Not So Classical Mechanics – Unexpected Symmetries of Classical Motion

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A survey of topics of recent interest in Hamiltonian and Lagrangian dynamical systems, including accessible discussions of regularization of the central force problem; inequivalent Lagrangians and Hamiltonians; constants of central force motion; a general discussion of higher-order Lagrangians and Hamiltonians with examples from Bohmian quantum mechanics, the Korteweg-de Vries equation and the logistic equation; gauge theories of Newtonian mechanics; classical spin, Grassmann numbers, and pseudomechanics.

I. INTRODUCTION

The study of classical mechanics is vast and ancient. Therefore, this collection of results, observations and questions necessarily omits most of the field and probably misses a number of older references even for topics covered in detail. We focus principally on issues of symmetry and subjects (old and new) which have appeared in the literature within recent decades. Nor should it be thought that we provide a complete survey of even the results we do discuss. Instead, our references for each topic are probably sufficient for the interested reader to gain a foothold on the relevant research.

Of course, the list of topics we do not examine is extensive. Certain topics such as nonlinear dynamics receive only a brief mention as an example of a higher order system in Section 5. We have chosen to omit any discussion of electromagnetism while touching on special and general relativity only to provide examples. Since our presentation is intended for a broad audience, we have also avoided the large body of formal work. Thus, while the formal study of symplectic manifolds, Kahler manifolds, Poincaré sections and so on make heavy use of modern differential geometry and field theory techniques, little mention is made of progress in these directions.

What remains is nonetheless filled with fascinating and diverse surprises in a field often
mistaken to be complete. Thus, what we do cover is a wide array of topics ranging from the Kepler problem to supersymmetry. The unifying theme, if there is one, is the occurrence of unexpected and surprising symmetries in classical physics, and especially in Lagrangian and Hamiltonian dynamics. Though we treat a few topics simply because there is recent reference to them in the literature, most of the topics concern symmetry in one way or another. The uses vary greatly, from the use of anticommuting numbers to the rotation group, from unusual constants of motion of the Kepler problem to the infinite hierarchy of constants of motion of the Korteweg-de Vries (KdV) equation. An additional guiding principle has been to treat topics that may not be familiar to many readers.

Curiously, quantum mechanics and quantum field theory have had a strong impact on current work in classical physics. As a result, a brief discussion of quantum mechanics appears in our examination of inequivalent Lagrangians in Section 3 and Bohmian quantum mechanics is discussed in Section 5. Further connections between quantum and classical mechanics are suggested in our treatment of Lagrangian and Hamiltonian dynamics as gauge theories in Section 6. Finally, Section 7 owes its entire existence to insights from supersymmetric quantum field theory.

The organization of the paper is simultaneously from old to new and from easy to difficult. Section 2 provides a warm-up exercise with some new thoughts on an old topic – the regularization of the central force problem. From there we move gradually to more recent and more mathematically challenging questions. In Section 3 we discuss inequivalent Lagrangians and Hamiltonians, a topic which begins with Lie and Dirac (if not earlier) and which received considerable new input in the 70s. Through the same period, old knowledge resurfaced with the rediscovery of the Laplace-Runge-Lenz and Hamilton vectors. A derivation of these rediscovered constants of the motion is given in Section 4, using a technique based on an old theorem. While the theorem will no doubt be familiar to mathematicians, its simple application to finding constants of the motion does not appear in classical physics textbooks.

Moving to more active areas of current interest, we look at the occurrence of higher order differential equations in classical physics. After a brief general introduction to these systems at the beginning of Section 5 is an example of such equations – Bohmian quantum mechanics.

The final two Sections deal with truly contemporary insights. In Section 6, a development
of both Lagrangian and Hamiltonian dynamics as gauge theories shows an interesting new connection between classical physics and conformal symmetry. Then, in Section 7, two further developments of field theory – spinors and anticommuting variables – are discussed in the context of particle mechanics. The KdV equation and the approach to chaos are treated in Appendices.

Before embarking, a few general comments are in order. First, observe that each section below is essentially independent of the others. Each section has its own brief introduction and references. Note that the end of most sections we have tried to provide a few stimulating questions. These questions do not reflect any consensus thinking and should not be taken to be the definitive puzzles facing the field. Rather, they are suggestions of some directions which might or might not prove fruitful. Finally, it should be noted that where derivations are given without citation, we have produced original calculations. We make no further mention of this fact since it is probable that many or all of these calculations have already appeared somewhere within the last few hundred years!

II. REGULARIZATION OF THE CENTRAL FORCE PROBLEM

We begin with some of the oldest problems of classical physics. In this Section and the next, we explore some interesting features of the Kepler problem and other central force motion. In this Section, we examine regularizations of the Kepler problem. In the next section we present a technique for finding constants of the motion \[1\], then, as an example, use the technique to find some recently rediscovered constants of the Kepler problem \[2\].

Regularizations of dynamical problems are transformations that turn the equations of motion into a simpler or less singular mechanical problem. Euler \[3\] and Levi-Civita \[4\] produced one- and two-dimensional regularizations, respectively, of the Kepler problem. These regularizations turn the Kepler/Coulomb equation of motion into an isotropic oscillator. It is not surprising that this is possible, because the transformations are time-dependent. Indeed, using similar transformations, it is possible locally to turn any central force problem into the isotropic oscillator. We present a general proof of this claim below. The Euler and Levi-Civita results are special cases.

There are some recent discussions in the literature extending these regularizations. Since the Levi-Civita result makes use of a complex variable, some authors have explored the
idea that the use of a vector space which is also a number field gives insights into the
problem. Thus, Kustaanheimo and Stiefel \([5],[6],[7]\) give a quaternionic transformation
from the 3-dim Kepler problem to a constrained 4-dim isotropic oscillator, thereby show-
ing that bounded Kepler orbits have an underlying \(SO(4)\) symmetry. Bartsch \([8]\) writes
the Kustaanheimo-Steifel result in terms of Hestenes’ geometric algebra \([9]\). Such use of
quaternionic, Clifford or Grassmann variables (see below) often extends, or streamlines, the
presentation of classical results.

However, it seems unlikely that number fields are necessary to transform the Kepler
problem into the oscillator. If that were the case, we would expect regularization to be
possible only using real, complex, quaternionic or octonionic variables and therefore only to
occur in dimensions less than or equal to eight. But since both the Kepler problem and the
isotropic oscillator are inherently two dimensional, the Levi-Civita solution should suffice in
any higher dimension as well. Our generalized solution below demonstrates this to be the
case.

\[A. \quad \text{Higher dimensions}\]

Consider the general central force motion in any dimension \(d \geq 2\). We begin from the
action
\[
S = \int dt \left[ \frac{1}{2} m \frac{dx_i}{dt} \frac{dx_i}{dt} - V(r) \right]
\]
where \(r = \sqrt{x_i x_i}\). It follows that
\[
m \frac{d^2 x_i}{dt^2} = -V' \frac{x_i}{r}
\]
We first compute the total angular momentum
\[
M_{ij} = x_i p_j - x_j p_i
= m (x_i \dot{x}_j - x_j \dot{x}_i)
\]
This is conserved, since
\[ \frac{d}{dt} M_{ij} = m \frac{d}{dt} (x_i \dot{x}_j - x_j \dot{x}_i) \]
\[ = m \left( x_j \frac{d^2 x_k}{dt^2} - x_k \frac{d^2 x_j}{dt^2} \right) \]
\[ = - \frac{V'}{r} (x_j x_k - x_k x_j) \]
\[ = 0. \]

To prove from this that the motion lies in a plane, let \( \mathbf{x}_0 \) and \( \mathbf{v}_0 \) be the initial position and velocity. Then the angular momentum is

\[ M_{ij} = x_{0i} v_{0j} - x_{0j} v_{0i} \neq 0 \]

Let \( \mathbf{w}^{(a)}, a = 1, \ldots, n - 2 \), be a collection of vectors perpendicular to the initial plane

\[ P = \{ \mathbf{v} = \alpha \mathbf{x}_0 + \beta \mathbf{v}_0 | \forall \alpha, \beta \} \]

\[ \mathbf{w}^{(a)} \mathbf{v} = 0 \]

so that the set \( \{ \mathbf{x}_0, \mathbf{v}_0, \mathbf{w}^{(a)} \} \) forms a basis. Then, for all \( a \)

\[ w^{(a)}_i M_{ij} = 0. \]

Now, at any time \( t \), \( M_{ij} \) is given by

\[ M_{ij} = m (x_i v_j - x_j v_i) \]

and since \( M_{ij} \) is constant we still have

\[ 0 = w^{(a)}_i m (x_i v_j - x_j v_i) \]
\[ 0 = (\mathbf{w}^{(a)} \cdot \mathbf{x}) \mathbf{v} - \mathbf{x} (\mathbf{w}^a \cdot \mathbf{v}) \]

Suppose, for some \( a_0 \), that

\[ \mathbf{w}^{(a_0)} \cdot \mathbf{x} \neq 0 \]

Then

\[ \mathbf{v} = \mathbf{x} \left( \frac{\mathbf{w}^{a_0} \cdot \mathbf{v}}{\mathbf{w}^{a_0} \cdot \mathbf{x}} \right) \]

and \( M_{ij} \) is identically zero, in contradiction to its constancy. Therefore, we conclude

\[ \mathbf{w}^{(a)} \cdot \mathbf{x} = 0 \]
for all $a$. A parallel argument shows that

$$\mathbf{w}^{(a)} \cdot \mathbf{v} = 0$$

for all $a$, so the motion continues to lie in the original plane.

Now we choose polar coordinates in the plane of motion, and the problem reduces to two dimensions. We next need to deal with the presence of angular momentum. With coordinates $x^{(a)}$ in the $\mathbf{w}^{(a)}$ directions, the central force equations of motion are

$$m \frac{d^2 x^{(a)}}{dt^2} = 0$$

$$m \left( \frac{d^2 r}{dt^2} - r \frac{d\varphi}{dt} \frac{d\varphi}{dt} \right) = -V'(r)$$

$$\frac{d (mr^2 \dot{\varphi})}{dt} = 0$$

We choose $x^{(a)} = 0$, and set $L = mr^2 \dot{\varphi} = \text{constant}$. Eliminating $\dot{\varphi}$, these reduce to the single equation

$$m \frac{d^2 r}{dt^2} - \frac{M^2}{mr^3} = -V'(r)$$

(1)

Notice that now any transform of $r$ will change the required form of the angular momentum term. What works to avoid this is to recombine the angular momentum and force terms. We again start with

$$r = f(u), \quad \frac{d}{dt} = \frac{1}{f'} \frac{d}{d\tau}.$$ 

Then eq. (1) becomes

$$\frac{1}{f'} \frac{d}{d\tau} \left( \frac{1}{f'} \frac{df}{d\tau} \right) - \frac{M^2}{m^2 f^3} = -\frac{dV}{df} \left[ f(u) \right].$$

Rearranging, we have

$$\frac{d^2 u}{d\tau^2} = \frac{M^2 f'}{m^2 f^3} - f' \frac{dV}{df} \left[ f(u) \right]$$

$$= \frac{M^2 f'}{m^2 f^3} - \frac{df}{du} \frac{dV}{df} \left[ f(u) \right]$$

$$= \frac{M^2 df}{m^2 f^3 du} - \frac{dV}{du} \left[ f(u) \right]$$

To obtain the isotropic harmonic oscillator we require the combined angular momentum and force terms to give the needed expression:

$$\frac{M^2}{m^2 f^3} \frac{df}{du} \left[ f(u) \right] - \frac{dV}{du} \left[ f(u) \right] = \frac{\dot{M}^2}{m^2 u^3} - ku$$
Integrating,
\[
\frac{M^2}{2m^2f^2} + V[f(u)] = \frac{\tilde{M}^2}{2m^2u^2} + \frac{1}{2}ku^2 + \frac{c}{2}.
\] 
(2)
If we define
\[g(f) \equiv \frac{M^2}{2m^2f^2} + V[f(u)]\]
the required function \(f\) is
\[f = g^{-1}\left(\frac{\tilde{M}^2}{2m^2u^2} + \frac{1}{2}ku^2 + \frac{c}{2}\right).
\]
Substituting this solution into the equation of motion, we obtain the equation for the isotropic oscillator,
\[m \frac{d^2u}{dt^2} - \frac{\tilde{M}^2}{mu^3} = -ku.
\]
Therefore, every central force problem is locally equivalent to the isotropic harmonic oscillator. Of course, the same result follows from Hamilton-Jacobi theory, since every pair of classical systems with the same number of degrees of freedom are related by some time-dependent canonical transformation.

The solution takes a particularly simple form for the Kepler problem, \(V = -\alpha/r\). In this case, eq. (2) becomes
\[
\frac{M^2}{2m^2f^2} - \frac{\alpha}{f} - \left(\frac{\tilde{M}^2}{2m^2u^2} + \frac{1}{2}ku^2 + \frac{c}{2}\right) = 0
\]
Solving the quadratic for \(1/f\), we take the positive solution
\[
\frac{1}{f} = \frac{m^2}{M^2} \left[\alpha + \sqrt{\alpha^2 + \frac{M^2}{m^2} \left( \frac{\tilde{M}^2}{m^2u^2} + ku^2 + c \right)}\right]
\]
\[
= \frac{am^2}{M^2} \left[1 + \frac{M}{amu} \sqrt{ku^4 + \left(c + \frac{\alpha^2m^2}{M^2}\right)u^2 + \frac{\tilde{M}^2}{m^2}}\right].
\]
There is also a negative solution.
We may choose \(c\) to complete the square under the radical and thereby simplify the solution. Setting
\[
c = \frac{2\sqrt{k\tilde{M}}}{m} - \frac{\alpha^2m^2}{M^2}
\]
the positive solution for \(f\) reduces to
\[
\frac{1}{f} = \frac{am^2}{M^2} + m\sqrt{ku} + \left(\tilde{M}/M\right)\frac{1}{u}
\]
or
\[ f = \frac{u}{(m\sqrt{k})u^2 + (\alpha m^2/M^2)u + M/M}. \]

The zeros of the denominator never occur for positive \( u \), so the transformation \( f \) is regular in the Kepler case. The regularity of the Kepler case is not typical – it is easy to see that the solution for \( f \) may have many branches. The singular points of the transformation in these cases should give information about the numbers of extrema of the orbits, the stability of orbits, and other global properties. The present calculation may provide a useful tool for studying these global properties in detail. The problem of global properties of orbits remains open – power law forces have been examined [10], but more complicated potentials allow arbitrarily many extrema. For example, the potential
\[ V = \alpha (r - r_0)^{2p} \]
gives an effective potential
\[ V_{\text{eff}} = \frac{M^2}{2mr^2} + \alpha (r - r_0)^{2p}. \]

Straightforward perturbation about circular orbits shows that, for arbitrary fixed angular momentum \( M \), the frequency of radial oscillations may be increased without bound by increasing \( p \). Such closed orbits will have arbitrarily many extrema.

**B. Euler’s regularization**

Essential features of the regularizing transformation are evident even in the 1-dim case. The Euler solution uses the substitutions
\[ x = -u^{-2}, \quad \frac{d}{dt} = u^3 \frac{d}{d\tau} \]
to turn the 1-dim Kepler equation of motion into the 1-dim harmonic oscillator. Before moving to a proof for the general \( n \)-dim case, we note that more general transformations are possible in the 1-dim case. Beginning with the equation of motion,
\[ m \frac{d^2x}{dt^2} = -\frac{\alpha}{x^2}. \]

let
\[ x = f(u), \quad \frac{d}{dt} = \frac{1}{f'} \frac{d}{d\tau}. \]
Then
\[ \dot{x} = f \frac{du}{dt} = \frac{du}{d\tau} \]
so the equation of motion becomes
\[ m\frac{d^2u}{d\tau^2} = -\frac{\alpha f'}{f^2}. \]

Now let \( V(u) \) be any potential. Demanding
\[ V' = \alpha \frac{f'}{f^2} \]
we integrate to find
\[ f = -\frac{\alpha}{V(u) - V_0}. \]
With this choice for \( f \), the equation of motion becomes simply
\[ m\frac{d^2u}{d\tau^2} = -V'. \]

Notice that \( u \) is not necessarily a monotonic function of \( x \) so the transformation at zeros of \( V' \) may be singular. We will not deal with such global issues here.

In higher dimensions the regularizing transformation is complicated by the presence of angular momentum. Still, the general proof is similar, involving a change of both the radial coordinate and the time. Once again, more general potentials can be treated. To begin, we eliminate the angular momentum variables to reduce the problem to a single independent variable. The only remaining difficulty is to handle the angular momentum term in the radial equation.

We end the Section with some questions:

1. To what degree can regularizations be accomplished by canonical transformations? What is the relationship between regularizations and canonical transformations?

2. What can be proved about extrema, boundedness and stability of orbits in monotonic central potentials bounded by various power law potentials? in monotonic central potentials? in arbitrary central potentials?
III. INEQUIVALENT LAGRANGIANS AND HAMILTONIANS

One of the more startling influences of quantum physics on the study of classical mechanics is the realization that there exist inequivalent Lagrangians determining a given set of classical paths. Inequivalent Lagrangians for a given problem are those whose difference is not a total derivative. While it is not too surprising that a given set of paths provides extremals for more than one functional, it is striking that some systems permit infinitely many Lagrangians for the same paths. There remain many open questions on this subject, with most of the results holding in only one dimension.

The existence of classically inequivalent Hamiltonians is not so clear, since there are far more transformations preserving Hamiltonian structure than there are preserving Lagrangian structure. However, distinct Hamiltonians abound in quantum theory, where equivalent Hamiltonians may lead to distinct quantum structures \[11\]. If there is more than one Hamiltonian for a system, which one do we quantize? Furthermore, while it is clear that distinct Hamiltonians can lead to different quantum theories, what about the converse? Do there exist distinct Hamiltonian operators, \( \hat{H}, \hat{H}' \) with identical expectation values for all observables? Can distinct Hamiltonian operators have the same energy spectra?

Here, we restrict our attention to classical questions. To begin our exploration of inequivalent Lagrangians, we describe classes of free particle Lagrangians and give some examples. Next we move to the theorems for 1-dim systems due to Yan, Kobussen and Leubner \([12, 13, 14, 15, 16]\) including a simple example. Then we consider inequivalent Lagrangians in higher dimensions. Finally, we briefly examine the possibilities for inequivalent Hamiltonians.

A. General free particle Lagrangians

There are distinct classes of Lagrangian even for free particle motion. We derive the classes and give an example of each, noting how Galilean invariance singles out the usual choice of Lagrangian.

The most general velocity dependent free particle Lagrangian is

\[ S = \int f(v)dt \]
We assume the Cartesian form of the Euclidean metric, so that \( v = \sqrt{\delta_{ij} v^i v^j} \). The equation of motion is
\[
\frac{d}{dt} \frac{\partial f}{\partial v^i} = 0
\]
so the conjugate momentum
\[
p_i = \frac{\partial f}{\partial v^i} = f' \frac{v_i}{v}
\]
is conserved. We need only solve this equation for the velocity. Separating the magnitude and direction, we have
\[
\frac{v_i}{v} = \frac{p_i}{p}
\]
\[
v = g(p) \equiv [f']^{-1}(p)
\]
This solution is well-defined on any region in which the mapping between velocity and momentum is \( 1 - 1 \). This means that velocity ranges may be any of four types: \( v \in (0, \infty), (0, v_1), (v_1, v_2), (v_1, \infty) \). Which of the four types occurs depends on the singularities of \( f' v^i / v \). Since \( v^i / v \) is a well-defined unit vector for all nonzero \( v_i \), it is \( f' \) which determines the range. Requiring the map from \( v_i \) to \( p_i \) to be single valued and finite, we restrict to regions of \( f' \) which are monotonic. Independent physical ranges of velocity will then be determined by each zero or pole of \( f' \). In general there will be \( n + 1 \) such ranges
\[
v \in (0, v_1), (v_1, v_2), \ldots, (v_n, \infty)
\]
if there are \( n \) singular points of \( f' \). Of course it is possible that \( v_1 = 0 \) (so that on the lowest range, \( (0, v_2) \), zero velocity is forbidden), or \( v_1 = \infty \) so that the full range of velocities is allowed. Within any of these regions, the Hamiltonian formulation is well-defined and gives the same equations of motion as the Lagrangian formulation.

Thus, the motion for general \( f \) may be described as follows. Picture the space of all velocities divided into a number of spheres centered on the origin. The radii of these spheres are given by the roots and poles of \( f' \). Between any pair of spheres, momentum and velocity are in \( 1 - 1 \) correspondence and the motion is uniquely determined by the initial conditions. In these regions the velocity remains constant and the resulting motion is in a straight line. On spheres corresponding to zeros of \( f' \), the direction of motion is not determined by the equation of motion. On spheres corresponding to poles of \( f' \), no solutions exist. It is amusing to note that all three cases occur in practice. We now give an example of each.
First, consider the regular situation when $f'$ is monotonic everywhere so the motion is uniquely determined to be straight lines for all possible initial velocities. The condition singles out the case of unconstrained Newtonian mechanics. This is the only case that is Galilean invariant, since Galilean boosts require the full range of velocities, $v \in (0, \infty)$.

When $f'$ has zeros, we have situations where a complete set of initial conditions is insufficient to determine the motion. Such a situation occurs in Lovelock, or extended, gravity, in which the action in $d$-dimensions (for $d$ even) is of the general form

$$S = \sum_{k=0}^{d/2} a_k \int R^{ab} \wedge R^{cd} \wedge \cdots \wedge R^{ef} \wedge e^g \wedge \cdots \wedge e^h \varepsilon_{abcdefgh}$$

where $R^{ab}$ is the curvature 2-form, $e^a$ the solder form and the $a_k$ are arbitrary constants. This is the most general curved spacetime gravity theory in which the field equations depend on no higher than second derivatives of the metric \[17\]. In general, the field equations depend on powers of the second derivatives of the metric, whereas in general relativity this dependence is linear. Among the solutions are certain special cases called “geometrically free” \[18\]. These arise as follows. For some choices of the constants $a_k$, we may rewrite $S$ in the form

$$S = \int \prod_{k=0}^{d/2} (R^{ab}_{a_k b_k} - \alpha e^{a_k e^{b_k}}) \varepsilon_{a_1 b_1 \cdots a_{d/2} b_{d/2}}$$

Suppose that for all $k = 1, \ldots, n$ for some $n$ in the range $2 < n < d/2$, we have

$$\alpha_k = \alpha$$

for some fixed value $\alpha$. Then the variational equations all contain at least $n - 1$ factors of

$$R^{a_k b_k} - \alpha e^{a_k e^{b_k}}$$

Therefore, if there is a subspace of dimension $m > d - n + 1$ of constant curvature

$$R^{ab} = \alpha e^a e^b$$

for $a, b = 1, \ldots, m$, then the field equations are satisfied regardless of the metric on the complementary subspace. This is similar to the case of vanishing $f'$, where the equation of motion is satisfied regardless of the direction of the velocity,

$$p_i = f' \frac{v_i}{v} \equiv 0$$
as long as $v$, but not $v_i$, is constant.

Finally, suppose $f'$ has a pole at some value $v_0$. Then the momentum diverges and motion never occurs at velocity $v_0$. Of course, this is the case in special relativity, where the action of a free particle may be written as

$$S = \int p_\alpha dx^\alpha = - \int E dt + p_i dx^i$$

$$= -mc^2 \int \sqrt{1 - \frac{v^2}{c^2}} dt.$$ With $f(v) = -mc^2 \sqrt{1 - v^2/c^2}$, we have

$$f' = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

with the well known pole in momentum at $v = c$.

Note that there is a complementary situation for Hamiltonians. From the Lagrangians for the free particle,

$$S = \int f(v) dt$$

we have the conjugate momenta

$$p_i = f' \frac{v_i}{v}, \quad p = f' = g$$

and Hamiltonians

$$H = vf' - f.$$ Hamilton’s equations are

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p^i}{p} g^{-1}(p)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} = 0$$

Once again, the constancy of the momentum is immediate. However, despite the apparent diversity of Hamiltonians, they are locally related by canonical transformations. The only distinctions are the global ones, and these exactly match those described above.

### B. Inequivalent Lagrangians

The existence of inequivalent Lagrangians for a given physical problem seems to trace back to Lie [19]. Dirac ([20], [21]) was certainly well aware of the ambiguities involved in
passing between the Lagrangian and Hamiltonian formulations of classical mechanics. Later, others (\cite{22,23,24,25}), identified certain non-canonical transformations which nonetheless preserve certain Hamiltonians. A specific non-canonical transformation of the 2-dim harmonic oscillator is provided by Gelman and Saletan \cite{26}. Bolza \cite{27} showed that independent Lagrangians can give the same equations of motion and, a few years later, Kobussen \cite{12}, Yan \cite{13,14} and Okubo \cite{28,29} independently gave systematic developments showing that an infinite number of inequivalent Lagrangians exist for 2-dim mechanical systems.

Shortly thereafter, Leubner \cite{16} generalized and streamlined Yan’s proof to include arbitrary functions of two constants of motion.

Leubner’s result, the most general to date, may be stated as follows. Given any two constants of motion, \((\alpha, \beta)\), associated with the solution to a given 1-dim equation of motion, the solution set for any Lagrangian of the form

\[
L(x, \dot{x}, t) = \int_v^x \frac{\dot{x} - v}{v} \left| \frac{\partial (\alpha, \beta)}{\partial (v, t)} \right| dv + \int_{x_0}^x f(\tilde{x}, v_0, t) \left| \frac{\partial (\alpha, \beta)}{\partial (v, t)} \right|_{v=v_0} d\tilde{x} + \frac{d\Omega}{dt} \tag{3}
\]

where \(|\frac{\partial (\alpha, \beta)}{\partial (v, t)}|\) is the Jacobian, includes the same solutions locally. Notice that \(\alpha\) and \(\beta\) are arbitrary constants of the motion – each may be an arbitrary function of simpler constants such as the Hamiltonian. We argue below that in 1-dim the solution sets are locally identical, though \cite{16} provides no explicit proof. In higher dimensions there are easy counterexamples.

We illustrate a special case of this formula, of the form

\[
L(x, v) = \dot{x} \int_v^x \frac{K(x, v)}{v^2} dv \tag{4}
\]

where \(K\) is any constant of the motion of the system. This expression is valid when the original Lagrangian has no explicit time dependence. Following Okubo \cite{29}, we prove that eq.\cite{41} leads to the constancy of \(K\). The result follows immediately from the Euler-Lagrange expression for \(L\):

\[
\begin{align*}
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= \frac{d}{dt} \left( \int_v^x \frac{K(x, v)}{v^2} dv + \frac{\dot{x}}{\dot{x}^2} \right) - \dot{x} \int_v^x \frac{1}{v^2} \frac{\partial K(x, v)}{\partial x} dv \\
&= \frac{\dot{x} \frac{\partial K(x, \dot{x})}{\partial \dot{x}}}{\dot{x} \frac{\partial K(x, \dot{x})}{\partial x}} + \frac{\partial K(x, \dot{x})}{\partial x} \\
&= \frac{1}{\dot{x}} \frac{dK(x, \dot{x})}{dt}.
\end{align*}
\]
Therefore, the Euler-Lagrange equation holds if and only if $K(x, \dot{x})$ is a constant of the motion.

The uniqueness in 1-dim follows from the fact that a single constant of the motion is sufficient to determine the solution curves up to the initial point. The uniqueness also depends on there being only a single Euler-Lagrange equation. These observations lead us to a higher dimensional result below.

It is interesting to notice that we can derive this form for $L$, but with $K$ replaced by the Hamiltonian, by inverting the usual expression,

$$H = \dot{x} \frac{\partial L}{\partial \dot{x}} - L$$

for the Hamiltonian in terms of the Lagrangian. First, rewrite the right side as:

$$H = \dot{x} \frac{\partial L}{\partial \dot{x}} - L = \dot{x}^2 \frac{\partial}{\partial \dot{x}} \left( \frac{L}{\dot{x}} \right).$$

Now, dividing by $\dot{x}$ and integrating (regarding $H$ as a function of the velocity) we find:

$$L = \dot{x} \int \frac{H(x, v)}{v^2} dv$$

The remarkable fact is that the Hamiltonian may be replaced by any constant of the motion in this expression. Conversely, suppose we begin with the Lagrangian in terms of an arbitrary constant of motion, $K$, according to eq.(4),

$$L(x, v) = \dot{x} \int \frac{K(x, v)}{v^2} dv$$

Then constructing the conserved Hamiltonian,

$$\tilde{H}(x, p) = \dot{x} \frac{\partial L}{\partial \dot{x}} - L$$

$$= \dot{x} \frac{\partial}{\partial \dot{x}} \left( \dot{x} \int \frac{K(x, v)}{v^2} dv \right) - \dot{x} \int \frac{K(x, v)}{v^2} dv$$

$$= \dot{x} \left( \int \frac{K(x, v)}{v^2} dv + \frac{K(x, \dot{x})}{\dot{x}} \right) - \dot{x} \int \frac{K(x, v)}{v^2} dv$$

$$= K(x, \dot{x})$$

we arrive at the chosen constant of motion! This proves the Gelman-Saletan-Currie conjecture [26]: any nontrivial time-independent constant of motion gives rise to a possible Hamiltonian. Proofs of the conjecture are due to Yan ([13], [14]) and Leubner [16].
The conjugate momentum to $L$ constructed according to eq. (4) is
\[ \tilde{p} = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left( \dot{x} \int_{v}^{\dot{x}} \frac{K(x, v)}{v^2} dv \right) = \int_{v}^{\dot{x}} \frac{K(x, v)}{v^2} dv + \frac{K(x, \dot{x})}{\dot{x}}. \]

Of course, if $K = \frac{1}{2}m\dot{x}^2 + V$, both $\tilde{H}$ and $\tilde{p}$ reduce to the usual expressions.

The simple harmonic oscillator is sufficient to illustrate the method ([30],[31]). Since the Hamiltonian, $H = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, is a constant of the motion so is $H^2$, so we write
\[ L = \frac{1}{4} \dot{x} \int_{v}^{\dot{x}} \frac{1}{v^2} \left( m^2 v^4 + 2kmv^2x^2 + k^2x^4 \right) dv = \frac{1}{12}m^2\dot{x}^4 + \frac{1}{2}km\dot{x}^2x^2 - \frac{1}{4}k^2x^4. \]

The Euler-Lagrange equation resulting from $L$ is
\[
0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt} \left( \frac{1}{3}m^2\dot{x}^3 + km\dot{x}x^2 \right) - \left( km\dot{x}^2x - k^2x^3 \right) = (m\ddot{x} + kx) \left( m\dot{x}^2 + kx^2 \right).
\]

Either of the two factors may be zero. Setting the first to zero is gives the usual equation for the oscillator, while setting the second to zero we find the same solutions in exponential form:
\[ x = Ae^{i\omega t} + Be^{-i\omega t} \]

The conjugate momentum and Hamiltonian are:
\[ \tilde{H}(x, \tilde{p}) = H^2(x, p) = \frac{1}{4} \left( m^2\dot{x}^4 + 2km\dot{x}^2x^2 + k^2x^4 \right) \]
\[ \tilde{p} = \frac{\partial L}{\partial \dot{x}} = \frac{1}{3}m^2\dot{x}^3 + km\dot{x}x^2. \]

While it is possible to solve the cubic equation to find $\dot{x}(\tilde{p})$, and then substitute to find $\tilde{H}(x, \tilde{p})$ as an explicit function of $\tilde{p}$, it is clear that the resulting expression is not of the same form as the original Hamiltonian. There remains the question of whether this effect could be achieved by a time-independent canonical transformation. The transformation of the momentum,
\[ \tilde{p} = \frac{p^3}{3m} + kpx^2 \]
is part of a canonical transformation, given by

\[
\tilde{x} = -\frac{1}{2k} \ln p \\
\tilde{p} = \frac{p^3}{3m} + kp\tilde{x}^2
\]

However, this does not simplify the form of the Hamiltonian. We find:

\[
\tilde{H}(x, \tilde{p}) = \frac{1}{4m^2} \left( \frac{4}{9} e^{-8k\tilde{x}} + m^2 p^2 e^{4k\tilde{x}} + 4m \tilde{p} e^{-2k\tilde{x}} \right)
\]

and the resulting Hamiltonian equations of motion are not transparent.

There does exist, of course, a time-dependent canonical transformation relating the two Hamiltonians. The systems are nonetheless distinct globally, since the cubic relationship between momentum and velocity limits the allowed ranges of the variables for the higher order Hamiltonian. It would be interesting to know if there is a unique Hamiltonian for which \( p(v) = 1 - 1 \).

1. Are inequivalent Lagrangians equivalent?

Inequivalent Lagrangians have been defined as Lagrangians which lead to the same equations of motion but differ by more than a total derivative. For the simple case above, the cubic order equation of motion factors into the energy times the usual equation of motion, and setting either factor to zero gives the usual solution and only the usual solution. However, is this true in general? The Yan-Leubner proof shows that the new Lagrangian has the same solutions, but how do we know that none of the higher order Lagrangians introduces spurious solutions? The proofs do not address this question explicitly. If some of these Lagrangians introduce extra solutions, then they are not really describing the same motions.

Suppose, for some time-independent Hamiltonian we write

\[
L = v \int_0^e f[\alpha(x, \xi)] \xi^2 d\xi
\]

where \( \alpha \) is any constant of the motion. Then we know that the Euler-Lagrange equation is satisfied by the usual equation of motion. But what is the Euler-Lagrange equation? We have shown that

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{1}{\dot{x}} \frac{dK(x, \dot{x})}{dt} = \frac{1}{\dot{x}} \int \rho d\alpha(x, \dot{x}).
\]
Setting this to zero, we have two types of solution

\[ f'(\alpha) = 0 \]
\[ \frac{d\alpha}{dt} = 0. \]

If spurious solutions could arise from motions with \( f' = 0 \), those motions would have to stay at the critical point, \( \alpha_0 \) say, of \( f \). But this means that \( \alpha = \alpha_0 \) remains constant. Therefore, the only way to introduce spurious solutions is if \( d\alpha/dt = 0 \) has solutions beyond the usual solutions. This may not be possible in one dimension. Finally, the inverse of the equation \( \alpha(x, t) = \alpha_0 \) may not exist at critical points, so the theorem must refer only to local equivalence of the solutions for inequivalent Lagrangians.

C. Inequivalent Lagrangians in higher dimensions

It is of interest to extend the results on inequivalent systems to higher dimension. Presumably, the theorems generalize in some way, but while one dimensional problems may be preferable “for simplicity”, this restricted case has many special properties that may not generalize. In any case, the method of proof of the Kobussen-Yan-Leubner theorem does not immediately generalize.

For 1-dim classical mechanics, there are only two independent constants of motion. The Kobussen-Yan-Leubner theorem, eq. (3), makes use of one or both to characterize the Lagrangian and, as noted above, one constant can completely determine the paths motion in 1-dim. The remaining constant is required only to specify the initial point of the motion. This leads to a simple conjecture for higher dimensions, namely, that the paths are in general determined by \( n \) of the \( 2n \) constants of motion. This is because \( n \) of the constants specify the initial position, while the remaining constants determine the paths.

We make these comments concrete with two examples. First, consider again the free particle in \( n \)-dim. The usual Hamiltonian is

\[ H = \frac{p_i^2}{2m} \]

and we immediately find that a complete solution is characterized by the initial components of the momentum, \( p_{0i} \) and the initial position, \( x_{0i} \). Clearly, knowledge of the momenta is
necessary and sufficient to determine a set of flows. If we consider inequivalent Lagrangians
\[ L = v \int^v \frac{f(\xi)}{\xi^2} d\xi = F(v) \]
where
\[ v = \sqrt{v^2} \]
then the momenta
\[ p_i = \frac{\partial L}{\partial v_i} = F' v_i \]
comprise a set of first integrals of the motion. Inverting for the velocity
\[ v^i = v^i(p_{i0}) \]
fixes the flow without fixing the initial point.

In general we will need at least this same set of relations, \( v^i = v^i(p_{i0}) \), to determine the flow, though the generic case will involve \( n \) relations depending on \( 2n \) constants:
\[ v^i = v^i(p_{i0}, x_0^i) \]
Notice that fewer relations does not determine the flow even for free motion in two dimensions. Thus, knowing only \( v_x = \frac{p_{0x}}{m} \) leaves the motion in the \( y \) direction fully arbitrary.

In an arbitrary number of dimensions, we find that expression for the energy in terms of the Lagrangian is still integrable as in the 1-dim case above, as long as \( v = \sqrt{v^2} \). If the Lagrangian does not depend explicitly on time, then energy is conserved. Then, letting \( \dot{\theta}^i = \dot{x}^i / v \), we can still write the Lagrangian as an integral over Hamiltonian:
\[ L(x, \dot{\theta}_v) = v \int^v \frac{H(x, \xi, \dot{\theta})}{\xi^2} d\xi + f(x, \dot{\theta}_v) \]
where \( f(x, \dot{\theta}_v) \) is now necessary in order for \( L \) to satisfy the Euler-Lagrange equations. The integral term of this expression satisfies one of the Euler-Lagrange equations. If we now define a new Lagrangian by replacing \( H \) by an arbitrary, time-independent constant of the motion, \( \alpha(x, v, \dot{\theta}) \),
\[ \tilde{L} = v \int^v \alpha(x, \xi, \dot{\theta}) d\xi + f(x, \dot{\theta}) \]
then the new Lagrangian, $\tilde{L}$, still satisfies the same Euler-Lagrange equation
\[
\dot{x}^i \left( \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{x}^i} - \frac{\partial \tilde{L}}{\partial x^i} \right) = 0.
\]

We conjecture that for a suitable choice of $f$, $\tilde{L}$ provides an inequivalent Lagrangian, thereby providing one of the $n$ relations required to specify the flow.

D. Inequivalent Hamiltonians

The question of inequivalent Hamiltonians is quite distinct from that of inequivalent Lagrangians, because the group of allowed transformations is much larger. Indeed, Hamilton-Jacobi theory shows that any Hamiltonian may be made trivial by a canonical transformation. This also means that any two Hamiltonians are related locally by a time-dependent canonical transformation. At least in this sense, all Hamiltonians with the same number of degrees of freedom are locally equivalent.

In light of this, the first question to be answered is the following. Since the properties of canonical transformations are defined by the demand that they change the action by no more than a total derivative (for example, in the derivation of the properties of generating functions), how can two Hamiltonians be locally equivalent while the corresponding Lagrangians are inequivalent? The answer is subtle. Canonical transformations are defined in such a way as to leave the Hilbert one-form
\[
L (p_i, \dot{x}^i, q^j, t) \, dt = p_i \frac{dx^i}{dt} \, dt - H (p_j, q^k) \, dt
\]
changed by no more than an exact form. But this $L$ is not quite the Lagrangian, since
\[
L = L (p_i, \dot{x}^i, q^j, t)
\]
while a true Lagrangian is a function of $x^i$ and $\dot{x}^i$ only. Since the Lagrangian formalism is invariant only under coordinate diffeomorphisms, $x^i = x^i (q^j, t)$, canonical transformations involving both $p_i$ and $x^i$ are not expected to preserve it.

Despite Hamilton-Jacobi theory, there are ways to define a notion of inequivalent Hamiltonians. First, if we restrict to time-independent canonical transformations, there exist distinct Hamiltonian systems related by diffeomorphisms, $H (q, \pi) = H [x (q, \pi), p (q, \pi)]$. This allows substantial variation in the functional form of the Hamiltonian and it may be
difficult to determine whether two Hamiltonians are related in this way. Second, we may define canonoid transformations, defined as preserving the canonical structure for one or more Hamiltonians. Presumably, these are related to symmetries of particular systems. Or, third, we may quantize the system and ask whether the quantum systems are equivalent.

These considerations point to a hierarchy of classifications of Hamiltonian systems, equivalent up to some set of transformations. It would be useful to know the exact set of transformations under which a given set of phase space paths is invariant. We know that the set is smaller than time-dependent transformations and larger than canonical transformations, and seems likely that the answer depends on the class of curves in some way. Since it is really the solution curves that define equivalence, it is clear that some systems will have more symmetry than others.

Even if we had a clear characterization of Hamiltonians for a given system, it is not clear that we know what the system is. For example, suppose a given system admits a time-independent canonical transformation taking the Hamiltonian to a constant. Such a system exists if we can find nontrivial solutions to the time-independent Hamilton-Jacobi equation

\[ \frac{m}{2} \mathbf{\nabla} S \cdot \mathbf{\nabla} S + V = 0 \]

and this is surely possible for some systems. But this means that one of the equivalent formulations of the problem describes straight-line motion. Clearly, finding the solution curves is not enough to describe such a system – we must also keep track of the sequence of transformations we used to trivialize that solution. Thus, specifying a classical system requires some statement about the correspondence between phase space coordinates and measurements in some physical system. Of course, there are many equivalent ways to set up such a correspondence, but at least one must be specified and the subsequent transformations tracked.

IV. CONSTANTS OF MOTION

Recent decades have seen interesting new techniques and revivals of known results for symmetries. Some of these have to do with the Kepler problem. The best-known rediscovery concerning the Kepler problem is that in addition to the energy, \( E \) and angular
momentum,
\[ E = \frac{1}{2} m \dot{x}^2 - \frac{\alpha}{r} \]
\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \] (5)
the Laplace-Runge-Lenz vector \([34], [35], [36], [37]\) is conserved. We define the Laplace-
Runge-Lenz vector by
\[ \mathbf{A} = \mathbf{p} \times \mathbf{L} - m \alpha \hat{\mathbf{r}}. \]
Geometrically, \(\mathbf{A}\) points in the direction of the periapsis, and may therefore be thought of
as giving specifying the orientation of the orbit within the orbital plane. Keplerian orbits
can be described completely in terms of six initial conditions, and since one of these is the
initial position on a given ellipse, only five remain among the energy, angular momentum
and Laplace-Runge-Lenz vector \([38]\). Two constraints – the orthogonality of \(\mathbf{A}\) and \(\mathbf{L}\), and a
relationship between the magnitudes \(A, L\) and \(E\) – give the correct count. Of course, these
three quantities are not the only set of constants we can choose. A number of fairly recent
authors \([2], [39], [40], [41]\) have identified a simpler conserved vector quantity, which (lacking
evidence for an earlier reference) we will call the Hamilton vector \([42]\). It is given by
\[ \mathbf{u} = \mathbf{v} - \frac{\alpha}{L} \dot{\phi} \]
and may be used together with the energy and angular momentum as a complete set of
constants. Apparently this vector was well-known in the 19th century, then dropped from
texts \([2]\). Its time constancy is a direct consequence of the force law since, for an arbitrary
central force \(f(r)\),
\[ \frac{d\mathbf{u}_i}{dt} = \frac{d\mathbf{v}_i}{dt} - \frac{\alpha d\dot{\phi}_i}{L} = -\frac{f(r) x_i}{mr} + \frac{\alpha}{mr^2 \dot{\phi}} \frac{d\dot{\phi} x_i}{dt} \]
\[ = \left[ \frac{\alpha}{r^2} - f(r) \right] \frac{x_i}{mr} \] (6)
and \(d\mathbf{u}_i/dt = 0\) precisely when \(f(r)\) is given by an inverse square law. Notice that it is the
balance between the radial dependences of angular momentum and the force that allow this
characterization. We give a derivation of the Hamilton vector below.

The Laplace-Runge-Lenz vector, the Hamilton vector and the angular momentum are
related by
\[ \mathbf{A} = m\mathbf{u} \times \mathbf{L} \]
and

\[ \mathbf{L} \times \mathbf{A} = m\mathbf{L} \times (\mathbf{u} \times \mathbf{L}) \]  
\[ = mL^2 \mathbf{u} \]  

where we have used the fact that \( \mathbf{u} \) lies in the plane of the orbit and is therefore perpendicular to \( \mathbf{L} \).

It might be of interest to study more general central force problems using time-dependent versions of the Laplace-Runge-Lenz and Hamiltonian vectors. While no longer conserved, they are more geometrical than the usual polar coordinates. This could result in some simplification. It would also be of interest to know if these vectors correspond to symmetries – perhaps they reflect symmetries of the corresponding phase space, or, since the bound state Kepler problem may be embedded with an \( SO(4) \) symmetry, perhaps they are part of that symmetry. It seems likely that the Kepler problem has an even larger symmetry – perhaps \( SO(4, 1) \) – since the open orbits have \( SO(3, 1) \) symmetry. This might be explored by writing the Kepler problem in terms of \( Spin(4, 1) \) conformal spinors.

Muñoz [2] shows that the Hamilton vector leads to an easy derivation of the equation of motion. Indeed, let the perihelion of the orbit occur at time \( t = 0 \) on the \( x \)-axis so that the velocity is given by

\[ \mathbf{v} = v_0 \hat{\phi} \]

Then \( \mathbf{u} = u \hat{\mathbf{j}} \), where the unit vector in the \( y \)-direction gives the initial direction of \( \hat{\phi} \). Dotting \( \mathbf{u} \) with \( \hat{\phi} \) we have

\[ \mathbf{u} \cdot \hat{\phi} = \mathbf{v} \cdot \hat{\phi} - \alpha/L \]

\[ u \cos \phi = r \dot{\phi} - \alpha/L \]

or replacing \( \dot{\phi} = L/(mr^2) \),

\[ \frac{1}{r} = \frac{mu}{L} \cos \varphi + \frac{am}{L^2} \]

\[ r = \frac{L^2/m\alpha}{1 + (Lu_0/\alpha) \cos \varphi} \]  

as usual.

Tjiang and Sutanto [1] describe a straightforward way to identify constants of the motion arising from the vanishing of

\[ \frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} \]  

(11)
based on a well-known theorem on the solution of differential equations. The theorem states that any equation of the form

\[ \sum_i P_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = R(x_1, \ldots, x_n, f) \]

has the general solution given by

\[ f = \Phi(u_1, \ldots, u_k) \]

where \( k \leq n \) and the \( u_i(x_1, \ldots, x_n, f) \) are solutions to

\[ \frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{df}{R} \]

Applied to eq. (11), the theorem implies that the functions \( u_i \) are all constants of the motion. Moreover, we can compute the possible constants of motion by solving the equations

\[ \frac{dx_1}{\frac{\partial H}{\partial p_1}} = \cdots = \frac{dx_n}{\frac{\partial H}{\partial p_n}} = \frac{dp_1}{\frac{\partial H}{\partial x_1}} = \cdots = \frac{dx_n}{\frac{\partial H}{\partial x_n}} = dt \]

To illustrate the method and at the same time derive the Hamilton vector as a constant of Keplerian motion, we apply the technique to the Kepler problem. First, the Hamiltonian is given, in any dimension \( n \geq 2 \), by

\[ H = \frac{p^2}{2m} - \frac{\alpha}{r} \]

so we must solve the equations

\[ \frac{m}{p_1} dx_1 = \cdots = \frac{m}{p_n} dx_n = - \frac{r^3}{\alpha x_1} dp_1 = \cdots = - \frac{r^3}{\alpha x_n} dp_n = dt \] (12)

First, for each \( i \), consider the equations of the form

\[ \frac{m}{p_i} dx_i = - \frac{r^3}{\alpha x_i} dp_i \]

\[ \frac{\alpha x_i}{r^3} dx_i + \frac{1}{m} p_i dp_i = 0. \]

Summing over \( i \) we have the constancy of the Hamiltonian:

\[ dH = d \left( \sum \frac{p_i^2}{2m} - \frac{\alpha}{r} \right) = 0 \]

Next, consider the equations among the \( dx_i \). For any pair of these \( i \neq j \) we have

\[ \frac{m}{p_i} dx_i = \frac{m}{p_j} dx_j \]
so that

\[ 0 = p_j dx_i - p_i dx_j \]  \hspace{1cm} (13)

\[ = d(p_j x_i - p_i x_j) - (dp_j x_i - dp_i x_j) \]  \hspace{1cm} (14)

We may replace the momentum differentials using the Tjiang-Sutanto equations, eqs. (12) to write

\[ dp_j x_i - dp_i x_j = -\frac{\alpha x_i m x_i}{r^3 p_1} dx_1 + \frac{\alpha x_i m x_j}{r^3 p_1} dx_1 = 0 \]

so we have conservation of all of the components

\[ M_{ij} = x_i p_j - x_j p_i \]

of the angular momentum. Of course, in 3-dimensions we may use the Levi-Civita tensor to write this as

\[ L_i = \frac{1}{2} \epsilon_{ijk} (x_i p_j - x_j p_i) = [\mathbf{x} \times \mathbf{p}]_i \]

In Sec. 2 we showed from the constancy of \( M_{ij} \) that the motion remains in a fixed plane for all time.

Finally, we study the additional constants of motion arising from equations of the form

\[ \frac{m}{p_1} dx_1 = -\frac{r^3}{\alpha x_i} dp_i \]

Since \( p_i = mdx_i/dt \), this may be written as

\[ dt + \frac{mr^3}{\alpha x_i} dv_i = 0 \]  \hspace{1cm} (15)

\[ \frac{\alpha x_i}{mr^3} dt + dv_i = 0 \]  \hspace{1cm} (16)

Now, using \( L = mr^2 \dot{\varphi} \) where \( L \) is the magnitude of the angular momentum, \( L = \sqrt{\sum M_{ij} M_{ij}} \), and \( \varphi \) is in the plane of the orbit, we have

\[ dv_1 + \frac{\alpha}{L r} d\varphi = 0 \]  \hspace{1cm} (17)

\[ d \left( v_1 + \frac{\alpha}{L} \sin \varphi \right) = 0 \]  \hspace{1cm} (18)

and similarly

\[ d \left( v_1 - \frac{\alpha}{L} \cos \varphi \right) = 0 \]
Adding these with unit vectors and noting that $\mathbf{v} = v_1 \hat{i} + v_2 \hat{j}$ comprises the entire velocity vector, establishes the constancy of the Hamilton vector:

$$\mathbf{u} = \mathbf{v} - \frac{\alpha}{L} \hat{\phi}$$

Thus, $\{H, M_{ij}, u_i\}$ is a complete set of constants of the motion for the $n$-dim Kepler problem. The solution, eq. (10), follows immediately.

V. HIGHER ORDER EQUATIONS OF MOTION

A. Euler-Lagrange and Hamiltonian systems of arbitrary order

There generalization of the Euler-Lagrange equation to systems for which the Lagrangian depends on higher than second derivatives of the position is immediate and well-known. If $L = L(x, \dot{x}, \ddot{x}, \ldots, x^{(n)}, t)$ the resulting variation leads to

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial x^{(k)}} = 0$$

This generalized Euler-Lagrange equation is generically of order $2n$. Such systems are, of course, allowed within even the Newtonian formulation. A simple electronic circuit and feedback mechanism can easily drive a motor in a way dependent upon rates of change of the acceleration. Moreover, there are systems of genuine physical and mathematical interest which require higher order differential equations for their description. After deriving a few general results for higher order systems, we look at one example in detail – Bohmian quantum mechanics.

Returning to generalised Euler-Lagrange systems, suppose $L$ is independent of time. Then

$$\frac{dL}{dt} = \sum_{k=0}^{n} x^{(k+1)} \frac{\partial L}{\partial x^{(k)}}$$
But

\[ x^{(k+1)} \frac{\partial L}{\partial x^{(k)}} = \frac{d}{dt} \left( x^{(k)} \frac{\partial L}{\partial x^{(k)}} \right) - x^{(k)} \frac{d}{dt} \frac{\partial L}{\partial v^{(k)}} \]  

(19)

\[ = \frac{d}{dt} \left( x^{(k)} \frac{\partial L}{\partial x^{(k)}} \right) - \frac{d}{dt} \left( x^{(k-1)} \frac{d}{dt} \frac{\partial L}{\partial x^{(k)}} \right) \]  

(20)

\[ + x^{(k-1)} \frac{d^2}{dt^2} \frac{\partial L}{\partial x^{(k)}} \]  

(21)

\[ + \cdots \]  

(22)

\[ = \sum_{m=0}^{k-1} (-m) \frac{d}{dt} \left( x^{(k-m)} \frac{d^m}{dt^m} \frac{\partial L}{\partial x^{(k)}} \right) - x^{(1)} \sum_{m=0}^{k-1} (-m) \frac{dk}{dt} \frac{d}{dt} \frac{\partial L}{\partial x^{(k)}} \]  

(23)

so

\[ \frac{dL}{dt} = \frac{d}{dt} \sum_{k=0}^{n} \sum_{m=0}^{k-1} (-m) \left( x^{(k-m)} \frac{d^m}{dt^m} \frac{\partial L}{\partial x^{(k)}} \right) + x^{(1)} \sum_{m=0}^{n} (-m) \frac{dk}{dt} \frac{d}{dt} \frac{\partial L}{\partial x^{(k)}} \]  

Using the equation of motion,

\[ \sum_{k=0}^{n} (-k) \frac{dk}{dt} \frac{d}{dt} \frac{\partial L}{\partial x^{(k)}} = 0 \]  

the final sum vanishes and we have the conserved energy

\[ E = \sum_{k=0}^{n} \sum_{m=0}^{k-1} (-m) \left( x^{(k-m)} \frac{d^m}{dt^m} \frac{\partial L}{\partial x^{(k)}} \right) - L \]  

The \( n = 3 \) case of this result is given in [43] and elsewhere.

Directly from the generalized equation we see immediately that if a coordinate \( x \) is cyclic, (i.e. \( \partial L/\partial x = 0 \)), we still get a conserved momentum,

\[ p = \sum_{m=0}^{n-1} (-m+1) \frac{d^m}{dt^m} \frac{\partial L}{\partial x^{(m+1)}} \]  

This follows from

\[ 0 = \sum_{k=0}^{n} (-k) \frac{dk}{dt} \frac{d}{dt} \frac{\partial L}{\partial x^{(k)}} \]  

(24)

\[ = \sum_{k=1}^{n} (-k) \frac{dk}{dt} \frac{d}{dt} \frac{\partial L}{\partial x^{(k)}} \]  

(25)

\[ = \frac{d}{dt} \sum_{m=0}^{n-1} (-m+1) \frac{d^m}{dt^m} \frac{\partial L}{\partial x^{(m+1)}} \]  

(26)

\[ = \frac{dp}{dt} \]  

(27)
With higher order Lagrangians, there are additional possibilities. Suppose the lowest \( m < n \) partials of \( L \) vanish:

\[
\frac{\partial L}{\partial x^{(k)}} = 0, \quad k = 0, 1, \ldots, m - 1
\]

Then the sum in the field equation starts at \( m \), and extracting \( m \) time derivatives

\[
0 = \frac{d^m}{dt^m} \left( \sum_{k=m}^n (-)^k \frac{d^{k-m}}{d t^{k-m}} \frac{\partial L}{\partial x^{(k)}} \right)
\]

so that the momentum

\[
p_m^m = \frac{d^{m-1}}{dt^{m-1}} \left( \sum_{k=m}^n (-)^k \frac{d^{k-m}}{d t^{k-m}} \frac{\partial L}{\partial x^{(k)}} \right)
\]

is conserved. Integrating \( m - 1 \) more times,

\[
\sum_{k=0}^{m-1} \frac{1}{k!} p_m^k = \sum_{k=m}^n (-)^k \frac{d^{k-m}}{d t^{k-m}} \frac{\partial L}{\partial x^{(k)}}
\]

where we now have \( m \) constants, \( p_m^k \).

Higher order systems also permit a Hamiltonian formulation. Let \( n = 2m - 1 \) be any odd integer. We divide the time derivatives of \( x \) into even and odd order, and replace the odd time derivatives with conjugate momenta. For even \( n \), Hamilton’s equations will be supplemented by one additional Euler-Lagrange equation. Thus, let

\[
y_k = \frac{d^{2k} x}{dt^{2k}} = x^{(2k)}
\]
\[
p_k = \frac{\partial L}{\partial x^{(2k+1)}}
\]

for \( k = 0, 1, \ldots, m - 1 \). The Legendre transformation is employed in the usual way to express the Hamiltonian in terms of \( y_k \) and the \( p_k \),

\[
H (y_k, p_k) = \sum_{k=1}^n p_k \frac{dy_k}{dt} - L
\]

As usual, \( H \) is independent of the odd accelerations, \( x^{(2k+1)} \), since

\[
\frac{\partial H}{\partial x^{(2k+1)}} = \frac{\partial H}{\partial \dot{y}} = p_k - \frac{\partial L}{\partial x^{(2k+1)}} = 0
\]

The variation of the Lagrangian with respect to \( y_k \) and \( p_k \) is straightforward:

\[
0 = \delta S = \int \sum_{k=1}^n p_k \delta \dot{y} + \sum_{k=1}^n \dot{y} \delta p_k - \sum_{k=1}^n \frac{\partial H}{\partial y_k} \delta y_k - \sum_{k=1}^n \frac{\partial H}{\partial p_k} \delta p_k
\]
Integrating by parts we find \(2m = n + 1\) first order equations:

\[
\begin{align*}
\frac{dy_k}{dt} &= \frac{\partial H}{\partial p_k} \\
\frac{dp_k}{dt} &= -\frac{\partial H}{\partial y_k}
\end{align*}
\]

These are the generalized Hamilton’s equations.

Naturally, higher order equations require more initial data than we usually have to specify to determine the motion of a classical system, so their occurrence is somewhat rare. But ultimately, any restriction to second order equations in classical physics is phenomenological, depending principally on the success of second order models for fitting measurements. If we take quantum physics into account, higher order equations in field theory may introduce ghosts or other undesirable features.

Nonetheless, there are situations where higher order equations are justified. We briefly discuss one of these below, Bohmian quantum mechanics. The KdV equation provides an example of an integrable system, having an infinity of independent constants of motion. It is presented in an Appendix, as is the approach to chaos.

**B. Bohmian quantum mechanics**

A central theme of the Bohmian approach to quantum mechanics is to give it a form which may be interpreted classically (44, 45, 46). The first step is to replace the complex wave function by pair of real valued functions. This is accomplished as follows. Let

\[
\psi = Ae^{\frac{i}{\hbar}S}
\]

where \(A\) and \(S\) are real valued functions. Substituting into the Schrödinger equation,

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t}
\]

and separating the real and imaginary parts give two equations:

\[
\begin{align*}
\frac{1}{2m} \nabla S \cdot \nabla S + V + \frac{\partial S}{\partial t} &= \frac{\hbar^2}{2mA} \nabla^2 A \\
\frac{\partial A}{\partial t} + \frac{1}{m} \nabla S \cdot \nabla A + \frac{1}{2m} A \nabla^2 S &= 0
\end{align*}
\]

We refer to this system as the Bohm equations. The first equation is the Hamilton-Jacobi equation with an additional term. This equation is frequently used to show how the classical
limit emerges from quantum mechanics when $\hbar \to 0$, but here we want to exactly replicate the content of the quantum theory while maintaining a classical viewpoint. Multiplying the second equation, eq.(29), by $A$, it may be rewritten as

$$\frac{\partial A^2}{\partial t} + \frac{1}{m} \vec{\nabla} \cdot \left( A^2 \vec{\nabla} S \right) = 0$$

(30)

Defining

$$\rho = A^2, \quad \mathbf{v} = \frac{1}{m} \vec{\nabla} S$$

the equation becomes the continuity equation for a current $\mathbf{J} = \rho \mathbf{v}$,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \mathbf{J} = 0$$

This is the usual conserved probability current of quantum mechanics, cast in classical guise.

Alternatively, we may view eq.(28) as a wave equation for $A$:

$$-\frac{\hbar^2}{2m} \nabla^2 A + \left( \frac{1}{2m} \vec{\nabla} S \cdot \vec{\nabla} S + \frac{\partial S}{\partial t} \right) A = 0$$

This is just a diffusion equation with a horrible potential.

In the one dimensional, stationary case we can reduce the Bohm equations to a single, higher order differential equation. In 1-dim, with the stationary conditions

$$\frac{\partial S}{\partial t} = -E, \quad \frac{\partial A}{\partial t} = 0$$

the continuity equation, eq.(30) is simply

$$\frac{1}{m} \left( A^2 S'' \right)' = 0$$

which integrates immediately to give

$$A^2 = \frac{a}{S'}$$

Now substituting this result into eq.(28) together with $\partial S/\partial t = -E$, results in a third-order, nonlinear equation

$$\frac{\hbar^2}{2m} \left[ \frac{S'''}{2S'} - \frac{3}{4} \left( \frac{S''}{S'} \right)^2 \right] + \frac{1}{2m} (S')^2 + V - E = 0$$

(31)

The interesting point is that this equation is rigorously equivalent to the 1-dim stationary state Schrödinger equation. The downside is that we have handled only the 1-dim, stationary case.
This equation lacks the simplicity of the Schrödinger equation. For example, suppose we solve
\[-\frac{\hbar^2}{2m} \left[ \frac{3}{4} \left( \frac{S''}{S'} \right)^2 - \frac{S'''}{2S'} \right] + \frac{1}{2m} (S')^2 + V - E = 0\]
to find a solution $S(x, E)$. Unfortunately, the nonlinearity means that we cannot take a superposition of stationary states to get a general time-dependent solution.

Consider the time-dependent case further. We can at least find a time dependent solution to the (linear!) second equation:
\[\frac{\partial A}{\partial t} + \frac{1}{m} S' A' + \frac{1}{2m} A S'' = 0\]
Direct integration shows that
\[A(x, t) = \sqrt{\frac{2\pi}{S'}} G \left( t - m \int dx \frac{S'}{S'} \right)\]
is a solution for any function $G$. The amplitude therefore propagates with fixed spatial form. It would be of interest to know if this type of solution for $A$ generalizes to higher dimensions.

Now we may substitute this into the first equation,
\[\frac{\partial S}{\partial t} + \frac{1}{2m} (S')^2 + V = \frac{\hbar^2}{2m} \frac{1}{\sqrt{\frac{2\pi}{S'}} G \left( t - m \int dx \frac{S'}{S'} \right)} \left[ \sqrt{\frac{2\pi}{S'}} G \left( t - m \int dx \frac{S'}{S'} \right) \right]''\]
Perhaps an appropriate separation can solve this equation as well.

1. **Bohmian Lagrangian**

The 1-dim, stationary state Bohm equation also follows from the variation of the usual Schrödinger action, reduced by the solution for $A$. Starting from the Schrödinger action,
\[S = \int \frac{\hbar^2}{2m} (\psi^*)' \psi' + V \psi^* \psi + \frac{i}{2} \hbar \left( \frac{\partial \psi^*}{\partial t} \psi - \frac{\partial \psi}{\partial t} \psi^* \right)\]
we substitute the polar expression $\psi = \frac{1}{\sqrt{S'}} e^{i[S(x) - Ex]}$ for the wave function. the action becomes
\[S = \int \frac{\hbar^2}{2m} \frac{(S'')^2}{4 (S')^3} + \frac{1}{2m} S' + \frac{V - E}{S'}\]
Varying, the equation of motion is found to be
\[0 = \frac{d}{dx} \left[ -\frac{\hbar^2}{2m} \frac{2S'''}{4 (S')^3} + \frac{\hbar^2}{2m} \frac{3S''S'''}{2 (S')^4} - \frac{\hbar^2}{2m} \frac{3 (S'')^2}{4 (S')^4} + \frac{1}{2m} - \frac{V - E}{(S')^2} \right]\]
The term in brackets is a constant. Choosing the constant to be $1/m$ correctly reproduces the stationary Bohm equation:

$$-\frac{\hbar^2}{4m} \frac{S'''}{S'} + \frac{3\hbar^2}{8m} \left( \frac{S''}{S'} \right)^2 - \frac{(S')^2}{2m} - (V - E) = 0$$

We could have avoided the need to pick this integration constant by taking the action to be

$$\tilde{S} = \int \frac{\hbar^2}{2m} \frac{(S'')^2}{4(S')^3} - \frac{1}{2m} S' + \frac{V - E}{S'}$$

This form differs from the previous one only by the integral of a total derivative.

Writing the equation of motion in the form,

$$0 = -\frac{\hbar^2}{4m} S' S''' + \frac{3\hbar^2}{8m} (S'')^2 - \frac{1}{2m} (S')^4 - (V - E) (S')^2$$

it has been observed ([47], [48]) that there are solutions with $S' = 0$. It is not difficult to see that these should not be considered to be the physical solutions. First, we arrived at eq. (32) by the substitution

$$A = 1/\sqrt{S'}$$

which is singular for constant $S$. Since wave functions with divergent amplitude are not considered physical, such points require a closer examination at the very least. Second, we can see that these are clearly not the physical solutions because they are independent of the potential $V$.

To eliminate the spurious solutions, while indicating a method of solution, we make a simple change of variable. Notice that the nonlinear terms in the Bohm equation, eq. (31), may be rewritten as

$$-2\sqrt{S'} \left( \frac{1}{\sqrt{S'}} \right)'' = \frac{S''}{S'} - \frac{3}{2} \left( \frac{S''}{S'} \right)^2$$

Substituting, the equation of motion becomes

$$-\frac{\hbar^2}{2m} \left( \frac{1}{\sqrt{S'}} \right)'' + \frac{(S')^{3/2}}{2m} + \frac{V - E}{\sqrt{S'}} = 0$$

Now let $r = 1/\sqrt{S'}$. Then

$$-\frac{\hbar^2}{2m} r'' + \frac{1}{2mr^3} + (V - E) r = 0$$

Rearranging,

$$mr'' - \frac{m^2/\hbar^2}{mr^3} = \frac{2m^2}{\hbar^2} (V - E) r$$
the form suggest that we think of the independent variable \( x \) as time. Replacing \( x \to t \), we interpret \( r(t) \) as a radial coordinate and \( V \) as a time-dependent potential \( V(t) \). Then we have

\[
m\ddot{r} - \frac{m^2/\hbar^2}{mr^3} = \frac{2m^2}{\hbar^2} [V(t) - E]r
\]

and we recognize the isotropic 2-dim harmonic oscillator with angular momentum \( m/\hbar \) with a time-dependent spring strength,

\[
k(t) = -\frac{2m^2}{\hbar^2} [V(t) - E]
\]

This clearly has bound state solutions for suitable energies and potentials. Notice that solutions with \( S' = 0 \) correspond to infinite radial coordinate, so bound state solutions automatically avoid this spurious case.

It is suggestive that the introduction of a time-dependent spring constant into the isotropic oscillator can lead to parametric resonance \[49\]. It would be of interest to find the relationship of such parametric resonances to the eigenmodes of the corresponding quantum problem.

VI. GAUGING NEWTON’S LAW

One surprising new result in classical mechanics is that both the Lagrangian and Hamiltonian formulations of Newton’s laws may be derived as gauge theories of Newton’s second law \([55, 56]\). To see how this comes about, and to understand the symmetries involved, we digress a moment to consider the essential elements of a physical theory. In particular, we want to distinguish two features: dynamical laws and measurement theory.

The distinction between these is easy to see. For example, in quantum mechanics the dynamical law is the Schrödinger equation

\[
\hat{H}\psi - i\hbar \frac{\partial \psi}{\partial t}
\]

which governs the time evolution of the wave function, \( \psi \). The measurement theory is what establishes the correspondence between calculations and measurable numbers. One of the chief elements of quantum measurement theory is therefore the Hermitian inner product on Hilbert space:

\[
\langle \psi | \psi \rangle = \int_V \psi^* \psi d^3x.
\]
As a second example, consider Newtonian mechanics. The dynamical variable for a particle is the position vector, \( \mathbf{x} \), and its motion is governed by the second law:

\[
\mathbf{F} = m \frac{d^2 \mathbf{x}}{dt^2}
\]

while the inner product allows us to extract measurable magnitudes

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}
\]

We are interested in the differences in the symmetries of dynamical laws and measurement. Generally, the differential or other equation governing dynamical evolution is invariant under some global symmetry. In contrast to this, a metric, an invariant product or some other real-valued mapping to measurable quantities is often invariant under a group of diffeomorphisms. Whatever the symmetries, it often occurs that the symmetry of dynamical evolution and the symmetry of the measurement theory are different. Which is the symmetry of the system?

For Newtonian measurements, the inner products allow local transformations and therefore have the larger symmetry. It makes sense to try to extend the symmetry of the dynamical law to agree with that of the measurement theory. Fortunately, there are standard techniques for accomplishing this extension – the methods of gauge theory.

Gauging takes a global symmetry, that is, a symmetry that is independent of position and time, and extends it to a local symmetry, i.e., one that may be different at different positions. We systematically extend to a local symmetry by introducing a connection: a one-form field valued in the Lie algebra of the symmetry we wish to gauge. Added to the usual partial derivative, the connection subtracts back out the extra terms arising from differentiating the local symmetry. The most familiar example is general relativity, in which the Christoffel connection, \( \Gamma^\alpha_{\mu\nu} \) added to partial derivative makes the derivative covariant with respect to general coordinate transformations. The \( U(1) \) gauge theory of electromagnetism is also familiar. In this case, the vector potential provides the connection.

What are the symmetries of the Newtonian dynamical and measurement theories? There is more than one answer. The dynamical law is invariant under the Galilean group, \( G \), consisting of rotations, translations, Galilean boosts and time translations. It is possible to extend the rotations to general linear transformations and still leave the second law invariant. For the measurement theory, the Euclidean line element is invariant under the set of 3-dimensional rotations and translations, \( ISO(3) \). This is called the Euclidean group. If
we regard Euclidean 3-space as a manifold instead of a vector space, these transformations may be local. Furthermore, recognizing that we only actually measure dimensionless ratios (for example, the ratio of the height of a tree to the length of a meter stick), we can require invariance of ratios of line elements. This gives the conformal group, $SO(4, 1)$.

We will consider two of the possible gaugings of Newton’s second law – Euclidean and conformal.

A. Euclidean gauge theory of Newton’s second law

The Euclidean gauging of Newton’s second law leads to Lagrangian mechanics. This is not a particularly surprising result. However, while it is well-known that Lagrangian mechanics provides a form of Newton’s second law valid for “generalized coordinates,” the construction by gauging has advantages: (1) it arrives at the Lagrangian formulation in a systematic way, (2) gauging displays explicitly the meaning of generalized coordinates, and (3) it illustrates the general techniques used for the gauging of conformal symmetry below. As we shall see, the results of conformal gauging in the next subsection are unexpected.

Proceeding, first recall that the transformations of the Euclidean group $ISO(3)$ include three rotations and three translations, and the Lie algebra has corresponding generators. Gauging therefore gives us two sets of 1-form gauge fields:

1. Three translational gauge fields, comprising the dreibein, $e^i$.

2. Three rotational gauge fields, the $SO(3)$ spin connection, $\omega^i_j$, antisymmetric under the interchange of indices.

The gauging proceeds just as when we gauge the Poincaré group to develop Riemannian geometry ([57], [58]). In Poincaré gauging, the vierbein, $e^a$, is identified with an orthonormal frame field on a 4-dim Riemannian manifold and the spin connection $\omega^a_b$ permits the use of local Lorentz transformations. For the present Euclidean case, the dreibein, $e^i$, is identified with an orthonormal basis of a 3-dim manifold and the $SO(3)$ spin connection, $\omega^i_j$, gives local rotational symmetry. The pair $(e^i, \omega^i_j)$ is equivalent to the metric and general coordinate connection, $(g_{mn}, \Gamma^m_{rs})$. 
The connection forms must satisfy the Lie algebra relations of the symmetry group, as encoded in the Maurer-Cartan structure equations:

\[
\begin{align*}
\mathbf{d} \omega^i_j &= \omega^k_j \omega^i_k \\
\mathbf{d} e^i &= e^j \omega^i_j
\end{align*}
\]

The solution of these is simple since we do not include curvature. The first equation is solved by the pure gauge form of the connection,

\[
\omega^i_k = (\mathbf{d} \Lambda^i_j) \left[ \Lambda^{-1} \right]^j_k
\]

where \( \Lambda^i_j(x) \) is a local rotation matrix. This means that there exists a choice of frames (say, \( \Lambda^i_j = \text{constant} \)) in which the spin connection is zero. Choosing this frame, the equation for the dreibein is satisfied by setting

\[
e^i = \mathbf{d} x^i
\]

From this we see that the equations describe Euclidean 3-space. Using the spin connection we can define a derivative operator which is covariant with respect to local rotations. If we cast the same equations in terms of a coordinate basis using the metric and Christoffel connection, \((g_{mn}, \Gamma^m_{rs})\), the derivative is covariant with respect to general coordinate changes, or diffeomorphisms.

We may find the new dynamical law using a variational principle. Using the coordinate metric,

\[
g_{mn} = e^i_m e^j_n \delta_{ij}
\]

we choose the squared norm of the velocity vector, plus a function of the coordinates to provide a source for the motion:

\[
S = \int [g_{mn} v^m v^n + \phi(x^m)] dt
\]

Because we have local symmetry, we can write the same thing in any coordinates. Notice that there is always some arbitrariness in the gauging procedure at this point. There are two properties we demand of this variational principle. First, it must be invariant under the local symmetry group. Second, we require the restriction of the new dynamical law to the original symmetry to reproduce the original law. The action \( S \) above satisfies these requirements.
Varying $S$, we find the new form of Newton’s law,

$$g_{mn} \frac{Dv^m}{dt} = \frac{\partial \phi}{\partial x^m}$$

where the covariant derivative of $v^m$ transforms as a vector under local rotations. When $\phi = 0$, this is the geodesic equation. Since the space is Euclidean, the geodesics are straight lines. The class of straight lines

$$x^\alpha = x_0^\alpha + v_0^\alpha t$$

is equivalent to the class of Newtonian inertial reference frames.

Writing $V = -\frac{m}{2} \phi$ for a potential $V$, we see that forces produce deviations from geodesic motion. This is the Lagrangian formulation of mechanics. Note that we get the same equation of motion if we substitute the Lagrangian in the form

$$L = g_{mn} v^m v^n + \phi(x^m)$$

into the usual Euler-Lagrange equation. The general coordinate invariance ("use of generalized coordinates") is, of course, one of the main reasons for the use of Lagrangian methods. The present approach, while principally intended to pave the way for the conformal gauging below, does have the advantage of systematically showing that the class of generalized coordinates is just the diffeomorphism group. In the usual formulation, this conclusion follows from the coordinate invariance of the action.

**B. Conformal gauge theory of Newton’s second law**

We now repeat the gauging process, but this time use the full conformal symmetry. The conformal group (for compactified 3-dim Euclidean space) contains ten transformations:

1. 3 rotations
2. 3 translations
3. 1 dilatation
4. 3 special conformal transformations
The first two sets of transformations reproduce the Euclidean group. Dilatations just rescale all lengths by a factor, while special conformal transformations are translations in inverse coordinates.

These global transformations preserve the Euclidean line element up to an overall multiple. As it stands, Newton’s second law is not invariant under even the \textit{global} form of these transformations – the special conformal transformations do not leave the law unchanged because they do not act linearly on Euclidean 3-space. This is easy to fix: we introduce a very limited covariant derivative with a connection specific to global special conformal transformations. It is unusual to require a connection in a dynamical law before gauging, but nothing forbids it and it gives us an equation with the symmetry we wish to gauge.

Now consider Newton’s law, modified just enough to let us perform all 10 global conformal transformations. Make those 10 global transformations local. There is more than one way to do this, but so far only one appears to be interesting – the \textit{biconformal} gauging described below.

There will now be ten gauge fields:

1. The dreibein, $e^i$
2. The (antisymmetric) $SO(3)$ spin connection, $\omega^i_j$
3. The Weyl vector, $W$.
4. The co-dreibein, $f_i$, from special conformal transformations

The local rotations, gauged by the spin connection, are as expected and we add local dilatations gauged by the Weyl vector. These allow general coordinate invariance and scale invariance. Employing the biconformal technique, we interpret $(e^i, f_i)$ as an orthonormal frame field of a \textit{six} dimensional space.

These gauge fields must satisfy the Maurer-Cartan structure equations:

\[
\begin{align*}
d\omega^i_j &= \omega^k_j \omega^i_k + e^i f_j - e_j f^i \\
d e^i &= e^j \omega^i_j + W e^i \\
d f_i &= \omega^j_i f_j + f_i W \\
d W &= 2 e^i f_i
\end{align*}
\]
This is just the conformal Lie algebra in a dual basis. Once again the equations are easily solved. The solution reveals a symplectic form,

\[ \theta = e^k f_k \]

\[ d\theta = 0 \]

The six dimensional space therefore has a similar structure to a one particle phase space. The units of the coordinates of this 6-dim space are not all the same. Three are correct for position \((x^i, \text{length})\) while the remaining three are geometric units for momentum \((y_i, 1/\text{length})\). Note that the conversion of units of momentum to units of inverse length may be accomplished using any conventional dimensional standards, e.g., meters, seconds and kilograms.

It also follows from the solution that the Weyl vector is given by

\[ W = -y_i dx^i \]

To find the new dynamical law we again write an action. Since the geometry is like phase space, the paths will not be anything like geodesics, so path length will not work. Instead, we have a new feature – the Weyl vector – that comes from the dilatations. We will base our dynamical law on the geometric interpretation of this vector field. We digress briefly to explore its properties.

It follows from the nature of conformal geometry that the integral of the Weyl vector along any path gives the relative physical size change along that path:

\[ l = l_0 \exp \int W_i v^i dt = l_0 \exp \int W \]

This means that magnitudes are not preserved – initially identical rods transported along different curves might be different sizes when they are returned together and compared. This possibility is the price we pay for the freedom to make local scale transformations, just as in a Riemannian geometry vectors may rotate even under “parallel” transport. We will return to this point below.

We take the action to be the integral of the Weyl vector. Then the physical paths will be paths of extremal size change. Notice that, while the exponential above is gauge dependent, its variation is not. Indeed, it is worth noting that the gauge freedom of the Weyl vector agrees exactly with the freedom to add a total derivative to a Lagrangian.
Once again we add a function to provide a source,

\[ S = \int (\mathbf{W} \cdot \mathbf{v} + \phi) \, dt \]

It is interesting that such a function is provided automatically in the relativistic version of this gauging, as the time component of the Weyl vector.

Now vary the action. There are six first-order equations:

\[ \frac{dx^i}{dt} = \frac{\partial \phi}{\partial y_i} \]
\[ \frac{dy_i}{dt} = -\frac{\partial \phi}{\partial x^i} \]

If we identify \( \phi \) with the Hamiltonian, these are Hamilton’s equations. Therefore, the gauge theory of Newton’s second law with respect to the conformal group is Hamiltonian mechanics.

There are a couple of points to be clarified. First, the multiparticle case works even though a single Weyl vector must account for the Hamiltonian and momentum of each particle as long as we assume that two particles never occupy exactly the same space. This is consistent with the usual requirements of Newtonian mechanics, by which matter is impenetrable.

Second, the extremal value of the integral of the Weyl vector is zero. Thus, no measurable size change occurs for classical motion, even though the Weyl vector does not have vanishing curl. The classical paths are precisely the ones along which no physical dilatation is ever measured.

It is also interesting to note that there is a 6-dim metric, of an unexpected form that is consistent with collisions. It follows from the solution to the structure equations that the line element is of the form

\[ ds^2 = dx^idx^i + dx^idy_i \]

Therefore, if we assume that the distance \( ds \) between two particles must vanish (or nearly so) in order for two particles to collide, we see we must have \( dx^i = 0 \), regardless of their separation in momentum, \( dy_i \). This would not be the case if we had simply imposed a Euclidean metric on the space.

These results provide a satisfying unification of classical mechanics. In addition, the relativistic version of biconformal gauging also turns out to be interesting. We have shown \((52, 60)\) that the method provides the best way to understand conformally invariant gravity. The results are consistent with general relativity, and improve previous conformal gravity
theories. The fact that we can satisfactorily gauge classical mechanics – and get something new – gives us a better understanding of, and more confidence in, the relativistic theory.

There is also a suggestion of something deeper. Notice that quantum mechanics requires both position and momentum variables to make sense, while biconformal gauging of Newton’s second law gives us a space which automatically has both sets of variables. Is it possible that quantum physics takes on a particularly simple form in biconformal space? Anderson and Wheeler claim it does, deriving a path integral formulation of quantum mechanics directly from a biconformal measurement theory \[56\]. It becomes possible to claim that the physical manifold is really a six- (or, relativistically, an eight-) dimensional place, in which quantum mechanics is a natural description of phenomena.

This interpretation of biconformal space works correctly. In particular, when we use the covering group of the conformal group, the Weyl vector is necessarily complex. The presence of an “\(i\)” in the Weyl vector makes an initially real, probabilistic evolution law into a unitary evolution. In addition, the requirement of the scale-invariant theory for taking ratios of lengths to produce a meaningful measurement leads directly to the use of the product of probability amplitudes in computing physically measurable probabilities. Naturally, the proportionality between the inverse-length \(y_i\)-coordinates and momenta is taken to be

\[
\hbar y_i = p_i
\]

Note that this factor drops out of Hamilton’s equations, making Planck’s constant classically unmeasurable.

Thus, we have another way to think of quantum phenomena in a classical context. In this formulation, however, we have the added advantage of a direct connection to general relativity. It becomes possible to ask questions about quantum measurement of curved spacetimes in a classical context.

VII. SPIN, STATISTICS, AND PSEUDOMECHANICS IN CLASSICAL PHYSICS

A. Spin

Now that we have a gauge theory of mechanics, we can ask further about the representation of the gauge symmetry. A representation of a group is the vector space on which
the group acts. The largest class of objects on which our symmetry acts will be the class
determining the covering group. This achieves the fullest realization of our symmetry. For
example, while the Euclidean group \( ISO(3) \) leads us to the usual formulation of Lagrangian
mechanics, we can ask if we might not achieve something new by gauging the covering group,
\( ISpin(3) \cong ISU(2) \). This extension, which places spinors in the context of classical physics,
depends only on symmetry, and therefore is completely independent of quantization.

There are numerous advantages to the spinorial extension of classical physics. After
Cartan’s discovery of spinors as linear representations of orthogonal groups in 1913 ([61],[62])
and Dirac’s use of spinors in the Dirac equation ([63],[64]), the use of spinors for other areas of
relativistic physics was pioneered by Penrose ([65],[66]). Penrose developed spinor notation
for general relativity that is one of the most powerful tools of the field. For example, the
use of spinors greatly simplifies Petrov’s classification of spacetimes (compare Petrov [67]
and Penrose [65],[68]), and tremendously shortens the proof of the positive mass theorem
(compare Schoen and Yau ([69],[70],[71]) and Witten [72]). Penrose also introduced the idea
and techniques of twistor spaces. While Dirac spinors are representations of the Lorentz
symmetry of Minkowski space, twistors are the spinors associated with larger conformal
symmetry of compactified Minkowski space. Their overlap with string theory as twistor
strings is an extremely active area of current research in quantum field theory (see [73] and
references thereto). In nonrelativistic classical physics, the use of Clifford algebras (which,
though they do not provide a spinor representation in themselves, underlie the definition
of the spin groups) has been advocated by Hestenes in the “geometric algebra” program [9].

It is straightforward to include spinors in a classical theory. We provide a simple example.
For the rotation subgroup of the Euclidean group, we can let the group act on complex
vectors, \( \chi^a, a = 1, 2 \). The resulting form of the group is \( SU(2) \). In this representation,
an ordinary 3-vector such as the position vector \( x^i \) is written as a traceless Hermitian matrix,

\[
X = x^i \sigma_i
\]

\[
[X]^{ab} = x^i [\sigma_i]^{ab}
\]

where \( \sigma_i \) are the Pauli matrices. It is easy to write the usual Lagrangian in terms of \( X \):

\[
L = \frac{m}{4} tr \left( \dot{X} \dot{X} \right) - V(X)
\]

where \( V \) is any scalar-valued function of \( X \). However, we now have the additional complex
2-vectors, $\chi^a$, available. Consider a Dirac-type kinetic term
\[ \lambda \chi^a (i\dot{\chi}^a - \mu \chi^a) \]
and potential
\[ V(\chi^a) = \lambda \bar{\chi}^a B^i \sigma_{iab} \chi^b + \ldots \]
Notice there is no necessity to introduce fermions and the concomitant anticommutation relations – we regard these spinors as commuting variables. A simple action therefore takes the form
\[ S = \int dt \left[ \frac{m}{4} tr(\dot{X} \dot{X}) + \bar{\chi}^a (i\dot{\chi}^a - \mu \chi^a) - V(X) - \lambda \bar{\chi}^a B^i \sigma_{iab} \chi^b \right] \]
The equations of motion are then
\[ m\ddot{x}^i = -\sigma^{iab} \frac{\partial V}{\partial X_{ab}} \]
\[ i\dot{\chi}^a = -i\mu \chi^a - i\lambda B^i \sigma_{iab} \chi^b \]

together with the complex conjugate of the second. The first reproduces the usual equation of motion for the position vector. Assuming a constant vector $B^i$, we can easily solve the second. Setting $\chi = \psi e^{-i\mu t}$, $\psi$ must satisfy
\[ \dot{\psi} = -i\lambda B^i \sigma^{iab} \psi_b \]
This describes steady rotation of the spinor,
\[ \psi = e^{-i\lambda B^i} \psi_0 \]
The important thing to note here is that, while the spinors $\psi$ rotate with a single factor of $e^{i\omega \sigma}$, a vector such as $X$ rotates as a matrix and therefore requires two factors of the rotation
\[ X' = e^{-i\omega^i} X e^{i\omega^i} \]
This illustrates the 2 : 1 ratio of rotation angle characteristic of spin 1/2. The new degrees of freedom therefore describe classical spin and we see that spin is best thought of as a result of the symmetries of classical physics, rather than as a necessarily quantum phenomenon. Similar results using the covering group of the Lorentz group introduce Dirac spinors naturally into relativity theory. Indeed, as noted above, 2-component spinor notation is a powerful tool in general relativity, where it makes such results as the Petrov classification or the positivity of mass transparent.
B. Statistics and pseudomechanics

The use of spinors brings immediately to mind the exclusion principle and the spin-statistics theorem. We stressed that spin and statistics are independent. Moreover, spin, as described above, follows from the use of the covering group of any given orthogonal group and is therefore classical. For statistics, on the other hand, the situation is not so simple. In quantum mechanics, the difference between Bose-Einstein and Fermi-Dirac statistics is a consequence of the combination of anticommuting variables with the use of discrete states. In classical physics we do not have discrete states. However, nothing prevents us from introducing anticommuting variables. In its newest form, the resulting area of study is called pseudomechanics.

The use of anticommuting, or Grassmann variables in classical physics actually has an even longer history than spin. The oldest and most ready example is the use of the wedge product for differential forms

\[ dx \wedge dy = -dy \wedge dx \]

This gives a grading of \((-)^p\) to all \(p\)-forms. Thus, if \(\omega\) is a \(p\)-form and \(\eta\) a \(q\)-form,

\[ \omega = \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \]
\[ \eta = \eta_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \]

Then their (wedge) product is even or odd depending on whether \(pq\) is even or odd:

\[ \omega \wedge \eta = (-)^{pq} \eta \wedge \omega \]

Nonetheless, \(p\)-forms rotate as covariant, rank-\(p\) tensors under \(SO(3)\) (or \(SO(n)\)), in violation of the familiar spin-statistics theorem. Under \(SU(2)\) they rotate as covariant, rank-2\(p\) tensors, not as spinors.

Another appearance of anticommuting variables in classical mechanics stems from the insights of supersymmetric field theory. Before supersymmetry, continuous symmetries in classical systems were characterized by Lie algebras, with each element of the Lie algebra generating a symmetry transformation. The Lie algebra is a vector space characterized by a closed commutator product and the Jacobi identity. Supersymmetries are extensions of the normal Lie symmetries of physical systems to include symmetry generators (Grassmann variables) that anticommute. Like the grading of differential forms, all transformations of
the graded Lie algebra are assigned a grading, 0 or 1, that determines whether a commutator or commutator is appropriate, according to

\[ [T_p, T_q] \equiv T_p T_q - (-)^{pq} T_q T_p \]

where \( p, q \in \{0, 1\} \). Thus, two transformations which both have grading 1 have anticommutation relations with one another, while all other combinations satisfy commutation relations.

Again, there is nothing intrinsically “quantum” about such generalized symmetries, so we can consider classical supersymmetric field theories and even supersymmetrized classical mechanics. Since anticommuting fields correspond to fermions in quantum mechanics, we may continue to call variables fermionic when used classically, even though their statistical properties may not be Fermi-Dirac. Perhaps more importantly, we arrive at a class of classical action functionals whose quantization leads directly to Pauli or Dirac spinor equations.

Casalbuoni pioneered the development of pseudomechanics, showing that it was possible to formulate an \( \hbar \to 0 \) limit of a quantum system in such a way that the spinors remain but their magnitude is no longer quantized (\[74\], \[75\], see also Freund \[76\]). Conversely, the resulting classical action leads to the Pauli-Schrödinger equation when quantized. Similarly, Berezin and Marinov \[77\], and Brink, Deser, Zumino, di Vecchia and Howe \[78\] introduced four anticommuting variables, \( \theta^a \) to write the pre-Dirac action. We display these actions below, after giving a simplified example. Since these approaches moved from quantum fields to classical equations, they already involved spinor representations. However, vector versions (having anticommuting variables without spinors) are possible as well. Our example below is of the latter type. Our development is a slight modification of that given by Freund \[76\].

To construct a simple pseudomechanical model, we introduce a superspace formulation, extending the usual “bosonic” 3-space coordinates \( x_i \) by three additional anticommuting coordinates, \( \theta^a \),

\[ \{\theta^a, \theta^b\} = 0 \]

Consider the motion of a particle described by \( [x_i (t), \theta^a (t)] \), and the action functional

\[ S = \int dt \left[ \frac{1}{2} m \dot{x}^i \dot{x}^i + i \frac{1}{2} \theta^a \dot{\theta}^a - V (x^i, \theta^b) \right] \]

Notice that \( \theta^2 = 0 \) for any anticommuting variable, so the linear velocity term is the best we can do. For the same reason, the Taylor series in \( \theta^a \) of the potential \( V (x^i, \theta^b) \) terminates:

\[ V (x^i, \theta^b) = V_0 (x^i) + \psi_a (x^i) \theta^a + \frac{1}{2} \varepsilon_{abc} B^a (x^i) \theta^b \theta^c + \frac{1}{3!} \kappa (x^i) \varepsilon_{abc} \theta^a \theta^b \theta^c \]
Since the coefficients remain functions of $x^i$, we have introduced four new fields into the problem. However, they are not all independent. If we change coordinates from $\theta^a$ to some new anticommuting variables, setting

$$\theta^a = \chi^a + \xi B_{bc}^a \chi^b \chi^c + C^a \varepsilon_{bcd} \chi^b \chi^c \chi^d$$

$$B_{bc}^a = B_{[bc]}^a$$

where $\zeta$ is an anticommuting constant, the component functions in $H(\theta^b)$ change according to

$$V = V_0 + \psi_\alpha \chi^\alpha + \left(\psi_\alpha \xi B_{bc}^a + \frac{1}{2} \varepsilon_{abc} B^a\right) \chi^b \chi^c$$

$$+ \left(\varepsilon_{afb} B_{cd}^a \xi B_{cd}^f + \frac{1}{3!} \kappa \varepsilon_{bcd} + \psi_\alpha C^a \varepsilon_{bcd}\right) \chi^b \chi^c \chi^d$$

The final term vanishes if we choose

$$\xi B_{bc}^a = \frac{\kappa + 6 \psi_\alpha C^a}{4B^2} (\delta^a_c B_e - \delta^a_e B_b)$$

while no choice of $B_{bc}^a$ can make the second term vanish because $\psi_\alpha \xi B_{bc}^a$ is nilpotent while $\frac{1}{2} \varepsilon_{abc} B^a$ is not. Renaming the coefficient functions, $V$ takes the form

$$V(\theta^b) = V_0 + \psi_\alpha \theta^\alpha + \frac{1}{2} \varepsilon_{abc} B^a \theta^b \theta^c.$$

Now, without loss of generality, the action takes the form

$$S = \int dt \left( \frac{1}{2} m \dot{x}^i \dot{x}_i - \frac{i}{2} \theta^a \dot{\theta}^a - V_0 - \psi_\alpha \theta^\alpha - \frac{1}{2} \varepsilon_{abc} B^a \theta^b \theta^c \right).$$

Varying, we get two sets of equations of motion:

$$m \ddot{x}^i = -\frac{\partial V}{\partial x^i} = -\frac{\partial V_0}{\partial x^i} + \frac{\partial \psi_\alpha}{\partial x^i} \theta^\alpha + \frac{1}{2} \varepsilon_{abc} \frac{\partial B^a}{\partial x^i} \theta^b \theta^c$$

$$\dot{\theta}^a = i\psi_\alpha + i\varepsilon_{abc} B^b \theta^c.$$

Clearly this generalizes Newton’s second law. The coefficients in the first equation depend only on $x^i$, so terms with different powers of $\theta^a$ must vanish separately. Therefore, $B^a$ and $\psi^a$ are constant and we can integrate the $\theta^a$ equation immediately. Since $[J_b]_c^a = \varepsilon_{cb}^a$ satisfies

$$[J_a, J_b]_d^c = \varepsilon_{ba}^e [J_e]_d^c$$
we see that $B^b\varepsilon_{bc}^a$ is an element of the Lie algebra of $SO(3)$. Exponentiating to get an element of the rotation group, the solution for $\theta^a$ is

$$\theta^a = iv^a t + e^{iB^b\varepsilon_{bc}^a} \theta_0^c$$

The solution for $x^i$ depends on the force, $-\partial V/\partial x^i$, in the usual way.

It is tempting to interpret the $\theta^a$ variables as spin degrees of freedom and $B^a$ as the magnetic field. Then the solution shows that the spin precesses in the magnetic field. However, notice that $B^b\varepsilon_{bc}^a$ is in $SO(3)$, not the spin group $SU(2)$. The coordinates $\theta^a$ therefore provide an example of fermionic, spin-1 objects.

One of the goals of early explorations of pseudomechanics was to ask what classical equations lead to the Pauli and Dirac equations when quantized. Casalbuoni (\cite{74},\cite{75}), see also \cite{76} showed how to introduce classical, anticommuting spinors using an $\hbar \rightarrow 0$ limit of a quantum system. Conversely, the action

$$S = \int dt \left[ \frac{1}{2} m \dot{x}^2 + \frac{i}{2} \theta^a \dot{\theta}^a - V_0(x) - (L \cdot S) V_{LS} - \kappa \frac{1}{2} (S \cdot B) \right]$$

where $L$ is the orbital angular momentum, $S = -i \varepsilon_{b c}^a \theta^b \theta^c$, and $V_{LS}$ is a spin-orbit potential, leads to the Pauli-Schrödinger equation when quantized. Similarly, Berezin and Marinov \cite{77}, Brink, Deser, Zumino, and di Vecchia and Howe \cite{78}, introduced four anticommuting variables, $\theta^\alpha$ to write the pre-Dirac action,

$$S_{Dirac} = \int d\lambda \left( -m \sqrt{-v^\alpha v^\alpha} + \frac{i}{2} \left[ \theta_3 \frac{d\theta^3}{d\lambda} + u_\alpha \theta^\alpha \frac{d\theta^\alpha}{d\lambda} - \alpha (u_\alpha \theta^\alpha + \theta_5) \right] \right)$$

where

$$v^\alpha = \frac{dx_\alpha}{d\lambda}, \quad u^\alpha = \frac{v^\alpha}{\sqrt{-v^\alpha v^\alpha}}$$

and $\alpha$ is a Lagrange multiplier. The action, $S_{Dirac}$, is both reparameterization invariant and Lorentz invariant. Its variation leads to the usual relativistic mass-energy-momentum relation together with a constraint. When the system is quantized, imposing the constraint on the physical states gives the Dirac equation.

Evidently, the quantization of these actions is also taken to include the entension to the relevant covering group.
C. Spin-statistics theorem

Despite the evident classical independence of spin and statistics, there exists a limited spin-statistics theorem due to Morgan [79]. The theorem is proved from Poincaré invariance, using extensive transcription of quantum methods into the language of Poisson brackets—an interesting accomplishment in itself. A brief statement of the theorem is the following:

**Theorem**: Let \( L \) be a pseudoclassical, Poincaré-invariant Lagrangian, built quadratically from the dynamical variables. If \( L \) is invariant under the combined action of charge conjugation (C) and time reversal (T) then integer spin variables are even Grassmann quantities while odd-half-integer spin variables are odd Grassmann quantities.

**Proof** relies on extending the quantum notions of charge conjugation and time reversal. As in quantum mechanics, charge conjugation is required to include complex conjugation. For fermionic variables, Morgan requires reversal of the order of Grassmann variables under conjugation

\[
(\eta \xi)^* = \xi^* \eta^*
\]

This insures the reality property \((\eta \xi)^* = \eta \xi^*\), but this is not a necessary condition for complex Grassmann numbers. For example, the conjugate of the complex 2-form

\[
dz \wedge dz^*
\]

is clearly just

\[
dz^* \wedge dz
\]

and is therefore pure imaginary. We must therefore regard the TC symmetry required by the proof as somewhat arbitrary.

Similarly, for time reversal, \[79\] requires both

\[
t \to -t
\]

\[
\tau \to -\tau
\]

Whether this is an allowed Poincaré transformation depends on the precise definition of the symmetry. If we define Poincaré transformations as those preserving the infinitesimal line element, \( dt \), then reversing proper time is not allowed. Of course, we could define Poincaré transformations as preserving the quadratic form, \( d\tau^2 = g_{\alpha\beta}dx^\alpha dx^\beta \), in which case the transformation is allowed.
Despite its shortcomings, the proof is interesting because it identifies a set of conditions under which a classical pseudomechanics action obeys the spin statistics theorem. This is an interesting class of theories and it would be worth investigating further. Surely there is some set of properties which can be associated with the classical version of the theorem. Perhaps a fruitful approach would be to assume the theorem and derive the maximal class of actions satisfying it.

There are other questions we might ask of spinorial and graded classical mechanics. A primary question is whether there are any actual physical systems which are well modeled by either spinors or graded variables. If such systems exist, are any of them supersymmetric? What symmetries are associated with spinorial and fermionic variables? Is there a generalization of the Noether theorem to these variables? What are the resulting conserved quantities? What is the supersymmetric extension of familiar problems such as the Kepler or harmonic oscillator?

The statistical behavior of fermionic classical systems is not clear. Quantum mechanically, of course, Fermi-Dirac statistics follow from the limitation of discrete states to single occupancy. This, in turn, follows from the action of an anticommuting raising operator on the vacuum:

\[
\begin{align*}
a^\dagger |0\rangle &= |1\rangle \\
a^\dagger a^\dagger &= 0
\end{align*}
\]

Since classical states are not discrete, there may be no such limitation. Do anticommuting classical variables therefore satisfy Bose-Einstein statistics? If so, how do Fermi-Dirac quantum states become Bose-Einstein in the classical limit?

The introduction of pseudomechanics has led to substantial formal work on supermanifolds and symplectic supermanifolds. See [80], [81] and references therein.

VIII. OBSERVATIONS

Clearly, the field of classical mechanics has no conclusion, and we do not provide one here. Within each topic we have tried to provide more questions than answers. However, in the process of collecting these results, we have observed a few patterns. In closing, we take note of those.
1. New elements in classical physics work their way into the field from fundamental research areas, notably quantum field theory and general relativity. The former has contributed spinors and anticommuting numbers while the latter lends the tools of differential geometry to the study of symplectic manifolds.

2. Many of the new insights have been seen only in one or two dimensions. In these cases, it remains an open question whether the properties even exist in higher dimensions, which higher dimensions, and why in those dimensions. This applies particularly to the study of inequivalent Lagrangians and Bohmian quantum mechanics.

3. Comparatively little use is made of the classical physics ArXiv. Researchers in the area would benefit by using this ready reference tool.

4. Classical mechanics is now strongly influenced by quantum mechanics. In addition to Bohmian quantum mechanics, which seeks to realize quantum physics as some sort of classical system, there is phase space quantization, which accomplishes much the same thing in a different way. An additional approach is suggested by the gauge theories of Section 6. These programs demonstrate broad overlap between the classical and quantum worlds.

**APPENDIX A: ALTERNATIVE LAGRANGIANS IN HIGHER DIMENSIONS**

We show that the Lagrangian
\[
L = v \int^{v} \alpha \left( x, \xi, \hat{\theta} \right) \frac{\xi^2}{\xi^2} d\xi + f \left( x, \hat{\theta} \right)
\]
where \( \alpha \left( x, \xi, \hat{\theta} \right) \) is a time-independent constant of the motion and \( \hat{\theta}_v = \frac{\dot{x}^i}{v} \) is a unit vector in the direction of the velocity, solves one of the Euler-Lagrange equations,
\[
\dot{x}^i \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \right) = 0
\]

First notice that
\[
\dot{x}_i \frac{d}{dt} \left( \frac{\dot{x}^i}{v} \right) = \dot{x}_i \dot{x}^j \frac{\partial}{\partial \dot{x}^i} \left( \frac{\dot{x}^i}{v} \right) = \frac{1}{v} \dot{x}_i \dot{x}^j \left( \delta^j_i - \frac{\dot{x}^j \dot{x}_j}{v^2} \right) = 0.
\]
We therefore have

\[
\dot{x}_i \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \right) = \dot{x}_i \frac{d}{dt} \left[ \frac{\dot{x}_i}{v} \int_0^v \frac{\alpha(x, \xi)}{\xi^2} d\xi + \frac{\alpha(x, v) \dot{x}_i}{v} + \frac{\partial f(x, \theta_v)}{\partial \dot{x}^i} \right]
\]

\[
- \dot{x}_i \int_0^v \frac{1}{\xi^2} \frac{\partial \alpha(x, \xi)}{\partial x^i} d\xi - \dot{x}_i \frac{\partial f(x, \theta_v)}{\partial x^i}.
\]

\[
\dot{x}_i \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \right) =
\frac{\dot{x}_i}{v} \frac{d}{dt} \int_0^v \frac{\alpha(x, \xi)}{\xi^2} d\xi + \frac{\dot{x}_i}{v} \frac{d}{dt} \frac{\alpha(x, v)}{v}
\]

\[
+ \dot{x}_i \frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \dot{x}_i \frac{\partial f}{\partial x^i}.
\]

Possibly the function \( f \) may be chosen so that the remaining equations of motion are satisfied.

**APPENDIX B: ARBITRARY NUMBER OF EXTREMA IN KEPLER ORBITS**

The problem of global properties of orbits remains open – power law forces have been studied \[ \text{[10]} \] and found to have limited numbers of extrema, but non-monotonic force laws allow arbitrarily many extrema. We provide a simple example here.

Consider the potential

\[
V = \alpha (r - r_0)^2 \]
The potential $V$ has energy and effective potential, 

$$E = \frac{1}{2} mr^2 + \frac{M^2}{2mr^2} + \alpha (r-r_0)^{2p}$$

$$V_{\text{eff}} = \frac{M^2}{2mr^2} + \alpha (r-r_0)^{2p}$$

This effective potential has an arbitrarily strong minimum near $r_0$. The exact location of the minimum is given by

$$0 = V'_{\text{eff}} = 2p\alpha (r-r_0)^{2p-1} - \frac{M^2}{mr^3}.$$ 

We solve this approximately as follows. Let $r = r_0 + a$. Then

$$0 = 2p\alpha r_0^3 \left(1 + 3\frac{a}{r_0} + 3\frac{a^2}{r_0^2} + \frac{a^3}{r_0^3}\right) a^{2p-1} - \frac{M^2}{m}$$

Now suppose $a << r_0$ so that we can neglect the $\frac{a}{r_0}$ terms. Then in order to have solutions we must have

$$a^{2p-1} = \frac{M^2}{2p\alpha mr_0^3}$$

or

$$\left(\frac{a}{r_0}\right)^{2p-1} = \frac{M^2}{2p\alpha mr_0^{2p+2}} << 1$$  \hspace{1cm} (B1)

This may be satisfied by choosing $p$ sufficiently large. Thus, $r = r_0 + a$ is the approximate position of the extremum. This solution for $r$ is a minimum since

$$V''_{\text{eff}} = \frac{3M^2}{mr^4} + 2p(2p-1)\alpha (r-r_0)^{2p-2} > 0$$

Now, setting $r = r_0 + a + \varepsilon$, and expanding the effective potential to second order about the minimum at $r_0 + a$,

$$V_{\text{eff}} = \frac{M^2}{2mr_0^2} \left(1 - \frac{\varepsilon}{r_0} + \varepsilon^2\right)^{2p}$$

$$= \frac{M^2}{2mr_0^2} \left[1 + \frac{2a}{r_0} - 2\varepsilon + \left(\frac{a}{r_0} + \varepsilon\right)^2\right]$$

$$+ \alpha a^{2p} \left(1 + \frac{2p\varepsilon}{a} + 2p(2p-1)\varepsilon^2\right)$$

$$= \frac{M^2}{2mr_0^2} \left(1 + \frac{2a}{r_0} + \frac{a^2}{r_0^2}\right) + \alpha a^{2p}$$

$$- \frac{M^2}{mr_0^2} \varepsilon + 2p\alpha a^{2p-1}\varepsilon$$

$$+ \frac{M^2}{2mr_0^2} \varepsilon^2 + 2p(2p-1)\alpha a^{2p-2}\varepsilon^2$$
The first term is just an overall constant, while the linear term vanishes because $r_0 + a$ is a minimum. The third term is a harmonic oscillator potential. The approximate equation of motion for the oscillator is

$$\frac{d^2 \varepsilon}{dt^2} + \frac{1}{m} \left( \frac{M^2}{2mr_0^2} + 2p(2p - 1) \alpha a^{2p-2} \right) \varepsilon^2 = 0$$

By eq. (B1), the squared frequency becomes

$$\frac{M^2}{2m^2r_0^2} + \frac{2p}{m} (2p - 1) \alpha a^{2p-2} = \frac{M^2}{2m^2r_0^2} + \frac{2p}{ma} (2p - 1) \alpha \frac{M^2}{2pmr_0^2}$$

$$= \frac{M^2}{2m^2r_0^2} \left[ 1 + \frac{2}{ar_0} (2p - 1) \right]$$

The frequency may be made arbitrarily large, at fixed angular momentum $M$, by increasing $p$. This means we may have arbitrarily many extrema per orbit.

**APPENDIX C: THE KORTEWEG-DE VRIES EQUATION**

While we have so far stayed within particle mechanics, many classical field theories also have interesting properties. Of particular interest are the “integrable systems” such as the KdV and the sine-Gordon equations. These differential equations turn out to have infinitely many constants of motion.

The KdV equation is a one dimensional, third order field equation that provides a good example of hidden symmetries. Here we briefly examine some of its properties. The interesting history of the equation stretches over more than a century [50]. We will begin with a modern form of the equation,

$$u_t = -6uu_x + u_{xxx}$$

Consider any function of the form $u = f(z) = f(x - vt)$. Substituting, we find that $f$ must satisfy

$$0 = \left( -6f \partial + v \partial + \partial^3 \right) f$$

Let $g = f + c$, this becomes

$$0 = \left( -6g \partial + (v + 6c) \partial + \partial^3 \right) g$$
so choosing \( c = -v/6 \) we have simply

\[
0 = -6gg_x + g_{xxx}
\]

Integrating twice we find the quadrature,

\[
\int \frac{df}{\sqrt{2af - 2f^3 - vf^2 + 2b}} = x - vt
\]

These solutions for \( u \) propagate with unchanging shape \( f \) and constant velocity \( v \). It can be shown that pairs of solitary waves can pass through one another and emerge unchanged. It has been suggested that the infinite hierarchy of constants of the motion of this system is related to the existence of such soliton solutions. Showing how these constants arise will simultaneously illustrate some techniques of classical field theory. Our treatment follows that of Abraham and Marsden [50], which is recommended for further detail.

First, we show that the KdV equation may be described as a Hamiltonian system. For particle motion expressed in canonical coordinates, we can define the Hamiltonian vector field, \( X_H \) which is everywhere tangent to the phase space motion. Conversely, the classical motion of the system is along the integral curves of this vector field. Restricted to any solution curve, \( X_H \) is therefore given by

\[
| X^A_H | = \left( \frac{dx_i}{dt}, \frac{dp_i}{dt} \right)
\]

\[
= \left( \frac{\partial H}{\partial p_j}, -\frac{\partial H}{\partial x^j} \right)
\]

\[
= | \Omega^{AB}_{\xi B} \frac{\partial H}{\partial \xi^B} |_{C=(x(t),p(t))}
\]

We therefore can characterize \( X_H \) everywhere by writing

\[
\frac{\partial H}{\partial \xi^B} = \Omega^{BA}_{\xi A} X^A_H
\]

or more simply using differential forms,

\[
dH (v) = \omega (X_H, v)
\]

for any vector field, \( v \).

The same relationship holds in classical field theory. For the KdV equation, we can define a symplectic form as follows:

\[
\omega (u, v) = \frac{1}{2} \int dx \int^x dy [u (y) v (x) - u (x) v (y)]
\]
Here \( u \) and \( v \) are arbitrary vector fields. Now suppose the Hamiltonian is given as an integral over a Hamiltonian density,

\[
H = \int f[u(x)]dx
\]

Hamilton’s equations are then involve functional derivatives. For the differential of the Hamiltonian,

\[
dH(v) = \int_{-\infty}^{\infty} dx \frac{\delta f}{\delta u}(x) v(x)
\]

so equating to the symplectic form,

\[
dH(v) = \omega(X_H, v)
\]

We seek an expression for \( X_H \). First, write \( X_H \) in the form

\[
X_H = \frac{\partial G}{\partial x}
\]

Then, integrating by parts and disregarding surface terms, we obtain,

\[
\int_{-\infty}^{\infty} dx \frac{\delta f}{\delta u}(x) v(x) = \frac{1}{2} \int dx \int^x dy [X_H(y) v(x) - X_H(x) v(y)]
\]

and since \( v \) is arbitrary we have

\[
G(x) = \frac{\delta f}{\delta u}(x)
\]

Therefore,

\[
X_H = \frac{\partial}{\partial x} \frac{\delta f}{\delta u}.
\]

Now consider the KdV equation,

\[
u_t = 6uu_x - u_{xxx}
\]

The time evolution of \( u \) is given by the tangent vector field \( u_t \). We ask if we can write this vector field as a Hamiltonian vector field \( X_H \) for some Hamiltonian \( H \). Equating

\[
u_t = X_H
\]
we require

\[ X_H = \frac{\partial}{\partial x} \frac{\delta f}{\delta u} = 6uu_x - u_{xxx} = \frac{\partial}{\partial x} (3u^2 - u_{xx}) \]
\[ \frac{\delta f}{\delta u} = 3u^2 - u_{xx} \]

and it is easy to see that we can take

\[ f = u^3 + \frac{1}{2} u^2_x \]
\[ H = \int dx \left( u^3 + \frac{1}{2} u^2_x \right). \]

We therefore have a Hamiltonian system, and the KdV equation may be studied in terms of a Hamiltonian flow.

We now show that the KdV equation possesses infinitely many constants of motion. Define an infinite set of Hamiltonian vector fields and Hamiltonian densities by acting repeatedly on \( X_1 \) and \( f_1 \) according to

\[ X_{n+1} = (2au \partial_x + au_x + b \partial^3_x) \frac{\delta f_n}{\delta u} \]
\[ \frac{\partial}{\partial x} \frac{\delta f_{n+1}}{\delta u} = X_{n+1} \]

The first expression always exists, but the second is possible as long as each new \( X_{n+1} \) is a Hamiltonian flow. In order for there to exist a Hamiltonian such that

\[ dH (v) = \omega (X_H, v) \]

we require the integrability condition

\[ 0 \equiv d^2 H (v) = d\omega (X_H, v) \]

Essentially, this condition reduces to the equality of mixed partial functional derivatives. We omit the inductive proof that shows that the condition is satisfied for all \( X_n \), as long as it holds for the initial set. Letting \( f_1 = u^2/2 \) it follows that

\[ X_2 = \partial_x (3u^2 - u_{xx}) = 6uu_x - \partial^3_x u \]
\[ \frac{\partial}{\partial x} \frac{\delta f_2}{\delta u} = \partial_x (3u^2 - u_{xx}) \]
\[ \frac{\delta f_2}{\delta u} = 3u^2 - u_{xx} \]
\[ f_2 = u^3 + \frac{1}{2} u^2_x \]
so $f_2$ is the Hamiltonian density for the KdV equation,

$$H = \int dx \left( u^3 + \frac{1}{2} u_x^2 \right)$$

Since the inductive hypothesis holds, the entire set of Hamiltonian vector fields $X_n$ and Hamiltonian densities $f_n$ exists.

Finally, we are ready for the proof that there exist an infinite number of constants of the motion of the KdV equation. Consider the higher order “Hamiltonians” given by integrating the $f_n$:

$$H_n = \int f_n(x) \, dx$$

We compute their Poisson brackets with one another by integration by parts,

$$\{H_n, H_m\} = \Omega (X_n, X_m)$$

$$= \frac{1}{2} \int dx \int dy \left[ X_n(y) X_m(x) - X_n(x) X_m(y) \right]$$

$$= \frac{1}{2} \int dx \int dy \left[ \partial_y \frac{\delta f_n}{\delta u}(y) X_m(x) - \partial_x \frac{\delta f_n}{\delta u}(x) X_m(y) \right]$$

$$= \int dx \frac{\delta f_n}{\delta u} X_m$$

$$= \int dx \frac{\delta f_n}{\delta u} (2au \partial_x + au_x + b \partial_x^3) \frac{\delta f_{m-1}}{\delta u}$$

$$= \int dx \left( 2a \frac{\delta f_n}{\delta u} u \partial_x \frac{\delta f_{m-1}}{\delta u} + a \frac{\delta f_n}{\delta u} u_x \frac{\delta f_{m-1}}{\delta u} + b \frac{\delta f_n}{\delta u} \partial_x^3 \frac{\delta f_{m-1}}{\delta u} \right)$$

$$= \int dx \left[ -2a \partial_x \left( \frac{\delta f_n}{\delta u} \right) \frac{\delta f_{m-1}}{\delta u} \right]$$

$$+ \int dx \left( a u_x \frac{\delta f_n}{\delta u} \frac{\delta f_{m-1}}{\delta u} - b \partial_x^2 \frac{\delta f_n}{\delta u} \frac{\delta f_{m-1}}{\delta u} \right)$$

$$= - \int dx \frac{\delta f_{m-1}}{\delta u} \left( a u_x + 2au \partial_x + b \partial_x^3 \right) \frac{\delta f_n}{\delta u}$$

$$= - \int dx \frac{\delta f_{m-1}}{\delta u} X_{n+1}$$

$$= - \{H_{m-1}, H_{n+1}\}$$

$$= \{H_{n+1}, H_{m-1}\}$$

Now iterate this relationship. First suppose $n$ and $m$ are either both even or both odd. Then without loss of generality we take $m - n = 2k > 0$. Setting $m = n + 2k$ and iterating
\( k \) times we have:
\[
\{ H_n, H_m \} = \{ H_n, H_{n+2k} \} \\
= \{ H_{n+k}, H_{n+2k-k} \} \\
= \{ H_{n+k}, H_{n+k} \} \\
= 0
\]

where the last step follows by antisymmetry of the bracket. Now let \( m = n + 2k + 1 \). Again iterating \( k - 1 \) times, and then one more time, give
\[
\{ H_n, H_m \} = \{ H_n, H_{n+2k+1} \} \\
= \{ H_{n+k-1}, H_{n+k} \} \\
= \{ H_{n+k}, H_{n+k-1} \}
\]

But the last two lines are negatives of one another, and therefore vanish. Therefore, all of the \( H_n \) have vanishing Poisson brackets with one another. In particular, since \( H_2 \) is the original Hamiltonian, \( \{ H_2, H_m \} = 0 \) for all \( m \), and the evolution generated by \( H_2 \) leaves all \( H_m \) constant. Since the evolution by \( H_2 \) generates solutions to the KdV equation, all \( H_n \) are constants of integration of the KdV system.

The KdV equation has interesting quantum properties as well. It can be shown that the Schrödinger equation with time-dependent potential \( u(t) \) has solutions with a fixed energy spectrum – the time-dependence of the potential does not change the energies of the solutions. The proof hinges on the Lax theorem, which states that the KdV equation is equivalent to the equation
\[
\frac{du}{dt} = [L, A]
\]
where
\[
A = 4\partial_x^3 + 6u\partial_x + 3u_x
\]
and
\[
L = \partial_x^2 + u
\]
This latter operator \( L \) is just the Schrödinger operator with potential \( u \). The proof of the theorem follows by direct calculation:
\[
\frac{du}{dt} f = \left[ 4\partial_x^3 + 6u\partial_x + 3u_x, \partial_x^2 + u \right] f \\
= (u_{xxx} + 6uu_x) f
\]
This is satisfied if $-u$ satisfies the KdV equation. The full proof of the resulting isospectral theorem may be found in [50]. In light of this relationship between a remarkable classical system and its equally striking quantum properties, one wonders whether the relationship between the classical and quantum mechanics may be much like the relationship between the real line and the complex plane. Just as real functions often display their full character only when analytically extended to the complex plane, many classical systems may show their true natures when quantized. The KdV equation provides an excellent example of this – solutions to the KdV equation, when used as quantum potentials, are isospectral despite time-dependent potentials, and there may be a profound connection between the KdV and Schrödinger systems.

The existence of equations such as the KdV equation, which have infinitely many independent conserved quantities is remarkable in several regards. For example, Goldstein, Poole and Safko observe that the KdV equation provides a counterexample to the converse of the Noether theorem [38]. Thus, while symmetries of an action lead to conserved quantities, the KdV and other equations have infinitely many conserved quantities without corresponding symmetries.

There is little systematic theory of these so-called “integrable systems.” In fact, we lack even a clear definition of this concept of integrability. Still, there has considerable recent progress (see, for example, [51] and references therein).

A related question is whether such systems exist in higher dimensions. As with many modern results in classical mechanics, examples are limited to one or two dimensions, and it is unclear whether we are seeing properties of geometries or only of the real and complex number systems.

APPENDIX D: CHAOS

While the study of nonlinear and chaotic systems is beyond the scope of this review, one common example provides an interesting case of a higher order differential equation for a classical system. The onset of chaos may be visualized by studying the fixed points of the logistic equation,

$$x_{k+1} = ax_k (1 + bx_k)$$
for varying values of the parameters $a$ and $b$ (see the review articles by May [52], May and Oster [53], as well as May [54] ) As the values of these parameters change, the number of fixed points passes through bifurcation points, leading to more and more frequent doubling of the their number. At a finite value of the parameters, the number of fixed points diverges and the behavior of the system is said to become chaotic.

This equation may be converted into a nonlocal equation of continuous motion for a one dimensional system. The fixed points of the discrete equation then become periodic solutions for the continuous system. Replace the discrete sequence $x_k$ with a function $x(t)$, which must satisfy

$$x(t + 1) = ax(t) [1 + bx(t)]$$

The left hand side may be expanded in a Taylor series as

$$x(t + 1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n x}{dt^n} \right|_t \cdot (1)^n$$

so at any time $t$, the function $x$ must satisfy

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n x}{dt^n} - ax - abx^2 = 0$$

This certainly qualifies as a higher order differential equation!

We can find the fixed points from the continuous representation as well as from the discrete one. At $k^{th}$-order fixed points of the discrete system, we require periodicity of the form $x(t + k) = x(t)$. To examine the consequences of this condition, we employ a common technique for periodic systems [49].

Suppose $x_i(t)$ are independent solutions to $x(t + k) = x(t)$. Then a general solution may be written as a superposition of these, so $x_i(t + k)$ must be some superposition:

$$x_i(t + k) = \sum_j a_{ij} x_j(t)$$

Periodic solutions then satisfy

$$x_i(t) = \sum_j a_{ij} x_j(t)$$

Now diagonalize $a_{ij}$. If $y_i$ are the new basis functions and $\lambda_i$ the eigenvalues, then

$$y_i(t + k) = \lambda_i y_i(t)$$
implies
\[ y_i(t) = \lambda_i^{t/k} \pi_i(t) \]
where \( \pi_i \) is any periodic function with period \( k \). Now, the \( y_i \) satisfy the equations
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n y_i}{dt^n} - a y_i - aby_i^2 = 0 \]
\[ \sum_{n=0}^{\infty} \frac{k^n}{n!} \frac{d^n y_i}{dt^n} = \lambda_i y_i(t) \]
Consider the long-term behavior of \( y_i \). Suppose \( \lambda_i > 1 \) so that \( y_i(t) \) diverges at late times. From the first equation \( y_i \) must satisfy
\[ -aby_i^2 = 0 \]
as \( t \to \infty \), so the limiting value of \( y_i \) is zero, which is inconsistent. Therefore, we require \( \lambda_i < 1 \) so that \( y_i \) is also converging to zero at late times, and must approximately satisfy both
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n y_i}{dt^n} = a y_i(t) \]
\[ \sum_{n=0}^{\infty} \frac{k^n}{n!} \frac{d^n y_i}{dt^n} = \lambda_i y_i(t) \]
Since these are linear, we may write
\[ y_i = e^{\alpha_i t} \]
Then
\[ \sum_{n=0}^{\infty} \frac{(\alpha_i)^n}{n!} = a \]
\[ \sum_{n=0}^{\infty} \frac{(\alpha_i k)^n}{n!} = \lambda_i \]
and we need both
\[ e^{\alpha_i} = a \]
\[ e^{\alpha_i k} = \lambda_i = a^k \]
thereby determining the (asymptotic) eigenvalues.

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