THE CYCLE STRUCTURE OF PERMUTATIONS WITHOUT LONG CYCLES

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ABSTRACT. We consider the cycle structure of a random permutation $\sigma$ chosen uniformly from the symmetric group, subject to the constraint that $\sigma$ does not contain cycles of length exceeding $r$. We prove that under suitable conditions the distribution of the cycle counts is approximately Poisson and obtain an upper bound on the total variation distance between the distributions using Stein’s method of exchangeable pairs. Our results extend the recent work of Betz, Schäfer, and Zeindler in [2].

1. Introduction

The matching problem, which asks for the probability that a uniformly selected permutation $\sigma \in S_n$ has no fixed points, is one of the oldest problems in probability theory. It is not difficult to show, using the inclusion-exclusion principle, that this probability converges to $1/e$ as $n \to \infty$, a result that was first given in [4] in 1708. More generally, it is a classical result that for large $n$ the number of fixed points of a randomly chosen permutation on $n$ letters is approximately distributed as a Poisson random variable with mean 1; see, for example, [5].

One can also rephrase the matching problem in terms of cycles by noting that the number of matches is exactly the number of 1-cycles. A natural extension is to then consider the expected value and distribution of the number of $k$-cycles in $\sigma$, a problem which was considered in [1]. The authors showed that if $C_j$ is the number of cycles of length $j$ in $\sigma$ and $(Y_1, Y_2, ..., Y_b)$ is a vector of independent Poisson random variables with $Y_j \sim \text{Poi}(1/j)$, then the total variation distance between the distributions of $(C_1, C_2, ..., C_b)$ and $(Y_1, Y_2, ..., Y_b)$ decays to zero super-exponentially fast as a function of $n/b$ if and only if $b = o(n)$. In the same paper the authors showed that when conditioning on cycle counts $C_j = c_j$ for $j \in J \subset \{1, 2, ..., b\}$ with $\sum_{j \in J} jc_j = s \leq n$, the limiting distribution is $(Z_1, Z_2, ..., Z_b)$ where

$$Z_j = \begin{cases} 
    c_j & j \in J, \\
    Y_j & \text{otherwise}
\end{cases}$$

if $(n-s)/b \to \infty$ and $Y_j \sim \text{Poi}(1/j)$ as before. While this result applies to permutations without short cycles, it does not allow for conditioning on the absence of long cycles; this opposite regime is the subject of this paper. Let

$$S_n^r = \{\sigma \in S_n : \sigma \text{ does not contain a cycle of length exceeding } r\}.$$
The set $S_n^r$ was first studied by Goncharov in [5] in 1944. His estimates of the size of $S_n^r$ were further refined in [7], and we will make use of these results in what follows. More broadly, the structure of different subsets of the symmetric group under the uniform measure has been an active area of study. For example, the set of permutations with cycle lengths belonging to a subset $A \subset \mathbb{N}$ is considered in [9].

Recently, in [2], it was shown that if $W = (W_1, W_2, ..., W_d)$ is a vector of independent Poisson random variables with $Y_i \sim \text{Poi}_{1/i}$, $r > n^\alpha$ for some $0 < \alpha < 1$, and $d = o(r/\log(n))$ then

$$\|\mathcal{L}(W) - \mathcal{L}(Y)\|_{TV} \leq C \left( \frac{r}{n} + \frac{d \log(n)}{r} \right)$$

for some constant $C$. In this paper we will prove the following result.

**Theorem 1.** Suppose $\sqrt{n \log(n)} \leq r \leq n$ and $d = o(r/\log(n/r))$. Let $W_k(\sigma)$ be the number of $k$-cycles in a permutation $\sigma$ chosen uniformly from $S_n^r$ and let $W = (W_1, W_2, ..., W_d)$. Let $Y = (Y_1, Y_2, ..., Y_d)$ have independent coordinates with $Y_i \sim \text{Poi}_{1/i}$. Let $u = n/r$. Then there exists a constant $C > 0$, independent of $d, n, r$, such that

$$\|\mathcal{L}(W) - \mathcal{L}(Y)\|_{TV} \leq C \left( \frac{2d \log(d) + 10d}{n - 1} + C \frac{(d^2 + du) \log(u + 1)}{nr} \right).$$

In particular, when $r \geq \sqrt{n \log(n)}$ Theorem 1 provides a sharper bound than (1).

2. **Background**

In this section we state several lemmas and propositions that will be instrumental in proving Theorem 1. The following result from [7] gives an estimate of the size of the set $S_n^r$.

**Proposition 2 ([7], Theorem 4).** Suppose $\sqrt{n \log(n)} \leq r \leq n$. Let $u = n/r$. Then

$$\nu(n, r) := \frac{|S_n^r|}{|S_n|} = \rho(u) \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right).$$

where $\rho$ is the Dickman function, the continuous solution to the difference-differential equation

$$t \rho'(t) + \rho(t - 1) = 0.$$

with initial condition $\rho(t) = 1$ for $0 \leq t \leq 1$.

**Remark.** It is worth noting that in Proposition 2 the Big-O term does not approach 0 if $r = \sqrt{n \log(n)}$; it is $\Theta(1)$. However, then $u = \sqrt{n/\log(n)} \to \infty$ as $n \to \infty$, and $\rho(u) \to 0$ rapidly. Indeed, it was shown in [6] that

$$\rho(t) \leq \frac{1}{\Gamma(t + 1)}.$$
Lemma 3 ([8], Lemma 3). If \( t \geq 1 \) and \( v > 0, v = O(1) \), then
\[
\frac{\rho(t - v)}{\rho(t)} = e^{\xi(t)}(1 + O(v/t))
\]
where \( \xi := \xi(t) \) is the positive solution to the equation
\[
e^{\xi} = 1 + t\xi.
\]
It was shown in [7] that if \( t > 1 \) then \( \log(t) < \xi(t) \leq 2\log(t) \).

Lemma 4. If \( \sqrt{n \log(n)} \leq r \leq n \),
\[
\frac{\nu(n - k, r)}{\nu(n, r)} = e^{\frac{\xi(u)}{r}} \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right).
\]
In particular,
\[
\frac{\nu(n - k, r)}{\nu(n, r)} = 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right).
\]

Proof. Recall that \( u = n/r \). By Proposition 2,
\[
\frac{\nu(n - k, r)}{\nu(n, r)} = \rho \left( \frac{u - \frac{k}{r}}{n} \right) \left( 1 + O \left( \frac{(u-(k/r)) \log(u-(k/r)+1)}{r} \right) \right) \rho(u) \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right).
\]
Applying Lemma 3 yields
\[
\frac{\nu(n - k, r)}{\nu(n, r)} = e^{\frac{k}{r} \xi(u)} \left( 1 + O \left( \frac{k}{ru} \right) \right) \left( 1 + O \left( \frac{(u-(k/r)) \log(u-(k/r)+1)}{r} \right) \right) \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right).
\]
Expanding \( e^{\frac{k}{r} \xi(u)} \) in a Taylor series gives that
\[
\frac{\nu(n - k, r)}{\nu(n, r)} = \left( 1 + \frac{\xi k/r}{1!} + \frac{\xi^2 (k/r)^2}{2!} + \ldots \right) \left( 1 + O \left( \frac{k}{n} \right) \right) \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right) \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right) \left( 1 + O \left( \frac{u \log(u + 1)}{r} \right) \right).
\]
completing the proof. \( \square \)

3. Results

Let \( C_a \) denote the cycle containing \( a \in \{1, \ldots, n\} \) and let \( L(C_a) \) be its length.

Lemma 5. Let \( a \in \{1, 2, \ldots, n\} \). Then for \( \sigma \) chosen uniformly from \( S_n^r \),
\[
P[L(C_a) = k] = \frac{1}{n} \left( 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right) \right).
\]
Proof. There are \( \binom{n-1}{k-1} \) ways to select the remaining elements to be in the cycle \( C_a \), \((k-1)!\) cycles that can be formed from these \( k \) elements, and \( |S_{n-k}^r| \) ways to permute the remaining elements. Dividing by \( |S_n^r| \) yields
\[
\frac{(n-1)(k-1)!(n-k)!\nu(n-k, r)}{n!\nu(n, r)} = \frac{1}{n} \left( 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right) \right)
\]
by Lemma 4. \(\square\)

Proposition 6. Let \( \sigma \) be a uniformly chosen permutation from \( S_n^r \) and let \( W \) be the number of \( k \)-cycles in \( \sigma \). Then
\[
\mathbb{E}[W] = \frac{1}{k} \left( 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right) \right).
\]

Proof. Let
\[
X_i(\sigma) = \begin{cases} 1 & \text{if } i \text{ is contained in a } k\text{-cycle} \\ 0 & \text{if } i \text{ is not contained in a } k\text{-cycle} \end{cases}
\]
for \( 1 \leq i \leq n \). Because the \( X_i \) are identically distributed for each \( i \in \{1, 2, ..., n\} \), by the method of indicators and Lemma 5
\[
\mathbb{E}W = \frac{1}{k} \sum_{i=1}^{n} \mathbb{E}X_i = \frac{n}{k} \mathbb{E}X_1 = \frac{1}{k} \left( 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right) \right). \quad \square
\]

The proof of Theorem 1 uses the following multivariate version of Stein’s method of exchangeable pairs.

Theorem 7 ([3], Proposition 10). Let \( W = (W_1, W_2, ..., W_d) \) with \( W_i \) a non-negative integer valued random variable. Let \( Y = (Y_1, Y_2, ..., Y_d) \) have independent coordinates with \( Y_i \) a Poisson random variable with mean \( \lambda_i \). Let \( W' = (W_1', W_2', ..., W_d') \) be defined on the same probability space as \( W \) with \( (W, W') \) an exchangeable pair. Then
\[
\|\mathcal{L}(W) - \mathcal{L}(Y)\|_{TV} \leq \sum_{k=1}^{d} \frac{\alpha_k}{2} \left( \mathbb{E}|\lambda_k - c_k\mathbb{P}[A_k]| + \mathbb{E}|W_k - c_k\mathbb{P}[B_k]| \right)
\]
with \( \alpha_k = \min\{1, 1.4\lambda^{-1/2}\} \), \( c_k > 0 \) for each \( k \), and
\[
A_k = \{W_k' = W_k + 1, W_j = W_j' \text{ for } k + 1 \leq j \leq d\}, \\
B_k = \{W_k' = W_k - 1, W_j = W_j' \text{ for } k + 1 \leq j \leq d\}.
\]

Remark. Although the estimate of Proposition 6 is only correct up to a constant factor if \( r = \sqrt{n \log(n)} \), we nevertheless take \( \lambda_k = 1/k \) in what follows. The remainder of the proof shows that this is indeed the correct parameter.

Construct an exchangeable pair \((\sigma, \sigma')\) by choosing \( \sigma \) uniformly from \( S_n \), choosing a transposition \( \tau \) uniformly from \( S_n \) and independently of \( \sigma \), and setting
\[
\sigma' = \begin{cases} \tau \circ \sigma & \tau \circ \sigma \in S_n^r \\
\sigma & \text{otherwise} \end{cases}
\]
Write \( W' = W(\sigma') \).
Lemma 8. For the random variables \( W, W' \) as defined above,

\[
P[A_k] = \frac{2}{n-1} + \frac{2}{n(n-1)} \sum_{a=1}^{n} \left( -\mathbb{1}_{L(C_a) \leq d+k} + \mathbb{1}_{d < L(C_a) < 2k} \right) \\
+ \frac{1}{n(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \mathbb{1}_{C_a \neq C_b} \mathbb{1}_{L(C_a) + L(C_b) = k}.
\]

(2)

If \( \lambda_k = 1/k \) and \( c_k = n/2k \), then

\[
E[|\lambda_k - c_k P[A_k]|] \leq \frac{1}{2(n-k)} + \frac{d + 3k + 1}{k(n-1)} + O \left( \frac{\xi(u)k + u \log(u+1)}{nr} \right).
\]

(3)

Proof. We count the number of transpositions \( \tau = (ab) \) such that the event \( A_k \) occurs. Suppose first that \( a, b \) are contained in different cycles in the cycle structure of \( \sigma \). Then the number of \( k \)-cycles will increase by 1 if \( L(C_a) + L(C_b) = k \).

Suppose now that \( a, b \) are contained in the same cycle in the cycle structure of \( \sigma \) and that \( L(C_a) > k \). There are \( L(C_a) \) transpositions that will break \( C_a \) into two cycles, one of which has length \( k \). Note, however, that if \( L(C_a) \in \{k+1, k+2, ..., d\} \) and \( C_a \) is broken into two cycles, then \( W'_{L(C_a)} = W_{L(C_a)} - 1 \). Also, if \( L(C_a) \in \{2k+1, 2k+2, ..., d+k\} \) then \( W'_{L(C_a)-k} = W_{L(C_a)-k} + 1 \). Furthermore, if \( L(C_a) = 2k \) then \( W' = W + 2 \). Putting this together yields

\[
P[A_k] = \frac{2}{n(n-1)} \sum_{a=1}^{n} \left( \mathbb{1}_{L(C_a) > d+k} + \mathbb{1}_{d < L(C_a) < 2k} \right) \\
+ \frac{1}{n(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \mathbb{1}_{C_a \neq C_b} \mathbb{1}_{L(C_a) + L(C_b) = k}.
\]

Equation (2) then follows by writing \( \mathbb{1}_{L(C_a) > d+k} = 1 - \mathbb{1}_{L(C_a) \leq d+k} \).

By the triangle inequality,

\[
E[|\lambda_k - c_k P[A_k]|] \leq \frac{1}{k(n-1)} + E \left[ \frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \mathbb{1}_{C_a \neq C_b} \mathbb{1}_{L(C_a) + L(C_b) = k} \right] \\
+ E \left[ \frac{1}{k(n-1)} \sum_{a=1}^{n} \mathbb{1}_{L(C_a) \leq d+k} + \mathbb{1}_{d < L(C_a) < 2k} \right].
\]
For the first expectation we have

\[
\mathbb{E}
\left[
\frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \mathbbm{1}_{C_a \neq C_b} \mathbbm{1}_{L(C_a) + L(C_b) = k}
\right]
\]

\[
= \mathbb{E}
\left[
\frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \sum_{j=1}^{k-1} \mathbbm{1}_{C_a \neq C_b} \mathbbm{1}_{L(C_a) = j} \mathbbm{1}_{L(C_b) = k-j}
\right]
\]

\[
= \frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \sum_{j=1}^{k-1} \mathbb{P}[L(C_b) = k-j | C_a \neq C_b, L(C_a) = j] \mathbb{P}[L(C_a) = j, C_a \neq C_b]
\]

\[
\leq \frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \sum_{j=1}^{k-1} \mathbb{P}[L(C_b) = k-j | C_a \neq C_b, L(C_a) = j] \mathbb{P}[L(C_a) = j]. \quad (4)
\]

Applying Lemma 5 to (4) yields

\[
\mathbb{E}
\left[
\frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \mathbbm{1}_{C_a \neq C_b} \mathbbm{1}_{L(C_a) + L(C_b) = k}
\right]
\]

\[
\leq \frac{1}{2k(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \sum_{j=1}^{k-1} \left[ \frac{1}{n(n-k)} \left( 1 + O \left( \frac{\xi(u)k + u \log(u+1)}{r} \right) \right) \right]
\]

\[
\leq \frac{1}{2(n-k)} \left( 1 + O \left( \frac{\xi(u)k + u \log(u+1)}{r} \right) \right).
\]

Furthermore, by Lemma 5,

\[
\mathbb{E}
\left[
\frac{1}{k(n-1)} \sum_{a=1}^{n} \mathbbm{1}_{L(C_a) \leq d+k + \mathbbm{1}_{d<L(C_a) \leq 2k}}
\right]
\]

\[
\leq \frac{d+3k}{k(n-1)} \left( 1 + O \left( \frac{\xi(u)k + u \log(u+1)}{r} \right) \right). \quad \Box
\]

**Lemma 9.** For the random variables \(W, W'\) as defined above,

\[
\mathbb{P}[B_k] = W_k - \frac{2}{n(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \left( - \mathbbm{1}_{L(C_a) = k} \mathbbm{1}_{L(C_b) \leq d} - \mathbbm{1}_{L(C_a) = k} \mathbbm{1}_{L(C_b) > r-k} + \mathbbm{1}_{L(C_a) = k} \mathbbm{1}_{L(C_b) < k} \mathbbm{1}_{L(C_b) > d-k} \right) + \frac{k-1}{n(n-1)} \sum_{a=1}^{n} \mathbbm{1}_{L(C_a) = k}.
\]

\[
(5)
\]

If \(c_k = n/2k\), then

\[
\mathbb{E}[W_k - c_k \mathbb{P}[B_k]] \leq \frac{d+k-1}{k(n-k)} + \frac{4}{n-k} + O \left( \frac{\xi(u)k + u \log(u+1)}{nr} \right).
\]

\[
(6)
\]

**Proof.** To decrease \(W\) by 1, an existing \(k\)-cycle must be destroyed and no new \(k\)-cycles can be created. If \(a \in \{1, 2, ..., n\}\) is in a \(k\)-cycle, then the \(k\)-cycle \(C_a\) is destroyed by any transposition \(\tau = (ab)\) with \(b \not\in C_a\) or \(b \in C_a, b \neq a\).
In the first case, $C_a$ can be combined with any cycle $C_b$ such that $L(C_b) > d$ and $L(C_b) \leq r - k$. Also, $C_a$ can be combined with any cycle $C_b$ with $L(C_b) < d$ if $L(C_b) < k$ and $L(C_b) + k > d$. In the second case, where $b \in C_a$, there are $k - 1$ elements in the same cycle that can be paired with $a$. This yields

$$
\mathbb{P}[B_k] = \frac{2}{n(n-1)} \sum_{a=1}^{n} \sum_{b \neq a} \mathbb{1}_{L(C_a) = k} \left( \mathbb{1}_{L(C_b) > d} \mathbb{1}_{L(C_b) \leq r - k} + \mathbb{1}_{L(C_b) < k} \mathbb{1}_{L(C_b) > d - k} \right)
+ \frac{k - 1}{n(n-1)} \sum_{a=1}^{n} \mathbb{1}_{L(C_a) = k}.
$$

Noting that

$$
\mathbb{1}_{L(C_b) > d} \mathbb{1}_{L(C_b) \leq r - k} = (1 - \mathbb{1}_{L(C_b) \leq d}) (1 - \mathbb{1}_{L(C_b) > r - k}) = 1 - \mathbb{1}_{L(C_b) \leq d} - \mathbb{1}_{L(C_b) > r - k}
$$

because $\mathbb{1}_{L(C_b) \leq d} \mathbb{1}_{L(C_b) > r - k} = 0$ yields (5).

By the triangle inequality,

$$
\mathbb{E}[W_k - c_k \mathbb{P}[B_k]] \leq \mathbb{E} \left[ \frac{k - 1}{2(n-1)k} \sum_{a=1}^{n} \mathbb{1}_{L(C_a) = k} \right]
+ \mathbb{E} \left[ \frac{1}{(n-1)k} \sum_{a=1}^{n} \sum_{b \neq a} \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) > r - k} \right]
+ \mathbb{E} \left[ \frac{1}{(n-1)k} \sum_{a=1}^{n} \sum_{b \neq a} (\mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) \leq d} + \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) < k} \mathbb{1}_{L(C_b) > d - k}) \right].
$$

By Lemma 5

$$
\mathbb{E} \left[ \frac{k - 1}{2(n-1)k} \sum_{a=1}^{n} \mathbb{1}_{L(C_a) = k} \right] \leq \frac{1}{2k(n-1)} \left( 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right) \right)
$$

and

$$
\mathbb{E} \left[ \frac{1}{(n-1)k} \sum_{a=1}^{n} \sum_{b \neq a} \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) > r - k} \right] \leq \frac{1}{n - k} \left( 1 + O \left( \frac{\xi(u)k + u \log(u + 1)}{r} \right) \right).
$$

Furthermore,

$$
\mathbb{E} \left[ \frac{1}{(n-1)k} \sum_{a=1}^{n} \sum_{b \neq a} (\mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) \leq d} + \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) < k} \mathbb{1}_{L(C_b) > d - k}) \right]
= \mathbb{E} \left[ \frac{1}{(n-1)k} \sum_{a=1}^{n} \sum_{b \neq a} \left( \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) \leq d, L(C_b) \neq k} + \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) = k} + \mathbb{1}_{L(C_a) = k} \mathbb{1}_{L(C_b) < k} \mathbb{1}_{L(C_b) > d - k} \right) \right]
$$
and by Lemma 5
\[ E \left[ \frac{1}{l(n-1)^k} \sum_{a=1}^{n} \sum_{b \neq a} \left( \mathbb{1}_{L(C_a)=k} \mathbb{1}_{L(C_b) \leq d, L(C_b) \neq k} + \mathbb{1}_{L(C_a)=k} \mathbb{1}_{L(C_b)=d-k} \right) \right] \leq \left[ \frac{d-1}{k(n-k)} + \frac{2}{n-1} + \frac{k}{k(n-k)} \right] \left( 1 + O \left( \frac{\xi(u)k + u \log(u+1)}{r} \right) \right) \]
\[ = \left[ \frac{d+k-1}{k(n-k)} + \frac{2}{n-1} \right] \left( 1 + O \left( \frac{\xi(u)k + u \log(u+1)}{r} \right) \right). \]□

**Proof of Theorem 1.** By Theorem 7 and Lemmas 8 and 9,
\[ \| \mathcal{L}(W) - \mathcal{L}(Y) \|_{TV} \leq \sum_{k=1}^{d} \frac{\alpha_k}{2} \left( E|\lambda_k - c_k \mathbb{P}[A_k]| + E|W_k - c_k \mathbb{P}[B_k]| \right) \]
\[ \leq \sum_{k=1}^{d} \frac{1}{2} \left[ \frac{2d + 4k}{k(n-k)} + \frac{4}{n-k} + O \left( \frac{\xi(u)k + u \log(u+1)}{nr} \right) \right] \]
\[ \leq \sum_{k=1}^{d} \frac{2d}{kn} + \frac{8}{n} + O \left( \frac{\xi(u)k + u \log(u+1)}{nr} \right) \]
\[ = \frac{2dH_d}{n} + \frac{8d}{n} + O \left( \frac{d^2 \xi(u) + d \log(u+1)}{nr} \right) \]
where
\[ H_d = \sum_{k=1}^{d} \frac{1}{k} \leq \log(d) + 1 \]
is the \( d \)th harmonic number. □

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