THE GENUS AND THE
LYUSTERNIK-SCHNIRELMANN CATEGORY OF
PREIMAGES

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Abstract. In this paper some axiomatic generalization (function of open subsets) of the relative Lyusternik-Schnirelmann category is considered, incorporating the sectional category and the Schwarz genus as well. For this function and a given continuous map of the underlying space to a finite-dimensional metric space some lower bounds on the value of this function on the (neighborhood of) preimage of some point are given.

1. Introduction

Let us introduce a notion that generalizes the relative Lyusternik-Schnirelmann category.

Definition 1. Let $X$ be a topological space. Let the function $\kappa$ takes the set of nonempty open subsets of $X$ to positive integers and has the following properties:

1) if $U \subseteq V$, then $\kappa(U) \leq \kappa(V)$ (monotonicity);
2) $\kappa(U_1 \cup \cdots \cup U_n) \leq \kappa(U_1) + \cdots + \kappa(U_n)$ (subadditivity);
3) $\kappa(U_1 \cup \cdots \cup U_n) \leq \max\{\kappa(U_1), \ldots, \kappa(U_n)\}$, if the sets $\text{cl} U_1, \ldots, \text{cl} U_n$ are pairwise disjoint.

We call such function $\kappa$ a generalized relative category.

The relative Lyusternik-Schnirelmann category $\text{cat}_X U$ \cite{6} is an example of a generalized relative category (when $X$ is arcwise connected to satisfy Property 3).

Another example is the sectional category of a map $f : Y \to X$, the definition from \cite{8} (see also \cite{4}) is as follows.

Definition 2. Let $f : Y \to X$ be a continuous surjective map. For an open subset $U \subseteq X$ we put

$$\text{secat}_f U = 1$$

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if $f|_{f^{-1}(U)}$ has a section $U \to Y$. We put
\[
\text{secat}_f U = k
\]
if $k$ is the smallest size of an open cover $V_1 \cup \cdots \cup V_k \supseteq U$ by open sets with $\text{secat} V_i = 1$.

The function $\text{secat}_f U$ is a generalized relative category, Properties 1 and 2 are obvious, Property 3 holds because if $V_1, \ldots, V_n$ are open and pairwise disjoint with $\text{secat} V_i = 1$, then
\[
\text{secat} V_1 \cup \cdots \cup V_n = 1,
\]
since the section of $f$ is defined on every $V_i$ separately.

A particular case of the sectional category is the Schwarz genus, introduced for $Z_2$-action by Krasnosel’skii, see [5, 8].

**Definition 3.** Let $G$ be a finite group, $Y$ be a free $G$-space. Define the **Schwarz genus** $g_G(Y)$ as the sectional category of the natural projection $Y \to Y/G$.

If the space $Y$ is fixed, and $\pi : Y \to Y/G = X$ is the natural projection, then $g_G(\pi^{-1}(U))$ is a generalized relative category on $X$. For paracompact $Y$ and its $G$-invariant open subset $U$ the property $g_G(U) \leq n$ is equivalent to existence of a $G$-equivariant map $U \to G * \cdots * G = G^n$. In this case the genus also makes sense for closed invariant subsets. See Section 3 for further discussion of the Schwarz genus.

Let us define the cohomology length for a ring of coefficients $A$.

**Definition 4.** Let $X$ be a topological space, $\iota : U \to X$ the inclusion of an open subset. Define the **relative cohomology length** of $U$ by
\[
\text{hlen}_X U = \max\{k : \exists u_1, \ldots, u_k \in H^*(X, A), \forall i \dim u_i > 0, \iota^*(u_1) \iota^*(u_2) \cdots \iota^*(u_k) \neq 0\}.
\]

In Section 4 we discuss the cohomology length and show that $\text{hlen}_X U + 1$ is a generalized relative category. The subadditivity of length also implies the well-known bound $\text{cat}_X U \geq \text{hlen}_X U + 1$. In fact, if a generalized relative category $\kappa$ takes contractible in $X$ subsets of $X$ to 1, then $\kappa(U) \leq \text{cat}_X U$.

There are other examples of a generalized relative category for arcwise connected $X$. For any continuous map $f : X \to Y$ in [1] the (restriction) category of a map is defined by the rule $\text{cat}_f(U) = 1$ iff $f|_U$ is null-homotopic. For any (generalized) cohomology class $\xi$ in [1] the restriction category is defined by the rule $\text{cat}_\xi U = 1$ iff $\xi|_U = 0$.

See the review [4] for different generalizations of the Lyusternik-Schnirelmann category. Note that the **strong category** of $U$, which requires the contractibility of $U$ in $U$ (not in $X$), is not a generalized relative category, the Property 3 obviously fails.
Now let us state the main result.

**Theorem 1.** Let $\kappa$ be a generalized relative category on $X$ with $\kappa(X) > n(d + 1)$. For any continuous map $f : X \to Y$ to a compact metric space of covering dimension $d$ there exists a point $c \in Y$ such that for any neighborhood $U \ni c$

$$\kappa(f^{-1}(U)) > n.$$ 

This statement also holds if $Y$ is not compact, but $X$ is compact, in this case the image $f(X)$ is compact.

If $\kappa$ can be defined for closed sets and homotopy invariant, and the preimage $f^{-1}(c)$ is a retract of its neighborhood in $X$, then the theorem claims that $\kappa(f^{-1}(c)) > n$. This is true for analytic maps from a real analytic variety $X$.

If we consider the sectional category of a fiber bundle with ANR fibers (e.g. the Schwarz genus), then for closed $F \subseteq X$ for some neighborhood $U \supseteq F$ we have $\kappa(F) \geq \kappa(U)$. If, in addition, $X$ is compact, then the theorem claims that $\kappa(f^{-1}(c)) > n$.

Note that a particular case of Theorem 1 for the $\mathbb{Z}_2$-genus is proved and used in [11, Lemma 3.1]. A close result on a certain homological analogue of the genus is proved in [10, Lemma 9.1 and Corollary 9.1]. Actually, the idea of the proof for the cohomology length is contained in [10].

Another result in this direction if obtained in [9] for fiber bundles, and the number $d + 1$ is replaced by the category of the map. Let us state that result.

**Theorem 2.** Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fiber bundle with connected $B$ and $F$. Then

$$\text{cat } E \leq \iota \cdot \text{cat } p.$$ 

2. **Proof of the main theorem**

We need a lemma [7, Lemma 2.4].

**Lemma 1.** Let $X$ be paracompact space of covering dimension $d$. Then any open covering $\mathcal{V}$ of $X$ has a refinement

$$\mathcal{U} = \bigcup_{i=1}^{d+1} \mathcal{U}_i,$$

where each $\mathcal{U}_i$ consists of the sets with pairwise disjoint closures.

**Proof of Theorem 1.** The space $Y$ is metric and therefore paracompact. By Lemma 1 we can cover $Y$ by a family $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{d+1}$ of open sets with diameters at most $\varepsilon$. Put $V_i = \bigcup \mathcal{U}_i$.

By Property 2 of a generalized relative category for some $i$ we have $\kappa(f^{-1}(V_i)) > n$. By Property 3 there exists a set $V \in \mathcal{U}_i$, such that $\kappa(f^{-1}(V)) > n$. 

If \( \varepsilon \) tends to zero, we find such sets \( V \subset Y \) with diameters tending to zero. By the compactness considerations, the sets \( V \) tend to a point \( c \in Y \). Let us show that \( c \) is the required point. If there is a neighborhood \( U \ni c \) such that \( \kappa(f^{-1}(U)) \leq n \), then for a fine enough covering we have \( V \subseteq U \), which contradicts Property 1 and the fact that \( \kappa(f^{-1}(V)) > n \).

\[ \square \]

3. **Properties of the Schwarz genus**

Remind the properties of the genus from [5, 12, 8]. A review on the genus and its generalizations for non-free actions can be found in [10]. We assume the spaces to be paracompact.

**Lemma 2** (Monotonicity). If there exists a \( G \)-equivariant map \( X \to Y \) between free \( G \)-spaces, then

\[ g_G(X) \leq g_G(Y). \]

The following property obviously follows from \( k - 2 \)-connectivity of \( G^k \) and the obstruction theory.

**Lemma 3.** For a paracompact \( G \)-space \( X \) we have

\[ g_G(X) \leq \dim X + 1. \]

**Lemma 4.** For an \( (n - 1) \)-connected free \( G \)-space \( X \) we have

\[ g_G(X) \geq n + 1. \]

The Borsuk-Ulam theorem [2] (in the version of Lusternik-Schnirelmann [6]) follows from the above two lemmas:

**Lemma 5.** If \( G \) acts freely on \( S^d \), then

\[ g_G(S^d) = d + 1. \]

Let us state another property of the genus, that is not listed in the properties of a generalized relative category.

**Theorem 3.** Let a free paracompact \( G \)-space \( Y \) be covered by \( G \)-invariant open subsets \( \{Y_i\}_{i=1}^l \). Then there exists \( x \in Y \) such that

\[ \sum_{Y_i \ni x} g_G(Y_i) \geq g_G(Y). \]

Let us give an example of applying this theorem to describing the critical values of a smooth function. If \( Y \) is a compact closed manifold with free action of \( G \), \( f \) is a smooth function on \( X = Y/G \), \( \xi \) is a gradient-like vector field for \( f \), then either there are infinitely many critical points of \( f \), or for every \( \varepsilon > 0 \) there exists an integral curve of \( \xi \), passing through \( \varepsilon \)-neighborhoods of at least \( g_G(X) \) critical points of \( f \).
Proof of theorem 3. Put \( g_G(Y_i) = k_i \), we may assume these numbers to be finite, otherwise the statement trivially holds. Consider the equivariant maps \( f_i : Y_i \to G^{*k_i} \), that exist by the definition of the genus (for paracompact spaces), and consider the \( G \)-invariant partition of unity \( \{ \rho_i \}_{i=1}^l \), subordinated to \( \{ Y_i \} \).

Define the map \( f : x \mapsto \rho_1(x)f_1(x) \oplus \cdots \oplus \rho_l(x)f_l(x) \), which maps \( X \) to the join \( G^{*}(k_1+\cdots+k_l) \). If the claim is not true, the image of this map is in the \((g_G(X) - 2)\)-dimensional skeleton of \( G^{*}(k_1+\cdots+k_l) \), which contradicts the monotonicity of the genus and Lemma 3. \( \square \)

4. Properties of the cohomology length

The cohomology length is widely used to estimate the Lyusternik-Schnirelmann category. The following lemma is from [3], see also the review [4]. Here \( A \) is the ring of coefficients.

**Lemma 6.** Let a space \( X \) be covered by open sets \( U_1, U_2, \ldots, U_m \), and let the elements \( a_1, a_2, \ldots, a_m \in H^*(X, A) \) be given. If for any \( i = 1, \ldots, m \) the image of \( a_i \) in \( H^*(U_i, A) \) is zero, then the product \( a_1a_2\cdots a_m = 0 \) in \( H^*(X, A) \).

This lemma imply Property 2 of a generalized relative category for \( \text{hlen}_X U + 1 \). Indeed, if a product \( N = \sum_{i=1}^n \text{hlen}_X U_i + n \) of classes of positive dimension \( u_1, \ldots, u_N \in H^*(X, A) \) is nonzero on \( \bigcup_{i=1}^n U_i \), then this product can be partitioned into \( n \) segments of length \( \text{hlen}_X U_i + 1 \) respectively. By Lemma 6, one of the segments is nonzero on its respective \( U_i \), which contradicts the definition of \( \text{hlen}_X U_i \).

The Properties 1 and 3 are obvious for the cohomology length.

5. Some corollaries

Let us give a corollary of Theorem 1. This corollary can also be deduced from the version of Theorem 1 in [12].

**Corollary 4** (A generalized Borsuk-Ulam theorem for functions). Let \( k, l, n \) be positive integers such that \( k(l+1) \leq n \). Then for any \( l \) continuous even functions \( (f_1, \ldots, f_l) \) on \( S^n \) there exist numbers \( (c_1, \ldots, c_l) \) such that for any \( k \) continuous odd functions \( (g_1, \ldots, g_k) \) the system of equations

\[
\begin{align*}
f_1(x) = c_1, & \quad f_2(x) = c_2, \ldots, f_l(x) = c_l \\
g_1(x) = 0, & \quad g_2(x) = 0, \ldots, g_k(x) = 0
\end{align*}
\]

has a solution.

**Proof.** By Lemma 5, \( Z_2 \)-genus of \( S^n \) under the involution \( x \mapsto -x \) equals \( n+1 \). Let us apply Theorem 1 to the map \( f : S^n/Z_2 \to \mathbb{R}^l \), given
by the functions \((f_1, \ldots, f_l)\). We obtain a set of numbers \((c_1, \ldots, c_l)\) such that the subset of \(S^n\) (see the comment after Theorem 1)

\[ Y = \{ x \in S^n : f_1(x) = c_1, \ f_2(x) = c_2, \ldots, f_l(x) = c_l \} \]

has genus at least \(k\). Similarly to the proof of the original Borsuk-Ulam theorem, Lemma 5 and the monotonicity of the genus imply, that any set of continuous odd functions \((g_1, \ldots, g_k)\) has a common zero on \(Y\).

If we take the functions \((g_1, \ldots, g_k)\) to be linear, we obtain the following result.

**Corollary 5.** Let \(k, l, n\) be positive integers such that \(k(l + 1) \leq n\). Then for any \(l\) continuous even functions \((f_1, \ldots, f_l)\) on \(S^n\) there exist numbers \((c_1, \ldots, c_l)\) such that the set

\[ Y = \{ x \in S^n : f_1(x) = c_1, \ f_2(x) = c_2, \ldots, f_l(x) = c_l \} \]

intersects with any linear subspace of dimension \(n + 1 - k\) in \(\mathbb{R}^{n+1}\).

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