Quantum chaos in a non-Hermitian $PT$-symmetric kicked top

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Abstract. A non-Hermitian $PT$-symmetric version of the kicked top is introduced to study the interplay of quantum chaos with balanced loss and gain. The classical dynamics arising from the quantum dynamics of the angular momentum expectation values are derived. It is demonstrated that the presence of $PT$-symmetry can lead to “stable” mixed regular chaotic behaviour without sinks or sources. This is an example of what is known in classical dynamical systems as reversible dynamical systems. For large values of the kicking strength a strange attractor is observed that also persists if $PT$-symmetry is broken. The amplitude dynamics of the classical map is investigated, and found to provide the main structure for the Husimi distributions of the subspaces of the quantum system belonging to certain ranges of the imaginary parts of the quasienergies, as well as in the quantum dynamics. Finally, the statistics of the eigenvalues of the quantum system are analysed and it is shown that if most of the eigenvalues are complex (which is the case already for fairly small non-Hermiticity parameters) the nearest-neighbour distances of the (unfolded) quasienergies follow GOE statistics when the classical dynamics is regular, and show a cubic level repulsion when the classical dynamics is chaotic as previously identified as universal behaviour for other types of open quantum systems. The $PT$-symmetry of the system does not seem to influence this statistical feature. Similar features are observed for the spectrum of a $PT$-symmetric extension of the triadic Baker map.

1. Introduction
Non-Hermitian and in particular $PT$-symmetric quantum systems, that can have either complex conjugate or purely real eigenvalues, have been the subject of extensive investigations over the past two decades. The interest reaches from their spectral theory to technological applications [1,2]. Yet, apart from a few exceptions [3–5], what has been little investigated hitherto is the interplay between non-Hermiticity and chaos. Since mixed regular-chaotic systems are the norm rather than the exception, it is clearly of great importance to study this interplay to gain deeper insights into the typical behaviour of $PT$-symmetric quantum systems and their classical counterparts.

For this purpose we introduce a $PT$-symmetric extension of one of the most prominent example systems of traditional quantum chaos: the kicked top. In dependence on the parameters it can show regular, mixed or (almost) entirely chaotic behaviour, and is faithful to random matrix theory [6–8]. It has been experimentally realised using cold atoms [9]. A dissipative version of the kicked top modelling decoherence via Lindblad dynamics has been studied in the literature before [10], and a $PT$-symmetric version of the non-kicked system has been studied in [11–13]. Here we present the first study of the $PT$-symmetric kicked system showing a wealth of
new behaviour on both the quantum and the classical side, compared to both the non-Hermitian system without PT-symmetry and the Hermitian system. In the classical limit, the dynamics shows a mixture of what may be termed quasi-Hamiltonian chaos and dissipative chaos. This is in fact an example of what is known as reversible dynamical systems [14] in classical dynamics.

The paper is organised as follows. We introduce the model of a PT-symmetric kicked top, adding a linear gain-loss profile to the non-kicked part, in section 2, and briefly discuss a possible experimental realisation. In section 3 we derive the classical dynamics using the dynamics of the expectation values of the angular momentum operators, and analyse it in some detail. This PT-symmetric classical dynamics shows stable Poincare dynamics without any sinks or sources, if the gain-loss parameter is not too large. There is no counterpart of this behaviour in non-Hermitian or other open systems without PT-symmetry. Furthermore, it is found that the presence of gain and loss leads to an early onset of chaos. Compared to the unitary system, the non-Hermitian PT-symmetric kicked top has an additional classical variable resulting from the norm of the quantum wave function, that can be viewed as a classical intensity. The behaviour of this intensity variable is analysed as a function of the initial state. In section 4 we then study some of the features of the quantum system, starting with the phase-space structure arising from the Husimi representation of the subspaces corresponding to certain lifetimes, which we find to be closely related to the classical intensity as a function of the initial state. Furthermore, the spectral statistics of the system are analysed to gain an insight into potentially universal features. While the spectrum is clearly PT-symmetric, with complex conjugate eigenvalue pairs and a (small) number of real eigenvalues, the nearest neighbour spacings in the complex plane seem unaffected by the symmetry, and show the same behaviour that has been identified by Haake and coworkers for regular and chaotic systems with complex eigenvalues [15–17]. That is, the (unfolded) nearest neighbour spacings exhibit a cubic level repulsion in the chaotic case, while they follow a GUE distribution when the system is regular. To verify that this is not a specific feature of the PT-symmetric kicked top we finally consider the eigenvalues statistics of a PT-symmetric generalisation of a triadic Baker map. This system exhibits the same level spacing characteristics as observed in the PT-symmetric kicked top. We finish with a brief summary and conclusion in section 5.

2. The model

We introduce a non-Hermitian kicked top, described by the time-dependent Hamiltonian

$$\dot{H}(t) = p\hat{L}_x + (\epsilon + i\gamma)\hat{L}_z + \frac{k}{L}L_z^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau),$$

(1)

where $\hat{L}_j$ are angular momentum operators, fulfilling the $SU(2)$ commutation relations

$$[\hat{L}_z, \hat{L}_\pm] = \pm \hat{L}_\pm, \quad [\hat{L}_\mu, \hat{L}_\nu] = \hat{L}_\kappa,$$

(2)

with $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$. The system has five real-valued parameters $p$, $\epsilon$, $\gamma$, $k$ and $\tau$. For $\gamma = 0$ we recover the familiar Hermitian kicked top [6][8]. The first two terms describe a linear precession of a quantum angular momentum; the quadratic delta-kicked term leads to the onset of chaos in the classical dynamics. The total
anglular momentum $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ is a conserved quantity. Thus, one can restrict
the analysis to a $(2L + 1)$-dimensional sector of the Hilbert space belonging to the
eigenvalue $L(L + 1)$ of $\hat{L}^2$. In the classical picture this restricts the system to a
spherical phase space. The kicked top has been experimentally realised using the spin
angular momentum of a single $^{133}$Cs atom as reported in [9]. Another interpretation
of the system is an ensemble of bosonic particles populating two modes (either in an
external potential or using internal degrees of freedom) where the interaction between
the particles is periodically switched on for infinitesimally short times [18], which could
be experimentally realised in cold atoms similar to the experiments reported in [19].

For non-vanishing $\gamma$ the Hamiltonian is no longer Hermitian, that is, the dynamics
is not unitary, and the norm is not conserved. The $\gamma$ term describes a linear gain/loss
profile along the $L_z$ direction. In the bosonic two-mode context this could be realised
(up to an imaginary energy shift) by a loss of particles from one of the modes. A
related model without explicit time-dependence, where the quadratic part is constantly
switched on, has been studied theoretically in detail in [12][13].

There are various definitions of $PT$-symmetry in use throughout the literature,
that differ in small details. Here we use the definition of [20]. A non-Hermitian
Hamiltonian is called $PT$-symmetric if it satisfies $[PT, H] = 0$ where $T$ is an antilinear
operator and $P$ is a linear involution that commutes with $T$. $T$ is usually set to be the
time-reversal operator, which, in our case fulfils $T^2 = +1$ (sometimes referred to as
even $PT$-symmetry). A $PT$-symmetric Hamiltonian $\hat{H}$ with even $PT$-symmetry thus
satisfies $\hat{H} = \hat{P}\hat{H}\hat{P}$, where the bar denotes complex conjugation. As a consequence $\hat{H}$
has either real eigenvalues when the corresponding eigenstates are also eigenstates of the
operator $PT$, or otherwise the eigenvalues come in complex conjugate pairs. For $\epsilon = 0$ for any given time the Hamiltonian [1] is $PT$-symmetric, with

$$PT: \quad \hat{L}_x \mapsto \hat{L}_x, \quad \hat{L}_y \mapsto -\hat{L}_y, \quad \hat{L}_z \mapsto -\hat{L}_z. \quad (3)$$

The investigation of kicked systems is most easily facilitated by the study of the
Floquet operator, the time-evolution of a state over one time period, $\hat{F} = \hat{U}(t_0, t_0 + \tau)$. Here,
we choose a time-interval symmetric around the kick, that is, we consider a free
evolution with $\hat{H}_0 = p\hat{L}_x + (\epsilon + i\gamma)\hat{L}_z$ for time length $\bar{\tau}$, followed by an instantaneous
kick with $\hat{H}_1 = \frac{k}{\bar{\tau}}\hat{L}_z^2$, and then another free evolution for time $\bar{\tau}$. That is, the Floquet
operator is given by

$$\hat{F} = e^{-i(p\hat{L}_x + (\epsilon + i\gamma)\hat{L}_z)\bar{\tau}} e^{-i\frac{k}{\bar{\tau}}\hat{L}_z^2} e^{-i(p\hat{L}_x + (\epsilon + i\gamma)\hat{L}_z)\bar{\tau}}. \quad (4)$$

The choice of the time interval ensures that the Floquet operator is indeed $PT$-
symmetric with respect to the $PT$ operator above for $\epsilon = 0$. Here we define a quantum
map as $PT$-symmetric if it fulfils the relation $\hat{F} = P(\hat{F}^*)^{-1}P$, as in [21]. Denoting
the eigenvalues of a $PT$-symmetric quantum map by $\lambda = re^{-i\theta} = e^{-i\theta}$, the eigenvalues are
either unimodular, when the corresponding eigenstates are also eigenstates of the $PT$
operator, or come in pairs $\lambda_1 = \lambda_2^{-1}$, in which case the $PT$ operator acting on one of
the eigenvectors yields the other. That is the quasienergies $\phi$ are either real or come in
complex conjugate pairs, just as the eigenvalues of time-independent $PT$-symmetric
Hamiltonians. Note that for another choice of the time interval the Floquet operator
would be isospectral to the one considered here and still be $PT$-symmetric, however,
with a non-trivial $P$ operator.

In what follows we shall study the occurrence of chaos in the classical dynamics
and its fingerprints in the quantum system for the $PT$-symmetric quantum map [1].
Quantum chaos in a non-Hermitian PT-symmetric kicked top

For this purpose, we shall first introduce the classical map, arising in the limit of large Hilbert space dimension $L \to \infty$, and analyse the resulting dynamics, before we return to its quantum features.

3. Classical dynamics

To derive the classical dynamics we consider the quantum evolution of the expectation values of the three angular momentum operators in the limit $L \to \infty$. We will deduce the mapping of the expectation values of the angular momentum operators over one period, by expressing

$$
\langle \hat{L}_j \rangle_{n+1} = \frac{\langle \psi^{(n+1)} | \hat{L}_j | \psi^{(n+1)} \rangle}{\langle \psi^{(n+1)} | \psi^{(n+1)} \rangle} \tag{5}
$$

as functions of the $\langle \hat{L}_j \rangle_n$, where

$$
|\psi^{(n+1)}\rangle = \hat{F}|\psi^{(n)}\rangle = \hat{F}^{(n+1)}|\psi^{(0)}\rangle, \tag{6}
$$

is the quantum state after $n+1$ evolution periods.

Let us first consider the free evolution with the Hamiltonian $\hat{H}_0 = p\hat{L}_x + (\epsilon + i\gamma)\hat{L}_z$. The equations of motion for the expectation values of the angular momentum operators can for example be deduced from the $2 \times 2$ representation of the system, which, due to the linearity in the $\hat{L}_j$ provides the full dynamics for all values of the total angular momentum. The time evolution of the expectation values of the rescaled angular momentum expectation values $s_j = \langle \hat{L}_j \rangle / L$ for the free evolution is given by [12,13]

$$
\dot{s}_x = -\epsilon s_y - \gamma s_z s_x,
\dot{s}_y = \epsilon s_x - p s_z - \gamma s_z s_y,
\dot{s}_z = p s_y + \gamma (1 - s_z^2). \tag{7}
$$

In the Hermitian case $\gamma = 0$ this describes a rigid rotation around the axis $(p, 0, \epsilon)^T$.

Some examples of the phase-space trajectories for non-vanishing $\gamma$ are depicted in figure [1]. For more details on the dynamics see, e.g., [13]. Figure [1] depicts both the dynamics on the sphere (in the top row) as well as the projection onto a flat phase space in (canonical) cylindrical coordinates $s_z$ and $\phi = \text{atan}\left(\frac{s_y}{s_x}\right)$ (in the bottom row). In the PT-symmetric case, $\gamma \neq 0$, $\epsilon = 0$, there are two distinct phases of motion for subcritical and supercritical values of $\gamma$, respectively, where the critical value is given by $\gamma_{\text{crit}} = p$. For $\gamma < p$, an example of which is shown in the left panel of figure [1], the motion is a combination of a rotation around the x-axis and a Lorentz boost around the z-axis, with two stable elliptic fixed points that, for increasing values of $\gamma$ move from the opposite poles at $s_x = \pm 1$ along the equator, approaching $s_x = 0$, where they meet at the critical value. Note that while there are no sinks and sources in this evolution the dynamics is not area preserving, but has balanced amounts of contraction and expansion over the phase sphere. For supercritical increasing values of $\gamma > \gamma_{\text{crit}}$ the fixed points separate and travel along the great circle with $s_x = 0$ towards the north and south pole. The southern fixed point is a source and the northern fixed point is a sink of the dynamics. An example of this motion is depicted in the middle panel of figure [1]. This classical behaviour is directly related to the PT-symmetry and its breaking in the quantum system [13]. Without the PT-symmetry, for $\epsilon \neq 0$ the elliptic fixed points present for the Hermitian system turn into a sink and a source for arbitrarily small values of $\gamma$, and the rigid rotation around the $(p, 0, \epsilon)^T$-axis turns
Quantum chaos in a non-Hermitian PT-symmetric kicked top

Figure 1: Trajectories of the free motion for $p = 1$ and different values of $\epsilon$ and $\gamma$, $\gamma = 0.5$ and $\epsilon = 0$ (left), $\gamma = 1.5$ and $\epsilon = 0$ (middle), and $\gamma = 0.5$ and $\epsilon = 0.5$ (right).

into a spiralling phase-space flow from source to sink, where again the sink and the source move closer to the poles of the sphere for increasing values of $\gamma$. An example of the resulting trajectories is depicted in the right panel of figure 1.

To deduce the phase-space map generated by the free motion over half a period $\tau/2$ we need to integrate the dynamical equations (7). For the PT-symmetric case $\epsilon = 0$, that explicitly yields

$$\tilde{s}_x = \frac{1}{\Gamma} s_x$$

$$\tilde{s}_y = \frac{1}{\Gamma} \left( \left( \frac{p^2}{\omega^2} \cos \left( \frac{\omega \tau}{2} \right) - \frac{\gamma^2}{\omega^2} \right) s_y - \frac{p}{\omega} \sin \left( \frac{\omega \tau}{2} \right) s_z + \frac{p \gamma}{\omega^2} \cos \left( \frac{\omega \tau}{2} \right) - \frac{p \gamma}{\omega^2} \right)$$

$$\tilde{s}_z = \frac{1}{\Gamma} \left( \frac{p}{\omega} \sin \left( \frac{\omega \tau}{2} \right) s_y + \cos \left( \frac{\omega \tau}{2} \right) s_z + \frac{\gamma}{\omega} \sin \left( \frac{\omega \tau}{2} \right) \right),$$

with

$$\Gamma = \frac{p^2}{\omega^2} - \frac{\gamma^2}{\omega^2} \cos \left( \frac{\omega \tau}{2} \right) + \frac{p \gamma}{\omega^2} \left( 1 - \cos \left( \frac{\omega \tau}{2} \right) \right) s_y + \frac{\gamma}{\omega} \sin \left( \frac{\omega \tau}{2} \right) s_z,$$

and $\omega = \sqrt{p^2 - \gamma^2}$. It is straightforward to verify that this indeed reduces to a rigid rotation by an angle $\frac{p \tau}{2}$ around the x-axis, for $\gamma = 0$.

The full kicked top map is composed of the mapping resulting from the free motion followed by the mapping arising from the kick, which is not altered by the non-Hermiticity of the free evolution, and then another free motion of half a period. The kicked part of the Floquet operator alone yields the map [7]

$$\tilde{s}_x = \cos(2ks_z)s_x - \sin(2ks_z)s_y$$

$$\tilde{s}_y = \sin(2ks_z)s_x + \cos(2ks_z)s_y$$

$$\tilde{s}_z = s_z.$$
that is, a torsion around the z-axis. Without loss of generality we shall set the period \( \tau \) to unity in what follows.

![Figure 2: Poincare plots of the classical map for \( \epsilon = 0 \) and different values of \( k \) and \( \gamma \). The top two rows show the Poincare dynamics on the sphere (top) and the projection to the (canonical) cylindrical variables (bottom) for \( k = 0.5 \) and \( \gamma = 0, 0.1, 0.5 \) (from left to right). The bottom two rows show the same for \( k = 1 \).](image)

In the Hermitian case \( \gamma = 0 \) an increasing kicking strength \( k \) leads to the onset of chaos and for large kicking strength \( k \gg p \) the dynamics is almost entirely chaotic. Here we shall investigate how the non-Hermiticity changes this behaviour, in particular in the PT-symmetric case \( \epsilon = 0 \).

Note that even for infinitesimal values of \( \gamma \) the dynamics is no longer area preserving. Since the kicking part of the dynamics remains Hamiltonian, the rate of contraction and expansion does not depend on the value of the kicking strength.
Some examples of Poincare plots of the phase-space dynamics for $\epsilon = 0$, $p = 2$, and relatively small $\gamma$ and $k$ are depicted in figure 2. We observe that the non-Hermiticity seems to enhance the chaos, but does not lead to the appearance of sinks and sources in the dynamics for the small values depicted here. This behaviour is due to the $PT$-symmetry of the system, and an inclusion of a small symmetry breaking parameter $\epsilon$ would lead to a fully contracting phase-space flow with a single point sink. Similar pseudo-Hamiltonian behaviour has been discussed in some detail in the classical dynamical systems literature, where the behaviour is related to the reversibility of a non-Hamiltonian dynamical system [14]. This seems to be the natural classical counterpart of the $PT$-symmetry of the non-Hermitian quantum system.

Figure 3: Lyapunov exponents on the projection of the phase space for $k = 1$, $p = 2$ and three values of $\gamma$ ($\gamma = 0, 0.1, 0.5$, from left to right). The Lyapunov exponent is estimated numerically using 2000 iterations of the map and a variation of $10^{-7}$ of the initial phase-space coordinates.

The enhancement of chaos in the classical phase-space dynamics observed in the Poincare sections is confirmed by the analysis of the Lyapunov exponent of the dynamics, depicted in false colours for $p = 2$, $k = 1$ and three different values of $\gamma$ in Figure 3. A positive Lyapunov exponent indicates exponential sensitivity to small variations in the initial conditions, that is, chaos. In the examples depicted, while most of the phase space is associated to a vanishing Lyapunov exponent, corresponding to regular motion, for vanishing $\gamma$, the region of positive Lyapunov exponent grows significantly with increasing $\gamma$ and for $\gamma = 0.5$ already a large proportion of the phase space is covered by a chaotic sea.

The most prominent fixed point in the Poincare sections in figure 2 is the one that originates from the elliptic fixed point at $\phi = 0$ for vanishing $\gamma$ and $k$. It moves along the equator with increasing $\gamma$, much as it would have done for $k = 0$, when it would have been approached from the other side from its partner stemming from $\phi = \pi$, which, however, in the case of non-vanishing $k$ bifurcates before they meet, and gets swallowed up by an increasing chaotic sea. This region is separated from the regular elliptic orbits around the more stable fixed point by increasing numbers of layers of chains of small regular islands.

For larger, but still subcritical values of $\gamma < p$ the prominent central fixed point moves further along the equator and finally bifurcates into a sink and a source for a small volume surrounding it. An example of the Poincare section just above this bifurcation, and a magnification of the area around the central fixed point, are depicted in figure 4 in the left and middle panels. At this value of $\gamma$ also other of the countless fixed points of the system have turned into sinks for tiny phase-space volumes, as is revealed by a zoom into the Poincare plot, depicted in the right panel of the same
Quantum chaos in a non-Hermitian \( \text{PT} \)-symmetric kicked top figure. To generate these figures both the forwards and backwards time-evolution have been plotted, to get a more complete picture of the phase-space flow.

![Figure 4: Poincare plot of the classical map for \( \epsilon = 0, p = 2, k = 1, \) and \( \gamma = 1.25 \) (left). The middle and right panels depict magnifications of smaller regions of phase space.](image)

For further increasing \( \gamma \) the basin of attraction of the sink in the central region grows. At another critical value of \( \gamma \), finally this region of attraction reaches the chaotic sea, and now the volume of the latter is also attracted by the sink, which by now is the only sink of the flow. An example of the resulting Poincare map and a magnification of the area around the sink and source are depicted in the top left and middle panels of figure 5. The right panel on the top of the same figure shows the same magnified area as the figure in the middle, however, for the backwards propagation, revealing the source corresponding to the sink. The sink attracts all of the phase-space volume but the small volume in the remaining regular island at \( s_z = 0 \) and \( \phi \ll -2 \).

This can be seen in the bottom panels of the same figure, that depict the distribution of 20000 phase-space points that were initially chosen from a uniform distribution on phase space and propagated for 100, 500, and 10,000 iterations, respectively. It can be clearly seen that while the points that started out in the regular island remain within this volume, all other points are eventually mapped into the sink.

For even larger values of \( \gamma \) the remaining regular island shrinks further and moves further along the equator approaching the hyperbolic fixed point between the sink and the source that moves towards it along the equator from the other side. At the critical value \( \gamma = p \) the stable island finally vanishes when it meets the hyperbolic fixed point, and the whole phase-space volume is attracted to the single remaining sink. This is the case, irrespectively of the value of the kicking strength.

As we shall see in what follows, the dynamics for smaller values of \( \gamma \) is quite different in nature for larger values of \( k \). In the Hermitian case \( \gamma = 0 \) increasing values of the kicking strength \( k \) lead to enlarged chaotic seas spreading over the phase sphere that eventually almost cover the sphere for sufficiently large values of \( k \). An example of the Poincare section for \( p = 2 \) and \( k = 7 \) is depicted on the left in figure 6. Introducing gain and loss, \( \gamma \neq 0 \), in this case leads to the appearance of spiral-shaped attractors in the phase space, as can be seen in the middle and right panels of the same figure. These spirals appear to be strange attractors. That is, over time the dynamics continues to move around the spiral in a random manner, and zooms into the Poincare plots, as depicted for example in figure 7, indeed reveal a fractal structure of the attractor.

The shape of these strange attractors can be easily understood as being structured around the skeleton of the orbits arising from the single period map, that is, the free
Quantum chaos in a non-Hermitian PT-symmetric kicked top

Figure 5: Dynamics of the classical map for $\epsilon = 0$, $p = 2$, $k = 1$, and $\gamma = 1.35$. The left plot on the top shows a Poincare section of the dynamics and the middle plot on the top shows a magnification of the region around the sink and the source. The right panel on the top shows a magnification of the same area for the backwards propagation. The bottom row shows the distribution of 20000 phase-space points that were initially chosen from a uniform distribution on phase space after propagation for 100 (left), 500 (middle), and 10,000 (right) iterations.

Figure 6: Poincare plots of the classical map for $\epsilon = 0$, $p = 2$, $k = 7$, and different values of $\gamma$ (From left to right: $\gamma = 0$, 0.5, 1).

motion distorted by the torsion of the kicking dynamics and then followed by another free motion. The left panels in figure 6 depict the image of the great circle under this
Quantum chaos in a non-Hermitian PT-symmetric kicked top

Figure 7: Consecutive zooms into the Poincare phase-space plot for $\gamma = 1$, $p = 2$, and $k = 7$.

transformation on the sphere as well as on the projection for the same values of $p$ and $k$ as in figure 6. We have chosen $\gamma = 0$ for this purpose, since the deformation of the spiral for small values of $\gamma$ is negligible. This demonstrates clearly, that the seed of the strange attractor actually lies within the Hermitian system, but is made visible only by the non-Hermiticity. Using the image of the great circle under this transformation to describe the strange attractor, we can easily deduce that the winding number of the spiral is given by $\frac{4k\pi}{\pi}$, since the north pole is rotated by $2k\pi$ in the positive direction, and the south pole is rotated by the same angle in the negative direction by the kicking torsion. We can further deduce that the “centre” of the spiral, is located at $\phi = -\pi$ (i.e. $s_x = 0$), and $s_z = \cos(\frac{p\pi}{2})$. In the middle and right panels of figure 6, this dependence of the winding number and orientation of the attractor is demonstrated using two different sets of parameters. The top row shows the Poincare sections of the map, while the bottom row shows the image of the great circle under the transformation of the kicking torsion followed by the free motion.

Note that in contrast to the behaviour for smaller $k$, where an arbitrarily small value of the PT-symmetry breaking parameter $\epsilon$ leads to appearance of a single sink in the phase-space dynamics, these strange attractors are stable and also appear in the non-Hermitian system without PT-symmetry. We have verified for a large range of parameters that the motion indeed continues to erratically move around the attractor rather than going into a sink for large time scales with or without PT-symmetry.

Finally, the classical limit of non-Hermitian systems has an additional variable on top of the phase-space coordinates, that describes the behaviour of the norm of the quantum state in the semiclassical limit,

$$n_{j+1} := \langle \psi^{(j+1)} | \hat{\psi}^{(j+1)} \rangle = \langle \psi^{(j)} | \hat{F} | \psi^{(j)} \rangle.$$

We have already encountered the map for this variable in the $2 \times 2$ representation as

$$n_{j+1} = \Gamma(s_x, s_y, s_z)n_j,$$

with $\Gamma$ defined in equation (11). This additional parameter can be viewed as an intensity of the classical phase-space trajectory, which growth and decays locally exponentially under the free dynamics proportionally to the momentary $z$-component, but is not affected by the (unitary) kick. In general the intensity is a complicated function of the iteration time for most initial states. To provide an overview of its behaviour over time, we show false colour plots of the final intensity as a function of the initial phase-space variable for different parameter values and different numbers of iterations in figure 9. We observe that as the number of iterations is increased
Figure 8: Structure of the strange attractor. The left panel shows the image of the great circle under the torsion around the z-axis with \( k = 7 \) followed by the rotation around the x-axis by an angle \( p/2 \), for \( p = 2 \) (corresponding to \( \gamma = 0 \)). The top shows the image on the sphere, the bottom on the flat projection. The left panel should be compared to the strange attractors in figure 6. The middle and right panels show the comparison between the Poincare section of the classical map (top) with \( p = 1, k = 7, \) and \( \gamma = 0.5 \) (middle) and \( p = 2, k = 4, \) and \( \gamma = 0.5 \) (right), and the attractor skeleton arising from the image of the great circle for the same parameters (bottom).

finer and finer structures emerge. There is an obvious correspondence to the phase-space portraits. In particular in the bottom panel, belonging to the case of the strange attractor, the fractal structure is very clearly visible in the final intensity after increasing numbers of iterations. Note however, that the structure visible in the norm dynamics does not correspond to the attractor, but the repeller of the motion, that would be obtained from a backwards evolution in time.

We further find that for large values of \( k \) for which the unitary map is chaotic, even for arbitrarily small values of \( \gamma \) the fractal repeller is clearly visible in the iterated intensity, while it is not visible in the dynamics, with the bare eye. This can be seen for example in the middle panel of figure 11 where the classical intensity after three iterations as a function of the initial state is depicted for \( p = 2, k = 10, \) and \( \gamma = 0.01 \).

After having uncovered some of the rich behaviour of the classical dynamics of this \( PT \)-symmetric chaotic system, let us now turn our attention to the quantum system.

4. The quantum system

Let us now consider how the classical features observed in the previous section manifest themselves in the full quantum description. In particular, we will investigate the spectral behaviour of the Floquet operator and the phase space structures associated to its eigenspaces. The quantum system is very sensitive to non-Hermitian perturbations, and even accurate numerical diagonalisation is problematic for medium sized values
Quantum chaos in a non-Hermitian PT-symmetric kicked top

Figure 9: Classical intensity behaviour as a function of the initial phase-space position - false colour plots of the final intensity after \( j \) iterations of the classical map. The top row shows the result after \( j = 5, 10, \) and 20 iterations (left to right), respectively, for the parameter values \( p = 2, k = 1, \) and \( \gamma = 0.1 \). The bottom panel shows the same for \( j = 2, 5, \) and 10 iterations for \( p = 2, k = 7, \) and \( \gamma = 0.5 \).

Figure 10: Eigenvalues and eigenstates in comparison to the classical Poincare sections, for \( p = 2, k = 0.5, \) and \( \gamma = 0.01 \) for \( L = 200 \). The left panel in the top row shows the quasi energies in the complex plane. The middle panel shows the classical Poincare section. The remaining four panels show the Husimi distribution of selected Floquet states.
Quantum chaos in a non-Hermitian $PT$-symmetric kicked top of $L$ if the gain-loss parameter is too large. This sensitivity to perturbations is related to the presence of high order exceptional points in the non-kicked system $^{[1]}$. Thus, we confine ourselves to small values of $\gamma$ in the analysis of the quantum system.

Let us begin with a small value of $k$ for which the classical system is in the regular regime. The left panel in the top row of figure 10 shows the quasi energies in the complex plane for $p = 2$, and $k = 0.5$ and a very small loss-gain parameter $\gamma = 0.01$ for $L = 200$. The corresponding regular classical dynamics is visualised by a Poincare section in the middle panel of the top row. While the dynamics does not show any sinks and sources, the intensity is not conserved for most initial conditions, but varies very slowly, due to the small loss-gain parameter, which introduces an asymmetry in the trajectories. This is reflected in the quantum quasi energies, a handful of which lie visibly off the real axis. The $PT$-symmetry of the system is clearly obvious from the symmetry of the quasi energies with respect to the real axis. Since the system is only mildly non-Hermitian most eigenstates are still approximately orthogonal to each other, and their phase-space distributions are organised according to the structure of the classical phase-space dynamics. This is illustrated in the remaining four panels in figure 10 which show the Husimi distributions of four different Floquet states.

Figure 11: Eigenvalues and quantum phase space structures in comparison to the classical intensity patterns, for $p = 2$, $k = 10$, and $\gamma = 0.01$ for $L = 200$. The left panel on the top shows the quasi energies in the complex plane. The other two top panels show the Husimi functions of two selected Floquet states. The left panel in the bottom row shows the Husimi function of the fastest growing Floquet state. The middle panel in the bottom row the Husimi representation of the subspace of growing states, i.e. the space spanned by those Floquet states with negative imaginary parts of the quasi energies. The right panel shows shows the classical intensity as a function of the initial phase space position after three iterations of the classical map.

In the regime which is chaotic in the unitary case, on the other hand, a small non-Hermitian perturbation has a very different effect. Figure 11 illustrates the case $p = 2$, $k = 10$ and $\gamma = 0.01$, again for $L = 200$. In this case almost all the quasi energies (depicted in the complex plane in the left panel of the figure) are complex.
Quantum chaos in a non-Hermitian PT-symmetric kicked top

Further, the Floquet matrix is far from normal, and most eigenvectors strongly overlap with other eigenvectors. Nevertheless, one can see some structures reminiscent of the classical attractor and repeller in the Husimi functions of individual Floquet states, three of which are depicted in the middle and right panel on the top and the left panel in the bottom row. To obtain a better insight into the quantum phase-space properties, rather than analysing individual eigenstates, it is useful to consider the Husimi representation of certain invariant subspaces, the Husimi-Schur representation as introduced in [22]. In the middle panel in the bottom row of figure 11 we depict the Husimi representation of the invariant subspace of states with growing norm, that is, we use the Schur vectors of the Floquet operator where the diagonal elements are sorted according to the imaginary part of the quasi energies, and average over the Husimi distributions of the Schur vectors belonging to the positive imaginary parts. The resulting picture shows a striking resemblance to the false colour plot arising from the classical intensity as a function of the initial phase-space position, which is depicted after 3 iterations in the right panel. This correspondence takes the place of the familiar quantum phase-space localisation on classical Poincaré structures in phase space for the unitary case for strongly open quantum systems. Similar observations have been made for other types of openings, for example in [22] and [23, 24]. It is an interesting open question how the eigenspaces of the quantum system are organised in phase space in regions where the real energy structures and the norm dynamics are competing and neither dominates. This question, however, goes beyond the scope of the present paper.

Figure 12: Quantum dynamics in phase space for $p = 2$, $k = 10$, and $\gamma = 0.01$ for $L = 200$. The figure on the top left shows the Husimi representation of the initial state, the remaining panels show the Husimi representations of the state after 1, 2, 3, 4, 5, 10, and 30 iterations, respectively, from left to right and top to bottom.

The classical phase space structure is also visible in the quantum dynamics, as demonstrated in figure [12] which depicts the Husimi representation of an initial eigenstate of $\hat{L}_z$ after various iterations. The spiral structure is clearly visible in the iterated states. After a large number of iterations all that remains is the state with the fastest growing state. Indeed for the present example after approximately 20 iterations the state has approached the final configuration. This can be seen from the similarity of the Husimi representation of the state after 30 iterations, depicted in the lower right panel in figure [12] and the Husimi function of the fastest growing state, depicted in
the lower left panel in figure 11.

Let us finally turn to the spectral statistics of our PT-symmetric kicked top. In the Hermitian case the level spacing statistics of the quasi energies of the kicked top follow GOE statistics in the chaotic regime, and show a Poisson distribution in the regular regime, in line with the Bohigas-Giannoni-Schmit conjecture. In the non-Hermitian case, the quasi-energies become complex, and a generalisation of level spacings needs to be considered. Here we follow the approach in [15–17] and analyse the distributions of Euclidean nearest neighbour distances in the complex plane, after unfolding the spectra. For the unfolding we make use of the fact that the quasi energy spectrum is approximately uniform in the real parts, and use the integrated staircase function of the imaginary parts to approximate the underlying smooth distribution and use this to unfold the imaginary parts, while leaving the real parts unscaled. To obtain a suitable amount of quasienergies we combine the data from a number of realisations with slightly altered parameters, similar to the common practice in the Hermitian case. In figure 13 we depict the nearest-neighbour distance distributions in the regular regime, for \( k \) varying in 0.05 increments between \( k = 0.1 \) and \( k = 1 \), for \( L = 200 \) and a small \( \gamma = 0.01 \). The nearest neighbour distance distribution in the chaotic case, for values of \( k = 9 \) to \( k = 10 \) and \( \gamma = 0.01 \) is depicted in figure 14. For comparison we also plot the GOE curve (green) that is expected for independent random numbers in the plane and associated to regular behaviour for complex eigenvalues [15–17], as well as the curve for the \( 2 \times 2 \) complex Ginibre ensemble (magenta), which shows cubic level repulsion described analytically by the equation

\[
P(s) = \frac{3^4 \pi^2}{2^{17} s^3} e^{-\frac{3^2}{2\pi^2} s^2},
\]

and has been associated to the behaviour of complex eigenvalues in the chaotic regime [15–17]. We observe an excellent agreement of the data in the regular regime with the GOE curve, and with the cubic level repulsion [15] in the chaotic regime.

Figure 13: Kicked top, \( \tau = 1, l = 200, p = 2, \epsilon = 0, \gamma = 0.01, k \) between 0.1 and 1

While this is compatible with the universal behaviour expected for non-Hermitian systems, it is rather surprising, that the PT-symmetry of the system, that is clearly observable in the spectra, does not seem to influence this statistical feature. To rule out that this behaviour is due to special properties of the kicked top, we have further analysed the spectra of a PT-symmetric generalisations of another quantum chaotic models showing GOE statistics in the Hermitian case, a triadic Baker map.
The unitary triadic baker map is given by the Floquet operator (see, e.g. [25] and references therein)
\[
\hat{\mathcal{B}} = F_N^{-1} \begin{pmatrix} F_{N/3} & 0 & 0 \\ 0 & F_{N/3} & 0 \\ 0 & 0 & F_{N/3} \end{pmatrix},
\]
where \(F_N\) is a discrete Fourier transform,
\[
(F_N)_{mn} = \frac{1}{\sqrt{N}} e^{-\frac{2\pi i}{N} (m+\frac{1}{2})(n+\frac{1}{2})} ; \quad m,n \in \{0,1,\ldots,N-1\}. \tag{17}
\]
This map is parameter free, the classical counterpart is fully chaotic, and the quantum quasi energies display random matrix statistics. We generalise this to a non-unitary PT-symmetric map (with \(T^2 = 1\)) by introducing a parameter \(\gamma \in (0,1)\) that amplifies phase space trajectories in one third of the phase space, leaves the central third unchanged, and introduces a loss (of equal strength to the gain) in the remaining third, yielding the map
\[
\hat{\mathcal{B}}_{PT} = F_N^{-1} \text{diag}(\gamma F_{N/3}, F_{N/3}, \frac{1}{\gamma} F_{N/3}). \tag{18}
\]
Figure (15) shows the statistics of the nearest neighbour distances of the quasi energies for the PT-symmetric Baker map with matrix size 6000 averaged over different values of \(\gamma\) ranging in 0.01 steps from \(\gamma = 0.5\) to \(\gamma = 0.55\). Also in this case we clearly observe the cubic level repulsion described by the 2 \times 2 complex Ginibre ensemble (15).
Quantum chaos in a non-Hermitian PT-symmetric kicked top

Preliminary studies of a PT-symmetric generalisation of an M-mode Bose-Hubbard model with intermediate interaction strength and unit filling factor, the eigenvalues of which also follow GOE statistics in the Hermitian case [26] also show similar behaviour. Thus, we conclude that the statistics of the nearest neighbour spacings in the complex plane for a number of PT-symmetric models are indeed universal, but do not differ from the non-Hermitian case without PT-symmetry. It would be interesting to investigate whether other spectral statistics differ between PT-symmetric and more general non-Hermitian models. A natural starting point for this would be the investigation of the statistical features of the real eigenvalues alone, that do not have a counterpart in generic non-Hermitian systems. This, however, would require a much larger data set, and goes beyond the scope of the present paper.

5. Summary and Conclusions

In summary, we have introduced a simple PT-symmetric generalisation of a quantum kicked top. We have derived the classical analogue of the dynamics and found that it leads to chaotic phase space dynamics without sinks or sources for a wide parameter range, and a mixture of stable orbits and dissipative behaviour for larger gain-loss parameters, and strange attractors similar to conventional dissipative chaotic systems when the corresponding unitary system is deep in the chaotic regime. Similar behaviour has been discussed before in the theory of classically reversible dynamical systems, which indeed are the natural counterpart of PT-symmetric quantum systems. The classical dynamics arising here has an additional degree of freedom, which can be interpreted as an intensity, that shows intricate dynamical behaviour. In the full quantum system, the nature of the classical counterpart is reflected in the eigenvalue behaviour as well as in quantum phase space structures. Finally, we have considered the spectral statistics of the quasi energies, and found that the nearest neighbour spacings in the complex plane follow the same universal statistics as those of the eigenvalues of other open quantum systems, following a two-dimensional Poisson distribution in the regular regime, and the distribution of the $2 \times 2$ complex Ginibre model in the chaotic regime. We have verified the same behaviour for a PT-symmetric generalisation of a triadic Baker map. It is perhaps surprising that this spectral feature is not influenced by the PT-symmetry of the models, and it would be an interesting task for future investigations to investigate possible universal spectral features that are due to the PT-symmetry.

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Quantum chaos in a non-Hermitian PT-symmetric kicked top

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