DECAY OF SOLUTIONS TO ONE DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATIONS WITH WHITE NOISE DISPERSION

SERGE DUMONT*

IMAG UMR 5149 CNRS, Université de Nîmes
Place Gabriel Péri, 30000 Nîmes, France

OLIVIER GOUBET

Laboratoire Paul Painlevé CNRS UMR 8524
et équipe projet INRIA PARADYSE, Université de Lille
59 655 Villeneuve d’Ascq cedex, France

YOUCEF MAMMERI

LAMFA UMR 7352 CNRS, Université de Picardie Jules Verne
33, rue Saint-Leu, 80039 Amiens, France

Abstract. In this article, the asymptotic behavior of the solution to the following one dimensional Schrödinger equations with white noise dispersion

\[ idu + u_{xx} \circ dW + |u|^{p-1}udt = 0 \]

is studied. Here the equation is written in the Stratonovich formulation, and \( W(t) \) is a standard real valued Brownian motion. After establishing the global well-posedness, theoretical proof and numerical investigations are provided showing that, for a deterministic small enough initial data in \( L^1_x \cap H^1_x \), the expectation of the \( L^\infty_x \) norm of the solutions decay to zero at \( O(t^{-\frac{1}{4}}) \) as \( t \) goes to \( +\infty \), as soon as \( p > 7 \).

1. Introduction. Stochasticity has been recently introduced in the dispersion for a particular case of nonlinear Schrödinger equations [7, 8]. The linear term of the equation oscillates with respect to the variations of a Brownian motion, and this stochastic equation reads as

\[ idu + u_{xx} \circ dW + |u|^{p-1}udt = 0. \]

In [7, 8] the authors studied nonlinear Schrödinger equation (NLS) with white noise modulation. They showed that it describes the homogenization of the deterministic NLS equations with time dependent dispersion satisfying some ergodicity properties. The initial value problem with initial data a.e. in \( L^2(\mathbb{R}) \) for \( p \leq 5 \) was addressed in [7, 8] using ad hoc Strichartz-like estimates that are valid for stochastic modulations. Another insights into stochastic Strichartz estimates appeared in [10]. Such
investigation furthers the study where the dispersion is driven by a deterministic oscillating function $\gamma(t)$ \[1\]

$$iu_t + \gamma(t)u_{xx} + |u|^{p-1}u = 0.$$  

For the sake of completeness, we refer also to \[6\] where the authors address the initial value problem through the rough path theory.

In this article we are interested in the asymptotic behavior of solution for large powers $p$ in the nonlinearity. We first establish that there exists a local in time solution in the energy space $H^1(\mathbb{R})$ for \(1\). Since we are in the supercritical case, then the blow up of solution may occur. Besides, when considering sufficiently small initial data, the solution remains small and the dynamics works as if the nonlinearity is negligible; we are here in the so-called asymptotically linear regime \[12\]. We prove that under suitable conditions on $p$ and on the smallness of the initial data then the solution decays towards 0 with the same decay rate than the solution of the linear stochastic equation. It is worth to point out that the decay rate of solutions is twice as slow as the decay rate of solutions to deterministic nonlinear Schrödinger equations

$$u_t + iu_{xx} + i|u|^pu,$$

for large $p$ and small initial data (see \[4\] and the references therein). We follow in this article a method introduced in \[5\] for generalized Benjamin-Bona-Mahony equations. We finish this article by some numerical computations that illustrate our theoretical results. These computations are performed using a Strang splitting scheme in time combined with pseudo-spectral discretization in space.

We complete this introduction with some notations. In general, the function $f(s)$ is a random function in the filtration $\mathcal{F}_s$. The numbers $s,t$ are assumed to satisfy $s < t$. The Wiener process $W(t)$ is such that $W(t) - W(s) \sim \mathcal{N}(0,t-s)$, where $\mathcal{N}(0,t-s)$ is a normally distributed random numbers with mean 0 and variance $t-s$, and that $W(t) - W(s)$ is independent of the past $\mathcal{F}_s$. We denote by $H^m_x(\mathbb{R})$ or $H^m_{x,W}$ the standard Hilbert space with respect to the $x$-variable. The notation $L^p_x$ will be used for standard Lebesgue spaces with respect to $x$. The Banach space $L^1_x \cap H^m_{x,W}$ will be endowed with the sum of the $L^1_x$ and $H^m_{x,W}$ norms. The scalar product on $L^2_x$ will be denoted as $(u,v) = \text{Re} \int uv$. The notation $\hat{u}$ denotes the Fourier transform of $u$ in space. We set $A$ for the self-adjoint operator $-\partial_{xx}$ which maps $H^m_x$ into $H^{m-2}_x$ for any $m$. For the sake of simplicity, the random variable $\omega$ and the space variable $x$ may be omitted throughout this article. The constant $c$ is a numerical positive constant that may vary from one line to another and we set $\langle x \rangle = \sqrt{1+x^2}$.

2. Preliminaries and main results.

2.1. White noise modulated dispersion. Let $W(t)$ be a standard real valued Wiener process associated with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$.

We first consider the linear Schrödinger equation. Let $u_s$ be a random $\mathcal{F}_s$-variable which takes values in some Hilbert space such as $H^2_x$. We seek for $s < t$ and $x \in \mathbb{R}$ a process $u(t)$ that is a solution of the Stochastic Partial Differential Equation (SPDE) in $H^2_x(\mathbb{R})$

$$du + iA^2udt = 0,$$

$$u(s) = u_s.$$ (2)
According to [7] the process $u(t)$ takes values in a functional space as $H^1_x$, and equation (2) means that, for any test function $\varphi$ say in the Schwartz class, the scalar process $X = (u, \varphi)$ is solution to the SDE
\begin{align*}
d(u, \varphi) + (iu, A\varphi) dW + \frac{1}{2} (u, A^2 \varphi) dt &= 0, \quad X(s) = (u_s, \varphi). 
\end{align*}
(3)

We refer to this solution as a weak solution to the SPDE, weak being understood in the PDE sense. This SPDE, written in its Ito formulation, reads in its formally equivalent Stratonovich formulation as
\begin{align*}
du + iAu \circ dW &= 0, 
\quad u(s) = u_s.
\end{align*}

It is well known (see [14]) that a solution of (2) is defined through its Fourier transform in space as follows
\begin{align*}
\hat{u}(t, \xi) &= e^{-i\xi^2 (W(t) - W(s))} \hat{u}_s(\xi).
\end{align*}
(4)

In comparison, the solution of usual linear Schrödinger equation with deterministic dispersion is
\begin{align*}
\hat{u}(t, \xi) &= e^{-i\xi^2 (t-s)} \hat{u}_s(\xi).
\end{align*}

In this work, we are dealing with the nonlinear stochastic equation written in Stratonovich’s formulation as
\begin{align*}
du + iAu \circ dW - i |u|^{p-1} u dt &= 0.
\end{align*}
(5)

Once again this is a short-hand notation for an equation in Ito’s formulation that reads
\begin{align*}
du + iAu dW + \frac{1}{2} A^2 u dt - i |u|^{p-1} u dt &= 0.
\end{align*}
(6)

2.2. Statements of the main results. We first prove by standard arguments the well posedness of the initial value problem that is valid for any $p \geq 1$ for data a.e. in $H^1_x$.

**Theorem 2.1.** Let $u_0$ be $F_0$ measurable and in $L^2(\Omega; H^1_x)$. Then there exists a unique weak pathwise solution $u(t)$ of the equation (6), adapted to the filtration $F_t$, with paths almost surely in $C(0, \tau_{\infty}; H^1_x)$. Here $\tau_{\infty}(\omega)$ is the blow up time in $H^1(\mathbb{R})$ of the trajectory $t \mapsto u(t, \omega)$, which is also a stopping time. Moreover a.s. the $L^2_x$ norm of the solution is conserved, that is $\|u(t)\|_{L^2_x} = |u_0|_{L^2_x}$ for all $t < \tau_{\infty}$.

Our main result concerns with the decay rate of solutions for large $p$ and initial data in a neighborhood of the rest state 0.

**Theorem 2.2.** Assume $p > 7$. There exists $\varepsilon_0 > 0$ such that for the deterministic initial data $u_0$ satisfying
\begin{align*}
\|u_0\|_{H^1_x \cap H^1_{T_0}} &\leq \varepsilon_0,
\end{align*}

the solution is global in time, i.e. $\tau_{\infty}(\omega) = \infty$ almost everywhere, and verifies the average decay estimate for all $t > 0$,
\begin{align*}
\mathbb{E}(\|u(t)\|_{L^\infty_x}) &\leq C(\varepsilon_0)(t)^{-\frac{1}{4}}.
\end{align*}
3. The initial value problem. Denote the solution of equation (2) by $u(t) = S(t,s)u(s)$ given by (4). Then almost surely in $\omega$ the linear operator $S(t,s)$ defines an isometry in any Sobolev space $H^m_x$.

Our goal is to seek a mild solution of the equation (5) with initial condition $u(x,0) = u_0$, namely a solution of the following Duhamel form

$$u(t) = S(t,0)u_0 + i \int_0^t S(t,s)(|u|^{p-1}u(s))ds.$$  

(7)

To prove Theorem 2.1, we follow a standard strategy for stochastic PDEs (see [7, 8, 14]). The nonlinearity is truncated to provide a globally Lipschitz mapping acting on the Banach algebra $H^1_x$, and we pass to the limit.

3.1. Solving a truncated equation. To begin with, we recall that for any $m \geq 1$, there exists a constant $c_{m,p} > 0$, that depends on $m$ and $p$, such that for any function $f$ in $H^m_x$

$$||f^{p+1}||_{H^m_x} \leq c_{m,p}||f||_{L^p_{\infty}}||f||_{H^m_x}.\quad(8)$$

A general proof of (8) for Lebesgue spaces appeared in [13]. In addition, due to the Sobolev embedding $H^1_x \hookrightarrow L^\infty_x$, it is straightforward to check that the map $u \rightarrow |u|^{p-1}u$ is a locally Lipschitz mapping in $H^1_x$.

We now introduce a smooth monotonous decreasing cutoff function $\theta$ that satisfies $\theta(s) = 1$ for $|s| \leq 1$ and $\theta = 0$ for $|s| \geq 2$ and we investigate the solution of the equation

$$u_R(t) = S(t,0)u_0 + i \int_0^t S(t,s)\theta_R(s)(|u_R(s)|^{p-1}u_R(s))ds, \quad(9)$$

where $\theta_R(s) = \theta(\frac{|u_R(s)|}{R})$. Note that $\theta_R$ depends on the $H^1_x$ norm of $u_R$.

**Proposition 3.1.** Let $u_0$ be $\mathcal{F}_0-$measurable and belong to $L^2(\Omega; H^1_x)$. For any $T > 0$, there exists a unique solution $u_R$ of the equation (9) in $L^2(\Omega; C([0,T]; H^1_x))$, and $u_R(s)$ is adapted to the filtration $\mathcal{F}_s$.

**Proof.** We use here the following classical Banach fixed point theorem.

We now omit the subscript $R$ to write $u_R = u$ for the sake of simplicity. Consider $T > 0$ fixed, and define the mapping, for $t < T$,

$$\mathcal{T}(u) = S(t,0)u_0 + i \int_0^t S(t,s)F_R(u(s))ds, \quad(10)$$

where $F_R(u) = \theta_R(|u(s)|^{p-1}u(s))$. Consider the sequences $u^k = \mathcal{T}(u^{k-1})$, $v^k = \mathcal{T}(v^{k-1})$ starting from $u^0, v^0 \in L^2(\Omega; C(0,T; H^1_x))$.

Introduce the nondecreasing function

$$\varphi_k(t) = \mathbb{E}(\sup_{s \leq t}||u^k(s) - v^k(s)||_{H^1_x}^2).$$

Observe that $\varphi_k(t)$ is the square of the norm of $u^k - v^k$ in $L^2(\Omega; C(0,T; H^1_x))$. It follows from (10), since $S(t,s)$ is an isometry, that for $\tau \leq t$ and for any $u, v$

$$||\mathcal{T}(u)(\tau) - \mathcal{T}(v)(\tau)||_{H^1_x}^2 \leq t \int_0^t ||F_R(u) - F_R(v)||_{H^1_x}^2 ds. \quad(11)$$
Since $F_R$ is globally Lipschitzian in $H^1_2$ with Lipschitz constant $L_R = c_p R^{p-1}$ then
\[ \| T(u)(\tau) - T(v)(\tau) \|_{H^1_2}^2 \leq tL_R^2 \int_0^t \sup_{r \leq s} |u(r) - v(r)|^2_{H^1_2} ds. \quad (12) \]
Taking $u = u^k$, $v = v^k$, the supremum in $\tau$ and the expectation, we obtain
\[ \varphi_{k+1}(t) \leq tL_R^2 \int_0^t \varphi_k(s) ds. \quad (13) \]

We now prove by induction that for any $t \leq T$
\[ \varphi_k(t) \leq \frac{(tL_R)^{2k} k!}{(2k)!} \varphi_0(t). \quad (14) \]
Therefore for any given $T$, there exists $k$ large enough such that
\[ \frac{(tL_R)^{2k} k!}{(2k)!} < 1. \]

Then the map $T^k$ is a contraction and therefore there exists a unique fixed point for $T$ in $L^2(\Omega; C(0,T); H^1_2)$.

3.2. Conservation of the mass. We recall from [7, 14] a result that yields that the $L^2$ norm of a solution (the mass) is conserved along the flow.

**Proposition 3.2.** The mild solution given by Proposition 3.1 is the weak solution of the truncated SPDE
\[ du_R + iAu_R + \frac{1}{2} A^2 u_R dt - i\theta_R(|u_R|^{p-1} u_R) dt = 0. \quad (15) \]
Moreover almost surely in $\omega$, $\| u_R(t) \|_{L^2_2} = \| u_0 \|_{L^2_2}$.

**Proof.** We refer to the articles above for the proof of this Proposition. Let us sketch the proof of the second assertion (that has to be rigorously proven as in [7] by a limiting argument using a Galerkin approximation of the solution). Introduce the process $X = \| u_R(t) \|_{L^2_2}^2$. Recall that loosely speaking (15) reads
\[ du_R = \sigma(u_R)dW + \mu_R(u_R) dt, \]
where $\sigma(u) = -iAu$ is skew-symmetric and $\mu_R(u) = -\frac{1}{2} A^2 u + i\theta_R(|u|^{p-1} u)$. Invoking Ito’s chain rule we then have
\[ dX = 2(u_R, du_R) + \| \sigma(u_R) \|_{L^2_2}^2 dt. \quad (16) \]
Due to cancelations $2(u_R, du_R) = -(A^2 u_R, u_R) dt$ that balances $\| \sigma(u_R) \|_{L^2_2}^2 dt$ and $dX = 0$. \qed

3.3. Passing to the limit $R \to +\infty$. Consider an initial data $u_0$ in $L^2(\Omega; H^1_2)$. Then a.s. in $\omega$ we have that $\| u_0 \|_{H^1_2} < +\infty$. Pick such an $\omega$. Consider now the sequence $R \mapsto u_R$. The trajectory $t \mapsto u_R(t)$ is continuous. Introduce the stopping time
\[ \tau_R = \inf \{ t > 0; \| u_R(t) \|_{H^1_2} > R \}. \quad (17) \]
For $R > \| u_0 \|_{H^1_2}$, we have that $\tau_R > 0$ and that for $t \leq \tau_R$ the function $u_R$ is solution to the non-truncated original equation, then solution of the truncated equation at level $R + m$ for nonnegative $m$. By uniqueness we then have that $u_R = u_{R+m}$ on $[0, \tau_R)$. It follows that $\tau_R \leq \tau_{R+m}$. Therefore we deduce that when $R$ goes to $+\infty$ then $\tau_R \to \tau_\infty$ that is also a stopping time. For $t < \tau_\infty$ there exists $R$ such that
We now prove that $u_\infty$ is solution to the original equation on the random interval of time $(0, \tau_\infty)$. We introduce the new variables that are $v_R = u_R 1_{(t<\tau_\infty)}$ and $v_\infty = u_\infty 1_{(t<\tau_\infty)}$. We first prove

**Lemma 3.3.** For any $T > 0$, the sequence $v_R$ converges strongly towards $v_\infty$ in $L^2(\Omega \times (0, T) \times \mathbb{R})$.

**Proof.** Thanks to Proposition 3.2, the $L^2$ norm of solutions is conserved along the flow of solutions and we have a.s.

$$||u_\infty - u_R||_{L^2}^2 \leq 4||u_0||_{L^2}^2,$$

that is integrable on $\Omega \times [0, T]$. On the other hand

$$E(\int_0^T ||v_\infty - v_R||_{L^2}^2 ds) = E(\int_{\min(\tau_\infty, T)}^{\min(\tau_R, T)} ||u_\infty - u_R||_{L^2}^2 ds).$$

Gathering (18),(19) we pass to the limit thanks to Lebesgue dominated convergence Theorem since $(u_R - u_\infty)1_{[\min(\tau_\infty, T), \min(\tau_R, T)]}$ converges a.e. to 0. \hfill \Box

For the nonlinear term, we know that $f_R(u_R) 1_{(\tau_\infty < +\infty)}$ converges a.e. towards $f(u_\infty) 1_{(\tau_\infty < +\infty)}$. At this stage we know that for $t < \tau_\infty$ the function $u_R$ is a weak solution to the stochastic PDE that reads in Itô’s formulation, for any test function $\varphi$ in the Schwartz space

$$(u_R(t) - u_0, \varphi) = -i \int_0^t (u_R(s), A\varphi) dW_s - \frac{1}{2} \int_0^t (u_R(s), A^2 \varphi) ds + i \int_0^t (f_R(u_R(s), \varphi) ds.$$

(20)

Due to Lemma 3.3 we can pass to the limit into each linear term in (20), i.e. the term in right hand side first line. About the last term, we prove a pathwise convergence result. Fix $\omega$ in $\Omega$ such that $||u_0||_{H^1} < +\infty$. For $t < \tau_\infty$ there exists $R$ such that $t < \tau_R < \tau_\infty$ thus $\int_0^t (f_R(u_R(s), \varphi) ds = \int_0^t (f(u_\infty(s)), \varphi) ds$. Therefore we have a.e. in $\omega$ a weak solution of the equation.

We now complete the proof of Theorem 2.1 proving that $\tau_\infty < +\infty$ if and only if the associated trajectory for this given $\omega$ blows up in finite time.

For $\omega$ such that $\tau_\infty = +\infty$ we know that for any $T > 0$ there exists $R$ such that $T < \tau_R$ and then $||u_\infty(t)||_{H^1} \leq R$ for $t \leq T$. Then $u_\infty$ cannot blow up in finite time. Conversely if $\tau_\infty < +\infty$, then the sequence $||u_\infty(\tau_R)||_{H^1}$ diverges to $+\infty$ when $R$ diverges towards $+\infty$.

4. Decay of solutions.

4.1. Linear Schrödinger equation with white noise dispersion. Consider a solution of (2). We have

**Lemma 4.1.** Assume $u(s)$ be $\mathcal{F}_s$ measurable and in $L^1_x \cap H^1_x$ a.s., then

$$||S(t, s)u(s)||_{L^\infty_x} \leq c \frac{||u(s)||_{L^1_x \cap H^1_x}}{\langle W(t) - W(s) \rangle^\frac{1}{2}}.$$  

(21)
Proof. Due to Van Der Corput Lemma, we have
\[ |S(t,s)u(s)|_{L^p} \leq c \frac{|u(s)|_{L^p}^t}{|t-s|^{1/4}}. \] (22)
This provides the estimate (21) for $|W(t) - W(s)|$ larger than 1. We also have
\[ |S(t,s)u(s)|_{L^p} \leq c |S(t,s)u(s)|_{H^s_x} \leq c |u(s)|_{H^s_x}, \]
which gives the estimate for $|W(t) - W(s)|$ smaller than 1 and completes the proof.

Using estimate (22), the following theorem can be obtained.

**Theorem 4.2.** Assume $u(s)$ be $\mathcal{F}_s$-measurable and in $L^1(\Omega;L^1_x \cap H^s_x)$, then for $t > s$
\[ E(|S(t,s)u(s)|_{L^p}) \leq c \frac{E(|u(s)|_{L^1_x \cap H^s_x})}{|t-s|^{1/4}}. \] (23)

Proof. Since $W(t) - W(s)$ is independent of $\mathcal{F}_s$,
\[ E \left( \frac{|u(s)|_{L^1_x \cap H^s_x}}{|W(t) - W(s)|^{1/2}} \right) = E(|W(t) - W(s)|^{-1/2}) E(|u(s)|_{L^1_x \cap H^s_x}). \]
Now because $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ which has probability density function
\[ \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)}\right), \]
\[ E(|W(t) - W(s)|^{-1/2}) = c \int \exp\left(-\frac{x^2}{2(t-s)}\right) \frac{dx}{|t-s|^{1/2} |x|^2} \leq c \frac{1}{|t-s|^{1/4}}. \]
Using (22), we obtain the desired result.

**Remark 4.3.** We point out that the average decay rate of the solution is $\frac{1}{4}$ which is half of the rate for the Schrödinger equation with deterministic dispersion. This may be explained because $E(|W(t) - W(s)|^2) = t-s$. Then the oscillations of solutions propagate in average twice as slow as in the deterministic case.

4.2. **Decay for the nonlinear problem.** We now handle the proof of Theorem 2.2. The proof is divided into several lemmata. Throughout this section $u_0$ is a deterministic function. We first bound above the $H^s_x$ norm of a solution. Set $v(t) = u(t)1_{t < \tau_\infty}$, where $\tau_\infty$ is the stopping time defined in Theorem 2.1.

**Lemma 4.4.** Consider $u(t)$ the solution of the equation given by Theorem 2.1. If $u_0$ belongs to $H^s_x$ and satisfies $|u_0|_{H^s_x} \leq \varepsilon_0$, there exists a constant $c > 0$ that does not depend on $t$ such that a.s.
\[ \|v_x(t)\|_{L^2_x} \leq \varepsilon_0 \exp(c \int_0^t \|v(s)\|_{L^2_x}^{p-1} ds). \] (24)

Proof. Let us observe that for $t < \tau_\infty$
\[ v_x(t) = S(t,0)(u_0)_x + i \int_0^t S(t,s) \partial_x (|v|^{p-1}v) ds, \]
therefore
\[ \|v_x(t)\|_{L^2_x} \leq \varepsilon_0 + c \int_0^t \|v(s)\|_{L^2_x}^{p-1} \|v_x(s)\|_{L^2_x} ds. \] (25)
We now apply the classical Gronwall lemma that finishes the proof.

\[ \square \]
Remark 4.5. This shows that solutions that decay fast enough in $L^\infty_\sigma$ are global in $H^1$. Besides, let us recall that the $L^2_x$ norm of solutions is conserved along the flow, and then $\|v(t)\|_{L^2_x} \leq \varepsilon_0$.

We now prove a key inequality (26) which provides a link between the modulus of the Wiener process $(W(t))$ and $\|u(t)\|_{L^\infty_x}$.

**Lemma 4.6.** Consider $u(t)$ the solution of the equation given by Theorem 2.1. Assume $u_0 \in L^1_x \cap H^1_x$ and $\|u_0\|_{L^1_x \cap H^1_x} \leq \varepsilon_0$. Let set

$$\rho(s) = \|v(s)\|^p_{L^\infty_x}(W(s))^{\frac{p-2}{2}} + \|v(s)\|^{p-1}_{L^2_x}(W(s))^\frac{p-1}{2},$$

Then we have a.s.

$$\|v(t)\|_{L^\infty_x} \leq c \frac{\varepsilon_0}{\langle W(t) \rangle} \left(1 + \varepsilon_0 \exp(c \int_0^t \rho(s) ds)\right).\tag{26}$$

**Proof.** Using the linear estimate (21) we obtain that for a solution of (5), and for $t \leq \tau_\infty$,

$$\|v(t)\|_{L^\infty_x} \leq c \frac{\|u_0\|_{L^1_x \cap H^1_x}}{\langle W(t) \rangle} + c \int_0^t \frac{\|v^{p-1}v(s)\|_{L^1_x \cap H^1_x}}{\langle W(t) - W(s) \rangle^{\frac{p-2}{2}}} ds.$$

The right hand side can be bounded by observing

$$\|v^{p-1}v\|_{L^1_x} \leq \|v\|^2_{L^2_x} \|v\|^{p-2}_{L^\infty_x},$$

and using (8) and the Sobolev embedding $H^1_x \hookrightarrow L^\infty_x$,

$$\|v^{p-1}v\|_{H^1_x} \leq c \|v\|^{p-1}_{L^\infty_x} \|v\|_{H^1_x} \leq c \|v\|^2_{L^\infty_x} \|v\|^{p-2}_{H^1_x}.$$

These inequalities, combined with the $H^1$-bound (24), give

$$\|v(t)\|_{L^\infty_x} \leq c \frac{\varepsilon_0}{\langle W(t) \rangle} + c \varepsilon_0^2 \int_0^t \frac{\|v(s)\|^{p-2}_{L^\infty_x} \exp(c \int_0^s \|v(\sigma)\|_{L^\infty_x} d\sigma)}{\langle W(t) - W(s) \rangle^{\frac{p-2}{2}}} ds.\tag{27}$$

From the triangular inequality, we have

$$\langle W(t) \rangle \leq \sqrt{2} \langle W(t) - W(s) \rangle \langle W(s) \rangle.$$  

Therefore (27) implies

$$\|v(t)\|_{L^\infty_x} \leq c \frac{\varepsilon_0}{\langle W(t) \rangle} \left(1 + c \varepsilon_0 \int_0^t \rho(s) \exp(c \int_0^s \rho(\sigma) d\sigma) ds\right).$$

This completes the proof of the Lemma by mere computations. \hfill \Box

We now gather the previous results to derive an upper bound for the solutions. We introduce two auxiliary functions that read respectively

$$y(t) = \langle t \rangle^{\frac{p-3}{p-2}} \langle W(t) \rangle^{\frac{1}{p-2}} \|v(t)\|_{L^\infty_x},\tag{28}$$

$$a(t) = \left(\frac{\langle t \rangle^{\frac{1}{2}}}{\langle W(t) \rangle}\right)^{\frac{p-3}{2p-2}},\tag{29}$$

with the corresponding maximal functions $y^*(t) = \sup_{s \leq t} y(s)$, $a^*(t) = \sup_{s \leq t} a(s)$. We now prove a preliminary upper bound.
Lemma 4.7. Assume $\varepsilon_0$ is small enough. Then there exists $c > 0$ that depends on $p$ such that for any $t \in \mathbb{R}_+$,
\[
\mathbb{E} \left( (y^*(t))^2 \right) \leq c(-\log(\varepsilon_0))^{-\frac{2}{p-2}}.
\] (30)
Proof. We infer from (26) that the following inequality is valid
\[
y(t) \leq c\varepsilon_0 a(t) \left( 1 + \varepsilon_0 \exp(c \int_0^t \frac{y(s)^{p-2} + y(s)^{p-1}}{(s)^{\frac{p-2}{2}}} ds) \right). \tag{31}
\]
Assuming that $p > 7$ in order to ensure that $\langle s \rangle^{-\frac{p-3}{2}}$ is integrable, and setting
\[
b(y) = c_p(y(s)^{\frac{p-2}{2}} + y(s)^{\frac{p-1}{2}}),
\]
we infer that
\[
\exp(\int_0^t \frac{b(y(s))}{\langle s \rangle^{\frac{p-2}{2}}} ds) \leq \exp(b(y^*(s))^2),
\]
and then
\[
y^*(t)^2 \leq c(\varepsilon_0 a^*(t))^2 \left( 1 + c\varepsilon_0 \exp(b(y^*(s))^2)) \right). \tag{32}
\]
Introduce now the auxiliary function, for a positive number $z$
\[
f(z) = \frac{z}{1 + c\varepsilon_0^2 \exp(b(z))}. \tag{33}
\]
We now proceed as follows

Lemma 4.8. There exists $Z_0$ such that the positive function $z \mapsto f(z)$ is convex if \(z \geq Z_0\). Moreover $Z_0 \simeq c(-\log(\varepsilon_0))^{-\frac{2}{p-2}}$.

Proof. A simple computation shows
\[
f'(z) = \frac{1}{1 + c\varepsilon_0^2 \exp(b(z))} - \frac{c\varepsilon_0^2 b'(z) z \exp(b(z))}{(1 + c\varepsilon_0^2 \exp(b(z)))^2},
\]
and then that
\[
f''(z) = \frac{c\varepsilon_0^2 b'(z) f(z) \exp(b(z))}{(1 + c\varepsilon_0^2 \exp(b(z)))^2} \left( -\frac{2}{z} - \frac{b''(z)}{b'(z)} + \frac{b'(z)(c\varepsilon_0^2 \exp(b(z)) - 1)}{1 + c\varepsilon_0^2 \exp(b(z))} \right).
\]
Therefore for $\varepsilon_0$ small enough $f$ is convex for $z > Z_0$ where $c\varepsilon_0^2 \exp(b(Z_0)) \simeq 1$. \hfill \Box

We consider a decreasing convex function $g$ as follows (see Figure 1)
\[
g(z) = \begin{cases} 
    f(Z_0) + f'(Z_0)(z - Z_0), & \text{for } z \leq Z_0, \\
    f(z), & \text{for } z \geq Z_0.
\end{cases}
\]

We now compute $\mathbb{E} \left( (y^*(t))^2 \right) = G(t) + B(t)$, where
\[
G(t) = \mathbb{E}(y^*(t)^2 | y^*(t)^2 \leq Z_0) = \int_{\{y^*(t)^2 \leq Z_0\}} y^*(t)^2 d\mathbb{P}(\omega) \leq Z_0,
\]
and
\[
B(t) = \mathbb{E}(y^*(t)^2 | y^*(t)^2 \geq Z_0).
\]

The Jensen inequality (see [3]) gives
\[
g(B(t)) \leq \mathbb{E}(g(y^*(t)^2) | y^*(t)^2 \geq Z_0) \leq \mathbb{E}(f(y^*(t)^2) \leq c\varepsilon_0^2 \mathbb{E}(a^*(t)^2),
\]
and thanks to Lemma 4.9 below (that is a consequence of Doob's maximal inequality; see [5] and the references therein)
\[
g(B(t)) \leq cc_p\varepsilon_0^2. \tag{34}
\]
Lemma 4.9. There exists $c_p > 0$ that does not depend on $t$ such that
\[ \mathbb{E}(a^*(t)^2) \leq c_p. \]  
(35)

At this stage we have proven that $g(B(t)) \leq c\varepsilon_0^2 \leq g(Z_0)$ if $\varepsilon_0$ small enough. Since $g$ is a decreasing function then $B(t) \geq Z_0$. Therefore $f(B(t)) = g(B(t)) \leq c\varepsilon_0^2$ and we have that $B(t)$ satisfies for any $t$ the inequality
\[ f(B(t)) = \frac{B(t)}{1 + c\varepsilon_0^2 \exp(b(B(t)))} \leq c\varepsilon_0^2. \]  
(36)

Finally we have
\[ \mathbb{E}(y^*(t)^2) = G(t) + B(t) \leq Z_0 + c\varepsilon_0^2(1 + c\varepsilon_0^2 \exp(b(\mathbb{E}(y^*(t)^2)))). \]  
(37)

By a continuity argument the continuous function $t \mapsto \mathbb{E}(y^*(t)^2)$ that is less than $\varepsilon_0^2$ at $t = 0$ remains trapped into $[0, 2(-\log(\varepsilon_0)^{-\frac{1}{cb-1}})]$ since the map $z \mapsto Z_0 + c\varepsilon_0^2(1 + c\varepsilon_0^2 \exp(b(z)) - z$ is negative for $2(-\log(\varepsilon_0)^{-\frac{1}{cb-1}}}$.

We finally improve the previous estimate

Proposition 4.10. Assume $\varepsilon_0$ is small enough. Then there exists $c > 0$ that depends on $p$ such that for any $t \in \mathbb{R}$,
\[ \mathbb{E}(y^*(t)^2) \leq c\varepsilon_0^3. \]  
(38)

Proof. Let us introduce the stopping time
\[ \tilde{\tau}_\omega = \inf\{t > 0; y^*(t) > 2c\varepsilon_0 a^*(t)\}. \]  
(39)

Either $\tilde{\tau}_\omega = +\infty$ or $y^*(\tilde{\tau}_\omega) = 2c\varepsilon_0 a^*(\tilde{\tau}_\omega)$. In this second case we know from (32) that
\[ y^*(\tilde{\tau}_\omega) = 2c\varepsilon_0 a^*(\tilde{\tau}_\omega) \leq c\varepsilon_0 a^*(\tilde{\tau}_\omega)(1 + c\varepsilon_0 \exp(b(y^*(\tilde{\tau}_\omega)))), \]  
(40)

and then
\[ 1 \leq c\varepsilon_0 \exp(b(2c\varepsilon_0 a^*(\tilde{\tau}_\omega))), \]  
(41)
or, if $\varepsilon_0$ small enough,
\[ -\log(\varepsilon_0) \leq cb(2c\varepsilon_0 a^*(\tilde{\tau}_\omega)). \]  
(42)
Therefore for any $T > 0$,
\[-\log(\varepsilon_0)\mathbb{P}(\tilde{\tau}_\omega \leq T) \leq \int_{\{\tilde{\tau}_\omega \leq T\}} -\log(\varepsilon_0) d\mathbb{P}(\omega) \leq c \int_\Omega b(2c\varepsilon_0 a^*(T)) d\mathbb{P}(\omega) \leq c\varepsilon_0^{p-2},
\] (43)
thanks to Doob’s inequality (see Lemma 4.9 above).
We now write
\[E(y^*(t)^{\frac{3}{2}}) = E(y^*(t)^{\frac{3}{2}} | \tilde{\tau}_\omega > t) + E(y^*(t)^{\frac{3}{2}} | \tilde{\tau}_\omega \leq t).
\] (44)
The first term of the right hand side of (44) is bounded by above by $c\varepsilon_0^{\frac{3}{2}} E(a^*(t)^{\frac{3}{2}}) \leq \tilde{c}\varepsilon_0^{\frac{3}{2}}$. We bound the second term by Young inequality as
\[E(y^*(t)^{\frac{3}{2}} | \tilde{\tau}_\omega \leq t) \leq E(y^*(t)^{\frac{3}{2}})^{\frac{3}{4}} \mathbb{P}(\tilde{\tau}_\omega \leq t)^{\frac{1}{4}}.
\]
Using Lemma 4.7 and estimate (43) concludes the proof of the proposition.

5. Numerical simulations. In this section, we consider the following equation
\[idu + \nu u_{xx} \circ dW + \gamma |u|^{p-1}udt = 0 \quad (46)
\]
where $\nu$ and $\gamma$ are positive real numbers, and we propose a discretization scheme to approximate the solution of this equation. For the time discretization we propose as in [14] a Strang splitting method that decouples the computation of the linear stochastic part and the nonlinear part. For this scheme the mass is almost surely preserved along the discrete flow. Moreover this scheme is of strong order 1 for cubic Schrödinger equation with white noise dispersion (see [2]). Then some numerical simulations are presented.

5.1. Discretization. In order to discretize the problem, we consider equation (46) on a bounded domain $[-K; K]$, where $K$ is large enough to avoid boundary effects.

Then, a pseudo-spectral discretization in space is considered, with basis functions
\[e^{i\frac{l\pi}{K}x}, \quad -M_x \leq l \leq M_x - 1 \]
where $M_x$ represents the number of modes.

The spatial approximation of $u$ is defined by
\[u(x) = \sum_{l=-M_x}^{M_x-1} \hat{u}(\xi_l)e^{i\xi_lx}
\]
with $\xi_l = \frac{l\pi}{K}$, and
\[\hat{u}(\xi_l) = \frac{1}{2K} \int_{-K}^{K} u(x)e^{-i\xi_lx} dx.
\]
The time interval $[0, T_{\text{max}}]$ of simulation is discretized considering $t_n = n\Delta t$, $n = 0, \ldots, N_t$ where $\Delta t = \frac{T_{\text{max}}}{N_t}$ and $N_t$ is the number of time step.

Equation (46) is then solved using the classical Strang splitting scheme (see for example [14, 9, 2]). This method can be summarized as follow:

- On the half time step $[t_n, t_n + \frac{\Delta t}{2}]$, one solves
\[idu + \nu u_{xx} \circ dW = 0 \quad (47)
\]
• Then, on a complete time step \([t_n, t_{n+1}]\), one solves

\[
\begin{cases}
    idu + \gamma |u|^{p-1}u \, dt = 0 \\
    u_0 = u_{n+1/2}
\end{cases}
\]  

(48)

where \(u_{n+1/2}\) is the solution of (47) at time \(t_n + \frac{\Delta t}{2}\).

• Finally, problem (47) is solved once again on interval \([t_n + \frac{\Delta t}{2}, t_{n+1}]\) with initial data given by the solution of (48) at time \(t_{n+1}\). More precisely, considering a spacial mesh time step size \(\delta x = \frac{2K}{N_x}\) where \(N_x = 2M_x\) is an even integer, the grid points in space are defined by \(x_k = -K + k\delta x, \quad k = 0, \ldots, N_x\). The approximation of \(u(t_n, x_k)\) is denoted by \(u_{n,k}\). Using these notations, for the first step of the splitting, we compute

\[
\hat{u}_{n,l} = \frac{1}{N_x} \sum_{i=1}^{N_x} u_{i,l} e^{-i\xi_l x_i}, \quad \text{for } l = -M_x, \ldots, M_x - 1,
\]

\[
\hat{u}_{n+1/2,l} = \hat{u}_{n,l} \exp \left( -i\nu \sqrt{\frac{\Delta t}{2}} \xi_l^2 \chi_n \right)
\]

\[
u_{n+1/2,k} = \sum_{l=-M_x}^{M_x-1} \hat{u}_{n+1/2,l} e^{i\xi_l x_k} \quad \text{for } k = 1, \ldots, N_x
\]

where \(\sqrt{\frac{\Delta t}{2}} \chi_n = W(t_n + \frac{\Delta t}{2}) - W(t_n)\). Consequently, \((\chi_n)_{n=0,1,\ldots}\) is a sequence of independent normal \(\mathcal{N}(0, 1)\) random variables.

For the second step, using the conservation of the \(L^2\)-norm of the solution of (48) on a time step, using a Fourier spectral discretization, the solution of (48) verify

\[
idu_t + \gamma |u|^{n+1/2}|^{p-1}u = 0
\]

and (48) can be exactly solved in time. More precisely, the solution of (48) at time \(t_{n+1}\) is

\[
u_{n+1,k} = \nu_{n+1/2,k} \exp(i\gamma \Delta t |u_{n+1/2}|^{p-1}).
\]

Finally, for the last step:

\[
u_{n+1,k} = \sum_{l=-N_x}^{N_x-1} \tilde{u}_{n+1,l} \exp \left( i(\xi_l x_k - \nu \sqrt{\frac{\Delta t}{2}} \xi_l^2 \chi_n) \right).
\]

5.2. Numerical results. We present in this paragraph some numerical results on problem (46), where \(\nu = 0.1, \quad \gamma = 1\). The initial datum is defined by \(u_0(x) = 0.1e^{-x^2}\).

The data \(\nu\) and \(\gamma\) are chosen such that the decay rate of the solution can be numerically obtained on a reasonable spatial domain, a final time not too large to observe the expected decay rate and total number of stochastic not too large to reach a sufficiently confidence Besides, note that the change of variables \(u(x, t) = \gamma^{-\frac{1}{p-1}}v\left(\frac{x}{\sqrt{\nu}}, t\right)\) allows to fix \(\nu = \gamma = 1\).

Firstly, we investigate the \(L^2\) convergence rate with respect to the time step of discretization \(\Delta t\). For that purpose, since there is no exact stochastic solution, we consider a reference solution computed with a time step of discretization \(\Delta t = 10^{-5}\), with a given white noise \(W_1\) (see [11]). Then, for various time step \(\Delta t > 10^{-5}\), we compute the corresponding approximate solution and the relative difference with the reference solution. The result is plotted in Figure 2, and a linear regression provides a slope equals to 0.96.
In Figures 3 and 4, are presented the space and time evolution of the approximate solution of the nonlinear problem (46), respectively for \( p = 5 \) and \( p = 13 \). The time step is equal to \( \Delta t = 0.1 \) and the final time is \( T_{\text{max}} = 1 \). The spatial variable belongs to the interval \([-500, 500]\) which is discretized with \( N_x = 2^{12} \) nodes.

Then, in Figure 5 is presented the average \( L^\infty \)-decay rate with respect to time for \( p = 5 \), for both the deterministic and the stochastic problem. For that purpose, the number of stochastic processes is determined with a Monte-Carlo method with 95% confidence, the size of the spatial interval is \( K = 500 \) and \( M_x = 2^8 \) Fourier modes are considered. Concerning the time discretization, the time step is equal to \( \Delta t = 0.1 \) and the final time is \( T_{\text{max}} = 1000 \).

One can observe in Figure 5 that, as expected even if \( p \) is less than 12, the average convergence rate is close to \( \alpha = \frac{1}{2} \) in the deterministic case, and it is close to \( \alpha = \frac{1}{4} \) in the stochastic case.
Figure 4. Space and time evolution of the approximate solution of the nonlinear equation with $p = 13$ for one stochastic process (left: real part; right: imaginary part).

Figure 5. $L^\infty$ decay rate with respect to time for the deterministic and the stochastic problem.

6. Concluding remarks. In this article we have proved theoretically and illustrated numerically the decay rate towards 0 of solutions to a nonlinear Schrödinger equation with a white noise dispersion. This decay rate was established for initial data that are in a neighborhood of the rest state. A challenging issue is to remove the smallness assumption on the initial data. Another interesting issue is to prove decay rate estimates for lower nonlinearities, i.e. to improve the assumption $p > 7$. This issues require new methods that will be addressed in a forthcoming work.

Moreover, another important issue is to study the properties of the numerical scheme used in this article. We may address the numerical analysis of the scheme, as in [14] where it is proved that this scheme is of strong order 1 (in time). An issue is to prove that this scheme is of weak order 2. This will be addressed elsewhere for the equation.
Acknowledgments. The second and third authors acknowledge the support of research project SODDA supported by the Conseil Régional des Hauts-de-France. Thanks also to the referees for their remarks and suggestions. We dedicate this article to the memory of our friend Ezzeddine Zahrouni, a good man and mathematician, who passed away in December 2018.

REFERENCES

[1] P. Antonelli, J.-C. Saut and C. Sparber, Well-posedness and averaging of NLS with time-periodic dispersion management, Adv. Diff. Eq., 18 (2013), 49–68.
[2] R. Belaouar, A. de Bouard and A. Debussche, Numerical analysis of the nonlinear Schrödinger equation with white noise dispersion, A. Stoch PDE: Anal Comp, 3 (2015), 103–132.
[3] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext. Springer, New York, 2011.
[4] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
[5] M. Chen, O. Goubet and Y. Mammeri, Generalized regularized long waves equations with white noise dispersion, Stoch. Partial Differ. Eq. Anal. Comput., 5 (2017), 319–342.
[6] K. Chouk and M. Gubinelli, Nonlinear PDEs with modulated dispersion I: Nonlinear Schrödinger equations, Comm. Partial Differential Equations, 40 (2015), 2047–2081.
[7] A. de Bouard and A. Debussche, The nonlinear Schrödinger equation with white noise dispersion, J. Func. Anal., 259 (2010), 1300–1321.
[8] A. Debussche and Y. Tsutsumi, 1D quintic nonlinear Schrodinger equation with white noise dispersion, J. Math. Pures Appli., 96 (2011), 363–376.
[9] R. Duboscq and R. Marty, Analysis of a splitting scheme for a class of random nonlinear partial differential equations, ESAIM: PS, 20 (2016), 572–589.
[10] R. Duboscq and A. Reveillac, On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for a noisy dispersion, arXiv:1711.07188v1 [math.AP], 2017.
[11] G. Fenger, O. Goubet and Y. Mammeri, Numerical analysis of the midpoint scheme for the generalized Benjamin-Bona-Mahony equation with white noise dispersion, CACP, 26 (2019), 1307–1414.
[12] N. Hayashi, E. Kaikina, P. Naumkin and A. Shishmarev, Asymptotics for Dissipative Nonlinear Equations, Lecture Notes in Mathematics, 1884. Springer-Verlag, Berlin, 2006.
[13] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41 (1988), 891–907.
[14] R. Marty, On a splitting scheme for the nonlinear Schrödinger equation in a random medium, Comm. Math. Sci., 4 (2006), 679–705.

Received February 2020; revised September 2020.

E-mail address: serge.dumont@unimes.fr
E-mail address: olivier.goubet@univ-lille.fr
E-mail address: youcef.mammeri@u-picardie.fr