Time-Efficient Algorithms for Nash-Bargaining-Based Matching Market Models

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October 28, 2024

Abstract

In the area of matching-based market design, existing models using cardinal utilities suffer from two deficiencies: First, the Hylland-Zeckhauser (HZ) mechanism [HZ79], which has remained a classic in economics for one-sided matching markets, is intractable; computation of even an approximate equilibrium is PPAD-complete [VY21, CCPY22]. Second, there is an extreme paucity of such models. This led [HV22] to define a rich collection of Nash-bargaining-based models for one-sided and two-sided matching markets, in both Fisher and Arrow-Debreu settings, together with very fast implementations using available solvers and very encouraging experimental results.

In this paper, we give fast algorithms with proven running times for the models of [HV22] using the techniques of multiplicative weights update (MWU) and conditional gradient descent (CGD). Additionally, we make the following contributions:

1. By [TV24], a linear one-sided Nash-bargaining-based matching market satisfies envy-freeness within factor two. We show that the other models satisfy approximate equal-share fairness, where the exact factor depends on the utility function being used in the particular model.
2. We define a Nash-bargaining-based model for non-bipartite matching markets and give fast algorithms for it using conditional gradient descent.

1 Introduction

General mechanisms for matching markets belong to two classes: whether they use ordinal or cardinal utility functions. Whereas the theory of the former is very well developed, both from an algorithmic and economic perspective, and several of its mechanisms have found exceedingly important applications (see Section 1.4.1), the latter has major deficiencies, which have restricted their applicability. This paper, which builds on [HV22], is an attempt at addressing these deficiencies.

The Hylland-Zeckhauser mechanism [HZ79] has remained a classic in economics for one-sided matching markets under cardinal utilities. It uses the power of a pricing mechanism to produce allocations that are Pareto optimal and envy-free [HZ79], and the mechanism is incentive compatible in the large [HMPY18]. However, the recent works [VY21, CCPY22] show that computation of even an approximate HZ equilibrium is PPAD-complete, thereby severely limiting its practical applicability.

More generally, the area of matching-based market design suffers from the following deficiency: There is an extreme paucity of models using cardinal utilities. This stands in sharp contrast with general equilibrium theory, which has defined and extensively studied several fundamental market models to address a number of specialized and realistic situations, e.g., see [MCWG+95]. If we were to draw a parallel with models in general equilibrium theory, HZ would correspond to the most elementary model, namely the

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1For a discussion of prominent solutions using cardinal and ordinal utilities for matching markets as well as a comparison of the two, see Section 1.4.1.
linear Fisher model. Hylland and Zeckhauser [HZ79] attempted an extension of their model to the Arrow-Debreu setting; however, they found instances that do not admit an equilibrium. Recently, [GTV24] found a way of salvaging this model by resorting to an $\epsilon$-approximate equilibrium. We are not aware of any other one-sided or two-sided matching market models using cardinal utilities.

The recent paper [HV22] addressed both these issues by defining a rich collection of Nash-bargaining-based models for one-sided and two-sided matching markets, in both Fisher and Arrow-Debreu settings. A solution to Nash bargaining [Nas53] satisfies Pareto optimality and symmetry, and since it maximizes the product of utilities of agents, its allocations satisfy proportional fairness. Additionally, since the solution is captured by a convex program, it is efficiently computable [GLS12, Vis21].

[HV22] gave implementations of several of their models using available solvers, and experimental results; see Section 1.4 for details. They also raised the question of finding efficient combinatorial algorithms with proven running times for these models. We address this question. Additionally, we make the following conceptual contributions to the proposal of [HV22] in order to set it on a more firm foundation:

1. We establish a connection between HZ and Nash-bargaining-based models via the celebrated Eisenberg-Gale convex program [EG59], thereby providing a theoretical ratification, see Section 1.1.
2. Whereas HZ satisfies envy-freeness, due to the presence of demand constraints, the Nash-bargaining-based models do not, and in its absence, Pareto optimality does not mean much. We rectify this to the extent possible by showing that these models satisfy an approximate notion of equal-share fairness, where the exact guarantee depends on the utility function being used in the particular model, see Lemmas 9, 11, 24, 27 and 29.
3. We define, for the first time, a model for non-bipartite matching markets under cardinal utilities; it is also Nash-bargaining-based.

We give combinatorial algorithms based on the techniques of multiplicative weights update (MWU) and conditional gradient descent (CGD) for several one-sided and two-sided models defined in [HV22]. We solve our model for non-bipartite matching markets using CGD. In every case, we study not only the Fisher but also the Arrow-Debreu version; the latter is also called the exchange version. Unlike the difficulty of generalizing HZ to the Arrow-Debreu setting, for Nash bargaining formulations this generalization is fairly easy, since the disagreement utility encodes, in a natural way, the utility of the initial endowment of each agent.

The seed idea for the Nash-bargaining-based models, proposed in [HV22], came from [Vaz12] which, for the linear case of the Arrow-Debreu market model, used Nash bargaining instead of pricing to arrive at an efficient algorithm. Since these models are inspired by models in general equilibrium theory, we borrow terminology from that theory to describe them. For each model, we give standard applications as well as the algorithmic technique(s) we use in this paper. As the models get more general, we will only state their additional features.

1. **One-sided linear Fisher (MWU and CGD):** The setup is analogous to that of HZ, and its standard application is matching agents to goods, with only agents having utilities for goods.
2. **One-sided linear Arrow-Debreu (MWU and CGD):** This is the exchange or Arrow-Debreu version of the previous model. Thus agents start with an initial endowment of goods and exchange them to improve their happiness.
3. **One-sided SPLC Fisher and Arrow-Debreu (MWU and CGD):** Economists model diminishing marginal utilities via concave utility functions. Since we are in a fixed-precision model of computation, we will consider separable, piecewise-linear concave (SPLC) utility functions, thus generalizing both previous models.
4. **Two-sided linear Fisher and Arrow-Debreu (CGD):** The standard application is matching workers with firms, with each side having utility functions over the other.
5. **Two-sided SPLC Fisher and Arrow-Debreu (CGD):** These are the SPLC generalizations of the previous models.

6. **Non-bipartite linear Fisher and Arrow-Debreu (CGD):** The standard application is the roommates problem, i.e., pairing up students for rooms in a dormitory. In the Arrow-Debreu version, the agents currently have rooms in the dormitory – their initial endowment – and wish to move to better rooms for the next academic year.

The recent computer science revolutions of the Internet and mobile computing led to the launching of highly impactful and innovative matching markets such as Adwords, Uber and Airbnb, and in turn led to a major revival of the area of matching markets, e.g., see [ftToC19] and [EIV23]. With that came the realization that the algorithmic aspects of this area are far from settled, e.g., see [Vaz22, GTV24] for basic algorithmic insights obtained recently. Our paper is motivated by this challenge and opportunity.

### 1.1 Connection between HZ and Nash-Bargaining-Based Models

In Definition 1, we state the linear Fisher problem (LFP). Theorem 2 proves that the pricing and Nash bargaining mechanisms coincide at this problem. In Definition 3, we add unit demand constraints to this problem to get a matching market, called LFUP. Observation 4 states that the two mechanisms diverge at this problem, with the pricing mechanism yielding HZ and the Nash bargaining mechanism yielding LiF; the latter is the most elementary Nash-bargaining-based matching market model defined in [HV22]. Hence there is a connection between HZ and the Nash-bargaining-based models.

Next, in Section 1.2, we characterize the optimal bundle of an agent in all three models, linear Fisher with unit money, HZ and LiF, thereby giving further relationships between these models. Finally, we give an informal reason for the intractability of HZ.

**Definition 1.** The **linear Fisher problem (LFP)** consists of a set $A = \{1, 2, \ldots, n\}$ of agents and a set $G = \{1, 2, \ldots, m\}$ of infinitely divisible goods, each of which is available to the extent of one unit. Each agent has a linear utility function over goods. If $x_{ij}$, for $1 \leq j \leq m$, represents a bundle of goods allocated to agent $i$, then the utility accrued by $i$ is $u_i(x) = \sum_{j \in G} u_{ij} x_{ij}$, where $u_{ij}$ is the utility accrued by $i$ for one unit of $j$. We will assume that the instance is non-degenerate in that each agent derives positive utility from some good and for each agent, there is a good which gives her positive utility; clearly, otherwise we may remove that agent or good from consideration.

The problem is to design a polynomial time mechanism for distributing the goods among the agents so that the allocation satisfies nice game-theoretic properties. In particular, it should satisfy Pareto optimality, an essential requirement in economics.

Let us consider two possible mechanisms for LFP. The first is to define a market model whose equilibrium allocation yields the desired solution. This is a special case of the linear Fisher market model in which each agent has 1 Dollar of money, see also Section 2. Prices $p$ for the goods form an *equilibrium* if they satisfy:

1. Each agent obtains a utility maximizing bundle of goods w.r.t. prices $p$.
2. The *market clears*, i.e., each agent fully spends her money and each good is fully sold.

Thus, an equilibrium involves creating parity between supply and demand, and such a mechanism is called a *pricing mechanism*. The allocations satisfy Pareto optimality and the mechanism is incentive compatible in the large. In addition, it also prevents artificial scarcity of goods and ensures that scarce goods go to agents who have the most need for them [MCWG+95]. Furthermore, this mechanism can be implemented in polynomial time, via a combinatorial algorithm [DPSV08], or by expressing it as a convex
program. The latter is the celebrated Eisenberg-Gale convex program [EG59], given in (1). Such a program can be solved to \( \epsilon \) precision in time that is polynomial in the size of the input and \( \log 1/\epsilon \) via ellipsoid-based methods [GS62, Vis21]. Additionally, since the Eisenberg-Gale program is a rational convex program, see [Vaz12] for definition, it can be solved exactly in polynomial time using ideas from [Jai07].

\[
\begin{align*}
\text{max} & \quad \sum_{i \in A} \log \left( \sum_{j \in G} u_{ij}x_{ij} \right) \\
\text{s.t.} & \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \quad (1a) \\
& \quad x \geq 0. \quad (1c)
\end{align*}
\]

Let \( p \) denote the dual variables corresponding to constraint (1b); these will be interpreted as prices of goods. The KKT conditions for (1) can be obtained from those for (4) given below by setting each \( q_i = 0 \) and therefore will not be repeated. Using these conditions, it is straightforward to prove that optimal primal and dual solutions, \( x \) and \( p \), form equilibrium allocations and prices, respectively, e.g. see [Vaz07]. In particular, the optimal bundle of agent \( i \) will be a solution to LP (2).

\[
\begin{align*}
\text{max} & \quad \sum_{j \in G} u_{ij}x_{ij} \\
\text{s.t.} & \quad \sum_{j} p_jx_{ij} \leq 1, \\
& \quad x_{ij} \geq 0 \quad \forall j \in G \\
\end{align*}
\]

The second mechanism for LFP is obtained by viewing it as a Nash bargaining problem; see Section 2 for details and for the properties possessed by the Nash bargaining solution. This entails defining a convex, compact set \( N \subseteq \mathbb{R}^n_+ \), called the feasible set, and a point \( c \in N \), called the disagreement point. As stated in Theorem 6, the Nash bargaining solution is given by the optimal solution to the convex program (3).

\[
\begin{align*}
\text{max} & \quad \sum_{i \in A} \log(v_i - c_i) \\
\text{s.t.} & \quad v \in N \quad (3)
\end{align*}
\]

In our case, \( c \) is the origin, i.e., \( c = 0 \), and \( N \) will consist of all possible vectors of utilities to the \( n \) agents that can be obtained by partitioning 1 unit each of all \( m \) goods among the agents. It is easy to see that the resulting convex program will be precisely (1). Therefore, the two mechanisms are identical. Hence we get:

**Theorem 2.** For the linear Fisher problem, the pricing mechanism and the Nash bargaining mechanism are identical.

**Definition 3.** The linear Fisher unit demand problem (LFUP) is the linear Fisher problem in which we require that the number of agents equals the number of goods, i.e., \( m = n \), and we impose the additional constraint that each agent should get a total of one unit of goods. As a result, every feasible allocation is a fractional perfect matching over the \( n \) agents and \( n \) goods.

The setup of LFUP is identical to that of HZ, see Section 2.1. Therefore, if we use the pricing mechanism for solving LFUP, we get HZ, provided we impose the constraint on optimal bundles stated in Section 2.1,
Figure 1: Figure illustrating Observation 4.

namely if there are several optimal bundles within budget, then the agent must choose the one that costs the least amount.

The second way of solving LFUP is using the Nash bargaining mechanism. For this we will first define the most basic Nash-bargaining-based matching market model, which is called LiF; the rest of the models are defined in Section 3.

The setup for the linear Fisher Nash bargaining one-sided matching market, abbreviated LiF, is the same as that for LFUP. Any feasible allocation \( x \) is a fractional perfect matching over agents and goods. Corresponding to each such \( x \), let vector \( v_x \) be the utilities accrued under the \( n \) agents under \( x \). Then, the feasible set \( \mathcal{N} \) consists of all such vectors \( v_x \). As before, the disagreement point is the origin. Therefore, the Nash bargaining solution for LiF is given by the convex program given in (4).

\[
\begin{align*}
\max & \quad \sum_{i \in A} \log \left( \sum_{j \in G} u_{ij} x_{ij} \right) \\
\text{s.t.} & \quad \sum_{i \in A} x_{ij} = 1 \quad \forall j \in G, \quad (4a) \\
& \quad \sum_{j \in G} x_{ij} = 1 \quad \forall i \in A, \quad (4b) \\
& \quad x \geq 0. \quad (4d)
\end{align*}
\]

Hence we get:

**Observation 4.** For the linear Fisher unit demand problem, the following holds: When solved via the pricing mechanism, it is identical to HZ and when solved via the Nash bargaining mechanism it is identical to LiF.

### 1.2 Characterizing Optimal Bundles in the Three Models

In this section, we will characterize the optimal bundle of an agent in all three models, linear Fisher market with unit budget, HZ and LiF, to further highlight the relationships between the models.

The optimal dual variables of the constraint in (2) can be interpreted as equilibrium prices \( p \) of goods under this linear Fisher market. Given prices \( p \), define the maximum bang-per-buck of \( i \) to be

\[
\gamma_i = \max_j \left\{ \frac{u_{ij}}{p_j} \right\}.
\]
Then agent $i$’s optimal bundle consists of goods costing 1 Dollar from the set

$$S_i = \arg \max_j \left\{ \frac{u_{ij}}{p_j} \right\}.$$  

The set $S_i$ constitutes $i$’s maximum bang-per-buck goods.

Next, we characterize optimal bundles for an agent under HZ. Taking $\mu_i$ and $\alpha_i$ to be the dual variables corresponding to the two constraints in LP (7), we get the following dual LP:

$$\begin{align*}
\min & \quad \alpha_i + \mu_i \\
\text{s.t.} & \quad \alpha_i p_j + \mu_i \geq u_{ij} \quad \forall i \in A, \forall j \in G, \\
& \quad \alpha_i \geq 0 \quad \forall i \in A \\
\end{align*}$$

Clearly $\mu_i$ is unconstrained; we will call it the utility-offset for $i$. By complementary slackness, if $x_{ij}$ is positive then $\alpha_i p_j = u_{ij} - \mu_i$. All goods $j$ satisfying this equality will be called optimal goods for agent $i$. The rest of the goods, called suboptimal, will satisfy $\alpha_i p_j > u_{ij} - \mu_i$. An optimal bundle for $i$ must contain only optimal goods.

The utility-offset plays a crucial role in ensuring that $i$’s optimal bundle satisfies both constraints of LP (7), i.e., size and cost. Assume that prices of goods have been fixed and we wish to compute $i$’s optimal bundle. Then, in general, different goods may better for maximizing utility, from the viewpoint of size and cost. Now, $\mu_i$ attains the right value so both types of goods become optimal for $i$ and if she buys them in an appropriate ratio, her size and cost constraints are both met; see [VY21] for illustrative examples.

Define the utility-adjusted bang-per-buck of good $j$ to be $\frac{(u_{ij} - \mu_i)}{p_j}$. Then the maximum utility-adjusted bang-per-buck of agent $i$ is:

$$\alpha_i = \max_j \left\{ \frac{u_{ij} - \mu_i}{p_j} \right\}.$$  

Then agent $i$’s optimal bundle consists of goods costing 1 Dollar from the set; it will automatically satisfy the size constraint of one unit of goods.

$$S_i = \arg \max_j \left\{ \frac{u_{ij} - \mu_i}{p_j} \right\}.$$  

For the model LiF, optimal bundles are given by optimal solutions to convex program (4). Using $p_j$ and $q_i$ as dual variables corresponding to constraints 4b and 4c, respectively, we get the KKT conditions for program (4) as follows.

1. $\forall j \in G: \quad p_j > 0 \implies \sum_{i \in A} x_{ij} = 1.$
2. $\forall i \in A: \quad q_i > 0 \implies \sum_{j \in G} x_{ij} = 1.$
3. $\forall i \in A, \forall j \in G: \quad p_j + q_i \geq \frac{u_{ij}}{u_i(x)}.$
4. $\forall i \in A, \forall j \in G: x_{ij} > 0 \implies p_j + q_i = \frac{u_{ij}}{u_i(x)}.$

Thus in the model LiF, in addition to each good $j$ having a price $p_j$, each agent $i$ has a price-offset $q_i$. The cost of one unit of $j$ for $i$ is defined to be $p_j + q_i$, and the cost of $i$’s bundle under allocation $x$ is defined to be $\sum_j (p_j + q_i) x_{ij}$. Define the price-adjusted bang-per-buck of $i$ for good $j$ to be $\frac{u_{ij}}{p_j+q_i}$.
Given prices $p$ and price-offset $q_i$, define the maximum price-adjusted bang-per-buck of $i$ to be

$$\gamma_i = \max_j \left\{ \frac{u_{ij}}{p_j + q_i} \right\}.$$ 

By KKT conditions (3) and (4), $i$’s optimal bundle will contain goods from the set $S_i$ given below; these constitute $i$’s maximum price-adjusted bang-per-buck goods.

$$S_i = \arg \max_j \left\{ \frac{u_{ij}}{p_j + q_i} \right\}.$$ 

Observe how the notion of maximum bang-per-buck differs in the three market models, despite having the same “form”.

Why is an HZ equilibrium difficult to compute? An informal reason is that there is no mathematical “object” that yields equilibrium prices for HZ. In contrast, for the linear Fisher model as well as LiF, the optimal duals to the convex programs capturing the primal solutions constitute such mathematical objects. Furthermore, under HZ, there is no mathematical “object” that yields the utility offsets, $\mu_i$. Another observation is that whereas the linear Fisher model as well as LiF satisfy weak gross substitutability, which leads to efficient computability, HZ does not; see [VY21] for an example.

### 1.3 Technical Contributions

In this section, we highlight the novel technical ideas of this paper. These fall into four categories.

1). Economic ratification of the new models: Section 1.1 gives a connection between HZ and the Nash-bargaining-based models. In Section 3 we show that Nash-bargaining-based models retain an approximate equal-share fairness (a weakened version of envy-freeness) in some cases, most notably under separable, concave utilities.

2). Modeling: Because of the difficulties encountered in generalizing HZ, mentioned in the Introduction, the economics literature has not defined a model of matching markets for general graphs, in particular, non-bipartite graphs, under cardinal utilities, despite the fact that it has natural applications. Our Nash-bargaining-based model for non-bipartite matching markets, given in Section 3, is the first of its kind.

3). Multiplicative Weights Update (MWU):

We provide a MWU algorithm which gives an $\epsilon$-approximate Nash bargaining solution\(^2\) in time $\frac{n^3 \log n}{\epsilon^2}$. The algorithm follows the primal-dual scheme and therefore yields both allocations and prices. Our techniques deviate from the literature due to additional constraints, namely matching, and the presence of endowments.

The main idea underlying our approach is to create a feasibility program, see (F-LiAD), the feasible solutions of which are exactly the optimal primal and dual values of the variables of Program (8). We then find an approximate feasible solution to this program using MWU, yielding an approximate Nash bargaining solution. Although other works in the past have also followed this approach of first finding a feasibility program, e.g., to solve linear programs and find equilibria in markets [AHK12], [FGK+08], in our case this step was much more challenging as expounded below.

In (F-LiAD), each agent aims to maximize her utility subject to budget constraints, i.e., for each agent $i$ the following holds: $\sum_j x_{ij}(p_j + q_i) \leq 1 + c_i \min_j \frac{u_{ij}}{p_j + q_i}$. Observe that both the left and the right hand sides of the inequality involve prices; in particular, in the case of Linear Fisher Program (1), the right hand side is

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\(^2\)For a formal definition, see Section 4.
just 1. To come up with the right budget constraints, we used the KKT conditions. The KKT conditions would yield that each agent $i$ maximizes her utility $u_i$ subject to budget constraints that would involve the utility $u_i$ itself. This self-reference had to be removed.

Note that the aforementioned challenge becomes even harder for the case of SPLC utilities where the correct constraints would involve prices in addition to allocations for all segments in the utility functions. The rest of the analysis of MWU algorithm follows a potential function argument adapted appropriately to our setting.

4). Conditional Gradient Descent (CGD):

First order methods, in particular projected gradient descent, have proven successful for solving large scale market equilibrium problems. The challenges to extending this approach to our models are two-fold. We get around these hurdles by resorting to a conditional gradient method instead.

a). The objective function of each of the convex programs we study is logarithmic. Therefore, its gradient is neither Lipschitz nor smooth. It must therefore be modified in order to guarantee these properties. [GK20] circumvent this problem for the Fisher market by replacing the objective by its quadratic extension below a certain point. This is obtained through the well-known equal-share fairness property, i.e. the fact that in an equilibrium we have $u_i \geq \frac{1}{n} \sum_{j \in G} u_{ij}$ for all agents $i$. Since our models are more involved, we need to extend this property by introducing approximations. We show that our models satisfy approximate notions of equal-share fairness, which allows us to bound the equilibrium solution away from the boundary. In particular, this holds even in the case of Arrow-Debreu extension of models, i.e., with initial endowments. We also show that these bounds are sharp up to a constant factor, which may be of independent interest.

b). In the Fisher setting, the efficiency of the projected gradient descent method relies on projections onto the simplex which are not time-consuming. On the other hand, in our models, the feasible region is generally given by matching or flow polytopes, for which projections are significantly more expensive. Therefore, we employ a conditional gradient method instead which relies on combinatorial matching and flow algorithms that are very efficient in practice. In the case of SPLC utilities we show how an additional “shifting step” can be used to decrease the dependence on the (rather large) diameter of the feasible region.

1.4 Related Results

The first comprehensive study of the computational complexity of the HZ scheme was undertaken in [VY21]. On the positive side, they gave a combinatorial, strongly polynomial time algorithm for the dichotomous case, i.e., 0/1 utilities, and extended it to the case that the utilities of each agent come from a set of cardinality two, i.e., a bivalued set; however, the set is allowed to be different for different agents. Clearly, for this case, the equilibrium is expressible using rational numbers. In contrast, they gave an example in which the utilities of agents come from a set of cardinality four for which an HZ equilibrium is not be expressible using rational numbers. Most importantly, they showed that the problem of approximately computing an equilibrium is in PPAD, though they left open the problem of showing PPAD-hardness. In particular, they stated that the problem should be PPAD-hard even if utilities of agents come from a set of cardinality three and that there should be an irrational example for this case as well. [CCPY22] established PPAD-hardness, even if utilities of agents come from a set of cardinality four.

This development opened up the question of finding tractable models of matching markets under cardinal utilities. [HV22] addressed this question as detailed in the Introduction. In particular, they gave very general models of matching markets for which the resulting convex program has linear constraints thereby ensuring zero duality gap and polynomial time solvability. However, as is well known, polynomial time solvability is often just the beginning of the process of obtaining an “industrial grade” implementation. Towards this end, [HV22] gave very fast implementations, using available solvers, and left the open prob-
lem of giving implementations with proven running times. The implementation given in [HV22] can solve
very large instances, with \( n = 2000 \), in one hour even for a two-sided matching market.

To deal with the fact that the Arrow-Debreu extension of HZ does not always admit an equilibrium,
[EMZ21] defined the notion of an \( \alpha \)-slack Walrasian equilibrium. This is a hybrid between the Fisher and
Arrow-Debreu settings. Agents have initial endowments of goods and for a fixed \( \alpha \in (0, 1] \), the budget of
each agent, for given prices of goods, is \( \alpha + (1 - \alpha) \cdot m \), where \( m \) is the value for her initial endowment;
the agent spends this budget to obtain an optimal bundle of goods. Via a non-trivial proof, they showed
that for \( \alpha > 0 \), an \( \alpha \)-slack Walrasian equilibrium always exists.

[GTV24] gave the notion of an \( \epsilon \)-approximate Arrow-Debreu HZ equilibrium and using the result stated
above on \( \alpha \)-slack Walrasian equilibrium, they showed that such an equilibrium always exists. They also
gave a polynomial time algorithm for computing such an equilibrium for the dichotomous and bivalued
utility functions.

An interesting recent paper [ACGH20] defines the notion of a random partial improvement mechanism
for a one-sided matching market. This mechanism truthfully elicits the cardinal preferences of the agents
and outputs a distribution over matchings that approximates every agent’s utility in the Nash bargaining
solution.

In recent years, several researchers have proposed Hylland-Zeckhauser-type mechanisms for a number
of applications, e.g., see [Bud11, HMPY18, Le17, McL18]. The basic scheme has also been generalized in
several different directions, including two-sided matching markets, adding quantitative constraints, and
to the setting in which agents have initial endowments of goods instead of money, see [EMZ21, EMZ19].

A large number of algorithms have been developed for computing a Fisher market equilibrium. Notable
examples are the DPSV algorithm [DPSV08], which is combinatorial, and an algorithm based on interior
point method [Ye08], which converges in time \( O(\text{poly}(n) \log(1/\epsilon)) \).

**Related works using Multiplicative Weights Update (MWU).** The Multiplicative Weights Update (MWU)
is a ubiquitous meta-algorithm with numerous applications in different fields [AHK12]. More specifically,
it has been used in max-flow problems [CKM+11], discrepancy minimization [LRR17], learning graphical
models [KM17], even in evolution [CLPV14, MPP15]. It is particularly useful in algorithmic game theory
due to its regret-minimizing properties [FL98, CBL06], i.e., the time average behavior of MWU leads to
(approximate) coarse correlated equilibria (CCE).

MWU has also been used to compute market equilibria in the Linear Fisher model (1); most notably is the
work in [FGK+08]. They give a simple and decentralized algorithm based on the multiplicative weights
update method, though its running time is \( O(\text{poly}(n)/\epsilon^2) \). Due to the general nature of multiplicative
weights, their algorithm extends to much more general classes of utilities including some that do not
satisfy weak gross substitutability. We will use similar techniques, since MWU can be adapted to handle
the matching constraints in our models. However, a non-trivial extension of the algorithm is required in
order to deal with initial endowments.

**Related works using Gradient Descent (GD).** Arguably one of the most commonly used first-order
methods for minimizing differentiable objectives, which has become very popular in optimization, ma-
chine learning and computer science communities is Gradient Descent. The main reason behind this fact
lies in GD’s simplicity and nice properties. For convex objectives, one can show that as long as the func-
tion is Lipschitz, \( O(1/\epsilon^2) \) steps suffice to get an \( \epsilon \)-approximate solution. Moreover, if the function has
Lipschitz gradient\(^3\) then \( O(1/\epsilon) \) steps suffice and finally if the function is strongly-convex, one can get an
\( \epsilon \)-approximate optimum in \( O(\log(1/\epsilon)) \) iterations, see [Bub15] for more information.

Most recently, GD and its stochastic counterpart have been extensively studied and used for optimiz-
ing non-convex landscapes with the guarantee of convergence almost always to local optima [GHJY15,
LPP+19, JNG+21]. These results shed light on why GD works well in practice.

\(^3\)Which we enforce in our case.
GD has also found numerous applications for computing market equilibria, most notably the recent work of [GK20], see references therein. [GK20] studied first-order methods for the Fisher market by considering gradient ascent type algorithms for the various convex programming formulations. They show that one may exploit the fairness of the Fisher market in order to bound the market equilibrium away from the boundary of the feasible region. Using this they develop a projected gradient ascent algorithm that also converges in \(O(poly(n) \log(1/\epsilon))\) time and requires only simplex projections. However, the efficiency of this approach depends quite critically on the simple structure of the convex programs for Fisher markets. A more general algorithm of this form would depend on geometric properties of the feasible region and use projections which are expensive. For these reasons we resort to a conditional gradient ascent algorithm instead.

### 1.4.1 Matching Markets under Ordinal vs Cardinal Utilities

Under ordinal utilities, the agents provide a total preference order over the goods and under cardinal utilities, they provide a non-negative real-valued function. Both forms have their own pros and cons and neither dominates the other. Whereas the former is easier to elicit from agents, the latter is far more expressive, enabling an agent to not only report if she prefers good \(A\) to good \(B\) but also by how much. [ACY15] exploit this greater expressivity of cardinal utilities to give mechanisms for school choice which are superior to ordinal-utility-based mechanisms.

**Example 5.** taken from [GTV24], provides a very vivid illustration of the advantage of cardinal utilities over ordinal ones in one-sided matching markets.

**Example 5.** The following example illustrates the advantage of cardinal vs ordinal utilities. The instance has three types of goods, \(T_1, T_2, T_3\), and these goods are present in the proportion of \((1\%, 97\%, 2\%)\). Based on their utility functions, the agents are partitioned into two sets \(A_1\) and \(A_2\), where \(A_1\) constitute 1\% of the agents and \(A_2\) 99\% of the agents. The utility functions of agents in \(A_1\) and \(A_2\) for the three types of goods are \((1, \epsilon, 0)\) and \((1, 1-\epsilon, 0)\), respectively, for a small number \(\epsilon > 0\). The main point is that whereas agents in \(A_2\) marginally prefer \(T_1\) to \(T_2\), those in \(A_1\) overwhelmingly prefer \(T_1\) to \(T_2\).

Clearly, the ordinal utilities of all agents in \(A_1 \cup A_2\) are the same. Therefore, a mechanism based on such utilities will not be able to make a distinction between the two types of agents. On the other hand, the HZ mechanism, which uses cardinal utilities, will fix the price of goods in \(T_3\) to be zero and those in \(T_1\) and \(T_2\) appropriately so that by-and-large the bundles of \(A_1\) and \(A_2\) consist of goods from \(T_1\) and \(T_2\), respectively.

The space of general\(^4\) mechanisms for one-sided matching markets can be classified according to two criteria: whether they use cardinal or ordinal utility functions, and whether they are in the Fisher or Arrow-Debreu\(^5\) setting. The HZ scheme is (cardinal, Fisher).

Prominent mechanisms for one-sided matching markets that use ordinal utilities are Random Priority [Mou18], which is strategyproof though not efficient or envy-free; Probabilistic Serial [BM01] which is efficient and envy-free but not strategyproof; and Top Trading Cycles which [SS74] is efficient, strategyproof and core-stable.

The HZ scheme is (cardinal, Fisher). Prominent mechanisms for the remaining three possibilities are covered as follows: (ordinal, Fisher) by Probabilistic Serial [BM01] and Random Priority [Mou18]; (cardinal, Arrow-Debreu) by \(\epsilon\)-Approximate Arrow-Debreu HZ [GTV24]; and (ordinal, Arrow-Debreu) by Top Trading Cycles [SS74]. The properties of these mechanisms are: Random Priority [Mou18] is strategyproof though not efficient or envy-free; Probabilistic Serial is efficient and envy-free but not strategyproof; and Top Trading Cycles is efficient, strategyproof and core-stable.

\(^{4}\)As opposed to mechanisms for specific one-sided matching markets.

\(^{5}\)This is also called the Walrasian or exchange setting.
For two-sided markets, the Gale-Shapley Deferred-Acceptance Algorithm which finds a stable matching [GS62] is (ordinal, Fisher). The possibilities (cardinal, Fisher) and (cardinal, Arrow-Debreu) are covered for the first time in this paper and in [HV22] via Nash-bargaining-based models, see Section 3.

2 Basic Solution Concepts

**Nash Bargaining Game:** An \( n \)-person Nash bargaining game consists of a pair \((\mathcal{N}, c)\), where \(\mathcal{N} \subseteq \mathbb{R}_+^n\) is a compact, convex set and \(c \in \mathcal{N}\). The set \(\mathcal{N}\) is called the feasible set – its elements are vectors whose components are utilities that the \(n\) players can simultaneously accrue. Point \(c\) is the disagreement point – its components are utilities which the \(n\) players accrue if they decide not to participate in the proposed solution.

The set of \(n\) agents will be denoted by \(A\) and the agents will be numbered 1, 2, \ldots \(n\). Instance \((\mathcal{N}, c)\) is said to be feasible if there is a point in \(\mathcal{N}\) at which each agent does strictly better than her disagreement utility, i.e., \(\exists v \in \mathcal{N}\) such that \(\forall i \in A, v_i > c_i\), and infeasible otherwise. In game theory it is customary to assume that the given Nash bargaining problem \((\mathcal{N}, c)\) is feasible; we will make this assumption as well.

The solution to a feasible instance is the point \(v \in \mathcal{N}\) that satisfies the following four axioms:

1. **Pareto optimality:** No point in \(\mathcal{N}\) weakly dominates \(v\).
2. **Symmetry:** If the players are renumbered, then a corresponding renumber the coordinates of \(v\) is a solution to the new instance.
3. **Invariance under affine transformations of utilities:** If the utilities of any player are redefined by multiplying by a scalar and adding a constant, then the solution to the transformed problem is obtained by applying these operations to the particular coordinate of \(v\).
4. **Independence of irrelevant alternatives:** If \(v\) is the solution to \((\mathcal{N}, c)\), and \(\mathcal{S} \subseteq \mathbb{R}_+^n\) is a compact, convex set satisfying \(c \in \mathcal{S}\) and \(v \in \mathcal{S} \subseteq \mathcal{N}\), then \(v\) is also the solution to \((\mathcal{S}, c)\).

Via an elegant proof, Nash proved:

**Theorem 6** (Nash [Nas53]). If the game \((\mathcal{N}, c)\) is feasible then there is a unique point in \(\mathcal{N}\) satisfying the axioms stated above. Moreover, this point is obtained by maximizing \(\Pi_{i \in A} (v_i - c_i)\over v \in \mathcal{N}\) over \(v \in \mathcal{N}\).

Nash’s solution to his bargaining game involves maximizing a concave function over a convex domain, and is therefore the optimal solution to the following convex program.

\[
\max_{v \in \mathcal{N}} \sum_{i \in A} \log(v_i - c_i)
\]

Further, Nash showed that the unique solution to his game is obtained by maximizing the product of utilities of agents over the feasible set. As a consequence, the allocations it produced by this solution are remarkably fair, see [CKM+19, ACGH20, Mou18] for remarks to this effect. This issue has been further explored under the name of Nash Social Welfare [CG18, CDG+17]. Additionally, since the Nash bargaining solution is captured via a convex program, if for a specific game, a separation oracle can be implemented in polynomial time, then using the ellipsoid algorithm one can get as good an approximation as desired in time polynomial in the number of bits of accuracy needed [GLS12, Vis21].

**Fisher Market Model:** The Fisher market model consists of a set \(A = \{1, 2, \ldots n\}\) of agents and a set \(G = \{1, 2, \ldots, m\}\) of infinitely divisible goods. By fixing the units for each good, we may assume without loss of generality that there is a unit of each good in the market. Each agent \(i\) has money \(m_i \in \mathbb{Q}_+\).
Let $x_{ij}$, $1 \leq j \leq m$ represent a bundle of goods allocated to agent $i$. Each agent $i$ has a utility function $u : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ giving the utility accrued by $i$ from a bundle of goods. We will assume that $u$ is concave and weakly monotonic. Each good $j$ is assigned a non-negative price, $p_j$. Allocations and prices, $x$ and $p$, are said to form an equilibrium if each agent obtains a utility maximizing bundle of goods at prices $p$ and the market clears, i.e., each good is fully sold to the extent of one unit and all money of agents is fully spent. We will assume that each agent derives positive utility from some good and for each agent, there is a good which gives her positive utility; clearly, otherwise we may remove that agent or good from consideration.

Arrow-Debreu Market Model: The Arrow-Debreu market model, also known as the exchange model differs from Fisher’s model in that agents come to the market with initial endowments of good instead of money. The union of all goods in initial endowments are all the goods in the market. Once again, by redefining the units of each good, we may assume that there is a total of one unit of each good in the market. The utility functions of agents are as before. The problem now is to find non-negative prices for all goods so that if each agent sells her initial endowment and buys an optimal bundle of goods, the market clears. Clearly, if $p$ is equilibrium prices then so is any scaling of $p$ by a positive factor.

One-Sided Matching Market: Let $A = \{1, 2, \ldots n\}$ be a set of $n$ agents and $G = \{1, 2, \ldots, n\}$ be a set of $n$ indivisible goods. Each agent $i$ has a utility function $u_{ij}$ over goods $j \in G$. The goal is to design a mechanism to allocate exactly one good to each agent, so that properties such as Pareto optimality are satisfied. Goods are rendered divisible by assuming that there is one unit of probability share of each good. Let $x_{ij}$ be the allocation of probability share that agent $i$ receives of good $j$. Then, $\sum_j u_{ij} x_{ij}$ is the expected utility accrued by agent $i$.

To make it a matching market, an additional constraint is that the total probability share allocated to each agent is one unit, i.e., the entire allocation must form a fractional perfect matching in the complete bipartite graph over vertex sets $A$ and $G$. Now, a solution can be viewed as a doubly stochastic matrix. The Birkhoff-von Neumann procedure [Bir46, VN53] then extracts a random underlying perfect matching in such a way that the expected utility accrued to each agent from the integral perfect matching is the same as from the fractional perfect matching. Since ex ante Pareto optimality implies ex post Pareto optimality, the integral allocation will also be Pareto optimal.

Two-Sided Matching Market: Our two-sided matching market model consist of a set $A = \{1, 2, \ldots n\}$ of agents and a set $J = \{1, 2, \ldots, n\}$ of jobs. Each agent $i$ has a utility function $u_{ij}$ over goods $j \in G$ and each job $j$ has a utility function $w_{ij}$ over agents $i \in A$. Let $x_{ij}$ be the allocation of probability share that agent $i$ receives of good $j$. Then, $\sum_j u_{ij} x_{ij}$ is the expected utility accrued by agent $i$ and $\sum_i w_{ij} x_{ij}$ is the expected utility accrued by job $j$. As in the one-sided case, the Birkhoff-von Neumann procedure [Bir46, VN53] then extracts a random underlying perfect matching from $x$.

Non-Bipartite Matching Market: Our non-bipartite matching market model consist of a set $A = \{1, 2, \ldots, n\}$ of agents and utilities for each pair $u_{ij} \in \mathbb{R}_+$ which $i$ and $j$ accrue on getting matched to each other. Let $x_{ij}$ be the fractional extent to which $i$ and $j$ are matched. Then, $\sum_j u_{ij} x_{ij}$ is the utility accrued by agent $i$. The allocation vector $x \in \mathbb{R}_+^n$ is said to be a fractional matching in $G$ if it is a convex combination of (integral) matchings in $G$. As in the bipartite case, it is possible to take a fractional matching and, in combinatorial, polynomial time, decompose it into a convex combination of most $n^2$ integral matchings. This was shown by Padberg and Wolsey [PW84]; for a modern proof see [Vaz20]. This allows a similar rounding strategy if one desires integral allocations.

2.1 Hylland-Zeckhauser Mechanism

A one-sided matching market consists of two types of entities, say agents and goods, with only one side having preferences over the other, i.e., agents over goods. Let $\hat{A} = \{1, 2, \ldots, n\}$ be a set of $n$ agents and $G = \{1, 2, \ldots, n\}$ be a set of $n$ indivisible goods. The goal of the HZ mechanism is to allocate exactly one
good to each agent. However, in order to use the power of a pricing mechanism, which endows the HZ mechanism with the properties of Pareto optimality and incentive compatibility in the large, it casts this one-sided matching market in the mold of a linear Fisher market as follows.

Goods are rendered divisible by assuming that there is one unit of probability share of each good, and utilities $u_{ij}$s are defined as in a linear Fisher market. Let $x_{ij}$ be the allocation of probability share that agent $i$ receives of good $j$. Then, $\sum_j u_{ij}x_{ij}$ is the expected utility accrued by agent $i$. Each agent has 1 dollar for buying these probability shares and each good $j$ has a price $p_j \geq 0$.

Beyond a Fisher market, an additional constraint is that the total probability share allocated to each agent is one unit, i.e., the entire allocation must form a fractional perfect matching in the complete bipartite graph over vertex sets $A$ and $G$. Subject to these constraints, each agent buys a utility maximizing bundle of goods. Another point of departure from a linear Fisher market is that in general, an agent’s optimal bundle may cost less than one dollar, i.e., the agents are not required to spend all their money. Since each good is fully sold, the market clears. Hence these are defined to be equilibrium allocation and prices.

Clearly, an equilibrium allocation can be viewed as a doubly stochastic matrix. The Birkhoff-von Neumann procedure then extracts a random underlying perfect matching in such a way that the expected utility accrued to each agent from the integral perfect matching is the same as from the fractional perfect matching. Since ex ante Pareto optimality implies ex post Pareto optimality, the integral allocation will also be Pareto optimal.

Next we describe the structure of optimal bundles from [VY21]. Let $p$ be given prices which are not necessarily equilibrium prices. By the definition of an optimal bundle given above, the optimal bundle for agent $i$ is a solution to LP (7).

$$\begin{align*}
\text{max} & \quad \sum_{j \in G} u_{ij}x_{ij} \\
\text{s.t.} & \quad \sum_j x_{ij} = 1, \\
& \quad \sum_j p_jx_{ij} \leq 1, \\
& \quad x_{ij} \geq 0 \quad \forall j \in G
\end{align*}$$

(7)

3 Nash-Bargaining-Based Models

We will define four one-sided matching market models based on our Nash bargaining approach. For the case of linear utilities, we have defined both Fisher and Arrow-Debreu versions, namely $LiF$ and $LiAD$. For more general utility functions we have defined only the Arrow-Debreu version; the Fisher version is obtained by setting all disagreement utilities to zero. For each model, we give the convex program whose optimal solution captures the Nash bargaining solution.

Our one-sided matching market models consist of a set $A = \{1,2,\ldots,n\}$ of agents and a set $G = \{1,2,\ldots,n\}$ of infinitely divisible goods; observe that there is an equal number of agents and goods. There is one unit of each good and each agent needs to be allocated a total of one unit of goods. Hence the allocation needs to be a fractional perfect matching, as defined next.

**Definition 7.** Let us name the coordinates of a vector $x \in \mathbb{R}_+^n$ by pairs $i,j$ for $i \in A$ and $j \in G$. Then $x$ is said to be a fractional perfect matching if

$$\forall i \in A : \sum_j x_{ij} = 1 \quad \text{and} \quad \forall j \in G : \sum_i x_{ij} = 1.$$
As mentioned in Section 2, an equilibrium allocation can be viewed as a doubly stochastic matrix, and the Birkhoff-von Neumann procedure [Bir46, VN53] can be used to extract a random underlying perfect matching in a way that the expected utility accrued to each agent from the integral perfect matching is the same as from the fractional perfect matching.

1). Under the linear Fisher Nash bargaining one-sided matching market, abbreviated \(1LF\), each agent \(i \in A\) has a linear utility function, as defined in Section 1.1. Corresponding to each fractional perfect matching \(x\), there is a vector \(v_x\) in the feasible set \(N\); its components are the utilities derived by the agents under the allocation given by \(x\). The disagreement point is the origin. Observe that the setup of \(1LF\) is identical to that of the HZ mechanism; the difference lies in the definition of the solution to an instance. (4) is a convex program for \(1LF\) and its KKT conditions are presented in Section 1.1.

2). Under the linear Arrow-Debreu Nash bargaining one-sided matching market, abbreviated \(LiAD\), each agent \(i \in A\) has a linear utility function, as above. Additionally, we are specified an initial fractional perfect matching \(x_I\) which gives the initial endowments of the agents. Each agent has one unit of initial endowment over all the goods and the total endowment of each good over all the agents is one unit, as given by \(x_I\). These two pieces of information define the utility accrued by each agent from her initial endowment; this is her disagreement point \(c_i\). We will assume that the problem is feasible, i.e., there is a fractional perfect matching, defining a redistribution of the goods, under which each agent \(i\) derives strictly more utility than \(c_i\). Each vector \(v \in N\) is as defined in \(1LF\). Henceforth, we will consider the slightly more general problem in which are specified the disagreement point \(c\) and not the initial endowments \(x_I\). There is no guarantee that \(c\) comes from a valid fractional perfect matching of initial endowments. However, we still want the problem to be feasible.

Remark 8. Note that in all the algorithms presented in this paper, we will provide explicit efficient algorithms for testing for feasibility. Secondly, we will relax the equalities in the convex programs to inequalities in order to ensure that the corresponding dual variables are constrained to be non-negative. It is easy to see that the left-over goods can be “packed” into the left-over demand of agents, without changing the objective function, to yield a fractional perfect matching.

The convex program for \(LiAD\) is given in (8).

\[
\begin{align*}
\max & \quad \sum_{i \in A} \log(u_i(x) - c_i) \\
\text{s.t.} & \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\
& \quad x \geq 0.
\end{align*}
\]

The KKT conditions for program (8) with non-negative dual variables \(p_j\) and \(q_i\) are given below. By setting \(c_i = 0\) for \(i \in A\), we get KKT conditions for program (4) as well.

1. \(\forall j \in G : p_j > 0 \implies \sum_{i \in A} x_{ij} = 1.\)
2. \(\forall i \in A : q_i > 0 \implies \sum_{j \in G} x_{ij} = 1.\)
3. \(\forall i \in A, \forall j \in G : p_j + q_i \geq \frac{u_{ij}}{u_i(x) - c_i}.\)
4. \(\forall i \in A, \forall j \in G : x_{ij} > 0 \implies p_j + q_i = \frac{u_{ij}}{u_i(x) - c_i}.\)

For \(c_i = 0\), this allows us to prove the following approximate equal-share fairness property which guarantees that every agent achieves at least 1/2 of the utility that they would get under the equal share matching that assigns \(\frac{1}{n}\) to all edges.
Lemma 9. Let \( x \) be an optimal solution to (4) (or rather (8) with \( c_i = 0 \)), then for all agents \( i \) we have

\[
u_i(x) = \sum_{j \in G} u_{ij} x_{ij} \geq \frac{1}{2n} \sum_{j \in G} u_{ij} .
\]

Proof. By the KKT conditions we know that \( \frac{u_{ij}}{u_i(x)} \leq p_j + q_i \) with equality if \( x_{ij} > 0 \). Moreover, if \( p_j > 0 \) then \( \sum_{i \in A} x_{ij} = 1 \) and likewise for \( q_i \). Thus

\[
\sum_{j \in G} p_j + \sum_{i \in A} q_i = \sum_{i \in A} \sum_{j \in G} x_{ij} (p_j + q_i) \leq n.
\]

In addition, note that \( q_i \leq 1 \) since otherwise we cannot have \( \sum_{j \in G} x_{ij} = 1 \) and \( \sum_{j \in G} x_{ij} (p_j + q_i) = 1 \). Finally, we conclude

\[
u_i(x) = \max \left\{ \frac{u_{ij}}{p_j + q_i} \mid j \in G \right\}
\geq \frac{\sum_{j \in G} u_{ij}}{\sum_{j \in G} p_j + q_i} \cdot \frac{p_j + q_i}{\sum_{j' \in G} p_{j'} + q_i}
\geq \frac{\sum_{j \in G} u_{ij}}{nq_i + \sum_{j \in G} p_j}
\geq \frac{1}{2n} \sum_{j \in G} u_{ij} .
\]

In the case of non-zero \( c_j \), the equal share matching may no longer be feasible and so this notion loses some meaning. However, we will give an analogous weaker bound in Section 5, specifically in Lemma 27.

3). Economists like to model diminishing marginal utilities for goods by considering concave utility functions. Since we are in a fixed-precision model of computing, we will consider separable, piecewise-linear concave (SPLC) utility functions.

The separable, piecewise-linear concave Arrow-Debreu Nash bargaining one-sided matching market, abbreviated SAD, is analogous to LiAD, with the difference that each agent has a separable, piecewise-linear concave utility function, hence generalizing the linear utility functions specified in LiAD. When there are no disagreement utilities \( c_i \), we call this model SF. We next define these functions in detail.

For each agent \( i \) and good \( j \), function \( f^j_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) gives the utility derived by \( i \) as a function of the amount of good \( j \) she receives. Each \( f^j_i \) is a non-negative, non-decreasing, piecewise-linear, concave function. The overall utility of buyer \( i \), \( u_i(x) \), for bundle \( x = (x_1, \ldots, x_n) \) of goods, is additively separable over the goods, i.e., \( u_i(x) = \sum_{j \in G} f^j_i(x_j) \).

We will call each piece of \( f^j_i \) a segment. Number the segments of \( f^j_i \) in order of decreasing slope; throughout we will assume that these segments are indexed by \( k \) and that \( S_{ij} \) is the set of all such indices. Let \( \sigma_{ijk}, k \in S_{ij} \), denote the \( k \)th segment, \( l_{ijk} \) denote the amount of good \( j \) represented by this segment; we will assume that the last segment in each function is of unbounded length. Let \( u_{ijk} \) denote the rate at which \( i \) accrues utility per unit of good \( j \) received, when she is getting an allocation corresponding to this segment. Clearly, the maximum utility she can receive corresponding to this segment is \( u_{ijk} \cdot l_{ijk} \). We will assume that \( u_{ijk} \) and \( l_{ijk} \) are rational numbers. Finally, let \( S^i_c \) be the set of all indices \( (j,k) \) corresponding to the segments in all utility functions of agent \( i \) under the given instance, i.e.,

\[
S^i_c = \{(j,k) \mid j \in G, k \in S_{ij}\}.
\]
Remark 10. Throughout this paper, we will index elements of $A$, $G$ and $S_{ij}$ by $i$, $j$ and $k$, respectively. When the domain of $i$, $j$ or $k$ is not specified, especially in summations, it should be assumed to be $A$, $G$ and $S_{ij}$, respectively.

Program (9) is a convex program for SAD.

$$
\begin{align*}
\text{max} & \quad \sum_{i \in A} \log (u_i(x) - c_i) \\
\text{s.t.} & \quad \sum_{i \in A} \sum_{k \in S_{ij}} x_{ijk} = 1 \quad \forall j \in G, \\
& \quad \sum_{j \in G} \sum_{k \in S_{ij}} x_{ijk} = 1 \quad \forall i \in A, \\
& \quad x_{ijk} \leq l_{ijk} \quad \forall i \in A, \forall j \in G, \forall k \in S_{ij}, \\
& \quad x \geq 0.
\end{align*}
$$

(9)

The KKT conditions for program (8) with non-negative dual variables $p$, $q$, and $h$ respectively are given below. By setting $c_i = 0$ for $i \in A$, we get KKT conditions for program (4) as well.

1. $\forall j \in G : \quad p_j > 0 \quad \implies \quad \sum_{i \in A} \sum_{k \in S_{ij}} x_{ijk} = 1.$
2. $\forall i \in A : \quad q_i > 0 \quad \implies \quad \sum_{j \in G} \sum_{k \in S_{ij}} x_{ijk} = 1.$
3. $\forall i \in A, \forall j \in G, \forall k \in S_{ij} : \quad h_{ijk} > 0 \quad \implies \quad x_{ijk} = l_{ijk}.$
4. $\forall i \in A, \forall j \in G, \forall k \in S_{ij} : \quad p_j + q_i + h_{ijk} \geq \frac{u_{ijk}}{u_i(x) - c_i}.$
5. $\forall i \in A, \forall j \in G, \forall k \in S_{ijk} : \quad x_{ijk} > 0 \quad \implies \quad p_j + q_i + h_{ijk} = \frac{u_{ijk}}{u_i(x) - c_i}.$

As was the case for linear utilities, a $\frac{1}{2}$-approximate notion of equal-share fairness holds if $c_i = 0$ for all $i$. For simplicity assume that there is some $\ell$ such that for all $i, j$, we have $S_{ij} = \{1, \ldots, \ell + 1\}$ and $\sum_{k=1}^{\ell} l_{ijk} = 1$. Note that due to the concavity of the utilities, the utility of player $i$ in the equal-share allocation is at most $\frac{1}{n} \sum_{j \in G} \sum_{k=1}^{\ell} u_{ijk} l_{ijk}$.

Lemma 11. Let $x$ be an optimal solution to (4) (or (8) with $c_i = 0$), then for all agents $i$ we have

$$
u_i(x) \geq \frac{1}{2n} \sum_{j \in G} \sum_{k=1}^{\ell} u_{ijk} l_{ijk}.$$

Proof. The proof is similar to that of Lemma 9. By the KKT conditions, we know that $\frac{u_{ijk}}{u_i(x)} \leq p_j + q_i + h_{ijk}$ with equality if $x_{ijk} > 0$.

Once again, one can then see that $q_i \leq 1$ and

$$
\sum_{i \in A} \sum_{j \in G} \sum_{k=1}^{\ell} l_{ijk} h_{ijk} + \sum_{j \in G} p_j + \sum_{i \in A} q_i = \sum_{i \in A} \sum_{j \in G} \sum_{k=1}^{\ell} x_{ijk} (p_j + q_i + h_{ijk}) = n
$$
and therefore

\[ u_i = \max \left\{ \frac{u_{ijk}}{p_j + q_i + h_{ijk}} \right\}_{j \in G, k \in [\ell]} \]

\[ \geq \sum_{j \in G} \sum_{k=1}^{\ell} \frac{u_{ijk}}{p_j + q_i + h_{ijk}} \cdot \frac{l_{ijk} \cdot (p_j + q_i + h_{ijk})}{\sum_{j' \in G} p_{j'} + nq_i + \sum_{j' \in G} \sum_{k'=1}^{\ell} l_{ijk'} h_{ijk'}} \]

\[ \geq \frac{\sum_{j \in G} \sum_{k=1}^{\ell} l_{ijk} u_{ijk}}{2n}. \]

For the remaining models we will not get such strong equal-share fairness properties. However, weakened versions are shown in Section 5. In particular, see Lemma 29.

### 3.1 Two-Sided Matching Markets

1.) Our two-sided matching market model consist of a set \( A = \{1, 2, \ldots n\} \) of agents and a set \( J = \{1, 2, \ldots, n\} \) of jobs. For uniformity, we have assumed that there is an equal number of agents and jobs, though the model can be easily enhanced and made more general. Our goal is to find an integral perfect matching between agents and jobs; however, we will relax this to finding a fractional perfect matching, \( x \), followed by rounding as described above. We will explicitly define only the simplest case of two-sided markets; more general models follow along the same lines as one-sided markets.

Under the linear Arrow-Debreu bargaining two-sided matching market, abbreviated \( 2AD \), the utility accrued by agent \( i \in A \) under allocation \( x \),

\[ u_i(x) = \sum_{j \in J} u_{ij} x_{ij}, \]

where \( u_{ij} \) is the utility accrued by \( i \) if she were assigned job \( j \) integrally. Analogously, the utility accrued by job \( j \in J \) under allocation \( x \),

\[ w_j(x) = \sum_{i \in A} w_{ij} x_{ij}, \]

where \( w_{ij} \) is the utility accrued by \( j \) if it were assigned to \( i \) integrally.

In keeping with the axiom of symmetry under Nash bargaining, we will posit that the desires of agents and jobs are equally important and we are led to defining the feasible set in a \( 2n \) dimensional space, i.e., \( \mathcal{N} \subseteq \mathbb{R}_{+}^{2n} \). The first \( n \) components of feasible point \( v \in \mathcal{N} \) represent the utilities derived by the \( n \) agents, i.e., \( u_i(x) \), and the last \( n \) components the utilities derived by the \( n \) jobs, i.e., \( w_j(x) \), under a fractional perfect matching \( x \). We seek the Nash bargaining point with respect to disagreement utilities \( c_i \) for all agents \( i \in G \) and \( d_j \) for jobs \( j \in J \). If all \( c_i \) and \( d_j \) are 0, we call this model reduces to the linear Fisher bargaining two-sided matching market or \( 2LF \). A convex program of \( 2AD \) is given in (10).

Program (10) is a convex program for \( 2AD \),

\[
\begin{align*}
\max & \quad \sum_{i \in A} \log(u_i(x) - c_i) + \sum_{j \in J} \log(w_j(x) - d_j) \\
\text{s.t.} & \quad \sum_{j \in J} x_{ij} \leq 1 \quad \forall j \in J, \\
& \quad \sum_{j \in J} x_{ij} \leq 1 \quad \forall i \in A, \\
& \quad x \geq 0.
\end{align*}
\]

(10)

The KKT conditions for program (10) with non-negative dual variables \( p \) and \( q \) are:
∀j ∈ J : p_j > 0 \implies \sum_{i \in A} x_{ij} = 1.

∀i ∈ A : q_i > 0 \implies \sum_{j \in J} x_{ij} = 1.

∀i ∈ A, ∀j ∈ J : p_j + q_i \geq \frac{u_{ij}}{u_i(x) - c_i} + w_j \frac{w_j}{w_j(x) - d_j}.

∀i ∈ A, ∀j ∈ J : x_{ij} > 0 \implies p_j + q_i = \frac{u_{ij}}{u_i(x) - c_i} + w_j \frac{w_j}{w_j(x) - d_j}.

2.) We may also extend this model even further by allowing SPLC utilities on both sides. This defines the SPLC Arrow-Debreu bargaining two-sided matching market or 2SAD. In this setting, every edge (i, j) comes with segments k ∈ S_{ij} that have a length of l_{ijk} as well as utilities u_{ijk} for agent i and w_{ijk} for job j. Note that we assume that the piecewise-linear, concave utility functions for i and j have the same breakpoints. This is without loss of generality since we may always simply take the union of the breakpoints of both sides.

As before we have u_i(x) = \sum_{j \in J} \sum_{k \in S_{ij}} u_{ijk} x_{ijk} and w_j(x) = \sum_{i \in A} \sum_{k \in S_{ij}} w_{ijk} x_{ijk}. The convex programming formulation is then given by

\begin{align*}
\text{max} & \quad \sum_{i \in A} \log(u_i(x) - c_i) + \sum_{j \in J} \log(w_j(x) - d_j) \\
\text{s.t.} & \quad \sum_{i \in A} \sum_{k \in S_{ij}} x_{ijk} \leq 1 \quad \forall j \in J, \\
& \quad \sum_{j \in J} \sum_{k \in S_{ij}} x_{ijk} \leq 1 \quad \forall i \in A, \\
& \quad x_{ijk} \leq l_{ijk} \quad \forall i \in A, j \in J, k \in S_{ij}, \\
& \quad x \geq 0
\end{align*}

(11)

Its KKT conditions with non-negative dual variables p, q, and h are:

1. ∀j ∈ J : p_j > 0 \implies \sum_{i \in A} \sum_{k \in S_{ij}} x_{ijk} = 1.

2. ∀i ∈ A : q_i > 0 \implies \sum_{j \in J} \sum_{k \in S_{ij}} x_{ijk} = 1.

3. ∀i ∈ A, ∀j ∈ J, ∀k ∈ S_{ij} : h_{ijk} > 0 \implies x_{ijk} = l_{ijk}.

4. ∀i ∈ A, ∀j ∈ J, ∀k ∈ S_{ij} : p_j + q_i + h_{ijk} \geq \frac{u_{ijk}}{u_i(x) - c_i} + w_j \frac{w_{ijk}}{w_j(x) - d_j}.

5. ∀i ∈ A, ∀j ∈ J, ∀k ∈ S_{ij} : x_{ijk} > 0 \implies p_j + q_i + h_{ijk} = \frac{u_{ijk}}{u_i(x) - c_i} + w_{ijk} \frac{w_{ijk}}{w_j(x) - d_j}.

3.2 Non-Bipartite Matching Market

Our non-bipartite matching market model consist of a set A = \{1, 2, \ldots, n\} of agents and utilities for each pair u_{ij} \in \mathbb{R}_+ which i and j accrue on getting matched to each other. Let G = (A, E) be the complete graph on n vertices, i.e., all edges are present. The vertices correspond to agents. A vector x \in \mathbb{R}^E_+ is said to be a fractional matching in G if it is a convex combination of (integral) matchings in G. The goal is to find a Nash-bargaining solution with respect to disagreement utilities c_i for all agents i. This is a linear Arrow-Debreu bargaining non-bipartite matching market or NBAD. If c_i = 0 for all agents, then we call this the linear Fisher bargaining non-bipartite matching market or NBLF.

Due to a classic result by Edmonds we have that x is a fractional matching if and only if x(\delta(i)) \leq 1 for all i and x(E(B)) \leq \frac{|B|-1}{2} for all sets B ⊆ A of odd cardinality. For ease of notation let us denote the
collection of all such odd subsets $B$ of $A$ by $\mathcal{O}$. This motivates the convex program:

$$\begin{align*}
\text{max} & \quad \sum_{i \in A} \log(u_i(x) - c_i) \\
\text{s.t.} & \quad x(\delta(i)) \leq 1 \quad \forall i \in A, \\
& \quad x(E(B)) \leq \frac{|B| - 1}{2} \quad \forall B \in \mathcal{O}, \\
& \quad x \geq 0.
\end{align*}$$

(12)

The KKT conditions for program (12) with non-negative dual variables $p$ and $z$ are

1. $\forall i \in A: \ p_i > 0 \implies x(\delta(i)) = 1.$
2. $\forall B \in \mathcal{O}: \ z_B > 0 \implies x(E(B)) = \frac{|B| - 1}{2}.$
3. $\forall i, j \in A: \ p_i + p_j + \sum_{(i,j) \in B} z_B \geq \frac{u_i}{u_i(x) - c_i} + \frac{u_j}{u_j(x) - c_j}.$
4. $\forall i, j \in A: \ x_{ij} > 0 \implies p_i + p_j + \sum_{(i,j) \in B} z_B = \frac{u_i}{u_i(x) - c_i} + \frac{u_j}{u_j(x) - c_j}.$

We remark that as in the bipartite case, it is possible to take a fractional matching and (in a combinatorial, polynomial time manner), decompose it into a convex combination of most $n^2$ integral matchings. This was shown by Padberg and Wolsey [PW84]; for a modern proof see [Vaz20]. This allows a similar rounding strategy if one desires integral allocations.

### 3.3 Leontief Utilities

In principle, one may also consider models with other classes of utilities. Another common type of utility function in market models is that of Leontief utilities. In this setting, each agent $i \in A$ demands goods at some fixed ratio $a_{ij}$ for all $j \in G$. For example, if the goods consist of items such as eggs, milk, flour, etc. and agent $i$ wishes to bake a cake, then they would require a very specific ratio of these ingredients as dictated by the recipe. The utility of agent $i$ is then given simply by the amount of cake that they can make.

More formally, one may define a Nash-bargaining based matching market with Leontief utilities by the convex program

$$\begin{align*}
\text{max} & \quad \sum_{i \in A} \log(u_i) \\
\text{s.t.} & \quad \sum_{j \in G} u_i a_{ij} \leq 1 \quad \forall i \in A, \\
& \quad \sum_{i \in A} u_i a_{ij} \leq 1 \quad \forall j \in G, \\
& \quad x \geq 0.
\end{align*}$$

However, these kinds of utilities does not really fit in with our other models for two main reasons:

1. The “unit demand constraints” for each agent $i$ do not alter the problem in any way. One can simply add an extra good $\tilde{j}_i$ for each agent $i$ and set $a_{ij} = \sum_{j \in G} a_{ij}$. This implies that this model is really just a special case of a Fisher market with Leontief utilities which is well-studied.

2. The feasible region of the polytope is not closely related to a matching polytope. Since each agent demands certain ratios of goods, a matching between goods and agents is generally a poor allocation. After all, getting one unit of eggs and nothing else is hardly useful if one wishes to bake a cake.

---

6 The latter paper was written without knowledge of [PW84], which has gone largely unknown in the research community.
For these reasons we do not consider models with Leontief utilities in this paper.

4 Multiplicative Weights Update

In this section we prove that MWU converges to an \( \epsilon \)-approximate Nash bargaining solution. The main result is given in the following theorem:

**Theorem 12.** The following hold:

- MWUA (Algorithm 1) computes an \( \epsilon \)-approximate Nash bargaining solution (see Definition 14) of Program (8), i.e., model LiAD, after \( O \left( \frac{n \log n}{\epsilon^2} \right) \) iterations and each iterate can be implemented in \( O(n^2) \) time.
- MWUA (Algorithm 2) computes an \( \epsilon \)-approximate Nash bargaining solution of Program (9), i.e., model SAD, after \( O \left( \frac{n \log n}{\epsilon^2} \right) \) iterations and each iterate can be implemented in \( O(l \cdot n^2) \) time.

We provide the analysis for the case of LiAD, i.e., linear utilities with endowments. The modified Lemmas so that the analysis carries over for piecewise linear utilities can be found in the end of the section.

4.1 From optimization to feasibility

Multiplicative Weights Update Algorithm (MWUA) [AHK12] has found numerous applications in Game Theory, e.g., has been used for computing Nash Equilibrium in zero-sum games and potential games or Correlated Equilibrium in general games. Another surprising application, is that MWUA can be used to find approximately feasible points for linear programs. Inspired by the latter, we use MWUA to find an (approximately) feasible solution to the feasibility program below (see Definition 14 for approximate feasibility):

**Feasibility program.**

\[
\begin{align*}
\text{CP}_i(p, q) &\leq u_i(x) \quad \forall i \in A, \\
\sum_{i \in A} x_{ij} &\leq 1 \quad \forall j \in G, \\
\sum_{j \in G} x_{ij} &\leq 1 \quad \forall i \in A, \\
\sum_{j \in G} p_j + \sum_{i \in A} q_i &= n + \sum_{i \in A} c_i \min_{j \in G} \frac{p_j + q_i}{u_{ij}}, \\
x, p, q &\geq 0
\end{align*}
\]

where

\[
\text{CP}_i(p, q) := \max \sum_{j \in G} u_{ij} y_j \\
\text{s.t.} \quad \sum_{j \in G} (p_j + q_i) y_j \leq 1 + c_i \min_{j \in G} \frac{p_j + q_i}{u_{ij}}, \\
y &\geq 0.
\]

We are able to show that the Feasibility Program (F-LiAD) actually contains only the optimal primal-dual solution of the convex program (8), namely any feasible point of (F-LiAD) satisfies the KKT of (8). This implies that as long as MWUA finds a solution that is approximately feasible for (F-LiAD), the solution
will satisfy approximately the KKT conditions and hence will be approximately optimal. The first step towards the proof is to argue that if (F-LiAD) is feasible, that is there exist \((x^*, p^*, q^*)\) that satisfy the constraints in (F-LiAD), then and only then \((x^*, p^*, q^*)\) are the primal-dual variables that satisfy the KKT conditions. This is given in the following Lemma:

**Lemma 13 (Optimization to Feasibility).** Let \(x^*\) be an optimal solution of Program (8), then there exist \(p^*, q^*\) so that \((x^*, p^*, q^*)\) is feasible for (F-LiAD). Conversely, assume that \((x^*, p^*, q^*)\) is a feasible solution for (F-LiAD), then \(x^*\) is an optimal solution for Program (8).

**Proof.** We first prove the direct, namely if \(x^*\) is an optimal solution of Program (8), then there exist \(p^*, q^*\) so that \((x^*, p^*, q^*)\) is feasible for (F-LiAD).

Since \(x^*\) is an optimal solution of Program (8), there must be dual variables \(w^*, z^*\) so that \((x^*, w^*, z^*)\) satisfies the KKT conditions, that is

\[
\begin{align*}
\forall j \in G : w^*_j &\geq 0, \forall i \in G : z^*_i \geq 0, \\
\forall j \in G : w^*_j = 0 &\Rightarrow \sum_{i \in A} x^*_i \leq 1 \text{ and } w^*_j > 0 \Rightarrow \sum_{i \in A} x^*_i = 1, \\
\forall i \in A : z^*_i = 0 &\Rightarrow \sum_{j \in G} x^*_i \leq 1 \text{ and } z^*_i > 0 \Rightarrow \sum_{j \in G} x^*_i = 1, \\
\forall i \in A, \forall j \in G : x^*_i = 0 &\Rightarrow u_i(x^*) \geq \frac{u_{ij}}{w^*_j + z^*_i} + c_i, \\
\forall i \in A, \forall j \in G : x^*_i > 0 &\Rightarrow u_i(x^*) = \frac{u_{ij}}{w^*_j + z^*_i} + c_i.
\end{align*}
\tag{13}
\]

We shall show that \((x^*, p^*, q^*)\) is feasible for (F-LiAD), by choosing \(p^* = w^*\) and \(q^* = z^*\). Observe that by definition of \(CP_i\) we get that

\[
\begin{align*}
CP_i(p, q) &= \left( \max_{j \in G} \frac{u_{ij}}{p_j + q_i} \right) \times \left( 1 + c_i \min_{j \in G} \frac{p_j + q_i}{u_{ij}} \right) \\
&= \frac{1}{\min_{j \in G} \frac{p_j + q_i}{u_{ij}}} \times \left( 1 + c_i \min_{j \in G} \frac{p_j + q_i}{u_{ij}} \right) \\
&= c_i + \max_{j \in G} \frac{u_{ij}}{p_j + q_i},
\end{align*}
\tag{14}
\]

and hence \(CP_i(p^*, q^*) = c_i + \max_{j \in G} \frac{u_{ij}}{p^*_j + q^*_i}\). From KKT Equations (13) we also have that \(u_i(x^*) \geq c_i + \frac{u_{ij}}{p^*_j + q^*_i}\) for all \(j \in G\), i.e., \(u_i(x^*) \geq c_i + \max_{j \in G} \frac{u_{ij}}{p^*_j + q^*_i}\). We conclude that \(u_i(x^*) \geq CP_i(p^*, q^*)\).

To finish the first part of the proof, i.e., that \((x^*, p^*, q^*)\) satisfies the constraints for (F-LiAD), it suffices to show that

\[
\sum_{j \in G} p^*_j + \sum_{i \in A} q^*_i = n + \sum_{i \in A} c_i \min_{j \in G} \frac{p^*_j + q^*_i}{u_{ij}}.
\]
By KKT Equations (13), one has that

\[
\sum_{j \in G} p_j^* + \sum_{i \in A} q_i^* = \sum_{j \in G} p_j^* \sum_{i \in A} x_{ij}^* + \sum_{i \in A} q_i^* \sum_{j \in G} x_{ij}^*
\]

\[
= \sum_{i \in A} \sum_{j \in G} x_{ij}^* (p_j^* + q_i^*)
\]

\[
= \sum_{i \in A} \sum_{j \in G} x_{ij}^* \frac{u_{ij}}{u_i(x^*)} - c_i
\]

\[
= \sum_{i \in A} \frac{1}{u_i(x^*) - c_i} \sum_{j \in G} u_{ij} x_{ij}^*
\]

\[
= \sum_{i \in A} \frac{u_i(x^*)}{u_i(x^*) - c_i}
\]

\[
= \sum_{i \in A} 1 + \frac{c_i}{u_i(x^*) - c_i}
\]

\[
= n + \sum_{i \in A} \frac{c_i}{u_i(x^*) - c_i} = n + \sum_{i \in A} \frac{c_i \min_{j \in G} u_{ij}}{u_{ij}}.
\]

The first part of the proof is complete.

For the converse direction, assuming that \((x^*, p^*, q^*)\) satisfies the constraints of (F-LiAD), we shall show that \(x^*\) is a maximizer for Program (8). By assumption, one has \(u_i(x^*) \geq \text{CP}_i(p^*, q^*)\) for all \(i \in A\), therefore since \(\text{CP}_i(p^*, q^*)\) is defined to be the maximum utility \(i\) can get with prices \(p^*, q^*\), \(i\) must spend at least all his budget, that is for all agents \(i \in A\)

\[
\sum_{j \in G} (p_j^* + q_i^*) x_{ij}^* \geq 1 + c_i \min_{j \in G} \frac{p_j^* + q_i^*}{u_{ij}}.
\]  \hspace{1cm} (15)

By summing the above inequality (15) for all \(i \in A\) we conclude that

\[
\sum_{i \in A} \sum_{j \in G} (p_j^* + q_i^*) x_{ij}^* \geq n + \sum_{i \in A} \frac{c_i \min_{j \in G} p_j^* + q_i^*}{u_{ij}}.
\]  \hspace{1cm} (16)

Moreover, using (F-LiAD) it holds

\[
n + \sum_{i \in A} \frac{c_i \min_{j \in G} p_j^* + q_i^*}{u_{ij}} = \sum_{j \in G} p_j^* + \sum_{i \in A} q_i^*
\]

\[
\geq \sum_{j \in G} p_j^* \left( \sum_{i \in A} x_{ij}^* \right) + \sum_{i \in A} q_i^* \left( \sum_{j \in G} x_{ij}^* \right)
\]

\[
= \sum_{i \in A} \sum_{j \in G} (p_j^* + q_i^*) x_{ij}^*,
\]  \hspace{1cm} (17)

where the inequality comes from the fact \(\sum_{i \in A} x_{ij}^* \leq 1\) for all \(j \in G\) and \(\sum_{j \in G} x_{ij}^* \leq 1\) for all \(i \in A\).

Using (16) and (17) it follows that we actually have equality in (16), that is
Corollary 15. \(\sum_{i \in A} \sum_{j \in G} (p^*_i + q^*_i) x^*_i = n + \sum_{i \in A} c_i \min_{j \in G} \frac{p^*_i + q^*_i}{u_{ij}},\)
and hence (15) is also equality, i.e.,
\[
\sum_{j \in G} (p^*_i + q^*_i) x^*_i = 1 + c_i \min_{j \in G} \frac{p^*_i + q^*_i}{u_{ij}} \quad \text{for all } i \in A.
\] (18)
Additionally, the inequality in (17) indicates that whenever \(p^*_j > 0\) we get \(\sum_{i \in A} x^*_i = 1\) and whenever \(q^*_i > 0\) we get \(\sum_{j \in G} x^*_i = 1\). Thus, \(x^*\) is feasible for \(CP_i(p^*, q^*)\) and since \(CP_i(p^*, q^*) \leq u_i(x^*)\), \(x^*\) must be a maximizer of \(CP_i(p^*, q^*)\) for all \(i \in A\).
Finally, since \(x^*\) is the maximizer of \(CP_i(p^*, q^*)\) it should hold for all \(i \in A\)
\[
u_i(x^*) = CP_i(p^*, q^*) = \left( \max_{j \in G} \frac{u_{ij}}{p^*_j + q^*_j} \right) \times \left( 1 + c_i \min_{j \in G} \frac{p^*_j + q^*_i}{u_{ij}} \right) = c_i + \max_{j \in G} \frac{u_{ij}}{p^*_j + q^*_j} \quad \text{for all } j \in G \text{ s.t. } x^*_i > 0.
\]
We conclude that \(u_i(x^*) \geq c_i + \frac{u_{ij}}{p^*_j + q^*_j}\), with equality only if \(x^*_i > 0\). The proof is complete because we showed that the KKT conditions (13) are satisfied.

A corollary that can be derived from Lemma 13 is that an approximate feasible solution of \((F-LiAD)\) is an approximately optimal Nash bargaining solution of Program (8). Before we state the corollary, we provide the formal definition of an \(\epsilon\)-approximate feasible solution to \((F-LiAD)\) and \(\epsilon\)-approximate Nash bargaining solution.

**Definition 14** (Approximately feasible). A point \((\tilde{x}, \tilde{p}, \tilde{q})\) is called \(\epsilon\)-approximate feasible of \((F-LiAD)\), if it satisfies non-negativity, price and max-utility constraints but might violate allocation constraints by an additive \(\epsilon\); that is we have \(\sum_{i \in A} \tilde{x}_{ij} \leq 1 + \epsilon\) for all \(j \in G\) and \(\sum_{j \in G} \tilde{x}_{ij} \leq 1 + \epsilon\), for all \(i \in A\). Moreover, we define an \(\epsilon\)-approximate Nash bargaining solution for program (8) to be a feasible allocation \(x\) such that for each agent \(i\), if \(i\) changes his allocation from \(x_i\) to some feasible allocation \(x_i'\), he cannot gain more than an additive \(\epsilon\) in his utility.

**Corollary 15.** Let \((\tilde{x}, \tilde{p}, \tilde{q})\) be a \(\epsilon\)-approximate feasible point of \((F-LiAD)\). Then \(\frac{1}{1+\epsilon} \cdot \tilde{x}\) is a \(O(\epsilon)\)-approximate Nash bargaining solution of Program (8).

**Proof.** Let \(\bar{x}, \bar{p}, \bar{q}\) be the time averages of allocations and prices respectively that the Algorithm 1 returns and let \(x^*\) denote the optimal solution of Program (8). Using the fact that \(CP_i(p, q)\) is convex (Claim 19), we showed Inequality 23, that is \(CP_i(\bar{p}, \bar{q}) \leq u_i(\bar{x})\). As a result, it must hold for each agent \(i\)
\[
\log(u_i(x^*) - c_i) \leq \log(u_i(\bar{x}) - c_i).
\]
Moreover, it holds $\ln(1 + \epsilon) \leq \epsilon$, hence

$$\log \left( u_i(\bar{x}) - c_i \right) - \log \left( \frac{1}{\epsilon + 1} \cdot u_i(\bar{x}) - c_i \right) = \log \left( 1 + \frac{\epsilon}{\epsilon + 1} \cdot u_i(\bar{x}) \right) \leq \frac{\epsilon \cdot u_i(\bar{x})}{u_i(\bar{x}) - (1 + \epsilon)c_i}$$

which is $O(\epsilon)$.

\[\blackslug\]

**Algorithm 1** Multiplicative Weight Update

\[
\hat{\rho}^{(0)} \leftarrow 1, \hat{\sigma}^{(0)} \leftarrow 1 \text{ initialization}
\]

for $t = 1$ to $T$ do

Rescale $\hat{\rho}^{(t)}, \hat{\sigma}^{(t)}$ so that $\sum_{j \in G} \rho_j^{(t)} + \sum_{i \in A} q_i^{(t)} = n + \sum_{i \in A} c_i \min_{j \in G} \frac{p_j^{(t)} + q_i^{(t)}}{u_{ij}}$.

for $i = 1$ to $n$ do

\[
x_i^{(t)} \leftarrow \arg \max_y \left\{ \sum_{j \in G} u_{ij} y_j : \sum_{j \in G} \left( \rho_j^{(t)} + q_i^{(t)} \right) y_j \leq 1 + c_i \min_{j \in G} \frac{p_j^{(t)} + q_i^{(t)}}{u_{ij}} \right\}.
\]

for $j = 1$ to $n$ do

\[
d_j^{(t)} \leftarrow \sum_{i \in A} x_i^{(t)}
\]

\[
h_j^{(t)} \leftarrow \sum_{i \in G} x_i^{(t)}
\]

$\sigma^{(t)} \leftarrow \min \left( \frac{1}{\max_{j \in G} d_j^{(t)}}, \frac{1}{\max_{i \in A} h_i^{(t)}} \right)$

for $j = 1$ to $n$ do

\[
p_j^{(t+1)} \leftarrow \hat{p}_j^{(t)} (1 + \epsilon \sigma^{(t)} d_j^{(t)})
\]

for $i = 1$ to $n$ do

\[
q_i^{(t+1)} \leftarrow \hat{q}_i^{(t)} (1 + \epsilon \sigma^{(t)} h_i^{(t)})
\]

return $\bar{x} \leftarrow \frac{\sum_{t=1}^{T} \sigma^{(t)} x_i^{(t)}}{\sum_{t=1}^{T} \sigma^{(t)}}$ (matrix), $\bar{p} \leftarrow \frac{\sum_{t=1}^{T} \sigma^{(t)} p_j^{(t)}}{\sum_{t=1}^{T} \sigma^{(t)}}, \bar{q} \leftarrow \frac{\sum_{t=1}^{T} \sigma^{(t)} q_i^{(t)}}{\sum_{t=1}^{T} \sigma^{(t)}}$.

### 4.2 MWUA Analysis for LiAD via (F-LiAD)

Since we showed equivalence (Lemma 13) between Program (8) and (F-LiAD), we will apply MWUA on the latter (see Algorithm 1). The rest of the section will be focusing on proving convergence and bounding the rate of convergence of MWUA. We shall show that after $O \left( \frac{n \log n}{\epsilon^2} \right)$ iterations, MWUA will reach a point $(\bar{x}, \bar{p}, \bar{q})$ that is an $\epsilon$-approximate feasible solution for (F-LiAD) and hence $\frac{1}{\epsilon^2} \cdot \bar{x}$ will be an $O(\epsilon)$-approximate optimal solution for each agent in Program (8), as stated in the aforementioned Corollary 15.

Before we proceed with the analysis of Algorithm 1, we need to argue that the rescaling step in the For-loop is well-defined, i.e., we can always rescale $\hat{\rho}^{(t)}, \hat{\sigma}^{(t)}$ to $p^{(t)}, q^{(t)}$ so that $\sum_{j \in G} p_j^{(t)} + \sum_{i \in A} q_i^{(t)} = n + \sum_{i \in A} c_i \min_{j \in G} \frac{p_j^{(t)} + q_i^{(t)}}{u_{ij}}$. This is captured by the following lemma:

**Lemma 16** (Rescaling). Assume (F-LiAD) is feasible. Given $\bar{p}, \bar{q} \geq 1$, we can always rescale them to $p, q$ so that $\sum_{j \in G} p_j + \sum_{i \in A} q_i = n + \sum_{i \in A} c_i \min_{j \in G} \frac{p_j + q_i}{u_{ij}}$.
Proof. Let \( \sum_{j \in G} \hat{p}_j + \sum_{i \in A} \hat{q}_i - \sum_{i \in A} c_i \min_{j \in G} \frac{\hat{p}_j + \hat{q}_i}{u_{ij}} = \lambda \). As long as \( \lambda > 0 \) then we can set \( p = \frac{\lambda}{\epsilon} \cdot \hat{p}, q = \frac{\lambda}{\epsilon} \cdot \hat{q} \) and the lemma would follow. Therefore it suffices to show \( \lambda \) is positive. Using the assumption (F-LiAD) is feasible we get that there exists \( x^* \) with \( u_i(x^*) > c_i \) for all \( i \in A \), thus \( \max_{j \in G} u_{ij} \geq u_i(x^*) > c_i \). We conclude

\[
\sum_{i \in A} c_i \min_{j \in G} \frac{\hat{p}_j + \hat{q}_i}{u_{ij}} \leq \sum_{i \in A} c_i \min_{j \in G} \frac{\hat{p}_j + \hat{q}_i}{u_{ij}(x^*)} \\
\leq \frac{1}{\epsilon} \sum_{i \in A} c_i \sum_{j \in G} \frac{\hat{p}_j + \hat{q}_i}{u_{ij}(x^*)} \\
< \frac{1}{\epsilon} \sum_{i \in A} \sum_{j \in G} \hat{p}_j + \hat{q}_i = \sum_{j \in G} \hat{p}_j + \sum_{i \in A} \hat{q}_i,
\]

from which follows \( \lambda > 0 \).

Potential function. To bound the rate of convergence of MWUA and show that it actually reaches an \( O(\epsilon) \)-approximate feasible point of (F-LiAD), we use a potential function argument, an idea that is quite common in online learning literature [AHK12]. The potential function at iterate \( t \) is defined to be (sum of prices at iterate \( t \))

\[
\Phi(t) = \sum_{j \in G} \hat{p}^{(t)}_j + \sum_{i \in A} \hat{q}^{(t)}_i.
\]

(19)

We need to find upper and lower bounds on \( \Phi(T) \) which will enable us to bound the convergence rate of MWUA (\( T \) denotes the number of iterations of MWUA). The first lemma gives an upper bound on the potential function \( \Phi \).

Lemma 17 (Upper bound on \( \Phi(T) \)). The following holds:

\[
\Phi(T) \leq 2n \cdot \exp \left( \epsilon \sum_{t=1}^{T-1} \sigma^{(t)} \right),
\]

(20)

where \( \sigma^{(t)} := \min \left( \frac{1}{\max_{j \in G} \sum_{i \in A} x_j^{(t)}}, \frac{1}{\max_{i \in A} \sum_{j \in G} x_j^{(t)}} \right) \) \( \epsilon \) (see Algorithm 1).

Proof. Fix a time index \( t \). We get that \( \frac{\Phi(t+1)}{\Phi(t)} \) is equal to

\[
\frac{\sum_{j \in G} \hat{p}^{(t+1)}_j + \sum_{i \in A} \hat{q}^{(t+1)}_i}{\Phi(t)} = \frac{\sum_{j \in G} (1 + \epsilon \sigma^{(t)} d_j^{(t)}) \hat{p}^{(t)}_j + \sum_{i \in A} (1 + \epsilon \sigma^{(t)} h_i^{(t)}) \hat{q}^{(t)}_i}{\Phi(t)}
\]

(definition of \( \Phi(t+1) \))

\[
= \frac{1 + \epsilon \sigma^{(t)} \sum_{j \in G} d_j^{(t)} \hat{p}^{(t)}_j + \sum_{i \in A} h_i^{(t)} \hat{q}^{(t)}_i}{\Phi(t)}
\]

(1 + \epsilon \sigma^{(t)} \sum_{j \in G} d_j^{(t)} \hat{p}^{(t)}_j + \sum_{i \in A} h_i^{(t)} \hat{q}^{(t)}_i)

\[
\Phi(t)
\]

\( \sigma^{(t)} \) is essentially the renormalization parameter so that the “utility” of each constraint/expert in MWU is bounded by one.
Lemma 18

In what follows, we prove a lower bound on the potential function \( \Phi \).

Claim. We will need the following straightforward auxiliary fact:

\[
\epsilon \geq 1 + \sum_{i \in A} \sum_{j \in G} x_{ij}^{(l)} \cdot p_j^{(l)} + \sum_{i \in A} \sum_{j \in G} x_{ij}^{(l)} \cdot q_i^{(l)}
\]

(rescale \( a^{(l)} \) and def. of \( d^{(l)}, h_i^{(l)} \))

\[
= 1 + \epsilon \sigma^{(l)} \sum_{i \in G} \sum_{j \in A} x_{ij}^{(l)} \cdot p_j^{(l)} + \sum_{i \in A} \sum_{j \in G} x_{ij}^{(l)} \cdot q_i^{(l)}
\]

(change order of sum)

\[
\leq 1 + \epsilon \sigma^{(l)} \sum_{i \in A} \sum_{j \in G} \frac{p_j^{(l)} + q_i^{(l)}}{a_{ij}^{(l)}}
\]

\[
= 1 + \epsilon \sigma^{(l)} \sum_{i \in A} \sum_{j \in G} \frac{p_j^{(l)} + q_i^{(l)}}{a_{ij}^{(l)}}
\]

\[
= 1 + \epsilon \sigma^{(l)} \sum_{i \in A} \sum_{j \in G} \frac{p_j^{(l)} + q_i^{(l)}}{a_{ij}^{(l)}}
\]

\[
= 1 + \epsilon \sigma^{(l)} \sum_{i \in A} \sum_{j \in G} \frac{p_j^{(l)} + q_i^{(l)}}{a_{ij}^{(l)}}
\]

\[
= 1 + \epsilon \sigma^{(l)} \sum_{i \in A} \sum_{j \in G} \frac{p_j^{(l)} + q_i^{(l)}}{a_{ij}^{(l)}}
\]

(definition of \( p_j^{(l)}, q_i^{(l)} \) in Algorithm 1)

\[
= 1 + \epsilon \sigma^{(l)} \leq 1 + \exp(\epsilon \sigma^{(l)}).
\]

Multiplying telescopically we have that

\[
\Phi(T) \leq \Phi(1) \cdot \exp \left( \epsilon \sum_{t=1}^{T-1} \sigma^{(t)} \right) = 2n \cdot \exp \left( \epsilon \sum_{t=1}^{T-1} \sigma^{(t)} \right)
\]

and the claim follows. \( \square \)

In what follows, we prove a lower bound on the potential function \( \Phi \).

Lemma 18 (Lower bound on \( \Phi(T) \)). The following holds:

\[
\Phi(T) \geq \exp \left( \epsilon (1 - \epsilon) \max \left( \sum_{t=1}^{T} \sigma^{(t)} \cdot d_j^{(t)}, \max_{i \in A} \sum_{t=1}^{T} \sigma^{(t)} \cdot h_i^{(t)} \right) \right),
\]

(21)

where \( d_j^{(t)} := \sum_{i \in A} x_{ij}^{(t)} \) and \( h_i^{(t)} := \sum_{j \in G} x_{ij}^{(t)} \) (see Algorithm 1).

Proof. We will need the following straightforward auxiliary fact:

Claim. \( e^{\mu(1-y)} \leq 1 + \mu \) for \( 0 < \mu \leq y < 1 \). By definition of \( \Phi \) we get

\[
\Phi(T) = \sum_{j \in G} p_j^{(T)} + \sum_{i \in A} q_i^{(T)}
\]

\[
\geq \sum_{j \in G} \prod_{t=1}^{T-1} (1 + \epsilon \sigma^{(t)} \cdot d_j^{(t)}) + \sum_{i \in A} \prod_{t=1}^{T-1} (1 + \epsilon \sigma^{(t)} \cdot h_i^{(t)})
\]

(definition of Algorithm 1)

\[
\geq \sum_{j \in G} \exp \left( \epsilon (1 - \epsilon) \sum_{t=1}^{T-1} \sigma^{(t)} \cdot d_j^{(t)} \right) + \sum_{i \in A} \exp \left( \epsilon (1 - \epsilon) \sum_{t=1}^{T-1} \sigma^{(t)} \cdot h_i^{(t)} \right)
\]

(use of Claim)

\[
\geq \max \left( \sum_{j \in G} \exp \left( \epsilon (1 - \epsilon) \sum_{t=1}^{T-1} \sigma^{(t)} \cdot d_j^{(t)} \right), \max_{i \in A} \sum_{t=1}^{T-1} \sigma^{(t)} \cdot h_i^{(t)} \right), \right)
\]

\[
= \exp \left( \epsilon (1 - \epsilon) \max \left( \sum_{j \in G} \sigma^{(t)} \cdot d_j^{(t)}, \max_{i \in A} \sum_{t=1}^{T} \sigma^{(t)} \cdot h_i^{(t)} \right) \right).
\]

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Combining Lemmas 17 and 18 we conclude that
\[
e(1 - \epsilon) \max \left( \max_{i \in G} \sum_{t=1}^{T} \sigma^{(t)} d_{ij}^{(t)}, \max_{i \in A} \sum_{t=1}^{T} \sigma^{(t)} h_{ij}^{(t)} \right) \leq \ln \Phi(T) \leq \ln(2n) + \epsilon \sum_{t=1}^{T-1} \sigma^{(t)}.
\] 
(22)

Proof of part (a) of Theorem 12. Let \( \bar{x} = \frac{\sum_{t=1}^{T} \sigma^{(t)} x_{ij}(t)}{\sum_{t=1}^{T} \sigma^{(t)}} \) and \( \bar{p} = \frac{\sum_{t=1}^{T} \sigma^{(t)} p_{ij}(t)}{\sum_{t=1}^{T} \sigma^{(t)}} \) and \( \bar{q} = \frac{\sum_{t=1}^{T} \sigma^{(t)} q_{ij}(t)}{\sum_{t=1}^{T} \sigma^{(t)}} \) (average among all allocations, prices at all times, the output of MWUA 1). Observe that \( \bar{x}, \bar{p}, \bar{q} \) are non-negative and satisfy the prices constraints (because of the rescaling step). We will show, as long as \( T \) is chosen to be \( \Theta \left( \frac{2n \log 2n}{\epsilon^2} \right) \), that
\[
CP_{\epsilon}(\bar{p}, \bar{q}) \leq u_{\epsilon}(\bar{x}) \quad \text{for all } i \in A,
\]
(23)
and moreover
\[
\sum_{i \in A} \bar{x}_{ij} \leq 1 + O(\epsilon) \quad \text{for all } j \in G \quad \text{and} \quad \sum_{j \in G} \bar{x}_{ij} \leq 1 + O(\epsilon) \quad \text{for all } i \in A.
\]
(24)

By showing (23) and (24) we will get that \( (\bar{x}, \bar{p}, \bar{q}) \) is an \( O(\epsilon) \)-approximate feasible of (F-LiAD) and thus from Corollary 15 \( \frac{1}{1 + O(\epsilon)} \cdot \bar{x} \) will be \( O(\epsilon) \)-approximate optimal solution for each agent in Program (8).

**Showing (23).** To show that \( CP_{\epsilon}(\bar{p}, \bar{q}) \leq u_{\epsilon}(\bar{x}) \) for all \( i \in A \), observe that
\[
CP_{\epsilon}(p^{(t)}, q^{(t)}) \leq u_{\epsilon}(x^{(t)}) \quad \forall i \in A \quad \text{(by choice of } x^{(t)} \text{ in Algorithm 1)}.
\]
(25)

It is not hard to see that \( CP_{\epsilon}(p, q) \) is a convex function for all \( i \in A \):

**Claim 19 (Convexity of \( CP_{\epsilon} \)).** \( CP_{\epsilon}(p, q) \) is convex for \( p > 0 \).

**Proof.** It is a well-known fact that the maximum of convex functions is convex. Moreover the function \( f(x_1, ..., x_k) = \frac{1}{\sum_{s=1}^{k} a_s x_s} \) with \( a_s \) non-negative is convex, therefore \( CP_{\epsilon}(p, q) = c_i + \max_{j \in G} \left\{ \frac{u_{ij}}{p_j + q_i} \right\} \) is also convex. \( \square \)

Using Claim 19 and Jensen’s inequality, it follows that
\[
CP_{\epsilon}(\bar{p}, \bar{q})^{19} \leq \sum_{t=1}^{T} \sigma^{(t)} CP_{\epsilon}(p^{(t)}, q^{(t)}) \leq \sum_{t=1}^{T} \sigma^{(t)} u_{\epsilon}(x_{ij}^{(t)}) = u_{\epsilon}(\bar{x}),
\]
i.e., inequality (23) is proved.

**Showing (24).** For the rest of the section, we will focus on showing (24), that is (for definition of \( \sigma^{(t)}, d^{(t)}, h^{(t)} \) see Algorithm 1) we need to show
\[
\sum_{t=1}^{T} \sigma^{(t)} d_{ij}^{(t)} := \sum_{i \in A} \bar{x}_{ij} \leq 1 + O(\epsilon) \quad \text{for all } j \in G
\]
and
\[
\sum_{t=1}^{T} \sigma^{(t)} h_{ij}^{(t)} := \sum_{j \in G} \bar{x}_{ij} \leq 1 + O(\epsilon) \quad \text{for all } i \in A
\]
Hence, \[
\max \left( \max_{j \in G} \sum_{i \in A} x_{ij}, \max_{i \in A} \sum_{j \in G} x_{ij} \right) \leq 1 + O(\epsilon),
\]
and (24) follows. Using Inequality (22) we get
\[
\max \left( \max_{j \in G} \sum_{i \in A} x_{ij}, \max_{i \in A} \sum_{j \in G} x_{ij} \right) = \max \left( \max_{j \in G} \sum_{i \in A} \sigma^{(t)} d_{ij}^{(t)}, \max_{i \in A} \sum_{j \in G} \sigma^{(t)} h_{ij}^{(t)} \right) \leq \ln(2n) + \epsilon \sum_{t=1}^{T} \sigma^{(t)} = 1 - \epsilon + \ln(2n) / \epsilon (1 - \epsilon) \sum_{t=1}^{T} \sigma^{(t)}.
\]
Observe that at every iteration \( t \), there exists a \( j \in G \) or \( i \in A \) so that \( p_j^{(t)} \) or \( q_i^{(t)} \) increases by a factor of \( (1 + \epsilon) \) (for iteration \( t \), it should be the arg max \( \sum_{j \in G} (d_{ij}^{(t)}, h_{ij}^{(t)}) \)). Hence after \( T \) iterations it follows that (Pigeonhole Principle)
\[
\max \left( \max_{j \in G} \sum_{i \in A} x_{ij}, \max_{i \in A} \sum_{j \in G} x_{ij} \right) \geq (1 + \epsilon)^{T/2n} \approx (2n)^{1/\epsilon}, \tag{26}
\]
where the last approximation holds by setting \( T = \frac{2n \log 2n}{\epsilon^2} \).

Finally it holds (using the fact that \( e^x \geq x + 1 \) for all \( x \))
\[
\max \left( \max_{j \in G} \sum_{i \in A} \sigma^{(t)} d_{ij}^{(t)}, \max_{i \in A} \sum_{j \in G} \sigma^{(t)} h_{ij}^{(t)} \right) = \max \left( \max_{j \in G} \frac{\ln \prod_{t=1}^{T} \exp \left( e \sigma^{(t)} d_{ij}^{(t)} \right)}{\epsilon \sum_{t=1}^{T} \sigma^{(t)}}, \max_{i \in A} \frac{\ln \prod_{t=1}^{T} \exp \left( e \sigma^{(t)} h_{ij}^{(t)} \right)}{\epsilon \sum_{t=1}^{T} \sigma^{(t)}} \right) \geq \max \left( \max_{j \in G} \frac{\ln \prod_{t=1}^{T} (1 + e \sigma^{(t)} d_{ij}^{(t)})}{\epsilon \sum_{t=1}^{T} \sigma^{(t)}}, \max_{i \in A} \frac{\ln \prod_{t=1}^{T} (1 + e \sigma^{(t)} h_{ij}^{(t)})}{\epsilon \sum_{t=1}^{T} \sigma^{(t)}} \right) \approx \ln 2n / \epsilon^2 \sum_{t=1}^{T} \sigma^{(t)} \text{ using (26)}. \]

Hence,
\[
\max \left( \max_{j \in G} \sum_{i \in A} x_{ij}, \max_{i \in A} \sum_{j \in G} x_{ij} \right) \leq \frac{1}{1 - \epsilon} + \frac{\epsilon}{1 - \epsilon} \cdot \max \left( \max_{j \in G} \sum_{i \in A} x_{ij}, \max_{i \in A} \sum_{j \in G} x_{ij} \right),
\]

or equivalently
\[
\max \left( \max_{j \in G} \sum_{i \in A} x_{ij}, \max_{i \in A} \sum_{j \in G} x_{ij} \right) \leq \frac{1}{1 - 2\epsilon} = 1 + O(\epsilon).
\]

Having shown (23) and (24), we conclude that \((\vec{x}, \vec{p}, \vec{q})\) is an \(O(\epsilon)\)-approximate feasible for F-LiAD and the proof of part (a) of Theorem 12 is complete.

### 4.3 SPLC utilities (SAD)

The previous approach can be generalized even for model SAD with piecewise linear concave utilities. We present the parts of the proof that need modification and the modified MWUA Algorithm. In a nutshell, we modify the price (dual variables) constraints to account for the extra constraints on the slopes. The feasibility program (analogously with F-LiAD) will have the form:

**Feasibility program.**

\[
\begin{align*}
CP_i(p, q, h) \leq u_i(x - c_i) & \quad \forall i \in A, \\
x_{ijk} \leq l_{ijk} & \quad \forall i, j, k \in A \times G \times [\ell], \\
\sum_{i \in A} \sum_{k \in [\ell]} x_{ijk} \leq 1 & \quad \forall j \in G, \\
\sum_{j \in G} \sum_{k \in [\ell]} x_{ijk} \leq 1 & \quad \forall i \in A, \\
\sum_{j \in G} \sum_{k \in [\ell]} (p_j + q_i + h_{ijk}) y_{jk} \leq 1 + c_i \min_{j,k \in G \times [\ell]} \frac{p_j + q_i + h_{ijk}}{u_{ijk}}, \\
x, p, q, h \geq 0.
\end{align*}
\]

where
\[
CP_i(p, q, h) := \max_{j \in G \times [\ell]} u_{ijk} y_{jk}
\]

\[
\text{s.t. } \sum_{j,k \in G \times [\ell]} (p_j + q_i + h_{ijk}) y_{jk} \leq 1 + c_i \min_{j,k \in G \times [\ell]} \frac{p_j + q_i + h_{ijk}}{u_{ijk}},
\]

\[
y \geq 0.
\]

Following the exact proof steps of Lemma 13 (using KKT conditions for SPLC utilities), it can also be shown that feasibility of (F-SPLC) is equivalent to solving 9.

**Lemma 20** (Optimization to Feasibility (SPLC)). Let \(x^*\) be an optimal solution of Program (9), then there exist \(p^*, q^*, h^*\) so that \((x^*, p^*, q^*, h^*)\) is feasible for (F-SPLC). Moreover assume that \((x^*, p^*, q^*, h^*)\) is a feasible solution for (F-SPLC), then \(x^*\) is an optimal solution for Program (9).

Moreover, Multiplicative Weights Update Algorithm (see Algorithm 2) is modified so that it accounts for the constraints on segments (i.e., \(x_{ijk} \leq l_{ijk}\)), i.e., we introduce extra variables \(h\) that penalize the allocations \(x_{ijk}\) that exceed the corresponding upper bound \(l_{ijk}\).

Before we proceed with the analysis of Algorithm 2, we must note that the rescaling step (first step inside the outer For-loop) is well-defined and the proof of the claim is identical to the proof of Lemma 16.
Algorithm 2: Multiplicative Weight Update for SPLC

\[ \tilde{p}(0) \leftarrow 1, \tilde{q}(0) \leftarrow 1, \tilde{h}(0) \leftarrow 1 \] initialization

for \( t = 1 \) to \( T \) do

Rescale \( \tilde{p}(t), \tilde{q}(t), \tilde{h}(t) \) so that

\[ \sum_{j \in G} \tilde{p}_j(t) + \sum_{i \in A} \tilde{q}_i(t) + \sum_{i,j,k \in A \times G \times [l]} \tilde{h}_{ijk}(t) = n + \sum_{i \in A} c_i \min_{j,k \in G \times [l]} \frac{\tilde{p}_j(t) + \tilde{q}_i(t) + \tilde{h}_{ijk}(t)}{u_{ijk}}. \]

for \( i = 1 \) to \( n \) do

\[ x_{ijk}(t) \leftarrow \arg \max_y \left\{ \sum_{j,k \in G \times [l]} u_{ijk} y_{jk} \colon \sum_{j,k \in G \times [l]} \left( \tilde{p}_j(t) + \tilde{q}_i(t) + \tilde{h}_{ijk}(t) \right) y_{jk} \leq 1 + c_i \min_{j,k \in G \times [l]} \frac{\tilde{p}_j(t) + \tilde{q}_i(t) + \tilde{h}_{ijk}(t)}{u_{ijk}} \right\}. \]

For all \( j, d_{G,j}(t) \leftarrow \sum_{i \in A} \sum_{k \in [l]} x_{ijk}(t) \) and for all \( i, d_{\Lambda,i}(t) \leftarrow \sum_{j \in G} \sum_{k \in [l]} x_{ijk}(t) \)

\[ r(t) \leftarrow \min \left( \frac{1}{\max_{j \in G} d_{G,j}(t)}, \frac{1}{\max_{i \in A} d_{\Lambda,i}(t)}, \min_{i,j,k \in A \times G \times [l]} \frac{1}{x_{ijk}} \right) \]

for \( j = 1 \) to \( n \) do

\[ \tilde{p}_j(t+1) \leftarrow \frac{\tilde{p}_j(t)}{t} \left( 1 + r(t) d_{G,j}(t) \right) \]

for \( i = 1 \) to \( n \) do

\[ \tilde{q}_i(t+1) \leftarrow \frac{\tilde{q}_i(t)}{t} \left( 1 + r(t) d_{\Lambda,i}(t) \right) \]

for \( i = 1 \) to \( n \) do

for \( k = 1 \) to \( l \) do

\[ \tilde{h}_{ijk}^{(t+1)} \leftarrow \tilde{h}_{ijk}^{(t)} \left( 1 + \epsilon r(t) x_{ijk}(t) \right) \]

return \( \chi = \frac{\sum_{t=1}^T r(t) x(t)}{\sum_{t=1}^T r(t)} \) (matrix), \( \overline{\chi} = \frac{\sum_{t=1}^T r(t)}{T} \), \( \overline{\eta} = \frac{\sum_{t=1}^T r(t) q(t)}{\sum_{t=1}^T r(t)} \), \( \overline{\eta} = \frac{\sum_{t=1}^T r(t) h(t)}{\sum_{t=1}^T r(t)} \) (average among all allocations, prices at all times, the output of MWUA 2). \( \chi, \overline{\chi}, \overline{\eta}, \overline{h} \) are non-negative and satisfy the prices constraints (because of the rescaling step). Moreover functions \( CP_i(p, q, h) \) are convex. As long as \( T \) is chosen to be \( \Theta \left( \frac{2n \log 2n}{\epsilon^2} \right) \), it holds

\[ CP_i(\overline{\chi}, \overline{\eta}, \overline{h}) \leq u_i(\chi) \] for all \( i \in A \),

and moreover

\[ \sum_{i \in A} \sum_{k \in [l]} \chi_{ijk} \leq 1 + O(\epsilon) \] for all \( j \in G \), \( \sum_{j \in G} \sum_{k \in [l]} \chi_{ijk} \leq 1 + O(\epsilon) \) for all \( i \in A \) and

\[ \chi_{ijk} \leq l_{ijk} + O(\epsilon) \] for all \( i, j, k \in A \times G \times [l] \).
As long as we have (30), we rescale appropriately so that \( \sum_{jk} x_{ijk} \leq 1 \) for all \( j \in G \) and \( \sum_{jk} x_{ijk} \leq 1 \) for all \( i \in A \). After rescaling, we push the allocations greedily for every agent \( i \) so that if segment \( 1 \leq t \leq l \) is not fully occupied (i.e., \( 0 < x_{ijt} < l_{ijt} \)) then \( x_{ijt-1} = \ell_{ijt-1} \). It turns out that the resulting allocation \( x \) will be an \( O(\epsilon) \)-approximate solution for each agent in Program (9).

5 Conditional Gradient Algorithms

Since the problem of finding Nash-bargaining points can be captured by a convex program, we can leverage techniques from convex optimization such as gradient descent to compute rapidly converging approximations. In this section we will provide a conditional gradient type algorithm which is relatively simple and projection free while converging at a rate of \( O(1/\epsilon) \).

5.1 Linear Utilities (LiF)

Let us first consider the simplest setting of a one-sided market with linear utilities, i.e. the model LiF. We are given a set \( A \) of agents and \( G \) of goods with \( |A| = |G| = n \) and utilities \( u_{ij} \) for all \( i \in A \) and \( j \in G \). Our goal is to solve the convex program

\[
\begin{align*}
\max & \quad \sum_{i \in A} \log u_i(x) \\
\text{s.t.} & \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\
& \quad x \geq 0,
\end{align*}
\]

where \( u_i(x) := \sum_{j \in G} u_{ij}x_{ij} \).

In the following we will assume that the utilities have been rescaled so that \( \max\{u_{ij} \mid j \in G\} = 1 \) for all \( i \) and \( \sum_{j \in G} u_{ij} \geq \frac{1}{\kappa} \) for some value \( \kappa \). In particular, the objective function is 1-strongly concave with respect to \( u_i(x) \) but not \( x_{ij} \). Note that since \( u_i(x) \) is not bounded from below, the objective function is neither Lipschitz nor smooth.

Recall also that we have shown in Lemma 9 that

\[
u_i(x) = \sum_{j \in G} u_{ij}x_{ij} \geq \frac{1}{2n} \sum_{j \in G} u_{ij} \geq \frac{1}{2\kappa n}.
\]

for any optimum solution \( x \). This implies that we could restrict ourselves to the problem

\[
\begin{align*}
\max & \quad \sum_{i \in A} \log u_i(x) \\
\text{s.t.} & \quad u_i(x) \geq \frac{1}{2\kappa n} \quad \forall i \in A, \\
& \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\
& \quad x \geq 0.
\end{align*}
\]

without changing the optimum solution. However, while general purpose projected gradient descent algorithms can achieve \( O(\log(1/\epsilon)) \) convergence rates on this modified problem, we wish to exploit the combinatorial structure of \( P \). Therefore we will modify the objective function instead.
Define the quadratic extension of the logarithm at $x_0 > 0$ by

$$
\eta(x; x_0) := \begin{cases} 
\log(x_0) + (x - x_0) \frac{1}{x_0} - \frac{1}{2} (x - x_0)^2 \frac{1}{x_0^2} & \text{if } x \leq x_0, \\
\log(x) & \text{otherwise.}
\end{cases}
$$

Then $\eta(x; x_0)$ is $\frac{1}{x_0^2}$-smooth everywhere, i.e. its gradient is $\frac{1}{x_0}$-Lipschitz.

Consider now the modified convex program

$$
\max \sum_{i \in A} \eta_u(x_i; \frac{1}{2kn}) \\
\text{s.t.} \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
\sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\
x \geq 0.
$$

(32)

**Lemma 21.** Every optimum solution to (32) is also optimum for (31).

**Proof.** First let $x$ be the optimum solution for (31). By Lemma 9, the objective function of the two programs agrees up to the first-order at $x$ and thus $x$ is also optimum for (32). But now any other optimum solution $x'$ for the modified problem must satisfy $u_i(x') = u_i(x)$ for all $i$ since the objective is strictly concave in $u_i$. Thus it is also an optimum solution for (31).

In the following let

$$
\phi(x) := \sum_{i \in A} \log u_i(x), \\
\psi(x) := \sum_{i \in A} \eta \left( u_i(x); \frac{1}{2kn} \right)
$$

be the objective functions of (31) and (32) respectively.

**Lemma 22.** Let $x$ be an arbitrary allocation and $x^*$ an optimal one. Let $\psi(x^*) - \psi(x) \leq \epsilon$ for some $\epsilon > 0$. Then $||u(x) - u(x^*)||^2 \leq 2\epsilon$. Moreover, if $\delta \in (0, 1/2)$, and $\epsilon = \min \left\{ \delta, \frac{\epsilon^{2/3}}{8nk} \right\}$, then $\phi(x^*) - \phi(x) \leq O(1)\delta$.

**Proof.** The first part follows directly from the fact that $\psi$ is 1-strongly concave over the feasible region. Now observe that by Taylor’s theorem we have

$$
\eta \left( u_i(x); \frac{1}{2nk} \right) - \log u_i(x) \leq \frac{2}{3} \left| \frac{u_i(x) - \frac{1}{2nk}}{u_i(x)^3} \right|^3.
$$

But because $u(x^*) \geq \frac{1}{2nk}$ and $||u(x) - u(x^*)||^2 \leq 2\epsilon$, we know that $\left| \frac{u_i(x) - \frac{1}{2nk}}{u_i(x)^3} \right| \leq |u_i(x) - u_i(x^*)|$ and

$$
u_i(x) \geq \frac{1}{2nk} - \sqrt{2\epsilon} \geq (1 - \delta^{1/3}) \frac{1}{2nk}.
$$
Finally, we compute
\[
\phi(x^*) - \phi(x) = \psi(x^*) - \psi(x) + \psi(x) - \phi(x) \\
\leq \delta + \sum_{i \in A} \left( \eta \left( u_i(x); \frac{1}{2n\kappa} \right) - \log u_i(x) \right) \\
\leq \delta + O(1)n^3\kappa^3 \sum_{i \in A} |u_i(x) - u_i(x^*)|^3 \\
\leq \delta + O(1)n^3\kappa^3 ||u(x) - u(x^*)||^{3/2} \\
\leq O(1)\delta.
\]

These two lemmas together imply that it suffices to solve (32). Approximate solutions to this modified program are also approximate solutions for (31), both wrt. the Euclidean norm on the utility vectors and the original objective function. Finding an approximate solution in a combinatorial way can be achieved using the conditional gradient method over the matching polytope; see Algorithm 3. Note that the gradient of \(\psi\) is easily computable.

Algorithm 3 Conditional Gradient for program (32)

1: \(x^{(0)} \equiv 0\)
2: for \(t \leftarrow 1, \ldots, T\) do
3: \(\text{for } i \in A, j \in G\) do
4: \(w_{ij}^{(t)} \leftarrow \partial_{ij}\psi(x^{(t-1)})\)
5: \(y^{(t)} \leftarrow \text{max-weight matching with weights } w^{(t)}\)
6: \(x^{(t)} \leftarrow (1 - \frac{2}{T+1}) x^{(t-1)} + \frac{2}{T+1} y^{(t)}\)
7: return \(x^{(T)}\)

Theorem 23. Algorithm 3 returns some \(x\) with \(\psi(x^*) - \psi(x) \leq \epsilon\) in \(O\left(\frac{n^3\kappa^2}{\epsilon}\right)\) many iterations. Each iteration can be implemented in \(O(n^3)\) time.

Proof. The conditional gradient algorithm converges in \(O\left(\frac{D^2L}{\epsilon}\right)\) iterations where \(D\) is the diameter of the polytope that is being optimized over and \(L\) is the smoothness of the objective function. For a modern proof of this fact, see for example [Jag13]. In this case, the diameter of the matching polytope is \(2\sqrt{n}\) and \(L = 4n^2\kappa^2\) since \(\eta \left( u_i; \frac{1}{2n\kappa} \right)\) is clearly \(4n^2\kappa^2\)-smooth. The amount of work in each iteration is \(O(n^2)\) except for the computation of the max weight matching which can be done in \(O(n^3)\) using the Hungarian method.

We remark that it is in principle possible to achieve faster convergence rates for conditional gradient type algorithms by leveraging strong concavity in addition to smoothness [GH16]. Note that while \(\psi\) is strongly concave in the utilities, it is unfortunately not strongly concave in the allocation \(x\).

Nevertheless, \(\psi\) can be written as \(g(Ux)\) where \(g\) is strongly concave and \(Ux\) is the linear transformation of allocations to utilities. There are more complex variants of conditional gradient methods, for example those which involve taking “away steps” [JLJ15], which can be shown to converge in \(O(\log(1/\epsilon))\) phases even in this more general setting. However, these algorithms depend on difficult to compute Hoffman-type constants of \(U\) relative to the matching polytope for which there is no known polynomial bound in the instance parameters.
5.2 Non-Bipartite Matching Markets (NBLF)

The result from the previous section can be extended to the non-bipartite setting, i.e. model NBLF. Recall that we have a set of $n$ agents $A$ with non-negative utilities $u_{ij}$ for all $i, j \in A$. The goal is then to solve the convex program

$$\begin{align*}
\max & \quad \sum_{i \in A} \log u_i(x) \\
\text{s.t.} & \quad x(\delta(i)) \leq 1 \quad \forall i \in A, \\
& \quad x(E(B)) \leq \frac{|B| - 1}{2} \quad \forall B \in \mathcal{O}, \\
& \quad x \geq 0
\end{align*}$$

(33)

where $u_i(x) = \sum_{j \in A} u_{ij} x_{ij}$ and $\mathcal{O}$ is the collection of all odd cardinality subsets of $A$.

Assume that the utilities have been rescaled so that $\max \{u_{ij} \mid j \in A\} = 1$ for all $i$ and $\sum_{j \in A} u_{ij} \geq \frac{1}{\kappa}$ for some value $\kappa$. Since it is possible to optimize over the matching polytope using combinatorial methods, the primary ingredient is to once again bound the utilities away from 0.

**Lemma 24.** Let $x$ be an optimal solution to (33), then for all agents $i$ we have

$$u_i(x) \geq \frac{1}{2n^2} \sum_{j \in \mathcal{G}} u_{ij} \geq \frac{1}{2n^2}.$$

**Proof.** The proof will be similar to the proof of Lemma 9, however the more complicated KKT conditions will yield a weaker bound. So let $p$ and $z$ be the optimal dual variables for our solution of (33).

Using the KKT conditions, we know that

$$\frac{u_{ij}}{u_i(x)} + \frac{u_{ji}}{u_j(x)} \leq p_i + p_j + \sum_{\{i,j\} \subseteq B \in \mathcal{O}} z_B$$

with equality when $x_{ij} > 0$. Therefore

$$\begin{align*}
2 \sum_{i \in A} p_i + \sum_{B \in \mathcal{O}} (|B| - 1)z_B &= \sum_{i \in A} \sum_{j \in A} \left( p_i + p_j + \sum_{\{i,j\} \subseteq B \in \mathcal{O}} z_B \right) x_{ij} \\
&= \sum_{i \in A} \sum_{j \in A} \left( \frac{u_{ij} x_{ij}}{u_i(x)} + \frac{u_{ji} x_{ij}}{u_j(x)} \right) \\
&= 2n.
\end{align*}$$

In addition, we can observe from the KKT conditions that

$$u_i(x) = \max \left\{ \frac{u_{ij}}{p_i + p_j + \sum_{\{i,j\} \subseteq B \in \mathcal{O}} z_B - u_{ji}/u_i(x)} \mid j \in A \right\}$$

$$\geq \max \left\{ \frac{u_{ij}}{p_i + p_j + \sum_{\{i,j\} \subseteq B \in \mathcal{O}} z_B} \mid j \in A \right\}$$

$$\geq \frac{\sum_{j \in A} u_{ij}}{np_i + \sum_{j \in A} p_j + \sum_{B \in \mathcal{O}} (|B| - 1)z_B}$$

$$\geq \frac{1}{2n^2} \sum_{j \in A} u_{ij}.$$
This bound may initially seem weak when compared to the $\frac{1}{2n} \sum_{j \in G} u_{ij}$ bound from Lemma 9 of the previous section. However, one can show that this bound is tight up to a constant factor.

More specifically, consider for some parameter $\ell \in \mathbb{N}$, an instance that has $2\ell + 1$ agents. Agents $1, \ldots, \ell$ have utility 1 for agent $2\ell + 1$ and 0 everywhere else. Agents $\ell + 1, \ldots, 2\ell$ are indifferent, i.e. have utility 1 for all agents. Finally, agent $2\ell + 1$ has utility 1 for agents $\ell + 1, \ldots, 2\ell$. One can then show that in the optimal allocation, agent $2\ell + 1$ will only get $\frac{1}{\ell + 1}$ units from their utility 1 edges. Therefore

$$\frac{\sum_{j \in A} u_{ij}}{u_i(x^*)} = \ell (\ell + 1) \approx \frac{n}{4}.$$ 

However, this utility bound is still adequate in order to show convergence of the conditional gradient method with the modified objective function $\psi(x) := \sum_{i \in A} \eta \left( u_i(x); \frac{1}{2n^2 \kappa} \right)$.

**Theorem 25.** Algorithm 3 returns some $x$ with $\psi(x^*) - \psi(x) \leq \epsilon$ in $O\left( \frac{n^3}{\epsilon^2} \right)$ many iterations. Each iteration can be implemented in $O(n^3)$ time.

**Proof.** The difference in the number of iterations compared to Theorem 23 comes from the fact that $\psi$ is now $4n^4 \kappa^2$-smooth. Each iteration can still be implemented in $O(n^3)$ time by using a weighted matching algorithm, now for non-bipartite graphs.

### 5.3 SPLC Utilities (SF)

The approach from the previous sections can be readily extended to SPLC utilities, i.e. model SF. In this case each agent $i$ has utilities $u_{ijk}$ for all $j \in G$ and $k \in \{1, \ldots, \ell\}$. Note that for ease of notation we assume here that all utility functions have the same number of segments. Moreover, each “segment” $i, j, k$ comes with a limit $l_{ijk}$ such that $\sum_{k=1}^{m} l_{ijk} = 1$ for all agents $i$ and goods $j$.\(^8\) Recall that our goal is to solve the convex program

\begin{align*}
\max & \quad \sum_{i \in A} \log u_i(x) \quad (34a) \\
\text{s.t.} & \quad \sum_{j \in G} \sum_{k=1}^{m} x_{ijk} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{i \in A} \sum_{k=1}^{m} x_{ijk} \leq 1 \quad \forall i \in A, \\
& \quad x_{ijk} \leq l_{ijk} \quad \forall i \in A, j \in G, k \in [m], \\
& \quad x \geq 0 \quad (34e)
\end{align*}

where $u_i(x) = \sum_{j \in G} \sum_{k=1}^{\ell} u_{ijk} x_{ijk}$.

Again we will assume that each agents utilities have been rescaled such that $\max\{u_{ijk} \mid j \in G, k \in [\ell]\} = 1$. Moreover, assume that $\sum_{j \in G} \sum_{k=1}^{m} u_{ijk} l_{ijk} \geq \frac{1}{\kappa}$ for some value $\kappa > 0$. Recall that we showed

$$u_i(x) \geq \frac{1}{2n} \sum_{j \in G} \sum_{k=1}^{\ell} u_{ijk} l_{ijk} \geq \frac{1}{4n^2 \kappa}.$$ 

\(^8\) We previously assumed that there is also a last segment that has unbounded length. However, since we will stay feasible at every step, this does not matter in this section.
for any optimum solution \( x \) in Lemma 11. From here, the modified convex program is given by

\[
\begin{align*}
\max & \quad \sum_{i \in A} \eta \left( u_i(x); \frac{1}{2kn} \right) \\
\text{s.t.} & \quad \sum_{i \in A} \sum_{k=1}^\ell x_{ijk} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{j \in G} \sum_{k=1}^\ell x_{ijk} \leq 1 \quad \forall i \in A, \\
& \quad x_{ijk} \leq t_{ijk} \quad \forall i \in A, j \in G, k \in [\ell], \\
& \quad x \geq 0
\end{align*}
\]

and we may use the conditional gradient method to get an \( \epsilon \)-approximate solution; see Algorithm 4. Here we make use of the operation \( \text{shift}(x) \) which takes a feasible solution \( x \) to (35) and for each agent \( i \) and good \( j \) it makes sure that the allocation uses segments with the largest value of \( u_{ijk} \) first. Assume wlog. that \( u_{ij1} > \cdots > u_{ij\ell} \), then \( \text{shift}(x) \) can be recursively defined via

\[
\text{shift}(x)_{ijk} := \min \left\{ t_{ijk}, \left( \sum_{k'=1}^\ell x_{ijk'} - \sum_{k'=1}^{k-1} \text{shift}(x)_{ijk'} \right) \right\}.
\]

Moreover, we remark that the gradient is given by

\[
\partial_{ijk} \psi(x) = \begin{cases} 
\frac{u_{ijk}}{u_i(x)} & \text{if } u_i \geq \frac{1}{2nk} \\
\frac{1}{4n^2k^2u_i(x)u_{ijk}} & \text{otherwise.}
\end{cases}
\]

**Algorithm 4** Conditional gradient for program (35)

1: \( x^{(0)} \equiv 0 \)
2: for \( t \leftarrow 1, \ldots, T \) do
3: \quad for \( i \in A, j \in G, k \in [\ell] \) do
4: \quad \quad \( w_{ijk}^{(t)} \leftarrow \partial_{ijk} \psi(x^{(t-1)}) \)
5: \quad \quad \( y^{(t)} \leftarrow \text{max-weight assignment with weights } w^{(t)} \text{ and edge capacities } l \)
6: \quad \quad \( x^{(t)} \leftarrow \text{shift}\left(\left(1 - \frac{2}{t+1}\right)x^{(t-1)} + \frac{2}{t+1}\text{shift}(y^{(t)})\right)\)
7: return \( x^{(T)} \)

**Theorem 26.** Algorithm 4 returns some \( x \) with \( \psi(x^*) - \psi(x) \leq \epsilon \) in \( O\left(\frac{n^3\kappa^2}{\epsilon}\right) \) many iterations. Each iteration can be implemented in \( O(n^4\ell^2 \log n) \) time.

**Proof.** Recall that we generally have convergence in \( O\left(\frac{D^2L}{\epsilon}\right) \) iterations where \( D \) is the diameter of the polytope that is being optimized over and \( L \) is the smoothness of the objective function. Normally, i.e. without the shifting, the key ingredient the proof is observing that

\[
\psi(x^{(t)}) - \psi(x^{(t-1)}) \geq a_t \nabla \psi(x^{(t-1)}) \cdot (y^{(t)} - x^{(t-1)}) - \frac{L}{2} a_t^2 ||y^{(t)} - x^{(t-1)}||^2
\]

where \( a_t = \frac{2}{t+1} \). Then one bounds

\[
\psi(x^{(t-1)}) \cdot (y^{(t)} - x^{(t-1)}) \geq \psi(x^{(t-1)}) \cdot (x^* - x^{(t-1)}) \geq \psi(x^*) - \psi(x^{(t-1)})
\]

36
and \(||y(t) - x(t-1)|| \leq D\) from which one can then deduce the stated rate of convergence.

Now note that \(\psi(\text{shift}(x)) \geq \psi(x)\) and \(\nabla \psi(x) \cdot \text{shift}(y) \geq \nabla \psi(x) \cdot y\). Thus we have

\[
\psi(x(t)) - \psi(x(t-1)) \geq \psi((1 - a_t)x(t-1) + a_t \text{shift}(y(t))) - \psi(x(t-1)) \\
\geq a_t \nabla \psi(x(t-1)) \cdot (\text{shift}(y(t)) - x(t-1)) \\
- \frac{L}{2} a_t^2 ||\text{shift}(y(t)) - x(t-1)||^2 \\
\geq a_t \nabla \psi(x(t-1)) \cdot (y(t) - x(t-1)) \\
- \frac{L}{2} a_t^2 ||\text{shift}(y(t)) - x(t-1)||^2.
\]

The advantage of this approach is that we now need to bound \(||\text{shift}(y(t)) - x(t-1)||^2\) instead of the distance between two potentially arbitrary points in the polytope. Indeed by the subadditivity of the square function for positive values, it now follows that

\[
||\text{shift}(y(t)) - x(t-1)||^2 \leq \sum_{i \in A} \sum_{j \in G} \left( \sum_{k=1}^{\ell} y_{ijk}^{(t)} - \sum_{k=1}^{\ell} x_{ijk}^{(t-1)} \right)^2.
\]

This allows us to conclude \(||\text{shift}(y(t)) - x(t-1)||^2 \leq 2\sqrt{n}\) because the vectors \(\left( \sum_{k=1}^{\ell} y_{ijk}^{(t)} \right)_{ij}\) and \(\left( \sum_{k=1}^{\ell} x_{ijk}^{(t-1)} \right)_{ij}\) are part of the bipartite matching polytope on agents \(A\) and goods \(G\).

As in the case of linear utilities, the objective function is \(4n^2\kappa^2\)-smooth and so we get convergence in \(O\left(\frac{n^3\kappa^2}{\epsilon}\right)\). The time per iteration is clearly bottlenecked by the computation of the max-weight assignment with edge capacities. This problem is a special case of computing a min-cost flow with \(2n\) vertices and \(n\ell\) capacitated edges which may be solved in \(O(n^4\ell^2 \log(n))\) time using Orlin’s algorithm via a classic reduction from the transshipment problem \([Orl93]\) or Vygen’s algorithm \([Vyg02]\). \(\square\)

We remark that in practice, computing min-cost flows is far more efficient than the stated worst case running time. This is especially the case when one has a “warm start”, i.e. a solution that is already close to optimal. In the case of algorithm 4, one may use \(y(t-1)\) as a starting solution for the min-cost flow algorithm that computes \(y(t)\).

### 5.4 Extension to Endowments (LiAD)

Let us now return to the case of linear utilities but in which agents come with preexisting endowments, i.e. each agent \(i\) has some disagreement utility \(c_i\) and the goal is then to solve the convex program

\[
\begin{align*}
\max_{i \in A} \ & \sum_{i \in A} \log(u_i(x) - c_i) \\
\st \ & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
& \sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\
& x \geq 0.
\end{align*}
\tag{36}
\]

where \(u_i(x) = \sum_{j \in G} u_{ij}x_{ij}\). This is model LiAD.
The added difficulty of this setting is that, in general, a feasible solution may not exist. We thus assume that we are given a feasible solution \((\hat{x}, \delta)\) of the LP

\[
\begin{align*}
\text{max} & \quad \delta \\
\text{s.t.} & \quad u_i(x) \geq (1 + \delta)c_i \quad \forall i \in A, \\
& \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\
& \quad x \geq 0, \\
& \quad \delta \geq 0.
\end{align*}
\]

with \(\delta > 0\). We call this \(\delta\) the feasibility gap of the instance.

In practice there are two likely scenarios as to how one might obtain \(\delta\). Either one solves the above linear program to find the optimal \(\delta\) or the disagreement utilities \(c_i\) are defined as some constant fraction of the agents’ utilities over their initial endowments. More precisely, assume that each agent \(i\) comes to the market with endowments \(e_{ij}\) over the goods \(j\). Then one may simply define \(c_i \coloneqq \frac{1}{1 + \delta} \sum_{j \in G} u_{ij} e_{ij}\). The market will then guarantee that no agent gets worse by a factor of more than \((1 + \delta)\) which is an approximate notion of individual rationality.

Regardless of how one obtains a feasible solution to (37), we may use it to derive a similar lower bound as in Lemma 9. Once again, assume that the utilities have been rescaled so that \(\max\{u_{ij} \mid j \in G\} = 1\) for all \(i\) and \(\sum_{j \in G} u_{ij} \geq \frac{1}{k}\) for some \(k > 0\).

**Lemma 27.** Let \(x\) be an optimal solution to (36), then for all agents \(i\) we have

\[
\frac{u_i(x)}{u_i(x) - c_i} \leq \frac{1}{c_i} \left( \frac{1}{2n^2(1 + 1/\delta)} \sum_{j \in G} u_{ij} \geq c_i + \frac{1}{2n^2(1 + 1/\delta)} \right) - c_i.
\]

**Proof.** Let \(p_j\) and \(q_i\) as always be optimum dual variables for the matching constraints in (36). Then by the KKT conditions we have that \(\frac{u_{ij}}{u_i(x) - c_i} \leq p_j + q_i\) with equality if \(x_{ij} > 0\).

From this complementarity we can deduce that

\[
\frac{u_i(x)}{u_i(x) - c_i} = \max \left\{ \frac{u_{ij}}{p_j + q_i} \mid j \in G \right\}
\]

and

\[
\frac{u_i(x)}{u_i(x) - c_i} = \sum_{j \in G} x_{ij}(p_j + q_i).
\]

Together this implies that

\[
\sum_{j \in G} x_{ij}(p_j + q_i) = 1 + c_i \min \left\{ \frac{p_j + q_i}{u_{ij}} \mid j \in G \right\}.
\]
Now we may use our feasible solution \((\hat{x}, \delta)\) to (37) in order to bound
\[
\sum_{j \in G} p_j + \sum_{i \in A} q_i = \sum_{i \in A} \sum_{j \in G} x_{ij}(p_j + q_i)
\]
\[
= n + \sum_{i \in A} c_i \min \left\{ \frac{p_j + q_i}{u_{ij}} \quad | \quad j \in G \right\}
\]
\[
\leq n + \frac{1}{1 + \delta} \sum_{i \in A} \sum_{j \in G} u_{ij} \hat{x}_{ij} \min \left\{ \frac{p_{j'} + q_i}{u_{ij'}} \quad | \quad j' \in G \right\}
\]
\[
\leq n + \frac{1}{1 + \delta} \sum_{i \in A} \sum_{j \in G} \hat{x}_{ij}(p_j + q_i)
\]
\[
\leq n + \frac{1}{1 + \delta} \left( \sum_{i \in A} p_j + \sum_{i \in A} q_i \right)
\]
which implies in particular that
\[
\sum_{j \in G} p_j + \sum_{i \in A} q_i \leq \left( 1 + \frac{1}{\delta} \right) n.
\]
Finally, using the same idea as in Lemma 9, we get
\[
u_i(x) = c_i \max \left\{ \frac{u_{ij}}{p_j + q_i} \quad | \quad j \in G \right\}
\]
\[
\geq c_i + \frac{\sum_{j \in G} u_{ij}}{nq_i + \sum_{j \in G} p_j}
\]
\[
\geq c_i + \frac{1}{2n^2(1 + 1/\delta)} \sum_{j \in G} u_{ij}.
\]

The modified convex program will thus use the objective function
\[
\psi(x) := \sum_{i \in A} \eta \left( \nu_i(x) - c_i \frac{1}{2n^2(1 + 1/\delta)\kappa} \right)
\]
and we can now use Algorithm 3 to solve the modified convex program with the new gradients
\[
\partial_{ij} \psi(x) = \begin{cases} 
\frac{u_{ij}}{u_i(x)} & \text{if } u_i(x) \geq \frac{1}{2n^2(1 + 1/\delta)\kappa} \\
4n^4(1 + 1/\delta)^2\kappa^2 u_{ij} u_{ij} & \text{otherwise.}
\end{cases}
\]

**Theorem 28.** Algorithm 3 returns \(x\) with \(\psi(x^*) - \psi(x) \leq \epsilon\) in \(O\left( \frac{n^3(1+1/\delta)^2\kappa^2}{\epsilon} \right)\) many iterations. Each iteration can be implemented in \(O(n^3)\) time.

**Proof.** The only difference to Theorem 23 is that the objective function \(\psi\) is now \(4n^4(1 + 1/\delta)^2\kappa^2\)-smooth instead of \(4n^2\kappa^2\) as before. \(\square\)

We remark that an analogous result to Lemma 22 holds also in this setting, i.e. convergence in the modified objective \(\psi\) implies convergence of the agents’ utilities in the Euclidean norm. Thus, if one chooses \(\epsilon\) on the order of \(O\left( \frac{1}{n\sqrt{(1+1/\delta)\kappa}} \right)\), then one is guaranteed that \(u_i(x) \geq c_i\) for all agents \(i\).
5.5 Two-Sided SPLC Utilities with Endowments (2SAD)

Let us finally consider the most complex model which includes two-sided SPLC utilities and endowments, i.e. model 2SAD. There two sides of agents $A$ and jobs $J$ with $|A| = |J| = n$. Each agent $i \in A$ has SPLC utilities over agents $j \in J$, defined as before by segments with utility $u_{ijk}$ and allocation limit $l_{ijk}$ for $k \in [\ell]$.

Likewise, each agent $j \in J$ has SPLC utilities over the agents $i \in A$ and these are defined by segments with utilities $w_{ijk}$ and the same limits $l_{ijk}$ for $k \in [\ell]$. These segments should satisfy $u_{ij1} \geq \cdots \geq u_{ij\ell}$, $w_{ij1} \geq \cdots \geq w_{ij\ell}$ and $\sum_{k=1}^{\ell} l_{ijk} = 1$. Finally, we assume that we are given disagreement utilities $c_i$ for agents in $A$ and $d_j$ for agents in $J$.

The goal is then to solve the convex program

$$
\max_{x \in A} \sum_{i \in A} \log(u_i(x) - c_i) + \sum_{j \in J} \log(w_j(x) - d_j) \tag{38a}
$$

subject to

$$
\sum_{i \in A} \sum_{k=1}^{\ell} x_{ijk} \leq 1 \quad \forall j \in J, \tag{38b}
$$

$$
\sum_{j \in J} \sum_{k=1}^{\ell} x_{ijk} \leq 1 \quad \forall i \in A, \tag{38c}
$$

$$
x_{ijk} \leq l_{ijk} \quad \forall i \in A, j \in J, k \in [\ell], \tag{38d}
$$

$$
x \geq 0 \tag{38e}
$$

where $u_i(x) := \sum_{j \in G} \sum_{k=1}^{\ell} u_{ijk} x_{ijk}$ and likewise for $w_j$.

We assume that the utilities have been scaled so that for all $i \in A$, we have $\max\{u_{ijk} \mid j \in J, k \in [m]\} = 1$ and $\sum_{j \in J} \sum_{k=1}^{m} l_{ijk} u_{ijk} \geq \frac{1}{\kappa}$ for some $\kappa$. Symmetrically, we require $\max\{w_{ijk} \mid i \in A, k \in [m]\} = 1$ and $\sum_{i \in A} \sum_{k=1}^{m} l_{ijk} w_{ijk} \geq \frac{1}{\kappa}$ for all $j \in J$. Finally, we also assume that there exists a feasible solution $\hat{x}$ with $u_i(\hat{x}) \geq c_i(1+\delta)$ and $w_j(\hat{x}) \geq d_j(1+\delta)$ for all $i$ and $j$.

**Lemma 29.** Let $x$ be an optimal solution to (38a), then for all agents $i$ and $j$ we have

$$
u_i(x) \geq c_i + \frac{1}{2n^2(1+1/\delta)\kappa},$$

$$w_j(x) \geq d_j + \frac{1}{2n^2(1+1/\delta)\kappa}.$$  

**Proof.** We essentially need to combine the proofs of Lemma 11 and Lemma 27. So let $p_j$ and $q_i$ be optimum dual variables for the matching constraints and let $h_{ijk}$ be the optimum dual variables for the constraints (34d). The KKT conditions yield that $\frac{u_{ijk}}{u_i(x) - c_i} + \frac{w_{ijk}}{w_j(x) - d_j} \leq p_j + q_i + h_{ijk}$ with equality when $x_{ijk} > 0$.

In particular, we have

$$u_i(x) - c_i = \max \left\{ \frac{u_{ijk}}{p_j + q_i + h_{ijk} - w_{ijk}/(w_j(x) - d_j)} \mid j \in J, k \in [\ell] \right\}$$

and thus

$$\frac{u_i(x)}{u_i(x) - c_i} = 1 + c_i \min \left\{ \frac{p_j + q_i + h_{ijk} - w_{ijk}/(w_j(x) - d_j)}{u_{ijk}} \mid j \in J, k \in [\ell] \right\}.$$  

We once again assume for the sake of simplicity that all utility functions have an equal number of segments.
Making use of our feasible allocation \( \hat{x} \) which satisfies \( u_i(\hat{x}) \geq (1 + \delta)c_i \) and \( w_j(\hat{x}) \geq (1 + \delta)d_j \) we can then compute

\[
\sum_{i \in A} \sum_{j \in J} \sum_{k=1}^{\ell} l_{ijk} h_{ijk} + \sum_{j \in J} p_j + \sum_{i \in A} q_i \\
= \sum_{i \in A} \frac{u_i}{u_i - c_i} + \sum_{j \in J} \frac{w_j}{w_j - d_j} \\
= n + \sum_{i \in A} c_i \min \left\{ \frac{p_j + q_i + h_{ijk}}{u_{ijk}} \mid j \in J, k \in [\ell] \right\} + \sum_{j \in J} \frac{w_j}{w_j - d_j} \\
\leq n + \frac{1}{1 + \delta} \left( \sum_{i \in A} \sum_{j \in J} \sum_{k=1}^{\ell} l_{ijk} h_{ijk} + \sum_{j \in J} p_j + \sum_{i \in A} q_i \right) \\
- \sum_{j \in J} \frac{d_j}{w_j - d_j} + \sum_{j \in J} \frac{w_j}{w_j - d_j} \\
= 2n + \frac{1}{1 + \delta} \left( \sum_{i \in A} \sum_{j \in J} \sum_{k=1}^{\ell} l_{ijk} h_{ijk} + \sum_{j \in J} p_j + \sum_{i \in A} q_i \right).
\]

This implies that

\[
\sum_{i \in A} \sum_{j \in J} \sum_{k=1}^{m} l_{ijk} h_{ijk} + \sum_{j \in J} p_j + \sum_{i \in A} q_i \leq 2n \left( 1 + \frac{1}{\delta} \right)
\]

and thus

\[
u_i = c_i + \max \left\{ \frac{u_{ijk}}{p_j + q_i + h_{ijk} - h_{ijk}/(w_j - d_j)} \mid j \in J, k \in [m] \right\} \\
\geq c_i + \max \left\{ \frac{u_{ijk}}{p_j + q_i + h_{ijk}} \mid j \in J, k \in [m] \right\}
\]

\[
\geq \sum_{j \in J} \sum_{k=1}^{m} \frac{u_{ijk}}{p_j + q_i + h_{ijk}} \cdot \frac{l_{ijk} \cdot (p_j + q_i + h_{ijk})}{\sum_{j' \in J} \sum_{k'=1}^{m} l_{j'k'} h_{j'k'}} \\
\geq \frac{\sum_{j \in J} \sum_{k=1}^{m} l_{ijk} u_{ijk}}{2n^2 (1 + 1/\delta)}.
\]

By symmetry the same bound holds for all \( w_j \).

Thus the modified objective function is given by

\[
\psi(x) := \sum_{i \in A} \eta \left( u_i(x) - c_i; \frac{1}{2n^2 (1 + 1/\delta) \kappa} \right) + \sum_{j \in J} \eta \left( w_j(x) - d_j; \frac{1}{2n^2 (1 + 1/\delta) \kappa} \right)
\]

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and its gradient is easily computable. As such we may apply conditional gradient as before to obtain the following theorem.

**Theorem 30.** Algorithm 4 returns some \( x \) with \( \psi(x^*) - \psi(x) \leq \epsilon \) in \( O\left( \frac{n^5(1+1/\delta)^2\kappa^2}{\epsilon} \right) \) many iterations. Each iteration can be implemented in \( O(n^4 m^2 \log n) \) time.

*Proof.* The proof is identical to that of Theorem 26 with the only difference being the fact that \( \psi \) is \( 8n^4(1 + 1/\delta)^2\kappa^2 \)-smooth which is where the change in the number of iterations comes from. Note that the polytope over which we are optimizing has not changed so each iteration can be implemented in the same way as before. \( \square \)
References

[ACGH20] Rediet Abebe, Richard Cole, Vasilis Gkatzelis, and Jason D Hartline. A truthful cardinal mechanism for one-sided matching. In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2096–2113. SIAM, 2020.

[ACY15] Atila Abdulkadiroğlu, Yeon-Koo Che, and Yosuke Yasuda. Expanding “choice” in school choice. American Economic Journal: Microeconomics, 7(1):1–42, 2015.

[AHK12] Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. Theory Comput., 8(1):121–164, 2012.

[Bir46] Garrett Birkhoff. Tres observaciones sobre el algebra lineal. Univ. Nac. Tucuman, Ser. A, 5:147–154, 1946.

[BM01] Anna Bogomolnaia and Hervé Moulin. A new solution to the random assignment problem. Journal of Economic theory, 100(2):295–328, 2001.

[Bub15] Sébastien Bubeck. Convex optimization: Algorithms and complexity. Found. Trends Mach. Learn., 8(3-4):231–357, 2015.

[Bud11] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy, 119(6):1061–1103, 2011.

[CBL06] Nikolo Cesa-Bianchi and Gabor Lugosi. Prediction, Learning, and Games. Cambridge University Press, 2006.

[CCPY22] Thomas Chen, Xi Chen, Binghui Peng, and Mihalis Yannakakis. Computational hardness of the hylland-zeckhauser scheme. In Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, 2022.

[CDG+17] Richard Cole, Nikhil Devanur, Vasilis Gkatzelis, Kamal Jain, Tung Mai, Vijay V Vazirani, and Sadra Yazdanbod. Convex program duality, Fisher markets, and Nash social welfare. In Proceedings of the 2017 ACM Conference on Economics and Computation, pages 459–460, 2017.

[CG18] Richard Cole and Vasilis Gkatzelis. Approximating the Nash social welfare with indivisible items. SIAM Journal on Computing, 47(3):1211–1236, 2018.

[CKM+11] Paul F. Christiano, Jonathan A. Kelner, Aleksander Madry, Daniel A. Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In Lance Fortnow and Salil P. Vadhan, editors, Proceedings of the 43rd ACM Symposium on Theory of Computing, STOC 2011, San Jose, CA, USA, 6-8 June 2011, pages 273–282. ACM, 2011.

[CKM+19] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum Nash welfare. ACM Transactions on Economics and Computation (TEAC), 7(3):1–32, 2019.

[CLPV14] Erick Chastain, Adi Livnat, Christos H. Papadimitriou, and Umesh V. Vazirani. Algorithms, games, and evolution. Proc. Natl. Acad. Sci. USA, 111(29):10620–10623, 2014.

[DPSV08] Nikhil R Devanur, Christos H Papadimitriou, Amin Saberi, and Vijay V Vazirani. Market equilibrium via a primal–dual algorithm for a convex program. Journal of the ACM (JACM), 55(5):22, 2008.

[EG59] E. Eisenberg and D. Gale. Consensus of subjective probabilities: the Pari-Mutuel method. The Annals of Mathematical Statistics, 30:165–168, 1959.
[EIV23] F. Echenique, N. Immorlica, and V.V. Vazirani, editors. *Online and Matching-Based Market Design*. Cambridge University Press, 2023.

[EMZ19] Federico Echenique, Antonio Miralles, and Jun Zhang. Fairness and efficiency for probabilistic allocations with endowments. *arXiv preprint arXiv:1908.04336*, 2019.

[EMZ21] Federico Echenique, Antonio Miralles, and Jun Zhang. Constrained pseudo-market equilibrium. *American Economic Review*, 111(11):3699–3732, November 2021.

[FGK+08] Lisa Fleischer, Rahul Garg, Sanjiv Kapoor, Rohit Khandekar, and Amin Saberi. A fast and simple algorithm for computing market equilibria. In *International Workshop on Internet and Network Economics*, pages 19–30. Springer, 2008.

[FL98] Drew Fudenberg and David K. Levine. *The Theory of Learning in Games*. MIT Press Books. The MIT Press, 1998.

[ftToC19] Simons Institute for the Theory of Computing. Online and matching-based market design, 2019. https://simons.berkeley.edu/programs/market2019.

[GH16] Dan Garber and Elad Hazan. A linearly convergent variant of the conditional gradient algorithm under strong convexity, with applications to online and stochastic optimization. *SIAM Journal on Optimization*, 26(3):1493–1528, 2016.

[GHJY15] Rong Ge, Furuong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points - online stochastic gradient for tensor decomposition. In Peter Grünwald, Elad Hazan, and Satyen Kale, editors, *Proceedings of The 28th Conference on Learning Theory, COLT 2015, Paris, France, July 3-6, 2015*, volume 40 of *JMLR Workshop and Conference Proceedings*, pages 797–842. JMLR.org, 2015.

[GK20] Yuan Gao and Christian Kroer. First-order methods for large-scale market equilibrium computation. *Advances in Neural Information Processing Systems*, 33, 2020.

[GLS12] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.

[GS62] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[GT24] Jugal Garg, Thorben Tröbst, and Vijay Vazirani. One-sided matching markets with endowments: Equilibria and algorithms. *Autonomous Agents and Multi-Agent Systems*, 38(2):40, 2024.

[HMPY18] Yinghua He, Antonio Miralles, Marek Pycia, and Jianye Yan. A pseudo-market approach to allocation with priorities. *American Economic Journal: Microeconomics*, 10(3):272–314, 2018.

[HV22] Mojtaba Hosseini and Vijay V Vazirani. Nash-bargaining-based models for matching markets: One-sided and two-sided; fisher and arrow-debreu. In *Proceedings of the 2022 Conference on Innovations in Theoretical Computer Science*, ITCS, 2022.

[HZ79] Aanund Hylland and Richard Zeckhauser. The efficient allocation of individuals to positions. *Journal of Political economy*, 87(2):293–314, 1979.

[Jag13] Martin Jaggi. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. In Sanjoy Dasgupta and David McAllester, editors, *Proceedings of the 30th International Conference on Machine Learning*, volume 28 of *Proceedings of Machine Learning Research*, pages 427–435, Atlanta, Georgia, USA, 17–19 Jun 2013. PMLR.

[Jai07] Kamal Jain. A polynomial time algorithm for computing an Arrow-Debreu market equilibrium for linear utilities. *SIAM Journal on Computing*, 37(1):303–318, 2007.
[JLJ15] Martin Jaggi and Simon Lacoste-Julien. On the global linear convergence of frank-wolfe optimization variants. *Advances in Neural Information Processing Systems*, 28, 2015.

[JNG+21] Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M. Kakade, and Michael I. Jordan. On non-convex optimization for machine learning: Gradients, stochasticity, and saddle points. *J. ACM*, 68(2):11:1–11:29, 2021.

[KM17] Adam R. Klivans and Raghu Meka. Learning graphical models using multiplicative weights. In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 343–354. IEEE Computer Society, 2017.

[Le17] Phuong Le. Competitive equilibrium in the random assignment problem. *International Journal of Economic Theory*, 13(4):369–385, 2017.

[LPP+19] Jason D. Lee, Ioannis Panageas, Georgios Piliouras, Max Simchowitz, Michael I. Jordan, and Benjamin Recht. First-order methods almost always avoid strict saddle points. *Math. Program.*, 176(1-2):311–337, 2019.

[LRR17] Avi Levy, Harishchandra Ramadas, and Thomas Rothvoss. Deterministic discrepancy minimization via the multiplicative weight update method. In Friedrich Eisenbrand and Jochen Könemann, editors, *Integer Programming and Combinatorial Optimization - 19th International Conference, IPCO 2017, Waterloo, ON, Canada, June 26-28, 2017, Proceedings*, volume 10328 of *Lecture Notes in Computer Science*, pages 380–391. Springer, 2017.

[McL18] Andy McLennan. Efficient disposal equilibria of pseudomarkets. In *Workshop on Game Theory*, volume 4, page 8, 2018.

[MCWG+95] Andreu Mas-Colell, Michael Dennis Whinston, Jerry R Green, et al. *Microeconomic theory*, volume 1. Oxford university press New York, 1995.

[Mou18] Hervé Moulin. Fair division in the age of internet. *Annual Review of Economics*, 2018.

[MPP15] Ruta Mehta, Ioannis Panageas, and Georgios Piliouras. Natural selection as an inhibitor of genetic diversity: Multiplicative weights updates algorithm and a conjecture of haploid genetics [working paper abstract]. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science, ITCS 2015, Rehovot, Israel, January 11-13, 2015*, page 73. ACM, 2015.

[Nas53] John Nash. Two-person cooperative games. *Econometrica: Journal of the Econometric Society*, pages 128–140, 1953.

[Orl93] James B. Orlin. A faster strongly polynomial minimum cost flow algorithm. *Operations Research*, 41(2):338–350, 1993.

[PW84] Manfred W Padberg and Laurence A Wolsey. Fractional covers for forests and matchings. *Mathematical Programming*, 29(1):1–14, 1984.

[SS74] Lloyd Shapley and Herbert Scarf. On cores and indivisibility. *Journal of mathematical economics*, 1(1):23–37, 1974.

[TV24] Thorben Tröbst and Vijay V. Vazirani. Cardinal-utility matching markets: The quest for envy-freeness, pareto-optimality, and efficient computability. *arXiv preprint arXiv:2402.08851*, 2024.

[Vaz07] Vijay V. Vazirani. Combinatorial algorithms for market equilibria. In Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V. Vazirani, editors, *Algorithmic Game Theory*. Cambridge University Press, 2007.
[Vaz12] Vijay V. Vazirani. The notion of a rational convex program, and an algorithm for the Arrow-Debreu Nash bargaining game. *Journal of the ACM*, 59(2), 2012.

[Vaz20] Vijay V. Vazirani. An extension of the Birkhoff-von Neumann Theorem to non-bipartite graphs. *arXiv preprint arXiv:2010.05984*, 2020.

[Vaz22] Vijay V. Vazirani. The general graph matching game: Approximate core. *Games and Economic Behavior*, 132:478–486, 2022.

[Vis21] Nisheeth Vishnoi. *Algorithms for Convex Optimization*. Cambridge University Press, 2021. To appear.

[VN53] John Von Neumann. A certain zero-sum two-person game equivalent to the optimal assignment problem. *Contributions to the Theory of Games*, 2(0):5–12, 1953.

[VY21] Vijay V. Vazirani and Mihalis Yannakakis. Computational complexity of the Hylland-Zeckhauser scheme for one-sided matching markets. In *12th Innovations in Theoretical Computer Science Conference, ITCS 2021, January 6-8, 2021, Virtual Conference*, 2021.

[Vyg02] Jens Vygen. On dual minimum cost flow algorithms. *Mathematical Methods of Operations Research*, 56(1):101–126, 2002.

[Ye08] Yinyu Ye. A path to the arrow–debreu competitive market equilibrium. *Mathematical Programming*, 111(1):315–348, 2008.