DOES STRING THEORY HAVE A DUALITY SYMMETRY RELATING WEAK AND STRONG COUPLING?*

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ABSTRACT

The heterotic string theory, compactified to four dimensions, has been conjectured to have a duality symmetry (S duality) that transforms the dilaton non-linearly. If valid, this symmetry could provide an important means of obtaining information about nonperturbative features of the theory. Even though it is inherently nonperturbative, S duality exhibits many similarities with the well-established target-space duality symmetry (T duality), which does act perturbatively. These similarities are manifest in a new version of the low-energy effective field theory and in the soliton spectrum obtained by saturating the Bogomol’nyi bound. Curiously, there is evidence that the roles of the S and T dualities are interchanged in passing to a five-brane formulation.

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Introduction

This talk reports on work done recently in collaboration with Ashoke Sen [1] [2], which investigated various issues concerning two types of duality symmetries that have been considered in string theory. One, which is well-established to all orders in string perturbation theory, is known as “target space duality,” or more succinctly as “T duality.” This discrete symmetry group is a generic feature of theories with compactified spatial dimensions. It is actually a discrete gauge group, so it should remain valid when nonperturbative effects are taken into account. The simplest example is the $Z_2$ symmetry that arises when one dimension is compactified with the topology of a circle. In this case, it can be described as the equivalence of a circle of radius $R$ and one of radius $\alpha'/R$. In fact, it can be realized as a field transformation, since $R$ corresponds to the classical value of a scalar field. More generally, for the four-dimensional heterotic string, compactified on a torus that is dual to an even self-dual lattice in the manner proposed by Narain [3], the corresponding T duality group turns out to be O(6,22;Z). This example has N=4 supersymmetry, and is therefore certainly unrealistic, but it is a particularly nice example to study. Many, but not all, of its properties are expected to hold in more realistic settings, but that question will not be explored in detail here.

The second kind of duality, which is our main focus, is much more speculative. It is an SL(2,Z) group that was discovered many years ago as a symmetry of the classical field equations of N=4 supergravity [4]. (This paper actually identified an SL(2,R) symmetry. Instanton effects are expected to break the symmetry to the discrete subgroup.) N=4 supergravity contains a dilaton $\phi$ and an axion $\chi$, which can be combined in a complex scalar field $\lambda$ as follows

$$\lambda = \lambda_1 + i\lambda_2 = \chi + ie^{-\phi}.$$  (1)

This field transforms nonlinearly under SL(2,Z)

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d},$$  (2)

where $a, b, c, d$ are integers satisfying $ad - bc = 1$. Since the value of the field $\lambda$
determines the coupling constant $g$ and the vacuum angle $\theta$ according to

$$< \lambda > = \frac{\theta}{2\pi} + \frac{8\pi i}{g^2},$$

(3)

such a symmetry is necessarily nonperturbative. In particular, the transformation $\lambda \to -1/\lambda$, for $\theta = 0$, inverts the coupling constant. The complex field $\lambda$ is present in the massless spectrum even for compactifications that only leave N=1 supersymmetry. It is therefore possible to speculate that it might be a symmetry of the full nonperturbative string theory in that case, which is what was done in 1990 by Font et al.[5]. This was, to the say the least, a very bold conjecture. The proposed symmetry could be called “dilaton–axion duality” or “weak coupling–strong coupling duality.” These are rather cumbersome, so we propose to refer to it as “S duality.” While S duality is still far from established, we find that all our studies support it. Indeed, when analyzed in the proper way, S duality and T duality have a great deal in common.

4D Effective Field Theory

This section summarizes material I presented at Strings ’92 in Rome [6]. The low-energy effective field theory that describes the massless bosonic fields associated with Narain compactification of the heterotic string has a T duality symmetry group $G_T = O(6,22;\mathbb{Z})$ and scalar fields (moduli) that parametrize the moduli space $O(6,22)/O(6) \times O(22) \times G_T$. These fields are conveniently described by a $28 \times 28$ matrix-valued scalar field $M^{ab}$ satisfying the constraints

$$M^T = M, \quad M^T L M = L,$$

(4)

where $L$ is the $O(6,22)$ metric

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}.$$

(5)

For generic values of the moduli fields, the spectrum also contains 28 massless abelian gauge fields (so that the gauge group is $[U(1)]^{28}$). For special values of the...
moduli there are additional massless gauge fields and enhanced gauge symmetry. However, to keep things as simple as possible, we will only include the 28 gauge fields $A_\mu^a$ that are massless for all values of the moduli. The other massless bosons are the graviton (described by a metric tensor $g_{\mu\nu}$), the dilaton $\phi$, and an antisymmetric tensor $B_{\mu\nu}$. The action for this theory can be obtained in a variety of ways, one of the easiest of which is dimensional reduction from ten dimensions [7]. In terms of the “string metric,” the result is

$$S = \int_M dx \sqrt{-g} e^{-\phi}(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) ,$$

(6)

where

$$\mathcal{L}_1 = R$$

$$\mathcal{L}_2 = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$\mathcal{L}_3 = -\frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}$$

$$\mathcal{L}_4 = \frac{1}{8} g^{\mu\nu} \text{tr}(\partial_\mu M L \partial_\nu M L)$$

$$\mathcal{L}_5 = -\frac{1}{4} F_{\mu\nu}^a (M L L)_{ab} F^{b\mu\nu} ,$$

(7)

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a .$$

(8)

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \frac{1}{2} A_\mu^a L_{ab} F_{\nu\rho}^b + \text{(cyc. perms.)} .$$

(9)

This result has manifest T duality since the metric and dilaton are invariant under T transformations, and the gauge fields transform by the vector representation of O(6,22).

This theory also has S duality symmetry, though this is not at all apparent in the form given above. To exhibit this symmetry, it is convenient to replace the string metric by the canonical metric by means of the Weyl rescaling $g_{\mu\nu} \to e^\phi g_{\mu\nu}$,
since the canonical metric will be invariant under $S$ duality, but the dilaton field is not. Also, to exhibit the axion $\chi$, it is necessary to make a duality transformation

$$\sqrt{-g} \, e^{-2\phi} H^{\mu\nu\rho} \to e^{\mu\nu\rho\lambda} \partial_\lambda \chi,$$

which (as usual) interchanges the role of a field equation and a Bianchi identity. Then one can introduce the complex field $\lambda$ defined in eq. (1) and write a “dual action,” whose classical field equations are equivalent to those obtained from the original action.

$$S_{\text{dual}} = \int_M dx \sqrt{-g} (\mathcal{L}'_1 + \mathcal{L}'_{2,3} + \mathcal{L}'_4 + \mathcal{L}'_5),$$

where

$$\mathcal{L}'_1 = R$$

$$\mathcal{L}'_{2,3} = -\frac{1}{2\lambda^2} g^{\mu\nu} \partial_\mu \lambda \partial_\nu \bar{\lambda}$$

$$\mathcal{L}'_4 = \frac{1}{8} g^{\mu\nu} \text{tr} (\partial_\mu M L \partial_\nu M L)$$

$$\mathcal{L}'_5 = -\frac{\lambda_2}{4} F^a_{\mu\nu} (L M L)_{ab} F^{b\mu\nu} + \frac{\lambda_1}{4} F^{a}_{\mu\nu} F_{ab} \bar{F}^{b\mu\nu},$$

where

$$\bar{F}^{a\mu\nu} = \frac{1}{2\sqrt{-g}} e^{\mu\nu\rho\sigma} F^a_{\rho\sigma}.$$ 

All terms in the action $S_{\text{dual}}$ are invariant under $S$ duality, except for $\mathcal{L}'_5$. However, the equations of motion do transform covariantly under $S$ duality provided that when $\lambda \to \frac{a\lambda + b}{c\lambda + d}$ (and $ad - bc = 1$),

$$F^a_{\mu\nu} \to c\lambda_2 (M L)_{ab} \tilde{F}^b_{\mu\nu} + (c\lambda_1 + d) F^a_{\mu\nu}.$$ 

Note that in terms of the gauge fields themselves this is a nasty nonlocal transformation.

**Manifest S Duality**
The construction of the action $S_{\text{dual}}$ is a significant step towards exhibiting S duality, but because of the noninvariance of $\mathcal{L}_5'$, it is not the final form. We would like to have a third form of the action with both dualities (S and T) made manifest, thereby putting them on an equal footing. The main problem in realizing S duality in $S_{\text{dual}}$ is attributable to the gauge fields. The basic idea for overcoming this difficulty is to replace each gauge field $A_\mu$ by a pair of independent gauge fields $A^{(\alpha)}_\mu$, $\alpha = 1, 2$ and to obtain the relation $F^{(2)}_{\mu\nu} = \tilde{F}^{(1)}_{\mu\nu}$ as an equation of motion. The formulas will not have manifest Lorentz invariance, though they will have manifest rotational symmetry. Accordingly, it is convenient to introduce separate “electric” and “magnetic” fields

$$E_i^{(\alpha)} = \partial_0 A_i^{(\alpha)} - \partial_i A_0^{(\alpha)}, \quad B^{(\alpha)i} = \epsilon^{ijk} \partial_j A_k^{(\alpha)}, \quad 1 \leq i, j, k \leq 3. \quad (15)$$

Let us first describe a two-potential version of free Maxwell theory, and explain how that theory is coupled to gravity, before applying the results to the problem of making S duality manifest. Our action for free Maxwell theory is

$$S = -\frac{1}{2} \int d^4x \left( B^{(\alpha)i} \mathcal{L}_{\alpha\beta} E_i^{(\beta)} + B^{(\alpha)i} B^{(\alpha)i} \right), \quad (16)$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (17)$$

is the metric for $\text{SL}(2,\mathbb{R}) = \text{Sp}(2)$. The symbols $\mathcal{L}$ and $\mathcal{M}$ (introduced below) to describe S duality are chosen to emphasize the analogy with $L$ and $M$ used in the description of T duality. The action (16) has the following gauge invariances

$$\delta A_0^{(\alpha)} = \Psi^{(\alpha)}, \quad \delta A_i^{(\alpha)} = \partial_i \Lambda^{(\alpha)}. \quad (18)$$

It is easy to show that it describes a single propagating photon with two physical polarizations. To understand how it works, note that up to total derivatives the
first term is proportional to $\epsilon^{ijk} \partial_0 A^{(1)}_i \partial_j A^{(2)}_k$. Therefore the fields $A^{(a)}_0$ do not give any classical equations of motion. Cast in this form, the fields $A^{(2)}_i$ have no time derivatives and can be treated as auxiliary. Thus the (integrated) equation of motion $B^{(2)}_i = E^{(1)}_i$ can be used to eliminate $A^{(2)}_i$ from the action. This gives rise to the standard Maxwell action in the $A^{(1)}_0 = 0$ gauge. Gauss’s law ($\partial_i E^{(1)}_i = 0$) is implied by the Bianchi identity for $B^{(2)}_i$. The action (16) is manifestly invariant under the electric–magnetic duality symmetry

$$A^{(a)}_\mu \rightarrow \mathcal{L}_{\alpha\beta} A^{(\beta)}_\mu,$$

which corresponds to $F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}$. This is to be contrasted with the usual Maxwell action, $\frac{1}{2} \int (E^2 - B^2) d^4x$, which goes to its negative. While not manifestly Lorentz invariant, the action (16) is in fact invariant under the global transformations

$$\delta A^{(a)}_i = x^0 v^k \partial_k A^{(a)}_i + \bar{v} \cdot \bar{v} \mathcal{L}_{\alpha\beta} \epsilon^{ijk} \partial_j A^{(\beta)}_k.$$  

(20)

On the mass shell, this is identical to the usual Lorentz transformation formula with boost parameter $v^i$.

Let us now consider how to couple (16) to gravity. Since we do not have manifest Lorentz invariance before coupling to gravity, we do not expect manifest general coordinate invariance after coupling to gravity. Nonetheless it is not difficult to figure out which action gives the desired generally covariant field equations. The result is

$$S_g = -\frac{1}{2} \int d^4x \left[ B^{(a)i} \mathcal{L}_{\alpha\beta} E^{(\beta)}_i - \frac{g_{ij}}{\sqrt{-g}} B^{(a)i} B^{(a)j} + \epsilon^{ijk} \frac{g_{00}^0}{g_{ij}} B^{(a)i} \mathcal{L}_{\alpha\beta} B^{(\beta)j} \right].$$  

(21)

As usual, $\sqrt{-g} = \sqrt{-\text{det}(g_{\mu\nu})}$ and $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, the ordinary four-dimensional metric. These conventions are retained even when space and time components are enumerated separately. If one eliminates the fields $A^{(2)}_{\mu}$ by using their field equations as before, one obtains the standard action $-\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu}^{(1)} F^{(1)}_{\mu\nu}$.  

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Equation (21) is invariant under general coordinate transformations with the metric transforming in the standard way and
\[
\delta A_i^{(\alpha)} = \xi^j \partial_j A_i^{(\alpha)} + (\partial_i \xi^j) A_j^{(\alpha)} + \xi^0 \left\{ - \frac{g_{ij}}{\sqrt{-g}} g^{00} \mathcal{L}_{\alpha\beta} B^{(\beta)j} - \frac{g^{0k}}{g^{00}} \epsilon^{ijk} B^{(\alpha)j} \right\}. \quad (22)
\]
As in the case of Lorentz transformations, which corresponds to setting \(\xi^0 = \vec{v} \cdot \vec{x}\), \(\xi^i = v^i x^0\), and \(g_{\mu\nu} = \eta_{\mu\nu}\), this differs from the standard transformation formula by an amount that vanishes when the classical equations of motion are satisfied.

Let us now return to the case of flat space-time and consider the generalization of (16) that includes the coupling to the axion–dilaton field \(\lambda\). The appropriate formula is
\[
S_{\lambda} = -\frac{1}{2} \int d^4 x \left[ B^{(\alpha)i} \mathcal{L}_{\alpha\beta} E_i^{(\beta)} + B^{(\alpha)i} (\mathcal{L}^T \mathcal{M} \mathcal{L})_{\alpha\beta} B^{(\beta)i} \right], \quad (23)
\]
where
\[
\mathcal{M}(\lambda) = \frac{1}{\lambda_2} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix}. \quad (24)
\]
The matrix \(\mathcal{M}\) is a symmetric SL(2,R) matrix, which therefore satisfies \(\mathcal{M}^T = \mathcal{M}\), \(\mathcal{M} \mathcal{L} \mathcal{M}^T = \mathcal{L}\), and \(\mathcal{L}\) is given in eq. (17). Under the SL(2,R) transformation (2) of the field \(\lambda\), \(\mathcal{M} \rightarrow \omega^T \mathcal{M} \omega\), where \(\omega = \begin{pmatrix} d & b \\ c & a \end{pmatrix}\). The action (23) is manifestly invariant provided that at the same time \(A_i^{(\alpha)} \rightarrow (\omega^T)_{\alpha\beta} A_i^{(\beta)}\). If we eliminate \(A_i^{(2)}\) from (23) by using its classical equation of motion, then we obtain the covariant expression \(L_5'\) given in eq. (12).

We are now in a position to give a third version of our theory that is classically equivalent to the two versions given earlier, but with both S duality and T duality realized as manifest symmetries. The formula that does the job is
\[
S = \int d^4 x \left[ \sqrt{-g} \left\{ R - \frac{1}{4} g^{\mu\nu} \text{tr}(\partial_\mu \mathcal{M} \mathcal{L} \partial_\nu \mathcal{M} \mathcal{L}) + \frac{1}{8} g^{\mu\nu} \text{tr}(\partial_\mu \mathcal{M} \mathcal{L} \partial_\nu \mathcal{M} \mathcal{L}) \right\} \right.

- \frac{1}{2} \left\{ B^{(a,\alpha)i} \mathcal{L}_{\alpha\beta} L_{ab} E_i^{(b,\beta)} + \epsilon^{ijk} \frac{g^{0k}}{g^{00}} B^{(a,\alpha)i} \mathcal{L}_{\alpha\beta} L_{ab} B^{(b,\beta)j} \right\}

\left. - \frac{g_{ij}}{\sqrt{-g} g^{00}} B^{(a,\alpha)i} (\mathcal{L}^T \mathcal{M} \mathcal{L})_{\alpha\beta} (LML)_{ab} B^{(b,\beta)j} \right\}. \quad (25)
\]
In the above equation Tr denotes trace over the indices \( a, b \) and tr denotes trace over the indices \( \alpha, \beta \). In this expression we have recast the kinetic term of the \( \lambda \) field (\( \mathcal{L}_{2,3}' \) in eq. (12)) in terms of the matrix \( \mathcal{M} \).

Written in the form (25), it is clear that the S and T duality are realized in quite analogous ways in the low energy effective action, despite their profound difference from the point of view of compactified string theory. As we have seen, the price for making both of them manifest is the loss of manifest general coordinate invariance. The first form we presented (with manifest T duality) was derived by dimensional reduction of the ten-dimensional N=1 supergravity theory containing a two-form potential \( B_{\mu\nu} \). However, there is also a dual version of that theory in which the two-form potential is replaced by its dual, which is a six-form potential \( B_{\mu_1\mu_2\cdots\mu_6} \) [8]. Dimensional reduction of this dual theory gives a version of the four-dimensional theory that has manifest S duality but not T duality. This fact gives an interesting insight into the possible significance of S duality.

The two-form potential couples in a natural way to the world volume of the string and is therefore the natural choice from that point of view. However, in similar fashion the six-form potential couples to the world volume of a ‘five-brane,’ a five-dimensional extended object discovered as a soliton solution of the heterotic string [9] [10]. It has been conjectured in these references that there is a dual formulation of the heterotic string theory in which the five-brane is fundamental and the string is the soliton. It is an interesting possibility that in five-brane perturbation theory S duality is true order-by-order, whereas T duality becomes a non-perturbative symmetry. In other words, the roles of the S and T dualities are interchanged in passing from strings to five-branes, a phenomenon that we call “duality of dualities.” As evidence in support of this conjecture, it was shown that S duality describes the interchange of five-brane Kaluza–Klein modes and winding modes, just as T duality does for the compactified string theory.

Before continuing with our main theme, let us pause to describe briefly generalizations of the two-potential formalism we have used for recasting the Maxwell
action. What we have done is to introduce a pair of independent unconstrained potentials whose field strengths are dual to one another as a consequence of the equations of motion. This construction is easily generalized to \( m \)-forms in \( d \) dimensions. Let \( A \) denote an \( m \)-form potential and \( F \) its \( (m+1) \)-form field strength, so that

\[
F_{\mu_1...\mu_{m+1}} = \partial_{[\mu_1} A_{\mu_2...\mu_{m+1}]}, \quad 0 \leq \mu_l \leq d - 1.
\]  

Similarly, let \( B \) denote a \( (d-m-2) \)-form potential and \( G \) its \( (d-m-1) \)-form field strength. Then form the action

\[
S = \int d^d x \left\{ \frac{\epsilon_{i_1...i_m j_1...j_{d-m-1}}}{m!(d-m-1)!} F_{0i_1...im} G_{j_1...j_{d-m-1}} + \frac{1}{2 \cdot (m+1)!} F_{i_1...im+1} F_{i_1...im+1}
\]

\[
+ \frac{1}{2 \cdot (d-m-1)!} G_{i_1...i_{d-m-1}} G_{i_1...i_{d-m-1}} \right\}, \quad 1 \leq i_l, j_l \leq d.
\]  

This action is invariant under the following gauge transformations:

\[
\delta A_{0i_1...im} = \psi^{(1)}_{i_1...im}, \quad \delta B_{0i_1...i_{d-m-3}} = \psi^{(2)}_{i_1...i_{d-m-3}}
\]

\[
\delta A_{i_1...im} = \partial_{[i_1} A^{(1)}_{i_2...im]}, \quad \delta B_{i_1...i_{d-m-2}} = \partial_{[i_1} A^{(2)}_{i_2...i_{d-m-2}}.
\]  

The field equations of (27) imply that \( G \) is dual to \( F \). These equations provide a natural generalization of eqs. (16) and (18), which correspond to the case \( d = 4 \) and \( m = 1 \). When \( d = 4n + 2 \) and \( m = 2n \) it is possible to have a self-dual field strength (which amounts to equating \( F \) and \( G \)). An example of this occurs in type 2B supergravity in ten dimensions, which contains a self-dual five-form field strength. In this case our formulas reduce to ones considered previously by Henneaux and Teitelboim [11]. In the particular case of a self-dual boson in two dimensions, they agree with those of ref.[12]. This case will be utilized in the next section. These formulas also can be used to reformulate N=1 supergravity in ten dimensions in a version containing both the two-form \( B_{\mu \nu} \) and the six-form \( B_{\mu_1 \mu_2...\mu_6} \). This requires a slight generalization of the formulas given above to accommodate the Chern-Simons terms that are present.
T-Duality Symmetric World-Sheet Theory

Let us now consider the dynamics of strings propagating in the presence of arbitrary background values of the fields in the low-energy effective action. Specifically, they are the string metric $g_{\mu\nu}(x)$ (which includes the dilaton as a factor), the two-form potential $B_{\mu\nu}(x)$, the moduli $M^{ab}(x)$, and the 28 abelian gauge fields $A_{\mu}^{a}(x)$. As is well-known, this world-sheet theory is conformally invariant when the backgrounds satisfy the appropriate equations of motion. The dynamical ‘fields’ of the world-sheet theory are the space-time coordinates $x^{\mu}(\sigma, \tau)$, $\mu = 0, 1, 2, 3$ and 28 internal coordinates $y^{a}(\sigma, \tau)$, $a = 1, 2, \cdots, 28$, describing six compactified right-movers and 22 compactified left-movers. The $y^{a}$'s are periodically identified, *i.e.*, $y^{a} \sim y^{a} + 2\pi$. The fact that they are chiral bosons of the world-sheet theory is reflected in their world-sheet equations of motion

$$D_{0}y^{a} = -(ML)^{a}_{b}D_{1}y^{b}, \quad (29)$$

where

$$D_{\alpha}y^{a} = \partial_{\alpha}y^{a} + A_{\mu}^{a}\partial_{\alpha}x^{\mu}. \quad (30)$$

Note that $(ML)^{2} = 1$, and the matrix $ML$ has 22 eigenvalues that are $-1$ and 6 eigenvalues that are $+1$. In addition, the $x^{\mu}$ equations of motion are

$$g_{\mu\nu}\partial^{\alpha}x^{\nu}\partial_{\alpha}x^{\rho} + \Gamma_{\mu\nu\rho}\partial^{\alpha}x^{\nu}\partial_{\alpha}x^{\rho} = -\frac{1}{2}D_{1}y^{a}(L\partial_{\mu}ML)_{ab}D_{1}y^{b} - \epsilon^{\alpha\beta}\partial_{\alpha}x^{\nu}F_{\mu\nu}^{a}L_{ab}D_{\beta}y^{b} + \frac{1}{2}\epsilon^{\alpha\beta}H_{\mu\nu\rho}\partial_{\alpha}x^{\nu}\partial_{\beta}x^{\rho}, \quad (31)$$

where $\Gamma$ is the usual Christoffel symbol and $H$ is defined in eq. (9). These equations are written in a form having manifest $O(6,22,\mathbb{Z})$ symmetry (T duality). The restriction to integers arises from the periodicity properties of the $y^{a}$.

At this point it is natural to wonder whether these T duality symmetric equations can be obtained from a world-sheet action that has this symmetry. This is
certainly not true for the usual formulation. In fact, it is easy to write down such an action. The answer is

\[ S = \frac{1}{4\pi} \int d^2\sigma \left\{ g_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu - D_0 y^a L_{ab} D_1 y^b - D_1 y^a (LML)_{ab} D_1 y^b \right. \]

\[ \left. + \epsilon^{\alpha\beta} [B_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu - A_\mu^\alpha \partial_\alpha x^\mu L_{ab} D_1 y^b] \right\}. \]

(32)

This result contains chiral bosons in the manner discussed earlier. It is a generalization of a result given previously by Tseytlin [13].

To understand this theory better, it is important to exhibit the coupling to a world-sheet metric \( h_{\alpha\beta} \) that gives 2D Weyl invariance and reparametrization invariance. This is achieved by replacing the first term (as usual) by \( \sqrt{-h} \eta^{\alpha\beta} g_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu \), and the third term by

\[ \frac{1}{\sqrt{-h}} D_1 y^a (LML)_{ab} D_1 y^b + h_{\mu0} D_1 y^a L_{ab} D_1 y^b. \]

(33)

which is analogous to the \( B^2 \) terms in eq. (25). Having this form of the world-sheet action, it is straightforward to deduce the (traceless) energy-momentum tensor \( T_{\alpha\beta} \) and the corresponding Virasoro constraints. It is also possible to eliminate the world sheet metric to obtain a “Nambu form” that maintains all the symmetries (including T duality).

The existence of the version of the world-sheet theory presented above is actually quite remarkable. It has manifest T duality, which is a symmetry that relates Kaluza–Klein excitations of the string (which can be regarded as elementary ‘particles’ of the world-sheet theory) to winding-mode excitations (which are solitons of the world-sheet theory). In terms of a compactification scale \( R \) and the string scale \( \alpha' \), the corresponding masses are \( M_{\text{KK}} \sim 1/R \) and \( M_{\text{winding}} \sim R/\alpha' \). Thus T duality is a nonperturbative symmetry of the world-sheet theory, just as S duality is conjectured to be for the space-time theory. A world-sheet theory that makes such a nonperturbative symmetry manifest is necessarily strongly coupled. We know it is correct, however, since the \( y \) coordinates only appear quadratically and

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can therefore be treated exactly. The analogy with the space-time theory raises the very interesting question whether it is possible to formulate a string field theory with manifest S duality. (It is already known how to implement T duality in string field theory [14].)

Discussion

The toroidally compactified heterotic string has an infinite spectrum of excitations carrying electric and magnetic charges (with respect to the 28 abelian gauge fields). The states that carry electric charges only are elementary in the sense that they have a perturbative description in the space-time theory. States that carry magnetic charge, on the other hand, must be regarded as solitons of the string theory. The electric and magnetic charges of a state can be defined by the asymptotic behavior of the gauge fields

\[ F_{0i}^a \sim q_{\text{el}}^a \frac{x^i}{r^3}, \quad \tilde{F}_{0i}^a \sim q_{\text{mag}}^a \frac{x^i}{r^3}. \]  

(34)

The allowed values of the electric and magnetic charges are determined by the asymptotic values of the moduli fields \((M_{ab}^{(0)} \text{ and } \lambda^{(0)})\). In terms of these one has [15]

\[ q_{\text{el}}^a = \frac{1}{\lambda^{(0)}} M_{ab}^{(0)} (\alpha_0^b + \lambda^{(0)} \beta_0^b), \quad q_{\text{mag}}^a = L_{ab} \beta_0^b, \]  

(35)

where both \(\alpha_0^a\) and \(\beta_0^a\) are 28-component vectors belonging to a reference lattice \(P_0\), which is even and self-dual with respect to the metric \(L\). These charges automatically incorporate the Dirac quantization condition suitably generalized to allow for dyons and a vacuum angle [16]. The appearance of the electric and magnetic charges as central charges in the supersymmetry algebra allows one to deduce a Bogomol’nyi lower bound on the masses of states with specified electric and magnetic charges [17]. In the present context this bound is given by

\[ (m_0)^2 = \frac{1}{16} (\alpha_0^a \beta_0^b) M^{(0)} (M^{(0)} + L)_{ab} \left( \begin{array}{c} \alpha_0^b \\ \beta_0^b \end{array} \right). \]  

(36)

Remarkably, this formula turns out to be symmetric under both S and T dualities,
providing yet further evidence that at a fundamental level they should operate in much the same way.

The perturbative string spectrum contains all states with electric charges only, in other words all states with $\beta^a_0 = 0$. This is a T duality invariant (but not S duality invariant) subset of the complete spectrum. The perturbative five-brane spectrum, on the other hand, contains all states for which the last 22 components of $\alpha^a_0$ and $\beta^a_0$ vanish. This is an S duality invariant (but not T duality invariant) subset of the complete spectrum. In view of these facts, it is tempting to speculate that the classical five-brane theory has S duality symmetry much as the string world-sheet theory has T duality symmetry. However, the severe nonlinearities of the five-brane theory have so far prevented us from proving this.

Since the S duality group $\text{SL}(2,\mathbb{Z})$ relates electrically charged states to magnetically charged states, it relates perturbative states and nonperturbative states of the space-time theory, just as the T duality group $\text{O}(6,22;\mathbb{Z})$ did for the world-sheet theory. Specifically, the $\text{SL}(2,\mathbb{Z})$ group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps states with charges $(\vec{\alpha}_0, 0)$ to ones with charges $(a\vec{\alpha}_0, c\vec{\alpha}_0)$. It is possible to find group elements for any pair of relatively prime integers $a$ and $c$. If we simultaneously allow the transformations to act on the background fields of the world-sheet action to give transformed background fields, whose relationship to the original ones are nonlocal in general, then we obtain a “dual” world-sheet action that is isomorphic to the original one. From the point of view of this dual theory states with charges of the form $(a\vec{\alpha}_0, c\vec{\alpha}_0)$ arise perturbatively and all others are solitons. Thus we have a generalization of Olive–Montonen duality [18] to a situation where there are an infinite number of isomorphic dual descriptions of the same theory labeled by pairs of relatively prime integers.

To conclude, we have seen that S duality and T duality have many similarities. To the extent that a fundamental five-brane formulation of the heterotic string theory makes sense, S duality should be a fundamental symmetry. However, even if five-branes turn out to only make sense as solitons, the symmetry could still be
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