Electromagnetic Annihilation Rates of $\chi_{c0}$ and $\chi_{c2}$ with Both Relativistic and QCD Radiative Corrections

Han-Wen Huang$^{1,2}$ Cong-Feng Qiao$^{1,2}$ Kuang-Ta Chao$^{1,2}$

$^1$ CCAST (World Laboratory), Beijing 100080, P.R.China
$^2$ Department of Physics, Peking University, Beijing 100871, P.R.China

Abstract

We estimate the electromagnetic decay rates of $\chi_{c0} \rightarrow \gamma\gamma$ and $\chi_{c2} \rightarrow \gamma\gamma$ by taking into account both relativistic and QCD radiative corrections. The decay rates are derived in the Bethe-Salpeter formalism and the QCD radiative corrections are included in accordance with factorization assumption. Using a QCD-inspired interquark potential, we obtain relativistic BS wavefunctions of $\chi_{c0}$ and $\chi_{c2}$ by solving BS equations for the corresponding $^{2S+1}L_J$ states. Our numerical result for the ratio $R = \frac{\Gamma(\chi_{c0} \rightarrow \gamma\gamma)}{\Gamma(\chi_{c2} \rightarrow \gamma\gamma)}$ is about 11 − 13 which agrees with the update E760 experimental data. Explicit calculations show that the relativistic corrections due to spin-dependent interquark forces induced by gluon exchange enhance the ratio $R$ substantially and its value is insensitive to the choice of parameters that characterize the interquark potential. Our expressions for the decay widths are identical with that obtained in the NRQCD theory to the next-to-leading order in $v^2$ and $\alpha_s$. Moreover, we have determined two new coefficients in the nonperturbative matrix elements for these decay widths.
1 Introduction

Charmonium physics is in the boundary domain between perturbative and non-perturbative QCD. Charmonium decays may provide useful information on understanding the nature of interquark forces and decay mechanisms. Both QCD radiative corrections and relativistic corrections are important for charmonium decays, because for charmonium the strong coupling constant $\alpha_s \approx 0.3$ [defined in the $\overline{MS}$ scheme (the modified minimal subtraction scheme)] and the velocity squared of the quark in the meson rest frame $v^2 \approx 0.3$, both are not small. Decay rates of heavy quarkonium in the nonrelativistic limit with QCD radiative corrections have been studied (see, e.g., ref. [1, 2, 3, 4]). However, the decay rates of many processes are subject to substantial relativistic corrections [4]. With this goal in mind, people have studied relativistic corrections to the decay rates of S-wave charmonium $\eta_c$, $J/\psi$ and their radial excited states [5, 6, 7]. These results show that relativistic effects are significant in the $c\bar{c}$ systems especially for the hadronic decays of $J/\psi$. In the present paper, we will investigate the relativistic corrections to the electromagnetic decays of P-wave charmonium states $\chi_{c0} \rightarrow \gamma\gamma$ and $\chi_{c2} \rightarrow \gamma\gamma$.

The P-wave charmonium decays are interesting. Now their experimental results are quite uncertain. The Crystal Ball group (see [8, 16] and references therein) gives $\Gamma(\chi_{c0} \rightarrow \gamma\gamma) = 4.0 \pm 2.8keV$. But for $\Gamma(\chi_{c2} \rightarrow \gamma\gamma)$, its central value differs significantly among various experiments [4, 9, 10, 11], and the ratio of the photonic widths for $\chi_{c0}$ and $\chi_{c2}$ states measured by E760 is much larger than that measured by other two groups. Theoretically, in the nonrelativistic limit, the ratio is $\frac{15}{4}$ [1], and it will increase to about 7.4 if QCD radiative corrections are considered. Recently a rigorous factorization formula which is based on NRQCD has been developed for calculations of inclusive decay rates of heavy quarkonium. In this approach the decay widths factor into a set of long distance matrix elements of NRQCD with each multiplied by a short distance coefficient. To any given order of relative velocity $v$ of heavy quark and antiquark, the decay rates are determined by several nonperturbative factors which can be evaluated using QCD lattice calculations or extracted by fitting the data. The study of the photonic decays of $\chi_{c0}$ and $\chi_{c2}$ can also provide a determination for the nonperturbative factors in the decays of P-wave quarkonium.

In this paper, we will use the Bethe-Salpeter (BS) formalism [12] to derive the decay amplitudes and to calculate the decay widths of $\chi_{c0} \rightarrow \gamma\gamma$ and $\chi_{c2} \rightarrow \gamma\gamma$. The meson will be treated as a bound state consist of a pair of constituent quark and antiquark (higher Fock states such as
|Q\bar{Q}g> and |Q\bar{Q}gg> are neglected because they don't contribute to electromagnetic decays) and described by BS wavefunction which satisfies the BS equation. A phenomenological QCD-inspired interquark potential will be used to solve for the wavefunction and to calculate the decay widths. Both relativistic and QCD radiative corrections to next-to-leading order will be considered based on the factorization assumption for the long distance and short distance effects. The remainder of this paper is organized as follows. In Sec.2 we derive the reduced BS equation for any angular momentum state $2S+1L_J$ of heavy mesons. In Sec.3 we give out the decay amplitudes of $\chi_J \rightarrow \gamma\gamma (J = 0, 2)$ and use the solved relativistic BS wavefunctions to calculate the numerical results of decay widths. A summary and discussion will be given in the last section.

2 Reduced BS equations for any angular-momentum state $2S+1L_J$ of heavy mesons

Define the Bethe-Salpeter wavefunction, in general, for a $Q_1\bar{Q}_2$ bound state $|P \rangle$ with overall mass $M$ and momentum $P = (\sqrt{P^2 + M^2}, \vec{P})$

$$\chi(x_1, x_2) = \langle 0 | T\psi_1(x_1)\bar{\psi}_2(x_2) | P \rangle, \quad (1)$$

and transform it into momentum space

$$\chi_P(p) = e^{-iP \cdot X} \int d^4 x e^{-ip \cdot x} \chi(x_1, x_2). \quad (2)$$

Here $p_1(m_1)$ and $p_2(m_2)$ represent the momenta(masses) of quark and antiquark respectively,

$$X = \eta_1 x_1 + \eta_2 x_2, \quad x = x_1 - x_2;$$

$$P = p_1 + p_2, \quad p = \eta_2 p_1 - \eta_1 p_2,$$

where $\eta_i = \frac{m_i}{m_1 + m_2} (i = 1, 2)$.

We begin with the bound state BS equation in momentum space

$$(\phi_1 - m_1)\chi_P(p)(\phi_2 + m_2) = \frac{i}{2\pi} \int d^4 k G(P, p - k)\chi_P(k), \quad (3)$$

where $G(P, p - k)$ is the interaction kernel which dominates the interquark dynamics. In solving (3), we will employ the instantaneous approximation since for heavy quarks the interaction is
dominated by instantaneous potentials. Meanwhile, we will neglect negative energy projectors in the quark propagators which are of even higher orders. Defining three dimensional BS wavefunction

$$\Phi_P(\vec{p}) = \int dP^0 \chi_P(p),$$

we then get the reduced Salpeter equation for $$\Phi_P(\vec{p})$$

$$(M - E_1 - E_2) \Phi(\vec{p}) = \Lambda_+^1 \gamma_0 \int d^3k G(P, \vec{p} - \vec{k}) \Phi(\vec{k}) \gamma_0 \Lambda_-^2.$$  

(4)

Here $$G(P, \vec{p} - \vec{k})$$ represents the instantaneous potential, $$\Lambda_+ (\Lambda_-)$$ are the positive (negative) energy projector operators for quark and antiquark respectively

$$\Lambda_+^1 = \frac{E_1 + \gamma_0 \vec{p}_1 \cdot \vec{p}_1 + m \gamma_0}{2E_1}$$

$$\Lambda_-^2 = \frac{E_2 - \gamma_0 \vec{p}_2 \cdot \vec{p}_2 - m \gamma_0}{2E_2}$$

$$E_1 = \sqrt{\vec{p}_1^2 + m_1^2}, \quad E_2 = \sqrt{\vec{p}_2^2 + m_2^2}.$$

We will follow a phenomenological approach by using QCD inspired inter-quark potentials, which are supported by both lattice QCD and heavy quark phenomenology, as the kernel in the BS equation. The potentials include a long-ranged confinement potential (Lorentz scalar) and a short-ranged one-gluon exchange potential (Lorentz vector)

$$V(r) = V_S(r) + \gamma_\mu \otimes \gamma^\mu V_V(r),$$

$$V_S(r) = \lambda r \frac{(1 - e^{-\alpha r})}{\alpha r},$$

$$V_V(r) = -\frac{4\alpha_s(r)}{3r} e^{-\alpha r},$$

(5)

where the introduction of the factor $$e^{-\alpha r}$$ is to regulate the infrared divergence and also to incorporate the color screening effects of dynamical light quark pairs on the $$Q\bar{Q}$$ linear confinement potential. In momentum space the potentials become

$$G(\vec{p}) = G_S(\vec{p}) + \gamma_\mu \otimes \gamma^\mu G_V(\vec{p}),$$

$$G_S(\vec{p}) = -\frac{\lambda}{\alpha} \delta^3(\vec{p}) + \frac{\lambda}{\pi^2 (\vec{p}^2 + \alpha^2)^2},$$

$$G_V(\vec{p}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\vec{p})}{\vec{p}^2 + \alpha^2},$$

(6)
where $\alpha_s(\vec{p})$ is the quark-gluon running coupling constant and is assumed to become a constant of $O(1)$ as $\vec{p}^2 \to 0$

$$\alpha_s(\vec{p}) = \frac{12\pi}{27 \ln(a + \vec{p}^2/\Lambda_{QCD}^2)}.$$ 

The constants $\lambda$, $\alpha$, $a$ and $\Lambda_{QCD}$ are the parameters that characterize the potential.

For any given angular-momentum state $^{2S+1}L_J$ of mesons, its three dimensional wavefunction in the rest frame of mesons takes the following two forms:

(i) $S=0$, then $J=L$,

$$\Phi_{Lm}(\vec{p}) = \Lambda^1_+ \gamma_0 (1 + \gamma_0) \gamma_5 \gamma_0 \Lambda^2 \gamma_0 Y_{Lm}(\hat{\vec{p}}) \phi(p); (7)$$

(ii) $S=1$, then $J=L-1,L,L+1$ for $L \neq 0$ or $J=1$ for $L=0$,

$$\Phi_{JM}(\vec{p}) = \sum_{l,m} <JM|1lLm> \Lambda^1_+ \gamma_0 (1 + \gamma_0) \gamma_5 \gamma_0 \Lambda^2 \gamma_0 Y_{Lm}(\hat{\vec{p}}) \phi(p) (8)$$

where $Y_{Lm}(\hat{\vec{p}})$ is the spherical harmonic function and $<JM|1lLm>$ is the Clebsch–Gordan coefficient. Substituting Eq.(7) and (8) in Eq.(4), one derives the equations for the scalar wavefunction $\phi(p)$

(i) $S=0$

$$(M - E_1(p) - E_2(p)) g_1(p) \phi(p)$$

$$= - \frac{E_1(p)E_2(p) + m_1m_2 + \vec{p}^2}{4E_1(p)E_2(p)} \int d^3 k (G_S(\vec{p} - \vec{k}) - 4G_V(\vec{p} - \vec{k})) g_1(k) P_L(cos\Theta) \phi(k)$$

$$- \frac{E_1(p)m_2 + E_2(p)m_1}{4E_1(p)E_2(p)} \int d^3 k (G_S(\vec{p} - \vec{k}) + 2G_V(\vec{p} - \vec{k})) g_2(k) P_L(cos\Theta) \phi(k)$$

$$+ \frac{E_1(p) + E_2(p)}{4E_1(p)E_2(p)} \int d^3 k (G_S(\vec{p} - \vec{k}) + \vec{p} \cdot \vec{k} g_3(k) P_L(cos\Theta) \phi(k)$$

$$+ \frac{m_1 - m_2}{4E_1(p)E_2(p)} \int d^3 k (G_S(\vec{p} - \vec{k}) - 2G_V(\vec{p} - \vec{k})) \vec{p} \cdot \vec{k} g_4(k) P_L(cos\Theta) \phi(k) (9)$$

where

$$g_1(p) = \frac{(E_1(p) + m_1)(E_2(p) + m_2) + \vec{p}^2}{4E_1(p)E_2(p)}$$

$$g_2(p) = \frac{(E_1(p) + m_1)(E_2(p) + m_2) - \vec{p}^2}{4E_1(p)E_2(p)}$$
\(\begin{align*}
g_3(p) & = \frac{E_1(p) + m_1 + E_2(p) + m_2}{4E_1(p)E_2(p)} \\
g_4(p) & = \frac{E_1(p) + m_1 - E_2(p) - m_2}{4E_1(p)E_2(p)} \\
E_1(p) & = \sqrt{p^2 + m_1^2} \\
E_2(p) & = \sqrt{p^2 + m_2^2}
\end{align*}\)

(ii) \(S=1\)

\[
(M - E_1(p) - E_2(p))f_8(p)\phi(p)
= \frac{1}{4E_1(p)E_2(p)} \left\{ \int d^3k [2G_V (\vec{p} - \vec{k}) - G_S (\vec{p} - \vec{k})] f_1(k)(m_1 + m_2) \\
- G_s(\vec{p} - \vec{k})f_2(k)(E_1(p) + E_2(p))P_L(\cos\Theta)\phi(k) \\
+ \int d^3k [4G_V (\vec{p} - \vec{k}) + G_S (\vec{p} - \vec{k})] f_8(k)(E_1(p)E_2(p) - m_1m_2 + p^2) \\
+ (G_S(\vec{p} - \vec{k}) - 2G_V (\vec{p} - \vec{k}))f_7(k)(m_1E_2 - m_2E_1))P_J(\cos\Theta)\frac{k}{p} \phi(k) \\
+ \int d^3k [(2G_V (\vec{p} - \vec{k}) - G_S (\vec{p} - \vec{k}))f_5(k)(m_1 + m_2) \\
- G_S(\vec{p} - \vec{k})f_6(k)(E_1 + E_2)\vec{p} \cdot \vec{k} P_J(\cos\Theta)\frac{k}{p} \phi(k) \right\},
\tag{10}
\]

where

\[
\begin{align*}
f_1(p) & = \frac{1}{4E_1(p)E_2(p)} ((E_1(p) + M_1)(E_2(p) + M_2) + p^2) \\
f_2(p) & = \frac{1}{4E_1(p)E_2(p)} ((E_1(p) + m_1)(E_2(p) + m_2) - p^2) \\
f_3(p) & = f_4(p) = \frac{2(E_1(p) + m_1)}{4E_1(p)E_2(p)} \\
f_5(p) & = -f_6(p) = -\frac{2}{4E_1(p)E_2(p)} \\
f_7(p) & = \frac{1}{4E_1(p)E_2(p)}(E_1(p) + m_1 - E_2(p) - m_2) \\
f_8(p) & = \frac{1}{4E_1(p)E_2(p)}(E_1(p) + m_1 + E_2(p) + m_2).
\end{align*}
\]
The normalization condition \( \int d^3p Tr\{ \Phi^+(\vec{p})\Phi(\vec{p})\} = \frac{2M}{(2\pi)^3} \) for the BS wavefunction \( \phi(p) \) leads to

\[
\int d^3p \frac{(E_1(p) + m_1)(E_2(p) + m_2)}{4E_1E_2} \phi^2(p) = \frac{2M}{(4\pi)^3}.
\]

(11)

To the leading order in the nonrelativistic limit, Eqs. (7) and (8) are just the ordinary nonrelativistic Schrödinger equation for orbital angular momentum \( L \) with simply a spin-independent linear plus Coulomb potential. Solving equation (9) or (10), we can get the spectra and wavefunctions for any given angular-momentum state \( ^{2S+1}L_J \) of heavy mesons. With these wavefunctions we can calculate hadronic matrix elements of the processes involving corresponding states, and the relativistic corrections due to interquark dynamics are included automatically in them. This approach is different from conventional ones which start from Schrödinger equation with all relativistic effects considered perturbatively.

3 Decay rates of \( \Gamma(\chi_{c0} \to \gamma\gamma) \) and \( \Gamma(\chi_{c2} \to \gamma\gamma) \)

Electromagnetic decays of \( \chi_{c0} \) and \( \chi_{c2} \) proceed via the annihilation of \( c\bar{c} \) to two photons. Here only electromagnetic interactions are considered, and color-octet components which contribute dominantly in hadronic decays of P-wave quarkonium do not contribute to electromagnetic decay widths, because final states are the photons which can not be produced via the annihilation of color-octet \( Q\bar{Q} \) pair. So two photonic decays of \( \chi_{cJ} \) for \( J = 0, 2 \) can be well expressed in the BS formalism and relativistic corrections are incorporated systematically in the decay rates. In the BS formalism the annihilation matrix elements can be written as follows

\[
<0|\bar{Q}IQ|P>=\int d^4p Tr[I(p, P)\chi_P(p)],
\]

(12)

where \( I(p, P) \) is the interaction vertex of the \( QQ \) with other fields (e.g., the photons or gluons) which, in general, may also depend on the variable \( q^0 \) (the time-component of the relative momentum). If \( I(p, P) \) is independent of \( q^0 \) (e.g., if quarks are on their mass-shells in the annihilation), the equation can be written as

\[
<0|\bar{Q}IQ|P>=\int d^3p Tr[I(\vec{p}, P)\Phi_P(\vec{p})],
\]

(13)

For process \( \chi_{c0} \to \gamma\gamma \) or \( \chi_{c2} \to \gamma\gamma \) with the momenta and polarizations of photons \( k_1, \epsilon_1 \)
and $k_2, \epsilon_2$, the decay amplitude can be written as

$$T = \langle 0 | \bar{e} \Gamma_{\mu \nu} | \chi_{cJ} > \epsilon_1^\mu \epsilon_2^\nu$$

(14)

for $J=0,2$, where $p_1(p_2)$ is the charm quark(antiquark) momentum, and

$$\Gamma_{\mu \nu} = e^2 \left[ \gamma_\nu \frac{1}{p_1 - k_1 - m} \gamma_\mu + \gamma_\mu \frac{1}{k_1 - p_2 - m} \gamma_\nu \right]$$

Since $p_1^0 + p_2^0 = M$, as usual we take

$$p_1^0 = p_2^0 = \frac{M}{2}.$$ 

(15)

Therefore, the amplitude $T$ becomes independent of $p^0$. In terms of $T$, the decay rates can be written as

$$\Gamma(\chi_{cJ} \rightarrow \gamma \gamma) = \frac{1}{2!} \sum_{\text{spin}} \sum_{\text{polar}} \int |T|^2 d\Omega$$

(16)

for $J = 0, 2$, where the factor $1/2!$ is needed because $N!$ same graphs appear for $N$-photon final states. The photon polarization is summed over in the Feynman gauge,

$$\sum_{\text{helicity}} \epsilon^\mu (k_1) \epsilon^{*\nu} (k_1) = -g^{\mu \nu},$$

Substituting BS wavefunction (8) into (16), after summing over final states and averaging over initial states, we get

$$\Gamma(\chi_{c0} \rightarrow \gamma \gamma) = 24 e^4 Q^2 (c_1 + 3c_2 + 2c_3)^2$$

(17)

$$\Gamma(\chi_{c2} \rightarrow \gamma \gamma) = \frac{12 e^4 Q^2}{5} (c_1^2 - 2c_1c_3 + 7c_3^2)$$

(18)

where

$$c_1 = \int d^3 p \frac{1}{(p - k)^2 + m^2} \left\{ [-E^2 - mE - \frac{p^2}{2} + \frac{3(p \cdot k)^2}{2}]p \cdot k \right. + \left. [-\frac{p^4}{4} + \frac{3p^2(p \cdot k)}{2} - \frac{5(p \cdot k)^4}{4}] \right\} \phi(p)$$

$$c_2 = \int d^3 p \frac{1}{(p - k)^2 + m^2} \left\{ \frac{p \cdot k}{2} [p^2 - (p \cdot k)^2] + \frac{1}{4} [p^2 - (p \cdot k)^2]^2 \right\} \phi(p)$$

$$c_3 = \int d^3 p \frac{1}{(p - k)^2 + m^2} \left\{ \frac{E^2 - mE}{2} [p^2 - (p \cdot k)^2] + \frac{1}{4} [p^2 - (p \cdot k)^2]^2 \right\} \phi(p)$$
In the nonrelativistic limit, (17) and (18) reduce to

\[
\Gamma(\chi_{c0} \rightarrow \gamma\gamma) = \frac{24e_Q^4\alpha^2}{m^4}\left| \int d^3pp\phi_{\chi_{c0}}(p) \right|^2
\]

\[
\Gamma(\chi_{c2} \rightarrow \gamma\gamma) = \frac{32e_Q^4\alpha^2}{5m^4}\left| \int d^3pp\phi_{\chi_{c2}}(p) \right|^2
\]

Using the Fourier transformation of wavefunctions

\[
\int d^3pp\phi_{\chi_{cJ}}(p) = \frac{3}{\sqrt{8}}R'_{\chi_{cJ}}(0)
\]

we derive the well known results in coordinate space, which is consistent with that given in [1]

\[
\Gamma(\chi_{c0} \rightarrow \gamma\gamma) = \frac{27e_Q^4\alpha^2}{m^4}\left| R'_{\chi_{c0}}(0) \right|^2
\]

(19)

\[
\Gamma(\chi_{c} \rightarrow \gamma\gamma) = \frac{36e_Q^4\alpha^2}{5m^4}\left| R'_{\chi_{c2}}(0) \right|^2,
\]

(20)

where \( R'_{\chi_{cJ}}(0) \) is the derivative of radial wavefunction at the origin, and in the nonrelativistic limit, \( R'_{\chi_{c0}}(0) = R'_{\chi_{c2}}(0) \), due to the heavy quark spin symmetry.

Recently, in the framework of NRQCD the factorization formulas for the long distance and short distance effects were found to involve a double expansion in the quark relative velocity \( v \) and in the QCD coupling constant \( \alpha_s \) [13, 14]. To next-to-leading order in both \( v^2 \) and \( \alpha_s \), as an approximation, we may write

\[
\Gamma(\chi_{c0} \rightarrow \gamma\gamma) = 24e_Q^4\alpha^2(c_1 + 3c_2 + 2c_3)^2\left[ 1 + \frac{\alpha_s}{\pi}\left( \frac{\pi^2}{3} - \frac{28}{9} \right) \right]
\]

(21)

\[
\Gamma(\chi_{c2} \rightarrow \gamma\gamma) = \frac{12e_Q^4\alpha^2}{5}(c_1^2 - 2c_1c_3 + 7c_3^2)(1 - \frac{\alpha_s}{\pi}\frac{16}{3})
\]

(22)

where we have used QCD radiative corrections given in [13]. We must emphasize that above factorization formula are correct only to next-to-leading order in \( v^2 \) and \( \alpha_s \). If higher order effects are involved, the decay widths can not be factored into a integral of wavefunction and a coefficient that can be written as a series of \( \alpha_s \). NRQCD has applied a more general factorization formula for quarkonium decay rates, which will be discussed in detail later.
For the heavy quarkonium \( \bar{c}c \) systems, \( m_1 = m_2 = m_c \), Eqs. (9) and (10) become much simpler. We take the following parameters which appear in potential (3),

\[
m_c = 1.5 \text{GeV}, \quad \lambda = 0.23 \text{GeV}^2, \quad \Lambda_{QCD} = 0.18 \text{GeV},
\]

\[
\alpha = 0.06 \text{GeV}, \quad a = e = 2.7183.
\]

With these values the mass spectrum of charmonium are found to fit the data well. In Fig.1 and Fig.2 the solved scalar wavefunctions both in momentum and coordinate space for P-wave triplet \( \chi_{cJ} \) states are shown and we can see explicitly the differences between wave functions for \( J = 0, 2 \) but they are same in the nonrelativistic limit. Substituting \( \phi_{\chi_{cJ}}(p) \) and \( \phi_{\chi_{cJ}}(p) \) into (21) and (22), we get

\[
\Gamma(\chi_0 \rightarrow \gamma\gamma) = 5.32 \text{keV},
\]

\[
\Gamma(\chi_2 \rightarrow \gamma\gamma) = 0.44 \text{keV}
\]

their ratio is

\[
R = \frac{\Gamma(\chi_0 \rightarrow \gamma\gamma)}{\Gamma(\chi_2 \rightarrow \gamma\gamma)} = 12.1. \quad (23)
\]

Our results are satisfactory, as compared with the Particle Data Group experimental values \[16\]
\[\Gamma(\chi_{c0} \rightarrow \gamma\gamma) = 5.6 \pm 3.2 \text{keV}, \quad \Gamma(\chi_{c2} \rightarrow \gamma\gamma) = 0.32 \pm 0.1 \text{keV}.\] Here in above calculations the value of \( \alpha_s(m_c) \) in the QCD radiative correction factor in (21) and (22) is chosen to be 0.29 \[4\], which is also consistent with our determination from the ratio of \( B(J/\psi \rightarrow 3g) \) to \( B(J/\psi \rightarrow e^+e^-) \) \[5\].

Moreover, in order to see the sensitivity of the decay widths to the parameters, especially the charm quark mass, we use other two sets of parameters

\[
m_c = 1.4 \text{GeV}, \quad \lambda = 0.24 \text{GeV}^2,
\]

\[
m_c = 1.6 \text{GeV}, \quad \lambda = 0.22 \text{GeV}^2,
\]

with other parameters keeping unchanged (the heavy quarkonia mass spectra are not sensitive to \( a, \alpha \) for \( \alpha \leq 0.06 \text{GeV} \)), By the same procedure, we obtain

\[
\Gamma(\chi_0 \rightarrow \gamma\gamma) = 5.82(4.85) \text{keV},
\]

\[
\Gamma(\chi_2 \rightarrow \gamma\gamma) = 0.50(0.39) \text{keV},
\]
and the ratio

$$R = \frac{\Gamma(\chi_0 \to \gamma \gamma)}{\Gamma(\chi_2 \to \gamma \gamma)} = 11.8(12.5)$$

(24)

for $m_c = 1.4(1.6)GeV$.

We find the widths are decreased with the decreasing of $\lambda$. This is obvious since the wavefunction in coordinate space will become broader when the slope of linear potential is decreased and the corresponding wavefunction in momentum space will become narrower, so the effective couplings decay become smaller. It is interesting to note that the ratio of two photonic decay width of $\chi_{c0}$ and $\chi_{c2}$ is almost unchanged and is insensitive to the choice of parameters.

Finally we discuss the relation between our approach and the NRQCD theory. Recently, a general factorization formula which is based on nonrelativistic QCD (NRQCD) has been developed for studying the inclusive cross sections of production and decay of heavy quarkonium. In this formalism the quarkonium decay rates can be written as a sum of a set of matrix elements to any given order in $v^2$, with each matrix element multiplied by a coefficient which can be calculated in perturbative QCD. This approach has been proved successful in the application of some processes involving heavy quarkonium [17, 18]. In NRQCD, the electromagnetic decay rates of $\chi_{c0}$ and $\chi_{c2}$ to next-to-leading order in $v^2$ can be written as

$$\Gamma(\chi_{c0} \to \gamma \gamma) = \frac{2Im f_{EM}(3p_0)}{m^4} <\chi_{c0}|O_{EM}(3P_0)|\chi_{c0}> + \frac{2Im g_{EM}(3p_0)}{m^6} <\chi_{c0}|G_{EM}(3P_0)|\chi_{c0}>$$

(25)

$$\Gamma(\chi_{c2} \to \gamma \gamma) = \frac{2Im f_{EM}(3p_2)}{m^4} <\chi_{c2}|O_{EM}(3P_2)|\chi_{c2}> + \frac{2Im g_{EM}(3p_2)}{m^6} <\chi_{c2}|G_{EM}(3P_2)|\chi_{c2}>$$

(26)

where

$$O_{EM}(3P_0) = \frac{1}{3}\psi^+(-\frac{i}{2}\hat{D})\cdot \vec{\sigma}\chi|0> <\chi^+(-\frac{i}{2}\hat{D})\cdot \vec{\sigma}\psi$$

$$O_{EM}(3P_2) = \psi^+(-\frac{i}{2}\hat{D})^{(i\sigma^j)}\chi|0> <\chi^+(-\frac{i}{2}\hat{D})^{(i\sigma^j)}\psi$$

$$G_{EM}(3P_0) = \frac{1}{2}\frac{1}{3}\psi^+(-\frac{i}{2}\hat{D})^2(-\frac{i}{2}\hat{D})\cdot \vec{\sigma}\chi|0> <\chi^+(-\frac{i}{2}\hat{D})\cdot \vec{\sigma}\psi + h.c]$$

$$G_{EM}(3P_0) = \frac{1}{2}[\psi^+(-\frac{i}{2}\hat{D})^2(-\frac{i}{2}\hat{D})^{(i\sigma^j)}\chi|0> <\chi^+(-\frac{i}{2}\hat{D})^{(i\sigma^j)}\psi + h.c]$$

(27)
where $\vec{D}$ is the space component of covariant derivate $D^\mu$, $\psi$ and $\chi$ are two component operators of quark and antiquark respectively. If identifying the quark operator expectation values with the derivatives of wavefunctions at the origin, the decay widths can be written as

$$\Gamma(\chi_0 \rightarrow \gamma\gamma) = \frac{9Imf_{EM}(3p_0)}{\pi m^4}|R'_{\chi_0}(0)|^2 + \frac{15Img_{EM}(3p_0)}{\pi m^6}Re(R_{\chi_0}(0)R'_{\chi_0}(0))$$  \hspace{1cm} (28)

$$\Gamma(\chi_2 \rightarrow \gamma\gamma) = \frac{9Imf_{EM}(3p_2)}{\pi m^4}|R'_{\chi_2}(0)|^2 + \frac{15Img_{EM}(3p_2)}{\pi m^6}Re(R_{\chi_2}(0)R'_{\chi_2}(0))$$  \hspace{1cm} (29)

In comparison, we take the on-shell condition $p_1^0 = p_0^0 = E$ instead of (15) and expand the annihilation amplitudes in (17) and (18) in terms of $\vec{p}/m^2$. The leading order contribution of P-wave decay comes from terms linear in $\vec{p}$ and in order to take into account their higher order effects we retain those terms proportional to the third powers of $\vec{p}$ then have

$$\Gamma(\chi_0 \rightarrow \gamma\gamma) = \frac{3e^2\alpha^2}{m^4}\int d^3pp(1 - \frac{\vec{p}^2}{6m^2})|\phi_{Sch}(p)|^2[1 + \frac{\alpha_s}{\pi}(\frac{\pi^2}{3} - \frac{28}{9})]$$ \hspace{1cm} (30)

$$\Gamma(\chi_2 \rightarrow \gamma\gamma) = \frac{4e^2\alpha^2}{5m^4}\int d^3pp|\phi_{Sch}(p)|^2(1 - \frac{\alpha_s}{\pi}(\frac{16}{3}))$$ \hspace{1cm} (31)

where the standard Schrödinger wavefunction (with relativistic corrections) $\phi_{Sch}(p)$ is related to $\phi(p)$ through the normalization condition (11)

$$\phi_{Sch}(p) = \frac{1}{\sqrt{M}}(\frac{m + E}{E})\phi(p),$$

$$(2\pi)^3 \int dpp^2|\phi_{Sch}(p)|^2 = 1.$$

Using formula

$$\int d^3pp\phi_{Sch}(p) = 3R'_{Sch}(0),$$

$$\int d^3pp^3\phi_{Sch}(p) = 5R_{Sch}(0),$$

the expressions of (30) and (31) are transferred into coordinate space and comparing with that derived from NRQCD (28) and (29), we can easily determine the coefficients

$$Imf_{EM}(3p_0) = 3\pi e^2\alpha^2[1 + \frac{\alpha_s}{\pi}(\frac{\pi^2}{3} - \frac{28}{9})]$$
\begin{align*}
\text{Im} f_{EM}(3P_0) &= -\pi e_Q^4 \alpha_s^2 \\
\text{Im} f_{EM}(3P_2) &= \frac{4\pi e_Q^4 \alpha_s^2}{5} \left(1 - \frac{\alpha_s}{\pi} \frac{16}{3}\right) \\
\text{Im} f_{EM}(3P_2) &= 0
\end{align*}

Here we only consider the QCD radiative corrections to leading order coefficients \( \text{Im} f_{EM}(3P_0) \) and \( \text{Im} f_{EM}(3P_2) \) which are equal to the results derived in [13]. More, we have determined two new coefficients, i.e., the second one and fourth one in [12]. These two matrix elements have a suppression factor of \( v^2 \) so we need not to take into account higher order corrections to their coefficients any more.

4 Summary and Discussion

In this paper we provide an estimate for the photonic decays of P-wave charmonium. It is clear from above calculations that comparing with nonrelativistic results the relativistic effects enhance the ratio \( R \) substantially. We know that there are two sources of relativistic corrections: 1) the correction due to relativistic kinematics which appears explicitly in the decay amplitudes; 2) the correction due to inter-quark dynamical effects (e.g., the well known Breit-Fermi interactions), which mainly causes the correction to the bound state wave functions. From the expressions (30) and (31) of decay rates which have been expanded to the first order of \( \frac{p^2}{m^2} \), one might expect that the ratio \( R \) would become smaller after taking relativistic corrections into account, because the coefficient of the term \( \frac{p^2}{m^2} \) in (30) is smaller than that in (31). However, mainly due to the attractive spin-orbital force induced by one gluon exchange for the \( 0^{++} \) meson, the \( \chi_0 \) wavefunction becomes narrower than \( \chi_2 \) wavefunction in coordinate space, and therefore the derivative of wavefunction at the origin become larger for \( \chi_0 \) than that for \( \chi_2 \). As a result, the dynamic relativistic effect on \( R \) is in the opposite direction and can be even larger. The overall relativistic correction to \( R \) is found to be positive, and our result is in agreement with the E760 data and disagree with the values measured by CLEO and TPC2.

Our expressions for the decay widths are identical with that derived from the rigorous factorization formula to next-to-leading order in \( v^2 \) and in \( \alpha_s \). Moreover we have determined two new coefficients in the nonperturbative matrix elements for these decay widths. For a more accurate estimate, higher order corrections both in \( v^2 \) and in \( \alpha_s \) should be taken into account. For
electromagnetic decays, in general we can estimate them within the $|Q\bar{Q} >$ sector and avoid the difficult problem due to the effects of high Fock states such as $|Q\bar{Q}g >$ and $|Q\bar{Q}gg >$. But we must notice that if higher order matrix elements are included, the decay widths can not be factored in the way like (28) and (29) because the higher order coefficients are different for each nonperturbative factor.

We have solved the BS equation for the bound-state wave functions with QCD inspired interquark potentials (linear confinement potential plus one gluon exchange potential) as the BS kernel. With some popular parameters for the potentials we obtained the wave functions and used them to calculate the decay widths. From (28) and (29) it can be seen that three different wave functions lead to somewhat different photonic decay widths but give very close values for $R$. This might indicate that our estimate of $R$ is insensitive to the quark mass and potential parameters, and therefore could be a rather reliable result, despite the uncertainty in the estimate of the dynamical relativistic effects. We hope the lattice simulations will give more reliable estimates for these decays within the framework of NRQCD, and can be compared with our results.
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Fig.1 Wave functions $\phi_{Sch}(p)$ (normalized in momentum space) of $\chi_{c0}$ (solid line), $\chi_{c1}$ (dashed line) and $\chi_{c2}$ (dashed and dot line) by solving BS equations with $m_c = 1.5 GeV$

Fig.2 Wave functions $R_{Sch}(x)$ (normalized in coordinate space) of $\chi_{c0}$ (solid line), $\chi_{c1}$ (dashed line) and $\chi_{c2}$ (dashed and dot line) by solving BS equations with $m_c = 1.5 GeV$
