MIMO Zero-Forcing Performance Evaluation Using the Holonomic Gradient Method

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Abstract

Infinite-series expressions for performance measures have recently been derived for multiple-input multiple-output (MIMO) spatial-multiplexing transmission with zero-forcing detection (ZF) for Rician–Rayleigh fading, which is relevant in heterogeneous networks. These expressions arise from an expansion around origin of the confluent hypergeometric function \( _1F_1(\cdot, \cdot, \sigma) \). Although, theoretically, this expansion converges for any \( \sigma \), numerical convergence is difficult away from origin, and ensuing ZF performance-measure expressions diverge numerically at practically-relevant Rician \( K \)-factor values. However, it is also known that \( _1F_1(\cdot, \cdot, \sigma) \) satisfies a differential equation with polynomial coefficients, i.e., \( _1F_1(\cdot, \cdot, \sigma) \) is a holonomic function. This enables herein computation using the holonomic gradient method (HGM), i.e., by numerically solving the satisfied differential equation. First, we reveal that the moment generating function (m.g.f.) and probability density function (p.d.f.) of the ZF signal-to-noise ratio (SNR) are holonomic. Then, from the differential equation for \( _1F_1(\cdot, \cdot, \sigma) \), we deduce those satisfied by the SNR m.g.f. and p.d.f., and demonstrate that the HGM computes the p.d.f. accurately at practically-relevant values of \( K \). Finally, numerical integration of the SNR p.d.f. output by the HGM yields accurate ZF outage probability and ergodic capacity evaluation.

Index Terms

Holonomic gradient method, hypergeometric function, MIMO, Rayleigh–Rician fading, zero-forcing.

I. INTRODUCTION

A. Background, Previous Work, and Motivation

The performance evaluation for multiple-input multiple-output (MIMO) wireless communications systems has been attracting substantial interest [1] [2] [3] [4] [5]. Typically, this evaluation proceeds from expressions of performance measures (e.g., average error probability (AEP), outage probability, ergodic capacity) derived based on statistical assumptions about the channel-fading matrix [1] [2] [4]. However, MIMO analyses have often assumed zero-mean channel matrix, i.e., Rayleigh fading [2], for tractability, although state-of-the-art channel measurements and models, e.g., WINNER II [6], have revealed nonzero-mean channel, i.e., Rician fading [2]. MIMO performance analysis for Rician fading is complicated by the ensuing noncentral Wishart matrix distribution [4] [7] [8]. Then, even for linear, i.e., low-complexity, interference-mitigation approaches such as zero-forcing detection (ZF), the performance analysis of MIMO spatial-multiplexing transmission is much less tractable than for Rayleigh fading [8]. Nevertheless, with the advent of the massive-MIMO concept, it is likely that low-complexity detection methods such as ZF will remain practically and analytically relevant [9] [10].

Thus, we have recently analyzed ZF in [8] [11] for Rician–Rayleigh fading, i.e., when the intended Stream 1 undergoes Rician fading whereas the interfering streams undergo Rayleigh fading, which is relevant in macrocells, microcells, and heterogeneous networks — see [4] [8] and references therein. Then, [8] derived exact infinite-series expressions for ZF performance measures, and [11] proved that they converge everywhere. The performance-evaluation procedure employed in [8] for MIMO ZF under Rician–Rayleigh fading is as follows. First, we expressed the moment generating function (m.g.f.) of the

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signal-to-noise ratio (SNR) for Stream 1 in terms of the confluent hypergeometric function \( {}_1F_1(\cdot, \cdot, \sigma) \) with scalar argument \( \sigma \) [12 Ch. 13]. Then, the well-known expansion of \( {}_1F_1(\cdot, \cdot, \sigma) \) around the origin from [12 Eq. (13.2.2), p. 322] yielded, upon inverse-Laplace transformation, the infinite-series expression for the SNR probability density function (p.d.f.) from [8 Eq. (39)]. Finally, integration of this p.d.f. expression yielded the infinite-series expressions for ZF outage probability and ergodic capacity in [8 Eqs. (69), (71)].

However, for these infinite series, computation by truncation breaks down at \( K \) values that are much smaller than those considered realistic (e.g., averages of lognormal distributions proposed by WINNER II for \( K \) in [6]) [11 Figs. 4, 5]. This is because, as mentioned above, these series were derived in [8] from the expansion around the origin of \( {}_1F_1(\cdot, \cdot, \sigma) \) from [12 Eq. (13.2.2), p. 322] whose own computation by truncation is nontrivial: with increasing \( \sigma \), numerical convergence is increasingly difficult (i.e., slow, resource-intensive) and eventually fails — see [13, 14, 15, 16] and references therein. For example, Kim et al. have recently found in [13 Section IV] for a MIMO-precoding application that the \( {}_1F_1(\cdot, \cdot, \sigma) \) series requires truncation after as many as 70 terms for the reasonably-accurate computation of their SNR p.d.f..

Finally, the fact that hypergeometric functions\(^1\) characterize the performance for many other MIMO techniques under many fading types [2, 4, 15, 16], further motivates our seeking herein approaches that can compute reliably performance measures ensuing from such functions.

B. New Approach and Contribution

A little-known approach to computing \( {}_1F_1(\cdot, \cdot, \sigma) \) follows from the fact that this function satisfies, with respect to \( \sigma \), a linear differential equation with polynomial coefficients [12 Eq. (13.2.1), p. 322], i.e., this function is holonomic [17, p. 334] [18, p. 7] [19, p. 140] [20, Section 6.4]. Thus, holonomic functions can be computed at some \( \sigma \) by numerically solving their differential equation starting from some \( \sigma_0 \) where the function is either known analytically, e.g., \( {}_1F_1(\cdot, \cdot, 0) = 1 \), or can be approximated accurately, e.g., \( {}_1F_1(\cdot, \cdot, \sigma_0 \approx 0) \) from its infinite series. This computation approach is known as the holonomic gradient method (HGM) [21] [22]. It has recently been applied in statistics to evaluating the normalizing constant of the Bingham distribution [21] and the cumulative distribution function (c.d.f.) of the dominant eigenvalue of a real-valued Wishart-distributed matrix [22]. To the best of our knowledge, the HGM has not yet been applied specifically for MIMO performance evaluation.

In the current paper, we apply HGM for the evaluation of MIMO ZF under Rician–Rayleigh fading, as follows. Starting from the SNR m.g.f. expression derived in terms of \( {}_1F_1(\cdot, \cdot, \sigma) \) in [8 Eq. (31)] and the differential equation satisfied by \( {}_1F_1(\cdot, \cdot, \sigma) \) [12 Eq. (13.2.1), p. 322], we first deduce the differential equations satisfied by the m.g.f. Then, inverse-Laplace transformation yields the differential equations satisfied by the p.d.f. They are shown to enable the accurate HGM-based computation of the SNR p.d.f., starting from an initial value computed accurately by truncating the available infinite series for the p.d.f. [8 Eq. (39)]. Finally, numerical integration of the HGM output yields accurately, for the first time, the outage probability and ergodic capacity for MIMO ZF at \( K \) values relevant to WINNER II.

C. Paper Organization

Section II describes the MIMO signal, noise, and channel models. Section III introduces the SNR m.g.f. and p.d.f. infinite-series expressions derived in [8]. Section IV discusses difficulties encountered in the truncation-based computation of the infinite-series expression for \( {}_1F_1(\cdot, \cdot, \sigma) \) and of the ensuing infinite-series expression for the ZF SNR p.d.f. Section V defines holonomic functions and deduces from their properties that the SNR m.g.f. and p.d.f. are holonomic. This justifies our search in Section VI for the differential equations they satisfy. These differential equations are exploited using the HGM to generate the numerical results shown and discussed in Section VII. Finally, Section VIII discusses other possible

\(^1\)Of both scalar and matrix arguments.
HGM applications in MIMO evaluation. Note that in this paper we employ the notation introduced in [8, Section I.D].

II. SIGNAL, NOISE, AND FADING MODELS

Herein, the signal, noise, and channel models and assumptions follow closely the ones from [8]. Thus, we consider uncoded MIMO spatial-multiplexing over a frequency-flat fading channel and assume that there are \( N_T \) and \( N_R \) antenna elements at the transmitter(s) and receiver, respectively, with \( N_T \leq N_R \). Let us denote the number of degrees of freedom as

\[
N = N_R - N_T + 1.
\]  

(1)

Letting \( \mathbf{x} = [x_1 \ x_2 \ \cdots \ x_{N_T}]^T \) denote the \( N_T \times 1 \) zero-mean transmit-symbol vector with \( \mathbb{E}\{\mathbf{x}\mathbf{x}^H\} = \mathbf{I}_{N_T} \), the \( N_R \times 1 \) vector with the received signals can be represented as \([\mathbf{I}, \text{Eq. (8)}] [\mathbf{8}, \text{Eq. (1)}]:\)

\[
\mathbf{r} = \sqrt{\frac{E_s}{N_T}} \mathbf{H} \mathbf{x} + \mathbf{v} = \sqrt{\frac{E_s}{N_T}} \mathbf{h}_1 x_1 + \sqrt{\frac{E_s}{N_T}} \sum_{k=2}^{N_T} \mathbf{h}_k x_k + \mathbf{v}.
\]  

(2)

Above, \( E_s/N_T \) represents the energy transmitted per symbol (i.e., per antenna), so that \( E_s \) is the energy transmitted per channel use. The additive noise vector \( \mathbf{v} \) is zero-mean, uncorrelated, circularly-symmetric complex-valued Gaussian \( \mathcal{CN}(0, \mathbf{I}_{N_R}) \), and \( \mathbf{h}_k \) comprises the channel factors between transmit-antenna \( k \) and all receive-antennas. The deterministic (i.e., mean) and random components of \( \mathbf{H} \) are denoted as \( \mathbf{H}_d = (\mathbf{h}_d,1 \ \mathbf{h}_d,2 \ \cdots \ \mathbf{h}_d,N_T) \) and \( \mathbf{H}_r = (\mathbf{h}_r,1 \ \mathbf{h}_r,2 \ \cdots \ \mathbf{h}_r,N_R) \), respectively, so that \( \mathbf{H} = \mathbf{H}_d + \mathbf{H}_r \). If \( |\mathbf{H}_{d,i,j}| = 0 \) then \( |\mathbf{H}_{i,j}| \) has a Rayleigh distribution; otherwise, \( |\mathbf{H}_{i,j}| \) has a Rician distribution \([2]\). Typically, the channel matrix for Rician fading is written as \([1]\)

\[
\mathbf{H} = \mathbf{H}_d + \mathbf{H}_r = \sqrt{\frac{K}{K + 1}} \mathbf{H}_{d,n} + \sqrt{\frac{1}{K + 1}} \mathbf{H}_{r,n},
\]  

(5)

where, for normalization purposes \([23]\), it is assumed that \( \|\mathbf{H}_{d,n}\|^2 = N_T N_R \) and \( \mathbb{E}\{|\mathbf{H}_{r,n}\|^2\} = 1, \forall i, j \), so that \( \mathbb{E}\{\|\mathbf{H}\|^2\} = N_T N_R \). Power ratio

\[
K = \frac{\|\mathbf{H}_{d,n}\|^2}{\mathbb{E}\{\|\mathbf{H}_{r,n}\|^2\}} = \frac{K}{K + 1} \frac{\|\mathbf{H}_{d,n}\|^2}{\mathbb{E}\{\|\mathbf{H}_{r,n}\|^2\}}
\]  

(6)

is the Rician \( K \)-factor: \( K = 0 \) yields Rayleigh fading for all elements of \( \mathbf{H} \); \( K \neq 0 \) yields Rician fading if \( \mathbf{H}_{d,n} \neq 0 \).

In \([8]\), we viewed the channel matrix as partitioned into the column that affects the intended stream, i.e., Stream 1, and the matrix whose columns affect the interfering streams, i.e.,

\[
\mathbf{H} = (\mathbf{h}_1 \ \mathbf{H}_2) = (\mathbf{h}_{d,1} \ \mathbf{H}_{d,2}) + (\mathbf{h}_{r,1} \ \mathbf{H}_{r,2}),
\]  

(7)

and assumed, for analysis tractability, that \( \mathbf{h}_{d,1} \neq 0 \) and \( \mathbf{H}_{d,2} = 0 \), i.e., Rician–Rayleigh fading. Then, we can write

\[
\|\mathbf{h}_{d,1}\|^2 = \|\mathbf{h}_{d,1} \ 0_{N_R \times (N_T - 1)}\|^2 = \|\mathbf{H}_d\|^2 = \frac{K}{K + 1} N_R N_T.
\]  

(8)
As in [24], for tractability, we assume zero receive-correlation and we allow for nonzero transmit-correlation whereby all the conjugate-transposed rows of \( \mathbf{H}_{n} \) have the same distribution \( \mathcal{CN}(0, \mathbf{R}_{T}) \), with \( [\mathbf{R}_{T}]_{i,i} = 1, \forall i = 1 : N_{T} \). Thus, all conjugate-transposed rows of \( \mathbf{H} \) have distribution \( \mathcal{CN}(0, \mathbf{R}_{T,K} = \frac{1}{K+1} \mathbf{R}_{T}) \). The elements of \( \mathbf{R}_{T} \) can be computed from the azimuth spread (AS) as shown in [26, Section VI.A] for WINNER II, i.e., Laplacian, power azimuth spectrum. Note that WINNER II has modeled both \( K \) (in dB) and AS (in degrees) as random variables with scenario-dependent lognormal distributions [6].

### III. Infinite-Series Expressions for MIMO ZF M.G.F. and P.D.F.

For the received signal vector from (2), ZF means mapping the elements of the following vector into the closest modulation constellation symbols [1, Eq. (22)]:

\[
\sqrt{\frac{N_{T}}{E_{s}}} \mathbf{H}^{H}\mathbf{H}^{-1}\mathbf{H}^{H}\mathbf{r} = \mathbf{x} + \frac{1}{\sqrt{\Gamma_{s}}} \mathbf{H}^{H}\mathbf{v}_{n}. \tag{9}
\]

Then, the ZF SNR for Stream 1 is given by

\[
\gamma_{1} = \frac{\Gamma_{s}}{[\mathbf{H}^{H}\mathbf{H}^{-1}]_{1,1}}. \tag{10}
\]

The m.g.f. of \( \gamma_{1} \) is defined as [2, Eq. (1.2)]

\[
M_{\gamma_{1}}(s, a) = \mathbb{E}\{ e^{s\gamma_{1}} \} = \int_{0}^{\infty} e^{st} p_{\gamma_{1}}(t, a) dt, \tag{11}
\]

where \( \mathbb{E}\{\cdot\} \) stands for mean, and \( p_{\gamma_{1}}(t, a) \) is the p.d.f. of \( \gamma_{1} \). Thus, the m.g.f. is related to the Laplace transform [12, Eq. (1.14.17)] \( L_{\gamma_{1}}(s, a) \) of the p.d.f. simply by a sign change, i.e.,

\[
M_{\gamma_{1}}(-s, a) = L_{\gamma_{1}}(s, a) = \int_{0}^{\infty} e^{-st} p_{\gamma_{1}}(t, a) dt. \tag{12}
\]

For MIMO ZF under Rician–Rayleigh fading we have recently obtained the following exact expression for the m.g.f. of the SNR of the Rician-fading Stream 1 [8, Eq. (31)]:

\[
M_{\gamma_{1}}(s, a) = \frac{1}{(1 - \Gamma_{1}s)^{N}} {}_{1}F_{1}\left(N; N_{R}; a \frac{\Gamma_{1}s}{1 - \Gamma_{1}s} \right), \tag{13}
\]

where \( {}_{1}F_{1}(N; N_{R}; \sigma) \) is the confluent hypergeometric function of scalar argument \( \sigma \) [12, Ch. 13], and

\[
\Gamma_{1} = \frac{\Gamma_{s}}{[\mathbf{R}_{T,K}]_{1,1}} \propto \frac{\Gamma_{s}}{K+1}, \tag{14}
\]

\[
a = [\mathbf{R}_{T,K}]_{1,1} \|\mathbf{h}_{d,1}\|^{2} \propto KN_{R}N_{T}. \tag{15}
\]

The confluent hypergeometric function has the infinite-series expression [12, Eq. (13.2.2), p. 322]

\[
{}_{1}F_{1}(N; N_{R}; \sigma) = \sum_{n=0}^{\infty} \frac{(N)_{n} \sigma^{n}}{(N_{R})_{n}} \frac{\Gamma_{s}}{n!} \sum_{n=0}^{\infty} A_{n}(\sigma), \tag{16}
\]

where \( (N)_{n} \) is the Pochhammer symbol, i.e., \((N)_{0} = 1 \) and \((N)_{n} = N(N + 1) \ldots (N + n - 1), \forall n \geq 1 \) [12, p. xiv]. A proof of the fact (important herein) that expression (16) is the expansion of \( {}_{1}F_{1}(N; N_{R}; \sigma) \) around \( \sigma = 0 \) is provided, for completeness, in Appendix I.

Using (16), we have shown that (13) can be written as the infinite series [8, Eq. (37)]

\[
M_{\gamma_{1}}(s, a) = \sum_{n=0}^{\infty} A_{n}(a) \sum_{m=0}^{n} \binom{n}{m} (-1)^{m} \frac{1}{(1 - s\Gamma_{1})^{N+n-m}}. \tag{17}
\]
Then, the SNR p.d.f. is given by the following infinite series [8 Eq. (39)]

\[
p_{\gamma_1}(t, a) = \sum_{n=0}^{\infty} A_n(a) \sum_{m=0}^{n} \binom{n}{m} (-1)^m t^{n+m-1} e^{-t/T_1} \frac{t^{N-n-m} \Gamma_1^{N+n-m}}{((N+n-m)-1) \Gamma_1^{N+n-m}}.
\]  

(18)

For the special case with \(N = 1\), i.e., for \(N_R = N_T\), the above becomes

\[
p_{\gamma_1}(t, a) = \frac{e^{-t/T_1}}{\Gamma_1} \sum_{n=0}^{\infty} A_n(a) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{t^{n-m}}{(n-m)! \Gamma_1^{n-m}},
\]

which yields

\[
\lim_{t \to 0, t > 0} p_{\gamma_1}(t, a) = p_{\gamma_1}(0+, a) = \frac{1}{\Gamma_1} \sum_{n=0}^{\infty} A_n(a) (-1)^n = \frac{1}{\Gamma_1} \sum_{n=0}^{\infty} \frac{(N_R)_n}{(N)_n} (-a)^n.
\]

(19)

Thus, (19), (16), and (18) yield:

\[
p_{\gamma_1}(0+, a) = \begin{cases} \frac{1}{\Gamma_1} \gamma_1 F_1(N; N_R; -a), & N = 1 \\ 0, & N > 1. \end{cases}
\]

(20)

For the special case of Rayleigh-only fading, \(K = 0\) yields \(a = 0\) from (15), and then only the term for \(n = m = 0\) remains in (17) and (18), i.e.,

\[
M_{\gamma_1}(s, 0) = \frac{1}{(1 - s \Gamma_1)^N},
\]

(21)

\[
p_{\gamma_1}(t, 0) = \frac{t^{N-1} e^{-t/T_1}}{(N-1)! \Gamma_1^N}, \quad t \geq 0,
\]

(22)

so that the ZF SNR is gamma-distributed [24] [25].

On the other hand, for \(K \neq 0\), i.e., for Rician–Rayleigh fading, (17) and (18) reveal that the distribution of the ZF SNR is an infinite linear combination of gamma distributions. These m.g.f. and p.d.f. expressions have yielded the infinite-series expressions in [8 Eqs. (68), (71)], respectively, for the outage probability and ergodic capacity, which are defined as follows:

\[
P_0(\gamma_1, \sigma) = \text{Probability}(\gamma_1 \leq \gamma_{1, \sigma}) = \int_{0}^{\gamma_{1, \sigma}} p_{\gamma_1}(t, a) dt,
\]

(23)

\[
C(a) = \mathbb{E}_{\gamma_1} \{ C(\gamma_1, a) \} = \int_{0}^{\infty} \log_2 (1 + t) p_{\gamma_1}(t, a) dt.
\]

(24)

In [8] we have integrated analytically, based on (23) and (24), the infinite-series expression of \(p_{\gamma_1}(t)\) from (18) and obtained for the outage probability and ergodic capacity the infinite-series expressions from [8 Eqs. (69), (71)].

IV. CONVENTIONAL COMPUTATION OF \(_{\text{iF}_1} (\cdot; \cdot; \sigma)\)

A. Computation of Infinite-Series Expansion of \(_{\text{iF}_1} (\cdot; \cdot; \sigma)\)

Recall that the ZF SNR expression (13) depends on the confluent hypergeometric function \(_{\text{iF}_1} (\cdot; \cdot; \sigma)\). Conventionally, this function has been expressed as the infinite series given in (16), which is the result of expansion around \(\sigma = 0\), as detailed in Appendix [1]. Then, the truncation of this series has conventionally been employed to compute (i.e., accurately approximate) \(_{\text{iF}_1} (\cdot; \cdot; \sigma)\).

However, computing and adding one-by-one each term \(A_n(a)\) of (16) is not efficient. It also encounters numerical instability at relatively-low values of \(\sigma\) [14]. This is because, with increasing \(\sigma\), terms for higher

\[\text{In [23], } \gamma_{1, \sigma} \text{ is the threshold SNR.} \]
Accurate results for larger $\sigma$ can be computed more efficiently through recursive methods, e.g., [14, Method 1]:

- Starting from $A_0(\sigma) = 1$, recursively update $A_n(\sigma)$ with:
  $$A_n(\sigma) = A_{n-1}(\sigma) \frac{N + n - 1}{N_R + n - 1} \frac{\sigma}{n}, \quad n = 1, 2, 3, \ldots \quad (25)$$

- Stop at sufficiently large $n = n_{\text{max}}$ (to avoid excessive computation time or numerical instability caused by the inaccuracy in representing large numbers) or when
  $$\left| \frac{A_n(\sigma)}{\sum_{i=0}^{n} A_i(\sigma)} \right| \leq \xi, \quad (26)$$

where $\xi$ is the tolerance level. Then, $_{1}F_{1}(N; N_R; \sigma) \approx \sum_{i=0}^{n} A_i(\sigma)$.

Nevertheless, such recursive methods converge slowly and also incur numerical instability for sufficiently-large $\sigma$ [14].

We have implemented in MATLAB the recursive series-approximation method described above, for $n_{\text{max}} = 150$ and $\xi = 10^{-15}$. We have also employed the MATLAB function hypergeom, whose implementation details are inaccessible. Fig. 1 depicts the obtained results, along with HGM results described later. First, notice that all methods break down for sufficiently-large argument (i.e., $\sigma \approx 700$), which is because the value of the function becomes larger than the maximum value representable in MATLAB ($\approx 10^{308}$). Then, the results obtained by series truncation diverge significantly from the true value for $\sigma > 200$, which is because the series in (16) is an expansion around $\sigma = 0$, as discussed in [14] [15].

### B. Closed Form and Infinite Series for $_{1}F_{1}(N; N_R; \sigma)$

On the one hand, $_{1}F_{1}(\alpha; \beta; \sigma)$ can be represented, when $\alpha \leq \beta$ are positive integers, as a combination of two finite series [8 Eq. (35)]. Unfortunately, this closed-form expression for $_{1}F_{1}(N; N_R; \sigma)$ yields for the
MIMO ZF SNR m.g.f. a closed-form expression [8 Eq. (36)] that cannot be Laplace-inverted to express the p.d.f. in terms of finite series.

On the other hand, the infinite-series expression for $iF_1(N; N_R; \sigma)$ from (16) has yielded the infinite-series expression for the MIMO ZF SNR m.g.f. from (17), which, in turn, has readily yielded the infinite-series p.d.f. expression in (18). Unfortunately, the difficulties described above for the computation of $iF_1(N; N_R; \sigma)$ from its infinite series [16] also affect the computation of the infinite-series p.d.f. expression from (18), as demonstrated next.

C. Difficulties Computing the SNR P.D.F. from Infinite Series

As we have detailed in [8] [11], the infinite-series expression (18) cannot be computed accurately, or even at all, for large values of $a$ (i.e., $K$), by truncation. This is also illustrated in Fig. 2 for $N_R = 6$, $N_T = 2$, and AS = 51°, which is the average AS for WINNER II scenario A1, i.e., indoor office/residential [26, Table I] [6]. For Rayleigh-only fading, i.e., for $K = 0$, there is agreement between results (identified in legend with Ray-Ray) obtained from series (18) — which reduces to the single term from (22) — and from simulation.

On the other hand, for Rician–Rayleigh fading with $K = 7$ dB (identified in legend with Rice-Ray), the results from series (18) do not match the simulations. The value of $K$ set for this experiment is not arbitrary: it is the average of the lognormal distribution of measured $K$ for WINNER II indoor (office, residential) scenario A1 [26, Table I] [6]. Thus, accurately computing $p_{\gamma_1}(t)$ for $K = 7$ dB has practical relevance. However, for $N_R = 6$, $N_T = 2$, we have been able to accurately compute $p_{\gamma_1}(t)$, and, thus, the outage probability and ergodic capacity, only up to $K \approx 1.5$ dB, as depicted in [11, Fig. 2]. Increasing $N_T$ to 6 decreases the value of $K$ that yields accurate results to $\approx 3$ dB [11, Figs. 1, 2]. (For $N_R = N_T = 4$ accurate results can be obtained only up to $K \approx 1.2$ dB as depicted and explained in [8 Figs. 7-9] [11, Figs. 3-5].)

Furthermore, even at low $K$ values, truncating series (18) for $p_{\gamma_1}(t, a)$ may not always be accurate or may even break down at large values of $t$, because the alternating sign, i.e., $(-1)^m$, from the sum over $m$ in (18) causes numerical instability — see discussion of [14, Eq. (2.7)].

Since series truncation cannot help compute ZF performance measures for relevant SNR and parameter values, we pursue next an HGM-based approach.

V. Holonomic Functions and the Holonomic Gradient Method (HGM)

A. Differential Equation and HGM for $iF_1(N; N_R; \sigma)$ and Holonomicity

It is known that $iF_1(N; N_R; \sigma)$ satisfies the second-order ordinary differential equation with polynomial coefficients [12, Eq. (13.2.1), p. 322]

$$\sigma \cdot iF_1^{(2)}(N; N_R; \sigma) + (N_R - \sigma) \cdot iF_1^{(1)}(N; N_R; \sigma) - N \cdot iF_1(N; N_R; \sigma) = 0,$$

where $iF_1^{(k)}(N; N_R; \sigma)$ stands for the $k$th derivative w.r.t. the sole variable $\sigma$.

A function is holonomic if it satisfies, w.r.t. each variable, an ordinary differential equation with polynomial coefficients [21, Section 2]. Thus, the confluent hypergeometric function $iF_1(N; N_R; \sigma)$ is holonomic because it satisfies (27). Simpler examples of holonomic functions are the polynomial and exponential-polynomial functions [17, Section 2.5].

Now, (27) can be recast as the system of differential equations

$$\partial_\sigma \begin{pmatrix} iF_1(N; N_R; \sigma) \\ iF_1^{(1)}(N; N_R; \sigma) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{N}{\sigma} & 1 - \frac{N_R}{\sigma} \end{pmatrix} \begin{pmatrix} iF_1(N; N_R; \sigma) \\ iF_1^{(1)}(N; N_R; \sigma) \end{pmatrix}.$$  

(28)

If we denote the vector made of functions $iF_1(N; N_R; \sigma)$ and $iF_1^{(1)}(N; N_R; \sigma)$ as $\mathbf{f}(\sigma)$, and the $2 \times 2$ matrix on the right-hand side of (28) as $\mathbf{F}(\sigma)$, then (28) can be written simply as

$$\partial_\sigma \mathbf{f}(\sigma) = \mathbf{F}(\sigma)\mathbf{f}(\sigma),$$

(29)
Fig. 2. P.d.f. of the SNR (in linear units) for Stream 1, for $N_T=2, N_R=6, K=7\, \text{dB}, \ AS=51^\circ, \ \Gamma_s=5\, \text{dB}, \ \text{n}_{\text{max}}=149, \ Z_F$.

whose left-hand side is the gradient of $f(\sigma)$ w.r.t. the sole variable $\sigma$.

Let us assume that initial conditions $1F_1(N; N_R; \sigma_0)$ and $1F_1^{(1)}(N; N_R; \sigma_0)$ are known for some $\sigma_0$. Then, $1F_1(N; N_R; \sigma)$ can be computed for any $\sigma$ by numerically solving\[^{29}\] between $\sigma_0$ and $\sigma$. However, because $\sigma$ appears in $F(\sigma)$ denominators, one cannot use $\sigma_0 = 0$, for which it is known analytically, from [16] and [12, Eq. (13.3.15), p. 325], that $1F_1(N; N_R; 0) = 1$ and $1F_1^{(1)}(N; N_R; 0) = \frac{N}{N_R}$. Thus, initial conditions $1F_1(N; N_R; \sigma_0)$ and $1F_1^{(1)}(N; N_R; \sigma_0) = \frac{N}{N_R}1F_1(N+1; N_R+1; \sigma_0)$ [12, Eq. (13.3.15), p. 325] have to be obtained numerically by truncation of series [16] — see Section 4V-A — for some $\sigma_0 > 0$. Nevertheless, since $\sigma_0$ can be selected arbitrarily small, highly-accurate computation of the initial condition $f(\sigma_0)$ is possible.

The entire procedure is summarized below [21, Section 2.1]:

- First, compute accurate initial conditions, i.e., $1F_1(N; N_R; \sigma_0)$ and $1F_1^{(1)}(N; N_R; \sigma_0)$, for some sufficiently-small $\sigma_0 > 0$, by infinite-series truncation.
- Then, solve numerically the system of differential equations [29] between $\sigma_0$ and $\sigma$.

This procedure is referred to as the holonomic gradient method [21, 22] (HGM) because, after starting from an initial condition, the holonomic function is computed by updating its gradient, e.g., through the Runge-Kutta method [12, Section 3.7].

For example, the HGM approach starting at $\sigma_0 = 10^{-15}$ has yielded the results identified with HGM in Fig. 1. They agree with the results from the MATLAB hypergeom function. However, at large $\sigma$, the HGM has yielded accurate results orders-of-magnitude faster than hypergeom.

B. ZF SNR M.G.F. and P.D.F. Are Holonomic Functions

Holonomic functions satisfy the following properties:

- If $f(x)$ is a polynomial then $1/f(x)$ is holonomic [17, Prop. 2.1].
- If $f(x)$ is holonomic and $h(x)$ is rational then $f(h(x))$ is holonomic [18, Th. 1.4.2, p. 16].

\[^{3}\text{E.g., with the ode function, in MATLAB.}\]
If \( f(x) \) and \( g(x) \) are holonomic then \( f(x)g(x) \) is holonomic [17, Prop. 3.2].

If \( f(x) \) is holonomic then its Fourier transform is holonomic [17, p. 337].

Based on these properties, in the SNR m.g.f. expression from (13), the term \( 1/(1 - \Gamma_1 s)^N \) is holonomic w.r.t. \( s \), and the term \( _1F_1 \left( N; N_R; \frac{as}{1 - s} \right) \) is holonomic w.r.t. \( s \) and \( a \), yielding the following property.

**Lemma 1:** The SNR m.g.f. \( M_{\gamma_1}(s,a) \) is holonomic w.r.t. both \( s \) and \( a \), i.e., it must satisfy ordinary differential equations with polynomial coefficients w.r.t. both \( s \) and \( a \). Also, the SNR p.d.f. \( p_{\gamma_1}(t,a) \) is holonomic w.r.t. both \( t \) and \( a \), i.e., it must satisfy ordinary differential equations with polynomial coefficients w.r.t. both \( t \) and \( a \).

Therefore, the remainder of this work is devoted to:

- Deducing the differential equations (known to exist) satisfied by \( M_{\gamma_1}(s,a) \), w.r.t. both \( s \) and \( a \), as well as by \( p_{\gamma_1}(t,a) \), w.r.t. both \( t \) and \( a \).
- Exploiting the differential equations for the p.d.f. \( p_{\gamma_1}(t,a) \) to accurately compute it by the HGM at practically-relevant values of \( K \).
- Integrating numerically as in (23), (24) to obtain the outage probability and ergodic capacity.

**VI. DIFFERENTIAL EQUATIONS FOR ZF SNR M.G.F. AND P.D.F.**

**A. M.G.F. and P.D.F. Variable Scaling**

In order to simplify notation and derivations hereafter, let us denote the m.g.f. \( M_{\gamma_1}(s,a) \) and the p.d.f. \( p_{\gamma_1}(t,a) \) for \( \Gamma_1 = 1 \) as \( M(s,a) \) and \( p(t,a) \), respectively. Now, by definition, we have

\[
M_{\gamma_1}(s,a) = \mathbb{E}\{e^{st_{\gamma_1}}\} = \int_0^\infty e^{st}p_{\gamma_1}(t,a)\,dt, \tag{30}
\]

\[
M(s,a) = \int_0^\infty e^{st}p(t,a)\,dt. \tag{31}
\]

Then, because

\[
M_{\gamma_1}(s,a) = M(s\Gamma_1,a) = \int_0^\infty e^{s\Gamma_1 t}p(t,a)\,dt = \int_0^\infty e^{sy} \frac{1}{\Gamma_1} p\left( \frac{y}{\Gamma_1},a \right) dy,
\]

the p.d.f. \( p_{\gamma_1}(t,a) \) of \( \gamma_1 \) for any \( \Gamma_1 \) can be obtained from \( p(t,a) \) as follows:

\[
p_{\gamma_1}(t,a) = \frac{1}{\Gamma_1} p\left( \frac{t}{\Gamma_1},a \right). \tag{32}
\]

Thus, below, we first derive differential equations for \( M(s,a) \) w.r.t. both \( s \) and \( a \). From them we then deduce differential equations for \( p(t,a) \) w.r.t. both \( t \) and \( a \). They will help compute, by HGM, the function \( p(t,a) \) at desired values of \( t \) and \( a \) (i.e., \( K \)). Finally, the scaling transformation from (32) will return the value of the SNR p.d.f. \( p_{\gamma_1}(t,a) \), for any \( \Gamma_1 \).

**B. Differential Equation w.r.t. \( s \) for \( M(s,a) \)**

Based on (13) and (30) we can write

\[
M(s,a) = \frac{1}{(1-s)^N} _1F_1 \left( N; N_R; \frac{as}{1 - s} \right). \tag{33}
\]

In Appendix II manipulation and differentiation w.r.t. \( s \) of (33) followed by substitution into the differential equation for \( _1F_1 \left( N; N_R; \sigma \right) \) from (27) have yielded the following differential equation w.r.t. \( s \) for \( M(s,a) \), in (76):

\[
(s(1-s)^2\partial_s^2 - [2(N+1)s(1-s)-(1-s)N_R+as]\partial_s + N[(N+1)s-N_R-a])M(s,a) = 0. \tag{34}
\]
Because \( s^l \) appears in front of \( \partial_s^k \) in (34), the corresponding differential equation for \( p(t, a) \) cannot be obtained by inverse-Laplace transform. Therefore, we shall first employ the following order-changing rule, which can readily be deduced from [20] Th. 6.1.2 (Liebniz Formula), p. 282] [27] Th. 1.1.1, p. 3.

**Proposition 1:**

\[
s^l \partial^k_s = \sum_{m=0}^{\min(l,k)} \frac{(-1)^m}{m!} l(l-1) \ldots (l-m+1) \frac{1}{[k(k-1) \ldots (k-m+1)]} \partial^{k-m}_s s^{l-m}.
\]  

(35)

From (35) we obtain the following particular rules

\[
s \partial_s = \partial_s s - 1,
\]

(36)

\[
s \partial^2_s = \partial^2_s s - 2 \partial_s,
\]

(37)

\[
s^2 \partial_s = \partial_s s^2 - 2s,
\]

(38)

\[
s^2 \partial^2_s = \partial^2_s s^2 - 4 \partial_s s + 2,
\]

(39)

\[
s^3 \partial^2_s = \partial^2_s s^3 - 6 \partial_s s^2 + 6s,
\]

(40)

which, when applied in (34), yield for \( M(s, a) \) the following differential equation w.r.t. \( s \):

\[
[\partial^2_s s^3 - 2 \partial^2_s s^2 + \partial^2_s s + (2N - 4) \partial_s s^2 + (6 - 2N - N_R - a) \partial_s s + (N_R - 2) \partial_s s^2] M(s, a) = 0.
\]

Unlike Eq. (34), Eq. (41) can be employed based on the Laplace transform to deduce the differential equation w.r.t. \( t \) for \( p(t, a) \), as shown next.

**C. Differential Equation w.r.t. \( t \) for \( p(t, a) \)**

The following proposition helps transform an expression whereby the operator \( \partial^k_s \) is applied to the product \( s^l M(s, a) \) into a differential equation for \( p(t, a) \) w.r.t. \( t \). Hereafter, \( p^{(l)}(t, a) \) stands for the \( l \)th derivative w.r.t. \( t \) of \( p(t, a) \).

**Proposition 2:** The integral \( \int_0^\infty e^{st} [t^k p^{(l)}(t, a)] dt \), which represents the Laplace transform of \( t^k p^{(l)}(t, a) \) for argument \(-s\), is given by:

\[
\left\{ (-1)^l \partial^l_s [s^l M(s, a)] + \sum_{m=k+1}^{l} (-1)^m p^{(l-m)}(0+, a) \frac{(m-1)!}{(m-k-1)!} s^{m-k-1}, \quad l \geq 1,
\right.

\[
\left. \partial^k_s M(s, a), \quad l = 0. \right\}
\]  

(42)

**Proof:** Follows from the well-known Laplace-transform property for higher-order derivatives from [12] Eq. (1.14.29), p. 28] and the sign change from [12].

Applying (42) for the terms in (41) yields the following Laplace-transform pairs:

\[
\partial^2_s s^3 M(s, a) + 2! p(0+, a) \leftrightarrow -t^2 p^{(3)}(t, a)
\]

\[
2 \partial^2_s s^2 M(s, a) \leftrightarrow -2t^2 p^{(2)}(t, a)
\]

\[
\partial^2_s s M(s, a) \leftrightarrow -t^2 p^{(1)}(t, a)
\]

\[
(2N - 4) \partial^2_s M(s, a) + (2N - 4) p(0+, a) \leftrightarrow (2N - 4) t^2 p^{(2)}(t, a)
\]

\[
(6 - 2N - N_R - a) \partial_s M(s, a) \leftrightarrow -(6 - 2N - N_R - a) t^1 p^{(1)}(t, a)
\]

\[
(N_R - 2) \partial_s M(s, a) \leftrightarrow (N_R - 2) t p(t, a)
\]

\[
(N - 1)(N - 2) s M(s, a) + (N - 1)(N - 2) p(0+, a) \leftrightarrow -(N - 1)(N - 2) t p^{(1)}(t, a)
\]

\[
(N - 1)(2 - N_R - a) M(s, a) \leftrightarrow (N - 1)(2 - N_R - a) t p(t, a).
\]
Summing the left-hand-side terms (i.e., the \( s \)-domain terms) of the above transform pairs and accounting for (41) yields the constant \( N(N-1)p(0+, a) \), which reduces to 0 for any \( N \geq 1 \) because, based on (20) and (32), we have:

\[
p(0+, a) = \begin{cases} 
F_1(N; N_R; -a), & N = 1, \\
0, & N > 1. 
\end{cases} \tag{43}
\]

Then, by the uniqueness of the Laplace transform, the right-hand-side terms (i.e., the \( t \)-domain terms) of the above transform pairs also sum to 0, i.e.,

\[
-t^2p^{(3)}(t, a) - 2t^2p^{(2)}(t, a) - t^2p^{(1)}(t, a) + (2N - 4)tp^{(2)}(t, a) - (6 - 2N - N_R - a)t p^{(1)}(t, a) + (N_R - 2)t p(t, a) - (N - 1)(N - 2)p^{(1)}(t, a) + (N - 1)(2 - N_R - a) p(t, a) = 0, \tag{44}
\]

which can be rewritten as follows:

\[
p^{(3)}(t, a) = \frac{(N_R - 2)t + (N - 1)(2 - N_R - a)}{t^2} p(t, a) - \frac{t^2 + (6 - 2N - N_R - a)t + (N - 1)(N - 2)}{t^2} p^{(1)}(t, a) - \frac{2t^2 - (2N - 4)t}{t^2} p^{(2)}(t, a). \tag{45}
\]

Finally, by defining the function vector

\[
p(t, a) = \begin{pmatrix} p(t, a) & p^{(1)}(t, a) & p^{(2)}(t, a) \end{pmatrix}^T,
\]

we can recast (45) as the system of differential equations w.r.t. \( t \)

\[
\partial_t p(t, a) = P(t, a) p(t, a), \tag{47}
\]

where the elements of the \( 3 \times 3 \) companion matrix \( P(t, a) \) are:

\[
\begin{align*}
[P(t, a)]_{1,1} &= [P(t, a)]_{1,3} = 0, \\
[P(t, a)]_{2,1} &= [P(t, a)]_{2,2} = 0, \\
[P(t, a)]_{1,2} &= [P(t, a)]_{2,3} = 1, \\
[P(t, a)]_{3,1} &= \frac{(N_R - 2)t + (N - 1)(2 - N_R - a)}{t^2}, \\
[P(t, a)]_{3,2} &= -\frac{t^2 + (6 - 2N - N_R - a)t + (N - 1)(N - 2)}{t^2}, \\
[P(t, a)]_{3,3} &= \frac{-2t^2 - (2N - 4)t}{t^2}.
\end{align*}
\]

D. Computation of \( p(t, a) \) vs. \( t \), Given \( a \), by HGM w.r.t. \( t \)

Because we are interested in computing the SNR p.d.f. over a relevant range of \( t \), as depicted in Fig. 2, applying the HGM w.r.t. \( t \) based on (47) suits naturally. It yields \( p(t, a) \) vs. \( t \), given \( a \) and initial condition \( p(t_0, a) = \begin{pmatrix} p(t_0, a) & p^{(1)}(t_0, a) & p^{(2)}(t_0, a) \end{pmatrix}^T \) for some \( t_0 \neq 0 \), by solving the system of differential equations (47) between \( t_0 \) and \( t \).

On the one hand, computing the initial conditions \( p(t_0, a) \), \( p^{(1)}(t_0, a) \), and \( p^{(2)}(t_0, a) \) may be attempted by truncating their infinite series that we derived in Appendix III. Since computation by series truncation is required only at a small \( t_0 \) of our choice, the HGM also helps avoid the numerical instability incurred by series truncation at large \( t \) due to the alternating-sign issue mentioned in Section IV-C.

\(^4\)HGM requires \( t_0 \neq 0 \) because \( t \) appears in \( P(t, a) \) denominators.
On the other hand, as for truncating the series [18], truncating the series for \( p^{(1)}(t, a) \), and \( p^{(2)}(t, a) \) derived in Appendix III is reliable only for \( a \) sufficiently small (i.e., practically-irrelevant \( K \) value). In order to compute \( p(t_0, a) \) at relevant values of \( a \), an HGM-based approach is hereafter pursued also w.r.t. \( a \). Therefore, next, we deduce the differential equation w.r.t. \( a \) for \( p(t, a) \), which was shown to exist in Lemma I.

**E. Differential Equation w.r.t. \( a \) for \( p(t, a) \)**

In Appendix IV Eq. (87), we have deduced the relationship

\[
a \partial_a M(s, a) = (s \partial_s - s^2 \partial_s - Ns) \ M(s, a),
\]

which, by using the order-changing rules (36) and (38), becomes

\[
a \partial_a M(s, a) = (\partial_s s - 1 - \partial_s s^2 + 2s - Ns) \ M(s, a)
\]

\[
= [-1 + (-Ns + \partial_s s + 2s) - \partial_s s^2] \ M(s, a).
\]

Transforming the above to the \( t \)-domain based on (42) and further manipulation yield

\[
a \partial_a p(t, a) = \frac{(N - 1)p(0+, a) - p(t, a) + (N - t - 2)p^{(1)}(t, a) - t p^{(2)}(t, a)}{t} = -p(t, a) + (N - t - 2)p^{(1)}(t, a) - t p^{(2)}(t, a),
\]

which relates the derivative of \( p(t, a) \) w.r.t. \( a \) and the derivatives of \( p(t, a) \) w.r.t. \( t \), which are, in turn, related as shown in (45).

Now, twice differentiating (49) w.r.t. \( t \) and substituting \( p^{(3)}(t, a) \) from (45) yields:

\[
a \partial_a p^{(1)}(t, a) = -t p^{(3)}(t, a) + (N - t - 3)p^{(2)}(t, a) - 2p^{(1)}(t, a)
\]

\[
= \left(2 - N_R + \frac{2 - 2N - N_R - a + N N_R + Na}{t}\right) p(t, a)
\]

\[
+ \left(4 - 2N - N_R - a + t + \frac{2 + N^2 - 3N}{t}\right) p^{(1)}(t, a)
\]

\[
+ (1 - N + t) p^{(2)}(t, a),
\]

(50)

\[
a \partial_a p^{(2)}(t, a) = \left(-2 + N_R + \frac{-4 + 4N + 2N_R + a - 2N N_R - Na}{t}\right) p(t, a)
\]

\[
+ \frac{-4 + 6N + 2N_R + 2a - 3N N_R - 3Na + N^2 N_R + N^2 a - 2N^2}{t^2} p(t, a)
\]

\[
+ \left(3N - 4 + a - t + \frac{-6 - 3N^2 + 9N}{t} + \frac{-4 + 8N - 5N^2 + N^3}{t^2}\right) p^{(1)}(t, a)
\]

\[
+ \left(-1 + 2N - N_R - a - t + \frac{-2 - N^2 + 3N}{t}\right) p^{(2)}(t, a).
\]

(51)

Finally, collecting (49)–(51) yields for the function vector \( p(t, a) \) defined in (46) the system of differential equations w.r.t. \( a \)

\[
\partial_a p(t, a) = \frac{1}{a} Q(t, a) \ p(t, a),
\]

(52)
where the elements of $3 \times 3$ matrix $Q(t,a)$ are as follows:

\[
\begin{align*}
[Q(t,a)]_{1,1} &= -1, \\
[Q(t,a)]_{1,2} &= N - t - 2, \\
[Q(t,a)]_{1,3} &= -t, \\
[Q(t,a)]_{2,1} &= 2 - N + \frac{2 - 2N - N_R - a + NN_R + Na}{t}, \\
[Q(t,a)]_{2,2} &= 4 - 2N - N_R - a + t + \frac{2 + N^2 - 3N}{t}, \\
[Q(t,a)]_{2,3} &= 1 - N + t, \\
[Q(t,a)]_{3,1} &= -2 + N_R + \frac{-4 + 4N + 2N_R + a - 2NN_R - Na}{t} + \frac{-4 + 6N + 2N_R + 2a - 3NN_R - 3Na + N^2N_R + N^2a - 2N^2}{t^2}, \\
[Q(t,a)]_{3,2} &= 3N - 4 + a - t + \frac{-6 - 3N^2 + 9N}{t} + \frac{-4 + 8N - 5N^2 + N^3}{t^2}, \\
[Q(t,a)]_{3,3} &= -1 + 2N - N_R - a - t + \frac{-2 - N^2 + 3N}{t}.
\end{align*}
\]

F. Computation of $p(t,a)$ vs. $a$, Given $t$, by HGM w.r.t. $a$

The system of differential equations (52) may be solved numerically between some $a_0 \neq 0$ and the desired $a$, given $t$ and initial condition vector $p(t,a_0) = (p(t,a_0) \ p^{(1)}(t,a_0) \ p^{(2)}(t,a_0))^T$ with elements computed by truncating their infinite-series expressions derived in Appendix III. However, the fact that this truncation suffers from numerical issues caused by alternating sign, as mentioned in Section IV-C, prevents this HGM-based approach from accurately computing $p(t,a)$ in the upper range of $t$ shown in Fig. 2 (numerical results unshown due to length limitations).

Therefore, we propose next an approach that avoids the numerical issues caused by truncating the series at large values of either $a$ or $t$, by applying the HGM simultaneously w.r.t. $a$ and $t$.

G. Computation of $p(t,a)$ vs. $t$, by HGM w.r.t. $t$ for $a = ct$

In the systems of differential equations obtained in (47) and (52), i.e., in

\[
\begin{align*}
\partial_t p(t,a) &= P(t,a) p(t,a), \\
\partial_a p(t,a) &= \frac{1}{a} Q(t,a) p(t,a),
\end{align*}
\]

we now make the following changes of variables

\[
\begin{align*}
t &= c_1 u, \\
a &= c_2 u.
\end{align*}
\]

Then, the bivariate function vector from (46) becomes the univariate function vector

\[
p(c_1 u, c_2 u) = \begin{pmatrix} p(c_1 u, c_2 u) \\ p^{(1)}(c_1 u, c_2 u) \\ p^{(2)}(c_1 u, c_2 u) \end{pmatrix} = \tilde{p}(u).
\]

$^5$HGM requires $a_0 \neq 0$ because $a$ divides matrix $Q$ in (52).
Based on the chain rule [12] Eq. (1.5.7), p. 7] as well as on (53) and (54), we can write:

\[
\frac{d}{du} \tilde{p}(u) = \frac{d}{du} p(c_1 u, c_2 u) = \left[ \frac{\partial_t p(t, a)}{t} + \frac{\partial_a p(t, a) }{a} \right] \bigg|_{t = c_1 u, a = c_2 u} \\
= c_1 \mathbf{P}(c_1 u, c_2 u) \tilde{p}(u) + c_2 \frac{1}{c_2 u} \mathbf{Q}(c_1 u, c_2 u) \tilde{p}(u)
\]

Then, for example, \(c_1 = 1\) and \(c_2 = c\), i.e., \(a = ct\), yields the system of differential equations

\[
\frac{d}{du} \tilde{p}(u) = \left[ \mathbf{P}(u, cu) + \frac{1}{u} \mathbf{Q}(u, cu) \right] \tilde{p}(u),
\]

which helps apply HGM for \(a = ct\), i.e., simultaneously w.r.t. \(a\) and \(t\), as shown next.

VII. NUMERICAL RESULTS

For the numerical results described below, we have used the following procedure. Given the per-symbol input SNR \(\Gamma\), which is defined in [3], and the Rician \(K\)-factor, we have computed \(\Gamma_1\) and \(a\) with (14) and (15), respectively. Then, we computed the Stream-1 ZF SNR p.d.f. over the relevant SNR range by employing the HGM ensuing from (59) as follows:

- Compute accurately the initial condition \(\tilde{p}(u_0) = \left(p(u_0, u_0) p^{(1)}(u_0, u_0) p^{(2)}(u_0, u_0)\right)^T\) for a sufficiently-small \(u_0\), using the infinite series for \(p^{(i)}(t, a)\) derived in Appendix II.
- Sample the range of interest \([u_1, u_M]\) for \(u\), i.e., \(t\), as \(u_1, \ldots, u_M\).
- For each sample \(u = u_m, m = 1 : M\), set \(c = a/u\), solve the system of differential equations (59) from \(u_0\) to \(u\), and save \(p(u, cu)\), which represents \(p(t, a)\) on the line \(a = ct\).
- Finally, recover the ZF SNR p.d.f. based on (32), i.e., with \(p_{\gamma_1}(t, a) = p(t/\Gamma_1, a)/\Gamma_1\).

This approach avoids truncating the infinite series for \(p(t, a), p^{(i)}(t, a)\), and \(p^{(2)}(t, a)\) for either large \(a\) or large \(t\), and, thus, it avoids the ensuing numerical issues described in Section IV-C.

Figs. 3 and 4 depict, respectively, the SNR p.d.f. and c.d.f. computed with the HGM as above, and by simulation, for \(N_R = 6\), \(N_T = 2\), and \(K = 7\) dB. HGM performs well, i.e., the resulting p.d.f. and c.d.f. plots agree with the simulation results. Also, the c.d.f. shown in Fig. 4 (from numerically integrating the HGM output) goes to 1 for increasing \(t\), as required.

Figs. 5 and 6 depict, respectively, the outage probability and ergodic capacity for \(N_R = 6\) and \(N_T = 2\). Recall that [8] Eqs. (69), (71)] provide infinite series for the outage probability \(P_o\) and ergodic capacity \(C\) derived from that for \(p_{\gamma_1}(t)\) from (18). On the one hand, for Rayleigh-only fading, i.e., for \(a = 0\), these series reduce to their respective term for \(n = m = 0\), whose numerical results (indicated with Ray-Ray,Series) agree with simulation results (indicated with Ray-Ray,Simulation) in Figs. 5 and 6.

On the other hand, for Rician-Rayleigh fading with \(K = 7\) dB, Figs. 5 and 6 show results from the simulation (indicated with Rice-Ray,Simulation) and from the numerical integration according to (23) and (24), respectively, of \(p_{\gamma_1}(t, a)\) computed with HGM (indicated with Rice-Ray,HGM) as described above in this section. Note that the simulation and HGM-based results agree closely. Thus, unlike series-based computation, the proposed HGM-based computation circumvents numerical breakdown and evaluates accurately the SNR p.d.f., as well as \(P_o\) and \(C\), for realistic values of \(K\).

Note that both \(t\) and \(a\) are substituted with \(u_0\) in this step.

Recall that in [11] we had been able to accurately compute \(p_{\gamma_1}(t)\) based on its infinite series only up to the unrealistically-small \(K\) value of 1.5 dB. This has also been illustrated herein in Fig. 2 where the series-based computation breaks down for \(K = 7\) dB. Thus, we no longer try to plot series results in Figs. 3 and 4.

Results from \(P_o\) and \(C\) series [8] Eqs. (69), (71)] could not be shown because they break down for \(K = 7\) dB.

Unshown results have revealed that HGM yields accurate results even for \(K\) as high as 20 dB, also for other combinations of \(N_T\) and \(N_R\), without a noticeable increase in computation time.
Fig. 3. Stream-1 SNR p.d.f. computed with HGM and by simulation, for $N_R = 6$, $N_T = 2$, $K = 7$ dB.

Fig. 4. Stream-1 SNR c.d.f. computed with HGM and by simulation, for $N_R = 6$, $N_T = 2$, $K = 7$ dB.

VIII. OTHER RELEVANT HGM APPLICATIONS. FUTURE WORK

A. HGM-Based Computation of Conditioned Error Probability

Given the symbol-detection SNR, e.g., $\gamma_1$, the bit error probability for BPSK modulation is given by [28, Eq. (5.2-57), p. 268]

$$ P_e(\gamma_1) = Q\left(\sqrt{2\gamma_1}\right) = \frac{1}{2} \left[1 - \text{erf}\left(\sqrt{\gamma_1}\right)\right], $$

(60)
where $Q(\cdot)$ is the Gaussian Q-function [2, Eq. (4.1), p. 70], and erf is the error function [12, Eq. (7.2.1), p. 160]:

$$
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy = \frac{2}{\sqrt{\pi}} [F(x) - F(0)] = E(x), \quad E(0) = 0,
$$

\hspace{1cm} (61)

\[10\] Note that erf, i.e., the Q-function, can be expressed in terms of \( F_1(\cdot; \cdot) \), based on [12, Eq. (7.11.4), p. 164].
where $F'(x) = e^{-x^2}$. Then, by differentiating twice, we obtain:

$$E'(x) = \frac{2}{\sqrt{\pi}} F'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad E'(0) = \frac{2}{\sqrt{\pi}}, \quad (62)$$

$$E''(x) = -2x \frac{2}{\sqrt{\pi}} e^{-x^2} = -2x E'(x). \quad (63)$$

Finally, differential equation (63) can be solved numerically starting at $x_0 = 0$, using the initial conditions from (61) and (62), to obtain $E(x)$ at any value of $x$, i.e., $P_e(\gamma_1)$ from (60) at any $\gamma_1$. The results of this (HGM-based) approach have been found to agree closely with those obtained with the MATLAB function $\text{erf}$, whose implementation details are inaccessible. HGM-based error probability computation may thus be of use in adaptive modulation.

B. HGM-Based Framework for MIMO Study

For many fading types (e.g., Rayleigh, Rician, Nakagami, lognormal), and for many single and multi-antenna transmission techniques, [2] derived expressions for the SNR p.d.f. and m.g.f., as well as for the AEP and outage probability. Many such expressions from [2] involve basic or special (e.g., hypergeometric, modified-Bessel) scalar-argument functions that are holonomic.

On the other hand, also hypergeometric functions of matrix argument occur frequently in MIMO analyses due to statistical assumptions about the channel matrix [15] [16]. For example, the c.d.f. and m.g.f. of the dominant eigenvalue of a (real-valued) central-Wishart-distributed matrix in [22]. Therefore, we shall investigate the feasibility of a HGM-based framework for MIMO study.

C. Automated Deduction of Differential Equations

In this paper, we have applied the HGM after manually deducing, for generic $N$ and $N_R$, the differential equations satisfied by the ZF SNR m.g.f. from the differential equation satisfied by $\text{i}_1 F_1 N; N_R; \sigma$. Nevertheless, software tools that can automate the deduction of differential equations, for specific $N$ and $N_R$ values, for holonomic functions ensuing from $\text{i}_1 F_1 N; N_R; \sigma$ have recently become available [19, p. 171] [20, Ch. 7] [31]. We shall investigate enhancing such tools to derive automatically, for generic $N$ and $N_R$, the results from this paper. Furthermore, we shall attempt to derive differential equations w.r.t. $a$ for $P_o(a)$ and $C(a)$.

IX. SUMMARY AND CONCLUSIONS

For MIMO ZF under Rician–Rayleigh fading, this paper has demonstrated that performance-measure expressions ensuing from the confluent hypergeometric function can be evaluated accurately, by using the HGM, at (realistic) Rician $K$-factor values that render unusable the conventional method of truncating infinite series. For the SNR m.g.f., which has been known in terms of the confluent hypergeometric function, we have deduced the satisfied differential equations. They have yielded the differential equations satisfied by the SNR p.d.f., which, in turn, have helped compute the p.d.f. accurately using the HGM at values of $K$ relevant according to WINNER II (e.g., $K = 7$ dB). Finally, numerical integration of the SNR p.d.f. obtained by the HGM (unlike by infinite-series truncation) has yielded for the MIMO
ZF outage probability and ergodic capacity close agreement with simulations. Since holonomic special functions enter MIMO performance-measure expressions derived for a wide range of transceiver methods and fading types, the future development of an HGM-based analysis and evaluation framework promises to accurately characterize MIMO performance for realistic fading-parameter values.

APPENDIX I

INFINITE-SERIES EXPRESSION FOR \( _1F_1(N; N_R; \sigma) \) IS EXPANSION AROUND \( \sigma = 0 \)

For \( N_R = N \), (16) reduces to \( _1F_1(N; N_R; \sigma) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} = e^{\sigma} \). On the other hand, for \( N_R > N \), the confluent hypergeometric function has the integral form [12, Eq. (13.4.1), p. 326] [12, Eq. (13.6.1), p. 327]

\[
_iF_1(N; N_R; \sigma) = \frac{\Gamma(N_R)}{\Gamma(N)\Gamma(N_R - N)} \int_0^1 y^{N-1}(1 - y)^{N_R-N-1} dy,
\]

which yields its well known infinite-series form [16], by way of the Maclaurin expansion of \( e^{\sigma y} \) around \( \sigma = 0 \) [12, Eq. (13.2.4), p. 322] [12, Eq. (13.4.1), p. 326] [32, Eq. (3.191.3), p. 315] [32, Eq. (8.384.1), p. 909] as follows:

\[
_iF_1(N; N_R; \sigma) = \frac{\Gamma(N_R)}{\Gamma(N)\Gamma(N_R - N)} \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} y^{N-1}(1 - y)^{N_R-N-1} dy
\]

\[
= \frac{\Gamma(N_R)}{\Gamma(N)\Gamma(N_R - N)} \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \int_0^1 y^{n+N-1}(1 - y)^{N_R-N-1} dy
\]

\[
= \frac{\Gamma(N_R)}{\Gamma(N)\Gamma(N_R - N)} \sum_{n=0}^{\infty} \frac{\sigma^n \Gamma(n + N)\Gamma(N_R - N)}{\Gamma(n + N_R)} = \sum_{n=0}^{\infty} \frac{\sigma^n \Gamma(n + N)\Gamma(N_R)}{n! \Gamma(n + N_R)}
\]

\[
= \sum_{n=0}^{\infty} \frac{(N + n - 1)! (N_R - 1)! \sigma^n}{(N - 1)! (N_R + n - 1)! n!} = \sum_{n=0}^{\infty} \frac{(N)_n}{(N_R)_n} \frac{\sigma^n}{n!} = \sum_{n=0}^{\infty} A_n(\sigma).
\]

Finally, note that \( _1F_1(N; N_R; \sigma) \) is referred to as ‘hypergeometric’ because the ratio of two successive terms in series (65) is a rational function in \( n \), i.e.,

\[
\frac{A_n(\sigma)}{A_{n-1}(\sigma)} = \frac{N + n - 1}{N_R + n - 1} \frac{\sigma}{n}, \quad n \geq 1.
\]

On the other hand, in a ‘geometric’ series, the ratio of successive terms is a constant, e.g., in \( \sum_{n=0}^{\infty} \sigma^n \).

APPENDIX II

DIFFERENTIAL EQUATION W.R.T. \( s \) FOR \( M(s, a) \)

First, substituting \( \sigma \) with \( \frac{as}{1-s} \) in the differential equation for \( _1F_1(N; N_R; \sigma) \) from (27) yields

\[
\frac{as}{1-s} _1F_1^{(2)}(N; N_R; \frac{as}{1-s}) + \left(N_R - \frac{as}{1-s}\right) _1F_1^{(1)}(N; N_R; \frac{as}{1-s}) - N_1F_1(N; N_R; \frac{as}{1-s}) = 0.
\]

Then, from (33), we have

\[
M(s, a) = \frac{1}{(1-s)^N} _1F_1(N; N_R; \frac{as}{1-s}),
\]

which yields

\[
_iF_1(N; N_R; \frac{as}{1-s}) = (1-s)^N M(s, a).
\]
Differentiating (68) w.r.t. \( s \) yields:

\[
\partial_s M(s, a) = \frac{N}{(1-s)^{N+1}} {_{1}F_{1}} \left( N; N_R; \frac{as}{1-s} \right) + \frac{a}{(1-s)^{N+2}} {_{1}F_{1}}^{(1)} \left( N; N_R; \frac{as}{1-s} \right).
\]  

(70)

By first substituting (69) into (70) and then by differentiating the result w.r.t. \( s \) we obtain

\[
\partial_s M(s, a) = \frac{N}{(1-s)} M(s, a) + \frac{a}{(1-s)^{N+2}} {_{1}F_{1}}^{(1)} \left( N; N_R; \frac{as}{1-s} \right),
\]  

(71)

\[
\partial_s^2 M(s, a) = \frac{N}{(1-s)^2} M(s, a) + \frac{N}{(1-s)} \partial_s M(s, a)
\]  

\[
+ \frac{a(N+2)}{(1-s)^{N+1}} {_{1}F_{1}}^{(1)} \left( N; N_R; \frac{as}{1-s} \right) + \frac{a^2}{(1-s)^{N+2}} {_{1}F_{1}}^{(2)} \left( N; N_R; \frac{as}{1-s} \right).
\]  

(72)

which yield, respectively:

\[
{_{1}F_{1}}^{(1)} \left( N; N_R; \frac{as}{1-s} \right) = \left(1-s\right)^{N+2} \left[ \partial_s - \frac{N}{(1-s)} \right] M(s, a),
\]  

(73)

\[
{_{1}F_{1}}^{(2)} \left( N; N_R; \frac{as}{1-s} \right) = \left(1-s\right)^{N+4} \left[ \partial_s^2 M(s, a) - \frac{N}{(1-s)^2} M(s, a)
\right.
\]  

\[
\left. - \frac{N}{(1-s)} \partial_s M(s, a) - \frac{a(N+2)}{(1-s)^{N+3}} {_{1}F_{1}}^{(1)} \left( N; N_R; \frac{as}{1-s} \right) \right].
\]  

(74)

Substituting (73) into (74) yields:

\[
{_{1}F_{1}}^{(2)} \left( N; N_R; \frac{as}{1-s} \right) = \left(1-s\right)^{N+4} \left[ \partial_s^2 - \frac{2(N+1)}{(1-s)} \partial_s + \frac{N(N+1)}{(1-s)^2} \right] M(s, a).
\]  

(75)

Finally, substituting (69), (73), and (75) into the differential equation (67), and further manipulation, yield

\[
(s(1-s)^2 \partial_s^2 - [2(N+1)s(1-s) - (1-s)N_R + as] \partial_s + N [(N+1)s - N_R - a]) M(s, a) = 0,
\]  

(76)

which appears in the main text in (34).

**APPENDIX III**

**INFINITE-SERIES EXPRESSIONS OF DERIVATIVES OF \( p(t, a) \) W.R.T. \( t \)**

Based on (18) and (32), let us define the function

\[
f(t, a) = p(t, a) e^t = \sum_{n=0}^{\infty} a_n(a) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{t^{N+n-m-1}}{(N+n-m-1)!},
\]  

(77)

whose first two derivatives are given by

\[
f^{(1)}(t, a) = p^{(1)}(t, a) e^t + p(t, a) e^t,
\]  

\[
f^{(2)}(t, a) = p^{(2)}(t, a) e^t + 2p^{(1)}(t, a) e^t + p(t, a) e^t,
\]  

The above yield

\[
p(t, a) = f(t, a) e^{-t},
\]  

(78)

as well as

\[
p^{(1)}(t, a) = \left[ f^{(1)}(t, a) - f(t, a) \right] e^{-t},
\]  

(79)

\[
p^{(2)}(t, a) = \left[ f^{(2)}(t, a) - 2f^{(1)}(t, a) + f(t, a) \right] e^{-t},
\]  

(80)
which are the only derivatives of \( p(t, a) \) required for (46).

Now, if we rewrite \( f(t, a) \) from (77) further as

\[
f(t, a) = t^{N-1} \sum_{n=0}^{\infty} A_n(a) \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (-1)^{m} \frac{t^{n-m}}{(N-1+n-m)!} = t^{N-1}g(t, a),
\]

(81)

then its \( q \)th partial derivative w.r.t. \( t \) is\(^{11}\) based on Leibniz’s formula \([12, \text{Eq. (1.4.12), p. 5}]:\)

\[
f^{(q)}(t, a) = \partial_t^q \left[ t^{N-1}g(t, a) \right] = \sum_{k=0}^{q} \left( \begin{array}{c} q \\ k \end{array} \right) t^{N-1-k}g^{(q-k)}(t, a) \]

\[
= \sum_{k=0}^{q} \left( \begin{array}{c} q \\ k \end{array} \right) \frac{(N-1)!t^{N-1-k}g^{(q-k)}(t, a)}{(N-1-k)!}.
\]

(82)

(83)

If we rewrite \( g(t, a) \) from (81) as

\[
g(t, a) = \sum_{n=0}^{\infty} A_n(a) \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-1)^{n-r} \frac{t^{r}}{(N-1+r)!},
\]

then its partial derivative of order \( q \geq 1 \) w.r.t. \( t \) is given by

\[
g^{(q)}(t, a) = \sum_{n=q}^{\infty} A_n(a) \sum_{r=q}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-1)^{n-r} \frac{1}{(N-1+r)!} \frac{r!}{(r-q)!} t^{r-q},
\]

(84)

which, along with (83), yields \( f^{(q)}(t, a) \). Finally, substituting into (79) and (80) yields expressions for \( p^{(1)}(t, a) \) and \( p^{(2)}(t, a) \), respectively.

However, because (83) follows from (82) only for \( k \leq N-1 \), and because \( k \) goes from 0 to \( q \), it is required that \( N-1 \geq q \). Then, because (46) requires \( f^{(q)}(t, a) \) for \( q \) as high as 2, \( f^{(q)}(t, a) \) can be written as in (83) only if \( N \geq 3 \). Table I characterizes the remaining cases.

### APPENDIX IV

**RELATIONSHIP BETWEEN DERIVATIVES OF \( M(s, a) \) W.R.T. \( s \) AND \( a \)**

Differentiating (68) w.r.t. \( a \) yields

\[
\partial_a M(s, a) = \frac{s}{(1-s)^{N+1}}_1F_1^{(1)} \left( \begin{array}{c} N; N_R; \frac{as}{1-s} \end{array}; \right),
\]

(85)

so that

\[
_1F_1^{(1)} \left( \begin{array}{c} N; N_R; \frac{as}{1-s} \end{array}; \right) = \frac{(1-s)^{N+1}}{s} \partial_a M(s, a).
\]

(86)

Now, by substituting (69) and (86) into (70), and by further manipulation, we obtain

\[
a\partial_a M(s, a) = s(1-s)\partial_s M(s, a) - NsM(s, a),
\]

(87)

which appears in the main text in (48).

\(^{11}\)Note that (83) follows from (82) only for \( k \leq N-1 \).
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