Utility-based pricing and hedging of contingent claims in Almgren-Chriss model with temporary price impact.

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Abstract

In this paper, we construct the utility-based optimal hedging strategy for a European-type option in the Almgren-Chriss model with temporary price impact. The main mathematical challenge of this work stems from the degeneracy of the second order terms and the quadratic growth of the first order terms in the associated HJB equation, which makes it difficult to establish sufficient regularity of the value function needed to construct the optimal strategy in a feedback form. By combining the analytic and probabilistic tools for describing the value function and the optimal strategy, we establish the feedback representation of the latter. We use this representation to derive an explicit asymptotic expansion of the utility indifference price of the option, which allows us to quantify the price impact in options’ market via the price impact coefficient in the underlying market. Finally, we describe a game between competing market makers for the option and construct an equilibrium in which the option is traded at the utility indifference price.

1 Introduction

This paper is concerned with the problem of hedging and pricing of contingent claims in a model with price impact. More specifically, we restrict our analysis to European-type claims and assume the Almgren-Chriss model (see [1]) with linear temporary impact for the underlying asset. We also assume that the preferences of the agent (performing the hedging or pricing of the option) are given by an exponential utility. Then, the optimal hedging strategy is determined by maximizing the expected exponential utility of the terminal wealth generated by the dynamic trading in the underlying plus the payoff of the option. A natural notion of option price, in this setting, is the utility indifference price (see Definition 1), which can be computed via the value function of the aforementioned maximization problem.

The problem of hedging of contingent claims in the Almgren-Chriss model (and in its extensions with non-linear price impact) has been studied before. Much of the existing literature is concerned with the problems of replication and super-replication of contingent claims: see, e.g., [3, 17, 30, 11], and the references therein. However, the optimal (super-)replication strategies are only constructed in the models with permanent impact – i.e. without temporary one – and the exact replication strategies typically do not exist in the presence of temporary impact. An optimal hedging strategy is constructed in [28, 6, 7, 20, 21], but for an agent maximizing a linear-quadratic objective. The latter objective suffers from several shortfalls: in particular, it penalizes the hedger for making profits and may produce arbitrage prices in the equilibria of the associated games. Our setting is close to the one of [24], which poses the hedging problem as the maximization of expected exponential

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utility. However, [24] does not provide a complete well-posedness theory for the associated Hamilton-Jacobi-Bellman (HJB) equation (the validity of comparison principle is left open), and, more importantly, it does not provide a rigorous characterization of the optimal hedging strategy. The reason for the latter, as well as for the lack of characterization of the optimal super-replicating strategies in [11] and in other models including temporary impact, is that the associated HJB equation (BSDE) is degenerate and has a quadratic nonlinearity in the gradient. This makes it difficult to establish the desired regularity of its solution, needed to construct the optimal strategy in feedback form (and the well-posedness of the associated forward-backward systems is not even clear). The main contribution of this paper is in providing an explicit and computationally tractable characterization of the optimal hedging strategy in feedback form. The latter is achieved by combining the analysis of the associated HJB equation, the direct properties of the stochastic optimization problem (in particular, its strong convexity), and the representation of the optimal control via Backward Stochastic Differential equation (BSDE), in order to establish the so-called “endogenous boundedness”: i.e., the optimal control is bounded by a constant, even though no a priori constraints on its values are imposed in the optimization problem. The latter result is summarized in Theorem 1, and it allows us to complete the description of optimal control in the feedback form and obtain Theorem 2.

Another contribution of the present paper is in the analysis of utility indifference price of an option. In particular, we provide a computationally tractable description of this price via the HJB equation for the value function and, more importantly, develop rigorously the asymptotic expansion of this price in the regime where the price impact in the underlying market is small (see Theorem 3). To understand the value of this result, assume that the underlying market is sufficiently liquid, so that that price impact coefficient in this market, denoted $\eta$, can be measured. The option’s market, on the other hand, is less liquid, and the trading occurs via a market maker, who buys from, or sells to, a client a certain number of shares of the option and hedges her position by trading in the underlying market. Then, the market maker plays the role of the aforementioned agent, and it is natural to assume that she will trade the option’s shares at her utility indifference price (see the next paragraph for a justification of this assumption). Recall that the indifference price of the option depends on $\eta$, as the latter affects the hedging costs. In addition, the indifference price depends on the current number of option’s shares held by the marker maker, due to the nonlinearity of the utility function. By buying or selling options, the client changes the market maker’s inventory, affecting the indifference price and, thus, generating price impact in the option’s market. The expansion provided in this paper allows one to compute the price impact coefficient in the option’s market (which is hard to measure directly, due to the lack of liquidity and/or transparency) in terms of the price impact coefficient $\eta$ in the underlying market (which is easier to measure directly), assuming the latter is small – this connection is given explicitly by equation (61). To obtain the small impact expansion of the indifference price, unlike the existing literature [33, 32, 19, 8, 25, 15, 23, 10], where the authors obtain an expansion for the value function of the optimal hedging problem, we need an expansion for a partial derivative of the value function. As the existing methods are not sufficient to obtain such an expansion, we employ a more direct approach that relies on the properties of the optimal control and on the stochastic representations of the derivatives of the value function, established in the preceding part of the paper.

Finally, to justify the interpretation of indifference price as the option price quoted by the market makers, we show that it is indeed an equilibrium price in a game with competing market makers, who trade dynamically in options (with a client) and hedge their positions by trading in the underlying. In contrast with the existing models of this type (see, e.g., [8, 4]), we allow the market makers to explicitly control their price quotes, assuming that the most attractive quotes win client’s orders. This is in contrast with the more popular modeling approaches: (i) to assume that the client’s orders are split among market makers in an efficient (e.g., Pareto-optimal) way, or (ii) to assume that the marker makers have full control over the number of option shares they trade. Both of the latter approaches hide the exact mechanism of sharing the client’s order flow by the market

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1Herein, we limit our discussion to the games in which market makers have random endowment and are allowed to hedge it by trading in the underlying market. There exist other related models that do not have this feature: see, e.g., [3,13], and the references therein.
makers, which we want to highlight.

The rest of the paper is organized as follows. In Section 2, we solve the problem of optimal hedging of a static position in the option. This is done in several steps. First, we consider an approximation of the control problem with the ones in which the state process contains additional noise and the controls are bounded. The latter features allow us to avoid the degeneracy and quadratic growth mentioned above and to characterize the solutions to approximating problems via the HJB equation. Then, using the martingale duality principle, we derive a Forward-Backward Stochastic Differential equation (FBSDE) for the optimal control of the approximating problem. Using the direct analysis of the original and the approximating control problems, we establish certain a priori estimates, which, along with the BSDE methods, allow us to obtain, in Theorem 1, an upper bound on the absolute value of the optimal control that is uniform over the approximation parameters. Using these estimates, along with the martingale duality principle, we establish its feedback representation in Theorem 2 (the representation for indifference price follows easily from this result). In Section 3, we establish the asymptotic expansion of the value function established in Section 2. Section 4 describes the game between competing market makers, which we want to highlight.

### 2 Characterization of optimal hedging strategy and indifference price

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space generated by the Brownian motions \(W = (W_t)_{t \in [0,T]}\) and \(B = (B_t)_{t \in [0,T]}\), where \(B\) is only used for approximation purposes. Consider a liquid financial market consisting of an adapted stock price process \(S = (S_t)_{t \in [0,T]}\) and a constant riskless asset. In addition to the liquid market, we consider a contingent claim \(H(S_T)\), with maturity \(T\) and with payoff function \(H(s): \mathbb{R} \to \mathbb{R}\). In this section, we study the control problem of an individual agent with a static position in the option.

Our goal is to find a tractable representation of the utility indifference price of an option with total payoff \(Q_H(S_T)\), as well as the indifference price function and optimal strategy for the exponential-utility-based hedging of this option, assuming the Almgren-Chriss model with temporary price impact for the underlying market. Namely, within this section, for any initial condition \(\pi, s, x \in \mathbb{R}\), any \(0 \leq t \leq v < T\), and any \(\delta, \epsilon \geq 0\), we consider

\[
\pi_v = \pi + \int_t^v (\nu_u du + \delta dB_u),
\]

\[
S_v = s + \int_t^v \sigma dW_r,
\]

\[
X_v = x - \eta \int_t^v \nu_r^2 dr - \int_t^v S_r d\pi_r = x - \eta \int_t^v \nu_r^2 dr - \int_t^v S_r (\nu_r dr + \delta dB_r),
\]

where \(B\) and \(W\) are two independent Brownian motions, and \(\nu \in \mathcal{A}(t, T)\) is the set of \(\mathbb{R}\)-valued stochastic processes that are progressively measurable w.r.t. \(\mathcal{F}^t := \sigma \{W_r - W_t, 1_{\{\delta > 0\}}(B_r - B_t) : r \in [t, T]\}\) and are such that \(|\nu| \leq 1/\epsilon\) and \(\mathbb{E} \left[\int_t^T \nu_r^2 dr\right] < \infty\) (the latter, clearly, is only needed when \(\epsilon = 0\)). The agent, in this auxiliary problem, still aims to maximize the expected utility of her terminal wealth:

\[
\hat{V}^{\epsilon, \delta}(t, s, \pi, x, Q) := \sup_{\nu \in \mathcal{A}(t, T)} \mathbb{E} \left[-\gamma \left(X_T + \pi_T S_T - l \frac{(\pi_T)^2}{2} + QH(S_T)\right)\right],
\]

with the dynamics of the state processes given by (1)-(3). We are mainly interested in the case \(\delta = \epsilon = 0\), which has a clear financial interpretation as the problem of optimal utility-based hedging. The case of \(\delta, \epsilon > 0\) is included for technical reasons, as a way of regularizing the problem.
Denote \( P(t, s) = \mathbb{E}_{t,s}[H(S_T)]. \)

We make the following assumption on \( P \).

**Assumption 1.** \( P \) is \( C^{1,3}_b \) on \([0, T] \times \mathbb{R} \).

Note that we require the boundedness of the derivatives up to and including the boundary. This assumption is easily satisfied if \( H \in C^{2,2}_b(\mathbb{R}) \). Using \( P \) we can write the terminal wealth generated by a strategy \( \nu \) as

\[
X_T + \pi_T S_T - \frac{l(\pi_T)^2}{2} + QH(S_T) = x + \pi s + QP(t, s) + \int_t^T (\pi_r + Q\partial_s P(r, S_r))dS_r - \eta \int_t^T \nu_r^2 dr - \frac{l}{2} \pi_T^2.
\]

For any \((t, s, \pi, Q) \in [0, T] \times \mathbb{R}^3 \) and any \( \nu \in \mathcal{A}(t, T) \), we denote

\[
\Gamma(t, s; \omega) := -\gamma \left( P(T, S_T) - P(t, s) - \int_t^T \left( \partial_t P + \frac{\sigma^2}{2} \partial_s P \right)(r, S_r) dr \right) = -\gamma \left( P(T, S_T) - P(t, s) \right) - \gamma \int_t^T \partial_s P(r, S_r) dS_r,
\]

\[
\Psi(t, \pi, \nu; \omega) := \gamma \eta \int_t^T \nu^2 dr + \gamma \frac{l}{2} \left( \pi + \int_t^T (\nu_r dr + \delta dB_r) \right)^2 - \gamma \int_t^T \left( \pi + \int_t^T (\nu_r dl + \delta dB_l) \right) dS_r,
\]

\[
J(t, s, \pi, Q; \nu) := \mathbb{E} \left[ e^{\Psi(t, \pi, \nu; t, s) + Q\Gamma(t, s)} \right],
\]

\[
U^{\delta, \epsilon}(t, s, \pi, Q) := \inf_{\nu \in \mathcal{A}_i(t, T)} \log J(t, s, \pi, Q; \nu) = \inf_{\nu \in \mathcal{A}_i(t, T)} \mathbb{E} \left[ e^{\Psi(t, \pi, \nu; t, s) + Q\Gamma(t, s)} \right],
\]

\[
u^{\delta, \epsilon}(t, s, \pi, Q) := \log U^{\delta, \epsilon}(t, s, \pi, Q) = \log \left( \inf_{\nu \in \mathcal{A}_i(t, T)} J(t, s, \pi, Q; \nu) \right).
\]

For convenience, we often drop the dependence on \( \omega \). Note that \( \Gamma \) does not depend on \( \nu \), and, due to Assumption 1, \( \Gamma \) is uniformly bounded. Thus, the function \( \hat{V} \) (defined in (4)) satisfies

\[
\hat{V}^{\delta, \epsilon}(t, s, \pi, x, Q) = -e^{-\gamma(x+\pi s+Q P(t, s))} U^{\delta, \epsilon}(t, s, \pi, Q).
\]

As shown below, due to the presence of expectation of the exponential of a square of Brownian motion, we only prove the finiteness of the above value functions for \( \delta \geq 0 \) small enough.

### 2.1 PDE representation of the value function

The following proposition provides the value of \( u \) and, in turn, of \( \hat{V} \), for the case with no price impact (\( \eta = 0 \)), no extra noise (\( \delta = 0 \)), and no constraints (\( \epsilon = 0 \)). Its proof follows easily from the fact that the payoff \( H(S_T) \) can be replicated perfectly when \( \eta = 0 \) and that the replication strategy can be approximated by the absolutely continuous ones, so that the associated objective values of the agent converge.

**Lemma 1.** If \( \eta = \delta = \epsilon = 0 \), then, for all \( t < T \), \((s, \pi, Q) \in \mathbb{R}^3 \), we have \( u^{0,0}(t, s, \pi, Q) = 0 \) and \( u^{0,0}(T, s, \pi, Q) = \frac{l^2 \pi^2}{2} \).
Next, we return to the case \( \eta > 0 \) and general \( \delta, \epsilon \geq 0 \). Denote the (partial) Hamiltonian of the associated HJB equation by

\[
H_\epsilon(p) := \inf_{|\nu| \leq \frac{1}{\epsilon}} \{ \gamma \eta \nu^2 + p \nu \}, \quad p \in \mathbb{R}.
\]

Note that, for \( \epsilon = 0 \),

\[
H_0(p) = -\frac{1}{4\eta} p^2.
\]

**Proposition 1.** Let Assumption 1 hold and consider arbitrary \( T, \sigma, \gamma, \eta > 0 \) and \( Q \in \mathbb{R} \). Then, there exist constants \((\gamma, \pi, \delta, C) \in (0, \infty)^4 \) (depending only on \((T, \sigma, \gamma, \eta, Q)\)), such that, for all \((t, s, s', \pi) \in [0, T] \times \mathbb{R}^3\), all \( \delta \in [0, \delta] \), and all \( \epsilon \geq 0 \), we have

\[
\gamma \pi^2 \leq u^{\delta, \epsilon}(t, s, \pi, Q) \leq \gamma \left( \frac{\pi^2}{2} + 1 \right),
\]

\[
|u^{\delta, \epsilon}(t, s, \pi, Q) - u^{\delta, \epsilon}(t, s', \pi, Q)| \leq C |s - s'|.
\]

Moreover, for all \( \delta \in [0, \delta] \) and \( \epsilon \geq 0 \), \( u^{\delta, \epsilon}(\cdot, \cdot, \cdot, Q) \) is a (continuous on \([0, T] \times \mathbb{R}^2\)) viscosity solution of

\[
0 = \partial_t u^{\delta, \epsilon} + \frac{\sigma^2}{2} \partial_{ss} u^{\delta, \epsilon} + \frac{\delta^2}{2} \partial_{\pi \pi} u^{\delta, \epsilon} + H_\epsilon(\partial_{\pi} u^{\delta, \epsilon}) + \frac{\delta^2}{2} (\partial_{\pi} u^{\delta, \epsilon})^2 + \frac{\sigma^2}{2} (\partial_{s} u^{\delta, \epsilon} - \gamma (\pi + Q \partial_{s} P(t, s)))^2,
\]

\[
u^{\delta, \epsilon}(T, s, \pi, Q) = \frac{1}{\gamma} \pi^2.
\]

In addition, if \( \delta \epsilon = 0 \), the viscosity solution of (13) is unique in the class of functions satisfying (11)-(12); and if \( \delta \epsilon > 0 \), then \( u^{\delta, \epsilon}(\cdot, \cdot, \cdot, Q) \in C^{1,2}([0, T] \times \mathbb{R}^2) \).

**Proof:**

We drop the dependence of the functions on \( Q, \delta, \) and \( \epsilon \). By the Cauchy-Schwarz and Jensen inequalities we have that

\[
u(t, \pi) := \inf_{\nu \in A(t, T)} E \left[ \gamma \eta \int_t^T \nu^2 dr + \frac{\gamma l}{2} \pi^2_T \right]
\]

\[
\leq u(t, s, \pi)
\]

\[
\leq \frac{1}{2} \log \left( E \left[ e^{2Q \Gamma(t, s)} \right] \right) + \frac{1}{2} \inf_{\nu \in A(t, T)} \log E \left[ e^{2\Psi(t, \pi, \nu)} \right]
\]

\[
= \frac{1}{2} \log \left( E \left[ e^{2Q \Gamma(t, s)} \right] \right) + \frac{1}{2} \pi(t, \pi)
\]

Due to the boundedness of \( \partial_s P \), we can bound \( \frac{1}{2} \log \left( E \left[ e^{2Q \Gamma(t, s)} \right] \right) \) from above by a constant \( C \). It is a standard exercise to verify that

\[
u(t, \pi) := \frac{1}{\gamma} \pi^2,
\]

with \( \gamma \) being the solution to the Riccati equation

\[
\frac{\gamma'}{2} + H_0(\gamma) = 0, \quad \gamma_T = \frac{\gamma l}{2}.
\]
Indeed, the latter can be deduced from the fact that the proposed $u$ is a classical solution to the associated HJB equation

$$-\partial_t u - H_0(\partial_x u) = 0.$$ 

Note that $\gamma$ is bounded from below on $[0, T]$. Next, we deduce by a standard computation that

$$\pi \leq \log \mathbb{E} \left[ e^{2\Psi(t, \pi, 0)} \right] \leq \gamma \left( \frac{\pi^2}{2} + 1 \right),$$

for some constant $\gamma > 0$ and for all $\delta \in [0, \bar{\delta}]$, where $\bar{\delta}$ is chosen so that

$$\mathbb{E} \exp \left( \gamma \left( \frac{l}{2} + T \right) \delta \sup_{t \in [0, T]} B_t^2 \right) < \infty.$$ 

Thus, we have proved (11).

To show the Lipschitz continuity of $u$, we first observe that

$$s \rightarrow Q\Gamma(t, s) = -\gamma Q \left( P(T, S_T) - P(t, s) \right) = -\gamma Q \left( P(T, s + \sigma(W_T - W_0)) - P(t, s) \right)$$

is Lipschitz continuous. Thus,

$$u(t, s', \pi, Q) = \log \left( \inf_{\nu} \mathbb{E} \left[ e^{\Psi(t, \pi, \nu) + Q\Gamma(t, s')} \right] \right) \leq \log \left( \inf_{\nu} \mathbb{E} \left[ e^{L|Q||s-s'| + \Psi(t, \pi, \nu) + Q\Gamma(t, s)} \right] \right) \leq u(t, s, \pi) + L|Q||s-s'|,$$

with some constant $L > 0$ which only depends on $P$ and $\gamma$. Interchanging $s$ and $s'$, we obtain the Lipschitz continuity of $u$, stated in (12).

It remains to show that $u$ solves (13). To this end, we apply [12, Corollary 5.6] to conclude that the lower- and upper-semicontinuous envelopes of $\hat{V}$ (defined in (9)) are, respectively, viscosity super- and sub-solutions to the associated HJB equation:

$$\partial_t \hat{V} + \frac{\sigma^2}{2} \partial_{xx} \hat{V} + \frac{\delta^2}{2} \partial_{\pi \pi} \hat{V} + \delta^2 s \partial_{x \pi} \hat{V} + \delta^2 s^2 \partial_{xx} \hat{V} + \sup_{|\nu| \leq 1/\epsilon} [\nu \partial_x \hat{V} - \nu(s + \eta \nu) \partial_x \hat{V}] = 0,$$ (14)

$$\hat{V}(T, s, \pi, x) = -\exp \left( -\gamma \left( x + \pi s - \frac{\pi^2}{2} + QH(s) \right) \right).$$ (15)

In addition, [12, Proposition 5.4] yields the weak dynamic programming principle for $\hat{V}$. Note that the assumption of Lipschitz coefficients of the controlled state process, stated at the beginning of Section 5 of [12], is not satisfied herein, as the drift of $X$ in (3) is a quadratic function of $\nu$. Nevertheless, [12, Assumption A], required for [12, Corollary 5.6], holds in the present case. Indeed, [12, Assumptions A1–A3] hold trivially, and [12, Assumption A4] is verified exactly as in the proof of [12, Proposition 5.4], since the flow property still holds for the controlled state process $(\pi, S, X)$ (defined in (1)–(3)). Multiplying $\hat{V}$ by an exponential and taking a logarithmic transformation (to pass from $\hat{V}$ to $u$ via (8)–(9)), we conclude that the lower- and upper-semicontinuous envelopes of $u$ are, respectively, viscosity super- and sub-solutions to (13).

First, we analyze the case $\delta \epsilon > 0$. Using the dominated convergence, it is easy to show that, for any sufficiently small $\delta > 0$, $J(t, s, \pi, Q; \nu)$ is continuous in $(t, s, \pi)$, uniformly over $|\nu| \leq 1/\epsilon$. This implies the
continuity of $U$ in $(t, s, \pi)$ and, in turn, the continuity of $\hat{V}$ in $(t, s, \pi, x)$. The latter yields (via [12] Proposition 5.4) the strong dynamic principle for $V$ (i.e., ‘$V^{**}$’ and ‘$\phi$’ can be replaced by ‘$V$’ in equations (3.1) and (3.2) of [12]), which reads as follows: for any stopping time $\tau$ with values in $[t, T]$, we have

$$
\hat{V}(t, s, \pi, x) = \sup_{\nu \in \mathcal{A}^\nu(t, T)} \mathbb{E} \hat{V}(\tau, S_\tau, \pi_\tau^\nu, X_\tau^\nu).
$$

(16)

Next, we change the variables introducing $v := e^{-R_1(T-t)} \hat{V}$ and use (14) to derive the PDE for $v$. We restrict the domain of the latter equation to $(0, T) \times [-R_2, R_2]^3$ and equip it with the condition $v = e^{-R_1(T-t)} \hat{V}$ on the boundary of this domain (note that it is consistent with the terminal condition (15) due to continuity of $\hat{V}$). For sufficiently large $R_1$, the resulting boundary-value problem for $v$ falls within the scope of Theorem 3 in Section 6.4 of [27], which yields the existence of its classical solution. Undoing the change of variables and applying the standard verification argument (for which we use (16), we conclude that $e^{R_1(T-t)} v$ coincides with the value function $\hat{V}$. Multiplying by the appropriate exponential and taking logarithmic transformation (see [8]–[9]), we conclude that $u$ solves (13) on $(0, T) \times [-R_2, R_2]^2$ (which suffices, as $R_2 > 0$ is arbitrary).

For the aforementioned verification, we use (16), as well as the fact that the feedback optimal control is given by

$$
\nu_t = \left(\frac{-\partial_s u(t, S_t, \pi_t^\nu)}{2\gamma \eta}\right) \vee (-1/\epsilon) \wedge (1/\epsilon),
$$

and that the associated SDE for $\pi^\nu$ has a solution.

For the case $\delta \epsilon = 0$, in view of [12] Corollary 5.6, it suffices to prove a comparison principle for (13).

To this end, we fix $C > 0$ and without loss of generality we establish the comparison principle in the class of functions satisfying (11) and (12) for this given constant. This part of the proof is based on the results of [29]. Denote, for $(p, X, Y) \in \mathbb{R}^3$,

$$
\tilde{C} := 2C + \gamma |Q| \sup_{t,s} |\partial_s P(t, s)|,
$$

$$
G(t, s, p, X, Y) = \frac{\sigma^2}{2} \sup_{|\beta| \leq \tilde{C}} \left\{ -p(-2\beta + 2\gamma \pi) - \left(\beta^2 + 2\beta \gamma Q \partial_s P(t, s) - 2\gamma \pi \gamma Q \partial_s P(t, s) - \gamma^2 \pi^2\right) + X + Y \frac{\delta^2}{\sigma^2}\right\}.
$$

Note that, if $|p| \leq 2C$, we have that

$$
G(t, s, p, X) = \frac{\sigma^2}{2} \left(p - \gamma (\pi + Q \partial_s P(t, s))\right)^2 + \frac{\sigma^2 X}{2} + \frac{\delta^2 Y}{2}.
$$

Note that we want to characterize $u$ as a viscosity solution of (13), and to verify this property one needs to replace the derivatives of $u$ with the elements of sub- and super-jets. It is clear that, if $u$ is $C$-Lipschitz continuous in $s$, then its sub- and super-jets in $s$ are absolutely bounded by $C$. Thus, thanks to (12) and the Definition of $G$, any viscosity sub- or super-solution to (13), satisfying (11)–(12), is, respectively, a sub- or super-solution to the following PDE:

$$
0 = \partial_t u + H_\epsilon(\partial_s u) + \frac{\delta^2}{2} |\partial_s u|^2 + G(t, s, \partial_s u, \partial_{ss} u, \partial_{s\pi} u)
$$

(17)

$$
u(T, s, \pi) = \frac{\gamma l}{2} \pi^2.
$$

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Next, we consider $\delta = 0$. Then, the above PDE satisfies all the assumptions of [29, Theorem 2.1], hence, the comparison principle holds for this equation, which, in turn, yields the comparison principle for (13) (in the desired class).

Finally, we consider $\epsilon = 0$. Then, in view of the explicit form of $H_0$ (see (10)), equation (17) transforms into

$$0 = \partial_t u + \left( -\frac{1}{4\eta} + \frac{\delta^2}{2} \right) |\partial_\pi u|^2 + G(t, s, \partial_s u, \partial_{ss} u, \partial_\pi u) =: \partial_t u + \tilde{H}_0(\partial_\pi u) + G(t, s, \partial_s u, \partial_{ss} u, \partial_\pi u),$$

where

$$\tilde{H}_0(p) := \inf_{\nu \in \mathbb{R}} \left\{ \frac{\gamma\eta}{1 - 2\delta^2 \eta \gamma} \nu^2 + p\nu \right\},$$

and, by possibly decreasing $\delta$, we ensure that $1 - 2\delta^2 \gamma \eta > 0$. The above PDE, again, falls within the setting of [29, Theorem 2.1], which yields the desired comparison principle for (13).

Notice that, for any $p \in \mathbb{R}$, as $\epsilon \downarrow 0$, we have:

$$H_0(p) := \inf_{|\nu| \leq \frac{1}{\epsilon}} \{ \gamma\eta \nu^2 + p\nu \} \downarrow H_0(p).$$

Then, a standard application of the comparison principle for (13), with $\delta = 0$, yields the following corollary of Proposition 1.

**Corollary 1.** For any sequences $\delta_n \downarrow 0$ and $\epsilon_n \downarrow \epsilon_0 \geq 0$, $u_{\delta_n, \epsilon_n}$ decreases to $u_{0, \epsilon_0}$ locally uniformly.

### 2.2 Existence, uniqueness, and stability of the optimal control

We begin with the existence and uniqueness of the optimal control.

**Lemma 2.** There exists $\tilde{\delta} > 0$, such that, for any $(t, s, \pi, Q) \in [0, T] \times \mathbb{R}^3$, any $\delta \in [0, \tilde{\delta}]$, and any $\epsilon > 0$, there exists an optimizer $\nu^{\ast, t, s, \pi, Q, \delta, \epsilon}$ of (7).

**Remark 1.** The main contribution of this lemma is for $\delta = 0$, since for $\delta > 0$ we can easily obtain a feedback control from the maximizer of the Hamiltonian.

**Proof:**

The result follows from the convexity of the problem [8], the boundedness of its admissible controls, and the application of the Komlos’ lemma in [9].

**Lemma 3.** For any $\delta \geq 0$, there exist locally bounded functions $C_1$ and $C_2$ mapping, respectively, $(t, s, \pi) \in [0, T] \times \mathbb{R}^2$ and $(t, s, \pi, \epsilon) \in [0, T] \times \mathbb{R}^2 \times (0, \infty)$ into $(0, \infty)$, such that, for a.e. $\omega$, the mapping $A_{\omega}(t, T) \ni \nu \mapsto e^{\Psi(t, \pi, \nu; \omega)}$ is $\iota$-strong convex in the topology of $L^2_t[T, T]$, where

$$\iota := e^{-C_1 \delta^2 \sup_{r \in [t, T]} (B_T - B_r)^2 - C_2 \sup_{r \in [t, T]} |W_T - W_r| + \gamma \delta \int_t^T (B_r - B_t) dW_r} / C_2.$$

**Proof:**

A direct computation of the Hessian of $\nu \mapsto \Psi(t, \pi, \nu)$ yields

$$\partial_{\nu\nu} \Psi(t, \pi, \nu)(\nu', \nu'') = 2\gamma \gamma \int_t^T (\nu'_r)^2 dr + \gamma l \left( \int_t^T \nu'_r dr \right)^2.$$
Therefore,

\[\partial_{\nu^\prime} \left( e^{\Psi(t, \pi, \nu)} \right) (\nu^\prime, \nu^\prime) \geq e^{\Psi(t, \pi, \nu)} \left( 2\eta\gamma \int_{t}^{T} (\nu^\prime_r)^2 dr + \gamma l \left( \int_{t}^{T} \nu^\prime_r dr \right)^2 \right) \geq 2\eta\gamma e^{\Psi(t, \pi, \nu)} \int_{t}^{T} (\nu^\prime_r)^2 dr.\]

The following lower bound completes the proof:

\[
\inf_{\nu \in \mathcal{A}_s(t, T)} 2\eta\gamma e^{\Psi(t, \pi, \nu)} \geq 2\eta\gamma \exp \left( -\frac{\gamma l}{\epsilon^2} - \frac{\gamma l}{2} (2+\delta (B_T - B_t))^2 - \gamma \epsilon |\pi| |W_T - W_t| \right. \\
\left. - \frac{\gamma \epsilon}{\epsilon} \int_{t}^{T} |W_T - W_r| dr + \gamma \epsilon \int_{t}^{T} (B_r - B_t) dr \right). 
\]

\[\] \[\]

**Corollary 2.** There exists \(\delta > 0\), such that, for any \(\delta \in [0, \delta]\), \(\epsilon > 0\), and \((t, s, \pi, Q) \in [0, T] \times \mathbb{R}^3\), the optimizer \(\nu^{*, t, s, \pi, Q, \epsilon} \) of \((\ref{7})\) is unique.

**Proof:**

Consider the mapping \(\mathcal{A}_s(t, T) \ni \nu \mapsto J(t, s, \pi, Q; \nu) \in \mathbb{R}\), which is well defined for sufficiently small \(\delta > 0\). Using Lemma 3 and the strict positivity of \(\epsilon \exp(Q(t, s)) \) (with \(\epsilon\) defined in Lemma 3), it is easy to deduce the strict convexity of the above mapping. The latter implies uniqueness of the optimizer. \(\]

Throughout the remainder of this section, we denote by \(\nu^{*, t, s, \pi, Q, \delta, \epsilon}\) the optimizer of \((\ref{8})\).

The following proposition establishes the stability of the optimal control w.r.t. the initial condition \((s, \pi, Q)\).

**Proposition 2.** There exists \(\delta > 0\), s.t., for any fixed \(t \in [0, T]\), \(\delta \in [0, \delta]\), and \(\epsilon > 0\), there exist locally Lipchitz functions \(C_{1, \delta, \epsilon}\) and \(C_{2, \delta, \epsilon}\) with \(C_{2, \delta, \epsilon}(s, s, \pi, \pi, Q, Q) = 0\) such that for all \(s, s', \pi, \pi', Q, Q \in \mathbb{R}^6\),

\[
\mathbb{E} \left[ |\nu^{*, t, s, \pi, Q, \delta, \epsilon} - \nu^{*, t, s', \pi', Q, \delta, \epsilon}|^2 \right] \leq C_{1, \delta, \epsilon} \left( s, s', \pi, \pi', |Q - Q'| \right) |\nu^{*, t, s, \pi, Q, \delta, \epsilon} U^{\delta, \epsilon} \right| + C_{2, \delta, \epsilon} \left( s, s', \pi, \pi', |Q - Q'| \right) \left( |U^{\delta, \epsilon}| + 1 \right). 
\]

In particular

\[
\mathbb{R}^3 \ni (s, \pi, Q) \mapsto \nu^{*, t, s, \pi, Q, \delta, \epsilon} \in L^2([t, T] \times \Omega) 
\]

is continuous for \(\delta \in [0, \delta]\).

**Proof:**

We fix \((t, \delta, \epsilon)\) and drop the dependence on these variables when not needed. First, we notice that there exists a constant \(L > 0\), s.t.

\[
e^{L|Q - Q'| + L(|Q| + |Q'|)|s - s'|} U(s, \pi, Q) \geq \mathbb{E} \left[ e^{\Psi(s, \pi, \nu^{*, s, \pi, Q, \epsilon}) + Q' \Gamma(s')} \right] \\
\geq \mathbb{E} \left[ e^{\Psi(s, \pi, \nu^{*, s, \pi, Q, \epsilon}) + Q' \Gamma(s')} - e^{\Psi(s, \pi, \nu^{*, s, \pi, Q})} \right] e^{Q' L} \\
\geq \mathbb{E} \left[ e^{\Psi(s, \pi, \nu^{*, s, \pi, Q, \epsilon}) + Q' \Gamma(s')} \right] + \mathbb{E} \left[ \partial_{\nu^\prime} \left( e^{\Psi(s, \pi, \nu^{*, s, \pi, Q, \epsilon}) + Q' \Gamma(s')} \right) (s, \pi, \nu^{*, s, \pi, Q, \epsilon} - s, \pi, Q^{'}) \right] \\
+ \mathbb{E} \left[ \int_{t}^{T} (s, \pi, \nu^{*, s, \pi, Q, \epsilon} - s, \pi, Q^{'})^2 dr \right] - \mathbb{E} \left[ e^{\Psi(s, \pi, \nu^{*, s, \pi, Q, \epsilon}) - e^{\Psi(s, \pi, \nu^{*, s, \pi, Q})}} \right] e^{Q' L},
\]
where $t$ is defined in Lemma 3 and the last inequality in the above relies on the $t$-convexity of the mapping $\nu \mapsto e^{\Psi(t,\pi,\nu)}$. Due to the optimality of $\nu^*, s', \pi', Q'$ and the admissibility of $\nu^*, s, \pi, Q$, for the problem with initial condition $(s', \pi', Q')$, we have

$$E \left[ \partial_\nu \left( e^{\Psi(s', u^*, s', \pi', Q')} + Q' \Gamma(s') \right) (\nu^*, s, \pi, Q - \nu^*, s', \pi', Q') \right] \geq 0. \quad (19)$$

Therefore, recalling the definition of $U$ in (7), we obtain

$$e^{L|Q-Q'|+L(|Q|+|Q'|)|s-s'|} U(s, \pi, Q) - U(s', \pi', Q') + \sup_{\nu \in A_0} E \left[ e^{\Psi(s, \pi, Q)} - e^{\Psi(s', \pi, Q)} \right] e^{|Q'| L}
\geq E \left[ t \int_t^T (\nu^*, s, \pi, Q - \nu^*, s', \pi', Q')^2 dt \right].$$

Note that

$$\sup_{\nu \in A_0} E \left[ e^{\Psi(s, \pi, Q)} - e^{\Psi(s', \pi, Q)} \right] \leq E [e^{\Psi(s', \pi)} | e^{\Psi(s, \pi)} - 1],$$

with $\chi_{\epsilon, \delta} := \frac{2\epsilon}{\delta}(|\pi| + |\pi'| + C(\epsilon) + 2\delta |BT - B_t| + \gamma \sigma |W_T - W_t| > 0$, which has finite exponential moments. It is easy to see that there exists a sufficiently small $\delta > 0$, s.t. $E(1/\epsilon) < \infty$ for all $\delta \in [0, \delta]$. Thus, using the reverse Holder’s inequality and the above estimates, we obtain

$$E(1/\epsilon) \left( e^{L|Q-Q'|+L(|Q|+|Q'|)|s-s'|} U(s, \pi, Q) - U(s', \pi', Q') + e^{|Q'| L} E[e^{\Psi(s', \pi)} | e^{\Psi(s, \pi)} - 1] \right)
\geq E\|\nu^*, s, \pi, Q - \nu^*, s', \pi', Q'\|_{L^2}^2$$

and we easily identify $C_{1, t, \delta, \epsilon}$ and $C_{2, t, \delta, \epsilon}$ whose regularity is a direct consequence of the boundedness of $\nu$ and the exponential moments of $\chi_{\epsilon, \delta}$. The continuity of $\mathbb{R}^3 \ni (s, \pi, Q) \mapsto \nu^*, t, s, \pi, Q, \delta, \epsilon \in L^2([t, T] \times \Omega)$ is now a consequence of the continuity of $U$. \hfill \blacksquare

Throughout the remainder of this section, we fix $\delta > 0$ for which the conclusions of Propositions 1 and 2, Lemma 2 and Corollary 2 hold.

### 2.3 Sensitivities of the value function

Our next goal is to analyze the regularity of the partial derivatives of $U^{\delta, \epsilon}$, and hence $u^{5, \epsilon}$, w.r.t. $(s, \pi, Q)$. We begin with $J^{s, \epsilon}$. For any $\delta \in [0, \delta]$, $\epsilon > 0$, and $\nu \in A_0(t, T)$, we use Fubini’s theorem to deduce:

$$\partial_s J(t, s, \pi, Q; \nu) = Q E \left[ \partial_s \Gamma(t, s) e^{\Psi(t, \pi, \nu) + \Gamma(t, s)} \right]
= \gamma Q E \left[ (\partial_s P(T, s + \sigma(W_T - W_t)) - \partial_s P(\pi, s)) e^{\Psi(t, \pi, \nu) + \Gamma(t, s)} \right], \quad (20)$$

$$\partial_{\pi} J(t, s, \pi, Q; \nu) = E \left[ \partial_{\pi} \Psi(t, \pi, \nu) e^{\Psi(t, \pi, \nu) + \Gamma(t, s)} \right]
= \gamma l(\pi + \int_t^T \nu_r dr + \delta(B_T - B_t) - \gamma (S_T - s)) e^{\Psi(t, \pi, \nu) + \Gamma(t, s)}, \quad (21)$$

$$\partial_Q J(t, s, \pi, Q; \nu) = E \left[ \Gamma(t, s) e^{\Psi(t, \pi, \nu) + \Gamma(t, s)} \right]. \quad (22)$$

Recall the definition of equidifferentiability given in [31].
Lemma 4. For any $\delta \in (0, \delta^*]$, any $\epsilon > 0$, and any $t \in [0, T]$, the family
\[
\{(\partial_s J(t, \cdot, \cdot), \partial_\pi J(t, \cdot, \cdot), \partial_Q J(t, \cdot, \cdot)) : \nu \in \mathcal{A}_c(t, T)\}
\]
is uniformly bounded and equidifferentiable on any compact in $[0, T] \times \mathbb{R}^3$. In addition, for any $(t, s_0, \pi_0, Q_0) \in [0, T] \times \mathbb{R}^3$, any $\delta \in (0, \delta^*)$, and any $\epsilon > 0$, the mapping
\[
(s, \pi, Q) \mapsto (\partial_s J(t, s_0, \pi_0, Q_0; \nu^{s,t,s,\pi,Q,\delta,\epsilon}), \partial_\pi J(t, s_0, \pi_0, Q_0; \nu^{s,t,s,\pi,Q,\delta,\epsilon}), \partial_Q J(t, s_0, \pi_0, Q_0; \nu^{s,t,s,\pi,Q,\delta,\epsilon}))
\]
is continuous.

Proof:
The uniform boundedness of $(\partial_s J, \partial_\pi J, \partial_Q J)$ follows by direct estimates. Formally differentiating the expressions for $(\partial_s J, \partial_\pi J, \partial_Q J)$, we represent all partial derivatives of these terms as expectations of the quantities of the form
\[
\chi_{\alpha,\beta}(t, s, \pi, Q)e^{\Psi(t, s, \pi, Q)+Q\Gamma(t, s)} \text{ for } \alpha, \beta = s, \pi, Q,
\]
for some random weights $\chi_{\alpha,\beta}$. Using the boundedness of $\nu \in \mathcal{A}_c$, the fact that $\delta$ is small enough, and Fubini’s theorem, we verify these formal derivations and show that the second order derivatives can be bounded locally uniformly in $(s, \pi, Q, \nu)$. Using the dominated convergence, we also deduce that the second order derivatives are continuous in $(s, \pi, Q, \nu)$. This implies the equidifferentiability of $(\partial_s J, \partial_\pi J, \partial_Q J)$. Finally, the continuity of $(\partial_s J, \partial_\pi J, \partial_Q J)$ in $\nu$ and Proposition 2 imply the second statement of the lemma.

The above lemma and the general version of the Envelope Theorem given in [31] allow us to establish the existence and representation of the partial derivatives of $U^{\delta,\epsilon}$.

Proposition 3. For any $\delta \in (0, \delta^*], \epsilon > 0$, and any $t \in [0, T]$, $U^{\delta,\epsilon}(t, s, \pi, Q)$ is continuously differentiable in $(s, \pi, Q)$, uniformly over $t \in [0, T]$ and $\delta \in (0, \delta^*)$, with some $\delta^* \in (0, \delta^*)$.

Proof:
Lemma 4 and Theorem 3 imply the existence of partial derivatives of $U^{\delta,\epsilon}$ w.r.t. $s, \pi, Q$, and the representations (23)–(25). An application of dominated convergence theorem shows that these partial derivatives are jointly continuous in $(s, \pi, Q)$. Hence, $U^{\delta,\epsilon}$ is continuously differentiable w.r.t. $(s, \pi, Q)$.

Using (18) and the differentiability of $U^{\delta,\epsilon}$, we conclude that the mapping
\[
\mathbb{R}^3 \ni (s, \pi, Q) \mapsto \nu^{s,t,s,\pi,Q,\delta,\epsilon} \in L^2([t, T] \times \Omega)
\]
is locally $1/2$-Hölder-continuous, uniformly over small enough $\delta > 0$. The latter observation, the explicit form of $\Psi$, $\Gamma$, $\partial_s \Psi$, $\partial_s \Gamma$ (see (20)–(22)), and the Cauchy-Schwartz inequality, imply the desired Hölder-continuity of the partial derivatives. It is easy to see that the Hölder exponent and the associated coefficients are uniform over $t \in [0, T]$ and $\delta \in (0, \delta^*)$, with some $\delta^* \in (0, \delta^*)$. Since, for $\alpha = s, \pi, Q$, $\partial_\alpha U^{\delta,\epsilon}(t, \cdot, \cdot, \cdot)$ is continuous uniformly over $t$, and, for any $(s, \pi, Q)$, $U^{\delta,\epsilon}(\cdot, s, \pi, Q)$ is continuous, it is a standard exercise to check (by contradiction) that $\partial_\alpha U^{\delta,\epsilon}$ is jointly continuous.
Remark 2. Due to the presence of the exponent ‘2’ in the left hand side of (18), at this stage, we cannot establish additional regularity of the derivatives of $U$ (such as the existence of second order derivatives). Nevertheless further regularity is shown in the subsequent sections of the paper (cf. the proof of Lemma 10).

2.4 Feedback representation of the optimal control

In this subsection, we first derive a Forward-Backward Stochastic Differential Equation (FBSDE) for the optimal control assuming $\delta, \epsilon > 0$ and use it to establish a uniform absolute bound on the optimal control. Then, taking limits as $\delta, \epsilon \to 0$, we obtain an Ordinary Differential Equation (ODE) for the optimal inventory in the underlying, with $\delta = \epsilon = 0$. We suppress the dependence on $Q$ in many quantities appearing in this subsection, as $Q$ remains constant.

Before proceeding, we comment briefly on the measurability issues. Thanks to Proposition 3, for $\delta \in [0, \frac{\epsilon}{\delta}]$ and $\epsilon > 0$, $u_{\delta,\epsilon}$ is continuous in $(t, s, \pi)$ and continuously differentiable in $(s, \pi)$. Hence, $\partial_s u_{\delta,\epsilon}$ and $\partial_\pi u_{\delta,\epsilon}$ are Borel measurable in $(t, s, \pi)$. The progressive measurability of $(r, \omega) \mapsto \nu^*, t, s, \pi, \delta, \epsilon r$ implies the progressive measurability of the optimal inventory in the underlying, $(r, \omega) \mapsto \pi^*, t, s, \pi, \delta, \epsilon r$. Thus, we conclude that $(r, \omega) \mapsto \partial_\alpha u_{\delta,\epsilon} (r, S^t, s, \pi^*, t, s, \pi, \delta, \epsilon r)$, for $\alpha = s, \pi$, are progressively measurable, which allows us to define the relevant quantities below. Finally, the continuity of the mapping $(s, \pi) \mapsto \nu^*, t, s, \pi, \delta, \epsilon \in \mathcal{A}_\epsilon(t, T)$ implies the progressive measurability of $(r, \omega, s, \pi) \mapsto \nu^*, t, s, \pi, \delta, \epsilon r$.

We begin with the (one-sided) martingale optimality principle for $U_{\delta,\epsilon}$.

Lemma 5. For any $\delta \in [0, \frac{\epsilon}{\delta}]$, any $\epsilon > 0$, and any $(t, s, \pi) \in [0, T] \times \mathbb{R}^2$, the process $(M^t, s, \pi, \delta, \epsilon)_t \in [t, T]$, defined by

$$M^t, s, \pi, \delta, \epsilon := U_{\delta,\epsilon} (l, S^t, l, \pi^*, l, t, s, \pi, \delta, \epsilon) \exp \left( \int_l^T \gamma \eta (\nu^*, t, s, \pi, \delta, \epsilon r)^2 dr - \sigma \gamma \int_l^T (\nu^*, t, s, \pi, \delta, \epsilon r + Q \partial_s P(r, St, s)) dW_r \right),$$

is a martingale with the terminal value

$$M^t, s, \pi, \delta, \epsilon_T = e^{\Psi(t, \pi, \nu^*, t, s, \pi, \delta, \epsilon)} + Q \Gamma(t, s).$$

Proof:

Throughout this proof, we fix $\delta \in [0, \frac{\epsilon}{\delta}]$ and $\epsilon > 0$, and drop these superscripts. Due to (7), we have $U(T, S^T, S^T, \pi^*, T, s, \pi) = \exp((\pi^*, t, s, \pi)^2 \gamma l)/2$. Then, the fact that $M^t, s, \pi$ satisfies the desired terminal condition follows directly from the definitions of $\Psi$ and $\Gamma$ (preceding (7)). It remains to show the martingale property. To this end, we claim that the optimal control is consistent (i.e. satisfies the flow property): for any $t \leq l \leq T$, a.s.

$$\nu^*, t, s, \pi r = \nu^*, l, S^t, l, \pi^*, t, s, \pi, a.e. r \in [l, T].$$

To prove this claim, we use the tower property and obtain, for any $(t, s, \pi) \in [0, T] \times \mathbb{R}^2$, $\nu \in \mathcal{A}_\epsilon(t, T)$, and
with the associated \((S, \pi) = (S^{t,s}, \pi^{t,s})\),

\[
\mathbb{E} e^{\Psi(t,\pi,\nu)+Q\Gamma(t,\pi)} = \mathbb{E} \left[ \exp \left( \int_t^T \gamma \nu_r^2 dr - \gamma \int_t^T (\pi_r + Q \partial_\pi P(r, S_r)) dW_r \right) \mathbb{E} (e^{\Psi(l,\pi,\nu)+Q\Gamma(l,S)} | F_t^l) \right] = \mathbb{E} \left[ \exp \left( \int_t^T \gamma \nu_r^2 dr - \gamma \int_t^T (\pi_r + Q \partial_\pi P(r, S_r)) dW_r \right) \times \mathbb{E} \left( e^{\Psi(l,\pi',\nu_2(z;1) \otimes \nu_1(z;1)) + Q\Gamma(l,s')} \right)_{s'=t, \pi'=t, z=(W-W_t, B-B_t)} \right] \\
\geq \mathbb{E} \left[ \exp \left( \int_t^T \gamma \nu_r^2 dr - \gamma \int_t^T (\pi_r + Q \partial_\pi P(r, S_r)) dW_r \right) \times \mathbb{E} \left( e^{\Psi(l,\pi',\nu_2(z;1) \otimes \nu_1(z;1)) + Q\Gamma(l,s')} \right)_{s'=t, \pi'=t, z=\nu} \right] = \mathbb{E} e^{\Psi(l,\pi,\nu_1(z;1) \otimes \nu_1(z;1)) + Q\Gamma(l,t)},
\]

where \(\otimes\) denotes the concatenation of paths, and we view the admissible controls as functions of Brownian increments on the associated time intervals. The above inequality implies that the objective of the optimization problem (\ref{eq:opt_prob}) will not increase if we modify \(\nu^{t,s,\pi}\) on \([t, T]\) to be equal to the right hand side of (\ref{eq:ineq1}). Then, due to uniqueness of the optimal control with the initial condition \((s, \pi)\) at time \(t\), (\ref{eq:ineq1}) must hold.

The martingale property follows easily from (\ref{eq:ineq1}): for \(t \leq l \leq T\),

\[
\mathbb{E} \left( M^{l,s,\pi}_t | F_t^l \right) = \mathbb{E} \left( e^{\Psi(l,\pi,\nu^{l,t,s,\pi}) + Q\Gamma(l,s)} | F_t^l \right) = \mathbb{E} \left( e^{\Psi(t,\pi,\nu^{t,t,s,\pi}) + Q\Gamma(t,s)} \right) = e^{\Psi(t,\pi,\nu^{t,t,s,\pi}) + Q\Gamma(t,s)},
\]

\[= \exp \left( \int_t^l \gamma \nu_r^2 dr - \gamma \int_t^l (\pi_r + Q \partial_\pi P(r, S_r)) dW_r \right) \times \mathbb{E} \left( e^{\Psi(l,\pi,\nu^{l,t,s,\pi}) + Q\Gamma(l,s)} \right) = \mathbb{E} \left( e^{\Psi(t,\pi,\nu^{t,t,s,\pi}) + Q\Gamma(t,s)} \right) = \mathbb{E} \left( e^{\Psi(t,\pi,\nu^{t,t,s,\pi}) + Q\Gamma(t,s)} \right) = \mathbb{E} \left( e^{\Psi(l,\pi,\nu^{l,t,s,\pi}) + Q\Gamma(l,s)} \right) = M^{l,s,\pi}_t.
\]

In order to derive an FBSDE representation for the optimal control it is convenient to work under a different probability measure. To construct such a measure, we will use the martingale \(M^{l,s,\pi,\delta,\epsilon} \). However, in order to apply Girsanov’s theorem, it is convenient to use an alternative representation of this martingale via

\[
\mathbb{E} \left( e^{\Psi(l,\pi,\nu^{l,t,s,\pi}) + Q\Gamma(l,s)} \right) = \mathbb{E} \left( e^{\Psi(t,\pi,\nu^{t,t,s,\pi}) + Q\Gamma(t,s)} \right) = \mathbb{E} \left( e^{\Psi(l,\pi,\nu^{l,t,s,\pi}) + Q\Gamma(l,s)} \right) = \mathbb{E} \left( e^{\Psi(t,\pi,\nu^{t,t,s,\pi}) + Q\Gamma(t,s)} \right) = \mathbb{E} \left( e^{\Psi(l,\pi,\nu^{l,t,s,\pi}) + Q\Gamma(l,s)} \right) = M^{l,s,\pi}_t.
\]

\[
\mathbb{P}^{\delta,\epsilon}(t, s, \pi) := \sigma \left( \partial_\pi u^{\delta,\epsilon}(t, s, \pi) - \gamma (\pi + Q \partial_\pi P(t, s)) \right), \tag{29}
\]

provided in the following lemma.

**Lemma 6.** For any \(\delta \in [0, \delta]\), any \(\epsilon > 0\), and any \((t, s, \pi) \in [0, T] \times \mathbb{R}^2\), the continuous modification of the
martingale \((M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon})_{t \in [t,T]}\) is given by

\[
M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon} = U^{\delta,\epsilon}(t, s, \pi) \exp \left( \int_t^l \mathbb{Z}^{\delta,\epsilon} (r, S_{r,s}^{t,s,\pi,\delta,\epsilon}, \pi_{r}^{t,s,\pi,\delta,\epsilon}) \, dW_r + \delta \partial_\pi u^{\delta,\epsilon}(r, S_{r,s}^{t,s,\pi,\delta,\epsilon}, \pi_{r}^{t,s,\pi,\delta,\epsilon}) dB_r \\
- \frac{1}{2} \int_t^l \left( \mathbb{Z}^{\delta,\epsilon} (r, S_{r,s}^{t,s,\pi,\delta,\epsilon}, \pi_{r}^{t,s,\pi,\delta,\epsilon}) \right)^2 + \delta^2 (\partial_\pi u^{\delta,\epsilon}(r, S_{r,s}^{t,s,\pi,\delta,\epsilon}, \pi_{r}^{t,s,\pi,\delta,\epsilon}))^2 \, dr \right).
\]

**Proof:**

As a martingale on a Brownian filtration, \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon}\) has a continuous modification. Since it is also positive, it must have the representation

\[
M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon} = U^{\delta,\epsilon}(t, s, \pi) \exp \left( \int_t^l \phi^W_r \, dW_r + \phi^B_r \, dB_r - \frac{1}{2} \int_t^l (\phi^W_r)^2 + (\phi^B_r)^2 \, dr \right),
\]

for some \(\phi^W\) and \(\phi^B\) that are almost surely square integrable in time. For \(\delta > 0\), applying Itô’s formula to the above representation of \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon}\) (viewed as a process in \(l \in [t,T]\)) and to the right hand side of (26), and equating the martingale terms, we obtain:

\[
\phi^W_r = \sigma (\partial_\pi u^{\delta,\epsilon}(r, S_r, \pi_{r}^{t,s,\delta,\epsilon}) - \gamma (\pi_{r}^{t,s,\delta,\epsilon} + Q \partial_\pi P(r, S_r))) = \mathbb{Z}^{\delta,\epsilon} (r, S_r, \pi_{r}^{t,s,\delta,\epsilon}) \quad \text{and} \quad \phi^B_r = \delta \partial_\pi u^{\delta,\epsilon}(r, S_r, \pi_{r}^{t,s,\delta,\epsilon}).
\]

To justify the application of Itô’s formula to \(u^{\delta,\epsilon}\), we recall that it is \(C^{1,2}\) for \(\delta > 0\).

It remains to analyze the case \(\delta = 0\). We begin by observing that Corollary \[\(|\quad|\]\) and the Hölder-continuity of the partial derivatives of \(U^{\delta,\epsilon}(t, \cdot)\), uniform over \(\delta\) (see Proposition \[\(|\quad|\]\)), imply that, for every \(t, \partial_\alpha U^{\delta,\epsilon}(t, \cdot) \to \partial_\alpha U^{0,\epsilon}(t, \cdot)\) uniformly on compacts, as \(\delta \to 0\), for \(\alpha = s, \pi\). Next, for \(\delta > 0\), we define \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon}\) by the right hand side of (26) with \(\pi_{t}^{t,s,\pi,\delta,\epsilon}\) and \(\nu_{t}^{t,s,\pi,\delta,\epsilon}\) in place of \(\pi_{t}^{t,t,\pi,\delta,\epsilon}\) and \(\nu_{t}^{t,t,\pi,\delta,\epsilon}\). It is easy to see, using Itô’s formula and (13), that the finite variation component of \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon}\) is nondecreasing. Its local martingale term can be expressed via \(\partial_\alpha U^{\delta,\epsilon}\), for \(\alpha = s, \pi\). Passing to the limit, as \(\delta \to 0\), we conclude that the local martingale component of \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon}\) converges to a continuous local martingale given by the same expression as the local martingale component of \(M_{l,s,\pi}^{t,s,\pi,0,\epsilon}\), but with \(\partial_\alpha U^{0,\epsilon}\) in place of \(\partial_\alpha U^{\delta,\epsilon}\), for \(\alpha = s, \pi\). On the other hand, in view of Corollary \[\(|\quad|\]\), \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon} \to M_{l,s,\pi}^{t,s,\pi,0,\epsilon}\), as \(\delta \to 0\), which implies that the finite variation component of \(M_{l,s,\pi}^{t,s,\pi,\delta,\epsilon}\) converges to a nondecreasing process. Since \(M_{l,s,\pi}^{t,s,\pi,0,\epsilon}\) is a continuous martingale, the latter nondecreasing process must be zero, and we conclude that \(dM_{l,s,\pi}^{t,s,\pi,0,\epsilon}\) has the same form as given by an application of Itô’s formula, despite the lack of smoothness of \(U^{0,\epsilon}\).

Using the martingales defined in Lemma \[\(|\quad|\]\) for any \((t, s, \pi) \in [0,T] \times \mathbb{R}^2\) and \(\delta \in [0, \mathcal{I}]\), \(\epsilon > 0\), we introduce the probability measure \(\mathbb{Q}^{t,s,\pi,\delta,\epsilon}\) on \(\mathcal{F}^T\):

\[
\frac{d\mathbb{Q}^{t,s,\pi,\delta,\epsilon}}{d\mathbb{P}} := \frac{M_T^{t,s,\pi,\delta,\epsilon}}{U^{\delta,\epsilon}(t, s, \pi)},
\]

so that

\[
\tilde{W}_{t,s,\pi,\delta,\epsilon} := W_{t,s} - \int_t^l \mathbb{Z}^{\delta,\epsilon} (r, S_{r,s}^{t,s,\pi,\delta,\epsilon}, \pi_{r}^{t,s,\pi,\delta,\epsilon}) \, dr \quad \text{and} \quad \tilde{B}_{t,s,\pi,\delta,\epsilon} := B_{t,s} - \int_t^l \delta \partial_\pi u^{\delta,\epsilon}(r, S_{r,s}^{t,s,\pi,\delta,\epsilon}, \pi_{r}^{t,s,\pi,\delta,\epsilon}) \, dr.
\]
are independent standard Brownian motions on \([t, T]\) under \(\mathbb{Q}^{t, s, \pi, \delta, \epsilon}\). For convenience, we will often drop some (or all) of the superscript \((t, s, \pi, \delta, \epsilon)\) in the notation for \(\mathbb{Q}, \tilde{B},\) and \(\hat{W}\), when it causes no confusion.

We now derive a FBSDE characterization of the optimal control under \(\mathbb{Q}\), for \(\delta, \epsilon > 0\). For notational convenience, we introduce the truncation function

\[
\phi_{\epsilon}(x) = (\epsilon^{-1} \wedge (x)) \vee (-\epsilon^{-1}), \quad x \in \mathbb{R}.
\]

**Proposition 4.** Let us fix an arbitrary initial point \((t_0, s_0, \pi_0) \in [0, T] \times \mathbb{R}^2\), and constants \(\delta, \epsilon > 0\), \(\pi\). Then, the associated optimal control has a continuous modification satisfying

\[
u^*_{t_0, s_0, \pi_0, \delta, \epsilon} = \phi_{\epsilon}(-Y^1_t/(2\gamma)), \quad Y^1_t := \partial_{\pi} u^{\delta, \epsilon}(t, S^{t_0, s_0}_t, Y^2_t), \quad Y^2_t := \pi^*_{t_0, s_0, \pi_0, \delta, \epsilon},
\]

and \((Y^1, Y^2)\) solve the following FBSDE on \([t_0, T]\):

\[
\begin{align*}
Y^1_t &= \gamma l Y^2_T - \int_t^T \gamma l \delta^2 Y^1_r + \sigma Z^{\delta, \epsilon}(r, S^{t_0, s_0}_r, Y^2_r)dr - \int_t^T \tilde{Z}^W_r d\tilde{W}_r - \int_t^T \tilde{Z}^B_r d\tilde{B}_r, \\
Y^2_t &= \pi_0 + \int_{t_0}^t \phi_{\epsilon}(-Y^1_s/(2\gamma)) + \delta^2 Y^1_sdr + \delta(\tilde{B}_t - \tilde{B}_{t_0}).
\end{align*}
\]

**Remark 3.** It is important to note that we are not using BSDE tools to claim the existence of a solution for the above system. A solution exists by the existence of an optimizer.

**Proof:**

For convenience, we drop the dependence on \((\delta, \epsilon)\). The representations \((33)\) and \((35)\) follow from the fact that \(u^{\delta, \epsilon} \in C^{1,2}\) and from the existence of an optimal control in a feedback form (see Proposition 1 and its proof).

It remains to prove that \((34)\) holds. Note that the latter BSDE is equivalent to the statement that

\[
\partial_{\pi} u(t, S^{t_0, s_0}_t, \pi^*_{t_0, s_0, \pi_0}) - \int_{t_0}^t \gamma l \delta^2 \partial_{\pi} u(r, S^{t_0, s_0}_r, \pi^*_{t_0, s_0, \pi_0}) + \gamma \sigma Z(r, S^{t_0, s_0}_r, \pi^*_{t_0, s_0, \pi_0})dr
\]

is a \(\mathbb{Q}^{t_0, s_0, \pi_0}\)-martingale, with the terminal condition

\[
\gamma l \pi^*_{t, S^{t_0, s_0}_t, \pi^*_{t_0, s_0, \pi_0}} - \int_{t_0}^T \gamma l \delta^2 \partial_{\pi} u(r, S^{t_0, s_0}_r, \pi^*_{t_0, s_0, \pi_0}) + \gamma \sigma Z(r, S^{t_0, s_0}_r, \pi^*_{t_0, s_0, \pi_0})dr.
\]

The terminal condition holds due to the fact that \(u(T, s, \pi) = \gamma l \pi^2/2\). To prove the martingale property, we notice that the representation \((26)\) and the consistency property \((28)\) imply, for all \(t_0 \leq t \leq t_1 \leq T\),

\[
M^1_{t, S^{t_0, s_0}_t, \pi^*_{t_0, s_0, \pi_0}} = M^{t_1, S^{t_1, s_0}_t, \pi^*_{t_1, s_0, \pi_0}}_{t_1, S^{t_1, s_0}_t, \pi^*_{t_1, s_0, \pi_0}} M^{t_1, S^{t_1, s_0}_t, \pi^*_{t_1, s_0, \pi_0}}_{t_1, S^{t_1, s_0}_t, \pi^*_{t_1, s_0, \pi_0}} \cdot M^1_{t_1, S^{t_1, s_0}_t, \pi^*_{t_1, s_0, \pi_0}} \bigg| U \left( t_1, S^{t_1, s_0}_t, \pi^*_{t_1, s_0, \pi_0} \right).
\]

Using the above inequality, the standard properties of conditional expectation, the consistency property \((28)\),
and the representation (24), we obtain

$$\partial_t U(t, S_t^0, \pi_t^0, \pi_t^{t_0, 0, \pi_0}) = \mathbb{E}_t \left[ M_t^{t, S_t^0, \pi_t^0, \pi_t^{t_0, 0, \pi_0}, \gamma l \delta (B_T - B_t) - \gamma \sigma (W_T - W_t) \right]$$

$$+ \mathbb{E}_t \left[ M_t^{t, S_t^0, \pi_t^0, \pi_t^{t_0, 0, \pi_0}, \gamma l \delta (B_T - B_t) - \gamma \sigma (W_T - W_t) \right]$$

Next, we notice that (36) implies

$$\text{Remark 4. It is easy to deduce from (36) and from the measurability properties discussed at the beginning of this subsection, that, for any } \mathcal{F}_T^{\pi_0}-\text{measurable random variable } \xi \text{ and any } r \in [t_0, T],$$

$$\mathbb{E}_r^{\pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0} \left[ \xi \right] = \mathbb{E}_r^{\pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0, \pi_0} \left[ \xi \right] \bigg|_{(s, \pi) = \left( g_r^{\delta, \epsilon}, \pi_r^{\delta, \epsilon}, 0, 0, \pi_0^{\delta, \epsilon} \right)}.$$

(37)

Next, we use the FBSDE representation in Proposition 4 to estimate the optimal control uniformly in $\epsilon, \delta$. To ease the notation, we introduce

$$g_r^{\delta, \epsilon} := \sigma (\partial_\delta u^{\delta, \epsilon}(r, S_r, \pi_r^{\delta, \epsilon}) - \gamma Q \partial_\delta P(r, S_r)),$$
which is bounded, uniformly over \((ω, t_0, s_0, π_0, δ, ϵ)\), due to Assumption I and the Lipschitz continuity of \(u^{δ, ϵ}\) in \(s\). Then, (34) can be written as

\[
Y_t^1 = γ l Y_t^2 + \int_t^T γ^2 σ^2 Y_r^2 - γ l δ^2 Y_r^1 - γ σ g_r^δ ϵ dr - \int_t^T \tilde{Z}_r^W dW_r - \int_t^T \tilde{Z}_r^B dB_r. \tag{38}
\]

**Theorem 1.** There exist constants \(δ_0, C > 0\), such that

\[
\left| \partial_u u^{δ, ϵ}(t, \mathbf{S}^{t_0, s_0, π_t, t_0, s_0, π_0, δ, ϵ}) \right| \leq C \left( 1 + |π_0| + δ \sup_{t_0 ≤ r ≤ t} |B_r - B_{t_0}| \right), \quad t \in [t_0, T], \tag{39}
\]

for all \((t_0, s_0, π_0) \in [0, T] \times \mathbb{R}^2, ϵ > 0, \text{ and } δ ∈ (0, δ_0]\).

**Proof:**

For notational simplicity, we drop the dependence of the processes on \(δ, ϵ\). By the classical BSDE estimates applied to (38) we obtain:

\[
\mathbb{E}^Q \left[ \sup_{t_0 ≤ t ≤ T} |Y_t^1|^2 + \int_{t_0}^T |\tilde{Z}_r^W|^2 + |\tilde{Z}_r^B|^2 dr \right] ≤ C \mathbb{E}^Q \left[ γ l (Y_T^2)^2 + \int_{t_0}^T (Y_r^2)^2 + g_r^2 dr \right] \leq C \mathbb{E}^Q \left[ (Y_T^2)^2 + \int_{t_0}^T (Y_r^2)^2 + g_r^2 dr \right] \tag{40}
\]

where as a part of our standing convention we have omitted the dependence of \(Q\) on \((t_0, s_0, π_0, δ, ϵ)\). Making use of (35), we apply Ito’s formula to \(Y_t^1 Y_t^2\) to obtain

\[
γ l (Y_T^2)^2 = Y_{t_0}^1 π_0 + \int_{t_0}^T -γ^2 σ^2 (Y_r^2)^2 + γ l δ^2 Y_r^1 Y_r^2 - Y_r^1 φ_r \left( \frac{Y_r^1}{2γ} \right) + δ^2 (Y_r^1)^2 + γ σ Y_r^2 g_r + δ \tilde{Z}_r^B dr
\]

\[+ \int_{t_0}^T Y_r^2 \tilde{Z}_r^W dW_r + \int_{t_0}^T Y_r^1 δ + Y_r^2 \tilde{Z}_r^B dB_r. \tag{41}\]

Fix \(λ > 0\) to be determined. There exists \(C_λ\) so that for all \(a, b ∈ \mathbb{R}, ab ≤ λa^2 + C_λ b^2\). The above equality and (41) imply that, for all \(ϵ > 0\) and all small enough \(δ > 0\),

\[
\mathbb{E} \left[ γ l (Y_T^2)^2 + \int_{t_0}^T γ^2 σ^2 (Y_r^2)^2 + Y_r^1 φ_r \left( \frac{Y_r^1}{2γ} \right) dr \right] ≤ Y_{t_0}^1 π_0
\]

\[+ \mathbb{E} \left[ \int_{t_0}^T δ^2 (Y_r^1)^2 + γ l δ^2 Y_r^1 Y_r^2 + γ σ Y_r^2 g_r + δ \tilde{Z}_r^B dr \right]
\]

\[≤ λ(Y_{t_0}^1)^2 + C_λ π_0^2 + \mathbb{E} \left[ \int_{t_0}^T δ^2 (1 + γ l/2) (Y_r^1)^2 + \frac{1}{3} γ^2 σ^2 (Y_r^2)^2 + C_γ g_r^2 + δ \tilde{Z}_r^B dr \right]
\]

\[≤ C_λ π_0^2 + δ \mathbb{E} \left[ \int_{t_0}^T \tilde{Z}_r^B dr \right]
\]

\[+ \mathbb{E} \left[ (λC + δ(T - t_0)) (Y_T^2)^2 + \int_{t_0}^T \left( λC + δ(T - t_0) + \frac{1}{3} λC + δ(T - t_0) + C_γ g_r^2 \right) (Y_r^2)^2 + (λC + δ(T - t_0) + C_γ g_r^2) dW_r \right]
\]
As \( g \leq \epsilon > 0 \), then the previous estimate implies that, for all

\[
(\lambda + \delta)C + \delta(T - t_0) \leq \frac{\gamma l}{2} \quad \text{and} \quad (\lambda + \delta)C + \delta(T - t_0) + \frac{1}{3} \gamma^2 \sigma^2 \leq \frac{1}{2} \gamma^2 \sigma^2.
\]

Then, the previous estimate implies that, for all \( \epsilon > 0 \) and \( \delta \in (0, \delta_0] \),

\[
E \left[ \gamma l(Y_t^2) + \int_{t_0}^{T} \gamma^2 \sigma^2(Y_r^2)^2 + Y_r^1 \phi_r \left( \frac{Y_r^1}{2\eta \gamma} \right) dr \right] \leq C_1 \left( \pi_0^2 + \delta + E \left[ \int_{t_0}^{T} g_r^2 dr \right] \right).
\]

As \( g \) is absolutely bounded, the above inequality implies

\[
E \left[ \gamma l(Y_t^2) + \int_{t_0}^{T} \gamma^2 \sigma^2(Y_r^2)^2 + Y_r^1 \phi_r \left( \frac{Y_r^1}{2\eta \gamma} \right) dr \right] \leq C_2 \left( \pi_0^2 + 1 \right).
\]

The above estimate and (40) yield

\[
|Y_{t_0}^1|^2 \leq C_3(\pi_0^2 + 1) = C_3((Y_{t_0}^2)^2 + 1).
\]

Repeating the procedure for arbitrary \( t \in [t_0, T] \) in place of \( t_0 \) (and taking conditional, as opposed regular, expectations), we obtain

\[
|Y_t^1|^2 \leq C_3(\pi_t^2 + 1) = C_3((Y_t^2)^2 + 1), \quad t \in [t_0, T].
\]

Bringing back the superscript \((\delta, \epsilon)\), we deduce from the above estimate that

\[
\phi_r \left( -Y_t^1 \right) = \gamma l(Y_t^2) + \int_{t_0}^{T} \gamma^2 \sigma^2(Y_r^2)^2 + Y_r^1 \phi_r \left( \frac{Y_r^1}{2\eta \gamma} \right) dr \leq C_2 \left( \pi_0^2 + 1 \right), \quad t \in [t_0, T],
\]

with some progressively measurable bounded processes \( \bar{C}_t^\epsilon \) and \( \tilde{C}_t^\delta \), that are uniformly bounded for \( \epsilon > 0 \) and \( \delta \in (0, \delta_0] \). Using the above representation, we can write the solution to (38) as follows:

\[
Y_t^2 = \pi_0 e^{\int_{t_0}^{t} C_u^\epsilon du} + \int_{t_0}^{t} e^{\int_{t_0}^{r} C_u^\epsilon du} (\bar{C}_u^\delta ds + \delta dB_s)
\]

where the anticipating integral is to be understood as

\[
\int_{t_0}^{T} e^{\int_{t_0}^{r} C_u^\epsilon du} dB_s = e^{\int_{t_0}^{r} C_u^\epsilon du} \int_{t_0}^{T} e^{-\int_{t_0}^{r} C_u^\epsilon du} dB_s.
\]

Using the above, we can represent the solution to (38) as

\[
Y_t^1 = e^{-\gamma l(T-t)} \mathbb{E}_t^{Q_{t_0}} \left[ \gamma l \left( \pi_0 e^{\int_{t_0}^{T} C_u^\epsilon du} + \int_{t_0}^{T} e^{\int_{t_0}^{r} C_u^\epsilon du} (\bar{C}/\sigma ds) \right) \right] + \int_{t}^{T} e^{\gamma l(T-r)} \left( \frac{1}{2} \gamma^2 \sigma^2 \left( \pi_0 e^{\int_{t_0}^{T} C_u^\epsilon du} + \int_{t_0}^{T} e^{\int_{t_0}^{r} C_u^\epsilon du} (\bar{C}/\sigma ds) \right) \right) dr.
\]

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Then, the uniform boundedness of the processes $C^{h,\epsilon}, \tilde{C}^{h,\epsilon}, g^{h,\epsilon}$, the identity
\[
\int_{t_0}^{t} e^{\int_{s}^{t} C^{h,\epsilon}_u du} dB_s = \tilde{B}_t - e^{\int_{t_0}^{t} C^{h,\epsilon}_u du} \tilde{B}_{t_0} + \int_{t_0}^{t} C^{h,\epsilon}_u e^{\int_{s}^{t} C^{h,\epsilon}_u du} \tilde{B}_s ds
\]
and the fact that $\tilde{B}$ is a Brownian motion under $Q^{h,\epsilon}$, yield
\[
|Y^1_t| \leq \hat{C}(1 + |\pi_0| + \delta \sup_{t_0 \leq r \leq t} |\tilde{B}_r - \tilde{B}_{t_0}|), \quad t \in [t_0, T],
\]
with a constant $\hat{C}$ independent of $(t_0, s_0, \pi_0) \in [0, T] \times \mathbb{R}^2$, $\epsilon > 0$, and $\delta \in (0, \delta_0]$. Note that $\tilde{B}_t - \tilde{B}_{t_0} = B_t - B_{t_0} - \delta \int_{t_0}^{t} Y^1_r dr$. Thus,
\[
\sup_{t_0 \leq r \leq t} |Y^1_r| \leq \hat{C}(1 + |\pi_0| + \delta \sup_{t_0 \leq r \leq t} |B_r - B_{t_0}| + \delta T \sup_{t_0 \leq r \leq t} |Y^1_r|),
\]
which yields the desired estimate. $\blacksquare$

Next, we establish the monotonicity of the feedback optimal control function $\partial_s u^{h,\epsilon}$.

**Lemma 7.** For any $(t, s) \in [0, T] \times \mathbb{R}$, $\epsilon > 0$, and $\delta \in (0, \bar{\delta}]$, the functions $U^{h,\epsilon}(t, s, \cdot)$ and $u^{h,\epsilon}(t, s, \cdot)$ are convex.

**Proof:**

We omit the dependence of the functions on $\epsilon$, $\delta$. The convexity of $U$ is a direct consequence of convexity of the square function and exponential. Indeed for any $(\lambda, t, s, \pi_1, \pi_2) \in [0, 1] \times [0, T] \times \mathbb{R}^3$, and any optimizing sequences $(\nu^{1,k})_{k=1,2 \in \mathbb{N}}$ for the problem (7) started at $(t, s, \pi_1)$, we have the inequality
\[
U(t, s, \lambda \pi_1 + (1 - \lambda) \pi_2) \leq J(t, s, \lambda \pi_1 + (1 - \lambda) \pi_2; \lambda \nu^{1,k} + (1 - \lambda) \nu^{2,k})
\]
\[
\leq \lambda J(t, s, \pi_1; \nu^{1,k}) + (1 - \lambda) J(t, s, \pi_2; \nu^{2,k}).
\]
Taking $k$ to $\infty$, this leads to the convexity of $U$ in $\pi$.

In order to prove the convexity of $u$ we adapt the ideas in [22, Section 4]. First, we define the measure of convexity
\[
[0, T] \times \mathbb{R}^3 \ni (t, s, \pi_1, \pi_2) \mapsto C(t, s, \pi_1, \pi_2) = u(t, s, \pi_1) + u(t, s, \pi_2) - 2u \left( t, s, \frac{\pi_1 + \pi_2}{2} \right)
\]
Due to the continuity of $u$, it is convex in $\pi$ if and only if $C(t, s, \pi_1, \pi_2) \geq 0$ for all $(t, s, \pi_1, \pi_2) \in [0, T] \times \mathbb{R}^3$. Due to the convexity of $u$ at final time, we have that
\[
C(T, \cdot) \geq 0.
\]
Denoting $\bar{\pi} = \frac{\pi_1 + \pi_2}{2}$, we differentiate $C$ and use the PDE (13), to obtain
\[
\mathcal{L}C := \partial_t C + \frac{\sigma^2}{2} \partial_{ss} C + \frac{\delta^2}{2} (\partial_{\pi_1 \pi_1} C + \partial_{\pi_1 \pi_2} C + 2 \partial_{\pi_2 \pi_2} C)
\]
\[
= \left( \partial_t u + \frac{\sigma^2}{2} \partial_{ss} u + \frac{\delta^2}{2} \partial_{\pi \pi} u \right) (\pi_1) + \left( \partial_t u + \frac{\sigma^2}{2} \partial_{ss} u + \frac{\delta^2}{2} \partial_{\pi \pi} u \right) (\pi_2)
\]
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There exists an affine function of the optimal control for
Thus,

\[
-2 \left( \partial_t u + \frac{\sigma^2}{2} \partial_x u + \frac{\delta^2}{2} \partial_{\pi} u \right) (\bar{\pi}) = -H_c(\partial_{\pi} u(\pi_1)) - H_c(\partial_{\pi} u(\pi_2)) + 2H_c(\partial_{\pi} u(\bar{\pi})) \\
- \frac{\delta^2}{2} ((\partial_u u(\pi_1))^2 + (\partial_u u(\pi_2))^2 - 2(\partial_u u(\bar{\pi}))^2) \\
- \frac{\sigma^2}{2} (\partial_u u(\pi_1) - \gamma(\pi_1 + Q\partial_x P))^2 + (\partial_u u(\pi_2) - \gamma(\pi_2 + Q\partial_x P))^2 - 2(\partial_u u(\pi) - \gamma(\pi + Q\partial_x P))^2 .
\]

Thus we can define \( A_1^{\delta, \epsilon} \), bounded continuous functions such that

\[
\mathcal{L}^C_C = A_1^{\delta, \epsilon} \partial_{\pi_1} C + A_2^{\delta, \epsilon} \partial_{\pi_2} C - \sigma^2 (\partial_u u(\bar{\pi}) - \gamma(\pi + Q\partial_x P)) \partial_u C \\
- \frac{\delta^2}{2} ((\partial_u C)^2 + (\partial_x C)^2 + 2\partial_x u(\bar{\pi})(\partial_u C + \partial_x C)) \\
- \frac{\sigma^2}{2} \left( (\partial_u u(\pi_1) - \partial_u u(\bar{\pi}) - \gamma(\pi_1 - \pi_2))^2 + (\partial_u u(\pi_2) - \partial_u u(\bar{\pi}) - \gamma(\pi_2 - \pi_1))^2 \right) \\
\leq A_1^{\delta, \epsilon} \partial_{\pi_1} C + A_2^{\delta, \epsilon} \partial_{\pi_2} C - \sigma^2 \left( \partial_u \left( \frac{\pi_1 + \pi_2}{2} \right) - \gamma \left( \frac{\pi_1 + \pi_2}{2} + Q\partial_x P \right) \right) \partial_u C \\
- \frac{\delta^2}{2} \partial_x u \left( \frac{\pi_1 + \pi_2}{2} \right) (\partial_{\pi_1} C + \partial_{\pi_2} C).
\]

Thus, \( C \) is a supersolution, of at most quadratic growth, of a linear parabolic equation. Due to Theorem 1 and the boundedness of \( \partial_u u \), the coefficients of the generator of this linear PDE have at most linear growth, which is sufficient to claim that \( C \geq 0 \) (e.g., via the Feynman-Kac formula).

Recall that the main goal of this subsection is to establish a tractable representation and the key properties of the optimal control for \( \epsilon = \delta = 0 \), by taking limits as \( \epsilon, \delta \downarrow 0 \).

**Theorem 2.** There exists an affine function \( 1/\epsilon_0 : \mathbb{R}^+ \to (0, \infty) \), such that the following statements hold.

- For any \( (t, s, \pi) \in [0, T] \times \mathbb{R}^2 \) and any \( \epsilon \in [0, \epsilon_0(\mathbb{E})] \), the optimal control \( \nu^{s, t, s, \pi, 0, \epsilon} \) has a modification that is a.s. continuous in time and absolutely bounded (a.s., uniformly in \( t \)) by \( 1/\epsilon_0(\mathbb{E}) \).

- For any \( t \in [0, T] \) and any \( \epsilon \geq 0 \), the mapping \( (s, \pi) \mapsto u^{0, \epsilon}(t, s, \pi) \) is continuously differentiable.

- For any \( (t, s, \pi) \in [0, T] \times \mathbb{R}^2 \) and \( \epsilon \in [0, \epsilon_0(\mathbb{E})] \), the aforementioned modification of the optimal control is given by

\[
\nu^{s, t, s, \pi, 0, \epsilon} = -\frac{1}{2\eta \gamma} \partial_{\pi} u^{0, \epsilon} (r, S_{r}^{t, s}, \pi^{s, t, s, \pi, 0, \epsilon}) ,
\]

where \( \pi^{s, t, s, \pi, 0, \epsilon} \) is the a.s. unique solution to the ODE

\[
d\pi^{s, t, s, \pi, 0, \epsilon} = -\frac{1}{2\eta \gamma} \partial_{\pi} u^{0, \epsilon} (r, S_{r}^{t, s}, \pi^{s, t, s, \pi, 0, \epsilon}) dr, \quad \pi^{s, t, s, \pi, 0, \epsilon} = \pi. \tag{42}
\]

- For any \( (t, s, \pi) \in [0, T] \times \mathbb{R}^2 \) and any \( \epsilon \in (0, \epsilon_0(\mathbb{E})] \), we have, a.s.,

\[
\lim_{\delta \downarrow 0} \sup_{r \in [t, T]} |\nu^{s, t, s, \pi, \delta, \epsilon} - \nu^{s, t, s, \pi, 0, \epsilon}| = 0 = \lim_{\epsilon \downarrow 0} |\nu^{s, t, s, \pi, 0, \epsilon} - \nu^{s, t, s, \pi, 0, 0}| ,
\]

where every optimal control is understood as its continuous modification.

\(^3\text{Note that the case } \epsilon > 0 \text{ is covered by Proposition 3}\)
Proof:

We fix \((t_0, s_0, \pi_0) \in [0, T] \times \mathbb{R}^2\), and, in most instances, drop the dependence on these variables.

First, we prove the statement of the theorem excluding the case \(\epsilon = 0\). Consider \(\epsilon > 0\), a sequence \(\delta_n \downarrow 0\), and the associated \(\pi^{*,\delta_n,\epsilon}\), satisfying (35):

\[
\frac{d\pi^t_{\delta_n,\epsilon}}{dt} = \left[ -\phi_{\epsilon} \left( \partial_{t} \pi^{\delta_n,\epsilon} \left( t, S_t, \pi^{\delta_n,\epsilon} \right) / (2\gamma) \right) + \delta_n^2 \partial_{t} \pi^{\delta_n,\epsilon} \left( t, S_t, \pi^{\delta_n,\epsilon} \right) \right] dt + \delta_n dB_t.
\]

Due to Theorem 1 for a.e. random outcome, the drift in the above ODE is absolutely bounded by an affine function of \(|\pi_0|\) and \(\delta_n \sup_{t_0 \leq r \leq T} |B_r - B_{t_0}|\) (the same function for all \(n\)). Thus, the family of functions \(\{ t \mapsto \pi^{*,\delta_n,\epsilon} \}_n \) is relatively compact (for a fixed random outcome). Hence, up to a subsequence, we can assume that it converges as \(n \to \infty\). Recall now that, as shown in the second part of the proof of Lemma 6 for any \(t \in [t_0, T]\), \(\partial_{t} \pi^{*,\delta_n,\epsilon}(t, \cdot) \to \partial_{t} \pi^{*,0,\epsilon}(t, \cdot)\) locally uniformly, as \(n \to \infty\). Then, using the dominated convergence, it is easy to see that the limit of \(\{ \pi^{*,\delta_n,\epsilon} \}_n \) (for a fixed random outcome, along a subsequence), denoted \(\hat{\pi}^*\), satisfies (43) with \(\delta_n\) replaced by zero. Recall also that \(|\hat{\pi}^*|\) is bounded by an affine function of \(|\pi_0|\) (independent of anything else, including the random outcome and the choice of a subsequence), which we denote by \(1/\epsilon_0\). Hence, for \(\epsilon \in (0, \epsilon_0([\pi_0]))\), \(\phi_{\epsilon}\) can be replaced by identity, and we conclude that \(\hat{\pi}^*\) satisfies (42). Proposition 3 and Lemma 7 imply that \(\partial_{t} \pi^{*,0,\epsilon}\) is jointly measurable and continuously increasing in \(\pi\) (the latter property is only established for \(\delta > 0\), but it extends trivially to \(\delta = 0\) by taking a limit, as above). Then, a combination of Caratheodory’s existence theorem and [16 Theorem 3.1] implies that the solution to (42) is unique. Thus, the limits along all subsequences of \(\{ \pi^{*,\delta_n,\epsilon} \}_n\) must be the same, and we conclude that this sequence converges a.s., uniformly in \(t\), to \(\hat{\pi}^*\), and that \(\nu^{*,\delta_n,\epsilon}\) converges in the same way to

\[
\hat{\nu}^* := -\frac{1}{2\eta \gamma} \partial_{t} \nu^{*,0,\epsilon} (t, S_t, \hat{\pi}^*_t).
\]

It only remains to show that \(\hat{\nu}^* = \nu^{*,0,\epsilon}\). The latter follows easily from the aforementioned convergence and the continuity of \(J(t, s, \pi, Q; \nu)\) in \((\delta, \nu)\), for uniformly bounded \(\{\nu\}\) (see (39)).

Next, we consider the case \(\epsilon = 0\). Recall that, for all \(\epsilon \in (0, \epsilon_0([\pi_0]))\), we have \(\nu^{*,0,\epsilon} = \nu^{*,0,\epsilon_0([\pi_0])}\). The first consequence of this observation is the existence of

\[
\hat{\nu} := \lim_{\epsilon \downarrow 0} \nu^{*,0,\epsilon}, \quad \hat{\pi} := \lim_{\epsilon \downarrow 0} \pi^{*,0,\epsilon},
\]

and the absolute boundedness of both processes (a.s., uniformly in \(t\)). The second consequence is the existence of

\[
\hat{\nu} := \lim_{\epsilon \downarrow 0} \partial_{t} \pi^{0,\epsilon}, \quad \hat{\nu} = \nu^{*,0,0},
\]

where the convergence holds uniformly on all compacts. Corollary 1 implies that

\[
\hat{\nu} = \partial_{t} \nu^{0,0}, \quad \hat{\nu} = \nu^{*,0,0},
\]

and the dominated convergence shows that the statement of the theorem holds for \(\epsilon = 0\). \(\blacksquare\)

Remark 5. For \(\pi\) restricted to a compact, there is in fact no need to take the limit in (44) — it suffices to consider small enough \(\epsilon > 0\). Indeed, for all \(\epsilon \in (0, \epsilon_0([\pi]))\), we have \(\nu^{*,0,\epsilon} = \nu^{*,0,\epsilon_0([\pi])}\). Thus, an immediate corollary of Theorem 2 is the following: \(\partial_{t} \nu^{0,0}(t, s, \pi) = \partial_{t} \nu^{0,\epsilon}(t, s, \pi)\) for all \(\epsilon \in (0, \epsilon_0([\pi]))\) and \(\alpha = s, \pi\).
Throughout the rest of the paper, we interpret the optimal control \( \nu^{*,t,s,\pi,\delta,\epsilon} \), for \( \delta, \epsilon > 0 \) and \( \delta = 0, \epsilon \geq 0 \), as its continuous modification (appearing in Theorems 4 and 2).

Our final goal in this subsection is to establish a convenient BSDE-type representation of the optimal control for \( \delta = \epsilon = 0 \). To this end, we recall the probability measure \( Q^{t,s,\pi,\delta,\epsilon} \) defined in (30) for \( \delta \in [0, \delta] \) and \( \epsilon > 0 \). We define the probability measure \( Q^{t,s,\pi,0,0} \) in the same way:

\[
\frac{dQ^{t,s,\pi,0,0}}{dP} := M^{t,s,\pi,0,0},
\]

\[
M^{t,s,\pi,0,0} := \exp \left( \int_t^l Z^{0,0}(r, S^t_r, \pi^{t,s,\pi,0,0}_r) \, dW_r - \frac{1}{2} \int_t^l \left( Z^{0,0}(r, S^t_r, \pi^{t,s,\pi,0,0}_r) \right)^2 \, dr \right), \quad l \in [t, T],
\]

\[
Z^{0,0}(t, s, \pi) = \mathcal{Z}(t, s, \pi) := \sigma (\partial_x u^{0,0}(t, s, \pi) - \gamma (\pi + Q \partial_s P(t, s))).
\]

Indeed, estimate (12) and Theorem 2 imply that \( Z^{0,0}(r, S^t_r, \pi^{t,s,\pi,0,0}_r) \) is absolutely bounded, uniformly over \( r \in [t, T] \), which, in particular, yields the martingale property of \( M^{t,s,\pi,0,0} \). The following lemma shows that the latter process is a limit of \( M^{t,s,\pi,\delta,\epsilon} \), defined in (26) (recall also Lemma 6).

**Lemma 8.** For any \((t, s, \pi) \in [0, T] \times \mathbb{R}^2 \) and any \( r \in [t, T] \), we have

\[
(L^2) \lim_{\varnothing \downarrow 0} \lim_{\delta \downarrow 0} M^{t,s,\pi,\delta,\epsilon} = M^{t,s,\pi,0,0}.
\]

**Proof:**

First, we recall the representation in Lemma 9 and notice that Theorem 2 and estimates (12), (39) imply the existence of a function \( \delta_0 : [1, \infty) \to (0, \infty) \) s.t.

\[
\sup_{(\delta, \epsilon) \in \mathcal{C}} \mathbb{E} \left( M^{t,s,\pi,\delta,\epsilon} \right)^p < \infty, \quad \forall \ p \geq 1.
\]

In addition, the last statement of Theorem 2 and the estimates (12), (39), and the uniform boundedness of the optimal control (as in the first statement of Theorem 2), yield

\[
(L^4) \lim_{\varnothing \downarrow 0} \lim_{\delta \downarrow 0} \sup_{r \in [t, T]} |\pi^{*,t,s,\pi,\delta,\epsilon}_r - \pi^{*,t,s,\pi,0,0}_r| = 0
\]

\[
= \lim_{\varnothing \downarrow 0} \lim_{\delta \downarrow 0} \sup_{r \in [t, T]} |\partial_x u^{\delta,\epsilon}_r (r, S^t_r, \pi^{*,t,s,\pi,\delta,\epsilon}_r) - \partial_x u^{0,0}_r (r, S^t_r, \pi^{*,t,s,\pi,0,0}_r)|,
\]

for \( \alpha = s, \pi \). To estimate \( \mathbb{E} \left( M^{t,s,\pi,\delta,\epsilon}_r - M^{t,s,\pi,0,0}_r \right)^2 \) (and complete the proof), it suffices to apply the inequality

\[
|e^x - e^y| \leq C|x - y|(e^x + e^y), \quad x, y \in \mathbb{R}
\]

(which holds for a sufficiently large constant \( C > 0 \)), along with the Cauchy-Schwartz inequality and Itô’s formula (applied to the fourth power of a Brownian integral, to compute its expectation).

Using the above constructions and Lemma 8 we can now derive the desired representation of the optimal
control for \( \delta = \epsilon = 0 \) via a conditional expectation. To this end, we define
\[
\kappa := \sqrt{\frac{\sigma^2 \gamma}{2\eta}}, \quad m(t) := -\kappa + \frac{2\kappa}{1 - \frac{l - \sqrt{2\gamma^2 \sigma^2 \eta}}{l + \sqrt{2\gamma^2 \sigma^2 \eta}} e^{-2\kappa(T-t)}}\tag{49}
\]

Using the above convergence and Lemma 8, it is easy to deduce that \( Q \) is a \( \mathbb{R} \) in addition, we define the (Borel measurable) function
\[
\mathcal{m} := \kappa \coth \left( \kappa(T-t) + \frac{1}{2} \ln \left( \frac{l + \sqrt{2\gamma^2 \sigma^2 \eta}}{l - \sqrt{2\gamma^2 \sigma^2 \eta}} \right) \right) \quad \text{if } l - \sqrt{2\gamma^2 \sigma^2 \eta} > 0,
\]
\[
\mathcal{m} := \kappa \tanh \left( \kappa(T-t) + \frac{1}{2} \ln \left( \frac{l + \sqrt{2\gamma^2 \sigma^2 \eta}}{l - \sqrt{2\gamma^2 \sigma^2 \eta}} \right) \right) \quad \text{if } l - \sqrt{2\gamma^2 \sigma^2 \eta} \leq 0,
\]
and note that \( m \) is the continuous (i.e. non-exploding) solution to the ODE
\[
-m'(t) + m^2(t) = \kappa^2, \quad t \in [0,T], \quad m(T) = \frac{l}{2\eta}.
\]

In addition, we define the (Borel measurable) function \( R \) via
\[
\partial_t u^{0,0}(t, s, \pi) = 2\gamma \eta m(t) \pi + e^{\int_0^t m(r)dr} R(t, s, \pi). \tag{50}
\]

Notice that the optimal control for \( \delta = \epsilon = 0 \) can be expressed in a feedback form via \( R \) (see Theorem \[4\]).

**Proposition 5.** The function \( R \) is absolutely bounded on \([0,T] \times \mathbb{R}^2\). Moreover, for any \((t_0, s_0, \pi_0) \in [0,T] \times \mathbb{R}^2\), the process \( R_t := R(t, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0) \), for \( t \in [t_0, T] \), is continuous and satisfies:
\[
R_t = -\gamma \sigma^2 \mathbb{E}_t^{\mathbb{Q}^{t_0,s_0,\pi_0,0,0}} \left[ \int_{t_0}^{T} e^{-\int_{t_0}^{r} m(v)dv} \left( \partial_t u^{0,0}(r, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0) - \gamma Q \partial_x P(r, S^{t_0,s_0}_{\pi_t}) \right) dr \right]. \tag{51}
\]

**Proof:**

First, we note that the right hand side of \eqref{51} is absolutely bounded by a constant. Taking \( t = t_0 \), we deduce the absolute boundedness of the function \( R \). Thus, it only remains to establish \eqref{51}.

Recall that, as follows from Proposition \[4\] for \( \delta \in (0, \delta] \) and \( \epsilon > 0 \), the process \( V^{\delta, \epsilon} \),
\[
V_t^{\delta, \epsilon} := \partial_t u^{\delta, \epsilon}(t, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0, 0) - \int_{t_0}^{T} \gamma \partial_x \partial_t u^{\delta, \epsilon}(r, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0, 0) dr + \gamma \sigma Z^{\delta, \epsilon}(r, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0, 0) dr,
\]
is a \( \mathbb{Q}^{t_0,s_0,\pi_0,0,0} \)-martingale on \([t_0, T] \). Recalling the definition of \( Z^{\delta, \epsilon} \) (see \eqref{29}) and using \eqref{48}, we conclude that there exists
\[
V_t^{0,0} := (L^2) \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} V_t^{\delta, \epsilon} = \partial_t u^{0,0}(t, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0, 0) - \int_{t_0}^{T} \gamma \sigma Z^{0,0}(r, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0, 0) dr.
\]

Using the above convergence and Lemma \[8\] it is easy to deduce that \( V^{0,0} \) is a \( \mathbb{Q}^{t_0,s_0,\pi_0,0,0} \)-martingale on \([t_0, T] \). Since the filtration is Brownian, there exists a continuous modification of this martingale. Using \eqref{50}, we obtain
\[
R_t = e^{-\int_{t_0}^{t} m(r)dr} V_t^{0,0} - 2\gamma \eta e^{-\int_{t_0}^{t} m(r)dr} m(t) \pi_t^{*}, t_0, s_0, \pi_0, 0, 0
\]
\[
+ e^{-\int_{t_0}^{t} m(r)dr} \int_{t_0}^{T} \gamma \sigma Z^{0,0}(r, S^{t_0,s_0}_{\pi_t}, \pi_t^{*}, t_0, s_0, \pi_0, 0, 0, 0) dr.
\]
Recalling that defined as marginal utility indifference price (also known as Davis price, see [18], [26], and references therein), which is with payoff $\hat{V}(t,s,\pi,x) = \int^t_0 \gamma \sigma \mathbb{E}[0,0](r,S^t_{r,t},\pi^*,t_0,s_0,\pi_0,0,0) drdt + \int^t_0 \gamma \sigma \mathbb{E}[0,0](s,S^t_{s,t},\pi^*,t_0,s_0,\pi_0,0,0) dt$

$$dR_t = -e^{-\int^0_0 m(r) dr} V^0,0(t) \gamma \sigma \mathbb{E}[0,0](r,S^t_{r,t},\pi^*,t_0,s_0,\pi_0,0,0) drdt + e^{-\int^0_0 m(r) dr} \gamma \sigma \mathbb{E}[0,0](s,S^t_{s,t},\pi^*,t_0,s_0,\pi_0,0,0) dt$$

$$\int^t_0 \gamma \sigma \mathbb{E}[0,0](r,S^t_{r,t},\pi^*,t_0,s_0,\pi_0,0,0) drdt + e^{-\int^0_0 m(r) dr} \gamma \sigma \mathbb{E}[0,0](s,S^t_{s,t},\pi^*,t_0,s_0,\pi_0,0,0) dt$$

Remark 6. The representation (51) is to be compared to [27] Theorem 3.1 where the authors study a linear-quadratic optimization problem with price impact. Due to the local structure of the optimization objective, they are able to explicitly find the optimal strategy of the investor which consists in following a convolution of the future target position with an explicit kernel. In the exponential utility framework, considered herein, the problem is not linear-quadratic anymore. However, (51) indicates that the investor follows a similar convolution of the target position $-Q \partial_s P_t$ shifted with $Q \partial_s u$. The presence of $\partial_s u$ means that this equality does not provide an explicit solution to the optimization problem. However, the representation (51) allows us to control the effect of $\partial_s u$, and in Section 2 we show that the impact of $\partial_s u$ can be controlled for small $\eta$, without decreasing the objective value at the main order of accuracy.

2.5 Utility indifference price

Recall the definition of utility indifference price (cf. [14] and references therein).

Definition 1. For any initial condition $(s,\pi,x,Q) \in \mathbb{R}^4$ at time $t \in [0,T]$, and any purchase quantity of the option $\Delta Q \in \mathbb{R}$, the number $P^*(t,s,\pi,s,Q,\Delta Q)$ is the utility indifference price of $\Delta Q$ units of the option with payoff $H(S_T)$ if

$$\hat{V}(t,s,\pi,x,Q) = \hat{V}(t,s,\pi,x) + P^*(t,s,\pi,s,Q,\Delta Q), Q - \Delta Q,$$

where $\hat{V}$ is defined in (4).

In view of (6) and (9), we have

$$P^*(t,s,\pi,s,Q,\Delta Q) = \Delta Q P(t,s) - \frac{1}{\gamma} \left( u^{0,0}(t,s,\pi,Q) - u^{0,0}(t,s,\pi,Q - \Delta Q) \right),$$

where we bring back the dependence on $Q$ in related quantities. To reduce the number of variables, we can assume that the option orders received by the market makers are small. Then, we only need to study the marginal utility indifference price (also known as Davis price, see [13], [26], and references therein), which is defined as

$$p^*(t,s,\pi,Q) := \lim_{\Delta Q \to 0} \frac{P^*(t,s,\pi,s,Q,\Delta Q)}{\Delta Q} = P(t,s) - \frac{1}{\gamma} \partial_Q u^{0,0}(t,s,\pi,Q) = \mathbb{E}^{Q^t,\pi_0,Q_0,0,0}[H(S_T)],$$
where the last equality follows from (25) (which is valid for $\epsilon = 0$ in view of the first statement of Theorem 2) and the fact that

$$
\frac{dQ^{t,s,\pi,Q,0,0}}{dP} = e^{\Psi(t,\pi,\nu^*,t,s,\pi,Q,0,0)+Q'(t,s)}U^{0,0}(t,s,\pi,Q).
$$

The latter fact follows from (27), Lemma 8, and the last statement of Theorem 2. Thus, the equilibrium price is the expectation under an equivalent measure, similar to the classical theory. We note that this measure depends on the claim and on the aggregate position of market maker in both the option and the underlying. However, it does not depend on the market maker’s expectations about the future order flow of options. Thanks to the definition of $Q^{t,s,\pi,Q,0,0}$, we also have

$$
p^*(t,s,\pi,Q) = E^{Q^{t,s,\pi,Q,0,0}}[H(S_T)] = P(t,s)
$$

$$
+ \sigma E\left[e^{\int_t^T Z^{0,0}_r(r,S_t^r,\pi^*,t,s,\pi,Q^*,0,0,Q)dr} - \frac{1}{2} \int_t^T \partial_s P(r,S_t^r)dr\right],
$$

where $Z^{0,0}(t,s,\pi,Q) = \sigma(\partial_s u^{0,0}(t,s,\pi,Q) - \gamma(\pi + Q\partial_s P(t,s)))$ is the function defined in (47).

### 3 Small impact expansion

In the previous section, we have established various theoretical properties of the (log-) value function $u$, the optimal hedging strategy, and the marginal utility indifference price $p^*$, for an option with payoff $QH(S_T)$ in the Almgren-Chriss model. We have also derived useful representations for these quantities, which, in particular, allow for numerical approximations (see e.g. (13)). However, the explicit expressions, that would provide additional insights into the behavior of $u$ and $p^*$, are not available. In this section, we derive an explicit expansion for $p^*$ assuming $\eta \to 0$. Note that, for $\eta = 0$, the underlying market turns into the complete Bachelier model, where the option can be hedged perfectly by the standard delta-hedging strategy, and the marginal utility indifference price (as well as any reasonable notion of price) of the option is given by $P(t,s)$. Naturally, we would like to find the leading order of the difference between $P(t,s)$ and $p^*$ as $\eta \to 0$.

First, we make an additional modeling convention. Namely, we claim that it is important to rescale the penalty coefficient for non-liquidation, $l$, appearing in (4). Indeed, this coefficient is meant to reflect the losses associated with liquidating the remaining inventory in the underlying. The latter losses are due to the presence of price impact in the underlying market, hence, they should vanish as $\eta \to 0$. Thus, in this section we make the following convention:

$$
l = \tilde{l}\eta,
$$

for some $\tilde{l} \geq 0$. This convention implies that we should replace $l$ by $\tilde{l}\eta$ in the formulas established in the previous section. In particular, since $\eta$ is small, the function $m$ defined in (49) satisfies, in the new notation:

$$
m(t) = \kappa \tanh \left( \kappa(T-t) + \frac{1}{2} \ln \left( \frac{\tilde{l}\sqrt{\eta} + \sqrt{2\gamma\sigma^2}}{-l\sqrt{\eta} + \sqrt{2\gamma\sigma^2}} \right) \right),
$$

and it solves

$$
-m'(t) + m^2(t) = \kappa^2, \quad m(T) = \frac{\tilde{l}}{2},
$$

with $\kappa$ defined in (49),

$$
\kappa = \sqrt{\frac{\sigma^2\gamma}{2\eta}}.
$$
Note that $\eta \to 0$ is equivalent to $\kappa \to \infty$.

For convenience, we often drop the superscript ‘($\delta, \epsilon$)’, as we mostly consider $\delta = \epsilon = 0$ in this section (whenever this is not the case, the superscripts will appear). In addition, to simplify the derivations, we will often omit the dependence on the initial condition $(s, \pi, Q)$, when it caused no confusion, and introduce

$$P_t := P(t, S_t), \quad \partial_s P_t := \partial_s P(t, S_t).$$

Before proceeding, we need to establish a BSDE representation for $\partial_s u$, which is similar to (50)–(51) established for $\partial_n u$.

**Proposition 6.** Let us fix an arbitrary initial condition $(s_0, \pi_0, Q) \in \mathbb{R}^3$ at time $t_0 \in [0, T]$ and drop the superscript $(t_0, s_0, \pi_0, Q)$. The following representation holds for $\partial_s u_t := \partial_s u(t, S_t, \pi_t^\star)$:

$$\partial_s u_t = Q\sigma^2 \gamma^2 \mathbb{E}_t^{Q^{t,s,\pi}} \left[ \int_t^T e^{\int_t^r Q\sigma^2 \gamma \partial_s P_t \, dr} \partial_s P_t (\pi_t^\star + Q \partial_s P_t) \, dr \right]_{(s, \pi) = (S_t, \pi_t^\star)}, \quad t \in [t_0, T]. \quad (56)$$

where $Q^{t,s,\pi}$ is the probability defined in (45).

**Proof:**

First, recalling (43) and noticing that for all $\epsilon \leq \epsilon_0$, with some deterministic $\epsilon_0 > 0$, we have

$$\pi_t^{s,0,\epsilon} = \pi_t^{s,0,0}, \quad \mathbb{Z}_{t_0}^{t, s, \pi_t^{s,0,\epsilon}}(t, S_t, \pi_t^{s,0,\epsilon}) = \mathbb{Z}_{t_0}^{t, s, \pi_t^{s,0,0}}(t, S_t, \pi_t^{s,0,0}),$$

for all $t \in [t_0, T]$, we conclude that Remark 4 applies for $\epsilon = 0$. Then, using (23), (52), (37), and the fact that $W$ has a drift under $Q^{t_0, s_0, \pi_0}$, we obtain:

$$\partial_s u_t = -\sigma \gamma Q \mathbb{E}_t^{Q^{t,s,\pi}} \left[ \int_t^T \partial_{ss} P_t \, dW_r \right]_{(s, \pi) = (S_t, \pi_t^\star)} = \sigma \gamma Q \mathbb{E}_t^{Q^{t_0, s_0, \pi_0}} \left[ \int_t^T \partial_{ss} P_t \, dW_r \right] = -\sigma \gamma Q \mathbb{E}_t^{Q^{t_0, s_0, \pi_0}} \left[ \int_t^T \partial_{ss} P_t (\partial_s u_t - \gamma (\pi_t^\star + Q \partial_s P_t)) \, dr \right].$$

Thus, $\partial_s u$ satisfies

$$d(\partial_s u_t) = \sigma^2 \gamma Q \partial_{ss} P_t (\partial_s u_t - \gamma (\pi_t^\star + Q \partial_s P_t)) \, dt + d\tilde{M}_t, \quad \partial_s u_{t_0} = 0,$$

where $\tilde{M}$ is a $Q^{t_0, s_0, \pi_0}$ martingale. We solve this BSDE for $\partial_s u$ and apply Remark 4 once more to obtain (56).

The following Lemma constitutes the main technical result for computing the desired price expansion. Its proof is postponed to the appendix.

**Lemma 9.** Let $\alpha$ and $\beta$ be adapted continuous and bounded processes (independent of $\eta$). Denote by $u$ and $\pi^\star$, respectively, the log-value function (8) and the associated optimal strategy, for an arbitrary (fixed) initial condition $(s, \pi, Q) \in \mathbb{R}^3$ at time $t = 0$. Define $\Gamma$ by

$$d\Gamma_t = \alpha_t (\partial_s u_t - \gamma (\pi_t^\star + Q \partial_s P_t)) \, dt + \beta_t d\tilde{W}_t,$$

26
with arbitrary (fixed) \( \Gamma_0 \in \mathbb{R} \), and with \( \tilde{W}_t \) being a \( \mathbb{Q} := \mathbb{Q}^{0,s,\pi} \)-Brownian motion. Then, as \( \eta \to 0 \),
\[
\mathbb{E}^\mathbb{Q} \left[ \int_0^T \Gamma_t(\pi^*_t + Q \partial_s P_t) dt \right] = \frac{1}{\kappa} \Gamma_0(\pi + Q \partial_s P(0, s)) + \int_0^T \frac{1}{\kappa} \mathbb{E} \left[ Q \sigma \partial_{ss} P r \beta_r \right] dr \\
+ Q \gamma \sigma^2 \int_0^T \mathbb{E}^\mathbb{Q} \left[ (\pi_t^* + Q \partial_s P_t) \partial_{ss} P_t \int_0^t \Gamma_r e^{\tilde{r}\gamma} Q^2 \gamma \partial_{ss} P_r, dr \right] dt + o(\kappa^{-1}). \tag{57}
\]

**Remark 7.** Although it is omitted in the above notation, \( \mathbb{Q} \) also depends on \( \eta \). In particular, in the second line of (57), \( \Gamma_0, (\pi^*_t + Q \partial_s P_t) \) and \( \Gamma_r \) depend on \( \eta \).

Lemma 9 is the main tool for the small impact asymptotic expansion derived in this section. It describes the behavior of the functional
\[
\Gamma \mapsto \mathbb{E}^\mathbb{Q} \left[ \int_0^T \Delta_t \Gamma_t dt \right]
\]
in the small \( \eta \), or large \( \kappa \), regime. Note that, in this regime, the function \( m \) is large and, thanks to (86), the process \( \Delta_t \), which is the optimally controlled deviation from the frictionless hedge \( Q \partial_s P_t \), is strongly mean reverting around zero. In fact, the process \( \Delta_t / \eta^{1/2} \) is in fact the so called fast variable mentioned in [8, 32, 34]. However, unlike the latter papers, herein we do not use the viscosity solution methods to characterize the limiting behavior of \( \Delta_t \). In fact, such methods seem to be inapplicable in our framework, as we establish an expansion for the marginal utility indifference price \( p^* \), computed via \( \bar{u} \), and the PDE describing the derivatives of \( \bar{u} \) lacks the crucial non-degeneracy property in the state variable \( \pi \). Therefore, herein, we develop a novel methodology that relies on the direct probabilistic analysis of the associated optimal control problem, which, in particular, allows us to establish the existence of the optimal control \( \nu^* \) and to deduce its relevant properties, such as the decomposition in Proposition 5.

**Theorem 3.** Let Assumption 7 and convention 54 hold. Then, the marginal utility indifference price \( p^* \) has the following representation for all \( (t, s, \pi, Q) \in [0, T] \times \mathbb{R}^3 \), as \( \eta \to 0 \):
\[
p^*(t, s, \pi, Q) = P(t, s) - Q \sqrt{2(\gamma \sigma^2 - \int_t^T \mathbb{E}_{t,s} \left[ (\sigma \partial_{ss} P)^2 \right] dr} \\
- \sqrt{2(\gamma \sigma^2(\pi + Q \partial_s P(t, s)) \partial_s P(t, s) + o(\sqrt{\eta})). \tag{58}
\]

**Proof:** Without loss of generality we prove the expansion at \( t = 0 \). We fix \( (S_0, \pi_0, Q_0) \) and drop these superscripts. Due to \( \{53\} \), we have
\[
p^*(t, s_0, \pi_0, Q_0) = P(0, S_0) + \sigma \mathbb{E} \left[ \exp \left( \int_0^T Z(r, S_r, \pi_r) dW_r - \frac{1}{2} \int_0^T (Z_r)^2 (r, S_r, \pi_r) dr \right) \right] \left[ \int_0^T \partial_s P_r dW_r \right] \\
= P(0, S_0) + \sigma^2 \mathbb{E}^\mathbb{Q} \left[ \int_0^T \partial_s P_r (\partial_s u(r, S_r, \pi_r) - \gamma(\pi^*_r + Q_0 \partial_s P_r)) dr \right] \\
= P(0, S_0) - \gamma \mathbb{E}^\mathbb{Q} \left[ \int_0^T \tilde{\Gamma}_r(\pi^*_r + Q_0 \partial_s P_r) dr \right],
\]
where
\[
\tilde{\Gamma}_r := \partial_s P_r - Q_0 \sigma^2 \gamma \partial_{ss} P_r \int_0^r \partial_s P_h e^{\tilde{r}h} Q_0 \sigma^2 \gamma \partial_{ss} P_r dh,
\]
and
and we have used (56) to obtain the last equality.

Recall that \( \partial_s P_t \) follows
\[
d(\partial_s P_t) = \sigma^2 \partial_{ss} P_t (\partial_s u_t - \gamma (\pi^*_t + Q \partial_s P_t)) dt + \sigma \partial_{ss} P_t dW_t,
\]
with a \( \mathbb{Q} \)-Brownian motion \( \tilde{W} \). Applying Lemma 9 to \( \Gamma_t := \partial_s P_t \), we obtain
\[
\mathbb{E}^Q \left[ \int_0^T \partial_s P_t (\pi^*_t + Q_0 \partial_s P_t) dt \right] = \frac{1}{\kappa} \partial_s P_0 (\pi_0 + Q_0 \partial_s P_0) + \int_0^T \frac{1}{\kappa} \mathbb{E} \left[ Q_0 (\sigma \partial_{ss} P_t)^2 \right] dr
\]
\[+ Q_0 \gamma^2 \int_0^T \mathbb{E}^Q \left[ (\pi^*_t + Q_0 \partial_s P_t) \partial_{ss} P_t \int_0^t \partial_s P_t e^{\int_0^t \gamma \sigma^2 \partial_{ss} P_t dv} dr \right] dt + o(\kappa^{-1}). \tag{59}
\]
Therefore,
\[
\mathbb{E}^Q \left[ \int_0^T \tilde{\Gamma}_t (\pi^*_t + Q_0 \partial_s P(t, S_t)) dt \right]
= \mathbb{E}^Q \left[ \int_0^T \left( \partial_s P_t - Q_0 \gamma^2 \partial_{ss} P_t \int_0^t \partial_s P_t e^{\int_0^t \gamma \sigma^2 \partial_{ss} P_t dv} dr \right) (\pi^*_t + Q_0 \partial_s P_t) dt \right]
= \frac{1}{\kappa} \partial_s P(0, S_0) (\pi_0 + Q_0 \partial_s P(0, S_0)) + \int_0^T \frac{1}{\kappa} \mathbb{E} \left[ Q_0 (\sigma \partial_{ss} P_t)^2 \right] dr + o(\kappa^{-1}).
\]

Collecting the above and recalling (55) we complete the proof. \( \blacksquare \)

The asymptotic expansion of the marginal utility indifference price, given by the right hand side of (58), has three components.

i) The frictionless, or fundamental, price \( P(t, s) \).

ii) A term of order \( \sqrt{\eta} \) proportional to the expected cumulative (frictionless) Gamma of the option.

iii) Another term of order \( \sqrt{\eta} \) which is proportional to the (frictionless) Delta of the option multiplied by the deviation of the current position from the optimal frictionless one, \( (\pi + Q \partial_s P(t, s)) \).

It is important to note that, along the optimal inventory path \( \pi^* \), the deviation \( (\pi^*_t + Q \partial_s P(t, S_t)) \) in fact converges to zero as \( \eta \to 0 \). Hence, if the agent acts optimally, the last term in the expansion (58) for \( p^*(r, S_r, \pi^*_r, Q) \) becomes negligible compared to the second one. This term is only relevant for the cases where the inventory level \( \pi \) is chosen to be far from the target frictionless value: e.g., at the initial moment when the agent starts hedging.

Finally, we point out that the leading terms in the expansion (58) are affine in \( Q \), and they are in fact linear if the agent follows optimal policy. Recall that the (arbitrage-free) utility indifference price is a natural notion of price in the options’ markets. Assume that the agent is a market maker for the option with payoff \( H(S_T) \), and that she is willing to buy or sell an infinitesimal amount of option’s shares for the price \( p^*(t, S_t, \tilde{\pi}_t, Q_t) \), where \( Q = (Q_t) \) is a given option’s inventory process and \( \tilde{\pi} \) is defined analogously to (42), as a solution to
\[
d\tilde{\pi}_t = -\frac{1}{2} \eta \gamma \partial_{\tilde{\pi}} u (t, S_t, \tilde{\pi}_t, Q_t) dt, \quad \tilde{\pi}_0 = \pi_0. \tag{60}
\]

\footnote{Note that, if \( Q \) is constant, then \( \pi^* = \tilde{\pi} \).}
(In the next section we describe a setting where it is optimal for the market maker to post such a price.) Then, as $\eta \to 0$, the expansion \[58\] implies that, in the leading order, the price impact in the option’s market is linear and permanent, with the impact coefficient at time $t$ being

$$
\sqrt{2\eta \gamma} \int_t^T E_t \left[ \sigma^2 (\partial_{ss} P_r)^2 \right] \, dr \geq 0.
$$

(61)

In particular, the trading costs (due to frictions) in the option’s market are locally quadratic in the traded volume.

4 Equilibrium with competing market makers

In this section we assume that the shares of option with payoff $H(S_T)$ can be traded through market makers. Each market maker can trade dynamically in the underlying asset, aiming to maximize her exponential utility, similar to the agent analyzed in Section 2. However, unlike the setting in Section 2, each market-making agent also determines the price at which she is willing to buy or sell the contingent claim (from or to her clients). The demand for the claim from the clients arrives according to the (adapted) process $(\bar{Q}_t)$, which we assume to be of finite variation (negative $d\bar{Q}_t$ represents sales by the clients) and denote $d\bar{Q}_t = q_t \, dt$. Remarkably, the exact dynamics of $q$ will not play any role in the equilibrium we construct. For the sake of the presentation, the reader may assume that $q$ is given exogenously. We assume that there are $N \geq 2$ market makers, and that they are in perfect competition: i.e. the agent who offers the best price gets all the client orders. If there are several agents offering the best price, the order is split uniformly among them. In what follows, we will often use the superscript $i = 1, \ldots, N$ to denote the variables associated with the $i$-th market maker. We assume that the market makers are homogenous: i.e., they have the same risk tolerance and start with the same initial inventory in the underlying and in the option.

Denote the price offered by the $i$th market maker by $p^i = (p^i_t)$, and the best alternative bid and ask prices available in the market by $p^b,i$ and $p^a,i$. Then, the process $Q^i = (Q^i_t)_{t \in [0,T]}$, representing the number of shares of claim $H$ held by the market maker, satisfies

$$
Q^i_t = Q^i_0 + \int_0^t \frac{1}{1 + N^b,i_u} \mathbf{1}_{ \{ p^b,i_u = p^b,i \} } (q_u)^+ - \frac{1}{1 + N^a,i_u} \mathbf{1}_{ \{ p^a,i_u = p^a,i \} } (q_u)^+ \, du.
$$

(62)

In equilibrium, we expect

$$
p^b,i_u = \max_{j \neq i} p^j_u, \quad p^a,i_u = \min_{j \neq i} p^j_u, \quad N^b,i_u = \sum_{k \neq i} \mathbf{1}_{ \{ p^b,k_u = p^b,i_u \} }, \quad N^a,i_u = \sum_{k \neq i} \mathbf{1}_{ \{ p^a,k_u = p^a,i_u \} }.
$$

(63)

The cash process $X^{h,i}(t)_{t \in [0,T]}$, generated by the market-making in the claim $H$, is given by

$$
X^{h,i}_t = - \int_0^t p^i_u \, dQ^i_u.
$$

(64)

Let the $S$-integrable process $\pi^i = (\pi^i_t)_{t \in [0,T]}$ denote the number of shares of $S$ held by the $i$th market maker, and suppose

$$
\pi^i_t = \pi^i_0 + \int_0^t \nu^i_u \, du,
$$

(65)

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where the adapted process $\nu^i = (\nu^i_t)_{t \in [0,T]}$ is the trading intensity. Trading in $S$ incurs a temporary price impact of Almgren-Chriss type, hence, the cash process $X^{s,i} = (X^{s,i}_t)_{t \in [0,T]}$ generated by the $i$th market maker’s trading in $S$ is given by

$$X^{s,i}_t = -\int_0^t \nu^i_u (S_u + \eta(\nu^i_u + \tilde{\nu}^i_u))du,$$

where $\eta > 0$ is the coefficient of temporary price impact, and $\tilde{\nu}^i$ is a locally square integrable process representing the total temporary impact produced by other agents. We assume that the value of $\eta$ is known, as it can be measured from the liquid market for $S$. In addition, in equilibrium, we expect

$$\tilde{\nu}^i = \sum_{j \neq i}^N \nu^j.$$

For simplicity, we assume that each agent knows the true model for the order flow $q$, which is an exogenously specified locally integrable stochastic process, adapted to the common filtration $\mathbb{F}^W$. In addition, each agent models $(\tilde{\nu}^i, p^a,i, p^b,i, N^{a,i}, N^{b,i})$ as random functions of her inventory and cumulative order flow (more details are given in Definition 2).

Let the process $X^i = (X^i_t)_{t \in [0,T]}$ denote the total cash position:

$$X^i_t = x^i_0 + X^{s,i}_t + X^{b,i}_t.$$

As in Section 2, we assume that the contingent claim $H$ is cash-settled, and the stock holdings $\pi_T$ are marked-to-market with a quadratic liquidation penalty $l(\pi_T)^2/2$, with $l \geq 0$. The agents’ risk preferences are described by an exponential utility with the risk aversion coefficient $\gamma$. Thus, each market-making agent maximizes the expected utility of her terminal wealth:

$$\sup_{(p^i, \nu^i) \in \mathcal{A}} \mathbb{E} \left[ -\exp \left( -\gamma \left( X^i_T + \pi^i_T S_T - l(\pi^i_T)^2/2 + Q_T H(S_T) \right) \right) \right],$$

where $\gamma > 0$ is the risk aversion of the market maker, and $q$ is an essentially bounded and progressively measurable demand process. The set $\mathcal{A}$ consists of all admissible strategies $(p, \nu)$, which are progressively measurable stochastic processes, such that $p$ is locally integrable and $\nu$ is essentially bounded. As mentioned above, we assume that the agents are homogeneous: they start with the same inventory levels $(\pi_0, Q_0)$, optimize the same objective \[69\], and use the same model for the underlying $S$ and for the demand (or order flow) process $q$.

**Definition 2.** A family of admissible strategies $(p^i_t, \nu^i_t)_{t \in [0,T]}$, for $i = 1, \ldots, N$, along with progressively measurable functions $\tilde{\nu}, \tilde{p}^a, \tilde{p}^b : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $N^a, \tilde{N}^a : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{N}$, form an equilibrium if:

1. for any admissible underlying inventory process $\pi$, $\tilde{p}^a(t, \pi_t)$ and $\tilde{p}^b(t, \pi_t)$ are admissible price processes and $\tilde{\nu}(t, \pi_t)$ is locally square integrable,

2. and, for any $i = 1, \ldots, N$, $(p^i_t, \nu^i_t)$ is a maximizer of the problem \[69\], where the state process $(S, X^i, Q^i, \pi^i)$ evolves according to \[62\]–\[68\], with

$$\tilde{\nu}^i := \tilde{\nu}(t, \pi^i_t), \quad p^a,i = \tilde{p}^a(t, \pi^i_t), \quad p^b,i = \tilde{p}^b(t, \pi^i_t), \quad N^{a,i} = \tilde{N}^a(t, \pi^i_t), \quad N^{b,i} = \tilde{N}^b(t, \pi^i_t),$$

As mentioned earlier, the exact dynamics of $q$ will not play any role in the construction of equilibrium. The equilibrium strategies are given in a feedback form as functions of cumulative order flow, with the functions being independent of the dynamics of $q$. 

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3. and the model is consistent: i.e., (63) and (67) hold.

Remark 8. The feedback form of the market characteristics \((\tilde{\nu}^i, p^{a,i}, p^{b,i}, N^{a,i}, N^{b,i})\) in Definition (7) is interpreted as follows. The \(i\)-th agent is aware of the presence of other market makers. However, she may not be aware of their exact characteristics, hence, she cannot deduce their inventories, or their total price impact \(\tilde{\nu}\), precisely. Then, it is natural for the agent to model the relevant market characteristics as functions of the factors that she can observe – i.e., \((S, \pi^i)\). An alternative approach would be to restrict these characteristics to be (path-dependent) functions of \(S\) only (i.e., to be given as adapted stochastic processes) – this would correspond to the more standard open-loop Nash equilibrium. We do not pursue the latter approach herein for technical reasons: it would require a solution to an additional, and rather involved, fixed-point problem. On the other hand, the approach chosen herein allows us to reduce the construction of equilibrium to a single-agent control problem easily, as shown in the next subsection.

We will show that there exists an equilibrium in which the agents trade options at the marginal utility indifference price computed in Section 2. Our construction of equilibrium is based on the simple observation that, if all other agents trade at the marginal indifference price, any given agent is also forced to trade at this price, and the latter does not change the value function of her control problem.

4.1 Construction of equilibrium

First, we assume that there is no trading in options and consider the problem of equilibrium between agents that only perform hedging (i.e. \(Q\) is a fixed constant). The following lemma connects the latter problem to the single-agent problem analyzed in Section 2.

Lemma 10. Denote by \(\bar{V}(t, s, \pi, x, Q)\) the value function of (4), with \(\eta \geq 0\) replaced by \(\bar{\eta} := \frac{\eta(N+1)^2}{4}\). Then,

\[
\partial_t \bar{V} + \frac{\sigma^2}{2} \partial^2_s \bar{V} + \sup_y \left[ y \partial_\pi \bar{V} - y(s + \eta y + \eta(N-1)\tilde{\nu}) \partial_x \bar{V} \right] = 0, \tag{70}
\]

\[
\bar{V}(T, s, \pi, x, Q) = -\exp \left( -\gamma \left( x + \pi s - \frac{\pi^2}{2} + QH(s) \right) \right), \tag{71}
\]

where

\[
\tilde{\nu}(t, s, \pi, Q) := (1 + (N-1)/2) \frac{1}{2\bar{\eta}} \left( \frac{\partial_\pi \bar{V}(t, s, \pi, 0, Q)}{\partial_x \bar{V}(t, s, \pi, 0, Q)} - s \right). \tag{72}
\]

Moreover, the supremum in (70) is attained at \(y = \tilde{\nu}\).

Remark 9. The choice \(x = 0\) in the right hand side of (72) is arbitrary: we could (and will, in the proof) use any value of \(x\), as this expression does not depend on \(x\). In addition,

\[
\tilde{\nu} = -\frac{1 + (N-1)/2}{2\gamma\bar{\eta}} \partial_\pi \bar{u}, \quad \bar{V} = -\frac{1}{\gamma} \partial_x \bar{V} = -e^{-\gamma(x + \pi s + QP(t,s))} \bar{u}, \tag{73}
\]

where \(\bar{u}\) is defined in (8), with \(\bar{\eta}\) in place of \(\eta\). In particular, Theorem (2) implies that \(|\tilde{\nu}(t, s, \pi, Q)|\) is bounded by an affine function of \(|\pi|\), uniformly over all \((t, s) \in [0, T] \times \mathbb{R}\) and over \(Q\) changing in a compact (the theorem is proven for fixed \(Q\), but it is easy to see that the desired bound holds uniformly over \(Q\) changing in a compact).
Proof:
Proposition 4 yields that $\tilde{V}$ is a viscosity solution to
\[
\frac{\partial}{\partial t} \tilde{V} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \tilde{V} + \sup_y \{ y \partial_x \tilde{V} - y(s + \tilde{\eta} y) \partial_x \tilde{V} \} = 0, \tag{74}
\]
with the terminal condition (71). Let us show that $\tilde{V}$ is in fact a classical solution to the above equation. We begin by noticing that the supremum in (74) is attained at
\[
y = \hat{\nu} := \frac{1}{2 \tilde{\eta}} \left( \frac{\partial_x \tilde{V}}{\partial_x \tilde{V}} - s \right) = (1 + (N - 1)/2)^{-1} \hat{\nu},
\]
and
\[
\sup_y [y \partial_x \tilde{V} - y(s + \tilde{\eta} y) \partial_x \tilde{V}] = \hat{\nu}^2 \tilde{\eta} \partial_x \tilde{V} = \frac{1}{4 \tilde{\eta}} \left( \frac{\partial_x \tilde{V}}{\partial_x \tilde{V}} - s \right)^2 \partial_x \tilde{V}.
\]
Then, we deduce from (7)–(9) that
\[
\left( |\tilde{V}| + |\partial_x \tilde{V}| + |\partial_{xx} \tilde{V}| + |\partial_{xx} \tilde{V}| + |\partial_x \tilde{V}| \right) (t, s, \pi, x, Q) \leq C_1(Q) \exp(C_2(Q)(x + \pi s + \pi^2)), \tag{75}
\]
with some locally bounded $C_1, C_2 > 0$. Treating the nonlinear part of (74) as a given source term, we notice that the latter is measurable and absolutely bounded by the right hand side of (75), with possibly different constants (recall Remark 9). Then, for any fixed $(\pi, x, Q) \in \mathbb{R}^3$, the Feynman-Kac formula yields the existence of a classical solution $\tilde{V}(\cdot, \pi, x, Q) \in C^{1,2}([0, T] \times \mathbb{R}^2)$ to (74), satisfying
\[
\tilde{V}(t, s, \pi, x, Q) = \mathbb{E} \left( \int_t^T \left( \frac{1}{4 \tilde{\eta}} \left( \frac{\partial_x \tilde{V}}{\partial_x \tilde{V}} - s \right)^2 \partial_x \tilde{V} \right) (r, s + \sigma W_{r-t}, \pi, x, Q) dr + \tilde{V}(T, s + \sigma W_{T-t}, \pi, x, Q) \right). \tag{76}
\]
Indeed, using the fact that $\partial_x \tilde{V}, \partial_{xx} \tilde{V}$ and $\tilde{V}$ are continuous in all variables (see Proposition 5 and Theorem 2) and the explicit form of Gaussian transition density, along with the growth estimate (75) and Fubini’s theorem, we can show that $\partial_x \tilde{V}$ and $\partial_{xx} \tilde{V}$ are well defined and continuous in $(t, s, \pi, x, Q)$. Next, we recall the value function $\tilde{V}^{\delta, \epsilon}$ of (9) for general $\delta, \epsilon > 0$. We fix $\pi \in \mathbb{R}$ and choose $\epsilon = \epsilon_0(|\pi|)/2$, where $\epsilon_0$ is defined in Theorem 2—so that the optimal control $\nu^*$ of the unconstrained hedging problem is absolutely bounded by $1/\epsilon$, for all initial underlying inventories in an open neighborhood of $\pi$, and for all $(t, s, x) \in [0, T] \times \mathbb{R}^2$. Recall that, as shown in the proof of Proposition 4, $\tilde{V}^{\delta, \epsilon}$ is a classical solution to (14). In addition, it is easy to see that, for sufficiently small $\delta > 0$, (75) holds with $\tilde{V}^{\delta, \epsilon}$ in place of $\tilde{V}$. Then, the Feynman-Kac and Itô’s formulas imply the following representation (for sufficiently small $\delta > 0$):
\[
\tilde{V}^{\delta, \epsilon}(t, s, \pi, x, Q) = \mathbb{E} \left( \int_t^T \left( \frac{\partial^2}{\partial x^2} \tilde{V}^{\delta, \epsilon} + \frac{\sigma^2}{2} s^2 \partial_{xx} \tilde{V}^{\delta, \epsilon} \right) \right) (r, s + \sigma W_{r-t}, \pi + \delta B_{r-t}, x, Q) dr
\]
\[
+ \tilde{V}(T, s + \sigma W_{T-t}, \pi + \delta B_{T-t}, x, Q).
\]
It follows from the proof of Theorem 2 that, as $\delta \downarrow 0$,
\[
(\partial_x \tilde{V}^{\delta, \epsilon}, \partial_{xx} \tilde{V}^{\delta, \epsilon}, \partial_x \tilde{V}^{\delta, \epsilon}, \partial_{xx} \tilde{V}^{\delta, \epsilon}, \partial_x \tilde{V}^{\delta, \epsilon}, \partial_{xx} \tilde{V}^{\delta, \epsilon}, \partial_Q \tilde{V}^{\delta, \epsilon}, \tilde{V}^{\delta, \epsilon}) \to (\partial_x \tilde{V}, \partial_{xx} \tilde{V}, \partial_x \tilde{V}, \partial_{xx} \tilde{V}, \partial_x \tilde{V}, \partial_{xx} \tilde{V}, \partial_Q \tilde{V}, \tilde{V}), \tag{78}
\]
uniformly over all compacts in \((t, s, x, Q)\) and over an open neighborhood of \(\pi\). Then, equation (78), the dominated convergence theorem, and equations (76) and (77), yield
\[
\hat{V}^\delta,\epsilon(t, s, \pi, x, Q) \to \hat{V}(t, s, \pi, x, Q),
\]
as \(\delta \downarrow 0\). Using (78) again, we conclude that \(\hat{V} = \hat{V}\). In particular, we conclude that \(\partial_t \hat{V}\) and \(\partial_{s_\epsilon} \hat{V}\) are well defined and continuous in \((t, s, \pi, x, Q)\). On the other hand, Proposition 3 and Theorem 2 yield the same property for all other first order partial derivatives of \(\hat{V}\). Hence, \(\hat{V} \in C^{1,2,1,1}(0, T) \times \mathbb{R}^2\). The infinite differentiability of \(\hat{V}\) in \(x\) (and the continuity of each derivative) follow easily from the second equation in (73).

To complete the proof, it suffices to notice that the supremum in (70) is attained at
\[
y = \frac{1}{2\eta} \left( \frac{\partial_x \hat{V}}{\partial_x V} - s \right) - \frac{N - 1}{2} \nu = \hat{\nu},
\]
and that
\[
\sup_y \left[ y \partial_x \hat{V} - y(s + \eta y + (N - 1) \hat{\nu}) \partial_x \hat{V} \right] = \hat{\nu}^2 \eta \partial_x \hat{V} = \frac{1}{4\eta} \left( \frac{\partial_x \hat{V}}{\partial_x V} - s \right)^2 \partial_x V.
\]
The equality of the nonlinear parts of (74) and (70) yields the statement of the lemma.

Next, we include the dynamic trading in options. Our ansatz for the equilibrium is as follows. Given a process \((\tilde{Q}_t)\), with \(d\tilde{Q}_t = q_t dt\) and with essentially bounded and progressively measurable \(q\), we define for \(i = 1, \ldots, N\):
\[
\nu_t^{*;i} := \hat{\nu}(t, S_t, \pi_t^{*;i}, Q_0 - \tilde{Q}_t/N), \quad \pi_t^{*;i} = \pi_0 + \int_0^t \hat{\nu}(u, S_u, \pi_u^{*;i}, Q_0 - \tilde{Q}_u/N) du, \quad (79)
\]
\[
\hat{p}(t, s, \pi, Q) := \frac{\partial Q_t \hat{V}(t, s, \pi, 0, Q)}{\partial_x \hat{V}(t, s, \pi, 0, Q)}, \quad (80)
\]
\[
p_t^{*;i} := \hat{p}(t, S_t, \pi_t^{*;i}, Q_0 - \tilde{Q}_t/N), \quad \rho^o(t, \pi) := \hat{\rho}(t, S_t, \pi, Q_0 - \tilde{Q}_t/N), \quad \hat{\rho}(t, \pi) := (N - 1) \hat{\rho}(t, S_t, \pi, Q_0 - \tilde{Q}_t/N), \quad \hat{N}^{\theta}(t, \pi) := n^{\theta}(t, \pi) := N - 1. \quad (81)
\]
The first equation in (73) and Lemma 7 imply that \(\hat{\nu}(t, s, \pi, Q)\) is decreasing in \(\pi\). Then, a combination of Carathéodory’s existence theorem and [16, Theorem 3.1] implies that there exists a unique solution \(\pi_t^{*,i}\) to (79) (see the proof of Theorem 2 for a similar argument). In particular, we obtain that \(\pi_t^{*,i} = \pi_t^{*;i}\) and \(p_t^{*,i} = p_t^{*;i}\) for all \(i, j = 1, \ldots, N\). In addition, as shown in Remark 9, \(\hat{\nu}(t, s, \pi, Q)\) is bounded by an affine function of \(|\pi|\) uniformly over \((t, s)\) and over \(Q\) changing in a compact. Using the boundedness of \((\tilde{Q}_t)\), we apply Gronwall’s inequality to conclude that every \(\pi_t^{*,i}\), and hence \(\nu_t^{*,i}\), is absolutely bounded by a constant. Next, we recall (8) and notice that Jensen’s inequality yields: \(\hat{U} := \exp(\hat{u}) \geq 1\). Then, (25) and (9) imply that \(\hat{p}\) is absolutely bounded. The latter means that \((p_t^{*,i}, \nu_t^{*,i})\), defined above, are admissible controls and that the first property in Definition 2 is satisfied by \((\hat{p}^0, \hat{p}^b, \hat{v})\). It is also clear that the third property in Definition 2 is satisfied. Thus, it remains to show that, for each \(i = 1, \ldots, N\), \((p_t^{*,i}, \nu_t^{*,i})\) is a maximizer for (69). The latter is accomplished by the following lemma.

**Lemma 11.** For any initial \((s_0, \pi_0, x_0, Q_0) \in \mathbb{R}^4\) and any \(i = 1, \ldots, N\), we have:
\[
\sup_{(p^{*,i}, \nu^{*,i}) \in \mathcal{A}} \mathbb{E} \left[ - \exp \left( -\gamma \left( X_T^i + \pi_T^{*,i} S_T - \frac{(\pi_T^{*,i})^2}{2} + Q_T^i H(S_T) \right) \right) \right] = \hat{V}(0, s_0, \pi_0, x_0, Q_0), \quad (83)
\]
and the supremum is attained at \((p^{*,i}, \nu^{*,i})\).
Proof:
Denote by $\hat{V}(t, s, \pi, x, Q)$ the value function of (69). A heuristic derivation yields the following HJB equation for $V$:

$$
\partial_t \hat{V} + \frac{1}{2} \partial^2_s \hat{V} + \sup_{y} [y \partial_x \hat{V} - y(s + \eta y + \eta(N - 1)\hat{\nu})\partial_x \hat{V}]
$$

$$
+ \sup_{p} \left[ (\partial_q \hat{V} - p \partial_x \hat{V}) \left( (1 + (N - 1)1_{\{p = \bar{p}\}})^{-1}1_{\{p \geq \bar{p}\}}q - (1 + (N - 1)1_{\{p = \bar{p}\}})^{-1}1_{\{p \leq \bar{p}\}}q^+ \right) \right] = 0.
$$

(84)

The idea of the proof is based on the simple observation that $\hat{V}$ solves the above equation. Indeed, recalling (80), we easily verify that, for any $(t, s, \pi, x, Q, \omega)$,

$$
\sup_{p \in \mathbb{R}} \left[ (\partial_q \hat{V}(t, s, \pi, x, Q) - p \partial_x \hat{V}(t, s, \pi, x, Q)) \left( (1 + (N - 1)1_{\{p = \bar{p}(t, s, \pi, Q)\}})^{-1}1_{\{p \geq \bar{p}(t, s, \pi, Q)\}}q - (1 + (N - 1)1_{\{p = \bar{p}(t, s, \pi, Q)\}})^{-1}1_{\{p \leq \bar{p}(t, s, \pi, Q)\}}q^+ \right) \right] = 0,
$$

(85)

turning (84) into (70), and that the optimizer in (85) is attained at $p = \bar{p}$. It only remains to notice that (23) and (9) imply

$$
|\partial_x \hat{V}(t, s, \pi, x, Q)| \leq C_1(Q) \exp(C_2(Q)(x + \pi s + \pi^2)),
$$

with locally bounded $C_1, C_2 > 0$. Using the latter estimate, as well as the smoothness of $\hat{V}$ established in Lemma 10 we apply the standard verification argument to complete the proof. ■

The above lemma proves that $(\{p^{\ast, i}, \nu^{\ast, i}\}, \bar{p}, \bar{p}^b, \bar{N}^a, \bar{N}^b)$, defined in (79)–(82), form an equilibrium in the sense of Definition 2.

5 Appendix

Proof of Lemma 9 By direct computation we have

$$
e^{-\int_{r}^{t} m(v)dv} = \frac{\cosh(\kappa(T - t) + \phi)}{\cosh(\kappa(T - r) + \phi)}.
$$

Next, we denote

$$
\Delta_t := \pi_t^* + Q\partial_{\nu} P_t,
$$

$$
\phi(t) := \frac{1}{2} \ln \left( \frac{\frac{2\gamma\sigma^2}{\eta} + \sqrt{\frac{\eta}{2\gamma\sigma^2}}}{\frac{2\gamma\sigma^2}{\eta} - \sqrt{\frac{\eta}{2\gamma\sigma^2}}} \right) = \frac{\eta}{2\gamma\sigma^2} + o(\eta^{1/2}),
$$

$$
\frac{A_{rs}}{\kappa} := \int_{r}^{s} e^{-\int_{r}^{t^*} m(v)dv}dt = \frac{\tanh(\kappa(T - r) + \phi)}{\kappa} - \frac{\sinh(\kappa(T - s) + \phi)}{\kappa \cosh(\kappa(T - r) + \phi)},
$$

$$
\phi_t := m(t) - \kappa A_t, T = \frac{\sinh(\phi)}{\cosh(\kappa(T - t) + \phi)} \leq \kappa \sqrt{\frac{\eta}{2\gamma\sigma^2}} \leq \frac{\epsilon}{2},
$$

$$
\frac{\dot{A}_{rs}}{\kappa} := \int_{r}^{s} e^{-\int_{r}^{t^*} m(v)dv}dt.
$$

Notice that for all $0 \leq r \leq s \leq T$, $0 \leq A_{rs} \leq 1$, $0 \leq \dot{A}_{rs} \leq 1$, and for all $0 \leq r < s \leq T$, we have: $A_{rs} \to 1$ as $\eta \to 0$. Then, the feedback representation of $\pi^*$ in Theorem 2 representation (50), Proposition 9.
representation (47), and the fact that \( \partial_t P_t \) is a martingale under \( \mathbb{P} \), yield

\[
d\Delta_t = -m(t) \Delta_t dt + \frac{\sigma^2}{2\eta} \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_r^t m(v) dv} \partial_s u_r dr \right] dt \\
- Q\kappa^2 \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_r^t m(v) dv} (\partial_s P_r - \partial_s P_t) dr \right] dt + Q\partial_s P_t (m(t) - \kappa A_{t,T}) dt \\
+ Q\sigma^2 \partial_{ss} P_t (\partial_s u_t - \gamma \Delta_t) dt + \sigma Q \partial_{ss} P_t d\tilde{W}_t.
\]

Using the dynamics of \( \partial_t P_t \) and the definition of \( \phi_t \), we transform the above into

\[
d\Delta_t = -m(t) \Delta_t dt + \frac{\sigma^2}{2\eta} \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_r^t m(v) dv} \partial_s u_r dr \right] dt \\
- Q\kappa^2 \sigma^2 \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_r^t m(v) dv} \int_{t_s}^r \partial_{ss} P_{t_s} (\partial_s u_{t_s} - \gamma \Delta_{t_s}) db dr \right] dt + Q\partial_s P_t \phi_t dt \\
+ Q\sigma^2 \partial_{ss} P_t (\partial_s u_t - \gamma \Delta_t) dt + \sigma Q \partial_{ss} P_t d\tilde{W}_t. \tag{86}
\]

Therefore,

\[
d(\Gamma_t \Delta_t) = -m(t) \Gamma_t \Delta_t dt + \frac{\sigma^2}{2\eta} \Gamma_t \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_r^t m(v) dv} \partial_s u_r dr \right] dt \\
- Q\kappa^2 \Gamma_t \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_r^t m(v) dv} (\partial_s P_r - \partial_s P_t) dr \right] dt + \Gamma_t Q \partial_s P_t \phi_t dt \\
+ Q\sigma^2 \Gamma_t \partial_{ss} P_t (\partial_s u_t - \gamma \Delta_t) dt + \Gamma_t \partial_{ss} P_t d\tilde{W}_t \\
+ \alpha_t (\partial_s u_t - \gamma \Delta_t) \Delta_t dt + \Gamma_t \beta_t d\tilde{W}_t + Q \sigma \partial_{ss} P_t \beta_t dt. \tag{87}
\]

Due to the boundedness assumption on \( \alpha, \beta, P \), and on the partial derivatives of \( P \), as well as the boundedness of the optimal control \( \nu^* \), the local martingales in the decomposition (87) of \( \Gamma_t \Delta_t = \Gamma_t (\pi_t^* + Q \partial_t P_t) \) are martingales. This decomposition also shows that \( \Gamma_t \Delta_t \) solves a random linear ODE (to derive this ODE, treat the first term in the right hand side of (87) as a linear function of \( \Gamma_t \Delta_t \) and the rest as exogenously given source term), which we solve explicitly and integrate the solution over \([0, T]\) to obtain:

\[
\mathbb{E}_t^Q \left[ \int_0^T \Gamma_t \Delta_t dt \right] = \frac{A_0}{\kappa} \Gamma_0 \Delta_0 + \int_0^T \mathbb{E}_t^Q \left[ \frac{A_r}{\kappa} Q \Gamma_r \partial_s P_r \phi_r \right] dr \\
+ \int_0^T \mathbb{E}_t^Q \left[ \Gamma_r \int_r^T e^{-\int_s^r m(v) dv} \partial_s u_v dh \right] dr \\
- Q\kappa \int_0^T \mathbb{E}_t^Q \left[ \int_0^r \Gamma_r \left( \int_r^T e^{-\int_s^r m(v) dv} (\partial_s P_h - \partial_s P_r) dh \right) \right] dr \\
+ Q\sigma^2 \int_0^T \mathbb{E}_t^Q \left[ \int_0^r e^{-\int_s^r m(v) dv} \Gamma_r \partial_{ss} P_r (\partial_s u_r - \gamma \Delta_r) dr \right] dt \tag{88}
\]
We now denote $\hat{\Gamma}$, where we (as before) used the fact that due to (56), the last term in the right hand side of the above becomes

\[ \int_0^T \mathbb{E}^Q \left[ \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t(\beta_t \gamma) \Delta_s dr} \right] dt \]

\[ + \int_0^T \mathbb{E}^Q \left[ \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t(\beta_t \gamma) \Delta_s dr} \right] dt \]

\[ = \frac{A_{0,T}}{\kappa} \Gamma_0 + \int_0^T \mathbb{E}^Q \left[ \frac{A_{r,T}}{\kappa} Q\Gamma_r \partial_s P_r \phi_r \right] dr \]

\[ + \int_0^T \frac{A_r}{\kappa} \mathbb{E}^Q [Q\partial_s P_r \beta_r] dr + \int_0^T \frac{\sigma^2}{2\kappa \eta} \mathbb{E}^Q \left[ \partial_s u_h \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t(\beta_t \gamma) \Delta_s dr} A_{r,T} \Gamma_r dr \right] \]

\[ - Q\sigma^2 \int_0^T \frac{A_r}{\kappa} \mathbb{E}^Q \left[ \partial_s P_r(\partial_s u_r - \gamma \Delta_r) \right] \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t(\beta_t \gamma) \Delta_s dr} \Gamma_r \partial_s P_r \phi_r \] \[ + Q\sigma^2 \int_0^T \frac{A_r}{\kappa} \mathbb{E}^Q \left[ \Gamma_r \partial_s P_r(\partial_s u_r - \gamma \Delta_r) \right] dr + \int_0^T \frac{A_{r,T}}{\kappa} \mathbb{E}^Q \left[ \alpha_r(\partial_s u_r - \gamma \Delta_r) \right] dr, \]

where we (as before) used the fact that $d\partial_s P_t = \sigma \partial_s P_t dW_t$ to represent the term $\partial_s P_s - \partial_s P_r$.

We denote

\[ \bar{\Gamma}_t := \frac{\kappa}{\gamma} \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t} A_{r,T} dr, \]

\[ \tilde{\Gamma}_t := -Q\sigma^2 \partial_s P_t A_{r,T} \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t} \Gamma_r A_{r,T} dr + \frac{A_{r,T}(\alpha_i \Delta_i + Q\sigma^2 \Gamma_i \partial_s P_i)}{\kappa}, \]

and group the terms in the right hand side of (89) as follows:

\[ \mathbb{E}^Q \left[ \int_0^T \Gamma_t \Delta_t dt \right] = \frac{A_{0,T}}{\kappa} \Gamma_0 + \int_0^T \mathbb{E}^Q \left[ \frac{A_{r,T}}{\kappa} Q\Gamma_r \partial_s P_r \phi_r \right] dr \]

\[ + \int_0^T \frac{A_r}{\kappa} \mathbb{E}^Q [Q\partial_s P_r \beta_r] dr \]

\[ + Q\gamma^2 \int_0^T \mathbb{E}^Q \left[ \partial_s P_r A_{r,T} \Gamma_r \partial_s P_r \phi_r \right] \Gamma_t \partial_s P_r \phi_r \] \[ - \frac{\gamma}{\kappa} \int_0^T \mathbb{E}^Q \left[ A_{r,T} \alpha_r \Delta_r \right] dr + \int_0^T \mathbb{E}^Q \left[ (\Gamma_t + \tilde{\Gamma}_t) \partial_s u_t \right] dt \]

Due to (56), the last term in the right hand side of the above becomes

\[ Q\sigma^2 \gamma^2 \mathbb{E}^Q \left[ \int_0^T \partial_s P_t \Delta_t \int_0^r e^{-\int_0^s \sigma(v) \partial_s P_t} (\Gamma_t + \tilde{\Gamma}_t) dt dr \right]. \]

We now denote $\tilde{\Gamma}_t := \int_0^T \Gamma_t e^{\int_0^T Q\sigma^2 \gamma \partial_s P_r e^{dv} d\tau}$ and $\hat{\Gamma}_t := \int_0^T \tilde{\Gamma}_t e^{\int_0^T Q\sigma^2 \gamma \partial_s P_r e^{dv} d\tau}$, so that

\[ \mathbb{E}^Q \left[ \int_0^T \Gamma_t \Delta_t dt \right] = \frac{A_{0,T}}{\kappa} \Gamma_0 + \int_0^T \mathbb{E}^Q \left[ \frac{A_{r,T}}{\kappa} Q\Gamma_r \partial_s P_r \phi_r \right] dr \]

\[ + \int_0^T \frac{A_r}{\kappa} \mathbb{E}^Q [Q\partial_s P_r \beta_r] dr + Q\gamma^2 \int_0^T \mathbb{E}^Q \left[ \partial_s P_r \tilde{\Gamma}_r \right] dr \]
Lemma 12 (stated further in the appendix) easily yields

\[ \int_0^T E^Q \left[ \Delta_r \partial_{ss} P_r \left( \tilde{\Gamma}_r + A_{r,T} \int_0^r e^{-\int_s^r m(v)dv} \Gamma_t A_t, T dt - \frac{A_{r,T} \Gamma_r}{\kappa} \right) \right] dr \]

\[ - \frac{2}{\kappa} \int_0^T E^Q [A_{r,T} \alpha_r \Delta_r^2] dr. \]

Lemma 12 (stated further in the appendix) easily yields

\[ \int_0^T E^Q [\Delta_4] dr + \int_0^T E^Q [\Delta_r^2] dr = o(1). \quad (90) \]

We now use this result to show that

\[ I_1 := \int_0^T E^Q [A_{r,T} Q \Gamma_r \partial_s P_r \phi_r] dr = o(1), \]

\[ I_2 := \int_0^T A_{r,T} E^Q [Q \sigma \partial_{ss} P_r \beta_r] dr = \int_0^T E [Q \sigma \partial_{ss} P_r \beta_r] dr + o(1), \]

\[ I_3 := \int_0^T E^Q \left[ \Delta_r \partial_{ss} P_r \tilde{\Gamma}_r \right] dr \]

\[ = E^Q \left[ \int_0^T \partial_{ss} P_r \Delta_r \int_0^r e^{\int_s^r Q \sigma^2 \gamma \partial_r P_r \phi_r \Gamma_t dv} dh dr \right] + o(\kappa^{-1}), \]

\[ I_4 := \int_0^T E^Q \left[ \Delta_r \partial_{ss} P_r \left( \tilde{\Gamma}_r + A_{r,T} \int_0^r e^{-\int_s^r m(v)dv} \Gamma_t A_t, T dt - \frac{A_{r,T} \Gamma_r}{\kappa} \right) \right] dr = o(\kappa^{-1}), \]

\[ I_5 := \int_0^T E^Q [A_{r,T} \alpha_r \Delta_r^2] dr = o(1). \]

We treat each term separately. Recall that \( 0 \leq A_{r,T} \leq 1 \) and \( \alpha \) and \( \beta \) are uniformly bounded. Note also that \( \int_0^T \phi_r^2 dr = o(1) \). Direct estimates yield for \( I_1 \) and \( I_5 \) to

\[ |I_1| \leq C E^Q \left[ \int_0^T \Gamma_r^2 dr \right]^{1/2} \left( \int_0^T \phi_r^2 dr \right)^{1/2}, \]

\[ |I_5| \leq C E^Q \left[ \int_0^T \Delta_r^2 dr \right]. \]

We also estimate \( \tilde{\Gamma} \) so follows

\[ \left| \tilde{\Gamma}_r + A_{r,T} \int_0^r e^{-\int_s^r m(v)dv} \Gamma_t A_t, T dt - \frac{A_{r,T} \Gamma_r}{\kappa} \right| \]

\[ \leq C \left( \sup_t (|\tilde{\Gamma}_t|) \left( \int_0^r e^{-\int_s^r m(v)dv} dt + \frac{1 + |\Delta_t|}{\kappa} \right) \right) \]

\[ \leq C \sup_t (1 + |\Gamma_t|) \left( \sup_s \left[ \int_0^s e^{-\int_s^r m(v)dv} dt + \int_0^r e^{-\int_s^r m(v)dv} dt + \frac{1 + |\Delta_t|}{\kappa} \right] \right) \]

\[ \leq C \sup_t (1 + |\Gamma_t|)(1 + |\Delta_t|). \]

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so that

$$ |I_4| \leq C \mathbb{E}^Q \left[ \int_0^T |\Delta_t| \left( 1 + |\Delta_t| \right) \sup_s \left( 1 + |\Gamma_t| \right) \, ds \right] \leq \frac{C}{\kappa} \mathbb{E}^Q \left[ \int_0^T |\Delta_t^2| + |\Delta_t^4| \, ds \right] \frac{1}{\sqrt{\mathbb{E}^Q \left[ \sup_t |\Gamma_t|^2 \right]}}. $$

Thus, (90) and the boundedness of the characteristics of $\Gamma$ imply:

$$ |I_1| + \kappa |I_4| + |I_5| = o(1). $$

Next, we expand $I_3$ and write it as follows:

$$ I_3 = \kappa \int_0^T \mathbb{E}^Q \left[ \Delta_t \partial_{ss} P_r - \int_0^r e^{f_r^h} Q^{s^2} \partial_{ss} P_r \, dv \int_0^h e^{-f_r^h m(v) \, d\Gamma} \, d\Gamma \right] \, dr $$

$$ = \kappa \int_0^T \mathbb{E}^Q \left[ \Delta_t \partial_{ss} P_r - \int_0^r e^{f_r^h} Q^{s^2} \partial_{ss} P_r \, dv \int_0^r e^{-f_r^h m(v) \, d\Gamma} \, d\Gamma \, dh \Gamma_{t,T} \right] \, dr $$

$$ = \mathbb{E}^Q \left[ \int_0^T \partial_{ss} P_r \Delta_t \int_0^r e^{f_r^h} Q^{s^2} \partial_{ss} P_r \, dv \Gamma_{t,T} \right] $$

$$ + \kappa \mathbb{E}^Q \left[ \int_0^T \partial_{ss} P_r \Delta_t \int_0^r e^{f_r^h} Q^{s^2} \partial_{ss} P_r \, dv \Gamma_{t,T} \right] $$

$$ + \mathbb{E}^Q \Gamma_{t,T} \left. \left( \int_0^r e^{-f_r^h m(v) \, d\Gamma} \, d\Gamma \partial_{ss} P_r \, dv - \frac{1}{\kappa} \right) \right| dt dr $$

Due to the uniform boundedness of $Q^{s^2} \partial_{ss} P_r$, there exists $C_{t,s}$ which is uniformly bounded over $s, t \in [0, T]$ and such that

$$ |e^{f_r^h} Q^{s^2} \partial_{ss} P_r - 1| = C_{t,s} |t - s|. $$

Thus,

$$ \left| \int_t^r e^{-f_r^h m(v) \, d\Gamma} e^{f_r^h} Q^{s^2} \partial_{ss} P_r \, dv \, ds \right| \leq \int_t^r e^{-f_r^h m(v) \, d\Gamma} C_{s,t} |t - s| \, ds + \frac{1 - A_{t,r}}{\kappa} \leq C \int_t^r \cosh \left( \frac{\kappa(T - s) + \phi}{\cosh (\kappa(T - t) + \phi)} \right) \, t - s \, ds + \frac{1 - A_{t,r}}{\kappa} $$

$$ \leq C \int_t^r e^{-\kappa(s-t)} |t - s| \, ds + \frac{1 - A_{t,r}}{\kappa} \leq C \int_0^r e^{-\kappa u} du + \frac{1 - A_{t,r}}{\kappa} \leq C \frac{1}{\kappa^2} + \frac{1 - A_{t,r}}{\kappa}. $$

We also have the following bound

$$ \int_0^r 1 - A_{r,s} \, ds = \int_0^r 1 - \tanh \left( \frac{\kappa(T - s) + \phi}{\cosh (\kappa(T - s) + \phi)} \right) \, ds $$

$$ = r + \frac{1}{\kappa} \ln \left( \frac{\cosh (\kappa(T - r) + \phi)}{\cosh (\kappa(T + \phi))} \right) $$

$$ + \frac{1}{\kappa} (\arctan (\sinh (\kappa T + \phi)) - \arctan (\sinh (\kappa(T - r) + \phi))) \sinh (\kappa(T - r) + \phi)$$

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\[
\ln \left( e^{\kappa r} \cosh (\kappa (T - r) + \phi) \right) + \frac{1}{\kappa} \ln \left( 1 + \frac{e^{-\kappa r} - e^{2\kappa r}}{1 + e^{-2\kappa r}} \right)\] 
\leq \frac{\ln(2)}{\kappa} + 1.
\]

(91)

Given the definition of \( I_3 \) and the inequality \( 1 - A_{s,r} \geq 1 - A_{s,T} \geq 0 \), for \( s \leq r \leq T \), the above inequalities, along with Cauchy inequality, yield:

\[
\left| I_3 - E^Q \left[ \int_0^T \partial_{ss} P_r \Delta_r \int_0^r e^{\int_t^r Q \sigma^2 \gamma \partial_{ss} P_r \Delta_s dW_r} d\Gamma_s d\sigma d\tau \right] \right| 
\leq C E^Q \left[ \int_0^T |\partial_{ss} P_r \Delta_r| \sup_\tau |\Gamma_t| \int_0^r |1 - A_{t,r}| d\sigma d\tau \right] + o(\kappa^{-1})
\]
\[
\leq C \frac{1}{\kappa} E^Q \left[ \int_0^T \Delta_t^2 dr \right]^{1/2} E^Q \left[ \sup_\tau |\Gamma_t|^2 \right]^{1/2} + o(\kappa^{-1}) = o(1)
\]

where we have used one more time that \( E^Q \left[ \int_0^T \Delta_t^2 dr \right] = o(1) \).

To finish the proof of the lemma it now suffices to prove that

\[
I_2 = \int_0^T E \left[ Q \sigma \partial_{ss} P_r \beta_r \right] d\tau + o(1).
\]

In view of (91), the above is a direct consequence of the convergence

\[
E^Q [X] \to E [X],
\]

(92)

for all absolutely bounded \( X \), and the dominated convergence theorem. Let us prove (92). Thanks to martingale representation theorem, there exists a \( P \)-square integrable \( h \) such that

\[
E^Q [X] - E [X] = E^Q \left[ \int_0^T h_t dW_t \right] = \sigma E^Q \left[ \int_0^T h_t \partial_{ss} u(t, S_t, \pi_t^*) dt \right] - \gamma \sigma E^Q \left[ \int_0^T h_t \Delta_t dt \right]
\]
\[
= Q \sigma^3 \gamma^2 E^Q \left[ \int_0^T h_t \int_0^T e^{\int_t^r Q \sigma^2 \gamma \partial_{ss} P_r \Delta_s dW_r} d\tau \right] - \gamma \sigma E^Q \left[ \int_0^T h_t \Delta_t dt \right]
\]
\[
= \sigma \gamma E^Q \left[ \int_0^T \frac{dQ}{dP} \Delta_r \left( Q \sigma^2 \gamma \partial_{ss} P_r \int_0^T h_t e^{\int_t^r Q \sigma^2 \gamma \partial_{ss} P_r dW_r} d\tau - h_r \right) dr \right].
\]
Thus, using the generalized Hölder inequality with $\frac{1}{4} + \frac{1}{2} + \frac{1}{2} = 1$, we have

$$\left| \mathbb{E}^Q [X - X] \right| \leq C \left[ \mathbb{E}^Q \int_0^T \Delta_t^4 dr \right]^{1/4} \left[ \mathbb{E} \int_0^T \left( \frac{dQ}{dP} \right)^3 dr \right]^{1/4} \left[ \mathbb{E} \int_0^T h_t^2 dt \right]^{1/2}.$$  

Thanks to Lemma 12 to finish the proof of this lemma it now suffices to prove that $\mathbb{E} \left[ \int_0^T \left( \frac{dQ}{dP} \right)^3 dr \right]$ is bounded as $\eta \to 0$. Thanks to (52),

$$\left( \frac{dQ}{dP} \right)^3 = \frac{e^{3\Psi(t, \pi, \gamma, s_0, \pi_0)} + 3Q\Gamma(0, S_0)}{U^3(0, S_0, \pi_0)}.$$  

Thus,

$$\mathbb{E} \left[ \left( \frac{dQ}{dP} \right)^3 \right] \leq \frac{\tilde{U}(0, S_0, \pi_0)}{U^3(0, S_0, \pi_0)}$$

where $\tilde{U}$ is defined as in (7) but with $\gamma$ replaced by $3\gamma$. This implies the desired convergence

$$\left| \mathbb{E}^Q [X] - X \right| \leq C\mathbb{E}^Q \left[ \int_0^T \Delta_t^4 dr \right]^{1/4} \to 0 \text{ as } \eta \to 0.$$

\textbf{Lemma 12.} Under the assumptions of Lemma 9 we have:

$$\int_0^T \mathbb{E}^Q \left[ \Delta_t^4 \right] dt \leq C \frac{\Delta_t^4}{\kappa} + \frac{C}{\kappa^{1/2}},$$

for small enough $\eta$ (i.e., large enough $\kappa$), with some constant $C > 0$ independent of $\eta$.

\textbf{Proof:}  
Recall that $\Delta_t = (\pi_t + Q\partial_t P_t)$. Equation (86), viewed as a linear random ODE for $\Delta$, implies that, for $0 \leq t_0 \leq t_1 \leq T$,

$$\Delta_{t_1} = \Delta_{t_0} e^{-\int_{t_0}^{t_1} m \pi u_t dr} + \int_{t_0}^{t_1} e^{-\int_{s}^{t_1} m(v) dv} \mathbb{E}^Q \left[ \int_t^T e^{-\int_r^T m(v) dv} \partial_x u_r dr \right] dt + \int_{t_0}^{t_1} e^{-\int_{s}^{t_1} m \pi u_t dr} Q \partial_t P_t \partial_t dt$$

$$- \int_{t_0}^{t_1} e^{-\int_{s}^{t_1} m \pi u_t dr} Q \kappa^2 \sigma^2 \mathbb{E}^Q \left[ \int_t^T \partial x P_h (\partial_x u_h - \gamma \Delta_h) dh dr \right] dt$$

$$+ \int_{t_0}^{t_1} e^{-\int_{s}^{t_1} m \pi u_t dr} Q \sigma^2 \partial x P_t (\partial_x u_t - \gamma \Delta_t) dt + \int_{t_0}^{t_1} e^{-\int_{s}^{t_1} m \pi u_t dr} \sigma Q \partial x P_t dW_t. \quad (93)$$

Plugging the above into (56), we obtain

$$\partial_x u_{t_0} = \Delta_{t_0} Q \sigma^2 \gamma^2 \mathbb{E}^Q_t \left[ \int_t^T e^{\int_0^t Q \sigma^2 \gamma \partial x P_r \partial x P_t e^{-\int_{t_0}^{t_1} m \pi u_t dr} dt} \right]$$

$$+ Q \sigma^2 \gamma^2 \mathbb{E}^Q_t \left[ \int_t^T e^{\int_0^t Q \sigma^2 \gamma \partial x P_r \partial x P_t e^{-\int_{t_0}^{t_1} m \pi u_t dr} dt} \right]$$

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\[
\int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt dt_1 \leq C \sigma^2 E_{t_0}^{Q} \left[ \int_{t_0}^{T} e^{-f_{1+}^{m} \sigma \tau} \, dt \right] \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt dt_1.
\]

Recall the estimate
\[
\kappa \int_{r}^{s} e^{-f_{1+}^{m} \sigma \tau} \, dt + \kappa \int_{r}^{s} e^{-f_{1+}^{m} \sigma \tau} \, dt = A_{r,s} + A_{r,s} \leq C \text{ for } 0 \leq r \leq T.
\]

Thus,
\[
|\partial_s u_{t_0}| \leq \frac{C(1 + |\Delta_{t_0}|)}{\kappa} + C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt dt_1 \right]
\]
\[
+ C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} \int_{t}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt \int_{t}^{t_1} (|\partial_s u_{t_1}| + \gamma |\Delta_{t_1}|) \, dtdr \right]
\]
\[
+ C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} (|\partial_s u_{t_1}| + \gamma |\Delta_{t_1}|) \, dt + C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt \right] \int_{t_0}^{t_1} |\partial_s u_{t_1}| \, dt_1 \right]
\]
\[
\leq \frac{C(1 + |\Delta_{t_0}|)}{\kappa} + C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} |\partial_s u_{t_1}| \, dt \right]
\]
\[
+ C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} \gamma |\Delta_{t_1}| \, dt \right] + C \kappa E_{t_0}^{Q} \left[ \int_{t_0}^{T} \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt \right] \int_{t_0}^{t_1} |\partial_s u_{t_1}| \, dt_1 \right].
\]

Thus, there exists a family of random variables \((M^{Q_0})\), continuous in \(t_0\), satisfying \(E_{t_0}^{Q_0} M^{Q_0} = 0\), and such that
\[
|\partial_s u_{t_0}| \leq \frac{C(1 + |\Delta_{t_0}|)}{\kappa} + C \int_{t_0}^{T} |\partial_s u_{t_1}| \, dt + C \int_{t_0}^{T} |\Delta_{t_1}| \, dt + C \int_{t_0}^{T} \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt \int_{t_0}^{t_1} |\partial_s u_{t_1}| \, dt_1 + M^{Q_0}.
\]

Applying Gronwall’s lemma backwards in time on \([0, T]\), we deduce
\[
|\partial_s u_{t_0}| \leq \left( \frac{C(1 + |\Delta_{t_0}|)}{\kappa} + C \int_{t_0}^{T} |\Delta_{t_1}| dt + C \int_{t_0}^{T} \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt \int_{t_0}^{t_1} |\partial_s u_{t_1}| \, dt_1 + M^{Q_0} \right)
\]
\[
+ C \int_{t_0}^{T} \left( \frac{C(1 + |\Delta_{t_1}|)}{\kappa} + C \int_{t_0}^{T} |\Delta_{t_1}| dt + C \int_{t}^{T} \int_{t_0}^{t_1} e^{-f_{1+}^{m} \sigma \tau} \, dt \int_{t_0}^{t_1} |\partial_s u_{t_1}| \, dt_1 + M^{Q_0} \right) e^{C(T-t)} \, dt.
\]
Taking expectation, we obtain

$$|\partial_s u_t| \leq \frac{C(1 + |\Delta_{t_0}|)}{\kappa} + \frac{C}{\kappa} \mathbb{E}_{t_0}^{Q} \left[ \int_{t_0}^{T} |\Delta_t| dt \right] + C \int_{t_0}^{T} \mathbb{E}_{t}^{Q} \left[ \int_{t_0}^{t} e^{-f_t^{(3)} m_v dv} \partial_{ss} \sigma_v d\tilde{W}_t \right] dt_1$$

$$+ C \int_{t_0}^{T} \int_{r}^{T} \mathbb{E}_{r}^{Q} \left[ \int_{r}^{t} e^{-f_t^{(3)} m_v dv} \partial_{ss} \sigma_v d\tilde{W}_t \right] dt_1 dr. \tag{94}$$

Note that

$$\mathbb{E}_{t_0}^{Q} \left[ \int_{r}^{t} e^{-f_t^{(3)} m_v dv} \partial_{ss} \sigma_v d\tilde{W}_t \right] \leq C \left[ \int_{r}^{t} e^{-2f_t^{(3)} m_v dv} dt \right]^{1/2} \leq \frac{C}{\sqrt{\kappa}}.$$

We finally obtain

$$|\partial_s u_t| \leq \frac{C(\sqrt{\kappa} + |\Delta_{t_0}|)}{\kappa} + \frac{C}{\kappa} \mathbb{E}_{t_0}^{Q} \left[ \int_{t_0}^{T} |\Delta_t| dt \right]. \tag{94}$$

Injecting (94) into (93) yields

$$\int_{t_0}^{T} \mathbb{E}_{t_0}^{Q} \left[ |\Delta_t| dt \right] \leq C \frac{\sqrt{\kappa} + |\Delta_{t_0}|}{\kappa} + C \int_{t_0}^{T} \int_{r}^{T} e^{-f_t^{(3)} m_v dv} \mathbb{E}_{t_0}^{Q} |\Delta_r| dr dt$$

$$\leq C \frac{\sqrt{\kappa} + |\Delta_{t_0}|}{\kappa} + C \int_{t_0}^{T} \mathbb{E}_{t_0}^{Q} |\Delta_r| \int_{t_0}^{T} e^{-f_t^{(3)} m_v dv} dt dr$$

$$\leq C \frac{\sqrt{\kappa} + |\Delta_{t_0}|}{\kappa} + \frac{C}{\kappa} \int_{t_0}^{T} \mathbb{E}_{t_0}^{Q} |\Delta_r| dr,$$

which implies that

$$\int_{t_0}^{T} \mathbb{E}_{t_0}^{Q} \left[ |\Delta_t| dt \right] \leq C \frac{\sqrt{\kappa} + |\Delta_{t_0}|}{\kappa}.$$

Combined with (94) the above inequality yields

$$|\partial_s u_t| \leq \frac{C(\sqrt{\kappa} + |\Delta_{t_0}|)}{\kappa}. \tag{95}$$

Next, we denote

$$A_{r,s}^4 := 4\kappa \int_{r}^{s} e^{-4f_t^{(3)} m_v dv} dt.$$

Similarly to $A_{r,s}$, for $0 \leq r \leq s$, we have: $A_{r,s}^4 \leq 1$. Applying Itô’s lemma to $\Delta_{t_0}^3$ and using (86), we derive a linear random ODE for $\Delta_{t}^3$, similar to (87). We solve it and integrate over $[0, T]$ to obtain (similar to the derivation of (88));

$$\int_{0}^{T} \mathbb{E}_{0}^{Q} \left[ \Delta_{t_0}^3 \right] dr = \Delta_{0}^3 \frac{A_{0,T}^4 m_v}{\kappa} + \int_{0}^{T} \frac{A_{t,T}^4}{\kappa} \mathbb{E}_{t}^{Q} \left[ \frac{\sigma^2 \Delta_t^3}{2\eta} \int_{t}^{T} e^{-f_t^{(3)} m_v dv} \partial_{su} \sigma_v dr \right] dt$$

$$- \int_{0}^{T} \kappa A_{t,T}^4 \mathbb{E}_{t}^{Q} \left[ \Delta_{t_0}^3 Q \sigma^2 \int_{t}^{T} e^{-f_t^{(3)} m_v dv} \int_{t}^{h} \partial_{su} - \gamma \Delta_v dr dh \right] dt.$$
\[ + \int_0^T \frac{A_{4,T}}{\kappa} E^Q \left[ \Delta_{t}^2 Q \partial_s P_t \phi_t \right] dt + \int_0^T \frac{A_{4,T}}{\kappa} E^Q \left[ \Delta_{t}^2 Q \sigma^2 \partial_{ss} P_t (\partial_s u_t - \gamma \Delta_t) \right] dt \]

\[ + \int_0^T \frac{A_{4,T}}{\kappa} E^Q \left[ \frac{3}{2} \Delta_{t}^2 Q^2 (\partial_s P_t)^2 \right] dt. \]

By Fubini's theorem we have

\[ \int_t^T e^{-\int_t^s \mu(r) \, dr} \int_t^h (\partial_s u_r - \gamma \Delta_r) \, dr \, dh = \int_t^T e^{-\int_t^s \mu(r) \, dr} (\partial_s u_r - \gamma \Delta_r) \int_T^T e^{-\int_T^s \mu(r) \, dr} \, dr \, dh \]

\[ = \int_t^T e^{-\int_t^s \mu(r) \, dr} (\partial_s u_r - \gamma \Delta_r) \frac{A_{r,T}}{\kappa} \, dr. \]

Collecting the last two equations above, using (95), the boundedness of \( A_{r,T} \), and the definition of \( \kappa \), we obtain:

\[ \int_0^T E^Q [\Delta_t^4] \, dt \leq C \frac{\Delta_0^4}{\kappa} + \frac{C}{\kappa} \int_0^T E^Q [\Delta_t^3] \, dt + C \int_0^T E^Q \left[ |\Delta_t| \int_t^T e^{-\int_t^s \mu(r) \, dr} |\Delta_r| \, dr \right] \, dt \]

\[ + C \int_0^T E^Q \left[ |\Delta_t| \int_t^T e^{-\int_t^s \mu(r) \, dr} (|\partial_s u_r| + \gamma |\Delta_r|) \, dr \right] \, dt \]

\[ + \frac{C}{\kappa} \int_0^T E^Q \left[ |\Delta_t|^3 (1 + |\partial_s u_t| + \gamma |\Delta_t|) \right] \, dt. \]

Using (95), Hölder inequality, and Jensen’s inequality, we obtain:

\[ \int_0^T E^Q [\Delta_t^4] \, dt \leq C \frac{\Delta_0^4}{\kappa} + C \left[ E^Q \int_0^T |\Delta_t|^4 \, dt \right]^{3/4} \left( \frac{1}{\kappa^{1/4}} + \left[ E^Q \int_0^T e^{-\int_t^s \mu(r) \, dr} |\Delta_r|^4 \, dr \, dt \right]^{1/4} \right) \]

\[ + \frac{C}{\kappa} \left[ E^Q \int_0^T |\Delta_t|^4 \, dt \right]^{3/4} + \frac{C}{\kappa} \int_0^T E^Q [|\Delta_t|^4] \, dt. \]

We combine the last term with the left hand side and apply Fubini’s theorem one more time to conclude

\[ \int_0^T E^Q [\Delta_t^4] \, dt \leq C \frac{\Delta_0^4}{\kappa} + C \left[ E^Q \int_0^T |\Delta_t|^4 \, dt \right]^{3/4} \left( \frac{1}{\kappa^{1/4}} + \frac{1}{\kappa^{1/4}} \left[ E^Q \int_0^T |\Delta_r|^4 \, dr \right]^{1/4} \right). \]

The above estimate, for \( \kappa \) large enough, yields the desired inequality. \( \blacksquare \)

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