Particular integrability and (quasi)-exact-solvability

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Received 7 August 2012, in final form 17 October 2012
Published 10 December 2012
Online at stacks.iop.org/JPhysA/46/025203

Abstract

A notion of a particular integrability is introduced when two operators commute on a subspace of the space where they act. Particular integrals for one-dimensional (quasi)-exactly-solvable Schrödinger operators and Calogero–Sutherland Hamiltonians for all roots are found. In the classical case some special trajectories for which the corresponding particular constants of motion appear are indicated. Particular integrability manifests the existence of super-integrable substructures in an integrable system.

PACS numbers: 03.65.Fd, 03.65.Ge, 02.30.Ik, 45.20.D—, 45.50.Dd, 45.50.Jf

Assume that \( \phi_0 \) is a common eigenfunction for two algebraically-independent operators \( A \) and \( B \): \( A\phi_0 = a_0\phi_0 \) and \( B\phi_0 = b_0\phi_0 \). It is evident that the commutator \( [A, B]\phi_0 = 0 \). The best-known example is related to the harmonic oscillator Hamiltonian in the second quantization representation, \( H(\hat{p}, \hat{x}) = \{a^+a\}_x \equiv A \), then \( B(\hat{p}, \hat{x}) = a^- \) commutes with \( H \) over the vacuum, \( [H, a^-]|0> = 0 \). By taking a straightforward classical analogue \( H(\hat{p}, \hat{x}) \rightarrow H(p, x) \) and \( B(\hat{p}, \hat{x}) \rightarrow B(p, x) \), one can ask whether there exists a trajectory for which the Poisson bracket \( \{H, B\} \) vanishes. This trajectory really exists and corresponds to \( p = x = 0 \), when the particle stands at the bottom of the potential. For this trajectory the energy \( E \) is equal to zero as well as \( B = 0 \). It is not periodic, unlike all other trajectories at \( E \neq 0 \). Such a trajectory we will call special.

It is evident that if for operators \( A \) and \( B \) there exists a number of common eigenfunctions \( \{\phi_0\} \), then the commutator \( [A, B] \) annihilates a space \( V_p \) spanned by \( \{\phi_0\} \). At the same time \( V_p \) is the invariant subspace for both \( A \) and \( B \). In more general terms, if we take two operators \( A, B \) acting on a space \( V \) and assume that for a subspace \( V_p \subset V \), the commutator \( [A, B]: V_p \rightarrow \{0\} \) and both operators \( A, B \) have \( V_p \) as an invariant subspace, \( A: V_p \rightarrow V_p \), then the operators \( A, B \) are called particular integrals (\( \pi \)-integrals). It can easily be shown that if \( [A, B]: V_p \rightarrow \{0\} \), and the operator \( A \) has \( V_p \) as an invariant subspace and non-degenerate, with all eigenvalues of multiplicity 1, then the operator \( B \) has \( V_p \) as an invariant subspace. Thus, \( \pi \)-integrability implies the existence of the common eigenfunctions of two spectral problems \( A(B)\psi = a(b)\psi \), where \( a(b) \) are spectral parameters. If the dimension of \( V_p \) is finite, \( \dim V_p < \infty \), then the operators become quasi-exactly-solvable, for which a finite number of eigenfunctions can be
found explicitly by algebraic means (for a review, see [1]). It allows us to connect quasi-
extact-solvability with particular integrability. It is worth noting an important particular case of
\( \pi \)-integrals when \( A : V_p \to V_p \) and \( B : V_p \to \{0\} \), hence \( V_p \) is the kernel space or the space of
zero modes of \( B \). Another example of \( \pi \)-integrals is when \( A(B) \) are operators which depend
on different variables; the bispectral problem by Duistermaat–Grünewald [2] is a particular
case of it. If the common eigenfunctions form a complete basis, hence, \( V_p = V \), then the
commutator \( [A, B] = 0 \). This situation corresponds to standard integrability.

The classical counterpart of \( \pi \)-integrability implies that the Poisson bracket \( \{A, B\} \)
may vanish on some special trajectories. These trajectories can be called super-integrable:
they are characterized by a larger number of constants of motion than generic ones. It is
not clear whether quantum \( \pi \)-integrability implies always the existence of physical special
trajectories; they might be complex or might not exist at all. The goal of this paper is to study
quantum \( \pi \)-integrals of the one-dimensional (quasi)-exactly-solvable Schrodinger equations,
the Calogero–Sutherland systems for all roots and their classical counterparts, the classical
\( \pi \)-integrals and related special trajectories.

Take the algebra \( gl_2 \) in \( (n+1) \)-dimensional representation realized by the first order
differential operators
\[
J^- = \frac{d}{dt}, \\
J_0^0 = t \frac{d}{dt} - n, \quad T^0 = 1 \\
J_0^+ = t \frac{d}{dt} - nt = t J_0^0,
\]
where \( n = 0, 1, \ldots, T^0 \) is the center element. The finite-dimensional representation space is
the space of polynomials of degree not higher than \( n \),
\[
\mathcal{P}_n = \langle 1, t, t^2, \ldots, t^n \rangle \equiv \langle t^k \mid 0 \leq k \leq n \rangle.
\]
The generator \( J^- \) is the lowering operator or filtration (of the grading \( -1 \))
\[
J^- : \mathcal{P}_n \mapsto \mathcal{P}_{n-1}.
\]
Hence, \( \mathcal{P}_n \) is the kernel of \( (J^-)^{n+1} \). It is evident that any non-trivial element \( h \) of the universal
enveloping algebra \( U_{gl_2} \) taken in the realization (1) commutes with
\[
i^{(k)}_n \equiv (J^-)^{n+1},
\]
as well as with
\[
i^{(k)}_n \equiv (J^-)^{n-k} \prod_{j=0}^k \left( J_0^0 + j \right), \quad k = 0, 1, \ldots, n,
\]
over \( \mathcal{P}_n \), namely,
\[
[h, i^{(k)}_n] : \mathcal{P}_n \mapsto 0, \quad k = -1, 0, 1, \ldots, n.
\]
Hence, with respect to \( h \) any \( i^{(k)}_n \), \( k = -1, 0, 1, \ldots, n \) is \( \pi \)-integral over \( \mathcal{P}_n \) and its grading
\( \text{deg}(i^{(k)}_n) = -(n-k) \). For any \( i^{(k)}_n \) the space \( \mathcal{P}_n \) is the space of zero modes. Hence, all these
\( \pi \)-integrals are in involution,
\[
[i^{(k)}_n, i^{(m)}_n] : \mathcal{P}_n \mapsto 0.
\]
Operators \( i^{(m)}_n \), \( m = -1, 0, 1, \ldots, n \) are generating elements of some infinite-dimensional
algebra. This is the algebra of differential operators for which any element has the space \( \mathcal{P}_n \)
as the kernel.
Any element \( e \) of the universal enveloping algebra \( U_{gl(n)} \), that does not contain the generator \( J_n^+ \), preserves the infinite flag \( \mathcal{P} \),
\[
\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_p \subset \ldots ,
\]
therefore
\[
[e, i_m^{(k)}] : \mathcal{P}_p \mapsto 0, \quad k = -1, 0, 1, 2, \ldots, m
\]
for any \( m = 0, 1, \ldots \) and \( p \leq m \). It seems evident that there is no \( \pi \)-integral in a form of an element of \( U_{gl(n)} \) other than a polynomial \( f(e) \) which commutes with \( e \) over any \( \mathcal{P}_p \).

Take a quadratic element \( h_2 \) of the universal enveloping algebra \( U_{gl(n)} \) in the realization (1),
\[
h_2 = c_{a\beta} J^n_{a_\beta} + c_{a} J^n_{a} + c_{-} J^{-} + c_{0} J^{0} J^{-} + c_{1} J^{1} J^{-} + c_{2} J^{2} J^{-} + c,
\]
where \( c_{a\beta}, c_{a}, c \in \mathbb{R} \) are parameters, their number is \( \text{par}(h_2) = 9 \). Substituting (1) into (7) we obtain the Heun operator
\[
h_2(t) = -P_1(t) \frac{d^2}{dt^2} + P_3(t) \frac{d}{dt} + P_2(t),
\]
where the \( P_i(t) \) is a polynomial of \( j \)th order with coefficients related to \( c_{a\beta}, c_{a}, c \) that can be easily calculated. In general, the operator (8) is diagonalizable; it has \( (n + 1) \) polynomial eigenfunctions in the form of a polynomial in \( t \) of \( n \)th degree, hence, this is a quasi-exactly-solvable operator. Every polynomial eigenfunction is an element of \( \mathcal{P}_n \). Moreover, it is known that the operator \( h_2(t) \) is the most general quasi-exactly-solvable second-order differential operator acting in the space of monomials [3]. In explicit form, the \( \pi \)-integral \( i_n^{(-1)} \) for \( h_2(t) \) has the form,
\[
i_n^{(-1)} = \frac{d^{n+1}}{dt^{n+1}},
\]
while the \( \pi \)-integral \( i_n^{(k)} \) is
\[
i_n^{(k)} = \frac{d^{n-k}}{dt^{n-k}} \int_{i=0}^{k} \left( \frac{d}{dt} - n + i \right), \quad k = 0, 1, \ldots, n.
\]
Hence, \( h_2(t) \) and all \( i_n^{(k)}(t) \), \( k = -1, 0, 1, \ldots, n \) have \( (n + 1) \) common eigenfunctions in the form of a polynomial in \( t \) of \( n \)th degree. For any \( i_n^{(k)} \) these polynomial eigenfunctions are the zero modes. This explains quasi-exact-solvability of (8) as a consequence of the existence of the \( \pi \)-integral \( i_n^{(k)} \), \( k = -1, 0, 1, \ldots, n \).

The commutativity (5) remains unchanged under a gauge (similarity) transformation of operators and space \( \mathcal{P}_n \) with a consequential change of variable \( t \mapsto x \),
\[
[\mathcal{H}(x), \mathcal{T}_n^{(k)}(x)] : \mathcal{V}_n \mapsto 0,
\]
where
\[
\mathcal{H}(x) = e^{-A(t)} h(t) e^{A(t)_{|\text{at}(x)}}, \quad \mathcal{T}_n^{(k)}(x) = e^{-A(t)} i_n^{(k)}(t) e^{A(t)_{|\text{at}(x)}}
\]
and
\[
\mathcal{V}_n = e^{-A(t)} (i_n^{(k)}(x) | 0 \leq k \leq n).
\]
Taking the gauge phase
\[
A(t) = \int \left( \frac{P_3}{P_4} \right) dt - \log x',
\]
and changing the variable
\[ x(t) = \pm \int \frac{dt}{\sqrt{P_4}}, \]
we arrive at \( \mathcal{H}(x) \) in the form of the Schrödinger operator
\[ \mathcal{H}(x) = -\frac{d^2}{dx^2} + V(x), \quad V(x) = (A'_x)^2(t(x)) - A''_x(t(x)) + P_2(t(x)). \]  
(13)
Taking the set of coefficients \( c_{\alpha \beta}, c_{\alpha} \) in (7) accordingly, one can obtain the Hamiltonians of all ten known 1D quasi-exactly-solvable Schrödinger equations with hidden algebra \( gl_2 \) [4–6]. In this case \( V_n \) is the subspace of the Hilbert space. The gauge transformed \( \pi \)-integrals occur, \( I_n^\pi(p, x) \equiv I_n(p, x) \), in general, they have the form of a finite-degree polynomial in \( p \) with singular coefficient functions, hence, they are singular functions on the phase space and can cease to exist for some trajectories. Usually, the only classical \( \pi \)-integral function, which is non-singular, is \( I_n^\pi(p, x) \equiv I_n(p, x) \),
\[ I_n(p, x) = \prod_{j=0}^{k} \left( t \frac{d}{dr} + tA'_x - n + j \right) \bigg|_{t=0}. \]
(16)
A natural question to ask is whether there exists a special trajectory (es) \( s(\tau, x) \) for which \( I_n(p, x) \) is a constant. For those trajectories the Poisson bracket vanishes, \( \{H(p, x), I_n(p, x)\}_{\tau(\tau, x)} = 0 \). Needless to say that these trajectories must correspond to points in (real) phase space: the Hamiltonian and a \( \pi \)-integral fix a point in the phase space, hence no dynamics has to occur. For all known (quasi)-exactly-solvable Hamiltonians, special trajectories exist. Usually, these trajectories are complex implying a dynamics in a complex phase space. While among real trajectories appear those corresponding to a particle at rest at extreme of the potential.

Let us consider a particular case of QES sextic anharmonic oscillator, which is the only 1D polynomial potential with QES property. It is described by a quadratic combination in \( gl_2 \)-generators (1),
\[ h_{2, \rho} = -4l^0_nJ^+_n + 4al^+_n + 4bl^0_n - 2(n + 1 + 2q)J^-_n + 2bn \]
(17)
or, as the second order differential operator,
\[ h_{2, \rho}(t) = -4 \frac{d^2}{dt^2} + 2(2ar^2 + 2bt - 1 - 2q) \frac{d}{dt} - 4ant, \]
where \( t \in [0, \infty) \) and \( a > 0, \forall b \) or \( a > 0, b > 0 \). Putting \( t = x^2 \) and choosing the gauge phase as follows:
\[ A = \frac{at^2}{4} + \frac{bt}{2} - \frac{q}{2} \ln t, \]
we arrive at the Hamiltonian (13) with the potential [4] (see also [7])
\[ V_\rho(x) = a^2x^6 + 2ax^4 + \left[ b^2 - (4n + 3 + 2q)a \right]x^2 - b(1 + 2q). \]  
(18)
for which \( q = 0 \) (\( q = 1 \)) and the first \((n + 1)\) eigenfunctions, even (odd) in \( x \), can be found algebraically. Of course, the number of those ‘algebraic’ eigenfunctions is nothing but the dimension of the irreducible representation of the \( gl_2 \)-algebra \((1)\). These \((n + 1)\) ‘algebraic’ eigenfunctions have the form
\[
\Psi^{(n)}(x) = x^n p_n(x^2)e^{-\frac{x^4}{2} - \frac{x^2}{2}},
\]
where \( p_n \) is a polynomial of degree \( n \). Quantum \( \pi \)-integral \( I_n(x) \equiv I^{(n)}_n(x) \) (cf \((14)\)) becomes
\[
I_n(x) = \frac{1}{2\pi i} \prod_{j=0}^{n} \left( x + \frac{d}{dx} + ax^4 + bx^2 - q - 2n + 2j \right).
\]
It is easy to give a direct proof that the Hamiltonian \( H_6(x) \) with the potential \((18)\) commutes with \( I_n(x) \),
\[
\left[ -\frac{d^2}{dx^2} + V_6(x), I_n(x) \right] : V_n \mapsto 0.
\]
Replacing the operator of momentum by the classical momentum \((15)\) in \( H_6(x) \) and \( I_n(x) \) we obtain a classical Hamiltonian \( H_6 \) and a classical analogue of the \( \pi \)-integral,
\[
I_n(p, x) = \frac{1}{2\pi i} \prod_{j=0}^{n} \left( ip + ax^4 + bx^2 - q - 2n + 2j \right).
\]
The Poisson bracket has a form
\[
\{H_6(p, x), I_n(p, x)\} = 2p \frac{dI_n(p, x)}{dx} - \frac{dV_6(x)}{dx} \xi Q(p, x),
\]
where \( Q(p, x) \) is a polynomial in \( p, x \) of degree \( n \) in \( p \). It is evident that the Poisson bracket vanishes for trajectories given by
\[
p = 0, \quad \frac{dV_6(x)}{dx} = 0.
\]
Therefore, the special trajectories correspond to a system at rest standing at (un)stable points of equilibrum. Each of then can be marked by a definite value of energy and zero momentum. The potential \((18)\) has in total five extrema, depending on the values of parameters some of them can be complex while \( x = 0 \) always corresponds to the real extremum. Hence, \( I_n(p, x) \) is a constant for these trajectories; this is a classical \( \pi \)-integral. Of course, the values of \( H_6, I_n(p, x) \) for these trajectories can be found explicitly. In particular, for the trajectory \( p = x = 0 \), the energy \( H_6(0, 0) = 0 \) and the \( \pi \)-integral
\[
I_n(0, 0) = (-1)^{n+1} \prod_{j=0}^{n} \left( \frac{q}{2} + n - j \right)
\]
at \( q = 1 \) and \( I_n(0, 0) = 0 \) at \( q = 0 \). Perhaps, it is worth noting that if \( b = 0 \) the potential \((18)\) has two real and two complex minima at \((x^2)_{\text{min}} = \pm \sqrt{\frac{2n+2q+3}{3a}}\). The constants of motion are
\[
H_6(0, (x^2)_{\text{min}}, b = 0) = \mp 2\sqrt{a} \left( \frac{4n + 2q + 3}{3} \right)^{\frac{1}{2}};
\]
\[
I_n(0, (x^2)_{\text{min}}, b = 0) = \prod_{j=0}^{n} \left( j + \frac{3 - q - 2n}{6} \right).
\]
Similar analysis can be carried out for all other QES problems \((7), (12), (13)\).
There exists a class of exactly-solvable multi-dimensional quantum systems with rational and trigonometric potentials (the Calogero–Sutherland models). Each system is associated
with a Lie algebra $g$ of rank $N$, with root space $\Delta$. In the case of a rational potential the Hamiltonian in the Cartesian coordinates has the form,

$$
\mathcal{H}^{(r)} = \frac{1}{2} \sum_{k=1}^{N} \left[ \frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 \right] + \frac{1}{2} \sum_{a \in R_+} \nu_{|a|} (\nu_{|a|} - 1) \frac{|a|^2}{(\alpha \cdot x)^2},
$$

where $R_+ \in \Delta$ is a set of positive roots in the root space $\Delta$, $x$ is a position vector and $\nu_{|a|}$ are coupling constants (parameters) which depend on the root length. The Hamiltonian (23) is invariant with respect to the Weyl (Coxeter) group transformation, which is the discrete symmetry group of the corresponding root space. If some roots are of the same length, then their $\nu_{|a|}$ have to be equal, if all roots are of the same length like for $A_n$, then all $\nu_{|a|} = \nu$. The configuration space for (23) is the Weyl chamber. The ground state wavefunction is written explicitly,

$$
\Psi_0^{(r)}(x) = \prod_{a \in R_+} |(\alpha \cdot x)|^{\nu_{|a|}} e^{-a^2/2},
$$

where $\sum x^2 = t_2$ is the lowest order (quadratic) Weyl polynomial invariant. The Hamiltonian (23) is super-integrable with $(N - 1)$ integrals of the relative motion in involution (see e.g. [8]) and one extra integral of the second order due to a separation of the relative radial variable from the relative angular motion (see e.g. [9]).

In the case of trigonometric potential the Hamiltonian in the Cartesian coordinates has the form,

$$
\mathcal{H}^{(t)} = \frac{1}{2} \sum_{k=1}^{N} \left[ \frac{\partial^2}{\partial y_k^2} + \frac{\beta^2}{8} \sum_{a \in R_+} \mu_{|a|} (\mu_{|a|} - 1) \frac{|a|^2}{\sin^2 \frac{\beta}{2} (\alpha \cdot y)^2},
$$

where $R_+ \in \Delta$ is a set of positive roots in the root space $\Delta$, $y$ is a position vector and $\mu_{|a|}$ are coupling constants depending on the root length. The Hamiltonian (25) is invariant with respect to the affine Weyl (Coxeter) group transformation, which is the discrete symmetry group of the corresponding root space plus translations. For roots of the same length the coupling constants $\mu_{|a|}$ are equal. The configuration space for (25) is the Weyl alcove. The ground state wavefunction is written explicitly,

$$
\Psi_0^{(t)}(y) = \prod_{a \in R_+} \left| \sin \frac{\beta}{2} (\alpha \cdot y) \right|^{\mu_{|a|}}.
$$

The Hamiltonian (25) is completely integrable with $(N - 1)$ integrals of the relative motion in involution (see e.g. [8]).

Extensive studies [10–19] led to the conclusion that the gauge-rotated Hamiltonian

$$
(\Psi_0^{(r)}(H^{(r)} - E_0^{(r)})\Psi_0^{(r)} = H^{(r)},
$$

in the space of orbits of (affine) Weyl (Coxeter) group becomes an algebraic operator—a differential operator with polynomial coefficients. Here $E_0^{(r)}$ is the ground state energy of rational (trigonometric) model. It implies that a change of variables should be done from the Cartesian coordinates to the Weyl (Coxeter) polynomial invariants,

$$
\tau^{(r)}_a(x) = \sum_{a \in \Omega} (\alpha \cdot x)^a,
$$

where $a$s are the degrees of the Weyl (Coxeter) group and $\Omega$ is an orbit, for the case of a rational potential. In the case of a trigonometric potential the Weyl (Coxeter) trigonometric invariants have to be taken as new variables,

$$
\tau_a(y) = \sum_{a \in \Omega} e^{i\beta(\alpha \cdot y)},
$$

as
where $O_{a}$ is the orbit generated by fundamental weight $w_{a}$. Furthermore, it was shown that any such algebraic operator $h^{(c,t)}$ has infinitely-many finite-dimensional invariant subspaces in a form of the `triangular' linear space of polynomials

$$
\mathcal{P}_{n,f}^{(d)} = \{ t_{1}^{p_{1}}t_{2}^{p_{2}} \ldots t_{d}^{p_{d}} | 0 \leq f_{1}p_{1} + f_{2}p_{2} + \ldots + f_{d}p_{d} \leq n \},
$$

where the `grades' $f$s are positive integer numbers which are ordered in such a way that they are non-decreasing with the growth of $i$. If a notion of the characteristic vector is introduced

$$
\vec{f} = (f_{1}, f_{2}, \ldots f_{d}),
$$

it was shown that for a given root space $\Delta$, the vector $\vec{f}$ is either the highest root or the Weyl co-vector, with the only exception for the $E_{8}$ rational case where $\vec{f}$ is special (for a discussion see [19]). For any space (29) one can indicate an infinite-dimensional algebra of differential operators for which $\mathcal{P}_{n,f}^{(d)}$ is a common finite-dimensional invariant subspace. Such an algebra is called hidden. It is evident that the algebraic operator $h^{(c,t)}$ is an element of the hidden algebra. For all studied root spaces this algebra is finitely generated. It is found that among generating elements of the hidden algebra, the most symmetric generator (the Euler–Cartan operator) is always present

$$
\mathcal{J}_{n}^{0} = \sum_{i=1}^{d} f_{i}t_{i} \frac{\partial}{\partial t_{i}} - n, \tag{30}
$$

which has zero grading and plays the role of a constant acting as an identity operator. It defines the highest weight vector and acts as a filtration mapping $\mathcal{P}_{n,f}^{(d)} \rightarrow \mathcal{P}_{n-1,f}^{(d)}$. This generator allows us to construct the $\pi$-integral of zero grading

$$
i_{n}(t) = \prod_{j=0}^{n}(\mathcal{J}_{n}^{0} + j) \tag{31}
$$

such that

$$
[h^{(c,t)}, i_{n}(t)] : \mathcal{P}_{n,f}^{(d)} \mapsto 0. \tag{32}
$$

This $\pi$-integral has no finite-dimensional invariant subspaces of the dimension higher than $\dim(\mathcal{P}_{n,f}^{(d)})$. Making a gauge rotation of (31) with $\Psi_{0}^{(c,t)}$ (see (24) and (26), respectively) and changing variables $t, \tau$ (see (27) and (28), respectively) back to the Cartesian coordinates we arrive at the quantum $\pi$-integral $\mathcal{I}_{n,f}^{(c,t)}(x)$. It is a differential operator of the $(n + 1)$th order. Under such a transformation the triangular space of polynomials $\mathcal{P}_{n,f}^{(d)}$ becomes the space

$$
\mathcal{V}_{n,f}^{(d)} = \Psi_{0}^{(c,t)} \mathcal{P}_{n,f}^{(d)}.
$$

Needless to say that the Hamiltonian $\mathcal{H}^{(c,t)}(x)$ commutes with $\mathcal{I}_{n,f}^{(c,t)}(x)$,

$$
[\mathcal{H}^{(c,t)}(x), \mathcal{I}_{n,f}^{(c,t)}(x)] : \mathcal{V}_{n,f}^{(d)} \mapsto 0.
$$

Replacing the operator of momentum by the classical momentum (15) in $\mathcal{H}^{(c,t)}$ and $\mathcal{I}_{n,f}^{(c,t)}(x)$ we obtain a classical Hamiltonian $H^{(c,t)}(p, x)$ and a classical analogue of the $\pi$-integral $I_{n,f}^{(c,t)}(p, x)$. It can be easily checked that the Poisson bracket $[H^{(c,t)}, I_{n,f}^{(c,t)}]$ must vanish for trajectories which correspond to the particle at extremes of the potential at zero momentum similar to (22). It seems natural to guess the existence of extra $(N - 3)$ particular classical integrals for rational and $(N - 2)$ for trigonometric models which would allow us to fix a point

\footnote{1 For a given root space $\Delta$ and a fixed $\beta$, there exist $N$ functionally independent trigonometric Weyl invariants $r_{a}$ generated by $N$ fundamental weights $w_{a}$.}
in the phase space. One can expect the existence of other real trajectories for which the Poisson bracket $\{H(r,t), I_n(r,t)\}$ vanishes. It seems that the important class of possible trajectories is a class of closed, periodic trajectories. Namely, such trajectories appear for $A_2, BC_2, G_2$ rational and $I_2(k)$ models [20] making these models super-integrable and, eventually, their quantum counterparts exactly solvable.

In this paper we introduced a notion of particular integrability, both quantum and classical. In quantum mechanics particular integrability implies the existence of a certain number of common eigenfunctions of the Hamiltonian and a particular integral. In classical mechanics it realizes the idea of the existence of a number of (special) trajectories which are characterized by additional, particular constant of motion. Hence, in such a way the property of particular integrability is preserved under the quantization, although the number of common eigenfunctions does not seem related to the number of special trajectories. We find an explicit form for a particular integral for all one-dimensional (quasi)-exactly-solvable problems as well as for Calogero–Sutherland models for all root spaces. In both cases the particular integral looks especially easy in action on the space of polynomials parameterized by symmetric coordinates (it coincides with the space of orbits in the case of (affine) Weyl (Coxeter) symmetry). Its common eigenfunctions with the gauge-rotated Hamiltonian written in symmetric coordinates are given by finite-degree polynomials in those coordinates. We found special trajectories which correspond to standing particle at an extreme of the potential. We cannot exclude the existence of other special, super-integrable trajectories for Calogero–Sutherland models as well as extra particular integrals.

The situation is different for the case of the magnetic Hamiltonians. For example, for two charges on the plane subject to a constant magnetic field the special trajectories appear as ‘well organized’ among, in general, chaotic motion: they are either simply closed, concentric or parallel corresponding to a free motion [21]. For each of these trajectories a certain number of $\pi$-particular constants of motion can be indicated. In the quantum case the common eigenfunctions of the Hamiltonian and $\pi$-particular integral(s) have the form of polynomial multiplied by some factor [22].

Acknowledgments

The author is grateful to S P Novikov and P Winternitz for interest in the work and valuable discussions. This research is supported in part by DGAPA grant IN109512 (Mexico). The author also thanks the University Program FENOMEC (UNAM, Mexico) for partial support.

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