Optimal Design of Spring Characteristics of Damper for Subharmonic Vibration in Automatic Transmission Powertrain

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Abstract. In the torque converter, the damper of the lock-up clutch is used to effectively absorb the torsional vibration. The damper is designed using a piecewise-linear spring with three stiffness stages. However, a nonlinear vibration, referred to as a subharmonic vibration of order $1/2$, occurred around the switching point in the piecewise-linear restoring torque characteristics because of the nonlinearity. In the present study, we analyze vibration reduction for subharmonic vibration. The model used herein includes the torque converter, the gear train, and the differential gear. The damper is modeled by a nonlinear rotational spring of the piecewise-linear spring. We focus on the optimum design of the spring characteristics of the damper in order to suppress the subharmonic vibration. A piecewise-linear spring with five stiffness stages is proposed, and the effect of the distance between switching points on the subharmonic vibration is investigated. The results of our analysis indicate that the subharmonic vibration can be suppressed by designing a damper with five stiffness stages to have a small spring constant ratio between the neighboring springs. The distances between switching points must be designed to be large enough that the amplitude of the main frequency component of the systems does not reach the neighboring switching point.

1. Introduction
Recent attempts to increase fuel economy in automobiles have revealed a number of problems related to vibrations occurring in the powertrain. The main source of vibrations is engine forced vibration. Engine torque fluctuations create engine speed fluctuations as a result of combustion. A recent trend in engine technology has been the widespread use of diesel engines and higher power engines. Such engines contribute to strong torsional vibration in powertrain systems. In order to address this problem, the damper of the lock-up clutch must be designed so as to effectively absorb the torsional vibration. A damper with low stiffness reduces the fluctuation in rotational speed caused by combustion in the engine combustion chamber. However, low-stiffness springs, which are applicable to a wide range of static torque, are difficult to use because of space limitations. As such, the damper is designed using a
piecewise-linear spring with three stiffness stages. This type of spring can realize a wide range of restoring torque characteristics while remaining compact.

However, a nonlinear vibration, called a subharmonic vibration of order 1/2, occurs because of the nonlinearity of the piecewise-linear spring. In the subharmonic vibration of order 1/2, the main frequency component is half the forced vibration frequency.

Although a number of studies [1], [2], [3] have examined nonlinear vibrations in mechanical systems having piecewise-linear spring properties, these studies are not related to forced vibration analysis of the piecewise-linear spring used in the actual torque converter of a passenger car. The authors clarified the occurrence mechanism of the subharmonic vibration of order 1/2 [4].

In the present study, we performed a numerical analysis to clarify the effects of the stiffness ratio between the neighboring springs on the subharmonic vibration of order 1/2. Moreover, a piecewise-linear spring with five stiffness stages is proposed, and the effect of the distance between the switching points on the subharmonic vibration is investigated.

2. Outline of subharmonic vibration of order 1/2

An automatic transmission consists of a torque converter and a gear train. The torque converter is located between the engine and the gear train and transmits the torque induced by the engine to the gear train. The torque converter is filled with automatic transmission fluid (ATF). Figure 1 shows a schematic diagram of a torque converter. The torque converter consists of a pump impeller, a turbine runner, and a stator. As the torque converter transmits torque through the fluid, the rotational speed of the turbine runner is slower than that of the pump impeller, which causes the inefficiency associated with the torque converter. In order to overcome this disadvantage, a lock-up clutch, which connects the input and output sides through friction, is used. The piston is supported by springs, which are collectively referred to as a damper. The function of the damper is to reduce the fluctuation in the rotation speed caused by combustion in the engine combustion chamber.

![Figure 1. Schematic diagram of a torque converter.](image1)

![Figure 2. Campbell diagram at turbine runner.](image2)

![Figure 3. Waveform of subharmonic vibration.](image3)
A traveling test using an actual automobile was conducted in order to investigate the occurrence mechanism of the subharmonic vibration of order 1/2 (henceforth referred to simply as subharmonic vibration). Figure 2 shows the Campbell diagram at turbine runner. The abscissa indicates the engine speed, and the ordinate indicates the vibration frequency. The color scale represents the vibration amplitude of the rotational speed of the turbine runner. The engine frequency in figure 2 represents the component of the torsional vibration frequency due to the engine combustion. The engine frequency increases in association with the increase in the engine speed. The strong vibration inside the dashed line is the subharmonic vibration. The vibration frequency of the subharmonic vibration is half the engine excitation frequency. Figure 3 shows the waveform at the pump impeller (upper figure) and the turbine runner (lower figure) when the subharmonic vibration occurred. The vibration period of the pump impeller and the turbine runner is twice that of the engine torque fluctuation. A frequency analysis of the engine torque confirmed that the 1/2 order component of the main excitation frequency is very small. Subharmonic vibrations occur in all state of the gears when the lock-up clutch is locked. The engine frequency range when the subharmonic vibration occurred is almost twice the second mode's natural frequency of the system.

The damper is designed as a piecewise-linear spring with three stages. Figure 4 shows the spring restoring torque characteristics of the damper. The abscissa $\phi$ represents the angular displacement of the damper, and the ordinate $T(\phi)$ represents the restoring torque. The subharmonic vibration occurred when static torque is near the switching point between the second and third stages of the spring restoring torque (Point B in figure 4). However, the subharmonic vibration did not occur near the switching point between the first and second stages of the spring restoring torque (Point A in figure 4).

3. Theoretical analysis
In order to analyze the generation of the subharmonic vibration, a system based on the actual automatic transmission is modeled. In the experiment, the subharmonic vibration occurred in the powertrain of an FF vehicle. In the present study, we consider only the fifth gear structure in the modeling.

Figure 4. Piecewise-linear spring restoring torque characteristics.

Figure 5. Analytical model of fifth gear.
Figure 5 shows a model of the gear train in fifth gear. This model includes the torque converter, the gear train, and the differential gear. The gear train is composed of a Ring gear, Carrier 1, Carrier 2, a Counter driven gear, and Sun gear 2. The damper, the shafts in the gear train, and the drive shafts are modeled by rotational springs. The wheels are assumed to be fixed ends. The system is modeled by a four-degree-of-freedom system. In the figure, \( J_1, \ldots, J_9 \) are the moments of inertia of the components, \( k_1, k_2, \) and \( k_3 \) are the rotational spring constants, \( \theta_1, \theta_2, \theta_3, \) and \( \theta_4 \) are the angular displacements of each element, \( c_1, c_2, \) and \( c_3 \) are the damping coefficients of each damping element, \( F_s \) is the static engine torque, \( F_d \) is the dynamic engine torque, and \( \omega \) is the angular frequency of the dynamic torque. The inertia of the engine is included in the inertia of the pump impeller, \( J_1. \) The damper is modeled by the nonlinear rotational spring \( K \) of the piecewise-linear spring and a dashpot with linear damping coefficient \( C. \)

As shown in figure 4, the spring constants of the first, second, and third stages in the piecewise-linear spring are \( K_1, K_2, \) and \( K_3, \) respectively, and the angular displacement and the torque at Points A and B are \((\phi_1, T_1)\) and \((\phi_2, T_2)\), respectively.

The angular displacements of Carrier 1, Carrier 2, the Counter driven gear, Sun gear 2, and the differential gear are represented by

\[
i\theta_3 + m_i\theta_4 \quad (1, \ldots, 5)
\]

where \( \theta_1 \) and \( \theta_2 \) are the constants related to the gear ratios between the components. In this gear train, \( m_1 = 0 \) and \( m_5 = m_2. \)

The equation of motion for a four-degree-of-freedom system is written as

\[
M\ddot{x} + C\dot{x} + Kx = n(x) + F, \quad x = [\theta_1, \theta_2, \theta_3, \theta_4]^T
\]

(1)

where the right superscript \( T \) indicates transposed representation. In Eq. (1), \( x \) is the angular displacement vector, \( n(x) \) is the spring restoring torque vector of the damper, \( F \) is the external torque vector, \( M \) is the mass matrix, \( C \) is the damping matrix, and \( K \) is the stiffness matrix.

The mass matrix \( M \) is represented as follows:

\[
M = \begin{bmatrix}
M_{11} & 0 & 0 & 0 \\
0 & M_{22} & 0 & 0 \\
0 & 0 & M_{33} & M_{34} \\
0 & 0 & M_{43} & M_{44}
\end{bmatrix}
\]

(2)

In Eq. (2), the components \( M_{11}, M_{22}, M_{33}, M_{34}, M_{43}, \) and \( M_{44} \) are functions of \( n_i, m_i, \) and the inertia of each part, \( J_1, \ldots, J_9. \)

The damping matrix \( C, \) the stiffness matrix \( K, \) and the spring restoring torque vector \( n(x) \) are given by the following equations:

\[
C = \begin{bmatrix}
C_{11} & C_{12} & 0 & 0 \\
C_{21} & C_{22} & C_{23} & 0 \\
0 & C_{32} & C_{33} & C_{34} \\
0 & 0 & C_{43} & C_{44}
\end{bmatrix}
\]

(3)

\[
C_{11} = C, \quad C_{12} = C_{21} = -C, \quad C_{22} = C + C_1, \\
C_{23} = C_{32} = -c_1, \quad C_{33} = c_1 + c_2 + n_1^2c_3, \\
C_{34} = C_{43} = -c_2 + n_4m_4c_3, \quad C_{44} = c_2 + m_4^2c_3
\]
The external force vector \( \mathbf{F} \) is obtained as follows:

\[
\mathbf{F} = \begin{bmatrix} F_x & F_y & \cos \omega t & 0 & 0 & 0 \end{bmatrix}^T
\]

(5)

Equations (1) through (5) are used to calculate nonlinear vibration in the powertrain system. The shooting method [6] is used to solve the nonlinear equation of Eq. (1). The stability of the solution is determined by eigenvalue analysis of the transition matrix, which is given by the variational equation of Eq. (1). In the process of the numerical integration method in the shooting method, the switching time of the piecewise-linear spring between the time steps is calculated precisely using the Newton-Raphson method.

In order to calculate the natural frequencies and the natural modes of the powertrain system, linear equations are introduced for each stage of the damper stiffness. Equations (1) and (4) are modified as follows:

\[
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix}, \quad \mathbf{n}(x) = \begin{bmatrix} -T(\theta_1 - \theta_2) \\ T(\theta_2 - \theta_1) \\ 0 \\ 0 \end{bmatrix}
\]

(4)

\[
K_{22} = k_1, \quad K_{23} = K_{32} = -k_1, \\
K_{33} = k_1 + k_2 + n_2^2k_3, \quad K_{34} = K_{43} = -k_2 + n_4m_4k_3, \\
K_{44} = k_2 + m_2^2k_3
\]

The external force vector \( \mathbf{F} \) is obtained as follows:

\[
\mathbf{F} = \begin{bmatrix} F_x & F_y & \cos \omega t & 0 & 0 & 0 \end{bmatrix}^T
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\[
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0
\]

\[
\mathbf{\ddot{K}} = \mathbf{K} + \begin{bmatrix} K & -K & 0 & 0 \\ -K & K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3
\]

(6)

4. Results of numerical calculations and discussion

4.1. Natural frequencies and natural modes

Table 1 lists the natural frequencies when the spring constant of the damper \( \mathbf{K} \) in Eq. (6) is set to \( \mathbf{K}_1, \mathbf{K}_2, \) and \( \mathbf{K}_3, \) respectively. Here, the dynamic absorber is not attached. Since the subharmonic vibration is related to the second mode in the experiment, we focus on the second mode. In Table 1, the natural frequencies of the second modes for \( \mathbf{K}_1, \mathbf{K}_2, \) and \( \mathbf{K}_3, \) are 24.5, 26.97, and 39.80 Hz, respectively. Figures 6(a), 6(b), and 6(c) show up to the fourth natural mode for \( \mathbf{K}_1, \mathbf{K}_2, \) and \( \mathbf{K}_3, \) respectively. In the second modes for \( \mathbf{K}_1, \mathbf{K}_2, \) and \( \mathbf{K}_3, \) \( \theta_2, \theta_3, \) and \( \theta_4 \) vibrate out of phase with respect to \( \theta_1. \)

| Spring constant | Modes |
|-----------------|-------|
| \( \mathbf{K} \) | 1st  | 2nd  | 3rd  | 4th  |
| \( \mathbf{K}_1 \) | 7.88 | 24.5 | 95.3 | 671  |
| \( \mathbf{K}_2 \) | 9.95 | 26.97| 97.1 | 671  |
| \( \mathbf{K}_3 \) | 13.07| 39.80| 112  | 671  |
4.2. Forced vibration analysis for subharmonic vibration

In this nonlinear forced vibration analysis, we focused on the turbine runner amplitude ($\theta_2$). For the standard parameters, the dynamic torque $F_y$ is set to 100 Nm. In the numerical calculation at Point A in figure 4, the damping ratio $\zeta$ of the damper in the second stage is set to 0.0402, and the same damping coefficient, $C$, used in the second stage is used in the first stage. In the numerical calculation at Point B in figure 4, the damping ratio $\zeta$ of the damper in the third stage is set to 0.0402, and the same damping coefficient, $C$, used in the third stage is applied in the second stage.

Figures 7(a) and 7(b) show the frequency response curves at the turbine runner when the static torques are $F_s = \bar{T}$ and $T_2$ in figure 4. The abscissa indicates the excitation frequency due to engine combustion $f$, and the ordinate indicates the peak-to-peak amplitude of the angular displacement.

Figure 7. Frequency response curve at the turbine runner (a) $F_s = \bar{T}$, (b) $F_s = T_2$. 

Figure 6. Natural modes for each spring constant.

| Modes | 1st | 2nd | 3rd | 4th |
|-------|-----|-----|-----|-----|
| $\theta_1$ | ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) | ![Diagram](image4) |
| $\theta_2$ | ![Diagram](image5) | ![Diagram](image6) | ![Diagram](image7) | ![Diagram](image8) |
| $\theta_3$ | ![Diagram](image9) | ![Diagram](image10) | ![Diagram](image11) | ![Diagram](image12) |
| $\theta_4$ | ![Diagram](image13) | ![Diagram](image14) | ![Diagram](image15) | ![Diagram](image16) |

(a) $K = K_1$

| Modes | 1st | 2nd | 3rd | 4th |
|-------|-----|-----|-----|-----|
| $\theta_1$ | ![Diagram](image17) | ![Diagram](image18) | ![Diagram](image19) | ![Diagram](image20) |
| $\theta_2$ | ![Diagram](image21) | ![Diagram](image22) | ![Diagram](image23) | ![Diagram](image24) |
| $\theta_3$ | ![Diagram](image25) | ![Diagram](image26) | ![Diagram](image27) | ![Diagram](image28) |
| $\theta_4$ | ![Diagram](image29) | ![Diagram](image30) | ![Diagram](image31) | ![Diagram](image32) |

(b) $K = K_2$

| Modes | 1st | 2nd | 3rd | 4th |
|-------|-----|-----|-----|-----|
| $\theta_1$ | ![Diagram](image33) | ![Diagram](image34) | ![Diagram](image35) | ![Diagram](image36) |
| $\theta_2$ | ![Diagram](image37) | ![Diagram](image38) | ![Diagram](image39) | ![Diagram](image40) |
| $\theta_3$ | ![Diagram](image41) | ![Diagram](image42) | ![Diagram](image43) | ![Diagram](image44) |
| $\theta_4$ | ![Diagram](image45) | ![Diagram](image46) | ![Diagram](image47) | ![Diagram](image48) |

(c) $K = K_3$
The black and red lines show the stable and unstable solutions, respectively. The static torque $T$ is set far from the switching points of Points A and B, and $T_2$ is set to the switching point of Point B. The frequency response curve of figure 7(a) is the same result obtained by the linear forced vibration analysis using $K = K_2$. The resonance frequency around $f = 30$ Hz in figure 7(a) is the main resonance of the second mode. In the case of figure 7(b), since this vibration straddles two stages with spring constants $K_2$ and $K_3$, the main resonance frequency is between the natural frequencies of 26.97 and 39.80 Hz.

The large-amplitude area around $f = 60$ Hz in figure 7(b) is the subharmonic vibration of order 1/2. In the case of $F_x = T$, there is no resonance around this area. Even though there are no natural frequencies around this area in Table 1, the subharmonic resonance occurs because of the nonlinearity of the restoring torque characteristics. The maximum amplitude of the subharmonic vibration is 0.017 rad. Figure 8 shows the waveforms of the subharmonic vibration at Point P in figure 7(b). The excitation frequency by the engine is $f = 60$ Hz. The top, middle, and bottom figures represent the waveforms of the input dynamic torque, the pump impeller ($\theta_1$), and the turbine runner ($\theta_2$), respectively. It is found that the period of $\theta_2$ is twice the input dynamic torque and that the waveforms of $\theta_1$ and $\theta_2$ are similar to the experimental results shown in figure 3.

Figure 9 shows the frequency analysis of $\theta_2$ at Point P in figure 7(b). The fundamental frequency of $\theta_2$ is 30 Hz, which is the half the excitation frequency of the engine. This is characteristic of the subharmonic vibration of order 1/2.

The vibration mode of the subharmonic vibration at Point P in figure 7(b) is examined. The magnitude and phase corresponding to the component of the fundamental frequency of 30 Hz are shown in Table 2. It is found that the vibration mode is similar to the second natural mode shown in

![Figure 8](image1.png)  ![Figure 9](image2.png)

**Figure 8.** Vibration waveform of subharmonic vibration at Point P ($f = 60$ Hz).

**Figure 9.** Results of frequency analysis of the turbine runner ($f = 60$ Hz).

**Table 2.** Vibration mode of subharmonic vibration.

|       | Magnitude | Phase   |
|-------|-----------|---------|
| $\theta_1$ | 1         | 0       |
| $\theta_2$ | 2.3       | -179.6  |
| $\theta_3$ | 2.6       | -179.7  |
| $\theta_4$ | 2.7       | -179.7  |
The peak amplitude in figure 7(b) is approximately $f = 50$ Hz. This is the superharmonic resonance of order 2 related to the third mode, but the peak amplitude is small. This phenomenon is also caused by the nonlinearity of the piecewise-linear spring. No such resonance appears in the results shown in figure 7(a).

Calculations were also performed for the cases in which the static torque is set slightly larger and smaller than $T_2$. It was confirmed that subharmonic vibrations still occur, even though the angular displacements are not just at the spring stiffness switching points of the piecewise linear spring.

Figure 10 shows the frequency response curve at the turbine runner when the static torque is $F_s = T_1$ (Point A in figure 4). In the experiment, subharmonic vibration did not occur near the switching point between the first and second stages of the spring restoring torque $T_1$. In figure 10, subharmonic vibration does not occur even though Point A is the switching point between $K_1$ and $K_2$, and only the main resonance occurs. This result is in good agreement with the experimental results.

4.3. Effect of spring constant ratio between neighboring springs of the damper

In this section, we confirm the effect of the spring constant ratio between neighboring springs of the damper. The spring constant ratio of the third spring to the second spring $K_3/K_2$ is considered. In this analysis, under the condition in which the first switching point $(\phi_1, T_1)$, the maximum angular displacement and torque of the third spring $(\phi_3, T_3)$, and the torque of the second switching point $T_2$ are set to be constant, the angular displacement of the second switching point $\phi_2$ is set as a parameter in order to vary the value of $K_3/K_2$.

Figure 11 shows the effect of the spring stiffness ratio $K_3/K_2$ of the damper on the occurrence of the subharmonic vibration at Point B in figure 4. The ordinate in figure 11 represents the compliance [rad/kNm], which is the ratio of peak-to-peak amplitude of the angular displacement to the static torque $T_2$. The results indicate that a larger spring stiffness ratio can contribute to a larger amplitude of the subharmonic vibration, and the stiffness ratio should be designed to be less than 2 in order to suppress the subharmonic vibration.
4.4. Designing the damper with five stiffness stages

In Section 4.3, the subharmonic vibration is confirmed to be suppressed by a small spring constant ratio between the second and third springs. In other words, a larger spring constant ratio between the second and third springs contributes to the occurrence of the subharmonic vibration. In this section, a new damper with five stiffness stages is designed in order to suppress the subharmonic vibration by setting each spring constant ratio between neighboring springs to be small. Figure 12 shows the new spring restoring torque characteristics of the damper with five stages. Spring constants $K'_1$ and $K'_4$ are newly set by the following equation:

$$K'_3/K'_2 = K'_4/K'_3 = K'_3/K'_4 = (K_3/K_2)^{1/3}$$

(7)

The angular displacements of the damper at the new switching points $\phi'_0$ and $\phi'_1$ are set by the following:

$$\phi'_0 - \phi'_2 = \phi'_1 - \phi'_3 = d$$

(8)

Figure 12. Restoring torque characteristics of a piecewise-linear spring with five stiffness stages.

The effect of the angular displacement between the switching points $d$ of the damper on the subharmonic vibration is next investigated. Nonlinear forced vibration analysis is once again conducted. The static torque is set to $T'_4$, which is the torque at the switching point of Point C. In the numerical calculation at Point C in figure 12, the damping ratio $\zeta$ of the damper in the fourth stage is set to 0.0402, and the same damping coefficient, $C$, used in the fourth stage is used in the third stage.
Figures 13(a), 13(b), and 13(c) show the frequency response curves at the turbine runner when the angular displacements between the switching points are \(d = 0.001\), 0.003, and 0.005 rad (Figure 12), respectively. As shown in figure 13(a) and 13(b), subharmonic vibrations still occur because of the nonlinearity of the piecewise-linear spring, and the maximum amplitude of the subharmonic vibration decreases slightly as the angular displacement between the switching points \(d\) increases. Subharmonic vibration does not appear in figure 13(c). The subharmonic vibrations still occur because the vibration amplitude of the main frequency component of the systems reaches neighboring switching point B when the angular displacement between the switching points \(d\) is 0.001 and 0.003 rad, which means that the damper vibrates between the second and fourth stages of the spring restoring torque. On the other hand, when the angular displacement between the switching points \(d\) is 0.005 rad, the angular displacement between switching points B and C is so large that the amplitude of the main frequency component of the systems does not reach the neighboring switching point. Therefore, the spring restoring torque characteristics of the damper must be designed such that the spring constant ratio between each set of neighboring springs is less than 2 and the angular displacement between the switching points must be set large enough that the vibration amplitude of the damper does not straddle the three stages of spring constants \(K_2\), \(K_3\), and \(K_4\).

Figures 14(a), 14(b), and 14(c) show the frequency response curves at the turbine runner when the static torques are \(F_s = T'_s\) (Point B), \(T'_c\) (Point C), and \(T'_d\) (Point D), respectively, in figure 12 and when the angular displacement between the switching points \(d\) is 0.01 rad. Figure 14 confirms that the subharmonic vibration does not occur at all switching points in figure 12.
Figure 14. Frequency response curve at the turbine runner (\(d = 0.01\) rad)
(a) \(F_s = T_2\), (b) \(F_s = T_3'\), (c) \(F_s = T_4'\).

5. Conclusions
The present study performed a numerical analysis to clarify the effects of the stiffness ratio between neighboring springs in a damper with five stiffness stages on the subharmonic vibration of order 1/2. The subharmonic vibration of order 1/2 was suppressed by a small spring stiffness ratio between neighboring springs. A larger spring stiffness ratio contributes to the occurrence of the subharmonic vibration of order 1/2. In the damper considered in the present study, the stiffness ratio between neighboring springs should be designed to be less than 2 in order to suppress subharmonic vibrations. The angular displacement between switching points must be set to be large enough that the vibration amplitude of the damper does not straddle three stages of spring constants.

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