THE ROGUE WAVE AND BREATHER SOLUTION OF THE GERDJIKOV-IVANOV EQUATION

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ABSTRACT. The Gerdjikov-Ivanov (GI) system of $q$ and $r$ is defined by a quadratic polynomial spectral problem with $2 \times 2$ matrix coefficients. Each element of the matrix of $n$-fold Darboux transformation of this system is expressed by a ratio of $(n+1) \times (n+1)$ determinant and $n \times n$ determinant of eigenfunctions, which implies the determinant representation of $q^{[n]}$ and $r^{[n]}$ generated from known solution $q$ and $r$. By choosing some special eigenvalues and eigenfunctions according to the reduction conditions $q^{[n]} = -(r^{[n]})^*$, the determinant representation of $q^{[n]}$ provides some new solutions of the GI equation. As examples, the breather solutions and rogue wave of the GI is given explicitly by two-fold DT from a periodic “seed” with a constant amplitude.

Key words: Gerdjikov-Ivanov equation, Darboux transformation, breather solution, rogue wave.

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1. INTRODUCTION

Ablowitz-Kaup-Newell-Segur(AKNS) system [1] is one of the most important integrable systems associated with a linear spectrum problem, which consists of well known nonlinear Schrödinger (NLS) equation [2]. So it is a natural extension to consider a polynomial spectral problem(PSP) of arbitrary order [3]. Note that the corresponding Kirillov-Kostant symplectic form of the general PSP is degenerated, which leads to some special restrictions of the PSP in order to get interesting integrable partial differential equations. The quadratic PSP under different restrictions implies three kinds of the derivative nonlinear Schrödinger equations, which are called DNLSI equation, DNLSII equation and DNLSIII equation respectively. The DNLSI equation [4] is given by

$$ iq_t - q_{xx} + i(q^2 q^*)_x = 0, \quad (1) $$

and the DNLSII equation [5] is given by

$$ iq_t + q_{xx} + iqq^*_x = 0, \quad (2) $$

and the DNLSIII equation is in the form of [6]

$$ iq_t + q_{xx} - iq^2 q^*_x + \frac{1}{2}q^3 q^{*2} = 0. \quad (3) $$

The last equation is firstly found by Gerdjikov and Ivanov in reference [9], then it is also called GI equation. Here asterisk denotes the complex conjugation, and subscript of $x$ (or $t$) denotes the partial derivative with respect to $x$ (or $t$). On the other hand, GI equation also can be

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regarded as an extension of the NLS when certain higher-order nonlinear effects are taken into account.

Comparing with the intensive studies on the DNLSI \[4,7-20\] from the point of view of physics and mathematics, there are only few works on the GI including soliton constructed by Darboux transformation (DT) \[21\], Hamiltonian structures \[22\], algebro-geometric solutions \[23\], hierarchy of the GI equation from an extended version of Drinfel’d-Sokolov formulation \[24\], Wronskian type solution \[25\] without using affine Lie groups. In order to show more possible physical relevance of the GI equation, and inspired by the importance of breather (BA) solution and rogue wave (RW) of the NLS \[26-33\], so we shall find these two types of solutions for the GI equation by DT. Moreover, We also want to show the more differences between DNLSI \[4\] and GI \[6\] through solutions and DT.

The organization of this paper is as follows. In section 2, it provides a relatively simple approach to construct DT for the GI system, and then the determinant representation of the n-fold DT and formulae of \(q[n]\) and \(r[n]\) are given by eigenfunctions of spectral problem. The reduction of DT of the GI system to the GI equation is also discussed by choosing paired eigenvalues and eigenfunctions. In section 3, under specific reduction conditions, two types of particular solutions-BA and RW are given by DT from a periodic “seed” with a constant amplitude. The conclusions and discussions will be given in section 4.

2. Darboux transformation

Let us start from the first non-trivial flow of a special quadratic spectral problem \[6\],

\[
\begin{align*}
\partial_t q + q_{xx} - iq^2 r_x + \frac{1}{2} q^3 r^2 &= 0, \\
\partial_t r - r_{xx} + ir^2 q_x - \frac{1}{2} q^2 r^3 &= 0, 
\end{align*}
\]

which are exactly reduced to the GI eq.(1) for \(r = -q^*\) while the choice \(r = q^*\) would lead to eq.(3) with the sign of the nonlinear term changed. The Lax pairs corresponding to coupled GI equations(4) and (5) can be given by the GI system with a quadratic spectral \[6\]

\[
\begin{align*}
\partial_x \psi &= (J\lambda^2 + Q_1\lambda + Q_0)\psi = U\psi, \\
\partial_t \psi &= (2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0)\psi = V\psi. 
\end{align*}
\]

Here

\[
\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -\frac{1}{2}iqr & 0 \\ 0 & \frac{1}{2}iqr \end{pmatrix}, \\
V_3 = 2Q_1, \quad V_2 = Jqr, \quad V_1 = \begin{pmatrix} 0 & iq_x \\ -ir_x & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} \frac{1}{4}iq^2r^2 & 0 \\ 0 & -\frac{1}{4}iq^2r^2 \end{pmatrix},
\]

the spectral parameter \(\lambda\) is an arbitrary complex constant and its eigenfunction is \(\psi\). Equations(4) and (5) are equivalent to the integrability condition \(U_t - V_x + [U, V] = 0\) of (6) and (7).

The main task of this section is to present a detailed derivation of the Darboux transformation of the GI system and the determinant representation of the n-fold transformation. Based on the DT for the NLS \[34-36\] and the DNLSI \[10,11,20\], the main steps are : 1) to find a 2 × 2
matrix $T$ so that the GI spectral problem eq.(6) and eq.(7) is covariant, then get new solution $(q^{[1]}, r^{[1]})$ expressed by elements of $T$ and seed solution $(q, r)$; 2) to find expressions of elements of $T$ in terms of eigenfunctions of GI spectral problem corresponding to the seed solution $(q, r)$; 3) to get the determinant representation of n-fold DT $T_n$ and new solutions $(q^{[n]}, r^{[n]})$ by n-times iteration of the DT; 4) to consider the reduction condition: $q^{[n]} = -(r^{[n]})^*$ by choosing special eigenvalue $\lambda_k$ and its eigenfunction $\psi_k$, and then get $q^{[n]}$ of the GI equation expressed by its seed solution $q$ and its associated eigenfunctions $\{\psi_k, k = 1, 2, \ldots, n\}$. However, we shall use the kernel of n-fold DT$(T_n)$ to fix it in the third step instead of iteration.

It is easy to see that the spectral problem (6) and (7) are transformed to

$$
\psi^{[1]}_x = U^{[1]} \psi^{[1]}, \quad U^{[1]} = (T_x + T U)T^{-1}.
$$

$$
\psi^{[1]}_t = V^{[1]} \psi^{[1]}, \quad V^{[1]} = (T_t + T V)T^{-1}.
$$

under a gauge transformation

$$
\psi^{[1]} = T \psi.
$$

By cross differentiating (8) and (9), we obtain

$$
U^{[1]}_t - V^{[1]}_x + [U^{[1]}, V^{[1]}] = T(U_t - V_x + [U, V])T^{-1}.
$$

This implies that, in order to make eqs.(14) and eq.(5) invariant under the transformation (10), it is crucial to search a matrix $T$ so that $U^{[1]}, V^{[1]}$ have the same forms as $U, V$. At the same time the old potential (or seed solution) $(q, r)$ in spectral matrixes $U, V$ are mapped into new potentials (or new solution) $(q^{[1]}, r^{[1]})$ in transformed spectral matrixes $U^{[1]}, V^{[1]}$.

2.1 One-fold Darboux transformation of the GI system

Considering the universality of DT, suppose that the trial Darboux matrix $T$ in eq.(10) is of form

$$
T = T(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix},
$$

where $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$ are functions of $x, t$ to need be determined. From

$$
T_x + T U = U^{[1]} T,
$$

comparing the coefficients of $\lambda^j, j = 3, 2, 1, 0$, it yields

$$
\lambda^3: b_1 = 0, \quad c_1 = 0,
$$

$$
\lambda^2: q a_1 + 2 i b_0 - q^{[1]} d_1 = 0, \quad -r^{[1]} a_1 + r d_1 - 2 i c_0 = 0,
$$

$$
\lambda^1: a_{1x} - \frac{1}{2} i a_1 q r + r b_0 + \frac{1}{2} i q^{[1]} r^{[1]} a_1 - q^{[1]} c_0 = 0, \quad d_{1x} + \frac{1}{2} i d_1 q r - r^{[1]} b_0 - \frac{1}{2} i q^{[1]} r^{[1]} d_1 + q c_0 = 0,
$$

$$
q a_0 - q^{[1]} d_0 = 0, \quad -r^{[1]} a_0 + r d_0 = 0,
$$

$$
\lambda^0: a_{0x} - \frac{1}{2} i a_0 q r + \frac{1}{2} i q^{[1]} r^{[1]} a_0 = 0, b_{0x} + \frac{1}{2} i b_0 q r + \frac{1}{2} i q^{[1]} r^{[1]} b_0 = 0,
$$

$$
c_{0x} - \frac{1}{2} i c_0 q r - \frac{1}{2} i q^{[1]} r^{[1]} c_0 = 0, d_{0x} + \frac{1}{2} i d_0 q r - \frac{1}{2} i q^{[1]} r^{[1]} d_0 = 0.
$$

Similarly, from

$$
T_t + T V = V^{[1]} T,
$$

comparing the coefficients of $\lambda^j, j = 4, 3, 2, 1, 0$, it implies

$$
\lambda^4: 2 i b_0 - q^{[1]} d_1 + q a_1 = 0, \quad -2 i c_0 - 2 r^{[1]} a_1 + r d_1 = 0,
$$

$$
\lambda^3: b_1 = 0, \quad c_1 = 0,
$$

$$
\lambda^2: q a_1 + 2 i b_0 - q^{[1]} d_1 = 0, \quad -r^{[1]} a_1 + r d_1 - 2 i c_0 = 0,
$$

$$
\lambda^1: a_{1x} - \frac{1}{2} i a_1 q r + r b_0 + \frac{1}{2} i q^{[1]} r^{[1]} a_1 - q^{[1]} c_0 = 0, \quad d_{1x} + \frac{1}{2} i d_1 q r - r^{[1]} b_0 - \frac{1}{2} i q^{[1]} r^{[1]} d_1 + q c_0 = 0,
$$

$$
q a_0 - q^{[1]} d_0 = 0, \quad -r^{[1]} a_0 + r d_0 = 0,
$$

$$
\lambda^0: a_{0x} - \frac{1}{2} i a_0 q r + \frac{1}{2} i q^{[1]} r^{[1]} a_0 = 0, b_{0x} + \frac{1}{2} i b_0 q r + \frac{1}{2} i q^{[1]} r^{[1]} b_0 = 0,
$$

$$
c_{0x} - \frac{1}{2} i c_0 q r - \frac{1}{2} i q^{[1]} r^{[1]} c_0 = 0, d_{0x} + \frac{1}{2} i d_0 q r - \frac{1}{2} i q^{[1]} r^{[1]} d_0 = 0.
$$

Similarly, from

$$
T_t + T V = V^{[1]} T,
$$

comparing the coefficients of $\lambda^j, j = 4, 3, 2, 1, 0$, it implies

$$
\lambda^4: 2 i b_0 - q^{[1]} d_1 + q a_1 = 0, \quad -2 i c_0 - 2 r^{[1]} a_1 + r d_1 = 0,
\[ \lambda^3 : r^{[1]} q^{[1]} a_1 i - 2q^{[1]} c_0 - a_1 r q i + 2 r b_0 = 0, \quad q a_0 - q^{[1]} d_0 = 0, \]
\[ r d_0 - r^{[1]} a_0 = 0, d_1 r q i - r^{[1]} q^{[1]} d_1 i + 2 q c_0 - 2 r^{[1]} b_0 = 0, \]
\[ \lambda^2 : a_0 r q - a_0 r^{[1]} q^{[1]} = 0, -b_0 r q i + q^{[1]} d_1 i - a_1 q x i - r^{[1]} q^{[1]} b_0 i = 0, \]
\[ c_0 r q i + r^{[1]} q^{[1]} c_0 i + d_1 r x i - r^{[1]} a_1 i = 0, \quad r^{[1]} q^{[1]} d_0 - r q d_0 = 0, \]
\[ \lambda^1 : a_{1t} - q^{[1]} c_0 i - b_0 r x i + \frac{1}{4} a r^2 q^2 - \frac{1}{4} a r^{[1]} q^{[1]} i^2 - \frac{1}{2} a_1 q x i + \frac{1}{2} a_1 q^{[1]} r^{[1]} x - \frac{1}{2} a_1 r^{[1]} q^{[1]} x = 0, \]
\[ d_{1t} + q x c_0 i + b_0 r^{[1]} x i - \frac{1}{4} d r^2 q^2 + \frac{1}{4} d r^{[1]} q^{[1]} i^2 + \frac{1}{2} d_1 q r x - \frac{1}{2} d_1 r^{[1]} x + \frac{1}{2} d_1 r^{[1]} q^{[1]} x = 0, \]
\[ q^{[1]} x d_0 i - a_0 q x i = 0, \quad d_0 r x - r^{[1]} a_0 i = 0, \]
\[ \lambda^0 : a_{0t} + \frac{1}{4} a_0 r^2 q^2 - \frac{1}{4} a_0 r^{[1]} q^{[1]} i^2 - \frac{1}{2} a_0 q r x + \frac{1}{2} a_0 r^{[1]} x - \frac{1}{2} a_0 r^{[1]} q^{[1]} x = 0, \]
\[ b_{0t} - \frac{1}{4} a r^2 q^2 - \frac{1}{4} a r^{[1]} q^{[1]} i^2 + \frac{1}{2} b_0 q r x - \frac{1}{2} b_0 q^{[1]} r^{[1]} x + \frac{1}{2} b_0 r^{[1]} q^{[1]} x = 0, \]
\[ c_{0t} + \frac{1}{4} c_0 r^2 q^2 + \frac{1}{4} c_0 r^{[1]} q^{[1]} i^2 - \frac{1}{2} c_0 q r x + \frac{1}{2} c_0 q^{[1]} r^{[1]} x - \frac{1}{2} c_0 r^{[1]} q^{[1]} x = 0, \]
\[ d_{0t} - \frac{1}{4} d r^2 q^2 - \frac{1}{4} d r^{[1]} q^{[1]} i^2 + \frac{1}{2} d_0 q r x - \frac{1}{2} d_0 r^{[1]} x + \frac{1}{2} d_0 r^{[1]} q^{[1]} x = 0. \] (16)

Note that the coefficient of \( \lambda^5 \) in eq. (15) is an identity automatically because of the first formula of eq. (16).

In order to get the non-trivial solutions, we shall construct a basic (or one-fold) Darboux transformation matrix \( T \) under an assumption \( a_0 = 0, d_0 = 0 \). If we set \( a_0 \neq 0 \), then \( d_0 \) is not zero, and furthermore find that some coefficients \( (a_1, d_1, a_0, d_0) \) of \( T \) are constants except \( b_0 = \frac{1}{2} i q \left( -\frac{a_0 d_1}{d_0} + a_1 \right) \) and \( c_0 = \frac{1}{2} i r \left( \frac{a_1 d_0}{a_0} - d_1 \right) \), which gives a trivial DT. This means our assumption does not decrease the generality of the DT of the GI system. Based on eq. (16), eq. (10) and above analysis, let Darboux matrix \( T \) be the form of

\[ T_1 = T_1(\lambda; \lambda_1) = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}, \] (17)

without losing any generality. Here \( b_0, c_0 \) are undetermined function of \((x, t)\), which will be expressed by the eigenfunction associated with \( \lambda_1 \) in the GI spectral problem. However, \( a_1 \) and \( d_1 \) are constants, which will be verified later. First of all, we introduce \( n \) eigenfunctions \( \psi_j \) as

\[ \psi_j = \begin{pmatrix} \phi_j \\ \varphi_j \end{pmatrix}, \quad j = 1, 2, ..., n, \phi_j = \phi_j(x, t, \lambda_j), \varphi_j = \varphi_j(x, t, \lambda_j). \] (18)

**Theorem 1.** The elements of one-fold DT are parameterized by the eigenfunction \( \psi_1 \) associated with \( \lambda_1 \) as

\[ a_1 = d_1 = 1, \quad b_0 = -\frac{\lambda_1 \phi_1}{\varphi_1}, \quad c_0 = -\frac{\lambda_1 \varphi_1}{\phi_1}, \] (19)

\[ \Leftrightarrow T_1(\lambda; \lambda_1) = \begin{pmatrix} \lambda & -\frac{\lambda_1 \phi_1}{\varphi_1} \\ -\frac{\lambda_1 \varphi_1}{\phi_1} & \lambda_1 \end{pmatrix}. \] (20)

and then the new solutions \( q^{[1]} \) and \( r^{[1]} \) are given by

\[ q^{[1]} = q - 2i \frac{\lambda_1 \phi_1}{\varphi_1}, \quad r^{[1]} = r + 2i \frac{\lambda_1 \varphi_1}{\phi_1}. \] (21)
and the new eigenfunction $\psi_j^{[1]}$ corresponding to $\lambda_j$ is

$$
\psi_j^{[1]} = \begin{pmatrix} \frac{1}{\phi_1} \\ \frac{1}{\phi_1} \end{pmatrix} \begin{pmatrix} \varphi_j & -\lambda_j \phi_j \\ -\lambda_j \phi_j & \phi_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\phi_1} \\ \frac{1}{\phi_1} \end{pmatrix} \begin{pmatrix} \phi_j & -\lambda_j \phi_j \\ -\lambda_j \phi_j & \phi_1 \end{pmatrix} = \begin{pmatrix} \varphi_j & -\lambda_j \phi_j \\ -\lambda_j \phi_j & \phi_1 \end{pmatrix}.
$$

(22)

**Proof.** Note that $(b_0c_0)_x = 0, a_{1x} = 0$ and $d_{1x} = 0$ is derived from the eq.(14), then $(b_0c_0)_t = 0$, $a_{1t} = 0$ and $d_{1t} = 0$ is derived from the eq.(16). So GI spectral problem is covariant under transformation (21), and thus it is the DT of eq.(4) and eq.(5). Further, by using the explicit matrix representation eq.(20) of solutions are given by

$$
T_{n} = T_{n}(\lambda; \lambda_1, \cdots, \lambda_n) = \sum_{l=0}^{n} P_{l} \lambda^{l},
$$

(25)

with

$$
P_{n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{D}, \quad P_{n-1} = \begin{pmatrix} 0 & b_{n-1} \\ c_{n-1} & 0 \end{pmatrix} \in \mathbf{A}, \quad P_{l} \in \mathbf{D} \text{ (if } l - n \text{ is even)}, \quad P_{l} \in \mathbf{A} \text{ (if } l - n \text{ is odd)}.
$$

(26)

Here $P_{i}(0 \leq i \leq n - 1)$ is the function of $x$ and $t$. In particular, $P_{0} \in \mathbf{D}$ if $n$ is even and $P_{0} \in \mathbf{A}$ if $n$ is odd, which leads to the separate discussion on the determinant representation of $T_{n}$ in the next subsection.

2.2 N-fold Darboux transformation for GI system

The key task is to establish the determinant representation of the n-fold DT for GI system in this subsection. To this purpose, set

\[
\mathbf{D} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \text{ are complex functions of } x \text{ and } t \right\},
\]

\[
\mathbf{A} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \text{ are complex functions of } x \text{ and } t \right\},
\]

as ref. [10].

According to the form of $T_{1}$ in eq.(17), the n-fold DT should be the form of

$$
T_{n} = T_{n}(\lambda; \lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_{l=0}^{n} P_{l} \lambda^{l},
$$

(25)
the following by means of its kernel. Specifically, from algebraic equations,

\[
\psi_k^{[m]} = T_n(\lambda; \lambda_1, \cdots, \lambda_n)|_{\lambda=\lambda_k} \psi_k = \sum_{l=0}^{n} P_l \lambda_l^k \psi_k = 0, \ k = 1, 2, \cdots, n, \tag{27}
\]

coefficients \(P_l\) is solved by Cramer's rule. Thus we get determinant representation of the \(T_n\).

**Theorem 2.** (1) For \(n = 2k (k = 1, 2, 3, \cdots)\), the \(n\)-fold DT of the GI system can be expressed by

\[
T_n = T_n(\lambda; \lambda_1, \lambda_2, \cdots, \lambda_n) = \begin{pmatrix}
\frac{(T_n)_{11}}{W_n} & \frac{(T_n)_{12}}{W_n} \\
\frac{(T_n)_{21}}{W_n} & \frac{(T_n)_{22}}{W_n}
\end{pmatrix},
\]

with

\[
W_n = \begin{vmatrix}
\phi_1 & \lambda_1 \phi_1 & \cdots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^{n-1} \phi_1 \\
\phi_2 & \lambda_2 \phi_2 & \cdots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^{n-1} \phi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_n & \lambda_n \phi_n & \cdots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^{n-1} \phi_n
\end{vmatrix},
\]

\[
\widetilde{(T_n)_{11}} = \begin{vmatrix}
1 & 0 & \cdots & 0 & \lambda^{n-2} & 0 & \lambda^n \\
0 & \lambda & \cdots & \lambda^{n-3} & 0 & \lambda^{n-1} & 0 \\
\phi_1 & \lambda_1 \phi_1 & \cdots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^{n-1} \phi_1 & \lambda_1^n \phi_1 \\
\phi_2 & \lambda_2 \phi_2 & \cdots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^{n-1} \phi_2 & \lambda_2^n \phi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\phi_n & \lambda_n \phi_n & \cdots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^{n-1} \phi_n & \lambda_n^n \phi_n
\end{vmatrix},
\]

\[
\widetilde{(T_n)_{12}} = \begin{vmatrix}
\varphi_1 & \lambda_1 \varphi_1 & \cdots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\
\varphi_2 & \lambda_2 \varphi_2 & \cdots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \varphi_n & \cdots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n
\end{vmatrix},
\]

\[
\widetilde{(T_n)_{21}} = \begin{vmatrix}
0 & \lambda & \cdots & \lambda^{n-3} & 0 & \lambda^{n-1} \\
\varphi_1 & \lambda_1 \varphi_1 & \cdots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\
\varphi_2 & \lambda_2 \varphi_2 & \cdots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \varphi_n & \cdots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n
\end{vmatrix},
\]

\[
\widetilde{(T_n)_{22}} = \begin{vmatrix}
1 & 0 & \cdots & 0 & \lambda^{n-2} & 0 & \lambda^n \\
\varphi_1 & \lambda_1 \varphi_1 & \cdots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\
\varphi_2 & \lambda_2 \varphi_2 & \cdots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \varphi_n & \cdots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n
\end{vmatrix}.
\]
(2) For \( n = 2k + 1 (k = 1, 2, 3, \ldots) \), then

\[
T_n = T_n(\lambda; \lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix}
\frac{(T_n)_{11}}{Q_n} & \frac{(T_n)_{12}}{Q_n} \\
\frac{(T_n)_{21}}{Q_n} & \frac{(T_n)_{22}}{Q_n}
\end{pmatrix},
\]

with

\[
Q_n = \begin{vmatrix}
\varphi_1 & \lambda_1 \phi_1 & \ldots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\
\varphi_2 & \lambda_2 \phi_2 & \ldots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \phi_n & \ldots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n
\end{vmatrix},
\]

\[
\frac{(T_n)_{11}}{Q_n} = \begin{vmatrix}
0 & \lambda & \ldots & 0 & \lambda^{n-2} & 0 & \lambda^n \\
\varphi_1 & \lambda_1 \phi_1 & \ldots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 & -\lambda_1^n \phi_1 \\
\varphi_2 & \lambda_2 \phi_2 & \ldots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 & -\lambda_2^n \phi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \phi_n & \ldots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n & -\lambda_n^n \phi_n
\end{vmatrix},
\]

\[
\frac{(T_n)_{12}}{Q_n} = \begin{vmatrix}
\varphi_1 & \lambda_1 \varphi_1 & \ldots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\
\varphi_2 & \lambda_2 \varphi_2 & \ldots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \varphi_n & \ldots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n
\end{vmatrix},
\]

\[
\frac{(T_n)_{21}}{Q_n} = \begin{vmatrix}
1 & 0 & \ldots & \lambda^{n-3} & 0 & \lambda^{n-1} & 0 \\
\phi_1 & \lambda_1 \phi_1 & \ldots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^{n-1} \phi_1 & -\lambda_1^n \phi_1 \\
\phi_2 & \lambda_2 \phi_2 & \ldots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^{n-1} \phi_2 & -\lambda_2^n \phi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\phi_n & \lambda_n \phi_n & \ldots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^{n-1} \phi_n & -\lambda_n^n \phi_n
\end{vmatrix},
\]

\[
\frac{(T_n)_{22}}{Q_n} = \begin{vmatrix}
\varphi_1 & \lambda_1 \varphi_1 & \ldots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-1} \varphi_1 \\
\varphi_2 & \lambda_2 \varphi_2 & \ldots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-1} \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\varphi_n & \lambda_n \varphi_n & \ldots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-1} \varphi_n
\end{vmatrix}.
\]

Next, we consider the transformed new solutions \((q^n, r^n)\) of GI system corresponding to the \( n \)-fold DT. Under covariant requirement of spectral problem of the GI system, the transformed form should be

\[
\partial_x \psi^n = (J \lambda^2 + Q_1^n \lambda + Q_0^n) \psi = U^n \psi,
\]
Substituting $T$ with $J$, we have

$$Q_1^{[n]} = \begin{pmatrix} 0 & q^{[n]} \\ r^{[n]} & 0 \end{pmatrix}, \quad Q_0^{[n]} = \begin{pmatrix} -\frac{1}{2} i q^{[n]} r^{[n]} & 0 \\ 0 & \frac{1}{2} i q^{[n]} r^{[n]} \end{pmatrix},$$

and then

$$T_{nx} + T_n = U^{[n]} T_n.$$  \hspace{1cm} (31)

Substituting $T_n$ given by eq.(25) into eq.(31), and then comparing the coefficients of $\lambda^{n+1}$, it yields

$$q^{[n]} = q + 2ib_{n-1}, \quad r^{[n]} = r - 2ic_{n-1}. \hspace{1cm} (32)$$

Furthermore, taking $b_{n-1}, c_{n-1}$ which are obtained from eq.(28) for $n = 2k$ and from eq.(29) for $n = 2k + 1$, into (32), then new solutions $(q^{[n]}, r^{[n]})$ are given by

$$q^{[n]} = q + 2i \frac{\Omega_{11}}{\Omega_{12}}, \quad r^{[n]} = r - 2i \frac{\Omega_{21}}{\Omega_{22}}. \hspace{1cm} (33)$$

Here, (1) for $n = 2k,$

$$\Omega_{11} = \begin{pmatrix} \phi_1 & \lambda_1 \varphi_1 & \ldots & \lambda_1^{n-3} \varphi_1 & \lambda_1^{n-2} \varphi_1 & -\lambda_1^n \varphi_1 \\ \phi_2 & \lambda_2 \varphi_2 & \ldots & \lambda_2^{n-3} \varphi_2 & \lambda_2^{n-2} \varphi_2 & -\lambda_2^n \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_n & \lambda_n \varphi_n & \ldots & \lambda_n^{n-3} \varphi_n & \lambda_n^{n-2} \varphi_n & -\lambda_n^n \varphi_n \end{pmatrix}, \hspace{1cm} (34)$$

and

$$\Omega_{12} = \begin{pmatrix} \varphi_1 & \lambda_1 \phi_1 & \ldots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^n \phi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \ldots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^n \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \ldots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^n \phi_n \end{pmatrix}, \hspace{1cm} (34)$$

(2) for $n = 2k + 1,$

$$\Omega_{11} = \begin{pmatrix} \varphi_1 & \lambda_1 \phi_1 & \ldots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & -\lambda_1^n \phi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \ldots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & -\lambda_2^n \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \ldots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & -\lambda_n^n \phi_n \end{pmatrix}, \hspace{1cm} (35)$$

and

$$\Omega_{12} = \begin{pmatrix} \varphi_1 & \lambda_1 \phi_1 & \ldots & \lambda_1^{n-3} \phi_1 & \lambda_1^{n-2} \phi_1 & \lambda_1^n \phi_1 \\ \varphi_2 & \lambda_2 \phi_2 & \ldots & \lambda_2^{n-3} \phi_2 & \lambda_2^{n-2} \phi_2 & \lambda_2^n \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n & \lambda_n \phi_n & \ldots & \lambda_n^{n-3} \phi_n & \lambda_n^{n-2} \phi_n & \lambda_n^n \phi_n \end{pmatrix}. \hspace{1cm} (35)$$
We are now in a position to consider the reduction of the DT of the GI system so that $q^{[n]} = -(r^{[n]})^*$, then the DT of the GI is given. Under the reduction condition $q = -r^*$, the eigenfunction $\psi_k = \begin{pmatrix} \phi_k \\ \varphi_k \end{pmatrix}$ associated with eigenvalue $\lambda_k$ has following properties,

(i): $\phi_k^* = \varphi_k$, $\lambda_k = -\lambda_k^*$;

(ii): $\phi_k^* = \varphi_k$, $-\varphi_k^* = \phi_k$, $\lambda_k^* = \lambda_l$, where $k \neq l$.

Notice that the denominator $W_n$ of $q^{[n]}$ is a non-zero complex function under reduction condition when $\lambda_i \neq 0$, so the new solution $q^{[n]}$ is non-singular. We shall explain it clearly in following Case 1. For the one-fold DT $T_1$, set

$$\lambda_1 = i\beta_1 \text{(a pure imaginary), and its eigenfunction } \psi_1 = \begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix},$$

then $T_1$ in theorem 1 is the DT of the GI. We note that $q^{[1]} = -(r^{[1]})^*$ holds with the help of eq. (21), $q = -r^*$ and this special choice of $\psi_1$. This is an essential distinctness of DT between GI and NLS, because one-fold transformation of AKNS can not preserve the reduction condition to the NLS. Furthermore, for the two-fold DT, according to above property (ii), set

$$\lambda_2 = \lambda_1^* \text{ and its eigenfunction } \psi_2 = \begin{pmatrix} -\varphi_1^* \\ \phi_1^* \end{pmatrix}, \psi_1 = \begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix} \text{ associated with eigenvalue } \lambda_1,$$

then $q^{[2]} = -(r^{[2]})^*$ can be verified from eq. (33) and $T_2$ given by eq. (28) is the DT of the GI. Of course, in order to get $q^{[2]} = -(r^{[2]})^*$ so that $T_2$ becomes also the DT of the DNLS, we can also set

$$\lambda_l = i\beta_l \text{(a pure imaginary) and its eigenfunction } \psi_l = \begin{pmatrix} \varphi_l^* \\ \phi_l^* \end{pmatrix}, l = 1, 2. \quad (38)$$

There are many choices to guarantee $q^{[n]} = -(r^{[n]})^*$ for the n-fold DTs when $n > 2$. For example, setting $n = 2k$ and $l = 1, 3, \ldots, 2k - 1$, then choosing following $k$ distinct eigenvalues and eigenfunctions in n-fold DTs:

$$\lambda_l \leftrightarrow \psi_l = \begin{pmatrix} \phi_l \\ \varphi_l \end{pmatrix}, \text{and } \lambda_{2l} = \lambda_{2l-1}^*, \leftrightarrow \psi_{2l} = \begin{pmatrix} -\varphi_{2l-1}^* \\ \phi_{2l-1}^* \end{pmatrix} \quad (39)$$

so that $q^{[2k]} = -(r^{[2k]})^*$ in eq. (33). Then $T_{2k}$ with these paired-eigenvalue $\lambda_i$ and paired-eigenfunctions $\psi_i (i = 1, 3, \ldots, 2k - 1)$ is reduced to the (2k)-fold DT of the GI. Similarly, $T_{2k+1}$ in eq. (29) can also be reduced to the (2k+1)-fold DT of the GI by choosing one pure imaginary $\lambda_{2k+1} = i\beta_{2k+1}$ (pure imaginary) and $k$ paired-eigenvalues $\lambda_{2l} = \lambda_{2l-1}^* (l = 1, 2, \ldots, k)$ with corresponding eigenfunctions according to properties (i) and (ii).
3. BREATHER SOLUTION AND ROGUE WAVE

Similar to case of the DNLSI [10][11][20], it is not difficult to get new solutions of the GI equation from zero “seed” and nonzero “seed” by one-fold DT. We shall focus directly on BA solutions and RW from a periodic “seed” by 2-fold and 4-fold DT.

Set $a$ and $c$ be two complex constants, then $q = ce^{i[ax-(a^2+c^2-a^4)t]}$ is a periodic solution of the GI equation, which will be used as a “seed” solution of the DT. Substituting $q = ce^{i[ax-(a^2+c^2-a^4)t]}$ into the spectral problem eq.(6) and eq.(7), and using the method of separation of variables and the superposition principle, the eigenfunction $\psi_k$ associated with $\lambda_k$ is given by

$$
\begin{align*}
\phi_k(x,t,\lambda_k) &= \left(\varpi_1(x,t,\lambda_k)[1,k] + \varpi_2(x,t,\lambda_k)[1,k] - \varpi_1^*(x,t,\lambda_k^*)[2,k] - \varpi_2^*(x,t,\lambda_k^*)[2,k]\right), \\
\varphi_k(x,t,\lambda_k) &= \left(\varpi_1(x,t,\lambda_k)[2,k] + \varpi_2(x,t,\lambda_k)[2,k] + \varpi_1^*(x,t,\lambda_k^*)[1,k] + \varpi_2^*(x,t,\lambda_k^*)[1,k]\right).
\end{align*}
$$

Here

$$
\varpi_1(x,t,\lambda_k)[1,k] = \left(\frac{\exp(-\sqrt{\lambda_k(ta-x-2t\lambda_k^2)})}{2\lambda_k c} \exp\left(-\sqrt{\lambda_k(ta-x-2t\lambda_k^2)}\right)\right),
$$

$$
\varpi_2(x,t,\lambda_k)[1,k] = \left(\frac{\exp\left(\sqrt{\lambda_k(ta-x-2t\lambda_k^2)}\right)}{2\lambda_k c} \exp\left(-\sqrt{\lambda_k(ta-x-2t\lambda_k^2)}\right)\right),
$$

$$
\varpi_1(x,t,\lambda_k) = \left(\varpi_1(x,t,\lambda_k)[1,k] \varpi_1(x,t,\lambda_k)[2,k]\right), \quad \varpi_2(x,t,\lambda_k) = \left(\varpi_2(x,t,\lambda_k)[1,k] \varpi_2(x,t,\lambda_k)[2,k]\right),
$$

$$
s = -c^4 + 2c^2a - a^2 - 4a\lambda_k^2 - 4\lambda_k^4.
$$

Note that $\varpi_1(x,t,\lambda_k)$ and $\varpi_2(x,t,\lambda_k)$ are two different solutions of the spectral problem eq.(3) and eq.(7), but we can only get the trivial solutions through DT of the GI by setting eigenfunction $\psi_k$ be one of them.

Using eigenfunctions $\psi_k$ in eq.(40), following several examples give a transparent explanation of the DT of the GI equation.

Case 1. ($N = 2$). Under the choice in eq.(37) with one paired eigenvalue $\lambda_1 = \alpha_1 + i\beta_1$ and $\lambda_2 = \alpha_1 - i\beta_1$, the two-fold DT eq.(33) of the GI equation implies a solution

$$
q^{[2]} = \frac{\lambda_1^2 - \lambda_1^2}{\lambda_1^2} \frac{\phi_1^* \varphi_1^*}{\lambda_1 \varphi_1^* \varphi_1^* + \lambda_2 \bar{\phi}_1 \bar{\phi}_1^*},
$$

with $\phi_1$ and $\varphi_1$ given by eq.(40). Note the denominator in eq.(41) is non-zero except $\lambda_1 = 0$, because the imaginary part of denominator will be zero if and only if $\alpha_1 = \beta_1 = 0$. Two concrete examples of eq.(41) are given below.

(a) For simplicity, let $a = -2\alpha_1^2 + 2\beta_1^2$ so that $\text{Im}(-c^4 + 2c^2a - a^2 - 4a\lambda_1^2 - 4\lambda_1^4) = 0$, then

$$
q^{[2]} = \exp(2i(2xS + (-4S^2 - 2c^2S + \frac{c^4}{2}t)))(c + \frac{u_1}{u_2}),
$$

$$
u_1 = 4\alpha_1\beta_1((c - 2\beta_1)^2(c^2 + 4\alpha_1^2) - K_1^2) \cosh(f1) - 16i\alpha_1^2\beta_1(c - 2\beta_1)K_1 \sin(f2) + 4\alpha_1\beta_1((c - 2\beta_1)^2(c^2 + 4\alpha_1^2) + K_1^2) \cos(f2) - 8i\alpha_1\beta_1(c - 2\beta_1)K_1 \sinh(f1),
$$
\[ u_2 = \alpha_1 ((c - 2\beta_1)^2(c^2 + 4\alpha_1^2) + K1^2) \cosh(f1) - 2i\alpha_1(c - 2\beta_1)K1 \sin(f2) + \alpha_1((c - 2\beta_1)^2(c^2 + 4\alpha_2^2) - K1^2) \cosh(f2) + 4i\alpha_1(c - 2\beta_1)K1 \sin(f1), \]

\[ S = \beta_1^2 - \alpha_1^2, \quad f1 = K1(4(\alpha_1^2 - \beta_1^2)t + x), \quad f2 = 4K1\alpha_1\beta_1t, \]

\[ K1 = \sqrt{16\alpha_1^2\beta_1^2 - 4c^2\alpha_1^2 + 4c^2\beta_1^2 - c^4}. \]

By letting \( x \to \infty, \ t \to \infty \), so \( |q[2]|^2 \to c^2 \), the trajectory of this solution is defined explicitly by

\[ x = -4\alpha_1^2t + 4\beta_1^2t \quad (43) \]

from \( f_1 = 0 \) if \( K1^2 > 0 \), and by

\[ t = 0 \quad (44) \]

from \( f_2 = 0 \) if \( K1^2 < 0 \). According to eq.\((42)\), we can get the Ma breathers \[37\] (time periodic breather solution) and the Akhmediev breathers \[28, 29\] (space periodic breather solution) solution. In general, the solution in eq.\((42)\) evolves periodically along the straight line with a certain angle of \( x \) axis and \( t \) axis. The dynamical evolution of \( |q[2]|^2 \) in eq.\((42)\) for different parameters are plotted in Figure 1, Figure 2 and Figure 3, which give a visual verification of the three cases of trajectories. As we have done for the DNLSI \[20\], the rogue wave of the GI equation will be given from its BA solutions by a limit procedure in the following. By letting \( c \to 2\beta_1 \) in \((42)\) with \( \text{Im}(-c^4 + 2c^2a - a^2 - 4a\lambda_1^2 - 4\lambda_1^4) = 0 \), it becomes rogue wave

\[ q_{\text{rogue wave}}[2] = \frac{k1}{k2} + k3. \quad (45) \]

Here

\[ k1 = -4\beta_1(16i\alpha_1\beta_1^2(\beta_1^2 + \alpha_1^2)t + \beta_1i + \alpha_1 - 4\beta_1(\beta_1^2 + \alpha_1^2)x - 16\beta_1(\alpha_4 - \beta_1^4)t) \times (16i\alpha_1\beta_1^2(\beta_1^2 + \alpha_1^2)t + i\beta_1 + \alpha_1 + 4\beta_1(\beta_1^2 + \alpha_1^2)x + 16\beta_1(\alpha_4 - \beta_1^4)t) \times \exp(-2i((\alpha_1^2 - \beta_1^4)x + 2(\alpha_4 - 4\alpha_1^2\beta_1^2 + \beta_1^4)t)), \]

\[ k2 = -16\beta_1^2(\beta_1^6 + \alpha_1^2)(16(\alpha_4^5 + \beta_1^6)t^2 - 8(-\alpha_1^4 + \beta_1^4)xt + (\beta_1^2 + \alpha_1^2)x^2) - \beta_1^2 - \alpha_1^2 \]

\[ + 32i\beta_1^2(\alpha_1^4 - 2\beta_1^2)(\beta_1^2 + \alpha_1^2)t + 8i\beta_1^2(\beta_1^2 + \alpha_1^2)x, \]

\[ k3 = 2\beta_1\exp(-2i((\alpha_1^2 - \beta_1^4)x + 2(\alpha_4 - 4\alpha_1^2\beta_1^2 + \beta_1^4)t)). \]

By letting \( x \to \infty, \ t \to \infty \), so \( |q_{\text{rogue wave}}[2]|^2 \to 4\beta_1^2 \). The rogue wave reaches its maximum amplitude \( |q_{\text{rogue wave}}[2]|^2 = 36\beta_1^2 \) at \( t = 0 \) and \( x = 0 \) and minimum amplitude \( |q_{\text{rogue wave}}[2]|^2 = 0 \) at

\[ t = \pm \frac{1}{16\sqrt{3(4\alpha_1^2 + \beta_1^4)\beta_1(\alpha_1^2 + \beta_1^2)}}, \quad x = \mp \frac{9\alpha_1^2}{4\sqrt{3(4\alpha_1^2 + \beta_1^4)\beta_1(\alpha_1^2 + \beta_1^2)}}. \quad \text{Figure 4 and} \]

Figure 5 of \( |q_{\text{rogue wave}}[2]|^2 \) show the the maximum value of \( |q_{\text{rogue wave}}[2]|^2 \) is nine times of its asymptotic value and there exists only a very small domain on whole \((x,t)\) plane of non-constant value. Therefore \( q_{\text{rogue wave}}[2] \) is one kind of typical rogue wave.

(b) When \( a = \frac{c^2}{2} \), it is not difficult from eq.\((40)\) to find that there are two sets of collinear eigenfunctions,

\[
\begin{align*}
\varpi 1(x, t, \lambda_k)[1, k] & \quad \text{and} \quad -\varpi 2^*(x, t, \lambda_k^*)[2, k] \\
\varpi 1(x, t, \lambda_k)[2, k] & \quad \text{and} \quad \varpi 2^*(x, t, \lambda_k^*)[1, k]
\end{align*}
\]

\[
\begin{align*}
\varpi 2(x, t, \lambda_k)[1, k] & \quad \text{and} \quad -\varpi 1^*(x, t, \lambda_k^*)[2, k] \\
\varpi 2(x, t, \lambda_k)[2, k] & \quad \text{and} \quad \varpi 1^*(x, t, \lambda_k^*)[1, k]
\end{align*}
\]

(46)
Therefore, the eigenfunction $\psi_k$ associated with $\lambda_k$ for this case is given by

$$
\begin{pmatrix}
\varphi_k(x,t,\lambda_k) \\
\varphi_k(x,t,\lambda_k)
\end{pmatrix}
= \begin{pmatrix}
\varpi 1(x,t,\lambda_k)[1,k] - \varpi 1^*(x,t,\lambda_k^*)[2,k] \\
\varpi 1(x,t,\lambda_k)[2,k] + \varpi 1^*(x,t,\lambda_k^*)[1,k]
\end{pmatrix}.
$$

(48)

Here

$$
\begin{pmatrix}
\varpi 1(x,t,\lambda_k)[1,k] \\
\varpi 1(x,t,\lambda_k)[2,k]
\end{pmatrix}
= \begin{pmatrix}
\exp(i(\lambda_k^2 x + 2\lambda_k^4 t + \frac{1}{2}c^2 x - \frac{1}{4}c^4 t)) \\
\frac{2i\lambda_k}{c} \exp(i(\lambda_k^2 x + 2\lambda_k^4 t))
\end{pmatrix}.
$$

Under the choice in eq. (37) with $\lambda_1 = \alpha_1 + i\beta_1, \lambda_2 = \alpha_1 - i\beta_1$, and the $\psi_1$ given by eq. (18), the solution $q^{[2]}$ is given simply from eq. (33). Figure 6 is plotted for $|q^{[2]}|^2$, which shows the periodical evolution along a straight line on $(x-t)$ plane.

Case 2, ($N = 4$). According to the choice in eq. (39) with two distinct eigenvalues $\lambda_1 = \alpha_1 + i\beta_1, \lambda_3 = \alpha_3 + i\beta_3$, substituting $\psi_1$ and $\psi_3$ defined by eq. (10) into eq. (33), then the new solution $q^{[4]}$ generated by 4-fold DT is given. Its analytical expression is omitted because it is very complicated. But $|q^{[4]}|^2$ are plotted in Figure 7 and 8 to show the dynamical evolution on $(x-t)$ plane: (a) Let $a = -2a_i^2 + 2\beta_i^2, i = 1, 3$, so that $\text{Im}(-c^4 + 2c^2 a - a^2 - 4a\lambda_i^2 - 4\lambda_i^4) = 0$, then Figure 7 shows intuitively that two breathers may have parallel trajectories; (b) Two breathers have an elastic collision so that they can preserve their profiles after interaction, which is verified in Figure 8.

4. Conclusions and discussions

The determinant representation of DT for the GI system has been constructed in Thm1 and Thm2. By taking paired eigenvalue $\lambda_2 = \lambda_4^*$ and associated eigenfunctions in two-fold DT, then reduction condition $q^{[2]} = - (r^{[2]})^*$ is preserved and the BA solution and RW of the GI equation are obtained analytically from a periodic “seed”. In particular, the kernel of the one-fold DT of the GI system is one-dimensional, which is true for GI equation when the eigenvalue is pure imaginary. Furthermore, $q^{[n]}$ in eq. (33) is also a solution of the GI equation if the eigenvalues and eigenfunctions in n-fold DT are chosen according to reduction conditions given at the last paragraph of the section 2. Eight figures are plotted for the BA and RW of the GI equation to show their dynamical evolution on $(x-t)$ plane.

For the one-fold DT, the DNLSI and GI equation have same form as eq. (17). The coefficient of the first order $\lambda$ is a constant matrix and the other is a matrix of function of $(x,t)$ for GI, but it is in an opposite case for the DNLSI (see eq. (17) of reference [20]). This implies that $q^{[2]}$ eq. (11) of the GI is simpler than eq. (52) in reference [20] of the DNLSI. Furthermore, the RW of the GI is also simpler than the one (eq. (56) of reference [20]) of the DNLSI. Although the DT of the GI has also been given before in reference [21, 22], there are two new points in present paper: 1) the one-fold DT has a one-dimensional kernel; 2) the BA solution and RW are new, which are constructed from a non-zero “seed”, because Fan has just got solitons from zero “seed” by DT. Our results show several interesting differences between DNLSI and GI, which supports again that each of them deserves studying separately. Moreover, the appearance of BA and RW for GI also reminds us possibly that GI will be an interesting physical model in optical system under consideration of higher order nonlinear effects.

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Figure 1. The dynamical evolution of $|q|^2$ (time periodic breather) in eq. (42) on $(x-t)$ plane with specific parameters $\alpha_1 = \beta_1 = 0.5, c = 0.75$. The trajectory is a line $x = 0$. 

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Figure 2. The dynamical evolution of $|q|^2$ (space periodic breather) in eq.$(42)$ on $(x - t)$ plane with specific parameters $\alpha_1 = \beta_1 = 0.5, c = 1.2$. The trajectory is a line $t = 0$.

Figure 3. The dynamical evolution of solution $|q|^2$ in eq.$(42)$ for case 1 (a). It evolves periodically along a straight line with certain angle of $x$ axis and $t$ axis under specific parameters $\alpha_1 = 0.5, \beta_1 = 0.55, c = 0.75$.

Figure 4. The dynamical evolution of $|q_\text{rogue wave}|^2$ given by eq.$(45)$ on $(x - t)$ plane with specific parameters $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{1}{2}$. By letting $x \to \infty$, $t \to \infty$, so $|q_\text{rogue wave}|^2 \to 1$. The maximum amplitude of $|q_\text{rogue wave}|^2$ occurs at $t = 0$ and $x = 0$ and is equal to 9, and the minimum amplitude of $|q_\text{rogue wave}|^2$ occurs at $t = \pm \frac{\sqrt{15}}{10}$ and $x = \pm \frac{3\sqrt{15}}{10}$ and is equal to 0.
Figure 5. Contour plot of the wave amplitudes of $|q^{[2]}_{\text{rogue wave}}|^2$ in the $(x-t)$ plane is given by eq.(15) for $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{1}{2}$.

Figure 6. The dynamical evolution of $|q^{[2]}|^2$ in case 1(b) on $(x-t)$ plane with specific parameters $\alpha_1 = 0.65, \beta_1 = 0.5, c = 0.95$. It evolves periodically along a straight line on $(x-t)$ plane.

Figure 7. The dynamical evolution of periodic breather solution given by case 2(a) on $(x-t)$ plane with specific parameters $\alpha_1 = 0.5, \beta_1 = 0.4, c = 0.35, \alpha_3 = 0.6, \beta_3 = \frac{3\sqrt{3}}{10}$. This picture shows two breathers may parallelly propagate on $(x-t)$ plane.
Figure 8. The dynamical evolution of periodic breather solution given by case 2(b) on \((x - t)\) plane with specific parameters \(a = \frac{c^2}{2}, \alpha_1 = -0.5, \beta_1 = 0.5, \alpha_3 = 0.6, \beta_3 = 0.5, c = 0.75\). This picture shows the elastic interaction of the two breathers.