PERIODS AND GL$_2 \times$ GL$_2$ RECIPROCITY

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Abstract. Given $F$ a number field with ring of integers $\mathcal{O}_F$ and $p, q$ two distinct prime ideals of $\mathcal{O}_F$, we prove a reciprocity relation for the second moment of Rankin-Selberg $L$-functions $L(\pi \times \pi_1, \frac{1}{2})$ twisted by $\lambda_p(p)$, where $\pi_1$ is a fixed cuspidal automorphic representation of $\text{PGL}_2(\mathbb{A}_F)$ and $\pi$ runs through generic automorphic representations of conductor dividing $q$. The method uses adelic integral representations of $L$-functions and the symmetric identity is established for a particular period. Finally, the integral period is connected to the second moment via Parseval formula.

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1. Introduction

Reciprocity formulas in the context of automorphic $L$-functions have been studied first for the second moment of Dirichlet $L$-functions $L(\chi, \frac{1}{2})$ modulo $q$ twisted by $\chi(p)$ by Conrey [Con07], Young [You11] and Bettin [Bet16]. In [BK17], Blomer and Khan considered a mixed rank version $\sum_{\text{level } q} L(f \times g, \frac{1}{2}) L(\pi, \frac{1}{2}) \lambda_f(p)$, where $F$ is a fixed automorphic form for the group $\text{SL}_3(\mathbb{Z})$, $p, q$ are two prime numbers and $\lambda_f(p)$ is the $p^{th}$ Hecke eigenvalue of $f$. Their result is non trivial even in the case $p = q = 1$ since it also transforms the parameters $s$ and $w$. We also mention the work of Blomer, Li and Miller [BLM17] obtaining an identity involving the first moment of $L(\Pi \times f, \frac{1}{2})$ where $\Pi$ is a self-dual cusp form on $\text{GL}_4$ and $f$ runs over $\text{GL}_2$ modular forms.

In this paper, we provide an adelic treatment of the recent work of Andersen and Kiral [AK18] who proved a reciprocity relation for the second moment of Rankin-Selberg $L$-functions $\sum_{\text{level } q} L(f \times g, \frac{1}{2})^2 \lambda_f(p)$ with $g$ a fixed automorphic form for $\text{SL}_2(\mathbb{Z})$. They obtained the result by means of classical tools in the analytic theory of automorphic forms; namely the Kuznetsov trace formula. At the end, the relation that exchanges the two prime numbers $p$ and $q$ is established by an identity of sums of Kloosterman sums. The main advantage in using the adelic language is that we can deal in a uniform way with arbitrary number fields. For such a field $F$, we write $\mathcal{O}_F$ for the ring of integers, $\mathbb{A}$ for its adele ring and we let $p, q$ be two prime ideals of $\mathcal{O}_F$. We fix $\pi_1$ a cuspidal automorphic representation.

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of $\text{PGL}_2(\mathbb{A})$ which is unramified at all finite places. We will prove a symmetric identity for the twisted second moment of the central value of Rankin-Selberg $L$-functions, roughly of the shape

\begin{equation}
\sum_{\pi \in \text{cond}(\pi)\mid q} h(\pi)|L(\pi \times \pi_1, \frac{1}{2})|^2 \lambda_\pi(p) \leadsto \sum_{\pi \in \text{cond}(\pi)\mid p} h(\pi)|L(\pi \times \pi_1, \frac{1}{2})|^2 \lambda_\pi(q),
\end{equation}

where $\lambda_\pi(p)$ is the eigenvalue of the Hecke operator $T_p$ and $h(\pi)$ is a weight function depending on the infinite part $\pi_\infty$ and its finite ramified data (see (4.3) for the precise formulation and Theorem 4.2 for the complete result).

1.1. An Overview of the Method. Our approach is different from the methods of the previous papers as it uses the integral representation of $L$-functions. The key observation is that by the Rankin-Selberg method, the left handside of (1.1) can be transformed into

\[ \sum_{\varphi \in \text{level q}} \left( \int_X T_p(\varphi) \varphi_1 E \right) \left( \int_X \overline{\varphi_1} \overline{E} \right) = \sum_{\varphi \in \text{level q}} \langle T_p \varphi, \varphi_1 E \rangle \langle \varphi, \overline{\varphi_1} E \rangle, \]

where $X = \text{PGL}_2(F) \setminus \text{PGL}_2(\mathbb{A})$, $T_p$ is the Hecke operator at the place $p$, $\varphi_1$ and $\overline{\varphi_1}$ are some smooth vectors in the representation $\pi_1$, $E, \overline{E}$ are Eisenstein series and $\varphi$ runs through an orthonormal basis of the space of level $q$ and square integrable automorphic forms (of course the sum becomes an integral when $\pi$ belongs to the continuous spectrum). Using the self-adjointness of $T_p$ and Parseval formula, we see that the right handside is equal to

\[ \langle \overline{\varphi_1} E, T_p(\overline{\varphi_1} E) \rangle = \int_X \overline{\varphi_1}(g)E(g)T_p \left( \overline{\varphi_1} E \right) (g) dg := \mathcal{P}(p,q). \]

This is the typical period we consider in Section 3 with suitable vectors $\varphi_1, \overline{\varphi_1} \in \pi_1$ and Eisenstein series $E, \overline{E}$, depending on the prime ideal $q$. Using the definition of the Hecke operator $T_p$ as a convolution with the characteristic function of a compact double coset $H_p \subset \text{GL}_2(F_p)$ ($F_p$ denotes the completion of $F$ at the place $p$), we obtain

\[ \mathcal{P}(p,q) = \int_{H_p} \int_X \overline{\varphi_1}(g)E(g)\varphi_1(gh)\overline{E}(gh)dgdh = \sum_{i \in \mathcal{J}} \int_X \overline{\varphi_1}(g)E(g)\varphi_1(gh_i)\overline{E}(gh_i)dg. \]

The $h_i$'s are representatives of the quotient $H_p/K_p$ (we need the function $\overline{\varphi_1} \overline{E}$ to be right $K_p$-invariant and the measure $dh$ is normalized so that it gives mass 1 to $K_p$, see Section 2.1 for the notations). There is a distinguished partition $\mathcal{J} = \mathcal{J}_1 \cup S, |S| = 1$ that we can form by looking at the shape of the representatives $h_i$ (c.f. (3.2)). The vectors $\varphi_1, \overline{\varphi_1}$ and $E, \overline{E}$ are choosen so that the part corresponding to $S = \{s\}$, say $\mathcal{P}(p,q)$, presents an obvious symmetry in $p$ and $q$ (c.f. (3.6)). For the remaining terms, using a change of variables and the automorphic properties, we can pass from $\mathcal{J}_1$ to $S$ as follows (c.f. (3.8))

\[ \sum_{i \in \mathcal{J}_1} \int_X \overline{\varphi_1}(g)E(g)\varphi_1(gh_i)\overline{E}(gh_i)dg = |\mathcal{J}_1| \int_X \overline{\varphi_1}(gh_s)E(gh_s)\varphi_1(g)\overline{E}(g)dg. \]

The last integral is not symmetric in $p$ and $q$. However, we can extract the symmetric part by completing the Hecke operator acting on $\overline{\varphi_1}$ (c.f. (3.10))

\[ \int_X \overline{\varphi_1}(gh_s)E(gh_s)\varphi_1(g)\overline{E}(g)dg = \int_X T_p(\overline{\varphi_1}) E(gh_s)\varphi_1(g)\overline{E}(g)dg \]

\[ - \sum_{i \in \mathcal{J}_1} \int_X \overline{\varphi_1}(gh_i)E(gh_s)\varphi_1(g)\overline{E}(g)dg. \]
Finally, imposing the condition that \( \varphi_1 \) is an eigenfunction of \( T_p \) with eigenvalue \( \lambda_{\pi_1}(p) \), we connect in Proposition 3.1 the two last terms with the two first terms, namely
\[
\int_X T_p(\tau_1) E(gs) \hat{\varphi}_1(g) \hat{E}(g) dg \asymp \lambda_{\pi_1}(p) \mathcal{P}(1,q)
\]
and
\[
\sum_{i \in \mathbb{Z}_+} \int_X \overline{\varphi}(gh_i) \hat{E}(gs) \hat{\varphi}_1(g) \hat{E}(g) dg \asymp \mathcal{S}(p,q),
\]
where \( \asymp \) means here up to a constant lying in \( \mathbb{C}(p^{1/2}) \), \( p \) being the norm of the ideal \( p \).

1.2. Some Remarks. 1) We restrict ourselves to the case of prime ideals because it simplifies the treatment of the Hecke operators in Section 3. It is although possible to adapt the method and deal with two squarefree and coprime ideals.

2) This method cannot be applied directly in the case where \( \varphi_1 \) is Eisenstein, giving a reciprocity formula for a fourth moment of Hecke \( L \)-functions, because it is not \( L^2 \). However it should be possible to handle this difficulty by using a regularized version of the inner product, valid for not necessarily square integrable automorphic forms, as in [MV10]. We will return to this problem in a near future.

3) We mention the work in progress by Ramon Nunes to give an adele treatment of the paper [BK17], i.e. the case \( (GL_3 + GL_1) \times GL_2 \).

4) All previous results involve a moment of a degree 8 \( L \)-function. It would be interesting to investigate what happens if we adapt this kind of idea for \( GL_3 \times GL_3 \). In this case, the Hecke operators are no longer self-adjoint and the decomposition (3.2) of the double coset is much more complicated. Nevertheless, it seems that we can recover the same kind of symmetries between the representatives of \( T_p \) and its adjoint when we make suitable changes of variables in the integral period.

2. Automorphic Preliminaries

2.1. Notations and Conventions.

2.1.1. Number fields. In this paper, \( F/\mathbb{Q} \) will denote a fixed number field with ring of integers \( \mathfrak{o}_F \) and discriminant \( d_F \). We make the assumption that all prime ideals considering this paper do not divide \( d_F \). We let \( \Lambda_F \) be the complete \( \zeta \)-function of \( F \); it has a simple pole at \( s = 1 \) with residue \( \Lambda_F(1) \).

2.1.2. Local fields. For \( v \) a place of \( F \), we set \( F_v \) for the completion of \( F \) at the place \( v \). We will also write \( F_{\mathfrak{p}} \) if \( v \) is finite place that corresponds to a prime ideal \( \mathfrak{p} \) of \( \mathfrak{o}_F \). If \( v \) is non-Archimedean, we write \( \mathfrak{o}_{F_v} \) for the ring of integers in \( F_v \) with maximal \( \mathfrak{m}_v \) and uniformizer \( \varpi_v \). The size of the residue field is \( q_v = \mathfrak{o}_{F_v}/\mathfrak{m}_v \). For \( s \in \mathbb{C} \), we define the local zeta function \( \zeta_{F_v}(s) \) to be \((1 - q_v^{-s})^{-1}\) if \( v < \infty \), \( \zeta_{F_v}(s) = \pi^{-s/2}\Gamma(s/2) \) if \( v \) is real and \( \zeta_{F_v}(s) = 2(2\pi)^{-s}\Gamma(s) \) if \( v \) is complex.

2.1.3. Adele ring. The adele ring of \( F \) is denoted by \( \mathbb{A} \) and its unit group \( \mathbb{A}^\times \). We also set \( \mathfrak{o}_F = \prod_{v < \infty} \mathfrak{o}_{F_v} \) for the profinite completion of \( \mathfrak{o}_F \) and \( \mathbb{A}^1 = \{ x \in \mathbb{A}^\times : |x| = 1 \} \), where \( |\cdot|: \mathbb{A}^\times \to [0,\infty) \) is the adelic norm map.

2.1.4. Additive characters. We denote by \( \psi = \prod_v \psi_v \) the additive character \( \psi_Q \circ \text{Tr}_{F/Q} \) where \( \psi_Q \) is the additive character on \( \mathbb{A} \setminus \mathbb{A}_Q \) with value \( e^{2\pi i x} \) on \( \mathbb{R} \). For \( v < \infty \), we let \( d_v \) be the conductor of \( \psi_v \): this is the smallest non-negative integer such that \( \psi_v \) is trivial on \( \mathfrak{m}^{d_v}_v \). We have in this case \( d_F = \prod_{v < \infty} d_v \). We also set \( d_v = 0 \) for \( v \) Archimedean.
2.1.5. Subgroups. We let $G = \text{GL}_2$ considered as an algebraic group defined over $F$. If $R$ is any commutative ring, $G(R)$ is the group of $2 \times 2$ matrices with coefficients in $R$ and determinant in $R^*$. We also defined the following standard algebraic subgroups

$$B(R) = \left\{ \begin{pmatrix} a & b \\ d \\ \end{pmatrix} : a, d \in R^*, b \in R \right\}, \quad P(R) = \left\{ \begin{pmatrix} a & b \\ 1 \\ \end{pmatrix} : a \in R^*, b \in R \right\},$$

$$Z(R) = \left\{ \begin{pmatrix} z \\ z \\ \end{pmatrix} : z \in R^* \right\}, \quad A(R) = \left\{ \begin{pmatrix} a \\ 1 \\ \end{pmatrix} : a \in R^* \right\},$$

$$N(R) = \left\{ \begin{pmatrix} 1 & b \\ 1 \\ \end{pmatrix} : b \in R \right\}.$$

We also set

$$n(x) = \begin{pmatrix} 1 & x \\ 1 \\ \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ -1 \\ \end{pmatrix} \quad \text{and} \quad a(y) = \begin{pmatrix} y \\ 1 \\ \end{pmatrix}.$$

For any place $v$, we let $K_v$ be the maximal compact subgroup of $G(F_v)$ defined by

$$K_v = \begin{cases} G(\mathcal{O}_{F_v}) & \text{if $v$ is finite} \\ \text{O}_2(\mathbb{R}) & \text{if $v$ is real} \\ \text{U}_2(\mathbb{C}) & \text{if $v$ is complex.} \end{cases}$$

We also set $K := \prod_v K_v$. If $v < \infty$ and $n \geq 0$, we define

$$K_{v,0}(\mathcal{w}_v^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v : c \in m_v^n \right\}.$$

If $a$ is an ideal of $\mathcal{O}_F$ with prime decomposition $a = \prod_{v<\infty} p_v^{f_v(a)}$ ($p_v$ is the prime ideal corresponding to the finite place $v$), then we set

$$K_0(a) = \prod_{v<\infty} K_{v,0}(\mathcal{w}_v^{f_v(a)}).$$

2.1.6. Measures. We use the same measures normalizations as in [MV10, Sections 2.1.3.1]. At each place $v$, $dx_v$ denotes a self-dual measure on $F_v$ with respect to $\psi_v$. If $v < \infty$, $dx_v$ gives the measure $q_v^{-d_v/2}$ to $\mathcal{O}_{F_v}$. We define $dx = \prod_v dx_v$ on $\mathbb{A}$. We take $d^\times x_v = \zeta_{F_v}(1) \frac{dx_v}{|x_v|}$ as the Haar measure on the multiplicative group $F_v^\times$ and $d^\times x = \prod_v d^\times x_v$ as the Haar measure on the idele group $\mathbb{A}^\times$.

We provide $K_v$ with the probability Haar measure $dk_v$. We identify the subgroups $Z(F_v)$, $N(F_v)$ and $A(F_v)$ with respectively $F_v^\times$, $F_v$ and $F_v^\times$ and equipped them with the measure $d^\times z$, $dx_v$ and $d^\times y_v$. Using the Iwasawa decomposition $G(F_v) = Z(F_v)N(F_v)A(F_v)K_v$, a Haar measure on $G(F_v)$ is given by

$$dg_v = d^\times z dx_v \frac{d^\times y_v}{|y_v|} dk_v.$$  \hspace{1cm} (2.1)

The measure on the adelic points of the various subgroups are just the product of the local measures defined above. We also denote by $dg$ the quotient measure on

$$X := \mathcal{Z}(\mathbb{A})G(F) \setminus G(\mathbb{A}),$$

with total mass $\text{vol}(X) < \infty$. 

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2.2. Automorphic Forms and Representations. Let $L^2(X)$ be the Hilbert space of square integrable functions $\varphi : X \to \mathbb{C}$. The $L^2$-norm is denoted by
\begin{equation}
||\varphi||_{L^2(X)}^2 = \int_X |\varphi(g)|^2 \, dg.
\end{equation}
We denote by $L^2_{\text{cusp}}(X)$ the closed subspace of cusp forms, i.e. the functions $\varphi \in L^2(X)$ with the additional property that
\[ \int_{F \setminus \mathbb{A}} \varphi(n(x)g) \, dg = 0, \text{ a.e. } g \in G(\mathbb{A}). \]
Each $\varphi \in L^2_{\text{cusp}}(X)$ admits a Fourier expansion
\begin{equation}
\varphi(g) = \sum_{\alpha \in F^*} W_{\varphi}(a(\alpha)g),
\end{equation}
with
\begin{equation}
W_{\varphi}(g) = \int_{F \setminus \mathbb{A}} \varphi(n(x)g)\psi(-x) \, dx.
\end{equation}
The group $G(\mathbb{A})$ acts by right translations on both spaces $L^2(X)$ and $L^2_{\text{cusp}}(X)$ and the resulting representation is unitary with respect to (2.2). It is well known that each irreducible component $\pi$ decomposes into $\pi = \bigotimes_v \pi_v$ where $\pi_v$ are irreducible and unitary representations of the local groups $G(F_v)$. The spectral decomposition is established in the first four chapters of [GJ79] and gives the orthogonal decomposition
\begin{equation}
L^2(X) = L^2_{\text{cusp}}(X) \oplus L^2_{\text{res}}(X) \oplus L^2_{\text{cont}}(X).
\end{equation}
$L^2_{\text{cusp}}(X)$ decomposes as a direct sum of irreducible $G(\mathbb{A})$-representations which are called the cuspidal automorphic representations. $L^2_{\text{res}}(X)$ is the sum of all one dimensional subrepresentations of $L^2(X)$. Finally the continuous part $L^2_{\text{cont}}(X)$ is a direct integral of irreducible $G(\mathbb{A})$-representations and it is expressed via the Eisenstein series. In this paper, we call the irreducible components of $L^2_{\text{cusp}}$ and $L^2_{\text{cont}}$ the generic automorphic representations. If $\pi$ is a generic representation appearing in the continuous part, we say that $\pi$ is Eisenstein. In this case, the sentence “let $\varphi \in \pi$” means that $\varphi$ is the Eisenstein series associated to some $\phi \in \pi$ (c.f. Section 2.3 below).

For any ideal $\mathfrak{a}$ of $\mathfrak{o}_F$, we write $L^2(X, \mathfrak{a}) := L^2(X)^{K_0(\mathfrak{a})}$ for the subspace of level $\mathfrak{a}$ automorphic forms: this is the closed subspace of functions that are invariant under the subgroup $K_0(\mathfrak{a})$.

2.2.1. Principal series representations. Let $k$ be a local field with ring of integers $\mathfrak{o}$. For $\langle \mu_1, \mu_2 \rangle$ a pair of characters of $k^\times$, we denote by $\mu_1 \boxplus \mu_2$ the principal series representation of $G(k)$ which is unitarily induced from the representation $b = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mapsto \mu_1(a)\mu_2(d)$ of $B(k)$: This is the $L^2$-space of functions $f : G(k) \to \mathbb{C}$ such that
\[ f \left( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g \right) = \left| \frac{a}{d} \right|^{1/2} \mu_1(a)\mu_2(d) f(g) \]
on which $G(k)$ acts by right translation and the inner product is given by
\begin{equation}
\langle f, g \rangle = \int_{k \in G(\mathfrak{o})} f(k)\overline{g(k)} \, dk.
\end{equation}
If $\mu_1, \mu_2$ are unitary, the inner product (2.6) is $G(k)$-invariant and the resulting representation $\mu_1 \boxplus \mu_2$ is thus a unitary representation.
2.3. **Eisenstein Series.** Let $\omega_1, \omega_2$ be Hecke characters, i.e. characters of $F^\times \setminus \mathbb{A}^\times$. As in Section 2.2.1, we denote by $\omega_1 \boxplus \omega_2$ the principal series representation of $G(\mathbb{A})$, which consists of $L^2$-functions $f : G(\mathbb{A}) \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a & b \\ d & c \end{pmatrix} g\right) = \left|\frac{a}{d}\right|^{1/2} \omega_1(a)\omega_2(d)f(g),$$

on which $G(\mathbb{A})$ acts by right translation and the inner product is given by

$$(f, g) = \int_K f(k)\overline{g(k)}\,dk.$$  

This representation is again unitary when $\omega_1, \omega_2$ are unitary characters. Moreover, if $\omega_i = \otimes_v \omega_i, v$, then $\omega_1 \boxplus \omega_2$ is the restricted tensor product of the local principal series representations of $G(F_v)$, i.e.

$$\omega_1 \boxplus \omega_2 = \bigotimes_v \omega_{1,v} \boxplus \omega_{2,v}.$$ 

Let $s \in \mathbb{C}$; there is a $K$-equivariant isomorphism of Hilbert spaces (but not $G(\mathbb{A})$-equivariant) (see [KL13, Section 5])

$$\omega_1 \boxplus \omega_2 \overset{s}{\longrightarrow} \omega_1|^{s} \boxplus \omega_2 |^{-s}, \quad \phi \mapsto \phi_s\left(\begin{pmatrix} a & b \\ d & c \end{pmatrix} k\right) = \left|\frac{a}{d}\right|^{s+1/2} \omega_1(a)\omega_2(d)\phi(k).$$

We assume from now that $\omega_1, \omega_2$ are unitary. Given a smooth function $\phi \in \omega_1 \boxplus \omega_2$ and $s \in \mathbb{C}$, we define the associated Eisenstein series by

$$E(g, \phi, s) = E(g, \phi_s) := \sum_{\gamma \in B(F) \setminus G(F)} \phi_s(\gamma g),$$

which is absolutely convergent for $\Re(s) > 1/2$, extends meromorphically to $\mathbb{C}$ and it is holomorphic on the line $\Re(s) = 0$. In fact it is holomorphic unless $\omega_1 = \omega_2$ in which case we call the Eisenstein series **singular**. Observe that the map $\phi \mapsto E(\cdot, \phi, s)$ is $\mathbb{C}$-linear and $G(\mathbb{A})$-equivariant from the dense subset of smooth functions of $\omega_1 \boxplus \omega_2$ to $C^\infty(G(F) \setminus G(\mathbb{A}))$, where the last space is endowed with the action of $G(\mathbb{A})$ given by right translation. We define the norm of the Eisenstein series to be

$${||E(\cdot, \phi_s)||}_E^2 := \langle \phi_s, \phi_s \rangle = \langle \phi, \phi \rangle.$$

2.3.1. **The Whittaker Function of an Eisenstein series.** For $\omega_1, \omega_2$ Hecke unitary characters, $\phi \in \omega_1 \boxplus \omega_2$ and $s \in \mathbb{C}$, we define the Whittaker function of $E(g, \phi, s)$ as in (2.4)

$$W_\phi(g, s) := \int_{G(\mathbb{A})} E(n(x)g, \phi, s) \psi(-x)\,dx.$$ 

**Lemma 2.1.** For $\Re(s) > 1/2$, we have

$$W_\phi(g, s) = \int_k \phi_s(wn(x)g)\psi(-x)\,dx.$$ 

**Proof.** The proof is a consequence of Bruhat decomposition [Ven10, Lemma 10.5].

Assume further that $\phi$ is a factorizable function, i.e. $\phi = \prod_v \phi_v$ with each $\phi_v$ in the local principal series $\omega_{1,v} \boxplus \omega_{2,v}$. In this case Lemma 2.1 implies the factorizability of the Whittaker function $W_\phi = \prod_v W_{\phi_v}$ with

$$W_{\phi_v}(g_v, s) = \int_{F_v} (\phi_s)_v(wn(x)g_v)\psi_v(-x)\,dx.$$
Remark 2.2. The factorization $W_{\phi}(g,s)$ into a product $\prod_{v} W_{\phi_{v}}(g_{v}, s)$ is established for $\Re(e(s)) > 1/2$, but remains true for $\Re(e(s)) = 0$ by analytic continuation. The map $(\phi_{v})_{v} \mapsto W_{\phi}(g,s) = \int_{F_{v}} (\phi_{v})_{v}(w) \psi_{v}(x) dx$ is the usual intertwiner map between the local principal series $\omega_{1,v}{|}{\cdot} \otimes \omega_{2,v}{|}{\cdot}^{-s}$ and its Whittaker model.

2.3.2. Some explicit Eisenstein series. In this paragraph, we define explicitly some Eisenstein series that we will need to construct the global Rankin-Selberg $L$-function $L(\pi_{1} \times \pi_{2}, s)$ associated respectively to $E_{1}^{1}(a,b), E_{1}^{2}(a,b),$ and $E_{1}^{(0,1)}$. We also write $\Psi_{1}^{(0,1)}, \Psi_{2}^{(0,1)},$ and $\Psi_{3}^{(0,1)}$ in the obvious way $\Psi_{1}^{(0,1)} \Psi_{2}^{(0,1)} \otimes \Psi_{3}^{(0,1)}$.

Let $\Phi \in S(A^{2})$; the space of Bruhat-Schwartz functions on $A^{2}$, and $\omega_{1}, \omega_{2}$ be unitary characters of $F^{\times} \setminus A^{\times}$. As in [Jac72, § 19], we define for $s \in \mathbb{C}$,

$$f_{\Phi}(g,s) := \omega_{1}(\det(g)) \mathcal{N}(g)^{s+1/2} \int_{A^{\times}} \Phi((0,t)g) \omega_{1}^{-1}(t)|t|^{2s+1}dt,$$

and the integral converges for $\Re(s) > 0$. Note that for any $a, d \in A^{\times}$ and $b \in A$, we have

$$f_{\Phi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} g, s \right) = \left| \frac{g_{d}}{d} \right|^{1/2+s} \omega_{1}(a) \omega_{2}(d) f_{\Phi}(g, s).$$

In other words, $f_{\Phi}(g,s)$ belongs to the principal series representation $\omega_{1}{|}{\cdot} \otimes \omega_{2}{|}{\cdot}^{-s}$. We simply denote by $E_{\Phi}(g,s)$ the associated Eisenstein series, i.e.

$$E_{\Phi}(g,s) = \sum_{\gamma \in B(F) \setminus GL_{2}(F)} f_{\Phi}(\gamma g,s),$$

which converges absolutely for $\Re(s) > 1/2$.

Definition 2.3. Let $q \in \text{Spec}(\mathcal{O}_{F})$. We define three factorizable functions $\Psi$, $\Psi^{q}$, and $\Psi_{q}$ in $S(A^{2})$ as follows:

1. For $v$ Archimedean, they have the same $v$-component, which is a fixed element of $S(F_{v}^{2})$ with values in $\mathbb{R}$.
2. If $v$ is finite, but $v \neq q$, we set $\Psi_{v} = (\Psi^{q})_{v} = (\Psi_{q})_{v} = 1_{\mathcal{O}_{F}}$.
3. Finally, at the place $v$ corresponding to the prime ideal $q$, we define

$$\Psi_{v} = 1_{\mathcal{O}_{F}}, \quad (\Psi^{q})_{v} = 1_{(m_{\mathcal{O}_{F}})} \quad \text{and} \quad (\Psi_{q})_{v} = 1_{(m_{\mathcal{O}_{F}})}.$$}

We define similarly $\Psi_{a}^{q}$ or $\Psi_{a}^{q}$ for $n > 1$ and by multiplicativity $\Psi_{a}^{q}$ or $\Psi_{q}$ for any ideal $a$ of $\mathcal{O}_{F}$. Further, if $a, b$ are coprime ideals, we define in the obvious way $\Psi_{b}^{q}$.

Definition 2.4. Let $a$ be an ideal of $\mathcal{O}_{F}$. We define $E_{a}(g,s)$, $E_{1}(g,s)$, and $E_{0}(g,s)$ to be the Eisenstein series associated respectively to $f_{\Psi}(g,s), f_{\Psi^{(0,1)}}(g,s)$ and $f_{\Psi^{(0,1)}}(g,s) \in |{\cdot}|^{1/2} \otimes |{\cdot}|^{-s}$. We also write $E_{b}(g,s)$ if $a$ is coprime with $b$.

Lemma 2.5. Let $q$ be a prime ideal of $\mathcal{O}_{F}$. Then both functions $g \mapsto \Psi_{q}((0,1)g)$ and $g \mapsto \Psi_{q}((0,1)g)$ are right $K_{0}(q)$-invariant.

Proof. Write $k = F_{q}, o = \mathcal{O}_{F_{q}}, m = m_{q}$ and $\varpi = \varpi_{q}$. Then it is enough to show that the functions $g \in G(k) \mapsto f_{m}(g) := 1_{(m,\mathcal{O}_{F})}((0,1)g)$ and $g \in G(k) \mapsto f_{m}(g) := 1_{(m,\mathcal{O}_{F})}((0,1)g)$ are right $K_{0}(\varpi)$-invariant. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(k)$ and $k = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in K_{0}(\varpi)$, so that

$$gk = \begin{pmatrix} * & * \\ (ca + d\gamma) & (c\beta + d\delta) \end{pmatrix}.$$
Observe that $f^m(gk) = f^m(g)$ if and only if we have the equivalence
\[(2.13) \quad c \in m \text{ and } d \in o \iff c\alpha + d\gamma \in m \text{ and } c\beta + d\delta \in o.\]
The part $\Rightarrow$ is clear since $\gamma \in m$. This proves also the other direction with the matrix $k^{-1}$ since $K_0(\varpi)$ is a group. We show that $f_m(gk) = f_m(g)$ exactly in the same way.

From (1) in Definition 2.3, it is easy to see that the three Eisenstein series defined in 2.4 satisfy $E(g, s) = E(g, \pi)$. Looking at (2.11), we can also derive that $E^1(g, s)$ is invariant under the subgroup $G(\mathfrak{o}_F)$ and by Lemma 2.5 both $E_2(g, s)$, $E_3(g, s)$ are invariant under $K_0(a)$. Furthermore, it is not difficult to deduce that for any $q \in \text{Spec}(\mathfrak{o}_F)$, we have the relations
\[(2.14) \quad E^n\left(g\left(1 \overline{\varpi}_q\right), s\right) = q^{s+1/2}E^{n+1}(g, s), \quad n \geq 0\]
and
\[(2.15) \quad E^n\left(g\left(1 \overline{\varpi}_q\right), s\right) = q^{s+1/2}E^{n+1}(g, s), \quad n \geq 1,\]
where $\overline{\varpi}_q$ is a uniformizer of the local field $F_v$ at the place $v$ corresponding to $q$ and the matrix $\left(1 \overline{\varpi}_q\right)$ has to be understood under the inclusion $G(F_v) \hookrightarrow G(\mathbb{A})$, $g_v \mapsto (1, \ldots, 1, g_v, 1, \ldots)$.

2.4. Invariant Inner Product on the Whittaker Model. Let $\pi = \otimes_v \pi_v$ be a generic automorphic representation of $\text{PGL}_2(\mathbb{A})$ as defined in Section 2.2. The intertwiner
\[(2.16) \quad \pi \ni \varphi \mapsto W_\varphi(g) = \int_{F_{\mathbb{A}}/\mathbb{A}} \varphi(n(x)g)\psi(-x)dx,\]
realizes a $G(\mathbb{A})$-equivariant embedding of $\pi$ into a space of functions $W : G(\mathbb{A}) \to \mathbb{C}$ satisfying $W(n(x)g)) = \psi(x)W(g)$. The image is called the Whittaker model of $\pi$ with respect to $\psi$ and it is denoted by $\mathcal{W}(\pi, \psi)$. This space has a factorization $\otimes_v \mathcal{W}(\pi_v, \psi_v)$ into local Whittaker models of $\pi_v$. A pure tensor $\otimes_v \varphi_v$ has a corresponding decomposition $\prod_v W_{\varphi_v}$ where $W_{\varphi_v}(1) = 1$ and is $K_v$-invariant for almost all place $v$.

We define a normalized inner product on the space $\mathcal{W}(\pi_v, \psi_v)$ by the rule
\[(2.17) \quad \vartheta_v(W_v, W'_v) := \zeta_{F_v}(2) \frac{\int_{F_v} W_v(a(y))W'_v(a(y))dy}{\zeta_{F_v}(1)L(\pi_v, Ad, 1)}.\]
This normalization has the good property that $\vartheta_v(W_v, W_v) = 1$ for $\pi_v$ and $\psi_v$ unramified and $W_v(1) = 1$; which is true for almost all $v$. This is a consequence of [JS81], Proposition 2.3, but can also be deduced from (5.2) in the non-Archimedean case. We can therefore define an invariant inner product on $\mathcal{W}(\pi, \psi)$ by setting $\vartheta(W, W) = \prod_v \vartheta_v(W_v, W_v)$ if $W = \prod_v W_v$ is a pure tensor.

Recall that we have a unitary structure on $\pi$ given by (2.2) in the cuspidal case and (2.10) if $\pi$ is Eisenstein. Using [MV10, Lemma 2.2.3], we can compare the two inner products under the intertwiner (2.16): for $\varphi \in \pi$, we have
\[(2.18) \quad 2\Lambda_F(2)d_F^{1/2}\Lambda^*(\pi, Ad, 1)\vartheta(W_\varphi, W_\varphi) = \begin{cases} ||\varphi||^2_{L^2(X)} & \text{if } \pi \text{ is cuspidal} \\ 2\Lambda_F(2)||\varphi||^2_{Eis} & \text{if } \pi \text{ is Eisenstein}, \end{cases}\]
2.5. Rankin-Selberg Method. We give in this section a brief review of the Rankin-Selberg method. The two principal references are [Cog04, Section 2.3] and [Ven10, Section 11.2].

Let \( \pi, \pi_1 \) be generic automorphic representations of \( \text{PGL}_2(\mathbb{A}) \) with \( \pi_1 \) cuspidal. Let \( \Phi = \prod_v \Phi_v \) be a Bruhat-Schwartz function on \( \mathbb{A} \) with \( \Phi_v = 1_{S_{F_v}} \) for almost all \( v \) and \( f_\Phi(g, s) \in \mathcal{C} \) be the flat section defined in (2.11). We write \( E(g, \Phi, s) \) for the Eisenstein series associated to \( f_\Phi(g, s) \) (c.f. (2.9)). For \( \varphi, \varphi_1 \in \pi \), we consider for \( \Re(s) > 1 \) the following integral period

\[
\mathcal{I}(\varphi, \varphi_1, \Phi, s) := \int_X \varphi(g) \varphi_1(g) E(g, \Phi, s - \frac{1}{2}) dg,
\]

which has a meromorphic continuation to \( \mathbb{C} \) and is holomorphic on the vertical line \( \Re(s) = 1/2 \). Since \( \varphi_1 \) is cuspidal, we obtain by the unfolding method

\[
\mathcal{I}(\varphi, \varphi_1, \Phi, s) = \int_{N(\mathbb{A}) \backslash \text{PGL}_2(\mathbb{A})} W_{\varphi}(g) W_{\varphi_1}(g) f_\Phi(g, s - \frac{1}{2}) dg,
\]

where \( W_{\varphi} \in \mathcal{M}(\pi, \psi) \) (resp. \( W_{\varphi_1} \in \mathcal{M}(\pi_1, \psi_1) \)) is the Whittaker function associated to \( \varphi \) (resp. to \( \varphi_1 \)) under the intertwiner (2.16). If we assume moreover that \( \varphi \) and \( \varphi_1 \) are pure tensors \( \otimes_v \varphi_v \) and \( \otimes_v \varphi_1,v \), then we know from Section 2.4 that there is a corresponding decomposition into products \( W_\varphi = \prod_v W_{\varphi_v} \) and \( W_{\varphi_1} = \prod_v W_{\varphi_1,v} \) with each \( W_{\varphi_1,v} \) (resp. \( W_{\varphi_1,v} \)) belongs to the local Whittaker model of \( \pi_v \) (resp. of \( \pi_{1,v} \)). Therefore we obtain the Euler factorization of the integral (2.20)

\[
\mathcal{I}(\varphi, \varphi_1, \Phi, s) = \prod_v \mathcal{Z}_v(W_{\varphi_v}, W_{\varphi_1,v}, \Phi_v, s),
\]

where the local zeta integral is defined by

\[
\mathcal{Z}_v(W_{\varphi_v}, W_{\varphi_1,v}, \Phi_v, s) := \int_{N(F_v) \backslash G(F_v)} W_{\varphi_v}(g_v) W_{\varphi_1,v}(g_v) \Phi((0,1)g_v) |\det(g_v)|^s dg_v.
\]

**Proposition 2.6.** Assume that \( \pi_v, \pi_{1,v} \) and \( \psi_v \) are unramified, that \( W_{\varphi_v}, W_{\varphi_1,v} \) are the new vectors normalized so that \( W_{\varphi_v}(1) = W_{\varphi_1,v}(1) = 1 \) and \( \Phi_v \) is the characteristic function of the lattice \( 1_{F_v} \). Then

\[
\mathcal{Z}_v(W_{\varphi_v}, W_{\varphi_1,v}, \Phi_v, s) = L(\pi_v \times \pi_{1,v}, s).
\]

If \( \psi_v \) is ramified with conductor \( d_v \) and \( W_{\varphi_v}, W_{\varphi_1,v} \) are nonzero \( K_v \)-invariant vectors, then

\[
\mathcal{Z}_v(W_{\varphi_v}, W_{\varphi_1,v}, \Phi_v, s) = a_v q_v^{sd_v} L(\pi_v \times \pi_{1,v}, s).
\]

Moreover, \( a_v = 1 \) if \( W_{\varphi_v}(\varpi_v^{d_v}) = W_{\varphi_1,v}(\varpi_v^{d_v}) = 1 \).

**Proof.** The first assertion is a classical result [Cog04, Theorem 3.3]. The second part is [Ven10, Lemma 11.1]. \( \square \)

2.6. The Spectral Decomposition.
2.6.1. The Parseval Formula. We state here a version of Parseval Formula with level restriction. Let \( \mathfrak{a} \) be an ideal of \( \mathcal{O}_F \). We start with the orthogonal decomposition which is a direct consequence of (2.5)

\[
L^2(X, \mathfrak{a}) = L^2_{\text{cusp}}(X, \mathfrak{a}) \oplus L^2_{\text{res}}(X, \mathfrak{a}) \oplus L^2_{\text{cont}}(X, \mathfrak{a}).
\]

From the classical orthogonal decomposition of the space of cusp forms into a direct sum, we obtain

\[
L^2_{\text{cusp}}(X, \mathfrak{a}) := L^2_{\text{cusp}}(X)^{K_0(\mathfrak{a})} = \bigoplus_{\pi \text{ cuspidal}} \pi^{K_0(\mathfrak{a})}.
\]

Observe that the cuspidal representations \( \pi \) with \( \pi^{K_0(\mathfrak{a})} \neq 0 \) are those whose conductor (in the sense of [Cog04, Section 3.1.2]) divides \( \mathfrak{a} \). For each such \( \pi \), we let \( \mathcal{B}(\pi, \mathfrak{a}) \) be an orthonormal basis of \( \pi^{K_0(\mathfrak{a})} \). The space \( L^2_{\text{res}}(X, \mathfrak{a}) \) is the orthogonal direct sum [KL13, Proposition 6.1]

\[
\bigoplus_{\omega^2 = 1} \mathbb{C} : h_\omega, \ h_\omega(g) := \omega(\det(g)),
\]

The parametrization of the continuous part \( L^2_{\text{cont}}(X, \mathfrak{a}) \) passes by the induced representations \( \omega \boxplus \varpi \) of \( \mathcal{B}(\mathfrak{a}) \) (see Section 2.3) for \( \mathfrak{a} \) a finite order Hecke character with conductor satisfying \( \text{cond}(\omega)^2 \mathfrak{a} \), and it is made through the Eisenstein series. For such a \( \omega \) and \( t \in \mathbb{R} \), we let \( \mathcal{D}(\omega, it, \mathfrak{a}) \) be an orthonormal basis of the space of \( K_0(\mathfrak{a}) \)-invariant vectors in \( \omega \cdot |it| \boxplus \varpi \cdot |it|^{-1} \). Then the Parseval formula holds [KL13, Theorem 6.2]

**Proposition 2.7.** Let \( f, g \in L^2(X, \mathfrak{a}) \) and put \( c := \text{vol}(X)^{-1} > 0 \). Then

\[
\langle f, g \rangle = \sum_{\pi \text{ cuspidal}} \sum_{\text{cond}(\pi) | \mathfrak{a}} \langle f, \psi \rangle \overline{\langle g, \psi \rangle} + c \sum_{\omega^2 = 1} \langle f, \phi_\omega \rangle \overline{\langle g, \phi_\omega \rangle}
\]

\[
+ \frac{1}{4\pi} \sum_{\mathfrak{a} \text{ of finite order}} \int_{-\infty}^{+\infty} \sum_{\phi_\mathfrak{a} \in \mathcal{D}(\omega, it, \mathfrak{a})} \langle f, E(\cdot, \phi_\mathfrak{a}) \rangle \overline{\langle g, E(\cdot, \phi_\mathfrak{a}) \rangle} dt.
\]

2.6.2. Choice of an Orthogonal Basis. Let \( \pi \) be a generic automorphic representation of \( \text{PGL}_2(\mathfrak{a}) \) of finite conductor dividing \( \mathfrak{a} \). Let \( \mathcal{W} : \pi \xrightarrow{\cong} \mathcal{W}(\pi, \psi) = \otimes_v \mathcal{W}(\pi_v, \psi_v) \) be the intertwiner defined in (2.16). We have

\[
\mathcal{W}(\pi, \psi)^{K_0(\mathfrak{a})} = \mathcal{W}(\pi_\infty, \psi_\infty) \otimes_{v < \infty} \mathcal{W}(\pi_v, \psi_v)^{K_0(\mathfrak{a})}.
\]

Therefore, it is natural to set

\[
\mathcal{B}_W(\pi, \mathfrak{a}) = \mathcal{B}_W(\pi_\infty) \otimes_{v < \infty} \mathcal{B}_W(\pi_v, m_v^{f_v(\mathfrak{a})}),
\]

where \( \mathcal{B}_W(\pi_\infty) \) (resp. \( \mathcal{B}_W(\pi_v, m_v^{f_v(\mathfrak{a})}) \)) is an orthonormal basis of \( \mathcal{W}(\pi_\infty, \psi_\infty) \) (resp. of \( \mathcal{W}(\pi_v, \psi_v)^{K_0(\mathfrak{a})} \)). For \( v \) non-Archimedean and not dividing \( \mathfrak{a} \), \( K_0(\mathfrak{a})^{(\mathfrak{a})} \) is one dimensional, so \( \mathcal{B}_W(\pi_v, \mathfrak{a}) \) has a single element \( W_v \) having norm 1 with respect to (2.17). If \( v \) corresponds to a prime dividing \( \mathfrak{a} \), the number of elements in \( \mathcal{B}_W(\pi_v, m_v^{f_v(\mathfrak{a})}) \) given by \( f_v(\mathfrak{a}) - c(\pi_v) + 1 \), assuming that \( c(\pi_v) \leq f_v(\mathfrak{a}) \) [Cas73]. The explicit shape of these local basis will be determined in Section 5.1 in the simplest case where \( \mathfrak{a} \) is squarefree. Finally, the subset

\[
(2.23) \quad \mathcal{W}_W^{-1}(\mathcal{B}_W(\pi, \mathfrak{a}))
\]
is an orthogonal basis of $\pi^{K_0(q)}$ because of the compatibility property (2.18). By construction, for any $\varphi \in W^{-1}(\mathcal{B}_W(\pi, a))$, the corresponding $W_\varphi$ is a pure tensor $\prod W_\varphi$. Moreover, we can compute its norm using (2.18) according to whether $\pi$ is cuspidal or Eisenstein.

2.6.3. Hecke Operators. It is enough for our purpose to restrict ourselves to Hecke operators of prime level because only squarefree ideals are considered in this paper. Let $p \in \text{Spec}(\mathfrak{a}_F)$ and let $p := |\mathfrak{a}_{F_p}/(\mathfrak{a}_p)|$ be the size of the residue field. We define the normalized Hecke operator $T_p$ as follows: for $f \in \mathcal{E}^\infty(G(\mathbb{A}))$,

$$
T_p(f)(g) := \frac{1}{p^{1/2}} \int_{H_p} f(gh)dh,
$$

where $H_p$ is the double coset

$$
H_p = K_p \left( \frac{\mathfrak{a}_p}{1} \right) K_p \subset G(F_p),
$$

and the function $h \in H_p \mapsto f(gh)$ as to be understood under the inclusion $G(F_p) \hookrightarrow G(\mathbb{A})$, $x \mapsto (1, ..., 1, x, 1, ...)$. We state a classical result:

**Proposition 2.8.** Let $p$ be a prime ideal not dividing $a$, $\pi$ a generic automorphic representation of $\text{PGL}_2(\mathbb{A})$ of conductor dividing $a$. Then each element of the basis (2.23) is an eigenfunction of $T_p$. The eigenvalues depend only on $\pi$ and are denoted by $\lambda_\pi(p)$. Moreover, if $\pi_p$ is an irreducible principal series $\mu_1 \boxplus \mu_2$, then

$$
\lambda_\pi(p) = \mu_1(\mathfrak{a}_p) + \mu_2(\mathfrak{a}_p).
$$

**Proof.** See for example [Bum97, Proposition 4.6.6].

### 3. A Symmetric Period

Let $\pi_1$ be a cuspidal automorphic representation of $\text{PGL}_2(\mathbb{A})$ with finite conductor $\mathfrak{a}_F$ and let $\varphi_1 \in \pi_1$. For $p, q$ two distinct prime ideals in $\mathfrak{a}_F$ with respective residue fields of size $p$, $q$, $T_p$ be the Hecke operator defined in (2.24) and $E_q^\varphi$, $E_q$ be the Eisenstein series defined in 2.4. We assume moreover that $\varphi_1$ is $G(\mathfrak{a}_F)$-invariant, so that $\varphi_1$ is an eigenfunction of $T_p$ with eigenvalue $\lambda_{\pi_1}(p) \in \mathbb{R}$. Write

$$
\varphi_1^q := \pi_1 \left( \frac{1}{\mathfrak{a}_q} \right) \varphi_1,
$$

for the right translate of $\varphi_1$ and note that $\varphi_1^q$ is invariant under the group $K_0(q)$. For $\Re(s) > 1/2$, consider the following integral period

$$
\mathcal{P}(\pi_1, p, q, s) := \int_X E_p(g, s)\varphi_1(g)T_p \left(E_q(\cdot, s)\varphi_1^q \right)(g)dg,
$$

which extends meromorphically to $\mathbb{C}$ and is in particular holomorphic on the line $\Re(s) = 0$. Using the definition of the Hecke operator $T_p$ as a convolution with the characteristic function of the double coset $H_p$ and the decomposition (c.f. [Bum97, (6.6)])

$$
K_p \left( \frac{\mathfrak{a}_p}{1} \right) K_p = \bigsqcup_{b \pmod{\mathfrak{a}_p}} \left( \frac{\mathfrak{a}_p}{b} \right) K_p \sqcup \left( \frac{1}{\mathfrak{a}_p} \right) K_p,
$$

we find since $E_q(\cdot, s)\varphi_1^q$ is right $K_p$-invariant,

$$
\mathcal{P}(\pi_1, p, q, s) = p^{-1/2} \left( \mathcal{P}_1(\pi_1, p, q, s) + \mathcal{P}(\pi_1, p, q, s) \right),
$$
Since our Eisenstein series satisfies
\begin{equation}
(3.6)
\end{equation}
and
\begin{equation}
(3.7)
\end{equation}
By \((2.14)\), we obtain that \((3.5)\) is equal to
\begin{equation}
(3.8)
\end{equation}
where we used in the last line the factorization
\begin{equation}
(3.9)
\end{equation}
and the fact that \(\left(1 - b \atop 1\right) \in K_0(p^n)\) for any \(n \geq 0\). We note that the term \(\varphi_1 \left( g \left( \frac{1}{a_p} \right) \right)\) is also the last part of the Hecke operator \(T_p\) acting on \(\varphi_1\) since \(\varphi_1\) is \(K_p\)-invariant. Therefore, completing this operator and using the fact that \(\varphi\) is an eigenfunction with eigenvalue \(\lambda_{\pi_1}(p)\), we obtain
\begin{equation}
(3.10)
\end{equation}
For the second term \(\mathcal{R}_3(\pi_1, p, q, s)\), we apply the same technique as before, namely the change of variables \(g \left( \frac{1}{a_p} \right) \leftrightarrow g\), the invariance by the center, the factorization \((3.9)\) and the scaling properties \((2.14)-(2.15)\) for the Eisenstein series to get
\begin{equation}
(3.11)
\end{equation}
The first term \(\mathcal{R}_2(\pi_1, p, q, s)\) (resp. the second expression \(\mathcal{R}_3(\pi_1, p, q, s)\)) is connected to \((3.1)\) (resp. to \((3.6)\)) in the following way:

**Proposition 3.1.** With the above notations, we have
\[\mathcal{R}_2(\pi_1, p, q, s) = \frac{1 - p^{-(2\tau + 1)}}{p(p + 1)} \mathcal{R}(\pi_1, 1, q, s)\] and \(\mathcal{R}_3(\pi_1, p, q, s) = p^{-\tau + 1} \mathcal{R}(\pi_1, p, q, s)\).
Proof. We begin with $\mathcal{P}_2(\pi_1, p, q, s)$. The strategy is to imitate Rankin-Selberg unfolding method in order to make the domain of integration factorizable. Using the definition of $X := \mathcal{Z}(\mathbb{A})G(F) \backslash G(\mathbb{A})$ and $E_{p^2}(g, s)$, we have

$$
\mathcal{P}_2(\pi_1, p, q, s) = \int_{\mathcal{Z}(\mathbb{A})B(F) \backslash G(\mathbb{A})} \varphi_1(g) \varphi_q^2(g) E_q(g, s) \psi_{p^2}(g, s) dg,
$$

where $f_{\psi_{p^2}}$ is defined in (2.11) and Definition 2.3. Noting that $\mathcal{Z}(\mathbb{A})B(F) = \mathcal{Z}(\mathbb{A})P(F)$ and expanding $\varphi_1$ into Fourier series yields (c.f. (2.3))

$$
\mathcal{P}_2(\pi_1, p, q, s) = \int_{\mathcal{Z}(\mathbb{A})N(F) \backslash G(\mathbb{A})} W_{\varphi_1}(g) \varphi_q^2(g) E_q(g, s) \psi_{p^2}(g, s) dg.
$$

Writing $W(g, s)$ for the inner integral, using the Iwasawa decomposition and the fact that $f_{\psi_{p^2}}(g, s) \in \mathcal{K}_p$ we obtain (recall the Haar measure (2.1))

$$
(3.12) \quad \mathcal{P}_2(\pi_1, p, q, s) = \int_{\mathbb{A}^\times} \left( \int_K W_{\varphi_1}(a(y)k) \mathcal{W}(a(y)k) \psi_{p^2}(k, s) dk \right) |y|^{-1/2} d^\times y.
$$

Even if $W$ is not factorizable as the classical Whittaker functions, it is at least $K_p$-invariant as well as $W_{\varphi_1}$ so the $K$-integral may be written in the form

$$
\int_{k_1 \in \prod_{v \neq p} K_v} W_{\varphi_1}(a(y)k_1) \mathcal{W}(a(y)k_1) \left( \int_{k_2 \in K_p} \psi_{p^2}(k_1k_2, s) dk_2 \right) dk_1.
$$

We observe that for fixed $k_1 \in \prod_{v \neq p} K_v$, the function

$$
k_2 \in K_p \mapsto \psi_{p^2}(k_1k_2, s)
$$

vanishes unless $k_2$ lies in $K_{p,0}(\mathbb{A}_p^2)$ and its $p$-component takes in this case the value 1. On the other hand, the function $k_2 \mapsto \psi_{p^2}(k_1k_2, s)$ is constant on $K_p$ with $p$-component equal to $(1 - p^{-2\pi} - 1)^{-1}$ (see also (5.6)-(5.7)). It follows that for any $k_1 \in \prod_{v \neq p} K_v$,

$$
\int_{k_2 \in K_p} \psi_{p^2}(k_1k_2, s) dk_2 = |K_0(p^2)|(1 - p^{-(2\pi) + 1}) \int_{k_2 \in K_p} \psi_{q^4}(k_1k_2, s) dk_2
$$

$$
= \frac{1 - p^{-(2\pi) + 1}}{p(p + 1)} \int_{k_2 \in K_p} \psi_{q^4}(k_1k_2, s) dk_2,
$$

which completes the proof of the first assertion.

We may play the same game with the second term $\mathcal{P}_3(\pi_1, p, q, s)$. The difference is that the function $W_{\varphi_1}$ is no longer $K_p$-invariant, but rather $K_{p,0}(p)$-invariant. Therefore, the $K$-integral in (3.12) becomes

$$
\int_{k_1 \in \prod_{v \neq p} K_v} \left( \int_{k_2 \in K_p} \varphi_1(a(y)k_1k_2) \mathcal{W}(a(y)k_1k_2) \psi_{p^2}(k_1k_2, s) dk_2 \right) dk_1.
$$

Using the same argumentation as before, namely that the function

$$
k_2 \in K_p \mapsto f_{\psi_{p^2}}(k_1k_2, s)
$$

vanishes unless $k_2$ lies in $K_{p,0}(\mathbb{A}_p^2)$, the above integral is the same, up to multiplication by the factor $|K_0(p^2)|/|K_0(p)| = p^{-2}$, that the one with $\Psi_{p^2}$ instead of $\Psi_{p^3}$. \qed
To summarize what we have done until here, we proved the decomposition

\[ M(\pi_1, p, q, s) := \mathcal{P}(\pi_1, p, q, s) - \lambda_{\pi_1}(p) \frac{p^{\frac{q}{2}} - p^{-s}}{p^{1/2}(p + 1)} \mathcal{P}(\pi_1, 1, q, s) \]

\[ = p^{-1/2}(1 - p^{2s+1}) \mathcal{P}(\pi_1, p, q, s). \]

Hence (3.7) yields to:

**Proposition 3.2.** Let \( \pi_1 \) be a cuspidal automorphic representation of PGL\(_2(\mathbb{A})\) unramified at all finite places. Let \( p, q \) be two distinct prime ideals of \( \mathcal{O}_F \), \( \varphi_1 \in \pi_1 \) be a \( G(\mathfrak{o}_F) \)-invariant vector and \( M(\pi_1, p, q, s) \) be defined in (3.13). Then we have the symmetric relation

\[ q^2(1 - q^{2s+1})M(\pi_1, p, q, s) = p^{s}(1 - p^{2s+1})M(\pi_1, q, p, s) . \]

4. A Reciprocity Formula

Let \( \pi_1 = \otimes_v \pi_{1,v} \) be a cuspidal automorphic representation of PGL\(_2(\mathbb{A})\) which is unramified at all finite places and let \( \varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1 \) be a \( G(\mathfrak{o}_F) \)-invariant vector. We assume that for \( v \) non-Archimedean, \( \varphi_{1,v} \) is the new vector in \( \pi_{1,v} \) normalized so that the associated local Whittaker function \( W_{\varphi_{1,v}} \) satisfies \( W_{\varphi_{1,v}}(1) = 1 \). Let \( p, q \) be two distinct prime ideals of \( \mathcal{O}_F \). In this section, we connect the integral period (3.1) with a certain second moment of Rankin-Selberg L-functions and use the symmetrical identity established in Proposition 3.2 to deduce a reciprocity formula in the conductor aspect for this moment.

Using the same arguments as in Proposition 3.1 and the fact that the function \( g \mapsto \varphi_1(g)T_p(E_q(\cdot, s)\varphi_1^q)(g) \) is right \( K_\varphi \)-invariant, we can replace the Eisenstein series \( E_\varphi^q \) by \( E_\varphi \) as follows:

\[ \mathcal{P}(\pi_1, p, q, s) = \frac{1 - p^{-(2s+1)}}{p + 1} \int_\mathbb{X} E_\varphi(g, s)\varphi_1(g)T_p(E_q(\cdot, s)\varphi_1^q)(g) dg \]

\[ = \frac{1 - p^{-(2s+1)}}{p + 1} \left( E_\varphi(\cdot, s)\varphi_1, T_p(E_q(\cdot, s)\varphi_1^q) \right) . \]

Observe that both functions \( E_\varphi(\cdot, s)\varphi_1 \) and \( T_p(E_q(\cdot, s)\varphi_1^q) \) are \( K_0(\mathfrak{q}) \)-invariant and square integrable since \( \varphi_1 \) is cuspidal. The Parseval formula in the space \( L^2(\mathbb{X}, \mathfrak{q}) \) (c.f. Proposition 2.7) implies that the above inner product decomposes as

\[ M^{\text{cusp}}(\pi_1, p, q, s) + M^{\text{res}}(\pi_1, p, q, s) + M^{\text{cont}}(\pi_1, p, q, s) , \]

where

\[ M^{\text{cusp}}(\pi_1, p, q, s) = \sum_{\varphi \in \mathcal{B}(\mathfrak{q})} \left( E_\varphi(\cdot, s)\varphi_1, \varphi \right) \left( T_p(E_q(\cdot, s)\varphi_1^q), \varphi \right) \]

\[ M^{\text{res}}(\pi_1, p, q, s) = c \sum_{\omega^2 = 1} \left( E_\varphi(\cdot, s)\varphi_1, h_\omega \right) \left( T_p(E_q(\cdot, s)\varphi_1^q), h_\omega \right) \]

\[ M^{\text{cont}}(\pi_1, p, q, s) = \frac{1}{4\pi} \sum_{\text{finite order } \omega} \int_{-\infty}^{\infty} \sum_{\phi_1, \omega \in \mathcal{D}(\mathfrak{w}, it, \mathfrak{q})} \left( E_\varphi(\cdot, s)\varphi_1, E(\phi_1, \cdot) \right) \left( T_p(E_q(\cdot, s)\varphi_1^q), E(\phi_1, \cdot) \right) dt . \]

Here \( \mathcal{B}(\mathfrak{q}) \) (resp. \( \mathcal{D}(\omega, it, \mathfrak{q}) \)) is an orthonormal basis of the space of level \( \mathfrak{q} \) cusp forms (reps. of \( K_0(\mathfrak{q}) \)-invariant vectors in the representation \( \omega \cdot | \cdot^t \oplus \mathfrak{q} | \cdot ^{-it} \)). It turns out
that the contribution of the one dimensional representations $\mathcal{M}_{\text{cusp}}^\text{res}$ is zero because of the following orthogonality property:

**Lemma 4.1.** Let $E(g, \phi, s)$ be an Eisenstein series associated to $\phi \in \omega_1 \boxplus \omega_2$ with $\omega_i$ unitary and $\omega_1 \omega_2 = 1$. Let $\omega$ be a Hecke character with $\omega^2 = 1$ and $h_\omega(g) = \omega(\det(g))$. Then we have

$$\int_X \varphi_1(g)h_\omega(g)E(g, \phi, s)dg = 0.$$ 

*Proof.* Let $W_{\varphi_1}$ be the Whittaker function of $\varphi_1$ which transforms by $\psi$ under $N(\mathbb{A})$. Observe that $h_\omega$ is $N(\mathbb{A})$-invariant, so that the function

$$g \mapsto W_{h_\omega}(g) := \int_{F \backslash \mathbb{A}} h_\omega(n(x)g)\psi(x)dx = h_\omega(g)\int_{F \backslash \mathbb{A}} \psi(x)dx$$

is identically zero since $\psi$ is non trivial. Hence using Rankin-Selberg unfolding method in the region $\Re(s) > 1/2$, we obtain

$$\int_{Z(\mathbb{A})G(F) \backslash G(\mathbb{A})} \varphi_1(g)h_\omega(g)E(g, \phi, s)dg = \int_{Z(\mathbb{A})B(F) \backslash G(\mathbb{A})} \varphi_1(g)h_\omega(g)\phi_s(g)dg$$

$$= \int_{Z(\mathbb{A})N(F) \backslash G(\mathbb{A})} W_{\varphi_1}(g)h_\omega(g)\phi_s(g)dg$$

$$= \int_{Z(\mathbb{A})N(\mathbb{A}) \backslash G(\mathbb{A})} W_{\varphi_1}(g)\phi_s(g)W_{h_\omega}(g)dg$$

$$= 0.$$ 

\[ \square \]

### 4.1. Evaluation of $\mathcal{M}_{\text{cusp}}$ and $\mathcal{M}_{\text{cont}}$

Fix $\pi$ a generic automorphic representation of finite conductor dividing $q$, i.e. $\pi$ is either cuspidal or Eisenstein $\omega \cdot |t| \oplus |t| \cdot |t|$ for some unramified finite order Hecke character $\omega$ and $t \in \mathbb{R}$. Let $\mathcal{B}(\pi, q)$ be the orthogonal basis of the space $\pi^{K_0(q)}$ constructed in (2.23). By construction and (2.18) each element $\varphi$ in this basis has a norm (cuspidal or Eisenstein) depending only on $\pi$ and the field $F$. Writing $N(\pi)$ for this norm, we have

$$N(\pi) = \begin{cases} \frac{2\Lambda_F(2)d_F^{1/2}\Lambda^*(\pi, \text{Ad}, 1)}{\Lambda_F^*(1)} & \text{if } \pi \text{ cuspidal} \noindent \frac{2d_F^{1/2}\Lambda^*(\pi, \text{Ad}, 1)}{\Lambda_F^*(1)} & \text{if } \pi \text{ Eisenstein.} \end{cases}$$

We now compute, using the self-adjointness of $T_p$, the fact that each $\varphi \in \mathcal{B}(\pi, q)$ is an eigenfunction of $T_p$ (c.f. Proposition 2.8) and Rankin-Selberg theory from Section 2.5 (recall the factorizability (2.21) and Definition (2.22))

$$\mathcal{L}(\pi, \pi_1, s) := \sum_{\varphi \in \mathcal{B}(\pi, q)} \langle \overline{\mathcal{E}}'(\cdot, s)\varphi_1, \varphi \rangle \langle T_p(\mathcal{E}'_{q, s})\varphi_1, \varphi \rangle = \lambda_{\pi}(p)|L(\pi \times \pi_1, s + \frac{1}{2})|^2 \mathcal{C}(\pi, \pi_1, q, s + \frac{1}{2})\Gamma(\pi_{\infty}, \pi_{1, \infty}, s + \frac{1}{2}),$$

where

$$\Gamma(\pi_{\infty}, \pi_{1, \infty}, s) = \sum_{W \in \mathcal{B}_W(\pi_{\infty})} |Z(\pi, W, \varphi_{1, \infty}, \Psi, s)|^2,$$

$$\mathcal{C}(\pi, \pi_1, q, s) = \Omega(\pi, \pi_1, q, s) \prod_{v < \infty} \frac{|Z(\pi, W_{\varphi, v}, \varphi_{1, v}, \Psi, s)|^2}{|L(\pi \times \pi_1, s)|^2},$$

and
and
\[ Q(\pi_q, \pi_{1,q}, s) = \sum_{W \in \mathcal{B}_W(\pi_q,q)} \frac{\mathcal{Z}(W, W^{\Psi}_{\phi_{1,q}}, (\Psi_q)_{q,s}) \mathcal{Z}(W, W^{\Psi}_{\phi_{1,q}}, (\Psi_q)^{\Psi}_{q,s})}{|L(\pi_q \times \pi_{1,q}, s)|^2}. \]

Here we recall that \( \mathcal{B}_W(\pi_q, q) \) is an orthonormal basis of the space of \( K_{q,0}(\mathbb{C}_q) \)-invariant vectors in the local Whittaker model \( \mathcal{W}(\pi_q) \), \( (\Psi_q)^{\Psi}_{q,s} \) is the characteristic function of the lattice \( (m_q, \sigma_F^{\Psi}_{q,s}) \) (resp. \( (m_q, \sigma_F) \)) and for \( v \neq \infty \) and finite, \( W_{\phi_v} \) is the unique \( G(a_F) \)-invariant vector having norm 1 w.r.t. \( 2.17 \). As a consequence of Proposition 2.6, we obtain
\[ \prod_{v < \infty, v \neq q} \frac{|\mathcal{Z}(W_{\phi_v}, W^{\Psi}_{\phi_{1,v}}, \Psi_v, s)|^2}{|L(\pi_v \times \pi_{1,v}, s)|^2} = 1. \]
Moreover, it is a general fact [Cog04, Theorem 3.1] that
\[ Q(\pi_q, \pi_{1,q}, s) \in \mathbb{C}[q^q, q^{-s}]. \]
This local factor will be computed explicitly in Section 5. Hence we obtain, for the cuspidal part
\[ M^{\text{cusp}}(\pi_1, p, q, s - \frac{1}{2}) = \sum_{\pi \text{ cuspidal }} \lambda_\pi(p) \frac{Q(\pi_q, \pi_{1,q}, s)}{N(\pi)} |L(\pi \times \pi_1, s)|^2 \Gamma(\pi_\infty, \pi_{1,\infty}, s). \]

For the continuous part, write \( \pi_{\omega}(it) \) for \( \omega : [it, i\omega_{\infty} - it] \) and \( \pi_{\omega_{\infty}}(it) \) for the local component of this representation at the place \( v \). Observe that we have the factorization
\[ L(\pi_{\omega}(it) \times \pi_1, s) = L(\pi_1 \times \omega, s + it)L(\pi_1 \otimes \overline{\omega}, s - it). \]

Then the Eisenstein part takes the form
\[ M^{\text{cont}}(\pi_1, p, q, s - \frac{1}{2}) = \frac{1}{4\pi} \sum_{\omega \text{ finite order \( \text{cond}(\omega) = p \)}} \int_{-\infty}^{\infty} \lambda_{\omega_\infty(it)}(p) \frac{Q(\pi_{\omega_\infty(it)}, \pi_{1,q}, s)}{N(\pi_{\omega_\infty(it)})} \times |L(\pi_1 \otimes \omega, s + it)|^2 |L(\pi_1 \otimes \overline{\omega}, s - it)|^2 \Gamma(\pi_{\omega_\infty(it)}, \pi_{1,\infty}, s) dt. \]

We define
\[ M(\pi_1, p, q, s) := 1 - p^{-2(\bar{\sigma}+1)} \left( M^{\text{cusp}}(\pi_1, p, q, s) + M^{\text{cont}}(\pi_1, p, q, s) \right), \]
\[ M(\pi_1, 1, q, s) := M^{\text{cusp}}(\pi_1, 1, q, s) + M^{\text{cont}}(\pi_1, 1, q, s), \]
and
\[ S(\pi_1, p, q, s) := M(\pi_1, p, q, s) - \lambda_{\pi_1}(p) p^{\bar{\sigma}+1} - p^{-\bar{\sigma}} \frac{p}{p^{\frac{1}{2}}(p+1)} M(\pi_1, 1, q, s). \]

Note that \( S(\pi_1, p, q, s) \) is holomorphic on \( \Re(s) = 0 \) and the evaluation at \( s = 0 \) involves the central values \( L(\pi \times \pi_1, \frac{1}{2}) \). Furthermore, since \( S(\pi_1, p, q, s) = \mathcal{M}(\pi_1, p, q, s) \) (c.f. eq. (3.13)), we get by Proposition 3.2

**Theorem 4.2.** Let \( \pi_1 \) be a cuspidal automorphic representation of \( \text{PGL}_2(\mathbb{A}) \) unramified at all finite places. Let \( p, q \) be two distinct prime ideals of \( \mathcal{O}_F \). Let \( S(\pi_1, p, q, s) \) be the twisted second moment defined in (4.3) and set \( S(\pi_1, p, q) := S(\pi_1, p, q, 0) \). Then we have
\[ (1-q)S(\pi_1, p, q) = (1-p)S(\pi_1, q, p). \]
5. Local Computations

Through this section, we fix \( k \) a non-Archimedean local field and we denote by \( \mathfrak{o} \) the ring of integers, \( \mathfrak{m} \) the maximal ideal, \( \varpi \) for a uniformizer, i.e. a generator of \( \mathfrak{m} \) and let \( q \) be the cardinality of the residue field \( \mathfrak{o}/\mathfrak{m} \). We also fix \( \psi \) a non-trivial unramified character of \( (k,+) \). Let \( \pi \) and \( \pi_1 \) be irreducible smooth admissible and infinite dimensional representations of \( G(k) \) with trivial central character. We assume that \( \pi_1 \) is unramified and that the conductor of \( \pi \) is either \( \mathfrak{o} \) or \( \mathfrak{m} \). In particular, both representations are self-dual, \( \pi_1 \) is a principal series and \( \pi \) is either principal or special depending on whether the conductor is \( \mathfrak{o} \) or \( \mathfrak{m} \) (see for example [Gel75, Remark 4.25]). Such representations appear as local factors of the generic automorphic representations considered in Sections 3 and 4. We denote by \( W \) (resp. \( W_1 \)) the new vector in the Whittaker model of \( \pi \) (resp. of \( \pi_1 \)) with respect to \( \psi \) (resp. of \( \overline{\psi} \)), normalized so that \( W(1) = W_1(1) = 1 \).

5.1. A Local Orthonormal Basis. We assume that \( \pi \) is unramified and we denote by \( \mathcal{W}(\pi,\psi) \) the whittaker model of \( \pi \) with respect to \( \psi \). The goal here is to determine an explicit orthonormal basis of \( \mathcal{W}(\pi,\psi)_{K_0(\varpi)} \) where we recall that

\[
K_0(\varpi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathfrak{o}) \mid c \in \mathfrak{m} \right\}.
\]

Let \( W^* := \pi(1/\varpi) \cdot W \) and note that \( W^* \) is invariant under \( K_0(\varpi) \), as well as \( W \). In fact, it is a famous result of Casselman (see [Cas73]) that the space \( \mathcal{W}(\pi,\psi)_{K_0(\varpi)} \) has dimension 2 and it is generated by \( W \) and \( W^* \). Unfortunately, these vectors are not orthogonal with respect to the inner product \( \vartheta \) defined in (2.17) but both have norm 1. In order to compute the inner product \( \vartheta(W, W^*) \), we first use the classical theory of GL\(_2\) automorphic \( L \)-functions [Gel75, Proposition 6.17]

\[
\int_{k^\times} W(a(y))|y|^{-1/2}d^\times y = L(\pi, s)
\]

\[
\int_{k^\times} W^*(a(y))|y|^{-1/2}d^\times y = q^{1/2}L(\pi, s).
\]

Let \( \alpha_\pi, \beta_\pi \in \mathbb{C} \) be such that \( L(\pi, s) = (1 - \alpha_\pi q^{-s})^{-1}(1 - \beta_\pi q^{-s})^{-1} \) (\( \alpha_\pi \beta_\pi = 1 \) because \( \pi \) is a principal series). Note that both functions in (5.1) are invariant by the maximal compact subgroup of the diagonal torus of GL\(_2\), so these relations determine completely the values of \( W \) and \( W^* \); we have (see [Ven10, (11.14)])

\[
W(a(\varpi^r)) = \begin{cases} q^{-r/2} \sum_{r_1 + r_2 = r} \alpha_\pi^{r_1} \beta_\pi^{r_2} & \text{if } r \geq 0 \\ 0 & \text{if } r < 0, \end{cases}
\]

\[
W^*(a(\varpi^r)) = \begin{cases} q^{-r+1} \sum_{r_1 + r_2 = r-1} \alpha_\pi^{r_1} \beta_\pi^{r_2} & \text{if } r \geq 1 \\ 0 & \text{if } r \leq 0. \end{cases}
\]

From this we can deduce that

\[
\vartheta(W, W^*) = \zeta_k(2) \frac{\lambda_\pi - \overline{\lambda_\pi} q^{-1}}{q^{1/2}} := \kappa(\pi), \quad \lambda_\pi := \alpha_\pi + \beta_\pi.
\]

Setting

\[
W^\psi := \pi(\pi)W - W^*,
\]
we get
\[ \vartheta(W, W) = 1 - |\kappa(\pi)|^2 =: \varsigma(\pi)^2 > 0. \]
Therefore, an orthonormal basis of \( \mathcal{W}(\pi, \psi)K_0(\pi) \) is given by
\[ \left\{ W, \varsigma(\pi)^{-1} W^b \right\}. \]

**Remark 5.1.** If \( \pi \) has conductor \( m \), then \( \mathcal{W}(\pi, \psi)K_0(\pi) \) has dimension 1, generated by \( W \). Since the local factor is in this case of the form \( L(\pi, s) = (1 - \gamma q^{-s})^{-1} \) for some \( \gamma \in \mathbb{C} \), one easily deduce (c.f. (5.12) below) that \( \vartheta(W, W) = \varsigma_k(2) \) and thus \( \left\{ \varsigma_k(2)^{-1/2} W \right\} \) is the correct orthonormal basis.

### 5.2. Evaluating the Zeta Integral

Let \( \varphi \in \pi, \varphi_1 \in \pi_1 \) be \( K_0(\pi) \)-invariant vectors and denote by \( W_\varphi \in \mathcal{W}(\pi, \psi) \) and \( W_{\varphi_1} \in \mathcal{W}(\pi_1, \psi) \) their associated Whittaker functions which transform by \( \psi \) and \( \overline{\psi} \) respectively under the left action of \( N(k) \). All this section is inspired by [Ven10, Section 11.3].

For \( \Phi \in S(k^2) \), we are interested in evaluating the following local integral (c.f. Section 2.5 and Proposition 2.6)
\[ \mathcal{Z}(W_\varphi, W_{\varphi_1}, \Phi, s) := \int_{N(k)\backslash G(k)} W_\varphi(g)W_{\varphi_1}(g)\Phi((0, 1)g)\det(g)^s dg, \]
depending on the nature of \( \varphi, \varphi_1 \) and \( \Phi \).

Using the invariance under the center \( Z(k) \) and Iwasawa decomposition, we have (recall the Haar measure (2.1))
\[ \mathcal{Z}(W_\varphi, W_{\varphi_1}, \Phi, s) = \int_{Z(k)N(k)\backslash G(k)} W_\varphi(g)W_{\varphi_1}(g)\det(g)^s \left( \int_{k^\times} \Phi((0, t)g)|t|^{2s} dt \right) dg = \int_{G(o)} \left( \int k^\times \Phi((0, t)k)|t|^{2s} d^s t \right) d^s ydk = \int_{G(o)} \mathcal{F}(k; W_\varphi, W_{\varphi_1}, \Phi, s) dk. \]

Note that the function \( k \mapsto \mathcal{F}(k; W_\varphi, W_{\varphi_1}, \Phi, s) \) is left invariant under the subgroups \( N(k) \cap G(o), A(k) \cap G(o) \) and \( Z(k) \cap G(o) \).

Let \( k_0, k_1, \ldots, k_q \) be representatives of the quotient \( G(o)/K_0(\pi) \) with \( k_0 \in K_0(\pi) \); the last integral can be cut into two pieces
\[ \int_{K_0(q)} \mathcal{F}(k; W_\varphi, W_{\varphi_1}, \Phi, s) dk + \sum_{i=1}^{q} \int_{K_0(q)} \mathcal{F}(k_i k; W_\varphi, W_{\varphi_1}, \Phi, s) dk. \]

**Lemma 5.2.** For all \( 1 \leq i \leq q \), we have
\[ \int_{K_0(q)} \mathcal{F}(k_i k; W_\varphi, W_{\varphi_1}, \Phi, s) dk = \int_{K_0} \mathcal{F}(wk; W_\varphi, W_{\varphi_1}, \Phi, s) dk. \]

**Proof.** Fix \( 1 \leq i \leq q \) and write \( k_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( c \in o^\times \). Since the function \( k \mapsto \mathcal{F}(k; W_\varphi, W_{\varphi_1}, \Phi, s) \) is invariant under \( Z(k) \cap G(o) \), we can assume that \( c = 1 \). Using the left invariance by \( N(k) \cap G(o) \), we see that we can replace \( k_i \) by \( \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \).

Since \( b - ad \in o^\times \), we can multiply on the left this matrix by \( \begin{pmatrix} (ad-b)^{-1} \\ a & b-
\end{pmatrix} \) \( A(k) \cap G(o) \), obtaining \( \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} \). Finally, we make the change of variables \( k \leftrightarrow \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \) in the \( k \)-integral, arriving at the desired result. \( \square \)
By Lemma 5.2, we obtain the following decomposition
\[
\mathcal{Z}(W_\varphi, W_{\varphi_1}, \Phi, s) = \int_{K_0(q)} \mathcal{F}(k; W_\varphi, W_{\varphi_1}, \Phi, s) dk + q \int_{K_0(q)} \mathcal{F}(wk; W_\varphi, W_{\varphi_1}, \Phi, s) dk.
\]

We now make the additional assumption that
\[
\Phi = \Psi_m := 1_{(m,a)} \text{ or } \Phi = \Psi_m := 1_{(m,0)},
\]
(c.f. Definition 2.3). In particular, the function \( k \mapsto \mathcal{F}(k; W_\varphi, W_{\varphi_1}, \Phi, s) \) is right \( K_0(\varpi) \)-invariant. Moreover, it is easy to compute for any \( k \in K_0(\varpi) \)
\[
(5.6) \quad \int_{k^\times} \Phi((0, t)k)|t|^{2s} d^\times t = \begin{cases} 
\zeta_k(2s) & \text{if } \Phi = \Psi_m \\
1 & \text{if } \Phi = \Psi_m,
\end{cases}
\]
and
\[
(5.7) \quad \int_{k^\times} \Phi((0, t)wk)|t|^{2s} d^\times t = \begin{cases} 
q^{-2s} \zeta_k(2s) & \text{if } \Phi = \Psi_m \\
0 & \text{if } \Phi = \Psi_m.
\end{cases}
\]

We conclude with:

**Proposition 5.3.** Assume that \( \varphi \) and \( \varphi_1 \) are right \( K_0(\varpi) \)-invariant.

1. If \( \Phi = \Psi_m \), we have
\[
\frac{\mathcal{Z}(W_\varphi, W_{\varphi_1}, \Phi, s)}{\zeta_k(2s)} = \frac{1}{q + 1} \int_{k^\times} W_\varphi(a(y)) W_{\varphi_1}(a(y)) |y|^{s-1} d^\times y + \frac{q^{1-2s}}{q + 1} \int_{k^\times} W_\varphi(a(y)w) W_{\varphi_1}(a(y)w) |y|^{s-1} d^\times y.
\]

2. If \( \Phi = \Psi_m \), we obtain
\[
\mathcal{Z}(W_\varphi, W_{\varphi_1}, \Phi, s) = \frac{1}{q + 1} \int_{k^\times} W_\varphi(a(y)) W_{\varphi_1}(a(y)) |y|^{s-1} d^\times y.
\]

The goal now is to connect the integral \( \mathcal{Z}(W_\varphi, W_{\varphi_1}, \Phi, s) \) with the local \( L \)-factor \( L(\pi \times \pi_1, s) \) for various choices of vectors \( \varphi \) and \( \varphi_1 \). In addition to relations (5.1) and (5.2), which are of course valid for \( \pi_1 \) since it is unramified, the functional equation for \( L(\pi_1, s) \) yields also (see [Gel75, Theorem 6.14] and [Ven10, (11.13)])
\[
(5.8) \quad \int_{k^\times} W_1^*(a(y)w) |y|^{s-1/2} d^\times y = q^{s-1/2} L(\pi_1, s),
\]
and thus
\[
(5.9) \quad W_1^*(a(\varpi^r)w) = \begin{cases} 
q^{-\frac{r+1}{2}} \sum_{r_1 + r_2 = r + 1} \alpha_{\pi_1}^{r_1} \beta_{\pi_2}^{r_2} & \text{if } r \geq -1 \\
0 & \text{if } r < -1,
\end{cases}
\]
where we recall that \( W_1^* = \pi_1(1_{\varpi}) \cdot W_1 \).
5.2.1. The case \( \text{cond}(\pi) = m \). We assume that \( \pi \) is ramified with conductor \( m \), so that \( \pi \) is a special representation (since the central character is trivial). Let \( \varepsilon \) be the local root number of \( \pi \) \( (\varepsilon \in \{-1, +1\}) \) because \( \pi \) is self-dual. By the properties of the new vector \( W \) and the functional equation, we have as in the previous section

\[
\int_{k^\times} W(a(y)) |y|^{s-1/2} d^\times y = L(\pi, s)
\]

(5.10)

\[
\int_{k^\times} W(a(y)w) |y|^{s-1/2} d^\times y = \varepsilon q^{s-1/2} L(\pi, s).
\]

If \( \gamma \in \mathbb{C} \) is such that \( L(\pi, s) = (1 - \gamma q^{-s})^{-1} \), we obtain as in (5.2)

\[
W(a(\varpi^r)) = \begin{cases} 
\gamma^r q^{-r/2} & \text{if } r \geq 0 \\
0 & \text{if } r < 0,
\end{cases}
\]

(5.11)

\[
W(a(\varpi^r)w) = \begin{cases} 
\varepsilon \gamma^{r+1} q^{r+1-2s} & \text{if } r \geq -1 \\
0 & \text{if } r < -1.
\end{cases}
\]

From this, we easily deduce

\[
\int_{k^\times} W(a(y)) W_1(a(y)) |y|^{s-1} d^\times y = \frac{1}{(1 - \gamma \alpha q^{-s})(1 - \gamma \beta q^{-s})}
\]

(5.12)

\[
\int_{k^\times} W(a(y)w) W_1(a(y)) |y|^{s-1} d^\times y = \frac{\varepsilon q^{-1/2}}{(1 - \gamma \alpha q^{-s})(1 - \gamma \beta q^{-s})},
\]

and also

\[
\int_{k^\times} W(a(y)) W^*_1(a(y)) |y|^{s-1} d^\times y = \frac{\gamma q^{1/2-s}}{(1 - \gamma \alpha q^{-s})(1 - \gamma \beta q^{-s})}
\]

(5.13)

\[
\int_{k^\times} W(a(y)w) W^*_1(a(y)w) |y|^{s-1} d^\times y = \frac{\varepsilon q^{s-1}}{(1 - \gamma \alpha q^{-s})(1 - \gamma \beta q^{-s})}.
\]

Using the fact that

\[
L(\pi \times \pi_1, s) = (1 - \gamma \alpha q^{-s})^{-1}(1 - \gamma \beta q^{-s})^{-1}
\]

(see [Jac72, Theorem 15.1]) and

\[
\gamma = -\varepsilon q^{-1/2}
\]

(see [JL70, Proposition 3.6]), we conclude by Proposition 5.3:

**Proposition 5.4.** Let \( \pi, \pi_1 \) be irreducible, smooth, admissible and infinite dimensional representations of \( G(k) \) with trivial central character. Assume that \( \pi_1 \) is unramified and that \( \pi \) has conductor \( m \) and local root number \( \varepsilon \in \{-1, +1\} \). Let \( W \) (resp. \( W_1 \)) be the new vector in the Whittaker model of \( \pi \) (resp. of \( \pi_1 \)) normalized so that \( W_1(1) = W(1) = 1 \). Then

\[
\frac{\mathcal{Z}(W; W', \Phi, s)}{L(\pi \times \pi_1, s)} = \begin{cases} 
\frac{1}{q+1} & \text{if } W' = W_1, \ \Phi = \Psi^m \\
-\frac{\varepsilon q^{-s}}{q+1} & \text{if } W' = W^*_1, \ \Phi = \Psi_m.
\end{cases}
\]
5.2.2. The case \( \text{cond}(\pi) = \varnothing \). We suppose that \( \pi \) is an unramified representation as in Section 5.1: so it is a principal series with \( L(\pi, s) = (1 - \alpha_\pi q^{-s})^{-1}(1 - \beta_\pi q^{-s})^{-1} \) and \( \alpha_\pi \beta_\pi = 1 \). Observe that we have in this case

\[
L(\pi \times \pi_1, s) = \frac{1}{(1 - \alpha_\pi \alpha_\pi q^{-s})(1 - \alpha_\pi \beta_\pi q^{-s})(1 - \beta_\pi \alpha_\pi q^{-s})(1 - \beta_\pi \beta_\pi q^{-s})}.
\]

For \( \tau \in \{ \pi, \pi_1 \} \), write \( \lambda_\tau = \alpha_\tau + \beta_\tau \). The same computations as in the previous section, but more tedious, leads to

\[
\int_{B^*} W'(a(y))W''(a(y))|y|^{s-1}d^s y \quad \frac{\zeta_k(2s)^{-1}}{\zeta_k(2s)} \quad \text{if} \quad W' = W, W'' = W_1
\]

\[
\int_{B^*} W'(a(y))W''(a(y))|y|^{s-1}d^s y \quad \frac{q^{1/2-s}(\lambda_\tau - \frac{\lambda_1}{\tau})}{q^{1/2-s}} \quad \text{if} \quad W' = W, W'' = W_1^*.
\]

We also have

\[
\int_{B^*} W'(a(y))W''(a(y))|y|^{s-1}d^s y \quad \frac{q^{1/2-s}(\lambda_\tau - \frac{\lambda_1}{\tau})}{q^{1/2-s}} \quad \text{if} \quad W' = W, W'' = W_1^*.
\]

Hence we conclude, together with Proposition 5.3

**Proposition 5.5.** Let \( \pi, \pi_1 \) be irreducible, smooth, admissible, infinite dimensional and unramified representations of \( G(k) \) with trivial central character. Let \( \pi \) (resp. \( W_1 \)) be the unique \( G(\varnothing) \)-invariant vector in the Whittaker model \( \mathcal{W}(\pi, \psi) \) (resp. in \( \mathcal{W}(\pi_1, \psi) \)) normalized by \( W(1 = W_1(1 = 1) = 1 \). For \( \tau \in \{ \pi, \pi_1 \} \), let \( \lambda_\tau := \alpha_\tau + \beta_\tau \). Then

\[
\frac{Z(W, W', \Phi, s)}{L(\pi \times \pi_1, s)} = \begin{cases} 
\frac{1+q^{1/2-s}}{q^{1/2}} & \text{if} \quad W' = W_1, \quad \Phi = \Psi^m \\
\frac{q^{1/2-s}(\lambda_\tau - \lambda_{\pi_1} q^{-s})}{q^{1/2}} & \text{if} \quad W' = W_1^*, \quad \Phi = \Psi^m,
\end{cases}
\]

and

\[
\frac{Z(W^*, W', \Phi, s)}{L(\pi \times \pi_1, s)} = \begin{cases} 
\frac{q^{1/2-s}(\lambda_{\pi_1})}{q^{1/2}} & \text{if} \quad W' = W_1, \quad \Phi = \Psi^m \\
\frac{q^{1/2-s}(\zeta_k(2s)^{-1})}{q^{1/2}} & \text{if} \quad W' = W_1^*, \quad \Phi = \Psi^m.
\end{cases}
\]

5.3. Computing the \( \Omega \)-Factor. Let \( \mathcal{B}(\pi, m) \) be an orthonormal basis of the space \( \mathcal{W}(\pi, \psi)^{K_0(\pi)} \) with respect to the invariant inner product \( (2.17) \). By Remark 5.1, \( \mathcal{B}(\pi, m) = \{ \zeta_k(2s)^{-1/2} \} \) if the conductor of \( \pi \) is \( m \) while for \( \pi \) unramified, \( \mathcal{B}(\pi, m) \) is given by \( (5.5) \).

According to the Definition (4.2), we define

\[
\Omega(\pi, \pi_1; m) := \sum_{f \in \mathcal{B}(\pi, m)} \frac{Z(f, W_1^*, \Phi_m, 1/2) Z(f, W_1, \Psi^m, 1/2)}{[L(\pi \times \pi_1, 1/2)]^2}.
\]

If \( \text{cond}(\pi) = m \), we use Proposition 5.4 and we get

\[
\Omega(\pi, \pi_1; m) = -\varepsilon q^{1/2}(q+1)^{-2} \zeta_k(2^{-1}).
\]

If the conductor of \( \pi \) is \( \varnothing \), we use Proposition 5.5 together with the notations from Section 5.1 to obtain

\[
\Omega(\pi, \pi_1; m) = \frac{1}{(q + 1)^2} \left\{ (\lambda_\pi - \lambda_{\pi_1} q^{-1/2}) \left( 2 |\kappa(\pi)|^{-2} - \pi(\pi) \lambda_{\pi_1} \right) + q^{1/2} \zeta_k(1)^{-1} (\lambda_\pi - 2 \kappa(\pi)) \right\}.
\]
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