Exact solutions for charged spheres and their stability.
II. Anisotropic Fluids.

K. Dev

Department of Physics and Astronomy, Dickinson College, Carlisle PA

E-mail: devk@dickinson.edu

Abstract. We study exact solutions of the Einstein-Maxwell equations for the interior gravitational field of static spherically symmetric charged compact spheres. The spheres consist of an anisotropic fluid with a charge distribution that gives rise to a static radial electric field. The density of the fluid has the form \( \rho(r) = \rho_0 + \alpha r^2 \) (here \( \rho_0 \) and \( \alpha \) are constants) and the total charge \( q(r) \) within a sphere of radius \( r \) has the form \( q = \beta r^3 \) (with \( \beta \) a constant). We evaluate the critical values of \( M/R \) for these spheres as a function of \( Q/R \) and compare these values with those given by the Andréasson formula.

Keywords: Einstein-Maxwell equations - exact solutions - black-hole physics

1. Introduction

In a recent paper [1], we studied exact solutions of the Einstein-Maxwell equations for the interior gravitational field of static spherically symmetric charged perfect fluid compact objects. Here, we extend our study to include charged spheres with anisotropic pressure. The term anisotropic pressure is used to describe physical situations in which the pressures associated with different directions differ from each other. Anisotropic spheres are compact distributions of matter in which the radial pressure is not equal to the tangential pressure, \( p_r \neq p_t \).

The pioneering study of the effects of anisotropic pressure on the properties of compact spheres was done by Bowers and Liang [2]. The Tolman-Oppenheimer-Volkoff (TOV) equation for an anisotropic sphere with constant density, \( \rho = \rho_0 = \text{const} \) can be
written as

$$\frac{dp_r}{dr} = \frac{2}{r} (p_t - p_r) - \frac{4\pi r \left( p_r^2 + \frac{4}{3} p_r \rho_o + \frac{1}{3} \rho_o^2 \right)}{\left( 1 - \frac{8}{3} \pi \rho_0 r^2 \right)}$$

(1.1)

In the case of isotropic spheres where \( p_r = p_t \) this equation can be integrated. Bowers and Liang proposed for anisotropic spheres with \( \rho = \rho_0 = \text{const} \) the following ansatz:

$$\frac{2}{r} (p_t - p_r) = C r^2 \frac{\left( p_r^2 + \frac{4}{3} p_r \rho_o + \frac{1}{3} \rho_o^2 \right)}{\left( 1 - \frac{8}{3} \pi \rho_0 r^2 \right)}$$

(1.2)

They were then able to integrate (1.1) and obtain an expression for \( p_r \). They studied the resulting equation for \( p_r \) and showed that the maximum value of \( M/R \) (where \( M \) is total mass and \( R \) is the radius of the sphere) for anisotropic spheres can be greater than \( 4/9 \) before gravitational collapse occurs. Buchdahl [19], had earlier shown that gravitational collapse will occur if \( M/R \) for a compact object with spherical symmetry exceeds \( 4/9 \). However, it must be noted that Buchdahl studied perfect fluid spheres.

Since Bowers and Liang’s paper many relativists have contributed to the study and development of the physics of anisotropic spheres. Bayin [3], considered generalizations of the perfect fluid solution with \( p = \alpha \rho \) to the anisotropic case. Herrera and his co-workers [4]-[8], have studied conformally flat anisotropic spheres, dissipative anisotropic fluids, anisotropic double polytrope fluids, cracking of anisotropic polytropes, anisotropic geodesic fluids and shear-free anisotropic fluids. Boonserm et al. [9], have shown that an anisotropic fluid can be modeled as a classical (charged) isotropic perfect fluid, a classical electromagnetic field and a classical (minimally coupled) scalar field. There have been at least two articles that claim to have developed a formalism for generating all anisotropic solutions, [10] and [11].

In this paper we are studying interior solutions for charged anisotropic spheres. These solutions must match the Reissner-Nordström metric (Reissner [12], Weyl [13] and Nordström [14]) at the surface of the sphere. The Reissner-Nordström metric is the solution of the Einstein-Maxwell equations that represents the exterior gravitational field of a spherically symmetric charged body. In fact, it is the unique asymptotically flat vacuum solution around any charged spherically symmetric object, irrespective of how that body may be composed or how it may evolve in time (Carter [15], Ruback [16] and Chruściel [17]).

The internal structure of a charged anisotropic sphere is established by solving the coupled Einstein-Maxwell equations. A solution of these equations establishes the gravitational field inside the sphere as a function of the distribution of matter and energy within the sphere. The Einstein-Maxwell equations for a charged static anisotropic
spherically distribution of matter reduce to a set of four independent non-linear second
order differential equations that connect the metric coefficients $g_{tt}$ and $g_{rr}$ with the
physical quantities that represent the matter content of the sphere. The matter content
of charged anisotropic fluid is given by functions that describes the inertial density
$\rho_{fl}$, the radial pressure $p_r$, the tangential pressure $p_t$ and the electromagnetic energy density
$\rho_{em}$ or the electromagnetic charge distribution $q(r)$. The complete system of equations
to be solved consists of four linearly independent equations with six unknowns.

An important question that arises in the study of the structure of spherically
symmetric compact objects is the following: what is the maximum of value $M/R$ that is
allowed before gravitational collapse occurs? (We will call this maximum value of $M/R$
the critical value of $M/R$). We have already mentioned that for neutral perfect fluid
spheres that the stability limit is given by the Buchdahl limit \[19\]:

\[
\left(\frac{M}{R}\right) \leq \frac{4}{9} \quad \text{for a perfect fluid.} \tag{1.3}
\]

In charged spheres the critical value of $M/R$ becomes dependent on the total charge
$Q$, since the addition of charge to the system increases its total energy and hence its
total mass. Andréasson [20], has published a remarkable result that claims that for
any compact spherically symmetric charged distribution the following relationship holds
between $M/R$ and $Q/R$:

\[
\frac{M}{R} \leq \left(\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{1}{3} \frac{Q^2}{R^2}}\right)^2 \quad \text{for all charged spheres.} \tag{1.4}
\]

This formula generalizes the Buchdahl result for perfect fluid neutral spheres. It is
worth noting that in his derivation, Buchdahl placed the following constraints on the
physical properties of the fluid: (i) $d\rho/dr < 0$ i.e., the density decreases outward from
the centre of the sphere and (ii) the pressure is isotropic. In the derivation of his formula,
Andréasson made the following assumptions about the matter content of the sphere: (i)
$p_r + 2p_t < \rho$ and (ii) $\rho > 0$. We note that unlike the neutral case Andréasson did not
require the condition $d\rho/dr < 0$ for the derivation of his formula. There have been
several numerical investigations of the upper bound of $M/R$ as a function of $Q/R$ and
they have all verified that the Andréasson formula provides a valid upper bound of $M/R$
for charged spheres [21] and [22].

In our previous paper [11] we showed that if the condition $d\rho/dr > 0$ is allowed in
charged perfect fluid spheres then the Andréasson limit can be violated. One of our aims
in this study is to continue the investigation of the the validity of the Andréasson limit
as it applies to charged anisotropic spheres using a mixture of analytical and numerical
techniques.
This paper is organized as follows in the next section we will briefly review the derivation of Einstein-Maxwell equations for the interior gravitational field of a charged anisotropic sphere. In section 3 we solve the field equations and consider their stability properties and the formation of extremal black holes in these solutions. In section 4 discuss our results and formulate our conclusions. We note that a prime (′) or a comma (,) denotes derivatives with respect \( r \), and a semi-colon (;) is used to represent covariant derivatives. We will work in units where \( c = G = 1 \).

2. The field equations

We are interested in studying the interior gravitational field of charged spheres, therefore we will assume a spherically symmetric metric of the form

\[
ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.\tag{2.1}
\]

In this paper we are concerned only with spherically symmetric static solutions of the coupled Einstein - Maxwell equations, therefore \( \nu = \nu(r) \) and \( \lambda = \lambda(r) \) are functions of \( r \) only. The Einstein field equations are

\[
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi T_{\alpha\beta}.\tag{2.2}
\]

The spheres that we propose to study have a matter content that consists of an anisotropic fluid with a charge distribution that gives rise to static radial electric field. The energy-momentum tensor \( T_{\alpha\beta} \), will thus written as

\[
T_{\alpha\beta} = (T_{\alpha\beta})_{af} + (T_{\alpha\beta})_{em},\tag{2.3}
\]

with \( (T_{\alpha\beta})_{af} \) the energy-momentum tensor for an anisotropic fluid and \( (T_{\alpha\beta})_{em} \) the energy-momentum tensor associated with the electric field.

The most general energy momentum tensor for a static spherically symmetric anisotropic fluid is

\[
(T_{\alpha\beta})_{af} = (\rho + p_t)u_\alpha u_\beta + p_t g_{\alpha\beta} + (p_r - p_t)v_\alpha v_\beta, \tag{2.4}
\]

where \( \rho, p_r \), and \( p_t \) are the density, the radial pressure and the tangential pressure of the fluid respectively and are functions of \( r \) only, \( u^\alpha \) is the 4-velocity of the fluid and \( v^\alpha \) is a unit space-like vector in the radial direction.
The details pertaining to the construction of the energy-momentum tensors and be found in [1], here we will quote the result:

\[(T_{\alpha\beta})_{af} = \text{diag}(-\rho e^{-2\nu}, p_r e^{-2\lambda}, r^2 p_t, r^2 p_t \sin^2 \theta)).\]  

(2.5)

The electromagnetic energy-momentum tensor has the form

\[(T_{\alpha\beta})_{em} = \frac{q(r)^2}{8\pi r^4} \text{diag}(e^{-2\nu}, -e^{-2\lambda}, r^2, r^2 \sin^2 \theta)).\]  

(2.6)

where

\[q(r) = \int_V j^t \sqrt{-g} dV,\]  

(2.7)

is the total charge contained in a sphere of radius and \(j^t\) is the time component of the four-current density.

The complete set of equations that describe the interior gravitational field of static anisotropic charged spheres are:

\[G^t_t = e^{-2\lambda} \left(\frac{2\nu'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi \rho + \frac{q^2}{r^4};\]  

(2.8)

\[G^r_r = e^{-2\lambda} \left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p_r - \frac{q^2}{r^4};\]  

(2.9)

and

\[G^\theta_\theta = G^\phi_\phi = e^{-2\lambda} \left(\nu'' + \nu'^2 - \nu' \lambda' + \frac{\nu'}{r} - \frac{\lambda'}{r}\right) = 8\pi p_t + \frac{q^2}{r^4}.\]  

(2.10)

3. Solutions for the field equations

We will now develop solutions for the field equations. We start by noting that (2.8), can be written as

\[\frac{d}{dr}(re^{-2\lambda}) = 1 - 8\pi \rho r^2 - \frac{q^2}{r^2}.\]  

(3.1)

This equation can be immediately integrated to give

\[e^{-2\lambda(r)} = 1 - \frac{2m_i(r)}{r} - \frac{f(r)}{r},\]  

(3.2)
with
\[ m_i(r) = 4\pi \int_0^r \rho(s)s^2ds \quad \text{and} \quad f(r) = \int_0^r \frac{q(s)^2}{s^2}ds. \] (3.3)

The quantity \( m_i(r) \) is the inertial mass of the fluid in a sphere of radius \( r \). The requirement that \( e^{-2\lambda(r)} \) matches the Reissner-Nordström metric at surface of the sphere, \( r = R \) gives
\[ 1 - \frac{M}{R} + \frac{Q^2}{R^2} = 1 - \frac{1}{R} \int_0^R (8\pi\rho r^2 + \frac{q^2}{r^2})dr, \] (3.4)

where \( M \) is the total mass and \( Q \) is the total charge. An expression for the total mass can be found from this equation:
\[ M = \frac{1}{2} \int_0^R (8\pi\rho r^2 + \frac{q^2}{r^2})dr + \frac{Q^2}{2R}. \] (3.5)

We note that \( e^{-2\lambda(r)} \) can also be written as
\[ e^{-2\lambda(r)} = 1 - \frac{2m_g(r)}{r} + \frac{q^2(r)}{r^2}. \] (3.6)

where the function \( q(r) \) is the total charge in a sphere of radius \( r \). It is the charge function defined in (2.7). In order for (3.2) to be equal to (3.6), \( m_g(r) \) must be defined in the following manner:
\[ m_g(r) \equiv \frac{1}{2} \int_0^r \left(8\pi\rho(s)s^2 + \frac{q^2(s)}{s^2}\right)ds + \frac{q(r)^2}{2r}. \] (3.7)

\( m_g(r) \) is the total gravitating mass in a sphere of radius \( r \). At the surface of the sphere \( m_g(r) \) is equal \( M \) the total mass. We note that in the absence of the electric field \( m_g = m_i \).

We will study charged spheres with the following density, \( \rho(r) \) and charge, \( q(r) \) profiles:
\[ \rho(r) = \rho_o - \frac{b}{8\pi} \frac{Q^2}{R^6}r^2 \quad \text{and} \quad q(r) = \frac{Q}{R^3} r^3, \] (3.8)

in the expression for \( \rho(r) \), \( b \) is a number. In our models
\[ e^{-2\lambda(r)} = 1 - \frac{8\pi\rho_o}{3}r^2 + \frac{(b - 1)Q^2}{5R^6}r^4 \] (3.9)
Exact solutions for charged spheres and their stability. II. Anisotropic Fluids.

\( m_g(r) = \frac{4\pi \rho_o}{3} r^3 + \frac{(6 - b) Q^2}{10} \frac{r^5}{R R^3}. \)  (3.10)

and

\[ M = \frac{4\pi \rho_o}{3} R^3 + \frac{(6 - b) Q^2}{10} \frac{R}{r}. \]  (3.11)

With the assumed fluid and charge density profiles here, when \( b = 6 \)

\[ \frac{m_g(r)}{r^3} = \frac{M}{R^3} = \text{const}, \]  (3.12)

thus, the \( b = 6 \) model is a sphere with a constant gravitational mass density. Also all models have

\[ \frac{q(r)}{r^3} = \left( \frac{Q}{R^3} \right) = \text{const.} \]  (3.13)

thus, the charge density is constant for all our models. We will study in detail three charged configurations:

(i) \( b = 0 \) - a sphere with a constant density fluid and constant charge density.
(ii) \( b = 1 \) - a sphere with constant total energy density and constant charge density.
(iii) \( b = 6 \) - a sphere with constant gravitational mass density and constant charge density.

We can solve for \( \rho_o \) from (3.11) and rewrite (3.9) as

\[ e^{-2\lambda(r)} = 1 - \left( \frac{2M}{R} + \frac{(b - 6) Q^2}{5} \frac{r^5}{R^2} \right) \frac{r^2}{r^2} + \frac{(b - 1) Q^2}{5} \frac{r^4}{R^2}. \]  (3.14)

Thus, if we are given \( \rho(r) \) and \( q(r) \) we have found \( \lambda(r) \).

We now need to solve for \( \nu(r) \). We start by transforming (2.10). First we subtract (2.9) from (2.10) to get

\[ e^{-2\lambda} \left( \nu'' + \nu'^2 - \nu' \lambda' - \frac{\nu'}{r} \right) - \frac{\lambda' e^{-2\lambda}}{r} - \frac{e^{-2\lambda}}{r^2} + \frac{1}{r^2} = \frac{2 q^2}{r^4} + 8\pi (p_t - p_r). \]  (3.15)

Then we substitute for \( 1/r^2 \) from (2.8) to get the following equation

\[ \nu'' + \nu'^2 - \nu' \lambda' - \frac{\nu'}{r} = 3 \frac{\lambda'}{r} - \left( 8\pi \rho - \frac{q^2}{r^4} - 8\pi (p_t - p_r) \right) e^{2\lambda}. \]  (3.16)
We next multiply both sides of this equation by $e^{-\lambda+\nu}/r$, and we find that the left hand-side becomes an exact differential: $(\nu'e^{-\lambda+\nu})'/r$. Introducing $\zeta(r) \equiv e^{\nu(r)}$, we can write (3.16) in the following form

$$\left(\frac{1}{r}e^{-\lambda}\zeta'\right)' = \left[\frac{3\lambda'e^{-2\lambda}}{r^2} - \frac{8\pi \rho}{r} + \frac{q^2}{r^5} + \frac{8\pi}{r}(p_t - p_r)\right] e^{\lambda}\zeta. \quad (3.17)$$

A similar equation was derived by Giuliani and Rothman [25] in their study of charged perfect fluid spheres. The transformation of the left hand-side (3.16) into that of (3.17) was introduced by Weinberg [24] in deriving the equation he used to prove the Buchdahl limit of $M/R$ for neutral perfect fluid spheres.

We need to define the form of the anisotropy in order to solve (3.17). Here we propose that the anisotropy is proportional to the electromagnetic energy density:

$$8\pi(p_t - p_r) = j\frac{q^2}{r^5} = j\frac{Q^2}{R^6} r^2, \quad (3.18)$$

with $j$ is a constant. Substituting this form of the anisotropy, $\rho(r)$ and $q(r)$ from (3.8) and $\lambda'e^{-2\lambda}$ from (3.14) we find that (3.17) becomes

$$\left(\frac{1}{r}e^{-\lambda}\zeta'\right)' = \left(\frac{11}{5} - \frac{b}{5} + j\right) \frac{Q^2}{R^6} r e^{\lambda}\zeta. \quad (3.19)$$

We now define a new variable

$$\tilde{\zeta}(u(r)) \equiv \zeta(r) \quad \text{with} \quad u(r) = \frac{1}{R^2} \int_0^r s e^{\lambda(s)} ds. \quad (3.20)$$

Then (3.19) becomes

$$\frac{d^2\tilde{\zeta}}{du^2} = a Q^2 \tilde{\zeta}(u). \quad (3.21)$$

with $(11/5 - b/5 + j) = a$. This is our master equation. Our task now is to find solutions for it given values of $b$ and $j$. There are three types of solutions of (3.21) depending on the value of $a$:

(i) $a > 0$: $\tilde{\zeta}(u(r)) = A e^{\sqrt{a}Q Ru(r)} + B e^{-\sqrt{a}Q Ru(r)} \quad (3.22)$

(ii) $a = 0$: $\tilde{\zeta}(u(r)) = A + B u(r), \quad (3.23)$

(iii) $a < 0$: $\tilde{\zeta}(u(r)) = A \sin \left(\sqrt{|a|} \frac{Q}{R} u(r)\right) + B \cos \left(\sqrt{|a|} \frac{Q}{R} u(r)\right) \quad (3.24)$
The values of the constants $A$ and $B$ in (3.22), (3.23) and (3.24) are fixed by the the boundary conditions imposed of $\tilde{\zeta}(u)$:

$$\tilde{\zeta}(u(R)) = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}$$ \hspace{1cm} (3.25)

and

$$\frac{d\tilde{\zeta}(u(R))}{du(R)} = \left(\frac{M}{R} - \frac{Q^2}{R^2}\right).$$ \hspace{1cm} (3.26)

The condition for stability is $\zeta(r = 0) = 0$. The critical values of $M/R$ as a function of $Q/R$ are found from solving the equation $\zeta(r = 0) = 0$. We will call the $\zeta(r = 0) = 0$ equation the critical values equation.

The introduction of anisotropy into the field equations results in an extra degree of freedom in the equation that determines $\zeta(r)$. In case of perfect fluids, a given value of $b$ uniquely determines $a$ in the master equation and this gives a one to one correspondence between $b$ and $\zeta(r)$. In the anisotropic case, the presence of the free parameter $j$ in the expression for $a$ allows us to generate a very large number of values for $a$ for a given value $b$. Thus for each $b$ there are now a very large number of $\zeta(r)$'s. It is clearly not possible to study all of the solutions, therefore here we will restrict ourselves to studying the changes that occur when anisotropy is added to some of the perfect fluid models that we studied in [1]. In particular, we will study three models in detail here: $b = 0$ (a fluid with constant inertial density plus constant charge density), $b = 1$ (a fluid with constant total energy density and constant charge density) and $b = 6$ (a fluid with constant gravitational mass density and constant charge density).

3.1. Solutions with $b = 0$. 

When $b = 0$,

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} - \frac{6Q^2}{5R^2}\right)\frac{r^2}{R^2} - \frac{1}{5}\frac{Q^2}{R^2}\frac{r^4}{R^4},$$ \hspace{1cm} (3.27)

$$u(r) = \frac{1}{R^2}\int_0^r e^{\lambda(s)} ds = \frac{\sqrt{5}R}{2Q}[\tan^{-1}\left(\left(\frac{\sqrt{5}M}{Q} - \frac{3Q}{\sqrt{5}R} + \frac{1}{\sqrt{5}R^3}\right)\frac{e^{\lambda(r)}}{r^2}\right)] - \tan^{-1}\left(\frac{\sqrt{5}M}{Q} - \frac{3}{\sqrt{5}R}\right),$$ \hspace{1cm} (3.28)
and the master equation becomes
\[
\frac{d^2 \tilde{\zeta}}{du^2} = \left( \frac{11}{5} + j \right) \frac{Q^2}{R^2} \tilde{\zeta}(u). \tag{3.29}
\]

There are three distinct types solutions here, solutions with \( j > -11/5 \), \( j = -11/5 \) and \( j < -11/5 \). We will consider the \( j > -11/5 \) case first.

### 3.1.1. Solutions with \( b = 0 \), and \( j > -11/5 \).

For \( b = 0 \), and \( j > -11/5 \), the master equation is
\[
\frac{d^2 \tilde{\zeta}}{du^2} = k^2 \frac{Q^2}{R^2} \tilde{\zeta}(u) \quad \text{with} \quad k = \left( \frac{11}{5} + j \right)^{\frac{1}{2}} \tag{3.30}
\]

The solutions here are
\[
\tilde{\zeta}(r) = A \exp \left[ k \frac{Q}{R} u(r) \right] + B \exp \left[ -k \frac{Q}{R} u(r) \right] \tag{3.31}
\]

The constants \( A \) and \( B \) are found using the boundary conditions. Solving for them we find we find,
\[
A = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} + \frac{1}{k} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left[ -k \frac{Q}{R} u(R) \right] \tag{3.32}
\]
and
\[
B = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} - \frac{1}{k} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left[ k \frac{Q}{R} u(R) \right] \tag{3.33}
\]

The stability condition is \( \tilde{\zeta}(u(r = 0) = 0) \). Here \( \tilde{\zeta}(u(r = 0) = 0) \) requires \( A + B = 0 \).

This leads to the following critical values equation:
\[
\left[ \frac{1}{k} \left( \frac{M}{Q} - \frac{Q}{R} \right) + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right] = \left[ \frac{1}{k} \left( \frac{M}{Q} - \frac{Q}{R} \right) - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right] \exp \left[ \sqrt{5k} \tan^{-1} \left( \left( \frac{\sqrt{5} M}{Q} - \frac{2}{\sqrt{5}} \frac{Q}{R} \right) e^{\chi(R)} \right) - \tan^{-1} \left( \frac{\sqrt{5} M}{Q} - \frac{3}{\sqrt{5}} \frac{Q}{R} \right) \right]. \tag{3.34}
\]

We studied this equation numerically and found that when \( j > 3 \) the critical values of \( M/R \) are greater than the corresponding values of \( M/R \) from the Andréasson formula for a given \( Q/R \). Figure show that the critical values here for \( j = 8 \) are greater than the corresponding values from Andréasson formula and the isotropic model.
Exact solutions for charged spheres and their stability. II. Anisotropic Fluids.

3.1.2. Solution for $b = 0$ and $j = -11/5$.

When $b = 0$ and $j = -11/5$ the master equation becomes

$$\frac{d^2 \zeta}{du^2} = 0$$

(3.35)

The solution for $\zeta(r)$ here is

$$\zeta(r) = \frac{\sqrt{5}}{2} \left( \frac{M}{Q} - \frac{Q}{R} \right) \left[ \tan^{-1} \left( \left( \sqrt{5} \frac{M}{Q} - \frac{3}{\sqrt{5}} \frac{Q}{R} + \frac{1}{\sqrt{5}} \frac{Q}{R^2} r^2 \right) e^{\lambda(r)} \right) - \tan^{-1} \left( \left( \sqrt{5} \frac{M}{Q} - \frac{2}{\sqrt{5}} \frac{Q}{R} \right) e^{\lambda(R)} \right) \right] + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{1/2}$$

(3.36)

The stability condition $\zeta(r = 0) = 0$ gives the following implicit equation for the dependence of the critical values of $M/R$ on $Q/R$:

$$\left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{1/2} = \frac{\sqrt{5}}{2} \left( \frac{M}{Q} - \frac{Q}{R} \right) \left[ \tan^{-1} \left( \left( \sqrt{5} \frac{M}{Q} - \frac{2}{\sqrt{5}} \frac{Q}{R} \right) e^{\lambda(R)} \right) - \tan^{-1} \left( \sqrt{5} \frac{M}{Q} - \frac{3}{\sqrt{5}} \frac{Q}{R} \right) \right].$$

(3.37)

This equation was solved numerically and the results were plotted in Figure 2. We observe that the critical values for this model are less than those from the the Andréasson formula and the $j = 0$ isotropic model.
3.1.3. Solutions with $b = 0$, and $j < -11/5$.

When $b = 0$ and $j < -11/5$, the master equation becomes

$$\frac{d^2 \tilde{\zeta}}{du^2} = -d^2 \frac{Q^2}{R^2} \tilde{\zeta}(u), \quad \text{with} \quad d = \sqrt{|11/5 + j|}$$

(3.38)

The solution for $\tilde{\zeta}(u(r))$ here is

$$\tilde{\zeta}(u(r)) = A \sin \left( d \frac{Q}{R} u(r) \right) + B \cos \left( d \frac{Q}{R} u(r) \right)$$

(3.39)

with

$$A = e^{-\lambda(R)} \sin \left( d \frac{Q}{R} u(R) \right) + \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \cos \left( d \frac{Q}{R} u(R) \right),$$

(3.40)

$$B = e^{-\lambda(R)} \cos \left( d \frac{Q}{R} u(R) \right) - \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \sin \left( d \frac{Q}{R} u(R) \right).$$

(3.41)

The stability condition $\zeta(r = 0) = 0$ requires

$$A \sin \left( d \frac{Q}{R} u(0) \right) + B \cos \left( d \frac{Q}{R} u(0) \right) = 0,$$

(3.42)
Exact solutions for charged spheres and their stability. II. Anisotropic Fluids.

However, since \( u(0) \equiv 0 \), then \( B \) must be equal to zero here. This results in the following critical values equation:

\[
e^{-\lambda(R)} \cos \left( d \frac{Q}{R} u(R) \right) = \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \sin \left( d \frac{Q}{R} u(R) \right)
\]

(3.43)

Here

\[
u(R) = \frac{\sqrt{5} R}{2 Q} \left[ \tan^{-1} \left( \left( \frac{\sqrt{5} M}{Q} - \frac{2 Q}{\sqrt{5} R} \right) e^{\lambda(R)} \right) - \tan^{-1} \left( \frac{\sqrt{5} M}{Q} - \frac{3 Q}{\sqrt{5} R} \right) \right],
\]

(3.44)

thus critical values of \( M/R \) vs \( Q/R \) are solutions of the following equation:

\[
\left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^\frac{1}{2} = \frac{1}{d} \left( \frac{M}{Q} - \frac{Q}{R} \right) \tan \left[ \frac{\sqrt{5}}{2} d \left[ \tan^{-1} \left( \left( \frac{\sqrt{5} M}{Q} - \frac{2 Q}{\sqrt{5} R} \right) e^{\lambda(R)} \right) - \tan^{-1} \left( \frac{\sqrt{5} M}{Q} - \frac{3 Q}{\sqrt{5} R} \right) \right] \right].
\]

(3.45)

This equation has an exact solution for \( d = \sqrt{4/5} \) i.e., \( j = -3 \). The expression is

\[
\frac{M}{R} = \frac{1}{2} \left[ 1 + \frac{Q^2}{R^2} - \left( \frac{16}{81Q} - \frac{128}{10935 R^2} \right)^2 \right]
\]

(3.46)

with

\[
\tilde{Q} = \left[ \frac{64}{729} - \frac{Q^2}{9 R^2} + \left( \frac{1}{3375 R^3} + \frac{59}{6075 R^4} - \frac{128}{10935 R^2} \right)^\frac{1}{2} \right]^\frac{1}{3}
\]

(3.47)

We solved (3.45) numerically for various values of \( d = \sqrt{|11/5 + j|} \) and some results are plotted in Figure 3. We found that for \(-9 < j < -3\) the curves are similar to the curve from the Andréasson formula but the values of \( M/R \) from these curves are less that those from the Andréasson formula. Also the \( M/R \) values for a given \( Q/R \) decreases with decreasing \( j \). When \( j \approx -10 \) the critical values curves start to turn downward and away from the Andréasson formula curve and when \( j \approx -20.3 \) the critical value of \( M/R \) for \( Q/R = 1 \) is zero. This behavior of the critical values curves with increasing negative values of \( j \) is similar to the isotropic case with increasing positive values of \( b \).

Here, for large negative \( j \) we found a new feature in the critical values curves that was not present in the isotropic models. For large negative \( j \) (or positive \( d = \sqrt{|11/5 + j|} \)) the critical values curves instead of being continuous now becomes disjointed and the segments have a quasi-periodic behavior. This quasi-periodic behavior results in there being several extremal values for each \( j \). This behavior is plotted in Figure 4.
Figure 3: The critical values of $M/R$ vs $Q/R$ for $b = 0$ and $j = -9$ (−−−−), $j = -15$ (· · · ·) and $j = -20.3$ (· · · ·) respectively from (3.45) and the Andréasson formula (---).

3.2. Solutions with $b = 1$

When $b = 1$,

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right) \frac{r^2}{R^2},$$

(3.48)

$$u(r) = \frac{1}{R^2} \int_0^r e^{\lambda(s)} s ds = \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right)^{-1} \left[1 - \left[1 - \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right) \frac{r^2}{R^2}\right]^{1/2}\right]$$

(3.49)

and the master equation becomes

$$\frac{d^2 \tilde{\zeta}}{du^2} = (2 + j) \frac{Q^2}{R^2} \tilde{\zeta}(u).$$

(3.50)

The solutions are divided into three types: (i) $j > -2$, $j = -2$ and $j > -2$.

3.2.1. Solutions for $b = 1$ and $j > -2$.

The solution for (3.50) with $j > -2$ is

$$\tilde{\zeta}(u(r)) = A \exp \left(\frac{Q}{R} u(r)\right) + B \exp \left(\frac{Q}{R} u(r)\right) \quad \text{with} \quad h = (2 + j)^{1/2}$$

(3.51)
Figure 4: The critical values curves (---) that are obtained from (3.45) by varying $d$. The dotted line (· · · ·) is the $M/R = Q/R$ extremal line and the curve from Andréasson formula is the solid line (---).

with

$$A = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} + \frac{1}{h} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left( -h \frac{Q}{R} u(R) \right),$$  \hspace{1cm} (3.52)

$$B = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} - \frac{1}{h} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left( h \frac{Q}{R} u(R) \right)$$ \hspace{1cm} (3.53)

and

$$u(R) = \left( \frac{2M}{R} - \frac{Q^2}{R^2} \right)^{-1} \left[ 1 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right].$$ \hspace{1cm} (3.54)
Exact solutions for charged spheres and their stability. II. Anisotropic Fluids.

Figure 5: The critical values of $M/R$ vs $Q/R$ from (3.55) for $j = 8$ (---), the Andréasson formula (-----) and the isotropic model (……..).

Here the critical values of $M/R$ are found from the following equation:

$$
\left( \frac{M}{Q} - \frac{Q}{R} \right) + h \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} = \left( \frac{M}{Q} - \frac{Q}{R} \right) - h \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \exp \left[ \frac{2h}{Q} \left( 1 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{3}{2}} \right) \right]
$$

The numerical results from (3.55) shows that critical values of $M/R$ for curves with $j > 4$ exceeds the values from the Andréasson formula. Also the critical values of $M/R$ for a given $Q/R$ increases with increasing $j$. The critical values curve for $j = 8$, the Andréasson formula and the isotropic $b = 1$ are shown in Figure 5.

3.2.2. Solution for $b = 1$ and $j = -2$

When $b = 1$ and $j = -2$ there is only one solution for the master equation. Here

$$
\frac{d^2 \tilde{\zeta}}{du^2} = 0.
$$

(3.56)

The solution for $\zeta(r)$ is

$$
\zeta(r) = \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} + \left( \frac{M}{R} - \frac{Q^2}{R^2} \right) \left( \frac{2M}{R} - \frac{Q^2}{R^2} \right)^{-1}
$$

(3.57)
Figure 6: The critical values of $M/R$ vs $Q/R$ for the $b = 1$ and $j = 2$ model (---), the Andréasson formula (——) and the isotropic model $b = 1$ model (……) .

The critical values equation from the stability condition $\zeta(r = 0) = 0$, has a very simple form here:

$$\left(1 - \frac{2M}{R} - \frac{Q^2}{R^2}\right)^{\frac{1}{2}} - \left(1 - \left(\frac{2M}{R} - \frac{Q^2}{R^2}\right)\frac{r^2}{R^2}\right)^{\frac{1}{2}} = 0$$

The roots of this equation are

$$M = \frac{1}{2} Q^2 - \frac{2}{9} + \frac{2}{3} \frac{Q^2}{R^2} - \frac{1}{9} \left(4 - \frac{3Q^2}{R^2}\right)^{\frac{1}{2}}, \quad \frac{2}{9} + \frac{2}{3} \frac{Q^2}{R^2} + \frac{1}{9} \left(4 - \frac{3Q^2}{R^2}\right)^{\frac{1}{2}}.$$ (3.58)

3.2.3. Solutions for $b = 1$ and $j < -2$
When \( b = 1 \) and \( j < -2 \) the master equation becomes

\[
\frac{d^2 \tilde{\zeta}}{du^2} = -l^2 \frac{Q^2}{R^2} \tilde{\zeta}(u) \quad \text{with} \quad l = \sqrt{|2 + j|}
\] (3.60)

The solution for \( \tilde{\zeta}(u(r)) \) here is

\[
\tilde{\zeta}(u(r)) = A \sin \left( \frac{lQ}{R} u(r) \right) + B \cos \left( \frac{lQ}{R} u(r) \right)
\] (3.61)

with

\[
A = e^{-\lambda(R)} \sin \left( \frac{lQ}{R} u(R) \right) + \frac{1}{l} \left( \frac{M}{Q} - \frac{Q}{R} \right) \cos \left( \frac{lQ}{R} u(R) \right),
\] (3.62)

\[
B = e^{-\lambda(R)} \cos \left( \frac{lQ}{R} u(R) \right) - \frac{1}{l} \left( \frac{M}{Q} - \frac{Q}{R} \right) \sin \left( \frac{lQ}{R} u(R) \right).
\] (3.63)

The stability condition \( \zeta(r = 0) = 0 \) requires

\[
A \sin \left( \frac{lQ}{R} u(0) \right) + B \cos \left( \frac{lQ}{R} u(0) \right) = 0,
\] (3.64)

however, since \( u(0) \equiv 0 \), then \( B \) must be equal to zero here. This results in the following critical values equation:

\[
e^{-\lambda(R)} \cos \left( \frac{lQ}{R} u(R) \right) = \frac{1}{l} \left( \frac{M}{Q} - \frac{Q}{R} \right) \sin \left( \frac{lQ}{R} u(R) \right)
\] (3.65)

Here

\[
u(R) = \left( \frac{2M}{R} - \frac{Q^2}{R^2} \right)^{-1} \left[ 1 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right].
\] (3.66)

thus critical values of \( M/R \) vs \( Q/R \) are solutions of the following equation:

\[
\left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} = \frac{1}{l} \left( \frac{M}{Q} - \frac{Q}{R} \right) \tan \left[ \frac{l \left[ 1 - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{3}{2}} \right]}{\left( \frac{2M}{Q} - \frac{Q}{R} \right)} \right].
\] (3.67)

We solved (3.67) numerically for various values of \( l = \sqrt{|2 + j|} \) and some results are plotted in Figure 7. We found that the curves are similar to the \( b = 0 \) model, however for a given \( j \) here the values of \( M/R \) are less than the \( b = 0 \) model and \( j \approx -19.7 \) when the critical value of \( M/R \) for \( Q/R = 1 \) is zero. The quasi-periodic behavior for large negative \( j \) that was found for the \( b = 0 \) models is also repeated and is shown in Figure 8. In the plots in Figure 8 we varied \( l = \sqrt{|6/5 + j|} \).
Figure 7: The critical values of $M/R$ vs $Q/R$ for $j = -9$ (---), $j = -15$ (· · · ·) and $j = -19.7$ (· · · ·) from (3.67) and the Andréasson formula (____).

3.3. Solutions with $b = 6$.

When $b = 6$,

$$e^{-2\lambda(r)} = 1 - \frac{2M}{R} \frac{r^2}{R^2} + \frac{Q^2}{R^2} \frac{r^4}{R^4},$$

(3.68)

and

$$u(r) = \frac{R}{2Q} \left[ \log \left( \frac{M}{Q} + 1 \right) - \log \left( \frac{M}{Q} - \frac{Q}{R^3} r^2 + e^{-\lambda(r)} \right) \right],$$

(3.69)

and the master equation becomes

$$\frac{d^2 \tilde{\zeta}}{du^2} = (1 + j) \frac{Q^2}{R^2} \tilde{\zeta}(u),$$

(3.70)

The 3 distinct cases to be considered here are $j > -1$, $j = -1$ and $j < -1$.

3.3.1. Solutions for $b = 6$ and $j > -1$

The solution for (3.70) with $j > -1$ is

$$\tilde{\zeta}(u(r)) = A \exp \left( p \frac{Q}{R} u(r) \right) + B \exp \left( p \frac{Q}{R} u(r) \right) \quad \text{with} \quad p = (1 + j)^{\frac{1}{2}},$$

(3.71)
Exact solutions for charged spheres and their stability. II. Anisotropic Fluids.

Figure 8: The critical values curves (---) that are obtained from (3.67) by varying \( l \).
The dotted line (\cdots\cdots) is the \( M/R = Q/R \) extremal line and the curve from Andréasson formula is the solid line (---).

\[
A = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} + \frac{1}{p} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left( -p \frac{Q}{R} u(R) \right), \quad (3.72)
\]

\[
B = \frac{1}{2} \left[ \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} - \frac{1}{p} \left( \frac{M}{Q} - \frac{Q}{R} \right) \right] \exp \left( p \frac{Q}{R} u(R) \right) \quad (3.73)
\]

and

\[
u(R) = \frac{R}{2Q} \left[ \log \left( \frac{M}{Q} + 1 \right) - \log \left( \frac{M}{Q} - \frac{Q}{R} + e^{-\lambda(R)} \right) \right] \quad (3.74)
Figure 9: The critical values of $M/R$ vs $Q/R$ from (3.75) for $j = 8$ (- - - - -), the Andréasson formula (-----) and the isotropic model (········).

The critical values of $M/R$ are solutions of the following equation:

$$\left[\left(\frac{M}{Q} - \frac{Q}{R}\right) + p\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{3}{2}}\right] = \left[\frac{(M/Q) + 1}{M_Q - \frac{Q}{R} + (1 + \frac{2M}{R} + \frac{Q^2}{R^2})^{\frac{3}{2}}}\right]^p$$

(3.75)

This critical values equations can be written as a polynomials and thus we were able to obtain values of $M/R$ for all $0 < Q/R < 1$. The critical values equation here has an exact solution for $p = 2$ i.e., $j = 3$. The expression is

$$\frac{M}{R} = \frac{1}{2} + \frac{Q^2}{2R^2} - \frac{1}{2} \left[\frac{\tilde{Q}}{9} - \frac{9}{Q} \left(\frac{Q^2}{3R^2} - \frac{16}{81}\right) - \frac{5}{9}\right]^2$$

(3.76)

with

$$\tilde{Q} = \left[64 + 81\frac{Q^2}{R^2} + 9 \left(243\frac{Q^6}{R^6} - 351\frac{Q^4}{R^4} + 384\frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

(3.77)

The critical values of $M/R$ from this equation are slightly less that the corresponding values from the Andréasson formula. The results from the numerical solutions for (3.75) indicates that the $j = 4$ model critical values saturates the Andréasson formula. The $j = 8$ curve is shown in Figure 9 and we see that the critical values for this model are greater than the critical values from the Andréasson formula.
3.3.2. Solutions for $b = 6$ and $j = -1$.

When $b = 6$ and $j = -1$ there is only one solution for the master equation. Here

$$\frac{d^2 \zeta}{du^2} = 0. \quad (3.78)$$

The solution for $\zeta(r)$ is

$$\zeta(r) = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} + \frac{1}{2} \left(\frac{M}{Q} - \frac{Q}{R}\right) \log \left[\frac{M}{Q} - \frac{Q}{R} + e^{\lambda(R)}\right] \quad (3.79)$$

Here the critical values of $M/R$ are solutions of the following equation:

$$\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{M}{Q} - \frac{Q}{R}\right) \log \left[\frac{M}{Q} + 1 + \frac{M}{Q} - \frac{Q}{R} + \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}}\right] \quad (3.80)$$

This equation was solved numerically and the results are plotted in Figure 10. We found that the critical values from this model is less than values from both the the Andréasson formula and the $b = 1$ isotropic model.
3.3.3. Solutions for \( b = 6 \) and \( j < -1 \). When \( b = 6 \) and \( j = -1 \) the master equation becomes

\[
\frac{d^2 \tilde{\zeta}}{du^2} = -t^2 \frac{Q^2}{R^2} \tilde{\zeta}(u), \quad \text{with} \quad t = \sqrt{|1 + j|} \tag{3.81}
\]

The solution for \( \tilde{\zeta}(u(r)) \) here is

\[
\tilde{\zeta}(u(r)) = A \sin \left( t \frac{Q}{R} u(r) \right) + B \cos \left( t \frac{Q}{R} u(r) \right) \tag{3.82}
\]

with

\[
A = e^{-\lambda(R)} \sin \left( t \frac{Q}{R} u(R) \right) + \frac{1}{t} \left( M - \frac{Q}{R} \right) \cos \left( t \frac{Q}{R} u(R) \right), \tag{3.83}
\]

\[
B = e^{-\lambda(R)} \cos \left( t \frac{Q}{R} u(R) \right) - \frac{1}{t} \left( M - \frac{Q}{R} \right) \sin \left( t \frac{Q}{R} u(R) \right). \tag{3.84}
\]

The stability condition \( \zeta(r = 0) = 0 \) requires

\[
A \sin \left( t \frac{Q}{R} u(0) \right) + B \cos \left( t \frac{Q}{R} u(0) \right) = 0, \tag{3.85}
\]

however, since \( u(0) \equiv 0 \), then \( B \) must be equal to zero here. This results in the following critical values equation:

\[
e^{-\lambda(R)} \cos \left( t \frac{Q}{R} u(R) \right) = \frac{1}{t} \left( M - \frac{Q}{R} \right) \sin \left( t \frac{Q}{R} u(R) \right) \tag{3.86}
\]

Since for \( b = 6 \)

\[
u(R) = \frac{R}{2Q} \left[ \log \left( \frac{M}{Q} + 1 \right) - \log \left( \frac{M}{Q} - \frac{Q}{R} + e^{-\lambda(R)} \right) \right], \tag{3.87}
\]

The critical values of \( M/R \) here are solutions of the following equation:

\[
\left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} = \frac{1}{t} \left( \frac{M}{Q} - \frac{Q}{R} \right) \tan \left( \frac{t}{2} \log \left[ \frac{\left( \frac{M}{Q} + 1 \right)}{\left( \frac{M}{Q} - \frac{Q}{R} + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} \right)} \right] \right) \tag{3.88}
\]

The numerical solution of this equation is plotted in Figure 11 for various values of \( j \). The behavior of the critical curves here are similar to the \( b = 0 \) and \( b = 1 \) models. However, here the critical values of \( M/R \) for a given \( Q/R \) and \( j \) are smaller than either the \( b = 0 \) or the \( b = 1 \) models. In particular here when \( j = -16.8 \) the critical value of \( M/R \) for \( Q/R = 1 \) is zero. The quasi-periodic behavior for large negative \( j \) or \( t = \sqrt{|1/5 + j|} \) is shown in Figure 12.
3.4. Solutions with 11/5 - b/5 + j = 0.

We now consider a class of solutions that satisfies the condition $11/5 - b/5 + j = 0$. The special cases $b = 0$, $b = 1$ and $b = 6$ were already studied above. Here and below we will use

$$\beta \equiv \frac{b}{5} - \frac{1}{5},$$  \hspace{1cm}  \text{(3.89)}

thus $11/5 - b/5 + j$ becomes $2 - \beta + j$. We will start with the case $\beta > 0$.

3.4.1. Solutions with $(2 - \beta + j) = 0$ and $\beta > 0$ \hspace{1cm} The following general expressions exists for $\beta > 0$:

$$e^{-2\lambda(r)} = 1 - \left( \frac{2M}{R} + (\beta - 1)\frac{Q^2}{R^2} \right) \frac{r^2}{R^2} + \beta \frac{Q^2}{R^2} \frac{r^4}{R^4}, \hspace{1cm} \text{(3.90)}$$

$$u(r) = \frac{1}{R^2} \int_0^r se^{\lambda(s)} ds = \frac{1}{2\sqrt{\beta} Q} \left[ \log \left( \frac{1}{\sqrt{\beta}} \frac{M}{Q} + \frac{(\beta - 1)Q}{2\sqrt{\beta}} \frac{Q}{R} + 1 \right) \right]$$

$$- \frac{1}{2\sqrt{\beta} Q} \left[ \log \left( \frac{1}{\sqrt{\beta}} \frac{M}{Q} + \frac{(\beta - 1)Q}{2\sqrt{\beta}} \frac{Q}{R} - \frac{1}{\sqrt{\beta}} \frac{Qr^2}{R^3} + e^{-\lambda(r)} \right) \right] \hspace{1cm} \text{(3.91)}$$
Figure 12: The critical values curves (---) that are obtained from (3.88) by varying $t$. The dotted line (· · · ·) is the $M/R = Q/R$ extremal line and the curve from Andréasson formula is the solid line (——).

and

$$
\zeta(u(r)) = A + Bu(r) = \frac{1}{2\sqrt{\beta}} \left( \frac{M}{Q} - \frac{Q}{R} \right) \left[ \log \left( \frac{1}{\sqrt{\beta}} \frac{M}{Q} + \frac{(\beta - 3)Q}{2\sqrt{\beta} R} + e^{-\lambda(R)} \right) \right] - \log \left( \frac{1}{\sqrt{\beta}} \frac{M}{Q} + \frac{(\beta - 1)Q}{2\sqrt{\beta} R} - \frac{1}{\sqrt{\beta}} \frac{Qr^2}{R^2} + e^{-\lambda(r)} \right) + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}}
$$

The condition $\zeta(r = 0) = 0$ gives the following transcendental for the critical values of $M/R$ as a function of $Q/R$:

$$
\left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{\frac{1}{2}} = \frac{1}{2\sqrt{\beta}} \left( \frac{M}{Q} - \frac{Q}{R} \right) \left[ \log \left( \frac{1}{\sqrt{\beta}} \frac{M}{Q} + \frac{(\beta - 1)Q}{2\sqrt{\beta} R} + 1 \right) - \log \left( \frac{1}{\sqrt{\beta}} \frac{M}{Q} + \frac{(\beta - 3)Q}{2\sqrt{\beta} R} + e^{-\lambda(R)} \right) \right].
$$
Figure 13: The critical values of $M/R$ vs $Q/R$ for $\beta = 16$ (---) and $\beta = 4$ (· · · · · ·), from (3.93) and the Andréasson formula (___).

Figure 14: The critical values of $M/R$ vs $Q/R$ for $\beta = 20000$ (___), $\beta = 1000$ (---), and $\beta = 500$ (· · · · · ·), from (3.93) and the Andréasson formula (___).
We solved this equation numerically and the results are plotted in Figure 13 and Figure 14. We found that when $0 < \beta < 24$ the curves (Figure 13) are similar to the curve from the Andréasson’s formula. The curves with $\beta < 1.5$ have critical values of $M/R$ that are less than the corresponding values from the Andréasson’s formula. The critical values of $M/R$ for $\beta = 1.5$ almost matches the values from the Andréasson’s formula and the curves for $\beta > 1.5$ have critical values of $M/R$ that are greater than the values from the Andréasson’s formula. We note however the range of $Q/R$ for which a solution exists get progressively smaller with increasing $\beta$. For large $\beta$ ($\beta > 500$) in Figure 14 we observe that the critical values of $M/R$ for a given $\beta$ increase very rapidly with increasing values of $Q/R$ but as we have just noted the range of values of $Q/R$ for which a solution exists is small, for example when $\beta = 20000$, $M/R$ values exists only for the $0 < Q/R < 0.04$.

3.4.2. Solutions with $(2 - \beta + j) = 0$ and $\beta < 0$.

We will now develop solutions for 3.22 for with $\beta < 0$. We can write the following general expressions for these solutions using $|\beta| = k$

$$e^{-2\lambda(r)} = 1 - \left(\frac{2M}{R} - (k + 1)\frac{Q^2}{R^2}\right)\frac{r^2}{R^2} - k\frac{Q^2}{R^2}\frac{r^4}{R^4}, \quad k = |\beta| \tag{3.94}$$

and

$$u(r) = \frac{1}{R^2} \int_0^r e^{\lambda(s)} \, ds = \frac{1}{R^2} \int_0^r \frac{sd}{[1 - \left(\frac{2M}{R} - (k + 1)\frac{Q^2}{R^2}\right)\frac{s^2}{R^2} - k\frac{Q^2}{R^2}\frac{s^4}{R^4}]^{\frac{3}{2}}} \tag{3.95}$$

$$= \frac{1}{2\sqrt{k}} \frac{R}{Q} \left[\tan^{-1}\left(\left(\frac{1}{\sqrt{k}} \frac{M}{Q} - k + 1\frac{Q}{2\sqrt{k} R} + \sqrt{\frac{k}{R}} \frac{Q}{R^3} \frac{r^2}{R^2}\right) \frac{e^{\lambda(r)}}{e^{\lambda(r)}} \right) \right.$$

$$\left. - \tan^{-1}\left(\frac{1}{\sqrt{k}} \frac{M}{Q} - \frac{k + 1}{2\sqrt{k} R} \right) \right] \tag{3.96}$$

$$\zeta(u(r)) = \frac{1}{2\sqrt{k}} \left(\frac{M}{Q} - \frac{Q}{R}\right) \left[\tan^{-1}\left(\left(\frac{1}{\sqrt{k}} \frac{M}{Q} - \frac{k + 1}{2\sqrt{k} R} + \sqrt{\frac{k}{R}} \frac{Q}{R^3} \frac{r^2}{R^2}\right) \frac{e^{\lambda(r)}}{e^{\lambda(r)}} \right) \right.$$

$$\left. - \tan^{-1}\left(\frac{1}{\sqrt{k}} \frac{M}{Q} + \frac{k - 1}{2\sqrt{k} R} \right) \right] + \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right) \frac{1}{2} \tag{3.97}$$

The stability condition $\zeta(r = 0) = 0$ gives the following implicit equation for the dependence of the critical values of $M/R$ on $Q/R$:

$$\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right) = \frac{1}{2\sqrt{k}} \left(\frac{M}{Q} - \frac{Q}{R}\right) \left[\tan^{-1}\left(\left(\frac{1}{\sqrt{k}} \frac{M}{Q} + \frac{k - 1}{2\sqrt{k} R} \right) \frac{e^{\lambda(r)}}{e^{\lambda(r)}} \right) \right.$$

$$\left. - \tan^{-1}\left(\frac{1}{\sqrt{k}} \frac{M}{Q} - \frac{k + 1}{2\sqrt{k} R} \right) \right] \tag{3.98}$$
We solved this equation numerically and the results are plotted in Figure 15. We found that when $|\beta| \approx 8$ the critical values of $M/R$ approximately equal to corresponding values of $M/R$ from the Andréasson formula and curves with $|\beta| > 8$ have critical $M/R$ values greater than the values from the Andréasson formula. We also found that the curves for large $-\beta$ values have a similar pattern to the large $\beta$ values curves. We note however that for large $-\beta$ values the range of $Q/R$ for which a solution for $(3.98)$
exists is larger.

3.5. Solutions with \((2 - \beta + j)\frac{1}{2} = a\) and \(\beta = a^2/4\).

We studied a set of solutions with the condition \((2 - \beta + j)\frac{1}{2} = a\) and \(\beta = a^2/4\). The general expressions for \(e^{-2\lambda}\) and \(u(r)\) here are same as (3.90) and (3.91). The solution for \(\zeta(r)\) here is

\[
\tilde{\zeta}(u(r)) = A \exp\left(\frac{a}{R}Q u(r)\right) + B \exp\left(-\frac{a}{R}Q u(r)\right)
\]

The constants \(A\) and \(B\) are evaluated using the boundary conditions. Applying the boundary conditions we find that

\[
A = \frac{1}{2} \left[ \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} + \frac{1}{a} \left(\frac{M}{Q} - Q\right) \right] \exp\left(-\frac{a}{R}Q u(r)\right)
\]

and

\[
B = \frac{1}{2} \left[ \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} - \frac{1}{a} \left(\frac{M}{Q} - Q\right) \right] \exp\left(\frac{a}{R}Q u(r)\right)
\]

The stability condition \(\tilde{\zeta}(r = 0) = 0\) requires \(A = -B\). Substituting \(\beta = a^2/4\) and the expression for \(u(R)\) into the equation \(A = -B\) results in the following equation for the critical values of \(M/R\) as a function of \(Q/R\):

\[
\left[ \left(\frac{M}{Q} - Q\right) + 2\sqrt{\beta} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} \right] / \left[ \left(\frac{M}{Q} - Q\right) - 2\sqrt{\beta} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} \right] = \left[ \frac{M}{Q} + \frac{(\beta-1)Q}{R} + \sqrt{\beta} e^{-\lambda R} \right]^2.
\]

The \(\beta = 1\) equation here is the same as the \(b = 6\) with \(j = 3\) model, (3.75) and thus has the exact solution (3.76).

The expressions from which the critical values of \(M/R\) are found for other values of \(\beta\) can be written as polynomials, for example when \(\beta = 2\) the expression is

\[
\left(-11\sqrt{2} \frac{M^2}{R^2} \frac{Q}{R} + 12\sqrt{2} \frac{M^3}{R^3} + 4\sqrt{2} \frac{M}{R} \left(\frac{Q}{R}\right)^2 - 3\sqrt{2} \left(\frac{Q}{R}\right)^3 - 2\sqrt{2} \frac{Q^3}{R^3} - 2 \frac{M^2}{R^2} + \frac{M Q^2}{R^2} + \frac{Q^4}{R^4}\right)^2 =
\]

\[
\left(-6 \frac{M^2}{R^2} + 11 \frac{M Q^2}{R^2} - 4\sqrt{2} \frac{M}{R} \left(\frac{Q}{R}\right)^2 - 6 \frac{Q^3}{R^3} - 2 \frac{Q^4}{R^4} - 8 \frac{Q^2}{R^2}\right) \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)
\]

\[
\left(\frac{M}{Q} - Q\right) + 2\sqrt{\beta} \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{\frac{1}{2}} = \left[ \frac{M}{Q} + \frac{(\beta-1)Q}{R} + \sqrt{\beta} e^{-\lambda R} \right]^2.
\]
Figure 17: The critical values of M/R vs Q/R for $\beta = 6$ (− − −) and $\beta = 4$ (· · · · ·), from (3.101) and the Andréasson formula (____). The numerical solution of (3.101) shows that for $\beta = 1$ the M/R values are less than those from the Andréasson formula but the for $\beta \geq 2$ the M/R values from from this model are greater than those from the Andréasson formula for a given Q/R. We plot the curves for $\beta = 4$ and $\beta = 6$ in Figure 17.

4. Conclusion

In this paper we studied a plethora of exact solutions for anisotropic charged spheres with the fluid density, the charge density and the anisotropy given by the following functions respectively:

$$\rho(r) = \rho_o - \frac{b}{8\pi} \frac{Q^2}{R^6} r^2, \quad q(r) = \frac{Q}{R^3} r^3, \quad \text{and} \quad 8\pi(p_t - p_r) = j \frac{Q^2}{R^6} r^2. \quad (4.1)$$

These exact charge density and fluid density functions were considered in our study of charged perfect fluid spheres [11]. The choice of anisotropy has no effect on the metric function $g_{rr} = e^{2\lambda(r)}$. It remains the same as in the charged perfect fluid case. The addition of anisotropy to the system however results in an extra degree of freedom in the equation that determines $g_{tt} = e^{2\nu(r)}$. The equation that determines $e^{\nu(r)} \equiv \zeta(r)$ is
now written as

\[
\left( \frac{1}{r} e^{-\lambda \zeta^j} \right)' = \left( \frac{11}{5} - \frac{b}{5} + j \right) \frac{Q^2}{R^6} r \lambda \zeta. \tag{4.2}
\]

In the isotropic case we found solutions for \( \zeta(r) \) by varying \( b \). The addition of anisotropy allows us to fix \( b \) and vary \( j \) or vice versa when generating solutions for \( \zeta(r) \). Thus for each \( b \) there are now a very large number of solutions for \( \zeta(r) \). Since \( b \) appears on equal footing as \( j \) in (4.2) (with the exception of a sign difference) we found the effect of fixing \( b \) and varying \( j \) to be similar to the results for the isotropic case when \( b \) was varied.

A summary of our results are as follows:

- We considered in detail the effects of varying \( j \) on a spheres with \( b = 0 \), \( b = 1 \) and \( b = 6 \). We found that by varying \( j \) for a fixed \( b \) a set of critical values curves can be generated that are similar to critical values curves in the isotropic case when we varied \( b \). In particular we found that for a sufficiently large positive \( j \) we can generate critical values curves with values of \( M/R \) greater than those given by the Andréasson formula.

- We found in the \( b = 0 \), 1 and 6 models when \( j \) is large and negative the critical values curve becomes quasi-periodic and this behavior gives more than one value for the extremal condition \( M/R = Q/R \).

- We also studied solutions with the condition \( (2 - \beta + j) = 0 \) for both \( \beta < 0 \) and \( \beta > 0 \), \( (\beta = b/5 - 1/5) \). Interestingly the critical values curves in both cases are similar to each other and for a sufficiently large \( |\beta| \) we can generate curves with critical values of \( M/R \) greater than those from the Andréasson formula.

- Finally we considered solutions with the condition \( (2 - \beta + j) = a^2 \), and \( a^2 = 4\beta \). Here we were able to reduce the critical values equations to polynomials. This allowed us to find \( M/R \) values for all \( 0 < Q/R < 1 \). When \( \beta > 2 \) we found that the critical values curves have values of \( M/R \) that are larger than those from the Andréasson formula.

References

[1] K. Dev, “Exact solutions for charged spheres and their stability. I. Perfect Fluids” arXiv: 2202.07819 [gr-qc].

[2] R. L. Bowers and E. P. T. Liang, “Anisotropic spheres in General Relativity”, Ap. J. 188, 657 (1974).

[3] S. S. Bayin, “Anisotropic Fluid Spheres in General Relativity”, Phys. Rev. D 26, 1262 (1982).

[4] G. Abellan, E Fuenmayor, E, Conteras, L. Herrera, “The general relativistic double polytope for anisotropic matter”, Physics of the Dark Universe. Volume 30, 100632 (2020)

[5] L. Herrera, E. Fuenmayor, P. Leon, “Creacking of general relativistic polytopes”, Phys. Rev. D 93, 024047, (2016).
Exact solutions for charged spheres and their stability. II. Anisotropic Fluids.

[6] L. Herrera, A. Di Prisco, W. Barreto, J. Ospino, “Conformally flat polytropes for anisotropic matter”, Gen. Rel. Grav. 46, 1827, (2014).
[7] L. Herrera, A. Di Prisco, J. Ospino, “On the stability of the shear-free condition”, Gen. Rel. Grav. 42, 1585, (2010).
[8] L. Herrera, A. Di Prisco, E. Fuenmayor, O. Troconis, “Dynamics of viscous dissipative gravitational collapse: A full causal approach”, Int. J. Mod. Phys. D 18, 129-145, (2009).
[9] Petarpa Boonserm, Tritos Ngampitipan, Matt Visser, “Modelling anisotropic fluid spheres in general relativity”, Int. J. Mod. Phys. D 25, 2, 1650019, (2019).
[10] M M Akbar and R Solanki, “Algorithms for Generating All Static Spherically Symmetric (An)isotropic Fluid Solutions of Einstein’s Equations”, arXiv:2012.15479 [gr-qc].
[11] L. Herrera, J. Ospino, A. Di Prisco, “All static spherically symmetric anisotropic solutions of Einstein’s equations”, Phys. Rev. D 77, 027502, (2008).
[12] H. Reissner, “Uber die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie”, Ann. Physik, 50, 106-120 (1916).
[13] H. Weyl, “Zur Gravitationstheorie”, Ann. Physik, 54 (359), 117-145 (1917).
[14] G. Nordström, “On the energy of the gravitational field in Einstein’s theory”, Proc. Kon. Ned. Akad. Wet., 20, 1238 (1918).
[15] B. Carter, “Mathematical foundations of the theory of relativistic stellar and black hole configurations”, in Gravitation in astrophysics, eds. B. Carter and J. B. Hartle, (Plenum Press), 63-122 (1987).
[16] P. Ruback, “A new uniqueness theorem for charged black holes”, Class. Quantum Grav., 5, L155-L159 (1988).
[17] P. T. Chruściel, “Uniqueness of stationary, electro-vacuum black holes revisited”, Helvet. Phys. Acta, 69, 529-543 (1996).
[18] B. V. Ivanov, “Static charged perfect fluid spheres in general relativity”, Phys. Rev. D 65, 104,001 (2002).
[19] H. A. Buchdahl, “General relativistic fluid spheres”, Phys. Rev. 116, 1027-1034 (1959).
[20] H. Andréasson, “Sharp bounds on the critical stability radius for relativistic charged spheres”, Commun. Math. Phys. 288, 715 (2009).
[21] J. D. V. Arbanil, J. P. S. Lemos, and V. T. Zanchin, “Polytropic spheres with electric charge: compact stars, the Oppenheimer-Volkov and Buchdahl limits, and quasiblack holes”, Phys. Rev. D 88, 084023 (2013).
[22] Jose P. S. Lemos, Wilson T. Zanchin, “Plethora of relativistic charged spheres: The full spectrum of Guilfoyle’s static, electrically charged spherical solutions”, Phys Rev D, 95, 104040 (2017).
[23] A. Strominger and C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy”, Phys. Lett. B 379, 99-104 (1996).
[24] S. Weinberg, (1972). Gravitation and Cosmology: Principles and applications of the General Theory of Relativity, (John Wiley, NY, 1972).
[25] A. Giuliani and T. Rothman, “Absolute stability limit for relativistic charged spheres”, Gen. Relativ. Gravit. 40, 1427 (2008).
[26] J. Bekenstein, “Hydrostatic equilibrium and gravitational collapse of relativistic charged fluid balls”, Phys. Rev. D 4, 2185 (1971).