SYMMETRIC POLYNOMIALS AND NON-FINITELY GENERATED $\text{Sym}(\mathbb{N})$-INVARIANT IDEALS

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Abstract. Let $K$ be a field and let $\mathbb{N} = \{1, 2, \ldots\}$. Let $R_n = K[x_{ij} | 1 \leq i \leq n, j \in \mathbb{N}]$ be the ring of polynomials in $x_{ij}$ $(1 \leq i \leq n, j \in \mathbb{N})$ over $K$. Let $S_n = \text{Sym}(\{1, 2, \ldots, n\})$ and $\text{Sym}(\mathbb{N})$ be the groups of the permutations of the sets $\{1, 2, \ldots, n\}$ and $\mathbb{N}$, respectively. Then $S_n$ and $\text{Sym}(\mathbb{N})$ act on $R_n$ in a natural way: $\tau(x_{ij}) = x_{\tau(i)j}$ and $\sigma(x_{ij}) = x_{i\sigma(j)}$ for all $\tau \in S_n$ and $\sigma \in \text{Sym}(\mathbb{N})$. Let $\overline{R}_n$ be the subalgebra of the symmetric polynomials in $R_n$.

Let $I$ be a non-empty set, let $\text{Sym}(I)$ denote the group of all permutations of $I$. The group $\text{Sym}(\mathbb{N})$ acts on $R_n$ in a natural way: $\sigma(x_{ij}) = x_{i\sigma(j)}$ if $\sigma \in \text{Sym}(\mathbb{N})$. An ideal $I$ of $R_n$ is called $\text{Sym}(\mathbb{N})$-invariant if $\sigma(I) = I$ for all $\sigma \in \text{Sym}(\mathbb{N})$.

It is clear that $R_n$ contains ideals that are not finitely generated. However, the following theorem holds.

Theorem 1 (see [1, 3, 4, 13]). Let $K$ be a Noetherian unital associative and commutative ring. Then each $\text{Sym}(\mathbb{N})$-invariant ideal of $R_n = K[x_{ij} | 1 \leq i \leq n, j \in \mathbb{N}]$ is finitely generated (as such). In this note we prove that this is not the case if $\text{char}(K) = p \leq n$.

We also survey some results about $\text{Sym}(\mathbb{N})$-invariant ideals in polynomial algebras and some related results.

1. Introduction

Let $K$ be a unital associative and commutative ring and let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all positive integers. Let $R_n = K[x_{ij} | 1 \leq i \leq n, j \in \mathbb{N}]$ be the ring of polynomials in $x_{ij}$ $(1 \leq i \leq n, j \in \mathbb{N})$ over $K$.

For a non-empty set $A$, let $\text{Sym}(A)$ denote the group of all permutations of $A$. The group $\text{Sym}(\mathbb{N})$ acts on $R_n$ in a natural way: $\sigma(x_{ij}) = x_{i\sigma(j)}$ if $\sigma \in \text{Sym}(\mathbb{N})$. An ideal $I$ of $R_n$ is called $\text{Sym}(\mathbb{N})$-invariant if $\sigma(I) = I$ for all $\sigma \in \text{Sym}(\mathbb{N})$.

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Theorem 1 (see [1, 3, 4, 13]). Let $K$ be a Noetherian unital associative and commutative ring. Then each $\text{Sym}(\mathbb{N})$-invariant ideal of $R_n = K[x_{ij} | 1 \leq i \leq n, j \in \mathbb{N}]$ is finitely generated (as a $\text{Sym}(\mathbb{N})$-invariant ideal).

For $n = 1$ this theorem was proved by Cohen [3] in 1967 and rediscovered independently by Aschenbrenner and Hillar [1] in 2007; for an arbitrary positive $n$ this was proved by Cohen [4] in 1987 and rediscovered independently by Hillar and Sullivant [13] in 2012. Cohen’s results were motivated by the finite basis problem for identities of metabelian groups and the results of Aschenbrenner, Hillar and Sullivant by applications to chemistry and algebraic statistics.
Let
\[ d_{i_1i_2\ldots i_n} = \det \begin{pmatrix} x_{i_1} & x_{i_2} & \ldots & x_{i_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_{i_1}} & x_{n_{i_2}} & \ldots & x_{n_{i_n}} \end{pmatrix} \]
be the determinant of the matrix above. It is clear that \( d_{i_1i_2\ldots i_n} \) is a polynomial contained in \( R_n \). Let \( L_n \) be the subalgebra in \( R_n \) generated by all polynomials \( d_{i_1i_2\ldots i_n} \) (\( i_\ell \in \mathbb{N} \)). The group \( \text{Sym}(\mathbb{N}) \) acts on \( L_n \) in a natural way: \( \sigma(d_{i_1i_2\ldots i_n}) = d_{\sigma(i_1)\sigma(i_2)\ldots\sigma(i_n)} \) (\( \sigma \in \text{Sym}(\mathbb{N}) \)). \( \text{Sym}(\mathbb{N}) \)-invariant ideals of \( L_n \) are also defined in a natural way.

The following theorem was proved by Draisma \[6\] in 2010; it solves a problem arising from applications to algebraic statistics and chemistry posed in \[1\].

**Theorem 2** (see \[6\]). Let \( K \) be a field of characteristic 0. Then every \( \text{Sym}(\mathbb{N}) \)-invariant ideal in \( L_n \) is finitely generated (as such).

However, Theorem 2 is not valid over a field \( K \) of characteristic 2: the algebra \( L_2 \) over such \( K \) contains \( \text{Sym}(\mathbb{N}) \)-invariant ideals that are not finitely generated. More precisely, the following theorem holds.

**Theorem 3** (see \[9\]). Suppose that \( K \) is a field of characteristic 2. Let \( I \) be the \( \text{Sym}(\mathbb{N}) \)-invariant ideal in \( L_2 \) generated by the set
\[ \{ d_{12}d_{23}\ldots d_{(k-1)k}d_{1k} \mid k = 3,4,\ldots \} \]
Then \( I \) is not finitely generated (as a \( \text{Sym}(\mathbb{N}) \)-invariant ideal in \( L_2 \)).

The proof of Theorem 3 is based on the ideas of Vaughan-Lee \[17\] developed in order to construct an example of a non-finitely based variety of abelian-by-nilpotent Lie algebras.

Theorems 2 and 3 raise the following problem:

**Problem.** For which \( n \) and \( p \) the algebra \( L_n \) over a field \( K \) of characteristic \( p > 0 \) is \( \text{Sym}(\mathbb{N}) \)-Noetherian (that is, each \( \text{Sym}(\mathbb{N}) \)-invariant ideal in \( L_n \) is finitely generated) ?

In order to find an approach to this problem one can consider a simpler (but similar in a certain sense) question about \( \text{Sym}(\mathbb{N}) \)-Noetherianity of the subalgebra \( \overline{R}_n \) of symmetric polynomials (defined below).

Let \( S_n = \text{Sym}(\{1,2,\ldots,n\}) \). The group \( S_n \) acts on \( R_n \) in a natural way: \( \tau(x_{ij}) = x_{\tau(i)j} \). In other words, \( S_n \) acts on the infinite matrix
\[ \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1i} & \ldots \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ x_{n1} & x_{n2} & \ldots & x_{ni} & \ldots \end{pmatrix} \]
permuting its lines.

We call a polynomial \( f \in R_n \) symmetric if \( \tau(f) = f \) for all \( \tau \in S_n \). Let \( \overline{R}_n \) be the set of all symmetric polynomials of \( R_n \); it is clear that \( \overline{R}_n \) is a \( \text{Sym}(\mathbb{N}) \)-invariant \( K \)-subalgebra in \( R_n \). \( \text{Sym}(\mathbb{N}) \)-invariant ideals of \( \overline{R}_n \) are defined in a natural way.
The following theorem was proved by the second author of the present note [14, Lemma 7] in 1992 in order to prove the finite basis property for certain varieties of nilpotent-by-abelian Lie algebras.

**Theorem 4** (see [14]). Let $K$ be a field of characteristic 0 or of characteristic $p > n$. Then every $\text{Sym}(\mathbb{N})$-invariant ideal in $R_n$ is finitely generated (as such).

The proof of Theorem 4 is very simple. Let $\pi : R_n \to \overline{R}_n$ be the symmetrization map:

$$\pi(f) = \pi(f(x_{11}, x_{12}, \ldots; x_{n1}, x_{n2}, \ldots)) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\tau(1)1}, x_{\tau(1)2}, \ldots; x_{\tau(n)1}, x_{\tau(n)2}, \ldots)$$

for all $f \in R_n$. Note that $\pi$ is $K$-linear, commutes with each permutation $\sigma \in \text{Sym}(\mathbb{N})$ and if $\alpha \in \overline{R}_n$, $f \in R_n$ then $\pi(\alpha) = \alpha$ and $\pi(\alpha f) = \alpha \pi(f)$.

Let $I$ be a $\text{Sym}(\mathbb{N})$-invariant ideal in $\overline{R}_n$. Then $I \cdot R_n$ is a $\text{Sym}(\mathbb{N})$-invariant ideal in $R_n$ that, by Theorem 4, is generated (as a $\text{Sym}(\mathbb{N})$-invariant ideal in $\overline{R}_n$) by a finite set, say, by $b_1, \ldots, b_k$. We may assume that $b_i \in I \subseteq \overline{R}_n$ for all$i$.

We claim that $b_1, \ldots, b_k$ generate $I$ as a $\text{Sym}(\mathbb{N})$-invariant ideal in $\overline{R}_n$. Indeed, take an arbitrary element $g \in I$. Since $b_1, \ldots, b_k$ generate $I \cdot R_n$ as a $\text{Sym}(\mathbb{N})$-invariant ideal in $R_n$, we have $g = \sum_i \sigma_i(b_{\ell_i})f_i$ for some $\sigma_i \in \text{Sym}(\mathbb{N})$ and $f_i \in R_n$. It follows that

$$g = \pi(g) = \pi(\sum_i \sigma_i(b_{\ell_i})f_i) = \sum_i \pi(\sigma_i(b_{\ell_i})f_i) = \sum_i \sigma_i(b_{\ell_i})\pi(f_i),$$

that is, $g$ belongs to the $\text{Sym}(\mathbb{N})$-invariant ideal of $\overline{R}_n$ generated by $b_1, \ldots, b_k$, as claimed.

The proof of Theorem 4 is much more sophisticated than that of Theorem 5 but follows the same plan, with the group $S_n$ acting on the matrix [1] by line permutations replaced by the group $\text{SL}_n(K)$ acting on [1] by multiplications from the left and with the symmetrization map replaced by the Reynolds operator.

The aim of the present note is to prove the following theorem.

**Theorem 5.** Let $K$ be a field of characteristic $p$ such that $0 < p \leq n$. Then the algebra $\overline{R}_n$ contains $\text{Sym}(\mathbb{N})$-invariant ideals that are not finitely generated.

Thus, the algebra $\overline{R}_n$ is $\text{Sym}(\mathbb{N})$-Noetherian if $\text{char} K = 0$ or $\text{char} K > n$ and is not $\text{Sym}(\mathbb{N})$-Noetherian if $0 < \text{char} K \leq n$. Possibly (although it still remains unknown), this is also the case for the algebra $L_n$ if $\text{char} K > 2$.

Theorem 5 is a corollary of the following result. For each $k \in \mathbb{N}$, let $h_k$ be the polynomial in $R_n$ defined as follows:

$$h_k = x_{11}x_{12} \ldots x_{1k} + x_{21}x_{22} \ldots x_{2k} + \cdots + x_{n1}x_{n2} \ldots x_{nk}.$$
For example, \( h_1 = x_{11} + x_{21} + \cdots + x_{n1}, \) \( h_2 = x_{11}x_{12} + x_{21}x_{22} + \cdots + x_{n1}x_{n2}. \) It is clear that \( h_k \in \mathcal{R}_n \) for all \( k. \)

Let \( U \) be the \( \text{Sym}(\mathbb{N}) \)-invariant ideal of \( \mathcal{R}_n \) generated by the set
\[
\{ h_k \in \mathcal{R}_n \mid k = 1, 2, \ldots \}.
\]

Our main result is as follows.

**Theorem 6.** Let \( K \) be a field of characteristic \( p > 0. \) Suppose that \( n \geq p. \) Then the ideal \( U \) is not finitely generated as a \( \text{Sym}(\mathbb{N}) \)-invariant ideal in \( \mathcal{R}_n. \)

We will prove Theorem 6 by proving the following slightly stronger result:

**Theorem 7.** Let \( K \) be a field of characteristic \( p > 0. \) Suppose that \( n \geq p. \) Then, for each \( k \in \mathbb{N}, \) the polynomial \( h_k \) is not contained in the \( \text{Sym}(\mathbb{N}) \)-invariant ideal of \( \mathcal{R}_n \) generated by the set \( \{ h_l \mid l \in \mathbb{N}, l \neq k \}. \)

**Remarks.**

1. For other recent results about \( \text{Sym}(\mathbb{N}) \)-Noetherian polynomial algebras see, for instance, articles [2, 8, 10, 11, 12] and a survey [7].

2. The main result of [14] about polynomial rings is, in fact, stronger than Theorem 4. It has been proved there that if \( \text{char} K = 0 \) or \( \text{char} K > n \) then each \( \text{Sym}(\mathbb{N}) \)-invariant \( \mathcal{R}_n \)-submodule in \( \mathcal{R}_n \) is finitely generated (as a \( \text{Sym}(\mathbb{N}) \)-invariant \( \mathcal{R}_n \)-submodule).

3. Some results about Noetherian properties of polynomial rings in infinitely many variables have been proved in [15, 16] in order to solve the finite basis problem for certain varieties of groups and group representations.

Let \( \Psi \) be the set of endomorphisms \( \psi_{k\ell} \) \((k \neq \ell)\) of \( \mathcal{R}_n = K[x_{ij} \mid 1 \leq i \leq n, j \in \mathbb{N}] \) such that
\[
\psi_{k\ell}(x_{ij}) = \begin{cases} x_{ik}x_{i\ell} & \text{if } j = \ell; \\ x_{ij} & \text{otherwise.} \end{cases}
\]

We say that a subset \( S \) of \( \mathcal{R}_n \) is \( \Psi \)-closed if \( \psi_{k\ell}(S) \subseteq S \) for all \( \psi_{k\ell} \in \Psi. \) It has been proved in [15] that, over a Noetherian associative and commutative unital ring \( K, \) each \( \text{Sym}(\mathbb{N}) \)-invariant \( \Psi \)-closed \( \mathcal{R}_n \)-submodule in \( \mathcal{R}_n \) is finitely generated (as such). Moreover, over such \( K \) each \( \text{Sym}(\mathbb{N}) \)-invariant \( \Psi \)-closed \( \mathcal{R}_n \)-submodule in \( \mathcal{R}_n \) is finitely generated [16]. Here \( \mathcal{R}_n \) is a \( K \)-subalgebra of \( \mathcal{R}_n \) generated by all products of the form
\[
f(x_{11}, x_{12}, \ldots) f(x_{21}, x_{22}, \ldots) \cdots f(x_{n1}, x_{n2}, \ldots)
\]
where \( f(t_1, t_2, \ldots) \in K[t_i \mid i \in \mathbb{N}]. \)

Note that the result about \( \mathcal{R}_n \)-modules has been proved using techniques similar to one used by Cohen [3] while the proof of the result about \( \mathcal{R}_n \)-modules is based on different ideas.

4. Let \( M \) be the free metabelian group on a free generating set \( \{ x_i \mid i \in \mathbb{N} \}. \) Then the group \( \text{Sym}(\mathbb{N}) \) acts on \( M \) permuting the free generators \( x_i. \) Cohen [3] has proved that in \( M \) all \( \text{Sym}(\mathbb{N}) \)-invariant normal subgroups are finitely generated (as such). For an application of this result see, for example, [5].
Note that $Sym(\mathbb{N})$-invariant normal subgroups are finitely generated (as $Sym(\mathbb{N})$-invariant normal subgroups) not just in the variety of metabelian groups but also in some larger varieties (Vaughan-Lee [18]). However, it can be easily shown that in the variety of centre-by-metabelian groups (and, therefore, in all larger varieties) this property does not hold.

2. Proof of Theorem 7

We claim that to prove Theorem 7 one may assume without loss of generality that $n = p$. Indeed, suppose that $n > p$. Let $\psi : R_n \to R_p$ be the homomorphism of $R_n$ onto $R_p$ such that
\[
\psi(x_{ij}) = \begin{cases} x_{ij} & \text{if } 1 \leq i \leq p; \\ 0 & \text{if } p < i \leq n. \end{cases}
\]
It is clear that $\psi(R_n) = R_p$ and $\psi \sigma = \sigma \psi$ for all $\sigma \in Sym(\mathbb{N})$. Hence, to prove that the polynomial
\[ h_k = x_{11}x_{12} \ldots x_{1k} + x_{21}x_{22} \ldots x_{2k} + \cdots + x_{n1}x_{n2} \ldots x_{nk} \]
is not contained in the $Sym(\mathbb{N})$-invariant ideal of $R_n$ generated by the set \{\(h_\ell \mid \ell \in \mathbb{N}, \ell \neq k\)\} it suffices to prove that the polynomial
\[ \psi(h_k) = x_{11}x_{12} \ldots x_{1k} + x_{21}x_{22} \ldots x_{2k} + \cdots + x_{p1}x_{p2} \ldots x_{pk} \]
is not contained in the $Sym(\mathbb{N})$-invariant ideal of $\psi(R_n) = R_p$ generated by the set \{\(\psi(h_\ell) \mid \ell \in \mathbb{N}, \ell \neq k\)\} where
\[ \psi(h_\ell) = x_{11}x_{12} \ldots x_{1\ell} + x_{21}x_{22} \ldots x_{2\ell} + \cdots + x_{p1}x_{p2} \ldots x_{p\ell}. \]
The claim follows.

Further, let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the prime subfield of $K$, $\mathbb{F}_p < K$. We claim that to prove Theorem 7 one can assume without loss of generality that $K = \mathbb{F}_p$. Indeed, let $R_{p,\mathbb{F}_p} = \mathbb{F}_p[x_{ij} \mid 1 \leq i \leq p; j \in \mathbb{N}]$ and let $\mathbb{R}_{p,\mathbb{F}_p}$ be the subalgebra of symmetric polynomials of $R_{p,\mathbb{F}_p}$. Then $R_{p,\mathbb{F}_p} < R_p = K[x_{ij} \mid 1 \leq i \leq p; j \in \mathbb{N}]$, $\mathbb{R}_{p,\mathbb{F}_p} < R_p$ and $h_k \in \mathbb{R}_{p,\mathbb{F}_p}$ for all $k$.

Suppose that $h_k$ is contained in the $Sym(\mathbb{N})$-invariant ideal of $R_p$ generated by the set \{\(h_\ell \mid \ell \in \mathbb{N}, \ell \neq k\)\}. Then
\[ h_k = \sum_s \sigma_s(h_{\ell_s})f_s \]
where $\sigma_s \in Sym(\mathbb{N})$, $f_s \in \mathbb{R}_p$, $\ell_s < k$ for all $s$. Let $\mathcal{B}$ be a basis of $K$ viewed as a vector space over $\mathbb{F}_p$ such that $1 \in \mathcal{B}$. Then, for each $s$, $f_s = f_{s,0} + \sum_t b_tf_{s,t}$ where $f_{s,0} \in R_{p,\mathbb{F}_p}$ and, for all $t$, $b_t \in \mathbb{B} \setminus \{1\}$, $f_{s,t} \in R_{p,\mathbb{F}_p}$. It follows that
\[ h_k = \sum_s \sigma_s(h_{\ell_s})f_{s,0} + \sum_s \sum_t b_t \sigma_s(h_{\ell_s})f_{s,t}. \]
Since $h_k$, $\sigma_s(h_{\ell_s})f_{s,0}$, $\sigma_s(h_{\ell_s})f_{s,t} \in R_{p,\mathbb{F}_p}$, we have
\[ (2) \quad h_k = \sum_s \sigma_s(h_{\ell_s})f_{s,0}. \]
Note that $f_{s,0} \in \overline{R}_{p,F_p}$ for all $s$. Indeed, for each $\tau \in S_n$,
$$f_{s,0} + \sum_{t} b_t f_{s,t} = f_s = \tau(f_s) = \tau(f_{s,0}) + \sum_{t} b_t \tau(f_{s,t})$$
so $\tau(f_{s,0}) = f_{s,0}$. Thus, if $h_k$ is contained in the $\text{Sym}(\mathbb{N})$-invariant ideal of $\overline{R}_p$ generated by the set $\{h_{l}\mid l \in \mathbb{N}, \ell \neq k\}$ then, by (2), $h_k$ belongs to the $\text{Sym}(\mathbb{N})$-invariant ideal of $\overline{R}_{p,F_p}$ generated by $\{h_{l}\mid l \in \mathbb{N}, \ell \neq k\}$. Hence, one may assume that $K = \mathbb{F}_p$, as claimed.

From now on, we assume $K = \mathbb{F}_p$ and write $R_p$ and $\overline{R}_p$ for $R_{p,F_p}$ and $\overline{R}_{p,F_p}$, respectively.

Let $R = \mathbb{Z}[x_{ij} \mid 1 \leq i \leq p, j \in \mathbb{N}]$. Let $\mu : R \to R_p$ be the homomorphism of $R$ onto $R_p$ such that $\mu(x_{ij}) = x_{ij}$ for all $i, j$. Suppose, in order to get a contradiction, that $h_k$ is contained in the $\text{Sym}(\mathbb{N})$-invariant ideal of $\overline{R}_p$ generated by the set $\{h_{l}\mid l \in \mathbb{N}, \ell \neq k\}$. Then in $\overline{R}_p$ we have
$$h_k = \sum_s \sigma_s(h_{\ell_s}) f_s$$
where $\sigma_s \in \text{Sym}(\mathbb{N})$, $f_s \in \overline{R}_p$, $\ell_s < k$ for all $s$. Hence, in $R$ we have

$$h_k = \sum_s \sigma_s(h_{\ell_s}) g_s + p g$$

where $g_s, g \in R$, $\mu(g_s) = f_s$ for all $s$. It is clear that we may assume that $g_s \in \overline{R}$ for all $s$. It follows that $g = \frac{1}{p}(h_k - \sum_s \sigma_s(h_{\ell_s}) g_s) \in \overline{R}$.

Let
$$m = x_{11}^{u_{11}} x_{21}^{u_{21}} \cdots x_{p1}^{u_{p1}} x_{12}^{u_{12}} x_{22}^{u_{22}} \cdots x_{p2}^{u_{p2}} \cdots x_{1k}^{u_{1k}} x_{2k}^{u_{2k}} \cdots x_{pk}^{u_{pk}} \in R$$
be a monomial. Define a multi-degree $d(m)$ of the monomial $m$ as follows:
$$d(m) = (u_1, u_2, \ldots, u_k, 0, 0, \ldots),$$
where $u_k = u_{1k} + u_{2k} + \cdots + u_{pk}$ for all $k \in \mathbb{N}$. Note that $d(m \cdot m') = d(m) + d(m')$ for all monomials $m, m'$ in $R$. Note also that we may assume that the polynomials $h_k, \sigma_s(h_{\ell_s}) g_s$ and $g$ in (3) are of the same multi-degree $(1, 1, \ldots, 1, 0, 0, \ldots)$.

Let $S = \mathbb{Z}[t_j \mid j \in \mathbb{N}]$. Define a homomorphism $\eta : R \to S$ of $R$ onto $S$ by $\eta(x_{ij}) = t_j$. It is clear that $\eta(h_k) = p t_{11} t_{21} \cdots t_k$, $\eta(\sigma_s(h_{\ell_s})) = p t_{\sigma_s(1)} t_{\sigma_s(2)} \cdots t_{\sigma_s(\ell_s)}$.

Note that if $m \in R$ is a monomial of multi-degree $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, 0, 0, \ldots)$ where $\varepsilon_j = 0, 1$ then the number of elements of the $S_p$-orbit of $m$ is a multiple of $p$. Hence, $\eta(m) = p(\varepsilon_1 m')$ for some monomial $m' \in S$ and some integer $q \in \mathbb{Z}$. It follows that, for all $s$, $\eta(g_s) = p u_s$ and $\eta(g) = p u$ for some $u_s, u \in S$.

Applying $\eta$ to both sides of the equality (3), we get
$$p t_{11} t_{21} \cdots t_k = p^2 \sum_s t_{\sigma_s(1)} t_{\sigma_s(2)} \cdots t_{\sigma_s(\ell_s)} u_s + p^2 u$$
for some $u_s, u \in S$. This is a contradiction because the right hand side of the equality above is a (non-zero) multiple of $p^2$ in $S = \mathbb{Z}[t_j \mid j \in \mathbb{N}]$ and the left hand side is not. It follows that $h_k$ is not contained in the $\text{Sym}(\mathbb{N})$-invariant ideal of $\mathbb{R}_p$ generated by the set $\{h_\ell \mid \ell \in \mathbb{N}, \ell \neq k\}$, as required.

The proof of Theorem [7] is completed.

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