Fourier transforms, generic vanishing theorems and polarizations of abelian varieties

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Abstract. The purpose of this paper is to give two applications of Fourier transforms and generic vanishing theorems:
— we give a cohomological characterization of principal polarizations
— we prove that if \( X \) an abelian variety and \( \Theta \) a polarization of type \((1,\ldots,1,2)\), then a general pair \((X,\Theta)\) is log canonical.

Introduction

There is a well known connection between the geometry of principally polarized abelian varieties (PPAVs), and the singularities of their theta divisors. This was first discovered by Andreotti and Mayer, [1] in their work on the Schottky locus, and has since found applications in a variety of contexts. Subsequently Kollár proved that the singularities of the theta divisors of a PPAV are mild in the sense that the pair \((A,\Theta)\) is log canonical. This implies for example that \( \Sigma_k(\Theta) := \{ x \in A | \text{mult}_x(\Theta) \geq k \} \) is of codimension at least \( k \) in \( A \). Ein and Lazarsfeld [2] prove that if \( \Theta \) is irreducible then \( \Theta \) is normal and has only rational singularities. In particular they show that if the codimension is exactly equal to \( k \), then \((A,\Theta)\) splits as a \( k \)-fold product of PPAVs. In [2], they also generalize the result of Kollár to \( \mathbb{Q} \)-divisors. They prove that if \( D \) is a divisor in the linear series \( |m\Theta| \) then the pair \((A,(1/m)D)\) is log canonical. In [6], the first result of Ein and Lazarsfeld was extended to the case of \( \mathbb{Q} \)-divisors. Let \((A,\Theta)\) be a PPAV, and \( D \) a divisor as above, such that \(|(1/m)D| = 0\). Then the pair \((A,(1/m)D)\) is log terminal. These results immediately generalize to arbitrary polarizations. Let \((X,L)\) be a polarization of type \((d_1,\ldots,d_g)\), (as usual assume that \( d_i|d_{i+1} \)), then there exists an isogeny \( X \rightarrow X_1 \) of degree \( d_g \), such that \( L \) is the pull back of a principal polarization \( L_1 \) on \( X_1 \). If \( D \) is a divisor in the linear series \( |mL| \), then by pulling back by the corresponding isogeny \( X_1 \rightarrow X \), one deduces for example that the pair \((A,(1/md_g)D)\) is log canonical.

In this paper, exploiting the theory of Fourier transforms, we prove that a coherent sheaf which is generically of rank 1 on an abelian variety and is "cohomologically" a principal polarization, must in fact be a line bundle and hence a principal polarization. A surprising consequence is that if \((X,L)\) is a general polarized abelian variety of type \((1,\ldots,1,2)\), and \( D \) is a divisor in the linear series \( |mL| \), then the pair \((A,(1/m)D)\) is log canonical. When \((X,D)\) is an abelian surface of type \((1,2)\), and \( D \) is irreducible, then the pair \((X,D)\) is log terminal. It is not clear whether the last statement generalizes to \( \mathbb{Q} \)-divisors. Using the same

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technique, we are also able to give a cohomological criterion for a subvariety of an abelian variety to be a principal polarization.

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0. Notation and conventions

\( f^*D \) pull-back \\
\(|D|\) linear series associated to the divisor \( D \) \\
\( Bs|D|\) the base locus of the linear series associated to the divisor \( D \) \\
\( \equiv \) numerical equivalence \\
\( \omega_X = \Omega^n_X \) canonical sheaf of \( X \) \\
\( \Omega^i_X \) sheaf of holomorphic \( i \)-forms on \( X \) \\
\( K_X \) linear equivalence class of a canonical divisor on \( X \) \\
\( \omega_{X/Y} := \omega_X \otimes (f^*\omega_Y)^* \) relative canonical sheaf of a morphism \( f: X \to Y \) \\
\( \kappa(X) \) Kodaira dimension of \( X \) \\
\( k(x) \) the sheaf \( O_X/m_x \) \\
\( \text{ALB}(Z) \) abelian variety of \( Z \) \\
\( \text{alb}_Z: Z \to \text{ALB}(Z) \) abelian morphism \\
\( \bar{Z} := \text{alb}_Z(Z) \) abelian image of \( Z \) \\
\( T^0 \) connected component containing the origin of the subgroup \( T \)

Unless otherwise stated \( X, Y \) will denote smooth complex projective varieties. If \( D \) is a \( \mathbb{Q} \)-divisor we will denote by \( \lfloor D \rfloor \) and by \( \lceil D \rceil \) the round down and the round up of \( D \) respectively.

1. Preliminaries

Let \( X \) be a smooth complex projective variety and let \( D \) be a \( \mathbb{Q} \)-divisor on \( X \). We will say that \( f: Y \to X \) is a log-resolution of the pair \( (X, D) \), if \( f \) is a proper birational morphism such that \( f^{-1}D \cup \{ \text{exceptional set of } f \} \) is a divisor with normal crossing support. Given a log-resolution of the pair \( (X, D) \), we can define the multiplier ideal sheaf associated to the divisor \( D \).

\[ \mathcal{I}(D) := f_*(O_Y(K_{Y/X} - \lfloor f^*D \rfloor)) \]

The definition is independent of the choice of log-resolution. Multiplier ideal sheaves may be defined in much greater generality. Their properties have been extensively studied eg. [N], [Sk], [De] and [E]. Under the above assumptions, we will say that the pair \( (X, D) \) is log canonical (respectively log terminal) if the multiplier ideal sheaf associated to the divisor \( (1 - \epsilon)D \) is trivial, for \( 0 < \epsilon < 1 \) (respectively \( 0 \leq \epsilon < 1 \)).
One of the main tools that we will use is the theory of deformation of cohomology groups developed by Green and Lazarsfeld [4], [5]. For the convenience of the reader, we include a brief summary of the main results. We wish to study the loci

\[ V^i := \{ P \in \text{Pic}^0(X) \mid h^i(\omega_X \otimes P) \neq 0 \} \]

The geometry of these loci is governed by the following theorem [4], [5, 2]:

**Theorem 1.1 (Generic vanishing).**

(i) \( \text{Pic}^0(X) \supset V^0(\omega_X) \supset V^1(\omega_X) \supset \ldots \supset V^n(\omega_X) = \{ \mathcal{O}_X \} \)

(ii) Any irreducible component of \( V^i \) is a translate of a torus and is of codimension at least \( i - (\dim(X) - \dim \text{alb}_X(X)) \) in \( \text{Pic}^0(X) \)

(iii) Given \( P \) a general point of an irreducible component \( T \) of \( V^i \). Let \( \phi \in H^0(X, \Omega_X^1) \). Suppose \( \overline{\phi} \in H^1(X, \mathcal{O}_X) \cong T_P \text{Pic}^0(X) \) is not tangent to \( T \). Then the sequence

\[
\begin{align*}
H^0(X, \Omega_X^{n-i-1} \otimes P) \xrightarrow{\wedge \phi} H^0(X, \Omega_X^{n-i} \otimes P) \xrightarrow{\wedge \phi} H^0(X, \Omega_X^{n-i+1} \otimes P)
\end{align*}
\]

is exact. If \( \overline{\phi} \) is tangent to \( T \), then the maps in the above sequence vanish.

**Remark 1.2.** It is possible to generalize the above definitions and results, to the more general case of a morphism

\[ \nu : X \longrightarrow A \]

from \( X \) to an abelian variety \( A \) see [2]. One must then consider the loci

\[ V^i(\omega_X, A) := \{ P \in \text{Pic}^0(A) \mid h^i(\omega_X \otimes \nu^* P) \neq 0 \} \]

The generic vanishing theorem still holds, with \( \text{Pic}^0(A) \) instead of \( \text{Pic}^0(X) \) and \( \dim(\nu(X)) \) instead of \( \dim(\text{alb}_X(X)) \).

2. A characterization of principal polarizations

In this section we will assume the notation and results of [10]. In particular \( X \) will always be an abelian variety and \( \tilde{X} \) will denote the corresponding dual abelian variety. For any point \( y \in \tilde{X} \) let \( P_y \) denote the associated topological trivial line bundle. Let \( \mathcal{P} \) be the normalized Poincaré bundle on \( X \times \tilde{X} \). One may define a functor \( \hat{S} \) of \( \mathcal{O}_X \)-modules into the category of \( \mathcal{O}_{\tilde{X}} \)-modules by

\[ \hat{S}(M) = \pi_{X,*} (\mathcal{P} \otimes \pi_{\tilde{X}}^* M) \]

The derived functor \( R\hat{S} \) of \( \hat{S} \) then induces an equivalence of categories between the two derived categories \( D(X) \) and \( D(\tilde{X}) \) [10] theorem 2.2:

**Theorem 2.1 (Mukai).** There are isomorphisms of functors:

\[ R\hat{S} \circ R\hat{S} \cong (-1_X)^{*}[-g] \]
and
\[ R\hat{S} \circ RS \cong (-1)^{\ast}[-g], \]
where \([-g]\) denotes “shift the complex \(g\) places to the right”.

The index theorem (I.T.) is said to hold for a coherent sheaf \(F\) on \(X\) if there exists an integer \(i(F)\) such that for all \(j \neq i(F)\), \(H^j(X, F \otimes P) = 0\) for all \(P \in \text{Pic}^0(X)\). The weak index theorem (W.I.T.) holds for a coherent sheaf \(F\) if there exists an integer which we again denote by \(i(F)\) such that for all \(j \neq i(F)\), \(R^j\hat{S}(F) = 0\). It is easily seen that the I.T. implies the W.I.T. We will denote the coherent sheaf \(R^i(F)\hat{S}(F)\) on \(X\) by \(\hat{F}\). One of the main themes of \([10]\) is that information on the cohomology groups \(H^i(X, F \otimes P)\) may be interpreted as information on the coherent sheaves \(F\), \(R^i\hat{S}(F)\) and \(\hat{F}\) (if \(F\) satisfies the W.I.T.). In this spirit we prove:

**Proposition 2.2.** Let \(X\) be an abelian variety, \(F\) a coherent sheaf generically of of rank 1 (e.g. \(F \cong L \otimes \mathcal{I}\) where \(L\) is a line bundle and \(\mathcal{I} \subset \mathcal{O}_X\) is an ideal sheaf). For all \(P \in \text{Pic}^0(X)\), suppose that \(h^0(X, F \otimes P) = 1\), and \(h^i(X, F \otimes P) = 0\) for all \(1 \leq i \leq n = \dim(X)\). Then \(F\) is a line bundle with \(h^0(X, F) = 1\), hence a principal polarization.

**Proof.** We will use the theory of Fourier functors developed by Mukai in \([10]\). \(F\) satisfies the I.T. and hence the W.I.T. and \(i(F) = 0\). We will need the following:

**Claim 2.3.** \(\hat{F}\) is a line bundle and \(h^n(\hat{X}, \hat{F}) = 1\).

**Proof of claim 2.3.** We will use the isomorphism of \([10]\) proposition 2.7
\[ \text{Ext}^i_{\mathcal{O}_X}(k(x), F) \cong H^i(\hat{X}, \hat{F} \otimes P_x) \]

\(\text{to compute } h^i(\hat{X}, \hat{F})\). Let \(K^{p,q}\) be the bicomplex \(\mathcal{C}^p[X, \mathcal{H}om(k(x), \mathcal{I}^q(F))]\) where \(\mathcal{I}^q(F)\) is an injective resolution of the sheaf \(F\) (see \([3]\) II 7.3). Since \(F\) is generically of rank 1, a local computation shows that for generic \(x \in X\),
\[ \text{Ext}^i_{\mathcal{O}_X}(k(x), F) \cong k(x), \]
and that for all \(0 \leq i \leq \dim(X) - 1\)
\[ \text{Ext}^i_{\mathcal{O}_X}(k(x), F) \cong 0. \]

The corresponding spectral sequence therefore degenerates at the \(E_2\) term and we have \(\text{Ext}^{p,q}_{\mathcal{O}_X}(k(x), F)\) and hence \(\text{Ext}^{p,q}_{\mathcal{O}_X}(k(x), F) \cong \mathbb{C}\) for \(p = 0\) and \(q = \dim(X)\), and \(\text{Ext}^{p,q}_{\mathcal{O}_X}(k(x), F) \cong 0\) otherwise. Since the \(E_{\infty}\) term gives a filtration of \(\text{Ext}^i_{\mathcal{O}_X}(k(x), F)\), it follows that for \(0 \leq i \leq \dim(X) - 1\) we have \(\text{Ext}^i_{\mathcal{O}_X}(k(x), F) \cong 0\). On the other hand, for \(i = \dim(X)\)
\[ \text{Ext}^n_{\mathcal{O}_X}(k(x), F) \cong H^0(X, k(x)) \cong \mathbb{C}. \]

The claim now follows since for a generic point \(x \in X\) we have \(h^n(\hat{X}, \hat{F} \otimes P_x) = 1\). QED
Consequently \( \hat{F} \cong \mathcal{O}_X(-\Theta) \) for an appropriate theta divisor \( \Theta \). \( \hat{F} \) also satisfies the I.T., and in fact \( h^i(\hat{X}, \hat{F} \otimes P_{\pi}) = 0 \) for all \( 0 \leq i \leq \dim(X) - 1 \) and hence \( F \cong (-1_X)^*\hat{F} \) is also a line bundle with only one section (see also [10] proposition 3.11). QED

**Remark 2.4.** The same proof also shows that if \( F \) is a coherent sheaf generically of rank \( r \) such that for all \( P \in \text{Pic}^0(X) \), \( h^0(X, F \otimes P) = 1 \) and \( h^i(X, F \otimes P) = 0 \) for all \( 1 \leq i \leq n = \dim(X) \). Then \( \hat{F} \) is dual to an ample line bundle with \( r \) sections, and \( F \) is a locally free sheaf. We will say that a sheaf satisfying the above properties is a cohomological principal polarization. Our proposition states that a cohomological principal polarization of generic rank 1 is a principal polarization.

**Remark 2.5.** The hypothesis on the cohomological groups may not be weakened. Consider an irreducible theta divisor \( \Theta \subset X \). Let \( f : \tilde{\Theta} \rightarrow \Theta \) be an appropriate smooth birational model. Since \( \Theta \) has only log-terminal singularities, \( h^0(f_*(\omega_{\tilde{\Theta}} \otimes P)) = 1 \) if \( P \neq \mathcal{O}_X \) and is \( n \) if \( P = \mathcal{O}_X \). Similarly for \( 1 \leq i \leq n - 1 \), \( h^i(f_*(\omega_{\tilde{\Theta}} \otimes P)) = 0 \) if \( P \neq \mathcal{O}_X \) and is \( \binom{n}{i+1} \) if \( P = \mathcal{O}_X \). It follows that the sheaf \( f_*(\omega_{\tilde{\Theta}}) \oplus \mathcal{O}_X \) is coherent, generically of rank 1 and satisfies the conditions for being a cohomological principal polarization, for all \( P \neq \mathcal{O}_X \). Analogously one could also consider the sheaf \( k(0) \oplus \mathcal{O}_X \). It is not clear however whether one can find a similar example for a sheaf of the form \( L \otimes I \) where \( L \) is a line bundle and \( I \subset \mathcal{O}_X \) is an ideal sheaf.

**Remark 2.6.** One might expect that an analogue of proposition 2.2 might hold for more general polarizations. This is however not the case. Consider an abelian surface with polarization \( L \) of type \((1,3)\). It is easy to show that for any point \( x \), and any topologically trivial line bundle \( P \), \( h^i(L \otimes \mathcal{I}_x \otimes P) = 0 \) for all \( i \geq 1 \). Of course it follows that the sheaf \( L \otimes I \) satisfies the W.I.T., and in general has all the cohomological properties of a polarization of type \((1,2)\). However the sheaf \( L \otimes \mathcal{I}_x \) is not a line bundle.

### 3. Sub-varieties of abelian varieties

Let \( X \) be an abelian variety of dimension \( n \). We will say that a reduced irreducible divisor \( Z \) is a cohomological theta divisor if for any desingularization \( \nu : \tilde{Z} \rightarrow Z \)

i) \( h^i(\omega_{\tilde{Z}}) = \binom{n}{i+1} \) for all \( 0 \leq i \leq n - 1 \)

ii) \( h^i(\omega_{\tilde{Z}} \otimes \nu^* P) = 0 \) for all \( 1 \leq i \leq n - 1 \) and all \( \mathcal{O}_X \neq P \in \text{Pic}^0(X) \).

By the Hodge and Serre dualities, condition i) is equivalent to \( h^0(\Omega^j_{\tilde{Z}}) = \binom{n}{j} \) for all \( 0 \leq j \leq n - 1 \), and condition ii) is equivalent to \( h^0(\Omega^j_{\tilde{Z}} \otimes \nu^* P) = 0 \) for all \( 0 \leq j \leq n - 2 \) and all \( \mathcal{O}_X \neq P \in \text{Pic}^0(X) \). The above definition, immediately generalizes to the case of a map \( \nu : Z \rightarrow X \) from a smooth \( n - 1 \) dimensional variety \( Z \) to a \( n \) dimensional abelian variety \( X \). The following lemma will be useful:
Lemma 3.1. Let $Z$ be a reduced irreducible divisor of a $n$ dimensional abelian variety $X$. Let $\tilde{Z}$ be a smooth birational model of $Z$. Then $Z$ is of general type if and only if $|\chi(\mathcal{O}_{\tilde{Z}})| > 0$.

Proof. If $Z$ is of general type, then $|\chi(\mathcal{O}_{\tilde{Z}})| > 0$ by \cite{2} theorem 3 and theorem 3.3. If $Z$ is not of general type, then one may consider the abelian subvariety $B := \{x \in X | x + Z \subset Z\}^0$. By \cite{11}, we have

(i) $Z \rightarrow Z/B$ is an étale fiber bundle with fiber $B$,
(ii) $Z \rightarrow Z/B$ is birational to the Iitaka fibering of $Z$, and
(iii) $Z/B$ is of general type.

We may assume that $\tilde{Z} \rightarrow Z/B$ is the Iitaka fibration. Restricting to a general fiber, since $(\omega_Z \otimes P)|_B \cong P|_B$, it follows that $|\omega_Z \otimes P|$ is empty unless $P |_B \cong \mathcal{O}_B$. So $H^0(\tilde{Z}, \omega_Z \otimes P) = 0$ for general $P \in \text{Pic}^0(X)$, and hence $|\chi(\mathcal{O}_{\tilde{Z}})| = 0$. QED

Lemma 3.2. Let $\nu : Z \rightarrow X$ be a morphism from a smooth variety to an $n$ dimensional abelian variety. Assume that $\nu$ restricted to $\tilde{Z}$ is generically finite onto a divisor $\tilde{Z} = \nu(Z)$. Then

(i) If $\tilde{Z}$ is of general type, then $|\chi(\mathcal{O}_{\tilde{Z}})| > 0$.
(ii) If $|\chi(\mathcal{O}_{\tilde{Z}})| > 0$, then $Z$ is of general type.
(iii) If $h^1(\omega_Z \otimes \nu^* P) = 0$ for all but finitely many $P \in \text{Pic}^0(X)$, then $Z$ is of general type if and only if $Z$ is of general type.

Proof. i) If $\chi(\omega_Z) = 0$, then $h^0(\omega_Z \otimes \nu^* P) = 0$ for general $P \in \text{Pic}^0(X)$, so $h^0(\omega_Z \otimes P)$ also vanishes for generic $P \in \text{Pic}^0(X)$. Hence $\chi(\omega_Z) = 0$, i.e. $Z$ is not of general type.

ii) If $Z$ is not of general type, then $\tilde{Z}$ is not of general type. Let $\beta$ be the generic fiber of the Iitaka fibration $Z \rightarrow Y$, then $\beta$ is étale onto its image $B \subset X$. In fact $\kappa(\beta) = \kappa(B) = 0$, and hence $\beta$ and $B$ are abelian varieties of positive dimension. Now $(\omega_Z \otimes \nu^* P)|_\beta = \nu^* P|_\beta$, and since $\beta \rightarrow B$ is an étale map of abelian varieties, we see that $h^0(\nu^* P|_\beta) = 0$ for generic $P \in \text{Pic}^0(X)$. So $h^0(\omega_Z \otimes \nu^* P)$ also vanishes for generic $P \in \text{Pic}^0(X)$ and by the generic vanishing theorem $\chi(\omega_Z) = 0$.

iii) Assume now that $Z$ is of general type, and $h^1(\omega_Z \otimes \nu^* P) = 0$ for all but finitely many $P \in \text{Pic}^0(X)$. If $\tilde{Z}$ is not of general type, then $\chi(\mathcal{O}_{\tilde{Z}}) = 0$. Let $\tilde{Z}$ be a desingularization of $Z$. Since $\tilde{Z}$ is a divisor of $X$, by \cite{12} proposition 2.2, there exists a positive dimensional subgroup $T$ of $\text{Pic}^0(X)$ such that for all $P \in T$, $h^0(\omega_{\tilde{Z}} \otimes P) \geq 1$. Since $\chi(\mathcal{O}_{\tilde{Z}}) = 0$, then also $h^0(\Omega_{\tilde{Z}}^{-2} \otimes P) = h^1(\omega_{\tilde{Z}} \otimes P) \geq 1$ for $P \in T$. It follows that also $h^0(\Omega_{\tilde{Z}}^{-2} \otimes \nu^* P) \geq 1$ for $P \in T$ which is a contradiction. QED

Theorem 3.3. Let $\Theta$ be a reduced irreducible divisor on an abelian variety $X$. Then $\Theta$ is a principal polarization if and only if $\Theta$ is a cohomological theta divisor.
Proof. We may assume that $(\bar{X}, \bar{\Theta})$ is a log resolution of the pair $(X, \Theta)$. One has the exact sequence of sheaves:

$$0 \rightarrow \omega_{\bar{X}} \rightarrow \omega_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(\bar{\Theta}) \rightarrow \omega_{\Theta} \rightarrow 0.$$ 

Pushing forward this sequence yields the exact sequence of sheaves on $X$:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(\Theta) \otimes \mathcal{I}(\Theta) \rightarrow f_* \omega_{\Theta} \rightarrow 0.$$ 

If $\Theta$ is an irreducible principal polarization, then by the pair $(X, \Theta)$ is log terminal, so the multiplier ideal sheaf $\mathcal{I}(\Theta)$ is trivial. The cohomology groups of $\omega_{\Theta}$ may now be easily computed from the second exact sequence, and we conclude that $\Theta$ is a cohomological theta divisor.

Assume now that $\Theta$ is a cohomological theta divisor. The goal here is to show that the sheaf $\mathcal{O}_X(\Theta) \otimes \mathcal{I}(\Theta)$ is a cohomological principal polarization and then to apply proposition 2.2. Of course this will follow from the equivalent statement for the sheaf $\omega_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(\bar{\Theta})$. For all $\mathcal{O}_X \neq P \in \text{Pic}^0(X)$ and $0 \leq i \leq n$, we have $h^i(\mathcal{O}_X \otimes P) = 0$, so $H^i(X, \mathcal{O}_X(\Theta) \otimes \mathcal{I}(\Theta) \otimes P) \cong H^i(X, f_* \omega_{\Theta} \otimes P)$. We therefore need only consider the case $P = \mathcal{O}_X$. There is an injection $H^0(X, \mathcal{O}_X) \hookrightarrow H^0(\bar{\Theta}, \mathcal{O}_{\bar{X}})$. Using the Hodge and Serre dualities, one may translate this into the isomorphisms $H^{n-i-1}(\bar{\Theta}, \omega_{\Theta}) \cong H^{n-i}(\bar{X}, \omega_{\bar{X}})$. Consequently $h^0(\omega_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(\bar{\Theta})) = 1$ and $h^i(\omega_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(\bar{\Theta})) = 0$ for all $1 \leq i \leq n$. QED

Corollary 3.4. Let $Z$ be a smooth variety mapping finitely onto a divisor $\tilde{Z}$ of an $n$ dimensional abelian variety $X$. Assume that $\tilde{Z}$ generates $X$, and that $Z$ satisfies cohomological properties analogous to those of a cohomological theta divisor. Then $\tilde{Z}$ is a theta divisor.

Proof. — Let $\tilde{Z}$ be the image of $Z$ by the map $\nu : Z \rightarrow X$. By assumption $h^i(\omega_Z) = \binom{n}{i+1}$ for all $0 \leq i \leq n-1$ and $h^i(\omega_Z \otimes \nu^* P) = 0$ for all $1 \leq i \leq n-1$ and $\mathcal{O}_X \neq P \in \text{Pic}^0(X)$. The induced map $H^i(X, \mathcal{O}_X) \hookrightarrow H^i(\tilde{Z}, \mathcal{O}_Z)$ is an isomorphism. Let $\tilde{Z}$ be a smooth birational model of $Z$. We may assume that $Z \rightarrow \tilde{Z}$ factors through $\tilde{Z}$. By lemma 3.2, $\tilde{Z}$ is of general type. By theorem 1, $h^i(\omega_{\tilde{Z}}) \geq \binom{n}{i+1}$ for all $0 \leq i \leq n-1$. Clearly

$$h^i(\omega_Z \otimes \nu^* P) = h^0(\Omega_{Z}^{n-i-1} \otimes \nu^* P) \geq h^0(\Omega_{\tilde{Z}}^{n-i-1} \otimes P) = h^i(\omega_{\tilde{Z}} \otimes P)$$

for all $0 \leq i \leq n-1$ and $P \in \text{Pic}^0(X)$. It follows that $Z$ is also a cohomological theta divisor. Theorem 3.3 now proves the corollary. QED

If $Z$ is a reduced and irreducible divisor of an abelian variety, then the cohomological theta divisor condition may be substantially weakened.

Corollary 3.5. An irreducible reduced divisor $Z$ of an $n$ dimensional abelian variety $X$ is a principal polarization if and only if one of the following equivalent conditions holds

i) $Z$ is a cohomological theta divisor

ii) $Z$ is of general type, $h^0(\omega_Z) = n$, $h^i(\omega_Z \otimes P) = 0$ for all $\mathcal{O}_X \neq P \in \text{Pic}^0(X)$
iii) $h^0(\omega_Z) = n$, $h^0(\omega_Z \otimes P) = 1$ for all $\mathcal{O}_X \neq P \in \text{Pic}^0(X)$

**Proof.** — The equivalence of condition i) is proposition 2. Condition i), clearly implies the other two conditions (see lemma 3.1).

Assume that condition ii) is satisfied. Let $\tilde{Z}$ be a smooth birational model of $Z$. By 8 theorem 1, it follows that $h^0(\Omega^i_Z) = \binom{n}{i}$ for all $0 \leq i \leq n - 1$, and $|\chi(\mathcal{O}_Z)| = 1$. By 2 lemma 1.8, for any $P \in \text{Pic}^0(X)$, the condition $h^1(\omega_Z \otimes P) = 0$ implies $h^i(\omega_Z \otimes P) = 0$ for all $1 \leq i \leq n - 1$. Since $|\chi(\mathcal{O}_Z)| = 1$, we conclude that $h^0(\omega_Z \otimes P) = 1$ for all $\mathcal{O}_X \neq P \in \text{Pic}^0(X)$. Thus condition ii) implies that $\tilde{Z}$ is a cohomological theta divisor.

Assume now that condition iii) is satisfied. Since $h^0(\omega_Z \otimes P) > 0$ for generic $P \in \text{Pic}^0(X)$, then $|\chi(\mathcal{O}_Z)| > 0$ and by lemma 3.1, $Z$ is of general type. We will now argue that condition ii) must also hold. Suppose therefore that there exists some irreducible component $T$ of $V^1 := \{ P \in \text{Pic}^0(X) \text{ s.t. } H^1(Z, \omega_Z \otimes P) \neq 0 \}$. By 2 or 4, $T$ is a subtourus of $\text{Pic}^0(X)$ of codimension at least 1. Let $P \neq \mathcal{O}_Z$ be a general point in $T$, let $\phi \in H^0(X, \Omega^n_X)$ be any holomorphic 1-form which is not the pullback of a 1-form on $S = T^*$. Consider the following complex of vector spaces:

$$
H^0(X, \Omega^n_{Z} \otimes P) \xrightarrow{\Delta \phi} H^0(X, \Omega^{n-i}_{Z} \otimes P) \xrightarrow{\Delta \phi} H^0(X, \Omega^{n-i}_{Z} \otimes P).
$$

By 2, this is exact for all $i \geq 1$ (notice that any component of $V^{1}$, $i \geq 2$ through the general point $P$ is also contained in $T$ and hence may be assumed to be either equal to $T$ or empty. We may of course also assume that $P$ is a general point of $V^n$, $i \geq 2$). Since $\chi$ is zero on exact sequences, and $|\chi(\mathcal{O}_X)| = 1$ it follows that $H^0(X, \Omega^n_{Z} \otimes P) \xrightarrow{\Delta \phi} 0$. We may choose local coordinates $\{ z_1, \ldots, z_{n-1} \}$ on $Z$, centered at a general point $z \in Z$. Choose $\phi_i \in H^0(X, \Omega^n_X)$ such that $dz_1, \ldots, dz_{n-1}$ correspond to the restriction of $\phi_i$ to $T^*_i(Z)^*$. Let $\eta \in H^0(X, \Omega^n_X)$ be a non zero holomorphic 1 form such that $\eta |_{T^*_i(Z)^*} = 0$. Since $Z$ is of general type, $Z$ is not vertical with respect to $\pi : X \rightarrow S$, and for general $z \in Z$, $\eta$ is not in $\pi^*(\text{Pic}^0(S, \Omega^n_S))$. We may further assume that the forms $\phi_i$ are not pulled back from $S$. (If this is not the case, then consider instead $\phi_i + \epsilon \eta$.) By a local computation, it follows that given any non zero $\omega \in H^0(X, \Omega^n_{Z}^{-2})$, there exists a $\phi_i$ such that $\omega \wedge \phi_i$ is not zero. Since this is a contradiction, $h^1(Z, \omega_Z \otimes P) = 0$ for all $\mathcal{O}_X \neq P \in \text{Pic}^0(X)$.

**QED**

4. Polarizations of type $(1, \ldots, 1, 2)$

**Theorem 4.1.** Let $(A, Z)$ be an $n$ dimensional abelian variety with a polarization of type $(1, \ldots, 1, 2)$ (i.e. $Z$ is the class of an ample line bundle with two sections), and let $D$ be any divisor in the linear series $[mZ]$. Then either

i) $(X, L)$ splits as the product of a PPAV and an elliptic curve, or

ii) $(A, (1/m)D)$ is log canonical.
Proof. Consider the exact sequence associated to the multiplier ideal sheaf $\mathcal{I} := \mathcal{I}((1 - (\epsilon/m))D)$ for an appropriate rational number $0 < \epsilon << 1$:

$$0 \longrightarrow L \otimes \mathcal{I} \longrightarrow L \longrightarrow L \otimes \mathcal{O}_X/\mathcal{I} \longrightarrow 0.$$ 

Since $h^i(L \otimes \mathcal{I} \otimes P) = 0$ for all $1 \leq i \leq n$ and all $P \in \text{Pic}^0(X)$, it follows that the above sequence is exact on global sections, and that $h^0(L \otimes \mathcal{I} \otimes P) = \chi(L \otimes \mathcal{I})$ for all $P \in \text{Pic}^0(X)$. Since $L$ is of type $(1,\ldots,1,2)$, we must have $0 \leq \chi(L \otimes \mathcal{I}) \leq 2$. If $\chi(L \otimes \mathcal{I}) = 0$ then $h^i(L \otimes \mathcal{I} \otimes P) = 0$ for all $0 \leq i \leq n$ and all $P \in \text{Pic}^0(X)$, and hence by a result of Mukai $L \otimes \mathcal{I} = 0$ (as sheaves) which is impossible. If $\chi(L \otimes \mathcal{I}) = 1$, then $L \otimes \mathcal{I}$ is a cohomological principal polarization and hence $L \otimes \mathcal{I}$ is a line bundle with only one section. This is again impossible, unless $\mathcal{I} \cong \mathcal{O}_X(-Z')$ where $Z'$ is a divisor of type $(0,\ldots,0,1)$. In this case there is a map $X \longrightarrow E$ of $X$ onto some elliptic curve $E$ such that $Z'$ is the pull back of a principal polarization on $E$. Let $L' := L \otimes \mathcal{O}_X(-Z')$. Then $(X, L')$ is a PPAV that splits as the product of $(E, \mathcal{O}_E(p))$ and the PPAV $(X, L)$. Finally if $\chi(L \otimes \mathcal{I}) = 2$, all sections of $L$ vanish along any translate of the cosupport of $\mathcal{I}$. This is only possible if the cosupport of $\mathcal{I}$ is empty, i.e. if $\mathcal{I} = \mathcal{O}_X$. QED

Remark 4.2. This statement is optimal in the sense that one may consider $X = E_1 \times E_2$, the product of two elliptic curves, endowed with the polarization of type $(1,2)$ given by $Z = p_1^*(p) + p_2^*(q_1 + q_2)$ where $p$ and $q_i$ are points of $E_1$ and of $E_2$ respectively. If the $q_i$ are distinct points, then $(X, Z)$ is log canonical but not log terminal. If $q_1 = q_2$, then $(X, Z)$ is not log canonical. One may ask whether for $D$ a reduced and irreducible divisor of type $(1,\ldots,1,2)$, the pair $(X, D)$ is log terminal. While this is not clear in higher dimensions, it does hold if $X$ is an abelian surface.

Theorem 4.3. Let $(X, D)$ be a polarized abelian surface with $D$ an ample reduced irreducible divisor such that $h^0(\mathcal{O}_A(D)) = 2$. Then the pair $(X, D)$ is log terminal.

Proof. As in the proof of proposition 2, one may consider an appropriate smooth birational model $f : \bar{D} \longrightarrow D$ and the induced exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(\bar{D}) \otimes \mathcal{I}(D) \longrightarrow f_* \omega_{\bar{D}} \longrightarrow 0.$$ 

Assume that $(X, D)$ is not log terminal, then the cosupport of $\mathcal{I} := \mathcal{I}(D)$ is not empty and has codimension two in $X$. Pick a general divisor $D' \in |D|$ which is distinct from $D$. These two divisors will intersect along a finite set of points. It follows that $h^0(\mathcal{O}_X(D) \otimes \mathcal{I} \otimes P) = 2$ only if the appropriate translate of the cosupport of $\mathcal{I}$ is contained in this finite set of points. (By [9] 10.1.2, $|D|$ has exactly four base points, corresponding to $D' \cap D''$ the intersection of two general members of $|D|$.) This means that $h^0(\mathcal{O}_X(D) \otimes \mathcal{I} \otimes P) \leq 1$ for all but finitely many $P \in \text{Pic}^0(X)$. For all topologically trivial line bundles $P / = \mathcal{O}_X$ there is an isomorphism $H^0(\mathcal{O}_X(D) \otimes \mathcal{I} \otimes P) \cong H^0(f_* \omega_{\bar{D}} \otimes P)$. Since $D$ is ample and hence of general type, $h^0(f_* \omega_{\bar{D}} \otimes P) > 0$ for all $P \in \text{Pic}^0(X)$. So $h^0(f_* \omega_{\bar{D}} \otimes P) = h^0(\mathcal{O}_X(D) \otimes \mathcal{I} \otimes P) = 1$ for all but finitely many $P \in \text{Pic}^0(X)$. Hence $g(\bar{D}) = 2$ and
\( h^0(f_*\omega_p \otimes P) = h^0(\mathcal{O}_X(D) \otimes \mathcal{I}(D) \otimes P) = 1 \) for all \( \mathcal{O}_X \neq P \in \text{Pic}^0(X) \). By corollary 3.5 the proposition now follows. QED

**Question 4.4.** Let \((X, D)\) be a general polarized abelian of type \((1, \ldots, 1, 2)\). Assume that \(D\) is irreducible. Is \((X, D)\) a log terminal pair? Similarly let \((X, D)\) be a general polarized abelian of type \((1, \ldots, 1, d)\), with \(d = 3, 4\). Assume that \(D\) is irreducible. Is \((X, D)\) a log canonical pair?

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