Remarks on localized sharp functions on certain sets in $\mathbb{R}^n$

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Abstract The aim of this note is to define localized sharp functions on certain domains in $\mathbb{R}^n$ and prove $L^p$ estimates analogue to that of Fefferman–Stein. The proofs go by modifications of the good lambda inequality.

Keywords Local maximal function · Sharp function · Good lambda inequality · Whitney decomposition

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1 Introduction

On $\mathbb{R}^n$ let $f_\Delta^\#(x)$ and $M^\Delta f(x)$ denote the classical dyadic sharp function and dyadic maximal function respectively, that is,

$$f_\Delta^\#(x) = \sup_{x \in Q \in \Delta} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

$$M^\Delta f(x) = \sup_{x \in Q \in \Delta} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$
where here and subsequently, \( \Delta \) denotes the collection of all dyadic cubes in \( \mathbb{R}^n \) and

\[
f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx.
\]

Suppose that \( f \in L^{p_0}(\mathbb{R}^n) \) for some \( p_0 \). The well-known Fefferman–Stein inequality asserts that if \( 1 < p < \infty \), \( 1 \leq p_0 \leq p \), and \( f^\#_{\Delta} \in L^p(\mathbb{R}^n) \), then \( M^\Delta f \in L^p(\mathbb{R}^n) \) and

\[
\| M^\Delta f \|_{L^p(\mathbb{R}^n)} \leq C_n(p) \| f^\#_{\Delta} \|_{L^p(\mathbb{R}^n)} \tag{1.1}
\]

(see [2, Sect. 3], [4, Chapter 4]). The inequality (1.1) implies that for every \( 1 < p < \infty \) one has

\[
\| f \|_{L^p(\mathbb{R}^n)} \leq C_n(p) \| f^\#_{\Delta} \|_{L^p(\mathbb{R}^n)}.
\]

The estimate (1.1) is a consequence of the following good lambda distributional inequality

\[
\left| \{ x \in \mathbb{R}^n : M^\Delta f(x) > \lambda, \ f^\#_{\Delta}(x) \leq c\lambda \} \right| \leq a \left| \{ x \in \mathbb{R}^n : M^\Delta f(x) > b\lambda \} \right|, \tag{1.2}
\]

where \( \lambda > 0, c > 0, 0 < b < 1, a = 2^n c/(1 - b) \), and \( f \in L^{1,\text{loc}}(\mathbb{R}^n) \) (see [4]).

Let \( \Omega \) be a domain in \( \mathbb{R}^n \). Our goal is to define for \( f \in L^{1,\text{loc}}(\Omega) \) a localized version \( f^\#_{\text{loc}} \) of the sharp function which will satisfy

\[
\| f \|_{L^p(\Omega)} \leq C_p \| f^\#_{\text{loc}} \|_{L^p(\Omega)} \text{ for } f \in L^p(\Omega). \tag{1.3}
\]

By localized we mean that the cubes which are taken in the definition of \( f^\#_{\text{loc}}(x) \) are contained in a bounded set \( B_x \subset \Omega \). So one possible definition can be taken as follows. Let \( \tau : \Omega \to (0, \infty) \). For \( f \in L^{1,\text{loc}}(\Omega) \) we set

\[
f^\#_{\text{loc}, \tau}(x) = \sup_{x \in Q \subset \Omega, \ \ell(Q) < \tau(x)} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,
\]

where \( Q \) is any cube (not necessarily dyadic) and \( \ell(Q) \) denotes its side-length. Note that \( \tau \) cannot be taken arbitrarily. For example, if \( \Omega = (0, \infty) \) and \( \tau \) is such that \( \lim_{x \to \infty} \tau(x)/x = 0 \), then taking \( f(x) = \chi_{(0,R)}(x) \) we have \( \| f \|_{L^p(\Omega)} = R^{1/p} \) while \( \lim_{R \to \infty} R^{-1/p} \| f^\#_{\text{loc}, \tau} \|_{L^p(\Omega)} = 0 \). On the other hand, we shall show that for certain sets \( \Omega \) in \( \mathbb{R}^n \) if \( \tau(x) \) behaves like \( \frac{1}{2} \text{dist}(x, \partial \Omega) \), then \( f^\#_{\text{loc}, \tau} \) satisfies (1.3). Moreover, the inequality (1.3) holds for \( p = 1 \), provided \( f \) is supported by a bounded set and \( |f| \log(2 + |f|) \) is integrable. These will be obtained by proving modifications of the good lambda inequality (see Propositions 2.4, 3.7, and 4.5).
2 Localized sharp function on $\mathbb{R}^n \setminus \{0\}$

Let $\Omega = \mathbb{R}^n \setminus \{0\}$. We define the localized sharp function on $\Omega$ as

$$f_{\text{loc}, \Omega}^\#(x) = \sup_{x \in K \subset \{y \in \Omega : \frac{|x|}{4\sqrt{n}} \leq |y| \leq 4\sqrt{n}|x|\}} \frac{1}{|K|} \int_K |f(y) - f_K| dy,$$

where the supremum is taken over all cubes $K$ (not necessarily dyadic) contained in the set

$$\left\{ y \in \Omega : \frac{|x|}{4\sqrt{n}} \leq |y| \leq 4\sqrt{n}|x| \right\}.$$

We now turn to define the local dyadic maximal function associated with a Whitney decomposition of $\Omega$. For this purpose, put

$$\rho(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|).$$

Set

$$\mathcal{L} = \{Q \in \Delta : \ell(Q) = \rho(0, Q)\}.$$

The set $\mathcal{L}$ forms a Whitney covering of $\Omega$. For every integer $k$ we define the $k$-th layer $L_k$ of $\mathcal{L}$ as

$$L_k = \{Q \in \mathcal{L} : \ell(Q) = 2^{-k}\}.$$

Clearly, $Q \in L_k$ if and only if $2^{-m}Q \in L_{k+m}$. Figure 1 shows three $k$-layers for $n = 2$. Here and subsequently, $\alpha Q = \{\alpha x : x \in Q\}, \alpha > 0$.

Fig. 1 Cubes from three $k$-layers of $\mathcal{L}$ for $n = 2$ which are contained in $(0, \infty)^n$
For every positive integer $m$ the partition $L_m$ of $\Omega$ is obtained by dividing each cube $Q$ from $\mathcal{L}$ into $2^{nm}$ dyadic cubes each of side-length $2^{-m} \ell(Q)$. Let

$$
\mathcal{D} = \mathcal{L} \cup \bigcup_{m=1}^{\infty} L_m.
$$

The local dyadic maximal function associated with the Whitney covering $\mathcal{L}$ of $\Omega$ is defined by

$$
M_{\mathcal{D}} f(x) = \sup_{x \in K \in \mathcal{D}} \frac{1}{|K|} \int_K |f(y)| \, dy.
$$

Our goal of this section is to prove the following theorem.

**Theorem 2.1** For every $1 \leq p < \infty$ there is a constant $C > 0$ such that for every locally integrable function $f$ on $\Omega$ for which there exists $0 < p_0 \leq p$ such that $M_{\mathcal{D}} f \in L^{p_0}(\Omega)$ one has

$$
\|M_{\mathcal{D}} f\|_{L^p(\Omega)} \leq C \|f_{\text{loc}, \Omega}\|_{L^{p_0}(\Omega)}.
$$

**Corollary 2.2** For every $1 < p < \infty$ there is a constant $C_p > 0$ such that

$$
\|f\|_{L^p(\Omega)} \leq C_p \|f_{\text{loc}, \Omega}\|_{L^{p}(\Omega)} \quad \text{for } f \in L^{p}(\Omega).
$$

**Corollary 2.3** There is a constant $C > 0$ such that if $f$ is supported by a bounded set and $|f| \log(2 + |f|)$ is integrable, then

$$
\|f\|_{L^1(\Omega)} \leq C \|f_{\text{loc}, \Omega}\|_{L^1(\Omega)}.
$$

There is no loss of generality if we assume that all the functions under consideration take values in $\mathbb{R}$. Clearly, for almost every $x \in \Omega$ there is a unique cube $Q \in \mathcal{L}$ such that $x \in Q$. For such an $x$ let

$$
Sf(x) = |f|_Q, \quad S'f(x) = f_Q. \quad (2.1)
$$

The proof of Theorem 2.1 is a consequence of the following modified version of the good lambda inequality, which is stated in the proposition below.

**Proposition 2.4** (modified good lambda inequality) For every constant $0 < b < 1$ there is a constant $C > 0$ such that for all $c, \alpha > 0$, and every locally integrable function $f$ which satisfies $\lim_{|x| \to \infty} Sf(x) = 0$ we have

$$
\left| \{x \in \Omega : M_{\mathcal{D}} f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c \alpha \} \right| \leq C c \left| \{x \in \Omega : M_{\mathcal{D}} f(x) > b \alpha \} \right| + \left| \{x \in \Omega : Sf(x) > b \alpha \} \right|.
$$
where here and subsequently,

\[ S f(x) = 2^{1-n} \cdot 2^{2n} \sum_{j=1}^{\infty} f_{\text{loc}, \Omega}^\#(2^j x). \]

**Proof of Theorem 2.1** If we assume Proposition 2.4, the proof of the theorem is a slight modification of that in the classical case (see [2–4]). For the convenience of the reader, we provide details. We may assume that \( \| f_{\text{loc}, \Omega}^\# \|_{L^p(\Omega)} \) is finite. Then, by the Minkowski inequality,

\[
\| S f \|_{L^p(\Omega)} \leq 2^{1-n} \cdot 2^{2n} \sum_{j=1}^{\infty} 2^{-jn/p} \| f_{\text{loc}, \Omega}^\# \|_{L^p(\Omega)} = C_p \| f_{\text{loc}, \Omega}^\# \|_{L^p(\Omega)}. \tag{2.2}
\]

Since \( M_D f \in L^{p_0}(\Omega) \), \( \lim_{|x| \to \infty} S f(x) = 0 \). Let

\[
I_R = p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : M_D f(x) > \alpha \} \right| d\alpha
\]

\[
= p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha \} \right| d\alpha
\]

\[
+ p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) \geq c\alpha \} \right| d\alpha.
\]

Applying Proposition 2.4, we obtain

\[
I_R \leq Ccp \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : M_D f(x) > b\alpha \} \right| d\alpha
\]

\[
+ p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : S f(x) > b\alpha \} \right| d\alpha
\]

\[
+ p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : f_{\text{loc}, \Omega}(x) \leq c\alpha \} \right| d\alpha
\]

\[
\leq Ccb^{-p} I_{bR} + p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : S f(x) > b\alpha \} \right| d\alpha
\]

\[
+ p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : f_{\text{loc}, \Omega}(x) \geq c\alpha \} \right| d\alpha.
\]

Clearly, \( I_R < \infty \), since, by assumption, \( M_D f \in L^{p_0}(\Omega) \) and \( 0 < p_0 \leq p \). Moreover, \( I_{bR} \leq I_b \) because \( 0 < b < 1 \). Taking \( c \) small enough such that \( Ccb^{-p} < 1 \) we obtain

\[
I_R \leq C' p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : S f(x) > b\alpha \} \right| d\alpha
\]

\[
+ C' p \int_0^R \alpha^{p-1} \left| \{ x \in \Omega : f_{\text{loc}, \Omega}(x) \geq c\alpha \} \right| d\alpha.
\]
Letting $R \rightarrow \infty$, we conclude

$$\|M_D f\|_{L^p(\Omega)} \leq C_1 \|S f\|_{L^p(\Omega)} + C_2 \|f^\#_{\text{loc}, \Omega}\|_{L^p(\Omega)} \leq C_3 \|f^\#_{\text{loc}, \Omega}\|_{L^p(\Omega)},$$

where in the last inequality we have used (2.2). \hfill \Box

The remaining part of the section is devoted to proving Proposition 2.4. The following two lemmas will play a crucial role in the proof.

**Lemma 2.5** For every locally integrable function $f$ and almost every $x \in \Omega$ one has

$$S' f(x) \leq S' f(2x) + 2^{-n} \cdot 3^2 f^\#_{\text{loc}, \Omega}(2x).$$

**Proof** It suffices to prove the lemma for $x = (x_1, x_2, \ldots, x_n)$ such that $x_j > 0$ for every $j = 1, 2, \ldots, n$. Let $Q_1$ be the unique dyadic cube from $\mathcal{L}$ which contains $x$. Let $k$ be such that $2^{-(k+1)} = \ell(Q_1)$. Set $Q_2 = 2Q_1$. Let

$$Q_0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_j \leq 1 \text{ for all } j = 1, 2, \ldots, n\}.$$  \hfill (2.3)

Then there is the unique vector $p = (p_1, p_2, \ldots, p_n) \neq \emptyset$, $p_j \in \{0, 1\}$, such that $Q_1 = 2^{-k-1}Q_0 + 2^{-k-1}p$. Set $K = 3 \cdot 2^{-k-1}Q_0 + 2^{-k-1}p$. Then

$$Q_1 \cup Q_2 \subset K \subset \left\{y \in \Omega : \frac{|2x|}{4\sqrt{n}} \leq |y| \leq 4\sqrt{n}|2x|\right\}.$$

We shall call the set $M = K \setminus (Q_1 \cup Q_2)$ the complementary neighborhood of the pair of cubes $(Q_1, Q_2)$.

Let us remark that for $n = 1$ the complementary neighborhood of two intervals is the empty set. For $n = 2$ the complementary neighborhoods are presented in Fig. 2.

We have

$$\frac{|Q_1|}{|K|} = \frac{1}{3^n}, \quad \frac{|Q_2|}{|K|} = \frac{2^n}{3^n}, \quad \frac{|M|}{|K|} = \frac{3^n - 2^n - 1}{3^n},$$

$$f_K = \frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M.$$  \hfill (2.4)
We consider two cases.

**Case 1** \( S' f(x) \geq f_M \). Then

\[
\begin{align*}
\text{f}_{\text{loc}, \Omega} (2x) &\geq \frac{1}{|K|} \int_{K} |f(y) - f_K| \, dy \\
&= \frac{1}{|K|} \int_{K} |f(y) - \left( \frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M \right) | \, dy \\
&\geq \frac{1}{|K|} \int_{Q_1} |f(y) - \left( \frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M \right) | \, dy \\
&\geq \frac{1}{|K|} \int_{Q_1} f(y) - \left( \frac{1}{3^n} f_{Q_1} + \frac{2^n}{3^n} f_{Q_2} + \frac{3^n - 2^n - 1}{3^n} f_M \right) \, dy \\
&= \left| f_{Q_1} \frac{3^n - 1}{3^{2n}} - f_{Q_1} \frac{2^n}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_M \frac{3^n - 1 - 2^n}{3^{2n}} \right|
\end{align*}
\]

By the assumption \(-f_M \geq -S' f(x) = -f_{Q_1}\), hence

\[
\begin{align*}
f_{Q_1} \frac{3^n - 1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_M \frac{3^n - 1 - 2^n}{3^{2n}} &\geq f_{Q_1} \frac{3^n - 1}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} - f_{Q_1} \frac{3^n - 1 - 2^n}{3^{2n}} \\
&= f_{Q_1} \frac{2^n}{3^{2n}} - f_{Q_2} \frac{2^n}{3^{2n}} = S' f(x) \frac{2^n}{3^{2n}} - S' f(2x) \frac{2^n}{3^{2n}},
\end{align*}
\]

which gives the lemma.

**Case 2** \( S' f(x) \leq f_M \). The proof in this case is similar to that in Case 1. The only difference is that we diminish the area of integration to \( Q_2 \) instead of \( Q_1 \). We omit the details. If \( n = 1 \), then \( M = \emptyset \). In this case we set \( f_M = 0 \) and proceed as in Case 1. 

**Remark 2.6** If we apply the lemma to the function \(-f\), we obtain the inequality

\[
S' f(2x) \leq S' f(x) + 2^{-n} \cdot 3^{2n} f_{\text{loc}, \Omega} (2x).
\]

**Lemma 2.7** For every locally integrable function \( f \) and almost every \( x \in \Omega \) one has

\[
S f(x) \leq S f(2x) + 2^{1-n} \cdot 3^{2n} f_{\text{loc}, \Omega} (2x).
\]  

**Proof** Using Lemma 2.5 to \(|f|\) we get \( S f(x) \leq S f(2x) + 2^{-n} \cdot 3^{2n} |f|_{\text{loc}, \Omega} (2x) \). The inequality (2.5) holds because \(|f|_{\text{loc}, \Omega} (2x) \leq 2 f_{\text{loc}, \Omega} (2x)\). 

Iterating the inequality (2.5) we obtain the following corollary.
Corollary 2.8 Assume that a locally integrable function $f$ on $\Omega$ satisfies

$$\lim_{|x| \to \infty} Sf(x) = 0.$$ 

Then

$$Sf(x) \leq Sf(x) \text{ for every } x \in \Omega.$$ 

Proof of Proposition 2.4 The proof of the proposition is a modification of that of the classical good lambda inequality (cf. [1,4]). Let $\{Q_j\}$ be the partition of the set $\{x \in \Omega : M_D f(x) > b\alpha\}$ which consists of maximal dyadic cubes $Q_j$ from $\mathcal{D}$ which satisfy $|f|_{Q_j} > b\alpha$. Obviously, the cubes $Q_j$ have disjoint interiors. Further,

$$\{x \in \Omega : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha\} \subset \{x \in \Omega : M_D f(x) > b\alpha\} = \bigcup_j Q_j,$$

$$\{x \in \Omega : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha\} = \bigcup_j \{x \in Q_j : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha\}.$$ 

Thus, it suffices to show that either

$$\left| \left\{ x \in Q_j : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha \right\} \right| \leq cC|Q_j| \quad (2.6)$$

or

$$\left\{ x \in Q_j : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha \right\} \subset \{x \in \Omega : Sf(x) > b\alpha\}. \quad (2.7)$$

Assume that the set $A_j = \{x \in Q_j : M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha\}$ is not empty, otherwise there is nothing to prove. Fix $x_0 \in A_j$. We consider two cases.

Case 1 $Q_j \in \mathcal{L}$. Then $b\alpha < |f|_{Q_j} = Sf(y) \leq Sf(y)$ for every $y \in Q_j$, where in the last inequality we have used Corollary 2.8. Thus $Q_j \subset \{x \in \Omega : Sf(x) > b\alpha\}$ and (2.7) holds in this case.

Case 2 $Q_j \in \mathcal{L}_m$ for $m \geq 1$. In this case the proof follows the pattern from [4]. Indeed, first observe that for every $Q \in \mathcal{D}$ such that $Q_j \not\subset Q$ one has $|f|_{Q} \leq b\alpha$. Thus, for every $x \in Q_j$ such that $M_D f(x) > \alpha$ one has $M_D|(f - f_{Q})\chi_{Q_j}|(x) > (1-b)\alpha$. Let $\tilde{Q}_j$ be the parent of $Q_j$. Clearly, $\tilde{Q}_j \in \mathcal{D}$ and

$$A_j = \{x \in Q_j : M_D(f - f_{\tilde{Q}_j})\chi_{Q_j}(x) > (1-b)\alpha, M_D f(x) > \alpha, f_{\text{loc}, \Omega}(x) < c\alpha\}.$$ 

Since $M_D$ satisfies the weak type $(1,1)$ inequality with the constant $C' = 1$, we have

$$|A_j| \leq \frac{1}{(1-b)\alpha} \int_{Q_j} |f - f_{\tilde{Q}_j}| \, dx \leq \frac{1}{(1-b)\alpha} \int_{\tilde{Q}_j} |f - f_{\tilde{Q}_j}| \, dx \leq \frac{\tilde{Q}_j}{(1-b)\alpha} f_{\text{loc}, \Omega}(\alpha_0)$$

$$\leq \frac{|\tilde{Q}_j|}{(1-b)\alpha} c\alpha = \frac{2^n c}{1-b} |Q_j|,$$

so (2.6) holds in this case with $C = 2^n (1-b)^{-1}$. \qed
Localised sharp function for $(0, \infty)^n$

In this section $\tilde{\Omega} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j > 0, \ j = 1, 2, \ldots, n \}$ denotes the generalized first quoter in $\mathbb{R}^n$. The distance of $x \in \tilde{\Omega}$ from the boundary is given by

$$\tilde{\rho}(x, \partial \tilde{\Omega}) = \min \{ x_j : j = 1, 2, \ldots, n \}.$$ 

We define the partition $\tilde{\mathcal{L}}$ of $\tilde{\Omega}$:

$$\tilde{\mathcal{L}} = \{ Q \in \Delta : Q \subset \tilde{\Omega} \ \text{and} \ \ell(Q) = \tilde{\rho}(Q, \partial \tilde{\Omega}) \}.$$ 

Clearly,

$$\tilde{\mathcal{L}} = \bigcup_{k \in \mathbb{Z}} \tilde{L}_k, \ \text{where} \ \tilde{L}_k = \{ Q \in \tilde{\mathcal{L}} : \ell(Q) = 2^{-k} \}.$$ 

Similarly to the previous section, for every positive integer $m$ the partition $\tilde{\mathcal{L}}_m$ consists of dyadic cubes which are obtained by dividing each cube $Q$ from $\tilde{\mathcal{L}}$ into $2^{mn}$ dyadic cubes each of the side-length $2^{-m}\ell(Q)$ (Fig. 3). Set

$$\tilde{\mathcal{D}} = \tilde{\mathcal{L}} \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{L}}_m.$$ 

Define the local maximal dyadic function $M_{\tilde{\mathcal{D}}}$ and localized sharp function $f^\#_{\text{loc}, \tilde{\Omega}}$ associated with the Whitney covering $\tilde{\mathcal{L}}$ of $\tilde{\Omega}$ as
$M_{\overline{\mathcal{D}}}f(x) = \sup_{x \in Q \in \overline{\mathcal{D}}} |f|_Q,$

$f^\#_{\text{loc}, \Omega}(x) = \sup_{x \in K \subset \tilde{\Omega}, 1/3 \rho(x, \partial \tilde{\Omega}) \leq \rho(y, \partial \tilde{\Omega}) \leq 4\rho(x, \partial \tilde{\Omega})} \frac{1}{|K|} \int_K |f(y) - f_K| \, dy,$

where the supremum is taken over all cubes (not necessarily dyadic).

It turns out that the following theorem analogue to Theorem 2.1 holds.

**Theorem 3.1** For every $1 \leq p < \infty$ there is a constant $C > 0$ such that for every locally integrable function $f$ on $\tilde{\Omega}$ for which there exists $0 < p_0 \leq p$ such that $M_{\overline{\mathcal{D}}}f \in L^{p_0}(\tilde{\Omega})$ one has

$$\|M_{\overline{\mathcal{D}}}f\|_{L^p(\tilde{\Omega})} \leq C \|f^\#_{\text{loc}, \tilde{\Omega}}\|_{L^p(\tilde{\Omega})}.$$ 

**Corollary 3.2** For every $1 < p < \infty$ there is a constant $C > 0$ such that

$$\|f\|_{L^p(\tilde{\Omega})} \leq C \|f^\#_{\text{loc}, \tilde{\Omega}}\|_{L^p(\tilde{\Omega})} \quad \text{for } f \in L^p(\tilde{\Omega}).$$

The remaining part of this section is devoted to proving Theorem 3.1. Similarly to the previous section [see (2.1)] we set

$$\tilde{S}f(x) = |f|_Q \quad \text{and} \quad \tilde{S}'f(x) = f_Q,$$ (3.1)

where $Q$ is the unique cube from $\tilde{\Omega}$ which contains $x$ (such a $Q$ is well-defined for almost every $x$). Let $k$ be such that $Q \in \tilde{L}_k$. Our goal is to define the successors $x'$ and $Q' \in \tilde{L}_{k-1}$ of $x$ and $Q$ respectively in such a way that $x' \in Q'$ and

$$\tilde{S}f(x) \leq \tilde{S}f(x') + 2^{1-n} 3^{2n} f^\#_{\text{loc}, \tilde{\Omega}}(x').$$

To this end, observe that there is a unique vector $q = (q_1, q_2, \ldots, q_n)$, where $q_j$ are non-negative integers such that at least one $q_j$ equals 0, and $Q = 2^{-k} Q_0 + 2^{-k} (q + 1)$, where here and subsequently, $1 = (1, 1, \ldots, 1)$. Consider the coordinates $x_j$ of $x$ for which $q_j = 0$. There is no loss of generality if we assume these are the first $m$ coordinates, $m \in \{1, 2, \ldots, n\}$. So $Q = 2^{-k} Q_0 + 2^{-k} (1, 1, \ldots, 1, 1+q_{m+1}, \ldots, 1+q_n), q_{m+1}, \ldots, q_n \geq 1$. Define

$$x' = F(x) = (2x_1, 2x_2, \ldots, 2x_m, x_{m+1}, \ldots, x_n) \text{ for } x \in Q.$$ (3.2)

Then, for almost every $x$, the point $x'$ belongs to the unique $Q' \in \tilde{L}_{k-1}$ and

$$Q' = 2^{-k+1} Q_0 + 2^{-k+1} (q' + 1),$$

$$q' = (0, \ldots, 0, q'_{m+1}, \ldots, q_n), \quad q'_j = \lfloor (q_j - 1) / 2 \rfloor, \quad j = m + 1, \ldots, n.$$
Fig. 4  Cubes which contain points $x, F(x), F(F(x)), \ldots$ for $n = 2$

Lemma 3.3  For every $f \in L^1_{\text{loc}}(\tilde{\Omega})$ and almost every $x \in \tilde{\Omega}$ one has

\[
\begin{align*}
\tilde{S}' f(x) &\leq \tilde{S}' f(x') + 2^{-n} \cdot 3^{2n} f_{\text{loc}, \tilde{\Omega}}^\#(x') , \\
\tilde{S} f(x) &\leq \tilde{S} f(x') + 2^{-n} \cdot 3^{2n} f_{\text{loc}, \tilde{\Omega}}^\#(x') , \\
\tilde{S}' f(x') &\leq \tilde{S}' f(x) + 2^{-n} \cdot 3^{2n} f_{\text{loc}, \tilde{\Omega}}^\#(x) .
\end{align*}
\]

Proof  Define (non-dyadic) cube $K'' = [2^{-k}, 2^{-k+2}]^m \times I_{m+1} \times \cdots \times I_n \subset \tilde{\Omega}$, where

\[
I_j = \left[ 2^{-k+1} q_j' + 2^{-k+2} - 3 \cdot 2^{-k}, 2^{-k+1} q_j' + 2^{-k+2} \right].
\]

We have $\ell(K'') = 3 \cdot 2^{-k}, Q \cup F(Q) \cup Q' \subset K''$. Moreover, $K''$ is taken into account if we compute $f_{\text{loc}, \tilde{\Omega}}^\#(x')$ and $f_{\text{loc}, \tilde{\Omega}}^\#(x)$. Set $M = K'' \setminus (Q \cup Q')$. We have [(cf. (2.4)]

\[
\begin{align*}
\frac{|Q|}{|K''|} &= \frac{1}{3^n} , \\
\frac{|Q'|}{|K''|} &= \frac{2^n}{3^n} , \\
\frac{|M|}{|K''|} &= \frac{3^n - 2^n - 1}{3^n} ,
\end{align*}
\]

hence the proof of the lemma is the same as those of Lemmata 2.5 and 2.7.

\[\square\]

Corollary 3.4  Assume that $f \in L^1_{\text{loc}}(\tilde{\Omega})$ and $\lim_{m \to \infty} \tilde{S} f(F^m(x)) = 0$ for almost every $x \in \tilde{\Omega}$, where $F^m(x) = F(F^{m-1}(x))$ (Fig. 4). Then

\[
\tilde{S} f(x) \leq \tilde{S} f(x),
\]

\[\square\] Springer
where
\[
\tilde{S}f(x) = 2^{1-n} \cdot 3^{2n} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \sum_{m=0}^{\infty} f^\#_{loc, \tilde{\Omega}} \left( 2^{k_1+m}x_1, 2^{k_2+m}x_2, \ldots, 2^{k_n+m}x_n \right).
\]

**Proof** It suffices to apply Lemma 3.3 and note that
\[
2^{1-n} 3^{2n} \sum_{m=1}^{\infty} f^\#_{loc, \tilde{\Omega}}(F^m(x)) \leq \tilde{S}f(x).
\]

**Remark 3.5** Let us note that \(\lim_{m \to \infty} \tilde{S}f(F^m(x)) = 0\) for \(f \in L^p(\tilde{\Omega})\). This is a consequence of the fact that \(\tilde{\rho}(F^m(x), \partial \tilde{\Omega}) \to \infty\) and \(\ell(Q^m) \to \infty\), where \(Q^m\) is the unique cube from \(\tilde{L}\) such that \(F^m(x) \in Q^m\).

**Lemma 3.6** For every \(1 \leq p < \infty\) there is a constant \(C > 0\) such that for every \(f \in L^1_{loc}(\tilde{\Omega})\) one has
\[
\|\tilde{S}f\|_{L^p(\tilde{\Omega})} \leq C \|f^\#_{loc, \tilde{\Omega}}\|_{L^p(\tilde{\Omega})}.
\]

**Proof** This follows from the Minkowski inequality and the summability of the series
\[
\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \sum_{m=0}^{\infty} 2^{-k_1/p-m/p} 2^{-k_2/p-m/p} \cdots 2^{-k_n/p-m/p}.
\]

\(\square\)

**Proposition 3.7** For every constant \(0 < b < 1\) there is a constant \(C > 0\) such that for all \(c, \alpha > 0\), and every \(f \in L^1_{loc}(\tilde{\Omega})\) satisfying
\[
\lim_{m \to \infty} \tilde{S}f(F^m(x)) = 0
\]
we have
\[
\left| \left\{ x \in \tilde{\Omega} : M_{\tilde{\Omega}} f(x) > \alpha, f^\#_{loc, \tilde{\Omega}}(x) < c\alpha \right\} \right| \leq C c \left| \left\{ x \in \tilde{\Omega} : M_{\tilde{\Omega}} f(x) > b\alpha \right\} \right| + \left| \left\{ x \in \tilde{\Omega} : \tilde{S}f(x) > b\alpha \right\} \right|.
\]

**Proof** The proof is identical to that of Proposition 2.4, and uses Corollary 3.4 instead of Corollary 2.8. \(\square\)

**Proof of Theorem 3.1** The theorem follows from Lemma 3.6 and Proposition 3.7. Its proof is identical to that of Theorem 2.1. \(\square\)
4 Localized sharp function for cube

In this section we consider the cube \((0, 2)^n \subset \mathbb{R}^n\) and its Whitney decomposition \(\tilde{\mathcal{L}}''\) which is defined in the following way. Let \(\tilde{\mathcal{L}}'\) be the restriction of the decomposition \(\tilde{\mathcal{L}}\) defined in the previous section into the unit cube \((0, 1)^n\). Let us denote by \(\tilde{\mathcal{L}}''\) the set of cubes which is obtained from \(\tilde{\mathcal{L}}'\) under the action of the group \(G\) of transformation generated by the reflections with respect to planes \(x_j = 1\). Let \(\tilde{\mathcal{L}}_k\) be the set of cubes from \(\tilde{\mathcal{L}}''\) of the side-length \(2^{-k}\). Clearly, \(\tilde{\mathcal{L}}'' = \tilde{\mathcal{L}}_1 \cup \tilde{\mathcal{L}}_2 \cup \cdots\) (Fig. 5).

We define the partition \(\tilde{\mathcal{L}}''_1\) by dividing each cube \(K\) from \(\tilde{\mathcal{L}}''\) into \(2^n\) dyadic cubes each of the side-length \(2^{-1}\ell(K)\). Inductively, \(\tilde{\mathcal{L}}''_m+1\) is defined by dividing each cube \(K\) from \(\tilde{\mathcal{L}}''_m\) into \(2^n\) dyadic cubes of side-length \(2^{-1}\ell(K)\). Set

\[
\tilde{\mathcal{D}} = \tilde{\mathcal{L}}'' \cup \bigcup_{m=1}^{\infty} \tilde{\mathcal{L}}''_m, \quad M_{\tilde{\mathcal{D}}} f(x) = \sup_{x \in K \subset \tilde{\mathcal{D}}} \frac{1}{|K|} \int_K |f(x)| \, dx.
\]

The localized sharp function is defined by

\[
f^\#_{\text{loc}, (0, 2)^n}(x) = \sup_{x \in K \subset B_x} \frac{1}{|K|} \int_K |f(y) - f_K| \, dy,
\]

where

\[
B_x = \left\{ y \in (0, 2)^n : \frac{1}{4} \text{dist}(x, \partial [0, 2]^n) \leq \text{dist}(y, \partial [0, 2]^n) \leq 4 \text{dist}(x, \partial [0, 2]^n) \right\}
\]

and the supremum is taken over all cubes \(K\) not necessarily dyadic.

Our aim of this section is to prove the following theorem.
Theorem 4.1 For every $1 \leq p < \infty$ there is a constant $C_p > 0$ such that if $f \in L^1((0,2)^n)$, $\int_{(0,2)^n} f(x) \, dx = 0$ and $M_{\mathcal{G}}f \in L^1((0,2)^n)$, then

$$\|M_{\mathcal{G}}f\|_{L^p((0,2)^n)} \leq C_p \|f\|_{L^p((0,2)^n)}.$$

The proof requires preparations.

For each $K \in \tilde{\mathcal{L}}''$ there is a unique $\sigma \in G$ such that $\sigma(K) \subset [0,1]^n$. Therefore in our considerations we shall deal with cubes contained in $[0,1]^n$ and then use the group action for other cubes.

From now on, let $Q_1 = \frac{1}{2} Q_0 + \frac{1}{2} 1$.

For $x \in (0,1)^n$ let $F(x)$ be defined by (3.2). Clearly, for every $R \in \tilde{\mathcal{L}}' \cap \tilde{\mathcal{L}}_m$ with $k \geq 2$ there is a unique $K \in \tilde{\mathcal{L}}' \cap \tilde{\mathcal{L}}_{k-1}$ such that $F(R) \subset K$. For $K \in \tilde{\mathcal{L}}'$ we set

$$\text{Pre}(K) = \{R \in \tilde{\mathcal{L}}' : F^j(R) \subset K \text{ for a certain positive integer } j\}.$$

Figure 6 shows examples of $\bigcup \text{Pre}(K)$ for $n = 2$. We have

$$\left| \bigcup \text{Pre}(K) \right| \leq (2^n - 1)|K|. \quad (4.1)$$

Lemma 4.2 Let $f$ be an integrable function on $(0,2)^n$. Assume that $K \subset \tilde{\mathcal{L}}' \cap \tilde{\mathcal{L}}_m$. Then

$$\left| \bigcup \text{Pre}(K) \right| f_K \leq \sum_{R \in \tilde{\mathcal{L}}' \cap \tilde{\mathcal{L}}_{m+1} \cap \text{Pre}(K)} |R| f_R + 3^{2n} \sum_{R \in \tilde{\mathcal{L}}' \cap \tilde{\mathcal{L}}_{m+1} \cap \text{Pre}(K)} |R| \inf_{y \in R} f^\#_{\text{loc},(0,2)^n}(y) + \sum_{R \in \tilde{\mathcal{L}}' \cap \tilde{\mathcal{L}}_{m+1} \cap \text{Pre}(K)} \left| \bigcup \text{Pre}(R) \right| f_R.$$
Remarks on localized sharp functions on certain sets in $\mathbb{R}^n$

**Proof** Set $C_n = 3^{2^n}2^{-n}$. By the same arguments we used to prove (3.5), we get $f_K \leq f_R + C_n \inf_{y \in R} f_{\text{loc}}^\#(y)$ for $R \in \tilde{L}_{m+1}$ such that $F(R) \subset K$. Hence,

$$\left| \bigcup \text{Pre}(K) \right| f_K = \sum_{R \in \tilde{L} \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left( |R| f_K + \left| \bigcup \text{Pre}(R) \right| f_K \right) \leq \sum_{R \in \tilde{L} \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left( |R| f_R + C_n |R| \inf_{y \in R} f_{\text{loc}}^\#(y) \right) + \sum_{R \in \tilde{L} \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left( C_n \left| \bigcup \text{Pre}(R) \right| \inf_{y \in R} f_{\text{loc}}^\#(y) \right) + \sum_{R \in \tilde{L} \cap \tilde{L}_{m+1} \cap \text{Pre}(K)} \left| \bigcup \text{Pre}(R) \right| f_R,$$

which, by (4.1), finishes the proof. \qed

**Corollary 4.3** Assume that $f$ is an integrable function on $(0, 2)^n$. Then

$$f_{Q_1} \leq \int_{(0,1)^n} f(y) \, dy + 3^{2^n} \int_{(0,1)^n} f_{\text{loc}}^\#(y) \, dy. \quad (4.2)$$

**Proof** Observe that $1 = |Q_1| + \left| \bigcup \text{Pre}(Q_1) \right|$. Thus

$$f_{Q_1} = |Q_1| f_{Q_1} + \left| \bigcup \text{Pre}(Q_1) \right| f_{Q_1}.$$

By iterating Lemma 4.2 we obtain that for every positive integer $m \geq 2$ one has

$$f_{Q_1} \leq \sum_{j=1}^{m} \left| K \right| f_K + 3^{2^n} \sum_{j=2}^{m} \left| K \right| \inf_{y \in K} f_{\text{loc}}^\#(y) + \sum_{K \in \tilde{L}' \cap \tilde{L}_{m}} \left| \bigcup \text{Pre}(K) \right| f_K.$$

Letting $m \rightarrow \infty$, we obtain the corollary, since the last summand tends to 0. \qed

**Corollary 4.4** There is a constant $C' > 0$ such that for every integrable function $f$ on $(0, 2)^n$ such that $\int_{(0,2)^n} f(x) \, dx = 0$ one has

$$\sum_{\sigma \in G} f_{\sigma}(Q_1) \leq C' \int_{(0,2)^n} f_{\text{loc}}^\#(y) \, dy, \quad (4.3)$$

$$\left| f_{\bigcup_{\sigma \in G} \sigma(Q_1)} \right| = 2^{-n} \left| \sum_{\sigma \in G} f_{\sigma}(Q_1) \right| \leq C' \int_{(0,2)^n} f_{\text{loc}}^\#(y) \, dy, \quad (4.4)$$
\[ |f|_{\sigma(G)} \leq C |f|_{(0,2)^n}, \quad (4.5) \]
\[ |f|_{\sigma(Q_2)} \leq C |f|_{(0,2)^n}, \quad (4.6) \]

**Proof** Clearly, \( \sum_{\sigma \in G} \int_{\sigma((0,1]^n)} f \, dy = \int_{(0,2)^n} f \, dy = 0 \). Hence (4.3) follows from (4.2). The inequality (4.4) is a direct consequence of (4.3). Further we write
\[ \int_{\sigma(Q_2)} |f| \, dy \leq \int_{\sigma(Q_2)} |f|_{\sigma(Q_2)} \, dy + |f|_{\sigma(Q_2)} \]

and apply (4.4) to obtain (4.5) and then (4.6). \( \square \)

Assume that \( f \in L^1((0,2)^n) \). For \( x \in (0,1]^n \) we define the function \( S f(x) \) as follows
\[ S f(x) = |f|_{Q_1} + 2^{1-n} 3^2 \sum_{j=0}^{k-1} f_{l,(0,2)^n}(F^j(x)) \quad \text{for} \quad x \in K \in \tilde{L}_k \cap \tilde{L}'. \]

**Proposition 4.5** For every constant \( 0 < b < 1 \) there is a constant \( C > 0 \) such that for all \( c, \alpha > 0 \), and every \( f \in L^1((0,2)^n) \) we have
\[ \left| \{ x \in (0, 1]^n : M_{\tilde{f}} f(x) > \alpha, f_{l,(0,2)^n}(x) < c\alpha \} \right| \leq C c \left| \{ x \in (0, 1]^n : M_{\tilde{f}} f(x) > b\alpha \} \right| + \left| \{ x \in (0, 1]^n : S f(x) > b\alpha \} \right| . \]

**Proof** For \( x \in (0,1]^n \) let \( \tilde{S} f(x) \) be defined by (3.1). The same arguments we used to prove Lemma 3.3 give
\[ \tilde{S} f(x) \leq \tilde{S}(F(x)) + 2^{1-n} 3^2 \sum_{j=0}^{k-1} f_{l,(0,2)^n}(F^j(x)) \quad \text{for} \quad x \in (0,1]^n \setminus Q_1. \quad (4.7) \]

Iteration of (4.7) leads to \( \tilde{S} f(x) \leq S f(x) \) for \( x \in (0,1]^n \). Now the proof is the same as that of Proposition 2.4. \( \square \)

**Proof of Theorem 4.1** For \( f \in L^1_{l,(0,2)^n}((0,2)^n) \) and \( \sigma \in G \) let \( f_{o}(x) = f(\sigma(x)) \). Since \( M_{\tilde{f}} f_{o} = (M_{\tilde{f}} f)_{\sigma}, (f_{o})_{l,(0,2)^n} = (f_{l,(0,2)^n})_{\sigma} \), and \( (0,2)^n = \bigcup_{\sigma \in G} \sigma((0,1]^n) \), it suffices to prove that
\[ \| M_{\tilde{f}} f \|_{L^p((0,1]^n)} \leq C_p \| f_{l,(0,2)^n} \|_{L^p((0,2)^n)} \quad (4.8) \]
for $f \in L^1((0, 2)^n)$, $\int_{(0,2)^n} f = 0$. Repeating the proof of Theorem 2.1 with the use of Proposition 4.5 we arrive at

$$\|M \tilde{\varphi}_f\|_{L^p((0,1)^n)} \leq C\|Sf\|_{L^p((0,1)^n)} + C\|f^\#\|_{L^p((0,1)^n)} \leq C'|f|_{Q_1} + C'|f^\#|_{L^p((0,1)^n)}.$$ (4.9)

Recall that the integral of $f$ is zero. Hence, applying (4.6), we obtain the desired inequality (4.8). □

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