On a constructive characterization of a class of trees related to pairs of disjoint matchings

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Abstract

For a graph consider the pairs of disjoint matchings which union contains as many
edges as possible, and define a parameter $\alpha$ which equals the cardinality of the largest
matching in those pairs. Also, define $\beta$ to be the cardinality of a maximum matching
of the graph.

We give a constructive characterization of trees which satisfy the $\alpha = \beta$ equality.
The proof of our main theorem is based on a new decomposition algorithm obtained
for trees.

Keywords: tree, pair of disjoint matchings, maximum matching

Introduction

Let $Z^+$ denote the set of non-negative integers. We consider finite, undirected
graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the
sets of vertices and edges of a graph $G$, respectively.

If $v \in V(G)$ then let $d_G(v)$ denote the degree of a vertex $v$ in a graph $G$.
For a bridge $e = (v_1, v_2)$ of a connected graph $G$, let $G_1, G_2$ be the connected
components of $G - e$. Define the graphs $G_1e, G_2e$ as follows:

$$G_1e \equiv G \setminus (V(G_2) \setminus \{v_2\}),$$

$$G_2e \equiv G \setminus (V(G_1) \setminus \{v_1\}),$$

where, without loss of generality, it is assumed, that $v_i \in V(G_i), i = 1, 2$.

For a graph $G$, let $\beta(G)$ denote the cardinality of a maximum matching of
$G$. Define:

$$M(G) \equiv \{F : F \text{ is a maximum matching of } G\},$$

$$L(G) \equiv \max\{\beta(G \setminus F) : F \in M(G)\},$$

$$M'(G) \equiv \{F \in M(G) : \beta(G \setminus F) = L(G)\}.$$
\[ \lambda(G) \equiv \max \{|H| + |H'| : H, H' \text{ are matchings of } G \text{ with } H \cap H' = \emptyset \}, \]
\[ M_2(G) \equiv \{(H, H') : |H| + |H'| = \lambda(G) \text{ and } H \cap H' = \emptyset \}, \]
\[ \alpha(G) \equiv \max \{|H|, |H'| : (H, H') \in M_2(G) \}, \]
\[ M_2'(G) \equiv \{(H, H') : (H, H') \in M_2(G), |H| = \alpha(G) \}. \]

It is known that every graph \( G \) contains a maximum 2-matching that includes a maximum matching of \( G \) (see [7]). In contrast with the theory of 2-matchings, in an arbitrary graph \( G \) we cannot always guarantee the existence of a "maximum" pair of disjoint matchings (i.e. pair of disjoint matchings the union of which contains \( \lambda(G) \) edges), which includes a maximum matching. The following is the best we can do here: for every graph \( G \) the following inequality is true [10]:

\[ 1 \leq \frac{\beta(G)}{\alpha(G)} \leq \frac{5}{4}. \]

Let us also note that in her master thesis [11] Tserunyan gave an elegant and very deep characterization of graphs which achieve the bound \( \frac{5}{4} \). Her theorem particularly implies that these graphs contain a spanning subgraph every component of which is isomorphic to the minimal graph that satisfies the \( \frac{\beta}{\alpha} = \frac{5}{4} \) equality.

In the light of this fact, the characterization of graphs which satisfy the \( \alpha = \beta \) equality becomes a problem of notable importance. Moreover, the problem is interesting not only because of its own but also because of the equivalence:

a graph \( G \) satisfies the equality \( \alpha(G) = \beta(G) \) if and only if
\[ \lambda(G) = \beta(G) + L(G). \]

Though, the calculation of \( \lambda(G) \) is \( NP \)-hard in general [4], the Ford-Fulkerson algorithm for finding a maximum flow in a network implies that it is indeed polynomial-time calculable for bipartite graphs. And, once we are given a bipartite graph \( G \) satisfying the equality \( \alpha(G) = \beta(G) \), we can calculate \( L(G) \) easily. This is important, since \( L(G) \) remains \( NP \)-hard calculable even for connected bipartite graphs \( G \) with maximum degree three [5]. Let us also note that there is a polynomial algorithm which constructs a maximum matching \( F \) of a tree \( G \) such that \( \beta(G \setminus F) = L(G) \) (to be presented in [6]).

The aim of present paper is the characterization of trees that satisfy the \( \alpha = \beta \) equality. An early result in this direction is given in [8]: for every matching covered tree \( G \) the equality \( \alpha(G) = \beta(G) \) holds (a graph \( G \) is referred to be matching covered if its every edge belongs to a maximum matching of the graph [7, 9], complete characterization of those trees can be found in [2, 3]). The characterization given in the paper is constructive, more specifically, we define four operations, with the help of which we prove that a tree \( G \) satisfies the equality \( \alpha = \beta \) if and only if it can be built from \( K_1 \) or \( K_2 \) (the trees containing one or two vertices, respectively) by using these operations. Our proof is based on a new decomposition algorithm obtained for the class of trees.

Non-defined terms and concepts can be found in [1, 7, 12].
Lemma 1 Let $G$ be a graph, $v$ be a vertex with $d_G(v) = 1$, and $e$ be the edge incident to it. Then

1. There is $(H,H') \in M'_2(G)$, such that $e \in H$.
2. There is $F \in M'(G)$, such that $e \in F$.

Lemma 2 [8]. Let $G$ be a graph, $U = \{u_0, ..., u_4\} \subseteq V(G)$ satisfying the conditions: $d_G(u_0) = d_G(u_4) = 1$, $d_G(u_1) = d_G(u_3) = 2$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4$ (fig 1). Then the following is true:

$$\lambda(G) = \lambda(G\backslash U) + 4,$$

$$\alpha(G) \geq 2 + \alpha(G\backslash U).$$

Lemma 3. Let $G$ be a graph and let $e \in E(G)$. Then

1. $\lambda(G) \geq \lambda(G - e)$;
2. if $(H,H') \in M_2(G)$ and $e \notin H \cup H'$ then $\lambda(G) = \lambda(G - e)$ and $\alpha(G) \geq \alpha(G - e)$;
3. if $(H,H') \in M'_2(G)$ and $e \notin H \cup H'$ then $\alpha(G) = \alpha(G - e)$.

Lemma 4. Let $G$ be a connected graph, $e$ be a bridge of $G$, and let $G_1, G_2$ be the connected components of $G - e$. Then

1. $\lambda(G) \geq \lambda(G_1e) + \lambda(G_2e) - 1$;
2. if there is $(H,H') \in M_2(G)$ with $e \in H \cup H'$ then $\lambda(G) = \lambda(G_1e) + \lambda(G_2e) - 1$ and $\alpha(G) \geq \alpha(G_1e) + \alpha(G_2e) - 1$;
3. if there is $(H,H') \in M'_2(G)$ with $e \in H$ then $\alpha(G) = \alpha(G_1e) + \alpha(G_2e) - 1$. 

Figure 1: Some auxiliary results about $\lambda(G)$, $\alpha(G)$ and $L(G)$
Proof. (1) Choose \((H_1, H_1') \in M_2'(G_1e), (H_2, H_2') \in M_2'(G_2e)\) with \(e \in H_1, H_2\) (1) of lemma 1). Define:

\[
H \equiv H_1 \cup H_2,
H' \equiv H_1' \cup H_2'.
\]

Clearly, \(H\) and \(H'\) are disjoint, and

\[
\lambda(G) \geq |H| + |H'| = |H_1| + |H_2| - 1 + |H_1'| + |H_2'| = \lambda(G_1e) + \lambda(G_2e) - 1.
\]

(2) Note that \((H \cap E(G_1e), H' \cap E(G_1e))\) and \((H \cap E(G_2e), H' \cap E(G_2e))\) are pairs of disjoint matchings in \(G_1e\) and \(G_2e\), respectively. Hence

\[
\lambda(G) = |H| + |H'| = |H \cap E(G_1e)| + |H' \cap E(G_1e)| + |H \cap E(G_2e)| + |H' \cap E(G_2e)| - 1 \leq \lambda(G_1e) + \lambda(G_2e) - 1,
\]

therefore

\[
\lambda(G) = \lambda(G_1e) + \lambda(G_2e) - 1.
\]

Note that this and lemma 1 imply that

\[
\alpha(G) \geq \alpha(G_1e) + \alpha(G_2e) - 1.
\]

(3) (2) implies that

\[
(H \cap E(G_1e), H' \cap E(G_1e)) \in M_2(G_1e) \quad \text{and} \quad (H \cap E(G_2e), H' \cap E(G_2e)) \in M_2(G_2e),
\]

hence

\[
\alpha(G) = |H| + |H \cap E(G_1e)| + |H \cap E(G_2e)| - 1 \leq \alpha(G_1e) + \alpha(G_2e) - 1, \quad \text{or}
\]

\[
\alpha(G) = \alpha(G_1e) + \alpha(G_2e) - 1.
\]

The proof of lemma 4 is completed.

**Lemma 5** [6]. Let \(G\) be a connected graph, \(e\) be a bridge of \(G\), and let \(G_1, G_2\) be the connected components of \(G - e\). Then

\[
L(G) = L(G_1e) + L(G_2e).
\]

**The main result**

In this section we introduce four elementary operations. They have the property of preserving the equality \(\beta = \alpha\), that is, if the graph satisfies the equality then so does the graph obtained from original one by the application of any of them. In the end of the section we prove that the tree \(G\) satisfying \(\beta(G) = \alpha(G)\) can be built from \(K_1\) or \(K_2\) by using only these operations.

**Operation A.** Let \(v_1, v_k (k \geq 1)\) be different vertices of a graph \(G\). Consider the graphs \(G'\) and \(G''\) obtained from \(G\) in the following way (figure 2):
Since there are \((H_1, H'_1) \in M'_2(G')\) and \((H_2, H'_2) \in M'_2(G'')\) such that 
\((u, v_i) \notin H_j \cup H'_j, 1 \leq i \leq k\) and \(j = 1, 2\), we imply that (lemma 3)
\[
\alpha(G') = 1 + \alpha(G), \tag{*}
\]
\[
\alpha(G'') = 2 + \alpha(G).
\]

Note that the following equalities are also true [6]:
\[
\beta(G') = 1 + \beta(G), L(G') = 1 + L(G), \tag{**}
\]
\[
\beta(G'') = 2 + \beta(G), L(G'') = 1 + L(G).
\]

Hence

**Lemma 6.** Either the graphs \(G, G', G''\) satisfy the equality \(\beta = \alpha\) or none of them does.

Now, we proceed to the definitions of the three other operations. In contrast with operation A, these ones are not always defined. This is the main reason why the description of each operation is preceded by the description of the cases when the operation is applicable.

**Operation B.**
Definition 1. A vertex $v$ of a graph $G$ is referred to be applicable for the operation B if either $d_G(v) \leq 1$ or there is $U = \{u_0, \ldots, u_4\} \subseteq V(G)$ satisfying the conditions:

(a) $v = u_2$;
(b) $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 4$;
(c) $d_G(u_0) = 1$, $d_G(u_1) = d_G(u_3) = 2$ (figure 3)

Figure 3

If $G$ is a graph, and $v$ is an applicable vertex for operation B, then $G'$ (the result of operation B) is defined as follows (figure 4):

Figure 4

Lemma 7. $\beta(G') = \alpha(G')$ if and only if $\beta(G) = \alpha(G)$.

Proof. First of all note that $\beta(G') = 1 + \beta(G)$. The statement is true if $d_G(v) = 0$. Assume that $d_G(v) = 1$. Then

$$\lambda(G') = 2 + \lambda(G),$$

and due to (1) of lemma 1 and (3) of lemma 4

$$\alpha(G') = 1 + \alpha(G).$$
This shows that the statement of lemma 6 is true for the case of $d_G(v) = 1$.

Therefore, we may assume that $d_G(v) \geq 2$. Since $v$ is applicable for operation B, there is $U = \{u_0, \ldots, u_4\} \subseteq V(G)$ satisfying the conditions (a), (b), (c) of definition 1. Let $\{w, w'\} = V(G') \setminus V(G)$ and $d_{G'}(w) = 2$, $d_{G'}(w') = 1$.

Lemma 3 implies that to complete the proof it suffices to show that there is $(H, H') \in M_2(G')$, such that $(w, v) / \in H \cup H'$, or $(u_1, v) / \in H \cup H'$.

Choose any $(H, H') \in M_2(G')$, and assume that $\{(w, v), (u_1, v)\} \subseteq H \cup H'$. Without loss of generality, we may assume that $\{(u_0, u_1), (w, v)\} \subseteq H$ and $\{(w, w'), (u_1, v)\} \subseteq H'$. We claim that $(u_3, u_4) \in H$. Suppose that $(u_3, u_4) / \in H$.

Define:

$$\bar{H} \equiv (H \setminus \{(w, v)\}) \cup \{(u_3, v), (w', w)\}, \quad \bar{H}' \equiv H' \setminus \{(w', w)\}.$$ 

Note that

$$|\bar{H}| + |\bar{H}'| = |H| + |H'| = \lambda(G') \quad \text{and} \quad |\bar{H}| > |H| = \alpha(G'),$$

which is impossible. Thus $(u_3, u_4) \in H$. Define:

$$H'' \equiv (H \setminus \{(u_1, v)\}) \cup \{(u_2, u_3)\}.$$

Note that $(H, H'') \in M_2(G')$ and $\{(w, v), (u_1, v)\} / \in H \cup H''$. The proof of lemma 7 is completed.

**Operation C.**
**Definition 2.** A vertex $v$ of a graph $G$ is referred to be applicable for the operation $C$ if either

1. there is $U = \{u_0, ..., u_6\} \subseteq V(G)$ with
   
   (1a) $v = u_0$;
   
   (1b) $d_G(u_0) = d_G(u_3) = d_G(u_5) = 1$, $d_G(u_2) = d_G(u_4) = 2$, $d_G(u_1) = 4$,
   
   $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3, 5$; $(u_1, u_4) \in E(G)$, $(u_1, u_6) \in E(G)$ (figure 5a);
   
   (1c) $\beta(He) = \beta(H) + 1$, $L(He) = L(H)$, where $H \equiv G \setminus (U \setminus \{u_6\})$ and $e = (u_1, u_6)$;

or

2. there is $U = \{u_0, ..., u_4\} \subseteq V(G)$ with
   
   (2a) $v = u_0$;
   
   (2b) $d_G(u_0) = d_G(u_3) = 1$, $d_G(u_2) = 2$, $d_G(u_1) = 3$,
   
   $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2, 3$; $(u_1, u_4) \in E(G)$ (figure 5b);
   
   (2c) $\lambda(He) = \lambda(H)$, where $H \equiv G \setminus (U \setminus u_4)$, and $e = (u_1, u_4)$.

If $G$ is a graph, and $v$ is an applicable vertex for operation $C$, then $G'$ (the result of operation $C$) is defined as follows (figure 6):

**Lemma 8.** If $\beta(G) = \alpha(G)$ then $\beta(G') = \alpha(G')$. 

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Proof. Case 1: There is $U = \{u_0, \ldots, u_6\} \subseteq V(G)$ satisfying (1) of definition 2 (figure 5a).

Note that $\beta(G') = 2 + \beta(G) = 5 + \beta(H)$ and due to (*)
$\alpha(H) = \alpha(G) = \beta(G) = 3 + \beta(H)$, hence
$\alpha(H) = \beta(H)$, or $\lambda(H) = \beta(H) + L(H)$.

Let $\{g_0, \ldots, g_3\} = E(G') \setminus E(G)$ (figure 7).

We claim that there is no $F \in M'(G')$ containing the edge $(u_0, u_1)$. Assume the contrary, and let $F \in M'(G')$ contain the edge $(u_0, u_1)$.

Due to lemma 5
$L(G') = \beta(G' \setminus F) = 2 + L(H)$.

Choose a maximum matching $F'_1 \in M'(He)$ (lemma 1). Note that $e \in F'_1$.

Define:
$F' \equiv F'_1 \cup \{g_1, g_3, (u_2, u_3), (u_4, u_5)\}$.

Note that
Figure 7:

$|F'| = 4 + \beta(He) = 5 + \beta(H) = \beta(G')$ and

$\beta(G' \setminus F') = 3 + L(He) = 3 + L(H) > L(G')$

which is a contradiction.

This implies that there is $F' \in M'(G')$ containing $g_1$. Note that $e \in F'$ (otherwise we would have an augmenting path), therefore due to lemma 5

$L(G') = \beta(G' \setminus F') = 3 + L(He)$.

On the other hand, lemma 2 implies that

$\lambda(G') = 8 + \lambda(H) = 8 + \beta(H) + L(H) = 8 + \beta(H) + L(He) = \beta(G') + L(G')$

hence

$\alpha(G') = \beta(G')$.

Case 2: There is $U = \{u_0, \ldots, u_4\} \subseteq V(G)$ satisfying (2) of definition 2 (figure 5b).

Note that

$\beta(G') = 2 + \beta(G) = 4 + \beta(H)$ and due to (*)

$2 + \alpha(H) = \alpha(G) = \beta(G) = 2 + \beta(H)$, hence

$\alpha(H) = \beta(H)$, and

$\beta(He) + L(He) \leq \lambda(He) = \lambda(H) = \beta(H) + L(H)$.

Let $\{f_0, \ldots, f_3\} = E(G') \setminus E(G)$ (figure 8).
Let us show that there is $F' \subseteq M'(G')$ containing $(u_0, u_1)$. Take any $F \subseteq M'(G')$, and assume that $(u_0, u_1) \notin F$. Note that $F \cap \{f_1, f_2\} \neq \emptyset$ and $e \in F$ (otherwise we would have an augmenting path).

Without loss of generality we may assume that $f_1 \in F$. It is not hard to see that

$$L(G') = \beta(G' \setminus F) = 3 + L(He) \text{ (lemma 5)},$$

$$4 + \beta(H) = \beta(G') = 3 + \beta(He), \text{ and therefore } L(He) \leq L(H) - 1.$$ 

Let $F'_1 \subseteq M'(H)$. Define $F'$ as follows:

$$F' \equiv F'_1 \cup \{f_0, f_3, (u_0, u_1), (u_2, u_3)\}.$$ 

Clearly

$$|F'| = 4 + \beta(H) = \beta(G') \text{ hence } F' \subseteq M(G'), \text{ and }$$

$$\beta(G' \setminus F') = 2 + L(H) \geq 3 + L(He) = L(G'),$$

hence $F' \subseteq M'(G')$ and $(u_0, u_1) \in F'$. Lemma 5 and (**) imply that

$$L(G') = 2 + L(H).$$
Figure 9:

Lemmata 2, 4 imply that
\[ \lambda(G') = 6 + \lambda(He) = 6 + \lambda(H) + L(H) = \beta(G') + L(G'), \]
hence
\[ \alpha(G') = \beta(G'). \]

The proof of lemma 8 is completed.

**Operation D.**

**Definition 3.** A vertex \( v \) of a graph \( G \) is referred to be applicable for the operation \( D \) if either

1. there is \( U = \{u_0, u_1, u_2\} \subseteq V(G) \) with
   - (1a) \( v = u_1; \)
   - (1b) \( d_G(u_0) = 1, d_G(u_1) = 2, (u_{i-1}, u_i) \in E(G) \) for \( i = 1, 2 \) (figure 9a);

or

2. there is \( U = \{u_0, ..., u_5\} \subseteq V(G) \) with
   - (2a) \( v = u_5; \)
   - (2b) \( d_G(u_0) = d_G(u_4) = 1, d_G(u_1) = d_G(u_3) = 2, d_G(u_2) = 3, \)
   - \( (u_{i-1}, u_i) \in E(G) \) for \( i = 1, 2, 3, 4; (u_2, u_5) \in E(G) \) (figure 9b);

or

3. there is \( U = \{u_0, ..., u_3\} \subseteq V(G) \) with
   - (3a) \( v = u_2; \)
   - (3b) \( d_G(u_0) = 1, d_G(u_1) = d_G(u_2) = 2, (u_{i-1}, u_i) \in E(G) \) for \( i = 1, 2, 3 \) (figure 9c);
   - (3c) \( \beta(He) = \beta(H) + 1, L(He) = L(H), \)
   where \( H \equiv G \setminus \{u_3\} \), and \( e = (u_2, u_3). \)
If $G$ is a graph, and $v$ is an applicable vertex for operation D, then $G'$ (the result of operation D) is defined as follows (figure 10):

**Figure 10**

**Lemma 9.** If $\beta(G) = \alpha(G)$ then $\beta(G') = \alpha(G')$.

**Proof.** Case 1: There is $U = \{u_0, u_1, u_2\} \subseteq V(G)$ satisfying (1) of definition 3 (figure 9a).

Note that lemma 2 implies that

$$\beta(G') = 2 + \beta(G),$$
$$\lambda(G') = 4 + \lambda(G),$$
therefore

$$\alpha(G') \geq 2 + \alpha(G) = 2 + \beta(G) = \beta(G'),$$
or

$$\alpha(G') = \beta(G').$$

Case 2: There is $U = \{u_0, ..., u_5\} \subseteq V(G)$ satisfying (2) of definition 3 (figure 9b).

From lemma 2 we have

$$\beta(G') = 2 + \beta(G),$$
$$\lambda(G') = 4 + \lambda(G),$$
therefore

$$\alpha(G') \geq 2 + \alpha(G) = 2 + \beta(G) = \beta(G'),$$
or

$$\alpha(G') = \beta(G').$$
Case 3: There is \( U = \{u_0, ..., u_3\} \subseteq V(G) \) satisfying (3) of definition 3 (figure 9c).

Note that

\[
\beta(G') = 4 + \beta(H), \\
\beta(G) = 1 + \beta(He), \\
\alpha(G) = 1 + \alpha(He) \text{ (lemma 4), hence} \\
\alpha(He) = \beta(He), \text{ or } \lambda(He) = \beta(He) + L(He)
\]

Let \( \{g_0, ..., g_4\} = E(G') \setminus E(G) \) (figure 11).

Figure 11

We claim that there is no \( F' \in M'(G') \) containing the edge \( g_4 \). On the opposite assumption, consider \( F' \in M'(G') \) with \( g_4 \in F' \). Note that

\[
\{g_0, g_3, (u_0, u_1)\} \subseteq F' \text{ and } L(G') = \beta(G' \setminus F') = 2 + L(H) \text{ (lemma 5).}
\]

Let \( F_1 \in M'(He) \). Note that \( e \in F_1 \). Define \( F \) as follows:

\[
F = F_1 \cup \{g_1, g_3, (u_0, u_1)\}.
\]

Clearly
\[ |F| = 3 + \beta(He) = 4 + \beta(H) = \beta(G') \text{ and } \beta(G'\backslash F) = 3 + L(H) > 2 + L(H) = L(G'), \]

which is a contradiction. This implies that there is \( F \in M'(G') \) containing \( g_1 \). Note that as \( e \in F \) (otherwise we would have an augmenting path), we imply that

\[ L(G') = \beta(G'\backslash F) = 3 + L(He) = 3 + L(H) \text{ (lemma 5)}. \]

On the other hand, lemma 2 implies that (see the definition of operation B)

\[ \lambda(G') = 4 + \lambda(G) = 6 + \lambda(He) = 6 + \beta(He) + L(He) = 7 + \beta(H) + L(H) = \beta(G') + L(G'), \]

hence

\[ \alpha(G') = \beta(G'). \]

The proof of lemma 9 is completed.

**Theorem.** A tree \( G \) satisfies the equality \( \beta(G) = \alpha(G) \) if and only if it is either \( K_1 \) or \( K_2 \), or can be obtained from them by the application of the operations A, B, C or D.

**Proof.** Note that \( K_1 \) and \( K_2 \) satisfy the equality \( \beta = \alpha \), and lemmata 6,7,8,9 imply that the operations A, B, C or D preserve this property, that is, whatever tree \( G \) we build from \( K_1 \) or \( K_2 \) by these operations we will always have \( \beta(G) = \alpha(G) \).

Let us show that the converse is also true, i.e. every tree \( G \) satisfying \( \beta(G) = \alpha(G) \) can be built from \( K_1 \) or \( K_2 \) by A, B, C or D.

The proof is on induction. Clearly, the statement is true if \( |E(G)| \leq 1 \). Assume that the statement is true for all trees \( G' \) which satisfy the equality \( \beta(G') = \alpha(G') \) and \( |E(G')| < |E(G)| \), and let us show that it also holds for the tree \( G \) satisfying \( \beta(G) = \alpha(G) \).

First of all note that we may always assume that there is no \( U = \{u_0, u_1, u_2\} \subseteq V(G) \) with \( d_G(u_0) = 1, d_G(u_1) = d_G(u_2) = 2, (u_{i-1}, u_i) \in E(G) \) for \( i = 1, 2 \).

On the opposite assumption, consider the set \( U \) comprised of vertices \( u_0, u_1, u_2 \) satisfying these conditions. Set:

\[ G' \equiv G\backslash\{u_0, u_1\}. \]

The definition of operation B and lemma 4 imply that \( \beta(G') = \alpha(G') \). The induction hypothesis implies that \( G' \) can be built from \( K_1 \) or \( K_2 \) by A, B, C or D, and since \( G \) can be built from \( G' \) by operation B, we are done.

Now let us show that we may also assume that there is no \( U = \{u_0, ..., u_6\} \subseteq V(G) \) with \( (u_{i-1}, u_i) \in E(G) \) for \( i = 1, 2, 3, 4, 6; (u_2, u_3) \in E(G) \), \( d_G(u_0) = d_G(u_4) = d_G(u_6) = 1, d_G(u_1) = d_G(u_3) = d_G(u_5) = 2 \). If \( U = \{u_0, ..., u_6\} \) is such a set, then set.
\[ G' \equiv G\setminus\{u_0, u_1\}. \]

The definition of operation B and lemma 7 imply that \( \beta(G') = \alpha(G') \) and therefore due to induction hypothesis, \( G' \) can be built from \( K_1 \) or \( K_2 \) by A, B, C or D. As \( u_2 \in V(G\setminus\{u_0, u_1\}) \) is applicable for B and \( G \) is built from \( G' \) by applying B, we conclude that \( G \) can be built from \( K_1 \) or \( K_2 \) by A, B, C or D.

Define:

\[ V_G(0) \equiv \{ v \in V(G) : d_G(v) = 1 \}, \]

and for \( i \geq 1 \) let

\[ V_G(i) \equiv \{ v \in V(G) : d_H(v) = 1, \text{ where } H \equiv G\setminus(\bigcup_{j=0}^{i-1} V_G(j))\}. \]

Consider a mapping \( k_G : V(G) \rightarrow \mathbb{Z}^+ \) defined as:

\[ \text{for } v \in V(G) \quad v \in V_G(k_G(v)). \]

Note that for each vertex \( v \) there is at most one vertex \( v' \) with \( (v, v') \in E(G) \) and \( k_G(v') > k_G(v) \).

Since \( G \) is not a path, we imply that it contains a vertex of degree at least three. Now, choose a vertex \( v \in V(G) \) satisfying the conditions:

\[ d_G(v) \geq 3 \text{ and } k_G(v) \rightarrow \min. \]

Note that the choice of \( v \) implies that there are paths \( P_1, \ldots, P_r \) \((r \geq 2)\) of \( G \) satisfying the conditions:

\[ \text{for every } w \in V(P_i) 1 \leq i \leq r, w \neq v, d_G(w) \leq 2 \text{ and } k_G(w) < k_G(v); \]

\[ d_G(v) = r + 1 \quad (\text{figure } 12). \]

Figure 12

We claim that without loss of generality we may assume that \( r = 2 \) and \( P_1, P_2 \) are of length two for every vertex \( v \in V(G) \) satisfying the conditions \( d_G(v) \geq 3 \) and \( k_G(v) \rightarrow \min. \)

Note that every path from \( P_1, \ldots, P_r \) is of length at most two. Now, let us show that paths \( P_1, \ldots, P_r \) may be assumed to have lengths equal to two. Let \( P_1 \) have a length equal to one, and let \( V(P_1) = \{u, v\} \). Consider the trees \( G_1, \ldots, G_{r-1} \) - the connected components of \( G\setminus(V(P_1) \cup V(P_2)) \). Note that (see operation A)

\[ \beta(G) - \alpha(G) = \sum_{i=1}^{r-1} (\beta(G_i) - \alpha(G_i)), \]
and since $\beta(G) = \alpha(G)$ we imply that $\beta(G_i) = \alpha(G_i)$, $i = 1, ..., r - 1$. Due to hypothesis of induction we conclude that $G_i$, $1 \leq i \leq r - 1$, can be built from $K_1$ or $K_2$ by A, B, C or D. Note that since $G$ is built from $G \setminus (V(P_1) \cup V(P_2))$ by operation A, we are done. This shows that the lengths of paths $P_1, ..., P_r$ may be assumed to be equal to two, and therefore we may also assume that $r = 2$ for every vertex $v \in V(G)$ satisfying the conditions: $d_G(v) \geq 3$ and $k_G(v) \to \min$.

As $G$ is not a path and $\beta(G) = \alpha(G)$ we imply that for every vertex $v \in V(G)$ with $d_G(v) \geq 3$ and $k_G(v) \to \min$ there is a unique $v' \in V(G)$ such that $(v, v') \in E(G)$ and $k_G(v) < k_G(v')$.

Now, choose a vertex $v \in V(G)$ satisfying the conditions:

$$d_G(v) \geq 3, \quad k_G(v) \to \min \quad \text{and} \quad k_G(v') \to \min,$$

where $v'$ is the abovementioned vertex corresponding to $v$.

Note that the choice of $v$ implies that $d_G(v') \geq 2$ and $k_G(v') = k_G(v) + 1$. Let us show that we may also assume that $d_G(v') \geq 3$. Suppose that $d_G(v') = 2$, and let $u_1, ..., u_4, v''$ be vertices shown in the figure below:

![Figure 13](image)

Let us show that there is $(H, H') \in M_2^2(G)$ such that

$$\{(u_2, v), (u_3, v)\} \not\subseteq H \cup H'.$$
Choose \((H, H') \in M_2(G)\) and, without loss of generality, assume that \((u_2, v) \in H, (u_3, v) \in H'\). Define: \(H_1\) and \(H'_1\) as follows:

\[
H_1 \equiv H, \quad H'_1 \equiv (H' \setminus \{(u_3, v)\}) \cup \{(v, v')\} \quad \text{if} \quad (v', v'') \in H,
\]

\[
H_1 \equiv (H \setminus \{(u_2, v)\}) \cup \{(v, v')\}, \quad H'_1 \equiv H' \quad \text{if} \quad (v', v'') \in H'.
\]

Note that \((H_1, H'_1) \in M_2(G)\) and \(\{(u_2, v), (u_3, v)\} \notin H_1 \cup H'_1\).

It is not hard to see that this implies that there is \((H_2, H'_2) \in M_2(G)\) such that \((u_2, v) \notin H_2 \cup H'_2\). Lemma 3 implies that

\[
\alpha(G \setminus \{u_1, u_2\}) = \alpha(G \setminus \{(u_2, v)\}) - 1 = \alpha(G) - 1 = \beta(G) - 1 = \beta(G \setminus \{u_1, u_2\}),
\]

hence the tree \(G \setminus \{u_1, u_2\}\) also satisfies the \(\beta = \alpha\) equality. Due to hypothesis of induction \(G \setminus \{u_1, u_2\}\) can be built from \(K_1\) or \(K_2\) by A, B, C or D. Note that \(G\) is obtained from \(G \setminus \{u_1, u_2\}\) by operation B since the vertex \(v\) is applicable for it. This shows that \(G\) can also be built from \(K_1\) or \(K_2\) by A, B, C or D.

Thus, we may assume that \(d_G(v') \geq 3\). Let us show that we may also assume that \(v'\) is not adjacent to a vertex \(u\) with \(d_G(u) = 1\). On the opposite assumption, consider a vertex \(u\) satisfying conditions: \(d_G(u) = 1\) and \((u, v') \in E(G)\). Let \(u_1, \ldots, u_4\) be vertices shown in the figure below:

Figure 14
We claim that there is \((H, H') \in M'_2(G)\) with \((v, v') \notin H \cup H'\). Take any \((H, H') \in M'_2(G)\) with \((u, v') \in H\) (lemma 1), and suppose that \((v, v') \in H'\). Note that one of the edges \((u_2, v)\) and \((u_3, v)\) does not belong to \(H \cup H'\). Assume that \((u_2, v) \notin H \cup H'\). Since \(|H| = \alpha(G)\) we have \((u_1, u_2) \in H\). Define:

\[
H'' = (H' \setminus \{(v, v')\}) \cup \{(u_2, v)\}.
\]

Note that \((H, H'') \in M'_2(G)\) and \((v, v') \notin H \cup H''\). This and lemma 3 imply that

\[
\alpha(G'\{v, u_1, \ldots, u_4\}) = \alpha(G'\{(v, v')\}) - 2 = \alpha(G) - 2 = \beta(G) - 2 = \beta(G'\{v, u_1, \ldots, u_4\}),
\]

hence the tree \(G'\{v, u_1, \ldots, u_4\}\) also satisfies the \(\beta = \alpha\) equality. Due to hypothesis of induction \(G'\{v, u_1, \ldots, u_4\}\) can be built from \(K_1\) or \(K_2\) by A, B, C or D. Note that \(G\) is obtained from \(G'\{v, u_1, \ldots, u_4\}\) by operation D since the vertex \(v\) is applicable for it. This shows that \(G\) can also be built from \(K_1\) or \(K_2\) by A, B, C or D.

Thus, we may assume that \(v'\) is not adjacent to a vertex \(u\) with \(d_G(u) = 1\). Now, we claim that we may assume that there is no a vertex \(\bar{v} \neq v\) such that

\[
d_G(\bar{v}) \geq 3, \ k_G(\bar{v}) = k_G(v) \rightarrow \min \text{ and } (\bar{v}, v') \in E(G).
\]

On the opposite assumption, consider a vertex \(\bar{v}\) satisfying these conditions, and let \(u_1, \ldots, u_8\) be vertices shown in the figure below:
We claim that there is \((H, H') \in M_2'(G)\) with \((v, v') \notin H \cup H'\). Take any \((H, H') \in M_2'(G)\).

Case 1: \((v, v') \in H'\). Note that one of the edges \((u_2, v)\) and \((u_3, v)\) does not belong to \(H \cup H'\). Assume that \((u_2, v) \notin H \cup H'\). Since \(|H| = \alpha(G)\) we have \((u_1, u_2) \in H\). Define:

\[
H'' \equiv (H' \setminus \{(v, v')\}) \cup \{(u_2, v)\}.
\]

Note that \((H, H'') \in M_2'(G)\) and \((v, v') \notin H \cup H''\).

Case 2: \((v, v') \in H\). Define \(H, H'\) as follows:

\[
H \equiv (H \cap E(G \setminus \{v, v', \bar{v}, u_1, \ldots, u_8\})) \cup \{(v', \bar{v}), (u_2, v), (u_3, u_4), (u_5, u_6), (u_7, u_8)\},
\]

\[
H' \equiv (H' \cap E(G \setminus \{v, v', \bar{v}, u_1, \ldots, u_8\})) \cup \{(u_7, \bar{v}), (u_1, u_2), (u_3, v)\}.
\]

Clearly, \((H, H') \in M_2'(G)\) and \((v, v') \notin H \cup H'\).

This and lemma 3 imply that

\[
\alpha(G \setminus \{v, u_1, \ldots, u_4\}) = \alpha(G \setminus \{(v, v')\}) - 2 = \alpha(G) - 2 = \beta(G) - 2 = \beta(G \setminus \{v, u_1, \ldots, u_4\}),
\]

hence the tree \(G \setminus \{v, u_1, \ldots, u_4\}\) also satisfies the \(\beta = \alpha\) equality. Due to hypothesis of induction \(G \setminus \{v, u_1, \ldots, u_4\}\) can be built from \(K_1\) or \(K_2\) by A, B,
Figure 16:

C or D. Note that $G$ is obtained from $G \setminus \{v, u_1, ..., u_4\}$ by operation D since the vertex $v$ is applicable for it. This shows that $G$ can also be built from $K_1$ or $K_2$ by A, B, C or D.

Thus, we may assume that $v'$ is not adjacent to another vertex $\bar{v}$ satisfying the conditions:

$$
\begin{align*}
  d_G(\bar{v}) &\geq 3, \\
  k_G(\bar{v}) &\leq k_G(v) \rightarrow \min.
\end{align*}
$$

It is not hard to see that there are paths $P_1, ..., P_r (1 \leq r \leq 2)$ starting from the vertex $v'$ and satisfying the conditions:

for every $w \in V(P_i) 1 \leq i \leq r$, $w \neq v'$, $d_G(w) \leq 2$ and $k_G(w) < k_G(v)$.

Now, we will consider the remaining two cases:
Case 1: $r = 2$. Let $v'', u_1, ..., u_8$ be vertices shown in the figure below:

Figure 16

Assume:

$$
H \equiv G \setminus \{v, v', u_1, ..., u_8\}, e \equiv (v', v'').
$$

We claim that there is no $F \in M'(G)$ containing the edge $(v, v')$. Suppose there is. Note that
\[
\beta(G) = 5 + \beta(H), \\
L(G) = \beta(G \setminus F) = 2 + L(H) \text{ (lemma 5)},
\]
and since \(\alpha(G) = \beta(G)\), we have
\[
\lambda(G) = \beta(G) + L(G) = 7 + \lambda(H),
\]
contradicting lemma 2 which implies that
\[
\lambda(G) = 8 + \lambda(H).
\]
This immediately implies that \(\beta(He) = 1 + \beta(H)\) and, consequently, \(L(He) \leq L(H)\). Let us show that \(L(He) = L(H)\). Suppose that \(L(He) \leq L(H) - 1\). Choose \(F \in M'(G)\). Since \((v, v') \notin F\) we have \(\{(u_2, v), (u_3, v)\} \cap F \neq \emptyset\), therefore \(\{e, (u_5, u_6), (u_7, u_8)\} \subseteq F\), hence
\[
L(G) = 3 + L(He) \text{ (lemma 5)}.
\]
Choose \(F'_1 \in M'(H)\), and define \(F'\) as follows:
\[
F' \equiv F'_1 \cup \{(u_1, u_2), (u_3, u_4), (u_5, u_6), (u_7, u_8), (v, v')\}.
\]
Note that \(F' \in M(G)\) and
\[
\beta(G \setminus F') = 2 + L(H) \geq 3 + L(He) = L(G),
\]
\[
F' \in M'(G) \text{ and } (v, v') \in F',
\]
which is impossible.
Hence \(L(H) = L(He)\) and \(L(G) = 3 + L(He) = 3 + L(H)\). Let us show that \(\alpha(G \setminus \{u_1, \ldots, u_4\}) = \beta(G \setminus \{u_1, \ldots, u_4\})\). Note that
\[
8 + \beta(H) + L(H) = 8 + \beta(H) + L(He) = \beta(G) + L(G) = \lambda(G) = 8 + \lambda(H) \geq 8 + \beta(H) + L(H),
\]
hence
\[
\lambda(H) = \beta(H) + L(H) \text{ or } \beta(H) = \alpha(H).
\]
(*) and (**) imply that
\[
\alpha(G \setminus \{u_1, \ldots, u_4\}) = 3 + \alpha(H),
\]
\[
\beta(G \setminus \{u_1, \ldots, u_4\}) = 3 + \beta(H),
\]
we imply that \(\alpha(G \setminus \{u_1, \ldots, u_4\}) = \beta(G \setminus \{u_1, \ldots, u_4\})\), and therefore due to hypothesis of induction \(G \setminus \{u_1, \ldots, u_4\}\) can be built from \(K_1\) or \(K_2\) by A, B, C or D. Note that \(G\) is obtained from \(G \setminus \{u_1, \ldots, u_4\}\) by operation C since the vertex \(v\) is applicable for it. This shows that \(G\) can also be built from \(K_1\) or \(K_2\) by A, B, C or D.

Case 2: \(r = 1\). Let \(v'', u_1, \ldots, u_6\) be vertices shown in the figure below:
Figure 17:

Assume:

\[ H = G \setminus \{v, v', u_1, ..., u_6\}, e = (v', v''). \]

We need to consider two cases:

Case 2a: there is no \( F \in M'(G) \) with \( (v, v') \in F \).

First of all note that since there is a maximum matching of \( G \) which does not contain the edge \((v, v')\), we have

\[ \beta(G) = 4 + \beta(H) = 3 + \beta(He), \]

\[ \beta(He) = \beta(H) + 1 \]

and

\[ L(He) \leq L(H). \]

Let us show that \( L(He) = L(H) \). Suppose that \( L(He) \leq L(H) - 1 \). Choose \( F \in M'(G) \). Since \( (v, v') \notin F \) we have \( \{(u_2, v), (u_3, v)\} \cap F \neq \emptyset \), therefore \( \{(u_5, u_6), e\} \subseteq F \), hence

\[ L(G) = 3 + L(He) \] (lemma 5).

Choose \( F'_1 \in M'(H) \), and define \( F' \) as follows:

\[ F' \equiv F'_1 \cup \{(u_1, u_2), (u_3, u_4), (u_5, u_6), (v, v')\}. \]

Note that \( F' \in M(G) \) and
\[ \beta(G' \setminus F') = 2 + L(H) \geq 3 + L(He) = L(G), \text{ hence} \]
\[ F' \in M'(G) \text{ and } (v, v') \in F', \]

which is impossible.

Hence \( L(H) = L(He) \) and \( L(G) = 3 + L(He) = 3 + L(H) \). Let us show that \( \alpha(G' \setminus \{v, u_1, ..., u_4\}) = \beta(G' \setminus \{v, u_1, ..., u_4\}) \). Note that lemma 2 implies that
\[ 6 + \beta(He) + L(He) = 7 + \beta(H) + L(H) = \beta(G) + L(G) = \lambda(G) = 6 + \lambda(He) \geq 6 + \beta(He) + L(He), \]

hence
\[ \lambda(He) = \beta(He) + L(He) \text{ or } \beta(He) = \alpha(He). \]

As (see operation B, lemma 4)
\[ \alpha(G' \setminus \{v, u_1, ..., u_4\}) = 1 + \alpha(He), \]
\[ \beta(G' \setminus \{v, u_1, ..., u_4\}) = 1 + \beta(He), \]

we imply that \( \alpha(G' \setminus \{v, u_1, ..., u_4\}) = \beta(G' \setminus \{v, u_1, ..., u_4\}) \), and therefore due to hypothesis of induction \( G' \setminus \{v, u_1, ..., u_4\} \) can be built from \( K_1 \) or \( K_2 \) by A, B, C or D. Note that \( G \) is obtained from \( G' \setminus \{v, u_1, ..., u_4\} \) by operation D since the vertex \( v \) is applicable for it. This shows that \( G \) can also be built from \( K_1 \) or \( K_2 \) by A, B, C or D.

Case 2b: there is \( F \in M'(G) \) with \( (v, v') \in F \).

Clearly,
\[ \beta(G) = 4 + \beta(H) \text{ and, due to lemma 5, } L(G) = 2 + L(H). \]

Let us show that \( \lambda(He) = \lambda(H) \). Lemma 2 implies that
\[ 6 + \beta(H) + L(H) = \beta(G) + L(G) = \lambda(G) = 6 + \lambda(He) \geq 6 + \beta(H) + L(H), \]

therefore \( \lambda(He) = \lambda(H) = \beta(H) + L(H). \) (**) imply that
\[ \alpha(G' \setminus \{u_1, ..., u_4\}) = 2 + \alpha(H), \]
\[ \beta(G' \setminus \{u_1, ..., u_4\}) = 2 + \beta(H), \]

therefore \( \alpha(G' \setminus \{u_1, ..., u_4\}) = \beta(G' \setminus \{u_1, ..., u_4\}) \), and due to hypothesis of induction \( G' \setminus \{u_1, ..., u_4\} \) can be built from \( K_1 \) or \( K_2 \) by A, B, C or D. Note that \( G \) is obtained from \( G' \setminus \{u_1, ..., u_4\} \) by operation C since the vertex \( v \) is applicable for it. This shows that \( G \) can also be built from \( K_1 \) or \( K_2 \) by A, B, C or D.

The proof of the Theorem is completed.

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