THE DUISTERMAAT-HECKMAN MEASURE FOR THE COADJOINT ORBITS OF COMPACT SEMISIMPLE LIE GROUPS

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This is a draft. Your comments are welcome.

Abstract. We apply the Guillemin-Lerman-Sternberg theorem to reprove a formula of Heckman for the Duistermaat-Heckman measure associated to the coadjoint action of $T$, a maximal torus of a compact semisimple Lie group $G$, on a regular coadjoint $G$-orbit in $\mathfrak{g}^\ast$, the dual space of the Lie algebra of $G$. This formula is, in an appropriate sense, a limiting case of the Kostant multiplicity formula.

Contents

1. Introduction 1
2. The Symplectic Structure of the Coadjoint Orbit 2
3. The Guillemin-Lerman-Sternberg Theorem 4
4. Generalities about Compact Semisimple Lie Groups 5
5. Computation for a Coadjoint Orbit 7
6. Comparison with the Kostant Multiplicity Formula 9
References 11

1. Introduction

Let $G$ be a compact connected semisimple Lie group and let $T$ be a maximal torus of $G$. In [H], Heckman considered the asymptotic behavior of the multiplicities of the representations of $T$ occurring in the restriction to $T$ of an highest weight representation of $G$, as given by the Kostant multiplicity formula. He obtained an ‘asymptotic multiplicity function’ and, using an integration formula of Harish-Chandra, proved this function is closely related to the push forward of the Liouville measure of the coadjoint orbit passing through the highest weight of the representation. This push forward measure was later generalized by Duistermaat and Heckman to any Hamiltonian torus action with a proper moment map.

Guillemin and Sternberg ([GS]) were able to derive Heckman’s results in the framework of symplectic geometry. Later, together with Lerman, they found a general formula for the Duistermaat-Heckman measure ([GLS]). This formula of Guillemin-Lerman-Sternberg is closely related to the exact stationary phase formula of Duistermaat-Heckman ([DH]).

We apply the Guillemin-Lerman-Sternberg formula in the case of coadjoint orbits, and reprove that the asymptotic multiplicity function is (the Radon-Nikodym derivative of)

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1Actually, Heckman works in a more general situation of restriction to any closed subgroup $K$ of $G$. We treat only the case $K = T$. 

the Duistermaat-Heckman measure. The idea of this computation was known to specialists, but it may have not been written up before. Some impetus to this approach is gained from the recent proof of the Guillemin-Lerman-Sternberg theorem via cobordism ([GGK1], [GGK2], [K1]).

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2. The Symplectic Structure of the Coadjoint Orbit

Let $V$ be a real vector space. The tangent space of $V$ at a point $p$, $\mathcal{T}_p(V)$, is identified with $V$ via the map $\tau_p : V \rightarrow \mathcal{T}_p(V)$, which assigns to a vector $v \in V$ the derivation in the direction of $v$. If $M \subset V$ is an (immersed) submanifold and $p \in M$, we identify $\mathcal{T}_p(M)$ with $W_p$, the subspace of $V$ which is the image of the composition

$$\mathcal{T}_p(M) \xrightarrow{i_*} \mathcal{T}_p(V) \xrightarrow{\tau_p^{-1}} V,$$

where $i_*$ is the differential of the inclusion map $i : M \rightarrow V$.

Assume a Lie group $G$ acts linearly on $V$, such that the $G$-action preserves $M$. The following proposition is easy to prove.

Proposition 2.1. The identification of the tangent spaces of $M$ with subspaces of $V$ is $G$-equivariant. That is, the following diagram is commutative:

$$\begin{array}{ccc}
W_p & \xrightarrow{\tau_p} & \mathcal{T}_p(M) \\
\downarrow g & & \downarrow g_* \\
W_{g.p} & \xrightarrow{\tau_{g.p}} & \mathcal{T}_{g.p}(M)
\end{array}$$

$p \in M$, $g \in G$.

Let $G$ be a compact connected Lie group with a Lie algebra $\mathfrak{g}$. Recall that for an action of $G$ on a manifold $M$, the generating vector field corresponding to an element $\xi \in \mathfrak{g}$ is the vector field on $M$ whose value at a point $p \in M$ is

$$\xi_M(p) = \frac{d}{dt} \bigg|_{t=0} (\exp(t\xi).p) \in \mathcal{T}_p(M).$$

$G$ acts on $\mathfrak{g}$ by the adjoint action, hence also on $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$, by the coadjoint action:

$$g \mapsto \text{Ad}^*(g^{-1}), \quad g \in G.$$ Fix $\lambda \in \mathfrak{g}^*$ and let $\mathcal{O} = G.\lambda$ be the orbit of $G$ through $\lambda$. $\mathcal{O}$ is an embedded submanifold of $\mathfrak{g}^*$ (by compactness of $G$). Its dimension is $\dim G - \dim G_{\lambda}$ ($G_{\lambda}$ is the stabilizer of $\lambda$).

Proposition 2.2. The generating vector field for the action of $G$ on $\mathcal{O}$, corresponding to an element $\xi \in \mathfrak{g}$, is given by the formula

$$(1) \quad \xi_{\mathcal{O}}(f) = -f([\xi, \cdot]), \quad f \in \mathcal{O}.$$ 

$(-f([\xi, \cdot]))$ is an element of $\mathfrak{g}^*$ upon replacing the dot with elements from $\mathfrak{g}$.)
Proof. As discussed before, we identify $T_f(O)$ ($f \in O$) with a linear subspace of $g^*$. Then, for any $\eta \in g$,

$$
\left. \frac{d}{dt} \right|_{t=0} \exp(t\xi).f(\eta) = \left. \frac{d}{dt} \right|_{t=0} f(\text{Ad}(\exp(-t\xi))\eta)
= f \left( \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp(-t\xi))\eta \right) \quad \text{(by linearity of } f) \\
= -f([\xi, \eta]).
$$

From general theory (see, for example, [B]), we know that for each $f \in O$ the map

$$
g \to T_f(O), \quad \xi \mapsto \xi_O(f) = -f([\xi, \cdot])
$$

is onto, and its kernel is

$$
g_f = \{ \xi \in g \mid f([\xi, \cdot]) = 0 \},
$$

the Lie algebra of $G_f$. Thus, $T_f(O) \cong g/g_f$.

If $g$ is equipped with a $G$-invariant inner product, denoted $(\cdot, \cdot)$, then it follows that the orthogonal complement of $g_f$ with respect to that inner product, $g_f^\perp$, is isomorphic to $T_f(O)$. This isomorphism is denoted $\Psi_f$:

$$
\Psi_f : g_f^\perp \to T_f(O), \quad f \in O.
$$

Moreover, by proposition [2.3] and the $G$-invariance of $(\cdot, \cdot)$, this ‘subspace model’ for the space tangent to $O$ at $f$ is compatible with the $G$-actions on $g$ and $O$: the diagram

$$
\begin{array}{ccc}
g_f^\perp & \xrightarrow{\Psi_f} & T_f(O) \\
\downarrow^{\text{Ad}(g)} & & \downarrow^{\text{Ad}^*(g^{-1})} \\
g_{g,f}^\perp & \xrightarrow{\Psi_{g,f}} & T_{g,f}(O)
\end{array}
$$

is commutative. We will use this model in the sequel.

The Kirillov-Kostant-Souriau 2-form on $O$ is defined by

$$
\omega_f(\xi_O(f), \eta_O(f)) = -f([\xi, \eta]), \quad f \in O, \xi, \eta \in g.
$$

The basic properties of $\omega$ are summarized in

**Proposition 2.3.** $\omega$ is a well-defined smooth non-degenerate closed 2-form on $O$ (so that $(O, \omega)$ is a symplectic manifold). Furthermore, $\omega$ is $G$-invariant, and the inclusion map $\Phi_G : O \hookrightarrow g^*$ is a moment map (that is, $\Phi_G$ is $G$-equivariant and satisfies the equations

$$
d\Phi_G^\xi = -\iota(\xi_O)\omega, \quad \xi \in g,
$$

where $\Phi_G^\xi = \langle \Phi_G, \xi \rangle : O \to \mathbb{R}$ is the $\xi$-coordinate of $\Phi_G$).

\[2\] Since $G$ is compact, such an invariant inner product always exists. In our application, we can (and will) take it to be the Killing form of $g$. 
The proof of this proposition may be found, for example, in [BGV, Lemma 7.22].

Note that since the manifold $\mathcal{O}$ has a symplectic structure, its dimension is even. We set $n = \frac{1}{2} \dim \mathcal{O}$.

From now on we assume that $G$ is semisimple — its center is a discrete (hence finite) subgroup of $G$. Let $T = G_{\lambda}$ be the stabilizer of $\lambda$. We assume that $\lambda$ is regular, which means that $T$ is a maximal torus in $G$. We denote the Lie algebra of $T$ by $\mathfrak{t}$.

The coadjoint action of $G$ restricts to an action of $T$ on $(\mathcal{O}, \omega)$. This action has a moment map

$$\Phi_T : \mathcal{O} \to \mathfrak{t}^*$$

where $\Phi_T = i^* \circ \Phi_G = i^*$, where $i^*$ is the (restriction to $\mathcal{O}$ of the) projection $i^* : g^* \to \mathfrak{t}^*$, which is dual to the inclusion map $i : \mathfrak{t} \to g$.

The Duistermaat-Heckman measure corresponding to the action of $T$ on the symplectic manifold $(\mathcal{O}, \omega)$ is the signed measure on $\mathfrak{t}^*$ defined by

$$DH_T(W) = \int_{\Phi_T^{-1}(W)} \omega^n / n! = \int_{(i^*)^{-1}(W)} \omega^n / n! ,$$

for an open subset $W \subset \mathfrak{t}^*$ with compact closure.

3. The Guillemin-Lerman-Sternberg Theorem

The Guillemin-Lerman-Sternberg theorem ([GLS, Theorem 3.3.3]) provides a formula for $DH_T$. We recall the statement of the theorem, with slight changes of notation, from [K2, Lecture 18] (see also [GGK2]).

Let $T$ be a compact torus acting on a compact symplectic manifold $(M^{2n}, \omega)$, with a moment map $\Phi_T : M \to \mathfrak{t}^*$. We assume that the set of $T$-fixed points, $M^T$, is finite.

For each $p \in M^T$, $T$ acts on $T_p(M)$ via the isotropy action. There is a unique decomposition of $T_p(M)$ into direct sum of irreducible $T$-representation. The trivial representation does not occur in this decomposition (this follows from the finiteness of $M^T$ and the equivariant slice theorem), so each summand is 2-dimensional. We recall how such irreducible 2-dimensional representations of $T$ are parameterized. Let

$$L = \ker(\exp : \mathfrak{t} \to T) .$$

$L$ is a lattice in $\mathfrak{t}$. Its dual lattice (the weight lattice) is defined by

$$L^* = \{ \alpha \in \mathfrak{t}^* \mid \alpha(\xi) \in 2\pi \mathbb{Z} , \text{ for all } \xi \in L \} .$$

For $\alpha \in L^* - \{0\}$, $T$ acts irreducibly on $\mathbb{R}^2$ by

$$R_\alpha : \exp(\xi) \mapsto \left( \begin{array}{cc} \cos \alpha(\xi) & - \sin \alpha(\xi) \\ \sin \alpha(\xi) & \cos \alpha(\xi) \end{array} \right) , \quad \xi \in \mathfrak{t} .$$

The representations corresponding to $\pm \alpha$ are equivalent (via conjugation by $(\frac{1}{\alpha} 1)$). Thus, the irreducible 2-dimensional representations of $T$ are parameterized by $(L^* - \{0\}) / \pm 1$.

We can also attach to $\alpha \in L^*$ a 1-dimensional complex representation of $T$, namely

$$\tilde{R}_\alpha : \exp(\xi) \mapsto \exp(\sqrt{-1} \alpha(\xi)) , \quad \xi \in \mathfrak{t} .$$

There is an $\mathbb{R}$-linear isomorphism $\mathbb{R}^2 \to \mathbb{C}$ (sending $(1,0) \mapsto 1$, $(0,1) \mapsto \sqrt{-1}$), which intertwines the representations $R_\alpha$ and $\tilde{R}_\alpha$. When composing this isomorphism with the complex conjugation, we get an intertwiner between $R_{-\alpha}$ and $R_\alpha$. Moreover, an $\mathbb{R}$-linear isomorphism
\[ \mathbb{R}^2 \to \mathbb{C} \] with these intertwining properties is unique up to multiplication by a non-zero real scalar. It follows that choosing one element from the unordered pair \( \pm \alpha \) is the same thing as endowing \( \mathbb{R}^2 \) with an invariant complex structure.

Returning to the isotropy action of \( T \) on \( \mathcal{T}_p(M) \), we denote by \( \pm \alpha_{p,j}, j = 1, \ldots, n \) the parameters of the irreducible summands. (\( \pm \alpha_{p,j} \) are called the isotropy weights at \( p \).) We now pick a ‘polarizing vector’ \( \Lambda \in \mathfrak{t} \) such that \( \alpha_{p,j}(\Lambda) \neq 0 \) for all \( p \in M^T \) and for all \( j \), and choose out of each pair \( \pm \alpha_{p,j} \) the one taking a positive value at \( \Lambda \) (that is, we fix the notation so that \( \alpha_{p,j}(\Lambda) > 0 \)). As explained before, the choices we have made determine \( \mathbb{R} \)-linear isomorphisms

\[ \Theta_p : \mathcal{T}_p(M) \rightarrow \mathbb{C}^n, \quad p \in M^T. \]

\( \Theta_p \) intertwines the isotropy action of \( T \) on \( \mathcal{T}_p(M) \) with the \( n \)-dimensional complex representation of \( T \)

\[ \exp(\xi) \mapsto \text{diag}(\exp(\sqrt{-1} \alpha_{p,1}(\xi)), \ldots, \exp(\sqrt{-1} \alpha_{p,n}(\xi))), \quad \xi \in \mathfrak{t}. \]

We endow \( \mathcal{T}_p(M) \) with the orientation determined by the symplectic structure of \( M \), and give \( \mathbb{C}^n \) the usual complex orientation. Set

\[ \epsilon_p = \begin{cases} +1 & \text{if } \Theta_p \text{ preserves orientation,} \\ -1 & \text{otherwise.} \end{cases} \]

We can now state the Guillemin-Lerman-Sternberg theorem.

**Theorem 1.** The Duistermaat-Heckman measure \( DH_T \) is absolutely continuous with respect to the Lebesgue measure on \( \mathfrak{t}^* \), and its density function (Radon-Nikodym derivative, also called the Duistermaat-Heckman function) \( \rho(x) \) is given by the formula

\[ \rho(x) = \sum_{p \in M^T} \epsilon_p \rho_p(x), \quad x \in \mathfrak{t}^*, \]

where

\[ \rho_p(x) = \text{Volume}\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_+ | \Phi_T(p) + \sum_{j=1}^n x_j \alpha_{p,j} = x \}. \]

(The volume here is \((n - \dim T)\)-dimensional; we shall recall its precise definition in Section 6 below.)

4. **Generalities about Compact Semisimple Lie Groups**

Before going on, we need to collect several facts from the structure theory of compact semisimple Lie groups and their Lie algebras. (The proofs of these facts may be found, for example, in [FH], Lecture 26 in particular.)

Let \( G \) be a compact connected semisimple Lie group with a maximal torus \( T \). Let \( \mathfrak{g}_C = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \) be the complexification of \( \mathfrak{g} \). It is a complex semisimple Lie algebra. \( \mathfrak{t}_C = \mathfrak{t} \oplus \sqrt{-1} \mathfrak{t} \) is a Cartan subalgebra of \( \mathfrak{g}_C \). The adjoint action of \( G \) on \( \mathfrak{g} \) (resp. of \( T \) on \( \mathfrak{t} \)) extends to an action on \( \mathfrak{g}_C \) (resp. \( \mathfrak{t}_C \)). The coadjoint actions extend to actions on \( \mathfrak{g}^*_C, \mathfrak{t}^*_C \), the complex duals of \( \mathfrak{g}_C, \mathfrak{t}_C \).

The Killing form, defined by

\[ (\xi, \eta) = \text{tr}(\text{ad} \xi \circ \text{ad} \eta), \quad \xi, \eta \in \mathfrak{g}_C, \]
is a non-degenerate bilinear form on $g_C$. It is \emph{invariant}, that is,
\begin{align*}
([\xi, \eta], \zeta) &= ([\xi, \eta], \zeta), \quad \xi, \eta, \zeta \in g_C, \\
(g, \xi, g, \eta) &= (\xi, \eta), \quad g \in G, \xi, \eta \in g_C.
\end{align*}

The restrictions of $(\cdot, \cdot)$ to $t_C$, $g$, and $t$ are non-degenerate as well; in fact, on $g$ (hence also on $t$) the restriction of $(\cdot, \cdot)$ is negative definite (so that its negation is an invariant inner product on $g$).

Define, for $\alpha \in t_C^*$,
\[(g_C)_\alpha = \{\xi \in g_C \mid [\tau, \xi] = \alpha(\tau)\xi \text{ for all } \tau \in t_C\}.
\]

The set of roots of $(g_C, t_C)$ is $\Delta = \{\alpha \in t_C^* - \{0\} \mid (g_C)_\alpha \neq 0\}$. If $\alpha \in \Delta$ is a root, then $\dim_C(g_C)_\alpha = 1$. The values of $\alpha$ on $t$ are pure imaginary, so that $\Delta \subset \sqrt{-1}t^*$; furthermore, $\Delta \subset \sqrt{-1}L^*$.

$\Delta$ can be decomposed (in several ways) into a disjoint union
\[\Delta = \Delta_+ \cup \Delta_-
\]
such that $\Delta_+$ (the \emph{positive roots}) and $\Delta_-$ (the \emph{negative roots}) satisfy the following conditions:
1. $\Delta_- = \{-\alpha \mid \alpha \in \Delta_+\}$.
2. If $\alpha, \beta \in \Delta_+$ and $\gamma = \alpha + \beta \in \Delta$, then $\gamma \in \Delta_+$.

Any such a disjoint union decomposition can be specified by a vector $\Lambda \in \sqrt{-1}t^*$ such that
\[
\alpha \in \Delta_+ \iff \alpha(\Lambda) > 0.
\]

We fix a choice of a set $\Delta_+$ of positive roots. The cardinality of $\Delta_+$ is\footnote{Note that this agrees with the former meaning of the notation $n$, because $\dim O = \dim G - \dim T$.}$n = \frac{1}{2}(\dim G - \dim T)$.

$g_C$ has a direct sum decomposition (the \emph{Cartan decomposition})
\[g_C = t_C \oplus \bigoplus_{\alpha \in \Delta_+} ((g_C)_\alpha \oplus (g_C)_{-\alpha}).
\]

For each $\alpha \in \Delta_+$, there is a triplet of vectors $X_\alpha, Y_\alpha, H_\alpha \in g_C$, such that:
1. $X_\alpha \in (g_C)_\alpha$, $Y_\alpha \in (g_C)_{-\alpha}$, and $H_\alpha \in \sqrt{-1}t$.
2. The standard $\mathfrak{sl}_2\mathbb{C}$ commutation relations hold:
\begin{align*}
[H_\alpha, X_\alpha] &= 2X_\alpha, & [H_\alpha, Y_\alpha] &= -2Y_\alpha, & [X_\alpha, Y_\alpha] &= H_\alpha.
\end{align*}
3. $\alpha(H_\alpha) = 2$.

The vectors $\{\sqrt{-1}H_\alpha \mid \alpha \in \Delta_+\}$ span the Lie algebra $t$ (over $\mathbb{R}$).

$g$ inherits from $g_C$ the \emph{real Cartan decomposition}
\[g = t \oplus \bigoplus_{\alpha \in \Delta_+} l_\alpha,
\]
where $l_\alpha = g \cap ((g_C)_\alpha \oplus (g_C)_{-\alpha})$. $l_\alpha$ is a real plane, which is spanned (over $\mathbb{R}$) by the vectors
\[U_\alpha = X_\alpha - Y_\alpha, \quad V_\alpha = \sqrt{-1}(X_\alpha + Y_\alpha).
\]

Let $N(T) = \{g \in G \mid g^{-1}Tg = T\}$ be the normalizer of $T$. The \emph{Weyl group} $\mathfrak{W} = N(T)/T$ is finite. We fix a set $\mathfrak{W} \subset N(T)$ of representatives for the elements of $\mathfrak{W}$. $\mathfrak{W}$ acts naturally
on $T$, $t$, $t_C$, and their duals. The sign of $w \in \mathfrak{m}$ (denoted $(-1)^w$) is $\det(w : t \to t)$. The action of $w \in \mathfrak{m}$ on $t$ permutes the vectors $\{\sqrt{-1}H_\alpha \mid \alpha \in \Delta\}$, and

\[(-1)^w = \text{parity of } \text{Card}\{\alpha \in \Delta_+ \mid w. (\sqrt{-1}H_\alpha) = \sqrt{-1}H_\beta, \text{ for some } \beta \in \Delta_+\}.\]

5. Computation for a Coadjoint Orbit

We work out the various ingredients of Theorem 1 for $M = \mathcal{O}$, the coadjoint orbit of $G$ through a regular element $\lambda \in \mathfrak{g}^*$, with the action of $T = G_\lambda$.

A point $p = g. \lambda \in \mathcal{O}$ is fixed by $T$ if and only if $t. g. \lambda = g. \lambda$ for all $t \in T$.

This equality says that $g^{-1}tg \in G_\lambda = T$, that is, $g \in N(T)$. Similarly, $g_1, g_2 \in N(T)$ yield the same orbit element $p$ precisely when $g_1T = g_2T$. We have shown:

**Proposition 5.1.** The fixed points of $T$ in $\mathcal{O}$ are $\mathcal{O}^T = \{w. \lambda \mid w \in \hat{\mathfrak{m}}\}$.

Hence, the formula of Theorem 1 in this case takes the form

\[\rho_\mathcal{O}(x) = \sum_{w \in \hat{\mathfrak{m}}} \epsilon_{w.\lambda} \rho_{w.\lambda}(x), \quad x \in t^*,\]

We turn to the analysis of the isotropy actions of $T$ on $T_{w,\lambda}(\mathcal{O})$. Bearing in mind that $\mathfrak{g}_{w,\lambda} = t$, the commutative diagram shows that, for all $t \in T$,

\[\text{Ad } t(\xi) = \Psi_{w,\lambda}^{-1}(t. \Psi_{w,\lambda}(\xi)), \quad \xi \in t^+.\]

This means that all the isotropy actions are equivalent to the adjoint action of $T$ on the orthogonal complement of $t$ in $\mathfrak{g}$ (with respect to the Killing form of $\mathfrak{g}$).

**Lemma 5.2.** The orthogonal complement of $t$ in $\mathfrak{g}$ is

\[t^+ = \bigoplus_{\alpha \in \Delta_+} l_\alpha.\]

**Proof.** By dimension considerations, it is enough to prove that $l_\alpha \subset t^+$, for all $\alpha \in \Delta_+$. Pick $\xi_0 \in t$ such that $\alpha(\xi_0) \neq 0$. For each $\xi \in t$,

\[\alpha(\xi_0)(\xi, X_\alpha) = (\xi, [\xi_0, X_\alpha]) = ([\xi, \xi_0], X_\alpha) \quad \text{(by invariance of } (\cdot, \cdot)) = 0,\]

hence $(\xi, X_\alpha) = 0$. Similarly, $t$ is orthogonal to $Y_\alpha$ (inside $\mathfrak{g}_C$), so that $l_\alpha \subset t$.

Let us determine how $T$ acts on $l_\alpha$. Put $\alpha' = -\sqrt{-1}\alpha \in t^*$. For $t \in T$, $t = \exp(\xi)$ for some $\xi \in t$, and then $\text{Ad } t = \exp(\text{ad } \xi)$. Since

\[\text{ad } \xi \begin{pmatrix} U_\alpha \\ V_\alpha \end{pmatrix} = \alpha(\xi) \begin{pmatrix} (X_\alpha + Y_\alpha) \\ \sqrt{-1}(X_\alpha - Y_\alpha) \end{pmatrix} = \alpha'(\xi) \begin{pmatrix} V_\alpha \\ -U_\alpha \end{pmatrix},\]
the action of $\text{Ad}_t$ on $\mathfrak{t}_a$ is represented (with respect to the basis $(U_\alpha, V_\alpha)$) by the matrix

$$\exp \left( \begin{array}{cc} 0 & -\alpha'(\xi) \\ \alpha'(\xi) & 0 \end{array} \right) = \begin{pmatrix} \cos \alpha'(\xi) & -\sin \alpha'(\xi) \\ \sin \alpha'(\xi) & \cos \alpha'(\xi) \end{pmatrix},$$

Thus, the $T$-action on $\mathfrak{t}_a$ is equivalent to the representation $R_{\alpha'}$ of $T$ (defined by (3)). By (3), the isotropy weights (for each fixed point) are $\{\pm \alpha' \mid \alpha \in \Delta_+\}$. We can take $\Lambda' = \sqrt{-1}\Lambda \in \mathfrak{t}$ as a polarizing vector, since by (3),

$$\alpha'(\Lambda') = -\sqrt{-1}\alpha(\sqrt{-1}\Lambda) = \alpha(\Lambda) > 0.$$

We conclude:

**Proposition 5.3.** In (8),

$$\rho_{w.\lambda}(x) = \text{Volume}\{ (x_\alpha)_{\alpha \in \Delta_+} \in \mathbb{R}^n_+ \mid w. i^*(\lambda) + \sum_{\alpha \in \Delta_+} x_\alpha \alpha' = x \}, \quad x \in \mathfrak{t}^*.$$

It remains to compute $\epsilon_{w.\lambda}$. It follows from the discussion preceding the proposition that the complex structure of $\mathfrak{t}_a$, determined by the polarization, is given by

$$U_\alpha \mapsto 1, \quad V_\alpha \mapsto \sqrt{-1}.$$

Denote this map by $\Theta: \mathfrak{t}^+ \rightarrow \mathbb{C}^n$. We have to check whether the maps

$$\Theta_{w.\lambda}: \mathcal{T}_{w.\lambda}(O) \rightarrow \mathbb{C}^n, \quad \Theta_{w.\lambda} = \Theta \circ \Psi_{w.\lambda}^{-1} \quad (w \in \check{\mathfrak{g}})$$

are orientation-preserving. By linear algebra, the definition of symplectic orientation, and (10), this is the case if and only if the Pfaffian of the $2n \times 2n$ matrix, which is made of the $2 \times 2$ blocks

$$\begin{pmatrix} \omega_{w.\lambda}(\Psi_{w.\lambda}(U_{\alpha_i}), \Psi_{w.\lambda}(U_{\alpha_j})) & \omega_{w.\lambda}(\Psi_{w.\lambda}(U_{\alpha_i}), \Psi_{w.\lambda}(V_{\alpha_j})) \\ \omega_{w.\lambda}(\Psi_{w.\lambda}(V_{\alpha_i}), \Psi_{w.\lambda}(U_{\alpha_j})) & \omega_{w.\lambda}(\Psi_{w.\lambda}(V_{\alpha_i}), \Psi_{w.\lambda}(V_{\alpha_j})) \end{pmatrix}, \quad i, j = 1, \ldots, n,$$

is positive. (We fixed an arbitrary ordering $\alpha_1, \ldots, \alpha_n$ of the positive roots.)

We claim that only the diagonal blocks are non-zero. This is a consequence of the following simple remarks:

- By the definition of $\omega$, a typical element in the matrix is of the form $-\omega(\lambda)([U_{\alpha_i}, U_{\alpha_j}]) = -\lambda(w^{-1}. [U_{\alpha_i}, U_{\alpha_j}]).$

- For all $\alpha, \beta \in \Delta$, $[(\mathfrak{g}_C)_{\alpha}, (\mathfrak{g}_C)_{\beta}] \subset (\mathfrak{g}_C)_{\alpha + \beta}$, so that the commutator may have non-zero intersection with $\mathfrak{t}_C$ only if $\beta = -\alpha$.

- Translating the previous remark to the real planes $\mathfrak{t}_a$, we find that

  $$\text{if } i \neq j, \text{ then } [\mathfrak{t}_{\alpha_i}, \mathfrak{t}_{\alpha_j}] \subset \mathfrak{t}^+. $$

- Cartan decomposition and (4) imply that $\mathfrak{t}^+ = [\mathfrak{t}, \mathfrak{g}]$.

- As $T = G_\lambda$, $\lambda$ is killed by the generating vector fields $\{\xi_O \mid \xi \in \mathfrak{t}\}$. This means (using (4)) that $\lambda([\mathfrak{t}, \mathfrak{g}]) = 0$.

- Conclusion: an element of a non-diagonal block belongs to

  $$\lambda(w^{-1}. [\mathfrak{t}_{\alpha_i}, \mathfrak{t}_{\alpha_j}]) \subset \lambda(w^{-1}. \mathfrak{t}^+) = \lambda(\mathfrak{t}^+) = \lambda([\mathfrak{t}, \mathfrak{g}]) = \{0\}.$$
It remains to deal with the diagonal blocks. By the commutation relations \((6)\),
\[
[U_{\alpha_i}, V_{\alpha_i}] = 2\sqrt{-1}H_{\alpha_i}, \quad i = 1, \ldots, n.
\]
Hence, the diagonal blocks are of the form
\[
\begin{pmatrix}
0 & -2\lambda(w^{-1}.(\sqrt{-1}H_{\alpha})) \\
-2\lambda(w^{-1}.(\sqrt{-1}H_{\alpha})) & 0
\end{pmatrix}.
\]
The Pfaffian is the product of the upper-right block elements, that is
\[
P_w.\lambda = \prod_{\alpha \in \Delta^+} (-2\lambda(w^{-1}.(\sqrt{-1}H_{\alpha}))).
\]
Let us concentrate on the case when \(-\sqrt{-1}(i^*\lambda) \in t_c^*\) is strongly dominant, that is the inequality
\[-\sqrt{-1}\lambda(H_{\alpha}) > 0\]
holds for all \(\alpha \in \Delta^+\). (We also term \(\lambda\) as strongly dominant in this case.) Using \((7)\) we get:

**Proposition 5.4.** If \(\lambda\) is strongly dominant, then \(P_{w,\lambda}\) is non-zero and its sign equals \((-1)^w\).

This finishes the proof of

**Theorem 2.** Assume \(\lambda \in g^*\) is strongly dominant. The Duistermaat-Heckman function for 
\(O = G.\lambda\) is
\[
\rho_O(x) = \sum_{w \in W} (-1)^w \rho_{w,\lambda}(x), \quad x \in t^*,
\]
where \(\rho_{w,\lambda}(x)\) is given by the formula in Proposition 5.3.

6. Comparison with the Kostant Multiplicity Formula

In the preceding sections we considered a fixed linear functional \(\lambda \in g^*\) and took \(T = G.\lambda\). We now fix a maximal torus \(T\) of \(G\) and consider linear functionals \(\lambda \in t^*\). Any such \(\lambda\) can be extended to an element \(\bar{\lambda} \in g^*\) via the direct sum decomposition \(g = t \oplus t^*\). Clearly, \(i^*\bar{\lambda} = \lambda\). Also, \(T \subset G.\bar{\lambda}\), with equality (meaning that \(\bar{\lambda}\) is regular) if and only if \(\lambda(\sqrt{-1}H_{\alpha}) \neq 0\) for all \(\alpha \in \Delta\).

Assuming that \(\bar{\lambda}\) is strongly dominant (recall that this means that
\[
\bar{\lambda}(\sqrt{-1}H_{\alpha}) = \lambda(\sqrt{-1}H_{\alpha}) > 0,
\]
for all \(\alpha \in \Delta^+\)), we deduce from Theorem 2 (applied to \(\bar{\lambda}\)) the following formula:
\[
\rho_{G.\bar{\lambda}}(\mu) = \sum_{w \in W} (-1)^w \text{Volume}\{(x_{\alpha})_{\alpha \in \Delta^+} \in \mathbb{R}^n_+ \mid \sum_{\alpha \in \Delta^+} x_{\alpha} \alpha' = \mu - w.\lambda\}, \quad \mu \in t^*.
\]
Recall that \(\alpha' = -\sqrt{-1}\alpha\). Put
\[
\lambda' = -\sqrt{-1}\lambda, \quad \mu' = -\sqrt{-1}\mu,
\]
and \(\tilde{\rho}_\lambda(\mu) = \rho_{G.\bar{\lambda}}(\mu)\). With this notation we have:
\[
(11) \quad \tilde{\rho}_\lambda(\mu) = \sum_{w \in W} (-1)^w \text{Volume}\{(x_{\alpha})_{\alpha \in \Delta^+} \in \mathbb{R}^n_+ \mid \sum_{\alpha \in \Delta^+} x_{\alpha} \alpha = w.\lambda' - \mu'\}, \quad \mu' \in \sqrt{-1}t^*.
\]
We now specialize to the case when $\lambda'$ is a strongly dominant weight (in the ‘complex’ sense), that is, $\lambda$ is strongly dominant, and $\lambda \in L^*$. The fundamental fact of the representation theory of $G$ is that there exists a unique irreducible complex representation, $V(\lambda')$, of $g_\mathbb{C}$ whose highest weight is $\lambda'$, and the restriction of $V(\lambda')$ to $g$ lifts to an irreducible unitary representation (also denoted $V(\lambda')$) of $G$. The restriction of $V(\lambda')$ from $G$ to $T$ is the direct sum of irreducible (1-dimensional, complex) representations $\tilde{R}_\mu$ of $T$ (defined by \((4)\)). Each representation $\tilde{R}_\mu$ occurs with a (possibly zero) multiplicity which is denoted by $m_\lambda(\mu)$:

$$\text{Res}_T V(\lambda') = \bigoplus_{\mu \in L^*} m_\lambda(\mu) \tilde{R}_\mu.$$ 

We let $Y(\lambda')$ be the set of $\mu \in L^*$ such that $\tilde{R}_\mu$ has non-zero multiplicity in $\text{Res}_T V(\lambda')$.

The Kostant multiplicity formula (see, for example, [S, Theorem IX.6.3]) asserts that, for $\mu \in Y(\lambda')$,

$$m_\lambda(\mu) = \sum_{w \in W} (-1)^w \text{Card}\{(x_\alpha)_{\alpha \in A} \in \mathbb{Z}^n_+ \mid \sum_{\alpha \in \Delta^+} x_\alpha \alpha = w.(\lambda'+\delta) - (\mu'+\delta)\}, \quad \lambda' = -\sqrt{-1} \lambda, \mu' = -\sqrt{-1} \mu, \text{ and } \delta \text{ is half the sum of the positive roots.}$$

There is an apparent similarity between the formulas \((11)\) and \((12)\). We follow Heckman in quantifying this similarity and quote several definitions and results from [H, Section 2].

Let $\mathbf{A}$ be a finite set contained in an open half space of a finite dimensional real vector space $E$. We also assume that $\mathbf{A}$ is contained in a lattice of maximal rank in $E$. Put $m = \text{Card}(\mathbf{A})$, $r = \text{rank}(\mathbf{A})$. The partition function of $\mathbf{A}$ is

$$p_\mathbf{A}(x) = \text{Card}\{(x_\alpha)_{\alpha \in \mathbf{A}} \in \mathbb{Z}^m_+ \mid \sum_{\alpha \in \mathbf{A}} x_\alpha \alpha = x\}, \quad x \in E.$$ 

The asymptotic partition function of $\mathbf{A}$ is

$$P_\mathbf{A}(x) = \text{Volume}\{(x_\alpha)_{\alpha \in \mathbf{A}} \in \mathbb{R}^m_+ \mid \sum_{\alpha \in \mathbf{A}} x_\alpha \alpha = x\}, \quad x \in E,$$

where the volume function is defined as follows. Fix an ordering $\mathbf{A} = (\alpha_1, \ldots, \alpha_m)$ of $\mathbf{A}$. Let $W_\mathbf{A} : \mathbb{R}^m \to E$ be the linear map sending $e_i$, the $i$-th standard basis vector of $\mathbb{R}^m$, to $\alpha_i$. $W_\mathbf{A}^{-1}(0)$, the kernel of $W_\mathbf{A}$, is an $(m-r)$-dimensional subspace of $\mathbb{R}^m$. By an easy argument, the integral points in $W_\mathbf{A}^{-1}(0)$ form a lattice of maximal rank. Normalize the Lesbegue measure on $W_\mathbf{A}^{-1}(0)$ so that the fundamental cell of this lattice has measure one. Call this normalized measure $\nu_0$. For every $x \in E$, $W_\mathbf{A}^{-1}(x)$ is a translation of $W_\mathbf{A}^{-1}(0)$, so we can translate the measure $\nu_0$ to get a measure $\nu_x$ on $W_\mathbf{A}^{-1}(x)$. (The translation invariance of Lesbegue measure guarantees that there is no ambiguity in the definition of $\nu_x$.) Since $\mathbf{A}$ is contained in an open half space, $W_\mathbf{A}^{-1}(x) \cap \mathbb{R}^m_+$ is a compact convex polytope. The $\nu_x$-measure of this polytope is $P_\mathbf{A}(x)$.

**Lemma 6.1** (Lemma 2.4 in [H]). Suppose $\text{rank}(\mathbf{A} - \{\alpha\}) = \text{rank}(\mathbf{A})$ for all $\alpha \in \mathbf{A}$. Fix $y$ in the $\mathbb{Z}$-span of $\mathbf{A}$. Then there exists a constant $C > 0$, depending on $y$, such that for all $x$ in the $\mathbb{Z}$-span of $\mathbf{A}$

$$|p_\mathbf{A}(x+y) - P_\mathbf{A}(x)| \leq C(1 + |x|)^{m-r-1}.$$
We want to apply this lemma with $E = \sqrt{-1}t^r$, $A = \Delta_+$, $m = n(=\frac{1}{2}(\dim G - \dim T))$, $r = \dim T$, $x_w = w. \lambda' - \mu'$ (with $\mu' \in Y(\lambda')$), and $y_w = w. \delta - \delta$ (for a fixed $w \in W$). Note that the $\mathbb{Z}$-span of $\Delta_+$ is the root lattice of $G$. Let us examine the validity of the assumptions in the lemma:

- The condition $\text{rank}(\Delta_+ - \{\alpha\}) = \text{rank}(\Delta_+)$, in the case that $G$ is simple, merely rules out the possibility $\mathfrak{g} = \mathfrak{su}_2$. A similar restriction applies when $G$ is semisimple (no $\mathfrak{su}_2$ direct factors). Since the result we wish to prove (Theorem 3 below) holds rather trivially in this case, this poses no serious difficulty.
- Although $\delta$ may not be a weight, $y_w$ does belong to the root lattice. In fact, it is the sum of distinct negative roots. (This can be easily deduced from, for example, the proof of Theorem IX.2.7 in [S].)
- The set $Y(\lambda')$ is invariant under the action of the Weyl group and the differences between its elements and $\lambda'$ lie in the root lattice ([S, Section IX.4]). These properties imply that $x_w$ belongs to the root lattice. Thus, there is a constant $C_w$ such that the estimate (13) holds. Summing over $w \in W$, and using the obvious identities

$$m_\lambda(\mu) = \sum_{w \in W} (-1)^w p_A(x_w + y_w),$$

$$\tilde{\rho}_\lambda(\mu) = \sum_{w \in W} (-1)^w P_A(x_w),$$

we infer that

$$|m_\lambda(\mu) - \tilde{\rho}_\lambda(\mu)| \leq C \sum_{w \in W} (1 + |w. \lambda' - \mu'|)^n - \dim T - 1$$

(with $C = \max_{w \in W}(C_w)$).

Set $s = n - \dim T$. The function $\tilde{\rho}_\lambda(\mu)$ has a simple homogeneity property:

$$\tilde{\rho}_{(k\lambda)}(k\mu) = k^s \tilde{\rho}_\lambda(\mu), \quad k \in \mathbb{Z}_+.$$ 

Now, for a constant $D$ depending on $\lambda$, $\mu$ (but not on $k$), we have

$$|m_{(k\lambda)}(k\mu) - \tilde{\rho}_{(k\lambda)}(k\mu)| \leq C \text{Card}(W)(1 + kD)^{s-1},$$

so that

$$k^s \left| \frac{m_{(k\lambda)}(k\mu)}{k^s} - \tilde{\rho}_\lambda(\mu) \right| \leq D_1 k^{s-1},$$

for another constant $D_1$. Dividing by $k^s$ and sending $k$ to infinity, we obtain

**Theorem 3.** If $\lambda'$ is a strongly dominant weight and $\mu \in Y(\lambda')$, then

$$\tilde{\rho}_\lambda(\mu) = \lim_{k \to \infty} \frac{m_{(k\lambda)}(k\mu)}{k^s}.$$

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