ASYMPTOTIC ANALYSIS OF MULTISCALE MARKOV CHAIN

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Abstract. We consider continuous-time Markov chain on a finite state space $X$. We assume $X$ can be clustered into several subsets such that the intra-transition rates within these subsets are of order $O(\frac{1}{\epsilon})$ comparing to the inter-transition rates among them, where $0 < \epsilon \ll 1$. Several asymptotic results are obtained as $\epsilon \to 0$ concerning the convergence of Kolmogorov backward equation, Poincaré constant, (modified) logarithmic Sobolev constant to their counterparts of certain reduced Markov chain. Both reversible and irreversible Markov chains are considered.

Key words. multiple time scale, continuous-time Markov chain, asymptotic analysis, Poincaré constant, logarithmic Sobolev constant.

AMS subject classifications. 60J27, 34E13, 34E05

1. Introduction.

1.1. Multiscale Markov chain. In recent decades, Markov chains have been intensively investigated due to their effectiveness in modeling systems arising from biology, physics, economics et al. [17, 14, 19]. Inspired by new phenomena from these disciplines, new topics related to Markov chains are continuously emerging and attracting researchers’ attentions. Metastability in Markov chains is one such interesting topic which tries to understand systems’ behaviors on large time scales by eliminating systems’ oscillations on short time scales and identifying certain effective dynamics on large time scales [22, 4].

In this work, we consider a continuous-time Markov chain $C$ on finite state space $X = \{x_1, x_2, \cdots , x_n\}$. We assume $C$ is irreducible and therefore has a unique invariant measure [14, 19]. Suppose state space $X$ can be clustered into $m$ ($m > 1$) nonempty disjoint subsets $X_1, X_2, \cdots , X_m$, with $|X_i| = n_i > 0$, $\sum_{i=1}^{m} n_i = n$. We will be interested in the situation when transitions of system’s states within the same subset occur much more frequently than transitions between states belonging to different subsets. Precisely, let $n \times n$ matrix $Q$ be the infinitesimal generator of Markov chain $C$, which we assume can be written as

$$Q = \frac{1}{\epsilon}Q_0 + Q_1,$$

for some parameter $0 < \epsilon \ll 1$. Matrices $Q_0$ and $Q_1$ satisfy that

1. $Q_0(x, y) \geq 0$, $Q_1(x, y) \geq 0$, if $x \neq y$.
2. Each row sum of their entries equals zero, i.e.

$$\sum_{y \in X} Q_0(x, y) = \sum_{y \in X} Q_1(x, y) = 0, \quad \forall x \in X.$$

3. $Q_0(x, y) = 0$, if $x \in X_i$, $y \in X_j$ for some $1 \leq i \neq j \leq m$.
4. $Q_1(x, y) = 0$, if $x, y \in X_i$ for some $1 \leq i \leq m$ and $x \neq y$.

That is, $Q_0$ and $Q_1$ describe the intra- and inter-transition rates among subsets $X_i$, respectively. Notice that comparing to Chapter 5 and 9 of [21], our setting is more general in that each subset

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may contain different number of states and transitions between states are less restrictive. From
the above assumptions, we can rearrange states in $X$ such that

$$Q_0 = \text{diag}\{Q_{0,1}, Q_{0,2}, \cdots, Q_{0,m}\} \quad (1.2)$$

is a block diagonal matrix consisting of $m$ submatrices $Q_{0,i}$, $1 \leq i \leq m$, where $Q_{0,i}$ is an $n_i \times n_i$
matrix and defines a Markov chain $\mathcal{C}_i$ on subset $X_i$. We further assume that for each $1 \leq i \leq m$,
Markov chain $\mathcal{C}_i$ is irreducible and therefore has a unique invariant measure $\pi_i$. Let $\pi^e$ be the
unique invariant measure of Markov chain $\mathcal{C}$ on $X$. Given $1 \leq i, j \leq m$, we define

$$Q(i, j) = \sum_{x \in X_i, y \in X_j} Q_1(x, y)\pi_i(x). \quad (1.3)$$

It is direct to verify that matrix $Q$ in (1.3) defines an infinitesimal generator (non-negative off-
diagonal elements with zero row sums) of Markov chain $\mathcal{C}$ on $X = \{1, 2, \cdots, m\}$. We will call $\mathcal{C}$
the reduced Markov chain and assume it has a unique invariant measure $\pi^e$.

The main aim of this paper is to consider several objects associated with Markov chain $\mathcal{C}$
and their counterparts associated with Markov chain $\mathcal{C}$. For this purpose, we first introduce the
Kolmogorov backward equations

$$\frac{d}{dt}\rho_t = Q\rho_t = \left(\frac{1}{\epsilon}Q_0 + Q_1\right)\rho_t, \quad \frac{d}{dt}\bar{\rho}_t = \bar{Q}\bar{\rho}_t, \quad (1.4)$$

where $\rho_t : X \to \mathbb{R}$ and $\bar{\rho}_t : \bar{X} \to \mathbb{R}$, $t \geq 0$. These equations play an important role in
understanding the dynamical behaviors of Markov chain $\mathcal{C}$ and $\mathcal{C}$ [19, 21].

We will also consider constants characterizing the speed of Markov chain converging to
equilibrium [19, 21, 22]. Let $\mathbb{E}_{\pi^e}$, $\text{Var}_{\pi^e}$ denote the expectation and variance with respect to
measure $\pi^e$ respectively. First recall the definition of Poincaré constant and logarithmic Sobolev constant for Markov chain $\mathcal{C}$, which are defined as

$$\lambda_e = \inf_{f} \left\{ \frac{\mathbb{E}_e(f, f)}{\text{Var}_{\pi^e} f} \left| \text{Var}_{\pi^e} f > 0, f : X \to \mathbb{R} \right. \right\} \quad (1.5)$$

$$\alpha_e = \inf_{f} \left\{ \frac{\mathbb{E}_e(f, f)}{\text{Ent}_{\pi^e}(f^2)} \left| \text{Ent}_{\pi^e}(f^2) > 0, f : X \to \mathbb{R} \right. \right\}, \quad (1.6)$$

where the infima are taken among all non-constant functions, $\mathbb{E}_e$, $\text{Ent}_{\pi^e}$ are the Dirichlet form
and relative entropy with respect to $\pi^e$, defined as

$$\mathbb{E}_e(f, g) = -\langle f, Qg \rangle_{\pi^e} = -\left\langle f, \left(\frac{Q_0}{\epsilon} + Q_1\right)g \right\rangle_{\pi^e}, \quad f, g : X \to \mathbb{R}, \quad (1.7)$$

$$\text{Ent}_{\pi^e}(f) = \sum_{x \in X} f(x) \ln \frac{f(x)}{\mathbb{E}_{\pi^e} f} \pi^e(x), \quad f : X \to \mathbb{R}^+. \quad (1.8)$$

For function $f : X \to \mathbb{R}$, we have

$$\mathbb{E}_e(f, f) = \frac{1}{2\epsilon} \sum_{i=1}^{m} \sum_{x, x' \in X_i} (f(x') - f(x))^2 Q_{0,i}(x, x') \pi^e(x)$$

$$+ \frac{1}{2} \sum_{x \in X, y \in X, y \neq x} (f(y) - f(x))^2 Q_1(x, y) \pi^e(x), \quad (1.9)$$

which holds in both reversible and non-reversible case [5].
The modified logarithmic Sobolev constant is defined as
\[
\gamma_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, \ln f)}{\operatorname{Ent}_\epsilon(f)} \middle| \operatorname{Ent}_\epsilon(f) > 0, f : X \to \mathbb{R}^+ \right\},
\] (1.10)
where the infimum is taken among all non-constant and non-negative functions. It is known that these constants satisfy
\[
4\alpha_\epsilon \leq \gamma_\epsilon \leq 2\lambda_\epsilon,
\]
in the reversible case, and
\[
2\alpha_\epsilon \leq \gamma_\epsilon \leq 2\lambda_\epsilon, \quad 2\alpha_\epsilon \leq \lambda_\epsilon,
\]
in the non-reversible case. See [2, 3, 5, 10] and references therein for more details. Let \( \bar{\mathcal{E}} \) denote the Dirichlet form of the reduced Markov chain \( \bar{\mathcal{C}} \). Its Poincaré constant \( \bar{\lambda} \), logarithmic Sobolev constant \( \bar{\alpha} \), as well as the modified logarithmic Sobolev constant \( \bar{\gamma} \) can be defined similarly as in (1.5), (1.6), (1.10), by replacing \( \mathcal{E}_\epsilon, \pi_\epsilon, Q \) with \( \bar{\mathcal{E}}, \bar{\pi}, \bar{Q} \) respectively. Correspondingly, they satisfy the inequality
\[
4\bar{\alpha} \leq \bar{\gamma} \leq 2\bar{\lambda},
\]
in the reversible case, and
\[
2\bar{\alpha} \leq \bar{\gamma} \leq 2\bar{\lambda}, \quad 2\bar{\alpha} \leq \bar{\lambda},
\]
in the non-reversible case.

Briefly speaking, in this paper we will establish the convergence of \( \rho_t \) to \( \bar{\rho}_t \) in (1.4), and the convergence of constants \( \lambda_\epsilon, \alpha_\epsilon, \gamma_\epsilon \) in (1.5), (1.6), (1.10) to their counterpart \( \bar{\lambda}, \bar{\alpha} \) and \( \bar{\gamma} \) respectively.

1.2. Notations. In this subsection we collect some notations and definitions used in this paper. Let \( \Omega \) be a finite set. For function \( f : \Omega \to \mathbb{R} \),
\[
|f|_\infty := \max_{x \in \Omega} |f(x)|, \quad |f|_2 := \left( \sum_{x \in \Omega} f^2(x) \right)^{\frac{1}{2}}
\]
are the \( L^\infty \) norm and \( L^2 \) norm of \( f \). Given a matrix \( A \) of order \( k \times l \), denote its infinity norm as \( \|A\|_\infty \), i.e. \( \|A\|_\infty = \sup_{1 \leq i \leq k} \sum_{j=1}^l |a_{ij}| \). For matrix \( Q_1 \) in (1.3), we define
\[
Q_\infty := \|Q_1\|_\infty = \max_{x \in X} \sum_{x' \in X} |Q_1(x, x')| = 2 \max_{x \in X} \sum_{x' \neq x} Q_1(x, x'),
\]
where we have used the fact that the off-diagonal entries of \( Q_1 \) are non-negative, and \( Q_1(x, x') = -\sum_{x' \neq x} Q_1(x, x') < 0, \forall x \in X \). From definition (1.3) of matrix \( \bar{Q} \), it is direct to check \( \|\bar{Q}\|_\infty \leq Q_\infty \).

Let \( \mu \) be a probability measure over set \( \Omega \), \( L^2(\mu) \) is the Hilbert space consisting of all real functions on \( \Omega \) with inner product
\[
\langle f, g \rangle_\mu = \sum_{x \in \Omega} f(x)g(x)\mu(x), \quad \forall f, g : \Omega \to \mathbb{R},
\]
and its norm is denoted as $|\cdot|_{2,\mu}$. We write
\begin{equation}
E_{\mu}f = \sum_{x \in \Omega} f(x)\mu(x), \quad \text{Var}_{\mu}f = \sum_{x \in \Omega} (f(x) - E_{\mu}f)^2\mu(x),
\end{equation}
as the expectation and the variance of function $f$ with respect to $\mu$.

For Markov chain $C_i$ whose infinitesimal generator is $Q_{0,i}$, we denote its Dirichlet form, Poincaré constant, logarithmic Sobolev constant, modified logarithmic Sobolev constant as $E_i$, $\lambda_i$, $\alpha_i$ and $\gamma_i$, respectively. Also set
\begin{equation}
\lambda_{\text{min}} = \min_i \lambda_i, \quad \alpha_{\text{min}} = \min_i \alpha_i, \quad \gamma_{\text{min}} = \min_i \gamma_i.
\end{equation}

Given function $f : X \rightarrow \mathbb{R}$ and $1 \leq i \leq m$, $f(i, \cdot)$ denotes the vector of length $n_i$ consisting of components $f(x)$ for $x \in X_i$, while $\tilde{f}$ denotes a function on $\bar{X}$, defined by $\tilde{f}(i) = \sum_{x \in X_i} f(x)\pi_i(x)$, $1 \leq i \leq m$.

We also need some notations when studying the general non-reversible case. Define
\begin{equation}
\Gamma := \text{tr}(Q_1Q_1^T) = \sum_{x \in \Omega} \sum_{y \in \Omega} Q_1(y,x)^2, \quad d := \max_x \left| \left\{ y \in X \mid Q_1(x,y) \neq 0 \right\} \right|,
\end{equation}
where $|\cdot|$ denotes the cardinality of a given set. $\sigma_i$ and $\bar{\sigma}$ denote the smallest nonzero singular value of matrix $Q_{0,i}$ and $Q_i$, respectively. Also set $\sigma_{\text{min}} = \min_i \sigma_i$.

The paper is organized as follows. Section 2 is devoted to obtain several asymptotic results when Markov chain $C$ is reversible. The general Markov chain without reversibility assumption is studied in Section 3. In Section 4, we discuss our results and make conclusions. Appendix A collects some useful facts related to continuous-time Markov chain. Appendix B contains formal arguments which motivates our asymptotic results.

2. Asymptotic analysis : reversible case. In this section, we establish several asymptotic convergence results under the assumption that Markov chain $C$ is reversible.

2.1. Invariant measure. We start with the invariant measure $\pi^\epsilon$. Taking the structure of matrix $Q$ in (1.1), (1.2) into consideration, the detailed balance condition reads
\begin{equation}
\pi^\epsilon(x)Q_{0,i}(x,x') = \pi^\epsilon(x')Q_{0,i}(x',x), \quad \text{if} \quad x,x' \in X_i,
\pi^\epsilon(x)Q_1(x,y) = \pi^\epsilon(y)Q_1(y,x), \quad \text{if} \quad x \in X_i, \ y \in X_j, \ i \neq j.
\end{equation}
Since we assume Markov chain $C_i$ has a unique invariant measure, the first equation above implies that $C_i$ is also reversible, for $1 \leq i \leq m$, and $\exists w^\epsilon(i) > 0$ s.t.
\begin{equation}
\pi^\epsilon(x) = w^\epsilon(i)\pi_i(x), \quad \text{for} \ x \in X_i.
\end{equation}
We have
\begin{equation}
\sum_{i=1}^m w^\epsilon(i) = \sum_{x \in \Omega} \pi^\epsilon(x) = 1.
\end{equation}
Substituting relation (2.2) into the second equation of (2.1) and summing up all states $x \in X_i, y \in X_j$, we obtain
\begin{equation}
w^\epsilon(i)\bar{Q}(i,j) = w^\epsilon(j)\bar{Q}(j,i), \quad 1 \leq i \neq j \leq m,
\end{equation}
where matrix $\bar{Q}$ is defined in (1.3). Equation (2.3) and (2.4) imply that $w^\epsilon$ coincides with the invariant measure $w$ of Markov chain $C$ and furthermore, $C$ is reversible with respect to $w$. From (2.2) we also know that $\pi^\epsilon$ is independent of parameter $\epsilon$. In the following of this section we will denote it as $\pi$ for simplicity.

### 2.2. Kolmogorov backward equation

We consider the Kolmogorov backward equation

$$\frac{d}{dt} \rho_t = Q \rho_t = \frac{1}{\epsilon} (Q_0 + Q_1) \rho_t$$

with initial condition $\rho_0$ ($\rho_0$ can be negative), or more explicitly,

$$\frac{d}{dt} \rho_t(x) = \frac{1}{\epsilon} \sum_{x' \neq x, x' \in X} (\rho_t(x') - \rho_t(x)) Q_{0,i}(x, x') + \sum_{y \not= X} (\rho_t(y) - \rho_t(x)) Q_{1}(x, y),$$

for $x \in X$, $1 \leq i \leq m$. Multiplying both sides of (2.6) by $\pi_i(x)$, summing up states $x \in X$, and noticing that $Q_{0,i} \pi_i = 0$, we obtain the equation of $\tilde{\rho}_t(i) = \sum_{x \in X} \rho_t(x) \pi_i(x)$ as

$$\frac{d}{dt} \tilde{\rho}_t(i) = \sum_{x \in X} \sum_{y \not= X} (\rho_t(y) - \rho_t(x)) Q_{1}(x, y) \pi_i(x), \quad 1 \leq i \leq m. \tag{2.7}$$

We also introduce the Kolmogorov backward equation of the reduced Markov chain $\bar{C}$

$$\frac{d}{dt} \bar{\rho}_t = \bar{Q} \bar{\rho}_t = \sum_{j \neq i} (\bar{\rho}_t(j) - \bar{\rho}_t(i)) \bar{Q}(i, j)$$

with initial condition $\bar{\rho}_0 = \bar{\rho}_0$, where matrix $\bar{Q}$ is defined in (1.3). We have

**Theorem 2.1.** Assume Markov chain $C$ is reversible. Consider functions $\rho_t$, $\tilde{\rho}_t$ and $\bar{\rho}_t$, which are solutions of equation (2.5), (2.7) and (2.8), respectively. For $t \geq 0$, we have

$$|\rho_t(x) \pi_i - \tilde{\rho}_t(x) \pi_i|_{L^2,\pi_i} \leq \left( e^{-\frac{2\epsilon}{\lambda_{\min}}} + \frac{2\epsilon}{\lambda_{\min}} |Q|_{L^\infty} \right) |\rho_0|_{L^\infty}, \tag{2.9}$$

where constants involved are defined in Section 7.

Before entering the proof, we would like to reinterpret the results of Theorem 2.1 by considering the corresponding Markov chain processes. Let $x_t \in X$ and $\tilde{x}_t \in \bar{X}$ be the Markov chain $C$ and $\bar{C}$, respectively. Given function $f : X \to \mathbb{R}$ and defining $\tilde{f}(i) = \sum_{x \in X} f(x) \pi_i(x)$ as before, we consider quantities

$$f_t(x) = \mathbb{E}(f(x_t) \mid x_0 = x), \quad \tilde{f}_t(i) = \mathbb{E}(f(x_t) \mid x_0 \sim \pi_i), \quad x \in X, \tag{2.11}$$

then we know $f_t$ satisfies equation (2.5) with initial condition $f_0 = f$, while $\tilde{f}_t(i)$ satisfies (2.7) with $\rho_t$ replaced by $f_t$. Similarly define

$$\tilde{f}_t(i) = \mathbb{E}(\tilde{f}(\tilde{x}_t) \mid \tilde{x}_0 = i), \quad 1 \leq i \leq m, \tag{2.12}$$

then $f$ satisfies (2.8) with initial condition $f_0 = \tilde{f}_0 = \tilde{f}$. Theorem 2.1 implies

$$|$
COROLLARY 2.2. Consider reversible Markov chains $x_t \in X$ and $\bar{x}_t \in \bar{X}$ defined by infinitesimal generator $Q$ and $\tilde{Q}$, respectively. Given $f : X \to \mathbb{R}$, define the quantities $f_t, \tilde{f}_t, \bar{f}_t$ by (2.11) and (2.12). We have $\forall t \geq 0$,

$$
|f_t(i, \cdot) - f_t(i)|_{2, \pi} \leq \left(e^{-\lambda t} + \frac{2r}{\lambda_i}Q_\infty \right)|f|_\infty, \quad 1 \leq i \leq m,
$$

$$
|\tilde{f}_t - \bar{f}_t|_{2, w} \leq \frac{Q_\infty|f|_\infty}{\lambda_{\min}}\left(\min \left\{ \frac{1}{\min_{i,x} \pi_i(x)}, \frac{m}{2} \right\}\right)^\frac{1}{2} \left(2Q_\infty + 1\right)\epsilon. \quad (2.13)
$$

Now consider a probability measure $\mu$ on $X$ and define probability measure $\tilde{\mu}$ on $\bar{X}$ by $\tilde{\mu}(i) = \sum_{x \in X_i} \mu(x), 1 \leq i \leq m$. Also define

$$
\rho_t(x) = P(x_t = x | x_0 \sim \mu), \quad x \in X,
$$

$$
\tilde{\rho}_t(i) = P(x_t \in X_i | x_0 \sim \mu) = \sum_{x \in X_i} \rho_t(x), \quad 1 \leq i \leq m, \quad (2.14)
$$

$$
\bar{\rho}_t(i) = P(\bar{x}_t = i | \bar{x}_0 \sim \tilde{\mu}), \quad 1 \leq i \leq m,
$$

and the probability densities with respect to invariant measures $\pi$ and $w$

$$
\rho_t = \frac{d\rho_t}{d\pi}, \quad \tilde{\rho}_t = \frac{d\tilde{\rho}_t}{dw}, \quad \bar{\rho}_t = \frac{d\bar{\rho}_t}{dw}. \quad (2.15)
$$

Recalling the detailed balance condition (2.1), we can check that functions $\rho_t, \tilde{\rho}_t$ and $\bar{\rho}_t$ satisfy equation (2.5), (2.7) and (2.8), respectively (see Appendix A). Therefore Theorem 2.1 implies Corollary 2.2.

COROLLARY 2.3. Consider reversible Markov chains $x_t \in X$ and $\bar{x}_t \in \bar{X}$ defined by infinitesimal generator $Q$ and $\tilde{Q}$, respectively. Given probability measure $\mu$ on space $X$. Let $\rho_t, \tilde{\rho}_t$ and $\bar{\rho}_t$ be the density of probability measures defined by (2.14) and (2.13). For $t \geq 0$, we have

$$
|\rho_t(i, \cdot) - \tilde{\rho}_t(i)|_{2, \pi} \leq \left(e^{-\lambda t} + \frac{2r}{\lambda_i}Q_\infty \right)|\rho_0|_\infty, \quad 1 \leq i \leq m,
$$

$$
|\tilde{\rho} - \tilde{\rho}|_{2, w} \leq \frac{Q_\infty|\rho_0|_\infty}{\lambda_{\min}}\left(\min \left\{ \frac{1}{\min_{i,x} \pi_i(x)}, \frac{m}{2} \right\}\right)^\frac{1}{2} \left(2Q_\infty + 1\right)\epsilon. \quad (2.16)
$$

Proof of Theorem 2.1:

1. We start with the first inequality (2.9) concerning $\rho_t$ and $\tilde{\rho}_t$. For $\rho_t$ satisfying (2.5), we know $|\rho_t|_{\infty} \leq |\rho_0|_\infty$, $t \geq 0$ (can be easily seen from (2.11)). For the right hand side of (2.7), we have

$$
\left| \sum_{x \in X_i} \sum_{y \in X_i} (\rho_t(y) - \rho_t(x))Q_1(x, y)\pi_i(x) \right| 
\leq \sum_{x \in X_i} \sum_{y \in X} |\rho_t(y)Q_1(x, y)| \pi_i(x) \leq |\rho_t|_{\infty} \max_{x \in X} \sum_{y \in X} |Q_1(x, y)| \leq Q_\infty |\rho_0|_\infty.
$$

Therefore (2.7) implies

$$
|\tilde{\rho}_t(i) - \tilde{\rho}_s(i)| \leq |t - s|Q_\infty |\rho_0|_\infty, \quad 1 \leq i \leq m. \quad (2.17)
$$

For equation (2.5) which is written in matrix form, using variation of constants formula, we can obtain

$$
\rho_t = e^{(t-s)Q_\infty} \rho_s + \int_s^t e^{(t-r)Q_\infty} Q_1 \rho_r dr, \quad 0 \leq s \leq t. \quad (2.18)
$$
Since $e^{(t-r)Q_{0}/\epsilon}$ is a stochastic matrix, we have
\[
\left| \int_{s}^{t} e^{(t-r)Q_{0}/\epsilon} Q_{1} \rho_{r} \, dr \right|_{\infty}^{\infty} \leq \int_{s}^{t} \left\| Q_{1} \right\|_{\infty} |\rho_{r}|_{\infty} \, dr \leq (t-s)Q_{\infty}|\rho_{0}|_{\infty} .
\]
(2.19)

For the first term on the right hand side of (2.18), noticing $Q_{0}$ (therefore also $e^{(t-s)Q_{0}/\epsilon}$) is a block diagonal matrix and applying Poincaré inequality [2, 3, 5], we deduce
\[
|e^{(t-s)Q_{0}/\epsilon} \rho_{s}(i, \cdot) - \tilde{\rho}_{s}(i) 1|_{2, \pi_{t}} \leq e^{-\frac{(t-s)\lambda_{\epsilon}}{2}} |\rho_{s}(i, \cdot) - \tilde{\rho}_{s}(i) 1|_{2, \pi_{t}},
\]
(2.20)
where $1 \leq i \leq m$, 1 denotes the constant vector over subset $X_{i}$ and $\rho_{s}(i, \cdot)$ denotes the vector consisting of $\rho_{s}(x)$ for $x \in X_{i}$. Combining estimates (2.17)-(2.20) together, we have
\[
|\rho_{t}(i, \cdot) - \tilde{\rho}_{t}(i) 1|_{2, \pi_{t}} \leq |\rho_{t}(i) - \tilde{\rho}_{t}(i) 1|_{2, \pi_{t}} + 2(t-s)Q_{\infty}|\rho_{0}|_{\infty}.
\]

Fix index $i$ and define $G(t) = |\rho_{t}(i, \cdot) - \tilde{\rho}_{t}(i) 1|_{2, \pi_{t}}$. We subtract $G(s)$ and then divide $(t-s)$ on both sides of the inequality above. Let $t \to s+$, we obtain
\[
\frac{d^{+}G(t)}{dt} + \frac{\lambda_{i}}{\epsilon}G(t) \leq 2Q_{\infty}|\rho_{0}|_{\infty}, \quad t \geq 0 .
\]
(2.21)

Gronwall’s inequality then implies
\[
|\rho_{t}(i, \cdot) - \tilde{\rho}_{t}(i) 1|_{2, \pi_{t}} \leq e^{-\frac{\lambda_{i}}{\epsilon}(t-s)} |\rho_{0}(i, \cdot) - \tilde{\rho}_{0}(i) 1|_{2, \pi_{t}} + \frac{2\epsilon}{\lambda_{i}} Q_{\infty}|\rho_{0}|_{\infty} \leq \frac{2\epsilon}{\lambda_{i}} Q_{\infty}|\rho_{0}|_{\infty} .
\]
(2.22)

where we have used
\[
|\rho_{0}(i, \cdot) - \tilde{\rho}_{0}(i) 1|_{2, \pi_{t}} = \left[ \sum_{x \in X_{i}} \left( \rho_{0}(x) - \sum_{x' \in X_{i}} \rho_{0}(x') \pi_{i}(x') \right)^{2} \right]^{\frac{1}{2}} = \left[ \sum_{x \in X_{i}} \rho_{0}(x)^{2} \pi_{i}(x) - \left( \sum_{x \in X_{i}} \rho_{0}(x) \pi_{i}(x) \right)^{2} \right]^{\frac{1}{2}} \leq |\rho_{0}|_{\infty} .
\]

2. Now we turn to the second inequality (2.10). First notice that the equation of $\tilde{\rho}_{i}$ in (2.7)
can be rewritten as

\[ \frac{d}{dt} \tilde{\rho}_t(i) = \sum_{j \neq i} \sum_{x \in X_i, y \in X_j} (\rho_t(y) - \rho_t(x))Q_1(x, y)\pi_i(x) \]

\[ = \sum_{j \neq i} (\tilde{\rho}_t(j) - \tilde{\rho}_t(i))Q(i, j) \]

\[ + \sum_{j \neq i} \sum_{x \in X_i, y \in X_j} \left[ (\rho_t(y) - \tilde{\rho}_t(j)) - (\rho_t(x) - \tilde{\rho}_t(i)) \right]Q_1(x, y)\pi_i(x) \]

\[ = \sum_{j \neq i} (\tilde{\rho}_t(j) - \tilde{\rho}_t(i))Q(i, j) + \phi_t(i) = Q\tilde{\rho}_t + \phi_t, \quad (2.23) \]

where

\[ \phi_t(i) = \sum_{j \neq i} \sum_{x \in X_i, y \in X_j} \left[ (\rho_t(y) - \tilde{\rho}_t(j)) - (\rho_t(x) - \tilde{\rho}_t(i)) \right]Q_1(x, y)\pi_i(x) \]

\[ = \sum_{j=1}^{m} \sum_{x \in X_i, y \in X_j} (\rho_t(y) - \tilde{\rho}_t(j))Q_1(x, y)\pi_i(x), \]

since the row sums of \( Q_1 \) are zero. Using detailed balance condition (2.21), we can obtain

\[ E_w\phi_t = \sum_{i=1}^{m} \phi_t(i)w(i) \]

\[ = \sum_{i=1}^{m} \left[ \sum_{j=1}^{m} \sum_{x \in X_i, y \in X_j} (\rho_t(y) - \tilde{\rho}_t(j))Q_1(x, y)\pi_i(x) \right]w(i) \]

\[ = \sum_{x \in X} \sum_{y \in X} (\rho_t(y) - \tilde{\rho}_t(j))\pi(y)Q_1(y, x) = 0. \]

We also need to estimate \( |\phi_t|_{2, w} \). On one hand, applying inequality (2.22), we can deduce a pointwise estimate

\[ |\rho_t(i, x) - \tilde{\rho}_t(i)| \leq \sqrt{\min_{x' \in X_i} \pi_i(x')} \left( e^{-\frac{\Delta t}{\lambda_{\min}}} + \frac{2\epsilon}{\lambda_{\min}Q_\infty} \right) |\rho_0|_\infty, \quad \forall x \in X_i. \quad (2.24) \]

Therefore

\[ |\phi_t|_{2, w} \leq \max_i |\phi_t(i)| \leq Q_\infty \max_i |\rho_t(i, \cdot) - \tilde{\rho}_t(i)|_\infty \]

\[ \leq Q_\infty \sqrt{\min_{i, x \in X_i} \pi_i(x')} \left( e^{-\frac{\Delta t}{\lambda_{\min}}} + \frac{2\epsilon}{\lambda_{\min}Q_\infty} \right) |\rho_0|_\infty. \quad (2.25) \]
On the other hand, we can avoid using pointwise estimate (2.24) and compute

\[ |\phi_i|_{2,w}^2 = \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \left( \sum_{x \in X_j} \sum_{y \in x_j} \left( \rho_i(y) - \tilde{\rho}_i(j) \right) Q_i(x,y) \right)^2 \right) w(i) \]

\[ \leq \left[ \sum_{j=1}^{m} \left( \rho_i(y) - \tilde{\rho}_i(j) \right)^2 \right] \left[ \sum_{i=1}^{m} \left( \sum_{x \in X_i} \left( \sum_{y \in x_i} Q_i(x,y) \right)^2 \right) \right] \frac{1}{w(i)} \]

where we have used detailed balance condition (2.1) and relation (1.16). Together with (2.25), we could deduce

\[ |\phi_i|_{2,w} \leq \left( \min \left\{ \frac{1}{\min_{x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} Q_{\infty} \left( e^{-\frac{\lambda_{\min}}{\lambda_{\max}}} + \frac{2\epsilon}{\lambda_{\min}} Q_{\infty} \right) |\rho_0|_{\infty}. \tag{2.26} \]

Now subtract (2.26) by equation (2.25), we obtain

\[ \frac{d}{dt} (\tilde{\rho}_t - \tilde{\rho}_t) = \tilde{Q}(\tilde{\rho}_t - \tilde{\rho}_t) + \phi_t, \tag{2.27} \]

together with initial condition \( \tilde{\rho}_0 = \tilde{\rho}_0 \). Therefore we have

\[ \tilde{\rho}_t - \tilde{\rho}_t = \int_0^t e^{(t-s)\tilde{Q}} \phi_s ds. \tag{2.28} \]

Since \( E \phi = 0 \), Poincaré inequality implies

\[ |e^{(t-s)\tilde{Q}} \phi_s|_{2,w} \leq e^{-\lambda(t-s)} |\phi_s|_{2,w}. \tag{2.29} \]

Therefore, using (2.28), we have

\[ |\tilde{\rho}_t - \tilde{\rho}_t|_{2,w} \leq \int_0^t e^{-\lambda(t-s)} |\phi_s|_{2,w} ds \leq \left( \min \left\{ \frac{1}{\min_{x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} Q_{\infty} \int_0^t e^{-\lambda(t-s)} |\rho_0|_{\infty} \left( e^{-\frac{\lambda_{\min}}{\lambda_{\max}}} + \frac{2\epsilon}{\lambda_{\min}} Q_{\infty} \right) ds \]

\[ \leq \frac{Q_{\infty} |\rho_0|_{\infty}}{\lambda_{\min}} \left( \min \left\{ \frac{1}{\min_{x \in X_i} \pi_i(x)}, \frac{m}{2} \right\} \right)^{\frac{1}{2}} \left( \frac{2Q_{\infty}}{\lambda} + 1 \right) \epsilon. \]

2.3. Poincaré constant, (modified) logarithmic Sobolev constants. In this subsection we consider the asymptotic behavior of the Poincaré constant \( \lambda_{\epsilon} \), logarithmic Sobolev constant \( \alpha_{\epsilon} \), and modified logarithmic Sobolev constant \( \gamma_{\epsilon} \) defined in (1.5), (1.6) and (1.10),
for all \( f, g \in X \rightarrow \mathbb{R} \). See [3, 4] and Appendix A for more details.

We start with the Poincaré constant.

**Theorem 2.4.** Assume Markov chain \( C \) is reversible and \( \epsilon \leq 1 \). Let \( \lambda_\epsilon, \hat{\lambda} \) be the Poincaré constants of Markov chain \( C \) and \( \hat{C} \) corresponding to infinitesimal generator \( \lambda \) and \( \hat{\lambda} \), respectively. We have

\[
\lambda_\epsilon = \inf \left\{ \frac{\mathcal{E}_\epsilon(f, f)}{\text{Var}_\pi f} \mid \text{Var}_\pi f > 0, f : X \rightarrow \mathbb{R} \right\},
\]

(2.31)

and the Dirichlet form \( \mathcal{E}_\epsilon \) in \( L^2_\pi \).

1. First choose function \( f : X \rightarrow \mathbb{R} \), s.t. \( f(x) = g(i) \) for \( x \in X_i \), where \( g \) is a function on \( \bar{X} \). Using the fact \( \pi(x) = \pi_i(x)w(i) \) when \( x \in X_i \), from (2.30) we know

\[
\mathcal{E}_\epsilon(f, f) = \frac{1}{2} \sum_{1 \leq i,j \leq m} (g(j) - g(i))^2 \hat{\lambda} Q(i, j) w(i) = \hat{\mathcal{E}}(g, g).
\]

It is also straightforward to check \( \mathcal{E}_\epsilon f = \mathcal{E}_w g \) and \( \text{Var}_\pi f = \text{Var}_w g \). Allowing \( g \) to vary among all functions from \( \bar{X} \) to \( \mathbb{R} \), we obtain \( \lambda_\epsilon \leq \hat{\lambda} \), i.e. the upper bound of the theorem.

2. For the lower bound, we assume the minimum in (2.31) is obtained by function \( f \), i.e.

\[
\frac{\mathcal{E}_\epsilon(f, f)}{\text{Var}_\pi f} = \lambda_\epsilon \leq \hat{\lambda}.
\]

The estimation of \( \lambda_\epsilon \) can be obtained if we could estimate \( \mathcal{E}_\epsilon(f, f) \) and the variance of \( f \).

From (2.30), we easily obtain

\[
\frac{1}{2\epsilon} \sum_{i=1}^m \left[ \sum_{x,x' \in X_i} (f(x') - f(x))^2 Q_0,i(x, x') \pi_i(x) \right] w(i) \leq \hat{\lambda} \text{Var}_\pi f .
\]

(2.32)

Applying Poincaré inequality to Markov chain \( C \), for each fixed \( i \), we obtain

\[
\sum_{i=1}^m \left[ \sum_{x \in X_i} (f(x) - \bar{f}(i))^2 \pi_i(x) \right] w(i) \leq \frac{\hat{\lambda}}{\lambda_{\text{min}}} \text{Var}_\pi f \epsilon ,
\]

(2.33)

where \( \bar{f}(i) = \sum_{x \in X_i} f(x) \pi_i(x) \). Using the elementary inequality

\[
(1 + \epsilon^2) \epsilon^2 \geq (a + b + c)^2 - 2(1 + \epsilon^{-\frac{1}{2}})(a^2 + b^2) , \quad \forall a, b, c \in \mathbb{R} ,
\]

(2.34)
and the detailed balance condition (2.1), we can estimate the Dirichlet form $\mathcal{E}_\epsilon$ in (2.30)

$$
\mathcal{E}_\epsilon(f, f) = \frac{1}{2} \sum_{i \neq j} \sum_{x_i, y_j} (f(y) - f(x))^2 Q_1(x, y) \pi(x)
$$

Recalling the definition of $Q_\infty$ in (1.16) and applying (2.33), we can estimate the second term on the right hand side of (2.35) and obtain

$$
\sum_{i \neq j} \sum_{x_i, y_j} (f(x) - \bar{f}(i))^2 Q_1(x, y) \pi(x) \leq \frac{Q_\infty}{2} \sum_{i=1}^m \sum_{x_i} (f(x) - \bar{f}(i))^2 \pi(x) \leq \frac{\lambda Q_\infty}{2 \lambda_{\min}} \text{Var}_\pi f \epsilon.
$$

To estimate the variance of $f$, we apply inequality (2.33), the elementary inequality

$$(a + b)^2 \leq (1 + \epsilon^{-\frac{1}{2}})a^2 + (1 + \epsilon^{\frac{1}{2}})b^2, \quad \forall a, b \in \mathbb{R},$$

together with the Poincaré inequality for the reduced Markov chain $\tilde{C}$. It gives

$$
\text{Var}_\pi f = \sum_{x \in X} (f(x) - \sum_{x' \in X} f(x') \pi(x'))^2 \pi(x)
$$

$$
= \sum_{x \in X} \left( f(x) - \sum_{i=1}^m \bar{f}(i) w(i) \right)^2 \pi(x)
$$

$$
\leq (1 + \epsilon^{-\frac{1}{2}}) \sum_{x \in X} \sum_{i=1}^m (f(x) - \bar{f}(i))^2 \pi(x) + (1 + \epsilon^{\frac{1}{2}}) \sum_{i=1}^m \sum_{j=1}^m (\bar{f}(i) - \sum_{j=1}^m \bar{f}(j) w(j))^2 w(i)
$$

$$
\leq (1 + \epsilon^{-\frac{1}{2}}) \frac{\lambda \epsilon}{\lambda_{\min}} \text{Var}_\pi f + \frac{1}{2} (1 + \epsilon^{\frac{1}{2}}) \lambda^{-1} \sum_{1 \leq i, j \leq m} (\bar{f}(j) - \bar{f}(i))^2 Q(i, j) w(i).
$$

Combining (2.35)–(2.37), we arrive at

$$
\mathcal{E}_\epsilon(f, f) \geq \frac{\lambda}{(1 + \epsilon^{\frac{1}{2}})^2} \text{Var}_\pi f \left[ 1 - (1 + \epsilon^{-\frac{1}{2}}) \frac{\lambda \epsilon}{\lambda_{\min}} \right] - \epsilon^2 \frac{\lambda Q_\infty}{\lambda_{\min}} \text{Var}_\pi f,
$$

which implies

$$
\lambda \epsilon \geq \frac{\lambda}{(1 + \epsilon^{\frac{1}{2}})^2} \left( 1 - \frac{2 \lambda \epsilon}{\lambda_{\min}} \right) - \lambda Q_\infty \epsilon^2
$$

when $\epsilon \leq 1$. 

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We continue to study the logarithmic Sobolev constant.

**Theorem 2.5.** Assume Markov chain $C$ is reversible. Let $\alpha, \alpha$ be the logarithmic Sobolev constants of Markov chain $C$ and $\bar{C}$ corresponding to infinitesimal generator $Q$ and $\bar{Q}$, respectively. We have

$$\frac{\bar{\alpha}}{1 + \epsilon^2} \left(1 - \frac{\alpha + \alpha Q_{\infty} \epsilon}{\alpha_{\min}}\right) - \frac{\bar{\alpha} Q_{\infty} \epsilon}{2\alpha_{\min}} \leq \alpha \leq \bar{\alpha}.$$ 

**Proof.** Recall the logarithmic Sobolev constant defined in (1.6)

$$\alpha = \inf \left\{ \frac{\mathcal{E}_i(f, f)}{\mathcal{E}_i(f^2)} \mid \mathcal{E}_i(f^2) > 0, f : X \to \mathbb{R} \right\}. \quad (2.38)$$

1. The upper bound follows directly by considering functions $f(x) = g(i)$ when $x \in X_i$, $g : X \to \mathbb{R}$, and noticing that $\mathcal{E}_i(f^2) = \mathcal{E}_w(g^2)$, $\mathcal{E}_i(f, f) = \mathcal{E}(g, g)$.

2. For the lower bound, we assume the minimum in (2.38) is achieved with function $f$, i.e.

$$\alpha = \frac{\mathcal{E}_i(f, f)}{\mathcal{E}_i(f^2)} \leq \bar{\alpha}.$$ 

Estimation of $\alpha$ can be obtained if we could estimate both the numerator and denominator. For the Dirichlet form $\mathcal{E}_i(f, f)$, from (2.30) we have

$$\frac{1}{2\epsilon} \sum_{i=1}^{m} \sum_{x, x' \in X_i} \left(f(x') - f(x)\right)^2 Q_{0, i}(x, x') \pi(x) \leq \bar{\alpha} \mathcal{E}_i(f^2).$$

Applying Poincaré inequality and logarithmic Sobolev inequality for Markov chain $C_i$ for each fixed $i$, and noticing $2\alpha_i \leq \lambda_i$ (see (1.11)), we obtain

$$\sum_{i=1}^{m} \left[ \sum_{x \in X_i} \left(f(x) - \bar{f}(i)\right)^2 \pi_i(x) \right] w(i) \leq \frac{\bar{\alpha}}{2\alpha_{\min}} \mathcal{E}_i(f^2) \epsilon,$$

$$\sum_{i=1}^{m} \left[ \sum_{x \in X_i} |f(x)|^2 \ln \frac{|f(x)|^2}{F(i)^2} \pi_i(x) \right] w(i) \leq \frac{\bar{\alpha}}{\alpha_{\min}} \mathcal{E}_i(f^2) \epsilon,$$ 

respectively. In the above, $F(i) = |f(i, \cdot)|_{L^2, \pi_i} = \sum_{x \in X_i} |f(x)|^2 \pi_i(x)$ and we have $|\bar{f}(i)| \leq F(i)$, $1 \leq i \leq m$. We proceed similarly as in the proof of Theorem 2.4. Applying the detailed balance condition (2.31), inequalities (2.34) and (2.39) to the Dirichlet form (2.30), we can obtain

\begin{align*}
\mathcal{E}_i(f, f) & \geq \frac{1}{2} \left( f(y) - f(x) \right)^2 Q_{1}(x, y) \pi(x) \\
& \geq \frac{1}{2} \left( f(j) - f(i) \right)^2 Q(i, j) w(i) - 2 \epsilon^{-\frac{1}{2}} \sum_{i=1}^{m} \sum_{x \in X_i, y \in X_j} \left( f(x) - f(i) \right)^2 Q_1(x, y) \pi(x) \\
& \geq \frac{1}{2} \left( f(j) - f(i) \right)^2 Q(i, j) w(i) - Q_{\infty} \epsilon^{-\frac{1}{2}} \sum_{i=1}^{m} \sum_{x \in X_i} \left( f(x) - f(i) \right)^2 \pi(x) \\
& \geq \frac{1}{2} \left( f(j) - f(i) \right)^2 Q(i, j) w(i) - \frac{\bar{\alpha} Q_{\infty} \mathcal{E}_i(f^2) \epsilon^{\frac{1}{2}}}{2\alpha_{\min}}. \quad (2.41)
\end{align*}
Finally, we study the modified logarithmic Sobolev constant. Using (2.39), the logarithmic Sobolev inequality for the reduced Markov chain \( \bar{\mathcal{C}} \), and noticing \( |f|_{\bar{\mathcal{C}},\pi}^2 = |F|_{\bar{\mathcal{C}},\bar{\pi}}^2 \), we have

\[
\text{Ent}_\pi(f^2) = \sum_{i=1}^{m} \sum_{x \in X_i} |f(x)|^2 \ln \frac{|f(x)|^2}{|F(i)|^2} \pi_i(x)w(i) + \sum_{i=1}^{m} |F(i)|^2 \ln \frac{|F(i)|^2}{|F|_{\bar{\mathcal{C}},\bar{\pi}}^2} w(i)
\]

\[
\leq \frac{\bar{\alpha}}{\alpha_{\min}} \text{Ent}_\pi(f^2) + \frac{1}{2\bar{\alpha}} \sum_{1 \leq i,j \leq m} |F(j) - F(i)|^2 \bar{Q}(i,j)w(i). \quad (2.42)
\]

The first inequality in (2.39) and the definition in (1.3) imply

\[
0 \leq \sum_{i=1}^{m} \left( F(i)^2 - \bar{f}(i)^2 \right) w(i) \leq \frac{\bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon ,
\]

\[
0 < \sum_{j \neq i} \bar{Q}(i,j) = \sum_{x \in X_i, y \notin X_i} \bar{Q}_i(x,y)\pi_i(x) \leq \frac{Q_{\infty}}{2}.
\]

Then the second term on the right hand side of (2.42) can be bounded as

\[
\sum_{1 \leq i,j \leq m} |F(j) - F(i)|^2 \bar{Q}(i,j)w(i)
\]

\[
= \sum_{1 \leq i,j \leq m} \left( F(i)^2 - 2F(i)F(j) + F(j)^2 \right) \bar{Q}(i,j)w(i)
\]

\[
\leq 2 \sum_{1 \leq i,j \leq m} F(i)^2 \bar{Q}(i,j)w(i) - 2 \sum_{1 \leq i,j \leq m} \bar{f}(i)\bar{f}(j) \bar{Q}(i,j)w(i)
\]

\[
= \sum_{1 \leq i,j \leq m} \left( \bar{f}(j) - \bar{f}(i) \right)^2 \bar{Q}(i,j)w(i) + 2 \sum_{1 \leq i,j \leq m} \left( F(i)^2 - \bar{f}(i)^2 \right) \bar{Q}(i,j)w(i)
\]

\[
\leq \sum_{1 \leq i,j \leq m} \left( \bar{f}(j) - \bar{f}(i) \right)^2 \bar{Q}(i,j)w(i) + \frac{Q_{\infty}\bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon ,
\]

where the detailed balance condition (2.28) for Markov chain \( \bar{\mathcal{C}} \) has been used. Therefore (2.42) implies

\[
\text{Ent}_\pi(f^2) \leq \frac{\bar{\alpha}}{\alpha_{\min}} \text{Ent}_\pi(f^2) + \frac{1}{2\bar{\alpha}} \left[ \sum_{1 \leq i,j \leq m} \left( \bar{f}(j) - \bar{f}(i) \right)^2 \bar{Q}(i,j)w(i) + \frac{Q_{\infty}\bar{\alpha}}{2\alpha_{\min}} \text{Ent}_\pi(f^2) \epsilon \right],
\]

or equivalently

\[
\frac{1}{2} \sum_{1 \leq i,j \leq m} \left( \bar{f}(j) - \bar{f}(i) \right)^2 \bar{Q}(i,j)w(i) \geq \bar{\alpha} \text{Ent}_\pi(f^2) \left( 1 - \frac{(\bar{\alpha} + \frac{1}{4}Q_{\infty})\epsilon}{\alpha_{\min}} \right).
\]

Substituting the above inequality into (2.31), we obtain

\[
\mathcal{E}_*(f, f) \geq \frac{\bar{\alpha} \text{Ent}_\pi(f^2)}{1 + \frac{\epsilon}{2}} \left( 1 - \frac{(\bar{\alpha} + \frac{1}{4}Q_{\infty})\epsilon}{\alpha_{\min}} \right) - \frac{\bar{\alpha}Q_{\infty} \text{Ent}_\pi(f^2) \epsilon^2}{2\alpha_{\min}},
\]

which indicates the lower bound.

Finally, we study the modified logarithmic Sobolev constant.
Theorem 2.6. Let $\gamma_\epsilon, \bar{\gamma}$ be the modified logarithmic Sobolev constants of the reversible Markov chain $C$ and $\bar{C}$ corresponding to infinitesimal generator $Q$ and $Q$, respectively. We have

$$\lim_{\epsilon \to 0} \gamma_\epsilon = \bar{\gamma}. \quad (2.43)$$

Proof. We argue by contradiction. Suppose the conclusion is not true. First recall the modified logarithmic Sobolev constant defined in (1.10)

$$\gamma_\epsilon = \inf_f \left\{ \frac{\mathcal{E}_\epsilon(f, \ln f)}{\text{Ent}_\pi(f)} \mid \text{Ent}_\pi(f) > 0, f : X \to \mathbb{R}^+ \right\}. \quad (2.44)$$

Take functions $f(x) = g(i)$ for $x \in X_i$, then it is straightforward to check $\mathcal{E}_\epsilon(f, \ln f) = \mathcal{E}(g, \ln g)$ and $\text{Ent}_\pi(f) = \text{Ent}_\gamma(g)$, therefore we can deduce $\gamma_\epsilon \leq \bar{\gamma}$ by allowing function $g$ to vary among all functions $g : X \to \mathbb{R}^+$.

Since $\gamma_\epsilon \leq \bar{\gamma}$, we can find a sequence $\epsilon^{(k)}$, $\lim_{k \to +\infty} \epsilon^{(k)} = 0$, s.t. $\lim_{k \to +\infty} \gamma^{(k)} < \bar{\gamma}$. Notice that in this proof, we will use notations $\gamma^{(k)}$, $\mathcal{E}^{(k)}$ instead of $\gamma_{\epsilon^{(k)}}$ and $\mathcal{E}_{\epsilon^{(k)}}$. We assume the infima in (2.41) are achieved with functions $f_k : X \to \mathbb{R}^+$, i.e.

$$\gamma^{(k)} = \frac{\mathcal{E}^{(k)}(f_k, \ln f_k)}{\text{Ent}_\pi(f_k)}, \quad \text{Ent}_\pi f_k = \sum_{x \in X} f_k(x) \pi(x) = 1. \quad (2.45)$$

Let $\pi_{\min} = \min_{x \in X} \pi(x)$. Clearly we have $0 < f_k(x) \leq \pi_{\min}^{-1}, \forall x \in X$ (see [2] for the positivity), and therefore

$$0 \leq \text{Ent}_\pi(f_k) \leq \ln \frac{1}{\pi_{\min}}. \quad (2.46)$$

Since $f_k$ are bounded, we further assume they converge to some function $\bar{f} : X \to \mathbb{R}$ for each $x \in X$ (This can be achieved by considering a convergent subsequence).

From Dirichlet form (2.30) and (2.45), we have

$$\frac{1}{\epsilon^{(k)}} \sum_{i=1}^m \mathcal{E}_i(f_k(i, \cdot), \ln f_k(i, \cdot)) w(i) = \frac{1}{2\epsilon^{(k)}} \sum_{i=1}^m \sum_{x, x' \in X_i} (f_k(x') - f_k(x))(\ln f_k(x') - \ln f_k(x))Q_{0,i}(x, x') \pi(x) \quad (2.47)$$

$$\leq \mathcal{E}^{(k)}(f_k, \ln f_k) = \gamma^{(k)} \text{Ent}_\pi(f_k) \leq \bar{\gamma} \ln \frac{1}{\pi_{\min}}.$$  

Recall $\gamma_i > 0$ is the modified logarithmic Sobolev constant of Markov chain $C_i$, together with (2.47), we can obtain

$$\sum_{i=1}^m \text{Ent}_{\pi_i}(f_k(i, \cdot)) w(i) \leq \frac{\bar{\gamma}}{\pi_{\min}} \frac{1}{\epsilon^{(k)}}. \quad (2.48)$$

Let $\mu^{(k)}_i$ be the probability measure on $X_i$ s.t. $\mu^{(k)}_i(x) = \frac{f_k(x)}{f_k(i)} \pi_i(x), \forall x \in X_i$. Applying Csiszár-
Kullback-Pinsker inequality, we can obtain
\[
\sum_{i=1}^{m} \left[ \sum_{x \in X_i} |f_k(x) - \hat{f}_k(i)| \pi_i(x) \right] \mu_i^{(k)}(\mathbf{w}) = \sum_{i=1}^{m} \hat{f}_k(i) ||\mathbf{w}|| TV w(i)
\]
\[
\leq \sum_{i=1}^{m} \hat{f}_k(i) \sqrt{2 \text{Ent}_{\pi_i}(\hat{f}_k(i))} \mu_i^{(k)}(\mathbf{w}) = \sum_{i=1}^{m} \sqrt{2 \hat{f}_k(i) \text{Ent}_{\pi_i}(\hat{f}_k(i))} \mu_i^{(k)}(\mathbf{w})
\]
\[
\leq \sqrt{2} \left[ \sum_{i=1}^{m} \text{Ent}_{\pi_i}(\hat{f}_k(i)) \mu_i^{(k)}(\mathbf{w}) \right] \leq \left( \frac{2\gamma \ln \frac{\epsilon}{\gamma_{\min}}}{\pi_{\min}} \right)^{1/2}, \tag{2.49}
\]
where \( \| \cdot \|_{TV} \) denotes the total variation distance of two probability measures, and we have used the fact
\[
\sum_{i=1}^{m} \hat{f}_k(i) \mu_i^{(k)}(\mathbf{w}) = \sum_{x \in X} f_k(x) \pi(x) = 1. \tag{2.50}
\]
Taking the limit \( k \to +\infty \), inequality \((2.49)\) indicates that \( \hat{f} \) is constant on each subset \( X_i \), i.e. \( f(x) = \hat{g}(i) \) if \( x \in X_i \), where \( \hat{g} : X \to \mathbb{R}^+ \). We argue that \( \hat{f} \) is both positive and non-constant (this argument is adapted from \cite{2}). Suppose \( \hat{f} \) is constant. Notice that
\[
E_{\pi} \hat{f} = E_{\pi} \hat{g} = \lim_{k \to +\infty} E_{\pi} f_k = 1, \tag{2.51}
\]
therefore we must have \( \hat{g} \equiv 1 \). Let \( f_k = 1 + f_k' \), where \( E_{\pi}(f_k') = 0 \) and \( \lim_{k \to +\infty} f_k'(x) = 0 \), \( \forall x \in X \).

Using Taylor expansion, we can verify
\[
\mathcal{E}^{(k)}(f_k, \ln f_k) = \left( 1 + O(|f_k'|_{\infty}) \right) \mathcal{E}^{(k)}(f_k', f_k'),
\]
\[
\text{Ent}_{\pi}(f_k) = \frac{1}{2} \text{Var}_{\pi}(f_k') + O(|f_k'|_{\infty}).
\]
Because \( \text{Var}_{\pi}(f_k') \geq |f_k'|_{\infty}^2 \pi_{\min} \), we know \( \text{Ent}_{\pi}(f_k) = \frac{1 + o(1)}{2} \text{Var}_{\pi}(f_k') \). Applying Poincaré inequality and Theorem \cite{2} (which implies \( \lambda(k) \) converges to \( \lambda \)), we can estimate
\[
\gamma > \lim_{k \to +\infty} \gamma^{(k)} \geq \lim_{k \to +\infty} \frac{\mathcal{E}^{(k)}(f_k', f_k')}{2 \text{Var}_{\pi}(f_k')} \geq 2 \lim_{k \to +\infty} \lambda^{(k)} = 2 \lambda, \tag{2.52}
\]
which contradicts to the relation \((1.13)\). Therefore function \( \hat{f} \) is non-constant. Now we consider the positiveness. Define two disjoint sets
\[
M = \left\{ x \in X \mid \hat{f}(x) = 0 \right\}, \quad M' = \left\{ x \in X \mid \hat{f}(x) > 0 \right\}. \tag{2.53}
\]
Since \( E_{\pi} \hat{f} = 1 \), we know \( M' \) is not empty. Now assume set \( M \) is also nonempty. Applying the irreducibility of Markov chain \( \mathcal{C} \) to subset \( M \) and \( M' \), we conclude that \( \exists x \in M, y \in M' \) s.t. \( Q(x, y) > 0 \). Since \( \hat{f} \) is constant on each subset \( X_i \), we know \( x \in X_i, y \in X_j \), for some \( i \neq j \) and \( Q_1(x, y) > 0 \). Then we have
\[
+\infty = \lim_{k \to +\infty} (f_k(y) - f_k(x)) (\ln f_k(y) - \ln f_k(x)) Q_1(x, y) \pi(x)
\leq \lim \sup_{k \to +\infty} \mathcal{E}^{(k)}(f_k, \ln f_k) = \lim \sup_{k \to +\infty} \gamma^{(k)} \text{Ent}_{\pi}(f_k) \leq \gamma \ln \frac{1}{\pi_{\min}}.
\]
This contradiction shows that $M$ is empty and therefore $\bar{f}$ is positive. Now we can take the limits

$$\liminf_{k \to +\infty} \mathcal{E}^{(k)}(f_k, \ln f_k) \geq \mathcal{E}(\bar{g}, \ln \bar{g})$$

$$\lim_{k \to +\infty} \text{Ent}_\pi(f_k) = \text{Ent}_\pi(\bar{f}) = \text{Ent}_w(\bar{g}) ,$$

and

$$\bar{\gamma} > \lim_{k \to +\infty} \gamma^{(k)} = \lim_{k \to +\infty} \frac{\mathcal{E}^{(k)}(f_k, \ln f_k)}{\text{Ent}_\pi(f_k)} \geq \frac{\mathcal{E}(\bar{g}, \ln \bar{g})}{\text{Ent}_w(\bar{g})} . \quad (2.54)$$

But the above inequality is in contradiction with the fact that $\bar{\gamma}$ is the modified logarithmic Sobolev constant of the reduced Markov chain $\bar{C}$. Therefore we conclude $\lim_{\epsilon \to 0} \gamma_\epsilon = \bar{\gamma}$. \[\square\]

**Remark 1.** Theorem 2.4 and Theorem 2.5 imply that both the Poincaré constant $\lambda_\epsilon$ and logarithmic Sobolev constant $\alpha_\epsilon$ of Markov chain $C$ converge to their counterparts $\bar{\lambda}$, $\bar{\alpha}$ of the reduced Markov chain $\bar{C}$ and the convergence order is $O(\epsilon^2)$.

On the other hand, the result of Theorem 2.6 is weaker in that we obtain convergence without convergence order.

3. **Asymptotic analysis : general case.** In this section we consider the general case without assuming reversibility. A convergence result of Kolmogorov backward equation can be found in [21] and will not be discussed here (also see Appendix B). Unlike the reversible case in Section 2, relation (2.2) does not hold in general and the invariant measure $\pi^\epsilon$ will depend on parameter $\epsilon$. Instead, we have the following result (recall the notations in Subsection 1.2).

**Theorem 3.1.** Let $\pi^\epsilon$, $w$, $\pi_i$ be the invariant measures of Markov chain $C$, $\bar{C}$, and $C_i$, respectively. We have

$$\left( \sum_{i=1}^m \sum_{x \in X_i} |\pi^\epsilon(x) - w(i)\pi_i(x)|^2 \right)^{\frac{1}{2}} \leq 3\frac{\sigma^2}{2} + \left( \frac{1}{2} + m \right) \bar{\sigma}^{-2} d \Gamma + n \right) \frac{1}{2} \sigma^{-1} Q_{\infty} \epsilon .$$

**Proof.** We first study the invariant measure $\pi^\epsilon$, which satisfies equation

$$\left( \frac{Q_0^T}{\epsilon} + Q_1^T \right) \pi^\epsilon = 0 , \quad (3.1)$$

i.e. $Q_0^T \pi^\epsilon = -\epsilon Q_1^T \pi^\epsilon$. Since matrix $Q_0$ is a block diagonal matrix given in (1.2), we obtain linear systems

$$Q_{0,i}^T \pi^\epsilon(i, \cdot) = -\epsilon b(i, \cdot) , \quad 1 \leq i \leq m , \quad (3.2)$$

where $b(i, x) = \sum_{y \in X} Q_1(y, x) \pi^\epsilon(y)$, $x \in X_i$. Summing up $x \in X_i$ in (3.2) and noticing that each matrix $Q_{0,i}$ have zero row sums, we obtain

$$\sum_{x \in X_i} b(i, x) = \sum_{x \in X_i} \sum_{y \in X} Q_1(y, x) \pi^\epsilon(y) = 0 , \quad 1 \leq i \leq m . \quad (3.3)$$
Using (1.16), we can estimate

\[ |b(i, \cdot)|_2 = \left[ \sum_{x \in X_i} \left( \sum_{y \in X} Q_1(y, x) \pi'(y) \right)^2 \right]^{\frac{1}{2}} \leq \left[ \sum_{x \in X_i} \sum_{y \in X} Q_1^2(y, x) \pi'(y) \right]^{\frac{1}{2}} \]

\[ \leq \left[ \max_{x, y \in X} |Q_1(x, y)| \sum_{x \in X_i} \sum_{y \in X} |Q_1(y, x)| \pi'(y) \right]^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} Q_{\infty} \sum_{x \in X_i} \sum_{y \in X} |Q_1(y, x)| \pi'(y) \right]^{\frac{1}{2}} =: g(i), \]

which implies

\[ \sum_{i=1}^m |b(i)|_2^2 \leq \sum_{i=1}^m g(i)^2 \leq \frac{1}{2} Q_{\infty}^2. \] (3.4)

Applying Lemma [A.1] to equation (3.2), we have

\[ |\pi'(i, \cdot) - w'(i)\pi_{1,\infty}| \leq |\pi'(i, \cdot) - w'(i)\pi_i|_2 \leq \epsilon \sigma_i^{-1} g(i) \] (3.5)

for some \(w'(i) \in \mathbb{R}\). Recall that \(\sigma_i\) is the smallest nonzero singular value of matrix \(Q_{0,i}\). Let \(\pi'(i, \cdot) = w'(i)\pi_i(\cdot) + r'(i, \cdot), 1 \leq i \leq m\), using (3.3), we have

\[ \sum_{j=1}^m Q(j, i)w'(j) = \sum_{x \in X_i} \sum_{j=1}^m \sum_{y \in X_j} Q_1(y, x) \pi_i(y) w'(j) \]

\[ = - \sum_{x \in X_i} \sum_{j=1}^m \sum_{y \in X_j} Q_1(y, x) r'(j, y) =: b'(i). \] (3.6)

We also have

\[ |b'|_2 = \left[ \sum_{i=1}^m \left( \sum_{x \in X_i} \sum_{j=1}^m \sum_{y \in X_j} Q_1(y, x) r'(j, y) \right)^2 \right]^{\frac{1}{2}} \]

\[ \leq \left[ \sum_{i=1}^m \left( \sum_{x \in X_i} \sum_{y \in X_j} Q_1(y, x)^2 \right) \left( \sum_{j=1}^m \sum_{y \in X_j} \sum_{x \in X_i} r'(j, y)^2 \right) \right]^{\frac{1}{2}}, \] (3.7)

where \(y \sim x\) means \(Q_1(y, x) \neq 0\). From (3.4) and (3.3),

\[ \left( \sum_{j=1}^m \sum_{y \in X_j} \sum_{x \in X_i} r'(j, y)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^m |r'(j, \cdot)|_2^2 \right)^{\frac{1}{2}} \leq \left( \frac{d}{2} \right)^{\frac{1}{2}} \sigma_{\min}^{-1} Q_{\infty} \epsilon, \]

therefore (3.7) becomes

\[ |b'|_2 \leq \epsilon \sigma_{\min}^{-1} Q_{\infty} \left( \frac{d \Gamma}{2} \right)^{\frac{1}{2}}, \]

where \(\Gamma = \text{tr}(Q_1 Q_1^T)\). Now applying Lemma [A.1] to equation (3.6), we obtain

\[ |w' - \lambda w|_{\infty} \leq |w' - \lambda w|_2 \leq (\tilde{\sigma} \sigma_{\min})^{-1} Q_{\infty} \left( \frac{d \Gamma}{2} \right)^{\frac{1}{2}} \epsilon, \] (3.8)

where \(\lambda \in \mathbb{R}, w\) is the invariant measure of the reduced Markov chain \(\hat{\mathcal{C}}\), i.e. \(Q^T w = 0\) and \(\tilde{\sigma}\) is the smallest nonzero singular value of \(Q\). Therefore

\[ \sum_{i=1}^m |w'(i) - \lambda w(i)| \leq m^2 |w' - \lambda w|_2 \leq (\tilde{\sigma} \sigma_{\min})^{-1} Q_{\infty} \left( \frac{m d \Gamma}{2} \right)^{\frac{1}{2}} \epsilon. \] (3.9)
From (3.4), (3.5) and
\[ \sum_{i=1}^{m} w^e(i) = 1 - \sum_{i=1}^{m} \sum_{x \in X_i} r^e(i, x), \quad \sum_{i=1}^{m} w(i) = 1, \]
we can estimate
\[ \left| \sum_{i=1}^{m} \left( w^e(i) - \lambda w(i) \right) \right| = \left| 1 - \lambda - \sum_{i=1}^{m} \sum_{x \in X_i} r^e(i, x) \right| \geq |1 - \lambda| - \left| \sum_{i=1}^{m} \sum_{x \in X_i} r^e(i, x) \right|, \]
\[ \sum_{i=1}^{m} \sum_{x \in X_i} r^e(i, x) \leq \sum_{i=1}^{m} \sum_{x \in X_i} r^e(i, x) \leq \sum_{i=1}^{m} n_i \sigma_i^{-1} g(i) \epsilon \leq \left( \frac{n}{2} \right)^{\frac{1}{2}} \sigma^{-1}_{\min} Q_{\infty} \epsilon. \]
Together with (3.9), we know
\[ |1 - \lambda| \leq \left( \sigma^{-1} \left( \frac{m d \Gamma}{2} \right)^{\frac{1}{2}} + \left( \frac{n}{2} \right)^{\frac{1}{2}} \right) \sigma^{-1}_{\min} Q_{\infty} \epsilon. \]
Combining (5.5), (8.8) and (5.10), we have
\[ \sum_{i=1}^{m} \sum_{x \in X_i} \left| \sum_{i} |x^e(x) - w(i) \pi_i(x)|^2 \right|^{\frac{1}{2}} \]
\[ \leq 3^{\frac{1}{2}} \sum_{i=1}^{m} \left( |r^e(i, x) - w^e(i) \pi_i(x)|^2 + |w^e - \lambda w|^2 + |\lambda - 1|^2 \sum_{i=1}^{m} w(i)^2 \right)^{\frac{1}{2}} \]
\[ \leq 3^{\frac{1}{2}} \left( \frac{1}{2} + \frac{\bar{\sigma}^{-2} d \Gamma}{2} + \left( \frac{1}{2} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \sigma^{-1}_{\min} Q_{\infty} \epsilon \]
\[ \leq 3^{\frac{1}{2}} \left( \frac{1}{2} + \left( \frac{1}{2} + m \right) \sigma^{-2} d \Gamma + n \right)^{\frac{1}{2}} \sigma^{-1}_{\min} Q_{\infty} \epsilon. \]

\[ \square \]

Based on Theorem 3.1, we can obtain convergence results of various constants of Markov chain \( C \). In the proof of the following result, we will use the fact that the infima in definitions (1.5), (1.6) and (1.10) can be attained by some functions. This fact can be verified using arguments in [2], which is also valid in non-reversible case.

**Theorem 3.2.** Let \( \lambda_e, \alpha_e, \gamma_e \) be the Poincaré constant, logarithmic Sobolev constant and modified logarithmic Sobolev constant of Markov chain \( C \). Also let \( \bar{\lambda}, \bar{\alpha}, \bar{\gamma} \) be their counterparts of Markov chain \( \bar{C} \). We have
\[ \lim_{\epsilon \to 0} \lambda_e = \bar{\lambda}, \quad \lim_{\epsilon \to 0} \alpha_e = \bar{\alpha}, \quad \lim_{\epsilon \to 0} \gamma_e = \bar{\gamma}. \]  
\[ (3.11) \]

**Proof.** We will sketch the proof, since the argument is similar to Theorem 2.6.

1. First consider the Poincaré constant. Let function \( g : X \to \mathbb{R} \) satisfy \( \mathbb{E}_w g = 0 \) and \( \text{Var}_w g = 1 \). Define \( f(x) = g(i) \) for \( x \in X_i \). We know \( |f|_{\infty} = |g|_{\infty} \) is bounded. Applying Theorem 5.1 and using (1.9), we know
\[ \mathbb{E}_e(f, f) = \frac{1}{2} \sum_{i \neq j} \sum_{x \in X_i, y \in X_j} (f(y) - f(x))^2 \mathbb{P}_i(x, y) \pi^e(x) = \bar{\mathbb{E}}(g, g) + o(\epsilon), \]
\[ \text{Var}_e f = \sum_{i=1}^{m} g^e(i) \left( \sum_{x \in X_i} \pi^e(x) \right)^2 - \left( \sum_{i=1}^{m} g(i) \sum_{x \in X_i} \pi^e(x) \right)^2 = \text{Var}_w g + o(\epsilon). \]  
\[ (3.12) \]
\[ (3.13) \]
2. We continue to prove the convergence of the modified logarithmic Sobolev constant and lim
\[ \sup_{\varepsilon} \lambda \leq \bar{\lambda} \]
and therefore we can find a subsequence (also denoted as \( f_k \) for simplicity) s.t. \( f_k \)
converges to \( f : X \to \mathbb{R} \). Using \( \lim_{k \to +\infty} \mathcal{E}(k) (f_k, f_k) \leq \bar{\lambda} \), we can deduce that \( f(x) = g(i) \)
if \( x \in X_i \), for some \( g : X \to \mathbb{R} \). And \( \text{Var}_{u} g = 1, \text{Ent}_{u} g = 0 \). Therefore
\[ \bar{\lambda} \leq \mathcal{E}(g, g) \leq \lim_{k \to +\infty} \mathcal{E}(k)(f_k, f_k) = \lim_{k \to +\infty} \lambda(k) < \bar{\lambda}. \] (3.14)
This contradiction shows that \( \lim_{\varepsilon \to 0} \lambda_\varepsilon = \bar{\lambda} \).

2. We continue to prove the convergence of the modified logarithmic Sobolev constant \( \gamma_\varepsilon \)
(proof for the convergence of \( \alpha_\varepsilon \) is similar and is omitted). First of all, using a
similar argument as above, we can obtain
\[ \lim_{\varepsilon \to 0} \sup \gamma_\varepsilon \leq \bar{\gamma}. \] (3.15)
Suppose the conclusion is not true and then we can find sequence \( \epsilon(k) \), lim
\[ \lim_{k \to +\infty} \epsilon(k) = 0 \]
and \( \lim_{k \to +\infty} \gamma(k) < \bar{\gamma} \). Let function \( f_k \) be the extreme functions in (1.5) and satisfy \( \text{Var}_{\pi(k)} f_k = 1 \) and \( \text{Ent}_{\pi(k)} f_k = 0 \). Then
\[ \lambda(k) = \mathcal{E}(k)(f_k, f_k). \] It is easy to see
\[ \lim_{k \to +\infty} \|f_k\|_{\infty} < +\infty, \]
and therefore we can find a subsequence (also denoted as \( f_k \) for simplicity) s.t.
\[ f_k \]
converges to \( f : X \to \mathbb{R} \). Using \( \lim_{k \to +\infty} \mathcal{E}(k)(f_k, f_k) \leq \bar{\lambda} \), we can deduce that \( f(x) = g(i) \)
if \( x \in X_i \), for some \( g : X \to \mathbb{R} \). And \( \text{Var}_{u} g = 1, \text{Ent}_{u} g = 0 \). Therefore
\[ \bar{\lambda} \leq \mathcal{E}(g, g) \leq \lim_{k \to +\infty} \mathcal{E}(k)(f_k, f_k) = \lim_{k \to +\infty} \lambda(k) < \bar{\lambda}. \] (3.14)
This contradiction shows that \( \lim_{\varepsilon \to 0} \lambda_\varepsilon = \bar{\lambda} \).

Taking limit \( k \to +\infty \), applying Theorem 3.1 and the boundness of \( \text{Ent}_{\pi(k)} (f_k) \),
we can deduce that \( f \) is constant on each subset \( X_i \), i.e. \( f(x) = g(i) \) if \( x \in X_i \), for some
\( g : X \to \mathbb{R}^+ \). We have \( \text{Ent}_{u} g = \lim_{k \to +\infty} \text{Ent}_{\pi(k)} f_k = 1 \). The same argument as in Theorem 2.6
shows that \( g \) is positive. Now we show \( g \) is non-constant. Assume \( g \) is constant and let
\( f_k = 1 + f_k' \), then we have \( \text{Ent}_{\pi(k)} f_k' = 0 \) and \( \lim_{k \to +\infty} f_k'(x) = 0, \forall x \in X \). Using Taylor expansion, we have
\[ \mathcal{E}(k)(f_k, \ln f_k) = - \sum_{x \in X} f_k(x) \left( \sum_{y \in X, y \neq x} Q(x, y)(\ln f_k(y) - \ln f_k(x)) \right) \pi(k)(x) \]
\[ = \left( 1 + O(|f_k'|_{\infty}) \right) \mathcal{E}(k)(f_k', f_k'). \]
and \( \text{Ent}_{\pi^{(k)}}(f_k) = \frac{(1+o(1))\text{Var}_{\pi^{(k)}} f_k'}{2} \). We can deduce a contradiction as in Theorem 2.6. Therefore \( g \) is non-constant, i.e. \( \text{Ent}_w g > 0 \). Taking the limit, we obtain

\[
\bar{\gamma} \leq \frac{\bar{\gamma}(g, \ln g)}{\text{Ent}_w (g)} \leq \lim_{k \to +\infty} \frac{\gamma^{(k)}(f_k, \ln f_k)}{\text{Ent}_{\pi^{(k)}}(f_k)} = \lim_{k \to +\infty} \gamma^{(k)} < \bar{\gamma}.
\]

This contradiction shows that \( \lim_{\epsilon \to 0} \gamma_{\epsilon} = \bar{\gamma} \).

\[ \square \]

4. Conclusion. In this paper we consider continuous-time Markov chains on finite state space and focus on the situation when systems’ transitions within clusters are much faster than transitions among clusters. Several asymptotic results are obtained concerning Kolmogorov backward equation, Poincaré constant, and (modified) logarithmic Sobolev constants. These results validate the reduced Markov chain as an approximation of the multiscale Markov chain in the asymptotic limit. Especially, when understanding the multiscale Markov chain becomes infeasible, either due to an extremely large state space or limited information to identify all transition rates, our results will be instructive as it suggests that the reduced Markov chain can be a useful approximation of the original one.

On the other hand, while we assume that there are several subsets of the state space such that transitions between them are relatively slow, in applications it might be the case that these subsets are not known a priori and need to be identified. How to identify (clustering) the slow subsets is an important problem in the studies of proteins \([4, 22]\), principal component analysis \([11]\), climates \([16]\) and network \([3, 15]\) et al. Readers are referred to those literatures for more details.

In future work, it might be interesting to consider asymptotic behaviors of other constants in \([7, 15]\). As more and more real data become available nowadays, it is also interesting to quantify the approximation error of the reduced Markov chain using a data-based approach.

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Appendix A. Some useful facts. In this section we collect some results related to continuous-time Markov chain. Let \( n > 1 \), \( Q \) be an \( n \times n \) matrix satisfying \( Q_{ij} \geq 0 \) for \( 1 \leq i \neq j \leq n \) and \( Q_{ii} = -\sum_{j \neq i} Q_{ij}, 1 \leq i \leq n \). We will call such matrix as transition rate matrix. Clearly, \( Q\mathbf{1} = 0 \), where \( \mathbf{1} = (1, 1, \cdots, 1)^T \in \mathbb{R}^n \). Define \( P(t) = e^{tQ} \), then \( P(t)_{ij} \geq 0 \) for \( 1 \leq i, j \leq n \), \( P(t)P(s) = P(t+s) \), for \( t, s \geq 0 \) and \( P(0)\mathbf{1} = \mathbf{1} \). Therefore \( P(t) \) are stochastic matrices and satisfy semigroup property. Let \( \Omega = \{1, 2, \cdots, n\} \) and \( \mathcal{F} = \{f \mid f : \Omega \to \mathbb{R}\} \), which can be viewed as \( \mathbb{R}^n \). For \( f \in \mathcal{F} \), denote \( ||f||_{\infty} = \max_{i \in \Omega} |f(i)| \). Then \( P(t) \) defines a semigroup on \( \mathcal{F} \) with infinitesimal generator \( Q \). It also defines a continuous-time Markov chain \( x(t) \) on \( \Omega \) such that \( P(x(t) = j \mid x(0) = i) = P(t)_{ij}, 1 \leq i, j \leq n \). Define \( f_t = P(t)f \in \mathcal{F} \) for function \( f \in \mathcal{F} \), we have

\[
 f_t(i) = \mathbf{E}(f(x(t)) \mid x(0) = i) \quad (A.1)
\]

and it satisfies the Kolmogorov backward equation

\[
 \frac{d}{dt} f_t = Q f_t, \quad f_0 = f. \quad (A.2)
\]
From (A.1), we know \( |f_t|_\infty \leq |f_0|_\infty \).

Assume the probability distribution of \( x(t) \) at time \( t \geq 0 \) is \( \mu_t \), then it is known that \( \mu_t \) satisfies Kolmogorov forward (Fokker-Planck) equation

\[
\dot{\mu}_t = Q^T \mu_t \tag{A.3}
\]

with initial value \( \mu_0 \). Therefore we have \( \mu_t = P(t)^T \mu_0 \). A probability measure \( \pi \) is called the invariant measure of Markov chain \( x_t \) iff \( Q^T \pi = 0 \). If we further assume the Markov chain is irreducible, then the invariant measure is unique. Also define the \( \pi \)-weighted inner product on \( F \) as

\[
\langle f, g \rangle_\pi = \sum_{i \in \Omega} f(i)g(i)\pi(i), \quad f, g \in F,
\]

and the Dirichlet form \( \mathcal{E} \)

\[
\mathcal{E}(f, g) = -\langle f, Qg \rangle_\pi.
\]

Let \( Q^* \) be the adjoint matrix under \( (\cdot, \cdot)_\pi \), we have \( Q^*_{ij} = \frac{Q_{ij} \pi(j)}{\pi(i)} \). We can check that \( Q^* \) is also a transition rate matrix and \( (Q^*)^T \pi = 0 \). The corresponding Markov chain defined by \( Q^* \) is called the time-reversed Markov chain. For \( f \in F \), we have

\[
\mathcal{E}(f, f) = -\frac{1}{2} \sum_{i,j \in \Omega} Q_{ij} + Q^*_{ij} \frac{(f(i) - f(j))^2 \pi(i)}{2} \tag{A.4}
\]

Define matrix

\[
\Pi = \text{diag}\{\pi(1), \pi(2), \ldots, \pi(n)\},
\]

then we have the matrix equation \( Q^* = \Pi^{-1}Q^T \Pi \). It is direct to see that

Dirichlet form \( \mathcal{E} \) is symmetric

\[
\iff Q = Q^* \iff \pi(i)Q_{ij} = \pi(j)Q_{ji}, \forall i, j \in \Omega.
\]

In this case, we say \( \pi \) satisfies the detailed balance condition and the Markov chain is reversible.

For \( \mu_t \) satisfying (A.3), we define \( \rho_t = \mu_t \pi \), i.e. \( \mu_t(i) = \rho_t(i)\pi(i), i \in \Omega \), then

\[
\frac{d}{dt}\rho_t = Q^* \rho_t. \tag{A.5}
\]

When \( Q^* = Q \), i.e. the detailed balance condition holds, equation (A.5) coincides with the Kolmogorov backward equation (A.2).

Consider the singular value decomposition (SVD) \( Q = UDV^T \), where \( U, V \) are \( n \times n \) orthogonal matrix. \( D = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) is a diagonal matrix consisting of singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \). Denote \( i \)th column of matrix \( U, V \) as \( U_i, V_i \), respectively, i.e.
\( U = [U_1, U_2, \ldots, U_n], V = [V_1, V_2, \ldots, V_n]. \) Then \( \{U_i\}_{1 \leq i \leq n} \) and \( \{V_i\}_{1 \leq i \leq n} \) are two orthonormal basis of \( \mathbb{R}^n. \) Since \( Q1 = 0, \) \( Q^T \pi = 0 \) and using the fact that the invariant measure \( \pi \) is unique, we know \( \sigma_n = 0 < \sigma_{n-1}. \) We can further deduce that \( V_n \parallel 1 \) and \( U_n \parallel \pi. \) The linear system \( Q^T x = b \) can be studied based on the SVD decomposition. We have

**Lemma A.1.** Consider linear system \( Q^T x = b, \) where \( x, b \in \mathbb{R}^n. \)

1. There is a solution if and only if \( b^T 1 = 0. \)
2. Assume \( b^T 1 = 0, \) then the solutions can be written as

\[
x = a \pi + \sum_{i=1}^{n-1} \sigma_i^{-1} (V_i^T b) U_i,
\]

for \( \forall a \in \mathbb{R}. \) Furthermore,

\[
|x - a \pi|_\infty \leq |x - a \pi|_2 \leq \sigma_n^{-1} |b|_2.
\]

**Proof.**

1. Assume \( Q^T x = b, \) we have \( b^T 1 = x^T Q 1 = 0. \) The sufficiency follows from the second conclusion.
2. We directly verify that expression \( (A.6) \) satisfies the equation \( Q^T x = b, \) \( \forall a \in \mathbb{R}, \) using \( Q^T \pi = 0, \) \( Q = UDU^T \) and \( U_i, V_i \) are orthonormal, we have

\[
Q^T x = Q^T \left( a \pi + \sum_{i=1}^{n-1} \sigma^{-1}_i (V_i^T b) U_i \right)
= \sum_{i=1}^{n-1} \sigma^{-1}_i (V_i^T b) VDU^T U_i
= \sum_{i=1}^{n-1} (V_i^T b) V_i = b.
\]

In the last equality, we have used \( b^T V_n = b^T 1 = 0. \) On the contrary, suppose \( x \in \mathbb{R}^n \) satisfies the equation \( Q^T x = b. \) Since \( U_i \) is orthonormal and \( U_n \parallel \pi, \) we can assume \( x = a \pi + \sum_{i=1}^{n-1} a_i U_i. \) Substituting it into \( Q^T x = b, \) we obtain

\[
a_i = \sigma^{-1}_i (V_i^T b), \quad 1 \leq i \leq n-1, \text{ i.e. } x \text{ is given by } (A.6).
\]

And we can estimate

\[
|x - a \pi|_2 = \left| \sum_{i=1}^{n-1} \sigma^{-1}_i (V_i^T b) U_i \right|_2 = \left( \sum_{i=1}^{n-1} \sigma^{-2}_i (V_i^T b)^2 \right)^{\frac{1}{2}} \leq \sigma_n^{-1} \left( \sum_{i=1}^{n-1} (V_i^T b)^2 \right)^{\frac{1}{2}} = \sigma_n^{-1} |b|_2,
\]

i.e. \( (A.7) \) is proved.

In the reversible case, we have \( Q = Q^* \) and \( \Pi Q = Q^T \Pi, \) which indicates that \( \Pi^{\frac{1}{2}} Q \Pi^{\frac{1}{2}} \) is symmetric. Consider the eigenvalue decomposition \( \Pi^{\frac{1}{2}} Q \Pi^{\frac{1}{2}} = UDU^T \) with \( U^T U = UU^T = I, \)

\( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a diagonal matrix consisting of real eigenvalues. Then \( Q = TDT^{-1}, \) with \( T = \Pi^{\frac{1}{2}} U. \) Denote \( i \)th column vector of \( T \) as \( T_i, \) i.e. \( T = [T_1, T_2, \ldots, T_n], \) then we have \( QT_i = \lambda_i T_i, \) \( \langle T_i, T_i \rangle_\pi = \delta_{ij}. \) Therefore \( T_i \) is the eigenvector of \( Q \) corresponding to eigenvalue
\( \lambda_i \) and \( \{T_i\}_{1 \leq i \leq n} \) forms an orthonormal basis of \( L^2(\Omega) \). For two functions \( f, g \) on \( \Omega \) written as 
\[
 f = \sum_{i=1}^{n} f_i T_i, \quad g = \sum_{i=1}^{n} g_i T_i,
\]
we have 
\[
 \mathcal{E}(f, g) = -\langle f, Qg \rangle_{\pi} = -\sum_{i=1}^{n} \lambda_i f_i g_i.
\]
From (A.4), we could assume \( 0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \). It is clear that the Poincaré constant \( \lambda = -\lambda_2 > 0 \).

**Appendix B. Asymptotic expansion method: formal argument.** Asymptotic expansion method has been widely used in studying dynamical systems, partial differential equations in certain limiting regime, see [6, 23, 20, 21]. In this section, we consider the Kolmogorov backward equation and various constants studied in Section 2-3 using this method. First consider the equation 
\[
 \frac{d}{dt} \rho_t = Q \rho_t = \left( \frac{1}{\epsilon} Q_0 + Q_1 \right) \rho_t. \tag{B.1}
\]
Assume we have the expansion 
\[
 \rho_t = \rho_{t,0} + \epsilon \rho_{t,1} + \epsilon^2 \rho_{t,2} + \cdots. \tag{B.2}
\]
Substitute it into (B.1), we obtain 
\[
 \frac{d\rho_{t,0}}{dt} + \epsilon \frac{d\rho_{t,1}}{dt} + \mathcal{O}(\epsilon^2) = \frac{Q_0 \rho_{t,0}}{\epsilon} + Q_0 \rho_{t,1} + Q_1 \rho_{t,0} + \epsilon Q_0 \rho_{t,2} + \epsilon Q_1 \rho_{t,1} + \mathcal{O}(\epsilon^2). \tag{B.3}
\]
Collecting terms of order \( \mathcal{O}(\frac{1}{\epsilon}) \) and \( \mathcal{O}(1) \) with respect to parameter \( \epsilon \), we obtain equations 
\[
 Q_0 \rho_{t,0} = 0, \quad \frac{d\rho_{t,0}}{dt} = Q_0 \rho_{t,1} + Q_1 \rho_{t,0}. \tag{B.4}
\]
Since \( Q_0 \) is a block diagonal matrix of form (1.2), the first equation in (B.4) can be written as 
\[
 Q_{0,i} \rho_{t,0}(i, \cdot) = 0, \quad 1 \leq i \leq m. \]
It follows from the irreducibility of each Markov chain \( C_i \) that \( \rho_{t,0} \) is constant on each subset \( X_i \). And we can assume \( \rho_{t,0}(x) = \hat{\rho}_t(i) \) for \( x \in X_i \), where function \( \hat{\rho}_t : \hat{X} \to \mathbb{R} \). Then the second equation of (B.4) can be written more explicitly as 
\[
 \frac{d\hat{\rho}_t}{dt} = \sum_{x' \in X_i} Q_{0,i}(x, x') \rho_{t,1}(x') + \sum_{y \not\in X_i} \sum_{x' \in X_i} Q_1(x, y) (\hat{\rho}_t(j) - \hat{\rho}_t(i)), \tag{B.5}
\]
where \( 1 \leq i \leq m \) and \( x \in X_i \). Now we multiply both sides of the above equation by \( \pi_i(x) \) and sum up \( x \in X_i \). Using \( \pi_i^T Q_{0,i} = 0 \) and the definition \( \hat{Q} \) in (1.3), we arrive at 
\[
 \frac{d\hat{\rho}_t}{dt} = \hat{Q} \hat{\rho}_t. \tag{B.6}
\]
From expansion (B.2), the above reasoning indicates the convergence of \( \rho_t \) in (B.1) to \( \hat{\rho}_t \) in (B.6). See Theorem 2.1 in Section 2.

The asymptotic behavior of Poincaré constant \( \lambda_\epsilon \), logarithmic Sobolev constant \( \alpha_\epsilon \) and modified logarithmic Sobolev constant \( \gamma_\epsilon \) can be studied as well. Let \( f^* \) be a function where the infimum in (1.5) is achieved. Then standard variation method implies 
\[
 -\frac{Q + Q^*}{2} f^* = \lambda_\epsilon f^*, \tag{B.7}
\]
with $|f'|_{L^2(\pi^\epsilon)} = 1$. Similarly, the minimizer of $f^\epsilon$ satisfies
\[
-\frac{Q + Q^*}{2}f^\epsilon = \alpha_\epsilon f^\epsilon \ln(f^\epsilon)^2,
\] (B.8)
with $|f'|_{L^2(\pi^\epsilon)} = 1$, while the minimizer of $f^\epsilon$ satisfies
\[
-Q^*f^\epsilon - f^\epsilon Q^* f = \gamma_\epsilon f^\epsilon \ln(f^\epsilon),
\] (B.9)
with $\mathbb{E} f^\epsilon = 1$. For simplicity, we only consider the modified logarithmic Sobolev constant $\gamma_\epsilon$ using (B.9), since constants $\lambda_\epsilon$ and $\alpha_\epsilon$ can be studied in a similar way. Assume we have the expansion
\[
f^\epsilon = f_0 + \epsilon f_1 + \cdots, \quad \gamma_\epsilon = \gamma_0 + \epsilon \gamma_1 + \cdots.
\] (B.10)
Substituting it into (B.9), we have
\[
-\frac{Q_0^* f_0}{\epsilon} - Q_1^* f_0 - Q_0^* f_1 - \frac{f_0 Q_0 \ln f_0}{\epsilon} - f_0 Q_1 \ln f_0 - f_1 Q_0 \ln f_0 - f_0 Q_0 \left(\frac{f_1}{f_0}\right) + O(\epsilon)
= \gamma_0 f_0 \ln f_0 + O(\epsilon).
\]
Collecting terms of order $O(\frac{1}{\epsilon})$ and $O(1)$ with respect to parameter $\epsilon$, we obtain equations
\[
Q_0^* f_0 + f_0 Q_0 \ln f_0 = 0,
- Q_1^* f_0 - Q_0^* f_1 - f_0 Q_1 \ln f_0 - f_1 Q_0 \ln f_0 - f_0 Q_0 \left(\frac{f_1}{f_0}\right) = \gamma_0 f_0 \ln f_0.
\] (B.11)
Now for each $1 \leq i \leq m$, we multiply both sides of the first equation of (B.11) by $\pi_i(x)$ and sum up $x \in X_i$. Using $(Q_{0,i}^*)^T \pi_i = 0$, we can obtain
\[
\mathcal{E}_i(f_0(\cdot, \cdot), \ln f_0(\cdot, \cdot)) = 0,
\]
where $\mathcal{E}_i$ is the Dirichlet form of Markov chain $\mathcal{C}_i$. From Lemma 2.7 of [5], we know
\[
\mathcal{E}_i(f_0^\frac{1}{2}(\cdot, \cdot), f_0^\frac{1}{2}(\cdot, \cdot)) \leq \frac{1}{2} \mathcal{E}_i(f_0(\cdot, \cdot), \ln f_0(\cdot, \cdot)) = 0.
\]
Since Markov chain $\mathcal{C}_i$ is irreducible, we can deduce that $f_0$ is constant on each subset $X_i$, i.e. we have $f_0(x) = \bar{f}(i)$ when $x \in X_i$, where $\bar{f} : X \to \mathbb{R}$. Now we multiply both sides of the second equation in (B.11) by $\pi_i(x)$ and sum up $x \in X_i$. Using the fact that $Q_0 \ln f_0 = 0$, $Q_{0,i}^* \pi_i = (Q_{0,i}^*)^T \pi_i = 0$, we can deduce that
\[
-\bar{Q}^* \bar{f} - \bar{f} \bar{Q} \ln \bar{f} = \gamma_0 \bar{f} \ln \bar{f},
\] (B.12)
with $\mathbb{E}_w \bar{f} = 1$ (see Theorem 3.1). Comparing to (B.9), this equation shows that function $\bar{f}$ is a minimizer of
\[
\mathcal{E}(\bar{f}, \ln \bar{f})
\]
and takes value $\gamma_0$. Using the fact that $\bar{\gamma}$ is the infimum of (B.13) and the fact $\lim_{\epsilon \to 0} \gamma_\epsilon \leq \bar{\gamma}$ (see Theorem 3.2), we have
\[
\bar{\gamma} \leq \gamma_0 = \lim_{\epsilon \to 0} \gamma_\epsilon \leq \bar{\gamma}.
\]
Therefore we conclude that $\lim_{\epsilon \to 0} \gamma_\epsilon = \bar{\gamma}$.
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