New features in a time dependent magnetic field lens

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Abstract. In this work we study the noise effect in a time dependent magnetic lens. We focus our analysis in a magnetic field, such that, it can be separate in two terms, one a constant and the other a random field. Using the Furutsu-Novikov-Shapiro-Logunov method, for dichotomous colorized noise, we find explicit differential equations for the first and second momenta and the correlation functions. Finally, we show that for some range of parameters is possible to find stable behaviors of these quantities.

1. Introduction

Recently we described the theory of a time dependent magnetic field lens (TDMFL) \cite{1}. The lens consists basically of an axially symmetric cylindrical coil connected to a current source of controllable intensity. If an electron beam is injected into this coil, it was shown that the system acts as a lens by either focusing a beam of parallel electrons or by forming an image of some object emitting electrons.

For recording purposes if a stationary image or focal point is desired it is necessary to employ a pulsed electrons beam injected in some appropriate synchronism with the time dependent electromagnetic field inside the coil. Electron dynamics is then described with no approximations, such s the paraxial expansion, and therefore spherical aberration is absent in this case \cite{2}.

The goal of this paper is to communicate our analysis of the noise influence in the TDMFL, for the for dichotomous colorized noise \cite{3}. The paper is organized as follows: In Sec. II, the theoretical model is presented In Sec. III the noise effect is performed. Finally, conclusions are presented in Sec. IV.

2. Theoretical Model

In this section we introduce the bases of this TDMFL. For this propose, let us use the main results obtained before \cite{1, 2} for the dynamics of an electron, assuming that the coil axis is oriented along the z-axis. Let $\mathbf{B}(t) = B(t)\hat{z}$ be the general expression for magnetic field and $\mathbf{E}(r, t) = \mathbf{r} \times \mathbf{B}(t)/2c$ the resulting electric field. Clearly, the Lorentz force will act in the transverse $x$-$y$ plane, so that the particle will drift freely along the $z$-axis with a constant speed $v_z$. The electron trajectories satisfying the initial conditions $\mathbf{r}(t_0) = \mathbf{r}_0$, $\mathbf{v}(t_0) = \mathbf{v}_0$ can be written as:
\[ \mathbf{r}(t) = R\left(\frac{\theta(t)}{2}\right)(\mathbf{u}_1(t)\mathbf{v}_{in} + \mathbf{u}_2(t)\mathbf{r}_{in} - \frac{1}{2}\mathbf{u}_1(t)\omega(t_0) \times \mathbf{r}_{in}) \]  

(1)

where \( \omega = (eB(t)/(mc))\dot{z} \) and \( \mathbf{r} \) represent the transverse position. The functions \( \mathbf{u}_{1,2}(t) \) are two solutions of

\[ \ddot{u}(t) + \left(\frac{\omega(t)}{2}\right)^2 u(t) = 0 \]  

(2)

satisfying the initial conditions \( \mathbf{u}_1(t_0) = \mathbf{\dot{u}}_2(t_0) = 0, \mathbf{\dot{u}}_1(t_0) = u_2(t_0) = 1 \) and

\[ R(\theta/2) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \]  

(3)

with \( \theta(t) = \int_{t_0}^{t} \omega(t')dt' \). In this form the problem is reduced to finding the solutions \( \mathbf{u}_{1,2}(t) \) and, in general the dynamical behavior is containing in the its solutions.

Let us note some general properties of Eq. (1). First, if \( \mathbf{v}_{in} = 0 \) and \( \omega(t_0) = 0 \), then every time \( t_2 \) for which \( u_2(t_2) = 0 \) produces \( \mathbf{r}(t_2) = 0 \), irrespective of \( \mathbf{r}_{in} \). On the other hand, for every \( t_1 \) for which \( u_1(t_1) = 0 \), Eq. (1) yields

\[ \mathbf{r}(t_1) = R(\theta(t_1)/2)u_2(t_1)\mathbf{r}_{in} \]  

(4)

which implies that, irrespective of \( \mathbf{v}_{in} \), the position at \( t_1 \) is obtained from \( \mathbf{r}_{in} \) by a rotation by \( \theta(t_1)/2 \) and re-scaling by \( u_2(t) \). These two properties constitute respectively the basis for beam focusing and image formation of our system.

### 3. Magnetic Noise Effect

We analyze the effect of the noise in the magnetic field, so we shall consider the case of a time random magnetic fields for which there is a strong enhancement of the instabilities. Let us assume that \( \mathbf{B}(t) \) has the form

\[ \mathbf{B}(t) = \mathbf{B}_0 + \mathbf{B}_1(t) \]  

(5)

where \( \mathbf{B}_0 \) is a constant field and \( \mathbf{B}_1(t) \) is a random function of \( t \). We assume that

\[ \langle \mathbf{B}(t) \rangle = 0 \]  

(6)

\[ \langle \mathbf{B}_1(t)\mathbf{B}_1(t') \rangle = \Gamma(t, t') \]  

(7)

where \( \langle \ldots \rangle \) indicates averaging over some distribution and \( \Gamma(t, t') \approx 0 \) for \( t' > \tau_c \), with \( \tau_c \) being the correlation time of the fluctuating field. If we denote \( \omega_{0,1} = (eB_{0,1}/(mc)) \) and substitute it into Eq. (2), we obtain a stochastic differential equation determining the functions \( u_{1,2}(t) \), which also corresponds to the equation of motion of a harmonic oscillator with a time-random frequency

\[ \ddot{u}(t) + \frac{1}{4}\left[\omega_0^2 + 2\omega_0\omega_1(t) + \omega_1^2(t)\right]u(t) = 0. \]  

(8)

Let us note that this type of equation arises in other contexts such as, for example, wave propagation through random media and has been amply studied in the literature [4, 5]. It is possible to develop approximate methods for this problem, in the case of white and colorized dichotomous noise [3]. Clearly, a non divergent behavior of \( u_{1,2}(t) \) implies a similar behavior for the electron trajectories; on the other hand, the divergence of \( u_{1,2}(t) \) and of their products may lead to dynamical instabilities.
In order to find analytical close solution we will restrict to the following color noise with the exponential correlator

$$\langle \omega_1(t) \omega_1(t') \rangle = \sigma^2 e^{\lambda |t-t'|}. \tag{9}$$

This special type of color noise is the symmetric dichotomous noise (random telegraph signal) where the random variable \( \omega_1(t) \) may take one of the two values \( \omega = \pm \sigma \) with the mean waiting time \( \lambda^{-1} \) in each of these two states.

Firstly, we note that the second-order differential equation (8) can be rewritten as two first-order differential equations which, after averaging, take the following form:

$$\frac{d}{dt} \langle u(t) \rangle = \langle y \rangle \tag{10}$$

$$\frac{d}{dt} \langle y(t) \rangle = -\frac{1}{4} \left[ \omega_0^2 \langle u \rangle + 2 \omega_0 \langle \omega_1(t) u \rangle + \sigma^2 \langle u \rangle \right] \tag{11}$$

In order to find the new correlator \( \langle \omega_1(t) u \rangle \), we use the Furutsu-Novikov-Shapiro-Logunov method[6, 7] which for exponentially correlated noise (9) yields

$$\frac{d}{dt} \langle \omega_1(t) g \rangle = \langle \omega_1(t) \frac{d}{dt} g \rangle - \lambda \langle \omega_1(t) g \rangle \tag{12}$$

where \( g \) is some function of noise, \( g = g[\omega_1(t)] \).

After some algebraic manipulation and using the same procedure for all the unknown correlators, we obtain a system of four equations for four variables \( \langle u \rangle \), \( \langle y \rangle \), \( \langle \omega_1 u \rangle \) and \( \langle \omega_1 y \rangle \). From these equations one can find the fourth-order differential equation for \( \langle u \rangle \):

$$\frac{d^4}{dt^4} \langle u \rangle + 2 \lambda \frac{d^3}{dt^3} \langle u \rangle + \Omega^2 \frac{d^2}{dt^2} \langle u \rangle + \frac{\lambda}{2} \frac{d}{dt} \langle u \rangle + \frac{1}{16} \left[ (\epsilon^2_-)^2 + \lambda^2 \epsilon^2_+ \right] \langle u \rangle = 0 \tag{13}$$

where \( \Omega^2 \equiv \frac{1}{2} (\omega_0^2 + \sigma^2 + 2 \lambda \sigma) \), and \( \epsilon^2_\pm \equiv \omega_0^2 \pm \sigma^2 \).

For the second moment and the correlation function we proceed in a similar way, obtaining:

$$\frac{d^3}{dt^3} \langle u^2 \rangle + \epsilon^2 \frac{d^2}{dt^2} \langle u^2 \rangle + 2 \omega_0 \frac{d}{dt} \langle \omega_1 u^2 \rangle + \lambda \omega_0 \langle \omega_1 u^2 \rangle = 0 \tag{14}$$

and for the correlation function, \( z = \langle u(t_1) u(t) \rangle \), is easily proved that has the following differential equation:

$$\frac{d^4}{dt^4} z + 2 \lambda \frac{d^3}{dt^3} z + \Omega^2 \frac{d^2}{dt^2} z + \lambda \frac{d}{dt} z + \frac{1}{16} \left[ (\epsilon^2_-)^2 + 4 \lambda^2 \epsilon^2_+ \right] z = 0 \tag{16}$$

These equations can be solved for different set of values for the parameters \( \lambda, \sigma \) and \( \omega_0 \). The main results are displayed in Fig. 1 where we plot \( \langle u_1 \rangle \) (a), \( \langle u_2^2 \rangle \) (b) and \( \langle u_2(t_1) u_2(t) \rangle \) (c) as function of time, at two set of values of control parameters. We note, that, for \( \lambda = 0.01, \sigma = 0.3 \) and \( \omega_0 = 1 \) the second moment diverge and consequently this is not a physically accepted solution. However there exist a set of values for which the behavior is stable (See Fig. 1(b)). In addition, an exhaustive revision for ten values of \( \lambda \) between 0.001 and 0.01, ten values of \( \sigma \) between 0.01 and 0.1 and ten values of \( \omega_0 \) between 1 and 10, has confirm us that the convergence has a critical dependence of this parameters.

Finally, let us remark that for other kinds of noise, for example white or Gaussian noise, the dynamic behavior of the second momenta and the correlation functions are in general divergent [3].
4. Conclusions
In this work is studied in detail, the noise effect in a time dependent magnetic field lens and it was shown that it have an important role in the performance of these lenses. In the case of dicotummuos colorized noise, we find the corresponding differential equations for the first moment, second moment and the correlation function. In the studied range of parameters, appears instable and stable regions which affect the expected trajectories of the particles. In fact, it is possible to find a set parameters in which produce stable trajectories and thus the stochastic dynamic for the temporal average has similar behavior to the pure lens.

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