The Generalization of the Decomposition of
Functions by Energy operators (Part II) and some
Applications

May 11, 2014

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Abstract

This work introduces the families of generalized energy operators
$[[x^p]_k]_{k\in \mathbb{Z}}$ and $[[x^{-p}]_k]_{k\in \mathbb{Z}}$ ($p$ in $\mathbb{Z}^+$). One shows that with Lemma 1, the successive derivatives of $([[(f^{p-1})^n]_1]^n$ ($n$ in $\mathbb{Z}$, $n \neq 0$) can be decomposed with the generalized energy operators $([[[x^p]_k]_{k\in \mathbb{Z}}$ when $f$ is in the subspace $S^{-p}(\mathbb{R})$. With Theorem 1 and $f$ in $s^{-p}(\mathbb{R})$, one can decompose uniquely the successive derivatives of $([[(f^{p-1})^n]_1]^n$ ($n$ in $\mathbb{Z}$, $n \neq 0$) with the generalized energy operators $([[[x^p]_k]_{k\in \mathbb{Z}}$ and $([[[x^{-p}]_k]_{k\in \mathbb{Z}}$. $S^{-p}(\mathbb{R})$ and $s^{-p}(\mathbb{R})$ ($p$ in $\mathbb{Z}^+$) are subspaces of the Schwartz space $S^{-}(\mathbb{R})$. These results generalize the work of [14]. The second fold of this work is the application of the generalized energy operator families onto the solutions of linear partial differential equations. As an example, the theory is applied to the Helmholtz equation. Note that in this specific case, the use of generalized energy operators in the general solution of this PDE extends the results of [13]. Finally, this work ends with some numerical examples. In particular, when defining the Poynting vector and intensity with generalized energy operators applied onto the planar electromagnetic waves, this allows to define a linear relationship with the radiation pressure force.

1 Introduction and guidelines

Two decades ago, an energy operator ($\Psi^{-1}_R$) was first defined in [7]. Since then, this work has been extensively used in signal processing (e.g., [3] or [4]) and image processing [4]. The bilinearity properties of this operator were studied in [2]. More recently, the author in [13] introduced the energy operators ($\Psi^+_R$, $\Psi^-_R$) and developed a general method for separating smooth real-valued finite energy functions ($f$) in time and space with application to the wave equation. In [14], the author introduced the family of differential
energy operators (DEOs) \((\Psi_k^-)_{k \in \mathbb{Z}^+}\) and \((\Psi_k^+)_{k \in \mathbb{Z}^+}\), and showed the decomposition of the \(s\)-th derivative of \(f^n\) (\(n \in \mathbb{Z}, n \neq 0\)) with the DEO families. In addition, this work demonstrated some properties of the family of energy operators and the application to the energy function \(\mathcal{E}(f^n)\).

This work starts with recalling the notations used with energy operators, important definitions (e.g., Definition 1 and Definition 2) and results (e.g., Lemma 0 and Theorem 0) from the recently published work of [14]. It emphasizes the notion of decomposing finite energy functions \(f\) in a Schwartz space \((S^-(\mathbb{R}))\) with families of energy operator. In the second part, Definition 3 defines an energy space \((\mathcal{E})\) subset of \(S^-(\mathbb{R})\) which is used to define other subspaces (e.g., \(S^-_p(\mathbb{R}), s^-_p(\mathbb{R})\)). In Section 1, one allows to define families of generalized energy operators. Their properties are also shown in Proposition 3 (e.g., bilinearity and derivative chain rule property). To extend the work in [14], the statements of Lemma 0 and Theorem 0 are generalized with Lemma 1 and Theorem 1. Thus in Lemma 1, the work emphasizes in particular \(S^-_p(\mathbb{R})\) a subset of \(S^-(\mathbb{R})\) where the decomposition of \(((|f|^{p-1})^n)_{k \in \mathbb{Z}}\) (\(n \in \mathbb{Z}, n \neq 0\)) with the generalized energy operators \(((|f|^{p-1})^n)_{k \in \mathbb{Z}}\) is valid. Whereas in Theorem 1, one shows the unique decomposition of \(((|f|^{p-1})^n)_{k \in \mathbb{Z}}\) (\(n \in \mathbb{Z}, n \neq 0\)) with the generalized energy operators \(((|f|^{p-1})^n)_{k \in \mathbb{Z}}\) and \(((|f|^{p-1})^n)_{k \in \mathbb{Z}}\) in \(S^-_p(\mathbb{R})\) subset of \(S^-(\mathbb{R})\). Note that in Section 2 to 4, \(f\) is a function of time (\(\partial_t\)). In Section 5 and 6 when dealing with PDEs, \(f\) and its derivatives are functions of time (\(t\)) and space (\(x\)).

In Section 5 we investigate the sets of solutions of linear Partial Differential Equations (PDEs) of the form \(g(x, t) = \partial^\alpha_x u^n(x, t)\) or \(g(x, t) = \partial^\alpha_x u^n(x, t)\) \((g, u \in S^-(\mathbb{R}), n \in \mathbb{Z}^+, n \neq 0, \alpha \in \mathbb{Z}^+)\). The results are further extended with the families of generalized energy operators. It allows us to establish the conditions for the energy operators and the generalized energy operators to be solution of linear Partial Differential Equations (PDEs). The manuscript ends with the application of the families of energy operator and generalized energy operator in the special case of the homogeneous Helmholtz equation and some numerical examples.

2 Preliminaries

Throughout this work, \(f^n\) for any \(n \in \mathbb{Z}^+ - \{0\}\) is supposed to be a smooth real-valued and finite energy function, and in the Schwartz space \(S^-(\mathbb{R})\) defined as:

\[
S^-(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}), \quad sup_{t < 0} |t^k |\partial^\alpha_t f(t)| < \infty, \quad \forall k \in \mathbb{Z}^+, \quad \forall j \in \mathbb{Z}^+ \}
\]

Sometime \(f^n\) can also be analytic if its development in Taylor-Series is relevant to this work. The choice of \(f^n\) (for any \(n \in \mathbb{Z}^+ - \{0\}\)) in the Schwartz space \(S^-(\mathbb{R})\) is based on the work developed in [14], as we are dealing with
multiple integrations or derivatives of \( f^n \) when applying the energy operators \((\Psi_k^\pm)_{k \in \mathbb{Z}^+}\), \((\Psi_k^\mp)_{k \in \mathbb{Z}^+}\) and later on the generalized energy operators.

In the following, let us call the set \( \mathcal{F}(\mathbf{S}^-(\mathbb{R}), \mathbf{S}^-(\mathbb{R})) \) all functions/operators defined such as \( \gamma : \mathbf{S}^-(\mathbb{R}) \to \mathbf{S}^-(\mathbb{R}) \). Let us recall some definitions and important results given in [14].

**Definition 1:** for all \( f \) in \( \mathbf{S}^-(\mathbb{R}) \), for all \( v \in \mathbb{Z}^+ - \{0\} \), for all \( n \in \mathbb{Z}^+ \) and \( n > 1 \), the family of operators \((\Psi_k)_{k \in \mathbb{Z}}\) (with \((\Psi_k)_{k \in \mathbb{Z}} \subseteq \mathcal{F}(\mathbf{S}^-(\mathbb{R}), \mathbf{S}^-(\mathbb{R}))\)) decomposes \( \partial_v^n f^n \) in \( \mathbb{R} \), if it exists \( N \) in \( \mathbb{Z} \), \((C_i)_{i=-N}^N \subseteq \mathbb{R} \), and it exists \((\alpha_j)\) in \( \mathbb{Z}^+ \cup \{0\} \) such as \( \partial_v^n f^n = \sum_{k=-N}^N C_k \Psi_k(\partial_v^k f) \).

However, one has to define \( s^-(\mathbb{R}) \) as:

\[
s^-(\mathbb{R}) = \{ f \in \mathbf{S}^-(\mathbb{R}) | f \notin (\cup_{k \in \mathbb{Z}} \text{Ker}(\Psi_k^+)) \cup (\cup_{k \in \mathbb{Z} - \{1\}} \text{Ker}(\Psi_k^-)) \} \tag{2}
\]

\( \text{Ker}(\Psi_k^+) \) and \( \text{Ker}(\Psi_k^-) \) are the kernels of the operators \( \Psi_k^+ \) and \( \Psi_k^- \) (\( k \) in \( \mathbb{Z} \)) as defined in [14]. Following Definition 1, the *uniqueness* of the decomposition in \( s^-(\mathbb{R}) \) with the families of differential operators can be stated as:

**Definition 2:** for all \( f \) in \( s^-(\mathbb{R}) \), for all \( v \in \mathbb{Z}^+ - \{0\} \), for all \( n \in \mathbb{Z}^+ \) and \( n > 1 \), the families of operators \((\Psi_k^+)|_{k \in \mathbb{Z}}\) and \((\Psi_k^-)|_{k \in \mathbb{Z}}\) \((\Psi_k^+)|_{k \in \mathbb{Z}}\) decompose uniquely \( \partial_v^n f^n \) in \( \mathbb{R} \), if for any family of operators \((S_k)_{k \in \mathbb{Z}} \subseteq \mathcal{F}(\mathbf{S}^-(\mathbb{R}), \mathbf{S}^-(\mathbb{R}))\) decomposing \( \partial_v^n f^n \) in \( \mathbb{R} \), there exists a unique couple \((\beta_1, \beta_2)\) in \( \mathbb{R}^2 \) such as:

\[
S_k(f) = \beta_1 \Psi_k^+(f) + \beta_2 \Psi_k^-(f), \quad \forall k \in \mathbb{Z} \tag{3}
\]

Two important results shown in [14] are:

**Lemma 0:** for \( f \) in \( \mathbf{S}^-(\mathbb{R}) \), the family of DEO \( \Psi_k^+ \) \((k = \{0, \pm 1, \pm 2, \ldots\})\) decomposes the successive derivatives of the \( n \)-th power of \( f \) for \( n \in \mathbb{Z}^+ \) and \( n > 1 \).

**Theorem 0:** for \( f \) in \( s^-(\mathbb{R}) \), the families of DEO \( \Psi_k^+ \) and \( \Psi_k^- \) \((k = \{0, \pm 1, \pm 2, \ldots\})\) decompose uniquely the successive derivatives of the \( n \)-th power of \( f \) for \( n \in \mathbb{Z}^+ \) and \( n > 1 \).

By definition if \( f^n \) is analytic, there are \((p,q)\) \((p > q)\) in \( \mathbb{R}^2 \) such as \( f^n \) can be developed in Taylor Series [8]:

\[
f^n(p) = f^n(q) + \sum_{k=1}^{\infty} \partial_v^k f^n(q) \frac{(p-q)^k}{k!}
\]

\[
\tag{4}
\]
Let us define for \( n \in \mathbb{Z}^+ - \{0\} \), for \( f^n \) in \( S^- (\mathbb{R}) \) and finite energy, \( \mathcal{E}(f^n) \) (in \( S^- (\mathbb{R}) \)) the energy function defined for \((\tau, q) \ (q < \tau)\) in \( \mathbb{R}^2 \) such as:

\[
\mathcal{E}(f^n(\tau)) = \int_q^\tau (f^n(t))^2 \, dt < \infty
\]  

(5)

**Proposition 1:** If for any \( n \in \mathbb{Z}^+ \), \( f^n \) in \( S^- (\mathbb{R}) \) is analytic and finite energy; for any \((p, q) \) in \( \mathbb{R}^2 \) (with \( p > q \)) and \( \mathcal{E}(f^n) \) in \( S^- (\mathbb{R}) \) is analytic, then \( \mathcal{E}(f^n(p)) \) is a convergent series.

**Proof.** From Equation (5) and for \((p, q) \) in \( \mathbb{R}^2 \) (with \( p > q \)), the development in Taylor series of \( \mathcal{E}(f^n(p)) \) is convergent and can be written as:

\[
\mathcal{E}(f^n(p)) = \mathcal{E}(f^n(q)) + \sum_{k=0}^{\infty} \partial_k^i (f^n(q))^2 \frac{(p-q)^k}{k!} < \infty
\]  

(6)

Now, let us assume that this series is divergent then from [8],

\[
\lim_{k \to \infty} \left| \frac{\partial_k^i f^{2n}(q)}{\partial_{k-1}^i f^{2n}(q)} \frac{(p-q)}{k+1} \right| > 1
\]  

(7)

for \( k_1 \) and \( k_2 \) in \( \mathbb{Z}^+ - \{0\} \) such as \( k_1 >> k_2 \),

\[
\left| \frac{\partial_{k_1}^i f^{2n}(q)}{\partial_{k_2}^i f^{2n}(q)} \frac{(p-q)^{k_1-k_2}}{(k_1+1)...(k_1-k_2-1)} \right| > 1
\]  

(8)

and thus we can conclude clearly that \( \mathcal{E}(f^n(p)) > \infty \). Finally, equation (6) is valid if and only if the development in Taylor series of \( \mathcal{E}(f^n(p)) \) is convergent for \( p \) in \( \mathbb{R} \).

\[ \square \]

### 3 Energy Space

Let us define the open sets \( M^i \subseteq S^- (\mathbb{R}) \) for \( i \) in \( \mathbb{Z}^+ \) such as:

\[
M^i = \{ g \in S^- (\mathbb{R}) \mid g = \partial_i^l f^n, \ f \in S^- (\mathbb{R}), \ n \in \mathbb{Z}^+ - \{0\} \}
\]  

(9)

Following **Theorem 0**, \( \partial_i^l f^n \) (in \( \mathbb{Z}^+ - \{0\} \)) can be decomposed uniquely with the family of energy operators \( (\Psi^+_k)_{k \in \mathbb{Z}^+} \) and \( (\Psi^-_k)_{k \in \mathbb{Z}^+} \). It then exists \( \alpha_n \) in \( \mathbb{R} \) such as:

\[
\partial_i^l f^n = \alpha_n (\partial_i^{l-1} \Psi^+_k(f) + \partial_i^{l-1} \Psi^-_k(f))
\]  

(10)

By definition \( \Psi^-_1(f) \) is equal to 0 for any \( f \) in \( S^- (\mathbb{R}) \). We can conclude that \( \partial_i^{l-1} \Psi^+_1(f) \subseteq \mathcal{F}(M^i, M^i) \). Hence, the definition of an energy space is:
Definition 3: for $f^n$ in $S^-(\mathbb{R})$ and analytic, for all $n \in \mathbb{Z}^+$ and $n > 1$, the energy space $E$ is equal to $E = \bigcup_{i \in \mathbb{Z}^+} M^i$ and associated with $E(.)$.

The reader may wonder why the energy space is associated with $E(.)$. As we consider for $f^n$ in $S^-(\mathbb{R})$, $E(f^n)$ in $S^-(\mathbb{R})$ and analytic, one can develop the energy function in Taylor-Series on the interval $[\tau_0, \tau]$ for any $\tau$ and $\tau_0$ in $\mathbb{R}$ ($\tau_0 < \tau$). Following [14]:

$$E(f^n(\tau)) = E(f^n(\tau_0)) + \sum_{k=1}^{\infty} \frac{\partial^{k-1} f^{2n}(\tau_0)}{k!} (\tau - \tau_0)^k$$

$$= E(f(\tau_0)) + f^{2n}(\tau_0)(\tau - \tau_0) + \alpha_n \sum_{k=2}^{\infty} \frac{\partial^{k-2} \Psi_1^+(f)(\tau_0)}{k!} (\tau - \tau_0)^k$$

$$+ \Psi_1^+(f)(\tau_0)) (\tau - \tau_0)^k$$

$$= E(f(\tau_0)) + f^{2n}(\tau_0)(\tau - \tau_0) + \alpha_n \sum_{k=2}^{\infty} \frac{\partial^{k-2} \Psi_1^+(f)(\tau_0)}{k!} (\tau - \tau_0)^k$$

(11)

and if the series is convergent in $\tau_0$, then by definition it exists $C \geq 1$ such as for $k > 1$, $|\partial^k f^n(\tau_0)| \leq C|\partial^{k-1} f^n(\tau_0)|$. Or from Equation (11) with $k > 1$, we have $E(\partial^k f^n(\tau_0)) \leq C^2 E(\partial^{k-1} f^n(\tau_0))$. In other words, $E(\partial^k \Psi_1^+(f)(\tau_0)) \leq C^2 E(\partial^{k-1} \Psi_1^+(f)(\tau_0))$. One can see that if for any $\tau$ in $\mathbb{R}$ and $f$ in $S^-(\mathbb{R})$, $E(\partial^k \Psi_1^+(f)(\tau))$ is equal to 0.

Note that the definition of $M^0$ does not involve the family of DEO $\Psi_k^+$ and $\Psi_k^-$ ($k = \{0, \pm 1, \pm 2, \ldots\}$). In this particular case, if $g$ is a general solution of some PDEs, then $f^n$ can be assimilated as some special form of the solution (if it exists). That will be investigated in the last section of this document.

Furthermore from [3], the energy of the function $f^n$ is directly connected to the $L1$ norm with the Cauchy-Schwartz inequality (for $(p, q)$ and $q < p$, $\mathbb{R}^2$, $(\int_p^q f^n dt)^2 \leq (p - q)E(f^n)$).

Proposition 2: for all $f$ in $S^-(\mathbb{R})$, for all $k \in \mathbb{Z}^+$, $E(\Psi_{k+1}^+(f)) \leq E(\partial^k \Psi_k^+(f))$

Proof. From the properties of the derivative chain rules and with Cauchy-
Schwartz inequality [8] one can write:

\[
\begin{align*}
\Psi_{k+1}(f) &= \partial_t \Psi_k(f) - \Psi_{k-1}(\partial_t f) \\
\int_{-\infty}^{+\infty} |\Psi_{k+1}(f)|^2 dt &\leq \int_{-\infty}^{+\infty} |\partial_t \Psi_k(f)|^2 dt \\
\mathcal{E}(\Psi_{k+1}(f)) &\leq \mathcal{E}(\partial_t \Psi_k(f))
\end{align*}
\]

(12)

4 Generalized Energy Operators

From the introduction of the DEO families in [11] and as recalled in Section 1, it is possible to generalize the definition of the energy operators \((\Psi_k^+)_{k \in \mathbb{Z}}\) and \((\Psi_k^-)_{k \in \mathbb{Z}}\) based on some operators defined as:

\[
[\cdot, \cdot]_k^- = \partial_t \partial_t^{k-1} - \partial_t^k, \quad k \in \mathbb{Z}
\]

(13)

\[
[\cdot, \cdot]_k^+ = [\cdot]_k^-
\]

(14)

Let us call it the generalized energy operator \([\cdot, \cdot]_k^- \subseteq \mathcal{F}(\mathbb{S}^-)\). Note that in [2], the authors defined a similar operator using Lie bracket restricted to signal processing applications (e.g., signal and speech AM-FM demodulation). Similarly to the DEO families, one can introduce the conjugate \([\cdot, \cdot]_k^+ \subseteq \mathcal{F}(\mathbb{S}^-)\) defined as:

\[
[\cdot, \cdot]_k^+ = \partial_t \partial_t^{k-1} + \partial_t^k, \quad k \in \mathbb{Z}
\]

(15)

\[
[\cdot, \cdot]_k^- = [\cdot]_k^+
\]

(16)

To obtain the families of DEOs \((\Psi_k^+)_{k \in \mathbb{Z}}\) and \((\Psi_k^-)_{k \in \mathbb{Z}}\) defined in [14], one can then apply \([\cdot, \cdot]_k^+\) and \([\cdot, \cdot]_k^-\) to \(f\) in \(\mathbb{S}^-\) such as:

\[
\begin{align*}
[f, f]_k^+ &= \partial_t f \partial_t^{k-1} f + f \partial_t^k f \\
[f, f]_k^- &= \Psi_k^+(f) \\
[f, f]_k^- &= \partial_t f \partial_t^{k-1} f - f \partial_t^k f \\
[f, f]_k^- &= \Psi_k^+(f)
\end{align*}
\]

(17)
Furthermore, one can write:

\[
\begin{align*}
[[f, f]_k^+, [f, f]_k^-]_k^+ &= \partial_k \psi_k^+(f) \partial_k^{-1} \psi_k^-(f) + \psi_k^+(f) \partial_k^0 \psi_k^+(f) \\
[[f, f]_k^+, [f, f]_k^-]_k^- &= \partial_k \psi_k^-(f) \partial_k^{-1} \psi_k^-(f) - \psi_k^+(f) \partial_k^0 \psi_k^-(f) \\
&= [[f]_k^1]^-
\end{align*}
\]

(18)

Here, we define the notation \([[:)]_k^p\) with \(k \in \mathbb{Z}\) and \(p \in \mathbb{Z}^+\). \(k\) is the degree of the derivative similar to the definition of \(\psi_k^+(.)\), and \(p\) is the number of ”iterations” of the operator \([[,)]_k^\pm\). Thus following the equation above, \(\psi_k^+(f)\) is equal to \([[f]_k^0]^+\), and \(\psi_k^-(f)\) equal to \([[f]_k^0]^\pm\). Now, let us show the proposition:

**Proposition 3**: for all \(p \in \mathbb{Z}^+\), for all \(k \in \mathbb{Z}\), the generalized energy operators \([[[:)]_k^p]^+\) and \([[[:)]_k^p]^\pm\) are bilinear and follow the derivative chain rule property.

**Proof.** First, let us recall the properties of a bilinear map according to [12]. Let us define the set \(\mathbf{V} \subseteq \mathbf{S}^-(\mathbb{R})\), and the map \(B : \mathbf{V} \times \mathbf{V} \to \mathbf{S}^-(\mathbb{R})\). \(B\) is a bilinear map if and only if:

\[
\begin{align*}
B(v_1 + v_2, w) &= B(v_1, w) + B(v_2, w), \forall v_1, v_2, w \in \mathbf{V} \\
B(v_1, w_1 + w_2) &= B(v_1, w_1) + B(v_1, w_2), \forall v_1, w_1, w_2 \in \mathbf{V} \\
B(cv_1, w) &= B(v_1, cw) = cB(v_1, w), \forall v_1, w \in \mathbf{V}, c \in \mathbb{R}
\end{align*}
\]

(19)

Previous works such as [2] and [14] showed that for any \(k \in \mathbb{Z}\), \(\psi_k^+(.)\) and \(\psi_k^-(.)\) are quadratic forms of a specific bilinear operator. Thus for any \(k \in \mathbb{Z}\), \(\psi_k^+(.)\) and \(\psi_k^-(.)\) are bilinear operators due to the quadratic superposition principle [2]. In addition, with the definition of the family of energy operators \((\psi_k^+)_{k \in \mathbb{Z}}\) and \((\psi_k^-)_{k \in \mathbb{Z}}\), it is straightforward that \(([,)]_k^+\) \(k \in \mathbb{Z}\) and \(([,)]_k^-\) \(k \in \mathbb{Z}\) are families of bilinear operators. In the following, the bilinearity and the derivative chain rule property of the generalized energy operator families are shown by induction on the index \(p\) in \(\mathbb{Z}^+\).

A - Bilinearity

- Case \(p = 0\)

To refer to the previous paragraph, based on \([[,)]_k^+\) \(([,)]_k^-\) in Equation (15), this is the generalization of the quadratic operator \(\psi_k^+(.)\) \((\psi_k^-(.)\). Therefore, \([[[:)]_k^0]^+\) and \([[[:)]_k^0]^\pm\) are bilinear operators.
• Case $p = 1$

By definition,

$$[[\cdot]^1]_k^+ = [[\cdot]^0]_k^+$$
$$[[\cdot]^1]_k^- = [\Psi^+_k, \Psi^+_k]^+$$ \hspace{1cm} (20)

With $\Psi^+_k(.)$ ($k$ in $\mathbb{Z}$) and $[.,.]^+_k$ bilinear operators, we can conclude that $[[\cdot]^1]_k^+$ is a bilinear operator as well for any $k$ in $\mathbb{Z}$.

• Case $p = h + 1$

Now, we assume that $[[\cdot]^h]_k^+$ is a bilinear operator for any $k$ in $\mathbb{Z}$. We can write:

$$[[\cdot]^{h+1}]_k^+ = [[\cdot]_k^h, [\cdot]_k^h]^+$$ \hspace{1cm} (21)

As mentioned before, $[[\cdot]^h]_k^+$ and $[.,.]_k^+$ are bilinear operators. Thus, we can conclude that $[[\cdot]^{h+1}]_k^+$ is also a bilinear operator for any $k$ in $\mathbb{Z}$. By replacing $+$ with $-$ in the previous equations, it shows that the bilinearity of the conjugate operator $[[\cdot]^p]_k^-$. 

**B - Derivative Chain Rule**

Now, let us show the derivative chain rule property of $[[\cdot]^p]_k^-$ and $[[\cdot]^p]_k^+$ for any $k$ in $\mathbb{Z}$ and $p$ in $\mathbb{Z}^+$ with induction.

• Case $p = 0$

It was shown in [14] that $[[\cdot]^0]_k^+$ and $[[\cdot]^0]_k^-$ ($k$ in $\mathbb{Z}$) follow the chain rules derivative rule such as for any $f$ in $\mathbf{S}^-(\mathbb{R})$:

$$[[f]^0]_k^+ = [[f]^0]_{k+1}^+ + [\partial_t [f]^0]_{k-1}^-$$
$$[[f]^0]_k^- = [[f]^0]_{k+1}^- + [\partial_t [f]^0]_{k-1}^+$$ \hspace{1cm} (22)

• Case $p = 1$
By definition of the generalized energy operator and with Equation (21), one can write for any \( f \) in \( S^-(\mathbb{R}) \):

\[
\begin{align*}
[[f]^{1+}]_k^n &= [\Psi^+_k(f), \Psi^+_k(f)]_k^n \\
[[f]^{1+}]_k &= \partial^{k-1}_t \Psi^+_k(f) \partial_t \Psi^+_k(f) + \Psi^+_k(f) \partial^k_t \Psi^+_k(f) \\
\partial_t [[f]^{1+}]_k &= \partial^k_t \Psi^+_k(f) \partial_t \Psi^+_k(f) + \partial_t \Psi^+_k(f) \partial^k_t \Psi^+_k(f) + \Psi^+_k(f) \partial_t \Psi^+_k(f) + \Psi^+_k(f) \partial^k_t \Psi^+_k(f)
\end{align*}
\]

\( (23) \)

- Case \( p = h + 1 \)

Let us assume that the derivative chain rule works for the generalized operator \( [[f]^{h+1}]_k \) \((k \in \mathbb{Z})\). Following the previous case, we write:

\[
\begin{align*}
[[f]^{h+1+}]_k^n &= [[f]^{h+1}]_k^n, [[f]^{h+1}]_k^n \\
[[f]^{h+1+}]_k &= \partial^{k-1}_t [[f]^{h+1}]_k \partial_t [[f]^{h+1}]_k + [[f]^{h+1}]_k \partial^k_t [[f]^{h+1}]_k \\
\partial_t [[f]^{h+1+}]_k &= \partial^k_t [[f]^{h+1}]_k \partial_t [[f]^{h+1}]_k + \partial_t [[f]^{h+1}]_k \partial^k_t [[f]^{h+1}]_k + \partial_t [[f]^{h+1}]_k \partial^k_t [[f]^{h+1}]_k
\end{align*}
\]

\( (24) \)

This is the end of the proof by induction of the derivative chain rule for the generalized operator \( [[f]^{h+1}]_k \). The same induction can be done for the generalized energy operator \( [[f]^{h+1}]_k \) by simply replacing the sign. Note that the derivative chain rule property of the generalized energy operators comes from the general Leibniz derivative rules. This can be compared to similar properties of fractal operators such as mentioned in [17].

Note that for \( p \) in \( \mathbb{Z}^+ \) and \( f \) in \( S^-(\mathbb{R}) \), \( [[f]^p]_k^n \) in \( S^-(\mathbb{R}) \), and for \( n \in \mathbb{Z}^+ \) and \( n > 1 \) \(([[f]^p]_k^n) \) in \( S^-(\mathbb{R}) \). With this property, it is possible to extent Lemma 0 and Theorem 0 (e.g., [14]) established for the families of DEOs \((\Psi^+_k)_{k \in \mathbb{Z}}\) and \((\Psi^-_k)_{k \in \mathbb{Z}}\) when using the generalized energy operators.

However following the energy space definition in Section 3 let us introduce the energy space \( E_p = \bigcup_{i \in \mathbb{Z}^+} H_i \) \((p \in \mathbb{Z}^+ \) with \( H_i \subseteq S^-(\mathbb{R}) \) with \( i \) in \( \mathbb{Z}^+ \) such as:

\[
H_i = \{ g \in S^-(\mathbb{R}), \ p \in \mathbb{Z}^+ | \ g = \partial^i_t ([[f]^p]_1^n), [f]^p_1^n \in S^-(\mathbb{R}), \ n \in \mathbb{Z}^+ - \{0\} \} 
\]

\( (25) \)

One can define for \( p \) in \( \mathbb{Z}^+ \), \( S^-_p(\mathbb{R}) \subseteq S^-(\mathbb{R}) \), such as:

\[
S^-_p(\mathbb{R}) = \{ f \in S^-(\mathbb{R}), \ p \in \mathbb{Z}^+ | E_p = \bigcup_{i \in \mathbb{Z}^+} H_i \neq \{0\} \} 
\]

\( (26) \)
With this definition, the energy space $E_p$ is associated with $E([t]_p)$. Note that Definition 3 does not involve directly the energy operators. In other words, $E$ is not equal to $E_0$. The case $p = 0$ can be defined such as ($i$ in $\mathbb{Z}^+$):

$$H^i = \{ g \in S^-(\mathbb{R}), \ p \in \mathbb{Z}^+ | g = \partial^i_t (\Psi^+_1(f))^n, \Psi^+_1(f) \in S^-(\mathbb{R}), \ n \in \mathbb{Z}^+ - \{0\} \}$$

(27)

Furthermore, one can define:

$$S^-_0(\mathbb{R}) = \{ f \in S^-(\mathbb{R}) | E_0 = \bigcup_{i \in \mathbb{Z}^+} H^i \neq \{0\} \}$$

(28)

Note that for the case $i = 0$, it is similar as in the previous case (e.g, definition of $M^0$) where $[[f]|_1]$ could be considered as a special solution of some PDEs.

Here is the Lemma:

**Lemma 1**: for $f$ in $S^-_p(\mathbb{R})$, $p$ in $\mathbb{Z}^+$, the families of generalized energy operators $[[|]^k_1] (k = \{0, \pm 1, \pm 2, ...\})$ decompose the successive derivatives of the $n$-th power of $[[f]|_1]$ for $n \in \mathbb{Z}^+$ and $n > 1$.

**Proof.** With the convention that $[[f]|_1]$ equal $f$, one can see that if $p$ equal 0, then Lemma 1 is reduced to the case with the families of generalized energy operators $[[|]^k_1] (k = \{0, \pm 1, \pm 2, ...\})$ decomposing the successive derivatives of the $n$-th power of $f$ for $n \in \mathbb{Z}^+$ and $n > 1$. This is exactly the statement of Lemma 0. Thus, the proof of the Lemma 1 is given by induction on the index $p$ and the the $n$-th power of $[[f]|_1]$. The induction is used to show the decomposition, and in a separated part on the non-uniqueness. However, this can be long and repetitive compared with the work already published in [14]. Thus, the Lemma 1 is demonstrated for the case $p = \{0, 1, N\} (N$ in $\mathbb{Z}^+)$ and $n \in \{2, 3, L\} (L > 1, L$ in $\mathbb{Z}^+)$.

>> A - Decomposition with generalized energy operators

- **Case** $p = 0$

Following exactly [14], the induction on $n$ is separated in two parts: the decomposition with the energy operator families and the non-uniqueness of the decomposition. As all the results are already properly shown in a previous work, we only remind here the main results. Note that $f$ is in $S^-_0(\mathbb{R})$ which according to [14], is equal to $S^-(\mathbb{R})$.

**Case** $n = 2$
We showed that when \( n = 2 \), one can decompose \( \partial_t^s f^2 \) (\( s \) in \( \mathbb{Z}^+ \), \( s > 0 \)) with the energy operators \([\cdot]_k^{0+} (k = \{0, \pm1, \pm2, \ldots\})\) following the Equation (15) in [14]:

\[
\begin{align*}
\partial_t^s f^2 &= a_s^+(f) \\
\partial_t f^2 &= \sum_{k=0}^{s-1} \left( \partial_t^{s-k-1} \left[ f \right]_2^{0+} \right)_{(k+1)-s}, \quad \forall s \in \mathbb{Z}^+ - \{0\} \quad (29)
\end{align*}
\]

**Case \( n = 3 \)**

From the Equation (20) in [14] and Equation (29), one can write:

\[
\begin{align*}
\partial_t f^3 &= f^3 \frac{3}{2} a_1^+(f) \\
\partial_t f^3 &= f^3 \frac{3}{2} \left[ f \right]_1^{0+} \\
\partial_t f^3 &= f A_1^+(f) \\
A_s^+(f) &= \frac{3}{2} \partial_t^{s-1} a_1^+(f), \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
A_s^+(f) &= \frac{3}{2} \partial_t^{s-1} \left[ f \right]_1^{0+}, \quad \forall s \in \mathbb{Z}^+ - \{0\}
\end{align*}
\]

\[
\partial_t f^3 = \sum_{k=0}^{s-1} \left( \partial_t^{s-k-1} A_{k+1}^+(f) \right) \partial_t^{s-k} f, \quad \forall s \in \mathbb{Z}^+ - \{0\} \quad (30)
\]

**Case \( n = L, L > 1 \)**

With the notation of the generalized energy operators, as shown in Equation (28) in [14]:

\[
\begin{align*}
\partial_t f^L &= p f^{L-1} \partial_t f \\
\partial_t f^L &= \frac{L}{2} f^{L-2} \left[ f \right]_1^{0+} \\
\partial_t f^L &= \frac{L}{L-1} B_1^+(f) f^{L-2} \\
B_s^+(f) &= \frac{L-1}{2} \partial_t^{s-1} \left[ f \right]_1^{0+}, \quad \forall s \in \mathbb{Z}^+ - \{0\}
\end{align*}
\]

\[
\begin{align*}
\partial_t^2 f^L &= \frac{L}{L-1} B_1^+(f) \partial_t f^{L-2} + \frac{L}{L-1} B_2^+(f) f^{L-2} \\
\partial_t^{s+1} f^L &= \sum_{k=0}^{s} \left( \frac{L}{L-1} B_{k+1}^+(f) \partial_t^{s-k} f^{L-2} \right), \quad \forall s \in \mathbb{Z}^+ \quad (31)
\end{align*}
\]
That is all the main results shown in the case of Lemma 0. This then ends the case \( p = 0 \).

- **Case** \( p = 1 \)

In this case, the Lemma 1 should be stated as: for \( f \) in \( S_1^- (\mathbb{R}) \), the families of generalized energy operators \([[[f]]_k^+] (k = 0, \pm 1, \pm 2, \ldots)\) decompose the successive derivatives of the \( n \)-th power of \([ [f] ]_1^+ \) for \( n \in \mathbb{Z}^+ \) and \( n > 1 \).

In other words, one can substitute \( f^n \) with \(([[f]]_0^+]_1^n)\), and the family of generalized operators \(([[f]]_k^+]_{k \in \mathbb{Z}^+})\). It is then possible to write according to the previous development:

**Case** \( n = 2 \)

Following the same development as in the previous case,

\[
\partial_t^s ( [[f] ]_1^+ )^2 = a_1^+ (f)
\]

\[
\partial_t^s ( [[f] ]_1^+ )^2 = \sum_{k=0}^{s-1} (s-1)(s-k-1)[\partial_t^{s-k-1}[[f] ]_1^{+}]_{2(k+1)-s}, \quad \forall s \in \mathbb{Z}^+ - \{0\} \tag{32}
\]

**Case** \( n = 3 \)

From the Equation (30), one can write:

\[
\partial_t ( [[f] ]_1^+ )^3 = f \frac{3}{2} [[f] ]_1^+
\]

\[
\partial_t ( [[f] ]_1^+ )^3 = f \frac{3}{2} a_1^+ (f)
\]

\[
\partial_t ( [[f] ]_1^+ )^3 = f A_1^+ (f)
\]

\[
A_1^+(f) = \frac{3}{2} \partial_t^{s-1} a_1^+(f), \quad \forall s \in \mathbb{Z}^+ - \{0\}
\]

\[
A_1^+(f) = \frac{3}{2} \partial_t^{s-1}[[f] ]_1^{+}, \quad \forall s \in \mathbb{Z}^+ - \{0\}
\]

\[
\partial_t^s ( [[f] ]_1^+ )^3 = \sum_{k=0}^{s-1} \binom{s-1}{k} A_{k+1}^+(f) \partial_t^{s-k-1} f, \quad \forall s \in \mathbb{Z}^+ - \{0\} \tag{33}
\]

**Case** \( n = L, L > 1 \)

With the notation of the generalized energy operators, it was shown in
Equation (31):

\[
\partial_t([f]_1^{0+})^L = L([f]_1^{0+})^{L-1}\partial_t([f]_1^{0+}) \\
\partial_t([f]_1^{0+})^L = \frac{L}{2}([f]_1^{0+})^{L-2}[f]_1^{1+} \\
\partial_t([f]_1^{0+})^L = \frac{L}{L-1}B^+_t(f)([f]_1^{0+})^{L-2} \\
B^+_t(f) = \frac{L-1}{2}\partial_t^{-1}([f]_1^{1+}), \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
\partial^2_t([f]_1^{0+})^L = \frac{L}{L-1}B^+_t(f)\partial_t([f]_1^{0+})^{L-2} + \frac{L}{L-1}B^+_t(f)([f]_1^{0+})^{L-2} \\
\partial^{t+1}_t([f]_1^{0+})^L = \sum_{k=0}^{s} \binom{s}{k} \frac{L}{L-1}B^+_{k+1}(f)\partial_t^{-k}([f]_1^{0+})^{L-2}, \quad \forall s \in \mathbb{Z}^+ \quad (34)
\]

- **Case** \( p = N \)

In this case, the Lemma 1 states that: for \( f \) in \( S_N^{-}(\mathbb{R}) \), the families of generalized energy operators \([ [\cdot]_k^N ]_1^{1+} (k = \{0, \pm 1, \pm 2, ...\})\) decompose the successive derivatives of the \( n \)-th power of \([ [f]_1^{N-1} ]_1^{1+}\) for \( n \in \mathbb{Z}^+ \) and \( n > 1 \).

As we see in the statement of the Lemma 1, one has to assume that \( S_N^{-}(\mathbb{R}) \) is not reduced to \( \{0\} \). Thus, following the previous development:

**Case** \( n = 2 \)

Following the same development as in the previous case,

\[
\partial^2_t([f]_1^{N-1})^2 = a^+_s(f) \\
\partial^2_t([f]_1^{N-1})^2 = \sum_{k=0}^{s-1} \binom{s-1}{k}[\partial_t^{-k-1}[f]_1^{N-1}]_{2(k+1)-s}, \quad \forall s \in \mathbb{Z}^+ - \{0\} \quad (35)
\]

**Case** \( n = 3 \)
From the Equation (30), one can write:

\[
\partial_t ([f]^{N-1})^3 = \frac{3}{2} a^+_1(f) \partial_t a^+_1(f) \\
\partial_t ([f]^{N-1})^3 = \frac{3}{2} a^+_1(f) \\
\partial_t ([f]^{N-1})^3 = f A^+_1(f)
\]

\[
A^+_s(f) = \frac{3}{2} \partial_t^{s-1} a^+_1(f), \quad \forall s \in \mathbb{Z}^+ - \{0\}
\]

\[
A^+_s(f) = \frac{3}{2} \partial_t^{s-1} [f]_1^{N+}, \quad \forall s \in \mathbb{Z}^+ - \{0\}
\]

\[
\partial_t^k ([f]^{N-1})^3 = \sum_{k=0}^{s-1} (s-1) A^+_k(f) \partial_t^{s-1-k} f, \quad \forall s \in \mathbb{Z}^+ - \{0\} \tag{36}
\]

**Case** \( n = L, L > 1 \) With the notation of the generalized energy operators, it was shown in Equation (31):

\[
\partial_t ([f]^{N-1})^L = L ([f]^{N-1})^{L-1} \partial_t ([f]^{N-1}) \\
\partial_t ([f]^{N-1})^L = \frac{L}{2} ([f]^{N-1})^{L-2} [f]_1^{N+} \\
\partial_t ([f]^{N-1})^L = \frac{L}{L-1} B^+_1(f) ([f]^{N-1})^{L-2} \\
B^+_s(f) = \frac{L-1}{2} \partial_t^{s-1} [f]_1^{N+}, \quad \forall s \in \mathbb{Z}^+ - \{0\}
\]

\[
\partial_t^L ([f]^{N-1})^L = \frac{L}{L-1} B^+_1(f) \partial_t ([f]^{N-1})^{L-2} + \frac{L}{L-1} B^+_2(f) ([f]^{N-1})^{L-2} \\
\partial_t^{s+1} ([f]^{N-1})^L = \sum_{k=0}^{s} (s-k) \frac{L}{L-1} B^+_{k+1}(f) \partial_t^{s-k} ([f]^{N-1})^{L-2}, \quad \forall s \in \mathbb{Z}^+ \tag{37}
\]

This ends the first part on the decomposition of functions with generalized energy operators.

**>> B - Non-uniqueness of the decomposition with generalized energy operators**

- **Case** \( p = 0 \)

In [14] in the proof of the Lemma 0, the non-uniqueness of the decomposition of the successive derivatives \( \partial_t^n f^i (n \in \mathbb{Z}^+, n > 1, i \in \mathbb{Z}^+ \) was shown with
a simple counter example for $f$ in $S^{-0}_0(\mathbb{R})$:

$$
\eta_{k}(f) = 3(\partial_t f \partial_t^{k-1} f) - f \partial_t^{k} f, \ \forall k \in \mathbb{Z}
$$

(38)

Note that the derivative chain rule property is applied to this operator. One can verify:

$$
\begin{align*}
\partial_t f^2 &= 2f \partial_t f \\
\partial_t^2 f^2 &= \eta_{1}(f) \\
\eta_{1}(f) &= [[f]^{0}]_{1}^{+} \\
\partial_t^2 f^2 &= 2(\partial_t f)^2 + 2f \partial_t^2 f \\
\partial_t^4 f^2 &= \partial_t \eta_{1}^{+}(f) \\
\partial_t [[f]^{0}]_{1}^{+} &= \partial_t \eta_{1}^{+}(f) \\
\partial_t^2 f^2 &= \eta_{2}^{+}(f) + \eta_{0}^{+}(\partial_t f)
\end{align*}
$$

(39)

- **Case $p = 1$**

In the same way, one can also define the generalized energy operator $[[\cdot]_{nk}^{1}]^{+}$ ($k = \{0, \pm 1, \pm 2, \ldots\}$) decomposing the successive derivatives of the $n$-th power of $[[f]^{0}]_{1}^{+}$ for $f$ in $S^{-1}_1(\mathbb{R})$, for $n \in \mathbb{Z}^{+}$ and $n > 1$.

$$
[[f]^{1}]_{nk} = 3(\partial_t [[f]^{0}]_{1}^{+} \partial_t^{k-1} [[f]^{0}]_{1}^{+}) - [[f]^{0}]_{1}^{+} \partial_t^k [[f]^{0}]_{1}^{+}, \quad \forall k \in \mathbb{Z}
$$

$$
\begin{align*}
\partial_t ([[f]^{0}]_{1}^{1+})^2 &= 2[[f]^{0}]_{1}^{1+} \partial_t f \\
\partial_t ([[f]^{0}]_{1}^{1+})^2 &= [[f]^{1}]_{n1} \\
[[f]^{1}]_{n1} &= [[f]^{1}]_{1}^{1+} \\
\partial_t^2 ([[f]^{0}]_{1}^{1+})^2 &= 2(\partial_t [[f]^{0}]_{1}^{1+})^2 + 2[[f]^{0}]_{1}^{1+} \partial_t^2 [[f]^{0}]_{1}^{1+} \\
\partial_t^2 ([[f]^{0}]_{1}^{1+})^2 &= \partial_t [f]_{1}^{1+} \\
\partial_t [[f]^{1}]_{n1} &= \partial_t [[f]^{1}]_{n1} \\
\partial_t^2 ([[f]^{0}]_{1}^{1+})^2 &= [[f]^{1}]_{n2} + [\partial_t [f]^{1}]_{n0}^{+}
\end{align*}
$$

(40)
• Case $p = N$

One can generalize the case $p = \{0, 1\}$ with the generalized energy operator for $f$ in $S^N_{\mathbb{R}}$:
\[
[[f]]^N_1\eta_k^+ = 3(\partial_t[[f]]^{N-1}_1\partial_t^{k-1}[[f]]^{N-1}_1) - [[f]]^{N-1}_1\partial_t^k[[f]]^{N-1}_1, \quad \forall k \in \mathbb{Z}
\]

Following the same development,
\[
\begin{align*}
\partial_t([[f]]^{N-1}_1)^{2} &= 2[[f]]^{N-1}_1\partial_t f \\
\partial_t([[f]]^{N-1}_1)^{2} &= [[f]]^{N}_1 \eta_1 \\
[[f]]^{N}_1 &= [[f]]^{N}_1 \\
\partial_t^2([[f]]^{N-1}_1)^{2} &= 2(\partial_t[[f]]^{N-1}_1)^2 + 2[[f]]^{N-1}_1\partial_t^2[[f]]^{N-1}_1 \\
\partial_t^2([[f]]^{N-1}_1)^{2} &= \partial_t[[f]]^{N}_1 \eta_1 \\
\partial_t^2([[f]]^{N-1}_1)^{2} &= [[f]]^{N}_1 \eta_0 + [\partial_t[[f]]^{N}_1 \eta_0] \\
\end{align*}
\]

(41)

Now, one can define the subset $s^*_p(\mathbb{R})$ defined as:
\[
s^*_p(\mathbb{R}) = \{ f \in S^*_p(\mathbb{R}), \ p \in \mathbb{Z}^+, f \notin (\cup_{k \in \mathbb{Z}} Ker([[f]]^{p}_k)) \cup (\cup_{k \in \mathbb{Z}} (1) Ker([[f]]^{p}_k)) \}
\]

(42)

The subset $s^*_p(\mathbb{R})$ is also defined such as $E_p \neq \{0\}$. Thus, one can see that $s^*_p(\mathbb{R}) \subseteq S^*_p(\mathbb{R})$.

**Theorem 1**: for $f$ in $s^*_p(\mathbb{R})$, for $p$ in $\mathbb{Z}^+$, the families of generalized operators $[[f]]^+_k$ and $[[f]]^-_k (k = \{0, \pm 1, \pm 2, \ldots\})$ decompose uniquely the successive derivatives of the $n$-th power of $[[f]]^{p-1}_1$ for $n \in \mathbb{Z}^+$ and $n > 1$.

**Proof.** The proof is an induction on the index $p$ and the $n$-th power of $[[f]]^{p-1}_1$. It is separated in three parts: the decomposition with the generalized energy operators, the existence and the uniqueness of the decomposition. However, that follows exactly the work of [13]. Similarly to the proof of **Lemma 1**, one should notice that for the case $p = 0$ the **Theorem 1** is exactly the statement of **Theorem 0** with $s^*_0(\mathbb{R})$ equal to $s^-(\mathbb{R})$ defined in the first section. To keep the demonstration short, the induction is done for $n$ in $\{2, L\}$ and $p$ in $\{0, N\}$.

>> A - Decomposition with generalized energy operators

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Case $p = 0$

Recall the definition of $[[.0]^-_1]$ from Equation (18) and the proof of Theorem 0 (see [14]), one can write for $f$ in $s^-_0(R)$:

**Case $n = 2$:**

$$
\begin{align*}
\partial_t f^2 &= f \partial_t f + f \partial_t f + f \partial_t f - f \partial_t f \\
\partial_t f^2 &= [[f]^0]+[[f]^0] \\
\partial^2_t f^2 &= 2(\partial_t f)^2 + 2f \partial^2_t f \\
\partial^2_t f^2 &= \partial_t([[f]^0]+[[f]^0] \\
\partial^2_t f^2 &= [[f]^0]+2[\partial_t[f]^0]+[[f]^0]+2[\partial_t[f]^0] \\
\partial^2_t f^2 &= [[f]^0]+2[\partial_t[f]^0]
\end{align*}
$$

(43)

There is a symmetry with the proof of the previous lemma. From Equations (43), one can define $a^-_s(f)$ in the same way that $a^+_s(f)$ was defined in Equations (29) as:

$$
\begin{align*}
\partial^s_t f^2 &= \sum_{k=0}^{s-1} \binom{s-1}{k} [\partial^s-k-1[f]^0]_{2(k+1)-s}, \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
\,(44)
\end{align*}
$$

With the property of the derivative chain rule, it is easy to calculate the first terms of the DEO family $a^-_s(f)$ such as :

$$
\begin{align*}
a^-_1(f) &= [[f]^0]^- = 0 \\
a^-_2(f) &= \partial_t [[f]^0]^- = [[[f]^0]^- + [\partial_t[f]^0]^- \\
[[f]^0]^- &= -[\partial_t[f]^0]^- \\
\end{align*}
$$

(45)

The family of DEO $[[f]^0]^- (k \in \mathbb{Z})$ has the same derivative properties as $[[f]^0]^+_0$. A similar equation can then be established for $a^-_s(f)$ following the development written in Equation (29) as:

$$
\begin{align*}
a^-_s(f) &= \sum_{k=0}^{s-1} \binom{s-1}{k} [\partial^s-k-1[f]^0]_{2(k+1)-s}, \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
a^-_s(f) &= 0 \\
\end{align*}
$$

(46)
This formula has just been checked for \( s = \{1, 2, 3, 4\} \) with the Equation (45). The generalization of the formula for \( s = m \) is very similar to the one described in the Equations (46) literally by changing + and − in the definition of the energy operator. It follows that the decomposition of the successive derivatives of \( f^2 \) is generalized for any \( n \) in \( \mathbb{Z}^+ - \{0\} \) as:

\[
\partial^m_t f^2 = \partial^{m-1}_t ([f]^0_0^{+} + [f]^0_0^{-}) = \sum_{k=0}^{m-1} \binom{m-1}{k} [\partial^{m-k-1}_t [f]^0_{2(k+1)-m}^{+}] + \sum_{k=0}^{m-1} \binom{m-1}{k} [\partial^{m-k-1}_t [f]^0_{2(k+1)-m}^{-}]
\]

(47)

Case \( n = L, L > 1 \)

Following the same step as in the proof of Lemma 1, let us define \( B^+_s (f) \) and \( B^-_s (f) \) ( \( s \) in \( \mathbb{Z}^+ - \{0\} \)) with the assumption that they decompose the successive derivatives of \( f^{p-1} \) as:

\[
\begin{align*}
\partial f^L &= pf^{L-1}\partial f, \\
\partial^2 f^L &= \frac{L}{2} f^{L-2}[[f]^0_1^+] \\
B^+_s (f) &= \frac{L-1}{2} \partial^{s-1}_t [[f]^0_1^+], \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
B^-_s (f) &= -\frac{L-1}{2} \partial^{s-1}_t [[f]^0_1^-], \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
\partial f^L &= \frac{L}{L-1} (B^+_1 (f) + B^-_1 (f)) f^{L-2} \\
\partial^2 f^L &= \frac{L}{L-1} (B^+_1 (f) + B^-_1 (f)) \partial f^{L-2} + \frac{L}{L-1} (B^+_2 (f) + B^-_2 (f)) f^{L-2}
\end{align*}
\]

(48)

There is again a symmetry with the proof of Lemma 1. One can define the \( s + 1 \)-th derivative of \( ([f]^{N-1}]^+_1) \) using \( B^+_{k+1} (f) \) and \( B^-_{k+1} (f) \):

\[
\partial^{s+1}_t f^L = \sum_{k=0}^{s} \binom{s}{k} \frac{L}{L-1} (B^+_{k+1} (f) + B^-_{k+1} (f)) \partial^{s-k}_t f^{L-2}, \quad \forall s \in \mathbb{Z}^+ \quad (49)
\]

This equation has just been checked for \( s = \{0, 1\} \). As the induction proof follows exactly the proof of Lemma 1 as in Equation (51) by only adding
$B_{k+1}^- (f)$ which has the same properties as $B_{k+1}^+ (f)$. It allows then to assume the generalization to the case $s + 2$.

Thus, $(B_k^+)_{k \in \mathbb{Z}}$ and $(B_k^-)_{k \in \mathbb{Z}}$ decompose the $s$-th derivative of $f^L$. From their definition, one can conclude that $(\partial_t^s [f]_k^N)_{k \in \mathbb{Z}}$ and $(\partial_t^{-s} [f]_k^N)_{k \in \mathbb{Z}}$ decompose $\partial_t^s f^L$.

- **Case $p = N$**

This case follows the proof of the Lemma 1 and in particular the Equations (35) to (37) and (45). One can then write for $f$ in $s_N (\mathbb{R})$:

**Case $n = 2$:**

\[
\begin{align*}
\partial_t^2 (\partial_t^{s-1} [f]_1^N)^2 &= a_+^s (f) + a_-^s (f) \\
\partial_t^2 (\partial_t^{s-1} [f]_1^N)^2 &= \sum_{k=0}^{s-1} (s-1) (\partial_t^{s-k-1} [f]_1^N) L^2 (\partial_t^{s-k-1} [f]_1^N + \partial_t^{s-k-1} [f]_1^{-N}) \\
&+ \partial_t^{s-k-1} [f]_1^{-N}, \quad \forall s \in \mathbb{Z}^+ - \{0\}
\end{align*}
\]  

(50)

**Case $n = L$, $L > 1$** With the notation of the generalized energy operators, it was shown in Equation (34):

\[
\begin{align*}
\partial_t (\partial_t^{s-1} [f]_1^N)^L &= L (\partial_t^{s-1} [f]_1^N)^L - 1 \partial_t (\partial_t^{s-1} [f]_1^N) \\
\partial_t (\partial_t^{s-1} [f]_1^N)^L &= \frac{L}{2} (\partial_t^{s-1} [f]_1^N)^L - 2 (\partial_t^{s-1} [f]_1^N + \partial_t^{s-1} [f]_1^{-N}) \\
\partial_t (\partial_t^{s-1} [f]_1^N)^L &= \frac{L}{L-1} (B_k^+ (f) + B_k^- (f)) (\partial_t^{s-1} [f]_1^N)^L - 2 \\
B_k^+ (f) &= \frac{L-1}{2} \partial_t^{s-1} [f]_1^N, \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
B_k^- (f) &= \frac{L-1}{2} \partial_t^{s-1} [f]_1^{-N}, \quad \forall s \in \mathbb{Z}^+ - \{0\} \\
\partial_t^2 (\partial_t^{s-1} [f]_1^N)^L &= \frac{L}{L-1} (B_k^+ (f) + B_k^- (f)) \partial_t (\partial_t^{s-1} [f]_1^N)^L - 2 \\
&+ \frac{L}{L-1} (B_k^+ (f) + B_k^- (f)) (\partial_t^{s-1} [f]_1^N)^L - 2 \\
\partial_t^{s + 1} (\partial_t^{s-1} [f]_1^N)^L &= \sum_{k=0}^{s} \frac{L}{L-1} (B_k^{+1} (f) + B_k^{-1} (f)) \partial_t^{s-k} (\partial_t^{s-1} [f]_1^N)^L - 2, \quad \forall s \in \mathbb{Z}^+
\end{align*}
\]  

(51)
Existence of the decomposition with generalized energy operators

In the proof of Lemma 1, it was shown that the non-uniqueness of the decomposition using a counter-example. Here, the proof re-investigate these examples. Then, it is generalized for $p > 0$ via induction on $p$.

- **Case $p = 0$**

With $f$ in $s_0^{-}(\mathbb{R})$,

**Case $n = 2$**:

It was shown that the family of operators $(\eta_k)_{k \in \mathbb{Z}}$ (proof of the Lemma, Equation (38)), decomposes $\partial^s f^2$ ($s \in \mathbb{Z}^+ - \{0\}$). One can rewrite it as a sum of the DEO family $(\Psi_k^+)_{k \in \mathbb{Z}}$ and $(\Psi_k^-)_{k \in \mathbb{Z}}$ as:

$$
\eta_k(f) = \Psi_k^+(f) + 2\Psi_k^-(f), \quad k \in \mathbb{Z}
$$

$$
\eta_k(f) = [[f]^0]_k^+ + 2[[f]^0]_k^-, \quad k \in \mathbb{Z}
$$

(52)

**Case $n = L$ with $L > 1$**:

Previously, Equation (38) defined the decomposition of $\partial^s f^L$ ($s \in \mathbb{Z}^+ - \{0\}$) with the generalized energy operator $[[.]^0]_1^+$. With the definition of $[[.]^0]_1^-$, one can define the operator $\theta_k^+$ and $\theta_k^-$ as:

$$
\partial_t^s f^L = L f^{L-1} \partial_t f
$$

$$
\partial_t^s f^L = \frac{L}{2} f^{L-2}[[f]^0]_1^+
$$

$$
\theta_k^+(f) = \frac{L}{2} [[f]^0]_k^+
$$

$$
\theta_k^-(f) = \frac{L}{2} [[f]^0]_k^-
$$

$$
\partial_t^s f^L = \frac{L}{2} f^{L-2}([[f]^0]_1^+ + [[f]^0]_1^-)
$$

$$
= \frac{L}{L-1} f^{L-2} (\theta_1^+(f) + \theta_1^-(f))
$$

(53)

Following the development in Equations (48) and (51), one can see that $B_k^+(f) = \partial_t^{-1} \theta_1^+(f)$ and $B_k^-(f) = \partial_t^{-1} \theta_1^-(f)$ ($s \in \mathbb{Z}^+ - \{0\}$). Note that $\theta_k^+$ and $\theta_k^-$ are bilinear operators and follow the derivative chain rule property.
by definition.
Using Equation \((51)\), one can easily show that \(\theta^-_k \) and \(\theta^+_k \) decomposes \(\partial_t^s f^L \) \((s \in \mathbb{Z}^+ - \{0\})\). It is then possible to conclude the existence of the decomposition of any operator by using \((\lceil . \rceil^0)^-_k \) and \((\lceil . \rceil^1_k)^+ \).

- **Case** \(p = N\)

With \(f\) in \(s_N^{-} (\mathbb{R})\),

**Case** \(n = 2\):

It was shown that the family of operators \((\lceil . \rceil^N)^+_k \) \((\text{proof of Lemma 1, Equation } (11)\)) decomposes \(\partial_t^s (\lceil . \rceil^{N-1})^+_1 \) \((s \in \mathbb{Z}^+ - \{0\})\). One can rewrite it as a sum of the DEO family \((\lceil . \rceil^N)^-_k \) and \((\lceil . \rceil^N)^+_k \) as:

\[
\begin{align*}
\lceil f \rceil_{\eta k}^N &= 3(\partial_t (\lceil f \rceil^{N-1})^+_1 \partial_t^{k-1} (\lceil f \rceil^{N-1})^+_1) - \lceil f \rceil^{N-1}^+_1 \partial_t^k (\lceil f \rceil^{N-1})^+_1 \\
&= \lceil f \rceil^+_1 + 2\lceil f \rceil^1_1, \quad k \in \mathbb{Z}
\end{align*}
\]

\((54)\)

**Case** \(n = L\) with \(L > 1\):

Previously, Equation \((48)\) defined the decomposition of \(\partial_t^s (\lceil . \rceil^{N-1})^+_1 \) \((s \in \mathbb{Z}^+ - \{0\})\) with the generalized energy operator \((\lceil . \rceil^N)^+_k \). With the definition of \((\lceil . \rceil^N)^+_1 \), one can define the operator \((\lceil . \rceil^N)^+_{\delta_k} \) and \((\lceil . \rceil^N)^-_k \) as:

\[
\begin{align*}
\partial_t (\lceil f \rceil^{N-1})^+_1 \rceil^L &= L \lceil (\lceil f \rceil^{N-1})^+_1 \rceil^L - 1 \partial_t (\lceil f \rceil^{N-1})^+_1 \\
\partial_t (\lceil f \rceil^{N-1})^-_1 \rceil^L &= \frac{L}{2} (\lceil f \rceil^{N-1})^L - 2 \lceil f \rceil^1_1 \\
\lceil f \rceil^+_k \rceil &= \frac{L - 1}{2} \lceil f \rceil^+_k \\
\lceil f \rceil^-_k \rceil &= \frac{L - 1}{2} \lceil f \rceil^-_k \\
\partial_t (\lceil f \rceil^{N-1})^+_1 \rceil^L &= \frac{L}{L - 1} f^{L-2} f^{L-2} (\lceil f \rceil^+_1 + \lceil f \rceil^-_1) \\
\end{align*}
\]

\((55)\)

Following the development in Equations \((48)\) and \((51)\), one can see that \(B^+_s (f) = \partial_t^{s-1} \lceil f \rceil^+_1 \rceil \) and \(B^-_s (f) = \partial_t^{s-1} \lceil f \rceil^-_1 \rceil \) \((s \in \mathbb{Z}^+ - \{0\})\). Note that \((\lceil f \rceil^N)^-_k \) and \((\lceil f \rceil^N)^+ \) are bilinear operators and follow the derivative chain rule property by definition.

With Equation \((55)\), one can easily show that \((\lceil . \rceil^N)^-_k \) and \((\lceil . \rceil^N)^+_k \) decomposes \(\partial_t^s (\lceil . \rceil^{N-1})^+_1 \) \((s \in \mathbb{Z}^+ - \{0\})\). It is then possible to conclude the existence
of the decomposition of any operator by using \(((f^N)_k^+\))_{k \in \mathbb{Z}}\) and \(((f^N)_k^-\))_{k \in \mathbb{Z}}.\)

\[\text{C - About the Uniqueness of the decomposition with generalized energy operators}\]

Following the previous sections, the proof by induction on the index \(p\) in \(\mathbb{Z}^+\) shows the uniqueness of the decomposition of any family of operators decomposing \(\partial_{t}^{p+1}((f)^{p-1}_k^+)^n\) \((f \in s^+_p(\mathbb{R}), s \in \mathbb{Z}^+, n \in \mathbb{Z}^+ \text{ and } n > 1)\) with the families of generalized operators \((\mathbb{Z})^n_k^+\) and \((\mathbb{Z})^n_k^-\) \((k = \{0, \pm 1, \pm 2, ...\})\). In other words for example for \(p\) is equal to 0, one wants to show that if a family of operators \((S_k)_{k \in \mathbb{Z}}\) \((S_k \subseteq \mathbb{F}(s_{0}^{-} (\mathbb{R}), S_{-}^{-} (\mathbb{R}))\)) decomposes \(\partial_{t}^{p+1} f^n\) \((s \in \mathbb{Z}^+, n \in \mathbb{Z}^+ \text{ and } n > 1)\), \(S_k\) \((k \in \mathbb{Z})\) can be written with an unique sum of \((\mathbb{Z})^n_k^+\) and \((\mathbb{Z})^n_k^-\) \((k = \{0, \pm 1, \pm 2, ...\})\). Thus, the induction is on the index \(p\) and the \(k\)-th order of the generalized energy operators. Note that for a matter of clarity, \(p\) is restricted to the case \(\{0, N\}\).

- **Case** \(p = 0\)

This is the case already shown in the proof of Theorem 0. The same logic of the proof is applied for the case \(p > 0\).

**Case** \(k = 2\): For \(f \in s_{0}^{-} (\mathbb{R})\) \((\text{or } s^{-} (\mathbb{R}))\) and \(n \in \mathbb{Z}^+ \text{ and } n > 1\) one can assume that \((\alpha_1, \alpha_2, \beta_1, \beta_2)\) exist in \(\mathbb{R}^4\) such as:

\[
\begin{align*}
\partial_t f^n & = \partial_t^{n-1} S_1(f) \\
\partial_t^2 f^n & = \alpha_1 \partial_t^{n-1} ([f]_1^0^+) + \alpha_2 \partial_t^{n-1} ([f]_1^0^-) \\
\partial_t^3 f^n & = \beta_1 \partial_t^{n-1} ([f]_1^0^+) + \beta_2 \partial_t^{n-1} ([f]_1^0^-)
\end{align*}
\]

\(\text{(56)}\)

As with the operator family \((S_k)_{k \in \mathbb{Z}}\) follows the derivative chain rule property:

\[
\begin{align*}
\partial_t S_1(f) & = S_2(f) + S_0(\partial_t f) \\
\partial_t S_1(f) & = \alpha_1 \partial_t ([f]_1^0^+) + \alpha_2 \partial_t ([f]_1^0^-) \\
\partial_t S_1(f) & = \alpha_1 ([f]_1^0^+) + \partial_t ([f]_1^0^+) + \alpha_2 ([f]_1^0^- + \partial_t ([f]_1^0^-)
\end{align*}
\]

\(\text{(57)}\)

And then,

\[
\begin{align*}
S_2(f) & = \alpha_1 ([f]_1^0^+) + \alpha_2 ([f]_1^0^-) \\
S_2(f) & = \beta_1 ([f]_1^0^+) + \beta_2 ([f]_1^0^-)
\end{align*}
\]

\(\alpha_1 - \beta_1 ([f]_1^0^+) + (\alpha_2 - \beta_2) ([f]_1^0^-) = 0\)

\(\text{(58)}\)
As $f$ in $s_0^0(\mathbb{R})$, the images of $[[.]_1^0]^+$ and $[[.]_2^0]$ ($Im([[.]_1^0]^+)$ and $Im([[.]_2^0])$ are not reduced to $\{0\}$. It follows that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Note that it is not possible to do this simple check for $k = 1$ as $Im([[.]_1^0]) = \{0\}$ by definition of the family $([[[.]_k^p])_{k \in \mathbb{Z}}$.

**Case $k = L$:** For $f$ in $s_0^0(\mathbb{R})$, let us assume the uniqueness of the decomposition for $k = L - 1$ (with $k \neq 1$). For $k = L$, following Equation (57):

\[
\begin{align*}
\partial_t S_{L-1}(f) &= S_{L}(f) + S_{L-2}(\partial_t f) \\
\partial_t S_{L-1}(f) &= \alpha_1 \partial_t [[f]_1^0]^+_L + \alpha_2 \partial_t [[f]_1^0]_{L-1} \\
\partial_t S_{L-1}(f) &= \alpha_1([[f]_1^0]^+_L + [\partial_t [f]_1^0]_{L-2}) + \alpha_2([[f]_1^0]_{L-1} + [\partial_t [f]_1^0]_{L-2})
\end{align*}
\]

(59)

And then,

\[
\begin{align*}
S_{L}(f) &= \alpha_1 [[f]_1^0]^+_L + \alpha_2 [[f]_1^0]_{L-1} \\
S_{L}(f) &= \beta_1 [[f]_1^0]^+_L + \beta_2 [[f]_1^0]_{L-1} \\
(\alpha_1 - \beta_1) [[f]_1^0]^+_L + (\alpha_2 - \beta_2) [[f]_1^0]_{L-1} &= 0
\end{align*}
\]

(60)

By definition for $L \neq 1$, $Im([[f]_1^0]^+_L)$ and $Im([[f]_1^0]_{L-1})$ are not reduced to $\{0\}$, and it follows that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.

**Special Case $k = 1$:** To complete the proof with the assumption that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ for $k \in \mathbb{Z}$ and $k \neq 1$, the special case $k = 1$ can be solved as:

\[
\begin{align*}
\partial_t (\alpha_1 [[f]_1^0]^+_1) &= \alpha_1 ([[f]_1^0]^+_2 + [\partial_t [f]_1^0]_0) \\
&= \beta_1 ([[f]_1^0]^+_2 + [\partial_t [f]_1^0]_0) \\
&= \partial_t (\beta_1 [[f]_1^0]^+_1) \\
\partial_t (\alpha_2 [[f]_1^0]^-_1) &= \alpha_2 ([[f]_1^0]^-_2 + [\partial_t [f]_1^0]_0) \\
&= \beta_2 ([[f]_1^0]^-_2 + [\partial_t [f]_1^0]_0) \\
&= \partial_t (\beta_2 [[f]_1^0]^-_1)
\end{align*}
\]

(61)

To conclude in equation (53) in [14], it is shown that $\alpha_1 = \beta_1 = 1$.

- **Case $p = N$**

In this case, $f$ is in $s_0^N(\mathbb{R})$. Following the previous development, one can assume that there is a family of energy operators $(V_k)_{k \in \mathbb{Z}}$ and $V_k \subseteq F(s_N^0(\mathbb{R}), S^0(\mathbb{R}))$.
which decomposes \(((f)^{N-1})_1\)^n.

**Case k = 2:** For n in \(\mathbb{Z}^+\) and \(n > 1\), one can assume that \((\alpha_1, \alpha_2, \beta_1, \beta_2)\) exist in \(\mathbb{R}^4\) such as:

\[
\begin{align*}
\frac{\partial^k}{\partial t^k}((f)^{N-1})_1^n &= \frac{\partial^k}{\partial t^k}V_1(f) \\
\frac{\partial^k}{\partial t^k}f^n &= \alpha_1\frac{\partial^k}{\partial t^k}([f]^N)_1^+ + \alpha_2\frac{\partial^k}{\partial t^k}([f]^N)_{-1}^-
\frac{\partial^k}{\partial t^k}f^n &= \beta_1\frac{\partial^k}{\partial t^k}([f]^N)_1^+ + \beta_2\frac{\partial^k}{\partial t^k}([f]^N)_{-1}^-
\end{align*}
\]

(62)

In addition, the family \((V_k)_{k \in \mathbb{Z}}\) follows the derivative chain rule property:

\[
\begin{align*}
\hat{\partial}_t V_1(f) &= V_2(f) + V_0(\hat{\partial}_t f) \\
\hat{\partial}_t V_1(f) &= \alpha_1\hat{\partial}_t ([f]^N)_1^+ + \alpha_2\hat{\partial}_t ([f]^N)_{-1}^- \\
\hat{\partial}_t V_1(f) &= \alpha_1([f]^N)_2^+ + [\hat{\partial}_t ([f]^N)_0^+] + \alpha_2([f]^N)_{-2}^- + [\hat{\partial}_t ([f]^N)_{-0}^-]
\end{align*}
\]

(63)

And then,

\[
\begin{align*}
V_2(f) &= \alpha_1([f]^N)_2^+ + \alpha_2([f]^N)_{-2}^- \\
V_2(f) &= \beta_1([f]^N)_2^+ + \beta_2([f]^N)_{-2}^- \\
(\alpha_1 - \beta_1)([f]^N)_2^+ + (\alpha_2 - \beta_2)([f]^N)_2^- &= 0
\end{align*}
\]

(64)

As \(f\) in \(s_N^\infty(\mathbb{R})\), the images of \([f]^N)_2^+\) and \([f]^N)_{-2}^-\) are not reduced to \(\{0\}\). It follows that \(\alpha_1 = \beta_1\) and \(\alpha_2 = \beta_2\). Note that it is not possible to do this simple check for \(k = 1\) as \(Im([f]^N)_{-1}^-\) = \(\{0\}\) by definition.

**Case k = L:** For \(f\) in \(s_N^\infty(\mathbb{R})\), let us assume the uniqueness of the decomposition for \(k = L - 1\) (with \(k \neq 1\)). For \(k = L\), following Equation (57):

\[
\begin{align*}
\hat{\partial}_t V_{L-1}(f) &= V_L(f) + V_{L-2}(\hat{\partial}_t f) \\
\hat{\partial}_t V_{L-1}(f) &= \alpha_1\hat{\partial}_t ([f]^N)_{L-1}^+ + \alpha_2\hat{\partial}_t ([f]^N)_{L-1}^- \\
\hat{\partial}_t V_{L-1}(f) &= \alpha_1([f]^N)_L^+ + [\hat{\partial}_t ([f]^N)_{L-2}^-] + \alpha_2([f]^N)_{L}^- + [\hat{\partial}_t ([f]^N)_{L-2}^-]
\end{align*}
\]

(65)

And then,

\[
\begin{align*}
V_L(f) &= \alpha_1([f]^N)_L^+ + \alpha_2([f]^N)_{L}^- \\
V_L(f) &= \beta_1([f]^N)_L^+ + \beta_2([f]^N)_{L}^- \\
(\alpha_1 - \beta_1)([f]^N)_L^+ + (\alpha_2 - \beta_2)([f]^N)_{L}^- &= 0
\end{align*}
\]

(66)

By definition for \(L \neq 1\), \(Im([f]^N)_L^+\) and \(Im([f]^N)_L^-\) are not reduced to \(\{0\}\), and it follows that \(\alpha_1 = \beta_1\) and \(\alpha_2 = \beta_2\).
Special Case \( k = 1 \): To complete the proof with the assumption that \( \alpha_1 = \beta_1 \) and \( \alpha_2 = \beta_2 \) for \( k \in \mathbb{Z} \) and \( k \neq 1 \), the special case \( k = 1 \) can be solved as:

\[
\begin{align*}
\partial_t(\alpha_1([f]^{N_1}_{1+})) &= \alpha_1(([f]^{N_1}_{2+} + [\partial_t [f]^{N_1}_{1+}]) \\
&= \beta_1(([f]^{N_1}_{2+} + [\partial_t [f]^{N_1}_{1+}]) \\
&= \partial_t(\beta_1([f]^{N_1}_{1+})) \\
\partial_t(\alpha_2([f]^{N_1}_{1+})) &= \alpha_2(([f]^{N_1}_{2+} + [\partial_t [f]^{N_1}_{1+}]) \\
&= \beta_2(([f]^{N_1}_{2+} + [\partial_t [f]^{N_1}_{1+}]) \\
&= \partial_t(\beta_2([f]^{N_1}_{1+}))
\end{align*}
\]

(67)

This concludes the proof of Theorem 1.

\[\Box\]

Discussion \( n < -1 \): In this case, one can define:

\[
\forall f \in S_p^-(\mathbb{R}), \forall t \in \mathbb{R}, \ p \in \mathbb{Z}^+, \ ([|f(t)|^p]_1^+)^n \neq 0, \ \forall n \in \mathbb{Z}^+, \ n > 1, \ \frac{1}{([f(t)]^p_1^+)^n} \quad (68)
\]

This set of functions can also be described as: \( f \) in \( S_p^-(\mathbb{R}) \) and \( f \) not in \( \text{Ker}([|f(t)|^p]_1^+) \) for \( p \in \mathbb{Z}^+ \). Note that one could also chose to have \( f \) in \( s_p^-(\mathbb{R}) \). However, this is more restrictive than the set defined in (68).

Using an intermediary function, \( h \) such as \( h = \frac{1}{([f(t)]^p_1^+)} \), the problem of decomposing \( \partial_t^n([|f(t)|^p]_1^+) \) \( (s \in \mathbb{Z}^+ - \{0\}) \) is equivalent to resolving \( \partial_t^n h^n \), which has been demonstrated in the Lemma 1 and Theorem 1.

Discussion \( n = 1 \) or \( n = -1 \): As already underlined in [14], one can use a general formula for \( f \) in the set defined in equation (68):

\[
\partial_t^s([|f(t)|^p]_1^+) = \partial_t^s \left( \frac{([|f(t)|^p]_1^+)^3}{([|f(t)|^p]_1^+)^2} \right)
\]

\[
s = 1, \quad \partial_t([|f(t)|^p]_1^+) = ([|f(t)|^p]_1^+)^{-2} \partial_t([|f(t)|^p]_1^+)^3 + ([|f(t)|^p]_1^+)^3 \partial_t([|f(t)|^p]_1^+)^{-2}
\]

\[
s = 2, \quad \partial_t^2([|f(t)|^p]_1^+) = 2\partial_t([|f(t)|^p]_1^+)^{-2} \partial_t([|f(t)|^p]_1^+)^3 + ([|f(t)|^p]_1^+)^3 \partial_t^2([|f(t)|^p]_1^+)^{-2} + ([|f(t)|^p]_1^+)^2 \partial_t^2([|f(t)|^p]_1^+)^3
\]

(69)

The example for \( s = \{1, 2\} \) in Equation (69) shows that \( \partial_t^s([|f(t)|^p]_1^+) \) can be decomposed into a product of successive derivatives of \( ([|f(t)|^p]_1^+)^3 \) and \( ([|f(t)|^p]_1^+)^{-2} \). Those derivatives can be decomposed into a sum of generalized energy operators based on the Lemma 1 and Theorem 1 plus the
previous discussion (for the case \( n < -1 \)).
Now for the case \( n = -1 \), it is easy to see that:

\[
\partial_t^\alpha \left( \left[ [f(t)]^\alpha \right]_{1+}^{-1} \right) = \partial_t^\alpha \left( \left( \left[ [f(t)]^\alpha \right]_{1+} \right)^2 \right) \quad (70)
\]

With the discussion for the case \( n = 1 \), we can conclude that \( \partial_t^\alpha \left( \left[ [f(t)]^\alpha \right]_{1+}^{-1} \right) \) can be decomposed into a product of successive derivatives of \( \left( \left[ [f(t)]^\alpha \right]_{1+} \right)^2 \) and \( \left( \left[ [f(t)]^\alpha \right]_{1+} \right)^{-3} \).

5 Application to linear PDEs and the definition of some particular solutions:

In this section and the remainder of this work, the finite energy functions of one variable described in Section 2 (e.g., Equation (1)), are now functions of two variables referring to the space dimension \( (x) \) and time \( (t) \). Thus, one has to add in the notation of the operators the symbol \( t \) or \( x \) to indicate to which variable the derivatives refer to. For example, the operators \( \Psi_{-t}^k(.) \) and \( [[.]^1]_{k}^{t,x} \) (\( k \) in \( \mathbb{Z}^+ \)) refer to their derivatives in space, whereas \( \Psi_{-t}^k(.) \) and \( [[.]^1]_{k}^{t,t} \) (\( k \) in \( \mathbb{Z}^+ \)) refer to their derivatives in time.

5.1 With the families of DEOs

Let us consider the linear partial differential equation \( F(x,t,\alpha) \) with \( \alpha \) in \( \mathbb{Z}^+ - \{0\} \) such as:

\[
F(x,t,\alpha) = a_1 \partial_x^\alpha g(x,t) + a_2 \partial_t^\alpha g(x,t) - h(x,t) \quad (71)
\]

with \( a_1, a_2 \) in \( \mathbb{R} \), and \( h(x,t) \) a linear function. \( g \) is the general solution of \( F(x,t) \), and \( g \) is in \( C^\infty(\mathbb{R}) \). However, if one defines the set of solutions \( \mathcal{G} = \{ g \in S^-(\mathbb{R}) | g(x,t) = u^n(x,t), u^n \in S^-(\mathbb{R}), \ n \in \mathbb{Z}^+ - \{0\}, \ (x,t) \in \mathbb{R}^2 \} \), then \( \mathcal{G} \subseteq M^0 \) with \( M^0 \) the subset of the energy space \( E \) defined in Definition 3.

Equation (71) can be written with the family of energy operators \( (\Psi_{k}^{t,f})_{k \in \mathbb{Z}}, (\Psi_{k}^{-t,f})_{k \in \mathbb{Z}}, (\Psi_{k}^{t,x})_{k \in \mathbb{Z}}, \) and \( (\Psi_{k}^{-t,x})_{k \in \mathbb{Z}} \) following Theorem 0 and the development of the proof of Theorem 1 (see equations (47) and (49)). With the development in the previous section, one can write:

\[
\partial_t^\alpha u^n = \sum_{k=0}^{\alpha-1} \binom{\alpha-1}{k} \frac{n}{2} (\partial_t^k \Psi_{1}^{t,f}(u) + \partial_t^k \Psi_{1}^{-t,f}(u)) \partial_t^{\alpha-1-k} u^{n-2}, \ \forall \alpha \in \mathbb{Z}^+, \ n > 1 \quad (72)
\]
substituting in Equation (71):

$$F(x, t, \alpha) = a_1 \sum_{k=0}^{\alpha-1} \left( \frac{n-1}{2} \right) \left( \partial_t^k \Psi_1^x(t)(u(x,t)) + \partial_t^k \Psi_1^{-x}(u(x,t)) \right) \partial_t^{\alpha-1-k} u^{n-2}(x,t) + a_2 \sum_{k=0}^{\alpha-1} \left( \frac{n-1}{2} \right) \left( \partial_t^k \Psi_1^x(t)(u(x,t)) + \partial_t^k \Psi_1^{-x}(u(x,t)) \right) \partial_t^{\alpha-1-k} u^{n-2}(x,t) - h(x,t)$$

$$F(x, t, \alpha) = a_1 \sum_{k=0}^{\alpha-1} \left( \frac{n-1}{2} \right) \left( \partial_t^k \Psi_1^x(t)(u(x,t)) \right) \partial_t^{\alpha-1-k} u^{n-2}(x,t) + a_2 \sum_{k=0}^{\alpha-1} \left( \frac{n-1}{2} \right) \left( \partial_t^k \Psi_1^{-x}(u(x,t)) \right) \partial_t^{\alpha-1-k} u^{n-2}(x,t) - h(x,t)$$

(73)

Equation (73) shows that for $\alpha > 1$ and $n = 2$, $(\Psi_1^{\pm, t}(u))_{k \in \mathbb{Z}}, (\Psi_1^{-x}(u))_{k \in \mathbb{Z}}$ and $(\Psi_1^{-x}(u))_{k \in \mathbb{Z}}$ can be solutions of $F(x, t, \alpha)$. If we define, $\mathcal{H} = \{ g \in S^{-}(\mathbb{R}) \mid g(x, t) = \partial_t u^n(x, t), \, u \in \mathcal{S}_0^1(\mathbb{R}), \, n \in \mathbb{Z}^+ - \{0\}, \, (x, t) \in \mathbb{R}^2 \}$, then $\mathcal{H} \subseteq \mathcal{H}$. However, for $n > 2$, the general solutions are nonlinear functions of $(\Psi_1^{\pm, t}(u))_{k \in \mathbb{Z}}, (\Psi_1^{-x}(u))_{k \in \mathbb{Z}}$ and $(\Psi_1^{-x}(u))_{k \in \mathbb{Z}}$ such as $(\partial_t^k \Psi_1^x(u(x,t))) \partial_t^{\alpha-1-k} u^{n-2}(x,t)$ and $(\partial_t^k \Psi_1^{-x}(u(x,t))) \partial_t^{\alpha-1-k} u^{n-2}(x,t)$.

5.2 Beyond the families of Energy operators

Based on the generalized formula of the energy operators (e.g., Equation (18)), one can wonder what is the set of solutions $\mathcal{G}_2 = \{ g_1 \text{ and } g_2 \in S^{-}(\mathbb{R}) \mid g_1(x, t) = (\Psi_1^{+, t}(u(x,t)))^n, g_2(x, t) = (\Psi_1^{-, t}(u(x,t)))^n, u \in \mathcal{S}_0^1(\mathbb{R}), \, n \in \mathbb{Z}^+ - \{0\}, \, (x, t) \in \mathbb{R}^2 \}$, also written as (with the notation of the generalized energy operators) $\mathcal{G}_2 = \{ g_1 \text{ and } g_2 \in S^{-}(\mathbb{R}) \mid g_1(x, t) = ((u(x,t))_{1}^{0} + ((u(x,t))_{1}^{0}))^n, g_2(x, t) = ((u(x,t))_{1}^{0} + ((u(x,t))_{1}^{0}))^n, u \in \mathcal{S}_0^1(\mathbb{R}), \, n \in \mathbb{Z}^+ - \{0\}, \, (x, t) \in \mathbb{R}^2 \}$. With the notation of the energy space $\mathcal{H}$ (e.g. equation (27)), $\mathcal{G}_2 \subseteq \mathcal{H}$ with $p = 0$. In addition, we define in the preliminaries $\mathcal{S}_1^1(\mathbb{R})$ for the family of DEO with the derivatives over time or space. In this particular case, $\mathcal{S}_1^1(\mathbb{R})$ is the union of the subspaces (of $S^{-}(\mathbb{R})$) for each family of DEO with the derivatives over time or space.

Let us consider the linear PDE $F_2(x, t, \alpha)$:

$$F_2(x, t, \alpha) = a_1 \partial_t^\alpha g_1(x,t) + a_2 \partial_t^\alpha g_2(x,t) - h(x,t)$$

(74)
Using Theorem 1 and the development in the previous section, one can write with the generalized energy operator families:

\[
F_2(x, t, \alpha) = a_1 \sum_{k=0}^{a_1-1} \frac{(-1)^k}{2} (\partial_t^k [u]^{1+x}) + \frac{[u]^{1-x}}{2} \partial_t^{a-1-k} ([u]^{0+x})^{n-2} \\
+ a_2 \sum_{k=0}^{a_2-1} \frac{(-1)^k}{2} (\partial_t^k [u]^{1+x}) + \frac{[u]^{1-x}}{2} \partial_t^{a-1-k} ([u]^{0+x})^{n-2} - h(x, t)
\]

\[
F_2(x, t, \alpha) = a_1 \sum_{k=0}^{a_1-1} \frac{(-1)^k}{2} (\partial_t^k [u]^{1+x}) + \frac{[u]^{1-x}}{2} \partial_t^{a-1-k} ([u]^{0+x})^{n-2} \\
+ a_2 \sum_{k=0}^{a_2-1} \frac{(-1)^k}{2} (\partial_t^k [u]^{1+x}) + \frac{[u]^{1-x}}{2} \partial_t^{a-1-k} ([u]^{0+x})^{n-2} - h(x, t)
\]

(75)

There is a symmetry with the previous section. Equation (75) shows that for \( n = 2 \), \(((u)^{1+x})_{k \in Z}, ([u]^{1-x})_{k \in Z}\) and \(((u)^{1+x})_{k \in Z}\) can be solutions of \( F_2(x, t, \alpha) \). Thus, one can define a set of solutions (see \( G_2 \) in the previous section) such as it is included in \( H^1 \) with \( p = 0 \). However, for \( n > 2 \), the general solutions are nonlinear functions of \(((u)^{0+x})_{k \in Z}, ([u]^{0-x})_{k \in Z}\) such as \((\partial_t^k [u]^{1+x}) \partial_t^{a-1-k} ([u]^{0+x})^{n-2} \) and \((\partial_t^k [u]^{1+x}) \partial_t^{a-1-k} ([u]^{0+x})^{n-2} \).

Discussion: It is easy to generalize the analysis of the PDE \( F_2(x, t) \) to the set of solutions \((l \in \mathbb{Z}^+, l > 1) G_l = \{g_1 \ and \ g_2 \in S^-(\mathbb{R}) \} \) \( g_1(x, t) = \langle [u(x,t)]^{1-2} \rangle, g_2(x, t) = ([u(x,t)]^{1-2} \rangle, u \in s_{-1}^- (\mathbb{R}), \ n \in \mathbb{Z}^+ - \{0\}, \ (x, t) \in \mathbb{R}^2 \) Following the development in the previous case, \( G_l \subseteq H^0 \) with \( p = l - 2 \). For the same reasons as before, \( s_{-1}^- \) is the union of the subspaces (of \( S^-(\mathbb{R}) \)) for each family of DEO with the derivatives over time and space. Finally, the solutions can be written as a sum of \((\partial_t^k [u]^{1+x}) \partial_t^{a-1-k} ([u]^{0+x})^{n-2} \) and \((\partial_t^k [u]^{1+x}) \partial_t^{a-1-k} ([u]^{0+x})^{n-2} \) for \( k \) in \( \mathbb{Z} \) and \( n \geq 2 \).

6 Applications

In the previous sections, we showed that it is possible to define the solutions of the linear PDE in Equation (71) with the families of energy operator and to some extent the families of generalized energy operators.

6.1 The homogeneous Helmholtz equation

From [14] and [15], the homogeneous equation can be deduced from Equation (71) with \( \alpha \) equal 2, \( h(x, t) = 0, a_1 = 1, a_2 = \frac{1}{c}, \) and \( F_1(x, t, \alpha) \) equal to 0.
such as:
\[ \partial_x^2 g(x,t) - \frac{1}{c^2} \partial_t^2 g(x,t) = 0 \] (76)
c is the speed of light. It is well-known that the general solution \( g(x,t) \) of this equation is a sum of two waves travelling in opposite direction such as \( g(x,t) = u_1(t-x/c) + u_1(t+x/c) \) (e.g., [13]). Applying the same development as in the previous section and looking for the solutions \( u(x,t)^n \) (\( n \) in \( \mathbb{Z}^+ \), \( n > 1 \)) in \( G \), one can write the PDE with the energy operator:
\[ \partial_x^2 u(x,t)^n - \frac{1}{c^2} \partial_t^2 u(x,t)^n = 0 \] (77)
if \( n = 2 \),
\[ \partial_x^2 u(x,t)^2 - \frac{1}{c^2} \partial_t^2 u(x,t)^2 = 0 \]
\[ -\frac{1}{c^2} (\partial_t \Psi_{1+}^{+,t}(u(x,t))) + (\partial_x \Psi_{1+}^{+,x}(u(x,t))) = 0 \] (78)
With the derivation chain rules property of the DEOs (e.g, [14]), we have the equality \( \frac{\partial (\Psi_{1+}^{+,x}(g))}{\partial x} = \Psi_{k+}^{+,x}(g) + \Psi_{k-}^{+,x}(\partial_x g) \). Then, the previous equation becomes:
\[ \Psi_{2+}^{+,x}(g) - \frac{1}{c^2} \Psi_{2+}^{+,t}(g) = 0 \] (79)
Note that we are using the families of energy operators and generalized energy operators whereas in [13] it was only used the operators \( \Psi_{1+}^{+,t} \) and \( \Psi_{1+}^{+,x} \) (according to the notation used in this work) when rewriting the wave equation. This last equation shows how the results agreed between this presented work and [13] for \( n \) equal 2.
Furthermore, one can use the generalized energy operator as shown in Section 5.2 when looking at the solutions in \( G_2 \):
\[ \partial_x^2 (\Psi_{1+}^{+,x}(u))^n - \frac{1}{c^2} \partial_t^2 (\Psi_{1+}^{+,t}(u))^n = 0 \]
\[ \partial_x^2 (\{u(x,t)^{0,+,l}\}_{1+}^n) - \frac{1}{c^2} \partial_t^2 (\{u(x,t)^{0,+,l}\}_{1+}^n) = 0 \] (80)
It is also possible to generalize to the solutions in \( G_l \) (\( l > 1 \)).
6.2 Numerical Example

6.2.1 A sinusoidal wave in one dimension

Let us take a numerical example. From \[15\], a solution of Equation (76) is
\[g(x,t) = A \cos(\omega t - Kx)\] with \(\omega = Kc\) and \(c\) the speed of light. Now, one can solve the Equation (78) in the particular case \(n = 2\) with \((\Psi^{+,t}_k(g))_{k \in \{1, 2\}}\), \((\Psi^{-,t}_k(g))_{k \in \{1\}}\) in \(G\) or:

\[
\begin{align*}
\Psi^{+,t}_1(g(x,t)) &= -2\omega A^2 \sin(\omega t - Kx) \cos(\omega t - Kx) \\
\Psi^{-,t}_1(g(x,t)) &= 0 \\
\Psi^{+,t}_2(g(x,t)) &= \omega^2 A^2 (\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx))
\end{align*}
\]

It is interesting to underline that the energy is equal to:

\[
\begin{align*}
\mathcal{E}(g(x,t)) &= \int_0^{\pi/2} |u(x,t)|^2 \, dt \\
&= \frac{A^2}{4} \\
\mathcal{E}(\Psi^{+,t}_1(g(x,t))) &= \frac{A^4 \omega^2}{4} \\
\mathcal{E}(\Psi^{+,t}_2(g(x,t))) &= \frac{A^4 \omega^4}{4}
\end{align*}
\]

(81)

Now, if we generalize it to \(\Psi^{+,t}_k(g(x,t))\):

\[
\begin{align*}
\Psi^{+,t}_k(g(x,t)) &= (-1)^{p+1} A^2 \omega^{2p+1} \sin(\omega t - Kx) \cos(\omega t - Kx), \ k = 2p + 1 \\
\Psi^{+,t}_k(g(x,t)) &= (-1)^{p+1} A^2 \omega^{2p} (\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx)), \ k = 2p
\end{align*}
\]

(83)

and the associated energy is equal to:

\[
\mathcal{E}(\Psi^{+,t}_k(g(x,t))) = \frac{A^4 \omega^{2k}}{4}
\]

(84)

Thus, if \(\omega > 1\), for \(k >> 1\) then \(\mathcal{E}(\Psi^{+,t}_k(g(x,t))) \gg 1\). In this case, the energy \(\mathcal{E}(\Psi^{+,t}_k(g(x,t)))\) is diverging. However, it is not diverging in the case \(\omega < 1\).

6.2.2 Application to the Electromagnetic waves and Poynting vector

From the previous example, one can apply this development for planar electromagnetic waves, simple solutions of the wave equation. From \[1\], we have
two electromagnetic waves solution of the electric and magnetic fields such as \( \vec{E}(x,t) = E_0 \cos(\omega t - Kx) \hat{j} \) and \( \vec{B}(x,t) = B_0 \cos(\omega t - Kx) \hat{k} \) respectively. Note that \((\hat{i}, \hat{j}, \hat{k})\) are the unitary vectors in the Cartesian coordinates referential \((x, y, z)\). The Poynting vector \( \vec{S} \) and the intensity \( <S> \) can be then calculated as (e.g., [1]):

\[
\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu}
\]

\[
\vec{S} = \frac{E_0 B_0 \cos^2(\omega t - Kx) \hat{i}}{\mu_0}
\]

\[
<S> = \frac{E_0 B_0}{\mu_0} \int_{-\pi/2}^{\pi/2} \cos^2(\omega t - Kx) \, dt
\]

\[
<S> = \frac{E_0 B_0}{\mu_0} <\cos^2(\omega t - Kx)>
\]

\[
<S> = \frac{E_0 B_0}{2\mu_0}
\]

If we now define the electromagnetic waves with \( \Psi^+_k \) in \( \mathbb{Z}^+ - \{0\} \) based on Equation [17], then:

\[
\Psi^+_k(E(x,t)) = (-1)^{p+1} E_0^2 \omega^{2p+1} 2 \sin(\omega t - Kx) \cos(\omega t - Kx) \hat{j}, \ k = 2p + 1
\]

\[
\Psi^+_k(B(x,t)) = (-1)^{p+1} E_0^2 \omega^{2p} (\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx)) \hat{j}, \ k = 2p
\]

and,

\[
\vec{S} = \frac{\Psi^+_k(E(x,t)) \times \Psi^+_k(B(x,t))}{\mu_0}
\]

\[
\vec{S} = \frac{(-1)^{2(p+1)} \omega E_0^2 B_0^2 (\sin(\omega t - Kx) \cos(\omega t - Kx))^2 \hat{i}}{\mu_0}, \ k = 2p + 1
\]

\[
\vec{S} = \frac{(-1)^{2(p+1)} \omega^{4p} E_0^2 B_0^2 (\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx))^2 \hat{i}}{\mu_0}, \ k = 2p
\]

(87)

\[31\]
and the Intensity,

\[
<S> = \frac{4E_0^2 \omega^{2(2p+1)} B_0^2}{\mu_0} <(\sin(\omega t - Kx) \cos(\omega t - Kx))^2>, \quad k = 2p + 1
\]

\[
<S> = \frac{4E_0^2 \omega^{2(2p+1)} B_0^2}{2\mu_0}, \quad k = 2p + 1
\]

\[
<S> = \frac{\omega^4 p E_0^2 B_0^2}{\mu_0} <(\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx))^2>, \quad k = 2p
\]

\[
<S> = \frac{\omega^4 p E_0^2 B_0^2}{2\mu_0}, \quad k = 2p
\]

\[(88)\]

Furthermore, it is possible to define the electromagnetic wave with the generalized energy operators. Let us simply give some examples for the derivative in time:

\[
[[\vec{E}]]_{1}^{+t} = 2\Psi_1^{+t}(\vec{E}(x,t)) \times \partial_t \Psi_1^{+t}(\vec{E}(x,t))^i
\]

\[
<S> = \frac{E_0^2 B_0^2 \omega^6}{2\mu_0}
\]

\[(89)\]

or

\[
[[\vec{E}]]_{1}^{+t} = 2(2\Psi_1^{+t}(\vec{E}(x,t)) \times \partial_t \Psi_1^{+t}(\vec{E}(x,t)))
\]

\[
\times ((\partial_t \Psi_1^{+t}(\vec{E}(x,t)))^2 + \Psi_1^{+t}(\vec{E}(x,t)) \times \partial_t \Psi_1^{+t}(\vec{E}(x,t)))^i
\]

\[(90)\]

\[
[[\vec{E}]]_{1}^{+t} = -32(E_0^8 \omega^7)(\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx))\cos(\omega t - Kx)\sin(\omega t - Kx)
\]

\[
((\sin^2(\omega t - Kx) - \cos^2(\omega t - Kx))^2 - 4(\cos(\omega t - Kx)\sin(\omega t - Kx))^2)^i
\]

\[
<S> = \frac{8E_0^8 \omega^7 (B_0^8 \omega^7)}{\mu_0}
\]

\[(91)\]

To conclude this section, the intensity \(<S>\) is directly linked to the radiation pressure force \(P = <S> \times s\) (s the surface of the reflective incident waves e.g., \([15]\) or \([1]\)).

7 Conclusions

This work generalizes the Lemma 0 and Theorem 0 shown in \([13]\) using the families of generalized energy operators \(([[\cdot]]_{k}^{+})_{k \in \mathbb{Z}}\) and \(([[\cdot]]_{k}^{-})_{k \in \mathbb{Z}}\) (p
Lemma 1 shows that the successive derivatives of \((([[f]^{p-1}]_1^n, n \in \mathbb{Z}^+, n > 1)\) can be decomposed with the generalized energy operators \(\left[[[.]^n]^+\right]_k \in \mathbb{Z}\) when \(f\) is in the subspace \(S_p^- (\mathbb{R})\). With Theorem 1 and \(f\) in \(s_p^- (\mathbb{R})\), one can decompose uniquely the successive derivatives of \(\left[[[f]^{p-1}]_1^n, n \in \mathbb{Z}^+, n > 1\right)\) with the generalized energy operators \(\left[[[.]^n]^+\right]_k \in \mathbb{Z}\) and \(\left[[[.]^n]^+\right]_k \in \mathbb{Z}\). It is important to understand that \(S_p^- (\mathbb{R})\) and \(s_p^- (\mathbb{R})\) \((p \in \mathbb{Z}^+)\) are subspaces of the Schwartz space \(S^- (\mathbb{R})\), and how their definitions involve the so-called energy spaces (e.g., Definition 3 in Section 3). The proofs of Lemma 1 and Theorem 1 follow a similar structure: an induction on both \(p\) and \(n\). Note that the special case \(n < -1\) and \(n = \pm 1\) are discussed at the end of Section 4. It is worth emphasizing that in the case \(p = 0\), Lemma 1 and Theorem 1 are the same statements as Lemma 0 and Theorem 0. This demonstrates that this work generalizes the previous work of [14].

The second part of this work focuses on \(u\) as a function of two variables in \(s_p^- (\mathbb{R})\) and solution of a linear PDEs \((F(x,t), F_2(x,t))\). One shows what are the conditions for \(\left[[[u]^+_k,t]_{k \in \mathbb{Z}}, \left[[[u]^+_k,-t]_{k \in \mathbb{Z}}, \left[[[u]^+_k,+x]_{k \in \mathbb{Z}}, \left[[[u]^+_k,-x]_{k \in \mathbb{Z}}\right)\right)\right)\) to be also solutions of \(F(x,t)\) or \(F_2(x,t)\). One can define the sets of solutions \(G_1, G_2\) and \(G_l\) \((l \in \mathbb{Z}^+ \text{ and } l > 1)\) when applying the results of Theorem 1. Note that \(G_1, G_2\) and \(G_l\) \((l \in \mathbb{Z}^+ \text{ and } l > 1)\) are included in the subspace of the energy spaces \((E \text{ and } H \text{ for } p \in \mathbb{Z}^+)\). As a practical example, the theory is applied to the Helmholtz equation.

Finally, this work ends with some numerical examples when applying the generalized energy operators to planar electromagnetic waves \((\vec{E}(x,t), \vec{B}(x,t))\). In particular, when defining the Poynting vector \((\vec{S})\) an the intensity \(<S>\) with generalized energy operators, it allows to define a linear relationship with the radiation pressure force \((P)\). This opens possible applications of the theory of generalized energy operators in astronautic and astrophysics for the special case \(\omega > 1\) in demultiplying the radiation pressure force in astronautical and astrophysical applications (e.g., [9], [16]).

**Acknowledgment**

A special thanks is addressed to Professor Alan McIntosh at the Centre for Mathematics and its Applications at the Australian National University (ANU) for its inputs and discussions when writing this manuscript. The author also acknowledges the comments from Dr. Igor Ivanov from the Atomic and Molecular Physics laboratory at the ANU, and Dr. Malcolm S. Woolfson from the School of Electrical Engineering at the University of Nottingham (UK).
References

[1] E. Amzallag, J. Cipriani, N. Piccioli, Ondes, 1st Ed., Ediscience international, Paris, 1997.

[2] A. O. Boudraa, J. C. Cexus, S. Benramdane, T. Chonavel, Some useful properties of cross-PsiB-energy operator, International Journal of Electronics and Communications, vol. 63, pp. 728-735, 2009.

[3] A. C. Bovik, J. P. Havlicek, and M. D. Desai, Theorems for Discrete Filtered Modulated Signals, in Proc. IEEE Int. Conference on Acoustics, Speech, and Signal Processing, 1993 (ICASSP-93), Vol.3, pp. 153-156.

[4] M. Felsberg and E. Jonsson, Energy Tensors: Quadratic, Phase Invariant Image Operators, Pattern Recognition, Lectures Note in Computer Science, vol. 3663, pp. 493-500, 2005.

[5] R. Hamila, M. Renfors, G. Gunnarsson, M. Alanen, Data processing for mobile phone positioning, in Proc. IEEE Int. Vehicular Technology Conference, 1999. VTC 1999 - Fall. IEEE VTS 50th, vol.1, pp. 446-449.

[6] R. Hamila, J. Astola, F. Alaya Cheikh, M. Gabbour, and M. Renfors, Teager Energy and the Ambiguity Function, IEEE Transactions on Signal Processing, 1999, Vol. 47, No. 1, pp. 260-262.

[7] J. F. Kaiser, On a simple algorithm to calculate the ’energy’ of a signal, in Proc. IEEE Int. Conference on Acoustics, Speech, and Signal Processing (ICASSP-90), vol. 1, pp. 381-384.

[8] E. Kreizig, Advanced Engineering Mathematics, 8th Edition, John Wiley & Sons, 2003.

[9] L. Labun and J. Rafelski, Nonlinear Electromagnetic Forces in Astrophysics, Acta Physica Polonica B, vol. 43, no 12, pp. 2237-2250, 2012.

[10] W. Lin, C. Hamilton, P. Chitrapu, A generalization to the Teager-Kaiser energy function and application to resolving two closely-spaced tones, in Proc. IEEE Int. Conference on Acoustics, Speech, and Signal Processing, 1995 (ICASSP-95), Vol.3, pp. 1637-1640.

[11] P. Maragos and A. Potamianos, Higher Order Differential Energy Operators, IEEE Signal Processing Letters, vol. 2, No 8, 1995, pp 152-154.

[12] N. Jacobson, Basic Algebra I (2nd ed.), 2009, ISBN 978-0-486-47189-1.

[13] J.P. Montillet, On a novel approach to decompose finite energy functions by energy operators and its application to the general wave equation, International Mathematical forum, 2010, issue 5, no 48, pp. 2387-2400.
[14] J.P. Montillet, *The Generalization of the Decomposition of Functions by Energy Operators*, Acta Applicandae Mathematicae, doi: 10.1007/s10440-013-9829-0 (or also available in: http://arxiv.org/abs/1208.3385).

[15] H. J. Pain, *The Physics of Vibrations and Waves*, 6th ed. John Wiley & Sons, Ltd, Chichester England, 2005.

[16] R. Shawyer, *The EM Drive a New Satellite Propulsion Technology*, 2nd Conference on Disruptive Technology in Space Activities, 2010. [available at: http://emdrive.com/]

[17] B.J. West, M. Bologna, P. Grigolini, *Physics of Fractal Operators*, 1st Edition, Springer-Verlag New-York, 2003.