A simple upper bound for trace function of a hypergraph with applications

Farhad Shahrokhi
Department of Computer Science and Engineering, UNT
P.O.Box 13886, Denton, TX 76203-3886, USA farhad@cs.unt.edu

Abstract

Let $H = (V, E)$ be a hypergraph on the vertex set $V$ and edge set $E \subseteq 2^V$. We show that number of distinct traces on any $k$-subset of $V$, is most $k \hat{\alpha}(H)$, where $\hat{\alpha}(H)$ is the degeneracy of $H$. The result significantly improves/generalizes some of related results. For instance, the $vc$ dimension $H$ (or $vc(H)$) is shown to be at most $\log(\hat{\alpha}(H)) + 1$ which was not known before. As a consequence $vc(H)$ can be computed in $O(n \log(\hat{\alpha}(H)))$ time. When applied to the neighborhood systems of a graphs excluding a fixed minor, it reduces the known linear upper bound on the VC dimension to a logarithmic one, in the size of the minor. When applied to the location domination and identifying code numbers of any $n$ vertex graph $G$, one gets the new lower bound of $\Omega(n/\hat{\alpha}(G))$, where $\hat{\alpha}(G)$ is the degeneracy of $G$.

1 Introduction and Summary

Many important combinatorial problems in computer science, mathematics and operations research arise from the set systems or hypergraphs. Formally, a hypergraph $H = (V, E)$ has the vertex set $V$ and the edge set $E$, where each $e \in E$ is a subset of $V$. We do not allow multiple edges in our definition of a hypergraph, unless explicitly stated. When multiple edges exist, we slightly modify the concept. Let $S \subseteq V$ and $e \in E$. The trace of $e$ on $S$ is $e \cap S$. The restriction of $H$ to $S$, denoted by $HS$, is the hypergraph on vertex set $S$ whose edges are set of all distinct traces of edges in $E$ on $S$. $H[S]$ is also referred to as the induced subhypergraph of $H$ on $S$. $S$ is shattered in $H$, if any $X \subseteq S$ is a trace. Thus if $S$ is shattered then it has $2^{|S|}$ traces, that is $H[S]$ has $2^{|S|}$ edges.

The trace function of $H$ denoted by $T[H, k]$ is the largest number traces of $H$ on a subset $S, |S| = k$. The trace function of $H$ denoted by $T[H, k]$ is the largest number traces of $H$ subset $S, |S| = k$. The Vapnik-Chervonenkis (VC) dimension of a hypergraph $H$, denoted by $vc(H)$ is the cardinality of a largest subset of $V$ which is shattered in $H$. It was originally introduced for its applications in statistical learning theory [22] but has shown to be of crucial importance in combinatorics and discrete geometry [11]. The concept of a trace function is also studied as the Max Partial VC Dimension [12].

A powerful tool in studying hypergraph problems is the Sauer-Shelah Lemma [20, 21]. The Lemma asserts for any hypergraph $H$, and any $S \subseteq V, |S| = k$, one has

$$T(H, k) \leq \sum_{i=0}^{d} \binom{k}{i} = O(k^d) \quad \text{for any } k \geq 0 \quad \text{where} \quad d = vc(H).$$

Our main result is to show $T(H, k) \leq k \hat{\alpha}(H)$. Here $\hat{\alpha}(H)$ is the degeneracy of $H$, or the largest minimum degree of a vertex in any induced subhypergraph of $H$. The result generalizes and significantly improves some of the related existing results. We write $H = (V, E)$ for a hypergraph; For a graph we write $G = (V, E)$. 


1.1 Background and definitions

Let $S \subseteq V$, then, $S$ is a transversal, or a hitting set, if $e \cap S \neq \emptyset$, for all $e \in E$. A transversal set $S$ is a distinguishing transversal if any two distinct edges of $H$ have different intersections with $S$. The distinguishing transversal number of $H$ is the minimum size of any $U \subseteq V$, so that $U$ has precisely $|E|$ traces [10].

For any $x \in V$, let degree of $x$, denoted by $d_H(x)$, denote the number of edges that contain $x$. We denote by $\delta(H)$, the minimum degree of $H$. Thus $\delta(H)$ is the smallest degree of any vertex in $H$.

Any definition for a hypergraph, readily extends to a subhypergraph. A Hypergraph $I$ is a subhypergraph of $H$ if it can be obtained by deleting some edges in $H[S]$ for some $S \subseteq V$. (Note that there are subhypergraphs of $H$ that may not be induced.) Particularly, for any $x \in S$, the degree of $x$ in $I$ is denoted by $d_I(x)$. Furthermore $\delta(I)$ denotes the minimum degree of $I$. The degeneracy of $H$, denoted by $\delta(H)$, is the largest minimum degree of any subhypergraph of $H$.

Observe that one can define $\hat{\delta}(H)$ as the largest minimum degree of any induced subhypergraph of $H$, since the addition of new edges to a hypergraph does not decrease the degrees of vertices. The degeneracy of a graph $G$, denoted by $\delta(G)$, is the largest minimum degree of any induced subgraph of $G$.

All graphs considered here are undirected and finite and simple. For a graph $G = (V,E)$ and a vertex $x$, $N(x)$ denotes the open neighborhood of $x$, that is the set of all vertices adjacent to $x$, not including $x$. The closed neighborhood of $x$ is $N[x] = N(x) \cup \{x\}$. The neighborhood hypergraph of an $n$ vertex graph $G$ is a hypergraph on same vertices as $G$ whose edges are all $n$ closed neighborhoods of $G$. A subset of vertices $S$ in $G$ is a dominating set [6], if for every vertex $x$ in $G$, $N[x] \cap S \neq \emptyset$. $S$ is a total domination set [2] if, $N(x) \cap S \neq \emptyset$. Let $x$ and $y, z, y \neq z$ be vertices in $G$. Then $x$ separates (distinguishes) $y$ from $z$, if $x$ is adjacent to either $y$ or $z$ but not to both. Let $S, T \subseteq V$, then $S$ separates $T$, if for any vertex pair in $T$, there is a vertex in $S$ that separates them. A subset $S$ of vertices in $G$ is a locating domination if it is a dominating set and it separates the vertices of $V - S$. $S$ is an identifying code if it is dominating set and it separates the vertices in $V$. Let $\gamma^{\text{Loc}}$ and $\gamma^{\text{ID}}$ denote the sizes of a smallest location domination and Identifying code sets in $G$, respectively.

1.2 Previous related Results

Results for number of traces. It was shown in [12] that when $H$ does not have multiple edges, given an integer $k$, one can construct in $O(k(|V| + |E|))$ time a set $S, |S| = k$ so that number of traces on $S$ is at least $\min\{|E|, k + 1\}$. It was also shown [12] that $T(H, k) \leq k(\Delta(H) + 1)/2 + 1$ where $\Delta(H)$ is the maximum degree of $H$. Consequence $T(H, k)$ can be approximated within a factor of $\frac{\Delta(H)+1}{2}$ in polynomial time [12].

Results on VC dimension. The known upper bound on $vc(H)$ is $\log(|E|)$. When $H$ is the neighborhood hypergraph of a graph excluding a minor on $t$ vertices, it is known that $vc(H) \leq t - 1$ [11]. It is also known that, when $H$ has an explicit representation by an $m \times n$ incident matrix, then $vc(H)$ can be computed in $n^{O(\log(n))}$ [5]. Furthermore, it is known that decision version of the problem is LOGNP-complete [23] and remains in this complexity class for neighborhood hypergraphs of graphs [17]. For the neighborhood hypergraphs of graphs with bounded degree one can compute in polynomial time; More precisely in $O(n^{2\Delta(G)})$ time, where $\Delta(G)$ is the maximum degree of $G$ [17].
Results on location domination and identifying code numbers. A number of results are known [14]. Particularly is known that $\gamma^{\text{Loc}}(G), \gamma^{\text{ID}}(G) = \Omega(\log(n))$ when $G$ is chordal [14], $\gamma^{\text{Loc}}(G), \gamma^{\text{ID}}(G) = \Omega(n^{1/2})$ when $G$ is interval [16], and $\gamma^{\text{Loc}}(G), \gamma^{\text{ID}}(G) = \Omega(n)$ when $G$ is planar [11] and approximation factor of 7 was obtained.

1.3 Our Results

In this paper we show

$$T(H, k) \leq k\hat{\delta}(H),$$

and explore the applications of this simple and new upper bound. An important simple consequence is that $T(H, k)$ can be approximated (for any $k$) within a factor of $\hat{\delta}(H))$, in polynomial time, from the lower bound provided in [12], and hence improving the quality of approximation in [12]. It also follows that the VC dimension of any hypergraph $H$, or $vc(H)$, is at most $\log(\hat{\delta}(H)) + 1$. The known previous result was $\log(|E|)$. Furthermore, it was known that, when $H$ has an explicit representation by an $m \times n$ matrix then $vc(H)$ can be computed in $n^{O(\log(n))}$ and furthermore, it is known that [13] that the decision version of the problem is LOGNP-complete.

As a consequence of our result, for any hypergraph with bounded degeneracy, $vc(H)$ can be computed in polynomial time which significantly advances the state of knowledge. Additionally, when applied to the neighborhood systems of graphs excluding a fixed minor, it reduces the best known bound on the VC Dimension which was linear in the size of minor [11] to a logarithmic one. When applied to the location domination and identifying code numbers of any $n$ vertex graph $G$, it gives the lower bound of $\Omega(n/\hat{\delta}(G))$, where $\hat{\delta}(G)$ is the degeneracy of $G$. This significantly generalizes and improves some of the related existing results in this area, since it implies that for graphs of bounded degeneracy any approximation for location domination and identifying code numbers is a good approximation. No such general result had been known in the past.

2 Main lemma and applications

For a subhypergraph $I = (U, F)$ of $H$, and any $x \in U$, let $F_x$ denote the set of edges in $F$ containing $x$.

**Lemma 2.1.** Let $H = (V, E)$, and let $k \geq 1$, then $T(H, k) \leq k\hat{\delta}(H)$, and consequently $vc(H) - \log(vc(H)) \leq \log(\hat{\delta}(H))$.

**Proof.** To prove the claim fro $T(H, k)$, let $S \subseteq V, |S| = k$, and let $I = H[S] = (S, F)$ be the restriction of $H$ to $S$. We will prove $|F| \leq k\hat{\delta}(I)$.

Let $x$ be a vertex with $d_I(x) = \delta(I)$ in $I$, let $x_1 = x, I_1 = I$ and $F_1 = F$, and for $i = 2, ..., k$, let $x_i$ be a vertex with $d_{I_i}(x_i) = \delta(I_i)$ in subhypergraph $I_i$ on the vertex set $S_i = S - \{x_1, x_2, ..., x_{i-1}\}$ and edge set $F_i = F - \{F_{x_1}, F_{x_2}, ..., F_{x_{i-1}}\}$. Now observe that

$$|F| = \sum_{i=1}^{k} d_{I_i}(x_i)$$  \hspace{1cm} (1)

$$= \sum_{i=1}^{k} \delta(I_i)$$  \hspace{1cm} (2)

$$\leq k\hat{\delta}(I).$$  \hspace{1cm} (3)

It is important to note that although $I_i$ may not be an induced subhypergraph of $I$, we still have $\delta(I_i) \leq \hat{\delta}(I)$ and hence the last inequality holds. Now the claim follows, since $\hat{\delta}(I) \leq \hat{\delta}(H)$. 


Next, let $S, |S| = \text{vc}(H)$ be a largest shattered set in $H$, let $I = H[S]$ and apply the upper bound on number of traces. Then, $2^{\text{vc}(H)} \leq \text{vc}(H) \hat{\delta}(H)$, and consequently $\text{vc}(H) - \log(\text{vc}(H)) \leq \log(\hat{\delta}(H))$.

\begin{remark}
The upper bound of lemma 2.1 on $\text{vc}(H)$ can be improved by removing $\log(\text{vc}(H))$ term as follows. Since $S$ is shattered, for any $x \in S$, we have $d_I(x) = 2^{|S|-1} = 2^{\text{vc}(H)-1}$, since there are exactly $2^{|S|-1}$ subsets of $S$ that contain $x$. Therefore, $\delta(I) = 2^{\text{vc}(H)-1}$, and therefore, $2^{\text{vc}(H)-1} \leq \hat{\delta}(H)$. Consequently, $\text{vc}(H) \leq \log(\hat{\delta}(H)) + 1$.
\end{remark}

Lemma 2.2. Let $H$ be the neighborhood hypergraph of a graph $G = (V, E)$, then $\hat{\delta}(H) \leq \hat{\delta}(G) + 1$.

Theorem 2.1. (i) For any $n$ vertex hypergraph $H$, $\text{vc}(H)$ can be computed in $n^{O(\log(\hat{\delta}(H)))}$. If $H$ is the neighborhood hypergraph of a graph, then, $\text{vc}(H)$ can be computed in $n^{O(\log(\hat{\delta}(G)))}$ time.

(ii) Let $H$ be the neighborhood hypergraph of a graph $G$ that excluded a fixed minor on $t$ vertices, then $\text{vc}(H) = O(\log(t))$. Furthermore, if $G$ is a chordal graph, then $\text{vc}(H) = O(\log(\omega))$, where $\omega$ is the size of the largest clique in $G$.

\begin{proof}
For (i), the general claim follows from the upper bound for $\text{vc}(H)$ in Lemma 2.1 (or Remark 2.1). When $H$ is a neighborhood hypergraph, one needs to apply Lemma 2.2 as well. For (ii), it is known that $\hat{\delta}(G) = O(t\sqrt{\log(t)})$ [1], and consequently the claim follows from Lemma 2.2 when $G$ does not have a fixed minor. It is further easy to verify that when $G$ is chordal, one has $\omega = \hat{\delta}(G)$, and hence the claim also follows from Lemma 2.2.
\end{proof}

Theorem 2.2. Let $G = (V, E), |V| = n$ be a graph, then $\gamma_{Loc}(G) = \Omega(n/(\hat{\delta}(G)))$ and $\gamma_{ID}(G) = \Omega(n/(\hat{\delta}(G)))$.

\begin{proof}
We apply Lemma 2.1 to the neighborhood hypergraph $H$ of $G$ with $|E| = n$ and use Lemma 2.2. Details are omitted.
\end{proof}

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