Research Article

Stacked Central Configurations for the Spatial Nine-Body Problem

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We show the existence of the twisted stacked central configurations for the 9-body problem. More precisely, the position vectors $x_1, x_2, x_3, x_4,$ and $x_5$ are at the vertices of a square pyramid $\Sigma$; the position vectors $x_6, x_7, x_8,$ and $x_9$ are at the vertices of a square $\Pi$.

1. Introduction and Main Results

The classical $n$-body problem \cite{1,2} concerns the motion of $n$ mass points moving in space according to Newton’s law:

$$m_i \ddot{x}_i = - \sum_{j=1, j \neq i}^{n} \frac{m_j (x_j - x_i)}{r_{ij}^3}, \quad i = 1, 2, \ldots, n. \quad (1)$$

Here, $x_i \in \mathbb{R}^d$ is the position of mass $m_i > 0$, the gravitational constant is taken equal to 1, and $r_{ij} = |x_i - x_j|$ is the Euclidean distance between $x_i$ and $x_j$.

The space of configuration is defined by

$$X = \left\{ (x_1, \ldots, x_n) \in \left( \mathbb{R}^d \right)^n : x_j \neq x_i \quad \forall i \neq j \right\}, \quad (2)$$

while the center of mass is given by

$$c = \frac{m_1 x_1 + \cdots + m_n x_n}{M}, \quad (3)$$

where $M = m_1 + \cdots + m_n$ is the total mass.

A configuration $x = (x_1, \ldots, x_n) \in X$ is called a central configuration \cite{2,3} if there exists a constant $\lambda$, called the multiplier, such that

$$-\lambda (x_i - c) = \sum_{j=1, j \neq i}^{n} \frac{m_j (x_j - x_i)}{r_{ij}^3}, \quad i = 1, 2, \ldots, n. \quad (4)$$

It is easy to see that a central configuration remains a central configuration after a rotation in $\mathbb{R}^d$ and a scalar multiplication. More precisely, let $A \in \text{SO}(d)$ and $a > 0$, if $x = (x_1, \ldots, x_n)$ is a central configuration, so are $A x = (Ax_1, \ldots, Ax_n)$ and $a x = (ax_1, \ldots, ax_n)$.

Two central configurations are said to be equivalent if one can be transformed to the other by a scalar multiplication and a rotation. In this paper, when we say a central configuration, we mean a class of central configurations as defined by the above equivalent relation.

Central configurations of the $n$-body problem are important because they allow the computation of homographic solutions; if the $n$ bodies are heading for a simultaneous collision, then the bodies tend to a central configuration (see \cite{3,4}); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see \cite{5}).

In this paper, we are interested in spatial central configurations, that is, $d = 3$. In 2005, Hampton \cite{6} provides a new family of planar central configurations for the 5-body problem with an interesting property: the central configuration has a subset of three bodies forming a central configuration of the 3-body problem. The authors \cite{7} find new classes of central configurations of the 5-body problem which are the ones studied by Hampton \cite{6} having three bodies in the vertices of an equilateral triangle, but the other two, instead of being located symmetrically with respect to a perpendicular bisector, are on the perpendicular bisector. The
stacked central configurations studied by Hampton [6] were completed by Llibre et al. [8] (see also [9]).

Zhang and Zhou [10] showed the existence of double pyramidal central configurations of \( N + 2 \)-body problem. The authors [11–13] provided new examples of stacked central configurations for the spatial 7-body problem where four bodies are at the vertices of a regular tetrahedron and the other three bodies are located at the vertices of an equilateral triangle.

In this paper, we find new classes of stacked spatial central configurations for the 9-body problem which have five bodies at the vertices of a square pyramid, and the other four bodies are located at the vertices of a square. More precisely, the spatial central configurations considered here satisfy the following (see Figure 1): the position vectors \( x_1, x_2, x_3, x_4, \) and \( x_5 \) are at the vertices of a square pyramid \( \Sigma \); the position vectors \( x_6, x_7, x_8, \) and \( x_9 \) are at the vertices of a square \( \Pi \).

Without loss of generality, we can assume that
\[
\begin{align*}
  x_1 &= (1,0,0), & x_2 &= (0,1,0), & x_3 &= (-1,0,0), \\
  x_4 &= (0,-1,0), & x_5 &= (0,0,h), & x_6 &= (x,0,y), \\
  x_7 &= (0,x,y), & x_8 &= (-x,0,y), & x_9 &= (0,-x,y),
\end{align*}
\]

\( (5) \)

where \( x > 0, y \in \mathbb{R}, \) and \( y \neq 0; \) the positive constant \( h \) satisfies the equation
\[
\frac{2}{r_{15}^3} = \frac{1}{r_{12}^3} + \frac{1}{r_{13}^3},
\]

\( (6) \)

(see [10] and the references therein); that is, \( h = 1.26276522 \).

The main results of this paper are the following.

**Theorem 1.** Consider the spatial configurations according to Figure 1, in order that the nine mass points are in a central configuration, the following statements are necessary:

1. The masses \( m_1, m_2, m_3, \) and \( m_4 \) must be equal;
2. The masses \( m_6, m_7, m_8, \) and \( m_9 \) must be equal.

**Theorem 2.** There exist points \( (x_0, y_0) \in T^{-1}(0) \cap D \) (see Figure 2) such that the nine bodies take the coordinates
\[
\begin{align*}
  x_1 &= (1,0,0), & x_2 &= (0,1,0), \\
  x_3 &= (-1,0,0), & x_4 &= (0,-1,0), \\
  x_5 &= (0,0,h), & x_6 &= (x_0,0,y_0), \\
  x_7 &= (0,x_0,y_0), & x_8 &= (-x_0,0,y_0), \\
  x_9 &= (0,-x_0,y_0).
\end{align*}
\]

\( (7) \)

Then, there are positive solutions of \( m_1, m_5, m_6 \) such that these bodies form a spatial central configuration according to Figure 1.

The proofs of the theorems are given in the next sections.

\[ \text{Figure 1: The configuration for the 9-body problem.} \]

\[ \text{Figure 2: The region } D. \]

\[ \text{2. Proof of Theorem 1} \]

For the spatial central configurations, instead of working with (4), we consider the Dziobek-Laura-Andoyer equations (see [9, 11–13] and the references therein):
\[
f_{ijk} = \sum_{l=1,l \neq i,j,k}^{n} m_l (d_{ij} - d_{il}) \Delta_{ijk} = 0 \quad (8)
\]

for \( 1 \leq i < j \leq n, k = 1, \ldots, n, k \neq i, j. \) Here, \( d_{ij} = 1/r_{ij}^3 \) and \( \Delta_{ijk} = (x_i - x_j) \cdot (x_i - x_k) \cdot (x_i - x_l). \) Thus, \( \Delta_{ijk} \) gives six times the signed volume of the tetrahedron formed by the bodies with positions \( x_i, x_j, x_k, \) and \( x_l; \) (8) is a system of \( n(n-1)(n-2)/2 \) equations.
For the 9-body problem, (8) is a system of 252 equations. According to Figure 1, our class of configurations with nine bodies must satisfy

\begin{align*}
    r_{12} &= r_{23} = r_{34} = r_{14} = \sqrt{2}, \quad r_{13} = r_{24} = 2, \\
    r_{67} &= r_{78} = r_{89} = r_{69} = \sqrt{2}x, \quad r_{68} = r_{79} = 2x, \\
    r_{16} &= r_{27} = r_{38} = r_{49} = \sqrt{(x-1)^2 + y^2}, \\
    r_{17} &= r_{19} = r_{26} = r_{28} = r_{37} = r_{39} = r_{46} \\
    &= r_{48} = \sqrt{x^2 + 1 + y^2}, \\
    r_{18} &= r_{29} = r_{36} = r_{47} = \sqrt{(x+1)^2 + y^2}, \\
    r_{15} &= r_{25} = r_{35} = r_{45} = \sqrt{1 + h^2}, \\
    r_{56} &= r_{57} = r_{58} = r_{59} = \sqrt{x^2 + (y-h)^2}.
\end{align*}

(9)

Due to assumption (5) and the definition of \( \Delta_{ijkl} \), we have several symmetries in the signed volumes.

By using the symmetries and the properties of \( \Delta_{ijkl} \), we obtain the following results.

Lemma 3. In order to have a spatial central configuration according to Figure 1, a necessary condition is that the masses \( m_1, m_2, m_3, \) and \( m_4 \) must be equal.

Proof. It is sufficient to consider the equations \( f_{687} = 0 \) and \( f_{796} = 0 \):

\begin{align*}
    f_{687} &= (m_1 - m_3)(d_{16} - d_{18}) \Delta_{6871} = 0, \\
    f_{796} &= (m_2 - m_4)(d_{16} - d_{18}) \Delta_{7962} = 0.
\end{align*}

(10)

For our class of central configurations, we have \( d_{16} - d_{18} \neq 0 \), \( \Delta_{6871} \neq 0 \), and \( \Delta_{7962} \neq 0 \). So the above equations hold if and only if \( m_1 = m_3, m_2 = m_4 \). Consider the expression of \( f_{678} = 0 \):

\begin{align*}
    f_{678} &= (m_1 - m_2)(d_{16} - d_{17}) \Delta_{6781} \\
    &\quad + (m_3 - m_4)(d_{18} - d_{17}) \Delta_{6783} = 0.
\end{align*}

(11)

Substituting \( m_1 = m_3, m_2 = m_4 \) into the above equation, we have

\[ f_{678} = (m_1 - m_2)(d_{16} + d_{18} - 2d_{17}) \Delta_{6781} = 0. \]

(12)

For our class of central configurations, we have \( d_{16} + d_{18} - 2d_{17} \neq 0 \), since the function \( g(x) = x^{-3/2} \) is convex for all \( x > 0 \), and \( \Delta_{6781} \neq 0 \). So the above equation holds if and only if \( m_1 = m_2 \). So statement 1 of Theorem 1 is proved. \( \square \)

Lemma 4. If the configuration, according to Figure 1, is a central configuration, a necessary condition is that the masses \( m_6, m_7, m_8, \) and \( m_9 \) must be equal.

Proof. It is sufficient to consider the equations \( f_{132} = 0 \) and \( f_{241} = 0 \):

\begin{align*}
    f_{132} &= (m_6 - m_8)(d_{16} - d_{18}) \Delta_{1326} = 0, \\
    f_{241} &= (m_7 - m_9)(d_{16} - d_{18}) \Delta_{2417} = 0.
\end{align*}

(13)

For our class of central configurations, we have \( d_{16} - d_{18} \neq 0 \), \( \Delta_{1326} \neq 0 \), and \( \Delta_{2417} \neq 0 \). So the above equations hold if and only if \( m_6 = m_8, m_7 = m_9 \). Consider the expression of \( f_{123} = 0 \):

\begin{align*}
    f_{123} &= (m_6 - m_7)(d_{16} - d_{17}) \Delta_{1236} \\
    &\quad + (m_6 - m_9)(d_{16} - d_{17}) \Delta_{1238} = 0.
\end{align*}

(14)

Substituting \( m_6 = m_8, m_7 = m_9 \) into the above equation, we have

\[ f_{123} = (m_6 - m_7)(d_{16} + d_{18} - 2d_{17}) \Delta_{1236} = 0. \]

(15)

For our class of central configurations, we have \( d_{16} + d_{18} - 2d_{17} \neq 0 \), and \( \Delta_{1236} \neq 0 \). So the above equation holds if and only if \( m_6 = m_7 \). Hence, statement 2 of Theorem 1 is proved.

The proof Theorem 1 is completed. \( \square \)

We restrict the set of admissible masses to \( m_1 = m_5 = m_3 = m_4 = \alpha \) and \( m_6 = m_7 = m_8 = m_9 = \beta \). Substituting \( m_1 = m_5 = m_3 = m_4 = \alpha \) and \( m_6 = m_7 = m_8 = m_9 = \beta \) into (8), they reduce to the following 4 equations:

\begin{align*}
    f_{152} &= \beta (d_{16} + d_{17} - 2d_{56}) \Delta_{1526} \\
    &\quad + (d_{17} + d_{18} - 2d_{56}) \Delta_{1528} = 0, \\
    f_{162} &= \alpha (d_{12} + d_{14} - d_{17} - d_{18}) \Delta_{1623} \\
    &\quad + m_5 (d_{15} - d_{56}) \Delta_{1625} + \beta (d_{17} + d_{18} - d_{67} - d_{58}) \Delta_{1628} = 0, \\
    f_{175} &= \alpha ((d_{12} - d_{16}) \Delta_{1752} + (d_{13} - d_{17}) \Delta_{1753} + (d_{12} - d_{18}) \Delta_{1754} \\
    &\quad + \beta ((d_{16} - d_{67}) \Delta_{1756} + (d_{18} - d_{67}) \Delta_{1758} + (d_{17} - d_{58}) \Delta_{1759} = 0, \\
    f_{562} &= \alpha ((d_{15} - d_{16}) \Delta_{5621} + (d_{15} - d_{18}) \Delta_{5623} + (d_{15} - d_{17}) \Delta_{5624} \\
    &\quad + (d_{56} - d_{58}) \Delta_{5629} + \beta ((d_{56} - d_{67}) \Delta_{5627} + (d_{56} - d_{68}) \Delta_{5628} + (d_{56} - d_{58}) \Delta_{5629} = 0. \quad (16)
\end{align*}

If we write \( f_{152} = \beta T = \beta (d_{16} + d_{17} - 2d_{56}) \Delta_{1526} + (d_{17} + d_{18} - 2d_{56}) \Delta_{1528} = 0 \), it follows that \( T = 0 \) in order to have central configurations. So in the following, we restrict our central configurations to the set \( T^{-1}(0) \).

Lemma 5. According to one’s assumptions and the set \( T^{-1}(0) \), (8) is satisfied if (17) and (18) are satisfied.
Proof. Under the assumptions (5), we have
\[ T = (d_{16} + 2d_{17} + d_{18} - 4d_{56}) (y - h) + hx (d_{16} - d_{18}) = 0; \]
(20)
that is,
\[ 4(y - h)d_{56} = (y - h)(d_{16} + 2d_{17} + d_{18}) + hx (d_{16} - d_{18}). \]
(21)
Substituting (21) into (19), we obtain the equation
\[ f_{175} = 0. \]
Hence in the set \( T^{-1}(0) \), \( f_{175} = 0 \) implies \( f_{562} = 0 \). This completes the proof.

From Lemma 5, in order to study central configurations according to Figure 1 in the set \( T^{-1}(0) \), it is sufficient to study the following 2 equations:
\[ f_{162} = 0, \quad f_{175} = 0. \]
(22)
Denote by \( A = (a_{ij}) \) the matrix of the coefficients of the homogeneous linear system in the variables \( \alpha, m_i, \beta \) defined by (22). Thus,
\[ a_{11} = (d_{12} + d_{13} - d_{17} - d_{18}) \Delta_{1623} \]
\[ = -2y \left( \frac{1}{2\sqrt{2}} + \frac{1}{8} - \frac{1}{(x^2 + 1 + y^2)^{3/2}} \right) \]
\[ - \frac{1}{(x + 1)^2 + y^2} - \frac{1}{(x - 1)^2 + y^2} \),
\[ a_{12} = (d_{15} - d_{56}) \Delta_{1625} \]
\[ = (-y - hx + h) \times \left( \frac{1}{(1 + h^2)^{3/2}} - \frac{1}{(x^2 + (y - h)^2)^{3/2}} \right), \]
\[ a_{13} = (d_{17} + d_{18} - d_{67} - d_{68}) \Delta_{1628} \]
\[ = -2xy \left( \frac{1}{(x^2 + 1 + y^2)^{3/2}} + \frac{1}{(x + 1)^2 + y^2} \right)^{3/2} - \frac{1}{8x^3} - \frac{1}{2\sqrt{2}x^2} \)
\[ a_{21} = (d_{12} - d_{16}) \Delta_{1752} + (d_{13} - d_{17}) \Delta_{1753} \]
\[ + (d_{12} - d_{18}) \Delta_{1754} \]
\[ = (y - h) \left( \frac{1}{(x + 1)^2 + y^2} - \frac{1}{(x - 1)^2 + y^2} \right) \]
\[ + hx \left( \frac{1}{4} + \frac{1}{\sqrt{2}} - \frac{1}{(x + 1)^2 + y^2} - \frac{2}{(x^2 + 1 + y^2)^{3/2}} \right) , \]
Let $\alpha = \left( \frac{\alpha}{m_4} \right)$. Then in order to get the spatial central configuration as Figure 1, we need to find a positive solution $\alpha, m_4, \beta$ of the following system:

$$Ax = 0,$$

where $T = 0$.

### 3. The Existence of Spatial Central Configurations

In order to prove the existence of positive solutions of (24) in the set $T^{-1}(0)$, it is sufficient to prove that the entries in each row of $A$ change the signs. So if the entries of some row of $A$ have the same signs, there are no admissible masses such that the bodies are in a central configuration according to Figure 1.

**Proof of Theorem 2.** Since the rank of matrix $A$ is two in the set $T^{-1}(0)$, there are nontrivial solutions of (24) in the set $T^{-1}(0)$.

Now we prove the existence of spatial central configurations according to Figure 1 for some points in the set $D$ (see Figure 2). In order to prove the existence of positive solutions of (24) in the set $T^{-1}(0)$, the entries $a_{21}, a_{23}$ of the second line in the matrix $A$ should have opposite signs. Thus, we consider the following set $D$, where $D$ is surrounded by curves $x = 0$, $y = 0$, $a_{21} = 0$, and $a_{23} = 0$.

In the set $D$, the entries of matrix $A$ have the following signs: $a_{21} > 0, a_{23} < 0$ (see Figures 3 and 4); $a_{11} > 0, a_{12} < 0$, $a_{13} > 0$ because the set $D$ is included in the set $E$, where $E$ is
where \( x_1 \in (0, 1) \). Obviously \( L \) is a segment with endpoints
\[
P_1 = (x_1, 0), \quad P_2 = (x_1, y_1),
\]
(see Figure 9), and the point \((x_1, y_1)\) satisfies the equation \( a_{21} = 0 \). Evaluating the function \( T \) at these points, we have
\[
T(P_1) < 0, \quad T(P_2) > 0.
\]
Thus, there exists a point \( P_0 = (x_0, y_0) \in L \), such that \( T(P_0) = 0 \). So at the point \( P_0 \) we have nontrivial positive solutions of (24), since the signs of the entries of the matrix \( A \) at this point are the following:
\[
A(P_0) = \left( \begin{array}{ccc} + & - & + \\ + & 0 & - \end{array} \right).
\]
Thus, the proof of Theorem 2 is completed. \( \square \)

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