Anomaly cancellation in
\( D = 4, \mathcal{N} = 1 \) orientifolds
and linear/chiral multiplet duality

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ABSTRACT

It has been proposed that gauge and Kähler anomalies in four-dimensional type IIB orientifolds are cancelled by a generalized Green-Schwarz mechanism involving exchange of twisted RR-fields. We explain how this can be understood using the well-known duality between linear and chiral multiplets. We find that all the twisted fields associated to the \( \mathcal{N} = 1 \) sectors and some of the fields associated to the \( \mathcal{N} = 2 \) sectors reside in linear multiplets. But there are no linear multiplets associated to order-two twists. Only the linear multiplets contribute to anomaly cancellation. This suffices to cancel all \( U(1) \) anomalies. In the case of Kähler symmetries the complete \( SL(2, \mathbb{R}) \) can be restored at the quantum level for all planes that are not fixed by an order-two twist.

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1 Introduction

Anomaly freedom is one of the basic requirements for gauge theories. At first sight, a
gauge theory can only be consistent if the charged matter content is precisely such that
all anomalous one-loop diagrams vanish. However, in low-energy effective theories of string
theory there is another possibility. The anomalous one-loop diagram can be cancelled by an
additional interaction which in the effective field theory appears as the tree-level exchange
of a field coupling to the gauge fields in exactly the right manner. In the seminal work of
Green and Schwarz [1] it is shown how the hexagon anomalies in
\[ D = 10 \]
are cancelled by the exchange of the NSNS 2-form \( B \). In \( D = 4 \) \( B \) is dual to a scalar \( \phi \) which has axionic
couplings. The special interactions between \( \phi \) and the gauge fields indeed cancel the triangle
anomalies [2]. Only \( U(1) \) anomalies that are universal with respect to the different gauge

group factors \( G_a \) (i.e. the coefficient of the \( G_a^2 U(1) \) triangle diagram is independent of \( a \))
can be cancelled by this mechanism. Incidentally, all gauge anomalies in four-dimensional
heterotic vacua are of this form.

The situation is different in type I vacua where non-universal \( U(1) \) gauge anomalies
appear. As string theory is believed to be consistent, there must be additional interactions
that cancel these anomalies. In [3, 4] a generalized version of the Green-Schwarz mechanism
was proposed which cancels the \( U(1) \) anomalies in \( D = 6 \) via exchange of twisted RR fields
(explicit calculations for many models are performed in [5]). The authors of [3] applied this
idea to type IIB orientifold vacua in \( D = 4 \) and showed that all anomalies can be cancelled
this way. They also pointed out that anomalies of the invariance of the effective theory under
Kähler transformations can be cancelled by the same mechanism [7].

In the present article we explain how the anomalous transformation of the twisted RR
axions, which are responsible for anomaly cancellation, can be understood by considering
the Chern-Simons couplings of the RR 2-forms. In the next section we show that the RR
2-forms which appear in the twisted spectrum of type IIB orientifolds belong to \( \mathcal{N} = 1 \)
linear multiplets \( L^{(k)} \), where \( k \) labels the twisted sectors. We find that all sectors except
the \( k = N/2 \) sector (for even \( N \)) contain linear multiplets. The two crucial ingredients for
this generalized Green-Schwarz mechanism to work are a Chern-Simons modification of the
field strength associated to the RR 2-form and an additional term \( \int d^2 \theta d^2 \bar{\theta} \sum_k L^{(k)} \text{tr}(\gamma_k V) \)
in the \( D = 4 \) effective Lagrangian (\( V \) is the vector multiplet corresponding to the anomalous
\( U(1) \) and \( \gamma_k \) is the matrix representing the twist on the gauge indices). Both terms can be
derived from string theory by considering either “branes within branes” [11] or by using the
inflow mechanism [12] (for \( D = 6 \) the relevant couplings are calculated in [13]).

A linear multiplet \( L^{(k)} \) is dual to a chiral multiplet \( M^{(k)} \) in the sense that there exists an
equivalent description of the theory in terms of \( M^{(k)} \) [14, 10]. Translating the Chern-Simons
modification of the RR field strength and the \( L^{(k)} V \) term to the chiral basis, one finds that
the \( M^{(k)} \) couple to the gauge fields just in the right way to cancel the anomalies. This is
discussed in section 3.

In section 4 we apply this method to Kähler anomalies and argue that a coupling of the form
\[ \int d^2 \theta d^2 \bar{\theta} \sum_{i,k} \alpha^i_k L^{(k)} \ln(T_i + \bar{T}_i) \] must be present, where \( T_i \) is the chiral superfield which

\footnote{Some useful facts about linear multiplets are assembled in appendix A; see also [3, 9, 11].}
parameterizes the Kähler class of the $i$-th complex compact plane. This leads to a non-trivial transformation of the dual chiral multiplets of twisted states $M^{(k)}$ and cancels the Kähler anomalies. More precisely, only the Kähler symmetries corresponding to complex planes that are not fixed under an order-two twist can be restored at the quantum level by this mechanism. However, an order-two twist implies the existence of D5-branes. The coupling of the $T_3$ (we chose the third complex plane to be fixed under the order-two twist) to the D5-brane gauge fields explicitly breaks the Kähler symmetry for the third plane. Therefore one cannot expect an anomaly cancellation for this plane. In contrast to the situation in heterotic string vacua, the full $SL(2,\mathbb{R})$ symmetry seems to be preserved for fixed planes which are not fixed under an order-two twist.

In section 5 we give the low-energy effective Lagrangian describing the interactions of the moduli (dilaton, twisted and untwisted moduli) and the gauge fields. We determine the values of the Fayet-Iliopoulos terms and the masses of the (pseudo-)anomalous gauge bosons.

An appendix summarizes useful results of $\mathcal{N} = 1$ supersymmetry, concerning linear multiplets and D-terms of general chiral Lagrangians.

## 2 Massless twisted spectrum of type IIB orientifolds

In this article we consider compact $\mathbb{Z}_N$ orientifolds (see [13, 16] and references therein). They are obtained from the ten-dimensional type IIB theory by compactifying on a six-dimensional torus and projecting onto $\mathbb{Z}_N$- and $\Omega'$-invariant states. Here $\Omega' = \Omega J$, with $\Omega$ the world-sheet parity and $J$ an operator acting on the twisted sectors [17] as explained below. For consistency one has to add twisted closed strings and open strings. The compact dimensions form a six-torus $T^6 = \mathbb{R}^6/\Lambda$, where $\Lambda$ is a six-dimensional lattice chosen such that the generator $\theta$ of $\mathbb{Z}_N$ acts on it as an automorphism. Acting on $\Lambda$, $\theta$ is a $(6 \times 6)$ integer matrix. Its action on the coordinates of $T^6$ can be diagonalized over the complex numbers: $\theta = \text{diag}(e^{2\pi i v_1}, e^{2\pi i v_2}, e^{2\pi i v_3})$, where we grouped the coordinates of $T^6$ into three complex planes. The twist vector $v = (v_1, v_2, v_3)$ is chosen such that $0 < |v_i| < 1$ and $\sum_i v_i = 0$. In general there are different lattices $\Lambda$, on which a twist $\theta(v)$ acts as an automorphism. The orbifold is defined to be the space $T^6/\mathbb{Z}_N$, whereas the orientifold includes an additional projection of the strings moving on the orbifold onto $\Omega'$-invariant states. All possible $\mathbb{Z}_N$-twists $\theta$ and lattices $\Lambda$ leading to $D = 4$, $\mathcal{N} = 1$ orbifolds have been classified in [18]. In [16] it was found that of the possible $\mathbb{Z}_N$ models only the ones shown in table 1 lead to consistent type IIB orientifolds (i.e. the tadpoles can be cancelled).

The massless spectrum of these models has been determined in [16]. However in the twisted sector only the number of states is given. In addition, for each twist $\theta(v)$ only one possible lattice (implicitly assumed as factorizable) is considered. As we are interested in the precise $\mathcal{N} = 1$ multiplets containing the twisted states, we rederive the spectrum by considering the cohomology of the compact orbifold space [18, 19, 20]. This method was used in [21, 22] to obtain the massless spectrum of twisted states of orientifold models in $D = 6$ and $D = 4$. The orbifold cohomology is encoded in the Hodge numbers $h^{p,q} = \dim H^{p,q}$, where $H^{p,q}$ is the space of harmonic $(p,q)$-forms. For six-dimensional compact orbifolds (i.e.
From the remaining harmonic forms one finds \( \phi \) \( h \). From the latter two fields (together with the corresponding fermion) form a n-dimensional subspace \( M \) of the complex structure.

More generally, for twists \( k \) \( (with two exceptions, explained below, where \( k = \pm 1 \)) \( \theta \)-invariant \((1,1)\)-forms and \((2,1)\)-forms of the six-torus. There are additional contributions \( h_{tw}^{1,1} \) and \( h_{tw}^{2,1} \) from the twisted sector.

The bosonic fields of type IIB theory in \( D = 10 \) are the dilaton \( \varphi \), the metric \( g \) and a 2-form \( B \) from the NSNS sector and a scalar \( C_{(0)} \), a 2-form \( C_{(2)} \) and a 4-form \( C_{(4)} \) from the RR sector. Under the world-sheet parity \( \varphi \), \( g \) and \( C_{(2)} \) are even, the other fields being odd. In the untwisted sector \( \Omega' = \Omega \) and therefore the bosonic fields in \( D = 4 \) are obtained by contracting the Lorentz indices of \( \varphi \), \( g \) and \( C_{(2)} \) with the harmonic forms of the orbifold. From \( h^{0,0}, h^{3,0}, h^{0,3}, h^{3,3} \) one gets the \( D = 4 \) graviton, dilaton and antisymmetric tensor. The latter two fields (together with the corresponding fermion) form an \( N = 1 \) linear multiplet. From the remaining harmonic forms one finds \( h^{1,1} \) chiral multiplets \( T_i \) corresponding to deformations of the Kähler class and \( h^{2,1} \) chiral multiplets \( U_i \) corresponding to deformations of the complex structure.

Let us now consider the twisted sectors. For the \( k \)-th twisted sector there is a singular subspace \( \mathcal{M}_k \) which is fixed under the action of \( \theta^k \). The cohomology \( H_{k-twisted}^{p,q} \) of this space contributes to \( H^{p+n_k,q+n_k} \) of the orbifold, where \( n_k = \sum_i k \cdot v_i \) and \( k \cdot v_i = k \cdot v_i \mod \mathbb{Z} \), such that \( 0 \leq k \cdot v_i < 1 \). The spaces \( \mathcal{M}_1 \) and \( \mathcal{M}_{N-1} \) consist of the set of fixed points of the orbifold. More generally, for twists \( \theta^k \), such that all \( k \cdot v_i \neq 0 \), the spaces \( \mathcal{M}_k \) and \( \mathcal{M}_{N-k} \) consist of the set of fixed points under the action of \( \theta^k \). For such \( k < N/2 \) one finds \( n_k = 1 \), \( n_{N-k} = 2 \) (with two exceptions, explained below, where \( n_k = 2 \), \( n_{N-k} = 1 \)). If \( k \cdot v_i = 0 \) for some \( i \), then the \( i \)-th complex plane is fixed under \( \theta^k \). In this case the spaces \( \mathcal{M}_k \) and \( \mathcal{M}_{N-k} \) consist of the set of fixed planes.

For such \( k \) one finds \( n_k = n_{N-k} = 1 \). Denote the number of \( \theta^k \) fixed points by \( f_k \) (if \( k \cdot v_i \neq 0 \)). If \( k \cdot v_i = 0 \) for some \( i \) let \( f'_k \) be the number of \( \theta^k \) fixed planes. On the other hand we define \( f_k^{(i)} \) to be the number of \( \theta^k \) fixed points of the four-dimensional space consisting only of the two rotated planes (i.e. the compact space without the \( i \)-th plane).

If the lattice \( L \) splits into a direct sum (over the integers) of sublattices, \( L = I \oplus J \), such that \( I \) is fixed under \( \theta^k \) and this block structure is preserved under \( \theta^k \), then \( f'_k \) is just given by \( f_k^{(i)} \). As noted by the authors of [18], this condition is not always satisfied, leading (in some cases) to a smaller value of \( f'_k \). However, we will restrict ourselves to lattices satisfying this condition. These are the ones that were discussed in [16] [17].

Write the contribution of the twisted sectors to the cohomology as \( h_{tw}^{p,q} = \sum_{k=1}^{N-1} h_k^{p,q} \). If

| \( Z_3 \) | \( \frac{1}{2} (1, 1, -2) \) | \( Z_6 \) | \( \frac{1}{2} (1, 1, -2) \) |
| \( Z_7 \) | \( \frac{1}{2} (1, 2, -3) \) | \( Z_6' \) | \( \frac{1}{2} (1, -3, 2) \) |
| \( Z_{12} \) | \( \frac{1}{12} (1, -5, 4) \) |

Table 1: \( Z_N \) actions in \( D = 4 \).
the greatest common divisor of \( k \) and \( N \), \( \gcd(k, N) \), is a prime number and for \( k < N/2 \) such that all \( ^k v_i \neq 0 \), we find

\[
h_{k}^{1,1} = h_{N-k}^{2,2} = f_1 + \frac{f_k - f_1}{\gcd(k, N)}, \quad h_k^{p,q} = 0 \text{ for all other } (p, q). \tag{2.2}\]

If \( \gcd(k, N) = pq \), with \( p, q \) prime numbers (which is only possible for \( N = 12 \)), this is modified to

\[
h_{k}^{1,1} = h_{N-k}^{2,2} = f_1 + \frac{f_p - f_1}{p} + \frac{f_q - f_1}{q} + \frac{f_k - f_1 - (f_p - f_1) - (f_q - f_1)}{pq}. \tag{2.3}\]

Here we used the fact that on a point one can only define a \((0,0)\)-form. If \( \gcd(k, N) \neq 1 \), then only \( f_1 \) of the \( f_i \) \( \theta^k \) fixed points are invariant under \( \mathbb{Z}_N \). The remaining \( (f_k - f_1) \) fixed points transform under a \( \mathbb{Z}_{\gcd(k,N)} \) (resp. \( \mathbb{Z}_p \) or \( \mathbb{Z}_q \) if \( \gcd(k, N) = pq \)) subgroup of \( \mathbb{Z}_N \). One can however form linear combinations of \( \gcd(k, N) \) fixed points that are invariant under the whole \( \mathbb{Z}_N \). Note that \( h_{k}^{1,1} = 0 \) for \( k > N/2 \) and \( h_{k}^{2,2} = 0 \) for \( k < N/2 \), with two exceptions. This means that for \( k < N/2 \) the \( h_{N-k}^{2,2} \) forms furnish the antiparticles corresponding to the particles from the \( h_{k}^{1,1} \) forms. The two exceptions are the \( k = 3 \) sector of \( \mathbb{Z}_7 \) and the \( k = 5 \) sector of \( \mathbb{Z}_{12} \), where the particles come from the \((N - k)\)-th sector (i.e. \( h_{N-k}^{1,1} \neq 0 \)) and the antiparticles from \( h_{k}^{2,2} \).

In the case of fixed planes one has to consider the torus cohomology which is \( h^{0,0} = h^{1,0} = h^{0,1} = h^{1,1} = 1 \). The \((0,0)\)-form and the \((1,1)\)-form are \( \mathbb{Z}_N \)-invariant but the \((1,0)\)-form and the \((0,1)\)-form generically transform under a \( \mathbb{Z}_q \) subgroup of \( \mathbb{Z}_N \). If \( ^k v_i = 0 \), then \( f_1^{(i)} \) is the number of tori that are fixed under the whole \( \mathbb{Z}_N \). Again one can form linear combinations of the \( \mathrm{the} \) \( \theta^k \) fixed tori and of the forms defined on them that are invariant under the whole \( \mathbb{Z}_N \). In total, one finds for \( \gcd(k, N) = pq \) (if \( \gcd(k, N) = \text{prime, set } q = 1 \))

\[
h_{k}^{1,1} = h_{k}^{2,2} = f_1^{(i)} + \frac{f_p^{(i)} - f_1^{(i)}}{p} + \frac{f_q^{(i)} - f_1^{(i)}}{q} + \frac{f_k^{(i)} - f_1^{(i)} - (f_p^{(i)} - f_1^{(i)}) - (f_q^{(i)} - f_1^{(i)})}{pq}, \tag{2.4}\]

\[
h_{k}^{2,1} = h_{k}^{1,2} = h_{k}^{1,1} - f_1^{(i)}, \quad h_{k}^{p,q} = 0 \text{ for all other } (p, q).
\]

For the orbifolds of table 1 we find the twisted Hodge numbers listed in table 2. (For completeness we added the untwisted Hodge numbers in the last column.)

The twisted bosonic fields of the \( D = 4 \) theory are obtained by contracting the bosonic fields in \( D = 10 \) with the additional harmonic forms from the twisted sectors. Now the operator \( J \) in \( \Omega' = \Omega J \) is important because it exchanges the sector twisted by \( \theta^k \) with the one twisted by \( \theta^{N-k} \). To get an \( \Omega' \)-invariant result, one has to contract the \( \Omega \)-even fields \( g \) and \( C_{(2)} \) with the \( J \)-even linear combinations of harmonic forms from the \( k \)-th and \( (N - k) \)-th twisted sector and the \( \Omega \)-odd fields \( B \) and \( C_{(4)} \) with \( J \)-odd linear combinations of harmonic forms.

From the twisted sectors with no fixed planes one finds \( h_{k}^{1,1} \) scalars in \( D = 4 \) from the \( J \)-even sector and the same number of antisymmetric tensors from the \( J \)-odd sector. Together
with their fermionic partners they form $h^{1,1}_k \mathcal{N} = 1$ linear multiplets. If there are fixed planes, one has $2(h^{1,1}_k+h^{2,1}_k)$ scalars from the $J$-even sector and $h^{1,1}_k$ scalars, $h^{1,1}_k$ antisymmetric tensors and $h^{2,1}_k$ vectors from the $J$-odd sector. Together with the corresponding fermions these fields form $(h^{1,1}_k+h^{2,1}_k)$ chiral multiplets, $h^{1,1}_k$ linear multiplets and $h^{2,1}_k$ vector multiplets if $k \neq N/2$. These fit into $h^{2,1}_k \mathcal{N} = 2$ hyper multiplets (consisting of two $\mathcal{N} = 1$ chiral multiplets), $h^{2,1}_k \mathcal{N} = 2$ vector-tensor multiplets [23] (consisting of an $\mathcal{N} = 1$ vector and an $\mathcal{N} = 1$ linear multiplet) and $(h^{1,1}_k-h^{2,1}_k) \mathcal{N} = 2$ linear hyper multiplets (consisting of an $\mathcal{N} = 1$ chiral and an $\mathcal{N} = 1$ linear multiplet).\footnote{The spectrum of the $k$-th twisted sector with $k \nu_j = 0$ can also be understood by compactifying on a four-dimensional $\mathbb{Z}_m$ orientifold [13-21], with $m = N/\text{gcd}(k,N)$, and then dimensionally reducing on a two-torus. For the $k'$-th twisted sector of the $\mathbb{Z}_m$ orientifold the authors find $f^{i(j)}_k+(f^{i(j)}_{kk'}-f^{i(j)}_k)/\text{gcd}(k',m)$ $D = 6$, $\mathcal{N} = 1$ hypers and, if $k' \neq m/2$, the same number of additional tensors (a $k' \neq m/2$ sector exists if $m \neq 2$, i.e. $k \neq N/2$). From these, $f^{i(j)}_1(h^{1,1}_k-h^{2,1}_k)$ correspond to $\mathbb{Z}_N$ fixed points in the four-dimensional compact space. The remaining multiplets can be grouped into $h^{2,1}_k$ different $\mathbb{Z}_N$-invariant linear combinations. From each of the $f^{i(j)}_1$ $D = 6$, $\mathcal{N} = 1$ hypers and tensors only half the states are $\mathbb{Z}_N$-invariant (in terms of $D = 4$, $\mathcal{N} = 1$: a linear and a chiral multiplet). The $h^{2,1}_k$ linear combinations of twisted states reduce to $N = 2$ hypers and vector-tensors in $D = 4$. (The latter are equivalent to $\mathcal{N} = 2$ vectors because an antisymmetric tensor is dual to a scalar in $D = 4$.)} For $k = N/2$ there are no $J$-odd linear combination of harmonic forms and therefore in this case only the $(h^{1,1}_N+h^{2,1}_{N/2})$ chiral multiplets appear in the $D = 4$ twisted spectrum.

Adding the contributions from the twisted sectors $k = 1, \ldots, [N/2]$ (because of the orientifold projection the sectors $k > N/2$ give no independent degrees of freedom) we find the following twisted fields, table 3.

| $\mathbb{Z}_3$ | $h^{1,1}_1 = 27$, $h^{1,1}_2 = 0$ | $h^{1,1}_{tw} = 27$ | $h^{1,1}_1 = 9$ |
|----------------|---------------------------------|------------------|------------------|
| $h^{1,1}_1 = 3$, $h^{1,1}_2 = 3+12$, $h^{1,1}_3 = 1+5$, $h^{1,1}_{4/5} = 0$ | $h^{1,1}_{tw} = 24$ | $h^{1,1}_1 = 5$ |
| $h^{2,1}_3 = 5$, $h^{2,1}_{1/2/4/5} = 0$ | $h^{2,1}_{tw} = 5$ | $h^{2,1}_1 = 1$ |
| $h^{1,1}_1 = 12$, $h^{1,1}_2 = h^{1,1}_4 = 3+3$, $h^{1,1}_3 = 4+4$, $h^{1,1}_5 = 0$ | $h^{1,1}_{tw} = 32$ | $h^{1,1}_1 = 3$ |
| $h^{2,1}_2 = h^{2,1}_4 = 3$, $h^{2,1}_3 = 4$, $h^{2,1}_{1/5} = 0$ | $h^{2,1}_{tw} = 10$ | $h^{2,1}_1 = 1$ |
| $h^{1,1}_1 = 7$, $h^{1,1}_2 = 7$, $h^{1,1}_4 = 7$, $h^{1,1}_{3/5/6} = 0$ | $h^{1,1}_{tw} = 21$ | $h^{1,1}_1 = 3$ |
| $h^{1,1}_1 = 3$, $h^{1,1}_2 = 3$, $h^{1,1}_3 = h^{1,1}_9 = 1+1$, $h^{1,1}_4 = 3+6$ | $h^{1,1}_{tw} = 26$ | $h^{1,1}_1 = 3$ |
| $h^{2,1}_6 = 1+1+2$, $h^{1,1}_7 = 3$, $h^{1,1}_{8/10/11} = 0$ | $h^{2,1}_{tw} = 5$ | $h^{2,1}_1 = 1$ |
| $h^{2,1}_9 = 1$, $h^{2,1}_6 = 1+2$, $h^{2,1}_{1/2/4/5/7/8/10/11} = 0$ | |

Table 2: Twisted (and untwisted) Hodge numbers of $\mathbb{Z}_N$ orbifolds.
|                | sectors without fixed planes | sectors with fixed planes |
|----------------|-----------------------------|---------------------------|
| $\mathbb{Z}_3$ | 27 lin. mult.               | –                         |
| $\mathbb{Z}_6$ | 18 lin. mult.               | 11 chir. mult.            |
| $\mathbb{Z}_6'$| 12 lin. mult.               | 21 chir. mult., 6 lin. mult., 3 vector mult. |
| $\mathbb{Z}_7$ | 21 lin. mult.               | –                         |
| $\mathbb{Z}_{12}$ | 18 lin. mult.               | 10 chir. mult., 2 lin. mult., 1 vector mult. |

Table 3: Twisted fields ($\mathcal{N} = 1$ multiplets) of $\mathbb{Z}_N$ orientifolds.

We will see that only $\mathcal{N} = 1$ linear multiplets can contribute to anomaly cancellation.

3 Gauge anomaly cancellation

It is well known that the $D = 4$ version of the Green-Schwarz anomaly cancellation mechanism [2] is closely related to the equivalence between the linear and the chiral $\mathcal{N} = 1$ supermultiplets. More precisely, in the heterotic string the transformation of the chiral dilaton superfield that cancels the anomaly can be understood by starting with the linear dilaton multiplet that appears in the string spectrum and then translating the Lagrangian carefully to the description in terms of a chiral field (see e.g. [24, 25, 26]). In the present article we apply this method to the anomaly cancellation mechanism in $\mathcal{N} = 1$, $D = 4$ type IIB orientifolds [6, 7]. The new features are that several linear multiplets are involved and that their vacuum expectation value is not related to the string loop expansion but rather to the blowing-up of the orbifold singularities.

3.1 Purely bosonic case

We start by reviewing the $D = 4$ Green-Schwarz mechanism as it can be derived from the equivalence between an antisymmetric tensor $B_{\mu\nu}$ and a scalar $\phi$ without making any use of supersymmetry. Start from an auxiliary Lagrangian describing the interactions of a non-dynamical field $Y^\mu$ with the antisymmetric tensor and several gauge fields:

$$\mathcal{L}' = -\frac{2}{b_1} Y^\mu Y_\mu + Y^\mu \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} + 2b_2^{(a)} (Y^\mu + b_3 A^{(0)}_\mu) \bar{Q}^{(a)\mu}$$

$$+ b_3 F^{(0)}_{\mu\nu} B^{\mu\nu} - \frac{1}{4g^2_{(a)}} \text{tr}(F^{(a)}_{\mu\nu} F^{(a)\mu\nu}).$$

(3.1)

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3In this and in the next subsection we follow the line of reasoning of [10].
Here $b_1$, $b_2^{(a)}$, $b_3$ are dimensionful coupling constants. The gauge fields $A^{(a)}_\mu$ correspond to different gauge group factors $G_a$ (which can be Abelian or non-Abelian); a sum over $a$ is understood. We choose $G$ the Chern-Simons 3-form associated to $G$

$$Q^{(a)}_{\mu\nu\rho} = \text{tr} \left( A^{(a)}_{\mu} \partial_\nu A^{(a)}_\rho - \frac{2i}{3} A^{(a)}_{[\mu} A^{(a)}_{\nu]} A^{(a)}_\rho \right). \quad (3.2)$$

A tilde denotes the Poincaré dual

$$\tilde{F}^{(a)\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^{(a)}_{\rho\sigma}, \quad \tilde{Q}^{(a)\mu} = \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} Q^{(a)}_{\nu\rho\sigma}, \quad \tilde{H}^{\mu} = \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}, \quad (3.3)$$

where $H$ is the modified antisymmetric tensor field strength

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} + b_2^{(a)} Q^{(a)\mu\nu\rho}. \quad (3.4)$$

Useful identities are

$$\partial_\mu \tilde{Q}^{(a)\mu} = \frac{1}{2} \text{tr}(F^{(a)}_{\mu\nu} \tilde{F}^{(a)\mu\nu}), \quad (3.5)$$

$$\tilde{H}^{\mu} \tilde{H}_{\mu} = -\frac{1}{3!} H^{\mu\nu\rho} H_{\mu\nu\rho}. \quad (3.6)$$

The Lagrangian \((3.1)\) contains no kinetic terms for $Y$ and $B$. Therefore these fields can be integrated out using the equations of motion. By varying with respect to $Y$ or $B$ two equivalent ("dual") descriptions are obtained from $\mathcal{L}'$.

1. From $\delta_Y \mathcal{L}' = 0$ one finds

$$Y^\mu = \frac{b_1}{4} \left( \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} + 2 b_2^{(a)} \tilde{Q}^{(a)\mu} \right) = \frac{1}{2} b_1 \tilde{H}^{\mu}. \quad (3.7)$$

Inserting this into \((3.1)\) gives

$$\mathcal{L}^B = \frac{1}{2} b_1 \tilde{H}^{\mu} \tilde{H}_\mu + b_3 \tilde{F}^{(0)\mu\nu} B^{\mu\nu} + 2 b_2^{(a)} \tilde{Q}^{(a)\mu} A^{(0)\mu} - \frac{1}{4 g_{(a)}} \text{tr}(F^{(a)}_{\mu\nu} F^{(a)\mu\nu}), \quad (3.8)$$

with $H = dB + b_2^{(a)} Q^{(a)}$. This is the usual Lagrangian describing an antisymmetric tensor coupled to gauge fields via Chern-Simons terms plus two additional terms proportional to $b_3$. They are crucial to cancel the $U(1)_X$ anomalies.

2. If instead one requires $\delta_B \mathcal{L}' = 0$, one finds $\epsilon^{\mu\nu\rho\sigma} \left( -\partial_\nu Y^\mu + b_3 \partial_\nu A^{(0)\mu} \right) = 0$, which is solved by

$$Y^\mu = \frac{1}{2} \partial_\mu \phi - b_3 A^{(0)\mu}, \quad (3.9)$$

where $\phi$ is an arbitrary function. Inserting this into \((3.1)\) gives after some simple algebra

$$\mathcal{L}^\phi = -\frac{1}{2 b_3} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} \text{tr}(F^{(a)}_{\mu\nu} F^{(a)\mu\nu}) - \frac{1}{4 g_{(a)}} \text{tr}(F^{(a)}_{\mu\nu} F^{(a)\mu\nu}) - \frac{1}{2 b_2^{(a)}} \phi \text{tr}(F^{(a)}_{\mu\nu} \tilde{F}^{(a)\mu\nu}). \quad (3.10)$$

The theory described by this Lagrangian is equivalent to the one described by \((3.8)\), and in this sense the scalar $\phi$ is dual to the antisymmetric tensor $B_{\mu\nu}$. The normalization for
\( \phi \) in (3.9) is chosen such that a canonically normalized kinetic term for \( B_{\mu \nu} \) in (3.8), i.e. \( b_1 = 1 \), implies a canonically normalized kinetic term for \( \phi \) in (3.10). In our conventions \( B_{\mu \nu} \) is dimensionless, \( A_\mu \) and \( H_{\mu \rho \sigma} \) have dimension of mass, \( F_{\mu \nu} \) and \( \phi \) have dimension of mass squared, \( Q_{\mu \nu \rho} \) and \( Y_\mu \) have dimension of mass cubed. This implies that the coefficients \( b_1 \) and \( b_3 \) have dimension of mass squared and \( b_2 \) has dimension of inverse mass squared.

The first term in (3.10) tells us that \( \phi \) transforms non-trivially under \( U(1)_X \) gauge transformations:

\[
\delta_\epsilon A_\mu^{(0)} = \partial_\mu \epsilon \quad \Rightarrow \quad \delta_\epsilon \phi = 2b_3 \epsilon. \tag{3.11}
\]

One observes that the presence of the Chern-Simons term in the antisymmetric tensor field strength \( H = dB + b_2^{(a)} Q^{(a)} \) in (3.8) leads to the \( \phi \) \( \text{tr}(F\tilde{F}) \) term in (3.10), and the \( b_3 \tilde{F}^{(0)} B \) term in (3.8) leads to the transformation law (3.11).

The variation of \( \mathcal{L}_\phi \) under a \( U(1)_X \) transformation is given by

\[
\delta_\epsilon \mathcal{L}_\phi = \left( \frac{A^{(a)}}{32\pi^2} \epsilon - \frac{1}{2} b_2^{(a)} \delta_\epsilon \phi \right) \text{tr}(F^{(a)}\tilde{F}^{(a)\mu\nu}). \tag{3.12}
\]

Here \( A^{(a)} \) is the coefficient of the \( (G_a)^2 U(1)_X \) triangle diagram, \( A^{(a)} = \sum_R q_R T(R) \), where the sum goes over all fields transforming in representations \( R \) under \( G^{(a)} \) and carrying \( U(1)_X \) charge \( q_R \). The generators \( \lambda^i_{(a)} \) of \( G_a \) are normalized to \( \text{tr}(\lambda^i_{(a)}\lambda^j_{(a)}) = \delta^{ij} \).

Using (3.11) one sees that all \( U(1)_X \) anomalies are cancelled if

\[
\frac{A^{(a)}}{32\pi^2} = b_2^{(a)} b_3. \tag{3.13}
\]

In the heterotic string in \( D = 10 \) the Chern-Simons modification of the antisymmetric tensor field strength is fixed by the supergravity algebra [27] and leads to an almost universal coefficient \( b_2^{(a)} = -k^{(a)} \alpha' / 4 \) in \( D = 4 \), depending only on the Kac-Moody level \( k^{(a)} \). The explicit string calculation [28] shows that at one loop the terms proportional to \( b_3 \) in (3.8) are indeed generated and that the coefficient is \( b_3 = -A^{(a)}/(k^{(a)} 8\pi^2 \alpha') \), independently of \( a \), which is just the right value to cancel the anomalies.

For the case of type IIB orientifolds we need to generalize the above formulae to include several antisymmetric tensors \( B_{\mu \nu}^{(k,f)} \), where \( k \) labels the different twisted sectors \( k = 1, \ldots, \left\lceil \frac{N-1}{2} \right\rceil \) and \( f = 1, \ldots, f_k \) counts the fixed points\(^4\) of each twisted sector (in general not all of these fields are independent). The \( k = N/2 \) sector (for even \( N \)) is excluded, because we saw in the previous section that it contains no antisymmetric tensors. Only certain linear combinations of the antisymmetric tensors are involved in the anomaly cancellation mechanism. We define

\[
B_{\mu \nu}^{(k)} = \frac{1}{\sqrt{f_k}} \sum_f B_{\mu \nu}^{(k,f)}. \tag{3.14}
\]

\(^4\)Here we restrict ourselves to the the antisymmetric tensors from the twisted sectors with no fixed planes. We will see that only in one case, namely the second twisted sector of \( Z_6' \), fields associated to fixed planes contribute to anomaly cancellation.
From the study of RR forms and their coupling to D-branes it has been found that in the presence of gauge fields on the D-branes the RR field strengths are modified by Chern-Simons terms (see e.g. \[11,13\]). In our case of twisted RR 2-forms we have

\[ H^{(k,f)}_{\mu\nu\rho} = \partial_{[\mu} B^{(k,f)}_{\nu]_{\rho]} + \epsilon_k c_2 \sum_a Q^{(k,a)}_{\mu\nu\rho} \Rightarrow H^{(k)}_{\mu\nu\rho} = \partial_{[\mu} B^{(k)}_{\nu]_{\rho]} + \epsilon_k c_2 \sum_a \sqrt{f_k} Q^{(k,a)}_{\mu\nu\rho}, \]  

(3.15)

where \(c_2\) is a normalization factor of mass dimension \(-2\) and \(\epsilon_k\) is a sign, that will be determined below. The Chern-Simons form now depends on the twist matrix \(\gamma_k\) that represents the action of \(\theta^k\) on the gauge indices,

\[ Q^{(k,a)}_{\mu\nu\rho} = \text{tr} \left( \gamma_k \left( A^{(a)}_{[\mu} \partial_{\nu] A^{(a)}_{\rho]} - \frac{2i}{3} A^{(a)}_{[\mu} A^{(a)}_{\nu]} A^{(a)}_{\rho]} \right) \right), \]  

(3.16)

The study of RR forms also shows \[11,13\] that terms of the form \(\text{tr}(\gamma_k \exp(iF)) \wedge C\) appear in the effective action, where \(F\) is the gauge field strength 2-form, \(C\) is the sum over RR forms of different degrees and it is understood that after expansion of the exponential only the terms with the correct total form degree are kept. For us of interest is the term coupling the RR 2-form to the (pseudo-)anomalous \(U(1)_X\) \[1\]:

\[ c_3 \text{tr}(\gamma_k i \tilde{F}^{(0)}_{\mu\nu}) B^{(k)\mu\nu}, \]  

(3.17)

where \(c_3\) is a normalization factor of mass dimension \(2\).

In the conventions of \[13\] the \(\gamma_k\)'s are parametrized by a 16-dimensional shift vector \(V\) and a 16-dimensional vector \(H\) containing the Cartan generators of \(SO(32)\): \(\gamma_k = \exp(-2\pi i k H \cdot V)\). The trace over the product of \(\gamma_k\) with the gauge group generators \(\lambda^{(a)}\) is easily calculated to give \[13\]: \(\text{tr}(\gamma_k \lambda^{(a)}_{(a)}) = \cos(2\pi k V_a)\) and \(\text{tr}(\gamma_k i \lambda^{(0)}) = 2 n_X \sin(2\pi k V_X)\). Here \(V_a\) is the component of the shift vector \(V\) corresponding to the gauge group \(G_a\) and \(n_X\) is the rank of the non-Abelian group in which \(U(1)_X\) is embedded. Comparison with (3.8) where we normalized the traces to one leads to the identification

\[ b^{(k,a)}_2 = \epsilon_k c_2 \sqrt{f_k} \cos(2\pi k V_a), \quad b^{(k)}_3 = 2 n_X c_3 \sin(2\pi k V_X). \]  

(3.18)

In direct generalization of the equivalence between the Lagrangians (3.8) and (3.10), we find that the effective theory describing the RR 2-forms,

\[ \mathcal{L}^B = \sum_k \left( \frac{1}{2} b^{(k)}_2 \tilde{F}^{(k)}_{\mu\nu} \tilde{F}^{(k)\mu\nu} + b^{(k)}_3 \tilde{F}^{(0)}_{\mu\nu} B^{(k)\mu\nu} + 2 b^{(k)}_3 b^{(k,0)} - \frac{1}{4} (Q^{(a)} F^{(a)\mu\nu}) \right) \]  

(3.19)

where \(Q^{(a)}\) (in contrast to \(Q^{(k,a)}\) in (3.15)) is defined and normalized as in (3.2), can equivalently be described in terms of scalars \(\phi^{(k)}\),

\[ \mathcal{L}^\phi = - \frac{1}{2 b^{(k)}_3} \sum_k \left( \partial_{\mu} \phi^{(k)} - 2 b^{(k)}_3 A^{(0)}_{\mu} \right) - \frac{1}{4 b^{(k,0)}_2} \text{tr}(F^{(a)} F^{(a)\mu\nu}) \]  

(3.20)

\[ \text{For simplicity we assume that only one } U(1) \text{ factor is anomalous. The generalization to several anomalous } U(1)'s \text{ is straightforward.} \]
Here and in the following the sum over the twisted sectors runs over \( k = 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \).

It is easy to see that all \( U(1)_X \) anomalies are cancelled if

\[
\mathcal{A}^{(a)} = \frac{A}{32\pi^2} = \sum_k b^{(k,a)}_2 b^{(k)}_3.
\]  

(3.21)

For \( D = 4, \mathcal{N} = 1 \) type IIB orientifolds the anomaly coefficient \( \mathcal{A}^{(a)} \) introduced above can be calculated as a function of the shift vector \( V \) and is given by

\[
\mathcal{A}^{(a)} = \frac{-4}{N} \sum_k C_k n_X \sin(2\pi k V_X) \cos(2\pi k V_a),
\]  

(3.22)

where \( C_k = \prod_{i=1}^3 2 \sin(\pi k v_i) \) if the gauge groups \( G_a \) and \( U(1)_X \) live either both on the D9-branes or both on the D5-branes \(^3\). In this case one has \((C_k)^2 = f_k\) if the \( k\)-th sector has no fixed planes and \( C_k = 0\) else. An explicit calculation of the various \( C_k^\prime\)’s for the considered models shows that \( C_k = -\sqrt{f_k} \) for the sectors \( k < N/2 \) of all models, except the \( k = 3 \) sector of \( \mathbb{Z}_7 \) and the \( k = 5 \) sector of \( \mathbb{Z}_{12} \). This is related to the fact, that these two sectors furnish antiparticles, whereas in all other cases the antiparticles come from \( k > N/2 \), as we saw in the previous section. Let us define \( \epsilon_k = 1 \) for the third sector of \( \mathbb{Z}_7 \) and the fifth sector of \( \mathbb{Z}_{12} \) and \( \epsilon_k = -1 \) for all other sectors. Then we have

\[
C_k = \epsilon_k \sqrt{f_k}.
\]  

(3.23)

If \( G_a \) and \( U(1)_X \) live on different types of branes (this is called the 95-sector), then \( C_k = 2 \sin(\pi k v_3) \), where it was assumed that the D5-branes are extended in the four-dimensional space-time and in the third complex plane and are at the origin in the first and second complex planes. It turns out that the only contribution to \( \mathcal{A}^{(a)} \) from twisted sectors with fixed planes is from the 95-sector and is only in the \( k = 2 \) sector of \( \mathbb{Z}_6^\prime \) non-vanishing. In the previous section we saw that there are six antisymmetric tensors from this sector associated to the second plane. But only three of them can couple to the D5-branes because they are associated to fixed planes located at the origin in the first complex direction. In analogy to (3.14) we therefore define \( B^{(2)}_{\mu\nu} = (1/\sqrt{3}) \sum_{f=1}^3 B^{(2,f)}_{\mu\nu} \) (and \( \epsilon_2 = 1 \)). In the \( k = 3 \) sector of \( \mathbb{Z}_{12} \) there are two more antisymmetric tensors, associated to the third complex plane. They do not seem to couple to the D5-branes. From (3.18) one therefore finds that all gauge anomalies are cancelled if

\[
c_2 c_3 = -\frac{1}{16\pi^2 N}.
\]  

(3.24)

It seems difficult to obtain the separate values of the coefficients \( c_2 \) and \( c_3 \) because, under a field redefinition \( B^{(k)}_{\mu\nu} \rightarrow \alpha B^{(k)}_{\mu\nu} \), these coefficients scale as \( c_2 \rightarrow \alpha c_2 \) and \( c_3 \rightarrow c_3/\alpha \). If one chooses a normalization such that the term in front of \( \frac{1}{4} tr(F F) \) in (3.20) is

\[
\frac{\alpha'}{N} \sum_{k,f} \phi^{(k,f)} \epsilon_k tr(\gamma_k \lambda^2 (a)) = \frac{\alpha'}{N} \sum_k \phi^{(k)} \epsilon_k \sqrt{f_k} \cos(2\pi k V_a),
\]  

(3.25)

then \( c_2 = \alpha'/2N \). The string tension \( \alpha' \) is inferred from dimensional analysis (recall that \( \phi \) has dimension of mass squared). A careful determination of the coefficient appearing in (3.17) should then yield \( c_3 = -1/(8\pi^2 \alpha') \) to satisfy the condition (3.24).

\(^6\)A factor 2 compared to the result of \(^4\) is due to our choice of normalization of gauge group generators, \( tr(\lambda \lambda') = 0^3 \).
3.2 Supersymmetric case

The supersymmetric generalization of the auxiliary Lagrangian (3.1) is

\[ \mathcal{L}' = \int d^2 \theta d^2 \bar{\theta} \left( \frac{1}{4b_1} Y^2 + Y L + 4b_3 V^{(0)} L \right), \]  

(3.26)

where \( Y \) is a real but otherwise unconstrained superfield, \( V^{(0)} \) is the vector multiplet corresponding to the \( U(1) \) gauge symmetry and \( L \) is a (modified) linear multiplet containing the antisymmetric tensor field strength. In components one has (see eqs. (A.10), (A.13) of appendix A)

\[ L = l + \theta \chi + \bar{\theta} \bar{\chi} + \theta \sigma^\mu \bar{\theta} \bar{H}_\mu + \ldots, \]

(3.27)

with \( \bar{H}_\mu = \frac{i}{2} \epsilon_{\mu \nu \rho \sigma} \partial^\nu B^{\sigma \rho} + b_2^{(a)} \tilde{Q}_\mu^{(a)}. \)

It satisfies the constraints

\[ \mathcal{D}^2 L = 2b_2^{(a)} \text{tr}(W^{(a)} W^{(a)}), \quad \bar{\mathcal{D}}^2 L = 2b_2^{(a)} \text{tr}(W^{(a)} W^{(a)}), \]

(3.28)

where \( \mathcal{D}^2 = \mathcal{D}^a \mathcal{D}_a, \bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_a \bar{\mathcal{D}}^a \) are the supercovariant derivatives and \( W^{(a)}_\alpha \) is the chiral field strength multiplet associated to the gauge symmetry \( G_a \). These equations are the supersymmetric generalization of the modified Bianchi identity \( \partial^\alpha \bar{H}_\mu = \frac{i}{2} b_2^{(a)} \text{tr}(F^{(a)} F^{(a)}) \), which follows from (3.3).

Again two equivalent descriptions can be obtained from (3.28) by varying with respect to \( Y \) or \( L \).

1. From \( \delta_Y \mathcal{L}' = 0 \) one finds \( Y = -2b_1 L \) and, inserting this into (3.26),

\[ \mathcal{L}^L = \int d^2 \theta d^2 \bar{\theta} \left( -b_1 L^2 + 4b_3 V^{(0)} L \right), \]

(3.29)

with \( \mathcal{D}^2 L = 2b_2^{(a)} \text{tr}(W^{(a)} W^{(a)}), \bar{\mathcal{D}}^2 L = 2b_2^{(a)} \text{tr}(W^{(a)} W^{(a)}). \) Expanding this Lagrangian in component fields (see eqs. (A.10), (A.13)), one finds that the bosonic part coincides with the Lagrangian (3.8) if we set the additional boson \( l \) to a constant value \( \langle l \rangle \) and identify the gauge coupling constants as \( g_{(a)}^{-2} = 2b_1 b_2^{(a)} \langle l \rangle \).

2. The variation with respect to \( L \) is more subtle because it is a constrained superfield, satisfying (3.28). A modified linear multiplet can always be written in terms of unconstrained superfield spinors \( \xi^\alpha, \bar{\xi}_\dot{\alpha} \) and a 3-form multiplet \( \Omega \) containing the Chern-Simons 3-form (see e.g. [10, 9]), such that \( L - \Omega \) contains no couplings to gauge fields and satisfies the usual linear constraint \( \mathcal{D}^2 (L - \Omega) = \bar{\mathcal{D}}^2 (L - \Omega) = 0: \)

\[ L = \mathcal{D}^a \bar{\mathcal{D}}_a \xi^\alpha + \bar{\mathcal{D}}_\dot{\alpha} \mathcal{D}^\alpha \bar{\xi} + \Omega, \]

(3.30)

with \( \mathcal{D}^2 \Omega = 2b_2^{(a)} \text{tr}(W^{(a)} W^{(a)}), \bar{\mathcal{D}}^2 \Omega = 2b_2^{(a)} \text{tr}(W^{(a)} W^{(a)}). \) An integration by parts and variation with respect to \( \xi, \bar{\xi} \) leads to

\[ \bar{\mathcal{D}}^2 \mathcal{D}_a (Y + 4b_3 V^{(0)}) = 0 = \mathcal{D}^2 \bar{\mathcal{D}}_\dot{\alpha} (Y + 4b_3 V^{(0)}), \]

(3.31)
which is solved by
\[ Y = S + \tilde{S} - 4b_3 V^{(0)}, \quad (3.32) \]
where \( S \) is an arbitrary chiral superfield, with components \( S = s + \sqrt{2} \theta \psi + \theta \psi F + i \theta \sigma^a \tilde{\theta} \partial_a s + \frac{1}{2} \sqrt{2} \theta \tilde{\theta} \sigma^a \partial_a \psi + \frac{1}{4} \theta \tilde{\theta} \theta \tilde{\theta} \partial s \). If we identify \( \text{Im}(s) = \phi \), then \( (3.32) \) reads \( Y = -4s \), which is solved by
\[ -S \]
\[ \text{where} \quad b_3 \text{string (see below). Inserting} \ (3.32) \text{into} \ (3.26) \text{and using the identity} \quad f d^2 \theta L = -\frac{1}{2} b_2^{(a)} S tr(W^{(a)} W^{(a)}) \], one finds
\[ \mathcal{L}^S = \int d^2 \theta d^2 \bar{\theta} \left(\frac{1}{4b_1} (S + \tilde{S} - 4b_3 V^{(0)}) \right)^2 - \frac{1}{2} \int d^2 \theta b_2^{(a)} S tr(W^{(a)} W^{(a)}) - \frac{1}{2} \int d^2 \bar{\theta} b_2^{(a)} \tilde{S} tr(\bar{W}^{(a)} \bar{W}^{(a)}). \quad (3.33) \]
The bosonic part of this Lagrangian coincides with \( (3.10) \) if we set the additional boson \( \text{Re}(s) \) to a constant value and identify \( g_{(a)}^{-2} = -2b_2^{(a)} \langle \text{Re}(s) \rangle \) (for consistency one must have \( b_2^{(a)} \langle \text{Re}(s) \rangle < 0 \)).

In the low-energy effective theory of string theory one has a logarithmic Kähler potential for the dilaton superfield. We therefore introduce the auxiliary Lagrangian
\[ \mathcal{L}'' = \int d^2 \theta d^2 \bar{\theta} \left( -b_1 \ln(Y/b_1) + Y L_{dil} + 4b_3 V^{(0)} L_{dil} \right). \quad (3.34) \]
In exactly the same manner as above one derives the two equivalent descriptions.

(1) From \( \delta_Y \mathcal{L}'' = 0 \) one finds \( Y = b_1 / L_{dil} \), which (using \( \int d^2 \theta d^2 \bar{\theta} b_1 = 0 \)) leads to
\[ \mathcal{L}^{L_{dil}} = \int d^2 \theta d^2 \bar{\theta} \left( b_1 \ln(L_{dil}) + 4b_3 V^{(0)} L_{dil} \right), \quad (3.35) \]
with \( D^2 L_{dil} = 2b_2^{(a)} tr(\bar{W}^{(a)} W^{(a)}) \), \( \bar{D}^2 L_{dil} = 2b_2^{(a)} tr(W^{(a)} \bar{W}^{(a)}) \). An expansion in components (see eqs. \( (A.10), (A.13) \)) shows that (in the limit where gravity decouples) this is part of the effective Lagrangian for heterotic string vacua in \( D = 4 \) with gauge group \( G = \prod_a G_a \). It describes the NSNS 2-form \( B_{\mu \nu} \), the dilaton \( \varphi \) in the lowest component of \( L_{dil} \), \( l_{dil} = e^{2\varphi} \), and the gauge fields.

(2) Writing \( L_{dil} \) again as in \( (3.30) \) and varying with respect to \( \xi, \bar{\xi} \), one finds \( Y = S_{dil} + \tilde{S}_{dil} - 4b_3 V^{(0)} \), as in eq. \( (3.32) \). Note that the sign of \( S_{dil} + \tilde{S}_{dil} \) in \( Y \) is fixed because we want to identify \( S_{dil} \) with the chiral dilaton multiplet, which implies \( \text{Re}(s) > 0 \) and \( b_2^{(a)} < 0 \). Inserting \( Y \) in \( (3.34) \) yields
\[ \mathcal{L}^{S_{dil}} = -\int d^2 \theta d^2 \bar{\theta} b_1 \ln \left( S_{dil} + \tilde{S}_{dil} - 4b_3 V^{(0)} \right) - \frac{1}{2} \int d^2 \theta b_2^{(a)} S_{dil} tr(W^{(a)} W^{(a)}) - \frac{1}{2} \int d^2 \bar{\theta} b_2^{(a)} \tilde{S}_{dil} tr(\bar{W}^{(a)} \bar{W}^{(a)}). \quad (3.36) \]
This dual description of heterotic vacua is more familiar because it is easier to deal with a chiral superfield than with a linear multiplet. As mentioned above (in the paragraph below
(3.13), in the heterotic string the coefficient $b_2^{(a)}$ is universal up to the Kac-Moody level $k^{(a)}$: $b_2^{(a)} = -k^{(a)} \alpha / 4$. We identify $\frac{\alpha}{2} s = e^{-2\varphi} + ia$, where $\varphi$ is the dilaton and $a$ the scalar field dual to the antisymmetric tensor. This leads to gauge couplings $g^{(a)} = e^{(a)} / \sqrt{k^{(a)}}$.

From the kinetic terms for $S_{\text{dil}}$ one sees that, under a $U(1)_X$ gauge transformation $V^{(0)} \rightarrow V^{(0)} + \frac{1}{2} (\Lambda + \bar{\Lambda})$, $S_{\text{dil}}$ transforms as $S_{\text{dil}} \rightarrow S_{\text{dil}} + 2b_3 \Lambda$. The variation of the Lagrangian is thus given by

$$\delta_A L^{S_{\text{dil}}} = \int d^2 \theta \Lambda \left( \frac{A^{(a)}}{32 \pi^2} - b_2^{(a)} b_3 \right) \text{tr}(W^{(a)} W^{(a)}) + \text{h.c.}$$

(3.37)

This vanishes if the condition (3.13) is satisfied, which, as we saw above, does indeed happen for heterotic string vacua in $D = 4$.

In type IIB orientifolds one has several linear multiplets $L^{(k,f)}$ from the twisted sectors, where $k = 1, \ldots, \left[ \frac{N-1}{2} \right]$, $f = 1, \ldots, f_k$ (not all of them are independent). In generalization of (3.14) we define the linear combinations

$$L^{(k)} = \frac{1}{\sqrt{f_k}} \sum_f L^{(k,f)}.$$  

(3.38)

The supersymmetric generalization of the Chern-Simons coupling (3.17) reads

$$D^2 L^{(k)} = 2c_2 C_k \text{tr}(\gamma_3 W^{(a)} W^{(a)}), \quad D^2 L^{(k)} = 2c_2 C_k \text{tr}(\gamma_3 W^{(a)} W^{(a)}).$$

(3.39)

Here $C_k = \prod_{i=1}^3 \sin(\pi k v_i)$ for all sectors except the second sector of $Z^t_0$, where $C_k = \sqrt{3}$. The coupling (3.17) of the linear multiplet to the anomalous $U(1)_X$ takes the form

$$4c_3 \text{tr}(i \gamma_k V^{(0)}) L^{(k)}.$$  

(3.40)

Again the traces over the product of $\gamma_k$ with the gauge group generators can be calculated and lead to the identification $b_2^{(k,a)} = c_2 C_k \cos(2\pi k V_a)$, $b_3^{(k)} = 2n_X c_3 \sin(2\pi k V_X)$, exactly as in the purely bosonic case, eq. (3.18).

Assuming a quadratic Kähler potential for the twisted fields, we find, in direct generalization of (3.29) and (3.33), that the effective theory describing the linear multiplets

$$L^L = \int d^2 \theta d^2 \bar{\theta} \sum_k \left( -b_1 (L^{(k)})^2 + 4b_3^{(k)} V^{(0)} L^{(k)} \right),$$

(3.41)

with $D^2 L^{(k)} = 2b_2^{(k,a)} \text{tr}(\bar{W}^{(a)} W^{(a)})$, $D^2 L^{(k)} = 2b_2^{(k,a)} \text{tr}(W^{(a)} W^{(a)})$, can equivalently be described in terms of chiral superfields $M^{(k)}$:

$$L^M = \int d^2 \theta d^2 \bar{\theta} \sum_k \left( M^{(k)} + \bar{M}^{(k)} - 4b_3^{(k)} V^{(0)} \right)^2$$

$$-\frac{1}{2} \int d^2 \theta \sum_k b_2^{(a,k)} M^{(k)} \text{tr}(W^{(a)} W^{(a)}) - \frac{1}{2} \int d^2 \bar{\theta} \sum_k b_2^{(a,k)} \bar{M}^{(k)} \text{tr}(\bar{W}^{(a)} W^{(a)}).$$

(3.42)

From the transformation law under $U(1)_X$ gauge transformations,

$$V^{(0)} \rightarrow V^{(0)} + \frac{1}{2} (\Lambda + \bar{\Lambda}) \quad \Rightarrow \quad M^{(k)} \rightarrow M^{(k)} + 2b_3^{(k)} \Lambda,$$

(3.43)
one sees that the variation of the Lagrangian is

$$
\delta \Lambda L^M = \int d^2 \theta \Lambda \left( \frac{A^{(a)}}{32\pi^2} - \sum_k b_2^{(k,a)} b_3^{(k)} \right) \text{tr}(W^{(a)}W^{(a)}) + \text{h.c.} \quad (3.44)
$$

The condition for gauge anomaly cancellation is therefore the same that was found in the purely bosonic case \((3.21)\).

Let us now analyze the question which twisted fields can contribute to anomaly cancellation. The twisted sectors with no fixed planes are commonly denoted as \(\mathcal{N} = 1\) sectors and the ones with fixed planes as \(\mathcal{N} = 2\) sectors. As we saw in section 2, all the fields from the \(\mathcal{N} = 1\) sectors are in \(\mathcal{N} = 1\) linear multiplets. The mechanism described above works well for the fields from the \(\mathcal{N} = 1\) sectors. The twisted fields of the \(\mathcal{N} = 2\) sectors fit into \(\mathcal{N} = 1\) chiral and \(\mathcal{N} = 2\) hyper, linear hyper and vector-tensor multiplets. We can think of the \(\mathcal{N} = 2\) sectors as follows. If the \(k\)-th twisted sector has a fixed plane, then \(\gcd(k,N) \neq 1\). Consider the \(Z_m\) orientifold, with \(m = N/\gcd(k,N)\), generated by \(\theta^k\). This leads to \(\mathcal{N} = 2\) supersymmetry in \(D = 4\) (compare the discussion in footnote 2). In particular, the gauge fields are in \(\mathcal{N} = 2\) vector multiplets, the untwisted matter fields in hyper multiplets and the twisted fields in hyper and vector-tensor multiplets. This constrains the possible couplings. The projection on \(Z_N\)-invariant states eliminates some of the fields (e.g. the the scalar partners of the gauge fields are projected out), but the remaining terms in the Lagrangian are inherited from the \(\mathcal{N} = 2\) theory. Now, it is important to note that in the \(\mathcal{N} = 2\) \(Z_m\) theory all of the twisted fields in linear multiplets belong to vector-tensor multiplets. In the \(\mathcal{N} = 1\) \(Z_N\) theory some of the vectors are projected out, but their linear partners still couple as required by \(\mathcal{N} = 2\) supersymmetry. In \([23]\) it was shown that in an \(\mathcal{N} = 2\) supersymmetric theory only fields belonging to vector multiplets can couple to the gauge kinetic terms. Therefore a coupling \(M^{(k)} \text{tr}(W^{(a)}W^{(a)})\) as in \((3.42)\) is only possible if one either has only \(\mathcal{N} = 1\) supersymmetry (i.e. no fixed planes) or the \(M^{(k)}\) belong to \(\mathcal{N} = 2\) vectors, which means that their duals \(L^{(k)}\) are in \(\mathcal{N} = 2\) vector-tensor multiplets. We conclude, that besides the linear multiplets from the \(\mathcal{N} = 1\) sectors all the linear multiplets from the \(\mathcal{N} = 2\) sectors, but none of the chiral multiplets, can contribute to anomaly cancellation. Otherwise stated, fields from all twisted sectors except the \(k = N/2\) sector can contribute to anomaly cancellation.\footnote{7} As we noted above these fields suffice to cancel all the \(U(1)\) anomalies in \(D = 4\), \(\mathcal{N} = 1\) type IIB orientifolds.

### 4 Kähler anomalies

In compact factorizable orbifolds of the heterotic string the tree-level Lagrangian is invariant under \(\prod_{i=1}^3 SL(2, \mathbb{R})_i\) transformations acting on the Kähler moduli \(T_i\) as

$$
T_i \rightarrow \frac{a_i T_i - ib_i}{ic_i T_i + d_i}, \quad \text{with} \quad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2, \mathbb{R})_i. \quad (4.1)
$$

\footnote{7}{It is interesting to note that the coefficient \(b_2^{(k,a)}\) in \((1.18)\) vanishes for \(k = N/2\), because, for even \(N\), \(V_a\) is of the form \(V_a = \frac{2j+1}{2N}\), with \(j \in \mathbb{Z}\) (see \([16]\)).}
The Kähler potential for the \( T_i \), 
\[ K^{(T)} = - \sum_i \ln(T_i + \bar{T}_i), \]
transforms as 
\[ K^{(T)} \rightarrow K^{(T)} + \ln(i c_i T_i + d_i) + \ln(-i c_i \bar{T}_i + d_i), \tag{4.2} \]
i.e. the kinetic terms of the \( T_i \) are not modified ((4.2) is a Kähler transformation). At one-loop one finds an anomalous variation of the Lagrangian (3.36) \[ \delta L_{\text{dil one-loop}} = \frac{1}{32\pi^2} \int d^2 \theta d^2 \bar{\theta} \sum_{i=1}^{3} b^i_a \ln(i c_i T_i + d_i) \text{tr}(W^{(a)} W^{(a)}) + \text{h.c.} \tag{4.3} \]
The anomaly coefficients \( b^i_a \) are defined by 
\[ b^i_a = -T(G_a) + \sum_{R_a} T(R_a)(1 + 2n^i R_a), \tag{4.4} \]
where the sum is over all fields charged under \( G_a \) (transforming in a representation \( R_a \)) and the modular weights \( n^i R_a \) can be read off from the Kähler potential of the charged fields \( \Phi_r \), which to second order in \( \Phi_r \) has the form 
\[ K^{\text{matter}} = \sum_r \prod_{i=1}^{3} (T_i + \bar{T}_i)^{n^i R_a} \Phi_r \bar{\Phi}_r. \tag{4.5} \]

However, the variation (4.3) is cancelled by an opposite variation of \( S_{\text{dil}} \) like in the case of \( U(1) \) gauge anomalies. The authors of [26] found that at one-loop the effective Lagrangian for the linear dilaton multiplet (3.35) contains an additional term 
\[ \int d^2 \theta d^2 \bar{\theta} \frac{1}{8\pi^2} \sum_{i=1}^{3} \tilde{b}^i_a \ln(T_i + \bar{T}_i) L_{\text{dil}}, \tag{4.6} \]
with \( b^i_a = k^{(a)} \tilde{b}^i_a \). The dual Lagrangian (3.36) is then modified to 
\[ L^{S_{\text{dil}}} = -\int d^2 \theta d^2 \bar{\theta} b_1 \ln \left( S_{\text{dil}} + \tilde{S}_{\text{dil}} - 4b_2 V^{(0)} - \frac{1}{8\pi^2} \sum_{i=1}^{3} \tilde{b}^i_a \ln(T_i + \bar{T}_i) \right) 
+ \frac{1}{4} \int d^2 \theta k^{(a)} S_{\text{dil}} \text{tr}(W^{(a)} W^{(a)}) + \frac{1}{4} \int d^2 \bar{\theta} k^{(a)} \tilde{S}_{\text{dil}} \text{tr}(\bar{W}^{(a)} \bar{W}^{(a)}). \tag{4.7} \]
We replaced \( b_2^{(a)} \) by its value \(-\frac{1}{2} k^{(a)}\) (in units of \( \alpha'/2 \)). This leads to the transformation law 
\[ S_{\text{dil}} \rightarrow S_{\text{dil}} - \frac{1}{8\pi^2} \sum_{i=1}^{3} \tilde{b}^i_a \ln(i c_i T_i + d_i) \tag{4.8} \]
under (4.1) and cancels the anomaly. (For a detailed discussion of Kähler anomalies and the restrictions they impose on heterotic string vacua see [31]).

The tree-level Lagrangian of \( D = 4, N = 1 \) type IIB orientifolds shows the same symmetry under \( \prod_{i=1}^{3} SL(2, \mathbb{R})_i \) transformations and again there are one-loop anomalies. In [7] it has been proposed that the anomalies can be cancelled by the same mechanism that cancels
$U(1)$ gauge anomalies via exchange of twisted fields $M^{(k)}$. To cancel also the Kähler anomalies these fields need to transform under the Kähler transformations (4.1). In the description in terms of linear multiplets this amounts to additional terms

$$
\int d^2\theta d^2\bar{\theta} \frac{1}{8\pi^2} \sum_{i=1}^{3} \alpha_k^i \ln(T_i + \bar{T}_i)L^{(k)}
$$

in the Lagrangian (3.41) such that

$$\sum_k 2b_{2}^{(k,a)}\alpha_k^i = -b_a^i.
$$

The Lagrangian for the chiral multiplets (3.42) is then modified to

$$\mathcal{L}^M = \int d^2\theta d^2\bar{\theta} \frac{1}{4b_1} \sum_k \left(M^{(k)} + \bar{M}^{(k)} - 4b_3^{(k)}V^{(0)} - \frac{1}{8\pi^2} \sum_{i=1}^{3} \alpha_k^i \ln(T_i + \bar{T}_i)\right)^2
$$

$$-\frac{1}{2} \int d^2\theta \sum_k b_{2}^{(a,k)}M^{(k)} \text{tr}(W^{(a)}\bar{W}^{(a)}) - \frac{1}{2} \int d^2\bar{\theta} \sum_k b_{2}^{(a,k)}\bar{M}^{(k)} \text{tr}(\bar{W}^{(a)}W^{(a)}),
$$

which leads to the transformation

$$M^{(k)} \rightarrow M^{(k)} - \frac{1}{8\pi^2} \sum_{i=1}^{3} \alpha_k^i \ln(ic\bar{T}_i + d_i)
$$

under (4.1) and cancels the anomaly. Let us again make a remark on mass dimensions. In our conventions $L^{(k)}$, $V^{(0)}$ and $T_i$ are dimensionless superfields, $M^{(k)}$, $b_1$, $b_3^{(k)}$ and $\alpha_k^i$ have mass dimension 2, $b_{2}^{(k,a)}$ has dimension $-2$ and $W^{(a)}$ has dimension $3/2$.

It is interesting to introduce the composite $U(1)$ connection associated to Kähler transformations\footnote{A general definition of the composite Kähler connection is given in [32], chapter 23; the application to Kähler anomalies is discussed in [24, 30].}

$$A^{(K)}_{\mu} = \frac{1}{4} \left( \frac{\partial K}{\partial T_i} \partial_{\mu} T_i - \frac{\partial K}{\partial \bar{T}_i} \partial_{\mu} \bar{T}_i \right).
$$

The field strength associated to this connection (with $K = -\sum_i \ln(T_i + \bar{T}_i) + \ldots$) is

$$F^{(K)}_{\mu\nu} = \frac{1}{2} \frac{\partial_{\mu} T_i \partial_{\nu} T_i - \partial_{\nu} T_i \partial_{\mu} T_i}{(\bar{T}_i + \bar{T}_i)^2}.
$$

An expansion of (4.9) in components, eq. (A.14), shows that it contains the term

$$
\int d^2\theta d^2\bar{\theta} \frac{1}{8\pi^2} \sum_{i=1}^{3} \alpha_k^i \ln(T_i + \bar{T}_i)L^{(k)} = \frac{1}{8\pi^2} \sum_{i=1}^{3} \alpha_k^i (i\bar{F}^{(K)}_{\mu\nu} B^{(k)\mu\nu}) + \ldots,
$$

where we used a partial integration for the term $\bar{H}^{(k)}_{\mu} \partial^\mu \text{Im}(T)/(T + \bar{T})$. This result is just the analogue of the coupling (3.17) between the twisted RR-fields and the $U(1)_X$ gauge fields
living on the D-branes, with \( \alpha^i_k i \tilde{F}^{(k)}_{\mu \nu} \) corresponding to tr\((\gamma_i i \tilde{F}^{(0)}_{\mu \nu})\) and \( c_3 = 1/8\pi^2 \). Up to factor of \(-\alpha'\), which is implicitly contained in \( \alpha^i_k \), this is the same normalization as the one chosen at the end of section 3.1.

In [7] it was shown that the anomaly coefficients \( a' \) indeed factorize as required in (4.10):

\[
b_{ai}^i = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{\alpha}_k^i \cos(4\pi k V_a) \tag{4.16}
\]

with

\[
\tilde{\alpha}_k^i = \begin{cases} 
-C_k \tan(\pi k v_i) & \text{if } N \text{ odd} \\
\eta_k C_{4k} \cot(2\pi k v_i) - C_k \cot(\pi k v_i) + \delta_3^i 2\eta_k \frac{C_{4k}}{C_{2k}} \cos(2\pi k v_3) & \text{if } N \text{ even}
\end{cases}
\]

\[
\eta_k = \begin{cases} 
(-1)^k & \text{if } N = 6 \\
(-1)^{k/2} & \text{if } N = 12, \ k \text{ even} \\
0 & \text{if } N = 12, \ k \text{ odd}
\end{cases}
\]

\[
C_k = \prod_{i=1}^{3} 2 \sin(\pi k v_i).
\]

For odd \( N \) this can be transformed to

\[
b_{ai}^i = \frac{2}{N} \sum_{k=1}^{(N-1)/2} \tilde{\alpha}_k^i \cos(2\pi k V_a), \quad \text{with } \tilde{k} = \begin{cases} 
k/2 & \text{if } k \text{ even} \\
\frac{N-k^2}{2} & \text{if } k \text{ odd}
\end{cases}
\tag{4.17}
\]

Comparison with (4.10) gives

\[
\alpha_k^i = \frac{-\tilde{\alpha}_k^i}{c_2 C_k N} = \pm \frac{\tan(\pi \tilde{k} v_i)}{c_2 N},
\tag{4.18}
\]

where we used \( b_{2(a,k)} = c_2 C_k \cos(2\pi k V_a) \) from (3.18) and \( C_k = \pm C_k \) which is valid for all odd \( N \) orientifolds. The sign in (4.18) is + for \( Z_3 \) and \( Z_7 \), \( k = 2 \), and − for \( Z_7 \), \( k = 1, 3 \).

For even \( N \) (4.16) can be transformed to

\[
b_{ai}^i = \frac{24}{N} \delta_3^i + \frac{2}{N} \sum_{k=1}^{N/2-1} \tilde{\alpha}_k^i \cos(4\pi k V_a) \tag{4.19}
\]

Two conclusions can be drawn from this expression: First, only for even \( k \) is \( \alpha_k^i \) in (4.10) non-vanishing. Second, the anomalies corresponding to the third complex plane cannot be cancelled by the exchange of twisted fields \( M^{(k)} \), because their coupling to the field strengths tr\((W^{(a)} W^{(a)})\) is proportional to \( \cos(2\pi k V_a) \), which is only for \( k = N \) independent of \( V_a \) (one has \( V_a = \frac{\pi + 1}{2N}, j \in \mathbb{Z} \)). The best way to compare the expression (4.19) for \( b_{ai}^i \) with eq. (4.10) is to consider each orientifold separately. We find

\[
Z_6 : \quad b_{ai}^i = 4\delta_3^i - (2 + 2\delta_3^i) \cos(4\pi V_a),
\]

\[
Z_6' : \quad b_{ai}^i = 4\delta_3^i + (2 + 4\delta_2^i) \cos(4\pi V_a),
\]

\[
Z_{12} : \quad b_{ai}^i = 2\delta_3^i + \frac{2}{3} \sqrt{3}(1 - 2\delta_2^i - \delta_3^i) \cos(4\pi V_a) - (1 + \delta_3^i) \cos(8\pi V_a).
\]
Comparing this to (4.10), one obtains, for $i = 1, 2$:

\[
Z_6 : \quad \alpha_1^i = 0, \quad \alpha_2^i = \frac{-1}{3\sqrt{3}c_2},
\]

\[
Z_6' : \quad \alpha_1^i = 0, \quad \alpha_2^i = \frac{-(1 + 2\delta_2^i)}{\sqrt{3}c_2},
\]

\[
Z_{12} : \quad \alpha_1^i = 0, \quad \alpha_2^i = \frac{-(1 - 2\delta_2^i)}{3c_2}, \quad \alpha_3^i = 0, \quad \alpha_4^i = \frac{-1}{6\sqrt{3}c_2}, \quad \alpha_5^i = 0.
\]

This shows that all Kähler anomalies, except those associated to the third complex plane in even $N$ orientifolds, can indeed be cancelled by the mechanism proposed in [7]. Whether the crucial coupling coupling (4.9) indeed occurs and whether the coefficients $\alpha_k^i$ really have the required values, should be verified by an explicit string calculation.

There is good reason to believe that the couplings (4.9) are generated with the correct coefficients. In four-dimensional heterotic string vacua, there is an exact symmetry, called T-duality, which acts on the moduli $T_i$ as the Kähler transformation (4.1) but with integer coefficients $a_i, b_i, c_i, d_i$. The Green-Schwarz mechanism guarantees that this symmetry is preserved at the quantum level. Because of heterotic-type I duality (see e.g. [33]) the same should be true for type IIB orientifolds. The type I moduli $T_i$ are defined by [16, 34]

\[
T_i = \frac{R_i^2}{\lambda_i \alpha'} + iC_{(2)i+2,2i+3}^{(2)}, \quad \alpha' = \alpha + iB_{2i+2,2i+3}.
\]

where $R_i$ is the radius of the $i$-th torus, $C_{(2)i,j}$ are the internal components of the ten-dimensional RR 2-form and $\lambda_i = e^{\phi}$ is the ten-dimensional type I string coupling. Under the duality mapping given in [33] the type I $T_i$ are mapped to the heterotic $T_i$, defined by $T_i = R_i^2/\alpha' + iB_{2i+2,2i+3}$. The heterotic T-duality should therefore be realized as a symmetry of four-dimensional type I vacua (and in particular of type IIB orientifolds) acting on the type I moduli $T_i$ as the $\prod_{i=1}^3 SL(2,\mathbb{Z})_i$ subgroup of (4.1). Note that on the type I side, this symmetry is not T-duality.

If D5-branes are present in the considered orientifold, the above reasoning has to be slightly modified. The field strengths corresponding to the gauge fields living on the D5-branes do not couple to $S_{dil}$ but rather to $T_3$ (here we assume that the D5-branes are extended in the four non-compact dimensions and in the third complex plane). The corresponding term in the Lagrangian is

\[
\int d^2\theta T_3 \text{tr}(W^{(a,5)} W^{(a,5)}) + \text{h.c.}
\]

Clearly, the $SL(2,\mathbb{Z})_3$ subgroup of (1.1) is explicitly broken by this coupling. Therefore one cannot expect the Green-Schwarz mechanism to work for the Kähler symmetries associated to the third complex plane if D5-branes are present. The existence of D5-branes in type I vacua exchanges different types of D-branes. From the T-duality rules derived in [34] one sees that it also exchanges the three $T_i$ and the type I dilaton multiplet $S_{dil}$. This is clearly different from (4.1).

\[9\text{T-duality in type I vacua exchanges different types of D-branes. From the T-duality rules derived in [34] one sees that it also exchanges the three }T_i\text{ and the type I dilaton multiplet }S_{dil}.\text{ This is clearly different from (4.1).}\]
IIB orientifolds is intimately linked to the existence of an order-two twist. If the considered orientifold contains an order-two twist, then tadpole cancellation requires D5-branes that are extended in the complex plane which is fixed by this twist. Thus, whenever a complex plane is fixed under the action of $\theta^{N/2}$, then the Kähler symmetry associated to this plane is explicitly broken.

There is a striking difference between the cancellation of Kähler anomalies in heterotic and type I vacua. In the heterotic string, only the anomalies in the $\mathcal{N} = 1$ sectors (i.e. the sectors without fixed planes) are cancelled by a Green-Schwarz mechanism. The anomalies in the $\mathcal{N} = 2$ sectors are cancelled by a $T_i$ dependent one-loop correction to the gauge kinetic function. The authors of [35] calculated the one-loop correction $f^{(1)}$ to the gauge kinetic function $f$ (as defined in (B.1), see [36] for a pedagogical presentation) for $D = 4, \mathcal{N} = 1$ heterotic orbifolds. They find

$$f^{(1)} = \frac{-1}{8\pi^2} \sum_{i=1}^{3} b_i^{(N=2)} \ln \eta^2(iT_i).$$  (4.23)

Because of the transformation property of the Dedekind $\eta$-function under (4.1), $\eta^2(iT_i) \to (ic_iT_i + d_i)\eta^2(iT_i)$, the $\prod_{i=1}^{3} SL(2, \mathbb{Z})_i$ symmetry is restored at the quantum level.

In contrast to this, the gauge kinetic function of type I vacua can only depend linearly on the $T_i$ [37]. This can be understood from the fact that the imaginary part of $T_i$ in eq. (4.21) is a RR-field, which leads to an invariance of the action under a Peccei-Quinn symmetry $T_i \to T_i + ia_i$. As a consequence, the mechanism for the cancellation of Kähler anomalies in the $\mathcal{N} = 2$ sectors in type IIB orientifolds must be different from the mechanism that achieves anomaly cancellation in heterotic orbifolds. Indeed, we saw above that the generalized Green-Schwarz mechanism also works in the $\mathcal{N} = 2$ sectors of type IIB orientifolds. The only orientifold that has fixed planes under a twist $\theta^k$, with $k \neq N/2$, is the $\mathbb{Z}'_6$ orientifold. It would therefore be interesting to obtain the coupling (4.3) for the $k = 2$ sector of $\mathbb{Z}'_6$ from an explicit string calculation to see if the Green-Schwarz mechanism works in the expected way. If the Kähler anomalies associated to the second complex plane of $\mathbb{Z}'_6$ are cancelled by the mechanism described above, then the complete $SL(2, \mathbb{R})_2$ symmetry is preserved at the quantum level. This differs from the situation in the corresponding $\mathbb{Z}_6'$ orbifold of the heterotic string where the classical $SL(2, \mathbb{R})_2$ symmetry is broken to $SL(2, \mathbb{Z})_2$ by quantum effects. Presumably the $SL(2, \mathbb{R})_2$ of the $\mathbb{Z}_6'$ orientifold is broken to its discrete subgroup by some non-perturbative effect.

The authors of [41] find that there are also mixed Kähler-gravitational anomalies proportional to the coefficient $b_{grav}^i = b_{cl}^i + b_{op}^i$.

The first contribution is from closed string modes and the second from open string modes. It turns out that the open string contribution can be cancelled by the twisted fields $M^{(k)}$ transforming as in (4.12). But the $b_{cl}^i$ contribution can only be cancelled by a transformation of the type I dilaton $S_{dil}$ under Kähler transformations:

$$S_{dil} \to S_{dil} - \frac{1}{8\pi^2} \sum_{i=1}^{3} b_{cl}^i \ln(ic_iT_i + d_i).$$  (4.24)

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The Kähler potential of the dilaton is therefore modified to

$$- \int d^2 \theta d^2 \bar{\theta} \ln \left( S_{\text{dil}} + \bar{S}_{\text{dil}} - \frac{1}{8\pi^2} \sum_{i=1}^{3} b_{\text{cl}}^{i} \ln(T_i + \bar{T}_i) \right). \quad (4.25)$$

As noted in [7], the additional transformation of $S_{\text{dil}}$ does not spoil the Kähler anomaly cancellation described above, because the coefficient $b_{\text{cl}}^{i}$ is of higher order than $\alpha_i$ in the string loop expansion.

5 Fayet-Iliopoulos terms

The generalized Green-Schwarz mechanism described in the proceeding sections cancels the gauge and Kähler anomalies, but it also gives mass to the gauge bosons (of the anomalous $U(1)'s$), thus breaking gauge invariance. In addition, it generically leads to a non-trivial contribution to the D-term of the (pseudo-)anomalous vector field. This contribution is called Fayet-Iliopoulos term and it induces non-vanishing vacuum expectation values for some of the charged fields. (Fayet-Iliopoulos terms in type IIB orientifolds have been discussed in [38, 39].)

The Lagrangian describing the interactions of the dilaton $S_{\text{dil}}$, the untwisted moduli $T_i$, the twisted moduli $M^{(k)}$ and the gauge fields of type IIB orientifolds is obtained from (4.11), (4.22) and (4.25):

$$\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \left( - \ln \left( S_{\text{dil}} + \bar{S}_{\text{dil}} - \frac{1}{8\pi^2} \sum_{i=1}^{3} b_{\text{cl}}^{i} \ln(T_i + \bar{T}_i) \right) 
+ \frac{1}{4b_1} \sum_{k} \left( M^{(k)} + \bar{M}^{(k)} - 4b_3^{(k)} V^{(0)} - \frac{1}{8\pi^2} \sum_{i=1}^{3} \alpha_i^{k} \ln(T_i + \bar{T}_i) \right)^2 \right) 
- \frac{1}{2} \int d^2 \theta \left( b_{2,\text{dil}}^{(a)} S_{\text{dil}} + \sum_{k} b_{2}^{(a,k)} M^{(k)} \right) \text{tr}(W^{(a,9)}W^{(a,9)}) + \text{h.c.} 
- \frac{1}{2} \int d^2 \theta \left( b_{2,\gamma}^{(a)} T_3 + \sum_{k} b_{2}^{(a,k)} M^{(k)} \right) \text{tr}(W^{(a,5)}W^{(a,5)}) + \text{h.c.}$$

The coefficient $b_{2,\text{dil}}^{(a)}$ is defined by $\mathcal{D}^2 L_{\text{dil}} = 2b_{2,\text{dil}}^{(a)} \text{tr}(W^{(a,9)}W^{(a,9)})$. As in the low-energy effective action of the heterotic string, this coefficient is uniquely fixed by the supergravity algebra in $D = 10$. In the conventions of [40], one has $b_{2,\text{dil}}^{(a)} = -\alpha'/(2\sqrt{2})$. The indices 9 and 5 on the field strengths $W^{(a)}$ indicate on which type of D-branes (9-branes or 5-branes) the corresponding gauge fields live. The coupling of $T_3$ to the D5-branes cannot be understood from linear/chiral multiplet duality, but its coefficient can be determined by performing a T-duality transformation which interchanges D9-branes and D5-branes. This yields $b_{2,\gamma}^{(a)} T_3 = b_{2,\text{dil}}^{(a)}$. Usually the chiral dilaton multiplet is defined to be the coefficient of $\frac{1}{4} \text{tr}(WW)$. This is obtained by redefining $S_{\text{dil}} \rightarrow (\alpha'/\sqrt{2}) S_{\text{dil}}$. 

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From the results (B.7), (B.8) of appendix B, one easily finds the Fayet-Iliopoulos term $\xi_{\text{FI}}^2$ and the gauge boson mass $m_A$ for an anomalous $U(1)_X$ embedded in $U(n_X)$ living on the D9-branes (the result for a D5-brane $U(1)_X$ is obtained by replacing $S_{\text{dil}}$ by $T$):

$$\xi_{\text{FI}}^2 = -\frac{g}{b_1} \sum_k b_3^{(k)} \left( M^{(k)} | + \bar{M}^{(k)} | - \frac{1}{8\pi^2} \sum_i^3 \alpha_i^2 \ln(T_i | + \bar{T}_i |) \right), \quad (5.2)$$

$$m_A^2 = \frac{4g^2}{b_1} \sum_k (b_3^{(k)})^2 = \frac{4n_X^2 c_3^2 (N - \rho) g^2}{b_1}, \quad \text{with} \quad \rho = \begin{cases} 
0 & \text{if } N \text{ odd} \\
2 & \text{if } N \text{ even}
\end{cases}, \quad (5.3)$$

where

$$g^{-2} = \text{Re}(f) = \frac{1}{2} (S_{\text{dil}} | + \bar{S}_{\text{dil}} |) - \sum_k b_2^{(X,k)} (M^{(k)} | + \bar{M}^{(k)} |). \quad (5.4)$$

A vertical slash denotes the lowest component of a superfield. In the second equality of (5.3) we replaced $b_3^{(k)}$ by its value determined in (3.18). The index $X$ refers to $U(1)_X$. The coefficients $b_1$ and $c_3$ are both proportional to $\alpha'^{-1} = M_{\text{str}}^2$. Thus, the (pseudo-)anomalous gauge bosons have masses comparable in size to the typical masses found in heterotic orbifolds. In the normalization $b_1 = \alpha'^{-1}, c_3 = -(8\pi^2 \alpha')^{-1}$, we find

$$m_A^2 = \frac{f_1}{\pi^4} g^2 M_{\text{str}}^2 \quad (5.5)$$

for the two odd $N$ orientifolds, with $f_1 = 27, 7$ for $Z_3, Z_7$. The Fayet-Iliopoulos term for $Z_3$ reads (in the same normalization)

$$\xi_{\text{FI}}^2 = \frac{3\sqrt{3}}{2\pi^2} g \left( M^{(1)} | + \bar{M}^{(1)} | - \frac{\sqrt{3}}{4\pi^2 \alpha'} \sum_i \ln(T_i | + \bar{T}_i |) \right). \quad (5.6)$$

At the supersymmetric minimum the D-term (B.6) vanishes. As explained in the appendix B, this relates $\xi_{\text{FI}}^2$ to the vacuum expectation values of the charged fields. If we assume canonical kinetic terms and minimal coupling to the gauge fields, $\Phi_r e^{2\varphi_r V} \Phi^*_r$, for the charged fields $\Phi_r$, then the Fayet-Iliopoulos term is given by

$$\xi_{\text{FI}}^2 = g \sum_r |\Phi_r|^2. \quad (5.7)$$

In most orientifold models there exist no fields which are charged under $U(1)_X$ but neutral under the non-Abelian gauge group factors. Therefore a non-vanishing $\xi_{\text{FI}}^2$ implies a breaking of the non-Abelian gauge symmetry by the Higgs effect. As noted in [33] this differs from the situation in heterotic orbifolds, where non-Abelian singlets charged under $U(1)_X$ generically exist. A second difference to heterotic vacua is the possibility of having a vanishing Fayet-Iliopoulos in type IIB orientifolds [I]. This is possible because of the dependence of $\xi_{\text{FI}}^2$ on the expectation value of the twisted fields. For type I vacua that are obtained by compactifying the ten-dimensional theory on a smooth Calabi-Yau manifold instead of an orientifold there is no such dependence on twisted fields. Indeed, it has been shown in [II] that the Fayet-Iliopoulos in smooth type I vacua has exactly the same form as in heterotic vacua. It is
proportional to the universal anomaly coefficient $A$ and given by $\xi_{\text{FI}, \text{CY}}^2 = A g^2 M_{\text{str}}^2 / 192 \pi^2$.

Comparing this to the result for orientifolds, we see that the anomaly coefficient $A$ is split into a sum of contributions from the different twisted sectors, as in (8.21).

If we insist on unbroken non-Abelian gauge symmetry, then in most orientifold models we must require $\xi_{\text{FI}}^2 = 0$. This relates the expectation value of the $M^{(k)}$ to $\sum_i \ln(T_i + \bar{T}_i)$. In the simplest example of the $Z_3$ orientifold, one has from (5.6)

$$M^{(1)}| + \bar{M}^{(1)}| = \frac{\sqrt{3}}{4\pi^2 \alpha'} \sum_i \ln(T_i + \bar{T}_i).$$

(5.8)

The blowing-up of the orientifold, however, is related to the expectation values of the lowest components of the linear multiplets $L^{(k)}$. From the discussion in sections 3 and 4, it can be seen that in terms of linear multiplets the Fayet-Iliopoulos term (5.2) is given by $\xi_{\text{FI}}^2 = -g \sum_k b_3^{(k)} L^{(k)}$. Therefore, a vanishing Fayet-Iliopoulos term corresponds to the orientifold limit.

6 Conclusions and outlook

We have seen that the duality between linear and chiral multiplets provides an easy way to understand anomaly cancellation in $D = 4, N = 1$ type IIB orientifolds. Due to the existence of RR forms there appear antisymmetric tensors (embedded in linear supermultiplets) in the twisted spectrum. Their coupling to the gauge fields on the D-branes leads to a non-trivial transformation under the (pseudo-)anomalous gauge symmetry of the scalars (embedded in chiral supermultiplets) which are dual to these antisymmetric tensors. This transformation cancels the anomaly. The crucial twist-number non-conserving couplings, which are not present in heterotic orbifolds, are possible in type IIB orientifolds because the corresponding world-sheets have boundaries and/or are non-orientable.

It is interesting to compare type IIB orientifolds to heterotic orbifolds. There are several differences, a better understanding of which could lead to a deeper insight into heterotic-type I duality [38, 39]. In $D = 4, N = 1$ heterotic orbifolds, $U(1)$ gauge anomalies are cancelled by a Green-Schwarz mechanism involving only one linear multiplet: the dilaton superfield. This is possible because of the universal nature of the anomalies. The non-universality of the anomalies in type IIB orientifolds is accounted for by several linear multiplets that contribute to anomaly cancellation. However their expectation values are not related to the string loop expansion but rather to the blowing-up of the orbifold singularities. This has interesting phenomenological implications. Whereas in heterotic orbifolds the value of the Fayet-Iliopoulos term is uniquely determined once the expectation value of the dilaton is fixed, the same term in type IIB orientifolds depends on the expectation value of the twisted moduli and can even vanish in the orientifold limit.

In heterotic orbifolds, the gauge kinetic function receives $T_i$-dependent threshold corrections from the $N = 2$ sectors. These are crucial to cancel the anomalies in the discrete Kähler symmetries associated to fixed planes. From general arguments such $T_i$-dependent threshold corrections cannot exist in type IIB orientifolds, because of a Peccei-Quinn symmetry
related to the $T_i$ \cite{37}. As a consequence, the continuous $SL(2, \mathbb{R})$ Kähler symmetries associated to fixed planes of type IIB orientifolds are either completely broken by the anomaly or completely preserved by the Green-Schwarz mechanism. In this article we showed that the Green-Schwarz mechanism works even for the fixed planes (which are not fixed by an order-two twist), but the precise coefficients of the required couplings have to be determined by an explicit calculation. A break-down of the continuous Kähler symmetries to their discrete subgroups seems to require some non-perturbative effect. In the $\mathcal{N} = 2$ sectors of heterotic orbifolds there are additional gauge group independent corrections to the gauge kinetic function \cite{23, 42}. It would be interesting to see if such corrections are also present in type IIB orientifolds. By the argument given above, they can however only depend on the $U_i$ but not on the $T_i$ to all orders in perturbation theory.

The twisted spectrum of some type IIB orientifolds contains vector fields. The corresponding gauge symmetry is inherited from the gauge invariance of the RR forms. But it is not clear what is the role of this gauge symmetry in $D = 4$. In heterotic orbifolds this does not happen. There, all twisted states are in chiral multiplets. Unfortunately, no heterotic dual is known for the $\mathbb{Z}_6'$ or $\mathbb{Z}_{12}$ orientifolds, which possess twisted vectors.

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Appendix

In this appendix some useful results of $\mathcal{N} = 1$ supersymmetry are summarized. We use the notation and conventions of [32].

A Linear Multiplets

An $\mathcal{N} = 1$ linear multiplet $L$ is defined to be the most general real superfield which satisfies the constraints

$$D^2 L = \bar{D}^2 L = 0,$$

where

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \sigma^\mu_{\alpha \dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu,$$

are the supercovariant derivatives. The solution of (A.1) has the component expansion

$$L = l + \theta \chi + \bar{\theta} \bar{\chi} + \theta \sigma^\mu \bar{\theta} \tilde{H}^\mu + \frac{i}{2} \theta \bar{\theta} \chi \sigma^\mu \bar{\chi} - \frac{i}{2} \theta \bar{\theta} \sigma^\mu \partial_\mu \bar{\chi} - \frac{1}{4} \theta \bar{\theta} \theta \bar{\theta} \Box l,$$

with $\partial_\mu \tilde{H}^\mu = 0$,

where $l$ is a real scalar, $\chi$ a Weyl spinor, $\bar{\chi}$ its complex conjugate and $\tilde{H}^\mu$ a conserved vector. The constraint on $\tilde{H}^\mu$ can be satisfied by taking it to be the Hodge-dual of an antisymmetric tensor, $\tilde{H}^\mu = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \partial^\nu B^{\rho \sigma}$. For an arbitrary function $K(L)$ one finds

$$\int d^2 \theta d^2 \bar{\theta} K(L) = - \frac{1}{4} \frac{\partial K}{\partial l} \bigg|_{\theta = \bar{\theta} = 0} \Box l - \frac{1}{4} \frac{\partial^2 K}{\partial l^2} \bigg|_{\theta = \bar{\theta} = 0} (\tilde{H}^\mu \tilde{H}_\mu - i \chi \sigma^\mu \partial_\mu \bar{\chi} - i \bar{\chi} \sigma^\mu \partial_\mu \chi)
+ \frac{1}{4} \frac{\partial^3 K}{\partial l^3} \bigg|_{\theta = \bar{\theta} = 0} \chi \sigma^\mu \bar{\chi} \tilde{H}_\mu + \frac{1}{16} \frac{\partial^4 K}{\partial l^4} \bigg|_{\theta = \bar{\theta} = 0} \chi \bar{\chi} \chi \bar{\chi}.$$ (A.4)

Thus, a quadratic Kähler potential for $L$ yields the usual kinetic terms for $l, B_{\mu \nu}$ and $\chi$:

$$- \int d^2 \theta d^2 \bar{\theta} \ L^2 = - \frac{1}{4} \partial_\mu l \partial^\mu l + \frac{1}{2} \tilde{H}_\mu \tilde{H}^\mu - i \chi \sigma^\mu \partial_\mu \bar{\chi} + \text{total derivatives.}$$ (A.5)

To describe an antisymmetric tensor whose field strength is modified by a Chern-Simons 3-form as in (3.4), $\partial^\mu \tilde{H}_\mu = \frac{1}{2} b_2 \text{tr}(F^{\mu \nu} F_{\mu \nu})$, one needs a modified linear multiplet, which satisfies

$$D^2 L = 2 b_2 \text{tr}(\tilde{W}^a \tilde{W}_a), \quad \bar{D}^2 L = 2 b_2 \text{tr}(W^a W_a),$$ (A.6)
where $W_\alpha$ is the chiral field strength multiplet corresponding to the gauge field that is contained in the Chern-Simons form,

$$W_\alpha = -i\lambda_\alpha + \left(\delta_\alpha^\beta D - \frac{i}{2}(\sigma^\mu\sigma^\nu)_{\alpha}^\beta F_{\mu\nu}\right)\theta_\beta + \theta\sigma^\mu\bar{\theta}\partial_\mu\lambda_\alpha + \theta\theta\sigma^\nu_{\alpha\dot{\alpha}} \left(\partial_\mu\bar{\lambda}^{\dot{\alpha}} + i[A_\mu, \bar{\lambda}^{\dot{\alpha}}]\right)$$

$$-\frac{1}{2}\theta\bar{\theta}(\sigma^\mu\bar{\theta})_{\alpha}(i\partial_\mu D - \partial^\nu F_{\mu\nu}) - \frac{i}{4}\theta\bar{\theta}\bar{\theta}\partial\lambda_\alpha.$$  

(A.7)

The constraint (A.6) can be solved by setting $L = L_0 + b_2\Omega$, where $L_0$ is an unmodified linear multiplet (i.e. $D^2L_0 = D^2L_0 = 0$) with component expansion as in (A.3) and

$$\Omega = -2i\theta\sigma^\mu\text{tr}(\bar{\lambda}A_\mu) - 2i\bar{\theta}\bar{\sigma}^\mu\text{tr}(\lambda A_\mu) + \frac{1}{2}\theta\bar{\theta}\text{tr}(\lambda\bar{\lambda}) + \frac{1}{2}\bar{\theta}\theta\text{tr}(\lambda\bar{\lambda}) + \theta\sigma^\mu\bar{\theta}Q_\mu$$

(A.8)

$$+\theta\bar{\theta}\left(-i\text{tr}(D\bar{\lambda}) - \bar{\sigma}^\mu\sigma^\nu\text{tr}(A_\nu(\partial_\mu\bar{\lambda} + i\frac{1}{2}[A_\mu, \bar{\lambda}])) + \text{tr}(\bar{\lambda}\partial^\mu A_\mu)\right)$$

$$+\bar{\theta}\theta\left(i\text{tr}(D\lambda) - \sigma^\mu\bar{\sigma}^\nu\text{tr}(A_\nu(\partial_\mu\lambda + i\frac{1}{2}[A_\mu, \lambda])) + \text{tr}(\lambda\partial^\mu A_\mu)\right)$$

$$+\theta\bar{\theta}\bar{\theta}\left(-\frac{1}{2}\text{tr}(D^2) + i\text{tr}(\lambda\sigma^\mu(\partial_\mu\bar{\lambda} + i[A_\mu, \bar{\lambda}]) + \frac{1}{4}\text{tr}(F_{\mu\nu}F_{\mu\nu})\right).$$

Here $\hat{Q}_\mu$ is the Hodge dual of the Chern-Simons 3-form $Q_{\mu\nu}\rho = \text{tr}(A_\mu\rho A_\nu - \frac{a}{2}A_\mu A_\nu A_\rho)$, and the traces are over the adjoint representation of the gauge group. It is easy to check that

$$D^2\Omega = -2\text{tr}(\lambda\lambda) - 4\theta\left(i\text{tr}(D\lambda) + \frac{1}{2}\sigma^\mu\bar{\sigma}^\nu\text{tr}(F_{\mu\nu}\lambda) - 4i\theta\sigma^\mu\bar{\theta}\text{tr}(\lambda\partial_\mu\lambda)\right)$$

$$+\theta\left(2\text{tr}(D^2) - 4i\theta\text{tr}(\lambda\sigma^\mu(\partial_\mu\bar{\lambda} + i[A_\mu, \bar{\lambda}]) - \text{tr}(F_{\mu\nu}F_{\mu\nu}) - i\text{tr}(F_{\mu\nu}\hat{F}_{\mu\nu})\right)$$

$$+\theta\bar{\theta}\partial_\mu (i\text{tr}(\sigma^\mu\bar{\sigma}^\nu F_{\mu\nu}\lambda) - 2\text{tr}(D\lambda))\sigma^\mu\bar{\theta} - \frac{1}{2}\theta\bar{\theta}\bar{\theta}\partial\text{tr}(\lambda\lambda)$$

(A.9)

and analogously $D^2\Omega = 2\text{tr}(\bar{W}_\alpha\hat{W}^{\dot{\alpha}})$. Consequently, the constraint (A.6) is satisfied. The generalization of (A.4) is given by

$$\int d^2\theta d^2\bar{\theta} K(L) =$$

$$-\frac{1}{4}\frac{\partial K}{\partial l}\bigg|_{\theta = \bar{\theta} = 0} \left(\square l + b_2\left(2\text{tr}(D^2) - 4i\text{tr}(\lambda\sigma^\mu(\partial_\mu\bar{\lambda} + i[A_\mu, \bar{\lambda}]) - \text{tr}(F_{\mu\nu}F_{\mu\nu})\right)\right)$$

$$-\frac{1}{4}\frac{\partial^2 K}{\partial l^2}\bigg|_{\theta = \bar{\theta} = 0} \left(\hat{H}_\mu\hat{H}^\mu - i\hat{\chi}\sigma^\mu\partial_\mu\hat{\chi} - i\hat{\chi}\sigma^\mu\partial_\mu\hat{\chi} - b_2^2\text{tr}(\lambda\lambda)\text{tr}(\bar{\lambda}\bar{\lambda})\right)$$

$$+b_2\hat{\chi}\left(2i\text{tr}(D\lambda) + \sigma^\mu\bar{\sigma}^\nu\text{tr}(F_{\mu\nu}\lambda)\right) + b_2\hat{\chi}\left(2i\text{tr}(D\bar{\lambda}) + \sigma^\mu\bar{\sigma}^\nu\text{tr}(F_{\mu\nu}\bar{\lambda})\right)$$

$$+\frac{1}{4}\frac{\partial^3 K}{\partial l^3}\bigg|_{\theta = \bar{\theta} = 0} \left(\hat{\chi}\sigma^\mu\hat{H}_\mu - b_2^2\hat{\chi}\hat{\chi}\text{tr}(\lambda\lambda) - b_2^2\hat{\chi}\hat{\chi}\text{tr}(\bar{\lambda}\bar{\lambda})\right) + \frac{1}{16}\frac{\partial^4 K}{\partial l^4}\bigg|_{\theta = \bar{\theta} = 0} \hat{\chi}\hat{\chi}\hat{\chi}\hat{\chi},$$

where $\hat{\chi} = \chi - 2b_2i\sigma^\mu\text{tr}(\bar{\lambda}A_\mu)$. 

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We also need the coupling of $L$ to an Abelian vector field $V^{(0)}$ and to chiral and antichiral fields $\Phi_i$, $\bar{\Phi}_i$. These have the component expansion (in Wess-Zumino gauge)

$$\Phi_i = \phi_i + \sqrt{2}\theta\psi_i + i\theta\sigma^\mu \partial_\mu \phi_i + \theta\theta F_i - \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\overline{\psi}_i\sigma^\mu \bar{\theta} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box \phi_i,$$  \hspace{1cm} (A.11)

$$V^{(0)} = -\theta\sigma^\mu \partial_\mu A_\mu^{(0)} + i\theta\theta\bar{\theta}\lambda^{(0)} - \bar{\theta}\theta\theta\lambda^{(0)} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D^{(0)}.$$  \hspace{1cm} (A.12)

We find

$$\int d^2\theta d^2\bar{\theta} \; V^{(0)}L = \frac{1}{4}A^{(0)\mu} \tilde{\bar{H}}_\mu - \frac{1}{4}\hat{\chi}\sigma^\mu \hat{\chi}A_\mu^{(0)} + i\frac{1}{2}\hat{\chi}\lambda^{(0)} - \frac{i}{2}\hat{\lambda}\bar{\lambda}^{(0)} + \frac{1}{2}D^{(0)}l$$  \hspace{1cm} (A.13)

$$= \frac{1}{4}\tilde{\bar{f}}^{(0)} B^{\mu\nu} + b_2 A^{(0)\mu} \tilde{Q}_\mu - \frac{1}{4}\hat{\chi}\sigma^\mu \hat{\chi}A_\mu^{(0)} + i\frac{1}{2}\hat{\chi}\lambda^{(0)} - \frac{i}{2}\hat{\lambda}\bar{\lambda}^{(0)} + \frac{1}{2}D^{(0)}l$$

and

$$\int d^2\theta d^2\bar{\theta} \; \ln(\Phi + \bar{\Phi})L = \ln(f + \bar{f}) \left( -\frac{1}{4}\Box - \frac{1}{2}b_2 \Tr(D^2) + \frac{1}{4}b_2 \Tr(F_\mu F^{\mu\nu}) \right)$$

$$+ \frac{\partial^\mu \Im(f) \tilde{\bar{H}}_\mu}{f + \bar{f}} - \frac{FF}{(f + \bar{f})^2}l + \frac{\partial_\mu f \partial^\mu \bar{f}}{(f + \bar{f})^2}l$$  \hspace{1cm} (A.14)

+ fermionic.

**B  D-terms of chiral models**

Consider a general Lagrangian depending on chiral and antichiral fields $\Phi_i$, $\bar{\Phi}_i$ and on an Abelian vector field $V$:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \; K(\Phi_i, \bar{\Phi}_i, V) + \frac{1}{4} \int d^2\theta f(\Phi_i) W^\alpha W_\alpha + \frac{1}{4} \int d^2\bar{\theta} f(\bar{\Phi}_i) \bar{W}_\alpha \bar{W}^\alpha$$

$$+ \int d^2\theta W(\Phi_i) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}_i),$$  \hspace{1cm} (B.1)

where the gauge kinetic function $f$ and the superpotential $W$ are holomorphic functions of $\Phi_i$ and the Kähler potential $K$ is an arbitrary real function. The component expansion of $\Phi_i$ and $V$ was given in (A.11), (A.12). Expanding the Lagrangian (B.1) in components (see e.g. [32]),

$$\mathcal{L} = (-\partial_\mu \phi_i \partial^\mu \bar{\phi}_j + F_i \tilde{\bar{F}}_j) \frac{\partial^2 K}{\partial \phi_i \partial \bar{\phi}_j} \bigg|_{V=0, \theta=0} - \left( \text{Im}(A^{\mu} \partial_\mu \phi_i \frac{\partial}{\partial \phi_i}) - \frac{1}{2} D \right) \left. \frac{\partial}{\partial \bar{V}} K \right|_{V=0, \theta=0}$$

$$- \frac{1}{4} A^{\mu} A^{\nu} \frac{\partial^2}{\partial \bar{V}^2} K \bigg|_{V=0, \theta=0} - \frac{1}{4} \Re(f(\phi_i)) F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} \Im(f(\phi_i)) F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$+ \frac{1}{2} \Re(f(\phi_i)) D^2 + \frac{\partial}{\partial \phi_i} W \bigg|_{\theta=0} F_i + \frac{\partial}{\partial \bar{\phi}_i} \bar{W} \bigg|_{\bar{\theta}=0} \tilde{F}_i$$  \hspace{1cm} (B.2)

+ fermionic,
one finds the equations of motion for the auxiliary fields $F_i, D$:

$$\frac{\partial^2 K}{\partial \phi_i \partial \phi_i} \bigg|_{V=0, \theta=\bar{\theta}=0} \bar{F}_j = -\frac{\partial}{\partial \phi_i} W \bigg|_{\theta=\bar{\theta}=0}, \quad D = \frac{-1}{2\text{Re}(f(\phi_i))} \frac{\partial}{\partial V} K \bigg|_{V=0, \theta=\bar{\theta}=0}. \quad (B.3)$$

We are interested in the case where one of the fields (say $\Phi_0$) appears as an axion like in the Green-Schwarz mechanism and the other fields ($\Phi_i, i \neq 0$) couple to $V$ in the usual way,

$$K = K(\Phi_0 + \bar{\Phi}_0 + 4b_3 V, \Phi_i e^{2q_i V} \Phi_i), \quad f = f(\Phi_0). \quad (B.4)$$

Substituting into (B.2) we find the component Lagrangian

$$\mathcal{L} = -K'' \left( \partial_\mu \text{Re}(\phi_0) \partial^\mu \text{Re}(\phi_0) + (\partial_\mu \text{Im}(\phi_0) + 2b_3 A_\mu)(\partial^\mu \text{Im}(\phi_0) + 2b_3 A^\mu) \right) \quad (B.5)$$

$$-K_{ij} (\partial_\mu + iq_i A_\mu) \phi_i (\partial^\mu - iq_i A^\mu) \phi_j + 2b_3 D K' + \sum_i q_i D K_i \phi_i + \frac{1}{4} \text{Re}(f(\phi_0)) D^2$$

$$-\frac{1}{4} \text{Re}(f(\phi_0)) F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \text{Im}(f(\phi_0)) F_{\mu\nu} \tilde{F}^{\mu\nu} + K_{ij} F_i F_j + W_i F_i + \bar{W}_j \bar{F}_j,$$

where a prime denotes the derivative with respect to $\phi_0$ evaluated at $V = 0 = \theta = \bar{\theta}$ and an index $i$ on $K$ or $W$ denotes the derivative with respect to $\phi_i$. The equation of motion for $D$ now reads

$$D = \frac{-1}{\text{Re}(f(\phi_0))}(2b_3 K' + \sum_i q_i K_i \phi_i). \quad (B.6)$$

Comparison of (B.3) with a Lagrangian of the form (3.10) relates $f(\phi_0)$ to the gauge coupling constant, $g = f(\langle \phi_0 \rangle)^{-\frac{1}{2}}$. To obtain the physical gauge fields, one has to rescale the vector field: $V \rightarrow gV$. The vacuum expectation value of the first term in the rescaled expression for $D$, eq. (B.6), is called the Fayet-Iliopoulos term:

$$\xi^2_{FI} = \frac{-2b_3 K'}{\sqrt{\text{Re}(f(\langle \phi_0 \rangle))}}. \quad (B.7)$$

(The square indicates that it has mass dimension 2.) At the supersymmetric minimum $D$ vanishes, and therefore some of the charged fields acquire vacuum expectation values, $\xi^2_{FI} = \text{Re}(f)^{-\frac{1}{2}} \sum_i q_i K_i \langle \phi_i \rangle$.

Another interesting fact is that (B.5) contains a mass term for the gauge fields. After rescaling one finds

$$m^2_A = \frac{8b_3^2 K''}{\text{Re}(f(\langle \phi_0 \rangle))}. \quad (B.8)$$

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10More precisely, we write $V = g \hat{V}$ and express the whole Lagrangian in terms of the physical field $\hat{V}$. 

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References

[1] M. B. Green, J. H. Schwarz, *Phys. Lett.* **149B** (1984) 117.

[2] E. Witten, *Phys. Lett.* **149B** (1984) 351; M. Dine, N. Seiberg, E. Witten, *Nucl. Phys.* **B289** (1987) 589.

[3] A. Sagnotti, *Phys. Lett.* **294B** (1992) 196, hep-th/9210127.

[4] M. Berkooz, R. G. Leigh, J. Polchinski, J. H. Schwarz, N. Seiberg, E. Witten, *Nucl. Phys.* **B475** (1996) 115, hep-th/9605184.

[5] C. A. Scrucca, M. Serone, hep-th/9907112.

[6] L. E. Ibáñez, R. Rabadán, A. M. Uranga, *Nucl. Phys.* **B542** (1999) 112, hep-th/9808139.

[7] L. E. Ibáñez, R. Rabadán, A. M. Uranga, hep-th/9905098.

[8] S. J. Gates, M. T. Grisaru, M. Roček, W. Siegel, “Superspace”, Frontiers in Physics 58, Benjamin/Cummings Publishing Company (1983).

[9] J.-P. Derendinger, hep-th/9412086.

[10] G. Girardi, R. Grimm, *Ann. Phys.* **272** (1999) 49, hep-th/9801201.

[11] M. R. Douglas, hep-th/9512077.

[12] M. B. Green, J. A. Harvey, G. Moore, *Class. Quant. Grav.* **14** (1997) 47, hep-th/9605033.

[13] J. F. Morales, C. A. Scrucca, M. Serone, *Nucl. Phys.* **B552** (1999) 291, hep-th/98121071; C. A. Scrucca, M. Serone, *Nucl. Phys.* **B556** (1999) 197, hep-th/9903145.

[14] U. Lindström, M. Roček, *Nucl. Phys.* **B222** (1983) 285; H. Nicolai, P. K. Townsend, *Phys. Lett.* **98B** (1981) 257; S. Ferrara, M. Villasante, *Phys. Lett.* **186B** (1987) 85.

[15] E. G. Gimon, C. V. Johnson, *Nucl. Phys.* **B477** (1996) 715, hep-th/9604129.

[16] G. Aldazabal, A. Font, L. E. Ibáñez, G. Violero, *Nucl. Phys.* **B536** (1998) 29, hep-th/9804026.

[17] J. Polchinski, *Phys. Rev.* **D55** (1997) 6423, hep-th/9606165.

[18] J. Erler, A. Klemm, *Comm. Math. Phys.* **153** (1993) 579, hep-th/9207111.

[19] E. Zaslow, *Comm. Math. Phys.* **156** (1993) 301, hep-th/9211119.

[20] C. Vafa, E. Witten, *J. Geom. Phys.* **15** (1995) 189, hep-th/9409188.
[21] A. Dabholkar, J. Park, *Nucl. Phys.* **B477** (1996) 701, hep-th/9604178.

[22] G. Zwart, *Nucl. Phys.* **B526** (1998) 378, hep-th/9708040.

[23] B. de Wit, V. Kaplunovsky, J. Louis, D. Lüst, *Nucl. Phys.* **B451** (1995) 53, hep-th/9504006.

[24] G. Lopes Cardoso, B. A. Ovrut, *Nucl. Phys.* **B369** (1992) 351; *Nucl. Phys.* **B392** (1993) 315, hep-th/9205009.

[25] J.-P. Derendinger, S. Ferrara, C. Kounnas, F. Zwirner, *Nucl. Phys.* **B372** (1992) 145.

[26] I. Antoniadis, E. Gava, K. S. Narain, T. R. Taylor, *Nucl. Phys.* **B407** (1993) 706, hep-th/9212045.

[27] E. Bergshoeff, M. de Roo, B. de Wit, P. van Nieuwenhuizen, *Nucl. Phys.* **B195** (1982) 97.

[28] J. J. Atick, L. J. Dixon, A. Sen, *Nucl. Phys.* **B292** (1987) 109; M. Dine, I. Ichinose, N. Seiberg, *Nucl. Phys.* **B293** (1987) 253.

[29] B. de Wit, P. G. Lauwers, A. van Proeyen, *Nucl. Phys.* **B255** (1984) 569.

[30] J. Louis, in *Particles, Strings and Cosmology*, ed. P. Nath and S. Reucroft, World Scientific (1992).

[31] L. E. Ibáñez, D. Lüst, *Nucl. Phys.* **B382** (1992) 305, hep-th/9202046.

[32] J. Wess, J. Bagger, “*Supersymmetry and Supergravity*”, 2nd ed., Princeton University Press (1992).

[33] J. Polchinski, E. Witten, *Nucl. Phys.* **B460** (1996) 525, hep-th/9510169.

[34] L. E. Ibáñez, C. Muñoz, S. Rigolin, *Nucl. Phys.* **B553** (1999) 43, hep-ph/9812397.

[35] L. J. Dixon, V. S. Kaplunovsky, J. Louis, *Nucl. Phys.* **B355** (1991) 649.

[36] J. Louis, K. Förger, *Nucl. Phys. Proc. Suppl.* **55B** (1997) 33, hep-th/9611184.

[37] I. Antoniadis, C. Bachas, E. Dudas, hep-th/9906039.

[38] M. Cvetič, L. Everett, P. Langacker, J. Wang, *JHEP* **9904** (1999) 020, hep-th/9903051.

[39] Z. Lalak, S. Lavignac, H. P. Nilles, hep-th/9903160.

[40] J. Polchinski, “*String Theory*”, Cambridge University Press (1998).

[41] J. March-Russell, *Phys. Lett.* **437B** (1998) 318, hep-ph/9806426.

[42] H. P. Nilles, S. Stieberger, *Nucl. Phys.* **B499** (1997) 3, hep-th/9702110.

[43] F. Quevedo, private communication.