A random matrix model for chiral symmetry breaking

A.D. Jackson and J.J.M. Verbaarschot
Department of Physics
SUNY, Stony Brook, New York 11794

Abstract
We formulate a random matrix model which mimics the chiral phase transition in QCD with two light flavors. Two critical exponents are calculated. We obtain the mean field values $\beta = \frac{1}{2}$ and $\delta = 3$. We also find that the chiral phase transition can be characterized by the dynamics of the smallest eigenvalue of the Dirac operator. This suggests an alternative order parameter which may be of relevance for lattice QCD simulations.
1 Introduction

In recent years the QCD phase transition has been studied in a variety of ways using both numerical simulations and analytical methods. Such studies have led to the conviction that it may be a second-order phase transition \cite{1,2}. As has been stressed, in particular in \cite{3,4}, this has important consequences because the transition can then be characterized by critical exponents corresponding to a specific universality class. In particular it was argued that the critical exponents are those of an $O(4)$ Heisenberg spin model. (On a lattice and with Kogut-Susskind fermions, the relevant model might rather be an $O(2)$ spin model.) However, as always, universality arguments must be used with care. According to a recent suggestion in \cite{5} based on simulations of the three dimensional Gross-Neveu model, they may not be valid for phase transitions involving soft modes composed of fermions. The reason is that the lowest Matsubara frequency suppresses infrared divergencies which lead to the universal critical exponents. Indeed, Kocic and Kogut found that the critical exponents in their model are given by mean field theory.

We wish to study the chiral phase transition from the perspective of the spectrum of the Euclidean Dirac operator. Although initial numerical lattice results regarding this issue have become available \cite{6,7,2}, a systematic study is beyond the reach of present day computers. Thus, we prefer to construct a simple model which contains the main ingredients of QCD related to chiral symmetry breaking. The zero temperature version of this model was considered previously \cite{8}, and it was shown \cite{8} that it is equivalent to the finite volume effective partition function. In particular, it was shown \cite{8} that the spectrum of the Dirac operator in this random matrix model obeys the so-called Leutwyler-Smilga sum rules \cite{10}.

In this work we consider the non-zero temperature results of two versions of the random matrix model, for $SU(2)$ and for $SU(N_c)$, $N_c \geq 3$ both with fundamental fermions, which are characterized by real and complex matrix elements, respectively. We shall give analytical results for the critical exponents $\beta$ and $\delta$ as well as numerical results (for the case of complex matrices only). We also study the temperature dependence of the
spectrum. Finally, we calculate the dependence of the chiral condensate on the valence quark mass. These results will be shown to be in qualitative agreement with recent lattice calculations [2].

2 Finite temperature chiral random matrix model

The QCD partition function for vacuum angle $\theta$ is defined as

$$Z_{QCD} = \sum_\nu e^{i\nu \theta} Z_{QCD}(\nu),$$

where the partition function in a sector with topological charge $\nu$ and $N_f$ fermionic flavors is given by

$$Z_{QCD}(\nu) = \langle \prod_{f=1}^{N_f} m_{\nu}^f \prod_{\lambda_n > 0} (\lambda_n^2 + m_{\nu}^2) \rangle_A.$$ (2.2)

Here, $\langle \cdots \rangle_A$ denotes the averaging over the gauge field configurations, with topological charge $\nu$, weighted according the QCD action. The eigenvalues, $\lambda_k$, of the Dirac operator fluctuate over the ensemble of gauge field configurations. In general, the complete eigenvalue density is determined in a nontrivial way by the QCD dynamics. However, fluctuations of the eigenvalues on the microscopic scale (on the scale of the average level spacing) are believed to show universal characteristics. It is our conjecture that the eigenvalues near zero virtuality obey such microscopic universality. This implies that the detailed dynamics of the QCD partition function are not important for such fluctuations, and can be described equally well with a random matrix ensemble which respects the global symmetries of the Dirac operator. In particular, the following properties are included: (i) The $U_A(1)$ symmetry leading to an eigenvalue spectrum $\pm \lambda_n$. (ii) The zero mode structure of the Dirac operator. (In the sector of topological charge $\nu$ the Dirac operator has exactly $\nu$ zero eigenvalues all of the same chirality.) (iii) The flavor chiral symmetry and its spontaneous or explicit breaking. (iv) The reality type of the representation of the gauge group. For $SU(N_c)$, $N_c \geq 3$ in the fundamental representation, the gauge field is complex and so are the matrix elements of the Dirac operator. The gauge group $SU(2)$ in the fundamental representation is pseudoreal leading to matrix elements
of the Dirac operator that are real. Finally, for gauge group $SU(N_c)$ in the adjoint representation, the gauge field is real and the matrix elements of the Dirac operator can be organized into real quaternions.

In this work we want to construct a model which describes the fluctuations of the smallest eigenvalues as a function of the temperature. Near $T_c$ the theory of critical phenomena tells us that the fluctuations are universal with non-trivial critical exponents which are related to the propagation of soft modes. However, the recent work of Kocic and Kogut [5] suggests that this scenario may not be valid for phase transitions involving bosons composed of fermions. Instead, the lowest Matsubara frequency, $\pi T$, cures the infrared divergences and leads to a mean-field-like second-order phase transition. According to their work on the three dimensional Gross-Neveu model, the dynamics of the phase transition are determined by the lowest Matsubara frequency. In this spirit, the only temperature effect we include in our model is that of the lowest Matsubara frequency between each pair of suitably chosen basis states $(1 \pm \gamma_5) \phi_n$. In the sector of topological charge $\nu$ our basis must be complemented by $\nu$ unpaired basis states of the same chirality. Together with the symmetries mentioned above, this leads to the random matrix model,

$$ Z_{\beta}(\nu, N_f) = \int D W \det^{N_f} \left( \begin{array}{cc} m^* & iW + i\pi T \\ iW^\dagger + i\pi T & m \end{array} \right) \exp \left[ -\frac{n\beta\Sigma^2}{2} \text{Tr} WW^\dagger \right] , $$

(2.3)

where $W$ is an $n \times (n + \nu)$ matrix. The integration over $W$ is to be performed according to the Haar measure. We also include an arbitrary complex mass matrix $m$ with mass eigenvalues equal to $m_f$. For QCD with $N_c = 2$ the matrix elements of $W$ are real ($\beta = 1$). They are complex for $N_c \geq 3$ ($\beta = 2$). In each case we include fermions in the fundamental representation.

The fermion determinant in (2.3) can be written as a Grassmann integral.

$$ Z_{\beta}(\nu, N_f) = \int D \psi D \psi^* D \psi \exp \left[ i \sum_{k=1}^{N_f} \psi_k^* \left( \begin{array}{cc} -im^* & W + \pi T \\ W^\dagger + \pi T & -im \end{array} \right) \psi_k \right] \exp \left[ -\frac{n\beta\Sigma^2}{2} \text{Tr} WW^\dagger \right] . $$

(2.4)

The quenched approximation is obtained from this model as in the replica trick. We calculate a property for arbitrary $N_f$ and take the limit $N_f \to 0$ at the end of the calculation. In particular, quantities which are $N_f$-independent are valid for $N_f = 0$ as well.
It should be stressed that the partition function (2.3) represents a schematic model for the chiral phase transition. Although the temperature dependence of this model does not coincide with that of chiral perturbation theory [11], it will be shown below that there is considerable qualitative agreement.

3 Analytical results

In this section we evaluate the partition function (2.4) using methods which are standard in the supersymmetric formulation of random matrix theory [12, 13]. The first step is to perform the average over \( W \) by performing a Gaussian integral. This leads to a four-fermion interaction. After averaging over the matrix elements of the Dirac operator, the partition function becomes

\[
Z_1(\nu, N_f) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ -\frac{1}{2n\Sigma^2 \beta} \left( \psi^f_R^* \psi^f_L \psi^g_R^* \psi^g_L + 2\psi^f_R^* \psi^f_L \psi^g_R^* \psi^g_L + \psi^f_L \psi^f_R \psi^g_L \psi^g_R \right) + m^*_f \psi^f_R \psi^g_R + m_f \psi^f_L \psi^g_L + i\pi T (\psi^f_R \psi^f_L + \psi^f_L \psi^f_R) \right],
\]

for \( \beta = 1 \), and

\[
Z_2(\nu, N_f) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ -\frac{2}{n\Sigma^2 \beta} \psi^f_R^* \psi^f_L \psi^g_R^* \psi^g_L + m^*_f \psi^f_R \psi^g_R + m_f \psi^f_L \psi^g_L + i\pi T (\psi^f_R \psi^f_L + \psi^f_L \psi^f_R) \right],
\]

for \( \beta = 2 \). In both cases, each of the four-fermion terms can be written as the difference of two squares. Each square can be linearized by the Hubbard-Stratonovitch transformation according to

\[
\exp(-AQ^2) \sim \int d\sigma \exp \left( -\frac{\sigma^2}{4A} - iQ\sigma \right).
\]

For \( \beta = 2 \) the partition function, expressed in terms of the two bosonic variables, can be combined into a single complex \( N_f \times N_f \) matrix, \( A \), resulting in

\[
Z_2(\nu, N_f) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ -\frac{n\Sigma^2 \beta}{2} \text{Tr} AA^\dagger \right. + \psi^f_L \psi^g_L (A + m) + \psi^f_R \psi^g_R (A^\dagger + m^*) + i\pi T (\psi^f_R \psi^f_L + \psi^f_L \psi^f_R) \]
\]
For $\beta = 1$ six new bosonic matrix variables are required. This is related to the fact that, for two colors, baryons are composed of two quarks and are bosons. They can be combined into one antisymmetric, complex $2N_f \times 2N_f$ matrix $A$ resulting in the partition function

$$Z_1(\nu, N_f) = \int DA \mathcal{D}\psi \mathcal{D}\psi^* \exp\left[-\frac{n\Sigma^2 \beta}{2} \text{Tr} AA^\dagger\right]$$
\begin{align*}
&\times \exp\left(\frac{1}{2}\begin{pmatrix} \psi_R^* & \psi_R^* \\ \psi_R^* & \psi_R^* \end{pmatrix}(A^\dagger + M^*) \begin{pmatrix} \psi_R^* \\ \psi_R^* \end{pmatrix}\right) \\
&\times \exp\left(\frac{i\pi T}{2}\begin{pmatrix} \psi_L^* & \psi_L^* \\ \psi_L^* & \psi_L^* \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(-A + M) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_L^* \\ \psi_L^* \end{pmatrix}\right) \\
&\times \exp[i\pi T(\psi_R^f \psi_L^f + \psi_L^f \psi_R^f)] .
\end{align*}

(3.5)

In this case the mass matrix is an antisymmetric matrix given by

$$M = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} .$$

(3.6)

Note also that the temperature-dependent term can be rewritten as

$$-\frac{i\pi T}{2}\begin{pmatrix} \psi_L^* & \psi_L^* \\ \psi_L^* & \psi_L^* \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(-A + M) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_L^* \\ \psi_L^* \end{pmatrix} \rightarrow (L \leftrightarrow R)$$

(3.7)

Using this, the fermionic integrals can be performed, and the partition function is given by

$$Z_2(\nu, N_f) = \int DA \exp\left[-\frac{n\Sigma^2 \beta}{2} \text{Tr} AA^\dagger\right]|\nu\rangle \langle \nu| (A + m) \text{det}^n\left(\begin{array}{cc} A + m & \pi i T \\ \pi i T & A^\dagger + m^* \end{array}\right)$$

(3.8)

for $\beta = 2$, and

$$Z_1(\nu, N_f) = \int DA \exp\left[-\frac{n\Sigma^2 \beta}{2} \text{Tr} AA^\dagger\right]\text{Pf} \langle \nu\rangle \langle \nu| (-A + M) \text{Pf}^n\left(\begin{array}{cc} A^\dagger + M^* & \pi i T \\ -\pi i T & -A + M \end{array}\right)$$

(3.9)

for $\beta = 1$. In (3.8) $A$ is an arbitrary complex matrix whereas in (3.9) $A$ is an arbitrary antisymmetric complex matrix.

In each case the condensate is given by

$$\langle \bar qq \rangle = \frac{1}{2nN_f} \partial_m \log Z ,$$

(3.10)

where $Z$ is evaluated for a diagonal mass matrix with equal diagonal matrix elements. In the limit $n \to \infty$, the condensate can be determined with the aid of a saddle point approximation. The saddle point equations for $\beta = 2$ are given by

$$-\frac{n\beta \Sigma^2}{2} A + n(A + m) \left((A^\dagger + m)(A + m) + \pi^2 T^2\right)^{-1} = 0 .$$

(3.11)
An arbitrary complex matrix can be diagonalized by performing the decomposition

\[ A = U \Lambda V^{-1}, \]  

(3.12)

with all eigenvalues positive and \( U \) and \( V \) unitary matrices. We find that the solution of (3.11) yields \( U = V = 1 \) with eigenvalues \( \lambda \) given by the positive roots of

\[ \Sigma^2 \lambda((\lambda + m)^2 + \pi^2 T^2) - \lambda - m = 0. \]  

(3.13)

In the chiral limit we find a critical point at

\[ T_c = \frac{1}{\pi \Sigma}. \]  

(3.14)

In order to calculate the condensate, we express the derivative of the partition function in (3.10) in terms of an average over \( A \),

\[ \langle \bar{q} q \rangle = \frac{1}{2nN_f} \langle \text{Tr} \left( \frac{A^\dagger}{\pi i T} \pi i T + A \right) \rangle^{-1}. \]  

(3.15)

Below \( T_c \) the mass \( m \) can be neglected in the saddle point equation, and we find

\[ \langle \bar{q} q \rangle = \Sigma(1 - \pi^2 T^2 \Sigma^2)^{1/2}. \]  

(3.16)

At \( T_c \) the solution of the saddle point equation develops a non-analytic dependence on \( m \) resulting in the condensate

\[ \langle \bar{q} q \rangle = \Sigma^{\frac{4}{3}} m^{\frac{1}{3}}. \]  

(3.17)

Therefore, we reproduce the mean field value for the critical exponent \( \delta = 3 \).

For \( \beta = 1 \) the saddle point equation for the \( A \) integration is

\[ -\frac{\beta \Sigma^2}{2} A + \frac{1}{2}(A + \mathcal{M}) \left( (A^\dagger + \mathcal{M})(A - \mathcal{M}) + \pi^2 T^2 \right)^{-1} = 0. \]  

(3.18)

This equation can be solved by diagonalizing the complex antisymmetric matrix \( A \) as \( A = U \Lambda \tilde{U} \), where \( U \) is a unitary matrix. Here, \( \Lambda \) is an antisymmetric, standard matrix such that \( \Lambda_{k,k+1} = -\Lambda_{k+1,k} = \lambda_k \) for \( k = 1, \cdots, 2N_f - 1 \) with all other matrix elements equal to zero. A suitable re-definition of \( U \) can always made such that all \( \lambda_k \geq 0 \). The condensate is calculated as in the case of \( \beta = 2 \) with the same result both below and at \( T_c \). Of course, it comes as no surprise that we obtain the mean field value for the critical exponent in this case as well.
4 The phase transition

In the remainder of this paper we describe the results of numerical investigations of the random matrix model (2.3). For \( N_f \neq 0 \), the determinantal weight must be included in the integration measure which is extremely costly. However, for \( N_f = 0 \) the distribution functions are simple Gaussians, and only this case will be studied. Instead of a Gaussian distribution of matrix elements, we have used a rectangular distribution centered about zero with variance given by

\[
|W_{ij}|^2 = \frac{1}{6n},
\]

where \( n \) is the size of the off-diagonal blocks. If we compare this to the variance of the Gaussian distribution (2.3), we can make the identification

\[
\frac{1}{6n} = \frac{2}{n\beta \Sigma^2},
\]

which yields \( \Sigma^2 = 6 \) for \( \beta = 2 \). According to (3.14) the critical temperature is thus \( \pi T_c = 1/\sqrt{6} = 0.40824 \). General universality arguments [14] imply that the shape of the distribution of the matrix elements does not affect the properties of our random matrix model, and we expect that the results for the rectangular distribution will be in complete agreement with those for the Gaussian distribution.

The chiral order parameter of our schematic model is the spectral density \( \rho(0) \). It is related the the chiral condensate via the Banks-Casher formula [15]

\[
\langle \bar{q}q \rangle = \frac{1}{2n} \pi \rho(0).
\]

Let us first consider the complete spectral density \( \rho(\lambda) \). At \( T = 0 \) it can be shown, using arguments familiar from random matrix theory [16], that this density has a semicircular shape. For non-zero temperature, it is somewhat more difficult to obtain the level density analytically [18]. In Figs. 1a, 1b, and 1c, we show numerical results for the complete spectral density for \( \pi T = 0, \pi T = 0.4, \) and \( \pi T = 1.0 \). The deviations from a semi-circle at \( T = 0 \) are not statistical fluctuations but rather well-understood finite-\( n \) corrections [17]. When \( T \) is large, we find that the distribution splits into two semicircles with centers at \( +\pi T \) and \( -\pi T \). It is elementary to demonstrate that all eigenvalues come in pairs located symmetrically about zero. When \( T = 0 \), the distribution of the eigenvalue of smallest
magnitude is also known analytically. For the moment, we note only that $\rho(\lambda) \sim \lambda$ for values of $\lambda$ less than the average value of this eigenvalue of smallest magnitude. Thus, some care is required in order to extract a meaningful (i.e., non-zero) value of $\rho(0)$. The simplest method is to construct $\rho(0)$ from the fraction of eigenvalues in an interval $2 \Delta \lambda$ centered about 0. When $\Delta \lambda$ is less than the average of eigenvalue of smallest magnitude, this estimate of $\rho(0)$ is proportional to $\Delta \lambda$.

Figure 1: Histograms of the complete spectrum of eigenvalues obtained for $n = 20$ at temperatures of (a) $\pi T = 0$ (top curve), (b) $\pi T = 0.4$ (middle curve), and (c) $\pi T = 1.0$ (bottom curve). Each spectrum was obtained from $10^5$ matrices.

For somewhat larger values of $\Delta \lambda$ there is a relatively rapid saturation of $\rho(0)$ to its desired asymptotic value. In principle, it is better to obtain $\rho(0)$ by performing a smearing of the full spectrum with a Gaussian whose width is roughly comparable to the spacing between adjacent eigenvalues. This avoids the small-amplitude oscillations associated with the first approach. In practice, there is little difference between the two methods.

According to the mean field argument presented in section 3, our model shows a
second-order phase transition with

\[ \rho(0, T) \sim \sqrt{T_c^2 - T^2}. \]  

(4.4)

In order to account for finite \( n \) effects approximately, we convolute this expression with a Gaussian

\[ \rho_\sigma(0, T) \sim \int_{-T_c}^{T_c} dx \sqrt{T_c^2 - x^2} \exp\left[-(x - T)^2/\sigma^2\right]. \]  

(4.5)

While we offer no analytic justification for this form, we note that it provides an excellent fit to the results of simulations. A best fit of this expression to our results for \( n = 20 \) is shown in Fig. 2. The spectral density was obtained from a bin size of 0.02. In this case, the fitted values for \( T_c \) and \( \sigma \) are 0.389 and 0.074, respectively. Using data for \( n \geq 50 \) we performed an extrapolation to \( n \to \infty \). We find a critical temperature of \( T_c = 0.3967 \) which is in good agrees with the theoretical value of \( \pi T_c = 1/\sqrt{6} \) for our present numerical parameters.

Figure 2: The \( \langle \bar{q}q \rangle \) condensate as calculated from (4.3) for \( n = 20 \). Each point was obtained on the basis of \( 2 \times 10^4 \) matrices, and \( \rho(0) \) was obtained by counting eigenvalues in the range \( 0 \leq \lambda \leq 0.02 \). The solid curve is a fit according to (4.5) using the parameters \( T_c = 0.3890 \) and \( \sigma = 0.7414 \).

Recent lattice QCD calculations [2] have studied the condensate as a function of the so-called valence mass. In these lattice simulations, the sea-quark mass was much larger
than the smallest eigenvalue so that the problem is effectively equivalent to taking $N_f = 0$. It is possible to simulate these calculations within the framework of the present model by defining

$$\langle \bar{\psi} \psi \rangle = \frac{1}{n} \sum_{k=1}^{n} \frac{m}{\lambda_k^2 + m^2}. \quad (4.6)$$

Fig. 3 shows a log-log plot of the resulting values for $\langle \bar{\psi} \psi \rangle$ as a function of $m$ for $n = 20$.

Figure 3: The condensate as a function of the so-called valence quark mass calculated according to (4.6) at $\pi T = 0$, $\pi T = 0.4$ and $\pi T = 1.0$. The calculations were performed for $n = 20$ and $2 \times 10^4$ matrices were used for each temperature.

Our results are qualitatively similar to the lattice calculations. The $m \to 0$ limit is clearly ambiguous. These results show that the spectral density yields a much more accurate determination of the chiral condensate [19].

5 Dynamics of the smallest eigenvalue

In this section we study the distribution of the eigenvalue of smallest magnitude, denoted by $\lambda_1$, as a function of temperature. This distribution is known analytically for a few special cases of the Laguerre ensemble. The present case, defined by the random matrix model (2.3) for $N_f = 0$, at $T = 0$ happens to be one of them. With our choice of
parameters, the distribution is given by
\[ f(\lambda_1) \sim \lambda_1 \exp \left( -\frac{\left( n\beta \Sigma \lambda_1 \right)^2}{2} \right). \]  
(5.1)

This \( T = 0 \) result is exact even for finite \( n \). (The factor of \( \lambda_1 \) is readily understood. For every eigenvalue \(+\lambda\) there is always a corresponding eigenvalue of \(-\lambda\). This factor is simply a consequence of level repulsion.) In order to be able to describe the temperature dependence, we introduce a more general distribution
\[ f(\lambda_1) = \lambda_1 \exp \left( -\frac{(\lambda_1 - x_0)^2}{\sigma^2} \right). \]  
(5.2)

At zero temperature \( x_0 = 0 \) and \( \sigma = 2/n\beta\Sigma \). The average value of the smallest eigenvalue is given as \( \bar{\lambda}_1 = \sqrt{\pi}/n\beta\Sigma \). While this expression is not, in general, exact, it is serviceable.

\[ \text{Figure 4: The distribution of the eigenvalue of smallest magnitude determined from} \]
\[ 2 \times 10^5 \text{ trials with } n = 20 \text{ for a bin size of 0.02 (points). The histogram was obtained from (5.2) with } x_0 = 0. \text{ The value of } \sigma = 0.020685 \text{ was determined from the average value of } \bar{\lambda}_1 \text{ obtained from the simulations. This fitted value is in excellent agreement with the analytic value of } 2/n\beta\Sigma = 0.02041. \text{ The form of (5.2) provides an excellent fit to the simulation data with } \chi^2 = 0.8. \]

\[ \text{In Fig. 4 we show a histogram of the distribution of the smallest eigenvalue at } T = 0 \]
\[ \text{for the case } n = 20. \text{ The results are in perfect agreement with (4.1). The temperature dependence of the parameters of the distribution is shown in Fig. 5, where we show the} \]
ratio $\bar{\lambda}_1/\Delta \lambda_1$ as a function of the temperature for $n = 10$. When $x_0 = 0$ this ratio is equal to $[\pi/(4 - \pi)]^{1/2} = 1.913 \ldots$. Surprisingly, we find that this ratio is constant for $T < T_c$. In terms of $x_0$, this implies that $x_0$ is strictly zero for $T < T_c$. Inspection of the corresponding distribution indicates that (5.1) remains quantitatively valid in this region. Above this temperature (and ignoring some finite $n$ effects near threshold), $x_0$ grows linearly with $T$. This behavior is expected since the entire distribution moves linearly with $T$ for sufficiently large $T$. What is surprising is that $x_0$ goes to zero below some finite $T$ and that the effect is so pronounced for a matrix of such small dimension.

Figure 5: The ratio of the smallest eigenvalue to its rms deviation, $\bar{\lambda}_1/\Delta \lambda_1$, as a function of temperature for $n = 10$. Each point represents $2 \times 10^4$ matrices. The average value of this ratio for $T < T_c$ is 1.9107 which is in good agreement with the expected value of $(\pi/(4 - \pi))^{1/2} = 1.913 \ldots$.

The behavior of $x_0(T)$ versus $T$ (shown in Fig. 6) allows us to extract $T_c$ from $x(T_c) = 0$. In practice we use a linear extrapolation of our data for $T > T_c$. Our results for different size matrices can be summarized by the expression

$$\pi T_c^{(n)} = 0.40859 \frac{n + 2.51316}{n + 4.10346}, \tag{5.3}$$

which yields an asymptotic result of $\pi T_c = 0.40859$, which is very close to the theoretical result of $1/\sqrt{6} = 0.40824$. As indicated, the form of (5.2) is not exact. Similar results can
be obtained in a model-independent fashion by straight-line extrapolation of the observed values of $\frac{\bar{\lambda}_1}{\Delta \lambda_1}$ as a function of the temperature.

![Graph](image)

Figure 6: The parameter $x_0$ of (5.2) as a function of $T$ for $n = 10$. Each point represents $2 \times 10^4$ matrices. The parameters $x_0$ and $\sigma$ were determined from the ratio $\bar{\lambda}_1/\Delta \lambda_1$ (shown in Fig. 5) and the corresponding values of $\bar{\lambda}_1$. Linear extrapolation suggests $\pi T_c = 0.3625$ for this case.

Finally, we study the scaling of the smallest eigenvalue with $N$. For $T < T_c$, the Banks-Casher formula suggests that $\lambda_{\text{min}} \sim 1/n$. Indeed, this is what is found numerically both for the expectation value and the variance of $\lambda_{\text{min}}$.

At $T = T_c$ we have that $\Sigma(m) \sim m^{1/\delta}$. This leads to the eigenvalue density (for $\lambda \to 0$)

$$\rho(\lambda) \sim m^{\alpha_1} \lambda^{\alpha_2} \quad \text{with} \quad \alpha_1 + \alpha_2 = \frac{1}{\delta}. \quad (5.4)$$

Smaller masses suppress the eigenvalue density near zero, so we must have $\alpha_1 > 0$. For a fixed mass, the eigenvalue density should not diverge, so we must also have $\alpha_2 > 0$. It is possible that $\alpha_1$ and $\alpha_2$ depend on $N_f$, but we were not able to investigate this point within the present framework. The scaling behavior of the smallest eigenvalue is obtained from $\int_0^{\lambda_{\text{min}}} \rho(\lambda) d\lambda \sim 1/N$ which leads to

$$\langle \lambda_{\text{min}} \rangle \sim N^{-1/(\alpha_2+1)} \quad . \quad (5.5)$$
For $N_f = 0$, the eigenvalue density cannot depend on $m$ and we thus have
\[ \alpha_2 = \frac{1}{3} \quad \text{and} \quad \alpha_1 = 0 \quad \text{for} \quad N_f = 0. \quad (5.6) \]

This yields $\langle \lambda_{\text{min}} \rangle \sim N^{-3/4}$, which has been verified numerically to very high accuracy. It was also shown that the variance of $\lambda_{\text{min}}$ has the same scaling behavior.

For $T > T_c$ the average position of the smallest eigenvalue departs from zero and scales like $N^0$. We expect [21] that its variance well have the scaling behavior $N^{-2/3}$ which is typical for eigenvalues near the edge of a semicircle. Again, this is in perfect agreement with our numerical simulations.

\section{Conclusions}

In this paper we have studied a random matrix model possessing the global symmetries of the QCD-action and the temperature dependence suggested by the form of the lowest Matsubara frequency. At $T = 0$ this model is completely soluble; it reduces to what is known in the mathematical literature as the Laguerre ensemble. For nonzero temperatures, we have succeeded in extracting some interesting properties analytically. In particular, we have shown that the model undergoes a second-order phase transition. Two critical exponents were obtained.

\[ \beta = \frac{1}{2}, \]
\[ \delta = 3. \quad (6.1) \]

The lattice result [1] for QCD with two light flavors for $1/\beta\delta$ is $0.77 \pm 0.14$, which is closer to these mean field values than to the results for either the $O(4)$ or $O(2)$ Heisenberg spin models. For three or more light flavors, QCD shows a first-order chiral phase transition, whereas for one flavor there is no transition at all. Our model does not contain such flavor dependence: It has a second-order phase transition for any number of flavors. If our model has anything to say about QCD, it is for QCD with two light flavors.

Numerically, we found one surprising result: The distribution of the smallest eigenvalue below $T_c$ is (numerically) equivalent to the distribution obtained for $T = 0$. For $T > T_c$,
the centroid of the Gaussian distribution grows linearly with $T$. The behavior at $T = 0$ agrees well with known analytical results.

Our results suggest an alternative method for obtaining the critical temperature; namely from the distribution of the eigenvalue of smallest magnitude. It would be interesting to study the dynamics of the smallest eigenvalues in lattice QCD as well.

Acknowledgements

The reported work was partially supported by the US DOE grant DE-FG-88ER40388. We would like to thank H.A. Weidenmüller and T. Wettig for useful discussions and for communicating their results on a related model prior to publication.
References

[1] F. Karsch and E. Laermann, *Susceptibilities, the specific heat and a cumulant in two-flavor QCD*, Bielefeld preprint BI-TP 94/29.

[2] S. Chandrasekharan, in *Continuous Advances in QCD*, eds. A. Smilga, World Scientific, Singapore 1994.

[3] R.D. Pisarski and F. Wilczek, Phys. Rev. **D29** (1984) 338.

[4] K. Rajagopal and F. Wilczek, Nucl. Phys. **B399** (1993) 395.

[5] A. Kocic and J. Kogut, Phys. Rev. Lett. **74** (1995) 3109; *Phase transitions at finite temperature and dimensional reduction for fermions and boson*, University of Illinois preprint, 1995 (hep-lat/9507012).

[6] S.J. Hands and M. Teper, Nucl. Phys. **B347** (1990) 819.

[7] T. Kalkreuter, Phys. Rev. **D**, (in press), hep-lat/9408013; Phys. Lett. **B276** (1992) 485; Phys. Rev. **D48** (1993) 1926.

[8] J. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531; Phys. Lett. **B329** (1994) 351; Nucl. Phys. **B426** [FS] (1994) 559; Nucl. Phys. **B427** (1994) 434.

[9] A. Halasz and J.J.M. Verbaarschot, Effective Lagrangians and chiral random matrix theory, Phys. Rev. **D** (1995) (in press).

[10] H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.

[11] J. Gasser and H. Leutwyler, Phys. Lett. **B188** (1987) 477.

[12] K.B. Efetov, Adv. Mod. Phys. **32** (1983)53.

[13] J. Verbaarschot, H. Weidenmüller, and M. Zirnbauer, Phys. Rep. **129** (1985) 367.

[14] E. Brézin and A. Zee, Nucl. Phys. **B402** (1993) 613; C. Beenakker, Nucl. Phys. **B422** (1994) 515; G. Hackenbroich and H. Weidenmüller, Phys. Rev. Lett. (1995).

[15] T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.
[16] M. Mehta, *Random Matrices*, Academic Press, San Diego, 1991

[17] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70** (1993) 3852.

[18] A.D. Jackson, M. Sener and J.J.M. Verbaarschot, in preparation.

[19] J.J.M. Verbaarschot, *Universal scaling of the valence quark mass dependence of the chiral condensate*, preprint SUNY-NTG-95/32.

[20] P.J. Forrester, Nucl. Phys. **B 402**[FS] (1993) 709.

[21] J.J.M. Verbaarschot and M.R. Zirnbauer, Ann. Phys. **158** (1984) 78.