Matter bounce cosmology with the $f(T)$ gravity

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Abstract

We show that the $f(T)$ gravitational paradigm, in which gravity is described by an arbitrary function of the torsion scalar, can provide a mechanism for realizing bouncing cosmologies, thereby avoiding the Big Bang singularity. After constructing the simplest version of an $f(T)$ matter bounce, we investigate the scalar and tensor modes of cosmological perturbations. Our results show that metric perturbations in the scalar sector lead to a background-dependent sound speed, which is a distinguishable feature from Einstein gravity. Additionally, we obtain a scale-invariant primordial power spectrum, which is consistent with cosmological observations, but suffers from the problem of a large tensor-to-scalar ratio. However, this can be avoided by introducing extra fields, such as a matter bounce curvaton.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Inflation is now considered to be a crucial part of the cosmological history of the universe [1]; however, the so-called standard model of the universe still faces the problem of the initial singularity. Such a singularity is unavoidable if inflation is realized using a scalar field while the background spacetime is described by the standard Einstein action [2]. As a consequence, there has been much energy expended toward resolving this problem, e.g. through null energy condition violating quantum fluctuations (‘island cosmology models’ [3, 4]), quantum gravity effects or effective field theory techniques.
A potential solution to the cosmological singularity problem may be provided by non-singular bouncing cosmologies [5]. Such scenarios have been constructed through various approaches to modified gravity, such as the pre-Big Bang [6] and the ekpyrotic [7] models, gravitational actions with higher order corrections [8], braneworld scenarios [9], non-relativistic gravity [10, 11], loop quantum cosmology [12] or in the frame of a closed universe [13]. For a review on models of modifying higher derivative gravity, which solve cosmic singularities in an efficient way, we refer to [14]. Non-singular bounces may be alternatively investigated using effective field theory techniques, introducing matter fields which violate the null energy condition [15, 16] or introducing non-conventional mixing terms [17]. The extension of all the above bouncing scenarios is the (old) paradigm of cyclic cosmology [18], in which the universe experiences a periodic sequence of contractions and expansions, which has been reawakened recently [19], since it brings different insights to the origin of the observable universe [20, 21, 22, 23] (see [24] for a review).

Along separate lines, $f(T)$ gravity has recently received attention in the literature [25–27], mostly in the context of explaining the observed acceleration of the universe. It is based on the old idea of the ‘teleparallel’ equivalent of general relativity (TEGR) [28, 29], which, instead of using the curvature defined via the Lévi-Civita connection, uses the Weitzenböck connection that has no curvature but only torsion. The dynamical objects in such a framework are the four linearly independent vierbeins. The advantage of this framework is that the torsion tensor is formed solely from products of first derivatives of the tetrad. As described in [29], the Lagrangian density, $T$, can then be constructed from this torsion tensor under the assumptions of invariance under general coordinate transformations, global Lorentz transformations and the parity operation, along with requiring the Lagrangian density to be second order in the torsion tensor. However, instead of using the tensor scalar $T$, the authors of [26, 27] generalized the above formalism to a modified $f(T)$ version, thus making the Lagrangian density a function of $T$, similar to the well-known extension of $f(R)$ Einstein–Hilbert action. In comparison with $f(R)$ gravity, whose fourth-order equations may lead to pathologies, $f(T)$ gravity has the significant advantage of possessing second-order field equations. This feature has led to a rapidly increasing interest in the literature, and apart from obtaining acceleration [26, 27] one can reconstruct a variety of cosmological evolutions [30] and solutions [31], add a scalar field [32], use observational data in order to constrain the model parameters [33], examine the dynamical behavior of the scenario [34], and proceed beyond the background evolution, investigating the vacuum and matter perturbations [35, 36] as well as the large-scale structure [37]. Note, however, that there is a discussion whether forms of $f(T)$ other than linear should be expected [38].

One interesting feature of the $f(T)$ theory is that the null energy condition could be effectively violated. Accompanied with this feature, one expects to obtain a series of nontrivial phenomenological solutions. Particularly, it was observed that the violation of null energy condition is related to the nonsingular bouncing solution in the early universe [15]. Therefore, the singularity avoidance can be obtained in general in $f(T)$ gravity. In this work, we are interested in searching for a scenario of bouncing cosmology in the early universe, which is governed by $f(T)$ gravity. As we show, the realization of a big bounce and the avoidance of singularities are straightforward. Additionally, we investigate in detail the evolution of perturbations in a specific model of the matter bounce. Our example illustrates that it is possible to generate a scale-invariant primordial power spectrum and pass through the nonsingular bouncing point in the context of $f(T)$ bounce cosmology.

This paper is organized as follows. In section 2, we briefly review the basic idea of $f(T)$ gravity. In section 3, we investigate the realization of bouncing cosmology by virtue of $f(T)$ gravity. Specifically, we postulate an explicit form for the background scale factor,
and following this ansatz we reconstruct the corresponding form of $f(T)$ both analytically and numerically. In section 4, we study the evolution of cosmological perturbations of scalar and tensor types along with the matter bounce scenario. Finally, section 5 is devoted to the summary of the results.

2. $f(T)$ gravity and cosmology

In this section, we briefly review $f(T)$ gravity and we provide the background cosmological equations in a universe governed by such a modified gravitational sector. Throughout the work, we consider a flat Friedmann–Robertson–Walker (FRW) background geometry with the metric

$$ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j,$$

where $t$ is the cosmic time, $x^i$ are the comoving spatial coordinates, and $a(t)$ is the scale factor. In this paper, our notation is as follows: Greek indices $\mu, \nu, \ldots$ run over all coordinate spacetimes 0, 1, 2, 3, lowercase Latin indices (from the middle of the alphabet) $i, j, \ldots$ run over spatial coordinates 1, 2, 3, capital Latin indices $A, B, \ldots$ run over the tangent spacetimes 1, 2, 3, and lowercase Latin indices (from the beginning of the alphabet) $a, b, \ldots$ will run over the tangent space spatial coordinates 1, 2, 3.

2.1. $f(T)$ gravity

As stated in the introduction, the dynamical variable of the old ‘teleparallel’ gravity, as well as its $f(T)$ extension, is the vierbein field $e^A(x^\mu)$. This forms an orthonormal basis for the tangent space at each point $x^\mu$ of the manifold, that is, $e^A \cdot e^B = \eta_{AB}$, where $\eta_{AB} = \text{diag}(1, -1, -1, -1)$. Furthermore, the vector $e_A$ can be analyzed with the use of its components $e^\mu_A$ in a coordinate basis, that is, $e_A = e^\mu_A \partial_\mu$.

In such a construction, the metric tensor is obtained from the dual vierbein as

$$g_{\mu \nu}(x) = \eta_{AB} e_A^\mu(x) e_B^\nu(x).$$

Contrary to general relativity, which uses the torsionless Lévi-Civita connection, in the present formalism ones uses the curvatureless Weitzenböck connection [39], whose torsion tensor reads

$$T^\lambda_{\mu \nu} = \Gamma^\lambda_{\nu \mu} - \Gamma^\lambda_{\mu \nu} = e_A^\lambda \left( \partial_\mu e^A_\nu - \partial_\nu e^A_\mu \right).$$

Moreover, the contorsion tensor, which equals the difference between the Weitzenböck and Lévi-Civita connections, is defined as

$$K^{\rho \nu}_{\mu} = -\frac{1}{2} \left( T^{\rho \nu}_{\mu} - T^{\nu \rho}_{\mu} - T^{\nu \mu}_{\rho} \right).$$

Finally, it proves useful to define

$$S^{\mu \nu}_{\rho} = \frac{1}{2} \left( K^{\mu \nu}_{\rho} + \delta^{\mu}_{\rho} T^{\nu a}_{a} - \delta^{\nu}_{\rho} T^{\mu a}_{a} \right).$$

Using these quantities, one can define the so-called teleparallel Lagrangian, which is nothing other than the torsion scalar, as [29, 40, 41]

$$T \equiv S^{\mu \nu}_{\rho} T^\rho_{\mu \nu}.$$

In summary, in the present formalism, all the information concerning the gravitational field is included in the torsion tensor $T^\lambda_{\mu \nu}$, and the torsion scalar $T$ arises from it in a similar way as the curvature scalar arises from the curvature (Riemann) tensor. Finally, the torsion scalar...
gives rise to the dynamical equations for the vierbein, which imply the Einstein equations for the metric.

While in teleparallel gravity the action is constructed by the teleparallel Lagrangian $T$, the idea of $f(T)$ gravity is to generalize $T$ to a function $T + f(T)$, which is similar in spirit to the generalization of the Ricci scalar $R$ in the Einstein–Hilbert action to a function $f(R)$. In particular, the action in a universe governed by $f(T)$ gravity reads

$$I = \frac{1}{16\pi G} \int d^4x e [T + f(T) + L_m],$$

(7)

where $e = \det(e^\mu_\alpha) = \sqrt{-g}$ and $L_m$ stands for the matter Lagrangian. We mention here that since the Ricci scalar $R$ and the torsion scalar $T$ differ only by a total derivative [42], in the case where $f(T)$ is a constant (which will play the role of a cosmological constant) the action (7) is equivalent to general relativity with a cosmological constant.

Lastly, we stress that throughout this work we use the common choice for the form of the vierbein, namely

$$e^A_\mu = \text{diag}(1, a, a, a).$$

(8)

It can be easily found that the family of vierbiens related to (8) through global Lorentz transformations lead to the same equations of motion. Note however that, as was shown in [43], $f(T)$ gravity does not preserve local Lorentz invariance. Thus, one should in principle study the cosmological consequences of a more general vierbien ansatz, but for simplicity we remain with choice (8) (see also [44]).

2.2. Background $f(T)$ cosmology

Let us now present the background cosmological equations in a universe governed by $f(T)$ gravity. Variation of the action (7) with respect to the vierbein gives the equations of motion

$$e^{-1} \partial_\mu (e S_\alpha^{\mu\nu})[1 + f, T] - e^A_\nu T^\rho_{\mu\lambda} S_\rho^{\nu\mu} + S_\alpha^{\mu\nu} \partial_\mu (T + f(T)) = 4\pi G e^{em \mu \nu},$$

(9)

where $f, T$ and $f, TT$ denote respectively the first and second derivatives of the function $f(T)$ with respect to $T$, and the mixed indices are used as in $S_\alpha^{\mu\nu} = e^A_{\alpha} S_\rho^{\mu\nu}$. Note that the tensor $e^{em \mu \nu}$ on the right-hand side is the usual energy–momentum tensor.

If we assume the background to be a perfect fluid, then the energy–momentum tensor takes the form $e^{em \mu \nu} = p g_{\mu \nu} - (\rho + p) u_\mu u_\nu$, where $u^\mu$ is the fluid four-velocity. Under this, one sees that equations (9) lead to the background (Friedmann) equations

$$H^2 = \frac{8\pi G}{3} \rho_m - f(T) \frac{2 f, T}{6}$$

(10)

$$\dot{H} = -\frac{4\pi G (\rho_m + p_m)}{1 + f, T - 12H^2 f, TT}.$$  

(11)

In these expressions, we have introduced the Hubble parameter $H \equiv \dot{a}/a$, where a dot denotes a derivative with respect to coordinate time $t$. Moreover, $\rho_m$ and $p_m$ stand respectively for the energy density and pressure of the matter content of the universe, with the equation-of-state parameter $w_m = p_m/\rho_m$. Finally, we have employed the very useful relation

$$T = -6H^2,$$

(12)

which straightforwardly arises from the evaluation of (6) for the unperturbed metric.
3. The background solution of $f(T)$ matter bounce

In this section, we examine how cosmological scenarios governed by $f(T)$ gravity can produce a cosmological bounce. There are two distinct points in such an investigation. The first is to examine whether the background evolution allows for bouncing solutions. If this is indeed possible, then the second point is to examine the evolution of perturbations through the bounce. The first task is the subject of this section, while the second one will be investigated in the next section. Finally, we mention that in order to be closer to the convention of the literature on this field, we use $M_{Pl}$ instead of $G$ when necessary, using the relation $M_{Pl} = 1/\sqrt{8\pi G}$.

In principle, whether a universe is expanding or contracting depends on the positivity of the Hubble parameter. In the contracting phase that exists prior to the bounce, the Hubble parameter $H$ is negative, while in the expanding one that exists after it, we have $H > 0$. By making use of the continuity equations, it follows that at the bounce point $H = 0$. Finally, it is easy to see that throughout this transition $\dot{H} > 0$. On the other hand, for the transition from expansion to contraction, that is, for the cosmological turnaround, we have $H > 0$ before and $H < 0$ after, while exactly at the turnaround point we have $H = 0$. Throughout this transition, $\dot{H} < 0$.

Having in mind the above general requirements for a cosmological bounce, and observing the Friedmann equations (10) and (11), we deduce that such a behavior can be easily obtained in principle in the context of $f(T)$ cosmology. In particular, one starts with a specific, desirable form of the bouncing scale factor $a(t)$, and thus one immediately knows $H(t)$. Concerning the matter fluid content of the universe, with the equation-of-state parameter $w_m$, its evolution equation $\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0$ straightforwardly gives the solution $\rho_m(t)$ since $a(t)$ is known. Thus, we can insert these relations in (10), and determine $f(T)$, that is, $f(-6H^2)$, which generates such an $H(t)$ solution. Although this procedure can always be performed numerically, in the following we present a simple example of a bouncing solution that allows for analytical results.

We start with a bouncing scale factor of the form

$$a(t) = a_B(1 + \frac{2}{3} \sigma t^2)^{1/3},$$

(13)

where $a_B$ is the scale factor at the bouncing point, and $\sigma$ is a positive parameter which describes how fast the bounce takes place. Such an ansatz presents the bouncing behavior, corresponding to matter-dominated contraction and expansion, and additionally it exhibits the advantage of allowing for semi-analytic solutions. In such a scenario, $t$ varies between $-\infty$ and $+\infty$, with $t = 0$ the bounce point. Finally, in the following, we normalize the bounce scale factor $a_B$ to unity.

Straightforwardly we find

$$H(t) = \frac{\sigma t}{(1 + 3\sigma t^2/2)}, \quad T(t) = -\frac{6\sigma^2 t^2}{(1 + \frac{1}{3} \sigma t^2)^{5/2}}.$$  

Therefore, provided $-\sqrt{\frac{2}{3\sigma}} \leq t \leq \sqrt{\frac{2}{3\sigma}}$, the inversion of this expression gives the $t(T)$ relation as

$$t(T) = \pm \left( -\frac{4}{3T} - \frac{2}{3\sigma} + \frac{4\sqrt{T\sigma^3 + \sigma^4}}{3T\sigma^2} \right)^{1/2},$$

(14)

where we have kept the solution pair that gives the correct ($t = 0$ at $T = 0$) behavior. Note that when $t > \sqrt{\frac{2}{3\sigma}}$ and $t < -\sqrt{\frac{2}{3\sigma}}$ we have assumed the usual Einstein gravity or the TEGR to be the prevailing framework, thus negating the need to pursue an $f(T)$ action in that region. Furthermore, we assume the matter content of the universe to be dust, namely $w_m \approx 0$. When
Figure 1. The form of $f$ as a function of the torsion scalar $T$ in matter bounce cosmology. The parameters $\sigma$ and $\rho_{mB}$ were chosen to be $\sigma = 7 \times 10^{-6} M_{pl}^2$ and $\rho_{mB} = 1.41 \times 10^{-5} M_{pl}^4$, respectively, and the graph is in units of $M_{pl}$.

inserted in the evolution equation, this leads to the usual dust evolution $\rho_m = \rho_{mB} a^3$, with $\rho_{mB}$ its value at the bouncing point.

Inserting the above expressions into (10), we obtain a differential equation for $f(t)$, which can be easily solved analytically as

$$f(t) = \frac{4t}{(2 + 3t^2) M_{pl}^2} \left[ \frac{\rho_{mB}}{t} + \frac{6tM_{pl}^2\sigma^2}{2 + 3t^2\sigma} + \sqrt{6\sigma\rho_{mB}} \text{ArcTan} \left( \sqrt{\frac{3\sigma}{2}t} \right) \right].$$

(15)

We mention that in the calculation, we have set the integration constant to be zero so that the solution is consistent with the Friedmann equation. Thus, the corresponding $f(T)$ expression that generates a bouncing scale factor of the form (13) arises from expression (15) with the insertion of the $t(T)$ relation from (14). Note that the solution (15) is an even function of $t$, and thus the $\pm$ solutions of (14) correspond to the contraction and expansion phase, respectively. Obviously they give the same form of $f(T)$.

In order to present this behavior more transparently, in figure 1 we depict $f(T)$ that generates the matter-dominated bounce in $f(T)$ gravity, with $a_B = 1$, $\sigma = 7 \times 10^{-6} M_{pl}^2$ and $\rho_{mB} = 1.41 \times 10^{-5} M_{pl}^4$. Note that the value of $\sigma$ is roughly determined by the amplitude of the CMB spectrum, and that of $\rho_{mB}$ depends on how fast the standard Einstein gravity is recovered in the $f(T)$ theory. Their physical meaning will be discussed in the next section in detail.

Furthermore, we numerically derived the evolution of the $f(T)$ and the Hubble parameter $H$ as functions of the cosmic time in figures 2 and 3, respectively. Particularly, figure 2 shows that the evolution of $f(T)$ is symmetric with respect to the bouncing point $t = 0$. At the bouncing point $f$ arrives at a minimal value $16\pi G \rho_{mB}$, which happens to cancel the contribution of normal matter fields, and thus leads to the nonsingular bounce. From figure 3, one can read that the background evolution of the universe follows the usual Einstein gravity away from the bouncing phase, but it is dominated by $f(T)$ in the middle period. These features are completely consistent with the designs of the scenario as expected.
4. Cosmological perturbations in the $f(T)$ matter bounce

In the previous section, we presented a simple realization of the cosmological bounce in $f(T)$ cosmology, at the background level. In this section, we extend our analysis to an investigation of the perturbations.

We begin with a brief discussion of the cosmological evolution of primordial perturbations in the framework of a flat FRW universe. A standard process for generating a primordial power spectrum suggests that cosmological fluctuations should initially emerge inside a Hubble radius, then exit it in the primordial epoch, and finally re-enter at late times. This process
can be achieved in the matter bounce cosmology (see, for example, [45]). In this scenario, there exist quantum fluctuations around the initial vacuum state, well in advance of the time when the bouncing point is reached. Along with the matter-like contraction, the quantum fluctuations would exit the Hubble radius, since the Hubble radius decreases faster than the wavelengths of the primordial fluctuations. When passing through the bouncing point, the background evolution could affect the scale dependence of the perturbations at ultraviolet scales. However, the observable primordial perturbations, responsible for the large scale structure of our universe, are mainly originated in the infrared regime, where the modified gravity effect becomes very limited [46]. Consequently, in a generic case, one can estimate the formation of primordial power spectrum with the standard cosmological perturbation theory. In the following, we study the perturbation theory in $f(T)$ cosmology in detail and we verify this statement in a specific model of matter bounce cosmology.

4.1. Generic analysis

To begin with, we will work in the longitudinal gauge which only involves scalar-type metric fluctuations as

$$\text{ds}^2 = (1 + 2\Phi)\text{dt}^2 - a^2(t)(1 - 2\Psi)\text{dx}^2;$$

thus, as usual, the scalar metric fluctuations are characterized by two functions $\Phi$ and $\Psi$. Correspondingly, the fluctuation of torsion scalar at leading order is given by

$$\delta T = 12H(\Phi + H\Psi),$$

which will be widely used in the following calculation.

By expanding the gravitational equations of motion to linear order, we obtain the following perturbation equations [35]:

$$\left(1 + f_{,T}\right)\frac{\nabla^2}{a^2}\Psi - 3\left(1 + f_{,T}\right)H\Psi - 3\left(1 + f_{,T}\right)H^2\Phi + 36f_{,TT}H^3(\Psi + H\Phi) = 4\pi G\delta \rho,$$

$$\left(1 + f_{,T}\right)(\Psi - \Phi) = 8\pi G\delta s,$$

$$\left(1 + f_{,T}\right)(\phi - \phi) = 8\pi G\delta q,$$

$$\left(1 + f_{,T}\right)(\phi - \phi) = 8\pi G\delta s,$$

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$$\left(1 + f_{,T}\right)(\phi - \phi) = 8\pi G\delta s,$$

The functions $\delta \rho, \delta \rho, \delta q$ and $\delta s$ are the fluctuations of energy density, pressure, fluid velocity, and anisotropic stress, respectively. We take the matter component to be a canonical scalar field $\phi$ with a Lagrangian in the form of

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi),$$

and thus we acquire

$$\delta \rho = \phi(\delta \dot{\phi} - \dot{\phi} \Phi) + V_\phi \delta \phi,$$

Note that the perturbed gravitational equations shown in [35] are incomplete due to appropriate approximations made in that paper, but are irrelevant to the analysis. All the missing terms have been included in the following equations.
\[ \delta q = \dot{\phi} \delta \phi, \quad (23) \]

\[ \delta s = 0, \quad (24) \]

\[ \delta p = \dot{\phi} (\delta \dot{\phi} - \ddot{\phi} \Phi) - V,\phi \delta \phi, \quad (25) \]

respectively.

Inserting relation (24) into (20), one can obtain \( \Psi = \Phi \) due to a vanishing anisotropic stress, which is also widely found in the standard theory of cosmological perturbations (for example, see [47]). Moreover, combining (19) and (23) implies that the gravitational potential \( \Phi \) can be completely determined by the scalar field fluctuation \( \delta \phi \). Therefore, we conclude that for our choice of the tetrad given in (8), there exists only a single degree of freedom in the scenario of \( f(T) \) gravity minimally coupled to a canonical scalar field. Note that there is another evolution equation to describe the dynamics of cosmological perturbations, namely the perturbed equation of motion for \( \delta \phi \). However, using (19), it can be shown that this second equation is consistent with (18) and (21).

In order to understand the evolution of scalar-sector metric perturbations, we use the perturbed equation of motion for the gravitational potential \( \Phi \) instead of the scalar field fluctuation \( \delta \phi \). Combining (18), (21) and (19), we obtain the complete form of the equation of motion of one Fourier mode \( \Phi_k \) as

\[ \ddot{\Phi}_k + \alpha \dot{\Phi}_k + \mu^2 \Phi_k + c_s^2 k^2 a^2 \Phi_k = 0, \quad (26) \]

with

\[ \alpha = 7H + \frac{2V,\phi}{\Phi} - \frac{36H \dot{H} (f,TT - 4H^2 f,TTT)}{1 + f,T - 12H^2 f,TT}, \quad (27) \]

\[ \mu^2 = 6H^2 + 2H + \frac{2HV,\phi}{\Phi} - \frac{36H^2 \dot{H} (f,TT - 4H^2 f,TTT)}{1 + f,T - 12H^2 f,TT}, \quad (28) \]

\[ c_s^2 = \frac{1 + f,T}{1 + f,T - 12H^2 f,TT}. \quad (29) \]

where we have applied the relation \( \Psi = \Phi \) holding in the absence of anisotropic stress. The functions \( \alpha, \mu^2 \) and \( c_s^2 \) are respectively the frictional term, the effective mass and the sound speed parameter for the gravitational potential \( \Phi \). Moreover, we recall the scalar-field background equation

\[ \dot{\phi} + 3H \phi + V,\phi = 0, \quad (30) \]

and the second Friedmann equation (11), which in our case reads

\[ (1 + f,T - 12H^2 f,TT) \dot{H} = -4\pi G \dot{\phi}. \quad (31) \]

Consequently, to make use of these two background equations, (26) can be further simplified as

\[ \ddot{\Phi}_k + \left( H - \frac{\dot{H}}{H} \right) \dot{\Phi}_k + \left( 2\dot{H} - \frac{H \ddot{H}}{H} \right) \Phi_k + \frac{c_s^2 k^2}{a^2} \Phi_k = 0. \quad (32) \]

Surprisingly, we find that the equation of motion for the gravitational potential in the present \( f(T) \) scenario is the same as that in the standard Einstein gravity [47], except the newly introduced sound speed parameter. This important feature could be a key for us to explore potential clues of the \( f(T) \) theory in cosmological surveys.
4.2. Variables of perturbations in bouncing cosmology

After having derived the equation of motion for the gravitational potential, we proceed in solving it in the detailed bouncing cosmology. As we showed in the background analysis of section 3, we can achieve the exact matter bounce scenario in which the universe evolves as a matter-dominated one in the contracting phase. In order to accommodate this picture, the matter component could be a massive scalar field similar to the mechanism of the Lee–Wick bounce [46], or a scalar field with a fine-tuned exponential potential [48].

One often uses a gauge-invariant variable $\zeta$, the curvature fluctuation in comoving coordinates, to characterize the cosmological inhomogeneities. In the case of $f(T)$ cosmology, we assume that the form of $\zeta$ is the same as that defined in the standard cosmological perturbation theory, which is given by

$$\zeta = \Phi - \frac{H}{H} (\dot{\Phi} + \dot{H}).$$

(33)

A useful relation for the time derivative of $\zeta$ can be derived upon making use of equation (32), namely

$$\dot{\zeta}_k = \frac{H}{H} c_s^2 k^2 a^2 \Phi_k.$$

(34)

In a generic expanding universe $\dot{\zeta}_k$ approaches zero at large length scales, $k \to 0$, since the dominant mode of $\Phi_k$ is then nearly constant. However, in the matter bounce cosmology, the metric perturbation $\Phi_k$ in the contracting phase is dominated by a growing mode with $\Phi_k \sim k^{-7/2}$, and thus $\zeta$ keeps increasing before arriving at the bouncing point [49].

Note that one may be concerned of the variable $\zeta$ that becomes ill-defined when $\dot{H}$ changes sign. In fact, this specious trouble was extensively studied in many aspects of cosmological perturbation theory. At present, our understanding on a well-defined cosmological perturbation theory is to require the metric perturbation and the corresponding extrinsic curvature behave smoothly throughout the background evolution. The pioneer discussion on this topic appeared in [48], and we refer to [46] for recent detailed analysis by tracking the evolution of metric perturbation step by step in matter bounce cosmology. In our calculation, we still make use of $\zeta$ merely since its analytic analysis is very convenient to be performed away from the bouncing phase. In addition, we would like to point out that the knowledge obtained in general relativity can also be applied to $f(T)$ gravity, which is a modification of Einstein gravity, in the frame of bouncing cosmology. The reason is that the observable modes of perturbations in bouncing cosmology are distributed in the infrared regime and thus the effect caused by the modification of Einstein gravity has to be very limited.

In order to simplify the calculation, we further introduce a canonical variable to characterize the cosmological perturbations

$$v = z \zeta,$$

(35)

where

$$z = a \sqrt{2\epsilon},$$

(36)

with $\epsilon \equiv -\frac{\dot{H}}{H^2}$. It can then be found that the equation of motion

$$v''_k + \left( c_s^2 k^2 - \frac{z''}{z} \right) v_k = 0,$$

(37)

where the prime denotes the derivative with respect to the comoving time $\tau \equiv \int dt/a$, is still available in a contracting universe.
4.3. The primordial power spectrum of the $f(T)$ matter bounce

In order to perform a specific analysis, we recall that in the matter-like contracting phase, the scalar factor evolves as

$$ a \sim t^{2/3} \sim \tau^2 $$

(38)

and $z \propto a$. In this period, by solving the background equations of motion, one obtains the approximate relations

$$ H \simeq \frac{2}{3t}, \quad f(T) \simeq \left(-1 + \frac{\rho_{mB}}{2M_P^2\sigma}T\right) $$

(39)

that hold far before the beginning of the bouncing era $t_m = -\sqrt{\frac{2}{3}\sigma}$. As we mentioned above, $\rho_{mB}$ is the energy density of the matter field at the bouncing point and $\sigma$ describes how fast the bounce takes place. Therefore, the sound speed of the curvature perturbation reverts to $c_s^2 \simeq 1$ in the matter-like contracting phase.

We mention here that inserting the $f(T)$ form of (39) into the action (7), we find that the standard Einstein gravity will automatically be recovered when $\rho_{mB} \simeq 2M_P^2\sigma$. Particularly, when $\rho_{mB}$ exactly equals $2M_P^2\sigma$, we get $f(T) \sim O(T^2)$, which will dilute faster than the Ricci scalar during late-time evolution. Even when $\rho_{mB}$ is not equal to $2M_P^2\sigma$, it is clear that the system satisfies general relativity with a rescaled gravitational constant. Thus, the combination of $\rho_{mB}$ and $\sigma$ could, in principle, be constrained by measurements of the gravitational constant. In our computation, we choose $\rho_{mB}$ to be slightly different from $2M_P^2\sigma$; thus, our model is able to approach the standard Einstein theory far away from the bounce.

As a consequence, the perturbation equation becomes

$$ v''_k + \left(k^2 - \frac{2}{\tau^2}\right)v_k \simeq 0, $$

(40)

in the contracting phase. Initially the $k^2$-term dominates in (40) and thus we can neglect the gravitational term. From this point of view, the fluctuation corresponds to a free scalar propagating in a flat spacetime, and naturally the initial condition takes the form of the Bunch–Davies vacuum:

$$ v_k \simeq e^{-i k \tau} \sqrt{2k} $$

Making use of the vacuum initial condition, we solve the perturbation equation exactly, obtaining the solution

$$ v_k \simeq e^{-i k \tau} \left(1 - \frac{i}{k \tau}\right). $$

From this result, we deduce that the quantum fluctuations could become classical perturbations, after exiting the Hubble radius, due to the gravitational term in equation (40). Moreover, the amplitude of the metric perturbations will keep increasing until the universe arrives at the bouncing phase at the moment $t_m$.

From the definition of the power spectrum, we see that $\zeta \sim k^{3/2} |v_k|$ is scale-invariant in our model, which can also be achieved in inflationary cosmology. However, the coefficient $\epsilon$ takes the value $\frac{3}{4}$ in the matter-like contraction and thus it is unable to amplify the power

As discussed in the previous subsections, the generic form of the perturbation equation involves a sound speed parameter, and correspondingly the Bunch–Davies vacuum initial condition should take the form $v_k \simeq e^{-ik\tau} / \sqrt{2k}$.
spectrum of metric perturbation as in inflation. A detailed calculation provides the expression of the primordial power spectrum for the $f(T)$ matter bounce as

$$P_\zeta \equiv \frac{k^3}{2\pi^2} \left| \frac{v_k}{z} \right|^2 = \frac{H_m^2}{48\pi^2 M_{Pl}^2}, \quad (41)$$

where $H_m = \sqrt{\sigma/6}$ is the absolute value of the Hubble parameter at the beginning of the bouncing phase. It should be noted that the $M_{Pl}^2$ which appears in the power spectrum will become rescaled when $\rho_{mB}$ is not equal to $2M_{Pl}^2 \sigma$.

### 4.4. The tensor-to-scalar ratio and the matter bounce curvaton scenario

We can now study the amplitude of tensor perturbations $h_{ij}$ in the $f(T)$ matter bounce. Following [35], the perturbation equation for the tensor modes can be expressed as

$$\left( \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} \right) - \frac{12H \dot{H} f_{TT}}{1 + f_T}h_{ij} = 0, \quad (42)$$

while the tensor modes are transverse and traceless, namely

$$\partial_i h^{ij} = h_i^i = 0.$$

In the specific case of matter bounce cosmology, equation (42) is the same as the standard one since $f_{TT}$ vanishes in the contracting phase. Hence, following the analysis done in [46], the primordial power spectrum of tensor fluctuations is also scale-invariant, but the amplitude is $h \sim H_m/M_{Pl}$, which is of the same order of the scalar perturbation. Therefore, the $f(T)$ matter bounce scenario suffers from the usual problem of all matter bounce models, namely that the ratio of tensor to scalar $r \equiv P_T/P_\zeta$ is difficult to accommodate with current observations. In particular, from the current CMB data [50], this ratio is required to be less than 0.2.

Consequently, in order to make our model consistent with cosmological observations on the value of $r$, it is necessary to introduce a mechanism to magnify the amplitude of scalar-type metric perturbations. This issue can be resolved by introducing additional light scalar fields, as in the matter bounce curvaton scenario [51]. These scalars are able to seed isocurvature fluctuations, and then transfer to a scale-invariant spectrum of the adiabatic fluctuations during the nonsingular bouncing phase, through the so-called kinetic amplification. Thus, we obtain a mechanism for enhancing the primordial adiabatic fluctuations and suppressing the tensor-to-scalar ratio.

Following the analysis performed in [51], we can introduce one massless scalar field $\chi$, coupling to the background matter field through the term $g^2 \phi^2 \chi^2$. Therefore, the tensor-to-scalar ratio in our model can be suppressed by the kinetic amplification factor in the bouncing phase, which can be expressed as $r \simeq F^{-2}$ as shown in equation (58) of [51]. Specifically, in our case, the kinetic amplification factor $F$ is determined to be $F \simeq 18.66$, and so $r \simeq 2.87 \times 10^{-3}$. Finally, by virtue of the matter bounce curvaton mechanism, our model is able to satisfy the constraints from current observations.

### 5. Conclusions and discussion

In this work, we investigated the realization of matter bounce cosmology in the framework of $f(T)$ gravity. Considering an explicit scale-factor form, which links the matter-like contraction and matter-like expansion through a smooth nonsingular bouncing phase, we reconstructed the specific form of $f(T)$ that generates it. Our analysis has illustrated the possibility of combining
the $f(T)$ gravity with bounce cosmologies, and thus exploring an alternative approach to avoid the Big Bang singularity encountered in the standard inflationary cosmology. Additionally, the constructed $f(T)$ matter bounce is free of ghost degrees of freedom and other potential problems of many matter bounce scenarios, which is a significant advantage.

At the background level, we found that the theory of torsion gravity is difficult to be verified by experiments. This is in agreement with the mechanism described in [52], where the torsion dynamics is highly degenerated with quantum corrections to the classical gravitational action caused by vacuum effects. However, going beyond the background level, we performed a detailed analysis of the evolution of metric perturbations in this work. For our choice of the tetrad, we found that the scalar-type metric perturbations posses a single degree of freedom, but they have a time-dependent sound speed, a feature that is generic in any $f(T)$ scenario. We expect that this behavior could be an important signature to be detected or constrained by observations.

The present $f(T)$ matter bounce scenario suffers from the usual problem, widely existing in many other bounce models, namely it predicts a value for the tensor-to-scalar ratio larger than observational constraints [46, 53]. This undesirable feature can be cured by the mechanism of matter bounce curvaton, which introduces an additional scalar field tracking the background evolution before the bounce, but with fluctuations obtaining a kinetic amplification during the bouncing phase.

Finally, we would like to point out that in order to establish the existence of matter bounce in $f(T)$ gravity and perform a first examination of its basic features, we constructed a simplified scenario, without the incorporation of the contribution of radiation, dark energy and other matter components. Therefore, the investigated scenario involves only two parameters, namely the energy density at the bouncing point $\rho_{mB}$ and the bouncing parameter $\sigma$. Through the perturbation analysis we found that these two parameters could be tightly constrained by CMB observations and experimental bounds on Newton’s constant. It would be interesting to construct bouncing models in $f(T)$ gravity going beyond this scenario, in which the contributions of other components would be taken into account and various cosmological observations would be satisfied. We leave such a construction of a phenomenologically realistic model and its observational constraints for a future investigation.

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