Vector opinion dynamics in a model for social influence

M. F. Laguna, Guillermo Abramson, and Damián H. Zanette

Consejo Nacional de Investigaciones Científicas y Técnicas, Centro Atómico Bariloche and Instituto Balseiro, 8400 Bariloche, Río Negro, Argentina

Abstract

We present numerical simulations of a model of social influence, where the opinion of each agent is represented by a binary vector. Agents adjust their opinions as a result of random encounters, whenever the difference between opinions is below a given threshold. Evolution leads to a steady state, which highly depends on the threshold and a convergence parameter of the model. We analyze the transition between clustered and homogeneous steady states. Results of the cases of complete mixing and small-world networks are compared.

Key words: social dynamics, opinion formation
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1 Introduction

Much attention has been paid in recent years to the potential applications of the methods of statistical physics to complex systems and phenomena that were traditionally considered to lie far outside the interests of physicists [1]. Modeling such systems as sets of interacting dynamical elements has proven to be a fruitful procedure to capture the essential mechanisms that lead to generic forms of emergent collective behavior –such as pattern formation and synchronization [2,3]– actually observed in many natural processes. This program has been successfully applied to a large variety of systems in the fields of physics, chemistry, and biology [2,4], and has also been extended to the study...
of social and economical systems [5]. In this class of systems, where elementary agents are human beings, models aim at describing rational behavior, such as decision making or the adoption of a strategy.

In this work we study a generic model of social influence inspired in Axelrod’s proposal [5], in which the interaction between individuals tends to homogenize the state of the population, in the presence of some pre-existing homogeneity. In Axelrod’s model, each agent is characterized by a “cultural” state defined as a string of features, each one of them being a trait of integer value. The agents are arranged on a bidimensional lattice, and can interact with nearest neighbors. The dynamical rules of the model prescribe that two agents can interact if they have some cultural similarity, v.g. if some of their features have equal traits. This pre-existing similarity defines a probability that allows the active agent to further approach the state of her partner, by adopting one of her neighbor’s cultural features. The system may evolve into an inhomogeneous state of “cultural regions” whose number and sizes depend on the parameters of the model. Interesting analysis are given in Refs. [6,7], where statistical properties of the dynamics are shown.

In this context, we use the term opinion for the set of individual attributes that are subject to social influence and can be updated as a result of the interaction between agents. Opinions can be modeled, in a simple approach, as binary numbers [8] and their dynamics have been first studied in economy [9]. Axelrod’s “cultural” state, restricted to features made up of binary traits (0 or 1), corresponds to what we call here “opinion” state, and has been used in Refs. [10,11]. It may be interpreted as the answers to a survey with dichotomous (yes/no) questions aimed at defining the agent’s position on a given subject. Alternatively, opinions can be considered as continuous variables. The conditions to reach consensus in this case were analyzed in several works [12,10].

In our model, we use opinions defined by a binary string, i.e., the state of each agent is a vector of components 0 or 1. Since we are interested in the propagation of opinions within a population, agents are randomly selected in pairs, instead of the regular lattice of Axelrod’s model, which is more appropriate for the geographical spread of culture. If the opinions of the selected agents are close enough, they interact. The probability of interaction is the same for all agents, and is controlled by a threshold-like parameter. This is also in contrast with Axelrod’s model, where the probability of interaction increases strictly with the cultural similarity. As for the underlying structure of the population, we analyze two cases. The first corresponds to the situation of complete mixing, where any pair of individuals can interact and adjust their opinions. In the second, we assume that the agents are in the nodes of a small-world network and the interaction process occurs only between connected agents.
In the following section we introduce the model for the opinion dynamics for the complete mixing case, and study the dynamics of the system. In Sec. III, the properties of the stationary state are analyzed. In Sec. IV, we introduce the small-world model and its main results, and emphasize similarities and differences between both cases. Finally, we summarize and discuss our most important results.

2 Social influence and the dynamics of opinion

Our model consists of a set of agents that update their opinions as a result of mutual influence. We consider a population of \( N \) agents, each of them characterized by a binary vector of \( k \) components representing her opinion. The number of possible different opinions is \( 2^k \), and \( k \) is the dimension of the opinion space. The initial condition is a random and uniform distribution of opinions. At each time step \( t \) two randomly chosen agents meet. If the difference between their opinions is lower than a threshold \( u \), one of them copies each opinion component from the other with probability \( \mu \). This event is called an “interaction” in the rest of the paper. Clearly, its occurrence is governed by \( u \), the parameter that characterizes the existence of some –even if partial– previous coincidence of opinions. The probability of change of an agent’s opinion is \( p = 1 - (1 - \mu)^d \), where \( d \leq u \) is the distance between the opinions of the interacting pair. Thus, \( \mu \) plays the role of a convergence parameter of the model.

To quantify the difference between two opinions we use the Hamming distance, i.e., the number of different components between the two binary vectors. In consequence, the interaction process occurs when agents agree in at least \( k - u \) opinion components. The threshold \( u \) is taken as constant in time and across the whole population.

In our simulations, we consider \( 16 \leq N \leq 4096 \) and \( k = 10 \), so that \( 2^k = 1024 \). Systems with \( N < 2^k \) are said to be “diluted,” whereas systems with \( N > 2^k \) are “saturated,” since there will always be at least two agents with the same opinion. Our numerical calculation thus cover both extremes of saturation. We have used several values of the convergence parameter, between \( \mu = 0.066 \) and \( \mu = 1 \). We show here the results corresponding to these two values, that serve well as characterization of the two observed dynamics. As we show below, the number of time steps needed to reach the stationary state, \( t_f \), strongly depends on \( \mu \) and scales with \( N^2 \). We take \( t_f = 500N^2 \) for the lowest value \( \mu \) used and \( t_f = 100N^2 \) for the highest value.

Our analysis begins with the study of opinion evolution from a random distribution in a well mixed population, where any two agents are allowed to
interact. We find that the system evolves, at long times, toward clusters of homogeneous opinions. As a characterization of the state of the system we compute, at each time step, the number of agents with the same opinion and calculate the number of clusters of different opinions, $N_c$. In Fig. 1 we plot the evolution of $N_c$ for several values of the threshold. Figure 1(a) shows the results for the convergence parameter $\mu = 0.066$, and Fig. 1(b) those for $\mu = 1$. A low value of $\mu$ insures a much slower approach to a stationary state. At short times we find that $N_c \gg 1$, as expected for a system with a random distribution of opinions. At long times, for sufficiently large thresholds, the number of clusters is $N_c = 1$, i.e. full consensus is achieved. If the threshold is low enough we have, instead, $N_c > 1$ at long times. This means that clusters of different opinions exist in the stationary state, in spite of the fact that complete mixing of the interaction mechanism and the homogeneity of the initial configuration allow the propagation of an opinion through the whole population. The transition from a multi-clustered state to a single cluster one occurs at higher values of $u$ for the fast $\mu = 1$ system than for the slower $\mu = 0.066$ one. This transition is also sharp for $\mu = 0.066$ (Fig. 1(a)) and smooth for $\mu = 1$ (Fig. 1(b)), a point to which we will return below.

Figure 1(a) shows, moreover, that the evolution in the number of clusters is not monotonous. There are randomly distributed events where $N_c$ increases, caused by the opinion rearrangement that takes place before two clusters collapse. This feature disappears when averaging over different realizations of the system, as is the case of the rest of the results shown in this paper.
Another characterization of the opinion dynamics is provided by the evolution of the cumulative number of interactions in the system, $N_i(t)$. If the threshold is large (of the order of $k$), every pair of agents selected in a given time step does interact, i.e. $N_i(t) = t$. In the opposite limit, if $u$ is zero there are no interactions and $N_i(t) = 0$ for all $t$. For intermediate values of $u$, the quantity $N_i(t)$ illustrates the dynamical state of the system. In Figs. 2(a) and 3 we plot $N_i$ as a function of time for different values of the threshold $u$, and for $\mu = 0.066$ and $\mu = 1$ respectively.

In Fig. 2(a) we find three regimes. An initial one is characterized by a linear growth of $N_i$, with a slope that depends on the threshold $u$. This regime is relatively short, and can be seen more clearly in the double logarithmic plot in the inset of Fig. 2(a), where the vertical separation of the lines represent the difference in their slopes. This transient regime corresponds to the first stage of clusterization. Since the initial condition is uniform over the phase space, every agent finds her randomly picked potential partner at a random distance, uniformly distributed over the phase space. Due to the fact that the interaction becomes effective only if the partner lies within a ball of radius $u$ from the agent, the probability of interactions is proportional to the fraction of the phase space covered by such balls. If $u \approx k$, these balls cover almost
all the space and the activity is the fastest possible, with \( N_i \approx t \), as can be seen in the figure, for the case \( u = 9 \). For smaller \( u \), the activity is slower and the slope of \( N_i(t) \) tends to 0. This initial regime continues during the first stage of the clusterization process, and Fig. 2(b) allows a visualization of its extent. Here we show the complement of the number of clusters, \( 2^k - N_c(t) \), for \( k = 10 \), corresponding to the simulation run of \( u = 3 \) shown in Fig. 2(a). A small arrow indicates the end of this regime, where the clusterization reaches a plateau. At this point, the system is composed of a great number of clusters distributed over the phase space. Since the convergence parameter \( \mu = 0.066 \) allows an interacting pair to approach their opinions without collapsing them into a single one, most of these clusters lie close to each other. This is in contrast to what we find for \( \mu = 1 \), as discussed below. After this plateau, the clusters have become near enough to interact more often, and an acceleration of the activity appears for all values of \( u \). This is apparent in Fig. 2(a) as an upward bending of \( N_i(t) \), and in Fig. 2(b) as a nearly linear growth of \( 2^k - N_c \). This mechanism rapidly reduces the number of clusters, until there are only a few of them remaining (typically around 10 for \( u = 3 \)). This marks the beginning of the third and final regime, indicated by the second arrow for the case \( u = 3 \) in both figures. The remaining clusters are very close to each other, and the dynamics proceeds at the maximum possible speed of one interaction per time step until the final state of a single opinion is reached, for all \( u > 2 \).

In Fig. 3 we show the curves \( N_i \) vs. \( t \) that correspond to the highest convergence parameter \( \mu = 1 \). For all values of \( u \) we find two regimes, a linear one at short times with a slope dependent of \( u \) (for the same underlying reasons as in the case \( \mu = 0.066 \)), and a long time behavior with a slower increase with time. After the initial linear regime in which there is an intense activity, the interactions become more spaced in time and finally stop, when the stationary state is reached. The reason for this slowing down, at variance with the acceleration displayed when \( \mu = 0.066 \), resides in the extreme value of the convergence parameter. Since \( \mu = 1 \), the opinions of a pair of interacting agents become identical after the interaction. The clusters, in consequence, become effectively isolated from one another, and intermediate opinions do not exist after the first clusterization stage. Eventually, and depending on the value of \( u \), a final state with several clusters becomes stationary.

In Fig. 4 we plot the average over 100 realizations of the total number of interactions \( N_i \) needed to achieve the stationary state, as a function of the system size \( N \) for two values of the threshold \( u \), \( k = 10 \) and \( \mu = 1 \). We observe an approximately linear increase of \( \langle N_i \rangle \) with \( N^2 \), with a slope depending on the threshold, as shown in the inset. For all system sizes we find the same dependence with \( N \), which coincides with the one observed for the time \( t_f \) needed to reach the stationary state. This means that the ratio \( N_i/t_f \), measuring the average number of interactions per time unit during the complete evolution of the system, is independent of the system size. For \( \mu = 0.066 \) the dependence
Fig. 3. Number of interactions $N_i$ as a function of time for a system of $N = 1024$ agents, $k = 10$, $\mu = 1$ and four values of the threshold: $u = 2$, $3$, $4$ and $9$. Inset: The same quantities in log-log scale.

Fig. 4. Total number of interactions as a function of system size, averaged over 100 realizations, with $k = 10$, $\mu = 1$ and two values of the threshold: $u = 5$ (diamonds), $u = 9$ (open circles). Inset: linear-linear plot of the same data; lines are linear fits.

on $N$ is the same, even though there is a factor between 10 and 100 in the total number of interactions needed to reach the final state. Since the behavior of $\langle N_i \rangle$ does not depend on $N$ in a qualitative way, we use an intermediate representative value, $N = 2^k$, in the rest of the paper.
Fig. 5. Average number of clusters $\langle N_c \rangle$ as a function of the threshold, for 100 realizations with $N = 1024$ and two values of the convergence parameter: $\mu = 0.066$ (squares) and $\mu = 1$ (open circles).

3 Characterization of stationary states

As shown in the previous section, a stationary state is achieved after a number of interactions that depends on the convergence parameter $\mu$ and on the particular realization. We are interested now in the dependence of the final state of the system on the threshold and the convergence parameter.

In Fig. 5 we plot the number of clusters in the stationary state, $\langle N_c \rangle$, as a function of the threshold $u$ for two values of the convergence parameter, $\mu = 0.066$ and $\mu = 1$. Each dot of the curve corresponds to an average over 100 realizations. In both cases the value of $\langle N_c \rangle$ for $u < 2$ is very similar and corresponds to a situation where several clusters of different opinions coexist in equilibrium. In the region $u > 7$, both cases are also similar, with $\langle N_c \rangle = 1$. This is a situation where a single opinion always prevails. In the region $2 < u < 7$ the mean number of clusters depends on the convergence parameter. The transition from the multi-cluster state to the single opinion state is smooth for $\mu = 1$, whereas for $\mu = 0.066$ there is a sharp transition at $u \lesssim 3$. For the system size shown here this last transition is found for $\mu < 0.99$. A signature of the deep difference between the two transitions is their behavior in a finite size analysis. We show this in Fig. 6 for $\mu = 0.066$ and $\mu = 1$, for a range of system sizes that correspond to increasing saturation, since $k$ remains fixed. It is clear that the transition becomes sharper with increasing system size when $\mu < 1$, and that it preserves its smoothness when $\mu = 1$. This is
Fig. 6. Average number of clusters $\langle N_c \rangle$ as a function of the threshold for 200 realizations and two values of $\mu$, as indicated in the plots. Different curves correspond to system sizes $N = 16, 32, 64, 128, 384, 1024, 2048$, and $4096$.

an indication that there is a further transition at $\mu = 1$ between the two types of behavior. Only the fastest systems ($\mu = 1$) allow the persistence of multi-opinion populations at intermediate values of the threshold $u$. In Figs. 6(a) and (b) we show a diversity of behaviors that correspond to the different regimes of saturation. This analysis does not allow a further characterization of the transitions. Indeed, the thermodynamic limit would require not only that $N \to \infty$ but also that $k \to \infty$, at a fixed saturation in opinion space. Such systems, with opinions defined by infinite features, go beyond the interest of the present study. In a related context, this has been carried out in Ref. [6], where the switch from the monocultural to the multicultural states in Axelrod’s model is identified as a phase transition. The roles of noise and of network structure have been analyzed in Refs. [7,13].

The transition as a function of the threshold can be further characterized analyzing the distribution of the population of clusters, $P(x)$, for different values of $u$. If the system is in a homogeneous state, all the agents having the same opinion, then $N_c = 1$ and the distribution has a peak at $x \simeq N$. A situation of many clusters with a few agents in each cluster is characterized by a distribution $P(x)$ with a dominant peak in $x \sim 1$. In Fig. 7 we show the results for the case $\mu = 0.066$ and two values of $u$ near the transition. For $u < 2$ the histogram has a single peak in $x = 1$ whereas for $u > 3$ it has a single peak in $x = N$. The threshold at which the transition between a clusterized state and a homogeneous state takes place is between $u = 2$ and 3. This result coincides with that of Fig. 5.
A very different sequence of histograms is obtained for $\mu = 1$, as we show in Fig. 8. The transition from a highly clustered state (the histogram with a peak in $x = 1$, at $u = 2$) to a homogeneous state (the histogram with a peak in $x = N$, at $u = 7$), takes place gradually for intermediate values of $u$. This result is also in agreement with the behavior observed in the $\langle N_c \rangle$ curve.

Note that to describe completely the state of the system we need both quantities, $\langle N_c \rangle$ and the sequence of histograms. As an example, observe that in Fig. 5, $\langle N_c \rangle \sim 4$ for both $u = 2$ with $\mu = 0.066$ and $u = 4$ with $\mu = 1$. However, the histograms are very different in Fig. 8 with $u = 2$ and Fig. 9 with $u = 4$.

The results of this section indicate that a well-defined transition between a homogeneous state and a highly clustered state takes place in this model as a function of $u$. A complementary result is that the parameter $\mu$ not only modifies the convergence speed towards equilibrium but also changes the structure of the stationary state. This last result contrasts with Ref. [8], where the authors report that the convergence parameter only modifies the time needed to reach the equilibrium.

### 4 Complex mixing vs. small-world network

The two previous sections have dealt with the case of complete mixing, in which any agent can interact with any other one in the system. How do the
Fig. 8. Distribution of cluster population $P(x)$, for 100 realizations of a system with $N = 1024$, $\mu = 1$ and six values of the threshold, as indicated in each plot.

dynamical and steady states change if the underlying structure of the population is different? Here we analyze the case in which the agents are situated at the nodes of a small-world network [14], which represents more accurately some aspects of real social populations. At variance with a complete mixing system, a small-world network exhibits some degree of local clustering in the neighborhood of each agent. We introduce the parameter $p$, the reconnection probability, that measures the randomness of the small-world and interpolates between an ordered lattice ($p = 0$) and a random graph ($p = 1$). Another property that characterizes the small-world is the network connectivity $K$. The small-world network is built from a one-dimensional regular lattice in which each node is linked to its $2K$ nearest neighbors. Then, each link is rewired with probability $p$ to a randomly chosen node. Multiple links are forbidden and disconnected networks are discarded. Every agent in the small-world net-
The interaction process occurs only between linked sites. Now, we randomly choose an agent and then we select, also at random, one of his neighbors to interact with. The mechanism of the interaction is the same as in the complete mixing model. To ease the comparison with previous results, we use $N = 1024$ and $k = 10$ in all the figures of this section.

Figure 9 shows the average number of clusters $\langle N_c \rangle$ as a function of the threshold $u$ for a small-world network with $K = 3$, $\mu = 1$ and three different values of the reconnection probability $p$. We compare this results with those obtained in the complete mixing case of Fig. 5 (open circles). The behavior of $\langle N_c \rangle$ for $p \neq 0$ is very similar to the results of the complete mixing case in all the range of thresholds. The only qualitatively different situation is the case $p = 0$, that corresponds to the agents in a regular lattice. In this case, and for $u < 7$, the number of clusters with different opinions is significantly larger than those corresponding to other values of $p$. The regular lattice— and only the one which is completely regular—is less effective at disseminating the opinions. The same dependence with $p$ was observed for the case of $\mu = 0.066$ (not shown here).

These results indicate that the dynamics in a small-world configuration does not differ from that of a well mixed system, and the only different case is the one corresponding to $p = 0$, actually, not a small-world network but a
Fig. 10. Number of clusters $N_c$ as a function of time, for a system arranged on a regular lattice of $N = 1024$ agents, $k = 10$, $\mu = 0.066$ and three values of the threshold. This situation corresponds to a small-world network with $p = 0$.

Fig. 11. Mean number of clusters as a function of network connectivity $K$ for 200 realizations in a small-world with $p = 0$, $\mu = 1$ and $N = 1024$. (a) The threshold is $u = 5$ (b) In this picture, $u = 3$. Dashed horizontal lines indicate the complete mixing (CM) case.
regular lattice. The transient evolution of single realizations of such a system is shown in Fig. 10, where we plot $N_c(t)$ for three values of the threshold. The parameters are the same as in Fig. 1, where we use the lowest value of $\mu$ to have the time evolution in detail. Note that the system on the lattice needs roughly 10 times more iterations than the completely mixed one. We also observe the random fluctuations originated by opinion rearrangements in the clustering process.

Finally, in order to analyze the effect of the connectivity in the system, we change the values of $K$. As can be observed in Fig. 11, the mean number of clusters tends to the complete mixing case at high values of $K$. The number of nearest neighbors necessary to recover that behavior depends on the threshold.

5 Conclusion

We have studied a generic model of social influence in which agents adjust their opinions as a result of random binary encounters whenever their difference in opinion is below a given threshold.

We have analyzed the dynamical behavior of the system and found that the cases $\mu = 1$ and $\mu < 1$ behave differently. For the extreme case $\mu = 1$ we observe an initial linear regime of intense activity that ceases when intermediate opinions disappear. This first stage is followed by a different regime in which the interactions become more spaced in time because the opinion clusters are effectively isolated, even for $u$ very near $k$. Finally, a saturation occurs when the stationary state is reached. The slower case $\mu = 0.066$ shows three regimes. The first stage of clusterization is similar to the one observed in the previous case, but it produces a great number of clusters that lie close to each other. A second stage starts, in which the clusters interact more often, giving rise to an acceleration of the activity until a few clusters remain in the system. Finally, in the third stage the dynamics proceeds at the maximum speed due to the interaction between clusters that are close enough. The dynamical process stops when a single opinion is reached (for $u > 2$) or several isolated clusters are formed (for $u < 2$). We can see, then, that the speed of the convergent dynamics plays a relevant role not only in the final composition of the population but also in the intermediate stages of evolution.

We also found that both the total number of interactions between individuals and total time needed to reach the equilibrium, increase quadratically with the size of the system. Consequently, the ratio between these two quantities, $N_i/t_f$, is independent of the system size. We obtained this result for all the values of $\mu$ used.
As for the stationary state properties, we have found that the interaction between individuals restricted by a proximity threshold results into clustering of opinions with a number of clusters decreasing with the increase of the threshold. Our model presents a transition driven by the threshold, from a state with homogeneous opinion (consensus) to a phase in which several clusters with different opinions coexist. The nature of the transition depends on the convergence parameter $\mu$, that changes the structure of the stationary state. This was also observed in the sequence of histograms of population of clusters near the transition. Finite size scaling supports the existence of such transition.

Finally, we have studied the influence of the underlying structure on the behavior of the system. We found that small-world networks and the complete mixing case have a very similar behavior, except for the limiting case of regular graphs ($p = 0$ in the small-world model). This last case is less effective to disseminate the opinions than the others. The peculiarity of the case $p = 0$ is found also in other contexts, within the study of small-world networks. There has been found that the small-world phenomenon appears at any value of the network disorder $p$, for sufficiently large systems. Geometrical properties such as the average path length [15,16], and some dynamical ones, such as the ferromagnetic properties of an Ising model [15], have been shown to behave critically at $p = 0$. This is not the case in some other dynamical models, such as the propagation of an epidemic [17], and of a rumor [18], that behave critically at a finite value of $p$ in the thermodynamical limit. In our model of opinion dynamics, the effect of disorder is even stronger, since any nonzero value of $p$ induces a behavior that is indistinguishable from both the fully disordered $p = 1$ case and the complete mixing case. We also found that greater network connectivity gives rise to a decrease in the number of clusters. The number of connected neighbors needed to reach the complete mixing ones depends on the threshold.

Several open problems associated with opinion dynamics are worth mentioning. For instance, the interplay between long-range effects, such as mass-media broadcasting of individual or collective opinions, and local interactions is an essential mechanism in modern opinion formation. Boundary effects and inhomogeneity conditions, including spatial variation of population density, may also strongly affect the dynamics of social influence and the properties of the resulting stationary state. Finally, the effects of noise, communication errors, and in general random fluctuations, should not be disregarded in any realistic representation of information spreading. The present model is expected provide a versatile tool to analyze these questions.
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