Bounds on entanglement dimensions and quantum graph parameters via noncommutative polynomial optimization

Sander Gribling\textsuperscript{*} David de Laat\textsuperscript{†} Monique Laurent\textsuperscript{‡}

Abstract

In this paper we study optimization problems related to bipartite quantum correlations using techniques from tracial noncommutative polynomial optimization. First we consider the problem of finding the minimal entanglement dimension of such correlations. We construct a hierarchy of semidefinite programming lower bounds and show convergence to a new parameter: the minimal average entanglement dimension, which measures the amount of entanglement needed to reproduce a quantum correlation when access to shared randomness is free. Then we study optimization problems over synchronous quantum correlations arising from quantum graph parameters. We introduce semidefinite programming hierarchies and unify existing bounds on quantum chromatic and quantum stability numbers by placing them in the framework of tracial optimization.

1 Introduction

1.1 Bipartite quantum correlations

One of the distinguishing features of quantum mechanics is quantum entanglement, which allows for nonclassical correlations between spatially separated parties. In this paper we consider the problems of quantifying the advantage entanglement can bring (first investigated through Bell inequalities in the seminal work \cite{Bell}) and quantifying the minimal amount of entanglement necessary for generating a given correlation (initiated in \cite{CLP} and continued, e.g., in \cite{Lewenstein, Peres, Horodecki}).

Quantum entanglement has been widely studied in the bipartite correlation setting (for a survey, see, e.g., \cite{Berta}). Here we have two parties, Alice and Bob, where Alice receives a question $s$ taken from a finite set $S$ and Bob receives a question $t$ taken from a finite set $T$. The parties do not know each other’s questions, and after receiving the questions they do not communicate. Then, according to some predetermined protocol, Alice returns an answer $a$ from a finite set $A$ and Bob returns an answer $b$ from a finite set $B$. The probability that the parties answer $(a,b)$ to questions $(s,t)$ is given by a bipartite correlation $P(a,b|s,t)$, which satisfies $P(a,b|s,t) \geq 0$ for all $(a,b,s,t) \in \Gamma$ and $\sum_{a,b} P(a,b|s,t) = 1$ for all $(s,t) \in S \times T$. Throughout we set $\Gamma = A \times B \times S \times T$. Which bipartite correlations $P = (P(a,b|s,t)) \in \mathbb{R}^\Gamma$ are possible depends on the additional resources available to the two parties Alice and Bob.

If the parties do not have access to additional resources, then the correlation is deterministic, which means it is of the form $P(a,b|s,t) = P_A(a|s) P_B(b|t)$, with $P_A(a|s)$ and $P_B(b|t)$ taking values in \{0,1\} and $\sum_a P_A(a|s) = \sum_b P_B(b|t) = 1$ for all $s,t$. If the parties have access to local randomness, then $P_A$ and $P_B$ take values in [0,1]. If the parties have access to shared randomness, then the resulting correlation is a convex combination of deterministic correlations.
and is said to be a classical correlation. The classical correlations form a polytope, denoted \( C_{\text{loc}}(\Gamma) \), whose valid inequalities are known as Bell inequalities [3].

We are interested in the quantum setting, where the parties have access to a shared quantum state on which they can perform measurements. The quantum setting can be modeled in different ways, leading to the so-called tensor and commuting models; see the discussion, e.g., in [52, 31, 11].

In the tensor model, Alice and Bob each have access to “one half” of a finite dimensional quantum state, which is modeled by a unit vector \( \psi \in \mathbb{C}^d \otimes \mathbb{C}^d \). Alice and Bob determine their answers by performing a measurement on their part of the state. Such a measurement is modeled by a positive operator valued measure (POVM), which consists of a set of \( d \times d \) Hermitian positive semidefinite matrices labeled by the possible answers and summing to the identity matrix. If Alice uses the POVM \( \{E^a_s\}_{a \in A} \) when she gets question \( s \in S \) and Bob uses the POVM \( \{F^b_t\}_{b \in B} \) when he gets question \( t \in T \), then the probability of obtaining the answers \((a, b)\) is given by

\[
P(a, b|s, t) = \text{Tr}((E^a_s \otimes F^b_t)\psi\psi^*) = \psi^*(E^a_s \otimes F^b_t)\psi.
\]

If the state \( \psi \) cannot be written as a single tensor product \( \psi_A \otimes \psi_B \), then \( \psi \) is entangled, which means it can be used to produce a nonclassical correlation \( P \).

A correlation of the above form (1) is a quantum correlation, realizable in the tensor model in local dimension \( d \) (or in dimension \( d^2 \)). Let \( C_q^d(\Gamma) \) be the set of such correlations and define

\[
C_q(\Gamma) = \bigcup_{d \in \mathbb{N}} C_q^d(\Gamma).
\]

Denote the smallest dimension needed to realize \( P \in C_q(\Gamma) \) in the tensor model by

\[
D_q(P) = \min\{d^2 : d \in \mathbb{N}, P \in C_q^d(\Gamma)\}.
\]

The set \( C_q^d(\Gamma) \) contains the deterministic correlations. Hence, by Carathéodory’s theorem, \( C_{\text{loc}}(\Gamma) \subseteq C_q^\Gamma(\Gamma) \) holds for \( c = |\Gamma| + 1 - |S||T| \); that is, quantum entanglement can be used as an alternative to shared randomness. If \( A, B, S, \) and \( T \) all contain at least two elements, then Bell [3] shows the inclusion \( C_{\text{loc}}(\Gamma) \subseteq C_q(\Gamma) \) is strict; that is, quantum entanglement can be used to obtain nonclassical correlations.

The second commonly used model to define quantum correlations is the commuting model (or relativistic field theory model). Here a correlation \( P \in \mathbb{R}^T \) is called a commuting quantum correlation if it is of the form

\[
P(a, b|s, t) = \text{Tr}(X^{ab}_s Y^{ab}_t \psi\psi^*) = \psi^*(X^{ab}_s Y^{ab}_t)\psi,
\]

where \( \{X^a_s\}_a \) and \( \{Y^b_t\}_b \) are POVMs consisting of bounded operators on a separable Hilbert space \( H \), satisfying \( [X^a_s, Y^b_t] = X^a_s Y^b_t - Y^b_t X^a_s = 0 \) for all \((a, b, s, t) \in \Gamma\), and where \( \psi \) is a unit vector in \( H \). Such a correlation is said to be realizable in dimension \( d = \dim(H) \) in the commuting model. Denote the set of such correlations by \( C_{qc}^d(\Gamma) \) and set \( C_{qc}(\Gamma) = C_{qc}^\infty(\Gamma) \). The smallest dimension needed to realize a quantum correlation \( P \in C_{qc}(\Gamma) \) is given by

\[
D_{qc}(P) = \min\{d \in \mathbb{N} \cup \{\infty\} : P \in C_{qc}^d(\Gamma)\}.
\]

We have \( C_q^d(\Gamma) \subseteq C_{qc}^{d2}(\Gamma) \), which follows by setting \( X^a_s = E^a_s \otimes I \) and \( Y^b_t = I \otimes F^b_t \). This shows \( D_{qc}(P) \leq D_q(P) \) for all \( P \in C_q(\Gamma) \).

The minimum Hilbert space dimension in which a given quantum correlation \( P \) can be realized quantifies the minimal amount of entanglement needed to represent \( P \). Computing \( D_q(P) \) is NP-hard [49], so a natural question is to find good lower bounds for the parameters \( D_q(P) \) and \( D_{qc}(P) \). A main contribution of this paper is proposing a hierarchy of semidefinite programming lower bounds for these parameters.
As said above we have $C_{q}^{d}(\Gamma) \subseteq C_{qc}^{d}(\Gamma)$. Conversely, each finite dimensional commuting quantum correlation can be realized in the tensor model, although not necessarily in the same dimension [52] (see, e.g., [11] for a proof). This shows

$$C_{q}(\Gamma) = \bigcup_{d \in \mathbb{R}} C_{qc}^{d}(\Gamma) \subseteq C_{qc}(\Gamma).$$

Using a direct sum construction one can show $\cup_{d \in \mathbb{N}} C_{qc}^{d}(\Gamma)$ and $C_{qc}(\Gamma)$ are convex. Whether the two sets $C_{q}(\Gamma)$ and $C_{qc}(\Gamma)$ coincide is known as Tsirelson’s problem. In a recent breakthrough Slofstra [48] showed that $C_{q}(\Gamma)$ is not closed for $|A| \geq 8$, $|B| \geq 2$, $|S| \geq 184$, $|T| \geq 235$. More recently it was shown in [13] that the same holds for $|A| \geq 2$, $|B| \geq 2$, $|S| \geq 5$, $|T| \geq 5$. Hence, for such $\Gamma$ there is a sequence $\{P_{i}\} \subseteq C_{q}(\Gamma)$ with $D_{q}(P_{i}) \rightarrow \infty$. Moreover, since $C_{qc}(\Gamma)$ is closed [14, Prop. 3.4], the inclusion $C_{q}(\Gamma) \subseteq C_{qc}(\Gamma)$ is strict, thus settling Tsirelson’s problem. Whether the closure of $C_{q}(\Gamma)$ equals $C_{qc}(\Gamma)$ for all $\Gamma$ is equivalent to Connes’ embedding conjecture in operator theory [20, 37].

Further variations on the above definitions are possible. For instance, we can consider a mixed state $\rho$ (a Hermitian positive semidefinite matrix $\rho$ with $\text{Tr}(\rho) = 1$) instead of a pure state $\psi$, where we replace the rank 1 matrix $\psi \psi^{*}$ by $\rho$ in the above definitions. By convexity this does not change the sets $C_{q}(\Gamma)$ and $C_{qc}(\Gamma)$. It is shown in [47] that this also does not change the parameter $D_{q}(P)$, but it is unclear whether or not $D_{qc}(P)$ might decrease. Another variation would be to use projection valued measures (PVMs) instead of POVMs, where the operators are projectors instead of positive semidefinite matrices. This again does not change the sets $C_{q}(\Gamma)$ and $C_{qc}(\Gamma)$ [35], but the dimension parameters can be larger when restricting to PVMs.

When the two parties have the same question sets ($S = T$) and the same answer sets ($A = B$), a bipartite correlation $P \in \mathbb{R}^{\Gamma}$ is called synchronous if $P(a, b|s, s) = 0$ for all $s$ and $a \neq b$. The sets of synchronous (commuting) quantum correlations, denoted $C_{q,s}(\Gamma)$ and $C_{qc,s}(\Gamma)$, are rich enough, so that Connes’ embedding conjecture still holds if and only if $\text{cl}(C_{q,s}(\Gamma)) = (C_{qc,s}(\Gamma)$ for all $\Gamma$ [12, Thm. 3.7]. The quantum graph parameters discussed in Section 1.3 will be defined through optimization problems over these sets.

A matrix $M \in \mathbb{R}^{n \times n}$ is completely positive semidefinite if there exist $d \in \mathbb{N}$ and Hermitian positive semidefinite matrices $X_{1}, \ldots, X_{n} \in \mathbb{C}^{d \times d}$ with $M = (\text{Tr}(X_{i}X_{j}))$. The minimal such $d$ is its completely positive semidefinite rank, denoted $\text{cpsd-rank}(M)$. Completely positive semidefinite matrices are used in [25] to model quantum graph parameters and the completely positive semidefinite rank is investigated in [43, 16, 44, 15]. By combining the proofs from [46] (see also [28]) and [41] one can show the following link between synchronous correlations and completely positive semidefinite matrices.

**Proposition A.1.** The smallest local dimension in which a synchronous quantum correlation $P$ can be realized is given by the completely positive semidefinite rank of the matrix $M_{P}$ indexed by $S \times A$ with entries $(M_{P})(s,a),(t,b) = P(a,b|s,t)$.

In [15] we use techniques from tracial polynomial optimization to define a semidefinite programming hierarchy $\{\xi^{\text{cpsd}}(M)\}$ of lower bounds on $\text{cpsd-rank}(M)$. By the above result this hierarchy gives lower bounds on the smallest local dimension in which a synchronous correlation can be realized in the tensor model. However, in [15] we show that the hierarchy typically does not converge to $\text{cpsd-rank}(M)$ but instead (under a certain flatness condition) to a parameter $\xi^{\text{cpsd}}(M)$, which can be seen as a block-diagonal version of the completely positive semidefinite rank.

Here we use similar techniques, now exploiting the special structure of quantum correlations, to construct a hierarchy $\{\xi^{q}(P)\}$ of lower bounds on the minimal dimension $D_{q}(P)$ of any – not necessarily synchronous – quantum correlation $P$. The hierarchy converges (under flatness)

---

1See Appendix A for a proof.
to a parameter $\xi^q_i(P)$, and using the additional structure we can show that $\xi^q_i(P)$ is equal to an interesting parameter $A_q(P) \leq D_q(P)$. This parameter describes the minimal average entanglement dimension of a correlation when the parties have free access to shared randomness; see Section 1.2.

In the rest of the introduction we give a road map through the contents of the paper and state the main results. We will introduce the necessary background along the way.

### 1.2 A hierarchy for the average entanglement dimension

We are interested in the minimal entanglement dimension needed to realize a given correlation $P \in C_q(\Gamma)$. If $P$ is deterministic or only uses local randomness, then $D_q(P) = D_{qc}(P) = 1$. But other classical correlations (which use shared randomness) have $D_q(P) > D_{qc}(P) = 1$, which means the shared quantum state is used as a shared randomness resource. In [5] the concept of dimension witness is introduced, where a $d$-dimensional witness is defined as a halfspace containing $\text{conv}(C^q_d(\Gamma))$, but not the full set $C_q(\Gamma)$. As a measure of entanglement this suggests the parameter

$$
\inf \left\{ \max_{i \in [l]} D_q(P_i) : I \in \mathbb{N}, \lambda \in \mathbb{R}^l_+, \sum_{i=1}^l \lambda_i = 1, P = \sum_{i=1}^l \lambda_i P_i, P_i \in C_q(\Gamma) \right\},
$$

(5)

Observe that, for a bipartite correlation $P$, this parameter is equal to 1 if and only if $P$ is classical. Hence, it more closely measures the minimal entanglement dimension when the parties have free access to shared randomness. From an operational point of view, (5) can be interpreted as follows. Before the game starts the parties select a finite number of pure states $\psi_i (i \in I)$ (instead of a single one), in possibly different dimensions $d_i$, and POVMs $\{E^a_s(i)\}_a$, $\{F^b_t(i)\}_b$ for each $i \in I$ and $(s,t) \in S \times T$. As before, we assume that the parties cannot communicate after receiving their questions $(s,t)$, but now they do have access to shared randomness, which they use to decide on which state $\psi_i$ to use. The parties proceed to measure state $\psi_i$ using POVMs $\{E^a_s(i)\}_a$, $\{F^b_t(i)\}_b$, so that the probability of answers $(a,b)$ is given by the quantum correlation $P_i$. Equation (5) then asks for the largest dimension needed in order to generate $P$ when access to shared randomness is free.

It is not clear how to compute (5). Here we propose a variation of (5), and we provide a hierarchy of semidefinite programs that converges to it under flatness. Instead of considering the largest dimension needed to generate $P$, we consider the average dimension. That is, we minimize $\sum_{i \in I} \lambda_i D_q(P_i)$ over all convex combinations $P = \sum_{i \in I} \lambda_i P_i$. Hence, the minimal average entanglement dimension is given by

$$
A_q(P) = \inf \left\{ \sum_{i=1}^l \lambda_i D_q(P_i) : I \in \mathbb{N}, \lambda \in \mathbb{R}^l_+, \sum_{i=1}^l \lambda_i = 1, P = \sum_{i=1}^l \lambda_i P_i, P_i \in C_q(\Gamma) \right\}
$$

in the tensor model. In the commuting model, $A_{qc}(P)$ is given by the same expression with $D_q(P_i)$ replaced by $D_{qc}(P_i)$. Observe that we need not replace $C_q(\Gamma)$ by $C_{qc}(\Gamma)$ since $D_{qc}(P) = \infty$ for any $P \in C_{qc}(\Gamma) \setminus C_q(\Gamma)$.

It follows by convexity for the above definitions it does not matter whether we use pure or mixed states. We show that for the average minimal entanglement dimension it also does not matter whether we use the tensor or commuting model.

**Proposition 2.1.** For any $P \in C_q(\Gamma)$ we have $A_q(P) = A_{qc}(P)$.

We have $A_q(P) \leq D_q(P)$ and $A_{qc}(P) \leq D_{qc}(P)$ for $P \in C_q(\Gamma)$, with equality if $P$ is an extreme point of $C_q(\Gamma)$. Hence, we have $D_q(P) = D_{qc}(P)$ if $P$ is an extreme point of $C_q(\Gamma)$. We show that the parameter $A_q(P)$ can be used to distinguish between classical and nonclassical correlations.
Proposition 2.2. For a correlation $P \in \mathbb{R}^\Gamma$ we have $A_q(P) = 1$ if and only if $P \in C_{loc}(\Gamma)$.

As mentioned before, there exist $\Gamma$ for which $C_q(\Gamma)$ is not closed [48, 13], which implies the existence of a sequence $\{P_i\} \subseteq C_q(\Gamma)$ such that $D_q(P) \to \infty$. We show this also implies the existence of such a sequence with $A_q(P_i) \to \infty$.

Proposition 2.3. If $C_q(\Gamma)$ is not closed, there exists $\{P_i\} \subseteq C_q(\Gamma)$ with $A_q(P_i) \to \infty$.

Using tracial polynomial optimization we construct a hierarchy $\{\xi^q_i(P)\}$ of lower bounds on $A_q(P)$. For each $r \in \mathbb{N}$ this is a semidefinite program, and for $r = \infty$ it is an infinite dimensional semidefinite program. We further define a (hyperfinite) variation $\xi^q_i(\Gamma)$ of $\xi^q_i(P)$ by adding a finite rank constraint, so that

$$
\xi_1^q(P) \leq \xi_2^q(P) \leq \ldots \leq \xi_n^q(P) \leq \xi_\infty^q(P) \leq A_{qc}(P).
$$

We do not know whether $\xi_\infty^q(P) = \xi^q(P)$ always holds; this question is related to Connes’ embedding conjecture [22]. First we show that we imposed enough constraints in the bounds $\xi_i^q(P)$ so that $\xi_i^q(P) = A_{qc}(P)$.

Proposition 2.8. For any $P \in C_q(\Gamma)$ we have $\xi_i^q(P) = A_{qc}(P)$.

Then we show that the infinite dimensional semidefinite program $\xi_\infty^q(P)$ is the limit of the finite dimensional semidefinite programs.

Proposition 2.9. For any $P \in C_q(\Gamma)$ we have $\xi_i^q(P) \to \xi_\infty^q(P)$ as $r \to \infty$.

Finally we give a criterion under which finite convergence $\xi_i^q(P) = \xi_\infty^q(P)$ holds. The definition of flatness follows later in the paper; here we only note that it is an easy to check criterion given the output of the semidefinite programming solver.

Proposition 2.10. If $\xi_i^q(P)$ admits a $(\lceil r/3 \rceil + 1)$-flat optimal solution, $\xi_i^q(P) = \xi_\infty^q(P)$.

1.3 Quantum graph parameters

Nonlocal games have been introduced in quantum information theory as abstract models to quantify the power of entanglement, in particular, in how much the sets $C_q(\Gamma)$ and $C_{qc}(\Gamma)$ differ from $C_{loc}(\Gamma)$. A nonlocal game is defined by a probability distribution $\pi : S \times T \to [0, 1]$ and a predicate $f : A \times B \times S \times T \to \{0, 1\}$. Alice and Bob receive a question pair $(s, t) \in S \times T$ with probability $\pi(s, t)$. They know the game parameters $\pi$ and $f$, but they do not know each other’s questions, and they cannot communicate after they receive their questions. Their answers $(a, b)$ are determined according to some correlation $P \in \mathbb{R}^\Gamma$, called their strategy, on which they may agree before the start of the game, and which can be classical or quantum depending on whether $P$ belongs to $C_{loc}(\Gamma)$, $C_q(\Gamma)$, or $C_{qc}(\Gamma)$. Then their corresponding winning probability is given by

$$
\sum_{(s, t) \in S \times T} \pi(s, t) \sum_{(a, b) \in A \times B} P(a, b|s, t) f(a, b, s, t).
$$

A strategy $P$ is called perfect if the above winning probability is equal to one, that is, if for all $(a, b, s, t) \in \Gamma$ we have

$$(\pi(s, t) > 0 \text{ and } f(a, b, s, t) = 0) \implies P(a, b|s, t) = 0.$$

Computing the maximum winning probability of a nonlocal game is an instance of linear optimization over $C_{loc}(\Gamma)$ in the classical setting, and over $C_q(\Gamma)$ or $C_{qc}(\Gamma)$ in the quantum setting. Since the inclusion $C_{loc}(\Gamma) \subseteq C_q(\Gamma)$ can be strict, the winning probability can be higher when the parties have access to entanglement. In fact there are nonlocal games that can
be won with probability 1 by using entanglement, but with probability strictly less than 1 in the classical setting.

The quantum graph parameters are analogues of the classical parameters defined through the coloring and stability number games as described below. These nonlocal games use the set \([k]\) (whose elements are denoted as \(a, b\)) and the set \(V\) of vertices of \(G\) (whose elements are denoted as \(i, j\)) as question and answer sets.

In the quantum coloring game, introduced in \([1, 9]\), we have a graph \(G = (V, E)\) and an integer \(k\). Here we have question sets \(S = T = V\) and answer sets \(A = B = [k]\), and the distribution \(\pi\) is strictly positive on \(V \times V\). The predicate \(f\) is such that the players’ answers have to be consistent with having a \(k\)-coloring of \(G\); that is, \(f(a, b, i, j) = 0\) precisely when \((i = j\) and \(a \neq b\)) or \((\{i, j\} \in E\) and \(a = b\)). This expresses the fact that if Alice and Bob receive the same vertex they should return the same color and if they receive adjacent vertices they should return distinct colors. A perfect classical strategy exists if and only if a perfect deterministic strategy exists, and a perfect deterministic strategy corresponds to a \(k\)-coloring of \(G\). Hence the smallest number \(k\) of colors for which there exists a perfect classical strategy is equal to the classical chromatic number \(\chi(G)\). It is therefore natural to define the quantum chromatic number as the smallest \(k\) for which there exists a perfect quantum strategy. Since such a strategy is necessarily synchronous we get the following definition.

**Definition 1.1.** The (commuting) quantum chromatic number \(\chi_q(G)\) (resp., \(\chi_{qc}(G)\)) is the smallest integer \(k \in \mathbb{N}\) for which there exists a synchronous correlation \(P = (P(a, b| i, j))\) in \(C_q, s([k]^2 \times V^2)\) (resp., \(C_{qc, s}([k]^2 \times V^2)\)) such that

\[
P(a, a| i, j) = 0 \quad \text{for all} \quad a \in [k], \{i, j\} \in E.
\]

In the quantum stability number game, introduced in \([28, 45]\), we again have a graph \(G = (V, E)\) and \(k \in \mathbb{N}\), but now we use the question set \([k] \times [k]\) and the answer set \(V \times V\). The distribution \(\pi\) is again strictly positive on the question set and now the predicate \(f\) of the game is such that the players’ answers have to be consistent with having a stable set of size \(k\), that is, \(f(i, j, a, b) = 0\) precisely when \((a = b\) and \(i \neq j\)) or \((a \neq b\) and \((i = j\) or \(\{i, j\} \in E\))). This expresses the fact that if Alice and Bob receive the same index \(a = b \in [k]\) they should answer with the same vertex \(i = j\) of \(G\), and if they receive distinct indices \(a \neq b\) from \([k]\) they should answer with distinct nonadjacent vertices \(i\) and \(j\) of \(G\). There is a perfect classical strategy precisely when there exists a stable set of size \(k\), so that the largest integer \(k\) for which there exists a perfect classical strategy is equal to the stability number \(\alpha(G)\). Again, such a strategy is necessarily synchronous, so we get the following definition.

**Definition 1.2.** The (commuting) stability number \(\alpha_q(G)\) (resp., \(\alpha_{qc}(G)\)) is the largest integer \(k \in \mathbb{N}\) for which there exists a synchronous correlation \(P = (P(i, j| a, b))\) in \(C_q, s(V^2 \times [k]^2)\) (resp., \(C_{qc, s}(V^2 \times [k]^2)\)) such that

\[
P(i, j| a, b) = 0 \quad \text{whenever} \quad (i = j \text{ or } \{i, j\} \in E) \text{ and } a \neq b \in [k].
\]

The classical parameters \(\chi(G)\) and \(\alpha(G)\) are NP-hard. The same holds for the quantum coloring number \(\chi_q(G)\) \([19]\) and also for the quantum stability number \(\alpha_q(G)\) in view of the following reduction to coloring shown in \([28]\):

\[
\chi_q(G) = \min\{k \in \mathbb{N} : \alpha_q(G \square K_k) = |V|\}.
\] (7)

Here \(G \square K_k\) is the Cartesian product of the graph \(G = (V, E)\) and the complete graph \(K_k\). By construction we have \(\chi_{qc}(G) \leq \chi_q(G) \leq \chi(G)\) and \(\alpha(G) \leq \alpha_q(G) \leq \alpha_{qc}(G)\). The separations between \(\chi_q(G)\) and \(\chi(G)\), and between \(\alpha_q(G)\) and \(\alpha(G)\), can be exponentially large in the number of vertices; this is the case for the graphs with vertex set \([\pm 1]^n\) for \(n\) a multiple of 4, where two vertices are adjacent if they are orthogonal \([1, 28, 29]\). While it was recently shown
that the sets $C_{q,s}(\Gamma)$ and $C_{qc,s}(\Gamma)$ can be different, it is not known whether there is a separation between the parameters $\chi_q(G)$ and $\chi_{qc}(G)$, and between $\alpha_q(G)$ and $\alpha_{qc}(G)$.

We now give an overview of the results of Section 3 and refer to that section for formal definitions. We first reformulate the quantum graph parameters in terms of $C^*$-algebras, which allows us to use techniques from tracial polynomial optimization to formulate bounds on the quantum graph parameters. We define a hierarchy $\{\gamma_r^{\text{col}}(G)\}$ of lower bounds on the commuting quantum chromatic number and a hierarchy $\{\gamma_r^{\text{stab}}(G)\}$ of upper bounds on the commuting quantum stability number. We show the following convergence results for these hierarchies.

**Proposition 3.2.** There is an $r_0 \in \mathbb{N}$ such that $\gamma_r^{\text{col}}(G) = \chi_{qc}(G)$ and $\gamma_r^{\text{stab}}(G) = \alpha_{qc}(G)$ for all $r \geq r_0$. Moreover, if $\gamma_r^{\text{col}}(G)$ admits a flat optimal solution, then $\gamma_r^{\text{col}}(G) = \chi_q(G)$, and if $\gamma_r^{\text{stab}}(G)$ admits a flat optimal solution, then $\gamma_r^{\text{stab}}(G) = \alpha_q(G)$.

Then we define tracial analogues $\{\xi_r^{\text{stab}}(G)\}$ and $\{\xi_r^{\text{col}}(G)\}$ of Lasserre type bounds on $\alpha(G)$ and $\chi(G)$ that provide hierarchies of bounds for their quantum analogues. These bounds are more economical than the bounds $\gamma_r^{\text{col}}(G)$ and $\gamma_r^{\text{stab}}(G)$ (since they use less variables) and also permit to recover some known bounds for the quantum parameters. We show that $\xi_r^{\text{stab}}(G)$, which is the parameter $\xi_r^{\text{stab}}(G)$ with an additional rank constraint on the matrix variable, coincides with the projective packing number $\alpha_p(G)$ from [45] and that $\xi_r^{\text{stab}}(G)$ upper bounds $\alpha_{qc}(G)$.

**Proposition 3.4.** We have $\xi_r^{\text{stab}}(G) = \alpha_p(G) \geq \alpha_q(G)$ and $\xi_r^{\text{stab}}(G) \geq \alpha_{qc}(G)$.

Next, we consider the chromatic number. The tracial hierarchy $\{\xi_r^{\text{col}}(G)\}$ unifies two known bounds: the projective rank $\xi_f(G)$, a lower bound on the quantum chromatic number from [28], and the tracial rank $\xi_r(G)$, a lower bound on the commuting quantum chromatic number from [41]. In [12, Cor. 3.10] it is shown that the projective rank and the tracial rank coincide if Connes’ embedding conjecture is true.

**Proposition 3.6.** We have $\xi_r^{\text{col}}(G) = \xi_f(G) \leq \chi_q(G)$ and $\xi_r^{\text{col}}(G) = \xi_r(G) \leq \chi_{qc}(G)$.

We compare the hierarchies $\xi_r^{\text{col}}(G)$ and $\gamma_r^{\text{col}}(G)$, and the hierarchies $\xi_r^{\text{stab}}(G)$ and $\gamma_r^{\text{stab}}(G)$. For the coloring parameters, we show the analogue of reduction (7).

**Proposition 3.10.** For $r \in \mathbb{N} \cup \{\infty\}$ we have $\gamma_r^{\text{col}}(G) = \min\{k : \xi_r^{\text{stab}}(G \square K_k) = |V|\}$.

We show an analogous statement for the stability parameters, when using the homomorphic graph product of $K_k$ with the complement of $G$, denoted here as $K_k \ast G$, and the following reduction shown in [28]:

$$\alpha_q(G) = \max\{k \in \mathbb{N} : \alpha_q(K_k \ast G) = k\}.$$ 

**Proposition 3.11.** For $r \in \mathbb{N} \cup \{\infty\}$ we have $\gamma_r^{\text{stab}}(G) = \max\{k : \xi_r^{\text{stab}}(K_k \ast G) = k\}$.

Finally, we show that the hierarchies $\{\gamma_r^{\text{col}}(G)\}$ and $\{\gamma_r^{\text{stab}}(G)\}$ refine the hierarchies $\{\xi_r^{\text{col}}(G)\}$ and $\{\xi_r^{\text{stab}}(G)\}$.

**Proposition 3.12.** For $r \in \mathbb{N} \cup \{\infty, \ast\}$, $\xi_r^{\text{col}}(G) \leq \gamma_r^{\text{col}}(G)$ and $\xi_r^{\text{stab}}(G) \geq \gamma_r^{\text{stab}}(G)$.

### 1.4 Techniques from noncommutative polynomial optimization

To derive our bounds we use techniques from tracial polynomial optimization. This is a noncommutative extension of the widely used moment and sum-of-squares techniques from Lasserre [23] and Parrilo [40] in polynomial optimization, dealing with the problem of minimizing a multivariate polynomial over a feasible region defined by polynomial inequalities. These techniques
have been adapted to the noncommutative setting in [31] and [11] for approximating the set $C_{qc}(\Gamma)$ of commuting quantum correlations and the winning probability of nonlocal games over $C_{qc}(\Gamma)$ (and, more generally, computing Bell inequality violations). In [42, 32] this approach has been extended to the general eigenvalue optimization problem, of the form

$$\inf \{ \psi^* f(X_1, \ldots, X_n) \psi : d \in \mathbb{N}, \psi \in \mathbb{C}^d \text{ unit vector}, X_1, \ldots, X_n \in \mathbb{C}^{d \times d},$$

$$g(X_1, \ldots, X_n) \geq 0 \text{ for } g \in \mathcal{G} \}. $$

Here, the matrix variables $X_i$ have free dimension $d \in \mathbb{N}$ and $\{f\} \cup \mathcal{G} \subseteq \mathbb{R}^{x_1, \ldots, x_n}$ is a set of symmetric polynomials in noncommutative variables. In tracial optimization, instead of minimizing the smallest eigenvalue of $f(X_1, \ldots, X_n)$, we minimize its normalized trace $\text{Tr}(f(X_1, \ldots, X_n))/d$ (so that the identity matrix has trace one) [7, 6, 8, 21]. The moment approach for these problems relies on minimizing $L(f)$, where $L$ is a linear functional on the space of noncommutative polynomials satisfying some necessary conditions, and $L(f)$ models $f^*(X_1, \ldots, X_n) \psi$ or $\text{Tr}(f(X_1, \ldots, X_n))/d$. By truncating the degrees one gets hierarchies of lower bounds for the original problem. The asymptotic limit of these bounds involves operators $X_i$ on a Hilbert space (possibly of infinite dimension). In tracial optimization this leads to allowing solutions $X_i$ in a $C^*$-algebra $\mathcal{A}$ equipped with a tracial state $\tau$, where $\tau(f(X_1, \ldots, X_n))$ is minimized.

An important feature in noncommutative optimization is the dimension independence: the optimization is over all possible matrix sizes $d \in \mathbb{N}$. In some applications one may want to restrict to optimizing over matrices with restricted size $d$. In [33, 30] techniques are developed that allow to incorporate this dimension restriction by suitably selecting the linear functionals $L$ in a specified space; this is used to give bounds on the maximum violation of a Bell inequality in a fixed dimension. A related natural problem is to decide what is the minimum dimension $d$ needed to realize a given algebraically defined object, such as a (commuting) quantum correlation $P$.

We propose an approach based on tracial optimization: starting from the observation that the trace of the $d \times d$ identity matrix gives its size $d$, we consider the problem of minimizing $L(1)$ where $L$ is a linear functional modeling the non-normalized matrix trace. This approach has been used in several recent works [51, 34, 15] for lower bounding factorization ranks of matrices and tensors.

## 2 A hierarchy for the minimal entanglement dimension

### 2.1 The minimal average entanglement dimension

We start by showing that we do not matter whether we use the tensor or the commuting model when defining the average entanglement dimension.

**Proposition 2.1.** For any $P \in C_q(\Gamma)$ we have $A_q(P) = A_{qc}(P)$.

**Proof.** The easy inequality $A_{qc}(P) \leq A_q(P)$ follows from $E_a^a \otimes F_t^b = (E_a^a \otimes I)(I \otimes F_t^b)$.

For the other inequality we suppose $P = \sum_{i=1}^I \lambda_i P_i$ is feasible for $A_{qc}(P)$. This means we have POVMs $\{X^a_a(i)\}_a$ and $\{Y^b_b(i)\}_b$ in $\mathbb{C}^{d_a \times d_a}$ with $[X^a_a(i), Y^b_b(i)] = 0$ and unit vectors $\psi_i \in \mathbb{C}^{d_a}$ such that $P_i(a, b|s, t) = \psi_i^* X^a_a(i) Y^b_b(i) \psi_i$ for all $(a, b, s, t) \in \Gamma$ and $i \in [I]$. We will construct a feasible solution to $A_q(P)$ with value at most $\sum \lambda_i d_i$.

Fix some index $i \in [I]$. By Artin-Wedderburn theory applied to $\mathbb{C}\langle \{X^a_a(i)\}_{a,s} \rangle$, the $*$-algebra generated by the matrices $X^a_a(i)$ for $(a, s) \in A \times S$, there exists a unitary matrix $U_i$ and integers $K_i, m_k, n_k$ such that

$$U_i \mathbb{C}\langle \{X^a_a(i)\}_{a,s} \rangle U_i^* = \bigoplus_{k=1}^{K_i} (\mathbb{C}^{n_k \times n_k} \otimes I_{m_k}) \quad \text{and} \quad d_i = \sum_{k=1}^{K_i} m_k n_k.$$
By the commutation relations each matrix $Y^b_t(i)$ commutes with all the matrices in $\mathbb{C}\langle \{X^a_s(i)\}_{a,s} \rangle$, and thus $U_t Y^b_t(i) U_t^*$ lies in the algebra $\bigoplus_k (I_{n_k} \otimes \mathbb{C}^{m_k \times m_k})$. Hence, we may assume

$$X^a_s(i) = \bigoplus_{k=1}^{K_i} E^a_s(i, k) \otimes I_{m_k}, \quad Y^b_t(i) = \bigoplus_{k=1}^{K_i} I_{n_k} \otimes F^b_t(i, k), \quad \psi_i = \bigoplus_{k=1}^{K_i} \psi_{i,k},$$

with $E^a_s(i, k) \in \mathbb{C}^{n_k \times n_k}$, $F^b_t(i, k) \in \mathbb{C}^{m_k \times m_k}$, and $\psi_{i,k} \in \mathbb{C}^{m_k n_k}$. Then we have

$$P_t(a, b|s, t) = \text{Tr}(X^a_s(i) Y^b_t(i) \psi_i \psi_i^*) = \sum_{k=1}^{K_i} \|\psi_{i,k}\|^2 \text{Tr} \left( E^a_s(i, k) \otimes F^b_t(i, k) \frac{\psi_{i,k} \psi_{i,k}^*}{\|\psi_{i,k}\|^2} \right),$$

where $Q_{i,k} \in C_q(\Gamma)$. As $\sum_k \|\psi_{i,k}\|^2 = \|\psi_i\|^2 = 1$, we have that $P_t = \sum_k \|\psi_{i,k}\|^2 Q_{i,k}$ is a convex combination of the $Q_{i,k}$’s.

We now show that $Q_{i,k} \in C_q^{\min\{m_k, n_k\}}(\Gamma)$. Consider the Schmidt decomposition $\psi_{i,k} / \|\psi_{i,k}\| = \sum_{l=1}^{\min\{m_k, n_k\}} \lambda_{i,k,l} v_{i,k,l} \otimes w_{i,k,l}$, where $\lambda_{i,k,l} \geq 0$ and $\{v_{i,k,l}\}_{l=1}^{n_k} \subseteq \mathbb{C}^{n_k}$ and $\{w_{i,k,l}\}_{l=1}^{m_k} \subseteq \mathbb{C}^{m_k}$ are orthonormal bases. Define unitary matrices $V_k \in \mathbb{C}^{n_k \times n_k}$ and $W_k \in \mathbb{C}^{m_k \times m_k}$ such that $V_k v_{i,k,l}$ is the $l$th unit vector in $\mathbb{R}^{n_k}$ and $W_k w_{i,k,l}$ is the $l$th unit vector in $\mathbb{R}^{m_k}$ for $l \leq \min\{m_k, n_k\}$. Let $E^a_s(i, k)'$ (resp., $F^b_t(i, k)'$) be the leading principal submatrices of $V_k E^a_s(i, k) V_k^*$ (resp., $W_k F^b_t(i, k) W_k^*$) of size $\min\{m_k, n_k\}$. Moreover, set $\phi_{i,k} = \sum_{l=1}^{\min\{m_k, n_k\}} \lambda_{i,k,l} e_l \otimes e_l$, where $e_l$ is the $l$th unit vector in $\mathbb{R}^{\min\{m_k, n_k\}}$. Then we have

$$Q_{i,k}(a, b|s, t) = \text{Tr} \left( E^a_s(i, k) \otimes F^b_t(i, k) \frac{\psi_{i,k} \psi_{i,k}^*}{\|\psi_{i,k}\|^2} \right) = \sum_{l,l'=1}^{\min\{m_k, n_k\}} \lambda_{i,k,l} \lambda_{i,k,l'} v_{i,k,l}^* E^a_s(i, k) v_{i,k,l'} w_{i,k,l}^* F^b_t(i, k) w_{i,k,l'}$$

$$= \sum_{l,l'=1}^{\min\{m_k, n_k\}} \lambda_{i,k,l} \lambda_{i,k,l'} e_l^* E^a_s(i, k)' e_l e_l^* F^b_t(i, k)' e_l'$$

$$= \text{Tr}((E^a_s(i, k)' \otimes F^b_t(i, k)') \phi_{i,k} \phi_{i,k}^*),$$

thus showing $Q_{i,k} \in C_q^{\min\{m_k, n_k\}}(\Gamma)$. Since $P = \sum_{i,k} \lambda_{i,k} \|\psi_{i,k}\|^2 Q_{i,k}$ is a convex decomposition, we obtain

$$A_q(P) \leq \sum_{i,k} \lambda_{i,k} \|\psi_{i,k}\|^2 \min\{m_k, n_k\} \leq \sum_{i,k} \lambda_{i,k} \min\{m_k, n_k\}^2 \leq \sum_{i,k} \lambda_{i,k} m_k n_k = \sum_i \lambda_i d_i. \quad \square$$

We now show the parameter $A_q(\cdot)$ permits to characterize classical correlations.

**Proposition 2.2.** For a correlation $P \in \mathbb{R}^\Gamma$ we have $A_q(P) = 1$ if and only if $P \in C_{\text{loc}}(\Gamma)$.

**Proof.** If $P \in C_{\text{loc}}(\Gamma)$, then $P$ can be written as a convex combination of deterministic correlations (which are contained in $C_q(\Gamma)$), hence $A_q(P) = 1$.

On the other hand, if $A_q(P) = 1$, then there exist convex decompositions indexed by $l \in \mathbb{N}$: $P = \sum_{l \in I^l} \lambda^l_1 P^l_1$ with $\{P^l_1\} \subseteq C_q(\Gamma)$ and $\lim_{l \to \infty} \sum_{l \in I^l} \lambda_l D_q(P^l_1) = 1$. Decompose $I^l$ as the disjoint union $I^l_+ \cup I^l_-$, so that $D_q(P^l_1)$ is equal to 1 for $i \in I^l_+$ and strictly greater than 1 for $i \in I^l_-$. Let $\varepsilon > 0$. For all $l$ sufficiently large we have

$$(1 - \sum_{i \in I^l_-} \lambda^l_1) + 2 \sum_{i \in I^l_+} \lambda^l_1 \leq \sum_{i \in I^l_-} \lambda^l_1 + \sum_{i \in I^l_+} \lambda^l_1 D_q(P^l_1) \leq 1 + \varepsilon,$$

which shows that $\sum_{i \in I^l_-} \lambda^l_1 \leq \varepsilon$. This shows that $P$ is the limit of convex combinations of deterministic correlations, which implies that $P \in C_{\text{loc}}(\Gamma)$. $\square$
Proposition 2.3. If \( C_q(\Gamma) \) is not closed, there exists \( \{ P_i \} \subseteq C_q(\Gamma) \) with \( A_q(P_i) \to \infty \).

Proof. Assume for contradiction there exists an integer \( K \) such that \( A_q(P) \leq K \) for all \( P \in C_q(\Gamma) \); we show this results in a uniform upper bound \( K' \) on \( D_{qc}(P) \), which implies \( C_q(\Gamma) = C^K_{qc}(\Gamma) \) is closed. For this, we will first show that \( P \in \text{conv}(C^K_{qc}(\Gamma)) \).

In a first step observe that any \( P \in C_q(\Gamma) \setminus \text{conv}(C^K_{qc}(\Gamma)) \) can be decomposed as

\[ P = \mu_1 R_1 + (1 - \mu_1) Q_1, \]

where \( R_1 \in C_q(\Gamma), \) \( Q_1 \in \text{conv}(C^K_{qc}(\Gamma)) \), and \( 0 < \mu_1 \leq K/(K + 1) \). Indeed, by assumption and using Proposition 2.1, \( A_q(P) = A_q(P) \leq K, \) so \( P \) can be written as a convex combination

\[ P = \sum_{i \in C_q(\Gamma)} \lambda_i P_i \] 

with \( \{ P_i \} \subseteq C_q(\Gamma) \) and \( \sum_{i \in I} \lambda_i D_{qc}(P_i) \leq K \). As \( P \notin \text{conv}(C^K_{qc}(\Gamma)) \), the set \( J \) of indices \( i \in I \) with \( D_{qc}(P_i) \geq K + 1 \) is non empty. Then \( (K + 1) \sum_{i \in J} \lambda_i \leq \sum_{i \in J} \lambda_i D_{qc}(P_i) \leq K, \) and thus \( 0 < \mu_1 := \sum_{i \in I \setminus J} \lambda_i \leq K/(K + 1) \). Hence (8) holds after setting \( R_1 = (\sum_{i \in J} \lambda_i P_i)/\mu_1 \) and \( Q_1 = (\sum_{i \in I \setminus J} \lambda_i P_i)/(1 - \mu_1) \).

As \( R_1 \in C_q(\Gamma) \setminus C^K_{qc}(\Gamma) \), we may repeat the same argument for \( R_1 \). By iterating we obtain for each integer \( k \in \mathbb{N} \) a decomposition

\[ P = \mu_1 \mu_2 \cdots \mu_k R_k + (1 - \mu_1) Q_1 + \mu_1 (1 - \mu_2) Q_2 + \cdots + \mu_1 \mu_2 \cdots \mu_k - 1 (1 - \mu_k) Q_k, \]

where \( R_k \in C_q(\Gamma), \) \( \hat{Q}_k \in \text{conv}(C^K_{qc}(\Gamma)) \) and \( \mu_1 \mu_2 \cdots \mu_k \leq (K/(K + 1))^k \). As the entries of \( R_k \) lie in \([0, 1] \), we know that \( \mu_1 \mu_2 \cdots \mu_k R_k \) tends to 0 as \( k \to \infty \). Hence the sequence \( (\hat{Q}_k)_k \) has a limit \( \hat{Q} \) and \( P = \hat{Q} \) holds. As all \( \hat{Q}_k \) lie in the compact set \( \text{conv}(C^K_{qc}(\Gamma)) \), we also have \( P \in \text{conv}(C^K_{qc}(\Gamma)) \).

The extreme points of the compact convex set \( \text{conv}(C^K_{qc}(\Gamma)) \) lie in \( C^K_{qc}(\Gamma) \), so, by the Carathéodory theorem, any \( P \in \text{conv}(C^K_{qc}(\Gamma)) \) is a convex combination of \( c \) elements from \( C^K_{qc}(\Gamma) \), where \( c = |\Gamma| + 1 - |S||T| \). By using a direct sum construction one can obtain \( D_{qc}(P) \leq cK \), which shows \( K' := cK \) is a uniform upper bound on \( D_{qc}(P) \) for all \( P \in C_q(\Gamma) \).

2.2 Setup of the hierarchy

We will now construct a hierarchy of lower bounds on the minimal entanglement dimension, using its formulation via \( A_{qc}(P) \). Our approach is based on noncommutative polynomial optimization, thus similar to the approach in [15] for bounding matrix factorization ranks.

We first need some notation. Set \( \mathbf{x} = \{ x^a_s : (a, s) \in A \times S \} \) and \( \mathbf{y} = \{ y^b_t : (b, t) \in B \times T \} \), and let \( (\mathbf{x}, \mathbf{y}, z)_r \) be the set of all words of length at most \( r \) in the order \( |S||A| + |T||B| + 1 \) symbols \( x^a_s, y^b_t, z \). Moreover, set \( (\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y}, z)_\infty \). We equip \( (\mathbf{x}, \mathbf{y}, z)_r \) with an involution \( w \mapsto w^* \) that reverses the order of the symbols in the words and leaves the symbols \( x^a_s, y^b_t, z \) invariant; e.g., \( (xa^s z)_r = zax^a_s \). Let \( \mathbb{R}(\mathbf{x}, \mathbf{y}, z)_r \) be the vector space of all real linear combinations of the words of length (aka degree) at most \( r \). The space \( \mathbb{R}(\mathbf{x}, \mathbf{y}, z) = \mathbb{R}(\mathbf{x}, \mathbf{y}, z)_\infty \) is the *-algebra with Hermitian generators \( x^a_s, y^b_t, z \), and the elements in this algebra are called noncommutative polynomials in the variables \( \{ x^a_s \} \), \( \{ y^b_t \} \), \( z \).

The hierarchy is based on the following idea: For any feasible solution to \( A_{qc}(P) \), its objective value can be modeled as \( L(1) \) for a certain tracial linear form \( L \) on the space of noncommutative polynomials (truncated to degree \( 2r \)).

Indeed, assume \( \{ (P_i, \lambda_i) \} \) is a feasible solution to the program \( A_{qc}(P) \) defined in Section 1.2, where \( P_i(a, b, s, t) = \text{Tr}(X^a_s(i) Y^b_t(i) \psi_i \psi_i^*) \) with \( X^a_s(i), Y^b_t(i) \in \mathbb{C}^{d_i \times d_i} \), \( \psi_i \in \mathbb{C}^{d_i} \), and \( d_i = D_{qc}(P_i) \). For \( r \in \mathbb{N} \cup \{ \infty \} \), consider the linear functional \( L \in \mathbb{R}(\mathbf{x}, \mathbf{y}, z)_r \), defined by

\[ L(p) = \sum_i \lambda_i \text{Re}(\text{Tr}(p(\mathbf{x}(i), \mathbf{y}(i), \psi_i \psi_i^*))) \quad \text{for} \quad p \in \mathbb{R}(\mathbf{x}, \mathbf{y}, z)_r. \]

Here, for each index \( i \), we set \( \mathbf{x}(i) = (X^a_s(i) : (a, s) \in A \times S), \) \( \mathbf{y}(i) = (Y^b_t(i) : (b, t) \in B \times T) \), and replace the variables \( x^a_s, y^b_t, z \) by \( X^a_s(i), Y^b_t(i), \) and \( \psi_i \psi_i^* \). Then \( L(1) = \sum_i \lambda_i d_i \).
is, \( L(1) \) is the objective value of the feasible solution \( \{ (P_i, \lambda_i) \} \) to \( A_{qc}(P) \). We will identify several computationally tractable properties that this \( L \) satisfies. Then the hierarchy of lower bounds on \( A_{qc}(P) \) consists of optimization problems where we minimize \( L(1) \) over the set of linear functionals that satisfy these properties.

First note that \( L \) is symmetric, that is, \( L(w) = L(w^*) \) for all \( w \in \langle x, y, z \rangle_{2r} \), and tracial, that is, \( L(ww') = L(w'w) \) for all \( w, w' \in \langle x, y, z \rangle \) with \( \deg(ww') \leq 2r \).

For all \( p \in \mathbb{R} \langle x, y, z \rangle_{r-1} \) we have
\[
L(p^*x_0^ap) = \sum_i \lambda_i \text{Re}(\text{Tr}(M(i)^*X_2^a(i)M(i))) \geq 0, \quad \text{where} \quad M(i) = p(X(i), Y(i), \psi_i\psi_i^*),
\]
as \( M(i)^*X_2^a(i)M(i) \) is positive semidefinite since \( X_2^a(i) \) is positive semidefinite. In the same way we have \( L(p^*y_0^bp) \geq 0 \) and \( L(p^*zp) \geq 0 \). That is, if we set
\[
G = \{ x_0^a : s \in S, a \in A \} \cup \{ y_0^b : t \in T, b \in B \} \cup \{ z \},
\]
then \( L \) is nonnegative (denoted as \( L \geq 0 \)) on the truncated quadratic module
\[
\mathcal{M}_{2r}(G) = \text{cone}\{ p^*gp : p \in \mathbb{R} \langle x, y, z \rangle, \ g \in G \cup \{ 1 \}, \ \deg(p^*gp) \leq 2r \}. \tag{9}
\]
Similarly, setting
\[
H = \{ z - z^2 \} \cup \{ 1 - \sum_{a \in A} x_0^a : s \in S \} \cup \{ 1 - \sum_{b \in B} y_0^b : t \in T \} \cup \{ [x_0^a, y_0^b] : (s, t, a, b) \in \Gamma \},
\]
we have \( L = 0 \) on the truncated ideal
\[
\mathcal{I}_{2r}(H) = \{ ph : p \in \mathbb{R} \langle x, y, z \rangle, \ h \in H, \ \deg(ph) \leq 2r \}. \tag{10}
\]
Moreover, we have \( L(z) = \sum_i \lambda_i \text{Re}(\text{Tr}(\psi_i\psi_i^*)) = 1 \). In addition, for any matrices \( U, V \in \mathbb{C}^{d \times d} \), we have
\[
\psi_i\psi_i^*U\psi_i\psi_i^*V\psi_i\psi_i^* = \psi_i\psi_i^*V\psi_i\psi_i^*U\psi_i\psi_i^*,
\]
and therefore, in particular,
\[
L(wzu vz) = L(wzv uz) \quad \text{for all} \quad u, v, w \in \langle x, y, z \rangle \quad \text{with} \quad \deg(wzu vz) \leq 2r.
\]
That is, we have \( L = 0 \) on \( \mathcal{I}_{2r}(\mathcal{R}_r) \), where
\[
\mathcal{R}_r = \{ zuvz - vzuz : u, v, w \in \langle x, y, z \rangle \text{ with } \deg(zuvz) \leq 2r \}.
\]
We get the idea of adding these last constraints from [32], where this is used to study the mutually unbiased bases problem.

We call \( \mathcal{M}(G) = \mathcal{M}_\infty(G) \) the quadratic module generated by \( G \), and we call \( \mathcal{I}(H \cup \mathcal{R}_\infty) = \mathcal{I}_\infty(H \cup \mathcal{R}_\infty) \) the ideal generated by \( H \cup \mathcal{R}_\infty \).

For \( r \in \mathbb{N} \cup \{ \infty \} \) we can now define the parameter:
\[
\xi_r^a(P) = \min\left\{ L(1) : L \in \mathbb{R} \langle x, y, z \rangle_{2r}^*, \text{ tracial and symmetric,} \right. \quad \left. L(z) = 1, L(x_0^ay_0^bz) = P(a, b, s, t) \text{ for all } (a, b, s, t) \in \Gamma, \right.
\]
\[
L \geq 0 \text{ on } \mathcal{M}_{2r}(G), \ L = 0 \text{ on } \mathcal{I}_{2r}(H \cup \mathcal{R}_r) \right\}. \tag{11}
\]
Additionally, we define \( \xi_\infty^a(P) \) by adding the constraint \( \text{rank}(M(L)) < \infty \) to \( \xi_r^a(P) \). By construction this gives a hierarchy of lower bounds for \( A_{qc}(P) \):
\[
\xi_r^a(P) \leq \ldots \leq \xi_\infty^a(P) \leq \xi_r^\infty(P) \leq \xi_\infty^\infty(P) \leq A_{qc}(P).
\]
Note that for order \( r = 1 \) we get the trivial lower bound \( \xi^\alpha_1(P) = 1 \).

For each finite \( r \in \mathbb{N} \) the parameter \( \xi^\alpha_r(P) \) can be computed by semidefinite programming. Indeed, the condition \( L \geq 0 \) on \( M_{2r}(\mathcal{G}) \) means that \( L(p^*gp) \geq 0 \) for all \( g \in \mathcal{G} \cup \{1\} \) and all polynomials \( p \in \mathbb{R}(x,y,z) \) with degree at most \( r - \lceil \text{deg}(g)/2 \rceil \). This is equivalent to requiring that the matrices \( (L(w^*gw')) \), indexed by all words \( w,w' \) with degree at most \( r - \lceil \text{deg}(g)/2 \rceil \), are positive semidefinite. To see this, write \( p = \sum a_w w \) and let \( \hat{p} = (a_w) \) denote the vector of coefficients, then \( L(p^*gp) \geq 0 \) is equivalent to \( \hat{p}^*(L(w^*gw'))\hat{p} \geq 0 \). When \( g = 1 \), the matrix \( (L(w^*gw')) \) is indexed by the words of degree at most \( r \), it is called the moment matrix of \( L \) and denoted by \( M_r(L) \) (or \( M(L) \) when \( r = \infty \)). The entries of the matrices \( (L(w^*gw')) \) are linear combinations of the entries of \( M_r(L) \), and the constraint \( L = 0 \) on \( T_{2r}(\mathcal{H} \cup \mathcal{R}) \) can be written as a set of linear constraints on the entries of \( M_r(L) \). It follows that for finite \( r \in \mathbb{N} \), the parameter \( \xi^\alpha_r(P) \) is indeed computable by a semidefinite program.

### 2.3 Background on positive tracial linear forms

Before we show the convergence results we give some background on positive tracial linear forms, which we use again in Section 3. We state these results using the variables \( x_1, \ldots, x_n \), where we use the notation \( \langle x \rangle = \langle x_1, \ldots, x_n \rangle \). The results stated below do not always appear in this way in the sources cited; we follow the presentation of [15], where full proofs for these results are also provided.

First we need a few more definitions. A polynomial \( p \in \mathbb{R}(x) \) is called symmetric if \( p^* = p \), and we denote the set of symmetric polynomials by \( \text{Sym} \mathbb{R}(x) \). Given \( \mathcal{G} \subseteq \text{Sym} \mathbb{R}(x) \) and \( \mathcal{H} \subseteq \mathbb{R}(x) \), the set \( \mathcal{M}(\mathcal{G})+\mathcal{I}(\mathcal{H}) \) is called Archimedean if it contains the polynomial \( R - \sum_{i=1}^n x_i^2 \) for some \( R > 0 \). We will use the concept of a \( C^* \)-algebra, which for our purposes can be defined as a norm closed \( * \)-subalgebra of the space \( B(\mathcal{H}) \) of bounded operators on a complex Hilbert space \( \mathcal{H} \). We say that \( \mathcal{A} \) is unital if it contains the identity operator (denoted 1). An element \( a \in \mathcal{A} \) is called positive if \( a = b^*b \) for some \( b \in \mathcal{A} \). A linear form \( \tau \) on a unital \( C^* \)-algebra \( \mathcal{A} \) is said to be a state if \( \tau(1) = 1 \) and \( \tau \) is positive; that is, \( \tau(a) \geq 0 \) for all positive elements \( a \in \mathcal{A} \).

We say that a state \( \tau \) is tracial if \( \tau(ab) = \tau(ba) \) for all \( a, b \in \mathcal{A} \). See, for example, [4] for more information on \( C^* \)-algebras.

The first result relates positive tracial linear forms to \( C^* \)-algebras; see [32] for the noncommutative (eigenvalue) setting and [8] for the tracial setting.

**Theorem 2.4.** Let \( \mathcal{G} \subseteq \text{Sym} \mathbb{R}(x) \) and \( \mathcal{H} \subseteq \mathbb{R}(x) \) and assume that \( \mathcal{M}(\mathcal{G})+\mathcal{I}(\mathcal{H}) \) is Archimedean. For a linear form \( L \in \mathbb{R}(x)^* \), the following are equivalent:

1. \( L \) is symmetric, tracial, nonnegative on \( \mathcal{M}(\mathcal{G}) \), zero on \( \mathcal{I}(\mathcal{H}) \), and \( L(1) = 1 \);

2. there is a unital \( C^* \)-algebra \( \mathcal{A} \) with tracial state \( \tau \) and \( x \in \mathcal{A}^n \) such that \( g(X) \) is positive in \( \mathcal{A} \) for all \( g \in \mathcal{G} \), and \( h(X) = 0 \) for all \( h \in \mathcal{H} \), with

\[
L(p) = \tau(p(X)) \quad \text{for all} \quad p \in \mathbb{R}(x).
\]

The following can be seen as the finite dimensional analogue of the above result. The proof of the unconstrained case (\( \mathcal{G} = \mathcal{H} = \emptyset \)) can be found in [7], and for the constrained case in [8]. Given a linear form \( L \in \mathbb{R}(x)^* \), recall that the moment matrix \( M(L) \) is given by \( M(L)_{u,v} = L(u^*v) \) for \( u, v \in \langle x \rangle \).

**Theorem 2.5.** Let \( \mathcal{G} \subseteq \text{Sym} \mathbb{R}(x) \) and \( \mathcal{H} \subseteq \mathbb{R}(x) \). For \( L \in \mathbb{R}(x)^* \), the following are equivalent:

1. \( L \) is a symmetric, tracial, linear form with \( L(1) = 1 \) that is nonnegative on \( \mathcal{M}(\mathcal{G}) \), zero on \( \mathcal{I}(\mathcal{H}) \), and has rank \( \text{rank}(M(L)) < \infty \);

2. there is a finite dimensional \( C^* \)-algebra \( \mathcal{A} \) with a tracial state \( \tau \) and \( X \in \mathcal{A}^n \) satisfying (11), with \( g(X) \) positive in \( \mathcal{A} \) for all \( g \in \mathcal{G} \), and \( h(X) = 0 \) for all \( h \in \mathcal{H} \).
(3) $L$ is a convex combination of normalized trace evaluations at tuples $X$ of Hermitian matrices that satisfy $g(X) \geq 0$ for all $g \in \mathcal{G}$ and $h(X) = 0$ for all $h \in \mathcal{H}$.

A truncated linear functional $L \in \mathbb{R}(\mathcal{X})_{2r}$ is called $\delta$-flat if the principal submatrix $M_{r-\delta}(L)$ of $M_r(L)$ indexed by monomials up to degree $r - \delta$ has the same rank as $M_r(L)$; $L$ is flat if it is $\delta$-flat for some $\delta \geq 1$. The following result claims that any flat linear functional on a truncated polynomial space can be extended to a linear functional $L$ on the full algebra of polynomials. It is due to Curto and Fialkow [10] in the commutative case and extensions to the noncommutative case can be found in [42] (for eigenvalue optimization) and [7, 21] (for trace optimization).

Theorem 2.6. Let $1 \leq \delta \leq t < \infty$, $\mathcal{G} \subseteq \text{Sym} \mathbb{R}(\mathcal{X})_{2r}$, and $\mathcal{H} \subseteq \mathbb{R}(\mathcal{X})_{2r}$. If $L \in \mathbb{R}(\mathcal{X})_{2r}$ is symmetric, tracial, $\delta$-flat, nonnegative on $\mathcal{M}_{2r}(\mathcal{G})$, and zero on $\mathcal{I}_{2r}(\mathcal{H})$, then $L$ extends to a symmetric, tracial, linear form on $\mathbb{R}(\mathcal{X})$ that is nonnegative on $\mathcal{M}(\mathcal{G})$, zero on $\mathcal{I}(\mathcal{H})$, and whose moment matrix has finite rank.

The following technical lemma, based on the Banach-Alaoglu theorem, is a well-known tool to show asymptotic convergence results in polynomial optimization.

Lemma 2.7. Let $\mathcal{G} \subseteq \text{Sym} \mathbb{R}(\mathcal{X})$, $\mathcal{H} \subseteq \mathbb{R}(\mathcal{X})$, and assume that for some $d \in \mathbb{N}$ and $R > 0$ we have $R - (x_1^2 + \cdots + x_n^2) \in M_{2d}(\mathcal{G}) + I_{2d}(\mathcal{H})$. For $r \in \mathbb{N}$ assume $L_r \in \mathbb{R}(\mathcal{X})_{2r}$ is tracial, nonnegative on $\mathcal{M}_{2r}(\mathcal{G})$ and zero on $\mathcal{I}_{2r}(\mathcal{H})$. Then $|L_r(w)| \leq R^{|w|/2}L_r(1)$ for all $w \in (\mathbb{X}_{2r-2d+2}^r)$. In addition, if $\sup \{L_r(1) < \infty\}$, then $\{L_r\}_r$ has a pointwise converging subsequence in $\mathbb{R}(\mathcal{X})^\mathbb{N}$.

2.4 Convergence results

We first show equality $\xi^\mathbb{H}_t(P) = A_{qe}(P)$, and then we consider convergence properties of the bounds $\xi^\mathbb{H}_t(P)$ to the parameters $\xi^\mathbb{H}_t(P)$ and $\xi^\mathbb{H}_t(P)$.

Proposition 2.8. For any $P \in C_q(\Gamma)$ we have $\xi^\mathbb{H}_t(P) = A_{qe}(P)$.

Proof. We already know that $\xi^\mathbb{H}_t(P) \leq A_{qe}(P)$. To show $\xi^\mathbb{H}_t(P) \geq A_{qe}(P)$ let $L$ be feasible for $\xi^\mathbb{H}_t(P)$, so that $L \geq 0$ on $\mathcal{M}(\mathcal{G})$ and $L = 0$ on $\mathcal{I}(\mathcal{H} \cup \mathcal{R}_\infty)$. By Theorem 2.5, there exist finitely many scalar $\lambda_i \geq 0$, Hermitian matrix tuples $X(i) = (X^a(i))_{a \in A}$ and $Y(i) = (Y^a(i))_{a \in A}$, and Hermitian matrices $Z_i$, so that $g(X(i), Y(i), Z_i) \geq 0$ for all $g \in \mathcal{G}$, $h(X(i), Y(i), Z_i) = 0$ for all $h \in \mathcal{H} \cup \mathcal{R}_\infty$, and

$$L(p) = \sum_i \lambda_i \text{Tr}(p(X(i), Y(i), Z_i)) \quad \text{for all } p \in \mathbb{C}(x,y,z).$$

(12)

By Artin–Wedderburn theory we know that for each $i$ there is a unitary matrix $U_i$ such that $U_i \mathbb{C}(X(i), Y(i), Z_i) U_i^* = \bigoplus_k \mathbb{C}^{d_k \times d_k} \otimes I_{m_k}$. Hence, after applying this further block diagonalization we may assume that in the decomposition (12), for each $i$, $\mathbb{C}(X(i), Y(i), Z_i)$ is a full matrix algebra $\mathbb{C}^{d_i \times d_i}$.

Since $h(E(i), F(i), Z_i) = 0$ for all $h \in \mathcal{R}_\infty \cup \{z - z^2\}$, the commutator $[Z_i u Z_i, Z_i v Z_i]$ vanishes for all $u, v \in \mathbb{C}(E(i), F(i), Z_i)$ and hence for all $u, v \in \mathbb{C}(E(i), F(i), Z_i)$. This means that $[\mathcal{Z}_1 T_1 Z_1, \mathcal{Z}_2 T_2 Z_2] = 0$ for all $T_1, T_2 \in \mathbb{C}^{d_i \times d_i}$. Since $Z_i$ is a projector, there exists a unitary matrix $U_i$ such that $U_i Z_i U_i^* = \text{Diag}(1, \ldots, 1, 0, \ldots, 0)$. The above then implies that for all $T_1$ and $T_2$, the leading principal submatrices of size rank($Z_i$) of $U_i T_1 U_i^*$ and $U_i T_2 U_i^*$ commute. This implies rank($Z_i$) $\leq 1$ and thus $\text{Tr}(Z_i) \in \{0, 1\}$. Let $I$ be the set of indices with $\text{Tr}(Z_i) = 1$. Then $\sum_{i \in I} \lambda_i = \sum_{i \in I} \lambda_i \text{Tr}(Z_i) = \text{L}(z) = 1$.

For each $i \in I$ define $P_i = (\text{Tr}(E^a(i) F^b(i) Z_i))$, which is a quantum correlation in $C_q^d(\Gamma)$ because $\text{Tr}(Z_i) = 1$, $\sum_i X^a(i) = \sum_i Y^a(i) = I$, and $[X^a(i), Y^a(i)] = 0$ by the ideal conditions. We have $P = \sum_{i \in I} \lambda_i P_i$, so that $(P, \lambda_i)_{i \in I}$ forms a feasible solution to $A_{qe}(P)$ with objective value $\sum_{i \in I} \lambda_i D_{qe}(P_i) \leq \sum_{i \in I} \lambda_i d_i \leq \sum_i \lambda_i d_i = L(1)$. \qed
The problem $\xi^0(P)$ differs in two ways from a standard tracial optimization problem. It does not have the normalization $L(1) = 1$ (and instead minimizes $L(1)$), and it has ideal constraints $L = 0$ on $\mathcal{I}_{2r}(\mathcal{R}_r)$ where $\mathcal{R}_r$ depends on $r$. We show asymptotic convergence still holds.

**Proposition 2.9.** For any $P \in C_q(\Gamma)$ we have $\xi^0(P) \rightarrow \xi^0_\infty(P)$ as $r \rightarrow \infty$.

**Proof.** First observe that $1 - z^2$, $1 - (x_s^a)^2$, $1 - (y_t^b)^2 \in \mathcal{M}_4(\mathcal{G} \cup \mathcal{H}_0)$, where $\mathcal{H}_0$ contains the symmetric polynomials in $\mathcal{H}$; i.e., omitting the commutators $[x_s^a, y_t^b]$. Indeed, we have $1 - z^2 = (1-z)^2 + 2(1-z^2)$ and $1 - (x_s^a)^2 = (1-x_s^a)^2 + 2(1-x_s^a)x_s^a - x_s^a$, and the same for $y_t^b$. Hence $R - z^2 - \sum_{a,s}(x_s^a)^2 - \sum_{b,t}(y_t^b)^2 \in \mathcal{M}_4(\mathcal{G} \cup \mathcal{H}_0)$ for some $R > 0$. Fix $\varepsilon > 0$ and for each $r \in \mathbb{N}$ let $L_r$ be feasible for $\xi^0(P)$ with value $L_r(1) \leq \xi^0_\infty(P) + \varepsilon$. As $L_r$ is tracial and zero on $\mathcal{I}_{2r}(\mathcal{H}_0)$, it follows (using the identity $p^* gp = pp^* g + [p^* g, p]$) that $L = 0$ on $\mathcal{I}_{2r}(\mathcal{H}_0)$. Hence, $L_r \geq 0$ on $\mathcal{M}_4(\mathcal{G} \cup \mathcal{H}_0)$. Since $\sup_r L_r(1) \leq A_q(P) + \varepsilon$, we can apply Lemma 2.7 and conclude that $\{L_r\}_r$ has a converging subsequence; denote its limit by $L_\varepsilon \in \mathbb{R}(x)^\ast$. One can verify that $L_\varepsilon$ is feasible for $\xi^0_\infty(P)$, and $\xi^0_\infty(P) \leq L_\varepsilon(1) \leq \lim_{r \rightarrow \infty} \xi^0_\varepsilon(P) + \varepsilon$. Letting $\varepsilon \rightarrow 0$ we obtain that $\xi^0_\infty(P) = \lim_{r \rightarrow \infty} \xi^0_\varepsilon(P)$. \hfill $\square$

Next we show that if $\xi^0(P)$ admits a $\delta$-flat optimal solution with $\delta = \lceil r/3 \rceil + 1$, then we have $\xi^0(P) = \xi^0_\varepsilon(P)$. This result is a variation of the flat extension result from Theorem 2.6, where $\delta$ now depends on the order $r$ because the ideal constraints in $\xi^0(P)$ depend on $r$.

**Proposition 2.10.** If $\xi^0(P)$ admits a $\lceil r/3 \rceil + 1$-flat optimal solution, $\xi^0(P) = \xi^0_\varepsilon(P)$.

**Proof.** Let $\delta = \lceil r/3 \rceil + 1$ and let $L$ be a $\delta$-flat optimal solution to $\xi^0_\varepsilon(P)$, which we do by constructing a feasible solution to $\xi^0_\varepsilon(P)$ with the same objective value. The main step in the proof of Theorem 2.6 consists of extending the linear form $L$ to a tracial symmetric linear form $\hat{L}$ on $\mathbb{R}(x,y,z)$ that is nonnegative on $\mathcal{M}(\mathcal{G})$, zero on $\mathcal{I}(\mathcal{H})$, and satisfies rank$(\mathcal{M}(\hat{L})) < \infty$ (see the proof of [15, Thm. 2.3] for a detailed exposition). To do this a subset $W$ of $\langle x,y,z \rangle_{\mathcal{R}_\varepsilon}$ is found such that we have the vector space direct sum $\mathbb{R}(x,y,z) = \text{span}(W) \oplus \mathcal{I}(N_r(L))$, where $N_r(L)$ is the vector space $N_r(L) = \{p \in \mathbb{R}(x,y,z)_r : L(q)p = 0 \text{ for all } q \in \mathbb{R}(x,y,z)_r\}$. It is moreover shown that $\mathcal{I}(N_r(L)) \subseteq N(\hat{L})$. For $p \in \mathbb{R}(x,y,z)$ we denote by $r_p$ the unique element in span$(W)$ such that $p - r_p \in \mathcal{I}(N_r(L))$. We now show that $\hat{L}$ is zero on $\mathcal{I}(\mathcal{R}_\infty)$. For this fix $u,v \in \mathbb{R}(x,y,z)$. Then we have

$$\hat{L}(w(zuzvz - vzuvz)) = \hat{L}(wzuvzv) - \hat{L}(wzuvzu).$$

Since $\hat{L}$ is tracial and $u - r_u, v - r_v, w - r_w \in \mathcal{I}(N_r(L)) \subseteq N(\hat{L})$, we have

$$\hat{L}(wzuvzv) = \hat{L}(r_wzruvr_vz) \quad \text{and} \quad \hat{L}(wzuvzu) = \hat{L}(r_wzruvr_uz).$$

Since $\deg(r_wzruvr_wz) = \deg(r_wzruvr_uz) \leq 2r$ we have

$$\hat{L}(r_wzruvr_u) = L(r_wzruvr_u) \quad \text{and} \quad \hat{L}(r_wzruvr_u) = L(r_wzruvr_u).$$

So $L = 0$ on $\mathcal{I}_{2r}(\mathcal{R}_r)$ implies $\hat{L} = 0$ on $\mathcal{I}(\mathcal{R}_\infty)$. Since $\hat{L}$ extends $L$ we have $\hat{L}(z) = L(z) = 1$ and $\hat{L}(x_s^ay_t^b) = L(x_s^ay_t^b) = P(a,b|s,t)$ for all $a,b,s,t$. So, $\hat{L}$ is feasible for $\xi^0_\varepsilon(P)$ and has the same objective value $\hat{L}(1) = L(1)$.
3 Bounding quantum graph parameters

3.1 Hierarchies $\gamma_r^{\text{col}}(G)$ and $\gamma_r^{\text{stab}}(G)$ based on synchronous correlations

In Section 1.3 we introduced quantum chromatic numbers (Definition 1.1) and quantum stability numbers (Definition 1.2) in terms of synchronous quantum correlations satisfying certain linear constraints. We first give (known) reformulations in terms of $C^*$-algebras, and then we reformulate those in terms of tracial optimization, which leads to the hierarchies $\gamma_r^{\text{col}}(G)$ and $\gamma_r^{\text{stab}}(G)$.

The following result from [41] allows us to write a synchronous quantum correlation in terms of synchronous quantum correlations satisfying certain feasibility problems recovering the classical graph parameters (Theorem 3.1).

Theorem 3.1 ([41]). Let $G = A^2 \times S^2$ and $P \in \mathbb{R}^G$. We have $P \in C_{qc,H}(\Gamma)$ (resp., $P \in C_{q,H}(\Gamma)$) if and only if there exists a unital (resp., finite dimensional) $C^*$-algebra $A$ with a faithful tracial state $\tau$ and a set of projectors $\{X^a_s : s \in S, a \in A\} \subseteq A$ satisfying $\sum_{a \in A} X^a_s = 1$ for all $s \in S$ and $P(a,b,s,t) = \tau(X^a_s X^b_t)$ for all $s,t \in S, a,b \in A$.

Here we add the condition that $\tau$ is faithful, that is, $\tau(X^*X) = 0$ implies $X = 0$, since it follows from the GNS construction in the proof of [41]. This means that

$$0 = P(a,b,s,t) = \tau(X^a_s X^b_t) = \tau((X^a_s)^2(X^b_t)^2) = \tau((X^a_s X^b_t)^* X^a_s X^b_t)$$

implies $X^a_s X^b_t = 0$. It follows that $\chi_{qc}(G)$ is equal to the smallest $k \in \mathbb{N}$ for which there exists a $C^*$-algebra $A$, a tracial state $\tau$ on $A$, and a family of projectors $\{X^a_i : i \in V, c \in [k]\} \subseteq A$ satisfying

$$\sum_{c \in [k]} X^c_i - 1 = 0 \quad \text{for all} \quad i \in V,$$

$$X^c_i X^d_j = 0 \quad \text{if} \quad (c \neq d' \text{ and } i = j) \quad \text{or} \quad (c = d' \text{ and } \{i,j\} \in E).$$

The quantum chromatic number $\chi_q(G)$ is equal to the smallest finite dimensional $C^*$-algebra $A$ with the above properties.

Analogously, $\alpha_{qc}(G)$ is equal to the largest $k \in \mathbb{N}$ for which there is a $C^*$-algebra $A$, a tracial state $\tau$ on $A$, and a set of projectors $\{X^a_i : c \in [k], i \in V\} \subseteq A$ satisfying

$$\sum_{i \in V} X^c_i - 1 = 0 \quad \text{for all} \quad c \in [k],$$

$$X^c_i X^d_{i'} = 0 \quad \text{if} \quad (i \neq i' \text{ and } c = c') \quad \text{or} \quad ((i = i' \text{ or } \{i,j\} \in E) \text{ and } c \neq c'),$$

and $\alpha_q(G)$ is equal to the largest $k \in \mathbb{N}$ for which $A$ can be taken finite dimensional.

These reformulations of $\chi_q(G), \chi_{qc}(G), \alpha_q(G)$ and $\alpha_{qc}(G)$ also follow from [36, Thm. 4.7], where general quantum graph homomorphisms are considered; the formulations of $\chi_q(G)$ and $\chi_{qc}(G)$ are also made explicit in [36, Thm. 4.12].

By Artin-Wedderburn theory [53, 2], a finite dimensional $C^*$-algebra is isomorphic to a matrix algebra. So the above reformulations of $\chi_q(G)$ and $\alpha_q(G)$ can be seen as feasibility problems of systems of equations in matrix variables of unspecified (but finite) dimension; such formulations are given in [9, 28, 46]. Restricting to scalar solutions ($1 \times 1$ matrices) in these feasibility problems recovers the classical graph parameters $\chi(G)$ and $\alpha(G)$.

We now reinterpret the above formulations in terms of tracial optimization. Given a graph $G = (V,E)$, let $i \sim j$ denote $\{i,j\} \in E$ or $i = j$. For $k \in \mathbb{N}$, let $\mathcal{H}_{G,k}^{\text{col}}$ and $\mathcal{H}_{G,k}^{\text{stab}}$ denote the sets of polynomials corresponding to equations (13)–(14) and (15)–(16):

$$\mathcal{H}_{G,k}^{\text{col}} = \left\{ 1 - \sum_{c \in [k]} x^c_i : i \in V \right\} \cup \left\{ x^c_i x^c_{i'} : (c \neq c' \text{ and } i = j) \text{ or } (c = c' \text{ and } \{i,j\} \in E) \right\},$$

and
\[ \mathcal{H}_{G,k}^{\text{stab}} = \{ 1 - \sum_{i \in V} x_i^c : c \in [k] \} \cup \{ x_i^c x_j^{c'} : (i \neq j \text{ and } c = c') \text{ or } (i = j \text{ and } c \neq c') \}. \]

We have \( 1 - (x_i^c)^2 \in M_2(\emptyset) + I_2(\mathcal{H}_{G,k}^{\text{col}}) \), since \( 1 - (x_i^c)^2 = (1 - x_i^c)^2 + 2(x_i^c - x_i^c)^2 \) and \( x_i^c - x_i^c = x_i^c (1 - \sum_{c' \neq c} x_i^{c'}) + \sum_{c' \neq c} x_i^{c'} x_i^{c'} \) \( \in I_2(\mathcal{H}_{G,k}^{\text{col}}) \), and the analogous statements hold for \( \mathcal{H}_{G,k}^{\text{stab}} \). Hence, both \( \mathcal{M}(\emptyset) + \mathcal{I}(\mathcal{H}_{G,k}^{\text{col}}) \) and \( \mathcal{M}(\emptyset) + \mathcal{I}(\mathcal{H}_{G,k}^{\text{stab}}) \) are Archimedean and we can apply Theorems 2.4 and 2.5 to express the quantum graph parameters in terms of positive tracial linear functionals. Namely,

\[ \chi_{qc}(G) = \min \{ k \in \mathbb{N} : L \in \mathbb{R}\{\{ x_i^c : i \in V, c \in [k]\}\}^* \text{ symmetric, tracial, positive,} \]

\[ L(1) = 1, \; L = 0 \text{ on } \mathcal{I}(\mathcal{H}_{G,k}^{\text{col}}), \]

and \( \chi_q(G) \) is obtained by adding the constraint rank(\( M(L) \)) < \( \infty \). Likewise,

\[ \alpha_{qc}(G) = \min \{ k \in \mathbb{N} : L \in \mathbb{R}\{\{ x_i^c : c \in [k], i \in V\}\}^* \text{ symmetric, tracial, positive,} \]

\[ L(1) = 1, \; L = 0 \text{ on } \mathcal{I}(\mathcal{H}_{G,k}^{\text{stab}}), \]

and \( \alpha_q(G) \) is given by this program with the additional constraint rank(\( M(L) \)) < \( \infty \).

Starting from these formulations it is natural to define a hierarchy \( \{ \gamma_r^{\text{col}}(G) \} \) of lower bounds on \( \chi_{qc}(G) \) and a hierarchy \( \{ \gamma_r^{\text{stab}}(G) \} \) of upper bounds on \( \alpha_{qc}(G) \), where the bounds of order \( r \in \mathbb{N} \) are obtained by truncating \( L \) to polynomials of degree at most \( 2r \) and truncating the ideal to degree \( 2r \). Then, by defining \( \gamma_r^{\col}(G) \) and \( \gamma_r^{\stab}(G) \) by adding the constraint rank(\( M(L) \)) < \( \infty \) to \( \gamma_\infty^{\col}(G) \) and \( \gamma_\infty^{\stab}(G) \), we have

\[ \gamma_\infty^{\col}(G) = \chi_{qc}(G), \; \gamma_\infty^{\stab}(G) = \alpha_{qc}(G), \; \gamma_r^{\col}(G) = \chi_q(G), \; \text{and} \; \gamma_r^{\stab}(G) = \alpha_q(G). \]

The optimization problems \( \gamma_r^{\col}(G) \), for \( r \in \mathbb{N} \), can be computed by semidefinite programming and binary search on \( k \), since the positivity condition on \( L \) can be expressed by requiring that its truncated moment matrix \( M_r(L) = (L(w^* w')) \) (indexed by words with degree at most \( r \)) is positive semidefinite. If there is an optimal solution \( (k, L) \) to \( \gamma_r^{\col}(G) \) with \( L \) flat, then, by Theorem 2.6, we have equality \( \gamma_r^{\col}(G) = \chi_q(G) \). Since \( \{ \gamma_r^{\col}(G) \}_{r \in \mathbb{N}} \) is a monotone nondecreasing sequence of lower bounds on \( \chi_q(G) \), there exists an \( r_0 \) such that for all \( r \geq r_0 \) we have \( \gamma_r^{\col}(G) = \gamma_{r_0}^{\col}(G) \), which is equal to \( \gamma_\infty^{\col}(G) = \chi_{qc}(G) \) by Lemma 2.7. The analogous statements hold for the parameters \( \gamma_r^{\stab}(G) \). Hence, we have shown the following result.

**Proposition 3.2.** There is an \( r_0 \in \mathbb{N} \) such that \( \gamma_r^{\col}(G) = \chi_q(G) \) and \( \gamma_r^{\stab}(G) = \alpha_q(G) \) for all \( r \geq r_0 \). Moreover, if \( \gamma_r^{\col}(G) \) admits a flat optimal solution, then \( \gamma_r^{\col}(G) = \chi_q(G) \), and if \( \gamma_r^{\stab}(G) \) admits a flat optimal solution, then \( \gamma_r^{\stab}(G) = \alpha_q(G) \).

**Remark 3.3.** A hierarchy \( \{ Q_r(\Gamma) \} \) of outer semidefinite approximations for the set \( C_{qc}(\Gamma) \) of commuting quantum correlations was constructed in [41], revisiting the approach in [31, 42]. This hierarchy is converging, that is,

\[ C_{qc}(\Gamma) = Q_\infty(\Gamma) = \bigcap_{r \in \mathbb{N}} Q_r(\Gamma). \]

The approximations \( Q_r(\Gamma) \) are based on the eigenvalue optimization approach, applied to the formulation (3) of commuting quantum correlations, and thus they use linear functionals on polynomials involving two sets of variables \( x_{s}^{a}, y_{t}^{b} \) for \( (a, b, s, t) \in \Gamma \). The authors of [41] use these outer approximations of \( C_{qc}(\Gamma) \) to define a converging hierarchy of lower bounds on \( \chi_{qc}(G) \) in terms of feasibility problems over the sets \( Q_r(\Gamma) \).

For synchronous correlations we can use the result of Theorem 3.1 and the tracial optimization approach used here to define directly a converging hierarchy \( \{ Q_{r,s}(\Gamma) \} \) of outer semidefinite
approximations for the set \( C_{qc,s}(\Gamma) \) of synchronous commuting quantum correlations. These approximations now use linear functionals on polynomials involving only one set of variables \( x_s^a \). Namely, define \( Q_{r,s}(\Gamma) \) as the set of \( P \in \mathbb{R}^1 \) for which there exists a symmetric, tracial, positive linear functional \( L \in \mathbb{R}\{x_s^a : (a, s) \in A \times S\}_{2r}^* \) such that \( L(1) = 1 \) and \( L = 0 \) on the ideal generated by the polynomials \( x_s^a - (x_s^a)^2 \) for \( (a, s) \in A \times S \) and \( 1 - \sum_a x_s^a \) (\( s \in S \)), truncated at degree \( 2r \). Then we have

\[
C_{qc,s}(\Gamma) = Q_{\infty,s}(\Gamma) = \bigcap_{r \in \mathbb{N}} Q_{r,s}(\Gamma).
\]

The synchronous value of a nonlocal game is defined in [12] as the maximum value of the objective function (6) over the set \( C_{qc,s}(\Gamma) \). By maximizing the objective (6) over the relaxations \( Q_{r,s}(\Gamma) \) we get a hierarchy of semidefinite programming upper bounds that converges to the synchronous value of the game. Finally note that one can also view the parameters \( \gamma_{r}^{col}(G) \) as solving feasibility problems over the sets \( Q_{r,s}(\Gamma) \).

### 3.2 Hierarchies \( \xi_r^{col}(G) \) and \( \xi_r^{stab}(G) \) based on Lasserre type bounds

Here we revisit some known Lasserre type hierarchies for the classical stability number \( \alpha(G) \) and chromatic number \( \chi(G) \) and we show that their tracial noncommutative analogues can be used to recover known parameters such as the projective packing number \( \alpha_p(G) \), the projective rank \( \xi_{p}(G) \), and the tracial rank \( \xi_{tr}(G) \). Compared to the hierarchies defined in the previous section, these Lasserre type hierarchies use less variables (they only use variables indexed by the vertices of the graph \( G \)), but they also do not converge to the (commuting) quantum chromatic or stability number.

Given a graph \( G = (V, E) \), define the set of polynomials

\[
\mathcal{H}_G = \{ x_i - x_i^2 : i \in V \} \cup \{ x_i x_j : \{i, j\} \in E \}
\]

in the variables \( x = (x_i : i \in V) \) (which are commutative or noncommutative depending on the context). Note that \( 1 - x_i^2 \) is \( \mathcal{M}_2(\emptyset) + I(\mathcal{H}_G) \) for all \( i \in V \), so \( \mathcal{M}(\emptyset) + I(\mathcal{H}_G) \) is Archimedean.

#### 3.2.1 Semidefinite programming bounds on the projective packing number

We first recall the Lasserre hierarchy of bounds for the classical stability number \( \alpha(G) \). Starting from the formulation of \( \alpha(G) \) via the optimization problem

\[
\alpha(G) = \sup\left\{ \sum_{i \in V} x_i : x \in \mathbb{R}^n, h(x) = 0 \text{ for } h \in \mathcal{H}_G \right\},
\]

the \( r \)-th level of the Lasserre hierarchy for \( \alpha(G) \) (introduced in [23, 24]) is defined by

\[
\text{las}_r^{\text{stab}}(G) = \sup\left\{ L\left( \sum_{i \in V} x_i \right) : L \in \mathbb{R}[x]_{2r}^*, \text{positive, } L(1) = 1, L = 0 \text{ on } I_{2r}(\mathcal{H}_G) \right\}.
\]

Then \( \text{las}_r^{\text{stab}}(G) \leq \text{las}_r^{\text{stab}}(G) \), the first bound is Lovász’ theta number: \( \text{las}_1^{\text{stab}}(G) = \theta(G) \), and finite convergence to \( \alpha(G) \) is shown in [24]: \( \text{las}_{\infty}^{\text{stab}}(G) = \alpha(G) \).

Roberson [45] introduces the projective packing number

\[
\alpha_p(G) = \sup\left\{ \frac{1}{d} \sum_{i \in V} \text{rank} \, X_i : d \in \mathbb{N}, X \in (S^d)^n \text{ projectors, } X_i X_j = 0 \text{ for } \{i, j\} \in E \right\}
\]

\[
= \sup\left\{ \frac{1}{d} \text{Tr} \left( \sum_{i \in V} X_i \right) : d \in \mathbb{N}, X \in (S^d)^n, h(X) = 0 \text{ for } h \in \mathcal{H}_G \right\}
\]

(17)
as an upper bound for the quantum stability number \( \alpha_q(G) \); the inequality \( \alpha_q(G) \leq \alpha_p(G) \) also follows from Proposition 3.4 below. In view of (17), the parameter \( \alpha_p(G) \) can be seen as a noncommutative analogue of \( \alpha(G) \).

For \( r \in \mathbb{N} \cup \{ \infty \} \) we define the noncommutative analogue of \( \text{las}^{\text{stab}}_r(G) \) by

\[
\xi_r^{\text{stab}}(G) = \sup \left\{ L \left( \sum_{i \in V} x_i \right) : L \in \mathbb{R} \langle x_i \rangle_{2r} \text{ tracial, symmetric, and positive,} \right. \\
\left. L(1) = 1, L = 0 \text{ on } I_{2r}(\mathcal{H}_G) \right\},
\]

and \( \xi_r^{\text{stab}}(G) \) by adding the constraint rank(\( M(L) \)) < \( \infty \) to the definition of \( \xi_r^{\text{stab}}(G) \).

In view of Theorems 2.4 and 2.5, both \( \xi_r^{\text{stab}}(G) \) and \( \xi_r^{\text{stab}}(G) \) can be reformulated in terms of \( C^* \)-algebras: \( \xi_r^{\text{stab}}(G) \) (resp., \( \xi_r^{\text{stab}}(G) \)) is the largest value of \( \tau(\sum_{i \in V} x_i) \), where \( A \) is a (resp., finite-dimensional) \( C^* \)-algebra with tracial state \( \tau \) and \( X_i \in A \) (\( i \in [n] \)) are projectors satisfying \( X_iX_j = 0 \) for all \( \{i, j\} \in E \). Moreover, as we now see, the parameter \( \xi_r^{\text{stab}}(G) \) coincides with the projective packing number and the parameters \( \xi_r^{\text{stab}}(G) \) and \( \xi_r^{\text{stab}}(G) \) upper bound the quantum stability numbers.

**Proposition 3.4.** We have \( \xi_r^{\text{stab}}(G) = \alpha_p(G) \geq \alpha_q(G) \) and \( \xi_r^{\text{stab}}(G) \geq \alpha_q(G) \).

**Proof.** By (17), \( \alpha_p(G) \) is the largest value of \( L(\sum_{i \in V} x_i) \) over linear functionals \( L \) that are normalized trace evaluations at projectors \( X \in \{ S^d \}^n \) (for some \( d \in \mathbb{N} \)) with \( X_iX_j = 0 \) for \( \{i, j\} \in E \). By convexity the optimum remains unchanged when considering a convex combination of such trace evaluations. In view of Theorem 2.5(3), this optimum is precisely the parameter \( \xi_r^{\text{stab}}(G) \). This shows equality \( \alpha_p(G) = \xi_r^{\text{stab}}(G) \).

Consider a \( C^* \)-algebra \( A \) with tracial state \( \tau \) and projectors \( X_c^i \in A \) (\( i \in V, c \in [k] \)) satisfying (15)-(16). Then, setting \( X_i = \sum_{c \in [k]} x_i^c \) for \( i \in V \), we obtain projectors \( X_i \in A \) that satisfy \( X_iX_j = 0 \) if \( \{i, j\} \in E \). Moreover, \( \tau(\sum_{i \in V} x_i) = \sum_{c \in [k]} \tau(\sum_{i \in V} x_i^c) = k \). This shows \( \xi_r^{\text{stab}}(G) \geq \alpha_q(G) \) and, when restricting \( A \) to be finite dimensional, \( \xi_r^{\text{stab}}(G) \geq \alpha_q(G) \).

Using Lemma 2.7 one can verify that \( \xi_r^{\text{stab}}(G) \) converges to \( \xi_r^{\text{stab}}(G) \) as \( r \to \infty \), and for \( r \in \mathbb{N} \cup \{ \infty \} \) the infimum in \( \xi_r^{\text{stab}}(G) \) is attained. Moreover, by Theorem 2.6, if \( \xi_r^{\text{stab}}(G) \) admits a flat optimal solution, then \( \xi_r^{\text{stab}}(G) \). Also, the first bound \( \xi_r^{\text{stab}}(G) \) coincides with the theta number, since \( \xi_r^{\text{stab}}(G) = \text{las}^{\text{stab}}_1(G) = \vartheta(G) \). Summarizing we have \( \alpha_q(G) \leq \xi_r^{\text{stab}}(G) \) and the following chain of inequalities

\[
\alpha_q(G) \leq \alpha_p(G) = \xi_r^{\text{stab}}(G) \leq \xi_r^{\text{stab}}(G) \leq \xi_r^{\text{stab}}(G) = \vartheta(G).
\]

### 3.2.2 Semidefinite programming bounds on the projective rank and tracial rank

We now turn to the (quantum) chromatic numbers. First recall the definition of the fractional chromatic number:

\[
\chi_f(G) := \min \left\{ \sum_{S \subseteq V} \lambda_S : \lambda \in \mathbb{R}^S, \sum_{S \subseteq V} \lambda_S = 1 \text{ for all } i \in V \right\},
\]

where \( S \) is the set of stable sets of \( G \). Clearly, \( \chi_f(G) \leq \chi(G) \). The following Lasserre type lower bounds for the classical chromatic number \( \chi(G) \) are defined in [18]:

\[
\text{las}^{\text{col}}_r(G) = \inf \{ L(1) : L \in \mathbb{R} \langle x \rangle_{2r} \text{ positive, } L(x_i) = 1 \text{ for } i \in V, L = 0 \text{ on } I_{2r}(\mathcal{H}_G) \}.
\]

By viewing \( \chi_f(G) \) as minimizing \( L(1) \) over linear functionals \( L \in \mathbb{R} \langle x \rangle^* \) that are conic combinations of evaluations at characteristic vectors of stable sets, we see that \( \text{las}^{\text{col}}_r(G) \leq \chi_f(G) \) for all \( r \geq 1 \). In [18] it is shown that \( \text{las}^{\text{col}}_{\alpha(G)}(G) = \chi_f(G) \). Moreover, the order 1 bound coincides with the theta number: \( \text{las}^{\text{col}}_1(G) = \vartheta(G) \).
The following parameter $\xi_f(G)$, called the projective rank of $G$, was introduced in [28] as a lower bound on the quantum chromatic number $\chi_q(G)$:

$$\xi_f(G) := \inf \left\{ \frac{d}{r} : d, r \in \mathbb{N}, X_1, \ldots, X_n \in S^d, \, \text{Tr}(X_i) = r \ (i \in V), \right.$$ 

$$\left. X_i^2 = X_i \ (i \in V), \ X_iX_j = 0 \ (\{i, j\} \in E) \right\}.$$ 

**Proposition 3.5 ([28]).** For any graph $G$ we have $\xi_f(G) \leq \chi_q(G)$.

**Proof.** Set $k = \chi_q(G)$. It is shown in [9] that in the definition of $\chi_q(G)$ from (13)–(14), one may assume w.l.o.g. that all matrices $X_i^c$ have the same rank, say, $r$. Then, for any given color $c \in [k]$, the matrices $X_i^c$ ($i \in V$) provide a feasible solution to $\xi_f(G)$ with value $d/r$. Finally, $d/r = k$ holds since by (13)–(14) we have $d = \text{rank}(I) = \sum_{c=1}^k \text{rank}(X_i^c) = kr$. \hfill $\square$

In [41, Prop. 5.11] it is shown that the projective rank can equivalently be defined as

$$\xi_f(G) = \inf \{ \lambda : \mathcal{A} \text{ is a finite dimensional } C^*\text{-algebra with tracial state } \tau, \ X_i \in \mathcal{A} \text{ projector with } \tau(X_i) = 1/\lambda \ (i \in V), \ X_iX_j = 0 \ (\{i, j\} \in E) \}.$$ 

They also define the tracial rank $\xi_{tr}(G)$ of $G$ as the parameter obtained by omitting in the above definition of $\xi_f(G)$ the restriction that $\mathcal{A}$ has to be finite dimensional. The motivation for the parameter $\xi_{tr}(G)$ is that it lower bounds the commuting quantum chromatic number [41, Thm. 5.11]: $\xi_{tr}(G) \leq \chi_q(G)$.

In view of Theorems 2.4 and 2.5, we obtain the following reformulations:

$$\xi_f(G) = \inf \{ L(1) : L \in \mathbb{R}(\mathcal{A})^* \text{ tracial, symmetric, positive, rank}(M(L)) < \infty, \right.$$ 

$$\left. L(x_i) = 1 \ (i \in V), \ L = 0 \text{ on } \mathcal{I}(\mathcal{H}_G) \}.$$ 

and $\xi_{tr}(G)$ is obtained by the same program without the restriction $\text{rank}(M(L)) < \infty$. In addition, using Theorem 2.5(3), we see that in this formulation of $\xi_f(G)$ we can equivalently optimize over all $L$ that are conic combinations of trace evaluations at projectors $X_i \in S^d$ (for some $d \in \mathbb{N}$) satisfying $X_iX_j = 0$ for all $\{i, j\} \in E$. If we restrict the optimization to scalar evaluations ($d = 1$) we obtain the fractional chromatic number. This shows that the projective rank can be seen as the noncommutative analogue of the fractional chromatic number, as was already observed in [28, 41].

The above formulations of the parameters $\xi_{tr}(G)$ and $\xi_f(G)$ in terms of linear functionals also show that they fit within the following hierarchy $\{\xi_{\text{col}}^r(G)\}_{r \in \mathbb{N} \cup \{\infty\}}$, defined as the noncommutative tracial analogue of the hierarchy $\{\text{las}_{\text{col}}^r(G)\}_{r}$:

$$\xi_{\text{col}}^r(G) = \inf \{ L(1) : L \in \mathbb{R}(\mathcal{A})_{2r}^\text{tracial, symmetric, positive,}$$

$$L(x_i) = 1 \ (i \in V), \ L = 0 \text{ on } \mathcal{I}_{2r}(\mathcal{H}_G) \}.$$ 

Again, $\xi_{\text{col}}^r(G)$ is the parameter obtained by adding the constraint $\text{rank}(M(L)) < \infty$ to the program defining $\xi_{\text{col}}(G)$. By the above discussion the following holds.

**Proposition 3.6.** We have $\xi_{\text{col}}^r(G) = \xi_f(G) \leq \chi_q(G)$ and $\xi_{\text{col}}^\infty(G) = \xi_{tr}(G) \leq \chi_q(G)$.

Using Lemma 2.7 one can verify that the parameters $\xi_{\text{col}}^r(G)$ converge to $\xi_{\text{col}}^\infty(G)$. Moreover, by Theorem 2.6, if $\xi_{\text{col}}^r(G)$ admits a flat optimal solution, then we have $\xi_{\text{col}}^r = \xi_{\text{col}}(G)$. Also, the parameter $\xi_{\text{col}}^1(G)$ coincides with $\text{las}_{\text{col}}^1(G) = \vartheta(\mathcal{G})$. Summarizing we have $\xi_{\text{col}}(G) = \xi_{tr}(G) \leq \chi_q(G)$ and the following chain of inequalities

$$\vartheta(\mathcal{G}) = \xi_{\text{col}}^1(G) \leq \xi_{\text{col}}^r(G) \leq \xi_{\text{col}}^\infty(G) = \xi_{tr}(G) \leq \xi_{\text{col}}(G) = \xi_f(G) \leq \chi_q(G).$$
Observe that the bounds $\text{las}_{r_{*}}^{\text{stab}}(G)$ and $\xi_{r}^{\text{col}}(G)$ remain below the fractional chromatic number $\chi_{f}(G)$, since $\xi_{f}(G) = \xi_{*}^{\text{col}}(G) \leq \text{las}_{*}^{\text{stab}}(G) = \chi_{f}(G)$. Hence, these bounds are weak if $\chi_{f}(G)$ is close to $\vartheta(G)$ and far from $\chi(G)$ or $\chi_{q}(G)$. In the classical setting this is the case, e.g., for the class of Kneser graphs $G = K(n, r)$, with vertex set the set of all $r$-subsets of $[n]$ and having an edge between any two disjoint $r$-subsets. By results of Lovász [26, 27], the fractional chromatic number is $n/r$, which is known to be equal to $\vartheta(K(n, r))$, while the chromatic number is $n - 2r + 2$. In [18] this was used as a motivation to define a new hierarchy of lower bounds $\{\lambda_{i}(G)\}$ on the chromatic number that can go beyond the fractional chromatic number. In Section 3.3 we recall this approach and show that its extension to the tracial setting recovers the hierarchy $\{\gamma_{r}^{\text{col}}(G)\}$ introduced in Section 3.1. We also show how a similar technique can be used to recover the hierarchy $\{\gamma_{r}^{\text{stab}}(G)\}$.

### 3.2.3 A link between $\xi_{r}^{\text{stab}}(G)$ and $\xi_{r}^{\text{col}}(G)$

In [18, Thm. 3.1] it is shown that, for any $r \geq 1$, the bounds $\text{las}_{r_{*}}^{\text{stab}}(G)$ and $\text{las}_{r}^{\text{col}}(G)$ satisfy $\text{las}_{r}^{\text{stab}}(G)\text{las}_{r}^{\text{col}}(G) \geq |V|$, with equality if $G$ is vertex-transitive, which extends a well-known property of the theta number (case $r = 1$). The same holds for the noncommutative analogues $\xi_{r}^{\text{stab}}(G)$ and $\xi_{r}^{\text{col}}(G)$.

**Lemma 3.7.** For any graph $G = (V, E)$ and $r \in \mathbb{N} \cup \{\infty, *\}$ we have $\xi_{r}^{\text{stab}}(G)\xi_{r}^{\text{col}}(G) \geq |V|$, with equality if $G$ is vertex-transitive.

**Proof.** Let $L$ be feasible for $\xi_{r}^{\text{col}}(G)$. Then $\tilde{L} = L/L(1)$ provides a solution to $\xi_{r}^{\text{stab}}(G)$ with value $\tilde{L}(\sum_{i \in V} x_{i}) = |V|/L(1)$, implying that $\xi_{r}^{\text{stab}}(G) \geq |V|/L(1)$ and therefore $\xi_{r}^{\text{stab}}(G)\xi_{r}^{\text{col}}(G) \geq |V|$. Assume $G$ is vertex-transitive. Let $L$ be a feasible solution for $\xi_{r}^{\text{stab}}(G)$. As $G$ is vertex-transitive we may assume (after symmetrization) that $L(x_{i})$ is constant, set $L(x_{i}) := 1/\lambda$ for all $i \in V$, so that the objective value of $L$ for $\xi_{r}^{\text{stab}}(G)$ is $|V|/\lambda$. Then $\tilde{L} = \lambda L$ provides a feasible solution for $\xi_{r}^{\text{col}}(G)$ with value $\lambda$, implying $\xi_{r}^{\text{col}}(G) \leq \lambda$. This implies $\xi_{r}^{\text{col}}(G)\xi_{r}^{\text{stab}}(G) \leq |V|$. \(\square\)

For vertex-transitive $G$, the inequality $\xi_{f}(G)\alpha_{q}(G) \leq |V|$ is shown in [28, Lem. 6.5]; it can be recovered from the $r = *$ case of Lemma 3.7 and $\alpha_{q}(G) \leq \alpha_{p}(G)$.

### 3.2.4 Comparison to existing semidefinite programming bounds

By adding the inequalities $L(x_{i}x_{j}) \geq 0$, for all $i, j \in V$, to $\xi_{1}^{\text{col}}(G)$, we obtain the strengthened theta number $\vartheta^{+}(G)$ (from [50]). Moreover, if we add the constraints

$$
L(x_{i}x_{j}) \geq 0 \quad \text{for } i \neq j \in V, \quad (18)
$$

$$
\sum_{j \in C} L(x_{i}x_{j}) \leq 1 \quad \text{for } i \in V, \quad (19)
$$

$$
L(1) + \sum_{i \in C, j \in C'} L(x_{i}x_{j}) \geq |C| + |C'| \quad \text{for } C, C' \text{ distinct cliques in } G \quad (20)
$$

to the program defining the parameter $\xi_{1}^{\text{col}}(G)$, then we obtain the parameter $\xi_{\text{SDP}}^{\text{col}}(G)$, which is introduced in [41, Thm. 7.3] as a lower bound on $\xi_{\text{tr}}(G)$. We will now show that the inequalities (18)–(20) are in fact valid for $\xi_{2}^{\text{col}}(G)$, which implies

$$
\xi_{2}^{\text{col}}(G) \geq \xi_{\text{SDP}}^{\text{col}}(G) \geq \vartheta^{+}(G).
$$

For this, given a clique $C$ in $G$, we define the polynomial $g_{C} := 1 - \sum_{i \in C} x_{i} \in \mathbb{R}(x)$. Then (19) and (20) can be reformulated as $L(x_{i}g_{C}) \geq 0$ and $L(g_{C}g_{C'}) \geq 0$, respectively, using the fact that $L(x_{i}) = L(x_{i}^{2}) = 1$ for all $i \in V$. Hence, to show that any feasible $L$ for $\xi_{2}^{\text{col}}(G)$ satisfies (18)-(20), it suffices to show Lemma 3.8 below. Recall that a commutator is a polynomial of the
form $[p, q] = pq - qp$ with $p, q \in \mathbb{R}[x]$. We denote the set of linear combinations of commutators $[p, q]$ with deg$(pq) \leq r$ by $\Theta_r$.

Lemma 3.8. Let $C$ and $C'$ be cliques in a graph $G$ and let $i, j \in V$. Then we have

$$g_C \in \mathcal{M}_2(\emptyset) + \mathcal{I}_2(\mathcal{H}(G)),$$

and $x_ix_j, x_jg_C, g_CC' \in \mathcal{M}_4(\emptyset) + \mathcal{I}_4(\mathcal{H}(G)) + \Theta_4$.

Proof. The claim $g_C \in \mathcal{M}_2(\emptyset) + \mathcal{I}_2(\mathcal{H}(G))$ follows from the identity

$$g_C = (1 - \sum_{i \in C} x_i)^2 + \sum_{i \in C}(x_i - x_i^2) + \sum_{i \neq j \in C} x_ix_j = g_C^2 + h,$$

where $h \in \mathcal{I}_2(\mathcal{H}(G))$. We also have

$$x_ix_j = x_ix_j^3x_i + x_j(x_i - x_i^2) + x_i^2(x_j - x_j^2) + [x_i, x_i x_j] + [x_i - x_i^2, x_j],$$

$$x_jg_C = x_jg_C^2x_i + g_C^2(x_i - x_i^2) + [x_i - x_i^2, g_C] + [x_i, x_j g_C^2],$$

and, writing analogously $g_C'^2 = g_C^2 + h'$ with $h' \in \mathcal{I}_2(\mathcal{H}(G))$, we have

$$g_Cg_C' = g_Cg_C^2 + [g_C, g_Cg_C^2] + [h, g_C^2] + g_C^2h' + hh' + g_C^2h.$$

Using $\xi_{\text{SDP}}(G)$, it is shown in [41, Thm. 7.4] that for the odd cycle $C_{2n+1}$, the tracial rank satisfies $\xi_{\text{col}}^G(C_{2n+1}) = (2n + 1)/n$. Combining this with Lemma 3.7 gives $n = \xi_{\text{stab}}^G(C_{2n+1}) \geq \alpha_{\text{qc}}(C_{2n+1})$. Equality holds since $\alpha_{\text{qc}}(C_{2n+1}) \geq \alpha(C_{2n+1}) = n$.

3.3 Links between the bounds $\gamma_r^\text{col}(G)$, $\xi_r^\text{col}(G)$, $\gamma_r^\text{stab}(G)$, and $\xi_r^\text{stab}(G)$

Here, in this last section, we make the link between the hierarchies $\{\xi_r^\text{stab}(G)\}$ (resp. $\{\xi_r^\text{col}(G)\}$) and $\{\gamma_r^\text{stab}(G)\}$ (resp. $\{\gamma_r^\text{col}(G)\}$). The key fact is the interpretation of the coloring and stability numbers in terms of certain graph products.

We start with the (quantum) coloring number. For an integer $k$, recall that the Cartesian product $G \square K_k$ is the graph with vertex set $V \times [k]$, where the vertices $(i, c)$ and $(j, c')$ are adjacent if $(i, j) \in E$ and $c = c'$ or $(i = j$ and $c \neq c')$. The following is a well-known reduction of the chromatic number $\chi(G)$ to the stability number of the Cartesian product $G \square K_k$: $\chi(G) = \min\{k \in \mathbb{N} : \alpha(G \square K_k) = |V|\}$. It was used in [18] to define the following lower bounds on the chromatic number:

$$\Lambda_r(G) = \min\{k \in \mathbb{N} : \text{las}_r^\text{stab}(G \square K_k) = |V|\},$$

where it was also shown that $\text{las}_r^\text{col}(G) \leq \Lambda_r(G) \leq \chi(G)$ for all $r \geq 1$, with equality $\Lambda_{|V|}(G) = \chi(G)$. Hence the bounds $\Lambda_r(G)$ may go beyond the fractional chromatic number. This is the case for the above mentioned Kneser graphs; see [17] for other graph instances.

The above reduction from coloring to stability number has been extended to the quantum setting by [28], where it is shown that $\chi_q(G) = \min\{k \in \mathbb{N} : \alpha_q(G \square K_k) = |V|\}$. It is therefore natural to use the upper bounds $\xi_r^\text{stab}(G \square K_k)$ on $\alpha_q(G \square K_k)$ in order to get the following lower bounds on the quantum coloring number:

$$\min\{k : \xi_r^\text{stab}(G \square K_k) = |V|\},$$

which are thus the noncommutative analogues of the bounds $\Lambda_r(G)$. Observe that, for any integer $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty, *\}$, we have $\xi_r^\text{stab}(G \square K_k) \leq |V|$, which follows from Lemma 3.8 and the fact that the cliques $C_i = \{(i, c) : c \in [k]\}$, for $i \in V$, cover all vertices in $G \square K_k$. Let $C_{i, K_k} = \{gc_i : i \in V\}$, where $gc_i = 1 - \sum_{c \in [k]} x_i^c$, denote the set of polynomials corresponding to these cliques. We now show that the parameters (22) coincide in fact with $\gamma_r^\text{col}(G)$ for all $r \in \mathbb{N} \cup \{\infty\}$. For this observe first that the quadratic polynomials in the set $\mathcal{H}_{G,k}^\text{col}$ correspond precisely to the edges of $G \square K_k$, and the projector constraints are included in $\mathcal{I}_2(\mathcal{H}_{G,k}^\text{col})$ (see Section 3.1), so that $\mathcal{I}_2(\mathcal{H}_{G,k}^\text{col}) = \mathcal{I}_2(\mathcal{H}_{G,k} \cup C_{G \square K_k})$. We will also use the following result.

21
Lemma 3.9. Let $r \in \mathbb{N} \cup \{\infty, *\}$ and assume $L$ is feasible for $\xi_r^{\text{stab}}(G \square K_k)$. Then, we have $L(\sum_{i \in V, c \in [k]} x_i^c) = |V|$ if and only if $L = 0$ on $I_{2r}(C_{G \square K_k})$.

Proof. First: If $L = 0$ on $I_{2r}(C_{G \square K_k})$, then $0 = \sum_{i \in V} L(g_{ci}) = |V| - \sum_{i,c} L(x_i^c)$. Conversely assume that $0 = L(\sum_{i \in V, c \in [k]} x_i^c) - |V| = \sum_{i \in V} L(g_{ci})$. We will show $L = 0$ on $I_{2r}(C_{G \square K_k})$. For this we first observe that $g_{ci} = (g_i)^2 \in I_2(H_{G \square K_k})$ by (21). Hence $L(g_{ci}) = L(g_{ci}^2) \geq 0$, which, combined with $\sum_{i} L(g_{ci}) = 0$, implies $L(g_{ci}) = 0$ for all $i \in V$. Next we show $L(\lambda g_{ci}) = 0$ for all words $\lambda$ with degree at most $2r - 1$, using induction on $\deg(\lambda)$. The base case $w = 1$ holds by the above. Assume now $w = uv$, where $\deg(v) < \deg(u) \leq r$. Using the positivity of $L$, the Cauchy-Schwarz inequality gives $|L(\lambda g_{ci})| \leq L(u^*u)^{1/2}L(v^*g_{ci}^2)^{1/2}$. Note that it suffices to show $L(v^*g_{ci}, v) = 0$ since, using again (21), this implies $L(v^*g_{ci}^2, v) = 0$ and thus $L(\lambda g_{ci}) = 0$. Using the tracial property of $L$ and the induction assumption, we see that $L(v^*g_{ci}, v) = L(vu^*g_{ci}) = 0$ since $\deg(v^*u) < \deg(w)$.

$\square$

Proposition 3.10. For $r \in \mathbb{N} \cup \{\infty\}$ we have $\gamma_r^{\text{col}}(G) = \min\{k : \xi_r^{\text{stab}}(G \square K_k) = |V|\}$.\[\]

Proof. Let $L$ be a linear functional certifying $\gamma_r^{\text{col}}(G) \leq k$. Then $L$ is feasible for $\xi_r^{\text{stab}}(G \square K_k)$ and, as $L = 0$ on $I_{2r}(C_{G \square K_k})$, Lemma 3.9 shows that $L(\sum_{i,c} x_i^c) = |V|$. This shows that $\xi_r^{\text{stab}}(G \square K_k) = |V|$ and thus $\min\{k : \xi_r^{\text{stab}}(G \square K_k) = |V|\} \leq k$.

Conversely, assume $\xi_r^{\text{stab}}(G \square K_k) = |V|$. Since the optimum is attained, there exists a linear functional $L$ feasible for $\xi_r^{\text{stab}}(G \square K_k)$ with $L(\sum_{i,c} x_i^c) = |V|$. Using Lemma 3.9 we can conclude that $L$ is zero on $I_{2r}(C_{G \square K_k})$ and thus also on $I_{2r}(\mathcal{H}_{G \square K_k}^\text{stab})$. This shows $\gamma_r^{\text{col}}(G) \leq k$.

$\square$

Note that the proof of Proposition 3.10 also works in the commutative setting; this shows that the sequence $\Lambda_r(G)$ corresponds to the usual Lasserre hierarchy for the feasibility problem defined by the equations (13)–(14), which is another way of showing $\Lambda_r(G) = \chi_r(G)$.

We now turn to the (quantum) stability number. For $k \in \mathbb{N}$, consider the graph product $K_k \star G$, with vertex set $[k] \times G$, and with an edge between $(c, i)$ and $(c', j)$ when $(c \neq c', i = j)$ or $(c = c', i \neq j)$ or $(c \neq c', i, j) \in E$. The product $K_k \star G$ coincides with the homomorphic product $K_k \times \overline{G}$ used in [28, Sec. 4.2], where it is shown that $\alpha_q(G) = \max\{k \in \mathbb{N} : \alpha_q(K_k \star G) = k\}$. This suggests using the upper bounds $\xi_r^{\text{stab}}(K_k \star G)$ on $\alpha_q(K_k \star G)$ to define the following upper bounds on $\alpha_q(G)$:

$$\max\{k \in \mathbb{N} : \xi_r^{\text{stab}}(K_k \star G) = k\}.$$ \hspace{1cm} (23)

For each $c \in [k]$, the set $C^c = \{(c, i) : i \in V\}$ is a clique in $K_k \star G$ and we let $C_{K_k \star G} = \{g_{C^c} : c \in [k]\}$, where $g_{C^c} = 1 - \sum_{i \in V} x_i^c$ denote the set of polynomials corresponding to these cliques. Since these $k$ cliques cover the vertex set of $K_k \star G$, we can use Lemma 3.8 to conclude $\xi_r^{\text{stab}}(K_k \star G) \leq k$ for all $r \in \mathbb{N} \cup \{\infty, *\}$. Again, observe that the quadratic polynomials in the set $\mathcal{H}_{G \square k}^\text{stab}$ correspond precisely to the edges of $K_k \star G$ and that we have $I_{2r}(\mathcal{H}_{G \square k}^\text{stab}) = I_{2r}(\mathcal{H}_{K_k \star G} \cup C_{K_k \star G})$. Based on this, one can show the analogue of Lemma 3.9: If $L$ is feasible for the program $\xi_r^{\text{stab}}(K_k \star G)$, then we have $L(\sum_{i,c} x_i^c) = k$ if and only if $L = 0$ on $I_{2r}(C_{K_k \star G})$, which implies the following result.

Proposition 3.11. For $r \in \mathbb{N} \cup \{\infty\}$ we have $\gamma_r^{\text{stab}}(G) = \max\{k : \xi_r^{\text{stab}}(K_k \star G) = k\}$.

We do not know whether the results of Propositions 3.10 and 3.11 hold for $r = \ast$, since we do not know whether the supremum is attained in the parameter $\xi_r^{\text{stab}}(\ast) = \alpha_r(\ast)$ (as was already observed in [45, p. 120]). Hence we can only claim the inequalities

$$\gamma_r^{\text{col}}(G) \geq \min\{k : \xi_r^{\text{stab}}(G \square K_k) = |V|\} \quad \text{and} \quad \gamma_r^{\text{stab}}(G) \leq \max\{k : \xi_r^{\text{stab}}(K_k \star G) = k\}.$$

As mentioned above, we have $\Lambda_r^{\text{col}}(G) \leq \Lambda_r(G)$ for any $r \in \mathbb{N}$ [18, Prop. 3.3]. This result extends to the noncommutative setting and the analogous result holds for the stability parameters. In other words the hierarchies $\{\gamma_r^{\text{col}}(G)\}$ and $\{\gamma_r^{\text{stab}}(G)\}$ refine the hierarchies $\{\xi_r^{\text{col}}(G)\}$ and $\{\xi_r^{\text{stab}}(G)\}$.\[\]

22
Proposition 3.12. For \( r \in \mathbb{N} \cup \{\infty, *\} \), \( \xi_r^{\text{col}}(G) \leq \gamma_r^{\text{col}}(G) \) and \( \xi_r^{\text{stab}}(G) \geq \gamma_r^{\text{stab}}(G) \).

Proof. We may restrict to \( r \in \mathbb{N} \) since we have seen earlier that the inequalities hold for \( r \in \{\infty, *\} \). The proof for the coloring parameters is similar to the proof of [18, Prop. 3.3] in the classical case and thus omitted. We show the inequality \( \xi_r^{\text{stab}}(G) \geq \gamma_r^{\text{stab}}(G) \). Set \( k = \gamma_r^{\text{stab}}(G) \) and, using Proposition 3.11, let \( L \in \mathbb{R}(x_c^i : i \in V, c \in [k])_{2r}^2 \) be optimal for \( \xi_r^{\text{stab}}(K_k \ast G) = k \). That is, \( L \) is tracial, symmetric, positive, and satisfies \( L(1) = 1, \sum_{i \in V} x_c^i = k \), and \( L = 0 \) on \( \mathcal{I}(\mathcal{H}_{K_k \ast G}) \). It suffices now to construct a tracial symmetric positive linear form \( \hat{\mathcal{L}} \in \mathbb{R}(x_i : i \in V \gamma 2r) \) such that \( \hat{\mathcal{L}}(1) = 1 \), \( \hat{\mathcal{L}}(\sum_{i \in V} x_i^c) = k \), and \( \hat{\mathcal{L}} = 0 \) on \( \mathcal{I}_G(\mathcal{H}_G) \), since this will imply \( \xi_r^{\text{stab}}(G) \geq k \). For this, for any word \( x_{i_1} \cdots x_{i_t} \) with degree \( 1 \leq t \leq 2r \), we define \( \hat{\mathcal{L}}(x_{i_1} \cdots x_{i_t}) := \sum_{c \in [k]} L(x_{i_1}^c \cdots x_{i_t}^c) \). Also, we set \( \hat{\mathcal{L}}(1) = L(1) = 1 \). Then, we have \( \hat{\mathcal{L}}(\sum_{i \in V} x_i^c) = k \). Moreover, one can easily check that \( \hat{\mathcal{L}} \) is indeed tracial, symmetric, positive, and vanishes on \( \mathcal{I}_G(\mathcal{H}_G) \). \( \square \)

References

[1] D. Avis, J. Hasegawa, Y. Kikuchi, and Y. Sasaki. A quantum protocol to win the graph coloring game on all Hadamard graphs. JEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, E89-A(5):1378–1381, 2006.
[2] G. P. Barker, L. Q. Eifler, and T. P. Kezlan. A non-commutative spectral theorem. Operator Algebras and Von Neumann Algebras. Encyclopaedia of Mathematical Sciences. Springer, 2006.
[3] J. S. Bell. On the Einstein Podolsky Rosen paradox. Physics, 1(3):195–200, 1964.
[4] B. Blackadar. Operator Algebras: Theory of \( C^* \)-Algebras and Von Neumann Algebras. Encyclopaedia of Mathematical Sciences. Springer, 2006.
[5] N. Brunner, S. Pironio, A. Acin, N. Gisin, A. A. Méthot, and V. Scarani. Testing the dimension of Hilbert spaces. Physical Review Letters, 100:210503, 2008.
[6] S. Burgdorf, K. Cafuta, I. Klep, and J. Povh. The tracial moment problem and trace-optimization of polynomials. Mathematical Programming, 137(1):557–578, 2013.
[7] S. Burgdorf and I. Klep. The truncated tracial moment problem. Journal of Operator Theory, 68(1):141–163, 2012.
[8] S. Burgdorf, I. Klep, and J. Povh. Optimization of Polynomials in Non-Commutative Variables. Springer Briefs in Mathematics. Springer, 2016.
[9] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini, and A. Winter. On the quantum chromatic number of a graph. The Electronic Journal of Combinatorics, 14(1), 2007.
[10] R. E. Curto and L. A. Fialkow. Solution of the Truncated Complex Moment Problem for Flat Data, volume 568 of Memoirs of the American Mathematical Society. American Mathematical Society, 1996.
[11] A.C. Doherty, Y.-C. Liang, B. Toner, and S. Wehner. The quantum moment problem and bounds on entangled multiprover games. Proceedings of the 2008 IEEE 23rd Annual Conference on Computational Complexity, pages 199–210, 2008.
[12] K. J. Dykema and V. I. Paulsen. Synchronous correlation matrices and Connes’ embedding conjecture. Journal of Mathematical Physics, 57:015214, 2016.
[13] K. J. Dykema, V. I. Paulsen, and J. Prakash. Non-closure of the set of quantum correlations via graphs. arXiv:1709.05032, 2017.
[14] T. Fritz. Tsirelson’s problem and Kirchberg’s conjecture. Reviews in Mathematical Physics, 24(05), 2012.
[15] S. Gribling, D. de Laat, and M. Laurent. Lower bounds on matrix factorization ranks via noncommutative polynomial optimization. arXiv:1708.01573, 2017.
[16] S. Gribling, D. de Laat, and M. Laurent. Matrices with high completely positive semidefinite rank. Linear Algebra and its Applications, 513:122–148, 2017.
[17] N. Gvozdenović and M. Laurent. Computing semidefinite programming lower bounds for the (fractional) chromatic number via block-diagonalization. SIAM Journal on Optimization, 19(2):592–615, 2008.

[18] N. Gvozdenović and M. Laurent. The operator \( \psi \) for the chromatic number of a graph. SIAM Journal on Optimization, 19(2):572–591, 2008.

[19] Z. Ji. Binary constraint system games and locally commutative reductions. arXiv:1310.3794, 2013.

[20] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V.B. Scholtz, and R.F. Werner. Connes’ embedding problem and Tsirelson’s problem. Journal of Mathematical Physics, 52:012102, 2011.

[21] I. Klep and J. Povh. Constrained trace-optimization of polynomials in freely noncommuting variables. Journal of Global Optimization, 64(2):325–348, 2016.

[22] I. Klep and M. Schweighofer. Connes’ embedding conjecture and sums of Hermitian squares. Advances in Mathematics, 217(4):1816–1837, 2008.

[23] J. B. Lasserre. Global optimization with polynomials and the problem of moments. SIAM Journal on Optimization, 11(3):796–817, 2001.

[24] M. Laurent. A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0-1 programming. Mathematics of Operations Research, 28(3):470–496, 2003.

[25] M. Laurent and T. Piovesan. Conic approach to quantum graph parameters using linear optimization over the completely positive semidefinite cone. SIAM Journal on Optimization, 25(4):2461–2493, 2015.

[26] L. Lovász. Kneser’s conjecture, chromatic number, and homotopy. Journal of Combinatorial Theory, Series A, 25(3):319–324, 1978.

[27] L. Lovász. On the shannon capacity of a graph. IEEE Transactions on Information Theory, 25(1):1–7, 2006.

[28] L. Mančinska and D. E. Roberson. Quantum homomorphisms. Journal of Combinatorial Theory, Series B, 118:228–267, 2016.

[29] L. Mančinska, G. Scarpa, and S. Severini. New separations in zero-error channel capacity through projective Kochen-Specker sets and quantum coloring. IEEE Transactions on Information Theory, 59(6):4025–4032, 2013.

[30] M. Navascués, A. Feix, M. Araujo, and T. Vértesi. Characterizing finite-dimensional quantum behavior. Physical Review A, 92, 2015.

[31] M. Navascués, S. Pironio, and A. Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. New Journal of Physics, 10(7):073013, 2008.

[32] M. Navascués, S. Pironio, and A. Acín. SDP relaxations for non-commutative polynomial optimization. In M. F. Anjos and J. B. Lasserre, editors, Handbook on Semidefinite, Conic and Polynomial Optimization, pages 601–634. Springer, 2012.

[33] M. Navascués and T. Vértesi. Bounding the set of finite dimensional quantum correlations. Physical Review Letters, 115(2):020501, 2015.

[34] J. Nie. Symmetric tensor nuclear norms. SIAM Journal on Applied Algebra and Geometry, 1(1):599–625, 2017.

[35] M. A. Nielsen and I. L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[36] C. M. Ortiz and V. I. Paulsen. Quantum graph homomorphisms via operator systems. Linear Algebra and its Applications, 497:23–43, 2016.

[37] N. Ozawa. About the Connes’ embedding problem–algebraic approaches. Japanese Journal of Mathematics, 8(1):147–183, 2013.

[38] K F. Pál and T. Vértesi. Efficiency of higher-dimensional Hilbert spaces for the violation of Bell inequalities. Physical Review A, 77:042105, 2008.

[39] C. Palazuelos and T. Vidick. Survey on nonlocal games and operator space theory. Journal of Mathematical Physics, 57(1):015220, 2016.
A Synchronous quantum correlations

We prove the following result by combining proofs from [46] (see also [28]) and [41].

Proposition A.1. The smallest local dimension in which a synchronous quantum correlation $P$ can be given is realised by the completely positive semidefinite rank of the matrix $M_P$ indexed by $S \times A$ with entries $(M_P)_{(s,a),(t,b)} = P(a,b|s,t)$.

Proof. Suppose first that $(\psi, E^a_s, F^b_t)$ is a realization of $P$ in local dimension $d$. That is, $\psi$ is a unit vector in $\mathbb{C}^d \otimes \mathbb{C}^d$, $E^a_s, F^b_t$ are $d \times d$ Hermitian positive semidefinite matrices such that $\sum_a E^a_s = \sum_b F^b_t = I$ for all $s, t$ and $P(a,b|s,t) = \psi^* (E^a_s \otimes F^b_t) \psi$ for all $(a,b,s,t) \in \Gamma$. We will show $\text{cpsd-rank}_C(A_P) \leq d$.

The Schmidt decomposition of the unit vector $\psi$ gives nonnegative scalars $\{\lambda_i\}$ and orthonormal bases $\{u_i\}$ and $\{v_i\}$ of $\mathbb{C}^d$ such that $\psi = \sum_{i=1}^d \sqrt{\lambda_i} u_i \otimes v_i$. If we replace $\psi$ by $\sum_{i=1}^d \sqrt{\lambda_i} v_i \otimes v_i$ and $E^a_s$ by $U^* E^a_s U$, where $U$ is the unitary matrix for which $u_i = U v_i$ for all $i$, then we obtain a new realization $(\sum_{i=1}^d \sqrt{\lambda_i} v_i \otimes v_i, U^* E^a_s U, F^b_t)$ of $P$ still in local dimension $d$. For the simplicity of notation we rename $U^* E^a_s U$ as $E^a_s$. Then we define the matrices

$$K = \sum_{i=1}^d \sqrt{\lambda_i} v_i v_i^*, \quad X^a_s = K^{1/2} E^a_s K^{1/2}, \quad Y^b_t = K^{1/2} F^b_t K^{1/2}.$$
By using the identities \( \text{vec}(K) = \psi \) and
\[
\text{vec}(K)^* (E_a^s \otimes F_t^b) \text{vec}(K) = \text{Tr}(KE_a^s KF_t^b) = \text{Tr}(K^{1/2}E_a^s K^{1/2}K^{1/2}F_t^b K^{1/2}),
\]
we see that
\[
P(a,b|s,t) = \langle X_a^s, Y_t^b \rangle \quad \text{for all} \quad a, b, s, t,
\]
and
\[
\langle K, K \rangle = 1, \quad \sum_a X_a^s = \sum_b Y_t^b = K \quad \text{for all} \quad s, t.
\]

For each \( s \), by applying twice the Cauchy–Schwarz inequality gives
\[
1 = \sum_a P(a,a|s,s) = \sum_a \langle X_a^s, X_a^s \rangle \leq \sum_a \langle X_a^s, X_a^s \rangle^{1/2} \langle Y_a^s, Y_a^s \rangle^{1/2}
\]
\[
\leq \left( \sum_a \langle X_a^s, X_a^s \rangle \right)^{1/2} \left( \sum_a \langle Y_a^s, Y_a^s \rangle \right)^{1/2}
\]
\[
\leq \left( \sum_a X_a^s, \sum_a X_a^s \right)^{1/2} \left( \sum_a Y_a^s, \sum_a Y_a^s \right)^{1/2} = \langle K, K \rangle = 1.
\]

Thus all inequalities above are equalities. The first inequality being an equality shows that there exist scalars \( \alpha_{s,a} \) such that \( X_a^s = \alpha_{s,a} Y_a^s \) for all \( a, s \). The second inequality being an equality shows that there exist scalars \( \beta_s \) such that \( \|X_a^s\| = \beta_s ||Y_a^s|| \) for all \( a, s \). Hence,
\[
\beta_s \|Y_a^s\| = \|X_a^s\| = \|\alpha_{s,a} Y_a^s\| = \alpha_{s,a} \|Y_a^s\| = \alpha_{s,a} \|Y_a^s\| \quad \text{for all} \quad s, a,
\]
which shows \( X_a^s = \beta_s Y_a^s \) for all \( s \). Since \( \sum_a X_a^s = K = \sum_a Y_a^s \), we have \( \beta_s = 1 \) for all \( s \). Thus \( X_a^s = Y_a^s \) for all \( a, s \). Therefore,
\[
(A_P)_{(s,a),(t,b)} = \langle X_a^s, X_t^b \rangle \quad \text{for all} \quad a, b, s, t,
\]
which shows \( \text{cpsd-rank}_C(A_P) \leq d \).

For the other direction we suppose \( \{X_a^s\} \) are Hermitian positive semidefinite matrices with the smallest possible size such that \( (A_P)_{(s,a),(t,b)} = \langle X_a^s, X_t^b \rangle \) for all \( a, s, t, b \). Then,
\[
1 = \sum_{a,b} P(a,b|s,t) = \sum_{a,b} \langle X_a^s, X_t^b \rangle = \left( \sum_a X_a^s, \sum_b X_t^b \right) \quad \text{for all} \quad s, t,
\]
which shows the existence of a matrix \( K \) such that \( K = \sum_a X_a^s \) for all \( s \). We have \( \langle K, K \rangle = 1 \) so that \( \text{vec}(K) \) is a unit vector, and since the factorization is smallest possible, \( K \) is invertible. Set \( E_a^s = K^{-1/2}X_a^s K^{-1/2} \) for all \( s, a \), so that \( \sum_a E_a^s = I \) for all \( s \). Then,
\[
P(a,b|s,t) = (A_P)_{(s,a),(t,b)} = \langle X_a^s, X_t^b \rangle = \text{vec}(K)^* (E_a^s \otimes E_t^b) \text{vec}(K),
\]
which shows \( P \) has a realization of local dimension \( \text{cpsd-rank}_C(A_P) \).

\( \square \)