Holographic heat engines

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Abstract
It is shown that in theories of gravity where the cosmological constant is considered a thermodynamic variable, it is natural to use black holes as heat engines. Two examples are presented in detail using AdS charged black holes as the working substance. We notice that for static black holes, the maximally efficient traditional Carnot engine is also a Stirling engine. The case of negative cosmological constant supplies a natural realization of these engines in terms of the field theory description of the fluids to which they are holographically dual. We first propose a precise picture of how the traditional thermodynamic dictionary of holography is extended when the cosmological constant is dynamical and then conjecture that the engine cycles can be performed by using renormalization group flow. We speculate about the existence of a natural dual field theory counterpart to the gravitational thermodynamic volume.

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(Some figures may appear in colour only in the online journal)

1. Extended black hole thermodynamics

Recently, the classic subject of black hole thermodynamics [1–4], which relates the mass $M$, surface gravity $\kappa$, and area $A$ of a black hole to the energy $U$, temperature $T$, and entropy $S$, according to

$$ M = U, \quad T = \frac{\kappa}{2\pi}, \quad S = \frac{A}{4}, \quad (1) $$
has been extended\(^1\) to include black hole counterparts for the pressure \(p\) and volume \(V\). The cosmological constant of the spacetime in question supplies the pressure via \(p = -\Lambda/8\pi\), while the thermodynamic volume \(V\) is associated with the volume occupied by the black hole itself\(^2\). (Here we are using geometrical units where \(G, c, \hbar, k_B\) have been set to unity. We may restore them using dimensional analysis when required later.) The formalism works in multiple dimensions, and our remarks will apply to those situations too, although for clarity we will mostly write four-dimensional formulae. The black holes may have other parameters such as gauge charges \(q_i\) and angular momenta \(J_i\), and these, with their conjugates the potentials \(\Phi_i\) and angular velocities \(\Omega_j\), enter additively into the first law in the usual manner.

In the presence of a variable pressure \(p\) (now identified with the cosmological constant), [9] proposed that the extension shifts the identification of the mass \(M\) from being the energy \(U\) to being the enthalpy, to wit: \(M = H \equiv U + pV\), so the first law now becomes

\[
dM = TdS + Vdp + \Phi dq + \Omega dJ,
\]

in four dimensions with an electric charge and rotation. When \(p\) is removed from the list of variables, we return to the usual situation.

In the case of static black holes, the thermodynamic volume \(V\) is simply the ‘geometric’ volume constructed by naive use of the radius of the black hole horizon to form the associated volume\(^3\). For example, in four dimensions, for a Schwarzschild or Reissner–Nordström black hole with horizon radius \(r_h\), we have

\[
V = \frac{4}{3} \pi r_h^3.
\]

Enthalpy is very natural here: the cosmological constant is a spacetime energy density of \(-p = \Lambda/8\pi\) per unit volume. Forming a black hole of volume \(V\) requires cutting out a region of spacetime of that volume, at cost \(pV\), and this energy of formation is naturally captured by the enthalpy. It is important to note that the entropy \(S\) is already related to the horizon radius through its relation to area via the Bekenstein area law. So in this case of static black holes, the thermodynamic volume \(V\) and the entropy \(S\) are simply related to each other. This is key to the simplicity of one of the results concerning thermodynamic cycles presented below. The lack of independence of \(S\) and \(V\) would be a concern if studying problems that use the internal energy \(U(S, V)\) as the central thermodynamic potential, but we are in a situation where it is the enthalpy \(H(S, p)\) that is natural. Pressure and entropy are the key variables here, and they are independent for the holes in question.

Note that for rotating black holes, the thermodynamic volume \(V\) and the entropy \(S\) are independent (the situation is resolved by non-zero angular momentum \(J\)), and there are no special subtleties involving \(U\) as a result. In fact, the thermodynamic volume is no longer the naive geometric volume occupied by the black hole in this case. [5, 11, 13] expand upon these issues.

### 2. Thermodynamic cycles and heat engines

With pressure and volume in play alongside temperature and entropy, the possibility of extracting mechanical useful work from heat energy naturally springs to mind. (We may also consider heat pumps or refrigerators, where instead work is done to transfer heat from a cold

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1. For a selection of references, see [5–13], including the reviews in [14, 15]. See also the early work in [16–18].
2. The term ‘associated’ is used since there is a subtlety to be discussed later.
3. Such a definition agrees with the definition of the volume of a static black hole proposed in [19].
We can start with an equation of state, e.g. a function \( p(V, T) \), and define an engine as a closed path in the \( p-V \) plane, allowing for the input of an amount of heat \( Q_H \), and the exhaust of an amount \( Q_C \). The total mechanical work done, by the first law, is of course \( W = Q_H - Q_C \). So the efficiency of the heat engine is defined to be \( \eta = W/Q_H = 1 - Q_C/Q_H \). Figure 1 shows the standard logic of the energy flows for one cycle of the engine.

The precise engine we make depends upon the choice of path in the \( p-V \) plane, and possibly the equation of state of the black hole in question. Let us make a simple cycle as follows: some of the classic cycles involve a pair of isotherms at temperatures \( T_H \) and \( T_C \), where \( T_H > T_C \), where there is an isothermal expansion while some heat is being absorbed, and an isothermal compression during the expulsion of some heat. We can connect these in a variety of ways, but two simple choices are natural. We can do isochoric paths to connect the two temperatures, as in the classic Stirling cycle, or we can do adiabatic paths, as in the classic Carnot cycle. For the latter, all the heat flows of the engine take place during those two isotherms, and (with the usual assumption that we do things slowly enough to be in the quasistatic regime) these are reversible. The whole heat engine is fully reversible (since the total entropy flow is zero) and so the engine should have the Carnot efficiency, which is set simply by the temperature difference: \( \eta = 1 - T_C/T_H \). This the maximum efficiency any heat engine can have when operating between these temperatures. Any higher efficiency would violate the second law.

This is therefore the gold standard engine, and so it is interesting to explore how it is precisely realized in black hole thermodynamics since any other black hole heat engine that
might be made will be measured against this one. Now, it is comforting to keep in mind that whatever the equation of state, the above described Carnot path will yield the Carnot efficiency\(^4\), but nevertheless it is important and useful to know exactly what the shape of the paths are for a given system. For a general black hole, working out the explicit equation of state can be a difficult task (it is usually easier to define \(p\), \(V\), and \(T\) in terms of another natural variable such as the horizon radius, or the entropy, which implies the equation of state upon elimination of that intermediate variable), and it is additionally complex (for a sufficiently complicated equation of state) to have a closed form equation for both the isotherms and adiabats. So it is a daunting task to determine the shapes of the Carnot cycle for the black holes explicitly.

This is where, for the static holes, the fact that the thermodynamic volume \(V\) and the entropy \(S\) are not independent is key. It means that adiabats and isochores are the same! Carnot and Stirling coincide. So the efficiency of our cycle may be simply computed, and some of the path known explicitly, without knowledge of the detailed equation of state. All that’s needed is that the entropy and volume are related.

Figure 2. Our Carnot engine, which for static black holes is also a Stirling engine.

So along the upper isotherm (subscripts refer to the labelling in figure 2) we have the following heat flow

\[ Q_H = T_H \Delta S_{1 \rightarrow 2} = T_H \left( \frac{3}{4\pi} \right)^2 \pi \left( V_{2}^3 - V_{1}^3 \right), \]

and along the lower

\[ Q_C = T_C \Delta S_{1 \rightarrow 4} = T_C \left( \frac{3}{4\pi} \right)^2 \pi \left( V_{4}^3 - V_{3}^3 \right). \]

\(^4\) This is a general result in thermodynamics following from the second law and the vanishing of the net entropy flow. For a recent alternative interesting (i.e. not directly appealing to the second law) proof, see [20].
Since $V_1 = V_4$ and $V_2 = V_3$ (we moved along isochores), the efficiency becomes

$$\eta = 1 - \frac{T_C}{T_H}. \quad (6)$$

Happily, for static black holes the equation of state can be made explicit too (as we will show in the example of the next section) and so the full shape of the Carnot cycle for these cases can be fully characterized. We can make Carnot engines for non-static black holes too, but now the adiabats will not be isochores, and the full equation of state must be used to determine the shapes of the paths. Using isochores will give the Stirling engine which will have a lower efficiency than Carnot since there'll be additional (non-reversible) heat flows.

Notice that the engines can also include non-trivial phase structure somewhere along the path we chose. If a phase transition between large and small black holes occurs as the pressure varies along the isotherm, as is well known to take place for such holes [21, 22] (see the example below), the Carnot result is robust since all it relies on are the volume differences. It does not matter whether those differences took place as a result of a discontinuous jump (as in a first order transition) or the milder change of a critical point (as in a second order transition).

### 3. An example: charged black holes in $\text{AdS}_4$

Just for clarity, it is worth exhibiting a concrete example that has all the elements we’ve discussed, so let us take static black holes in four dimensions with negative cosmological constant. The black hole is a Reissner–Nordström solution of the Einstein–Maxwell system with bulk action

$$I = -\frac{1}{16\pi} \int d^4 x \sqrt{-g} \left( R - 2\Lambda - F^2 \right), \quad (7)$$

where $\Lambda = -3/\ell^2$, the cosmological constant, sets a length scale $\ell$. The black hole has mass and charge $M$ and $q$, with metric

$$ds^2 = -Y(r)dt^2 + \frac{dr^2}{Y(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where $Y(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{r^2}{\ell^2}$. \quad (8)

and there is a gauge potential that is chosen to vanish on the horizon located at $r = r_+$, the largest positive real root of $Y(r)$: $A_t = q(r - r_+)^2/r_+$. The requirement of regularity of the Euclidean section fixes the temperature $T$ according to

$$\frac{1}{T} = 4\pi Y'|_{r=r_+} = \frac{4\pi \ell^2 r_+^3}{3r_+^4 + l^2 r_+^2 - q^2 \ell^2}. \quad (9)$$

and the entropy is $S = \pi r_+^2$. We can define the pressure $p = 3/(8\pi \ell^2)$ and re-arrange the temperature expression above into an equation of state [21, 23] for a given charge $q$

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5 We’ve chosen to work with a spacetime which is asymptotic to global AdS in this example. Our general remarks in this paper are not restricted to such situations, and choices for AdS with flat or hyperbolic slicings are also relevant.
where we substituted $r_c$ for the thermodynamic volume using $V = 4\pi r_c^3/3$. For our heat engine discussion, since we are interested in the mechanical work, we will fix at a specific value of the charge and so we will turn off the $\Phi dq$ term, leaving

$$dH = dM = TdS + Vdp.$$  \hfill (11)

The function $H(S, p)$ can be easily computed (for example by converting the potentials computed in the action computations of [21, 22], or by other methods—see e.g., [13])

$$H(S, p) = \frac{1}{2} \left( \frac{S}{\pi} \right)^{\frac{3}{2}} \left( 1 + \frac{\pi q^2}{S} + \frac{8Sp}{3} \right),$$  \hfill (12)

from which we can recover $V$ and $T$ by partial differentiation. It is easy to check that the consistency conditions for $dH$ to be exact (Maxwell’s relations) are satisfied

$$\left( \frac{\partial T}{\partial p} \right)_S = 2 \left( \frac{S}{\pi} \right)^{\frac{1}{3}} = \left( \frac{\partial V}{\partial S} \right)_p.$$  \hfill (13)

Figure 3 shows some sample (uncorrected) isotherms. (The structure in the $p-V$ plane is of course the same, but much more horizontally stretched.)

We use the qualifier ‘uncorrected’ above for the following reason: as discovered in [21] (and studied with variable pressure in [23]), this fixed charge ensemble has phase transitions due to the multi-valued nature of the equation of state that appears at low enough temperature. There is a first order phase transition between small and large black holes (as pressure is changed) reminiscent of the Van der Waals liquid–gas system. The resulting jumps between
large and small black holes corrects the naive isotherms given by equation (10), and removes the negative pressure regions that are evident in figure 3. The resulting line of first order transitions in the \((p, T)\) plane ends in a second order critical point. Many properties of these transitions have been worked out in [21–23], and they won’t be a focus here.

An important result that underlies the simple observation that the isochores are adiabats can be derived from first writing the temperature in equation (9) in terms of \(S\) and \(p\) as follows

\[
T = \frac{1}{4\sqrt{\pi}} \frac{1}{\sqrt{S}} \left( 1 - \frac{\pi q^2}{S} + 8pS \right). \tag{14}
\]

Then differentiation gives the specific heat

\[
C = T \frac{\partial S}{\partial T} = \left( 1 - \frac{2S^2 + \partial p}{\sqrt{\pi} \partial T} \right) 2S \left( \frac{8pS^2 + S - \pi q^2}{8pS^2 - S - 3\pi q^2} \right). \tag{15}
\]

which shows (since \(\partial p/\partial T\)_\(c\) = \(\pi^2/2S^2\)) that the specific heat at constant volume vanishes \(C_V = 0\), while \(C_p\) is given by setting \(\partial p/\partial T = 0\) in the expression above [10, 23]. The vanishing of \(C_V\) is the ‘isochore equals adiabat’ result, specific to static black holes, making our Carnot cycles particularly simple to make explicit. We can put a Carnot cycle on the diagram by picking two isotherms for \(T_H\) and \(T_C\), and then dropping two vertical lines between them to close the loop as we did in figure 2. The loop can include the jumps in volume as the pressure changes along an isotherm.

Actually, an explicit expression for \(C_p\) would suggest that we ought to have a new engine that we can analyze simply, involving two isobars and two isochores/adiabats. See figure 4. The work done along the isobars is very easy to compute

\[
W = \frac{4}{3\sqrt{\pi}} \left( \frac{3}{2}\left( p_2 - p_1 \right) \right). \tag{16}
\]

where the subscripts refer to the quantities evaluated at the corners labeled (1, 2, 3, 4) and we’ve written the volume in terms of the entropy to reduce the number of variables in the final expression for the efficiency. The heat flows take place along the top and bottom. The upper
isobar will give the net inflow of heat, which is therefore $Q_H$, so we may write

$$Q_H = \int_{T_i}^{T_f} C_p(p_i, T) dT,$$

(17)

where the non-trivial entropy dependence of $C_p$ gives a non-trivial $T$ dependence, which makes the integral messy. In any case, the efficiency is then $\eta = W/Q_H$, where the previous two quantities can be substituted. As a check on our methods we can take a limit where the cycle is at high pressure and temperature. Then our expressions simplify and allow us to perform the integral. We can focus on the large volume branch of solutions and therefore neglect $q$ to leading order, expanding at large $T$ and $p$ to get

$$S = \frac{\pi}{4}T^2 - \frac{1}{4p} - \frac{1}{16\pi T^2} + \cdots, \quad C_p = \frac{\pi}{2p^2}T^2 + \frac{1}{8\pi T^2} + \cdots,$$

(18)

which yields

$$Q_H = \frac{\pi}{6p_i^2}(T_f^3 - T_i^3) + \frac{1}{8\pi}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + \cdots$$

$$= \frac{4}{3\sqrt{\pi}}p_i\left(S_f^\frac{1}{2} - S_i^\frac{1}{2}\right) + \frac{1}{2\sqrt{\pi}}\left(S_f^\frac{1}{2} - S_i^\frac{1}{2}\right) + O\left(\frac{1}{p_i^2}\right).$$

(19)

So dividing the work in equation (16) by this, we have our expression for the efficiency, which we can write as

$$\eta = \left(1 - \frac{p_f}{p_i}\right)\left(1 - \frac{3}{8}p_i\left(\frac{S_f^\frac{1}{2} - S_i^\frac{1}{2}}{S_f^\frac{1}{2} - S_i^\frac{1}{2}}\right) + O\left(\frac{1}{p_i^2}\right)\right).$$

(20)

At leading order, since $p \sim T/V + \cdots$ in this limit, we can see that the efficiency becomes $\eta = 1 - \frac{2\epsilon}{T_i}\left(\frac{V_f}{V_i}\right)^\frac{1}{2}$, where we use $T_C = T_i$ and $T_H = T_f$ since those are the lowest and highest temperatures the engine will need to operate at in order to exchange heat with its environment. So we can approach the Carnot efficiency (6) at leading order only if we also make the cycle an extremely narrow rectangle (which in the limit would produce no work at all). This makes sense since the heat exchanges (along the isobars that change the volume to perform the work) are irreversible and comparable in magnitude to the work done, even at high pressure and temperature. Of course, the corrections to the high pressure and temperature limit shown in the expression take us even further away from the ideal.

### 4. Renormalization group engineering

While our remarks apply to both positive and negative cosmological constant, at least formally, we can imagineer the kind of engine proposed in section 2 quite naturally in the case of negative cosmological constant $\Lambda$, since we have the AdS/CFT holographic correspondence [24–27] to help us. The black hole in $D$-dimensional gravity is dual to a non-gravitational field theory of a fluid in $D - 1$ dimensions.

However, first we must solve a puzzle. In the extended thermodynamics we’ve been discussing, how are we to interpret the pressure and the volume in the dual field theory? Are they the pressure and volume of the fluid exchange heat with its environment or are they the pressure and volume of the fluid exchange heat with its environment. So we can approach the Carnot efficiency (6) at leading order only if we also make the cycle an extremely narrow rectangle (which in the limit would produce no work at all). This makes sense since the heat exchanges (along the isobars that change the volume to perform the work) are irreversible and comparable in magnitude to the work done, even at high pressure and temperature. Of course, the corrections to the high pressure and temperature limit shown in the expression take us even further away from the ideal.
The stress tensor’s properties are consistent with that of a conformally invariant fluid with density $\rho$ proportional to pressure (see [29] for a review). Both are set by the energy (mass $M$ of the black hole, plus the Casimir energy, if we’re in global AdS). So that fluid pressure is not the $p$ of the AdS thermodynamics that is set by the cosmological constant. They simply do not match.

As it stands, therefore, we have the standard black hole thermodynamics of the gravity theory, where $(M, T, S)$ map (after putting in the value of Newton’s constant $G$) to $(U, T, S)$ of the dual field theory. This is the translation that is used in standard holographic discussions. On the other hand, we have the extended black hole thermodynamics where $p$ and $V$ are dynamical, and then $M$ is the enthalpy $H = U + pV$ of the gravity theory instead. Should we use this instead for discussing holography? They agree only when $p$ is not a thermodynamic variable. However, they seem to contradict each other otherwise. Which system is correct? Is the conclusion that we should never have $p$ dynamical in holographic discussions? This is an issue that does not seem to have been addressed in the literature, and we now propose a resolution.

There is a way that we can extend holography to include dynamical $p$, by recognizing that both relations can be correct at the same time. The mass of the black hole $M$ remains as the energy $U$ in the dual field theory, but it is also the enthalpy $H = U + pV$ in the gravity theory. On the gravity side, $p$ is dynamical and plays the role of a pressure, while on the non-gravitational side, although it has meaning, it is not a thermodynamic variable. The same relationships will be true for the other thermodynamic potentials. The Euclidean path integral $I^{\beta}$/\(\beta \equiv -\log (Z_{\text{grav}})/\beta\), in the usual (fixed $\Lambda$) gravitational thermodynamics (here, $\beta = 1/T$) is to be identified with the Helmholtz free energy $F = U - TS$ of the dual field theory. When we allow the cosmological constant $(\rho)$ to vary, the natural quantity it equates to on the gravity side should be the Gibbs free energy $G = U - TS - pV$. See table 1 for a summary.

So on the field theory side, what is the meaning of $p$, given that it is not a thermodynamic variable? The answer remains what it is in the standard AdS/CFT dictionary. In the extended thermodynamics $p = -\Lambda/8\pi G$ (where here $G$ is Newton’s constant and not Gibbs of the previous paragraph) and $\Lambda = -(D - 2)(D - 1)/2\ell^2$, in $D$ dimensions. Recall that the value of the length scale $\ell$ is set by the Planck length of the underlying uncompactified theory (e.g., the 11-dimensional Planck length or 10-dimensional string length) and in the simplest examples, a pure number, $N$, related to the number of coincident branes (M-branes or D-
branes). Larger \( N \) means larger \( \ell' \), and as is well known the gauge/gravity correspondence becomes very useful for large \( N \), where the curvatures are small. On the field theory side, \( N \) is typically the rank of a gauge group of the theory, and as such it also determines the maximum number of available degrees of freedom. (For example, for \( U(N) \) it would be \( N^2 \).) Table 2 gives a summary of the \( N \) dependences in the simplest cases of \( \text{AdS}_D \) (\( D = 4, 5, 7 \)), where the \( N \) dependence of Newton’s constant \( G \) in those dimensions (obtained by dimensional reduction) is shown since it is needed to compute the pressure via: \( p = -\Lambda/8\pi G \). Overall, we see that the pressure \( p \), scales with \( N \). So dynamical \( p \) must mean dynamical \( N \).

From the perspective of the \( D \)-dimensional gravity theory under discussion, \( \Lambda \) (and hence the positive pressure \( p \)) is set by the value of the potential \( V(q) \) of the scalars \( q_i \) of the gravity theory. The pure \( \text{AdS} \) case is the highly symmetric fixed point of the gravity theory where \( q_i = 0 \), but there are other fixed points of the potential corresponding to other \( \text{AdS} \) spaces. They have different values of the potential, and hence different values of \( \Lambda \)—i.e., different values for the effective \( N \) measuring the available degrees of freedom. This is the core idea in the holographic renormalization group [31, 32] (see [29] for a review), and there are many explicit examples, although they are hard to construct (even at zero temperature) in general since the scalar dynamics are highly nonlinear. So turning on relevant operators in the field theory (dual to the scalars \( q_i \)), driving the theory to new IR fixed points, is one way to dynamically change \( \Lambda \).

However, in order to explore the full space of available values of negative cosmological constant (and hence of positive \( p \)) will require more than just turning on relevant operators to trigger flows to IR fixed points. We expect that irrelevant operators will be needed as well (dual to higher mass Kaluza–Klein fields on the gravity side), allowing us to explore the extremes of the Coulomb branch (and the finite temperature deformation thereof) corresponding to moving some number of branes off the collection of \( N \) coincident branes whose throat is the \( \text{AdS}_D \) and moving them off to infinity, or vice versa, thereby changing \( N \). (This can also be discussed in terms of changing the number of units of flux on the compact directions in the dimensionally reduced picture.) Such operators have been discussed in the literature (mostly at zero temperature) where they correspond to motions on the Coulomb branch that go beyond what is accessible by holographic RG flow with relevant operators.

The point for us is that the required operators in field theory correspond to dynamical fields in the full gravity theory and so exploring them is a full dynamical problem. In other words, the value of \( p \) changes dynamically as a result of the dynamics of these supergravity fields. On the other hand it is the asymptotic values of the fields on the \( \text{AdS} \) boundary that have precise meaning in the field theory, where they are masses and expectation values of field theory operators. The field theory does not know about the full dynamics of the fields in the bulk, although it knows about the value of \( p \) through the effective \( N \) that sets the number of degrees of freedom. It is in this precise sense that \( p \), while it has meaning in both theories (connected to pressure in one and number of degrees of freedom in the other), is naturally dynamical in the gravity theory, while on the field theory side changing it is rather more akin

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6 Note that although they did not pursue the connection, [9, 30] also mention the link between \( p \) and \( N \), and wondered as to its significance. We thank B Dolan and D Kastor for pointing this out after the first version of this manuscript appeared.
7 See e.g., [33–37] for some early discussion at zero temperature [38, 39] for later clarification of the connection between gravity and field theory for this sector, and [40] for some exploration of the finite temperature physics of the kind of irrelevant operator involved. The early work on Coulomb branch solutions accessible by holographic RG flow triggered by low dimension operators are [41, 42]. The solutions correspond to smeared distributions of branes in higher dimensions [33]. Our application needs access to fully multi-centered solutions which can be widely separated.
to motion on the space of field theories, allowing \( N \) to change. So \( p \) is not dynamical in a particular field theory. This is how the black hole mass \( M \) can be energy \( U \) in the field theory, and the enthalpy \( U + pV \) in the gravity at the same time.

So the stage is set for how to realize our heat engines. Using the flow on the space of field theories just described we can perform thermodynamic cycles of the type described in section 2, exploring different values of \( p \), by turning on appropriate choices of operators in the dual field theory. Our heat engines are truly holographic in that we can describe their operation using a dual holographic description in the field theory.

This leads us to the matter of the mechanical work done over the cycle. What is the meaning of this work? Normally when we conceive of an engine and the mechanical work it does we have in mind coupling (via say, a piston) the volume to some external environment on which we are doing work. This means we must try to understand what the meaning of \( V \) is in the field theory. It is not the field theory volume since it has dependence on parameters other than the AdS scale \( \ell \). (In global AdS\(_D\), for example, the finite volume the dual field theory is on is an \( S^{D-2} \) of radius \( \ell \).) We should look for something analogous to what we saw above with the pressure, which is a meaningful quantity in the field theory (without actually being a pressure) and was set by some power of \( N \), the number of degrees of freedom. Our \( V \) should be a sort of conjugate to that. So this implies that it would be a chemical potential, it seems, for (some positive power of) the number of degrees of freedom (although since in thermodynamics \( p \) is the intensive variable and \( V \) the extensive, it may well be that \( p \) is more akin to a chemical potential). It would be interesting to identify such a quantity in field theory. If this is possible, one’s expectation is that it might be a geometrically defined quantity. Important geometrically defined quantities that are related to measures of degrees of freedom in a theory are not unfamiliar in this field. The entanglement entropy is an example [43, 44].

Whatever non-volume quantity it is that \( V \) turns out to compute in a field theory, it will get changed when mechanical work is done by on it one of our heat engines. The picture would be that field theory \( A \) is used as a holographic heat engine that can operate on field theory \( B \) by coupling them together appropriately. After a cycle, performed by renormalization group flows in theory \( A \) as described above, the \( V \) (remember, \textit{not} volume) of theory \( B \) has changed. Mechanical work was performed using heat.

It is probably wise to stop speculating at this point, but it does seem possible that these holographic heat engines may serve as new tools for the study of gauge theories in some enlarged framework yet to be understood.

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References

[1] Bekenstein J D 1973 Black holes and entropy Phys. Rev. D 7 2333–46
[2] Bekenstein J D 1974 Generalized second law of thermodynamics in black hole physics Phys. Rev. D 9 3292–300
[3] Hawking S 1975 Particle creation by black holes Commun. Math. Phys. 43 199–220
[4] Hawking S 1976 Black holes and thermodynamics Phys. Rev. D 13 191–7
[5] Caldarelli M M, Cognola G and Klemm D 2000 Thermodynamics of Kerr–Newman–AdS black holes and conformal field theories Class. Quantum Grav. 17 399–420

[6] Wang S, Wu S-Q, Xie F and Dan L 2006 The first laws of thermodynamics of the (2+1)-dimensional BTZ black holes and Kerr–de Sitter spacetimes Chin. Phys. Lett. 23 1096–8

[7] Sekiwa Y 2006 Thermodynamics of de Sitter black holes: thermal cosmological constant Phys. Rev. D 73 084009

[8] Larranaga A 2007 Stringy generalization of the first law of thermodynamics for rotating BTZ black hole with a cosmological constant as state parameter arXiv:0711.0012 [gr-qc]

[9] Kastor D, Ray S and Traschen J 2009 Enthalpy and the mechanics of AdS black holes Class. Quantum Grav. 26 195011

[10] Dolan B P 2011 The cosmological constant and the black hole equation of state Class. Quantum Grav. 28 125020

[11] Cvetic M, Gibbons G, Kubiznak D and Pope C 2011 Black hole enthalpy and an entropy inequality for the thermodynamic volume Phys. Rev. D 84 024037

[12] Dolan B P 2011 Compressibility of rotating black holes Phys. Rev. D 84 127503

[13] Dolan B P 2011 Pressure and volume in the first law of black hole thermodynamics Class. Quantum Grav. 28 235017

[14] Dolan B P 2012 Where is the PdV term in the first law of black hole thermodynamics? arXiv:1209.1272 [gr-qc]

[15] Altamirano N, Kubiznak D, Mann R B and Sherkatghanad Z 2014 Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume Galaxies 2 89–159

[16] Henneaux M and Teitelboim C 1984 The cosmological constant as a canonical variable Phys. Lett. B 143 415–20

[17] Teitelboim C 1985 The cosmological constant as a thermodynamic black hole parameter Phys. Lett. B 158 293–7

[18] Henneaux M and Teitelboim C 1989 The cosmological constant and general covariance Phys. Lett. B 222 195–9

[19] Parikh M K 2006 The volume of black holes Phys. Rev. D 73 124021

[20] Tjiang P C and Sutanto S H 2006 The efficiency of the Carnot cycle with arbitrary gas equations of state Eur. J. Phys. 27 719–26

[21] Chamblin A, Emparan R, Johnson C V and Myers R C 1999 Charged AdS black holes and catastrophic holography Phys. Rev. D 60 064018

[22] Chamblin A, Emparan R, Johnson C V and Myers R C 1999 Holography, thermodynamics and fluctuations of charged AdS black holes Phys. Rev. D 60 104026

[23] Kubiznak D and Mann R B 2012 P-V criticality of charged AdS black holes J. High Energy Phys. JHEP07(2012)033

[24] Maldacena J M 1998 The large n limit of superconformal field theories and supergravity Adv. Theor. Math. Phys. 2 231–52

[25] Gubser S S, Klebanov I R and Polyakov A M 1998 Gauge theory correlators from non-critical string theory Phys. Lett. B 428 105–14

[26] Witten E 1998 Anti-de Sitter space and holography Adv. Theor. Math. Phys. 2 253–91

[27] Witten E 1998 Anti-de Sitter space, thermal phase transition, and confinement in gauge theories Adv. Theor. Math. Phys. 2 505–32

[28] Balasubramanian V and Kraus P 1999 A stress tensor for Anti-de Sitter gravity Commun. Math. Phys. 208 413–28

[29] Johnson C V 2003 D-branes (Cambridge: Cambridge University Press)

[30] Dolan B P 2014 The compressibility of rotating black holes in D-dimensions Class. Quantum Grav. 31 035022

[31] Girardello L, Petrimi M, Porrati M and Zaffaroni A 1998 Novel local CFT and exact results on perturbations of N = 4 superYang Mills from AdS dynamics J. High Energy Phys. JHEP12 (1998)022

[32] Distler J and Zamora F 1999 Nonsupersymmetric conformal field theories from stable Anti-de Sitter spaces Adv. Theor. Math. Phys. 2 1405–39

[33] Kraus P, Larsen F and Trivedi S P 1999 The Coulomb branch of gauge theory from rotating branes J. High Energy Phys. JHEP03(1999)003

[34] Klebanov I R and Witten E 1999 AdS/CFT correspondence and symmetry breaking Nucl. Phys. B 556 89–114
[35] Intriligator K A 2000 Maximally supersymmetric RG flows and AdS duality *Nucl. Phys.* B 580 99–120

[36] Costa M S 2000 Absorption by double centered D3-branes and the Coulomb branch of N = 4 SYM theory *J. High Energy Phys.* JHEP05(2000)041

[37] Costa M S 2000 A test of the AdS/CFT duality on the Coulomb branch *Phys. Lett.* B 482 287–92

[38] Skenderis K and Taylor M 2006 Kaluza–Klein holography *J. High Energy Phys.* JHEP05 (2006)057

[39] Skenderis K and Taylor M 2006 Holographic Coulomb branch vevs *J. High Energy Phys.* JHEP08 (2006)001

[40] Evans N J, Johnson C V and Petrini M 2002 Clearing the throat: irrelevant operators and finite temperature in large N gauge theory *J. High Energy Phys.* JHEP05(2002)002

[41] Freedman D, Gubser S, Pilch K and Warner N 2000 Continuous distributions of D3-branes and gauged supergravity *J. High Energy Phys.* JHEP07(2000)038

[42] Brandhuber A and Sfetsos K 1999 Wilson loops from multicenter and rotating branes, mass gaps and phase structure in gauge theories *Adv. Theor. Math. Phys.* 3 851–87

[43] Bombelli L, Koul R K, Lee J and Sorkin R D 1986 A quantum source of entropy for black holes *Phys. Rev.* D 34 373–83

[44] Srednicki M 1993 Entropy and area *Phys. Rev. Lett.* 71 666–9