Sensitivity of collective outcomes identifies pivotal components

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A social system is susceptible to perturbation when its collective properties depend sensitively on a few pivotal components. Using the information geometry of minimal models from statistical physics, we develop an approach to identify pivotal components to which coarse-grained, or aggregate, properties are sensitive. As an example, we introduce our approach on a reduced toy model with a median voter who always votes in the majority. The sensitivity of majority-minority divisions to changing voter behavior pinpoints the unique role of the median. More generally, the sensitivity identifies pivotal components that precisely determine collective outcomes generated by a complex network of interactions. Using perturbations to target pivotal components in the models, we analyze data sets from political voting, finance, and Twitter. Across these systems, we find remarkable variety, from systems dominated by a median-like component to those whose components behave more equally. In the context of political institutions such as courts or legislatures, our methodology can help describe how changes in voters map to new collective voting outcomes. For economic indices, differing system response reflects varying fiscal conditions across time. Thus, our information-geometric approach provides a principled, quantitative framework that may help assess the robustness of institutions to targeted perturbation and compare institutions with one another and across time.

When collective outcomes are highly sensitive to the behavior of few individual components, these components are pivotal. Collective outcomes could be the partition of voters into blocs, the pattern of co-moving financial indices, or the coalescence of shared vocabulary in a social community. A classic example is the swing, or median, voter, prominent in political science and economics: if voters can be deterministically ranked according to preference, the median will always vote in the majority and thus is predictive of the outcome (1–3). In real systems, this simple picture becomes much more complicated because the median might change depending on the contested issue (4), multiple issues may be at stake simultaneously (5), voters might exchange votes strategically (6, 7), etc. In other words, competing interactions between voters imply that changes in individual voting behavior may cascade into alignment or antagonistic changes in others resulting from direct physical interactions or indirect ones, i.e., mediated through a new compromise on the contents of a legislative bill. In contrast with an idealized notion of a median, we consider a “pivotal” voter one that could change collective outcomes even when accounting for such complexity. Here, we develop this generalized notion and use it to identify components that are especially indicative of collective changes in political voting, financial indices, and social media on Twitter.

Information geometry provides a natural framework for measuring how sensitive collective properties are to change in component behavior. The Fisher information, the fundamental quantity of information geometry (8), establishes a metric over probability distributions \( p(s; \{ J_{ij} \} ) \) of states \( s \) (e.g., a voting configuration, stock and sector price movement, behavior on social networks) determined by the set of parameters \( \{ J_{ij} \} \) indexed by \( ij \). When the parameters are infinitesimally changed to \( \{ J^\prime_{ij} \} \), the distribution becomes \( p(s; \{ J^\prime_{ij} \} ) \), and the distance between the two distributions is given by the Fisher information (FI) (9). By measuring the FI for perturbations to each pair of variables \( J_{ij} \) and \( J_{ij}^\prime \) in turn, we construct the Fisher information matrix (FIM) \( F_{ij,ij} \), whose eigenvectors describe how changes to the parameters lead either to sharp change in the model (large eigenvalues) or slow change (small eigenvalues) (10, 11). Often, collective outcomes are a coarse-graining of the high-dimensional state \( s \) to a lower-dimensional variable \( f(s) \). An example is when the full vector of individual votes is compressed into a binary variable indicating the direction of the majority outcome. By coarse-graining, we map the probability distribution to one over the coarse-grained state \( p(s) \rightarrow q(f(s)) \). By calculating the FIM for \( q \) instead of \( p \), we consider the sensitivity of collective outcomes. When the FIM indicates that aggregate properties are highly sensitive to a few components, we determine that they are key contributors to the collective properties of the system.

We outline our approach in Figure 1, where we fit a minimal statistical model to a data set, measure the FIM, and extract properties of the local information geometry. We first discuss this approach on a toy Median Voter Model to build intuition, and we extend detailed analysis to voting on an example from the US Supreme Court (SCOTUS) and State Street Global Advisors SPDR exchange-traded funds (12, 13). Then, we perform a survey across multiple systems in society including examples of judicial voting across US state high courts (14),

**Significance Statement**

The relation between collective outcomes and individual behavior is a central question in social science and statistical physics. Using the information geometry of minimal models from statistical physics, we develop a general approach for identifying key “pivotal” components on which aggregate statistics depend most sensitively. For political voting, pivotal blocs are like swing voters on whom the distribution of majority-minority divisions depends most sensitively. Analogously, collective market movement may be characterized by a few important stock indices, or the identity of a community on Twitter may hinge on a few individuals. Our approach may help evaluate how political bodies change with membership or analyze the robustness of political, financial, and social institutions to targeted perturbations.
1. Median Voter Model (MVM)

The role of the median derives from the fact that in a majority-rule voting system, the voting outcome depends on a coarse-graining instead of the detailed nature of every individual’s vote. The margin by which the majority wins, as is captured in the probability $q(k)$ that $k$ voters of the system are in the majority, can reflect the appeal of the voting outcome or even its legitimacy. These perceptions feed back into the decision process (18). Thus, $q(k)$ serves as an aggregate measure of underlying decision dynamics that we will use to identify pivotal blocs.

To outline our approach, we study the sensitivity of $q(k)$ in the context of a reduced toy model that captures the essence of a median voter. The ideal median voter exists in a majority-rule system where voters’ preferences are unidimensional. By virtue of a unique ranking of preference, the median is always in the majority (1). We propose a statistical generalization, the Median Voter Model (MVM), with an odd number of $N$ voters. The MVM consists of $N-1$ Random voters and one Median voter who always joins the majority. The binary vote of voter $i$, $s_i$, is equally likely to be $-1$ and $1$ such that only majority-minority divisions are relevant. Thus, the average votes are all the same, but the set of pairwise correlations as shown in Figure 1A display nonzero correlations between $M$ and $R_i$, $\langle s_M s_R \rangle = 0.3125$, and no correlations between $R_i$, $\langle s_R s_R \rangle = 0$. Thus, this model consists of a special voter, the Median (M), who after a voting sample has been taken, is perfectly correlated with the majority, whereas Random (R) voters all behave in a statistically uniform and random way.

To capture the network of interactions between individuals from which majority-minority coalitions emerge, we take a pairwise maximum entropy (maxent) approach (19). The maxent principle describes a way of building minimal models based on data. We maximize the information entropy $S = -\sum p(s) \ln p(s)$ while fixing the model to match the pairwise correlations from the data, $\langle s_i s_j \rangle_{\text{data}} = \langle s_i s_j \rangle$ as defined in Figure 1A. The result is a minimal model parameterized by statistical interactions between voters, or “couplings” $J_{ij}$ in Figure 1B (20). For each pair of voters with pairwise correlations in Figure 1A, there is a corresponding coupling such that the set of couplings is specified exactly by the pairwise correlation matrix. For the MVM, the $5^2 - 5 = 20$ couplings only take two possible values, one for each of the two unique correlations. The couplings for the MVM indicate that all R’s tend to vote with M (agreement between M and R leads to an increase in the log-probability $\ln r(s_M = s_R) \propto J_{MR}$ as in Figure 1B) with a slight tendency for R’s to disagree with each other more than would be expected given their shared correlation with M (disagreement between R$_i$ and R$_j$ decreases the log-probability of the vote by $\ln r(s_R_i = s_R_j) \propto J_{RR}$).

In principle, any probabilistic graph model is a viable alternative for the approach we outline, but the pairwise maxent model has been shown to capture voting statistics better than other models of voting with surprisingly few parameters (21, 22), fits the data well (SI Appendix B), and presents a particularly tractable model for calculating information quantities.

To probe how the collective properties captured by the distribution $q(k)$ depend on the voters, we ask how the distribution changes if the voters were slightly different. In this example of majority-minority voting, any change in voting behavior is reflected in the pairwise correlations and preserves the symmetry between the two possible outcomes $-1$ and $1$. A natural endpoint for the set of possible $q(k)$ as we increase the pairwise correlations is when all voters are perfectly correlated, so we consider perturbations that take us towards this endpoint: with some small probability, voter $y$’s votes are replaced by $x$’s,

$$\tilde{r}(s_y = s_x) = (1 - \epsilon) r(s_y = s_x) + \epsilon.$$  \[1\]

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Eq 1 is a weighted average that interpolates from the current probability of agreement between x and y, \( r(s_y = s_x) \), when \( \epsilon = 0 \) to perfect agreement when \( \epsilon = 1 \). We then account for the changes to y’s correlations with the remaining voters:

\[
f(s_y = s_x' | x') = (1 - \epsilon)r(s_y = s_x') + \epsilon r(s_y = s_x).
\]  

Eq 2 interpolates from the current probability of agreement between y and x’ when \( \epsilon = 0 \) to that between x and x’ when \( \epsilon = 1 \). If replacing M with any R voter such that y = M and x = R, the operation defined in Eq 1 increases the pairwise correlation \( \langle s_M s_R \rangle \) while simultaneously changing M’s correlations with the others to be more like those with R, pushing them to zero. When the statistical model exactly manifests in the comparison of local neighborhood with the eigenmatrix, the total asymmetry over all voters is made more similar to the corresponding row voter \( x \). Since each column corresponds to a directed change where voter y disagreement to the voting configuration where i and j agree, the perturbation described in Eqs 1 and 2 is equivalent to shifting entries represent the relative amount by which pairs should be simultaneously varied for maximal local change to \( y \)’s correlations with the remaining voters: changes to y’s correlations with the remaining voters:

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\]  

For larger N, the MVM asymmetry \( a_M \) grows as the role of M more visibly skews the distribution. Thus, both the asymmetry in the roles of voters and the growing importance of a median with system size is reflected in the symmetry of the eigenmatrices.

To measure the sensitivity of \( q(k) \) to each voter, we inspect the subspace eigenvalues \( \lambda_i \), specifying the sensitivity of \( q(k) \) to change in a single voter’s behavior. These values are calculated from the subspace of the FIM describing localized perturbations — the diagonal blocks of the FIM as outlined in SI Figure A.2C and whose eigenvectors are projected into Figure 1C. The upper leftmost block corresponds to M and the remaining blocks correspond to each R in turn. For each subspace, we retrieve the principal eigenvalue. To compare the eigenvalues across voters, we calculate the normalized eigenvalue as defined in Figure 1D, \( \lambda_i \), defining our measure of how “pivotal” a component is relative to others. For the \( N = 7 \) MVM, the principal eigenvalues are \( \lambda_M = 0.70 \) and \( \lambda_R = 0.05 \). This large difference indicates that \( q(k) \) is over 10x more sensitive to variation in M than R, again reaffirming the special role of the median. It is important to note that voters with strong asymmetry are not necessarily the most pivotal — clearly because eigenvalues and eigenvectors present different information. Still, asymmetry in the eigenmatrix indicates heterogeneity amongst the voters; thus, large asymmetry is necessary, if insufficient, for the pivotal measure to vary across a wide range. Overall, the information geometry of this minimal class of models provides a way of quantifying the role of individual components on collective outcomes, identifying key components with pivotal roles that can emerge given strong heterogeneity in the population.

2. US Supreme Court (SCOTUS) and S&P 500

We perform the same analysis on an example from SCOTUS of \( N = 9 \) voters, \( K = 909 \) votes, and between the years 1994–2005 (see SI Appendix I for details about data sets). We show the principal eigenmatrix in Figure 2 that consists of perturbations primarily increasing similarity across ideological wings given by the positive values connecting liberals and conservatives. The principal mode has a total asymmetry of \( A = 0.10 \) compared to \( A = 0.25 \) for the \( N = 9 \) MVM, indicating the absence of a median-like, pivotal voter. This absence is surprising because discussion of mediars A. Kennedy and S. O’Connor is prominent in the context of this court. When we consider voter-subspace eigenvalues shown in Figure 2, we find the justices in ranked order: C. Thomas, S. Breyer, and Chief Justice W. Rehnquist. A change in C.T., given his strongly conservative voting record, would naturally constitute consequential change, but the roles of W.R. and S.B. are more subtle (21, 24, 25). Despite A.K. and S.O.’s prominent role in the narrative of Supreme Court voting, we find that other justices come to the foreground when we consider the sensitivity of the Court to behavioral change.

The principal mode can be projected into the more intuitive space of dissenting coalitions in terms of the rate of change of the probabilities for dissenting blocs (SI Appendix E). Though the eigenmatrix in Figure 2 shows increasing similarity between ideological wings, suggesting suppression of partisan 5–4 divides, the frequency of any 5–4 divide actually increases

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Since the recovered eigenvector is arbitrary with respect to sign, we could just as well consider the negative eigenvector that would reverse the sign but preserve the magnitude of the elements.
To visualize changes in the existing 5–4 conservative-liberal votes where a single liberal vote is missing, and likewise defec-
tions from the liberal bloc, consistent with the balance of power favoring conservatives. For the liberal bloc, the most promi-
nent change entails R.G. defecting, leaving D.S., J.S., S.B., which reflects the central role of R.G. in the liberal coalition.
On the other side, increasing the probability of S.O. or A.K. defecting is important though not as much as the defection of W.R., which reflects his often-understated, unusual statistical role in the Court (21). Consistent with pundits’ understanding is the large surprise associated with C.T.’s defection from the conservative majority, a change that would represent a funda-
mental shift in the established partisan dynamics. Overall, this individual variation in the context of the partisan 5–4 dynamic reveals a portrait of much deeper subtlety than that suggested by unidimensional partisan intuition (4, 21, 26).

Thus, the information geometry of statistical models of social systems can provide detailed insight into specific components or blocs in direct connection to their role in collective modes of the system.

In Figure 3, we analyze the founding set of State Street Global Advisors SPDR exchange-traded funds (N = 9; K = 4,779; 2000–2018), which replicate the indices and provide daily price data (binarized to positive s_i = 1 or negative daily changes including no change s_i = −1 in analogy to votes). In contrast with SCOTUS, the collective behavior of each index reflects the aggregation of many individual investors: no stock index is monolithic in the sense of an individual voter. Given this aggregate nature, it is natural to consider the eigenvectors as the most surprising set of unanticipated global changes — although entire sectors might be “perturbed” by government policy like sector-specific regulation or tariffs. From this point of view, fluctuations in the pivotal blocs might reveal notable shifts in economic conditions or collective perceptions thereof (SI Appendix G). Taking a look at the model, we find that the principal mode displays large asymmetry across every index, reflecting the diversity of roles played by the various sectors of the economy as captured in price movements. Rela-
tively large subspace eigenvalues highlight XLE (energy) and XLU (utilities), in agreement with their role as drivers of the economy on whose outputs many of the other sectors depend (27, 28). Perhaps unsurprisingly, we also find a “bellweather” XLP (consumer staples) and XLV (healthcare) as notably pivotal whereas XLF (financial) and XLI (industrials) seem strongly along with a decrease in lone and pair dissents as in the bottom of Figure 2. Seven of the nine most common pair dissents found in the data decrease in likelihood. Thus, this shift reflects an increasing tendency for justices to join larger blocs, reflected in the suppression of every Justice’s lone dissents, in a way that breaks the typical partisan divide.

To visualize changes in the existing 5–4 conservative-liberal dynamic, we inspect defections from the liberal bloc, or 6–3 votes where a single liberal vote is missing, and likewise defections from the five-member conservative bloc. On the whole, defections from the liberal bloc are less surprising than those for the conservative bloc, consistent with the balance of power favoring conservatives. For the liberal bloc, the most promi-
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We explore other examples of social systems, including votes on Twitter and CA Assembly. Below each eigenmatrix, we show the pivotal measure and the asymmetry per component (see Figure 1). For the CA Assembly, voter are grouped into nine blocs after rank ordering by the first W-Nominate dimension (see SI Appendix Figure I.14). Large error bars in the pivotal measure indicate that the unique pivotal role ofBloc 8 depends on a few crucial votes in this session.

(D) Asymmetry of the principal eigenmatrix of the FIM. Error bars represent 95% bootstrapped confidence intervals.

3. Pivotal components in society

We explore other examples of social systems, including votes from US state high courts (14), the California State Assembly and Senate (15), the US federal legislature (16), and communities on Twitter (17). As with the previous two examples, we map behavior in these systems to binary form. To reduce the larger legislative bodies to a comparable number of blocs, we first separate voters into 9 blocs nearly-equally-sized blocs by ranked similarity according to a standard political science measure of ideology, the first W-Nominate dimension (29).

The bloc vote is given by the majority vote of the members and is randomly chosen if equally divided (see SI Appendix Figure I.14). For Twitter communities, we identify individuals as high-dimensional binary vectors where an element is positive if they used a corresponding keyword, or else negative, such that the pairwise correlations reflect overlap in their use of keywords. Thus, our analysis of the information geometry involves the same procedure outlined above, but for a wider variety of social systems.

Considering the principal eigenmatrix of the Alaskan (AK) Supreme Court (N = 5; K = 1,021; 1998–2007), we find a remarkable degree of symmetry between justices and a small value for the total asymmetry A = 0.01. Such symmetry implies that the justices on this court all dissent in a statistically uniform way as described by the set of their pairwise correlations. Though this could be trivially true if all pairwise couplings were the same, this is not the case, a fact that is mirrored in the spread of positive and negative values in the eigenmatrix in Figure 4. Checking the local interaction networks described by the set of couplings to every neighbor j for justice i (SI Appendix Figure B.3), we find that the sets are all similar for every justice i. This symmetry is mirrored in the similarity of the individual subspace eigenvalues shown in Figure 4. Consistent with this symmetry extracted from the voting record, four out of the five justices served as Chief Justice during this period, a regular rotation of roles imposed by the state constitution stipulating that the Chief Justice only serve for three consecutive years at a time. In contrast, we show that the New Jersey (NJ) Supreme Court (N = 7, K = 185, 2007–2010) has strong asymmetry of A = 0.5 (SI Appendix Figure B.8). Appointments to the NJ Court follow a tradition of maintaining partisan balance, apparently codifying a median role into the institution, and we find two nearly equal pivotal voters. Despite the seeming alignment between each of these two examples and the institutional norms, AK Supreme Courts are not always less symmetric than their NJ counterparts. The asymmetry is highly variable for previous years, suggesting that codified institutional rules only partially determine the role of pivotal voters (SI Appendix I).

We also show the eigenmatrices of the 1999 session of the CA State Assembly (N = 77; K = 5,424; 1999–2000) and a K-pop Twitter community (N = 10; K = 7,940; 2009–2017). The CA Assembly is an example of strong asymmetry (A = 0.46). For the sessions starting between the years 1993–2017, we find that the Assembly displays stronger signatures of asymmetry (average total asymmetry (A) = 0.4 ± 0.1) compared to the Senate ((A) = 0.3 ± 0.1), showing how the rules of the

W. Matthews, D. Fabe, and A. Bryner rotated as Chief Justice during the period 1997–2009 and W. Carpeneti from 2009–2012 (following the period of analysis).
institution might be reflected in the distribution of pivotal blocs. We then compare the distribution of the single largest pivotal measure $\lambda_{\text{max}}$ with that of the similarly coarse-grained US House of Representatives and Senate. Though the CA distributions for $\lambda_{\text{max}}$ are statistically indistinguishable from each other (Kolmogorov-Smirnov test statistic $k = 0.31$ and significance level $p = 0.5$) and the federal bodies’ distributions are similar ($k = 0.31$, $p = 0.05$), the state vs. federal levels show larger differences ($k > 0.46$, $p < 0.03$). This separation between the behavior at state and federal legislatures reflects institutional differences that are captured in the sensitivity of majority-minority coalitions (see SI Appendix H).

As for the Twitter Kpop community, we find much heterogeneity amongst users with total asymmetry $A = 0.30$, exceeding that of the $N = 9$ MVM. In contrast with the MVM, this community contains multiple pivotal members but wide variation in the strength of their subspace eigenvalues. Twitter communities may be on average sensitive to the wide variation in the strength of their subspace eigenvalues. Intriguingly, we find hints that institutional differences may contribute to structuring individual roles in social networks, generalizing the intuition behind the median to pivotal components on which each aggregate properties, measured by majority-minority divisions, depend strongly. If the subspace eigenvectors are knobs, the pivotal measure is inversely proportional to the spacing of the dials such that for large eigenvalues the smallest turn results in the strongest effect. Since each pivotal component only considers the effects of perturbations localized to a single component, these knobs are independent. If $n$ components were accessible simultaneously, however, we would consider the joint space of multiple pivotal components, and the principal subspace eigenvalue must increase beyond (or stay at) the maximum eigenvalue over the set of component subspaces: this reflects the fact that enhancing the breadth of control only increases the range of possible outcomes (36, 37).

4. Discussion

An important question in the study of social institutions is whether or not collective decisions are robust to perturbation targeting individual components. Robustness is reciprocal to sensitivity: when a system is highly sensitive to small changes to components, its collective properties are not robust. In neural networks with avalanches of firing activity (31–33), in bird flocks with propagating velocity fluctuations (34), or in macaque societies with conflict cascades (35), such sensitivity might have an adaptive functional role. In the context of human society, questions of robustness are relevant to the stability of voting coalitions or the susceptibility of a population to disease or disinformation. For example, we might be interested in comparing the impact of different judicial nominees on the dynamics of voting on a judicial bench or the impact of modified user behavior on the spread of disinformation in social networks. By relying on the formal framework of information geometry to investigate statistical signatures of sensitivity, we present a data-driven and general approach to characterizing robustness. As a result, our approach is not model-specific, only relying on the calculation of how sensitive a model is to changes in observable individual behavior (Figure 1D).

In voting systems, median voters are conventionally considered to be power brokers who have outsized influence (1, 7, 23). Building on this idea, we propose a reduced toy model to extract features of the Fisher information matrix that correspond to signatures of a median voter. We show how to identify and interpret signatures of strong sensitivity on individual components in multiple social contexts, generalizing the intuition behind the median to pivotal components on which aggregate properties, measured by majority-minority divisions, depend strongly. Intriguingly, we find hints that institutional differences may contribute to structuring individual roles in collective outcomes both in courts and legislatures. Though it is unsurprising that the particular rules of a voting body may structure bloc dynamics, pivotal components provide a principled way of comparing social systems with differing composition, from different eras, and across different institutions in a unified, quantitative framework.

We might think of pivotal components as “knobs” that could drive a system out of its current configuration described by the ensemble $p(s)$. If the subspace eigenvectors are knobs, the pivotal measure is inversely proportional to the spacing of the dials such that for large eigenvalues the smallest turn results in the strongest effect. Since each pivotal component only considers the effects of perturbations localized to a single component, these knobs are independent. If $n$ components were accessible simultaneously, however, we would consider the joint space of multiple pivotal components, and the principal subspace eigenvalue must increase beyond (or stay at) the maximum eigenvalue over the set of component subspaces: this reflects the fact that enhancing the breadth of control only increases the range of possible outcomes (36, 37). By considering which knobs are accessible experimentally, our analysis could be extended to measuring signs of statistical control in real systems. For judicial voting, the realizable knobs that change judicial voting behavior may be the submission of amicus curiae briefs, choice of litigating cases, or lobbying. Those trying to craft a legislative coalition might “perturb” aspects of proposed policy to affect its acceptability to potential supporters (38). In controlled biological systems, localized perturbations to single components could include manipulation of single neurons or the upregulation of specific genes. Our work presents the possibility of informing the direction of such external perturbations in the broader context of control.

The understanding of the interplay between components and social structures across social and biological examples remains nascent at mesoscopic and macroscopic scales. With this principled, quantitative approach for measuring pivotal components, we might, by comparing systems, better understand how institutional and environmental factors shape the emergence of social structure.

ACKNOWLEDGMENTS. We thank Katherine Quinn, Guru Khalsa, Bryan Daniels, Jess Flack, and David Krakauer for helpful conversations. We thank Gavin Hall with help with the Twitter data. EDL acknowledges a dissertation grant from the Dirksen Congressional Center, an NSF GRFP under grant no. DGE-1650441, and the Omega Miller Program at the Santa Fe Institute. DMK & MJB thank Illinois Tech - Chicago Kent College of Law for support of this project.

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1. We are careful to point out that the ensemble of votes for political systems already include such effects so it is important to distinguish between endogenous and exogenous factors.

2. For example, manipulation of single neurons is possible by electrical stimulation, optogenetic techniques, or chemical stimulation, all ways of enacting the localized perturbations of neural “votes” (31). Analogously, gene expression might be perturbed by switching genes on and off or by adding protein directly to simulate changed expression levels (39).
A. Median Voter Model (MVM)

In the canonical median voter model (1), we assume that voters are described by single-peaked, unidimensional preference function and that they vote according to them. This has been proposed as a model for voting in political science and for how supply-demand curves determine market prices. The median position is special because it determines the only outcome for which it is possible to obtain a simple majority against all alternatives. Here, we map this idea to a reduced, statistical model.

The MVM consists of an odd number of $N$ Random (R) voters with a single special voter, labeled Median (M), that is guaranteed to vote in the majority. More generally, we can interpolate between a perfect median and set of $N$ random voters by setting a probability $\alpha$ that M votes in the majority and $1 - \alpha$ that M votes randomly. The remaining $N - 1$ R’s are random. This model presents a simple testing ground for exploring the information geometry of a system with a unique, statistically well-defined median voter that can be range from random ($\alpha = 0$) to perfect median ($\alpha = 1$).

The probability distribution defined by the MVM cannot be exactly captured by a pairwise maximum entropy (maxent) model (see SI Appendix B). We recall that pairwise maxent models can be derived by maximizing the entropy of the model $S = -\sum p(s) \ln p(s)$ while constraining the single component and pair component distributions, $p(s_i)$ and $p(s_i, s_j)$, respectively, to match the data. From the maxent perspective, the MVM is equivalent to specifying that $M$ perfectly correlated with the majority of R’s, a correlation of $\alpha = 1$.

We model the probability distribution of votes using a pairwise maxent model (B). We solve a pairwise maxent model to learn the probability distribution $p(s; \{J_{ij}\})$ parameterized by the couplings $J_{ij}$ (2). We calculate the FIM for $q(k)$, the probability of $k$ votes in the majority, measuring the sensitivity of $q(k)$ to changes in voter behavior. The perturbation to the vector of couplings determined by Eqs 1 and 2 are denoted as $\Delta_{xy} J$. The matrix is segmented into 6x6 blocks for readability. The variation in the entries of the FIM clearly indicates the unique role of the median. Each entry $F_{xy}^{\text{Median}}$ of the FIM shows how quickly $q(k)$ changes when two pairs of voters (y becomes x and y’ becomes x’) are changed together. When at least one index is M, we find values different from when only R’s are involved. (D) The principal eigenvector of the FIM $v_{xy}$, reshaped into an “eigenmatrix.” (E) The asymmetry $\alpha_y$ measures the difference in perturbations localized to a specific voter vs. all its neighbors in turn. The principal subspace eigenvalues, computed from each outlined diagonal block in panel C, give our pivotal measure after normalization.

B. Fitting the pairwise maxent model

We model the probability distribution of votes using a pairwise maxent approach (19). We begin by maximizing the information entropy $S = -\sum s p(s) \ln p(s)$ while ensuring that the model match

\[
\langle s_i s_j \rangle = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \frac{s_i s_j}{s_i + s_j} = 1. \tag{A.1}
\]

In general, this nonlinear correlation cannot be written as the linear combination of pairwise correlations. Thus, the MVM serves as an example of a model for which the pairwise maxent model, by definition, could only be an approximation, and it provides a test to see if our approach can capture the essential features of the system.

The MVM can be solved numerically for large $N$ by exploiting the symmetry between the Random voters, allowing for fast enumeration of the entire partition function in linear time. This solution will be exact captured by a pairwise maximum entropy (maxent) model (see SI Appendix B). We recall that pairwise maxent models can be derived by maximizing the entropy of the model $S = -\sum p(s) \ln p(s)$ while constraining the single component and pair component distributions, $p(s_i)$ and $p(s_i, s_j)$, respectively, to match the data. From the maxent perspective, the MVM is equivalent to specifying that $M$ be perfectly correlated with the majority of R’s, a correlation of $\alpha = 1$.

We model the probability distribution of votes using a pairwise maxent model (B). We solve a pairwise maxent model to learn the probability distribution $p(s; \{J_{ij}\})$ parameterized by the couplings $J_{ij}$ (2). We calculate the FIM for $q(k)$, the probability of $k$ votes in the majority, measuring the sensitivity of $q(k)$ to changes in voter behavior. The perturbation to the vector of couplings determined by Eqs 1 and 2 are denoted as $\Delta_{xy} J$. The matrix is segmented into 6x6 blocks for readability. The variation in the entries of the FIM clearly indicates the unique role of the median. Each entry $F_{xy}^{\text{Median}}$ of the FIM shows how quickly $q(k)$ changes when two pairs of voters (y becomes x and y’ becomes x’) are changed together. When at least one index is M, we find values different from when only R’s are involved. (D) The principal eigenvector of the FIM $v_{xy}$, reshaped into an “eigenmatrix.” (E) The asymmetry $\alpha_y$ measures the difference in perturbations localized to a specific voter vs. all its neighbors in turn. The principal subspace eigenvalues, computed from each outlined diagonal block in panel C, give our pivotal measure after normalization.
the pairwise correlations calculated over the $K$ data points (20),

$$\langle s_i s_j \rangle = \langle s_i s_j \rangle_{\text{data}} \quad [B.2]$$

$$\sum_j s_i s_j \, p(s) = \frac{1}{K} \sum_{k=1}^{K} s_i^k s_j^k, \quad [B.3]$$

where the left hand sum is over all possible configurations of the binary vector $s$, and the right hand sum is over all $K$ observations in the data. Going through the usual calculation (41), we derive the pairwise maxent model

$$p(s) = e^{-E(s)} / Z. \quad [B.4]$$

Eq B.4 is normalized by the “partition function”

$$Z = \sum_s e^{-E(s)}, \quad [B.5]$$

and contains the energy functional of the form

$$E(s) = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij}s_i s_j. \quad [B.6]$$

The “couplings” $J_{ij}$ are numerically solved such that the model matches the pairwise correlations. In this sense, the couplings are not fit to the data, but are given exactly by the pairwise correlations in the data. For the small systems that we consider, these couplings can be found exactly by explicit calculation of the pairwise correlations and standard numerical optimization techniques as implemented in the Convenient Interface for Inverse Ising, or ConII (41). For the MVM, the problem can be simplified because there are only two types of couplings corresponding to the two types of correlations between Median and Random and between Random and Random voters. All the shown examples from Figs. B.3–B.8 have least-squares fit norm errors to the pairwise correlations of $<10^{-9}$. Thus, the relatively small system size ensures that straightforward enumeration techniques can be used to solve directly for the pairwise maxent models that correspond to the data.

Across all the system that we study, we find that the pairwise maxent model captures well higher-order features of the data even when only fit to the pairwise correlations. To characterize this fit, we use a property of maxent models. The entropy of the maxent model always decreases with the inclusion of additional constraints such that entropy is largest when all $N$ components are treated as independent $S_1 = \log_2 N$ bits and minimized when the entire probability distribution of the data is fit exactly $S_M = S_{\text{data}}$, where $S_1$ is the entropy of the model matching all correlations up to and including order 1. Thus, as we increase the number of parameters, we impose higher-order structure, and we monotonically approach the entropy of the data. This suggests as a measure for comparing maxent models, the total multi-information captured (21)

$$I = \frac{S_1 - S_2}{S_1 - S_N}, \quad [B.7]$$

which varies from 0 (no improvement beyond the independent model) to 1 (exact fit to the data). For the examples considered in the main text, the pairwise maxent model serves as an excellent fit, capturing over 94% of the multi-information in all cases. The pairwise maxent model captures over 98% of the multi-information for the $N = 7$ MVM. Overall, the pairwise maxent model is a minimal but convincing approximation of the ensemble statistics of the examples we consider (43, 44).

---

The keen reader may note that we did not constrain the average of each element of the vector $\langle s_i \rangle$, so we did not explicitly fix the individual marginal distributions in this abbreviated derivation. Indeed, we assumed that the averages were 0 because we were only interested in majority-minority dynamics so each vote equally likely to be either up or down, $\langle s_i \rangle = 0$. As a result, the “fields” $h_i$ in the complete energy function $E(s) = -\frac{1}{2} \sum_{i,j} J_{ij}s_i s_j - \sum_i h_i s_i$ are 0 as is assumed in Eq B.6. In other words, the maxent distribution when fixing only the pairwise correlations shows symmetry about the two possible orientations $-1$ and 1.

For how to estimate the entropy of the data see Refs (21) or (42). Calculating an unbiased estimate entropy of the entropy of the data can be an issue for sparse samples, but is straightforward for the relatively large number of samples we have.

Fig. B.3. Pairwise correlations and couplings for AK Supreme Court.

Fig. B.4. Pairwise correlations and couplings for US Supreme Court.

Fig. B.5. Pairwise correlations and couplings for Twitter K-pop community.

Fig. B.6. Pairwise correlations and couplings for SPDR.

Fig. B.7. Pairwise correlations and couplings for CA Assembly 1999 session. Composition of blocs is given in Figure I.14.
The Kullback-Leibler (KL) divergence is one such measure that tells corresponding changes in the set of pairwise correlations for a other words, we do not know how to access the coupling parameter terms in the energy functional are opaque and often nontrivial. In changing temperature that modulates all couplings by a factor). For the maximum entropy formulation, as the parameters by which couplings provide transparent insight into how perturbation of the parameters behavior collective statistics depend the most sensitively.

Here, we focus on the sensitive degrees of freedom, using them to shared is high, whereas when it is fairly insensitive the informa-

extremely sensitive to the distribution of data, then the information to orthogonal directions in the tangent space of the model manifold, the Fisher information, the curvature of the divergence, the Hessian, which is also known as

If we know \( p(s; \theta) \) over a set of discrete states \( s \) and parameterized by parameters \( \theta \), how do we measure how different one model is from another? The Kullback-Leibler (KL) divergence is one such measure that tells us how much information is necessary to reach a distribution \( p(s; \theta) \) if we know \( p(s; \theta) \) (9, 45).

\[
D_{KL}[p(s; \theta) || p(s; \theta')] = \sum_s p(s; \theta) \ln \left( \frac{p(s; \theta)}{p(s; \theta')} \right). \tag{C.8}
\]

In the limit where the two distributions are infinitesimally close to one another, the KL divergence becomes a metric. The constant and linear terms go to zero, and the first nonzero term is the curvature of the divergence, the Hessian, which is also known as the Fisher information,

\[
F_{ij} = \left. \frac{\partial^2 D_{KL}[p(s; \theta) || p(s; \theta')] \div \partial \theta_i \partial \theta_j}{\theta_i = \theta'_i, \theta_j = \theta'_j} \right|_{\theta = \theta}. \tag{C.9}
\]

Thus, the FI is a description of how quickly the probability distribution changes if we move along various directions in parameter space. Because it is a metric, the eigenvectors of the Hessian correspond to orthogonal directions in the tangent space of the model manifold, where the eigenvalues describe how quickly the manifold is varying along these directions.

The Fisher information also measures how much information about a parameter is in a random sample (8). When a parameter is extremely sensitive to the distribution of data, then the information shared is high, whereas when it is fairly insensitive the information shared is low. This is described formally by the Cramér-Rao bound, which sets a lower bound on the precision of an unbiased estimator for the parameters (8). This picture, more formally, has been used as technique for model reduction, removing degrees of freedom in parameter space to which the system is insensitive (46). Here, we focus on the sensitive degrees of freedom, using them to identify components interesting because their behavior is precisely determined by the statistics of the data, or equivalently on whose behavior collective statistics depend the most sensitively.

We propose using as parameters aspects of the system that provide transparent insight into how perturbation of the parameters affects the system. In physical systems, it is natural to consider the couplings \( J_{ij} \), or more generally the Langrangian multipliers from the maximum entropy formulation, as the parameters by which to control the system because they are experimentally accessible (e.g., a magnetic system can be tuned by an applied field or by changing temperature that modulates all couplings by a factor). For a statistical model of a social system, however, the meaning of the terms in the energy functional are opaque and often nontrivial. In other words, we do not know how to access the coupling parameter \( J_{ij} \). Certainly, we could be methodical about it, calculate the corresponding changes in the set of pairwise correlations for a perturbation in \( J_{ij} \), but that would result in changes across all pairwise correlations in varying amounts. This change may be difficult to effect in a social system when opportunities for control are often limited.

This impracticality suggests a different approach, where we instead consider how the measured behavior of the system might be perturbed directly since these are straightforward to measure. This reasoning leads us to consider as parameters the observables. Formally, the observables for a maxent model are the conjugate variables to the Langrangian multipliers as given by the Legendre transform (47). There is no difference in knowing one or the other: the transformation is a one-to-one mapping. For the pairwise maxent model, the “natural parameters” are the couplings and their conjugate the pairwise correlations

\[
J_{ij} \leftrightarrow \langle s_i s_j \rangle
\]

The Fisher information diverges at the boundaries of the parameter space \( p = 0 \) and \( p = 1 \) because that is where a finite change in \( p \) can lead to a diverging information distance. More closely with the perturbation considered in the main text, we could insist that the coin “mimic” a perfectly biased coin such that

\[
\tilde{p} = \epsilon + (1 - \epsilon)p
\]

Under this scenario, the Jacobian captures the fact that the coin’s bias makes no finite jump near \( p = 1 \), but changes ever more slowly when it is almost a perfectly biased coin. As a result,

\[
F = \frac{1}{\ln 2} \left( \frac{1}{p} - 1 \right).
\]

which goes to zero at \( p = 1 \). Now, consider a maxent version of this problem which is the nonlinear transformation

\[
p = \frac{\tanh(x) + 1}{2}
\]

where the field determining the bias is \( x \) and

\[
F = \frac{\text{sech}(x^2)}{2 \ln 2}.
\]
In contrast with using \( p \) as the parameter, the quantity in Eq. 17 peaks at \( p = 1/2 \) and decays to 0 at the boundaries, and an infinite change in \( x \) is necessary to reach \( p = 0 \) and \( p = 1 \). Of course, we could have chosen any possible model, choosing instead of the maxent transformation in Eq. 16, our favorite nonlinear transformation. Thus, by choosing our favorite model, we would end up effectively specifying the FI. This is generally not an issue if one cares about measuring the relationship between a particular model and the data, but it does become an issue if one cares more about the statistics of the data rather than of the particular model specified.

If we restrict ourselves to perturbing \( p \), we ensure that the choice of perturbation does not depend on the choice of model. Additionally when the probability distribution is matched exactly, there is no dependence on the model class. In the special case of the biased coin, the probability distribution is specified exactly by a single parameter. Assuming we can measure it with infinite precision and we have a model that can fit the measured \( p \) exactly (e.g., a model limited to \( 0 \leq p \leq 1/2 \) does not count), the calculation of FI — whether Eq. 13, 15, or some other choice — is concretely defined. More generally, data resolution is not perfect and so we must infer probabilities for configurations that we have not observed. As an example, the pairwise maxent approach assumes that the pairwise marginal distributions are known exactly, but the higher-order joint probabilities are assumed to conform to the maxent principle. When any such model feature is used in the calculation of the FI, clearly the FI will depend on the assumptions of the model. For the pairwise maxent model, this means that the calculation of FI on features of the distribution that are constructed explicitly from pairwise marginals matches the data exactly, but in general the distribution of majority-minority divisions does depend on the model assumptions. This is not always unfavorable: if the higher-order terms decrease in importance such that they behave as small perturbations on top of the pairwise model, the FI will show weaker dependence on these corrections (44). Thus, we propose a generalizable approach that links minimal, maxent models with a simple class of perturbations defined on the range of relative component behavior.

**D. Calculation of the Fisher information matrix (FIM)**

Here, we calculate the FIM for the transformation described in Eqs 1 and 2 and go through some examples to show how to calculate the FIM. We go into some detail with the derivation to make clear how to perform such a calculation for those less familiar with maxent models and information geometry.

In Eqs 1 and 2, we consider how the correlations between component \( y \) and all other components \( x' \) change when component \( y \) appears to vote more like component \( x \). To effect this perturbation, we use a parameter \( \epsilon \to 0 \) that leads to a linear change in the couplings \( J_{y'y'} \) as described by the rate of change \( dd_{y'y'} \), where we are taking a total derivative with respect to the change in the pairwise probabilities described by the vector \( r(s_k = s_y) \). To obtain this derivative, we perturb to first order in \( \epsilon \) the expression for the pairwise correlations (Eq B.2) to obtain the self-consistent equation

\[
\langle s_k s_y \rangle_{pert} - \langle s_k s_y \rangle = \frac{1}{2} \sum_{x'x''} J_{x'y'} \left( \left( \langle s_k s_x s_y s_{x''} \rangle - \langle s_k s_y \rangle_{pert} \langle s_x s_{x''} \rangle \right) \right). \tag{D.18}
\]

By self-consistent, we are referring to the fact that the new pairwise correlations after perturbation \( \langle s_k s_y \rangle_{pert} \) depend on the change in the couplings \( \Delta J_{xy} \), so the perturbation to the couplings determine quantitatively on both sides of Eq D.18.** The coupling perturbations, \( \Delta J_{xy} \), are related to the linear response of the couplings to change in the collective statistics induced by perturbing the pair of components \( x \) and \( y \).

\[
d J_{y'y'} = \lim_{\epsilon \to 0} \Delta J_{xy} J_{x'y'}/\epsilon. \tag{D.19}
\]

The resulting matrix of new couplings is

\[
J_{y'y'} = J_{y'y'} + \Delta J_{xy} J_{x'y'}. \tag{D.20}
\]

\[
p'(s'; \{J_{y'y'} \}) = p(s'; \{J_{y'y'} + \Delta J_{xy} J_{x'y'} \}). \tag{D.21}
\]

The set of the perturbations \( \Delta J_{xy} J \equiv \{ \Delta J_{xy} J_{x'y'} \} \) appear in Figure A.2C.

Eqs D.18–D.21 describe the numerical algorithm for calculating the changes in the statistics of the system under the perturbation described in Eqs 1 and 2. Note that the algorithm implicitly depends on \( \epsilon \), which must be taken to zero. The remaining calculations are to coarsen the full distribution \( p(s) \) to \( q(k) \), the distribution of \( k \) votes in the majority and to calculate the FIM on \( q \). For pedagogical clarity, we will first show how to calculate the FIM without coarse-graining.

There is a simple, intuitive form for the FIM for maxent models. Under an infinitesimal change in the parameters such that the energy of each voting configuration \( E(s) \to E(s) + \Delta E(s) \), we can expand

\[
p'(s) = \sum_{s'} e^{-E(s') - \Delta E(s')} \approx \sum_{s'} e^{-E(s)} \left( 1 - \Delta E(s) \right) \tag{D.22}
\]

\[
\approx p(s) [1 - \Delta E(s)] [1 + (\Delta E)]
\]

\[
= p(s) [1 + \Delta E - \Delta E^2] + O(\Delta E^3).
\]

Now calculating the Kullback-Leibler divergence to second order,

\[
D_{KL}[p||\tilde{p}] = \sum_s p(s) \log[p(s) - \tilde{p}(s)]
\]

\[
= \frac{1}{2} \big( (\Delta E - \Delta E^2)^2 \big) + O(\Delta E^3). \tag{D.23}
\]

The constant term is zero because the \( D_{KL}[p||\tilde{p}] = 0 \) and the linear term is zero because the changes to the probability function under perturbation must sum to zero to preserve normalization. The FI is the second order term, or the curvature, so we must send the norm change in the energy \( |\Delta E| \to 0 \),

\[
F = \lim_{\epsilon \to 0} \epsilon^{-2} \left( \Delta E^2 - \Delta E^2 \right), \tag{D.24}
\]

where \( \epsilon \) is defined in Eq D.19. For the symmetrized pairwise maxent model class, \( \Delta E \) is the sum over all couplings. Thus, the FI of the distribution over the full state space \( p(s) \) has a simple form in terms of the change in energy for maxent models.

Alternatively, the result from Eq D.24 can be expressed as a matrix of correlation functions. In other words, the correlation functions are the linear response functions for perturbations to the natural parameters, here the couplings \( J_{xy} \). As the simplest example,

**Another way to describe Eq D.18 is as the linear combination of the linear response functions of every pairwise correlation to a change in the corresponding coupling, also known as the susceptibility.**
consider an Ising model under perturbation to a particular coupling \( J_{xy} \). Using our form Eq D.24 for the FI and \( \Delta J_{xy} \equiv J_{xy} - J_{xy}^\prime \),

\[
F_{xy} = \lim_{\Delta J_{xy} \to 0} \frac{\Delta J_{xy}^2}{2} \left( (\Delta J_{xy} s_x s_y - \Delta J_{xy} \langle s_x s_y \rangle)^2 \right) . \tag{D.25}
\]

The perturbations to the couplings \( \Delta J_{xy} \) do not depend on the state \( s \), so they can be pulled out of the averages to obtain

\[
\begin{align*}
( s_x^2 s_y^2) - \langle s_x s_y \rangle^2 \\
1 - \langle s_x s_y \rangle^2 .
\end{align*} \tag{D.26, D.27}
\]

The diagonal entries of the FIM are the variance of the pairwise agreement probabilities, while the off-diagonal elements are the covariance of these. Working through the same calculation as before but with this general formulation of a maxent model, we find for the FI,

\[
F = \lim_{\epsilon \to 0} \epsilon^{-2} \left\{ \sum_n \Delta \theta_n^2 \left[ \langle f_n^2 \rangle - \langle f_n \rangle^2 \right] + \sum_{n,m} \Delta \theta_n \Delta \theta_m \left[ \langle f_n f_m \rangle - \langle f_n \rangle \langle f_m \rangle \right] \right\} , \tag{D.28}
\]

where the \( \Delta \theta_n \) depend implicitly on \( \epsilon \). As noted earlier, the perturbation in the pairwise agreement probabilities leads to a nontrivial combination of changes to the entire vector of couplings. As a result, the FI in Eq D.28 contains cross terms between all pairwise correlations and the change in the Lagrangian multipliers \( \Delta \theta_n \) each come with a factor of the Jacobian relating changes in the pairwise marginals to the couplings as described by Eq D.18.

For the analysis in the main text, however, there is an additional step. We do not consider the full state space, but coarse-grain each \( p(s) \) to the distribution of \( k \) votes in the majority \( q(k) \). As a result, we are not calculating the variance in the energies for the pairwise maxent model as described in Eq D.24, but the variance in the logarithm of the sum of all terms in the partition function with \( k \) voters in the majority. We label the set of all states with \( s_k \) to write

\[
q(k) = \sum_{s \in S_k} p(s) = \frac{1}{Z} \sum_{s \in S_k} e^{-E(s)} = \frac{1}{Z} e^{-E_{maj}(k)} \tag{D.29}
\]

\[
E_{maj}(k) \equiv - \ln \left( \sum_{s \in S_k} e^{-E(s)} \right) . \tag{D.30}
\]

Eq D.30 defines an effective "k majority" energy such that under perturbation to a probability \( p(s) \) as indicated by \( \Delta_{xy} \)

\[
F_{xy} = \lim_{\epsilon \to 0} \epsilon^{-2} \left( \langle \Delta_{xy} E_{maj}^2 \rangle - \langle \Delta_{xy} E_{maj} \rangle^2 \right) . \tag{D.31}
\]

Eq D.31 is the form that the limit in Figure 1C takes. To summarize the algorithm, we first find the total derivative of each coupling \( dJ_{xy}^\prime \) with respect to the change in the pairwise marginals as explained in Eq D.19. Then, we calculate the change in the distribution of \( k \) votes in the majority for both the model without the perturbation \( q(k) \) and with \( q(k) \) for a range of small values \( \epsilon \) as in Eq D.23. By comparing these two distributions for increasingly smaller \( \epsilon \), we estimate numerically the FIM, relying on the definition of an "effective" energy \( E_{maj} \) as in Eq D.30 to deal with issues in numerical precision that may arise when comparing ratios of floating point numbers. These steps generate the FIM as shown in Figure 1C with which we calculate the eigenvalue spectrum to measure our pivotal voters.

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**E. Dissenting coalitions**

In Figure 2, we project the eigenvectors onto the probabilities of dissenting coalitions to obtain a detailed picture of how the parameter directions obtained from the FIM affect dissenting coalitions. Such a projection involves taking the sum over all the probabilities of the states with the particular dissenting bloc and calculating the effective energy. Expanding the log-likelihood to first order, we calculate the rate at which this probability changes to be

\[
d \ln q(k) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\Delta E_{maj}(k) - \langle \Delta E_{maj}(k) \rangle) . \tag{E.32}
\]

The limit \( \epsilon \to 0 \) refers to an infinitesimal perturbation of \( q(k) \) along the first eigenvector of the FIM. Then, the rate of change in the log-likelihood simplifies to comparing the change in the effective majority energy \( E_{maj}(k) \) with the average change across \( p(s) \).

**F. Measure of asymmetry**

As a way of measuring the heterogeneity between the components of a system, we calculate the asymmetry of the "eigenmatrices" of the Fisher information matrix as defined in Figure 1D. Given that the matrices are normalized \( \sum_{xy} v_{xy}^2 = 1 \), the total asymmetry can be written

\[
A = \frac{1}{2} \left( 1 - \sum_{xy} v_{xy} v_{yx} \right) . \tag{F.33}
\]

Thus, we might think of the asymmetry as a measure of correlation between the entries in the upper triangle and lower triangle of the eigenmatrix. When they are perfectly correlated such that the matrix is symmetric, \( A = 0 \). If the entries \( v_{xy} \) are completely uncorrelated with their partners in the transpose \( v_{yx} \), then \( A = 1/2 \). When they are anti-correlated, the summation in Eq F.33 can become negative and \( A > 1/2 \).

As an example, consider the asymmetry for a 2x2 matrix

\[
\begin{pmatrix}
0 & \beta \\
\beta & \alpha
\end{pmatrix} . \tag{F.34}
\]

The normalization constrains \( \alpha \) and \( \beta \) to the unit circle,

\[
\alpha = \pm \sqrt{1 - \beta^2} . \tag{F.35}
\]
Fig. G.11. Retrospective time series analysis of the SPDR for the two most pivotal indices XLE and XLU and least pivotal XLY. We use a moving windowed window of duration \( t = 256 \) days and a shift of \( \Delta t = 50 \) days. The width of the moving window is delimited by the gray box. We compare the maximum of the normalized eigenvalues of the covariance matrix with \( \eta^* \), the projection of the windowed time series fluctuation onto the stock index principal subspace eigenvector (Eq. G.37). Lines are drawn for readability.

Fig. G.12. Comparison of the cumulative distribution functions (CDF) of (left) the total asymmetry \( A \) and (right) the dominant pivotal measure across CA state legislatures and the US House of Representatives and Senate. We compare the distributions of \( \lambda_{\text{max}} \) in Figure G.13.

In Figure F.10A, we have colored the upper and lower halves of this circle (for positive and negative values of \( \alpha \)) by different colors. Now calculating the total asymmetry,

\[
A = \frac{1}{2} \pm \frac{1}{2} \beta \sqrt{1 - \beta^2}. \tag{F.36}
\]

We plot Eq. F.36 in Figure F.10B and again color the curves differently depending on the half of the unit circle that we are tracing out. When \( \beta = -1 \), normalization asserts that \( \alpha = 0 \) and the total asymmetry \( A = 1/2 \). As we increase \( \beta \), we can follow \( \alpha \) along the positive (negative) route which leads to maximization (minimization) of \( A \).

\[
\beta \rightarrow \infty \quad \Rightarrow \quad J_{ij} = \lambda_{\text{max}} \quad \Rightarrow \quad A = 1/2 \tag{F.37}
\]

\[
\beta \rightarrow -1 \quad \Rightarrow \quad J_{ij} = 0 \quad \Rightarrow \quad A = 0 \tag{F.38}
\]

As we keep increasing to \( \beta = 0 \), we return to \( A = 1/2 \) and have effectively swapped the roles of \( \alpha \to -\beta \) and \( \beta \to \alpha \) (a rotation of the matrix in Eq. F.34 by \(-\pi/2\)).

G. Time series analysis of SPDR

The temporal fluctuations of the sector indices in the SPDR represent potentially useful information about changing economic conditions. As a preliminary demonstration of the type of analysis that may be interesting in this context, we consider a retrospective analysis of how local temporal fluctuations in the market are reflected in the subspace eigenvectors of the FIM.

To do this, we take a long temporal window \( t = 256 \) days that allows us to obtain a precise estimate of the distribution of configurations in a time window \( p_{\text{win}}(s) \). Then, we minimize the KL divergence between \( p_{\text{win}}(s) \) and the pairwise maxent model solved on the entire data set with change in the couplings \( J_{ij} = J_{ij} + \eta \Delta J_{ij} \) constrained to be along the principal stock index subspace eigenvector.

\[\Delta J_{ij} \text{ by adjusting the coefficient } \eta. \]

\[\eta^* = \arg \min_{\eta} \sum_{s} p_{\text{win}}(s) \ln \left( \frac{p_{\text{win}}(s)}{p(s; \{J_{ij} + \eta \Delta J_{ij}\})} \right). \tag{G.37}\]

The magnitude of this coefficient \(|\eta^*|\) is a measure of how strongly the fluctuations in the windowed time series are reflected in the direction of parameter space specified by the subspace eigenvector. As we show in Figure G.11, the fluctuations show patterns that diverge at many points from the maximum of the normalized eigenvalue of the windowed covariance matrix, a measure used to determine when economic conditions are changing (13). In particular, we note that periods of time where the best fit value \( \eta^* \) between the various stock indices are correlated or anti-correlated may be useful indicators.

Although it remains to relate these patterns to recognized features of the time series, this presents a potentially useful complement to existing tools for analyzing market data.

H. Comparison of CA state and federal legislatures

Are institutional differences captured in our measures of the eigenvectors of the FIM? As an example, we compare the CA Assembly and Senate with the US House of Representatives and Senate using our measures of the dominant pivotal measure and total asymmetry defined in Figure G.12.

When we inspect the distribution of the principal pivotal component \( \lambda_{\text{max}} \), we find significant distinction between the state vs. federal levels. With the Kolmogorov-Smirnov (KS) test — testing whether or not the largest difference between the two sample cumulative distribution functions (CDF) rules out a coincident underlying distribution — summarized in Figure G.13, we show the KS statistics to be larger and the \( p \)-values smaller between state and federal levels. Thus, the distribution of the principal pivotal measure is one way of distinguishing between the different levels of legislature. Notwithstanding further questions about the choice of coarse-graining — which was implemented following the same

\[
\Delta J_{ij} \text{ by adjusting the coefficient } \eta. \]

\[\eta^* = \arg \min_{\eta} \sum_{s} p_{\text{win}}(s) \ln \left( \frac{p_{\text{win}}(s)}{p(s; \{J_{ij} + \eta \Delta J_{ij}\})} \right). \tag{G.37}\]

\[\Delta J_{ij} \text{ by adjusting the coefficient } \eta. \]

\[\eta^* = \arg \min_{\eta} \sum_{s} p_{\text{win}}(s) \ln \left( \frac{p_{\text{win}}(s)}{p(s; \{J_{ij} + \eta \Delta J_{ij}\})} \right). \tag{G.37}\]

\[\Delta J_{ij} \text{ by adjusting the coefficient } \eta. \]

\[\eta^* = \arg \min_{\eta} \sum_{s} p_{\text{win}}(s) \ln \left( \frac{p_{\text{win}}(s)}{p(s; \{J_{ij} + \eta \Delta J_{ij}\})} \right). \tag{G.37}\]

\[\Delta J_{ij} \text{ by adjusting the coefficient } \eta. \]

\[\eta^* = \arg \min_{\eta} \sum_{s} p_{\text{win}}(s) \ln \left( \frac{p_{\text{win}}(s)}{p(s; \{J_{ij} + \eta \Delta J_{ij}\})} \right). \tag{G.37}\]

\[\Delta J_{ij} \text{ by adjusting the coefficient } \eta. \]

\[\eta^* = \arg \min_{\eta} \sum_{s} p_{\text{win}}(s) \ln \left( \frac{p_{\text{win}}(s)}{p(s; \{J_{ij} + \eta \Delta J_{ij}\})} \right). \tag{G.37}\]
procedure outlined for the CA legislatures in Section I — our findings suggest that the structure of CA state legislatures makes them more conducive to the presence of pivotal voting blocs that precisely determine the distribution of majority-minority coalitions.

After measuring the total asymmetry for all the sessions we solved, we find that the distributions of total asymmetry for the House and the Senate are strongly similar, but those of the CA Assembly and CA Senate are not. We find that the large difference between the House and the CA Senate and between the Senate and the CA Senate is not statistically significant, given the number of data points. However, all three are significantly different from the CA Assembly with $p < 0.05$, suggesting that some systematic, institutional pattern might be gleaned from inspecting total asymmetry.

I. Additional notes on data sets

A. US state supreme courts. We obtained the latest data set from the State Supreme Court Data Project (SSCDP) and used their binary coding of justice votes (14).

We show the total asymmetry for all the natural courts on the Alaska and New Jersey Supreme Courts we considered in Figure I.15. We only considered natural courts with at least 100 where the full complement of justices were voting. As we mention in the main text, there is variation in the measured value of asymmetry that makes it unclear whether or not there is relationship between the total asymmetry and the codified institutional rules of voting.

B. SCOTUS. We use data from the Supreme Court Database Version 2016 Release 1, taking their binary coding of majority-minority votes (12). This same data set and version has been analyzed previously. See Refs (21, 22).

C. SPDR. The SPDR Select Sector indices track the Standard & Poor’s (S&P) and Morgan Stanley Capital International (MSCI) Global Industry Classification Standard (GICS) sectors. As described on Wikipedia, GICS “is an industry taxonomy developed in 1999 by MSCI and S&P for use by the global financial community. The GICS structure consists of 11 sectors, 24 industry groups, 69 industries and 158 sub-industries into which S&P has categorized all major public companies. The system is similar to ICB (Industry Classification Benchmark), a classification structure maintained by FTSE [Financial Times Stock Exchange] Group.”

We focus on these assets and their adjusted price action because (1) they are the most heavily-traded and representative sector assets in the world; so their prices and volumes reflect actual interest in exposure to the sectors, (2) they have been traded daily without exception for over 20 years, and (3) unlike the Dow indices, the S&P indices are not subject to effects of price-weighting such as reverse-split over-weighting. The historical price data is available online on Yahoo Finance.

D. Twitter. We analyze one of the communities from the data considered in Ref (17). In this work, the authors divide the Twitter community into smaller subcommunities using the CNM algorithm (49). We take one example from their K-pop community with 10 individuals.

E. CA Assembly and Senate. Session records were obtained from Prof. Jeff Lewis’ scrape of the CA legislature’s public data API (15). For all sessions from 1993–2017, we solved the W-Nominate model using the code provided in Ref (29). We then removed any voter who did not participate in more than 20% of the votes, rank-ordered the voters by the first W-Nominate dimension, and divided them as equally as possible into 9 groups as shown for the 1999–2000 session in Figure I.14.

For the results of bootstrap sampling to calculate error bars, we found that 3% of the Assembly samples showed significant error from the fit correlations because of numerical precision issues. This is generally an issue for systems that are poised near the boundaries of the model manifold where the couplings become large. For the error bars on the normalized subspace eigenvalues, however, the contribution from these three missing samples is negligible.

In Figure 4, the most pivotal bloc that we observe, Bloc 8, is constituted of Republicans whilst the State Assembly’s majority is held by the Democratic party. Indeed, we find that the mutual information between the vote of Bloc 8 with the majority vote across the blocs is $I_{8} = 0.96$ bits versus that of a Democratic Bloc 1, $I_{1} = 0.17$ bits. This measure of correlation indicates that this Republican bloc is, like a median, highly predictive of the majority outcome across all of these blocs and, additionally, is pivotal.

F. US House of Representatives and Senate. Data was obtained from Voteview (16). We analyze the 80–113th Senate sessions and the 80–115th House sessions. The 80th congressional session started in 1947 and each session lasts two years.

See Refs (21, 22).
