Multivariate doubly truncated moments for generalized skew-elliptical distributions with application to multivariate tail conditional risk measures

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Abstract

In this paper, we focus on multivariate doubly truncated first two moments of generalized skew-elliptical (GSE) distributions and derive explicit expressions for them. It includes many useful distributions, for examples, generalized skew-normal (GSN), generalized skew-Laplace (GSLa), generalized skew-logistic (GSLo) and generalized skew student-t (GSSt) distributions, all as special cases. We also give formulas of multivariate doubly truncated expectation and covariance for GSE distributions. As applications, we show the results of multivariate tail conditional expectation (MTCE) and multivariate tail covariance (MTCov) for GSE distributions.

Keywords: Generalized skew-elliptical distribution; Multivariate doubly truncated moments; Multivariate doubly truncated expectation; Multivariate doubly truncated covariance

1. Introduction

Truncated moments’ expansions are applied to many areas including the design of experiment (see Thompson, 1976), robust estimation (see Cuesta-Albertos et al., 2008), outlier detections (see, e.g., Riani et al., 2009; Cerioli, 2010), robust regression (see Torti et al., 2012), robust detection (see Cerioli et al. (2014)), statistical estates’ estimation (see Shi et al., 2014), risk averse selection (see Hanasusanto et al., 2015), entropy computation and application (see, e.g., Milev et al., 2012; Zellinger and Moser, 2021). Therefore, the related research of truncated moment are developed to different distributions by many scholars.

Since Tallis (1961) has given an explicit formula for the first two moments of a lower truncated multivariate standard normal distribution by moment generating function, Amemiya (1974) and Lee (1979) used Tallis’ results (see Tallis, 1961) for extending Tobin’s model (see Tobin, 1958) to the multivariate regression and simultaneous equations models when the dependent variables are truncated normal. Lien (1985) provided the expressions for the moments of lower truncated bivariate log-normal distributions. Manjunath and Wilhelm (2012) computed the first and second moments for the rectangularly double truncated multivariate
normal density, and extended the derivation of Tallis (1961) to general \( \mu, \Sigma \) and for doubly truncation. Arismendi (2013) derived formulae for the higher order tail moments of the lower truncated multivariate standard normal, Student’s \( t \), lognormal and a finite-mixture of multivariate normal distributions. Arismendi and Broda (2017) deeply derived multivariate elliptical lower truncated moment generating function, and first second-order moments of quadratic forms of the multivariate normal, Student and generalized hyperbolic distributions. Moreover, Ho et al. (2012) presented general formulae for computing the first two moments of the truncated multivariate \( t \) distribution under the doubly truncation. Recently, Kan and Robotti (2017) provided expressions of the moments for folded and doubly truncated multivariate normal distribution. Galarza et al. (2021) and Morales et al. (2022) generalized moments of folded and doubly truncated to multivariate Student-\( t \) and extended skew-normal distributions, respectively. Also, Ogasawara (2021) derived a non-recursive formula for various moments of the multivariate normal distribution with sectional truncation, and introduced the importance of truncated moments in biological field, such as animals or plants breeding programs (see Herrendörfer and Tuchscherer, 1996) and medical treatments with risk variables as blood pressures and pulses, where low and high values of the variables are of primary concern. Galarza et al. (2022) further computed doubly truncated moments for the selection elliptical class of distributions, and established sufficient and necessary conditions for the existence of these truncated moments.

From a practical viewpoint, skewed distribution model is more used than non skewed distribution model because of data sets possessing large skewness and/or kurtosis measures (for instant, in economic and financial data sets). Based on this reason, Roozegar et al. (2020) derived explicit expressions of the first two moments for doubly truncated multivariate normal mean-variance mixture distributions.

Inspired by those works, we derived multivariate doubly truncated first two moments for GSE distributions, and provided expressions of multivariate doubly truncated expectation and covariance for those distribution. Some important cases of those distributions, including GSN, GSSt, GSLo and GSLa distributions, also were presented. As applications, formulas of MTCE and MTCov for GSE are given in Section 5. Numerical illustration is shown in Section 6. Finally, the paper closes with the
concluding remarks.

2. Preliminaries

We start introducing the definition of GSE distributions as follows.

2.1. Generalized skew-elliptical distributions

Let \( X \sim E_n(\mu, \Sigma, g_n) \) (if it exists) be an \( n \)-dimensional elliptical random vector with location vector \( \mu \), scale matrix \( \Sigma \) and density generator \( g_n(u), u \geq 0 \). Its probability density function (pdf) takes the form (see, for instance, Landsman and Valdez, 2003)

\[
fx(x) := \frac{c_n}{\sqrt{|\Sigma|}} g_n \left\{ \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}, \quad x \in \mathbb{R}^n,
\]

where

\[
c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty s^{n/2-1} g_n(s) ds \right]^{-1}
\]

is normalizing constant. The density generator \( g_n(u) \) satisfies

\[
\int_0^\infty s^{n/2-1} g_n(s) ds < \infty.
\]

We will call the random vector \( Y \sim GSE_n(\mu, \Sigma, g_n, H) \) an \( n \)-dimensional generalized skew-elliptical (GSE) random vector if its pdf (exists) takes the form (see McNeil et al., 2005; Adcock et al., 2019)

\[
f_Y(y) = \frac{2c_n}{\sqrt{|\Sigma|}} g_n \left\{ \frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu) \right\} H(\Sigma^{-1/2}(y - \mu)), \quad y \in \mathbb{R}^n,
\]

where \( H \) is the skewing function, and it satisfies \( H(-t) = 1 - H(t) \) and \( 0 \leq H(t) \leq 1 \) for \( t \in \mathbb{R}^n \). Note that \( H : \mathbb{R}^n \to \mathbb{R} \), we can also define skewing function \( J : \mathbb{R} \to \mathbb{R} \) through \( H(t) = J(\gamma^T t) \) for \( t \in \mathbb{R}^n \).

In Shushi (2016), it was proved that the characteristic function of any \( n \times 1 \) random vector \( Y \sim GSE_n(\mu, \Sigma, g_n, H) \) takes the following form

\[
c(t) = 2e^{it^T \mu} \psi_n \left( \frac{1}{2} \gamma^T \Sigma \gamma t \right) k_n(t), \quad t \in \mathbb{R}^n,
\]

where \( \psi_n(t) : [0, \infty) \to \mathbb{R} \) is the characteristic generator of elliptical random vector \( X \) (see Fang et al., 1990), \( k_n \) is a function, and it satisfies \( k_n(-t) = 1 - k_n(t) \).

We define cumulative generator \( G_n(u) \) and \( \overline{G}_n(u) \) as follows (see Landsman et al., 2018):

\[
\overline{G}_n(u) = \int_u^\infty g_n(v) dv
\]
and

\[ \overline{G}_n(u) = \int_u^\infty G_n(v) \, dv, \]

and their normalizing constants are, respectively, written as (see Zuo et al., 2021):

\[ c_n^* = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty s^{n/2-1} \overline{G}_n(s) \, ds \right]^{-1} \]

and

\[ c_n^{**} = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty s^{n/2-1} \overline{G}_n(s) \, ds \right]^{-1}. \]

For those density generators, it is necessary to meet the following conditions

\[ \int_0^\infty s^{n/2-1} \overline{G}_n(s) \, ds < \infty \]  \hfill (4)

and

\[ \int_0^\infty s^{n/2-1} \overline{G}_n(s) \, ds < \infty. \]  \hfill (5)

Now, we define elliptical random vectors \( X^* \sim E_n(\mu, \Sigma, \overline{G}_n) \) and \( X^{**} \sim E_n(\mu, \Sigma, \overline{G}_n) \). Their form of pdf (if them exist) are, respectively:

\[ f_{X^*}(x) = \frac{c_n^*}{\sqrt{|\Sigma|}} \overline{G}_n \left\{ \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}, \ x \in \mathbb{R}^n \]

\[ f_{X^{**}}(x) = \frac{c_n^{**}}{\sqrt{|\Sigma|}} \overline{G}_n \left\{ \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}, \ x \in \mathbb{R}^n. \]

So \( Y^* \sim GSE_n(\mu, \Sigma, \overline{G}_n, H) \) and \( Y^{**} \sim GSE_n(\mu, \Sigma, \overline{G}_n, H) \) are corresponding generalized skew-elliptical random vectors.

Next, we introduce some notations and present some concepts that will be used in our proposed theory.

2.2. Notation

Assume \( W = (W_1, W_2, \ldots, W_n)^T \in \mathbb{R}^n \) is an arbitrary random vector with probability density function \( f_W(w) \), then for any \( a = (a_1, a_2, \ldots, a_n)^T, b = (b_1, b_2, \ldots, b_n)^T \) and \( a < b \), i.e., \( a_k < b_k \), for \( k \in \{1, 2, \ldots, n\} \), we denote the doubly truncated random vector \( W|a < W \leq b \) by \( W_{(a,b)} \), and \( \Pr(a < W \leq b) = \Pr(a_1 < W_1 \leq b_1, a_2 < W_2 \leq b_2, \ldots, a_n < W_n \leq b_n) \) by \( F_W(a, b) \).

For any \( v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n \), writing

\[ v_{-k} = (v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n)^T, \ k \in \{1, 2, \ldots, n\} \]

\[ v_{-kl} = (v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_l, \cdots, v_n)^T, \quad k, l \in \{1, 2, \cdots, n\}. \]

And denoting
\[ W_{vk} = (W_1, W_2, \cdots, W_{k-1}, v_k, W_{k+1}, \cdots, W_n)^T, \quad k \in \{1, 2, \cdots, n\} \]
and
\[ W_{vk,v_l} = (W_1, \cdots, W_{k-1}, v_k, W_{k+1}, \cdots, W_{l-1}, v_l, W_{l+1}, \cdots, W_n)^T, \quad k, l \in \{1, 2, \cdots, n\}. \]

In addition,
\[ W_{-vk} = (W_1, W_2, \cdots, W_{k-1}, W_{k+1}, \cdots, W_n)^T \in \mathbb{R}^{n-1}, \quad k \in \{1, 2, \cdots, n\} \]
and
\[ W_{-vk,v_l} = (W_1, \cdots, W_{k-1}, W_{k+1}, \cdots, W_{l-1}, W_{l+1}, \cdots, W_n)^T \in \mathbb{R}^{n-2}, \quad k, l \in \{1, 2, \cdots, n\}. \]

To give expression of multivariate doubly truncated moment, we denote doubly truncated expectation
\[ E_{W}^{(t,v)}[h(W)] \] of \( n \)-dimensional random vector \( W \) with pdf \( f_W(w) \) as
\[ E_{W}^{(t,v)}[h(W)] = \int_{t}^{v} h(w)f_W(w)dw, \quad w, t, v \in \mathbb{R}^n, \]
where \( h : \mathbb{R}^n \to \mathbb{R} \).

**Remark 1.** When \( v \to +\infty \), the doubly truncated expectation will be tail expectation:
\[ E_{W}[h(W)] = \int_{t}^{+\infty} h(w)f_W(w)dw, \quad w, t \in \mathbb{R}^n, \]
which is defined in Zuo and Yin (2021b).

The following notation will be used throughout this paper: \( \Phi(\cdot), T(\cdot) \) and \( Lo(\cdot) \) denote the cumulative distribution function (cdf) of the univariate standard normal, student-\( t \) (with degrees of freedom \( m \)) and logistic distributions; \( \phi(\cdot), t(\cdot) \) and \( lo(\cdot) \) denote the pdf of the univariate standard normal, student-\( t \) (with degrees of freedom \( m \)) and logistic distributions, respectively.

3. **Multivariate doubly truncated moments**

Let \( Y \sim GSE_n (\mu, \Sigma, g_n, H) \) be a random vector with finite fixed vector \( \mu = (\mu_1, \cdots, \mu_n)^T \), positive defined fixed matrix \( \Sigma = (\sigma_{ij})_{i,j=1}^{n} \) and pdf \( f_Y(y) \). Let \( Z = \Sigma^{-\frac{1}{2}}(Y - \mu) \sim GSE_n (0, I_n, g_n, H) \). Writing \( \xi_v = (\xi_{v,1}, \xi_{v,2}, \cdots, \xi_{v,n})^T = \Sigma^{-\frac{1}{2}}(v - \mu), \quad v \in \{a, b\} \). Denoting
\[ M^*_{-\xi_v} = (M^*_1, \cdots, M^*_k, \cdots, M^*_n) \in \mathbb{R}^{n-1}, \]

\[ M^{*-\xi_k} = (M^{*+\xi_k}, \cdots, M_{k-1}^{*+\xi_k}, M_k^{*+\xi_k}, \cdots, M_n^{*+\xi_k})^T \in \mathbb{R}^{n-1} \]

and
\[ M^{*-\xi_k,\xi_l} = (M_1^{*+\xi_k,\xi_l}, \cdots, M_{k-1}^{*+\xi_k,\xi_l}, M_k^{*+\xi_k,\xi_l}, \cdots, M_n^{*+\xi_k,\xi_l})^T \in \mathbb{R}^{n-2}. \]

Now, we define \( f_{M^{*-\xi_k}}(w) \), \( f_{M^{*-\xi_k}}(v) \) and \( f_{M^{*-\xi_k,\xi_l}}(u) \), the pdf associated with elliptical random vectors \( M^{*-\xi_k} \), \( M^{*-\xi_k,\xi_l} \) and \( M^{*-\xi_k,\xi_l} \), respectively:

\[
f_{M^{*-\xi_k}}(w) = c_{n-1,\xi_k}^* \mathcal{G}_n \left\{ \frac{1}{2} w^T w + \frac{1}{2} \xi_k^2 \right\}, \quad w \in \mathbb{R}^{n-1},
\]

\[
f_{M^{*-\xi_k}}(v) = c_{n-1,\xi_k}^* \mathcal{G}_n \left\{ \frac{1}{2} v^T v + \frac{1}{2} \xi_k^2 \right\}, \quad v \in \mathbb{R}^{n-1},
\]

\[
f_{M^{*-\xi_k,\xi_l}}(u) = c_{n-2,\xi_k,\xi_l}^* \mathcal{G}_n \left\{ \frac{1}{2} u^T u + \frac{1}{2} \xi_k^2 + \frac{1}{2} \xi_l^2 \right\}, \quad u \in \mathbb{R}^{n-2},
\]

where \( s, t \in \{a, b\}, k, l \in \{1, 2, \ldots, n\} \). In addition, \( c_{n-1,\xi_k}^*, c_{n-1,\xi_k}^{**}, \) and \( c_{n-2,\xi_k,\xi_l}^{**} \) are corresponding normalizing constants of \( f_{M^{*-\xi_k}}(w) \), \( f_{M^{*-\xi_k}}(v) \) and \( f_{M^{*-\xi_k,\xi_l}}(u) \), which are written as:

\[
c_{n-1,\xi_k}^* = \frac{\Gamma((n-1)/2)}{(2\pi)^{(n-1)/2}} \int_0^\infty x^{(n-3)/2} \mathcal{G}_n \left( \frac{1}{2} \xi_k^2 + x \right) dx^{-1},
\]

\[
c_{n-1,\xi_k}^{**} = \frac{\Gamma((n-1)/2)}{(2\pi)^{(n-1)/2}} \int_0^\infty x^{(n-3)/2} \mathcal{G}_n \left( \frac{1}{2} \xi_k^2 + x \right) dx^{-1}
\]

and

\[
c_{n-2,\xi_k,\xi_l}^{**} = \frac{\Gamma((n-2)/2)}{(2\pi)^{(n-2)/2}} \int_0^\infty x^{(n-4)/2} \mathcal{G}_n \left( \frac{1}{2} \xi_k^2 + \frac{1}{2} \xi_l^2 + x \right) dx^{-1}.
\]

Next, we present explicit expressions of multivariate doubly truncated (first two) moments for generalized skew-elliptical distributions.

**Theorem 1.** Let \( Y \sim GSE_n(\mu, \Sigma, g_n, H) \) \((n \geq 2)\) be as in \( \mathcal{G} \). Suppose that it satisfies conditions \( \mathcal{G} \), \( \mathcal{H} \) and \( \mathcal{Q} \). Further, assume \( \partial_i H \) and \( \partial_{ij} H \) exist for \( i, j \in \{1, 2, \ldots, n\} \). Then

(i) \( E[Y|a < Y \leq b] = \mu + \frac{\Sigma \delta}{F_Z(\xi_a, \xi_b)} \).

(ii) \( E[YY^T|a < Y \leq b] = \mu \mu^T + \frac{\Sigma \delta \mu^T}{F_Z(\xi_a, \xi_b)} + \frac{\mu \delta^T \Sigma \delta}{F_Z(\xi_a, \xi_b)} + \frac{\Sigma \delta \Omega \Sigma \delta}{F_Z(\xi_a, \xi_b)} \).
where $\Omega = (\Omega_{ij})_{i,j=1}^n$ is an $n \times n$ symmetric matrix, and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$ is an $n \times 1$ vector. Here

$$\Omega_{ij} = 2\left(\frac{c_n}{c_n^{(n-1,\xi_{a_{ij}})}} \mathbb{E}[\xi_{a_{ij}} - \xi_{b_{ij}}] [H(M^{**}_{\xi_a}, \xi)] - \frac{c_n}{c_n^{(n-2,\xi_{a_{ij}})}} \mathbb{E}[\xi_{a_{ij}} - \xi_{b_{ij}}] [H(M^{**}_{\xi_a_{ij}}, \xi)] \right)$$

$$+ \frac{c_n}{c_n^{(n-1,\xi_{b_{ij}})}} \mathbb{E}[\xi_{b_{ij}} - \xi_{b_{ij}}] [\partial_j H(M^{**}_{\xi_b})]$$

and (11), respectively. In addition, the corresponding normalizing constants

$$M^{**}_{\xi_a}, \xi]$$

are same as in (6), (7) and (8), respectively. The corresponding normalizing constants $c_n^{(n-1,\xi_{a_{ij}})}$, $c_n^{(n-2,\xi_{a_{ij}})}$ and $c_n^{(n-2,\xi_{a_{ij}})}$ are as in (9), (10) and (11), respectively. In addition, $\partial_i H(z) = \frac{\partial H(z)}{\partial z_i}$ and $\partial_{ij} H(z) = \frac{\partial^2 H(z)}{\partial z_i \partial z_j}$.

**Proof.** (i) Using the transformation $Z = \Sigma^{-\frac{1}{2}}(Y - \mu)$ and basic algebraic calculations, we have

$$E[Y|a < Y < b] = E \left[ (\Sigma^{\frac{1}{2}}Z + \mu) | \xi_a < Z < \xi_b \right]$$

$$= \mu + \Sigma^{\frac{1}{2}}E[Z|\xi_a < Z < \xi_b],$$

where $\xi = (\xi_{v,1}, \xi_{v,2}, \ldots, \xi_{v,n})^T = \Sigma^{-\frac{1}{2}}(v - \mu)$, $v \in \{a, b\}$.

By definition of conditional expectation, we obtain

$$E[Z|\xi_a < Z < \xi_b] = \frac{1}{F_Z(\xi_a, \xi_b)} \int_{\xi_a}^{\xi_b} \frac{1}{z} \mathbb{E}[Z|Z < \xi_a]z^2 \mu_g(\frac{1}{2}z^T z) H(z)dz$$

$$= \frac{\delta}{F_Z(\xi_a, \xi_b)},$$

where $\mu_g$ is a probability density function.
where $\delta = (\delta_1, \delta_2, \cdots, \delta_n)^T$.

Note that, for $\Omega$

$$\int_{\xi_a}^{\xi_b} z_k 2c_n g_n \left( \frac{1}{2} z^T z \right) H(z) dz$$

$$= 2c_n \int_{\xi_a - k}^{\xi_b - k} \int_{\xi_a, k}^{\xi_b, k} - H(z) \partial_k \overline{G_n} \left( \frac{1}{2} z^T_k z_{-k} + \frac{1}{2} z_k^2 \right) dz_{-k}$$

$$= 2c_n \int_{\xi_a - k}^{\xi_b - k} \left\{ [H(z_{\xi_a, k})] \overline{G_n} \left( \frac{1}{2} z^T_k z_{-k} + \frac{1}{2} z^2_{a, k} \right) - [H(z_{\xi_b, k})] \overline{G_n} \left( \frac{1}{2} z^T_k z_{-k} + \frac{1}{2} z^2_{b, k} \right) \right\} dz_{-k}$$

$$+ 2c_n \int_{\xi_a}^{\xi_b} \partial_k H(z) \overline{G_n} \left( \frac{1}{2} z^T z \right) dz$$

$$= 2 \left\{ \frac{c_n}{e^{n - 1, \xi_{ak}}} \overline{F(M_\xi^*_{-\xi_{ak}})} [H(M_\xi^*_{\xi_a})] - \frac{c_n}{e^{n - 1, \xi_{bk}}} \overline{F(M_\xi^*_{-\xi_{bk}})} [H(M_\xi^*_{\xi_b})] + \frac{c_n}{e^{n - 1, \xi_{ak}}} \overline{F(M_\xi^*)} [\partial_k H(M^*)] \right\},$$

where $z_{\xi_{vk}} = (z_1, \cdots, z_{k-1}, \xi_{vk}, z_{k+1}, \cdots, z_n)^T$, $v \in \{a, b\}$, and we have used integration by parts in the third equality. Therefore, we get $12$, as required.

(ii) Similarly, using the transformation $Z = \Sigma^{\frac{1}{2}} (Y - \mu)$ and basic algebraic calculations, we have

$$E[YY^T | a < Y \leq b] = E \left[ (\Sigma^{\frac{1}{2}} Z + \mu) (\Sigma^{\frac{1}{2}} Z + \mu)^T | \xi_a < Z \leq \xi_b \right]$$

$$= \Sigma^{\frac{1}{2}} E[Z^T | \xi_a < Z \leq \xi_b] \Sigma^{\frac{1}{2}} + \Sigma^{\frac{1}{2}} E[Z | \xi_a < Z \leq \xi_b] \mu^T$$

$$+ \mu E[Z^T | \xi_a < Z \leq \xi_b] \Sigma^{\frac{1}{2}} + \mu \mu^T.$$
\[
2c_n \int_{\xi_{a,i}}^{\xi_{b,i}} \int_{\xi_{a,i}}^{\xi_{b,i}} \left[ -H(z_{\xi_{a,i}}) \partial_j \overline{G}_n \left( \frac{1}{2} z_{\xi_{a,i}}^T z_{\xi_{a,i}} + \frac{1}{2} \zeta_{a,i}^2 \right) + H(z_{\xi_{b,i}}) \partial_j \overline{G}_n \left( \frac{1}{2} z_{\xi_{b,i}}^T z_{\xi_{b,i}} + \frac{1}{2} \zeta_{b,i}^2 \right) \right] dz_{-ij}
\]
\[
+ 2c_n \int_{\xi_{a,-j}}^{\xi_{b,-j}} \int_{\xi_{a,-j}}^{\xi_{b,-j}} \partial_i H(z) \partial_j \overline{G}_n \left( \frac{1}{2} z_{\xi_{b,-j}}^T z_{\xi_{b,-j}} + \frac{1}{2} \zeta_{b,-j}^2 \right) dz_{-j}
\]
\[
= 2c_n \int_{\xi_{a,-j}}^{\xi_{b,-j}} \int_{\xi_{a,-j}}^{\xi_{b,-j}} \left[ H(z_{\xi_{a,-j}}) \overline{G}_n \left( \frac{1}{2} z_{\xi_{a,-j}}^T z_{-j} + \frac{1}{2} \zeta_{a,-j}^2 \right) \right. \]
\[
- H(z_{\xi_{b,-j}}) \overline{G}_n \left( \frac{1}{2} z_{\xi_{b,-j}}^T z_{-j} + \frac{1}{2} \zeta_{b,-j}^2 \right) - H(z_{\xi_{a,-j}}) \overline{G}_n \left( \frac{1}{2} z_{\xi_{a,-j}}^T z_{-j} + \frac{1}{2} \zeta_{a,-j}^2 \right) \]
\[
+ \partial_j H(z_{\xi_{a,-j}}) \overline{G}_n \left( \frac{1}{2} z_{\xi_{a,-j}}^T z_{-j} + \frac{1}{2} \zeta_{a,-j}^2 \right) dz_{-j}
\]\nand we used integration by parts in the third and fifth equalities.

Where

\[
z_{\xi_{a},\xi_{b}} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, \xi_{v,j}, z_{j+1}, \ldots, z_n)^T, \ v \in \{a, b\},
\]
where we have used integration by parts in the third and fourth equalities.

As for $E[Z|\xi_a < Z \leq \xi_b]$, using (i) we directly obtain

$E[Z|\xi_a < Z \leq \xi_b] = \frac{\delta}{F_Z(\xi_a, \xi_b)}$.

Consequently, we obtain (13), ending the proof of (ii).

Zuo and Yin (2021c) introduced multivariate doubly truncated expectation (MDTE) and covariance (MDTCov) for any an $n \times 1$ vector $X$ as follows, respectively:

$$MDTE_{(a,b)}(X) = E[X|a < X \leq b]$$

and

$$MDTCov_{(a,b)}(X) = E[(X - MDTE_{(a,b)}(X))(X - MDTE_{(a,b)}(X))^T|a < X \leq b].$$

Now, we can give following proposition of MDTE and MDTCov for GSE distributions.

**Proposition 1.** Under the conditions of Theorem 1, we have

(I) $MDTE_{(a,b)}(Y) = \mu + \frac{\Sigma \delta}{F_Z(\xi_a, \xi_b)}, \tag{17}$

(II) $MDTCov_{(a,b)}(Y) = \Sigma \left[ \frac{\Omega}{F_Z(\xi_a, \xi_b)} - \frac{\delta \delta^T}{F_Z(\xi_a, \xi_b)} \right] \Sigma^T, \tag{18}$

where $\Omega$ and $\delta$ are the same as those in Theorem 1.

**Proof.** (I) Using (i) of Theorem 1, we immediately obtain (17).

(II) By the definition of MDTCov, we have

$$MDTCov_{(a,b)}(Y) = E[(Y - MDTE_{(a,b)}(Y))(Y - MDTE_{(a,b)}(Y))^T|a < Y \leq b]$$

$$= E[YY^T|a < Y \leq b] - MDTE_{(a,b)}(Y)MDTE_{(a,b)}(Y)^T.$$
Applying (i) and (ii) of Theorem 1, and using basic algebraic calculations we instantly obtain (18).

**Remark 2.** When \( H(\cdot) = \frac{1}{2} \), we note that above results coincide with the results of Theorem 1 and Theorem 2 in Zuo and Yin (2021e).

**Remark 3.** When \( a \to -\infty \) and \( b \to +\infty \), MDTE and MDTCov are reduced to expectation and covariance, respectively.

4. Special cases

This section focuses \( E[Y|a < Y \leq b] \), \( E[YY^T|a < Y \leq b] \) and MDTCov\((a,b)(Y)\) for important cases of the GSE distributions, including GSN, GSSt, GSLo and GSLa distributions. Their forms of \( E[Y|a < Y \leq b] \), \( E[YY^T|a < Y \leq b] \) and MDTCov\((a,b)(Y)\) are given in (12), (13) and (13), respectively, so that we merely present \( \delta_k \), \( k \in \{1, 2, \ldots, n\} \), \( \Omega_{ij} \), \( i = j \) and \( i \neq j \).

**Corollary 1** (GSN distribution). Let \( Y \sim GSN_n(\mu, \Sigma, \gamma, J) \). In this case,

\[
\begin{align*}
\bar{G}_n(u) &= G_n(u) = g_n(u) = \exp(-u), \quad c_n^{**} = c_n = (2\pi)^{-\frac{d}{2}} \quad \text{and} \quad H \left( \Sigma^{-\frac{1}{2}}(y - \mu) \right) = J \left( \gamma^T \Sigma^{-\frac{1}{2}}(y - \mu) \right),
\end{align*}
\]

where \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T \in \mathbb{R}^n \) and \( J : \mathbb{R} \to \mathbb{R} \). Since

\[
\begin{align*}
&f_{M^{-1}\cdot\xi_k}(w) = c_{n-1,\xi_k} \exp \left\{ -\frac{1}{2} w^T w - \frac{1}{2} \xi_k^2 \right\} = \phi_{n-1}(w), \quad w \in \mathbb{R}^{n-1}, \\
&f_{M^{-1}\cdot\xi_k}(u) = c_{n-1,\xi_k} \exp \left\{ -\frac{1}{2} u^T u - \frac{1}{2} \xi_k^2 \right\} = \phi_{n-1}(u), \quad u \in \mathbb{R}^{n-1}, \\
&f_{M^{-1}\cdot\xi_k,\xi_l}(u) = c_{n-2,\xi_k,\xi_l} \exp \left\{ -\frac{1}{2} u^T u - \frac{1}{2} \xi_k^2 - \frac{1}{2} \xi_l^2 \right\} = \phi_{n-2}(u), \quad u \in \mathbb{R}^{n-2},
\end{align*}
\]

\( \phi_k(\cdot) \) denotes the pdf of \( N_k(0, \Sigma) \) (the \( k \)-variate normal distribution with mean vector \( 0 \) and covariance matrix \( \Sigma \)). So \( c_{n-1,\xi_k} = c_{n-1,\xi_k} = \frac{(2\pi)^{-\frac{d}{2}}}{\phi(\xi_k)} \) and \( c_{n-2,\xi_k,\xi_l} = \frac{(2\pi)^{-\frac{d}{2}}}{\phi(\xi_k) \phi(\xi_l)} \). Thus,

\[
\Omega_{ij} = 2 \left\{ \phi(\xi_{ai}) \phi(\xi_{aj}) \mathbb{E}_{M^{-1}\cdot\xi_k,\xi_l} \left\{ (J_2 - M_{\xi_{ai},\xi_{aj}})^T [J', (\gamma^T M_{\xi_{ai},\xi_{aj}})] \right\} \right. \\
+ \gamma_j \phi(\xi_{ai}) \mathbb{E}_{M^{-1}\cdot\xi_k,\xi_l} \left\{ J' (\gamma^T M_{\xi_{ai},\xi_{ai}}) \right\} \\
+ \phi(\xi_{bi}) \phi(\xi_{bj}) \mathbb{E}_{M^{-1}\cdot\xi_k,\xi_l} \left\{ (J_2 - M_{\xi_{bi},\xi_{bj}})^T [J', (\gamma^T M_{\xi_{bi},\xi_{bj}})] \right\} \right.
\]

\[
\left. \left. - \gamma_j \phi(\xi_{bi}) \mathbb{E}_{M^{-1}\cdot\xi_k,\xi_l} \left\{ J' (\gamma^T M_{\xi_{bi},\xi_{bi}}) \right\} \\
+ \gamma_i \phi(\xi_{aj}) \mathbb{E}_{M^{-1}\cdot\xi_k,\xi_l} \left\{ J' (\gamma^T M_{\xi_{aj},\xi_{aj}}) \right\} \right\}, \quad i \neq j,
\right.
\]

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Corollary 2 (GSSt distribution). Let \( Y \sim GSSt_n(\mu, \Sigma, m, \gamma, J) \). Here \( m > 0 \) is degrees of freedom, \( J : \mathbb{R} \to \mathbb{R} \) and \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T \in \mathbb{R}^n \). In this case,

\[
g_n(u) = \left(1 + \frac{2u}{m}\right)^{-(m+n)/2}, \quad c_n = \frac{\Gamma \left(\frac{m+n}{2}\right)}{\Gamma(m/2)(m/2)^{m/2}},
\]

and

\[
H\left(\Sigma^{-\frac{1}{2}}(y - \mu)\right) = J \left(\gamma^T \Sigma^{-\frac{1}{2}}(y - \mu)\right).
\]
So $\overline{G}_n(t)$ and $\overline{g}_n(t)$ can be expressed, respectively, as
\[
\overline{G}_n(t) = \frac{m}{m + n - 2} \left(1 + \frac{2t}{m}\right)^{-(m+n-2)/2}
\]
and
\[
\overline{g}_n(t) = \frac{m^2}{(m + n - 2)(m + n - 4)} \left(1 + \frac{2t}{m}\right)^{-(m+n-4)/2}.
\]
In addition,
\[
c_n^* = \frac{(m + n - 2)\Gamma(n/2)}{(2\pi)^{n/2} m} \left[ \int_0^{\infty} t^{n/2 - 1} \left(1 + \frac{2t}{m}\right)^{-(m+n-4)/2} \, dt \right]^{-1}
\]
\[
= \frac{(m + n - 2)\Gamma(n/2)}{(m\pi)^{n/2} m B\left(\frac{n}{2}, \frac{m-2}{2}\right)}, \quad \text{if } m > 2
\]
and
\[
c_n^{**} = \frac{(m + n - 2)(m + n - 4)\Gamma(n/2)}{(2\pi)^{n/2} m^2} \left[ \int_0^{\infty} t^{n/2 - 1} \left(1 + \frac{2t}{m}\right)^{-(m+n-4)/2} \, dt \right]^{-1}
\]
\[
= \frac{(m + n - 2)(m + n - 4)\Gamma(n/2)}{(m\pi)^{n/2} m B\left(\frac{n}{2}, \frac{m-2}{2}\right)}, \quad \text{if } m > 4,
\]
where $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ denote Gamma function and Beta function, respectively. Since
\[
f_{M^* - \xi_{sk}}(w) = \frac{c_{n-1, \xi_{sk}}}{m + n - 2} \left(1 + \frac{\xi_{sk}^2}{m}\right)^{-\frac{m+n-2}{2}} \left(1 + \frac{w^T \Delta_{\xi_{sk}}^{-1} w}{m - 1}\right)^{-\frac{m+n-2}{2}}
\]
\[
= \text{St}_{n-1}(0, \Delta_{\xi_{sk}}, m - 1), \quad w \in \mathbb{R}^{n-1},
\]
\[
f_{M^{**} - \xi_{sk}}(v) = \frac{c_{n, \xi_{sk}}}{m + n - 2} \left(1 + \frac{\xi_{sk}^2}{m}\right)^{-\frac{m+n-4}{2}} \left(1 + \frac{v^T \Lambda_{\xi_{sk}}^{-1} v}{m - 3}\right)^{-\frac{m+n-4}{2}}
\]
\[
= \text{St}_{n-1}(0, \Lambda_{\xi_{sk}}, m - 3), \quad v \in \mathbb{R}^{n-1}
\]
and
\[
f_{M^{**} - \xi_{sk}, \xi_{lT}}(u) = \frac{c_{n, \xi_{sk}, \xi_{lT}}}{m + n - 2} \left(1 + \frac{\xi_{sk}^2 + \xi_{lT}^2}{m}\right)^{-\frac{m+n-4}{2}} \left(1 + \frac{u^T \Theta_{\xi_{sk}, \xi_{lT}}^{-1} u}{m - 2}\right)^{-\frac{m+n-4}{2}}
\]
\[
= \text{St}_{n-2}(0, \Theta_{\xi_{sk}, \xi_{lT}}, m - 2), \quad u \in \mathbb{R}^{n-2},
\]
so that
\[
c_{n, \xi_{sk}} = \frac{\Gamma\left(\frac{m+n-2}{2}\right) \Gamma\left(\frac{m+n-4}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right) \Gamma\left(\frac{m-3}{2}\right)} \left(\frac{m+\xi_{sk}^2}{m}\right)^{\frac{m+n-2}{2}}
\]
\[
c_{n, \xi_{sk}, \xi_{lT}} = \frac{\Gamma\left(\frac{m+n-4}{2}\right)}{\Gamma\left(\frac{m-3}{2}\right)} \left(\frac{m+\xi_{sk}^2}{m}\right)^{\frac{m+n-4}{2}}
\]
\[ c_{n,\xi_k} = \frac{n+\xi_k}{\Gamma(n+2)\pi^{(n-1)/2}} \left( 1 + \frac{\xi_k^2}{m} \right)^{(m-n-2)/2} \]

Then

\[ \Omega_{ij} = 2\left\{ \frac{c_n}{c_{n-2,\xi_i,\xi_j}} \left[ \gamma^T \bar{M}_{\xi_i,\xi_j} \right] - \frac{c_n}{c_{n-2,\xi_i,\xi_j}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_j} \right] \right\} + \frac{c_n\gamma_j}{c_{n-1,\xi_k}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_j} \right] - \frac{c_n\gamma_i}{c_{n-1,\xi_k}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_j} \right] + \frac{m^2\gamma_i\gamma_j}{(m-2)(m-4)} \left[ \gamma^T \bar{M}_{\xi_k,\xi_j} \right], i \neq j, m > 4, \]

\[ \Omega_{ii} = 2\left\{ \frac{c_n\xi_{i,1}}{c_{n-1,\xi_i}} \left[ \gamma^T \bar{M}_{\xi_i,\xi_i} \right] - \frac{c_n\xi_{i,1}}{c_{n-1,\xi_i}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_i} \right] \right\} + \frac{c_n\gamma_i}{c_{n-1,\xi_k}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_i} \right] - \frac{c_n\gamma_i}{c_{n-1,\xi_k}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_i} \right] + \frac{m^2\gamma_i^2}{(m-2)(m-4)} \left[ \gamma^T \bar{M}_{\xi_k,\xi_i} \right] + \frac{m}{m-2} FZ(\xi_i,\xi_k), m > 4, \]

\[ \delta_k = 2\left\{ \frac{c_n\xi_{i,1}}{c_{n-1,\xi_k}} \left[ \gamma^T \bar{M}_{\xi_i,\xi_k} \right] - \frac{c_n\xi_{i,1}}{c_{n-1,\xi_k}} \left[ \gamma^T \bar{M}_{\xi_k,\xi_k} \right] \right\} + \frac{m^2\gamma_k}{m-2} \left[ \gamma^T \bar{M}_{\xi_k,\xi_k} \right], m > 2, \]

\[ i, j, k, l \in \{1, 2, \ldots, n\}, Z \sim GSST_{n_0}(0, I_n, m, \gamma, J), Z^* \sim GSST_{n_0}(0, I_n, \Gamma_n, \gamma, J), M^* - E_n(0, I_n, \bar{G}_n), M^{**} - E_n(0, I_n, \bar{G}_n), M^{**-\xi_k,\xi_l} - St_{n-2}(0, \Theta_{\xi_k,\xi_l}, m - 2), \]

\[ M^{**-\xi_k} - St_{n-1}(0, \Lambda_{\xi_k}, m - 3), M^{**-\xi_k,\xi_l} - St_{n-1}(0, \Delta_{\xi_k}, m - 1), \Delta_{\xi_k} = (m+\xi_k^2)I_{n-1}. \]

\[ \Lambda_{\xi_k} = \left( \frac{m^2+\xi_k^2}{m-3} \right) I_{n-1}, \Theta_{\xi_k,\xi_l} = \left( \frac{m^2+\xi_k^2+\xi_l^2}{m-2} \right) I_{n-2} \text{ and } M \sim St_{n}(0, I_n, m). \]

Therefore, \( c_{n}, c_{n-1,\xi_k}, c_{n-2,\xi_k,\xi_l} \) can be further simplified as

\[ \frac{c_n}{c_{n-1,\xi_k}} = \frac{\Gamma(m-1)}{\Gamma(m)\sqrt{\pi}m} \left( 1 + \frac{\xi_k^2}{m} \right)^{(m-n-2)/2} \]

\[ \frac{c_n}{c_{n-1,\xi_k}} = \frac{\Gamma(m)}{\Gamma(m+1)\sqrt{\pi}m} \left( m + \xi_k^2 \right)^{(m-n-4)/2} \]

and

\[ \frac{c_n}{c_{n-2,\xi_k,\xi_l}} = \frac{(m+n)m^2}{2(m+n-4)} \left( m + \xi_k^2 + \xi_l^2 \right)^{(m-n-4)/2}. \]
Example 2 (Skew student-t distribution) Letting $J(\cdot) = T(\cdot) \sim St_1(0, 1, m)$ in Corollary 2. Thus,

$$
\Omega_{ij} = 2\left\{ \frac{c_n}{c_{n-2, \xi_a, \xi_b}} \frac{1}{E_{M^{***}_{\xi_a, \xi_b}}[T(\gamma^T M^{***}_{\xi_a, \xi_b})]} - \frac{c_n}{c_{n-2, -\xi_a, -\xi_b}} \frac{1}{E_{M^{***}_{-\xi_a, -\xi_b}}[T(\gamma^T M^{***}_{-\xi_a, -\xi_b})]} \right\} + \frac{c_n}{c_{n-2, \xi_a, \xi_b}} \frac{1}{E_{M^{***}_{\xi_a, \xi_b}}[t(\gamma^T M^{***}_{\xi_a, \xi_b})]} \cdot \Omega_{ii} \left[ \left( 1 + \frac{(\gamma^T M^{***})^2}{m} \right)^{-1} \right] \gamma^T M^{**} t(\gamma^T M^{**}) \right\}, \quad i \neq j, \ m > 4,
$$

$$
\delta_k = 2\left\{ \frac{c_n}{c_{n-1, -\xi_a}} \frac{1}{E_{M^*_{-\xi_a}}[T(\gamma^T M^*_{-\xi_a})]} - \frac{c_n}{c_{n-1, \xi_a}} \frac{1}{E_{M^*_{\xi_a}}[T(\gamma^T M^*_{\xi_a})]} \right\} + \frac{c_n}{c_{n-1, -\xi_a}} \frac{1}{E_{M^*_{-\xi_a}}[t(\gamma^T M^*)]} \cdot \delta_{kk} \left[ \left( 1 + \frac{(\gamma^T M^*)^2}{m} \right)^{-1} \right] \gamma^T M^* t(\gamma^T M^*) \right\}, \quad m > 2,
$$

$i, j, k \in \{1, 2, \ldots, n\}, \ Z \sim SS_{n}(0, 1_n, m, \gamma), \ Z^* \sim GSE_n (0, 1_n, m, \overline{\gamma}, \gamma, T)$. In addition, $M^*$, $M^{***}$, $M^{**-\xi_a, \xi_b}, M^{**-\xi_a}, M^{**-\xi_b}$ and $M^{*-\xi_a}$ are given in Corollary 2.

Corollary 3 (GSLo distribution). Let $Y \sim GSLo_n(\mu, \Sigma, \gamma, J)$, where $J : \mathbb{R} \to \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T \in \mathbb{R}^n$. In this case,

$$
g_n(u) = \frac{\exp\{-u\}}{\left( 1 + \exp\{-u\} \right)^2}; \quad c_n = \frac{1}{(2\pi)^{n/2} \Psi_1^2(-1, -1, 1)}
$$

and $H \left( \Sigma^{-1/2}(y - \mu) \right) = J \left( \gamma^T \Sigma^{-1/2}(y - \mu) \right)$. $\gamma_n(t)$ and $\overline{\gamma}_n(t)$ are expressed as

$$
\gamma_n(t) = \frac{\exp(-t)}{1 + \exp(-t)}; \quad \overline{\gamma}_n(t) = \ln \left[ 1 + \exp(-t) \right].
$$

In addition,

$$
c_n = \frac{1}{(2\pi)^{n/2} \Psi_1^2(-1, -1, 1)}
$$
\[ c_n^{**} = \frac{1}{(2\pi)^{n/2} \Psi_1(-1, \frac{n}{2} + 1, 1)} \]

Since

\[ f_{M^+, \xi_k, \xi_k}(w) = c_{n-1, \xi_k} \exp \left( -\frac{1}{2} w^T \left( \begin{array}{c} \vec{c} \\ \vec{\xi} \end{array} \right) ^2 \right) \left[ 1 + \exp \left( -\frac{1}{2} \left( \begin{array}{c} \vec{u} \\ \vec{\xi} \end{array} \right)^T \vec{v} - \frac{1}{2} \vec{c}^2 \right) \right] , \quad w \in \mathbb{R}^{n-1}, \]

\[ f_{M^+, \xi_k, \xi_k}(v) = c_n^{**} \ln \left[ 1 + \exp \left( -\frac{1}{2} \left( \begin{array}{c} \vec{u} \\ \vec{\xi} \end{array} \right)^T \vec{v} - \frac{1}{2} \vec{c}^2 \right) \right] , \quad v \in \mathbb{R}^{n-1} \]

and

\[ f_{M^+, \xi_k, \xi_k, \xi_k}(u) = c_n^{**} \ln \left[ 1 + \exp \left( -\frac{1}{2} \left( \begin{array}{c} \vec{u} \\ \vec{\xi} \end{array} \right)^T \vec{v} - \frac{1}{2} \vec{c}^2 \right) \right] , \quad u \in \mathbb{R}^{n-2}, \]

so that

\[ c_{n-1, \xi_k} = \frac{\Gamma((n-1)/2) \exp(\frac{\xi^2_k}{2})}{(2\pi)^{(n-1)/2} \Psi_1} \left[ \int_0^\infty \frac{\Gamma((n-3)/2) \exp(-t)}{1 + \exp(-\frac{\xi^2_k}{2}) \exp(-t)} dt \right]^{-1} \]

\[ = \frac{\exp(\xi_k^2)}{(2\pi)^{(n-1)/2} \Psi_1 \left( \exp(-\frac{\xi_k^2}{2}), \frac{n-1}{2}, 1 \right)} \]

\[ c_n^{**} = \frac{\Gamma((n-1)/2)}{(2\pi)^{(n-1)/2}} \left\{ \int_0^\infty \int_0^\infty \frac{\Gamma((n-3)/2) \exp(\xi_k^2)}{1 + \exp(-\frac{\xi_k^2}{2}) \exp(-t)} dt \right\}^{-1}, \]

and

\[ c_{n-2, \xi_k, \xi_k} = \frac{\Gamma((n-2)/2)}{(2\pi)^{(n-2)/2}} \left\{ \int_0^\infty \int_0^\infty \frac{\Gamma((n-4)/2) \exp(\xi_k^2)}{1 + \exp(-\frac{\xi_k^2}{2}) \exp(-t)} dt \right\}^{-1}, \]

Then

\[ \Omega_{ij} = 2 \left\{ \begin{array}{l} -\frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] - \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
+ \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
+ \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
- \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
+ \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
+ \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
+ \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \\
+ \frac{c_i \gamma_i}{c_{n-1, \xi_i, \xi_i}} \mathbb{E}_{\xi_{M^*, \xi_{\xi_i}, \xi_i}} \left[ J(\gamma^T M^{**}_{\xi_{\xi_i}, \xi_i}) \right] \right\}, \quad i \neq j, \]
Therefore, 

\[
\Omega_{ii} = 2 \left\{ \frac{c_n}{c_{n-1, \xi_{ak}}} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J(\gamma^T M^*_\alpha)] - \frac{c_{n-1}}{c_{n-1, \xi_{ak}}} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J(\gamma^T M^*_\beta)] + \Psi_1^*(-1, n, 1) \gamma_{ab} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J'(\gamma^T M^*_\gamma)] \right\}
\]

\[
\delta_k = 2 \left\{ \frac{c_n}{c_{n-1, \xi_{ak}}} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J(\gamma^T M^*_\alpha)] - \frac{c_{n-1}}{c_{n-1, \xi_{ak}}} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J(\gamma^T M^*_\beta)] + \Psi_1^*(-1, n, 1) \gamma_{ab} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J'(\gamma^T M^*_\gamma)] \right\}
\]

i, j, k \in \{1, 2, \cdots, n\}, Z \sim GSLo_0(0, I_n, \gamma, J), Z^* \sim GSE_n(0, I_n, \Theta_n, \gamma, J), M^* \sim E_n(0, I_n, \Theta_n), \ M^{**} \sim E_n(0, I_n, \Theta_n), \ M \sim Lo_n(0, I_n). \ \Psi^*_\mu(z, s, a) \text{ denotes generalized Hurwitz-Lerch zeta function (see, for instance, Lin et al., 2006).}

Therefore,

\[
c_n = \frac{\Psi_1^* \left( -\exp \left( -\frac{\xi_{a,k}^2}{2} \right), \frac{n+1}{2}, 1 \right) \phi(\xi_{a,k})}{\Psi_2^*(-1, n, 1)}
\]

\[
c_n = \frac{\int_0^\infty t^{(n-3)/2} \ln \left[ 1 + \exp \left( -\frac{\xi_{a,k}^2}{2} \right) \exp(-t) \right] dt}{\Gamma((n-1)/2)\sqrt{2\pi}\Psi_2^*(-1, n, 1)}
\]

and

\[
c_n = \frac{\int_0^\infty t^{(n-4)/2} \ln \left[ 1 + \exp \left( -\frac{\xi_{a,k}^2}{2} - \frac{\xi_{a,k}^2}{t} \right) \exp(-t) \right] dt}{2\Gamma((n-1)/2)\sqrt{2\pi}\Psi_2^*(-1, n, 1)}
\]

Example 3 (Skew-logistic distribution) Letting \( J(\cdot) = Lo(\cdot) \) in Corollary 3. Thus,

\[
\Omega_{ij} = 2 \left\{ \frac{c_n}{c_{n-1, \xi_{ak}}} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [Lo(\gamma^T M^{**})] - \frac{c_{n-1}}{c_{n-1, \xi_{ak}}} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [Lo(\gamma^T M^{**})] + \Psi_1^*(-1, n, 1) \gamma_{ab} \mathbb{E}_{M_{\xi_{ak}}}^{\xi_{a}, \xi_{b}, \xi_{\gamma}} [J'\gamma^{(T M^{**})}] \right\}
\]

\[
\Psi_2^*(-1, n, 1) \mathbb{E}_{M^{**}}^\xi [J'^{\gamma(T M^{**})}] \left( \frac{2\phi(\gamma^T M^{**})}{(2\pi)^{-1/2} + \phi(\gamma^T M^{**})} - 1 \right) \}
\]

i \neq j,
\[ \Omega_{ii} = 2 \left[ \frac{c_n \xi_{a,i}}{c_{n-1,\xi_{a,i}}} \mathbb{E}_{M^a_{\xi_{a,i}}} \left[ \log(\gamma^T M_{\xi_{a,i}}^a) \right] - \frac{c_n \xi_{b,i}}{c_{n-1,\xi_{b,i}}} \mathbb{E}_{M^b_{\xi_{b,i}}} \left[ \log(\gamma^T M_{\xi_{b,i}}^b) \right] \right] \\
+ \frac{c_n \gamma_i}{c_{n-1,\xi_{a,i}}} \mathbb{E}_{M^a_{\xi_{a,i}}} \left[ \log(\gamma^T M_{\xi_{a,i}}^a) \right] - \frac{c_n \gamma_i}{c_{n-1,\xi_{b,i}}} \mathbb{E}_{M^b_{\xi_{b,i}}} \left[ \log(\gamma^T M_{\xi_{b,i}}^b) \right] \\
+ \frac{\Psi^*_1(-1, \xi, 1)}{\Psi^*_2(-1, \xi, 1)} \left[ \gamma^T M^a \log(\gamma^T M^a) \left( \frac{2\phi(\gamma^T M^a)}{(2\pi)^{1/2} + \phi(\gamma^T M^a)} - 1 \right) \right] \\
+ \frac{\Psi^*_1(-1, \xi, 1)}{\Psi^*_2(-1, \xi, 1)} F_{Z^*}(\xi_a, \xi_b), \right] \]

where \( i, j, k \in \{1, 2, \ldots, n\} \), \( Z \sim SL(a, 0, I_n, \gamma) \), \( Z^* \sim GSE(a, 0, I_n, \gamma, Lo) \). Moreover, \( M^a, M^b, M^{a* - \xi_a, \xi_b}, M^{b* - \xi_a, \xi_b} \) and \( M^{a* - \xi_a, \xi_b} \) are given in Corollary 3.

**Corollary 4 (GSLa distribution).** Let \( Y \sim GSLCa(\mu, \Sigma, \gamma, J) \), where \( J : \mathbb{R} \to \mathbb{R} \) and \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T \).

In this case, \( g_n(u) = \exp\{-\sqrt{2u}\} \), \( c_n = \frac{n(\pi/2)^{n/2}}{2\pi^{n/2}} \) and

\[ H \left( \Sigma^{-1} (y - \mu) \right) = J \left( \gamma^T \Sigma^{-1} (y - \mu) \right). \]

So

\[ G_n(t) = (1 + \sqrt{2t}) \exp(-\sqrt{2t}), \]

\[ G_n(t) = (3 + 2t + 3\sqrt{2t}) \exp(-\sqrt{2t}). \]

In addition,

\[ c_n^* = \frac{n(\pi/2)^{n/2}}{2\pi^{n/2}} \]

\[ c_n^{**} = \frac{n(n + 2)\Gamma(n/2)}{2\pi^{n/2} \Gamma(n + 4)} \]

Since

\[ f_{M^{a* - \xi_{a,k}}} (w) = c_n^{* -1, \xi_{a,k}} \left( 1 + \sqrt{w^T w + \xi_{a,k}^2} \right) \exp \left\{ -\sqrt{w^T w + \xi_{a,k}^2} \right\}, \ w \in \mathbb{R}^{n-1}, \]

\[ f_{M^{b* - \xi_{b,k}}} (v) = c_n^{* -1, \xi_{b,k}} \left( 1 + \frac{3}{\sqrt{2}} \right) \left( v^T v + \xi_{b,k}^2 \right) \exp \left\{ -\sqrt{v^T v + \xi_{b,k}^2} \right\}, \ v \in \mathbb{R}^{n-1} \]

and

\[ f_{M^{a* - \xi_{a,k, \xi_{a_1}}} (u) = c_n^{* -2, ij} \left( 1 + \frac{3}{\sqrt{2}} \right) \left( u^T u + \xi_{a,k}^2 + \xi_{a_1}^2 \right) \exp \left\{ -\sqrt{u^T u + \xi_{a,k}^2 + \xi_{a_1}^2} \right\}, \ u \in \mathbb{R}^{n-2}, \]

thus

\[ c_n^{* -1, \xi_{a,k}} = \frac{\Gamma \left( \frac{n-1}{2} \right)}{(2\pi)^{(n-1)/2}} \left\{ \int_0^\infty e^{-t} \left( 1 + \sqrt{2t + \xi_{a,k}^2} \right) \exp \left\{ -\sqrt{2t + \xi_{a,k}^2} \right\} dt \right\}^{-1}, \]
\[ c_{n-1, \xi_k}^{*} = \frac{\Gamma \left( \frac{n+1}{2} \right)}{(2\pi)^{(n-1)/2}} \left( \int_{0}^{\infty} \frac{t^{n-2}}{\sqrt{2}} \left( t + \frac{\xi_{s,k}}{2} \right)^{n-1} \exp \left( -\sqrt{2t + \xi_{s,k}^2} \right) dt \right)^{-1} \]

and

\[ c_{n-2, \xi_k, \xi_l}^{*} = \frac{\Gamma \left( \frac{n+2}{2} \right)}{(2\pi)^{(n-2)/2}} \left( \int_{0}^{\infty} \frac{t^{n-3}}{\sqrt{2}} \left( t + \frac{\xi_{s,k}}{2} + \frac{\xi_{s,l}}{2} \right)^{n-2} \exp \left( -\sqrt{2t + \xi_{s,k}^2 + \xi_{s,l}^2} \right) dt \right)^{-1}. \]

Then

\[ \Omega_{ij} = 2 \left\{ \frac{c_{n+1, \xi_i, \xi_j}}{c_{n+1, \xi_i, \xi_j}} E_{M^*}^{\xi_{i-1}, \xi_{j-1}} \left[ J(\gamma^T M_n^{*} \xi_{i}, \xi_{j}) \right] - \frac{c_{n+1, \xi_i, \xi_j}}{c_{n+1, \xi_i, \xi_j}} E_{M^*}^{\xi_{i-1}, \xi_{j-1}} \left[ J(\gamma^T M_n^{*} \xi_{i}, \xi_{j}) \right] \right\}, \quad i \neq j, \]

\[ \Omega_{ii} = 2 \left\{ \frac{c_{n+1, \xi_i, \xi_i}}{c_{n+1, \xi_i, \xi_i}} E_{M^*}^{\xi_{i-1}, \xi_{i-1}} \left[ J(\gamma^T M_n^{*} \xi_{i}, \xi_{i}) \right] - \frac{c_{n+1, \xi_i, \xi_i}}{c_{n+1, \xi_i, \xi_i}} E_{M^*}^{\xi_{i-1}, \xi_{i-1}} \left[ J(\gamma^T M_n^{*} \xi_{i}, \xi_{i}) \right] \right\} + (n+1) F^{*} \left( \xi_{i}, \xi_{i} \right), \]

\[ \delta_k = 2 \left\{ \frac{c_{n+1, \xi_i, \xi_i}}{c_{n+1, \xi_i, \xi_i}} E_{M^*}^{\xi_{i-1}, \xi_{i-1}} \left[ J(\gamma^T M_n^{*} \xi_{i}, \xi_{i}) \right] - \frac{c_{n+1, \xi_i, \xi_i}}{c_{n+1, \xi_i, \xi_i}} E_{M^*}^{\xi_{i-1}, \xi_{i-1}} \left[ J(\gamma^T M_n^{*} \xi_{i}, \xi_{i}) \right] \right\} + (n+1) F^{*} \left( \xi_{i}, \xi_{i} \right), \]

\[ i, j, k \in \{1, 2, \cdots, n\}, Z \sim GSLa_{00} \left( I_{n}, \gamma, J \right), \xi_{i} \sim \text{GSE}_{n} \left( I_{n}, \gamma, J \right), \text{M}^* \sim E_{n0} \left( I_{n}, \text{G}_{n} \right), \]
Example 4 (Skew-Laplace-normal distribution) Letting $J(\cdot) = \Phi(\cdot)$ in Corollary 4. Thus,

$$
\Omega_{ij} = 2 \left\{ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] \right. \\
+ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)]] \\
+ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)]] \\
- \frac{c_n\gamma_i}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n\gamma_i}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)]] \\
- (n+1)F_{Z_{ij}}(\xi, \xi) \right\}, \quad i \neq j,
$$

$$
\Omega_{ii} = 2 \left\{ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] \\
+ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)]] \\
+ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)]] \\
- (n+1)\gamma_i \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] \\
- (n+1)F_{Z_{ii}}(\xi, \xi, \xi) \right\}, \quad i \neq j,
$$

$$
\delta_k = 2 \left\{ \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] - \frac{c_n}{c^*_n} \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] \\
+ (n+1)\gamma_k \mathbb{E}_{M^*_{\xi, \xi}}[\Phi(\gamma^* M_{\xi, \xi} \xi)] \right\},
$$

where \( i, j, k \in \{1, 2, \ldots, n\} \), \( Z \sim SLan(0, I_n, \gamma) \), \( Z^* \sim GSE_n (0, I_n, \gamma, \Phi) \). In addition, \( M^* \), \( M^{**} \), \( M^{**+\xi_k, \xi+k} \), and \( M^{**-\xi_k, \xi-k} \) are given in Corollary 4.

5. Multivariate tail conditional risk measures for GSE distributions

In this section, we mainly consider multivariate tail expectation (MTCE) and multivariate tail covariance (MTCov) for GSE distributions.

Landsman et al. (2018) proposed a new multivariate tail conditional expectation (MTCE) for an \( n \times 1 \) vector of risks \( X = (X_1, X_2, \ldots, X_n)^T \) with cdf \( F_X(x) \):

$$
\text{MTCE}_q(X) = E[X | X > VaR_q(X)] \\
= E[X | X_1 > VaR_{q_1}(X_1), \ldots, X_n > VaR_{q_n}(X_n)],
$$

where \( q = (q_1, \ldots, q_n) \in (0, 1)^n \), \( VaR_q(X) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2), \ldots, VaR_{q_n}(X_n))^T \), and \( VaR_{q_k}(X_k) \), \( k \in \{1, 2, \ldots, n\} \), denotes the \( q_k \)-th quantile of \( X_k \). Specially, Landsman et al. (2016) is a special case as
\( q = (q, \cdots, q) \in (0, 1)^n \). Landsman et al. (2018) further proposed a novel form of multivariate tail covariance (MTCov):

\[
\text{MTCov}_q(X) = E \left[ (X - \text{MTE}_q(X))(X - \text{MTE}_q(X))^T \right] = \inf_{c \in \mathbb{R}^n} E \left[ (X - c)(X - c)^T \right] \geq \text{VaR}_q(X).
\]

From Proposition 1, we give following corollary.

**Corollary 5** Let \( Y \sim \text{GSE}_n(\mu, \Sigma, g_n, H) \) \((n \geq 2)\) be as in (3). Suppose that it satisfies conditions

\[
\lim_{z_k \to +\infty} z_k H(z) G_n \left( \frac{1}{2} z^T z \right) = 0, \quad k \in \{1, 2, \cdots, n\}
\]

and

\[
\lim_{z_k \to +\infty} H(z) G_n \left( \frac{1}{2} z^T z \right) = 0, \quad k \in \{1, 2, \cdots, n\}.
\]

Further, assume \( \partial_i H \) and \( \partial_{ij} H \) exist for \( i, j \in \{1, 2, \cdots, n\} \). Then

\[
(\overline{I}) \text{ MTE}_q(Y) = \mu + \Sigma^\delta \frac{\delta}{F_{\text{Z}}(\xi_q)},
\]

\[
(\overline{II}) \text{ MTDTE}_{(a,b)}(Y) = \Sigma^\delta \left[ \frac{\Omega}{F_{\text{Z}}(\xi_q)} - \frac{\delta \delta^T}{F_{\text{Z}}(\xi_q)} \right] \Sigma^\delta,
\]

where \( \Omega = (\Omega_{ij})_{i,j=1}^n \) is an \( n \times n \) symmetric matrix, and \( \delta = (\delta_1, \delta_2, \cdots, \delta_n)^T \) is an \( n \times 1 \) vector. Here

\[
\Omega_{ij} = \left\{ \frac{c_n}{c_{n-1, \xi_q}} \frac{q_{ij}}{q_{ij}^*} \right\} \left[ H(M_{\xi_q}^{s*}) + \frac{c_n}{c_{n-1, \xi_q}} E_{M_{\xi_q}^{s*}} \left[ \partial_i H(M_{\xi_q}^{s*}) \right] \right]
\]

\[
+ \left\{ \frac{c_n}{c_{n-1, \xi_q}} \frac{q_{ij}^*}{q_{ij}} \right\} \left[ \partial_{ij} H(M_{\xi_q}^{s*}) \right], \quad i \neq j,
\]

\[
\Omega_{ii} = \left\{ \frac{c_n}{c_{n-1, \xi_q}} \frac{q_{ii}}{q_{ii}^*} \right\} \left[ H(M_{\xi_q}^{s*}) + \frac{c_n}{c_{n-1, \xi_q}} E_{M_{\xi_q}^{s*}} \left[ \partial_i H(M_{\xi_q}^{s*}) \right] + \frac{c_n}{c_{n-1, \xi_q}} E_{M_{\xi_q}^{s*}} \left[ \partial_{ii} H(M_{\xi_q}^{s*}) \right] \right]
\]

\[
\delta_k = \left\{ \frac{c_n}{c_{n-1, \xi_q}} \frac{q_{kk}^*}{q_{kk}} \right\} \left[ H(M_{\xi_q}^{s*}) + \frac{c_n}{c_{n-1, \xi_q}} E_{M_{\xi_q}^{s*}} \left[ \partial_k H(M_{\xi_q}^{s*}) \right] \right],
\]

where \( q_{ij} = (\xi_{q1}, \xi_{q2}, \cdots, \xi_{qn})^T = \Sigma^{-\frac{1}{2}}(\text{VaR}_q(Y) - \mu) \), \( Z, M^*, Z^*, M^{s*}, M^{s* - \xi_k} \) and \( M^{s* - \xi_k} \) are given in Theorem 1. In addition, \( F_{\mathbf{W}}(\cdot) \) is tail function of \( \mathbf{W} \).

**Proof.** Letting \( a = \text{VaR}_q(Y) \) and \( b \to +\infty \) in Proposition 1, and combining with conditions (21) and (22), we can instantly obtain above results.

Note that the results \( \overline{I} \) and \( \overline{II} \) of Theorem 2 are coincide with Theorem 3.1 and Theorem 1 in Zuo and Yin (2020, 2021b), respectively.
6. Numerical illustration

In this section, we examine multivariate doubly truncated expectation, multivariate doubly truncated covariance and MTCE for skew-normal (SN) distributions.

Now, we compute multivariate doubly truncated expectation, multivariate doubly truncated covariance for SN distributions. Let \( \mathbf{a} = (2, 2)^T \), \( \mathbf{b} = (6, 7)^T \), \( \mathbf{P} = (P_1, P_2)^T \sim SN_2(\mu, \Sigma, \gamma) \) with parameters

\[
\mu = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.9 & 0.5 \\ 0.5 & 0.7 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

From Proposition 1 and Example 1, firstly, we give \( \delta \) and \( \Omega \):

\[
\delta = \begin{pmatrix} 0.4172 \\ 0.4875 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0.7886 & 0.0993 \\ 0.0993 & 0.6688 \end{pmatrix}.
\]

Then the multivariate doubly truncated expectation and multivariate doubly truncated covariance matrix of \( \mathbf{P} \) for \( \mathbf{a} \) and \( \mathbf{b} \) can be obtained as

\[
MDTE_{(a, b)}(\mathbf{P}) = \begin{pmatrix} 3.4459 \\ 4.6047 \end{pmatrix}, \quad MDTCov_{(a, b)}(\mathbf{P}) = \begin{pmatrix} 0.6008 & 0.2436 \\ 0.2436 & 0.2760 \end{pmatrix}.
\]

Next, we compute MTCE for skew-normal (SN) distributions. Let \( \mathbf{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5)^T \sim SN_5(\mu, \Sigma, \gamma) \) with parameters

\[
\mu = \begin{pmatrix} 0.4 \\ 0.1 \\ 1.4 \\ 1.1 \\ 1.0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.84 & -0.27 & 0.13 & -0.09 & -0.08 \\ -0.27 & 1.32 & 0.11 & -0.07 & -0.15 \\ 0.13 & 0.11 & 0.82 & 0.02 & -0.02 \\ -0.09 & -0.07 & 0.02 & 0.64 & 0.22 \\ -0.08 & -0.15 & -0.02 & 0.22 & 0.67 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ 0.2 \\ -0.3 \\ 0.1 \\ -0.2 \end{pmatrix}.
\]

From Corollary 5 and Example 1, VaRs and MTCEs of \( \mathbf{Q} \) for \( \mathbf{q}_1 \) and \( \mathbf{q}_2 \) are shown in Tables 1 and 2, respectively. As we see in Table 2 and Figure 1, there is a clear difference between the MTCE of \( \mathbf{q}_1 \) and \( \mathbf{q}_2 \).

| \( i \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) |
|-------|-------|-------|-------|-------|-------|
| VaR\( q_1(Q_i) \) | 1.6435 | 1.7330 | 2.3206 | 2.1851 | 1.9080 |
| VaR\( q_2(Q_i) \) | 1.6435 | 1.2370 | 1.6554 | 1.3691 | 0.8722 |

Table 1: The VaRs of \( \mathbf{Q} \) for \( \mathbf{q}_1 = (0.90, 0.90, 0.90, 0.90, 0.90)^T \) and \( \mathbf{q}_2 = (0.90, 0.80, 0.70, 0.60, 0.50)^T \).

| \( i \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) |
|-------|-------|-------|-------|-------|-------|
| MTCE\( q_1(Q_i) \) | 7.0066 | 10.6656 | -3.0526 | 5.1563 | -2.2325 |
| MTCE\( q_2(Q_i) \) | 3.8282 | 3.3809 | 3.0977 | 2.5537 | 1.6741 |

Table 2: The MTCEs of \( \mathbf{Q} \) for \( \mathbf{q}_1 = (0.90, 0.90, 0.90, 0.90, 0.90)^T \) and \( \mathbf{q}_2 = (0.90, 0.80, 0.70, 0.60, 0.50)^T \).

MTCE\( q_2(Q) \). The MTCE measures of Risks 1, 2 and 4 for \( \mathbf{q}_1 \) are greater than the MTCE measure of corresponding Risks for \( \mathbf{q}_2 \). However, Risks 3 and 5 are opposite.
7. Concluding remarks

In this paper, we have studied multivariate doubly truncated first two moments of GSE distributions, which provides further generalisation of the moments for doubly truncated multivariate normal mean-variance mixture distributions (Roozegar et al., 2020). Several important cases, for examples, GSN, GSSt, GSlO and GSLa distributions, are given. We also have presented multivariate doubly truncated expectation (MDTE) and covariance (MDTCov) for GSE distributions providing further generalisation of the MDTE and MDTCov for elliptical distributions, considered in Zuo and Yin (2021c). Note that when $a \to -\infty$, those doubly truncated moments will degenerate into lower truncated moments; When $b \to +\infty$, those doubly truncated moments will degenerate into upper truncated moments; When $a \to -\infty$ and $b \to +\infty$, those doubly truncated moments will degenerate into usual moments. As applications of our results, the MTCE and MTCov for generalized skew-elliptical distributions are given. Aim to examine established results, we have presented the numerical illustrations. Moreover, Ignatieva and Landsman (2021) computed tail conditional expectation (TCE) for generalised hyper-elliptical distributions. In Zuo and Yin (2021a), the authors derived formula of MTCE for location-scale mixture of elliptical distributions. It will, of course, be of interest to generalize the results (established here) to those mixture distributions and we hope to report the findings in a future paper.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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