Helicity and alpha-effect by current-driven instabilities of helical magnetic fields

M. Gellert1*, G. Rüdiger1,2, R. Hollerbach3

1Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany
2Forschungszentrum Dresden Rossendorf, P.O. Box 510119, D-01314 Dresden, Germany
3Department of Applied Mathematics, University of Leeds, Leeds, LS2 9JT, UK

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ABSTRACT

Helical magnetic background fields with adjustable pitch angle are imposed on a conducting fluid in a differentially rotating cylindrical container. The small-scale kinetic and current helicities are calculated for various field geometries, and shown to have the opposite sign as the helicity of the large-scale field. These helicities and also the corresponding α-effect scale with the current helicity of the background field. The α-tensor is highly anisotropic as the components αφφ and αzz have opposite signs. The amplitudes of the azimuthal α-effect computed with the cylindrical 3D MHD code are so small that the operation of an αΩ dynamo on the basis of the current-driven, kink-type instabilities of toroidal fields is highly questionable. In any case the low value of the α-effect would lead to very long growth times of a dynamo in the radiation zone of the Sun and early-type stars of the order of mega-years.

Key words: magnetic fields - instabilities - stars: magnetic field - dynamo

1 INTRODUCTION

No hydromagnetic dynamo can exist driven only by differential rotation (Elsasser 1946), but it is known that such dynamos can exist if the turbulence is helical in the sense that its kinetic helicity

\[ H_{\text{kin}} = \langle u \cdot \text{curl} u \rangle \]  (1)

and/or its current helicity

\[ H_{\text{curr}} = \frac{1}{\mu_0 \rho} \langle b \cdot \text{curl} b \rangle \]  (2)

do not vanish. Here \( u \) and \( b \) are the fluctuating parts of the flow \( U \) and magnetic field \( B \). This condition of non-vanishing helicity is clearly fulfilled if the turbulence is rotating and stratified. In such turbulence a pseudo-scalar exists which allows the pseudo-scalars [1] and [2] to take finite values. The same is true for linear shear flows where the stratified turbulence in the presence of the shear also can form a kinetic helicity (see Rüdiger & Kitchatinov 2006). The simplest pseudo-scalar is the scalar product \( g \cdot \Omega \) with \( g \) as the gradient vector of the turbulence (or the fluid density) and \( \Omega \) the rotation vector. In spheres the gradient vector \( g \) is mainly radial so the pseudo-scalar \( g \cdot \Omega \) has opposite signs in the two hemispheres, and vanishes at the equator. Because of the close relationship of the helicity to the α-effect in the mean-field electrodynamics of turbulent media,

\[ (u \times b) = \alpha B_0 + \ldots \]  (3)

the dots represent higher derivatives of \( B_0 \) the above-mentioned sign rules are also the sign rules of the α-effect i.e. \( \alpha \propto g \cdot \Omega \).

This sort of α-effect only exists for inhomogeneous turbulence. In planetary cores, however, and also in laboratory experiments the only inhomogeneities result from boundary conditions, as the density gradients are negligible. One can show that under the presence of magnetic background fields other inhomogeneities also form pseudo-scalars and, as a consequence, lead to new mechanisms for an α-effect (e.g. Gellert, Rüdiger & Elstner 2008). In the present study we demonstrate that instabilities due to inhomogeneous magnetic fields also lead to finite values of the helicities [1] and [2], and in accord with [3] also to finite values of \( \alpha \). Indeed, in the presence of electric currents the simplest existing pseudo-scalar is \( B_0 \cdot \text{curl} B_0 \) which does not vanish for helical field geometries. We show that for such background fields the small-scale helicities obtain final values with the opposite sign as the helicity of the background field.

According to the Rayleigh criterion, in the absence of MHD effects an ideal flow is stable against axisymmetric perturbations whenever the specific angular momentum increases outward

\[ \frac{\text{d}}{\text{d}R}(R^2 \Omega)^2 > 0, \]  (4)

where \( \Omega \) is the angular velocity, and \((R, \phi, z)\) are cylindrical coordinates in a right-handed system. In the presence of an azimuthal magnetic field \( B_\phi \), this criterion is modified to

\[ \frac{1}{R^3} \frac{\text{d}}{\text{d}R}(R^2 \Omega)^2 - \frac{R}{\mu_0 \rho} \frac{\text{d}}{\text{d}R} \left( \frac{B_0 \phi}{R} \right)^2 > 0, \]  (5)
where $\mu_0$ is the permeability and $\rho$ the density [Michael 1954].

Note also that this criterion is both necessary and sufficient for (axisymmetric) stability. In particular, all ideal flows can thus be destabilized, by azimuthal magnetic fields with the right profiles (steeply increasing outwards) and amplitudes.

On the other hand, for nonaxisymmetric modes one has

$$\frac{d}{dR}(RB_3) < 0$$

as the necessary and sufficient condition for stability of an ideal fluid at rest [Taylor 1973]. Again, outwardly increasing fields are thus unstable. If (6) is violated, the most unstable mode has azimuthal wave numbers of $m = \pm 1$.

## 2 EQUATIONS

We are interested in the stability of the background field $\mathbf{B}_0 = (0, B_\phi(R), B_0)$, with $B_0 = \text{const}$, and the flow $\mathbf{u}_0 = (0, R\Omega(R), 0)$. The governing equations for the flow $\mathbf{U}$ and the field $\mathbf{B}$ are

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{U} + \frac{1}{\mu_0 \rho} \nabla \times (\mathbf{B} \times \mathbf{B}),$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \Delta \mathbf{B},$$

and $\text{div} \mathbf{U} = \text{div} \mathbf{B} = 0$, where $\rho$ is the pressure, $\nu$ the kinematic viscosity and $\eta$ the magnetic diffusivity. Their ratio is the magnetic Prandtl number

$$Pm = \frac{\nu}{\eta}.$$  

From now on we drop the subscripts from the large-scale values so that the total flow is $\mathbf{U} + \mathbf{u}$ and the total field is $\mathbf{B} + \mathbf{b}$. The stationary background solution is

$$\Omega = a_\Omega + \frac{b_\Omega}{R^2}, \quad B_\phi = a_B R + \frac{b_B}{R},$$

where $a_\Omega, b_\Omega, a_B$ and $b_B$ are constants defined by

$$a_\Omega = \Omega_\text{in} \frac{\mu_\Omega - \tilde{\eta}^2}{1 - \tilde{\eta}^2}, \quad b_\Omega = \Omega_\text{in} R^2 \frac{1 - \mu_\Omega}{1 - \tilde{\eta}^2},$$

$$a_B = \frac{B_\text{in} (\tilde{\eta} (\mu_B - \tilde{\eta}))}{R_\text{in}} \frac{1 - \mu_B}{1 - \tilde{\eta}^2}, \quad b_B = \frac{B_\text{in} R_\text{in} \mu_B}{R_\text{out}} \frac{1 - \mu_B \tilde{\eta}}{1 - \tilde{\eta}^2},$$

with

$$\tilde{\eta} = \frac{R_\text{out}}{R_\text{in}}, \quad \mu_\Omega = \frac{\Omega_\text{out}}{\Omega_\text{in}}, \quad \mu_B = \frac{B_\text{out}}{B_\text{in}}.$$  

$R_\text{in}$ and $R_\text{out}$ are the radii of the inner and outer cylinders, $\Omega_\text{in}$ and $\Omega_\text{out}$ are their rotation rates, and $B_\text{in}$ and $B_\text{out}$ are the azimuthal magnetic fields at the inner and outer cylinders. A field of the form $b_B/R$ is generated by running an axial current only through the inner region $R < R_\text{in}$, whereas a field of the form $a_B R$ is generated by running a uniform axial current through the entire region $R < R_\text{out}$, including the fluid.

Given the $z$-component of the electric current, curl$_z \mathbf{B} = 2a_\Omega$, one finds for the current helicity of the background field $\mathbf{B} \cdot \nabla \times (\mathbf{B} \times \mathbf{B}) = 2a_\Omega B_\text{in} B_0$, which may be positive, negative, or zero.

The inner value $B_\text{in}$ is normalized by the uniform vertical field, i.e.

$$\beta = \frac{B_\text{in}}{B_0}.$$  

For our standard profile $\mu_B = 1$ we have $B_0 \cdot \nabla \times B_0 = 2B_0^2/\mu_\Omega R_\text{in}$ for the helicity of the background field. For fixed toroidal field amplitude this quantity scales as $\beta^{-1}$:

$$B_0 \cdot \nabla \times B_0 = \frac{2}{3\beta} \frac{B_0^2}{R_\text{in}}.$$  

The sign of $\beta$ determines the sign of the current helicity. If the toroidal field is due to the interaction of a poloidal field with a differential rotation with negative shear then $\beta$ is negative and vice versa. Interchanging $\pm \beta$ simply interchanges left and right spirals, $m \rightarrow -m$.

As usual, the toroidal field amplitude is measured by the Hartmann number and the global rotation by the Reynolds number, i.e.

$$Ha = \frac{B_\text{in} D}{\sqrt{\mu_\rho |\mathbf{B}|}}, \quad Re = \frac{\Omega_\text{in} D^2}{\nu}.$$  

$D = R_\text{out} - R_\text{in}$ is used as the unit of length, $\nu/D$ as the unit of velocity and $B_\text{in}$ as the unit of the azimuthal fields. Frequencies, including the rotation $\Omega$, are normalized with the inner rotation rate $\Omega_\text{in}$. The Lundquist number $S$ is defined by $S = \sqrt{Pm} Ha$. The magnetic-diffusion frequency is $\omega_\eta = \eta/\sqrt{D}$ and then the Alfvén frequency $\Omega_A = S \omega_\eta$ is

$$\Omega_A = \frac{B_\text{in}^2}{\mu_\rho D}.$$  

Throughout the whole paper numerical values of helicities are given in units of $\Omega_A^2 D$. In this notation the helicity (14) of the background field can be written as

$$\frac{1}{\mu_\rho} B_0 \cdot \nabla \times B_0 = \frac{2}{3\beta} \Omega_A^2 D D_\text{in} \Omega_A \rightarrow -\frac{\Omega_A^4 D \beta}{3\beta},$$

The boundary conditions associated with the perturbation equations are no-slip for $\mathbf{u}$ and perfectly conducting for $\mathbf{B}$, at both $R = R_\text{in}$ and $R = R_\text{out}$, where we fix $R_\text{out} = 2R_\text{in}$, i.e. $\tilde{\eta} = 0.5$. The computational domain is periodic in $z$. The nonlinear MHD code used for the solution of Eqs. (7) and (8) has been described in detail by Gellert, Rüdiger & Fournier (2007) (see also Fournier et al. 2004, 2005).

## 3 RESULTS

Fig. 1 shows the growth rates for a purely toroidal field ($\beta = \infty$), and no rotation. We see that beyond the critical Lundquist number, the growth rate is essentially linear, i.e.

$$\omega_{kr} \propto \Omega_A,$$

where $Pm = 1$ has the steepest slope, and is thus more unstable than both $Pm < 1$ and $Pm > 1$.

The azimuthal wavenumber of the modes shown in Fig. 1 is $m = \pm 1$. For a purely toroidal field, ±m, corresponding to left- and right-handed spirals, are degenerate, and necessarily have exactly the same growth rate curves. See also Hollerbach, Teeluck & Rüdiger (2009), who obtained the same effect in magnetorotational instabilities, and Rüdiger, Kitchatinov & Elstner (2011a), who consider instabilities of toroidal fields in spheres.

We next consider the nonlinear equilibration of these modes. As Fig. 2 shows, even though $m = \pm 1$ are degenerate, the equilibrated solutions do not consist of equal mixtures of both modes. Instead, either the left or the right mode wins out, and completely
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Figure 1. Growth rate curves for various magnetic Prandtl numbers, for stationary cylinders and a purely toroidal field, with $\mu_B = 1$. The growth rate and the magnetic field are normalized with the magnetic-dissipation frequency $\omega_\eta$.

Figure 2. The left and right spirals that can be excited by a purely toroidal basic state. Apart from having the opposite handedness, the two modes are exactly equivalent; their kinetic helicity is $\pm 6.0 \cdot 10^{-4}$ and their current helicity is $\pm 3.5 \cdot 10^{-3}$ (both in units of $\Omega^2 D_0^2$), where the left spiral is positive and the right spiral negative. These modes do not drift in the azimuthal direction. The parameter values are $\beta = \infty$, $\mu_B = 1$, $\mu_\Omega = 0$, $Pm = 1$. suppresses the other. Which mode one obtains depends on the precise initial conditions. If these already favor one mode, then (not surprisingly) that one wins, but if the initial condition is evenly balanced between the two modes, it is ultimately just numerical noise that determines which mode wins. Eventually though one mode always wins; the solution consisting of an equal mixture of both is unstable.

Spontaneous parity-breaking bifurcations of this type are well known in classical, non-magnetic Taylor-Couette flow (e.g. Hoffmann et al. 2009, Altmeyer et al. 2010 and reference therein), but are almost unknown in magnetohydrodynamic problems. To the best of our knowledge, the only other example is in the very recent work by [Chatterjee et al. 2010]. Given the importance of helicity in mean-field dynamics, any effect that generates helicity from an underlying basic state without helicity could be significant.

Figure 3. The turbulence intensities for flow (top) and field (bottom) fluctuations in units of $D^2 \Omega^2$. Note that once $\beta$ exceeds $\sim 100$ it has virtually no further influence. $\mu_B = 1$, $\mu_\Omega = 0.5$, $Pm = 1$.

We next present two series of solutions where $B_0$ includes an axial component ($\beta < \infty$). In contrast to Figs. [1] and [2] we also include a differential rotation here. The profiles of the basic state field and flow are fixed at $\mu_B = 1$ and $\mu_\Omega = 0.5$. Their amplitudes are $Ha = 100$ and $Re = 200$ for the first series, and $Ha = 200$, $Re = 20$ for the second. The first series is thus rotationally dominated ($\Omega > \Omega_A$), whereas the second is magnetically dominated ($\Omega_A > \Omega$). The astrophysically relevant case is rotationally dominated, which is not the classical realization of the Tayler instability.

We have called this instability the Azimuthal Magnetorotational Instability (AMRI, see Hollerbach, Teeluck & Rüdiger 2009).

For both series of runs Fig. [4] shows the kinetic and magnetic turbulence intensities $\langle u^2 \rangle$ and $\langle b^2 \rangle$. For sufficiently large $\beta$ its influence is very small; the axial component of $B_0$ is then so weak that it has no further influence. This is not true for small $\beta$, where the axial field starts to dominate. For $\beta < 1$ the kink-instability is strongly stabilized [Rüdiger, Schultz & Elstner 2011b] and the resulting energies of the perturbations are reduced.

Fig. [4] shows the kinetic [1] and current [2] helicities for the two series of runs. For both series, both helicities have the opposite sign as $\beta$ (see also Käpylä & Brandenburg 2009 for comparison). In stating this result, it is important though to specify carefully the nature of the initial conditions used in each run. For $\beta = O(1)$, the basic state has a sufficiently strong handedness that it forces the instabilities to have a particular parity as well, which as indicated turns out to be opposite to that of the basic state. If one then gradually increases $\beta$, each time using the previous solution as the new
initial condition, this parity of the instabilities is preserved all way to \( \beta \to \infty \), where the basic state no longer has a handedness, and both left and right instabilities could exist equally well, as in Fig. 2.

That is, by the time one reaches \( \beta = 500 \), say, the basic state makes sufficiently little distinction between left and right modes that both could exist, but because of the way we have reached \( \beta = 500 \), we consistently obtain the right mode. However, suppose one does the following experiment now: Take the right mode at \( \beta = 500 \), swap its parity to be left, and use that as a new initial condition for a series of runs where \( \beta \) is now gradually reduced. Eventually there comes a point where the basic state’s handedness is sufficiently great that it no longer allows the instability to have the ‘wrong’ parity, and the solution reverts back to the right mode. This feature that both left and right modes are allowed for sufficiently large \( \beta \) (where the degeneracy between the two modes is only weakly broken) but not for smaller \( \beta \) (where the degeneracy is strongly broken) is in many ways analogous to an imperfect pitchfork bifurcation.

4 ALPHA-EFFECT AND DYNAMO THEORY

We have also calculated the \( \alpha \)-effect in (3) with the same averaging procedure over the azimuth. Because of the complex structure of the background field it is even possible to determine parts of the tensorial structure of the \( \alpha \)-tensor. In particular we are interested in the signs and amplitudes of the \( \alpha \)-effect in both azimuthal and axial directions. According to the general rule that the azimuthal \( \alpha \)-effect is anticorrelated with the (kinetic) helicity we expect the azimuthal \( \alpha \)-effect to be positive for \( \beta > 0 \). The expected sign of the axial \( \alpha \)-effect is not clear. There are theories and simulations leading to \( \alpha_{\phi\phi} \) and \( \alpha_{zz} \) with opposite signs (see Rüdiger & Hollerbach 2004 for an overview). We should not be surprised to find a similar behavior in the present simulations. It also means that any dynamo with very weak differential rotation cannot be treated with a scalar \( \alpha \)-effect.

Figs. 5 and 6 give the results for slow and rapid rotation. On the basis of Eq. (3) the dimensionless \( \alpha \)-effect in the form

\[
C_{\alpha} = \frac{\alpha D}{\eta}
\]

is plotted for the components \( \alpha_{\phi\phi} \) and \( \alpha_{zz} \). In both cases, these two components have opposite signs, with \( \alpha_{\phi\phi} > 0 \) and \( \alpha_{zz} < 0 \) almost everywhere in the meridional plane. This anti-correlation between the two components is also strongest in the center of the gap, and weakest near the boundaries. It is therefore not caused by the boundaries.
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That Figs. 5 and 6 show such similar results is surprising, and is one of the basic results of this paper. The influence of rotation on \( \alpha \) is evidently rather weak. The fact that – contrary to previous results for rotating convection – \( \alpha_{\phi \phi} \) is actually smaller for rapid rotation than for slow rotation illustrates just how different these magnetic-induced helicities are from some of the previous results. Finally, Fig. 7 shows how the amplitudes of \( \alpha_{\phi \phi} \) vary with \( \beta \), being roughly inversely proportional in both cases.

To consider some possible astrophysical implications of these results, imagine a disk dynamo with dominant field components \( B_\phi \) and \( B_R \). Dynamo waves of \( \alpha \Omega \)-type require for self-excitation that the product of (19) and

\[
C_R = - \frac{D^3}{\eta} \frac{d\Omega}{dR}
\]

exceeds a critical value of order unity, i.e. \( C_\alpha C_\Omega \gtrsim 1 \). The ratio of the amplitudes of the field components \( B_\phi \) and \( B_R \) follows the simple rule

\[
\frac{|B_\phi|}{|B_R|} \approx \sqrt{\frac{C_R}{C_\alpha}}
\]

so that dynamo excitation requires

\[
\frac{|B_\phi|}{|B_R|} C_\alpha \gtrsim 1.
\]

We know from Fig. 7 that \( C_\alpha \gtrsim C/\beta \) with \( C \ll 1 \), so that (22) gives, at least as an order-of-magnitude estimate, the condition

\[
C > \frac{|B_\phi|}{|B_z|}
\]

for self-excitation of a dynamo with differential rotation and current-driven \( \alpha \)-effect. For disk dynamos \( B_R \) dominates \( B_\phi \), and for spherical dynamos \( B_R \) is comparable to \( B_\phi \). In both cases the condition for self-excitation becomes \( C > 1 \), which cannot be fulfilled according to Fig. 7 which suggests instead that \( C \lesssim 0.05 \). The \( \alpha \)-effect due to the current helicity of the background field appears as much too small to allow the operation of an \( \alpha \Omega \)-dynamo.

Another argument concerns the growth rate of such a dynamo (if it exists at all) in relation to the very long magnetic diffusion times in radiative zones. Assume that for self-excitation \( C_\alpha C_\Omega > 1 \), then the growth rate \( \omega_{gr} \) is given by

\[
\omega_{gr} = \frac{\eta}{D^2} \sqrt{C_\alpha C_\Omega} \approx \frac{\eta}{D^2} C \frac{|B_\phi|}{|B_R|},
\]

Hence, only for \( C > 1 \) the growth time of the dynamo would be shorter than the magnetic diffusion time \( D^2/\eta \), which is known to be of order Gyr for the radiative interior of stars. One can also argue as follows. The relation (24) also reads

\[
\omega_{gr} \approx \sqrt{\alpha \Omega}
\]

independent of the magnetic diffusivity. On the other hand, for given \( C_\alpha \) (25) states

\[
\omega_{gr} \approx \frac{1}{D} \sqrt{C_\alpha \eta \Omega},
\]

which for the computed value \( C_\alpha \approx 0.01 \) taken from Fig. 7 and \( \eta \approx 500 \, \text{cm}^2/\text{s} \) for the solar core leads to values of order \( 10^{-15} \, \text{s}^{-1} \), i.e. to growth times of order 10 Myr. As it is typical for \( \alpha \Omega \)-dynamos their growth times are only slightly shorter than the basic magnetic decay time.

5 SUMMARY

We have shown that the current-driven instability of helical large-scale fields does produce small-scale helicity (kinetic plus current helicity) and even \( \alpha \)-effects, but the resulting numerical values...
seem to be too small for the operation of large-scale dynamos in radiative zones of early-type stars.

REFERENCES

Altmeyer S., Hoffmann C., Heise M., Abshagen J., Pinter A., Lücke M., Pfister G., 2010, Phys. Rev. E, 81, 066313
Chatterjee P., Mitra D., Brandenburg A., Rheinhardt M., 2010, PRL, submitted, astro-ph/1011.1251
Elsasser W.M., 1946, Phys. Rev., 69, 106
Fournier A., Bunge H.-P., Hollerbach R., Vilotte J.-P., 2004, Geophys. J. Int., 156, 682
Fournier A., Bunge H.-P., Hollerbach R., Vilotte J.-P., 2005, J. Comp. Phys., 204, 462
Gellert M., Rüdiger G., Fournier A., 2007, Astron. Nachr., 328, 1162
Gellert M., Rüdiger G., Elstner D., 2008, A&A, 479, L33
Hoffmann C., Heise M., Altmeyer S., Abshagen J., Pinter A., Pfister G., Lücke M., 2009, Phys. Rev. E, 80, 066308
Hollerbach R., Teeluck V., Rüdiger G., 2010, Phys. Rev. Lett., 104, 44502
Käpylä P., Brandenburg A., 2009, ApJ, 699, 1059
Michael D., 1954, Mathematica, 1, 45
Rüdiger G., Hollerbach R., 2004, The Magnetic Universe, Wiley, Berlin
Rüdiger G., Kitchatinov L.L., 2006, Astron. Nachr., 327, 298
Rüdiger G., Kitchatinov L.L., Elstner D., 2011a, MNRAS, in preparation
Rüdiger G., Schultz M., Elstner D., 2011b, A&A, submitted
Tayler R.J., 1973, MNRAS, 161, 365