An Adaptive Pilot Model with Reaction Time-Delay

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Abstract—Practical adaptive control implementations where human pilots coexist in the loop are still uncommon, despite their success in handling uncertain dynamical systems. This is owing to their special nonlinear characteristics which lead to unfavorable interactions between pilots and adaptive controllers. To pave the way for the implementation of adaptive controllers in piloted applications, we propose an adaptive human pilot model that takes into account the time delay in the pilot’s response while operating on an adaptive control system. The model can be utilized in the evaluation of adaptive controllers through the simulation environment and guide in their design.

I. INTRODUCTION

Adaptive controllers are one of the major advancements in the field of control theory when it comes to addressing the control of dynamical systems that are prone to failures and uncertainties. While their design techniques are simple, and the theory behind their stability and performance is well established, their wide-spread use in real-life applications, where human pilots are in the loop, is yet to be seen. Several flight tests showed unfavorable interactions between human pilots and adaptive controllers due to their special nonlinear characteristics [1]. Hence, to aid in the design of adaptive controllers for piloted applications, a human-in-the-loop analysis is deemed necessary.

Human pilot models play a crucial role in the evaluation of human-in-the-loop control systems as they allow the designer to test a controller through a simulation environment. Prominent models such as McRuer’s crossover model [2] and its extensions [3], [4], provide a simple fixed representation of human pilots in the loop with time-invariant control systems. However, such models fail to capture the adaptive behavior of human pilots when faced with unexpected anomalies.

A few adaptive human pilot models have been proposed in the literature. In [5] and [6], an adaptive pilot model is proposed, where the adaptation laws are based on expert knowledge, aiming to make the adaptive pilot model follow the dictates of the crossover model. Inspired by this idea, an experimentally-validated adaptive pilot model is recently developed in [7] and [8], by resorting to model reference adaptive control (MRAC) techniques which allow a rigorous stability analysis using the Lyapunov-Krasovskii stability criteria. These models assume that the pilot is operating on a linear control system, making them unsuitable for the evaluation of adaptive control systems. Although there exist studies such as [9] and [10], where an adaptive controller is in the loop, the pilot model used is not adaptive.

The first adaptive human pilot model that is used in the loop with an adaptive controller has been recently proposed in [11] and [12]. The development of the model is carried out based on MRAC architecture, with a rigorous Lyapunov stability analysis. The model does not explicitly take into account the time delay in human pilot’s response, which narrows down the class of suitable applications.

In this paper, we build upon the works in [11] and [12], by proposing an adaptive pilot model that considers the human internal time delay while operating on an adaptive control system. The model can be used for the evaluation of adaptive controllers in piloted applications and aid in their design. The inclusion of time delay in the human’s response forms a major difficulty, which necessitates the prediction of the future states of a time-varying uncertain adaptive control system. We propose a novel approach by resorting to the fundamental theory of linear systems, and MRAC to provide a rigorous Lyapunov-Krasovskii stability analysis.

The notation used here is standard, where \( \mathbb{R}^{p \times q} \) denotes the set of real \{symmetric\} \{diagonal real\} \( p \) by \( q \) matrices, and \( \| \cdot \| \) refers to the euclidean norm for vectors \( (q = 1) \), and the induced-2 norm for matrices. \( \| \cdot \|_F \) refers to the Frobenius norm for matrices, \( \text{Tr}\{\cdot \} \) refers to the trace operator, and \( (\cdot)^T(\cdot)^{-1} \) denotes the transpose [inverse] operator. \( \text{Proj}\{\hat{\theta}(t), Y\} \) is the element-wise projection operator, defined in [13], used to bound each element \( \hat{\theta}_{i,j}(t) \) of an adaptive parameter \( \hat{\theta}(t) \) in a compact set \([\theta_{\min,i,j}, \theta_{\max,i,j}]\). Finally, we write \( \lambda_{\text{min}}(A) \) for the minimum eigenvalue of the matrix \( A \) and we denote the set of positive definite real matrices by \( \mathbb{R}^{p \times p}_+ \).

II. PROBLEM STATEMENT

To model the human’s adaptive control behavior with an adaptive controller in the loop, we start with a block diagram given in Fig. 1. In the figure, the block diagram is divided into inner and outer loops. The inner loop consists of an adaptive controller controlling a plant with uncertain dynamics such that the plant states follow those of a reference model by adjusting the control parameters using an adaptive law.

The outer-loop consists of the human controlling the inner-loop such that the plant output follows a reference input. The human is assumed to be well trained, i.e., familiar with the nominal plant-controller dynamics. However, he/she is not aware of the uncertainties in the plant dynamics. This motivates modeling the human as an adaptive outer-loop.
controller, where an adaptive law is utilized to force the plant states to follow the states of the crossover-reference model.

III. INNER LOOP

Consider the following uncertain plant dynamics

$$\begin{align}
\dot{x}_p(t) &= A_px_p(t) + B_py_u(t), \\
y_1(t) &= C_1^T x_p(t), \\
y_2(t) &= C_2^T x_p(t),
\end{align}$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is the accessible state vector, $u_p(t) \in \mathbb{R}^m$ is the plant control input, $\Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{D}^{m \times m}$ is an unknown control effectiveness matrix with the diagonal elements $\lambda_{i,i} \in (0,1]$. $A_p \in \mathbb{R}^{n_p \times n_p}$ is an unknown system matrix, $B_p \in \mathbb{R}^{n_p \times m}$ is a known control input matrix, and $C_1 \in \mathbb{R}^{n_r \times n_p}$ and $C_2 \in \mathbb{R}^{n_r \times n_p}$ are both known output matrices. The outputs $y_1(t) \in \mathbb{R}^m$ and $y_2(t) \in \mathbb{R}^m$ are the outputs of interest for the inner and outer loops, respectively. Furthermore, it is assumed that the pair $(A_p, B_p)$ is controllable.

Let the nominal plant dynamics be given as

$$\dot{x}_n(t) = A_n x_n(t) + B_p u_n(t),$$

where $u_n(t) \in \mathbb{R}^m$ is a nominal controller given as

$$u_n(t) = -L_x x_n(t) + L_r y_h(t - \tau),$$

where $y_h(t - \tau) \in \mathbb{R}^m$ is the human command to the inner-loop with an internal human time delay $\tau \in \mathbb{R}_+$ and $L_x \in \mathbb{R}^{m \times n_p}$ is such that $A_x \triangleq A_n - B_p L_x$ is Hurwitz. It is noted that the human input $y_h(t)$ is bounded due to physical manipulator limits. In the design of the outer loop, given in the following section, human input saturation bounds imposed by the manipulator limits are considered in the stability analysis. Defining $B_r \triangleq B_p L_r$, the reference model is assigned as

$$\begin{align}
\dot{x}_r(t) &= A_r x_r(t) + B_r y_h(t - \tau), \\
x_r(t_0) &= 0.
\end{align}$$

For a constant $y_h$, at steady state, it is obtained using (4) that

$$\dot{x}_r(\infty) = 0 = A_r x_r(\infty) + B_r y_h,$$

and therefore $x_r(\infty) = -A_r^{-1} B_r L_r y_h$. This means that once the reference model state tracking is achieved, i.e., $\lim_{t \to \infty} x_p(t) = x_r(t)$, the plant output $y_1(t)$, given in (1), takes the form

$$y_1(\infty) = -C_1^T A_r^{-1} B_r L_r y_h.$$

To achieve $\lim_{t \to \infty} y_1(t) = y_h$, we select

$$L_r = -(C_1^T A_r^{-1} B_p)^{-1}.$$  

Considering the uncertain plant dynamics (1), we assume that there exist $K^*_x \in \mathbb{R}^{m \times n_p}$ and $K^*_r \in \mathbb{R}^{m \times m}$ such that the matching conditions

$$A_p - B_p AK^*_x = A_r, \\
B_p AK^*_r = B_r \triangleq B_p L_r$$

are satisfied, where the second matching condition implies that $K^*_r = \Lambda^{-1} L_r$. We define the plant control law as

$$u_p(t) = -\hat{K}_x(t)x_p(t) + \text{diag}(\hat{\lambda}(t)) L_r y_h(t - \tau),$$

where $\hat{K}_x(t) \in \mathbb{R}^{m \times n_p}$ and $\hat{\lambda}(t) \in \mathbb{R}^m$ are adjustable adaptive parameters serving as estimates for the ideal values $K^*_x$ and $\lambda^*$, respectively. It is noted that $\text{diag}(\lambda^*) = \Lambda^{-1}$ exists since $\Lambda$ is diagonal positive definite.

Substituting (9) into (1), one can rewrite (1) as

$$\begin{align}
\dot{x}_p(t) &= A_r x_p(t) + B_r y_h(t - \tau) \\
&\quad + B_p \text{diag}(\hat{\lambda}(t)) L_r y_h(t - \tau) - B_p \hat{K}_x(t)x_p(t),
\end{align}$$

where $\hat{K}_x(t) \triangleq \hat{K}_x(t) - K^*_x$ and $\hat{\lambda}(t) \triangleq \hat{\lambda}(t) - \lambda^*$ are the inner-loop adaptive parameters errors.

By subtracting (4) from (10), and using

$$\text{diag}(\hat{\lambda}(t)) L_r y_h(t - \tau) = \text{diag}(L_r y_h(t - \tau)) \Lambda \hat{\lambda}(t),$$

Fig. 1: Block Diagram
we obtain that
\[
\dot{e}_1(t) = A_r e_1(t) + B_p \text{diag}(L_r y_h(t - \tau)) \hat{\lambda}(t) - B_p \Lambda \hat{K}_x(t) x_p(t),
\]
where \(e_1(t) \equiv x_p(t) - x_r(t)\) is the inner-loop tracking error.

We define the inner-loop adaptive laws as
\[
\dot{\hat{K}}_x^T(t) = \dot{K}_x^T(t) = \gamma_x x_p(t) e_1(t) e_1^T(t) P_1 B_p,
\]
\[
\dot{\hat{\lambda}}(t) = \dot{\lambda}(t) = \gamma_{\lambda} \text{Proj} \left( \lambda(t), -\text{diag}(L_r y_h(t - \tau)) B_p^T P_1 e_1(t) \right),
\]
for some \(Q_1 \in \mathbb{R}^{n_p \times n_p} \cap \mathbb{S}^{n_p 	imes n_p}_{+}\). In this paper, without loss of generality, all learning rates are taken as scalars, instead of diagonal positive definite matrices, for simplicity of notation.

**Lemma 1:** Consider the uncertain dynamical system (1), the reference model (4), and the feedback control law given by (9) and (13). The solution \((e_1(t), K_x(t), \lambda(t))\) is Lyapunov stable in the large. Furthermore, since the human command \(y_h(t)\) is bounded, due to imposed saturation limits by the physical manipulator, \(\lim_{\tau \to \infty} e_1(t) = 0\) and \(\dot{K}_x(t)\) and \(\dot{\lambda}(t)\) remain bounded along with all the signals in the inner-loop.

The proof of Lemma 1 can be found in [12].

### IV. OUTER LOOP

Using the matching conditions given in (8), and the fact that \(\text{diag}(\Lambda^*) = \Lambda^{-1}\), the inner-loop dynamics given in (10) can be rewritten as
\[
\dot{\hat{x}}_p(t) = A_r x_p(t) + B_p \text{Adiag}(\hat{\lambda}(t)) L_r y_h(t - \tau) - B_p \Lambda \hat{K}_x(t) x_p(t).
\]  

Since we assume that the human operator is familiar with the nominal dynamics (2) and (3), the only unknowns in (15) are \(\Lambda, \hat{\lambda}(t)\) and \(\hat{K}_x(t)\). Furthermore, it is assumed that the internal time delay \(\tau\) is known by the human pilot.

Defining the unknown time-varying parameters as
\[
H^T(t) \equiv -\Lambda \hat{K}_x(t),
\]
\[
\Lambda_2(t) \equiv \text{Adiag}(\hat{\lambda}(t)),
\]
equation (15) can be rewritten as
\[
\dot{\hat{x}}_p(t) = (A_r + B_p H^T(t)) x_p(t) + B_p \Lambda_2(t) L_r y_h(t - \tau).
\]  

It is noted that although (17) is a non-linear control system, it is viewed by the pilot as a linear-time-varying system whose state matrix is represented by \(A(t) = A_r + B_p H^T(t)\).

The goal of the human is to control the system such that the plant states follow that of a unity feedback reference model with an open loop crossover model transfer function. We refer to the latter as the **crossover-reference model** (Fig. 1). Let the crossover-reference model be given as
\[
\dot{x}_m(t) = A_m x_m(t) + B_m r(t - \tau),
\]
where \(x_m(t) \in \mathbb{R}^{n_p}\) is the crossover-reference model state vector, \(r(t) \in \mathbb{R}^n\) is a bounded reference input, \(A_m \in \mathbb{R}^{n_p \times n_p}\) is Hurwitz and \(B_m \equiv B_r \in \mathbb{R}^{n_p \times m}\). Similar to the inner-loop, and for a constant reference input \(r\), the nominal feed-forward gain \(\theta_r \in \mathbb{R}^n\) is selected as
\[
\theta_r = -(C^T A_m^{-1} B_r)^{-1},
\]  

to achieve \(\lim_{t \to \infty} y_2(t) = r\), whenever \(\lim_{t \to \infty} x_p(t) = x_m(t)\).

In an ideal case where the human input is not saturated, and both \(H(t)\) and \(\Lambda_2(t)\) are known, the following non-causal control law achieves the crossover-reference model dynamics
\[
G^*(t) = -\theta_r x_p(t + \tau) + \theta_r r(t)
\]
\[
- L_r^{-1} H(t + \tau) x_p(t + \tau),
\]
\[
y_h(t) = L_r^{-1} \Lambda_2^{-1}(t + \tau) L_r G^*(t),
\]
where we assume that there exists \(\theta_x \in \mathbb{R}^{m \times n_p}\) such that
\[
A_m = A_r - B_p L_r \theta_x.
\]

The future state of the plant is predicted by solving the time-varying differential equation (17) as
\[
x_p(t + \tau) = \Phi(t + \tau, t) x_p(t)
\]
\[
+ \int_0^0 \Phi(t + \tau, t + \eta + \tau) B_p \Lambda_2(t + \eta + \tau) L_r y_h(t + \eta) d\eta,
\]
where \(\Phi(t_2, t_1) \in \mathbb{R}^{n_p \times n_p}\) is the state transition matrix of (17). Motivated by (20) and (22), we define the human control input as
\[
G(t) = \Phi_1(t) x_p(t) + \theta_r r(t)
\]
\[
+ \int_0^0 \Phi_2(t, \eta) L_r y_h(t + \eta) d\eta,
\]
\[
v(t) = L_r^{-1} \text{diag}(\hat{\lambda}_2(t)) L_r G(t),
\]
\[
y_h(t) = \left\{ \begin{array}{ll}
  v_1(t), & \text{if } |v_1(t)| \leq y_{o_1}, \\
  y_{o_1} \text{sgn}(v_1(t)), & \text{if } |v_1(t)| > y_{o_1},
\end{array} \right.
\]
where \(\Phi_1(t) \in \mathbb{R}^{m \times n_p}\), \(\Phi_2(t, \eta) \in \mathbb{R}^{m \times m}\) and \(\hat{\lambda}_2(t) \in \mathbb{R}^m\)

are adaptive parameters serving as estimates for the ideal values
\[
\Phi_1(t) = \tilde{H}(t) \Phi(t + \tau, t),
\]
\[
\Phi_2(t, \eta) = \tilde{H}(t) \Phi(t + \tau, t + \eta + \tau) B_p \Lambda_2(t + \eta + \tau),
\]
and \(\lambda_2(t)\), respectively, where
\[
\tilde{H}(t) \equiv -(\theta_x + L_r^{-1} H(t + \tau))
\]
\[
\text{It is noted that diag } (\lambda_2(t)) = \Lambda_2^{-1}(t + \tau) \text{ exists for all } t \geq 0.
\]

It is guaranteed due to the positive
lower bounds imposed by the projection operator in (13b) on
the inner-loop adaptive parameter $\hat{\lambda}(t)$. Furthermore, (23c)
is an element-wise saturation function where $y_{a_{i}} \in \mathbb{R}^+$
is the saturation limit of $y_{h_{i}}(t)$ (the $i^{th}$ element of $y_{h}(t)$).

Substituting (23) into (17), and with some algebraic manipulations, we obtain that

$$\dot{x}_{p}(t) = A_{m}x_{p}(t) + B_{m}r(t - \tau) + B_{p}\lambda_{2}(t)L_{r}\Delta y(t - \tau) + B_{p}\lambda_{2}(t)\omega_{2}(t)G(y(t - \tau)) + B_{p}L_{r}\Phi_{1}(t - \tau)x_{p}(t - \tau) + B_{p}L_{r}\int_{-\tau}^{0}\Phi_{2}(t - \tau, \eta)L_{r}y_{h}(t + \eta - \tau)d\eta,$$

(26)

where $\dot{\Phi}_{1}(t) \equiv \dot{\Phi}_{1}(t) - \dot{\Phi}_{1}^{*}(t, \eta)$, $\dot{\Phi}_{2}(t, \eta) \equiv \dot{\Phi}_{2}(t, \eta) - \Phi_{2}^{*}(t, \eta)$ and $\dot{\lambda}_{2}(t) \equiv \dot{\lambda}_{2}(t) - \lambda_{2}^{*}(t)$ are outer-loop adaptive parameters, and $\Delta y(t) \equiv y_{h}(t) - y(t)$ is the control deficiency due to human input saturation.

Subtracting (18) from (26), and using

$$\Lambda_{2}\text{diag}(\dot{\lambda}_{2})L_{r}G = \text{diag}(L_{r}G)\Lambda_{2}\dot{\lambda}_{2},$$

(27)

results in the outer-loop error dynamics

$$\dot{e}_{2}(t) = A_{m}e_{2}(t) + B_{p}\text{diag}(\dot{\lambda}_{2}(t))L_{r}\Delta y(t - \tau) + B_{p}L_{r}\Phi_{1}(t - \tau)x_{p}(t - \tau) + B_{p}L_{r}\int_{-\tau}^{0}\Phi_{2}(t - \tau, \eta)L_{r}y_{h}(t + \eta - \tau)d\eta,$$

(28)

where $e_{2}(t) \equiv x_{p}(t) - x(t)$ is the outer-loop tracking error.

To eliminate the effect of the control deficiency $\Delta y(t)$ from the outer-loop error dynamics (28), we generate an auxiliary signal $e_{\Delta}(t)$ as in [14], [15]

$$\dot{e}_{\Delta}(t) = A_{m}e_{\Delta}(t) + B_{p}\text{diag}(\dot{\lambda}_{3}(t))L_{r}\Delta y(t - \tau) + B_{p}L_{r}\Phi_{1}(t - \tau)x_{p}(t - \tau) + B_{p}L_{r}\int_{-\tau}^{0}\Phi_{2}(t - \tau, \eta)L_{r}y_{h}(t + \eta - \tau)d\eta,$$

(29)

where $\dot{\lambda}_{3}(t) \in \mathbb{R}^{m}$ is an adjustable adaptive parameter serving as an estimate for the ideal value $\lambda_{3}(t)$, and $\text{diag}(\dot{\lambda}_{3}(t)) = \Lambda_{3}(t)$. Defining an augmented error signal as $e_{y}(t) \equiv e_{2}(t) - e_{\Delta}(t)$, and exploiting the fact that

$$\text{diag}(\dot{\lambda}_{3}(t))L_{r}\Delta y(t - \tau) = \text{diag}(L_{r}\Delta y(t - \tau))\dot{\lambda}_{3}(t),$$

(30)

yields

$$\dot{e}_{y}(t) = A_{m}e_{y}(t) + B_{p}\text{diag}(L_{r}\Delta y(t - \tau))\dot{\lambda}_{3}(t) + B_{p}\text{diag}(L_{r}\Delta y(t - \tau))\lambda_{2}(t) + B_{p}L_{r}\Phi_{1}(t - \tau)x_{p}(t - \tau) + B_{p}L_{r}\int_{-\tau}^{0}\Phi_{2}(t - \tau, \eta)L_{r}y_{h}(t + \eta - \tau)d\eta,$$

(31)

where $\dot{\lambda}_{3}(t) \equiv \dot{\lambda}_{3}(t) - \lambda_{3}^{*}(t)$. Equation (31) is in a standard error model form [16], [7]. We propose the adaptive laws

$$\dot{\lambda}_{2}(t) = \gamma_{2}\text{Proj}\left(\dot{\lambda}_{2}(t), -\text{diag}(L_{r}\Delta y(t - \tau))B_{p}^{T}P_{2}e_{y}(t)\right),$$

(32a)

$$\dot{\lambda}_{3}(t) = \gamma_{3}\text{Proj}\left(\dot{\lambda}_{3}(t), \text{diag}(L_{r}\Delta y(t - \tau))B_{p}^{T}P_{2}e_{y}(t)\right),$$

(32b)

$$\dot{\Phi}_{1}^{*}(t) = \gamma_{\phi_{1}}\text{Proj}\left(\Phi_{1}^{*}(t), -x_{p}(t - \tau)c^{T}_{y}(t)P_{2}B_{p}L_{r}\right),$$

(32c)

$$\dot{\Phi}_{2}^{*}(t, \eta) = \gamma_{\phi_{2}}\text{Proj}\left(\Phi_{2}^{*}(t, \eta), -L_{r}y_{h}(t + \eta - \tau)c^{T}_{y}(t)P_{2}B_{p}L_{r}\right),$$

(32d)

where $\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{\phi_{1}}, \gamma_{\phi_{2}} \in \mathbb{R}^{+}$ are learning rates, and $P_{2} \in \mathbb{R}^{n_{p} \times n_{p}} \cap \mathbb{S}^{n_{p} \times n_{p}}$ is the solution of the Lyapunov equation

$$A_{m}^{T}P_{2} + P_{2}A_{m} = -Q_{2},$$

(33)

for some $Q_{2} \in \mathbb{R}^{n_{p} \times n_{p}} \cap \mathbb{S}^{n_{p} \times n_{p}}$.

The following Lemma establishes key bounds on the state transition matrix of (17) and its time derivative, which is then utilized in the remarks that follow to show that all ideal values, of the outer-loop adaptive parameters, and their time derivatives are bounded.

**Lemma 2:** There exist $\phi \in \mathbb{R}^{+}$ and $\Phi \in \mathbb{R}^{+}$ such that

$$\|\Phi(t + \tau, t + \eta + \tau)\|_{F} \leq \phi, \text{ for all } t \geq t_{0}, \quad \|\Phi(t + \tau, t + \eta + \tau)\|_{F} \geq \phi, \text{ for all } t \geq t_{0}, -\tau \leq \eta \leq 0,$$

(34a)

and

$$\|\Phi(t + \tau, t + \eta + \tau)\|_{F} \leq \phi, \text{ for all } t \geq t_{0}, \quad \|\Phi(t + \tau, t + \eta + \tau)\|_{F} \geq \phi, \text{ for all } t \geq t_{0}, -\tau \leq \eta \leq 0,$$

(34b)

We provide the proof of this lemma in the extended online version of this paper in [17].

**Remark 1:** It follows from Lemma 1 that $\tilde{K}(t)$, $\lambda(t)$, $\tilde{K}(t)$ and $\lambda(t)$ are bounded, which implies the boundedness of $H(t)$, $\Lambda_{2}(t)$, $\tilde{H}(t)$ and $\tilde{\Lambda}_{2}(t)$. Therefore, there exist $\tilde{h} \in \mathbb{R}^{+}$, $\tilde{\lambda}_{2} \in \mathbb{R}^{+}$ and $\beta_{3} \in \mathbb{R}^{+}$ such that $\|H(t)\| \leq \tilde{h}$, $\|\Lambda_{2}(t)\|_{F} \leq \beta_{3}$ and $\|\tilde{\Lambda}_{2}(t)\|_{F} \leq \beta_{3}$ for all $t \geq t_{0}$. The latter implies that $\|\lambda_{2}^{*}(t)\| \leq \beta_{3}$ and $\|\lambda_{3}^{*}(t)\| \leq \beta_{3}$. Moreover, as $\lambda_{\min_{i}} > 0$ for $i = 1, \ldots, m$, there exists $\beta_{2} \in \mathbb{R}^{+}$ such that $\|\lambda_{2}^{*}(t)\|_{F} \leq \beta_{2}$. And since

$$\frac{d\lambda_{3}^{*}}{dt} = -\lambda_{3}^{*}L_{r}\lambda_{2}^{*},$$

(35)

then there exists $\beta_{2} \in \mathbb{R}^{+}$ such that $\|\lambda_{3}^{*}(t)\|_{F} \leq \beta_{2}$. This implies that $\|\lambda_{2}^{*}(t)\| \leq \beta_{2}$ and $\|\lambda_{3}^{*}(t)\| \leq \beta_{2}$, for all $t \geq t_{0}$.

**Remark 2:** Together with Remark 1, the bounds (34) and (35), established in Lemma 2, show that all the terms of the ideal values (24) and their time derivatives are bounded. Hence, there exist $\phi_{1}, \phi_{1}, \phi_{2}, \phi_{2} \in \mathbb{R}^{+}$ such that $\|\Phi_{1}^{*}(t)\|_{F} \leq \phi_{1}$, $\|\Phi_{1}^{*}(t)\|_{F} \leq \phi_{2}$ for all $t \geq t_{0},$ and $\|\Phi_{2}^{*}(t, \eta)\|_{F} \leq \phi_{2}$, $\|\Phi_{2}^{*}(t, \eta)\|_{F} \leq \phi_{2}$, for all $t \geq t_{0}$.
Theorem 1: Consider the uncertain dynamical system given by (1), the adaptive controller given by (4), (9) and (13), and the adaptive human pilot model given by (18), (23) and (32). Then, there exists $\tau^* \in \mathbb{R}_+$ satisfying
\[
\tau^* \left\{ (\mu + 2\gamma^2_1)\alpha_3 + (1 + 2\gamma^2_1)\alpha_1 + (1 + 2\gamma^2_1)\tau^*\alpha_2 \right\} < q,
\]
such that for all $\tau \in [0, \tau^*]$, the solution $(e_p(t), \lambda_2(t), \lambda_3(t), \hat{\Phi}_1(t), \hat{\Phi}_2(t, \eta))$ remains bounded for all $t \geq t_0$ and converges to the compact set
\[
E \triangleq \left\{ (e_p(t), \lambda_2(t), \lambda_3(t), \hat{\Phi}_1(t), \hat{\Phi}_2(t, \eta)) : \right. \\
\|e_p(t)\|^2 \leq \frac{z_1}{2}, \left\|\lambda_2(t)\right\| \leq \beta_2, \left\|\lambda_3(t)\right\| \leq \beta_3, \\
\left\|\hat{\Phi}_1(t)\right\| \leq \phi_1, \left\|\hat{\Phi}_2(t, \eta)\right\| \leq \phi_2 \},
\]
where
\[
z_1 \triangleq 2\tau\beta_2^2 + \phi_2^2 + \tau\phi_2^2 + 2\gamma_1 - 1\beta_2\beta_3 + 2\gamma_2 - 1\beta_2\beta_3 + 2\gamma_3 - 1\beta_2\beta_3 + 2\gamma_1\phi_1 \phi_2 + 2\gamma_2\phi_1 \phi_2 - \tau\phi_2\phi_2, \\
z_2 \triangleq p - \tau \left\{ (\mu + 2\gamma^2_1)\alpha_3 + (1 + 2\gamma^2_1)\alpha_1 \\
+ (1 + 2\gamma^2_1)\tau\alpha_2 \right\}
\]
and
\[
\mu \triangleq \max_i \left( \alpha_i, \lambda_{\text{max}} \right)^2, \\
p \triangleq \max \{ \| P_2B_p \|^2, \| P_2B_pL_r \|^2 \}, \\
\beta_2 \triangleq \left\| \lambda_{\text{max}} \right\| + \beta_2, \\
\beta_3 \triangleq \left\| \lambda_{\text{max}} \right\| + \beta_3, \\
\phi_1 \triangleq \left\| \hat{\Phi}_{1\text{max}} \right\| + \phi_1, \\
\phi_2 \triangleq \left\| \hat{\Phi}_{2\text{max}} \right\| + \phi_2
\]
and $q \triangleq \lambda_{\text{min}}(Q_2)/p$. It is noted that the variables $\beta_2, \beta_3, \beta_3$ and $\beta_3$ are defined in Remark 1, while the variables $\phi_1, \phi_1, \phi_2$ and $\phi_2$ are defined in Remark 2. The scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+$ are such that $\|x_p(t)\|^2 \leq \alpha_1$, $\|L_y\eta(t)\|^2 \leq \alpha_2$, and $\|\text{diag} (L_r \tilde{\mathbf{G}}(t))\|^2 \leq \alpha_3$ for all $t \geq t_0$. Furthermore, the closed-loop system is stable in the large, and all signals are bounded.

We provide the proof of this theorem in the extended online version of this paper in [17].

Remark 3: While the existence of the upper bound on the time delay $\tau^*$ is guaranteed, its value depends on the selection of the outer-loop learning rates $\gamma_{out}$ and $\gamma_{opt}$. Note from (37) that as larger values of $\gamma_{out}$ are used, the allowable maximum time delay $\tau^*$ becomes smaller. On the other hand, in the limit where $\gamma_{out} \to 0$, which corresponds to no adaptation, $\tau^*$ approaches its ultimate value $\tau_{\text{max}}$ satisfying
\[
\tau_{\text{max}} \left( \mu\alpha_3 + \alpha_1 + \tau_{\text{max}}\alpha_2 \right) < q.
\]
On the contrary, the ultimate bound $z \triangleq z_1/z_2$ on the error $e_p(t)$, which is defined by the set (38), (39) and (40), is inversely proportional to the values of $\gamma_{out}$. That is, to achieve a better tracking performance, which corresponds to smaller values of $z$, the outer-loop learning rates $\gamma_{out}$ should be selected as large as possible. And in the limit where $\gamma_{out} \to \infty$, the upper bound $z \to 0$. Therefore, given any delay value $\tau < \tau_{\text{max}}$, the optimal outer-loop learning rates $\gamma_{out, opt}$ are the ones that satisfy $\tau^* = \tau$. A further increase in $\gamma_{out} > \gamma_{out, opt}$ renders our stability analysis inapplicable due to $\tau > \tau^*$, while a decrease in $\gamma_{out} < \gamma_{out, opt}$ allows for higher delay values to be tolerated at the expense of a deteriorated tracking performance.

V. SIMULATIONS

Consider the perturbation equations of the longitudinal motion for the 747 airplane [18] cruising in level flight at an altitude of 40 kft and a velocity of 774 ft/sec with the dynamics given in the form of (1). The state vector is
\[
x_p(t) = [x_{p1}(t) \ x_{p2}(t) \ x_{p3}(t) \ x_{p4}(t)]^T,
\]
where $x_{p1}(t)$ and $x_{p2}(t)$ are the components of the aircraft’s velocity along the $x$ and $z$-axes, respectively, with respect to the reference axis (in ft/sec), $x_{p3}(t)$ is the aircraft’s pitch rate (in rad/sec), and $x_{p4}(t)$ is the pitch angle of the aircraft (in rad). The input $u_p(t)$ represents the elevator deflection (in rad), and the nominal system and control input matrices are given by
\[
A_n = \begin{bmatrix}
-0.0030 & 0.0390 & 0 & -0.3220 \\
-0.0650 & -0.3190 & 7.7400 & 0 \\
0.0200 & -0.1010 & -0.4290 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]
\[
B_p = \begin{bmatrix}
0.0100 & -0.1800 & -1.1600 & 0
\end{bmatrix}^T,
\]
with the eigenvalues at $-0.3750 \pm 0.8818i$ and $-0.0005 \pm 0.0674i$. We consider an uncertainty in the system matrix $A_p$ constructed as
\[
A_p = \begin{bmatrix}
-0.0029 & 0.0389 & -0.0047 & -0.3220 \\
-0.0661 & -0.3171 & 7.8254 & 0.0008 \\
0.0129 & -0.0888 & 0.1210 & 0.0051 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]
such that the eigenvalues are placed at 0.1, 0.2, 0.3 and 0.4 in the right-half complex plane. A pilot is controlling the aircraft to achieve a desired pitch angle by feeding pitch rate commands to the inner-loop controller, i.e., $y_1(t) = x_{p3}$, and $y_2(t) = x_{p4}(t)$, where the pilot command is saturated as in (23c), with $y_0 = 10\text{ deg/s}$. The reference model dynamics are assigned in the form of (4), where $A_r = A_n$ and $B_r = B_pL_r$. Achieving pilot command following by assigning the feed-forward gain $L_r$ as in (7) is not possible since the transfer function $x_{p3}(s)/u_p(s)$ has a zero at the origin. Instead, we design the inner-loop feed-forward controller by assuming short-period dynamics approximation of the nominal dynamics given by
\[
A_{sp} = \begin{bmatrix}
-0.3190 & 7.7400 \\
-0.1010 & -0.4290 \\
-0.3190 & 7.7400 \\
-0.1010 & -0.4290
\end{bmatrix},
B_{sp} = \begin{bmatrix}
-0.1800 & -1.1600
\end{bmatrix},
\]
which is a common approach in the design of control augmentation systems (CAS) for flight control [19].

The eigenvalues are at $-0.3740 \pm 0.8824i$ which makes a good approximation of the fast dynamics (eigenvalues) of (43). Then, the feed-forward gain is selected as $L_r =$
Fig. 2: Pitch angle, pilot commands and controller input for \( \gamma_x = 1 \).

Fig. 3: Pitch angle, pilot commands and controller input for \( \gamma_x = 0.01 \).

\[-(C_{sp}^T A_{sp}^{-1} B_{sp})^{-1}, \text{ where } C_{sp} = [0, 1]^{T}.\]

For the outer-loop human pilot model, the LQR method is used to design the crossover-reference model (18) by calculating \( \theta_x \) using \( Q_{LQR} = \text{diag}([0, 0, 0, 3]) \) and \( R_{LQR} = 3 \), with \( \theta_x \) assigned as in (19). We use \( \tau = 0.3 \) s for the human internal time delay, which is determined by averaging the operators’ delay in an adaptive pilot experiment [7]. The Lyapunov matrices are taken as \( Q_1 = Q_2 = 0.001 I_{4 \times 4} \). The finite integral term in (23a) and the adaptive law (32d) are implemented by discretizing the integral into 5 intervals as illustrated in [20].

The adaptive parameters \( \lambda(t), \lambda_2(t) \) and \( \lambda_3(t) \) are initialized at 1, \( \Phi_1(t) \) is initialized at \( -\theta_x e^{A_{sp} \tau} \), and the rest are initialized at zero. Finally, the outer-loop learning rates are taken as \( \gamma_2 = 1, \gamma_3 = 5, \gamma_{\phi_1} = \text{diag}([0.01, 0.001, 0.01, 0.01]) \) and \( \gamma_{\phi_2} = 0.1 \), and the inner loop learning rate \( \gamma_{\lambda} \) is fixed at \( \gamma_{\lambda} = 1 \).

Figs. 2 and 3 show the aircraft pitch angle, the evolution of the pilot commands and the plant control input for different inner-loop learning rates. We start with \( \Lambda = 1 \), and we introduce a failure into the system by making \( \Lambda = 0.6 \) for \( t \geq 35 \) s. Good tracking performance is achieved with a reasonable control effort of both the pilot and the controller in Fig. 2, where \( \gamma_x = 1 \). As the inner-loop learning rate is decreased to \( \gamma_x = 0.01 \) in Fig. 3, a significant deterioration in the tracking performance is observed, accompanied by saturating high frequency oscillations in the pilot commands and the controller input. This showcases an example of a poor adaptive controller design, where the pilot spends a large control effort to maintain a satisfactory performance.

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