A REACTION-DIFFUSION-ADVECTION TWO-SPECIES COMPETITION SYSTEM WITH A FREE BOUNDARY IN HETEROGENEOUS ENVIRONMENT

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Abstract. In this paper, we investigate a reaction-diffusion-advection two-species competition system with a free boundary in heterogeneous environment. The primary aim is to study the impact of small advection terms and heterogeneous environment, which is on two species’ dynamics via a free boundary. The function \( m(x) \) represents heterogeneous environment, and it can satisfy positive everywhere condition or changeable sign condition. Firstly, on one hand, we provide long time behaviors of the solution in vanishing case when \( m(x) \) satisfies both conditions above; on the other hand, long time behaviors of the solution in spreading case are got when \( m(x) \) satisfies positive everywhere condition. Secondly, a spreading-vanishing dichotomy and several sufficient conditions through the initial data and the moving parameters are obtained to determine whether spreading or vanishing of two species happens when \( m(x) \) satisfies both conditions above. Furthermore, we derive estimates of spreading speed of the free boundary when \( m(x) \) satisfies positive everywhere condition and two species spreading occurs.

1. Introduction. In population dynamics, invasive species is a grave threat to bio-diversity conservation and the development of economy. Because of that, it is primarily important to forecast and stem biological invasion. A large number of researchers have been devoting to deeper comprehension of the process of controlling the invasion and spread of species.

Mathematically, a vast number of ecological phenomena, such as species invasion, can be described and researched in the form of differential equations. There has been some research on reaction-diffusion models to comprehend the nature of spreading. For this, we can refer to [3] and the references cited therein.

In recent years, a growing number of researchers have discovered that heterogeneous environment plays a significant role in evolution and so on. Particularly, environmentally heterogeneous reaction-diffusion models, studied for interactive biological species in a bounded isolated habitat, are especially suitable for depicting

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biogeographic characteristics of species’ interactions. In past several decades, it has
draw extensive attention of mathematicians and so on that environmental hetero-
genicity has important impact on the overall size of a single invasive species or the
coeexistence of multiple interactive species. There are various forms of environmental
heterogeneity, many of which lead to the spatial variation of coefficients.

In order to demonstrate how the environmental heterogeneity affects persistence
or extinction of species, Lou [32], Lam and Ni [28] investigated the following two-
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\[ u_t - d_1 \Delta u = u(a_1(r) - b_1(r)u - c_1(r)v), \quad t > 0, \quad 0 \leq r < h(t), \]
\[ v_t - d_2 \Delta v = v(a_2(r) - b_2(r)u - c_2(r)v), \quad t > 0, \quad 0 \leq r < h(t), \]
\[ u(t, 0) = v(t, 0) = 0, \quad t > 0, \]
\[ u(t, h(t)) = v(t, h(t)) = 0, \quad t > 0, \]
\[ h'(t) = -\mu[u_r(t, h(t)) + \beta v_r(t, h(t))], \quad t > 0, \]
\[ u(0, r) = u_0(r), \quad v(0, r) = v_0(r), \quad 0 \leq r \leq h_0 = h(0), \]

where \( u(x, t) \) and \( v(x, t) \) represent the population densities of two competing species
with diffusion rate \( \mu \) and \( \nu \), respectively. The function \( m(x) \) is their common in-
trinsic growth rate, or it represents that the environment is heterogeneous. Besides,
it is positive on favorable habitats and is negative on unfavorable ones. \( b \) and \( c \)
represent interspecific competition coefficients. The initial data \( u_0 \) and \( v_0 \) are non-
negative functions which are not identically equal to zero. The bounded habitat \( \Omega \)
in \( \mathbb{R}^N \) has smooth boundary \( \partial \Omega \). The zero Neumann boundary condition implies
that no individual can pass through the boundary of the habitat. \( n \) is the outward
unit normal vector on \( \partial \Omega \). Lou [32] studied the effect of diffusion and heterogeneous
environment on two species in weak competition case. Lam and Ni [28] also investi-
tigated interactive relations between diffusion and environmental heterogeneity in
weak competition case. They obtained global asymptotic stabilies of coexistence
steady states under diverse circumstances, and hence got a complete comprehen-
sion of dynamic change when one of interspecific competition coefficients is small.
More results with fixed boundary conditions can be found in the references cited in
[28, 32].

Recently, Du and Lin [9] proposed a different approach to establish a model which
depicts the spreading phenomenon for a single species by giving an assumption that
the spreading front is regarded as a free boundary, in which the key is that the
population density vanishes at the front. Near the spreading front, a mathematical

\[ \partial u \quad \partial v \]
\[ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega \times (0, \infty), \]
\[ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{in } \Omega, \]

where \( u(x, t) \) and \( v(x, t) \) represent the population densities of two competing species
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population density vanishes at the front. Near the spreading front, a mathematical
deductive process is to take into account the population loss, and Allen effect [2]
is under consideration. Following the approach in [9], there are different biological
results on Lotka-Volterra type two-species competition models, such as [20, 42].
In [46], Zhao and Wang studied the evolution of positive radially solution to the
following free boundary problem for a Lotka-Volterra type competition system in
heterogeneous environment

\[ \begin{cases} u_t - d_1 \Delta u = u(a_1(r) - b_1(r)u - c_1(r)v), & t > 0, \quad 0 \leq r < h(t), \\ v_t - d_2 \Delta v = v(a_2(r) - b_2(r)u - c_2(r)v), & t > 0, \quad 0 \leq r < h(t), \\ u(t, 0) = v(t, 0) = 0, & t > 0, \\ u(t, h(t)) = v(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu[u_r(t, h(t)) + \beta v_r(t, h(t))], & t > 0, \\ u(0, r) = u_0(r), \quad v(0, r) = v_0(r), & 0 \leq r \leq h_0 = h(0), \end{cases} \]
where $h_0$ represents the size of initial habitat, $r = h(t)$ is a free boundary, and it fits the widely-known one-phase Stefan type condition. The parameters $\mu$ and $\mu\beta$ represent the capacity for expanding into a new habitat of the two species $u$ and $v$, respectively. They also can be considered as the “moving parameter” of the free boundary. For $r = 0$ and $r = h(t)$, Neumann boundary condition and Dirichlet boundary condition were imposed, respectively. Besides, the initial datum $(u_0, v_0, h_0)$ satisfies

\[
\begin{aligned}
& h_0 > 0, \quad u_0(r), \quad v_0(r) \in C^2([0, h_0]), \quad u_0(r), \quad v_0(r) > 0 \text{ in } [0, h_0), \\
& u_0'(0) = v_0'(0) = u_0(h_0) = v_0(h_0) = 0.
\end{aligned}
\]

They got the asymptotic behaviors of two competing species when spreading happens, through a free boundary, and obtained several sufficient conditions for spreading or vanishing. Moreover, long time behaviors of the solution were provided when spreading happens.

In ecology field, in addition to the random dispersal, organisms can often make perception and response to local environmental signals by moving to favorable habitats, which usually depends on the integration of local living and nonliving factors such as food, climate, etc. In mathematic models, advection term is universally utilized to describe the directional movement mentioned above. In 2009, Maidana and Yang [33] investigated the spreading of West Nile Virus from New York to California. It was found that West Nile Virus the first time appeared in New York in summer, 1999. Next year, the wave front propagated 187km to the north and 1100km to the south. Therefore, they took in consideration of advection motion and demonstrated that bird advection made a significant factor in reducing mosquito bite rate. Recently, Averill [1] took account of the influence of intermediate advection on the dynamics of two-species competition system, and provided a specific advection intensity range for the coexistence of two competing species. Moreover, three different sorts of conversions from small advection to large one were demonstrated theoretically and numerically.

For free boundary problems about the influence of advection term, Gu et al. [16, 17] studied the single case as follows

\[
\begin{aligned}
& u_t = u_{xx} - \beta u_x + u(1 - u), \quad t > 0, \quad g(t) < x < h(t), \\
& u(t, g(t)) = 0, \quad g'(t) = -\mu u_x(t, g(t)), \quad t > 0, \\
& u(t, h(t)) = 0, \quad h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
& u(0, x) = u_0(x), \quad -h_0 \leq x \leq h_0, \\
& g(0) = h(0) = h_0,
\end{aligned}
\]

where $\beta$ is a positive advection coefficient that measures the tendency of slanted motion of $u$. The initial datum $(u_0, -h_0, h_0)$ satisfies

\[
\begin{aligned}
& h_0 > 0, \quad u_0(x) \in C^2([-h_0, h_0]), \quad u_0(x) > 0 \text{ in } (-h_0, h_0), \\
& u_0(-h_0) = u_0(h_0) = 0.
\end{aligned}
\]

In their paper, a spreading-vanishing dichotomy and its criteria depending on initial value were given, and estimates of the asymptotic spreading speeds were obtained when the advection coefficient $\beta$ is small. Subsequently, Kaneko and Matsumuza [24] took this problem one step further. They used general nonlinearity $f(u)$ to replace $u(1-u)$, and obtained sharp estimates of spreading speeds for three types of $f(u)$, which is monostable, bistable and combustion.
In 2015, Gu et al. [19] studied long-time behaviors of the solution to the problem under the conditions that the reaction term is Fisher-KPP type and the advection coefficient \( \beta \in (0, \infty) \). They discovered a critical value \( \beta^* \) which satisfies \( \beta^* > c_0 \), where \( c_0 = 2\sqrt{T(0)} \) is the minimum speed of the travelling waves of the problem

\[
\begin{align*}
q''(z) - cq'(z) + f(q) &= 0, & -\infty < z < \infty, \\
q(-\infty) &= 0, & q(\infty) = 1, \\
q(0) &= \frac{1}{2}, & q'(z) > 0, & -\infty < z < \infty.
\end{align*}
\]

They gave the results: when \( \beta \geq \beta^* \), only vanishing occurs; when \( \beta \in [c_0, \beta^*) \), a trichotomy was derived. They also provided sufficient conditions for different behaviors. Zhao and Wang [47] investigated a reaction-diffusion-advection equation in one-dimensional heterogeneous environment.

In 2019, Duan and Zhang [14, 15] studied the following reaction-diffusion-advection competition system

\[
\begin{align*}
 u_t &= u_{xx} - \beta_1 u_x + u(1 - u - cv), & t > 0, & g(t) < x < h(t), \\
v_t &= Dv_{xx} - \beta_2 v_x + rv(1 - v - bu), & t > 0, & g(t) < x < h(t), \\
u(t, g(t)) &= u(t, h(t)) = 0, & t > 0, \\
v(t, g(t)) &= v(t, h(t)) = 0, & t > 0, \\
g'(t) &= -\mu[u_x(t, g(t)) + rv_x(t, g(t))], & t > 0, \\
h'(t) &= -\mu[u_x(t, h(t)) + rv_x(t, h(t))], & t > 0, \\
u(0, x) &= u_0(x), & v(0, x) = v_0(x), & -h_0 \leq x \leq h_0, \\
-g(0) &= h(0) = h_0,
\end{align*}
\]

where \( D, r, c, b, \mu, \rho, \beta_1, \beta_2 > 0 \). The initial datum \((u_0, v_0, -h_0, h_0)\) satisfies

\[
\begin{align*}
h_0 > 0, & u_0(x), & v_0(x) \in C^2([-h_0, h_0]), \\
u_0(x) > 0 \text{ in } (-h_0, h_0), \\
u_0(-h_0) = v_0(-h_0) = u_0(h_0) = v_0(h_0) = 0.
\end{align*}
\]

They draw complete conclusions: advection terms and double free boundaries have an effect on four aspects, such as long time behaviors of the solution, a spreading-vanishing dichotomy, criteria governing spreading or vanishing, and spreading speeds in different cases of \( 0 < b, c < 1 \) [14] and \( 0 < b < 1 \leq c \) (or \( 0 < c < 1 \leq b \)) [15]. More results with the effect of advection terms can be found in [4, 5, 18, 30, 44] and references cited therein.

Motivated by above works, in this paper, we consider the following free boundary problem (P), which is a reaction-diffusion-advection two-species competition system in one-dimensional heterogeneous environment

\[
\begin{align*}
 u_t &= u_{xx} - \beta_1 u_x + u(m(x) - u - cv), & t > 0, & 0 < x < h(t), \\
v_t &= Dv_{xx} - \beta_2 v_x + v(m(x) - v - bu), & t > 0, & 0 < x < h(t), \\
u_x(t, 0) &= u(t, h(t)) = 0, & t > 0, \\
v_x(t, 0) &= v(t, h(t)) = 0, & t > 0, \\
h'(t) &= -\mu[u_x(t, h(t)) + rv_x(t, h(t))], & t > 0, \\
u(0, x) &= u_0(x), & v(0, x) = v_0(x), & 0 \leq x \leq h_0, \\
h(0) &= h_0,
\end{align*}
\]

where \( D, c, b, \mu, \rho \) are positive constants, and they represent the same meaning as above. \( \beta_1 \) and \( \beta_2 \) are positive advection coefficients that measure the tendency
of slanted motion of $u$ and $v$, respectively. The function $m(x)$ not only shows the intrinsic growth rate of the species $u$ and $v$, but also represents that the environment is heterogeneous.

Throughout this paper, $m(x)$, $\beta_1$ and $\beta_2$ satisfy

(\textbf{H0}) $m(x)$ is positive somewhere in $[0, \infty)$, and $m(x) \in C^1([0, \infty)) \cap L^\infty([0, \infty))$, $m'(x) \in L^\infty([0, \infty))$, $0 < \beta_1 < 2\sqrt{\|m\|_{L^\infty([0, \infty))}}$, $0 < \beta_2 < 2\sqrt{|D|\|m\|_{L^\infty([0, \infty))}}$.

The initial datum $(u_0, v_0, h_0)$ satisfies

\[
\begin{cases}
h_0 > 0, \quad u_0(x), \quad v_0(x) \in C^2([0, h_0]), \quad u_0(x), \quad v_0(x) > 0 \text{ in } [0, h_0), \\
u'_0(0) = v'_0(0) = u_0(h_0) = v_0(h_0) = 0.
\end{cases}
\]  

From a biological standpoint, the problem (P) with (1) can be applied to explain how the two new species $(u(t, x), v(t, x))$ invade when they are in an interval $0 < x < h_0$ initially. Both species tend to invade a new habitat further. The free boundary $x = h(t)$ means spreading front, which is proportionate to normalized gradients of population densities at expanding front, i.e., $h'(t) = -\mu(u_x(t, h(t)) + \rho v_x(t, h(t)))$. More results with the Stefan condition can be found in [2, 22, 23, 31].

Besides, $m(x)$ can satisfy the following different conditions.

(\textbf{H1}) $m_0 := \inf_{x \geq 0} m(x) > 0$, $m_1 := \sup_{x \geq 0} m(x) > 0$.

(\textbf{H2}) There is a constant $\rho > 0$ and two sequences $\{x_n\}$, $\{y_n\}$ which satisfies $y_n > x_n > 0$, and $y_n - x_n \to \infty$ as $n \to \infty$, such that $m(x) \geq \rho$ for $x_n \leq x \leq y_n$.

Throughout this paper, we investigate nonnegative solutions in small advection case. Besides, in hypothesis of (H1), the following cases are considered (see [34, 38]).

(\textbf{A1}) weak competition: $0 < b < c < \frac{m_0}{m_1}$, $0 < \beta_1 < 2\sqrt{m_0 - cm_1}$, $0 < \beta_2 < 2\sqrt{(m_0 - bm_1)D}$.

(\textbf{A2}) u is an inferior competitor and v is a superior competitor: $0 < b < \frac{m_0}{m_1}$, $c > \frac{m_0}{m_1}$, $0 < \beta_1 < 2\sqrt{m_0}$, $0 < \beta_2 < 2\sqrt{(m_0 - bm_1)D}$.

(\textbf{A3}) u is a superior competitor and v is an inferior competitor: $0 < c < \frac{m_0}{m_1}$, $b > \frac{m_0}{m_1}$, $0 < \beta_1 < 2\sqrt{m_0 - cm_1}$, $0 < \beta_2 < 2\sqrt{m_0D}$.

(A2) and (A3) are called weak-strong competition case. Besides, we give some notations and definitions which are frequently used in what follows.

$h_\infty := \lim_{t \to \infty} h(t)$; \textbf{Spreading of the two species}: $h_\infty = \infty$; \textbf{Vanishing of the two species}: $h_\infty < \infty$.

The main purpose of the present paper is to analyze the effect of advection terms and heterogeneous environment on two species’ dynamics via a free boundary. The skills of handling two advection coefficients and heterogeneous environment are more complex than single species case.

Firstly, when $m(x)$ satisfies (H1) or (H2), we derive long time behaviors of the solution in vanishing case. When $m(x)$ satisfies (H1), we obtain long time behaviors of the solution in spreading case. On one hand, if (A1) holds, we demonstrate that competitors $u$ and $v$ coexist in the long run. The method is an iterative scheme to gradually construct better supersolutions and subsolutions to squeeze the solution. On the other hand, if (A2) holds, we prove $(u, v) \to (0, V)$ as $t \to \infty$. Similarly, if (A3) holds, we get $(u, v) \to (U, 0)$ as $t \to \infty$. We compare the PDE system with corresponding ODE system by using the comparison principle.
Secondly, when \( m(x) \) satisfies (H1) or (H2), we get a criterion which governs spreading or vanishing, with the threshold \( h_\ast \). Then we obtain a spreading-vanishing dichotomy immediately. On this basis, by utilizing the comparison principles, several sufficient conditions for spreading or vanishing through \( h_0 \) and \( \mu \) are given. In order to get over the difficulties caused by advection terms and heterogeneous environment, we calculate the new problem’s principal eigenvalue, and utilize it to get relevant sharp threshold which is related to spreading or vanishing of two species. Simultaneously, we also discover that both advection coefficients and heterogeneous environment can affect the sharp threshold.

Furthermore, when \( m(x) \) satisfies (H1), for asymptotic spreading speed, we conclude that both advection terms and heterogeneous environment can affect it. The tool we utilize is modifying single species case.

When studying the eigenvalue problem, due to the heterogeneous environment represented by \( m(x) \), we can’t calculate the threshold explicitly, which determines the sign of principal eigenvalue. Therefore, when studying the monotonicity of eigenvalue on domain, we can only deal with the case of a single free boundary. In the case of replacing left boundary \( x = 0 \) by free boundary \( x = g(t) \), i.e., double free boundaries, we leave it as a future research.

At the end of the introduction, we mention some related research. In weak competition case, Guo and Wu [20] investigated the dynamics of two competing species system with identical single free boundary. Later, Wu [42] derived estimates of spreading speed and existence of traveling wave solutions. Then, Wu [43] studied a weak competition system, which has different free boundaries, and minimal habitat size for spreading was got. In weak-strong competition case, Guo and Wu [21] investigated a competition system with two free boundaries. The native species’ systems experiencing diffusion and growth in \( \mathbb{R}^N \) were studied in [10] and [38]. Later on, Du et al. [13] investigated a semi-wave problem and obtained the system’s asymptotic spreading speed. Wang and Zhao [40] gave two different competition models with same free boundary, and the two models have different boundary conditions respectively at \( x = 0 \): one is Neumann boundary and the other is Dirichlet type. Complete descriptions of spreading-vanishing dichotomy were provided for both weak and weak-strong competition case. Some predator-prey models and mutualistic models also used such free boundary conditions, such as [26, 29, 30, 36, 39, 41, 44, 45, 49]. More results can be found in the references cited therein.

If \( v \) is absent, Du and Guo investigated logistic models in [6] and [7]. Du et al. [8] investigated a model with time-periodic environment. Peng and Zhao [35] studied the seasonal succession case. For more mathematical problems with free boundary conditions, we refer to [11, 25] and the references cited therein.

The rest of this paper is organized as follows. In Section 2, we demonstrate existence and uniqueness of a global solution to the problem (P) with (1). In Section 3, we prepare several preliminaries which include an eigenvalue problem and the comparison principles. In Section 4, we obtain long time behaviors of the solution. In Section 5, we provide criteria governing spreading or vanishing. In Section 6, we give estimates of spreading speed of the free boundary when spreading occurs.

2. **Existence and uniqueness.** In this section, we obtain global existence and uniqueness of the solution to the problem (P) with (1).
We give two lemmas first.

**Lemma 2.1.** *(Local existence and uniqueness)* The problem \((P)\) with \((1)\) admits a unique local solution

\[
(u, v, h) \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_T) \times C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_T) \times C^{1+\frac{\alpha}{2}}([0, T]),
\]

for any \(\alpha \in (0, 1)\) and some \(T > 0\), where \(\Omega_T := \{(t, x) : 0 < t \leq T, 0 \leq x \leq h(t)\}\).

**Proof.** Due to heterogeneous environment, by modifying Theorem 1.1 of [37], we omit the process. \(\square\)

**Lemma 2.2.** Let \((u, v, h)\) be a solution to the problem \((P)\) with \((1)\) for \(t \in [0, T]\) for some \(T > 0\), then

\[
0 < u(t, x) \leq \max\{\|m\|_{L^{\infty}([0, \infty])}, \|u_0\|_{L^{\infty}([0, h_0])}\} \text{ for } t \in [0, T], \ x \in (0, h(t)), \quad (2)
\]

\[
0 < v(t, x) \leq \max\{\|m\|_{L^{\infty}([0, \infty])}, \|v_0\|_{L^{\infty}([0, h_0])}\} \text{ for } t \in [0, T], \ x \in (0, h(t)), \quad (3)
\]

\[
0 < h'(t) \leq \mu \Lambda \text{ for } t \in [0, T], \quad (4)
\]

where \(\Lambda > 0\) depends only on \(D, \rho, \beta_1, \beta_2, h_0, \|m\|_{L^{\infty}([0, \infty])}, \|u_0\|_{L^{\infty}([0, h_0])}, \|v_0\|_{L^{\infty}([0, h_0])}, \|u_0\|_{C^1([0, h_0])} \text{ and } \|v_0\|_{C^1([0, h_0])}\).

**Proof.** The proof is similar to Lemma 2.3 of [14], so we omit it. \(\square\)

Now, we give Theorem 2.3.

**Theorem 2.3.** The problem \((P)\) with \((1)\) admits a unique global solution

\[
(u, v) \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega) \times C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega) \times C^{1+\frac{\alpha}{2}}([0, \infty]),
\]

for any \(\alpha \in (0, 1)\), where \(\Omega := \{(t, x) : t > 0, 0 \leq x \leq h(t)\}\).

**Proof.** By Lemma 2.1, we define \(T_{\text{max}} > 0\) as the maximal existence time of the solution. On the contrary, assuming \(T_{\text{max}} < \infty\). Then by Lemma 2.2, there exists a positive constant \(K\), which does not depend on \(T_{\text{max}}\) such that

\[
0 \leq u(t, x), \ v(t, x), \ h'(t) \leq K \text{ for all } t \in [0, T_{\text{max}}], \ x \in [0, h(t)].
\]

In particular,

\[
h_0 \leq h(t) \leq h_0 + Kt \text{ for all } t \in [0, T_{\text{max}}).
\]

Fix \(\epsilon \in (0, T_{\text{max}})\) and \(\Lambda > T_{\text{max}}\), according to the standard regularity theory, there exists \(K' > 0\), which depends only on \(\epsilon, \Lambda\) and \(K\) such that

\[
\|u(t, \cdot)\|_{C^2([0, h(t)])}, \ \|v(t, \cdot)\|_{C^2([0, h(t)])} \leq K', \ \forall t \in [\epsilon, T_{\text{max}}).
\]

Due to the proof of Lemma 2.1, there exists a constant \(\tau > 0\) depending only on \(K\) and \(K'\) such that the solution to the problem \((P)\) with \((1)\) with any initial time \(t \in [\epsilon, T_{\text{max}}]\) can be extended to the interval \([t, t + \tau]\) uniquely. Thus, the solution with the initial time \(T_{\text{max}} - \frac{\tau}{2}\) can be extended to the time \(T_{\text{max}} + \frac{\tau}{2}\) uniquely, which contradicts with the definition of \(T_{\text{max}}\). Therefore, \(T_{\text{max}} = \infty\). Thus the proof of Theorem 2.3 is ended. \(\square\)

3. **Preliminaries.** In this section, we will give an eigenvalue problem and some comparison principles, which will be utilized in the following part.
3.1. An eigenvalue problem. In this subsection, we study an eigenvalue problem, and the properties of its principal eigenvalue are obtained.

Consider the following eigenvalue problem (P0)
\[
\begin{cases}
-Du_{xx} + \beta u_x - m(x)u = \lambda u, & 0 < x < L, \\
u_x(0) = 0, & u(L) = 0.
\end{cases}
\]

Let \( \lambda_1(D, \beta, m(x), L) \) be the principal eigenvalue of the problem (P0). Then it is well-known that \( \lambda_1(D, \beta, m(x), L) \) exists uniquely, and the associated eigenfunction, denoted by \( \phi_1 \), can be chosen positive in \([0, L]\) and normalized by \( \|\phi_1\|_{L^2([0, L])} = 1 \). According to the variational method, \( \lambda_1 \) has the following form (for detailed derivation, refer to [3])
\[
\lambda_1 = \inf_{\phi \in S(L)} \left\{ \frac{\int_0^L e^{-\frac{\beta}{m}(D\phi_x^2 - m(x)\phi^2)}dx}{\int_0^L e^{-\frac{\beta}{m}\phi^2}dx} \right\}
= \inf_{\phi \in S(L)} \left\{ \frac{\int_0^L (\phi_x^2 + \beta \phi \phi_x \phi - \int_0^L m(x)\phi^2)dx}{\int_0^L \phi^2dx} \right\},
\]
where \( S(L) = \{ \phi \in W^{1, 2}([0, L]) : \phi'(0) = 0, \phi(L) = 0 \} \). Then we get the following theorem.

**Theorem 3.1.** For (5), fix \( D, \beta, m(x) \). When \( m(x) \) satisfies (H1) or (H2), we have
(i) \( \lambda_1(D, \beta, m(x), L) \) is strictly monotone decreasing in \( L \).
(ii) \( \lim_{L \to 0^+} \lambda_1(D, \beta, m(x), L) = \infty \).
(iii) \( \lim_{L \to \infty} \lambda_1(D, \beta, m(x), L) < 0 \).

**Proof.** The proof of (i) is similar to Theorem 3.2 of [48] with some minor modifications. Here, we omit it.

Now we prove (ii) and (iii). When \( m(x) \) satisfies (H1), from (5), it is easy to obtain that \( \lambda_1 \) is monotone decreasing with respect to \( m(x) \) in the following sense
\[
\lambda_1(D, \beta, m(x), L) \geq \lambda_1(D, \beta, m(x), L) \geq \lambda_1(D, \beta, \bar{m}(x), L)
\]
for any \( m(x) \leq m(x) \leq \bar{m}(x) \). In particular, we get
\[
\lambda_1(D, \beta, m_0, L) \geq \lambda_1(D, \beta, m(x), L) \geq \lambda_1(D, \beta, m_1, L),
\]
from which we can obtain the desired results directly.

When \( m(x) \) satisfies (H2), since \( m(x) \in C^1([0, \infty)) \cap L^\infty([0, \infty)) \), we get (ii) also holds by using the same argument as above. For (iii), taking a function \( \phi_n \in H^1((0, y_n)) \) such that \( \phi_n(x) = 0 \) in \([0, x_n]\), \( \phi_n(x) = x - x_n \) in \([x_n, x_n + 1]\), \( \phi_n(x) = 1 \) in \([x_n + 1, y_n - 1]\) and \( \phi_n(x) = y_n - x \) in \([y_n - 1, y_n]\). Then
\[
\phi_n'(0) = \phi_n(y_n) = 0, \quad \int_0^{y_n} (\phi_n'(x))^2dx = 2, \quad \int_0^{y_n} \phi_n'(x)\phi_n(x)dx = 0, \\
\int_0^{y_n} \phi_n^2(x)dx = y_n - x_n - \frac{4}{3}, \\
\int_0^{y_n} m(x)\phi_n^2(x)dx = \int_{x_n}^{y_n} m(x)\phi_n^2(x)dx \geq \rho(y_n - x_n - \frac{4}{3}).
\]
Hence, we get
\[
\lim_{L \to \infty} \lambda_1(D, \beta, m(x), L) < \lambda_1(D, \beta, m(x), y_n) \leq \frac{2D - \rho(y_n - x_n - \frac{\lambda}{3})}{y_n - x_n - \frac{\lambda}{3}} \to -\rho < 0,
\]
as \(n \to \infty\). Thus the proof is completed. \(\square\)

As a sequence of the above theorem, we derive the following corollary.

**Corollary 1.** For (5), fix \(D, \beta, m(x)\). When \(m(x)\) satisfies (H1) or (H2), there exists \(L^* = L^*(D, \beta, m(x)) > 0\) such that
\[
\begin{align*}
\lambda_1(D, \beta, m(x), L) &> 0 \quad \text{if } 0 < L < L^*; \\
\lambda_1(D, \beta, m(x), L^*) &< 0;
\end{align*}
\]
(6)

If we replace \(L\) in (5) by \(h(t)\), then the following consequence holds true.

**Theorem 3.2.** For (3), fix \(D, \beta, m(x)\). When \(m(x)\) satisfies (H1) or (H2), \(\lambda_1(D, \beta, m(x), h(t))\) is strictly monotone decreasing of \(t\), it is equivalent to \(\lambda_1(D, \beta, m(x), h(t))\) being a strictly monotone decreasing function of \(h(t)\).

### 3.2. The comparison principles.

**Lemma 3.3.** (Comparison Principle I) (see [6], Lemma 2.6) Suppose \(h_1, h_2 \in C^1([0, \infty)), u \in C(D_1) \cap C^{1,2}(D_1)\) with \(D_1 = \{(t, x) \in \mathbb{R}^2 : t > 0, \ 0 < x < h(t)\}\), \(\bar{u} \in C(D_2) \cap C^{1,2}(D_2)\) with \(D_2 = \{(t, x) \in \mathbb{R}^2 : t > 0, \ 0 < x < \bar{h}(t)\}\), and
\[
\begin{cases}
u_t \leq Du_{xx} \ - \ \beta u_x + u(m(x) - u) & \text{in } D_1, \\
u_x(t, 0) \geq 0, & t > 0, \\
u(t, h(t)) = 0, & t > 0, \\
h'(t) \leq -\mu u_x(t, h(t)), & t > 0.
\end{cases}
\]
(7)

If \(\bar{u}(0, x) \geq u(0, x)\) for \(x \in [0, h(0)]\), \(\bar{h}(0) \geq h(0)\), then \(\bar{h}(t) \geq h(t)\) for \(t \geq 0\), \(\bar{u}(t, x) \geq u(t, x)\) in \(D_1\).

**Lemma 3.4.** (Comparison Principle II) (see [9, 20]) Suppose \(h \in C^1([0, \infty)), w_1, w_2 \in C(D_1) \cap C^{1,2}(D_1)\) with \(D_1 = \{(t, x) \in \mathbb{R}^2 : t > 0, \ 0 < x < h(t)\}\), and
\[
\begin{cases}
w_{1,t} \geq w_{1,xx} - \beta_1 w_{1,x} + w_1(m(x) - w_1) & \text{in } D_1, \\
w_{2,t} \geq Dw_{2,xx} - \beta_2 w_{2,x} + w_2(m(x) - w_2) & \text{in } D_1, \\
w_{i,x}(t, 0) \leq 0, & w_i(t, h(t)) = 0, & t > 0, \ i = 1, 2, \\
h'(t) \leq -\mu w_{i,x}(t, h(t)), & t > 0, \ i = 1, 2.
\end{cases}
\]
(9)

If \(w_1(0, x) \geq w_0(x), w_2(0, x) \geq v_0(x)\) for \(x \in [0, h_0]\), \(\bar{h}(0) \geq h_0\), then the solution \((u, v, h)\) to the problem (P) with (1) satisfies: \(h(t) \geq \bar{h}(t)\) for \(t \geq 0\), \(w_1(t, x) \geq u(t, x)\), and \(w_2(t, x) \geq v(t, x)\) for \(t \geq 0, \ 0 \leq x \leq h(t)\).

**Remark 1.** Lemma 3.4 still holds if the left boundary condition \(w_{i,x}(t, 0) \leq 0\) \((i = 1, 2)\) is replaced by \(w_1(t, 0) \geq u(t, 0), w_2(t, 0) \geq v(t, 0)\).
Motivated by Lemma 4.1 of [48], a contradiction argument is used. Suppose they will extinct eventually.

\[ u \leq u_{xx} - \beta_1 u_x + u(m(x) - u - c\bar{u}) \quad \text{in } D_1, \]
\[ \bar{u}_t \geq \bar{u}_{xx} - \beta_1 \bar{u}_x + \bar{u}(m(x) - \bar{u} - c\bar{u}) \quad \text{in } D_2, \]
\[ v \leq Dv_{xx} - \beta_2 v_x + v(m(x) - v - b\bar{v}) \quad \text{in } D_1, \]
\[ \bar{v}_t \geq D\bar{v}_{xx} - \beta_2 \bar{v}_x + \bar{v}(m(x) - \bar{v} - b\bar{v}) \quad \text{in } D_2, \]
\[ u_{x}(t, 0) = \bar{u}_{x}(t, 0) = u(t, \bar{h}(t)) = \bar{u}(t, \bar{h}(t)) = 0, \quad t > 0, \]
\[ v_{x}(t, 0) = \bar{v}_{x}(t, 0) = v(t, \bar{h}(t)) = \bar{v}(t, \bar{h}(t)) = 0, \quad t > 0, \]
\[ \bar{h}'(t) \geq -\mu(1 + \rho)\bar{u}_x(t, \bar{h}(t)), \quad \bar{h}'(t) \geq -\mu(1 + \rho)\bar{v}_x(t, \bar{h}(t)), \quad t > 0, \]
\[ \bar{h}'(t) \leq -\mu(1 + \rho)u_x(t, \bar{h}(t)), \quad \bar{h}'(t) \leq -\mu(1 + \rho)v_x(t, \bar{h}(t)), \quad t > 0. \]

If \( u(0, x) \geq u_0(x) \geq \bar{u}(0, x) \geq v_0(x) \geq v(0, x) \) for \( x \in [0, \bar{h}(0)] \), \( \bar{h}(0) \geq h_0 \geq \bar{h}(0) \), then the solution \( (u, v, h) \) to the problem (P) with (1) satisfies: \( \bar{h}(t) \geq h(t) \geq \bar{h}(t) \) for \( t \geq 0 \), \( \bar{u}(t, x) \geq u(t, x) \geq \bar{u}(t, x) \), and \( \bar{v}(t, x) \geq v(t, x) \geq v(t, x) \) in \( D_1 \).

**Lemma 3.6.** (Comparison Principle IV) (see [12], Lemma 2.1) Suppose that \( (0, L) \) is a bounded domain in \( \mathbb{R} \), \( \alpha(x) \) and \( \gamma(x) \) are continuous functions in \( (0, L) \) with \( \|\alpha\|_{L^\infty} < \infty, \gamma(x) \geq 0 \) and \( \gamma(x) \neq 0 \). Let \( u, \bar{u} \in C^2((0, L)) \) be positive in \( (0, L) \) and satisfy
\[ -D_{xx} u + \beta u_x - \alpha(x)u + \gamma(x)g(u) \leq 0, \]
\[ -D_{xx} \bar{u} + \beta \bar{u}_x - \alpha(x)\bar{u} + \gamma(x)g(\bar{u}) \geq 0, \quad x \in (0, L), \]
and
\[ \limsup_{x \to 0} u - \bar{u} \leq 0, \quad \limsup_{x \to L} u - \bar{u} \leq 0, \]
where \( g(u) \) is continuous and \( \frac{g(u)}{u} \) is strictly increasing for \( u \) in the range \( \inf_{(0, L)} \{ u, \bar{u} \} < u < \sup_{(0, L)} \{ u, \bar{u} \} \).

Then \( u \leq \bar{u} \) in \( (0, L) \).

4. Long time behaviors of the solution. In this section, we will discuss long time behaviors of the solution. By (4), the free boundary \( x = h(t) \) is monotone increasing. Therefore, we have \( h_\infty \in (0, \infty) \).

4.1. Vanishing case (\( h_\infty < \infty \)).

**Theorem 4.1.** Let \( (u, v, h) \) be a solution to the problem (P) with (1), \( h_\infty < \infty \). When \( m(x) \) satisfies (H1) or (H2), we have
\[ \lim_{t \to \infty} \|u(t, \cdot)\|_{C(\{0, h(t)\})} = 0, \quad \lim_{t \to \infty} \|v(t, \cdot)\|_{C(\{0, h(t)\})} = 0. \]

This result indicates that: if the two species can not spread into the infinity, then they will extinct eventually.

**Proof.** Motivated by Lemma 4.1 of [48], a contradiction argument is used. Suppose
\[ \delta := \limsup_{t \to \infty} \|u(t, \cdot)\|_{C(\{0, h(t)\})} > 0, \]
then there exists a sequence \( \{(t_k, x_k)\} \in (0, \infty) \times [0, h(t)) \) with \( t_k \to \infty \) as \( k \to \infty \) such that
\[
 u(t_k, x_k) \geq \frac{\delta}{2} \quad \text{for all } k \in \mathbb{N}.
\]
Since \( 0 \leq x_k < h(t) < h_\infty < \infty \), if necessary, passing to a subsequence, we get \( x_k \to x_0 \in [0, h_\infty) \) as \( k \to \infty \).

Define \( u_k(t, x) = u(t + t_k, x) \), \( v_k(t, x) = v(t + t_k, x) \) in \( G_k := \{(t, x) : t \in (-t_k, \infty), x \in [0, h(t + t_k))\} \). As \( u \) is uniformly bounded for any \( t \geq 0 \), \( \{(u_k, v_k)\} \) has a subsequence \( \{(u_{k_i}, v_{k_i})\} \) such that
\[
\|(u_{k_i}, v_{k_i}) - (\tilde{u}, \tilde{v})\|_{C^{1,2}(G_{k_i}) \times C^{1,2}(G_{k_i})} \to 0
\]
as \( i \to \infty \), and \( (\tilde{u}, \tilde{v}) \) satisfies
\[
\begin{align*}
\tilde{u}_t &= \tilde{u}_{xx} - \beta_1 \tilde{u}_x + \tilde{u}(m(x) - \tilde{u} - c\tilde{v}), \quad -\infty < t < \infty, \quad 0 < x < h_\infty, \\
\tilde{v}_t &= D\tilde{v}_{xx} - \beta_2 \tilde{v}_x + \tilde{v}(m(x) - \tilde{v} - b\tilde{u}), \quad -\infty < t < \infty, \quad 0 < x < h_\infty.
\end{align*}
\]
Since
\[
\tilde{u}(0, x_0) = \lim_{k_i \to \infty} u_{k_i}(0, x_{k_i}) = \lim_{k_i \to \infty} u(t_{k_i}, x_{k_i}) \geq \frac{\delta}{2},
\]
according to the strong maximum principle, \( \tilde{u} > 0 \) in \( (-\infty, \infty) \times [0, h_\infty) \). As
\[
\tilde{u}(0, h_\infty) = \lim_{k_i \to \infty} u_{k_i}(0, h(t_{k_i})) = \lim_{k_i \to \infty} u(t_{k_i}, h(t_{k_i})) = 0,
\]
by Hopf’s lemma, \( \tilde{u}_x(0, h_\infty) \leq -\sigma < 0 \) for some \( \sigma > 0 \), then
\[
u_x(t_{k_i}, h(t_{k_i})) = u_{k_i, x}(0, h(t_{k_i})) \leq -\frac{\sigma}{2} < 0,
\]
for all large \( k_i \), therefore,
\[
h'(t_{k_i}) = -\mu \nu_x(t_{k_i}, h(t_{k_i})) \geq \frac{\mu \sigma}{2} > 0,
\]
then we derive \( h_\infty = \infty \), which is a contradiction to \( h_\infty < \infty \). Thus, we have finished the proof. \( \square \)

4.2. Spreading case \( (h_\infty = \infty) \).

**Theorem 4.2.** Let \((u, v, h)\) be a solution to the problem \((P)\) with \((1)\), \( h_\infty = \infty \). When \( m(x) \) satisfies \((H1)\), we have the following conclusions.

(i) If \((A1)\) holds, then

\[
\begin{align*}
\frac{m_0 - cm_1}{1 - bc} &\leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \frac{m_1 - cm_0}{1 - bc}, \\
\frac{m_0 - bm_1}{1 - bc} &\leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \frac{m_1 - bm_0}{1 - bc},
\end{align*}
\]
uniformly in any compact subset of \([0, \infty)\).

(ii) If \((A2)\) holds, then \( \lim_{t \to \infty} u(t, x) = 0, \lim_{t \to \infty} v(t, x) = V(x) \) uniformly in any compact subset of \([0, \infty)\). Where \( V(x) \) is the unique positive solution of

\[
\begin{align*}
- DV_{xx} + \beta_2 V_x &= V(m(x) - V), \quad 0 < x < \infty, \\
V_x(0) &= 0.
\end{align*}
\]
(iii) If (A3) holds, then \( \lim_{t \to \infty} v(t, x) = 0, \lim_{t \to \infty} u(t, x) = U(x) \) uniformly in any compact subset of \([0, \infty)\). Where \( U(x) \) is the unique positive solution of
\[
\begin{cases}
-U_{xx} + \beta_1 U_x = U(m(x) - U), \quad 0 < x < \infty, \\
U_x(0) = 0.
\end{cases}
\] (15)

Before giving the proof of Theorem 4.2, some lemmas are presented first.

**Lemma 4.3.** Let \((u, v, h)\) be a solution to the problem (P) with (1), \( h_\infty = \infty \). When \( m(x) \) satisfies (H1), and (A1) holds, then
(i) \( \limsup_{t \to \infty} u(t, x) \leq m_1 \) and \( \limsup_{t \to \infty} v(t, x) \leq m_1 \) uniformly in \([0, \infty)\).
(ii) \( \liminf_{t \to \infty} u(t, x) \geq m_0 - cm_1 \) and \( \liminf_{t \to \infty} v(t, x) \geq m_0 - bm_1 \) uniformly in any compact subset of \([0, \infty)\).

**Proof.** The process is a minor modification of Lemma 4.1 of [20], so we omit the process. \( \square \)

**Lemma 4.4.** Let \((u, v, h)\) be a solution to the problem (P) with (1), \( h_\infty = \infty \). When \( m(x) \) satisfies (H1), and (A1) holds, then
(i) Consider sequences \( \{\bar{u}_n\}_{n \in \mathbb{N}} \) and \( \{\bar{v}_n\}_{n \in \mathbb{N}} \)
\[
(\bar{u}_1, \bar{v}_1) := (m_1, m_0 - bm_1), \quad (\bar{u}_{n+1}, \bar{v}_{n+1}) := (m_1 - cv_n, m_0 - b(m_1 - cv_n)).
\]
Then we derive \( \bar{u}_n > \bar{u}_{n+1} > 0, \) and \( \bar{v}_n < \bar{v}_{n+1} < 1 \) for all \( n \in \mathbb{N} \). Moreover,
\[
\lim_{n \to \infty} \bar{u}_n = \frac{m_1 - cm_0}{1 - bc}, \quad \lim_{n \to \infty} \bar{v}_n = \frac{m_0 - bm_1}{1 - bc}.
\]
(ii) Consider sequences \( \{u_n\}_{n \in \mathbb{N}} \) and \( \{v_n\}_{n \in \mathbb{N}} \)
\[
(u_1, v_1) := (m_0 - cm_1, m_1), \quad (u_{n+1}, v_{n+1}) := (m_0 - c(m_1 - bu_n), m_1 - bu_n).
\]
Then we derive \( u_n < u_{n+1} < 1, \) and \( v_n > v_{n+1} > 0 \) for all \( n \in \mathbb{N} \). Moreover,
\[
\lim_{n \to \infty} u_n = \frac{m_0 - cm_1}{1 - bc}, \quad \lim_{n \to \infty} v_n = \frac{m_1 - bm_0}{1 - bc}.
\]

**Proof.** The process is similar to Lemma 4.2 of [20], so we omit the details. \( \square \)

**Lemma 4.5.** Let \((u, v, h)\) be a solution to the problem (P) with (1), \( h_\infty = \infty \). When \( m(x) \) satisfies (H1), and (A1) holds, then
\[
\begin{align*}
\underline{u}_2 &\leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_2, \\
\underline{v}_2 &\leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_2,
\end{align*}
\]
uniformly in any compact subset of \([0, \infty)\).

**Proof.** The proof is similar to Lemma 4.5 of [14], so we omit it. \( \square \)

In order to obtain Theorem 4.2 (i), we continue the strategy as in the proof of Lemma 4.5 to derive the following corollary.

**Corollary 2.** Let \((u, v, h)\) be a solution to the problem (P) with (1), \( h_\infty = \infty \). When \( m(x) \) satisfies (H1), and (A1) holds, then for each \( n \in \mathbb{N} \),
\[
\begin{align*}
\underline{u}_n &\leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{u}_n, \\
\underline{v}_n &\leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \bar{v}_n,
\end{align*}
\]
uniformly in any compact subset of \([0, \infty)\).
Lemma 4.6. For free boundary problem
\[
\begin{align*}
    u_t &= Du_{xx} - \beta u_x + u(m(x) - u), \quad t > 0, \quad 0 < x < h(t), \\
    u_x(t, 0) &= u(t, h(t)) = 0, \quad t > 0, \\
    h'(t) &= -\mu u_x(t, h(t)), \quad t > 0, \\
    u(0, x) &= u_0(x), \quad h(0) = h_0, \quad 0 \leq x \leq h_0.
\end{align*}
\]
(16)

When \( m(x) \) satisfies (H1), \( h_0 \geq L^*(D, \beta, m(x)) \), we have \( \lim_{t \to \infty} h(t) = \infty \).

Proof. The proof is a minor modification of Theorem 2.5 of [6], so we omit the process. \( \square \)

Now, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. (i) Letting \( n \to \infty \) in Corollary 2, and utilizing Lemma 4.4, we obtain the results.

We only prove (ii), since the proof of (iii) is parallel. We adopt the idea of Lemma 5.3 and Theorem 4.1 of [38].

Step 1. We prove that \( \lim_{t \to \infty} u(t, x) = 0 \) uniformly in any compact subset of \([0, \infty)\).

By the comparison principle, we get \( u(t, x) \leq u^*(t) \) for \( t > 0, \ x \in [0, \infty) \), where \( u^*(t) \) satisfies
\[
\begin{align*}
    u^*_t &= u^*(m_1 - u^*), \quad t > 0, \\
    u^*(0) &= \|u_0\|_{L^\infty([0,h_0])}.
\end{align*}
\]

Since \( u^*(t) \to m_1 \) as \( t \to \infty \),
\[
\limsup_{t \to \infty} u(t, x) \leq m_1 \text{ uniformly in } [0, \infty)
\]
is obtained. Therefore, for \( \epsilon_1 = \frac{1}{2}(\frac{m}{\beta} - m_1) \), there exists \( T_1 > 0 \) large enough such that
\[
u(t, x) \leq m_1 + \epsilon_1
\]
for any \( t \geq T_1 \) and \( x \in [0, \infty) \). Therefore, \( v \) satisfies
\[
\begin{align*}
v_t &\geq Dv_{xx} - \beta_2 v_x + v(b\epsilon_1 - v), \quad t > T_1, \quad 0 < x < h(t), \\
v_x(t, 0) &= v(t, h(t)) = 0, \quad t > T_1, \\
h'(t) &\geq -\mu v_x(t, h(t)), \quad t > T_1, \\
v(T_1, x) &> 0, \quad 0 \leq x < h(T_1).
\end{align*}
\]

Now we prove for any \( L > L^*(D, \beta_2, b\epsilon_1) > 0 \), there exists \( T_L \geq T_1 \) such that for any \( \epsilon > 0, v(t, x) > b\epsilon_1 - \epsilon \) for \( t \geq T_L, \quad 0 \leq x \leq L \). In fact, since \( h_\infty = \infty \), there always exists \( T_2 \geq T_1 \) such that \( h(T_2) \geq L \). Consider the following problem
\[
\begin{align*}
    \underline{v}_t &= D\underline{v}_{xx} - \beta_2 \underline{v}_x + \underline{v}(b\epsilon_1 - \underline{v}), \quad t > T_2, \quad 0 < x < h(t), \\
    \underline{v}_x(t, 0) &= \underline{v}(t, h(t)) = 0, \quad t > T_2, \\
    \underline{h}'(t) &= -\mu \underline{v}_x(t, \underline{h}(t)), \quad t > T_2, \\
    \underline{v}(T_2, x) &= v(T_2, x), \quad \underline{h}(T_2) = h(T_2), \quad 0 \leq x \leq \underline{h}(T_2).
\end{align*}
\]

Since \( \underline{h}(T_2) = h(T_2) \geq L \geq L^*(D, \beta_2, b\epsilon_1) \), it is easy to see that the above problem has a unique solution \( (\underline{v}, \underline{h}) \) for all \( t \geq T_2 \). Furthermore, by Lemma 4.6, \( \lim_{t \to \infty} \underline{h}(t) = \infty \) and \( \lim_{t \to \infty} \underline{v}(t, x) = b\epsilon_1 \) uniformly in \([0, L]\) as \( t \to \infty \). Taking \( (v, h) \) as a supersolution, from Lemma 3.3, we can get \( v(t, x) \geq \underline{v}(t, x) \) for \( t \in [T_2, \infty) \),
for \( x \in [0, h(t)] \), and \( h(t) \geq h(t) \) for \( t \in [T_2, \infty) \). These facts show that for some \( T_L \geq T_1 \), our claim holds.

Therefore, we find \((u, v)\) satisfies
\[
\begin{aligned}
    u_t &= u_{xx} - \beta_1 u_x + u(m(x) - u - cv), & t \geq T_L, & 0 \leq x \leq L, \\
    v_t &= Dv_{xx} - \beta_2 v_x + v(m(x) - v - bu), & t \geq T_L, & 0 \leq x \leq L, \\
    u_x(t, 0) &= v_x(t, 0) = 0, & t \geq T_L, \\
    u(t, x) \leq m_1 + \epsilon_1, & v(t, x) \geq b\epsilon_1, & t \geq T_L, & 0 \leq x \leq L.
\end{aligned}
\]

Since \( h_\infty = \infty \), similar to Lemma 3.5, we have \( u \leq \bar{u} \) and \( v \geq \bar{v} \) for \( t \geq T_L, 0 \leq x \leq L \), where \((\bar{u}, \bar{v})\) is a solution to
\[
\begin{aligned}
    \bar{u}_t &= \bar{u}_{xx} - \beta_1 \bar{u}_x + \bar{u}(m(x) - \bar{u} - cv), & t \geq T_L, & 0 \leq x \leq L, \\
    \bar{v}_t &= D\bar{v}_{xx} - \beta_2 \bar{v}_x + \bar{v}(m(x) - \bar{v} - b\bar{u}), & t \geq T_L, & 0 \leq x \leq L, \\
    \bar{u}_x(t, 0) &= \bar{v}_x(t, 0) = 0, & t \geq T_L, \\
    \bar{u}(T_L, x) &= m_1 + \epsilon_1, & \bar{v}(T_L, x) = b\epsilon_1, & 0 \leq x \leq L.
\end{aligned}
\]

The above system generates a monotone dynamical system with respect to the competition order, which means that
\[
(u_1, v_1) \leq_c (u_2, v_2) \quad \text{if and only if} \quad u_1 \leq u_2 \text{ and } v_1 \geq v_2.
\]

In particular, the initial datum \((m_1 + \epsilon_1, \epsilon_1)\) is a supersolution with respect to the competition order. By the theory of monotone dynamical systems, we derive \( \lim_{t \to \infty} \bar{u}(t, x) = \bar{u}_L(x) \) and \( \lim_{t \to \infty} \bar{v}(t, x) = \bar{v}_L(x) \) uniformly in \([0, L]\), where \((\bar{u}_L(x), \bar{v}_L(x))\) is the maximal solution under \((m_1 + \epsilon_1, \epsilon_1)\) under the competition order for the following system
\[
\begin{aligned}
    - (\bar{u}_L)_{xx} + \beta_1 \bar{u}_x &= \bar{u}_L(m(x) - \bar{u}_L - c\bar{v}_L), & 0 \leq x \leq L, \\
    - D(\bar{v}_L)_{xx} + \beta_2 \bar{v}_x &= \bar{v}_L(m(x) - \bar{v}_L - b\bar{u}_L), & 0 \leq x \leq L, \\
    (\bar{u}_L)(0) &= (\bar{v}_L)(0) = 0, \\
    \bar{u}_L(L) &= m_1 + \epsilon_1, \quad \bar{v}_L(L) = b\epsilon_1.
\end{aligned}
\]

Let \( L \to \infty \), according to the classical elliptic regularity theory and a diagonal procedure, we derive \((\bar{u}_L(x), \bar{v}_L(x))\) converges to \((\bar{u}_\infty(x), \bar{v}_\infty(x))\) uniformly in any compact subset of \([0, \infty)\), and \((\bar{u}_\infty(x), \bar{v}_\infty(x))\) satisfies
\[
\begin{aligned}
    - (\bar{u}_\infty)_{xx} + \beta_1 \bar{u}_x &= \bar{u}_\infty(m(x) - \bar{u}_\infty - c\bar{v}_\infty), & 0 \leq x < \infty, \\
    - D(\bar{v}_\infty)_{xx} + \beta_2 \bar{v}_x &= \bar{v}_\infty(m(x) - \bar{v}_\infty - b\bar{u}_\infty), & 0 \leq x < \infty, \\
    (\bar{u}_\infty)(0) &= (\bar{v}_\infty)(0) = 0, \\
    \bar{u}_\infty(x) \leq m_1 + \epsilon_1, & \bar{v}_\infty(x) \geq b\epsilon_1, & 0 \leq x < \infty.
\end{aligned}
\]

Next, we prove \( \bar{u}_\infty(x) = 0 \). Consider the following ODE problem
\[
\begin{aligned}
    z_t &= z(m_1 - z - cw), & t > 0, \\
    w_t &= w(m_0 - w - bz), & t > 0, \\
    z(0) &= m_1 + \epsilon_1, \quad w(0) = b\epsilon_1.
\end{aligned}
\]
Since (A2) holds, we get \((z(t), w(t)) \to (0, m_0)\) as \(t \to \infty\). Therefore, the solution to the following system
\[
\begin{align*}
Z_t &= Z_{xx} - \beta_1 Z_x + Z(m_1 - Z - cW), && t > 0, \quad 0 \leq x < \infty, \\
W_t &= DW_{xx} - \beta_2 W_x + W(m_0 - W - bZ), && t > 0, \quad 0 \leq x < \infty, \\
Z_x(t, 0) &= W_x(t, 0) = 0, && t > 0, \\
Z(0, x) &= m_1 + \epsilon_1, \quad W(0, x) = b\epsilon_1, \quad 0 \leq x < \infty,
\end{align*}
\]
satisfies \((Z(t, x), W(t, x)) \to (0, m_0)\) uniformly in \([0, \infty)\) as \(t \to \infty\). Moreover, by the comparison principle, we have \(\bar{u}_\infty(x) \leq Z(t, x)\) and \(\underline{v}_\infty(x) \geq W(t, x)\) for \(t > 0, \ x \in [0, \infty)\). Immediately, \(\bar{u}_\infty(x) = 0\) is obtained.

Now, as \(\limsup_{t \to \infty} u(t, x) \leq \bar{u}_L(x)\) and \(\liminf_{t \to \infty} v(t, x) \geq \underline{v}_L(x)\) in \([0, L]\), it can be seen \(\limsup_{t \to \infty} u(t, x) \leq 0\) uniformly in any compact subset of \([0, \infty)\), which directly indicates that \(\limsup_{t \to \infty} u(t, x) = 0\) uniformly in any compact subset of \([0, \infty)\).

**Step 2.** We prove that \(\lim_{t \to \infty} v(t, x) = V(x)\) uniformly in any compact subset of \([0, \infty)\).

We utilize the squeezing argument, which is introduced in [12]. Consider the following Dirichlet boundary problem
\[
\begin{align*}
- D_{xx} v + \beta_2 v_x &= v(m(x)(1 - \epsilon) - \nu), \quad 0 < x < L, \\
v_x(0) &= v(L) = 0,
\end{align*}
\]
and the boundary blow-up problem
\[
\begin{align*}
- D_{xx} \bar{v} + \beta_2 \bar{v}_x &= \bar{v}(m(x) - \nu), \quad 0 < x < L, \\
\bar{v}_x(0) &= 0, \quad \bar{v}(L) = +\infty.
\end{align*}
\]
As is known to all, when \(\epsilon > 0\) is small and \(L > 0\) is large, these two problems have positive solutions, denoted by \(V'_L\) and \(\bar{V}_L\), respectively. By Lemma 3.6, we get that as \(\epsilon \to 0^+\) and \(L \to +\infty\), \(V'_L\) increases to the unique positive solution \(V\) of (14), and \(\bar{V}_L\) decreases to \(V\), respectively. Therefore, a decreasing sequence \(\{k_L\}\) and an increasing sequence \(\{L_k\}\) can be selected such that \(\epsilon_k \to 0^+\), \(L_k \to +\infty\) as \(k \to \infty\), and \(\lambda_1(D, \beta_2, m(x)(1 - \epsilon_k), L_k) < 0\) holds for all \(k \in \mathbb{N}\). Due to step 1, we get that for each \(\epsilon_k, L_k > 0\), there exists \(T_k > 0\) large enough such that
\[
u_k(t, x) \leq \epsilon_k \frac{m_0}{b} \leq \epsilon_k \frac{m(x)}{b}, \quad h(t) \geq L_k
\]
for all \(t \geq T_k\) and \(x \in [0, L_k]\). For this \(T_k\), according to the choice of \(\epsilon_k\) and \(L_k\), we derive that the following problem
\[
\begin{align*}
\nu_k &= D_{xx} \nu_k - \beta_2 \nu_k + v(m(x)(1 - \epsilon_k) - \nu), \quad t > T_k, \quad 0 < x < L_k, \\
\nu_x(t, 0) &= \nu(t, L_k) = 0, \quad t > T_k, \\
\nu(T_k, x) &= v(T_k, x), \quad 0 \leq x \leq L_k,
\end{align*}
\]
admits a unique positive solution \(\nu_k(t, x)\), and it satisfies
\[
\nu_k(t, x) \to \underline{V}'_{L_k}(x)\ 	ext{uniformly in} \ [0, L_k] \text{as} \ t \to \infty.
\]
By the comparison principle, for each \(k \in \mathbb{N}\),
\[
v(t, x) \geq \nu_k(t, x) \text{ for } t \geq T_k \text{ and } x \in [0, L_k].
\]
Therefore, we get
\[ \liminf_{t \to \infty} v(t, x) \geq V'_{L_k}(x) \text{ uniformly in } [0, L_k]. \]
Letting \( k \to \infty \), we have
\[ \liminf_{t \to \infty} v(t, x) \geq V(x) \text{ uniformly in any compact subset of } [0, \infty). \]
By similar arguments as in the proof of Theorem 4.1 of [12],
\[ \limsup_{t \to \infty} v(t, x) \leq \bar{V}_{L_k}(x) \text{ uniformly in } [0, L_k] \]
can be proved, which implies (by setting \( k \to \infty \)) that
\[ \limsup_{t \to \infty} v(t, x) \leq V(x) \text{ uniformly in any compact subset of } [0, \infty). \]
Then the desired result follows from the inequalities above directly. \( \square \)

5. The criteria governing spreading or vanishing. In this section, we assume \( m(x) \) satisfies (H1) or (H2). By denoting
\[ h_* := \min\{L^*(D, \beta_2, m(x)), L^*(1, \beta_1, m(x))\}, \]
we first give a spreading-vanishing dichotomy, then the criteria governing spreading or vanishing are derived when \( h_0 \) and \( \mu \) are variable factors.

**Theorem 5.1.** Let \((u, v, h)\) be a solution to the problem (P) with (1). If \( h_\infty > h_* \), then \( h_\infty = \infty \).

**Proof.** We give a proof by contradiction. Assume \( h_* < h_\infty < \infty \), then according to Theorem 4.1, \( \lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h_\infty])} = 0 \), \( \lim_{t \to \infty} \|v(t, \cdot)\|_{C([0, h_\infty])} = 0 \).

If \( h_\infty > L^*(D, \beta_2, m(x)) \), then there exist \( \epsilon_1 > 0 \) small enough and \( T_1 > 0 \) large enough such that \( h(T_1) > L^*(D, \beta_2, m(x) - b\epsilon_1) \), and \( u(t, x) < \epsilon_1 \), for any \( t \geq T_1 \), \( 0 \leq x \leq h(T_1) \).

Set \( l_1 = h(T_1) \), and let \( w(t, x) \) be the positive solution of the following problem
\[
\begin{aligned}
&w_t = Dw_{xx} - \beta_2 w_x + w(m(x) - b\epsilon_1 - w), \quad t > T_1, \quad 0 < x < l_1, \\
&w_x(t, 0) = w(t, l_1) = 0, \quad t > T_1, \\
&w(T_1, x) = v(T_1, x), \quad 0 \leq x \leq l_1.
\end{aligned}
\]
Using the comparison principle, \( w(t, x) \leq v(t, x) \), for \( t \geq T_1 \), \( 0 \leq x \leq l_1 \).

Since \( l_1 > L^*(D, \beta_2, m(x) - b\epsilon_1) \), then \( \lambda_1(D, \beta_2, m(x) - b\epsilon_1, l_1) < 0 \), as is known to all, \( w(t, x) \to W(x) \) uniformly in \([0, l_1]\) as \( t \to \infty \), where \( W(x) \) is the unique positive solution to the following problem
\[
\begin{aligned}
&-DW_{xx} + \beta_2 W_x = W(m(x) - b\epsilon_1 - W), \quad 0 < x < l_1, \\
&W_x(0) = W(l_1) = 0.
\end{aligned}
\]
Thus, we have \( \liminf_{t \to \infty} v(t, x) \geq \lim_{t \to \infty} w(t, x) = W(x) > 0 \) in \([0, l_1]\), and it is in contradiction to \( \lim_{t \to \infty} \|v(t, \cdot)\|_{C([0, h_\infty])} = 0 \). If \( h_\infty > L^*(1, \beta_1, m(x)) \), using a process similar above, a contradiction of \( \lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h_\infty])} = 0 \) is obtained. \( \square \)

Then the following corollary is given immediately.

**Corollary 3.** Let \((u, v, h)\) be a solution to the problem (P) with (1). If \( h_0 \geq h_* \), then \( h_\infty = \infty \).
From Corollary 3, we can find that if $h_0 \geq h_*$, regardless of the initial value $(u_0, v_0)$, the moving parameters $\mu$ and $\rho$, spreading of two species always happens. In what follows, when $h_0$ is small, fix $(u_0, v_0)$ and $\rho$, the effect of $\mu$ on spreading or vanishing of two species is discussed.

Firstly, when $\mu$ is large, we give Lemma 5.2, then Theorem 5.3 is obtained.

**Lemma 5.2.** Let $D$, $\beta$ and $C$ be positive constants. For any given $g_0$, $L > 0$, and any $w_0 \in C^2([0, g_0])$ satisfying $w_0(0) = w_0(g_0) = 0$ and $w_0 > 0$ in $[0, g_0)$, if there exists $\bar{\mu} > 0$ such that when $\mu > \bar{\mu}$, $(w, g)$ satisfies the following problem

\[
\begin{align*}
\begin{cases}
\frac{w_t}{\mu} \geq D w_{xx} - \beta w_x - Cw, & t > 0, \ 0 < x < g(t), \\
w_x(t, 0) = w(t, g(t)) = 0, & t > 0, \\
g'(t) = -\mu w_x(t, g(t)), & t > 0, \\
w(0, x) = u_0(x), g(0) = g_0, & 0 \leq x \leq g_0,
\end{cases}
\end{align*}
\]

then we get $\liminf_{t \to \infty} g(t) > L$.

**Proof.** The proof is similar to proposition 3.1 of [36].

**Theorem 5.3.** Assume $h_0 < h_*$, then there exists $\mu^0 > 0$ depending on $(u_0, v_0, h_0)$ such that if $\mu \geq \mu^0$, then $h_{\infty} = \infty$.

**Proof.** Consider the following auxiliary problem

\[
\begin{align*}
\begin{cases}
\frac{u_t}{\mu} = u_{xx} - \beta_1 u_x - Cu, & t > 0, \ 0 < x < h(t), \\
u_x(t, 0) = u(t, h(t)) = 0, & t > 0, \\
u'(t) = -\mu u_x(t, h(t)), & t > 0, \\
u(0, x) = u_0(x), h(0) = h_0, & 0 \leq x \leq h_0,
\end{cases}
\end{align*}
\]

where $C = \|m - u + cv\|_{L^\infty([0, \infty))}$. Utilizing Lemma 5.2, we know that for any given $L > h_*$, there exists $\mu^0 > 0$ such that $\frac{h_{\infty}}{\mu^0} > L$ for all $\mu \geq \mu^0$. By Lemma 3.3, we have $h_{\infty} \geq h_{\infty} > h_*$. Therefore, by the process of proof of Theorem 5.1, the desired result follows.

Secondly, when $\mu$ is small, we get Theorem 5.4.

**Theorem 5.4.** Denote

\[
h_{*,1} := \min\{L^*(D, \beta_2, \|m\|_{L^\infty([0, \infty))}), L^*(1, \beta_1, \|m\|_{L^\infty([0, \infty))})\}. \tag{18}
\]

Assume $h_0 < h_{*,1}$, then there exists $\mu_0 > 0$ depending on $(u_0, v_0, h_0)$ such that if $0 < \mu \leq \mu_0$, then $h_{\infty} < \infty$.

**Proof.** We will construct some suitable supersolution to derive the desired results. Due to the difficulties induced by combination of advection terms and Neumann boundary condition at $x = 0$, we adopt the idea of Theorem 3.1 of [44]. By simple calculations, we get

\[
h_{*,1} = \min\left\{ \frac{2D}{\sqrt{4D\|m\|_{L^\infty([0, \infty))}} - \beta_2^2} \arctan \frac{4D\|m\|_{L^\infty([0, \infty))} - \beta_2^2}{\beta_2}, \right. \\
\left. \frac{2}{\sqrt{4\|m\|_{L^\infty([0, \infty))}} - \beta_1^2} \arctan \frac{4\|m\|_{L^\infty([0, \infty))} - \beta_1^2}{\beta_1} \right\}.
\]
For simplicity, denote
\[ a_1 = \frac{\beta_2}{2D}, \quad a_2 = \frac{\sqrt{4D\|m\|_{L^\infty([0,\infty))}} - \beta_2}{2D}, \quad \theta = \arctan \frac{a_2}{a_1}, \tag{19} \]
we have \( 0 < \theta < \frac{\pi}{2} \) and \( \frac{a_2}{a_1} \geq h_{*,1} > h_0 \). Let \( \phi(z) = a_2 \cos z - a_1 \sin z \) for \( 0 \leq z \leq \theta \), then we get
\[ \phi(z) > 0 \text{ in } [0, \theta], \quad \phi(\theta) = 0, \quad \text{and } \phi'(z) < 0 \text{ in } [0, \theta]. \]

Let \( 0 < \delta \ll 1 \) and \( M > 0 \) be constants to be determined. Setting
\[ \sigma(t) = h_0(1 + \delta - \frac{\delta}{2}e^{-\delta t}), \quad f(t,x) = e^{-\delta t - a_1(h_0 - h_0(1 + \delta))}, \quad q(t,x) = \frac{\theta x}{\sigma(t)}, \]
and
\[ w_2(t,x) = Mf(t,x)\phi(q(t,x)). \]
It is clear that
\[ \sigma(0) = h_0(1 + \delta - \frac{\delta}{2}) \geq h_0, \quad w_2(t,\sigma(t)) = Me^{a_1h_0\delta - \delta t}\phi(\theta) = 0, \]
and
\[ w_{2,x}(t,0) = \frac{a_1a_2f_0(1 + \delta) - a_1\theta}{\sigma(t)}Me^{-(a_1h_0 + \delta t)} \]
\[ = \frac{a_1[(1 + \delta)a_2h_0 - \theta]}{\sigma(t)}Me^{-(a_1h_0 + \delta t)} < 0 \]
provided \( 0 < \delta \ll 1 \) since \( a_2h_0 < \theta \). By routine calculations, we obtain
\[ w_{2,t} - Dw_{2,xx} + \beta_2w_{2,x} - w_2(\|m\|_{L^\infty([0,\infty))} - w_2) \]
\[ \geq w_{2,t} - Dw_{2,xx} + \beta_2w_{2,x} - \|m\|_{L^\infty([0,\infty))}w_2 := W_1(t,x) + W_2(t,x), \]
where
\[ W_1(t,x) = w_2\left(-\delta - \frac{a_1h_0(1 + \delta)x\sigma'(t)}{\sigma^2(t)} - \frac{a_1\beta_2h_0^2(1 + \delta)^2}{2\sigma^2(t)} + \frac{D\theta^2}{\sigma^2(t)} \right) \]
\[ + \frac{a_1\beta_2h_0(1 + \delta)}{\sigma(t)} - \|m\|_{L^\infty([0,\infty))}, \]
\[ W_2(t,x) = -M\theta f(t,x)\left(\frac{x\sigma'(t)}{\sigma^2(t)} + \frac{\beta_2h_0(1 + \delta)}{\sigma^2(t)} - \frac{\beta_2}{\sigma(t)}\right)\phi'(q(t,x)). \]

Notice \( \phi'(q(t,x)) < 0, \ f(t,x) > 0, \ \sigma'(t) > 0 \) and \( \sigma(t) < h_0(1 + \delta) \), one can see \( W_2(t,x) \geq 0 \).

Taking advantage of \( \theta > a_2h_0 \), and
\[ h_0 < h_0(1 + \delta) \leq \sigma(t) \leq h_0(1 + \delta) < 2h_0, \quad \frac{\sigma'(t)}{\sigma(t)} \leq \frac{\delta^2}{2}, \quad 0 < \frac{x}{\sigma(t)} < 1 \]
along with
\[ \frac{2\sigma(t) - h_0(1 + \delta)}{\sigma^2(t)} \geq \frac{1}{h_0(1 + \delta)^2} > 0 \]
for all \( t \geq 0 \) and \( 0 < x < \sigma(t) \), we get
\[
W_1(t, x) \geq w_2\left(-\delta - \frac{a_1h_0(1+\delta)\delta^2}{2} + \frac{a_1\beta_2h_0(1+\delta)(2\sigma(t) - h_0(1+\delta))}{2\sigma^2(t)} + \frac{D\theta^2}{\sigma^2(t)} - \|m\|_{L^\infty([0,\infty))}\right)
\geq w_2\left(-\delta - \frac{a_1h_0(1+\delta)\delta^2}{2} + \frac{a_1\beta_2}{2(1+\delta)} + \frac{D\theta^2}{\sigma^2(t)} - \|m\|_{L^\infty([0,\infty))}\right)
:= w_2\xi(\delta).
\]
Due to (19) and \( \theta^2 > a_2^2h_0^2 \), by a direct calculation,
\[
\xi(0) = \frac{a_1\beta_2}{2} + \frac{D\theta^2}{\sigma^2(t)} - \|m\|_{L^\infty([0,\infty))}
= \frac{\beta_2^2 - 4D\|m\|_{L^\infty([0,\infty))}}{4D} + \frac{D\theta^2}{\sigma^2(t)} - \frac{D\theta^2}{\sigma^2(t)} - Da_2^2 > 0,
\]
which implies \( W_1(t, x) > 0 \) provided \( 0 < \delta \ll 1 \).

Through the conclusions above, we obtain
\[
w_{2,t} \geq Dw_{2,xx} - \beta_2w_{2,x} + w_2(\|m\|_{L^\infty([0,\infty))} - w_2), \quad \forall t \geq 0, \quad 0 < x < \sigma(t),
\]
provided \( 0 < \delta \ll 1 \). As \( 0 < \theta < \frac{\pi}{2} \), it is obvious that
\[
\phi(q(0, x)) = a_2 \cos \left( \frac{\theta_x}{\sigma(0)} \right) - a_1 \sin \left( \frac{\theta_x}{\sigma(0)} \right) > 0 \quad \text{for} \quad 0 \leq x \leq \sigma_0.
\]
According to the regularities of \( v_0(x) \) and \( \phi(q(0, x)) \), we can select \( M \gg 1 \) satisfying
\[
v_0(x) \leq Me^{-a_1(\sigma(t) - \frac{1+\delta}{\epsilon_3})} \phi(q(0, x)) = w_2(0, x).
\]
Since
\[
w_{2,x}(t, \sigma(t)) = -\frac{M\theta}{\sigma(t)} e^{a_1h_0\delta - \delta t}(a_2 \sin \theta + a_1 \cos \theta)
\geq -\frac{M\theta}{h_0} e^{a_1h_0\delta - \delta t}(a_2 \sin \theta + a_1 \cos \theta),
\]
for fixed \( \delta \) and \( M \) above, we have \( \mu_0 > 0 \) such that, for all \( 0 < \mu \leq \mu_0 \),
\[
\sigma'(t) = \frac{h_0\delta^2}{2}e^{-\delta t} \geq \frac{\mu(1+\rho)M\theta}{h_0} e^{a_1h_0\delta - \delta t}(a_2 \sin \theta + a_1 \cos \theta)
\geq -\mu(1+\rho)w_{2,x}(t, \sigma(t)), \quad \forall t \geq 0.
\]
Similarly, we derive
\[
w_{1,t} \geq w_{1,xx} - \beta_1w_{1,x} + w_1(\|m\|_{L^\infty([0,\infty))} - w_1), \quad \forall t \geq 0, \quad 0 < x < \sigma(t),
\]
\[
w_{1,x}(t, 0) < 0, \quad w_1(t, \sigma(t)) = 0, \quad \forall t \geq 0,
\]
\[
w_0(x) \leq w_1(0, x), \quad \sigma'(t) \geq -\mu(1+\rho)w_{1,x}(t, \sigma(t)), \quad \forall t \geq 0.
\]
Then by Lemma 3.4, \( h(t) \leq \sigma(t) \). Therefore, \( h_\infty \leq h_0(1+\delta) < \infty \) for all \( 0 < \mu \leq \mu_0 \).

\[\square\]

**Remark 2.** In Theorem 5.4, we obtain the result only for the case \( h_0 < h_{*,1} \). When \( m(x) \neq \text{constant}, h_{*,1} < h_* \). For a general \( m(x) \), we have not found a universal way to obtain \( h_\infty < \infty \) for the case \( h_{*,1} \leq h_0 < h_* \). But, we can find an example of \( m(x) \) as follows, then we use the method in Theorem 5.4 to obtain \( h_\infty < \infty \) when
\( \mu \) is small, by constructing a suitable supersolution.

**Example.** In problem (P0), let \( D = D, \beta = \beta_2, m(x) \equiv m_2 \), where \( m_2 \) is a positive constant. From this, it is determined that the corresponding \( L^*(D, \beta_2, m(x)) = L^*(D, \beta_2, m_2) \equiv \text{constant} \), we denote it by \( L_1 \). Similarly, \( L^*(1, \beta_1, m(x)) = L^*(1, \beta_1, m_2) \equiv \text{constant} \), we denote it by \( L_2 \). Let \( m_3 \) be a positive constant, and \( m_3 > m_2 \).

Define a cut-off function as

\[
m(x) := \begin{cases} m_2, & 0 \leq x \leq \max\{L_1, L_2\}, \\ f(x), & \max\{L_1, L_2\} < x < \max\{L_1, L_2\} + 1, \\ m_3, & x \geq \max\{L_1, L_2\} + 1, \end{cases}
\]

where \( f(x) \) is a smooth function such that \( m(x) \) satisfies the hypothesis (H0).

Lastly, base on Theorems 5.3 and 5.4, we have the following theorem.

**Theorem 5.5.** Suppose \( h_0 < h_{*1} \), then there exist two constants \( \mu^* \) and \( \mu_* \), satisfying \( \mu^* \geq \mu_* > 0 \), depending on \( (u_0, v_0, h_0) \), such that

(i) If \( \mu > \mu^* \), then \( h_\infty = \infty \).

(ii) If \( 0 < \mu \leq \mu_* \) or \( \mu = \mu^* \), then \( h_\infty < \infty \).

**Proof.** We adopt the idea of Theorem 3.2 of [46].

Define \( \chi^* := \{ \mu > 0 : h_\infty < \infty \} \). By Theorems 5.3 and 5.4, we get \((0, \mu_0] \subset \chi^* \) and \( \chi^* \cap [\mu^0, \infty) = \emptyset \). Therefore, \( \mu^* := \sup \chi^* \in [\mu_0, \mu^0] \). In consideration of this definition, we have \( h_\infty = \infty \) when \( \mu > \mu^* \). Consequently, \( \chi^* \subset (0, \mu^* \).

We claim \( \mu^* \in \chi^* \). Assume \( h_\infty = \infty \) for \( \mu = \mu^* \). Then we can choose \( T > 0 \) such that \( h(T) > h_* \). For the sake of emphasizing the dependence of the solution \((u, v, h)\) to the problem (P) with (1) on \( \mu \), we write \((u_\mu, v_\mu, h_\mu)\) instead of \((u, v, h)\). Then we get \( h_\mu(T) > h_* \). By means of the continuous dependence of \((u_\mu, v_\mu, h_\mu)\) on \( \mu \), we can select \( \varepsilon > 0 \) small enough such that \( h_\mu(T) > h_* \) for any \( \mu \in [\mu^* - \varepsilon, \mu^* + \varepsilon] \). Then we can obtain

\[
\lim_{t \to \infty} h_\mu(t) > h_\mu(T) > h_* \quad \forall \mu \in [\mu^* - \varepsilon, \mu^* + \varepsilon].
\]

By Theorem 5.1, it implies \([\mu^* - \varepsilon, \mu^* + \varepsilon] \cap \chi^* = \emptyset \). Hence we get \( \sup \chi^* \leq \mu^* - \varepsilon \), which contradicts to the definition of \( \mu^* \). This proves the claim \( \mu^* \in \chi^* \).

Define \( \chi_* := \{ \kappa : \kappa \geq \mu_0 \text{ such that } h_\infty < \infty \text{ for any } 0 < \mu \leq \kappa \} \). Evidently, we have \( \mu_* := \sup \chi_* \leq \mu^* \) and \((0, \mu_*] \subset \chi_* \). Similar to the process above, we can derive \( \mu_* \in \chi_* \). Then the proof is finished. \( \square \)

**Remark 3.** By Comparison Principles II and III (Lemmas 3.4 and 3.5), the free boundary \( h_\mu(t) \) of problem (P) may be not monotone with respect to \( \mu \), then whether \( \mu_* \) equals \( \mu^* \) or not in Theorem 5.5 is unknown. Therefore, if \( \mu < \mu^* \), for the case \( \mu_* < \mu \), at present we could not show \( h_\infty < \infty \) or \( h_\infty = \infty \) by a comparison principle. We leave it as an open problem to the interested readers.

6. **Spreading speed.** In this section, in spreading case, we give upper and lower bounds for spreading speed of \( h(t) \).

We provide two lemmas first.
Lemma 6.1. Suppose $0 < \beta < 2\sqrt{aD}$. Let $(u, g, h)$ be a solution to the problem
\[
\begin{aligned}
&u_t = Du_{xx} - \beta u_x + u(a - bu), \quad t > 0, \quad g(t) < x < h(t), \\
u(t, g(t)) = u(t, h(t)) = 0, \quad t > 0, \\
g'(t) = -\mu u_x(t, g(t)), \quad t > 0, \\
h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
u(0, x) = u_0(x), \quad -h_0 \leq x \leq h_0, \\
-\beta u_x(0) = h(0) = h_0.
\end{aligned}
\] (20)

If spreading happens, then we get the existence of the leftward and rightward asymptotic spreading speeds
\[
-c_l^* := \lim_{t \to \infty} \frac{g(t)}{t}, \quad c_r^* := \lim_{t \to \infty} \frac{h(t)}{t}.
\] (21)

Furthermore, $0 < c_l^* < c^* < c_r^*$, where $c^*$ is the spreading speed of both $g(t)$ and $h(t)$ with the case of $\beta = 0$. In order to underline the dependence on the parameters of the problem, we write $c_l^* = c_l^* (a, b, D, \mu, \beta)$ and $c_r^* = c_r^* (a, b, D, \mu, \beta)$.

Proof. The proof is similar to Theorem 1.1 of [17].

Lemma 6.2. We denote $q(z)$ a semi-wave with speed $c$, if $(c, q(z))$ satisfies
\[
\begin{aligned}
&Du'' - (c - \beta)q' + q(a - bq) = 0, \quad 0 < z < \infty, \\
&q(0) = 0, \quad q(\infty) = \frac{a}{b}, \quad q'(z) > 0, \quad 0 < z < \infty.
\end{aligned}
\] (22)

Then for each $\mu > 0$, problem (22) has exactly a solution $(c, q) = (c^*_r, q^*_r)$ such that
\[
\mu [(q^*_r)'(0)] = c^*_r,
\] (23)

where $c^*_r$ is defined in (21). Furthermore, $c^*_r \in (0, 2\sqrt{aD} + \beta)$.

Proof. The proof is similar to Proposition 2.2 of [17].

Now, we give Theorem 6.3.

Theorem 6.3. Let $(u, v, h)$ be a solution to the problem (P) with (1), $h_\infty = \infty$. When $m(x)$ satisfies (H1), we have
\begin{enumerate}[(i)]
\item If (A1) holds, then
\[
\max\left\{c_{r,u}, c_{r,v}^*\right\} \leq \liminf_{t \to \infty} \frac{h(t)}{t} \leq \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_{\min} := \max\{2\sqrt{m_1} + \beta_1, 2\sqrt{m_1 D} + \beta_2\}.
\]
\item If (A2) holds, then
\[
c_{r,u}^* \leq \liminf_{t \to \infty} \frac{h(t)}{t} \leq \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_{\min} := \max\{2\sqrt{m_1} + \beta_1, 2\sqrt{m_1 D} + \beta_2\}.
\]
\item If (A3) holds, then
\[
c_{r,u}^* \leq \liminf_{t \to \infty} \frac{h(t)}{t} \leq \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_{\min} := \max\{2\sqrt{m_1} + \beta_1, 2\sqrt{m_1 D} + \beta_2\}.
\]
\end{enumerate}

Where $c_{r,u} = c^*_r (m_0 - cm_1, 1, 1, \mu, \beta_1)$, $c_{r,v}^* = c^*_r (m_0 - bm_1, 1, D, \mu \rho, \beta_2)$. 

Proof. We only prove (i). Firstly, we prove that \( \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_{\min} := \max\{2\sqrt{m_1} + \beta_1, 2\sqrt{m_1 D} + \beta_2\} \). A supersolution is constructed in order to utilize Lemma 3.4. Let \( \sigma(t) := \sigma_0 + c_{\min} t \) for \( t > 0 \), where \( \sigma_0 \gg 1 \) is to be determined.

Let \((U_1(\xi), U_2(\xi))\), \( \xi := x - c_{\min} t \) be a solution to

\[
\begin{aligned}
(c_{\min} - \beta_1)U_1' + U_1'' + U_1(m_1 - U_1) &= 0, & -\infty < \xi < \infty, \\
(c_{\min} - \beta_2)U_2' + DU_2'' + U_2(m_1 - U_2) &= 0, & -\infty < \xi < \infty, \\
U_1(-\infty) &= m_1, & U_1(0) = \frac{m_1}{2}, & U_1(\infty) = 0, \\
U_2(-\infty) &= m_1, & U_2(0) = \frac{m_1}{2}, & U_2(\infty) = 0, \\
U_1' &< 0, & U_2' &< 0, & -\infty < \xi < \infty.
\end{aligned}
\]

As \( c_{\min} - \beta_1 \geq 2\sqrt{m_1}, c_{\min} - \beta_2 \geq 2\sqrt{m_1 D}, (U_1, U_2) \) exists ([27]). Setting \( w_1(t, x) := kU_1(x - c_{\min} t) - kU_1(\sigma_0), \ w_2(t, x) := kU_2(x - c_{\min} t) - kU_2(\sigma_0), \) then select \( k > 1 \) satisfying \( kU_1(\xi) > \|v_0\|_{L^\infty([0, \sigma_0])} \) and \( kU_2(\xi) > \|v_0\|_{L^\infty([0, \sigma_0])} \), \( \xi \in [0, \sigma_0] \). We fix \( \sigma_0 > \sigma_0 \) such that

\[
U_1(\sigma_0) < \min_{x \in [0, \sigma_0]} \left[U_1(x) - \frac{u_0(x)}{k}\right], \quad U_2(\sigma_0) < \min_{x \in [0, \sigma_0]} \left[U_2(x) - \frac{v_0(x)}{k}\right],
\]

\[
U_1(\sigma_0) \leq m_1 - \frac{m_1}{k}, \quad U_2(\sigma_0) \leq m_1 - \frac{m_1}{k},
\]

\[
-k\mu(1 + \rho)\min\{U_1'(\sigma_0), U_2'(\sigma_0)\} < c_{\min}.
\]

By (25), we obtain

\[
\begin{aligned}
w_{1,t} - w_{1,xx} + \beta_1 w_{1,x} - w_1(m_1 - w_1) \\
= k[(k - 1)(U_1 - \frac{kU_1(\sigma_0)}{k - 1})^2 + m_1U_1(\sigma_0) - \frac{k}{k - 1}U_1^2(\sigma_0)] \geq 0,
\end{aligned}
\]

\[
\begin{aligned}
w_{2,t} - Dw_{2,xx} + \beta_2 w_{2,x} - w_2(m_1 - w_2) \\
= k[(k - 1)(U_2 - \frac{kU_2(\sigma_0)}{k - 1})^2 + m_1U_2(\sigma_0) - \frac{k}{k - 1}U_2^2(\sigma_0)] \geq 0.
\end{aligned}
\]

Apparently, \( w_1(t, \sigma(t)) = 0, w_2(t, \sigma(t)) = 0 \). By means of the monotonicity of \( U_1 \) and \( U_2 \), we have

\( w_{1,x}(t, 0) = kU_1'(-c_{\min} t) < 0, \ w_{2,x}(t, 0) = kU_2'(-c_{\min} t) < 0 \).

The last inequality of (9) follows from (26). In addition, (24) implies \( w_1(0, x) \geq u_0(x) \) and \( w_2(0, x) \geq v_0(x) \) for \( x \in [0, \sigma_0] \). Because \( \sigma(0) = \sigma_0 > \sigma_0 \), Lemma 3.4 is utilized to make sure \( h(t) \leq \sigma(t) \) for \( t > 0 \). Hence,

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{\sigma(t)}{t} = c_{\min}.
\]

Secondly, we prove \( \liminf_{t \to \infty} \frac{h(t)}{t} \geq \max\{c_{r, u}, c_{r, v}\} \). We easily get \( \limsup_{t \to \infty} v(t, x) \leq V(x) \), where \( V(x) \) is the unique positive solution of (14). Then for any given \( \epsilon > 0 \), there exists \( T_\epsilon > 0 \) large enough such that \( v(t, x) \leq (1 + \epsilon)V(x) \) for any \( t \geq T_\epsilon \) and \( x \geq 0 \). Since \( h_\infty = \infty \), there still exists \( T'_\epsilon > 0 \) large enough such that

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{\sigma(t)}{t} = c_{\min}.
\]
h(T'_e) > L^*(1, \beta_1, m(x) - c(1 + \epsilon)V(x))$. By denoting $\bar{T}_e := \max\{T_e, T'_e\}$, we get $(u, h)$ satisfies
\[
\begin{cases}
u_t \geq \nu_{xx} - \beta_1 \nu_x + \nu(m - c(1 + \epsilon)V - \nu), & t > \bar{T}_e, \quad 0 < x < h(t), \\
u_x(t, 0) = \nu(t, h(t)) = 0, & t > \bar{T}_e, \\
h'(t) \geq -\mu u_x(t, h(t)), & t > \hat{T}_e, \\
u(\bar{T}_e, x) > 0, & 0 \leq x < h(\bar{T}_e).
\end{cases}
\]

By Lemma 3.3, $(u, h)$ is a supersolution to the following problem
\[
\begin{cases}
u_t = \nu_{xx} - \beta_1 \nu_x + \nu(m - c(1 + \epsilon)V - \nu), & t > \hat{T}_e, \quad 0 < x < h(t), \\
u_x(t, 0) = \nu(t, h(t)) = 0, & t > \hat{T}_e, \\
h'(t) = -\mu u_x(t, h(t)), & t > \hat{T}_e, \\
u(\hat{T}_e, x) = u(\hat{T}_e, x), \quad h(\hat{T}_e) = h(\hat{T}_e), & 0 \leq x \leq h(\hat{T}_e).
\end{cases}
\]

As $h(\hat{T}_e) > L^*(1, \beta_1, m(x) - c(1 + \epsilon)V(x)), \hat{T}_e \to \infty$ as $t \to \infty$ is obtained. Furthermore, as is known to all, $\limsup_{x \to \infty} V(x) \leq m_1$. Therefore,
\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq c_1^*(m_0 - c(1 + \epsilon)m_1, 1, 1, \mu, \beta_1).
\]

Because of the arbitrariness of $\epsilon$, and the continuity of $c_1^*$ with respect to its parameters, we derive $\liminf_{t \to \infty} \frac{h(t)}{t} \geq c_{1, v}^*$. The proof of $\liminf_{t \to \infty} \frac{h(t)}{t} \geq c_{1, v}^*$ is parallel, so we omit the details. \qed

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