EXTENSION PROBLEM FOR THE FRACTIONAL PARABOLIC LAMÉ OPERATOR AND UNIQUE CONTINUATION

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Abstract. In this paper, we introduce and analyse an explicit formulation of fractional powers of the parabolic Lamé operator $H$ (see (1.3) below) and we then study the extension problem associated to such non-local operators. We also study the various regularity properties of solutions to such an extension problem via a transformation as in [26, 5, 22, 7], which reduces the extension problem for the parabolic Lamé operator to another system that resembles the extension problem for the fractional heat operator. Finally in the case when $s \geq 1/2$, by proving a conditional doubling property for solutions to the corresponding reduced system followed by a blowup argument, we establish a space-like strong unique continuation result for $H^s u = V u$.

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1. Introduction and statement of the main results

Let $\mu, \lambda$ be two constants satisfying the following non-degeneracy condition for some $\delta_0 > 0$
\( \mu \geq \delta_0 \) and \( 2 \mu + \lambda \geq \delta_0 \).

Corresponding to such constants, we consider the isotropic Lamé operator in \( \mathbb{R}^n, n \geq 1 \), defined by

\[
Lu = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u.
\]

where \( u : \mathbb{R}^n \to \mathbb{R}^n \) and is say twice continuously differentiable.

Here \( \mu \) and \( \lambda \) are referred to as Lamé parameters. The Lamé operator appears quite prominently in the theory of elasticity (see [48]). Quite interestingly, this operator appears quite distinctly in the Signorini problem of elasticity (see [42, 43]). We refer to the works [1, 46, 40] for various regularity results for the Signorini problem of elasticity and other references can be found therein.

In this work, for the parabolic Lamé operator

\[
\mathbb{H} := \partial_t - L,
\]

we are interested in studying the fractional powers \( \mathbb{H}^s \) for \( s \in (0, 1) \). The parabolic Lamé operator has been investigated for instance in [44] where the Dirichlet problem has been studied in Lipschitz domains. Our goal is to give a meaning framework of an appropriate extension problem for \( \mathbb{H}^s \) in the spirit of [19, 36, 47]. We refer to the recent work [45] where the extension problem for the time-independent fractional operator \( (-L)^s \) has been extensively studied. We also refer to the introduction in [45] where various possible applications of such nonlocal systems in the theory of continuum mechanics has been discussed.

Our main result concerning the extension problem for \( \mathbb{H}^s \) is as follows. See Section 2 for the relevant notations.

1.1. Main results.

**Theorem 1.1.** Let \( u \in H^s \) and \( \tilde{u} \) be defined by

\[
\tilde{u}(x, y, t) := \int_0^\infty \int_{\mathbb{R}^n} P_y^{(a)}(x, z, \tau) u(z, t - \tau) dz d\tau,
\]

where \( P_y^{(a)} \) is as in (4.4). Then \( \tilde{u} \) solves the Dirichlet problem (4.1) and satisfies

\[
\| \tilde{u}(\cdot, y, \cdot) - u(\cdot, \cdot) \|_{H^s} \to 0, \quad y \to 0+,
\]

\[
\left\| \frac{2^{2s-1} \Gamma(s)}{\Gamma(1-s)} y^a \partial_y \tilde{u} + \mathbb{H}^s u \right\|_{H^{-s}} \to 0, \quad y \to 0+,
\]

(1.4)  \( \mu \geq \delta_0 \) and \( 2 \mu + \lambda \geq \delta_0 \).

(1.5)  \( \| \tilde{u}(\cdot, y, \cdot) - u(\cdot, \cdot) \|_{H^s} \to 0, \quad y \to 0+ \),

(1.6)  \( \left\| \frac{2^{2s-1} \Gamma(s)}{\Gamma(1-s)} y^a \partial_y \tilde{u} + \mathbb{H}^s u \right\|_{H^{-s}} \to 0, \quad y \to 0+ \),
Furthermore for a fixed $M > 0$, we have
\begin{equation}
(1.7) \int_0^M \int_{\mathbb{R}^{n+1}} y^a |\tilde{u}|^2 dxdt dy \leq M^{1+a} \|u\|_{L^2(\mathbb{R}^{n+1})}^2, \quad \int_0^\infty \int_{\mathbb{R}^{n+1}} y^a |\nabla_{x,y}\tilde{u}|^2 dxdt dy \leq \|u\|_{H^s}.
\end{equation}

Theorem 1.1 constitutes the parabolic counterpart of [45, Theorem 8.5]. We also mention over here that the extension problem for the fractional heat operator has been independently developed in [36] and [47].

We then study the qualitative properties of solutions to nonlocal equations of the type
\begin{equation}
(1.8) \quad \mathbb{H}^s u = V u,
\end{equation}
via such an extension problem, where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a scalar potential. The various regularity properties of solutions to the extension problem are obtained via a transformation inspired by [26, 5, 22, 7] which reduces the extension problem for $\mathbb{H}^s u = V u$ to a system of the following type
\begin{equation}
(1.9) \begin{cases}
y^a \partial_t U^* - \text{div} \left( y^a \nabla_{x,y} U^* \right) - y^a B \nabla U^* = 0 & y > 0, \\
\lim_{y \rightarrow 0} y^a \partial_y U^* = \tilde{V} U^* \text{ at } \{y = 0\}.
\end{cases}
\end{equation}

where $\tilde{V}$ is a matrix valued potential of the type (5.13) below which involves the first derivatives of $V$.

Now due to asymmetric nature of the potential matrix $\tilde{V}$ as in (5.13), unlike that in scalar valued case as in [4, 14] where space-like and space-time strong unique continuation results have been obtained for non-local equations of the type
\begin{equation}
(\partial_t - \Delta)^s = V u,
\end{equation}
it is not obvious as to how the Poon type frequency function approach as in [37] can be adapted to obtain conditional doubling properties or non-degeneracy estimates for solutions to the corresponding extension problem which then facilitates a blowup argument that leads to the desired unique continuation property. This is a somewhat delicate aspect and we refer to subsection 6.1 for a further discussion on such an obstruction. However in the case when $s \geq 1/2$ and where $V$ satisfies a $C^3$ regularity assumption as in (1.11) below, via a transformation as in (6.17), it turns out that one can get rid of the Neumann datum and then apply the methods in [4] to obtain the space-like strong unique continuation result that is stated in Theorem 1.2 below. Over here, we cannot stress enough the fact that although the matrix potential $\tilde{V}$ of the reduced system (1.9) is not symmetric, nevertheless quite remarkably, it has a specific structure that ensures that it commutes with its derivative matrices. This plays a key role in some of the computations in (6.17)-(6.21) below that is used to assert that the transformed function $W$ in (6.17) satisfies a differential inequality of the type
(6.22). It is to be noted that the precise structure of $\tilde{V}$ that we obtain in (5.13) is due to the nature of the reduction scheme that is employed to reduce the Lamé extension problem to a fractional heat type extension system. This is indeed a new feature in the vectorial case and it also indicates that the question of strong unique continuation for nonlocal systems in general is somewhat more subtle than for nonlocal scalar equations.

We now state our unique continuation result.

**Theorem 1.2.** Let $s \in [1/2, 1)$ and $u : \mathbb{R}^n \to \mathbb{R}^n \in H^s$ be a solution to (1.8) in $B_1 \times (-1, 0]$ where the potential $V$ satisfies the growth assumption in (1.11) below. Then if $u$ vanishes to infinite order at $(0, 0)$ in the following sense

$$\int_{B_r \times (-r^2, 0]} |u|^2 = O(r^k) \text{ as } r \to 0,$$

for all $k \in \mathbb{N}$, then $u(\cdot, 0) \equiv 0$.

**Remark 1.3.** For $u \in H^s$, the equation (1.8) is to be understood in the $H^{-s}$ sense.

The precise assumption on the potential $V$ is as follows:

$$||V||_{C^3_{(x,t)}(\mathbb{R}^n \times \mathbb{R})} \leq K, \text{ for some } K \geq 0.$$ (1.11)

It is worth emphasizing that there is an example due to Frank Jones (see [29]) of a global caloric function (i.e. solution to the heat equation) in $\mathbb{R}^n \times \mathbb{R}$ which is supported in a strip of the type $\mathbb{R}^n \times (t_1, t_2)$. In view of such an example, the space-like propagation of zeros as claimed in Theorem 1.2 is the best possible. In this connection, we mention that for local solutions to second order parabolic equations space-like strong unique continuation results were proven in the remarkable works [20, 21].

In order to provide a better perspective on our unique continuation result, we mention that for nonlocal elliptic equations of the type $(-\Delta)^s + V$, strong unique continuation results have been obtained in [23, 24, 38, 39, 41]. The method in [23] combined the frequency function approach in [27, 28] with the Caffarelli-Silvestre extension method in [19]. The approach in [38, 41] is instead on Carleman estimates.

In the time dependent case, for global solutions of

$$\left(\partial_t - \Delta\right)^s u = V u,$$ (1.12)

a backward space-time strong unique continuation theorem has been established in [14] with appropriate assumptions on the potential $V$ which represents the non-local counterpart of the one first obtained by Poon in [37] for the local case $s = 1$. See also [10] for related work on regularity.
of nodal sets of solutions to (1.12). More recently, a space like strong unique continuation result for local solutions to (1.12) has been obtained in [4] which constitutes the nonlocal counterpart of the unique continuation results in the aforementioned works [20, 21]. We also refer to [16] for a unique continuation result for fractional powers of variable coefficient parabolic operators and [3] for a related quantitative uniqueness result.

In closing, we refer to the works [7, 8, 9, 30, 34, 35] for various qualitative and quantitative results on strong unique continuation for variable coefficient Lamé operators in the local case, i.e. when \( \lambda \) and \( \mu \) are functions of \( x \).

The paper is organized as follows. In Section 2, we introduce some basic notations and notions and gather some preliminary results that relevant to this work. In Section 3, we define the fractional powers \( \mathbb{H}^s \) via the Fourier transform and then obtain a pointwise representation of such non-local operators in terms of the associated semigroup. See Theorem 3.1 below. Section 4 is devoted to proving Theorem 1.1. In Section 5, we obtain various quantitative regularity estimates for the associated extension problem that are needed for justifying the Poon type computations as well as for the blowup argument in the proof of the space-like strong unique continuation result Theorem 1.2.

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2. Notations and Preliminaries

In this section we introduce the relevant notation and gather some auxiliary results that will be useful in the rest of the paper. Generic points in \( \mathbb{R}^n \times \mathbb{R} \) will be denoted by \((x_0, t_0), (x, t)\), etc. For an open set \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \) we indicate with \( C_0^\infty(\Omega) \) the set of compactly supported smooth functions in \( \Omega \). We also indicate by \( H^\alpha(\Omega) \) the non-isotropic parabolic Hölder space with exponent \( \alpha \) defined in [33, p. 46]. The symbol \( \mathcal{S}(\mathbb{R}^{n+1}) \) will denote the Schwartz space of rapidly decreasing functions in \( \mathbb{R}^{n+1} \).

For \( f \in L^1(\mathbb{R}^n) \), we denote by \( \hat{f} \) its Fourier transform as below

\[
\mathcal{F}_{x\rightarrow \xi}(f) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot \xi} f(x) dx
\]

Also then, we have the translation property which says

\[
\mathcal{F}_{x\rightarrow \xi}(f(\cdot + h)) = e^{ih\cdot \xi} \mathcal{F}_{x\rightarrow \xi}(f), \quad h \in \mathbb{R}^n.
\]

In the setting of the extension problem (4.1), we will deal with the thick half-space \( \mathbb{R}^{n+1} \times \mathbb{R}^+_y \). At times it will be convenient to combine the additional variable \( y > 0 \) with \( x \in \mathbb{R}^n \) and denote
the generic point in the thick space $\mathbb{R}_+^n \times \mathbb{R}_y^+$ with the letter $X = (x, y)$. For $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}$ and $r > 0$ we let $B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$, $B_r(X_0) = \{X = (x, y) \in \mathbb{R}^n \times \mathbb{R} \mid |x - x_0|^2 + |y - y_0|^2 < r^2\}$, $B^+_r(x_0, y_0) = B_r(X_0) \cap \{y > 0\}$ (note that this is the upper half-ball), and $Q^+_r((x_0, t_0), 0) = B^+_r(x_0, 0) \times [t_0, t_0 + r^2)$. When the center $x_0$ of $B_r(x_0)$ is not explicitly indicated, then we are taking $x_0 = 0$. Similar agreement for the thick half-balls $B^+_r(x_0, 0)$. Unless otherwise specified, for notational ease, $\nabla U$ and $\text{div} U$ will respectively refer to the quantities $\nabla_X U$ and $\text{div}_X U$. The partial derivative in $t$ will be denoted by $\partial_t U$ and also at times by $U_t$. The partial derivative $\partial_x U$ will be denoted by $U_i$. The distance between two points $(X_1, t_1)$ and $(X_2, t_2)$ in the space time is defined as
\begin{equation}
|(X_1, t_1) - (X_2, t_2)| \overset{\text{def}}{=} |X_1 - X_2| + |t_1 - t_2|^{1/2}.
\end{equation}

We will need the following coercivity result in our analysis which can be found in [25, Lemma 5.1].

**Lemma 2.1.** Let $A = (A^{ij}_{\alpha \beta})$ be a constant matrix satisfying the following Legendre-Hadamard condition
\begin{equation}
\sum_{i,j,\alpha,\beta} A^{ij}_{\alpha \beta} \xi_i \xi_j \eta^\alpha \eta^\beta \geq \nu |\xi|^2 |\eta|^2
\end{equation}
with $\nu \geq 0$. Then one has for every $\tau \in C^\infty_0(\mathbb{R}^n)$
\begin{equation}
\sum_{i,j,\alpha,\beta} \int A^{ij}_{\alpha \beta} \partial_\alpha \tau^i \partial_\beta \tau^j \geq \nu \int |\nabla \tau|^2.
\end{equation}

Such a coercivity result is crucial in proving the energy estimate in Theorem 5.1 for the Lamé extension problem.

In the final step of the proof of Theorem 1.2, when we analyse the blowup limit, we will need the following weak unique continuation result from [31, Proposition 5.6].

**Proposition 2.2.** Let $U_0$ be a weak solution to
\begin{equation}
\begin{cases}
g^a \partial_t U_0 + \text{div}(g^a \nabla U_0) = 0 \quad \text{in } B^+_1 \times [0, 1), \\
l_{\text{lim}_{y \to 0}} g^a \partial_y U_0((x, 0), t) \equiv 0 \quad \text{for all } (x, t) \in B_1 \times [0, 1),
\end{cases}
\end{equation}
such that $U_0((x, 0), t) \equiv 0$ for all $(x, t) \in B_1 \times [0, 1)$. Then $U_0 \equiv 0$ in $B^+_1 \times [0, 1)$.

We mention over here that Proposition 2.2 also follows from the space-like analyticity of solutions to (2.4) up to $\{y = 0\}$ as established in the recent work [15].
3. The fractional powers of the parabolic Lamé operator

For $f \in L^1(\mathbb{R}^n)$, we denote by $\hat{f}$ its Fourier transform as below

$$\mathcal{F}_{x \to \xi}(f) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$$

Also then, we have the translation property which says

$$\mathcal{F}_{x \to \xi}(f(\cdot + h)) = e^{ih \cdot \xi} \mathcal{F}_{x \to \xi}(f), \quad h \in \mathbb{R}^n.$$ 

From (1.2), it is not hard to notice that

$$\hat{\hat{L}u}(\xi) = (\mu|\xi|^2 I_n + (\mu + \lambda)\xi \otimes \xi) \hat{u}(\xi)$$

Having observed that, we define the heat semigroup on $\mathbb{R}^n$ with generator $-\hat{L}$ as

$$\widehat{P_t u}(\xi) = e^{-t(\mu|\xi|^2 I_n + (\mu + \lambda)\xi \otimes \xi)} \hat{u}(\xi), \quad t > 0.$$ 

At this point, we also introduce the evolutive semigroup $P_t^{\mathbb{H}} = \{ e^{\mathbb{H}t} \}_{t>0}$ on $\mathbb{R}^{n+1}$ as follows

$$(3.1) \quad \widehat{P_t^{\mathbb{H}} u}(\xi, \sigma) = e^{-\mathbb{H}t} \hat{u}(\xi, \sigma) = e^{-\tau((\mu|\xi|^2 + i\sigma) I_n + (\mu + \lambda)\xi \otimes \xi)} \hat{u}(\xi, \sigma).$$

From (3.1), one can verify that $\{ P_t^{\mathbb{H}} \}_{t>0}$ defines a self-adjoint strongly continuous contraction semigroup on $L^2(\mathbb{R}^{n+1})$. Before proceeding further, let us mention an important identity which says

$$(3.2) \quad e^{-t(\mu|\xi|^2 I_n + (\mu + \lambda)\xi \otimes \xi)} = e^{-\mu|\xi|^2 I_n + (e^{-2(\mu - \lambda)} - e^{-\mu|\xi|^2}) \frac{\xi \otimes \xi}{|\xi|^2}}.$$ 

To prove the identity (3.2), we first write

$$(3.3) \quad \mu|\xi|^2 I_n + (\mu + \lambda)\xi \otimes \xi = \mu|\xi|^2 \left( I_n - \frac{\xi \otimes \xi}{|\xi|^2} \right) + |\xi|^2 (2\mu + \lambda) \frac{\xi \otimes \xi}{|\xi|^2}.$$ 

The fact that the matrix $\frac{\xi \otimes \xi}{|\xi|^2}$ is idempotent implies that $(I_n - \frac{\xi \otimes \xi}{|\xi|^2})$ is also idempotent and

$$(3.4) \quad \left( I_n - \frac{\xi \otimes \xi}{|\xi|^2} \right) \frac{\xi \otimes \xi}{|\xi|^2} = \frac{\xi \otimes \xi}{|\xi|^2} \left( I_n - \frac{\xi \otimes \xi}{|\xi|^2} \right) = O_n.$$ 

We now combine (3.3) and (3.4) to obtain

$$e^{-t(\mu|\xi|^2 I_n + (\mu + \lambda)\xi \otimes \xi)} = e^{-t|\xi|^2 (I_n - \frac{\xi \otimes \xi}{|\xi|^2}) - t|\xi|^2 (2\mu + \lambda) \frac{\xi \otimes \xi}{|\xi|^2}}$$

$$= \sum_{k=0}^{\infty} \left( \frac{(-t\mu|\xi|^2)^k}{k!} \right) \left( I_n - \frac{\xi \otimes \xi}{|\xi|^2} \right) + \sum_{k=0}^{\infty} \left( \frac{(-t|\xi|^2 (2\mu + \lambda))^k}{k!} \right) \frac{\xi \otimes \xi}{|\xi|^2}$$

$$= e^{-t\mu|\xi|^2 I_n} + e^{-t|\xi|^2 (2\mu + \lambda)} \frac{\xi \otimes \xi}{|\xi|^2}$$

$$= e^{-t\mu|\xi|^2 I_n} + \left( e^{-t|\xi|^2 (2\mu + \lambda)} - e^{-t\mu|\xi|^2} \right) \frac{\xi \otimes \xi}{|\xi|^2},$$
which proves (3.2). In view of (3.2), we equivalently express

\[ \widetilde{P}_t \mathbf{u}(\xi) = e^{-t\mu|\xi|^2} \mathbf{u}(\xi) + \left( e^{-t|\xi|^2(2\mu+\lambda)} - e^{-t\mu|\xi|^2} \right) \frac{\xi \otimes \xi}{|\xi|^2} \mathbf{u}(\xi). \]

Following similar steps, we also have

\[ \widetilde{P}_{\tau}^s \mathbf{u}(\xi, \sigma) = e^{-\tau(\mu|\xi|^2+i\sigma)} \mathbf{u}(\xi, \sigma) + \left( e^{-\tau((2\mu+\lambda)|\xi|^2+i\sigma)} - e^{-\tau(\mu|\xi|^2+i\sigma)} \right) \frac{\xi \otimes \xi}{|\xi|^2} \mathbf{u}(\xi, \sigma). \]

**Definition 1.** Let \( s \in (0, 1) \) and \( \mathbf{u} \in \mathcal{S}(\mathbb{R}^{n+1}) \). We define

\[ \mathcal{H}_s \mathbf{u}(\xi, \sigma) = ((\mu|\xi|^2 + i\sigma)I_n + (\mu + \lambda)|\xi| \otimes |\xi|)^s \mathbf{u}(\xi, \sigma). \]

We note that the fractional power of the matrix

\[ A(\xi, \sigma) := (\mu|\xi|^2 + i\sigma)I_n + (\mu + \lambda)|\xi| \otimes |\xi| \]

can also be realized in simple terms by means of diagonalization employed in [45]. To see that, we first consider a rotation matrix \( R(\xi) \) in \( \mathbb{R}^n \) which takes the unit vector \( \frac{\mathbf{e}_1}{|\xi|} \) to \( \mathbf{e}_1 \). Here we assumed \( \xi \neq 0 \) and \( e_1 = (1, 0, \ldots, 0) \). Summarizing, we have

\[ R^t(\xi)R(\xi) = I_n, \quad R(\xi)\xi = |\xi|\mathbf{e}_1. \]

Consequently, we express

\[ A(\xi, \sigma) = (\mu|\xi|^2 + i\sigma)I_n + (\mu + \lambda)|\xi|^2 \left( R^t(\xi) \mathbf{e}_1 \otimes R(\xi) \mathbf{e}_1 \right) \]

\[ = R^t(\xi) \left( (\mu|\xi|^2 + i\sigma)I_n + (\mu + \lambda)|\xi|^2 \mathbf{e}_1 \otimes \mathbf{e}_1 \right) R(\xi) \]

\[ = R^t(\xi) \begin{pmatrix} (2\mu + \lambda)|\xi|^2 + i\sigma & 0 & \ldots & 0 \\ 0 & (\mu|\xi|^2 + i\sigma) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (\mu|\xi|^2 + i\sigma) \end{pmatrix} R(\xi) \]

and therefore we can write

\[ A^s(\xi, \sigma) = R^t(\xi) \begin{pmatrix} (2\mu + \lambda)|\xi|^2 + i\sigma & 0 & \ldots & 0 \\ 0 & (\mu|\xi|^2 + i\sigma)^s & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (\mu|\xi|^2 + i\sigma)^s \end{pmatrix} R(\xi) \]

\[ = (\mu|\xi|^2 + i\sigma)^s I_n + \left( (2\mu + \lambda)|\xi|^2 + i\sigma \right)^s \left( (\mu|\xi|^2 + i\sigma)^s \right) R^t(\xi) \mathbf{e}_1 \otimes \mathbf{e}_1 R(\xi) \]

\[ = (\mu|\xi|^2 + i\sigma)^s I_n + \left( ((2\mu + \lambda)|\xi|^2 + i\sigma)^s - (\mu|\xi|^2 + i\sigma)^s \right) \frac{\xi \otimes \xi}{|\xi|^2}. \]

Now for a given \( \xi \neq 0 \), since the matrices \( \frac{\xi \otimes \xi}{|\xi|^2} \) and \( I_n - \frac{\xi \otimes \xi}{|\xi|^2} \) maps a vector into orthogonal components (which follows from (3.4)), it turns out from the representation of the fractional powers as in (3.7).
that
\[
|\mathbb{H}^s u| \geq \min(|\mu|\xi^2 + i\sigma|^s|\hat{u}|, |(2\mu + \lambda)|\xi^2 + i\sigma|^s|\hat{u}|).
\]
From (3.8) it thus follows that the natural domain for the definition of \( \mathbb{H}^s \) is
\[
\text{Dom}(\mathbb{H}^s) = H^{2s} \overset{\text{def}}{=} \{ u \in L^2(\mathbb{R}^{n+1}); (\mu|\xi^2 + i\sigma)^s \hat{u}(\xi, \sigma) \in L^2(\mathbb{R}^{n+1}) \}.
\]
We now have the following Balakrishnan type representation for \( \mathbb{H}^s \) based on Bochner subordination principle. See [12].

**Theorem 3.1.** For \( s \in (0, 1) \) and \( u \in \mathcal{S}(\mathbb{R}^{n+1}) \), we have
\[
(3.10) \quad \mathbb{H}^s u(x,t) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \left( P_\tau (\Lambda_{-\tau} u)(x,t) - u(x,t) \right) \frac{d\tau}{\tau^{1+s}}
\]
where \( \Lambda_h \) is the translation operator defined as \( \Lambda_h u(x,t) := u(x,t+h), \ h \in \mathbb{R} \).

**Proof.** We will consider fourier transform in both the space and time variables in order to prove the theorem. We observe that
\[
\mathcal{F}_{x,t}(P_\tau (\Lambda_{-\tau} u))(\xi,\sigma) = e^{-i\sigma \tau} \mathcal{F}_x (P_\tau (\mathcal{F}_t u))(\xi,\sigma) \\
= e^{-i\sigma \tau} e^{-\tau(\mu|\xi^2 I_n + (\mu+\lambda)\xi \otimes \xi)} \hat{u}(\xi,\sigma) \\
= e^{-\tau((2\mu + \lambda)|\xi^2 + i\sigma) - (\mu|\xi^2 + i\sigma)^s} \hat{u}(\xi,\sigma).
\]
Denoting RHS of (3.10) by \( h(x,t) \) and then taking its Fourier transform and also by using (3.5) we find
\[
\hat{h}(\xi,\sigma) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \left( e^{-\tau((2\mu + \lambda)|\xi^2 + i\sigma) - (\mu|\xi^2 + i\sigma)^s} \hat{u}(\xi,\sigma) \right) \frac{d\tau}{\tau^{1+s}} \\
= -\frac{s}{\Gamma(1-s)} \int_0^\infty \left( e^{-\tau(\mu|\xi^2 + i\sigma) - 1} \hat{u}(\xi,\sigma) \right) \frac{d\tau}{\tau^{1+s}} \\
- \frac{s}{\Gamma(1-s)} \int_0^\infty \left( e^{-\tau((2\mu + \lambda)|\xi^2 + i\sigma) - (\mu|\xi^2 + i\sigma)^s} \right) \frac{d\tau}{\tau^{1+s}} \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}(\xi,\sigma) \\
= (\mu|\xi^2 + i\sigma)^s \hat{u}(\xi,\sigma) + (((2\mu + \lambda)|\xi^2 + i\sigma)^s - (\mu|\xi^2 + i\sigma)^s) \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}(\xi,\sigma) \\
= A^s(\xi,\sigma) \hat{u}(\xi,\sigma),
\]
where in the last inequality above, we used (3.7). \( \Box \)
4. The extension problem and energy estimates

In this section, we build the framework for the related extension problem for $H^s$. Given $u$, we look for $\tilde{u}$ satisfying

\[
\begin{cases}
\partial_t \tilde{u} = \left( \partial_y^2 + \frac{a}{y} \partial_y + L \right) \tilde{u}, & \text{for } (x, t) \in \mathbb{R}^{n+1}, \ y > 0,
\end{cases}
\]

where

\[
a = 1 - 2s.
\]

Here the Dirichlet data i.e. $u$ is prescribed on the boundary $y = 0$. More precisely, we show that $H^s$ can be realized as a certain weighted Dirichlet to Neumann map corresponding to (4.1) which is precisely the content of Theorem 1.1.

A key ingredient in our analysis is the Poisson representation formula for $\tilde{u}$. To describe this, let $W$ denote the heat kernel associated to the Lamé operator $L$ i.e.

\[
(\partial_t - L) W(x - x_0, t - t_0) = \delta(x - x_0, t - t_0) I_n.
\]

Then following \[45, Lemma A.1\], we have

\[
F_x W(\xi, t) = e^{-\mu||\xi||^2 t} \frac{\Gamma(s)}{2^{2s} \Gamma(s)} \frac{y^{2s}}{t^{1+s}} e^{-\frac{y^2}{4t}} \frac{\xi \otimes \xi}{||\xi||^2}.
\]

Now similar to that in \[18\], for $y > 0$, we define

\[
P_y^{(s)}(x, z, t) = \frac{1}{2^{2s} \Gamma(s)} \frac{y^{2s}}{t^{1+s}} e^{-\frac{y^2}{4t}} W(x - z, t)
\]

which represents the Poisson kernel for the problem (4.1). For the sake of completeness, we briefly justify this claim. From straightforward computations, we obtain

\[
\begin{aligned}
\left( \partial_y^2 + \frac{a}{y} \partial_y \right) P_y^{(s)}(x, z, t) &= \left( \frac{y^2}{4t^2} - \frac{1 + s}{t} \right) P_y^{(s)}(x, z, t).
\end{aligned}
\]

Now for $x \neq z$, using the fact that $W$ is the Heat kernel for $L$, we have

\[
(\partial_t - L) P_y^{(s)}(x, z, t) = \left( \frac{y^2}{4t^2} - \frac{1 + s}{t} \right) P_y^{(s)}(x, z, t).
\]

Combining (4.5) and (4.6), we can conclude that

\[
\partial_t P_y^{(s)}(x, z, t) = \left( \partial_y^2 + \frac{a}{y} \partial_y + L \right) P_y^{(s)}(x, z, t), \text{ for } x \neq z, \ y > 0.
\]

We now proved the central result of this section which is Theorem 1.1.
Proof. We first verify that \( \hat{u} \) indeed solves the degenerate PDE in (4.1). To do so, we differentiate (4.4) under the integral sign and make use of the facts that \( P_y^{(a)} \) satisfies (4.7) and

\[
\lim_{\tau \to 0^+} \frac{e^{-\frac{y^2}{4\tau}}}{\tau^{1+s}} = \lim_{\tau \to \infty} \frac{e^{-\frac{y^2}{4\tau}}}{\tau^{1+s}} = 0,
\]

to arrive at

\[
\partial_\tau \hat{u} = \left( \partial_y^2 + \frac{a}{y} \partial_y + \mathbb{L} \right) \hat{u}, \quad \text{for } (x, t) \in \mathbb{R}^{n+1} \text{ and } y > 0.
\]

We now prove (1.5), i.e.

\[
\lim_{y \to 0^+} \hat{u}(\cdot, y, \cdot) = u(\cdot, \cdot), \quad \text{in } H^s.
\]

Taking Fourier transform w.r.t time variable in (4.4), we have

\[
\mathcal{F}_t \hat{u}(x, y, \sigma) = \int_0^\infty \int_{\mathbb{R}^n} P_y^{(a)}(x, z, \tau) e^{-i\sigma \tau} \mathcal{F}_t u(x, \sigma) dz d\tau,
\]

which by subsequently taking the Fourier transform in \( x \) variable further reduces to

\[
\hat{u}(x, y, \sigma) = \frac{1}{2\pi \Gamma(s)} \int_0^\infty \int_{\mathbb{R}^n} \left( e^{-\frac{y^2}{4\tau} + (-i\sigma - \mu|\xi|^2)\tau} - e^{-\frac{y^2}{4\tau} + (-i\sigma + \mu|\xi|^2)\tau} \right) \frac{d\tau}{\tau^{1+s}} e^{-\frac{y^2}{4\tau} + (-i\sigma - \mu|\xi|^2)\tau} \frac{d\tau}{\tau^{1+s}} \hat{u}(\xi, \sigma).
\]

Now by using (4.3) and (4.4) we find

\[
\hat{u}(x, y, \sigma) = \frac{y^{2s}}{2\pi \Gamma(s)} \int_0^\infty \frac{d\tau}{\tau^{1+s}} \hat{u}(\xi, \sigma)
\]

\[
\hat{u}(x, y, \sigma) = \frac{y^{2s}}{2\pi \Gamma(s)} \int_0^\infty \left( e^{-\frac{y^2}{4\tau} + (-i\sigma - \mu|\xi|^2)\tau} - e^{-\frac{y^2}{4\tau} + (-i\sigma + \mu|\xi|^2)\tau} \right) \frac{d\tau}{\tau^{1+s}} \hat{u}(\xi, \sigma).
\]

Now, we recall the following important identity

\[
\int_0^{\infty} \tau^{\nu - 1} e^{-\left(\frac{a}{\tau} + A\gamma\right)} d\tau = \frac{2}{\gamma} K_{\nu}(2\sqrt{A\gamma})
\]

where \( \text{Re}(\beta), \text{Re}(\gamma) > 0 \) and \( K_{\nu}(\cdot) \) denotes the Macdonald’s function. Using (4.9), we find

\[
\int_0^{\infty} e^{-\frac{y^2}{4\tau} + (-i\sigma - \mu|\xi|^2)\tau} \frac{d\tau}{\tau^{1+s}} = \int_0^{\infty} \tau^{1-s} e^{-\left(\frac{a}{\tau} + \mu|\xi|^2 + i\sigma\right)\tau} d\tau
\]

\[
= \frac{2^{1+s}}{y^s} \left( \mu|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y \sqrt{\mu|\xi|^2 + i\sigma} \right),
\]

where we have used (4.9) for \( \nu = -s, \beta = \frac{y^2}{4}, \gamma = \mu|\xi|^2 + i\sigma \) and the fact \( K_s = K_{-s} \). Following an exactly similar set of arguments, we also obtain

\[
\int_0^{\infty} e^{-\frac{y^2}{4\tau} + (i\sigma + (2\mu + \lambda)|\xi|^2)\tau} \frac{d\tau}{\tau^{1+s}} = \frac{2^{1+s}}{y^s} \left( (2\mu + \lambda)|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y \sqrt{(2\mu + \lambda)|\xi|^2 + i\sigma} \right).
\]

In light of (4.10) and (4.11), we obtain from (4.8)

\[
\hat{u}(\xi, \sigma) = \frac{2^{1-s} y^s}{\Gamma(s)} \left( \mu|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y \sqrt{\mu|\xi|^2 + i\sigma} \right) \hat{u}(\xi, \sigma).
\]
\[ + \frac{2^{1-s}}{\Gamma(s)} \left( (2\mu + \lambda)|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y\sqrt{(2\mu + \lambda)|\xi|^2 + i\sigma} \right) \]
\[ - \left( \mu|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y\sqrt{\mu|\xi|^2 + i\sigma} \right) \left| \xi \right|^2 \hat{u}(\xi, \sigma). \]

(4.12)

We now recall the following property of the Macdonald’s function
\[ \lim_{z \to 0} z^s \mathcal{K}_s(z) = 2^{s-1} \Gamma(s) \]
which will be important in the forthcoming steps.

We have using (4.12)
\[ ||\hat{u}(\cdot, y, \cdot) - u(\cdot, \cdot)||_{H^s(\mathbb{R}^{n+1})} = \int_{\mathbb{R}^{n+1}} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s |\hat{u}(\xi, y, \sigma) - \hat{u}(\xi, \sigma)|^2 d\xi d\sigma \]
\[ \leq \int_{\mathbb{R}^{n+1}} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s \left| \frac{2^{1-s}}{\Gamma(s)} (\mu|\xi|^2 + i\sigma)^{\frac{s}{2}} K_s \left( y\sqrt{\mu|\xi|^2 + i\sigma} \right) - 1 \right|^2 |\hat{u}(\xi, \sigma)|^2 d\xi d\sigma \]
\[ + \int_{\mathbb{R}^{n+1}} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s \left| y^s (2\mu + \lambda)|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y\sqrt{(2\mu + \lambda)|\xi|^2 + i\sigma} \right) \]
\[ - y^s (\mu|\xi|^2 + i\sigma)^{\frac{s}{2}} K_s \left( y\sqrt{\mu|\xi|^2 + i\sigma} \right) \left| \xi \right|^2 |\hat{u}(\xi, \sigma)|^2. \]

We first look at the first term in RHS of (4.14). For \( \epsilon > 0 \), let us choose \( \delta > 0 \) small such that we have
\[ \left| \frac{2^{1-s}}{\Gamma(s)} z^s \mathcal{K}_s(z) - 1 \right| \leq \frac{\epsilon}{2}, \quad \text{when} \quad |z| \leq \delta \]
from (4.13). Next, we compute
\[ \int_{\mathbb{R}^{n+1}} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s \left| \frac{2^{1-s}}{\Gamma(s)} (\mu|\xi|^2 + i\sigma)^{\frac{s}{2}} K_s \left( y\sqrt{\mu|\xi|^2 + i\sigma} \right) - 1 \right|^2 |\hat{u}(\xi, \sigma)|^2 d\xi d\sigma \]
\[ \leq \sup_{|z| \leq \delta} \left| \frac{2^{1-s}}{\Gamma(s)} \mathcal{K}_s(z) - 1 \right|^2 \int_{\mathbb{R}^{n+1}} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s |\hat{u}(\xi, \sigma)|^2 d\xi d\sigma \]
\[ + \sup_{|z| > \delta} \left| \frac{2^{1-s}}{\Gamma(s)} \mathcal{K}_s(z) - 1 \right|^2 \int_{\mathbb{R}^{n+1}} \chi_{|y|\mu|\xi|^2 + i\sigma|^2 > \delta} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s |\hat{u}(\xi, \sigma)|^2 d\xi d\sigma. \]

Using our choice of \( \delta \), we can show that the term in (I) can be upper bounded by \( \frac{\epsilon}{2} ||u||_{H^s(\mathbb{R}^{n+1})} \). To handle the term (II), we use boundedness of \( z^s \mathcal{K}_s(z) \) when \( z \) is away from zero. An application of Lebesgue dominated convergence theorem along with the fact that \( u \in H^s(\mathbb{R}^{n+1}) \) implies that
\[ \lim_{y \to 0^+} \int_{\mathbb{R}^{n+1}} \chi_{|y|\mu|\xi|^2 + i\sigma|^2 > \delta} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s |\hat{u}(\xi, \sigma)|^2 d\xi d\sigma = 0. \]

We can proceed similarly to establish
\[ \lim_{y \to 0^+} \int_{\mathbb{R}^{n+1}} \left( 1 + ||\xi|^2 + i\sigma|^2 \right)^s \left| y^s (2\mu + \lambda)|\xi|^2 + i\sigma \right)^{\frac{s}{2}} K_s \left( y\sqrt{(2\mu + \lambda)|\xi|^2 + i\sigma} \right) \]
\[ - y^s (\mu|\xi|^2 + i\sigma)^{\frac{s}{2}} K_s \left( y\sqrt{\mu|\xi|^2 + i\sigma} \right) \left| \xi \right|^2 |\hat{u}(\xi, \sigma)|^2 = 0. \]
Hence we have proved (1.5). Now we proceed to prove (1.6). Taking $y$ derivative in (4.12), we find

\[
\partial_y \hat{u}(\xi, \sigma) = \frac{s}{y} \hat{u}(\xi, \sigma) + \frac{2^{1-s}y^s}{\Gamma(s)} \left( \mu |\xi|^2 + i\sigma \right)^{1+s} \frac{\mathcal{K}_s(y)}{|\xi|^2} \hat{u}(\xi, \sigma) + \frac{2^{1-s}y^s}{\Gamma(s)} \left( \left( 2\mu + \lambda \right) |\xi|^2 + i\sigma \right)^{1+s} \frac{\mathcal{K}_s(y)}{|\xi|^2} \hat{u}(\xi, \sigma) - \left( \mu |\xi|^2 + i\sigma \right)^{1+s} \frac{\mathcal{K}_s(y)}{|\xi|^2} \hat{u}(\xi, \sigma).
\]

For notational convenience, we denote

\[
L_1 = \sqrt{\mu |\xi|^2 + i\sigma}, \quad L_2 = \sqrt{(2\mu + \lambda) |\xi|^2 + i\sigma}.
\]

Now we use the following recurrence and derivative formula for the Macdoland’s function

\[
\mathcal{K}_s'(z) = \frac{s}{z} \mathcal{K}_s(z) - \mathcal{K}_{s+1}(z), \quad \mathcal{K}_{s+1}(z) - \mathcal{K}_{s-1}(z) = \frac{2s}{z} \mathcal{K}_s(z).
\]

Using the first recurrence relation in (4.15) above, we have

\[
\partial_y \hat{u}(\xi, \sigma) = \frac{2^{1-s}y^s}{\Gamma(s)} \left( \frac{2s}{y} L_1^{1+s} \mathcal{K}_s(L_1 y) - L_1^{1+s} \mathcal{K}_{1+s}(L_1 y) \right) \hat{u}(\xi, \tau) + \frac{2^{1-s}y^s}{\Gamma(s)} \left\{ \left( \frac{2s}{y} L_2^{1+s} \mathcal{K}_s(L_2 y) - L_2^{1+s} \mathcal{K}_{1+s}(L_2 y) \right) + \left( \frac{2s}{y} L_1^{1+s} \mathcal{K}_s(L_1 y) - L_1^{1+s} \mathcal{K}_{1+s}(L_1 y) \right) \right\} \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}(\xi, \sigma).
\]

Now using the second recurrence relation in (4.15) we find

\[
\frac{\Gamma(s)}{2^{1-s}} y^{1-2s} \partial_y \hat{u}(\xi, \sigma) = -L_1^{1+s} y^{1-s} \mathcal{K}_{1-s}(L_1 y) \hat{u}(\xi, \tau) - \left\{ L_2^{1+s} y^{1-s} \mathcal{K}_{1-s}(L_2 y) - L_3^{1+s} y^{1-s} \mathcal{K}_{1-s}(L_3 y) \right\} \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}(\xi, \sigma)
\]

\[
\longrightarrow \quad -2^{-s} \frac{\Gamma(1-s)}{2^{1-s}} \mathcal{K}_{1-s}(L_1 y) \hat{u}(\xi, \tau) - L_2^{2s} y^{2s} \hat{u}(\xi, \tau) - 2^{-s} \frac{\Gamma(1-s)}{2^{1-s}} \mathcal{K}_{1-s}(L_2 y) \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}(\xi, \sigma) = -\frac{\Gamma(1-s)}{2^s} \mathcal{K}_{1-s}(y) \hat{u}(\xi, \sigma).
\]

In the last equality above, we used (3.7). Along with the above pointwise limit, we can imitate the arguments in the proof of (1.5) using duality (see for instance the proof of Theorem 3.1 in [16]) to guarantee \( \lim_{y \to 0^+} \frac{2^{s-1} \Gamma(s)}{1-1-s} y^s \partial_y \hat{u} = (\partial_y - \mathbb{L})^s \hat{u} \) in \( H^{-s} \) topology.

Next we turn our attention to proving (1.7). Using the Fourier representation of \( u \) from (4.8) and Plancherel theorem, we easily note that

\[
||\hat{u}(\cdot, y, \cdot)||^2_{L^2(\mathbb{R}^{n+1})} \leq y^{2s} \int_{\mathbb{R}^{n+1}} \int_0^\infty \frac{e^{-\frac{y^2}{\tau}}}{\tau^{1+s}} d\tau |\hat{u}(\xi, \sigma)|^2 d\xi d\sigma \leq ||u||^2_{L^2(\mathbb{R}^{n+1})}
\]
which results in

\[
\int_0^M \int_{\mathbb{R}^{n+1}} y^a |\hat{u}|^2 \, dx \, dy \, dt \leq \|u\|_{L^2(\mathbb{R}^{n+1})}^2 M^{1+a}.
\]

Now we want to show that \( \int_0^1 \int_{\mathbb{R}^{n+1}} y^a |\nabla_x y \hat{u}|^2 \, dx \, dy \, dt \leq \|u\|_{H^s} \). In order to do so, we again use the Plancherel theorem, (4.12) and (4.16) to obtain

\[
\int_0^1 \int_{\mathbb{R}^{n+1}} y^a |\nabla_x y \hat{u}|^2 \, dx \, dy \, dt = \int_0^1 \int_{\mathbb{R}^{n+1}} y^a \left( |\xi|^2 |\hat{u}|^2 + |\partial_y \hat{u}|^2 \right) (\xi, y, \sigma) \, d\xi \, d\sigma \, dy \, dt \\
\leq \int_0^1 \int_{\mathbb{R}^{n+1}} y \left( |\xi|^2 L_1^{2s} K_2^2 (L_1 y) + |\xi|^2 L_2^{2s} K_2^2 (L_2 y) \right) |\hat{u}(\xi, \sigma)|^2 \, d\xi \, d\sigma \, dy \, dt \\
+ \int_0^1 \int_{\mathbb{R}^{n+1}} y \left( L_1^{2+2s} K_1^{2-2s} (L_1 y) + L_2^{2+2s} K_1^{2-2s} (L_2 y) \right) |\hat{u}(\xi, \sigma)|^2 \, d\xi \, d\sigma \, dy \, dt \\
\leq \int_0^1 \int_{\mathbb{R}^{n+1}} \sum_{i=1}^2 y L_i^{2+2s} \left( K_1^{2}(L_1 y) + K_1^{2-2s} (L_1 y) \right) |\hat{u}(\xi, \sigma)|^2 \, d\xi \, d\sigma \, dy \, dt.
\]

(4.17)

In the very last step, we have used \(|\xi| \leq L_i, i = 1, 2\). Now similar to [14, Lemma 4.5], we can use the asymptotics of Bessel functions to obtain

\[
\int_0^1 y L_i^{2} \left( K_1^{2}(L_1 y) + K_1^{2-2s} (L_1 y) \right) \, dy \leq C_s, \quad i = 1, 2.
\]

(4.18)

Using (4.18) in (4.17) we thus conclude

\[
\int_0^1 \int_{\mathbb{R}^{n+1}} y^a \nabla_x y \hat{u} \, dx \, dy \, dt \leq \sum_{i=1}^2 \int_{\mathbb{R}^{n+1}} L_i^{2s} |\hat{u}(\xi, \sigma)|^2 \, d\xi \, d\sigma \leq \|u\|_{H^s}.
\]

This finishes the proof of the theorem. \( \square \)

5. Regularity theory for the extension problem

It follows from Theorem 1.1 that if \( u \in H^s \) solves \( H^s u \overset{def}{=} (\partial_t - L)^s u = Vu \), then we have by analogous arguments as in the proof of Lemma 4.6 in [16] that \( \hat{u} \) is a weak solution to the following extension problem

\[
\begin{aligned}
\partial_t \hat{u} &= \left( \partial_y^2 + \frac{a}{y} \partial_y + L \right) \hat{u}, \quad \text{for } y > 0, \\
\hat{u}(x, t, 0) &= u(x, t), \quad \text{for } (x, t) \in \mathbb{R}^{n+1}, \\
\lim_{y \to 0} y^a \partial_y \hat{u} &= Vu \quad \text{in } B_1 \times (-1, 0].
\end{aligned}
\]

(5.1)

We refer to Definition 4.3 in [14] for the precise notion of weak solutions. See also [17]. We now show the smoothness of weak solutions to (5.1). As an intermediate step, we prove the following \( W^{2,2} \) type estimate.

**Theorem 5.1.** Let \( \hat{u} \) be a weak solution to (5.1) in \( \mathbb{B}_1^+ \times (-1, 0] \) where \( V \) satisfies (1.11). Then it follows that \( \nabla^2 \hat{u} \in L^2_{loc}(\mathbb{B}_1^+ \times (-1, 0], y^a \, dX \, dt). \)
Proof. It suffices to show that the following estimate holds

$$\int_{B^+_{1/2} \times (-1/4,0]} \left( |\nabla \tilde{u}|^2 + |\nabla \nabla_x \tilde{u}|^2 \right) y^a dX dt \leq C \int_{B^+_{1} \times (-1,0]} |\tilde{u}|^2 y^a dX dt,$$

where $C$ depends on $\delta_0$ in (1.1) and the constant in (1.11).

We first note that $\tilde{u}$ solves the following parabolic system in $\{y > 0\}$ for $i = 1, \ldots, n$

$$(5.3) \quad \sum_{j,\alpha,\beta=1}^n A^{ij}_{\alpha\beta} \tilde{u}_{\alpha\beta}^j + \left( \frac{a}{y} \partial_y + \partial_y^2 \right) \tilde{u}^i = \tilde{u}^i,$$

where

$$A^{ij}_{\alpha\beta} = \left( \mu \delta^{ij}_{\alpha\beta} + (\mu + \lambda) \delta_{\alpha\beta}^{\alpha\beta} \right).$$

Now using (5.3) and the boundary condition in (5.1), from the proof of Theorem 5.1 in [14] we find that the following inequality holds for all $r < 1$

$$\int_{B^+_{1} \times (-1,0]} |(\phi \tilde{u})_y^a|^2 y^a + \int_{B^+_{1} \times (-1,0]} \sum_{i,j,\alpha,\beta=1}^n A^{ij}_{\alpha\beta}(\phi \tilde{u})_\alpha(\phi \tilde{u})_\beta y^a \leq \frac{C}{(1-r)^2} \int_{B^+_{1} \times (-1,0]} |\tilde{u}|^2 y^a,$$

where $\phi$ is a suitable cut-off such that $\phi \equiv 1$ in $B_r \times (-r^2,0]$ and vanishes outside $B_1 \times (-1,0]$.

Then from (1.1), we note that the constant matrix $(A^{ij}_{\alpha\beta})$ satisfies the Legendre-Hadamard condition (2.2) with $\nu = \delta_0$. Thus using the coercivity result in Lemma 2.1 we deduce the following energy estimate from (5.4) for all $r < 1$

$$\int_{B^+_{1} \times (-r^2,0]} |\nabla \tilde{u}|^2 y^a dX dt \leq \frac{C}{(1-r)^2} \int_{B^+_{1} \times (-1,0]} |\tilde{u}|^2 y^a.$$

Moreover for $i = 1, \ldots, n$, if we let

$$\tau_{h,i} \tilde{u}(X, t) = \frac{\tilde{u}(X + he_i, t) - \tilde{u}(X, t)}{h},$$

then we note that $\mathbf{v} = \tau_{h,i} \tilde{u}$ solves the following problem

$$(5.7) \quad \left\{ \begin{array}{ll}
\sum_{j,\alpha,\beta=1}^n A^{ij}_{\alpha\beta} \mathbf{v}_\alpha^j + \left( \frac{a}{y} \partial_y + \partial_y^2 \right) \mathbf{v}_t^i = \mathbf{v}_t^i, & \text{for } i = 1, \ldots, n \text{ and } y > 0, \\
\lim_{y \to 0} y^a \partial_y \mathbf{v} = (\tau_{h,1} \mathbf{V}) \mathbf{u} + \mathbf{V} \cdot (\mathbf{\cdot} + h \mathbf{e}_1 \cdot) \mathbf{v} & \text{in } B_1 \times (-1,0].
\end{array} \right.$$

Since $V$ satisfies (1.11), we find that $\tau_{h,i} V$ is bounded independent of $h$. Moreover we have that $\mathbf{v}$ satisfies the same equation as $\mathbf{u}$ in the "bulk" (i.e. in $\{y > 0\}$). Therefore it again follows from the proof of Theorem 5.1 in [14] using also the coercivity result in Lemma 2.1 that $\tau_{h,i} \mathbf{u} = \mathbf{v}$ satisfies the following energy estimate

$$\int_{B^+_{1/2} \times (-1/4,0]} |\nabla \tau_{h,i} \mathbf{u}|^2 y^a dX dt \leq C \int_{B^+_{3/4} \times (-9/16,0]} (|\tau_{h,i} \mathbf{u}|^2 + |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2) y^a dX dt.$$

Now for all $h$ small enough, it follows that
\[ \int_{B_{3/4}^+ \times (-9/16, 0]} \tau_{h,i} \tilde{u}^2 y^a dX dt \leq C \int_{B_{5/6}^+ \times (-25/36, 0]} \nabla \tilde{u}^2 y^a. \]

Using this inequality along with (5.5) in (5.8), we can finally assert the following estimate holds

\[ \int_{B_{1/2}^+ \times (-1/4, 0]} \nabla \tau_{h,i} \tilde{u}^2 y^a dX dt \leq C \int_{B_{1}^+ \times (-1, 0]} \tilde{u}^2 y^a. \]

By letting \( h \to 0 \) in (5.9), we find that (5.2) follows.

\[ \square \]

**Reduction to a fractional heat type extension problem.** In order to obtain further regularity of solutions, we use a reduction technique in our analysis which is borrowed from [26, 5, 22, 7]. Such a reduction is also crucial for the proof of our space-like strong unique continuation result in Theorem 1.2.

Note that \( \tilde{u} \) solves the extension problem (5.1). Now by formally applying the divergence operator (with respect to \( x \) variable) in (5.1), we find

\[ \begin{cases} 
\partial_t (\text{div} \tilde{u}) = \left( \partial_y^2 + \frac{a}{y} \partial_y \right) \text{div} \tilde{u} + \text{div} L (\tilde{u}), & \text{for } y > 0, \\
\text{div} \tilde{u}(x, t, 0) = \text{div} u(x, t), & \text{for } (x, t) \in \mathbb{R}^{n+1}.
\end{cases} \]

We then compute the term \( \text{div} L (\tilde{u}) \) in the following way

\[ \text{div} L (\tilde{u}) = \mu \Delta (\text{div} \tilde{u}) + (\mu + \lambda) \Delta (\text{div} \tilde{u}) = (2\mu + \lambda) \Delta (\text{div} \tilde{u}). \]

Before proceeding further, we would like to alert the reader that in this subsection, \( \text{div} \tilde{u} \) will refer to the tangential divergence \( \text{div}_x \tilde{u} \).

Now let

\[ U = \begin{pmatrix} \tilde{u} \\ \text{div} \tilde{u} \end{pmatrix}, \quad U = \begin{pmatrix} U^1 \\ U^2 \\ \vdots \\ U^{n+1} \end{pmatrix}. \]

Then by combining (5.1) and (5.11) we observe that

\[ \left( \partial_t - \left( \partial_y^2 + \frac{a}{y} \partial_y \right) \right) U = B_0 \Delta U + B_1 \nabla U. \]
where \( B_0 \) and \( B_1 \) are as follows

\[
B_0 = \begin{pmatrix}
\mu & 0 & 0 & \ldots & 0 \\
0 & \mu & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (2\mu + \lambda)
\end{pmatrix}, \quad
B_1 \nabla U = \begin{pmatrix}
(\mu + \lambda)\partial_{x_1} U^{n+1} \\
(\mu + \lambda)\partial_{x_2} U^{n+1} \\
\vdots \\
0
\end{pmatrix}.
\]

Moreover, formally one has

\[
U(x, t, 0) = \begin{pmatrix}
u \\
\text{div} u
\end{pmatrix}.
\]

Now taking divergence w.r.t. \( x \) variables we find

\[
\lim_{y \to 0^+} y^a \partial_y (\text{div} u) = V(x, t) \text{div} u + \nabla_x V(x, t) \cdot u
\]

As a result, we have

\[
\lim_{y \to 0^+} y^a \partial_y U = \begin{pmatrix}
V(x, t) & 0 & 0 & \ldots & 0 \\
0 & V(x, t) & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\partial_1 V & \partial_2 V & \partial_3 V & \ldots & V(x, t)
\end{pmatrix} \times U = \hat{V} U.
\]

To reduce (5.12) into a further accessible form, we make the following change of variables

\[
U^*(x, y, t) = \begin{pmatrix}
U^1(\sqrt{\mu}x_1, \ldots, \sqrt{\mu}x_n, y, t) \\
U^2(\sqrt{\mu}x_1, \ldots, \sqrt{\mu}x_n, y, t) \\
\vdots \\
U^{n+1}(\sqrt{2\mu + \lambda}x_1, \ldots, \sqrt{2\mu + \lambda}x_n, y, t)
\end{pmatrix}
\]

We then notice that in \( \{y > 0\} \), \( U^* \) solves

\[
y^a \partial_t U^* - \text{div} (y^a \nabla_{x,y} U^*) - y^a B \nabla U^* = 0,
\]

where

\[
B \nabla U^* = \begin{pmatrix}
\frac{(\mu + \lambda)}{\sqrt{\mu}} \partial_1 (U^*)^{n+1} \\
\frac{(\mu + \lambda)}{\sqrt{\mu}} \partial_2 (U^*)^{n+1} \\
\vdots \\
0
\end{pmatrix}.
\]

Moreover, the weighted Neumann condition also gets transformed as

\[
\lim_{y \to 0} y^a \partial_y U^* = \hat{V} U^*.
\]
where $\tilde{V}$ has a similar structure as the matrix valued function in (5.13). Now from the twice Sobolev differentiability result as in Theorem 5.1, we find that the formal computations in (5.10)-(5.16) above can be justified by weak type arguments and we thus have the following result on the reduction of the parabolic Lamé extension problem to an almost decoupled parabolic system.

**Lemma 5.2.** Let $U^*$ be as in (5.14) corresponding to $\tilde{u}$ which solves the extension problem (5.1). Then $U^*$ is a weak solution to the following problem

\begin{equation}
\begin{align*}
y^a \partial_t U^* - \text{div} \left(y^a \nabla_{x,y} U^*\right) - y^a B \nabla U^* &= 0 \quad \text{in } B_1^+ \times (-1, 0], \\
\lim_{y \to 0} y^a \partial_y U^* &= \tilde{V} U^* \quad \text{in } B_1 \times (-1, 0].
\end{align*}
\end{equation}

Further regularity for the parabolic Lamé extension problem (5.1) will follow from the regularity results for (5.17) that we develop subsequently.

In this direction, we first prove the following Hölder continuity result for a general class of extension problem modelled on (5.17) via compactness arguments similar to that employed in [17].

**Theorem 5.3.** Let $U$ be a weak solution to the following problem

\begin{equation}
\begin{align*}
y^a \partial_t U - \text{div} \left(y^a \nabla_{x,y} U\right) - y^a B \nabla U &= 0 \quad \text{for } y > 0 \cap \mathbb{R}^n_+ \times (-1, 0], \\
\lim_{y \to 0} y^a \partial_y U &= W U + W_1 \quad \text{in } B_1^+ \times (-1, 0],
\end{align*}
\end{equation}

where $B, W, W_1$ are bounded. Then there exists $\alpha = \alpha(a)$ such that $U \in H^\alpha(B_1/2 \times (-1, 4, 0]).$

**Proof.** Step 1: Claim 1: We first show that given $\varepsilon > 0$, there exists $\delta > 0$ such that if

\begin{equation}
\begin{align*}
\int_{B^+ \times (-1, 0]} |U|^2 y^a &\leq 1, \\
|B|, |W|, |W_1| &\leq \delta,
\end{align*}
\end{equation}

then there exists $U_0$ which solves the fractional heat extension system

\begin{equation}
\begin{align*}
y^a \partial_t U_0 - \text{div} \left(y^a \nabla_{x,y} U_0\right) &= 0 \quad \text{for } y > 0 \cap \mathbb{R}^n_+ \times (-1, 4, 0], \\
\lim_{y \to 0} y^a \partial_y U_0 &= 0 \quad \text{in } B_1/2 \times (-1/4, 0],
\end{align*}
\end{equation}

such that

\begin{equation}
\int_{B_1/2 \times (-1, 4, 0]} |U - U_0|^2 y^a \leq \varepsilon.
\end{equation}
We argue by contradiction. If not, there exists an \( \varepsilon_0 > 0 \) such that no choice of \( \delta \) works. That means for each \( k = 1, 2, \ldots \), there exists \( U^k \) which solves (5.18) corresponding \( B^k, W^k, W^k_1 \) satisfies
\[
\begin{cases}
\int_{B^+_1 \times (-1,0)} |U^k|^2 y^a \leq 1,
|B^k|, |W^k|, |W^k_1| \leq \frac{1}{k},
\end{cases}
\]
such that for all \( k \), we have
\[
\int_{B^+_{1/2} \times (-1/4,0)} |U^k - U_0|^2 y^a > \varepsilon_0
\]
for all such \( U_0 \) which solves (5.20). Now from the proof of Theorem 5.1 in [14] and the bounds in (5.22), it follows that
\[
\int_{B^+_{1/2} \times (-1/4,0)} |\nabla U^k|^2 y^a \leq C.
\]
Moreover from (5.22) and (5.24), it follows in a standard way (see for instance the proof of Theorem 4.1 in [6]) that \( \partial_t U^k \in L^2((-1/4,0); (W^{1,2}(B^+_{1/2}, y^a dX))^*) \) are uniformly bounded independent of \( k \). Here \( (W^{1,2}(B^+_{1/2}, y^a dX))^* \) denotes the dual space. Therefore by the Aubin-Lions compactness lemma (2), we have that up to a subsequence \( k \to \infty \), \( U^k \)'s converge to some \( U^0 \) in \( L^2(B^+_{1/2} \times (-1/4,0)) \) which by standard weak type arguments, is a weak solution to (5.20). This contradicts (5.23) for large enough \( k \)'s and proves Claim 1.

Step 2: We now show that there exists universal \( r, \delta, \alpha \in (0,1) \) such that if (5.19) holds, then there exists a vector \( b_1 \) with universal bounds such that
\[
\int_{B^+_1 \times (-r^2,0)} |U - b_1|^2 y^a \leq r^{n+3+a+2\alpha}.
\]
From Step 1, we obtain the existence of \( U_0 \). Since \( U_0 \) is smooth (see for instance [15, Theorem 1.1]), we have that the following inequality holds
\[
\int_{B^+_1 \times (-r^2,0)} |U_0 - U_0(0,0)|^2 y^a \leq C r^{n+3+a+2}.
\]
We now choose \( \alpha \) such that
\[
\alpha < \min(2s, 1).
\]
We then choose \( r < 1/2 \) such that
\[
C r^{n+3+a+2} = \frac{1}{4} r^{n+3+a+2\alpha}.
\]
Subsequently we let
\[
\varepsilon = \frac{1}{4} r^{n+3+a+2\alpha}.
\]
which decides the choice of $\delta$. We thus have
\[
(5.29) \quad \int_{B^+ \times (-r^2, 0]} |U - U_0|^2 \, y^a d\nu \leq \frac{1}{4} r^{n+3+a+2\alpha}.
\]
(5.25) thus follows from (5.26) and (5.29) with $b_1 = U_0(0, 0)$ in view of the fact that (5.28) holds.

*Step 3:* With the smallness assumption as in *Step 1* and *Step 2*, we now show that for $k = 1, 2, \ldots$, we have that there exists $b_k$ such that
\[
(5.30) \quad \int_{B^+ \times (-r^{2k}, 0]} |U - b_k|^2 \, y^a \leq r^{k(n+3+a+2\alpha)},
\]
such that
\[
(5.31) \quad |b_k - b_{k+1}| \leq Cr^{k\alpha}.
\]
For $k = 1$, it is as in *Step 2*. We assume that (5.30) holds up to $k$. We now show this implies that (5.30) holds for $k + 1$. By letting
\[
\tilde{U} = \frac{U(r^k X, r^{2k} t) - b_k}{r^{k\alpha}}
\]
we find that $\tilde{U}$ solves (5.18) corresponding to
\[
\tilde{B}(X, t) = r^k B(r^k X, r^{2k} t), \quad \tilde{W}(X, t) = r^{2ks} W(r^k X, r^{2k} t), \quad \tilde{W}_1(X, t) = r^{k(2s-\alpha)} W_1(r^k X, r^{2k} t).
\]
By change of variable it is seen from (5.30) that the following holds
\[
\int_{B^+ \times (-1, 0]} |\tilde{U}|^2 \, y^a \leq 1.
\]
Furthermore the smallness condition in (5.19) is seen to be verified since $r < 1$ and $\alpha$ is chosen as in (5.27). Therefore by *Step 2*, there exists $\tilde{b}$, such that
\[
(5.32) \quad \int_{B^+ \times (-r^2, 0]} |\tilde{U} - \tilde{b}|^2 \, y^a \leq r^{n+3+a+2\alpha}.
\]
By rescaling back to $U$, we find that (5.30) follows from (5.32) with $b_{k+1} = b + r^{k\alpha} \tilde{b}$.

*Step 4:* (Conclusion) We first note that by rescaling as follows
\[
U_{r_0}(X, t) = U_{r_0}(r_0 X, r_0^2 t)
\]
we find that $U_{r_0}$ solves (5.18) with $B_{r_0}(X, t) = r_0 B(r_0 X, r_0^2 t)$, $W_{r_0}(X, t) = r_0^{2s} W(r_0 X, r_0^2 t)$ and $(W_1)_{r_0}(X, t) = r_0^{2s} W_1(r_0 X, r_0^2 t)$. Thus by choosing $r_0$ small enough, we can ensure that the smallness condition in (5.19) can be ensured.

Now from (5.30), it follows by a standard real analysis argument that the following estimate holds for all $r < 1/2$ with $b_0 = \lim_{k \to \infty} b_k$
\[
(5.33) \quad \int_{B^+ \times (-r^2, 0]} |U - b_0|^2 \, y^a \leq Cr^{n+3+a+2\alpha}.
\]
Similarly we have by translation in the tangential directions that for every \((x_0, 0, t_0) \in B_{1/2} \times \{0\} \times (-1/4, 0]\), there exists \(b(x_0, t_0)\) such that
\[
(5.34) \quad \int_{B^+_{1/2}((x_0,0) \times (t_0-r^2,t_0)]} |U - b(x_0, t_0)|^2 y^a \leq C r^{n+3+a+2\alpha}.
\]
Moreover the fact that assignment \((x_0, t_0) \rightarrow b(x_0, t_0)\) is \(H^\alpha\) regular also follows in a standard way from (5.34). Now given the boundary decay estimate in (5.34), combined with the fact that \(U\) solves a uniformly parabolic PDE away from \(y > 0\), one can combine (5.34) with interior estimates (see for instance the proof of Theorem 2.3 in [13]) to assert that \(U\) is in \(H^\alpha(\mathbb{B}^+_{1/2} \times (-1/4, 0])\). We nevertheless provide the details for the sake of completeness.

Let now \((X_1, t_1)\) and \((X_2, t_2)\) be two points in \(\mathbb{B}^+_{1/2} \times (-1/4, 0]\). Our objective is to show that the following estimate holds
\[
(5.35) \quad |U(X_1, t_1) - U(X_2, t_2)| \leq C |(X_1, t_1) - X_2, t_2)|^\alpha.
\]
Without loss of generality, we assume that \(y_1 \leq y_2\). There are two cases:

1. \(|(X_1, t_1) - (X_2, t_2)| \leq \frac{1}{4} y_1\);
2. \(|(X_1, t_1) - (X_2, t_2)| \geq \frac{1}{4} y_1\).

If (1) occurs, then applying (5.34) with \(r = \frac{y_1}{2}\), it ensues that the following \(L^2\) bound is satisfied by \(U^0(X, t) = U(X, t) - b(x_1, t_1)\)
\[
(5.36) \quad \int_{\mathbb{B}^+_{1/2}(X_1) \times (t_1 - \frac{y_1^2}{4}, t_1]} (U^0)^2 y^a \leq C (y_1)^{n+3+a+2\alpha}.
\]
We then observe that the rescaled function
\[
(5.37) \quad \tilde{U}_0(X, t) = U_0(x_1 + y_1 x, y_1 y, t_1 + y_1^2 t)
\]
satisfies in \(\mathbb{B}_{1/2}((0,1)) \times (-1/4, 0]\) a uniformly parabolic system with bounded drift. Thus from the classical parabolic theory using (5.36) one has
\[
(5.38) \quad |\tilde{U}^0(X, t) - \tilde{U}^0(0, 1, 0)| \leq C \left( \int_{\mathbb{B}^+_{1/2}((0,1)) \times (-1/4, 0]} (\tilde{U}^0)^2 \right)^{1/2} |(X, t) - (0, 1, 0)|^\alpha
\]
\[
\leq \frac{C}{y_1^{n+3+a}} \left( \int_{\mathbb{B}^+_{1/2}(X_1) \times (t_1 - \frac{y_1^2}{4}, t_1]} (U^0)^2 y^a \right)^{1/2} |(X, t) - (0, 1, 0)|^\alpha \quad \text{(change of variable)}
\]
\[
\leq \frac{C}{y_1^{n+3+a}} \left( \int_{\mathbb{B}^+_{1/2}(X_1) \times (t_1 - \frac{y_1^2}{4}, t_1]} (U^0)^2 y^a \right)^{1/2} |(X, t) - (0, 1, 0)|^\alpha
\]
\[
\leq C y_1^\alpha |(X, t) - (0, 1, 0)|^\alpha.
\]
Thus from \((5.40)\), the triangle inequality gives
\[
|y_2| = |X_2 - x_2| \leq |X_2 - X_1| + |X_1 - x_1| + |x_1 - x_2| \leq 6|X_1 - X_2|.
\]

Moreover, the triangle inequality gives
\[
|y| = |X_2 - x_2| \leq |X_2 - X_1| + |X_1 - x_1| + |x_1 - x_2| \leq 6|X_1 - X_2|.
\]

Thus from \((5.39)\) and \((5.40)\) we have
\[
|U(X_1, t_1) - U(X_2, t_2)| \leq \sum_{i=1}^2 |U(X_i, t_i) - b(x_i, t_i)| + |b(x_1, t_1) - b(x_2, t_2)| \leq C|(X_1, t_1) - (X_2, t_2)|^\alpha.
\]

This finishes the proof of the theorem.

Now following the Hölder continuity result in Theorem 5.3 which is the analogue of Theorem 5.1 in [14], given such an \(\alpha\), starting with \(k = 1, 2, ... \lfloor \frac{1}{\alpha} \rfloor + 1\), we now take iterated difference quotients of \(U^*\) in \(x\) and \(t\) of the type
\[
U^*_{h,i} = \frac{U^*(x + he_i, y, t) - U^*(x, y, t)}{h^{k\alpha}}, \quad U^*_{h,t} = \frac{U^*(x, y, t + h) - U^*(x, y, t)}{h^{\frac{k\alpha}{2}}},
\]
for \(i = 1, \ldots, n\) similar to in the proof of Lemma 5.6 in [14]. Note that \(U^*_{h,i}\) and \(U^*_{h,t}\) solves a problem of the type \((5.18)\) thanks to \((1.11)\), thus by applying Theorem 5.3 to these difference quotients we can conclude that \(\nabla_x U^*, U^*_t\) are in \(H^\alpha\) up to \(\{y = 0\}\). Now by applying the method of difference quotients as in the proof of Theorem 5.1, we can assert that \(\nabla \nabla_x U^* \in L^2(\mathbb{B}^+_1 \times (-r^2, 0], y^a dXdt)\) for all \(r < 1\). Moreover since we have that \(U^*_t \in H^\alpha\) up to \(\{y = 0\}\), it now follows by using the equation \((5.17)\) that \((y^a U^*_y)_y \in L^2(\mathbb{B}^+_1 \times (-r^2, 0], y^{-a} dXdt)\) for all \(r < 1\). We record all such observations in the following lemma.

**Lemma 5.4.** Let \(U^*\) be as in Lemma 5.2. Then we have that the \(\nabla_x U^*, \partial_t U^* \in H^\alpha\) up to \(\{y = 0\}\) for all \(\alpha\) as in Theorem 5.3. Moreover, the following estimates hold

\[
||U^*||_{L^\infty(\mathbb{B}^+_1 \times (-1/4,0])} + ||\nabla_x U^*, \partial_t U^*||_{H^\alpha(\mathbb{B}^+_1 \times (-1/4,0])} \leq C \left( \int_{\mathbb{B}^+_1 \times (-1,0]} |U^*|^2 y^a \right)^{1/2}
\]
and
\[
(5.43) \quad \int_{B^{+}_{1/2} \times (-1/4,0)} y^a |\nabla U^*|^2 + y^a |\nabla \nabla_x U^*|^2 + y^a |\partial_t U^*|^2 + y^{-a} (y^a U_y^*)_y|^2 \leq C \int_{B^{+}_1 \times (-1,0)} |U^*|^2 y^a.
\]

We now show the Hölder continuity of $y^a U_y^*$ up to the thin set $\{y = 0\}$.

**Lemma 5.5.** Let $U^*$ be as in Lemma 5.2. Then we have that $y^a U_y^* \in H^{3}(\overline{B^{+}_{1/2} \times (-1/4,0)})$ for some $\beta > 0$. Moreover the following estimate holds
\[
(5.44) \quad ||y^a U_y^*||_{H^{3}(\overline{B^{+}_{1/2} \times (-1/4,0)})} \leq C \left( \int_{B^{+}_1 \times (-1,0)} |U^*|^2 y^a \right)^{1/2}.
\]

**Proof.** We first note that $U^*$ solves
\[
(5.45) \quad \begin{cases}
    y^a \partial_t U^* - \text{div} \left( y^a \nabla_{x,y} U^* \right) - y^a B \nabla U^* = 0 \text{ in } B^{+}_1 \times (-1,0), \\
    \lim_{y \to 0} y^a \partial_y U^* = \tilde{V} U^* = \phi \in H^{\alpha_0} \text{ in } B_1 \times (-1,0) \text{ for } \alpha_0 = 1 \text{ (thanks to Lemma 5.4)}. 
\end{cases}
\]

Now by an analogous compactness argument as in the proof of Lemma 5.6 in [17], it follows that there exists $\alpha > 0$ such that $1 < \alpha + 2s < 2$ for which the following estimate holds
\[
(5.46) \quad \int_{B^{+}_t((x_0,0)) \times (0-t^2,t_0)} \left( U^* - \frac{1}{1-a} \phi(x_0,t_0) y^{1-a} - \langle \nabla_x U^*(x_0,t_0), x - x_0 \rangle \right)^2 y^a \leq C r^{\alpha+a+2(\alpha+2s)}.
\]

Given $(x_0,t_0) \in B_{1/2} \times (-1/4,0)$, Let $H = y^a U_y^* - \phi(x_0,t_0)$. Then from the action of the drift $B$ as in (5.15) and Lemma 5.4 it follows from a direction calculation that $H$ is a weak solution of the following conjugate system
\[
(5.47) \quad y^{-a} \partial_t H - \text{div} \left( y^{-a} \nabla_{x,y} H \right) - y^{-a} B \nabla H = 0.
\]

Now for a point $(X_0,t_0) = (x_0,y_0,t_0)$, let $H^* = U^* - \frac{1}{1-a} \phi(x_0,t_0) y^{1-a} - \langle \nabla_x U^*(x_0,t_0), x - x_0 \rangle$. Then we note that $H^*$ solves the system of the type
\[
(5.48) \quad y^a \partial_t H^* - \text{div} \left( y^a \nabla_{x,y} H^* \right) - y^a B \nabla H^* = y^a F
\]
in $B_{1/2}(X_0) \times (t_0 - y_0^2/4,t_0)$. Then from the energy estimate applied to $H^*$ it follows that the following inequality holds
\[
(5.49) \quad \int_{B_{1/2}(X_0) \times (t_0 - y_0^2/4,t_0)} |H|^2 y^{-a} \leq \int_{B_{1/2}(X_0) \times (t_0 - y_0^2/4,t_0)} |\nabla H^*|^2 y^a \]
\[
\leq \frac{C}{y_0^4} \int_{B_{3/2}(X_0) \times (t_0 - y_0^3/16,t_0)} (|H|^2 + y_0^4 |F|^2) y^a.
\]
Now using the decay estimate (5.46), we obtain the following bound from (5.49)

\[(5.50) \quad \int_{B_{y_0}(X_0) \times (t_0 - y_0^2/4, t_0)} |H|^{2}y^{-a} \leq Cy_0^{n+1+a+2(\alpha+2s)}.\]

Now since \(H\) solves (5.47), by rescaling as in (5.37) we note that the rescaled function solves a uniformly parabolic system in \(B_{1/2}((0,1)) \times (-1/4,0]\) and thus by applying the classical estimates to the rescaled function and by scaling back we obtain the following bound

\[(5.51) \quad |H(X_0, t_0)| = |y^aU^*_y(x_0, t_0) - \phi(x_0, t_0)| \leq Cy_0^\alpha.\]

Again let \((X_1, t_1)\) and \((X_2, t_2)\) be two points in \(B_{1/2}^+(0,1) \times (-1/4,0]\). Our objective is to show that the following estimate holds

\[(5.52) \quad |y^aU^*_y(X_1, t_1) - y^aU^*_y(X_2, t_2)| \leq C|(X_1, t_1) - X_2, t_2)|^\alpha.\]

Without loss of generality, we assume that \(y_1 \leq y_2\). There are two cases:

1. \(|(X_1, t_1) - (X_2, t_2)| \leq \frac{1}{4}y_1;\)
2. \(|(X_1, t_1) - (X_2, t_2)| \geq \frac{1}{4}y_1.\)

If (1) occurs, then the function \(H_1 = y^aU^*_y - \phi(x_1, t_1)\) satisfies an equation of the type (5.47) in \(B_{y_1}(X_1) \times (t_1 - y_1^2/4, t_1]\). Again by rescaling as in (5.37), we note that the rescaled function satisfies a uniformly parabolic system in \(B_{1/2}((0,1)) \times (-1/4,0]\). Arguing as in (5.36)-(5.38) we thus obtain

\[(5.53) \quad |y^aU^*_y(X_1, t_1) - y^aU^*_y(X_2, t_2)| = |H_1(X_1, t_1) - H_1(X_2, t_2)|\]
\[\leq \frac{C}{y_1^\alpha} \left(\frac{1}{y_1^{n+3-a}} \int_{B_{y_1/2}(X_1) \times (t_1 - y_1^2/4, t_1]} H^2 y^{-a}\right)^{1/2} |(X_1, t_1) - (X_2, t_2)|^\alpha\]
\[\leq C|(X_1, t_1) - (X_2, t_2)|^\alpha.\]

In the first inequality in (5.53) we used that in \(B_{y,1/2}(X_1), y \sim y_1\) and in the last inequality, we used the decay estimate in (5.50) with \(y_0\) replaced by \(y_1\). Suppose now (2) occurs. In this case, we note that (5.51) holds with \((X_0, t_0)\) replaced by \((X_1, t_1)\) and \((X_2, t_2)\). More precisely we have the following inequality

\[(5.54) \quad |y^aU^*_y(x_i, t_i) - \phi(x, t_i)| \leq C y_i^\alpha\]

for \(i = 1, 2\). Moreover from (5.39) we have that \(|y_2| \leq 6|(X_1, t_1) - (X_2, t_2)|\). Consequently, we obtain

\[(5.55) \quad |y^aU^*_y(X_1, t_1) - y^aU^*_y(X_2, t_2)| \leq \sum_{i=1}^{2} |y^aU^*_y(X_i, t_i) - \phi(x, t_i)| + |\phi(x_1, t_1) - \phi(x_2, t_2)|\]
\[\leq Cy_1^\alpha + Cy_2^\alpha + C|(x_1, t_1) - (x_2, t_2)|^\alpha \leq C|(X_1, t_1) - (X_2, t_2)|^\alpha.\]
This finishes the proof of the lemma.

Finally by combining the energy estimate in Theorem 5.1 along with the regularity results in Lemma 5.4, we can assert that the following regularity estimate holds for solutions to the original extension problem (5.1) which will be needed subsequently in the proof of Theorem 1.2.

**Lemma 5.6.** Let $\tilde{u}$ be as in (5.1). Then the following estimates hold for $\alpha$ as in Lemma 5.4.

\begin{align}
(5.56) \quad & ||\tilde{u}||_{L^\infty(B_{1/2}^+ \times (-1/4,0))} + ||\nabla_x \tilde{u}, \partial_t \tilde{u}||_{H^\alpha(B_{1/2}^+ \times (-1/4,0))} \leq C \left( \int_{B_{1/2}^+ \times (-1,0)} |\tilde{u}|^2 y^{\alpha} \right)^{1/2} \\
(5.57) \quad & \int_{B_{1/2}^+ \times (-1/4,0)} y^\alpha |\nabla \tilde{u}|^2 + y^\alpha |\nabla \nabla \tilde{u}|^2 + y^\alpha |\partial_t \tilde{u}|^2 + y^{-\alpha} |(y^\alpha \partial_y \tilde{u})_y|^2 \leq C \int_{B_{1/2}^+ \times (-1,0)} |\tilde{u}|^2 y^{\alpha}.
\end{align}

6. **Proof of Theorem 1.2**

As previously mentioned in the introduction, it turns out that some subtle obstructions which will be described below doesn’t allow us to adapt the Poon type frequency approach to the reduced system (5.17). Such an obstruction is very specific to the vectorial case. However with the smoothness assumptions on the potential $V$ as in (1.11), it turns out that in the range $s \geq 1/2$, a certain change of variables can be carried out which reduces the system (5.17) further to another system with zero Neumann conditions.

Before proceeding further, we mention that for notational convenience, we will denote $U^*$ as in (5.14) by $U$ throughout this section. Similar to that in [4], by letting $t \rightarrow -t$ and by rescaling, we may assume that $U$ solves the following backward parabolic problem

\begin{align}
(6.1) \quad & \begin{cases}
y^\alpha \partial_t U + \text{div} (y^\alpha \nabla U) - y^\alpha B \nabla U = 0 \text{ in } Q_4^+ \overset{\text{def}}{=} B_4^+ \times [0,16), \\
\lim_{y \rightarrow 0} y^\alpha \partial_y U = \tilde{V} U \text{ in } B_4 \times [0,16).
\end{cases}
\end{align}

Our first lemma is a monotonicity in time result which allows the passage of information to $t = 0$ for the extension problem (6.1). We now introduce an assumption that will remain in force till the proof of Theorem 1.2. We will assume that

\begin{align}
(6.2) \quad & \int_{B_4^+} |U(\cdot,0)|^2 y^{\alpha} > 0.
\end{align}
As a consequence of such hypothesis the number
\begin{equation}
\theta \overset{d}{=} \frac{\int_{B_1^+ \times (0,16)} |U|^2 y^a dX dt}{\int_{B_1^+} |U(\cdot, 0)|^2 y^a dX}
\end{equation}
is well defined. We now state and prove the relevant monotonicity in time result. Before proceeding further, we would like to mention throughout this section, for a vector valued function \( f \), we will denote its components by \( f^i \).

**Lemma 6.1.** Let \( U \) be a solution of (6.1) in \( Q^+_4 \). Then there exists a constant \( N = N(n, a, |B|, ||\bar{V}||_{C^1}) > 2 \) such that \( N \log(N \theta) \geq 1 \), and for which the following inequality holds for \( 0 \leq t \leq \frac{1}{N \log(N \theta)} \)
\begin{equation}
N \int_{B_2^+} |U(x, t)|^2 y^a dX \geq \int_{B_1^+} |U(x, 0)|^2 y^a dX.
\end{equation}

**Proof.** The proof is similar to that in [4]. Let \( f = \phi U \), where \( \phi \in C_0^\infty(\mathbb{B}_2) \) is a spherically symmetric cutoff such that \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on \( \mathbb{B}_{3/2} \). Since \( U \) solves (6.1) and \( \phi \) is independent of \( t \), it is easily seen that the function \( f \) solves the problem
\begin{equation}
\begin{cases}
y^a \partial_t f + \text{div}(y^a \nabla_x y f) - y^a B \nabla f = 2y^a \langle \nabla \phi, \nabla U \rangle + \text{div}(y^a \nabla \phi) U - y^a B \nabla \phi U, & \text{for } (x, t) \in Q^+_4, \\
\lim_{y \to 0} y^a \partial_y f = \bar{V}(x, t) f(x, t) \text{ in } B_1 \times [0, 16). 
\end{cases}
\end{equation}
As in [4], \( \phi \) has the following properties:
\begin{equation}
\left\{ \begin{array}{l}
\text{supp}(\nabla \phi) \cap \{ y > 0 \} \subset \mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+ \\
|\text{div}(y^a \nabla \phi)| \leq Cy^a 1_{\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+},
\end{array} \right.
\end{equation}
where for a set \( E \) we have denoted by \( 1_E \) its indicator function.

Let us fix a point \( X_1 = (x_1, y_1) \in \mathbb{R}^{n+1}_+ \) and introduce the quantity
\begin{equation}
H(t) = \int_{\mathbb{R}^{n+1}_+} |f(X, t)|^2 \mathcal{G}(X, t) y^a dX
\end{equation}
where
\begin{equation}
\mathcal{G}(X, t) = p(x_1, x, t) p^{(a)}(y_1, y, t)
\end{equation}
is the product of the standard Gauss-Weierstrass kernel \( p(x_1, x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x_1-x|^2}{4t}} \) in \( \mathbb{R}^n \times \mathbb{R}^+ \) with the heat kernel of the Bessel operator \( \mathcal{B}_a = \partial_{yy} + \frac{a}{y} \partial_y \) with Neumann boundary condition in \( y = 0 \) on \( (\mathbb{R}^+, y^a dy) \) (reflected Brownian motion)
\begin{equation}
p^{(a)}(y_1, y, t) = (2t)^{-\frac{n+1}{2}} \left( \frac{y_1 y}{2t} \right)^{\frac{a}{2}} I_{\frac{a-1}{2}} \left( \frac{y_1 y}{2t} \right) e^{-\frac{y_1^2 + y^2}{4t}}.
\end{equation}
From the asymptotic behaviour of $I_{\frac{a-1}{2}}(z)$ near $z = 0$ and at infinity one immediately obtains the following estimate for some $C(a), c(a) > 0$ (see e.g. [32, formulas (5.7.1) and (5.11.8)]),

\begin{equation}
I_{\frac{a-1}{2}}(z) \leq C(a)z^{\frac{2a-1}{2}} \quad \text{if } 0 < z \leq c(a), \quad I_{\frac{a-1}{2}}(z) \leq C(a)z^{-1/2}e^z \quad \text{if } z \geq c(a).
\end{equation}

Note that as $z \to 0^+$

\begin{equation}
z^{\frac{1-a}{2}}I_{\frac{a-1}{2}}(z) \approx \frac{2^{1-a}}{\Gamma((1+a)/2)}.
\end{equation}

Observe that (6.6) and (6.7) imply that for every $x, x_1 \in \mathbb{R}^n$ and $t > 0$ one has

\begin{equation}
\lim_{y \to 0^+} y^a \partial_y \mathcal{G}((x, y), (x_1, 0), t) = 0.
\end{equation}

We observe explicitly that by the semigroup property of $P_t$ and the fact that $\phi \equiv 1$ in $B_1$, we have for every $X_1 \in B_1^+$

\begin{equation}
\lim_{t \to 0^+} H(t) = |f(X_1, 0)|^2 = |U(X_1, 0)|^2.
\end{equation}

We want to establish the following.

**Claim:** There exist constants $C = C(n, a, |B|_{\infty}, ||\tilde{V}||_1)$ and $0 < t_0 = t_0(n, a, ||\tilde{V}||_{\infty}) < 1$ such that for $X_1 \in B_1^+$ and $0 < t < t_0$ one has

\begin{equation}
H'(t) \geq -C||\tilde{V}||_{\infty}t^{-\frac{1+a}{2}}H(t) - Ce^{-\frac{1}{c}t}||U||_{L^2(Q^n_+; e^{\omega}dXdt)}^2.
\end{equation}

Using the divergence theorem, (6.4) and the Neumann condition (6.10) above we obtain for any fixed $X_1 \in B_1$ and $0 < t \leq 1$

\begin{equation}
H'(t) = \sum_i \int 2f^i f^i G y^a dX + \int f^i f^i G y^a dX = \sum_i \int 2f^i f^i G y^a dX + \int f^i f^i \text{div}(y^a \nabla G) dX
\end{equation}

\begin{equation}
= \sum_i \int 2f^i f^i G y^a dX - \int (\nabla f^2, \nabla G) y^a dX = \sum_i \int 2f^i f^i G y^a dX + \int \text{div}(y^a \nabla f^2) G dX
\end{equation}

\begin{equation}
+ \int_{\{y=0\}} 2\tilde{V} f \cdot f G dx
\end{equation}

\begin{equation}
= \sum_i \int 2f^i (y^a f^i + \text{div}(y^a \nabla f^i)) G dX + \int 2|\nabla f^2|^2 G y^a dX + \int_{\{y=0\}} 2\tilde{V} f \cdot f G dx
\end{equation}

\begin{equation}
= I_1 + I_2 + I_3.
\end{equation}

We first prove that
(i) for every $X_1 \in B_1^+$, $0 < t \leq 1$ and $\epsilon > 0$, there exists $C = C(n, a, |B|_{\infty})$ such that the following inequality holds

$$I_1 \geq -Ce^{-\frac{\epsilon}{t^3}} \int_{Q_4^+} |U|^2 y^a dX dt + \left[ \sum_i |C e | \int \left| \nabla f_i^1 y^a \right| dX + \frac{C}{\epsilon} \int f_i^2 y^a dX \right].$$

(ii) there exists $t_0 < 1$ such that for every $X_1 \in B_1^+$ and $0 < t \leq t_0$ one has

$$|I_3| \leq C(n, a) \|\tilde{V}\|_{\infty} \sum_i \left( t^{-\frac{1+a}{2}} \int f_i^1 \tilde{G} y^a dX + t^{\frac{1+a}{2}} \int \left| \nabla f_i^1 \tilde{G} y^a \right| dX \right).$$

With (6.14) and (6.15) in hands, we return to (6.13) to find

$$H'(t) \geq -Ce^{-\frac{\epsilon}{t^3}} \int_{Q_4} U^2 y^a dX dt + \sum_i 2 \int \left| \nabla f_i^1 \tilde{G} y^a \right| dX$$

$$- C(n, a) \|\tilde{V}\|_{\infty} \left( t^{-\frac{1+a}{2}} H(t) + t^{\frac{1+a}{2}} \sum_i \int \left| \nabla f_i^1 \tilde{G} y^a \right| dX \right)$$

$$- \left[ \sum_i \left| C e \int \left| \nabla f_i^1 \tilde{G} y^a \right| dX + \frac{C}{\epsilon} \int f_i^2 y^a dX \right] \right].$$

If at this point in this inequality we choose $t_0 < 1$ and $\epsilon > 0$ such that $C\epsilon + C(n, a) \|\tilde{V}\|_{\infty} t_0^{\frac{1+a}{2}} \leq 1$, it is clear that for $X_1 \in B_1^+$ and $0 < t < t_0$ we obtain the Claim (6.12).

We further note that (6.15) in fact follows from the inequality [4, (3.14)]. Therefore, it suffices to prove the estimate (6.14). Next when $X_1 \in B_1^+$, $X \in B_2^+ \setminus B_{3/2}^+$ and $0 < t \leq 1$, the following bound holds for some universal $M > 0$ (See [4, (3.16)])

$$\tilde{G}(X_1, X, t) \leq e^{-\frac{\epsilon}{t^3}}.$$

With this estimate in hand, we now insert such inequality in the definition of $I_1$ and using (6.4) and (6.5) we finally obtain for $\epsilon > 0$

$$|I_1| \leq Ce^{-\frac{\epsilon}{t^3}} \sum_i \int_{B_2^+} \left( |\nabla U^i| + |U^i| \right) |U^a| y^a + C \sum_i \int \left| f^i(y^a [B \nabla f^i]) \tilde{G} y^a \right| dX$$

$$\leq Ce^{-\frac{\epsilon}{t^3}} \sum_i \int_{B_2^+} \left( |\nabla U^i| + |U^i| \right) |U^a| y^a + \sum_i C e \int \left| \nabla f_i^1 \tilde{G} y^a \right| dX + \frac{C}{\epsilon} \int f_i^2 y^a dX,$$

by Cauchy Schwarz inequality. Here $C$ depends on $|B|_{\infty}$. At this point we invoke the $L^\infty$ bounds for $U, \nabla_x U^i, U_t$ and $y^a U_y$ in Lemma 5.4 and Lemma 5.5 to finally conclude that for every $X_1 \in B_1^+$ and $0 < t \leq 1$ the inequality (6.14) holds and this completes the proof of (6.12).

Once the Claim (6.12) is proved which is the analogue of [4, (3.7)], we can now repeat the arguments as in [4] using the regularity estimate in Lemma 5.4 to complete the proof of the lemma.

\[\square\]
6.1. A further reduction of (6.1). Now it turns out that due to the asymmetric nature of the matrix potential $\tilde{V}$ which has a similar structure as in (5.13), it is not obvious to adapt the Poon type approach as in the scalar case in [4] to obtain the analogues of the first variation results in [4, Lemma 3.3]. More precisely, due to the structure of the matrix potential in (5.13), one cannot ensure certain key cancellations in the proof of the variation of the energy in Lemma 3.3 in [4] which is one of the important ingredients in the proof of the doubling inequality. However in the case when $s \geq 1/2$, with a higher $C^2$ regularity assumption on $\tilde{V}$ which in turn is guaranteed by (1.11), we show that by a certain transformation, we can get rid of the Neumann datum. This transformation also exploits very crucially the special structure of $\tilde{V}$ as in (5.13). Over here it is to be said that although $\tilde{V}$ is not symmetric, nevertheless it has a specific structure that ensures that $\tilde{V}$ commutes with $\partial_t \tilde{V}, \nabla_x \tilde{V}$ which in turn allows for some of the ensuing computations below. As previously mentioned in the introduction, this feature is very specific to the vectorial case.

Corresponding to $U$ as in (6.1), we let

\begin{equation}
(6.17) \quad W = e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U}.
\end{equation}

Therefore we have for $i \in \{1, 2, \cdots, n\}$

\begin{equation}
\partial_t W = e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} \left( \partial_t U - \frac{y^{1-a}}{1-a} V_t U \right), \quad \partial_x_i W = e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} \left( \partial_{x_i} U - \frac{y^{1-a}}{1-a} V_{x_i} U \right)
\end{equation}

and

\begin{equation}
(6.18) \quad y^a \partial_y W = e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} (-\tilde{V}) U + e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} y^a U_y.
\end{equation}

By letting $y \to 0^+$, it thus follows from the Neumann condition in (6.1) that

\begin{equation}
(6.19) \quad \lim_{y \to 0^+} y^a \partial_y W = 0.
\end{equation}

After a further computation, we find that

\begin{equation}
\partial^2_{x_i x_j} W = e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} \left( \partial^2_{x_i x_j} U - \frac{2y^{1-a} \tilde{V}_{x_i} \partial_{x_i} U}{1-a} \right) + \left( \tilde{V}^2 \frac{y^{2(1-a)}}{(1-a)^2} U - \tilde{V} \frac{y^{1-a} W}{1-a} \right),
\end{equation}

\begin{equation}
\partial_y (y^a W_y) = -\tilde{V} W_y + e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} \partial_y (y^a U_y) - \tilde{V} e^{-\frac{\tilde{V}(x,t)y^{1-a}}{1-a}U} U_y.
\end{equation}

An important constituent in the above derivations lies in the observation that $\tilde{V}$ and $\tilde{V}_t$ (or $\tilde{V}_{x_i}, \tilde{V}_{x_i x_j}$) commute and thus we can compute the derivatives of exponential matrices. This crucial aspect is easily verified from the structure of $\tilde{V}$ as in (5.13).

In view of the above relations, we obtain using the equation (6.1) satisfied by $U$ that the following holds
\[
(6.20) \quad y^a W_t + \text{div}(y^a \nabla W) = e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} y^a \nabla U - \left( \tilde{V}_t y + \frac{y}{1-a} \Delta \tilde{V} \right) W \\
+ \frac{y^{2-a}}{(1-a)^2} \sum_i \tilde{V}^2 \frac{1}{x_i} W - \frac{2y}{1-a} \sum_{x_i=1}^n \tilde{V}_{x_i} e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \partial_{x_i} U \\
- \tilde{V} W_y - \tilde{V} e^{-\frac{\alpha}{1-a} \tilde{V}(x,t)} U_y.
\]

Now again from (6.17) we have
\[
\partial_{x_i} U = e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \left( \partial_{x_i} W + \frac{y^{1-a}}{1-a} \tilde{V}_{x_i} W \right), \quad \partial_y U = e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \left( \partial_y W + y^{-a} \tilde{V} W \right).
\]

Using this in (6.20) above, we get
\[
(6.21) \quad y^a W_t + \text{div}(y^a \nabla W) = y^a e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} B e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla W - \left( \tilde{V}_t y + \frac{y}{1-a} \Delta \tilde{V} \right) W \\
- \frac{y^{2-a}}{(1-a)^2} \sum_i \tilde{V}^2 \frac{1}{x_i} W - \frac{2y}{1-a} \sum_{x_i=1}^n \tilde{V}_{x_i} \partial_{x_i} W \\
- 2\tilde{V} W_y - y^{-a} (\tilde{V})^2 W + ye^{-\frac{\alpha}{1-a} \tilde{V}(x,t)} B e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla_x \tilde{V} W.
\]

Over here the term \( B e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla W \) is to be interpreted as
\[
B e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla W := B(e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla W).
\]

Note that \( B \) acts as a linear map from \( \mathbb{R}^{(n+1) \times (n+1)} \to \mathbb{R}^{n+1} \). Similarly the following term in (6.21) above, i.e.
\[
B e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla_x \tilde{V} W,
\]

is to be understood as
\[
B e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \nabla_x \tilde{V} W := \frac{1}{1-a} B(e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \tilde{V}_{x_1} W \ldots e^{\frac{\alpha}{1-a} \tilde{V}(x,t)} \tilde{V}_{x_n} W, 0).
\]

It thus follows from (6.21) that when \( a \leq 0 \) (or equivalently when \( s \geq 1/2 \)), then one has that \( W \) solves the following differential inequality \( B^+_1 \times [0,16) \)
\[
(6.22) \quad |y^a W_t + \text{div}(y^a \nabla W)| \leq C y^a(|W| + |\nabla W|),
\]

with zero Neumann conditions.

Furthermore, we note that since \( W \) and \( U \) are related by the transformation as in (6.17) which is invertible in a bounded manner as \( \tilde{V} \) is bounded, therefore we have that the assumption (6.2) as well as the monotonicity result in Lemma 6.1 holds for \( W \). Moreover since the zero Neumann condition
(6.19) holds, at this point we can apply the approach as in the classical case in [21, Section 2] based on the Poon’s frequency function (more precisely we use the adaptation of such an approach as in [4] where \( a \neq 0 \) is considered) combined with the regularity estimates in Lemma 5.4 and Lemma 5.5 to finally conclude that a conditional doubling property similar to that in [4, Theorem 3.5] holds for \( W \) and consequently for \( U \) which can be stated as follows.

**Theorem 6.2.** Let \( U \) be a solution of (6.1) in \( Q^+_4 \) and assume that \( a \leq 0 \). There exists \( N > 2 \), depending on \( n, a, \) sup norm of \( B \) and the \( C^2 \)-norm of \( \tilde{V} \), for which \( N \log(N\theta) \geq 1 \) and such that:

(i) For \( r \leq 1/2 \), we have

\[
\int_{B^+_2} |U(X,0)|^2 y^a dX \leq (N\theta)^N \int_{B^+_1} |U(X,0)|^2 y^a dX.
\]

Moreover for \( r \leq 1/\sqrt{N \log(N\theta)} \) the following two inequalities hold:

(ii) \[
\int_{Q^+_2} |U(X,t)|^2 y^a dX dt \leq \exp(N \log(N\theta) \log(N \log(N\theta))) r^2 \int_{B^+_1} |U(X,0)|^2 y^a dX.
\]

(iii) \[
\int_{Q^+_2} |U(X,t)|^2 y^a dX dt \leq \exp(N \log(N\theta) \log(N \log(N\theta))) \int_{Q^+_1} |U(X,t)|^2 y^a dX dt.
\]

With the doubling property in Theorem 6.2 in hand, we now proceed with the proof of Theorem 1.2.

**Proof of Theorem 1.2.** The proof is divided into the following steps.

**Step 1:** We first show that \( |U| \) (keeping in mind that we are denoting \( U^* \) in Lemma 5.2 by \( U \)) vanishes to infinite order at \((0,0)\) in the sense of (1.10). Now since \( U \) (i.e. \( U^* \)) is of the type

\[
\begin{pmatrix}
\tilde{u} \\
\text{div}_x \tilde{u}
\end{pmatrix},
\]

and already by our assumption, \( \tilde{u} \) vanishes to infinite order in \( x \) and \( t \) at \((0,0)\), therefore we only need to additionally show that \( \text{div}_x \tilde{u} \) vanishes to infinite order in the tangential variables \( x, t \) at \((0,0)\). It suffices to show that given \( r > 0 \),

\[
(6.23) \quad ||\text{div}_x \tilde{u}||_{L^2(B_r \times (0,r^2))} = O(r^k),
\]

for all \( k > 0 \).

In order to show this, we use an idea based on an interpolation type argument as in [3]. See also [39] for the stationary case.
We first state the relevant interpolation inequality that we need which is Lemma 2.3 in [3]. Given \( s \in (0, 1) \) and \( f \in C^0_0(\mathbb{R}^n \times \mathbb{R}_+) \), there exists a universal constant \( C \) such that for any \( 0 < \eta < 1 \) the following holds
\[
\|\nabla_x f\|_{L^2(\mathbb{R}^n)} \leq C \eta^s \left( \|y^{a/2} \nabla_x f\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+)} + \|y^{a/2} \nabla_t f\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+)} \right) + C \eta^{-1} \|f\|_{L^2(\mathbb{R}^n)}.
\]

Let \( \phi \) be a smooth function of \( X \) supported in \( \mathbb{B}_r \) such that \( \phi \equiv 1 \) in \( \mathbb{B}_r \). We now apply the interpolation inequality (6.24) to \( f = \phi \hat{u} \) and obtain for any \( 0 < \eta < 1 \) that the following estimate holds
\[
\|\nabla_x \hat{u}\|_{L^2(\mathbb{R}^n \times [0, r^2])} \leq C \left( \eta^s \left( \|y^{a/2} \nabla_x \hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times [0, r^2])} + \|y^{a/2} \nabla_t \hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times [0, r^2])} \right) \right) + \eta^{-1} \|\hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times (0, r^2))}
\]
\[
\leq C \eta^s \|y^{a/2} \nabla_x \hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times [0, r^2])} + \eta^{-1} \|\hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times (0, r^2))},
\]

where in the last inequality, we used the rescaled versions of the regularity estimate in Lemma 5.6.

From (6.25) it follows
\[
\|\nabla_x \hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times (0, r^2))} \leq C \eta^s r^{-2} \|y^{a/2} \nabla_x \hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times [0, r^2])} + \eta^{-1} \|\hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times (0, r^2))}.
\]

Now since \( \hat{u} \) vanishes to infinite order in the sense of (1.10), we have that given \( k \in \mathbb{N} \) large enough, there exists \( C_{k,s} \) such that
\[
\|\hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times (0, r^2))} \leq \eta^{-1} \|\hat{u}\|_{L^2(\mathbb{R}^n \times \mathbb{R}_+ \times (0, r^2))}.
\]

By letting \( \eta = r^{k/s} \) and using (6.27) in (6.26) we find for a new \( \tilde{C}_{k,s} \),
\[
\|\nabla_x \hat{u}\|_{L^2(\mathbb{R}^n \times [0, r^2])} \leq \tilde{C}_{k,s} r^{k-2}.
\]

Since \( k \) is arbitrary, (6.23) follows.

**Step 2:** From **Step 1**, we obtain that \( |U| \) vanishes to infinite order in the sense of (1.10) at \( (0, 0) \) in the tangential variables \( x \) and \( t \). We can now argue as in the proof of Theorem 1.1 in [4].

We first show that we must have
\[
(6.29) \quad U(X, 0) = 0, \quad \text{for every } X \in \mathbb{B}_r^+.
\]
We argue by contradiction and assume that (6.29) is not true. Consequently, (6.2) does hold. In particular, from (6.2) and (i) in Theorem 6.2 it follows that \( \int_{\mathbb{B}_r^+} |U(X, 0)|^2 y^adX > 0 \) for all
0 < r ≤ \frac{1}{2}. From this fact and the continuity of \( U \) up to the thin set \( \{ y = 0 \} \) we deduce that

\[
\int_{Q_r^+} |U|^2 y^a dX dt > 0,
\]

for all \( 0 < r \leq 1/2 \). Moreover, the inequality (iii) in Theorem 6.2 holds, i.e. there exist \( r_0 \) and \( C \) depending on \( \theta \) in (6.3) such that for all \( r \leq r_0 \) one has

\[
\int_{Q_r^+} |U|^2 y^a dX dt \leq C \int_{Q_{r/2}^+} |U|^2 y^a dX dt.
\]

From this doubling estimate we can derive in a standard fashion the following inequality for all \( r \leq \frac{r_0}{2} \)

\[
\int_{Q_r^+} |U|^2 y^a dX dt \geq \frac{r L}{C} \int_{Q_{r_0}^+} |U|^2 y^a dX dt,
\]

where \( L = \log_2 C \). Letting \( c_0 = \frac{1}{C} \int_{Q_{r_0}^+} |U|^2 y^a dX dt \), and noting that \( c_0 > 0 \) in view of (6.30), we can rewrite the latter inequality as

\[
\int_{Q_r^+} |U|^2 y^a dX dt \geq c_0 r L.
\]

Let now \( r_j \searrow 0 \) be a sequence such that \( r_j \leq r_0 \) for every \( j \in \mathbb{N} \), and define

\[
U_j(X, t) = \frac{U(r_j X, r_j^2 t)}{\left( \frac{1}{r_j^{1-a} + \alpha} \int_{Q_{r_j}} |U|^2 y^a dX dt \right)^{1/2}}.
\]

Note that on account of (6.30) the functions \( U_j \)'s are well defined. Furthermore, a change of variable gives for every \( j \in \mathbb{N} \)

\[
\int_{Q_{r_j}^+} |U_j|^2 y^a dX dt = 1.
\]

Moreover from (6.31) using change of variables we have

\[
\int_{Q_{r_j/2}^+} |U_j|^2 y^a dX dt \geq C^{-1}.
\]

Moreover \( U_j \) solves the following problem in \( Q_{r_j}^+ \)

\[
\left\{ \begin{array}{l}
\text{div}(y^a \nabla U_j) + y^a \partial_t U_j = y^a r_j B \nabla U_j, \\
\lim_{y \to 0} y^a \partial_y U_j((x, 0), t) = r_j^{1-a} \tilde{V}(r_j x, r_j^2 t) U_j((x, 0), t).
\end{array} \right.
\]

From (6.33) and the regularity estimates in Lemma 5.4 and Lemma 5.5 we infer that, possibly passing to a subsequence which we continue to indicate with \( U_j \), we have \( U_j \to U_0 \) in \( H^a(Q_{3/4}^+) \) up to \( \{ y = 0 \} \) and also \( \lim_{y \to 0} y^a \partial_y U_j((x, 0), t) \to \lim_{y \to 0} y^a \partial_y U_0((x, 0), t) \) uniformly in \( Q_{3/4}^+ \cap \{ y = 0 \} \).
Consequently, from (6.35) we infer that the blowup limit \( U_0 \) solves in \( Q_{3/4}^+ \)

\[
\begin{align*}
\text{div}(y^a \nabla U_0) + y^a \partial_t U_0 &= 0, \\
\lim_{y \to 0} y^a \partial_y U_0((x,0), t) &= 0.
\end{align*}
\]

We now observe that a change of variable and (6.32) give

\[
\int_{B_1 \times [0,1)} |U_j((x,0), t)|^2 \, dxdt \leq c_0^{-1} r_j^{n+1-L} \int_{B_{r_j} \times [0,r_j^2)} |U((x,0), t)|^2 \, dxdt.
\]

Now using (6.23) we infer that

\[
\int_{B_1 \times [0,1)} |U_j((x,0), t)|^2 \, dxdt \to 0.
\]

Again, since \( U_j \to U_0 \) uniformly in \( Q_{1/2}^+ \) up to \( \{y = 0\} \), we deduce that it must be \( U_0 \equiv 0 \) in \( Q_{1/2}^+ \cap \{y = 0\} \). Moreover, since \( U_0 \) solves the problem (6.36), we can now apply the weak unique continuation result in Proposition 2.2 to infer that \( U_0 \equiv 0 \) in \( Q_{1/2}^+ \). On the other hand, from the uniform convergence of \( U_j \)'s in \( Q_{1/2}^+ \) and the non-degeneracy estimate (6.34) we also have

\[
\int_{Q_{1/2}^+} |U_0|^2 y^a \, dX dt \geq C^{-1},
\]

and thus \( U_0 \not\equiv 0 \) in \( Q_{1/2}^+ \). This contradiction leads to the conclusion that (6.29) must be true. We now note that, away from the thin set \( \{y = 0\} \), \( U \) solves a uniformly parabolic PDE with smooth coefficients and vanishes identically in the half-ball \( B_1^+ \). We can thus appeal to [11, Theorem 1] to assert that \( U \) vanishes to infinite order both in space and time in the sense of (1.10) at every \((X,0)\) for \( X \in B_1 \). At this point, we can use the strong unique continuation result in [20, Theorem 1] to finally conclude that \( U(X,0) \equiv 0 \) for \( X \in \mathbb{R}^{n+1}_+ \). The conclusion of the theorem thus follows. It is to be noted that although the results in [11, 20] are stated for scalar equations, nevertheless the results are valid for systems which have decoupled principal part.

\[\square\]

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