Measures and geometric probabilities for ellipses intersecting circles

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Abstract

Santaló calculated the measures for all positions of a moving line segment in which it lies inside a fixed circle and intersects this circle in one or two points. From these measures he concluded hitting probabilities for a line segment thrown randomly onto an unbounded lattice of circles. In the present paper these results are generalized to ellipses instead of line segments. The respective measures for all positions of a moving ellipse in which it lies completely inside a fixed circle, encloses it, and intersects it in two or four points are derived. Then the hitting probabilities for lattices of circles are deduced. It is shown that the results for a line segment follow as special cases from those of the ellipse.

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1 Introduction

Santaló [11] calculated the measure $M_i$ of all oriented line segments of length $\ell$ that are completely contained in a disk of radius $r$. If $\ell \geq 2r$, $M_i$ is obviously equal to zero; if $\ell \leq 2r$, then

$$M_i = 2\pi \left( \pi r^2 - 2r^2 \arcsin \frac{\ell}{2r} - \ell \sqrt{r^2 - \frac{\ell^2}{4}} \right).$$

(1.1)

From this he concluded the measures

$$M_1 = 4\pi^2 r^2 - 2M_i, \quad M_2 = 4\pi r\ell - 2\pi^2 r^2 + M_i$$

(1.2)

for all line segments intersecting the circle in one and two points, respectively. Then he considered the random throw of a line segment onto an unbounded lattice of circles, as shown as an example in Fig. 1. Such a lattice consists of circles of radius $r$ whose center points are placed at the vertices of parallelograms with sides of length $s$ and $t$, and angle $\sigma$. Under the assumption that the line segment can hit only one circle of the lattice at the same time, he derived the geometric probabilities that this line segment, placed randomly onto the lattice, hits a circle in one point, in two points, lies completely inside a circle, and outside all circles (see Fig. 1).

Duma and Stoka calculated the probability that an ellipse hits a lattice of parallelograms [6].

Most recently, Böttcher calculated the measures and geometric probabilities that an arbitrarily long line segment hits exactly one or two sides of a triangle [1], and that this segment hits two non-overlapping circles [2].

In this paper we generalize Santaló’s above mentioned results for ellipses (see Fig. 1) instead of line segments.

We denote by $K_0$ the closed disk bounded by the circle $C_0 := \partial K_0$ of radius $r$, and by $K_1$ the closed set of points bounded by the ellipse $C_1 := \partial K_1$ with semi-major axis of length $a$ and semi-minor axis of length $b$. 

1
The present paper is organized as follows:

- In Section 2 we first consider a moving ellipse $C_1^*$ with fixed direction (indicated by the asterisk) and a fixed circle $C_0$, and discuss the possible intersection cases depending on the position of the center point $M_1$ of $C_1^*$. We show that it is possible to consider the inverse translation with $C_0$ moving and $C_1^*$ fixed instead of the original motion. Then, the outer and the inner parallel curve of $C_1^*$ in the distance $r$ (= radius of $C_0$) bound sets which are essential for the following investigations.

- In Section 3 we calculate areas which are the measures for all positions of $C_1^*$ with $C_0 \subset K_1^*$ or $C_1^* \subset K_0$, and all positions of $C_1^*$ in which it intersects $C_0$ in two or four points.

- In Section 4 we consider the motion of $C_1$ without the restriction to a fixed direction, and derive the respective measures for all positions of $C_1$ (= all congruent copies of $C_1$) intersecting the fixed circle $C_0$.

- In Section 5 the measures from Section 4 are used to find the hitting probabilities for an ellipse $C_1$ which is thrown randomly onto a lattice of circles $C_0$ as it is shown in Fig. 1.

- In Section 6 we show that Santaló’s measures (1.1), (1.2) and probabilities for a line segment follow from the results in Sections 4 and 5, respectively.

2 Intersections of a circle and an ellipse

From Bézout’s theorem (see e. g. [4, pp. 291-304], [8, pp. 51-69]) we know that an ellipse and a circle always have four intersection points if each point is counted with its intersection multiplicity. We have the following cases:

a) All intersection points are real.

b) There are two real and two conjugate complex intersection points.

c) There are two pairs of conjugate complex intersection points.

Even in the case that the ellipse is also a circle, Bézout’s theorem holds true if homogeneous coordinates are used. Here two of the four intersection points are the so called circular points at infinity. All circles pass through this complex pair of points. (See e. g. [9, p. 94] and [10].)

Here we are interested only in real intersection points.
**Example 2.1.** Let us consider the fixed circle $C_0$ and the moving ellipse $C_1$ with fixed direction in Fig. 2. We write $C_1^*$ in order to indicate that $C_1$ has fixed direction. $C_1^*$ intersects $C_0$ in two distinct points if the center point $M_1$ of $C_1^*$ lies in the open set bounded by the curves $C^*$ and $C^{**}$ (positions 1 and 2 of $C_1^*$). $C_1^*$ does not intersect $C_0$ if $M_1$ is outside $C^*$ (position 3) or inside the middle loop of $C^{**}$ (position 4). $C_1^*$ intersects $C_0$ in four distinct points if $M_1$ lies inside the upper or lower loop of $C^{**}$ (position 5). Clearly, the closed set bounded by $C^*$ is the Minkowski sum $K_0 + K_1^*$, where $K_1^*$ is the closed set bounded by $C_1^*$.

![Fig. 2: Fixed circle $C_0$, and moving ellipse $C_1^*$ in five positions 1, 2, ..., 5](image1)

![Fig. 3: Fixed ellipse $C_1^*$, and moving circle $C_0$ in five positions 1, 2, ..., 5](image2)

Now we consider the inverse translation which means that $C_1^*$ is fixed and $C_0$ is moving (see Fig. 3). Here we have the situation that $C_0$ intersects $C_1^*$ in two distinct points if the center point $M_0$ of $C_0$ lies in the open set bounded by the curves $C_1^{++}$ and $C_1^-$ (positions 1 and 2 of $C_0$). Due
to the commutative property of the Minkowski addition, $K_0 + K^*_1 = K^*_1 + K_0$, we immediately know that $C^+_1 \equiv C^*$. Since we consider only translations, we also have $C^-_1 \equiv C^{**}$. $C^+_1$ and $C^-_1$ are, respectively, the outer and inner parallel curve of $C^*_1$ in the distance $r$. Analogous to the original motion, one finds that $C_0$ does not intersect $C^*_1$ if $M_0$ is outside $C^+_1$ (position 3) or inside the middle loop of $C^-_1$ (position 4), and $C_0$ intersects $C^*_1$ in four distinct points if $M_0$ lies inside the upper or lower loop of $C^-_1$ (position 5).

![Fig. 4: Support functions $p(\phi)$ and $p(\phi) - r$ of $C_1$ and $C^-_1$, respectively](image)

Now, we will provide parametric representations of $C_1 = C^*_1$, $C^+_1$ and $C^-_1$ with the support function $p(\phi)$ of $C_1$ (see Fig. 4) that is given by

$$p(\phi) = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}, \quad 0 \leq \phi \leq 2\pi,$$

(2.1)

with derivatives

$$p'(\phi) = \frac{(a^2 - b^2) \cos \phi \sin \phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}, \quad p''(\phi) = -\frac{(a^2 - b^2) \left( a^2 \cos^4 \phi - b^2 \sin^4 \phi \right)}{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)^{3/2}}.$$

Using the formulas

$$x = p(\phi) \cos \phi - p'(\phi) \sin \phi, \quad y = p(\phi) \sin \phi + p'(\phi) \cos \phi,$$

(see e.g. [12, p. 3]), we get

$$x = \frac{a^2 \cos \phi}{p(\phi)}, \quad y = \frac{b^2 \sin \phi}{p(\phi)}, \quad 0 \leq \phi \leq 2\pi,$$

as parametric parametric representation of $C_1$. $p(\phi) + r$ is the support function of $C^+_1$, and $p(\phi) - r$ that of $C^-_1$ (Fig. 4.) Hence parametric representations of $C^+_1$ and $C^-_1$ are given by

$$x = (p(\phi) + kr) \cos \phi - p'(\phi) \sin \phi = \left( \frac{a^2}{p(\phi)} + kr \right) \cos \phi, \quad 0 \leq \phi \leq 2\pi,$$

$$y = (p(\phi) + kr) \sin \phi + p'(\phi) \cos \phi = \left( \frac{b^2}{p(\phi)} + kr \right) \sin \phi,$$

(2.2)

where $k = 1$ for $C^+_1$, and $k = -1$ for $C^-_1$. 

4
Now we are looking for the singularities (cusps) of $C_1^-$ as may be seen in Figures 2, 3 and 4. The derivatives of the parametric representation of $C_1^-$ are

$$x' = p' \cos \phi - (p - r) \sin \phi - p'' \sin \phi - p' \cos \phi = -(p - r + p'') \sin \phi,$$

$$y' = p' \sin \phi + (p - r) \cos \phi + p'' \cos \phi - p' \sin \phi = (p - r + p'') \cos \phi.$$ 

So in order to find the singularities of $C_1^-$ we have to solve the equation

$$p(\phi) - r + p''(\phi) = 0$$

for $\phi$. One finds the solutions

$$\phi = \pm \frac{1}{2} \arccos \left( \frac{2(a^2b^2/r)^{2/3} - a^2 - b^2}{a^2 - b^2} \right).$$

These solutions are real if

$$-1 \leq \frac{2(a^2b^2/r)^{2/3} - a^2 - b^2}{a^2 - b^2} \leq 1.$$ 

From the left and the right inequality it follows $r \leq a^2/b$ and $r \geq b^2/a$, respectively. This means that singularities of $C_1^-$ occur only if $b^2/a \leq r \leq a^2/b$, where $b^2/a$ and $a^2/b$ are, respectively, the minimum and maximum radius of curvature of the ellipse $C_1$. The solutions in the interval $0 \leq \phi \leq 2\pi$ are given by

$$\phi = \lambda, \; \phi = \pi - \lambda, \; \phi = \pi + \lambda, \; \phi = 2\pi - \lambda$$

with

$$\lambda = \frac{1}{2} \arccos \left( \frac{2(a^2b^2/r)^{2/3} - a^2 - b^2}{a^2 - b^2} \right) \text{ if } \frac{b^2}{a} \leq r \leq \frac{a^2}{b}.$$ 

One gets the parametric presentation of the evolute $C_1^*$ of $C_1$:

$$x = \frac{a^2 \cos \phi}{p(\phi)} \left( 1 - \frac{b^2}{p^2(\phi)} \right), \quad y = \frac{b^2 \sin \phi}{p(\phi)} \left( 1 - \frac{a^2}{p^2(\phi)} \right).$$

Now we set the parameter functions of the evolute equal to that of $C_1^-$. This yields

$$\frac{a^2}{p} - r = \frac{a^2}{p} - \frac{a^2b^2}{p^3} \implies a^2b^2/p^3 = r, \quad \frac{b^2}{p} - r = \frac{b^2}{p} - \frac{b^2a^2}{p^3} \implies a^2b^2/p^3 = r.$$ 

We see that $C_1^-$ and $C_1^*$ have common points. Solving the equation

$$a^2b^2 = rp^3(\phi)$$

for $\phi$, one finds that the singularities of $C_1^-$ are these common points.

Fig. 5 shows all possible types of inner parallel curves $C_1^-$ of $C_1 = C_1^*$ for fixed values of $a$ and $b$. We denote by $\#(C_0 \cap C_1)$ the number of intersection points of $C_0$ and $C_1$ counted with its multiplicity. If $M_0$ lies in the open set bounded by $C_1^-$ and $C_1^+$, then there are two distinct intersection points, hence $\#(C_0 \cap C_1) = 2$. If $M_0 \in C_1^+$, then $C_1$ touches $C_0$ in one point with multiplicity 2, hence also $\#(C_0 \cap C_1) = 2$. We give some comments to the shown cases:

a) Since the smallest radius of curvature of $C_1$ is equal to $b^2/a$, it follows that $C_0$ and $C_1$ cannot intersect in four points. If $M_0$ lies in the open set bounded by $C_1^-$, then $\#(C_0 \cap C_1) = 0$. If $M_0 \in C_1^+$, then $C_0$ touches $C_1$ in one point with multiplicity two, hence $\#(C_0 \cap C_1) = 2$.

b) The situation is the same as in Case (a) if $M_0$ does not coincide with a cusp. If $M_0$ lies in one of the two cusps, then $C_0$ touches $C_1$ in one of the two points with smallest radius of curvature, and $C_0$ is the osculating circle in such a point (multiplicity four $\Rightarrow \#(C_0 \cap C_1) = 4$).
c) We have \( \#(C_0 \cap C_1) = 0 \) if \( M_0 \) lies in the open set bounded by the inner loop of \( C_1^- \), and four distinct intersection points, hence \( \#(C_0 \cap C_1) = 4 \), if \( M_0 \) lies in one of the open sets bounded by the outer loops of \( C_1^- \). If \( M_0 \) lies on the inner loop without the double points, then \( C_0 \) touches \( C_1 \) in one point with multiplicity two (\( \#(C_0 \cap C_1) = 2 \)). If \( M_0 \) coincides with one double point, then \( C_0 \) touches \( C_1 \) in two points, each with multiplicity two, hence \( \#(C_0 \cap C_1) = 4 \). If \( M_0 \) lies on an outer loop without the cusps (and the double point), then \( C_0 \) touches \( C_1 \) in one point and there are two distinct intersection points, hence \( \#(C_0 \cap C_1) = 4 \). If \( M_0 \) coincides with one of the cusps, then \( C_0 \) is the osculating circle in the touching point (multiplicity three) and there is a distinct intersection point (multiplicity one), hence \( \#(C_0 \cap C_1) = 4 \).

d) There are four distinct intersection points if \( M_0 \) lies in the open set bounded by \( C_1^- \). If \( M_0 \) lies on \( C_1^- \) but not in the cusps and the self touching point, then \( C_0 \) touches \( C_1 \) in one point with multiplicity two and there are two distinct intersection points. If \( M_0 \) coincides with one cusp, then \( C_0 \) touches \( C_1 \) in one point where \( C_0 \) is the osculating circle, and there is one distinct intersection point. If \( M_0 \) coincides with the self touching point, then \( C_0 \) touches \( C_1 \) in two points, each with multiplicity two. In all of these subcases we have \( \#(C_0 \cap C_1) = 4 \).

e) The situation is the same as in Case (d) with the exception that there is no self touching point.

f) See Case (d).

g) See Case (c).
h) The situation is the same as in Case (i) if \( M_0 \) does not coincide with a cusp. If \( M_0 \) lies in one of the two cusps, then \( C_0 \) touches \( C_1 \) in one of the two points with largest radius of curvature, and \( C_0 \) is the osculating circle in such a point (multiplicity four \( \Rightarrow \#(C_0 \cap C_1) = 4 \)).

i) Since the largest radius of curvature of \( C_1 \) is equal to \( a^2/b \), it follows that \( C_0 \) and \( C_1 \) cannot intersect in four points. If \( M_0 \) lies in the open set bounded by \( C_0^1 \), then \( \#(C_0 \cap C_1) = 0 \). If \( M_0 \in C_0^1 \), then \( C_0 \) touches \( C_1 \) in one point with multiplicity two, hence \( \#(C_0 \cap C_1) = 2 \).

The orientations of \( C_0^1 \) shown in Fig. 5 are the orientations resulting from (2.2). The starting points with value \( \phi = 0 \) are marked with small line segments.

### 3 Areas and measures for ellipses with fixed direction

We consider the moving ellipse \( C_1^* \) (with fixed direction) and center point \( M_1 \), and derive expressions for the following areas of sets of positions of \( M_1 \)

\[
A_i^{01} := A(\{ M_1 : C_0 \subset K_1^* \}) \quad A_i^{10} := A(\{ M_1 : C_1^* \subset K_0 \})
\]  

with property \( X \). Since \( C_1^* \) has fixed direction,

\[
m(\{ C_1^* : C_0 \subset K_1^* \}) = A_i^{01}, \quad m(\{ C_1^* : C_1^* \subset K_0 \}) = A_i^{10}, \quad m(\{ C_1^* : \#(C_0 \cap C_1^*) = 2j \}) = A_{2j}.
\]

In the following, \( E(\phi, \varepsilon) \) denotes the incomplete elliptic integral of the second kind,

\[
E(\phi, \varepsilon) = \int_0^\phi \sqrt{1 - \varepsilon^2 \sin^2 \theta} \, d\theta,
\]

and \( E(\varepsilon) := E(\varepsilon, \pi/2) \) the complete elliptic integral of the second kind.

**Lemma 3.1.** Depending on the following cases, the areas (3.1) and (3.2) are given by

| Case | Interval | \( A_i^{01} \) | \( A_i^{10} \) | \( A_2 \) | \( A_4 \) |
|------|----------|----------------|----------------|--------|--------|
| 1    | \( 0 < r \leq \frac{b^2}{a} \) | \( A^* \) | 0 | \( 8raE(\varepsilon) \) | 0 |
| 2    | \( \frac{b^2}{a} < r < b \) | \( \tilde{F}(\alpha) \) | 0 | \( 2\pi r^2 + 2\pi ab - 2\tilde{F}(\alpha) \) | \( \tilde{F}(\alpha) - A^* \) |
| 3    | \( b \leq r \leq a \) | 0 | 0 | \( 2\pi r^2 + 2\pi ab \) | \(-A^* \) |
| 4    | \( a < r < \frac{a^2}{b} \) | 0 | \( F(\beta) \) | \( 2\pi r^2 + 2\pi ab - 2F(\beta) \) | \( F(\beta) - A^* \) |
| 5    | \( \frac{a^2}{b} \leq r < \infty \) | 0 | \( A^* \) | \( 8raE(\varepsilon) \) | 0 |

where

\[
F(\phi) = 2r^2\phi + 2ab \arctan \left( \frac{b}{a} \tan \phi \right) - 4raE(\phi, \varepsilon) + \frac{ra\varepsilon^2 \sin 2\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}},
\]

\[
A^* = F(\pi/2) = \pi r^2 + \pi ab - 4raE(\varepsilon), \quad \tilde{F}(\phi) = A^* - F(\phi),
\]

and

\[
\alpha = \arctan \frac{\sqrt{r^2a^2 - b^4}}{b\sqrt{b^2 - r^2}}, \quad \beta = \arctan \frac{a\sqrt{r^2 - a^2}}{\sqrt{a^4 - r^2b^2}}.
\]
Proof. We consider the inverse translation with fixed ellipse \( C_1 = C_1^+ \) and moving circle \( C_0 \) (of radius \( r \)). From Section 2 (see Figures 2 and 3) it follows that the areas (3.1) and (3.2) are also given by

\[
A_{01}^i = A(\{M_0: C_0 \subset K_1\}) \quad \text{and} \quad A_{10}^i = A(\{M_0: C_1 \subset K_0\}), \quad A_{2j} = A(\{M_0: \#(C_0 \cap C_1) = 2j\}).
\]

Intersection (touching) points with multiplicity \( >1 \) only occur if \( M_0 \subset C^+_1 \) and \( M_0 \subset C^-_1 \), so the sets of positions of \( M_0 \) with intersection multiplicity \( >1 \) always have area zero. Therefore, it suffices to consider only positions of \( M_0 \) with distinct intersection points.

The area \( A^+ \) of the set enclosed by the outer parallel curve \( C^+_1 \) of \( C_1 \) in the distance \( r \) is given by the integral

\[
A^+ = \frac{1}{2} \int_0^{2\pi} [p(\phi) + r] [p(\phi) + r + p''(\phi)] \, d\phi
\]

(see p. 3, Eq. (1.7) and p. 7 in [12]). The area and the perimeter of \( K_1 \) are given by \( \pi ab \) and \( 4aE(\varepsilon) \), respectively, where

\[
\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}
\]

is the eccentricity of \( C_1 \) (see e.g. [3, pp. 230-232]). From Eq. (1.18) on p. 8 in [12] it follows that

\[
A^+ = \pi r^2 + \pi ab + 4raE(\varepsilon).
\]

(This result also follows from Eq. (14.5) or (14.6) on p. 600 in [13].) The (signed) area of the set enclosed by the inner parallel curve \( C^-_1 \) in the distance \( r \) is given by the integral

\[
A^- = \frac{1}{2} \int_0^{2\pi} [p(\phi) - r] [p(\phi) - r + p''(\phi)] \, d\phi
\]

if \( C^+_1 \) has no self intersections, even in the cases in which \( C^-_1 \) is not convex (cp. 12, p. 8]). The sign depends on the orientation of the curve. (Only if \( r \leq b^2/a \), \( A_r \) is the area of the interior parallel set.) For a curve \( C^+_1 \) with self intersections, (3.5) gives the sum of the signed areas of its loops depending on the orientation of each loop (see Figures 4 and 5). Therefore, instead of (3.5), in the following we use

\[
\tilde{A}^- = 2 \int_{\phi_1}^{\phi_2} [p(\phi) - r] [p(\phi) - r + p''(\phi)] \, d\phi, \quad 0 \leq \phi_1 < \phi_2 \leq \frac{\pi}{2},
\]

with suitable limits \( \phi_1 \), \( \phi_2 \) in order to derive areas \( \tilde{A}^- \) of sets enclosed by loops of \( C^-_1 \), where the factor 2 results from the symmetry of \( C^-_1 \).

So for the function

\[
f(\phi) = 2 [p(\phi) - r] [p(\phi) - r + p''(\phi)]
\]

we have to determine one of its antiderivatives \( F(\phi) \),

\[
F(\phi) = 2 \int [p(\phi) - r] [p(\phi) - r + p''(\phi)] \, d\phi
\]

\[
= 2 \int [p(\phi) - r]^2 \, d\phi + 2 \int [p(\phi) - r] p''(\phi) \, d\phi.
\]

Since we want to determine one antiderivative, we omit the constant of integration. Using integration by parts in the last integral with \( u = p - r, \ u' = p', \ v' = p'', \ v = p' \), one gets

\[
F(\phi) = 2 \int [p(\phi) - r]^2 \, d\phi + 2[p(\phi) - r] p'(\phi) - 2 \int p''(\phi) \, d\phi
\]

\[
= 2r^2 \phi + 2 \int [p^2(\phi) - p'^2(\phi)] \, d\phi - 4r \int p(\phi) \, d\phi + 2 [p(\phi) - r] p'(\phi).
\]
After the rearrangement
\[
p^2(\phi) - p'^2(\phi) = p^2(\phi) - \frac{(a^2 - b^2)^2 \cos^2 \phi \sin^2 \phi}{p^2(\phi)}
\]
\[
= p^2(\phi) - a^2 \sin^2 \phi - b^2 \cos^2 \phi - \frac{(a^2 - b^2)^2 \cos^2 \phi \sin^2 \phi}{p^2(\phi)} + a^2 \sin^2 \phi + b^2 \cos^2 \phi
\]
\[
= (a^2 - b^2)(\cos^2 \phi - \sin^2 \phi) + \frac{a^2 b^2}{p^2(\phi)} = (a^2 - b^2) \cos 2\phi + \frac{a^2 b^2}{p^2(\phi)}
\]
we find
\[
\int \left[ p^2(\phi) - p'^2(\phi) \right] d\phi = \frac{(a^2 - b^2) \sin 2\phi}{2} + a^2 b^2 \int \frac{d\phi}{p^2(\phi)}.
\] (3.7)

Now we consider the last integral
\[
\int \frac{d\phi}{p^2(\phi)} = \int \frac{d\phi}{a^2 \cos^2 \phi + b^2 \sin^2 \phi}
\]
which may be written as
\[
\frac{1}{a^2} \int \frac{1}{1 + (b/a)^2 \tan^2 \phi} \frac{d\phi}{\cos^2 \phi}
\]
(see [5, p. 115, Eq. (235)]). The substitution
\[
z = \frac{b}{a} \tan \phi, \quad \frac{dz}{d\phi} = \frac{b}{a \cos^2 \phi}, \quad \frac{d\phi}{\cos^2 \phi} = \frac{a}{b} dz
\]
gives
\[
\int \frac{d\phi}{p^2(\phi)} = \frac{1}{ab} \int \frac{dz}{1 + z^2} = \frac{1}{ab} \arctan z = \frac{1}{ab} \arctan \left( \frac{b}{a} \tan \phi \right).
\] (3.8)

The support function (2.1) may be written as
\[
p(\phi) = a \sqrt{1 - \varepsilon^2 \sin^2 \phi}
\]
with \(\varepsilon\) according to (3.3). Now one gets
\[
\int p(\phi) d\phi = \int \sqrt{1 - \varepsilon^2 \sin^2 \phi} \, d\phi = a E(\phi, \varepsilon),
\] (3.9)
and
\[
[p(\phi) - r] p'(\phi) = \frac{ra \varepsilon^2 \sin 2\phi}{2\sqrt{1 - \varepsilon^2 \sin^2 \phi}} - \frac{1}{2} a^2 \varepsilon^2 \sin 2\phi.
\] (3.10)

Taking (3.7), (3.8), (3.9) and (3.10) into account, an antiderivative \(F(\phi)\) (see (3.6)) is given by
\[
F(\phi) = 2r^2 \phi + (a^2 - b^2) \sin 2\phi + 2ab \arctan \left( \frac{b}{a} \tan \phi \right) - 4ra E(\alpha, \varepsilon)
\]
\[
+ \frac{ra \varepsilon^2 \sin 2\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}} - a^2 \varepsilon^2 \sin 2\phi,
\]
hence
\[
F(\phi) = 2r^2 \phi + 2ab \arctan \left( \frac{b}{a} \tan \phi \right) - 4ra E(\alpha, \varepsilon) + \frac{ra \varepsilon^2 \sin 2\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}}.
\]
Now we are able to investigate the five cases according to Lemma 3.1. For this purpose we use Fig. 5.

1) (Figures 5 a and b) We have \( A_4 = 0 \). \( C_1^- \) is positively oriented, hence
\[
A_1^{01} = F(\pi/2) - F(0) = F(\pi/2) = \pi r^2 + \pi a b - 4 r a E(\varepsilon) = A^*.
\]
Since \( b > r \), \( C_1 \) cannot be contained in \( K_0 \), hence \( A_1^{10} = 0 \). With \( A^+ \) according to (3.4), we get
\[
A_2 = A^+ - A_1^{01} = \pi r^2 + \pi a b + 4 r a E(\varepsilon) - \left( \pi r^2 + \pi a b - 4 r a E(\varepsilon) \right) = 8 r a E(\varepsilon).
\]

2) (Fig. 5 c) Here, as in Case 1, \( A_1^{10} = 0 \). We denote by \( \alpha \) the first value of \( \phi \) belonging to a self intersection point. Here the \( y \)-coordinate of \( C_1^- \) is equal to zero. From (2.2) we get
\[
\frac{b^2}{p(\alpha)} - r = 0
\]
which yields
\[
\alpha = \arctan \frac{\sqrt{r^2 a^2 - b^2}}{b \sqrt{b^2 - r^2}}.
\]
The middle loop of \( C_1^- \) is positively oriented, hence
\[
A_1^{01} = F(\pi/2) - F(\alpha) = A^* - F(\alpha) = \tilde{F}(\alpha) - A^*.
\]
The outer loops of \( C_1^- \) are negatively oriented, hence
\[
A_4 = - (F(\alpha) - F(0)) = -F(\alpha) = - \left( A^* - \tilde{F}(\alpha) \right) = \tilde{F}(\alpha) - A^*.
\]
It follows that
\[
A_2 = A^+ - A_1^{01} - A_4
= \pi r^2 + \pi a b + 4 r a E(\varepsilon) - \tilde{F}(\alpha) - \left[ \tilde{F}(\alpha) - (\pi r^2 + \pi a b - 4 r a E(\varepsilon)) \right]
= 2 \pi r^2 + 2 \pi a b - 2 \tilde{F}(\alpha).
\]

3) (Fig. 5 d, e, f) Here one easily sees that \( A_1^{01} = 0 = A_1^{10} \). \( C_1^- \) is negatively oriented, hence
\[
A_4 = -(F(\pi/2) - F(0)) = -F(\pi/2) = -A^*.
\]
It follows that
\[
A_2 = A^+ - A_4 = \pi r^2 + \pi a b + 4 r a E(\varepsilon) - (4 r a E(\varepsilon) - \pi r^2 - \pi a b) = 2 \pi r^2 + 2 \pi a b.
\]

4) (Fig. 5 g) Since \( r > a \), \( C_0 \) cannot be contained in \( K_1 \), hence \( A_1^{01} = 0 \). We denote by \( \beta \) the first value of \( \phi \) belonging to a self intersection point. Here the \( x \)-coordinate of \( C_1^- \) is equal to zero. From (2.2) we get
\[
\frac{a^2}{p(\beta)} - r = 0
\]
which yields
\[
\beta = \arctan \frac{a \sqrt{r^2 - a^2}}{\sqrt{a^4 - r^2 b^2}}.
\]
The middle loop of $C_1^-$ is positively oriented, hence
\[ A_{10}^i = F(\beta) - F(0) = F(\beta) \]
The outer loops of $C_1^-$ are negatively oriented, hence
\[ A_4 = -(F(\pi/2) - F(\beta)) = F(\beta) - A^*, \]
and
\[ A_2 = A^+ - A_{10}^i - A_4 \\
= \pi r^2 + \pi ab + 4raE(\varepsilon) - F(\beta) - \left[ F(\beta) - (\pi r^2 + \pi ab - 4raE(\varepsilon)) \right] \\
= 2\pi r^2 + 2\pi ab - 2F(\beta). \]

5) (Fig. 5 i) We have $A_4 = 0$. $C_1^-$ is positively oriented, hence
\[ A_{10}^i = F(\pi/2) - F(0) = A^*. \]

Finally, we get
\[ A_2 = A^+ - A_{10}^i = \pi r^2 + \pi ab + 4raE(\varepsilon) - (\pi r^2 + \pi ab - 4raE(\varepsilon)) = 8raE(\varepsilon). \]

4 Measures for oriented ellipses

Now let us go back to the original motion of the ellipse $C_1$ with respect to the fixed circle $C_0$. We give up the assumption that the direction of $C_1$ is fixed. In the following we consider $C_1$ as oriented. For this orientation we attach a frame $\xi, \eta$ to $C_1$ with its origin in the center point $M_1$ of $C_1$ (see Fig. 6). Furthermore, we define these measures
\[ m_{10}^i := m(\{ C_1 : C_1 \subset K_0 \}), \quad m_{01}^i := m(\{ C_1 : C_0 \subset K_1 \}), \]
\[ m_2 := m(\{ C_1 : \#(C_0 \cap C_1) = 2 \}), \quad m_4 := m(\{ C_1 : \#(C_0 \cap C_1) = 4 \}). \]
for $C_1$.

**Theorem 4.1.** The measure for all oriented ellipses with semi-major axis of length $a$ and semi-minor axis of length $b$ which intersect a fixed circle of radius $r$ in two and four points are given by
\[ m_2 = 4\pi^2 r^2 + 4\pi^2 ab - 2m_i \quad \text{and} \quad m_4 = 8\pi raE(\varepsilon) - 2\pi^2 r^2 - 2\pi^2 ab + m_i, \] respectively, where, with the cases of Lemma 3.1,
\[ m_i = \begin{cases} 
2\pi A_{10}^i & \text{if } a < r \quad \text{(Cases 4 and 5)}, \\
0 & \text{if } (a \geq r) \wedge (b \leq r) \quad \text{(Case 3)}, \\
2\pi A_{01}^i & \text{if } b > r \quad \text{(Cases 1 and 2)}.
\end{cases} \] (4.2)

and
\[ A_{10}^i = \begin{cases} 
\pi r^2 + \pi ab - 4raE(\varepsilon) & \text{if } a^2 / b \leq r \quad \text{(Case 5)}, \\
F(\beta) & \text{if } a^2 / b > r \quad \text{(Case 4)},
\end{cases} \]
\[ A_{01}^i = \begin{cases} 
\tilde{F}(\alpha) & \text{if } b^2 / a < r \quad \text{(Case 2)}, \\
\pi r^2 + \pi ab - 4raE(\varepsilon) & \text{if } b^2 / a \geq r \quad \text{(Case 1)}. 
\end{cases} \]
Proof. We have
\[
m_{10}^i = m(\{C_1 : C_1 \subset K_0\}) = m(\{K_1 : K_1 \subset K_0\}) = \int_{\{K_1 : K_1 \subset K_0\}} dK_1,
\]
where \(dK_1\) denotes the kinematic density of \(K_1\) (see [12, pp. 85-89]). We can write it as \(dK_1 = dx_1 \wedge dy_1 \wedge d\psi\), where \(x_1, y_1\) are the coordinates of the center point \(M_1\) of \(K_1\) with respect to the fixed \(x,y\)-frame and \(\psi\) is the angle between the \(x\)-axis of the fixed frame and the \(\xi\)-axis of the moving frame (see Fig. 6). We get
\[
m_{10}^i = 2\pi \int dx_1 \wedge dy_1,
\]
where the integral has to be taken over the points \(M_1\) such that \(K_1 \subset K_0\) for fixed angle \(\psi\), hence
\[
m_{10}^i = 2\pi A(\{M_1 : M_1^* \subset K_0\}) = 2\pi A(\{M_0 : C_1^* \subset K_0\}) = 2\pi A_{10}^i,
\]
where
\[
A_{10}^i = \begin{cases} 
\pi r^2 + \pi ab - 4raE(\varepsilon) & \text{if } a^2/b \leq r \quad \text{(Case 5 in Lemma 3.1)}, \\
F(\beta) & \text{if } a^2/b > r \quad \text{(Case 4)}, \\
0 & \text{in Cases 1-3}.
\end{cases}
\]
Analogously, one finds \(m_{01}^i = 2\pi A_{01}^i\), \(m_2 = 2\pi A_2\) and \(m_4 = 2\pi A_4\), with \(A_{01}^i, A_2, A_4\) from Lemma 3.1.

5 Hitting probabilities

Now we consider the random throw of an ellipse \(C_1\) onto an unbounded lattice of circles \(C_0\) of radius \(r\) as it is shown in Fig. 1. The center points \(M_0\) of the circles \(C_0\) lie on the vertices of parallelograms with sides of length \(s\) and \(t\), and angle \(\sigma\), \(0 < \sigma \leq \pi/2\). We assume that \(C_1\) can hit at most one \(C_0\) at the same time. Due to the periodicity of the lattice it suffices to consider only one parallelogram \(\mathcal{P}\) for the calculation of the hitting probabilities, for which we choose
\[
\mathcal{P} = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq t \sin \sigma, \ y \cot \sigma \leq x \leq s + y \cot \sigma\}.
\]
Now, we define the random throw as follows: The coordinates \( x_1, y_1 \) of \( M_1 \) are random variables uniformly distributed in \([y \cot \sigma, s + y \cot \sigma]\) and \([0, t \sin \sigma]\), respectively; the angle \( \psi \) between the \( x \)-axis of the fixed frame and the \( \xi \)-axis of the frame attached to \( C_1 \) is uniformly distributed in \([0, 2\pi]\). All three random variables are stochastically independent.

The total measure for all positions of the oriented ellipse \( C_1 \) with center point \( M_1 \) in \( P \) is given by

\[
m_t = \int_{\{C_1 : M_1 \in P\}} dx_1 \wedge dy_1 \wedge d\psi = 2\pi \int_{\{M_1 : M_1 \in P\}} dx_1 \wedge dy_1 = 2\pi st \sin \sigma.
\]

The events \( C_1 \subset K_0 \) and \( C_0 \subset K_1 \) are mutually exclusive, 

\[(C_1 \subset K_0) \land (C_0 \subset K_1) = \emptyset.
\]

So we may write the measure \( m_t \) (see (4.2)) as

\[
m_t = m(\{C_1 : (C_1 \subset K_0) \lor (C_0 \subset K_1) \neq \emptyset\}).
\]

We define the measure

\[
m_e = m(\{C_1 : K_0 \cap K_1 = \emptyset\}) = m(\{K_1 : K_0 \cap K_1 = \emptyset\})
\]

and find

\[
m_e = m_t - (m_2 + m_4 + m_i) = m_t - m(\{K_1 : K_0 \cap K_1 \neq \emptyset\})
\]

\[
= 2\pi st \sin \sigma - \left[2\pi^2 r^2 + 2\pi^2 ab + 8\pi ra E(\varepsilon)\right].
\]

Now we define the hitting probabilities

\[
p_i = P((C_1 \subset K_0) \lor (C_0 \subset K_1) \neq \emptyset), \quad p_e = P(K_0 \cap K_1 = \emptyset) = 1 - P(K_0 \cap K_1 \neq \emptyset)
\]

and, for the number of intersection points,

\[
p_{2j} = P(\#(C_0 \cap C_1) = 2j), \quad j \in \{0, 1, 2\}.
\]

So with

\[
p_2 = \frac{m_2}{m_t}, \quad p_4 = \frac{m_4}{m_t}, \quad p_i = \frac{m_i}{m_t}, \quad p_e = \frac{m_e}{m_t}
\]

and

\[
p_0 = 1 - p_2 - p_4 = p_i + p_e
\]

we get the following corollary.

**Corollary 5.1.** Under the assumption \( 2(a + r) \leq \min(s, t) \),

\[
p_0 = 1 - \frac{2\pi^2 r^2 + 2\pi^2 ab + 8\pi ra E(\varepsilon) - m_i}{2\pi st \sin \sigma}, \quad p_2 = \frac{2\pi^2 r^2 + 2\pi^2 ab - m_i}{\pi st \sin \sigma},
\]

\[
p_4 = \frac{8\pi ra E(\varepsilon) - 2\pi^2 r^2 - 2\pi^2 ab + m_i}{2\pi st \sin \sigma}, \quad p_i = \frac{m_i}{2\pi st \sin \sigma}, \quad p_e = 1 - \frac{2\pi^2 r^2 + 2\pi^2 ab + 8\pi ra E(\varepsilon)}{2\pi st \sin \sigma}
\]

with \( m_t \) according to Theorem 4.1.
6 Special case: Line segment

Now we consider for the ellipse \( C_1 \) the special case \( a \neq 0, b = 0 \) so that it degenerates to a pair of coinciding line segments of length \( \ell = 2a, r = \infty > r \), the Case 5 cannot occur, so Case 3 and Case 4 are the remaining cases. We have

\[
\varepsilon = 1 \quad \text{and} \quad E(1) = 1. \]

Therefore, considering the two coinciding line segments as one line segment, from (4.1) follows

\[
m_1 = 4\pi r^2 - 2m_1, \quad m_2 = 4\pi r\ell - 2\pi^2 r^2 + m_1. \tag{6.1}
\]

In Case 3 we have \( \ell \geq 2r, m_1 = 0 \). Therefore, \( m_1 = \mathcal{M}_1 \) and \( m_2 = \mathcal{M}_2, \) where \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are Santaló’s terms for the measures in (1.2) with \( \mathcal{M}_i = 0 \). In Case 4 it remains to show that

\[
m_1 = 2\pi A_1 = 2\pi F(\beta) = 2\pi \left[ 2r^2 \beta + 2ab \arctan \left( \frac{b}{a} \tan \beta \right) - 4raE(\beta, \varepsilon) + \frac{ra\varepsilon^2 \sin 2\beta}{\sqrt{1 - \varepsilon^2 \sin^2 \beta}} \right]
\]

with

\[
\beta = \arctan \frac{\sqrt{r^2 - a^2}}{a} = \arctan \frac{\sqrt{1 - (a/r)^2}}{a/r}
\]

is equal to \( \mathcal{M}_i \) according to (1.1). The expression for the angle \( \beta \) may be written as

\[
\beta = \frac{\pi}{2} - \arcsin \frac{a}{r} = \frac{\pi}{2} - \arcsin \frac{\ell}{2r}. \]

(In [7, Vol. 1, p. 76], Eq. (1.624.3) states incorrectly that

\[
\arctan \frac{\sqrt{1 - x^2}}{x} = \arcsin x, \quad 0 < x \leq 1. \]

The correct formula, which we use, is

\[
\arctan \frac{\sqrt{1 - x^2}}{x} = \frac{\pi}{2} - \arcsin x. \]

Next we have

\[
\arctan \left( \frac{b}{a} \tan \beta \right) = \arctan (0 \tan \beta) = 0.
\]

With

\[
\sin \beta = \sin \left( \arctan \frac{\sqrt{r^2 - a^2}}{a} \right) = \sqrt{1 - \frac{a^2}{r^2}},
\]

one finds that

\[
E(\beta, \varepsilon) = E(\beta, 1) = E \left( \arctan \frac{\sqrt{r^2 - a^2}}{a}, 1 \right) = \sqrt{1 - \frac{a^2}{r^2}} = \frac{1}{r} \sqrt{r^2 - \frac{\ell^2}{4}}
\]

as well as

\[
\frac{\varepsilon^2 \sin 2\beta}{\sqrt{1 - \varepsilon^2 \sin^2 \beta}} = \frac{2\varepsilon^2 \sin \beta \cos \beta}{\sqrt{1 - \varepsilon^2 \sin^2 \beta}} = \frac{2 \sin \beta \cos \beta}{\sqrt{1 - \varepsilon^2 \sin^2 \beta}} = 2 \sin \beta = 2 \sqrt{1 - \frac{a^2}{r^2}} = \frac{2}{r} \sqrt{r^2 - \frac{\ell^2}{4}}.
\]
It follows that
\[
F(\beta) = 2r^2 \left( \frac{\pi}{2} - \arcsin \frac{\ell}{2r} \right) + 0 - 4r^2 \frac{\ell}{2r} \sqrt{r^2 - \frac{\ell^2}{4}} - 2r \frac{\ell}{2r} \sqrt{r^2 - \frac{\ell^2}{4}}
= \pi r^2 - 2r^2 \arcsin \frac{\ell}{2r} - \ell \sqrt{r^2 - \frac{\ell^2}{4}},
\]
so \( m_i = 2\pi F(\beta) \) is indeed Santaló’s formula for \( M_i \) (see (1.1), or (32) in [11, p. 165]) for our Case 4 in which \( \ell < 2r \) holds.

The hitting probabilities in Corollary 5.1 turn into
\[
P_0 = 1 - \frac{2\pi r^2 + 4\pi r\ell - m_i}{2\pi st \sin \sigma}, \quad p_1 = \frac{2\pi^2 r^2 - m_i}{\pi st \sin \sigma}, \quad p_2 = \frac{4\pi r\ell - 2\pi^2 r^2 + m_i}{2\pi st \sin \sigma}, \quad p_c = \frac{2\pi^2 r^2 + 4\pi r\ell}{2\pi st \sin \sigma}.
\]
(6.2)
These are Santaló’s probabilities for the line segment under the assumption that the line segment can hit only one circle [11, p. 166, Eq. (36)].

7 Some comments
In all five cases of Theorem 4.1 one finds that \( 2A_2 + 4A_4 = 16raE(\varepsilon) \), hence
\[
2m_2 + 4m_4 = 2\pi (2A_2 + 4A_4) = 32\pi raE(\varepsilon) = 4 \times 2\pi r \times 4aE(\varepsilon)
= 4 \times (\text{length of } C_0) \times (\text{length of } C_1).
\]
This generally follows from Poincaré’s formula (see Eq. (7.10) in [12, p. 111]). Analogously, from Corollary 5.1 it follows that the expected value for the random number \( Z := \#(C_0 \cap C_1) \) of intersection points is given by
\[
E(Z) = \sum_{j=1}^{4} j p_j = 2p_2 + 4p_4 = \frac{16raE(\varepsilon)}{st \sin \sigma} = \frac{2 \cdot 2\pi r \cdot 4aE(\varepsilon)}{\pi st \sin \sigma}
= \frac{2 \times (\text{length of } C_0) \times (\text{length of } C_1)}{\pi st \sin \sigma}.
\]
One also gets \( E(Z) \) from Eq. (8.11) in [12, p. 134].

The measure \( m(\{K_1: K_0 \cap K_1 \neq \emptyset\}) \) also follows from the principal kinematic formula (see Theorem 5.1.3 in [13, p. 175]).

Santaló only calculated the measure \( M_i \) and derived the measures \( M_1 \) and \( M_2 \) using Poincaré’s formula in the form
\[
M_1 + 2M_2 = 4 \times (\text{length of } C_0) \times (\text{length of line segment}) = 8\pi r\ell
\]
(see [11, pp. 164-165], Equations (27) and (30)).

Our approach was different. We first calculated all areas from geometrical considerations. It was easy to find the area of the set enclosed by the outer parallel curve \( C_1^+ \), but it required some effort to calculate the areas of the sets enclosed by the loops of the inner parallel curve \( C_1^- \). Then, we derived the respective measures.

Clearly, the hitting probabilities in Corollary 5.1 remain unchanged if we throw a circle \( C_0 \) onto a lattice as in Fig. 1 where each circle is replaced by a congruent copy of an ellipse \( C_1 \).
References

[1] Roger Böttcher: *Geometrical probability for an arbitrarily long segment hitting a triangle exactly once or twice*, FernUniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik, 88 (2016) 65-90. https://www.fernuni-hagen.de/imperia/md/content/fakultaetfuermathematikundinformatik/forschung/berichte_matematik/berichte_band_88.pdf

[2] Roger Böttcher: *Measure and geometrical probabilities for arbitrarily long segments hitting two non-overlapping circles*, 19 January 2015, unpublished.

[3] Karl Bosch: *Mathematik-Taschenbuch*, 3., verbesserte Aufl., R. Oldenbourg Verlag, München/Wien, 1991.

[4] Egbert Brieskorn, Horst Knörrer: *Ebene algebraische Kurven*, Birkhäuser Verlag, Basel/Boston/Stuttgart, 1981.

[5] Fritz Chemnitius: *Differentiation und Integration ausgewählter Beispiele*, 4. Aufl., VEB Verlag Technik, Berlin, 1959.

[6] Andrei Duma, Marius Stoka: *Hitting probabilities for random ellipses and ellipsoids*, J. Appl. Prob. 30 (1993), 971-974. http://www.jstor.org/stable/3214526

[7] Israel S. Gradstein, Joniss M. Ryshik: *Tables of Series, Products, and Integrals*, 2 Volumes, Verlag Harri Deutsch, Thun/Frankfurt a. M., 1981.

[8] Frances Kirwan: *Complex Algebraic Curves*, Cambridge University Press, 1992.

[9] Gerhard Kowalewski: *Einführung in die Analytische Geometrie*, 4., verbesserte Auflage, Walter de Gruyter, Berlin, 1953.

[10] Floor van Lamoen: *Circular Point at Infinity*, From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/CircularPointatInfinity.html

[11] Luis A. Santaló: *Sur quelques problèmes des probabilités géométriques*, Tôhoku Math. J., First Series, 47 (1940), 159-171.

[12] Luis A. Santaló: *Integral Geometry and Geometric Probability*, Addison-Wesley, London, 1976.

[13] Rolf Schneider, Wolfgang Weil: *Stochastic and Integral Geometry*, Springer-Verlag, Berlin/Heidelberg, 2008.

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