The CNM-Hypermultiplet Nexus

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ABSTRACT

We consider additional properties of CNM (chiral-nonminimal) models. We show how 4D, \( N = 2 \) nonlinear \( \sigma \)-models can be described solely in terms of \( N = 1 \) superfield CNM doublets. These actions are described by a Kähler potential together with an infinite number (in the general case) of terms involving its successively higher derivatives. We briefly discuss how \( N = 2 \) supersymmetric extension of the previously proposed \( N = 1 \) CNM low-energy QCD effective action can be achieved.

1Supported in part by NSF Grant PHY-98-02551 and Deutsche Forschungsgemeinschaft.
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1 Introduction

The nonminimal scalar multiplet [1], long ignored, is contained in a complex linear superfield and like the chiral scalar multiplet only describes physical fields of spin-0 and spin-1/2. Previously [2], it was suggested that using chiral-nonminimal (CNM) doublets leads to a natural introduction of Dirac particles in 4D, \( N = 1 \) supersymmetrical theories. A hallmark of CNM models is that the left-handed components of Dirac spinors appear in distinct supersymmetry representations from the corresponding right-handed components. In fact, CNM doublets can never be used to describe Majorana particles, unlike theories with solely chiral superfields. Recently nonminimal scalar multiplets have appeared in a number of investigations. It has been proposed to construct 4D, \( N = 1 \) supersymmetric extensions of “soft pion physics” [3] by use of CNM doublets [4]. Similarly CNM doublets have been used to discuss 2D, \( N = 2, 4 \) models (obtainable simply from dimensional reduction) [5]. In conjunction with performing explicit calculations [6] within the context of matter coupled to 4D, \( N = 2 \) supersymmetric Yang-Mills theory and the Seiberg-Witten conjecture [7], CNM doublets have emerged. The CNM doublets also underlie the parallel development of these theories via harmonic superspace [8] and their role has been recently clarified [9] within the harmonic superspace approach [10, 11] to the manifest realization of 4D, \( N = 2 \) supersymmetry. The work of ref. [9] also provides a simple answer to why only Dirac particles (not Majorana particles) arise in CNM models. The ‘q-hypermultiplets’ of harmonic superspace can only describe Dirac particles and breaking the \( N = 2 \) supersymmetry to \( N = 1 \) preserves this result. The dynamical \( N = 1 \) superfield content of ‘q-hypermultiplets’ consists of CNM doublets.

There is a very old result similar to the observations above. In the nomenclature of harmonic superspace, there is a second hypermultiplet, called the “\( \omega \)-hypermultiplet” [10, 11]. This manifest 4D, \( N = 2 \) representation is closely related to the “relaxed hypermultiplet” [12]. It is known that the relaxed hypermultiplet can also be decomposed into \( N = 1 \) superfields. When this is done [13], the dynamical \( N = 1 \) superfields that emerge also involves a CNM doublet. The same is true for higher relaxed hypermultiplets [14, 15]. Moreover, any consistent projective truncation of the \( \omega \)-hypermultiplet, which can be carried out along the line described in [9], leads to a matter multiplet involving a CNM doublet. We can thus make the following observation.

*Every known formalism which possesses manifest off-shell 4D, \( N = 2 \) supersymmetry and describes minimal irreducible hypermultiplet representations, decomposes into CNM doublets under \( N = 1 \) superfield reduction.*
At this point, it is worth mentioning other historical antecedents. A 4D, \(N = 1\) superfield action involving the nonminimal multiplet and arbitrary numbers of nondynamical auxiliary superfields was proposed in [2]. Even more closely related, expansions of the type above, but for \((in)finite\) series in \(w\) were first suggested in [16]. This work was also the first to propose projective expansions in terms of \(N = 1\) superfields as a way to realize \(N = 2\) supersymmetry. However, this discussion did not extend to nonlinear \(\sigma\)-models.

With these new developments, we believe that CNM models will play an increasingly important role in our understanding of supersymmetry. In this letter, we wish to add to this growing inventory of applications of CNM models by presenting more such results.

These may be viewed as a simplification in 4D, \(N = 2\) nonlinear \(\sigma\)-models. It has long been asserted that such models must parametrize hyper-Kähler manifolds. Such theories have been discussed completely within the context of the harmonic superspace approach and as well within the confines of projective superspace [16]. We shall show that a large class of \(N = 2\) nonlinear \(\sigma\)-models in the projective approach, are in fact completely expressible as CNM nonlinear \(\sigma\)-models.

We also briefly discuss how our observations provide the starting point for the construction of an \(N = 2\) supersymmetric extension to the previous work [4] of the class of \(N = 1\) supersymmetric actions based on the idea of using CNM doublets to extend non-supersymmetric chiral perturbation theory (“soft pion physics”) [3].

2 \(N = 1\) CNM Formulation of 4D, \(N = 2\) Non-linear \(\sigma\)-model Actions

Presently, there are only two known formalisms that make manifest 4D, \(N = 2\) supersymmetry; (a.) the harmonic superspace approach [10] and (b.) the projective superspace approach [16]. Recently it was explicitly demonstrated [9] how these two approaches (not surprisingly) are related. Any theory that is formulated in projective superspace can be re-written as an equivalent theory in harmonic superspace. More unexpectedly, however, is that this recent explicit demonstration makes it clear that the \(N = 1\) superspace decomposition of the “q-hypermultiplet” of the harmonic approach, necessarily leads to the appearance of CNM doublets. From this view, the appearance of 4D, \(N = 1\) CNM models is closely related to the potential realization of manifest 4D, \(N = 2\) supersymmetry. In our prior work [4] on CNM models, the issue of 4D, \(N = 2\) supersymmetry has played no part in our considerations. Accordingly the progress of the two lines of research of [4]
and [10] developed independently. However, with the observation of [9] it is clear that these two can coalesce around a nexus. In particular the projective representations first suggested in [16] and more completely proffered in [6] provide a convenient set of tools for finding where in the model space of CNM actions it becomes possible to describe 4D, $N = 2$ hypermultiplet nonlinear $\sigma$-models. Although the necessary presence of CNM doublets arising from manifest 4D, $N = 2$ theories is surprising, the close relation between 4D, $N = 1$ CNM models and 4D, $N = 2$ supersymmetry is not. In fact, it was noted [2] long ago that the CNM model defined on a “flat” Kähler metric automatically possesses an on-shell $N = 2$ supersymmetry. The use of projective superspace techniques (the minimal reduction of harmonic superspace) permits the realization of this $N = 2$ supersymmetry beyond the limit of “flat” Kähler metrics in 4D, $N = 1$ CNM models.

2.1 Lifting $\sigma$-models: $N = 1 \to N = 2$

As is well known, the $N = 1$ superfield action

$$S_\sigma = \int d^8 z \ K(\Phi^I, \Phi^I) ,$$

leads to the component result (in the notation of ‘Superspace’ [13])

$$S_\sigma = \int d^4 x \left[ -\frac{1}{2} g_{IJ} (\partial^2 T^I) (\partial_2 A^I) - i g_{IJ} \bar{\psi}^i j D_\alpha \psi \alpha^I + \frac{1}{4} \psi \alpha^I \bar{\psi}^i j \psi \alpha^I R_{JKI} \right] ,$$

after elimination of the auxiliary $F$-fields by their equations of motion.

We consider the following 4D, $N = 2$ nonlinear $\sigma$-model given in $N = 1$ superspace

$$S_\sigma[\Upsilon, \bar{\Upsilon}] = \int d^8 z \left[ \frac{1}{2\pi i} \oint \frac{dw}{w} K(T^I, \bar{T}^I) \right] .$$

For $\Upsilon$ we have

$$\Upsilon^I = \sum_{n=0}^\infty \Upsilon^I_n(z) w^n = \Phi^I(z) + w \Sigma^I(z) + O(w^2) ,$$

$$\bar{\Upsilon}^I = \sum_{n=0}^\infty \bar{\Upsilon}^I_n(z) \left( \frac{1}{w} \right)^n = \bar{\Phi}^I(z) - \frac{1}{w} \bar{\Sigma}^I(z) + O((\frac{1}{w})^2) ,$$

with $\Phi$ being chiral, $\Sigma$ being complex linear, and the remaining component superfields being unconstrained complex superfields.

$$\overline{D}_\alpha \Phi = 0 , \quad (\overline{D})^2 \Sigma = 0 .$$
The expansions in (2.4) describe “polar” multiplets in the nomenclature of [6]. If we terminate the series in (2.4) at a finite value, say $p$, and place a reality constraint on the highest $N = 1$ superfield in the expansion, this maintains $N = 2$ supersymmetry. Such multiplets, first presented in [17], are called $O(p)$ multiplets in the nomenclature of [6].

The action possesses a linearly realized $N = 2$ supersymmetry (see Appendix A) even if one replaces $K(\Upsilon, \bar{\Upsilon})$ by a more general $w$-dependent Lagrangian $K(\Upsilon, \bar{\Upsilon}, w)$ of the form

$$K(\Upsilon, \bar{\Upsilon}, w) = \sum_n K_n(\Upsilon, \bar{\Upsilon}) w^n, \quad \bar{K}_n = (-1)^n K_{-n}.$$  \hspace{1cm} (2.6)

A specific feature of the above action is that it remains invariant under rigid U(1) transformations

$$\Upsilon(w) \to \Upsilon(e^{i\alpha}w) \iff \Upsilon_n(z) \to e^{i\alpha} \Upsilon_n(z).$$  \hspace{1cm} (2.7)

Such transformations are compatible with the conjugation (A.10) and can be treated as chiral rotations of those fermionic coordinates of the $N = 2$ superspace, which are eliminated in the $N = 1$ approach.

The $N = 2$ sigma-model introduced respects all the geometric features of its $N = 1$ predecessor in (2.1). The Kähler invariance of (2.1)

$$K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + \left( \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \right)$$  \hspace{1cm} (2.8)

turns into

$$K(\Upsilon, \bar{\Upsilon}) \to K(\Upsilon, \bar{\Upsilon}) + \left( \Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon}) \right)$$  \hspace{1cm} (2.9)

for the model (2.3). A holomorphic reparametrization $A^I \to f^I(A)$ of the Kähler manifold has the following counterparts

$$\Phi^I \to f^I(\Phi)$$
$$\Upsilon^I(w) \to f^I(\Upsilon(w))$$  \hspace{1cm} (2.10)

in the $N = 1$ and $N = 2$ cases, respectively. Therefore, the physical superfields of the $N = 2$ theory

$$\Upsilon^I(w)\big|_{w=0} = \Phi^I, \quad \frac{d\Upsilon^I(w)}{dw}\big|_{w=0} = \Sigma^I$$  \hspace{1cm} (2.11)

should be regarded, respectively, as a coordinate of the Kähler manifold and a tangent vector at point $\Phi$ of the same manifold [4, 9]. That is why the variables $(\Phi^I, \Sigma^I)$ parametrize the tangent bundle of the Kähler manifold. Thus, the ad hoc geometrical properties assigned in [4] can be derived from projective superspace. Our discussion to this point seems to have the following rather surprising implication.
Every 4D, \( N = 1 \) supersymmetric nonlinear \( \sigma \)-model described by (2.1) possesses a minimal extension to a 4D, \( N = 2 \) supersymmetric nonlinear \( \sigma \)-model described by (2.3).

Let us represent \( \Upsilon(w) \) in the form:

\[
\Upsilon(w) = [\Phi + \Sigma w] + A(w), \quad (2.12)
\]

where the quantity \( A(w) \) contains all the auxiliary superfields at quadratic and higher powers in \( w \). Since off-shell the dynamical superfields \( \Phi, \Sigma \) and the auxiliary ones \( A(w) \) are functionally independent, we can write (2.3) in the form

\[
S_{\sigma}[\Upsilon, \bar{\Upsilon}] = \int d^8z \left\{ \frac{1}{2\pi i} \oint dw \frac{e^{[A_0 + A_{\bar{0}}]}}{w} K(\Phi + \Sigma w, \bar{\Phi} - \frac{1}{w} \Sigma) \right\},
\]

\[
= \int d^8z \left\{ \frac{1}{2\pi i} \oint dw \frac{e^{[A_0 + A_{\bar{0}}]}}{w} \exp\left[w \Sigma \partial - \frac{1}{w} \Sigma \overline{\partial} K(\Phi, \bar{\Phi})\right] \right\}, \quad (2.13)
\]

where we have introduced the notation \( \partial \equiv \partial/\partial \Phi \) and \( \bar{\partial} \equiv \partial/\partial \bar{\Phi} \). This last form of writing the action makes clear a number of features. For example if \( K(\Phi, \bar{\Phi}) \) is a finite polynomial of order \( p \), then the highest power to which the auxiliary superfields appear is \( p \).

Given the Kähler potential \( K(\Upsilon^I, \bar{\Upsilon}^I) \), let us introduce the notation

\[
K_I(\Upsilon, \bar{\Upsilon}) = \frac{\partial}{\partial \Upsilon^I} K(\Upsilon, \bar{\Upsilon}), \quad (2.14)
\]

and suppose that \( \Upsilon_*(w) \) is a solution to the equations of motion for the auxiliary superfields. These equations read

\[
\frac{1}{2\pi i} \oint dw \frac{w^n}{w} K_I(\Upsilon_*, \bar{\Upsilon}_*) = 0, \quad n \geq 2. \quad (2.15)
\]

The full mass shell is described by adding the equation of motion for \( \Sigma \)

\[
\overline{\mathcal{D}}_{\dot{a}} \oint dw K_I(\Upsilon_*, \bar{\Upsilon}_*) = 0, \quad (2.16)
\]

and the equation of motion for \( \Phi \)

\[
(\overline{\mathcal{D}})^2 \oint dw K_I(\Upsilon_*, \bar{\Upsilon}_*) = 0. \quad (2.17)
\]

Our goal is to eliminate the infinite number of \( N = 1 \) auxiliary superfields and reveal the explicit presence of a CNM model. Although it is relatively simple to write the equations of motion for the auxiliary superfields (2.15), the actual elimination of the infinite number of auxiliary fields is problematical in the general case. In fact to this point, we have only a perturbative approach and special exact solutions as examples. We will now turn to the discussion of these.
2.2 Perturbative Elimination of Auxiliary Superfields

Upon writing $K = \Upsilon \bar{\Upsilon}$, we see that the action describes a free, massless CNM doublet $(\Phi_I, \Sigma_I)$ together with an infinite number of $N = 1$ auxiliary superfields whose purpose is to linearly realize an $N = 2$ supersymmetry. Let us specialize the Kähler potential in (2.3) to the form (field indices are suppressed)

$$K(\Upsilon, \bar{\Upsilon}) = \Upsilon \bar{\Upsilon} + \varepsilon K(\Upsilon, \bar{\Upsilon}) ,$$

where $\varepsilon$ is a small parameter, and $K(\Upsilon, \bar{\Upsilon})$ is an analytic function of $\Upsilon$ and $\bar{\Upsilon}$ (possessing a convergent Taylor series in an open vicinity of each point). The partial derivative of $K(\Upsilon, \bar{\Upsilon})$ with respect to $\Upsilon$ reads

$$K_\Upsilon(\Upsilon, \bar{\Upsilon}) = \bar{\Upsilon} + \varepsilon K_\Upsilon(\Upsilon, \bar{\Upsilon}) .$$

We continue by expanding $A_\star$ (assumed to be a solution to the equations (2.15)) as a power series in $\varepsilon$

$$A_\star(w) \equiv \sum_{p=1}^{\infty} \varepsilon^p \Delta \Upsilon_p(w) , \quad \Delta \Upsilon_p(w) = \sum_{m=2}^{\infty} \Upsilon_{p,m} w^m .$$

Inserting eqs. (2.12, 2.20) into (2.17), we obtain

$$(-1)^n \sum_{p=1}^{\infty} \varepsilon^p \bar{\Upsilon}_{p,n} + \varepsilon \left( \frac{1}{2\pi i} \oint \frac{dw}{w} w^n K_\Upsilon(\Upsilon_\star, \bar{\Upsilon}_\star) \right) = 0 , \quad n > 1 .$$

These equations allow us to solve for $\Delta \Upsilon_p(w)$ iteratively. To linear in $\varepsilon$ order we obtain

$$(-1)^n \bar{\Upsilon}_{1,n} + \frac{1}{2\pi i} \oint \frac{dw}{w} w^n K_\Upsilon(\Phi + \Sigma w, \bar{\Phi} - \bar{\Sigma}/w) = 0 , \quad n > 1 .$$

Having determined $\Delta \Upsilon_1(w)$, equations (2.21) allow us to derive $\Delta \Upsilon_2(w)$,

$$(-1)^n \bar{\Upsilon}_{2,n} + \frac{1}{2\pi i} \oint \frac{dw}{w} w^n \left\{ \Delta \Upsilon_1(w) \bar{\Upsilon}_1(w) \bar{\Upsilon}_1(w) \right\}$$

and on and on (at least in principle) to all orders. The result of this process is that the auxiliary superfields are replaced by their on-shell values, i.e. $A(w) \to A_\star(w; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$. We thus have a proof that there exists a large class of 4D, $N = 2$ nonlinear $\sigma$-models that permit the “perturbatively elimination” of an infinite number of $N = 1$ auxiliary fields leaving behind a dynamical system described solely in terms of the CNM doublet $(\Phi, \Sigma)$. 

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The above method can be applied for eliminating the auxiliary superfields in more general \( N = 2 \) models described by \( w \)-dependent potentials (2.6). A particular class of \( w \)-dependent Lagrangians on which we plan further study takes the form (below \( \bar{w} = -1/w \))

\[
K \equiv K(\Upsilon, \bar{\Upsilon}) + \left[ J(\Upsilon, \bar{w}) + \overline{J}(\bar{\Upsilon}, w) \right],
\]

Provided \( K(\Upsilon, \bar{\Upsilon}) \) is \( R \)-invariant, \( K(e^{i\phi} \Upsilon, e^{-i\phi} \bar{\Upsilon}) = K(\Upsilon, \bar{\Upsilon}) \), the action may be invariant under generalized U(1) transformations (2.7)

\[
\Upsilon(w) \rightarrow e^{i\phi} \Upsilon(e^{i\alpha} w),
\]

where the parameter \( \phi \) is determined by the holomorphic potential \( J(\Upsilon, \bar{w}) \).

Finally, we observe that although our method of proof required that the parameter \( \varepsilon \) be small in some sense, depending on the convergence properties of the iterative solution, it may be possible to drop this requirement and simply regard \( \varepsilon \) as a device for deriving the pure CNM formulation of the theory.

### 2.3 Special Exact Solutions

Below we shall describe a technique to solve equations (2.15) for a large family of Kähler manifolds including the complex projective spaces \( \mathbb{CP}^n \) and the complex Grassmann spaces \( G_{m,n} \).

Let us choose the two-sphere \( \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \) in the role of our target Kähler manifold. Introducing a complex local coordinate \( z \), the Kähler potential and the corresponding metric read

\[
K(z, \bar{z}) = r^2 \ln \left( 1 + \frac{z \bar{z}}{r^2} \right), \quad g_{zz}(z, \bar{z}) = \left( 1 + \frac{z \bar{z}}{r^2} \right)^{-2},
\]

with \( 1/r^2 \) being proportional to the scalar curvature of the manifold. It can be easily checked that a solution to (2.16) is given by

\[
\Upsilon_\star(w) = \Phi(1 + \Phi \bar{\Phi}/r^2) + w \Sigma \\
\frac{1}{1 + \Phi \bar{\Phi}/r^2 - w \Phi \Sigma/r^2}.
\]

All the auxiliary superfields are now determined in terms of the physical ones, \( \Phi \) and \( \Sigma \). At first sight, \( \Upsilon_\star(w) \) appears to develop an unexpected simple pole at

\[
w = \frac{r^2}{\Phi \Sigma} (1 + \Phi \bar{\Phi}/r^2).
\]
However, this is misleading. The point is that $\Upsilon_*(w)$ passes through the North Pole of the two-sphere at $w$ given as in (2.28); but we must replace the coordinate system chosen by another one in a vicinity of the North Pole.

With the use of (2.27) we find

$$ K \left( \Upsilon_*(w), \tilde{\Upsilon}_*(w) \right) = r^2 \ln \left\{ \left( 1 + \Phi \bar{\Phi}/r^2 \right) \left( 1 - \frac{1}{r^2 (1 + \Phi \bar{\Phi}/r^2)^2} \right) \right\} + \Psi(\Upsilon_*(w)) + \tilde{\Psi}(\tilde{\Upsilon}_*(w)) , \tag{2.29} $$

where

$$ \Psi(\Upsilon_*(w)) = - r^2 \ln \left( 1 - \frac{\bar{\Phi} \Sigma}{r^2 (1 + \Phi \bar{\Phi}/r^2)} \right) . $$

Therefore, the action (2.3) turns into

$$ S[\Upsilon_*, \tilde{\Upsilon}_*] = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) + r^2 \ln \left( 1 - \frac{1}{r^2 g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \bar{\Sigma}} \right) \right\} . \tag{2.30} $$

The action is real under the following global restriction

$$ r^2 > g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \bar{\Sigma} , \tag{2.31} $$

which constitutes the upper bound for admissible values of $\Sigma$. Under the above restriction, the second term in (2.30) can be expanded in a power series in $\Sigma \bar{\Sigma}$ which is in fact an expansion in powers of the curvature.

Let us comment upon how the solution (2.27) has been obtained. Keeping in mind the explicit structure of the Kähler potential (2.26), one readily observes that the particular curve

$$ \Upsilon_0(w) = \Sigma w , \quad \tilde{\Upsilon}_0(w) = - \frac{\bar{\Sigma}}{w} \tag{2.32} $$

solves the equations (2.15). A specific feature of this curve is that $\Upsilon_0(w = 0) = 0$. Further, one has to take into account two more observations: (i) $\mathbb{C}P^1$ is a homogeneous space of SU(2); (ii) the Kähler metric (2.28) is invariant with respect to the SU(2) transformations. A simple consequence of (ii) is that the equations (2.15) are SU(2) invariant. Now, to construct a curve $\Upsilon_*(w)$ under the boundary condition $\Upsilon_*(0) = \Phi$, it is sufficient to apply a special SU(2) transformation to the curve $\Upsilon_0(w)$.

The technique of elimination of the auxiliary superfields, which we have just described, remains effective for a large class of Kähler manifolds. It perfectly works if the following conditions are satisfied:
(i) the Kähler potential can be chosen to be $R$-invariant
\[ K(e^{i\varphi}\Phi, e^{-i\varphi}\bar{\Phi}) = K(\Phi, \bar{\Phi}) \quad ; \] (2.33)

(ii) the Kähler manifold is a homogeneous space of a Lie group $G$;

(iii) the group $G$ leaves the Kähler metric invariant.

Nontrivial examples of such Kähler manifolds are the complex projective spaces $\mathbb{C}P^n$ and the complex Grassmann spaces $G_{m,n}$. Under the above conditions, the particular curve (2.32) solves the equations (2.15); to obtain $Y^I_*(w)$ under the boundary condition $Y^I_*(0) = \Phi^I$, it remains to apply a special group transformation to $Y_0(w)$.

For arbitrary initial conditions (2.11), the corresponding complex curve (2.27) turns out to be a solution of the holomorphic geodesic equation
\[ \frac{d^2 Y^I_*(w)}{dw^2} + \Gamma^I_{JK}(Y_*(w), \bar{\Phi}) \frac{dY^K_*(w)}{dw} \frac{dY^J_*(w)}{dw} = 0 \quad , \] (2.34)

The above geodesic equation can be equivalently rewritten as follows
\[ \frac{d^2}{dw^2} K^I_*(Y_*(w), \bar{\Phi}) = 0 \quad . \] (2.35)

This is the equation which eliminates the auxiliary superfields in case of the manifolds $\mathbb{C}P^n$ and $G_{m,n}$. In general, eq. (2.34) is not equivalent to the original equations of motion for the auxiliary superfields (2.15).

In accordance with (2.31), the physical variables $(\Phi, \Sigma)$ of the $N = 2$ CNM model (2.30) parametrize an open domain of the tangent bundle to the Kähler manifold. The nonminimal sector can be dualized into a chiral model if we replace the action
\[ S[\Sigma, \bar{\Sigma}] = r^2 \int d^8 z \ln \left( 1 - \frac{1}{r^2} g_{\Phi\bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \bar{\Sigma} \right) \] (2.36)

by the following one
\[ S[U, \bar{U}, \Psi, \bar{\Psi}] = \int d^8 z \left\{ r^2 \ln \left( 1 - \frac{1}{r^2} g_{\Phi\bar{\Phi}}(\Phi, \bar{\Phi}) U \bar{U} \right) + U \Psi + \bar{U} \bar{\Psi} \right\} \quad . \] (2.37)

Here $U$ is a complex unconstrained superfield, and $\Psi$ is a chiral superfield. It is worth pointing out that $U$ is to be treated as a tangent vector to the point $\Phi$ of the Kähler manifold, while $\Psi$ as a one-form. If we vary the latter action with respect to $\Psi$, $U$ becomes a linear superfield, and we return to the nonminimal action (2.36). On the other hand, we can eliminate $U$ with the aid of its equation of motion
\[ \frac{g_{\Phi\Phi}(\Phi, \bar{\Phi}) \bar{U}}{1 - g_{\Phi\Phi}(\Phi, \bar{\Phi}) U \bar{U} / r^2} = \Psi \quad , \] (2.38)
and this results in a purely chiral model. When $U$ spans the open ball (2.31), $\Psi$ spans the whole cotangent space to the point $\Phi$. Therefore, the complex dynamical variables $(\Phi, \Psi)$ parametrize the cotangent bundle to $\mathbb{CP}^1$, which is known to be a hyper-Kähler manifold [18].

Upon implementing this duality transformation, the action (2.30) turns into

$$S[\Phi, \bar{\Phi}, \Psi, \bar{\Psi}] = \int d^8z \left\{ K(\Phi, \bar{\Phi}) + F(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \right\} \quad (2.39)$$

where

$$F(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = r^2 \ln \left( f(\kappa) \right) - 2r^2 \frac{\kappa}{f(\kappa)} \quad , \quad f(\kappa) = \frac{1}{2} \left( 1 + \sqrt{1 + 4\kappa} \right) \quad (2.40)$$

and

$$\kappa = \frac{1}{r^2} g^{\Phi \bar{\Phi}} (\Phi, \bar{\Phi}) \Psi \bar{\Psi} \quad . \quad (2.41)$$

The lagrangian in (2.39) is the hyper-Kähler potential of the cotangent bundle to $\mathbb{CP}^1$. One can easily check the identity

$$F'(\kappa) = \frac{r^2}{2\kappa} \left( -1 + \sqrt{1 + 4\kappa} \right) \quad (2.42)$$

which is in complete agreement with ref. [18].

### 2.4 General Structure of $S_\sigma[\Upsilon_*, \bar{\Upsilon}_*]$

The results of subsections 2.2 and 2.3 yield some insights into the structure of $S_\sigma[\Upsilon_*, \bar{\Upsilon}_*]$ in the general case. This action, in general, turns out to be an infinite power series in covariant derivatives of the curvature tensor of the Kähler manifold.

If we set $A = 0$ in (2.13), we obtain

$$S_\sigma[\Upsilon_*, \bar{\Upsilon}_*] \approx \int d^8z \left\{ K(\Phi, \bar{\Phi}) - g_{I,J}(\Phi, \bar{\Phi}) \Sigma^I \Sigma^J + \frac{1}{4} K_{I,J,K,L} \Sigma^I \Sigma^J \Sigma^K \Sigma^L 
- \sum_{p=0}^{\infty} (-1)^p \left[ \frac{1}{(p+3)!} \right]^2 \left( \prod_{\ell=1}^{p+3} \Sigma^I \Sigma^K \Sigma^L \right) \right\} \quad . \quad (2.43)$$

The first two terms above have the characteristic form of the 4D, $N = 1$ CNM nonlinear $\sigma$-model [4]. For the terms quartic in $\Sigma$ we find

$$\frac{1}{4} \left[ \partial_I \partial_K \partial_J \partial_L K \right] \Sigma^I \Sigma^J \Sigma^K \Sigma^L \quad . \quad (2.44)$$
This is proportional to the “dominant term” in the full curvature tensor for a Kähler manifold.

\[ R_{I\bar{K}J\bar{L}} = K_{I,\bar{K},J,\bar{L}} - g^{M\bar{N}} K_{I,J,M} K_{\bar{K},\bar{L},N} \quad . \] (2.45)

It must be the role of the infinite number of auxiliary superfields to covariantize the dominant terms so that these terms above are replaced according to

\[ K_{I_1,\bar{K}_1,J_1,\bar{J}_1} \rightarrow R_{I_1\bar{K}_1I_2\bar{K}_2} \quad , \]

\[ K_{I_1,...,I_{p+3},\bar{K}_1,...,\bar{K}_{p+3}} \rightarrow \left( \prod_{\ell=1}^{p+1} \nabla_{I_{\ell}} \right) \left( \prod_{\ell'=1}^{p+1} \nabla_{\bar{K}_{\ell'}} \right) R_{I_{p+2}\bar{K}_{p+2}I_{p+3}\bar{K}_{p+3}} \quad . \] (2.46)

Above in (2.46), the operators \( \nabla_I \) and \( \nabla_{\bar{K}} \) denote covariant derivatives with respect to the Kähler metric \( g_{I\bar{J}} \). We emphasize that at present, we have no proof of this proposal. However, this is suggested by demanding covariance with respect to the Kähler geometry of the model. As well this is also suggested by the role of the auxiliary field in ordinary 4D, \( N = 1 \) supersymmetric nonlinear \( \sigma \)-models.

However, the results in (2.30) imply that the results in (2.43) and (2.46) cannot be the complete answer for the removal of the auxiliary fields, for the structures in the second line of (2.46) vanish for covariantly curvature. The obvious reason that we know this is because our second explicit example in (2.30) describes a Kähler metric with covariantly constant curvature. The second term in (2.30) may be expanded in \( \Sigma \bar{\Sigma} \) and contains terms to all orders; this is in fact an expansion in powers of curvature. It must be the case that the correct expression for the action takes the form

\[ S_\sigma[\Upsilon_*, \bar{\Upsilon}_*] = \int d^8z \left\{ K(\Phi, \bar{\Phi}) - g_{I\bar{J}}(\Phi, \bar{\Phi}) \Sigma^I \Sigma^{\bar{J}} + \frac{1}{4} R_{I\bar{K}J\bar{L}} \Sigma^I \Sigma^{\bar{J}} \Sigma^\bar{K} \Sigma^\bar{L} \right. \]

\[ - \sum_{p=0}^{\infty} (-1)^p \left[ \frac{1}{(p+3)!} \right]^2 \left( \prod_{\ell=1}^{p+3} \Sigma^{I_{\ell}} \Sigma^{\bar{K}_{\ell}} \right) G_{I_1,...,I_{p+3}\bar{K}_1,...,\bar{K}_{p+3}} \left( R_{I\bar{K}J\bar{L}}, \nabla_M, \nabla_{\bar{M}} \right) \quad , \] (2.47)

The terms on the last line of (2.43) are simply the leading terms in the (yet to be completely determined) functions \( G_{I_1,...,I_{p+3}\bar{K}_1,...,\bar{K}_{p+3}} \). The perturbative description of these is provided by the sequence of steps in (2.18 - 2.22). It is worth noting that each term in the action \( S_\sigma[\Upsilon_*, \bar{\Upsilon}_*] \) contains an equal number of \( \Sigma \)- and \( \Sigma \)-multiplets, as a consequence of the \( U(1) \) symmetry (2.7).

The naive dimensional compactification of our 4D, \( N = 2 \) results leads to new CNM formulations of 2D, \( N = 4 \) theories. Similar theories have recently been discussed by
Penati et. al. [5]. One result shown is that CNM models possess a natural encoding of the three complex structures one expects of a hyper-Kähler model. Although their 2D, \( N = 4 \) models do not describe unitary quantum field theories, the encoding of the complex structures they describe remains true in our construction as well which in addition is ghost-free. The 2D, \( N = 4 \) form of our results follow by applying the toroidal compactification to our 4D models with \( N = 2 \) supersymmetry.

3 Special Hyper-Kähler Geometries

Our approach to 4D, \( N = 2 \) supersymmetrical non-linear \( \sigma \)-models yields an interesting viewpoint on issues surrounding hyper-Kähler manifolds. Metrics describing such manifolds are in general difficult to explicitly construct. Remarkably, \( N = 2 \) supersymmetry provides us with a regular procedure to construct hyper-Kähler structures associated with a family of hyper-Kähler manifolds – the cotangent bundles to Kähler manifolds. In fact, it was conjectured for a long time that every total space \( T^* \mathcal{M} \) of the cotangent bundle to a Kähler manifold \( \mathcal{M} \) admits a hyper-Kähler structure\(^3\), and a lot of relevant examples were elaborated, see, for instance, [18]. This conjecture has been recently proved by Kaledin [20], as an existence theorem, for an open neighborhood \( U \subset T^* \mathcal{M} \) of the zero section \( \mathcal{M} \subset T^* \mathcal{M} \). Using the supersymmetric model (2.1) allows us to explicitly construct the hyper-Kähler metrics on \( T^* \mathcal{M} \).

Upon elimination of the auxiliary superfields, the model (2.1) turns into \( N = 2 \) supersymmetric CNM \( \sigma \)-model (2.47),

\[
\mathcal{S}_\sigma[\Upsilon_*, \bar{\Upsilon}_*] = \mathcal{S}_\sigma[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}] = \int d^8 z \mathcal{L}_{\text{CNM}}(\Phi^I, \bar{\Phi}^J, \Sigma^K, \bar{\Sigma}^L) \tag{3.1}
\]

which possesses a nonlinearly realized \( N = 2 \) supersymmetry and is defined on an open neighborhood \( U \subset T \mathcal{M} \) (of the zero section \( \mathcal{M} \subset T \mathcal{M} \)) of the tangent bundle \( T \mathcal{M} \) to the Kähler manifold \( \mathcal{M} \). The tangent-space variables are realized in terms of the complex linear superfield \( \Sigma \) and its conjugate. In the case of \( \mathbb{C}P^1 \), the domain \( U \) is defined by eq. (2.31). The action (3.1) can be replaced by a dually equivalent one with the action

\[
\mathcal{S}_\sigma[\Phi, \bar{\Phi}, G, \bar{G}, \Psi, \bar{\Psi}] = \int d^8 z \left\{ \mathcal{L}_{\text{CNM}}(\Phi^I, \bar{\Phi}^J, G^K, \bar{G}^L) + G^K \Psi^L + \bar{G}^L \bar{\Psi}_K \right\} , \tag{3.2}
\]

where \( G \)'s are complex unconstrained superfields, and \( \Psi \)'s are chiral superfield. It is obvious that \( G^I \) should be treated as a tangent vector, while \( \Psi_I \) as a one-form to the

\(^3\)We are grateful to Andrei Lossev for enlightening discussions on this conjecture.
point $\Phi$ of $\mathcal{M}$. The auxiliary superfields $G$ and $\bar{G}$ can be eliminated with the aid of their equations of motion. Then, we stay with a purely chiral-chiral model

$$S_\sigma[\Phi, \bar{\Phi}, \Psi, \bar{\Psi}] = \int d^8z \mathcal{L}_{CC}(\Phi^I, \bar{\Phi}^J, \Psi^K, \bar{\Psi}_L)$$

(3.3)

which possesses a nonlinearily realized $N = 2$ supersymmetry and is defined on an open submanifold $V \subset T^*\mathcal{M}$ of the cotangent bundle $T^*\mathcal{M}$ to the Kähler manifold $\mathcal{M}$. Now, the most important point is that $\mathcal{L}_{CC}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ is nothing else but the hyper-Kähler potential on $T^*\mathcal{M}$!

Our consideration shows how to construct hyper-Kähler metrics on cotangent bundles to Kähler manifolds. The procedure is as follows:

- given a Kähler manifold $\mathcal{M}$, construct the $N = 2$ models (2.1) with linearly realized supersymmetry;
- eliminate the infinite number of auxiliary superfields;
- fulfill the duality transform described in this section.

This is a long but constructive way to generate hyper-Kähler metrics. The above construction defines a special family of hyper-Kähler metrics and, hence, “special hyper-Kähler geometries”.

In accordance with the results of [20], the hyper-Kähler metric on $T^*\mathcal{M}$ is invariant under the action of the group $U(1)$ on $T^*\mathcal{M}$ given by dilatations along the fibers of the canonical projection $\pi : T^*\mathcal{M} \to \mathcal{M}$. This invariance group is nothing else but the $U(1)$ fiber symmetry (2.7) of our $N = 2$ $\sigma$-model (2.3).

4 4D Soft Pion Physics, $N = 2$ Supersymmetry and Holomorphic Higher Derivative Actions

In previous works [4], the use of well-known aspects of chiral perturbation theory [3] was advocated for the construction of a “phenomenologically based” 4D, $N = 1$ CNM low-energy QCD supersymmetric effective action. The advent of the “polar multiplet formalism” now permits a further $N = 2$ extension of these ideas.

We are inspired to use this name in analogy with “special Kähler geometries” associated the KVM in [19].
In the final work of [4], it was shown that any higher derivative term which occurs in a non-supersymmetric action of the type described in chiral perturbation theory can be mapped into a 4D, \( N = 1 \) superfield action under the action of an operator denoted by \( G_S \). After an integration-by-parts to form actions integrated over \( d^8z \) such superfield actions take forms very similar to that of (2.24). However, (2.24) does not produce higher derivative terms. A way to produce such terms within the polar multiplet formalism seems to begin with the following observation. The action

\[
S[\mathcal{Y}, \mathcal{Y}] = \frac{1}{2\pi i} \int \frac{dw}{w} \int d^8z \mathcal{H}\left( \mathcal{Y}, \partial_a \mathcal{Y}, ..., \bar{\mathcal{Y}}, \partial_a \bar{\mathcal{Y}}, ..., w \right)
\]

(4.1)

necessarily generates higher derivative terms. Combining this with our previous suggestion seems to have the following implication. Since we believe that holomorphy is of critical importance for the higher derivative terms, it is only a change

\[
\mathcal{J}(\mathcal{Y}, \bar{w}) \rightarrow \mathcal{J}(\mathcal{Y}, \partial_a \mathcal{Y}, \bar{w})
\]

(4.2)

in (2.24) that will generate holomorphic higher derivative terms consistent with an \( N = 2 \) supersymmetry. In particular, we believe the first term in (2.24) should not be modified. However, there must also be addition changes that correspond to inserting the factors of \( D_\alpha \) and \( \bar{D}_{\dot{\alpha}} \) which occurred in the \( N = 1 \) case and. Such higher derivative terms can come from a manifestly \( N = 2 \) supersymmetric functional of the general form

\[
S[\mathcal{Y}, \bar{\mathcal{Y}}] = \int d^{12}z \mathcal{G}\left( \mathcal{Y}_m, \bar{\mathcal{Y}}_n \right), \quad m, n = 1, 2, \ldots
\]

(4.3)

where the integration is carried out over the full \( N = 2 \) superspace. This topic is still under study.

Since manifest realization of \( N = 2 \) supersymmetry forces the use of the CNM doublet, this provides an “\( N = 2 \) answer” to yet another seemingly ad hoc feature of the \( N = 1 \) CNM low-energy QCD supersymmetric model. In our present work, the pion octet must be embedded in the superfield pair \( (\Phi', \Sigma') \) as a consequence of \( N = 2 \) supersymmetry. In turn the \textit{a priori} existence of the mixing angle \( \gamma_S \) [4], which expresses the possibility that the pion octet can occur as a linear combination of the two \( 0^- \) octets that appear in the CNM doublet, is seen to have an \( N = 2 \) supersymmetry origin.
5 Conclusion

In a sense, the action in (2.3) together with (4.2) completes several cycles of our past work. Many years ago, we [19] proposed a 4D, $N = 1$ superfield action of the form

$$S_{KVM} = \left\{ \int d^4x \left[ \int d^2\theta d^2\bar{\theta} F^I \partial_I + \frac{1}{4} \int d^2\theta W^\alpha W^K \partial_I \partial_K \right] H(\Phi) + \text{h.c.} \right\}. \quad (5.1)$$

to describe a 4D, $N = 2$ supersymmetric $\sigma$-model which we named the “Kählerian Vector Multiplet” model. With (2.3) and the subsequent elimination of the auxiliary superfields, we have succeeded in writing a purely 4D, $N = 1$ supersymmetric $\sigma$-model action for the hypermultiplet. In the intervening years, of course, there has been the much heralded work of Seiberg and Witten [7] on $N = 2$ supersymmetrical Yang-Mills theory that completely determines the function $H(\Phi)$ above. Two years ago [4], we renewed our study of CNM models because they offered the opportunity to clearly inject geometrical aspects of real-world chiral perturbation theory [3] into the hypothetical world of 4D, $N = 1$ (and now likely $N = 2$) supersymmetry. The results in (2.3) and (4.2) hold out the prospect that these two worlds may not be as far removed as might have first seemed.

“Nullius in verba.” – Motto of the Royal Society of London

Acknowledgment

Both authors wish to acknowledge very helpful exchanges with I. Buchbinder, B. de Wit, N. Dragon, F. Gonzales-Rey, T. Hübsch, E. Ivanov, U. Lindström, D. Lüst, B. Ovrut, M. Roček, S. Theisen and M. Vasiliev. We thank especially A. Lossev for informing us about Ref. [20] and A. Nersessian for bringing Ref. [18] to our attention. One of us (S.K.) acknowledges the Bogoliubov Laboratory of Theoretical Physics at the Joint Institute for Nuclear Research, where part of this project was done, for hospitality. The work of S.K. was supported in part from RFBR grant, project No 96-02-16017; RFBR-DFG grant, project No 96-02-00180; INTAS grant, INTAS-96-0308.

A Projective Superspace

Superfields living in the $N = 2$ projective superspace [16] are parametrized by a complex bosonic variable $w$ along with the coordinates of $N = 2$ global superspace $\mathbb{R}^{4|8}$

$$z^M = (x^m, \theta_i^\alpha, \bar{\theta}_i^\dot{\alpha}), \quad \bar{\theta}_i^{\dot{\alpha}} = \bar{\theta}_i^{\dot{\alpha} i}, \quad i = 1, 2. \quad \text{(A.1)}$$
A superfield of the general form

$$\Xi(z, w) = \sum_{n=-\infty}^{+\infty} \Xi_n(z)w^n$$  \hspace{1cm} (A.2)

is said to be projective if it satisfies the constraints

$$\nabla_\alpha(w)\Xi(z, w) = 0 \ , \ \nabla_{\dot{\alpha}}(w)\Xi(z, w) = 0$$  \hspace{1cm} (A.3)

which involve the operators

$$\nabla_\alpha(w) \equiv wD^1_\alpha - D^2_\alpha \ , \ \nabla_{\dot{\alpha}}(w) \equiv \nabla_{\dot{\alpha}} \equiv \nabla_{\dot{\alpha}}(w)$$  \hspace{1cm} (A.4)

constructed from the $N = 2$ covariant derivatives $D_M = (\partial_m, D^i_\alpha, D^i_{\dot{\alpha}})$. The operators $\nabla_\alpha(w)$ and $\nabla_{\dot{\alpha}}(w)$ strictly anticommute with each other, as a consequence of the covariant derivative algebra

$$\{D^i_\alpha, D^j_\beta\} = \{\nabla_{\dot{\alpha}}i, \nabla_{\dot{\beta}}j\} = 0 \ , \ \{D^i_\alpha, \nabla_{\dot{\alpha}}j\} = -2i \delta^i_j D^i_{\dot{\alpha}} \ .$$  \hspace{1cm} (A.5)

With respect to the inner complex variable $w$, the projective superfields are holomorphic functions on the punctured complex plane $\mathbb{C}^*$

$$\partial_{\bar{w}} \Xi(z, w) = 0 \ .$$  \hspace{1cm} (A.6)

Constraints (A.3) rewritten in components

$$D^2_\alpha \Xi_n = D^1_\alpha \Xi_{n-1} \ , \ \nabla_{\dot{\alpha}} \Xi_n = -\nabla_{\dot{\alpha}} \Xi_{n+1}$$  \hspace{1cm} (A.7)

determine the dependence of the component $N = 2$ superfields $\Xi$’s on $\theta^\alpha_i$ and $\bar{\theta}^\alpha_{\dot{\alpha}}$ in terms of their dependence on $\theta_1^\alpha$ and $\bar{\theta}^1_{\dot{\alpha}}$. Therefore, the components $\Xi_n$ are effectively superfields over the $N = 1$ superspace parametrized by

$$\theta^\alpha = \theta^\alpha_1 \ , \ \bar{\theta}_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}}^1 \ .$$  \hspace{1cm} (A.8)

If the power series in (A.2) terminates, some of the $N = 1$ superfields satisfy constraints involving the $N = 1$ covariant derivatives

$$D_\alpha = D^1_\alpha \ , \ \nabla^i = \nabla^i_{\dot{\alpha}}$$  \hspace{1cm} (A.9)

A natural operation of conjugation, which brings every projective superfield into a projective one, reads as follows

$$\bar{\Xi}(z, w) = \sum_n (-1)^n \bar{\Xi}_n(z)w^n$$  \hspace{1cm} (A.10)
with \( \tilde{\Xi}_n \) being the complex conjugate of \( \Xi_n \). A real projective superfield is constrained by

\[
\tilde{\Xi} = \Xi \iff \tilde{\Xi}_n = (-1)^n \Xi_{-n}.
\]  
(A.11)

The component \( \Xi_0(z) \) is seen to be real. The following equations are also seen to be valid,

\[
\check{\nabla}_\alpha(w) = -\frac{1}{w} \nabla_\alpha(w) , \quad \check{\nabla}_{\dot{\alpha}}(w) = \frac{1}{w} \nabla_{\dot{\alpha}}(w)
\]  
(A.12)

so that application of the \( \check{\quad} \) operation twice is equivalent to the identity map on these operators. It follows the operators

\[
\tilde{\nabla}_\alpha(w) \equiv (i)^{\frac{3}{2}} \left( \frac{1}{w} \right)^{\frac{1}{2}} \nabla_\alpha(w) , \quad \tilde{\nabla}_{\dot{\alpha}}(w) \equiv - (i)^{\frac{3}{2}} \left( \frac{1}{w} \right)^{\frac{1}{2}} \nabla_{\dot{\alpha}}(w)
\]  
(A.13)

transform as

\[
\check{\nabla}_\alpha(w) \rightarrow \tilde{\nabla}_{\dot{\alpha}}(w) , \quad \check{\nabla}_{\dot{\alpha}}(w) \rightarrow \tilde{\nabla}_\alpha(w)
\]  
(A.14)

under \( \check{\quad} \)-conjugation.

Given a real projective superfield \( \mathcal{L}(z,w) \), \( \check{\mathcal{L}} = \mathcal{L} \), we can construct a \( N = 2 \) supersymmetric invariant by the following rule

\[
S = \int d^4x \, D^4 \mathcal{L}_0(z) | , \quad D^4 = \frac{1}{16} D^{\alpha_1} D^{\dot{\alpha}_1} \mathcal{D}_{\alpha_1} \mathcal{D}^{\dot{\alpha}_1}
\]  
(A.15)

where \( D^4 \) is the \( N = 1 \) superspace measure, \( \mathcal{L}_0(z) = \mathcal{L}(z,0) \) and \( U| \) means the \( \theta \)-independent component of a superfield \( U \). Actually, from the standard supersymmetric transformation law

\[
\delta \mathcal{L} = i \left( \epsilon^\alpha Q^\alpha_i + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^\dagger_{\dot{\alpha}} \right) \mathcal{L}
\]  
(A.16)

we find

\[
\delta S = i \int d^4x \left( \epsilon^\alpha Q^\alpha_i + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^\dagger_{\dot{\alpha}} \right) D^4 \mathcal{L}_0 = - \int d^4x \left( \epsilon^\alpha D^\alpha_i + \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\dagger_{\dot{\alpha}} \right) D^4 \mathcal{L}_0
\]

\[
= - \int d^4x \left( \epsilon^2 D^2_i + \bar{\epsilon}_{\dot{2}} \bar{D}^\dagger_{\dot{2}} \right) D^4 \mathcal{L}_0 = - \int d^4x D^4 \left( \epsilon^2 D^2_i \mathcal{L}_1 \bar{|} - \bar{\epsilon}_{\dot{2}} \bar{D}^\dagger_{\dot{2}} \mathcal{L}_1 \right) = 0
\]

The action can be rewritten in the form

\[
S = \frac{1}{2} \int_C d \frac{1}{w} \int d^4x \, D^4 \mathcal{L}_1 | , \quad \mathcal{L}_1 | = \mathcal{L}_0 | - \mathcal{L}_1 \]

(A.17)

where \( C \) is a contour around the origin.

The only projective superspace representation considered in the main body of this paper is the so-called polar multiplet (or \( \Upsilon \) hypermultiplet) which describes a charged \( N = 2 \) scalar multiplet and is defined by eq. (2.4). In Appendix B we also make use of a complex tropical multiplet.
Since the concept of duality is presently considered of importance, in this appendix we discuss the implementation of Poincaré Duality within the context of polar multiplets. The hypermultiplet model

\[ S[\Upsilon, \bar{\Upsilon}] = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^8z \bar{\Upsilon} \Upsilon \quad . \quad (B.1) \]

with

\[ \Upsilon = \sum_{n=0}^{\infty} \Upsilon_n(z)w^n = \Phi(z) + w\Sigma(z) + O(w^2) \quad , \]

\[ \bar{\Upsilon} = \sum_{n=0}^{\infty} (-1)^n \bar{\Upsilon}_n(z)w^{-n} = \bar{\Phi}(z) - \frac{1}{w}\bar{\Sigma}(z) + O\left(\frac{1}{w}\right)^2 \quad , \quad (B.2) \]

describes a free CNM doublet. The chiral scalar \( \Phi \) can be dualized into a complex linear one, and the complex linear scalar \( \Sigma \) can be dualized into a chiral one. As a result, we again obtain a CNM doublet. Therefore, the above theory can be said to be self-dual, in the sense of Poincaré duality. Below we will develop the relevant formalism to make this manifest.

First, let us introduce a complex tropical multiplet

\[ U = \sum_{n=-\infty}^{\infty} U_n(z)w^n \quad , \quad \bar{U} = \sum_{n=-\infty}^{\infty} (-1)^n \bar{U}_n(z)w^{-n} \quad , \quad (B.3) \]

with all the component \( U_n \) being complex unconstrained superfields. We also introduce a multiplet realized as follows

\[ \Xi = \sum_{n=-1}^{\infty} \Xi_n(z)w^n = \frac{1}{w}\Psi(z) + \Gamma(z) + O(w) \quad , \]

\[ \bar{\Xi} = \sum_{n=-1}^{\infty} (-1)^n \bar{\Xi}_n(z)w^{-n} = -w\bar{\Psi}(z) + \bar{\Gamma}(z) + O\left(\frac{1}{w}\right) \quad , \quad (B.4) \]

with \( \Psi \) being chiral, \( \Gamma \) being complex linear, and the rest component superfields being complex unconstrained. It is worth keeping in mind the difference between the definitions of \( \Upsilon \) and \( \Xi \).
Consider the following auxiliary action

\[ S = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^8 z \left( \bar{U} U - U \Xi - \bar{U} \Xi \right) \]

\[ = \int d^8 z \left( \sum_{n=-\infty}^{\infty} (-1)^n \bar{U}_n U_n - \sum_{n=-1}^{\infty} \Xi_n U_{-n} - \sum_{n=-1}^{\infty} \bar{\Xi}_n \bar{U}_{-n} \right) . \]  \hspace{1cm} (B.5)

One easily observes

\[ \frac{\delta S}{\delta \Xi} = 0 \iff U(w) = \Upsilon(w) = \sum_{n=0}^{\infty} \Upsilon_n(z) w^n . \]  \hspace{1cm} (B.6)

Therefore we return to the original model \((B.4)\). On the other hand, let us now consider the equation of motion for \(U(w)\). From

\[ \frac{\delta S}{\delta U} = 0 \]  \hspace{1cm} (B.7)

we obtain

\[ U_n = 0 , \quad n \geq 2 \]

\[ (-1)^n \bar{U}_n - \Xi_{-n} = 0 , \quad (-1)^n U_n - \bar{\Xi}_{-n} = 0 , \quad n \leq 1 . \]  \hspace{1cm} (B.8)

Therefore

\[ U(w) = \bar{\Xi}(w) \quad , \quad \bar{U}(w) = \Xi(w) \]  \hspace{1cm} (B.9)

and the action takes the form

\[ S[\Xi, \bar{\Xi}] = -\frac{1}{2\pi i} \oint \frac{dw}{w} \int d^8 z \bar{\Xi} \Xi . \]  \hspace{1cm} (B.10)

We see that the theory \((2.3)\) is self-dual.

In case of a general nonlinear sigma-model

\[ S[\Upsilon, \bar{\Upsilon}] = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^8 z K(\bar{\Upsilon}, \Upsilon) \]  \hspace{1cm} (B.11)

we can also formally implement the duality transformation described above. But now it replaces the potential \(K(\bar{\Upsilon}, \Upsilon)\) by a new one. The requirement that the sigma-model be self-dual can be put forward and it restricts, in a nontrivial way, the class of possible models. It is interesting to conjecture that for \(N = 4\) SYM effective theories realized in projective superspace, the hypermultiplet action must be self-dual with respect to the duality transformation above.
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