Quantifier-free descriptions for quantifier solutions to interval linear systems of relations

Irene A. Sharaya

Institute of Computational Technologies SB RAS
and Novosibirsk State University,
Novosibirsk, Russia

Abstract

We study systems of relations of the form \( Ax \sigma b \), where \( \sigma \) is a vector of binary relations with the components “=”, “\( \geq \)”, and “\( \leq \)”, and the parameters (elements of the matrix \( A \) and right-hand side vector \( b \)) can take values from prescribed intervals. What is considered to be the set of its solutions depends on which logical quantifier is associated with each interval-valued parameter and what is the order of the quantifier prefixes for certain parameters. For solution sets that correspond to the quantifier prefix of a general form, we present equivalent quantifier-free descriptions in the classical interval arithmetic, in Kaucher complete interval arithmetic and in the usual real arithmetic.

Keywords: interval linear systems of equations and inequalities, quantifier elimination, Kaucher interval arithmetic.

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1 Introduction

1.1 Quantifier solutions to interval linear systems

In the classical interval arithmetic \( \mathbb{I} \), an interval is a non-empty bounded connected closed subset of the real line \( \mathbb{R} \). According to the notation standard \[9\], we will denote interval objects in bold type (\( A, B, \ldots, y, z \)), in contrast to usual point (non-interval) quantities that are not specifically distinguished.

We consider a system of linear equations and inequalities of the form

\[
Ax \sigma b, \quad A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m, \quad \sigma \in \{=, \geq, \leq\}^m, \quad m, n \in \mathbb{N},
\]

where \( x \) is a vector of unknowns, \( \sigma \) is a vector of binary relations, with the components “=”, “\( \geq \)” and “\( \leq \)”, and every parameter \( u \in \mathbb{R} \) (which may be an element of the matrix \( A \) or of the right-hand side \( b \)) can take values within the prescribed eponymous interval \( u \) from \( \mathbb{I} \).

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As an example of such “mixed” systems, we can consider a $3 \times 2$-system:

\[
\begin{align*}
2x_1 - 3x_2 &\leq 4, \\
5x_1 + 6x_2 &= 7, \\
-x_1 + 4x_2 &\geq 5.
\end{align*}
\]

With each parameter $u$, we connect either the universal quantifier “$\forall$” or the existential quantifier “$\exists$”, as well as the corresponding elementary quantifier prefix ($\forall u \in u$) or ($\exists u \in u$). Such interval uncertainty of parameters can be specified by the interval matrix $A \in \mathbb{IR}^{m \times n}$, the quantifier matrix $A$ of the same size as $A$, the interval vector $b \in \mathbb{IR}^m$ and the quantifier vector $\beta$ of length $m$. All the elementary quantifier prefixes can be written down in an arbitrary order, and we denote the resulting prefix of the length $m(n + 1)$ as $Q(A, b, A, \beta)$.

**Definition 1** For given interval matrix $A$, interval vector $b$, and quantifier prefix $Q(A, b, A, \beta)$, the interval-quantifier linear system of relations, or interval-quantifier linear system in short, will be called the predicate of the form $Q(A, b, A, \beta)(Ax \sigma b)$. A vector $\tilde{x} \in \mathbb{R}^n$ is referred to as a solution to the interval-quantifier linear system if the predicate $Q(A, b, A, \beta)(Ax \sigma b)$ takes the value “true” in $\tilde{x}$.

This definition is, in fact, a further refinement of the general ideas expressed in [20]. Also, the above introduced interval-quantifier linear systems are a natural generalization of interval linear systems, which have long been studied in interval analysis. *Interval linear system* of the form $Ax \sigma b$ is a formal record, for which we specially stipulate what is considered a solution in each specific case. Usually, interval linear systems of only equations or of only inequalities of the same sign are considered, while their solutions are taken as formal (algebraic) solutions, AE-solutions, strong solutions, weak solutions, tolerable solutions, controllable solutions (sometimes called simply “control solutions”), etc. (see [20,21; 3, chapter 2] and references in these publications).

In order to agree with the existing terminology in this field, the solutions to the interval-quantifier linear system $Q(A, b, A, \beta)(Ax \sigma b)$ will also be called the quantifier solutions of the interval linear system $Ax \sigma b$. Notice that AE-solutions, strong solutions, weak solutions, tolerable solutions, controllable solutions to the system $Ax \sigma b$ are subsumed under the quantifier solutions. In particular, the AE-solutions to interval linear systems of equations are their quantifier solutions for which the determining predicate $Q(A, b, A, \beta)(Ax \sigma b)$ has the so-called AE-form, that is, in which all occurrences of the universal quantifier “$\forall$” precede the occurrences of the existential quantifier “$\exists$”.

The notation $Q(A, b, A, \beta)(Ax \sigma b)$ defines all possible interval-quantifier linear systems in parametric form. The parameters of the description are $A$, $b$, $A$, $\beta$, $\sigma$ and the order of elementary quantifier prefixes in $Q$ (since the elementary prefixes with different quantifiers do not always commute). Imposing additional constraints on the parameters, we obtain different classes (subsets) of interval-quantifier linear systems. For example, if we require equality to be the value of each component of the relation vector $\sigma$, then we obtain a class of interval-quantifier systems of linear equations.

1.2 Transition to quantifier-free descriptions

Interval-quantifier linear systems and their solutions were introduced in the previous section through a logical predicate of the first order. Predicative description is close to the formulation of practical problems, but it allows very limited means of theoretical investigation and is not at all suitable for calculations. As a consequence, the following problem arises:
Problem  For the widest possible subset of interval-quantifier linear systems, find a convenient quantifier-free description of their solutions in algebraic systems with sufficiently developed tools for equivalent transformations, study and computation.

Usually, the solution sets to interval systems of equations and inequalities are described using real arithmetic in \( \mathbb{R} \) [3, chapter 2; 2, pp. 93–95; 5, 10, 11, 15, 16, 22], since it is simple, familiar, has good properties, and we can apply developed numerical methods in \( \mathbb{R} \). For various subclasses of interval quantifier systems of linear equations, a number of quantifier-free descriptions have been obtained in interval arithmetics [1, 14, 20, 21]. Despite poor algebraic properties of the interval arithmetics (the absence of the distributivity, etc.), these descriptions turned out to be very useful. For instance, the description of the AE-solution sets of interval systems of linear equations made it possible to construct a general theory of these solutions and interval numerical methods for inner and outer estimation of the AE-solution sets [20, 21].

The features of the quantifier-free descriptions proposed in this paper are as follows:

1. They expand the class of the interval-quantifier linear systems for which a convenient description of solutions can be given in comparison with those known descriptions where non-negativity of \( x \) is not required. (The nonnegativity requirement on the vector of unknowns can be formulated as an additional restriction on the parameters \( A, b \) and \( \sigma \). A. Vatolin in [22] obtained quantifier-free descriptions for solutions of general interval-quantifier linear systems, but his result is only valid under nonnegativity condition on the variables, which is quite severe limitation in practice. The class \( Q^\sigma \) of interval-quantifier linear systems we discuss in the present paper has no restrictions on \( A, b \) and \( \sigma \), but it has a restriction on the order of the elementary quantifier prefixes.)

2. Quantifier-free descriptions of the solutions are obtained in ordinary real arithmetic \( \mathbb{R} \), classical interval arithmetic \( \mathbb{I}\mathbb{R} \), and in Kaucher interval arithmetic \( \mathbb{K}\mathbb{R} \). This enables us to carry out investigation of the solution sets and computation with them by both real and interval methods.

2 Necessary facts about interval arithmetics

In this section, we give the necessary information on various interval arithmetics. The desire to improve the properties of the classical interval arithmetic \( \mathbb{I}\mathbb{R} \) led to the appearance of its various extensions. The most popular of them is Kaucher interval arithmetic \( \mathbb{K}\mathbb{R} \) developed by E. Kaucher [8]. E. Gardeñes and A. Trepat [4] and then S. Markov [13] proposed another similar constructions. All these researchers constructed extensions of the classical interval arithmetic \( \mathbb{I}\mathbb{R} \) on the basis of different principles, which was reflected in the names of the corresponding algebraic structures: extended interval arithmetic [8], modal interval arithmetic [4], arithmetic of directed intervals [13]. However, despite the difference in their construction, all three algebraic systems coincide up to notation.

Interval in \( \mathbb{K}\mathbb{R} \) is just a record of the form \([a, b]\), where \( a, b \in \mathbb{R} \). In \( \mathbb{I}\mathbb{R} \), the values \( a \) and \( b \) should additionally satisfy the requirement \( a \leq b \). Intervals are also denoted by small boldface letters, e.g., \( \mathbf{u} \in \mathbb{K}\mathbb{R} \). If \( \mathbf{u} \) and \([a, b]\) denote the same interval, then \( a \) is called left (lower) endpoint of the interval, which is written as \( \underline{u} \), and \( b \) is called the right (upper) endpoint of the interval \( \mathbf{u} \), which is written as \( \overline{u} \). Therefore, \( \mathbf{u} \equiv [\underline{u}, \overline{u}] \). The intervals from \( \mathbb{I}\mathbb{R} \), as was stated in Introduction, can be considered as subsets of the real axis \( \mathbb{R} \):

\[
[\underline{u}, \overline{u}] = \{ u \in \mathbb{R} \mid \underline{u} \leq u \leq \overline{u} \}.
\]
In this paper, we will mainly use concepts and properties of Kaucher interval arithmetic, and we present them below.

Two intervals are considered equal if both their left and right endpoints coincide:

\[ u = v \iff \begin{cases} u = v, \\ \underline{u} = \underline{v}. \end{cases} \]

The inclusion relation "\( \subseteq \)" in \( \mathbb{K} \mathbb{R} \) continues the inclusion relation in \( \mathbb{R} \mathbb{R} \) that considers intervals as sets. So, we have:

\[ u \subseteq v \iff \begin{cases} u \geq v, \\ \underline{u} \leq \underline{v}. \end{cases} \quad (1) \]

In particular, \([6, 3] \subseteq [4, 5]\).

The operations of taking the least upper bound (supremum) and greatest lower bound (infimum) with respect to inclusion are introduced for families of intervals bounded from above and from below respectively, using the infimum and supremum in \( \mathbb{R} \):

\[ \bigvee_{i \in I} u_i := \sup_{i \in I} u_i := \left[ \inf_{i \in I} u_i, \sup_{i \in I} u_i \right], \]

\[ \bigwedge_{i \in I} u_i := \inf_{i \in I} u_i := \left[ \sup_{i \in I} u_i, \inf_{i \in I} u_i \right]. \]

We need the following unary operations on intervals:

- \( \text{mid} u := \hat{u} := (\underline{u} + \overline{u})/2 \) — the midpoint,
- \( \text{rad} u := \hat{u} := (\overline{u} - \underline{u})/2 \) — the radius,
- \( \text{dual} u := [\underline{u}, \overline{u}] \) — the dualization, i.e., swapping the endpoints of the interval,
- \( \text{pro} u := \begin{cases} u, & \text{if } u \leq \overline{u}, \\ \text{dual } u, & \text{if } u > \overline{u}, \end{cases} \) — the proper projection of the interval.

Notice that the dualization makes sense only in \( \mathbb{K} \mathbb{R} \).

Arithmetic operations of addition, subtraction, multiplication and division are determined through the corresponding real operations and taking exact lower and upper bounds by inclusion so that

\[ u \ast v = \bigwedge_{\text{pro } u} \bigvee_{\text{pro } u} (u \ast v), \quad \text{where } \bigwedge_{\text{pro } u} := \begin{cases} \bigvee_{\text{pro } u}, & \text{if } u \leq \overline{u}, \\ \bigwedge_{\text{pro } u}, & \text{if } u > \overline{u}, \end{cases} \]

for each \( \ast \in \{+, -, \cdot, /\} \). Naturally, division is determined only for such intervals \( v \) that \( 0 \not\in \text{pro } v \). The addition and multiplication are commutative. The addition is defined “by endpoints”:

\[ u + v = [\underline{u} + \underline{v}, \overline{u} + \overline{v}], \quad (2) \]
The real numbers $\lambda \in \mathbb{R}$ are identified with intervals of zero radius $[\lambda, \lambda]$. Multiplication of an interval by the number $\lambda \in \mathbb{R}$ satisfy the following properties:

$$
\lambda \mathbf{u} = \begin{cases} 
[\lambda \mathbf{u}, \lambda \mathbf{u}], & \text{for } \lambda \geq 0, \\
[\lambda \mathbf{u}, \lambda \mathbf{u}], & \text{for } \lambda \leq 0;
\end{cases}
$$

(3)

$$(\text{dual } \mathbf{u}) \lambda \equiv \text{dual}(\mathbf{u} \lambda) = [\mathbf{u} \lambda, \mathbf{u} \lambda].$$

(4)

The symbol $-\mathbf{u}$ means the result of multiplication $(-1) \cdot \mathbf{u}$, not taking the opposite interval for $\mathbf{u}$ with respect to the addition.

The matrices and vectors whose elements are intervals are called interval matrices and interval vectors respectively. We denote by $A_i$, the $i$-th row of the matrix $A$. For interval vectors and matrices, the endpoints, the relations $=$ and $\subseteq$, the operations mid, rad, dual, pro, as well as addition, subtraction and multiplication by numbers are introduced componentwise. For example, $(\text{dual } A)_{ij} := \text{dual}(A_{ij}), (A - B)_{ij} := A_{ij} - B_{ij}, (-A)_{ij} = -A_{ij}$. The multiplication rules for interval vectors and matrices are interval extensions of analogous rules for the non-interval case:

$$(AB)_{ij} := \sum_k A_{ik} B_{kj}.$$  

(5)

Also, we need the property

$$(\text{dual } A)x = \text{dual}(Ax) \quad \text{for } A \in \mathbb{K}R^{m \times n}, \ x \in \mathbb{R}^n,$$  

(6)

which can be easily derived from the definition of interval matrix-vector product (a particular case of (5)) with the use of (2) and (4).

3 Results

3.1 Quantifier-free descriptions in interval arithmetics

First of all, we are going to develop quantifier-free descriptions in interval arithmetics for interval-quantifier linear systems and their solutions. We need the following notation:

$Q_c(A, b, \mathcal{A}, \beta)$ will denote a quantifier prefix obtained from $Q(A, b, \mathcal{A}, \beta)$ by deleting all those elementary prefixes that are not related to the $i$-th row of the system;

$Q_{c\exists}(A, b, \mathcal{A}, \beta)$ will denote a quantifier prefix of the form $Q(A, b, \mathcal{A}, \beta)$ satisfying the additional condition: for each $i \in \{1, \ldots, m\}$ in $Q_c(A, b, \mathcal{A}, \beta)$, the universal quantifiers (if any) precede the existential quantifiers (if any);

$Q^{AE}(A, b, \mathcal{A}, \beta)$ will denote a quantifier prefix of the form $Q(A, b, \mathcal{A}, \beta)$ with the additional condition: all the universal quantifiers (if any) precede all the existential quantifiers (if there are such quantifiers);

$A^\forall, A^\exists \in \mathbb{I}R^{m \times n}, b^\forall, b^\exists \in \mathbb{I}R^m, A^c \in K\mathbb{R}^{m \times n}, b^c \in K\mathbb{R}^m$ will denote interval vectors and matrices defined by the rules

$$
A^\forall_{ij} := \begin{cases} 
A_{ij}, & \text{if } A_{ij} = \forall, \\
0, & \text{if } A_{ij} = \exists,
\end{cases} \quad A^\exists_{ij} := \begin{cases} 
A_{ij}, & \text{if } A_{ij} = \exists, \\
0, & \text{if } A_{ij} = \forall,
\end{cases}
$$

(7)

$$
b^\forall_i := \begin{cases} 
b_i, & \text{if } \beta_i = \forall, \\
0, & \text{if } \beta_i = \exists,
\end{cases} \quad b^\exists_i := \begin{cases} 
b_i, & \text{if } \beta_i = \exists, \\
0, & \text{if } \beta_i = \forall,
\end{cases}
$$
\[ A^c_{ij} := \begin{cases} A_{ij}, & \text{if } A_{ij} = \forall, \\ \text{dual } A_{ij}, & \text{if } A_{ij} = \exists \end{cases} \quad b^c_i := \begin{cases} \text{dual } b_i, & \text{if } \beta_i = \forall, \\ b_i, & \text{if } \beta_i = \exists. \end{cases} \] (8)

The Gothic letter “c” as the superscript of \( A \) and \( b \) in formula (8) means “characteristic”. Overall, the matrix \( A^c \) and vector \( b^c \) will be called characteristic matrix and characteristic vector that correspond to the distribution of interval uncertainty types (A-type or E-type) described by the quantifier matrix \( A \) and vector \( \beta \) in the system under study [20, 21].

We should write out the property of interval-quantifier linear systems, which we will repeatedly apply in the sequel: each elementary quantifier prefix from \( Q(A, b, A, \beta) \) can be carried to the row of the system in which the parameter of this prefix is present. This means

\[ Q(A, b, A, \beta) (Ax \sigma b) \iff \bigwedge_{i \in \{1, \ldots, m\}} Q_i(A, b, A, \beta) (A_i x \sigma_i b_i). \] (9)

The substantiation for this property is that the system of relations \( (Ax \sigma b) \) is, in terms of logic, the conjunction of the relations, that is,

\[ \bigwedge_{i} (A_i x \sigma_i b_i). \]

Additionally, for the conjunction, there hold equivalences

\[ \forall t \in S \ (P_1(t) & P_2) \iff (\forall t \in S \ P_1(t)) & P_2, \]
\[ \exists t \in S \ (P_1(t) & P_2) \iff (\exists t \in S \ P_1(t)) & P_2, \]

where \( S \) is the set of values of the variable \( t \), \( P_1, P_2 \) are formulas, and \( P_2 \) does not depend on \( t \).

In view of (9), it is obvious that

\[ Q^{\forall\exists}(A, b, A, \beta) (Ax \sigma b) \iff Q^{AE}(A, b, A, \beta) (Ax \sigma b), \] (10)

i.e., the vector \( x \) is a solution to the system \( Q^{\forall\exists}(A, b, A, \beta) (Ax \sigma b) \) if and only if it is a solution to the system \( Q^{AE}(A, b, A, \beta) (Ax \sigma b) \). Thus, although the class of systems of the form \( Q^{\forall\exists}(A, b, A, \beta) (Ax \sigma b) \) is wider than the class of systems of the form \( Q^{AE}(A, b, A, \beta) (Ax \sigma b) \), the statements proved for the solutions to the system \( Q^{AE}(A, b, A, \beta) (Ax \sigma b) \) are trivially generalized into statements for the solutions to the system \( Q^{\forall\exists}(A, b, A, \beta) (Ax \sigma b) \).

Now let us turn to the interval-quantifier systems of linear equations. Quantifier-free descriptions for the widest subset of such systems have been obtained by S.P. Shary. In [18, 19], he first proved that

\[ Q^{AE}(A, b, A, \beta) (Ax = b) \iff A^\forall x - b^\forall \subseteq b^\exists - A^\exists x \iff A^c x \subseteq b^c. \] (11)

(see also [20]). Equivalence (10) allows us to make the following generalization of (11).

**Theorem 1**

\[ Q^{\forall\exists}(A, b, A, \beta) (Ax = b) \iff A^\forall x - b^\forall \subseteq b^\exists - A^\exists x \iff A^c x \subseteq b^c. \] (12)
Theorem 1 for the interval-quantifier system of equations $Q^3(A, b, A, \beta) (Ax = b)$ gives equivalent quantifier-free inclusion systems in $\mathbb{IR}$

$$A^\forall x - b^\forall \subseteq b^3 - A^3x$$

and in $\mathbb{KR}$

$$A^c x \subseteq b^c.$$

**Definition 2** Let us agree to refer to interval quantifier systems of relations, in which the vector of relations $\sigma$ consists of the same components, as *relationally homogeneous systems*.

The results of Theorem 1 is intended for systems of equations, and our immediate goal is to obtain similar results for relationally homogeneous systems of inequalities.

**Theorem 2**

$$Q(A, b, A, \beta) (Ax \geq b) \iff A^\forall x + A^3x \geq b^\forall + b^3 \iff A^c x \geq b^c. \tag{13}$$

$$Q(A, b, A, \beta) (Ax \leq b) \iff \overline{A^\forall x} + \overline{A^3x} \leq \overline{b^\forall} + \overline{b^3} \iff \overline{A^c x} \leq \overline{b^c}. \tag{14}$$

**Proof.** We carry out the proof only for the chain of equivalences (13). For (14), the substantiation is completely similar.

1) From (9), it follows that

$$Q(A, b, A, \beta) (Ax \geq b) \iff &_{i \in \{1, \ldots, m\}} Q_i(A, b, A, \beta) (A_i: x \geq b_i). \tag{15}$$

2) Using the fact that

$$A_i: x \geq b_i \iff \sum_{j=1}^{n} A_{ij} x_j + (-b_i) \geq 0$$

and that, for any continuous functions $h : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and an interval $u \in \mathbb{IR}$, there holds

$$(\forall u \in u) \ (h(u, x) + g(x) \geq 0) \iff \min_{u \in u} h(u, x) + g(x) \geq 0,$$

$$(\exists u \in u) \ (h(u, x) + g(x) \geq 0) \iff \max_{u \in u} h(u, x) + g(x) \geq 0,$$

enables us to get a quantifier-free description for $Q_{i,}(A, b, A, \beta) (A_i: x \geq b_i)$:

$$Q_{i,}(A, b, A, \beta) (A_i: x \geq b_i) \iff \sum_{j=1}^{n} \text{extr}_{A_{ij}}^A (A_{ij} x_j) + \text{extr}_{b_i}^{A_j} (-b_i) \geq 0, \tag{16}$$

where “extr” means conditional extremum, such that

$$\text{extr}^\forall = \min, \quad \text{extr}^3 = \max.$$
which are valid for any interval \( u \in \mathbb{I} \), and taking into account \([2]\), the sum of the extrema in (16) can be expressed in terms of the matrices \( A^v, A^3 \) and the vectors \( b^v, b^3 \) from (17):

\[
\sum_{j=1}^{n} \text{extr}_{A_{ij}}^\beta (A_{ij}x_j) + \text{extr}_{b_i}^\beta (-b_i) \geq 0 \iff A^v x + A^3 x \geq b^v + b^3.
\]  

(17)

4) From (15)–(17), it follows that

\[
Q(A, b, A, \beta) (Ax \geq b) \iff A^v x + A^3 x \geq b^v + b^3.
\]

5) Let us prove the second equivalence in the chain (13). For the matrix \( A^c \) in (8), we have

\[
[A^c x, A^c x] = A^c x = A^v x + (\text{dual } A^3)_x \text{ of } A^c, A^v, A^3 = (\text{dual } A^v x, \text{dual } A^3 x)
\]

(18)

and therefore \( A^c x = A^v x + A^3 x \). The definitions of the vectors \( b^c, b^v, \) and \( b^3 \) give

\[
[b^c, b^v] = b^c = \text{dual}(b^v) + b^3 = [b^v + b^3, b^v + b^3],
\]

(19)

hence \( b^c = b^v + b^3 \). Overall, we get

\[
A^v x + A^3 x \geq b^v + b^3 \iff A^c x \geq b^c.
\]

The proof of Theorem 2 is complete.

In the interval arithmetics \( \mathbb{I} \mathbb{R} \) and \( \mathbb{K} \mathbb{R} \), the relations “\( \geq \)” and “\( \leq \)” are applicable, and they are continuations of the same relations over \( \mathbb{R} \). For vectors, “\( \geq \)” and “\( \leq \)” are introduced componentwise. This allows us to formally refer to the records with \( A^v, A^3, b^v, b^3 \) in (13) and (14) as inequalities in classical interval arithmetic, while the records with \( A^c \) and \( b^c \) will be called inequalities in the Kaucher arithmetic. Still, in practice it is more convenient to understand all inequalities from (13) and (14) as componentwise inequalities in \( \mathbb{R}^m \).

From (9) and Theorems 2 the following remarkable fact follows: the solution sets of interval-quantifier systems of linear inequalities with arbitrary \( \sigma \in \{\geq, \leq\}^m \) does not depend on the order of the elementary quantifier prefixes, that is, all interval-quantifier systems of linear inequalities with the same \( A, b, A, \beta \) and \( \sigma \) have the same solution sets. This property essentially distinguishes interval systems of inequalities from interval systems of equations.

We give a corollary of Theorems 1 and 2 which establishes the relation between AE-solution sets of interval systems of linear equations and quantifier solution sets of interval relationally homogeneous systems of linear inequalities.

**Corollary 1**

\[
Q^{AE}(A, b, A, \beta) (Ax = b) \iff Q^{\geq}(A, b, A, \beta) (Ax = b) \iff \begin{cases} Q(A, b, A, \beta)(Ax \geq b), \\ Q(A, b, A, \beta)(Ax \leq b). \end{cases}
\]

The proof is given by the following chain of equivalences:

\[
\begin{cases} Q(A, b, A, \beta) (Ax \geq b) \\ Q(A, b, A, \beta) (Ax \leq b) \end{cases} \iff \begin{cases} A^c x \geq b^c \\ A^c x \leq b^c \end{cases} \iff A^c x \subseteq b^c.
\]

\[
\begin{cases} \text{Theorem 2} \end{cases} \iff \begin{cases} Q^{\geq}(A, b, A, \beta) (Ax = b) \iff Q^{AE}(A, b, A, \beta) (Ax = b). \end{cases}
\]

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Theorems \(\text{(1)}\) and \(\text{(2)}\) give quantifier-free descriptions for relationally homogeneous systems. Next, we turn to the consideration of systems with an arbitrary relationship vector \(\sigma\).

**Definition 3** We denote by \(Q^\sigma(A, b, A, \beta)\) a quantifier prefix of the form \(Q(A, b, A, \beta)\) satisfying the following condition: if \(\sigma_i\) is “\(=\)”, then the universal quantifiers (if any) in \(Q_i^\sigma(A, b, A, \beta)\).

**Definition 4** The class \(Q^\sigma\) within the set of all interval-quantifier systems of linear relations is a subset consisting of all systems of the form \(Q^\sigma(A, b, A, \beta) (Ax \sigma b)\).

The following theorem gives a quantifier-free description of the class \(Q^\sigma\) in the interval arithmetics \(\mathbb{K}\mathbb{R}\) and \(\mathbb{I}\mathbb{R}\), with the use of componentwise inequalities from \(\mathbb{R}^m\), where \(\mathbb{R}\) denotes the extended real axis, i.e., \(\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}\).

**Theorem 3**

\[
Q^\sigma(A, b, A, \beta) (Ax \sigma b) \iff \begin{cases}
\frac{A^\tau x}{A^\tau x} \geq \frac{b^\tau}{b^\tau} + u, \\
\frac{A^\tau x}{A^\tau x} \leq \frac{b^\tau}{b^\tau} + v,
\end{cases} 
\iff \begin{cases}
\frac{A^\tau x + A^3 x}{A^\tau x + A^3 x} \geq \frac{b^\tau + b^3}{b^\tau + b^3} + u, \\
\frac{A^\tau x + A^3 x}{A^\tau x + A^3 x} \leq \frac{b^\tau + b^3}{b^\tau + b^3} + v,
\end{cases}
\tag{20}
\]

where \(A^\tau\) and \(b^\tau\) are from \(\mathbb{K}\), \(A^\tau, A^3, b^\tau, b^3\) are from \(\mathbb{I}\), while the vectors \(u, v \in \mathbb{R}^m\) are defined as

\[
u_i := \begin{cases}
0, & \text{if } \sigma_i \text{ is "} = \text{" or "} \geq \text{"}, \\
-\infty, & \text{if } \sigma_i \text{ is "} \leq \text{"}, \\
\infty, & \text{if } \sigma_i \text{ is "} \geq \text{"}.
\end{cases}
\]

**Proof (step by step).**

1) Due to the fact that each interval parameter (the element of the matrix \(A\) or the vector \(b\)) enters only one row of the system \(Ax \sigma b\), we have \(\text{(9)}\) and, in particular,

\[
Q^\sigma(A, b, A, \beta) (Ax \sigma b) \iff \bigwedge_{i \in \{1, \ldots, m\}} Q_i^\sigma(A, b, A, \beta) (A_i x \sigma_i b_i). \tag{21}
\]

2) We eliminate quantifier prefixes in the predicate \(Q_i^\sigma(A, b, A, \beta) (A_i x \sigma_i b_i)\) using Theorems \(\text{(3)}\) and \(\text{(4)}\) based on the specific values of \(\sigma_i\):

\[
Q_i^\sigma(A, b, A, \beta) (A_i x = b_i) \iff (A^\tau x)_i \subseteq b^\tau_i \iff ((A^\tau x)_i \geq b^\tau_i) \& ((A^\tau x)_i \leq b^\tau_i),
\]

\[
Q_i^\sigma(A, b, A, \beta) (A_i x \geq b_i) \iff (A^\tau x)_i \geq b^\tau_i \iff ((A^\tau x)_i \geq b^\tau_i) \& ((A^\tau x)_i \leq \infty),
\]

\[
Q_i^\sigma(A, b, A, \beta) (A_i x \leq b_i) \iff (A^\tau x)_i \leq b^\tau_i \iff ((A^\tau x)_i \geq -\infty) \& ((A^\tau x)_i \leq b^\tau_i).
\]

3) Introducing the vectors \(u\) and \(v\), we pass to the matrix-vector inequalities

\[
Q^\sigma(A, b, A, \beta) (Ax \sigma b) \iff \begin{cases}
\frac{A^\tau x}{A^\tau x} \geq \frac{b^\tau}{b^\tau} + u, \\
\frac{A^\tau x}{A^\tau x} \leq \frac{b^\tau}{b^\tau} + v,
\end{cases}
\]

4) The equivalence

\[
\begin{align*}
\frac{A^\tau x}{A^\tau x} \geq \frac{b^\tau}{b^\tau} + u, \\
\frac{A^\tau x}{A^\tau x} \leq \frac{b^\tau}{b^\tau} + v,
\end{align*} \iff \begin{align*}
\frac{A^\tau x + A^3 x}{A^\tau x + A^3 x} \geq \frac{b^\tau + b^3}{b^\tau + b^3} + u, \\
\frac{A^\tau x + A^3 x}{A^\tau x + A^3 x} \leq \frac{b^\tau + b^3}{b^\tau + b^3} + v,
\end{align*}
\]
is obvious in view of (18) and (19). The proof of Theorem 3 is complete.

Convenient quantifier-free representations for the class $Q^\sigma$ can be obtained from Theorem 3, if we introduce the sets of intervals $K_{\mathbb{R}} = \{ [z, \tau] \mid z, \tau \in \mathbb{R}, z \leq \tau \}$ and continue relation “⊆” according to rule (1). Then

\[
Q^\sigma (A, b, A, \beta) (Ax \sigma b) \iff A^\ell x \subseteq b^\ell + w \iff A^\gamma x - b^\gamma \subseteq b^\ell - A^\gamma x + w, \quad (22)
\]

where $A^\ell$ and $b^\ell$ from (8), $A^\gamma$, $b^\gamma$, $b^\ell$ from (7), and the interval vector $w \in \mathbb{I}^m$ is such that

\[
w_i := \begin{cases} 
0, & \text{if } \sigma_i \text{ is } = \\
[0, \infty], & \text{if } \sigma_i \text{ is } \geq \\
[-\infty, 0], & \text{if } \sigma_i \text{ is } \leq 
\end{cases}
\]

The inclusion

\[
A^\ell x \subseteq b^\ell + w
\]

provides a quantifier-free description of the solution set to the quantifier interval linear system $Q^\sigma (A, b, A, \beta) (Ax \sigma b)$ in any interval arithmetic that extends the Kaucher arithmetic to the set $K_{\mathbb{R}}$. An example of such an extension is given in [7]. We agree to denote the arithmetic extension, as well as its basic set, through $K_{\mathbb{R}}$.

Similarly, the inclusion

\[
A^\gamma x - b^\gamma \subseteq b^\ell - A^\gamma x + w
\]

provides a quantifier-free description of the solution set to the system $Q^\sigma (A, b, A, \beta) (Ax \sigma b)$ in interval arithmetic that extends $\mathbb{I}$ to the set $\mathbb{I}_{\mathbb{R}}$. Examples of the extension of the classical interval arithmetic to a set of intervals with infinite endpoints are described in [12]. Let us agree to refer to any such extension as arithmetic $\mathbb{I}_{\mathbb{R}}$. Thus, the relation (22) gives quantifier-free descriptions of the solution sets to quantifier interval linear systems of class $Q^\sigma$ in the interval arithmetics $K_{\mathbb{R}}$ and $\mathbb{I}_{\mathbb{R}}$.

Comparing the quantifier-free descriptions obtained for the solution sets to quantifier interval linear systems, we can see that,

- on the one hand, the quantifier-free description in $K_{\mathbb{R}}$ ($K_{\mathbb{R}}$) is much more remote from the initial data $A$, $b$, $A$ and $\beta$ due to multilevel notation, and,
- on the other hand, the description in $K_{\mathbb{R}}$ ($K_{\mathbb{R}}$) is more concise and convenient for analysis than a similar description in $\mathbb{I}$ ($\mathbb{I}$).

### 3.2 Quantifier-free descriptions in real arithmetic

In this section, we derive quantifier-free descriptions of the quantifier solution sets to interval linear systems in the real arithmetic $\mathbb{R}$. To do that, we will need Hadamard product of matrices (entrywise product), denoted by the symbol “$\odot$” (see e.g. [6]). Hadamard product is defined for two matrices of the same dimensions and produces another matrix in which the $ij$-th element is the product of the $ij$-th elements of the original matrices:

\[
(A \odot B)_{ij} = A_{ij}B_{ij}.
\]

Also, notice that the operation of taking the modulus of a vector is understood componentwise. If, for instance, $x \in \mathbb{R}^n$, then $|x|$ is a nonnegative vector with the components $|x_i| = |x_i|$. 


Theorem 4

\[ Q^{\equiv}(A, b, \mathcal{A}, \beta) \ (Ax = b) \iff |Ax - \tilde{b}| \leq (A^s \circ \hat{A})|x| + \beta^s \circ \tilde{b}, \]  
\[ Q(A, b, \mathcal{A}, \beta) \ (Ax \geq b) \iff \tilde{b} - \hat{A}x \leq (A^s \circ \hat{A})|x| + \beta^s \circ \tilde{b}, \]  
\[ Q(A, b, \mathcal{A}, \beta) \ (Ax \leq b) \iff \hat{A}x - \tilde{b} \leq (A^s \circ \hat{A})|x| + \beta^s \circ \tilde{b}, \]  
\[ Q^s(A, b, \mathcal{A}, \beta) \ (Ax \sigma b) \iff \text{abs}^s(\hat{A}x - \tilde{b}) \leq (A^s \circ \hat{A})|x| + \beta^s \circ \tilde{b}, \]

where

\[ A^s_{ij} = \begin{cases} 1, & \text{if } A_{ij} = \exists, \\ -1, & \text{if } A_{ij} = \forall \end{cases} \]
\[ \beta^s_i = \begin{cases} 1, & \text{if } \beta_i = \exists, \\ -1, & \text{if } \beta_i = \forall \end{cases} \]  
\[ \text{abs}^s(y) = \begin{cases} |y_i|, & \text{if } \sigma_i \text{ is } \equiv, \\ -y_i, & \text{if } \sigma_i \text{ is } \geq, \\ y_i, & \text{if } \sigma_i \text{ is } \leq . \end{cases} \]

**Proof.**

1) The equivalence (23) was proposed and proved by Jiri Rohn at the international conference INTERVAL’96 (September–October of 1996, Würzburg, Germany), in a private talk with Sergey Shary and Anatoly Lakeyev. Later, its reformulation with the use of Hadamard product was proposed by Anatoly Lakeyev in the work [11]. Below, we present our own proof.

In view of Theorem 1

\[ Q^{\equiv}(A, b, \mathcal{A}, \beta)(Ax = b) \iff A^s x \subseteq b^s . \]

Then, using the properties of Kaucier arithmetic

\[(\forall u, v \in \mathbb{KR}^m) \ (u \subseteq v \iff |u - \tilde{v}| \leq |\hat{u} - \hat{v}|), \]

\[ \text{mid}(A^s x) = \hat{A}^s x, \quad \text{rad}(A^s x) = \hat{A}^s |x|, \]  
and we get

\[ A^s x \subseteq b^s \iff |\hat{A}^s x - \tilde{b}^s| \leq |\hat{b}^s - \hat{A}^s |x|. \]

From the definitions of (28) and (27) for \( A^s, b^s, \mathcal{A}^s, \) and \( \beta^s, \) we have

\[ \hat{A}^s = A, \quad \hat{A}^s = -A^s \circ \hat{A}, \quad \tilde{b}^s = \tilde{b}, \quad \tilde{b}^s = \beta^s \circ \tilde{b} . \]  

2) Let us prove the equivalence (24). According to Theorem 2

\[ Q(A, b, \mathcal{A}, \beta)(Ax \geq b) \iff A^s x \geq b^s . \]

Drawing on the obvious property of the Kaucier arithmetic

\[(\forall u, v \in \mathbb{KR}^m) \ (u \geq v \iff \tilde{v} - \tilde{u} \leq \hat{v} - \hat{u}), \]

which allows us to replace the inequality between the endpoints by the inequality between centers and radii, and then involving (28), we get

\[ A^s x \geq b^s \iff \tilde{b}^s - A^s x \leq \tilde{b}^s - \hat{A}^s |x|. \]

Finally, we use (29).

3) The equivalence (25) is proved similarly to (24).
4) It remains to substantiate the equivalence (26). Just as in the item 1 of the proof of
Theorem 3, we have (21), i.e., the problem splits in rows. We apply, to each row, one of the
equivalences (23), (24) or (25), depending on the corresponding binary relation, and convolve
the resulting system of inequalities using the operation abs$^\sigma$.

The proof of Theorem 4 is complete.

From the equivalences (23)–(25), one more proof of Corollary 1 becomes obvious. In addi-
tion, it is not difficult to establish the following connection between relationally homogeneous
systems of inequalities of the opposite signs.

**Corollary 2**

\[
Q(A, b, A, \beta) (Ax \geq b) \iff Q(-A, -b, A, \beta) (Ax \leq b),
\]

\[
Q(-A, -b, A, \beta) (Ax \geq b) \iff Q(A, b, A, \beta) (Ax \leq b).
\]

**Proof.** Based on the properties of intervals

\[
\text{mid}(-u) = -\hat{u}\quad \text{and} \quad \text{rad}(-u) = \hat{u},
\]

we can show the validity of relation (31):

\[
Q(-A, -b, A, \beta) (Ax \geq b) \iff \text{mid}(-b) - \text{mid}(-A)x \leq (A^* \circ \text{rad}(-A))|x| + \beta^* \circ \text{rad}(-b),
\]

\[
\iff -b + Ax \leq (A^* \circ \hat{A})|x| + \beta^* \circ \hat{b},
\]

\[
\iff Q(A, b, A, \beta) (Ax \leq b).
\]

Relation (30) is proved similarly.

Corollary 2 means that, if the sign of the inequality and the signs of all intervals of the
parameter values are reversed to the opposite, then the set of quantifier solutions to the interval
system of linear inequalities does not change. For example, the sets of solutions to the systems
\((\forall A \in A) (\exists b \in b) (Ax \geq b)\) and \((\forall A \in -A) (\exists b \in -b) (Ax \leq b)\) coincide.

### 3.3 Quantifier-free descriptions in \(\mathbb{KR}, \mathbb{IR}\) and \(\mathbb{R}\)
for systems of basic types

So far, considering the interval-quantifier linear systems \(Q(A, b, A, \beta) (Ax \sigma b)\), we tried to
obtain results in which there are no constraints on the parameters \(A, b, A, \beta\) and \(\sigma\), and the
restrictions on the order of the elementary quantifier prefixes in \(Q\) are minimal. In this sense,
the most general descriptions were found for the class \(Q^\sigma\). In this section, we consider subsets
of interval-quantifier linear systems of class \(Q^\sigma\), which are distinguished by the requirement
of homogeneity of \(A\) and the homogeneity of \(\beta\). Elements of all these subsets will be called
systems of basic types, and their solutions will be referred to as quantifier solutions of basic
types for interval linear systems of the form \(Ax \sigma b\).

Depending on which quantifiers fill the matrix \(A\) and the vector \(\beta\), all the interval-quantifier
linear systems of the basic types are divided into 4 subsets, or 4 types. This subdivision is
presented in the last column of Table 1. For solutions to systems of each of the main
types, we give a proper name that continues the one used in [3, 21] for solutions of relationally
homogeneous systems of this type. The names of the solutions are listed in the first column

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Table 1

Basic types of quantifier solutions to the interval system $Ax \sigma b$

| Name of solution | Values of elements of the matrix $A$ of the vector $\beta$ | Interval-quantifier system of basic type |
|------------------|--------------------------------------------------------|----------------------------------------|
| weak             | $\exists$                                               | $(\exists A \in A)(\exists b \in b) \ (Ax \sigma b)$ |
| tolerable        | $\forall$                                               | $(\forall A \in A)(\exists b \in b) \ (Ax \sigma b)$ |
| controllable     | $\exists$                                               | $(\forall b \in b)(\exists A \in A) \ (Ax \sigma b)$ |
| strong           | $\forall$                                               | $(\forall A \in A)(\forall b \in b) \ (Ax \sigma b)$ |

of Table 1, the values of the elements of the matrix $A$ and components of the vector $\beta$ are listed in the second and third columns, and the fourth column gives the general form for the interval-quantifier systems of the corresponding basic type.

Quantifier-free descriptions in $\mathbb{K} \mathbb{R}$, $\mathbb{I} \mathbb{R}$ and $\mathbb{R}$ for systems of basic types can be obtained as corollaries of the corresponding descriptions for systems of class $Q^\sigma$. Let us explain that for relationally homogeneous systems using Table 2.

In the Table 2, columns 4–7, corresponding to the basic types of quantifier solutions, are obtained, in row-wise manner, from column 3 corresponding to quantifier solutions with the prefix $Q^\sigma$. It is necessary to use the definition of (8) of the matrix $A^s$ and vector $b^s$ in the rows corresponding to the Kaucher arithmetic. In the rows that correspond to the classical interval arithmetic, we have to use definition (7) of the matrices $A^\forall$, $A^\exists$ and the vectors $b^\forall$, $b^\exists$. Finally, the rows corresponding to real non-interval arithmetic, the definition (27) of the matrix $A^s$, vector $\beta^s$ and the definition of the product o should be used.

Approximately half of the descriptions of the basic types of quantifier solutions for interval linear systems, presented in columns 4–7 of Table 2, have been obtained earlier. The descriptions that were found first, obtained their own proper names. These are

- the Oettli-Prager characterization in $\mathbb{R}$ [15] and the Beeck characterization in $\mathbb{I} \mathbb{R}$ [1] for weak solutions of the equation $Ax = b$,

- the Gerlach description in $\mathbb{R}$ for weak solutions of the inequality $Ax \leq b$ [5].

The quantifier-free descriptions of the set of tolerable solutions to the equation $Ax = b$ was obtained in $\mathbb{R}$ by J. Rohn [16] and in $\mathbb{I} \mathbb{R}$ by A. Neumaier [14]. The description in $\mathbb{R}$ was further investigated by A.V. Lakeyev and S.I. Noskov in [10], and they also presented, as an evident one, a description, in $\mathbb{I} \mathbb{R}$, for the set of controllable solutions to the equation $Ax = b$ (see also [17]). The remaining descriptions for the basic types of quantifier solutions to the equation $Ax = b$ in the interval arithmetics $\mathbb{I} \mathbb{R}$ and $\mathbb{K} \mathbb{R}$ are also known, for example, as obvious corollaries of the statement (11), proved by S.P. Shary in [18, 19]. In Theorem 2.25 from the book [3], a quantifier-free description in $\mathbb{I} \mathbb{R}$ for strong solutions to the interval inequality $Ax \leq b$ was presented.

For interval-quantifier systems of basic types in which the relationship vector $\sigma$ is not homogeneous, quantifier-free descriptions in $\mathbb{K} \mathbb{R}$ can be obtained from (22) and (8). The descriptions in $\mathbb{I} \mathbb{R}$ can be derived from (22) and (17), and the descriptions in $\mathbb{R}$ follows from (20) and (27). Below, we give these descriptions only in $\mathbb{I} \mathbb{R}$ and $\mathbb{R}$ (in $\mathbb{K} \mathbb{R}$, they are less expressive and differ from the descriptions in $\mathbb{I} \mathbb{R}$ by obvious arithmetic transformations, in the same way as the
### Characterization of relationally homogeneous interval-quantifier linear systems and their main solution types

| Space of description | Type of solution and corresponding quantifier prefix $Q(A, b, A, \beta)$ | Basic types of solutions |
|----------------------|------------------------------------------------|--------------------------|
|                      | $Q^*(A, b, A, \beta)$ | Weak $(\exists A \in A) (\exists b \in b)$ | Tolerable $(\forall A \in A) (\exists b \in b)$ | Controllable $(\forall b \in b) (\exists A \in A)$ | Strong $(\forall A \in A) (\forall b \in b)$ |
| $Ax = b$             | $A^x x \subseteq b'$ | (dual $A)x \subseteq b$ | $Ax \subseteq b$ | (dual $A)x \subseteq \text{dual } b$ | $Ax \subseteq \text{dual } b$ |
| KR                   | $A^y x - b^y \subseteq b^3 - A^3 x$ | $0 \in b - Ax$ | $Ax \subseteq b$ | $b \subseteq Ax$ | $Ax - b \subseteq 0$ |
| IR                   | $|Ax - b| \leq (A^* A)|x| + \beta^* b$ | $|Ax - b| \leq -A|x| + b$ | $|Ax - b| \leq A|x| - b$ | $|Ax - b| \leq -A|x| - b$ |
| $Ax \geq b$          | $A^x x \geq b'$ | (dual $A)x \geq b$ | $Ax \geq b$ | (dual $A)x \geq \text{dual } b$ | $Ax \geq \text{dual } b$ |
| KR                   | $\overline{A^x x} + A^y x \geq b^3 + b^y$ | $AX \geq b$ | $Ax \geq b$ | $AX \geq \text{dual } b$ | $Ax \geq \text{dual } b$ |
| IR                   | $b - Ax \leq (A^* A)|x| + \beta^* b$ | $b - Ax \leq -A|x| + b$ | $b - Ax \leq A|x| - b$ | $b - Ax \leq -A|x| - b$ |
| $Ax \leq b$          | $\overline{A^x x} \leq b'$ | (dual $A)x \leq b$ | $AX \leq b$ | (dual $A)x \leq \text{dual } b$ | $AX \leq \text{dual } b$ |
| KR                   | $A^3 x + A^y x \leq b^3 + b^y$ | $Ax \leq b$ | $AX \leq b$ | $AX \leq \text{dual } b$ | $AX \leq \text{dual } b$ |
| IR                   | $AX - b \leq (A^* A)|x| + \beta^* b$ | $AX - b \leq -A|x| + b$ | $AX - b \leq A|x| - b$ | $AX - b \leq -A|x| - b$ |
descriptions in $\mathbb{KR}$ and $\mathbb{IR}$ differ from each other in Table 2):

\[(\exists A \in A)(\exists b \in b) \ (Ax \sigma b) \iff 0 \in b - Ax + w \iff \text{abs}^\sigma(\hat{A}x - \hat{b}) \leq \hat{A}|x| + \hat{b};\]

\[(\forall A \in A)(\exists b \in b) \ (Ax \sigma b) \iff Ax \subseteq b + w \iff \text{abs}^\sigma(\hat{A}x - \hat{b}) \leq -\hat{A}|x| + \hat{b};\]

\[(\forall b \in b)(\exists A \in A) \ (Ax \sigma b) \iff b \subseteq Ax + w \iff \text{abs}^\sigma(\hat{A}x - \hat{b}) \leq \hat{A}|x| - \hat{b};\]

\[(\forall A \in A)(\forall b \in b) \ (Ax \sigma b) \iff Ax - b \subseteq w \iff \text{abs}^\sigma(\hat{A}x - \hat{b}) \leq -\hat{A}|x| - \hat{b}.\]

4 Conclusion

The main results of the paper are presented in Theorems 2–4 (equivalence (23) was previously known) and in Corollary 1.

Among the statements that have no restrictions on the parameters $A$, $b$, $A$, $\beta$ and $\sigma$, those that give quantifier-free descriptions of interval-quantifier linear systems of class $Q^\sigma$ have the greatest generality. These are the relation (20), which provides a transition to $\mathbb{KR}$ and $\mathbb{IR}$, the relation (22) for the transition to $\mathbb{KR}$ and $\mathbb{IR}$, and equivalence (26) that allows us to go into $\mathbb{R}$.

The usefulness of quantifier-free descriptions from (20), (22) and (26) is that they give us the possibility

- to study all interval-quantifier linear systems of class $Q^\sigma$ simultaneously and in a uniform way, and to derive results for their subclasses (in particular, for interval-quantifier systems of basic types) as consequences of the general result;

- to design such solution methods for problems related to interval-quantifier linear systems that are suitable for all systems of class $Q^\sigma$ (an example is the author’s software package for visualization of quantifier solution sets to interval linear systems, available at [http://www.nsc.ru/interval/sharaya])

Quantifier-free descriptions, in interval arithmetic, for various classes of interval-quantifier linear systems and for their solutions, both previously known (for example, relation (11)) and those obtained in this paper in the form of relations (12)–(14), (20), (22), allow us

- to investigate interval-quantifier linear systems by interval methods, i.e., to reveal the properties of their solution sets, the relationships between systems with various conditions on the parameters $A$, $b$, $A$, $\beta$, $\sigma$ and the order of the quantifier prefixes (an example is the proof of Corollary 1);

- to construct interval (that is, essentially using interval arithmetic) solution methods for problems in which the formulation involves interval-quantifier linear systems (examples of such methods for systems of equations can be found in [21], while for inequalities and systems of class $Q^\sigma$ constructing such methods is a matter of the future).

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