Curvature Estimates for Four-Dimensional Gradient Steady Ricci Solitons

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Received: 18 August 2018 / Published online: 28 January 2019
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Abstract
In this paper, we derive certain curvature estimates for 4-dimensional gradient steady Ricci solitons either with positive Ricci curvature or with scalar curvature decay \( \lim_{x \to \infty} R(x) = 0 \).

Keywords Curvature estimate · Steady Ricci solitons

Mathematics Subject Classification 53C21 · 53C25 · 53C44

1 The Results

A complete Riemannian metric \( g_{ij} \) on a smooth manifold \( M^n \) is called a gradient steady Ricci soliton if there exists a smooth function \( f \) on \( M^n \) such that the Ricci tensor \( R_{ij} \) of the metric \( g_{ij} \) satisfies the equation

\[
R_{ij} + \nabla_i \nabla_j f = 0.
\] (1.1)

The function \( f \) is called a potential function of the gradient steady soliton. Clearly, when \( f \) is a constant the gradient steady Ricci soliton \( (M^n, g_{ij}, f) \) is simply a Ricci-flat manifold. Gradient steady solitons play an important role in Hamilton’s Ricci flow as they correspond to translating solutions, and often arise as Type II singularity models. Thus one is interested in either classifying them or understanding their geometry.
As is well known, compact steady solitons must be Ricci-flat. In dimension \( n = 2 \), Hamilton [21] discovered the first example of a complete non-compact gradient steady soliton on \( \mathbb{R}^2 \), called the *cigar soliton*, where the metric is given by
\[
\text{d}s^2 = \frac{\text{d}x^2 + \text{d}y^2}{1 + x^2 + y^2}.
\]
The cigar soliton has potential function \( f = -\log(1 + x^2 + y^2) \), positive (scalar) curvature \( R = 4e^f \), and is asymptotic to a round cylinder at infinity. Furthermore, Hamilton [21] showed that the only complete steady soliton on a two-dimensional manifold with bounded (scalar) curvature \( R \) which assumes its maximum at an origin is, up to scaling, the cigar soliton. For \( n \geq 3 \), Bryant [3] proved that there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on \( \mathbb{R}^n \) (see, e.g., Chow et al. [14] for more detail). The Bryant soliton has positive sectional curvature, linear curvature decay, and volume growth of geodesic balls \( B(0, r) \) on the order of \( r(n+1)/2 \).

In the Kähler case, the first author [5] constructed a complete \( U(m) \)-invariant gradient steady Kähler-Ricci soliton on \( \mathbb{C}^m \), for \( m \geq 2 \), with positive sectional curvature; its geodesic ball of radius \( r \) has volume growth on the order of \( r^m \) and its curvature has linear decay at infinity. Note that in each of these three examples, the maximum of the scalar curvature is attained at the origin. One can find additional examples of steady solitons, e.g., in [4, 16, 17, 19, 23] etc; see also [6] and the references therein.

In dimension \( n = 3 \), Perelman [27] conjectured that the Bryant soliton is the only complete non-compact, \( \kappa \)-non-collapsed, gradient steady soliton with positive sectional curvature. Recently, Brendle has affirmed this conjecture of Perelman (see [1]; and also [2] for an extension to the higher-dimensional case). On the other hand, for \( n \geq 3 \), Cao–Chen [8] proved that any \( n \)-dimensional complete non-compact locally conformally flat gradient steady Ricci soliton \((M^n, g_{ij}, f)\) is either flat or isometric to the Bryant soliton. We remark that Catino–Mantegazza [11] also gave an independent proof of the same results when \( n \geq 4 \). In addition, Bach-flat gradient steady solitons with positive Ricci curvature for all \( n \geq 3 \) and half-conformally flat ones for \( n = 4 \) have been classified in [7] and [13], respectively. However, it remains a challenge to understand the geometry of complete steady solitons, such as their asymptotic curvature behavior, for dimensions \( n \geq 4 \) in general. Inspired by the very recent work of Munteanu–Wang [25], in this paper we study curvature estimates of 4-dimensional complete non-compact gradient steady solitons.

In [25], Munteanu and Wang made an important observation that the curvature tensor of a four-dimensional gradient Ricci soliton \((M^4, g_{ij}, f)\) can be estimated in terms of the potential function \( f \), the Ricci tensor, and its first derivatives. In addition, the (optimal) asymptotic quadratic growth property of the potential function \( f \) proved in [9], and a key scalar curvature lower bound \( R \geq c/f \) shown in [15] are crucial in their work. While gradient steady Ricci solitons in general do not share these two special features (cf. [24, 30, 31] and [15, 20]), some of the arguments in [25] can still be adapted to prove certain curvature estimates for two classes of gradient steady solitons. Our main results are:

**Theorem 1.1** Let \((M^4, g_{ij}, f)\) be a complete non-compact 4-dimensional gradient steady Ricci soliton with positive Ricci curvature \( \text{Ric} > 0 \) such that the scalar cur-
Curvature $R$ attains its maximum at some point $x_0 \in M^4$. Then, $(M^4, g_{ij})$ has bounded Riemann curvature tensor, i.e.,

$$\sup_{x \in M} |Rm| \leq C$$

for some constant $C > 0$. Suppose in addition $R$ has at most linear decay, then

$$\sup_{x \in M} \frac{|Rm|}{R} \leq C.$$ 

**Theorem 1.2** Let $(M^4, g_{ij}, f)$, which is not Ricci-flat, be a complete non-compact 4-dimensional gradient steady Ricci soliton. If $\lim_{x \to \infty} R(x) = 0$, then, for each $0 < a < 1$, there exists a constant $C > 0$ such that

$$|Ric|^2 \leq CR^a$$

and

$$\sup_{x \in M} |Rm| \leq C.$$ 

Suppose in addition $R$ has at most polynomial decay. Then, for each $0 < a < 1$, there exists a constant $C > 0$ such that

$$|Rm|^2 \leq CR^a.$$ 

**Remark 1.1** We have learned that the first part of Theorem 1.2 was also known to O. Munteanu and J. Wang.

2 Preliminaries

In this section, we recall some basic facts and collect several known results about gradient steady solitons. Throughout the paper, we denote by

$$Rm = \{R_{ijkl}\}, \quad Ric = \{R_{ik}\}, \quad R$$

the Riemann curvature tensor, the Ricci tensor, and the scalar curvature of the metric $g_{ij}$, respectively.

**Lemma 2.1** (Hamilton [22]) Let $(M^n, g_{ij}, f)$ be a complete gradient steady soliton satisfying Eq. (1.1). Then

$$R = -\Delta f, \quad (2.1)$$

$$\nabla_i R = 2R_{ij} \nabla_j f, \quad (2.2)$$

$$R + |\nabla f|^2 = C_0 \quad (2.3)$$

for some constant $C_0$.

Next we need the following useful result by Chen [12].
Proposition 2.1  Let \( g_{ij}(t) \) be a complete ancient solution to the Ricci flow on a non-compact manifold \( M^n \). Then the scalar curvature \( R \) of \( g_{ij}(t) \) is non-negative for all \( t \).

As an immediate corollary, we have

Lemma 2.2  Let \((M^n, g_{ij}, f)\) be a complete gradient steady soliton. Then it has non-negative scalar curvature \( R \geq 0 \).

Remark 2.1  In fact, by Proposition 3.2 in [28], either \( R > 0 \) or \((M^n, g_{ij})\) is Ricci-flat.

It follows from Lemma 2.2 that the constant \( C_0 \) in (2.3) is positive whenever \( f \) is a non-constant function (i.e., the steady soliton is non-trivial). By a suitable scaling of the metric \( g_{ij} \), we can normalize \( C_0 = 1 \) so that

\[
R + |\nabla f|^2 = 1. \tag{2.4}
\]

In the rest of the paper, we shall always assume this normalization (2.4).

Combining (2.1) and (2.4), we obtain \(-\Delta f + |\nabla f|^2 = 1\). Thus, setting \( F = -f \), we have

\[
\Delta f F = 1, \tag{2.5}
\]

where

\[
\Delta f =: \Delta - \nabla f \cdot \nabla. \tag{2.6}
\]

For gradient steady solitons with positive Ricci curvature \( Ric > 0 \), we have the following useful fact by Cao–Chen [8] on asymptotic behavior of the potential function.

Proposition 2.2  Let \((M^n, g_{ij}, f)\) be a complete non-compact gradient steady soliton with positive Ricci curvature \( Ric > 0 \) such that the scalar curvature \( R \) attains its maximum \( R_{\text{max}} = 1 \) at some origin \( x_0 \in M^n \). Then, there exist some constants \( 0 < c_1 \leq 1 \) and \( c_2 > 0 \) such that \( F = -f \) satisfies the estimates

\[
c_1 r(x) - c_2 \leq F(x) \leq r(x) + |F(x_0)|, \tag{2.7}
\]

where \( r(x) = d(x_0, x) \) is the distance function from \( x_0 \).

Remark 2.2  In (2.7), only the lower bound on \( F \) requires the assumptions on \( Ric \) and \( R \).

Note that, under the assumption in Proposition 2.2, \( F(x) \) is proportional to the distance function \( r(x) = d(x_0, x) \) from above and below. Throughout the paper, we denote

\[
D(t) = \{ x \in M : F(x) \leq t \},
\]

\[
B(t) = B(x_0, t) = \{ x \in M : d(x_0, x) \leq t \}.
\]

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We also collect several differential identities on $R$, $Ric$, and $Rm$ which are special cases of the curvature evolution equations under the Ricci flow.

**Lemma 2.3**  Let $(M^n, g_{ij}, f)$ be a complete gradient steady soliton satisfying Eq. (1.1). Then, we have
\[
\Delta f R = -2|Ric|^2, \quad \Delta f Ric = -2R_{ijkl}R_{jl}, \quad \Delta f Rm = Rm \ast Rm,
\]
where the RHS of the last equation denotes (a finite number of) terms involving quadratics in $Rm$.

Based on Lemma 2.3, one can easily derive the following inequalities similar to [25]:

**Lemma 2.4**  Let $(M^n, g_{ij}, f)$ be a complete gradient steady soliton satisfying Eq. (1.1). Then
\[
\Delta f |Ric|^2 \geq 2|\nabla Ric|^2 - 4|Rm||Ric|^2, \\
\Delta f |Rm| \geq -c|Rm|^2, \\
\Delta f |Rm|^2 \geq 2|\nabla Rm|^2 - C|Rm|^3.
\]
Here $c > 0$ is some universal constant depending only on the dimension $n$.

**Remark 2.3**  To derive the second differential inequality, one needs to use the Kato inequality $|\nabla| Rm | \leq |\nabla Rm|$ as in [25].

## 3 4D Gradient Steady Solitons with Positive Ricci Curvature

First of all, we need the following key fact, valid for 4-dimensional gradient steady Ricci solitons in general, essentially due to Munteanu and Wang [25].

**Lemma 3.1**  (Munteanu-Wang [25]) Let $(M^4, g_{ij}, f)$ be a complete non-compact gradient steady soliton satisfying (1.1). Then, there exists some universal constant $c > 0$ such that
\[
|Rm| \leq c \left( \frac{|\nabla Ric|}{|\nabla f|} + \frac{|Ric|^2}{|\nabla f|^2} + |Ric| \right).
\]

**Proof**  This follows from the same arguments as in the proof of Proposition 1.1 of [25], but using Lemma 2.4 instead and without replacing $|\nabla f|^2$ by $f$ in their argument. □

**Proposition 3.1**  Let $(M^4, g_{ij}, f)$ be a complete non-compact gradient steady soliton with positive Ricci curvature and $R$ attains its maximum. Then, there exists some constant $C > 0$, depending on the constant $c_1$ in (2.7), such that outside a compact set,
\[
|Rm| \leq C \left( |\nabla Ric| + |Ric|^2 + |Ric| \right).
\]
Proof This easily follows from Lemma 3.1 and the following fact shown by Cao–Chen [8],

$$|\nabla f|^2 \geq c_1 > 0.$$  \hspace{1cm} (3.1)

Remark 3.1 Note that, combining (3.1) with (2.4) and Lemma 2.2, we have

$$0 < c_1 \leq |\nabla F|^2 = |\nabla f|^2 \leq 1.$$  \hspace{1cm} (3.2)

Now we are ready to prove the first main result of our paper.

Theorem 3.1 Let \((M^4, g_{ij}, f)\) be a complete non-compact gradient steady soliton with positive Ricci curvature \(\text{Ric} > 0\) such that \(R\) attains its maximum at some point \(x_0 \in M^4\). Then, there exists some constant \(C > 0\), depending on \(c_1\) in (2.7), such that

$$\sup_{x \in M} |Rm| \leq C.$$ 

Proof First of all, from (2.4), we have

$$0 < |\text{Ric}| \leq R \leq 1.$$  \hspace{1cm} (3.3)

Thus, by Proposition 3.1 and (3.3), we see that

$$|\nabla \text{Ric}|^2 \geq \frac{1}{2C^2} |Rm|^2 - \left( |\text{Ric}|^2 + |\text{Ric}| \right)^2 \geq \frac{1}{2C^2} |Rm|^2 - 4.$$  \hspace{1cm} (3.4)

Using the first two inequalities in Lemma 2.4, we obtain

$$\Delta_f \left( |Rm| + \lambda |\text{Ric}|^2 \right) \geq -C |Rm|^2 + 2\lambda \left( |\nabla \text{Ric}|^2 - 2|Rm||\text{Ric}|^2 \right).$$ \hspace{1cm} (3.5)

By (3.3), (3.4), and picking constant \(\lambda > 0\) sufficiently large (depending on the constant \(C\) in Proposition 3.1, hence on \(c_1\)), it follows that

$$\Delta_f \left( |Rm| + \lambda |\text{Ric}|^2 \right) \geq 2|Rm|^2 - 4\lambda |Rm| - C' \geq \left( |Rm| + \lambda |\text{Ric}|^2 \right)^2 - C.$$ \hspace{1cm} (3.6)

Next, let \(\varphi(t)\) be a smooth function on \(\mathbb{R}^+\) so that \(0 \leq \varphi(t) \leq 1\), \(\varphi(t) = 1\) for \(0 \leq t \leq R_0\), \(\varphi(t) = 0\) for \(t \geq 2R_0\), and

$$t^2 \left( |\varphi'(t)|^2 + |\varphi''(t)| \right) \leq c$$  \hspace{1cm} (3.7)

for some universal constant \(c\) and \(R_0 > 0\) arbitrary large. We now take \(\varphi = \varphi(F(x))\) as a cut-off function with support in \(D(2R_0)\). Note that

$$|\nabla \varphi| = |\varphi'||\nabla F| \leq \frac{c}{R_0} \text{ and } |\Delta_f \varphi| \leq |\varphi'||\Delta_f F| + |\varphi''||\nabla F|^2 \leq \frac{c}{R_0}$$ \hspace{1cm} (3.8)
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Setting $u = |Rm| + \lambda |Ric|^2$ and $G = \varphi^2 u$, then direct computations, (3.6) and (3.8) yield
\[
\varphi^2 \Delta_f G = \varphi^4 \Delta_f u + \varphi^2 u \Delta_f (\varphi^2) + 2\varphi^2 \nabla u \cdot \nabla \varphi^2 \\
\geq \varphi^4 \left( u^2 - C \right) + \varphi^2 u \left( 2\varphi \Delta_f \varphi + 2|\nabla \varphi|^2 \right) + 2\nabla G \cdot \nabla \varphi^2 - 8|\nabla \varphi|^2 G \\
\geq G^2 + 2\nabla G \cdot \nabla \varphi^2 - CG - C.
\]

Now it follows from the maximum principle that $G \leq C$ on $D(2R_0)$ by some constant $C > 0$ depending on $c_1$ but independent of $R_0$. Hence $u = |Rm| + \lambda |Ric|^2 \leq C$ on $D(R_0)$. Since $R_0 > 0$ is arbitrary large, we see that
\[
\sup_{x \in M} |Rm| \leq \sup_{x \in M} \left( |Rm| + \lambda |Ric|^2 \right) \leq C.
\]

This completes the proof of Theorem 3.1.

**Proposition 3.2** Let $(M^4, g_{ij}, f)$ be a complete non-compact gradient steady soliton with positive Ricci curvature $Ric > 0$ and $R$ attains its maximum at $x_0 \in M^4$. Then, function $u = \frac{|Rm| + \lambda |Ric|^2}{R}$, with $\lambda > 0$ sufficiently large, satisfies the differential inequality
\[
\Delta_f u \geq u^2 R - CR - 2\nabla u \cdot \nabla \log R
\]
for some constant $C > 0$.

**Proof** First of all, similar to deriving (3.4)–(3.6) in the proof of Theorem 3.1, by choosing $\lambda$ sufficiently large we have
\[
\Delta_f \left( |Rm| + \lambda |Ric|^2 \right) \geq \left( |Rm| + \lambda |Ric|^2 \right)^2 - 4\lambda^2 |Ric|^4 - \lambda \left( |Ric|^4 + |Ric|^2 \right) \\
\geq \left( |Rm| + \lambda |Ric|^2 \right)^2 - C|Ric|^2
\]
for some constant $C > 0$. Here we have also used the fact (3.3).

Thus, by a direct computation,
\[
\Delta_f u = R^{-1} \Delta_f \left( |Rm| + \lambda |Ric|^2 \right) + (uR) \Delta_f \left( R^{-1} \right) + 2\nabla (uR) \cdot \nabla \left( R^{-1} \right) \\
\geq \frac{\left( |Rm| + \lambda |Ric|^2 \right)^2}{R} - C|Ric|^2 + (uR) \left[ 2 \frac{|Ric|^2}{R^2} + 2 \frac{|\nabla R|^2}{R^3} \right] \\
- \frac{2}{R^2} \left( u|\nabla R|^2 + R\nabla u \cdot \nabla R \right) \\
\geq Ru^2 - CR - 2\nabla u \cdot \nabla \log R.
\]
Theorem 3.2 Let \((M^4, g_{ij}, f)\) be a complete non-compact gradient steady Ricci soliton with \(\text{Ric} > 0\) such that \(R\) attains its maximum. Suppose \(R\) has at most linear decay, \(R(x) \geq c/r(x)\), outside a compact set. Then
\[
\sup_{x \in M} \frac{|Rm|}{R} \leq C.
\]

Proof Fix \(\lambda\) sufficiently large so that Proposition 3.2 holds and set \(u = \frac{|Rm| + \lambda |\text{Ric}|^2}{R}\). Next, let \(\varphi(t)\) be a smooth function on \(\mathbb{R}^+\) so that \(\varphi(t) = \frac{d-t}{d}\) for \(0 \leq t \leq d\), \(\varphi(t) = 0\) for \(t \geq d\). Let \(\varphi = \varphi(F)\) and \(G = \varphi^2 u\). Then,
\[
|\nabla \varphi| = |\varphi' \nabla F| \leq \frac{1}{d},
\]
\[
\Delta_f \varphi = \varphi' \Delta_f F = -\frac{1}{d}.
\]
Then, outside \(D(1)\), we have
\[
\varphi^2 \Delta_f (G) = \varphi^4 (\Delta_f u) + \varphi^2 u \left( \Delta_f \varphi^2 \right) + 2 \varphi^2 (\nabla \varphi^2, \nabla u)
\geq \varphi^4 \left( R u^2 - cR - 2(\nabla u, \nabla (\log R)) \right)
+ 2 \left( \varphi \Delta_f \varphi + |\nabla \varphi|^2 \right) G + 2 \varphi^2 (\nabla \varphi^2, \nabla u)
\geq R G^2 - cR + 4 \varphi (\nabla \varphi, \nabla (\log R)) G
- \frac{8}{d} G + \left( 2 \nabla \varphi^2 - 2 \varphi^2 \nabla (\log R), \nabla G \right).
\]

Now by (2.2), (3.2), and \(\text{Ric} > 0\), we have \(|\nabla \log R| = 2 |\frac{\text{Ric}(\nabla f)}{R}| \leq 2\). Also, when \(R\) has at most linear decay outside some \(D(t_0)\) and for \(d > t_0\), we have \(R \geq \frac{a}{d}\) in \(D(d) \setminus D(t_0)\) for some constant \(a > 0\) independent of \(d\). Therefore, there exists \(c\) independent of \(d\) so that,
\[
\varphi^2 \Delta_f (G) \geq \frac{R G^2}{d} - cR - \frac{c}{d} G + \left( 2 \nabla \varphi^2 - 2 \varphi^2 \nabla (\log R), \nabla G \right)
\geq \frac{1}{2} \frac{R G^2}{d} - cR + \left( 2 \nabla \varphi^2 - 2 \varphi^2 \nabla (\log R), \nabla G \right).
\]

Therefore, it follows from standard maximum principle argument that \(u \leq C\) on \(M^4\), hence \(|Rm| \leq CR\) on \(M^4\). \(\square\)

4 The Proof of Theorem 1.2

In this section, we prove our second main result, Theorem 1.2 in the introduction. Throughout this section we assume \((M^4, g_{ij}, f)\) is a complete, non-compact, non-Ricci-flat, 4-dimensional gradient steady Ricci soliton such that
\[ \lim_{x \to \infty} R(x) = 0. \tag{4.1} \]

Note that, by Lemma 2.2 and Remark 2.1, \((M^4, g_{ij}, f)\) necessary has strictly positive scalar curvature \(R > 0\).

First of all, we need the following useful Laplacian comparison type result for gradient Ricci solitons.

**Lemma 4.1** Let \((M^n, g_{ij}, f)\) be any gradient steady Ricci soliton satisfying (1.1) and let \(r(x) = d(x_0, x)\) denote the distance function on \(M^n\) from a fixed base point \(x_0\). Suppose that

\[ \text{Ric} \leq (n - 1)K \]

on the geodesic ball \(B(x_0, r_0)\) for some constants \(r_0 > 0\) and \(K > 0\). Then, for any \(x \in M^n \setminus B(x_0, r_0)\), we have

\[ \Delta f (x) \leq (n - 1) \left( \frac{2}{3} Kr_0 + r_0^{-1} \right). \]

**Remark 4.1** Lemma 4.1 is a special case of a more general result valid for solutions to the Ricci flow due to Perelman [26], see, e.g., Lemma 3.4.1 in [10]. Also see [18] and [29] for a different version.

**Theorem 4.1** Let \((M^4, g_{ij}, f)\), which is not Ricci-flat, be a complete non-compact gradient steady Ricci soliton. If \(\lim_{x \to \infty} R(x) = 0\), then, for each \(0 < a < 1\), there exists a constant \(C > 0\) such that

\[ \sup_{x \in M} |\text{Ric}|^2 \leq CR^a \quad \text{and} \quad \sup_{x \in M} |\text{Rm}| \leq C. \]

**Proof** The proof is similar to that of Munteanu–Wang [25] except we need to use the distance function to cut-off rather than the potential function since the potential function may not be proper in our case.

Since \(\lim_{x \to \infty} R(x) = 0\), it follows from (2.4) that

\[ |\nabla f| \geq c_1 > 0 \]

for some \(0 < c_1 < 1\) outside a compact set. By Lemma 2.4 and Lemma 3.1, we have

\[ \Delta f |\text{Ric}|^2 \geq 2 |\nabla \text{Ric}|^2 - C |\text{Rm}||\text{Ric}|^2 \]

\[ \geq 2 |\nabla \text{Ric}|^2 - C \left( |\nabla \text{Ric}| + |\text{Ric}|^2 + |\text{Ric}| \right) |\text{Ric}|^2. \]

Also, since \(R > 0\) on \(M^4\), by using the first identity in Lemma 2.3 we have

\[ \Delta f \left( \frac{1}{R^a} \right) = 2a \frac{|\text{Ric}|^2}{R^{a+1}} + a(a + 1) \frac{|\nabla R|^2}{R^{a+2}}. \]
Hence,

\[
\Delta f \left( \frac{|\text{Ric}|^2}{R^a} \right) = \frac{\Delta_f |\text{Ric}|^2}{R^a} + |\text{Ric}|^2 \Delta_f \left( \frac{1}{R^a} \right) + 2\nabla |\text{Ric}|^2 \cdot \nabla \left( \frac{1}{R^a} \right) \\
\geq \frac{2|\nabla \text{Ric}|^2}{R^a} - C \left( |\nabla \text{Ric}| + |\text{Ric}|^2 + |\text{Ric}| \right) \frac{|\text{Ric}|^2}{R^a} \\
+ |\text{Ric}|^2 \left[ 2a \frac{|\text{Ric}|^2}{R^{a+1}} + a(a + 1) \frac{|\nabla R|^2}{R^{a+2}} \right] \\
- 4a \frac{|\text{Ric}| |\nabla |\text{Ric}|| |\nabla R|}{R^{a+1}}.
\]

Apply the Cauchy–Schwarz inequality to the last term, we have

\[
-4a \frac{|\text{Ric}| |\nabla |\text{Ric}|| |\nabla R|}{R^{a+1}} \geq -4a \frac{|\text{Ric}| |\nabla |\text{Ric}|| |\nabla R|}{R^{a+1}} \\
\geq -a(a + 1) \frac{|\text{Ric}|^2 |\nabla R|^2}{R^{a+2}} - 4a \frac{|\nabla \text{Ric}|^2}{R^a}.
\]

Thus,

\[
\Delta f \left( \frac{|\text{Ric}|^2 R^{-a}}{R^a} \right) \geq \frac{2(1 - a) |\nabla \text{Ric}|^2}{1 + a} \frac{R^a}{R^a} - C \frac{|\nabla |\text{Ric}|| |\text{Ric}|^2}{R^a} \\
- C \frac{|\text{Ric}|^4 + |\text{Ric}|^3}{R^a} + 2a \frac{|\text{Ric}|^4}{R^{a+1}} \\
\geq \left( 2a - \frac{CR}{1 - a} \right) \frac{|\text{Ric}|^4}{R^{a+1}} - C \frac{|\text{Ric}|^3}{R^a}.
\]

Therefore, for \( u = \frac{|\text{Ric}|^2}{R^a} \), we have derived the differential inequality

\[
\Delta f u \geq \left( 2a - \frac{CR}{1 - a} \right) u^2 R^{a-1} - Cu^{3/2} R^{a/2}. \quad (4.2)
\]

By assumption (4.1), for any \( 0 < a < 1 \), we can choose a fixed \( d_0 > 0 \) depending on \( a \) and sufficiently large so that

\[
\left( 2a - \frac{CR}{1 - a} \right) \geq a \quad (4.3)
\]

outside the geodesic ball \( B(x_0, d_0) \).

Next, for any \( D_0 > 2d_0 \), we choose a function \( \phi(t) \) as follows: \( 0 \leq \phi(t) \leq 1 \) is a smooth function on \( \mathbb{R} \) such that

\[
\phi(t) = \begin{cases} 
1, & \text{if } 2d_0 \leq t \leq D_0, \\
0, & \text{if } t \leq d_0 \text{ or } t \geq 2D_0.
\end{cases}
\]
Also,

\[ t^2 |\varphi''(t)| \leq c \quad \text{and} \quad 0 \geq \varphi'(t) \geq -\frac{c}{D_0}, \text{ if } 2d_0 \leq t \leq 2D_0. \quad (4.4) \]

Now we use \( \varphi = \varphi(r(x)) \) as a cut-off function whose support is in \( B(x_0, 2D_0) \setminus B(x_0, d_0) \). Note that, by (4.4) and Lemma 4.1, we get

\[ |\nabla \varphi|^2 = |\varphi'|^2 \leq \frac{c}{D_0^2} \quad \text{and} \quad \Delta f \varphi = \varphi' \Delta f r(x) + \varphi'' \geq -\frac{C}{D_0}. \quad (4.5) \]

on \( B(x_0, 2D_0) \setminus B(x_0, 2d_0) \), respectively.

Setting \( G = \varphi^2 u \), then by our choice of \( \varphi \) and (4.5), we see that

\[
\varphi^2 \Delta f G = \varphi^4 \Delta f u + \varphi^2 u \Delta f \varphi^2 + 2\varphi^2 (\nabla u \cdot \nabla \varphi^2) \\
\geq \varphi^4 \left( a u^2 R^{a-1} - Cu^{3/2} R^{a/2} \right) + 2\varphi^2 u (\Delta f \varphi^2) - 8 |\nabla \varphi|^2 G + 2 \nabla G \cdot \nabla \varphi^2 \\
\geq a G^2 R^{a-1} - C G^{3/2} R^{a/2} - C G + 2 \nabla G \cdot \nabla \varphi^2.
\]

Assume \( G \) achieves its maximum at some point \( p \in B(x_0, 2D_0) \). If \( p \in B(x_0, 2D_0) \setminus B(x_0, 2d_0) \), then it follows from the maximum principle that

\[ 0 \geq a G^2(p) R^{a-1}(p) - C G^{3/2}(p) R^{a/2}(p) - C G(p). \]

On the other hand, noticing that the fact \( 0 < a < 1 \) and \( R \) uniformly bounded from above, implies

\[ G(p) \leq C \]

for some constant \( C \) depending on \( a \) but independent of \( D_0 \).

Thus,

\[ \max_{B(x_0,D_0)} u \leq \max_{B(x_0,2D_0)} G \leq \max \left\{ C, \max_{B(2d_0)} u \right\} \leq C' \]

for some \( C' > 0 \) independent of \( D_0 \). Therefore \( |Ric|^2 \leq CR^a \) on \( M^4 \).

It remains to show \( |Rm| \leq C \) on \( M^4 \). However, once we know \( \sup_{x \in M} Ric \leq C \), \( |Rm| \leq C \) follows essentially from the same argument as in the proof of Theorem 3.1. We leave the details to the interested readers. \( \square \)

**Lemma 4.2** Let \( (M^4, g_{ij}, f) \), which is not Ricci-flat, be a complete non-compact gradient steady Ricci soliton with \( \lim_{x \to \infty} R(x) = 0 \). Then for each \( 0 < a < 1 \) and \( \mu > 0 \), there exist constants \( \lambda > 0 \) and \( D > 0 \) so that function

\[ v = \frac{|Rm|^2 + \lambda |Ric|^2}{R^a} \]
satisfies the differential inequality

$$\Delta_f v \geq \mu v - D.$$ 

**Proof** By Lemma 2.4 and Theorem 4.1,

$$\Delta_f v = \frac{\Delta_f (|Rm|^2 + \lambda|Ric|^2)}{R^a} + v R^a \Delta_f \left( \frac{1}{R^a} \right) + 2\nabla(v R^a) \cdot \nabla(R^{-a})$$

$$\geq \frac{2|\nabla Rm|^2 + 2\lambda|\nabla Ric|^2}{R^a} - c \frac{|Rm|^2 + \lambda|Ric|^2}{R^a}$$

$$+ \left( |Rm|^2 + \lambda|Ric|^2 \right) \left[ -a \frac{\Delta_f R}{R^{a+1}} + a(a + 1) \frac{|\nabla R|^2}{R^{a+2}} \right]$$

$$- 4a \frac{|Rm||\nabla Rm||\nabla R|}{R^{a+1}} - 4a \lambda \frac{|Ric||\nabla Ric||\nabla R|}{R^{a+1}}.$$ 

By using Cauchy’s inequality to terms with $|\nabla R|$, 

$$\Delta_f v \geq \frac{2|\nabla Rm|^2 + 2\lambda|\nabla Ric|^2}{R^a} - c \frac{|Rm|^2 + \lambda|Ric|^2}{R^a}$$

$$- \frac{4a}{a + 1} \frac{|\nabla Rm|^2}{R^a} - \frac{4a \lambda}{a + 1} \frac{|\nabla Ric|^2}{R^a}$$

$$\geq \frac{2\lambda(1 - a)|\nabla Ric|^2}{1 + a} - c \frac{|Rm|^2 + \lambda|Ric|^2}{R^a}.$$ 

Now by Proposition 3.1, for some constant $\epsilon > 0$, we have

$$2\epsilon |Rm|^2 \leq \left( |\nabla Ric| + |Ric|^2 + |Rc| \right)^2$$

$$\leq 2|\nabla Ric|^2 + 2 \left( |Ric|^2 + |Ric| \right)^2.$$ 

Thus,

$$\Delta_f v \geq \left[ \frac{2\epsilon \lambda(1 - a)}{1 + a} - c \right] \frac{|Rm|^2}{R^a} - \left[ \frac{2\lambda}{1 + a} (|Ric| + 1)^2 + c \lambda \right] \frac{|Ric|^2}{R^a}$$

$$\geq [\epsilon \lambda(1 - a) - c] \left( v - \lambda \frac{|Ric|^2}{R^a} \right) - \lambda \left[ 2(1 - a)(|Ric| + 1)^2 + c \right] \frac{|Ric|^2}{R^a}.$$ 

Therefore, by Theorem 4.1, for each $0 < a < 1$ and $\mu > 0$ one can choose $\lambda \geq C/(1 - a)$, with $C > 0$ depending on $\mu$ and sufficiently large, so that

$$\Delta_f v \geq \mu v - D$$

for some constant $D > 0$ depending on $\lambda$. $\square$
Theorem 4.2 Let \((M^4, g_{ij}, f)\), which is not Ricci-flat, be a complete non-compact gradient steady Ricci soliton with \(\lim_{r \to \infty} R = 0\). Suppose \(R\) has at most polynomial decay, i.e., \(R(x) \geq C/r^k(x)\) outside \(B(r_0)\) for some fixed \(r_0 > 1\), some constant \(c > 0\) and positive integer \(k\). Then, for each \(0 < a < 1\), there exists a constant \(C\) such that

\[ |Rm| \leq CR^{a/2}. \]

Proof Let \(p = \frac{k}{2}\). Consider the following function on \(\mathbb{R}^+\):

\[ \varphi(t) = \begin{cases} \left(\frac{d-t}{d}\right)^p & \text{if } 0 \leq t \leq d \\ 0 & \text{if } t \geq d. \end{cases} \]

Next, let \(\varphi = \varphi(r(x))\) on \(M^4\). Then we have

\[ |\nabla \varphi| = \frac{p}{d} \left(\frac{d-r}{d}\right)^{p-1} |\nabla r| = \frac{p}{d-r} \varphi, \]

\[ \triangle_f \varphi = -\frac{p}{d} \left(\frac{d-r}{d}\right)^{p-1} \triangle_f r + \frac{p(p-1)}{d^2} \left(\frac{d-r}{d}\right)^{p-2} |\nabla r|^2 \]

\[ = \left[\frac{p}{d-r} \triangle_f r + \frac{p(p-1)}{(d-r)^2}\right] \varphi \]

Consider \(w = v - \frac{D}{\mu} \) with \(v = |Rm|^2 + \lambda |Ric|^2 \), \(\mu\) and \(D\) as in Lemma 4.2. Then, \(w\) satisfies

\[ \triangle_f w \geq \mu w. \]

Let \(G = \varphi^2 w\), then outside \(B(r_0)\), we have

\[ \triangle_f G = (\triangle_f \varphi^2)w + \varphi^2 \triangle_f w + 2(\nabla \varphi^2) \cdot \nabla w \]

\[ \geq \left(2\varphi \triangle_f \varphi + 2|\nabla \varphi|^2\right) w + \mu \varphi^2 w + 4\varphi \nabla \varphi \cdot \nabla \frac{G}{\varphi^2} \]

\[ \geq \left(\mu + \frac{2\triangle_f \varphi}{\varphi} - 6 \frac{|\nabla \varphi|^2}{\varphi^2}\right) G + \frac{4}{\varphi} (\nabla G, \nabla \varphi). \quad (4.6) \]

Recall that \(G = 0\) outside \(B(d)\). Now consider a maximum point \(q\) of \(G\).

Case 1 \(G(q) \leq 0\). Then, \(\max_{B(d)} w \leq 0\).

Case 2 \(G(q) > 0\) and \(q \in B(r_0)\). Then, on \(\Omega = B((1 - \frac{1}{2^{1/p}})d)\), we have

\[ \max_{\Omega} w \leq \max_{\Omega} \frac{1}{\varphi^2} \cdot G(q) \]

\[ \leq 4G(q) \]

\[ \leq 4 \max_{B(r_0)} w. \]
Case 3 $G(q) > 0$ and $q \not\in B(r_0)$. Then, by (4.6) and Lemma 4.1, at $q$ we have

$$0 \geq \mu + 2 \frac{\triangle f \varphi}{\varphi} - 6 \frac{\nabla \varphi}{\varphi^2} \geq \mu - 2 p K_0 \frac{1}{d - r} - (4 p^2 + 2 p) \frac{1}{(d - r)^2}$$

for some constant $K_0 > 0$ depending on $r_0$ and $\max_{B(r_0)} |Ric|$. Hence $\frac{1}{d - r(q)} > C$ for some constant $C$ depending on $\mu$, $p = k/2$ and $K_0$. Thus, we have

$$d - r(q) \leq c$$

for some constant $c > 0$ independent of $d$.

Therefore,

$$\max_{\Omega} w \leq \max_{\Omega} \frac{1}{\varphi^2} \cdot G(q) \leq 4 G(q) \leq 4 (d - r(q))^2p \left( |Rm|^2 + \lambda |Ric|^2 \right) \leq C \frac{R^{a_k}(q)}{d^2p} \leq C d^{(a-1)k} \leq C$$

for some constant $C > 0$ independent of $d$. Since $d > r_0$ is arbitrary, we obtain $\sup_M w \leq C$, and hence $|Rm|^2 \leq CR^a$ on $M^4$ for each $0 < a < 1$.

Acknowledgements We are grateful to Ovidiu Munteanu and Jiaping Wang for sending us their paper [25], and its early version in July 2014, which motivated us to consider curvature estimates for 4D steady solitons. The first author also would like to thank Ovidiu Munteanu for very helpful discussions; part of the work was carried out when the first author was visiting University of Macau, where he was partially supported by Science and Technology Development Fund (Macao S.A.R.) Grant FDCT/016/2013/A1 and the RDG010 Project of the University of Macau.

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