Partial Factorization of Wave Function for A Quantum Dissipation System

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ABSTRACT

The microscopic approach quantum dissipation process presented by Yu and Sun [Phys.
Rev., A49(1994)592, A51(1995)1845] is developed to analyze the wave function structure
of dynamic evolution of a typical dissipative system, a single mode boson soaked in a
bath of many bosons. In this paper, the wave function of total system is explicitly
obtained as a product of two components of the system and the bath in the coherent
state representation. It not only describes the influence of the bath on the variable of the
system through the Brownian motion, but also manifests the back- action of the system
on the bath and the effects of the mutual interaction among the bosons of the bath. Due
to the back-action, the total wave function can only be partially factorizable even for the
Brownian motion can be ignored in certain senses, such as the cases with weak coupling
and large detuning.

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1. Introduction: The typical microscopic treatment for the model of quantum dissipation is to consider a harmonic oscillator interacting with a many-oscillator bath (e.g., in [1-6]). This simple and rather conventional model has been used to completely clarify the relation between two different approaches for quantum dissipation process frequently appearing in the literature, i.e., the system plus bath model and the time-dependent effective Hamiltonian by Kanai and Calderora [7,8], since Yu and one (Sun) of the authors wrote down the total wave function explicitly in a form of direct product of the bath component and the system component [1,2]. In the discussion, because the mixed variables were chosen to describe the system and the bath, the wave function only manifested the influence of the bath on the system through the Brownian broadening of the width of the wave function for the system, but the back-action of the system on the bath was not discussed. If there indeed exists the back-action of the system on the bath, it is reasonable to expect that, for the individual particles constituting the bath, the mutual couplings among them can indirectly appear in a second order through coupling the system as an intermediate process.

Another question is the relation between quantum and classical systems. In many real situations, the classical or macroscopic states can be represented by coherent states in the quantum optics and the macroscopic quantum mechanics. Therefore it is significant to study how the system with an initial coherent state evolves if it really has a macroscopic or classical meanings and to test if it can move in the classical orbits.

In this paper, the back-action and mutual couplings with a simple model is studied and manifested in the coherent representation. In the presence of both the back-action and the indirect mutual coupling, we also consider the meaning of the wave function of the dissipative system governed by the effective Hamiltonian, which is also to be determined in this paper. A very interesting result is that a factorizable evolution represented by a product of two coherent states is obtained with one component representing the system, which evolves according to a classical orbit and possess a classical meaning.

2. Structure of Wave Function: This paper will consider the problem with the simple model consisting of a single mode boson and a bath of many bosons and its Hamiltonian is written as

\[ H = \hbar \omega b^+ b + \sum_j \hbar \omega_j a_j^+ a_j + \hbar \sum_j \left[ \xi_j b^+ a_j + h.c \right], \]  

(1)

where \( \xi_j = |\xi_j| e^{i\sigma_j} \)'s are the complex coupling constants and \( b^+ \), \( b \), \( a_j^+ \) and \( a_j \) are the bosonic creation and annihilation operators for the system and the bath respectively. This model can be regarded as a rotating wave approximation of the original oscillator model with the linear coupling \( \sum_j \xi_j q x_j \sim \sum_j [\xi_j b^+ a_j + \xi_j b^+ a_j^+ + h.c] \) of the system coordinate \( q \) to the bath variables \( x_j \).

To obtain the explicit expression for the wave function of the total system formed by the system plus the bath, we invoke the well-known solutions [9]

\[ b(t) = u(t)b(0) + \sum_j v_j(t)a_j(0), \]  

(2)
The wave function \( \Psi(\lambda, \{\lambda_j\}, t) \) can be defined by its coherent state representation at time \( t \):

\[
\Psi(\lambda, \{\lambda_j\}, t) = \langle \lambda, \{\lambda_j\} | \Psi(t) \rangle = \langle \Psi(0) | U(t) \rangle |\lambda, \{\lambda_j\} \rangle^* .
\]

Here, we used the overcomplete basis

\[
|\lambda, \{\lambda_j\}\rangle = |\lambda\rangle \otimes \prod_j |\lambda_j\rangle = N(\lambda, \{\lambda_j\}) \exp \left( \lambda \sum_j a_j^\dagger(0) \right) |0\rangle
\]

constructed by the coherent states \( |\lambda\rangle \) and \( |\lambda_j\rangle \) for the annihilation operators \( b(0) \) and \( a_j(0) \) respectively. Here, the normalization constant \( N(\lambda, \{\lambda_j\}) = \exp \left( -\frac{1}{2} |\lambda|^2 - \sum_j \frac{1}{2} |\lambda_j|^2 \right) \).

Then, we turn to obtain an explicit expressions of \( U(t) \) by considering the role of
the evolution matrix $U(t)$ in the Heisenberg picture. In fact, since $U(t)O(0)U(t) = O(t)$ and $U(t)|0⟩ = |0⟩$ for a operator $O$, it is easy to obtain

$$U(t)† | λ, \{λ_j\}⟩ = N (λ, \{λ_j\}) \exp \left( λb^†(t) + \sum_j λ_j a_j^†(t) \right) |0⟩$$

$$= N (λ, \{λ_j\}) \exp \left( α(t)b^†(0) + \sum_j λβ(t)a_j^†(0) \right) |0⟩ = |α(t)⟩ \otimes \prod_j |β_j(t)⟩$$

(11)

where

$$α(t) = u(t)^*λ + \sum_j λ_j u_j(t)^*$$

$$β_j(t) = e^{iω_jt}λ_j + v_j(t)^*λ + \sum_{s(\neq j)} v_{s,j}(t)^*λ_s$$

Finally, we obtain a formal factorized wave function for the total system

$$Ψ (λ, \{λ_j\}, t) = φ \left( u(t)^*λ + \sum_j u_j(t)^*λ_j \right) \otimes \prod_j φ_j \left( e^{iω_jt}λ_j + v_j(t)^*λ + \sum_{s(\neq j)} v_{s,j}(t)^*λ_s \right).$$

(14)

3. Partial Factorization and Effective Hamiltonian: The above wave function (14) is not completely factorizable because of the entanglements of the variables $λ$ and $λ_j$, which are implied by the term, $\sum_j u_j(t)^*λ_j$, of the bath variables $λ_j$ modifying the system variable $λ$ and the term, $v_j(t)^*λ$, of the system modifying the bath one. This former represents the bath fluctuation due to the Brownian motion while the later the back-action of the system on the bath. In fact the term, $\sum_j u_j(t)^*λ_j$, is caused by the bath fluctuation operator $B(t) = \sum_j v_j(t)a_j(0)$ in the system operator $b(t)$, which has a zero thermal average, but a non-zero correlation

$$< B(t)^† B(t') > = \sum_j 4|ξ_j|^2 \frac{f_j(t,t')}{\gamma^2 + 4(ω_j - ω - Δω)^2} \left[ \exp \left( \frac{ℏω_j}{k_B T} \right) - 1 \right]^{-1}. \quad (15)$$

The term, $\sum_{j\neq s} v_{s,j}(t)^*λ_s$, shows the mutual interactions among the bosons of the bath through the system. Mathematically, if the coupling is weak with the small $ξ_j$, the mutual interactions are the second order as shown by $v_{s,j}(t)^* ∝ ξ_j^2ξ_s$. Notice that the main difference between the present result and that in refs. [1,2] is the back-action and the mutual interactions.

When the fluctuation can be ignored for certain cases, e.g., $λ$ is very large in the initial state and the coupling is weak enough, the entanglement disappears so that the wave function becomes a product

$$Ψ (λ, \{λ_j\}, t) ≈ φ (u(t)^*λ) \prod_{j=1}^N φ_j \left( e^{iω_jt}λ_j + v_j(t)^*λ \right).$$

(16)
In this case, all the influences of the bath on the system are represented by the damping constant \( \gamma \) and then the wave function is partially factorizable due to the term \( v_2(t)^* \lambda \). It is not difficult to prove that the system component \( \phi(u(t)^* \lambda) \) is governed by an effective Hamiltonian, which is also equivalent to the Calderora-Kanai Hamiltonian.

To prove it, we need to return into the Heisenberg picture by dropping of the bath operators \( a_j(0) \) in \( b(t) \), namely, \( b(t) \) is replaced by \( \tilde{b}(t) = u(t)b(0) \). However, \( \tilde{b}(t)^+ \) and \( \tilde{b}(t) \) are not the canonical operators since \( [\tilde{b}(t), \tilde{b}(t)^+] = e^{-\gamma t} \). However, the Bogoliubov transformation gives the general canonical operators

\[
A(t) = \alpha \tilde{b}(t) + \beta \tilde{b}(t)^+
\]

satisfying \( [A(t), A(t)^+] = 1 \), where

\[
|\alpha|^2 - |\beta|^2 = \exp(\gamma t).
\]

To given the correct Heisenberg equations for operators \( A(t) \) and \( A(t)^+ \), the effective Hamiltonian is determined by the definition (17) as time-dependent

\[
H_{\text{eff}} = i\hbar \exp(-\gamma t) \left[ (\tilde{\alpha}^* \beta^* - \tilde{\beta}^* \alpha^*) A(t)^+ A(t) + \frac{1}{2} (\beta \alpha - \bar{\alpha} \beta) A(t)^+ A(t)^+ + \frac{1}{2} (\alpha^* \beta^* - \beta^* \alpha^*) A(t) A(t) \right]
\]

where \( \tilde{\alpha} = \alpha - (\gamma/2 + i\bar{\omega})\alpha \) and \( \tilde{\beta} = \beta - (\gamma/2 - i\bar{\omega})\beta \). Notice that the number \( \tilde{\alpha} \alpha^* - \tilde{\beta} \beta^* = \delta \) should be a pure imaginary number, i.e. \( \delta^* = -\delta \). It is not exotic that the effective Hamiltonian is not unique because there is only one constraint (18). Its different forms correspond to different realizations of the canonical variables.

For instance, a specific solution of Eq. (18), \( \alpha = \exp(\gamma/2 + i\varphi)t \), \( \beta = 0 \), \( \varphi = \bar{\omega} - \sqrt{\gamma^2/4 + \bar{\omega}^2} \), gives \( H_{\text{eff}} = \hbar \exp(\gamma t)\Omega A(t)^+ A(t) \) with \( \Omega = \sqrt{\gamma^2/4 + \bar{\omega}^2} \). By formally introducing the canonical coordinate \( Q = \sqrt{\hbar/2\Omega} [A(t) + A(t)^+] \) and momentum \( P = -i\sqrt{\Omega M\hbar/2} [A(t) - A(t)^+] \) with the varying mass \( M = m \exp(\gamma t) \), this special effective Hamiltonian is just of the form by Calderora and Kanai. This result can be also given in purely quantized version in Schrödinger picture by the direct calculation of matrix elements, \( \langle \alpha | H_{\text{eff}} | \beta \rangle = \langle \alpha | i\hbar (\partial U(t)/\partial t) U(t)^+ | \beta \rangle \).

4. Motion of The Center of Wave Packet with Quantum Fluctuation: Let us consider the physical significance of the above wave function and its entanglements in details. In the representation of coordinate-momentum,

\[
q = \sqrt{\frac{\hbar}{2\omega}} (b + b^\dagger), \quad p = -i\sqrt{\frac{\hbar\omega}{2}} (b - b^\dagger),
\]

\[
x_j = \sqrt{\frac{\hbar}{2\omega_j}} (a_j + a_j^\dagger), \quad p_j = -i\sqrt{\frac{\hbar\omega_j}{2}} (a_j - a_j^\dagger),
\]
coherent states $|\lambda\rangle$ and $|\lambda_j\rangle$ are understood as the Gaussians of widths $\sqrt{\hbar/2\omega}$ centered in $q_0 = \sqrt{\hbar/2\omega} (\lambda + \lambda^*)$ and $x_{j0} = \sqrt{\hbar/2\omega} (\lambda_j + \lambda_j^*)$ respectively. If the initial state of the total system is a direct product of such Gaussians,

$$|\Psi(0)\rangle = |\lambda = \sqrt{\frac{\omega}{2\hbar}} q_0\rangle \otimes \prod_j |\lambda_j = \sqrt{\frac{\omega_j}{2\hbar}} x_{j0}\rangle,$$

the wave function at time $t$

$$|\Psi(t)\rangle = |\lambda(t)\rangle \otimes \prod_j |\lambda_j(t)\rangle = \left| u(t) \sqrt{\frac{\omega}{2\hbar}} q_0 + \sum_j u_j(t) \sqrt{\frac{\omega_j}{2\hbar}} x_{j0} \right\rangle \otimes \prod_j \left| \sqrt{\frac{\omega_j}{2\hbar}} x_{j0} e^{i\omega_j t} + v_j(t) \sqrt{\frac{\omega}{2\hbar}} q_0 + \sum_{j(\neq s)} v_{s,j}(t) \sqrt{\frac{\omega_s}{2\hbar}} x_{s0} \right\rangle$$

defines the position evolution of center of the Gaussian wave packet

$$q_c(t) = \sqrt{\frac{\hbar}{2\omega}} (\lambda(t) + \lambda^*(t))$$

$$= q_0 \exp \left( -\frac{1}{2} \gamma t \right) \cos (\tilde{\omega}) t + \sum_j |\xi_j| \sqrt{\frac{\omega_j}{\omega}} \frac{x_{j0} \Theta_j(t)}{\gamma^2/4 + (\omega_j - \tilde{\omega})^2},$$

$$\Theta_j(t) = \exp \left( -\frac{1}{2} \gamma t \right) \left[ \frac{\gamma}{2} \sin (\tilde{\omega} + \sigma_j) t + (\omega_j - \tilde{\omega}) \cos (\tilde{\omega} + \sigma_j) t \right]$$

$$- \left[ \frac{\gamma}{2} \sin (\omega_j + \sigma_j) t + (\omega_j - \tilde{\omega}) \cos (\omega_j + \sigma_j) t \right].$$

It is known from Eq. (16) that the center of the wave packet moves along the classical trajectory of a damping harmonic oscillator, which is described by first term in Eq. (16) and perturbed by the initial displacements $x_{j0}$ of the bath oscillators shown in the second term in Eq. (16). This fluctuation effect is just the explicit manifestation of the Brownian motion. For a very large initial displacement, $q_0$, the weak coupling with small $\xi_j$ or a sharp spectral distribution of the bath with a large detuning from the renormalized frequency $\tilde{\omega}$, one can ignore this fluctuation effect.

Finally, we consider the back-action of the system on the bath and the mutual interaction among the bosons of the bath through the motion law of the center of Gaussian for each boson of the bath

$$x_{jc}(t) = \sqrt{\frac{\hbar}{2\omega_j}} (\lambda_j(t) + \lambda_j^*(t))$$

$$= x_{j0} \cos \omega_j t + q_0 \sqrt{\frac{\omega}{\omega_j}} Re (v_j(t)) + \sum_{s(\neq j)} Re (v_{s,j}(t)) \sqrt{\frac{\omega_s}{\omega_j}} x_{s0}.$$
Notice that the back-action \( q_0 \sqrt{\frac{\omega}{\omega_j}} \text{Re}(v_j(t)) \) is proportional to both the initial displacement \( q_0 \) times a Lorentz function \( \propto \left[ \gamma^2/4 + (\omega_j - \tilde{\omega})^2 \right]^{-1} \). Thus it can not be neglected for large \( q_0 \) or in the quasi-resonance case that the bath spectral distribution \( \rho(\omega_j) \) peaked in the renormalized frequency \( \tilde{\omega} \) of the system. The last term \( \sum_{s(\neq j)} \text{Re}(v_{s,j}(t)) \sqrt{\omega_s/\omega_j} x_{s0} \) is of second order and explicitly reflects the mutual coupling among the bosons of the bath, which can be neglected in the first order approximation.

5. Summary and Discussions: In summary, we first mention that the Langevin approach is an standard treatment for quantum dissipation process in the present model Hamiltonian [4], but it hardly concerns the structure of wave function that is essentially important in zero temperature; the Markoff approximation is also a quite effective density operator method, but it only considers very a few intuitional picture based on the classical correspondence and its dissipation-fluctuation relation directly. This paper take both two aspects of this problem into account and thus gives a direct and clarified picture for quantum dissipation process with the most simple model. Our discussion not only concerns the necessary details in the dynamics of quantum dissipation, but also reveals the roles of the back-action of bath and the mutual coupling among the bosons of bath.

Notice that the Langevin approach is based on a stochastic equation
\[
\dot{b}(t) = \left[ -\gamma b(t) - i (\omega_a + \Delta \omega) \right] b(t) + F(t)
\]
with the stochastic force \( F(t) = -i \sum_j \xi_j a_j(0) e^{-i\omega_j t} \). Its explicit solution is the starting point finding the partially factorizable wave function of the total system in this paper. In this sense, our study is a generalization of quantum Langevin theory. As for the Markoff theory, our explicit solution somewhat can be regarded as the zero-temperature result of the density matrix approach, but it deal with the dissipation-fluctuation relation directly. This paper take both two aspects of this problem into account and thus gives a direct and clarified picture for quantum dissipation process with the most simple model. Our discussion not only concerns the necessary details in the dynamics of quantum dissipation, but also reveals the roles of the back-action of bath and the mutual coupling among the bosons of bath.

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To conclude this paper we point out that the method used both in this paper and the previous works [1,2] is very limited to a linear system we considered before, such as the harmonic oscillator, the inverse harmonic oscillator (the harmonic oscillator with image frequency, \( \omega \rightarrow i\omega \)) and a linear potential for a constant force. This is because, only for these systems, the solutions of canonical Heisenberg operators are the linear combinations of the system variable and the bath variables and thus the wave functions are factorizable in the appropriate representations. There still exists the difficulties in principle to generalize the idea and method of this paper to the nonlinear cases except those systems that can be reasonably linearized.
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