The Tachyon Inflationary Models with Exact Mode Functions

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We show two analytical solutions of the tachyon inflation for which the spectrum of curvature (density) perturbations can be calculated exactly to linear order, ignoring both gravity and the self-interactions of the tachyon field. The main feature of these solutions is that the spectral indices are independent with scale.

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I. INTRODUCTION

The inflationary scenario[1] provides a mechanism to produce the primordial fluctuations of spacetime and matter, which lead to the CMBR(cosmic microwave background radiation) anisotropies and to the large scale structure[2]. In standard inflationary models[3], the physics lies in the inflation potential. The underlying dynamics is simply that of a single scalar field rolling in the potential. The scenario is generically referred to as chaotic inflation in reference to its choice of initial conditions. This picture is widely favored because of its simplicity and has received by far the most attention to date. Some potentials that give the correct inflationary properties have been proposed[4] in the past two decades. This is very interesting to find the exact solutions of mode function for the tachyon inflation.

II. THE EQUATIONS OF MOTION

We consider spatially flat FRW line element given by:

\[ ds^2 = dt^2 - a^2 (t) \left( dx^2 + dy^2 + dz^2 \right) \]

\[ = a^2 (\tau) \left[ d\tau^2 - (dx^2 + dy^2 + dz^2) \right] \]

where \( \tau \) is the conformal time, with \( d\tau = adt \). As shown by Sen[7], a rolling tachyon condensate in either bosonic or supersymmetric string theory can be described by a fluid which in the homogeneous limit has energy density and pressure as follows

\[ \rho = \frac{V(T)}{\sqrt{1 - T^2}} \]

\[ p = -V(T)\sqrt{1 - T^2} \]

where \( T \) and \( V(T) \) are the tachyon field and potential, and an overdot denote a derivative with respect to the coordinate time \( t \). To take the gravitational field into account the effective lagrangian density in the Born-Infeld-type form is described by:

\[ L = \sqrt{-g} \left( \frac{R}{2\kappa} - V(T)\sqrt{1 + g^{\mu\nu}\partial_\mu T \partial_\nu T} \right) \]

where \( \kappa = 8\pi G = M_p^{-2} \). For a spatially homogeneous tachyon field \( T \), we have the equation of motion

\[ \ddot{T} + 3H \dot{T} \left( 1 - \dot{T}^2 \right) + \frac{V'}{V} \left( 1 - \dot{T}^2 \right) = 0 \]

which is equivalent to the entropy conservation equation. Here, the Hubble parameter \( H \) is defined as \( H \equiv \dot{a}/a \),

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and \( V' = dV/dT \). If the stress-energy of the universe is dominated by the tachyon field \( T \), the Einstein field equations for the evolution of the background metric, \( G_{\mu\nu} = \kappa T_{\mu\nu} \), can be written as

\[
H^2 = \frac{\kappa}{3} \frac{V(T)}{\sqrt{1 - T^2}}
\]

(5)

\[
\frac{\dot{a}}{a} = H^2 + \dot{H} = \frac{\kappa}{3} \frac{V(T)}{\sqrt{1 - T^2}} \left( 1 - \frac{3}{2} T^2 \right)
\]

(6)

From Eqs.(5) and (6) we deduce that

\[
\frac{dT}{dt} = -\frac{2}{3} \frac{H'}{H^2(T)}
\]

(7)

and leading to

\[
V^2(T) = \frac{9}{\kappa^2} H^4 - \frac{4}{\kappa^2} H^2
\]

(8)

\[
\frac{a}{a_0} = \exp \left[ -\frac{3}{2} \int_{T_0}^T \frac{H^3}{H'} dT \right]
\]

(9)

\[
t = -\frac{3}{2} \int_{T_0}^T \frac{H^2}{H'} dT
\]

(10)

where \( a_0 \) is the initial value of scale factor during inflation.

Bardeen et.al. [14] have shown that the general form of the metric for the background and scalar perturbation id given by

\[
a^{-2}(\tau) ds^2 = (1 + 2A) dt^2 - 2 \partial_i B dx^i d\tau - \left[ (1 - 2R) \delta_{ij} + 2 \partial_i \partial_j E \right] dx^i dx^j
\]

(11)

The intrinsic curvature perturbation \( R \) of the comoving hypersurfaces can be written as

\[
R = -R - \frac{H}{T} \delta T
\]

(12)

during evolution of the universe, where \( \delta T \) represents the fluctuation of the tachyon field and \( \dot{T} \) and \( H \) are determined by the background field equations Eqs.(7)-(8). On the analogy of discussion for the inflation driven ordinary scalar field with the standard kinetic term[15], we use the gauge-invariant potential

\[
u = a \left[ \delta T + \frac{\dot{T}}{H} R \right] \equiv -z R
\]

(13)

where we have introduced the new variable

\[
z \equiv \frac{a \dot{T}}{H}
\]

(14)

The evolution of the scalar perturbations is calculated by the Einstein action. The first-order perturbation equations of motion are given by a second-order action. Therefore, the gravitational and matter sectors are separated and each expanded to second-order in the perturbations. The action for the matter perturbations can be determined by expanding the Lagrangian as a Taylor series about the background equations and integrating by parts. Note that the inflationary requirement \( \ddot{a} > 0 \) as long as \( T^2 < \frac{1}{2} \). In the chaotic scenario, the inflation will slowly roll down its potential, i.e., \( T^2 \ll \frac{1}{2} \) and \( H^2 \approx \frac{1}{2} \frac{V(T)}{T} \). Hence, the tachyon equation of motion (4) is approximated to the one with the Lagrangian of the normal nearly quadratic form. Furthermore, we can state a small parameter \( \sqrt{\epsilon} = \sqrt{\frac{a^2}{H^2}} \) which suppressed the non-linear contributions. The action of linear scalar perturbation of tachyon field is given by

\[
S = \frac{1}{2} \int d\tau d^3x \left[ (\partial_\tau u)^2 - \delta^{ij} \partial_i u \partial_j u + \frac{\epsilon\tau}{z} u^2 \right]
\]

(15)

Eq.(15) is formally equivalent to the action for a scalar field with the standard kinetic term and a time-dependent effective mass \( m^2 = z \epsilon / z \) in flat space-time. Note that the Eq.(15) is formally the same as the case of ordinary scalar field, but it is new because the definition of variable \( z \) in Eq.(14) is different from the one in Ref.[15]. In fact \( z \) is dependent on the equation of tachyon motion(4). Quantizing \( u(\tau, x) \), we have

\[
\hat{u}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3/2} \left[ u_k(\tau) \hat{a}_k e^{ik \cdot \mathbf{x}} + u_k^*(\tau) \hat{a}_k^* e^{-ik \cdot \mathbf{x}} \right]
\]

\[
\left[ \hat{a}_k, \hat{a}_l^* \right] = \delta^3(\mathbf{k} - \mathbf{l}), \; \hat{a}_k(0) = 0, etc.
\]

(16)

From Eq.(15), the mode functions \( u_k \) satisfy following equation

\[
\frac{d^2 u_k}{d\tau^2} + \left( k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) u_k = 0
\]

(17)

Using Eqs.(4),(7) and (8), it is easy to find that

\[
\frac{1}{2a^2 H^2} \frac{1}{z} \frac{d^2 z}{d\tau^2} = 1 + 4\epsilon(T) - 3\eta(T) + 9\epsilon^2(T) - 14\epsilon(T)\eta(T) + 2\eta^2(T) + \frac{1}{2} \frac{\xi^2(T)}{T}
\]

(18)

where

\[
\epsilon(T) = \frac{2}{3} \left( \frac{H'(T)}{H^2(T)} \right)^2
\]

(19)
\[ \eta(T) = \frac{1}{3} \frac{H''(T)}{H^3(T)} \]  \quad (20)

and

\[ \xi(T) = \frac{2}{3} \left( \frac{H'H'''(T)}{H^6(T)} \right)^{\frac{1}{2}} \]  \quad (21)

Using Eqs.(7)-(10), we can give the exact expression in which the mode function \( u_k \) is related to the field \( T \). One has the boundary conditions

\[ u_k \rightarrow \frac{1}{2k} e^{-ik\tau}, \quad aH \ll k \]  \quad (22)

\[ u_k \propto z, \quad aH \gg k \]  \quad (23)

which guarantees that the perturbation behaves like a free field well inside the horizon and is fixed at super-horizon scales. Eqs.(14)-(17) make a difference between tachyon and ordinary scalar, because the curvature perturbations couple to the stress-energy of tachyon field.

On each scale \( R \) is constant well outside the horizon. Its spectrum is defined by

\[ R = \int \frac{d^3k}{(2\pi)^{3/2}} \mathcal{P}_R(k) e^{ik\cdot x}, \]  \quad (24)

\[ \langle R_k R^-_k \rangle = \frac{2 \pi}{\kappa^3} \mathcal{P}_R \delta^3(k-1), \]  \quad (25)

where \( \mathcal{P}_R(k) \) is the power spectrum. From eq.(25), we have

\[ \mathcal{P}_R \xi^2(k) = \sqrt{\frac{k^3}{2\pi^2}} \left| \frac{u_k}{z} \right| \]  \quad (26)

Furthermore, we find a simple relation as follows

\[ \frac{dz}{d\tau} = -a^2 \left( \frac{2}{3} \right)^{\frac{1}{2}} (1 - 2\eta + 3\epsilon) \]  \quad (27)

The exact inflationary solutions for the mode equations of curvature perturbations might be found from two cases. Type I: we start from the quantity \( z \) is a constant, which is equivalent to requiring \( 1 - 2\eta + 3\epsilon = 0 \). Type II: we also might start from the quantities \( \epsilon(T), \eta(T) \) and \( \xi(T) \) are constants.

### III. THE SOLUTION OF TYPE I

In this case, we should demand that \( H \) satisfies the differential equation

\[ H^4 - \frac{2}{3} HH'' + 2H^2 = 0 \]  \quad (28)

which has the solution

\[ H(T) = \frac{1}{\sqrt{AT + B - \frac{3}{2}T^2}} \]  \quad (29)

where \( A \) and \( B \) are arbitrary integration constants. However, we can chose \( A = 0 \) without loss of generality, as it can be recovered by making a translation of the field, \( T \rightarrow T + \frac{A}{B} \). From Eqs.(8)-(10), we have the corresponding tachyon cosmology,

\[ V(T) = \frac{3}{2} \sqrt{\frac{B - \frac{2}{3}T^2}{(B - \frac{2}{3}T^2)^3}} \]  \quad (30)

\[ \alpha(T) = \frac{a_0 T_0}{T} \]  \quad (31)

The conformal time is

\[ \tau(T) = \left( B - \frac{3}{2}T_0^2 \right)^{\frac{1}{2}} - \left( B - \frac{3}{2}T^2 \right)^{\frac{1}{2}} \]  \quad (32)

\[ + \frac{\sqrt{B}}{2} \ln \left[ \sqrt{B} - \left( B - \frac{3}{2}T_0^2 \right)^{\frac{1}{2}} \right] \sqrt{B + \left( B - \frac{3}{2}T_0^2 \right)^{\frac{1}{2}}} \]  \quad (33)

\[ + \frac{2}{3} a_0 T_0 \arcsin \sqrt{\frac{2}{3} \frac{B}{a_0 T_0}} \arcsin \sqrt{\frac{2}{3} \frac{B}{a_0 T_0}} \]  \quad (34)

It is easy to find that \( \tau \) tends to a constant value at late time, or as \( T \) goes to zero.

For this solution, the cosmological properties are easily derived. As the tachyon field \( T \) goes to zero or infinite, the potential \( V(T) \) tends to non-zero value \( \frac{3}{2} \) or zero. The motion is not inflationary at all time. From Eq.(6), we find that the period of accelerated expansion corresponds to \( T^2 < \frac{2}{3} \) and decelerate otherwise. Thus inflation occurs only when \( |T| < \sqrt{\frac{2}{3}} \). If this model was to produce all the 50 e-foldings of inflation needed to solve the initial conditions problems in the standard model of cosmology, tachyon field must evolve to be close to zero.
Next, we discuss the spectrum of curvature perturbations produced by this model, which is similar to one of the inflation model driven by ordinary scalar field\[16\]. The solution of mode equation (17) is simple,

$$u_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad (34)$$

for the growing mode, after we have imposed the boundary conditions. Since the conformal time \( \tau \) tends to a constant, mode function \( u_k \) is essentially fixed at superhorizon scales. The spectral index \( n_R \) is

$$n_R - 1 \equiv \frac{d \ln \mathcal{P}_R}{d \ln k} = 2 \quad (35)$$

Note that this result is exact and independent of tachyon field \( T \). By now, the inflationary universe is generally recognized to be the most likely scenario that explains the origin of the Big Bang. So far, its predictions of the flatness of the universe and the almost scale-invariant power spectrum of the curvature perturbation that seeds structure formations are in good agreement with the cosmic microwave background (CMB) observations. The key data of CMB are the curvature perturbation magnitude measured by COBE\[17\] and its power spectrum index \( n_R \[18\]

$$\delta_H \simeq 1.9 \times 10^{-5} \quad (36)$$

and

$$|n_R - 1| < 0.1 \quad (37)$$

Therefore, the spectrum of this model is “blue” as it possesses more power at large values of \( k \), or small scales, which is ruled out by the observable universe. However, this model can be used to probe the accuracy of the first and second order approximations for the mode equation.

**IV. THE SOLUTION OF TYPE II**

In this case, we should demand that \( H \) satisfies the differential equations

$$\epsilon(T) = \text{const.}, \quad \eta(T) = \text{const.} \quad \text{and} \quad \xi(T) = \text{const.} \quad (38)$$

which have the solution

$$H(T) = \frac{(\frac{2}{3}n)^\frac{1}{2}}{T - T_0} \quad (39)$$

and

$$\epsilon(T) = \frac{1}{n}, \quad \eta(T) = \frac{1}{n}, \quad \text{and} \quad \xi(T) = \frac{\sqrt{6}}{n} \quad (40)$$

From Eqs. (8)-(10), we have the corresponding tachyon cosmology,

$$V(T) = \frac{2}{k} \left( n^2 - \frac{n}{3} \right) \frac{1}{T - T_0} \quad (41)$$

$$a(T) = a_0(1 - n) \left( \frac{3}{2} n \right)^\frac{1}{2} (T - T_0) \quad (42)$$

$$t(T) = \left( \frac{3}{2} n \right)^\frac{1}{2} (T - T_0) \quad (43)$$

The conformal time is

$$\tau(T) = a_0^{1 - n} \left( \frac{3}{2} n \right)^\frac{1}{2} (T - T_0)^{1 - n} \quad (44)$$

It is easy to find that the condition for inflation is \( n > 1 \), which corresponds to \( T^2 < \frac{4}{n} \). This model actually inflates forever. The model equation (17) is solved in terms of Bessel functions, \( u_k(\tau) = (k\tau)^\frac{1}{2} [C_1 J_{\mu}(k\tau) + C_2 J_{-\mu}(k\tau)] \) where \( C_1 \) and \( C_2 \) are constants fixed from Eq. (22). The spectrum may be calculated exactly to read

$$\mathcal{P}_R(k) = \frac{\kappa}{\pi^2 A^2} 2^{2\mu - 1} \Gamma(\mu) k^{-2\mu} \quad (45)$$

where \( A \) is a constant

$$A = a_0^{\frac{1}{2} - \mu} (n - 1)^{\mu - \frac{1}{2}} \left( \frac{3}{2} n \right)^{\frac{1}{2} n} \quad (46)$$

The corresponding spectral index is

$$n_R = 1 - \frac{4}{n - 1} \quad (47)$$

In particular, we have \( n_R \approx 1 \) for \( n \gg 1 \). In this limit, the solution and slow roll inflation with \( \epsilon = \eta = \frac{1}{n} \) agree at the leading order in \( \epsilon \). Only for \( n > 201, \) i.e. for \( 1 > n_R > 0.98 \), the error of slow roll approximation is less than 1 percent.

We briefly conclude the main result of this paper. We find two family of exact solutions for the mode equation (17) on curvature perturbations of tachyon inflation. This calculation is done to linear order, ignoring both gravity and self-interactions of the tachyon field.
Since the observed anisotropies are small, this approximation is considerably more accurate than the slow-roll approximation, and we need not attempt to go beyond it, though it is possible to extend calculations beyond linear perturbation theory[19]. These models can be probe the accuracy of the first and second order expressions for the curvature perturbation spectra. In fact, almost all analytical predictions for perturbation spectra from inflation rely on the slow roll approximation. Furthermore, the parameter \( \eta \) becomes large near the top of the tachyon potential in type I model, indicating a breakdown of the slow roll assumption. In the cases that \( n \) are not large enough, the slow roll conditions are badly violated by type II model. Therefore, the first order expression does not give good agreement with the exact for all solutions. But we have confidence that we use slow roll approximation in more realistic situations, since the second order expression for the spectral index of curvature perturbation can match the exact result to within 10 percent over most of the inflationary epoch. Finally, we point out that the slow-roll approximation is the assumption that the field evolution is dominated by drag from the expansion is small parameters \( \epsilon = 0 \). We have found the solutions of mode equation are again Hankel function of the first kind \( H^{(1)}_\mu(-k\tau) \) with \( \mu = \frac{3}{2} + 4\epsilon - 2\eta \).

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