Dual-stage adaptive finite-time modified function projective multi-lag combined synchronization for multiple uncertain chaotic systems

Abstract: In this paper, for multiple different chaotic systems with unknown bounded disturbances and fully unknown parameters, a more general synchronization method called modified function projective multi-lag combined synchronization is proposed. This new method covers almost all of the synchronization methods available. As an advantage of the new method, the drive system is a linear combination of multiple chaotic systems, which makes the signal hidden channels more abundant and the signal hidden methods more flexible. Based on the finite-time stability theory and the sliding mode variable structure control technique, a dual-stage adaptive variable structure control scheme is established to realize the finite-time synchronization and to tackle the parameters well. The detailed theoretical derivation and representative numerical simulation is put forward to demonstrate the correctness and effectiveness of the advanced scheme.

Keywords: Finite-time adaptive control, Modified function projective multi-lag combined synchronization, Sliding mode variable structure control, Chaotic systems, Unknown parameter and disturbance

MSC: 93C10

1 System description

In our drive-response type combination synchronization scheme, $m$ different chaotic systems with unknown parameters and disturbance are considered as the drive systems. The $l$th drive system is given by

$$
\begin{align*}
\dot{x}_1^l(t) &= F_1^l(x^l(t))\theta^l + f_1^l(x^l(t)) + w_1^l(t), \\
\dot{x}_2^l(t) &= F_2^l(x^l(t))\theta^l + f_2^l(x^l(t)) + w_2^l(t), \\
\vdots \\
\dot{x}_n^l(t) &= F_n^l(x^l(t))\theta^l + f_n^l(x^l(t)) + w_n^l(t),
\end{align*}
$$

(1)
in which $l = 1, 2, \cdots, m$.
At the meantime, the response system is described as:

\[
\begin{align*}
\dot{y}_1(t) &= H_1(y(t))\phi + h_1(y(t)) + d_1(t) + u_1(t), \\
\dot{y}_2(t) &= H_2(y(t))\phi + h_2(y(t)) + d_2(t) + u_2(t), \\
&\vdots \\
\dot{y}_n(t) &= H_n(y(t))\phi + h_n(y(t)) + d_n(t) + u_n(t),
\end{align*}
\]

where \(x^l = [x^l_1, x^l_2, \cdots, x^l_n]^T\), \(y(t) = [y_1(t), y_2(t), \cdots, y_n(t)]^T \in \mathbb{R}^n\) are the state vectors of the drive system and the response system respectively, \(f^l_i(x^l(t)), l = 1, 2, \cdots, m\) and \(h_i(y(t))\), \(i = 1, 2, \cdots, n\) are continuous nonlinear functions, \(F^l_i(x^l(t))\) and \(H_i(y(t))\) are the \(i\)th row of the continuous linear function matrices \(F^l(x^l(t))\) and \(H(y(t))\), respectively, \(\theta^l = [\theta^l_1, \theta^l_2, \cdots, \theta^l_m]^T\) and \(\phi = [\phi_1, \phi_2, \cdots, \phi_n]^T\) are unknown parameter vectors, \(w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^T\), \(d(t) = [d_1(t), d_2(t), \cdots, d_n(t)]^T\) and \(p(t) = [p_1(t), p_2(t), \cdots, p_n(t)]^T\) are unknown external time-varying disturbances, \(u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T\) is the vector of control input.

\section{2 Preliminary definition and lemmas}

As the essence of finite-time synchronization, it means that the state trajectory of the response system can converge to the state trajectory of the drive system within a finite time. In this section, we introduce the precise definitions and several important lemmas, which are necessary for further study.

\textbf{Assumption 2.1.} The unknown parameters \(\theta^l\) and \(\phi\) are bounded, in another word, there exist known constants \(\tilde{\theta}^l \geq 0\) and \(\tilde{\phi} \geq 0\), such that

\[
\|\theta^l\| \leq \tilde{\theta}^l, \|\phi\| \leq \tilde{\phi},
\]

where \(l = 1, 2, \cdots, m\), and \(\|\cdot\|\) stands for the 2-norm.

\textbf{Assumption 2.2.} The unknown external time-varying disturbances \(w^l(t)\) and \(d^l_i(t)\) are bounded, that is to say, there exist non-negative constants \(\tilde{w}^l_i\) and \(\tilde{d}^l_i\) satisfy

\[
\left|w^l_i(t)\right| \leq \tilde{w}^l_i, \left|d^l_i(t)\right| \leq \tilde{d}^l_i,
\]

where \(l = 1, 2, \cdots, m\) and \(i = 1, 2, \cdots, n\).

\textbf{Lemma 2.3 ([40]).} Assume that a continuous and positive-definite function \(V(t)\) satisfies the following differential inequality:

\[
\dot{V}(t) \leq -b_1 V^\theta(t) - b_2 V(t), t \geq t_0, V(t_0) \geq 0.
\]

where \(b_1 > 0, b_2 > 0\) and \(0 < \theta < 1\) are constants.

Then, when \(V^{1-\theta}(t_0) \leq \frac{b_1}{b_2}\), the following results are true:

\[
V(t) \leq e^{b_2(t-t_0)}V^{1-\theta}(t_0) + \frac{b_1}{b_2} - \frac{b_1}{b_2} e^{-b_2(1-\theta)(t-t_0)} V^{1/(1-\theta)}, \text{if } t_0 \leq t < T,
\]

\[
V(t) = 0, \text{if } t \geq T.
\]

with \(T\) given by

\[
T = t_0 + \frac{1}{b_2(1-\theta)} \ln(1 + \frac{b_2 V^{1-\theta}(t_0)}{b_1}),
\]

\textbf{Lemma 2.4 ([35]).} Consider the system

\[
\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n
\]
where the mapping function $f : I \rightarrow \mathbb{R}^n$ is continuous. If there exists a continuous differential positive-definite function $V : I \rightarrow \mathbb{R}$, real constants $\zeta > 0, 0 < \varrho < 1$, satisfying

$$\dot{V}(x) \leq -\zeta V(x), \forall x \in I,$$

then, the origin of system (5) is a locally finite-time stable equilibrium, the settling time $T(x_0)$ depends on the initial state $x(0) = x_0$, and the following inequality holds

$$T(x_0) \leq \frac{V^{1-\varrho}(x_0)}{\zeta(1-\varrho)}. \quad (7)$$

**Lemma 2.5** ([15]). Suppose $a_1, a_2, \cdots, a_n$ and $0 < q < 2$ are all real numbers, then the inequality below holds

$$|a_1|^q + |a_2|^q + \cdots + |a_n|^q \geq (a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{q}{2}}. \quad (8)$$

**Lemma 2.6.** By choosing $q = 1$ in Lemma 2.5, we can obtain

$$|a_1| + |a_2| + \cdots + |a_n| \geq (a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}}. \quad (9)$$

**Definition 2.7.** It is said that the group of the drive systems (1) and the response system (2) are modified function projective multi-lag combined synchronization (MFPMLCS), if there exist $m$ different delay times $\tau^l$ and $m + 1$ scaling matrices $A^l (l = 1, 2, \cdots, m)$ and $\Lambda(t)$, such that

$$\lim_{t \to \infty} \left\| \sum_{l=1}^{m} A^l x^l(t - \tau^l) - \Lambda(t)y(t) \right\| = 0, \quad (10)$$

or

$$\lim_{t \to \infty} \left\| \sum_{l=1}^{m} \sum_{j=1}^{n} a^l_{ij} x^j(t - \tau^l) - \lambda_i(t)y_i(t) \right\| = 0, i = 1, 2, \cdots, n, \quad (11)$$

where $A^l = (a^l_{ij})_{n \times n}$ is constant matrix, $\Lambda(t) = \text{diag} \{\lambda_1(t), \cdots, \lambda_n(t)\}$ is a reversible function matrix whose elements are continuously differentiable nonzero function with bound.

**Definition 2.8.** If there exist a constant $T > 0$, such that

$$\lim_{t \to T} \left\| \sum_{l=1}^{m} A^l x^l(t - \tau^l) - \Lambda(t)y(t) \right\| = 0, \quad (12)$$

or

$$\lim_{t \to T} \left\| \sum_{l=1}^{m} \sum_{j=1}^{n} a^l_{ij} x^j(t - \tau^l) - \lambda_i(t)y_i(t) \right\| = 0, i = 1, 2, \cdots, n, \quad (13)$$

and $\left\| \sum_{l=1}^{m} A^l x^l(t - \tau^l) - \Lambda(t)y(t) \right\| = 0$ if $t \geq T$, then it is said that the group of the drive systems (1) and the response system (2) are finite-time modified function projective multi-lag combined synchronization.

**Remark 2.9.** As is shown in Table 1, the proposed MFPMLCS is more general, and it concludes a large class of the previous synchronization methods. Selecting specific scaling matrix $A^l$, $\Lambda(t)$ and specific delay times $\tau^l, l = 1, 2, \cdots, m$, the MFPMLCS will be simplified to specific synchronization. Here $CS^*$ represents combined synchronization, $CS$ means complete synchronization, $\Lambda = \text{diag} \{\lambda_1, \cdots, \lambda_n\}, I$ is a $n \times n$ unit matrix.
Table 1. The special cases of MFPMLCS.

| Case  | \( m = 2, \tau^1 = \tau^2 = 0, \Lambda(t) = \Lambda \) | \( e(t) = \sum_{i=1}^{m} A^l x^l (t - \tau^l) - \Lambda(t)y(t) \) | MFPMLCS |
|-------|---------------------------------|---------------------------------|-------|
| Case 1 | \( m = 1, A^1 = I \) | \( e(t) = A^1 x^1 (t) + A^2 x^2 (t) - \Lambda(t)y(t) \) | \( CS^* \) [38] |
| Case 2 | \( m = 1, \tau^1 = 0, A^1 = I \) | \( e(t) = x(t - \tau) - \Lambda(t)y(t) \) | MFPPLS [33] |
| Case 3 | \( m = 1, \tau^1 = 0, A^1 = I, \Lambda(t) = \Lambda \) | \( e(t) = x(t) - \Lambda(t)y(t) \) | MFPPLS [30] |
| Case 4 | \( m = 1, \tau^1 = 0, A^1 = I, \Lambda(t) = -I \) | \( e(t) = x(t) - y(t) \) | PS [26] |
| Case 5 | \( m = 1, \tau^1 = 0, A^1 = I, \Lambda(t) = -I \) | \( e(t) = x(t) + y(t) \) | CS [17] |
| Case 6 | \( m = 1, \tau^1 = 0, A^1 = I, \Lambda(t) = I \) | \( e(t) = x(t) + y(t) \) | AS [19] |

**Remark 2.10.** As another advantage of the new method, the drive system is a linear combination of the multiple chaotic systems, which means the signal hidden channels are more diversified and the signal hidden methods are more flexible. The complexity of this new synchronization scheme improves, to a great degree, the abilities to anti-attacking and anti-decoding in the process of signal transmission.

Notice that \( \lambda_i(t) \neq 0 \) is a continuously differentiable function with bound, we can further put forward the following assumption.

**Assumption 2.11.** There exist positive constants \( p_i \) and \( q_i \), \( i = 1, 2, \ldots, n \), i.e.

\[ p_i \leq |\lambda_i(t)| \leq q_i. \]

Let

\[ \rho_i(t) = \sum_{l=1}^{m} \sum_{j=1}^{n} a_{ij}^l w_j^l (t - \tau^l) - \lambda_i(t)d_l(t), \quad (14) \]

combining Assumption 2.2 with Assumption 2.11, we can obtain that \( \rho_i(t) \) is bounded.

Denote \( \rho = [\rho_1, \rho_2, \ldots, \rho_n]^T \) in which \( \rho_i = \sup |\rho_i(t)|, i = 1, 2, \ldots, n \). To deal with the more general case in which the bound \( \rho_i > 0 \) is unknown, the following assumption is needed.

**Assumption 2.12.** There exist definite positive constants \( \tilde{\rho}_i \) (\( i = 1, 2, \ldots, n \)) which are large enough, such that

\[ \rho_i < \tilde{\rho}_i. \]

In order to solve the finite-time synchronization problem, we now define the MFPMLCS error vector

\[ e(t) = \sum_{l=1}^{m} A^l x^l (t - \tau^l) - \Lambda(t)y(t), \quad (16) \]

that is to say

\[ e_i(t) = \sum_{l=1}^{m} \sum_{j=1}^{n} a_{ij}^l x_j^l (t - \tau^l) - \lambda_i(t)y_i(t), i = 1, 2, \ldots, n. \]

(17)

from which, the corresponding error dynamic system below can be obtained:

\[ \dot{e}_i(t) = \sum_{l=1}^{m} \sum_{j=1}^{n} a_{ij}^l x_j^l (t - \tau^l) - \lambda_i(t)y_i(t) - \dot{\lambda}_i(t)y_i(t) - \dot{\lambda}_i(t)y_i(t) \]

\[ = \left\{ \sum_{l=1}^{m} \sum_{j=1}^{n} a_{ij}^l f_j^l (x(t - \tau^l)) - \lambda_i(t)h_i(y(t)) - \dot{\lambda}_i(t)y_i(t) \right\} \]

\[ + \left\{ \sum_{l=1}^{m} \sum_{j=1}^{n} a_{ij}^l F_j^l (x(t - \tau^l))\theta^l - \lambda_i(t)H_i(y(t))\phi \right\} \]
\[ + \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} w_i(t - t^j) - \lambda_i(t) d_i(t) \right] - \dot{\lambda}_i(t) u_i(t). \]  

(18)

For convenience, let us denote
\[ \Omega_i = \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} f_j(x_i(t - t^j)) - \lambda_i(t) h_i(y(t)) - \dot{\lambda}_i(t) y_i(t), \right. \]
\[ \tilde{u}_i(t) = \lambda_i(t) u_i(t). \]  

(19)

Now, the error dynamics system (18) can be reduced as follows
\[ \dot{e}_i(t) = \Omega_i + \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} f_j(x_i(t - t^j)) \phi - \lambda_i(t) H_i(y(t)) \phi + \rho_i(t) - \tilde{u}_i(t). \]  

(20)

3 Design of dual-stage finite-time control scheme

It is clear that the finite-time MFPMMLCS problem is directly equivalent to the finite-time stabilization of the error system (20). In this section, we pay our attention to design an adaptive sliding mode variable structure control scheme to ensure the error trajectories converge to zero within a limited time. The finite-time control scheme is divided into the sliding mode stage and the sliding mode reaching stage. What is more, the time required for each stage is limited.

3.1 Sliding mode stage

In order to realize the desired finite-time sliding motion, let us establish a new nonsingular terminal sliding surface [41] as follows,
\[ s_i(t) = c_{i0} e_i(t) + \int_0^t \left( c_{i1} e_i + c_{i2} \text{sgn}(e_i(\sigma)) |e_i(\sigma)|^{2 - \alpha_i} + c_{i3} \text{sgn}(e_i(\sigma)) |e_i(\sigma)|^{\alpha_i} \right) \text{d}\sigma, \]  

(21)

where the constants 0 < \( \alpha_i < 1 \), \( c_{iv} > 0, v = 0, 1, 2, 3, i = 1, 2, \cdots, n. \)

Remark 3.1. Compared with the terminal sliding surface
\[ s_i(t) = c_i e_i(t) + \int_0^t \text{sgn}(e_i(\sigma)) |e_i(\sigma)|^{\alpha_i} \text{d}\sigma, i = 1, 2, \cdots, n, \]

which is proposed in [15], the terminal sliding surface (21) has the following advantage: the factor \( c_{i1} e_i + c_{i2} \text{sgn}(e_i(\sigma)) |e_i(\sigma)|^{2 - \alpha_i} \) plays a leading role to guarantee a fast convergence speed as \( |e_i(t)| \) is much larger than 1, while the factor \( c_{i3} \text{sgn}(e_i(\sigma)) |e_i|^{\alpha_i} \) is the dominant one ensuring the finite-time convergence as \( |e_i(t)| \) is much less than 1.

According to the sliding mode control theory, when the state trajectories of the error system are located on the sliding surface, it is necessary and sufficient that
\[ s_i(t) = \dot{s}_i(t) = 0, i = 1, 2, \cdots, n, \]

from which, we can obtain the following dynamics of sliding mode:
\[ \dot{e}_i(t) = - \frac{1}{c_{i0}} \left( c_{i1} e_i(t) + c_{i2} \text{sgn}(e_i(t)) |e_i(t)|^{2 - \alpha_i} + c_{i3} \text{sgn}(e_i(t)) |e_i(t)|^{\alpha_i} \right), i = 1, 2, \cdots, n. \]  

(22)
The error vector $e(t)$ of the sliding mode is finite-time stable and its trajectory converges to the equilibriums $e(t) = 0$ within a finite time $T_1$,

$$T_1 = \max\{T_{11}, T_{12}, \ldots, T_{1n}\},$$

with

$$T_{1i} = \frac{1}{b_{i2}(1 - \hat{\alpha}_i)} \ln(1 + \frac{\hat{b}_{i2}V^{1-\hat{\alpha}_i}(0)}{\hat{b}_{i1}}), i = 1, 2, \ldots, n,$$

and

$$\hat{b}_{i1} = \frac{2^{1+\alpha_i}}{c_{i3}}, \quad \hat{b}_{i2} = \frac{2c_{i1}}{c_{i0}}, \quad \hat{\alpha}_i = \frac{1 + \alpha_i}{2}.$$  

**Proof.** Design the following Lyapunov function for the dynamics of the proposed nonsingular terminal sliding mode

$$V_{1i}(t) = \frac{1}{2}e_i^2(t).$$

Taking the time derivative of $V_{1i}(t)$, we obtain

$$V_{1i}(t) = e_i(t)\dot{e}_i(t) = -\frac{1}{c_{i0}}(c_{i1}e_i(t)^2 + c_{i2}|e_i(t)|^{3-\alpha_i} + c_{i3}|e_i(t)|^{1+\alpha_i})$$

$$= -\frac{1}{c_{i0}}(2c_{i1}V_{1i} + 2^{3-\alpha_i}c_{i2}(V_{1i})^{\frac{3-\alpha_i}{2}} + 2^{1+\alpha_i}c_{i3}(V_{1i})^{\frac{1+\alpha_i}{2}})$$

$$\leq -\frac{2c_{i1}}{c_{i0}}V_{1i} - \frac{2^{1+\alpha_i}c_{i3}}{c_{i0}}(V_{1i})^{\frac{1+\alpha_i}{2}}.$$  

(27)

Applying the Lemma 2.3, we can directly deduce that during the sliding mode phase the error $e_i(t)$ converges to zero in the finite time $T_{1i}$ given by (24). This yields that the error vector $e(t)$ converges to $e(t) = 0$ in a finite time $T_1$ given by (23). Hence the proof is completed.



### 3.2 Sliding mode reaching stage

Until now, the suitable sliding surface is established and the finite-time convergence and stability in sliding mode stage has been proved. We now turn to design an adaptive controller to force the error trajectories move toward the sliding surface within a finite time and remain on it forever. In order to achieve the finite-time sliding mode reaching stage, the controller is given as follows:

$$u_i(t) = \frac{1}{k_i(t)}\{\Omega_i + \frac{1}{c_{i0}}(c_{i1}e_i + c_{i2}\text{sgn}(e_i)|e_i|^{2-\alpha_i} + c_{i3}\text{sgn}(e_i)|e_i|^\alpha_i) + (k_i + \hat{k}_i)\text{sgn}(s_i)$$

$$+ \sum_{j=1}^{m} \sum_{l=1}^{n} a_{ij}^l f_i^j(x^l(t - t^l))\hat{\theta}^l - \lambda_i(t)(y(t))\hat{\phi} + \frac{\lambda g}{n c_{i0}} \cdot \frac{\text{sgn}(s_i)}{|s_i|}\}$$

$$i = 1, 2, \ldots, n,$$

(28)

with

$$g = \|\hat{\phi}\| + \|\hat{\theta}\| + \|\hat{\rho}\| + \sum_{l=1}^{m} (\|\hat{\theta}^l\| + \|\hat{\rho}^l\|),$$

(29)

in which, the constants $\zeta > 0$ and $k_i > 0$ are the control gains, which can be designed according to the demands of the designer. $\hat{\rho} = [\hat{\rho}_1, \ldots, \hat{\rho}_n]^T$ is the estimation of the upper bound constant vector $\rho$, $\hat{\theta}^l$ and $\hat{\phi}$ are the estimations of the parameters $\theta^l$, $\phi$ respectively, and $\eta = [c_{10}s_1, c_{20}s_2, \ldots, c_{n0}s_n]^T$, $\mu = \min\{c_{10}k_1, c_{20}k_2, \ldots, c_{n0}k_n\}$. 

Meanwhile, the adaptive laws are given as follows to tackle the unknown parameters:

\[
\hat{\dot{\theta}}_l = c_{i0} |s_l|, \quad \hat{\dot{\dot{\theta}}}_l(0) = \hat{\dot{\theta}}_l(0),
\]

\[
\hat{\dot{\theta}}^l = [A^l F^l(x^l(t - \tau^l))]^{T} \eta, \quad \hat{\dot{\dot{\theta}}}_l(0) = \hat{\dot{\theta}}^l(0), \quad l = 1, 2, \ldots, m,
\]

\[
\hat{\phi} = -[\Lambda(t)H(y(t))]^{T} \eta, \quad \hat{\phi}(0) = \hat{\phi}_0.
\] (30)

**Theorem 3.3.** Using the controller (28) and the adaptive control laws (30), the state of the MFPMLCS error system (22) will reach to the sliding surface \(s = 0\) in a finite time \(T_2\), and remain on it forever. Meanwhile, the sliding mode reaching time \(T_2\) satisfies

\[
T_2 \leq \frac{\|[s(0)]^2 + \|\dot{s}\|^2 + \|\ddot{s}\|^2 + \frac{1}{2} \sum_{l=1}^{m} (\|\ddot{\theta}^l\|^2 + (\ddot{\phi})^2)}{\gamma},
\] (31)

in which, \(\gamma = \min\{\mu, \zeta\}\).

**Proof.** Choose the following Lyapunov function candidate

\[
V_2(t) = V_{21}(t) + V_{22}(t),
\] (32)

in which

\[
V_{21}(t) = \frac{1}{2} \|[s]^{2},
\]

\[
V_{22}(t) = \frac{1}{2} \|[\dot{s} - \ddot{\phi}]^{2} + \frac{1}{2} \|[\ddot{\theta} - \theta^l]^{2} + \sum_{l=1}^{m} (\|\ddot{\theta}^l\|^2 + (\ddot{\phi})^2).\]

(33)

Taking the time derivative of \(V_2(t)\), we get

\[
\dot{V}_2(t) = s^T \ddot{s} = \sum_{i=1}^{n} s_i \dot{s}_i = \sum_{i=1}^{n} s_i [c_{i0} \dot{e}_i + c_{i1} e_i + c_{i2} \text{sgn}(e_i)|e_i|^{2-\alpha_i} + c_{i3} \text{sgn}(e_i)|e_i|^{\alpha_i}].
\]

Along the error system, \(\dot{V}_2(t)\) can be described as

\[
\dot{V}_{21}(t) = -\sum_{i=1}^{n} s_i c_{i0} \cdot \frac{\|s_l\|}{n c_{i0}} \cdot \text{sgn}(s_l) - \sum_{i=1}^{n} c_{i0} k_i |s_l| + \sum_{i=1}^{n} [s_i c_{i0} \rho_i(t) - c_{i0} |s_l| \dot{\theta}_l]
\]

\[
+ \sum_{i=1}^{m} \sum_{l=1}^{n} n c_{i0} k_l F^l(x^l(t - \tau^l)) (\theta^l - \ddot{\theta}^l) + \sum_{i=1}^{n} s_i c_{i0} [-\lambda_i(t) H_i(y(t))(\phi - \ddot{\phi})]
\]

\[
= -\sum_{i=1}^{n} s_i c_{i0} \cdot \frac{\|s_l\|}{n c_{i0}} \cdot \text{sgn}(s_l) - \sum_{i=1}^{n} c_{i0} k_i |s_l| + \sum_{i=1}^{n} [s_i c_{i0} \rho_i(t) - c_{i0} |s_l| \dot{\theta}_l]
\]

\[
+ \sum_{i=1}^{m} (\theta^l - \ddot{\theta}^l) [A^l F^l(x^l(t - \tau^l))]^{T} \eta + (\phi - \ddot{\phi})^{T} [-\Lambda(t)H(y(t))]^{T} \eta.
\]

Using the fact

\[
s_i c_{i0} \rho_i(t) \leq |s_i c_{i0} \rho_i(t)| \leq c_{i0} |s_l| |\rho_i(t)| \leq c_{i0} |s_l| |\rho_l|,
\]

\[
\sum_{i=1}^{n} \left( s_i c_{i0} \cdot \frac{1}{n c_{i0}} \cdot \text{sgn}(s_l) \right) = 1,
\]

\[
\mu = \min \{c_{10} k_1, c_{20} k_2, \ldots, c_{n0} k_n\},
\]

we can derive

\[
\dot{V}_{21}(t) \leq \sum_{i=1}^{m} (\theta^l - \ddot{\theta}^l)^{T} [A^l F^l(x^l(t - \tau^l))]^{T} \eta + (\phi - \ddot{\phi})^{T} [-\Lambda(t)H(y(t))]^{T} \eta
\]
Combining (34) with (35), we can obtain

\[-\gamma g - \mu \sum_{i=1}^{n} |s_i| + \sum_{i=1}^{n} c_{i0} |s_i| (\varrho_i - \hat{\varrho}_i).\]  

(34)

The time derivative of \(V_{22}(t)\) can be calculated as

\[
\dot{V}_{22}(t) = \sum_{i=1}^{n} (\dot{\varrho}_i - \varrho_i) \dot{\varrho}_i + \sum_{l=1}^{m} (\ddot{\theta}_l - \theta_l)^T \dot{\varphi} + (\bar{\varphi} - \phi) \dot{\varphi}
\]

\[
= \sum_{i=1}^{n} c_{i0} |s_i| (\dot{\varrho}_i - \varrho_i) + \sum_{l=1}^{m} (\ddot{\theta}_l - \theta_l)^T [A_l F_l (x_l (t - \tau_l))]^T \eta
\]

\[
+ (\bar{\varphi} - \phi) [-\Lambda(t) H(y(t))]^T \eta.
\]

(35)

Combining (34) with (35), we can obtain

\[
\dot{V}_2(t) = \dot{V}_{21}(t) + \dot{V}_{22}(t) \leq -\mu \sum_{i=1}^{n} |s_i| - \gamma g
\]

\[
= -\mu \sum_{i=1}^{n} |s_i| - \gamma (2\|\hat{\varrho}\| + \|\varrho\| + \|\varphi\| + \|\ddot{\theta}\| + \|\theta\|)
\]

\[
\leq -\gamma (\sum_{i=1}^{n} |s_i| + \|\hat{\varrho}\| + \|\varrho\| + \|\varphi\| + \|\ddot{\theta}\| + \|\theta\|)
\]

\[
\leq -\gamma (\sum_{i=1}^{n} |s_i| + \|\varrho - \varphi\| + \|\varphi - \varrho\| + \|\ddot{\theta} - \theta\|).
\]

(36)

According to Lemma 2.6, we get

\[
\dot{V}_2(t) \leq -\gamma (\sum_{i=1}^{n} s_i^2 + \sum_{l=1}^{m} \|\ddot{\theta}_l - \theta_l\|^2 + \|\dot{\varrho} - \varrho\|^2 + \|\varphi - \phi\|)^{\frac{1}{2}}
\]

\[
= -\gamma (\|s\|^2 + \|\dot{\varrho} - \varrho\|^2 + \|\varphi - \phi\| + \sum_{l=1}^{m} \|\ddot{\theta}_l - \theta_l\|^2)^{\frac{1}{2}}
\]

\[
= -\sqrt{2} \gamma (\frac{1}{2} \|s\|^2 + \frac{1}{2} \|\dot{\varrho} - \varrho\|^2 + \frac{1}{2} \|\varphi - \phi\| + \frac{1}{2} \sum_{l=1}^{m} \|\ddot{\theta}_l - \theta_l\|^2)^{\frac{1}{2}}
\]

\[
= -\sqrt{2} \gamma V_2^{\frac{1}{2}}(t).
\]

(37)

Applying Lemma 2.4, it follows that the error trajectory \(e(t)\) converges to the sliding surface \(s(t) = \varrho = 0\) in the finite time \(\hat{T}_2\) and then remains on it forever, meanwhile the following inequality holds

\[
\frac{\|s(0)\|^2 + \|\dot{\varrho}_0 - \varrho\|^2 + \|\varphi_0 - \phi\|^2 + \sum_{l=1}^{m} \|\ddot{\theta}_l - \theta_l\|^2}{\gamma}
\]

\[
\leq \hat{T}_2.
\]

(38)

It is clear that \(\hat{T}_2 \leq T_2\) in which \(T_2\) is given by (31). This completes the proof. □

**Remark 3.4.** The results of Theorem 3.2 and Theorem 3.3 imply that the group of the drive systems (1) and the response system (2) are MFPMLCS in the finite time \(T_1 + T_2\) under the action of the adaptive control law (28)-(30).

**Remark 3.5.** According to the previous discussion, the convergence times \(T_1, T_2\) and the controller \(u_i(t)\) are dependent on the control gains \(c_{i0}\), \(k_i\) and \(\zeta\). On the one hand, \(T_1\) is proportional to the value of \(c_{i0}\), which means a smaller \(c_{i0}\) results in a shorter convergence times \(T_1\), on the other hand, the sliding mode reaching time \(T_2\) is inversely proportional to \(\gamma = \min\{\mu, \zeta\} = \min\{c_{10}k_1, c_{20}k_2, \ldots, c_{n0}k_n, \zeta\}\). At the same time, the control input \(u_i(t)\) is proportional to \(\frac{1}{c_{i0}}, k_i\) and \(\zeta\). Based on these relationships, the appropriate control gains above can be selected according to the specific requirements of designer.
Remark 3.6. According to Eqs. (28), the control input $u_i(t)$ contains the factor $\frac{\text{sgn}(s_i)}{|s_i|}$. In fact, during the sliding mode reaching phase, when the error trajectories $e_i(t)$ reach onto the sliding surfaces $s_i(t) = 0$, it is obvious that $\frac{\text{sgn}(s_i)}{|s_i|}$ is singular. In order to overcome this disadvantage, the control law (28) is modified as follows

$$u_i(t) = \frac{1}{\lambda_i(t)} \left\{ \Omega_i + \frac{1}{c_i} \left( c_{i1} e_i + c_{i2} \text{sgn}(e_i) |e_i|^{2-\alpha_i} + c_{i3} \text{sgn}(e_i) |e_i|^{\alpha_i} \right) + k_i \text{sgn}(s_i) + \hat{\lambda} \text{sgn}(s_i) \right\}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} F^j_i \left( x^j(t \pm \tau^j) \hat{\theta}^j - \lambda_i(t) H_i(y(t)) \right) + \frac{\varepsilon}{n c_{i0}} \cdot \Delta^i$$

$$i = 1, 2, \ldots, n,$$  \hspace{1cm} (39)

with

$$\Delta = \begin{cases} \frac{\text{sgn}(s_i)}{|s_i|}, & \text{if } \sum_{i=1}^{n} |s_i| \geq \delta, \\ 0, & \text{if } \sum_{i=1}^{n} |s_i| < \delta. \end{cases}$$  \hspace{1cm} (40)

where the switching gain $\delta$ is a sufficiently small positive constant which can be chosen according to the designer requirements.

Another effective approach is using the function $\frac{\text{sgn}(s_i)}{|s_i| + \varepsilon}$ ( $\varepsilon$ is a sufficiently small positive constant) to approximate $\frac{\text{sgn}(s_i)}{|s_i|}$, which is common in the sliding mode application.

4 Numerical simulation

In this section, we choose two famous chaotic systems: Lü system and Lorenz system with fully unknown parameters and unknown bounded disturbances as the drive systems. At the same time, another well-known chaotic system named Chen system is considered as the response system. They can be described as follows:

Lü system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 x_3 \\ 4x_1^2 \end{bmatrix} + \begin{bmatrix} x_1^2 - x_2^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0.5 \sin t \\ 2 \sin(2t) \\ 2 \cos t \end{bmatrix}.$$  \hspace{1cm} \text{f}^1(x(t))

Lorenz system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 x_2 - x_2^2 \\ x_1^2 x_2 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 10 \\ 8/3 \end{bmatrix} + \begin{bmatrix} \cos 2t \\ \sin 3t \\ \cos t \end{bmatrix}.$$  \hspace{1cm} \text{f}^2(x(t))

Chen system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{bmatrix} + \begin{bmatrix} y_2 - y_1 \\ -y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 35 \\ 28 \\ 3 \end{bmatrix} + \begin{bmatrix} \cos 2t \\ \sin 3t \\ \cos t \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}.$$  \hspace{1cm} \text{h}(y(t))

In the simulation, the drive systems are started with $x^1(0) = (2, 2, 2)$ and $x^2(0) = (3, 3, 3)$, and the response system is initialized with $y(0) = (-6, -6, -6)$, the control gains are selected as $k = (100, 80, 80), \alpha_i = 0.1, c_{i0} =$
2. \(c_{i1} = 10, c_{i2} = 30, c_{i3} = 50 \ (i = 1, 2, 3)\) and \(\zeta = 0.1\), it yields \(\mu = 40, \gamma = 0.1\). The bound vectors are chosen as \(\|\hat{\rho}\| = 15, \hat{\theta}^l = \hat{\phi} = 55\). Choosing delay times \(\tau^1 = 1, \tau^2 = 2\) and the following scaling matrices

\[
A^1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -1
\end{pmatrix},
A^2 = \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix},
A(t) = \begin{pmatrix}
2 + \sin t & 0 & 0 \\
0 & 1 + 0.5 \cos t & 0 \\
0 & 0 & 1 - 0.5 \sin t
\end{pmatrix}
\]

Using the modified controller (39)-(40) and the adaptive control law (30) with \(\delta = 0.1\), the MFPMLCS errors are revealed in Figure 1. It is observed that the MFPMLCS errors convergence to \(e_i(t) = 0\) within a very short time. The time responses of the adaptive parameter vectors \(\hat{\rho}, \hat{\theta}^l, \text{and} \hat{\phi}\), converge to the values \(\rho, \theta^l, \text{and} \phi\), respectively which can be shown in Figures 2-5. Meanwhile, Figure 6 shows the sliding surface can rapidly converge to zero. The simulation results illustrate the effectiveness of the proposed method.

Fig. 1. Time response of MFPLS error \(e\)

Fig. 2. Time response of \(\hat{\rho}\)

Fig. 3. Time response of \(\hat{\theta}^l\)
5 Conclusion

In this paper, we dealt with the problem of the finite-time modified function projective multi-lag combined synchronization (MFPMLCS) for a series of different chaotic systems with unknown bounded disturbances and fully unknown parameters. Based upon the sliding mode control technique and Lyapunov stability theory, we designed an adaptive dual-stage variable structure control scheme to realize the finite-time synchronization. The resulted systems are provided with fast convergence rate, strong robustness, small chattering and high accuracy. Finally, the numerical simulation demonstrated the correctness and effectiveness of the advanced scheme.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

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