ON THE FORELLI–RUDIN PROJECTION THEOREM

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Abstract. Motivated by the Forelli–Rudin projection theorem we give in this paper a criteria for boundedness of an integral operator on Lebesgue spaces in the interval (0, 1). We also give the precise norm of this integral operator. As a consequence, one can derive a generalization of the Dostanić result concerning the norm of the Berezin transform $B$ acting on the Lebesgue space $L^p(B)$ of the unit ball in $\mathbb{C}^n$ which says that

$$\|B : L^p(B) \to L^p(B)\| = \prod_{k=1}^n \left(1 + \frac{1}{k^p}\right) \frac{\pi}{\sin \frac{\pi}{p}}$$

for any real $p$ greater then 1. The result belong to Dostanić in the case $n = 1$.

1. Introduction

Introduce the notation which will be used in this paper.

If $T$ is a linear operator from a space with a norm $(X, \|\cdot\|_X)$ into a space $(Y, \|\cdot\|_Y)$, we denote by $\|T\|_{X \to Y}$ the norm of $T$, i.e.,

$$\|T : X \to Y\| = \sup_{a \in X, a \neq 0} \frac{\|T a\|_Y}{\|a\|_X}.$$

Throughout the whole paper $n$ will be a positive integer. Let $\langle \cdot, \cdot \rangle$ stands for the inner product in the complex $n$–dimensional space $\mathbb{C}^n$ given by

$$\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n},$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. The standard norm in $\mathbb{C}^n$, induced by the inner product, is denoted by $|\cdot|$.

Denote by $B$ the unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ in $\mathbb{C}^n$ and let $S = \partial B$ be the unit sphere. The normalized Lebesgue measure on the unit ball (sphere) is denoted by $dv (d\tau)$. Let $L^p(B)$, $1 \leq p < \infty$ stands for the Lebesgue space of all measurable functions in the unit ball of $\mathbb{C}^n$ which modulus with the exponent $p$ is integrable. For $p = \infty$ let it be the space of all essentially bounded measurable functions. Denote by $\|\cdot\|_p$ the usual norm on $L^p(B)$ ($1 \leq p \leq \infty$). Recall that

$$\|f\|_p^p = \int_B |f(z)|^p \, dv(z)$$

for $f \in L^p(B)$ ($1 \leq p < \infty$).

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Following the Rudin monograph \cite{15} as well as the Forelli and Rudin work \cite{7}, associate with each complex number $s = \sigma + it$, $\sigma > -1$ the integral kernel

$$K_s(z, w) = \frac{(1 - |w|^2)^s}{(1 - \langle z, w \rangle)^{n+1+s}},$$

and let

$$T_s f(z) = c_s \int_B K_s(z, w) f(w) \, dv(w), \quad z \in B.$$ \hfill

understand $f(w)$ is a such on function in $B$ that the previous integral is well defined, and the complex power is understood to be the principal branch. The coefficient $c_s$ is chosen in the way that for the weighted measure in the unit ball

$$dv_s(w) = c_s (1 - |w|^2)^s \, dv(w)$$

holds $v_s(B) = 1$, i.e., $T_s 1 = 1$. Using the polar coordinates

$$\int_B h(z) \, dv(z) = 2n \int_0^1 r^{2n-1} \, dr \int_B h(r\zeta) \, d\tau(\zeta),$$

one can show that

$$c_s^{-1} = n B(s + 1, n) = \frac{\Gamma(s + 1)\Gamma(n + 1)}{\Gamma(n + s + 1)},$$

where $\Gamma$ and $B$ are Euler functions. $T_s$ is the Bergman projection operator. Bergman type projections are central operators when dealing with questions related to analytic function spaces.

Forelli and Rudin \cite{7} proved that $T_s : L^p(B) \to L^p_0 = L^p \cap H(B)$, where $H(B)$ is the space of all analytic functions in the unit ball, is a bounded (and surjective) operator if and only if $\sigma > \frac{1}{p} - 1$, where $1 \leq p < \infty$. Moreover, they find $\|T_s : L^1(B) \to L^1_0(B)\|$ for $\sigma > 0$ and $\|T_s : L^2(B) \to L^2_0(B)\|$ for $\sigma > -\frac{1}{2}$. It seems that the calculation of $\|T_s : L^p(B) \to L^p_0(B)\|$ in other cases is not an easy problem. Mateljević and Pavlović extended the Forelli and Rudin result in \cite{12}.

On the other hand, if $p = \infty$, it is known \cite{8} that the operator $T_\sigma (\sigma > -1)$ projects $L^\infty (B)$ continuously onto the Bloch space $B$ of the unit ball in $C^n$. Recall that the Bloch space $B$ contains all functions $f$ analytic in $B$ for which the semi-norm $\|f\|_B = \sup_{z \in B} (1 - |z|^2) |\nabla f(z)|$ is finite. One can obtain a true norm by adding $|f(0)|$, more precisely in the following way

$$\|f\|_B = |f(0)| + \|f\|_B, \quad f \in B.$$ \hfill

The $\beta$-(semi-)norm of $T_\sigma : L^\infty (B) \to B$ is defined to be

$$\|T_\sigma\|_\beta = \sup_{\|f\|_\infty \leq 1} \|T_\sigma f\|_\beta.$$ \hfill

In \cite{8} we find the (semi-)norm of $T_\sigma$ w.r.t. $\beta$-(semi-)norm. We have

$$\|T_\sigma\|_\beta = \frac{\Gamma(\lambda + 1)}{\Gamma^2(\frac{1}{2} + \frac{\sigma}{2})},$$

where we have introduced (for the sake of simplicity) $\lambda = n + \sigma + 1$. Following the approach as in \cite{14}, one can derive

$$\|T_\sigma : L^\infty (B) \to B\| = 1 + \|T_\sigma\|_\beta = 1 + \frac{\Gamma(\lambda + 1)}{\Gamma^2(\frac{1}{2} + \frac{\sigma}{2})}.$$
Particularly, for $\sigma = 0$ and $n = 1$ we put $P = T_0$ and $B = U$ (then we have the original Bergman projection). Perälä proved that
\[ \|P\|_\beta = \frac{8}{\pi} \quad \text{and} \quad \|P : L^\infty(U) \to B\| = 1 + \frac{8}{\pi}, \]
which are the main results from [13] and [14], respectively. For related results we refer to [9].

2. The main result and preliminaries

In the consideration which follows we will assume that $\mu > 0$.

We denote by $L^p_{\mu}(0,1)$ the space of all measurable function $\varphi(t)$ in $(0,1)$ which satisfy the condition
\[ \|\varphi\|_{p,\mu} = \mu \int_0^1 |\varphi(t)|^p t^{\mu - 1} dt < \infty. \]
The measure $\mu t^{\mu - 1} dt$ will be denoted by $d_\mu(t)$.

The Gauss hypergeometric function is given by the series
\[ _2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!} \]
for all $c$ different from zero and negative integers; here
\[ (d)_k = d(d+1) \cdots (d+k-1) \]
stands for the Pochhammer symbol. The series converges at least for $|z| < 1$ and for $|z| = 1$ if $\Re(c - a - b) > 0$.

For a parameter $\sigma > -1$ we will consider the operator $F_\sigma$ given in the following way
\[ F_\sigma \varphi(s) = \mu \int_0^1 (1-t)^{\sigma} \ _2F_1\left(\frac{\lambda}{2}, \frac{\lambda}{2} + \mu; st\right) \varphi(t) t^{\mu - 1} dt, \]
where we have denoted $\lambda = \mu + \sigma + 1$.

It happens that the operator $F_\sigma$ is bounded on $L^p_{\mu}(0,1)$ if and only if $\sigma > \frac{1}{p} - 1$. This is the content of the following

**Theorem 2.1.** For $1 \leq p < \infty$ the operator $F_\sigma$ maps continuously the space $L^p_{\mu}(0,1)$ into itself if and only if $\sigma > \frac{1}{p} - 1$. Moreover, in this case we have
\[ \|F_\sigma : L^p_{\mu}(0,1) \to L^p_{\mu}(0,1)\| = \frac{\Gamma(\mu + 1)\Gamma\left(\frac{1}{p}\right)\Gamma(\sigma + 1 - \frac{1}{p})}{\Gamma\left(\frac{\lambda}{2}\right)^2}. \]

The case $p = 1$ is not difficult to consider. It will be derived from

**Lemma 2.2.** Let $\nu$ be a finite measure on $X$. Let $T$ be an integral operator which acts on $L^1 = L^1(X,\nu)$ with the non-negative kernel $K(x,y)$, i.e., let
\[ Tf(x) = \int_X K(x,y) f(y) d\nu(y). \]

Then $T$ maps $L^1$ into itself if and only if
\[ \sup_{y \in X} \int_X K(x,y) d\nu(x) < \infty. \]
In this case we have
\[ \|T : L^1 \to L^1\| = \sup_{y \in X} \int_X K(x, y) \, d\nu(x). \]

**Proof.** Namely, for all \( f \in L^1 \) holds
\[
\|Tf\|_1 = \int_X \left| \int_X K(x, y) f(y) \, d\nu(y) \right| \, d\nu(x) \\
\leq \int_X \left\{ \int_X K(x, y) \, d\nu(x) \right\} |f(y)| \, d\nu(y) \\
\leq \left\{ \sup_{y \in X} \int_X K(x, y) \, d\nu(x) \right\} \|f\|_1.
\]

If we take \( f \equiv 1 \), then we have the equality sign at each place above, what gives the necessary condition, as well as the norm of \( T \). \( \square \)

In the case \( 1 < p < \infty \) we will use the following well known result.

**Lemma 2.3 (The Schur test).** Suppose that \((X, \nu)\) is a \( \sigma \)-finite measure space and \( K(x, y) \) is a nonnegative measurable function on \( X \times X \), and \( T \) the associated integral operator
\[
Tf(x) = \int_X K(x, y) f(y) \, d\nu(y).
\]
Let \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If there exist a positive constant \( C \) and a positive measurable function \( f \) on \( X \) such that
\[
\int_X K(x, y) f(y)^q \, d\nu(y) \leq C f(x)^q,
\]
for almost every \( x \in X \) and
\[
\int_X K(x, y) f(x)^p \, d\nu(x) \leq C f(y)^p.
\]
for almost every \( y \in X \), then \( T \) is bounded on \( L^p = L^p(X, \nu) \) and the following estimate of the norm holds
\[
\|T : L^p \to L^p\| \leq C.
\]

Besides these lemmas we will use the following transformations and facts concerning the Gauss hypergeometric functions. We list them for the sake of easy reference. Some of them are well known.

**Lemma 2.4 (Gauss).** If \( \Re(c - a - b) > 0 \), then
\[
\binom{a, b}{c; 1} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\]

**Lemma 2.5 (The Euler formula).** For \( \Re c > \Re d > 0 \) holds
\[
\binom{a, b}{c; z} = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c - d)} \int_0^1 t^{d-1} (1 - t)^{c-d-1} \binom{a, b}{d; tz} \, dt,
\]
where \( z \) is different from 1, and \( |\arg(1 - z)| < \pi \).

**Lemma 2.6.** The Euler transform says that
\[
\binom{a, b}{c; z} = (1 - z)^{c-a-b} \binom{c-a, c-b}{c, z}.
\]
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The preceding facts may be found in the second chapter in [1].

**Lemma 2.7.** For reals $c > 0$ and $d$ the function $\, _2F_1(d, d, c, r) \,$ is bounded in $(0, 1)$ if and only if $c > 2d$ in which case we have

$$\sup_{0 < r < 1} \, _2F_1(d, d, c, r) = \, _2F_1(d, d, c, 1) = \frac{\Gamma(c)\Gamma(c - 2d)}{\Gamma^2(c - d)}.$$  

**Proof.** If $c > 2d$, then $\, _2F_1(d, d; c; r) \,$ is continuous in $[0, 1]$ and increasing in $0 < r < 1$. Therefore, we may apply the Gauss relation to obtain the maximum. In other cases holds

$$\, _2F_1(d, d, c, r) \sim \frac{\Gamma(c)\Gamma(2d - c)}{\Gamma^2(d)} \frac{1}{(1 - r)^{2d - c}} \quad \text{if} \quad c < 2d.$$  

For these asymptotic relations we refer to [1]. \hfill \square

**Lemma 2.8.**

$$\int_0^1 t^{c-1}(1-t)^{d-1} \, _2F_1(a, b; c; t) \, dt = \frac{\Gamma(c)\Gamma(d)\Gamma(c + d - a - b)}{\Gamma(c + d - a)\Gamma(c + d - b)}$$

for $\Re c > 0, \Re d > 0$ and $\Re(c + d - a - b) > 0$.

**Proof.** Under the assumption of the lemma, both sides of the Euler formula are continuous at $z = 1$. The lemma then follows by letting $z \to 1$ and applying the Gauss theorem. \hfill \square

3. **Proof of the main theorem**

Denote the kernel of $F_\sigma$ by

$$K_\sigma(s, t) = (1-t)^\sigma \, _2F_1\left(\frac{\lambda}{2}, \frac{\lambda}{2}, \mu; st\right).$$

Let us first discuss the simple case $p = 1$. We have

$$\sup_{t \in (0, 1)} \int_0^1 K_\sigma(s, t) \, d\mu(s) = \sup_{t \in (0, 1)} \mu(1-t)^\sigma \int_0^1 s^{\mu-1} \, _2F_1\left(\frac{\lambda}{2}, \frac{\lambda}{2}; \mu; st\right) \, ds.$$  

By the Euler formula we obtain

$$\int_0^1 s^{\mu-1} \, _2F_1\left(\frac{\lambda}{2}, \frac{\lambda}{2}; \mu; st\right) \, ds = \mu^{-1} \, _2F_1\left(\frac{\lambda}{2}, \frac{\lambda}{2}; \mu + 1; t\right).$$  

Regarding the Euler transform we have

$$(1-t)^\sigma \, _2F_1\left(\frac{\lambda}{2}, \frac{\lambda}{2}; \mu + 1; t\right) = \, _2F_1\left(\mu + 1 - \frac{\lambda}{2}, \mu + 1 - \frac{\lambda}{2}; \mu + 1; t\right).$$
It follows, in view of Lemma 2.7, that
\[
\sup_{t \in (0, 1)} \int_0^1 K_\sigma(s, t) \, d\mu(s) = \sup_{t \in (0, 1)} 2F_1 \left( \mu + 1 - \frac{\lambda}{2} \mu + 1 - \frac{\lambda}{2}; \mu + 1; t \right) = \frac{\Gamma(\mu + 1)\Gamma(\lambda - \mu - 1)}{\Gamma^2(\frac{\lambda}{2})} = \frac{\Gamma(\mu + 1)\Gamma(\sigma)}{\Gamma^2(\frac{\lambda}{2})} < \infty
\]
if and only if \( \sigma > 0 \). According to Lemma 2.2 we conclude
\[
\|F_\sigma : L^1_\mu(0, 1) \to L^1_\mu(0, 1)\| = \frac{\Gamma(\mu + 1)\Gamma(\sigma)}{\Gamma^2(\frac{\lambda}{2})}
\]
for all \( \sigma > 0 \).

The rest of the proof of our main result is devoted to the case \( 1 < p < \infty \).
We divide it into two parts. In the first one, using the Schur test, we prove that \( F_\sigma \) is bounded in the case \( \sigma > \frac{1}{p} - 1 \) and we obtain the estimate of the norm \( \|F_\sigma : L^p_\mu(0, 1) \to L^p_\mu(0, 1)\| \) from above. In the second part we deliver the proof that the same number is the estimate of \( \|F_\sigma : L^p_\mu(0, 1) \to L^p_\mu(0, 1)\| \) from below.
In this part we also obtain that the condition \( \sigma > \frac{1}{p} - 1 \) is necessary for the boundedness of \( F_\sigma \).

**Part I**

To start with this, denote \( \varphi(t) = (1 - t)^{-\frac{1}{q}} \). Assume that \( \sigma > \frac{1}{p} - 1 \).

We have
\[
\int_0^1 K_\sigma(s, t) \varphi(t)^q \, d\mu(t) = \mu \int_0^1 t^{\mu - 1} (1 - t)^{\sigma} 2F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \mu; st \right) \varphi(t)^q \, dt.
\]

Since
\[
(1 - t)^{\sigma} \varphi(t)^q = (1 - t)^{(\lambda - \frac{1}{q}) - \mu - 1},
\]
we may use the Euler formula given in Lemma 2.5 for \( c = \lambda - \frac{1}{q} \) and \( d = \mu \) (observe that \( c - d = \sigma + \frac{1}{q} > \frac{1}{p} - 1 + \frac{1}{q} = 0 \)). It follows
\[
\int_0^1 K_\sigma(s, t) \varphi(t)^q \, d\mu(t) = \mu \frac{\Gamma(\mu)\Gamma(\lambda - \mu - \frac{1}{p})}{\Gamma(\lambda - \frac{1}{p})} 2F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \lambda - \frac{1}{p}; s \right) \varphi(s)^q.
\]

Since
\[
2F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \lambda - \frac{1}{p}; s \right) = (1 - s)^{-\frac{1}{p}} 2F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \lambda - \frac{1}{p}; s \right) \varphi(s)^q = 2F_1 \left( \frac{\lambda}{2} - \frac{1}{p}, \frac{\lambda}{2} - \frac{1}{p}; \lambda - \frac{1}{p}; s \right) \varphi(s)^q
\]
(regarding Lemma 2.6), we obtain
\[
\int_0^1 K_\sigma(s, t) \varphi(t)^q \, d\mu(t) = \frac{\Gamma(\mu + 1)\Gamma(\lambda - \mu - \frac{1}{p})}{\Gamma(\lambda - \frac{1}{p})} 2F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \lambda - \frac{1}{p}; s \right) \varphi(s)^q.
\]
The function \(2F_1\left(\frac{\lambda}{2} - \frac{1}{p}, \frac{\lambda}{2} - \frac{1}{p}; \lambda - \frac{1}{p}; s\right)\) is increasing in \(0 < s < 1\), thus, by Lemma 2.7 we have
\[
\int_0^1 K_\sigma(s, t) \varphi(t)^q d\mu(t) \leq \frac{\Gamma(\mu + 1)\Gamma(\lambda - \mu - \frac{1}{p})}{\Gamma(\lambda - \frac{1}{p})} \frac{\Gamma(\lambda - \frac{1}{p})\Gamma(\lambda - \frac{1}{p})}{\Gamma^2(\lambda)} \varphi(s)^q
\]

\[
\leq \frac{\Gamma(\mu + 1)\Gamma(\frac{1}{p})\Gamma(\lambda - \mu - \frac{1}{p})}{\Gamma^2(\frac{1}{p})} \varphi(s)^q.
\]

Similarly, one has
\[
\int_0^1 K_\sigma(s, t) \varphi(s)^p d\mu(s) = \mu (1 - t)^\sigma \int_0^1 s^{\mu - 1} 2F_1\left(\frac{\lambda}{2}; \frac{\lambda}{2}; s t\right) \varphi(s)^p ds.
\]

Since
\[
\varphi(s)^p = (1 - s)^{\mu + \frac{1}{p} - \mu - 1},
\]
we obtain
\[
\int_0^1 K_\sigma(s, t) \varphi(s)^p d\mu(s) = \mu (1 - t)^\sigma \frac{\Gamma(\mu)\Gamma(\frac{1}{p})}{\Gamma(\mu + \frac{1}{p})} 2F_1\left(\frac{\lambda}{2}; \frac{\lambda}{2}; \mu + \frac{1}{p}; t\right)
\]

(see Lemma 2.5 for \(c = \mu + \frac{1}{p}\) and \(d = \mu\)). Since
\[
(1 - t)^\sigma 2F_1\left(\frac{\lambda}{2}; \frac{\lambda}{2}; \mu + \frac{1}{p}; t\right) = (1 - t)^{\lambda - \mu - \frac{1}{p}} 2F_1\left(\frac{\lambda}{2}; \frac{\lambda}{2}; \mu + \frac{1}{p}; t\right) \varphi(t)^p
\]
\[
= 2F_1\left(\mu - \frac{\lambda}{2} + \frac{1}{p}; \mu - \frac{\lambda}{2} + \frac{1}{p}; \mu + \frac{1}{p}; t\right) \varphi(t)^p,
\]
we transform
\[
\int_0^1 K_\sigma(s, t) \varphi(s)^p d\mu(s)
\]
\[
= \frac{\Gamma(\mu + 1)\Gamma(\frac{1}{p})}{\Gamma(\mu + \frac{1}{p})} 2F_1\left(\mu - \frac{\lambda}{2} + \frac{1}{p}; \mu - \frac{\lambda}{2} + \frac{1}{p}; \mu + \frac{1}{p}; t\right) \varphi(t)^p.
\]

Using now Lemma 2.7, we obtain
\[
\int_0^1 K_\sigma(s, t) \varphi(s)^p d\mu(s) \leq \frac{\Gamma(\mu + 1)\Gamma(\frac{1}{p})\Gamma(\mu + \frac{1}{p})\Gamma(\lambda - \mu - \frac{1}{p})}{\Gamma^2(\lambda)} \varphi(t)^p
\]
\[
= \frac{\Gamma(\mu + 1)\Gamma(\frac{1}{p})\Gamma(\sigma + 1 - \frac{1}{p})}{\Gamma^2(\frac{1}{p})} \varphi(t)^p.
\]

By the Schur test we finally obtain
\[
\|F_\sigma : L^p_\mu(0, 1) \to L^p_\mu(0, 1)\| \leq \frac{\Gamma(\mu + 1)\Gamma(\frac{1}{p})\Gamma(\sigma + 1 - \frac{1}{p})}{\Gamma^2(\frac{1}{p})}
\]

for every \(\sigma > \frac{1}{p} - 1\).

\textit{Part II}
As we have said, the aim of this part is to establish the norm estimate of $F_\sigma$ ($1 < p < \infty$, $\sigma > \frac{1}{p} - 1$) from below. The following two simple lemmas will be useful in that approach.

**Lemma 3.1.** If $H(t) = C t^{\frac{\sigma}{p}} (1-t)^{\frac{\theta}{p}}$, where $C$ is a positive constant, then $H \in L^p_\mu(0,1)$ ($1 < p < \infty$) and $\|H\|_{p,\mu} = 1$ if and only if $\theta > -\mu$, $\theta > -1$ and

$$C = \mu^{-\frac{\theta}{p}} B(\theta + \mu, \theta + 1)^{-\frac{\theta}{p}}.$$  

**Proof.** From

$$1 = \|H\|_{p,\mu} = \mu \int_0^1 H(t)^p \, t^{\mu-1} \, dt = \mu \int_0^1 t^{\theta+\mu-1} (1-t)^{\theta} \, dt$$

it follows that $H \in L^p_\mu(0,1)$ if and only if $\theta > -1$, $\theta > -\mu$, and the expression for the constant $C$. \hfill \Box

**Lemma 3.2.** Let $l > 0$ and let $G$ be any function defined in an interval $(0,1)$ with positive values. For every $1 < p < \infty$ we have

$$\limsup_{(\zeta, \eta) \to (0,0)} \frac{G(\eta)^{-\frac{\theta}{p}} G(\zeta + \eta)}{G(\zeta)^{-\frac{\theta}{p}} G(\frac{\zeta}{p} + 1)^{1-\frac{\theta}{p}}} \geq 1.$$  

**Proof.** To prove this part of lemma, it is enough to note that if we set $\eta = \frac{\zeta}{p-1}$, we obtain

$$\frac{G(\eta)^{-\frac{\theta}{p}} G(\zeta + \eta)}{G(\zeta)^{-\frac{\theta}{p}} G(\frac{\zeta}{p} + 1)^{1-\frac{\theta}{p}}} = \frac{G(\zeta)^{-\frac{\theta}{p}} G(\frac{\zeta}{p} + 1)^{1-\frac{\theta}{p}}}{G(\eta)^{-\frac{\theta}{p}} G(\frac{\zeta}{p} + 1)^{1-\frac{\theta}{p}}} = 1,$$

what immediately implies the statement of this lemma. \hfill \Box

If $L^p = L^p(X, \nu)$ is a Lebesgue space, recall that for an operator $T : L^p \to L^p$ ($1 < p < \infty$) we have

$$\|T : L^p \to L^p\| = \sup \left\{ \left( \int_X T \Phi(x) \overline{\Psi(x)} \, dx \right) : f \in \Phi \in L^p, \Psi \in L^q, \|\Phi\|_p = \|\Psi\|_q = 1 \right\},$$

where $q$ is conjugate to $p$, i.e., $q = \frac{p}{p-1}$.

For our operator $F_\sigma$ we will calculate

$$\int_0^1 F_\sigma \Phi(s) \overline{\Psi(s)} \, d\mu(s)$$

for appropriate $\Phi(s) \in L^p_\mu(0,1)$ and $\Psi(t) \in L^q_\mu(0,1)$.

Using the Fubini theorem we obtain

$$\int_0^1 F_\sigma \Phi(s) \overline{\Psi(s)} \, d\mu(s)$$

$$= \mu^2 \int_0^1 s^{\mu-1} \left\{ \int_0^1 t^{\mu-1} (1-t)^{\sigma} 2F_1 \left( \frac{\lambda}{2} \frac{\lambda}{2} ; \mu ; st \right) \Phi(t) \, dt \right\} \overline{\Psi(s)} \, ds$$

$$= \mu^2 \int_0^1 t^{\mu-1} (1-t)^{\sigma} \left\{ \int_0^1 s^{\mu-1} 2F_1 \left( \frac{\lambda}{2} \frac{\lambda}{2} ; \mu ; st \right) \overline{\Psi(s)} \, ds \right\} \Phi(t) \, dt.$$
In order to estimate the norm of $F_\sigma : L_\mu^p(0,1) \rightarrow L_\mu^p(0,1)$ from below in the preceding relation we will take for $\Phi(t)$ and $\Psi(s)$ the functions of the following form

$$\Phi(t) = C t^{\tilde{\theta}} (1-t)^{\tilde{\theta}}$$

and

$$\Psi(s) = C' s^{\tilde{\theta}} (1-s)^{\tilde{\theta}}.$$  

By Lemma 3.1 we must have $\theta, \tilde{\theta} > -\mu,$ and $\theta, \tilde{\theta} > -1,$ as well as

$$C_\mu = \mu^{-1} B(\theta + \mu, \tilde{\theta} + 1)^{-1} \quad \text{and} \quad C_\sigma = \mu^{-1} B(\theta + \mu, \tilde{\theta} + 1)^{-1}.$$  

In the sequel we will chose $\theta, \tilde{\theta}$ and $\sigma$ in the way that it makes simpler the calculation of integrals in the expression for $\int_0^1 F_\sigma \Phi(s) \Psi(s) \, d\mu(s).$

Introducing the preceding type of functions with $\tilde{\theta} = 0$ we obtain

$$\int_0^1 s^{\mu-1} 2 F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \mu; s t \right) \Psi(s) \, ds$$

$$= \tilde{C} \int_0^1 s^{\mu-1} (1-s)^{\tilde{\theta}} 2 F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \mu; s t \right) \, ds$$

$$= \tilde{C} \int_0^1 s^{\mu-1} (1-s)^{\tilde{\theta}+\mu+1} 2 F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \mu; s t \right) \, ds$$

$$= \tilde{C} \frac{\Gamma(\mu+\tilde{\theta}) \Gamma(\frac{\lambda}{2}+\sigma+1) \Gamma(\frac{\lambda}{2}+\frac{\lambda}{2}+1)}{\Gamma(\frac{\lambda}{2}+\mu+1) \Gamma(\frac{\lambda}{2}+\mu+\tilde{\theta}+1)} \right).$$

where we have used Lemma 2.8 for $c = \frac{\tilde{\theta}}{q} + \mu + 1$ and $d = \mu;$ note $c - d = \frac{\tilde{\theta}}{q} + 1 > -\frac{1}{q} + 1 = \frac{1}{p} > 0.$

For the sake of simplicity in the following calculation we set $\tilde{\theta} = \frac{\theta - q}{p-1}$. Then we have

$$\tilde{\theta} + \mu + 1 = \mu + \frac{\theta}{p}.$$  

Since we must have $\tilde{\theta} > -1,$ it follows that $\theta > 1.$ Now, it remains to transform

$$\int_0^1 t^{\mu+\tilde{\theta}-1} (1-t)^{\tilde{\theta}+\mu+1} 2 F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \mu+\tilde{\theta}+1; t \right) \Phi(t) \, dt$$

$$= C \int_0^1 t^{\mu+\tilde{\theta}+\lambda+\tilde{\theta}+1} 2 F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; \mu+\tilde{\theta}+1; t \right) \, dt$$

$$= C \frac{\Gamma(\mu+\tilde{\theta}) \Gamma(\frac{\lambda}{2}+\sigma+1) \Gamma(\frac{\lambda}{2}+\frac{\lambda}{2}+1)}{\Gamma(\frac{\lambda}{2}+\mu+1) \Gamma(\frac{\lambda}{2}+\mu+\tilde{\theta}+1)} \right).$$

We have used Lemma 2.8, note that $\lambda + \frac{\tilde{\theta}}{p} + \frac{\lambda}{2} + \frac{\lambda}{2} + 1 - \lambda = \frac{\tilde{\theta}}{p} + \frac{\lambda}{2} + 1 > 0.$

All together we have

$$\int_0^1 F_\sigma \Phi(s) \Psi(s) \, d\mu(s)$$

$$= \mu^2 \tilde{C} \frac{\Gamma(\mu+\tilde{\theta}) \Gamma(\frac{\lambda}{2}+\mu+1) \Gamma(\frac{\lambda}{2}+\mu+\tilde{\theta}+1) \Gamma(\frac{\lambda}{2}+\frac{\lambda}{2}+1)}{\Gamma(\frac{\lambda}{2}+\mu+1) \Gamma(\frac{\lambda}{2}+\mu+\tilde{\theta}+1) \Gamma(\frac{\lambda}{2}+\mu+\frac{\lambda}{2}+1)} \right).$$
In the sequel we assume that $\sigma > \frac{1}{p} - 1$ (note that in this case we have $\hat{\sigma} + \sigma + 1 > 0$). If this inequality does not hold, then $F_\sigma : L^p_\mu(0, 1) \to L^p_\mu(0, 1)$ is not well defined.

Since $\vartheta = 0$ and $\hat{\vartheta} = \frac{\theta - p}{p - 1}$ we obtain

$$\begin{align*}
C \hat{C} &= \mu^{-\frac{1}{p}} B(\theta + \mu, \hat{\theta} + 1)^{-\frac{1}{p}} \mu^{-\frac{1}{p}} B(\theta + \mu, \hat{\theta} + 1)^{\frac{1}{p} - 1} \\
&\sim \mu^{-1} \Gamma(\hat{\theta} + 1)^{-\frac{1}{p}} \\
&= \frac{\Gamma(\mu)^{\left(\frac{1}{p}\right)} \Gamma\left(-\frac{1}{p} + \sigma + 1\right)}{\Gamma^2\left(\frac{1}{p}\right)} \\
&= \frac{\Gamma(\mu + 1)^{\left(\frac{1}{p}\right)} \Gamma\left(-\frac{1}{p} + \sigma + 1\right)}{\Gamma^2\left(\frac{1}{p}\right)} \\
&\geq \frac{\Gamma(\mu + 1) \Gamma\left(-\frac{1}{p} + \sigma + 1\right)}{\Gamma^2\left(\frac{1}{p}\right)} \\
&= \frac{\Gamma(\mu + 1) \Gamma\left(-\frac{1}{p} + \sigma + 1\right)}{\Gamma^2\left(\frac{1}{p}\right)} \\
&\geq \frac{\Gamma(\mu + 1) \Gamma\left(-\frac{1}{p} + \sigma + 1\right)}{\Gamma^2\left(\frac{1}{p}\right)}.
\end{align*}$$

Thus, we have proved

$$\|F_\sigma : L^p_\mu(0, 1) \to L^p_\mu(0, 1)\| \geq \limsup_{(\theta, \hat{\theta}) \to (1, -1)} \int_0^1 F_\sigma \Phi(s) \Psi(s) d\mu(s)$$

$$\geq \frac{\Gamma(\mu + 1) \Gamma\left(-\frac{1}{p} + \sigma + 1\right)}{\Gamma^2\left(\frac{1}{p}\right)}$$

for $\sigma > \frac{1}{p} - 1$.

4. ON THE FORELLI–RUDIN THEOREM

We go back now to the operator $T_\sigma$ mentioned in Introduction. Beside that operator we will consider now the integral operator $\tilde{T}_\sigma$ ($\sigma > -1$) given by the kernel

$$|K_\sigma(z, w)| = \frac{(1 - |w|^2)^\sigma}{|1 - \langle z, w \rangle|^\lambda},$$

i.e.,

$$\tilde{T}_\sigma f(z) = c_\sigma \int_B \frac{(1 - |w|^2)^\sigma}{|1 - \langle z, w \rangle|^\lambda} f(w) dv(w), \quad z \in B;$$

where $\lambda = n + \sigma + 1$.

It is known that $\tilde{T}_\sigma$ maps $L^p(B)$ ($1 \leq p < \infty$) into itself continuously if and only if and $\sigma > \frac{1}{p} - 1$; see [7] where Forelli and Rudin used this operator in order to establish the continuity of $T_\sigma$. 
In order to connect our main result with the Forelli–Rudin result, we will first transform the integral $I_c(z)$ which appears in the first chapter of the Rudin monograph [15].

**Lemma 4.1.** Introduce

$$I_c(z) = \int_S \frac{d\tau(\zeta)}{|1 - (z, \zeta)|^{n+c}}$$

for $z \in B$ and for any real number $c$. Then

$$I_c(z) = 2F_1(\tilde{\lambda}, \tilde{\lambda}; n; |z|^2),$$

where $\tilde{\lambda} = \frac{n}{2} + \frac{c}{2}$. 

**Proof.** In order to prove this lemma we use the result from the proof of Proposition 1.4.10 in [15]. In this proposition the following fact is proved

$$I_c(z) = \frac{\Gamma(n)}{\Gamma(\lambda)} \sum_{k=0}^\infty \frac{\Gamma(\lambda + k)}{\Gamma(n + k)|z|^{2k}}.$$

Since

$$(d)_k = \frac{\Gamma(d + k)}{\Gamma(d)} = d(d + 1) \cdots (d + k - 1)$$

for $d \notin \{0, -1, -2, \ldots\}$, it follows now

$$I_c(z) = \sum_{k=0}^\infty \frac{\Gamma(\lambda + k)}{\Gamma(n + k)|z|^{2k}} \frac{1}{k!} = \sum_{k=0}^\infty \frac{(\lambda)_k (\tilde{\lambda})_k |z|^{2k}}{(n)_k} = 2F_1(\tilde{\lambda}, \tilde{\lambda}; n; |z|^2)$$

for all $z \in B$. 

We say that a function $h(y)$ defined in the unit ball is radially symmetric if there exist $H(r)$ defined for $0 < r < 1$ such that $h(y) = H(|y|^2)$ for all $y \in B$.

The following simple lemma will be useful.

**Lemma 4.2.** Let $h(w)$ be a radially symmetric function in the unit ball of the form $h(w) = H(|w|^2)$, where $H(t)$ is defined in the interval $(0, 1)$ and non–negative.

a) Norm of $h \in L^p(B)$ is given by

$$\|h\|_p^p = n \int_0^1 s^{n-1} H(s)^p \, ds = \|H\|_{p,n}^p$$

for $1 \leq p < \infty$.

b) If a function $h(w)$ defined in the unit ball is radially symmetric then so is $\tilde{T}_\sigma h(z)$, if it is defined. Moreover, if $h(w) = H(|w|^2)$, where $H(t)$ is a non–negative measurable function in $(0, 1)$, then

$$\tilde{T}_\sigma h(z) = c_\sigma n \int_0^1 t^{n-1} (1 - t)^\sigma \, 2F_1 \left( \frac{\lambda}{2}, \frac{\lambda}{2}; n; t |z|^2 \right) H(t) \, dt.$$

**Proof.** a) Using polar coordinates we obtain

$$\int_B h(z) \, dv(z) = 2n \int_0^1 r^{2n-1} \, dr \int_S h(r\zeta) \, d\tau(\zeta) = 2n \int_0^1 r^{2n-1} H(r^2) \, dr$$

$$= n \int_0^1 s^{n-1} H(s) \, ds.$$
b) Using polar coordinates and Lemma 4.1 we obtain
\[ c_\sigma^{-1} \hat{T}_\sigma h(z) = 2n \int_0^1 r^{2n-1} (1 - r^2)^\sigma I_{\lambda - n}(rz) H(r^2) \, dr \]
\[ = n \int_0^1 s^{n-1} (1 - s)^\sigma \, \gamma_1 \left( \frac{\lambda}{2} ; \lambda ; n ; s \right)^2 F_1 \left( \frac{\lambda}{2} ; \lambda ; n ; |z|^2 \right) H(s) \, ds \]

(we introduced \( s = r^2 \) to obtain the last integral). \( \square \)

Thus we have

**Lemma 4.3.** The operator \( \hat{T}_\sigma : L^p(B) \to L^p(B) \) is bounded if and only if \( F_\sigma : L^p_n(0,1) \to L^p_n(0,1) \) is bounded. Moreover,
\[ \|\hat{T}_\sigma : L^p(B) \to L^p(B)\| = c_\sigma \|F_\sigma : L^p_n(0,1) \to L^p_n(0,1)\| \]
for all \( 1 \leq p < \infty \) and \( \sigma > \frac{1}{p} - 1 \).

From this lemma we derive

**Theorem 4.4.** The operator \( \hat{T}_\sigma \) is bounded if and only if \( \sigma > \frac{1}{p} - 1 \). Norm of \( \hat{T}_\sigma : L^p(B) \to L^p(B) (1 \leq p < \infty) \) is
\[ \|\hat{T}_\sigma : L^p(B) \to L^p(B)\| = \frac{\Gamma(\lambda) \Gamma\left(\frac{1}{p}\right) \Gamma\left(\sigma + 1 - \frac{1}{p}\right)}{\Gamma(\sigma + 1) \Gamma^2\left(\frac{1}{2}\right)} \]
\[ = \frac{\Gamma(n + \sigma + 1) \Gamma\left(\frac{1}{p}\right) \Gamma\left(\sigma + 1 - \frac{1}{p}\right)}{\Gamma(\sigma + 1) \Gamma^2\left(\frac{n}{2} + \frac{q}{2} + \frac{1}{2}\right)} \]
for all \( \sigma > \frac{1}{p} - 1 \).

**Remark 4.5.** For \( n = 1 \) Theorem 4.4 reduces to the main result in [4]. See Theorem 1 there. See also [11].

The conjugate operator \( \hat{T}_\sigma^* : L^p(B) \to L^p(B) (1 < p \leq \infty) \) of \( \hat{T}_\sigma : L^q(B) \to L^q(B) (1 \leq q < \infty) \) is
\[ \hat{T}_\sigma^* g(z) = c_\sigma \int_B \frac{(1 - |z|^2)^\sigma}{|1 - \langle z, w \rangle|^\lambda} g(w) \, dv(w), \quad z \in B. \]

Since
\[ \|\hat{T}_\sigma^* : L^p(B) \to L^p(B)\| = \|\hat{T}_\sigma : L^q(B) \to L^q(B)\|, \]
we immediately deduce

**Corollary 4.6.** Norm of \( \hat{T}_\sigma^* : L^p(B) \to L^p(B) (1 < p \leq \infty) \) is
\[ \|\hat{T}_\sigma^* : L^p(B) \to L^p(B)\| = \frac{\Gamma(n + \sigma + 1) \Gamma\left(\frac{1}{q}\right) \Gamma\left(\sigma + 1 - \frac{1}{q}\right)}{\Gamma(\sigma + 1) \Gamma^2\left(\frac{n}{2} + \frac{q}{2} + \frac{1}{2}\right)} \]
\[ = c_\sigma \Gamma(n + 1) \frac{\Gamma\left(\frac{1}{q}\right) \Gamma\left(\sigma + 1 - \frac{1}{q}\right)}{\Gamma^2\left(\frac{n}{2} + \frac{q}{2} + \frac{1}{2}\right)} \]
for \( \sigma > \frac{1}{q} - 1 \).
5. An estimate of the norm of $T_\sigma$

Forelli and Rudin [7] proved that

$$\|T_\sigma : L^1(B) \to L^1_a(B)\| = \frac{\Gamma(\lambda) \Gamma(\sigma)}{\Gamma(\lambda/2) \Gamma(\sigma + 1)}, \quad \sigma > 0.$$ 

Note that $\|T_\sigma : L^1(B) \to L^1(B)\| = \|T_\sigma : L^1(B) \to L^1_a(B)\|$. They also proved

$$\|T_\sigma : L^2(B) \to L^2_a(B)\| = \frac{\sqrt{\Gamma(2\sigma + 1)}}{\Gamma(\sigma + 1)}^{1 - \frac{1}{2}}, \quad \sigma > -\frac{1}{2}.$$ 

By the Riesz–Thorin theorem we obtain

$$\|T_\sigma : L^p(B) \to L^p_a(B)\| \leq \left\{ \frac{\Gamma(\lambda)}{\Gamma(\lambda/2)} \frac{\Gamma(\sigma)}{\Gamma(\sigma + 1)} \right\}^{\frac{2}{p} - 1} \left\{ \frac{\sqrt{\Gamma(2\sigma + 1)}}{\Gamma(\sigma + 1)} \right\}^{2 - \frac{2}{p}}$$

for $1 \leq p \leq 2$ and $\sigma > 0$.

The estimate of $\|T_\sigma : L^p(B) \to L^p_a(B)\|$ given in the following corollary is better in some cases.

**Corollary 5.1.**

$$\|T_\sigma : L^p(B) \to L^p_a(B)\| \leq \frac{\Gamma(\lambda)}{\Gamma(\lambda/2)} \frac{\Gamma(\frac{1}{2} - \frac{1}{p}) \Gamma(\sigma + 1 - \frac{1}{p})}{\Gamma(\sigma + 1)}$$

for $1 \leq p < \infty$ and $\sigma > \frac{1}{p} - 1$.

Particularly, for $\sigma = 0$ we have $P = T_0$ and the norm estimate

$$\|P : L^p(B) \to L^p_a(B)\| \leq \frac{\Gamma(n + 1)}{\Gamma(\frac{n}{2} + \frac{1}{p})} \frac{\pi}{\sin \frac{\pi}{p}},$$

where $1 < p < \infty$. This estimate for $n = 1$ is also obtained in [5].

6. $L^p$–norm of the transform of Berezin

In the case of the unit ball, the Berezin transform takes the form

$$\mathcal{B} f(z) = \int_B \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2n+2}} f(w) dv(w), \quad z \in B.$$ 

Berezin [2] introduces the notion of covariant and contravariant symbols of an operator. The Berezin transform finds applications in the study of Hankel and Toeplitz operators. An interesting result says that if $f \in L^1(U)$, where $U = \{ z \in \mathbb{C} : |z| < 1 \}$ is the unit disc in the complex plane, then $f$ is a harmonic function in $U$ if and only if $\mathcal{B} f = f$. For this result see [9].

Observe that

$$\hat{T}_{n+1}^* = c_{n+1} \mathcal{B}.$$
Corollary 6.1. Norm of the Berezin transform $\mathfrak{B} : L^p(B) \to L^p(B)$ is
\[
\|\mathfrak{B} : L^p(B) \to L^p(B)\| = \left\{ \prod_{k=1}^n \left( 1 + \frac{1}{k} \right) \right\} \frac{\pi}{\sin \frac{\pi}{p}}
\]
for $1 < p < \infty$, and
\[
\|\mathfrak{B} : L^\infty(B) \to L^\infty(B)\| = 1.
\]
The result in this corollary is obtain in [11], but we give a proof for the sake of completeness.

Proof. Let $1 < p < \infty$. Since by Corollary 4.6 we have
\[
\|\tilde{T}_{n+1}^* : L^p(B) \to L^p(B)\| = c_{n+1} \Gamma(n+1) \frac{\Gamma(n+1) \Gamma(1 - \frac{1}{p})}{\Gamma^2(n+1)},
\]
it follows
\[
\|\mathfrak{B} : L^p(B) \to L^p(B)\| = \frac{\Gamma(n+1) \Gamma(1 - \frac{1}{p})}{\Gamma(n+1)}
\]
\[
= \frac{1}{n!} \left\{ \prod_{k=1}^n \left( k + \frac{1}{p} \right) \right\} \frac{\Gamma(\frac{1}{p}) \Gamma(1 - \frac{1}{p})}{\Gamma(n+1)}
\]
\[
= \left\{ \prod_{k=1}^n \left( 1 + \frac{1}{k} \right) \right\} \frac{\pi}{\sin \frac{\pi}{p}}
\]
We have used the Euler identity
\[
\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin \frac{\pi}{x}} \quad \text{for} \quad 0 < x < 1
\]
in order to obtain the last expression (for example see the first chapter in [1]).

The case $p = \infty$ follows also from Corollary 4.6. Introducing $q = 1$ we obtain
\[
\|\tilde{T}_{n+1}^* : L^\infty(B) \to L^\infty(B)\| = c_{n+1},
\]
what implies the result concerning the $L^\infty$--norm of the Berezin transform. \qed

Corollary 6.2.
\[
\|\mathfrak{B} : L^2(B) \to L^2(B)\| = \frac{(2n+1)!! \pi}{(2n)!!}.
\]

Corollary 6.3.
\[
\|\mathfrak{B} : L^p(B) \to L^p(B)\| \sim \frac{(n+1)\pi}{\sin \frac{\pi}{p}} \sim \frac{(n+1)\pi}{\pi - \frac{\pi}{p}} \sim \frac{n+1}{p-1} \quad \text{as} \quad p \to 1.
\]

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