DOI: https://doi.org/10.24297/jam.v20i.9075

**Metallic Ratios, Pythagorean Triples & \( p \equiv 1 \pmod{4} \) Primes: Metallic Means, Right Triangles and the Pythagoras Theorem**

Dr. Chetansing Rajput

M.B.B.S. Nair Hospital (Mumbai University) India, Asst. Commissioner (Govt. of Maharashtra)

Email: chetansingkrajput@gmail.com 
Website: https://goldenratiorajput.com/

Lecture Link 1 : https://youtu.be/vBfVDaFnA2k
Lecture Link 2 : https://youtu.be/74uF4sBqYjs
Lecture Link 3 : https://youtu.be/raosnlXwRhw
Lecture Link 4 : https://youtu.be/Qh2B1tMI8Bk

**Abstract**

This paper synergizes the newly discovered geometry of all Metallic Means and the recently published mathematical formulae those provide the precise correlations between different Metallic Ratios. The paper illustrates the concept of the “Triads of Metallic Means”, and aslo the close correspondence between Metallic Ratios and the Pythagorean Triples as well as Pythagorean Primes.

**Keywords:** Metallic Mean, Pythagoras Theorem, Fibonacci, Pell, Lucas, Pi, Phi, Silver Ratio, Right Triangle, Metallic Numbers, Metallic Ratio Triads, 3 6 9, Pythagorean Triples, Bronze Ratio, Golden Ratio, Pascal’s Triangle, Metallic Ratio

**Introduction**

This paper brings together following, recently discovered, new aspects of Metallic Ratios.

1) The Generalised Geometric Construction of all Metallic Ratios: cited by the Wikipedia in its page on “Metallic Mean” [6]. This generalised geometric substantiation of all Metallic Means was published in January 2021 [7].

2) The Mathematical Formula that provides the precise correlation between different Metallic Means. This explicit formula has been recently published in the month of May 2021 [1].

These couple of important aspects of Metallic Means: viz. the generalised geometric constructions of all Metallic Means and the concerned mathematical formulae, were brought together in the work mentioned in Reference [2].

The prime objective of current paper is to further synergize the these two newly discovered aspects of Metallic Means.

The *synergism* between above two features of Metallic Means unveils an intriguing pattern of Metallic Ratios, which asserts that the mathematical implications of these Means have not been fully appreciated so far. The
abovementioned Geometry and Mathematics synergically enable us to recognize the full worth of these Metallic Means, as described in this paper.

As a brief introduction, each Metallic Mean $\delta_n$ is the root of the simple Quadratic Equation $X^2 - nX - 1 = 0$, where $n$ is any positive natural number.

Thus, the fractional expression of the $n^{th}$ Metallic Ratio is $\delta_n = \frac{n + \sqrt{n^2 + 4}}{2}$

Moreover, each Metallic Ratio can be expressed as the continued fraction:

$\delta_n = n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \cdots}}}$; And hence, $\delta_n = n + \frac{1}{\delta_n}$

.....References: [3], [4], [5]

GEOMETRIC CONSTRUCTION OF ALL METALLIC MEANS:

Each Metallic Ratio can be constructed geometrically with a special Right Angled Triangle. Any $n^{th}$ Metallic Mean can be represented by the Right Triangle having its catheti 1 and $\frac{n}{2}$. Hence, the right triangle with one of its catheti = 1 may substantiate any Metallic Mean, having its second cathetus $= \frac{n}{2}$, where $n = 1$ for Golden Ratio, $n = 2$ for Silver Ratio, $n = 3$ for Bronze Ratio, and so on. Such Right Triangle provides the precise value of $n^{th}$ Metallic Mean by the generalised formula:

The $n^{th}$ Metallic Mean ($\delta_n$) = Hypotenuse + Cathetus $\frac{n}{2}$

Such Right Triangle not only provides for the accurate geometric construction and precise fractional expression of any $n^{th}$ Metallic Mean ($\delta_n$), but its every geometric feature is the prototypical form of that Metallic Mean [6], [7], [8], [9].

The characteristic geometry of such Right Triangle having its catheti 1 and $\frac{n}{2}$, is resplendent with the corresponding $n^{th}$ Metallic Mean ($\delta_n$) embedded in its every geometric aspect.

For example, the remarkable expression of Golden Ratio in every geometric feature of $1:2:\sqrt{5}$ triangle, including all its angles and side lengths, its ‘Incenter-Excenters Orthocentric system’, its Gergonne and Nagel triangles, and also the Nobbs points and the Gergonne line, various triangle centers as well as the Incircle of $1:2:\sqrt{5}$ triangle, make this triangle the quintessential form of the Golden Ratio ($\phi$) and also of the fourth Metallic Mean ($\phi^3$). [7]
Figure 1: Generalised Right Triangle to represent any $n^{th}$ Metallic Mean $\delta_n$

MATHEMATICAL CORRELATIONS AMONG DIFFERENT METALLIC RATIOS:

If $K$, $m$ and $n$ are three positive integers such that $n$ is the smallest of the three integers and $\frac{mn + 4}{m - n} = k$ then, it is observed that

$$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$$

where $\delta_k$, $\delta_m$ and $\delta_n$ are the $k^{th}$, $m^{th}$ and $n^{th}$ Metallic Means respectively.

This explicit formula, among several other formulae those give the precise mathematical relations between different Metallic Means, has been recently published in the work mentioned in References [1] and [2].

The “TRIADS” Of Metallic Means:

The abovementioned explicit formula gives the “Triads” of Metallic Means as $[\delta_n, \delta_m, \delta_k]$ where

$$\frac{mn + 4}{m - n} = k \quad \text{and} \quad \frac{kn + 4}{k - n} = m$$

hence,

$$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k \quad \text{and also} \quad \frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$$

Moreover,

$$\frac{km - 4}{k + m} = n \quad \text{and} \quad \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n$$
For example, if \( n=6 \), the three integers 6, 11 and 14 satisfy the prerequisite
\[
\frac{mn + 4}{m - n} = k;
\]

Hence, the three Metallic means \( \delta_6, \delta_{11} \) and \( \delta_{14} \) form a Triad \( [\delta_6, \delta_{11}, \delta_{14}] \) such that:
\[
\frac{\delta_{11} \times \delta_6 + 1}{\delta_{11} - \delta_6} = \delta_{14} \quad \text{and also} \quad \frac{\delta_{14} \times \delta_6 + 1}{\delta_{14} - \delta_6} = \delta_{11} \quad \text{Also,} \quad \frac{\delta_{14} \times \delta_{11} - 1}{\delta_{14} + \delta_{11}} = \delta_6
\]

Noticingly, \( n=6 \) forms such multiple triads:

| \( n \) | 6  | 6  | 6  | 6  | 6  | 6  | 6  |
|--------|----|----|----|----|----|----|----|
| \( m \) | 7  | 8  | 10 | 11 | 14 | 16 | 26 | 46 |
| \( k \) | 46 | 26 | 16 | 14 | 11 | 10 | 8  | 7  |

: Shaded Triads have been exemplified above.

And, just like \( n=6 \) exemplified above, every integer forms such multiple triads:

For example, \( n=10 \)

| \( n \) | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
|--------|----|----|----|----|----|----|----|
| \( m \) | 11 | 12 | 14 | 18 | 23 | 36 | 62 | 114 |
| \( k \) | 114 | 62 | 36 | 23 | 18 | 14 | 12 | 11 |

Noticeably, Odd \( n \) forms Fewer Triads

| \( n \) | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
|--------|----|----|----|----|----|----|----|
| \( m \) | 6  | 34 |
| \( k \) | 34 | 6  |

It may be noticed from above Tables that every \( n^{th} \) Metallic Mean can give precise values of various Metallic Means by the formula
\[
\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k, \text{ maximum upto } (n^2 + n + 4)^{th} \text{ Metallic Mean:}
\]
\[
m_{\text{max}} = k_{\text{max}} = (n^2 + n + 4)
\]
Also noticeably, the Even Integers (Even \( n \)) form comparatively more Triads than the Odd \( n \). Several such patterns about these Triads of Metallic Means have been discussed in detail in Reference [2]. Here, let us consider the classical correspondence of the abovementioned Formula and TRIADS with the Geometry of Metallic Ratios.

Remarkably, the abovementioned Triads of Metallic Means can be represented geometrically, as shown below.

For instance, the Triad \( [\delta_n, \delta_m, \delta_k] \) is illustrated geometrically in following Figure 2.

![Figure 2: Three Right Triangles representing the “Triad of Metallic Means”](image)

Remarkably, in above Figure 2: if the three Metallic Means \( \delta_n, \delta_m \) and \( \delta_k \) constitute a Triad as \( \frac{mn + 4}{m - n} = k \), then

\[
\theta_1 = \theta_2 + \theta_3 \quad \text{(and also } \Psi_n + 90^\circ = \Psi_m + \Psi_k \text{)}
\]

In other words,

\[
\arctan\frac{2}{n} = \arctan\frac{2}{m} + \arctan\frac{2}{k}
\]

Solving it gives the correlations:

\[
\frac{mn + 4}{m - n} = k \quad \text{and} \quad \frac{kn + 4}{k - n} = m
\]

And also

\[
\frac{km - 4}{k + m} = n
\]

And hence satisfy the prerequisite for

\[
\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k \quad \text{and} \quad \frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m
\]

And also

\[
\frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n
\]
Moreover, the Hypotenuses of these Triad-Triangles: \( H_n, H_m \) and \( H_k \) in Figure 2 exhibit following relations.

\[
\frac{H_k}{H_m} = \frac{2}{m-n} \times H_n
\]

Simplying this, we get

\[
\frac{k^2 + 4}{m^2 + 4} = \frac{n^2 + 4}{(m-n)^2}
\]

And solving it gives

\[
k = \frac{mn + 4}{m-n}
\]; which is the prerequisite for formation of a Triad \([\delta_n, \delta_m, \delta_k]\)

Moreover, entire geometry of such Triad-Triangles is resplendent with the precise correlation among the three Metallic Means \(\delta_n, \delta_m\) and \(\delta_k\). Following intriguing relations are observed in above Figure 2. Consider the larger acute angles of the three triangles;

\[
\Psi_m + \Psi_k = 2 \arctan \delta_n
\]

\[
\Psi_m - \Psi_n = 2 \arctan \frac{1}{\delta_k}
\]

\[
\Psi_k - \Psi_n = 2 \arctan \frac{1}{\delta_m}
\]

Similarly, \(\arctan \frac{1}{\delta_n} = \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k}\)

simplifying which we get

\[
\frac{km - 4}{k + m} = n \quad \text{and} \quad \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n
\]

And,

\[
\arctan \frac{1}{\delta_n} + \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k} = 2 \arctan \frac{1}{\delta_n} = (\theta_1) = [\theta_2 + \theta_3]
\]

\[
= \arctan \frac{2}{n} = \left[ 180^0 - (\Psi_m + \Psi_k) \right]
\]

= Half of the Smaller Acute Angle of the Pythagorean Triple associated with the Right Triangle for \(\delta_n\) (Described in following Section)
Similarly,
\[ \arctan \delta_n + 90^0 = \arctan \delta_m + \arctan \delta_k \]
And,
\[ \arctan \delta_n + \arctan \delta_m + \arctan \delta_k = \Psi_n + 180^0 = (\Psi_m + \Psi_k) + 90^0 \]

However, more interesting aspect of this geometry of Metallic Means is its classical correspondence with Primitive Pythagorean Triples and Pythagorean Primes.

**Right Triangle for \( n \)th Metallic Mean and the Primitive Pythagorean Triples**:

Each of the Right Triangles representing various Metallic Means, and hence the abovementioned Triads of Metallic Means, are associated with particular primitive Pythagorean triple, depending upon the value of \( n \). For example, Right Triangle for 6th Metallic Mean (\( \delta_6 \)), and hence, all Triads with \( n=6 \) are associated with 3-4-5 Pythagorean Triple. For instance, consider the Triad \( [\delta_6, \delta_{11}, \delta_{14}] \). If this Triad is constructed geometrically, as in above Figure 2, then it is observed that:

\[ 2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = \text{The Smaller Acute Angle of 3-4-5 Pythagorean Triangle i.e. arctan} \frac{3}{4} \]

\[ 2\Psi_n - 90^0 = (\Psi_n + \Psi_m + \Psi_k) - 180^0 = \text{The Larger Acute Angle of 3-4-5 Pythagorean Triangle i.e. arctan} \frac{4}{3} \]

Likewise, Right Triangle for 3rd Metallic Mean (and all Triads with \( n=3 \)), or Right Triangle for 10th Metallic Mean (and Triads with \( n=10 \)), are all related to 5-12-13 Pythagorean Triple, and so on.

Consider the fractional expression of the \( n \)th Metallic Mean : 
\[ \delta_n = \frac{n + \sqrt{n^2 + 4}}{2} \]

The radical \( (n^2 + 4) \) in this Fractional expression of \( \delta_n \) determines the Pythagorean Triple associated with the Right Triangle representing a Metallic Mean (\( \delta_n \)). Any such Right Triangle representing a Metallic Mean \( \delta_n \) is associated with such Primitive Pythagorean Triple whose Hypotenuse is Factor of the Radical \( (n^2 + 4) \), with following observed sub-rules:

**If \( n \) is Odd**: Hypotenuse of associated Pythagorean Triple = \( (n^2 + 4) \)

**If \( n \) is Even and multiple of Four (\( n=4x \))**: Hypotenuse of associated Pythagorean Triple = \( nx + 1 = \frac{(n^2+4)}{4} \)

**If \( n \) is Even but not Multiple of Four**: Hypotenuse of associated Pythagorean Triple = \( \frac{(n^2+4)}{8} \) and the Smaller Cathetus of associated Pythagorean Triple = \( \frac{n}{2} \)
Noticeably, the doubling of the smaller acute angle of the $1 : \frac{n}{2}$ Right Triangle for any $n^{th}$ Metallic Mean, produces the associated Pythagorean Triple; as illustrated below in Figure 3. For example, if the 26.565° angle of $1 : \frac{1}{2}$ triangle for Golden Ratio is doubled to 53.13°, the associated 3-4-5 Pythagorean Triple is produced.

![Figure 2: Doubling of the smaller acute angle of the Right Triangle for $n^{th}$ Metallic Mean](image)

**Figure 2:** Doubling of the smaller acute angle of the Right Triangle for $n^{th}$ Metallic Mean

The following Table 1 shows associated Pythagorean Triples for first few values of $n$:

**Table 1:** Primitive Pythagorean Triples associated with the Right Triangles for Metallic Means: depending upon $n$

| $n$ | Associated Pythagorean Triple |
|-----|-------------------------------|
| 1   | 3-4-5                         |
| 2   | None                          |
| 3   | 5-12-13                       |
| 4   | 3-4-5                         |
| 5   | 20-21-29                      |
| 6   | 3-4-5                         |
| 7   | 28-45-53                      |
| 8   | 8-15-17                       |
| 9   | 36-77-85                      |
| 10  | 5-12-13                       |
| 11  | 44-117-125                    |
| 12  | 12-35-37                      |
| 13  | 52-165-173                    |
| 14  | 7-24-25                       |
The classical correspondence between these Right Triangles for Metallic Means and the corresponding Primitive Pythagorean Triple is manifested as follows.

Consider again the Figure 2 representing the Triad of Metallic Means \([\delta_n, \delta_m, \delta_k]\).

\[2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = \text{The Smaller Acute Angle of associated Pythagorean Triple.}\]

Similarly, \[2\psi_n - 90^0 = (\psi_n + \psi_m + \psi_k) - 180^0 = \text{The Larger Acute Angle of associated Pythagorean Triple.}\]

Special cases:

1) For \(n=1\) : the right Triangle that represents Golden Ratio has its Cathetus 1 longer than its Second Cathetus \(\frac{n}{2}\). Hence, for the Right Triangle representing Golden Ratio, and the Triad formed with \(n=1\) \([\delta_1, \delta_2, \delta_6]\):

\[2\theta_1 - 90^0 = (\theta_1 + \theta_2 + \theta_3) - 90^0 = \text{The Smaller Acute Angle of associated Pythagorean Triple 3-4-5.}\]

And, \[2\psi_n = (\psi_n + \psi_m + \psi_k) - 90^0 = \text{The Larger Acute Angle of associated Pythagorean Triple}.\]

2) Similarly, in case of \(n=2\) : the Triangle representing Silver Ratio is an Isosceles Right Triangle. Hence, for Triad formed with \(n=2\), like \([\delta_2, \delta_3, \delta_{10}]\) or \([\delta_2, \delta_4, \delta_6]\):

\[2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = 90^0,\]

and \[2\psi_n + 90^0 = (\psi_n + \psi_m + \psi_k) = 180^0\]

And hence there is no associated Pythagorean Triple for \(n=2\).

3) For \(n=3\) or \(4\):

\[2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = \text{The Larger Acute Angle of associated Pythagorean Triple.}\]

And, \[2\psi_n - 90^0 = (\psi_n + \psi_m + \psi_k) - 180^0 = \text{The Smaller Acute Angle of associated Pythagorean Triple}.\]

Note the Right Triangles those represent the First and Fourth Metallic Means: \(\delta_1\) (that is Golden Ratio \(\phi\)) and \(\delta_4\) (which equals \(\phi^3\)), are similar triangles.

In all the Right Triangles those represent 5th Mean onwards, and Triads formed with \(n=5\) onwards, \((n = 5, 6, 7, \ldots\) and so on\):

\[2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = \text{The Smaller Acute Angle of associated Primitive Pythagorean Triple};\]

\[2\psi_n - 90^0 = (\psi_n + \psi_m + \psi_k) - 180^0 = \text{The Larger Acute Angle of associated Primitive Pythagorean Triple}.\]

Moreover, an intriguing relation is also observed between the Hypotenuses of the Three Triangles in Figure 2 (viz. \(H_n, H_m\) and \(H_k\)) and the Hypotenuse of associated Pythagorean Triple \(H\).

If \(n\) is Odd: \[
\frac{H_k}{H_m} \times (m-n) = \sqrt{H}
\]
If \( n \) is Even and multiple of Four:
\[
\frac{H_k}{H_m} \times (m-n) = 2\sqrt{H}
\]

If \( n \) is Even but not Multiple of Four:
\[
\frac{H_k}{H_m} \times (m-n) = 2\sqrt{2\sqrt{H}}
\]

Thus, consider the following couple of facts. First: the generalised geometric substantiation of all Metallic Means based upon Right Triangles, as elaborated in the work mentioned in References [6] and [7]. And second: the close correspondence between the Right Triangles representing Metallic Ratios and the Primitive Pythagorean Triples, as described here. These couple of facts clearly highlight the underlying proposition that the Metallic Means are more closely associated with; and more holistically represented by the “Right Angled Triangles”, rather than Pentagon, Octagon or any other \((n^2+4)\)-gon.

Metallic Means and Pythagorean Primes: The Prime Families of Metallic Means

From the close correspondence between Metallic Means and Pythagorean Triples described so far, it becomes obvious that various Metallic Means are also closely associated with different Pythagorean Primes.

Consider the radical \((n^2+4)\) in the Fractional expression of the \(n^{th}\) Metallic Mean \(\delta_n\). By Fermat’s Theorem on Sums of Two Squares, this radical is an integer multiple of a prime of the form \(p \equiv 1(\text{mod } 4)\). The Greatest Prime Factor (i.e. the Largest Prime Divisor) of this radical \((n^2 + 4)\) is a Pythagorean Prime, as shown below in Table 2.

| \(n\) | \(n^2+4\) | Greatest Prime Factor of \((n^2+4)\): A Pythagorean Prime |
|------|-----------|-----------------------------------------------------------|
| 1    | 5         | 5                                                          |
| 2    | 8         | 2                                                          |
| 3    | 13        | 13                                                         |
| 4    | 20        | 5                                                          |
| 5    | 29        | 29                                                         |
| 6    | 40        | 5                                                          |
| 7    | 53        | 53                                                         |
| 8    | 68        | 17                                                         |
| 9    | 85        | 17                                                         |
Table 2: The Greatest Prime Factors of the radical \((n^2 + 4)\)

| n  | \(n^2 + 4\) | Greatest Prime Factor |
|----|-------------|-----------------------|
| 10 | 104         | 13                    |
| 11 | 125         | 5                     |
| 12 | 148         | 37                    |
| 13 | 173         | 173                   |
| 14 | 200         | 5                     |
| 15 | 229         | 229                   |
| 16 | 260         | 13                    |

It is noticeable from above table that various values of \(n\) have common Greatest Prime Factor of \((n^2 + 4)\).

For example, for 3\(^{rd}\), 10\(^{th}\) and 16\(^{th}\) Metallic Means, the common Greatest Prime Factor of the radical \((n^2 + 4)\) is 13.

Also, for 1\(^{st}\), 4\(^{th}\), 6\(^{th}\), 11\(^{th}\) Metallic Means, the common Greatest Prime Factor of the radical \((n^2 + 4)\) is 5.

Hence, the different Metallic Means can be classified into various groups corresponding to the Greatest Prime Factors \((\text{GPF})\) of the radical \((n^2+4)\). This GPF is necessarily a Pythagorean Prime \((4x + 1)\), as shown below in Table 3.

Table 3: Prime Families of Metallic Means

| Greatest Prime Factor \((\text{GPF})\) of \([n^2+4]\) | \(n_s\) of the associated Metallic Means \((\delta_n)\) |
|--------------------------------|---------------------------------|
| 5                                | 1, 4, 6, 11                     |
| 13                               | 3, 10, 16, 29                   |
| 17                               | 8, 9, 26, 43                    |
| 29                               | 5, 24, 34                       |
Noticeably, as described in previous section: the Hypotenuse of associated Pythagorean Triple is a factor of \((n^2+4)\), and the associated Pythagorean Primes, as shown in Table 3 are the Greatest Prime Factors of \((n^2+4)\). Note: the 8\textsuperscript{th} and the 9\textsuperscript{th} Metallic Means both have Pythagorean Prime 17 as the GPF of their \((n^2+4)\), however they have different associated Primitive Pythagorean Triples, as shown in Table 1.

Moreover, consider the \textbf{Triads} of Metallic Means \([\delta_n, \delta_m, \delta_k]\). Noticeably Two of the Three Metallic Ratios forming such Triad, belong to same Prime Family i.e. Two of the Three Metallic Ratios have common Greatest Prime Factors of their respective \((n^2+4)\) radicals.

For example, consider the Triad \([\delta_4, \delta_5, \delta_{24}]\): \(\delta_5\) and \(\delta_{24}\) belong to the 29 prime family. Similarly, consider the Triad \([\delta_5, \delta_6, \delta_{34}]\): \(\delta_5\) and \(\delta_{34}\) belong to the 29 prime family.

Consider another example, the Triad \([\delta_2, \delta_3, \delta_{10}]\): \(\delta_3\) and \(\delta_{10}\) belong to the 13 prime family. Similarly, consider the Triad \([\delta_3, \delta_4, \delta_{16}]\): \(\delta_3\) and \(\delta_{16}\) belong to the 13 prime family.

Moreover, beside the Greatest Prime Factors indicated in above Table, multiple Pythagorean Primes constitute the factors of various \((n^2+4)\) radicals.

\[
\begin{align*}
(59)^2 + 4 &= (41 \times 17 \times 5) \\
(49)^2 + 4 &= (37 \times 13 \times 5), \text{ and so on.}
\end{align*}
\]

Noticeably, the Greatest Prime Factor of \((n^2+4)\) for \(n=43\) is 109

\[
(43)^2 + 4 = 109 \times 17
\]

However, why it has been included in the Prime 17 Family in above Table, that will become obvious with following couple of illustrations.

Remarkably, the Metallic Means belonging to same Prime Family exhibit very distinctive relations among themselves, as illustrated below.
For instance, consider the Prime Family of Metallic Means associated with the Pythagorean Prime 13 : δ₃, δ₁₀ and δ₁₆

| Metallic Means associated with the Pythagorean Prime 13 : δ₃, δ₁₀ and δ₁₆ |
|-------------------------------------------------------------------------|
| Prime 13                                                                |
| 10 + 3 = 13 = 16 - 3                                                   |
| δ₁₀   δ₃   δ₁₆                                                        |

Hence, the Pythagorean Prime of this Family 13 = \(\frac{3² + 4}{1}\) : Note the Digit 1 in Denominator.

Here, the 3rd and (3 + 1)th Metallic Means give the precise value of 16th Mean:
\[
\frac{δ₄ × δ₃ + 1}{δ₄ - δ₃} = δ₁₆
\]

Similarly, the 3rd and (3 - 1)th Metallic Means give the precise value of 10th Mean:
\[
\frac{δ₃ × δ₂ + 1}{δ₃ - δ₂} = δ₁₀
\]

Moreover, \(\frac{13 - 1}{2} = 6\) : this 6 forms a new Triad with members 10 and 16 of the family: [δ₆, δ₁₀, δ₁₆]:

\[
\frac{δ₁₆ × δ₆ + 1}{δ₁₆ - δ₆} = δ₁₀ \quad \text{and} \quad \frac{δ₁₀ × δ₆ + 1}{δ₁₀ - δ₆} = δ₁₆
\]

Further, beyond 3, 10 and 16, more members would be added in the family of Pythagorean Prime 17, next member in the family would be 29 which equals the sum of all smaller members in the family;

3 + 10 + 16 = 29 and, the Prime 13 = 29 - 16

And,

\(2 × 16 + 3 + 1 = 36\) : this 36 forms a new Triad with members 16 and 29 of the family [δ₁₆, δ₂₉, δ₃₆]:

\[
\frac{δ₃₆ × δ₁₆ + 1}{δ₃₆ - δ₁₆} = δ₂₉ \quad \text{and} \quad \frac{δ₂₉ × δ₁₆ + 1}{δ₂₉ - δ₁₆} = δ₃₆
\]
Likewise, consider another example for illustration and comparison. the Metallic Means associated with the Pythagorean Prime 17 : δ₈, δ₉, and δ₂₆

![Metallic Means associated with the Pythagorean Prime 17: δ₈, δ₉, and δ₂₆](image)

And, the Prime 17 = \( \frac{9^2 + 4}{5} \) : Note the Digit 5 in Denominator.

Here, the 9th and (9 + 5)th Metallic Means give the precise value of 26th Mean:

\[
\frac{\delta_9 \times \delta_{14} + 1}{\delta_{14} - \delta_9} = \delta_{26}
\]

Similarly, the 9th and (9 - 5)th Metallic Means give the precise value of 8th Mean:

\[
\frac{\delta_9 \times \delta_4 + 1}{\delta_4 - \delta_9} = \delta_8
\]

Moreover, \( \frac{17 - 5}{2} = 6 \) : this 6 forms a new Triad with members 8 and 26 of the family: [δ₆, δ₈, δ₂₆]

\[
\frac{\delta_{26} \times \delta_6 + 1}{\delta_{26} - \delta_6} = \delta_8 \quad \text{and} \quad \frac{\delta_8 \times \delta_6 + 1}{\delta_8 - \delta_6} = \delta_{26}
\]

Further, beyond 8, 9, and 26, more members would be added in the family of Pythagorean Prime 17, like next member in the family would be 43 which equals the sum of all smaller members in the family;

8 + 9 + 26 = 43 and, the Prime 17 = 43 - 26

And,

\[
2 \times 26 + 9 + 5 = 66 \, : \, \text{this 66 forms a new Triad with members 26 and 43 of the family: [δ₂₆, δ₄₃, δ₆₆]}
\]

\[
\frac{\delta_{66} \times \delta_{26} + 1}{\delta_{66} - \delta_{26}} = \delta_{43} \quad \text{and} \quad \frac{\delta_{43} \times \delta_{26} + 1}{\delta_{43} - \delta_{26}} = \delta_{66}
\]

Such several distinctive correlations are observed among the Metallic Means belonging to the same Pythagorean Prime Families, and these correlations are bound to generate more such intriguing mathematical formulae, which may provide the precise relations between different Metallic Ratios.
On the last note, it is worth mentioning here that several other intriguing properties of Metallic Means and their abovementioned TRIADS are described in details in the works mentioned in the References. For instance, these TRIADS of Metallic Means are found to be closely associated with the Pascal’s Triangle [11]; the geometric substantiation of Metallic Ratios and their TRIADS [6] [7] [8] [9]; and special positions of Integers 3, 6 and 9 in the realm of Metallic Means [2][12]. Further, all imperative formulae those provide the precise relations between different Metallic Means are described in the work mentioned in Reference [1].

**Conclusion:**

This paper brought together the generalised geometry of all Metallic Means and the mathematical formula that provides the precise correlations between different Metallic Ratios. The paper illustrated the concept of the “Triads of Metallic Means”, and also the close correspondence between Metallic Ratios and the Pythagorean Triples as well as Pythagorean Primes.

**References:**

[1] Rajput, Chetansing (2021). Metallic Ratios : Beyond the Golden Ratio: The Mathematical Relationships between different Metallic Means. JOURNAL OF ADVANCES IN MATHEMATICS, 20, 158-166. [https://doi.org/10.24297/jam.v20i.9023](https://doi.org/10.24297/jam.v20i.9023)

[2] Rajput, Chetansing (2021). “Metallic Means : Beyond the Golden Ratio, New Mathematics and Geometry of all Metallic Ratios based upon Right Triangles, The Formation of the Triples of Metallic Means, And their Classical Correspondence with Pythagorean Triples and p≡1(mod 4) Primes, Also the Correlation between Metallic Numbers and the Digits 3 6 9”, JOURNAL OF ADVANCES IN MATHEMATICS, 20, 250–266. [https://doi.org/10.24297/jam.v20i.9056](https://doi.org/10.24297/jam.v20i.9056)

[3] Vera W. de Spinadel (1999). The Family of Metallic Means, Vismath 1(3) from Mathematical Institute of Serbian Academy of Sciences and Arts.

[4] Weisstein, Eric W. "Table of Silver means", MathWorld.

[5] "An Introduction to Continued Fractions: The Silver Means", maths.surrey.ac.uk.

[6] Rajput, Chetansing (2021). "A Right Angled Triangle for each Metallic Mean". Journal of Advances in Mathematics. 20: 32–33. [https://en.wikipedia.org/wiki/Metallic_mean#cite_note-15](https://en.wikipedia.org/wiki/Metallic_mean#cite_note-15)

[7] Rajput, Chetansing (2021). Golden Ratio. JOURNAL OF ADVANCES IN MATHEMATICS, 20, 19–42. [https://doi.org/10.24297/jam.v20i.8945](https://doi.org/10.24297/jam.v20i.8945)

[8] Rajput, Chetansing (2021). Metallic Means and Right Triangles: The Geometric Substantiation of all Metallic Ratios JOURNAL OF ADVANCES IN MATHEMATICS, 20, 167-173. [https://doi.org/10.24297/jam.v20i.9029](https://doi.org/10.24297/jam.v20i.9029)

[9] Rajput, Chetansing (2021). Golden Ratio and other Metallic Means: The Geometric Substantiation of all Metallic Ratios with Right Triangles. JOURNAL OF ADVANCES IN MATHEMATICS, 20, 174-187. [https://doi.org/10.24297/jam.v20i.9034](https://doi.org/10.24297/jam.v20i.9034)

[10] Rajput, Chetansing (2021). Metallic Ratio Triads, The Mathematical and Geometric Relations between different Metallic Means, Metallic Numbers and the Right Angled Triangles. JOURNAL OF ADVANCES IN MATHEMATICS, 20, 167-173. [https://doi.org/10.24297/jam.v20i.9077](https://doi.org/10.24297/jam.v20i.9077)
[11] Rajput, Chetansing (2021). Metallic Ratios and Pascal’s Triangle: Triads of Metallic Means in the Pascal’s Triangle. JOURNAL OF ADVANCES IN MATHEMATICS, 20, 167-173. https://doi.org/10.24297/jam.v20i.9078

[12] Rajput, Chetansing (2021). Metallic Ratios and the Digits 3, 6, 9, Mathematical Relations between different Metallic Means, And the Special Significance of Digits 3, 6, 9 in the realm of Metallic Numbers. JOURNAL OF ADVANCES IN MATHEMATICS, 20, 167-173. https://doi.org/10.24297/jam.v20i.9076

| Select Lecture Videos of Author: | Contacts: |
|---------------------------------|-----------|
| 1) https://youtu.be/vBfVDaFna2k | Website: https://goldenratiorajput.com/ |
| 2) https://youtu.be/74uf4sbqyjs  | Email: chetansingrajput@gmail.com |
| 3) https://youtu.be/raosnixwRhw  | Email: chetansingrajput1129@gmail.com |
| 4) https://youtu.be/Qh2B1tMl8Bk  | Contacts: (+91) 7057521129 |

282