Effect of inhomogeneities on high precision measurements of cosmological distances

Austin Peel∗, M. A. Troxel†, Mustapha Ishak‡

Department of Physics, The University of Texas at Dallas, Richardson, TX 75080, USA

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We study exact relativistic effects of inhomogeneities on distance measures, focusing on the distance modulus, in a Swiss-cheese model of the universe with ΛCDM background dynamics, and where the ‘holes’ are non-symmetric structures described by the Szekeres metric. The Szekeres exact solution of Einstein’s equations, which is inhomogeneous and anisotropic, allows us to capture potentially relevant effects on light propagation due to nontrivial evolution of structures. Unlike Swiss-cheese model studies with spherically symmetric holes, we find a net shift in the distance modulus \( \mu \) to distant sources even when we average over many lines of sight with randomized hole orientations and impact parameters. The light also need not propagate strictly through underdense regions to result in an overall bias, which is an accumulated effect associated with the evolution of the anisotropic holes. We find the average shift \( \langle \Delta \mu \rangle \) relative to a pure ΛCDM model to increase with redshift in general, and the spread \( \sigma_{\Delta \mu} \) of the distribution to increase, as well. For sources at high redshifts, the bias can be as large as \( \langle \Delta \mu \rangle = -0.0337 \) mag with \( \sigma_{\Delta \mu} = 0.0093 \) mag, corresponding to a fractional error on the source luminosity distance of 1.56%, which in principle alters the Hubble diagram and our inference of cosmological parameters. Indeed, with the statistical improvements expected in upcoming surveys, systematic biases arising from nonlinear relativistic effects such as those studied here will become a limiting factor to consider in constraining cosmological parameters.

I. INTRODUCTION

Light from distant objects propagates through a cosmic web of structure on its way to us, and the distance we infer to various such sources helps us build a consistent picture of the expansion history of the universe. In particular, type Ia supernovae observations have become a standard probe of cosmological expansion, providing the most compelling evidence supporting a universe presently dominated by some form of dark energy or a cosmological constant. The Lambda cold dark matter (ΛCDM) paradigm, which is based on linear perturbations of a Friedmann-Lemaître-Robertson-Walker (FLRW) metric, constitutes our current best description of the cosmic dynamics and is so far consistent with the array of observations we have available. However, the universe is lumpy and inhomogeneous at nearly all scales we observe, and so the homogeneous ΛCDM model is assumed to represent a smoothing out of the overall dynamics. (The question of precisely how to smooth or average out spacetime inhomogeneities is still open [1–27], and it is not the subject of this paper.)

Varying density environments and local expansion along the path of a light beam distort and magnify (or demagnify) its cross section, which we can use to determine the distance to its source. Different approaches have been employed to account for the biases introduced by intervening structure between sources and observers. For example, one method is via weak gravitational lensing in the context of perturbation theory, where the magnification probability distribution of distant sources can be calculated from the probability distribution of the density contrast [28]. Effects on the distance-redshift relation due to perturbations at second order have been studied by, e.g., [29, 30], and nonlinear relativistic contributions in weak-lensing convergence has also been recently examined in [31]. Another approach is the Dyer-Roeder formula [32], which takes into account that the average density along a line of sight will be some fraction of the smoothed background value, thereby modifying the distance-redshift relation due to magnification by a beam partially filled with structure.

Swiss-cheese models provide a third way to incorporate inhomogeneities and have the advantage of retaining nonlinear effects of general relativity that impact on light propagation. In the general Swiss-cheese approach, spherical regions are removed (holes) from an FLRW background (cheese) and replaced by a different solution of general relativity. By construction, as long as hole-cheese junctions obey an exact matching and no holes overlap, the dynamics of the background model remains unaffected by the presence of the holes. The metric of the entire spacetime is also still

∗ austin.peel@utdallas.edu
† troxel@utdallas.edu
‡ mishak@utdallas.edu
an exact solution of Einstein’s equations, and it can be made statistically homogeneous by an appropriate distribution of the holes. The goal of such a model is to provide a more realistic environment for photons as they traverse the inhomogeneous universe, where most or all of the volume of the holes is underdense; indeed, most lines of sight in the universe are underdense [33]. Many authors have studied the effects on distance measures in Swiss-cheese models with varying degrees of realism in the distribution and structure of the holes, both with and without a cosmological constant [34–44].

In this work, we study the effects on distance measures in a Swiss-cheese model where spherical regions of constant comoving size are removed from a ΛCDM background and replaced by the exact inhomogeneous and nonsymmetric metric of Szekeres. Spherically symmetric Lemaître-Tolman (LT) holes have often been used in similar investigations with the aim of determining whether we might mistake a nonaccelerating inhomogeneous universe for a homogeneous one with accelerated expansion; for example, see [34, 35, 37, 41]. The Szekeres solution allows us to make the holes anisotropic as well as inhomogeneous, and they therefore evolve differently compared to LT structures, which is important for photons traveling through them [45, 46]. A Szekeres Swiss-cheese construction is still an idealized model of the universe, and it is worth clarifying that we are not advocating it as a realistic representation of the true cosmos. However, we do aim with this model to explore the effects of anisotropic and nonlinear matter clustering on scales that are important for real light beams. In Ref. [40], the authors studied light propagation in an axially symmetric Swiss-cheese universe with axial light rays. They found that structures of order ~500 Mpc are required in this scenario to explain the apparent dimming of the type Ia supernovae when the background model is Einstein-de Sitter (EdS). Using smaller ~50 Mpc structures did not significantly alter the distance-redshift relation from that of EdS, and including a cosmological constant improved their fits to supernova data in both cases.

In this investigation, we use a Swiss-cheese model based on a neither axial nor spherical Szekeres metric, which uses holes of size 30–90 Mpc comoving radius. We allow for random orientations of Szekeres holes and beam paths with random impact parameters. We adopt the viewpoint that the universe is indeed well approximated by the standard ΛCDM framework, but that nonlinear structures will induce biases in the inference of cosmological distances, and therefore in cosmological parameters, as well. In particular, these effects become increasingly important as we devise observation schemes to probe the universe with ever finer precision.

The layout of the paper is as follows. We first introduce the Szekeres metric in Sec. II which we use to build the inhomogeneous holes, and describe how we build the Swiss-cheese model. We then present the formalism of light propagation in an inhomogeneous universe and how we implement it in our code in Sec. III. In Sec. IV we quantify the effect on the distance modulus of sources at various redshifts and with varying degrees of inhomogeneity of the holes. We summarize and conclude in Sec. V with comments on the relevance of our results to upcoming precision surveys.

II. BUILDING NONSYMMETRIC LARGE-SCALE STRUCTURES IN AN EXACT GENERAL RELATIVISTIC FRAMEWORK

A. The Szekeres metric

The Szekeres metric is an exact solution of the Einstein field equations for irrotational dust [47, 48] that can be used to model inhomogeneous and anisotropic structures in the universe. Its general formulation has no Killing vector fields [49] and therefore no symmetries; however, imposing certain conditions on the metric functions can produce symmetries. Numerous authors have studied Szekeres models in a variety of cosmological and astrophysical contexts, and we refer the reader to the partial list [50–62] (and references therein) for further details and model-building techniques. We provide only an abridged introduction to the solution and its properties that are relevant to this work.

Two families of the solution exist, which are called class I and class II, and they are distinguished by a different dependence of the metric functions on the coordinates. We use the more general class I family in this work, as it provides a more intuitive framework for building inhomogeneous structures. The class I solution allows for three different types of 2-surface geometries to foliate the three-spaces of constant time. In the parameterization introduced by Hellaby [53], this is determined by the metric parameter ε, which takes values in the set {−1, 0, +1}. The three cases are called quasihyperbolic, quasiplanar, and quasispherical, with 2-metrics of hyperboloids, planes, and spheres, respectively. We specialize to the quasispherical metric where ε = +1 so that we can replace spherical holes excised from a background FLRW spacetime.

The line element in comoving and synchronous coordinates for the quasispherical case is

\[ ds^2 = -dt^2 + \frac{(\Phi_x - \Phi E_x / E)^2}{1 - k} \, dr^2 + \frac{\Phi^2}{E^2} (dp^2 + dq^2) \]  

in units where c = 1, and where a comma indicates partial derivative. The purely spatial function E can be written
as

\[ E(r,p,q) = \frac{S(r)}{2} \left[ \left( \frac{p - P(r)}{S(r)} \right)^2 + \left( \frac{q - Q(r)}{S(r)} \right)^2 + 1 \right], \quad (2) \]

where \( S(r) \), \( P(r) \), and \( Q(r) \) are arbitrary functions that control the anisotropy of the dust structure. Setting these three functions to constants specializes the metric to the spherically symmetric LT solution. In that case, the generally nonconcentric \((p,q)\) 2-spheres become concentric, thereby allowing \( r \) to be properly viewed as a radial coordinate.

The \( p \) and \( q \) coordinates vary over \((-\infty, \infty)\), and a coordinate transformation reveals that \( p \) and \( q \) are stereographically projected coordinates from \( \theta \) and \( \phi \) angular coordinates on the unit sphere:

\[
\begin{align*}
 p &= P(r) + S(r) \cot(\theta/2) \cos \phi, \quad \text{and} \quad q &= Q(r) + S(r) \cot(\theta/2) \sin \phi. \quad (3)
\end{align*}
\]

That is, for each value of \( r \), there is a different mapping (in general) of the \( p, q \) plane to the unit sphere that depends on the choice of \( S, P, \) and \( Q \). By fixing \((t,r)\) and using Eqs. \((2)\) and \((3)\) in Eq. \((1)\), one finds that \( E^{-2}(dp^2 + dq^2) \) is the metric of the unit sphere.

The evolution of the quantity \( \Phi(t,r) \), which is often called the generalized scale factor or areal radius, obeys

\[
(\Phi,t)^2 = -k(r) + \frac{2M(r)}{\Phi} + \frac{\Lambda}{3} \Phi^2. \quad (4)
\]

For a given \( t \), \( \Phi \) is the radius of the comoving sphere labeled by \( r \), the area of which is \( 4\pi \Phi^2 \). By Eq. \((4)\), it is clear then that each \((p,q)\) 2-sphere evolves independently according to its own Friedmann equation. The function \( M(r) \) appears as an arbitrary function of integration and is physically interpreted as the total gravitational mass inside a sphere of constant \( r \). A reasonable constraint on \( M \) is therefore that it should be non-negative. Finally, \( \Lambda \) is the cosmological constant.

Solutions for \( \Phi \) can be expressed in closed form when \( \Lambda = 0 \) in terms of a parameter. Three cases are possible depending on the sign of \( k(r) \), which determines the type of evolution of the 2-sphere at that \( r \). As in FLRW universes, \( k > 0 \) is elliptic, \( k = 0 \) is parabolic, and \( k < 0 \) is hyperbolic. When \( \Lambda \neq 0 \), analytic solutions for \( \Phi \) have been found in terms of elliptic functions (see, for example, \cite{64}), and we have implemented one such solution in our code in terms of the Weierstrass \( \wp \) and associated Jacobi \( \theta \) functions. A final arbitrary function \( t_B(r) \) called the bang time appears in all solutions for \( \Phi \) in the combination \((t - t_B(r))\). It arises by integration of Eq. \((4)\) and gives the local time of the big bang.

The field equations give one other unique relation, which is the expression for the mass-energy density:

\[
\kappa \rho = \frac{2(M_{0r} - 3M_{0}E_{0r} / E)}{\Phi^2(\Phi_{0r} - \Phi E_{0r} / E)}, \quad (5)
\]

where \( \kappa = 8\pi \) in units with \( c = G = 1 \). When \( E_{0r} \neq 0 \), the density varies as a dipole around each \((p,q)\) 2-sphere. The density is maximum at a single point \((p_0,q_0)\) for a given \( r \) and minimum at the antipodal point, while varying monotonically between.

The metric contains FLRW solutions as special cases, which arise by imposing two conditions on its functions. In this parameterization, we can take the first to be \( \Phi(t,r) = a(t)r \), where \( a(t) \) is the FLRW scale factor, and the second as \( k(r) = k_0r^2 \), where \( k_0 \) is the curvature index of the FLRW spacetime. It follows also in this limit that \( M(r) \) becomes proportional to \( r^3 \). The functions \( S, P, \) and \( Q \) need not assume any particular form, but different choices amount to different coordinate systems on the resulting FLRW spacetime.

**B. Description of the Swiss-cheese model**

We construct a model of the universe where spherical regions (holes) are removed from a homogeneous background FLRW metric (cheese) and replaced by inhomogeneous Szekeres structures. We choose the background metric to have \( \Lambda \)CDM parameters of \((\Omega_m, \Omega_\Lambda) = (0.3, 0.7) \) and \( H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \). The matching is exact across the spherical junction surfaces, which are comoving hypersurfaces of constant \( r = r_{\text{match}} \), so that the resulting Swiss-cheese spacetime is still an exact solution of Einstein’s equations. To achieve the matching, we satisfy the Darmois conditions \cite{62}, which require that on the hypersurface, the first and second fundamental forms are (independently) equal as evaluated by the metrics on either side. In practice, these conditions imply the same constraints on the metric functions as in the spherically symmetric LT case; they are that (i) the integrated mass of the hole is the same as FLRW region that it replaces; (ii) at the junction, the Szekeres curvature function \( k(r) \) becomes \( k_0r^2 \), where \( k_0 \)
is the curvature index of the FLRW exterior; (iii) the bang time function $t_B(r)$ becomes constant and equal to the FLRW value (usually 0) at the junction; and (iv) $A$ has the same value within the holes as in the FLRW background.

In our model, we impose also that the $S$, $P$, and $Q$ Szekeres functions become constant at the junction and for larger $r$, which is not necessary for proper matching, but a convenience for the coordinates to be naturally extended through the junction. As mentioned previously, by construction, the dynamics of the background spacetime is not affected by presence of the holes, since the average mass-energy density of the entire spacetime remains unchanged. We therefore do not address the issue of backreaction (e.g., see \cite{17,27}), which considers effects on cosmic dynamics and parameter determination due to averaging an inhomogeneous spacetime.

We take as the FLRW background, instead of its more usual metric form, the Szekeres solution specialized to its FLRW limit. This is more convenient for our code, since then we can work consistently within the Szekeres framework and do not have to change between metrics. The above matching conditions then imply that $k(r)$ and $t_B(r)$ take on the appropriate forms not only at the junction surface, but also beyond for $r > r_{\text{match}}$. $M(r)$ becomes proportional to $r^3$ in the FLRW limit, as noted above, and our parameterization of $M$ makes it straightforward to achieve this (see Eq. (7) below).

In practice, we are interested in light propagation through many randomly placed holes, but we do not construct the entire spacetime from the outset. We achieve the effect of an observer receiving a bundle of light rays that have propagated through many intervening holes by patching together alternating Szekeres and FLRW regions as we integrate along the path of the light. This is sufficient for analyzing the effects on redshift and distance for a single source, and by repeating the process for different realizations containing different distributions of holes, we simulate the effect of having many different lines of sight in a $\Lambda$CDM universe filled with Szekeres holes.

Figure 1 gives an illustration of our Swiss-cheese model setup. For simplicity, we take the holes at first to all have the same size and density distribution, but their orientations are not uniform along a line of sight. Adjacent Szekeres regions also do not touch, instead having $\Lambda$CDM spaces of approximately 40 Mpc between them, although this is not a requirement of the framework. We later test the effect of multiple hole sizes (see the end of Sec. IV C).

### C. Building the Szekeres holes

To build the Szekeres structures that serve as the holes in the $\Lambda$CDM spacetime, we start by designing the desired density profile at the present time $t_0$, which in our units is $t_0 = 4.13$ Gpc (equivalently, $\sim 13.5$ Gyr) in the background. The structures have radii of about 30 Mpc, and the interior void regions have fractional underdensities of $\delta = -0.8$.

We have chosen the size of the structures, as well as their interiors not to be completely empty ($\delta = -1$), in order to represent typical voids in the real universe (see, e.g., \cite{67,68}). We start with the spherically symmetric (LT) density and then introduce anisotropy via the $S$, $P$, and $Q$ functions—or, more importantly, their derivatives—which have the effect of redistributing the dust within shells. In particular, the structures we end up with are underdense and homogeneous in their interiors and have compensating mass shells of thickness about 8 Mpc. The matter in the outer shells is distributed anisotropically, which we achieve with $Q_{,r} \neq 0$ within this range of $r$; see \cite{61} for another example of this approach.
The spherically symmetric density is given by
\[ \rho_{LT}(t_0, r) = \rho_{bg,0} \left( 1 - A e^{-b(r/\sigma)^n} \left[ 1 - \frac{b}{3} \left( \frac{r}{\sigma} \right)^n \right] \right), \] (6)
where we use the parameter set \((n, b, \sigma, A) = (10, 0.8, 0.025, 0.8)\), and \(\rho_{bg,0} = (3/8\pi)\Omega_m H_0^2\) indicates the present background density. This parameterization is a modified version of one we and other authors have used previously [51, 58]. The additional parameter \(A\) allows the structure to have \(\delta = -0.8\) in the underdense central region while still satisfying the matching conditions. The term in square brackets of Eq. (6) results in a simple form of the associated \(M(r)\) function, which we obtain next.

The resulting mass function \(M(r)\) is obtained by integrating Eq. (5) with \(E, r\) set to zero and applying the gauge freedom in the choice of \(r\) coordinate: \(\Phi(t_0, r) = r\). The expression becomes
\[ M(r) = 4\pi \int_0^r \rho_{LT}(t_0, x) x^2 dx = \frac{4\pi}{3} \rho_{bg,0} r^3 \left( 1 - A e^{-b(r/\sigma)^n} \right). \] (7)

In the LT metric, these same profiles would describe a spherically symmetric structure with a maximum overdensity in the compensating shell of \(\rho/\rho_{bg,0} \approx 1.7\). To introduce anisotropy, we choose the additional Szekeres \(S, P,\) and \(Q,\) functions to be
\[ S(r) = 0.4, \quad P(r) = 0, \quad \text{and} \quad Q(r) = \frac{A Q}{2} \left[ 1 - \tanh \left( \frac{r - b Q}{\sigma Q} \right) \right], \] (8)
where \((A Q, b Q, \sigma Q) = (70, 26, 2.5)\) Mpc. Physically, these functions produce an overall density dipole-like structure, where mass from one side of the overdense shell has been transplanted to the opposite side. The maximum value of the overdensity is now \(\rho/\rho_{bg,0} \approx 12\), while the opposite side has been reduced to \(\rho/\rho_{bg,0} \approx 1.3\). Due to constraints of the matching and the nature of the metric functions, the peak in the overdensity cannot be made arbitrarily large without inducing shell crossings. We have therefore chosen a moderate value to work with at first and later test the impact of augmenting it (see Sec. IV C). Within the overdense shell, there is a monotonic and smooth decrease in the density from the maximum point to the minimum point. The left panel of Fig. 2 plots the present density profile of the holes along lines encountering the extreme values of \(\rho\) (in units of the background) at each \(r\). The maximum is dash-dotted (red), the minimum is dashed (blue), and the corresponding spherically symmetric LT curve is solid (green).

An exact matching across the comoving \(r_{match} = 35\) Mpc surface would generally require the use of two different metrics, which in our case is Szekeres within the holes and FLRW in the background. However, since the Szekeres
metric reduces to FLRW in limiting cases of the metric functions, we simply allow the functions to take on their FLRW values beyond $r_{\text{match}}$, and therefore the entire spacetime is in fact Szekeres. In practice, our functions have an abrupt but smooth transition between the holes and the background, instead of using piecewise functions patched together at exactly $r = r_{\text{match}}$. The density, for example, is different from the would-be FLRW background density by only $\sim 10^{-7}$ percent at $r = 35$ Mpc, and decreases rapidly with increasing $r$. We have verified that using a smooth transition instead of a sharp piecewise one does not affect our results.

The right panel of Fig. 2 shows the evolution of the density profile along a curve through the maximum density at each value of $r$. Starting from $t_0$ (darkest curve) and looking at progressively earlier times, the density peak at $r \approx 26$ Mpc decreases toward the background value $\rho_{\text{bg}}$, while the underdense central region increases toward it. By $t/t_0 = 0.01$ (lightest curve), the structure has been reduced to only a slight deviation from the homogeneous background. In the future, the model encounters a shell crossing singularity by $t/t_0 \approx 1.2$, since there is no pressure in these models to avoid such a nonphysical divergence in the density. Despite some caveats, like the absence of pressure and rotation, we expect the models to capture the density environments encountered by typical light beams that pass through mostly underdense, and occasionally overdense, regions of the universe before reaching us.

Szekeres models in general have 6 functional degrees of freedom with $\{M(r), k(r), t_B(r), S(r), P(r), Q(r)\}$, plus a gauge freedom to choose a new $r$ coordinate. By redefining $r$ in terms of one of these functions (e.g., $r' = M(r)$), the Szekeres model is completely determined, and its evolution can be solved. In this work, we have chosen explicit forms for $\{M(r), t_B(r), S(r), P(r), Q(r)\}$ along with $r \rightarrow r' = \Phi(t_0, r)$ (dropping primes for convenience) and therefore solve for $k(r)$ by the following integral,

$$t - t_B(r) = \int_0^\Phi \frac{dx}{\sqrt{-k(r) + \frac{2M(r)}{2} + \frac{4}{3} x^2}}. \tag{9}$$

Equation (9) comes from integrating Eq. (4) with respect to $t$, which has become a constraint equation on $k$ by our choice of how to use the degrees of freedom. Solving for $k$ is carried out in our code semi-analytically using the Carlson symmetric forms of elliptic integrals [70, 71].

III. LIGHT PROPAGATION IN AN INHOMOGENEOUS UNIVERSE

Determining the path of a light ray through a general spacetime requires solving the null geodesic equations for the metric. Notions of distance typically used in cosmology, including the angular diameter distance $d_A$ and luminosity distance $d_L$, further require knowledge of how neighboring geodesics spread or converge relative to each other along their parameterized paths. In other words, we must study light beams, which are bundles of light rays, in addition to single light rays in order to connect geodesics to physically meaningful measures of distance. The Sachs formalism provides a framework for this, where the evolution of the beam cross section, which gives the distance, is related to Ricci and Weyl focusing terms. For more details of these results and applications or extensions of the formalism, see [44, 45, 72, 73].

A. Sachs formalism

To develop the formalism, we first consider a fiducial null geodesic $\gamma$, affinely parameterized by $\lambda$, and with tangent vector components $k^\mu = dx^\mu/d\lambda$. We imagine $\gamma$ to belong to an infinitesimal bundle of geodesics that are all parameterized by $\lambda$. Neighboring geodesics to $\gamma$ (an infinitesimal distance away) can be described by a separation vector whose components are $\xi^\mu = dx^\mu/d\sigma$, where $\sigma$ is a parameter that labels the family of neighboring geodesics. Distance measures involve knowing how the shape or cross-sectional area of light beams evolve along the geodesic, so we expect that knowing how $\xi^\mu$ changes with $\lambda$ will provide this information. It can be shown that $\xi^\mu$ obeys

$$\frac{D^2 \xi^\mu}{d\lambda^2} = R^\mu_{\nu\alpha\beta} k^\nu k^\alpha \xi^\beta, \tag{10}$$

where $D/d\lambda = k^\mu \nabla_\mu$, or the covariant derivative along $k$, and $R^\mu_{\nu\alpha\beta}$ is the Riemann curvature tensor. Equation (10) is the well-known geodesic deviation equation.

For an observer with 4-velocity components $u^\mu$ observing the light ray, we can set up a tetrad of vectors called the Sachs basis consisting of $u^\mu$, $k^\mu$, $(E_1)^\mu$, and $(E_2)^\mu$. Vectors $(E_1)^\mu$ and $(E_2)^\mu$ are space-like and span the 2-space screen orthogonal to both $u^\mu$ and $k^\mu$; they therefore comprise a 2D basis on which to project $\xi^\mu$. It is only this projection
that is relevant to the observer, since his line of sight is directed along \( k^\mu \). The screen basis vectors are orthonormal as well as parallely propagated along the geodesic; that is,

\[
(E_a)^\mu (E_b)_\mu = \delta_{ab}, \quad \text{and} \quad k^\mu \nabla_\mu (E_a)^\nu = 0,
\]

where \( a, b \in \{1, 2\} \), and \( \delta_{ab} \) is the Kronecker delta.

We can write the decomposition of \( \xi \) relative to the Sachs basis as

\[
\xi = \xi^1 E_1 + \xi^2 E_2 + \xi^k k,
\]

where \( \xi^1 = \xi^\mu (E_1)_\mu \) is the component of \( \xi \) along \( E_1 \), and so on. There is no component along the observer’s \( u \) vector due to the orthogonality relations of the basis set and the fact that \( \xi^\mu k_\mu = 0 \). This can be seen by taking the inner product of \( \xi \) with \( k \) in Eq. (12).

We restrict our attention then to the components \( \xi^a \) that lie within the screen, since they characterize the cross section of the beam viewed by the observer. Projecting Eq. (10) onto the screen basis gives the evolution of \( \xi^a \) along the path of the beam:

\[
\frac{d^2 \xi^a}{d\lambda^2} = \mathcal{T}^a_b \xi^b,
\]

where \( \mathcal{T}_{ab} = R_{\mu\nu\alpha\beta} (E_a)^\mu k^\nu k^\alpha (E_b)^\beta \) is known as the optical tidal matrix. We note that index placement on \( \mathcal{T} \) and similar quantities is not important, except to keep with standard summation notation, since \( a \) and \( b \) are raised and lowered by \( \delta_{ab} \). The four components that comprise \( \mathcal{T}_{ab} \) can be written in terms of the Ricci and Weyl curvatures of the spacetime as follows.

\[
\mathcal{T} = \begin{pmatrix}
R & Re \mathcal{F} & Im \mathcal{F} \\
Im \mathcal{F} & R + Re \mathcal{F}
\end{pmatrix},
\]

where

\[
R = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu
\]

is the Ricci focusing term, and

\[
\mathcal{F} = -\frac{1}{2} C_{\alpha\beta\mu\nu} (\epsilon^\ast)^\alpha k^\beta k^\mu (\epsilon^\ast)^\nu
\]

is the Weyl focusing term. We have introduced a complex form of the screen basis with \( \epsilon = E_1 + iE_2 \), and \( R_{\mu\nu} \) is the Ricci tensor, while \( C_{\alpha\beta\mu\nu} \) is the Weyl tensor.

Since Eq. (13) is linear, there must exist 2-matrices that take the initial conditions (i.e., \( (\xi^a)_0 = \xi^a|_{\lambda=0} \) and \( (d\xi^a/d\lambda)_0 = d\xi^a/d\lambda|_{\lambda=0} \) at the observation event) to the solution \( \xi^a(\lambda) \) at some event farther along the geodesic in a linear way. Noting that \( (\xi^a)_0 = 0 \) at the observer, since the beam converges there, we can then write

\[
\xi^a(\lambda) = \mathcal{D}^a_b \left( \frac{d\xi^b}{d\lambda} \right)_0,
\]

where \( \mathcal{D} \) is known as the Jacobi matrix. By Eq. (13), this implies

\[
\frac{d^2 \mathcal{D}^a_b}{d\lambda^2} = \mathcal{T}^a_c \mathcal{D}^c_b,
\]

with the initial conditions

\[
\mathcal{D}_0 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad \text{and} \quad \left( \frac{d\mathcal{D}}{d\lambda} \right)_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

at \( \lambda = 0 \). It is the elements of the \( \mathcal{D} \) matrix, solved at some value of the parameter \( \lambda \), that determine the angular diameter distance to that event along the null geodesic. It turns out that the relation to \( d_A \) is simply

\[
d_A = \sqrt{|\det \mathcal{D}|}.
\]
The Jacobi matrix elements can further be related to the lensing convergence $\kappa$ and shear $\gamma$ by

$$\kappa = 1 - \frac{D_{11} + D_{22}}{2D_A},$$

$$\gamma_1 = \frac{D_{22} - D_{11}}{2D_A},$$

$$\gamma_2 = \frac{D_{12}}{D_A} = \frac{D_{21}}{D_A},$$

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2},$$

where $D_A$ is the angular diameter distance in the homogeneous background model.

For certain cosmological observations like supernovae, the luminosity distance $d_L$ and associated distance modulus $\mu$ are more practical. The connection between $d_A$ and $d_L$ is provided by the distance duality relation

$$d_L = (1 + z)^2 d_A,$$

which holds in any spacetime where photon number is conserved, and is based on the Etherington reciprocity relation \[76\]. The distance modulus is defined as

$$\mu = 5 \log_{10} \left( \frac{d_L}{\text{Mpc}} \right) + 25,$$

and the redshift in the geometric optic approximation is determined by the ratio of the light’s wavelength at observation to its wavelength at emission; that is,

$$1 + z = \left( \frac{k^\mu u_0}{k^\mu u_\mu} \right)_o,$$

where the subscripts $o$ and $e$ refer to observation and emission, respectively.

B. Code implementation

To solve the evolution and light propagation equations in Szekeres spacetimes, we have implemented a specific semi-analytic calculation using a Fortran code package developed by the present authors and others. The initial work in solving the geodesic equations and luminosity distance for observers at the origin (in terms of partial derivatives of the null vector components) for general Szekeres models was carried out in \[55\]. Subsequent works \[45, 58, 61\] used code developed to extend the initial approach and accommodate observers at arbitrary positions in and around a Szekeres structure. The code package solves the model evolution via elliptic functions and light propagation by the Sachs formalism as described in Sec. III A. For calculations in the present work, the latter code has been modified to meet the needs of a Swiss-cheese model, which includes allowing for multiple randomly oriented structures along a line of sight.

The strategy for obtaining $d_A$ in our Swiss-cheese model is to propagate a light beam from observer to source by solving the geodesic and Jacobi evolution equations simultaneously. In reality, the beam travels from source to observer, but our calculations proceed the other direction, since we want to take the observation event, where the beam is converged to a vertex, as boundary condition. We start with a single coordinate patch that contains a Szekeres inhomogeneous structure within $0 \leq r \leq r_{\text{match}}$. Beyond $r_{\text{match}}$, the spacetime is $\Lambda$CDM, and we place the observer in this outer region. The beam is propagated across the structure, where it encounters first the homogeneous background, then inhomogeneous structure, and then it returns to the homogeneous space. Once back in the $\Lambda$CDM background, the beam transitions to a new coordinate patch of the same type, but with a different orientation of the structure and a new impact parameter. The process is repeated until the desired redshift is reached, whereby $d_A$ is determined using Eq. (20) with the $D$ matrix elements at the source.

We first choose the coordinates of the observation event, which are $t = t_0$, $r = r_0 > r_{\text{match}}$, and $p_0$ and $q_0$ are given random values. In the code, we actually randomize the angular coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ and convert them to $(p, q)$ using Eq. (3) with $r = r_0$, since these coordinates have finite ranges, and their location relative to the structure is more intuitive. The choice of $r_0$ indirectly determines at what redshift the beam encounters the final structure along the line of sight before reaching the observer. (The geodesics are past directed so that the parameter decreases toward zero as $t \rightarrow t_0$.) Larger $r_0$ corresponds to greater distance between the observer and the
in the geometric center of the \( p, q \) integral for \( r \) defined in the general Szekeres metric; initially radial geodesic will not remain so as it traverses the inhomogeneity. (The notion of radial is not even well defined in the ΛCDM universe, but rather using an entirely Szekeres framework specialized to its FLRW limit.

FIG. 3: Illustration of a few possible beam paths through one hole. The beam may be almost radial, spending most of its time in the underdense central region and passing through the overdense shell in two places. Otherwise, it can pass near the edge through just the overdense region, or even miss the structure entirely, depending on initial conditions. Bending of the beams due to lensing has not been depicted.

IV. RESULTS

A. Single hole: particular paths through a Szekeres structure

As a first test of our theoretical approach and numerical code, we verify that the results are what we expect for light propagating along particular paths through a single hole. With the observer at \( x^\mu \) in the ΛCDM region, a light beam can travel many paths across a structure placed at some redshift \( z > 0 \), including the extreme cases of radially through the center, or alternatively missing the structure altogether. See Fig. 3 for an illustration of a few such possible paths. With anisotropic structures like the ones we use, a perfectly radial trajectory is not actually possible, since an initially radial geodesic will not remain so as it traverses the inhomogeneity. (The notion of radial is not even well defined in the general Szekeres metric; it is not truly a radial coordinate, and the coordinate origin need not reside at the geometric center of the \( p, q \) 2-spheres.) However, we do not expect the bending of the geodesics to be significant for structures of this size, and so by a radial path we will mean one that only deviates slightly from a line connecting two antipodal points on the \( r_{\text{match}} \) 2-sphere. In this case, the photons spend most of their time in the underdense central region of the structure and minimal time in the overdense shell.

For beams that do not cross the structure and instead propagate only in the background spacetime, we find no deviation in \( d_A \) from the ΛCDM result. This is an important confirmation, since we are not simply solving the usual integral for \( d_A \) in a ΛCDM universe, but rather using an entirely Szekeres framework specialized to its FLRW limit.
FIG. 4: Distance modulus shifts for particular paths through a single Szekeres hole at \( z = 0.1 \) and with source at \( z = 0.4 \). The upper dotted (red) curve shows the shift in \( \mu \) for a light beam that has crossed the hole essentially along an axial line connecting the maximum and minimum density locations of the compensating shell. On its way to the observer, the beam first encounters the maximum value, followed by the underdense interior, and finally exits through the minimum. The lower dashed (blue) curve represents nearly the same path across the structure, but traversed in the opposite direction. The different signs of \( \Delta \mu \) for these two cases reveals the asymmetric effects on observables that can arise using exact anisotropically evolving structures. The central solid (green) curve is for a beam crossing the hole along a path orthogonal to that of the other two, where the entrance and exit densities seen by the light in the shell are almost equivalent. The shift in \( \mu \) for this case is negligible.

We are solving the full set of geodesic and \( D \) matrix evolution equations, and the result is not sensitive to initial conditions \((x^\nu)_0\) and \((k^\nu)_0\), so long as they are chosen so that the beam avoids the structure.

Beams that cross the structure can experience quite varied density and local evolution environments, depending on the orientation of the hole and the beam’s impact parameter. For example, the beam might graze just the overdense shell, or it could travel closer to the axis of the overall dipole, where it would encounter both the maximum and minimum density values in the mass-compensating shell. As with spherically symmetric LT structures, we do not expect an exact cancellation effect in \( d_A \) for a typical beam path, even when the path is radial. This is because holes built using either metric evolve during the time it takes the light to cross them, so the photons experience a nonzero integrated density contrast passing from one side to the other. We do expect, however, certain beam paths across Szekeres structures to result in significantly different values of \( d_A \) than are possible in their corresponding LT structures. In addition to anisotropic density distributions that have locations of larger maximum \( \rho \), Szekeres structures have been shown to provide gravitational infall that is faster than that in their spherically symmetric counterparts \([51, 57, 58, 61]\), and this can have an appreciable effect on distances.

In Fig. 4, we show the shift in distance modulus \( \Delta \mu = \mu_{\Lambda CDM} - \mu_{SC} \), where \( \mu_{SC} \) is the Swiss-cheese value, for particular paths through a single Szekeres hole at \( z = 0.1 \) and with source at \( z = 0.4 \). The upper dotted (red) curve is for a beam path that passes essentially along the axis of the overall dipole. The light first encounters the maximum overdensity in the shell, then crosses the central underdense region, and finally exits the hole through the minimum shell overdensity before reaching the observer. The lower dashed (blue) curve is for a similarly axial beam, but one that traverses the hole in the opposite direction. In other words, the beam first encounters the minimum shell overdensity and exits through the maximum. \( \Delta \mu \) for the first case is approximately +0.006 mag, while for the latter it is about -0.008 mag. The central solid (green) curve in Fig. 4 is for a beam that traverses the hole along a line orthogonal to the dipole, and it experiences a negligible shift in \( \Delta \mu \), as in spherical symmetry. We have also verified that corrections to the redshift are negligible for a single hole.

The fact that \( \Delta \mu \) has different magnitudes with opposite signs for the two paths along the dipole axis is significant, because it exemplifies the effect that anisotropic structures evolving nonlinearly can have on distance determinations. This asymmetric effect for light has also been seen for lensing observables through overdense Szekeres structures \([45]\) embedded in an FLRW background, and we can understand the results here by reference to the discussion there. Sources that an observer perceives as farther in a Swiss-cheese model (i.e., \( \Delta \mu < 0 \)) have been demagnified, which follows mathematically from the convergence and shear definitions given in Eqs. \((21)-(23)\). The magnification of a source is \( \tilde{\mu} = \left[ 1 - \kappa^2 - \gamma^2 \right]^{-1} \), where \( \kappa \) and \( \gamma \) are the convergence and shear, respectively, of the light beam, and we have used a tilde on the usual symbol to avoid confusion with the distance modulus. After substituting \( \kappa \) and \( \gamma \) in terms of Jacobi matrix elements, this becomes \( \tilde{\mu} = \left( \frac{D_A}{d_A} \right)^2 \), where \( D_A \) is the background \( \Lambda \)CDM angular diameter distance. A demagnified source therefore has \( \tilde{\mu} < 1 \) and \( \Delta \mu < 0 \), while the inequalities are both \( > \) for a magnified source.
It was found in [45] that the convergence term in $\tilde{\mu}$ dominated over the shear for light paths parallel to the axis of a dipole cluster model. We have confirmed that this is also the case for the hole density profiles studied in this work, where the off-diagonal elements of $\mathcal{D}$ remain negligible compared to the diagonal terms, which are in turn only slightly different from each other along such paths. The asymmetry in $\Delta \mu$ we find for crossing the hole in opposite directions is therefore due to the same asymmetry in convergence that was found for the cluster model. We can understand the effect physically as one similar to the Rees-Sciama effect for CMB photons, whereby the gravitational potential wells and hills of the components of a structure evolve during the time it takes the light to cross it. The heights of the overdensities encountered by the beam for one orientation of the hole do not match those of the opposite orientation, as light passing through the same location within the structure does so at different times in the two cases. On the other hand, light crossing along paths orthogonal to the axis encounters a much more symmetric sequence of densities, producing a near cancellation of the effect on $\Delta \mu$ for those particular paths.

The results of this subsection indicate that the shift in $\mu$ (or $d_A$) for photons passing through anisotropic structures can be appreciable for particular paths that encounter anisotropic densities, and it also depends on the direction of crossing. Since $\Delta \mu$ can be positive or negative, averaging over an ensemble of randomly oriented structures with sources at a given redshift should wash out the effect to some extent. However, as the size of $\Delta \mu$ is not the same for paths traveled in opposite directions, there will be a residual net bias, which was pointed out in [45]. Furthermore, for light traversing a Swiss-cheese model with many such holes along the line of sight, the effect could accumulate and end up with a significant shift in $\mu$ for sources at high redshifts. We explore this possibility in the following sections.
B. Multiple holes: comparison to Swiss-cheese models with spherically symmetric holes and $\Lambda = 0$

We now consider the effect of placing multiple holes along the line of sight, and to verify our procedure, we first reproduce results from previous studies with spherically symmetric LT holes. In [34], the authors (hereafter MKMR) use single-sized structures of radius approximately 120 Mpc with evacuated interiors ($\delta = -1$) and compensating mass shells. The density profiles of the structures are specified at an initial time $\bar{t} = 0.2t_0$, where the expansion rate of the LT holes is taken to match that of the background EdS space. The overdense shell then accumulates matter from the underdense interior as the structure evolves toward the present, and the density becomes sharply peaked in the shell at $t_0$.

MKMR found that placing five such LT structures along the line of sight to distant supernovae ($z \approx 1.8$) would cause an observer to erroneously conclude that he/she lives in a FLRW universe with density parameters of $(\Omega_m, \Omega_\Lambda) = (0.6, 0.4)$. However, this universe in reality would be expanding as EdS without acceleration. In Fig. 5, we have reproduced the results of MKMR for this scenario, showing the effects on the distance modulus, and on the angular diameter and luminosity distances. The figure corresponds to Fig. 13 of [34]. Again, we have used our general Szekeres framework, reduced to its LT limit, in order to obtain the same results as MKMR, who used a different theoretical approach with a different numerical implementation.

It is of interest to note that Vanderveld et al. [36] (hereafter VFW) showed that randomizing the impact parameters of the MKMR model effectively eliminates the large effect seen in [34], and therefore such inhomogeneities cannot remove the need for dark energy. Finding that an inhomogeneous Swiss-cheese universe mimics an FLRW one with $\Omega_\Lambda = 0.4$ in this case is due to considering only a special subset of possible light paths, namely perfectly radial ones. The more realistic procedure of averaging over many lines of sight with various placements of the holes does not produce a significantly different Hubble diagram from that of just the background spacetime.

More recently, Szybka in [41] reached the same conclusion as VFW, but by a different method. While VFW used the perturbative weak field gravitational lensing formalism and only considered magnification effects, Szybka carried out the analysis by solving the set of fully relativistic equations and evaluated the effect of shear directly. In both cases, the overall effect of properly randomized inhomogeneities on $d_A$ is negligibly small, since the average density along a typical beam path is not significantly different from the background EdS value. We have also verified that this is the case by carrying out the analysis within our framework.

C. Multiple holes: a Szekeres Swiss-cheese model with $\Lambda$CDM background

For the general Szekeres Swiss-cheese case, we quantify the shifts in $\mu$ due to multiple anisotropic inhomogeneities along the line of sight for sources out to redshift 1.5. The left panel of Fig. 6 is an example showing $\Delta \mu$ as a function of redshift for a light beam traversing 38 randomly oriented Szekeres holes, where $\Lambda$CDM regions fill the spaces between them, and the source is at redshift 1.0. The determination of 38 holes for this case arises from using Szekeres structures 30 Mpc in size over the redshift range $0 \leq z \leq 1.0$ and spaced by 30 Mpc $\Lambda$CDM regions. The result is a typical realization of this scenario and reflects the size of the shift in distance modulus one would expect for
FIG. 7: Distribution of shift in distance modulus for 200 lines of sight to sources at $z = 1.0$ in a Szekeres Swiss-cheese universe. The inhomogeneous Szekeres holes are randomly oriented, and the impact parameters of the light beams through each hole are random. $\Lambda$CDM regions fill the spaces between the 38 holes. The mean of $\Delta \mu$ is $-0.0121$ mag, and the standard deviation $\sigma_{\Delta \mu}$ is 0.0039 mag, indicating a systematic bias due to inhomogeneities.

A random line of sight. The fact that $\Delta \mu$ is not zero at $z = 1.0$—a general result for this model—is significant, since there is then a systematic bias toward inferring smaller distances in such a Szekeres Swiss-cheese universe compared to pure $\Lambda$CDM. In other words, the effect of randomizing the holes in the Szekeres case does not entirely wash out the effect on $d_A$, in contrast to the results found in [36, 41] for five large LT holes.

In the right panel of Fig. 6, we plot the density encountered by the beam represented at left as a function of redshift. The Szekeres structures homogenize toward the past, and, correspondingly, the maximum possible overdensities able to be seen by the beam decrease with increasing $z$. Likewise, the density contrast of the void interior increases toward zero. In holes where the beam has passed through the central underdense region, the effect on $\Delta \mu$ is determined by whether the overdensity encountered at entrance is significantly greater or less than that at exit. As was seen in the single hole case, $\Delta \mu$ for that hole is positive if the beam first encounters the larger overdensity and negative if it first encounters the smaller overdensity. Other paths, such as those that merely graze the edge of a hole, do not have a substantial impact in shifting $\mu$.

The effect on light propagating in a Szekeres Swiss-cheese universe is then to produce a slight increase or decrease in $\Delta \mu$ for each hole crossed. The cumulative effect of many holes, however, is one that tends toward negative $\Delta \mu$, corresponding to a net dimming of sources. Figure 7 shows the distribution of $\Delta \mu$ resulting from 200 realizations of light beams emitted by sources at redshift 1.0. Each of the 200 lines of sight again contains 38 randomly oriented Szekeres holes with $\Lambda$CDM regions filling the spaces between them. The mean $\langle \Delta \mu \rangle$ of the distribution is $-0.0121$ mag, the standard deviation is 0.00393 mag, and we have checked that the statistics are not significantly different for 1000 runs. (We use 200 runs in the remainder of the work.) Statistics for sources in $0 < z \leq 1.5$ are presented in Table I where we see that the magnitude of $\langle \Delta \mu \rangle$ and $\sigma_{\Delta \mu}$ both increase with redshift. This is consistent with expectations, since the higher the redshift of the source, the more time the light spends within the holes where effects that shift the distance modulus occur.

We next explore the effect of strengthening the anisotropy of the holes by increasing the maximum density peak in the overdense shells. Since the holes are mass-compensated, sharpening the density peak comes at the expense of reducing the density elsewhere in the shell. We achieve this straightforwardly by decreasing the (constant) value of the $S$ metric function, which directly controls the strength of the anisotropy. Table II shows statistics for various cases of maximum $\rho/\rho_{bg}$ in the holes today and with sources at $z = 1.0$. When crossing each hole, the light now has the possibility of traveling through more extreme density environments, which in principle affects the size of $\langle \Delta \mu \rangle$ and its dispersion. However, it is clear that neither $\langle \Delta \mu \rangle$ nor $\sigma_{\Delta \mu}$ is very sensitive to augmenting the density in this manner.

Finally, as real voids in the universe vary in size, we also examine the impact of introducing two additional sizes of Szekeres holes (so we use 30, 60, and 90 Mpc radii) and cycle among the three along single lines of sight. The two new larger structure sizes have density distributions nearly identical to that of the smaller size we have been using, except that the overdense shell has merely been pushed to larger $r$. We keep the density peak in the shells the same, namely $\rho/\rho_{bg,0} \approx 12$ with $S(r) = 0.4$, and the underdense central regions still have $\delta = -0.8$. The results for sources at various redshifts are presented in Table III. We see that for each redshift except $z_{src} = 0.25$, $\langle \Delta \mu \rangle$ and $\sigma_{\Delta \mu}$ are larger in magnitude than their corresponding values in Table I. For example, for sources at $z = 1.5$, $\langle \Delta \mu \rangle$ and $\sigma_{\Delta \mu}$
TABLE I: Behavior of $\langle \Delta \mu \rangle$ and $\sigma_{\Delta \mu}$ as a function of redshift for 200 lines of sight with single hole size. For sources at higher redshift, the number of holes the light must pass through is greater than for lower redshift sources, since the holes have uniform comoving size and spacing. The average shift in distance modulus, as well as its spread, increase (in magnitude) with increasing redshift, which we expect due to the increased time the light spends within holes.

| $z_{\text{src}}$ | num. holes | $\langle \Delta \mu \rangle$ (mag) | $\sigma_{\Delta \mu}$ (mag) |
|-----------------|------------|-------------------------------|--------------------------|
| 0.25            | 12         | -0.0022                       | 0.0015                   |
| 0.50            | 22         | -0.0047                       | 0.0019                   |
| 0.75            | 31         | -0.0083                       | 0.0030                   |
| 1.00            | 38         | -0.0121                       | 0.0039                   |
| 1.25            | 45         | -0.0182                       | 0.0047                   |
| 1.50            | 50         | -0.0251                       | 0.0056                   |

Varying max. density peak

| $S$ | max. $p/p_{\text{max}}$ | $\langle \Delta \mu \rangle$ (mag) | $\sigma_{\Delta \mu}$ (mag) |
|-----|-------------------------|-------------------------------|--------------------------|
| 0.400 | 12                     | -0.0121                       | 0.0039                   |
| 0.385 | 19                     | -0.0117                       | 0.0040                   |
| 0.374 | 40                     | -0.0124                       | 0.0041                   |
| 0.368 | 112                    | -0.0123                       | 0.0039                   |
| 0.365 | 968                    | -0.0124                       | 0.0038                   |

TABLE II: Statistics on $\Delta \mu$ for 200 lines of sight with increasing maximum density in the compensating shell and sources at $z = 1.0$. Despite the possibility of encountering larger peak densities in each hole, the mean and standard deviation of $\Delta \mu$ do not change significantly as the density distribution is modified this way.
TABLE III: Statistics on $\Delta \mu$ (200 lines of sight) for sources at the same redshifts as in Table I but where hole sizes cycle between three different values of comoving radius: 30, 60, and 90 Mpc. The magnitude of both $\langle \Delta \mu \rangle$ and $\sigma_{\Delta \mu}$ are larger than for their corresponding values in Table I except for $z_{\text{src}} = 0.25$. For the same source redshift, light in this case travels through fewer holes but spends more time within the holes, which is the primary source of the difference.

| $z_{\text{src}}$ | num. holes | $\langle \Delta \mu \rangle$ (mag) | $\sigma_{\Delta \mu}$ (mag) |
|-------------------|------------|----------------------------------|----------------------------|
| 0.25              | 7          | -0.0022                          | 0.0014                     |
| 0.50              | 13         | -0.0051                          | 0.0026                     |
| 0.75              | 18         | -0.0106                          | 0.0037                     |
| 1.00              | 22         | -0.0197                          | 0.0054                     |
| 1.25              | 25         | -0.0246                          | 0.0077                     |
| 1.50              | 29         | -0.0337                          | 0.0093                     |

V. SUMMARY AND CONCLUSIONS

Understanding biases in observations due to nonlinear inhomogeneous structures is crucial in an age of precise and accurate cosmology. The universe we observe is populated on all scales by voids and overdense structures that affect the characteristics of light beams as they propagate to us. In this paper, we have explored bias in distance determinations by studying light propagation in a Swiss-cheese universe with $\Lambda$CDM as background and holes modeled by the Szekeres metric. By construction, the entire spacetime is an exact solution of Einstein’s equations, and its expansion history is identical to that of a pure $\Lambda$CDM universe. The Szekeres holes of radial size 30–90 Mpc are mass-compensated, where matter from the interior has been redistributed anisotropically around the overdense outer shells, and the underdense inner regions have $\delta = -0.8$ today to mimic voids in the real universe.

We have found that the shift in distance modulus $\Delta \mu$ for light traversing a single Szekeres structure can be positive or negative, depending on the particular path followed by the beam. For example, beams encountering the density maximum of the overdense shell at entrance and the minimum at exit resulted in $\Delta \mu > 0$, while beams crossing the structure in the opposite direction had $\Delta \mu < 0$. For fixed redshifts of the source and the hole, the magnitude of $\Delta \mu$ in the latter scenario was larger than for the former, even though the beam paths were nearly identical. We traced the physical interpretation of this asymmetric effect to the evolution of the anisotropic structures, producing a difference in the magnification for the two cases, and illustrating the nonlinear effects on light propagation that exact relativistic models can capture. We also verified that the effect on redshift for such inhomogeneities is negligible.

Contrary to studies with spherically symmetric LT holes, the shift in $\Delta \mu$ did not vanish when many randomly oriented holes were placed along the line of sight to a source. Instead, we have found a systematic residual bias...
toward negative $\Delta \mu$ that increases with redshift, as light can spend more time within holes the farther away the source is. For example, sources at $z = 1.5$ with 50 intervening single-sized holes experienced an average shift in distance modulus of $\langle \Delta \mu \rangle = -0.0251$ mag with standard deviation $\sigma_{\Delta \mu} = 0.0056$ mag, which implies a systematic dimming effect of the inhomogeneities. We found that these results were not sensitive to augmenting the density peak of the holes, where we increased it to nearly 100 times its original value, thereby allowing the light beams to encounter more extreme density gradients. Introducing two additional larger hole sizes of 60 and 90 Mpc radius and cycling among the three resulted in fewer holes being crossed to a given source, but more time overall being spent within holes. This increased the magnitude of the shift and the spread of the distance modulus distributions compared to the single small hole size; for example, $\langle \Delta \mu \rangle = -0.0337$ mag and $\sigma_{\Delta \mu} = 0.0093$ mag for $z_{\text{src}} = 1.5$ in this case. A systematic shift in $\Delta \mu$ of this size is associated with a fractional error on the source luminosity distance of 1.56%.

The models we have explored in this work induce corrections to $\mu$ on the order of 0.01 mag, while the uncertainties in supernova observations are of order 0.1 mag. Binning the supernova distance modulus data, which scales uncertainties by $N^{-1/2}$ with the number $N$ of objects per bin, could turn the corrections we have found into an important systematic effect. Indeed, the number of type Ia supernovae measured in experiments like the Dark Energy Survey (DES), Euclid, and the Large Synoptic Survey Telescope (LSST) will be large, and a typical uncertainty of 0.12 mag, for example, would become 0.012 assuming 100 supernovae per redshift bin. This is less than $\langle \Delta \mu \rangle$ we found in Table III for sources with $z_{\text{src}} \geq 1$.

While we expect our void profiles and Swiss-cheese model construction to reflect the properties important for realistic light beams, we have not calibrated our line of sight density fluctuations against specific simulations or survey data. However, assuming the results of such a calibrated model to be comparable to those obtained in this work, a comparison with expected statistical errors of upcoming surveys is instructive. As an example, we plot in Fig. 8 $\langle \Delta \mu \rangle$ as a function of $z$ for our three hole size Swiss-cheese model (Table III) and the predicted rms/$\sqrt{N}$ Hubble diagram scatter for the DES 10-field simulated survey [79]. For redshifts larger than about 0.6, $\langle \Delta \mu \rangle$ becomes comparable to the predicted scatter amplitude for the survey, indicating the potential importance of a systematic bias due to inhomogeneities like those we have studied here.

Finally, a further consequence is that such biases will in principle propagate to the inferred constraints on cosmological parameters entering distance and expansion measurements. The analysis of the effect on the corresponding cosmological parameters, the associated interpretations, and comparison to simulations will be presented in a follow-up paper.

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