THE VIRTUAL FUNDAMENTAL CLASS FOR THE MODULI SPACE OF SURFACES OF GENERAL TYPE

YUNFENG JIANG

ABSTRACT. We suggest a construction of obstruction theory on the moduli stack of index one covers over semi-log-canonical surfaces of general type. Comparing with the index one covering Deligne-Mumford stack of a semi-log-canonical surface, we define the lci covering Deligne-Mumford stack. The lci covering Deligne-Mumford stack only has locally complete intersection singularities.

We construct the moduli stack of lci covers over the moduli stack of surfaces of general type and a perfect obstruction theory. The perfect obstruction theory induces a virtual fundamental class on the Chow group of the moduli stack of surfaces of general type. Thus our construction proves a conjecture of Sir Simon Donaldson for the existence of virtual fundamental class. A tautological invariant is defined by taking integration of the power of first Chern class for the CM line bundle on the moduli stack over the virtual fundamental class. This can be taken as a generalization of the tautological invariants defined by the integration of tautological classes over the moduli space $\mathcal{M}_g$ of stable curves to the moduli space of stable surfaces.

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1. INTRODUCTION

The main goal of this paper is to construct a virtual fundamental class for the moduli space of semi-log-canonical (s.l.c.) general type surfaces.

1.1. The index one cover. Let $S$ be a projective surface, and $\omega_S$ be its dualizing sheaf. From [42, Definition 4.17] and [42, Theorem 4.24], roughly speaking a reduced Cohen-Macaulay projective surface $S$ is semi-log-canonical (s.l.c.) if it has only normal crossing singularities in codimension one, all the other singularities are finite set of isolated points, and there exists some $N > 0$ such that $\omega_S^\otimes N := (\omega_S^\otimes N)_{\text{inv}}$ is invertible; see §4.1 and Definition 4.1 for the formal definition. The least integer $N$ is called the index of the s.l.c. surface $S$.

Let $(S, x)$ be an s.l.c. surface germ. The index of the singular point $x \in S$ is, by definition, the least integer $r > 0$ such that $\omega_S^\otimes r$ is invertible around $x$. Note that if for $N > 0$ such that $\omega_S^\otimes N$ is globally invertible, then $r$ divides $N$. Thus, let lcm($S$) be the least common multiple of all the local indexes of the finite isolated singularity germs $(S, x)$ with local index bigger
than one, then \( \text{lcm}(S) \) divides \( N \). Fixing an isomorphism \( \theta : \omega^r_S \to \mathcal{O}_S \), then each semi-log-canonical germ \((S,x)\) defines a local cover \( Z := \text{Spec}_{\mathcal{O}_S}(\mathcal{O}_S \oplus \omega^1_S \oplus \cdots \oplus \omega^{r-1}_S) \to S \) under the \( \mathbb{Z}_r \)-action, where the multiplication is given by the isomorphism \( \theta \). The surface \( Z \) is Gorenstein, which implies that \( \omega_Z \) is invertible. This cover is uniquely determined by the étale topology which we call the index one cover. All of these data of index one covers for s.l.c. germs (which locally give the stacks \( [Z/\mathbb{Z}_r] \)) glue to define a Deligne-Mumford stack \( \pi : \mathcal{S} \to S \) which is called the index one covering Deligne-Mumford stack. The dualizing sheaf \( \omega_{\mathcal{S}} \), which is étale locally given by the \( \mathbb{Z}_r \)-equivariant \( \omega_Z \), is invertible.

Around the singularity germ \((S,x)\), a deformation \( S/T \) over a scheme \( T \) is called Q-Gorenstein if locally there is a \( \mathbb{Z}_r \)-equivariant deformation \( Z/T \) of \( Z \) whose quotient is \( S/T \). Let \( \omega_{S/T} \) be the relative dualizing sheaf of \( S/T \). We define \( \omega_{S/T}^r := (\omega_{S/T}^r)^{\vee \vee} = i_*\omega_S^{\otimes r} \) where \( i : S^0 \hookrightarrow S \) is the inclusion of the Gorenstein locus of \( S/T \), which is the locus where \( \omega_{S/T} \) is invertible; see \cite{27, §3.1} and \cite{42, §5.4}. The associated relative divisor of \( \omega_{S/T}^r \) is \( r \cdot K_{S/T} \). From Hacking \cite{27, §3.2}, let \( S/T \) be a Q-Gorenstein deformation family of s.l.c. surfaces and \( x \in S \) has index \( r \), then \( Z \) is given by \( Z := \text{Spec}_{\mathcal{O}_S}(\mathcal{O}_S \oplus \omega^{1}_S \oplus \cdots \oplus \omega^{r-1}_S) \), where the multiplication is given by fixing a trivialization of \( \omega_{S/T}^r \) at the point \( x \). The canonical covering \( Z \) of \( x \in S/T \) is uniquely determined by the étale topology. These data of local quotient stacks \( [Z/\mathbb{Z}_r] \) glue to define the index one covering Deligne-Mumford stack \( \mathcal{S}/T \) which is a flat family over \( T \) from \cite{27, Lemma 3.5].

An s.l.c. surface \( S \) is called stable if its dualizing sheaf \( \omega_S \) is ample. Let \( G \) be a finite group. We consider the stable s.l.c. surfaces together with a finite group \( G \) action. Fixing \( K^2 := K^2_{\mathcal{S}, \chi} := \chi(O_S), N \in \mathbb{Z}_{>0}, \) and we consider the moduli stack \( M := \mathcal{M}_{K^2_{\mathcal{S}, \chi}, N} \) which is defined by the moduli functor of Q-Gorenstein deformation families \( \{ S \to T \} \) of stable s.l.c. \( G \)-surfaces such that \( \omega^N_S \) is invertible. In the definition, \( \omega^N_S \otimes k(t) \cong \omega^N_S \) is an isomorphism for each \( t \in T \) which implies that \( \omega^N_S \) commutes with specialization. This ensures that the moduli space is separated. We should point out that for any family \( S \to T \) in the moduli stack, the index \( r \) of a singularity germ \( x \in S/T \) divides \( N \).

We consider \( G \)-equivariant s.l.c. surfaces, and we write s.l.c. \( G \)-surfaces just as s.l.c. surfaces. From \cite{46, Proposition 6.11], \( M \) is a Deligne-Mumford stack of finite type over \( k \). When we fix \( K^2_{\mathcal{S}, \chi} \) and \cite{28, Theorem 1.1] proved the boundedness of the moduli space, which implies that there exists a uniform bound \( N > 0 \) such that whenever we have a family \( S \to T \) of s.l.c. surfaces in the moduli space, the index of any s.l.c. surface in the family divides \( N \). Thus, from \cite{46, Theorem 1.1, §6.1, Remark 6.3], if \( N \) is large divisible enough, the stack \( \mathcal{M}_{K^2_{\mathcal{S}, \chi}, N}^G := \mathcal{M}_{K^2_{\mathcal{S}, \chi}, N}^G \) is a proper Deligne-Mumford stack with projective coarse moduli space.

The construction of the index one covering Deligne-Mumford stack is canonical. From \cite{27, Proposition 3.7], there is a one-to-one correspondence between the set of isomorphic classes of Q-Gorenstein deformation families of s.l.c. surfaces and the set of isomorphic classes of flat families of the index one covering Deligne-Mumford stacks. We have the following result.

**Theorem 1.1.** (Theorem 5.1) The moduli functor of the isomorphism classes of flat families of index one covering Deligne-Mumford stacks is represented by a Deligne-Mumford stack \( M^\text{ind} := \mathcal{M}_{K^2_{\mathcal{S}, \chi}, N}^\text{ind} \). There exists an isomorphism between Deligne-Mumford stacks \( f : M^\text{ind} \to M \).

If \( N \) is large divisible enough, then \( M^\text{ind} \) is a projective Deligne-Mumford stack and the isomorphism \( f : M^\text{ind} \to M \) induces an isomorphism on the projective coarse moduli spaces.
The moduli stack $M^{\text{ind}}$ of index one covers is a fine moduli Deligne-Mumford stack. Therefore, there exists a universal family $p^{\text{ind}} : \mathcal{M}^{\text{ind}} \to M^{\text{ind}}$, which is a projective, flat and relative Gorenstein morphism between Deligne-Mumford stacks. Let
\[
E^{\text{ind}} \cdot M^{\text{ind}} := R\mathcal{H}^{\text{ind}} \mathcal{L} \mathcal{L}^{\text{ind}} M^{\text{ind}} \otimes \omega^{\text{ind}} M^{\text{ind}} \mid [-1],
\]
where $\mathcal{L} \mathcal{L}^{\text{ind}} M^{\text{ind}}$ is the relative cotangent complex of $p^{\text{ind}}$, and $\omega^{\text{ind}} M^{\text{ind}}$ is the relative dualizing sheaf of $p^{\text{ind}}$ which is a line bundle. This is the case of the moduli space of projective Deligne-Mumford stacks satisfying the condition in Theorem 3.5 (see also [10, Proposition 6.1]). Thus, the Kodaira-Spencer map $\mathcal{L} \mathcal{L}^{\text{ind}} M^{\text{ind}} \to (p^{\text{ind}})^{\ast} E^{\text{ind}} \cdot M^{\text{ind}}$ induces an obstruction theory
(1.1.1)
\[
\phi^{\text{ind}} : E^{\text{ind}} \cdot M^{\text{ind}} \to E^{\text{ind}} \cdot M^{\text{ind}}
\]
on $M^{\text{ind}}$, see Theorem 3.6. In general the obstruction theory $\phi^{\text{ind}} : E^{\text{ind}} \cdot M^{\text{ind}} \to E^{\text{ind}} \cdot M^{\text{ind}}$ is not perfect due to the possible existence of higher obstruction spaces. Let $E_S^{\ast}$ be the cotangent complex of the index one covering Deligne-Mumford stack $\Sigma$ in [34] and [35]. The higher obstruction spaces $T^{1}_{\mathcal{O}}(S, O_{\Sigma}) := \text{Ext}^{i}(E_{\ast}, O_{\Sigma})$ in general do not vanish for $i \geq 3$, see [37]. The vanishing of the obstruction spaces $T^{1}_{\mathcal{O}}(S, O_{\Sigma})$ for $i \geq 3$ is necessary for the existence of a Behrend-Fantechi, Li-Tian style perfect obstruction theory.

1.2. Singularities of index one cover and lci cover. From [42] Theorem 4.23, Theorem 4.24], the singularities of an s.l.c. surface $S$, except the normal crossing singularities in codimension one, are all isolated singularities as follows: finite group quotient surface singularities, simple elliptic singularities, cusp singularities, degenerate cusp singularities, $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ quotients of simple elliptic singularities, and $\mathbb{Z}_2$ quotients of cusps, and degenerate cusps. We refer the readers to the proof of Proposition 4.3 for a list of the s.l.c. singularities.

From [42] Proposition 3.10], Kollár-Shepherd-Barron proved that if the quotient singularities admit Q-Gorenstein smoothings, then they must be class $T$-singularities. Therefore the index one covers of such singularities have $A_{\mu}$ type singularities which are l.c.i. For $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ quotients $(S, x)/\mathbb{Z}_r (r = 2, 3, 4, 6)$ of the simple elliptic singularities, and $\mathbb{Z}_2$ quotients $(S, x)/\mathbb{Z}_2$ of cusps, and degenerate cusps, the index of such singularities can only be $r = 2, 3, 4, 6$ and the index one covers are given by the germs $(S, x)$. Thus for an s.l.c. surface $S$, the possible singularities of the index one covering Deligne-Mumford stack $\Sigma$ are l.c.i. singularities, simple elliptic singularities, cusps, and degenerate cusp singularities. This is one of the key new ideas for the construction in the paper.

For l.c.i. singularity germs $(S, x)$, the local tangent sheaves $T^{q}(S) = 0$ for $q \geq 2$. A simple elliptic singularity, a cusp, or a degenerate cusp singularity germ $(S, x)$ who has local embedded dimension $\leq 4$ is l.c.i.; see [48] Theorem 3.13] and [70]. But if the simple elliptic singularity, the cusp, or the degenerate cusp singularity germ $(S, x)$ has embedded dimension $\geq 5$, [48] Theorem 3.13] and [70] showed that these singularities are never l.c.i. When the embedded dimension of the germ $(S, x)$ is $\geq 6$, then the higher tangent spaces $T^{q}(S)$ for $q \geq 0$ are not zero (see [37] Theorem 1.3]). From the local to global spectral sequence, the higher obstruction spaces $T^{i}_{\mathcal{O}}(S, O_{\Sigma})$ do not vanish for $i \geq 3$. For this type of singularities, we define the lci cover $(\hat{S}, x)$ of $S$. The lci cover is determined by the topological type of the link $\Sigma$ of the germ singularity. The link $\Sigma$ is the boundary (which is a real 3-dimensional oriented manifold) of a small neighborhood $U \subset S$ around the singular point $x$. For the link $\Sigma$, we work over the field of real numbers $\mathbb{R}$.

There are two cases for these singularities which are called log canonical in birational geometry. The first case is either a Gorenstein simple elliptic singularity, or a cusp and a degenerate cusp singularity which has index one. The link $\Sigma$ is not a $Q$-homology sphere. The second case is the the singularity $(S, x)$ which is either the $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$-quotient of a simple elliptic singularity, the $\mathbb{Z}_2$-quotient of a cusp, or the $\mathbb{Z}_2$-quotient of a degenerate
cusp singularity. The index of the singularity is the order of the cyclic group. In this case the quotient is a rational singularity and the link $\Sigma$ is a rational homology sphere. In both of these two cases we construct a finite cover $(\tilde{S}, x)$ of $S$ with transformation group $D$, using the theory of Newmann-Wahl in [54, Proposition 4.1 (2)] and [55]. This is called the lci cover of $(S, x)$. In the first case the cover works in analytic topology, and in the second case the cover is a Galois cover. The key issue is that $(\tilde{S}, x)$ is l.c.i. The map $(\tilde{S}, x) \to (S, x)$ from the lci cover to $S$ factors through the map $(Z, x) \to (S, x)$ from the index one cover (see §6.1, §6.2, and §6.3).

For the lci covers constructed above, the germ $(S, x)$ is an equisingular deformation of $(\tilde{S}/D, x)$. We consider the $Q$-Gorenstein deformations of $(S, x)$ as the $D$-equivariant deformations of $(\tilde{S}, x)$; i.e., the deformations of the Deligne-Mumford stack $[\tilde{S}/D]$. For a simple elliptic singularity $(S, x)$ with embedded dimension $d$, the smoothing of $(S, x)$ is induced from an equivariant smoothing of an lci cover $(\tilde{S}, x)$ (which is also an lci simple elliptic singularity) if and only if $1 \leq d \leq 9$ and $d \neq 5, 6, 7$, see [39, Theorem 1.3]. For a cusp singularity $(S, x)$ or the cyclic quotients of a simple elliptic singularity and a cusp singularity $(S, x)$, there exist criterions for the lci smoothing lifting of $(S, x)$, see [39, Theorem 1.4, Theorem 1.5].

The lci covering construction is canonical on each analytic germ of the singularities considered above. Thus the data of the local lci covers glue to give a Deligne-Mumford stack $\pi_\text{lci} : \mathcal{E}_\text{lci} \to S$ which is called the lci covering Deligne-Mumford stack. The lci covering Deligne-Mumford stack $\mathcal{E}_\text{lci}$ is s.l.c. and only has l.c.i. singularities. In particular, its dualizing sheaf $\omega_{\text{lci}}$ is invertible. This is the second key new idea in this paper.

1.3. The moduli stack of lci covers. Let $S/T$ be a $Q$-Gorenstein deformation family of s.l.c. surfaces, and $\mathcal{E}/T$ be the corresponding index one covering Deligne-Mumford stacks. It is not known to the author whether $\mathcal{E}/T$ can always be lifted to a flat family $\mathcal{E}_{\text{lci}}/T$ of lci covering Deligne-Mumford stacks. But suppose that there is a flat family $\mathcal{E}_{\text{lci}}/T$ of lci covering Deligne-Mumford stacks, then it induces a flat family $\mathcal{E}/T$ of index one covering Deligne-Mumford stacks, which in turn induces a flat family $\tilde{S}/T$ of $Q$-Gorenstein deformations of s.l.c. surfaces. The lci covering Deligne-Mumford stacks locally have the form $[\tilde{S}/D]$ where $\tilde{S}$ is the lci cover of a simple elliptic singularity, a cusp, or a degenerate cuspidal singularity. The singularities of $\tilde{S}$ are l.c.i. which always admit smoothings. If there is a $D$ equivariant smoothing of $\tilde{S}$, then it induces a $Q$-Gorenstein smoothing for the quotient surface $\tilde{S}$ (see §6.5).

Let $M_{\text{lci}} := \overline{\mathcal{M}}^G_{\text{lci}, X, N}$ be the moduli functor of flat families $\mathcal{E}_{\text{lci}}/T$ of lci covering Deligne-Mumford stacks. Any family $\mathcal{E}_{\text{lci}}/T$ induces a $Q$-Gorenstein deformation family $S \to T$ of s.l.c. surfaces, and we let $M = \overline{\mathcal{M}}^G_{\text{lci}, X, N}$ be the corresponding moduli functor induced from $M_{\text{lci}}$. Kollár’s result in [43, Theorem 2.6] implies that the moduli functor $M = \overline{\mathcal{M}}^G_{\text{lci}, X, N}$ is coarsely represented by a projective scheme.

We have the following result.

Theorem 1.2. (Theorem [5, 19]) Let $M$ be the moduli stack of stable s.l.c. surfaces such that any $Q$-Gorenstein deformation family of s.l.c. surfaces in $M$ is induced from the flat family of lci covering Deligne-Mumford stacks. Then there exists a moduli stack $M_{\text{lci}}$ of lci covers and a morphism between Deligne-Mumford stacks $f_{\text{lci}} : M_{\text{lci}} \to M$.

If $N$ is large divisible enough, then the stack $M_{\text{lci}}$ is a proper Deligne-Mumford stack and the morphism $f_{\text{lci}} : M_{\text{lci}} \to M$ is a finite morphism which induces a finite morphism on their projective coarse moduli spaces.

If the moduli stack $M$ contains s.l.c. surfaces $S$ with simple elliptic and cusp singularities with local embedded dimension $> 5$, then the lci covers work in the analytic topology. Thus
in this case the Deligne-Mumford $\mathcal{M}^{\text{lci}}$ is an analytic stack. In all the other cases, we get an algebraic Deligne-Mumford stack.

There exist examples of the moduli stack of lci covers. Donaldson’s example in §9.2 gives a compact example $M$ of the KSBA moduli space of sextic hypersurfaces of degree 6 in $\mathbb{P}^3$ under a finite group $G$ action. The surfaces in $M$ are all lci surfaces and $M$ is the same the moduli stack of lci covers. In [6] V. Alexeev constructed examples of moduli space of Campedelli and Burniat surfaces. Except lci singularities, and degenerate cusp singularities (which are always equivariantly smoothable), the only singularity in an s.l.c. Campedelli surface is a simple elliptic singularity of degree $d = 8$, from [39] Theorem 1.3], there exists a moduli stack of lci covers for the moduli space of Campedelli surfaces in [6], see also [5] for the calculation of Kappa classes on this space.

Theorem 1.2 implies an interesting result for the smoothing component $M^\text{sm} := \overline{\mathcal{M}_{K^2, X, N}}$ of $M = \overline{\mathcal{M}_{K^2, X, N}}$ for $N$ large divisible enough. The smoothing component $M^\text{sm} \subset M$ is the component containing smooth surfaces or surfaces with ADE type singularities. We let $M^\text{sm}_{\text{eq}} \subset M^\text{sm}$ be the components such that the smoothing of s.l.c. surfaces in $M^\text{sm}_{\text{eq}}$ can be obtained from the equivariant smoothing of lci covers.

Let $M^\circ \subset M$ be the open locus containing smooth surfaces or surfaces with ADE singularities, then the smoothing component $M^\text{sm} \subset M$ is the closure of $M^\circ$ inside $M$. Let $M^\text{sm} \subset M^\text{sm}$ be the singular locus of $M^\text{sm}$ containing the deformation limits of smooth surfaces so that $M^\text{sm} \setminus M^\text{eq} \text{sm}$ consists of connected smoothing components. Suppose that $M^\text{eq} \subset M^\circ$ is the open sub-locus such that all the smooth surfaces in $M^\text{eq} \subset M^\text{sm}$ can be obtained by the equivariant deformations of lci covers, then $M^\text{eq} \subset M^\circ$ is the closure of $M^\text{eq} \setminus (M^\text{sm} \setminus M^\text{eq})$ inside $M^\text{sm}$, see [39] for the explanation on the smoothing of surface singularities, and the equivariant smoothing of simple elliptic singularities and cusp singularities.

**Theorem 1.3. (Theorem 6.23)** Let $M = \overline{\mathcal{M}_{K^2, X, N}}$ be a KSBA moduli stack of s.l.c. surfaces, and let $M^\text{sm}_{\text{eq}} \subset M$ be the equivariant smoothing component. Then there exists a moduli stack $\mathcal{M}^\text{lci,sm}_{\text{eq}}$ of lci covers and a finite morphism $f^\text{lci} : \mathcal{M}^\text{lci,sm}_{\text{eq}} \to M^\text{sm}_{\text{eq}}$.

Theorem 1.2 and Theorem 1.3 imply that for any KSBA moduli stack, there is a closed projective substack $M$ containing the equivariant smoothing component $M^\text{sm}_{\text{eq}}$ such that there is a moduli stack $M^{\text{lci}}$ of lci covers over $M$ and a finite morphism $f^{\text{lci}} : M^{\text{lci}} \to M$.

### 1.4. Main results

For the Deligne-Mumford stack $M^{\text{lci}} = \overline{\mathcal{M}^{\text{lci}}_{K^2, X, N}}$ which is a fine moduli stack, there exists a universal family $p^{\text{lci}} : \mathcal{M}^{\text{lci}} \to M^{\text{lci}}$ which is a projective, flat and relative Gorenstein morphism. Let

$$E_{M^{\text{lci}}} = R^1 p^{\text{lci}}_* (\mathcal{L} \cdot \mathcal{O}_{M^{\text{lci}}} \otimes \mathcal{O}_{M^{\text{lci}}} [-1]),$$

where $\mathcal{L} \cdot \mathcal{O}_{M^{\text{lci}}} / M^{\text{lci}}$ is the relative cotangent complex of $p^{\text{lci}}$, and $\mathcal{O}_{M^{\text{lci}}} / M^{\text{lci}}$ is the relative dualizing sheaf of $p^{\text{lci}}$ which is a line bundle. Thus, from Theorem 3.5 (see also 10 Proposition 6.1), the Kodaira-Spencer map $\mathcal{L} \cdot \mathcal{O}_{M^{\text{lci}}} / M^{\text{lci}} \to (p^{\text{lci}})^* \mathcal{L} \cdot \mathcal{O}_{M^{\text{lci}}} [1]$ induces an obstruction theory

$$\phi^{\text{lci}} : E^{\text{lci}}_{M^{\text{lci}}} \to \mathcal{L} \cdot \mathcal{O}_{M^{\text{lci}}}$$

on $M^{\text{lci}}$.

Since the lci covering Deligne-Mumford stack $\mathcal{M}^{\text{lci}}$ has only l.c.i. singularities, its higher obstruction spaces $\mathcal{H}^i_{\mathcal{O}_{\mathcal{M}^{\text{lci}}}} / \mathcal{O}_{\mathcal{M}^{\text{lci}}}$ vanish when $i \geq 3$. The complex $E^{\text{lci}}_{M^{\text{lci}}}$ is a perfect complex with perfect amplitude contained in $[-1, 0]$.

Here is the main result in the paper.

**Theorem 1.4. (Theorem 7.7)** Let $M = \overline{\mathcal{M}^{\text{lci}}_{K^2, X, N}}$ be the moduli stack of stable s.l.c. surfaces of general type with invariants $K^2, X, N$, and $f^{\text{lci}} : M^{\text{lci}} \to M$ be the moduli stack of lci covers.
over $M$. Then the obstruction theory $\phi^{\text{ldt}} : E^{\bullet}_{M^{\text{ldt}}} \to L^\bullet_{M^{\text{ldt}}}$ in (1.4.1) is a perfect obstruction theory in the sense of Behrend-Fantechi. Restricting the morphism $\phi^{\text{ldt}}$ to the universal family $p^{\text{ldt}} : \mathcal{M}^{\text{ldt}} \to M^{\text{ldt}}$ we get a perfect obstruction theory on $M^{\text{ldt}}$ in Theorem 1.3.

Therefore, the perfect obstruction theory induces a virtual fundamental class

$$[M^{\text{ldt}}]^{\text{vir}} \in A_{\text{vd}}(M^{\text{ldt}}),$$

where the virtual dimension is given by

$$\text{vd} = \dim(H^1(S, T_S)^G) - \dim(H^2(S, T_S)^G)$$

for a smooth surface $S \in M$. In the case that $G = 1$, we have $\text{vd} = 10\chi - 2K^2$.

Let $f^{\text{ldt}} : M^{\text{ldt}} \to M$ be the canonical morphism between these two Deligne-Mumford stacks. The morphism $f^{\text{ldt}}$ is finite and is not necessary representable, but it induces a finite morphism on the coarse moduli spaces. From [75, Definition 3.6(iii)], we define

$$[M]^{\text{vir}} := f^{\text{ldt}}_*( [M^{\text{ldt}}]^{\text{vir}} ) \in A_{\text{vd}}(M)$$

(1.4.2)

to be the virtual fundamental class of the moduli stack $M$. Note that the virtual fundamental class is a cycle in the Chow group with $Q$-coefficient.

From [46, Theorem 1.1, Remark 6.3], for $N > 0$ large divisible enough, we get the virtual fundamental class $[\overline{M}_K^{G}]^{\text{vir}} \in A_{\text{vd}}(\overline{M}_K^{G})$. The main Theorem 1.4 induces some interesting results.

A Kawamata-log-terminal (k.l.t.) surface $S$ is a projective surface with only cyclic quotient singularities. We have:

**Theorem 1.5.** (Theorem 7.3) Let $M$ be the moduli stack of stable surfaces of general type with invariants $K^2, \chi, N$. If the moduli stack $M$ consists of k.l.t. surfaces, then the moduli stack $M^{\text{ldt}}$ of l.c.i covers is the same as the moduli stack $M^{\text{ind}}$, which is isomorphic to the moduli stack $M$.

Moreover, the obstruction theory for the moduli stack $M^{\text{ind}}$ of index one covers in (1.1.1) is perfect in the sense of Behrend-Fantechi, and is the same as the perfect obstruction theory on $M^{\text{ldt}}$ in (1.4.1).

Let $S$ be a surface with only locally complete intersection singularities. Then $S$ is Gorenstein and $\omega_S$ is invertible. In particular, the index one covering Deligne-Mumford stack and the lci covering Deligne-Mumford stack are all $S$ itself. Thus, if the moduli stack $M$ consists of l.c.i. surfaces, then the moduli stacks $M^{\text{ldt}}, M^{\text{ind}}$ and $M$ are all the same and the universal family $p : \mathcal{M} \to M$ is projective, flat and relatively Gorenstein; i.e., the relative dualizing sheaf $\omega_{\mathcal{M}/M}$ is a line bundle. We have that

**Corollary 1.6.** (Corollary 7.4) If the moduli stack $M$ only consists of l.c.i. surfaces, then $M$ admits a perfect obstruction theory

$$\phi : E^\bullet_M \to L^\bullet_{/M}$$

in the sense of [110], where

$$E^\bullet_M = R\pi_* (L^\bullet_{/M} \otimes \omega_{\mathcal{M}/M})[-1]$$

and $L^\bullet_{/M}$ is the relative cotangent complex of $p$. Therefore, the perfect obstruction theory induces a virtual fundamental class $[M]^{\text{vir}} \in A_{\text{vd}}(M)$. This proves Donaldson’s conjecture for the existence of virtual fundamental class in his example [17, §5].

1.5. **Tautological invariants.** Donaldson [17] also suggested to extend the MMM-classes (tautological classes) to the cohomology $H^*(M, Q)$ for $M = \overline{M}_K^{G}$. In algebraic geometry the CM line bundle on $M$ was proven to be ample by Patakfalvi and Xu in [59]. From Theorem 7.1 and (1.4.2), the moduli stack $M$ admits a virtual fundamental class $[M]^{\text{vir}}$, we use the CM line bundle on $M$ to define the tautological invariant by the integration of the power of the Chern class of the CM line bundle over the virtual fundamental class $[M]^{\text{vir}}$. 
This can be taken as a generalization of the tautological invariants on the moduli space $\overline{M}_g$ of stable curves to the moduli space of stable surfaces.

Thus, it is interesting to calculate the tautological invariants. We include Donaldson’s example in [5]. More interesting examples will be studied for the tautological invariants of the moduli space of log surfaces of general type. In [5], Alexeev calculated the Kappa classes and the tautological invariants for some moduli spaces of general type surfaces, including moduli space of product curves, moduli space of Burniat and Campedelli surfaces. The moduli spaces in the examples in [5] are all smooth.

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2. Overview, Convention and Structure

2.1. Motivation. The study of virtual fundamental class for the moduli space of s.l.c. surfaces is motivated by the study of moduli space of stable curves. The Deligne-Mumford moduli stack of dimension 3 of stable curves of genus $g \geq 2$ is a smooth projective Deligne-Mumford stack of dimension $3g-3$. It can be taken as the compactification of the moduli space of general type algebraic curves by adding nodal curves on the boundary. This moduli space, together with its variation $\overline{M}_{g,n}$, the moduli space of stable curves of genus $g$ with $n$-marked points, has been a rich geometric object with relations to many other theories in mathematics and physics.

There exists a universal family $\overline{M}_{g,1} \to \overline{M}_g$ such that, we can pushforward the relative dualizing sheaf $\omega_{\overline{M}_{g,1}/\overline{M}_g}$ to get a tautological class called the kappa class on $\overline{M}_g$. There exist other tautological classes like Hodge classes by taking the Chern classes of the Hodge bundle of $\overline{M}_g$. The study of the tautological ring $R^*(\overline{M}_g)$ or $R^*(\overline{M}_{g,n})$ is a hot subject; see [19], [63] and [64]. The integration of the tautological classes over the fundamental class $[\overline{M}_g]$ and $[\overline{M}_{g,n}]$ gives interesting tautological invariants which were studied in decades, for instance, Witten’s conjecture and Kontsevich’s theorem. Let $X$ be a smooth projective variety and let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of stable maps $(f : C \to X)$ from a genus $g$ curve $C$ with $n$ marked points to $X$, then $\overline{M}_{g,n}(X, \beta)$ is a singular Deligne-Mumford stack and admits a perfect obstruction theory in the sense of [50] and [10]. The Gromov-Witten invariants are defined using the virtual fundamental class of the perfect obstruction theory on $\overline{M}_{g,n}(X, \beta)$ (see [9]).

The two dimensional analogue of the moduli space of stable curves is the moduli space of stable general type surfaces. We fix invariants $K^2 := K^2_S$ and $\chi := \chi(\mathcal{O}_S)$ for a general type surface $S$, and an integer $N > 0$. Let $\overline{M}_{K^2,N}$ be the moduli stack in [11]. For $N$ large divisible enough, [16] Theorem 1.1, Definition 6.2, Remark 6.3 proved that the stack $\overline{M}_{K^2,N}$ is a proper Deligne-Mumford stack with projective coarse moduli space. In [17] Donaldson studied the Fredholm topology and enumerative geometry of general type surfaces, and proposed the following two premises:

(1) there exists a virtual fundamental class $[\overline{M}_{K^2,N}]^{\text{vir}} \in H_{\text{td}}(\overline{M}_{K^2,N}, \mathbb{Q})$ by the theory of Behrend-Fantechi [10] and Li-Tian [50].

(2) the Miller-Mumford-Morita (MMM)-classes can be extended to $H^*(\overline{M}_{K^2,N}, \mathbb{Q})$. 

Donaldson calculated the tautological invariant defined by integrating the Miller-Mumford-Morita (MMM)-classes over the (conjectural) virtual fundamental class $[\overline{M}_{K^2,X}]^{vir}$ in an example. Donaldson’s example provided a very interesting invariant defined by the complex structures of general type surfaces. This paper confirms the virtual fundamental class calculation in Donaldson’s example.

2.2. Discussion on the moduli stack. Theorem 1.4 provided a rigorous construction of virtual fundamental class $[\overline{M}_{K^2,X}]^{vir}$ for the moduli space $\overline{M}_{K^2,X}$. Thus, we prove Donaldson’s conjecture for his first premise. In the rest of the paper, the constructions are given for the moduli stack $M := \overline{M}_{K^2,X,N}$ for an arbitrary $N \in \mathbb{Z}_{>0}$. Fixing $K^2,X$, and when $N$ is large divisible enough, we get the results for $\overline{M}_{K^2,X}$.

One key construction is the moduli stack $M^{lei} = \overline{M}_{K^2,X,N}^{lei}$ of lci covers in Theorem 1.2 (Theorem 5.19). The lci covering Deligne-Mumford stack $\mathcal{E}^{lei} \to S$ is different from the index one covering Deligne-Mumford stack $\mathcal{E} \to S$ only when the s.l.c. surface $S$ has simple elliptic singularities, cusps, degenerate cusp singularities, or the cyclic quotients of them with local embedded dimension $\geq 5$. For the simple elliptic singularity germ $(S,x)$ with higher embedded dimension, the lci cover $(\tilde{S},x) \to (S,x)$ (with transform group $D$) always has lower embedded dimensions, hence must be smoothable. Thus, there exists a Gorenstein smoothing of $(S,x)$ which is induced from the $D$-equivariant smoothing of $(\tilde{S},x)$. Also the lci covering Deligne-Mumford stack $\mathcal{E}^{lei} \to S$ has only l.c.i. singularities, thus is smoothable. This means that there should exist the moduli stack $M^{lei}$ of lci covers such that it contains the lci cover $(\tilde{S},x) \to (S,x)$ mentioned above, see [6], [39] for an example containing simple elliptic singularities of degree 8.

More interesting case is the cusp singularity germ $(S,x)$. From Looijenga’s conjecture (now a theorem) in [18], [26] and [52], a cusp singularity $(S,x)$ is smoothable if and only if the resolution cycle $E$ of its dual cusp is an anti-canonical divisor on a smooth rational surface. Considering the lci cover $(\tilde{S},x) \to (S,x)$ with transformation group $D$, where $(\tilde{S},x)$ is an lci cusp, it is interesting to prove the equivariant Looijenga conjecture for cusp singularities and construct explicit moduli stacks of lci covers.

There are two cases: cusp singularities $(S,x)$ with index one and quotient cusp singularities $(S,x)/\mathbb{Z}_2$ with index two. These singularities are called log canonical surface singularities, and are the only log canonical singularities except weighted homogeneous singularities. Suppose that $(X,x) = (S,x)/\mathbb{Z}_2$ is a quotient-cusp singularity, and let $(\tilde{X},0) \to (X,x)$ be the universal abelian cover in [54] with transformation group $D$. Then [54] Theorem 5.1] gives the local equations of the lci cover $(\tilde{X},0)$. Since the lci cover $\tilde{X}$ obviously admits a $D$-equivariant smoothing such that the quotient gives a smoothing of the quotient-cusp $(X,x)$, this provides another evidence for the existence of our moduli stack of lci covers. For the equivariant Looijenga conjecture for cusp singularities, the case $(S,x)/\mathbb{Z}_2$ was studied in [67], see also [68].

2.3. Convention. We work over the field of complex numbers $k = \mathbb{C}$ throughout of the paper, although some parts work for any algebraically closed field $k$ of characteristic zero. For the notion of algebraic stack and Deligne-Mumford stack, we follow the book [49], [16] and [68]. All Deligne-Mumford stacks are quasi-projective which, from A. Kretch’s equivalence condition, means that they can be embedded into a smooth projective Deligne-Mumford stack. Let $D(O_M)$ be the derived category of coherent modules on the Deligne-Mumford stack $M$. The Chow group $A_*(M) := A_*(M,Q)$ of the Deligne-Mumford stack $M$ is under $Q$-coefficients as in [75].

We use lci to represent locally complete intersection and l.c.i. for locally complete intersection singularities. Class $T$-singularities are either rational double point or two dimensional cyclic quotient singularities of the form $\text{Spec} k[x,y]/\mu_2^{2s}$, where $\mu_2^{2s} = \langle\alpha\rangle$.
and there exists a primitive $r^2 s$-th root of unity $\eta$ such that the action is given by: $a(x, y) = (\eta x, \eta^{rst-1} y)$ and $(d, r) = 1$. When $s = 1$, these are called Wahl singularities.

Recall a normal surface singularity $(S, x)$ is a rational singularity if the exceptional divisor of the minimal resolution is a tree of rational curves. Simple elliptic surface singularities, cusp or degenerate cusp surface singularities were defined in [42, Definition 4.20]. A simple elliptic singularity is a normal Gorenstein surface singularity such that the exceptional divisor of the minimal resolution is a smooth elliptic curve. A normal Gorenstein surface singularity is called a cusp if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve. A degenerate cusp is a non-normal Gorenstein surface singularity $S$. If $f : X \to S$ is a minimal semi-resolution, then the exceptional divisor is a cycle of smooth rational curves or a rational nodal curve. In this case $S$ has no pinch points and the irreducible components of $S$ have cyclic quotient singularities.

2.4. Outline. Here is a short outline for this paper. In §3 basic materials about perfect obstruction theory in [10] and [50] are reviewed. §4 reviews the moduli stack of semi-log-canonical surfaces, and constructs the moduli stack of semi-log-canonical surfaces with a finite group action. In §5 we construct the moduli stack of index one covers over the moduli stack of s.l.c. surfaces. We define the moduli stack of ICI covers over the moduli stack of s.l.c. surfaces in §6 and in §7 we construct the perfect obstruction theory. The virtual fundamental class on the moduli stack of semi-log-canonical surfaces is constructed by the perfect obstruction theory. In §8 we construct the CM line bundle on the moduli stack of s.l.c. surfaces. We define the tautological invariant by integrating the power of the first Chern class of the CM line bundle over the virtual fundamental class. Finally, in §9 we calculate some examples: the moduli stack of quintic surfaces, and Donaldson’s example on sextic surfaces in $\mathbb{P}^3$ with a finite group action. We also give a short discussion on the moduli stack $\overline{M}_{24,11}$ of numerical minimal general type sextic surfaces with $K^2_S = 24 \chi(O_S) = 11$. The coarse moduli space of this moduli stack is a scheme with wrong dimension. We discuss the virtual fundamental class for this moduli stack, although we can not fully understand its construction.

3. Preliminaries on perfect obstruction theory

We review the basic construction of perfect obstruction theory in [10] and [50].

3.1. Perfect obstruction theory. Let $M$ be a quasi-projective Deligne-Mumford stack, which is an algebraic stack over $k$ in the sense of [10] and [49] with unramified diagonal. Let $\mathbb{L}^\bullet_M$ be the cotangent complex of $M$ in the sense of [34] and [35].

**Definition 3.1.** ([10, Definition 4.4]) An obstruction theory for $M$ is a morphism

$$\phi : E^\bullet_M \to \mathbb{L}^\bullet_M$$

in the derived category $D(O_M)$ such that

1. $E^i_M \in D(O_M)$ satisfies the condition that $h^i(E^i_M) = 0$ for all $i > 0$, and $h^i(E^0_M)$ is coherent for $i = 0, -1$.

2. $\phi$ induces an isomorphism on $h^0$ and an epimorphism on $h^{-1}$.

**Definition 3.2.** ([10, Definition 5.1]) An obstruction theory $\phi : E^\bullet_M \to \mathbb{L}^\bullet_M$ for $M$ is called perfect if $E^i_M$ is of perfect amplitude contained in $[-1, 0]$.

3.2. Bundle stack. Any complex $E^\bullet_M \in D(O_M)$ defines an algebraic stack $h^1/h^0((E^\bullet_M)^\vee_U)$ over $M$ as follows: locally around an étale chart $U \to M$, $(E^\bullet_M)^\vee|_U$ is a complex written as

$$(E^\bullet_M)^\vee|_U = [E_0 \to E_1 \to \cdots].$$

The stack $h^1/h^0((E^\bullet_M)^\vee)(U)$ is the groupoid of pairs $(P, f)$ where $P$ is an $E_0$-torsor (principle homogeneous $E_0$-bundle) on $U$ and $f : P \to E_1|_U$ is an $E_0$-equivariant morphism
of sheaves on \( U \). Thus \( h^1/h^0((E_M^\bullet)^\vee) \) is a fiber category fiber by groupoids which is an algebraic \( M \)-stack (called an abelian cone stack).

If \( E_M^\bullet \in D(\mathcal{O}_M) \) is perfect; i.e., of perfect amplitude contained in \([-1,0]_\bullet\), then \( h^1/h^0((E_M^\bullet)^\vee) \) is a vector bundle stack, since étale locally around \( U \to M \), \( (E_M^\bullet)^\vee|_U \) is a complex of vector bundles \( (E_M^\bullet)^\vee|_U = \left[ E_0 \to E_1 \right] \). The stack is \( h^1/h^0((E_M^\bullet)^\vee)|_U = \left[ E_1 \to E_0 \right] \).

3.3. **Intrinsic normal cone.** Let \( M \) be a quasi-projective Deligne-Mumford stack. Étale locally there exists a diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow \chi \\
M & \xrightarrow{g} & Y,
\end{array}
\]

where \( i : U \to M \) is an étale morphism and \( f : U \to Y \) is a closed immersion into a smooth scheme \( Y \). There is a cone stack \( [C_{U/Y}/T_Y]|_U \) where \( C_{U/Y} \) is the normal cone, and \( T_Y\)|\_U acts on the normal cone \( C_{U/Y} \). Whenever we have a morphism \( \chi : (U',Y') \to (U,Y) \) of the local embeddings, which means there exists a commutative diagram

\[
\begin{array}{ccc}
U' & \xrightarrow{f'} & Y' \\
\downarrow \phi_U & & \downarrow \phi_Y \\
U & \xrightarrow{f} & Y,
\end{array}
\]

where \( \phi_U \) is étale and \( \phi_Y \) is smooth, we have that \( (C_{U/Y} \hookrightarrow N_{U/Y})|_{U'} \) is the quotient of \( (C_{U'/Y'} \hookrightarrow N_{U'/Y'}) \) by the action of \( f'^*T_{Y'/Y} \). Here \( N_{U/Y} \) is the normal sheaf of \( U \) to \( Y \). Hence the isomorphism

\[
\tilde{\chi} : \left[ N_{U'/Y'}/f'^*T_{Y'} \right] \cong \left[ N_{U/Y}/f^*T_Y \right]|_U
\]

identifies the closed subcone stacks

\[
\tilde{\chi} : \left[ C_{U'/Y'}/f'^*T_{Y'} \right] \cong \left[ C_{U/Y}/f^*T_Y \right]|_U.
\]

The stacks \( \left[ N_{U/Y}/f^*T_Y \right] \) glue to give the stack \( h^1/h^0((\mathbb{L}_M^\bullet)^\vee) \), which is called the intrinsic normal sheaf; and the stacks \( \left[ C_{U/Y}/f^*T_Y \right] \) glue to give the stack \( c_M, \) which is called the **intrinsic normal cone** of \( M \).

3.4. **Infinitesimal obstruction theory.** We review a bit for the infinitesimal deformation and obstruction theory for a later use.

Let \( T \to \mathcal{T} \) be a square-zero extension of scheme with ideal \( J; \) i.e., \( J^2 = 0 \). For the Deligne-Mumford stack \( M \), let \( g : T \to M \) be a morphism, then there is a canonical morphism

\[
g^*\mathbb{L}_M^\bullet \to \mathbb{L}_T^\bullet \to \mathbb{L}_{\mathcal{T}/\mathcal{T}}^\bullet
\]

in \( D(\mathcal{O}_\mathcal{T}) \) by functoriality properties of the cotangent complex. One has \( \tau_{\geq 1}\mathbb{L}_{\mathcal{T}/\mathcal{T}}^\bullet = \mathcal{I}/[1] \), so the homomorphism (3.4.1) can be taken as an element

\[
\omega(g) \in \text{Ext}^1(g^*\mathbb{L}_M^\bullet,J).
\]

Basic fact about deformation theory says that an extension \( \mathcal{F} : \mathcal{T} \to M \) of \( g \) exists if and only if \( \omega(g) = 0 \), and if \( \omega(g) = 0 \) the extensions form a torsor under \( \text{Ext}^0(g^*\mathbb{L}_M^\bullet,J) = \text{Hom}(\Omega_{\mathcal{T}/\mathcal{T}},J) \).

Let \( \phi : E_M^\bullet \to \mathbb{L}_M^\bullet \) be an obstruction theory. Then [10, Proposition 2.6] tells us that

\[
\phi^\vee : h^1/h^0((\mathbb{L}_M^\bullet)^\vee) \to h^1/h^0((E_M^\bullet)^\vee)
\]
is a closed immersion. Since the intrinsic normal cone $c_M \hookrightarrow h^1/h^0((L^\bullet_M)^\vee)$ is embedded into the intrinsic normal sheaf, we have that $\phi^\vee(c_M) \hookrightarrow h^1/h^0((E^\bullet_M)^\vee)$ is a closed subcone stack. If $T \to \mathcal{T}$ is a square zero extension of $k$-schemes with ideal sheaf $\mathfrak{I}$ and $\mathfrak{g} : T \to M$ is a morphism, then $\omega(\mathfrak{g}) \in \text{Ext}^1(g^{*}E^\bullet_M, M)$ and we denote by $\phi^*\omega(\mathfrak{g}) \in \text{Ext}^1(g^{*}E^\bullet_M, M)$ the image of the obstruction $\omega(\mathfrak{g})$ in $\text{Ext}^1(g^{*}E^\bullet_M, M)$.

We have the following result in [10].

**Theorem 3.3.** ([10] Theorem 4.5) Let $M$ be a Deligne-Mumford stack. The following statements are equivalent:

1. $\phi : E^\bullet_M \to L^\bullet_M$ is an obstruction theory.
2. $\phi^\vee : h^1/h^0((L^\bullet_M)^\vee) \to h^1/h^0((E^\bullet_M)^\vee)$ is a closed immersion of cone stacks over $M$.
3. For any $(T, \mathcal{T}, \mathfrak{g})$ as above, the obstruction $\phi^*\omega(\mathfrak{g}) \in \text{Ext}^1(g^{*}E^\bullet_M, M)$ vanishes if and only if an extension $\mathfrak{g}$ of $\mathfrak{g}$ to $\mathcal{T}$ exists; and if $\phi^*\omega(\mathfrak{g}) = 0$, the extensions form a torsor under $\text{Ext}^1(g^{*}E^\bullet_M, M) = \text{Hom}(g^{*}h^0(E^\bullet_M), M)$.

**Remark 3.4.** [10] Theorem 4.5] has a fourth equivalent condition by using the stack $h^1/h^0(L^\bullet_M) = C(J)$ and the morphism $\text{ob}(\mathfrak{g}) : C(J) \to g^{*}L^\bullet_M$. Since we don’t use this in this paper, we refer the detailed discussion to [10] Theorem 4.5).

3.5. **Virtual fundamental class.** We construct the virtual fundamental class as in [10, §5] for a perfect obstruction theory $\phi : E^\bullet_M \to L^\bullet_M$. First the intrinsic normal cone

$$c_M \hookrightarrow h^1/h^0((L^\bullet_M)^\vee) \hookrightarrow h^1/h^0((E^\bullet_M)^\vee)$$

is a closed subcone stack of the vector bundle stack $h^1/h^0((E^\bullet_M)^\vee)$. Then intersection theory of Artin stacks in [17] gives the virtual fundamental class

$$[M]^\text{vir} = 0_{h^1/h^0((E^\bullet_M)^\vee)}(c_M) \in A_{\text{rk}(E^\bullet_M)}(M);$$

e.g., the intersection of the intrinsic normal cone $c_M$ with the zero section of the bundle stack $h^1/h^0((E^\bullet_M)^\vee)$. Readers may like to construct the virtual fundamental class by intersection theory on Deligne-Mumford stacks. For this, we take a global resolution of $E^\bullet_M$ ([8, Lemma 2.5]) given by

$$E = \begin{bmatrix} E^{-1} \to E^0 \end{bmatrix}$$

of two term vector bundles such that $E^\bullet_M \cong E$. Then we let $E_i := (E^{-i})^\vee$ and form $E^V = E_0 \to E_1$. We have the following Cartesian diagram

$$\begin{array}{ccc}
    C & \longrightarrow & E_1 \\
    \downarrow & & \downarrow \\
    c_M & \hookrightarrow & [E_1/E_0],
\end{array}$$

where $C \subset E_1$ is a subcone inside the vector bundle $E_1$ which can be taken as the lift of the intrinsic normal cone $c_M$. Then the virtual fundamental class

$$[M]^\text{vir} = 0_{E_1}(C) \in A_{\text{rk}(E)}(M)$$

is the intersection of the cone $C$ with the zero section of the vector bundle $E_1$. The construction of the virtual fundamental class $[M]^\text{vir}$ is a fundamental tool to define enumerative invariants in algebraic geometry for various of moduli spaces $M$, see [9], [74], [65] and [72].
3.6. Moduli space of projective Deligne-Mumford stacks. We recall one result in [10] §6 for the obstruction theory of the moduli space of projective varieties.

Let \( p : \mathcal{M} \rightarrow M \) be a projective, flat morphism between two Deligne-Mumford stacks. The morphism \( p \) is called relative Gorenstein if the relative dualizing complex \( \omega^\bullet_{\mathcal{M}/M} \) is a line bundle \( \omega \). Let \( \mathcal{L}^\bullet_{\mathcal{M}/M} \) be the relative cotangent complex of \( p \). We construct the following complex

\[
E^\bullet := Rp_* \left( \mathcal{L}^\bullet_{\mathcal{M}/M} \otimes \omega \right) [-1].
\]

The Kodaira-Spencer map \( \mathcal{L}^\bullet_{\mathcal{M}/M} \rightarrow p^* \mathcal{L}^\bullet_M[1] \) induces a map

\[
\phi : E^\bullet_M \rightarrow \mathcal{L}^\bullet_M.
\]

**Theorem 3.5.** ([10] Proposition 6.1) Let \( p : \mathcal{M} \rightarrow M \) be a projective, flat and relative Gorenstein morphism of Deligne-Mumford stacks. Assume that the family \( \mathcal{M} \) is universal at every point of \( M \). Then \( \phi : E^\bullet_M \rightarrow \mathcal{L}^\bullet_M \) is an obstruction theory for \( M \). Moreover, if \( E^\bullet_M \) is perfect; i.e., of perfect amplitude contained in \([-1, 0]\), then \( \phi \) is a perfect obstruction theory for \( M \).

**Proof.** The proof is in [10] Proposition 6.1. We provide the proof here for completeness and a later use.

We show an equivalence condition as in Theorem 3.3. Consider a scheme \( T \) and let \( f : T \rightarrow M \) be a morphism, then we have the following Cartesian diagram

\[
\begin{array}{ccc}
T & \xrightarrow{g} & \mathcal{M} \\
\downarrow q & & \downarrow p \\
T & \xrightarrow{f} & M
\end{array}
\]

given by the fiber product. Let \( T \rightarrow T \) be a square-zero extension with ideal sheaf \( J \), then the obstruction to extending \( \mathcal{T} \) to a flat family over \( T \) lies in \( \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{T}/T}, q^*J) \). If the extensions exist, they form a torsor under \( \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{T}/T}, q^*J) \). The flatness of \( p \) implies that \( \mathcal{L}^\bullet_{\mathcal{T}/T} = g^* \mathcal{L}^\bullet_{\mathcal{M}/M} \), we have that

\[
\text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{T}/T}, q^*J) = \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{M}/M}, Rg_*q^*J) = \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{M}/M}, p^*Rf_*J)
\]

and also

\[
\text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{M}/M}, p^*Rf_*J) = \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{M}/M}, Rg_*q^*J) = \text{Ext}^k(g^*\mathcal{L}^\bullet_{\mathcal{M}/M}, J).
\]

The family \( \mathcal{M} \) is universal, which means that the fibers of \( p \) have finite automorphism groups. Therefore, \( E^\bullet_M \) satisfies that \( h^i(E^\bullet_M) = 0 \) for \( i > 0 \) and \( h^0(E^\bullet_M) \) is coherent for \( i = 0, -1 \). The morphism \( \phi : E^\bullet_M \rightarrow \mathcal{L}^\bullet_M \) induces morphisms

\[
\phi_k : \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{T}/T}, q^*J) = \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{M}/M}, J) \rightarrow \text{Ext}^k(\mathcal{L}^\bullet_{\mathcal{M}/M}, J).
\]

Then if \( M \) is a moduli stack, then \( \phi_1 \) is an isomorphism and \( \phi_2 \) is injective. So from Theorem 3.5 \( \phi \) is an obstruction theory.

If \( E^\bullet_M \) is perfect which is of perfect amplitude contained in \([-1, 0]\), then \( \phi \) is a perfect obstruction theory from Definition 3.2. \( \square \)

**Remark 3.6.** If \( p \) is smooth and the relative fiber is of dimension \( \leq 2 \), then it is not hard to see that \( E^\bullet_M \) is a perfect obstruction theory. In the case that the relative fibers are all smooth projective surfaces, the cohomology \( H^*(M, (R^i\mathcal{L}^\bullet_{\mathcal{M}/M} \otimes \omega)) \) calculates the cohomology \( H^*(\mathcal{S}, \mathcal{T}_S) \) for each fiber \( S \) for the morphism \( p \). Let us further assume that all the surfaces in the fibers are of general type which means \( S \) has a finite automorphism group. Then \( M \) is a Deligne-Mumford stack. The cohomology \( H^1(\mathcal{S}, \mathcal{T}_S) \) classifies the deformations for the surface \( S \); and \( H^2(\mathcal{S}, \mathcal{T}_S) \) classifies the obstructions. Since there are no higher dimensional cohomology spaces, the obstruction theory is perfect.

In this paper, we apply Theorem 3.5 in the more general setting for the moduli stack where \( p : \mathcal{M} \rightarrow M \) is the universal family of the moduli of surfaces with semi-log-canonical singularities which is called the KSBA compactification of the moduli space of surfaces of general type.
4. Moduli stack of surfaces of general type

In this section we review the moduli stack of surfaces of general type with only semi-log-canonical (s.l.c.) singularities. The moduli space of varieties of general type has been studied for decades. Our main references are [24], [25], [44]. Let

Definition 4.1. Let S be a projective surface. We say that S has s.l.c. singularities if the following conditions hold:

1. the surface S is reduced, Cohen-Macaulay, and has only double normal crossing singularities (xy = 0) ⊂ \mathbb{A}^3 \setminus \text{finite set of points};
2. we use the notations above. Let the pair (S', \Delta') be the normalization of S with the inverse image of the double curve. Then (S', \Delta') has log canonical singularities;
3. for some N > 0 the N-th reflexive tensor power \omega_S[N] of the dualizing sheaf \omega_S is invertible.

Remark 4.2. Let us recall the type of surface singularities here. Let (S, P) be a \mathbb{Q}-Gorenstein singularity germ. Then there exists N > 0 such that we can write \omega_S[N] \equiv f^* \omega_S[N] \otimes O(\sum N_a E_i), where E_i are the exceptional divisors and all a_i are rational. Then (S, P) is called

1. semi-canonical if a_i \geq 0,
2. semi-log-terminal if a_i > -1,
3. semi-log-canonical if a_i \geq -1.

If (S, P) is normal, then we get the definition of canonical, log-terminal and log-canonical singularity with the above inequality unchanged.

Definition 4.3. A stable surface is a connected projective surface S such that S has s.l.c. singularities and the dualizing sheaf \omega_S is ample.

Let us recall the index one cover for a surface S with s.l.c. singularities as in [27] §2.3], [66] and [44]. Let (S, P) be an s.l.c. surface germ. The index of P ∈ S is the least integer r such that \omega_S[r] is invertible around P. Fix an isomorphism \theta : \omega_S[r] ∼ \mathcal{O}_S, we define

\[ Z := \text{Spec}_O \left( O_S \oplus \omega_S^{[1]} \oplus \cdots \oplus \omega_S^{[r-1]} \right), \]

where the multiplication on \mathcal{O}_Z is defined by the isomorphism \theta. Then \pi : Z → S is a cyclic cover of degree r which is called the index one cover of S. This cover satisfies the properties that the inverse image of the point P is a single point Q ∈ Z; the morphism \pi is étale over S \setminus P; and the surface Z is Gorenstein, which means that Z is Cohen-Macaulay and the dualizing sheaf \omega_Z is invertible. The germ (Z, P) is also s.l.c. This is uniquely determined locally in the étale topology.

Definition 4.4. Let (S, P) be an s.l.c. surface germ. We say a deformation (P ∈ S)\{0 \in T\} is \mathbb{Q}-Gorenstein if it is induced by an equivariant deformation of the index one cover of (P ∈ S). This means there exists a \mathbb{Z}_r-equivariant deformation Z/T of Z whose quotient is S/T.
Let us define the moduli stack of s.l.c. surfaces. Let \( T \) be a scheme of finite type over \( k \). A family of stable surfaces over \( T \) is a flat family \( S \to T \) such that each fiber is a stable surface and \( S/T \) is \( \mathbb{Q} \)-Gorenstein in the sense above; i.e., everywhere locally on \( S \) the family \( S/T \) is induced by an equivariant deformation of the index one cover of the fiber.

**Definition 4.5.** We fix three invariants \( K^2, \chi, \) and \( N \in \mathbb{Z}_{>0} \). Let \( \overline{M} := \overline{M}_{K^2, \chi, N} \) be the moduli functor

\[
\overline{M} : \text{Sch}_k \to \text{Groupoids}
\]

sending

\[
T \mapsto \begin{cases}
S \to T & \text{is a flat } \mathbb{Q} \text{-Gorenstein family of stable } \text{s.l.c. surfaces in Definition 4.4,} \\
\text{for each fiber } S_t, t \in T, \omega[^N_S]_t & \text{is invertible and ample,} \\
\text{for each fiber } S_t, t \in T, K^2_S = K^2_S(\mathcal{O}_S) = \chi, \\
\text{the natural map } \omega[^N_{S/T}]_t \otimes \mathbb{k}(t) & \text{is an isomorphism.}
\end{cases}
\]

where \( \omega[^N_{S/T}]_t = i_* (\omega[^N_{S_0/T}]_t \otimes 1) \) and \( i : S^0 \to S \) is the inclusion of the locus where \( f \) is a Gorenstein morphism. The isomorphism \( \omega[^N_{S/T}]_t \otimes \mathbb{k}(t) \cong \omega[^N_{S_0}]_t \) holds for each \( t \in T \), which implies that \( \omega[^N_{S/T}]_t \) commutes with specialization, and ensures that the moduli space is separated.

From [42 Corollary 5.7] we have that

**Theorem 4.6.** ([42 Corollary 5.7]) The functor \( \overline{M} \) is coarsely represented by a separated algebraic space \( \overline{M} \) of finite type.

If we fix \( K^2, \chi, \) [3] proved the boundedness of families of semi-log-canonical log surfaces of general type, and [28 Theorem 1.1] proved the boundedness of families of semi-log-canonical log varieties of general type of any dimension with fixed volume. Thus, in the surface case there is a uniform bound \( N \in \mathbb{Z}_{>0} \) such that whenever we have a family \( S \to T \) of s.l.c. surfaces such that the generic fiber has invariants \( K^2, \chi, \) the index \( r \) of the special fiber divides \( N \). Thus from [48 Theorem 1.1, Remark 6.3], we have that

**Theorem 4.7.** ([48 Theorem 1.1, Remark 6.3]) For fixed invariants \( K^2, \chi, \) if \( N > 0 \) is large divisible enough, then the functor \( \overline{M} \) is represented by a proper Deligne-Mumford stack \( M := \overline{M}_{K^2, \chi, N} \) of finite type over \( k \) with projective coarse moduli space \( \overline{M} \). In this case we just write \( M := \overline{M}_{K^2, \chi} = \overline{M}_{K^2, \chi, N} \).

### 4.2. Moduli of surfaces of general type with a finite group action.

We go further to define moduli stack of s.l.c. surfaces with finite group actions.

#### 4.2.1. S.L.C. surfaces with finite group action.

Let \( S \) be a surface of general type and \( G \) a finite group. We consider the action of \( G \) on \( S \) and form the quotient Deligne-Mumford stack \( \mathcal{G} = [S/G] \).

Here is one example of surface with a finite group action. Let \( S \subset \mathbb{P}^3 \) be a smooth quintic surface \( \{x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \). Let \( \zeta \in \mu_5 \) be a primitive generator of the cyclic group of order 5. Then we set the group action for the group \( G = (\mu_5)^2 \) with two generators \( \zeta_1, \zeta_2 \) by

\[
\zeta_1 \cdot (x_1, x_2, x_3, x_4) = (\zeta_1 x_1, \zeta_1^{-1} x_2, x_3, x_4),
\]

\[
\zeta_2 \cdot (x_1, x_2, x_3, x_4) = (x_1, x_2, \zeta_2 x_3, \zeta_2^{-1} x_4).
\]

Then \([S/G]\) is a quotient surface.

Let \( S \) be a stable surface; i.e., a surface with only s.l.c. singularities. Then a \( G \)-action on \( S \) is given by

\[
\sigma : G \times S \to S
\]

taken as a homomorphism such that it satisfies the group action conditions.
Proposition 4.8. Let $S$ be a stable surface with a finite group $G$-action. We call $[S/G]$ a global quotient surface Deligne-Mumford stack with only s.l.c. singularities. Then the $G$-action preserves the s.l.c. singularities in the sense that if $(S, P)$ is an s.l.c. germ, then the action locally sends s.l.c. germs to s.l.c. germs.

Proof. It is a good place to recall the classification of surface s.l.c. singularities in [42, Theorem 4.24]. The s.l.c. surface singularities are exactly as follows:

(1) the semi-log-terminal singularities;
(2) the Gorenstein surfaces such that every Gorenstein surface $S$ is either semi-canonical (which is smooth, normal crossing, a pinch point or a DuVal singularity), or has simple elliptic singularities, cusp, or degenerate cusp singularities;
(3) the $Z_2, Z_3, Z_4, Z_6$ quotients of simple elliptic singularities;
(4) the $Z_2$ quotient of cusps and degenerate cusps.

The semi-log-terminal surface singularities are exactly as follows:

(1) the quotient of $\mathbb{A}^2_k$ by Brieskorn [14];
(2) normal crossing or pinch points;
(3) $(xy = 0)$ modulo the group action given by $x \mapsto \zeta^a x, y \mapsto \zeta^b y$, and $z \mapsto \zeta z$, where $\zeta$ is a primitive $r$-th root of unity and $(a, r) = 1, (b, r) = 1$;
(4) $(xy = 0)$ modulo the group action $x \mapsto \zeta^a x, y \mapsto x$, and $z \mapsto \zeta z$, where $\zeta$ is a primitive $r$-th root of unity and $4| r, (a, r) = 2$;
(5) $x^2 = y z^2$ modulo the group action given by $x \mapsto \zeta^{1+a} x, y \mapsto \zeta^a y$, and $z \mapsto \zeta^2 z$, where $\zeta$ is a primitive $r$-th root of unity and $r$ odd, and $(a, r) = 1$;

see [42, Theorem 4.22, 4.23, 4.24].

The $G$-action on $S$ induces the action on s.l.c. germs. If $(S, P)$ and $(S, P')$ are two s.l.c. germs, then the $G$-action induces a morphism $(S, P) \to (S, P')$ on the s.l.c. germs which is a $G$-equivariant morphism under the above classification. □

4.2.2. Q-Gorenstein deformations. Next we generalize the Q-Gorenstein deformation of s.l.c. surfaces to the case with finite group actions. Everything is a routine generalization and we only list the basic results.

Definition 4.9. Let $S$ be a stable surface endowed with a finite group $G$-action, and $(S, P)$ be an s.l.c. surface germ. A $G$-equivariant deformation $(P \in S)/ (0 \in T)$ is $Q$-Gorenstein if it is induced by an equivariant deformation of the index one cover of $(P \in S)$ compatible with the $G$-action. This means there exists a $\mu_N$-equivariant deformation $Z/T$ of $Z$ whose quotient is $S/T$. Both $Z$ and $S$ admit $G$-actions compatible with the local $\mu_N$-action.

Here is a result in [42] for one-parameter deformation family which automatically holds for $G$-equivariant deformations.

Lemma 4.10. ([27, Lemma 3.4]) Let $S/(0 \in T)$ be a $G$-equivariant flat family of s.l.c. surfaces over a curve $T$. Assume that the generic fiber is canonical, which has only Du Val singularities and the canonical line bundle $K_S$ is $Q$-Cartier. Then $S/T$ is $Q$-Gorenstein.

We collect some facts for the $G$-equivariant Q-Gorenstein deformations. We omit the $G$-actions. For a flat family $S/T$ of s.l.c. surfaces, let $\omega_{S/T}$ be the relative dualizing sheaf. From [42, §5.4], [66, Appendix to §1] and [27, §3.1], we have that

$$\omega^{[N]}_{S/T} := (\omega_{S/T}^{[N]} \oplus i_* (\omega_{S_0/T}^{[N]})),$$

where $i : S^0 \to S$ is the inclusion of the Gorenstein locus; i.e., the locus where the relative dualizing sheaf $\omega_{S/T}$ is invertible. Suppose that $\omega_{S/T}^{[N]}$ is invertible, and if $(S, P)$ is an s.l.c. surface germ with index $r$ in the family $S/T$, then the index $r[N]$.

From [27, Lemma 3.5], let $(S, P)$ be an s.l.c. surface germ with index $r$, and $Z \to S$ be the index one cover under the cyclic group $Z_r$-action. Let $Z/(0 \in T)$ be a $Z_r$-equivariant
deformation of $Z$ inducing a $\mathbb{Q}$-Gorenstein deformation $S/(0 \in T)$ of $S$, then we have that

$$Z = \text{Spec}_{O_S}(O_S \oplus \omega^{[1]}_{S/T} \oplus \cdots \oplus \omega^{[r-1]}_{S/T}),$$

where the multiplication of $O_Z$ is given by fixing a trivialization of $\omega^{[r]}_{S/T}$. If the deformation $S/(0 \in T)$ admits a $G$-action, then every power $\omega^{[i]}_{S/T}$ is endowed with a $G$-action and the index one cover is also endowed with a $G$-action making this $Z/(0 \in T)$ $G$-equivariant.

The index one cover of the s.l.c. germ $(S, P)$ is uniquely determined in the étale topology. These data of index one covers everywhere locally on $S/T$ glue to define a Deligne-Mumford stack $\mathcal{S}/T$ which we call the canonical covering (Hacking) stack, or the index one covering Deligne-Mumford stack associated with $S/T$. The dualizing sheaf $\omega_{\mathcal{S}/T}$ is invertible.

Let us collect some deformation and obstruction facts about the index one covering Deligne-Mumford stacks. We replace $A$ by a $k$-algebra $A$, and consider infinitesimal extensions $A' \rightarrow A$. Let $S/A$ be a $\mathbb{Q}$-Gorenstein family of s.l.c. surfaces with $G$-action and $\mathcal{S}/A$ be its index one covering Deligne-Mumford stack.

**Definition 4.11.** A deformation of $\mathcal{S}/A$ over $A'$ is a Deligne-Mumford stack $\mathcal{S}'/A'$ which is flat over $A'$ such that $\mathcal{S}' \times_{\text{Spec} A'} \text{Spec} A \cong \mathcal{S}$.

Equivalently a deformation $\mathcal{S}'/A'$ of $\mathcal{S}/A$ is a sheaf $O_{\mathcal{S}'}$ of flat $A'$-algebras on the étale site of $\mathcal{S}$ such that $O_{\mathcal{S}'} \otimes_{A'} A = O_{\mathcal{S}}$. Thus the deformation theory of $\mathcal{S}$ is controlled by the cotangent complex $L_{\mathcal{S}/A}$ as in [27]. Let us fix the following notations.

Let $A$ be a $k$-algebra and $J$ be a finite $A$-module. For a flat family $S/A$ of schemes over $A$ let $L_{S/A}$ be the relative cotangent complex. Then we define

$$T^i(S/A, J) := \text{Ext}^i(L_{S/A}, O_S \otimes_A J),$$

and

$$T^i(S/A, J) := \mathcal{E}xt^i(L_{S/A}, O_S \otimes_A J).$$

The groups $T^i(S/A, J)$ control the deformation and obstruction theory of $S/A$.

We are actually working on the $G$-equivariant $\mathbb{Q}$-Gorenstein deformation theory of $S/A$. Thus for the $\mathbb{Q}$-Gorenstein family $S/A$ of s.l.c. surfaces, let $\mathcal{S}/A$ be the family of the index one covering Deligne-Mumford stacks, and $\pi : \mathcal{S} \rightarrow S$ be the map to its coarse moduli space. Define

$$T^i_{\mathbb{Q}}(S/A, J) := \text{Ext}^i(L_{\mathcal{S}/A}, O_{\mathcal{S}} \otimes_A J),$$

and

$$T^i_{\mathbb{Q}}(S/A, J) := \pi_* \mathcal{E}xt^i(L_{\mathcal{S}/A}, O_{\mathcal{S}} \otimes_A J).$$

We denote by $T^i_{\mathbb{Q}}(S/A, J)^G$ and $T^i_{\mathbb{Q}}(S/A, J)^G$ their $G$-invariant parts of the extension groups.

The following two results are proven by P. Hacking [27, Proposition 3.7, Theorem 3.9] which automatically work in the $G$-equivariant case.

**Proposition 4.12.** ([27, Proposition 3.7]) Let $S/A$ be a $G$-equivariant $\mathbb{Q}$-Gorenstein family of s.l.c. surfaces and $\mathcal{S}/A$ be its corresponding index one covering Deligne-Mumford stack. Consider the infinitesimal extension $A' \rightarrow A$, and let $S'/A'$ be a $G$-equivariant $\mathbb{Q}$-Gorenstein deformation of $S/A$, and $\mathcal{S}'/A'$ be the corresponding index one covering Deligne-Mumford stack. Then, there exists a one-to-one correspondence from the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformation families of $S/A$ over $A'$ to the set of isomorphism classes of flat deformation families $\mathcal{S}'/A'$ over $A'$.

**Proposition 4.13.** Let $S_0/A_0$ be a $G$-equivariant $\mathbb{Q}$-Gorenstein family of s.l.c. surfaces, and let $J$ be a finite $A_0$-module. Then we have that
Proposition 4.14. Let $S$ be a s.l.c. surface satisfying the following conditions:

(1) the set of isomorphism classes of $G$-equivariant $\mathbb{Q}$-Gorenstein deformations of $S_0/A_0$ over $A_0 + f$ is naturally an $A_0$-module and is canonically isomorphic to $T^1_{\mathcal{O}_G}(S/A,f)^G$. Here $A_0 + f$ means the ring $A_0[f]$ with $f^2 = 0$;

(2) let $A' \to A \to A_0$ be the infinitesimal extensions, and $f$ be the kernel of $A' \to A$. Let $S/A$ be a $G$-equivariant $\mathbb{Q}$-Gorenstein deformation of $S_0/A_0$. Then we have

(a) there exists a canonical element $ob(S/A,A') \in T^2_{\mathcal{O}_G}(S/A,f)^G$ called the obstruction class. It vanishes if and only if there exists a $G$-equivariant $\mathbb{Q}$-Gorenstein deformation $S'/A'$ of $S/A$ over $A'$.

(b) if $ob(S/A,A') = 0$, then the set of isomorphism classes of $G$-equivariant $\mathbb{Q}$-Gorenstein deformations $S'/A'$ is an affine space underlying $T^1_{\mathcal{O}_G}(S_0/A_0,f)^G$.

Proof. This is a basic result of deformation and obstruction theory of algebraic varieties; see [27, Theorem 3.9] and [35].

4.2.3. Higher obstruction spaces of the index one covering Deligne-Mumford stack. Let $S$ be an s.l.c. surface, and let $\mathfrak{S} \to S$ be the index one covering Deligne-Mumford (Hacking) stack in $\mathfrak{S}$ of stable log surfaces. The spaces $T^i_{\mathcal{O}_G}(S) = \text{Ext}^i(\mathcal{L}_\mathfrak{S}, \mathcal{O}_\mathfrak{S})$ can be calculated by the local to global spectral sequence

$$E_2^{p,q} = H^p(T^q_{\mathcal{O}_G}(S)) \Longrightarrow T^{p+q}_{\mathcal{O}_G}(S),$$

where $T^q_{\mathcal{O}_G}(S) = \pi_*\{E\mathcal{xt}^q(\mathcal{L}_\mathfrak{S}, \mathcal{O}_\mathfrak{S})\}$ and $\pi : \mathfrak{S} \to S$ is the map to its coarse moduli space. The spaces $T^i_{\mathcal{O}_G}(S)$ for $i \geq 3$ classify the higher obstruction spaces for the $\mathbb{Q}$-Gorenstein deformations of $S$. We have that

Proposition 4.14. Let $S$ be an s.l.c. surface satisfying the following conditions:

(1) $S$ is Kawamata-log-terminal (k.l.t.); or

(2) the possible simple elliptic singularity, the cusp and the degenerate cusp singularity of $S$, and the possible $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ quotients of the simple elliptic singularity, the $\mathbb{Z}_2$-quotient of the cusp and the degenerate cusp singularity of $S$ all have embedded dimension at most 4,

then the higher obstruction spaces $T^i_{\mathcal{O}_G}(S)$ vanish for $i \geq 3$.

Proof. From the classification of semi-log-canonical surface singularities in Proposition 4.8 and known fact in birational geometry, a k.l.t. surface $S$ only has cyclic quotient singularities, cyclic quotients of the normal crossing, and pinch point singularities, or DuVal singularities. Then if the surface $S$ admits a $\mathbb{Q}$-Gorenstein deformation, from [42, Proposition 3.10], the cyclic quotient singularities must have the form

$$\text{Spec } \mathbb{k}[x,y]/\mu_{2s},$$

where $\mu_{2s} = \langle \eta \rangle$ and there exists a primitive $r^2s$-th root of unity $\eta$ such that the action is given by

$$\alpha(x,y) = (\eta^r x, \eta^d y),$$

where $(d,r) = 1$. Thus the index one cover of $S$ locally has the quotient

$$\text{Spec } \mathbb{k}[x,y]/\mu_{rs},$$

given by $\alpha'(x,y) = (\eta^ry, (\eta^r)^{s-1}y)$, which is an $A_{rs-1}$-singularity, and therefore is l.c.i. The cotangent sheaf $\mathcal{L}_\mathfrak{S}$ only has two terms concentrated in degrees $-1,0$. Therefore, the tangent sheaf $T^q_{\mathcal{O}_G}(S)$ is zero for $q \geq 2$. By the local to global spectral sequence $T^1_{\mathcal{O}_G}(S) = 0$ for $i \geq 3$.

If an s.l.c. surface $S$ has a simple elliptic singularity, a cusp or a degenerate cusp singularity with embedded dimension at most 4, then from [45, Theorem 3.13], and [70], these singularities must be locally complete intersection singularities. For the s.l.c. surfaces with $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ quotients of a simple elliptic singularity, a cusp and or a degenerate cusp singularity such that the local embedded dimension $\leq 4$, their index one covers $\mathfrak{S}$
modulo equivalence. The Conditions (1)-(5) above are given by Remark 4.15.

Remark 4.15. Recall for an s.l.c. surface $S$, the tangent sheaves $T^q_{\mathcal{QC}}(S)$ satisfy the following properties (see for example [27]):

1. $T^0_{\mathcal{QC}}(S) = T_S$ is the tangent sheaf of $S$;
2. $T^i_{\mathcal{QC}}(S)$ supports on singular locus of $S$, which can be calculated as follows: if locally $\mathcal{S}$ is given by $[V/\mathbb{Z}_l] \rightarrow U$ for an open subset $U \subset S$, we have

$$T^1_{\mathcal{QC}}(S) = \left( p_*\mathcal{E}xt^1(\mathcal{O}_V, \mathcal{O}_V) \right)^{Z_r}$$

where $p : V \rightarrow U$ is the natural morphism;
3. $T^2_{\mathcal{QC}}(S)$ supports on the locus of the index one cover $Z$ which is not a local complete intersection;
4. $T^q_{\mathcal{QC}}(S)$ for $q \geq 3$ may support on non-complete intersection singularities of $S$.

Therefore, from the local to global spectral sequence, to determine the higher obstruction spaces $T^i_S$ it is sufficient to know $T^q_{\mathcal{QC}}(S)$ for $q \geq 3$ since for any coherent sheaf $F$ the cohomology spaces $H^p(S, F)$ only survive for $p = 1, 2$. From [27], if a cusp or a degenerate cusp singularity has embedded dimension $\geq 5$, then the singularity is definitely not a complete intersection singularity.

There should exist an example of degenerate cusp singularity $(S, p)$ such that its embedded dimension is $\geq 5$, and the tangent sheaves $T^q_{\mathcal{QC}}(S) \neq 0$ for some $q \geq 3$. It is likely that for a cusp or degenerate singularity germ $(S, p)$ with embedded dimension $\geq 5$, if the tangent sheaf $T^q_{\mathcal{QC}}(S, O_S) \neq 0$, then $T^q_{\mathcal{QC}}(S, O_S) \neq 0$; see [37]. In this situation, the obstruction spaces $T^q_{\mathcal{QC}}(S)$ are not zero for $i \geq 3$. These higher obstruction spaces for the s.l.c. surface $S$ imply that there is no natural Behrend-Fantechi style perfect obstruction theory on the moduli stack of surfaces of general type containing s.l.c. surfaces with such type of singularities.

From Remark 4.15 we make the following condition for s.l.c. surfaces.

Condition 4.16. If an s.l.c. surface $S$ has the following surface singularity $(S, x)$: a simple elliptic singularity, a cusp or a degenerate cusp singularity, or the $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ quotients of the simple elliptic singularity, and the $\mathbb{Z}_2$ quotient of a cusp or a degenerate cusp singularity, then $(S, x)$ has embedded dimension at most 4.

4.2.4. The moduli stack of s.l.c. surfaces with $G$-action. We define the moduli functor of s.l.c. surfaces with a finite group $G$-action. We still fix $K^2, \chi, N \in \mathbb{Z}_{>0}$. Let

$$\mathcal{M}^G := \mathcal{M}^G_{K^2, \chi, N} : \text{Sch}_k \rightarrow \text{Groupoids}$$

be the moduli functor sending

$$T \mapsto \left\{ \begin{array}{l}
\bullet \mathcal{S} \xrightarrow{f} T \text{ is a } G\text{-equivariant } \mathbb{Q}\text{-Gorenstein deformation family of stable s.l.c. surfaces;}
\bullet \text{ Conditions (1)-(5) hold for each geometric fiber;}
\bullet \text{ For each geometric point } t \in T, \text{ we have }
\omega_{S/T}^N \otimes k(t) \rightarrow \omega_{S_t}^N \\
\text{is an isomorphism, where } \omega_{S/T}^N = j_* (\omega_{S/T}^N), \text{ and } j : S^0 \rightarrow S \text{ is the inclusion of the locus where } f \text{ is Gorenstein.}
\end{array} \right\}$$

modulo equivalence. The Conditions (1)-(5) above are given by

1. each fiber of $f : \mathcal{S} \rightarrow T$ is a reduced projective surface with $G$-action, i.e., the quotient stack $[S_t/G]$;
2. each $S_t$ is connected with only s.l.c. singularities with a $G$-action;
3. the sheaf $\omega_{S_t}^N$ which is defined by $\omega_{S_t}^N = j_* (\omega_{S_t}^N)$ and $j : (S_t)^0 \rightarrow S_t$ is the inclusion of Gorenstein locus of $S_t$, is a $G$-equivariant ample line bundle;
(4) $K^2_{\nu_i} = \frac{1}{2\pi} (\omega^{[N]}_{\nu_i} \cdot \omega^{[N]}_{\nu_i}) = K^2$ for any $t \in T$;
(5) $\chi(O_{S_t}) = \chi$ for $t \in T$.

We have that

**Theorem 4.17.** When fixing $K^2, \chi, N \in \mathbb{Z}_{>0}$, the functor $\overline{\mathcal{M}}^G$ is represented by a Deligne-Mumford stack $M := \overline{\mathcal{M}}^G_{K^2, \chi, N}$ of finite type over $k$. Suppose that $N > 0$ is large divisible enough, then the stack $\overline{\mathcal{M}}^G_{K^2, \chi} := \overline{\mathcal{M}}^G_{K^2, \chi, N}$ is a proper Deligne-Mumford stack with projective coarse moduli space.

**Proof.** Since we consider s.l.c. surfaces with a finite group $G$-action, the moduli stack $M$ should exist as a closed substack of the stack $\overline{\mathcal{M}}^G_{K^2, \chi, N}$. Therefore, we get all the results in the theorem immediately.

We choose to provide more details here. From [3, 28 Theorem 1.1], after fixing the data $K^2, \chi$, any Q-Gorenstein family of s.l.c. surfaces with fixed volume is bounded, therefore there exists a uniform bound $N > 0$ such that $\omega^{[N]}_{S/T}$ is invertible for any flat Q-Gorenstein family $S \to T$ of s.l.c. surfaces. Note that [3] did the case of surfaces which is exactly what we want. [28 Theorem 1.1] proved the case of higher dimensional log general type varieties. Therefore, from [14 §4.21], to prove $M$ is a Deligne-Mumford stack, one needs to show that $M$ has representable and unramified diagonal, and there is a smooth étale surjection from a scheme of finite type to $M$.

We first show that the diagonal morphism $M \to M \times_k M$ is representable and unramified. Let $(f : S \to T), (f' : S' \to T)$ be two objects in $\overline{\mathcal{M}}^G(T)$. It is sufficient to show that the isomorphism functor $\text{Isom}_T(S, S')$ is represented by a quasi-projective group scheme over $T$. But this is just from [46 Proposition 6.8]. Since we only consider stable surfaces (while [46] studied the more general case of log stable varieties), the global line bundle $L$ in [46, Definition 6.2, Proposition 6.8] for the family $(f : S \to T)$ is just the invertible sheaf $\omega^{[N]}_{S/T}$. The first half of the proof in [46 Proposition 6.8] implies that the isomorphism functor $\text{Isom}_T(S, S')$ is represented by a quasi-projective group scheme over $T$.

To prove that there exists a smooth étale surjection from a scheme $\mathcal{E}$ of finite type to $M$, from [46 Proposition 6.11], we consider the Hilbert scheme $\text{Hilb}_{K^2, \chi}$ parametrizing closed two dimensional subschemes in a higher dimensional projective space with the same Hilbert polynomial determined by the invariants $K^2, \chi$. After fixing the necessary conditions for the stable s.l.c. surfaces in $\text{Hilb}_{K^2, \chi}$, techniques in [45 Theorem 10, Definition-Lemma 33] and [46 Proposition 6.11] imply that there exists a scheme $\mathcal{E}$ and a smooth étale morphism $\mathcal{E} \to M$. Thus, $M$ is a Deligne-Mumford stack of finite type over $k$.

If $N$ is large divisible enough, the properness of the stack $M$ is just from the boundedness result of [28 Theorem 1.1]. Thus, from the Nakai-Moishezon criterion, for any family $(f : S \to T)$ of stable s.l.c. surfaces we need to show that, for a large divisible enough $N > 0$, the determinant $\det(f_* \omega^{[N]}_{S/T})$ of the pushforward of the relative invertible sheaf $\omega^{[N]}_{S/T}$ is big. This is obtained in [45 Theorem 7.1, Corollary 7.3]. From [46 Theorem 1.1, Remark 6.3, Corollary 7.3], the Deligne-Mumford stack $M$ has a projective coarse moduli space. \hfill $\square$

5. MODULI STACK OF INDEX ONE COVERS

In this section we construct an obstruction theory on the moduli stack $\mathcal{M}^{\text{ind}} := \overline{\mathcal{M}}^{\text{ind}, G}_{K^2, \chi, N}$ of index one covers over one connected component $M = \overline{\mathcal{M}}^G_{K^2, \chi, N}$ of the moduli stack of s.l.c. surfaces with a finite group $G$-action. The obstruction theory is not perfect in general, but in some nice situation such that there is no higher obstruction spaces for the s.l.c. surfaces the obstruction theory is perfect.
5.1. The moduli space of index one covers. Let $G$ be a finite group. Recall from Section 4.2.2 a $G$-equivariant $\mathbb{Q}$-Gorenstein deformation family $\mathcal{S} \to T$ of s.l.c. surfaces is the same as the $G$-equivariant deformation $\mathcal{S} \to T$ of the index one covering Deligne-Mumford stacks. There is a canonical morphism $p : \mathcal{S} \to \mathcal{S}$ which makes the following diagram

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\pi} & \mathcal{S} \\
\downarrow{p} & & \downarrow{p} \\
\mathcal{T} & \xrightarrow{=} & \mathcal{T} \\
\end{array}
\]

commute. The scheme $\mathcal{S}$ is the coarse moduli space of the Deligne-Mumford stack $\mathcal{S}$. Thus, the canonical correspondence motivates us to define the moduli functor

$$M^{\text{ind}} = \overline{M}^{\text{ind}, G}_{K^2, \chi, N} : \text{Sch}_k \to \text{Groupoids}$$

which sends

$$T \mapsto \{f : \mathcal{S} \to T\}$$

where $\{f : \mathcal{S} \to T\}$ represents the isomorphism classes of families of index one covering Deligne-Mumford stacks $\mathcal{S} \to T$. The coarse moduli space of the family $\{f : \mathcal{S} \to T\}$ must satisfy the conditions in (4.2.1).

**Theorem 5.1.** The functor $M^{\text{ind}}$ has representable and unramified diagonal, therefore, is represented by a fine Deligne-Mumford stack $M^{\text{ind}}$. Moreover, there is a canonical isomorphism

$$f : M^{\text{ind}} \to M.$$ 

The isomorphism $f$ induces an isomorphism on the coarse moduli spaces.

Fixing $K^2, \chi$, if $N$ is large divisible enough, then the stack $M^{\text{ind}}$ is a proper Deligne-Mumford stack with projective coarse moduli space, and the isomorphism $f : M^{\text{ind}} \to M$ induces an isomorphism on the projective coarse moduli spaces.

**Proof.** Every s.l.c. surface and its index one covering Deligne-Mumford stack admit $G$-actions making the families $G$-equivariant. In the following we omit the $G$-action. We first show that the diagonal morphism

$$M^{\text{ind}} \to M^{\text{ind}} \times_k M^{\text{ind}}$$

is representable and unramified. Let $(f : \mathcal{S} \to T)$ and $(f' : \mathcal{S}' \to T)$ be two objects in $M^{\text{ind}}(T)$, then the isomorphism functor of the two families $\text{Isom}_T(\mathcal{S}, \mathcal{S}')$ is represented by a quasi-projective group scheme $\text{Isom}_T(\mathcal{S}, \mathcal{S}')$ over $T$. We prove this statement here. Let $(f : \mathcal{S} \to T)$ and $(f' : \mathcal{S}' \to T)$ be the $\mathbb{Q}$-Gorenstein families of the corresponding s.l.c. surfaces over $T$. From the proof of [46, Proposition 6.8] and Theorem 4.17, the isomorphism functor $\text{Isom}_T(\mathcal{S}, \mathcal{S}')$ is represented by a quasi-projective group scheme $\text{Isom}_T(\mathcal{S}, \mathcal{S}')$ over $T$. The canonical morphisms $\mathcal{S} \to \mathcal{S}$ and $\mathcal{S}' \to \mathcal{S}'$ are maps to their coarse moduli spaces. Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{S} & \cong & \mathcal{S}' \\
\downarrow{=} & & \downarrow{=} \\
\mathcal{S} & \cong & \mathcal{S}' \\
\end{array}
\]

any isomorphism $\cong \mathcal{S}'$ induces an isomorphism $\cong \mathcal{S}'$ on the coarse moduli spaces. Any isomorphism $\cong \mathcal{S}'$ of families of $\mathbb{Q}$-Gorenstein deformations implies the isomorphism $\cong \mathcal{S}'$. Therefore, the functor $\text{Isom}_T(\mathcal{S}, \mathcal{S}')$ is represented by a quasi-projective group scheme $\text{Isom}_T(\mathcal{S}, \mathcal{S}')$ and is also unramified over $T$ since its geometric fibers are finite (due to the automorphic group of each fiber $\cong$ is finite).
From [46, Proposition 6.11] and Theorem 4.17, there is a cover \( \varphi : \mathscr{U} \to M \) which is an \( \acute{e} \text{tale} \) surjective morphism onto \( M \) where \( \mathscr{U} \) is a scheme of finite type. This is because \( M \) is a projective Deligne-Mumford stack. Also from the construction of the moduli functor there is a canonical morphism \( f : M^{\text{ind}} \to M \) of stacks, which sends every flat family \( f : \mathcal{S} \to T \) of index one covering Deligne-Mumford stacks to the corresponding Q-Gorenstein deformation family \( S \to T \) of s.l.c. surfaces.

We construct the following diagram

\[
\begin{array}{ccc}
\mathscr{U} & \xrightarrow{\varphi} & M^{\text{ind}} \\
\downarrow{\varphi'} & & \downarrow{f} \\
M & & \\
\end{array}
\]

For each \( T = \text{Spec}(A) \to \mathscr{U} \), the Q-Gorenstein deformation family \( S \to T \) of the s.l.c. surfaces and the corresponding family \( \mathcal{S} \to T \) of index one covering Deligne-Mumford stacks induce the following diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\varphi'} & M^{\text{ind}} \\
\downarrow{\varphi} & & \downarrow{f} \\
M & & \\
\end{array}
\]

This induces the diagram (5.1.1). Thus, taken as Deligne-Mumford stacks, \( M^{\text{ind}} \) and \( M \) share the same cover \( \mathscr{U} \).

Now we show that the morphism \( f : M^{\text{ind}} \to M \) is proper, since from the diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{f^{\text{ind}}} & M^{\text{ind}} \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \xrightarrow{f^{\text{ind}}} & M \\
\end{array}
\]

where \( R \) is a valuation ring and \( K \) is the field of fractions, any family \( \{ S \to \text{Spec}(R) \} \) of s.l.c. surfaces corresponds to a unique flat family \( \{ \mathcal{S} \to \text{Spec}(R) \} \) of index one covering Deligne-Mumford stacks and the above dotted arrow exists and is unique. The morphism \( f : M^{\text{ind}} \to M \) is also quasi-finite, since for each geometric point \( S = \text{Spec}(k) \in M \), there is a unique \( \mathcal{S} \in M^{\text{ind}} \) in the preimage. Therefore, the morphism \( f : M^{\text{ind}} \to M \) is finite. To prove that the Deligne-Mumford stack \( M^{\text{ind}} \) is isomorphic to the Deligne-Mumford stack \( M \), it is sufficient to show that for any s.l.c. surface \( S \), the automorphism group \( \text{Aut}(S) \) is isomorphic to the automorphism group \( \text{Aut}(\mathcal{S}) \) of its index one covering Deligne-Mumford stack \( \mathcal{S} \to S \). From the canonical construction of the index one cover in §4.1 any automorphism \( \sigma : \mathcal{S} \to \mathcal{S} \) of the index one covering Deligne-Mumford stack \( \mathcal{S} \) induces an automorphism \( \sigma' : S \to S \). Thus, we get a map

\[ g : \text{Aut}(\mathcal{S}) \to \text{Aut}(S). \]

Conversely, for any automorphism \( \sigma' : S \to S \), from the canonical construction of the index one cover, we get an \( \sigma : \mathcal{S} \to \mathcal{S} \). Thus, we get a map

\[ h : \text{Aut}(S) \to \text{Aut}(\mathcal{S}). \]

The canonical construction of the index one cover implies that \( g \circ h = 1, h \circ g = 1 \). Thus we get \( \text{Aut}(S) \cong \text{Aut}(\mathcal{S}) \).

The canonical morphism \( f : M^{\text{ind}} \to M \) induces a bijection on the coarse moduli spaces since the index one covering Deligne-Mumford stack \( \mathcal{S} \) has coarse moduli space \( S \). If \( N \) is large divisible enough, then the stack \( M \) is a proper Deligne-Mumford stack with projective coarse moduli space. Therefore the stack \( M^{\text{ind}} \) is a proper Deligne-Mumford stack with
projective coarse moduli space and the isomorphism \( f : M^{\text{ind}} \to M \) induces a bijection on the projective coarse moduli spaces.

Remark 5.2. The author should point out that in the paper [1], Abramovich-Hassett has studied the moduli functor of index one covers and constructed the moduli stack of the index one covers of stable varieties.

Corollary 5.3. Let \( M \) be a connected component of the moduli stack of stable general type surfaces with invariants \( K_S^2, \chi, N \). If each s.l.c. surface \( S \) in \( M \) has only l.c.i. singularities, then the moduli stack \( M^{\text{ind}} \) of index one covers is just the moduli stack \( M \).

Proof. This is a special case. If an s.l.c. surface has at most l.c.i. singularities, it is Gorenstein and the dualizing sheaf \( \omega_S \) is a line bundle. From the construction in Section 4.2.2, the index one covering Deligne-Mumford stack \( \mathcal{S}_t \) is just \( S \). Therefore, from the construction of the moduli functor \( M^{\text{ind}} \), \( M^{\text{ind}} \) is the same as \( M \) as Deligne-Mumford stacks.

\[ \square \]

5.2. Obstruction theory. Let \( M \) be one connected component of the moduli stack of \( G \)-equivariant s.l.c. surfaces with fixed invariants \( K_S^2 = K, \chi(O_S) = \chi \) and \( N \in \mathbb{Z}_{>0} \) as in Theorem 4.17. Still from Theorem 4.17 there exists a universal family for the moduli stack \( p : \mathcal{M} \to M \), since the stack is a fine moduli stack. From Theorem 5.1, there also exists a universal family \( p^{\text{ind}} : \mathcal{M}^{\text{ind}} \to M^{\text{ind}} \), and a commutative diagram

\[ (5.2.1) \]

Lemma 5.4. The universal family \( p^{\text{ind}} : \mathcal{M}^{\text{ind}} \to M^{\text{ind}} \) is projective, flat and relative Gorenstein. Therefore the relative dualizing sheaf \( \omega_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \) is invertible.

Proof. Since \( p^{\text{ind}} \) is a universal family for the moduli stack \( M^{\text{ind}} \), it is flat and projective. The relative dualizing sheaf \( \omega_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \) is invertible since it gives the dualizing sheaf \( \omega_{\mathcal{S}_t} \) of the canonical index one covering Deligne-Mumford stack \( \mathcal{S}_t \) for each geometric point \( t \in M^{\text{ind}} \) and \( \omega_{\mathcal{S}_t} \) is invertible (due to \( \mathcal{S}_t \) Gorenstein).

\[ \square \]

Remark 5.5. In general, for the universal family \( p : \mathcal{M} \to M \), the relative dualizing sheaf \( \omega_{\mathcal{M}/M} \) is not a line bundle since the relative dualizing sheaf \( \omega_{\mathcal{M}/M} \) is not a line bundle on the non-Gorenstein locus.

Let \( \mathcal{L}_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \) be the relative cotangent complex of \( p^{\text{ind}} \) and we consider

\[ E^{\text{ind}}_{M^{\text{ind}}} := R^p_{\text{ind}} \left( \mathcal{L}_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \otimes \omega_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \right)[1]. \]

Here the relative dualizing sheaf \( \omega_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \) satisfies the property

\[ \omega_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}}(p^{\text{ind}})^{-1}(\mathcal{S}_t) \cong \omega_{\mathcal{S}_t}, \]

where the dualizing sheaf \( \omega_{\mathcal{S}_t} \) of the index one covering Deligne-Mumford stack \( \mathcal{S}_t \to S_t \), which is locally given by \( \omega_{S_t}^{[r]} \) at a singularity germ \( (r \text{ is the index of the singular germ}) \), is invertible.

Theorem 5.6. The complex \( E^{\text{ind}}_{M^{\text{ind}}} \) defines an obstruction theory (in the sense of Behrend-Fantechi) \( \phi^{\text{ind}} : E^{\text{ind}}_{M^{\text{ind}}} \to \mathcal{L}^{\text{ind}}_{M^{\text{ind}}} \) induced by the Kodaira-Spencer map \( \mathcal{L}_{\mathcal{M}^{\text{ind}}/M^{\text{ind}}} \to (p^{\text{ind}})^{*}\mathcal{L}_{M^{\text{ind}}}[1] \).
Proof. From Lemma 5.14 the universal family \( \pi^{\text{ind}} : \mathcal{M}^{\text{ind}} \to M^{\text{ind}} \) is a projective, flat, relative Gorenstein morphism between Deligne-Mumford stacks. Also \( M^{\text{ind}} \) is a fine moduli stack. Thus, \( q^{\text{ind}} : E_{M^{\text{ind}}}^{\text{ind}} \to \mathcal{L}^{\text{ind}}_{M^{\text{ind}}} \) gives an obstruction theory from Theorem 3.6 (also see [10, Proposition 6.1]). For completeness of the analysis of local deformation and obstruction theory of s.l.c. surfaces, we include the details here.

The basic observation is that the complex

\[
\tilde{E}^\bullet_{M^{\text{ind}}} := Rp_{s}^{\text{ind}} \left( \mathcal{L}^{\text{ind}}_{M^{\text{ind}}} \otimes \omega_{M^{\text{ind}}} \right),
\]

when restricted to a point \( t \in M^{\text{ind}} \), calculates the cohomology spaces \( H^*(\mathcal{E}_t, T_{\mathcal{E}_t})^G = T^*_G(S_t, \mathcal{O}_{S_t})^G \) for the index one covering Deligne-Mumford stack \( \mathcal{E}_t \). Since it is of general type, \( \dim H^0(\mathcal{E}_t, T_{\mathcal{E}_t}) = 0 \). Over a point \( t \in M^{\text{ind}} \), the complex \( \tilde{E}^\bullet_{M^{\text{ind}}} \) gives

\[
\tilde{E}^\bullet_{M^{\text{ind}}}|_t = Rp^{\text{ind}}_s(\mathcal{L}^\text{ind}_{\mathcal{E}_t} \otimes \omega_{\mathcal{E}_t}),
\]

and

\[
\left( \tilde{E}^\bullet_{M^{\text{ind}}}|_t \right)^\vee = Rp^{\text{ind}}_s(\mathcal{L}^\text{ind}_{\mathcal{E}_t} \otimes \mathcal{O}_{\mathcal{E}_t}).
\]

Thus \( \left( \tilde{E}^\bullet_{M^{\text{ind}}}|_t \right)^\vee \) is given by \( p^{\text{ind}}_s \mathcal{E}xt^t(\mathcal{L}^\text{ind}_{\mathcal{E}_t} \otimes \mathcal{O}_{\mathcal{E}_t}) \) which was studied in [27] §3, Proposition 4.12 and Proposition 4.13. Therefore, the cohomology spaces of \( \left( \tilde{E}^\bullet_{M^{\text{ind}}}|_t \right)^\vee \) give

\[
T^1_{QC}(S_t, \mathcal{O}_{S_t})^G; \quad T^2_{QC}(S_t, \mathcal{O}_{S_t})^G
\]

in Proposition 4.13.

If we have a diagram

\[
\begin{array}{ccc}
S_t & \xrightarrow{\pi^{\text{ind}}} & \mathcal{M}^{\text{ind}} \\
\downarrow & & \downarrow \pi^{\text{ind}} \\
t = \text{Spec}(k) & \xrightarrow{\rho^{\text{ind}}} & M^{\text{ind}},
\end{array}
\]

then from Proposition 4.13 the first order infinitesimal Q-Gorenstein deformation of \( \text{Spec}(k) \in M^{\text{ind}} \) (i.e., the Q-Gorenstein deformation of \( S_t \)) is given by \( T^1_{QC}(S_t, \mathcal{O}_{S_t})^G \), and the obstruction is given by \( T^2_{QC}(S_t, \mathcal{O}_{S_t})^G \). There may exist higher obstruction spaces \( T^i_{QC}(S_t, \mathcal{O}_{S_t})^G \) for \( i \geq 3 \). We make this more precise following Proposition 4.13. Let \( A \) be a finitely generated Artinian local \( k \)-algebra, and \( S_A/A \) be a Q-Gorenstein deformation of \( S \) over \( A \). Let \( \overline{A} \to A \) be an infinitesimal extension of \( A \) with kernel \( J \). We let \( \overline{m} \) be the maximal ideal of \( \overline{A} \) and assume that \( \overline{m} \cdot J = 0 \) (if \( J \) is a \( A/\overline{m} = k \) space). Then there is an obstruction class

\[
\text{ob}(S_A/A, \overline{A}) \in T^2_{QC}(S, \mathcal{O}_S)^G \otimes J,
\]

such that \( \text{ob}(S_A/A, \overline{A}) = 0 \) if and only if there exists a Q-Gorenstein deformation \( S_{\overline{A}} \) of \( S_A \) over \( \overline{A} \). Moreover, if \( \text{ob}(S_A/A, \overline{A}) = 0 \), then the isomorphism classes of such deformations form a torsor under \( T^1_{QC}(S, \mathcal{O}_S)^G \otimes J \).

One can make this argument into a family by considering a scheme \( T = \text{Spec}(A) \to M^{\text{ind}} \), and the diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{g} & \mathcal{M}^{\text{ind}} \\
\downarrow q & & \downarrow \rho^{\text{ind}} \\
T & \xrightarrow{f} & M^{\text{ind}},
\end{array}
\]

Let \( T \to \overline{T} \) be a square zero extension with ideal sheaf \( J \). The obstruction to extending \( \mathcal{M}_T \) to a flat family over \( \overline{T} \) lies in \( \text{Ext}^2(\mathcal{L}^\text{ind}_{\mathcal{M}_T/\overline{T}}, \mathcal{Q}_J) \) and if the extensions exist, they form a torsor.
under $\text{Ext}^1(L^{•}_{\mathcal{M}/T}, q^{•}f)$. Since $L^{•}_{\mathcal{M}/T} = \mathcal{O}^{\text{ind}}_{\mathcal{M}/\text{T}}$ and $p^{\text{ind}}$ is flat, we have that

$$\text{Ext}^i_{\mathcal{O}_{\mathcal{M}/T}}(L^{•}_{\mathcal{M}/T}, q^{•}f) = \text{Ext}^i_{\mathcal{O}_{\mathcal{M}/\text{T}}}(L^{•}_{\mathcal{M}/\text{T}}, Rg_*q^{•}f) = \text{Ext}^i_{\mathcal{O}_{\mathcal{M}/\text{T}}}(L^{•}_{\mathcal{M}/\text{T}}, (p^{\text{ind}})^*Rf_!f).$$

Thus,

$$\text{Ext}^i_{\mathcal{O}_{\mathcal{M}/\text{T}}}(L^{•}_{\mathcal{M}/\text{T}}, (p^{\text{ind}})^*Rf_!f) = \text{Ext}^{i-1}_{\mathcal{O}_{\mathcal{M}/\text{T}}}(E^{•}_{\mathcal{M}/\text{T}}, Rf_!f) = \text{Ext}^{i-1}_{\mathcal{O}_{\mathcal{M}/\text{T}}}(f^*E^{•}_{\mathcal{M}/\text{T}}, f),$$

where for the first isomorphism, we use Grothendieck duality since $(p^{\text{ind}})^*(\mathcal{O}_{\mathcal{M}/\text{T}})$ is the dualizing sheaf $\mathcal{O}_{\mathcal{M}/\text{T}}$ which is invertible.

Since $p^{\text{ind}} : \mathcal{M}^{\text{ind}} \rightarrow \mathcal{M}_{\text{lci}}$ is a universal family for the moduli stack $\mathcal{M}^{\text{ind}}$, the Kodaira-Spencer map $L^{•}_{\mathcal{M}/\text{T}} \rightarrow (p^{\text{ind}})^*L^{•}_{\mathcal{M}/\text{T}}[1]$ defines a morphism

$$\varphi^{\text{ind}} : E^{•}_{\mathcal{M}/\text{T}} \rightarrow L^{•}_{\mathcal{M}/\text{T}}.$$ 

From the above analysis, this morphism satisfies Condition (3) in Theorem 3.3. Therefore, $\varphi^{\text{ind}}$ defines an obstruction theory for $\mathcal{M}^{\text{ind}}$ in the sense of Behrend-Fantechi. \hfill \square

6. Moduli stack of ICI covers

In this section we construct the moduli stack $\mathcal{M}^{\text{ici}} := \mathcal{M}_{\text{K}^{\text{ici}}_{\mathcal{X}}/\mathcal{X}}$ of ICI covers over the moduli stack $\mathcal{M}$ such that there is a perfect obstruction theory on $\mathcal{M}^{\text{ici}}$.

6.1. Universal abelian cover of s.l.c. surface germs. Recall from Remark 4.16 in §4.2.3, let $S$ be an s.l.c. surface and $\pi : \mathcal{E} \rightarrow S$ be the corresponding index one covering Deligne-Mumford stack. Except l.c.i. singularities, the germs on the index one covering Deligne-Mumford stack $\mathcal{E}$ may have simple elliptic singularities, cusp or degenerate cusp singularities of embedded dimension $\geq 5$. Locally, the germ singularity is of the form $[Z/\mu_r]$, where $(Z, 0)$ is a germ singularity which is a simple elliptic singularity, a cusp or a degenerate cusp singularity and $r$ is the index. Note that $r = 1, 2, 3, 4, 6.$

From the classification result in [42, Theorem 4.24], we consider the simple elliptic singularity, the cusp or the degenerate cusp singularity $(S, 0)$, and the $Z_2, Z_3, Z_4, Z_6$-quotient of a simple elliptic singularity $(S, 0)$, the $Z_2$-quotient of a cusp singularity or a degenerate cusp singularity $(S, 0)$, the $\mathbb{Q}$-Gorenstein deformation of $(S, 0)$ is equivalent to the $\mathbb{Z}_r$-equivariant deformation of $(Z, 0)$.

Let us fix to the surface singularity germ $(S, 0, 0)$. Let

$$(6.1.1) \quad \sigma : X \rightarrow S$$

be a good resolution and $A = \bigcup_{i=1}^n A_i$ be the decomposition of exceptional set $\sigma^{-1}(0) = A$ such that $A$ is a divisor having only simple normal crossings. A divisor supported in $A$ is called a cycle. Let $\Sigma$ be the link of $(S, 0)$ which is, by definition, the boundary $\partial U$ of a small neighborhood $U$ of the singularity 0. The link $\Sigma$ is an oriented 3-manifold over the field $\mathbb{R}$ of real numbers. The neighborhood $U$ can be made to be a tubular neighborhood of the exceptional divisor so that $\partial U = \Sigma$ is the link of the singularity. This can be obtained by plumbing theory of surface singularities in [50]. Then, we have that

$$H_2(U, Z) \cong \mathbb{Z}^n \subseteq H_2(U, Q) \cong \mathbb{Q}^n,$$

where $n$ is the number of exceptional curves in $A$. Let $(,)$ be the intersection form on these groups and define

$$H_2(U)^\# = \{ v \in H_2(U, Q) : (v, w) \in \mathbb{Z} \text{ for all } w \in H_2(U, Z) \}.$$ 

Then the embedding $H_2(U, Z) \rightarrow H_2(U)^\#$ can be identified with the map $H_2(U, Z) \rightarrow H_2(U, \Sigma)$. So the long exact sequence in homology identifies the discriminant group

$$D := H_2(U)^\#/H_2(U, Z).$$
with the torsion subgroup \( H_1(\Sigma, \mathbb{Z})_{\text{tor}} \) of \( H_1(\Sigma, \mathbb{Z}) \). The intersection form \( \langle \cdot, \cdot \rangle \) induces on \( D \) a natural non-singular pairing:

\[
D \otimes D \to \mathbb{Q}/\mathbb{Z}; \quad v \otimes w \mapsto (v, w)/\mathbb{Z}
\]

which is the torsion link pairing of \( \Sigma \).

If \( K \subset D \) is a subgroup, then there is an induced non-singular pairing

\[
K \otimes (D/K^\perp) \to \mathbb{Q}/\mathbb{Z}
\]

where \( K^\perp \) is the orthogonal complement of \( K \) under the pairing. The group \( D/K^\perp \) is canonically isomorphic to the dual \( \hat{K} = \text{Hom}(K, \mathbb{Q}/\mathbb{Z}) \) and is non-canonically isomorphic to \( K \) itself.

If \( \Sigma \) is a rational homology sphere, then the universal abelian cover of \( \Sigma \) is the Galois cover of \( \Sigma \) determined by the natural homomorphism \( \pi_1(\Sigma) \to H_1(\Sigma) = D \). Thus, any subgroup \( K \subset D \) determines an abelian cover of \( \Sigma \); i.e., the Galois cover with covering transformation group \( D/K \). The Galois cover corresponding to \( K^\perp \) is called the dual cover for \( K \), with transformation group \( D/K^\perp \). The dual cover for \( D \) is thus the universal abelian cover.

Let us consider the \( \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \)-quotient of a simple elliptic singularity \((S, 0)\), the \( \mathbb{Z}_2 \)-quotient of a cusp singularity or a degenerate cusp singularity \((S, 0)\). Then \( A \) is a tree of rational curves since the \( \mathbb{Z}_r \)-quotient of simple elliptic singularity and cusp singularity are rational singularities. An explicit \( \mathbb{Z}_2 \)-action on cusps was given in \([54]\), and the \( \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \) actions on a simple elliptic singularity were given in \([42] \S5.2\), \([40] \S9.6\]. By the way all of these singularities are log-canonical. In particular, a cyclic group quotient of log-canonical singularity is a rational singularity.

For such an s.l.c. germ \((S, 0)\), its link \( \Sigma \) is a rational homology sphere. The group \( D = H_1(\Sigma, \mathbb{Z}) \) is a finite abelian group. From \([55]\), we take

\[
(\tilde{S}, 0) \to (S, 0)
\]

to be the universal abelian cover, where the topology of the cover is determined by the link \( \Sigma \). Let \((Z, 0) \to (S, 0)\) be the index one cover of the singularity germ \((S, 0)\) such that

\[
[Z/\mathbb{Z}_r] \cong S \quad \text{for} \quad r = 2, 3, 4, 6.
\]

Then the universal abelian cover \((\tilde{S}, 0) \to (S, 0)\) factors through the index one cover

\[
(\tilde{S}, 0) \to (Z, 0)
\]

since \((Z, 0) \to (S, 0)\) is an abelian cover.

The deformation of \((S, 0)\) can be given by the \( D \)-equivariant deformation of \((\tilde{S}, 0)\). Thus we have

**Theorem 6.1.** If \((S, 0)\) is the \( \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \) quotient of a simple elliptic singularity, or the \( \mathbb{Z}_2 \) quotient of a cusp or a degenerate cusp singularity germ, then there exists the universal abelian cover \((\tilde{S}, 0)\). Moreover, the \( D \)-equivariant deformations of \((\tilde{S}, 0)\) gives \( Q \)-Gorenstein deformations of \((S, 0)\).

**Proof.** The cases of the \( \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \) quotients of a simple elliptic singularity and the \( \mathbb{Z}_2 \) quotient of a cusp is from \([54]\), \([55]\), and \([6.1.2]\). The \( \mathbb{Z}_2 \)-quotient of degenerate cusp is given in \([40] \S9.6\], where the the degenerate cusp only has two irreducible components. In this case we consider the following diagram

\[
\begin{array}{ccc}
(S^\text{norm}) & \to & S^\text{norm} = S_1 \sqcup S_2 \\
\downarrow & & \downarrow \\
(\tilde{S}) & \to & S
\end{array}
\]

where \( S^\text{norm} \) is the normalization of \( S \), and the two components \( S_i \) have cyclic quotient singularities. From \([40], [55]\), \( S^\text{norm} \to S^\text{norm} \) is the universal abelian cover. Then, \( \tilde{S} \) is
obtained from \( \tilde{S} \) by identifying the double curves. We know that \( \tilde{S} \) is l.c.i., so is \( \tilde{S} \).

**Remark 6.2.** Suppose that \((S, 0)\) is the \( \mathbb{Z}_2^*, \mathbb{Z}_3^*, \mathbb{Z}_4^*, \mathbb{Z}_6^* \) quotient of a simple elliptic singularity, or the \( \mathbb{Z}_2^* \) quotient of a cusp singularity. Let \((\tilde{S}, 0)\) be the universal abelian cover. It is interesting to study if any \( Q \)-Gorenstein deformation of \((S, 0)\) gives a \( D \)-equivariant deformations of \((S, 0)\).

For instance, in the case of \( \mathbb{Z}_2^* \)-quotient of simple elliptic singularity \((S, 0)\), if the exceptional smooth elliptic curve \( E \) has self-intersection number \( \leq 8 \), \([67]\) proves that \((S, 0)\) always admits a \( \mathbb{Z}_2^* \)-equivariant smoothing. It is interesting to study if the universal abelian cover \((\tilde{S}, 0)\) of the quotient elliptic singularity admits \( D \)-equivariant smoothings.

**Example 1.** We provide an interesting example of the \( \mathbb{Z}_2 \)-quotient-cusp singularity. It is the \( \mathbb{Z}_2 \)-quotient of the cusp surface singularity \((Z, 0)\) whose resolution graph is given by

\[
(6.1.3)
\]

where \( k \geq 2, e_i \geq 2 \) and some \( e_j > 2 \). The quotient-cusp singularity \((S, 0)\) has resolution graph

\[
(6.1.4)
\]

There is an associated matrix

\[
B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B(e_1 - 1, e_2, \cdots, e_k - 1, e_k - 1)
\]

where

\[
B(e_1 - 1, e_2, \cdots, e_k - 1, e_k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_k - 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & e_1 - 1 \end{pmatrix}.
\]

From \([54]\) Theorem 5.1, the universal abelian l.c.i cover \((\tilde{S}, 0) \to (S, 0)\) has transformation abelian group \( D \) with order 16b. Let \( \zeta \) be a primitive 4b-th root of unity. We consider the following diagonal matrices:

\[
A_1 = \text{Diag}[\zeta^a, \zeta^a, \zeta^a, \zeta^a] \\
A_2 = \text{Diag}[\zeta^a, -\zeta^a, \zeta^a, \zeta^a] \\
A_3 = \text{Diag}[\zeta^a, \zeta^a, -\zeta^d, \zeta^d] \\
A_4 = \text{Diag}[\zeta^a, \zeta^a, \zeta^d, -\zeta^d].
\]

Then the finite abelian group is \( D = \langle A_1, A_2, A_3, A_4 \rangle \), which has order 16b. The group structure of \( D \) depends on the parity of \( c \), see \([54]\) Theorem 5.1.

The local equations of \((\tilde{S}, 0)\) are given by:

\[
x^2 + y^2 = u^\alpha v^\beta; \quad u^2 + v^\alpha = x^\gamma y^\delta,
\]

where \( \alpha, \beta, \gamma, \delta \geq 0 \) satisfy the conditions

\[
\alpha + \beta = 2a; \quad \gamma + \delta = 2d; \quad \alpha \equiv \beta \equiv \gamma \equiv \delta \equiv c \pmod{2}.
\]

The resolution graph of the universal abelian cover \((\tilde{S}, 0)\) is given by

\[
(6.1.5)
\]

where the four strings of \(-2\)'s are lengths \( 2a - 3, 2d - 3, 2a - 3, \) and \( 2d - 3 \) if \( a, d \neq 1 \).
If \( d = 1 \) or \( a = 1 \) the resolution graph is given by
\[
(6.1.6)
\]
\[
\begin{array}{c}
\text{-4} \\
\text{-2} \\
\text{2} \\
\text{4}
\end{array}
\]
where the top and bottom strings are of length \( 2a - 3 \) or \( 2d - 3 \).

From [54], Proposition 2.5, a cusp singularity with resolution graph \([-b_1, \ldots, -b_k]\) is a complete intersection cusp. From dual graph (of the dual cusp) of (6.1.5) and (6.1.6) is given by
\[
The anticanonical divisor \( E \) given by
\[
T
\]
The link above resolution graph of the universal abelian cover cusp has resolution cycle of its dual cusp is the anticanonical divisor of a smooth rational surface.

Example 2. Let us look at the \( \mathbb{Z}_2 \)-quotient cusp singularity \((S, 0)\) in Example 1. The universal abelian cover cusp \((\tilde{S}, 0)\) has resolution cycle given by (6.1.5) and (6.1.6), and it is a complete intersection cusp. From [26], this cusp singularity \((S, 0)\) is smoothable if and only if the resolution cycle of its dual cusp is the anticanonical divisor of a smooth rational surface.

From [52] (1.1) Theorem, for certain \( a, d \geq 1 \), there is a smooth rational surface \((X, E)\) with the anticanonical divisor \( E \) given by \([-2a - 2d - 2a - 2d]\). Thus from [26], the cusp singularity \((\tilde{S}, 0)\) is smoothable, which induces the \( \mathbb{Q} \)-Gorenstein deformation of \((S, 0)\).

Example 3. Recall that in Example 1, the quotient-cusp singularity \((S, 0)\) has resolution cycle (6.1.4), which associates with a matrix
\[
B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
The quotient \((\tilde{S}/D, 0)\) is isomorphic to \((S, 0)\). The lci singularity \((\tilde{S}, 0)\) admits a smoothing
\[
\tilde{S} \subset \mathbb{A}^4_k \times \mathbb{A}^1_k
\]
which is given by the equations:
\[
x^2 + y^2 - u^a v^d = t; \quad u^2 + v^2 - x^1 y^d = t.
\]
The group \( D \) acts on \( t \) trivially, and the quotient \( S = \tilde{S}/D \) gives a smoothing of the singularity \((S, 0)\).

6.2. Discriminant cover of s.l.c. surface germs. Now we assume that the s.l.c. germ \((S, 0)\) is a Gorenstein simple elliptic singularity, a cusp singularity or a degenerate cusp singularity. Note that simple elliptic singularities and cusps are normal surface singularities.

6.2.1. Cusp singularities. Let us first fix to the cusp singularity case. In this case the index one cover is just \((Z, 0) = (S, 0)\), and we have the good resolution \( \sigma : X \to S \), where \( \sigma^{-1}(0) = A \) is a cycle of rational curves. The link \( \Sigma \) is not a rational homology sphere. The link a \( T^2 \)-bundle over the circle \( S^1 \) and \( H_1(\Sigma, \mathbb{Z}) = \mathbb{Z} \oplus D \). Suppose that the type of the cusp singularity is given by \([-e_1, \ldots, -e_k]\) determined by the resolution graph of the cusp, where \( e_i \) are positive integers and \(-e_i\) are the self-intersection numbers of the the component curves in the exceptional divisor of the minimal resolution of \((S, 0)\). Then the monodromy of the link is given by the matrix
\[
A = \begin{pmatrix} 0 & 1 \\ -1 & e_k \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & e_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
such that \( \pi_1(\Sigma) = \mathbb{Z}^2 \times_A \mathbb{Z} \).

As in [54, §4], there is no natural epimorphism \( \pi_1(\Sigma) \to D \), hence no natural Galois cover with transformation group \( D \). But different epimorphisms of \( H_1(\Sigma, \mathbb{Z}) = \mathbb{Z} \oplus D \to D \) are related by automorphisms of \( \pi_1(\Sigma) \), and hence by automorphisms of \( (S, 0) \).

Therefore, there is a natural cover up to automorphisms, called the discriminant cover. Also for any subgroup \( K \subset D \) we still have the cover for \( K \) and the dual cover for \( K \), with transformation groups \( D/K \) and \( D/K^\perp \) respectively. From the proof in [54, §4], take \( K = \{1\} \) and let \( (\tilde{S}, 0) \to (S, 0) \) be the discriminant cover of \( (S, 0) \), which is also the dual cusp of \( (S, 0) \).

In [54] Proposition 4.1 (2), Neumann and Wahl constructed a finite cover \( (\tilde{S}, 0) \) of \( S \) with transformation group \( D' \) so that \( (\tilde{S}, 0) \) is a hypersurface cusp, which is l.c.i. Let \( H \) be the subspace of \( \mathbb{Z}^2 \) generated by \( \begin{pmatrix} a \\ c \\ d \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} \). Then the matrix \( A \) takes the subspace \( H \) itself by the matrix \( \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \) where \( t = \text{tr}(A) = a + d \). The finite transformation group \( D' \) is given as follows: first we take the quotient finite group \( \pi_1(\Sigma)/H \times \mathbb{Z} \), the subgroup \( H \times \mathbb{Z} \subset \pi_1(\Sigma) \) determines a cover of \( S \). This cover determined by \( H \times \mathbb{Z} \) is either the cusp with resolution graph consisting of a cycle with one vertex weighted \( -t \) or the dual cusp of this, according as the above basis is oriented correctly or not, i.e., whether \( a < 0 \) or \( a > 0 \). By taking the discriminant cover if necessary we get the cover \( (\tilde{S}, 0) \) of \( S \) with transformation group \( D' \). The key issue is that \( (\tilde{S}, 0) \) a complete intersection cusp. Thus, we obtain

**Lemma 6.3.** Let \( (S, 0) \) be a cusp singularity, then there exists a finite discriminant cover \( (\tilde{S}, 0) \) of \( S \) with transformation group \( D' \) and the cusp \( (\tilde{S}, 0) \) is a complete intersection cusp. A deformation of the Deligne-Mumford stack \( \tilde{S}/D' \); i.e., a \( D' \)-equivariant deformation of \( \tilde{S} \), induces a Gorenstein deformation of the cusp \( (S, 0) \).

### 6.2.2. Simple elliptic singularities.

Let \( (S, 0) \) be a simple elliptic singularity. Let \( \sigma : X \to S \) be the minimal resolution such that \( A = \sigma^{-1}(0) \) is the exceptional elliptic curve. The local embedded dimension of the singularity is given by \( \max(3, -A \cdot A) \). It is known from [45], that the simple elliptic singularity \( (S, 0) \) is a cusp singularity if the negative self-intersection \( -A \cdot A < 4 \). If \( -A \cdot A \geq 5 \), then \( (S, 0) \) is never lci. Let \( d := -A \cdot A \) be the degree of \( (S, 0) \).

From [63], [42], it admits a smoothing if and only of \( 1 \leq d \leq 9 \). We say that \( (S, 0) \) admits an lci lifting if there is an lci cover \( (\tilde{S}, 0) \to (S, 0) \) with transformation group \( D' \) such that \( (\tilde{S}, 0) \) is an lci singularity. We call the \( D' \)-equivariant smoothing the lci smoothing lifting and it induces a Gorenstein smoothing of \( (S, 0) \). We list the result in [39, Theorem 1.2] here.

**Theorem 6.4.** Let \( (X, 0) \) be a simple elliptic surface singularity, and \( (X, E) \) its minimal resolution. Then \( (X, 0) \) admits an lci smoothing lifting only when \( d \neq 5, 6, 7 \) and \( 1 \leq d \leq 9 \).

From the above analysis and Theorem [54] we have

**Theorem 6.5.** Let \( (S, 0) \) be a simple elliptic singularity, a cusp or a degenerate cusp singularity germ. Suppose that there exists a discriminant cover \( (\tilde{S}, 0) \) of \( (S, 0) \) with transformation group \( D' \).

Then, the \( D' \)-equivariant deformations of \( (\tilde{S}, 0) \) induce Gorenstein deformations of \( (S, 0) \).

**Proof.** We only need to prove the degenerate cusp singularity case. Let \( (S, 0) \) be a degenerate cusp singularity, which is a non-normal surface singularity sharing the same properties of cusp singularities. We construct the following diagram

\[
(\tilde{S}^{\text{norm}}, 0) \xrightarrow{(\tilde{S}^{\text{norm}}, 0)} (S^{\text{norm}}, 0)
\]

\[
(\tilde{S}, 0) \xrightarrow{(\tilde{S}, 0)} (S, 0),
\]
where the vertical maps are normalizations and the horizontal maps are universal abelian covers.

The cover \((\tilde{S}, 0)\) can be constructed as follows. From [69, §1], suppose that \(S_1, \ldots, S_r\) are the irreducible components of \(S\) that form a cycle if \(r \geq 3\). After reordering if necessary, \(S_i\) and \(S_{i+1}\) meet generically transversally in a smooth irreducible curve and for \(j \neq i, i \pm 1\), \(S_i \cap S_j = \{0\}\). If \(r = 2\), then \(S_1\) and \(S_2\) meet generically transversally in the union of two smooth curves meeting transversally at \(0\). If \(r = 1\), then the singular locus of \(S\) is smooth and irreducible. The normalization \(S^{\text{norm}}\) of \(S\) is a disjoint union of cyclic quotient singularities (which are rational singularities). Let \(\sigma^{\text{norm}} : X^{\text{norm}} \to S^{\text{norm}}\) be the minimal resolution of \(S^{\text{norm}}\). (Here \(S^{\text{norm}} = \sqcup S_i\) where \(\overline{S_i}\) is the normalization of \(S_i\). Then, \(X^{\text{norm}} = \sqcup X_i\), where \(X_i = \text{Bl}_{\overline{S_i}} \overline{S_i}\) if \(\overline{S_i}\) is smooth, and the minimal resolution of \(\overline{S_i}\) otherwise). Then we get the minimal resolution

\[
\sigma : X \to S
\]

by identifying \(X_i\) and \(X_{i+1}\) along the strict transform of the curve along which \(S_i\) and \(S_{i+1}\) meet in \(S\). Thus, \(\sigma^{-1}(0)\) is a cycle of rational curves. We construct the following diagram

\[
(6.2.2)
\]

where the vertical arrows are all normalizations, and the two top and bottom squares are fiber products. First the top square is constructed as follows: let \(\tilde{\sigma}^{\text{norm}} : \tilde{X}^{\text{norm}} \to \tilde{S}^{\text{norm}}\) be the minimal resolution of \(\tilde{S}^{\text{norm}}\) constructed above. Then we take the fiber product \(\tilde{X}^{\text{norm}}\). Since \(X\) is obtained by identifying \(X_i\) and \(X_{i+1}\) along the strict transform of the curve along which \(S_i\) and \(S_{i+1}\) meet in \(S\). Then, \(\tilde{X}\) is obtained by identifying \(\tilde{X}_i\) and \(\tilde{X}_{i+1}\) along the preimages of the transformation curves under the covering map \(\tilde{f}\) along which \(S_i\) and \(S_{i+1}\) meet in \(S\). Note that the cover map \(\tilde{f}\) may give different orders on different components, and we only identify same number of the preimage curves. The transformation group \(D\) of the universal abelian cover \(\tilde{f} : \tilde{S}^{\text{norm}} \to \tilde{S}^{\text{norm}}\) is the product of all the finite abelian groups in the components of \(f\). Thus, contracting down all the exceptional rational curves we get the cover \(\tilde{S} \to S\) with the same finite abelian transformation group \(D\). This constructs the diagram \(6.2.1\). Since \((\tilde{S}^{\text{norm}}, 0)\) is l.c.i., \((\tilde{S}, 0)\) is also l.c.i. \(\square\)

**Remark 6.6.** It is possible that not all of the Gorenstein deformations of \((S, 0)\) come from the deformations of \([\tilde{S} / D']\). From [26], a cusp singularity \((S, 0)\) is smoothable if and only if the resolution cycle of its dual cusp sits as an anticanonical divisor in a smooth rational surface. It is interesting to study under which condition the Gorenstein deformations of the cusp singularity \((S, 0)\) is given by the deformations \([\tilde{S} / D']\) of the discriminant cover. See [38], [39].

**6.2.3. Examples.**

**Example 4.** In Example [I] there is a universal abelian cover of the quotient-cusp which factors through the cusp in the quotient. We have examples of cusps which do not admit abelian covers by complete intersection cusps.

Let \((S, 0)\) be a cusp singularity whose resolution graph is given by \([6.1.3]\) in Example [I] Let

\[
k = 4, e_1 = 6, e_2 = 3, e_3 = 3, e_4 = 2.
\]
Then the resolution cycle of this specific cusp is $[-10, -3, -3, -3, -2]$. From [54] Lemma 2.4, the dual cusp has resolution cycle

$$[-4, -2, -2, -2, -2, -2, -2, -2].$$

The dual cusp of the cusp corresponding to $[-4, -2, -2, -2, -2, -2, -2, -2]$ has resolution cycle $[-2, -10]$, which is a complete intersection cusp. Thus the cusp $(S, 0)$ corresponding to the resolution cycle $[-10, -3, -3, -3, -3, -2]$ might be covered by a complete intersection cusp.

But if we choose

$$k = 5, e_1 = 4, e_2 = 2, e_3 = 2, e_4 = 2, e_5 = 3,$$

then the resolution cycle of this specific cusp is $[-6, -2, -2, -3, -3, -2, -2, -4]$. The dual cusp has resolution cycle

$$[-2, -2, -2, -5, -5, -2].$$

The dual cusp of the cusp corresponding to $[-2, -2, -2, -5, -5, -2]$ has resolution cycle $[-6, -2, -2, -3, -3, -2, -2, -4]$ which has length 8 (not a complete intersection).

Therefore, the dual cusp corresponding to $[-6, -2, -2, -3, -3, -2, -2, -4]$ and its dual cusp corresponding to $[-2, -2, -2, -5, -5, -2]$ are both complete intersection cusps. From [54] Proposition 2.5, the cusp corresponding to $[-6, -2, -2, -3, -3, -2, -2, -4]$ cannot have an abelian cover by a complete intersection cusp. We have to take the discriminant cover presented in Theorem 6.5.

In this case, we calculate the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} -40 & -211 \\ 131 & 691 \end{pmatrix}. $$

From the proof of [54] Proposition 4.1, the subspace $H \subset \mathbb{Z}^2$ generated by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -40 \\ 131 \end{pmatrix}$$

gives a subgroup $H \times \mathbb{Z} \subset \mathbb{Z}^2 \times \mathbb{Z} = \pi_1(\Sigma)$ (where $\Sigma$ is the link of the cusp singularity). The cover determined by $H \times \mathbb{Z} \subset \pi_1(\Sigma)$ is the cusp with resolution graph consisting of a cycle with one vertex weighted by $-651$. Then the discriminant group of this cusp has order 651. By taking the abelian cover again corresponding to this finite group we get a hypersurface cusp whose resolution graph is given by $651 - 3 = 648$ numbers of vertexes weighted by $-2$ and one vertex weighted by $-3$. The final cusp singularity is the discriminant cover of the original cusp $(S, 0)$.

**Example 5.** Here is an example of hypersurface cusp singularities with a finite abelian group action in [53] Corollary. Let $(\tilde{S}, x)$ be a hypersurface cusp given by:

$$\{x^p + y^q + z^r + xyz = 0\}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$ 

Here $p, q, r$ are positive integers. The resolution cycles of such a cusp is given in [53] Lemma 2.5. The dual cusp of this cusp has resolution cycle

$$(-(p - 1), -(q - 1), -(r - 1)).$$

Let $\Sigma$ be the link of $(\tilde{S}, x)$. The torsion subgroup $D = H_1(\Sigma, \mathbb{Z})_{tor}$ is isomorphic to the group

$$\{\lambda, \mu, \nu | \lambda^p = \mu^q = \nu^r = 1\}.$$ 

The group $D$ acts on the hypersurface cusp singularity by

$$x \mapsto \lambda x; \quad y \mapsto \mu y; \quad z \mapsto \nu z.$$ 

The quotient $(\tilde{S}, x)/D$ is the cusp $(S, x)$ whose resolution cycle is $(-(p - 1), -(q - 1), -(r - 1))$. Note that if $(p - 1) - 2 + (q - 1) - 2 + (r - 1) - 2 > 4$, then the dual cusp $(S, x)$ is not a complete intersection cusp.

The hypersurface cusp $(\tilde{S}, x)$ admits a $D$-equivariant smoothing which is given by the equation

$$\{x^p + y^q + z^r + xyz = 1\}$$

and the group $D$-action on $t$ is trivial. The quotient gives a smoothing of the cusp singularity $(S, x)$.  

6.3. More on equivariant smoothing of simple elliptic and cusp singularities. Let \((X,0)\) be a germ of simple elliptic or cusp singularity as in §6.2 and \((S,0) = (X,0)/\mathbb{Z}_r\) the quotient singularity germ in §6.1. Note that \(r = 2, 3, 4, 6\) in the simple elliptic singularity case and \(r = 2\) in the cusp singularity case.

Let \(\Sigma_X\) and \(\Sigma_S\) be the links of the singularity germs. Then \(\Sigma_X \rightarrow \Sigma_S\) is a unramified \(r\)-th fold cover. Since the link \(\Sigma_S\) of \((S,0)\) is a rational homology sphere, from §6.1 let \(\pi : (\tilde{S},0) \rightarrow (S,0)\) be the universal abelian cover with transformation finite abelian group \(D = H_1(\Sigma_S)\). Suppose that there is a subgroup \(K \subset D\) such that we have an exact sequence
\[
0 \rightarrow K \rightarrow H_1(\Sigma_S) \rightarrow \mathbb{Z}_r \rightarrow 0,
\]
then it determines a \(r\)-fold cover of germs \((S',0) \rightarrow (S,0)\) such that the map \(\Sigma_{S'} \rightarrow \Sigma_S\) is a unramified \(r\)-cover of the links. So this implies that \((S',0) \cong (X,0)\) and \(\Sigma_{S'} \cong \Sigma_X\). Also topologically we have the following diagram
\[
\Sigma_{\tilde{S}} \rightarrow \Sigma_X \\
\downarrow \quad \downarrow \\
\Sigma_S \rightarrow \Sigma_S
\]
which is from the following commutative diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & K' & \rightarrow & K'' & \rightarrow & K & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \pi_1(\Sigma_S) & \rightarrow & H_1(\Sigma_S) & \rightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z}_r & \rightarrow & \mathbb{Z}_r & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]
The cover \(\Sigma_{\tilde{S}} \rightarrow \Sigma_X\) has transformation group \(K\). Thus, this induces a finite abelian cover
\[
\pi : (\tilde{S},0) \rightarrow (X,0)
\]
with transformation group \(K\).

Comparing with Theorem 6.1, we have

**Theorem 6.7.** If \((X,0)\) is a simple elliptic singularity germ, or a cusp singularity germ such that there exists a quotient \(((X,0)/\mathbb{Z}_r,0) = (S,0)\) above, then the \(K\)-equivariant deformations of \((\tilde{S},0)\) induce \(\mathbb{Z}_r\)-equivariant deformations of \((X,0)\), which induce \(\mathbb{Q}\)-Gorenstein deformations of \((S,0)\).

**Proof.** We only need to check that in the simple elliptic singularity and cusp singularity cases, the cyclic group \(\mathbb{Z}_r\) for \(r = 2, 3, 4, 6\) can be taken as a quotient of \(H_1(\Sigma_S)\). This is from the direct calculations for the group \(H_1(\Sigma_S)\) for the simple elliptic singularities and cusps. The group \(H_1(\Sigma_S)\) can be calculated using the resolution graphs in [42, 40, Theorem 9.6, (3), (4)]. The cyclic group \(\mathbb{Z}_2\) is a summand of \(H_1(\Sigma_S)\) in the simple elliptic singularity case. From the calculation of \(H_1(\Sigma_S)\) in [54 §5] and Example [11 in the quotient-cusp case, the group \(\mathbb{Z}_2\) can definitely be taken as a quotient of \(H_1(\Sigma_S)\).

**Remark 6.8.** Theorem 6.7 is different from Theorem 6.5, since Gorenstein deformations of simple elliptic singularities and cusp singularities are different from their \(\mathbb{Z}_r\)-equivariant deformations.
Remark 6.9. As we talked about the cusp singularities in §6.2, not every cusp admits a \( \mathbb{Z}_2 \)-quotient. Thus, not every cusp has a finite abelian cover by a complete intersection cusp. From [54, Proof of Proposition 4.1], a necessary condition that a cusp singularity \( (X, 0) \) has no finite abelian cover by a complete intersection is that the cusp \( (X, 0) \) and its dual cusp are both not complete intersections. For instance, let \( (X, 0) \) be a cusp with resolution graph self-intersection sequence \([-2, -4, -2, -2, -5]\). This cycle is self-dual, is not a complete intersection from [54, Proposition 2.5]. Thus, there is no finite abelian cover by a complete intersection for \( (X, 0) \). We have to use Theorem 6.5 to get a finite (not abelian) cover which is a complete intersection.

6.4. The lci covering Deligne-Mumford stack over s.l.c. surfaces. Let \( S \) be an s.l.c. surface such that the possible elliptic singularities, cusp and degenerate cusp singularities in \( S \) all have embedded dimension \( \geq 5 \); i.e. they are not l.c.i. singularities. Then the argument in Theorem 6.5 and Theorem 6.1 constructed the universal abelian cover or the discriminant cover of the singularity germs so that their covers are l.c.i. The construction only depends on the local analytic structure of the singularity.

Similar to the construction of index one covering Deligne-Mumford stack \( \pi : \mathcal{S} \to S \), there are only finite singularity germs \( (S, 0) \) in \( S \), such that the corresponding simple elliptic singularities, cusp and degenerate cusp singularities have embedded dimension \( \geq 5 \) (i.e., not l.c.i.). Thus, for each such germ singularity, we perform the universal abelian cover or the discriminant cover construction in §6.1 and §6.2. We get another Deligne-Mumford stack

\[ \pi^{\text{lci}} : \mathcal{S}^{\text{lci}} \to S \]

with the coarse moduli space \( S \) such that \( \mathcal{S}^{\text{lci}} \) only has l.c.i. singularities. We call \( \mathcal{S}^{\text{lci}} \) the lci covering Deligne-Mumford stack of \( S \). Note that if \( [Z/\mu_N] \) is a germ chart of \( \mathcal{S} \), then \( \mathcal{S}^{\text{lci}} \) locally has the germ chart \([\mathcal{S}/D] \), where \( D \) is the transformation group of the lci cover.

The Deligne-Mumford stack \( \mathcal{S}^{\text{lci}} \) is Gorenstein since \( \mathcal{S}^{\text{lci}} \) only has l.c.i. singularities on each chart. Thus, we get a commutative diagram

\[
\begin{array}{ccc}
\mathcal{S}^{\text{lci}} & \xrightarrow{\tilde{\pi}} & \mathcal{S} \\
\downarrow{\pi^{\text{lci}}} & & \downarrow{\pi} \\
S & & S \\
\end{array}
\]

We make a summary here. Let \( (S, 0) \) be a singularity germ in an s.l.c. surface \( S \), then we have that

1. if \( (S, 0) \) is a simple elliptic singularity, a cusp or a degenerate cusp singularity with embedded dimension \( \geq 5 \), we have

\[ \mathcal{S}^{\text{lci}} \cong [(\tilde{Z}, 0)/D'] \to \mathcal{S} = (Z, 0), \]

where \( (Z, 0) \to (S, 0) \) is the index one cover. In this case \( (Z, 0) = (S, 0) \) and \( (\tilde{Z}, 0) \to (S, 0) \) is the discriminant cover.

2. if \( (S, 0) \) is the \( \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5 \)-quotient of a simple elliptic singularity, the \( \mathbb{Z}_2 \)-quotient of a cusp or a degenerate cusp singularity with embedded dimension \( \geq 5 \), then we have

\[ \mathcal{S}^{\text{lci}} \cong [(\tilde{Z}, 0)/D] \to S = (Z, 0), \]

where \( (\tilde{Z}, 0) \to (S, 0) \) is the universal abelian cover. The map factors through the index one cover map \( (Z, 0) \to (S, 0) \). Therefore we have the morphism \( \mathcal{S}^{\text{lci}} \cong [(\tilde{Z}, 0)/D] \to \mathcal{S} = [(Z, 0)/\mathbb{Z}_r] \) of stacks, where \( r \) is the local index of the quotient singularity.

6.5. Flat family of lci covering Deligne-Mumford stacks. From Theorem 6.5 and Theorem 6.1, the deformation of the lci covering Deligne-Mumford stack \( \mathcal{S}^{\text{lci}} \) induces a \( \mathbb{Q} \)-Gorenstein deformation of \( \mathcal{S} \). We make this more precise in families.
For an s.l.c. surface germ \((S, 0)\), we have the lci cover \(\tilde{S} \to S\) with transformation group \(D\) such that the lci-covering Deligne-Mumford stack \(\mathcal{S}^{\text{lc}}\) is given by \([\tilde{S}/D]\). First we have the following result.

**Proposition 6.10.** Suppose that we have a curve \(C\) and let \(S \to C\) be a \(\mathbb{Q}\)-Gorenstein one-parameter deformation of the s.l.c. surface \(S_0\) with only simple elliptic singularities, cusps or degenerate cusps (with local embedded dimension \(\geq 5\)), or the \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6\) quotients of simple elliptic singularities, \(\mathbb{Z}_2\) quotient of cusps, and \(S_1\) has RDP singularities. Then, if around \(P \in S_0 \subset S\), there exists a \(D\)-equivariant deformation \(\tilde{S}\) of \(\tilde{S}_0\) which induces the local deformation of \(S \to C\), then there exists a deformation \(\mathcal{S}^{\text{lc}} \to C\) of the lci-covering Deligne-Mumford stacks which induces the \(\mathbb{Q}\)-Gorenstein one-parameter deformation \(S \to C\).

**Proof.** The lci-covering Deligne-Mumford stack \(\mathcal{S}^{\text{lc}}\) and \(S\) are the same when removing the finite singular points of simple elliptic singularities, cusps or degenerate cusps. Thus if locally around the singular points the \(\mathbb{Q}\)-Gorenstein deformation is induced by the deformation of the lci-covering Deligne-Mumford stack, then the result is true globally. \(\Box\)

**Remark 6.11.** Comparing with Example[7] and Example[4], it is interesting to study the equivariant smoothing of cusp and quotient-cusp singularities. We hope the equivariant Looijenga’s conjecture also holds; see [18] and [26].

Let \(A\) be a \(k\)-algebra, and let \(S/A\) be a family of s.l.c. surfaces. Let \(\mathcal{S}/A\) be the family of the corresponding index one covering Deligne-Mumford stacks.

**Lemma 6.12.** Let \(S/A\) be a \(\mathbb{Q}\)-Gorenstein deformation family of s.l.c. surfaces. Let \(\pi : \mathcal{S} \to S\) be the corresponding index one covering Deligne-Mumford stack and \(\pi^{\text{lc}} : \mathcal{S}^{\text{lc}} \to S\) be the corresponding lci covering Deligne-Mumford stack. For the diagram

\[
\begin{array}{ccc}
\mathcal{S}^{\text{lc}} & \xrightarrow{\pi^{\text{lc}}} & \mathcal{S} \\
\downarrow \pi^{\text{lc}} & & \downarrow \pi \\
S, & & S,
\end{array}
\]

we have that \((\pi^{\text{lc}})^* \omega_{\mathcal{S}/A} \cong \omega_{\mathcal{S}^{\text{lc}}/A}\).

**Proof.** For the isomorphism of the dualizing sheaves, note that for each fiber \(S_t\) of the family \(S/A\), the dualizing sheaf of the index one covering Deligne-Mumford stack is \(\omega_{S_t} \cong \omega_{Z_t}^r\) for each singularity germ \((S_t, 0)\), where \(r\) is the index of the s.l.c. surface germ \((S_t, 0)\). We look at the diagram (6.4.1) at any singularity germ. For a germ singularity \((S, 0)\), let \(\pi : Z \to S\) be the index one cover such that \([Z/Z_r] \cong S\) and the diagram (6.5.1) is given by

\[
\begin{array}{ccc}
Z^{\text{lc}} & \xrightarrow{\pi^{\text{lc}}} & Z \\
\downarrow \pi^{\text{lc}} & & \downarrow \pi \\
S, & & S,
\end{array}
\]

In the case that \((S, 0)\) is a simple elliptic singularity, cusp or degenerate cusp singularity, since \((S, 0)\) is Gorenstein, then the index one cover is itself; i.e., \(Z = S\). In this case we only have the morphism \(\pi : Z^{\text{lc}} \to S\) where \(Z^{\text{lc}} \to S\) is the discriminant cover with transformation group \(D\) constructed in Theorem (6.5) and \(\mathcal{S}^{\text{lc}} = [Z^{\text{lc}}/D]\). Since \(Z^{\text{lc}}\) is l.c.i., it follows that \((\pi^{\text{lc}})^* \omega_S \cong \omega_{Z^{\text{lc}}}\). This is because the dualizing sheaves \(\omega_S, \omega_{Z^{\text{lc}}}\) can be
given by the minimal resolutions:

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi_{\text{li}}} & S \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
X & \xrightarrow{\pi} & X' \\
\end{array}
\]

see [69, Lemma 1.1].

In the case that \((S, 0)\) is the \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6\)-quotients of a simple elliptic singularity or the \(\mathbb{Z}_2\)-quotient of a cusp or a degenerate cusp singularity, we really have the diagram 6.5.2 such that \((S, 0)\) is a rational singularity. Then, \(Z_{\text{li}} \to S\) is the universal abelian cover with transformation group \(D = H_1(\Sigma, \mathbb{Z})\) where \(\Sigma\) is the link of the singularity. Therefore \(\mathcal{E}_{\text{li}} \cong [Z_{\text{li}}/D]\). In this case \(\omega_Z = \omega_S^{[r]}\) where \(r\) is the index of the singularity. The dualizing sheaf \(\omega_{\mathcal{E}}\) is the \(\mathbb{Z}_N\)-equivariant \(\omega_Z\). Thus, taken as the equivariant dualizing sheaves, \(\omega_{Z_{\text{li}}} \cong (\hat{\pi})^*\omega_Z\), which can be seen from the minimal resolutions in diagram 6.5.3 again and \(\omega_Z\) is constructed from \(\omega_X(A)\) where \(A\) is the exceptional divisor. \(\square\)

### 6.6. Deformation and obstruction of lci covering Deligne-Mumford stacks.

Let \(A\) be a \(k\)-algebra and \(\mathcal{E}_{\text{li}}/A\) be a flat family of lci covering Deligne-Mumford stacks. Let \(\Omega^*_{\mathcal{E}_{\text{li}}/A}\) be the cotangent complex of \(\mathcal{E}_{\text{li}}/A\) and let \(J\) be a finite \(A\)-module. We also let \(\pi_{\text{li}} : \mathcal{E}_{\text{li}} \to S\) be the map to its coarse moduli space. Define

\[
\hat{T}_{\mathcal{QG}}(S/A, J) := \text{Ext}^t(\Omega^*_{\mathcal{E}_{\text{li}}/A}, \mathcal{O}_{\mathcal{E}_{\text{li}}/A})
\]

\[
\hat{T}_{\mathcal{QG}}(S/A, J) := \pi_{\text{li}}^* \text{Ext}^t(\Omega^*_{\mathcal{E}_{\text{li}}/A}, \mathcal{O}_{\mathcal{E}_{\text{li}}/A}).
\]

If the s.l.c. surface \(S\) admits a finite group \(G\) action, then its index one covering Deligne-Mumford stack and lci covering Deligne-Mumford stack also admit \(G\)-actions. We let \(\left(\hat{T}_{\mathcal{QG}}(S/A, J)\right)^G\) and \(\left(\hat{T}_{\mathcal{QG}}(S/A, J)\right)^G\) be the \(G\)-invariant parts of the extension groups.

We have similar results as in Proposition 4.12 and Proposition 4.13 for lci covering Deligne-Mumford stacks.

**Proposition 6.13.** Let \(S/A\) be a \(\mathbb{Q}\)-Gorenstein family of s.l.c. surfaces. The corresponding index one covering Deligne-Mumford stack and lci covering Deligne-Mumford stack are denoted by \(\mathcal{G}/A\) and \(\mathcal{E}_{\text{li}}/A\) respectively. Suppose that \(A' \to A\) is an infinitesimal extension. Let \(S'/A'\) be a \(\mathbb{Q}\)-Gorenstein deformation of \(S/A\), and \(\mathcal{G}'/A'\) be the index one covering Deligne-Mumford stack. Then we have

1. \(S'/A' \to \mathcal{G}'/A'\)

   give a bijection between the set of isomorphism classes of \(\mathbb{Q}\)-Gorenstein deformations of \(S/A\) over \(A'\) and the set of isomorphism classes of deformations of \(\mathcal{G}/A\).

2. any isomorphism class of the deformations \((\mathcal{G}'_{\text{li}})/A'\) of the lci covering Deligne-Mumford stack induces an isomorphism class of deformations of the index one covering Deligne-Mumford stacks

\[
(\mathcal{G}'_{\text{li}})/A' \to \mathcal{G}'/A'
\]

which in turn induces an isomorphism class of \(\mathbb{Q}\)-Gorenstein deformations of \(S/A\) over \(A'\)

\[
(\mathcal{G}'_{\text{li}})/A' \to S'/A'.
\]

**Proof.** The case \(S'/A' \to \mathcal{G}'/A'\) for the index one covering Deligne-Mumford stack is Proposition 4.12. For the second case, from Remark 6.2 and Remark 6.6, any deformation of the lci covering Deligne-Mumford stack induces a \(\mathbb{Q}\)-Gorenstein deformation of the surface singularity \(S/A\). \(\square\)
**Remark 6.14.** We should point out again that it is not known whether any deformation of the index one covering Deligne-Mumford stack \( \mathcal{S} / A' \) is induced by the deformation \( (\mathcal{S}')^{\text{lci}} / A' \) of the lci covering Deligne-Mumford stack.

**Proposition 6.15.** Let \( S_0 / A_0 \) be a \( G \)-equivariant \( \mathbb{Q} \)-Gorenstein family of s.l.c. surfaces and let \( J \) be a finite \( A_0 \)-module. We let \( (\mathcal{S}_0)^{\text{lci}} / A_0 \) be the corresponding lci covering Deligne-Mumford stack. Then we have that

1. the set of isomorphism classes of \( G \)-equivariant \( \mathbb{Q} \)-Gorenstein deformations of \( S_0 / A_0 \) which are induced from the deformations of the lci covering Deligne-Mumford stack \( (\mathcal{S}_0)^{\text{lci}} / A_0 \) over \( A_0 + J \) is naturally an \( A_0 \)-module and is canonically isomorphic to \( \mathcal{T}^{\mathbb{Q}}_{\mathcal{O}}(S / A, J)^G \).

2. let \( A' \to A \to A_0 \) be the infinitesimal extensions and the kernel of \( A' \to A \) is \( J \). Let \( S / A \) be a \( G \)-equivariant \( \mathbb{Q} \)-Gorenstein of \( S_0 / A_0 \). Then we have

   (a) there exists a canonical element \( \text{ob}(S / A, A') \in \mathcal{T}^{\mathbb{Q}}_{\mathcal{O}}(S / A, J)^G \) called the obstruction class. It vanishes if and only if there exists a \( G \)-equivariant \( \mathbb{Q} \)-Gorenstein deformation \( S' / A' \) over \( S / A \) which are induced from the deformation of the lci covering Deligne-Mumford stack \( (S')^{\text{lci}} / A' \).

   (b) if \( \text{ob}(S / A, A') = 0 \), then the set of isomorphism classes of \( G \)-equivariant \( \mathbb{Q} \)-Gorenstein deformations \( S' / A' \) is an affine space underlying \( \mathcal{T}^{\mathbb{Q}}_{\mathcal{O}}(S_0 / A_0, J)^G \).

**Proof.** From Theorem 6.5, this is a basic result of deformation and obstruction theory of algebraic varieties; see [35]. □

**Lemma 6.16.** Let \( S \) be an s.l.c. surface, and \( S^{\text{lci}} \to S \) be the lci covering Deligne-Mumford stack. Then we have that \( \mathcal{T}^{\mathbb{Q}}_{\mathcal{O}}(S, \mathcal{O}_S) = 0 \) for \( i \geq 3 \).

**Proof.** There is also a local to global spectral sequence

\[
E_2^{p,q} = H^p(\mathcal{T}^q_{\mathcal{O}}(S, \mathcal{O}_S)) \Rightarrow \mathcal{T}^{p+q}_{\mathcal{O}}(S, \mathcal{O}_S).
\]

Since \( S \) is of general type, the higher cohomology \( H^p(F) = 0 \) for any sheaf \( F \) and \( p \geq 3 \). The sheaf \( \mathcal{T}^q_{\mathcal{O}}(S, \mathcal{O}_S) = 0 \) when \( q \geq 2 \) since \( S^{\text{lci}} \) only has l.c.i. singularities. Thus, from the local to global spectral sequence we get the result in the lemma. □

**6.7. The moduli stack of lci covers.** We consider the families \( S^{\text{lci}} / T \) of lci covering Deligne-Mumford stacks. Extending the result in §5.1 we define the moduli stack of lci covers over the moduli stack \( M \) of s.l.c. surfaces.

**Definition 6.17.** We define the families over a scheme \( T \) in the following diagram

\[
(6.7.1)
\]

which is the family version of Diagram [6.4.1] where

1. \( \mathcal{T} : S \to T \) is a \( \mathbb{Q} \)-Gorenstein deformation family of s.l.c. surfaces;
2. \( f : S \to T \) is the corresponding index one covering Deligne-Mumford stack;
3. \( f^{\text{lci}} : S^{\text{lci}} \to T \) is the lifting lci covering Deligne-Mumford stack of \( \mathcal{T} \), such that the morphism \( \pi^{\text{lci}} : S^{\text{lci}} \to S \) factors through the morphism \( \pi : S \to S \);
4. both \( S^{\text{lci}} \) and \( S \) have the same coarse moduli space \( S \); and
5. the set \( \{ \mathcal{T} : S \to T \} \) of the coarse moduli spaces of the families must satisfy the conditions in [6.2.7].
(6) for the flat family \( f^{\text{lci}}: \mathcal{E}^{\text{lci}} \to T \), let \((S, x)\) be a singularity germ in \( S = T^{-1}(0) \) such that \((\tilde{S}, x) \to (S, x)\) is the lci cover with transformation group \( D \). We make the following conditions.

(a) suppose that the flat family \( f: S \to T \) lies on the smoothing component \( M^\text{sm} \) (i.e., the component containing smooth surfaces) of \( M = \overline{M}_{k^2,\chi,N} \). We may assume that \( f: S \to T = \text{Spec}(k(t)) \) is a one-parameter smoothing of the singularity \((S, x)\). If the lci cover \((\tilde{S}, x)\) is locally given by

\[
\text{Spec } k[x_1, \ldots, x_\ell]/(h_1, \ldots, h_{\ell-2}),
\]

then the flat family \( f^{\text{lci}}: \mathcal{E}^{\text{lci}} \to T \) is given by the \( D \)-equivariant smoothing of the singularity \((\tilde{S}, x)\) which is given by:

\[
\text{Spec } k[x_1, \ldots, x_\ell, t]/(h_1 - t, \ldots, h_{\ell-2} - t),
\]

where \( D \) acts on \( t \) trivially. The detail definition of the smoothing component is in \( \S 6.8 \)

(b) suppose that the flat family \( f: S \to T \) lies on a deformation component of \( M = \overline{M}_{k^2,\chi,N} \) containing the same type of singularities as \((S, x)\), then we require that the flat family \( f^{\text{lci}}: \mathcal{E}^{\text{lci}} \to T \) induces the family \( f: S \to T \).

(7) for any singularity germ \((S, x)\) in a family \( \overline{f}: \overline{S} \to T \), if \((S, x)\) is a simple elliptic singularity, a cusp or a degenerate cusp singularity, or a cyclic quotient of them which does not satisfies the condition in Condition 4.10 then the lci lifting \((\tilde{S}, x)\) is nontrivial such that we have the Deligne-Mumford stack \([\overline{S}/D]\), we require that they all belong to the lci covers constructed in Theorem 6.4, Lemma 6.5, Lemma 7\?, Theorem 6.7 and Theorem 6.8.

If \((S, x)\) is any other singularity germ, then we only take the index one covering Deligne-Mumford stack \( f: \mathcal{E} \to T \).

We define the functor:

\[
M^{\text{lci}} = \overline{M}^{\text{lci},G}_{k^2,\chi,N}: \text{Sch}\_k \to \text{Groupoids}
\]

which sends

\[
T \mapsto \{ f^{\text{lci}}: \mathcal{E}^{\text{lci}} \to T \}
\]

where \( \{ f^{\text{lci}}: \mathcal{E}^{\text{lci}} \to T \} \) is the groupoid of isomorphism classes of families of lci covering Deligne-Mumford stacks \( \mathcal{E}^{\text{lci}} \to T \).

Remark 6.18. From the construction of lci covering Deligne-Mumford stack \( \mathcal{E}^{\text{lci}} \to S \) in \( \S 6.7 \) and \( \S 6.8 \) and the family of lci covering Deligne-Mumford stacks in \( \S 6.6 \), we only take the lci cover for an s.l.c. surface \( S \) with simple elliptic singularities, cusp or degenerate cusp singularities, or cyclic quotients of them with local embedded dimension > 5.

Let \( S_t \) be an s.l.c. surface such that its index one covering Deligne-Mumford stack \( \mathcal{E}_t \to S_t \) is a fiber of \( f: \mathcal{E} \to T \). Look at the diagram \( \S 6.7.2 \) again, from Lemma 6.12 we have \( (\tilde{\pi})^* \omega_{\mathcal{E}/A} \cong \omega_{\mathcal{E}^{\text{lci}}/A} \) (by taking \( T = \text{Spec}(A) \)). Thus, we have that

\[
K^2 = K^2_{S_t} = \frac{1}{N^2} (\omega_{S_t}^N \cdot \omega_{S_t}^N) = (\omega_{\mathcal{E}_t} \cdot \omega_{\mathcal{E}_t}) = (\omega_{\mathcal{E}^{\text{lci}}} \cdot \omega_{\mathcal{E}^{\text{lci}}}),
\]

where \( N \in \mathbb{Z}_{>0} \) can be chosen to satisfy that \( \omega_{S_t}^N \) is invertible.

Let \( M := \overline{M}^G_{k^2,\chi,N} \) be the moduli functor which parametrizes the flat families \( \overline{f}: \overline{S} \to T \) of Q-Gorenstein deformations of s.l.c. surfaces induced from the flat families \( f^{\text{lci}}: \mathcal{E}^{\text{lci}} \to T \) of lci covering Deligne-Mumford stacks in Definition 6.17. Then \( M \) is a Deligne-Mumford stack.

From Remark 6.18 our moduli space \( M \) may loose some components and branches in the KSBA compactification of moduli space of s.l.c. surfaces with invariants \( K^2, \chi \) such that the Q-Gorenstein deformations of simple elliptic singularities, cusp or degenerate cusp
singularities, or cyclic quotients of them with local embedded dimension > 5 cannot be obtained from the equivariant deformations of the lci covers.

**Theorem 6.19.** The functor $M^{\text{lci}}$ represents a Deligne-Mumford stack. Moreover, there exists a proper morphism

$$f^{\text{lci}} : M^{\text{lci}} \to M$$

which factors through the morphism $f : M^{\text{ind}} \to M$. Thus there is a commutative diagram

(6.7.2)

$$\begin{array}{ccc}
M^{\text{lci}} & \xrightarrow{f} & M^{\text{ind}} \\
\downarrow{f^{\text{lci}}} & & \downarrow{f} \\
M & & M
\end{array}$$

In particular, if $N$ is large divisible enough, the stack $M^{\text{lci}}$ is a proper Deligne-Mumford stack with projective coarse moduli space. The morphism $f^{\text{lci}}$ in the above diagram induces a finite morphism on their coarse moduli spaces.

**Proof.** The proof is from the above construction of lci covering Deligne-Mumford stacks, and has the same method as in Theorem 5.1. From [68], the functor $M^{\text{lci}}$ is a stack. There is a natural morphism $f^{\text{lci}} : M^{\text{lci}} \to M$ of stacks by sending any family $\{S^{\text{lci}} \to T\}$ to the corresponding family $\{S \to T\}$ in $M$.

To show $M^{\text{lci}}$ is a Deligne-Mumford stack, we show that the diagonal morphism

$$M^{\text{lci}} \to M^{\text{lci}} \times_k M^{\text{lci}}$$

is representable and unramified. This is from the following reason. If we have two objects $(f : \mathcal{E}^{\text{lci}} \to T)$ and $(f' : (\mathcal{E}')^{\text{lci}} \to T)$ in $M^{\text{lci}}(T)$, then the isomorphism functor of the two families $\text{Isom}_T(\mathcal{E}^{\text{lci}}, (\mathcal{E}')^{\text{lci}})$ is represented by a quasi-projective group scheme $\text{Isom}_T(\mathcal{E}^{\text{lci}}, (\mathcal{E}')^{\text{lci}})$ over $T$. Still let $(\mathcal{F} : S \to T)$ and $(\mathcal{F}' : S' \to T)$ be the corresponding Q-Gorenstein deformation families of s.l.c. surfaces over $T$. The isomorphism functor $\text{Isom}_T(S, S')$ is represented by a quasi-projective group scheme $\text{Isom}_T(S, S')$ over $T$. Look at the following diagram

$$\begin{array}{ccc}
\mathcal{E}^{\text{lci}} & \cong & (\mathcal{E}')^{\text{lci}} \\
\downarrow \cong & & \downarrow \cong \\
S & \cong & S'.
\end{array}$$

Any isomorphism $\mathcal{E}^{\text{lci}} \cong (\mathcal{E}')^{\text{lci}}$ induces an isomorphism $S \cong S'$ on the coarse moduli spaces and the isomorphisms coming from the local stacky isotropy groups induces the same isomorphism on the coarse moduli spaces. Thus, the functor is represented by a quasi-projective scheme $\text{Isom}_T(\mathcal{E}^{\text{lci}}, (\mathcal{E}')^{\text{lci}})$ over $\text{Isom}_T(S, S')$ and is also unramified over $T$ since its geometric fibers are finite.

From the proof of Theorem 5.1 there is a cover $\varphi : \mathcal{E} \to M$. Then the fiber product $\mathcal{E}^{\text{lci}}$ in the diagram

$$\begin{array}{ccc}
\mathcal{E}^{\text{lci}} & \to & M^{\text{lci}} \\
\downarrow & & \downarrow \\
\mathcal{E} & \to & M
\end{array}$$

serves as a cover over the stack $M^{\text{lci}}$. This is because for a given family of s.l.c. surface $S/T$, there is a family $\mathcal{E}^{\text{lci}}/T$ of lci covering Deligne-Mumford stacks.
We show that the morphism $f^{lci}: M^{lci} \to M$ is proper. We use the valuative criterion for properness and consider the following diagram

\[
\begin{array}{c}
\text{Spec}(K') \ar[r] \ar[d] & \text{Spec}(K) \ar[r] \ar[d] & M^{lci} \\
\text{Spec}(R') \ar[r] & \text{Spec}(R) & M
\end{array}
\]

where $R$ is a valuation ring with field of fractions $K$, and residue field $k$. In this case we can take $R = k[[t]]$ and $K = k((t))$. The morphism $\text{Spec}(R) \to M$ corresponds to a flat Q-Gorenstein family $f: S \to \text{Spec}(R)$ of s.l.c. surfaces. We may assume that $\text{Spec}(R) \to M$ lies on the smoothing component of the moduli stack $M$, since if $\text{Spec}(R) \to M$ lies in other component of $M$, then from condition (6) in Definition 6.17 we always have that the family $f: S \to \text{Spec}(R)$ is induced from a flat family of lci covering DM stacks.

Now let $S$ be the s.l.c. surface over $0 = \text{Spec}(k)$ in the family $f: S \to \text{Spec}(R)$. Over a singularity germ $(S, x)$ in $S$, we assume that the singularity is given by

\[\text{Spec}(k[x_1, \ldots, x_s]/I),\]

where $I$ is the ideal of the singularity. Let $I = (g_1, \ldots, g_l)$ be the generators. Then the singularity germ $(S, x)$ is given by

\[\text{Spec}(R[x_1, \ldots, x_s, t]/I_1),\]

where $I_1 = (g'_1, \ldots, g'_l)$ and $g'_i$ are polynomials involving $t$. Taking $t = 0$, we get $g'_i = g_i$. Since the family $f: S \to \text{Spec}(R)$ is flat, the parameter $t$ cannot not happen in the factors of the monomial terms of $g'_i$. For instance, we choose $I_1 = (g_1 - t, \ldots, g_l - t)$.

Suppose that $f^{lci}: S^{lci} \to \text{Spec}(K)$ is the lifting of $f: S \to \text{Spec}(R)$ to a lci-covering Deligne-Mumford stacks at the generic point. Then over the the singularity germ $(S, x)$, we have that the local lci cover $(\tilde{S}, x)$ is given by

\[\text{Spec}(K[x_1, \ldots, x_s]/J),\]

where $J = (h_1, \ldots, h_{\ell-2})$ and $h_j$ are polynomials involving the variable $t$. Here the ideal $J$ has $\ell - 2$ generators since the singularity $(\tilde{S}, x)$ is a l.c.i. singularity. The quotient of $\text{Spec}(K[x_1, \ldots, x_s]/J)$ by the finite transformation group $D$ gives $\text{Spec}(K[x_1, \ldots, x_s]/I_1)$, or equivalently, the invariant ring $(K[x_1, \ldots, x_s]/J)^D$ by the transformation group $D$ gives $K[x_1, \ldots, x_s]/I_1$.

The finite group $D$ acts on the variety $\text{Spec}(K[x_1, \ldots, x_s]/J)$. The field $K$ is the fraction field of $R$ with the uniformizer $\ell$. The generators $h^i_j$ for $1 \leq j \leq \ell - 2$ may contain powers of $t$. We let $I$ be the index set such that for $i \in I$, $c_i \in \mathbb{Z}$, and $t^{c_i}$ is a factor of some term in $h^i_j$. Note that $c_i$ may be negative at the moment. Let $d \in \mathbb{Z}_{>0}$ be a large integer depending on the set $\{c_i | i \in I\}$. We take the finite cover

\[\text{Spec}(K') \to \text{Spec}(R)\]

by

\[t \mapsto t^d.\]

Let $K'$ be the field of fractions of $R'$. We choose $d$ large enough so that the group $D$ acts on the parameter $t'$ trivially. Now the polynomials $h^i_j$ for $1 \leq j \leq \ell - 2$ become $h^i_j$ for $1 \leq j \leq \ell - 2$. Since the singularity germ $(\tilde{S}, x)$ is given by an lci cover $(\tilde{S}, x)$, and $D$ acts on the parameter $t'$ trivially, then from condition (6) in Definition 6.17 the $D$-equivariant smoothing of the lci cover $(\tilde{S}, x)$ is given by

\[\text{Spec}(K'[x_1, \ldots, x_s]/J').\]
The generators $h_i' = h_i - t'$. The morphism $\text{Spec}(K) \to M_{\text{lci}}$ naturally extends to the morphism $\text{Spec}(K') \to M_{\text{lci}}$. Therefore, taking $t' = 0$, we get the lci covering Deligne-Mumford stack $[\tilde{S}/D]$ for the s.l.c. surface $S$. This gives the unique morphism $\text{Spec}(R') \to M_{\text{lci}}$ which completes the valuative criterion for properness.

If $N$ is large divisible enough, then the stack $M$ is a proper Deligne-Mumford stack with projective coarse moduli space. Therefore, the morphism $f_{\text{lci}} : M_{\text{lci}} \to M$ in the diagram induces a proper morphism on their coarse moduli spaces since $f_{\text{lci}}$ is proper.

Finally we show that the morphism $f_{\text{lci}} : M_{\text{lci}} \to M$ is a finite morphism which induces a finite morphism on their coarse moduli spaces. Let $S \in M$ be an s.l.c. surface and $(S, x)$ be an s.l.c. singularity germ. It is sufficient to show that $(f_{\text{lci}})^{-1}(S)$ only contains finite elements. Therefore, it is enough to show that there are finite of lci liftings for the singularity germ $(S, x)$. Since there are only two cases for the log canonical surface singularities $(S, x)$ which need to take the lci covers. We prove by cases.

Case 1. If the singularity germ $(S, x)$ has index bigger than 1, then it is either the $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$-quotient of a simple elliptic singularity, the $\mathbb{Z}_2$-quotient of a cusp, or the $\mathbb{Z}_2$-quotient of a degenerate cusp singularity. Then from Theorem 6.1, the singularity $(S, x)$ is a rational singularity and the lci cover is the universal abelian cover which is unique. Thus $(f_{\text{lci}})^{-1}(S)$ only contains one geometric element.

Case 2. If the singularity germ $(S, x)$ has index 1, then it is either a simple elliptic singularity, a cusp, or a degenerate cusp singularity. From Theorem 6.7, since we take the lci cover for a degenerate cusp singularity $(S, x)$ using the universal abelian covers, thus the lci lifting is unique.

For the case of a simple elliptic singularity $(S, x)$, the lci covers are either from Lemma 6.20 or Theorem 6.7 and the lci covers $(\tilde{S}, x)$ will reduce the self-intersection number of the exceptional elliptic curve, therefore, there are only finite of lci covers $(\tilde{S}, x)$ such that the self-intersection number becomes $1, 2, 3, 4$ which imply the singularity is l.c.i.

The last case is the cusp singularity $(S, x)$ which is a bit complicated. If the lci covers are from Theorem 6.7, it is not hard to see that the lci lifting is unique.

In other cases, let $\Sigma$ be the link of the singularity $(S, x)$, and $\pi_1(\Sigma) = \mathbb{Z}^2 \rtimes \mathbb{Z}$ be the fundamental group. From the proof of Lemma 6.3 in §6.2 we form the following diagram:

$$
\begin{array}{ccc}
H \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \tau \\
\downarrow & & \downarrow \\
\pi_1(\Sigma) & \longrightarrow & H_1(\Sigma, \mathbb{Z}) \\
\downarrow & & \downarrow \\
D & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0,
\end{array}
$$

where $\mathbb{Z} \oplus \tau$ is the abelianization of $H \times \mathbb{Z}$, and $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z} \oplus (H_1(\Sigma, \mathbb{Z}))_{\text{tor}}$. The transformation group $D'$ for the lci cover $(\tilde{S}, x)$ is obtained from $D$ by taking discriminant cover. Since there are finite morphisms $\text{Hom}(\pi_1(\Sigma), D)$, we conclude that there are only finite possibilities for the covers determined by $H \times \mathbb{Z}$. Therefore, the preimage $(f_{\text{lci}})^{-1}(S)$ contains only finite elements. This proves that the morphism $f_{\text{lci}}$ is finite. □

Remark 6.20. (1) We only take lci covers $(\tilde{S}, x) \to (S, x)$ for the simple elliptic, cusp and degenerate cusp singularities $(S, x)$ with local embedded dimension $> 5$. For such singularities, from the construction in §6.1, Example §6.2 and Example §6.3, the lci cover $(\tilde{S}, x)$ is always a locally complete intersection singularity with the transformation group $D$-action. Since a locally complete intersection singularity admits a $D$-equivariant
Then we let \( M \) inducing the smoothing (which takes the action trivial on the parameter \( t \)), the quotient gives the Q-Gorenstein smoothing of \((S, x)\). The situation exactly matches the condition (6) in Definition 6.17. Thus the valuative criterion for properness always holds case by case for such singularities.

(2) The morphism \( f^{\text{lci}} : M^{\text{lci}} \to M \) is not necessarily representable.

We have the following corollary.

**Corollary 6.21.** The moduli stack \( M \) is a projective Deligne-Mumford stack.

**Proof.** From the conditions in Definition 6.17 the flat families \( \overline{f} : S \to T \) of Q-Gorenstein deformations of s.l.c. surfaces definitely satisfy the conditions in [43 Theorem 2.6], i.e., the moduli functor is separated, complete, semi-positive, and bounded. Separateness is from the definition of the flat families, and semi-positiveness, boundedness hold since \( M \) is a subfunctor of the KSBA moduli functor. For completeness, suppose that \( \overline{f} : S_{\text{gen}} \to K \) is a Q-Gorenstein family of s.l.c. surfaces over the generic point of the spectrum \( \text{Spec}(R) \) of a discrete valuation ring \( R \). Then after a finite cover \( \text{Spec}(R') \to \text{Spec}(R) \), from the above proof in Theorem 6.19 the lifting family \( \overline{f} \) of \( \text{Spec}(R') \) of lci covers induces a family \( \overline{f} : S \to \text{Spec}(R') \). Thus the moduli functor \( M \) is complete. Therefore the moduli functor \( M \) is represented by a proper Deligne-Mumford stack with projective coarse moduli spaces if \( N \) is large divisible enough.

**Corollary 6.22.** Let \( M = \overline{M}_{K^2, \chi, N} \) be a connected component of the moduli stack of stable s.l.c. surfaces with invariants \( K^2, \chi, N \). If each s.l.c. surface in \( M \) satisfies Condition 6.16 then the moduli stack \( M^{\text{lci}} \) of lci covers is the same as \( M^{\text{ind}} \). If moreover, every s.l.c. surface \( S \in M \) is l.c.i., then \( M^{\text{lci}} = M^{\text{ind}} = M \).

**Proof.** The corollary is from the construction of the lci covering Deligne-Mumford stacks.

### 6.8. The equivariant smoothing component

We fix the moduli stack \( M = \overline{M}_{K^2, \chi, N} \) for a large divisible enough \( N \in \mathbb{Z}_{>0} \). Recall that a stable surface \( S \in M \) is called **smoothable** if there exists a one-parameter family \( f : S \to T \) of stable s.l.c. surfaces such that \( f^{-1}(0) = S \), and the generic fiber \( f^{-1}(t) \) for \( t \neq 0 \) is either a smooth surface or an s.l.c. surface with only DuVal singularities. Let \( M^{\text{sm}} := \overline{M}_{K^2, \chi, N}^{\text{sm}} \) be the subfunctor of \( M = \overline{M}_{K^2, \chi, N} \) where all the fibers are smoothable surfaces. Then from [43, 5.6 Corollary], [3], [28] the moduli stack \( M^{\text{sm}} \subset M \) is a projective closed substack of \( M \) with projective coarse moduli space.

Let us consider s.l.c. surface singularity germs \((S, x)\) in \( M \) such that the singularities are the ones in Remark 6.18. We always consider the smoothings of the germs \((S, x)\) in \( M \) that are obtained from the equivariant smoothings of the lci cover

\[
\pi^{\text{lci}} : (\mathcal{S}, x) \to (S, x)
\]

with transformation group \( D \). An argument of the cusp singularities can be found in [38]. Then we let \( M^{\text{sm}}_{\text{eq}} \subset M^{\text{sm}} \) be the equivariant smoothing components of \( M \). Then similarly from condition (6) of Definition 6.17 and the boundedness of [43, 5.6 Corollary], [3], [28], \( M^{\text{sm}}_{\text{eq}} \subset M \) is a projective closed substack with projective coarse moduli space. The equivariant smoothing component \( M^{\text{sm}}_{\text{eq}} \) can be taken as the closure of the open locus of the component in \( M^{\text{sm}} \) consisting of smoothing components by lci-covers, see [39].

We include a review for the dimensions of the smoothing components. The lci cover \((\mathcal{S}, x)\) admits a \( D \)-equivariant one-parameter smoothing

\[
(6.8.1) \quad f^{\text{lci}} : (\mathcal{S}, x) \to \Delta
\]

inducing the smoothing \((S, x) \to \Delta \) of \((S, x)\), where \( \Delta \) is an analytic disc.
The germ \((S, x)\) has a miniversal deformation
\[
(S, x) \xrightarrow{\tilde{f}} (\tilde{S}, x)
\]
where \((T, t) \subset M\). We know that \((S, x)\) has non-zero obstruction spaces \(\mathcal{T}_{QC}^q(S)\) for \(q \geq 2\), see [37]. This implies that \((T, t)\) is in general singular and may contain irreducible components of various dimensions. Let
\[
(T', t) \subset (T, t)
\]
be the smoothing component, i.e., the component in \(T\) such that \(F\) has smooth generic fibers or generic fibers only with DuVal singularities. Let
\[
j : (\Delta, 0) \to (T', t)
\]
be the inclusion of the unit disc to \((T', t)\). Then we have the pullback
\[
f := F^*(j) : (X, x) \to (\Delta, 0)
\]
where we use \((X, x)\) as the one-parameter family.

Let \(\mathcal{O} := \mathcal{O}_{\Delta, 0}\) be the local ring and we have that
\[
\text{Hom}_{\mathcal{O}_{T', t}}(\Omega_{T', t}, \mathcal{O}_{T, t}) \otimes \mathcal{O} \cong \mathcal{T}_{T', t} \otimes_{\mathcal{O}_{T, t}} \mathcal{O}
\]
where \(\mathcal{T}_{T, t}\) is the tangent sheaf of \((T, t)\). For the singularity germ \((S, x)\), we need to work on the index one covers, and for the (higher) tangent sheaves \(\mathcal{T}_{S, x}^q\), we should use \(\mathcal{T}_{QC}^q(S)\).

All the arguments below work for tangent sheaves \(\mathcal{T}_{QC}^q(S)\) for the index one covers and we just fix to general tangent sheaves.

Let \(\mathcal{T}_{X/\Delta, x}^1\) be the relative (higher) tangent sheaves of \(X/\Delta\). From [25, §2], there is a morphism
\[
\Phi : \mathcal{T}_{X/\Delta, x} \to \mathcal{T}_{S, x}
\]
which is coming from the exact sequence:
\[
(6.8.2) \quad 0 \to \mathcal{T}_{X/\Delta, x} \to \mathcal{T}_{X/\Delta, x} \to \mathcal{T}_{S, x} \to \mathcal{T}_{X/\Delta, x}^1 \to \mathcal{T}_{S, x}^1
\]
as in [25, §2]. Then the main result in [25] is:
\[
(6.8.3) \quad \dim(T', t) = \dim_k(\text{Coker}(\Phi)).
\]

Now let
\[
(\bar{S}, x) \xrightarrow{\tilde{f}} (\bar{S}, x)
\]
be the \(D\)-equivariant miniversal deformation family such that \((\bar{T}, t) \subset (T, t)\), since any \(D\)-equivariant deformation family induces a deformation family of \((S, x)\). Let \(j : (\Delta, 0) \to (\bar{T}, t)\) be the inclusion and let
\[
\bar{f} := F^*(j) : (\bar{X}, x) \to (\Delta, 0)
\]
be the \(D\)-equivariant one-parameter family of \((S, x)\) such that \((\bar{X}, x)/D \cong (X, x)\). Thus we have the exact sequence:
\[
(6.8.4) \quad 0 \to \mathcal{T}_{\bar{X}/\Delta, x} \to \mathcal{T}_{\bar{X}/\Delta, x} \to \mathcal{T}_{S, x} \to \mathcal{T}_{\bar{X}/\Delta, x}^1 \to \mathcal{T}_{S, x}^1
\]
and we have the $D$-invariant part:
\[(6.8.5)\quad 0 \to T^D_{X/\Delta x} \xrightarrow{\tilde{f}} T^D_{X/\Delta x} \to T^D_{S_x} \to \left(\mathcal{T}^1_{X/\Delta x}\right)^D \to \left(\mathcal{T}^1_{S_x}\right)^D.
\]

We also have the morphism
\[\Phi^D : T^D_{X/\Delta x} \to T^D_{S_x}.
\]

**Lemma 6.23.** Let $(\tilde{T}, t) \subset (\tilde{T}, t)$ be the $D$-equivariant smoothing component of $(S, x)$, then \[\dim((T', t)) = \dim((\tilde{T}', t)).\]

**Proof.** Same proof as in [25, §2] implies that \[\dim((T', t)) = \dim((\tilde{T}', t)).\] Since $T^D_{X/\Delta x} \cong T_{X/\Delta x'}$ and $T^D_{S_x} \cong T_{S_x}$, we have $\Phi^D = \Phi$. Thus, the result follows from \[6.8.5.\] \[\square\]

Finally we have the following result:

**Theorem 6.24.** Let $M = \overline{M}^{G}_{2,2,2,2}$ be a KSBA moduli stack of s.l.c. surfaces, and let $M^\text{eq} \subset M$ be the equivariant smoothing component. Then there exists a moduli stack $M^{\text{cl,sm}}_\text{eq}$ of lci covers and a finite morphism $f^{\text{cl}} : M^{\text{cl,sm}}_\text{eq} \to M^\text{eq}$.

**Proof.** From Lemma 6.23, we know that the smoothing of bad singularity germs $(S, x)$ in Remark 6.18 are given by the equivariant smoothing of the lci covers. Thus we restrict our moduli functor of lci covers in Definition 6.17 and Theorem 6.19 to $M^{\text{cl,sm}}_\text{eq}$ such that it induces the functor of the smoothing component $M^{\text{sm}}_\text{eq}$. Then the proof in Theorem 6.19 implies the result. \[\square\]

### 7. The virtual fundamental class

#### 7.1. Perfect obstruction theory

In this section we prove that there is a perfect obstruction theory on the moduli stack $\mathcal{M}^{\text{cl}}$ of lci covers over the moduli stack $M$ of s.l.c. surfaces. Let \[\mathcal{M}^{\text{cl}} : \mathcal{M}^{\text{cl}} \to M^{\text{cl}}\]
be the universal family. Let $\mathbb{L} : \mathcal{M}^{\text{cl}} / \mathcal{M}^{\text{cl}}$ be the relative cotangent complex of $p^{\text{cl}}$ and
\[E^{\bullet}_{\mathcal{M}^{\text{cl}}} := Rp^{\text{cl}}_* \left( \mathbb{L} \otimes \mathcal{O}_{\mathcal{M}^{\text{cl}} / \mathcal{M}^{\text{cl}}} \right)[-1].\]

The relative dualizing sheaf $\omega_{\mathcal{M}^{\text{cl}} / \mathcal{M}^{\text{cl}}}$ satisfies the property
\[\omega_{\mathcal{M}^{\text{cl}} / \mathcal{M}^{\text{cl}}} |_{(p^{\text{cl}})^{-1}(t)} \cong \omega_{\mathcal{M}^{\text{cl}} / \mathcal{M}^{\text{cl}}},\]
where $\omega_{\mathcal{M}^{\text{cl}}}$ is the dualizing sheaf of the lci covering Deligne-Mumford stack $\mathcal{M}^{\text{cl}}$ which is invertible.

When restricting to the smoothing component $M^{\text{sm}} \subset M$, we get the universal family $p^{\text{sm}} : \mathcal{M}^{\text{sm}} \to M^{\text{sm}}$ and the complex
\[E^{\bullet}_{\mathcal{M}^{\text{sm}}} := Rp^{\text{sm}}_* \left( \mathbb{L} \otimes \mathcal{O}_{\mathcal{M}^{\text{sm}} / \mathcal{M}^{\text{sm}}} \otimes \omega_{\mathcal{M}^{\text{sm}} / \mathcal{M}^{\text{sm}}} \right)[-1].\]

**Theorem 7.1.** Let $M = \overline{M}^{G}_{2,2,2,2}$ be a connected component of the moduli space of $G$-equivariant stable s.l.c. general type surfaces with invariants $K^2, \chi$, and $f^{\text{cl}} : \mathcal{M}^{\text{cl}} \to M$ be the moduli stack of lci covers over $M$. Then the complex $E^{\bullet}_{\mathcal{M}^{\text{cl}}}$ defines a perfect obstruction theory (in the sense of Behrend-Fantechi)
\[\phi^{\text{cl}} : E^{\bullet}_{\mathcal{M}^{\text{cl}}} \to \mathbb{L}^{\bullet}_{\mathcal{M}^{\text{cl}}}
\]
induced by the Kodaira-Spencer map $\mathbb{L}^{\bullet}_{\mathcal{M}^{\text{cl}}} / \mathcal{M}^{\text{cl}} \to (p^{\text{cl}})^* \mathbb{L}^{\bullet}_{\mathcal{M}^{\text{cl}}}[1]$. 
If we restrict the perfect obstruction theory \( \phi_{\text{lei}} \) to the smoothing component, we get a perfect obstruction theory

\[
\phi_{\text{lei},\text{sm}} : E_{M_{\text{lei},\text{sm}}}^* \to L_{M_{\text{lei},\text{sm}}}^*.
\]

**Proof.** Since the universal family \( p_{\text{lei}} \) is a flat, projective and relative Gorenstein morphism between Deligne-Mumford stacks, Theorem 5.6 [10, Proposition 6.1] implies that \( \phi_{\text{lei}} \) is an obstruction theory. Detailed analysis is the same as Theorem 5.6.

To show that \( \phi_{\text{lei}} \) is a perfect obstruction theory, it is sufficient to show that the complex

\[
E_{M_{\text{lei}}}^* = Rp_{\text{lei}}^* \left( L_{lci/M_{\text{lei}}} \otimes \omega_{lci/M_{\text{lei}}} \right) [-1]
\]

is of perfect amplitude contained in \([-1, 0]\). The complex \( E_{M_{\text{lei}}}^* \), when restricted to every geometric point \( t \) in \( M_{\text{lei}} \), calculates the cohomology \( \tilde{T}_{\text{lei}}(S_t, \mathcal{O}_S) \), where \( \tilde{T}_{\text{lei}}(S_t, \mathcal{O}_S) \) is the lci covering Deligne-Mumford stack corresponding to the point \( t \). From Lemma 6.16, the cohomology spaces \( \tilde{T}_{\text{lei}}(S_t, \mathcal{O}_S) \) only survive when \( i = 1, 2 \), and all the higher obstruction spaces vanish. Therefore the obstruction theory \( \phi_{\text{lei}} \) is perfect. The last statement is similar. \( \square \)

**Corollary 7.2.** Let \( M = \overline{\mathcal{M}}_{K^2, \chi, N}^G \) be the moduli stack of stable \( G \)-surfaces of general type with invariants \( K^2, \chi, N \). If all the s.l.c. surfaces in \( M \) satisfy the Condition 4.16 then the moduli stack \( M_{\text{lei}} \) of lci covers is the same as the moduli stack \( M_{\text{ind}} \), and the obstruction theory for the moduli stack \( M_{\text{ind}} \) of index one covers in Theorem 5.6 is perfect in the sense of Behrend-Fantechi.

**Proof.** If the Condition 4.16 holds, then the index one covering Deligne-Mumford stack \( \mathfrak{S} \to S \) has only l.c.i. singularities. Therefore, the moduli stack \( M_{\text{lei}} = M_{\text{ind}} \), and the obstruction theory in Theorem 5.6 is the same as the obstruction theory in Theorem 7.1. \( \square \)

**Theorem 7.3.** Let \( M = \overline{\mathcal{M}}_{K^2, \chi, N}^G \) be the moduli stack of stable \( G \)-surfaces of general type with invariants \( K^2, \chi, N \). If the moduli stack \( M \) consists of k.l.t. surfaces, then the moduli stack \( M_{\text{lei}} \) of lci covers is the same as the moduli stack \( M_{\text{ind}} \) of index one covers in Diagram 6.7.2.

Moreover, the obstruction theory for the moduli stack \( M_{\text{ind}} \) of index one covers in Theorem 5.6 is perfect in the sense of Behrend-Fantechi, and is the same as the perfect obstruction theory on \( M_{\text{lei}} \) in Theorem 7.1.

**Proof.** If the s.l.c. surfaces \( S \) in \( M \) is k.l.t., then \( S \) must only have cyclic quotient singularities. From the argument in Proposition 3.10 and [12, Proposition 3.10], since the surface \( S \) admits a Q-Gorenstein deformation, the cyclic quotient singularities must have the form

\[
\text{Spec } k[x, y]/(\mu_{rs}^d) = \langle a \rangle, \quad \text{where } \mu_{rs} = \langle a \rangle \text{ and there exists a primitive } r^2s\text{-th root of unity } \eta \text{ such that the action is given by } \eta(x, y) = (\eta x, \eta^{d/s} y) \text{ and } (d, r) = 1.
\]

Thus, the index one cover of \( S \) locally has the quotient

\[
\text{Spec } k[x, y]/(\mu_{rs})
\]

given by \( \phi'(x, y) = (\eta y, (\eta')^{s-1} y) \), which is an \( A_{rs-1} \)-singularity. Therefore the index one covering Deligne-Mumford stack \( \mathfrak{S} \to S \) has only l.c.i. singularities. From the definition of moduli space of lci covering Deligne-Mumford stacks in §6.7, there is no need to take the lci covering for such an s.l.c. surface \( S \). Thus \( M_{\text{lei}} = M_{\text{ind}} \), and the obstruction theory in Theorem 5.6 is the same as the obstruction theory in Theorem 7.1 which is perfect. \( \square \)

**Corollary 7.4.** Let \( p : \mathcal{M} \to M \) be the universal family for the moduli stack \( M \) of stable s.l.c. G-stable surfaces, which is projective and flat. Assume that globally the stack \( M \) consists of l.c.i.
surfaces, then the relative dualizing sheaf $\omega_{/M}$ is relatively Gorenstein, which means $\omega_{/M}$ is a line bundle. The complex
\[ E^\bullet_M := R^p_*(-L^\bullet_{/M} \otimes \omega_{/M})[-1] \]
defines a perfect obstruction theory
\[ \phi : E^\bullet_M \to \mathcal{L}^\bullet_M. \]

Proof. From Corollary \[\ref{ref:4.2}\] $M^{\text{lc}} = M$. The complex $E^\bullet_M$ is of perfect amplitude contained in $[-1, 0]$. This is because $p$ is relative Gorenstein, which means each fiber surface of $p$ is Gorenstein and $R^p_*(-L^\bullet_{/M} \otimes \omega_{/M})$ gives the cohomology spaces $H^i(S, T_S)$ for any fiber of $p$ which vanish except $i = 1, 2$. \qed

Remark 7.5. Let $p : \mathcal{M} \to M$ be the universal family for the moduli stack $M$ of stable s.l.c. $G$-stable surfaces, which is projective and flat. Assume that in the stack $M$ there exist s.l.c. surfaces $S$ containing cyclic quotients of simple elliptic singularities, cusp or degenerate cusp singularities with embedded dimension $> 5$; or the moduli stack $M$ is constructed from non $\mathbb{Q}$-Gorenstein deformations containing s.l.c. surfaces with cyclic quotient singularities of order $> 3$, then from the existence of the higher obstruction spaces $T^i(S, \mathcal{O}_S)$ for such s.l.c. surfaces (see calculations in \[\ref{ref:37}\]), $M$ doesn’t admit a Behrend-Fantechi, Li-Tian style virtual fundamental class.

Remark 7.6. It is therefore interesting to construct explicit examples of the moduli stack of lc covers using birational geometry techniques. If some component of the moduli stack $M$ does not admit a lifting by a moduli stack $M^{\text{lc}}$ of lc covers, then there may exist cusp or degenerate cusp singularities with higher embedded dimension whose Gorenstein deformations can not be obtained from the deformations of the lc covering Deligne-Mumford stacks. For instance, there may exist cusp singularities $(S, x)$ with higher embedded dimension such that, there are boundary divisors in the moduli stack $M$ which are given by the Gorenstein deformation of $(S, x)$, but can not be obtained from the deformation of the corresponding lc covering Deligne-Mumford stack $\mathcal{G}^{\text{lc}} \to S$. Due to the existence of higher obstruction spaces calculated in \[\ref{ref:37}\]. There is no Li-Tian, Behrend-Fantechi style perfect obstruction theory on such components of the moduli stack, hence no such type virtual fundamental class.

For all of these cases above involving higher obstruction spaces, we need key new ideas for the construction of virtual fundamental class. A similar situation of virtual fundamental class for the moduli space of stable sheaves on Calabi-Yau 4-folds was given in \[\ref{ref:11}\] and \[\ref{ref:58}\] motivated by gauge theory in higher dimensions and the symmetric property on the obstruction theory.

7.2. Virtual fundamental class. Let $M = \overline{M}^{\text{lc}}_{K3, X, N}$ be a connected component of the moduli stack of s.l.c. surfaces. From Theorem \[\ref{ref:7.3}\] the moduli stack $M^{\text{lc}}$ of lc covers admits a perfect obstruction theory
\[ \phi^{\text{lc}} : E^{\bullet}_{M^{\text{lc}}} \to \mathcal{L}^{\bullet}_{M^{\text{lc}}}, \]
where
\[ E^{\bullet}_{M^{\text{lc}}} := R^p_{\text{ind}}(L^{\bullet}_{/M^{\text{lc}}} \otimes \omega_{/M^{\text{lc}}})[-1]. \]
We follow the method in Section \[\ref{ref:3.5}\] to construct the virtual fundamental class on $M^{\text{lc}}$.

Let $\mathfrak{c}_{M^{\text{lc}}}$ be the intrinsic normal cone of $M^{\text{lc}}$ such that étale locally on an open subset $U \subset M^{\text{lc}}$, there exists a closed immersion
\[ U \hookrightarrow Y \]
into a smooth scheme $Y$. Then we have $\mathfrak{c}_{M^{\text{lc}}}|_U = [\mathcal{C}_U/Y/T_Y|_U]$. Let $N_{M^{\text{lc}}} = h^1/h^0((L^{\bullet}_{M^{\text{lc}}})^\vee)$ be the intrinsic normal sheaf of $M^{\text{lc}}$, and there is a natural inclusion $\mathfrak{c}_{M^{\text{lc}}} \hookrightarrow N_{M^{\text{lc}}}$. The perfect obstruction theory complex $E^{\bullet}_{M^{\text{lc}}}$ is perfect, and we denote the corresponding bundle stack by $h^1/h^0((E^{\bullet}_{M^{\text{lc}}})^\vee)$. The perfect obstruction theory $\phi^{\text{lc}} :$
$$E^\bullet_{\text{M}^{\text{lc}}_i} \to \mathbb{L}^\bullet_{\text{M}^{\text{lc}}_i}$$ satisfies that $h^{-1}(\phi^{\text{lc}}_i)$ is surjective, and $h^0(\phi^{\text{lc}}_i)$ is isomorphic. Therefore it induces an inclusion of stacks $\text{N}_{\text{M}^{\text{lc}}_i} \hookrightarrow h^1/h^0((E^\bullet_{\text{M}^{\text{lc}}_i})^\text{vir})$.

**Definition 7.7.** The virtual fundamental class of the perfect obstruction theory $\phi^{\text{lc}}_i : E^\bullet_{\text{M}^{\text{lc}}_i} \to \mathbb{L}^\bullet_{\text{M}^{\text{lc}}_i}$ is defined as

$$[M^{\text{lc}}_i]^{\text{vir}} = [M^{\text{lc}}_i, \phi^{\text{lc}}_i]^{\text{vir}} := 0^!_{h^1/h^0((E^\bullet_{\text{M}^{\text{lc}}_i})^\text{vir})}(\mathbb{e}_{M^{\text{lc}}_i}) \in A_{\text{vd}}(M^{\text{lc}}_i),$$

where vd is the virtual dimension of $M^{\text{lc}}_i$, and $0^!_{h^1/h^0((E^\bullet_{\text{M}^{\text{lc}}_i})^\text{vir})}$ is the Gysin map in the intersection theory of Artin stacks [47].

For the morphism $f^{\text{lc}}_i : M^{\text{lc}}_i \to M$ which is a finite morphism (hence a proper morphism) as in Theorem 6.19, from [75, Definition 3.6 (iii)] we define

$$[M]^{\text{vir}} := f^{\text{lc}}_!(M^{\text{lc}}_i, \phi^{\text{lc}}_i) \in A_{\text{vd}}(M),$$

which is called the virtual fundamental class for the moduli stack $M$.

**Remark 7.8.** From Corollary 7.4 if the moduli stack $M^{G}_{K^2,X}$ consists of k.l.t. surfaces, then the morphism $f^{\text{lc}}_i : M^{\text{lc}}_i \to M$ is an isomorphism and the perfect obstruction theory induces a virtual fundamental class $[M]^{\text{vir}} \in A_{\text{vd}}(M)$.

**Corollary 7.9.** Suppose the moduli stack $M$ of s.l.c. $G$-stable surfaces only consists of l.c.i. surfaces, then the perfect obstruction theory in Corollary 7.4

$$\phi : E^\bullet_M \to \mathbb{L}^\bullet_M$$

induces a virtual fundamental class

$$[M]^{\text{vir}} \in A_{\text{vd}}(M).$$

**Proof.** This is from Corollary 7.4 and the construction of virtual fundamental class in this section. \hfill \Box

**Remark 7.10.** The virtual dimension of $M^{\text{lc}}_i$ is the same as the virtual dimension of the moduli stack $M$, which is

$$\text{vd} = \dim H^1(S, T_S)^G - \dim H^2(S, T_S)^G$$

for $S$ is a general smooth s.l.c. surface in $M$.

In the case that $G$ is trivial, the virtual dimension of $M^{\text{lc}}_i$ can be calculated by Grothendieck-Riemann-Roch theorem

$$\text{vd} = \text{rk}(E^\bullet_M) = \chi(S, T_S) = \int_S \text{Ch}(T_S) \cdot \text{Td}(T_S)$$

$$= \left(\frac{7}{6}c_1^2 - \frac{5}{6}c_2\right) = 10\chi - 2K^2.$$ 

Thus, if $10\chi - 2K^2 \geq 0$, the virtual dimension is nonnegative and one can define invariants by taking integration over the virtual fundamental class $[M^{K^2,X}]^{\text{vir}}$ by some tautological classes.

**Remark 7.11.** Our main results Theorem 7.4 and Definition 7.7 show that for the moduli stack $M = \overline{M^{G}_{K^2,X,N}}$ obtained from Q-Gorenstein deformations, the moduli stack $M^{\text{lc}}_i = \overline{M^{\text{lc},G}_{K^2,X,N}}$ of l.c.i. surfaces admits a virtual fundamental class. This provides a strong evidence on Donaldson’s conjecture for the existence of virtual fundamental class for a large class of moduli stacks of surfaces of general type. In practice people hope that there are many examples where the boundary divisors of the moduli stack $M$ consist of only L.c.i. surfaces; see for example $\overline{M^{\text{Cor}}_{1,3}}$ and $\overline{M^{\text{Cor}}_{1,2}}$ for the moduli stacks of Gorenstein surfaces in [21], and Donaldson’s example in [97]. Note that the moduli stacks $\overline{M^{\text{Cor}}_{1,3}}$ and $\overline{M^{\text{Cor}}_{1,2}}$ are open substacks in the moduli stack $\overline{M_{1,3}}$ and $\overline{M_{1,2}}$. Actually for the moduli stack $M$ obtained from Q-Gorenstein deformations, the boundary divisors may only contain the Q-Gorenstein deformation of class T-singularities. Almost for all of the known examples for $M$ in the literature the boundary divisors were constructed using Q-Gorenstein
deformation of class $T$-singularities; i.e., using the deformation of the corresponding index one covering Deligne-Mumford stacks. In this case, the moduli stack $M^{\text{ind}}$ of index one covers admits a virtual fundamental class. An interesting example is given by the moduli stack $\mathcal{M}_{1,3}$ of s.l.c. surfaces with $K^2 = 1, \chi = 3$ in [22], where some boundary divisors and other irreducible components in $\mathcal{M}_{1,3}$ were explicitly constructed by deformation of class $T$-singularities. We hope that the explicit components constructed in [22] completely determine the stack $\mathcal{M}_{1,3}$.

8. CM line bundle and tautological invariants

Let $M = \mathcal{M}_{K^2,\chi}^G$ be one connected component of KSBA moduli stack of s.l.c. $G$-stable surfaces. In this section we require that $N$ is large divisible enough so that $M = \mathcal{M}_{K^2,\chi}^G$ is the moduli stack of s.l.c. $G$-stable surfaces with invariants $K^2, \chi$.

8.1. CM line bundle on the moduli stack. From [17] §2.1], over smooth part $\mathcal{M}_{K^2,\chi}^G$ consisting of smooth general type surfaces $S$ with $K_S^2 = K_S^2 \chi(O_S) = \chi$, differential geometry can define Miller-Mumford-Morita (MMM)-classes on $H^*(\mathcal{M}_{K^2,\chi}^G, \mathbb{Q})$. Donaldson [17] §4] proposed a question to extend the MMM-classes to $H^*(\mathcal{M}_{K^2,\chi}^G, \mathbb{Q})$ of the whole KSBA compactification $M$.

In algebraic geometry there exists a CM line bundle on the moduli stack $M$ as defined in [73], [20] and [59]. We recall it here. Let $p : \mathcal{M} \to M$ be the universal family which is a projective, flat morphism with relative dimension 2. Then the relative canonical sheaf $K_{\mathcal{M}/M}$ is $\mathbb{Q}$-Cartier and relatively ample, see [28] and [46]. For any relatively ample line bundle $\mathcal{L}$ on $\mathcal{M}$, we have

$$\det(p_!(\mathcal{L}^k)) = \det(R^i p_*(\mathcal{L}^k)) = \bigotimes_i \left( \det \left( R^i p_*(\mathcal{L}^k) \right) \right)^{(-1)^i}.$$ 

As $\mathcal{L}$ is relatively ample, $R^i p_*(\mathcal{L}^k) = 0$ for $i > 0, k >> 0$, thus $\det(p_!(\mathcal{L}^k)) = \det p_*(\mathcal{L}^k)$. From [41], there exist line bundles $\lambda_i$ for $i = 0, 1, 2, 3$ on $\mathcal{M}_{K^2,\chi}^G$ such that for all $k$,

$$\det p_!(\mathcal{L}^k) \cong \lambda_3^k \otimes \lambda_2^k \otimes \lambda_1^k \otimes \lambda_0.$$ 

Let $\mu := - (K_{\mathcal{M}} \cdot \mathcal{L}|_{\mathcal{M}}) / \mathcal{L}^2|_{\mathcal{M}}$, then the CM line bundle (corresponding to $\mathcal{L}$) is

$$\lambda_{CM} = \lambda_{CM}(\mathcal{M} / M, \mathcal{L}) := \lambda_3^{2\mu+6} \otimes \lambda_2^{-6}.$$ 

Using Grothendieck-Riemann-Roch theorem in [20], we have that

$$\begin{cases} c_1(\lambda_3) = p_!(c_1(\mathcal{L})^3); \\ 2c_1(\lambda_3) - 2c_1(\lambda_2) = p_!(c_1(\mathcal{L}) \cdot c_1(K_{\mathcal{M}/M})). \end{cases}$$ 

Let $\mathcal{L} = K_{\mathcal{M}/M}$, then the CM line bundle is

$$\lambda_{CM}(\mathcal{M} / M, K_{\mathcal{M}/M}) := \lambda_3^4 \otimes \lambda_2^{-6} = \lambda_2^2$$

since Serre duality implies that $\lambda_3 = \lambda_2^2$. We have that

$$c_1(\lambda_{CM}(\mathcal{M} / M, K_{\mathcal{M}/M})) = p_!(\left(K_{\mathcal{M}/M}\right)^3).$$

We define

$$L_{CM} := \lambda_{CM}(\mathcal{M} / M, K_{\mathcal{M}/M}).$$

From [59] Theorem 1.1], the CM line bundle $L_{CM}$ is ample on the KSBA moduli stack $M$. 
8.2. Tautological invariants. Let \( M \) be one connected component of the moduli stack of \( G \)-equivariant stable general type surfaces with invariants \( K^2 S = K^2, \chi(O_S) = \chi \). From Theorem 7.1 the moduli stack \( M^{\text{lcI}} = \mathcal{M}^{\text{lcI}}_{K^2, \chi} \) of lci covers admits a perfect obstruction theory \( \phi_{\text{lcI}} : E_{\text{lcI}} \to I_{\text{lcI}} \), hence induces a virtual fundamental class \([M]^{\text{vir}}\) in Definition 7.7.

**Definition 8.1.** Let \( M \) be one connected component of the moduli stack of stable surfaces with fixed invariants \( K^2, \chi, N \). We define the tautological invariant by

\[
I_{CM} = \int_{[M]^{\text{vir}}}(c_1(L_{CM}))^{vd}.
\]

**Remark 8.2.** It is interesting to consider other tautological classes on the moduli stack \( \mathcal{M}_{K^2, \chi} \).

9. Examples

In this section we study several examples.

9.1. Moduli space of quintic surfaces.

9.1.1. General degree \( d \) hypersurfaces in \( \mathbb{P}^3 \). Let us first consider some basic invariants for smooth hypersurfaces in \( \mathbb{P}^3 \) of degree \( d \geq 5 \). Let \( i : S \subset \mathbb{P}^3 \) be a smooth hypersurface of degree \( d \), then we have the exact sequence

\[
(9.1.1) \quad 0 \to T_S \to T_{\mathbb{P}^3} \to N_{S/\mathbb{P}^3} \to 0,
\]

where \( N_{S/\mathbb{P}^3} = O_{\mathbb{P}^3}(d) \) is the normal bundle. When \( d \geq 5 \), the surfaces \( S \) is of general type. Therefore, \( H^i(S, T_S) = 0 \) only except \( i = 1, 2 \). We calculate the dimensions of the cohomology spaces of the tangent bundle of \( S \) for \( d = 5, 6 \),

\[
(9.1.2) \quad \begin{cases} 
\dim H^1(S, T_S) = 40; \\
\dim H^2(S, T_S) = 0,
\end{cases}
\]

and

\[
(9.1.3) \quad \begin{cases} 
\dim H^1(S, T_S) = 68; \\
\dim H^2(S, T_S) = 6.
\end{cases}
\]

The cohomology spaces \( H^*(S, T_S) \) are calculated by taking the long exact sequence of the cohomology of (9.1.1)

\[
0 \to H^0(S, T_S) \to H^0(S, T_{\mathbb{P}^3}|_S) \to H^0(S, N_{S/\mathbb{P}^3})
\]

\[
\to H^1(S, T_S) \to H^1(S, T_{\mathbb{P}^3}|_S) \to H^1(S, N_{S/\mathbb{P}^3})
\]

\[
\to H^2(S, T_S) \to H^2(S, T_{\mathbb{P}^3}|_S) \to H^2(S, N_{S/\mathbb{P}^3})
\]

\[
\to 0,
\]

and the long exact sequences on the cohomology of the following two exact sequences

\[
0 \to O_{\mathbb{P}^3} \to O_{\mathbb{P}^3}(d) \to i_* N_{S/\mathbb{P}^3} \to 0
\]

and

\[
0 \to T_{\mathbb{P}^3}(-d) \to T_{\mathbb{P}^3} \to i_*(T_{\mathbb{P}^3}|_S) \to 0.
\]

We omit the detailed calculation.
9.1.2. Moduli space of quintic surfaces. Let us first briefly recall the moduli space of quintic surfaces in $\mathbb{P}^3$. Let $S \subset \mathbb{P}^3$ be a smooth quintic surface defined by a homogeneous degree five polynomial. It is well-known that the topological invariants of $S$ are given by $K_S = O_S(1)$, and
$$K_S^2 = 5; \quad q = \dim H^1(S, O_S) = 0, \quad p_g = 4; \quad \chi(O_S) = 5.$$ Let $\overline{M}_{5,5}$ be the moduli stack of general type minimal surfaces $S$ with $K_S^2 = 5, \chi(O_S) = 5$. From [31], the coarse moduli space of the Gieseker’s moduli stack $M_{5,5} \subset \overline{M}_{5,5}$ is a 40-dimensional scheme with two irreducible components $M_0 \cup_{\mathcal{W}} M_1$ meeting transversally at a 39-dimensional scheme $\mathcal{W}$, where $M_0$ is the component containing quintic surfaces in $\mathbb{P}^3$ with rational double points singularities (RDP’s), and the other components $M_1$ and $\mathcal{W}$ consist of the following surfaces: first from [31] Theorem 1, for any minimal surface $S$ with $K_S^2 = 5; \quad q = \dim H^1(S, O_S) = 0, \quad p_g = 4$ and $\chi(O_S) = 5$, the canonical system $|K_S|$ has at most one base point. There are three types of surfaces:

- **Type I:** $|K_S|$ has no base point. The surface $S$ is birationally equivalent to $S'$, where $S' \subset \mathbb{P}^3$ is a quintic surface with only RDP’s singularities;

- **Type IIa:** $|K_S|$ has one base point. Let $\pi : \tilde{S} \to S$ be the quadric transformation with center at the base point $b \in |K_S|$, then there exists a surjective morphism $f : \tilde{S} \to \mathbb{P}^1 \times \mathbb{P}^1$ of degree 2;

- **Type IIb:** $|K_S|$ has one base point. In this case there exists a surjective map $f : \tilde{S} \to \Sigma_2$ of degree two, where $\Sigma_2$ is the Hirzebruch surface of degree two, and there also exists a diagram:

$$
\begin{array}{ccc}
\Sigma_2 & \xrightarrow{f} & \tilde{S} \\
\downarrow{\psi} & & \downarrow{\phi} \\
& & \mathbb{P}^3
\end{array}
$$

such that the image of $\phi$ and $\psi$ are the quadric cone in $\mathbb{P}^3$. Note that all the Type I, IIa and IIb surfaces are l.c.i. surfaces. The deformation of Type I, Type IIa and Type IIb surfaces are given by the deformation of the corresponding birational models in the description.

The component $M_0$ consists of Type I surfaces; the component $M_1$ consists of Type IIa surfaces and the intersection $\mathcal{W}$ parametrizes type IIb surfaces. For a surfaces $S$, from [29], $|\text{Aut}(S)| \leq 42 \cdot \text{Vol}(S, K_S)$ and if $S$ is minimal then $\text{Vol}(S, K_S) = K_S^2$ and $|\text{Aut}(S)| \leq 42 \cdot 5$. If we consider all the automorphism groups of $S$, we get the Deligne-Mumford stack $\overline{M}_{5,5}$.

The complete boundary divisors of $\overline{M}_{5,5}$ are still not explicitly constructed; see [62] for an explicit construction of one boundary divisor $D_{\frac{1}{4}(1,1)} \subset \overline{M}_{5,5}$ corresponding to a Wahl singularity of type $\frac{1}{4}(1,1)$. But the abstract KSBA compactification $\overline{M}_{5,5}$ was constructed and is a proper Deligne-Mumford stack; see [44].

Let us give an example for $\overline{M}_{5,5}$ on the boundary loci consisting of s.l.c. surfaces. In [62], Rana gave a construction of one boundary divisor $D_{\frac{1}{4}(1,1)} \subset \overline{M}_{5,5}$, which consists of s.l.c. surfaces $S$ with only one Wahl type $\frac{1}{4}(1,1)$ singularity. This singularity has index $r = 2$, and $S$ has global index $N = 2$. The boundary divisor $D_{\frac{1}{4}(1,1)} \supset D_1 \cup_{\mathcal{W}_2} D_2$ also contains two irreducible components, where $D_1$ is the component consisting of Type 1 surfaces; $D_2$ is the component consisting of Type 2a surfaces, and $\mathcal{W}_2$ is the component consisting of Type 2b surfaces. Type 1, 2a and 2b surfaces are classified as follows.: the minimal resolution of a Type 1 surface is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$, branched over a sextic intersecting a given diagonal tangentially at 6 points. The preimage of the diagonal is given by two $(−4)$-curves intersecting at 6 points. Contracting one of these $(−4)$-curves gives a stable numerical quintic surface of Type 1. The minimal resolutions of Type 2a (respectively Type 2b) surfaces are themselves minimal resolutions of double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ (respectively a quadric cone), the branch curve of which is a sextic $B$ intersecting a given ruling at two nodes of $B$ and transversally at two other points. There are relations: Type 1 (respectively
Type 2a, Type 2b) surfaces are the deformation limits of Type I (respectively Type II a, Type II b) surfaces. Thus $D_1, D_{2a}$ are 39-dimensional, and $W_{2b}$ is 38-dimensional.

The obstruction space at such a boundary divisor was calculated. Let us take a point $S \in D_{2b}$ such that there exist open étale neighborhoods $U_{2a} \subset \overline{M}_{5,5}$ and $U_{2b} \subset \overline{M}_{5,5}$ satisfying the condition that $U_{2b}$ contains elements in $D_1$ and the boundary $W_{2b}$, and $U_{2a}$ is only a neighborhood of $D_1$ which does not intersect with $W_{2b}$. From [62, Theorem 5.1], $U_{2b} \subset A_{5,1}$ is cut out by $H' = q'(t) \cdot r'(t) = 0$ for two holomorphic functions. For the surface germ $(S, P)$ of Type 2b, the obstruction space is calculated by the corresponding canonical index one covering Deligne-Mumford stack $S$ which contains only one orbifold point of type $\frac{1}{4}(1, 1)$. The obstruction space $T^*_{\overline{QG}}(L_{\overline{S}}, O_{\overline{S}}) = H^2(S, T_{\overline{S}})$ has dimension 1, which was calculated in [62].

Recall that in [62] a quotient $T$-singularity is given by a quotient 2-dimensional singularity of type $\frac{1}{dn}(1, dha - 1)$, where $n > 1$ and $d, a > 0$ are integers with $a$ and $n$ coprime. These are the quotient singularities that admit a Q-Gorenstein smoothing. When $d = 1$, these are called Wahl singularities. The s.l.c. minimal surfaces $S$ with $T$-singularities satisfying $K^2_S = 5, p_g = 4$ may give some other irreducible components of $\overline{M}_{5,5}$.

9.1.3. Discussion of the virtual fundamental class. For a large divisible $N > 0$, the KSBA compactification $\overline{M}_{5,5,N}$ may contain a lot of irreducible components. Let us only consider the following two irreducible components

$$\overline{P} := \overline{M}_0 \cup \overline{M}_1$$

where $\overline{M}_0 = \overline{M}^{\text{trin}}$ is the closure of the component in $\overline{M}_{5,5}$ containing the smooth quintics, and $\overline{M}_1$ is the closure of the component in $\overline{M}_{5,5}$ containing the smooth Type IIA surfaces in [31]. The two Deligne-Mumford stacks $\overline{M}_0$ and $\overline{M}_1$ are 40-dimensional Deligne-Mumford stacks meeting at a 39-dimensional closed substack $\overline{W}$. The above calculation on the cohomology spaces $H^*(\overline{S}, T_{\overline{S}})$ implies that the main component $\overline{M}_0$ is smooth on the open part $M_0$ consisting quintic surfaces. From [31], the open subset $M_1 \subset \overline{M}_1$ is also smooth. The singular locus of $\overline{P}$ only happens on $\overline{W}$. We assume that all the boundary loci of $\overline{P}$ contain stable surfaces with class $T$-singularities, so that their index one covers only have normal crossing and $A_n$-type singularities. Thus, from Corollary [62,22] $\overline{P}^\text{lei} = \overline{P}^\text{ind}$.

We construct the virtual fundamental class for $\overline{P}$. Let $f : \overline{P}^\text{ind} \to \overline{P}$ be the moduli stack of index one covers. Then

$$\overline{P}^\text{ind} = \overline{M}_0^\text{ind} \cup \overline{M}_1^\text{ind},$$

where $f^0 : \overline{M}_0^\text{ind} \to \overline{M}_0$ and $f^1 : \overline{M}_1^\text{ind} \to \overline{M}_1$ are the moduli stacks of index one covers over the components $\overline{M}_0$ and $\overline{M}_1$ respectively, and they intersect at $f_W : \overline{W}^\text{ind} \to \overline{W}$ which is the moduli stack of index one covers over $\overline{W}$. The morphisms $f^0, f^1$ and $f_W$ are isomorphisms except on the boundary divisors of $\overline{P}$ given by Q-Gorenstein smoothing of class $T$-singularities. For example, over the divisor $D_{\frac{4}{5}(1,1)} = D_1 \cup W_{2b} D_{2a}$ in $\overline{P}$, the fibers of $f^0, f^1$ and $f_W$ are the index one covering Deligne-Mumford stacks of the stable surfaces with one Wahl singularity $\frac{1}{4}(1, 1)$ of Type I, Type 2a and Type 2b respectively.

Let $p^\text{ind} : \overline{M}^\text{ind} \to \overline{P}^\text{ind}$ be the universal family. Then there is a perfect obstruction theory

$$q^\text{ind} : E^\text{ind} := F^\text{ind}_p \to \mathbb{L}^\text{ind}_{\overline{M}^\text{ind}, \overline{P}^\text{ind}} \cdot$$

where

$$E^\text{ind} := R^p_{\overline{M}^\text{ind}} \mathbb{L}^\text{ind}_{\overline{M}^\text{ind}, \overline{P}^\text{ind}} \otimes \omega_{\overline{M}^\text{ind}, \overline{P}^\text{ind}} [-1].$$

Let $\mathbb{C}^\text{ind}$ be the intrinsic normal cone of $\overline{P}^\text{ind}$. This intrinsic normal cone can be written as

$$\mathbb{C}^\text{ind} = \mathbb{C}^\text{ind}_{\overline{M}_0} + \mathbb{C}^\text{ind}_{\overline{M}_1}.$$
where \( c_{M_{\text{ind}}^0} \) and \( c_{M_{\text{ind}}^1} \) are the intrinsic normal cones of the components \( M_{\text{ind}}^0 \) and \( M_{\text{ind}}^1 \) respectively. This can be calculated by embedding \( P_{\text{ind}} \) into a higher dimensional smooth Deligne-Mumford stack \( Y \) and the normal cone of \( C_{P_{\text{ind}}/Y} \) contains two irreducible components given by the two irreducible components \( M_{\text{ind}}^0 \) and \( M_{\text{ind}}^1 \).

Look at the following diagram

\[
\begin{array}{ccc}
\mathcal{C}_{P_{\text{ind}}} & \rightarrow & h^1(h^0((E_{\text{ind}}^*)^\vee)) \\
\downarrow & & \downarrow \\
\mathcal{C}P_{\text{ind}} & \rightarrow & h^1((E_{\text{ind}}^*)^\vee) = \text{Ob}_{P_{\text{ind}}},
\end{array}
\]

where \( \mathcal{C}P_{\text{ind}} \) is the coarse moduli space of the intrinsic normal cone \( c_{P_{\text{ind}}} \), and \( \text{Ob}_{P_{\text{ind}}} \) is the obstruction sheaf of the perfect obstruction theory \( \phi_{P_{\text{ind}}} \).

Let \( s : P_{\text{ind}} \rightarrow \text{Ob}_{P_{\text{ind}}} \) be the zero section. From Definition 7.7 the virtual fundamental class \( [P_{\text{ind}}]_{\text{vir}} \in A_{40}(P_{\text{ind}}) \) is obtained from the intersection of the intrinsic normal cone with the zero section of the bundle stack \( h^1/h^0((E_{\text{ind}}^*)^\vee) \). From the decomposition of the intrinsic normal cone \( c_{P_{\text{ind}}} = c_{M_{\text{ind}}^0} + c_{M_{\text{ind}}^1} \), the intersection can be calculated separately.

Also note that both \( M_{\text{ind}}^0 \) and \( M_{\text{ind}}^1 \) are smooth, and the coarse moduli spaces of the intrinsic normal cones \( c_{M_{\text{ind}}^0} \) and \( c_{M_{\text{ind}}^1} \) are just \( M_{\text{ind}}^0 \) and \( M_{\text{ind}}^1 \) respectively. Therefore, the intersections are just the intersections of \( M_{\text{ind}}^0 \) and \( M_{\text{ind}}^1 \) with the zero sections of the obstruction sheaf. We obtain

\[
[P_{\text{ind}}]_{\text{vir}} = [M_{\text{ind}}^0] + [M_{\text{ind}}^1] \in A_{40}(P_{\text{ind}}).
\]

There is a canonical morphism

\[
f : P_{\text{ind}} \rightarrow P
\]

which is a finite morphism and is an isomorphism except on the boundaries. Thus we have that

\[
[P]_{\text{vir}} = f_*([P_{\text{ind}}]_{\text{vir}}) \in A_{40}(P).
\]

**Remark 9.1.** It is interesting to calculate the tautological invariants for the moduli stack of quintic surfaces.

### 9.2. Donaldson’s example on sextic hypersurfaces

In this section we talk about Donaldson’s example on sextic hypersurfaces in \( \mathbb{P}^3 \), and give an affirmative answer for the existence of virtual fundamental class on the moduli of \( G \)-equivariant sextic hypersurfaces in \( \mathbb{P}^3 \) for some finite group \( G \), thus proving Donaldson’s conjecture on the existence of virtual fundamental class of this example. In this section all the surfaces are l.c.i. and the index \( N = 1 \).

#### 9.2.1. The GIT moduli space

Let \( S \subset \mathbb{P}^3 \) be a smooth degree 6 hypersurface, then the formula of the cohomology of the tangent bundle of \( S \) are given in (9.1.3). Other topological invariants are given by:

\[
e(S) = 108; \quad p_g = 10; \quad K_S^2 = 24; \quad \chi(O_S) = 11.
\]

Let \( M_{24,11} \) be the moduli stack of stable surfaces \( S \) with invariants \( K_S^2 = 24, \chi(O_S) = 11 \). It is not known in the literature what this moduli stack looks like, but at least there exists one component of \( M_{24,11} \) containing sextic surfaces in \( \mathbb{P}^3 \).

In order to get an explicit moduli stack, Donaldson [17, §5] put more symmetries on the sextic surfaces. Let \( \mathbb{P}^3 = \text{Proj}(k[x_1, y_1, x_2, y_2]) \). Let \( \zeta \in \mu_6 \) be a primitive generator of the
cyclic group of order 6, and let $G$ be the subgroup of $GL(4, \mathbb{k})$ generated by

$$\begin{align*}
(x_1, y_1, x_2, y_2) &\mapsto (\zeta x_1, \zeta^{-1} y_1, x_2, y_2); \\
(x_1, y_1, x_2, y_2) &\mapsto (x_1, y_1, \zeta x_2, \zeta^{-1} y_2); \\
(x_1, y_1, x_2, y_2) &\mapsto (x_2, y_2, x_1, y_1),
\end{align*}$$

which are the actions on $\mathbb{A}^4_k$.

Then $G$ acts on the sextic hypersurfaces in $\mathbb{P}^3$. The invariant degree 6 homogeneous polynomials are given by

$$ax_1^6 + \beta y_1^6 + ax_2^6 + \beta y_2^6 + AQ_+^3 + BQ_+ Q^2,$$

where $Q_{\pm} = x_1y_1 \pm xy_2$. Then the $G_m = \mathbb{A}_k^1$-action by

$$(x_1, y_1, x_2, y_2) \mapsto (\lambda x_1, \lambda^{-1} y_1, \lambda x_2, \lambda^{-1} y_2)$$

induces homogeneous polynomials invariant under the action of $G$. All the invariant degree 6 polynomials under the $G$-action give the parameter space $$(a, \beta, A, B).$$

The $G_m$ acts on the parameter space by

$$(a, \beta, A, B) \mapsto (\lambda^6 a, \lambda^{-6} \beta, A, B).$$

Thus, the invariant degree 6 homogeneous polynomials in $\mathbb{P}^3$ are classified by

$$a(x_1^6 + y_1^6 + x_2^6 + y_2^6) + AQ_+^3 + BQ_+ Q^2,$$

which is unique up to change of the sign of $a$. The moduli space of GIT stable locus of sextic hypersurfaces with $G$-action is $\mathbb{A}^2_k / \{\pm\}$. Here $\mathbb{A}^2_k = \text{Spec} \, k[A, B]$ and each $(A, B)$ corresponds to a hypersurface

$$(9.2.1) \quad S_{AB} = \{x_1^6 + y_1^6 + x_2^6 + y_2^6 + AQ_+^3 + BQ_+ Q^2 = 0\} \subset \mathbb{P}^3.$$

We recall the KSBA compactification of $\mathbb{A}^2_k / \{\pm\}$ in [17 §5]. Before KSBA, there is a naive compactification by embedding $\mathbb{A}^2_k \hookrightarrow \mathbb{P}^2$ and then taking the quotient by $\mu_2 = \{\pm\}$-action. Modulo the automorphism group of the sextic surfaces, the moduli stack is the quotient $M_{GIT} = [\mathbb{P}^2 / \mu_2]$. The polytope description is given in [17 §5]. The stacky fan in the sense of [12, 35] is given by $\Sigma = (N, \Sigma, \beta)$, where $N = \mathbb{Z}^2$, $\Sigma$ is the fan in $\mathbb{R}^2$ generated by rays $R_{(2,0)}, R_{(0,1)}$ and $R_{(-2, -1)}$, and $\beta : \mathbb{Z}^2 \rightarrow N$ is given by $(2, 0), (0, 1), (-2, -1)$. The quotient action of $\mu_2$ on the homogeneous coordinates $[x : y : z]$ of $\mathbb{P}^2$ by

$$[x : y : z] \mapsto [x : -y : -z].$$

The fixed point locus are the point $[1 : 0 : 0]$ and $\mathbb{P}^1 = \text{Proj}(k[0 : y : z])$ which is the divisor at infinity. The divisor $\mathbb{P}^1$ in the moduli toric Deligne-Mumford stack $[\mathbb{P}^2 / \mu_2]$ corresponds to the following surfaces

$$\{AQ_+^3 + BQ_+ Q^2 = 0\}$$

for $A, B \neq 0$ at the same time. Note that there are three cases

1. $A, B \neq 0$, then $\{AQ_+^3 + BQ_+ Q^2 = 0\}$ corresponds to three quadrics meeting in four lines;
2. $B = 0, A \neq 0$, this corresponds to the quadric $\{Q_+ = 0\}$ with multiplicity 3;
3. $B \neq 0, A = 0$, this corresponds to the quadric $\{Q_+ = 0\}$ and the quadric $\{Q_- = 0\}$ with multiplicity 2.
9.2.2. The KSBA compactification. Let us consider the KSBA compactification of the moduli stack \([\mathcal{A}_6^2/\mu_2]\) of sextic hypersurfaces with \(G\)-action. We follow Donaldson’s argument using the fan structure of the toric Deligne-Mumford stack \(M^{GIT} = [\mathbb{P}^2/\mu_2]\).

- Let \(O := ((0,1),(-2,-1))\) be the top cone generated by \{(0,1),(-2,-1)\}, which corresponds to the affine toric Deligne-Mumford stack \([\mathcal{A}_6^2/\mu_2]\), and the sextic surfaces in \([92,\S 5]\). One can think of the ray \(R_{(2,0)}\) standing for the infinity divisor \(\mathbb{P}^1 \subset M^{GIT}\) which is fixed under \(\mu_2\). The ray \(R_{(-2,-1)}\) (which corresponds to \(\Omega II\) in Donaldson’s picture in \([17]\) Page 20) corresponds to the surfaces \(\{S_{A0}\}\) in \([7,\S 2.1]\).

- The toric Deligne-Mumford stack \([\mathbb{P}^2/\mu_2] = [\mathcal{A}_6^2/\mu_2] \sqcup \mathbb{P}^1\), where \(\mathbb{P}^1 = \chi(\Sigma/\mathbb{R}_{(2,0)})\); i.e., the toric Deligne-Mumford stack of the quotient fan \(\Sigma/\mathbb{R}(2,0)\) modulo the ray \(R_{(2,0)}\). So it is enough to know what sextic surfaces the ray \(R_{(2,0)}\) corresponds for. As pointed out in \([17]\) \S 5, this ray \(R_{(2,0)}\) corresponds to surfaces \(\{AQ_+^1 + BQ_+Q_-^2 = 0\}\) for \(A, B\) not zero at the same time. Let \(III := ((-2,-1),(2,0))\) be the top cone generated by \((-2,-1),(2,0)\) and \(II := (2,0),(0,1)\) be the top cone generated by \((2,0),(0,1)\). Note that the surface \(\{Q_+ = 0\}\) with multiplicity 3 corresponds to the origin in the cone \(III\), and the surface \(\{Q_+Q_-^2 = 0\}\), one quadric \(\{Q_+ = 0\}\) and one \(\{Q_- = 0\}\) with multiplicity 2 corresponds to the origin in the cone \(II\).

The surfaces corresponding to the infinity divisor \(\mathbb{P}^1\) are not s.l.c. surfaces, and we perform weighted blow-ups on \(M^{GIT} = [\mathbb{P}^2/\mu_2]\) to get the KSBA compactification. From \([17]\) \S 5, in the cone \(III\), the vertex corresponds to the surface \(\{S_{A0}\}\) when \(A \to \infty\). The construction is given as follows: let \(\pi : Y \to \mathbb{P}^3\) be the triple cover over \(\mathbb{P}^3\) branched over \(S_{00}\). There exists a section \(\eta \in \pi^*\mathcal{O}(2) \to Y\) such that \(\eta^3 = s\), and \(s\) is the section cutting out \(S_{00}\). Then let \(W \subset Y \times \mathbb{P}^1\) be the surface cut out by \(\eta = \lambda Q_+\). Let \(A = \lambda^3\). When \(A \to \infty\), we get a triple cover \(S_{III}\) over \(\{Q_+ = 0\}\) branched over \(\{S_{00} \cap \{Q_+ = 0\}\}\). This triple cover \(S_{III} \to \mathbb{P}^1 \times \mathbb{P}^1\) has an extra automorphism group \(\mu_3\). Therefore, we do the weighted blow-up on the toric Deligne-Mumford stack \([\mathbb{P}^2/\mu_2]\) by inserting the ray \(R_{(4,-1)}\) generated by \((4,-1) = 3(2,0) + (-2,-1)\). This ray splits the cone \(III\) into two top cones denoted by \(III = ((-2,-1),(4,-1))\) and \(IV' = ((4,-1),(2,0))\). From \([13]\), this gives a new stacky fan \(\Sigma'\) and a toric Deligne-Mumford stack \(h : \chi(\Sigma') \to M^{GIT}\), which is a weighted blow-up. The exceptional locus (divisor) of \(h\) corresponds to the following family of surfaces: taking affine coordinates \((s,t)\) of \(\mathbb{P}^1 \times \mathbb{P}^1\), and let \(C_\mu \subset |\mathcal{O}(6,6)|\) be the curve with affine equation:

\[
1 + s^5 + t^5 + s^6t^6 + \mu \cdot s^3t^3 = 0.
\]

Then the family of surfaces (corresponding to the exceptional locus \(\mathbb{P}^1\) by \(\mu\), but \(\mu \neq 0\)) is

\[
\mu : S_\mu \to \mathbb{P}^1 \times \mathbb{P}^1,
\]

which are triple covers over \(\mathbb{P}^1 \times \mathbb{P}^1\) with simple branching over \(C_\mu, C_{-\mu}\). The \(\mu = 0\) case corresponds to the surface \(S_{III}\) above. All of these surfaces are s.l.c. surfaces.

Now we perform on the top cone \(II\) similarly. Since the vertex of the cone \(II\) corresponds to \(\{Q_+Q_-^2 = 0\}\), there exists a \(\mathbb{Z}_2\)-symmetry. We do the weighted blow-up on the toric Deligne-Mumford stack \(\chi(\Sigma')\) by inserting (inside the cone \(II\)) a ray \(R_{(2,1)}\) generated by \((2,1) = (2,0) + (0,1)\). This ray splits the top cone \(II\) into two top cones \(II = ((2,1),(0,1))\) and \(II' = ((2,0),(2,1))\). Thus we get a new stacky fan \(\Sigma''\) such that the morphisms

\[
\chi(\Sigma'') \xrightarrow{h''} \chi(\Sigma') \xrightarrow{h} \chi(\Sigma) = M^{GIT}
\]

are all weighted blow-ups. The exceptional divisor \(\mathbb{P}^1\) of \(h' : \chi(\Sigma'') \to \chi(\Sigma')\) (also using affine coordinates \(\mu\), but \(\mu \neq 0\)) parametrizes the family of surfaces:

\[
\tilde{\mu} : S_\mu \to \mathbb{P}^1 \times \mathbb{P}^1,
\]
which are double covers over \( \mathbb{P}^1 \times \mathbb{P}^1 \) branched over a divisor in \( \mathcal{O}(8,8) \) given by \( C_\mu \cup \{ s = 0, s = \infty, t = 0, t = \infty \} \). Each of these four lines meets with \( C_\mu \) in 6 points. Let
\[
\text{Bl}_{24 \ p\ell s} S_\mu \rightarrow S_\mu \rightarrow \mathbb{P}^1 \times \mathbb{P}^1
\]
be the blow-up along these 24 points, and then let
\[
\overline{\text{Bl}}_{24 \ p\ell s} S_\mu \rightarrow S_\mu
\]
be the morphism by collapsing down the proper transformation of the four lines \( \{ s = 0, s = \infty, t = 0, t = \infty \} \). The \( \mu = 0 \) case corresponds to the surface \( S_{II} \) and all of these surfaces are s.l.c.

For the toric Deligne-Mumford stack \( \chi(\Sigma'') \rightarrow \chi(\Sigma) = M^{GIT} \), we collapse down the proper transformation of the locus \( \chi(\Sigma/R(2,0)) = \mathbb{P}^1 \) and obtain a toric Deligne-Mumford stack \( \chi(\Sigma) \), where the stacky fan is given by \( \Sigma = (\mathbb{Z}^2, \Sigma, \beta) \). The fan \( \Sigma \) contains four top cones, where \( O, III, IV, I \) are the same as before, and \( IV = ((4, -1), (2, 1)) \). This toric Deligne-Mumford stack \( \chi(\Sigma) \) is projective since the fan \( \Sigma \) is clearly complete.

To see that \( \chi(\Sigma) \) is the KSBA compactification of \( M^{GIT} \), note that in \( M^{GIT} \), the only non-KSBA surfaces are given by the infinity divisor \( \chi(\Sigma/R(2,0)) = \mathbb{P}^1 \). After doing weighted blow-ups and collapsing this infinity divisor, all surfaces parametrized by \( \chi(\Sigma) \) are KSBan s.l.c. surfaces. Also, the surfaces parametrized by the top cone \( IV \) are given by (see [17] §5.2) complete intersections in the weighted projective stack \( \mathbb{P}(1,1,1,2,2) \). More precisely, let \( \mathbb{P}(1,1,1,2,2) = \text{Proj}(k[x_1, y_1, x_2, y_2, h_+, h_-]) \) where \( x_1, y_1, x_2, y_2 \) have degree 1 and \( h_+, h_- \) have degree 2. We define the surfaces \( S^{\alpha, \beta} \subset \mathbb{P}(1,1,1,2,2) \) by
\[
(9.2.2) \quad S^{\alpha, \beta} = \left\{ \begin{aligned}
x_1^4 + y_1^4 + x_2^4 + y_2^4 + h_+^4 + h_-^2 &= 0; \\
x_1 y_1 &= a h_+ + b h_-; \\
x_2 y_2 &= a h_+ - b h_-.
\end{aligned} \right.
\]
The most singular one \( S^{0,0} \) corresponds to the vertex in \( IV \), which corresponds to the surface in \( III \) and \( II \) by taking \( \mu \rightarrow \infty \). Also from [17] §5.2, the surfaces in \( III \) and \( II \) can be obtained from the surfaces \( S^{\alpha, \beta} \). The surfaces \( S^{\alpha, \beta} \) are complete intersections, therefore are Gorenstein; i.e., the dualizing sheaf \( \omega_{S^{\alpha, \beta}} \) is a line bundle for any pair \((\alpha, \beta)\).

9.2.3. Virtual fundamental class. From the construction in §7.2.2 there exists a universal family
\[
p : \mathbb{M} \rightarrow \chi(\Sigma)
\]
which is projective, flat and relatively Gorenstein. It is relatively Gorenstein since every fiber surface \( S_t \) of \( p \) at \( t \in \chi(\Sigma) \) is a complete intersection surface. This implies that the relative dualizing sheaf \( \omega_{\mathbb{M}/\chi(\Sigma)} \) is a line bundle. Therefore from Corollary 7.4 and Corollary 7.9 we have that

**Proposition 9.2.** Let
\[
E_{\chi(\Sigma)} := Rp_* \left( L_{\mathbb{M}/\chi(\Sigma)} \otimes \omega_{\mathbb{M}/\chi(\Sigma)} \right)[-1].
\]
Then there exists a perfect obstruction theory
\[
\phi : E_{\chi(\Sigma)}^* \rightarrow \mathbb{L}_{\chi(\Sigma)}^*. \]
Therefore, there exists a virtual fundamental class \( [\chi(\Sigma)]^\text{vir} \in A_{\text{vir}}(\chi(\Sigma)) \). This proves Donaldson’s conjecture for the existence of virtual fundamental class in [17] §5.

The virtual dimension \( \text{vd} = 1 \) was calculated in [17] §5. The moduli stack \( \chi(\Sigma) \) is smooth of dimension 2, but has wrong dimension.

We briefly review the calculation of the virtual dimension. We actually have for a sextic hypersurface \( S_6 \),
\[
\dim H^1(S_6, T_{S_6})^G = 2; \quad \dim H^2(S_6, T_{S_6})^G = 1,
\]
where the calculation in [17] §5 is given as follows: look at the Euler sequence
\[ 0 \to T^*\mathbb{P}^3(1) \to O(\mathbb{P}^4) \to O(1) \to 0. \]
We have an exact sequence
\[ 0 \to T^*\mathbb{P}^3(2) \to O(\mathbb{P}^4) \to O(2) \to 0. \]
Taking sections gives
\[ 0 \to H^0(T^*\mathbb{P}^3(2)) \to O(\mathbb{P}^4) \otimes O(\mathbb{P}^4) \to S^2(O(\mathbb{P}^4)) \to 0. \]
Since the canonical line bundle \( K_{\mathbb{P}^6} \cong O_{\mathbb{P}^6}(2) \), we have
\[ H^0(T^*\mathbb{P}^3 \otimes K_{\mathbb{P}^6}) \cong \Lambda^2(O(\mathbb{P}^4)) \]
and the \( G \)-equivariant part of \( \Lambda^2(O(\mathbb{P}^4)) \) is 1-dimensional spanned by \( \omega = dx_1 dy_1 + dx_2 dy_2 \).

By Serre duality, the obstruction space has dimension \( \dim H^2(S_6, T_{S_6})^G = 1 \).

The moduli stack \( \chi(\Sigma) \) admits an obstruction bundle which is a line bundle such that, over a point \( t \in \chi(\Sigma) \) representing a sextic surface \( S_6 \), it is given by the obstruction space satisfying \( \dim H^2(S_6, T_{S_6})^G = 1 \). As proved in [17] Page 24, the obstruction bundle is given by studying the section \( S_6 \in T^*\mathbb{P}^3(2) \) defined by the symplectic form \( \omega \) on \( \mathbb{A}^4 \). We omit the details and for more precise proof, we refer to [17] Page 24. We denote by \( L_{\text{Ob}} \) the obstruction bundle. Since the moduli stack \( \chi(\Sigma) \) is a smooth toric Deligne-Mumford stack, standard perfect obstruction theory in [13] implies that the virtual fundamental class is
\[ [\chi(\Sigma)]^\text{vir} = c(L_{\text{Ob}}) \cap [\chi(\Sigma)] \in A_1(\chi(\Sigma)). \]

In the new toric Deligne-Mumford stack \( \chi(\Sigma) \), we have two divisors
\[ D_{II} := \chi(\Sigma/R_{(-2,-1)}); \quad D_{III} := \chi(\Sigma/R_{(0,1)}). \]
The coarse moduli space of these two substacks are all isomorphic to \( \mathbb{P}^1 \), and is the same as the closed substack in \( M_{\text{GIT}} = [\mathbb{P}^2/\mu_2] \) corresponding to the rays \( R_{(-2,-1)} \) and \( R_{(0,1)} \).

Donaldson [17] Formula (19] calculated that
\[ \langle -c_1(L_{\text{Ob}}), D_{II} \rangle = -1/4. \]
Also [17] Formula (17] calculated that
\[ \langle c_1(\lambda_2), D_{II} \rangle = 12. \]
So \( c_1(L_{\text{Ob}}) = \frac{1}{48} c_1(\lambda_2) \) and
\[ PD[\chi(\Sigma)]^\text{vir} = \frac{1}{48} c_1(\lambda_2). \]

Also Donaldson calculated
\[ \langle c_1(\lambda_2)^2, [\chi(\Sigma)] \rangle = 288 \]
in [17] Formula (18] using the property of the line bundle \( \lambda_2 \). Thus
\[ \langle c_1(\lambda_2), [\chi(\Sigma)]^\text{vir} \rangle = 6. \]

9.2.4. Tautological invariants. Let us calculate one tautological invariant following [17] §5.3. There are two MMM-classes associated to the characteristic classes \( c_1^3, c_1c_2^2 \). Donaldson calculated the integration of these classes against the virtual fundamental class \( [\chi(\Sigma)]^\text{vir} \).

Consider the CM line bundle \( L_{\text{CM}} := \lambda_{\text{CM}}(\mathcal{M}/\chi(\Sigma), K_{\mathcal{M}/\chi(\Sigma)}) \) in §8.1 We have
\[ \lambda_{\text{CM}}(\mathcal{M}/\chi(\Sigma), K_{\mathcal{M}/\chi(\Sigma)}) = \lambda_2^3 \otimes \lambda_2^6, \]
where \( \lambda_2, \lambda_3 \) are line bundles on \( \chi(\Sigma) \). Serre duality implies that \( \lambda_3 \cong \lambda_2^2 \). Thus
\[ \langle \lambda_{\text{CM}}(\mathcal{M}/\chi(\Sigma), K_{\mathcal{M}/\chi(\Sigma)}), [\chi(\Sigma)]^\text{vir} \rangle = \lambda_2^2. \]
Then \( L_{CM} = \lambda^2 \). The tautological invariant in Definition 8.1 is

\[
L_{CM} = \int_{[\mathcal{M}_{S^3}]} c_1(L_{CM}) = 12
\]

from (9.2.4).

**Remark 9.3.** Donaldson [17] §5.4 related the KSBA compactification \( \chi(\Sigma) \) to some moduli space of stable maps to \( \mathbb{P}^2 / (\mu_2 \times \mu_2) \) and probably Gromov-Witten invariants of \( \mathbb{P}^2 / (\mu_2 \times \mu_2) \). It is very interesting to explore its deep relationship.

9.3. Short discussion on the moduli stack of sextic surfaces. For a large divisible \( N > 0 \), let \( \mathcal{M}_{24,11,N} \) be the KSBA moduli stack of sextic surfaces \( S \) with \( K_S^2 = 24, \chi(O_S) = 11 \). Although it seems hard to obtain explicitly all the boundary divisors of \( \mathcal{M}_{24,11,N} \) which contain s.l.c. sextic surfaces with quotient singularities, in [32] Horikawa classified all the deformations of smooth sextic hypersurfaces; i.e., the substack for \( N = 1 \). Let us review [32] Theorem 1. Let \( S \) be a smooth sextic surface in \( \mathbb{P}^3 \), then the line bundle \( K_S \) is divisible by 2 which we denote by \( 2L = K_S \). From [32] Lemma 2.1, \( h^0(S,L) = 4 \), thus, the line bundle \( L \) determines a morphism

\[
\phi_L: S \to \mathbb{P}^3.
\]

Then from [32] Theorem 1, there are totally six deformations of \( S \) associated with the morphism \( \phi_L \).

Ia: \( S \) is birationally equivalent to a sextic surface in \( \mathbb{P}^3 \) with at most RDP’s as singularities;

Ib: \( \phi_L \) is a generically 2-fold map onto a cubic surface in \( \mathbb{P}^3 \);

IIa: \( \phi_L \) is a generically 3-fold map onto a quadratic surface in \( \mathbb{P}^3 \);

IIb: \( \phi_L \) is a generically 2-fold map onto a smooth quadratic surface in \( \mathbb{P}^3 \);

Ic: \( \phi_L \) is a generically 2-fold map onto a singular quadratic surface in \( \mathbb{P}^3 \);

IIc: \( \phi_L \) is composed of a pencil of curves of genus 3 of non-hyperelliptic type.

In [32] Horikawa gave explicit constructions for each possible deformation. We list all the constructions as complete intersection surfaces in weighted projective spaces.

Ia: The surface \( S \) of type Ia is a sextic hypersurface \( S \subset \mathbb{P}^3 \) given by a degree 6 homogeneous polynomial with only RDP’s as singularities.

Ib: The surface \( S \) of type Ib is a complete intersection surface in \( \mathbb{P}(3,1,1,1,1) \) with coordinates \((w,x_0,x_1,x_2,x_3)\) of weights \((3,1,1,1,1)\) given by

\[
g = g(x_0,x_1,x_2,x_3) = 0; \quad w^2 + f = 0,
\]

where \( g = g(x_0,x_1,x_2,x_3) \) is cubic function and \( f = f(x_0,x_1,x_2,x_3) \) is a degree 6 homogeneous polynomial.

Ic: The surface \( S \) of type Ic is a complete intersection surface in \( \mathbb{P}(2,1,1,1,1) \) with coordinates \((u,x_0,x_1,x_2,x_3)\) of weights \((2,1,1,1,1)\) given by

\[
g = g(x_0,x_1,x_2,x_3) = 0; \quad u^3 + A_2u^2 + A_4u + A_6 = 0,
\]

where \( g = g(x_0,x_1,x_2,x_3) \) is of degree 2 and \( A_{2j} = A_{2j}(x_0,x_1,x_2,x_3) \) are degree 2j homogeneous polynomials.

IIa and IIb: For a surface \( S \) of type IIa or IIb, its canonical model is in the weighted projective space \( \mathbb{P}(1,1,1,1,2,3) \) with coordinates \((x_0,x_1,x_2,x_3,u,w)\) of weights \((1,1,1,1,2,3)\) defined by

\[
q = 0; \quad x_0u = h; \quad w^2 = u^3 + A_2u^2 + A_4u + A_6,
\]

where \( q,h,A_{2j} \) are homogeneous polynomials in \( x_i \) of degree 2, 3, 2j respectively.

III: From [32] §6, the surface of type III can be given as a subspace in the weighted projective space \( \mathbb{P}(1,1,1,1,2,2,3) \) with coordinates \((x_0,x_1,x_2,x_3,y_1,y_2,z,w)\) of weights \((1,1,1,1,2,2,3)\) defined by

\[
\Phi_i = 0; \quad \Psi_i = 0; \quad \Gamma_i = 0; \quad \Delta = 0.
\]
Here $\Phi_i (1 \leq i \leq 3)$ are of degree 2, $\Psi_i (1 \leq i \leq 3)$ are of degree 3, $\Gamma_i (1 \leq i \leq 3)$ are of degree 4, and $\Delta$ has degree 6. These functions can be found in [32, §6]. Although it is hard to see if the surface of type III is a global complete intersection in $\mathbb{P} (1, 1, 1, 1, 2, 2, 2, 3)$, [32, §6] pointed out that this surface of type III is either smooth or with rational double points as singularities.

We can perform the same calculation as in [91, §3] to calculate the dimensions of the cohomology spaces of such complete intersection surfaces $S$,

$$\begin{align*}
\dim H^1 (S, T_S) &= 68; \\
\dim H^2 (S, T_S) &= 6.
\end{align*}$$

Let $\overline{M}_{\text{sextic}} \subset \overline{M}_{24,11,N}$ be the closure of Gieseker moduli stack $M_{24,11} \subset \overline{M}_{24,11,N}$.

**Theorem 9.4.** Suppose that we know all the boundary divisors consisting s.l.c. sextic surfaces in $\overline{M}_{\text{sextic}}$, then the moduli stack $\overline{M}_{\text{sextic}}$ is an irreducible Deligne-Mumford stack of dimension 68.

**Proof.** From [31, Theorem 2], the Gieseker moduli stack $M_{24,11}$ (without the KSBA compactification) is irreducible. There may have some other irreducible components in $\overline{M}_{24,11}$ consisting of singular s.l.c. sextic surfaces. But the closure $\overline{M}_{\text{sextic}}$ is irreducible. For the dimension of the moduli stack, note that the dimension of the homogeneous polynomials in $k[x_0, x_1, x_2, x_3]$ modulo equivalence is $84 - 16 = 68$.

This locus contains all the type Ia surfaces; i.e., the sextic hypersurfaces in $\mathbb{P}^3$. All the other types of deformation surfaces above should belong to the boundary divisor since the moduli stack is irreducible. Therefore the dimension of the moduli stack is 68. \qed

**Conjecture 9.5.** Over the s.l.c. sextic surfaces $S$ in all the boundary divisors of $\overline{M}_{\text{sextic}}$, the dimensions of the cohomology spaces of the tangent sheaf of $S$ are given by

$$\begin{align*}
\dim H^1 (S, T_S) &= 68; \\
\dim H^2 (S, T_S) &= 6.
\end{align*}$$

**Remark 9.6.** In the case of moduli stack $\overline{M}_{5,5}$ of numerical quintics, the boundary divisors consisting of a unique Wahl singularity $\frac{1}{5} (1, 1, 1, 1, 1)$ were found in [62], where the only cases of minimal surfaces with a unique Wahl singularity are of type $\frac{1}{5} (1, 1, 1, 1, 1)$ and $\frac{1}{5} (2, 5)$, and the case $\frac{1}{5} (2, 5)$ was proven in [62] to be impossible.

In the case of sextic surfaces, from calculation there are totally possible 29 cases of the unique Wahl singularity in the minimal surfaces in the boundary divisors, which makes the calculation much more complicated.

Let us only consider the moduli stack $\overline{M}_{\text{sextic}}$ such that all of its boundary divisors consist of Q-Gorenstein deformation of class 7-singularities. Let $f : \overline{M}_{\text{ind}} \rightarrow \overline{M}_{\text{sextic}}$ be the moduli stack of index one covers. Thus from the conjecture we have that

**Proposition 9.7.** Under the conjecture [91, §6] there exists a rank 6 nontrivial obstruction bundle $\text{Ob} \rightarrow \overline{M}_{\text{ind}}$ such that over any surface $S \in \overline{M}_{\text{sextic}}$, the fiber is given by $\text{T}^2_{\text{QC}} (S)$. Assume that the obstruction bundle $\text{Ob}$ is nontrivial, then the virtual fundamental class $\overline{M}_{\text{ind}}^{\text{vir}} \in \mathbb{A}_{\text{vir}} (\overline{M}_{\text{ind}})$ is given by

$$\overline{M}_{\text{ind}}^{\text{vir}} = \epsilon (\text{Ob}) \cap \overline{M}_{\text{ind}}^{\text{vir}}.$$

**Proof.** Since under the conjecture the moduli stack $\overline{M}_{\text{sextic}}$ and $\overline{M}_{\text{ind}}^{\text{vir}}$ are projective Deligne-Mumford stacks and the obstruction bundle $\text{Ob} \rightarrow \overline{M}_{\text{ind}}$ is nontrivial, then standard argument in the perfect obstruction theory shows that the virtual fundamental class is just the Euler class of the obstruction bundle. \qed

**Remark 9.8.** It is very interesting to check if Conjecture [91, §6] holds, and calculate the tautological invariants for the moduli stack $\overline{M}_{24,11,N}$. 
