Super Mario’s prison break — A proposal of object-intelligent-feedback–based classical Zeno and anti-Zeno effects

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Abstract — Super Mario is imprisoned by a demon in a finite potential well on his way to save Princess Peach. He can escape from the well with the help of a flight of magic stairs floating in the space. However, the hateful demon may occasionally check his status. At that time, he has to make a judgement of either jumping to the inside ground immediately in order to avoid the discovery of his escape intention, or speeding up his escape process. Therefore, if the demon checks him too frequently such that there is no probability for him to reach the top of the barrier, he will be always inside the well, then a classical Zeno effect occurs. On the other hand, if the time interval between two subsequent checks is large enough such that he has a higher probability of being beyond the demon’s controllable range already, then the demon’s check actually speeds up his escape and a classical anti-Zeno effect takes place.

The Fletcher’s paradox proposed by Zeno of Elea states that a flying arrow is motionless because it occupies an equal space when it is at rest and will be occupying an equal space in locomotion at any moment. The paradox is clearly impossible in Newtonian mechanics because our common sense tells us that the size of the arrow is fixed without any fluctuation and there is no projective measurement in Newtonian mechanics. Nevertheless, quantum mechanics provides such a possibility of “motionlessness”. That is an unstable quantum state would never decay if it is observed continuously. This phenomenon, called quantum Zeno effect, was first proposed by Misra and Sudarshan [1] in 1977. There are two key ingredients in the quantum Zeno effect. They are the informatic description of quantum states and the projective measurement. Both of them are among the fundamentals of quantum mechanics. Because of the absence of the projective measurement in classical physics, the quantum Zeno effect (see a review [2]) then becomes a fascinating game existing uniquely in the quantum world.

To have a clear picture and for the later comparison, we first give a simple prove of the quantum Zeno effect. Consider a general quantum system described by a Hamiltonian $H$, and suppose the system is initially at an excited state $|\Psi(0)\rangle$, the quantum state, according to quantum mechanics, will evolve like

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle.$$  

If we measure the state after a small time interval $\delta t$, the survival probability of $|\Psi(0)\rangle$ can be measured by

$$P = |\langle\Psi(0)|\Psi(\delta t)\rangle| \approx 1 - \frac{(\delta t)^2}{2} \bar{H}^2,$$  

where $\bar{H}^2$ denotes the fluctuation of $H$. Therefore, if we perform $N$ measurements during a fixed time interval $\Delta t = N\delta t$, the final survival probability of $|\Psi(0)\rangle$ becomes

$$P \simeq 1 - \frac{(\Delta t)^2 \bar{H}^2}{2N},$$  

which tends to 1 if $N \to \infty$. This concludes a simple prove of the quantum Zeno effect, i.e. the state $|\Psi(0)\rangle$ will never decay if it is observed continuously. The quantum anti-Zeno effect [3,4] takes place if $\delta t$ is large enough, then $|\Psi(\delta t)\rangle$ is far away from $|\Psi(0)\rangle$, for instance $P < 1/2$, then any projective measurement will speed up the decay of the state $|\Psi(0)\rangle$ to other states.

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The quantum Zeno and anti-Zeno effects show how environmental interferences speed up or slow down a physical process. One may wonder why such a paradigm becomes a paradox in classical physics. Actually, the first ingredient in the above Zeno effect is not unique in quantum mechanics. The fluctuation term can be realized via a statistical process though in this case it might be classical while the quantum Zeno effect is based on quantum fluctuation. However, the second ingredient about the projective measurement does not exist in a classical physics process. So it is almost impossible to image Zeno and anti-Zeno effects in classical physics. However, as we will show below, a projective measurement in classical world can be also realized by an inclined judgement made by the measured object. That is an environmental interference can lead to an intelligent feedback from the measured object. Though such a feedback is somehow beyond physical law, it provides a possibility of Zeno and anti-Zeno effects in our classical world.

We consider such a scenario about Super Mario’s prisoner break (see fig. 1). He is imprisoned by a demon in a finite potential well, and a flight of magic stairs, which can provide him a random potential to jump up or down with a certain probability distribution, floats in the space. Super Mario can escape from the well with his own efforts and the random potential provided by the stairs. His effort is to block up his standing height with gradual progress hence enhances his chance continuously to reach the top of barrier. The random potential ensures that his state is a probability distribution within a certain time interval. Therefore, he can finally escape from the well given the duration time is longer enough. However, the hateful demon feels worried about him and may occasionally check his status. Once Super Mario hears the demon’s step, he has to make a judgement of either jumping to the inside ground immediately in order to avoid the discovery of his escape intention, or speeding up the escape process. His judgement is made based on the effective distance of his current state from the initial state because the distance determines his chance of final successful escape. Therefore, if the demon checks him too frequently such that there is no probability for him to even reach the top of the barrier, he will be always inside the well, then a classical Zeno effect occurs. On the other hand, if the time interval between two subsequent checks is large enough such that he might have already been on those steps outside the well, then the demon’s check actually speeds up his escape and a classical anti-Zeno effect takes place. Such a scenario, though is beyond physical laws in physical textbooks, seems reasonable in our daily life.

Let us now simply compare the above classical Zeno effect with the quantum one. First, in both effects, the fluctuation plays an important role. In the classical Zeno effect, it is a classical fluctuation, while in the quantum Zeno effect, it becomes a quantum fluctuation. Both fluctuations ensure an algebraical decay of the physical state at the very beginning time. Meanwhile, we would like to point out also that the algebraical decay can be realized even without fluctuations, which therefore, is not a necessary condition. Secondly, the significant difference between both effects is the definition of the projective measurement. In quantum mechanics, the measurement performed in experiments is artificial, while the projection of quantum state is a postulate whose correctness is assumed to be checked by experiments only. In our scenario, how to measure to the object’s state is artificial (done by the demon), the “projective” output is also artificial (an intelligent feedback by Super Mario). The later is possible if and only if the measured object itself has an inclined selective response. Therefore, the “projection” of Super Mario to hide his escape or speed up his escape is somehow beyond physical science, but belongs to behavioral science. From this point of view, the classical Zeno and anti-Zeno effects we proposed here look like a combination of physical science and behavioral science.

The above game provides us only a sketch for the classical Zeno and anti-Zeno effects. To quantitatively illustrate how Super Mario escapes from the well, we need to establish an effective model to describe his process. Actually, Super Mario’s escape process is very similar to the decay of a classical metastable state to a final stable state in statistical physics except for that the projective feedback should be done artificially. That is the demon’s check and Super Mario’s projective feedback can be simulated by projecting artificially the system’s state to the original one. Then we can use a statistical system, like the two-dimensional Ising model as an effective model to clarify the classical Zeno and anti-Zeno effects.

The Hamiltonian of the two-dimensional Ising model defined on a square lattice reads

$$H = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^z,$$

where $\sigma_i^z = \pm 1$ denotes the spin state at site $i$, $h$ is the magnetic field, and $M = \sum_i \sigma_i^z$ is the magnetization. The model has been solved exactly by Onsager [5] in 1944. According to the exact solution, a thermal phase transitions occurs at $T_c \simeq 2.269185$ in the thermodynamic limit [6]. The system is in a paramagnetic state above the critical point. The thermal distribution is located in the middle of the magnetization space, and there is no symmetry breaking. Below the critical point, the system will be self-magnetized and be in an ordered phase with either positive $M$ or negative $M$. A sketch of thermal distributions at various temperatures is shown in fig. 2, which is simulated from a $100 \times 100$ sample and $2 \times 10^8$ importance samplings.

In order to study the classical Zeno effects, we focus on the ordered phase below the critical point for a finite system. To be precise, we will focus only on a finite

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1Super Mario is the protagonist in the popular game of Super Mario.
2Prison Break is an American serial drama television series by Paul Scheuring.
100 × 100 sample in this paper to ensure a finite barrier between two magnetized states. In terms of the free energy, there are two wells separated by a finite barrier in the magnetization space, and Super Mario is supposed to be in one of the two wells. Since the barrier is finite, Super Mario has a certain probability, though very very small, to finally transit to the another well. The probability is determined by both the Boltzmann distribution and density of state in the magnetization space. We assume that the magic stair shown in the left picture of fig. 1 offers such a probability to Super Mario to jump up or down. Moreover, he makes also his own effort, say blocking up his starting point \( h \) (corresponding to magnetic field) with some boxes, to enhance his potential hence the transition probability to the right well. So if we plot the free energy in the magnetization space, it looks schematically like the right picture of fig. 1.

We assume Super Mario blocks up his starting point \( h \) so slowly hence the adiabatic theorem works. He will be in a metastable state as \( h \) increases until a complete transition occurs. In fig. 3, we show the state of Super Mario with various small \( h \)s. From the figure, we can see the state of Super Mario changes from his initial position to the right well gradual. In fig. 4, we show his distance from the initial state as a function of \( h \). Therefore, it is certainly expected that Super Mario will finally escape to the right well if there is no other interference. However, if the demon checks his status after a time interval \( \delta t \), then Super Mario should make a decision to return to the initial state or speed up his escape. Her decision depends on the distance between the current state and the initial state. The distance can be measured by the fidelity between two thermal states [7].

To be precise, according to statistical physics, Super Mario’s initial state (with symmetry breaking) can be described by

\[
\rho(T, h = 0) = \frac{1}{Z} \sum_{M < 0} e^{-\beta E_M} \tag{5}
\]

where the partition function is

\[
Z = \sum_{M < 0} e^{-\beta E_M} \tag{6}
\]

When he blocks up his height with a velocity of \( v \), which corresponds to the increasing rate of the external magnetic field, i.e. \( h = vt \), his instant state becomes

\[
\rho(T, h) = \frac{1}{Z} \sum_{M < 0} e^{-\beta (E_M - hM)} \tag{7}
\]
The state has clearly been dragged from the initial state under the external field, as shown in fig. 3. In the parameter space of $h$, the distance between the initial state and the instant state at $h$ can be measured by the thermal-state fidelity [7],

$$F(0; h) = \frac{1}{\sqrt{Z(0)Z(h)}} \sum_{M < 0} e^{-\beta(2H_{M} - hM)/2}.$$  

Then after a time interval $\delta t$,

$$F(0; \delta h) \simeq 1 - \frac{(\delta h)^2}{2} \chi_{F}. \quad (9)$$

where $\chi_F$ is the thermal-state fidelity susceptibility which is defined as [8]

$$\chi_F = \frac{\beta^2 (\langle M^2 \rangle - \langle M \rangle^2)}{4}. \quad (10)$$

The fidelity susceptibility is proposed in recent studies on the fidelity approach to quantum phase transitions [9] (See also a review article [10]). In quantum mechanics, the fidelity susceptibility denotes the leading response of a quantum state to an adiabatic parameter [8]. The most important here is that the leading term in fidelity is quadratic. In fig. 4, we show the fidelity defined in eq. (9) as a function of $\delta h$. Clearly, though there are some fluctuations, the fidelity is almost parabolic, as being consistent with eq. (9).

Now we assume that the demon checks Super Mario’s status $N$ times uniformly in a time interval $\Delta t = t_f - t_i$. That is the time interval between two subsequent checks is

$$\delta t = \frac{\Delta t}{N}, \quad (11)$$

then

$$F(0; \delta h) = 1 - \frac{v^2(\Delta t)^2 \chi_F}{2N^2},$$

which defines also the probability that Super Mario want to hide himself. Then the final survival probability, after the time interval $\Delta t$, becomes

$$P = \left[ 1 - \frac{v^2(\Delta t)^2 \chi_F}{2N^2} \right]^N.$$  

In the continuous limit $N \to \infty$,

$$P \simeq 1 - \frac{v^2 \Delta t^2 \chi_F}{2N},$$

which becomes 1, then a classical Zeno effect takes place.

On the other hand, if the duration time $\delta t$ between two subsequent checks is very large, say $F(0, \delta h) > 1/2$, then Super Mario might judge that he has a very high probability to escape from the well successfully. He will speed up his escape process when he hears the demon’s step. A classical anti-Zeno effect then takes place.

In summary, we have presented a gedanken proposal of classical Zeno and anti-Zeno effects via a scenario of Super Mario’s escape progress. Bearing some analogy to the quantum Zeno and anti-Zeno effects, environmental interferences can either slow down or speed up such a classical progress. The significant difference between the quantum and classical Zeno effects is how to understand the projective feedback from the measured object. In the quantum Zeno effect, the projective measurement is a postulate; while in our classical Zeno effect, it is somehow an inclined feedback which is intelligent. Therefore, we can call these newly proposed effects as object-intelligent-feedback–based classical Zeno and anti-Zeno effects.

Finally, we would like to emphasize that the effects we proposed here is very popular in our daily life. An apparent example is: in order to prevent some unexpected crimes, we usually install various monitors. The existence of various monitors plays a role of continuous observation, hence we potential violators away from deregulation. From this point of view, our work clarified such scenarios in the classical world in terms of the famous paradox proposed by Zeno.

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