Equivariant dendroidal sets

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Abstract

We extend the Cisinski-Moerdijk-Weiss theory of $\infty$-operads to the equivariant setting to obtain a notion of $G$-$\infty$-operads that encode “equivariant operads with norm maps” up to homotopy. At the root of this work is the identification of a suitable category of $G$-trees together with a notion of $G$-inner horns capable of encoding the compositions of norm maps.

Additionally, we follow Blumberg and Hill by constructing suitable variants associated to each of the indexing systems featured in their work.

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1 Introduction

Operads encode a variety of algebraic structures, such as monoids, commutative monoids or (depending on the ambient category) Lie algebras, $E_n$-algebras, etc. Indeed, all such instances can be regarded as categories of algebras for some (fixed) suitable operad. Informally, an operad $O$ consists of “sets/spaces of $n$-ary operations” $O(n)$, $n \geq 0$, each of which carries a $\Sigma_n$-action encoding “reordering the inputs of the operations”, and a suitable notion of “composition of operations”.

From the homotopy theory point of view, one of the most important classes of operads is certainly that of the $E_\infty$-operads, which are “up to homotopy” replacements of the commutative operad $\text{Com}$. More concretely, while algebras for $\text{Com}$ are the usual commutative monoids, the algebras for an $E_\infty$-operad are “up to homotopy commutative monoids”, where associativity and commutativity are only enforced up to homotopy. Further, $E_\infty$-operads $O$ are characterized by the property that each space $O(n)$ is a contractible space with a free $\Sigma_n$-action.

This work lies at the intersection of operad theory and equivariant homotopy theory. Briefly, in $G$-equivariant homotopy theory a map of $G$-spaces $X \to Y$ is considered a $G$-weak equivalence only if all the induced fix point maps $X^H \to Y^H$, $H \leq G$ are weak equivalences. Therefore, it is no surprise that the characterization of $G$-$E_\infty$-operads, i.e. $G$-equivariant operads whose algebras are “$G$-equivariant up to homotopy commutative monoids” would need to be modified. Indeed, a naive first guess might be that a $G$-operad $O$ should be called $G$-$E_\infty$ if (i) each space $O(n)$ has a free $\Sigma_n$-action and (ii) $O(n)$ is $G$-contractible. Accepting this tentative characterization for the moment, such a $G$-operad is easily produced: simply taking a (non-equivariant) $E_\infty$-operad and giving it a trivial $G$-action yields such an example. However, it has long been known ([7]) that such “$G$-trivial $E_\infty$-operads” are not the correct replacement for the commutative operad in the equivariant setting. To see why, we consider the much studied example of $R$ a (strictly) commutative $G$-ring spectrum. For a finite $G$-set $T$ with $n$ elements it is possible to equip $R^T \cong R^n$ with a mixed $G$-action combining the actions on $R$ and $T$. One often writes $N^T R$ for $R^T$ together with this action and calls it a norm. Multiplication then induces norm maps

$$N^T R \to R$$

satisfying equivariance and associativity conditions. The flaw of “$G$-trivial $E_\infty$-operads” is then that they lack all norm maps (1.1) with $T$ a non-trivial $G$-set (or, after restriction to $H \leq G$, $T$ a non-trivial $H$-set).

In understanding this issue, note first that though $O(n)$ has a $G \times \Sigma_n$-action when $O$ is a $G$-operad, conditions (i) and (ii) above actually fail to determine a unique $G \times \Sigma_n$-homotopy type. Indeed, (i) implies that $O(n)^G = \emptyset$ whenever $\Gamma \cap \Sigma_n \neq \emptyset$ while (ii) implies that $O(n)^G \simeq \ast$ if $\Gamma \leq G$, but these conditions leave
out many subgroups $\Gamma \leq G \times \Sigma_n$. Indeed, there are identifications $\Gamma$ such that $\Gamma \cap \Sigma_n = \ast$ iff graph of $G \to \Sigma_n \Leftarrow H$-action on $\{1, \cdots, n\}$ (1.2) and (ii) covers only those $\Gamma$ encoding trivial $H$-actions. The correct characterization of $G\text{-}E_\infty$-operads is then that: (i) $O(n)$ is $\Sigma_n$-free; (ii’) $O(n)$ is graph-contractible, i.e. $O(n)^\Gamma \sim \ast$ for any $\Gamma \leq g \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = \ast$.

A key observation of Blumberg and Hill’s in [2] is that the reason a “$G$-trivial $E_\infty$-operad” induces only the norm maps for trivial sets is that it satisfies (ii’) only for those $\Gamma$ encoding trivial sets. Indeed, their work takes this observation much further. Motivated by the study of equivariant spectra in incomplete universes, they define a whole lattice of types of $G$-operads, which they dub $N_\infty$-operads, and which satisfy (ii’) only for $\Gamma$ encoding $H$-sets within certain special families, which they dub indexing systems. “$G$-trivial $E_\infty$” and $G\text{-}E_\infty$ are then the minimum and maximum types of $N_\infty$-operads, with the remaining types interpolating in between.

The departure point for this paper (and the larger project it belongs to) is the observation that the closure conditions for the $H$-sets in an indexing system identified in [2, Def. 3.22] admit a nice diagrammatic interpretation and that this in turn suggests the possibility of encoding equivariant operads with norm maps via suitable diagrammatic models.

Indeed, it is well known that composition of operations in an operad can be encoded using tree diagrams and work of Moerdijk and Weiss in [14] and follow up work of Cisinski and Moerdijk in [3] build a category $\Omega$ of trees and a model structure on the presheaf category $\text{dSet}_\text{op}$ which is shown in the follow up papers [4] and [5] to be Quillen equivalent to the category of simplicial operads.

The role of this paper is to provide the equivariant analogue of the work in [14] and [3]. We first identify a (non-obvious) category $\Omega_G$ of $G$-trees (introduced in §3 and formally defined in §4.3) capable of encoding norm maps and their compositions, and then adapt the proofs in [14] and [3] to prove the existence of a model structure on $\text{dSet}^G$ whose fibrant objects, which we call $G\text{-}\infty$-operads, are “up to homotopy $G$-operads with norms”. We note that our results are not formal: indeed, while our proofs closely follow those in [14] and [3] the presence of equivariance often requires significant modifications. Moreover, we note that alternative so called “genuine” model structures on $\text{dSet}^G$ built by formal methods (say, by mimicking the definition of the genuine model structure on $\text{Top}^G$) would instead only model $G$-operads without non trivial norm maps.

Acknowledgments

The current work owes much to Peter Bonventre: the category $\Omega_G$ of $G$-trees, which is at the root of this manuscript, is a joint discovery with him; the author first heard of the notion of broad posets (due to Weiss) from him; and finally, this work has also greatly benefited from extensive joint conversations.

2 Main results

Our main result follows. It is the equivariant analogue of [3, Thm. 2.4].

Theorem 2.1. There exists a model structure on $\text{dSet}^G$ such that
• the cofibrations are the $G$-normal monomorphisms;
• the fibrant objects are the $G$-$\infty$-operads;
• the weak equivalences are the smallest class containing the $G$-inner anodyne extensions, the trivial fibrations and closed under “2-out-of-3”.

Theorem 2.1 will be proven as the combination of Proposition 8.1, Theorem 8.13 and Corollary 8.14.

Further, letting $\mathcal{F}$ denote an indexing system (cf. [2, Def. 3.22], and as reinterpreted in Definition 9.5), we also prove the following more general result.

**Theorem 2.2.** For $\mathcal{F}$ a weak indexing system there exists a model structure on $\text{dSet}^G$ such that

• the cofibrations are the $\mathcal{F}$-normal monomorphisms;
• the fibrant objects are the $\mathcal{F}$-$\infty$-operads;
• the weak equivalences are the smallest class containing the $\mathcal{F}$-inner anodyne extensions, the $\mathcal{F}$-trivial fibrations and closed under “2-out-of-3”.

Theorem 2.2 is proven at the end of §9.

### 3 Outline

After reviewing the familiar types of trees found elsewhere in the literature (e.g. [13], [14], [3], among others), §4 provides an introductory look at the new equivariant trees that motivate this paper, focusing on examples. Most notably, each $G$-tree can be represented by two distinctly shaped tree diagrams, called the expanded and orbital representations, each capturing different key features. §5 lays the necessary framework for our work. Specifically, §5.1 recalls Weiss’ algebraic broad poset model (cf. [15]) for the category $\Omega$ of trees, which we prefer since planar representations of $G$-trees can easily get prohibitively large. §5.2 discusses forests, which play an auxiliary role. §5.3 formally introduces the category $\Omega_G$ of $G$-trees. Finally §5.4 introduces all the necessary presheaf categories, most notably the category $\text{dSet}^G$ featured in Theorems 2.1 and 2.2.

§6 discusses the notions of $G$-normal monomorphism and $G$-$\infty$-operad needed to state Theorem 2.1. The former of these is straightforward, but the latter requires the key (and more subtle) notion of $G$-inner horn (Definition 6.11).

§7 is the technical heart of the paper, extending the key technical results [14, Prop 9.2] and [3, Thms. 5.2 and 4.2] concerning tensor products of dendroidal sets and the dendroidal join to the equivariant setting (Theorems 7.1, 7.2, 7.4).

§8 then finishes the proof of Theorem 2.1 by combining the results of §7 with the arguments in the proof of the original non-equivariant result [3, Thm. 2.4].

Finally, 9 proves Theorem 2.2 by straightforward generalizations of our arguments to the framework of general indexing systems.
4 An introduction to equivariant trees

4.1 Planar trees

Operads are a tool for studying various types of algebraic structures that possess operations of several arities. More concretely, an operad $\mathcal{O}$ consists of a sequence of sets (or, more importantly to us, spaces/simplicial sets) $\mathcal{O}(n)$, $n \geq 0$ which behave as sets (spaces/simplicial sets) of $n$-ary operations. I.e., one should have composition product maps (cf. [12, Def. 1.1] or [8, Def. 1.4])

$$\mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \xrightarrow{\varphi, \psi} \mathcal{O}(n_1 + \cdots + n_k)$$

and an identity $id \in \mathcal{O}(1)$ satisfying suitable associativity and unital conditions.

A powerful tool for visualizing operadic compositions and their compatibilities is given by tree diagrams. For instance, the tree

encodes the composition of operations $\varphi \in \mathcal{O}(3)$, $\psi_1 \in \mathcal{O}(2)$, $\psi_2 \in \mathcal{O}(3)$ and $\psi_3 \in \mathcal{O}(0)$, and one has $\varphi(\psi_1, \psi_2, \psi_3) \in \mathcal{O}(5)$, with arity of the composite given by counting leaves (i.e. the edges at the top of the tree, not capped by a circle).

Alternatively, given the presence of the identity $id \in \mathcal{O}(1)$, one can instead define operads using so called partial composition products ([11, Def. 1.16])

$$\mathcal{O}(k) \otimes \mathcal{O}(n) \xrightarrow{\varphi, \psi} \mathcal{O}(n + k - 1)$$

which are also readily visualized using trees. For example,

encodes the partial composition $\varphi \circ_2 \psi = \varphi(id, \psi, id)$.

Heuristically, trees encoding iterations of $\circ$ are naturally tiered whereas trees encoding iterations of $\otimes$ operations are not, as exemplified by the following.
In the leftmost tree, encoding an iterated composition of \( \circ \), all leaves appear at the same height and the operations, encoded by nodes (i.e. the circles) are naturally divided into levels. On the other hand, this fails for the rightmost tree, which encodes iterated compositions of \( \circ_i \). Indeed, while the definition \( \varphi \circ_i \psi = \varphi(id, \ldots, id, \psi, id, \ldots, id) \) would allow us to convert the rightmost tree into a tiered tree by inserting nodes labeled by \( id \), there are multiple ways to do so (indeed, the leftmost tree represents one such possibility).

In practice, the second type of trees seems to be the most convenient and we will henceforth work only with such trees.

\[4.2 \text{ Symmetric trees}\]

The (planar) tree notation described above is suitable for working with so-called "non-\( \Sigma \) operads". In many applications, however, operads possess an additional piece of structure: each set (space/simplicial set) \( O \hat{\Sigma}^n \) has a left action of the symmetric group \( \Sigma_n \). Heuristically, the role of this action is to "change the order of the inputs of an operation": thinking of \( \varphi \in O(n) \) as an operation \( x_1, \ldots, x_n \mapsto \varphi(x_1, \ldots, x_n) \) and letting \( \sigma \in \Sigma_n \), then \( \sigma \varphi \in O(n) \) would correspond to the operation \( x_1, \ldots, x_n \mapsto \varphi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \).

When representing compositions on a (symmetric) operad, it thus becomes convenient to think of the edges above a node, which represent the inputs for the operation labeling the node, as not having a fixed order. One immediate drawback of this perspective, however, is that drawing a planar representation of such a tree on paper necessarily requires choosing an (arbitrary) order for the input edges of every node. Therefore, it is possible for different planar representations to encode the exact same information. For example, the pictures

\[\begin{align*}
  &a_1 \quad \psi \quad a_2 \\
  &\quad c_2 \\
  \varphi &\quad c_1 \\
  &b \\
  \quad c_3
\end{align*}\]  

\[\begin{align*}
  &a_1 \quad (12)\psi \\
  &\quad c_3 \\
  \varphi &\quad c_1 \\
  &b \\
  \quad c_2
\end{align*}\]

display two planar representations of the same tree that encode the same composition data. To explain why, we first point out that \( a_1, a_2, b, c_1, c_2, c_3 \) are simply the names of the edges of the tree (needed so as to distinguish different representations of the tree in the plane), by contrast with \( \psi, \chi, (123)\varphi \), etc. which are operations in \( O \). Next, for a finite set \( S \) of size \( n \), denote \( O(S) = (Iso(\{1, \ldots, n\}, S) \times O(n))_{\Sigma_n} \), where the orbits \( (-)_{\Sigma_n} \) are defined using the diagonal action, acting on the \( Iso(\{1, \ldots, n\}, S) \) component by precomposition with the inverse. Note that though \( O(S) \) is of course isomorphic to \( O(n) \), the isomorphism is not canonical, depending on the choice of an isomorphism \( \{1, \ldots, n\} \xrightarrow{\sim} S \). With this convention, both trees in (4.3) represent the same instance of a composition

\[O(\{c_1, c_2, c_3\}) \times O(\{a_1, a_2\}) \times O(\{b\}) \xrightarrow{(-)_{\Sigma_1} (-)_{\Sigma_2} (-)_{\Sigma_3}} O(\{a_1, a_2, c_2, b\}).\]

The reason for the differing labels on the nodes of the trees is that different planar representations correspond to different choices of isomorphisms \( \{1, \ldots, n\} \xrightarrow{\sim} \)}
These generators are (unique) morphisms $\varphi$ generated by morphisms associated to each node. Explicitly, for the tree in (4.3) regular operads are then "colored operads with a single color" satisfying natural associativity, unitality and symmetry conditions. Note that the automorphism group isomorphic to the wreath product $\Sigma_3 \wr \Sigma_2 = \Sigma_3 \times (\Sigma_2)^3$.

For reasons to become apparent when we discuss how to encode compositions in equivariant operads using equivariant trees, the fact that the labels in (4.3) change depending on the planar representative is rather inconvenient. Since the isomorphisms $\hat{O}$ change depending on the planar representative is rather inconvenient. For reasons to become apparent when we discuss how to encode compositions in equivariant operads using equivariant trees, the fact that the labels in (4.3) change depending on the planar representative is rather inconvenient.

Throughout the following we fix a finite group $G$ and the term operad will now refer to an operad $O$ together with a left $G$-action compatible will all the labels. For example, the leftmost tree uses the identification $\{1,2,3\} \cong \{c_1,c_2,c_3\}$ while the rightmost tree uses $\{1,2,3\} = \{c_3,c_1,c_2\}$ so that, for example, the classes $[(\{1,2,3\} \to \{c_1,c_2,c_3\}, \varphi)]$, $[(\{1,2,3\} \to \{c_3,c_1,c_2\}, (123) \varphi)]$ are in fact the same element of $O(c_1,c_2,c_3)$.

Remark 4.4. An alternative and more rigorous perspective on (4.3) is provided by [13, §3], where it is explained that any tree $T$ has an associated colored operad $\Omega(T)$. Briefly, colored operads generalize operads much in the way that categories generalize monoids: each colored operad $O$ has a collection of objects, morphism sets $O(b_1; a)$ for an ordered tuple of source objects $b = b_1, \ldots, b_n$ and target object $a$, units $id_a \in O(a; a)$, compositions

$$O(b_1, \ldots, b_n; a) \times O(c_1; b_1) \times \cdots \times O(c_n; b_n) \xrightarrow{\varphi} O(c_1, \ldots, c_n; a),$$

and isomorphisms $O(b_1, \ldots, b_n; a) \xrightarrow{\psi} O(b_{\sigma^{-1}(1)}; b_{\sigma^{-1}(2)}; \ldots; b_{\sigma^{-1}(n)}; a)$ for each $\sigma \in \Sigma_n$ satisfying natural associativity, unitality and symmetry conditions. Note that regular operads are then "colored operads with a single color".

$\Omega(T)$ is then the colored operad with objects the set of edges of $T$ and freely generated by morphisms associated to each node. Explicitly, for the tree in (4.3) these generators are (unique) morphisms $a_1a_2 \to c_1$, $b \to c_3$ and $c_1c_2c_3 \to d$ if using the left planar representation, or generators $a_2a_1 \to c_1$, $b \to c_3$, $c_3c_1c_2 \to d$ if using the right planar representation (that the two descriptions coincide follows from the symmetry isomorphisms).

(4.3) is then a diagrammatic representation of a morphism $\Omega(T) \to O$, between the colored operad $\Omega(T)$ and the regular operad $O$, with the node labels being the image of the associated generators of $\Omega(T)$.

4.3 Equivariant trees

Throughout the following we fix a finite group $G$ and the term operad will now refer to an operad $O$ together with a left $G$-action compatible will all the
structure. Notably, $\mathcal{O}(n)$ now has a $G \times \Sigma_n$-action, so that from an equivariant homotopy theory perspective it is natural to consider fixed points $\mathcal{O}(n)^\Gamma$ for $\Gamma \leq G \times \Sigma_n$. On the other hand, since operad theory often focuses on $\Sigma$-cofibrant operads (i.e. such that $\mathcal{O}(n)$ is $\Sigma_n$-free), it is natural to focus attention on $\Gamma$ such that $\Gamma \cap \Sigma_n = \ast$, since for such $\mathcal{O}$ it is $\mathcal{O}(n)^\Gamma = \emptyset$ otherwise. The key identifications

$\Gamma$ such that $\Gamma \cap \Sigma_n = \ast \iff$ graph of $G \geq H \to \Sigma_n \iff$ action of $H \leq G$ on $\{1, \ldots, n\}$

then hint at a deep connection between $G$-operads and $H$-sets that is at the core of Blumberg and Hill’s work in [2]. Briefly, to each $\Sigma$-cofibrant $G$-operad $\mathcal{O}$ they associate the family of those $\Gamma$ such that $\mathcal{O}(n)^\Gamma \neq \emptyset$ and, in turn, the family of the corresponding $H$-sets, $H \leq G$ ([2, Def. 4.5]). They then show that such families satisfy a number of novel closure conditions ([2, Lemmas 4.10, 4.11, 4.12, 4.15]), and dub such a family an indexing system [2, Def. 3.22]. Moreover, analyzing their proofs one sees that the key idea is that a careful choice of fixed point conditions on the source of the composition (4.1) will induce fixed point conditions on its target (for an explicit example, see (4.11) below).

The discovery of equivariant trees was the result of an attempt to encode the closure conditions of Blumberg and Hill diagrammatically, and we provide more details on how that works in §9. For now, however, we focus on examples.

As a first guess, one might attempt to define $G$-trees simply as symmetric trees together with a $G$-action (using the automorphisms mentioned in the previous section). As it turns out, such “trees with a $G$-action” are only a part of what is required, though we will choose such trees as our first examples.

**Example 4.5.** Let $G = \mathbb{Z}_4$. The following are two equivalent representations of a symmetric tree $T$ with a $G$-action.

The leftmost representation, which we call the expanded representation, is simply a planar representation of the corresponding equivariant tree, together with a naming convention for the edges that reflects the $G$ action. More concretely, $1 \in G$ acts on the tree by sending $a$ to $a + 1$, $a + 1$ to $a + 2$, $b$ to $b + 1$, etc (note that implicitly $b + 2 = b$, $c + 2 = c$, $d + 1 = d$).

The rightmost representation, which we call the orbital representation, is obtained from the expanded representation by “identifying edges which lie in the same $G$-orbit”, and then labeling the corresponding “edge orbit” by the $G$-set of the edges corresponding to it.

**Example 4.6.** Let $G = D_6 = \{e, r, r^2, r^3, r^4, r^5, s, sr, \ldots, sr^5\}$ denote the hexagonal dihedral group with generators $r, s$ such that $r^6 = e$, $s^2 = e$, $srs = r^5$. 

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Letting \( H_1 \geq H_2 \geq H_3 \) denote the subgroups \( H_1 = \langle r^2, s \rangle \), \( H_2 = \langle s \rangle \), \( H_3 = \langle e \rangle \) one has the following representations of a tree \( T \) with \( G \)-action.

We note that it is implicit in the orbital representation that, for example, the assignment \( b \mapsto c \) defines a \( G \)-set map \( (G/H_2) \cdot b \to (G/H_1) \cdot c \) (i.e. that \( H_1 \geq H_2 \)).

We can now ask what the analogue of the node labels in (4.2) are for such \( G \)-trees. For example, for \( G = \mathbb{Z}_4 \) consider the label \( \varphi \) in the leftmost expanded representation of the equivariant corolla (i.e. tree with a single node) below.

An immediate answer is provided by Remark 4.4: indeed, the corolla \( G \) assembles to an action of \( G \) (indeed, a label in \( \mathcal{O}(4) \), but an additional equivariance condition is to be expected. To make this explicit, note first that both \( G \) and \( \Sigma_4 \) act on the left on the set of all morphisms of \( \Omega(C) \) and that these actions commute, assembling to an action of \( G \times \Sigma_4 \). As concrete examples, \( 1 \in \mathbb{Z}_4 \) sends the morphism \( cb(b+1)(c+1) \to d \) to \( (c+1)(b+1)bc \to d \) while \( (124) \in \Sigma_4 \) sends \( cb(b+1)(c+1) \to d \) to \( (c+1)c(b+1)b \to d \). Further, one can readily check that the \( G \times \Sigma_4 \)-isotropy of the morphism \( cb(b+1)(c+1) \to d \) is precisely the subgroup \( \Gamma_{\{c,b,b+1,c+1\}} \) given by the graph of the homomorphism \( G \to \Sigma_4 \) encoding the \( G \)-set \( \{c,b,b+1,c+1\} \). And since \( \varphi \) is the image of that morphism, we get the sought condition \( \varphi \in \mathcal{O}(4)\Gamma_{\{c,b,b+1,c+1\}} \).

We now turn our attention to the orbital representation of \( C \) on the right side of (4.7), which is often preferable both for conceptual reasons and compactness. Writing \( S = \{b,c,b+1,c+1\} \), our node label is now the coordinate free label \([\varphi] \in \mathcal{O}(S)\) (indeed, a label in \( \mathcal{O}(4) \) can not be used since the orbital notation provides no ordering of \( S \)). Further, the equivariance condition is now straightforward: \( \mathcal{O}(S) \) has a \( G \)-action induced from that on \( S \) and it is simply \([\varphi] \in \mathcal{O}(S)^G \).

It seems helpful to make the equivalence between the previous two paragraphs explicit. The homomorphism \( \rho: G \to \Sigma_S \) encoding \( S \) induces a semi-direct product \( G \ltimes \Sigma_S \) with multiplication \( (g,\sigma)(\bar{g},\bar{\sigma}) = (g\bar{g},\sigma\rho(g)\bar{\sigma}\rho(g)^{-1}) \) and which naturally acts on \( S \) with the action of \((g,\sigma)\) given by \( \sigma\rho(g) \). The isomorphism \( \tau: \{c,b,b+1,c+1\} \to \{1,2,3,4\} \) then induces an isomorphism \( \tau_\ast: G \ltimes \Sigma_S \cong G \times \Sigma_4 \)
via \( \tau_\sigma(g, \sigma) = (g, \tau \sigma(g) \tau^{-1}) \). That the previous two paragraphs are equivalent is then simply the observation that \( \tau \) sends the subgroup \( G \leq G \times \Sigma_S \) to \( \Gamma_{\{s, b, b+1, c+1\}} \leq G \times \Sigma_4 \), i.e. that the graph subgroup encodes the canonical action when in coordinate dependent notation.

Now that we know that corollas with \( G \)-actions encode operations fixed by graphs of full homomorphisms \( G \to \Sigma_n \), we turn to the question of how to encode operations fixed by graphs of partial homomorphisms \( G \to H \to \Sigma_n \). A natural first guess might be that this role is played by corollas with a \( H \)-action. However, due to the lack of full \( G \)-actions this would not quite provide the necessary maps for the category \( \Omega_G \) of \( G \)-trees that we introduce in §5.3. The solution is both simple and surprising: one simply “induces a tree with a \( H \)-action into a \( G \)-object”. We start with an example where \( H = \ast \).

**Example 4.8.** Let \( G = \mathbb{Z}_3 \). The equivariant corolla \( C \) with orbital representation given on the right

has expanded representation given by the union of the three (non-equivariant) corollas on the left. For clarity, we stress that we refer to the three trees on the left together as a forming a single \( \mathbb{Z}_3 \)-tree. The legitimacy of this convention is born out of the role such \( G \)-trees play in the theory, though for now we simply point out that at least the orbital representation is a “honest” tree (for further discussion, see §5.3).

A map \( \Omega(C) \to \mathcal{O} \) is then determined by the image of the morphism \( abc \to d \), and hence by an arbitrary operation \( [\varphi] \in \mathcal{O}(\{a, b, c\}) = \mathcal{O}(3) \), which determines operations \( [\varphi] + 1 \in \mathcal{O}(\{a + 1, b + 1, c + 1\}) \), \( [\varphi] + 2 \in \mathcal{O}(\{a + 2, b + 2, c + 2\}) \).

**Example 4.9.** Keeping \( G = D_6 \) and \( H_1, H_2, H_3 \) as in Example 4.6, removing the root orbit (i.e. bottom orbit) from the \( G \)-tree \( T \) therein yields the \( D_6 \)-tree

We end this introduction by illustrating the kind of compositions that \( G \)-trees encode. Taking the \( D_6 \)-tree \( S \) from Example 4.9, a map \( \Omega(S) \to \mathcal{O} \) leads
to node labels

where $[\psi] \in \mathcal{O}((H_2/H_3) \cdot a)^{H_2}$ and $[\varphi] \in \mathcal{O}((H_1/H_3) \cdot b)^{H_1}$. We note that in particular $r[\varphi] \in \mathcal{O}(r(H_1/H_2) r^{-1} \cdot r b)^{r H_1 r^{-1}} = \mathcal{O}((H_1^h / H_3^h) \cdot r b)^{H_1^h}$ and likewise for $r \psi, r^2 \psi, ..., r^3 \psi$, so that we are adopting the convention that labels in the orbital notation are chosen according to the edge orbit generators $a, b, c$.

Further unpacking the map $\hat{\Omega}(S) \to \mathcal{O}$, $S$ encodes the fact that the composition product

$$\mathcal{O}(H_1/H_2) \times \prod_{[h] \in H_1/H_2} \mathcal{O}(H_2^h / H_3^h) \to \mathcal{O}(H_1/H_3)$$

restricts to

$$\left( \prod_{[h] \in H_1/H_2} \mathcal{O}(H_2^h / H_3^h)^{H_1} \right)^{H_1} \to \mathcal{O}(H_1/H_3)^{H_1} \quad (4.11)$$

or, using that $\left( \prod_{[h] \in H_1/H_2} \mathcal{O}(H_2^h / H_3^h)^{H_1} \right)^{H_1} = \mathcal{O}(H_2/H_3)^{H_2}$, simply

$$\mathcal{O}(H_1/H_2)^{H_1} \times \mathcal{O}(H_2/H_3)^{H_2} \to \mathcal{O}(H_1/H_3)^{H_3}.$$

## 5 Categories of trees and forests

In this section we introduce the several categories of trees, forests and presheafs we will be working with. We will make heavy use of the broad poset framework introduced by Weiss in [15], which provides an algebraically flavored model for the category $\Omega$ of trees (cf. [13], [14], [3], [9], among others). We will find this particularly convenient since, when using tree diagrams as in §4.3, representative examples of equivariant trees are typically quite large.

### 5.1 Broad posets

We start by recalling the key notions in [15] and establishing some basic results.

Given a set $T$ we denote by $T^*$ the free abelian monoid generated by $T$. Elements of $T^*$ will be written in tuple notation, such as $\epsilon = e_1 e_2 e_3 e_4 \in T^*$ for $e_1, e_2, e_3 \in T$. We will also write $e_i \in \epsilon$ whenever $e_i$ is a “letter” appearing in $\epsilon$, $f \notin \epsilon$ if $f g = \epsilon$ for some $g \in T^*$, and denote the “empty tuple” of $T^*$ by $\epsilon$.

**Definition 5.1.** A (commutative) broad poset structure ([15, Def. 3.2]) on $T$ is a relation $\preceq$ on $(T^*, T)$ satisfying

- **Reflexivity:** $e \preceq e$ (for $e \in T$);
- **Antisymmetry:** if $e \preceq f$ and $f \preceq e$ then $e = f$ (for $e, f \in T$);
• Broad transitivity: if \( f_1 f_2 \cdots f_n = f \leq e \) and \( g_i \leq f_i \), then \( q_i \cdots q_n \leq e \) (for \( e, f_i \in T \), \( f, q_i \in T^* \)).

Since the main examples of broad posets are induced by constructions involving trees, we will refer to the elements of a broad poset as its edges.

**Definition 5.2.** A broad poset \( P \) is called simple if for any broad relation \( e_1 \cdots e_n \leq e \) one has \( e_i = e_j \) only if \( i = j \).

**Notation 5.3.** A broad poset structure \( \leq \) on \( T \) naturally induces the following preorder relations on \( T \) and \( T^* \):

- for \( f, e \in T \) we say that \( f \) is a descendant of \( e \), written \( f \leq_d e \), if there exists a broad relation \( f \leq e \) such that \( f \in f \).
- for \( f, e \in T^* \), we write \( f \leq e \) if it is possible to write \( f = f_1 \cdots f_k \), \( e = e_1 \cdots e_k \) such that \( f_i \leq e_i \) for \( i = 1, \ldots, k \).

**Remark 5.4.** Generally, these preorders can be fairly counter-intuitive. For example, it is possible to have \( ab \leq a \), or even both \( aa \leq a \) and \( a \leq aa \) simultaneously. The case of simple broad posets, however, is much simpler.

**Proposition 5.5.** Let \( T \) be a simple broad poset. Then \( \leq_d \) (resp. \( \leq \)) is an order relation on \( T \) (resp. on \( T^* \)). Further, if \( f_1 \cdots f_k \leq e \) then the \( f_i \) are \( \leq_d \)-incomparable (in particular, \( e f \leq e \) only if \( f = e \)).

**Proof.** The “further” part is immediate: if two \( f_i \) were \( \leq_d \)-comparable then broad transitivity would produce a non simple broad relation.

To see that \( \leq_d \) satisfies antisymmetry, note that if \( e'f \leq e \) and \( eg \leq e' \) then \( egf \leq e \) so that it must be \( g = f = e' \) and the antisymmetry of \( \leq \) applies.

Finally, we show antisymmetry of \( \leq \) by induction on the size of the tuple \( e \) in a pair of relations \( f \leq e \) and \( e \leq f \). The \( e = f \) case is immediate. Otherwise let \( e \in e \) be \( \leq_d \)-minimal and choose \( eg \in e, f \leq f \) such that \( eg \leq f \) and choose \( hf \in f \) and \( e'f \in e \) such that \( hf \leq e'f \). Then \( e \leq_d f \leq e'f \) and by \( \leq_d \)-minimality of \( e \) it must be \( e = f \neq e' \) and hence, by the “further” claim, also \( g = e = h \). And since this must hold regardless of how \( f, e', g, h \) are chosen, one concludes that, writing \( e = e' \), \( f = e'f \), it must in fact also be \( e' \leq f' \) and \( e' \leq e' \), so that the induction hypothesis applies.

**Definition 5.6.** An edge \( e \in T \) is called

- a leaf if there are no \( f \in T^* \) such that \( f < e \) (i.e. \( f \leq e \) and \( f \neq e \));
- a node if there is a non empty maximum \( f \neq e \) such that \( f < e \);
- a stump if \( e \) is the maximum (in fact, only) \( f \) such that \( f < e \).

Further, in either the node or stump case the maximum such \( f \) is denoted \( e^* \).

**Remark 5.7.** While it is customary to regard stumps simply as a type of node, we find it convenient, in lieu of Proposition 7.12 and Lemma 7.21, to separate the two cases.

The following definition is the key purpose of [15].
Definition 5.8. A dendroidally ordered set is a finite simple broad poset $T$ such that

- each edge $e \in T$ is either a leaf, a node or a stump;
- there is a maximum $r_T \in T$ for $\preceq_d$, called the root of $T$.

Weiss proves in [15] that the category of dendroidally ordered sets (together with the obvious notion of monotonous function) is equivalent to the category $\Omega$ of trees ([13], [14], [3], [9] et al.). As such, we will henceforth refer to dendroidally ordered sets simply as trees and use them as our model for $\Omega$.

Example 5.9. The tree diagram

$$\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\text{e} \\
\text{f} \\
\text{g} \\
\text{r}
\end{array}$$

represents a broad poset structure on \{a, b, c, d, e, f, g, r\}. The nodes represent generating broad relations $\epsilon \leq b$, $ab \leq e$, $\epsilon \leq f$, $cd \leq g$, $efg \leq r$ with the other broad relations, such as $afcd \leq r$, obtained by “composition” (i.e. using broad transitivity). Note that, alternatively, one can also write $b^1 = \epsilon$, $e^1 = ab$, $f^1 = \epsilon$, $g^1 = cd$, $r^1 = efg$ to denote the generating broad relations.

We will make use of the following basic results.

Proposition 5.11. Let $T$ be a tree and $A$ any broad poset. A set map $\varphi:T \to A$ is a broad poset map if and only if $\varphi(e^1) \leq \varphi(e)$ for each node/stump $e \in T$.

Proof. Since for any non-identity relation $f \leq e$ one has $f \leq e^1 \leq e$ one can write $e^1 = e_1 \cdots e_k$, $f = f_1 \cdots f_k$ so that $f_j \leq e_i$ (k = 0 is allowed, in which case $e^1 = f = e$), so the result follows by upward $\preceq_d$-induction on $e$.

Lemma 5.12. Let $T$ be a tree. For any $e \in T$ there exists a minimum $e^\lambda \in T^*$ such that $e^\lambda \leq e$. In fact, $e^\lambda = l_1 \cdots l_k$ consists of those leaves $l_i$ such that $l_i \preceq_d e$.

Further, a broad relation $f = f_1 \cdots f_n \leq e$ holds if and only if $f_i \preceq_d e$, the $f_i$ are $\preceq_d$-incomparable and $f_1^\lambda \cdots f_n^\lambda = e^\lambda$.

Proof. The proof is by upward $\preceq_d$ induction on $e$. The leaf case is obvious. Otherwise, let $f \leq e^1 \leq e$ be any non identity relation and write $e^1 = e_1 \cdots e_k$ and $f = f_1 \cdots f_k$ so that $f_j \leq e_i$. By induction, $e^1_j \leq f_j \leq e_i$ where $e^1_i$ consists of the leaves $l$ such that $l \preceq_d e_i$ and hence indeed $e^\lambda = e_1^\lambda \cdots e_k^\lambda \leq f$.

Only the “if” half of the “further” statement needs proof. We use the same induction argument: incomparability yields $e \notin f$ for $f \neq e$ and, writing $e^1 = e_1 \cdots e_k$ and $f = f_1 \cdots f_k$ so that $s \leq e_i$ if and only if $s \preceq_d e_i$, the induction hypothesis applies.

Example 5.13. In Example 5.9 $e^\lambda = a$, $g^\lambda = cd$, $f^\lambda = \epsilon$, $r^\lambda = acd$.

Lemma 5.14. Let $T$ be a tree and $\epsilon$ a tuple of $\preceq_d$-incomparable edges of $T$. Then, letting $\tau$ be the root of $T$, there exists a broad relation of the form $\epsilon f \leq \tau$. 

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Proof. The proof is by induction on the sum of the distances (measured in \( \leq_d \) inequality chains) from the leaves to the root. Clearly \( r \in \mathcal{L} \) only if \( r = \mathcal{L} \). Otherwise one can write \( r^d = r_1 \cdots r_k, \mathcal{L} = \mathcal{L}_1 \cdots \mathcal{L}_k \) so that \( s \in \mathcal{L} \) if and only if \( s \leq_d r_i \). The induction hypothesis applies to the subtrees \( T_i = \{ e \in T | e \leq_d r_i \} \).

5.2 Categories of forests

We will also need to discuss forests, i.e. formal coproducts of trees. We start by generalizing Definition 5.8.

Definition 5.15. A forestly ordered set is a finite simple broad poset \( F \) such that

- each edge \( e \in F \) is either a leaf, a node or a stump;
- for each edge \( e \) there is a unique \( \leq_d \)-maximal element \( r \in T \) such that \( e \leq_d r \).

Further, we will denote by \( \mathcal{L}_F \) the tuple of \( \leq_d \)-maximal elements of \( F \) and refer to it as the root tuple.

The last condition guarantees that any forestly ordered set decomposes as a disjoint union of dendroidally ordered sets, i.e. trees. We shall hence refer to these simply as forests and denote by \( \Phi \) the category formed by them.

Definition 5.16. A map of forests \( F \xrightarrow{\varphi} F' \) is called

- wide if \( \varphi(\mathcal{L}_F) \leq \mathcal{L}_{F'} \);
- independent if there exists a tuple \( \mathcal{L} \in F^* \) such that \( \varphi(\mathcal{L}_F) \mathcal{L} \leq \mathcal{L}_{F'} \).

The subcategory of forests and independent (resp. wide) maps is denoted \( \Phi_i \) (resp. \( \Phi_w \)). Note that there are inclusions \( \Phi_w \hookrightarrow \Phi_i \hookrightarrow \Phi \).

Remark 5.17. The category \( \Phi_i \) nearly coincides with the category of forests discussed in [9, §3.1], the only difference being that here we include the empty forest \( \emptyset \).

Example 5.18. Consider the following tree \( T \) and subtrees (all labels are on edges).

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) [circle,draw] {a};
  \node (b) at (1,1) [circle,draw] {b};
  \node (c) at (1,0) [circle,draw] {c};
  \node (d) at (0,-1) [circle,draw] {d};
  \node (e) at (2,0) [circle,draw] {e};
  \node (f) at (1,-2) [circle,draw] {f};
  \node (g) at (2,-1) [circle,draw] {g};
  \node (h) at (3,-1) [circle,draw] {h};
  \node (i) at (2,-2) [circle,draw] {i};
  \path[](a)edge(b) edge (d) (b)edge (c) (d)edge (f) (f)edge (g) edge (e) (e)edge (h) (h)edge (i);
\end{tikzpicture}
\end{array}
\]

Further denoting by \( \eta_a, \eta_b, \cdots \) the “single edge” subtrees corresponding to each edge, some examples of wide morphisms are given by the inclusions

\[
U \hookrightarrow T, \quad V \sqcup \eta_f \hookrightarrow T, \quad V \sqcup \eta_e \hookrightarrow T
\]

\[
V \sqcup \eta_f \sqcup \eta_i \hookrightarrow T, \quad V \sqcup \eta_f \sqcup \eta_h \hookrightarrow T, \quad \eta_a \sqcup \eta_b \sqcup \eta_f \hookrightarrow T.
\]
some examples of non-wide independent maps are given by
\[ V \cup \eta_T, \eta_f \cup \eta_h \rightarrow T, \eta_g \cup \eta_h \rightarrow T, \]
and examples of non-independent maps are given by
\[ V \cup \eta_T, V \cup \eta_f \cup \eta_g \rightarrow T, \eta_h \cup \eta_h \rightarrow T. \]

The following is a useful “forestal” strengthening of Proposition 5.5, which follows by combining that result with Lemma 5.14.

**Proposition 5.19.** Let \( F \) be a forest. Then, if \( f \leq e = e_1 \cdots e_k \) with \( \preceq_d \)-incomparable \( e_i \), the decomposition \( f = f_1 \cdots f_k \) with \( f_i \leq e_i \) is unique. In fact, for \( s \in f \) one has \( s \in f_i \) iff \( s \preceq_d e_i \).

**Proof.** If \( s \preceq_d e_i, s \preceq_d e_j \) with \( i \neq j \) the unique root \( r \) such that \( s \preceq_d r, e_j \preceq_d r \) and Lemma 5.14 can be used to produce a non simple relation. Thus such \( e_i \) is unique and the result follows.

We now turn to describing the degeneracy-face decomposition of maps in our setting, which we obtain as Proposition 5.26 below.

**Definition 5.20.** A map \( F \xrightarrow{\varphi} F' \) is called
- a **face map** if the underlying set map is injective;
- a **degeneracy** if the underlying set map is surjective and for each leaf \( l \in F \) it is \( \epsilon \notin \varphi(l) \).

**Lemma 5.21.** For any map \( F \xrightarrow{\varphi} F' \) in \( \Phi_i \), \( e, \bar{e} \) are \( \preceq_d \)-comparable iff \( \varphi(e), \varphi(\bar{e}) \) are \( \preceq_d \)-comparable.

Further, if \( \varphi(e) = \varphi(\bar{e}) \) then \( e \preceq \bar{e} \) can hold only if \( f = e \) and thus either \( e \preceq \bar{e} \) or \( \bar{e} \preceq e \).

**Proof.** If \( e, \bar{e} \) are not \( \preceq_d \)-comparable, Lemma 5.14 ensures that there exists \( f \) such that \( e \preceq f \preceq \varphi\). Thus, by definition of \( \Phi_i \), there exists \( g \) such that \( \varphi(e) \preceq \varphi(\bar{e}) \preceq \varphi(f) \preceq \varphi(\bar{e}) \), hence \( \varphi(e), \varphi(\bar{e}) \) are \( \preceq_d \)-incomparable.

The “further” claim is immediate.

**Lemma 5.22.** If \( F \xrightarrow{\varphi} F' \) is both a face map and a degeneracy then \( \varphi \) is an isomorphism.

**Proof.** Bijectiveness allows us to assume that the underlying sets are the same and it thus follows by Lemma 5.21 that the relations \( \preceq_d F \) and \( \preceq_d F' \) coincide. Hence, both forest structures have the same roots (i.e. \( \preceq_d \)-maximal edges) and one needs only show that broad relations coincide in each of the composing trees. But noting that a leaf \( l \) is precisely a \( \preceq_d \)-minimal edge such that \( \epsilon \notin t \), the definition of degeneracy implies that \( F, F' \) have the same leaves and the result follows by Lemma 5.12.

**Lemma 5.23.** Let \( F \xrightarrow{\varphi} F' \) be a map in \( \Phi_i \) and \( \imath \preceq_f e \) be a broad relation in \( F \) such that \( \varphi(\imath) \neq \varphi(e) \). Then for any \( \imath', e' \) such that \( \varphi(\imath') = \varphi(\imath), \varphi(\imath') = \varphi(e) \) it is also \( \imath' \preceq_f e' \).
Proof. Consider first the case $e = e'$. The proof is by upward $\leq_d$-induction on $e$. Writing $e^i = e_1 \cdots e_k$, $f = \bigwedge \cdots \bigwedge f_k$, $f' = \bigwedge \cdots \bigwedge f_k'$ so that $f \leq e_i$, $\varphi(f) = \varphi(f')$, the induction hypothesis applies to establish each relation $f'_i \leq_F e_i$ unless it is $\varphi(f') = \varphi(e_i)$. But then $f'_i = f^i$ is in fact a simpleton and hence by the “further” part of Lemma 5.21 it is either $e_i \leq f'_j$ or $f'_j \leq e_i$. On the other hand, the first half of Lemma 5.21 together with the condition $\varphi(f) \neq \varphi(e)$ imply that $f'_i \leq_d e$ and thus that $f'_i \leq_d e_j$ for some $j$. But since the $e_m$ are pairwise $\leq_d$-incomparable, it must be that $j = i$ and $f'_i \leq e_i$, finishing the argument in this case.

Consider now the case $f = f'$, again proven by upward $\leq_d$-induction on $e$. By the “further” part of Lemma 5.21 it is either $e \leq e'$ or $e' < e$. The former case is immediate and in the later case $e' \leq e$ and hence by applying $\varphi$ we see that $e'$ must be a simpleton, so that the induction hypothesis finishes the proof. \hfill \square

Remark 5.24. It follows by the “further” part in Lemma 5.21 together with antisymmetry that the pre-image of any edge by a map $\varphi$ always consists of a linearly ordered subset of edges. As such, degeneracies are necessarily maps that “collapse linear sections of a tree”, and indeed that was their original description in [15]. A typical degeneracy (sending edges labeled $a_i$, $b_i$, $c_i$, $d_i$ to respective edges $a$, $b$, $c$, $d$) is pictured below.

\begin{center}
\begin{tikzpicture}
\node (a1) at (0,2) {$a_1$};
\node (a2) at (0,1) {$a_2$};
\node (b1) at (1,1) {$b_1$};
\node (b2) at (1,0) {$b_2$};
\node (c1) at (2,2) {$c_1$};
\node (c2) at (2,1) {$c_2$};
\node (d1) at (2,0) {$d_1$};
\node (e1) at (2,-1) {$e_1$};
\node (e2) at (2,-2) {$e_2$};

\draw (a1) -- (b1);
\draw (a2) -- (b2);
\draw (b1) -- (c1);
\draw (b2) -- (c2);
\draw (c1) -- (d1);
\draw (c2) -- (d2);
\draw (d1) -- (e1);
\draw (d2) -- (e2);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\node (b) at (0,1) {$b$};
\node (c) at (1,1) {$c$};
\node (d) at (1,0) {$d$};
\node (e) at (1,-1) {$e$};
\end{tikzpicture}
\end{center}

Lemma 5.25. Let $\varphi: F \rightarrow F'$ be a map in $\Phi_i$ and let $U \subset F$ be any sub-broad poset consisting of exactly one edge in each pre-image of $\varphi$ and the broad relations of $F$ between them. Then $U$ is a forest.

Proof. Simplicity of $U$ is inherited from $F$. Given $u \in U$, let $\bar{u} \in F$ be the $\leq_d$-minimal edge such that $\varphi(\bar{u}) = \varphi(u)$. Lemma 5.23 implies that $u$ is a leaf in $U$ if $\bar{u}$ is a leaf in $F$. Otherwise, $\bar{u} \not< \bar{u}$ and again by Lemma 5.23 the unique tuple $\bar{v}$ of $U$ such that $\varphi(\bar{v}) = \varphi(\bar{v}^\rightarrow)$ provides the desired tuple $\bar{v}^\rightarrow = v$ for $U$. Lastly, by Lemma 5.23 $s$ will be a root of $U$ iff $\varphi(s) = \varphi(r)$ for $r$ a root of $F$. \hfill \square

We now prove the following factorization result, which in the $\Omega$ case first appeared as [13, Lemma 3.1] and in the $\Phi_i$ case was proven in [9, Lem. 3.1.3].

Proposition 5.26. Each map $\varphi$ of $\Phi_i$ has a factorization $\varphi = \varphi^+ \circ \varphi^-$ as a degeneracy followed by a face. Further, this decomposition is unique up to unique isomorphism.

Finally, the decomposition restricts to the subcategories $\Omega$ and $\Phi_w$.

Proof. Given a map $F \rightarrow F'$ and picking any $U$ as in Lemma 5.25, the isomorphism $U \cong \varphi(F)$ allows us to equip $\varphi(F)$ with a forest structure. Moreover, by
Lemma 5.23 the broad relations on \( \varphi(F) \) are exactly the image of those in \( F \), and thus independent of \( U \). The existence of a factorization thus follows.

The uniqueness of the \( F \xrightarrow{s} G \xrightarrow{\varphi} F' \) factorization follows since by the description of the broad structure on \( \varphi(F) \) there is clearly a broad poset map \( \varphi(F) \to G \), which by Lemma 5.22 is an isomorphism.

The decomposition restricts to \( \Omega \) since the image by a degeneracy map of a tree is necessarily a tree and restricts to \( \Omega_w \) since any edge surjective map \( F \xrightarrow{s} G \) must map roots to roots and hence it must be \( \varphi^{-1}(\xi_F) = \xi_G \).

\[ \square \]

Corollary 5.27. If \( F \xrightarrow{s} F' \) is a degeneracy in any of \( \Omega, \Phi_i, \Phi_w \), then the broad relations in \( F' \) are precisely the image of the broad relations of \( F \). Further, any section \( \rho \xrightarrow{s} F \) of the underlying set map is a section in \( \Omega, \Phi_i, \Phi_w \).

The following will be needed in §5.4 when discussing dendroidal boundaries.

Corollary 5.28. In any of \( \Omega, \Phi_i, \Phi_w \), pairs of degeneracies with common domain have absolute pushouts\(^1\).

Proof. We will throughout write \( \Gamma \) for any of \( \Omega, \Phi_i, \Phi_w \) and \( \Gamma[F] \in \text{Set}^{\Gamma^{op}} \) for the presheaf represented by \( F \in \Gamma \). Given a diagram \( E \xrightarrow{s} F \xrightarrow{\rho} E' \) form a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{s} & F \\
\downarrow & & \downarrow \rho \\
D & \xrightarrow{\rho'} & E' \\
\end{array}
\]

where \( s \) is any chosen section of \( \rho \), the left bottom path is the factorization of \( \rho' \circ s \) given by Proposition 5.26 and the dashed arrow is the underlying surjection of sets. This last surjection is seen to also be a map in \( \Gamma \) by applying the first half of Corollary 5.27 to the maps around the right square. Applying the second half of Corollary 5.27 one sees that compatible sections can be chosen for the maps around the right square (possibly after changing the section \( s \)) showing that the presheaf map \( \Gamma[E] \xrightarrow{\rho} \Gamma[F] \xrightarrow{\rho'} \Gamma[E'] \to \Gamma[D] \) is surjective. Injectivity follows since broad posets map are determined by the underlying set map. \( \square \)

We now recall the usual ([15],[14],[3],[9]) description of faces as composites of maximal “codimension 1” faces. We first discuss some terminology.

Firstly, we will regard a face \( F' \) of \( F \) as a subset of \( F \) together with a subset of the broad relations of \( F \), and write \( F' \rightrightarrows F \) to indicate this. Further, if the broad relations between edges of \( F' \) in the broad posets \( F' \) and \( F \) coincide, then we will call \( F' \) a closed face of \( F \) and write \( F' \subseteq F \) instead.

Secondly, an edge \( e \in F \) is called external if \( e \) is either a leaf or a root and internal otherwise.

Finally, we denote a generating broad relation of \( F \) by \( v_e = (e^\ell \leq e) \) and refer to it as the vertex at \( e \).

Notation 5.29. The maximal faces of \( F \) in \( \Omega, \Phi_i, \Phi_w \) have the following types.

- The inner face (valid for any of \( \Omega, \Phi_i, \Phi_w \)) associated to an inner edge \( e \) is the closed face \( F - e \subseteq F \) obtained by removing \( e \);

\[^1\text{Recall that an absolute colimit can be described as either a colimit that is preserved by the Yoneda embedding or (equivalently) a colimit that is preserved by any functor.}\]
• The leaf vertex outer face (valid for any of $Ω, Φ_i, Φ_w$) associated to a vertex $v_e$ such that $e^\dagger$ consists of leaves is the closed face $F - v_e \in F$ obtained by removing the leafs in $e^\dagger$;

• The stump outer face (valid for any of $Ω, Φ_i, Φ_w$) associated to a stump vertex $v_e = (ε = e^\dagger \leq e)$ is the face $F - v_e \rightarrow F$ with the same edges as $F$ but removing $ε \leq e$ as a generating broad relation;

• The root vertex outer face (valid only for $Ω$) for an edge $e_r^\dagger$ such that the edges of $r^\dagger$ other than $e$ are leaves is the closed face $T_{_Bd}e \in T$ consisting of those edges $\bar{e}$ such that $\bar{e} \leq d$;

• The root face (valid only for $Φ_i, Φ_w$) associated to a root $r_i \in F$ (which, in the $Φ_w$ case, can not also be a leaf) is the closed face $F - r_i \in F$ obtained by removing $r_i$;

• The stick component face (valid only for $Φ_i$) associated to a stick $η \in F$ (i.e. an edge that is simultaneously a root and a leaf) is the closed face $F - η \in F$ obtained by removing $η$.

Remark 5.30. The implicit claim that an inner face $F - e$ is itself a forest can easily be checked using Lemma 5.12, which shows that $f^\dagger F - e$ can be defined to consist of the $\leq_d$-maximal edges $f_i \not\leq e$ such that $f_i \leq_d f$.

Similarly, Lemma 5.12 shows that a broad relation $f_1 \cdots f_n \leq f$ in $F$ holds in a stump outer face $F - v_e$ if the condition $e \leq_d f$ implies that $e \leq_d f_i$ for some $i$.

Example 5.31. Consider the trees in Example 5.18. One can write

$$ U = (((((T - v_c) - v_d) - f) - i) - g), $$

where the intermediate steps (from the inside out) are a stump face, a leaf vertex face and three inner faces. Further, both $η_a = V_{_Bd}a$ and $η_b = V_{_Bd}b$ are root vertex outer faces of $V$ when viewing $V$ as a tree and $η_a \sqcup η_b = V - d$ is a root face of $V$ when viewing $V$ as a forest.

5.3 The category of equivariant trees

Let $G$ be a finite group. We will denote by $Φ^G$ the category of $G$-forests, i.e. forests equipped with a $G$-action.

Definition 5.32. The category of $G$-trees, denoted $Ω_G$, is the full subcategory $Ω_G \in Φ^G$ of $G$-forests $F$ such that the root tuple $ε_F$ consists of a single $G$-orbit.

Examples of equivariant trees can be found throughout section 4.3. The author is aware that the fact that $G$-trees often “look like forests” is likely counter-intuitive at first (indeed, that was a major hurdle in the development of the theory presented in this paper). However, the following two facts may assuage such concerns: (i) similarly to how a non-equivariant tree is a forest that can not be decomposed as a coproduct of forests, so too a $G$-tree can not be equivariantly decomposed as a coproduct of $G$-forests; (ii) the orbital decomposition of a $G$-tree (cf. section 4.3) always does “look like a tree”.

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Remark 5.33. We note that $\Omega_G$, the category of $G$-trees, is rather different from $\Omega^G$, the category of trees with a $G$-action. In fact, each $G$-tree is (non-canonically) isomorphic to a forest of the form $G \cdot_H T$ for some $H \leq G$ and $T \in \Omega^H$. More precisely, one has the following elementary proposition.

**Proposition 5.34.** $\Omega_G$ is equivalent to the Grothendieck construction for the functor

\[
\begin{align*}
G/H & \longrightarrow \Omega^{G/H} \\
\end{align*}
\]

where $G/H$ denotes the groupoid with objects the cosets $gH$ and arrows $gH \xrightarrow{g}\bar{g}gH$.

Remark 5.35. There is a natural inclusion $G \Omega_0 \Omega_G$ given by regarding each object $(\hat{\varphi}, T) \in G \times \Omega$ as the $G$-tree given by the $G$-free forest $G \cdot_H T$. 

Remark 5.36. While maps in $\Omega$ can be built out of two types of maps, faces and degeneracies (Proposition 5.26), in $\Omega_G$ we need a third type of map: quotients. To see this, note that by Proposition 5.34, each $G$-tree $T$ sits inside of one of the subcategories of $\Omega^{G/H}$, and that $\Omega^H$ is a skeleton of the latter. Since it is immediate by (the proof of) Proposition 5.26 that the degeneracy-face decomposition extends to $\Omega^H$, Proposition 5.34 implies that any map $\Omega_G$ factors as a degeneracy followed by a face (both inside one of the fibers $\Omega^{G/H}$) followed by a cartesian map. We will prefer to refer to cartesian maps as quotients.

For a representative example, let $G = \mathbb{Z}/8$ and consider the map below (represented in orbital notation, cf. section 4.3), where we follow the following conventions: (i) edges in different trees with the same label are mapped to each other; (ii) if an edge is denoted $e^{+i}$, we assume that its orbit is disjoint from that of $e$ and that, if no edge labeled $e^{+i}$ appears in the target tree, then $\varphi(e^{+i}) = \varphi(e) + i$.

This map can be factored as (where, for the sake of brevity, we write $a + G/4G$ as $a/4$, etc.)
It is perhaps worth unpacking the last map in (5.37), which is an example of a quotient. The $G$-tree labeled $V$ can be written as $V = G \cdot \overline{V}$ where $\overline{V} \in \Omega^{G}$ is the tree with a $4G$-action pictured below.

In words, $V$ consists (non-equivariantly) of four trees identical to $\overline{V}$ which are interchanged by the action of elements of $G$ other than 0, 4. $W$, on the other hand, consists of a single (non-equivariant) tree, also shaped like $\overline{V}$, and can be thought of as the quotient of $V$ obtained by gluing the four trees so that the edges $e^i + j$ and $e + i + j$ are identified.

**Remark 5.38.** One particularly convenient property of $\Omega$ is that $\Omega^{\text{op}}$ is a generalized Reedy category, in the sense of [1]. In fact, $\Omega$ is a dualizable generalized Reedy category, so that both $\Omega$ and $\Omega^{\text{op}}$ are generalized Reedy.

Unfortunately, this is not the case for $\Omega^{G}$: while indeed $\Omega^{G}$ itself can be shown to be generalized Reedy, the opposite category $\Omega^{G, \text{op}}$ is not. The problem is readily apparent in the factorization in (5.37). Indeed, for the Reedy factorizations to hold (cf. [1, Defn. 1.1(iii)]), quotient maps would need to be considered the same type of maps as face maps, i.e. degree raising maps of $\Omega^{G}$. However, the quotient map in (5.37) fails [1, Defn. 1.1(iv')], since there is an automorphism of $V$ (given by $e^i \mapsto e^{i+1} - 1$, where $e^0$ is interpreted as $e$, and yet undefined $e^i$ labels are interpreted by regarding $i \in \mathbb{Z}_{pk}$ as needed) compatible with the quotient map to $W$.

### 5.4 Presheaf categories

We now establish some key terminology and notation concerning the presheaf categories we will use. Recall that the category of **dendroidal sets** is the presheaf category $\dSet = \mathbf{Set}^{\Omega^{\text{op}}}$. 

**Definition 5.39.** The category of **$G$-equivariant dendroidal sets** is the category $\dSet^{G} = \mathbf{Set}^{\Omega^{G, \text{op}}}$.

The category of **genuine $G$-equivariant dendroidal sets** is the category $\dSet_{G} = \mathbf{Set}^{G^{\text{op}}}$.

Twisting the inclusion in Remark 5.35 by the inverse map $G^{\text{op}} \xrightarrow{(-)^{+1}} G$ yields an inclusion $u^{*}: \Omega^{\text{op}} \times G \rightarrow \Omega_{G}$.

**Proposition 5.40.** The adjunction

$$u^{*}: \dSet_{G} \rightleftarrows \dSet^{G}: u_{*}$$

identifies $\dSet^{G}$ as a reflexive subcategory of $\dSet_{G}$.
Remark 5.42. Since Theorem 2.1 concerns $dSet^G$, that category will be our main focus throughout the present paper, although $dSet_G$ also plays a role in its proof (cf. §8.2).

Nonetheless, $dSet_G$ is arguably the most interesting category. Indeed, the adjunction (5.41) bears many similarities to the adjunction $sSet_G^{op} \rightleftarrows sSet^{G^{op}}$ and, as will be shown in upcoming work, the full structure of the “homotopy operad” of a $G$-operad is described as an object in $dSet_G$ rather than in $dSet^G$, similarly to how $\pi_n$ of a $G$-space forms a $G$-coefficient system rather than just a $G$-set. We conjecture that a model structure on $dSet_G$ making (5.41) into a Quillen equivalence exists, and that too is the subject of current work. The presence of extra technical difficulties when dealing with $\Omega_G$ (cf. Remark 5.38), however, make it preferable to address the $dSet_G$ case first.

Notation 5.43. Recall the usual notation $\Omega_T: \Omega \rightarrow dSet$ for the Yoneda embedding.

One can naturally extend this notation to the category $\Phi$ of forests: given $T = u_i T_i$, set $\Omega[F] = u_i \Omega[T_i]$. Passing to the $G$-equivariant object categories and using the inclusion $\Omega_G \hookrightarrow \Phi$ we will slightly abuse notation and write

$$\Omega_T: \Omega \rightarrow dSet^G.$$  (5.45)

More explicitly, if $T = G \cdot_H T_e$ for some $T_e \in \Omega^H$, then $\Omega[T] = G \cdot_H \Omega[T_e]$, where $\Omega[T_e]$ is just the Yoneda embedding of (5.44) together with the resulting $H$-action.

Remark 5.46. Note that while (5.45) defines “representable functors” for each $T \in \Omega_G$, given a presheaf $X \in dSet^G$ the evaluations $X(U)$ are defined only for $U \in \Omega$, i.e., for $U$ a non-equivariant tree.

This is in contrast with $dSet_G$, where both representables and evaluations are defined in terms of $\Omega_G$. We note that to reconcile this observation with the inclusion $u_*$ of (5.41) the non-equivariant tree $U \in \Omega$ should be reinterpreted as the free $G$-tree $G \cdot U \in \Omega_G$ (cf. Remark 5.35).

We end this section by introducing a category of “forestal sets” which, while secondary for our purposes, will greatly streamline our discussion of the dendroidal join in section 7.4.

Definition 5.47. The category of wide forestal sets is the category

$$fSet_w = \mathbf{Set}^{\Phi^{op}}.$$  

Remark 5.48. The category $fSet_i = \mathbf{Set}^{\Phi^{op}}$ of what we might call “independent forestal sets” was one of the main objects of study in [9], where they are called simply “forestal sets”.

Mimicking (5.44) by writing $\Phi[F] \in fSet_i$ for the representable functor of $F \in \Phi$, it is shown in [9] that one can define a formal boundary $\partial \Phi[F]$ possessing the usual properties one might expect.

We will find it desirable to be able to use the analogous construction for the representable $\Phi_u[F] \in fSet_w$, but this does not quite follow from the result in
[9], since while $\Phi_i[F] \in \mathfrak{fSet}_w$, one typically has a proper inclusion $\Phi_w[F] \hookrightarrow u^*\Phi_i[F]$.

We thus instead mimic the discussion in [1], making use of the key technical results established in section 5.2.

Letting $\Xi$ denote any of $\Omega, \Phi_i, \Phi_w$ and setting

$$|F| = \# \{\text{edges of } F\} + \# \{\text{stumps of } F\},$$

then Lemma 5.22 and Proposition 5.26 say that $\Xi$ is a generalized Reedy category (cf. [1, Defn. 1.1]). As in [1], call an element $x: \Xi[F] \to X$ of a presheaf $X \in \text{Set}^{\Xi^{op}}$ degenerate if it factors through a non invertible degeneracy operator and non-degenerate otherwise. Corollary 5.28 then allows us to adapt the proof of [1, Prop. 6.9] to obtain the following.

**Proposition 5.49.** Let $X \in \text{Set}^{\Xi^{op}}$ for $\Xi$ any of $\Omega, \Phi_i, \Phi_w$. Then any element $x: \Xi[F] \to X$ has a factorization, unique up to unique isomorphism,

$$\Xi[F] \xrightarrow{\rho_x} \Xi[G] \xrightarrow{\bar{x}} X$$

as a degeneracy operator $\rho_x$ followed by a non degenerate element $\bar{x}$.

Defining skeleta as in [1, §6] the proof of [1, Cor. 6.8] yields the following.

**Corollary 5.50.** Let $\Xi$ be any of $\Omega, \Phi_i, \Phi_w$. The counit $sk_n X \to X$ for $X \in \text{Set}^{\Xi^{op}}$ is a monomorphism whose image consists of those elements of $X$ that factor through some $\Xi[F] \to X$ for $|F| \leq n$.

**Definition 5.51.** Let $\Xi$ be any of $\Omega, \Phi_i, \Phi_w$. The formal boundary

$$\partial \Xi[F] \to \Xi[F]$$

is the subobject formed by those maps that factor through a non invertible map in $\Xi^*$, i.e. through a non invertible face map.

Note that by combining the Reedy axioms with Corollary 5.50 one has

$$\partial \Xi[F] = sk_{|F|-1} \Xi[F].$$

6 Normal monomorphisms and anodyne extensions

6.1 Equivariant normal monomorphisms

Recalling that the cofibrations in $\mathfrak{dSet}$ are not the full class of monomorphisms, but rather the subclass of so called normal monomorphisms, one should expect a similar phenomenon to take place in $\mathfrak{dSet}^G$.

We start by noting that for $X \in \mathfrak{dSet}^G$ and $U \in \Omega$, the set $X_U = X(U)$ is acted on by the group $G \times \Sigma_U$, where $\Sigma_U$ denotes the automorphism group of $U$.

**Definition 6.1.** A subgroup $N \leq G \times \Sigma_U$ is called non-arboreal if $N \cap \Sigma_U = \{e\}$. 

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It is straightforward to check that a non-arboreal subgroup $N$ can equivalently be described by a partial homomorphism $G \to H \to \Sigma U$. Further, since such a $\rho$ allows us to write $U \in \Omega H$, one has that each non-arboreal subgroup $N$ has an associated $G$-tree $G^\sim \Sigma U$. More precisely, one has the following result.

**Proposition 6.2.** The functor $\Omega [-]: \Omega_G \to \mathsf{dSet}^G$ induces an equivalence between $\Omega_G$ and the full subcategory of quotients of the form $(G \cdot \Omega[U])/N$ for $U \in \Omega$ and $N \leq G \times \Sigma U$ non-arboreal.

Recalling the discussion following (5.44) one can, for a forest $F = \sqcup_i T_i$ in $\Phi$, define $\partial \Omega[F] = \sqcup_i \partial \Omega[T_i]$. Carrying this discussion through to $G$-objects leads to the following definition.

**Definition 6.3.** The boundary inclusions of $\mathsf{dSet}^G$ are the maps of the form

$$\partial \Omega[T] \to \Omega[T]$$

for $T \in \Omega_G$.

More explicitly, if $T = G \cdot H T_e$ for some $T_e \in \Omega H$, then the (non-equivariant) presheaf $\Omega[T_e]$ inherits a $H$-action and (6.4) is isomorphic to the map

$$G \cdot \partial \Omega[T_e] \to \Omega[T_e]$$

or, letting $N \leq G \times \Sigma T_e$ denote non-arboreal subgroup associated to $T_e$,

$$(G \cdot \partial \Omega[T_e] \to \Omega[T_e])/N.$$  

The following is an immediate generalization of [1, Prop. 7.2].

**Proposition 6.5.** Let $\phi: X \to Y$ be a map in $\mathsf{dSet}^G$. Then the following are equivalent.

(i) for each tree $U \in \Omega$, the relative latching map $l_U(\phi): X_U \coprod_{l_U X} L_U Y \to Y_U$ is a $\Sigma_U$-free extension;

(ii) $\phi$ is a monomorphism and, for each $U \in \Omega$ and non degenerate $y \in Y_U - \phi(X)_U$, the isotropy group $\{g \in G \times \Sigma_U | gy = y\}$ is non-arboreal;

(iii) for each $n \geq 0$, the relative $n$-skeleton $sk_n(\phi) = X \coprod_{sk_n X} sk_n Y$ is obtained from the relative $(n-1)$-skeleton by attaching boundary inclusions.

**Definition 6.6.** A monomorphism satisfying any of the equivalent conditions in Proposition 6.5 will be called a normal monomorphism.

**Remark 6.7.** Note that by 6.5(i) a monomorphism is normal in $\mathsf{dSet}^G$ iff is it so as an underlying map in $\mathsf{dSet}$ (cf. [1, Prop. 7.2]).

### 6.2 Equivariant anodyne extensions

The key to the preceding section is the observation that if $T \in \Omega H$ then the usual boundary $\partial \Omega[T]$ inherits a $H$-action. However, such is not the case for inner horns: if $e \in T$ is an inner edge, then $\Lambda' e$ will inherit an $H$ action iff $e$ is an $H$-fixed edge.

Thus, to define $G$-inner horns, one must treat all inner edges in an *inner edge orbit* in a uniform way. To do so, we first recall the notion of generalized inner horns (cf. [14, Lemma 5.1]).
Definition 6.8. Let $E \in \text{Inn}(T)$ be a subset of the inner edges of $T \in \Omega$. We define
\[ \Lambda_{E}^{T}[T] \to \partial \Omega[T] \to \Omega[T] \]
as the subpresheaf formed by the union of those faces not of the form $T - E'$ for $E' \subset E$.

More generally, given a forest $F = \cup_{i} T_{i}$ and $E = \cup_{i} E_{i}$ with $E_{i} \subset \text{Inn}(T_{i})$ we set
\[ \Lambda_{E}^{F}[F] = \cup_{i} \Lambda_{E_{i}}^{T_{i}}[T_{i}]. \] (6.9)

Remark 6.10. The reader of [9] may note that (6.9) clashes with [9, §3.6]. This is because in [9] the presheaf being defined lives in $\mathbf{fSet}$, rather than in $\mathbf{dSet}$.

Definition 6.11. The generating $G$-inner horn inclusions are the maps in $\mathbf{dSet}^{G}$ of the form
\[ \Lambda_{G}^{T}[T] \to \Omega[T] \]
where $T \in \Omega_{G}$ is a $G$-tree and $Ge$ is the $G$-orbit of an inner edge $e$.

Definition 6.12. A dendroidal set $X$ is called a $G$-$\infty$-operad if $X$ has the right lifting property with respect to all generating $G$-inner horn inclusions.

\[ \Lambda_{G}^{T}[T] \rightarrow X \]
\[ \Omega[T] \]

Further, $A \rightarrow B$ is called an inner $G$-anodyne extension if it is in the saturation of the generating $G$-horn inclusions under pushouts, transfinite compositions and retracts.

Example 6.13. If one considers the $G = \mathbb{Z}_{4}$-tree $T$ in Example 4.5, one possible inner orbit edge is $Gb = \{b, b + 1\}$. The following are the (inner) faces of $T$ not included in $\Lambda_{G}^{T}[T]$.

We recall (cf. Remark 5.46) that since presheafs $X \in \mathbf{dSet}^{G}$ are only evaluated on non-equivariant trees $U \in \Omega$, the faces above are merely non-equivariant.
faces of the equivariant tree $T$: indeed, $T/\{b\}$ and $T/\{b+1\}$ do not admit a full compatible $G$-action. Rather, $G$ acts instead on the set of such faces and since $T/\{b\}$ admits a compatible $H = 2G$-action and one can think of $T/\{b\} \cup T/\{b+1\}$ as the $G$-tree $G \cdot_H T/\{b\}$.

We will need to develop for $G$-inner horns most of the key results of [14] and [3]. Our proofs will for the most part merely repeat the core ideas of the original proofs together with moderate equivariant modifications. Unfortunately, however, the presentation of the original proofs often makes the necessary modifications quite hard to describe, and we will hence also spend some time reworking the original results in more convenient language.

The hardest of these results, concerning the tensor product, are the subject of section 7. To finish this section, we collect a couple of easier results, starting with the analogue of [14, Lemma 5.1].

**Lemma 6.14.** Let $T \in \Omega_G$ be a $G$-equivariant tree and $E$ a $G$-equivariant subset of the edges of $T$. Then the generalized $G$-horn inclusion

$$\Lambda^E[T] \to \Omega[T]$$

is inner $G$-anodyne.

**Proof.** Since $E$ consists of a union of edge orbits, one immediately reduces to proving that maps of the form

$$\Lambda^E[T] \to \Lambda^{E\cdot G}[T]$$

are inner $G$-anodyne. In the non-equivariant case [14, Lemma 5.1] the analogous maps can be described as single pushouts, but here we require multiple pushouts, naturally indexed by an equivariant poset we now describe.

Firstly, let $T_e$ denote the (non-equivariant) tree component containing the edge $e$ and set $H \leq G$ to be its isotropy, resulting in a canonical identification $G \cdot_H T_e \simeq T$. Writing $\text{Inn}_{H_e}(T_e)$ for the $H$-poset (under inclusion) of the inner faces of $T_e$ collapsing only edges in $H_e$, it suffices to check that, for any $H$-equivariant convex\(^2\) subsets $B \subset B' \subset \text{Inn}_{H_e}(T_e)$ it is

$$\Lambda^E[T] \cup G \cdot_H \left( \bigcup_{T_e - \beta \in B} \Omega[T_e - \beta] \right) \to \Lambda^E[T] \cup G \cdot_H \left( \bigcup_{T_e - \beta \in B'} \Omega[T_e - \beta] \right) \quad (6.15)$$

inner $G$-anodyne. Without loss of generality, we may assume $B'$ is obtained from $B$ by adding a single orbit $H(T_e - \beta)$ and, setting $H \subset H'$ to be isotropy of (the edge set) $\beta$, we claim that (6.15) is a pushout of

$$G \cdot_B \left( \Lambda^{E\cdot G_e}[T_e - \beta] \to \Omega[T_e - \beta] \right),$$

where $(E - Ge) \cap T_e$ denotes the subset of inner edges of $T_e$ that are in $(E - Ge)$. This claim is straightforward except for the following: one needs to note that the $G$-isotropy of any faces in $\text{Inn}_{(E - Ge)}(T_e - \beta)$ (i.e. those faces missing from $\Lambda^{(E\cdot G_e)}[T_e - \beta]$) is indeed contained in $H$, and this follows since by definition $(E - Ge) \cap T_e$ contains none of the conjugates of the edges in the edge set $\beta$.

This concludes the proof by nested induction on the order of $G$ and the number of $G$-orbits of $E$.

---

\(^2\)Recall that a subset $E \subset P$ of a poset $P$ is called convex if $e \preceq e$ and $e \in E$ implies $\bar{e} \in E$. 

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The following is the equivariant analogue of [14, Lemma 5.2]. Note that edge orbits of a $G$-tree $T$ are encoded by maps $G/H \xrightarrow{\eta} T$ for some $H \leq G$.

**Proposition 6.16.** Suppose that $T$ has a leaf orbit and $U$ root orbit both isomorphic to $G/H$. Write $V = T \cup_{G/H} U$ for the grafted $G$-tree.

Then

$$\Omega[T] \cup_{\Omega[G/H \eta]} \Omega[U] \to \Omega[V]$$

is inner $G$-anodyne.

**Proof.** Let $\text{Out}(V)$ denote the $G$-poset of outer faces (defined as those faces which are not an inner face or any other face) of the grafted tree $V$, and $\text{Out}_{T,U}(V)$ the $G$-subposet of those outer faces contained in neither $T$ or $U$.

It now suffices to show that for all $G$-equivariant convex subsets $B \subset B'$ of $\text{Out}_{T,U}(V)$ it is

$$\Omega[T] \cup_{G/H} \Omega[U] \cup \bigcup_{R \in B} R \to \Omega[T] \cup_{G/H} \Omega[U] \cup \bigcup_{R \in B'} R \quad (6.17)$$

inner $G$-anodyne.

As usual, we can assume $B'$ is obtained from $B$ by adding the orbit $G \cdot S$ of a single outer face $S$. Letting $H \leq G$ denote the isotropy of $S$, so that $G \cdot H S$ can be regarded as a $G$-tree, one has that (6.17) is then the pushout (note that the $G$-isotropy of an inner face of the outer face $S$ is at most $H$) of

$$\Lambda^{\text{inn}(G \cdot H S)}[G \cdot H S] \to \Omega[G \cdot H S],$$

finishing the proof. \hfill \Box

7 Tensor products

Our goal in this section is to prove equivariant analogues of [14, Prop. 9.2], [3, Thm. 5.2] and [3, Thm. 4.2], which are the key technical results in their respective papers. These results concern the interaction of anodyne extensions with the tensor product and the join constructions, which we recall in sections 7.1 and 7.4. We now list our versions of the results, starting with the analogue of [14, Prop. 9.2].

**Theorem 7.1.** Let $S, T \in \Omega_G$ be $G$-trees and let $G \xi$ be an inner orbit edge of $T$. Then the map

$$\partial \Omega[S] \otimes \Omega[T] \bigsqcup_{\partial \Omega[S] \otimes \Lambda^{G \xi}[T]} \Omega[S] \otimes \Lambda^{G \xi}[T] \to \Omega[S] \otimes \Omega[T]$$

is an inner $G$-anodyne extension if either

(i) both $S$ and $T$ are open $G$-trees (i.e. have no stumps);

(ii) at least one of $S, T$ is a linear tree (i.e. isomorphic to $G \cdot H[n]$ for $[n] \in \Delta$).

The proof of Theorem 7.1 will be the subject of section 7.3. More specifically, the result will follow by Proposition 7.39 when $B = \emptyset$ and $B' = 1 \in G_\xi(S \otimes T)$.

The following is the equivariant analogue of [3, Thm. 5.2].
Theorem 7.2. Let $S \in \Omega^G$ be a tree with a $G$ action such that $S \neq \eta$ and denote by $A \to \Omega[S] \otimes \Omega[1]$ the pushout product map

$$(\partial \Omega[S] \to \Omega[S]) \sqcup (\{1\} \to \Omega[1]).$$

Then there is a factorization $A \to B \to \Omega[S] \otimes \Omega[1]$ such that

(a) $A \to B$ is an inner $G$-anodyne extension;

(b) there is a pushout (the join $S \ast \eta$ is introduced in Definition 7.41)

\[
\Lambda \Omega[\ast \eta] \to B \\
\Omega[\ast \eta] \to \Omega[S] \otimes \Omega[1];
\]

(7.3)

(c) letting $\eta \to S$ denote the root edge, the composite

$\Omega[1] \xrightarrow{\eta \ast \text{id}} \Omega[S \ast \eta] \to \Omega[S] \otimes \Omega[1]$,

coincides with the composite

$\Omega[1] \xrightarrow{\eta \otimes \text{id}} \Omega[S] \otimes \Omega[1]$.

Theorem 7.2 will be proven at the end of section 7.3 as a direct consequence of the arguments used in the proof of Proposition 7.39.

The following is the equivariant analogue of [3, Thm. 4.2].

Theorem 7.4. Let $T \in \Omega^G$ be a tree with a $G$-action. Assume further that $T$ has at least two vertices and unary root vertex $G/\eta \xrightarrow{v_r} T$. Then a lift exists in any commutative diagram

\[
\xymatrix{ \Lambda^* \Omega[T] \ar[r]^f \ar[d] & X \ar[d] \\
\Omega[T] \ar[r] & Y }
\]

(7.5)

such that $X \to Y$ is an inner $G$-fibration between $G$-$\infty$-operads and $f(v_r)$ is a weak equivalence in the $\infty$-category $\mathcal{P}(X^G)$.

We will prove Theorem 7.4 at the end of section 7.4.

7.1 Tensor product

In order to keep the proofs of Theorems 7.1 and 7.2 compact we will find it preferable to use the broad poset language throughout. We start by discussing tensor products in that framework.

Given $s \in S^*$, $t \in T^*$, we will let $s \times t \in (S \times T)^*$ denote the obvious tuple whose elements $(s, t) \in S \times T$ are those pairs with $s \in s$, $t \in t$. 

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Definition 7.6. Given broad posets $S$, $T$, their tensor product $S \otimes T$ is the broad poset whose underlying set is $S \times T$ and whose relations are generated by relations of the form $s \times t \leq (s, t)$ (resp. $s \times t \leq (s, t)$) for $s \in S$, $t \in T$ and $s \leq s$ (resp. $t \leq t$) a broad relation in $S$ (resp. $T$).

Proposition 7.7. Let $S, T$ be trees. Then:

(i) $S \otimes T$ is simple. Further, if $(s_1, t_1) \cdots (s_n, t_n) \leq (s, t)$ then the pairs $s_i, s_j$ and $t_i, t_j$ are both $\leq_d$ comparable only if $i = j$;

(ii) an edge $(s, t) \in S \otimes T$ has one of five types:

(leaf) it is a leaf if both $s \in S$, $t \in T$ are leaves;

(stump) it is a stump if $s \in S$ is a leaf and $t \in T$ is a stump or vice versa, or if both $s \in S$, $t \in T$ are stumps;

(regular node) it is a node if $s \in S$ is a leaf and $t \in T$ is a node or vice versa. In fact $(s, t)^t = s^t \times t$ or $(s, t)^d = s^d \times t$, accordingly;

(nullary node) it is a node such that $e \leq (s, t)$ if $s \in S$ is a node and $t \in T$ a stump or vice versa. In fact $(s, t)^t = s^t \times t$ or $(s, t)^d = s^d \times t$, accordingly;

(fork) if $s \in S$, $t \in T$ are both nodes then there are exactly two maximal $f$ such that $f \leq (s, t)$, namely $s \times t^d$ and $s^d \times t$. We call such a $(s, t)$ a fork.

Proof. We first show (i). The “further” condition suffices for simplicity. Noting that in a tree $e \leq_d e'$ and $e \leq_d e''$ can happen simultaneously only if $e'$, $e''$ are $\leq_d$ comparable (by Lemma 5.14) and using Propositions 5.5 and 5.19, one sees that composing broad relations satisfying the “further” condition with generating broad relations of $S \otimes T$ yields broad relations still satisfying that condition.

We now show (ii). Only the fork case requires proof. In fact, it is tautological that only $s \times t^d$ and $s^d \times t$ can possibly be maximal, hence one needs only verify that neither $s \times t^d \leq s^t \times t$ nor $s^d \times t \leq s \times t^d$. This follows since the $S$ coordinates of the pairs in the tuple $s^t \times t$ are $\leq_d$ than those in $s \times t^d$ and vice versa.

In order to simplify notation, we will henceforth write $e^S = s^d \times t$, $e^T = s \times t^d$ and $e^{ST} = s^d \times t^d$ whenever appropriate.

Proposition 7.8. Let $S, T$ be trees and consider the broad relation

$$\leq = (s_1, t_1)(s_2, t_2) \cdots (s_n, t_n) \leq (s, t) = e.$$

in $S \otimes T$. Then:

(i) $\leq \leq e^S$ (resp. $\leq \leq e^T$) if and only if $s_i \neq s, \forall i$ (resp. $t_i \neq t, \forall i$);

(ii) $\leq \leq e^{ST}$ if and only if both $s_i \neq s$ and $t_i \neq t, \forall i$.

Proof. Only the “if” directions need proof, and the proof follows by upward $\leq_d$ induction on $s, t$. The base cases of either $s$ or $t$ a leaf are obvious.

Otherwise, let $e$ satisfy the “if” condition in (i). Since it must be either $\leq \leq e^S$ or $\leq \leq e^T$ we can assume it is the latter case. Writing $t^d = u_0 \cdots u_k$, $e = e_1 \cdots e_k$ so that $e_i \leq (s_i, u_i)$ (note that possibly $k = 0$), the induction hypothesis now yields $e_1 \leq (s_i, u_i)^S = s^d \times u_i$, and hence

$$e = e_1 \cdots e_k \leq (s^d \times u_1) \cdots (s^d \times u_k) = s^d \times t^d \leq s^t \times t = e^S.$$ (7.9)
Remark 7.16. As discussed in Proposition 7.7, the relation $\epsilon \leq e$ is not a minimal relation in $S \otimes T$ whenever $e$ is a nullary node. As a consequence, any elementary tree containing such a generating relation is in fact an inner face of a larger elementary tree.
Remark 7.17. Given initial subtrees $U \to U' \to S \otimes T$ such that $U = U'$, one has $(r_S, r_T)^{\lambda_U} \leq (r_S, r_T)^{\lambda_{U'}}$ with $(r_S, r_T)^{\lambda_U}$ obtained from $(r_S, r_T)^{\lambda_U}$ by removing those leaves of $U$ which become stumps in $U'$.

**Lemma 7.18.** If $U$ is a closed face of $S$, then $U \otimes T \subset S \otimes T$, i.e., $U \otimes T$ contains all broad relations in its image.

**Proof.** Note first that by Proposition 7.12 any maximal elementary subtree $V$ of $S \otimes T$ is closed.

It therefore suffices, for each broad relation $e_1 \cdots e_n = e \leq S \otimes T$ between elements in the image of $U \otimes T$, to show that $e_1, \cdots, e_n, e$ lie in one such maximal elementary $V$ that contains a maximal elementary subtree of $U \otimes T$ as a closed subface. The outer face cases (cf. Notation 5.29) are straightforward, hence we focus on the harder case of $U = S - s$ for $s$ an inner edge. Letting $p_s \in S$ denote the parent of $s$, i.e. the unique edge such that $s \in (p_s)^T$, note that, setting $e = (p_s, t)$, $e_{15}^{(S-s)} \leq e$ is the only generating relation of $(S - s) \otimes T$ that is not also a generating relation of $S \otimes T$. It hence suffices to check that if $e_1 \cdots e_n \leq e_{15}$, then it is also $e_1 \cdots e_n \leq e_{15}^{(S-s)}$. This now follows by Proposition 7.8 applied to the edge $(s, t) \in e_{15}$. 

**Definition 7.19.** Let $S, T$ be trees and $A = \{a_i\}$, $B = \{b_j\}$ subsets of the sets of stumps of $S$, $T$, respectively and let $v_A = \{e \leq a_i\}$, $v_B = \{e \leq b_j\}$ denote the corresponding vertices.

We say that a subtree $U \to S \otimes T$ misses $v_A$ and $v_B$ if one has a factorization $U \to (S - v_A) \otimes (T - v_B) \to S \otimes T$.

Further, if $B = \emptyset$ (resp. $A = \emptyset$) we say simply that “$U$ misses $v_A$” (resp. “$U$ misses $v_B$”).

**Remark 7.20.** In [14] similar notions of “$U$ missing an inner edge/leaf vertex are also defined, but we note that those notions are far more straightforward. In fact, as implicit in the erratum to the follow up paper [3], the earlier treatment overlooked some subtle properties of stumps.

For instance, note that Corollary 7.23 below implies that the notion “$U$ misses $v_A$ and $v_B$” does not coincide with the notion “$U$ misses $v_A$ and $U$ misses $v_B$” whenever $A$ and $B$ are both non-empty.

**Lemma 7.21.** Let $S, T$ be trees and $A = \{a_i\}$, $B = \{b_j\}$ subsets of the stamps of $S$, $T$, respectively. Then a broad relation

$$f = f_1f_2\cdots f_k \leq e$$

in $S \otimes T$ is a broad relation in $(S - v_A) \otimes (T - v_B)$ if and only if

$$e^{\lambda_{(S-v_A)\otimes(T-v_B)}} \leq f.$$  

**Proof.** Only the “if” direction needs proof. We argue by upward $\leq_d$ induction on $e = (s, t)$. The base case, that of $s$ a leaf of $S - v_A$ and $T$ a leaf of $T - v_B$, is obvious (we note that the proof will follow even when this case is vacuous).

Otherwise, either $f \leq e_{15} \leq e$ or $f \leq e_{15}^T \leq e$ and our assumption ensures, accordingly, that $s \notin A$ or $t \notin B$. Writing $e^*$ to denote either $e_{15}^S$ or $e_{15}^T$ as appropriate, this last observation guarantees that the relation $e^* \leq e$ is in $(S - v_A) \otimes (T - v_B)$. Further, writing $f = f_1\cdots f_k$ and $e^* = e_1\cdots e_k$ so that $f_j \leq e_j$, the induction hypothesis shows that these last relations are also in $(S - v_A) \otimes (T - v_B)$. 

\[\square\]
Recalling Proposition 5.5 hence yields the following.

**Corollary 7.22.** A collection of broad relations of the form \( q_i \leq f_i, f_1 \cdots f_k \leq e \) are all in \( (S \cdot v_A) \otimes (T - v_B) \) if and only if the composite relation \( q_i \cdots q_k \leq e \) is.

**Corollary 7.23.** A face \( U \hookrightarrow S \otimes T \) misses \( v_A \) and \( v_B \) if and only if, for \( r \) the root of \( U \), one has

\[
\rho(U) \otimes (S \cdot v_A) \otimes (T - v_B) \leq \rho(U).
\]

**Proof.** This follows from Corollary 7.22 since any relation in \( U \) is a factor of \( \rho(U) \leq r \).

The following was first stated in [3, Prop 1.9 in erratum] and proven via a careful combinatorial analysis in [6]. We include here a short proof in our language.

**Proposition 7.24.** Let \( S \otimes T \in \Omega \) be trees which are either (i) both open; (ii) \( S = [n] \) is linear. Then the square

\[
\begin{array}{ccc}
\partial \Omega[S] \otimes \partial \Omega[T] & \to & \Omega[S] \otimes \partial \Omega[T] \\
\downarrow & & \downarrow \\
\partial \Omega[S] \otimes \Omega[T] & \to & \Omega[S] \otimes \Omega[T]
\end{array}
\]

(7.25)

consists of normal monomorphisms. Further,

\[
\partial \Omega[S] \otimes \Omega[T] \bigcup_{\partial \Omega[S] \otimes \partial \Omega[T]} \Omega[S] \otimes \partial \Omega[T] \to \Omega[S] \otimes \Omega[T]
\]

(7.26)

is also a normal monomorphism.

**Proof.** Note first that since \( \otimes \) commutes with colimits in each variable,

\[
\partial \Omega[S] \otimes \Omega[T] = \text{colim}_{F \in \text{Face}(S)} \Omega[F] \otimes \Omega[T].
\]

(7.27)

In the open case, since all faces are closed, Lemma 7.18 implies that if \( U \subset F_S \otimes T \) and \( U \subset S \otimes F_T \) then it will be \( U \subset F_S \otimes F_T \), showing that (7.26) is a monomorphism whenever (7.25) consists of monomorphisms. By skeletal induction, it thus suffices to check that the right and bottom maps in (7.25) are monomorphisms, and this will follow if for any \( U \subset S \otimes T \) there exists a minimal face \( F \subset S \) such that \( U \subset F \otimes T \). Clearly \( F = \{ s \mid (s, t) \in U \} \) will work once we show that this is indeed a face. This is clear when \( U \) is elementary (in which case each vertex of \( U \) either adds a vertex to \( F \) or nothing at all) and holds in general since any \( U \) is an inner face of an elementary.

In the \( S = [n] \) is linear case one still has that (7.26) being a monomorphism follows from (7.25) consisting of monomorphisms and the fact that \( \partial \Omega[n] \otimes \Omega[T] \to \Omega[n] \otimes \Omega[T] \) follows by the same argument building \( F \) in the same way. It remains to show that \( \Omega[n] \otimes \partial \Omega[T] \to \Omega[n] \otimes \Omega[T] \) is a monomorphism. We note that the projection \( \pi : [n] \otimes T \to T \) given by \( \pi(k, e) = e \) is a map of dendroidal sets. Given \( U \hookrightarrow [n] \otimes T \) we claim that \( \pi(U) \) is the minimal face such that \( U \hookrightarrow [n] \otimes \pi(U) \), and note that this is implied by the more general claim that \( U \hookrightarrow [n] \otimes F \) iff \( \pi(U) \hookrightarrow F \). It now suffices to check this when \( F \) is maximal, with the case of \( F \) closed being obvious and the stump outer face case following by Corollary 7.23.
7.3 Pushout product filtrations

This section features our main technical proofs, namely the proof of Theorem 7.1 and the related but easier proof of Theorem 7.2.

The majority of the ideas in this section are adapted from the (rather long) proof of [14, Prop. 9.2], but here we will need to significantly repackage those ideas. To explain why, we note that the filtrations in the proof of [14, Prop. 9.2] are actually divided into three nested tiers: an outermost tier described immediately following [14, Cor. 9.3], an intermediate tier described in the proof of [14, Lemma 9.9] and an innermost tier described in the proof of [14, Lemma 9.7]. However, in the equivariant case $G$ acts transversely to these tiers, i.e. one can not attach dendrices at an inner tier stage without also attaching dendrices in a different outer tier stage.

Our solution to this issue will be to encode the top two filtration tiers as a poset $\mathcal{I}G_{\hat{G}}$ on which $G$ acts (to handle the lower tier). To improve readability, however, we first describe our repackaged proof in the non-equivariant case, then indicate the (by then minor) necessary equivariant modifications.

We will make use of an order relation on initial elementary subtrees (cf. Definition 7.15) of $S \otimes T$. Write

$$U' \preceq_{\text{lex}} U$$

whenever $U$ is obtained from $U'$ by replacing the intermediate edges in a string of broad relations $e^{ST} \leq e^{S} \leq e$ occurring in $U'$ by the intermediate edges in $e^{ST} \leq e^{T} \leq e$ occurring in $U$. An illustrative diagram follows.

In what follows we refer to non stump generating relations of the form $e^{ST} \leq e$ (resp. $e^{T} \leq e$) in an elementary tree as $S$-vertices (resp. $T$-vertices).

Proposition 7.28. Suppose $T$ is open (i.e. has no stumps). Then $\preceq_{\text{lex}}$ induces a partial order on the set of initial elementary subtrees of $S \otimes T$.

Further, $\preceq_{\text{lex}}$ together with the inclusion $\Rightarrow$ assemble into a partial order as well, and we denote this latter order simply by $\preceq$.

Proof. One needs only check antisymmetry. Let $\varphi(U)$ count pairs $(v_S, v_T)$ of a $S$-vertex and $T$-vertex in $U$ such that $v_S$ occurs before $v_T$ (following $\preceq$). Since the generating relations of $\preceq_{\text{lex}}$ strictly decrease $\varphi$, $\preceq_{\text{lex}}$ is a partial order. Similarly, letting $h(U) = \#(\text{stumps of } U) + \sum_{\text{leaves and stumps of } U} d(e, r)$ (where $d(\cdot, r)$ denotes “distance to the root”, measured in generating $\leq_d$ relations), $\Rightarrow$ decreases $h$ and $\preceq_{\text{lex}}$ either (i) preserves $h$ is $e$ is a fork; (ii) is an instance of $\Rightarrow$ if $e$ is a nullary node (since $T$ is assumed open). Thus $\preceq$ is a partial order.

Remark 7.29. Note that if $T$ is not open, then by Remark 7.16, it is possible for the $\preceq_{\text{lex}}$ and $\Rightarrow$ to be in conflict.
Henceforth we will let $\xi$ denote a fixed inner edge of $T$.

**Definition 7.30.** An initial elementary subtree $U \bowtie S \otimes T$ is called $\xi$-internal if it contains an edge of the form $(s, \xi)$, abbreviated as $\xi$, and the $T$-vertex $\xi^T \leq \xi$.

For $T$ open, we will denote the subposet of such trees by $(\text{IE}_\xi(S \otimes T), \leq)$.

Further, when $S$ is open one can modify the order in $\text{IE}_\xi(S \otimes T)$ by reversing the $\leq_{\text{lex}}$ order (but not the $\bowtie$ order). The resulting poset will be denoted $\text{IE}^{\bowtie-\text{lex}}_\xi(S \otimes T)$.

**Lemma 7.31.** Suppose $T$ is open. Let $U \bowtie S \otimes T$ be an elementary subtree with root vertex $a$ a $T$-vertex $e^T \leq e$ and suppose that $e^\lambda \leq a^S$.

Then there exists an elementary subtree $U'$ such that $U' \leq_{\text{lex}} U$ and $U'$ contains the relations $e^{S,T} \leq e^T \leq e$.

**Proof.** We argue by induction on the sum of the distances (cf. proof of Proposition 7.28) between the leaves and stumps of $U$ and its root $e$. The base case, that of $U$ the elementary tree generated by $e^{S,T} \leq e^T \leq e$, is obvious.

Otherwise, writing $e = (s, t)$, for each $t_i \in T$ either $(s, t_i)^{T} \leq U_{\text{lex}} (s, t_i)$ or $(s, t_i)^{T} \leq U_{\text{lex}} (s, t_i)$ and iteratively applying the induction hypothesis to each $i$ in the latter case allows us find $U'' \leq_{\text{lex}} U$ such that $U''$ contains all relations $(s, t_i)^{S,T} \leq (s, t_i)^{S} \leq (s, t_i)$. But now $U''$ contains the relations $e^{S,T} \leq e^T \leq e$ and hence a final generating $\leq_{\text{lex}}$ relation yields $U' \leq_{\text{lex}} U'' \leq_{\text{lex}} U$.

**Lemma 7.32.** Suppose $U \bowtie S \otimes T$, $U' \bowtie S \otimes T$ are subtrees with common leaves and root. Then $F = U \cap U'$ defines a closed face of both $U$ and $U'$.

**Proof.** As a set, $F$ could alternatively be defined as the underlying set of the composite inner face of $U$ that removes all inner edges of $U$ not in $U'$, or vice versa. Thus, the real claim is that both constructions yield the same broad relations. Noting that the $\leq$ order relations on $U$, $U'$ are induced from $S \otimes T$ (as argued in the proof of Proposition 7.12), this now follows from Lemma 5.12.

**Lemma 7.33.** Suppose $T$ is open. If $F$ is a common face (resp. inner face) of two elementary subtrees $U, U'$, then $F$ is also a face (resp. inner face) of an initial elementary subtree $U''$ such that $U'' \leq_{\text{lex}} U$, $U'' \leq_{\text{lex}} U'$ (resp. $U'' \leq_{\text{lex}} U$, $U'' \leq_{\text{lex}} U'$). In fact, in the inner face case the $\leq_{\text{lex}}$ inequalities factor through generating $\leq_{\text{lex}}$ inequalities involving only trees having $F$ as an inner face.

**Proof.** One is free to remove any vertices of $U, U'$ that are descendant to leaves of $F$ (i.e. vertices $e^i \leq e$ such that $e \leq d$ for some leaf $l$ of $F$) and not descendant to the root of $F$, so that $F, U, U'$ now have exactly the same leaves and root, with the latter denoted $r$. Thus, by Lemma 7.32 we are free to assume $F = U \cap U'$.

If the root vertices $r^U \leq r$, $r^U \leq r$ coincide, the result follows by induction on $\leq$. Otherwise, we can assume that the root vertex of $U$ is $r^U \leq r$ and that of $U'$ is $r^{T} \leq r$. Letting $U''_{\text{lex}^T,r}$ be the smallest subtree of $U'$ containing the root vertex $r^T \leq r$ of $F$, one can now apply Lemma 7.31 to $U''_{\text{lex}^T,r}$, and hence build $U'' \leq_{\text{lex}} U'$ with a strictly larger intersection with $U$, finishing the proof.

Recall that a subset $B$ of a poset $P$ is called convex if $b \leq \bar{b}$ and $b \in B$ implies $\bar{b} \in B$.
Proposition 7.34. Let \(S, T\) be trees and \(\xi \in T\) an inner edge. Further, assume that either both \(S\) and \(T\) are open or that one of them is linear. Set
\[
A = \Omega[S] \otimes A^T \Omega[T] \bigcup_{\partial \Omega[S] \otimes \partial \Omega[T]} \partial \Omega[S] \otimes \Omega[T]
\]
and regard \(A\) and the \(\Omega[V]\) below as subpresheaves of \(\Omega[S] \otimes \Omega[T]\).

Then, for any convex subsets \(B \subset B'\) of the poset \(IE_\xi(S \otimes T)\) (or, in the special case of \(S\) a linear tree and \(T\) not open, of the poset \(IE^{op, lex}_\xi(S \otimes T)\)), one has that
\[
A \cup \bigcup_{V \in B} \Omega[V] \rightarrow A \cup \bigcup_{V \in B} \Omega[V]
\]
is an inner anodyne extension.

Remark 7.36. For the proof above to work it is key for the tree \(U_{s,\xi}^a\) to in fact have inner edges, as is insured by the assumptions on \(S, T\). It is the failure of this latter claim to hold when \(\xi\) is a stump edge that leads to the need for the following alternate lemma.

Lemma 7.37. Suppose that \(S\) is a linear tree (i.e., \(S \cong [n]\) for \([n] \in \Delta\)).

Then for \(U \in IE^{op, lex}_\xi(S \otimes T)\) the \(\xi_a\)-maximal edge of \(U\) of the form \(\xi_a\) is a characteristic edge (in the sense of Lemma 7.35).
Proof. Since $S$ is linear we will for simplicity label its edges as $0 \leq 1 \leq \cdots \leq n$.

Suppose first that $F - \xi_s$ is in $A$, so that repeating the argument in the proof of Lemma 7.35 we conclude it must be $F - \xi_s \cong (S - s) \otimes T$ for $s < n$ (note that this makes sense even if $s = 0$ is the leaf of $S$).

Since $s \neq n$, one can chose a $\leq_d$-minimal edge $a_{s+1}$ of $U$ such that $\xi_s \leq_d a_{s+1}$. The characterization of $\xi_s$ implies $a \neq \xi$ and the characterization of $a_{s+1}$ implies that $U$ contains the $S$-vertex $a_s = a_{s+1}^S \leq a_{s+1}$. $U$ therefore contains the relations $a_{s+1}^S \leq a_{s+1}^T \leq a_{s+1}$, which since $F$ must collapse $a_s = a_{s+1}^S \implies$ yields that $F$ is a subface of the tree $V \leq_{op-lex} U$ obtained by replacing $a_s = a_{s+1}^S$ with $a_{s+1}^T$.

The case of $F - \xi_s$ a subface of some $V \in B$ follows by an argument identical to that in the proof of Lemma 7.35, except now noting that generating $\leq_{op-lex}$ relations do not remove edges whose vertex is a $T$-vertex. $
abla$

Proof of Proposition 7.34. Without loss of generality we can assume that $B'$ is obtained from $B$ by adding a single $\xi$-internal initial elementary tree $U$ with $\xi_s$ its corresponding edge.

We first note that the outer faces of $U$ are in $A \cup \bigcup_{V \in B} V$. Since a maximal outer face $U' \implies U$ is always still elementary, $U'$ will be $\xi$-internal initial elementary unless (i) $U' = U_{s,\xi(e,r)}$ (resp. $U' = U_{s,\xi(r,e)}$) is a root vertex face, in which case $U' \cong (S \otimes T)$ (resp. $U' \cong S \otimes (T_{s,e})$) and is hence in $A$; (ii) $U' = U - v_\xi$, and $U'$ is no longer $\xi$-internal since it contains no $T$-vertices of the form $\xi^T \leq \xi$. But then it would be $U$ in $A$ unless $v_\xi$ is a leaf vertex or stump vertex of $T$ and thus $U' \cong S \otimes (T - v_\xi)$ (by either Lemma 7.18 or Corollary 7.23).

Finally, we let $I^{\xi_s}$ denote the subset of inner edges of $U$ distinct from $\xi_s$ and let $P(I^{\xi_s})$ denote its power poset, ordered by inclusion. We claim that for any concave\(^3\) subsets $C \subset C' \subset P(I^{\xi_s})$ the map

$$A \cup \bigcup_{V \in B} V \cup \bigcup_{E \in C} U - E \to A \cup \bigcup_{V \in B} V \cup \bigcup_{E \in C'} U - E$$

(7.38)

is inner anodyne.

We argue by induction on $C$ and again we can assume that $C'$ is obtained from $C$ by adding a single $D \in P(I^{\xi_s})$ such that $U - D$ is not in the domain of (7.38). The concavity of $C, C'$ and the characteristic edge condition in Lemmas 7.35, 7.37 then imply that the only faces of $U - D$ in the source of (7.38) are precisely $U - D$ and $U - (D \cup \xi_s)$, and thus we conclude that (7.38) is in fact a pushout of $A \cup \{[U - D] \to \Omega(U - D)$, finishing the proof. $
abla$

In the $G$-equivariant case, given an inner edge orbit $G\xi$, we write $\mathsf{IE}_{G\xi}(S \otimes T)$ for the poset of initial elementary trees containing at least one $T$-vertex of the form $(g\xi)^T \leq (g\xi)_s$ (alternatively, one has $\mathsf{IE}_{G\xi}(S \otimes T) = \bigcup_{g \in G} \mathsf{IE}_g(S \otimes T)$). Note that in this case the group $G$ acts on the poset $\mathsf{IE}_{G\xi}(S \otimes T)$ as well. The following is the equivariant version of the previous result.

**Proposition 7.39.** Let $S, T \in \Omega_G$ be $G$-trees and $\xi \in T$ an inner edge. Further, assume that either both $S$ and $T$ are open or that one of them is linear (i.e. of the form $G/H \cdot [n]$). Set

$$A = \bigwedge_{G\xi \in \Omega[G \otimes [S \otimes \Omega[T]]} \Omega[S] \otimes \partial \Omega[T].$$

\(^3\)This is the notion dual to “convex”. I.e. if $E \in E'$ and $E \in C$, then $E' \in C$. 

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and regard $A$ and the $\Omega[V]$ below as subpresheaves of $\Omega[\mathcal{S}] \otimes \Omega[\mathcal{T}]$.

Then for any $G$-equivariant convex subsets $B \subset \mathcal{B}'$ of $\mathcal{I}_G(S \otimes T)$ (or, in the special case of $S$ a linear tree and $T$ not open, of $\mathcal{I}_G^{\text{op-lex}}(S \otimes T)$) one has that

$$A \cup \bigcup_{V \in B} \Omega[V] \to A \cup \bigcup_{V \in B'} \Omega[V]$$

is an inner $G$-anodyne extension.

Again we will suppress $\Omega$ from the notation of the proof.

Proof. Note that we are free to assume $S, T \in \Omega^G \subset \Omega_G$, i.e. that $S, T$ are actual trees with a $G$-action rather than $G$-indecomposable forests. Indeed, otherwise $S = G \cdot H \ S_e, T = G \cdot K \ T_e$ for $S_e \in \Omega^H, T_e \in \Omega^K$, then

$$S \otimes T = \bigsqcup_{[g] \in H \backslash G / K} G \cdot H \cdot g \cdot K \cdot \hat{S}_e \otimes g T_e,$$

where when regarding $S_e, T_e \in \Omega^{H \cdot g \cdot K \cdot \hat{S}_e}$ we omit the forgetful functors.

In analogy with the regular case we can assume $B'$ is obtained from $B$ by adding the $G$-orbit of a single $\xi$-internal initial elementary tree $U$ with $\xi_s$ the corresponding edge. Let $H$ denote the $G$-isotomy of $U$.

That the outer faces of any of the conjugates $g \cdot U$ are in $A \cup \bigcup_{V \in B} V$ follows by the corresponding non-equivariant argument in the proof of Proposition 7.34.

The key is now to prove the equivariant analogues of Lemmata 7.35 and 7.37, stating that $H \cdot \xi_s$ is a characteristic edge orbit of $U$, i.e., that for each inner face $F$ of $U$ containing an edge $(h \xi)s \in H \xi_s$ with isotropy $\hat{H} \leq H$, then $F$ is in $A \cup \bigcup_{V \in B} V$ if $F - H(h \xi)s$ is (note the condition on isotropy).

By equivariance, we may without loss of generality assume that $F$ contains $\xi_s$ itself. When proving the equivariant analogue of Lemma 7.35, in the case of $F - H \xi_s$ in $A$ the argument in the proof yields that $F$ itself must already miss all the edges of at least one of the $H$-conjugates of the $U^{\geq \xi_s}$ subtree (and hence, by $H$-equivariance, all of them) and therefore applying Lemma 7.31 again shows that $F$ is a subface of some $V <_{\text{lex}} U$. For the case $F - H \xi_s$ in some $V < U$, repeating the argument in the proof we can again assume $V <_{\text{lex}} U$ via a generating $\xi_s$ relation and such a $V$ must contain all the edges in $H \xi_s$. Proving the equivariant analogue of Lemma 7.37 requires no changes to the proof, since defining $a_{s+1}$ in the same way one still concludes that $F$ must collapse $a_s$ (in fact, $F$ must collapse $H a_s$, though that fact in not needed).

Lastly, we equivariantly modify the last two paragraphs in the proof of Proposition 7.34: setting $\mathcal{I}^{H \xi_s}$ to be the $H$-set of inner edges of $U$ not in $H \xi_s$, we show by induction over $H$-equivariant concave subsets $A \subset A' \in \mathcal{P}(\mathcal{I}^{H \xi_s})$ that the map

$$A \cup \bigcup_{Q \in B} Q \cup \left( \bigcup_{g \in G, \xi \in A} gU - gE \right) \to A \cup \bigcup_{Q \in B} Q \cup \left( \bigcup_{g \in G, \xi \in A} gU - gE \right) \quad (7.40)$$

is inner $G$-anodyne. Again we can assume that $A'$ is obtained from $A$ by adding the $H$-orbit of a single $D$ with isotropy $\hat{H} \leq H$ (further, $D$ can be chosen to contain $\xi_s$) such that $U - D$ is not in the domain of (7.40). At this point an extra equivariant argument is needed: one needs to know that the $G$-isotomy of
$U$ coincides with its $H$-isotropy $\bar{H}$. To see this, note that if it was otherwise then $U$ would be contained in both $U$ and a distinct conjugate $gU$, and Lemma 7.33 would imply that $U$ is already in the domain of (7.40). Finally, repeating the “characteristic edge (orbit)” argument we see that the map (7.40) is a pushout of

$$G \bar{H} \left( A^B \varepsilon \cdot [U - D] \rightarrow \Omega [U - D] \right),$$

finishing the proof.

We now adapt the previous proof to deduce the easier Theorem 7.2.

**Proof of Theorem 7.2.** The argument follows by attempting to follow the proof of Proposition 7.39 when $T = [1]$ and $\xi = 1$ (note that Definition 7.30 still makes sense, although the edge $1_s$ of $U$ may now possibly be the root, hence not internal). The only case where (the equivariant analogue) of Lemma 7.35 does not provide a characteristic edge orbit is when the root vertex of $U$ is $1^T_r \leq 1_r$, in which case $U$ will miss at least one color/stump of $S$ (and hence be contained in $A$) unless it is $U = S \otimes \{0\} \cup \{1_r\}$. Denoting this latest $U$ as $S \ast \eta$ and noting that it is the maximum of the poset $IE_{\xi}(S \otimes [1])$ one concludes that

$$A \rightarrow B = A \cup \bigcup_{V \in IE_{\xi}(S \otimes [1]), V \ast S \ast \eta} V$$

is indeed inner $G$-anodyne. The pushout (7.3) follows by noting that the only face of $S \ast \eta$ not in $B$ is $(S \ast \eta) - \eta$ (in fact, the only face not missing a color/stump of $S$ is $(S \ast \eta) - 0_r$, which is a face of the tree obtained by applying a $<_{lex}$ relation to the root of $S \ast \eta$).

### 7.4 Dendroidal join

We now turn to the equivariant version of the dendroidal join $\ast$ discussed in [3, §4], which will be needed to understand the last piece of the filtration in Theorem 7.2. We recall that several categories of forests were discussed in section 5.2.

**Definition 7.41.** Given an object $F \in \Phi$ and $[n] \in \Delta$ we define $F \ast [n] \in \Omega$ as the broad poset having underlying set $F \cup [n]$ and relations

- $e_1 \cdots e_n \leq e$ if $e_i, e \in F$ and $e_1 \cdots e_n \leq_F e$;
- $i \leq j$ if $i, j \in [n]$ and $i \leq_{[n]} j$;
- $e_1 \cdots e_n \leq i$ if $e_j \in F$, $i \in [n]$ and $e_1 \cdots e_n \leq F \varepsilon_F$.

**Example 7.42.** As explained in [3, §4.3], one can readily visualize $\ast$ when using tree diagrams, such as in the following example.
Further, note that when $F = \emptyset$ is the empty forest, it is $e_F = \epsilon$, and since $\epsilon \leq \epsilon$, $\emptyset \ast [n]$ adds a stump at the top of $[n]$.

We now discuss the functoriality of $\ast$. As implicit in the discussion in [9, §4.5], $\ast$ is only functorial with respect to some maps of forests. Indeed, it is clear from the third condition in Definition 7.41 that $\ast$ will be functorial in $F$ precisely with respect to the maps in $\Phi_w$.

Moreover, the canonical inclusions $[n] \to F \ast [n]$, $F \to F \ast [n]$ can be encoded thusly: letting $\Delta$, $\Phi_w$, $\Phi_i$ denote the categories $\Delta$, $\Phi_w$, $\Phi_i$ together with an additional initial object, one has the following.

**Proposition 7.43.** $\ast$ defines a functor

$$
\Phi_w \times \Delta \xrightarrow{\ast} \Phi_i.
$$

Note that this functor “nearly lands” in $\Omega_*$, the only exceptions occurring when the second input is the additional initial object.

We now extend the join operation to presheaves by defining

$$
fSet_w \times sSet \xrightarrow{\otimes} dSet.
$$

(7.44)

**Remark 7.45.** Unpacking (7.44) one can write (cf. [10, Defn. 1.2.8.1])

$$(X \ast Y)(T) = X(T) \sqcup Y(T) \sqcup \bigsqcup_{F \to T, [n] \to T \ast F \ast [n]} X(F) \times Y([n]),
$$

(7.46)

where $Y(T) = \emptyset$ when $T$ is not linear.

**Remark 7.47.** Due to the passage through the $(\_)_*$ categories in (7.44), $fSet_w \times sSet \xrightarrow{\otimes} dSet$ does not preserve colimits in each variable. Rather, the functors $F \ast (-), (\_ \ast [n])$ preserve colimits when mapping into the categories $dSet_F, dSet_{[n]/}$. Therefore, $\ast$ does nonetheless preserve connected colimits in each variable.

The following is an equivariant generalization of the key technical lemma [3, Lemma 4.10] combined with key arguments in the proof of [3, Thm. 4.2].

**Proposition 7.48.** Let $A \xrightarrow{f} B$ be a normal monomorphism in $fSet^G_w$ and $C \xrightarrow{g} D$ be a left $G$-anodyne map in $sSet$. Then

$$
f \square^* g : A \ast D \bigsqcup_{A \ast C} B \ast C \to C \ast D
$$

is inner $G$-anodyne.
Proof. Note first that that \( f \circ g \) is indeed a normal monomorphism whenever \( g \) is a monomorphism follows directly from (7.46).

In lieu of Remark 7.47 concerning connected colimits, it now suffices to consider the case where \( f, g \) have the form \( \partial \Phi_m[F] \to \Phi_m[F] \) and \( \Lambda^i[n] \to \Delta[n] \), \( 0 \leq i < n \). But in that case \( f \circ g \) is simply the inner horn inclusion

\[
\Lambda^i \Omega(F \ast [n]) \to \Omega(F \ast [n]).
\]

\( \square \)

**Proof of Theorem 7.4.** First note that the conditions on \( T \) are equivalent to saying that \( T = F \ast [1] \) for some \( F \in \Phi^G_m \). One can thus rewrite the left vertical map as

\[
\Lambda^i[1] \to \Delta[1] \xrightarrow{\pi^G} \emptyset \to \Phi_m[F]
\]

and denoting by \( A \langle - \rangle \) the right adjoint to \( *_{\text{Set}} \xrightarrow{\Lambda^G(-)} *_{\text{Set}^G} \), standard adjunction arguments allows us to convert (7.5) into the equivalent lifting problem

\[
\begin{array}{ccc}
\Lambda^j[1] & \to & (\Phi_m[F] \ast +) \backslash X \\
\downarrow & & \downarrow \\
\Delta[1] & \to & \Omega[F] \backslash X \\
\end{array}
\]

(7.49)

where the right hand diagram merely simplifies the notation in the left: \( \Phi_m[F] \ast + \simeq u^* \Phi_m[F] = \Omega[F] \), where \( \Omega[F] \) denotes the coproduct of representable sheaves for the trees in the forest; \( (\emptyset \ast +) \backslash (Z) = i^*(X^G) \), where we caution that \( \emptyset \neq \Phi_m[\emptyset] \), the former being the *empty presheaf* and the latter the representable presheaf on the empty forest.

Standard repeated applications of Proposition 7.48 (setting \( A = \emptyset \) or \( Y = \ast \) as needed) yield that (i) \( \Omega[F] \backslash X \to i^*(X^G) \), \( \Omega[F] \backslash Y \to i^*(Y^G) \) are left fibrations (ii) \( \Omega[F] \backslash X, \Omega[F] \backslash Y \) are left fibrant and thus \( \infty \)-categories (iii) the rightmost map \( \Omega[F] \backslash X \to \Omega[F] \backslash Y \times_{\partial \ast (Y^G)} i^*(X^G) \) in (7.49) is a left fibration. Therefore, the map \( \Omega[F] \backslash Y \times_{\partial \ast (Y^G)} i^*(X^G) \to i^*(X^G) \) is itself a left fibration (it is a pullback of \( \Omega[F] \backslash Y \to i^*(Y^G) \)) and since left fibrations are conservative the image of the lower map in (7.49) is a weak equivalence. The result now follows since weak equivalences can be lifted over left fibrations between \( \infty \)-categories.

\( \square \)

**Remark 7.50.** The interested reader may note that Proposition 7.48 and Theorem 7.4 have notably shorter proofs than the analogue results [3, Lemma 4.10] and [3, Thm. 4.2]. In fact, our proof more closely resembles the proof for the simplicial case, such as in [10, Prop. 1.2.4.3]. This is the reason for our introduction of the forest category \( \Phi_m \); from the perspective presented here, many of the arguments in [3] are replaced with the task of showing that the boundary \( \partial \Phi_m[F] \) has the usual formal properties.
8 Model structure on equivariant dendroidal sets

8.1 Existence

We now adapt the treatment in [3, §3] to equip the category $\mathbf{dSet}^G$ with a model structure where the cofibrations are the normal monomorphisms (cf. Prop. 6.5) and the fibrant objects are the $G$-$\infty$-operads (cf. Defn. 6.12). Since we have already established the equivariant analogues of the main technical results needed (Theorems 7.1, 7.2 and Theorem 7.4), the proofs in [3] now carry over to our context with only minimal modifications.

Recall the notation $J = N(0 \not\cong 1)$ for the nerve of the groupoid generated by a single isomorphism between distinct objects. As in [3] we will write $J_d = i_!(J)$ when regarding $J$ as a dendroidal set, and we will further regard $J_d$ as a $G$-dendroidal set by equipping it with the trivial $G$-action.

Following [3, §3], we let $\mathcal{A}$, the class of $J$-anodyne extensions, consist of those maps in the saturation of

$$\{A^G[T] \to \Omega[T]: T \in \Omega_G, e \in \operatorname{Inn}(T)\} \cup\{\{0\} \to J_d \square (\partial \Omega[T] \to \Omega[T]): T \in \Omega_G\}.$$

A $G$-dendroidal set $X$ (resp. map $X \to Y$) is then called $J$-fibrant (resp. $J$-fibration) if it has the right left lifting property with respect to the maps in $\mathcal{A}$. Following the treatment in [3, §3] mutatis mutandis gives the following generalization of [3, Prop. 3.12].

**Proposition 8.1.** $\mathbf{dSet}^G$ is equipped with a left proper cofibrantly generated model structure such that

- cofibrations are the normal monomorphisms;
- anodyne extensions are trivial cofibrations;
- fibrant objects are the $J$-fibrant objects;
- fibrations between fibrant objects are the $J$-fibrations.

**Remark 8.2.** As in the case of the model structure on $\mathbf{dSet}$ or of the Joyal model structure on $\mathbf{sSet}$, the trivial cofibrations (resp. fibrations) in Proposition 8.1 do not coincide with the anodyne extensions (resp. $J$-fibrations), but merely contain (resp. are contained in) them.

8.2 Characterization of fibrant objects

Much as in [3], the bulk of the work is now that of characterizing the fibrant objects as indeed being the $G$-$\infty$-operads.

We will need to make use of the adjunction

$$u^*: \mathbf{dSet}_G \rightleftarrows \mathbf{dSet}^G: u_*$$

discussed in Proposition 5.40.

**Definition 8.3.** The class of normal monomorphisms of $\mathbf{dSet}_G$ is the saturation of the maps of the form $u_*(A \to B)$ for $A \to B$ a normal monomorphism in $\mathbf{dSet}^G$. Further, a map $X \to Y$ in $\mathbf{dSet}_G$ is called a trivial fibration if it has the right lifting property with respect to normal monomorphisms.
We now extend some notation from [3, §6.1].

As usual, for $X$ an $\infty$-category, $k(X)$ will denote the maximal Kan complex inside $X$.

**Notation 8.4.** For a $G\infty$-operad $X$ and simplicial set $K$ (thought of as having a trivial $G$-action), we define $X^{(K)} \in \mathsf{dSet}_G$ to have $T$-dendrices (recall $T \in \Omega_G$) the maps

$$i_!(K) \otimes \Omega[T] \xrightarrow{\sim} X$$

such that for each edge orbit $G/H \cdot \eta \xrightarrow{G/H \cdot e} \Omega$ the induced map

$$K \xrightarrow{a} i^*(X^H)$$

factors through $k(i^*(X^H)) = k((i^*(X))^H)$.

**Remark 8.5.** We note that for $X$ a $G\infty$-category one always has $k(X^G) \subset k(X)^G$, but that this inclusion is rarely an equality.

**Notation 8.6.** For a normal $G$-dendroidal set $A \in \mathsf{dSet}_G$ and a $G\infty$-operad $X$ we define $k(A, X) \in \mathsf{sSet}$ to have $n$-simplices the maps

$$i_!(\Delta[n]) \otimes A \xrightarrow{b} X$$

such that, for all vertex orbits $G/H \cdot \eta \xrightarrow{G/H \cdot a} \Omega$ the induced map

$$\Delta[n] \xrightarrow{b} i^*(X^H)$$

factors through $k(i^*(X^H)) = k((i^*(X))^H)$.

Note that there are canonical isomorphisms

$$\text{Hom}_{\mathsf{dSet}}(K, k(A, X)) = \text{Hom}_{\mathsf{Set}_G}(u_*(A), X^{(K)}).$$

The following is the analogue of [3, Thm. 6.4].

**Theorem 8.7.** Let $p:X \rightarrow Y$ be an inner $G$-fibration between $G\infty$-operads. The map $ev_1: X^{(\Delta[1])} \rightarrow Y^{(\Delta[1])} \times_Y X$ has the right lifting property with respect to inclusions $u_j(\partial \Omega[S] \rightarrow \Omega[S])$ for any $G$-tree $S \in \Omega_G$ with at least one vertex.

Consequently, $ev_1: X^{(\Delta[1])} \rightarrow Y^{(\Delta[1])} \times_Y X$ is a trivial fibration in $\mathsf{dSet}_G$ iff all the maps $\tau i^*(X^G \rightarrow Y^G)$ are categorical fibrations.

**Proof.** Noting that it is $S = G \cdot H S_c$ for some $S_c \in \Omega^H$, $H \leq G$, it suffices to deal with the case $S \in \Omega^G$.

The proof of the main claim now follows exactly as in the proof of [3, Thm. 6.4] by replacing uses of [3, Thm. 5.2] and [3, Thm. 4.2] by the equivariant analogues Theorem 7.2 and Thm. 7.4.

For the “consequently” part, one needs only note that in the equivariant context there are now multiple trees with no vertices, namely the trees of the form $G/H \cdot \eta$.

The following is the analogue of [3, Prop. 6.7].
Proposition 8.8. Let $p: X \to Y$ be an inner $G$-fibration between $G$-$\infty$-operads. If $\tau^i(X^H \to Y^H)$ is a categorical fibration for all $H \leq G$ then, for any monomorphism between normal dendroidal sets $A \to B$, the map

$$k(B, X) \to k(B, Y) \times_{k(A, Y)} k(A, X)$$

(8.9)

is a Kan fibration between Kan complexes.

Proof. We will mainly refer to the proof of [3, Prop. 6.7] while indicating the main changes. First, note that it follows by Proposition 7.34 that (8.9) is an inner $G$-fibration between $G$-categories. As in [3, Prop. 6.7], it suffices to check that (8.9) as the right lifting property against the “left pushout products”

$$\partial \Delta[n] \times \Delta[1] \cup \partial \Delta[n] \times \{1\} \to \Delta[n] \times \Delta[1].$$

A lifting problem

\[
\begin{array}{ccc}
\partial \Delta[n] \times \Delta[1] \cup \partial \Delta[n] \times \{1\} & \overset{\partial}{\longrightarrow} & k(B, X) \\
\downarrow & & \downarrow \\
\Delta[n] \times \Delta[1] & \overset{h}{\longrightarrow} & k(B, Y) \times_{k(A, Y)} k(A, X)
\end{array}
\]

(8.10)

induces a (a priori non equivalent) lifting problem

\[
\begin{array}{ccc}
\partial \Delta[n] \times \Delta[1] \cup \partial \Delta[n] \times \{1\} & \overset{\partial}{\longrightarrow} & \chi \Delta[1] \\
\downarrow & & \downarrow \\
\Delta[n] \times \Delta[1] & \overset{h}{\longrightarrow} & Y \Delta[1] \times_Y X.
\end{array}
\]

(8.11)

That the lifting $\tilde{h}$ exists for (8.11) follows by Theorem 8.7 and it hence remains to check that the associated map $\iota((\Delta[n] \times \Delta[1]) \otimes B \to X$ indeed provides the map $h$ in (8.10). I.e., one must check that for any vertex orbit $G[H \cdot b] \cdot B$ the induced map $\Delta[n] \times \Delta[1] \to \iota^* (X^H)$ factors through $k(i^*(X^H))$ (note that the existence of $h$ only guarantees such a factorization for the restriction along $\{0, 1, \ldots, n\} \times \Delta[1] \subset \Delta[n] \times \Delta[1]$). The $n = 0$ case is immediate and the $n > 0$ case follows by arguing using the “2 out of 3 property”, just as in the penultimate paragraph of the proof of [3, Prop. 6.7].

Standard arguments (setting $A = \emptyset$ of $Y = *$ just as in [3, Prop. 6.7] or at the end of the proof of Theorem 7.4) finish the proof.

Corollary 8.12. Let $p: X \to Y$ be an inner $G$-fibration between $G$-$\infty$-operads such that $\tau^i(X^H \to Y^H)$ is a categorical fibration for all $H \leq G$. Then, for any anodyne extension of simplicial sets $K \to L$,

$$X^K \to Y^K \times_{Y^L} X^L$$

is a trivial fibration in $dSet_G$.

Proof. This is a formal consequence of Proposition 8.8.\[\]
We now finally obtain our sought generalization of [3, Thm. 6.10].

**Theorem 8.13.** A $G$-dendroidal set $X$ is $J$-fibrant iff it is a $G$-$\infty$-operad. Further, an inner $G$-fibration $p: X \to Y$ between $G$-$\infty$-operads is a $J$-fibration iff $\tau^i(X^G \to Y^G)$ is a categorical fibration for all $H \leq G$.

**Proof.** Unwinding definitions one must show that the map

$$X^{J_d} \to Y^{J_d} \times_{Y^{(0)}} X^{[0]}$$

is a trivial fibration of $G$-dendroidal sets iff $\tau^i(X^G \to Y^G)$ is a categorical fibration for all $H \leq G$. Note now that

$$u_*(X^{J_d} \to Y^{J_d} \times_{Y^{(0)}} X^{[0]})$$

is in fact the map

$$X^{(J_d)} \to Y^{(J_d)} \times_{Y^{(0)}} X^{(0)}$$

which is a trivial fibration in $dSet_G$ by Corollary 8.12. The result now follows since $dSet^G$ is a reflexive subcategory of $dSet_G$, so that $u_*(f)$ is a trivial fibration iff $f$ is.

The following follows exactly as in [3, Cor. 6.11].

**Corollary 8.14.** The weak equivalences in $dSet^G$ are the smallest class containing the inner $G$-anodyne extensions, the trivial fibrations and closed under "2-out-of-3".

## 9 Indexing system analogue results

In this section we follow the lead of [2] and build variant model structures on $dSet^G$ associated to certain indexing systems, a notion originally introduced in [2, Def. 3.22], which we repackage (and slightly extend) as Definition 9.5.

**Definition 9.1.** A $G$-graph subgroup of $G \times \Sigma_n$ is a subgroup $K \leq G \times \Sigma_n$ such that $K \cap \Sigma_n = \ast$.

**Remark 9.2.** $G$-graph subgroups are graphs of homomorphisms $G \to H \to \Sigma_n$.

**Definition 9.3.** A $G$-vertex family is a collection

$$\mathcal{F} = \coprod_{n \geq 0} \mathcal{F}_n$$

where each $\mathcal{F}_n$ is a family of subgroups of $G \times \Sigma_n$ closed under subgroups and conjugation.

Further, a $H$-set $X$ for a subgroup $H \leq G$ is called a $\mathcal{F}$-set if for some (and hence any) choice of isomorphism $X \cong \{1, \cdots, n\}$ the induced graph subgroup of $G \times \Sigma_n$ is in $\mathcal{F}$.

**Definition 9.4.** Let $\mathcal{F}$ be a $G$-vertex family.

A $G$-tree $T$ is called an $\mathcal{F}$-tree if for all edges $e \in T$ with isotropy $H$ one has that the $H$-set $e^3$ is an $\mathcal{F}$-set.
It is clear that whenever \( T \rightarrow S \) is either an outer face or quotient, \( S \) being an \( F \)-tree implies that so is \( T \). However, the same is typically not true for inner faces and degeneracies.

**Definition 9.5.** A \( G \)-vertex family \( F \) is called a *weak indexing system* if \( F \) -trees form a sieve of \( \Omega_G \), i.e., for any map \( T \rightarrow S \) with \( S \) a \( F \)-tree then it is also \( T \) a \( F \)-tree.

Additionally, \( F \) is called an *indexing system* if all \( F \) contain all subgroups \( H \times \Sigma_n \preceq G \times \Sigma_n \) for \( H \preceq G \).

**Remark 9.6.** Closure under degeneracies is simply the statement that \( F \) contain all subgroups \( H \times \Sigma_1 \preceq G \times \Sigma_1 \) for \( H \preceq G \).

**Remark 9.7.** Since definition 9.5 may at first sight seem quite different from the original [2, Def. 3.22], we quickly address the equivalence between the two. To a \( H \)-set with orbital decomposition \( K_1/H \sqcup \cdots \sqcup K_n/H \) one can associate the \( G \)-corolla with orbital representation as follows.

\[
\begin{align*}
G/K & \\
\cdots & \\
G/K_n & \\
\rightarrow & \\
G/H & \\
\end{align*}
\]

(9.8)

Note that for any of the roots \( r \) one has that \( r^\dagger \) is a conjugate of the \( H \)-set \( K_1/H \sqcup \cdots \sqcup K_n/H \). The conditions (cf. [2, Def. 3.22]) that indexing systems are closed under disjoint unions ([2, Def. 3.19]) and sub-objects ([2, Def. 3.21]) of \( F \)-sets are then encoded by trees

\[
\begin{align*}
G/K_1 & \\
\cdots & \\
G/K_n & \\
\rightarrow & \\
G/H & \\
\end{align*}
\]

(9.9)

while the closure under self-induction ([2, Def. 3.20]) is encoded by trees as on the left below.

\[
\begin{align*}
G/L_1 & \\
\cdots & \\
G/L_n & \\
\rightarrow & \\
G/K & \\
\rightarrow & \\
G/H & \\
\end{align*}
\]

(9.10)

Closure under cartesian products ([2, Def. 3.22]) is in fact redundant, as the coset formula \( H/K \times H/L = \bigsqcup_{\alpha \in \Delta H/K} H \times \Delta H/K \times H/K \times L^\rho \) allows such \( H \)-sets to be built using self inductions as displayed by the right tree above (the case of products with multiple orbits being then obtained via disjoint units).

Definitions 6.3, 6.11 and 6.12 admit weak indexing system analogues.
Definition 9.11. Let $\mathcal{F}$ be a weak indexing system.

A $\mathcal{F}$-boundary inclusion (resp. $\mathcal{F}$-inner horn inclusion) is a boundary inclusion $\partial \Omega[T] \to \Omega[T]$ (resp. inner horn inclusion $\Lambda^G \Theta[T] \to \Theta[T]$) with $T \in \Omega_G$ a $\mathcal{F}$-tree.

A monomorphism is called $\mathcal{F}$-normal (resp. $\mathcal{F}$-anodyne) if it is in the saturation of $\mathcal{F}$-boundary inclusions (resp. $\mathcal{F}$-inner horn inclusions) under pushouts, transfinite compositions and retracts.

Finally, a dendroidal set $X$ is called a $\mathcal{F}$-analog of an operad if it has the right lifting property with respect to all $\mathcal{F}$-inner horn inclusions.

We now list the necessary modifications to extend the results in this paper to the indexing system case.

A direct analogue of Proposition 6.5 yields that $X \in \mathbf{dSet}^G$ is $\mathcal{F}$-normal (i.e. $\varnothing \to X$ is $\mathcal{F}$-normal monomorphism) iff all dendrices $x \in X(T)$ have $\mathcal{F}$-isotropy, i.e. isotropies $\Gamma \subseteq G \times \Sigma_T$ are graph subgroups for partial homomorphisms $G \rightrightarrows H \to \Sigma_T$ such that the induced $G$-tree $G : H T$ is an $\mathcal{F}$-tree.

It then follows that, much like normal dendroidal sets, $\mathcal{F}$-normal dendroidal sets form a sieve, i.e., for any map $X \to Y$ with $Y$ a $\mathcal{F}$-normal dendroidal set then so is $X$.

Noting that the subtrees of $S \otimes T$ are $\mathcal{F}$-trees whenever $S, T$ are $\mathcal{F}$-trees (since the generating vertices/broad relations of $S \otimes T$ are induced from those of $S, T$), it follows that $\Omega[S] \otimes \Omega[T]$ is then $\mathcal{F}$-normal so that the sieve condition implies that Proposition 7.24 generalizes to the $\mathcal{F}$-normal case.

Likewise, the key results Theorem 7.1 and 7.2 immediately generalize by replacing the terms “$G$-tree” and “$G$-anodyne” with “$\mathcal{F}$-tree” and “$\mathcal{F}$-anodyne”. This is because their proofs, while long, boil down to identifying suitable edge orbits of suitable subtrees of $S \otimes T$ and then attaching the corresponding equivariant horns.

Likewise, Proposition 7.48 generalizes to the $\mathcal{D}$-case for the same reason, and hence so does Theorem 7.4, since its proof is an application of Proposition 7.48.

We can now prove Theorem 2.2.

proof of Theorem 2.2. The proof of the existence of the model structure follows just as in the case of Proposition 8.1, the only change being that one defines a similar notion of $J_{\mathcal{F}}$-anodyne extensions by using only $\mathcal{F}$-trees before running through the argument in [3, Prop. 3.12] (we note that the fact that $\mathcal{F}$-normal dendroidal sets form a sieve is key).

The characterization of the $J_{\mathcal{F}}$-fibrant objects as being the $\mathcal{F}$-$\infty$-operads follows by repeating the arguments in §8.2, though some care is needed when adapting the definitions preceding Theorem 8.7. Firstly, letting $\Omega_{\mathcal{F}} \subseteq \Omega_G$ denote the sieve of $\mathcal{F}$-trees, one sets $\mathbf{dSet}_{\mathcal{F}} = \mathbf{dSet}^{\Omega_{\mathcal{F}}}$, leading to an adjunction

$$u^* : \mathbf{dSet}_{\mathcal{F}} \rightleftarrows \mathbf{dSet}^G : u_*$$

so that the $\mathcal{F}$-normal monomorphisms of $\mathbf{dSet}_{\mathcal{F}}$ are defined just as in Definition 8.3.
For a $\mathcal{F}$-$\infty$-operad and simplicial set $K$, one defines $X^{(K)} \in dSet_{/\mathcal{F}}$ just as in Notation 8.4 while for $A \in dSet^{G}$ a $\mathcal{F}$-normal dendroidal set and $\mathcal{F}$-$\infty$-operad $X$ one defines $k(A,X) \in sSet$ just as in Notation 8.6.

The proofs of Theorems 8.7, Proposition 8.8 and Theorem 8.13 now extend mutatis mutandis by using the $\mathcal{F}$ versions of Theorems 7.2 and 7.4.

References

[1] C. Berger and I. Moerdijk. On an extension of the notion of Reedy category. Math. Z., 269(3-4):977–1004, 2011.
[2] A. J. Blumberg and M. A. Hill. Operadic multiplications in equivariant spectra, norms, and transfers. Adv. Math., 285:658–708, 2015.
[3] D.-C. Cisinski and I. Moerdijk. Dendroidal sets as models for homotopy operads. J. Topol., 4(2):257–299, 2011.
[4] D.-C. Cisinski and I. Moerdijk. Dendroidal Segal spaces and $\infty$-operads. J. Topol., 6(3):675–704, 2013.
[5] D.-C. Cisinski and I. Moerdijk. Dendroidal sets and simplicial operads. J. Topol., 6(3):705–756, 2013.
[6] D.-C. Cisinski and I. Moerdijk. Note on the tensor product of dendroidal sets. Available at: https://arxiv.org/abs/1403.6507, 2014.
[7] S. R. Costenoble and S. Waner. Fixed set systems of equivariant infinite loop spaces. Trans. Amer. Math. Soc., 326(2):485–505, 1991.
[8] E. Getzler and J. Jones. Operads, homotopy algebra and iterated integrals. Available at: https://arxiv.org/abs/hep-th/9403055, 1995.
[9] G. Heuts, V. Hinich, and I. Moerdijk. On the equivalence between Lurie’s model and the dendroidal model for infinity-operads. Adv. Math., 302:869–1043, 2016.
[10] J. Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.
[11] M. Markl, S. Shnider, and J. Stasheff. Operads in algebra, topology and physics, volume 96 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2002.
[12] J. P. May. The geometry of iterated loop spaces. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.
[13] I. Moerdijk and I. Weiss. Dendroidal sets. Algebr. Geom. Topol., 7:1441–1470, 2007.
[14] I. Moerdijk and I. Weiss. On inner Kan complexes in the category of dendroidal sets. Adv. Math., 221(2):343–389, 2009.
[15] I. Weiss. Broad posets, trees, and the dendroidal category. Available at: https://arxiv.org/abs/1201.3987, 2012.