The \( q \)-boson–fermion realizations of the quantum superalgebra \( U_q(\mathfrak{gl}(2/1)) \)

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Abstract

We show that our construction of realizations for Lie algebras and quantum algebras can be generalized to quantum superalgebras, too. We study an example of quantum superalgebra \( U_q(\mathfrak{gl}(2/1)) \) and give the boson–fermion realization with respect to one pair of \( q \)-boson operators and 2 pairs of fermions.

1 Introduction

Boson–fermion realizations of a given set of operators via Bose–Fermion creation and annihilation operators are among the main tools of solving various quantum problems. The origin is linked with the Schwinger \([1]\), Dyson \([2]\) and Holstein–Primakoff \([3]\) realizations which are different boson realizations of the algebra \( \mathfrak{sl}(2) \).

Generalizations of the Dyson realization to the Lie algebra \( \mathfrak{sl}(n) \) were derived in \([4]\). In our paper \([5]\) we formulated the method starting from the Verma modules for obtaining boson realizations and in \([6]\) we obtained explicitly a braid class of realizations which generalized the results from \([7, 8]\).

Later the idea was extended to the Lie superalgebra, and the Dyson type boson–fermion realizations were explicitly given in \([9]\), generalizing the results to \( \mathfrak{sl}(2/1) \) \([10, 11]\).

Today these boson–fermion realizations become a standard technique in quantum many–body physics and we can also find several other applications in all fields of quantum physics.

Quantum groups and quantum supergroups or \( q \)-deformed Lie algebras and superalgebras imply some specific deformations of the classical Lie algebras and superalgebras. From a mathematical point of view, those are noncommutative associative Hopf algebras and superalgebras. The structure and representation theory of quantum groups were extensively developed by Jimbo \([12]\) and Drinfeld \([13]\). The first ”quantum” version of Holstein–Primakoff was worked out for \( U_q(\mathfrak{sl}(2)) \) \([14]\) and then for \( U_q((\mathfrak{sl}(3)) \) \([15]\). The Schwinger type realization was written in \([16]\) and \([17]\). These realizations found immediate applications \([18–23]\).

In our papers \([24, 25, 26]\) we studied the Dyson realizations of the series algebras \( U_q(\mathfrak{sl}(2)) \), \( U_q(\mathfrak{gl}(n)) \), \( U_q(B_n) \), \( U_q(C_n) \) and \( U_q(D_n) \). There is some special case \([25]\) for
which the realization of the subalgebra $U_q(\mathfrak{gl}(n-1))$ in the recurrence is trivial. Such special realizations of the quantum algebra $U_q(\mathfrak{sl}(n))$ of Dyson type were studied in [27].

The aim of the present paper is to show that there is a possibility of generalizing our method [5] for deriving the boson–fermion realization, too. This will be exemplified by the quantum superalgebra $U_q(\mathfrak{gl}(2/1))$. This superalgebra can be applied to physical problems such as strongly correlated electron systems [28, 29, 30]. We explicitly see the recurrence with respect to $U_q(\mathfrak{gl}(1/1))$ and consequently we will show that again it is a generalization of the result from [31].

Some preliminary results concerning the general case $U_q(\mathfrak{gl}(m/n))$ have already been obtained and prepared for publication.

## 2 Preliminaries

In this article, we will use the definition of a quantum superalgebra $U_q(\mathfrak{gl}(2/1))$ which can be found in [31].

Let $q$ be an independent variable, $\mathcal{A} = C[q, q^{-1}]$ and $C(q)$ be a division field of $\mathcal{A}$. The superalgebra $U_q(\mathfrak{gl}(2/1))$ is the associative superalgebra over $C(q)$ generated by even generators $K_i, K_i^{-1}, i = 1, 2, 3, E_{12}, E_{21}$ and odd generators $E_{32}, E_{32}$ which satisfy the following relations:

\[
\begin{align*}
K_i^{\pm 1} K_j^{\pm 1} &= K_j^{\pm 1} K_i^{\pm 1}, \quad K_i K_i^{-1} = 1 \\
K_i E_{jk} &= q^{\delta_{ij} - \delta_{ik}} E_{jk} K_i \\
[E_{12}, E_{32}] &= [E_{21}, E_{23}] = 0 \\
[E_{12}, E_{21}] &= \frac{K_1 K_2^{-1} - K_1^{-1} K_2}{q - q^{-1}} \\
\{E_{23}, E_{32}\} &= \frac{K_2 K_3 - K_2^{-1} K_3^{-1}}{q - q^{-1}} \\
E_{23}^2 &= E_{32}^2 = 0 \\
E_{12} E_{13} - q E_{13} E_{12} &= 0 \\
E_{21} E_{31} - q E_{31} E_{21} &= 0
\end{align*}
\]

where

\[
\begin{align*}
E_{13} &= E_{12} E_{23} - q^{-1} E_{23} E_{12} \\
E_{31} &= -E_{21} E_{32} + q^{-1} E_{32} E_{21}
\end{align*}
\]

The Hopf structure of this superalgebra is defined by the following operations:

1. **Coproduct** $\Delta$

\[
\begin{align*}
\Delta(1) &= 1 \otimes 1 \\
\Delta(K_i) &= K_i \otimes K_i \\
\Delta(E_{12}) &= E_{12} \otimes K_1 K_2^{-1} + 1 \otimes E_{12} \\
\Delta(E_{21}) &= E_{21} \otimes 1 + K_1^{-1} K_2 \otimes E_{21} \\
\Delta(E_{23}) &= E_{23} \otimes 1 + K_2^{-1} K_3^{-1} \otimes E_{32} \\
\Delta(E_{32}) &= E_{32} \otimes 1 + K_2^{-1} K_3^{-1} \otimes E_{32}
\end{align*}
\]

2. **Counit** $\varepsilon$

\[
\begin{align*}
\varepsilon(1) &= \varepsilon(K_i) = 1 \\
\varepsilon(E_{12}) &= \varepsilon(E_{23}) = \varepsilon(E_{21}) = \varepsilon(E_{32}) = 0
\end{align*}
\]
3. Antipode $S$

\[
S(1) = 1 \quad S(K_i) = K_i^{-1} \\
S(E_{12}) = -E_{12}K_1^{-1}K_2 \quad S(E_{23}) = -E_{12}K_2^{-1}K_3^{-1} \\
S(E_{21}) = -K_1K_2^{-1}E_{21} \quad S(E_{32}) = -K_2K_3E_{32}
\]

We do not use these operations for construction of the realization.

The method of construction used is the same as in the case of the Lie algebras [5] or quantum algebra [26] and is based on using the induced representation. The difference from quantum algebra is that together with $q$–deformed boson operators [16], [17] we also use fermion operators.

The algebra $\mathcal{H}$ of the $q$–deformed boson operators is the associative algebra over the field $\mathbb{C}(q)$ generated by the elements of $a^+, a^- = a, q^x$ and $q^{-x}$, satisfying the commutation relations

\[
q^x q^{-x} = q^{-x} q^x = 1, \quad q^x a^+ q^{-x} = qa^+, \quad q^x a q^{-x} = q^{-1}a,
\]

\[
aa^+ - q^{-1}a^+ a = q^x, \quad aa^+ - qa^+ a = q^{-x},
\]

(2)

The algebra $\mathcal{H}$ has faithful representation on vector space with basic elements $\{|n\rangle$, where $n = 0, 1, \ldots \}$ of the form

\[
q^x |n\rangle = q^n |n\rangle, \quad a^+ |n\rangle = |n+1\rangle, \quad a |n\rangle = |n| |n-1\rangle,
\]

(3)

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

Because of odd generators $E_{23}$ and $E_{32}$ we construct realization by means of the algebra $\mathcal{H}$ for even elements, and by fermion elements $b^+$ and $b$ for odd ones. These fermion elements commute with the elements of $\mathcal{H}$ and together fulfill the relations

\[
bb = b^+ b^+ = 0, \quad bb^+ + b^+ b = 1.
\]

(4)

As in the case of the Lie algebras or quantum groups, our realizations contain elements of quantum sub–superalgebra of $U_q(\mathfrak{gl}(2/1))$, namely, quantum superalgebra $U_q(\mathfrak{gl}(1/1))$. The element $x$ of this subalgebra commutes with the elements from $\mathcal{H}$, and for the fermion elements $b^\pm$ the relation

\[
xb^\pm = (-1)^{\deg x} b^\pm x,
\]

(5)

holds.

Realization of the quantum superalgebra $U_q(\mathfrak{gl}(2/1))$ is called the homomorphism $\rho$ of the $U_q(\mathfrak{gl}(2/1))$ to associative superalgebra $\mathcal{W}$ generated by $\mathcal{H}, b^\pm$ and $U_q(\mathfrak{gl}(1/1))$.

### 3 Construction of the realization of $U_q(\mathfrak{gl}(2/1))$

First, for construction of the realization we find the induced representation of $U_q(\mathfrak{gl}(2/1))$. As subalgebra $A_0$ of $U_q(\mathfrak{gl}(2/1))$ we choose a quantum superalgebra generated by $E_{23}, E_{21}, E_{32}, K_i$ and $K_i^{-1}, i = 1, 2, 3$. Let $\varphi$ be a representation of $A_0$ on vector space $V$. Let
\begin{align*}
\lambda(x)(y \otimes v) &= xy \otimes v. \\
\text{Let } \mathcal{I} \text{ be subspace of } U_q(\mathfrak{gl}(2/1)) \otimes V \text{ generated by the relations }
xy \otimes v &= x \otimes \varphi(y)v,
\end{align*}

for all \( x \in U_q(\mathfrak{gl}(2/1)), y \in \mathcal{A}_0 \) and \( v \in V \). It is easy to see that the subspace \( \mathcal{I} \) is \( \lambda \)-invariant. Therefore, (6) gives the representation on the factor–space \( W = [U_q(\mathfrak{gl}(2/1)) \otimes V]/\mathcal{I} \).

Let \( E_{12}^n E_{13}^M = |N, M\rangle \). Due to the Poincaré–Birkhoff-Witt theorem the space \( W \) of the induced representation is generated by the elements \( |N, M\rangle \otimes v \) where \( N = 0, 1, 2, \ldots, M = 0, 1 \) and \( v \in V \).

To obtain the explicit form of the induced representation, we give some relations. They can be proved by mathematical induction from relations (1).

**Lemma 1.** For any \( n = 0, 1, 2, \ldots \) the following formulae hold:

\begin{align*}
E_{12}^n E_{13}^n &= q^{-n} E_{12}^n E_{13} \\
E_{23}^n E_{12}^n &= q^n E_{12}^n E_{23} - q[n] E_{12}^{n-1} E_{13} \\
E_{23}^n &= (-q)^n E_{13}^n E_{23} \\
E_{32}^n E_{13}^n &= (-1)^n E_{13}^n E_{32} + \frac{1 - (-1)^n}{2} q^{-n} E_{12}^n E_{13}^{-1} K_2 K_3 \\
E_{21}^n E_{12}^n &= E_{12}^n E_{21} - \frac{|n|}{q - q^{-1}} E_{12}^{n-1} (q^{n-1} K_1 K_2^{-1} - q^{-n+1} K_1^{-1} K_2) \\
E_{21} E_{13}^n &= E_{13}^n E_{21} + \frac{1 - (-1)^n}{2} E_{13}^{n-1} E_{23} K_1^{-1} K_2 \\
E_{31}^n E_{12} &= E_{12}^n E_{31} + q^{-n} E_{12}^{n-1} K_1 K_2^{-1} E_{32} \\
E_{31} E_{13}^n &= (-1)^n E_{13}^n E_{31} + \frac{1 - (-1)^n}{2} q^{-1} E_{13}^{n-1} \frac{K_1 K_3 - K_1^{-1} K_3^{-1}}{q - q^{-1}} \\
E_{32} E_{23}^n &= (-1)^n E_{23}^n E_{32} + \frac{1 - (-1)^n}{2} E_{23}^{n-1} K_2 K_3^{-1} - \frac{K_2^{-1} K_3^{-1}}{q - q^{-1}}
\end{align*}

We omit the details of the calculations and write the result for the action of the induced representation on the basis elements \( |N, M\rangle \otimes v \).

**Theorem 1.** The formulae

\begin{align*}
E_{12} |N, M\rangle &\otimes v = |N + 1, M\rangle \otimes v \\
E_{13} |N, M\rangle &\otimes v = q^{-N} |N, M + 1\rangle \otimes v \\
E_{23} |N, M\rangle &\otimes v = -q[N] |N - 1, M + 1\rangle \otimes v + (-1)^M q^{N+M} |N, M\rangle \otimes \varphi(E_{23}) v \\
K_1 |N, M\rangle &\otimes v = q^{N+M} |N, M\rangle \otimes \varphi(K_1) v \\
K_2 |N, M\rangle &\otimes v = q^{-N} |N, M\rangle \otimes \varphi(K_2) v \\
K_3 |N, M\rangle &\otimes v = q^{-M} |N, M\rangle \otimes \varphi(K_3) v
\end{align*}
Theorem 2. The mapping

\[ E_{32}|N, M\rangle \otimes v = \frac{1}{2} \frac{(-1)^M}{q^{M}} q^{-M}|N + 1, M - 1\rangle \otimes \varphi(K_2K_3)v + \]
\[ +(-1)^M|N, M\rangle \otimes \varphi(E_{32})v \]
\[ E_{21}|N, M\rangle \otimes v = -\frac{[N]}{q} q^{N+M-1} |N - 1, M\rangle \otimes \varphi(K_1K_2^{-1})v + \]
\[ +\frac{[N]}{q} q^{-M-1} |N - 1, M\rangle \otimes \varphi(K_1^{-1}K_2)v + \]
\[ +\frac{1}{2} (-1)^M |N, M - 1\rangle \otimes \varphi(E_{23}K_1^{-1}K_2)v + |N, M\rangle \otimes \varphi(E_{21})v \]
\[ E_{31}|N, M\rangle \otimes v = \frac{1}{2} \frac{(-1)^M}{q^{N-1}} [N]|N, M - 1\rangle \otimes \varphi(K_1K_3)v + \]
\[ +(-1)^M q^{N+M-2}[N]|N - 1, M\rangle \otimes \varphi(K_1K_2^{-1}E_{32})v + \]
\[ +\frac{1}{2} (-1)^M \frac{q^{-1}}{q^{N-1}} |N, M - 1\rangle \otimes (\varphi(K_1K_3 - K_1^{-1}K_3^{-1})v + \]
\[ +(N, M)|N, M\rangle \otimes \varphi(E_{31})v \]
give the induced representation of the quantum superalgebra \( U_q(gl(2/1)) \).

We construct the realization of quantum superalgebra \( U_q(gl(2/1)) \) from the induced representation given in Theorem 1 as follows:

We chose the representation \( \varphi \), for which \( \varphi(E_{21})v = 0 \), \( \varphi(E_{31})v = 0 \), \( \varphi(K_1)v = q^\lambda v \) and substitute

\[ q^{\pm N} \rightarrow q^{\pm x} \]
\[ |N, M + 1\rangle \rightarrow b^+ \]
\[ |N + 1, M\rangle \rightarrow a^+ \]
\[ |N, M - 1\rangle \rightarrow b \]
\[ q^{\pm M} \rightarrow (bb^+ + q^{\pm 1}b^+b) \]
\[ \varphi(E_{21})v \rightarrow 0 \]
\[ \varphi(E_{31})v \rightarrow 0 \]
\[ \varphi(K_2^{\pm 1})v \rightarrow k_2^{\pm 1} \]
\[ \varphi(K_3^{\pm 1})v \rightarrow k_3^{\pm 1} \]
\[ (-1)^M \varphi(E_{23})v \rightarrow e_{23} \]
\[ (-1)^M \varphi(E_{32})v \rightarrow e_{32} \]

(the last two relations reflect the fact that \( e_{23} \) and \( e_{32} \) are fermions).

By this substitution we obtain the realization of the quantum superalgebra \( U_q(gl(2/1)) \).

**Theorem 2.** The mapping \( \rho : U_q(gl(2/1)) \rightarrow \mathcal{W} \) defined by the formulae

\[ \rho(E_{12}) = a^+ \]
\[ \rho(E_{13}) = q^{-x}b^+ \]
\[ \rho(E_{23}) = -qab^+ + q^x(bb^+ + q^{b^+})e_{23} \]
\[ \rho(K_1) = q^{\lambda_1 + x}(bb^+ + q^{b^+}) \]
\[ \rho(K_2) = q^{-x}k_2 \]
\[ \rho(K_3) = (bb^+ + q^{-1}b^+b)k_3 \]
\[ \rho(E_{32}) = q^{-1}a^+bk_2k_3 + e_{32} \]
\[ \rho(E_{21}) = \frac{-a}{q - q^{-1}} \left( q^{\lambda_1 + x - 1}(bb^+ + q^{b^+})k_2^{-1} - q^{-\lambda_1 - x + 1}(bb^+ + q^{-1}b^+b)k_2 \right) - q^{-\lambda_1}be_{23}k_2 \]
\[ \rho(E_{31}) = a^+abq^{\lambda_1 + x - 1}k_3 + aq^{\lambda_1 + x - 2}(bb^+ + q^{b^+})k_2^{-1}e_{32} + q^{-1}bq^{\lambda_1}k_3^{-1}q^{-\lambda_1}k_3^{-1}q^{-1} \]
is the realization of the quantum superalgebra $U_q(gl(2/1))$.

This theorem can be proved by a direct calculation.

4 Conclusion

In this paper we gave the method of construction of the $q$–boson–fermion realization of quantum superalgebras and applied it to the quantum superalgebra $U_q(gl(2/1))$. One of the advantages of this method, in comparison with [31], is that we automatically obtain a realization and we do not need to verify the generating relation. The reason is that the representation of $q$–bosons and fermions on the vector space $W$ with basis $|N, M\rangle$ is faithful.

The other advantage we see in the fact that our realization is expressed by means of polynomials of $q$–deformed bosons and fermions. On the other hand, we can easily obtain the Dyson realization of quantum superalgebra. For this purpose, it is sufficient to choose a realization of the generators of the algebra $H$ in the form

$$a^+ = A^+, \quad a = \frac{[N + 1]}{N + 1} A, \quad q^x = q^N,$$

where $[A, A^+] = 1$ and $N = A^+ A$. It is easy to verify that the realization of $U_q(gl(2/1))$ from Theorem 2 with realization (7) of the algebra $H$ and with a trivial realization of subalgebra $U_q(gl(1/1))$ leads, after homomorphism of $U_q(gl(2/1))$, to the realization given in [31]. In this case, the realization is of course expressed by means of a series in operators $A^+$ and $A$. Therefore, we prefer our form of realizations.

Finally, our realizations contain, in contrast with those in [31], quantum sub-superalgebras. Various forms of realizations of this sub-superalgebra give various realizations of the quantum superalgebra. In the studied case, this sub-superalgebra is $U_q(gl(1/1))$, and, therefore, is very simple. We can choose a realization of this superalgebra as

$$\rho(e_{23}) = \rho(e_{32}) = 0, \quad \rho(k_2) = \rho(k_3^{-1}) = q^{\lambda_2} \quad \text{and} \quad \rho(k_2^{-1}) = \rho(k_3) = q^{-\lambda_2}.$$ 

In this case, we obtain a realization with one $q$–deformed boson pair, one fermion pair and two parameters.

However, by means of our method we construct other realization of $U_q(gl(1/1))$, namely, realization of the form

$$\rho(e_{23}) = b_2^+, \quad \rho(k_2) = q^{\lambda_2}(b_2 b_2^++q b_2^+ b_2), \quad \rho(k_3) = q^{\lambda_3}(b_2 b_2^++q^{-1} b_2^+ b_2), \quad \rho(e_{32}) = \frac{q^{\lambda_2+\lambda_3} - q^{-\lambda_2-\lambda_3}}{q - q^{-1}} b_2 = [\lambda_2 + \lambda_3] b_2$$

where $b_2$ and $b_2^+$ are the fermion elements. If we use this realization of the quantum superalgebra in the realization of $U_q(gl(2/1))$ given in Theorem 2, we obtain realization with
one \(q\)-deformed boson pair, two fermion pairs and three parameters, which corresponds to the case of the Lie and quantum algebras.

As it is evident from [25, 26], this method of construction of realization is very successful for quantum groups. Therefore, we believe that it will be very useful for construction of realizations of quantum supergroups, too.

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References

[1] J. Schwinger, in Quantum Theory of Angular Momentum, Acad. Press, New York–London, 1965.

[2] F. J. Dyson, Phys. Rev. 102, (1956) 1217.

[3] T. Holstein and H. Primakoff, Phys. Rev. 58, (1949) 1098.

[4] S. Okubo, J. Math. Phys. 16, (1975) 528.

[5] Č. Burdík: J.Phys.A: Math.Gen. 18 (1985) 3101.

[6] Č. Burdík:, Czechoslovak J. Phys. B36 (1986), 1235.
— J.Phys.A: Math.Gen. 19 (1986) 2465.
— J.Phys.A: Math.Gen. 21 (1988) 289.

[7] M. Havlíček and W. Lassner,Rep. Mathematical Phys. 8 (1975), 391.
— Internal. J. Theoret. Phys. 15 (1976), 867.

[8] P. Exner and M. Havlíček Ann. Inst. H. Poincare Sect. A (N.S.) 23 (1975) 335.

[9] T. D. Palev, J. Phys. A: Math. Gen, 30 (1997) 8273, hep-th/9607222.

[10] A. Angelucci and R. Link, Phys. Rev. B46,(1992) 3809.

[11] N. I. Karchev, Teor, Mat. Fiz. 92, (1992) 988.

[12] M. Jimbo: Lett.Math.Phys. 10 (1985) 63; 11 (1986) 247.

[13] V. Drinfeld: Proc.Intern.Congress of Mathematicians, Berkeley, 1986, p.798.

[14] M. Chaichian, D. Ellinas and P. P. Kulish, Phys. Rev. Lett. 65,(1990) 980.

[15] J. da-Providencia, J. Phys, A: Math. Gen. 26, (1993) 5845.

[16] A.J. Macfarlane: J.Phys.A: Math.Gen. 22 (1989) 4581.

[17] L.C. Biedenharn: J.Phys.A: Math.Gen. 22 (1989) L873.
[18] C. Quesne, Phys. Lett. A153 (1991) 303.
[19] R. Chakrabarti and R. Jagannathan, J. Phys. A: Math. Gen. 24, (1991) L711.
[20] J. Katriel and A. I. Solomon, J. Phys. A: Math. Gen. 24, (1991) 2093.
[21] Z. R. Yu, J. Phys. A: Math. Gen. 24, (1991) L1321.
[22] A. Kundu and M. Basu Mallik, Phys. Lett. A156, (1991) 175.
[23] F. Pan. Own., Phys. Lett 8 (1991) 56.
[24] Č. Burdík and O. Navrátil: J.Phys.A: Math.Gen. 23 (1990) L1205.
[25] Č. Burdík, L. Černý, and O. Navrátil: J.Phys.A: Math.Gen. 26 (1993) L83.
[26] Č. Burdík and O. Navrátil: Czech.J.Phys. B47 (1998) 1301.
  — J.Phys.A: Math.Gen. 32 (1999) 6141.
  — Inter. Journ. of Mod. Phys. Lett. 14 (1999) 4491.
[27] T.D. Palev: J.Phys.A: Math.Gen. 31 (1998) 5145.
[28] A. J. Bracken, M. D. Gould and J. R. Links, Phys. Rev. Lett. 74, 2768 (1994); cond–mat/9410026.
[29] M. D. Gould, K. E. Hibberd and J. R. Links, Phys. Lett. A212, 156 (1996); cond–mat/9506119.
[30] A. Kümpner and K. Sakai, J. Phys. A34, 8015 (2001); cond–mat/0105416.
[31] T.D. Palev, Mod. Phys. Lett. A 14 (1999) 299.