Hamiltonian structure of the teleparallel formulation of GR

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Abstract

We apply Dirac’s Hamiltonian approach to study the canonical structure of the teleparallel form of general relativity without matter fields. It is shown, without any gauge fixing, that the Hamiltonian has the generalized Dirac–ADM form, and constraints satisfy all the consistency requirements. The set of constraints involves some extra first class constraints, which are used to find additional gauge symmetries and clarify the gauge structure of the theory.

I. INTRODUCTION

Among various attempts to overcome the problems of quantization and the existence of singular solutions in Einstein’s general relativity (GR), gauge theories of gravity are especially attractive, as they are based on the concept of gauge symmetry which has been very successful in the foundation of other fundamental interactions. The importance of the Poincaré symmetry in particle physics leads one to consider the Poincaré gauge theory (PGT) as a natural framework for description of the gravitational phenomena [1–5] (for more general attempts, see [6]).

Basic gravitational variables in PGT are tetrad field $b^k_{\mu}$ and Lorentz connection $A^{ij}_{\mu}$, which are associated to the translation and Lorentz subgroups of the Poincaré group, respectively. These gauge fields are coupled to the energy–momentum and spin of matter fields, and their field strengths are geometrically identified with the torsion and the curvature:

$$T^i_{\mu\nu} = \partial_\mu b^i_{\nu} + A^i_{s\mu} b^s_{\nu} - (\mu\nu), \quad R^{ij}_{\mu\nu} = \partial_\mu A^{ij}_{\nu} + A^i_{s\mu} A^j_{s\nu} - (\mu\nu).$$

The spacetime of PGT turns out to be Riemann–Cartan space $U_4$, equipped with metric and linear, metric compatible connection. Dynamical content of PGT is determined by the Lagrangian $\mathcal{L} \equiv b(\mathcal{L}_G + \mathcal{L}_M)$, where the gravitational part $\mathcal{L}_G$ is usually assumed to be at most quadratic in field strengths, and $\mathcal{L}_M$ describes minimally coupled matter fields.

General geometric arena of PGT, the Riemann–Cartan space $U_4$, may be a priori restricted by imposing certain conditions on the curvature and the torsion. Thus, Einstein’s GR is defined in Riemann space $V_4$, which is obtained from $U_4$ by the requirement of vanishing torsion. Another interesting limit of PGT is teleparallel or Weitzenböck geometry $T_4$, defined by the requirement of vanishing curvature:

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\[ R^{ij}_{\mu
u}(A) = 0. \]  

The vanishing of curvature means that parallel transport is path independent, hence we have an absolute parallelism. The teleparallel geometry is, in a sense, complementary to Riemannian: curvature vanishes, and torsion remains to characterize the parallel transport.

Of particular importance for the physical interpretation of the teleparallel geometry is the fact that there is a one-parameter family of teleparallel Lagrangians which is \textit{empirically} equivalent to GR. For the parameter value \( B = 1/2 \) the Lagrangian of the theory coincides, modulo a four-divergence, with the Einstein–Hilbert Lagrangian, and defines the teleparallel form of GR.

The teleparallel description of gravity has been one of the most promising alternatives to GR. However, analyzing this theory Kopczyński found a hidden gauge symmetry, and concluded that the torsion evolution is not completely determined by the field equations. Assuming, then, that the torsion should be a measurable physical quantity, he argued that this theory is internally inconsistent. Hayashi and Shirafuji tried to avoid this problem by interpreting certain different torsion configurations as physically equivalent, i.e. related to each other by a gauge transformation, but the consistency of this idea in the interacting theory seems to be questionable for non-scalar matter. Various modifications of the one-parameter teleparallel theory are proposed in order to avoid the above problems. Trying to re-examine the gauge structure of the one-parameter teleparallel geometry Nester improved the arguments of Kopczyński; the predictability problem was stated more precisely and bound to certain special solutions.

Hecht et al. traced the appearance of non-physical modes of torsion back to some symmetries which are necessarily present in the (3+1) decomposition of spacetime. Using certain geometric arguments they concluded that some components of the tetrad velocity are not suited to represent dynamical degrees of freedom. In other words, these velocities must not appear in the evolution equations, hence they should appear at most linear in the Lagrangian. The choice of parameters in the teleparallel Lagrangian that ensures this to happen is just the one corresponding to GR

The teleparallel geometry possesses many salient features. Thus, Nester succeeded in formulating a pure tensorial proof of the positivity of total energy for Einstein’s theory in terms of the teleparallel geometry. He found that special gauge features of GR, which are usually considered to be problematic, are quite beneficial for this purpose. Mielke used the teleparallel geometry of GR to give a transparent description of Ashtekar’s complex variables, while Andrade et al. formulated a five-dimensional teleparallel equivalent of Kaluza–Klein theory. There are also some attempts to understand the role of torsion at the quantum level.

The purpose of this paper is to investigate the canonical structure of GR using Dirac’s Hamiltonian approach, as this is, in our opinion, the best way to clarify both the nature of somewhat mysterious extra gauge symmetries, and the question of consistency of GR. We shall find that a specific choice of coupling constants in the teleparallel Lagrangian leads to the appearance of some additional first class constraints, and, consequently, to extra gauge symmetries, which clarify the meaning of non-dynamical torsion components and give us a complete picture of the gauge structure of GR.

We remark here that Maluf tried to analyze some aspects of the Hamiltonian structure of GR. However, his approach is based on some unnecessary gauge fixing conditions,
adopted at the level of Lagrangian in order to simplify the calculations, so that many specific
gauge features of the theory remained effectively hidden.

The layout of the paper is as follows. After recalling some basic elements of the La-
grangian teleparallel formulation of GR in Sec. II, we work out all the primary constraints
and construct the corresponding Hamiltonian density in Sec. III. It is shown that a specific
choice of parameters in the Lagrangian leads to additional primary constraints. Then, we
study the consistency conditions in Sec. IV, and derive the algebra of constraints in Sec. V.
These results are used in Sec. VI to construct extra gauge generators and clarify the nature
of the related gauge symmetries. Section VII is devoted to concluding remarks, while some
technical details are presented in the Appendix.

Our conventions are the same as in Ref. [22]: the Latin indices refer to the local Lorentz
frame, whereas the Greek indices refer to the coordinate frame; the first letters of both alphabets
(a, b, c,...; α, β, γ,...) run over 1,2,3, and the middle alphabet letters (i, j, k,...; μ, ν, λ,...)
run over 0,1,2,3; η_{ij} = \text{diag} (+, -, -, -), \varepsilon^{0123} = +1 and \delta = \delta(x - x').

II. THE TELEPARALLEL FORMULATION OF GR

Lagrangian. Gravitational field in the framework of the teleparallel geometry in PGT
is described by the tetrad b^k_\mu and Lorentz connection A^{ij}_\mu, subject to the condition of
vanishing curvature (1.1). We shall consider here the gravitational dynamics determined by
a class of Lagrangians quadratic in the torsion (2.1)

\[ \mathcal{L} = b(\mathcal{L}_T + \lambda^{\mu\nu}_ij R^{ij}_\mu\nu + \mathcal{L}_M), \]
\[ \mathcal{L}_T = a(\beta^{ijk}(T)T^{ijk} + BT_{ijk}T^{jik} + CT_kT^k), \]
where \lambda^{\mu\nu}_ij are Lagrange multipliers introduced to ensure the teleparallelism condition (1.1)
in the variational formalism, \[ a = 1/2\kappa \quad (\kappa = \text{Einstein's gravitational constant}), \]
T_k = T^m_{mk}, and \mathcal{L}_M is the Lagrangian of matter fields. The explicit form of \beta_{ijk} is

\[ \beta_{ijk} = a(\beta T_{ijk} + BT_{j[ik]} + C[\eta_{ij}T_k]). \]

The parameters A, B, C in the Lagrangian should be determined on physical grounds,
so as to obtain a consistent theory which could describe all the known gravitational ex-
periments. If we require that the theory (2.1) gives the same results as GR in the linear,
weak–field approximation, we can restrict our considerations to the one–parameter family
of Lagrangians, defined by the conditions (2.1)

\[ i) \ 2A + B + C = 0, \quad C = -1. \]

This family represents a viable gravitational theory for macroscopic, spinless matter, empir-
ically indistinguishable from GR. Von der Heyde [23] and Hehl [5] have given certain
theoretical arguments in favor of the choice B = 0. There is, however, another, particularly
interesting choice determined by the requirement

\[ ii) \ 2A - B = 0, \]
It leads effectively to the Einstein–Hilbert Lagrangian \( \mathcal{L}_{GR} = -abR(\Delta) \), defined in Riemann spacetime \( V_4 \) with Levi–Civita connection \( A = \Delta \), via the geometric identity (A3):

\[
bR(A) = bR(\Delta) + b(\frac{1}{2}T_{ijk}T^{ijk} + \frac{1}{2}T_{ijk}T^{jik} - T_kT^k) - 2\partial_\nu(bT^\nu) .
\]

Indeed, in Weitzenböck spacetime the above identity in conjunction with the condition (1.1) implies that the torsion Lagrangian in (2.1) is equivalent to the Einstein–Hilbert Lagrangian, up to a four–divergence, provided that

\[
A = \frac{1}{4}, \quad B = \frac{1}{2}, \quad C = -1 , \tag{2.2}
\]

which coincides with the conditions \( i \) and \( ii \) given above.

The theory defined by equations (2.1) and (2.2) is called the teleparallel formulation of \( GR \) (\( GR_{\parallel} \)). Note that the equivalence with \( GR \) holds certainly for scalar matter, while the gravitational couplings to spinning matter fields in \( T_4 \) and \( V_4 \) are in general different.

**Field equations.** By varying the Lagrangian (2.1) with respect to \( b^i_\mu, A^{ij}_\mu \) and \( \lambda^{ij}_\mu
\)

we obtain the gravitational field equations:

\[
-4\nabla_\mu(b^\beta_\mu^{\nu}) - 4b\beta^{nm\nu}T_{nmi} + h^{\nu}_i b\mathcal{L}_T = \tau^\nu_i , \tag{2.3a}
\]

\[
-8b\beta^{[ij}_\nu + 4\nabla_\mu(b\lambda^{ij}_\nu^{\mu}) = \sigma^{\nu}_{ij} , \tag{2.3b}
\]

\[
R^{ij}_\mu = 0 , \tag{2.3c}
\]

where \( \tau^\nu_i \) and \( \sigma^{\nu}_{ij} \) are the energy–momentum and spin currents of matter fields, respectively.

The first field equation can be rewritten as

\[
-4\nabla_\mu(b^{\beta}_{i\mu}^{\mu}) + 2b\beta^{nm\nu}T_{nmi} - 4b\beta^{nmk}T_{nmi} + \delta^{k}_i b\mathcal{L}_T = \tau^{k}_i . \tag{2.4a}
\]

Then, combined with the identity (A3), it takes the form of Einstein’s field equation:

\[
R^{ik}(\Delta) - \frac{1}{2}\eta^{ik}R(\Delta) = \tau^{ki}/2ab .
\]

Here, on the left hand side we have Einstein’s tensor of \( GR \), which is a symmetric tensor. Therefore, the dynamical energy–momentum tensor must be also symmetric, \( \tau^{ik} = \tau^{ki} \).

Using the identity (A4) the second field equation can be written in the form

\[
\nabla^{\mu}(2aH^{ij}_\mu + 4b\lambda^{ij}_\nu^{\mu}) = \sigma^{\nu}_{ij} , \tag{2.4b}
\]

where \( H^{ij}_\mu = b(h^{\nu}_i h^{\mu}_j - h^{\nu}_j h^{\mu}_i) \). The integrability condition for this equation is identically satisfied, because the covariant divergence of the left hand side vanishes on account of \( R^{ij}_\mu = 0 \), whereas

\[
\nabla_\nu\sigma^{\nu}_{ij} = \tau_{ij} - \tau_{ji} = 0 .
\]

The first equality in this relation is a consequence of the covariant conservation of angular momentum for matter fields (which holds when matter field equation is satisfied), and the vanishing of \( \tau_{ij} \) follows from the first field equation.

Simple counting shows that the number of independent field equations (2.4b) is 24 – 6 = 18. The multipliers \( \lambda^{ij}_\alpha^{\beta} \) remain arbitrary functions of time, as will be shown in the forthcoming Hamiltonian analysis. For any specific choice of \( \lambda^{ij}_\alpha^{\beta} \) (gauge fixing), equations (2.4b) can be used to determine (at least locally) the remaining 18 multipliers \( \lambda^{ij}_{\alpha} \).

In what follows we shall investigate the Hamiltonian structure and gauge properties of \( GR_{\parallel} \) without matter fields \( (\sigma^{\nu}_{ij} = \tau_{ij} = 0) \). We expect that the results obtained here will be also useful for the analysis of interacting \( GR_{\parallel} \).
III. PRIMARY CONSTRAINTS AND HAMILTONIAN

1. The basic Lagrangian dynamical variables of our theory are \((b^i, A^{ij}, \lambda_{ij}^{\mu\nu})\), and the corresponding momenta are denoted by \((\pi_i^\alpha, \pi_{ij}^\alpha, \pi^{ij\mu\nu})\). Due to the fact that the torsion and the curvature do not involve velocities \(b^k_\circ\) and \(A^{ij}_\circ\), one immediately obtains the following set of the so-called sure primary constraints:

\[
\phi^0_k \equiv \pi^0_k \approx 0, \quad \phi^0_{ij} \equiv \pi^0_{ij} \approx 0. \tag{3.1}
\]

Similarly, the absence of the time derivative of \(\lambda_{ij}^{\mu\nu}\) implies

\[
\phi^{ij\mu\nu} \equiv \pi^{ij\mu\nu} \approx 0. \tag{3.2}
\]

The next set of constraints follows from the linearity of the curvature in \(\dot{A}^{ij}\):

\[
\phi_{ij\alpha} \equiv \pi_{ij\alpha} - 4b\lambda_{ij}^{0\alpha} \approx 0. \tag{3.3}
\]

Before we continue, it is convenient introduce the so–called \((3+1)\) decomposition of spacetime \([22]\). If \(n\) is the unit normal to the hypersurface \(\Sigma_0 : x^0 = \text{const.}\), with \(n_k = h_k^0/\sqrt{g^{00}}\), the four vectors \((n, e_\alpha)\) define the ADM basis of tangent vector fields. Introducing the projectors on \(n\) and \(\Sigma_0\), \((P_\perp)_k = n^i n_k\), \((P_\parallel)_k = \delta^i_k - n^i n_k\), any tangent vector \(V\) can be decomposed in terms of its parallel and orthogonal components: \(V_k = V_\perp + n_k V_\parallel\), where \(V_\perp = b^k V_k, V_\parallel = V_k - n_k V_\perp\), and \(n^k V_k = 0\). Using an analogous decomposition of the torsion and the curvature in last two indices,

\[
T^i_{mk} = T^i_{\text{\perp}mk} + 2n^m T^i_{\text{\perp}\text{\perp}k}, \quad R^{ij}_{mk} = R^{ij}_{\text{\perp}mk} + 2n^m R^{ij}_{\text{\perp}\text{\perp}k},
\]

we find that the parallel components \(T^i_{\text{\perp}k}\) and \(R^{ij}_{\text{\perp}k}\) are independent of velocities. The replacement in the gravitational Lagrangian yields \(L = \bar{L}(T^i_{\text{\perp}k}, R^{ij}_{\text{\perp}k}, T^i_{\text{\perp}\text{\perp}l}, R^{ij}_{\text{\perp}\text{\perp}l}, n^k)\).

The decomposition of \(e_0\) in the ADM basis yields \(e_0 = N n + N^\alpha e_\alpha\), where \(N = n_k b^k_0\) and \(N^\alpha = h_k^\alpha b^k_0\) are lapse and shift functions, respectively. We note also that \(b\) satisfies the factorization property \(b = NJ\), where \(J\) does not depend on \(b^k_0\).

Now, we turn our attention to the remaining momenta \(\pi_i^\alpha\) \([22]\). The relations defining \(\pi_i^\alpha\) can be written in the form

\[
\hat{\pi}_i^k = J \frac{\partial \bar{L}_T}{\partial T^i_{\text{\perp}k}} = 4J \beta_{i\perp k}(T),
\]

where \(\hat{\pi}_i^k = \pi_i^\alpha b^\alpha\) are conveniently defined “parallel” gravitational momenta. Using now the fact that \(\beta\) is a linear function of \(T\) we can make the expansion \(\beta(T) = \beta(0) + \beta(1)\), where \(\beta(0)\) does not depend on “velocities” \(T^i_{\text{\perp}k}\) and \(\beta(1)\) is linear in them, and rewrite the above equation in the form

\[
P_{ik} \equiv \hat{\pi}_{ik}/J - 4\beta_{i\perp k}(0) = 4\beta_{i\perp k}(1).
\]

Here, the so–called “generalized momenta” \(P_{ik}\) do not depend on velocities, which appear only on the right hand side of the equation. Explicit calculation leads to the result
\[ P_{ik} \equiv \pi_{ik}/J - 4a[\frac{1}{2}BT_{\perp ik} + \frac{1}{2}Cn_iT_{\perp \perp i}^{\perp}] = 4a[AT_{i\perp k} + \frac{1}{2}BT_{k\perp i} + \frac{1}{2}CT_{ik}^{\perp} + \frac{1}{2}(B + C)n_iT_{\perp \perp k}]. \]

This system of equations can be decomposed into irreducible parts with respect to the group of three-dimensional rotations in \( \Sigma_0 \):

\[
P_{\perp k} \equiv \pi_{\perp k}/J - 2aCT_{\perp \perp k} = 2a(2A + B + C)T_{\perp \perp k},
\]
\[
P_{ik}^A \equiv \pi_{ik}^A/J - 2aBT_{\perp \perp k} = 2a(2A - B)T_{i\perp k}^A,
\]
\[
P_{ik}^T \equiv \pi_{ik}^T/J = 2a(2A + B)T_{i\perp k}^T,
\]
\[
P_{\perp \perp m} \equiv \pi_{\perp \perp m}/J = 2a(2A + B + 3C)T_{i\perp m}.
\]

where \( X_{ik}^A = X_{[ik]}, X_{ik}^T = X_{(ik)} - \eta_{ik}X_{\perp}^A/3 \). Taking now into account the special choice of parameters adopted in equation (2.2), we recognize here two sets of relations: the first set represents extra primary constraints,\[ P_{\perp k} = \pi_{\perp k}/J + 2aT_{\perp \perp k} \approx 0, \]
\[ P_{ik}^A = \pi_{ik}^A/J - aT_{i\perp k} \approx 0, \tag{3.4a} \]

usually called if–constraints, while the second set gives nonsingular equations,\[ P_{ik}^T = \pi_{ik}^T/J = 2aT_{i\perp k}^T, \]
\[ P_{\perp \perp m} = \pi_{\perp \perp m}/J = -4aT_{i\perp m} \tag{3.4b} \]

which can be solved for velocities.

Further calculations are greatly simplified by observing that both sets of extra constraints (3.4a) can be represented in a unified manner as\[ \phi_{ik} = \pi_{ik} - \pi_{ki} + a\nabla_aB_{ik}^0, \quad B_{ik}^0 \equiv \varepsilon_{ikmm}b^m_{\perp}b^m. \tag{3.5} \]

This is seen from the fact that relations (3.4a) can be equivalently written as\[ \pi_{ik} - \pi_{ki} \approx 2aJ(T_{\perp \perp k} - n_iT_{\perp \perp m} + n_kT_{\perp \perp i}) = 2a\nabla_aH_{0a}^i, \]

where the last equality follows from (A1), and the identity \( 2H_{ik}^{\mu\nu} = -B_{ik}^{\mu\nu} \).

2. Having found all the primary constraints, we now proceed to find the canonical Hamiltonian density [22]:\[ \mathcal{H}_c = \pi_{i}^{\alpha}b_{\alpha}^{i} + \frac{1}{2}\pi_{ij}^{\alpha}A_{ij}^{\alpha} - bL. \]

The velocities \( \dot{b}_{\alpha} \) and \( \dot{A}_{ij}^{\alpha} \) can be calculated from the relations defining \( T_{i0a}^{i} \) and \( R_{ij0a}^{ij} \):

\[
T_{i0a}^{i} \equiv \partial_{0}b_{\alpha}^{i} + A_{i0}b_{0}^{m} - \partial_{0}b_{\alpha}^{i} - A_{i0}b_{0}^{m} = NT_{i\perp 0}^{i} + N^{\alpha}T_{i\perp 0}^{i},
\]
\[
R_{ij0a}^{ij} \equiv \partial_{0}A_{ij}^{0} + A_{i0}b_{0}^{m}A_{ij}^{0} - \partial_{0}A_{ij}^{0} - A_{i0}A_{ij}^{0} = NR_{ij\perp 0}^{ij} + N^{\beta}R_{ij\perp 0}^{ij}. \]

After a simple algebra we find that the canonical Hamiltonian can be written as a linear function of unphysical variables \( (b_{k0}, A_{ij0}) \), up to a 3–divergence,\[ \mathcal{H}_c = NH + N\alpha H_{\alpha} - \frac{1}{2}A_{ij0}H_{ij} + \partial_{\alpha}D_{\alpha}, \tag{3.6a} \]
where
\[
\begin{align*}
\mathcal{H}_{ij} &= 2\pi_{[i}^{\beta}b_{j]^{\beta} + \nabla_{\alpha}\pi_{ij}^{\alpha}, \\
\mathcal{H}_{\alpha} &= \pi_{k}^{\beta}T_{\alpha\beta} - b_{\alpha}^{k}n_{k}^{\beta} + \frac{1}{2}\pi_{ij}^{\alpha}R_{\alpha\beta}^{ij}, \\
\mathcal{H}_{\perp} &= (\pi_{i}^{k}\tilde{T}_{\perp k} - J\tilde{\mathcal{L}}_{T} - n_{k}^{\beta}\nabla_{\beta}\pi_{k}^{\beta} - J\lambda_{ij}^{m\bar{m}}R_{ij}^{m\bar{m}}, \\
D^{\alpha} &= \frac{b_{k}^{0}\pi_{k}^{\alpha}}{a} + \frac{1}{2}aT_{ij}^{0}\pi_{ij}^{\alpha}.
\end{align*}
\]
(3.6b)
Here, \(\mathcal{H}_{ij}\) and \(\mathcal{H}_{\alpha}\) are purely kinematical terms whose form does not depend on the choice of the Lagrangian, and \(\mathcal{H}_{\perp}\) is the only dynamical part.

The explicit form of \(\mathcal{H}_{\perp}\) can be obtained by eliminating “velocities” \(T_{i\perp k}\) with the help of the relations defining momenta \(\pi_{i}^{k}\). To do that we first rewrite the first two terms of \(\mathcal{H}_{\perp}\) in the form
\[
\tilde{\pi}^{ik}T_{i\perp k} - J\tilde{\mathcal{L}}_{T} = \frac{1}{2}PT_{\perp}^kT_{i\perp k} - J\tilde{\mathcal{L}}_{T}(T),
\]
where \(\tilde{T}_{ik} = T_{ik}\). Then, taking into account the constraints (3.4a) one finds that \(T_{i\perp k}\) and \(T_{A}^{i\perp k}\) are absent from \(\mathcal{H}_{\perp}\), whereupon the relations (3.4b) can be used to eliminate the remaining “velocities” \(T_{T}^{i\perp k}\) and \(T_{m\perp m}\), leading directly to
\[
\mathcal{H}_{\perp} = (\frac{1}{2}P_{T}^2 - J\tilde{\mathcal{L}}_{T}(T) - n_{k}^{\beta}\nabla_{\beta}\pi_{k}^{\beta} - J\lambda_{ij}^{m\bar{m}}R_{ij}^{m\bar{m}},
\]
(3.7a)
where
\[
\begin{align*}
P_{T}^2 &= \frac{1}{2aJ} \left( \pi_{(ik)}\pi^{(ik)} - \frac{1}{2}\pi_{m}^{m}\pi_{\bar{n}}^{\bar{n}} \right), \\
\tilde{\mathcal{L}}_{T}(T) &= a \left( \frac{1}{2}T_{m\bar{n}k}T^{m\bar{n}k} + \frac{1}{2}T_{m\bar{n}k}T^{m\bar{n}k} - T_{m\bar{n}k}T_{n\bar{k}}^{m\bar{n}k} \right).
\end{align*}
\]
(3.7b)

The general Hamiltonian dynamics of the system is described by the total Hamiltonian, which is given as
\[
\mathcal{H}_{T} = \mathcal{H}_{c} + u_{i}^{0}\pi_{i}^{0} + \frac{1}{2}u_{ij}^{\mu\nu}\pi_{ij}^{\mu\nu} + \frac{1}{2}u_{ij}^{\mu\nu}\pi_{ij}^{\mu\nu} + \frac{1}{2}u_{ik}^{\alpha}\phi_{ik}^{\alpha} + \frac{1}{2}u_{ij}^{\alpha}\phi_{ij}^{\alpha},
\]
(3.8)
where \(u_{i}\)’s are, at this stage, arbitrary Hamiltonian multipliers.

Although the torsion components \(T_{i\perp k}\) and \(T_{A}^{i\perp k}\) are absent from the canonical Hamiltonian, they re–appear in the total Hamiltonian as the non–dynamical Hamiltonian multipliers. Indeed, the Hamiltonian field equations for \(b_{k}^{0}\alpha\) imply (Appendix B)
\[
NT_{i\perp k} = u_{i}^{k}, \quad NT_{A}^{i\perp k} = u_{i}^{k}.
\]
(3.9)
The presence of non–dynamical torsion components does not imply that GR\(\parallel\) is an inconsistent theory \([4]\), as it has very clear interpretation via the gauge structure of the theory: it is related to the existence of additional first–class constraints \(\phi_{ik}\), as we shall see in Sec. V.

**IV. CONSISTENCY CONDITIONS**

The consistency of the theory requires that the constraints do not change during the time evolution of the system governed by the total Hamiltonian:
\[
\chi(x) \equiv \frac{d}{dt}\phi(x) = \int d^{3}x'\{\phi, \mathcal{H}_{T}^{'}\} \approx 0,
\]
where \(\{A, B\}^{'}\) denotes the Poisson bracket (PB) of two variables \(A(x)\) and \(B(x')\), and \(x^{0} = (x')^{0}\). The integration sign will be often omitted for simplicity.
A. Consistency conditions of primary constraints

Having found the form of primary constraints in GR, displayed in Eqs. (3.1), (3.2), (3.3) and (3.5), we now consider the requirements for their consistency. Since the canonical Hamiltonian is linear in unphysical variables \((b^0_0, A^{ij}_0)\), the consistency conditions of the sure primary constraints (3.1) are given by

\[ \chi^\perp \equiv H^\perp \approx 0, \quad \chi^\alpha \equiv H^\alpha \approx 0, \quad \chi^{ij} \equiv H^{ij} \approx 0. \] (4.1)

By noting that the components \(\pi^{ij\alpha\beta}\) in Eq. (3.2) have vanishing PBs with all primary constraints, we easily obtain

\[ \chi^{ij\alpha\beta} \equiv R^{ij\alpha\beta} \approx 0. \] (4.2a)

On the other hand, the consistency of \(\pi^{ij}_0\) implies

\[ \chi^{ij}_0 \equiv u^{ij}_0 \to N^\alpha R^{ij\alpha}_0 \approx 0 \Rightarrow u^{ij}_0 \approx 0. \] (4.2b)

The dynamical meaning of the last condition can be seen more clearly if we note that the equation of motion for \(A^{ij}_\beta\), \(\partial_0A^{ij}_\beta = \{A^{ij}_\beta, H_c\} + u^{ij}_\beta\), can be transformed into the form

\[ R^{ij\beta}_0 \approx u^{ij}_\beta. \] (4.3)

Hence, Eqs. (4.2) tell us that all the components of the curvature tensor weakly vanish, as one could have expected.

Using the PB relation

\[ \{\phi^{ij}_\alpha, \phi_{kl}\} = a(\eta_{ik}\phi_{nj}^0 + \eta_{jm}\phi_{in}^0)\delta - (kl), \]

the consistency condition for \(\phi^{ij}_\alpha\) takes the form

\[ \chi^{ij}_\alpha = \{\phi^{ij}_\alpha, H_c\} + a(u^s_i B^{00}_{sj} + u^s_j B^{00}_{is}) - 4bu^{ij}_0 \approx 0. \]

It can be used to determine \(u^{ij}_0\):

\[ u^{ij}_0 = \frac{a}{4b}(u^s_i B^{00}_{sj} + u^s_j B^{00}_{is}) + \bar{u}^{ij}_0, \quad \bar{u}^{ij}_0 \equiv \frac{1}{4b}\{\pi^{ij}_0, H_c\}, \] (4.4)

where \(\{\pi^{ij}_0, H_c\}\) is calculated in Appendix C. The first part of \(u^{ij}_0\) contains \(u_{kl}\) and gives an additional contribution to \(u^{kl}\phi_{kl}\), so that the replacement of this result into \(H_T\) leads effectively to

\[ u^{ij}_0 \to \bar{u}^{ij}_0, \quad u^{kl}\phi_{kl} \to u^{kl}\tilde{\phi}_{kl}, \quad \tilde{\phi}_{kl} \equiv \phi_{kl} - \frac{a}{4b}(\pi^{s}_{k0a}B^{00}_{sl} + \pi^{s}_{l0a}B^{00}_{ka}). \] (4.5)

Note that \(\{\phi^{ij}_\alpha, \tilde{\phi}^{ij}_{kl}\} = 0\).

The most complicated consistency conditions are those for the tetrad constraints \(\phi_{ij}\) (or, equivalently, \(\tilde{\phi}_{ij}\)). First we note that their PB algebra has the form

\[ \{\phi_{ij}, \phi_{mn}'\} = (\eta_{mi}\phi_{nj} + \eta_{jm}\phi_{in})\delta - (mn), \] (4.6)

and that the term \(\frac{1}{2}u^{ij}_0\phi^{ij}_\alpha\) in \(H_T\) can be discarded according to (4.2b). Then, after showing that the Poisson bracket \(\{\phi_{ij}, H'_c\}\) vanishes weakly, we will be able to conclude that the consistency condition for \(\phi_{ij}\) is automatically fulfilled:

\[ \chi_{ij} \equiv \{\phi_{ij}, H'_c\} \approx \{\phi_{ij}, H'_c\} \approx 0. \] (4.7)
B. The consistency condition of $\phi_{ij}$

In order to simplify the derivation of the consistency condition for $\phi_{ij}$ we rewrite this constraint in the form

$$
\phi_{ij} = \mathcal{H}_{ij} - F_{ij},
$$

$$
F_{ij} = \nabla_\alpha (\pi_{ij}^\alpha - aB_{ij}^\alpha) \equiv \nabla_\alpha \Pi_{ij}^\alpha.
$$

(4.8)

General arguments in PGT, related to the local Lorentz symmetry of the theory, imply that the constraint $\mathcal{H}_{ij}$ is of the first class [22]. This is also clear from the PB algebra of constraints, discussed in the next section. As a consequence, the consistency of $\phi_{ij}$ follows from the consistency of $F_{ij}$.

We are now going to show that $\{ F_{ij}, \mathcal{H}'_{kl} \} \approx 0$. First, we note that $\{ F_{ij}, \phi'_{kl} \} = 0$.

(4.9)

Then, using the results (C.4a) we obtain

$$
\{ F_{ij}, \mathcal{H}'_{kl} \} = (\eta_{ik} F_{lj} + \eta_{jk} F_{il}) \delta - (kl),
$$

$$
\{ F_{ij}, \mathcal{H}'_{\beta} \} = -\nabla_\beta (\phi_{ij} \delta) + [\nabla_\alpha, \nabla_\beta](\Pi_{ij}^\alpha \delta).
$$

(4.10a)

while (C.4b) leads to

$$
\{ F_{ij}, \mathcal{H}'_{\perp} \} = \frac{1}{2}[\nabla_\beta ((n_i \phi_{jk} - n_j \phi_{ik}) h^{k3} \delta)] + \frac{1}{2}[\nabla_\alpha, \nabla_\beta](M_{ij}^{\alpha \beta} \delta),
$$

$$
NM_{ij}^{\alpha \beta} \equiv 2aH_{ij}^{\alpha \beta} - 4b\lambda_{ij}^{\beta \alpha} + N^\alpha (\Pi_{ij}^{\beta} - \phi_{ij}^{\beta}) - N^\beta (\Pi_{ij}^{\alpha} - \phi_{ij}^{\alpha}),
$$

(4.10b)

Hence, the consistency condition of $F_{ij}$, and consequently of $\phi_{ij}$, is automatically satisfied.

C. Consistency conditions of secondary constraints

In the process of investigating the consistency of primary constraints in GR$_{||}$, we obtained secondary constraints (4.1) and (4.2a).

Consider, first, the consistency condition of the secondary constraint $R_{ij}^{\alpha \beta}$. Since $R_{ij}^{\alpha \beta}$ depends only on $A_{ij}^{\alpha}$, one can express $dR_{ij}^{\alpha \beta}/dt$ in terms of $dA_{ij}^{\alpha}/dt$, use the equation of motion (4.3) for $A_{ij}^{\alpha}$, and rewrite the result in the form

$$
\nabla_0 R_{ij}^{\alpha \beta} \approx \nabla_\alpha u_{ij}^{\beta} - \nabla_\beta u_{ij}^{\alpha}.
$$

Hence, the consistency condition for $R_{ij}^{\alpha \beta}$ is identically satisfied.

The above relation has a very interesting geometric interpretation. Indeed, using equation (4.3) we see that it is a weak consequence of the second Bianchi identity.

General arguments in PGT show that the secondary constraints $\mathcal{H}_{ij}, \mathcal{H}_{\alpha}$ and $\mathcal{H}_{\perp}$ are related to Poincaré gauge symmetry [22,24]. Consequently, they are of the first class, and their consistency conditions are automatically satisfied. This will be explicitly seen in the next section, from the form of their PB algebra.

Finally, at the end of the consistency procedure, we give the final expression for the total Hamiltonian:
\[
\mathcal{H}_T = \mathcal{H}' + u^i_0 \pi_i^0 + \frac{1}{2} u^{ij}_0 \pi_{ij}^0 + \frac{1}{4} u_{ij} \alpha \beta \pi_{ij} \alpha \beta + \frac{1}{2} u^{ik} \tilde{\phi}_{ik}, \\
\mathcal{H}' = \mathcal{H}_c + \frac{1}{2} u_{ij}^{0 \beta} \pi_{ij}^{0 \beta}.
\]

The multipliers \( u^i_0, u^{ij}_0, u_{ij}^{\alpha \beta} \) and \( u^{ik} \) remained arbitrary functions of time, hence we expect that the related constraints \( \pi_i^0, \pi_{ij}^0, \pi_{ij}^{\alpha \beta} \) and \( \tilde{\phi}_{ik} \) are of the first class.

V. THE ALGEBRA OF CONSTRAINTS

In the previous analysis we found that GR\( || \) is characterized by the following set of constraints:

- primary: \( \pi_i^0, \pi_{ij}^0, \tilde{\phi}_{ij}, \pi_{ij}^{\alpha \beta}, \phi_{ij}^\alpha, \pi_{ij}^{0 \beta} \);
- secondary: \( \mathcal{H}_\perp, \mathcal{H}_\alpha, \mathcal{H}_{ij}, R_{ij}^{\alpha \beta} \).

It is simple to see that \( \phi_{ij}^\alpha \) and \( \pi_{ij}^{0 \beta} \) are second class constraints. They can and will be used as strong equalities to eliminate \( \lambda_{ij}^\beta, \mu \) and \( \pi_{ij}^{0 \beta} \) from the theory and simplify further exposition. In particular, the term with the determined multiplier \( \tilde{u}^{ij}_\alpha \) in the total Hamiltonian can be now neglected, and \( \tilde{\phi}_{ij} \) reduces to \( \phi_{ij} \), equation (4.15). Due to a simple form of these second class constraints, the related Dirac brackets have the form of PBs in the phase space of the remaining variables. All the remaining constraints are of the first class, as follows from their PB algebra.

Since the kinematical constraints \( \mathcal{H}_{ij}, \mathcal{H}_\alpha, \mathcal{H}_\perp \) have the same general form as in (22,24), their algebra remains the same:

\[
\{\mathcal{H}_{ij}, \mathcal{H}'_{kl}\} = (\eta_{ik} \mathcal{H}_{lj} + \eta_{jk} \mathcal{H}_{il}) \delta - (kl), \\
\{\mathcal{H}_{ij}, \mathcal{H}'_\alpha\} = 0, \\
\{\mathcal{H}_\alpha, \mathcal{H}'_\beta\} = (\mathcal{H}'_\alpha \partial_\beta + \mathcal{H}_\beta \partial_\alpha - \frac{1}{2} R_{ij}^{\alpha \beta} \mathcal{H}_{ij}) \delta.
\]

As a consequence of \( R_{ij}^{\alpha \beta} \approx 0 \), the last term in \( \{\mathcal{H}_\alpha, \mathcal{H}'_\beta\} \) is quadratic in constraints.

The brackets involving \( \mathcal{H}_\perp \) are found to have the form

\[
\{\mathcal{H}_{ij}, \mathcal{H}'_\perp\} = 0, \\
\{\mathcal{H}_\alpha, \mathcal{H}'_\perp\} = \mathcal{H}_\perp \partial_\alpha \delta, \\
\{\mathcal{H}_\perp, \mathcal{H}'_\perp\} = -3 g^{\alpha \beta} \mathcal{H}_\alpha + 3 g^{i \alpha \beta} \partial_\alpha \mathcal{H}'_{ij} \partial_\beta \delta.
\]

The first two brackets are most easily verified by taking into account that \( \mathcal{H}_\perp \) can be written in the form \( \mathcal{H}_\perp = J f(\xi^A) - n^k \nabla_\alpha \pi_k^\alpha \), where \( f \) is a Lorentz scalar formed from variables \( \xi^A = (T^m_n, R_{ij}^{\alpha \beta}, \pi_k^i / J, \tilde{\pi}_k^i / J, n^k) \), as shown in (24). The second bracket is obtained from the general formula based on the chain rule for PBs,

\[
\{\mathcal{H}_\alpha, \mathcal{H}'_\perp\} = \frac{\partial \mathcal{H}_\alpha}{\partial \xi^A} \{\xi^A, \xi'B\} \frac{\partial \mathcal{H}'_\perp}{\partial \xi'^B} = \left( \mathcal{H}_\perp \partial_\alpha + \frac{1}{2} \partial \mathcal{H}_\perp \partial_\alpha \mathcal{H}_{ij} \right) \delta,
\]

which explains why the second term is absent in (5.1b).

The last and most important bracket \( \{\mathcal{H}_\perp, \mathcal{H}'_\perp\} \) is evaluated using the chain rule and keeping only those terms that contain \( \partial_\alpha \delta \) (terms proportional to \( \delta \) do not have the correct symmetry under \( x \leftrightarrow x' \), hence they cancel each other).
In the next step we want to extend the above algebra by adding \( \tilde{\phi}_{ij} = \phi_{ij} \). The relevant PBs involving \( \phi_{ij} \) are given by

\[
\{ \phi_{ij}, \phi_{kl} \} = (\eta_{ik} \phi_{lj} + \eta_{jk} \phi_{il}) - (kl),
\]

\[
\{ \phi_{ij}, \mathcal{H}'_{kl} \} = \nabla_{\beta}(\phi_{ij}\delta) - [\nabla_{\alpha}, \nabla_{\beta}](\Pi^{\alpha \delta}_{ij}),
\]

\[
\{ \phi_{ij}, \mathcal{H}'_{\perp} \} = -\frac{1}{2}\nabla_{\beta}[(n_{i} \phi_{jk} - n_{j} \phi_{ik})h^{k\beta}\delta] - \frac{1}{2}[\nabla_{\alpha}, \nabla_{\beta}](M^{\alpha \beta}_{ij}).
\]

(5.2)

Finally, we display the non–vanishing PBs involving \( R_{ij \alpha \beta} \) and \( \pi_{ij \alpha \beta} \):

\[
\{ R^{ij}_{\alpha \beta}, \mathcal{H}'_{kl} \} = (\delta^{i}_{k} R^{j}_{l \alpha \beta} + \delta^{j}_{k} R^{i}_{l \alpha \beta})\delta - (kl),
\]

\[
\{ R^{ij}_{\alpha \beta}, \mathcal{H}'_{\gamma} \} = \nabla_{\alpha}(R^{ij}_{\gamma \beta}\delta) - (\alpha \beta),
\]

\[
\{ \pi^{ij}_{\alpha \beta}, \mathcal{H}'_{\perp} \} = 4J R^{ij}_{\alpha \beta}.
\]

(5.3)

Thus, all constraints except \( \phi^{ij}_{\alpha} \) and \( \pi^{ij}_{\alpha \beta} \) are of the first class. The fact that \( \phi_{ij} \) is first class is of particular importance for the consistent interpretation of the non–dynamical torsion components, as noted at the end of Sec. III.

### VI. EXTRA GAUGE SYMMETRIES

The presence of arbitrary multipliers in the total Hamiltonian is related to the existence of gauge symmetries in the theory. The general method of constructing the generators of such symmetries has been given by Castellani [25]. If we limit ourselves to gauge transformations given in terms of arbitrary parameters \( \varepsilon(t) \) and their first time derivative \( \dot{\varepsilon}(t) \), which is sufficient for the present analysis, the gauge generators take the form

\[
G = \int d^{3}x [\varepsilon(t)G^{(0)} + \dot{\varepsilon}(t)G^{(1)}],
\]

where \( G^{(0)} \) and \( G^{(1)} \) are phase space functions determined by the conditions

\[
G^{(1)} = C_{PFC},
\]

\[
G^{(0)} + \{ G^{(1)}, H_{T} \} = C_{PFC},
\]

\[
\{ G^{(0)}, H_{T} \} = C_{PFC},
\]

(6.1b)

and \( C_{PFC} \) denotes primary first class (PFC) constraint.

The Poincaré gauge symmetry is present in our formulation of GR\( _{\parallel} \) by construction, and the related gauge generator is based on the sure constraints \( \pi^{0}_{i}, \pi^{0}_{ij} \) and \( \mathcal{H}_{\perp}, \mathcal{H}_{\alpha}, \mathcal{H}_{ij} \) [26]. Here, we shall focus our attention on extra gauge symmetries based on \( \pi^{ij}_{\alpha \beta}, \phi_{ij} \) and \( R^{ij}_{\alpha \beta} \).

#### A. Extra gauge symmetry, 1

Starting with \( \pi^{ij}_{\alpha \beta} \) as \( G^{(1)} \) in (6.1b) we find that the related gauge generator is given by

\[
G = \int d^{3}x \left[ \frac{1}{4}(\nabla_{0}\varepsilon^{\alpha \beta}_{ij})\pi^{ij}_{\alpha \beta} + \frac{1}{4}\varepsilon^{\alpha \beta}_{ij}(-4bR^{ij}_{\alpha \beta} + (b/b)\pi^{ij}_{\alpha \beta}) \right].
\]

(6.2)
The only nontrivial gauge transformations \( \delta_0 X = \{X, G\} \) are

\[
\begin{align*}
\delta_0 (b \lambda_{ij}^{\alpha\beta}) &= \nabla_0 (b \varepsilon_{ij}^{\alpha\beta}) , \\
\delta_0 \pi_{ij}^{\alpha} &= 4 \nabla_\beta (b \varepsilon_{ij}^{\alpha\beta}) .
\end{align*}
\] (6.3)

To see the meaning of these transformations, consider the Hamiltonian equation for the variable \( \Pi_{ij}^{\alpha} = \pi_{ij}^{\alpha} - aB_{ij}^{\alpha} \). Introducing \( K_{ij}^{\alpha\beta} = 4b \lambda_{ij}^{\alpha\beta} - aB_{ij}^{\alpha\beta} \) and using the results of Appendix C we obtain the equation

\[
\nabla_0 \Pi_{ij}^{\alpha} - \nabla_\beta K_{ij}^{\alpha\beta} = 0 ,
\] (6.4)

which is the Hamiltonian analogue of (2.4b). The application of the above gauge transformation to this equation yields

\[
(\nabla_0 \nabla_\beta - \nabla_\beta \nabla_0) (4b \varepsilon_{ij}^{\alpha\beta}) = 0 .
\]

The invariance follows from the fact that the left hand side vanishes in Weitzenböck space, where \( R_{ij0\beta} = 0 \).

**B. Extra gauge symmetry, 2**

Starting with \( G_{ij}^{(1)} = \phi_{ij} \) in (6.1b), one finds that the gauge generator has the form

\[
G_{ij} = \int d^3 x [\frac{1}{2} \dot{\varepsilon}_{ij}^{\alpha\beta} G_{ij}^{(1)} + \frac{1}{2} \varepsilon_{ij}^{\alpha\beta} G_{ij}^{(0)}] ,
\] (6.5a)

where (Appendix D)

\[
G_{ij}^{(0)} = \frac{1}{2} R_{i}^{\alpha\beta}s_{j}^{\alpha\beta} + \frac{1}{24b} \left( A_{i}^{m} n_{0}s_{n}^{\alpha\beta} + A_{n}^{s} m_{0}n_{i}^{\alpha\beta} \right) K_{ij}^{\alpha\beta} - \pi_{ij}^{\alpha\beta} K_{ij}^{\alpha\beta} - (ij) .
\] (6.5b)

The corresponding gauge transformations are:

\[
\begin{align*}
\delta_0 b_{i}^{k} \alpha_{j}^{\alpha} &= \dot{\varepsilon}_{i}^{k} \alpha \cdot s_{j}^{\alpha} , \quad \delta_0 A_{ij}^{\alpha} &= 0 , \\
4b \delta_0 \lambda_{ij}^{\alpha\beta} &= \left[ \varepsilon_{i}^{n} n_{j}^{\alpha\beta} + A_{i}^{m} n_{0}^{\alpha\beta} \right] K_{ij}^{\alpha\beta} - \pi_{ij}^{\alpha\beta} K_{ij}^{\alpha\beta} - (ij) , \\
\delta_0 \pi_{ij}^{\alpha} &= \left[ a (\dot{\varepsilon}_{i}^{s} B_{j}^{0}) + \nabla_\beta (\varepsilon_{i}^{n} K_{ij}^{\alpha\beta}) \right] - (ij) ,
\end{align*}
\] (6.6)

and similarly for other variables.

Consider, again, equation (6.4). Using \( \delta_0 \Pi_{ij}^{\alpha} = \nabla_\beta (\varepsilon_{i}^{n} K_{ij}^{\alpha\beta}) - (ij) \), we easily obtain

\[
\begin{align*}
\delta_0 (\nabla_0 \Pi_{ij}^{\alpha}) &= \nabla_0 (\delta_0 \Pi_{ij}^{\alpha}) \approx \nabla_\beta \nabla_0 (\varepsilon_{i}^{n} K_{ij}^{\alpha\beta}) - (ij) \\
&= \nabla_\beta [\varepsilon_{i}^{n} K_{ij}^{\alpha\beta} + \varepsilon_{i}^{n} K_{ij}^{\alpha\beta} + A_{i}^{s} n_{0}\varepsilon_{s}^{n} K_{nj}^{\alpha\beta} + \varepsilon_{j}^{n} K_{sn}^{\alpha\beta}] - (ij) ,
\end{align*}
\]

where we made use of \( R_{ij0\beta} = 0 \). On the other hand,

\[
\delta_0 K_{ij}^{\alpha\beta} = [\varepsilon_{i}^{n} K_{nj}^{\alpha\beta} + \varepsilon_{i}^{n} K_{nj}^{\alpha\beta} + A_{i}^{s} n_{0}\varepsilon_{s}^{n} K_{nj}^{\alpha\beta} + \varepsilon_{j}^{n} K_{sn}^{\alpha\beta}] - (ij) ,
\]

and we see that equation (6.4) is gauge invariant.
VII. CONCLUDING REMARKS

The investigation of the Hamiltonian structure of the teleparallel formulation of GR presented here is based on Dirac’s general method for constrained dynamical systems [20].

To complete our results, we now discuss how the physical degrees of freedom of GR are counted. After the elimination of $\lambda_{ij0\alpha}$ and $\pi_{ij0\alpha}$, the reduced phase space is spanned by the $40 + 18$ field components ($b_{\mu}^{ij}, A_{ij\mu}^{\alpha\beta}, \lambda_{ij\alpha\beta}$), and the same number of momenta. The primary first class constraints $\pi_{ij\alpha\beta}$ diminish the number of independent variables for $2 \times 18$, leaving us with the phase space containing effectively $2 \times 40$ components. Before going on, we wish to clarify the counting of constraints $R_{ij\alpha\beta} \approx 0$. Note that here we have formally 18 equations, but they represent only 12 independent conditions on $A_{ij\alpha}$. Indeed, starting with the simplest solution $\bar{A}_{ij\alpha} = 0$ of $R_{ij\alpha\beta}(A) = 0$, one can construct a new, Lorentz rotated solution $\bar{A}_{ij\alpha}^\Lambda = \Lambda_{ik}^{\alpha} \partial_{\alpha} \bar{A}_{jk}^\Lambda$, containing 6 arbitrary parameters $\Lambda_{ik}$ [8], so that the number of independent conditions on $A_{ij\alpha}$ is $18 - 6 = 12$. Continuing now the counting, we find 20 sure first class constraints [ten primary ($\pi_{i0}, \pi_{ij0}$), and ten secondary ($\mathcal{H}_{\perp}, \mathcal{H}_{\alpha}, \mathcal{H}_{ij}$)], and $6 + 12 = 18$ additional first class constraints $\phi_{ij}$ and $R_{ij\alpha\beta}$, which leaves us with $2 \times 40 - 2 \times 38 = 4$ physical degrees of freedom, corresponding to massless graviton.

We found two types of extra gauge symmetries in the PGT formulation of GR. The first type is related to the primary constraints $\pi_{ij\alpha\beta}$. The related gauge transformations do not act on $b_{\mu}^{ij}$, hence they are irrelevant for the structure of the first field equation (2.4a). On the other hand, the gauge symmetry acts nontrivially on Lagrange multipliers. If we recall that the only role of the second field equation (2.4b) is to determine these multipliers [9], it becomes clear that this cannot be done uniquely without fixing the gauge.

The second type of extra gauge symmetry originates from the tetrad constraints $\phi_{ij}$. We note that Nester [14] derived these constraints in the form (3.4a), in his analysis of the positivity of energy in the teleparallel form of GR. Their existence may be interpreted as a consequence of the fact that the velocities contained in $T_{\perp\perp}^k - T_{\perp k}^A$ appear at most linear in the Lagrangian [15] and, consequently, remain arbitrary functions of time. The phenomenon that some velocities are dynamically undetermined is quite usual for constrained dynamical systems [20]. For the related initial value problem to be well defined, these undetermined velocities should be removed from the set of dynamical velocities [15].

The role of this symmetry is very clearly seen if we observe that the teleparallel geometry can be also formulated as the translational gauge theory, where local Lorentz symmetry is in general absent [13]. However, for the special choice of parameters corresponding to GR one finds that $\phi_{ij}$ is an additional first class constraint, which generates local Lorentz symmetry as an extra gauge symmetry [14]. This also clarifies the form (3.5) of $\phi_{ij}$, which is seen to “imitate” $\mathcal{H}_{ij}$ in the tetrad sector.

Maluf [21] studied GR by imposing the time gauge at the Lagrangian level. His arguments concerning the necessity of the time gauge in the canonical formalism are conceptually misleading: this gauge (as well as any other gauge) may be useful, but certainly not essential [20]. After fixing the time gauge, he found the Hamiltonian and derived the constraint corresponding to our $\phi_{ik}$ [Eq. (25) in his paper], while $\phi_{\perp k}$ is missed. Moreover, Maluf was not able to calculate the constraint algebra unless imposing another gauge condition. His constraint algebra [Eqs. (30)-(34)] does not agree with our results, which might be a consequence of the adopted gauge conditions. All this makes this analysis of the gauge structure
of GR ∥ rather unclear.

The results obtained in this paper refer to non–interacting GR ∥, and can be used to define and analyze the gravitational energy and other conserved quantities [27,16]. Interaction with matter fields may be included in a straightforward manner [7,28]. Studying consistency requirements imposed by extra gauge symmetries on the matter sector will tell us more about the existence and nature of consistent couplings [15].

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APPENDIX A: SOME GEOMETRIC IDENTITIES IN $T_4$

We begin with a simple but technically important identity

$$\nabla_\nu H^\mu_{ij} = bh_k^\mu(T^k_{ij} - \delta^k_i T_j + \delta^k_j T_i) = -4b\beta_{[ij]}^\mu/\alpha, \quad H^\mu_{ij} \equiv b(h^\mu_i h_j^\nu - h^\mu_j h_i^\nu), \quad (A1)$$

which implies $\nabla_\mu (b\beta_{[ij]}^\mu) = 0$ for $R^i_{\mu\nu} = 0$.

In Riemann–Cartan space $U_4$ the Lorentz connection can be expressed in the form $A = \Delta + K$, where $\Delta$ is Levi–Civita connection and $K$ the contortion. Substituting this expression into the definition of the curvature tensor $R^i_{\mu\nu}(A)$, we obtain the basic identity:

$$R^i_{\mu\nu}(A) = R^i_{\mu\nu}(\Delta) + \left[\nabla^i K'_{\nu} + K^i_{s\mu}K^{s\nu} - (\mu \leftrightarrow \nu)\right], \quad (A2)$$

where $\nabla' = \nabla(\Delta)$ is Riemannian covariant derivative. Then, multiplying this relation with $H^\mu_{ij}/2$ and using $\nabla_\mu H^\mu_{ij} = 0$, we find

$$bR(A) = bR(\Delta) + b\left(\frac{1}{4}T_{ijk}T^{ijk} + \frac{1}{2}T_{ik}T^{ijk} - T_k^k\right) + 2\partial_\mu(bK^\mu), \quad (A3)$$

where $K^\mu = K^{\mu n}_n = -T^\mu$.

Now, if we write equation (A2) in an equivalent form,

$$R^i_{\mu\nu}(A) = R^i_{\mu\nu}(\Delta) + \left[\nabla_\mu K^i_{\nu} - K^i_{s\mu}K^{s\nu} - (\mu \leftrightarrow \nu)\right],$$

and multiply it with $H^\mu_{ij}/2$, we obtain the result

$$bR^i_k(A) = bR^i_k(\Delta) + \left[\nabla_\mu K^i_{\nu} - K^i_{s\mu}K^{s\nu}\right] H^\mu_{jk},$$

which can be written as

$$abR^{ik}(A) = abR^{ik}(\Delta) + 2\nabla_\mu (b\beta^{ijk}) + 2b\beta_{mn}k^{mni} - b\beta^{imn}T^{k}_{mn} - \eta^{ik}a\partial_\mu (bT^\mu) - 4\nabla_\mu (b\beta^{ik}[\mu]). \quad (A4)$$

The last term on the right hand side vanishes for $R^i_{\mu\nu}(A) = 0$. In that case we find

$$2ab[R^{ik}(\Delta) - \frac{1}{2}\eta^{ik}R(\Delta)] = -4\nabla_\mu (b\beta^{ijk}) - 4b\beta_{mn}k^{mni} + 2b\beta^{imn}T^{k}_{mn} + \eta^{ik}b\mathcal{L}_T. \quad (A5)$$

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APPENDIX B: UNPHYSICAL TORSION COMPONENTS

In this Appendix we show that the unphysical torsion components $T_{i\perp k}$ and $T_{i\perp k}^A$ can be expressed in terms of the Hamiltonian multipliers $u_{kl}$.

Using the PB relations

\[
\{b^i_\alpha, \mathcal{H}^\perp_{kl}\} = \delta^i_k b^\alpha_\delta - (kl),
\]
\[
\{b^i_\alpha, \mathcal{H}^\perp_{k\beta}\} = (\nabla_\alpha b^i_\beta)_\delta - b^i_\delta \phi^\prime_\alpha \delta = T^i_{\beta\alpha} \delta + \nabla_i (b^i_\beta \delta),
\]
\[
\{b^i_\alpha, \mathcal{H}^\perp_{\perp}\} = \frac{1}{2a} \left( b_{ma} \hat{P}^m_T - \frac{1}{6} b^i_\alpha \hat{P}^m_T \right) \delta + \nabla_i (n^i \delta),
\]

one easily finds that the Hamiltonian equation for $b^k_\alpha$ can be written in the form

\[
\nabla_0 b^i_\alpha = \nabla_\alpha b^i_0 + N^\beta T^i_{\beta\alpha}
\]
\[
+ N b_{\kappa\alpha} \frac{1}{2aJ} \left( \hat{\pi}^{(ik)} - \frac{1}{2} \hat{\pi}^{(k)}_{\hat{\pi}} \right) + b_{\kappa\alpha} \left( n^i u^{\perp k} + u^\perp k \right).
\]

As a consequence,

\[
\hat{\pi}^{(i\bar{k})} - \frac{1}{2} \hat{\pi}^k_{\hat{\pi}} = 2aJ T^{(i\bar{k})},
\]
\[
u^{\perp k} = NT^{\perp \perp k}, \quad u^\perp k = NT^A_{\perp \perp k}.
\]

APPENDIX C: CONSISTENCY CONDITIONS

We collect here several technical relations which simplify the derivation of the consistency conditions for $\phi^i_\alpha$ and $F_{ij}$.

1. The term $\{\pi^i_\alpha, H_e\}$ in the consistency condition for the primary constraint $\phi^i_\alpha$ is calculated using the relations

\[
\{\pi^i_\alpha, \mathcal{H}^\perp_{kl}\} = (\eta_{i\kappa} \pi^j_\alpha + \eta_{j\kappa} \pi^i_\alpha) \delta - (kl),
\]
\[
\{\pi^i_\alpha, \mathcal{H}^\perp_{k\beta}\} = -\delta^\alpha_\beta (\pi^i_\alpha - \pi^j_\beta) \delta = \delta^\alpha_\beta \nabla_\gamma (\pi^i_\alpha \delta) + \nabla_\beta (\pi^i_\alpha \delta).
\]

and

\[
\{\pi^i_\alpha, \mathcal{H}^\perp_{\perp}\} = -4 \nabla_\beta (J \lambda^i_\beta \alpha) \delta + 4 \nabla_\beta [J (N^\alpha \lambda^0_\beta - N^\beta \lambda^0_\alpha) \delta],
\]
\[
- \left[ 8J \beta_{i[jk]} (0) + 2aJ n_{[i} T_{\perp jk]} \right] h^k \alpha \delta,
\]
\[
+ (n^i \pi^j_k - n^j \pi^i_k) h^k \alpha \delta.
\]

Using the identity $8J \beta_{i[jk]} (0) h^k \alpha = a \varepsilon^{\alpha \beta \gamma \delta} T^m_{\gamma \beta n},$ we obtain

\[
\{\pi^i_\alpha, H_e\} = -(A_i^s 0 \pi^s_\alpha + A_j^s 0 \pi^s_\alpha) - N^\alpha (\pi^i_\alpha - \pi^j_\beta)
\]
\[
- 4 \nabla_\beta (b \lambda^i_\beta \alpha) - a \varepsilon^{\alpha \beta \gamma \delta} T^m_{\gamma \beta n} + N (n_i \pi^j_k - n_j \pi^i_k) h^k \alpha
\]
\[
- \nabla_\beta [(N^\alpha \phi^i_\beta - N^\beta \phi^i_\alpha)] + \frac{1}{2} N (n_i \phi^j_k - n_j \phi^i_k) h^k \alpha.
\]
2. In order to calculate the Poisson brackets between \( F_{ij} \) and the Hamiltonian constraints, we also need the following relations:

\[
\{ B_{ij}^{\alpha\alpha}, \mathcal{H}_{kl} \} = (\eta_{ik} B_{ij}^{\alpha\alpha} + \eta_{jk} B_{ij}^{\alpha\alpha}) \delta - (kl) , \\
\{ B_{ij}^{\alpha\alpha}, \mathcal{H}_{j}^{\beta} \} = \nabla_{\beta}(B_{ij}^{\alpha\alpha} \delta) + \delta_{\beta}^{\alpha} B_{ij}^{\alpha\alpha} \partial_{\gamma} \delta ,
\]

and

\[
\{ B_{ij}^{\alpha\alpha}, \mathcal{H}_{j}^{\gamma} \} = \frac{1}{a}(n_{ij} \pi_{jk} - n_{ij} \pi_{kj}) h^{\gamma} \delta + 2 \varepsilon_{ijmn} b_{ij} \nabla_{\gamma}(n^{\delta}) .
\]

Combining \((C1a)\) and \((C3a)\) we find

\[
\{ \Pi_{ij}^{\alpha}, \mathcal{H}_{kl} \} = (\eta_{ik} \Pi_{ij}^{\alpha} + \eta_{jk} \Pi_{ij}^{\alpha}) \delta - (kl) , \\
\{ \Pi_{ij}^{\alpha}, \mathcal{H}_{j}^{\beta} \} = -\delta_{\beta}^{\alpha} \phi_{ij} \delta - \delta_{\beta}^{\alpha} \nabla_{\gamma}(\Pi_{ij}^{\gamma} \delta) + \nabla_{\beta}(\Pi_{ij}^{\alpha} \delta) ,
\]

which implies \((1.10a)\).

Similarly, combining \((C1b)\) and \((C3b)\), and using the identity

\[
B_{ij}^{\alpha\beta} = 2 N \xi_{ijmn} b_{ij} n^{\alpha} (N_{\alpha} B_{ij}^{\beta\beta} - N_{\beta} B_{ij}^{\alpha0} ) ,
\]

we obtain

\[
\{ \Pi_{ij}^{\alpha}, \mathcal{H}_{ij} \} = \frac{1}{2}(n_{ij} \dot{\phi}_{jk} - n_{ij} \dot{\phi}_{kj}) h^{\gamma} \delta + \nabla_{\beta}(M_{ij}^{\beta\delta}) , \\
N M_{ij}^{\alpha\beta} \equiv 2 a H_{ij}^{\alpha\beta} + 4 b \lambda_{ij}^{\alpha\beta} + N_{\alpha} (\Pi_{ij}^{\beta} - \phi_{ij}^{\beta}) - N_{\beta} (\Pi_{ij}^{\alpha} - \phi_{ij}^{\alpha}) ,
\]

which implies \((1.10b)\).

APPENDIX D: EXTRA GAUGE GENERATORS

In this Appendix we derive the form of the gauge generator \((3.3)\). We start with \( G_{ij}^{(1)} = \phi_{ij} \) in \((5.10)\). In order to find the form of the accompanying component \( G_{ij}^{(0)} \), we use the PB algebra given in Eq. \((5.2)\), and calculate

\[
\{ \phi_{ij}, H_{T} \} = -\frac{1}{2} [\nabla_{\alpha}, \nabla_{\beta}] K_{ij}^{\alpha\beta} , \\
K_{ij}^{\alpha\beta} \equiv 2 a H_{ij}^{\alpha\beta} + 4 b \lambda_{ij}^{\alpha\beta} ,
\]

where terms proportional to \( \phi_{ij}^{\alpha} \) are discarded. The second condition in \((5.13)\) implies

\[
G_{ij}^{(0)} = \frac{1}{2} [R_{i}^{s\alpha\beta} K_{s}^{\alpha\beta} - (ij)] + \theta_{ij} ,
\]

where \( \theta_{ij} \) is a primary FC constraint. The third condition in \((5.13)\) can be written in the form

\[
\frac{1}{2} [\dot{R}_{i}^{s\alpha\beta} K_{s}^{\alpha\beta} + R_{i}^{s\alpha\beta} \dot{K}_{s}^{\alpha\beta} - (ij)] + \dot{\theta}_{ij} = C_{PFC} ,
\]

where \( \dot{X} = \{ X, H_{T} \} \). Then, using the relations \( \nabla_{0} R_{i}^{s\alpha\beta} \approx 0 \) and \( \dot{\pi}_{i}^{s\alpha\beta} = 4 b \dot{R}_{i}^{s\alpha\beta} \) we obtain

\[
\theta_{ij} = \frac{1}{2} \frac{1}{2 b} \left[ (A_{i}^{n_{0}} \pi_{n}^{s\alpha\beta} + A_{i}^{n_{0}} \pi_{n}^{s\alpha\beta}) K_{s}^{\alpha\beta} - \pi_{i}^{s\alpha\beta} \dot{K}_{s}^{\alpha\beta} \right] - (ij) .
\]
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