Excessive backlog probabilities of two parallel queues

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Abstract
Let $X$ be the constrained random walk on $\mathbb{Z}^2_+$ with increments $(1, 0), (-1, 0), (0, 1)$ and $(0, -1)$; $X$ represents, at arrivals and service completions, the lengths of two queues (or two stacks in computer science applications) working in parallel whose service and interarrival times are exponentially distributed with arrival rates $\lambda_i$ and service rates $\mu_i$, $i = 1, 2$; we assume $\lambda_i < \mu_i$, $i = 1, 2$, i.e., $X$ is assumed stable. Without loss of generality we assume $\rho_1 = \lambda_1/\mu_1 \geq \rho_2 = \lambda_2/\mu_2$. Let $\tau_n$ be the first time $X$ hits the line $\partial A_n = \{x \in \mathbb{Z}^2 : x(1) + x(2) = n\}$, i.e., when the sum of the components of $X$ equals $n$ for the first time. Let $Y$ be the same random walk as $X$ but only constrained on $\{y \in \mathbb{Z}^2 : y(2) = 0\}$ and its jump probabilities for the first component reversed. Let $\partial B = \{y \in \mathbb{Z}^2 : y(1) = y(2)\}$ and let $\tau$ be the first time $Y$ hits $\partial B$. The probability $p_n = P_x(\tau_n < \tau_0)$ is a key performance measure of the queueing system (or the two stacks) represented by $X$ (if the queues/stacks share a common buffer, then $p_n$ is the probability that this buffer overflows during the system’s first busy cycle). Stability of the process implies that $p_n$ decays exponentially in $n$ when the process starts off the exit boundary $\partial A_n$. We show that, for $x_n = \lfloor nx \rfloor$, $x \in \mathbb{R}^2_+$, $x(1) + x(2) \leq 1$, $x(1) > 0$, $P_{(n-x_n(1), x_n(2))}(\tau < \infty)$ approximates $P_{x_n}(\tau_n < \tau_0)$ with exponentially vanishing relative error. Let $r = (\lambda_1 + \lambda_2)/(\mu_1 + \mu_2)$; for $r^2 < \rho_2$ and $\rho_1 \neq \rho_2$, we construct a class of harmonic functions from single and conjugate points on a related characteristic surface for $Y$ with which the probability $P_y(\tau < \infty)$ can be approximated with bounded relative error. For $r^2 = \rho_1 \rho_2$, we obtain the exact formula $P_y(\tau < \infty) = r^{y(1)-y(2)} + \frac{r(1-r)}{r^2 - \rho_2} \left( \rho_1^{y(1)} - r^{y(1)-y(2)} \rho_1^{y(2)} \right)$.

Keywords  Approximation of probabilities of rare events · Exit probabilities · Constrained random walks · Queueing systems · Large deviations

1 Introduction

This work concerns the random walk $X$ with independent and identically distributed increments $\{I_1, I_2, I_3, \ldots\}$, constrained to remain in $\mathbb{Z}^2_+$:
$X_0 = x \in \mathbb{Z}_+^2, \quad X_{k+1} = X_k + \pi(X_k, I_k), \; k = 1, 2, 3, \ldots$

\[ \pi(x, v) = \begin{cases} 
  v, & \text{if } x + v \in \mathbb{Z}_+^2, \\
  0, & \text{otherwise},
\end{cases} \]

$I_k \in \mathcal{V} \doteq \{(1, 0), (-1, 0), (0, 1), (0, -1)\}, \; P(I_k = (1, 0)) = \lambda_1, \; P(I_k = (0, 1)) = \lambda_2, \; P(I_k = (-1, 0)) = \mu_1, \; P(I_k = (0, -1)) = \mu_2.$

(1)

The dynamics of $X$ are depicted in Fig. 1. We denote the constraining boundaries by $\partial_i \doteq \{x \in \mathbb{Z}_+^2 : x(i) = 0\}, \; i = 1, 2.$ A well known interpretation for $X$ is as the embedded random walk of two parallel queues with Poisson arrivals and independent and exponentially distributed service times. This random walk appears in computer science as a model of two stacks running on the same memory (Knuth 1972; Yao 1981; Flajolet 1986; Maier 1991); these works assume the walk to be unstable or assume that the jump probabilities are the same for both queues/stacks. We comment further on this literature in Sect. 9. Define the region

$A_n = \{x \in \mathbb{Z}_+^2 : x(1) + x(2) \leq n\}$

and its boundary

$\partial A_n = \{x \in \mathbb{Z}_+^2 : x(1) + x(2) = n\}.$

(2)

(3)

Let $\tau_n$ be the first time $X$ hits $\partial A_n$:

$\tau_n \doteq \inf\{k : X_k \in \partial A_n\}.$

(4)

When $X$ is stable, i.e., when $\lambda_i < \mu_i$, a well known performance measure associated with $X$ is the probability

$p_n(x) \doteq P_x(\tau_n < \tau_0), \; x \in A_n.$

Fig. 1 Dynamics of $X$
If the queues share a common buffer, then \( p_n \) is the probability that this buffer overflows during the system’s first busy cycle. The stability assumption implies that \( p_n \) decays exponentially in \( n \), when the walk starts away from the exit boundary \( \partial A_n \). Hence \( \{ \tau_n < \tau_0 \} \) is a rare event for \( n \) large. There is no closed form formula for \( p_n \) in terms of the parameters of the problem. The goal of the present work is to construct approximations of \( p_n \) whose relative error decay exponentially in \( n \) when the initial position \( x \) of the random walk is off the boundary \( \partial_1 \).

In this we will use and extend the approach developed in Sezer (2015, 2018), which treats the two dimensional constrained random walk representing two queues in tandem, i.e., the constrained random walk with increments \((1, 0), (-1, 1) \) and \((0, -1)\).

Results currently available in the literature on the approximation of \( p_n \) are as follows. The works (Glasserman and Kou 1995; Ignatiouk-Robert 2000) compute the large deviations (LD) limit of \( p_n(x) \), for \( x = x_n, x_n/n \to 0 \), as

\[
\lim_{n \to \infty} \frac{1}{n} \log p_n(x_n) = \min(-\log \rho_1, -\log \rho_2),
\]

where \( \rho_i = \lambda_i/\mu_i \). Because \( p_n \) is a rare event probability, a natural idea is to estimate it by simulation with variance reduction. To the best of our knowledge, the article (Parekh and Walrand 1989) is the first to study the optimal importance sampling (IS) simulation of \( p_n \) for two tandem queues and the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. To tackle this problem (Parekh and Walrand 1989) introduced dynamic IS measures that depend on the position of \( X \). It was observed in Glasserman and Kou (1995) that static changes of measure implied by LD analysis can perform poorly for the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. To tackle this problem (Parekh and Walrand 1989) introduced dynamic IS measures that depend on the position of \( X \). It was observed in Glasserman and Kou (1995) that static changes of measure implied by LD analysis can perform poorly for the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. To tackle this problem (Parekh and Walrand 1989) introduced dynamic IS measures that depend on the position of \( X \). It was observed in Glasserman and Kou (1995) that static changes of measure implied by LD analysis can perform poorly for the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. To tackle this problem (Parekh and Walrand 1989) introduced dynamic IS measures that depend on the position of \( X \). It was observed in Glasserman and Kou (1995) that static changes of measure implied by LD analysis can perform poorly for the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. To tackle this problem (Parekh and Walrand 1989) introduced dynamic IS measures that depend on the position of \( X \). It was observed in Glasserman and Kou (1995) that static changes of measure implied by LD analysis can perform poorly for the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. To tackle this problem (Parekh and Walrand 1989) introduced dynamic IS measures that depend on the position of \( X \). It was observed in Glasserman and Kou (1995) that static changes of measure implied by LD analysis can perform poorly for the exit boundary \( \partial A_n \); Parekh and Walrand (1989) observes that static changes of measure implied by optimal LD sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process.
As already noted, the goal of the present work is to construct approximations of $p_n$ whose relative error decay exponentially in $n$ when the initial position $x$ of the random walk is off the boundary $\partial_1$. The approximation technique and the proof method are reviewed below (see Sect. 1.1). Although the main approach of the present work is parallel to that of Sezer (2015, 2018), many new challenges and ideas appear in the treatment of the present case; there are also significant differences in the assumptions made and the results obtained. These are discussed below and in Sect. 8.

First, several definitions; the utilization rates of the nodes are:

$$\rho_i = \frac{\lambda_i}{\mu_i}, \; i = 1, 2.$$  

We assume that $X$ is stable, i.e.,

$$\rho_1, \rho_2 < 1.$$  

The following quantity plays a central role in our analysis:

$$r = \frac{\lambda_1 + \lambda_2}{\mu_1 + \mu_2}.$$  

Without loss of generality we can assume

$$\rho_2 \leq r \leq \rho_1$$  

(if this doesn’t hold, rename the nodes). We will make two further technical assumptions:

$$\rho_1 \neq \rho_2, \frac{r^2}{\rho_2} < 1.$$  

The first of these is needed in the construction of the $Y$-harmonic functions in Sect. 2, see (17). The second is useful both in the computation of $P_Y(\tau < \infty)$ (see the proof of Proposition 7.2) and in the limit analysis (see the proof of Proposition 3.3). We further comment on these assumptions in Sect. 8 and in the Conclusion (Sect. 9).

Define the linear transformation

$$I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$  

and the affine transformation

$$T_n = ne_1 + I$$

where $(e_1, e_2)$ is the standard basis for $\mathbb{R}^2$. Furthermore, define the constraining map

$$\pi_1(x, y) = \begin{cases} 
  y, & \text{if } x + y \in \mathbb{Z} \times \mathbb{Z}_+; \\
  0, & \text{otherwise}.
\end{cases}$$

Define $Y$ to be a constrained random walk on $\mathbb{Z} \times \mathbb{Z}_+$ with increments

$$J_k \triangleq I I_k : \quad Y_{k+1} = Y_k + \pi_1(Y_k, J_k).$$  

(7)
Y has the same increments as X, but the probabilities of the increments \( e_1 \) and \(-e_1\) are reversed. Define

\[
B \subset \mathbb{Z} \times \mathbb{Z}^+, \quad B = \{ y : y(1) \geq y(2) \}, \quad \partial B \subset \mathbb{Z} \times \mathbb{Z}^+, \quad \partial B = \{ y : y(1) = y(2) \},
\]

and the hitting time \( \tau = \inf \{ k : Y_k \in \partial B \} \).

1.1 Summary of our analysis

Sezer (2015, Proposition 3.1) asserts, in a more general framework than the model given above, that for any \( y \in \mathbb{Z}^2_+ \), \( y(1) > y(2) \), \( P_{T_n(y)}(\tau_n < \tau_0) \rightarrow P_Y(\tau < \infty) \). The approximation idea connecting these two probabilities is shown in Fig. 2: by applying \( T_n \), we move the origin of the coordinate system to \((n, 0)\) and take limits, which leads to the limit problem of computing \( P_Y(\tau < \infty) \) where the limit \( Y \) process is the same process as \( X \) (observed from the point \((n, 0)\)) but not constrained on \( \partial_1 \). A more interesting convergence analysis is when the initial point is given in \( x \) coordinates. A convergence analysis from this point of view has only been performed so far for the constrained random walk with increments \((1, 0)\), \((-1, 1)\) and \((0, -1)\) (representing two tandem queues) in Sezer (2015, 2018). Our first main result extends this analysis to the two dimensional simple random walk:

**Theorem** (Theorem 6.1) For any \( x \in \mathbb{R}^2_+, x(1) + x(2) < 1, x(1) > 0 \), there exists \( C > 0 \) and \( N > 0 \) such that

\[
\frac{|P_{X_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)|}{P_{X_n}(\tau_n < \tau_0)} < e^{-C \gamma n}
\]

for \( n > N \), where \( x_n = \lfloor xn \rfloor \).

Thus, as \( n \) increases \( P_{T_n(x_n)}(\tau < \infty) \) approximates \( P_{X_n}(\tau_n < \tau_0) \) very well (with exponentially decaying relative error in \( n \)) if \( x(1) > 0 \).

The computation of \( P_Y(\tau < \infty) \) is treated in Sect. 7. In the tandem case treated in Sezer (2018) there is a simple explicit formula for \( P_Y(\tau < \infty) \). In the present case, there is an explicit formula only under the assumption

\[
\rho_1 \rho_2 = r^2.
\]

The formula for \( P_Y(\tau < \infty) \) under this condition is

\[
P_Y(\tau < \infty) = r^{y(1)-y(2)} + \frac{(1-r)r}{r - \rho_1} \left( \rho_1^{y(1)} - r^{y(1)-y(2)} \rho_1^{y(2)} \right).
\]
The main results of Sect. 7 are: (1) Proposition 7.2 which derives (10) under assumption (9), (2) Proposition 7.3 which proves that one can approximate $P_y(\tau < \infty)$ with bounded relative error using two explicit $Y$-harmonic functions when (9) doesn’t hold and (3) Proposition 7.6 which derives an upper bound on the relative error of the approximation of $P_y(\tau < \infty)$ by a linear combination of complex valued $Y$-harmonic functions based on the values the latter takes on $\partial B$. Section 7.1 demonstrates how one can construct improved approximations by increasing the number of $Y$-harmonic functions used in the approximation.

Define the stopping times

$$
\sigma_1 = \inf \{ k : X_k \in \partial_1 \}, \quad \bar{\sigma}_1 = \inf \{ k : T_n(Y_k) \in \partial_1 \}.
$$

If we set the initial position of $Y$ to $Y_0 = T_n(X_0)$, we have

$$
\{\tau_n < \tau_0\} \cap \{\tau_n < \sigma_1 \land \tau_0\} = \{\tau < \infty\} \cap \{\tau < \bar{\sigma}_1 < \infty\}.
$$

The main argument in the proof of Theorem 6.1 is that most of the probability of the events $\{\tau_n < \tau_0\}$ and $\{\tau < \infty\}$ come from the events $\{\tau_n < \sigma_1 \land \tau_0\}$ and $\{\tau < \bar{\sigma}_1 < \infty\}$ respectively, if the initial position $X_0$ of $X$ is away from $\partial_1$. The full implementation of this argument will require the following steps:

1. Construction of $Y$-harmonic functions, $Y - z$-harmonic functions and bounds on $E_y[z^{\tau_1}1_{[\tau < \infty]}]$ for $z > 1$ (Sects. 2, 3),
2. Large deviations (LD) analysis of $P_{x_n}(\tau_n < \tau_0)$ (Sect. 4),
3. LD analysis of $P_{x_n}(\sigma_1 < \tau_n < \tau_0)$ (Sect. 5),
4. LD analysis of $P_{x_n}(\bar{\sigma}_1 < \tau < \infty)$ (Sect. 5.1).

These steps are put together in Sect. 6.

The existence $z_0 > 1$, such that $E_y[z_0^{\tau_1}1_{[\tau < \infty]}] < \infty$ uniformly in $y$ (first step above) is needed in truncating time in the analysis of the probability $P_x(\bar{\sigma}_1 < \tau < \infty)$ in Sect. 5.1. Section 3 proves the existence of such $z_0 > 1$ based on explicit constructions of what we call $Y - z$-harmonic functions using points on $1/z$-level curves of the characteristic surface; all of these are novel aspects of the problem as compared to the tandem case treated in Sezer (2018). We are also not aware of previous works using these structures and this approach. Section 4.2 uses subharmonic functions constructed from maximums of power functions (where the base numbers come from components of points on the characteristic surface) to find an explicit upperbound on the probability $P_x(\tau_n < \tau_0)$; although the approach seems natural, we are not aware of previous works using this explicit construction of subharmonic functions for constrained processes. Section 8 gives a further comparison of the analysis of the constrained walk $X$ treated in the present work and the tandem walk treated in Sezer (2015, 2018). We comment on future work in the Conclusion (Sect. 9).

## 2 Harmonic functions of $Y$

A function $h$ on $\mathbb{Z} \times \mathbb{Z}_+$ is said to be $Y$-harmonic if

$$
E_y[h(Y_1)] = h(y), \; y \in \mathbb{Z} \times \mathbb{Z}_+.
$$

Following Sezer (2015), introduce the the interior characteristic polynomial of $Y$:

$$
p(\beta, \alpha) \triangleq \lambda_1 \frac{1}{\beta} + \mu_1 \beta + \lambda_2 \frac{\alpha}{\beta} + \mu_2 \frac{\beta}{\alpha}.
$$
Fig. 3 The real section of the characteristic surface \( \mathcal{H} \) for \( \lambda_1 = 0.15, \lambda_2 = 0.2, \mu_1 = 0.25, \mu_2 = 0.4 \); the end points of the dashed line are an example of a pair of conjugate points \((\beta, \alpha_1)\) and \((\beta, \alpha_2)\); Each such pair defines a \( Y \)-harmonic function, see Proposition 2.2.

and characteristic polynomial of \( Y \) on \( \partial_2 \):

\[
p_2(\beta, \alpha) = \lambda_1 \frac{1}{\beta} + \mu_1 \beta + \lambda_2 \frac{\alpha}{\beta} + \mu_2
\]

As in Sezer (2015), we will construct \( Y \)-harmonic functions from solutions of \( p = 1 \); the set of all solutions of this equation defines the characteristic surface

\[
\mathcal{H} \doteq \{ (\beta, \alpha) \in \mathbb{C}^2 : p(\beta, \alpha) = 1 \}
\]

define, similarly, the characteristic surface for \( \partial_2 \):

\[
\mathcal{H}_2 \doteq \{ (\beta, \alpha) \in \mathbb{C}^2 : p_2(\beta, \alpha) = 1 \}
\]

Multiplying both sides of \( p = 1 \) by \( \alpha \) transforms it to the quadratic equation

\[
\alpha \left( \lambda_1 \frac{1}{\beta} + \mu_1 \beta - 1 \right) + \lambda_2 \frac{\alpha^2}{\beta} + \mu_2 \beta = 0 \tag{12}
\]

Define

\[
\alpha(\beta, \alpha) \doteq \frac{1}{\beta^2} \frac{\beta^2}{\alpha^2} \tag{13}
\]

if for a fixed \( \beta, \alpha_1 \) and \( \alpha_2 \) are distinct roots of (12), they will satisfy

\[
\alpha_2 = \alpha(\beta, \alpha_1)
\]

by simple algebra; we will call the points \((\beta, \alpha_1) \in \mathcal{H} \) and \((\beta, \alpha_2) \in \mathcal{H} \) arising from such roots conjugate. Following Sezer (2015) we refer to the function \( \alpha \) as the conjugator. An example of two conjugate points are shown in Fig. 3.

For any point \((\beta, \alpha) \in \mathcal{H} \) define the following \( \mathbb{C} \)-valued function on \( \mathbb{Z}^2 \):

\[
z \mapsto [(\beta, \alpha), z], z \in \mathbb{Z}^2,
\]

\[
[(\beta, \alpha), z] \doteq \beta^{z(1)-z(2)} \alpha^{z(2)}.
\]

**Lemma 1** \([(\beta, \alpha), \cdot \] is \( Y \)-harmonic on \( \mathbb{Z} \times \mathbb{Z}_+ - \partial_2 \) when \((\beta, \alpha) \in \mathcal{H} \). In addition \( x \mapsto [(\beta, \alpha), T_n(x)], x \in \mathbb{Z}_+^2, \) is \( X \)-harmonic on \( \mathbb{Z}_+^2 - \partial_1 \cup \partial_2 \).
Proof As in Sezer (2015), the first claim follows from the definitions involved:

\[ E(z((\beta, \alpha), Z_1)) = \beta^{z(1)-z(2)} \alpha^{z_2} p(\beta, \alpha) = ((\beta, \alpha), z). \]

and the second claim follows from the first and the fact that \( J_k = I_k \) [see (7)]. \( \square \)

Proceeding parallel to Sezer (2015), one can define the following class of \( Y \)-harmonic functions from the functions \( (\beta, \alpha) \cdot \cdot \cdot \):

Proposition 2.1 Suppose \((\beta, \alpha) \in \mathcal{H} \cap \mathcal{H}_2\). Then \( (\beta, \alpha) \cdot \cdot \cdot \) is \( Y \)-harmonic.

Proof Lemma 1 says that for \((\beta, \alpha) \in \mathcal{H} \), \( (\beta, \alpha) \cdot \cdot \cdot \) is \( Y \)-harmonic on \( \mathbb{Z} \times \mathbb{Z}^+ - \partial_2 \). An argument parallel to the one given in the proof of Lemma 1 shows that \( (\beta, \alpha) \cdot \cdot \cdot \) is \( Y \)-harmonic on \( \partial_2 \) when \((\beta, \alpha) \in \mathcal{H}_2\). These two facts imply the statement of the proposition. \( \square \)

Define

\[ C(\beta, \alpha) = \left(1 - \frac{\beta}{\alpha}\right), (\beta, \alpha) \in \mathbb{C}^2, \alpha \neq 0. \]

The next proposition gives us another class of \( Y \)-harmonic functions constructed from conjugate points on \( \mathcal{H} \), it is a special case of Sezer (2015, Proposition 4.9):

Proposition 2.2 Suppose \((\beta, \alpha_1) \neq (\beta, \alpha_2)\), are conjugate points on \( \mathcal{H} \). Then

\[ h_\beta \doteq C(\beta, \alpha_2)((\beta, \alpha_1), \cdot) - C(\beta, \alpha_1)((\beta, \alpha_2), \cdot) \]

is \( Y \)-harmonic.

For sake of completeness and easy reference, let us reproduce the argument given in the proof of Sezer 2015, Proposition 4.9:

Proof That \( h_\beta \) is \( Y \)-harmonic on \( \mathbb{Z} \times \mathbb{Z}^+ - \partial_2 \) follows from Lemma 1. For \( y \in \partial_2 \) a direct computation gives:

\[ E_y[((\beta, \alpha_i), y + \pi_1(y, J_1))] - ((\beta, \alpha_i), y) = \mu_2 C(\beta, \alpha_i)\beta^{y(1)}. \]

It follows that

\[ E_y[h_\beta(y + \pi_1(y, J_1))] - h_\beta(y) = \mu_2 C(\beta, \alpha_1)C(\beta, \alpha_2)(\beta^{y(1)} - \beta^{y(1)}) = 0, \]

i.e., \( h_\beta \) is \( Y \)-harmonic on \( \partial_2 \) as well. \( \square \)

The intersection of \( \mathcal{H} \) and \( \mathcal{H}_2 \) consists of the points \((0, 0), (1, 1)\) and \((\rho_1, \rho_1)\) (the function \( p \) is singular at \((0, 0)\); to realize \((0, 0)\) as a point on \( \mathcal{H} \), rewrite \( \mathcal{H} \) as the 0 level set of the polynomial \( \beta \alpha \beta - \beta \alpha \); similar comments apply to \( p_2 \) and \( \mathcal{H}_2 \). The point \((0, 0)\) corresponds to the harmonic function that is identically 0.) The last of these gives us our first nontrivial loglinear \( Y \)-harmonic function:

Lemma 2 \( (\rho_1, \rho_1), \cdot \cdot \cdot \) is \( Y \)-harmonic.

The proof follows from Proposition 2.1 and the fact that \((\rho_1, \rho_1) \in \mathcal{H} \cap \mathcal{H}_2\).

Fixing \( \beta \in \mathbb{C} \) and solving (12) gives us the two conjugate points corresponding to \( \beta \). It is also natural to start the computation from a fixed \( \alpha \) and find its \( \beta \) and its conjugate. For this, one rewrites \( p = 1 \), now as a polynomial in \( \beta \):

\[ \left(\mu_1 + \frac{\mu_2}{\alpha}\right) \beta^2 - \beta + \lambda_1 + \lambda_2 \alpha = 0. \]
For $\alpha$ fixed, the roots of (14) are
\[
\beta_1(\alpha) = \frac{1 - \sqrt{\Delta(\alpha)}}{2(\frac{\mu_2}{\alpha} + \mu_1)}, \quad \beta_2(\alpha) = \frac{1 + \sqrt{\Delta(\alpha)}}{2(\frac{\mu_2}{\alpha} + \mu_1)},
\]
where
\[
\Delta(\alpha) = 1 - 4\left(\frac{\mu_2/\alpha + \mu_1}{\alpha}\right)(\lambda_1 + \lambda_2\alpha),
\]
and for $z \in \mathbb{C}$, $\sqrt{z}$ is the square root of $z$ satisfying $\Re(\sqrt{z}) \geq 0$.

The function $y \mapsto Py(\tau < \infty)$ takes the value 1 on $\partial B$; a function of the form $[(\beta, \alpha), \cdot]$ takes also the value 1 on $\partial B$. Therefore among the functions of the form $[(\beta, \alpha), \cdot]$, that for which $\alpha = 1$ plays a key role in the computation/approximation of $Py(\tau < \infty)$; in such a function we solve (14) with $\alpha = 1$. The roots (15) for $\alpha = 1$ are
\[
\beta_1 = r, \quad \beta_2 = 1.
\]
That $r \leq \rho_1 < 1$ implies $C(r, 1) = (1 - r) \neq 0$. The assumption $\rho_1 \neq \rho_2$ implies
\[
C(r, \alpha(r, 1)) = 1 - \rho_2/r \neq 0.
\]
Therefore, by Proposition 2.2, the root $\beta_1 = r$ above defines the $Y$-harmonic function
\[
h_r = C(r, \alpha(r, 1))[(r, 1), \cdot] - C(r, 1)[(r, \alpha(r, 1)), \cdot],
\]
\[
= (1 - \rho_2/r)[(r, 1), \cdot] - (1 - r)[(r, r^2/\rho_2), \cdot].
\]
For this function to be useful in our analysis, we need $r^2/\rho_2 < 1$ (see Proposition 7.1), therefore, we assume:
\[
\frac{r^2}{\rho_2} < 1. \tag{16}
\]
Finally, the following scalar multiple of $h_r$ is frequently used in the calculations, therefore, we will denote it in bold thus:
\[
h_r = \frac{1}{1 - \rho_2/r}h_r = [(r, 1), \cdot] - \frac{1 - r}{1 - \rho_2/r}[(r, r^2/\rho_2), \cdot]; \tag{17}
\]
the assumption $\rho_1 \neq \rho_2$ ensures that the denominator $1 - \rho_2/r$ is nonzero.

### 3 Moment generating function of $\tau$

To bound approximation errors we will have to argue that we can truncate time without losing much probability. For this, it will be useful to know that there exists $z > 1$ such that
\[
\mathbb{E}_Y \left[z^\tau 1_{[\tau < \infty]}\right] < \infty, \tag{18}
\]
note that the expectation on the left is the moment generating function of $\tau$ evaluated at $\log(z) > 0$. In Sezer (2015), Dupuis et al. (2007b), bounds similar to this are obtained using large deviations arguments, which are based on the ergodicity of the underlying chain. In Setayeshgar and Wang (2013), again a similar bound is obtained invoking the geometric ergodicity of the underlying process. The process underlying (18) is not stationary. For this reason, these arguments do not immediately generalize to the analysis of (18). To prove the existence of $z > 1$ such that (18) holds, we will extend the characteristic surface an additional
dimension to include a new parameter; points on the generalized surface will correspond to
discounted (in our case we are in fact interested in inflated costs) expected cost functions of
the process $Y$, i.e., points on this surface will give us functions of the form

$$E_y \left[ z^r g(\tau) 1_{\{\tau < \infty\}} \right].$$

We will use these functions to find our desired $z$.

### 3.1 1/z-Level characteristic surfaces and $Y$-z-harmonic functions

The development in this subsection is parallel to Sect. 2 with an additional variable $z \in \mathbb{C}$. A function $h$ on $\mathbb{Z} \times \mathbb{Z}_+$ is said to be $Y$-z-harmonic if

$$zE_y[h(Y_1)] = h(y), \ y \in \mathbb{Z} \times \mathbb{Z}_+.$$

As before, let $p$ denote the characteristic polynomial of $Y$; the set of all solutions of the equation $z p = 1$ defines the 1/z-level characteristic surface

$$\mathcal{H}_z \doteq \{(\beta, \alpha) \in \mathbb{C}^2 : z p(\beta, \alpha) = 1\}.$$

Similarly, define

$$\mathcal{H}_z^2 \doteq \{(\beta, \alpha) \in \mathbb{C}^2 : z p_2(\beta, \alpha) = 1\},$$

the 1/z-level characteristic surface on $\partial_2$. These surfaces reduce to the ordinary characteristic surfaces when $z = 1$.

Multiplying both sides of $z p = 1$ by $\frac{\beta}{z}$ transforms it to the quadratic (in $\alpha$) equation

$$\alpha \left( \frac{1}{\beta} + \frac{\mu_1 \beta - 1}{z} \right) + \alpha^2 \frac{\lambda_2}{\beta} + \mu_2 \beta = 0,$$

whose discriminant is

$$\Delta_z(\beta) \doteq \left( \frac{1}{\beta} + \frac{\mu_1 \beta - 1}{z} \right)^2 - 4\lambda_2 \mu_2.$$

Let $\alpha$ be the conjugator defined in (13). If $(\beta, \alpha_1) \in \mathcal{H}_z$ and $\alpha \neq 0$ then $(\beta, \alpha_2) \in \mathcal{H}_z$ for $\alpha_2 = \alpha(\beta, \alpha_1)$; if $\Delta_z(\beta) \neq 0$, $(\beta, \alpha_1)$ and $(\beta, \alpha_2)$ will be distinct points on $\mathcal{H}_z$ and we will call them conjugate.

**Lemma 3** $[(\beta, \alpha), \cdot]$ is $Y$-z-harmonic on $\mathbb{Z} \times \mathbb{Z}_+ - \partial_2$ when $(\beta, \alpha) \in \mathcal{H}_z$. In addition $x \mapsto [(\beta, \alpha), T_n(x)]$, $x \in \mathbb{Z}_+^2$, is $X$-z-harmonic on $\mathbb{Z}_+^2 - \partial_1 \cup \partial_2$.

The proof is parallel to that of Lemma 1 and follows from the definitions.

Define

$$C_z(\beta, \alpha) \doteq z \left( 1 - \frac{\beta}{\alpha} \right), \ (\beta, \alpha) \in \mathbb{C}^2, \ \alpha \neq 0.$$

Parallel to Sect. 2, the above definitions give us the following class of $Y$-z-harmonic functions;

**Proposition 3.1** Suppose $(\beta, \alpha) \in \mathcal{H}_z \cap \mathcal{H}_z^2$. Then $[(\beta, \alpha), \cdot]$ is $Y$-z-harmonic.

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Proposition 3.2 Suppose \((\beta, \alpha_1) \neq (\beta, \alpha_2)\), are conjugate points on \(\mathcal{H}^z\). Then

\[
h_{z, \beta} \doteq C_z(\beta, \alpha_2)((\beta, \alpha_1), \cdot) - C_z(\beta, \alpha_1)((\beta, \alpha_2), \cdot)
\]

(19)
is \(Y - z\)-harmonic.

The proofs are the same as those of the corresponding Propositions 2.1 and 2.2 of the previous section.

3.2 Bounds on the moment generating function of \(\tau\)

We next use the \(Y - z\)-harmonic functions constructed in Propositions 3.1 and 3.2 to get our existence result.

Proposition 3.3 There exist \(z_0 > 1\) and \(C_1\) such that

\[
\mathbb{E}_y[z_0^1_{\tau < \infty}] < C_1
\]

(20)

for all \(y \in B\).

Recall from (8) that \(B = \{y \in \mathbb{Z} \times \mathbb{Z}_+, y(1) \geq y(2)\}\).

**Proof** Let us first prove the following: if we can find, for some \(z_0 > 1\) and \(C_1 > 0\), a \(Y-z_0\)-harmonic function \(h\) satisfying \(h(y) \geq 1\) on \(\partial B\) and \(C_1 > h \geq 0\) on \(B\) we are done. The reason is as follows: that \(h\) is \(Y-z_0\)-harmonic and the option sampling theorem imply that \(h(\tau \wedge n)z_0^{\tau \wedge n}\) is a martingale. It follows that

\[
h(y) = \mathbb{E}_y[h(\tau \wedge n)z_0^{\tau \wedge n}],
\]

for \(y \in B\). Decompose the last expectation to \(\{\tau \leq n\}\) and \(\{\tau > n\}\):

\[
h(y) = \mathbb{E}_y[h(\tau)z_0^\tau 1_{\{\tau \leq n\}}] + \mathbb{E}_y[h(n)z_0^n 1_{\{\tau > n\}}].
\]

That \(h \geq 0\) on \(B\) implies

\[
h(y)\geq \mathbb{E}_y[h(\tau \wedge n)z_0^{\tau \wedge n}] 1_{\{\tau \leq n\}}.
\]

Now \(\lim_{n \to \infty} h(\tau)z_0^\tau 1_{\{\tau \leq n\}} = h(\tau)z_0^\tau 1_{\{\tau < \infty\}}\). This and Fatou’s lemma imply

\[
h(y)\geq \mathbb{E}_y[h(\tau)z_0^\tau 1_{\{\tau < \infty\}}].
\]

Finally, \(h \geq 1\) on \(\partial B\) and \(h < C_1\) give (20).

To get our desired \(h\) we start from the points \((r, 1)\) and \((\rho_1, \rho_1)\) on \(\mathcal{H}\). The first point gives us the root \((z, \beta) = (1, r)\) of the equation

\[
z p(\beta, 1) = 1.
\]

That

\[
\left.\frac{\partial z p(\beta, \alpha)}{\partial \beta}\right|_{(1, r, 1)} = (\lambda_1 + \lambda_2) \frac{r - 1}{r^2} \neq 0
\]

and the implicit function theorem give us a differentiable function \(\beta_1\) on an open interval \(I_1\) around \(z = 1\) that satisfies

\[
z p(\beta_1(z), 1) = 1, z \in I_1, \beta_1(1) = r.
\]
The conjugate of \((\beta_1, 1)\) on \(H^c\) is \((\beta_1, \alpha(\beta_1, 1))\) (whenever possible, we will omit the \(z\) variable and simply write \(\beta_1\); similarly, we will write \(\alpha\) for \(\alpha(\beta_1, 1)\)). These points give us the \(Y-z\)-harmonic function

\[
  h_z = [(\beta_1, 1), \cdot] - \frac{1 - \beta_1}{1 - \rho_2/\beta_1}[(\beta_1, \alpha), \cdot],
\]

where we used \(C(\beta_1, 1)/C(\beta_1, \alpha) = 1 - \beta_1/1 - \rho_2/\beta_1\). That \(\alpha(\beta_1(1), 1) = r^2/\rho_2 \in (0, 1)\) (Assumption 16) implies that \(0 < \alpha(\beta_1(z), 1) < 1\) if we choose \(z > 1\) close enough to 1. \(h_z\) will almost serve as our \(h\), except that it does take negative values on a small section of \(B\). To get a positive function we will add to \(h_z\) a constant multiple of the \(Y-z\)-harmonic function defined by a point on \(H^c \cap H_2^c\) that is the continuation of \((\rho_1, \rho_1)\) on \(H\). This point is \((\beta_2(z), \beta_2(z))\) where \(\beta_2(z)\) is the root of the equation

\[
  \left(\frac{\lambda_1}{\beta} + \mu_1\beta + \lambda_2 + \mu_2\right) = \frac{1}{z};
\]

satisfying \(\beta_2(1) = \rho_1\). The implicit function theorem (or direct calculation) shows that \(\beta_2\) is smooth in an open interval \(I_2\) containing 1. Now \((\beta_2, \beta_2) \in H^c \cap H_2^c\) and Proposition 3.1 imply that \([(\beta_2, \beta_2), \cdot] \geq 0\) is a \(Y-z\)-harmonic function. Now define

\[
  h' = h_z + C_0((\beta_2, \beta_2), \cdot).
\]

By its definition \(h'\) is \(Y-z\)-harmonic. We would like to choose \(C_0\) large enough so that \(h'\) is bounded below by 1 on \(\partial B\) and is nonnegative on \(B\). By our assumptions (5) and (6), \(\beta_1(1) = r < \beta_2(1) = \rho_1\); therefore, for \(z > 1\) close enough to 1, we will still have \(\beta_1(z) < \beta_2(z)\); let us assume that \(I_1\) and \(I_2\) are tight enough that this holds. By definition,

\[
  h'(y) = \beta_1^{y(1) - y(2)} \left(1 - \frac{1 - \beta_1}{1 - \rho_2/\beta_1} \alpha^{y(2)}\right) + C_0 \beta_2^{y(1) - y(2)} \beta_2^{y(2)}.
\]

\(\beta_2 > \beta_1\) implies that \(h'\) takes its most negative value for \(y(1) = y(2)\), i.e., on \(\partial B\) and if we can choose \(C_0 > 0\) so that \(h'\) is nonnegative on \(\partial B\), it will be so on all of \(B\). On \(\partial B\), \(h'\) reduces to

\[
  1 - \frac{1 - \beta_1}{1 - \rho_2/\beta_1} \alpha^{y(2)} + C_0 \beta_2^{y(2)}.
\]

If \(\alpha \leq \beta_2\), then setting \(C_0 = \frac{1 - \beta_1}{1 - \rho_2/\beta_1}\) would imply \(h' \geq 1\) on \(\partial B\); \(\alpha, \beta_1, \beta_2 \in (0, 1)\) imply

\[
  h' \leq 1 + C_0;
\]

then, \(h'\) can serve as our desired \(Y-z\)-harmonic function \(h\) with \(C_1 = 1 + C_0\).

Now let us consider the case \(\alpha > \beta_2\) : ordinary calculus implies that if we choose \(C_0\) large enough we can make the minimum \(m_0 < 0\) over \(y(2) > 0\) of

\[
  -\frac{1 - \beta_1}{1 - \rho_2/\beta_1} \alpha^{y(2)} + C_0 \beta_2^{y(2)}
\]

arbitrarily close to 0; then choosing \(h = \frac{1}{1 + m_0} h'\) gives us a \(Y-z\)-harmonic function that satisfies \(h \geq 1\) on \(\partial B, h \geq 0\) on \(B\) and \(h < C_1\) on \(B\) where \(C_1 = \frac{2}{1 + m_0}(1 + C_0)\).

We now use (20) to derive an upper bound on the probability that \(\tau\) is finite but too large:
**Proposition 3.4** For any $\delta > 0$, there exists $C_2 > 0$ such that
\[
P_y(nC_2 < \tau < \infty) \leq e^{-\delta n}
\] for any $y \in \mathbb{Z} \times \mathbb{Z}_+$, $y(1) > y(2)$ and $n > 1$.

**Proof** Let $z_0 > 1$ and $C_1$ be as in (20). For any $A > 0$, Chebyshev’s inequality gives
\[
P_y(nC_2 < \tau < \infty) = P_y(z_0^{nC_2} < z_0^\tau < \infty) \\
\leq E_y[z_0^\tau < \infty]z_0^{-nC_2} \\
\leq e^{-n(C_2 \log(z_0) - \log(C_1)/n)}.
\]
Choosing $C_2 = (\delta + \log(C_1))/\log(z_0)$ gives (21). \qed

### 4 LD limit for $P_x(\tau_n < \tau_0)$

Define
\[
V(x) \doteq \log \rho_1(x(1) - 1) \wedge \log(r)(x(1) + x(2) - 1) \wedge \log \rho_2(x(2) - 1), \ x \in \mathbb{R}^2_+.
\]
Assumption (5) implies
\[
-\log(\rho_2)(1 - x(2)) \geq -\log(r)(1 - (x(1) + x(2))),
\]
for $x(1) + x(2) < 1$, $x \in \mathbb{R}^2_+$, and therefore, over the same region $V$ reduces to
\[
V(x) = \log(r)(x(1) + x(2) - 1) \wedge \log \rho_1(x(1) - 1).
\]
(22)
The level curves of $V$ for for $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\mu_1 = 0.3$, $\mu_2 = 0.4$ are shown in Fig. 4.

The goal of this section is to prove
Theorem 4.1 \( V \) is the LD limit of \( P_x(\tau_n < \tau_0) \), i.e.,
\[
\lim_{n \to \infty} -\frac{1}{n} \log P_{\lfloor nx \rfloor}(\tau_n < \tau_0) = V(x).
\] (23)
for \( x(1) + x(2) < 1, x \in \mathbb{R}_+^2 \).

Proof Propositions 4.1 and 4.3 state
\[
\liminf_{n \to \infty} -\frac{1}{n} \log P_{\lfloor nx \rfloor}(\tau_n < \tau_0) \geq V(x).
\] (24)
and
\[
\limsup_{n \to \infty} -\frac{1}{n} \log P_{\lfloor nx \rfloor}(\tau_n < \tau_0) \leq V(x).
\] (25)
These imply (23). \( \square \)

Next two subsections prove (24) and (25). To prove the first, we will proceed parallel to Sezer (2005, 2015), Dupuis et al. (2007b) and construct a sequence of supermartingales \( M^n \) starting from a subsolution of a limit Hamilton Jacobi Bellman (HJB) equation associated with the problem. To prove the bound (25) we will directly construct a sequence of subharmonic functions of the process \( X \).

4.1 LD lowerbound for \( P_x(\tau_n < \tau_0) \)

Recall from (1) that \( \mathcal{V} = \{ \pm e_i, i = 1, 2 \} \) is the set of all possible increments of \( X \) and \( Y \). Let \( p : \mathcal{V} \mapsto [0, 1] \) denote the distribution of the increment \( I_1 \), i.e., \( p(e_i) = \lambda_i, p(-e_i) = -\mu_i, i = 1, 2 \). For \( a \subset \{ 1, 2 \} \) define \( \mathcal{V}_a = \bigcup_{i \in a} \{ v \in \mathcal{V} : v(i) = -1 \} \) and the Hamiltonian function
\[
H_a(q) = -\log \left( \sum_{v \in \mathcal{V}_a^c} p(v) e^{-\langle q, v \rangle} + \sum_{v \in \mathcal{V}_a} p(v) \right).
\]
We will denote \( H_{\varnothing} \) by \( H \). \( H_a \) is convex in \( q \). For \( x \in \mathbb{R}_+^2 \), define
\[
b(x) = \{ i : x(i) = 0 \}.
\]
Following Dupuis and Ellis (1997) one can represent \( V \) as the value function of a continuous time deterministic control problem; the HJB equation associated with this control problem is
\[
H_{b(x)}(D V(x)) = 0; \tag{26}
\]
where \( D \) denotes the gradient of \( V : \mathbb{R}^2 \mapsto \mathbb{R} \). A function \( W \in C^1 \) is said to be a classical subsolution of (26) if
\[
H_{b(x)}(D V(x)) \geq 0; \tag{27}
\]
supersolutions are defined by replacing \( \geq \) in (27) with \( \leq \).

To prove (24) will proceed parallel to Sezer (2015), Section 7: find an upperbound on \( P_x(\tau_n < \tau_0) \) by constructing a supermartingale associated with the process \( X \). To construct our supermartingale we will proceed parallel to Sezer (2005), Dupuis et al. (2007b) and use a subsolution of (26), i.e., a solution of (27).
Define

\[ r_0 \doteq (0, 0), \quad r_1 \doteq \log(\rho_1)(1, 0), \quad r_2 \doteq \log(\rho_1)(0, 1), \quad r_3 \doteq \log(\rho_1)(1, 1) \]  \hspace{1cm} (28)

and

\[ \tilde{V}_0(x, \epsilon) \doteq -\log(\rho_1) - 3\epsilon, \quad \tilde{V}_1(x, \epsilon) \doteq -\log(\rho_1) + \langle r_1, x \rangle - 2\epsilon, \]
\[ \tilde{V}_2(x, \epsilon) \doteq -\log(\rho_1) + \langle r_2, x \rangle - \epsilon, \quad \tilde{V}_3(x, \epsilon) \doteq -\log(\rho_1) + \langle r_3, x \rangle \]

and

\[ \tilde{V}(x, \epsilon) \doteq \bigwedge_{i=0}^3 \tilde{V}_i(x, \epsilon). \]  \hspace{1cm} (29)

A direct calculation gives

**Lemma 4** The gradients defined in (28) satisfy

\[ H(r_0) = H_1(r_0) = H_2(r_0) = 0, \quad H(r_1) = H_1(r_1) = 0, \]
\[ H(r_2) > 0, \quad H_1(r_2) > 0, \quad H(r_3) = 0 \]

The 0-level curve of the Hamiltonians and the gradients \( r_i \) are shown in Fig. 5a.

Define

\[ C_3 \doteq -\frac{1}{\log(\rho)}, \quad C_4 \doteq -\log(\rho_1)C_3 = \frac{\log(\rho_1)}{\log(\rho)}. \]  \hspace{1cm} (30)

\( \log(\rho) < \log(\rho_1) < 0 \) implies

\[ 1 > C_4 > 0. \]  \hspace{1cm} (31)
The functions $\tilde{V}_i$, $i = 1, 2, 3$ meet at
\begin{equation}
x^* = (C_3 \epsilon, 1 - C_4 + C_3 (1 + C_4) \epsilon)
\end{equation}
i.e.,
\begin{equation}
\tilde{V}_1(x^*) = \tilde{V}_2(x^*) = \tilde{V}_3(x^*) = -\log(\rho_1) - (2 + C_4) \epsilon.
\end{equation}
We assume that $\epsilon > 0$ is small enough so that $x^*$ satisfies
\begin{equation}
x^*(1), x^*(2) > 0, x^*(1) + x^*(2) < 1.
\end{equation}
$\tilde{V}(\cdot, \epsilon)$ equals $\tilde{V}_i(\cdot, \epsilon)$ in the region
\begin{equation}
R_i = \{ x \in \mathbb{R}^2 : \tilde{V}(x, \epsilon) = \tilde{V}_i(x, \epsilon) \},
\end{equation}
these regions are shown in Fig. 5b.

As in Sezer (2005) and Dupuis et al. (2007b), we will mollify $\tilde{V}(x, \epsilon)$ with
\begin{equation}
\eta(x) = 1_{|x| \leq 1} (|x|^2 - 1)^2, x \in \mathbb{R}^2, C_5 = \int_{\mathbb{R}^2} \eta(x) dx, \eta_{\delta}(x) = \frac{1}{\delta^2 C_5} \eta(x/\delta), \delta > 0.
\end{equation}
to get our smooth subsolution of (26):
\begin{equation}
V(x, \epsilon) = \int_{\mathbb{R}^2} \tilde{V}(x + y, \epsilon) \eta_{0.5C_3 \epsilon}(y) dy,
\end{equation}

**Lemma 5** The function $V(x, \epsilon)$ of (34) satisfies (27) (i.e., $V$ is a subsolution of (26)) and
\begin{equation}
\left| \frac{\partial^2 V(\cdot, \epsilon)}{\partial x_i \partial x_j} \right| \leq C_6 / \epsilon
\end{equation}
where $C_6$ is independent of $x$. Furthermore,
\begin{equation}
V(x, \epsilon) \leq \epsilon \text{ for } x(1) + x(2) = 1, x \in \mathbb{R}^2_+.
\end{equation}

**Proof** The proof is parallel to that of Sezer (2005, Lemma 2.3.2). $\tilde{V}$ is the minimum of four affine functions and hence is Lipschitz continuous and has a bounded (piecewise constant) gradient almost everywhere. This implies
\begin{equation}
DV(x, \epsilon) = \int_{\mathbb{R}^2} D\tilde{V}(x + y, \epsilon) \eta_{0.5C_3 \epsilon}(y) dy \\
= \sum_{i=0}^{3} w_i(x) D\tilde{V}_i = \sum_{i=1}^{3} w_i(x) r_i
\end{equation}
where
\begin{equation}
w_i(x) = \int_{\mathbb{R}^2} \eta_{0.5C_3 \epsilon}(x) 1_{R_i}(x) dx.
\end{equation}
This shows that $V(\cdot, \epsilon) \in C^1$. To show
\begin{equation}
H_{b(x)}(DV(\cdot, \epsilon)) \geq 0,
\end{equation}
one considers $x \in \mathbb{R}^{2\alpha}_+ \triangleq \{ x \in \mathbb{R}^2_+, x(1), x(2) > 0 \}, x \in \partial_1$ and $x \in \partial_2$ separately. We will provide the details only for the first two. For $x \in \mathbb{R}^{2\alpha}_+$,
\begin{equation}
H_{b(x)}(DV(\cdot, \epsilon)) = H(DV(\cdot, \epsilon)).
\end{equation}
By Lemma 4 we know that all $r_i$ satisfy $H(r_i) \geq 0$. That $H$ is a convex function and Jensen’s inequality imply that the $DV(x, \epsilon) = \sum_{i=0}^{3} w_i(x) r_i$ satisfies $H(DV(x, \epsilon)) \geq 0$; this proves (38) for $x \in \mathbb{R}^2_{+}$.

We know by (31) that $C_4 \in (0, 1)$; therefore, by (33)

$$\tilde{V}_0(\cdot, \epsilon) = -\log(\rho_1) - 3\epsilon < -\log(\rho_1) - (2 + C_4)\epsilon$$

$$= \tilde{V}_1(x^*, \epsilon) = \tilde{V}_2(x^*, \epsilon) = \tilde{V}_3(x^*, \epsilon).$$

This implies that the region $R_0$ intersects all of the $R_1$, $R_2$ and $R_3$ and in particular that the strip $\{x \in \mathbb{R}^2_{+} : x(1) < C_3\epsilon\}$ lies in $R_0 \cup R_2$. Then, for $x \in \partial_1$, the ball $B(x, C_3\epsilon/2)$ lies completely in $R_0 \cup R_2$, which implies

$$DV(x, \epsilon) = w_0(x)r_0 + w_1(x)r_2, w_0(x) + w_1(x) = 1.$$

By Lemma 4 we know that $H_1(r_0) = 0$ and $H_1(r_2) \geq 0$. These and the convexity of $H_i$ imply (38) for $x \in \partial_1$.

The bound (35) follows from the Lipschitz continuity of $\tilde{V}(\cdot, \epsilon)$, differentiation under the integral sign in (34) and bounds on the first derivative of $\eta$. Finally, (36) follows from

$$\tilde{V}_3(x, \epsilon) \leq \frac{\epsilon}{\sqrt{2}}, \text{ for any } x \in B(y, \epsilon C_3/2), y \text{ such that } y(1) + y(2) = 1, y \in \mathbb{R}^2_{+},$$

To get our upper bound on the probability $P_X(\tau_n < \tau_0)$ we define

$$M^{(n, \epsilon)}_k = e^{-nV(X/n, \epsilon)} - \frac{C_6}{n\epsilon},$$

where $C_6$ is as in (35).

**Lemma 6** $M^{(n, \epsilon)}$ is a supermartingale.

**Proof** The Markov property of $X$ implies that it suffices to show

$$\mathbb{E}_x \left[ M^{(n,\epsilon)}_1 \right] \leq e^{-nV(x/n, \epsilon)}$$

$$\mathbb{E}_x \left[ M^{(n,\epsilon)}_1 e^{nV(x/n, \epsilon)} \right] \leq 1$$

$$-\log \left( \mathbb{E}_x \left[ M^{(n,\epsilon)}_1 e^{nV(x/n, \epsilon)} \right] \right) \geq 0. \quad (39)$$

The expression on the left equals

$$-\log \left( \sum_{v \in \mathcal{V}_{b(n)}} e^{-n(V((x+v)/n, \epsilon) - V(x/n, \epsilon))} p(v) + \sum_{v \in \mathcal{V}_{b(n)}} p(v) \right) + \frac{C_6}{n\epsilon}. \quad (40)$$

A Taylor expansion and the bound (35) imply

$$|V((x+v)/n, \epsilon) - V(x/n, \epsilon) - \langle DV(x), v/n \rangle| \leq \frac{C_6}{n^2\epsilon}$$

Then, the expression in (40) is bounded below by

$$-\log \left( \sum_{v \in \mathcal{V}_{b(n)}} e^{-\langle DV(x), v \rangle} p(v) + \sum_{v \in \mathcal{V}_{b(n)}} p(v) \right) - \frac{C_6}{n\epsilon} + \frac{C_6}{n\epsilon}.$$
The log term above equals $H_{b(x)}(DV(x, \epsilon))$, which by Lemma 5 is nonnegative [see (27)]. This proves (39).

\[\square\]

**Proposition 4.1** Let $x \in \mathbb{R}_+^2$ with $x(1) + x(2) < 1$, $x_n = \lfloor nx \rfloor$ and let $V$ be as in (22). Then for any $\epsilon > 0$ there exists an integer $N$ such that for $n > N$

$$P_{x_n}(\tau_n < \tau_0) \leq e^{-n(V(x) - \epsilon)}.$$ \hspace{1cm} (41)

In particular,

$$\lim \inf -\frac{1}{n} \log P_{\lfloor nx \rfloor}(\tau_n < \tau_0) \geq V(x).$$ \hspace{1cm} (42)

The proof is parallel to that of Sezer (2018, Proposition 4.3).

**Proof** The inequality (42) follows from (41) upon taking limits. The rest of the proof focuses on (41). Let $\epsilon_n > 0$ be a sequence satisfying $\epsilon_n \to 0$ and $\epsilon_n n \to \infty$. Let $\tau_{0,n} = \tau_n \wedge \tau_0$. The optional sampling theorem (Durrett 2010, Theorem 5.7.6) applied to the supermartingale $M_k = M_k^{(n, \epsilon_n)}$ at time $\tau_{0,n}$ gives

$$E_{\tau_{0,n}}[M_{\tau_{0,n}}] \leq M_0 = e^{-nV(x_n/n, \epsilon_n)}.$$

Restricting the expectation on the left to $\{ \tau_n < \tau_0 \}$ makes it smaller:

$$E_{\tau_{0,n}}[1_{\{\tau_n < \tau_0\}}M_{\tau_n}] \leq e^{-nV(x_n/n, \epsilon_n)}$$

Expanding $M_{\tau_n}$ using its definition gives

$$E_{\tau_{0,n}}[1_{\{\tau_n < \tau_0\}}e^{-nV(X_{\tau_n}/n, \epsilon_n) + \frac{C_6 n}{\epsilon n}}] \leq e^{-nV(x_n/n, \epsilon_n)}$$

$X_{\tau_n} \in \partial A_n$ and the bound (36) reduce the last display to

$$E_{\tau_{0,n}}[1_{\{\tau_n < \tau_0\}}e^{-\frac{C_6 n}{\epsilon n}}] \leq e^{-nV(x_n/n, \epsilon_n)} e^{\frac{C_6 n}{\epsilon n}}$$ \hspace{1cm} (43)

By the definitions involved we have

$$\lim_{n \to \infty} V(x_n/n, \epsilon_n) = V(x).$$

This, $\epsilon_n \to 0$ and taking the lim inf $-\frac{1}{n} \log$ of both sides in (43) give

$$\lim \inf_{n \to \infty} -\frac{1}{n} \log E_{\tau_{0,n}}[1_{\{\tau_n < \tau_0\}}e^{-\frac{C_6 n}{\epsilon n}}] \geq V(x).$$ \hspace{1cm} (44)

Now suppose that (41) doesn’t hold, i.e., there exists $\epsilon > 0$ and a sequence $n_k$ such that

$$P_{x_{n_k}}(\tau_{n_k} < \tau_0) > e^{-n_k(V(x) - \epsilon)}$$ \hspace{1cm} (45)

for all $k$; we pass to this subsequence and omit the subscript $k$. Sezer (2005, Theorem A.1.1) implies that there is a $C_2 > 0$ such that

$$P_{x_n}(\tau_{0,n} > nC_2) \leq e^{-n(V(x) + 1)}$$ \hspace{1cm} (46)
for \( n \) large. Then
\[
\mathbb{E}_x \left[ 1_{\{\tau_n < \tau_0\}} e^{-\frac{c_{6,7}n}{\alpha_n}} \right] \geq \mathbb{E}_x \left[ 1_{\{\tau_n < \tau_0\}} e^{-\frac{c_{6,7}n}{\alpha_n}} 1_{\{\tau_0,n \leq nC_2\}} \right] \\
\geq e^{-\frac{c_{6,7}n}{\alpha_n}} \mathbb{E}_x \left[ 1_{\{\tau_n < \tau_0\}} 1_{\{\tau_0,n \leq nC_2\}} \right] \\
\geq e^{-\frac{c_{6,7}n}{\alpha_n}} (P_{x_n}(\tau_n < \tau_0) - P_{x_0}(\tau_{0,n} > nC_2)) \\
\geq e^{-\frac{c_{6,7}n}{\alpha_n}} (e^{-n(V(x)-\varepsilon)} - e^{-(V(x)+1)n}) .
\]

Now taking \( \limsup_{n \to \infty} \frac{1}{n} \log \) of both sides gives
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_x^n \left[ 1_{\{\tau_n < \tau_0\}} e^{-\frac{c_{6,7}n}{\alpha_n}} \right] \leq V(x) - \varepsilon .
\]

This contradicts (44). Therefore, the assumption (45) is false and there does exist \( N > 0 \) such that (41) holds for \( n > N \). This finishes the proof of this proposition. \( \square \)

### 4.2 LD upperbound for \( P_x(\tau_n < \tau_0) \)

The LD upperbound corresponds (because of the \(-\log\) transform) to a lower bound on the probability \( P_x(\tau_n < \tau_0) \). To get a lower bound on this probability, it suffices to have a submartingale of \( X \) with the right values when \( X \) hits \( \partial A_n \cup \{0\} \). As opposed to the analysis of the previous section (where we constructed a supermartingale from a subsolution to a limit HJB equation), one can directly construct a subharmonic function of \( X \) to get the desired submartingale. The next proposition gives this explicit subharmonic function. In its proof the following fact will be useful: if \( g_1 \) and \( g_2 \) are subharmonic functions of \( X \) at a point \( x \), then so is \( g_1 \vee g_2 \), this follows from the definitions involved.

**Proposition 4.2**

\[
f_n(x) := \rho_1^{n-x(1)} \vee r(n-x(1)-x(2)) \vee \rho_1^{n-1}
\]

is a subharmonic function of \( X \) on \( A_n - \partial A_n \).

**Proof** We note
\[
\rho_1^{n-x(1)} = \rho_1^{\tau(\tau^{(1)} - x(2))} \rho_1^{x(2)} = [(\rho_1, \rho_1), T_n(x)].
\]

Furthermore, \( (\rho_1, \rho_1) \in \mathcal{H} \). It follows from these and Lemma 1 that \( x \mapsto \rho_1^{n-x(1)} \) is \( X \)-harmonic for \( x \in \mathbb{Z}^2_+ \). A parallel argument proves the same for \( x \mapsto r(n-x(1)) - x(2) \). The constant function \( x \mapsto \rho_1^{n-1} \) is trivially \( X \)-harmonic for all \( x \in \mathbb{Z}^2_+ \). It follows that their maximum, \( f_n \), is subharmonic on \( \mathbb{Z}^2_+ \).

It remains to prove that \( f_n \) is subharmonic on \( \partial_1 \) and \( \partial_2 \). \( f_n(x) = r(n-x(1))-x(2) \vee \rho_1^{n-1} \) for \( x \in \partial_1 \cup \{x \in \mathbb{Z}^2_+ : x(1) = 1\} \). Both \( x \mapsto r(n-x(1))-x(2) \) and \( x \mapsto \rho_1^{n-1} \) are \( X \)-harmonic on \( \partial_1 \). It follows from these that \( f_n \) is subharmonic on \( \partial_1 \).

For \( x \in \partial_2 \cap \{x \in \mathbb{Z}^2_+: x(1) < n\} \) we have \( f_n(x) = \rho_1^{n-x(1)} \vee \rho_1^{n-1} \). By Lemma 2 and by fact that \( I_k = \mathbb{I}_J \) we know \( x \mapsto \rho_1^{n-x(1)} \) is harmonic on \( \partial_2 \); the same trivially holds for \( x \mapsto \rho_1^{n-1} \); therefore, \( x \mapsto \rho_1^{n-x(1)} \vee \rho_1^{n-1} \) is subharmonic on \( \partial_2 \cap \{x \in \mathbb{Z}^2_+: x(1) < n\} \). Furthermore, by definition \( f_n(x) \geq \rho_1^{n-x(1)} \vee \rho_1^{n-1} \). These imply that \( f_n \) is subharmonic on \( \partial_2 \cap \{x \in \mathbb{Z}^2_+: x(1) < n\} \).
The last three paragraphs together imply the statement of the proposition. □

**Proposition 4.3**

\[ P_x(\tau_n < \tau_0) \geq f_n(x) - f_n(0) \]  

(47)

and in particular

\[ \limsup \frac{1}{n} \log P_{[nx]}(\tau_n < \tau_0) \leq V(x), \]  

(48)

for \( x \in \mathbb{R}_+^2 \), \( x(1) + x(2) < 1 \), \( x(1) > 0 \).

**Proof** By Proposition 4.2, we know that \( f_n \) is a subharmonic function of \( X \). It follows that \( f_n(X_k) \) is a submartingale. This and the optional sampling theorem imply:

\[ f_n(x) \leq E_x[f_n(\tau_n \wedge \tau_0)], \]

\[ = P_x(\tau_n < \tau_0)(1 - f_n(0)) + f_n(0), \]

where we have used \( f_n(x) = 1 \) for \( x \in \partial A_n \); this gives (47). Taking \(-\frac{1}{n} \log\) of both sides and applying \( \limsup \) gives (48).

\[ \square \]

### 5 LD limit of \( P_x(\sigma_1 < \tau_n < \tau_0) \)

To implement the argument given in the introduction we need an LD lowerbound for the probability

\[ P_x(\sigma_1 < \tau_n < \tau_0). \]  

(49)

We will obtain the desired bound through a subsolution of the limit HJB equation associated with \( X \). This is parallel to the construction given in Sezer (2018, Proposition 4.3) and the argument of Sect. 4.1. The main difference is in the construction of the subsolution. Bounding (49) requires a subsolution consisting of two pieces, one piece for before \( \sigma_1 \) and one for after. For the first piece we need the following additional root of the limit Hamiltonian:

\[ r_4 \equiv (\log(\rho_1/r), \log(r)). \]  

(50)

Define

\[ \tilde{V}_4(x, \epsilon) \equiv -\log(r) + \langle r_4, x \rangle, \tilde{V}(0, x, \epsilon) \equiv \bigwedge_{i \in \{0, 2, 4\}} \tilde{V}_i(x, \epsilon) \]

\[ \tilde{V}(1, x, \epsilon) \equiv \tilde{V}(x, \epsilon) = \bigwedge_{i=0}^{3} \tilde{V}_i(x, \epsilon) \]  

(51)

and

\[ V_\sigma(0, x) \equiv \tilde{V}(0, x, 0) = (-\log(\rho_1)) \wedge (-\log(r) + \langle r_4, x \rangle) \]

\[ V_\sigma(1, x) \equiv \tilde{V}(1, x, 0) = (-\log(\rho_1) + \langle r_1, x \rangle) \wedge (-\log(r) + \langle r_3, x \rangle) \]

where the vectors \( r_i \) are as in (28). Now define the smoothed subsolution:

\[ V(i, x, \epsilon) \equiv \int_{\mathbb{R}^2} \tilde{V}(i, x + y, \epsilon) \eta_{0.5\epsilon^3}(y)dy, i = 0, 1. \]  

(52)
The function \( \bar{V}(0, \cdot, \cdot) \) is obtained from \( \bar{V}(1, \cdot, \cdot) \) by striking out \( \bar{V}_1 \) from the minimum and replacing \( \bar{V}_3 \) with \( \bar{V}_4 \). In particular, the components \( \bar{V}_0 \) and \( \bar{V}_2 \) are common to both \( \bar{V}(1, \cdot, \cdot) \) and \( \bar{V}(0, \cdot, \cdot) \); this ensures that these functions overlap around an open region along \( \partial_1 \), which implies in particular that

\[
V(1, x, \epsilon) = V(0, x, \epsilon)
\]  

(53)

for \( x \in \partial_1 \).

**Remark 1** The condition (53) allows one to think of \( V(\cdot, \cdot, \cdot) \) as a subsolution of the HJB equation on a manifold; the manifold consists of two copies of \( \mathbb{R}^2_+ \), glued to each other along \( \{ x \in \mathbb{R}_+, x(1) = 0 \} \).

We use \( V(\cdot, \cdot, \cdot) \) to construct the supermartingale

\[
M_k^{(n,\epsilon,\sigma)} = e^{-nV(1_{[k<\sigma]}),X_k/n,\epsilon)}C_k^\epsilon,
\]

where \( C_k^\epsilon/\epsilon \) is an upperbound on the second derivative of \( V(\cdot, \cdot, \cdot) \), which can be obtained by an argument parallel to the one used in the proof of (35) of Lemma 5. The main difference from Sect. 4.1 is that the smooth subsolution has an additional parameter \( i \) to keep track of whether \( X \) has touched \( \partial_1 \); this appears as the \( I_{[k<\sigma]} \) term in the definition of the supermartingale \( M_k^{(n,\epsilon,\sigma)} \). A three stage version of this argument appears in Sezer (2018, Proposition 4.3) to bound another related probability arising from the analysis of the two dimensional tandem random walk. The main result of this section is the following:

**Proposition 5.1** For any \( \epsilon > 0 \), there exists \( N > 0 \) such that

\[
P_{x_n}(\sigma_1 < \tau_n < \tau_0) \leq e^{-n(V(0,x) - \epsilon)},
\]

(54)

for \( n > N \), where \( x_n = \lfloor nx \rceil, 0 < x(1) + x(2) < 1, x \in \mathbb{R}_+^2 \).

**Proof** Parallel to the proof of Proposition 4.1, we choose a sequence \( \epsilon_n \rightarrow 0 \) with \( n\epsilon_n \rightarrow \infty \); (54) follows from an application of the optional sampling theorem to the supermartingale \( M_k^{(n,\epsilon_n,\sigma)} \) and the bound (46).

\[\square\]

**5.1 LD limit for \( P_x(\sigma_1 < \tau < \infty) \)**

For this subsection and the next section it will be convenient to express the \( Y \) process in \( x \) coordinates, we do this by setting, \( \bar{X}_k \equiv T_n(Y_k); \bar{X}_k \) has the following dynamics:

\[
\bar{X}_{k+1} = \bar{X}_k + \pi_1(\bar{X}_k, I_k).
\]

\( \sigma_1 \) of (11) in terms of \( \bar{X} \) is \( \bar{\sigma}_1 = \inf \{ k : \bar{X}_k \in \partial_1 \} \). The processes \( \bar{X} \) and \( X \) have the same dynamics except that \( \bar{X} \) is not constrained on \( \partial_1 \). By definition, \( \bar{X}_0 = X_0 \). Note the following: \( \bar{X} \) hits \( \{ x \in \mathbb{Z} \times \mathbb{Z}_+ : x(1) + x(2) = n \} \) exactly when \( Y \) hits \( \{ y \in \mathbb{Z} \times \mathbb{Z}_+ : y(1) = y(2) \} \); i.e., if we define

\[
\bar{\tau}_n = \inf \{ k : \bar{X}_k(1) + \bar{X}_k(2) = n \},
\]

then \( \tau = \bar{\tau}_n \).

**Proposition 5.2** For any \( \epsilon > 0 \), there exists \( N > 0 \) such that

\[
P_{x_n}(\bar{\sigma}_1 < \bar{\tau}_n < \infty) \leq e^{-n(V(0,x) - \epsilon)},
\]

(55)

for \( n > N \), where \( x_n = \lfloor nx \rceil, x(1) + x(2) < 1, x \in \mathbb{R} \times \mathbb{R}_+ \).
**Proof** The two stage subsolution $V(\cdot, \cdot, \cdot)$ of (52) is a subsolution for the $\bar{X}$ process as well because, $\bar{X}$ has identical dynamics as $X$ with one less constraint. Therefore, the proof of Proposition 5.1 applies verbatim to the current setup with one change: in the proof of (54) we truncate time with the bound (46) for $\tau_n$. We replace this with the corresponding bound (21) for $\tau$.

\[ \therefore \]

6 Completion of the limit analysis

We now combine Propositions 4.3, 5.1 and 5.2 to get the main approximation result of this work:

**Theorem 6.1** For any $x \in \mathbb{R}_+^2$, $x(1) + x(2) < 1$, $x(1) > 0$, there exists $C_7 > 0$ and $N > 0$ such that

\[ |P_{x_n}(\tau_n < \tau_0) - P_{\hat{t}_n}(\tau < \infty)| = \frac{|P_{x_n}(\tau_n < \tau_0) - P_{\hat{t}_n}(< \infty)|}{P_{x_n}(\tau_n < \tau_0)} < e^{-nC_7n} \]

for $n > N$, where $x_n = \lfloor x \rfloor$.

That $P_{x_n}(\bar{t}_n < \infty) = P_{\hat{t}_n}(\tau < \infty)$ follows from the definitions in Sect. 5.1.

**Proof** The definitions (22) and (51) imply that

\[ 2C_7 = V_\sigma(x, 0) - V(x) > 0 \]

for $x \in \mathbb{R}_+^2$, $x(1) + x(2) < 1$, $x(1) > 0$. Choose $\epsilon < C_7$. The processes $X$ and $\bar{X}$ follow exactly the same path until they hit $\partial_1$. It follows that

\[ |P_{x_n}(\tau_n < \tau_0) - P_{x_n}(\bar{t}_n < \infty)| \leq P_{x_n}(\sigma_1 < \tau_0) + P_{x_n}(\bar{t}_1 < \bar{t}_n < \infty) \] (56)

By Propositions 5.1, 5.2 and 4.3 there exists $N > 0$ such that

\[ P_{x_n}(\sigma_1 < \tau_0) + P_{x_n}(\bar{t}_1 < \bar{t}_n < \infty) \leq e^{-n(V_\sigma(0,x) - \epsilon/2)} \] (57)

and

\[ P_{x_n}(\tau_n < \tau_0) \geq e^{-n(V(x) - \epsilon/2)} \] (58)

for $n > N$. The bounds (56), (57) and (58) give

\[ \frac{|P_{x_n}(\tau_n < \tau_0) - P_{x_n}(\bar{t}_n < \infty)|}{P_{x_n}(\tau_n < \tau_0)} < e^{-nC_7}, \]

for $n > N$.

\[ \therefore \]

7 Computation of $P_y(\tau < \infty)$

We call a $Y$-harmonic function $\partial B$-determined if it has the representation

\[ y \mapsto \mathbb{E}_y[g(Y_\tau)1_{\{\tau < \infty\}}] \] (59)

for some function $g$ on $\partial B$. Theorem 6.1 tells us that $P_y(\tau < \infty)$, $y = T_n(x_n)$, approximates $P_{x_n}(\tau_n < \tau_0)$ with exponentially decaying relative error for $x(1) > 0$. To complete our

\[ \therefore \]

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analysis, it remains to compute $P_y(\tau < \infty)$. As a function of $y$, $P_y(\tau < \infty)$ is $Y$-harmonic and $\partial B$-determined (for $y \mapsto P_y(\tau < \infty)$, $g$ of (59) equals 1 identically). We will try to compute $P_y(\tau < \infty)$ as a superposition of the $Y$-harmonic functions expounded in Sect. 2; because $P_y(\tau < \infty)$ is 1 for $y \in \partial B$, we would like the superposition to be as close to 1 as possible on $\partial B$. We have two classes of $Y$-harmonic functions given in Propositions 2.1 (constructed from a single point on $\mathcal{H}$) and 2.2 (constructed from conjugate points on $\mathcal{H}$). The first class gives us only one nontrivial $Y$-harmonic function, computed in Lemma 2: $h_{\rho_1} = [(\rho_1, \rho_1), \cdot]$. Remember that we have assumed $\alpha(r, 1) = r^2/\rho_2 < 1$. This implies that, among the functions in the second class, the most relevant for the computation of $P_y(\tau < \infty)$ is

$$h_r = \frac{1}{1 - \rho_2/r} h_r = [(r, 1), \cdot] - \frac{1 - r}{1 - \rho_2/r} [(r, \rho_2/r^2), \cdot],$$

because this $Y$-harmonic function exponentially converges to 1 for $y = (k, k) \in \partial B$ and $k \to \infty$. A simple criterion to check whether a $Y$-harmonic function of the form $\sum_{i=1}^I c_i[(\beta_i, \alpha_i), \cdot]$ is $\partial B$-determined is as follows.

**Proposition 7.1** A $Y$-harmonic function of the form $\sum_{i=1}^I c_i[(\beta_i, \alpha_i), \cdot]$ is $\partial B$ determined if $|\beta_i| < 1$ and $|\alpha_i| \leq 1$, $i = 1, 2, 3, \ldots, I$.

The proof is given Sect. A. When $r$ is the harmonic mean of $\rho_1$ and $\rho_2$, $P_y(\tau < \infty)$ can be expressed perfectly as a linear combination of $h_{\rho_1}$ and $h_r$.

**Proposition 7.2** If

$$\rho_2 \rho_1 = r^2$$

then

$$P_y(\tau < \infty) = h_r(y) + \frac{1 - r}{1 - \rho_2/r} h_{\rho_1}(y)$$

for $y \in B$.

**Proof** The right side of (61) is $Y$-harmonic by construction. Furthermore, $\rho_2 \rho_1 = r^2$ implies $h_r(y) + \frac{1 - r}{1 - \rho_2/r} h_{\rho_1}(y) = 1$ for $y \in \partial B$. Therefore, to prove (61) it suffices to prove that

$$h_r + \frac{1 - r}{1 - \rho_2/r} h_{\rho_1}$$

is $\partial B$-determined. For this we will use Proposition 7.1; in the present case, the $\beta_i$ are $\rho_1$, $r < 1$ and the $\alpha_i$ are 1 and $\rho_1 \leq 1$. It follows that (62) is $\partial B$-determined. \square

If (60) doesn’t hold, i.e., if $r^2 \neq \rho_1 \rho_2$ then one can proceed in several ways. As a first step, one can use the functions $h_r$ and $h_{\rho_1}$ to construct lower and upper bounds on $P_y(\tau < \infty)$.

**Proposition 7.3** There exists positive constants $c_0$, $c_1$, and $C_8$

$$P_y(\tau < \infty) \leq h_{a, 0}(y) \leq C_8 P_y(\tau < \infty)$$

where

$$h_{a, 0} = c_0 h_r + c_1 h_{\rho_1}.$$

In particular, $h_{a, 0}$ approximates $P_y(\tau < \infty)$ with bounded relative error.
Proof If $\rho_1 > r^2 / \rho_2$, one can set $c_0 = 1$ and $c_1 = \frac{1 - r}{1 - \rho_2 / r}$ since, for these values

$$h^{a,0} = h_r + \frac{1 - r}{1 - \rho_2 / r} h_{\rho_1} \geq 1$$

(65)
on $\partial B$. Both $h^{a,0}$ and $y \mapsto P_y(\tau < \infty)$ are $\partial B$-determined; this and (65) imply $h^{a,0}(y) \geq P_y(\tau < \infty)$ for $y \in B$. To get the second bound in (63) set

$$C_8 = \max_{y \in \partial B} h^{a,0}(y) = 1 + \frac{1 - r}{1 - \rho_2 / r} \max_{x \geq 0} \left[ \rho_1^x - \left( \frac{r^2}{\rho_2} \right)^x \right].$$

(66)

With this choice of $C_8$ we get the second bound in (63) on $\partial B$; that both $y \mapsto P_y(\tau < \infty)$ and $h^{a,0}$ are $\partial B$-determined implies the same bound on all of $B$.

If $\rho_1 < r^2 / \rho_1$, first choose $C_0$ so that

$$1 + \min_{x \geq 0} \left[ C_0 \rho_1^x - \frac{1 - r}{1 - \rho_2 / r} \left( \frac{r^2}{\rho_2} \right)^x \right] \geq 1/2.$$  

(67)

Setting $c_0 = 2$ and $c_1 = 2C_0$, we have

$$h^{a,0}(y) = 2h_r(y) + 2C_0 h_{\rho_1}(y) \geq 1$$

for $y \in \partial B$, from which the first bound in (63) follows. To get the second bound in (63), set

$$C_8 = \max_{y \in \partial B} h^{a,0}(y) = 2 + \max_{x \geq 0} \left[ 2C_0 \rho_1^x - 2 \frac{1 - r}{1 - \rho_2 / r} \left( \frac{r^2}{\rho_2} \right)^x \right]$$

and proceed as above.  \[ Q.E.D. \]

Our choice of the constant 1/2 in (67) is arbitrary, any value between (0, 1) would suffice for the argument. Therefore, the constants $c_0$ and $c_1$ are not unique and they can be optimized to reduce relative error.

Proposition 7.4 For $x \in \mathbb{R}^2_+$, $x(1) + x(2) < 1$, $x(1) > 0$, $x_n = [nx]$, and for $n$ large, $h^{a,0}$ of (64) evaluated at $T_n(x_n)$ approximates $P_{x_n}(\tau_n < \tau_0)$ with bounded relative error.

Proof We know by Theorem 6.1 that, for $x \in \mathbb{R}^2_+$, $x(1) + x(2) < 1$ and $x(1) > 0$, $P_{T_n(x_n)}(\tau < \infty)$ approximates $P_{x_n}(\tau_n < \tau_0)$ with vanishing relative error. On the other hand, the above Proposition tells us that $h^{a,0}$ of (64) approximates $P_{\tau}(\tau < \infty)$ with bounded relative error. These imply that $h^{a,0}(T_n(x))$ approximates $P_{x_n}(\tau_n < \tau_0)$ with bounded relative error. \[ Q.E.D. \]

Proposition 3.2 gives not one but a one-complex-parameter family of $Y$-harmonic functions. A natural question is whether one can obtain finer approximations of $P_\tau(\tau < \infty)$ than what $h^{a,0}$ provides. In this, we need $B$-determined $Y$-harmonic functions. The next proposition (an adaptation of Sezer 2015, Proposition 4.13) to the current setting) identifies a class of these which are naturally suitable for the approximation of $P_\tau(\tau < \infty)$.

Proposition 7.5 There exists $0 < R < 1$ such that for all $\alpha \in \mathbb{C}$ with $R < |\alpha| \leq 1$, $\max(|\beta_1(\alpha)|, |\alpha(\beta_1(\alpha), \alpha)|) < 1$; in particular $h_{\beta_1(\alpha)}$ is $\partial B$-determined.

Proof We know by Sezer (2015, Proposition 4.7) that $|\beta_1(\alpha)| \leq r < 1$ for all $|\alpha| = 1$. Then

$$|\alpha(\beta_1(\alpha), \alpha)| = \left| \frac{\beta_1(\alpha)^2}{\alpha \rho_2} \right| \leq \frac{r^2}{\rho_2} < 1,$$
where the last inequality is the assumption (16). The functions $\beta_1$ and $\alpha$ are continuous; it follows that the inequality above holds also for $R < |\alpha| \leq 1$ if $R < 1$ is sufficiently close to 1. That $h_{\beta_1}$ is $\partial B$-determined follows from these and Proposition 7.1.

We can now use as many of the $\partial B$-determined $Y$-harmonic functions identified in Propositions 2.2 and 7.5 as we like to construct finer approximations of $P_y(\tau < \infty)$. Once the approximation is constructed upperbounds on its relative error can be computed from the maximum and the minimum of the approximation on $\partial B$, as was done in the proof of Proposition 7.3:

**Proposition 7.6** Let $R$ be as in Proposition 7.5. For $c_k \in \mathbb{C}$ and $R < |\alpha_k| \leq 1$, $k = 0, 1, 2, \ldots, K$, define

$$h^{a,K} = \mathfrak{R}(h^{a,K}), h^{a,K} = h_r + c_0 h_{\beta_1} + \sum_{i=1}^{K} c_k h_{\beta_i}(\alpha_k).$$

(68)

Then $h^{a,K}$ is $Y$-harmonic and $\partial B$-determined. Furthermore, for

$$c^* = \max_{y \in \partial B} |h^{a,K} - 1| < \infty$$

(69)

$h^{a,K}$ approximates $P_y(\tau < \infty)$ with relative error bounded by $c^*$.

**Proof** We know by Propositions 2.1 and 2.2 that $h^{a,K}$ is $Y$-harmonic. That $R < |\alpha_k| \leq 1$ and Proposition 7.1 imply that $h^{a,K}$ is also $\partial B$-determined, i.e.,

$$h^{a,K}(y) = E_y[h^{a,K}(Y_{\tau})1_{\tau<\infty}].$$

Taking the real part of both sides gives:

$$h^{a,K}(y) = \mathfrak{R}E_y[h^{a,K}(Y_{\tau})1_{\tau<\infty}],$$

(70)

i.e, $h^{a,K}$ is $Y$-harmonic and $\partial B$-determined. That $c^* < \infty$ follows from $\max(|\beta_1(\alpha_k)|, |\alpha(\beta_1(\alpha_k), \alpha_k)|) < 1$ (see Proposition 7.5). The inequality

$$1 - c^* < h^{a,K}(k, k) < 1 + c^*$$

(71)

follows from (69), $|\mathfrak{R}(z) - 1| \leq |z - 1|$ for any $z \in \mathbb{C}$. It follows from (71) and (70) that

$$(1 - c^*)E_y[1_{\tau<\infty}] \leq h^{a,K}(y) \leq (1 + c^*)E_y[1_{\tau<\infty}]$$

and

$$(1 - c^*)P_y(\tau < \infty) \leq h^{a,K}(y) \leq (1 + c^*)P_y(\tau < \infty).$$

This implies that $h^{a,K}$ approximates $P_y(\tau < \infty)$ with relative error bounded by $c^*$.

One of the key aspects of Proposition 7.6 is that it shows us how to compute an upper bound on the relative error of an approximation of the form (68) from the values it takes on $\partial B$. We can use this to choose the $\alpha_k$ and the $c_k$ to reduce relative error, the next subsection illustrates this procedure.

### 7.1 Finer approximations when $r^2 \neq \rho_1 \rho_2$

To illustrate how one can use approximations of the form (68) to improve on the approximation provided by Proposition 7.3, let us assign values to the parameters $\lambda_i$ and $\mu_i$ satisfying the assumptions (5), (6):

$$\lambda_1 = 0.1, \mu_1 = 0.2, \lambda_2 = 0.2, \mu_2 = 0.5;$$
for these choice of parameters we have
\[ r = \frac{\lambda_1 + \lambda_2}{\mu_1 + \mu_2} = \frac{3}{7}. \]
We note \( r^2 = 9/49 \neq 1/5 = \rho_1 \rho_2 \); therefore, we don’t have an explicit formula for \( P_\gamma(\tau < \infty) \). But Proposition 7.3 implies that
\[ h^{a,0}(y) = h_r + \frac{1 - r}{1 - \rho_2/r} h_{\rho_1} \]
approximates \( P_\gamma(\tau < \infty) \) with relative error bounded by
\[ C_8 - 1 = \frac{1 - r}{1 - \rho_2/r} (\rho_1^* \alpha_2^* - \alpha_2^*) = 0.3607, \]
where \( C_8 \) is computed as in (66). Then, by Theorem 6.1, \( h^{a,0}(T_\tau(x_0)) \) approximates \( P_{x_0}(\tau_\tau > \tau_0) \) with relative error converging to a level bounded by \( C_8 - 1 = 0.3607 \).

We can reduce this error by using further \( Y \)-harmonic functions given by Propositions 2.2 and 7.5 and constructing an approximation of the form (68). We note \( \beta_1(0.7) = 0.34610 \) and therefore, by an argument parallel to the proof of Proposition 7.5, we infer that \( |\beta_1(0.7)| \leq 0.34619 \), \(|(\beta_1(\alpha), \alpha)| < 1 \) for \(|\alpha| = 0.7\). Thus, \( h_{\beta_1(\alpha)} \) is \( Y \)-harmonic and \( \partial B \)-determined for all \( |\alpha| = 0.7 \), and we can use this class of functions in improving our approximation of \( P_\gamma(\tau < \infty) \). Let us begin with using \( K = 3 \) additional \( Y \)-harmonic functions of this form in our approximation: for the \( \alpha \)'s let us take
\[ \alpha_{1,j} = 0.7 e^{j \frac{4\pi}{3}}, j \in \{1, 2, 3 = K \}. \]

The resulting harmonic functions are
\[ h_{\beta_1(\alpha_{1,j})}, j \in \{1, 2, 3 = K \}. \]
[see (15) and (19)].

Our approximation \( h^{a,K} \) will be of the form
\[ h^{a,K} = \mathcal{R}(h^{a*,K}), \quad h^{a*,K} = h_r + c_{1,0} h_{\rho_1} + \sum_{j=1}^{K} c_{1,j} h_{\beta_1(\alpha_{1,j})}. \]

One can choose the coefficients \( c_{1,j}, j \in \{0, 1, 2, 3 = K \} \) in a number of ways, for example, by minimizing \( L_p \) errors. Here we will proceed in the following simple way: the ideal situation would be \( h^{a,K}(y) = 1 \) for all \( y \in \partial B \), which would mean \( h^{a,K}(y) = P_\gamma(\tau < \infty) \), but this will not hold in general. We will instead require that this identity holds for \( y = (k, k), k = 0, 1, 2, 3 \). This leads to the following four dimensional linear equation:
\[ 1 = h^{a,K}(k, k) = h_r((k, k)) + c_{1,0} h_{\rho_1}(k, k) + \sum_{j=1}^{3} c_{1,j} h_{\beta_1(\alpha_{1,j})}(k, k), \quad (73) \]
k = 0, 1, 2, 3; Solving (73) gives
\[ c_{1,0} = 7.80744 - 0.12974i, \quad c_{1,1} = -0.25880 + 1.46155i \]
\[ c_{1,2} = -0.26358 - 0.01349i, \quad c_{1,3} = 0.17597 + 0.01433i \]
Once the approximation is computed, following Proposition 7.6 one can easily compute its relative error in approximating $P_y(\tau < \infty)$ by computing
\[
\max_{y \in \partial B} |h^{a,K}_y(y) - 1|.
\]
That $\max_{j=1,2,..,K}(r, \rho_1, r^2/\rho_1, |\alpha_1,j|, |\beta_1(\alpha_1,j)|, |\alpha(\beta_1(\alpha_1,j), \alpha_1,j)|) < 1$ implies argmax$_{y \in \partial B}[h^{a,K}_y(y) - 1] = 1$ is finite. For $h^{a,K}$ computed above, the maximizer turns out to be $y^* = (4, 4) = (K + 1, K + 1)$ and the maximum approximation error is
\[
c^* = \max_{y \in \partial B} |h^{a,K}_y(y) - 1| = |h^{a,K}((y^*)) - 1| = 0.17764
\] (74)

The graph of the approximation error $|h^{a,K}_y(y) - 1|, y \in \partial B$, is shown in Fig. 6a.

By Proposition 7.6
\[
\left| P_y(\tau < \infty) - h^{a,0}(y) \right| \leq c^*.
\]

Theorem 6.1 now implies that $h^{a,K}(T_n(x_n))$ approximates $P_{x_n}(\tau_n < \tau_0)$ with relative error bounded by $c^* = 0.17764$ for $n$ large. Therefore, improving our approximation from $h^{a,0}$ of (72) to $h^{a,3}$ by adding three $Y$-harmonic functions of the form $h^{\beta_1(\alpha_1,j)}$ to the approximating basis, decreases the maximum relative error from $c^* = 0.3607$ to $c^* = 0.17764$. Figure 6b shows the level curves of $-\frac{1}{n} \log h^{a,3}(T_n(x))$ and $-\frac{1}{n} \log P_x(\tau_n < \tau_0)$ (the latter computed numerically via iteration of the harmonic equation satisfied by $P_x(\tau_n < \tau_0)$ for $n = 60$; the level curves overlap completely except along $\partial_1$, as suggested by our analysis.

To illustrate how the approximation error decreases when $K$ increases, let us repeat the computation above with $K = 20$. The resulting maximum relative error turns out to be:
\[
c^{20} = \max_{y \in \partial B} |h^{a,K}_{20}(y) - 1| = |h^{a,K}_{20}((21, 21)) - 1| = 1.6211 \times 10^{-3}.
\]

The probability $P_{(4,0)}(\tau_{60} < \tau_0)$, computed numerically, equals $4.6658 \times 10^{-17}$, the best approximation of this quantity computed above is $h^{a,20}(56, 0) = 5.2 \times 10^{-17}$. The discrepancy arises from the proximity of $(4, 0)$ to $\partial_1$. As we move away from the $\partial_1$, these quantities
get closer \( P_{(10,0)}(\tau_{60} < \tau_0) = 3.3303 \times 10^{-15}, h_{a,20}(50, 0) = 3.3358 \times 10^{-15} \), compatible with the maximum relative error computed above. Figure 7 shows how approximation improves as \( K \) increases:

![Figure 7](image.png)

**Fig. 7** \( K \mapsto h_{a,K}(50, 0) \) and \( P_{(10,0)}(\tau_{60} < \tau_0) \) (the flat line), drawn at \( 10^{-15} \) scale

## 8 Comparison with the tandem case

This section compares the analysis and results of the current work to those of Sezer (2018) treating the approximation of the probability \( P_x(\tau_n < \tau_0) \) for the constrained random walk representing two tandem queues, which has the increments \( (1, 0), (-1, 1) \) and \( (0, -1) \). The main idea is the same for both walks: i.e., approximation of \( P_x(\tau_n < \tau_0) \) by \( P_y(\tau < \infty) \) and computing/approximating the latter via harmonic functions constructed out of single and conjugate points on the characteristic surface. However, the assumptions, the results and the analysis manifest nontrivial differences. Let us begin with the assumptions:

**Assumption** \( r^2/\rho_2 < 1 \) In the tandem case \( \beta_1(1) = \rho_2 \) and the conjugate point of \( (\rho_2, \rho_1) \) is \( (\rho_2, \rho_1) \), therefore, the stability assumption automatically implies \( \alpha(r, 1) < 1 \). For the parallel case, \( \alpha(r, 1) \) can indeed be greater than 1 if \( r \) and \( \rho_1 \) are close and \( \rho_2 \) is small; we therefore explicitly assume \( r^2/\rho_2 < 1 \). This assumption appears in two places: 1) in the convergence analysis, in the derivation of the bound (20) and 2) in the computation of \( P_y(\tau < \infty) \) in Sect. 7. We think that the use of the assumption \( r^2/\rho_2 < 1 \) in the first case can be removed without much change from the arguments of the present and earlier works; the details remain for future work. The assumption \( r^2/\rho_2 < 1 \) enables the function \( h_r \) to be approximately 1 away from \( (0, 0) \); when \( r^2/\rho_2 > 1 \) \( h(r)(k, k) \) will grow exponentially in \( k \). For this reason, the assumption \( r^2/\rho_2 < 1 \) plays a key in the computation of \( P_y(\tau < \infty) \). This computation presents genuine difficulties when \( r^2/\rho_2 > 1 \); the treatment of this case remains for future work.

Next we point out the differences in results:
Region where \( \gamma (\tau < \infty) \) is a good approximation for \( P_x (\tau_n < \tau_0) \) That the tandem walk involves no jumps of the form \((-1, 0)\) implies that \( P_{\tau_n (\gamma a)} (\tau < \infty) \) provides an approximation of \( P_{\tau_n} (\tau_n < \tau_0) \) with exponentially decaying relative error for all \( x \) away from 0; in contrast, the presence of the jump \((-1, 0)\) in the parallel case, implies that the same approximation works only away from \( \partial_1 \) for the parallel walk case treated in the present work. This difference shows itself in the proofs of exponential decay of relative error, too, this is discussed below.

Explicit formula for \( \gamma (\tau < \infty) \) In the case of the tandem walk, the probability \( P_x (\tau < \infty) \) can be explicitly represented as a linear combination of the harmonic functions \( h_{\rho_1} \) and \( h_{\rho_2} \) for all stable parameter values as long as \( \mu_1 \neq \mu_2 \); in the parallel case this only happens when \( r^2 = \rho_1 \rho_2 \) (see Proposition 7.2). When \( r^2 \neq \rho_1 \rho_2 \), \( h_r \) and \( h_{\rho_1} \) can only provide an approximation of \( P_x (\tau < \infty) \) with bounded relative error (Proposition 7.3). This relative error can be reduced by adding into the approximation further \( \partial B \)-determined \( Y \)-harmonic functions (Proposition 7.6 and Sect. 7.1).

The changes in argument from the tandem walk to the parallel walk are as follows:

Analysis of \( P_x (\tau_n < \tau_0) \) In previous works (Dupuis et al. 2007b; Sezer 2005, 2009, 2015) the LD analysis of \( P_x (\tau_n < \tau_0) \) and similar quantities are based on sub and supersolutions of the limit HJB equation, similar to the analysis given in Sect. 4.1. In the present work, a novelty is the use of explicit subharmonic functions (Proposition 4.2) of the constrained random walk \( X \) in the proof of the upperbound Proposition 4.3.

Analysis of \( P_x (\sigma_1 < \tau_n < \tau_0) \) The probability corresponding to \( P_x (\sigma_1 < \tau_n < \tau_0) \) in the tandem case is \( P_x (\sigma_1 < \sigma_1, 2 < \sigma_1 < \tau_0) \). For the proof of the exponential decay of the relative error, we need upperbound on these probabilities. Both papers develop these upperbound from subsolutions to a limit HJB equation. The subsolution consists of three pieces (one for each of the stopping times \( \sigma_1, \sigma_1, 2 \) and \( \tau_n \)) for the tandem walk, and two pieces for the parallel walk (one for each of the times \( \sigma_1 \) and \( \tau_n \)). In the tandem case, the pieces of the subsolution are constructed from the subsolution for the probability \( P_x (\tau_n < \tau_0) \), whereas in the parallel case a new piece is introduced based on the gradient \( r_4 \) of (50).

Analysis of \( P_x (\tilde{\sigma}_1 < \tau < \infty) \) The probability corresponding to \( P_x (\tilde{\sigma}_1 < \tau < \infty) \) in the tandem case is \( P_x (\tilde{\sigma}_1 < \tilde{\sigma}_1, 2 < \tau < \infty) \). The special nature of the tandem walk allowed us to find upperbounds on this probability from the explicit formula we have for \( P_x (\tau < \infty) \); this significantly simplified the analysis of the tandem walk case. For the parallel walk, we extended the analysis of \( P_x (\sigma_1 < \tau < \tau_0) \), based on subsolutions, to \( P_x (\tilde{\sigma}_1 < \tau < \infty) \). In this, the most significant novelty is the analysis given Sect. 3, where we prove the existence of \( z > 1 \) such that \( \mathbb{E}_z [z^{\tau} 1_{\{\tau < \infty\}}] < \infty \). For this, we introduce what we call \( Y - z \)-harmonic functions and provide methods of construction of classes of them from points on 1/z-level characteristic surfaces, which are generalizations of characteristic surfaces.

9 Conclusion

The probability \( P_x (\tau < \infty) \) approximates \( P_x (\tau_n < \tau_0) \) well when \( x \) is away from \( \partial_1 \); as noted in the previous section, this is in contrast to the tandem case, where the approximation is good away from the origin. How can one extend the approximation to the region along \( \partial_1 \)? A natural idea, already pointed out in Sezer (2015) is to repeat the same analysis, but this time taking the corner \((0, n)\) as the origin of the \( Y \) process, i.e., to use the change of coordinate \( y = T_n (x) = (x(1), n - x(2)) \) to construct the \( Y \) process. Numerical calculations indicate
that the resulting approximation will be accurate (i.e., exponentially decaying relative error) along $\partial_1$ between the points $(0, n)$ and $(0, [(1 - C_4) n])$ [see (30)] for the definition of $C_4$.

We believe that arguments and computations parallel to the ones given in the present work would imply these results; the details are left for future work. We think that the extension of the approximation to the region along the line segment between $(0, 0)$ and $(0, [(1 - c_1) n])$ requires further ideas and computations.

We expect the analysis linking $P_y(\tau_n < \tau_0)$ to $P_y(\tau < \infty)$ when $\rho_1 = \rho_2$ to be parallel to the analysis given in the current work. For the computation of $P_y(\tau < \infty)$, when $\rho_1 = \rho_2$, the case $\lambda_1 = \lambda_2, \mu_1 = \mu_2$ appears to be particularly simple. In this case, upon taking limits in (61) one obtains

$$P_y(\tau < \infty) = r^{\gamma(1) - \gamma(2)} + (1 - r)r^{\gamma(1)}(y(1) - y(2)),$$

where $r = \rho_1 = \rho_2$. A complete analysis of the computation of $P_y(\tau < \infty)$ when $\rho_1 = \rho_2$ remains for future work.

In Sect. 7.1, the computation of $P_y(\tau < \infty)$ when $r^2 \neq \rho_1 \rho_2$ proceeds as follows: 1) we first construct a candidate approximation $h^{a, K} = \mathcal{H}(h^{a, K})$ of $P_y(\tau < \infty)$ 2) we find an upperbound on the relative error of the approximation by finding the maximum of $|h^{a, K} - 1|$ on $\partial B$. A natural question is the following: given a relative error bound, can we know apriori that an approximation having that maximum relative error can be constructed? If that is possible, how many $Y$-harmonic functions of the form given by Proposition 2.2 would we need? To answer these questions require a fine understanding of the functional analytic properties of the span of the $\partial B$-determined $Y$-harmonic functions given by Propositions 2.1 and 2.2. This appears to be a difficult problem because the functions given in these propositions don’t have simple geometric properties, such as the orthogonality of the Fourier basis in $L^2$. A study of this problem remains for future work.

The work (Sezer 2018) and the present manuscript study the approximation of $P_x(\tau_n < \tau_0)$ for two tandem and parallel queues and for the exit boundary $\partial A_n$. A natural direction for future research is the extension of these analyses to constrained random walks representing general two dimensional Jackson networks. The work (Miyazawa 2011) studies constrained processes on $\mathbb{Z}_+^2$ and on $\mathbb{Z} \times \mathbb{Z}_+$ taking the steps $V \cup \{+e_1 + e_2, \pm(e_1 - e_2)\}$ and having distinct increment distributions on the constraining boundaries or on different regions of the domain of the process. Miyazawa (2011) carries out an asymptotic analysis of the tails of the stationary distribution of this process using points on curves associated with the process. An extension of the analysis of exit probabilities of the form $P_x(\tau_n < \tau_0)$ to this class of processes is a further direction for future research.

The exact formula for $P_y(\tau < \infty)$ for the tandem case has a remarkable extension to $d$ dimensions; this is derived in Sezer (2015) and is based on harmonic-systems, a concept defined in that work. We think that it is also possible, in the case of parallel queues, to obtain nontrivial harmonic systems in higher dimensions. A complete characterization of such systems and the question of under what conditions they would give a rich class of $Y$-harmonic functions to approximate $P_y(\tau < \infty)$ also remain challenging problems for future research.

In the computer science literature, the simple random walk is used as a model of two stacks running on a finite memory. In this context the expectation

$$\mathbb{E} \left[ \max(X_1(\tau_n), X_2(\tau_n)) \right],$$

i.e., the expected size of the longest stack at the time of memory overflow has received much attention. It was first proposed in Knuth (1972) and computed in the same work for
Various versions of this problem has since been treated in Yao (1981), Flajolet (1986), Maier (1991) and Louchard et al. (1994). We refer the reader to Ünlü (2018), Chapter 5 for a detailed review. All of these works either assume that either the original walk is unstable, metastable (i.e., all jump probabilities are 1/4) or that the jump probabilities are symmetric (i.e., both queues have the same arrival and service rates). Another direction for future research is the treatment of this expectation for the general stable two dimensional simple random walk with the techniques used in the present manuscript.

A Proof of Proposition 7.1

Define

\[
\zeta_n \doteq \inf \{ k : Y_k(1) - Y_k(2) = n \};
\]

(76)

\(\zeta_n\) is the first time \(Y\) hits the line \(\{ y : y(1) - y(2) = n \}\). Starting inside \(B_n \doteq \{ y \in \mathbb{Z} \times \mathbb{Z}_+ : 0 \leq y(1) - y(2) \leq n \}\), \(Y\) cannot remain in \(B_n\) forever:

**Lemma 7** For \(y \in B_n\)

\[
P_y(\zeta_n \land \zeta_0 = \infty) = 0.
\]

(77)

**Proof** For \(y \in B_n\)

\[
P_y(\zeta_n \land \zeta_0 \leq n) > (\lambda_1 + \lambda_2)^n
\]

(78)

because, at least the sample paths whose increments consist only of \(\{-e_1, e_2\}\) push \(Y\) to \(\partial B\) in \(n\) steps and the total probability of such paths is \((\lambda_1 + \lambda_2)^n\).

\(\hat{Y}_k \doteq Y_{nk \land \zeta_n \land \zeta_0}\) is Markov on \(B_n\). The boundary of \(B_n\) is

\[
\partial B_n = \{ y \in \mathbb{Z} \times \mathbb{Z}_+ , y(1) = y(2) \text{ or } y(1) = y(2) + n \}.
\]

By definition

\[
P_y(\zeta_0 \land \zeta_n = \infty) \leq P_y(\hat{Y}_k \in B_n - \partial B_n).
\]

(79)

The bound (78) implies

\[
P_y\left(\hat{Y}_1 \in B_n - \partial B_n\right) \leq 1 - (\lambda_1 + \lambda_2)^n. \text{ This and that } \hat{Y} \text{ is Markov give}
\]

\[
P_y\left(\hat{Y}_k \in B_n - \partial B_n\right) \leq (1 - (\lambda_1 + \lambda_2)^n)^k.
\]

This and (79) imply

\[
P_y(\zeta_0 \land \zeta_n = \infty) \leq (1 - (\lambda_1 + \lambda_2)^n)^k.
\]

(80)

Letting \(k \rightarrow \infty\) gives (77).

\(\square\)

For reader’s convenience we repeat the statement of Proposition 7.1:

**Proposition A.1** A \(Y\)-harmonic function \(h\) of the form \(\sum_{i=1}^J c_i[(\beta_i, \alpha_i), .]\) is \(\partial B\) determined if

\[
|\beta_i| < 1, |\alpha_i| \leq 1, i = 1, 2, 3, ..., I.
\]

(81)
The assumption (81) and \[\{\beta_i, \alpha_i\}, y\] = \(\beta_i^{Y(1)} - \gamma(2), \alpha_i^{Y(2)}\) imply that \(h\) is bounded on \(B^a = \{y \in \mathbb{Z} \times \mathbb{Z}_+: y(1) > y(2)\}\). Then \(M_k = h(Y_t \wedge \zeta_n \wedge k)\) is a bounded martingale. This, Lemma 7 and the optional sampling theorem imply
\[
h(y) = \mathbb{E}_y \left[ h(Y_t 1_{\{\tau \leq \zeta_n, \tau < \infty\}}) \right] + \mathbb{E}_y \left[ h(Y_{\zeta_n} 1_{\{\zeta_n < \tau, \zeta_n < \infty\}}) \right], y \in B^a. \tag{82}
\]
\(Y_{\zeta_n}(1) = n\) for \(\zeta_n < \infty\) This, (81) and the form of \(h\) imply
\[
\lim_{n \to \infty} \mathbb{E}_y \left[ h(y) 1_{\{\tau \leq \zeta_n\}} \right] \leq \lim_{n \to \infty} \beta_n^a c_n = 0
\]
where \(\beta_n = \max_{i=1}^{l} |\beta_i|\) and \(c_n = \sum_{i=1}^{l} |c_i|\). This, \(\lim_n \zeta_n = \infty\) and letting \(n \to \infty\) in (82) give
\[
h(y) = \mathbb{E}_y \left[ h(Y_\tau) 1_{\{\tau < \infty\}} \right],
\]
i.e., \(h\) is \(\partial B\)-determined.

\[\square\]

References

Alanyali, M., & Hajek, B. (1998). On large deviations in load sharing networks. Annals of Applied Probability, 8, 67–97.

Aldous, D. (2013). Probability approximations via the poisson clumping heuristic (Vol. 77). Berlin: Springer.

Anantharam, J., Heidelberger, P., & Tsoucas, P. (1990). Analysis of state-independent importance-sampling measures for the two-node tandem queue. ACM Transactions on Modeling and Computer Simulation (TOMACS), 22(3), 698–719.

Atar, R., & Dupuis, P. (1999). Stochastic simulation: Algorithms and analysis (Vol. 51). Berlin: Springer.

Asmussen, S. (2008). Large deviations for markov chains via time reversal and fluid approximation. IBM Research: Tech Rep.

Atar, R., & Dupuis, P. (1999). Large deviations and queueing networks: Methods for rate function identification. Stochastic Processes and Their Applications, 84(2), 255–296.

Blanchet, J. (2013). Optimal sampling of overflow paths in Jackson networks. Mathematics of Operations Research, 38(4), 698–719.

Blanchet, J., Glynn, P., & Leder, K. (2008). Efficient simulation of light-tailed sums: An old folk song sung to a faster new tune. Monte Carlo and Quasi-Monte Carlo Methods, 2008, 227–258.

Blanchet, J., Glynn, P., & Leder, K. (2012). On Lyapunov inequalities and subsolutions for efficient importance sampling. ACM Transactions on Modeling and Computer Simulation (TOMACS), 22(3), 13.

Blanchet, J., & Mandjes, M. (2009). Rare event simulation for queues in Rare event simulation using Monte Carlo methods. In G. Rubino & B. Tuffin (Eds.), Rare event simulation for queues (pp. 87–124). Wiley.

Borovkov, A. A., & Mogul’skii, A. A. (2001). Large deviations for markov chains in the positive quadrant. Russian Mathematical Surveys, 56(5), 803–916.

Boué, M., Dupuis, P., & Ellis, R. S. (2000). Large deviations for small noise diffusions with discontinuous statistics. Probability Theory Related Fields, 116(1), 125–149.

Chang, C.-S., Heidelberger, P., Juneja, S., & Shahabuddin, P. (1994). Effective bandwidth and fast simulation of ATM intree networks. Performance Evaluation, 20, 45–66.

Collingwood, J., Foley, R. D., & McDonald, D. R. (2011). Networks with cascading overloads. In Proceedings of the 6th international conference on queueing theory and network applications (pp. 33–37). ACM.

Comets, F., Delarue, F., & Schott, R. (2007). Distributed algorithms in an ergodic Markovian environment. Random Structures & Algorithms, 30(1–2), 131–167.

Comets, F., Delarue, F., & Schott, R. (2009). Large deviations analysis for distributed algorithms in an ergodic Markovian environment. Applied Mathematics and Optimization, 60(3), 341–396.

Crane, M. A., & Iglehart, D. L. (1974). Simulating stable stochastic systems, I: General multiserver queue. Journal of the Association for Computing Machinery, 21(1), 103–113.

Dai, J. G., Miyazawa, M., et al. (2011). Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution. Stochastic Systems, 1(1), 146–208.

de Boer, P.-T. (2006). Analysis of state-independent importance-sampling measures for the two-node tandem queue. ACM Transactions on Modeling and Computer Simulation (TOMACS), 16(3), 225–250.

De Boer, P.-T., Kroese, D. P., & Rubenstein, R. Y. (2004). A fast cross-entropy method for estimating buffer overflows in queueing networks. Management Science, 50, 883–895.
McDonald, D. R. (1999). Asymptotics of first passage times for random walk in an orthant. *Annals of Applied Probability*, 9, 110–145.

Miretskiy, D., Scheinhardt, W., & Mandjes, M. (2010). State-dependent importance sampling for a Jackson tandem network. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 20(3), 15.

Miyazawa, M. (2009). Tail decay rates in double QBD processes and related reflected random walks. *Mathematics of Operations Research*, 34(3), 547–575.

Miyazawa, M. (2011). Light tail asymptotics in multidimensional reflecting processes for queueing networks. *Top*, 19(2), 233–299.

Ney, P., & Nummelin, E. (1987). Markov additive processes I. Eigenvalue properties and limit theorems. *The Annals of Probability*, 15, 561–592.

Nicola, V., & Zaburnenko, T. (2007). Efficient importance sampling heuristics for the simulation of population overflow in Jackson networks. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 17(2), 10.

Parekh, S., & Walrand, J. (1989). A quick simulation method for excessive backlogs in networks of queues. *IEEE Transactions on Automatic Control*, 34(1), 54–66.

Randhawa, R. S., & Juneja, S. (2004). Combining importance sampling and temporal difference control variates to simulate Markov chains. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 14(1), 1–30.

Riddler, A. (2009). Importance sampling algorithms for first passage time probabilities in the infinite server queue. *European Journal of Operational Research*, 199(1), 176–186.

Robert, P. (2003). *Stochastic networks and queues, stochastic modelling and applied probability series* (Vol. 52). New York: Springer.

Rubino, G., & Tuffin, B. (2009). *Rare event simulation using Monte Carlo methods*. New York: Wiley.

Setayeshgar, L., & Wang, H. (2013). Efficient importance sampling schemes for a feed-forward network. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 23(4), 21.

Sezer, A. D. (2005). *Dynamic importance sampling for queueing networks*. Ph.D. thesis, Brown University Division of Applied Mathematics.

Sezer, A. D. (2007). Asymptotically optimal importance sampling for Jackson networks with a tree topology. Preprint. [http://arxiv.org/abs/0708.3260](http://arxiv.org/abs/0708.3260).

Sezer, A. D. (2009). Importance sampling for a Markov modulated queuing network. *Stochastic Processes and Their Applications*, 119(2), 491–517.

Sezer, A. D. (2010). Asymptotically optimal importance sampling for Jackson networks with a tree topology. *Queueing Systems*, 64(2), 103–117. Longer (2007) version available at [http://arxiv.org/abs/0708.3260](http://arxiv.org/abs/0708.3260).

Sezer, A. D. (2015). Exit probabilities and balayage of constrained random walks. [https://arxiv.org/abs/1506.08674](https://arxiv.org/abs/1506.08674).

Sezer, A. D. (2018). Approximation of excessive backlog probabilities of two tandem queues. *Journal of Applied Probability*, 55(3), 968–997.

Shwartz, A., Weiss, A. (1995). *Large deviations for performance analysis. Stochastic modeling series*. London: Chapman & Hall, Queues, communications, and computing. With an appendix by Robert J. Vanderbei.

Ünlü, K. D. (2018). *Exit probabilities of constrained simple random walks*. Ph.D. thesis, Institute of Applied Mathematics, Middle East Technical University.

Yao, A. C. (1981). An analysis of a memory allocation scheme for implementing stacks. *SIAM Journal on Computing*, 10(2), 398–403.

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