How to compute the thermal quarkonium spectral function from first principles?

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Abstract

In the limit of a high temperature $T$ and a large quark-mass $M$, implying a small gauge coupling $g$, the heavy quark contribution to the spectral function of the electromagnetic current can be computed systematically in the weak-coupling expansion. We argue that the scale hierarchy relevant for addressing the disappearance (“melting”) of the resonance peak from the spectral function reads $M \gg T > g^2 M > gT \gg g^4 M$, and review how the heavy scales can be integrated out one-by-one, to construct a set of effective field theories describing the low-energy dynamics. The parametric behaviour of the melting temperature in the weak-coupling limit is specified.

Key words: Thermal field theory, Perturbative QCD, Quark–gluon plasma, Bottom mesons
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1. Introduction

Given possible applications in heavy ion collision experiments [1], the last 20 years have seen a huge amount of phenomenological work on the properties of heavy quarkonium at temperatures just above the deconfinement crossover (for a recent review, see ref. [2]). During the last couple of years, these phenomenological works have been complemented by more theoretical investigations, aiming at a QCD-based approach to the problem. The purpose of this talk is to distill a basic message from some of the latter studies [3–9].

Let us start by defining the observables we are interested in. The heavy quark contribution to the spectral function of the electromagnetic current can be written as

$$\rho_V(Q) = \int dt \int d^3x e^{iQ \cdot x} \left\langle \frac{1}{2} [\hat{J}^\mu(x), \hat{J}^\mu(0)] \right\rangle,$$

where $\hat{J}^\mu \equiv \hat{\psi} \gamma^\mu \hat{\psi}$; $\hat{\psi}$ is the heavy quark field operator in the Heisenberg picture; $\langle \ldots \rangle \equiv Z^{-1} \text{Tr}[\ldots e^{-\beta \hat{H}}]$ is the thermal expectation value; $\beta \equiv 1/T$ is the inverse temperature;
and we assume the metric convention (+−−−). This spectral function determines the production rate of muon–antimuon pairs from the system \([10]\),

\[
\frac{dN_{\mu^-\mu^+}}{d^4x \, d^4Q} = \frac{-2e^4 Z^2}{3(2\pi)^3 Q^2} \left( 1 + \frac{2m^2}{Q^2} \right) \left( 1 - \frac{4m^2}{Q^2} \right) n_B(q^0) \rho_V(Q),
\]

where \(Z\) is the heavy quark electric charge in units of \(e\), and \(n_B\) is the Bose-Einstein distribution function. In the following we assume, largely for notational simplicity, that the muon–antimuon pair is at rest with respect to the thermal medium, i.e. \(Q \equiv (\omega, \mathbf{0})\)\(^1\). Furthermore, \(M\) denotes the heavy quark (charm, bottom) pole mass.

The ultimate goal of the study would then be to compute \(\rho_V\) and \(dN_{\mu^-\mu^+}/d^4x \, d^4Q\) in a certain energy range around the two-particle threshold, say \(\omega \sim (1 \ldots 3)M\), and in a certain temperature range, say \(T \sim (0.1 \ldots 1.0)\) GeV. The phenomenological interest of the problem lies in the fact that the observables mentioned are believed to undergo a qualitative change in this temperature range, thus constituting a sensitive “thermometer” for the system; examples of possible patterns are shown in Fig. 1.

Now, given that we are interested in fairly low temperatures and that the charm quark mass is fairly moderate compared with the QCD scale, it would be interesting to tackle the problem with lattice techniques (see, e.g., refs. [11]). Unfortunately, lattice methods are Euclidean, and even for a perfectly known Euclidean correlator \(G_V(\tau), \tau \in [0, \beta]\), it is strictly speaking not possible to invert the relation to \(\rho_V\), viz.

\[
G_V(\tau) = \int_0^\infty \frac{d\omega}{\pi} \rho_V(\omega) \frac{\cosh \left( \frac{\beta}{2} - \tau \right) \omega}{\sinh \frac{\beta\omega}{2}},
\]

without inserting further input (for instance, in momentum space, one must set \(e^{i\beta\omega_n} \equiv 1\) in order to get the correct analytic continuation from \(\omega_n\) to \(\omega\)). Another possible strong

\(^1\) For a non-zero total spatial momentum \(q\), with \(0 < |q| \ll M\), the main modification would be a shift of the two-particle threshold, from \(\omega \approx 2M\) to \(\omega \approx 2M + q^2/4M\) (see, e.g., ref. [11]).
coupling method, an analysis through a gravity dual (see, e.g., ref. [12]), is not available for QCD. In the following we therefore resort to the weak-coupling expansion, which at least is a theoretically consistent framework; in fact, particularly for the bottomonium system, experience from zero temperature (see, e.g., ref. [13]) suggests that numerical convergence could be reasonable as well.

2. Scale hierarchy

A systematic weak-coupling computation can be carried out once the scales that enter the problem have been identified. In the heavy quark-mass and high-temperature limit, the QCD coupling constant \( g \) is small, and the quarkonium states resemble non-relativistic systems such as positronium. Thereby we treat the heavy quark mass \( M \) as the “hard” scale; the inverse Bohr radius, or relative momentum \( Mv \sim g^2 M \) as the “soft” scale; and the binding energy \( Mv^2 \sim g^4 M \) as the “ultrasoft” scale. We assume the QCD scale to be at most as large as the ultrasoft scale.

The energy scale with which we probe the system, \( \omega \), could in principle be anything. However, conceivably the most interesting range is \( |\omega - 2M| \sim g^4 M \), i.e. deviations from the two-particle threshold by at most the binding energy; this is where the quarkonium resonance is to be found. In the following, we concentrate on this regime.

Finally, we need to decide how temperature relates to the zero-temperature scales. Maybe quarkonium melts when \( T \sim g^2 M \), given that then hard particles from the plasma have enough momentum to kick constituents out of the bound state? However, such a temperature cannot be high enough, since kicking happens through interactions, and those are suppressed by \( g \). On the other hand, increasing the temperature just a little bit, to \( gT \sim g^2 M \), is certainly enough to dissociate the bound state, given that then even the Coulomb binding gets Debye screened. In fact, removing completely the Coulomb binding is an overkill, since a finite width could arise already earlier on. To summarise, the relevant range is somewhere between these two limits, \( g^2 M < T < gM \) [6]. Therefore we now assume the equivalent hierarchy

\[
M \gg T > g^2 M > gT \gg g^4 M ,
\]

and integrate out the hard scales one-by-one.

3. Integrating out the scale \( M \)

Integrating out the hard scale \( M \) yields an effective theory called NRQCD [14][15]. The topic is very well developed by now, and we only list some basic steps here; many details and references can be found, e.g., in the review [16].

Starting from the QCD Lagrangian, \( \mathcal{L}_{QCD} = \bar{\psi}(i\gamma^\mu D_\mu - M)\psi + \mathcal{L}_{\text{gluon}} \), one can first carry out a Foldy-Wouthuysen transformation:

\[
\bar{\psi} \equiv \begin{pmatrix} \theta \\ -\phi \end{pmatrix}^\dagger \exp \left( -i\gamma^j \overline{D}_j \right), \quad \psi \equiv \exp \left( i\gamma^j \overline{D}_j \right) \begin{pmatrix} \theta \\ \phi \end{pmatrix},
\]

where \( \overline{D}_j \equiv \overline{\partial}_j + igA_j \), \( \overline{D}_j \equiv \overline{\partial}_j - igA_j \), and \( \theta, \phi \) are two-component spinors. Expanding in \( 1/M \) and dropping total derivatives, this yields...
\[ \mathcal{L}_{\text{QCD}} = \theta^\dagger \left( iD_0 - M + \frac{D^2 + \sigma \cdot gB}{2M} \right) \theta + \phi^\dagger \left( iD_0 + M - \frac{D^2 + \sigma \cdot gB}{2M} \right) \phi \\
+ \frac{i}{2M} \left( \theta^\dagger \sigma \cdot gE \phi - \phi^\dagger \sigma \cdot gE \theta \right) + \mathcal{O} \left( \frac{1}{M^2} \right) + \mathcal{L}_{\text{gluon}}. \]

The heavy masses on the first row can be shifted away by \( \theta \to e^{-iMt} \theta, \phi \to e^{iMt} \phi \). The mixed terms on the second row become rapidly oscillating after this shift, \( e^{2iMt} \theta^\dagger \sigma \cdot gE \phi - e^{-2iMt} \phi^\dagger \sigma \cdot gE \theta \), and describe hard reactions; such oscillations can be integrated out yielding four-fermion operators. Finally, at the loop level, we need to correct for the local expansion through matching coefficients, which concerns both parameters, e.g. \( M = m_{\text{NRQCD}} \) (1 + \( g^2 C_F / 4\pi^2 + ... \)), as well as composite operators, e.g.

\[ \bar{\psi} \gamma^k \psi = \left[ \theta^\dagger \sigma_k \phi + \phi^\dagger \sigma_k \theta \right] \left( 1 - \frac{g^2 C_F}{2\pi^2} + ... \right). \]

4. Integrating out the scale \( T \)

The next step is to integrate out the hard thermal scale \( T \). In the heavy-quark sector, this does nothing dramatic, just corrects the parameters. In fact, since the thermal loops can be computed within NRQCD, all effects are power-suppressed in mass, e.g. \[ \delta M = \frac{g^2 T^2 C_F}{12M}. \]

In the gauge field sector, the same step yields nothing but the Hard Thermal Loop effective theory \[ \mathcal{L}_{\text{gluon}} \to \mathcal{L}_{\text{gluon}} + \mathcal{L}_{\text{HTL}} + .... \]

5. Integrating out the scale \( g^2 M \)

The crux of the problem is the integration out of the scale \( g^2 M \), yielding an effective theory called pNRQCD \[ \text{[19,20]. (To be more precise, all gluons of energy or momentum } \sim g^2 M \text{ are integrated out, while the composite quarkonium fields left over as dynamical degrees of freedom can still have relative momentum } \sim g^2 M; \text{ it is only their energy that is small, } \sim g^4 M.) \] Although well established, this step is in many ways more delicate than the previous ones; we only outline the basic ideas here (for reviews, see refs. \[ \text{[21,16].} \])

The pNRQCD setup is special in that it only applies to particular Green’s functions; in the thermal context, we can say that the thermal expectation value is restricted to the quark–antiquark sector of the Fock space, \( \text{Tr}[\ldots \exp(-\beta \hat{H})] \to \text{Tr}[P_3 \gamma_3 s(\ldots \exp(-\beta \hat{H})]. \) Thereby exponentially small effects \( \sim \exp(-\beta M) \) are omitted. For the quark–antiquark pair, were represent colour as \( 3^* \otimes 3 = 1 \oplus 8, \) and space coordinates as \( x_1 = X + r/2, \) \( x_2 = X - r/2. \) The singlet field, \( S, \) is diagonal in colour-space, \( S \equiv S 1_{N_c \times N_c} / \sqrt{N_c}, \) while the octet field is traceless, \( O \equiv O^{a T^a} / \sqrt{T_F}, \) where \( T_F \) defines the generator normalization through \( \text{Tr}[T^a T^b] = T_F \delta^{ab}. \) The interpolating operators for the component fields \( S, O^a \) can be chosen as

\[ \theta^\dagger (x_2, t) W_{x_2, x_1} \phi (x_1, t) \simeq Z_s^+(r) S(X, r, t), \]
\[ \theta^\dagger (x_2, t) W_{x_2, x} T^a W_{x, x_1} \phi (x_1, t) \simeq Z_o^+(r) O^a(X, r, t), \]
where $W$ is a straight Wilson line. Subsequently a general Lagrangian is written for the fields $S, O$ as well as the ultrasoft gauge fields, respecting gauge invariance and expanded in $1/M, 1/g^2 M \sim r$:

$$
\mathcal{L} \simeq \text{Tr} \left\{ S^\dagger \left[ i\partial_0 - V_s^{(0)}(r) + \frac{\nabla^2 r}{M} - \frac{V_s^{(1)}(r)}{M} \right] S + O^\dagger \left[ i\partial_0 - V_o^{(0)}(r) + \frac{\nabla^2 r}{M} - \frac{V_o^{(1)}(r)}{M} \right] O + V_s(r) \left[ O^\dagger r \cdot gE S + S^\dagger r \cdot gE O \right] + \frac{V_b(r)}{2} \left[ O^\dagger r \cdot gE O + O^\dagger O r \cdot gE \right] \right\} \mathcal{O}\left(r^2, \frac{1}{M^2}\right),
$$

where $D_0$ is the covariant derivative in the adjoint representation, i.e. $iD_0 O = i\partial_0 O + g[A_0(X, t), O]$. As indicated, gauge fields only depend on the center-of-mass coordinates $X, t$, whereas dependence on $r$ can be Taylor-expanded. As far as the structure on the second row is concerned, we remark that at leading order $V_A = V_B = 1$, as can be inferred from a tree-level matching of a three-object vertex to NRQCD, on which side the covariant derivative can be expanded as $iD_0 \to iD_0 + r \cdot gE + \ldots$.

Now, the effective theory contains many “potentials”, $V_s^{(i)}, V_o^{(i)}, V_A, V_B, \ldots$, which are to be treated as matching coefficients. They can be computed from various Wilson loops with $t \gg r$ (since $\partial_t \sim g^4 M \ll |\nabla| \sim g^2 M$), i.e. in the static limit. We return to the determination of $V_s^{(0)}$ presently.

For the computation of the spectral function $\rho_V$, a representation of the vector current within pNRQCD is also needed. The vector current is a local object, corresponding to $r = 0$. At zero distance, $Z_s(0) = N_c$, and the relation between pNRQCD and NRQCD fields becomes unambiguous. In addition, spin indices, which have been suppressed in the discussion above, can be added in a trivial way. Thereby the spectral function can indeed be computed within pNRQCD (for a concise yet explicit presentation at zero temperature, see ref. [22]).

6. Integrating out the scale $gT$

Even though our ultimate goal is to treat the situation $g^2 M > gT$, in which case the scale $gT$ is integrated out within pNRQCD, it will be illuminating to start by considering the case $g^2 M \sim gT$. Indeed the scale $gT$ can then be integrated out together with the scale $g^2 M$, and the problem boils down to determining the potentials of the previous section in the presence of non-zero Debye screening (since $m_D r \sim 1$).

Luckily, this problem was considered already in ref. [3]. In that paper the result was called a “real-time static potential”, and was in general denoted by $V_s(t, r)$; the potential $V_s^{(0)}(r)$ equals the infinite-time limit thereof, $V_s^{(0)}(r) = \lim_{t \to \infty} V_s(t, r)$. Thereby

$$
\text{Re} V_s^{(0)}(r) = -\frac{g^2 C_F}{4\pi} \left[ m_D + \frac{\exp(-m_D r)}{r} \right],
$$

$$
\text{Im} V_s^{(0)}(r) = -\frac{g^2 T C_F}{4\pi} \phi(m_D r),
$$

where $m_D \sim gT$ is the Debye mass, $C_F = 4/3$, and
φ(x) ≡ 2 \int_0^\infty \frac{dz z}{(z^2 + 1)^2} \left[ 1 - \frac{\sin(zx)}{zx} \right].

Let us comment, in passing, on the existence of an imaginary part in the potential. For \( r \to \infty \), the real part becomes a constant, corresponding to twice the thermal heavy quark mass shift \([23]\); in complete analogy, the imaginary part at \( r \to \infty \) equals \([7]\) twice the known heavy quark width \([24]\). In some sense, the physics of the imaginary part is also closely related to (single) heavy quark energy loss / drag force / diffusion coefficient, on which a lot of work has been carried out recently (see, e.g., refs. \([25]\)) : in fact the essential graph can be depicted in the same way, but the integral over the spatial momentum of the gluon is weighed differently.

What we really wanted to do in this section, however, was to integrate out the scale \( gT \) for \( g^2 M > gT \). One possibility would now be just to expand the previous potential \( gT \to \infty \), the real part becomes a constant, corresponding to twice the thermal heavy quark mass shift \([23]\); in complete analogy, the imaginary part at \( r \to \infty \) equals \([7]\) twice the known heavy quark width \([24]\). In some sense, the physics of the imaginary part is also closely related to (single) heavy quark energy loss / drag force / diffusion coefficient, on which a lot of work has been carried out recently (see, e.g., refs. \([25]\)) : in fact the essential graph can be depicted in the same way, but the integral over the spatial momentum of the gluon is weighed differently.

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Re $V_s^{(0)}(r) = \frac{g^2 C_F}{4\pi} \frac{1}{r} + ... $,

Im $V_s^{(0)}(r) = -\frac{g^2 C_F T m_D^2 r^2}{4\pi} \left( \ln \frac{1}{m_D r} - \gamma_E + \frac{4}{3} \right) + ... $.

7. Physics at the scale $g^4 M$

Having integrated out all but the ultrasoft scale, let us now inspect the singlet propagator at the lowest energies. We can expect a resonance to melt once the width appearing in the propagator becomes as large as the binding energy. Of course, this condition cannot be posed in an exact way, and for real-world applications we would indeed be interested in computing $\rho_V$ in some macroscopic energy and temperature range; nevertheless, for the purposes of this talk, we restrict to the task of estimating the melting temperature.

Parametrically (omitting $C_F/4\pi$ and numerical factors), the equality of the binding energy and width from the end of the previous section reads

$$\frac{g^2}{r} \sim g^2 T m_D^2 r^2 \ln \frac{1}{m_D r}.$$ 

Inserting $r \sim 1/g^2 M$ and $m_D \sim g T$, we get

$$g^4 M^3 \sim T^3 \ln \frac{g M}{T}.$$ 

For small enough $g$ this can be solved in a leading-logarithmic approximation, yielding

$$T \sim g^2 \left( \ln g M \right)^{-\frac{1}{3}} M.$$ 

This result was first obtained (without the logarithm) in ref. [8], and it indeed lies within the range $g^2 M < T < g M$ discussed in Sec. 2 as must be the case [6].

8. Conclusions

The main purpose of this talk has been to underline the question posed by the title. Indeed, it appears that quarkonium physics at high temperatures is important enough phenomenologically to deserve some theoretically-minded consideration as well.

As far as an answer to the question is concerned, we have tried to obtain one within the framework of the weak-coupling expansion. The most important point to realise is that a hierarchy can be found between the different dynamical scales affecting the problem. In fact, supplementing the discussion so far with the non-perturbative colour-magnetic scale $g^2 T$, the hierarchy relevant for the melting of quarkonium can be written as

$$M \gg T > g^2 M > gT > g^2 T > g^4 M.$$ 

Furthermore, for energies around the two-particle threshold, $|\omega - 2M| \sim g^4 M$, we have argued that the way to exploit the scale hierarchy is to generalize the effective theories known as NRQCD and pNRQCD to finite temperatures [89]. As a result of such an analysis, it can be argued that quarkonium melts at $T \sim g^{1/3} M$ [8]. We find it comforting that the phenomenon of melting can thus be confirmed model-independently.
For practical applications of the effective theory setup, the issue of the convergence of the weak-coupling series must be addressed. For this purpose, it would be helpful to use the framework to compute higher order corrections to the quarkonium spectral function. Also, the influence of the colour-magnetic scale $g^2 T$ needs to be estimated [5]. It appears reasonable to expect that as a result of such work, at least the study of the bottomonium resonance can ultimately be promoted to a quantitative level.

Finally, irrespective of the numerical convergence of the weak-coupling expansion, it is perhaps useful to emphasize the new qualitative features that this approach has unveiled. In particular, the fact that the real-time static potential has an unsuppressed $r$-dependent imaginary part at finite temperatures should in our opinion be taken into account also in phenomenological potential model studies.

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References

[1] T. Matsui and H. Satz, Phys. Lett. B 178 (1986) 416.
[2] R. Rapp, D. Blaschke and P. Crochet, [arXiv:0807.2470]
[3] M. Laine, O. Philipsen, P. Romatschke and M. Tassler, JHEP 03 (2007) 054.
[4] M. Laine, JHEP 05 (2007) 028.
[5] M. Laine, O. Philipsen and M. Tassler, JHEP 09 (2007) 066.
[6] Y. Burnier, M. Laine and M. Vepsäläinen, JHEP 01 (2008) 043.
[7] A. Beraudo, J.P. Blaizot and C. Ratti, Nucl. Phys. A 806 (2008) 312.
[8] M.A. Escobedo and J. Soto, [arXiv:0804.0601]
[9] N. Brambilla, J. Ghiglieri, A. Vairo and P. Petreczky, Phys. Rev. D 78 (2008) 014017.
[10] L.D. McLerran and T. Toimela, Phys. Rev. D 31 (1985) 545; H.A. Weldon, Phys. Rev. D 42 (1990) 2384; C. Gale and J.I. Kapusta, Nucl. Phys. B 357 (1991) 65.
[11] A. Jakovac, P. Petreczky, K. Petrov and A. Velytsky, Phys. Rev. D 75 (2007) 014506; G. Aarts, C. Allton, M.B. Oktay, M. Peardon and J.I. Skullerud, Phys. Rev. D 76 (2007) 094513.
[12] R.C. Myers, A.O. Starinets and R.M. Thomson, JHEP 11 (2007) 091.
[13] X. Garcia i Tormo and J. Soto, Phys. Rev. Lett. 96 (2006) 111801.
[14] W.E. Caswell and G.P. Lepage, Phys. Lett. B 167 (1986) 437.
[15] J.G. Körner and G. Thompson, Phys. Lett. B 264 (1991) 185.
[16] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. 77 (2005) 1423.
[17] J.P. Donoghue, B.R. Holstein and R.W. Robinett, Annals Phys. 164 (1985) 233.
[18] J. Frenkel and J.C. Taylor, Nucl. Phys. B 374 (1992) 156; E. Braaten and R.D. Pisarski, Phys. Rev. D 45 (1992) 1827.
[19] A. Pineda and J. Soto, Nucl. Phys. B (Proc. Suppl.) 64 (1998) 428.
[20] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B 566 (2000) 275.
[21] M. Beneke, [arXiv:hep-ph/9911490]
[22] M. Beneke, Y. Kiyo and A.A. Penin, Phys. Lett. B 653 (2007) 53.
[23] E. Gava and R. Jengo, Phys. Lett. B 105 (1981) 285.
[24] R.D. Pisarski, Phys. Rev. Lett. 63 (1989) 1129.
[25] E. Braaten and M.H. Thoma, Phys. Rev. D 44 (1991) 2625; G.D. Moore and D. Teaney, Phys. Rev. C 71 (2005) 064904; S. Caron-Huot and G.D. Moore, JHEP 02 (2008) 081.