Singularities and isotropy in a Brans-Dicke Bianchi-type VII universe

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Abstract.

We study the dynamical effects in the scale factors due to the scalar $\phi$-field at the early stages of a supposedly anisotropic Universe expansion in connection with the problem of the initial singularity in the scalar-tensor cosmology of Jordan-Brans-Dicke. This is done by considering the behaviour of the general analytical solutions for the homogeneous model of Bianchi type VII in the vacuum case. We conclude that the Bianchi-VII$_0$ model shows an isotropic expansion and that its only physical solution is equivalent to a Friedman-Robertson-Walker spacetime whose evolution begins in a singularity and ends in another; moreover, we obtain that the general Bianchi-VII$_h$ (with $h \neq 0$) models show strong curvature singularities that produces a complete collapse of the homogeneity surfaces to a 2-plane, to a string-like one-dimensional manifold, or to a single point.

Key words: Singularities, Scalar tensor theory, Bianchi VII cosmological model.

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§ 1 Introduction

In the Jordan-Brans-Dicke scalar-tensor theory of gravity (JBD)\[1–2\](Jordan
1959, Brans and Dicke 1961), a massless scalar $\phi$-field is introduced in addition to the
pseudo-Riemannian metric of the spacetime $R_4$ occurring in Einstein general
relativity (GR); supposedly, this long range field is generated by the whole of
matter in the Universe according to Mach’s principle \[3\](Dicke 1964); furthermore,
the inclusion of this field, allows Dirac’s idea about the secular variation of the
gravitational constant $G$ to occur in JBD cosmology. The scalar field in JBD
acts as an additional effective source of $R_4$ geometry, and it is coupled to the
tensorial degrees of freedom of the theory by a constant parameter $\omega$ \[4\](Ruban and
Finkelstein 1975). The value of $\omega$ can be estimated from astronomical observations
as $|\omega| \simeq 500$ to be in accord with current observations, but, as the GR limit of
the theory practically starts at such value, the theory does not seem to predict
anything too different from GR with enough observational evidence to support it.

However, despite what we have just said, there is a renewed interest in JBD
(and other scalar tensor theories), mainly because of superstring theories which
lead naturally to a dilaton theory of gravity where scalar fields are mandatory,
and also due to the emergence of extended inflation models and pre-big bang
cosmologies where scalar fields can provide solutions to some of the problems
of inflation \[5–6\](Steinhard 1993, Gasperini and Veneziano 1994). This comes
about since the JBD action functional already includes a string sector where the
dilaton field $\phi_D$ can be suitably related with the JBD scalar field $\phi \propto \exp(-\phi_D)$.
The important role of the $\phi$-field of JBD would especially occur at the strongly
relativistic stages of the Universe expansion in view of the important role that
the so-called scale factor duality plays in the pre-big bang scenario \[6,7\](Gasperini
and Veneziano 1994, Clancy et al 1998). Thence the importance of studying
Bianchi universes in JBD cosmology. In this work we analyse the Bianchi type
VII vacuum universes and show that these exhibit curvature singularities which
make their surfaces of homogeneity collapse to a plane, to a string or, even, to a
point, and that, despite the supposed anisotropic behaviour of a Bianchi type VII
universe, a Bianchi VII\(_0\) evolves isotropically and becomes essentially equivalent to a Friedman-Roberson-Walker universe starting from a singularity and ending in another.

§(2) Vacuum Bianchi-type VII equations

Let us write for the line element of the spacetime, using signature +2 and natural units \(c = G = 1\),

\[ ds^2 = -dt^2 + h_{ij}(t) \omega^i \omega^j, \tag{1} \]

where the \(h_{ij}\) is the metric on the surface of homogeneity, \(t\) is the synchronous time, \(\omega^i\) are the one-forms \cite{Ryan_y_Shepley_1975} expressing the properties of the 3-space; the specific one-forms appropriate for the homogeneous but anisotropic Bianchi VII model are

\[
\begin{align*}
\omega^1 &= a_1 \left( (\eta - k\nu)dy - \nu dz \right), \\
\omega^2 &= a_2 (\nu dy - (\eta + k\nu)dz), \\
\omega^3 &= a_3 dx, \\
\omega^4 &= dt,
\end{align*} \tag{2}
\]

where \(\eta = \exp(-kx)\cos(Mx)\), \(\nu = (-M^{-1})\exp(-kx)\sin(Mx)\), \(k = h/2\) and \(M = (1 - k^2)^{1/2}\); the parameter \(h\) distinguishes the case of the special model VII\(_0\) \((h = 0)\) from the generic model VII\(_h\) \((h \neq 0)\).

If we now assume a vacuum model and insert the line element (1), with the forms (2), into the JBD field equations in a vacuum, we get

\[
\frac{d^2}{dt^2} (\ln a_i) + \frac{d}{dt} (\ln a_i) \frac{d}{dt} (\ln a_1 a_2 a_3) + \frac{d}{dt} (\ln a_i) \left( \frac{\dot{\phi}}{\phi} \right) + A_i a_1^{-2} + \mathcal{E}_i \beta_2 + F_i \beta_3 = 0, \tag{3}
\]

where one of the coordinates has been chosen as the synchronous time \(t\). We additionally have what we have called the constriction equation, coming from off-diagonal terms in the JBD field equations,
\[
\frac{d}{dt}(\ln a_1) + \frac{d}{dt}(\ln a_2) + \frac{d}{dt}(\ln a_3) + \frac{d}{dt}(\ln a_2) \frac{d}{dt}(\ln a_3) \\
+ \frac{d}{dt}(\ln a_1 a_2 a_3) \left(\frac{\dot{\phi}}{\phi} \right) - \frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi} \right)^2 + A_4 a_1^{-2} + \mathcal{E}_2 + \mathcal{F}_3 = 0,
\]

this is an equation of Raychaudhuri-type [9]; finally, the scalar field comply with
\[
\frac{d}{dt} \left( [a_1 a_2 a_3] \frac{d\phi}{dt} \right) = 0,
\]
where \( \beta_i \equiv (a_i/(2a_j a_k))^2 \) and the indexes \( i, j, k \) are to be taken in cyclic order of 1,2,3. Equations (3), (4) and (5) are written in the standard form we introduced previously for solving the Bianchi-models [9](Chauvet et al 1992); the specific values for the constants appearing in them are: \( A_1 = 4M^2 - (5/2) \), \( A_2 = A_1 - 2 \), \( A_3 = 0 \), \( A_4 = A_1 - 1 \), \( \mathcal{E}_1 = \mathcal{E}_3 = \mathcal{F}_1 = \mathcal{F}_2 = -2 \), \( \mathcal{E}_2 = \mathcal{F}_3 = 2 \), \( \mathcal{E}_4 = \mathcal{F}_4 = -1 \), and \( M^2 = 1 - (h^2/4) \); compare with Table 1 in [9](Chauvet et al 1992). Notice that according to these relationships, \( h \) must be restricted to be \( |h| \leq 2 \) (this can also be seen from the differential forms in (2)). Also, the equations for the Bianchi VII\(_0\) model can be particularized from the general ones, just by taking \( h = 0 \).

Most Bianchi models, as consequence of the off-diagonal contributions to the field equations, lead to additional relationships between their scale factors; in this case, we have the following two additional relationships
\[
\frac{d}{dt}(\ln a_1) - \frac{d}{dt}(\ln a_2) = 0,
\]
\[
\frac{h a_2}{2a_1^2 a_3} = 0.
\]
Equation (6) always provides an additional relationship between the two scale factors, \( a_1 \) and \( a_2 \), not mattering what the value of the \( h \)-parameter; but, on the other hand, equation (7) becomes just a trivial identity when \( h = 0 \), not restricting in any way the values of the scale factors. The case \( h \neq 0 \) and some of its consequences are analysed in subsection 3.2.
2.1 Scaling the scalar field

From equation (5), we easily obtain

\[ a_1 a_2 a_3 \frac{d\phi}{dt} = \phi_0, \]  

(8)

where \( \phi_0 \) is an integration constant; thus we can introduce the scaled \( \phi \)-field, \( \Phi \), as \( \Phi \equiv \phi/\phi_0 \), which coincides with the so-called intrinsic time [10](Carretero-González et al 1994). From equation (8), we easily get

\[ \partial_t = (a_1 a_2 a_3)^{-1} \partial_{\Phi}. \]  

(9)

This shows that \( \Phi \) is a monotonic function of the synchronous time \( t \); \( \Phi \) can hence be used also as a time reparametrization useful for solving equations (3). In fact, \( \Phi \) has been found useful for analysing the Bianchi vacuum models in several situations [9–12](Chauvet et al 1991, 1992, Carretero-González et al 1994, Núñez-Yépez 1995). For the sake of convenience, let us introduce the notation \( (\cdot)' \equiv \partial_{\Phi} \) and, defining the Hubble expansion rates as \( H_i \equiv (\ln a_i)' \), the reparametrized field equations become

\[ H_i' + \frac{H_i}{\Phi} + \mathcal{J}_i a_i^4 + \mathcal{K}_i a_i^3 + \mathcal{N}_i a_i^2 a_i^2 = 0, \quad i = 1, 2, 3 \]  

(10)

and the constriction equation becomes

\[ H_1 H_2 + H_1 H_3 + H_2 H_3 + \frac{(\ln a_1 a_2 a_3)'}{\Phi} - \frac{\omega}{2 \Phi^2} + \mathcal{J}_4 a_4^4 + \mathcal{K}_4 a_4^3 + \mathcal{N}_4 a_4^2 a_4^2 = 0, \]  

(11)

the specific values for the constants appearing in equations (10) and (11) are combinations of the constants previously used: \( \mathcal{J}_1 = \mathcal{J}_3 = \mathcal{K}_1 = \mathcal{K}_2 = -1/2, \mathcal{J}_2 = \mathcal{K}_3 = 1/2, \mathcal{J}_4 = \mathcal{K}_4 = -1/4, \mathcal{N}_1 = 4M^2 - (5/2), \mathcal{N}_2 = \mathcal{N}_1 - 2, \mathcal{N}_3 = 0, \mathcal{N}_4 = \mathcal{N}_1 - 1 \). The specific form chosen to write the equations and the parameters just emphasizes the relationship with our previous work [9,11,12] (Chauvet et al 1991, 1992, Núñez-Yépez 1995) on exact solutions for vacuum Bianchi models in JBD.
§(3) Solutions for the vacuum Bianchi type VII universes

For solving the equations of the Bianchi-type VII model, we found convenient to address separately the specific VII$_0$ and the generic VII$_h$ Bianchi models. The exact solutions of the next subsections are obtained using essentially the method in Chauvet et al [9](1992, Appendix).

3.1 Bianchi type VII$_0$

In the reparametrized formulation of the equations for the anisotropic homogeneous metric of Bianchi type VII$_0$, solutions can be obtained for the case of a Bianchi-type VII$_0$; the specific solution depends on the sign of the quantity $\Delta \equiv -4(B + 1/4)$ where

$$B = \Phi^2 H_1^2 - \frac{\Phi}{2} H_1 - \frac{\Phi^2}{2} H_1',$$

(12)

is a constant, i.e. is a first integral of the reescaled system (3) and (4) that depends on the Hubble expansion rates; in this way, we can find that out of the possible solutions of equations (10), the only physically plausible is the one corresponding to the case $\Delta < 0$ (the $\Delta > 0$ or $\Delta = 0$ cases can be seen to led to negative or even complex scale factors); for some details see [12](Chauvet et al 1992). The only physical solution can be explicitly written as

$$a_1(\Phi) = \left(\frac{4B + 1}{c_0^4}\right)^{1/4} \left\{ \Phi \cosh \left[ -\sqrt{\frac{1}{4}(B + 1/4) \ln(f\Phi^2)} \right] \right\}^{-1/2},$$

(13)

where $c_0$ and $f$ are positive integration constants. The other two scale factors can be easily obtained from $a_1(\Phi)$, as follows from (6) and (10), they are

$$a_2(\Phi) = c_0 a_1(\Phi),$$

(14)

$$a_3(\Phi) = 2^{-1/2} c_0 a_1(\Phi);$$

(15)

the three scale factors are proportional to each other. This means that the Bianchi-VII$_0$ model, despite what we could have anticipated, shows an isotropic expansion;
it also implies that the shear, vorticity and acceleration of the reference congruence all vanish. The vacuum Bianchi-VII_{0} JBD universe behaves as a Friedmann-Roberson-Walker (FRW) space-time—in a way, this is not totally surprising since Bianchi cosmologies correspond to the simplest deviations from a FRW environment. The local volume on the surface of homogeneity is then

\[ V = \frac{c_0^2 (a_1)^3}{\sqrt{2}}. \]  

(16)

The constriction equation implies the following relationship in our case

\[ 12 \left( B + \frac{1}{4} \right) - (3 + 2\omega) = 0; \]  

(17)

we notice that to have meaningful solutions the coupling parameter has to be restricted to \( \omega > -3/2 \). Figure 1 shows the evolution of the scale factors of the model as a function of the intrinsic time (or reescaled \( \phi \)-field) \( \Phi \).

Using equations (9) and (16) we can obtain the dependence of \( \Phi \) on \( t \), as follows

\[ t = \frac{(4B + 1)^{3/4}}{\sqrt{2}c_0} \int \{\Phi \cosh[\sqrt{B + 1/4} \ln(f\Phi^2)]\}^{-3/2} d\Phi, \]  

(18)

although we can obtain \( t \) as a function of \( \Phi \), as in (18), we cannot invert it to obtain explicitly \( \Phi \) as a function of \( t \). Nevertheless, figure 2, exhibits the dependence of the field \( \Phi \) on \( t \), showing the enormous change that occurs in \( \Phi \) over a very small span of \( t \) values. This also shows that, asymptotically, \( \Phi \) grows without bound and that as \( t \to 0 \), \( \Phi \) vanishes too; notice that these conclusions can be applied with no changes to the scalar field \( \phi \), excepting when \( \phi_0 = 0 \). We can now relate the behaviour of \( \Phi \) with that of \( a_1 \); since, as figure 1 shows, the universe begins with a singularity and then, as \( \phi \) grows, the universe expands as \( \sim t^\alpha \), where \( \alpha \equiv 3\sqrt{B + 1/4} - 3/2 \)—it thus does not really expand unless \( B > 0 \)—rapidly reaching a maximum volume \( V_{\text{max}} \) that can be easily calculated from (13); and then, as \( t \to \infty \), it shrinks rapidly as \( \sim t^{-\alpha/3+1/2} \) until it reaches again a singularity; see figures 1 and 3.
3.2 Bianchi type VII$_h$

From equations (6) and (7) several relations can be obtained for the scale factors irrespective of the value of $h \neq 0$; from (6) we get (14) (the same equation than in the case $h = 0$). From equation (7), we might get $a_2 = 0$, implying that $a_1 = 0$ and $a_3$ can take any value; then the model collapses into essentially a spatial one-dimensional manifold: i.e. a string. Other possible options allowed by equation (7), imply that $a_2 a_3 = \infty$, thus the spatial 3-surfaces of homogeneity are seen to collapse into 2-surfaces. The important point here is that for any choice of values for the scale factors in equation (7), the model is found always to spatially collapse. Any Bianchi-type VII$_h$ ($h \neq 0$) universe is thus highly singular.

As we have been able to get the scale factors for the specific cases addressed in this paper, we have obtained exact vacuum solutions for the Bianchi VII models. The important conclusion is that all solutions found for the generic Bianchi VII model with $h = 0$, and $h \neq 0$—with the restriction $|h| \leq 2$—describe collapsing singular universes. The behaviour of the scale factors is exhibited in figure 1.

§(4) Curvature singularities in the Bianchi-type VII universes

In this section we study the curvature singularities present on the Bianchi-VII universes, though some very specific singularities were discussed in section 3.2.

We say that a universe is singular if the value of the Ricci scalar $R = g^{ab}R_{ab}$ along a congruence of the geodesics $|R| \to \infty$ whereas the associated affine parameter tends to a finite value. If this happens, then we say that there exists a curvature singularity and thus that the universe is also singular. Notice also that, due to the choice we made for the $\omega$ parameter, here the singularity occurs when $R \to -\infty$ (figure 3) rather than the other way round. The problem is basically how to choose an appropriate geodesic congruence. For our Bianchi VII models, we have chosen as the proper congruences the world lines of test observers (time-like geodesics) whose affine parameter is the synchronous time $t$, this is the simplest choice we could found. Thence, the scalar curvature is found to be
\[ R = 2 \left( \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{\phi}}{\phi} + \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 \right). \]  \hspace{1cm} (19)

Using only equations (5), (10) and (11) we can rewrite \( R \) for the vacuum Bianchi-VII model in terms of the coupling parameter \( \omega \) and the scalar field \( \phi \), as follows

\[ R = -\omega \left( \frac{\dot{\phi}}{\phi} \right)^2 = -\omega \left( \frac{1}{a_1 a_2 a_3 \Phi} \right)^2; \]  \hspace{1cm} (20)

it is important to notice that expressions (19) and (20) do not depend on the \( h \)-value, they are valid for all the Bianchi-type VII models we are discussing. We have expressed the scalar curvature in two different ways, equations (19) or (20), both are important because they exhibit the explicit dependence of \( R \) on the scale factors and \( \phi \) and its time derivatives, or on the coupling parameter \( \omega \) and the scale factors (20) and, besides, they have both a certain pleasant simplicity. For \( R \to \infty \) in (20), all that is needed is that at least one of the scale factors \((a_1, a_2, \text{ or } a_3)\) or the reescaled scalar field \( \Phi \), or just the scalar field \( \phi \), vanish at a finite value of the synchronous time \( t \). Notice also that we can regard the evolution of the Bianchi-type VII universe —equations (3) and (4)— as driven by the curvature \( R \), this is especially true near the singularity.

Notice that the plot of \( t \) against \( \Phi \) in figure 2, which does not depend on \( h \) in any way, shows the rather small range of \( \Phi \)-values in which the expansion of the model universe \( \text{VII}_0 \) occurs, as we can see comparing with figure 1; this also corresponds to the region free of singularities in the model \((\text{VII}_0)\), as can be seen on comparing with figure 3. On comparing figures 1 and 2, the role of the curvature scalar in governing the expansion can be qualitatively described as follows: a strong curvature prevents expansion; it is only when curvature is small that the full extent of expansion is reached but, as soon as the curvature is large in magnitude again, contraction sets in. This qualitative behaviour is in accord with current ideas [7, 13].
In the general \( \text{VII}_h \) JBD model, the universe is always singular, as can be easily seen from equation (7) and the discussion in section 3.2; in this sense, we say that the model \( \text{VII}_h \) is completely singular.

§(5) Concluding remarks

The supposed homogeneous and anisotropic Bianchi VII model in fact shows an isotropic expansion in the case in which \( h = 0 \). On the other hand, this article shows that the dynamics of the early stages of the expansion in the specific \( \text{VII}_0 \) JBD model depends on just one of the scale factors (that, here, we choose as \( a_1 \)). We have also obtained the dependence of the three scale factors \( a_i \) on the rescaled field \( \Phi \). As we have concluded that the three scale factors are proportional to each other, the expansion is necessarily isotropic and not anisotropic, as it is usually assumed to be in this model. In fact, we have shown that a Bianchi-VII\(_0\) JBD vacuum universe is basically equivalent to a FRW-spacetime. But the important point is that this shows that, even with supposedly anisotropic models, the inclusion of a scalar field can sometimes isotropize the behaviour, which thus offers the possibility of coordinating it with the observed isotropic properties of our Universe. Moreover, this feature does not seem to depend on \( \omega \) in any way.

For the case of the \( \text{VII}_h \) vacuum model, we can say that, not mattering the choice made in equation (7), the universe always collapses. According to the discussion in section 3.2, the universe may spatially collapse into a plane or into a one dimensional object or, even, into a singular point; this model universe always collapses to a permanent singularity. Such behaviour may make this model of little interest from a factual point of view unless one is interested in singular behaviour. In this respect, the dependence of the universe dynamics, especially near the singularity, on the curvature scalar is worth pinpointing.

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Figure Captions

Figure 1
The behaviour of the three scale factors $a_1$, $a_2$ and $a_3$ is shown in the Bianchi-type VII$_0$ universe. Notice that all scale factors are proportional to each other. This figure basically show graphs of equations (13), (14) and (15). The values used for the parameters are $B = 83.33$, $c_0 = 2$, $f = 1$.

Figure 2
The behaviour of the reescaled scalar field $\Phi$ is shown as a function of $t$. This graph can be also interpreted as the behaviour of the intrinsic time versus the synchronous time $t$, as follows from equation (18). Notice how the scalar field goes from a certain finite value $\phi_0$ at $t \sim 0$ to an infinite asymptotic value in a very small $t$-lapse.

Figure 3
The scalar curvature $R$ in the Bianchi-type VII$_0$ universe is shown in the range $-\infty < R < \infty$ versus $\Phi$ in the range $0 < \Phi < \infty$. What we really graph here is $\arctan(R)$ against $\arctan(\Phi)$. Small $\Phi$-values correspond to small $t$-values but large values of $\Phi$ correspond to $t \gtrsim 3.5$, as can be seen in figure 2.
