Wright-Fisher Diffusion in One Dimension

Charles L. Epstein,* and Rafe Mazzeo†

July 22, 2009

This paper is dedicated to the memory of Ralph S. Phillips (1913-1998), one of the seminal figures in the theory of semi-groups. Early in our careers, he was a profound and positive influence on both of us.

Abstract
We analyze the diffusion process associated to equations of Wright-Fisher type in one spatial dimension. These are associated to the degenerate heat equation
\[ \partial_t u = a(x) \partial_x^2 u + b(x) \partial_x u \] (1)
on the interval \([0, 1]\), where \(a(x) > 0\) on the interior and vanishes simply at the endpoints, and \(b(x) \partial_x\) is a vector field which is inward-pointing at both ends. We consider various aspects of this problem, motivated by their applications in biology, including a comparison of the natural boundary conditions from the probabilistic and analytic points of view, a sharp regularity theory for the “zero flux” boundary conditions, as well as a derivation of the precise asymptotics of solutions of this equation, both as \(t \to 0, \infty\) and as \(x \to 0, 1\). This is a precursor to our more complicated analysis of these same questions for Wright-Fisher type problems in higher dimensions.

1 Introduction

Consider the differential operator
\[ L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \] (2)

*Research partially supported by NSF grant DMS06-03973, DARPA HR00110510057, and the Thomas A. Scott Chair. Address: Department of Mathematics, University of Pennsylvania; e-mail: cle@math.upenn.edu
†Research partially supported by NSF grant DMS0805529, and DARPA HR00110510057. Address: Department of Mathematics, Stanford University; e-mail: mazzeo@math.stanford.edu
on the interval \([A, B]\), where the coefficient functions \(a(x), b(x) \in C^\infty([A, B])\). Our main assumptions on the coefficients are that

\[
a(x) = (x - A)(B - x)\tilde{a}(x),
\]
where \(\tilde{a}(x) \in C^\infty([A, B])\), and \(\tilde{a}(x) > 0 \forall x \in [A, B]\) \hfill (3)

and

\[
b(A) \geq 0, \quad b(B) \leq 0. \hfill (4)
\]

In other words, we assume that \(a(x)\) vanishes simply at the endpoints of the interval and nowhere else, and that the first order term \(b(x)d/dx\) is an inward pointing vector field. The diffusion associated to \(L\) is of importance in population biology. The basic example is the so-called Wright-Fisher operator

\[
L_{WF} = x(1 - x)\frac{d^2}{dx^2} \hfill (5)
\]

on the interval \([0, 1]\), which is the diffusion limit of a Markov chain modeling the frequency of a gene with 2 alleles, without mutation or selection. If the mutation rates between the alleles are \(\mu_{12}\) and \(\mu_{21}\), and the mutant allele has selective advantage \(s\), then

\[
b(x) = \mu_{12}(1 - x) - \mu_{21}x + sx(1 - x), \hfill (6)
\]

see [4]. Accordingly, we shall call any operator of the form (2) with coefficients satisfying (3) and (4) a generalized Wright-Fisher operator. By an affine transformation, we can also reduce to the case where \(L\) is defined on the interval \([0, 1]\).

In a seminal 1952 paper, [5], Feller considered the additional conditions one should impose on the domain of an operator like \(L\) to obtain the generator of a positivity preserving \(C^0\)-semi-group. Feller’s starting point was the Hille-Yosida theorem and his analysis centered on the construction of the resolvent kernel, \((L - \lambda)^{-1}\). Our approach, by contrast, focuses directly on the Schwartz kernel for \(e^{tL}\). In addition, we restrict to a particular choice of boundary condition, which arises in applications to population genetics.

The plan of this paper is as follows. In the next section we recall a change of variables introduced in [5] which reduces a general Wright-Fisher operator \(L\) to one with principal part \(x(1 - x)\partial_x^2\). We then describe the natural boundary conditions for this operator and its adjoint. The operator \(L\) is modeled near each endpoints, \(x = 0\) or \(x = 1\) by an operator of the form:

\[
L_b = x\partial_x^2 + b\partial_x, \hfill (7)
\]
where $b$ is a nonnegative constant. The model operators act on functions on $[0, \infty)$, with boundary conditions at $x = 0$ induced from those for $L$. The next several sections are devoted to a careful analysis of the solution operator for $\partial_t - L_b$. After a discussion in §5 of maximum principles in this setting, we derive in §6 an explicit formula for the Schwartz kernel $k^b_t$ of this solution operator, and then in §7 combine these ingredients to prove sharp mapping properties for the kernels $k^b_t$. The long §8 contains a variety of technical estimates needed to analyze perturbations of these model operators. After these extensive preliminaries, it is straightforward to assemble this information and express the fundamental solution for a general Wright-Fisher operator $L$ on $[0, 1]$ as a convergent Neumann series. This is done in §9; this leads directly to the description of precise asymptotics of solutions of $(\partial_t - L)u = 0$ as $t \to 0$ and as $x \to 0, 1$. An asymptotic expansion of the heat kernel for $L$ was obtained using other methods by Keller and Tier in [9]. The next section §10 uses this to provide a characterization of the infinitesimal generator of the induced semi-group on $C^0$ and its spectrum, which gives the long-time asymptotics for solutions of the diffusion problem. In §11 we then use the Hille-Yosida theorem to study the higher order regularity of solutions to the “elliptic” equation $(\lambda - L)w = f$, for $f \in C^m([0, 1])$. Finally, in §12 we characterize the adjoint semi-group, using various abstract results from the theory of semi-groups as well as the specific analytic information accrued to this point. This allows us to discuss the “forward” Kolmogorov equation, which is crucial for applications in population genetics.

Our focus throughout is on the $C^0$ (and $C^m$) theory, rather than the (simpler) $L^2$ theory. We shall return to a comparison between the $C^0$ and $L^2$ semi-groups elsewhere.

Remark 1 (Notational Remark). We let $I \subset \mathbb{R}$ be an interval or a ray. We use the notation $C^m_b(I)$ for the space of functions with $m$ continuous, bounded derivatives on $I$; the notation $C^m_c(I)$ for the space of functions with $m$ continuous derivatives, and compact support on $I$. Finally we use $C^m([a, b))$ to denote the space of functions with $m$ continuous derivatives on $[0, b)$, tending to zero at $b$.

Acknowledgements

We would like to thank Charlie Fefferman for showing us his construction of the heat kernel for the model problem on the half line, Dan Stroock for sharing his probabilistic approach to these questions, which appears in a recently completed preprint written jointly with Linan Chen, [2], and Nick Patterson for very helpful discussions on this subject. Finally, we are both grateful to Ben Mann and the DARPA FunBio program for giving providing the intellectual stimulus and opportunity to undertake this research.
2 A change of variables

The first task is to show that by a judicious change of variables, already known to Feller [5], any operator of the form (2) can be reduced to a first order perturbation of the exact Wright-Fisher operator \( L_{WF} \).

Given the operator (2), assume (by rescaling if necessary) that

\[
\int_0^1 \frac{ds}{\sqrt{a(s)}} = \pi.
\]

For the heat equation this amounts to rescaling the time variable. We define a diffeomorphism of \([0, 1]\) by setting:

\[
\xi(x) = \sin^2 \left( \frac{\eta(x)}{2} \right), \quad \text{where} \quad \eta(x) = \int_0^x \frac{ds}{\sqrt{a(s)}}.
\]

A short computation shows that \( L \) becomes

\[
\xi(1 - \xi) \frac{d^2}{d\xi^2} + \tilde{b}(\xi) \frac{d}{d\xi},
\]

where \( \tilde{b}(\xi) \) has the same properties as the original function \( b(x) \), i.e. it is smooth on the closed interval, and \( \tilde{b}(0) = b(0), \tilde{b}(1) = b(1) \), so in particular is inward pointing at the two boundary points, and vanishes at a boundary point if and only if \( b \) does.

Henceforth we return to using \( x \) as the independent variable and \( a \) and \( b \) as the coefficient functions. We assume that our general Wright-Fisher operator has the form

\[
L = x(1 - x) \partial_x^2 + b(x) \partial_x,
\]

where \( b(0), -b(1) \geq 0 \). As noted earlier, the case of principal interest in population genetics is when these values lie in \([0, 1)\), the so called “weak mutation” regime; however, it is no more difficult to treat the general case. In our analysis it is actually necessary to consider larger values of these constants, to obtain higher order estimate for the cases when \( |b(x)| \) lies in \([0, 1)\).

3 Natural boundary problems

Because \( L \) is degenerate at the boundary of the interval \([0, 1]\), we must carefully specify how boundary conditions for \( L \) are formulated. The main observation is
that operator \( \tilde{L} := x(1-x)L \) has regular (Fuchsian) singularities at the two boundary points \( x = 0 \) and \( x = 1 \), hence may be studied by standard ODE methods. This is something of a red herring, as it is the resolvent operator \((\lambda - L)^{-1}\) that governs the behavior of the heat kernel; the resolvents of \( L \) and \( \tilde{L} \) are fundamentally different. The simple vanishing of the coefficient of the second order term in \( \tilde{L} \) plays a crucial role in the analytic properties of the solutions to the associated heat equation and their applications in stochastic processes and population genetics. In particular, the diffusion processes associated to \( L \) are qualitatively quite different from those associated to \( \tilde{L} \).

We define the notion of indicial roots for \( L \). The complex number \( s \) is called an indicial root for this operator at \( x = 0 \) if

\[
L x^s = O(x^n).
\]

Note that for any \( s \), \( L x^s = x^{s-1}c(s, x) \) where \( c \) is smooth up to the boundary, so \( s \) is an indicial root only if some leading order cancellation takes place. Indeed,

\[
L x^s = (s(s - 1) + b(0)s)x^{s-1} + O(x^s)
\]

and this leading coefficient vanishes precisely when

\[
s(s + (b(0) - 1)) = 0 \iff s = 0, 1 - b(0).
\]

There are some subtle differences in the analysis of \( L \) when \( b(0) < 1 \), \( b(0) = 1 \) or \( b(0) > 1 \). The case of principal interest in biology is when \( 0 \leq b(0) < 1 \) (and similarly, \( 0 \geq b(1) > -1 \)). For the boundary conditions we are considering it is no more difficult to handle the general case where \( b(0) \geq 0 \), and, for technical reasons, it is actually very useful to do so. The case \( b(0) < 0 \) behaves quite differently, mainly due to the loss of the maximum principle.

It is well known that if \( b(0) \in \mathbb{R}^+, b(0) \notin \{0, 1, 2, \ldots\} \), then any solution of \( Lu = 0 \) satisfies

\[
u = u_1(x) + x^{1-b(0)}u_2(x), \quad \text{where} \quad u_1, u_2 \in C^\infty([0, 1)).
\]

We call the leading coefficients \( u_1(0) \) and \( u_2(0) \) the Dirichlet and Neumann data of \( u \), respectively. If \( u \) solves the equation formally, i.e. in the sense of Taylor series, we see that if \( u_j(0) = 0 \), then \( u_j(x) \) vanishes to infinite order at \( x = 0 \); in particular, if \( u_2(0) = 0 \), then \( u(x) \in C^\infty([0, 1)) \). When \( b(0) = 0 \), the expansion has the slightly different form

\[
u = u_1(x) + x \log x u_2(x), \quad u_1, u_2 \in C^\infty([0, 1)),
\]
and we again consider \( u_1(0) \) and \( u_2(0) \) as the Dirichlet and Neumann data, so \( u \in C^\infty([0,1]) \) if and only if \( u_2(0) = 0 \). If \( b(0) \in \mathbb{N} \), then there is a regular solution \( u_1 \), with \( u_1(0) \) non-vanishing, and a singular solution, \( u_2 \) of the form

\[
u_2(x) = x^{1-b(0)}u_2r(x) + \log x u_2l(x), \quad u_2r, u_2l \in C^\infty([0,1]),
\]  

with both \( u_2r(0) \) and \( u_2l(0) \) non-zero. A solution is regular at zero if and only if it is bounded as \( x \to 0^+ \).

There is an almost identical formal analysis of the asymptotics of solutions to \( Lu = f \) where \( f \in C^\infty([0,1]) \), and for each of these cases we have the

**Proposition 1.** Let \( u \) be a solution to \( Lu = f \) where \( b(0) \geq 0 \), \( b(1) \leq 0 \), and \( f \in C^\infty([0,1]) \). Then \( u \) is smooth up to \( x = 0 \) if \( u_2(0) = 0 \).

Later in this paper we also consider the adjoint \( L^t \) of the operator \( L \), defined formally by

\[
\int_0^1 (Lu)v \, dx = \int_0^1 u(L^tv) \, dx, \quad u, v \in C^\infty((0,1)).
\]

Integration by parts gives

\[
L^t u = \frac{d}{dx}((1-x)u) - \frac{d}{dx}(b(x)u) = \frac{d}{dx}((1-x)u) - b(x)u
\]

\[
= x(1-x)\frac{d^2}{dx^2}u + [2(1-2x) - b(x)]\frac{d}{dx}u - [2 + b'(x)]u.
\]

(11)

For historical reasons, \( \partial_t - L \) is called Kolmogorov’s backwards equation, while \( \partial_t - L^t \) is called Kolmogorov’s forward equation. A short calculation shows that the indicial roots of \( L^t \) at \( x = 0 \) are

\[
s = 0, b(0) - 1.
\]

(12)

Suppose that \( u \in C^\infty([0,1]) \). If \( b(0), -b(1) \notin \mathbb{N} \cup \{0\} \), then we see that \( \int_0^1 (Lu)v \, dx = \int_0^1 u(L^tv) \, dx \) holds if and only if

\[
\lim_{x \to 0^+} \partial_x((xv(x)) - b(0)v(x)) = 0 \quad \text{and} \quad \lim_{x \to 1^-} \partial_x((1-x)v(x) + b(1)v(x)) = 0.
\]

(13)

These limits characterize the adjoint boundary condition. Loosely speaking, a function satisfying these conditions is of the form: \( x^{b(0)-1}(1-x)^{-(1+b(1))}\tilde{v}(x) \), with \( \tilde{v} \) smooth at 0 and 1. In this connection, the boundary conditions for \( L \) are often formulated as “zero flux” conditions:

\[
\lim_{x \to 0^+} x^{b(0)}\partial_x u(x) = 0, \quad \lim_{x \to 1^-} (1-x)^{-b(1)}\partial_x u(x) = 0.
\]

(14)

We shall return to a more precise discussion of the adjoint operator in §[12]
4 Reduction to model problems on the half-line

The first step in the construction of the parametrix is to localize the analysis to neighborhoods of the boundary points. This leads us to study certain model problems, which are the focus of much of the rest of the paper. Indeed, we establish a maximum principle, and also derive explicit formulae for the fundamental solutions of these model operators, and using these results, prove sharp mapping properties for the heat kernels of these model problems. Finally, by perturbation and standard parametrix methods, we prove analogous results for general Wright-Fisher operators.

Our goal is the construction of the “heat kernel” or solution operator for the initial value problem for the diffusion equation:

\[ \frac{\partial u}{\partial t} - \left[ x(1-x)\partial_x^2 + b(x)\partial_x \right] u = 0 \text{ with } u(x,0) = f(x), \tag{15} \]

where we recall that an explicit change of variable has reduced the coefficient to \( \partial_x^2 \) to this normal form. The function \( b \) is assumed to be smooth on \([0,1]\), with \( b(0) \) and \(-b(1)\) non-negative. As above, we let \( L \) denote the differential operator

\[ Lu = \left[ x(1-x)\partial_x^2 + b(x)\partial_x \right] u. \]

The additional requirements that

\[ u \in C^0([0,1] \times [0,\infty)) \cap C^1([0,1] \times (0,\infty)) \text{ and,} \]

\[ \lim_{x \to 0^+,1^-} x(1-x)\partial_x^2 u(x,t) = 0 \text{ for } t > 0, \tag{16} \]

assure the uniqueness of the solution. This follows from a maximum principle for super-solutions.

**Proposition 2.** Let \( u \) satisfy (16), and \((\partial_t - L)u \leq 0\) in \([0,1] \times (0,\infty)\); for any \( T > 0 \), let \( D_T = [0,1] \times [0,T] \). Then

\[ \max_{\{(x,t) \in D_T\}} u(x,t) = \max_{\{x \in [0,1]\}} u(x,0). \]

**Proof.** Let \( \epsilon > 0 \), then \( u_\epsilon = u + \epsilon(1+t)^{-1} \) is a strict super-solution. The standard argument (see [8]) then shows that the maximum of \( u_\epsilon \) occurs along the distinguished boundary \( \partial' D_T = \partial D_T \setminus [0,1] \times \{T\} \). The regularity hypotheses in (16) show that the maximum cannot occur where \( x = 0 \) or 1. For example, if the maximum occurred at \((0,t_0)\), then \( \partial_t u_\epsilon(0,t_0) = 0 \), and \( \partial_x u_\epsilon(0,t_0) \leq 0 \). The conditions \( b(0) \geq 0 \), and

\[ \lim_{x \to 0^+} x(1-x)\partial_x^2 u_\epsilon(x,t_0) = 0, \]
contradict the fact that $u_\epsilon$ is a strict super-solution. Thus

$$\max_{\{(x,t)\in D_T\}} u(x,t) < \max_{\{(x,t)\in D_T\}} u_\epsilon(x,t) = \max_{\{x\in[0,1]\}} u_\epsilon(x,0) < \max_{\{x\in[0,1]\}} u(x,0) + \epsilon.$$  

This estimate holds for any $\epsilon > 0$, which completes the proof of the proposition.

The transformation

$$y = \sin^2 \sqrt{x}$$

maps the interval $[0,(\pi/2)^2]$ bijectively onto $[0,1]$, is regular at $x = 0$, but becomes singular at $(\pi/2)^2$. In the $x$-variable, the operator $L_{WF} = y(1 - y)\partial_y^2$ transforms to

$$L_{WF} = x\partial_x^2 + x\tilde{c}(x)\partial_x. \quad (17)$$

Here $\tilde{c}(x)$ is a real analytic function in a neighborhood of $x = 0$, which includes $[0,(\pi/2)^2)$, with $\tilde{c}(0) = -1/3$. More generally, $L$ is carried to

$$L = x\partial_x^2 + x\tilde{c}(x)\partial_x + \tilde{c}(x)\partial_x, \quad (18)$$

where $\tilde{c}$ is again a smooth function which satisfies

$$\tilde{c}(0) = b(0).$$

The coefficient $\tilde{b}$ of the first order term has the form

$$\tilde{b}(x) = b_0 + xc(x), \text{ where } b_0 = b(0).$$

This suggests introducing the half-line model operators

$$L_b = x\partial_x^2 + b\partial_x.$$  

In this coordinate the operator $L$ is represented in the form

$$L = L_{b_0} + xc(x)\partial_x.$$  

The first order term $xc(x)\partial_x$ is, in a precise sense, a lower order perturbation of $L_{b_0}$. We show, in the succeeding sections, how to construct the solution operator for (15) with regularity conditions (16), from the solution operators for $L_b$. The next several sections are devoted to the analysis of these solution operators.
5 Maximum principles and uniqueness for the model operators

For the heat equation on Euclidean space, the maximum principle and the resulting uniqueness of solutions on \( \mathbb{R}^n \times [0, \infty) \), satisfying given initial conditions, requires that we impose a growth hypothesis on solutions at \(|x| \to \infty\). In this section we establish analogous results for the model operators \( L_b, b \geq 0 \).

For any \( T > 0 \), we consider solutions on the domains

\[
D_T = [0, \infty) \times [0, T].
\]

**Proposition 3.** Suppose that \( b \geq 0 \), and

\[
u \in C^0(D_T) \cap C^1([0, \infty) \times (0, T]) \cap C^2((0, \infty) \times (0, T])
\]

is a super-solution for the heat equation associated to \( L_b \), so \( \partial_t u - L_b u \leq 0 \).

Suppose further that

\[
\lim_{x \to 0+} x \partial_x^2 u(x, t) = 0, \text{ for } 0 < t \leq T,
\]

and that for some \( a, M > 0 \), \(|u(x, t)| \leq Me^{ax}\) for all \((x, t) \in D_T\); then

\[
\sup_{(x, t) \in D_T} u(x, t) = \sup_{x \in [0, \infty)} u(x, 0).
\]

**Proof.** For \( k \) a non-negative integer, define the function

\[
v_{\tau, k}(x, t) = \frac{1}{(\tau - t)^k} e^{\frac{x}{\tau - t}} = \partial_x^k v_{\tau, 0}.
\]

Clearly \( v_{\tau, k}(x, t) > 0 \), is smooth up to \( x = 0 \), and one readily checks that \((\partial_t - L_k) v_{\tau, k} = 0\). We shall use the \( v_{\tau, k} \) as barriers.

The proof is much the same as in the Euclidean case. We first treat the case \( b = k \), a non-negative integer. Choose \( \epsilon_1, \epsilon_2 > 0 \) and \( \tau \in (0, 1/a) \), and consider the functions

\[
\begin{align*}
u_e(x, t) &= u(x, t) - \epsilon_1 v_{\tau, b}(x, t) + \frac{\epsilon_2}{1 + t}.
\end{align*}
\]

These are defined in \( D_\tau \cap D_T \) and satisfy

\[
\partial_t u_e - L_0 u_e < 0
\]

there, hence are strict supersolutions. Our choice of \( \tau \) ensures that \( v_{\tau, b} \to +\infty \) more rapidly than \( e^{ax} \) as \( x \to \infty \), so \( u_e < 0 \) for \( x \) sufficiently large, uniformly for all \( t \in [0, \tau] \); therefore, the standard proof of the maximum principle shows that:

\[
u_e(x, t) \leq \sup_{(x, t) \in \partial(D_\tau \cap D_T)} u_e(x, t) \quad \text{for } (x, t) \in D_\tau \cap D_T,
\]
where $\partial'D_T$ is the distinguished boundary $([0, \infty) \times \{0\}) \cup \{(0) \times [0,T)\}$. Clearly $\sup u_\epsilon$ is attained at some point in $D_T$.

First observe that the maximum of $u_\epsilon(x, t)$ cannot occur along $x = 0$ with $t > 0$. Indeed, suppose that the maximum occurs at $x = 0$, for some $t_0 \in (0, T)$. If $b = 0$, then the regularity of $u_\epsilon$ and (21) give $\partial_t u_\epsilon(0, t) < 0$, which contradicts that $u_\epsilon$ reaches a maximum at $(0, t_0)$. On the other hand, if $b > 0$, then (21) would imply that $\partial_t u_\epsilon(0, t_0) = 0$ and $\partial_x u_\epsilon(0, t_0) > 0$, which is also a contradiction since it shows that $u_\epsilon(0, t_0) < u_\epsilon(x, t_0)$ for $x$ small.

If the maximum were to occur at $u(x, T)$, then $\partial_t u_\epsilon(0, t_0) \geq 0$, so as before, $\partial_x u_\epsilon(0, t_0) > 0$ and hence $u_\epsilon(0, T) < u_\epsilon(x, T)$, for small positive $x$. Hence the maximum does not occur along $x = 0$. Note that this argument relies strongly on the assumption of $C^1$ regularity up to $x = 0$, as well as (20).

All of this shows that we can replace (22) by

$$u_\epsilon(x, t) \leq \sup_{x \in [0, \infty)} u_\epsilon(x, 0) \leq \sup_{x \in [0, \infty)} u(x, 0) + \epsilon_2.$$ 

Now letting $\epsilon_1, \epsilon_2 \to 0$, we conclude that

$$\sup_{(x, t) \in D_T} u(x, t) \leq \sup_{x \in [0, \infty)} u(x, 0). \quad (23)$$

If $\tau \geq T$, then this completes the proof of the proposition; otherwise repeat the argument recursively, with $u_1(x, t) = u(x, t + \tau)$ replacing $u(x, t)$. Finitely many iterates produce the desired conclusion.

We now turn to non-integral values of $b$. If $k \leq b < k + 1$, for a non-negative integer $k$, then

$$(\partial_t - L_b)(u - \epsilon_1 v_{\tau, k+1} + \frac{\epsilon_2}{1 + t}) =
(\partial_t - L_b)u + \epsilon_1 (b - (k + 1)) v_{\tau, k+2} - \frac{\epsilon_2}{(1 + t)^2} < 0.$$

The inequality uses that $v_{\tau, k+2} > 0$. We can argue exactly as before with

$$u_\epsilon(x, t) = u - \epsilon_1 v_{\tau, k+1} + \frac{\epsilon_2}{t + 1},$$

to obtain (23), and iterate, if needed, to complete the proof of the proposition. 

The uniqueness of classical, “tempered” solutions is an immediate corollary:
Corollary 1. Let \( u \) satisfy (19), (20), and be a solution to \( \partial_t u - L_b u = 0 \) for some constant \( b \geq 0 \). If \( u(x,0) = 0 \) and \( u(x,t) \) satisfies
\[
|u(x,t)| \leq M e^{ax} \text{ for } (x,t) \in D_T
\]
for some \( a, M > 0 \), then \( u \equiv 0 \) in \( D_T \).

Remark 2. The discussion of indicial roots in Section 3 shows that other solutions to the initial value problem for \( \partial_t - L_b \), have an asymptotic expansion at \( x = 0 \) with the term \( x^{1-b} \), so even if the initial data is smooth, those solutions are not \( C^1 \) up to \( x = 0 \) and do not satisfy (20). Following Feller, the local boundary condition that singles out the smooth solution is the zero flux condition:
\[
\lim_{x \to 0^+} x^b \partial_x u(x,t) = 0.
\]

6 Fundamental solutions for the model operator

Consider the heat equation
\[
\partial_t u = L_b u, \quad x \geq 0, \quad u(0,x) = f(x)
\]
with boundary condition at \( x = 0 \) for \( t > 0 \) dictated by the demand that \( u(t,x) \in C^m \) up to \( x = 0 \) for all \( m \in \mathbb{N} \) provided the initial condition \( f \) also lies in \( C^m \) (and has moderate growth as \( x \to \infty \)). This is clearly needed for the solution operator to (26) to define a semi-group on \( C^m \).

The goal in this section is to find an explicit expression for the fundamental solution of this problem. Our strategy is as follows: We first study the Fourier representation of the heat kernel for \( L_0 \); this turns out to be somewhat simpler to analyze, and it suggests the general form of the corresponding kernel when \( b > 0 \). It is useful to have both a Fourier representation and the explicit Schwartz kernels of these operators, so we include material describing both approaches.

Remark 3. We would again like to thank Charlie Fefferman for showing us his construction of the kernel \( k_0^0(x,y) \), essentially (28), on which all our subsequent development is based. This kernel also appears in [9].

6.1 Fundamental solution for \( \partial_t - L_0 \)

We begin by seeking a function \( E^0(x,\xi,t) \), which is a solution to \( \partial_t E^0 = L_0 E^0 \), with initial condition \( E^0(x,\xi,0) = e^{ix\xi} \). A first guess is that \( E^0 \) should involve a Gaussian, but because of the natural appearance of the variable \( \sqrt{x} \) in this problem (or, equivalently, the fact that the dilation \( (t, x) \mapsto (\lambda t, \lambda x) \) preserves the equation...
under consideration), we are led to the ansatz $E_0^0(x, \xi, t) = \exp(x\phi(\xi, t))$ for some as yet unknown function $\phi$. Inserting this into the equation leads to the initial value problem

$$\partial_t \phi = \phi^2, \quad \phi(\xi, 0) = ix\xi,$$

which yields that

$$E_0^0(x, \xi, t) = \exp\left(\frac{-x}{t + i\xi - 1}\right) = \exp\left(\frac{ix\xi}{1 - it\xi}\right).$$

As an oscillatory integral, the corresponding Schwartz kernel is

$$k_0^0(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0^0(x, \xi, t)e^{-iy\xi} d\xi.$$  

(28)

With a bit of algebra, one sees that the integrand decreases exponentially as a function of $\text{Im} \ \xi$ when $x, t > 0$ and $y < 0$. Thus, a contour deformation shows that $k_0^0$ vanishes in this region. In other words, $k_0^0(x, \cdot)$ is supported in $y \geq 0$ when $x, t > 0$.

For the special case $b = 0$, the condition $u(0, t) = 0$ defines a $C_0^0$-semi-group, which turns out to be a sub-semi-group of that defined by $k_0^0$. We call this the “Dirichlet semi-group.” Since $k_0^0(0, y) = \delta(y)$, the kernel we have constructed is not the Dirichlet heat kernel, but can be obtained by a simple modification:

$$k_{t}^{0,D}(x, y) = k_0^0(x, y) - e^{-x/t}\delta(y).$$

(29)

In other words, the fundamental solution for (26) with Dirichlet boundary conditions at $x = 0$ is

$$k_{t}^{0,D}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{(ix\xi)/(1-it\xi)-iy\xi} - e^{-x/t-iy\xi}\right) d\xi.$$  

(30)

The integrand here again diverges like $1/\xi$ as $|\xi| \to \infty$, so we interpret (30) as an oscillatory integral via the regularization (obtained by a formal integration by parts)

$$k_{t}^{0,D}(x, y) = -\left(\frac{x}{y}\right) \int_{-\infty}^{\infty} \frac{E(x, \xi; t)e^{-iy\xi}}{(t\xi + i)^2} \frac{d\xi}{2\pi}.$$  

(31)

This satisfies the same heat equation, but now vanishes $x = 0$, so is indeed the Dirichlet heat kernel.

It is possible to find an explicit expression for $k_{t}^{0,D}$ in terms of elementary transcendental functions. First note that $k_{t}^{0,D}$ is a function of

$$\alpha = \frac{x}{t} \text{ and } \beta = \frac{y}{t}.$$
Indeed, setting $\eta = \xi t$, and $z = \eta + i$, we find that

$$k^D_t(x, y) = -\left(\frac{\alpha}{t\beta}\right)\int_{-\infty}^{\infty} e^{\frac{-\alpha \eta}{\eta + i}} \frac{d\eta}{2\pi} = -e^{-(\alpha + \beta)} \left(\frac{\alpha}{t\beta}\right) \int_1^{\Gamma_1} \frac{e^{\frac{i\alpha}{z} - i\beta z}}{z^2} \frac{dz}{2\pi},$$

where, by definition, $\Gamma_\tau = \{z : \text{Im } z = \tau\}$. By an elementary contour deformation,

$$\int_{\Gamma_1} \frac{e^{\frac{i\alpha}{z} - i\beta z}}{z^2} \frac{dz}{2\pi} = \int_{\Gamma_\tau} \frac{e^{\frac{i\alpha}{z} - i\beta z}}{z^2} \frac{dz}{2\pi} - \int_{|z|=1} \frac{e^{\frac{i\alpha}{z} - i\beta z}}{z^2} \frac{dz}{2\pi}$$

for any $\tau < 0$. The first term on the right vanishes since it can be made arbitrarily small by letting $\tau \to -\infty$, so it suffices to compute the integral on the unit circle. For this, note that the coefficient of $1/z$ in

$$z^{-2} \exp(i\alpha/z) \times \exp(-i\beta z) = z^{-2} \sum_{j=0}^{\infty} \frac{1}{j!} (i\alpha/z)^j \sum_{k=0}^{\infty} \frac{1}{k!} (-i\beta z)^k$$

is equal to

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell! (\ell + 1)!} (-i\beta)^{\ell+1} (i\alpha)^{\ell} = -i\beta \sum_{\ell=0}^{\infty} \frac{(\alpha\beta)^{\ell}}{\ell!(\ell + 1)!} = -i\beta I_1(2\alpha\beta),$$

where $I_1$ is the modified Bessel function of order 1. This yields, finally, the explicit formula

$$k^D_t(x, y) = \frac{1}{t} e^{-\frac{x+y}{t}} \sqrt{\frac{x}{y}} I_1 \left(\frac{2\sqrt{xy}}{t}\right)$$

for the Dirichlet fundamental solution of the model heat equation. It is useful to represent this kernel in an alternate form. For $b > 0$ we define the entire functions

$$\psi_b(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(j + b)},$$

An elementary calculation shows that $\psi_b$ satisfies the ordinary differential equation:

$$z\psi''_b + b\psi'_b - \psi_b = 0$$

The Dirichlet heat kernel is then given by

$$k^D_t(x, y) dy = \left(\frac{x}{t}\right) e^{-\frac{x+y}{t}} \psi_2 \left(\frac{xy}{t^2}\right) \frac{dy}{t^2}.$$
The kernel $k^0_t$ can be expressed as

$$k^0_t(x, y) = e^{-\frac{x}{2}} \delta(y) + k^{0,D}_t(x, y). \quad (34)$$

The relationship between $\psi_2$ and the $I$-Bessel function implies the asymptotic expansion:

$$\psi_2(z) \sim e^{\frac{2\sqrt{z}}{\sqrt{4\pi z^3}}} \left[ 1 + \sum_{j=1}^{\infty} \frac{c_{2,j}}{z^{j}} \right].$$

See [6, 8.451.5]

### 6.2 Fundamental solution for $\partial_t - L_b$

We now undertake a similar analysis of the fundamental solution for the problem (26) for $0 < b$. As explained in §3 the boundary condition at $x = 0$ which should guarantee that solutions are smooth up to $x = 0$ is the analogue of the Neumann condition, i.e. the one which excludes the term $x^{1-b}$. We denote by $k^b_t(x, y)$ the heat kernel for this problem with this zero-flux conditions.

As before, we first determine the Fourier representation of $k^b_t$ using the ansatz that $E^b(t, x, \xi) = \psi(t \xi) e^{-\frac{ix}{2} \phi(t, \xi)}$; this leads fairly directly to the expression

$$E^b(t, x, \xi) = (1 - \frac{it}{2} \xi)^{-b} e^{\frac{ix}{2} \phi(t, \xi)}, \quad (35)$$

and hence, (when $b \leq 1$) as an oscillatory integral,

$$k^b_t(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{ixy}{2} - \frac{iy}{2} \xi} \frac{(1 - it \xi)^{-b}}{1 - it \xi} d\xi. \quad (36)$$

If $f \in C^0_0([0, \infty))$ has an absolutely integrable Fourier transform, then this formula can be interpreted to mean that

$$\int_0^\infty k^b_t(x, y) f(y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{ixy}{2} - \frac{iy}{2} \xi} \frac{\hat{f}(\xi)}{1 - it \xi} d\xi. \quad (37)$$

In order to evaluate (36), we start off as before, setting $\alpha = x/t$, $\beta = y/t$, and expressing (36) as a contour integral over $\text{Im } z = 1$:

$$k^b_t(x, y) = \frac{e^{\pi ib}}{2\pi t} \lim_{R_1, R_2 \to \infty} \lim_{R_1 \to -i} \int_{R_1 + i}^{R_2 + i} e^{i(\frac{x}{2} - \beta z)} z^{-b} dz.$$
To define \( z^{-b} \), we take \( \arg z = 0 \) on the positive real axis and consider \( z^{-b} \) as defined on the plane cut along the negative imaginary axis.

Now change variables, setting \( z = \sqrt{\alpha/\beta} \tau \), and for simplicity write \( \zeta = \sqrt{\alpha/\beta} \), to write

\[
\int_{\text{Im } \tau = 1} e^{i \zeta (\frac{1}{2} - \tau)} \tau^{-b} d\tau.
\]

Since the integrand decays as \( |\text{Re } \tau| \to \infty \) and as \( \text{Im } \tau \to -\infty \), we can deform the contour to yield

\[
\Lambda = \{ ye^{\frac{3\pi i}{2}} : y \in (\infty, 1] \} \cup \{ e^{i\theta} : \theta \in \left[ \frac{3\pi}{2}, -\frac{\pi}{2} \right] \} \cup \{ ye^{-\frac{\pi i}{2}} : y \in [1, \infty) \}.
\]

Inserting the explicit parameterizations of the various parts of this contour into the integrand \( e^{i \zeta (\frac{1}{2} - \tau)} \frac{d\tau}{\tau^{b}} \) and simplifying yields the absolutely convergent representation:

\[
\int_{\Lambda} e^{\frac{\pi i b}{2}} e^{i \zeta (\frac{1}{2} - \tau)} \frac{d\tau}{\tau^{b}} = \int_{-\pi}^{\pi} e^{2 \zeta \cos \phi} \cos(b-1)\phi d\phi - \sin \pi(b-1) \int_{1}^{\infty} e^{-\zeta(y+\frac{1}{2})} \frac{dy}{y^{b}}.
\]

Changing the variable to \( y = e^{t} \) in the second integral on the right, we get

\[
2 \int_{0}^{\pi} e^{2 \zeta \cos \phi} \cos(b-1)\phi d\phi - 2 \sin \pi(b-1) \int_{0}^{\infty} e^{-2 \zeta \cosh t} e^{-t(b-1)} dt.
\]

This combination of integrals appears as Formula 8.431.5 in [6], and is seen to equal \( 2\pi I_{b-1}(2\zeta) \). Putting all of these calculations together gives the two equivalent expressions

\[
k_{i}^{b}(x, y) = \frac{1}{t} \left( \frac{x}{y} \right)^{\frac{1-b}{2}} e^{-\frac{x+y}{t}} I_{b-1} \left( 2 \sqrt{\frac{xy}{t^{2}}} \right)
\]

and

\[
k_{i}^{b}(x, y) dy = \left( \frac{y}{t} \right)^{b} e^{-\frac{x+y}{t}} \psi_{b} \left( \frac{xy}{t^{2}} \right) \frac{dy}{y}.
\]

Using the classical asymptotics for \( I_{b-1} \) we see that as \( z \to \infty \), \( \psi_{b} \) has the asymptotic expansion

\[
\psi_{b}(z) \sim \frac{z^{\frac{1}{2} - \frac{b}{2}} e^{2\sqrt{z}}}{\sqrt{4\pi}} \left[ 1 + \sum_{j=1}^{\infty} c_{b,j} \frac{1}{z^{j/2}} \right].
\]
From these explicit formulæ it is evident that the kernels $k^b_t(x, y)$ are pointwise positive. Once it is shown that they generate semi-groups, it follows that these semi-groups are positivity improving. The Fourier representation has a very useful consequence:

**Lemma 1.** If $f \in C^m_c([0, \infty))$ satisfies $\partial^j_x f(0) = 0$, for $1 \leq j \leq m$, then

$$\partial^j_x \int_0^\infty k^b_t(x, y) f(y) \, dy = \int_0^\infty k^{b+j}_t(x, y) \partial^j_y f(y) \, dy, \text{ for } 1 \leq j \leq m.$$  

**Proof:** This follows easily from (37), as the hypothesis implies that we can differentiate under the integral sign. The vanishing conditions on $\partial^j_x f(0)$ give that

$$\hat{\partial^j_x f}(\xi) = (i\xi)^j \hat{f}(\xi), \text{ for } 1 \leq j \leq m-1.$$  

If $\xi^m \hat{f}(\xi) \in L^1(\mathbb{R})$, then we can immediately extend the formula to $j = m$. In general, approximate $f$ in the $C^m$-norm by a sequence $\{f_n\} \subset C^m$, which satisfies $\partial^j_x f(0) = 0$, for $1 \leq j \leq m$ and this additional integrability hypothesis. A limiting argument then establishes the truth of the lemma for $j = m$ in the stated generality.

**Remark 4.** If $0 < b < 1$, then, as shown in [5], one can define the Dirichlet problem for $\partial_t - L_b$. For this range of parameters this corresponds to having a non-zero coefficient of $x^{1-b}$, hence a non-smooth solution. For $b \geq 1$, Feller showed that (25) is the unique local boundary condition that defines a positivity preserving semigroup.

### 6.3 Representation formula

As a first application of the existence of these fundamental solutions, we prove a representation formula for the solution to the initial value problem

$$\partial_t u = L_b u \text{ with } u(x, 0) = f(x),$$  

where $f$ and $u$ satisfy (24) and in addition,

$$\lim_{x \to 0^+} x^b \partial_x u(x, t) = 0 \text{ for } b > 0, \quad \lim_{x \to 0^+} x \partial_x u(x, t) = 0 \text{ if } b = 0.$$  

Note that if $u$ is in $C^1(D_T)$, then it automatically satisfies this boundary condition.
**Proposition 4.** Suppose that \( u \in C^0(D_T) \cap C^2((0, \infty) \times (0, T)) \) satisfies (41), (42). If \( f, u, \partial_t u \) and \( \partial_x u \) satisfy (24) then for all \( t < a^{-1} \), \( u \) is given by the absolutely convergent integral:

\[
u(x, t) = \int_0^\infty k^b_t(x, y) f(y) dy.
\]

**Proof.** For \( b > 0 \), the proof of this proposition is a simple integration by parts argument. It uses the fact that \( k^b_t \) also satisfies the adjoint equation and boundary condition: if \( t > 0 \), then

\[
\partial_t k^b_t(x, y) = (\partial^2_y y - b\partial_y)k^b_t(x, y) \quad \text{and} \quad \lim_{y \to 0^+} [\partial_y y k^b_t(x, y) - bk^b_t(x, y)] = 0.
\]

Moreover, it is not difficult to show that as \( y \to \infty \), with \( x \) in a compact set, we have

\[
k^b_t(x, y) \leq Cy^{2b-3}e^{-\frac{y}{t}}.
\]  \( \tag{43} \)

With these preliminaries, we can integrate the equation satisfied by \( u \) to obtain

\[
0 = \int \int_0^{t-\epsilon} k^b_t(x, y) (\partial_s u(y, s) - Lu(y, s)) dy ds.
\]

Here \( \epsilon \) is a small positive number. Provided \( t < a^{-1} \), the estimates satisfied by \( u, \partial_t u, \partial_x u, \) and (43) justify the subsequent manipulations of this integral.

We first integrate by parts in \( s \) to obtain:

\[
\int_0^\infty [u(t-\epsilon, y)k^b_t(x, y) - u(0, y)k^b_t(x, y)] dy = \int_0^{t-\epsilon} \int_0^\infty [Lb u(s, y)k^b_t(x, y) - u(s, y)\partial_t k^b_t(x, y)] ds dy.
\]

Now integrate by parts in \( y \) using the boundary conditions satisfied by \( u \) and \( k^b_{t-s} \) at \( y = 0 \), and the estimates (24) and (43), to conclude that since \( t < a^{-1} \), then

\[
\int_0^\infty [u(t-\epsilon, y)k^b_t(x, y) - u(0, y)k^b_t(x, y)] dy = \int_0^{t-\epsilon} \int_0^\infty u(s, y) \left[ Lb k^b_t(x, y) - \partial_t k^b_t(x, y) \right] ds dy.
\]
For any $\epsilon > 0$, the right hand side of this equation vanishes identically, and therefore
\[ \int_0^\infty u(t - \epsilon, y)k^b_t(x, y)dy = \int_0^\infty u(0, y)k^b_t(x, y)dy. \]

Letting $\epsilon \to 0$ gives the result.

For $b = 0$ we proceed a little differently. If we write $u = u_0 + f(0)$, then $u_0$ also satisfies $(\partial_t - L_0)u_0 = 0$, $u_0(0, t) = 0$, and the same regularity conditions as those satisfied by $u$. Arguing as above,
\[ u_0(x, t) = \int_0^\infty k^0_D(t, y)(f(y) - f(0)) dy, \]
and hence
\[ u(x, t) = \int_0^\infty k^0_t(x, y)f(y) dy. \]

As a special case we can demonstrate that the kernels $\{k^b_t : t > 0\}$ have the semi-group property.

**Corollary 2.** If $t$, $s$ are positive numbers and $b > 0$, then
\[ k^b_{t+s}(x, y) = \int_0^\infty k^b_t(x, z)k^b_s(z, y)dz. \]

**Proof.** For fixed $y$, the kernel $k^b_t(\cdot, y)$ satisfies $\partial_t k^b_t(x, y) - L_b k^b_t(x, y) = 0$, and is $C^\infty$ in $[0, \infty) \times (0, \infty)$. This solution decays exponentially as $x \to \infty$, so the maximum principle implies $s \mapsto k^b_{t+s}(x, y)$ is the unique smooth solution of this PDE with respect to the variables $(s, x)$, with initial data $k^b_0(\cdot, y)$. The proof now follows from Proposition (4). \qed

**Remark 5.** A similar argument using the uniqueness of the solution to the Dirichlet problem when $b = 0$ leads to a proof that
\[ k^{0,D}_{t+s}(x, y) = \int_0^\infty k^{0,D}_t(x, z)k^{0,D}_s(z, y)dz. \]
It is also true that if \( f \in C^\infty_c([0, \infty)) \), then
\[
\int_0^\infty k^0_{t+s}(x,y)f(y)dy = \int_0^\infty k^0_t(x,z) \left[ \int_0^\infty k^0_s(z,y)f(y)dy \right] dz.
\]

We conclude this section with:

**Proposition 5.** If \( f \in C^0([0, \infty)) \) has an absolutely integrable Fourier transform, then
\[
\lim_{b \to 0} \int_0^\infty k^b_t(x,y)f(y)dy = \int_0^\infty k^0_t(x,y)f(y)dy.
\]

This convergence is obvious using the Fourier representations (28) and (35) of the kernels \( k^b_t \) and \( k^0_t \). If \( \hat{f} \) is absolutely integrable, then, for any fixed \( 0 < T \) the convergence is uniform on \([0, \infty) \times [0, T]\).

### 7 Mapping properties for the model operators

We can now prove the key fact that the diffusions associated to the operators \( L_b \) are semi-groups, which, for every \( m \in \mathbb{N} \cup \{0\} \), preserve the spaces \( \dot{C}^m([0, \infty)) \), see Remark 6. We prove this in two steps. The first is to show that the kernels map polynomials to polynomials; then, decomposing an arbitrary \( C^m \) initial condition into a polynomial (its Taylor series) and a remainder term that vanishes to order \( m \), the fact that the solution with initial condition given by this remainder term is also \( C^m \) follows from the maximum principle.

Suppose that the initial data is a polynomial, \( f(x) = \sum_{j=0}^n a_j x^j \). It is clear that the formal exponential
\[
u = e^{tL_b}f(x) = \sum_{\ell=0}^\infty \frac{t^\ell}{\ell!} L_b^\ell f(x)
\]
makes sense and in fact is a polynomial which is the unique moderate growth solution of (41). We simply observe that \( L_b^\ell x^j \) is a constant multiple of \( x^j \ell \), hence this vanishes as soon as \( \ell > j \). Thus the sum on the right above is finite and is a polynomial in \((x,t)\). It solves the equation by the usual elementary calculation. Therefore, by the uniqueness theorem, this is the only (exponentially bounded) solution of this problem, and hence, by the representation formula, we must also have that
\[
u(x,t) = \int_0^\infty k^b_t(x,y)f(y)dy.
\]

We now turn to the case of general \( C^m \) data.
**Proposition 6.** Suppose that \( f \in C^m([0, \infty)) \) has compact support. Then
\[
    u(x, t) = \int_0^\infty k^b_t(x, y) f(y) \, dy \in C^0([0, \infty); C^m([0, \infty)_x)),
\]
solves the initial value problem (26). The norm of the difference, \( \|u(\cdot, t) - f\|_{C^m} \) tends to zero as \( t \to 0 \), and therefore \( k^b_t \) extends to define a \( C^m \)-semi-group for each \( m = 0, 1, 2, \ldots \).

**Remark 6.** We let \( \dot{C}^m([0, \infty)) \) denote the closed subspace of \( C^m([0, \infty)) \) consisting of functions \( f \) with
\[
    \lim_{x \to \infty} \partial_j^x f(x) = 0 \text{ for } 0 \leq j \leq m.
\]
This is the closure of \( C^m_c([0, \infty)) \) in the \( C^m \)-norm.

To prove the proposition, the following two lemmas are useful

**Lemma 2.** For \( b \geq 0 \) and \( 0 < \ell < L \), there is a constant \( C_b \) so that if \( f \in C^m_c([0, L)) \), then, for \( x > L \), and \( t < 1 \), we have the estimate
\[
    \left| \int_0^\infty k^b_t(x, y) f(y) \, dy \right| \leq C_b \|f\|_{\infty} \sqrt{L} \left( \frac{L}{x} \right)^{\frac{3}{4}} e^{-\sqrt{\frac{(\sqrt{\ell} - \sqrt{L})^2}{4t}}}.
\]

If \( f \in C^0(l, \infty) \), is of tempered growth, then for \( x < l \), and \( t < 1 \),
\[
    \left| u(x, t) \right| \leq C_b, N \|f\|_{(N)} \frac{e^{-\frac{(\sqrt{\ell} - \sqrt{x})^2}{4t}}}{t^b}.
\]
Here \( N \in \mathbb{N} \) is chosen so that
\[
    \|f\|_{(N)} := \sup_{0 < x} (1 + |x|)^{-N} |f(x)| < \infty.
\]

The proofs are elementary using the asymptotic expansion for \( \psi_b \) and are left to the reader. It is also useful to have the following

**Lemma 3.** If \( b \geq 0 \), and \( f \in C^m_c((0, \infty)) \), then
\[
    u(x, t) = \int_0^\infty k^b_t(x, y) f(y) \, dy,
\]
solves (26) and
\[
    \lim_{t \to 0^+} \|u(\cdot, t) - f\|_{C^m} = 0.
\]
Proof. It is clear that $\partial_t u - L_b u = 0$ in $[0, \infty) \times (0, \infty)$. Suppose that $\text{supp} f \subset [l, L]$. The previous lemma shows that for any $\eta > 0$, the functions $u(\cdot, t)(1 - \chi_{[l-\eta, L+\eta]}(\cdot))$ converge uniformly to zero as $t \to 0^+$. If $\eta < l/2$, then $xy > l^2/2$, for $(x, y) \in [l - \eta, L + \eta] \times \text{supp} f$, and we can therefore use the asymptotic expansion for $\psi_b$ to conclude that $u(\cdot, t)\chi_{[l-\eta, L+\eta]}(\cdot)$ converges uniformly to $f$. As $f \in C^m_c((0, \infty))$, we know from Lemma[1] that

$$\partial_k^l u(x, t) = \int_0^\infty k_l^b(x, y)\partial_k^l f(y)dy, \text{ for } j \leq m.$$ 

The argument above applies to show the uniform convergence of the derivatives $\partial_k^l u(\cdot, t)$ to $\partial_k^l f$, for $1 \leq j \leq m$. \qed

Proof of Proposition[2] Fix $m$ and $f \in C^m_c(0, \infty)$. From the expressions (40) and (34) for $k_l^b$ it is clear that, if $f \in C^0$, then $u \in C^\infty([0, \infty) \times (0, \infty))$, and $\partial_t u - L_b u = 0$ in $[0, \infty) \times (0, \infty)$. Choose a smooth cutoff function $\chi(x)$ which equals 1 near $x = 0$, $|\chi(x)| \leq 1$, and which vanishes for $x \geq 1$. Let

$$q(x) = \sum_{j=0}^m \frac{f^{(j)}(0)}{j!} x^j$$

be the Taylor polynomial of order $m$ for $f$; thus

$$\tilde{f}(x) = f(x) - \chi(x)q(x) = o(x^m), \text{ as } x \downarrow 0.$$ 

We observe that

$$u_{\chi q}(x, t) = \int_0^\infty k_l^b(x, y)\chi(y)q(y) dy$$

$$= \int_0^\infty k_l^b(x, y)q(y) dy - \int_0^\infty k_l^b(x, y)(1 - \chi(y))q(y) dy.$$ 

By the remarks above, the first term on the right is a polynomial, and therefore in $C^\infty([0, \infty) \times [0, \infty))$. On the other hand, $(1 - \chi(y))q(y)$ is supported away from $y = 0$, and it is elementary from the regularity properties of $k_l^b$ proved in the last section and Lemma[2] that

$$\int_0^\infty k_l^b(x, y)(1 - \chi(y))q(y) dy \in C^\infty([0, \infty) \times [0, \infty)),$$

and tends locally uniformly to $(1 - \chi(x))q(x)$ as $t \to 0^+$. Together with Lemma[2] this shows that $u_{\chi q}(x, t)$ tends to $\chi q$ in $C^l([0, \infty))$ for all $l \in \mathbb{N}$. Thus it remains to prove that

$$\tilde{u}(x, t) = \int_0^\infty k_l^b(x, y)\tilde{f}(y) dy$$

21
is also in $C^m$ and tends to $\tilde{f}$ in the $C^m$-norm as $t \to 0^+$.

First consider the case $m = 0$. For every $\epsilon > 0$ we can choose $\delta > 0$ so that $\tilde{f}(x) < \epsilon$ when $x < \delta$. Decompose $\tilde{f}(x) = \chi(x/\delta)f(x) + (1 - \chi(x/\delta))\tilde{f}(x)$. Applying $k^b_t$ to the second term gives a smooth function $v(x, t)$ in $C^\infty([0, \infty) \times (0, \infty))$, which satisfies $\partial_t v - L_b v = 0$. Lemma 5 shows that $v \in C^0([0, \infty) \times [0, \infty))$, and tends uniformly to $(1 - \chi(x/\delta))\tilde{f}$ as $t \to 0^+$. As $k^b_t$ is pointwise positive, and has integral 1 for all $t > 0$, we see that

$$\left| \int_0^\infty k^b_t(x, y)\chi(y/\delta)\tilde{f}(y) \, dy \right| \leq \epsilon.$$  

Since $\epsilon$ is arbitrary, this shows that

$$\limsup_{t \to 0^+} \|\tilde{u}(\cdot, t) - f\|_{C^0} = 1.$$

If $m > 0$, then $\partial^j_x \tilde{f}(0) = 0$, for $1 \leq j \leq m$, and we can apply Lemma 1 to conclude that

$$\partial^j_x \tilde{u}(x, t) = \int_0^\infty k^{b+j}_t(x, y)\partial^j_y f(y) \, dy, \text{ for } 1 \leq j \leq m.$$  

The $C^0$-argument then applies to complete the proof.

As a corollary of these results we can extend Lemma 1 to general data in $C^m_c([0, \infty))$.

**Corollary 3.** If $b \geq 0$, and $f \in C^m_c([0, \infty))$, and

$$u(x, t) = \int_0^\infty k^b_t(x, y)f(y) \, dy$$

then

$$\partial^j_x u(x, t) = \int_0^\infty k^{b+j}_t(x, y)\partial^j_y f(y) \, dy, \text{ for } 1 \leq j \leq m. \quad (44)$$

**Proof.** The Proposition implies that $u$ is a solution to $\partial_t u - L_b u = 0$, which belongs to $C^m([0, \infty) \times [0, \infty))$, and satisfies

$$\lim_{t \to 0^+} u(\cdot, t) = f,$$
with convergence in the $C^m$-norm. One can therefore differentiate the equation satisfied by $u$ to conclude that

$$
\partial_t \partial_x^j u - L_{b,j} u = 0 \text{ for } 1 \leq j \leq m, \text{ and } \lim_{t \to 0^+} \partial_x^j u = \partial_x^j f.
$$

For each $j$, the right hand side of (44) is another solution, $u_j$, to this initial value problem, that also satisfies the hypotheses of the maximum principle. Thus $u_j = \partial_x^j u$ for $1 \leq j \leq m$.

We can use this result to study the regularity properties of the operator

$$
g \mapsto K_t^b g(x) = \int_0^t \int_0^\infty k_{t-s}(x,y) g(y,s) dy ds.
$$

A case of particular importance in applications is when $g \in C^\infty([0, \infty) \times (0, \infty))$, for example $g$ is any solution to (26). We begin with a lemma.

**Lemma 4.** Suppose that $b \geq 0$, and $f \in C^m_b([0, \infty))$; then

$$
\begin{aligned}
 u(x,t) &= \int_0^\infty k_t^b(x,y) f(y) dy \\
 \partial_t^j u(x,t) &= \int_0^\infty k_t^b(x,y) L^j_b f(y) dy
\end{aligned}
$$

for $2j \leq m$ and $t > 0$.

**Proof.** Suppose that $m = 2$ and $f$ vanishes to order 2 at $x = 0$. Using the fact that $(\partial_t - (L^b_y))^2 k_t(x,y) = 0$ and a simple integration by parts argument we easily see that

$$
\partial_t u(x,t) = \int_0^\infty k_t^b(x,y) L_b f(y) dy.
$$

The second order vanishing of $f$ at 0 implies that the boundary terms at $x = 0$ vanish.

Choose a smooth function $\chi$, which equals 1 in $[0, 1]$ and is supported in $[0, 2]$. If $f$ does not vanish at $x = 0$, then we let

$$
q_2(x) = \chi(x) \left( f(0) + f'(0)x + f''(0)\frac{x^2}{2} \right), \text{ and } \bar{f} = f - q_2.
$$
From the argument above,
\[
\partial_t \int_0^\infty k^b_t(x, y) \tilde{f}(y) dy = \int_0^\infty k^b_t(x, y) L_b \tilde{f}(y) dy,
\]
and then by Proposition 6,
\[
\partial_t \int_0^\infty k^b_t(x, y) q_2(y) dy = \int_0^\infty k^b_t(x, y) L_b q_2(y) dy.
\]
This argument can be applied inductively to obtain the case of general \( m \).

Combining this lemma with Corollary 3, we conclude

**Corollary 4.** Suppose that \( b \geq 0 \), and \( f \in C^m_b([0, \infty)) \), then
\[
u(x, t) = \int_0^\infty k^b_t(x, y) f(y) dy,
\]
satisfies
\[
\partial_x^j \partial_t^k \nu(x, t) = \int_0^\infty k^{b+k}_t(x, y) L_{b+k}^j \partial_y^k f(y) dy,
\]
provided \( 2j + k \leq m \) and \( t > 0 \).

We can now examine the regularity of \( K_t^b g \).

**Proposition 7.** If \( b \geq 0 \), \( T > 0 \), and
\[
g \in C^0([0, \infty) \times [0, T)) \cap C^m_b([0, \infty) \times (0, T)),
\]
then, for \( 0 < t < T \), the derivatives \( \partial_t^j \partial_x^k K_t^b g \) are continuous for \( 2j + k \leq m \); moreover, if \( m \geq 2 \), then
\[
(\partial_t - L_b) K_t^b g = g \tag{45}
\]

**Proof.** For \( t > 2\epsilon > 0 \), we define
\[
K_{t,\epsilon}^b g(x) = \int_0^{t-\epsilon} \int_0^\infty k^b_{t-s}(x, y) g(y, s) dy ds.
\]
For $\epsilon > 0$ it follows from the corollary, that the derivatives $\partial_t^j \partial_x^k K_{t,\epsilon}^b g$ exist, provided $2j + k \leq m$ and can be expressed as

$$
\partial_t^j \partial_x^k K_{t,\epsilon}^b g(x) = \int_0^\infty \int_0^\infty \partial_t^j \partial_x^k k_{t-s}^b(x, y) g(y, s) dyds + \int_0^{t-\epsilon} \int_0^\infty k_{t-s}^{b+k}(x, y) L_{b+k}^j \partial_y^k g(y, s) dyds.
$$

If $2j + k \leq m$, then $\partial_t^j \partial_x^k K_{t,\epsilon}^b g(x)$ converges locally uniformly to a continuous function. This establishes the existence of these derivatives. If $m \geq 2$, then differentiating, for any $0 < \epsilon < t$ we have that

$$
(\partial_t - L_b) K_{t,\epsilon}^b g = \int_0^\infty k_b^0(x, y) g(y, t - \epsilon) dy.
$$

As $m \geq 2$, the limiting function $K_b^0 g$ has one time, and two spatial derivatives, which are the limits of the corresponding derivatives of $K_{t,\epsilon}^b g$. Thus letting $\epsilon \to 0^+$ in (46) gives (45). \hfill \Box

We can finally extend the convergence result Proposition 5 to arbitrary data in $\dot{C}^0([0, \infty))$.

**Proposition 8.** Fix $f \in \dot{C}^0([0, \infty))$, then

$$
\lim_{b \to 0} \int_0^\infty k_b^0(x, y) f(y) dy = \int_0^\infty k_0^0(x, y) f(y) dy.
$$

For any $T > 0$ this convergence takes place in the $C^0([0, \infty) \times [0, T])$-topology.

**Proof.** Choose a sequence $\{f_n\} \subset C_c^0([0, \infty))$ with all $\hat{f}_n \in L^1$, and such that $f_n$ converges uniformly to $f$. Let $u^b_n$ denote the solutions to (26) with initial data $\hat{f}$ and $u^b_0$ the solutions with initial values $f_n$. The maximum principle implies that, for any $n$,

$$
||u^b - u^0||_{C^0([0,\infty) \times [0,T])} \leq 2||f - f_n||_{\infty} + ||u^b_n - u^0_n||_{C^0([0,\infty) \times [0,T])}.
$$

Given $\epsilon > 0$, fix some $n$ so that $||f - f_n||_{\infty} < \epsilon$. Applying Proposition 5 to the $u^b_n$ gives a positive $b_0$ so that if $b < b_0$, then

$$
||u^b_n - u^0_n||_{C^0([0,\infty) \times [0,T])} < \epsilon,
$$

as $\epsilon > 0$ is arbitrary, this completes the proof of the proposition. \hfill \Box
8 Perturbation estimates

In order to use the model heat kernels \(k^b_t\) in perturbative constructions for the heat kernels of general Wright-Fisher operators, it is necessary to prove estimates in \(C^\ell\) for every \(\ell \geq 0\) for operators of the form

\[
g \mapsto A^b_t g(x) = \int_0^t \int_0^\infty k^b_{t-s}(x, z) h(z) z \partial_z g(z, s) \, dz \, ds,
\]

where \(h \in C^\infty_c([0, \infty))\) is fixed.

Corollary 3, the Leibniz formula and the mapping results established in the previous section show that if \(g \in C^\ell\), then

\[
\partial^\ell_x A^b_t g = A^{b+\ell}_t \partial^\ell_x g + \sum_{j=1}^{\ell} \binom{\ell}{j} \int_0^t \int_0^\infty k^{b+\ell}_{t-s}(x, z) \partial^j_z (zh(z)) \partial^{\ell+1-j}_z g(z, s) \, dz \, ds.
\]

Hence to estimate \(A^b_t g\) on \(C^\ell\) for arbitrary \(\ell \geq 0\) it suffices to prove mapping properties for \(A^b_t\) on \(C^0\) for all \(b' \geq 0\).

The following lemma is the key to all that follows:

**Lemma 5.** There is a constant \(C_b\), defined for each \(b \geq 0\) and uniformly bounded for \(b\) in any compact interval \([0, B]\), such that if \(f \in C^0([0, \infty))\) and

\[
u(x, t) = \int_0^\infty k^b_t(x, y)f(y) \, dy.
\]

then

\[|\partial_z \nu(z, s)| \leq C_b \frac{\|f\|_\infty}{s + \sqrt{zs}}.
\]

**Proof.** It is enough to prove that for each \(b > 0\) there is a constant \(C_b\) so that

\[
\phi_{s, b}(z) := \int_0^\infty \left| \frac{\partial k^b_s}{\partial z}(z, y) \right| \, dy \leq \frac{C_b}{s + \sqrt{sz}} = \frac{C_b}{s(1 + \sqrt{z/s})},
\]

and that \(C_b\) is uniformly bounded above on any interval \((0, B]\), i.e. it does not depend on a positive lower bound for \(b\). The case \(b = 0\) is then obtained from a separate limiting argument.
We first compute that
\[ \phi_{s,b}(z) = \frac{1}{s} \int_{0}^{\infty} \left( \frac{y}{s} \right)^{b} e^{-\frac{y}{s}} \left| \left( \frac{y}{s} \right) \psi_{b}' \left( \frac{zy}{s^{2}} \right) - \psi_{b} \left( \frac{zy}{s^{2}} \right) \right| \frac{dy}{y}. \]

Set \( w = y/s \) and \( \lambda = z/s \), so that \( \phi_{s,b}(z) = \frac{1}{s} \phi \left( \frac{z}{s} \right) \), where
\[ \phi(\lambda) = \int_{0}^{\infty} w^{b-1} e^{-w} e^{-\lambda} \left| w \psi_{b}'(\lambda w) - \psi_{b}(\lambda w) \right| dw. \]

Hence we only need prove that \( \phi(\lambda) \leq C_{b}/(1 + \sqrt{\lambda}) \).

Using the asymptotic formulæ
\[ \psi_{b}(w) \sim \frac{w^{-\frac{1}{4}} e^{2\sqrt{w}}}{\sqrt{4\pi}} (1 + O(w^{-\frac{1}{2}})) \] and
\[ \psi_{b}'(w) \sim \frac{w^{-\frac{1}{4} + \frac{3}{4}} e^{2\sqrt{w}}}{\sqrt{4\pi}} (1 + O(w^{-\frac{1}{2}})), \]
we see that \( \phi(\lambda) \leq C_{b,\Lambda} \) for \( \lambda \leq \Lambda \). It is less obvious that this constant is bounded as \( b \to 0 \) since the integrand appears to become nonintegrable at \( w = 0 \) in that limit. To analyze this, define \( \tilde{\psi}_{b}(z) = \psi_{b}(z) - \psi_{b}(0) \) and recall that \( \psi_{b}(0) = 1/\Gamma(b) \), so that
\[ \phi(\lambda) \leq \int_{0}^{\infty} w^{b-1} e^{-w} e^{-\lambda} \left[ |w \tilde{\psi}'_{b}(\lambda w)| + |\tilde{\psi}_{b}(\lambda w)| + \frac{1}{\Gamma(b)} \right] dw. \]

The integral of the last term in brackets is identically equal to \( e^{-\lambda} \). As for the other two terms, note that the coefficients in the error terms in these asymptotic formulæ are bounded as \( b \to 0 \), so the integral from 1 to \( \infty \) converges uniformly, independently of \( b \). If \( w \leq 1 \) (and \( \lambda \) bounded), these terms are \( O(w) \), hence this part of the integral is also uniformly bounded. Hence \( C_{b,\Lambda} \) is uniform in \( b \in [0, B] \) for any fixed \( B, \Lambda \).

Now consider what happens when \( \lambda \to \infty \). Suppose first that \( b > 0 \). Break the integral defining \( \phi \) into the sum \( J_{b}' + J_{b}'' \), where \( J_{b}' \) is the integral from 0 to \( 1/\sqrt{\lambda} \) and \( J_{b}'' \) is the integral from \( 1/\sqrt{\lambda} \) to \( \infty \). It is straightforward that \( J_{b}' \leq C e^{-c\lambda} \), for \( c, C > 0 \) which are independent of \( b \). For the other part use the asymtotics of \( \psi_{b} \) and \( \psi_{b}' \) to get
\[ J_{b}'' \leq C_{b,\sqrt{\lambda}} \int_{0}^{\infty} \left( \frac{w}{\lambda} \right)^{\frac{1}{4} + \frac{3}{4}} e^{-\lambda(1 - \sqrt{w})^{2}} \left| 1 - \sqrt{\frac{w}{\lambda}} \right| \frac{dw}{w}. \]
Changing variables to $y = \sqrt{w/\lambda} - 1$ transforms this to

$$J''_b \leq C_b \sqrt{\lambda} \int_{-1}^{\infty} (1 + y)^{b-\frac{1}{2}} e^{-\lambda y^2} |y| \, dy,$$

which now, by Laplace’s method, satisfies $J''_b \leq \frac{C_b}{\sqrt{\lambda}}$. This proves the estimate for $b > 0$ bounded away from $\infty$.

To finish the proof, observe that for any $\eta > 0$ and fixed $z, s > 0$, we have

$$\lim_{b \to 0^+} \int_{\eta}^{\infty} |\partial_z k^b_s(z, y)| \, dy = \int_{\eta}^{\infty} |\partial_z k^0_P(z, y)| \, dy$$

Since the constant $C_b$ is uniformly bounded as $b \to 0$, there is a constant $C_0$ so that for any $\eta > 0$,

$$\int_{\eta}^{\infty} |\partial_z k^0_P(z, y)| \, dy \leq \frac{C_0}{s(1 + \sqrt{z/s})}.$$  

The right hand side is independent of $\eta$, so letting $\eta \to 0$ gives the same estimate for the integral on all of $\mathbb{R}^+$. Setting $z = 0$ shows that

$$\int_{\eta}^{\infty} |(\partial_z k^0_P)(0, y)| \, dy = \frac{\psi_2(0)}{s}.$$  

Since

$$\int_{0}^{\infty} k^0_s(z, y) f(y) \, dy = \int_{0}^{\infty} k^0_P(z, y) (f(y) - f(0)) \, dy + f(0),$$

this completes the case when $b = 0$ as well.  

**Mapping properties of $(A^b_t)^j$**

Now we turn to estimates of $A^b_t$ and its iterates. Because the proofs of the next two Propositions are quite technical, we state the results here and relegate their proofs to Appendix A at the end of the paper.

We assume that $g \in C^0([0, \infty) \times [0, T]) \cap C^1((0, \infty) \times (0, T])$ with a very specific blowup for $\partial_x g$ as $x \to 0^+$, as suggested by Lemma 5.
Proposition 9. Define the sequence of constants
\[ d_j = \frac{(\pi)^{j+1}}{\Gamma\left(\frac{j+1}{2}\right)}. \]

For any smooth \( h \) with \( \text{supp} \ h \subset [0, L] \), if \( g \in C^0([0, \infty) \times [0, T]) \cap C^1((0, \infty) \times (0, T]) \) satisfies
\[ |\partial_x g(x, t)| \leq \frac{M}{\sqrt{xt}}, \]
then
\[ |(A^b_t)^j g(x)| \leq \frac{2d_j^{-1} M j^{-1}}{t} \left(\sqrt{h}\|h\|_{\infty}\right)^j t^{\frac{j}{2}} \]  
and
\[ |\partial_x (A^b_t)^j g(x)| \leq \frac{d_j M j^{-1}}{\sqrt{x}} \left(\sqrt{h}\|h\|_{\infty}\right)^j t^{\frac{j}{2}-1}. \]

where \( C_b \) is the constant appearing in Lemma 5.

Remark 7. The hypotheses of this proposition imply that
\[ |yh(y)\partial_y g(y, s)| \leq \frac{\sqrt{g}\|h\|_{\infty} M}{\sqrt{s}}, \]
so we could replace \( k_{1-s}^0 \) by \( k_{1-s}^{0,D} \) in the definition of \( A^0_t \). With this choice, \( A^0_t g(0) = 0 \). However, \( k_{1-s}^{0,D} \) does not satisfy Corollary 4 and its use would also complicate the derivation of the estimate for \( \partial_x^j A^0_t g(x) \), so it is simpler to use the kernel \( k_{1-s}^0 \) in the definition of \( A^0_t \).

For the higher norm estimates for \( A^b_t \) and its iterates, it is useful to introduce, for \( T > 0 \) and \( \ell \in \mathbb{N} \), the norms
\[ \|g\|_{C^\ell;[0, T]} = \max_{0 \leq t \leq T} \|g(\cdot, t)\|_{C^\ell([0, \infty))}. \]
The maximum principle and (48) immediately give the

Lemma 6. If \( g \in C^\ell_b([0, \infty) \times [0, T]) \), and \( 1 \leq p \leq \ell - 1, j \in \mathbb{N} \), then
\[ |\partial_x^p (A^b_t)^j g(x)| \leq \frac{t^j}{j!} \left(2^p \|xh\|_{C^p}\right)^j \|g\|_{C^{p+1, \infty}[0, T]}. \]
Proposition 10. Define the constants $D_j$ inductively by

$$D_j = 2D_{j-1} \frac{\Gamma \left( \frac{j}{2} \right)}{\Gamma \left( \frac{j+1}{2} \right)} ,$$

and $D_0 = 1$, so that $D_j \leq C \frac{2^j}{\Gamma \left( \frac{j+1}{2} \right)}$ for all $j \geq 0$. Let $h$ be any smooth function with $\text{supp} \ h \subset [0, L]$ and suppose that the function

$$g \in C^\ell \left( [0, \infty) \times [0, T] \right) \cap C^{\ell+1} \left( (0, \infty) \times [0, T] \right),$$

satisfies

$$|\partial_x^{\ell+1} g(x, t)| \leq \frac{M}{\sqrt{xt}}$$

for some constant $M > 0$ (depending on $g$). Then there are constants $C_{T, L, \ell, b}, C'_{T, L, \ell, b}$ so that, for $j \in \mathbb{N}$, we have

$$|\partial_x^\ell [A^b_{t}]^j g(x)| \leq \frac{2D_{j-1}}{j} (M + \|g\|_{C^\ell([0, T])})(C'_{T, L, \ell, b})^j \frac{t^{\frac{j}{2}}}{x} ,$$

and

$$|\partial_x^{\ell+1} [A^b_{t}]^j g(x)| \leq \frac{D_j (M + \|g\|_{C^\ell([0, T])})(C_{T, L, \ell, b})^j t^{\frac{j+1}{2}}}{\sqrt{x}} .$$

Off-diagonal estimates for $(A^b_{t})^j$

The final set of estimates we derive for the operators $A^b_{t}$ involve the off-diagonal behavior of the Schwartz kernel of the infinite sum

$$\sum_{j=0}^{\infty} (A^b_{t})^j k^b_{t} (x, y) = \sum_{j=0}^{\infty} \int_0^t \cdots \int_0^{s_1} \cdots \int_0^{s_{j-1}} \int_0^\infty A^b_{t-s_j} (x, z_1) \times$$

$$A^b_{s_j-s_{j-1}} (z_1, z_2) \cdots A^b_{s_{j-2}-s_{j-1}} (z_{j-1}, z_j) k^b_{s_{j}} (z_j, y) \ ds_1 \cdots ds_j dz_1 \cdots dz_j .$$

The most precise bounds are best described in a blown-up space. For our purposes, the slightly cruder estimates derived here suffice.

We begin by observing that if $|x - y| \geq \alpha > 0$, then $k^b_{t} (x, y) \leq Cy^{b-1}e^{-c/t}$ (where $c$ is any number less than $(\sqrt{x} - \sqrt{y})^2$). In fact, $k^b_{t} (x, y) = y^{b-1}F(t, x, y)$ where $F$ is smooth away from the diagonal $x = y$ but up to $x = 0$ and $y = 0$, and satisfies $|F| \leq Ce^{-c/t}$ where $c$ depends only on $|x - y|$. This is proved using the explicit expression for $k^b_{t}$, and separately considering the behavior in the regions $xy > t^2$ and $xy < t^2$. Using this we can now prove the
Proposition 11. Fix $\alpha \geq 0$, and $k, \ell \in \mathbb{N}$; then there exist constants $C, c, B > 0$ depending on $\alpha$, $\|h\|_{\infty}$, $L$, $k$ and $\ell$ such that if $|x - y| \geq \alpha$ and $0 \leq x, y \leq L$, then

$$|\partial_x^k (y \partial_y)^\ell [(A_t^b)^j k_t^{1,b}(x, y)]| \leq CD_j B^j e^{-c/t} y^{b-1},$$

where $D_j$ are the constants appearing in Proposition 10. (These may be replaced by the constants $d_j$ from Proposition 9 when $k = \ell = 0$.)

Proof. As just indicated, this assertion is clear from the formula for $k_t^b$ when $j = 0$, so we proceed by induction, assuming that we have proved it for all powers up to some $j$.

First use a smooth partition of unity to decompose $k_t^b = k_t^{1,b} + k_t^{2,b}$ where $k_t^{1,b}$ is supported in the region $|x - y| \leq \alpha/2$ and $y^{1-b} k_t^{2,b}$ is $C^\infty$ when $x, y, t \geq 0$ and satisfies $|y^{1-b} k_t^{2,b}(x, y)| \leq C e^{-c/t}$ for all $x, y$. Using Proposition 9 (or Proposition 10 for the higher derivatives), we have that

$$|(A_t^b)^{j+1} k_t^{2,b}(x, y)| \leq C d_j B^{j+1} e^{-c/t} y^{b-1}.$$

To estimate the other term, let us begin by writing

$$A_t^b \circ k_t^{1,b}(x, y) = \int_0^t \int_0^\infty k_t^{1,b}(x, z) h(z) z \partial_z k_s^{1,b}(z, y) dzds \quad = \int_0^{t/2} \int_0^\infty - (z \partial_z + 1) (k_t^{1,b}(x, z) h(z)) k_s^{1,b}(z, y) dzds \quad + \quad \int_{t/2}^t \int_0^\infty k_t^{1,b}(x, z) h(z) z \partial_z k_s^{1,b}(z, y) dzds.$$

The integration by parts used to obtain the first term on the right is valid because $zk_t^{1,b}(x, z) k_s^{1,b}(z, y) \leq C s^{1,b}$ and $h(z)$ has compact support. For this first term, use that $|x - z| \geq \alpha/2$ and $|z - y| \leq \alpha/2$ to get

$$|(z \partial_z + 1) k_t^{b}(x, z)| \leq C e^{-c/t} y^{b-1}, \quad \text{and} \quad |k_s^{1,b}(z, y)| \leq C s^{-1/2} y^{b-1},$$

which shows that this term is bounded by $B_1 e^{-c/t} y^{b-1}$. The second term is bounded similarly, using

$$|\partial_z k_s^{1,b}(z, y)| \leq B_2 s^{-3/2} y^{b-1}.$$

Taken together, we obtain that

$$|A_t^b k_t^{b}(x, y)| \leq B e^{-c/t} y^{b-1}$$

where $B$ depends only on the quantities indicated. Finally, use Propositions 9 and 10 to obtain

$$|(A_t^b)^j \circ (A_t^b k_t^{1,b})| \leq C d_j B^{j+1} e^{-c/t} y^{b-1},$$

as claimed. The estimates for the higher derivatives are proved similarly. \qed
From these estimates we now obtain the

**Corollary 5.** The kernel \( \tilde{q}_t^b(x, y) \) of the operator

\[
f \mapsto \tilde{Q}_t^b f = \sum_{j=0}^{\infty} (A_t^b)^j \int_0^\infty k_t^b(\cdot, y) f(y) dy
\]

can be written as

\[
\tilde{q}_t^b = y^{b-1} \tilde{q}_t^{b,\text{reg}}(x, y), \text{ where } \tilde{q}_t^{b,\text{reg}} \in C^\infty([0, 1] \times [0, 1] \times (0, \infty)).
\]

For each \( d > 0 \), there is a constant \( c_d > 0 \), so that in the off-diagonal region \( \{(x, y) : |x - y| > d\} \), we have the estimate

\[
|\tilde{q}_t^b(x, y)| \leq C e^{-c_d t} y^{b-1}, \tag{55}
\]

with analogous estimates for the derivatives \( \partial^k_x (y \partial_y)^l \tilde{q}_t^b(x, y) \).

**9 Construction of the heat kernel for a general Wright-Fisher operator**

After this long excursion into the analysis of the model problems we are now prepared to construct the heat kernel for the full generalized Wright-Fisher operator

\[
L = y(1 - y) \partial_y^2 + b(y) \partial_y.
\]

Let

\[
y = \sin^2 \sqrt{x_\ell} \text{ and } 1 - y = \sin^2 \sqrt{x_r}.
\]

According to the discussion in Section 4 pulling back \( L \) to these coordinate charts gives

\[
L_\ell = x_\ell \partial_{x_\ell}^2 + b_0 \partial_{x_\ell} + x_\ell c_\ell(x_\ell) \partial_{x_\ell} \quad \text{and} \quad L_r = x_r \partial_{x_r}^2 + b_1 \partial_{x_r} + x_r c_r(x_r) \partial_{x_r},
\]

where

\[
b_0 = b(0) \text{ and } b_1 = -b(1).
\]

Suppose that \( u \) solves (15), with \( u(y, 0) = f(y) \). Then on the interval \([0, \left(\frac{\pi}{2}\right)^2]\) the functions

\[
u_\ell(x_\ell, t) = u(\sin^2 \sqrt{x_\ell}, t) \quad \text{and} \quad u_r(x_r, t) = u(1 - \sin^2 \sqrt{x_r}, t),
\]

32
satisfy

\[(\partial_t - L\ell) u\ell = 0 \text{ with } u\ell(x\ell, 0) = f(\sin^2 \sqrt{x\ell}) \text{ and} \]
\[(\partial_t - Lr) ur = 0 \text{ with } ur(xr, 0) = f(1 - \sin^2 \sqrt{xr}).\]

It is clear that the symmetry \(y \to 1 - y\) carries the left end to the right and vice-versa, so to simplify this discussion, we focus on the left end. We use \(x\) to denote \(x\ell, b\) to denote \(b_0\), and we choose a smooth cutoff function \(\varphi\) so that
\[
\varphi(x) = \begin{cases} 
1 & \text{for } x \in [0, (\frac{x}{2})^2 - 2\eta] \\
0 & \text{for } x > (\frac{x}{2})^2 - \eta,
\end{cases}
\]
where \(\eta > 0\) is small. With \(h(x) = c(x)\varphi(x)\), we now focus attention on
\[
\tilde{L} = L_{b_0} + xh(x)\partial_x.
\]
A solution to (15), pulled by \(x\ell\) satisfies:
\[(\partial_t - \tilde{L}) \tilde{u} = 0 \text{ on } [0, (\frac{x}{2})^2 - 2\eta].\]

To build a parametrix for the heat kernel that has the correct boundary behavior near to \(y = 0\), consider the initial value problem:
\[(\partial_t - \tilde{L}) \tilde{u} = 0 \text{ with } \tilde{u}(x, 0) = \tilde{f}(x) \in C^0_0([0, \infty)).\]  \hfill (56)

For our application \(\tilde{f}\) is obtained from \(f\) by pullback and multiplication by a smooth cut-off. Multiply the equation by \(k_b^t(x, y)\) and integrate to obtain:
\[
\tilde{u}(x,t) - \int_0^t \int_0^\infty k_b^t(x, y)yh(y)\partial_y \tilde{u}(y,s)dyds = \int_0^\infty k_b^t(x, y)\tilde{f}(y) dy,
\]

or equivalently
\[
\tilde{u}(x,t) - A_b^t\tilde{u}(y,s) = \int_0^\infty k_b^t(x, y)\tilde{f}(y) dy.
\]

If \(b = 0\), then as noted in Remark 7 we could replace \(k_{t-s}^0\) in the definition of \(A_t^0\) with \(k_{t-s}^{0,D}\), which shows that, as expected, \(\tilde{u}(0, t) = \tilde{f}(0)\) for all \(t \geq 0\).

It is straightforward to solve (56) using the estimates from Section 8. Indeed, the solution can be expressed as a convergent Neumann series:
\[
\tilde{u}(x, t) = (1 - A_t^b)^{-1}\tilde{g}(x, t) = \sum_{j=0}^\infty (A_t^b)^j\tilde{g},
\]
Let us denote the operator \( \tilde{f} \rightarrow (\text{Id} - A^b_{t})^{-1}\tilde{g} \) by \( \tilde{Q}^b_t\tilde{f} \). Later on we shall need to distinguish the operator constructed near \( x = 0 \) from the one near \( x = 1 \), and at that point we shall write them as \( \tilde{Q}^{b_0}_{t,\ell} \) and \( \tilde{Q}^{b_1}_{t,r} \), and denote their kernels by \( \tilde{q}^{b_0}_{t,\ell} \) and \( \tilde{q}^{b_1}_{t,r} \), respectively.

An immediate consequence of Propositions 9 and 10 is that if \( f \in C^\ell([0, \infty)) \), then this sum converges uniformly in the topology of \( C_0^0([0, T]; C^\ell([0, \infty))) \), for any \( T > 0 \), and Propositions 6, 9, and 10 show that for such data,

\[
\lim_{t \to 0^+} \partial_j^x \tilde{u}(.,t) = \partial_j^x \tilde{f} \quad \text{for} \quad j \leq l,
\]

where the convergence is with respect to the \( C^0 \) topology.

The regularity of these solutions shows that if \( \tilde{f} \in \dot{C}^\ell([0, \infty)) \), then

\[
\tilde{g} \in C^0([0, \infty); C^\ell([0, \infty)) \cap C^\infty([0, \infty) \times (0, \infty)).
\] (57)

As a consequence of Proposition 7, we see that for \( 0 < \epsilon < t \), the sum defining \( \tilde{Q}^b_t\tilde{f} \) actually converges uniformly in the \( C^\infty \)-topology. Hence

\[
\tilde{u} \in C^0([0, \infty); C^\ell([0, \infty)) \cap C^\infty([0, \infty) \times (0, \infty)),
\] (58)

as well, and differentiating shows that

\[
(\partial_t - \tilde{L})\tilde{u} = xh(x)\partial_x \tilde{u} \iff (\partial_t - \tilde{L})\tilde{u} = 0.
\]

We have now solved the problem ‘exactly’ near each endpoint, and the next step is to paste together these left and right solution operators to obtain an infinite order parametrix for \( \partial_t - L \), which has remainder term vanishing identically near the \( x = 0 \) and \( x = 1 \). To accomplish this, define maps \( \phi_\ell \) and \( \phi_r \) near \( x = 0 \) and \( 1 \) which put the leading part of \( L_{\text{WF}} \) into the model form \( x\partial_x^2 \). Pulling back the operator in (18) gives exactly the operator \( L \) in the intervals \( [0, \frac{5}{8}] \), and \( [\frac{1}{2}, 0] \). Now choose cutoffs \( \varphi_\ell, \varphi_r, \varphi_0 \in C^\infty[0, 1] \) so that

\[
\varphi_\ell(x) = 1 \quad \text{for} \quad x \in [0, \frac{11}{16}], \quad \supp \varphi_\ell \subset [0, \frac{3}{4}],
\]
\[
\varphi_r(x) = 1 \quad \text{for} \quad x \in \left[\frac{5}{16}, 1\right], \quad \supp \varphi_r \subset \left[\frac{1}{4}, 1\right],
\]
\[
\varphi_0(x) = 1 \quad \text{for} \quad x \in [0, \frac{3}{8}], \quad \supp \varphi_0 \subset \left[0, \frac{5}{8}\right].
\]
and define the parametrix
\[
q_t(x, y) = \varphi_0(x)\bar{q}_t^{b_0}(\phi_t(x), \phi_t(y))\varphi_t(y)|\phi_t'(y)| + (1 - \varphi_0(x))\bar{q}_t^{b_1}(\phi_t(x), \phi_t(y))\varphi_t(y)|\phi_t'(y)|.
\]

The heat kernel for \( L \) is determined symbolically to infinite order as \( t \to 0 \) near the diagonal away from \( x, y = 0, 1 \). Furthermore, the kernel and all its derivatives tend to zero like \( e^{-c/t} \) away from the diagonal and away from \( y = 0, 1 \). Hence in the overlap region, \( \text{supp } \varphi_0 \cap \text{supp}(1 - \varphi_0) \), the two terms agree to all orders.

Now set
\[
e_t(x, y) = [\partial_t - (x(1-x)\partial_x^2 + b(x)\partial_x)]q_t(x, y);
\]
by construction, this satisfies
\[
\text{supp } e_t(x, y) \subset \left[\frac{3}{8}, \frac{5}{8}\right] \times [0, 1] \times [0, \infty).
\]

Since this error term is obtained by applying derivatives in the first \( x \) variable to \( q_t(x, y) \), Corollary 5 shows that \( y^{1-b_0}(1-y)^{1-b_1}e_t \in C^\infty([0, 1] \times [0, 1] \times (0, \infty)) \) and vanishes, along with all its derivatives, like \( e^{-c/t} \) for some \( c > 0 \), on \([0, 1] \times [0, 1] \times (0, \infty)\) disjoint from the diagonal at \( t = 0 \). In a neighborhood including the diagonal it vanishes, in the \( C^\infty \)-topology, faster than \( O(t^N) \) for any \( N \in \mathbb{N} \).

Given a function \( f \in C^m([0, 1]) \), set
\[
u_0(x, t) = \frac{1}{t} \int_0^1 q_t(x, y)f(y) \, dy.
\]

Propositions 9 and 10 show that \( u_0(x, t) \in C^0([0, \infty); C^m([0, 1])) \), and
\[
[\partial_t - L]u_0(x, t) = v(x, t), \quad \lim_{t \to 0^+} u_0(x, t) = f(x),
\]
where \( v(x, t) \) is smooth, vanishes in \( \{[0, \frac{3}{8}] \cup [\frac{5}{8}, 1]\} \times [0, \infty) \) and tends rapidly to zero as \( t \to 0^+ \). This is even true for \( \partial^j_x v(x, t) \) for all \( j \in \mathbb{N} \).

We now complete the construction of the solution operator for the generalized Wright-Fisher operator. Define, for \( \epsilon > 0 \),
\[
Q_t^\epsilon g = \int_0^{t-\epsilon} \int_0^1 q_{t-s}(x, y)g(y, s) \, dyds,
\]
\[
E_t g = -\int_0^t \int_0^1 e_{t-s}(x, y)g(y, s) \, dyds,
\]

(59)
and write $Q^0_t$ simply as $Q_t$. If
\[ g \in C^0([0, \infty) \times [0, \infty)) \cap C^2([0, \infty) \times (0, \infty)), \] (60)
then the estimates above imply that $Q^\varepsilon_t g \to Q_t g$ in the $C^2$ topology for any $t > 0$, and hence
\[ (\partial_t - L)Q^\varepsilon_t g = \lim_{\varepsilon \to 0^+} (\partial_t - L)Q^\varepsilon_t g = (\Id - E_t)g. \]

The inverse of $(\Id - E_t)$ is an operator of the same type, and we write it as $(\Id - H_t)$; here $H_t$ is represented by a kernel $h_t(x, y)$. Note that
\[ (\Id - E_t)(\Id - H_t) = (\Id - H_t)(\Id - E_t) = \Id, \]
so
\[ H_t = -E_t + E_t H_t = -E_t + H_t E_t \implies H_t = -E_t - E_t^2 + E_t H_t E_t. \]
The last identity shows that $h_t$ have the same regularity properties as $e_t$. In particular, it vanishes, along with all derivatives, to infinite order as $t \to 0$, and behaves like $y^{b_0-1}$ along $y = 0$, and $(1 - y)^{b_1-1}$ along $y = 1$.

Setting $g = (\Id - H_t)v$, then $u_1 = Q_t g$ satisfies
\[ (\partial_t - L)u_1 = v \quad \text{and} \quad \lim_{t \to 0} u_1(x, t) = 0. \]
The true solution $u$ is the difference
\[ u = u_0 - u_1 = Q_t(\Id - E_t + H_t E_t)(f \otimes \delta(t)), \]
or equivalently,
\[ u = \int_0^1 q_t(x, y) f(y) \, dy + \int_0^t \int_0^1 q_{t-s}(x, z) \int_0^1 h_s(z, y) f(y) \, dy \, dz \, ds. \] (61)
This was derived under the assumption that $g$ satisfies (60). If $f \in C^0([0, 1])$, then the fact that the solution can be represented the same way follows from the mapping properties of $Q_t$.

**Definition 1.** We let $Q_t f$ denote the operator on the right hand side in (61), and $q_t(x, y)$ its kernel.
Note that since \( y^{1-b_0}(1 - y)^{1-b_1} h_t \) decays rapidly decreasing as \( t \searrow 0 \) in \( C^\infty([0, 1] \times [0, 1]) \), the kernel \( q_t \) already defines a complete parametrix for \( (\partial_t - L)^{-1} \) away from \( y = 0 \) and \( y = 1 \). The contributions of \( h_t \) along \( y = 0 \) and \( y = 1 \) are essential for \( Q_t \) to satisfy the adjoint boundary conditions (13), and hence to provide the solution operator for the adjoint problem.

Putting together all the information we have obtained about the kernel for \( Q_t \), and using the uniqueness for the regular solution of (15), we can now state the fundamental

**Theorem 1.** For each \( m \in \mathbb{N} \cup \{0\} \) the operators \( Q_t \) define positivity preserving semi-groups on \( C^m([0, 1]) \). The function \( u(x, t) = Q_t f(x) \) satisfies (15). Moreover, for \( f \in C^m \),

\[
\lim_{t \to 0^+} \|Q_t f - f\|_{C^m([0, 1])} = 0.
\]

One of the reasons for having worked hard to establish convergence of the Neumann series used in the construction of the solution operator \( Q_t \) is that we can estimate how good of an approximation the partial sums are.

**Proposition 12.** Fix \( N \geq 0 \) and define the operator \( Q_{t,N} \) by pasting together the finite sums

\[
\sum_{j=0}^{N} (A_t^b)^j k_t^b
\]

using the solution operators for the models at the left and right endpoints of \([0, 1]\), and denote its kernel by \( q_{t,N} \). For any \( f \in C^m([0, 1]) \), let \( u(x, t) \) denote the exact solution to (15); for \( N > m \) and \( 2j \leq m \), the function

\[
u_N(x, t) = \int_0^1 q_{t,N}(x, y) f(y) \, dy
\]

satisfies

\[
||\partial_x^j [u_N(\cdot, t) - u(\cdot, t)]||_{C^0([0, 1])} \leq C t^{N+1-2j}
\]

where the constant \( C \) can be estimated in terms of \( j, N \), and the coefficients of \( L \).

In particular, when \( m = N = 0 \) then

\[
\sup_{x \in [0, 1]} |u_0(x, t) - f(x)| \leq C \sqrt{t}.
\]

We recall the changes of variables

\[
\sqrt{x_l} = \sin^{-1} \sqrt{x} \quad \sqrt{y_l} = \sin^{-1} \sqrt{y} \\
\sqrt{x_r} = \sin^{-1} \sqrt{1-x} \quad \sqrt{y_r} = \sin^{-1} \sqrt{1-y};
\]

(63)
differentiating we see that
\[
\frac{dy_t}{dy} = \frac{\sin^{-1} \sqrt{y}}{\sqrt{y(1-y)}} \quad \frac{dy_r}{dy} = -\frac{\sin^{-1} \sqrt{1-y}}{\sqrt{y(1-y)}}.
\] (64)

Substituting, we see that, for \(b_0\) and \(b_1\) non-zero, the leading term in the solution operator \(q_t\) is given by:
\[
q_{t,0}(x, y) = \varphi_0(x) y_t^{-1} \left( \frac{y_t}{t} \right)^{b_0} e^{-x_t y_t \left( \frac{y_t}{t} \right)} \psi_0(x) y_t^{-1} \left( \frac{y_t}{t} \right)^{b_0} e^{-x_t y_t \left( \frac{y_t}{t} \right)} \varphi_t(y) \frac{\sin^{-1} \sqrt{y}}{\sqrt{y(1-y)}} + \]
\[
(1 - \varphi_0(x)) y_r^{-1} \left( \frac{y_r}{t} \right)^{b_1} e^{-x_r y_r \left( \frac{y_r}{t} \right)} \psi_1(x) y_r^{-1} \left( \frac{y_r}{t} \right)^{b_1} e^{-x_r y_r \left( \frac{y_r}{t} \right)} \varphi_t(y) \frac{\sin^{-1} \sqrt{1-y}}{\sqrt{y(1-y)}}.
\]

10 The infinitesimal generator and long time asymptotics of solutions

By Theorem 1, \(Q_t\) defines a semi-group on \(C^0\); indeed, it also defines a semi-group on \(C^m\) for every nonnegative integer \(m\). To understand this \(C^0\) semi-group better, and in particular to estimate the long-time asymptotics of solutions, we now seek a characterization of its infinitesimal generator \(A\) as an unbounded operator on \(C^0\), including some features of its spectrum and a description of the behavior at \(x = 0\) and \(x = 1\) of the elements in its domain. These facts will be proved using the various regularity results we have obtained. At various points in this discussion it will be necessary bring in the the adjoint operator \(L^t\), and in particular the infinitesimal generator \(A^*\) for the adjoint semi-group. Note that \(A^*\) is an unbounded operator on \([C^0([0, 1])]^\prime\), which we identify with \(\mathcal{M}([0, 1])\), the space of finite Borel measures on \([0, 1]\). Section 12 contains a more complete discussion of the adjoint semi-group.

The first step is to note that the infinitesimal generator has a compact resolvent.

**Proposition 13.** Let \(A\) be the infinitesimal generator associated to the \(C^0\) semi-group defined by \(Q_t\). Then as an unbounded operator on \(C^0([0, 1])\), the spectrum of \(A\) lies in the left half-plane \(\left\{ \lambda : \text{Re} \lambda \leq 0 \right\}\); furthermore, \(A\) has a compact resolvent.

**Proof.** The first statement follows from the maximum principle, which implies that \(\|Q_t f\|_{C^0} \leq \|f\|_{C^0}\). Next, observe that for any \(t > 0\) and \(f \in C^0([0, 1])\), \(Q_t f \in C^\infty([0, 1])\). The closed graph theorem and Arzela-Ascoli theorem now apply to show that \(Q_t\) is a compact operator on \(C^0\) for any \(t > 0\), so the second statement follows from the results in Section 8.2 of [3].
Of course, the full characterization of $A$ involves a detailed description of its domain. This will be based on a basic result from semi-group theory, due to Nelson:

**Proposition 14** (Nelson, [11]). *Let $B$ be a Banach space and $Z$ a closed operator on $B$ generating a semi-group $T_t$. If $D \subset \text{Dom}(Z)$ is a subspace which is dense in $B$, and if $T_tD \subset D$, for every $t > 0$, then $D$ is a core for $Z$, i.e.

$$Z = \overline{Z \mid D}.$$*

To apply this, recall that $Q_tf C^2([0,1]) \subset C^2([0,1])$ for $t > 0$, and also, if $f \in C^2([0,1])$, then $u = Q_tf$ satisfies $\partial_t u = Lu$ for $t \geq 0$.

**Proposition 15.** *If $A$ is the generator of the $C^0$ semi-group defined by $Q_t$, then

$$A = \overline{L} \mid _{C^2([0,1])}.$$

*Proof.* If $f \in C^2([0,1])$, then $u(x,t) = Q_tf(x)$ has one time derivative and two spatial derivatives, all of which are continuous on $[0,1] \times [0,\infty)$. Now integrate the equation satisfied by $u$ to compute that

$$\frac{Q_tf - f}{t} = \frac{1}{t} \int_0^t Lu(x,s)ds.$$*

Since $Lu \in C^0([0,1] \times [0,\infty))$ and equals $Lf(x)$ at $t = 0$, we have

$$\frac{Q_tf - f}{t} = Lf + o(1).$$

This implies that $C^2([0,1]) \subset \text{Dom}(A)$ and on this subspace, $Af = Lf$. Since $C^2([0,1])$ is dense is $C^0([0,1])$, the proposition now follows directly from Nelson’s theorem. 

On the other hand, if $f \in \text{Dom}(A)$ then $Lu \in C^0([0,1])$, so the one-dimensional version of “elliptic regularity” shows that $f \in C^2((0,1))$. In other words, the final characterization of $\text{Dom}(A)$ involves only the description of its elements at the boundaries.

**The case where neither $b(0)$ nor $b(1)$ vanish**

It turns out that the results in case either $b(0)$ or $b(1)$ vanish are slightly more complicated to state, so for the moment let us suppose that $0 < b(0), -b(1)$. As before, denote $b(0) = b_0$ and $-b(1) = b_1$. 


We begin by noting that there is a solution \( v \) to the adjoint equation \( L^t v = 0 \), where \( L^t \) is given in (11), satisfying the adjoint boundary conditions (13). Thus

\[
v_0(x) = x^{b_0 - 1}(1 - x)^{b_1 - 1}e^{B(x)},
\]

(65)

where \( B(x) \in C^\infty([0, 1]) \), and

\[
\partial_x [x(1 - x)v_0](x) - b(x)v_0(x) = 0.
\]

The existence of \( v_0 \) follows using standard ODE techniques.

Choose \( \varphi \in C^\infty([0, 1]) \) with support in \([0, \frac{1}{2}]\) such that \( \varphi(x) = 1 \), for \( x \) in \([0, \frac{1}{2}]\). If \( f \in C^2 \), then

\[
\frac{1}{0} \int (Lf(x)) \varphi(x)v_0(x) \, dx = \frac{1}{0} \int f(x)L^t(\varphi v_0)(x) \, dx.
\]

(66)

Since \( C^2 \) is dense in \( \text{Dom}(A) \), this identity also holds for the graph closure. If \( f \in \text{Dom}(A) \), and \( \delta \) is any small positive number, then

\[
\frac{1}{\delta} \int [Lf(x)\varphi(x)v_0(x) - f(x)L^t(\varphi v_0)(x)] \, dx = -\delta(1 - \delta)v_0(\delta)\partial_x f(\delta).
\]

Using (66) and the asymptotic form of \( v_0 \), we deduce that

\[
\lim_{x \to 0^+} x^{b_0} \partial_x f(x) = 0.
\]

(67)

A similar argument using a cutoff function with support near to 1 shows that

\[
\lim_{x \to 1^-} (1 - x)^{b_1} \partial_x f(x) = 0.
\]

(68)

Note that these are precisely the boundary conditions described in Section 3. They are also the ones described by Feller as defining a positivity preserving contraction semi-group on \( C^0([0, 1]) \).

If \( f \in C^0_c((0, 1)) \), then the definition of \( q_t \) and the maximum principle imply

\[
\lim_{t \to 0^+} \int_{0}^{1} q_t(x, y) f(y) \, dy = f(x) \quad \text{and} \quad \lim_{t \to 0^+} \int_{0}^{1} q_t(x, y) f(x) \, dx = f(y).
\]

Using this and the sharp maximum principle for parabolic operators in one dimension, see e.g. Theorem 2 in Chapter 3 of [12], by a straightforward limiting argument we obtain a strict pointwise lower bound for \( q_t \):
Proposition 16. If \( b(0) \) and \( b(1) \) are non-vanishing, then for \( t > 0 \) and \( x, y \in [0, 1] \),
\[
q_t(x, y) > 0.
\] (69)

In this case, for each \( t > 0 \), the kernel \( q_t(x, y) \) defines a strictly positive operator on \( C^0([0, 1]) \). In other words, if \( f \in C^0([0, 1]) \) is non-negative and not identically zero, then \( Q_t f(x) > 0 \) for \( x \in [0, 1] \) and \( t > 0 \). Consequently, we can now apply Theorem 23.1 from [10] (The Perron-Frobenius Theorem) to conclude that:

**Theorem 2.** If \( b(0) \) and \( b(1) \) are non-zero, then the infinitesimal generator \( A \) is a compact operator on \( C^0([0, 1]) \) with spectrum lying in \( \{ \lambda : \Re \lambda \leq 0 \} \). The only element in \( \sigma(A) \) on the imaginary axis is the point \( \lambda = 0 \), and the only associated eigenfunctions are the constant functions. The function \( v_0 \), defined in (65) spans the 0-eigenspace of \( A^* \).

One consequence of the theorem above is that
\[
\lambda_1 = \sup \{ \Re(\lambda) : \lambda \in \sigma(A) \setminus \{0\} \} < 0.
\]

Thus defining
\[
u(x, t) = \int_0^1 q_t(x, y)f(y) \, dy, \quad \text{and} \quad c_0 = \int_0^1 v_0(y)f(y) \, dy,
\]
where \( v_0 \) is normalized to have integral 1, then Theorem 2.1 in Section B-IV of [1] implies the

**Corollary 6.** Under the hypotheses of Theorem 2 for each \( \delta \in (0, |\lambda_1|) \), there is a constant \( M_\delta > 0 \) such that
\[
\|u(x, t) - c_0\|_{C^0} \leq M_\delta \|f\|_{C^0} e^{-\delta t}.
\]

**The case where either** \( b_0 = 0 \) **or** \( b_1 = 0 \)

We now turn to the characterization of \( \text{Dom}(A) \), and the corresponding decay results for solutions, when either \( b_0 \) or \( b_1 \) (or both) vanish.

If \( b_0 = 0 \), then by Proposition 15, \( Af(0) = 0 \), for every \( f \in \text{Dom}(A) \). This in turn implies that \( \delta(y) \in \text{Dom}(A^*) \), and that \( A^* \delta(y) = 0 \). This is a new feature, since if \( b_0 \neq 0 \), then clearly \( \delta(y) \notin \text{Dom}(A^*) \), and it is what complicates the discussion. Similar remarks apply at \( x = 1 \) if \( b_1 = 0 \), of course.
More generally, if \( \mu \in \text{Dom}(A^*) \) and \( A^* \mu = 0 \), then by elliptic regularity in the open interval \((0, 1)\), the measure \( \mu \) has the representation \( g(y) dy \) where \( g \) satisfies \( L g = 0 \) and the adjoint boundary conditions, (13),

\[
\partial_x [x(1-x)g(x)] - b(x)g(x) = 0,
\]

so in particular \( g \in C^\infty((0, 1)) \). Writing \( b(x) = b_0 (1-x) - b_1 x + x(1-x) \tilde{b}(x) \), then solutions to (70) have the form:

\[
g(x) = C \frac{e^{-\tilde{B}(x)}}{x^{1-b_0} (1-x)^{1-b_1}},
\]

where \( C \) is a constant and \( \tilde{B} \) is a primitive of \( \tilde{b} \). Clearly, \( g(x) dx \) is a measure of finite total variation if and only if both \( b_0, b_1 > 0 \).

If both \( b_0 = b_1 = 0 \), then \( b(x) = x(1-x) \tilde{b}(x) \), where \( \tilde{b} \in C^\infty([0, 1]) \). The functions,

\[
u_0(x) = C \int_0^x \exp \left[ - \int_0^y \tilde{b}(z) dz \right] dy
\]

are strictly monotonically increasing in \((0, 1)\), and solve \( Lu = 0 \). Choosing \( C > 0 \) appropriately, we can assume that

\[
u_0(0) = 0 \text{ and } \nu_0(1) = 1.
\]

If \( b_0 = 0 \), but \( b_1 > 0 \), then there is a non-negative solution \( u_0 \), such that

\[
u_0(0) = 0, \quad \text{ and } \lim_{x \to 1^-} (1-x)^{b_1} \partial_x u_0(x) \neq 0,
\]

while \( f b_0 > 0, b_1 = 0 \), then there is a non-negative solution \( u_0(x) \) with

\[
u_0(1) = 0, \quad \text{ and } \lim_{x \to 0^+} x^{b_0} \partial_x u_0(x) \neq 0.
\]

If either \( b_0 \) or \( b_1 \) vanish, then the solution operator to (15) can be expressed in a form analogous to (34). Suppose first that \( b \) vanishes at only one endpoint, say \( b(0) = 0 \), but \( b(1) < 0 \). We can use the kernel \( k_{t,0}^D \) to build a solution operator, \( q_t^D \), for the Dirichlet problem at \( x = 0 \) with the regular boundary condition at \( x = 1 \):

\[ q_t(x, y) = q_t^D(x, y) + \delta(y) c_0(x, t), \]

where \((\partial_t - L)c_0(x, t) = 0\). As before, \( q_t^D(x, y) > 0 \) for \( x > 0 \), and moreover,

\[ c_0(x, t) = 1 - \int_0^1 q_t^D(x, y) dy, \]

42
so $c_0 \geq 0$. A similar argument works if $b(0) > 0$, but $b(1) = 0$. Finally, if $b(0) = b(1) = 0$, then we can write

$$q_t(x,y) = q^D_t(x,y) + \delta(y)c_0(x,t) + \delta(1-y)c_1(x,t),$$  \hspace{1cm} (74)$$

where $q^D_t(x,y)$ is positive on $(0,1) \times (0,1)$ and

$$c_0(x,t) + c_1(x,t) = 1 - \int_0^1 q^D_t(x,y) \, dy,$$

$$c_1(x,t) = 1 - \int_0^1 q^D_t(x,y) u_0(y) \, dy.$$

In each of these cases, $C^0([0,1])$ splits into a finite dimensional subspace, invariant under $Q_t$, and an infinite dimensional complement, also invariant under $Q_t$. For example, if $b(0) = b(1) = 0$, then

$$C^0([0,1]) = C^0_0([0,1]) \oplus \text{span}\{1, u_0\}. \hspace{1cm} (75)$$

In all cases there is a corresponding splitting of the semi-group into the Dirichlet semi-group, $Q^D_t$, and a semi-group on a finite dimensional space. The infinitesimal generator $A^D$ of $Q^D_t$ is the graph closure of $L$ on the set of functions in $C^2([0,1])$ which vanish at the appropriate end-point, or -points. The fact that $q^D_t$ is positive for $0 < x$, or $x < 1$, or $0 < x < 1$, respectively, implies as before that the semi-groups $Q^D_t$ are positive and irreducible.

The next proposition follows from known results about positive, irreducible, compact semi-groups acting on $C^0(X)$, where $X = (0,1), (0,1], \text{or } [0,1)$, see [1].

**Proposition 17.** If either or both of the numbers $b(0), b(1)$ vanish, then $A^D$ is compact. There is an element $\lambda_1 \in \sigma(A^D)$ with $\lambda_1 \in (\infty, 0)$ and a unique corresponding eigenfunction $u_1$ which is smooth and positive in $(0,1)$. The remainder of the spectrum lies in $\operatorname{Re} \lambda < \lambda_1 - \eta$, for some $\eta > 0$.

**Proof.** The compactness follows from the fact that $Q^D_t$ is compact for every $t > 0$. The existence of the eigenfunction $u_1$ and the negativity of $\lambda_1$ is obtained by applying oscillation and comparison theorems for Sturm-Liouville operators. When $b(0) = b(1) = 0$, comparison with operators of the form $M\partial^2_x + \frac{mu}{x(1-x)}$ yields the existence of $\lambda_1 < 0$ and a unique associated eigenfunction $u_1 \in \text{Dom}(A)$ with $u_1 > 0$ in $(0,1)$. The cases where only one of $b(0)$ or $b(1)$ vanish are somewhat
easier. If \( b(0) = 0, b(1) < 1 \), then the eigenvalue problem can written as

\[
\partial_x \left[ (1 - x)^{b_1} e^{-B(x)} \partial_x u \right] + \frac{\mu e^{-B(x)} u(x)}{x(1 - x)^{1-b_1}} = 0,
\]

\( u(0) = 0 \) and \( \lim_{x \to 1^-} (1 - x)^{b_1} \partial_x u(x) = 0 \).

Since \((1 - x)^{b_1-1}\) is integrable near \( x = 1 \), we can apply standard oscillation theorems to obtain the desired conclusion. The result then follows from Corollary 2.2 in Section B-IV of [1] and the fact that \( A^D \) has a compact resolvent.

If \( b(0) = b(1) = 0 \), then any \( f \in C^0 \) can be decomposed, according to (75), as

\[
f = f_0 + [f(0) + (f(1) - f(0)) u_0],
\]

and then

\[
Q_t f = Q_t^D f_0 + [f(0) + (f(1) - f(0)) u_0].
\]  

The proposition above and Theorem 2.1, in Section B-IV of [1] imply that there is a constant, independent of \( f \) so that

\[
\| Q_t^D f_0 \|_{C^0} \leq C e^{\lambda_1 t} \| f \|_{C^0}.
\]

Comparing the representations of \( Q_t f \) in (74) and (76) shows that

\[
c_0(x, t) = 1 - u_0(x) + O(e^{\lambda_1 t}) \text{ and } c_1(x, t) = u_0(x) + O(e^{\lambda_1 t}).
\]  

There are similar results for the other two cases, where only one of \( b(0) \) or \( b(1) \) vanishes. In this case the solution tends asymptotically to the constant \( f(0) \), if \( b(0) = 0 \), and \( f(1) \), if \( b(1) = 0 \).

11 The Resolvent of \( A \)

The Hille-Yosida theorem states that if \( A \) is the infinitesimal generator of a contraction semi-group \( Q_t \), then the right half plane belongs to the resolvent set of \( A \). For \( \lambda \) with positive real part the resolvent, \((\lambda - A)^{-1}\), is given by the Laplace transform of \( Q_t \):

\[
(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} Q_t dt.
\]  

If \( A \) is the \( C^0 \)-graph closure of a generalized Wright-Fisher operator, \( L \), then \( Q_t \), defined above, is a contraction semi-group on \( C^0([0, 1]) \), and therefore the right
half-plane is in $\rho(A)$. In this section we consider the higher order regularity of solutions to $(\lambda - A)w = f$.

Our regularity results show that if $f \in C^m([0, 1])$ then the solution, $u$ to (15) satisfies

$$u \in C^0([0, \infty); C^m([0, 1]) \cap C^\infty([0, 1] \times (0, \infty)).$$

(79)

We can therefore differentiate the equation satisfied by $u$ with respect to $x$ to obtain that, for $1 \leq j \leq m$, and $t > 0$,

$$\partial_t \left[ \partial^j_x u \right] = x(1 - x)\partial^2_x \left[ \partial^j_x u \right] + (b(x) + j(1 - 2x))\partial_x \left[ \partial^j_x u \right] + c_j(x)\left[ \partial^j_x u \right],$$

(80)

for a function $c_j \in C^\infty([0, 1])$. The operator

$$L_{[j]} = x(1 - x)\partial^2_x + (b(x) + j(1 - 2x))\partial_x,$$

(81)

is a generalized Wright-Fisher operator. Applying a standard extension of the maximum principle to equations with a zero order term, see Theorem 4 in Chapter 3.3 of [12], we easily obtain

**Proposition 18.** For $m \in \mathbb{N}$, there are constants $C_m, \mu_m \geq 0$, so that for $f \in C^m([0, 1])$, the solution to (15) satisfies

$$\|u(\cdot, t)\|_{C^m([0,1])} \leq C_m e^{\mu_m t}\|f\|_{C^m([0,1])}.$$  

(82)

Combined with the continuity result in Theorem 1 this shows that $Q_t$ defines a semi-group on $C^m([0, 1])$ with $\|Q_t\|_{C^m} \leq C_m e^{\mu_m t}$. We let the infinitesimal generators be denoted by $A_m$. By Nelson’s theorem, these can be taken as the $C^m$-graph-norm closure of $L$ acting on $C^\infty([0, 1])$. As $Q_t$ is a compact operator on $C^m([0, 1])$ for $t > 0$, the operators $A_m$ are compact for all $m \in \mathbb{N}$.

Evidently the resolvent set of $A_m$ contains the half-plane $\{ \text{Re } \lambda > \mu_m \}$. Because $A_m$ is a compact operator, if $\lambda \in \sigma(A_m)$, then there is an eigenvector $u \in \text{Dom}(A_m)$ so that

$$Lu = \lambda u.$$  

(83)

Because $Q_t u = e^{\lambda t} u \in C^\infty([0, 1])$, we see that the eigenvectors all belong to $C^\infty([0, 1])$, and therefore an eigenvector of $A_m$ is also an eigenvector of $A$, and vice versa. This shows that, as a point-set,

$$\sigma(A_m) = \sigma(A) \subset \{ \lambda : \text{Re } \lambda \leq 0 \}.$$  

(84)

**Theorem 3.** If $m \in \mathbb{N}$, then for $f \in C^m([0, 1])$ and $\lambda$, with positive real part, the solution $w \in \text{Dom}(A)$ to the equation

$$(\lambda - L)w = f,$$

(85)
belongs to $C^m([0, 1])$ and if $\text{Re} \lambda > \mu_m$, then

$$\|w\|_{C^m} \leq \frac{\|f\|_{C^m}}{|\lambda - \mu_m|}, \quad (86)$$

**Proof.** The Hille-Yosida theorem shows that the estimate (82) implies that $\{\lambda : \text{Re} \lambda > \mu_m\}$ belongs to the resolvent set of $A_m$. For $f \in C^0$ we set

$$(\lambda - A)^{-1}f = \int_0^\infty e^{-\lambda t} Q_t f \, dt. \quad (87)$$

This is an analytic $C^0$-valued function in $\{\lambda : \text{Re} \lambda > 0\}$. If $f \in C^m$, then in $\{\lambda : \text{Re} \lambda > \mu_m\}$, this equals $((\lambda - A_m)^{-1}f$, which is an analytic $C^m$-valued function in, $\rho(A_m)$, the resolvent set of $A_m$. As noted in (84), $\rho(A_m)$, includes $\{\lambda : \text{Re} \lambda > 0\}$, which completes the proof of the theorem.

This is not, in any real sense, an elliptic estimate, as it only shows that the solution $w$ is at least as regular as $f$. Of course in the interior of the interval, $w$ has two more derivatives than $f$, but this may not be true, in a uniform sense, up to the boundary.

12 **The adjoint semi-group**

It is a consequence of the boundary behavior of $q_t$, which follows from Corollary 5 that for $t > 0$, and each $x \in [0, 1]$,

$$L_t^* q_t(x, \cdot) \in L^1([0, 1]) \quad (88)$$

The results in Section 9 show that, if $f \in C^m([0, 1])$ for $m \geq 0$, then

$$u(x,t) = \int_0^1 q_t(x,y) f(y) \, dy \in C^0([0, \infty); C^m([0,1]))$$

and satisfies $\partial_t u = Lu$ with $u(x,0) = f(x)$. Thus for $f \in C^2([0,1])$, uniqueness for this initial value problem shows that, for $t > 0$,

$$Lu(x,t) = \int_0^t q_t(x,y) Lf(y) \, dy.$$
By Corollary 5, \( q_t \) satisfies the adjoint boundary conditions (13), so the integrability of \( L_y^t q_t \), implies that we can integrate by parts to conclude that

\[
Lu(x,t) = \int_0^t L_y^t q_t(x,y) f(y) \, dy.
\]

Since

\[
\partial_t u = \int_0^1 \partial_t q_t(x,y) f(y) \, dy,
\]

it follows immediately that, for all \( f \in C^2([0,1]) \) and \( t > 0 \),

\[
\int_0^t (\partial_t - L_y^t)q_t(x,y) f(y) \, dy = 0,
\]

and hence we conclude by a straightforward limiting argument the following:

**Theorem 4.** If \( g \in C^0([0,1]) \), then, for \( t > 0 \)

\[
v(y,t) = \int_0^1 q_t(x,y)g(x) \, dx,
\]

satisfies the boundary conditions (13), and solves the initial value problem

\[
(\partial_t - L_y) v(y,t) = 0 \text{ and } \lim_{t \to 0^+} v(y,t) = g(y).
\]

The dual space of \( C^0([0,1]) \) is naturally identified with \( \mathcal{M}([0,1]) \), the space of Borel measures with finite total variation on \([0,1]\). The dual semi-group, \( Q'_t \), is thus canonically defined on this space by

\[
\langle Q_t f, d\mu \rangle = \langle f, Q'_t d\mu \rangle.
\]

However, \( C^0([0,1]) \) is not a reflexive Banach space, so the dual semi-group is weak\(^*\) -continuous, but not necessarily strongly continuous. The infinitesimal generator \( A \) of \( Q_t \) has a canonically defined adjoint, \( A^* \), whose domain is defined by the prescription: a measure \( d\nu \in \text{Dom}(A^*) \) if there exists a constant \( C \) so that for every \( f \in \text{Dom}(A) \),

\[
\left| \int_0^\infty A f(x) d\nu(x) \right| \leq C \| f \|_{C^0}.
\]
The subtlety is that $\text{Dom}(A^*)$ may not be dense in $\mathcal{M}([0, 1])$. Following Phillips, see [7], we define the adjoint semi-group as

$$Q_t^\ominus = Q_t |_{\mathcal{M}^\ominus}, \text{ where } \mathcal{M}^\ominus = \overline{\text{Dom}(A^*)} \cap A^* \text{Dom}(A^*)$$

Phillips shows that $Q_t^\ominus$ is a strongly continuous semi-group on $\mathcal{M}^\ominus$, with infinitesimal generator:

$$A^\ominus = A^* |_{\text{Dom}(A^*) \cap cM^\ominus}$$

Thus our task is to identify $\mathcal{M}^\ominus$.

By Theorem 4, measures of the form $d\mu = g(y)dy$ with $g \in C^2((0, 1))$ are in $\text{Dom}(A^\ominus)$ provided $L_t g \in L^1$ and $g$ satisfies the adjoint boundary conditions (13). The closure of all such measures with respect to the topology of $\mathcal{M}([0, 1])$ is $L^1([0, 1])$. Any classical eigenvector of $L_t$ satisfying (13) belongs to $\mathcal{M}^\ominus$. Elliptic regularity implies that a distributional solution to $L_t d\mu = d\nu$ is absolutely continuous on $(0, 1)$, and hence

$$L^1([0, 1]) \subset \mathcal{M}^\ominus.$$

In fact, except for the possibility of atomic measures at 0 and/or 1, these two spaces are equal.

If $b(0) \neq 0$, then for any $f \in C^2([0, 1])$,

$$\langle Lf, \delta(y) \rangle = b(0) \partial_x f(0).$$

Since the right hand side does not represent a bounded functional on $C^0([0, 1])$, we obtain that $\delta \notin \text{Dom}(A^*)$. A similar calculation can be done at $x = 1$. On the other hand, if $b$ vanishes at either end, then:

1. If $b(0) = b(1) = 0$, then the nullspace of $A^\ominus$ is spanned by $\delta(y)$ and $\delta(1-y)$, both of which belong to $\mathcal{M}^\ominus$.

2. If one of $b(0)$ or $b(1)$ is non-zero, then $A^\ominus$ has a 1-dimensional nullspace spanned by $\delta(y)$, if $b(0) = 0$, or $\delta(1-y)$, if $b(1) = 0$, and this nullspace again lies in $\mathcal{M}^\ominus$.

Summarizing these observations we have proved the

**Proposition 19.**

- If $b(0)$ and $b(1)$ are non-vanishing, then $\mathcal{M}^\ominus = L^1([0, 1])$.

- If $b(0) = b(1) = 0$, then $\mathcal{M}^\ominus = L^1([0, 1]) \oplus \text{span}\{\delta(y), \delta(1-y)\}$;

- If $b(0) = 0$, $b(1) \neq 0$, then $\mathcal{M}^\ominus = L^1([0, 1]) \oplus \text{span}\{\delta(y)\}$;

- If $b(0) \neq 0$, $b(1) = 0$, then $\mathcal{M}^\ominus = L^1([0, 1]) \oplus \text{span}\{\delta(1-y)\}$.
The spectrum of the operator $A^\circ$ equals that of $A$, see Theorem 14.3.3 in [7]; therefore we have the same sort of asymptotics for solutions to (89):

**Theorem 5.** If $0 < b(0)$ and $b(1) < 0$, then there exists a $\lambda_1 < 0$, so that the solution $v(x, t)$ to (89), with initial data $f \in L^1([0, 1])$ satisfies

$$v(x, t) = c_0 v_0(x) + O(e^{\lambda_1 t}),$$

where $c_0 = \int_0^1 f(y) \, dy$.

Here $v_0 \in \text{Dom}(A^*)$ is the positive solution to $L^t v_0 = 0$, normalized to have integral 1. If $b(0) = b(1) = 0$, then there is a $\lambda_1 < 0$ such that

$$v(x, t) = c_0 \delta(y) + c_1 \delta(1-y) + O(e^{\lambda_1 t}),$$

where $c_0 = \int_0^1 (1 - u_0(y)) f(y), \quad c_1 = \int_0^1 u_0(y) f(y) \, dy$.

where $u_0$ is defined in (72) and (73). Finally, if only one of $b(0)$ or $b(1)$ vanishes, then

$$v(x, t) = \begin{cases} c \delta(y) + O(e^{\lambda_1 t}), & b(0) = 0, \\ c \delta(1-y) + O(e^{\lambda_1 t}), & b(1) = 0 \end{cases}, \quad c = \int_0^1 u_0(y) f(y) \, dy.$$

**Proof.** The first case is immediate from the fact that 0 is an isolated point in the spectrum of $A^\circ$, and all other eigenvalues have strictly negative real part. The second case follows from (77) and the analogous fact about the spectrum of $A^{D^*}$; the last two assertions are similar.

As a simple special case of our regularity theorem for solutions of the backwards equation we have:

**Corollary 7.** If $u(x, t)$ is a solution to

$$\partial_t u = x(1-x) \partial_x^2 u \quad u(x, 0) = f(x),$$

for $f \in C^2([0, 1])$, then, for $(j + k) \leq l$, the functions $\partial_t^j L^k u$ are continuous on $[0, 1] \times [0, \infty)$.

If $v$ solves the forward Kolmogorov equation:

$$\partial_t v = \partial_y^2 [g(1-y)v] \quad \text{with} \quad v(y, 0) = g(y) \in C^2([0, 1]),$$

49
then it is a simple calculation to see that $u(x, t) = x(1 - x)v(x, t)$, solves the backwards equations, with $u(x, 0) = x(1 - x)g(x)$, and $u(0, t) = u(1, t) = 0$.

The theorem shows that $x(1 - x)v(x, t)$ is therefore a $C^{2l}$ function on $[0, 1] \times [0, \infty)$, vanishing at 0 and 1. This easily implies that $v$ itself is in $C^{2l}([0, 1] \times [0, \infty))$. If we let $L_{WF}^t g = \partial^2_x x(1 - x)g$, then this regularity shows that for any $1 \leq j \leq l$

$$\partial_t L_{WF}^{t(j-1)} v = L_{WF}^{tj} v. \quad (92)$$

We can integrate the forward equation, and use this formula, repeatedly integrating by parts, to obtain

$$v(x, t) = g(x) + \int_0^t L_{WF}^t v(x, s) ds$$

$$= g(x) + (s - t)L_{WF}^t v(x, s) \bigg|_{s=0}^{s=t} + \int_0^t (t - s)L_{WF}^{t2} v(x, s) ds \quad (93)$$

$$= \sum_{j=0}^{l-1} \frac{t^j L_{WF}^{tj}}{j!} g + \frac{1}{(l - 1)!} \int_0^t (t - s)^l L_{WF}^{tl} v(x, s) ds. \quad (94)$$

Using the same argument, we can show that if $f \in C^{2l}([0, 1])$ and $u$ solves the initial value problem: $(\partial_t - L)u = 0, u(x, 0) = f(x)$, for $L$ a generalized Wright-Fisher operator, then

$$u(x, t) = \sum_{j=0}^{l-1} \frac{t^j L^j f}{j!} + \frac{1}{(l - 1)!} \int_0^t (t - s)^l L^l u(x, s) ds. \quad (94)$$

Applying Proposition 18 show that there is a constant $M_{2l}$ so that

$$|L^l u(x, s)| \leq M_{2l} e^{\mu_2 t} \|f\|_{C^{2l}}, \quad (95)$$

and therefore:

$$\left| u(x, t) - \sum_{j=0}^{l-1} \frac{t^j L^j f}{j!} \right| \leq \frac{M_{2l} \|f\|_{C^{2l}} e^{\mu_2 t} t^l}{l!}. \quad (96)$$

A Appendix

Proof of Proposition 2
Consider the functions
\[ v(x, t; s) = \int_0^\infty k_{t-s}^b(x, y) y h(y) \partial_y g(y, s) \, dy, \]
defined when \( t \geq s \); these satisfy
\[ (\partial_t - L_b)v = 0 \text{ and } v(x, s; s) = x h(x) \partial_x g(x, s). \]

By the maximum principle and the derivative bound on \( g \) it follows that
\[ |v(x, t; s)| \leq \sup |x h(x) M/\sqrt{s x} \leq \sqrt{LM} \|h\|_\infty, \]
and hence
\[ |A_t^b g(x)| \leq \int_0^t M \sqrt{L} \|h\|_\infty \, ds = 2M \sqrt{Lt} \|h\|_\infty. \]

This establishes (50) for \( j = 1 \), with \( d_0 = 1 \). Now apply Lemma 5 to get
\[ |\partial_x v(x, t; s)| \leq \frac{C_b \|v(\cdot, s; s)\|_\infty}{\sqrt{x(t-s)}}, \]
so that
\[ |\partial_x A_t^b g(x)| \leq \int_0^t C_b M \sqrt{L} \|h\|_\infty \frac{ds}{\sqrt{s(t-s)}} = \frac{C_b \pi M \sqrt{L} \|h\|_\infty}{\sqrt{x}}; \]
setting \( d_1 = \pi \), this is (51) for \( j = 1 \).

Assume that we have chosen \( d_0, \ldots, d_{j-1} \). Now write
\[ v_j(x, t; s) = \int_0^\infty k_{t-s}^b(x, y) y \partial_y v_{j-1}(y, s) \, dy. \]
Using (51) for \( j - 1 \), this is bounded by
\[ d_{j-1} M C_b^{j-1} (\sqrt{L} \|h\|_\infty)^j \int_0^\infty k_{t-s}^b(x, y) y \frac{dy}{s^2} = 2 \frac{d_{j-1} M C_b^{j-1} (\sqrt{L} \|h\|_\infty)^j t^2}{j}. \]
since \( \int_0^\infty k_{t-s}^b(x,y) \, dy = 1 \) for \( s < t \), which establishes (50) for \( j \).

Using the induction hypothesis one more time, insert the result into the estimate of Lemma 5. Noting that \( 1/(t - s + \sqrt{x(t - s)}) \leq 1/\sqrt{x(t - s)} \), we get

\[
|\partial_x (A_j^b g(x))| \leq \int_0^t \frac{d_{j-1} \sqrt{L} \|h\|_\infty \rho_s^{j+1} \sqrt{x(t - s)}}{\sqrt{x(t - s)}} ds
\]

\[
= \frac{d_{j-1} \sqrt{L} \|h\|_\infty \rho_s^{j+1} \sqrt{x(t - s)}}{\sqrt{x}} \int_0^1 \frac{\sigma^{j+1/2}}{\sqrt{1 - \sigma}} d\sigma.
\]

Evaluating the integral shows that we should set

\[
d_j = \sqrt{\pi} d_{j-1} \frac{\Gamma \left( \frac{j}{2} \right)}{\Gamma \left( \frac{j + 1}{2} \right)},
\]

and this proves (51) for \( j \), thereby completing the induction. A straightforward calculation shows that \( d_j \) is given by the formula in the statement of the Proposition.

□

**Proof of Proposition 10**

We start with \( j = 1 \). These results follow from (48), the maximum principle, and Proposition 9. Combining these ingredients shows that when \( t < T \),

\[
|\partial_x^\ell A_1^b g(x)| \leq \sqrt{\ell} [2^{\ell+1} \|xh\|_{C^\ell} (M + \|g\|_{C^\ell,\infty ([0,T])})],
\]

and,

\[
|\partial_x^{\ell+1} A_1^b g(x)| \leq \frac{d_1 \sqrt{MCb+\ell\sqrt{L} \|h\|_\infty}}{\sqrt{x}} + \sqrt{\ell} [2^{\ell+1} \|xh\|_{C^{\ell+1}} (1 + \sqrt{L}) (\|g\|_{C^{\ell,\infty}} + M)],
\]

which establishes (53) and (54) when \( j=1 \). The main issue is to see how \( D_j \) decreases as \( j \) increases. We assume that these estimates have been established for \( \{1, \ldots, j - 1\} \).

Applying (48), we see that

\[
\partial_x^\ell (A_1^b)^j g(x) = A_1^{b+\ell} (\partial_x^\ell (A_1^b)^{j-1} g)(x) + \sum_{p=1}^\ell \binom{\ell}{p} \int_0^t \int_0^\infty k_{t-s}^{b+\ell} (x,z) \partial_z^p (zh(z)) \partial_z^{\ell+1-p} (A_1^b)^{j-1} g(z) \, dz \, ds.
\]

52
Using the induction hypothesis and the maximum principle, the first term on the right here is bounded by
\[
\frac{2D_{j-1}\sqrt{L}\Vert h\Vert_{\infty} t^j}{j} (M + \Vert g\Vert_{\ell,\infty}) [C_{T,L,\ell,b}]_{\ell+1}^{j-1}.
\]
The terms in the sum with \(2 \leq p \leq \ell\) can be bounded using the maximum principle and Lemma 6, while the term with \(p = 1\) is controlled using the induction hypothesis. Thus altogether, the sum is bounded by
\[
\frac{(2^\ell \Vert xh\Vert_{\ell})^j \Vert g\Vert_{\ell} t^j}{j^j} + \frac{2D_{j-2}t^{j+2}(M + \Vert g\Vert_{\ell,\infty}) \Vert xh\Vert_{\ell} [C_{T,L,\ell,b}]_{\ell+1}^{j-1}}{j(j-2)}.
\]
So long as
\[
D_{j-1} \geq \max\left\{\frac{1}{j^j}, \frac{2D_{j-2}}{j-1}\right\},
\]
then there is some \(C_{T,L,\ell,b}\) so that (53) holds for all \(j\). That (97) holds follows easily from the proof of (54), to which we now turn.

We next apply (48) to see that
\[
\partial_x^{\ell+1}(A^j_t)g(x) = \partial_x \int_0^t \int_0^\infty k_t^b(x,z)z h(z) \partial_z^{\ell+1}(A^j_t)g(z) dz ds + \]
\[
\binom{\ell}{1} \partial_x \int_0^t \int_0^\infty k_t^b(x,z)z h(z) \partial_z^\ell(A^j_t)g(z) dz ds +
\]
\[
\sum_{p=2}^\ell \binom{\ell}{p} \int_0^t \int_0^\infty k_t^b(x,z)z h(z) \partial_z^{p+1}(A^j_t)g(z) dz ds.
\]
By the induction hypothesis and Lemma 5, the first term is estimated by
\[
\frac{D_{j-1}(M + \Vert g\Vert_{\ell,\infty}) (C_{T,L,\ell,b})_{\ell+1}^{j-1}}{\sqrt{\pi} t^{j-1}} \int_0^t C_{b+\ell} \sqrt{L} \Vert h\Vert_{\infty} \frac{t^{j-1}}{\sqrt{t-s}} ds = \sqrt{\pi} D_{j-1} t^{j-1} \frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} (M + \Vert g\Vert_{\ell,\infty}) (C_{T,L,\ell,b})_{\ell+1}^{j-1} \sqrt{\pi},
\]
and the second term by:
\[
\frac{\sqrt{\pi} 2D_{j-2} t^{j-2} \frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} (M + \Vert g\Vert_{\ell,\infty}) (C_{T,L,\ell,b})_{\ell+1}^{j-1}}{\sqrt{\pi}}.
\]
Using the induction hypothesis, Lemma 6 and the maximum principle, the last term
is bounded by
\[
2^\ell \frac{\sqrt{\pi} D_{j-2}}{j(j-2)} (M + \|g\|_{C^{\ell, \infty}}) (C'_{T,L,\ell,b})^2 (C_{T,L,\ell} \|h\|_{C^{\ell+1}}).
\]

Hence if \( D_j \) satisfies the recursion relationship in the statement of the proposition,
then \( D_{j-1} \) satisfies (97), for \( j \geq 2 \) and there exist constants \( C_{T,L,\ell,b} \) and \( C'_{T,L,\ell,b} \)
so that (53) and (54) hold for all \( j \geq 2 \).

\[\square\]

References

[1] W. ARENDT, A. GRABOSH, G. GREINER, U. GROH, H. LOTZ, U. MOUSTAKAS, P. NAGEL, F. NEUBRANDER, AND U. SCHLOTTERBECK, One Parameter Semigroups of Positive Operators, vol. 1184 of Lecture Notes in Mathematics, Springer Verlag, Berlin, Heidelberg, New York and Tokyo, 1984.

[2] L. CHEN AND D. STROOCK, The fundamental solution to the Wright-Fisher equation, preprint, (2009), pp. 1–26.

[3] E. B. DAVIES, Linear Operators and Their Spectra, vol. 106 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK, 2007.

[4] W. EWENS, Mathematical Population Genetics, I, 2nd edition, vol. 27 of Interdisciplinary Applied Mathematics, Springer Verlag, Berlin and New York, 2004.

[5] W. FELLER, The parabolic differential equations and the associated semigroups of transformations, Ann. of Math., 55 (1952), pp. 468–519.

[6] I. GRADSHTYEIN AND I. RYZHIK, Table of Integrals, Series and Products, sixth edition, Academic Press, New York, 2000.

[7] E. HILLE AND R. PHILLIPS, Functional Analysis and Semi-groups, revised edition, vol. XXXI of AMS Colloquium Publications, American Mathematical Society, Providence, RI, 1957.

[8] F. JOHN, Partial Differential Equations, 3rd edition, vol. 1 of Applied Mathematical Sciences, Springer Verlag, Berlin Heidelberg New York, 1978.
[9] J. B. Keller and C. Tier, Asymptotic analysis of diffusion equations in population genetics, SIAM J. Appl. Math., 34 (1978), pp. 549–576.

[10] P. D. Lax, Functional Analysis, Pure and Applied Mathematics, John Wiley and Sons, New York, 2002.

[11] E. Nelson, Analytic vectors, Ann. of Math., 70 (1959), pp. 572–614.

[12] M. H. Protter and H. F. Weinberger, Maximum Principles and Differential Equations, Prentice Hall, Inc., Englewood Cliffs, NJ, 1967.