An extended formulation for the 1-wheel inequalities of the stable set polytope

Sven de Vries1 | Ulf Friedrich2 | Bernd Perscheid1

1 Operations Research, FB IV – Mathematics, Trier University, Trier, Germany
2 Operations Research, Department of Mathematics, Technical University of Munich, Munich, Germany

Correspondence
Bernd Perscheid, Operations Research, FB IV – Mathematics, Trier University, 54286 Trier, Germany.
Email: perscheid@uni-trier.de

Funding information
This research was supported by the German Research Foundation, RTG-2126. Federal Ministry of Education and Research. Alexander von Humboldt Foundation. Open Access funding provided by Projekt DEAL.

Abstract
The 1-wheel inequalities for the stable set polytope were introduced by Cheng and Cunningham. In general, there is an exponential number of these inequalities. We present a new polynomial size extended formulation of the stable set relaxation that includes the odd cycle and 1-wheel inequalities. This compact formulation allows one to polynomially optimize over a polyhedron instead of handling the separation problem for 1-wheel inequalities by solving many shortest walk problems and relying on the ellipsoid method.

KEYWORDS
extended formulation, graph product, polyhedral combinatorics, separation problem, stable set problem, wheel inequalities

1 INTRODUCTION
Let $G = (V, E)$ be a simple, connected, and undirected graph. The incidence vector of $N \subseteq V$ is denoted by $x^N \in \{0, 1\}^V$ where $x^N_i = 1$ if and only if $i \in N$. A stable set is a subset $N \subseteq V$ which does not contain adjacent vertices of $G$. The stable set polytope $\text{STAB}(G)$ is the convex hull of incidence vectors of stable sets in $G$. Since the task of finding a maximum stable set is $\mathcal{NP}$-hard, it is unlikely that there exists a formulation of polynomial size for $\text{STAB}(G)$.

Stable sets occur in many optimization problems of practical relevance. For example, Hoffman and Padberg [16] study the airline crew scheduling problem with base constraints (SPB).

$$\begin{align*}
& \min \sum_{j=1}^n c_j x_j \\
& \text{s.t.} \quad Ax = 1 \\
& \quad d_1 \leq Dx \leq d_2 \\
& \quad x_j \in \{0, 1\} \quad \forall j = 1, \ldots, n,
\end{align*}$$

(SPB)

where $A \in \{0, 1\}^{m \times n}$, $d_1, d_2 \in \mathbb{Q}^m$, and $D \in \mathbb{Q}^{d \times n}$.

The set packing polytope $\text{conv}\{x \in \mathbb{R}^n : Ax \leq 1, x \in \{0, 1\}^n\}$ constitutes a major part of the constraint set of (SPB). Hoffman and Padberg [16] analyze (SPB) by considering the intersection graph $G_A = (V_A, E_A)$ of the matrix $A$, see Padberg [24], where $V_A = \{1, \ldots, n\}$ for the column set of $A$ and the set $E_A$ contains an edge $uv$ if and only if $a_{*,u} + a_{*,v} \neq 1$. Every pair of adjacent vertices in $G_A$ represents two columns that cannot be concurrently present in the set packing polytope and vice versa. It follows directly that every valid binary packing must be a stable set.

Some well-known valid inequalities for $\text{STAB}(G)$ are the following, see Gröschel et al. ([15], Chapter 9):

$$\begin{align*}
0 \leq x_i & \leq 1 \quad \forall i \in V \quad \text{(trivial inequalities)}, \\
0 \leq x_i + x_j & \leq 1 \quad \forall ij \in E \quad \text{(edge inequalities)},
\end{align*}$$

(1) (2)
\[\sum_{i \in C} x_i \leq \frac{|C| - 1}{2} \quad \forall \text{ odd cycles } C \quad \text{(odd cycle inequalities)}, \tag{3}\]
\[\sum_{i \in K} x_i \leq 1 \quad \forall \text{ cliques } K \quad \text{(clique inequalities)}. \tag{4}\]

The clique inequalities (4) are defined by Padberg [24], who as well introduces the odd hole and odd antihole inequalities. Notice that the odd hole inequalities are a subset of the odd cycle inequalities (3), as odd holes are chordless odd cycles.

The edge-constrained stable set polytope \(P^E(G)\) is given by
\[P^E(G) = \{x \in \mathbb{R}^n : x \text{ satisfies (1) and (2)}\}\]
(also known as the fractional stable set polytope and denoted by \(\text{FRAC}(G)\)) and will be our starting point for the extended formulations in the following sections. It is of polynomial size since it has \(n\) variables and \(m + 2n\) inequalities. For polytopes depending on a graph \(G\) and some specific class of inequalities \(I\) we will use \(P^I(G)\) for simplicity of notation.

A generally exponential class of valid stable set inequalities is constituted by the odd cycle inequalities (3). By intersecting these inequalities with \(P^E(G)\), we obtain the odd cycle polytope
\[P^\text{OC}(G) = \{x \in \mathbb{R}^n : x \text{ satisfies (1), (2) and (3)}\}.

Yannakakis [29] constructs an extended formulation \(Q^\text{OC}(G)\) for \(P^\text{OC}(G)\), given in Section 4, by adding only polynomial many variables and inequalities to \(P^E(G)\). Although there could be an exponential number of odd cycles in \(G\), one finds an optimal solution for \(P^\text{OC}(G)\) in polynomial time by directly optimizing over \(Q^\text{OC}(G)\) instead. Optimal values of the original variables in \(Q^\text{OC}(G)\) are as well optimal for \(P^\text{OC}(G)\).

Cheng and Cunningham [5] introduce a new rich class of inequalities called 1-wheel inequalities. Starting with simple 1-wheels, they extend their results to the class of nonsimple 1-wheels. They prove necessary and sufficient conditions for 1-wheel inequalities to be facet-inducing. The separation problem for 1-wheels can be solved polynomially in \(O(n^4)\) with their algorithm, based on the application of shortest path algorithms in a complete auxiliary graph. We present a compact formulation with linear constraints for optimizing over the 1-wheel polytope \(P^W(G)\) that is based on an improved separation algorithm by de Vries [8]. The auxiliary digraphs required for this algorithm are categorical products (also known as tensor products) of the original graph and a special digraph and preserve the density of the original graph.

**Definition 1.1.** The categorical product \(G \cdot D\) of a graph \(G = (V_G, E_G)\) and a digraph \(D = (V_D, A_D)\) is given by the vertex set \(V_{G \cdot D} = V_G \times V_D\) and the arc set \(A_{G \cdot D} = \{(u, i), (v, j) : uv \in E_G \text{ and } (i, j) \in A_D\}\). An important application of categorical products is the shortest odd cycle problem for graphs without negative edges (cf. Grötschel et al. [15, 8.3.6]: let \(G = (V_G, E_G)\) be a graph and \(D = (V_D, A_D)\) be the digraph with vertex set \(V_D = \{1, 2\}\) and arc set \(A_D = \{(1, 2), (2, 1)\}\). The edge weight of \(uv \in E_G\) is inherited by every arc \(((u, i), (v, j)) \in A_{G \cdot D}\), that is, by \(((u, 1), (v, 2)), ((u, 2), (v, 1)), ((v, 1), (u, 2)), \text{ and } ((v, 2), (u, 1))\). Then a shortest path from \((u, 1)\) to \((u, 2)\) yields a shortest odd cycle in \(G\) which contains vertex \(u \in V_G\). Computing such a shortest path for all vertices in \(G\) and selecting a shortest of these paths solves the problem.

In Section 3, we define several types of 1-wheels and present some useful results on valid inequalities for 1-wheels. We review some basics on extended formulations and recap an extended formulation for the odd cycle polytope \(P^\text{OC}(G)\) in Section 4. In Section 5, we continue with some results from de Vries [8], such as an improvement of the separation algorithm of Cheng and Cunningham [5]. We construct and prove the new compact extended formulation for 1-wheels in Section 6.

### 2 RELATED WORK

In order to classify the results presented in this paper, we provide a literature overview of important work related to our topic.

We first turn to general techniques that allow polynomial time optimization. If a given LP has polynomial many variables and inequalities, it can be solved in polynomial time via the ellipsoid method, see Grötschel et al. [14]. Although many classes of valid inequalities of the stable set polytope are known that have exponential size, there exist polynomial time separation algorithms for some of them. Carr and Lancia [3] prove that for an exponential size LP, compact optimization is possible if and only if compact separation is possible.

A method for generating linear extended formulations by using cutting plane methods is developed by Martin [22], who beyond that gives the first polynomial size formulation of the spanning tree problem. For an extensive insight into the theoretical background of problems it can be applied to, we refer the reader to Lancia and Serafini [20]. Moreover, Kaibel [17] presents results on extension techniques and lower bounds on the extension size.
The method of Martin is used for various combinatorial optimization problems. Goemans and Myung [13], for example, apply it for Steiner tree formulations. Lancia and Serafini [19] use this approach to develop a compact extended formulation for the max cut problem. Quadratic size extended formulations for independence polytopes of graphic and cographic matroids by application of this method are presented by Kaibel et al. [18]. Afshari Rad and Kakhki [1] construct an extended formulation for the cardinality form of the maximum flow interdiction problem by using this technique.

Since efficient separation plays an important role in this framework, we survey literature from the last decades containing polynomial time separation algorithms for exponential classes of inequalities of the stable set polytope. Gerards and Schrijver [9] show how odd cycle inequalities of the stable set polytope can be separated in polynomial time. A polynomial time separation algorithm for a small subclass of the 1-wheel inequalities, which are defined in Section 3, is presented by Grötschel et al. [15]. This subclass includes all simple 1-wheels where \( S = R = \emptyset \), cf. Definition 3.1. Cheng and Cunningham [5] extend this result to separate all 1-wheel inequalities. Moreover, de Vries [8] shows how their result can be improved by constructing a faster separation algorithm. Cheng [4] shows how to separate the so-called generalized bicycle wheel inequalities, which are closely related to the 1-wheel inequalities, of the cut polytope. He points out the strength of the generalized bicycle wheel inequalities beyond the odd cycle inequalities with numerical experiments. Although he applies separation algorithms to the cut polytope, these results are relevant for the stable set polytope, since there exists a transformation between valid inequalities for the cut polytope and the stable set polytope. A polynomial time separation algorithm for a relaxation of the stable set polytope including all clique, odd cycle, odd antihole, and 1-wheel inequalities for simple 1-wheels where \( S = R = \emptyset \) is given by Lovász and Schrijver [21]. Giandomenico and Letchford [10] present a separation algorithm for all web inequalities of Trotter [28]. The antiweb inequalities of Trotter [28] are extended to the so-called antiweb-wheel inequalities by Cheng and de Vries [6]. Additionally, they give a separation algorithm for them.

Yannakakis [29] presents an extended formulation based on the separation algorithm for odd cycle inequalities of Gerards and Schrijver [9]. We recapitulate the construction of his formulation in Section 5. Our extended formulation for 1-wheels in Section 6 is based on the separation algorithm of de Vries [8].

Although we do not consider separation heuristics here, we want to mention some of them that can be very efficient. Nemhauser and Sigismondi [23] present a strong algorithm using cutting planes from clique and lifted odd hole inequalities. A separation heuristic for the so-called rank inequalities is presented by Rossi and Smriglio [26]. Furthermore, Giandomenico et al. [11, 12] and Rebennack et al. [25] propose algorithms for the maximum stable set problem and substantiate their strength with extensive computational results.

3 SIMPLE AND NONSIMPLE 1-WHEELS

In this section, we describe the results of Cheng and Cunningham [5] which are required for the extended formulation presented in Section 6. To keep our notation consistent throughout this paper, we slightly deviate from their notation and use the following equivalent definition.

**Definition 3.1.** A graph \( W \) with vertices \( V_W = C \cup S \cup R \cup \{ h \} \) is called a simple 1-wheel if \( W \) can be constructed in the following way:

1. \( C = \{ v_1, \ldots, v_{2k+1} \} \) is the vertex set of an odd cycle;
2. the edges \( v_i h \) are added for all \( i = 1, \ldots, 2k + 1 \);
3. some of the edges \( v_i h \) are (multiply) subdivided and \( S \) is constituted by the inner vertices of the subdivisions of these edges;
4. some of the edges \( v_i v_{i+1} \) are (multiply) subdivided and \( R \) is constituted by the inner vertices of the subdivisions of these edges;
5. every cycle \( h - \ldots - v_i - \ldots - v_{i+1} - \ldots - h \) of \( W \), which does not include any vertex \( v \in C \setminus \{ v_i, v_{i+1} \} \), is odd.

In the definition, we use the convention \( v_{2k+2} = v_1 \). We call the vertex \( h \) the hub and the elements in \( C \) the spoke ends. In simple 1-wheels, for every \( i \in \{ 1, \ldots, 2k + 1 \} \) the spoke path connecting the hub \( h \) to \( v_i \) in \( S \) is denoted by \( P_{h,v_i} \) and the rim path connecting the spoke end \( v_i \) to \( v_{i+1} \) in \( R \) is denoted by \( P_{v_i,v_{i+1}} \). Thus, all of the “face” cycles of \( W \) are given by the subdivision of \( C \) and the \( 2k + 1 \) cycles of the form \( P_{v_i,v_{i+1}} \cup P_{v_{i+1},h} \cup P_{h,v_i} \).

We partition the vertex set \( C = \mathcal{E} \cup \mathcal{O} \), such that the \( v_i \) for which \( P_{h,v_i} \) is of even length (in the number of edges) constitute \( \mathcal{E} \) and the \( v_i \) for which \( P_{h,v_i} \) is of odd length constitute \( \mathcal{O} \). Therefore, every vertex of a simple 1-wheel belongs to exactly one of the sets \( \mathcal{E}, \mathcal{O}, S, R, \{ h \} \). An example is given in Figure 1A, where \( \mathcal{E} = \{ v_3, v_4, v_5 \} \), \( \mathcal{O} = \{ v_1, v_2 \} \), \( S = \{ s_1, s_2, s_3, s_4, s_5 \} \), and \( R = \{ r_1, r_2, r_3, r_4 \} \).
**Definition 3.2.** A nonsimple 1-wheel is a graph constructed by identifying (zero or more) pairs of nonadjacent vertices of a simple 1-wheel.

Thus, nonsimple 1-wheels are a generalization of simple 1-wheels where \( E, \emptyset, S \), and \( R \) are multisets and one or more vertices can belong to more than one of these multisets. Note that we consider simple 1-wheels as a special case of nonsimple 1-wheels where the set of identified pairs is empty. We use the general term 1-wheel in what follows. Valid inequalities arise for the resulting nonsimple 1-wheel after the identification of vertices and addition of the respective coefficients. Identifying vertices may yield edges parallel to other edges that are irrelevant for stable set problems and are removed.

For example, the nonsimple 1-wheel in Figure 1B is obtained by identifying the pairs \( \{v_1, s_4\}, \{s_1, s_3\}, \) and \( \{r_4, s_5\} \). The odd closed walk \( a, c, v_5, c, h, a \) in Figure 1B originates from the odd face cycle \( v_1, r_4, v_5, s_5, h, v_1 \) in Figure 1A.

Cheng and Cunningham [5] show that the inequalities

\[
\begin{align*}
(k + 1)x_h + \sum_{i=1}^{2k+1} x_{v_i} + \sum_{v \in E} x_v + \sum_{v \in S \cup R} x_v & \leq k + \frac{|S| + |R| + |E|}{2} \quad (I_A) \\
(k + 1)x_h + \sum_{i=1}^{2k+1} x_{v_i} + \sum_{v \in E} x_v + \sum_{v \in S \cup R} x_v & \leq k + \frac{|S| + |R| + 1}{2} \quad (I_B)
\end{align*}
\]

for 1-wheels are valid for the stable set polytope of \( G \). Furthermore, they give sufficient conditions for \( I_A \) and \( I_B \) to be facet-inducing for \( \text{STAB}(G) \). Based on their results, de Vries [8] considers 1-wheels in which \( \emptyset \) or \( \emptyset \) is empty and calls the resulting inequalities \( I'_A \) and \( I'_B \), respectively. He proves the following results.

**Lemma 3.3.** ([8], Lemmas 3, 4). Every \( I_A \)-inequality is representable by an \( I'_A \)-inequality. Every \( I_B \)-inequality on a 1-wheel without odd spokes of length 1 is representable by an \( I'_B \)-inequality.

Having \( E = \emptyset \) in the first case implies that every spoke end belongs to \( \emptyset \). Therefore we call the inequalities \( I'_A \) odd 1-wheel inequalities. For the inequalities \( I'_B \) every spoke end is in \( E \) because \( \emptyset \) is empty. Hence, they are called even 1-wheel inequalities. For satisfying every 1-wheel inequality in \( G \) (except the \( I_B \)-inequalities for 1-wheels where at least one spoke is a single edge), Lemma 3.3 yields equivalent conditions due to the representation property. It is sufficient to ensure that no odd and no even 1-wheel inequality is violated.

**Definition 3.4.** The polytopes that contain the set of all \( x \in P^{OC}(G) \) which additionally satisfy all \( I'_A \)-inequalities and all \( I'_B \)-inequalities are denoted by \( P^{W_A}(G) \) and \( P^{W_B}(G) \), respectively.

## 4 | CYCLE INEQUALITIES: SEPARATION AND EXTENDED FORMULATION

Let \( P^X \subseteq P \). Then the separation problem for \( P^X \) and a given \( x \in P \) is to either show \( x \in P^X \) or to give a valid inequality of \( P^X \) that is violated by \( x \).

In an instance of the linear integer optimization problem, a given LP relaxation may have an exponential number of inequalities. The idea of an extended formulation is to add a polynomial number of variables to obtain a formulation with polynomial many constraints, but with a tighter projection in the original variables. For a detailed introduction to extended formulations we refer the reader to Conforti et al. [7].
Definition 4.1. Let \( P \subseteq \mathbb{R}^n \) be a polyhedron. The polyhedron \( Q \subseteq \mathbb{R}^{n+m} \) is an extended formulation of \( P \) if there exists a projection of \( Q \) into \( \mathbb{R}^n \) yielding \( P \).

An example is shown in Figure 2, where \( P \subseteq \mathbb{R}^2 \) has six facets, but \( Q \subseteq \mathbb{R}^3 \) has only five facets.

We recapitulate the idea of the extended formulation \( Q^{OC}(G) \) for \( P^{OC}(G) \) given by Yannakakis [29]. In our presentation edge weights \( w_{ij} \) are introduced for the separation of some given \( \bar{x} \). In the case of optimizing over \( Q^{OC}(G) \) the notation \( w_{ij} \) for the edge variables will be used, as the \( w_{ij} \) themselves are variables dependent on \( x_i \) and \( x_j \).

First of all, we take a closer look at the edge inequalities. Define variables \( w_{ij} = 1 - x_i - x_j \) for every edge \( ij \) in \( G \). This is the nonnegative slack of the edge inequality for \( ij \). For every odd cycle \( C \) the corresponding odd cycle inequality

\[
\sum_{i\in C} x_i \leq \frac{|C| - 1}{2}
\]

of \( P^{OC}(G) \) is equivalent to

\[
1 \leq |C| - 2 \sum_{i\in C} x_i = \sum_{ij \in E_C} (1 - x_i - x_j) = \sum_{ij \in E_C} w_{ij} =: w(E_C).
\]

In other words, for a given vector \( \bar{x} \in P^{OC}(G) \), every odd cycle \( C \) has weight \( \bar{w}(E_C) \) at least 1. In particular this holds for the shortest odd cycle.

Yannakakis [29] proves that the polytope

\[
Q^{OC}(G) = \{(x,f,g) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n : (x,f,g) \ satisfies \ (1),(5)-(8) \}
\]

with

\[
\begin{align*}
0 \leq f_{ij} & \leq w_{ij} & \forall \ ij \in E, \tag{5} \\
f_{ij} & \leq f_{ik} + g_{kj} & \forall \ ik \in E, \ j \in V, \tag{6} \\
g_{ij} & \leq f_{ik} + f_{kj} & \forall \ ik \in E, \ j \in V, \tag{7} \\
 f_i & \geq 1 & \forall \ i \in V, \tag{8}
\end{align*}
\]

and \( w_{ij} = 1 - x_i - x_j \), as above, is an extended formulation of \( P^{OC}(G) \). Inequalities (5) could be replaced by

\[
0 \leq f_{ij} \leq 1 - x_i - x_j \quad \forall \ ij \in E \quad \tag{5'}
\]

to express that defining variables \( w_{ij} \) is not necessary for \( Q^{OC}(G) \). However, we prefer using the shorter notation from above. Note that the edge inequalities (2) are a subset of inequalities (5), because \( 0 \leq w_{ij} \) is equivalent to \( x_i + x_j \leq 1 \) for all edges \( ij \in E \).

Inequalities (5), (6), and (7) imply that \( f_{ij} \) is bounded from above by the weight of a shortest odd walk between two vertices \( i \) and \( j \), while \( g_{ij} \) is bounded from above by the weight of a shortest even walk using at least two edges between \( i \) and \( j \) if \( i \neq j \). Finally, inequalities (8) ensure that all odd cycle inequalities hold.

We point out one special property of the variables \( g_{ij} \) that we use later in Section 6.

**Lemma 4.2.** Let \( (\bar{x}, \bar{f}, \bar{g}) \in Q^{OC}(G) \) and \( \bar{g}_{ij} := \bar{g}_{ij} \) for all \( i,j \in V \) with \( i \neq j \). Then for all \( \hat{g}_{ij} \) in the interval

\[
[0, 2 \min \{\bar{f}_{ik} : k \in E\}]
\]

with \( l \in V \) we have \((\bar{x}, \bar{f}, \hat{g}) \in Q^{OC}(G)\).

**Proof.** For each \( l \in V \) the variable \( g_{il} \) occurs in inequalities (6), that is, \( f_{il} \leq f_{il} + g_{il} \) when \( j = k = l \). Therefore, it is bounded from below by zero and there is no tighter lower bound given by the set of inequalities. On the other hand,
the variable \( g_{ij} \) is bounded from above by inequalities (7) when \( i = j = l \). In this case we obtain \( g_{ij} \leq f_{lk} + f_{kl} \) for every \( lk \in E \). There is no tighter upper bound for \( g_{ij} \) than \( f_{lk} + f_{kl} \) and we obviously have \( 2 \min \{ f_{lk}, f_{kl} \} \leq f_{lk} + f_{kl} \). Therefore \( (x, f, g) \in \mathcal{O}(G) \).

\[ \blacksquare \]

5 | SEPARATION OF THE 1-WHEEL INEQUALITIES VIA GRAPH PRODUCTS

For a given \( \bar{x} \in P^{OC}(G) \), de Vries [8] describes how to check if \( \bar{x} \in P^{Wx}(G) \) and if \( \bar{x} \in P^{Ww}(G) \) in polynomial time. This procedure involves computing the weights of shortest walks in auxiliary graphs. Using a graph product, he improves the \( O(n^4) \) runtime of the separation algorithm for 1-wheel inequalities by Cheng and Cunningham [5] and achieves an overall running time of \( O(n^2m + n^3 \log n) \). In contrast to the algorithm of Cheng and Cunningham, there is a dependence on the density of the graph because of the parameter \( m \) instead of the \( n^2 \)-term. This is beneficial for sparse graphs, whereas there are direct combinatorial methods that are fast in practice for dense graphs with \( m = \Omega(n^2) \), see Babel [2] and San Segundo et al. [27].

For the separation of the 1-wheel inequalities, we require \( \bar{x} \in P^{OC}(G) \) and the weights of shortest odd and even walks with respect to edge weights \( \bar{w}_{ij} = 1 - x_i - x_j \) for every \( ij \in E \) have to be known. Thus, we use the separation algorithm for odd cycles where \( \bar{f}_{uv} \) and \( \bar{g}_{uv} \) denote the weights of shortest odd and even walks (the latter having at least two edges), respectively, from vertex \( u \) to vertex \( v \) in \( G \). Let \( P_{u,v}^1 \) be a shortest odd walk between vertices \( u \) and \( v \) for fixed \( x \). Then \( P_{u,v}^o \) is defined as the set of its interior vertices, that is, the vertices on this path except the start and end point. Similarly, for a shortest even walk \( P_{u,v}^e \) we denote the set of interior vertices by \( P_{u,v}^o \). We can determine \( \bar{f} \) and \( \bar{g} \) via

\[
\bar{f}_{uv} = \sum_{ij \in P_{u,v}^o} \bar{w}_{ij} = 1 - x_u - x_v + |P_{u,v}^o| - 2 \sum_{i \in P_{u,v}^o} x_i
\]

and

\[
\bar{g}_{uv} = \sum_{ij \in P_{u,v}^o} \bar{w}_{ij} = 1 - x_u - x_v + |P_{u,v}^e| - 2 \sum_{i \in P_{u,v}^e} x_i.
\]

We consider the categorical product of the underlying graph \( G \) for the maximum stable set problem and \( F \), where \( F \) is the digraph with vertex set \( \{0, 1, 2, 3, 4, 5\} \) and arc set \( \{(0, 1), (1, 2), (2, 1), (2, 3), (3, 4), (4, 5), (5, 4), (5, 0), (0, 3), (3, 0)\} \), see Figure 3. Then, for every fixed \( h \in V \) as a candidate for the hub of a 1-wheel, \( D_h = (V_h, \Gamma_h) := G \cdot F \) consists of arcs \( ((u, i), (v, j)) \) for every edge \( uv \) in \( G \) and every arc \( (i, j) \) in \( F \).

Notice that a walk \( Z \) in \( D_h \) induces simultaneous walks \( Z_G \) in \( G \) and \( Z_F \) in \( F \). The walk \( Z_G \) represents the entire rim \( C \cup R \) of the 1-wheel while the walk \( Z_F \) ensures that all the parity conditions on the rim are fulfilled. Spoke ends of a wheel are represented in \( F \) by the vertices 0 and 3. A walk of a single edge between two spoke ends uses in \( F \) the arc \((0, 3)\) or \((3, 0)\). On the other hand, a truly subdivided walk between two spoke ends \( v_i \) and \( v_{i+1} \) yields in \( F \) a walk from 0 to 1 to 2 (then maybe a couple times back to 1 and 2) and finally to 3 or similarly in \( F \) from 3 via 4 and 5 to 0. Next we define weights as in [8], that ensure that the weight of a rim walk is represented correctly in this product.

For every vertex \( h \in V \) and arc \( ((u, i), (v, j)) \in \Gamma_h \) we define the following arc weights \( \bar{w}_{uvij}^{Ah} \) and \( \bar{w}_{uvij}^{bh} \):

\[
\begin{align*}
\bar{w}_{uvij}^{Ah} &= \bar{w}_{uvij}^{Ah0} = f_{hu} + f_{hv} - x_u - x_v \\
\bar{w}_{uvij}^{Ab} &= \bar{w}_{uvij}^{Ab1} = \bar{w}_{uvij}^{Ab2} = \bar{w}_{uvij}^{Ab3} = \bar{w}_{uvij}^{Ab4} = 2(1 - x_u - x_v)
\end{align*}
\]

\[
\begin{align*}
\bar{w}_{uvij}^{Ah} &= \bar{w}_{uvij}^{Ah3} = \bar{w}_{uvij}^{Ah0} = f_{hu} + f_{hv} - x_u - x_v \\
\bar{w}_{uvij}^{Ab} &= \bar{w}_{uvij}^{Ab1} = \bar{w}_{uvij}^{Ab2} = \bar{w}_{uvij}^{Ab3} = \bar{w}_{uvij}^{Ab4} = 2(1 - x_u - x_v)
\end{align*}
\]
\[
\begin{align*}
W_{abv0}^h &= W_{a3b0}^h = g_{hu} + g_{hv} - x_u - x_v \\
W_{abv1}^h &= W_{a3b3}^h = g_{hu} + 1 - x_a - 2x_v - x_h \\
W_{abv2}^h &= W_{a5b0}^h = g_{hv} + 1 - 2x_a - x_v + x_h \\
W_{abv12}^h &= W_{a4b5}^h = w_{a4v4}^h = 2(1 - x_a - x_v)
\end{align*}
\]

The superscript \(Ah\) refers to the 1-wheel inequalities of type \(I_A'\) with \(h\) fixed as a hub and \(Bh\) refers to \(I_B'\). The weights of all arcs but those of the fourth type in (9) and (10) depend on \(h\). These weights are taken from the arc weight construction of de Vries [8], in order to state the following technical lemma that will be central in the proof of our main theorem.

**Lemma 5.1.** (8), Corollary 11. For a graph \(G\) and \(\bar{x} \in P_{OC}(G)\) there is a violated inequality \(I_A'\) with hub \(h\) and rim starting in \(v\) if and only if \(D_h\) contains a walk from \((v, 0)\) to \((v, 3)\) of weight less than \(2 - 2\bar{x}_h\) with respect to \(W^{Ah}\). There is a violated inequality \(I_B'\) with hub \(h\) and rim starting in \(v\) if and only if \(D_h\) contains a walk from \((v, 0)\) to \((v, 3)\) of weight less than \(2\bar{x}_h\) with respect to \(W^{Bh}\).

In contrast to de Vries [8], we avoid using shortest walk algorithms for the separation of 1-wheel inequalities, but show how to optimize over both polytopes \(P^{W_A}(G)\) and \(P^{W_B}(G)\) simultaneously with a compact linear formulation in the next section.

### 6 AN EXTENDED FORMULATION FOR 1-WHEELS

In this section, we describe an extended formulation of polynomial size that implies the odd cycle inequalities as well as the odd and even 1-wheel inequalities. The 1-wheel polytope is denoted by

\[
P^W(G) := P^{W_A}(G) \cap P^{W_B}(G) \subseteq P_{OC}(G).
\]

Lemma 5.1 requires that \(\bar{x} \in P_{OC}(G)\). Thus, we start with inequalities (1) and (5)–(8), which constitute the polynomial size extended formulation \(Q_{OC}(G)\) from Section 4. Then we extend it further to derive \(Q^W(G)\), the extension of the 1-wheel polytope \(P^W(G)\). For this purpose we introduce arc variables \(w_{uihv}^{Ah}\) and \(w_{uihv}^{Bh}\) for every vertex \(h \in V\) and all \(O(m)\) arcs \((u, i), (v, j)\) \(\in \Gamma_h\) in the product graph \(D_h\). As for the variables \(w_{ij}\) in the extended formulation for odd cycles, they are defined as substitutes for larger expressions, which makes the representation of \(Q^W(G)\) much shorter and clearer. We define them similarly to (9) and (10) dependent on \(x, f,\) and \(g\):

\[
\begin{align*}
W_{abv0}^{Ah} &= W_{a3b0}^{Ah} = f_{hu} + f_{hv} - x_u - x_v \\
W_{abv1}^{Ah} &= W_{a3b3}^{Ah} = f_{hu} + 1 - x_a - 2x_v - x_h \\
W_{abv2}^{Ah} &= W_{a5b0}^{Ah} = f_{hv} + 1 - 2x_a - x_v + x_h \\
W_{abv12}^{Ah} &= W_{a4b5}^{Ah} = W_{a4v4}^{Ah} = 2(1 - x_a - x_v)
\end{align*}
\]

\[
\begin{align*}
W_{abv0}^{Bh} &= W_{a3b0}^{Bh} = g_{hu} + g_{hv} - x_u - x_v \\
W_{abv1}^{Bh} &= W_{a3b3}^{Bh} = g_{hu} + 1 - x_a - 2x_v - x_h \\
W_{abv2}^{Bh} &= W_{a5b0}^{Bh} = g_{hv} + 1 - 2x_a - x_v + x_h \\
W_{abv12}^{Bh} &= W_{a4b5}^{Bh} = W_{a4v4}^{Bh} = 2(1 - x_a - x_v)
\end{align*}
\]

The variables \(f\) and \(g\) in \(Q_{OC}(G)\) are constrained by inequalities in \(x\). As in Section 4, they are bounded from above by the weights of shortest odd and even walks, respectively, in \(G\) with respect to \(w_{ij} = 1 - x_i - x_j\). Although the variables \(w_{uihv}^{Ah}\) and \(w_{uihv}^{Bh}\) can obviously take negative values for \((i, j) \notin \{(1, 2), (2, 1), (4, 5), (5, 4)\}\), de Vries [8] shows that \(D_h\) contains no cycle with negative (total) weight.

**Theorem 6.1.** Consider the 1-wheel polytope

\[
P^W(G) = \{x \in \mathbb{R}^n : x \text{ satisfies } (1), (2), (3), (I_A'), (I_B')\}.
\]

Then an extended formulation \(Q^W(G)\) for \(P^W(G)\) is given by

\[
Q^W(G) = \{(x, f, g, p^A, p^B) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{\delta_{or}} \times \mathbb{R}^{\delta_{or}} : \\
(x, f, g, p^A, p^B) \text{ satisfies } (1), (5)–(8), (13)–(18)\}
\]
with \( w^{Ah}_{uij} \) and \( w^{Bh}_{uij} \) defined as in (11) and (12) and

\[
\begin{align*}
p^{Ah}_{uij} & \leq w^{Ah}_{uij} \quad \forall \; ((u, i), (v, j)) \in \Gamma_h, \; h \in V, \quad (13) \\
p^{Bh}_{uij} & \leq w^{Bh}_{uij} \quad \forall \; ((u, i), (v, j)) \in \Gamma_h, \; h \in V, \quad (14) \\
p^{Ah}_{uij} & \leq p^{Ah}_{uij} + p^{Ah}_{vij} \quad \forall \; ((u, i), (w, k)) \in \Gamma_h, \; (v, j) \in V_h, \; h \in V, \quad (15) \\
p^{Bh}_{uij} & \leq p^{Bh}_{uij} + p^{Bh}_{vij} \quad \forall \; ((u, i), (w, k)) \in \Gamma_h, \; (v, j) \in V_h, \; h \in V, \quad (16) \\
p^{Ah}_{vih} & \geq 2 - 2x_h \quad \forall \; v \in V, \; h \in V, \quad (17) \\
p^{Bh}_{vih} & \geq 2x_h \quad \forall \; v \in V, \; h \in V. \quad (18)
\end{align*}
\]

Remark 6.2. Our formulation \( Q^W(G) \) requires \( 72n^3 + 2n^2 + n \) variables. Notice that \( w^{Ah} \) and \( w^{Bh} \) can be replaced by their definition that include the variables \( x, f, \) and \( g. \) The number of inequalities is \( 240mn^2 + 42mn + 2n^2 + 2m + 3n. \)

We now prove that the extended formulation \( Q^W(G) \) for the 1-wheel polytope \( P^W(G), \) as given above, is correct.

**Proof of Theorem 6.1.** We first show that \( P^W(G) \subseteq \text{proj}^j(Q^W(G)). \) Let \( \pi \in P^W(G). \) We construct \( f^\pi, g^\pi, p^A, \) and \( p^B \) so that \((x, f^\pi, g^\pi, p^A, p^B) \in Q^W(G)\): define \( f^\pi_{ij} \) for all \( i, j \in V \) and \( g^\pi_{ij} \) for all \( i, j \in V \) with \( i \neq j \) as the weights of shortest odd and even walks, respectively, between vertices \( i \) and \( j \) in \( G \) (if no such path exists, then assign a large value to the corresponding variable). In particular, this implies \( f^\pi_{ij} = f_{ij} \) and \( g^\pi_{ij} = g_{ij}. \) With these definitions, inequalities (1) and (5)–(7) are fulfilled. Let \( \overline{f}_{ij} = 2 \min \{w_{lk} : l k \in E\} \) for every vertex \( l \in V. \) Then \( \overline{f}_{ij} \) is exactly the weight of a shortest even closed walk in \( G \) using at least two edges. Here, we use that \( 2 \min \{w_{lk} : l k \in E\} = 2 \min \{f_{lk} : l k \in E\} \) and \( f_{lk} = \overline{f}_{kl}. \)

These weights occur as terms in some arc weights \( w^{Ah}_{uij} \) with \( h = u \) or \( h = v. \) Assigning the weights to the respective variables is feasible for \( Q^{OC}(G) \) and for \( Q^W(G) \) by Lemma 4.2. For every \( h \in V \) and for every pair \( (u, i) \) and \( (v, j) \) of vertices in \( D_h \) define \( p^{Ah}_{uij} \) as the weight of a shortest walk in \( D_h \) from \( (u, i) \) to \( (v, j) \) with respect to arc weights \( w^{Ah}. \)

Since there is no cycle of negative weight in \( D_h, \) the shortest walks exist although arc weights in \( D_h \) can be negative. Analogously, define \( p^{Bh}_{uij} \) as the weight of a shortest walk in \( D_h \) from \( (u, i) \) to \( (v, j) \) with respect to arc weights \( w^{Bh}. \)

Now it remains to show that proj\((Q^W(G)) \subseteq P^W(G). \) Let \((x, f^\pi, g^\pi, p^A, p^B) \in Q^W(G). \) By inequalities (5)–(7) the variables \( f^\pi_{ij} \) and \( g^\pi_{ij} \) are bounded from above by the weight of a shortest odd and even walk, respectively, between vertices \( i \) and \( j \) in \( G. \) Moreover, inequality (8) is satisfied for every \( i \in V. \) Therefore, every odd cycle has weight at least 1. This is equivalent to the condition that no odd cycle inequality in \( x \) is violated by \( \pi. \)

The variables \( p^{Ah}_{uij} \) are bounded from above by the weight of a shortest walk in \( D_h \) with respect to arc variables \( w^{Ah} \) between vertices \( (u, i) \) and \( (v, j) \) in \( D_h \) by inequalities (15) together with inequalities (13). Analogously, the variables \( p^{Bh}_{uij} \) are bounded from above by the weight of a shortest walk between vertices \( (u, i) \) and \( (v, j) \) in \( D_h \) with respect to arc variables \( w^{Bh} \) by inequalities (14) and (16). Since all inequalities (17) are satisfied by \((x, f^\pi, g^\pi, p^A, p^B) \), no \( I'_c \)-inequality in \( x \) is violated by \( \pi \) due to Lemma 5.1. Similarly, all inequalities (18) are satisfied by \((x, f^\pi, g^\pi, p^A, p^B). \) This implies that no \( I'_c \)-inequality in \( x \) is violated by \( \pi. \)

7 | CONCLUSION

We have presented the first compact extended formulation for the 1-wheel relaxation \( P^W(G). \) It includes the 1-wheel inequalities \( I'_A \) and \( I'_B \) of the stable set polytope. Our formulation permits optimizing directly over the polytope \( Q^W(G), \) which has a polynomial number of variables and inequalities, instead of using the ellipsoid method for optimizing over the exponential polytope \( P^W(G). \)

**ACKNOWLEDGMENTS**

The authors like to thank the anonymous reviewers and editors for their insightful comments that greatly helped to improve previous versions of this manuscript. All authors were supported by the Research Training Group 2126 Algorithmic Optimization (ALOP), funded by the German Research Foundation DFG. Additionally, the second author was supported by the Alexander
von Humboldt Foundation with funds from the German Federal Ministry of Education and Research (BMBF). Open Access funding enabled and organized by Projekt DEAL.

**ORCID**

Sven de Vries [https://orcid.org/0000-0002-440-4937](https://orcid.org/0000-0002-440-4937)

Ulf Friedrich [https://orcid.org/0000-0001-6566-245X](https://orcid.org/0000-0001-6566-245X)

Bernd Perscheid [https://orcid.org/0000-0002-9756-6603](https://orcid.org/0000-0002-9756-6603)

**REFERENCES**

[1] M. Afshari Rad and H.T. Kakhki, Two extended formulations for cardinality maximum flow network interdiction problem, Networks 69 (2017), 367–377.

[2] L. Babel, Finding maximum cliques in arbitrary and in special graphs, Computing 46 (1991), 321–341.

[3] R.D. Carr and G. Lancia, Compact vs. exponential-size LP relaxations, Oper. Res. Lett. 30 (2002), 57–65.

[4] E. Cheng, Separating subdivision of bicycle wheel inequalities over cut polytopes, Oper. Res. Lett. 23 (1998), 13–19.

[5] E. Cheng and W.H. Cunningham, Wheel inequalities for stable set polytopes, Math. Program. 77 (1997), 389–421.

[6] E. Cheng and S. de Vries, Antiew-wheel inequalities and their separation problems over the stable set polytopes, Math. Program. 92 (2002), 153–175.

[7] M. Conforti, G. Cornuéjols, and G. Zambelli, Extended formulations in combinatorial optimization, 4OR 8 (2010), 1–48.

[8] S. de Vries, Faster separation of 1-wheel inequalities by graph products, Discrete Appl. Math. 195 (2015), 74–83.

[9] A.M. Gerards and A. Schrijver, Matrices with the Edmonds-Johnson property, Combinatorica 6 (1986), 365–379.

[10] M. Giandomenico and A.N. Letchford, Exploring the relationship between max-cut and stable set relaxations, Math. Program. 106 (2006), 159–175.

[11] M. Giandomenico, A.N. Letchford, F. Rossi, and S. Smriglio, An application of the Lovász-Schrijver M(K,K) operator to the stable set problem, Math. Program. 120 (2009), 381–401.

[12] M. Giandomenico, F. Rossi, and S. Smriglio, Strong lift-and-project cutting planes for the stable set problem, Math. Program. 141 (2013), 165–192.

[13] M.X. Goemans and Y.S. Myung, A catalog of Steiner tree formulations, Networks 23 (1993), 19–28.

[14] M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 169–197.

[15] M. Grötschel, L. Lovász, and A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer, Berlin, 1988.

[16] K.L. Hoffman and M. Padberg, Solving airline crew scheduling problems by branch-and-cut, Manage Sci. 39 (1993), 657–682.

[17] V. Kaibel, Extended formulations in combinatorial optimization, Optim. Math. Optim. Soc. Newslett. 85 (2011), 2–6.

[18] V. Kaibel, J. Lee, M. Walter, and S. Weltge, Extended formulations for independence polytopes of regular matroids, Graphs Combin. 32 (2016), 1931–1944.

[19] G. Lancia and P. Serafini, An effective compact formulation of the max cut problem on sparse graphs, Electron. Notes Discrete Math. 37 (2011), 111–116.

[20] G. Lancia and P. Serafini, Deriving compact extended formulations via LP-based separation techniques, 4OR 12 (2014), 201–234.

[21] L. Lovász and A. Schrijver, Cones of matrices and set-functions and 0-1 optimization, SIAM J. Optim. 1 (1991), 166–190.

[22] R.K. Martin, Using separation algorithms to generate mixed integer model reformulations, Oper. Res. Lett. 10 (1991), 119–128.

[23] G.L. Nemhauser and G. Sigismondi, A strong cutting plane/branch-and-bound algorithm for node packing, J. Oper. Res. Soc. 43 (1992), 443–457.

[24] M.W. Padberg, On the facial structure of set packing polyhedra, Math. Program. 5 (1973), 199–215.

[25] S. Rehennack, M. Oswald, D.O. Theis, H. Seitz, G. Reinelt, and P.M. Pardalos, A branch and cut solver for the maximum stable set problem, J. Combin. Optim. 21 (2011), 434–457.

[26] F. Rossi and S. Smriglio, A branch-and-cut algorithm for the maximum cardinality stable set problem, Oper. Res. Lett. 28 (2001), 63–74.

[27] P. San Segundo, A. Lopez, and P.M. Pardalos, A new exact maximum clique algorithm for large and massive sparse graphs, Comput. Oper. Res. 66 (2016), 81–94.

[28] L.E. Trotter Jr., A class of facet producing graphs for vertex packing polyhedra, Discrete Math. 12 (1975), 373–388.

[29] M. Yannakakis, Expressing combinatorial optimization problems by linear programs, J. Comput. Syst. Sci. 43 (1991), 441–466.

---

**How to cite this article:** de Vries S, Friedrich U, Perscheid B. An extended formulation for the 1-wheel inequalities of the stable set polytope. *Networks*. 2020;75:86–94. [https://doi.org/10.1002/net.21906](https://doi.org/10.1002/net.21906)