}, \check{\beta} > 0} \eta_{\check{\alpha}} \eta_{\check{\beta}} (\alpha, \beta) c t(\check{\alpha}) c t(\sigma(\check{\beta})) s_{\check{\beta}} s_{\check{\alpha}}, \tag{4.15}
\]

where \( c t(t) = \partial c t(t)/\partial t = -(\exp(t/2) - \exp(-t/2))^2 \). Following [Ch5], Lemma 2.7, we check that the contribution of the terms with \( \check{\alpha} \neq \check{\beta} \) in the last sum equals zero. Hence we arrived at the following theorem:

Theorem 4.4. The differential \( \mathbb{C}[W^a] \)-valued operators

\[
L_2^0 = \sum_{i=1}^n \partial_{\alpha_i} - \sum_{\check{\alpha} > 0} (\alpha, \alpha) \eta_{\check{\alpha}} c t(\check{\alpha})(\eta_{\check{\alpha}} - s_{\check{\alpha}}) \tag{4.16}
\]

and \( L_{\mu=0}^2 \) defined for \( p \in \mathbb{C}[x]^W \) are pairwise commutative. Moreover they are \( W^a \)-invariant with respect to the \( \delta \)-action on \( \check{z} \) and on \( \mathbb{C}[W^a] \) (by conjugations).
When considered in finite dimensional representations of the latter, the coefficients are convergent matrix-valued functions for sufficiently big $\Re(\xi)$.

We can obtain the scalar OP operators (for arbitrary root systems) from this construction as well. Let $\{s_{\alpha}\}$ be taken in one-dimensional representations $C_\varepsilon$ (see (1.8)). Then

$$L_2^0 = \sum_{i=1}^n \partial_{\beta_i} \partial_{\alpha_i} + \sum_{\alpha>0} (\alpha,\alpha) \eta_\alpha (\eta_\alpha - \varepsilon_\alpha) \zeta'(z_\alpha).$$

The corresponding $L_p$ are $W$-invariant and their coefficients are elliptic = $B$-periodic (cf. Theorem 4.2). Generally speaking, the coefficients are ”matrix” elliptic functions with the values in the endomorphisms of vector bundles over elliptic curves. Ignoring the differential operators in (4.16) and substituting ”good” $z$, we obtain ”periodic” generalizations of Haldane-Shastry hamiltonians. Presumably the points of finite order of the corresponding elliptic curve and the critical points of the scalar hamiltonians (4.17) without the differentiations and after a proper normalization) lead to integrable models (cf. [BGHP],[F]).

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