TWO-TERM SILTING AND $\tau$-CLUSTER MORPHISM CATEGORIES

ERLEND D. BØRVE

Abstract. We generalise $\tau$-cluster morphism categories to the setting of non-positive dg algebras with finite dimensional cohomology in all degrees. The compatibility of silting reduction with support $\tau$-tilting reduction will be an essential ingredient when linking our definition to those of Buan–Marsh and Buan–Hanson.

Acknowledgements. I would like to express my gratitude to my PhD supervisor, Aslak B. Buan, for introducing me to the topic, countless helpful discussions, and meticulous proofreading. I am also grateful to Eric J. Hanson, Yann Palu, and an anonymous referee for very helpful comments on previous versions of this manuscript. I would also like to thank Gustavo Jasso for useful conversations.

Statements and Declarations. A previous version of this manuscript is part of the author’s PhD thesis. The author was funded directly by his employer, NTNU.

Contents

0. Introduction 2
1. Notation and preliminaries 4
   1.1. t-structures and wide subcategories of the heart 5
   1.2. Silting and $\tau$-tilting 10
2. Silting t-structures and reduction 13
3. Compatibility of the Iyama–Yang and Buan–Marsh bijections 23
4. $\tau$-cluster morphism categories 30
References 34

Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway (Current address: Abteilung Mathematik, Department Mathematik/Informatik der Universität zu Köln, Weyertal 86–90, 50931 Cologne, Germany).

E-mail address: erlend.d.borve@gmail.com.

2020 Mathematics Subject Classification. 16G10, 18E10, 18E30.

Key words and phrases. Wide subcategory, t-structure, silting, $\tau$-tilting, $\tau$-cluster morphism category.
0. Introduction

In tilting theory, it is valuable to find all tilting modules over a fixed finite dimensional algebra $\Lambda$, or better still, all tilting objects in the bounded homotopy category $\mathcal{K}^b(\text{proj } \Lambda)$, where $\text{proj } \Lambda$ is the category of finitely generated projective $A$-modules. If all tilting objects in $\mathcal{K}^b(\text{proj } \Lambda)$ could be identified, one could construct all finite dimensional algebras $\Gamma$ that are derived Morita equivalent to $\Lambda$ [Ric89], which is to say that their derived categories are equivalent as triangulated categories. Many invariants of finite dimensional algebras are preserved by derived Morita equivalence, and it might be more efficient to compute these for some $\Gamma$ rather than for $\Lambda$ directly.

Finding all tilting objects turns out to be an unrealistic project in general. However, one could instead study the larger class of silting objects, for which there is a mutation process generating new examples in a reliable manner [AI12]. In addition, one has a reduction procedure for silting objects. Iyama–Yang show that if $P$ is a presilting object of $\mathcal{K}^b(\text{proj } \Lambda)$, then the (pre)silting objects in $\mathcal{K}^b(\text{proj } \Lambda)$ having $P$ as a direct summand bijectively correspond to the (pre)silting objects in the Verdier quotient $\mathcal{K}^b(\text{proj } \Lambda)/\text{thick}(P)$ [IY18, Theorem 3.7]. The class of two-term silting objects closely resembles that of tilting modules. Indeed, a generalisation of the Brenner–Butler Theorem applies [BZ16, Theorem 1.1] [BZ24, Theorem 2.1]. Also, when performing silting reduction with respect to a two-term presilting object, one can restrict Iyama–Yang’s bijection to a bijection of two-term (pre)silting objects [Jas15, Proposition 4.11].

The mutation of two-term silting objects is closely tied to cluster combinatorics. For path algebras of a quiver of simply laced Dynkin type, the two-term silting objects of the bounded homotopy category are in bijection with the cluster-tilting objects in the cluster category [AIR14, Theorem 4.1]. Moreover, this bijection respects the mutation processes for cluster-tilting objects [AIR14, Corollary 4.8]. Cluster-tilting objects are in turn in bijection with the clusters of the corresponding cluster algebra [BMR+06, Corollary 4.4].

One may also model cluster combinatorics using $\tau$-tilting theory [AIR14]. Adachi–Iyama–Reiten define support $\tau$-tilting modules and $\tau$-rigid pairs, and show that they correspond bijectively with two-term silting objects and two-term presilting objects, respectively [AIR14, Theorem 3.2]. The reduction procedure of Iyama–Yang is transferable to $\tau$-tilting theory. Indeed, given any finite dimensional algebra, Jasso provides a suitable bijection between classes of support $\tau$-tilting modules [Jas15, Theorem 3.16], and shows that it is compatible with Iyama–Yang’s silting reduction [Jas15, Theorem 4.12(b)]. Buan–Marsh later extended Jasso’s bijection to $\tau$-rigid pairs [BM21, Theorem 3.6].

The bijection of Buan–Marsh was constructed in order to define the $\tau$-cluster morphism category of a $\tau$-tilting finite algebra. This builds on Igusa–Todorov’s cluster morphism category, which is defined for representation finite hereditary algebras [IT17, Section 1], as well as a categorification of non-crossing partitions [Igu14]. The definition of $\tau$-cluster morphism categories has recently been extended to all finite dimensional algebras [BH23]. Although Buan–Marsh define the $\tau$-cluster morphism category in terms of $\tau$-tilting theory, they occasionally translate the setting to two-term silting in order to prove necessary results.
τ-cluster morphism categories are useful in the study of picture groups and picture spaces, defined by Igusa–Todorov–Weyman [ITW16]. In the case of representation finite hereditary algebras, we have that the geometric realisation of the τ-cluster morphism category is a \( K(\pi, 1) \) for the corresponding picture group [IT17 Theorem 3.1]. This result was later extended to include Nakayama algebras [HI21 Theorem 4.16].

Outline. Section 1 sets up the notation and covers the necessary preliminaries. We review how t-exact subcategories (see Definition 1.5) of a triangulated category with a bounded t-structure correspond with wide subcategories in the heart (Theorem 1.10), the interaction between two-term silting and τ-tilting theory, and τ-tilting reduction. In particular, we recall in Proposition-Definition 1.18 that a τ-rigid pair \((M, Q)\) in \(\text{mod}(\Lambda)\), where \(\Lambda\) is a finite dimensional algebra, gives rise to a wide subcategory \(J(M, Q) \overset{\text{def}}{=} M^\perp \cap \perp (\tau M) \cap Q^\perp \) of \(\text{mod}(\Lambda)\).

In Section 2, we work over a non-positive dg algebra \(A\) such that \(H^i A\) is finite dimensional for all \(i \in \mathbb{Z}\). Let \(\mathcal{D}_{fd}(A)\) be the subcategory of the derived category \(\mathcal{D}(A)\) spanned by complexes of finite dimensional total cohomology. In Proposition 2.8(2), we show that a two-term presilting object \(P \in \text{per}(A)\) determines a t-exact subcategory \(P^{< \perp} \overset{\text{def}}{=} \{ X \in \mathcal{D}_{fd}(A) \mid \text{Hom}_{\mathcal{D}(A)}(P, \Sigma^i X) = 0 \ \forall i \in \mathbb{Z} \}\) of \(\mathcal{D}_{fd}(A)\). By Proposition 2.9(1), the subcategory \(P^{< \perp} \cap \text{mod} H^0 A \subseteq \text{mod} H^0 A\) is equal to the \(J(M, Q)\), where \((M, Q)\) is the τ-rigid pair corresponding to \(P\) (see Theorem 1.17).

Section 3 presents a proof that Iyama–Yang’s silting reduction is compatible with Buan–Marsh’s support τ-tilting reduction, generalising Jasso’s compatibility result [Jas15 Theorem 4.12(b)].

Theorem 3.5 Let \(C\) be a Hom-finite Krull–Schmidt triangulated category and \(S\) be a silting object in \(C\). Consider a 2\(_S\)-term presilting object \(P\), let \(\Lambda = \text{End}_C(S)\), let \(T^+_P\) be the Bongartz completion of \(P\) (defined by the triangle \((1, C)\)), let \(Z_P = \text{presilt}(\text{presilt}(P))\), and let \(\Gamma\) be the τ-tilting reduction of \(\Lambda\) with respect to the τ-rigid pair in \(\text{mod}(\Lambda)\) corresponding to \(P\). We have a commutative diagram of bijections

\[
\begin{array}{ccc}
2_{S}\text{-presilt}_P(C) & \xrightarrow{H_S} & \text{τ-rigid pair } H_{S}(P) \langle \Lambda \rangle \\
\downarrow \varphi_P & & \downarrow \psi_{H_S(P)} \\
2_{T^+_P}\text{-presilt}(Z_P/[P]) & \xrightarrow{H_{T^+_P}} & \text{τ-rigid pair}(\Gamma)
\end{array}
\]

where \(\varphi_P\) is the Iyama–Yang bijection and \(\psi_{H_S(P)}\) is the Buan–Marsh bijection. The horizontal maps are the bijections between two-term presilting objects and τ-rigid pairs (see Theorem 1.17).

Our main result appears in Section 4. Given a non-positive dg algebra \(A\) such that \(H^i A\) is finite dimensional for all \(i \in \mathbb{Z}\), we construct a category \(\mathfrak{W}_A\), which we call the τ-cluster morphism category of \(A\). Developing the theory purely in terms of two-term silting, as opposed to τ-tilting, has a major advantage; since silting reduction is induced by a functor, it is easier to prove that the composition law in \(\mathfrak{W}_A\) is associative, as we do in Theorem 4.3. When we claim that the work of Buan–Marsh and Buan–Hanson has been generalised, it is in the following specific sense.

Theorem 4.4 Let \(A\) be a non-positive dg algebra such that \(H^i A\) is finite dimensional for all \(i \in \mathbb{Z}\), and let \(\mathfrak{W}_{H^0 A}^{BM}\) be the τ-cluster morphism category (as defined by Buan–Hanson [BH23]) of the zeroth
cohomology of $A$. There is an equivalence of categories

$$E: \mathcal{M}_A \longrightarrow \mathcal{M}_{H^nA}.$$  

In particular, if $A$ is a finite dimensional algebra, then $\mathcal{M}_A$ is equivalent to $\mathcal{M}_{BM}$.

1. Notation and preliminaries

Throughout, all subcategories we consider will be full and closed under isomorphism. We fix a field $k$, and declare all triangulated categories and their (not necessarily triangulated) subcategories to be $k$-linear. A $k$-linear category $\mathcal{C}$ is Hom-finite if the Hom-spaces $\mathcal{C}(X, X')$ (alternatively denoted $\text{Hom}_k(X, X')$) are finite dimensional for all $X, X' \in \mathcal{C}$. It is Krull–Schmidt if any object $X \in \mathcal{C}$ is isomorphic to a finite direct sum $\bigoplus_{i=1}^n X_i$, where the endomorphism $k$-algebra of each $X_i$ is a local. An object in a Krull–Schmidt $k$-linear category with a local endomorphism $k$-algebra is called indecomposable.

Recall that a triangulated subcategory is thick if it is closed under direct summands. For a triangulated category $\mathcal{T}$, we write $\text{thick}_\mathcal{T}(\mathcal{X})$ for the smallest thick subcategory containing the subcategory $\mathcal{X}$, or simply $\text{thick}(\mathcal{X})$ when there is no risk of confusion.

All modules are right modules, unless otherwise specified. For a $k$-algebra $\Lambda$, the category of finitely generated right $\Lambda$-modules will be denoted by $\text{mod} \ \Lambda$, and the category of finitely generated projective $\Lambda$-modules will be denoted by $\text{proj} \ \Lambda$. The triangulated category $\mathcal{D}(\Lambda)$ is the bounded derived category of the former, whereas $\mathcal{K}^b(\text{proj} \ \Lambda)$ is the bounded homotopy category of the latter. They are both subcategories of the derived category $\mathcal{D}(\Lambda)$. More generally, if $A$ is a differential graded (henceforth $dg$) $k$-algebra $A$, we denote its derived category by $\mathcal{D}(A)$, define $\text{per}(A)$ to be the thick closure of $A$ in $\mathcal{D}(A)$, and $\mathcal{D}_{\text{id}}(A)$ to be the (thick) subcategory of $\mathcal{D}(A)$ spanned by complexes with finite dimensional total cohomology.

A $dg$-$k$-algebra $A$ is said to be non-positive if the cohomology $k$-vector spaces $H^iA$ vanish for all $i > 0$.

Let $\mathcal{C}$ be a $k$-linear category and let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. The additive closure of $\mathcal{X}$ is the full subcategory of $\mathcal{C}$ consisting of all finite direct sums of direct summands of objects in $\mathcal{X}$. We denote this category by $\text{add}(\mathcal{X})$, or by $\text{add}(X)$ if $\mathcal{X}$ contains only a single object $X$.

Let $\mathcal{P}$ be an additive subcategory of $\mathcal{C}$. A morphism $Q \xrightarrow{\beta} Y$ (resp. $X \xrightarrow{\alpha} Q$) is $\mathcal{P}$-epic (resp. $\mathcal{P}$-monic) if the induced morphism $\mathcal{C}(P, \beta)$ (resp. $\mathcal{C}(\alpha, P)$) is surjective for all objects $P \in \mathcal{P}$. It is a right $\mathcal{P}$-approximation of $Y$ (resp. left $\mathcal{P}$-approximation of $X$) if, in addition, we have that $Q \in \mathcal{P}$. We say that $\mathcal{P}$ is a contravariantly finite subcategory (resp. covariantly finite subcategory) of $\mathcal{C}$ if every object in $\mathcal{C}$ admits a right (resp. left) $\mathcal{P}$-approximation. A functorially finite subcategory of $\mathcal{C}$ is one that is both contravariantly and covariantly finite.

The left and right perpendicular subcategories of $\mathcal{P}$ in $\mathcal{C}$ are the full subcategories

$$\mathcal{P}^\perp \overset{\text{def}}{=} \{ X \in \mathcal{C} \mid \mathcal{C}(X, \mathcal{P}) = 0 \},$$

$$\mathcal{P}^\perp \overset{\text{def}}{=} \{ Y \in \mathcal{C} \mid \mathcal{C}(\mathcal{P}, Y) = 0 \},$$

respectively. If $\mathcal{P} = \text{add}(P)$ for some object $P \in \mathcal{C}$, we write $\mathcal{P}^\perp \overset{\text{def}}{=} \mathcal{P}^\perp(\text{add}(P))$ and $\mathcal{P}^\perp \overset{\text{def}}{=} (\text{add}(P))^\perp$, respectively.

If $\mathcal{T}$ is a triangulated category, we denote its suspension functor by $\Sigma$, unless otherwise specified. Let $\mathcal{P}$ be an additive subcategory of $\mathcal{T}$. For each subset $I \subseteq \mathbb{Z}$, we define the left and right perpendicular
TWO-TERM SILTING AND $\tau$-CLUSTER MORPHISM CATEGORIES

subcategories

\[
\begin{align*}
\perp_I P & \overset{\text{def}}{=} \{ X \in \mathcal{T} \mid \mathcal{T}(X, \Sigma^i P) = 0 \quad \forall i \in I \}, \\
\mathcal{P}^{\perp_I} & \overset{\text{def}}{=} \{ Y \in \mathcal{T} \mid \mathcal{T}(P, \Sigma^i Y) = 0 \quad \forall i \in I \},
\end{align*}
\]

respectively. If $P = \text{add}(P)$ for some object $P \in \mathcal{C}$, we write $\perp_I P \overset{\text{def}}{=} \perp_I (\text{add}(P))$ and $P^{\perp_I} \overset{\text{def}}{=} (\text{add}(P))^{\perp_I}$, respectively. In the case where $I$ is the subset of $\mathbb{Z}$ consisting of positive (resp. non-positive, resp. non-zero) integers, we replace the subscript $I$ by $>0$ (resp. $\leq 0$, resp. $\neq 0$), which for example introduces the notation $P^{\perp_{>0}}$ (resp. $P^{\perp_{\leq 0}}$, resp. $P^{\perp_{\neq 0}}$) for the right perpendicular subcategory in question.

Suppose that $\mathcal{T}$ is triangulated and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be additive subcategories of $\mathcal{T}$. The category of extensions of $\mathcal{P}_2$ in $\mathcal{P}_1$ (also known as the Verdier product of $\mathcal{P}_1$ and $\mathcal{P}_2$) is the full subcategory $\mathcal{P}_1 \ast \mathcal{P}_2$ of $\mathcal{T}$ consisting of the objects $E$ that fit in a triangle

\[
X_1 \longrightarrow E \longrightarrow X_2 \longrightarrow \Sigma X_1,
\]

where $X_1 \in \mathcal{P}_1$ and $X_2 \in \mathcal{P}_2$. It can be shown using the octahedral axiom that $\ast$ is an associative operation [BBD82, Lemme 1.3.10]. A full subcategory $\mathcal{P} \subseteq \mathcal{T}$ is extension-closed in $\mathcal{T}$ if $\mathcal{P} = \mathcal{P} \ast \mathcal{P}$. One also has a notion of extension-closure for subcategories of Quillen exact categories (e.g. abelian categories), where exact sequences play the role of triangles.

An additive subcategory $\mathcal{P} \subseteq \mathcal{T}$ is suspended (resp. co-suspended) if it is extension-closed and closed under the suspension functor $\Sigma$ (resp. inverse suspension functor $\Sigma^{-1}$). The ideal $[\mathcal{P}]$ in $\mathcal{C}$ contains precisely the morphisms that factor through an object in $\mathcal{P}$. The ideal quotient will be denoted by $\mathcal{C}/[\mathcal{P}]$. If $\mathcal{P} = \text{add}(P)$ for some object $P$, we denote this quotient by $\mathcal{C}/[P]$.

We use Deligne’s convention, writing $F = G$ when we mean that these functors are naturally isomorphic.

1.1. t-structures and wide subcategories of the heart. Truncation structures (t-structures for short) were introduced by Beilinson–Bernstein–Deligne [BBD82] (and also Gabber [BBDG18]). We give a quick survey of their elementary properties. A triangulated category $\mathcal{D}$ with suspension functor $\Sigma$ is fixed.

**Definition 1.1.** A pair of full additive subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ of $\mathcal{D}$ constitute a truncation structure (henceforth *t-structure*) if the conditions (t.1) and (t.2) below are met.

(t.1) $(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ is a torsion pair in $\mathcal{D}$, i.e.

(t.1.1) The Hom-spaces $\mathcal{D}(X, Y)$ are trivial for all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{>0}$,

(t.1.2) every object in $\mathcal{D}$ is an extension of an object in $\mathcal{D}^{>0}$ with an object in $\mathcal{D}^{\leq 0}$. This is to say that $\mathcal{D}^{\leq 0} \ast \mathcal{D}^{>0} = \mathcal{D}$.

(t.2) The subcategory $\mathcal{D}^{\leq 0}$ is suspended (equivalently, the subcategory $\mathcal{D}^{>0}$ is co-suspended).
We define $\mathcal{D}^{\leq -1} \overset{\text{def}}{=} \Sigma \mathcal{D}^{\leq 0}$. More generally, we define

\begin{align*}
\mathcal{D}^{\leq n} & \overset{\text{def}}{=} \Sigma^{-n} \mathcal{D}^{\leq 0}, \\
\mathcal{D}^{> m} & \overset{\text{def}}{=} \Sigma^{-m} \mathcal{D}^{> 0}, \\
\mathcal{D}^{[m, n]} & \overset{\text{def}}{=} \mathcal{D}^{> m-1} \cap \mathcal{D}^{\leq n}, \\
\mathcal{D}^m & \overset{\text{def}}{=} \mathcal{D}^{[m, m]},
\end{align*}

for integers $n$ and $m$. The t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$ is bounded if $\bigcup_{m,n \in \mathbb{Z}} \mathcal{D}^{[m,n]} = \mathcal{D}$. The subcategory $\mathcal{D}^0$ of $\mathcal{D}$ is called the heart of the t-structure.

**Remark 1.2.** The subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{> 0}$ determine each other; we have that $(\mathcal{D}^{\leq 0})^\perp = \mathcal{D}^{> 0}$ and that $\perp^0(\mathcal{D}^{> 0}) = \mathcal{D}^{\leq 0}$. As a result of this fact, the subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{> 0}$ are closed under taking direct summands.

Let $\mathcal{A}$ be an abelian category. Then the derived category $\mathcal{D}(\mathcal{A})$ can be equipped with the standard t-structure:

\begin{align*}
\mathcal{D}^{\leq 0}(\mathcal{A}) & = \{ X \in \mathcal{D}(\mathcal{A}) \mid \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, X) = 0 \text{ for all } i > 0 \}, \\
\mathcal{D}^{> 0}(\mathcal{A}) & = \{ X \in \mathcal{D}(\mathcal{A}) \mid \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, X) = 0 \text{ for all } i \leq 0 \}.
\end{align*}

It restricts to a bounded t-structure on $\mathcal{D}_{\text{id}}(\mathcal{A})$, also called the standard t-structure. The heart of these t-structures are both equivalent to the abelian category $\mathcal{A}$. In general, the heart of any t-structure is an abelian category [BBD82, Théorème 1.3.6].

For a suspended subcategory $\mathcal{U} \subseteq \mathcal{D}$, the pair $(\mathcal{U}, \mathcal{U}^\perp)$ forms a t-structure on $\mathcal{D}$ if and only if $\mathcal{U}$ is co-reflective [KV88, 1.1 Proposition], i.e. the inclusion admits a right adjoint. A subcategory of $\mathcal{D}$ is called an aisle (resp. a co-aisle) if it is suspended (resp. co-suspended) and co-reflective (resp. reflective, i.e. the inclusion admits a left adjoint). Since the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$ is a t-structure on $\mathcal{D}$ precisely when $\mathcal{D}^{\leq 0}$ is an aisle in $\mathcal{D}$ (or, equivalently $\mathcal{D}^{> 0}$ is a co-aisle), we say that $\mathcal{D}^{\leq 0}$ is the aisle of the t-structure (and that $\mathcal{D}^{> 0}$ is the co-aisle).

We may hence define a truncation functor $\sigma^{\leq 0} : \mathcal{D} \to \mathcal{D}^{\leq 0}$ (resp. $\sigma^{> 0} : \mathcal{D} \to \mathcal{D}^{> 0}$) as the right adjoint of the inclusion functor $\iota^{\leq 0} : \mathcal{D}^{\leq 0} \to \mathcal{D}$ (resp. as the left adjoint of the inclusion functor $\iota^{> 0} : \mathcal{D}^{> 0} \to \mathcal{D}$). More generally, one defines truncation functors $\sigma^{\leq i} : \mathcal{D} \to \mathcal{D}^{\leq i}$ (resp. $\sigma^{> i} : \mathcal{D} \to \mathcal{D}^{> i}$) for any integer $i$, *mutatis mutandis*. The cohomology functor in degree $0$ of the t-structure is defined by $\sigma^0 \overset{\text{def}}{=} \sigma^{\leq 0} \sigma^{> -1}$.

For any integer $i$, we define the cohomology functor in degree $i$ by $\sigma^i \overset{\text{def}}{=} \sigma^0 \circ \Sigma^i$.

**Remark 1.3.** It is conventional to denote the truncation functors by $\tau^{\leq 0}$ and $\tau^{> 0}$. We have chosen to use $\sigma^{\leq 0}$ and $\sigma^{> 0}$ instead, so that they will not be confused with the Auslander–Reiten translation.

For all objects $X \in \mathcal{D}$, there is a unique morphism $\sigma^{> 0} X \xrightarrow{\partial_X} \Sigma \sigma^{\leq 0} X$ such that we have a triangle

\begin{equation}
\sigma^{\leq 0} X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \sigma^{> 0} X \xrightarrow{\partial_X} \Sigma \sigma^{\leq 0} X \tag{1.B}
\end{equation}

where $\varepsilon_X$ (resp. $\eta_X$) is induced by the co-unit of the adjunction $(\iota^{\leq 0}, \sigma^{\leq 0})$ (resp. by the unit of the adjunction $(\sigma^{> 0}, \iota^{> 0})$).
Proposition 1.4. Let $\mathcal{D}$ be a triangulated category equipped with a $t$-structure $(\mathcal{D}^{-\infty}, \mathcal{D}^{\infty})$.

1. [BBDS2 Théorème 1.3.6] The functor $\sigma^0 = \sigma^{\leq 0} \sigma^{> -1}: \mathcal{D} \to \mathcal{D}^0$ is cohomological, i.e. it sends distinguished triangles to long exact sequences. Since $\Sigma^i$ is an auto-equivalence on $\mathcal{D}$ for every integer $i$, it follows from Proposition 1.4 (1) that any cohomology functor $\sigma^i = \sigma^0 \circ \Sigma^i$ is cohomological.

2. [BBDS2 Remarque 3.1.17(ii)] A complex $U \to V \to W$ in $\mathcal{D}^0$ is an exact sequence if and only if there exists a morphism $W \to \Sigma U$ in $\mathcal{D}$ such that $U \to V \to W \to \Sigma U$ is a distinguished triangle in $\mathcal{D}$.

3. [BBDS2 Théorème 1.3.3(ii)] Let $X \in \mathcal{D}$. The truncation triangle $(\mathcal{T.B})$ is the unique triangle up to isomorphism where the first term is in $\mathcal{D}^{-\infty}$, the second term is $X$, and the third term is in $\mathcal{D}^{\infty}$.

Definition 1.5 ([BBDS2 Definition 1.3.16]). Let $\mathcal{D}$ be a triangulated category equipped with a $t$-structure $(\mathcal{D}^{-\infty}, \mathcal{D}^{\infty})$, and let $\mathcal{U}$ be a triangulated category equipped with a $t$-structure $(\mathcal{U}^{-\infty}, \mathcal{U}^{\infty})$. A triangle functor $F: \mathcal{U} \to \mathcal{D}$ is

1. left $t$-exact if $F(\mathcal{U}^{\infty}) \subseteq \mathcal{D}^{\infty}$,
2. right $t$-exact if $F(\mathcal{U}^{-\infty}) \subseteq \mathcal{D}^{-\infty}$,
3. $t$-exact if it is both left $t$-exact and right $t$-exact.

A $t$-exact subcategory of $\mathcal{D}$ with respect to the $t$-structure $(\mathcal{D}^{-\infty}, \mathcal{D}^{\infty})$, is a triangulated subcategory $\mathcal{S}$ of $\mathcal{D}$ such that $\sigma^{\leq 0}X \in \mathcal{S}$ for any $X \in \mathcal{S}$.

Remark 1.6. We might as well have defined a $t$-exact subcategory to be a triangulated subcategory which is closed under any truncation functor of the form $\sigma^{\leq i}$ or $\sigma^{> i}$, not just $\sigma^{\leq 0}$. First of all, a $t$-exact subcategory $\mathcal{S} \subseteq \mathcal{D}$ is closed under truncation by $\sigma^{\leq m}$, for all $m \in \mathbb{Z}$, since we have assumed that $\mathcal{S}$ is closed under suspension and desuspension. It is also closed under $\sigma^{> 0}$ (and thus $\sigma^{> m}$, for all $m \in \mathbb{Z}$), which one proves by forming a truncation triangle

$$
\sigma^{\leq 0}X \xrightarrow{\epsilon_X} X \to \sigma^{> 0}X \xrightarrow{\Sigma \epsilon^{\leq 0}} X,
$$
where $X \in \mathcal{S}$, and then pointing out that $\sigma^{> 0}X$ is a cone of $\epsilon_X$, a morphism in the triangulated subcategory $\mathcal{S}$.

The $t$-exact subcategories of $\mathcal{D}$ are precisely those on which we can induce a $t$-structure $(\mathcal{D}^{-\infty} \cap \mathcal{S}, \mathcal{D}^{\infty} \cap \mathcal{S})$ such that the inclusion functor $\mathcal{S} \to \mathcal{D}$ is $t$-exact [BBDS2 §1.3.19]. The heart of the induced $t$-structure is $\mathcal{S}^0 \overset{def}{=} \mathcal{D}^0 \cap \mathcal{S}$, and the cohomological functor $\mathcal{S} \to \mathcal{S}^0$ is naturally isomorphic to the restriction of $\sigma^0: \mathcal{D} \to \mathcal{D}^0$.

We include the following well-known result to emphasise that $t$-exact functors induce exact functor of hearts.

Proposition 1.7 ([BBDS2 Proposition 1.3.17]). Let $F: \mathcal{U} \to \mathcal{D}$ be a $t$-exact functor. Then the composite functor

$$
F \circ F : \mathcal{U} \to \mathcal{D}^0
$$

is $t$-exact.
is an exact functor of abelian categories. In particular, if $S$ is a $t$-exact subcategory of $D$, then $S^0$ is an exact abelian subcategory of $D^0$.

The subcategory $D^0 = D^{≤0} \cap D^{>0}$ is an intersection of extension-closed subcategories. If $S$ is a triangulated subcategory of $D$, the intersection $D^0 \cap S$ is extension-closed in the triangulated category $D$. It now follows from Proposition 1.4(2) that $S^0 = D^0 \cap S$ is extension-closed in the abelian category $D^0$. An extension-closed exact abelian subcategory is often called a wide subcategory (or a weak Serre subcategory). Equivalently, it is a subcategory which is closed under kernels, cokernels and extensions. In particular, every idempotent endomorphism admits a kernel, which is to say that the wide subcategory in question is idempotent complete. It follows that wide subcategories are closed under taking direct summands.

**Lemma 1.8.** Let $D$ be a triangulated category equipped with a bounded $t$-structure $(D^{≤0}, D^{>0})$, and let $S$ be a $t$-exact subcategory of $D$. Then $S$ is thick.

**Proof.** It is to be shown that $S$ is closed under direct summands. Let $X, Y \in D$ be such that $X \oplus Y \in S$. Since $(D^{≤0}, D^{>0})$ is bounded, only finitely many integers $i$ are such that $σ^i(X \oplus Y)$ is non-zero. We proceed by induction on the number $d$ of such integers $i$.

The case $d = 0$ concerns only zero objects, and is therefore trivial to establish. If $d = 1$, we have that $X \oplus Y \in S^i$ for some $i$. Since $S^i$ is equivalent to $S^0$, we may assume without loss of generality that $i = 0$. Having remarked above that wide subcategories are idempotent complete, it follows that the direct summand $X$ of $X \oplus Y$ is in the wide subcategory $S^0$, as desired.

Assume that the statement is true whenever $d < ℓ$ for some $ℓ ≥ 2$. If $d = ℓ$, suppose that $X \oplus Y \in S$ has non-vanishing cohomology in $ℓ$ positions, and let $m$ be the smallest integer such that $σ^{>m+1}(X \oplus Y) = 0$. Let $X_1 = σ^{≤m}X$ and $X_2 = σ^{>m}X$. In the truncation triangle

$$X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow ΣX_1,$$

the object $X_1$ has non-vanishing cohomology in $ℓ − 1$ positions, and $X_2$ has non-vanishing cohomology in one position. In the same vein, let $Y_1 = σ^{≤m}Y$ and $Y_2 = σ^{>m}Y$. In the truncation triangle

$$Y_1 \longrightarrow Y \longrightarrow Y_2 \longrightarrow ΣY_1,$$

the objects $Y_1$ and $Y_2$ have the same properties as $X_1$ and $X_2$ above. Then $X_1 \oplus Y_1 = σ^{≤m}(X \oplus Y)$ and $X_2 \oplus Y_2 = σ^{>m}(X \oplus Y)$ are objects in $S$ with cohomology in fewer than $ℓ$ positions, whence $X_1, X_2 \in S$ by the induction hypothesis. Since $S$ is a triangulated subcategory of $D$, it follows that $X \in S$, since it is an extension of $X_2$ in $X_1$. \hfill □

Following Zhang–Cai [ZC17 Definition 2.3], we say that a thick subcategory $S$ of $D$ is $σ^0$-stable if $σ^0X \in S$ for all $X \in S$. Having defined the cohomology functor $σ^i$ by $σ^0 \circ Σ^n$, it is equivalent to impose that $S$ is closed under all cohomology functors $σ^i$.

**Lemma 1.9.** Let $D$ be a triangulated category equipped with a $t$-structure $(D^{≤0}, D^{>0})$. A thick subcategory $S$ of $D$ is $σ^0$-stable if it is $t$-exact. If the $t$-structure is bounded, the converse also holds.
Proof. We explained in Remark 1.9 that a t-exact subcategory $S$ is closed under truncation by $\sigma^{\leq 0}$ and $\sigma^{> 1}$. Hence, it is closed under $\sigma^0 = \sigma^{\leq 0}\sigma^{> 1}$, which is to be $\sigma^0$-stable.

Conversely, suppose that the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$ is bounded and let $S$ be an $\sigma^0$-stable subcategory of $\mathcal{D}$. For any $X \in S$ and any integer $n$, we can form a truncation triangle

$$
\sigma^{\leq n-1} X \longrightarrow \sigma^{\leq n} X \longrightarrow \Sigma^{-n} H^n X \longrightarrow \Sigma \sigma^{\leq n-1} X.
$$

The object $\Sigma^{-n} H^n X$ lies in $S$, and hence $\sigma^{\leq n-1} X$ belongs to $S$ if and only if $\sigma^{\leq n} X$ does. By induction, it follows that $\sigma^{\leq 0} X \in S$ if and only if $\sigma^{\leq n} X \in S$, where $n$ is an arbitrary integer. The boundedness of the t-structure ensures that $\sigma^{\leq n} X = X$ for some $n$; indeed, the object $X$ lies in some subcategory of $\mathcal{D}$ of the form $\mathcal{D}^{[m,n]}$, where the truncation functor $\sigma^{\leq n}$ acts as the identity. We conclude that $S$ is t-exact, having shown that $\sigma^{\leq 0} X \in S$ for any $X \in S$.

Fix a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$ on $\mathcal{D}$. The set of t-exact subcategories of $\mathcal{D}$ with respect to $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$ is closed under arbitrary intersections of aisles. Ordering this set with respect to inclusion of the aisles, it has been endowed with the structure of a complete lattice. The join of a family $\{S_i\}_{i \in I}$ is the smallest t-exact subcategory containing all $S_i$, whereas the meet is computed by intersection. We denote this lattice by $\text{t-exact}_{(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})}(\mathcal{D})$, and we let $\text{wide}(\mathcal{D}^0)$ denote the lattice of wide subcategories in $\mathcal{D}^0$, ordered by inclusion. We conclude this subsection with a result providing mutually inverse isomorphisms between these lattices, and a lemma that will be handy for later use.

**Theorem 1.10** ([ZC17] Theorem 2.5]). Let $\mathcal{D}$ be a triangulated category equipped with a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$. The cohomology functor $\sigma^0: \mathcal{D} \to \mathcal{D}^0$ induces an isomorphism of lattices

$$
\sigma^0: \text{t-exact}_{(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})}(\mathcal{D}) \to \text{wide}(\mathcal{D}^0),
$$

with inverse $\mathcal{D}(-): \text{wide}(\mathcal{D}^0) \to \text{t-exact}_{(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})}(\mathcal{D})$ sending a wide subcategory $W \subseteq \mathcal{D}^0$ to the t-exact subcategory

$$
\mathcal{D}_W \overset{\text{def}}{=} \{ X \in \mathcal{D} \mid \sigma^i X \in W \text{ for all } i \in \mathbb{Z} \}
$$

of $\mathcal{D}$.

Note that $\mathcal{D}_W$ clearly is $\sigma^0$-stable by definition. By Lemma 1.9, it is indeed a t-exact subcategory of $\mathcal{D}$.

**Lemma 1.11.** Let $\mathcal{D}$ be a triangulated category equipped with a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$.

1. The map $\sigma^0: \text{t-exact}_{(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})}(\mathcal{D}) \to \text{wide}(\mathcal{D}^0)$, as defined in Theorem 1.10, can be expressed by $S \mapsto S \cap \mathcal{D}^0$, intersecting a t-exact subcategory $S$ with the heart $\mathcal{D}^0$.

2. For $W \in \text{wide}(\mathcal{D}^0)$, let $\mathcal{D}_W$ be as in Theorem 1.10. We have that $\mathcal{D}_W = \text{thick}_\mathcal{D}(W)$.

**Proof.** We first prove (1). Let $S$ be a t-exact subcategory of $\mathcal{D}$. For any $X \in S \cap \mathcal{D}^0$, we have that $\sigma^0 X$ is isomorphic to $X$, putting $X$ in $\sigma^0 S$, namely the essential image of $S$ under the cohomology functor. This shows that $S \cap \mathcal{D}^0 \subseteq \sigma^0 S$. The reverse inclusion $S \cap \mathcal{D}^0 \supseteq \sigma^0 S$ also holds; we have that $\sigma^0 S$ is contained in $\mathcal{D}^0$ by the definition of the cohomology functor, and in $S$ by the $\sigma^0$-stability of $S$ (see Lemma 1.9).
Secondly and finally, we prove (2). We will regard the wide subcategory \( \mathcal{W} \subseteq \mathcal{D}^0 \) as a subcategory of \( \mathcal{D} \). Since \( \mathcal{D}_\mathcal{W} \) is a thick subcategory of \( \mathcal{D} \) containing \( \mathcal{W} \), it immediately follows that \( \mathcal{D}_\mathcal{W} \supseteq \text{thick}_\mathcal{D}(\mathcal{W}) \). We complete the proof by showing that any object \( X \in \mathcal{D}_\mathcal{W} \) can be built from objects in \( \mathcal{W} \) using extensions, (de)suspensions and direct summands. Since the \( t \)-structure \( (\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0}) \) is bounded, there exists some interval \([m,n]\) of integers such that \( X \in \mathcal{D}^{[m,n]} \). We proceed by induction of the length of this interval, namely the integer \( \ell = n - m \). The case \( \ell = 0 \) concerns intervals of the form \([m,m]\). We can build any object in \( \mathcal{D}^{[m,m]} \cap \mathcal{D}_\mathcal{W} \) by (de)suspending objects in \( \mathcal{W} \), thus settling the anchor step. Suppose that the statement has been shown for all intervals of length shorter than \( \ell \), where \( \ell \ge 1 \). Supposing that \( X \in \mathcal{D}^{[m,n]} \cap \mathcal{D}_\mathcal{W} \), and that the interval \([m,n]\) is of length \( \ell \), one considers the truncation triangle

\[
\sigma^{\le n-1}X \longrightarrow X \longrightarrow \sigma^{> n-1}X \longrightarrow \Sigma \sigma^{\le n-1}X.
\]

By Lemma 1.9, the objects \( \sigma^{\le n-1}X \) and \( \sigma^{> n-1}X \) are in \( \mathcal{D}_\mathcal{W} \) by t-exactness. By the induction hypothesis, both objects are in \( \text{thick}_\mathcal{D}(\mathcal{W}) \). Since \( X \) is an extension of the two, we conclude that \( X \in \text{thick}_\mathcal{D}(\mathcal{W}) \). By the Principle of Induction, we conclude that \( \mathcal{D}_\mathcal{W} \subseteq \text{thick}_\mathcal{D}(\mathcal{W}) \), and in turn that these triangulated subcategories of \( \mathcal{D} \) are equal. \( \square \)

1.2. Silting and \( \tau \)-tilting.

**Definition 1.12.** Let \( \mathcal{C} \) be a triangulated category with suspension functor \( \Sigma \). An object \( X \in \mathcal{C} \) is **presilting** if \( \mathcal{C}(X, \Sigma^nX) = 0 \) for all positive integers \( n \). A presilting object \( X \) is a **silting object** if, in addition, \( \text{thick}(X) = \mathcal{C} \).

Recall that if \( \mathcal{C} \) is a Krull–Schmidt \( k \)-category, an object \( X \in \mathcal{C} \) is **basic** if any two distinct indecomposable direct summands of \( X \) are non-isomorphic. The sets of basic presilting and basic silting objects in \( \mathcal{C} \) are denoted by \( \text{presilt}(\mathcal{C}) \) and \( \text{silt}(\mathcal{C}) \), respectively. The subsets containing the objects having a fixed (basic) presilting object \( \mathcal{P} \) as a direct summand are denoted by \( \text{presilt}_\mathcal{P}(\mathcal{C}) \) and \( \text{silt}_\mathcal{P}(\mathcal{C}) \), respectively.

**Definition 1.13.** For a fixed silting object \( S \in \mathcal{C} \), we say that an object \( P \in \mathcal{C} \) is **2\(_S\)-term** if it is belongs to \( \text{add}(S) \ast \Sigma \text{add}(S) \), i.e if there exists a triangle

\[
S_1 \longrightarrow S_0 \longrightarrow P \longrightarrow \Sigma S_1,
\]

where \( S_1, S_0 \in \text{add}(S) \). The set of basic 2\(_S\)-term presilting objects is denoted by 2\(_S\)-presilt\( (\mathcal{C}) \), and the subset of basic 2\(_S\)-term silting objects by 2\(_S\)-silt\( (\mathcal{C}) \). The subsets containing those having a (basic) presilting object \( \mathcal{P} \) as a direct summand are denoted by 2\(_S\)-presilt\( _\mathcal{P}(\mathcal{C}) \) and 2\(_S\)-silt\( _\mathcal{P}(\mathcal{C}) \), respectively. When the choice of silting object \( S \) is clear from context, we refer to 2\(_S\)-term (pre)silting objects as **two-term (pre)silting objects**.

We note that \( S \) in the above definition does not have to be basic. However, if \( \mathcal{R} \) denotes the largest basic direct summand of \( S \), the set of basic 2\(_S\)-term silting objects in \( \mathcal{C} \) coincides with that of basic 2\(_\mathcal{R}\)-term silting objects in \( \mathcal{C} \). Thanks to this fact, we will not need to assume that \( S \) is basic in our results.

If \( \mathcal{C} = \text{K}^b(\text{proj} \Lambda) \), the bounded homotopy category of a finite dimensional \( k \)-algebra \( \Lambda \), then \( \Lambda \) is a silting object in \( \mathcal{C} \) (removing redundant direct summands will yield a basic silting object). More generally,
if $A$ is a non-positive dg $k$-algebra, then $A$ is silting in $\text{per}(A)$. In these cases, we write $2$-silt$(A)$ for the set of $2_\Lambda$-term basic silting objects in $\text{per}(A)$ (and so on), omitting the subscript beneath the symbol $2$.

The set $2_S$-silt$(\mathcal{C})$ admits a partial order $\geq$, where $P \geq Q$ provided that $\mathcal{C}(P, \Sigma^n Q) = 0$ whenever $n > 0$ [AI12 Theorem 2.11 and Proposition 2.14]. If $\mathcal{C}$ is Hom-finite and Krull–Schmidt, it turns out that any $2_S$-term presilting object $P \in \mathcal{C}$ can be completed to a $2_S$-term silting object. One such silting object is the Bongartz completion $T^+_p \overset{\text{def}}{=} P \oplus Q^+$ of $P$, where $Q^+$ is defined by the triangle

$$S \xrightarrow{\beta_{\Sigma S}} Q^+ \xrightarrow{P_0} \Sigma S,$$

(1.C)

in which $\beta_{\Sigma S}$ is a right $\text{add}(P)$-approximation of $\Sigma S$ [DYY14 Lemma 4.2]. Consequently, a $2_S$-term presilting object is precisely the same thing as a direct summand of a $2_S$-term silting object. If $|P|$ denotes the number of non-isomorphic indecomposable direct summands of a $2_S$-term presilting object $P$, we have that $P$ is silting if and only if $|P| = |S|$ [DYY14 Proposition 4.3]. If $P$ is basic, we make its Bongartz completion basic by removing superfluous direct summands, so that $T^+_p$ becomes an element in $2_S$-silt$(\mathcal{C})$. The Bongartz completion of $P$ is then the maximal element in $2_S$-silt$(\mathcal{C})$ (with respect to the partial order $\geq$ defined above) in which $P$ is a direct summand.

There is a close connection between the $2_S$-term silting theory of $\mathcal{C}$ and the support $\tau$-tilting theory of the finite dimensional $k$-algebra $\text{End}_C(S)$. We now recall the definitions and results of $\tau$-tilting theory [AIR14].

**Definition 1.14.** For a finite dimensional $k$-algebra $\Lambda$, a finitely generated right $\Lambda$-module $M$ is $\tau$-rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$, where $\tau$ is the Auslander–Reiten translation. A pair $(M, Q)$ is $\tau$-rigid if $M$ is a $\tau$-rigid module and $Q$ is a finitely generated projective right $\Lambda$-module such that $\text{Hom}_\Lambda(Q, \tau M) = 0$. A $\tau$-rigid pair $(M, Q)$ is support $\tau$-tilting if $|M| + |Q| = |\Lambda|$, where $|X|$ is the number of non-isomorphic indecomposable direct summands of $X$. An $\Lambda$-module $M$ is support $\tau$-tilting if there exists a projective $\Lambda$-module $Q$ such that $(M, Q)$ is a support $\tau$-tilting pair.

We identify the $\tau$-rigid pairs $(M, Q)$ and $(N, R)$ if $\text{add}(M) = \text{add}(N)$ and $\text{add}(Q) = \text{add}(R)$. We say that $(M, Q)$ is basic if no two indecomposable direct summands of $(M, Q)$ are isomorphic. By the Krull–Schmidt property of mod $\Lambda$, any $\tau$-rigid pair can be identified with a basic $\tau$-rigid pair. A $\tau$-rigid pair $(N, R)$ is a direct summand of the $\tau$-rigid pair $(M, Q)$ if $N$ is a direct summand of $M$ and $R$ is a direct summand of $Q$. Note that indecomposable $\tau$-rigid pairs necessarily have the zero module as exactly one of the two components. If $(M, Q)$ and $(M, R)$ are support $\tau$-tilting pairs, then $\text{add}(Q) = \text{add}(R)$ [AIR14 Proposition 2.3(b)], so $Q = R$ if these pairs are basic. It follows that we have a one-to-one correspondence between basic support $\tau$-tilting modules and basic support $\tau$-tilting pairs, which truncates a pair $(M, Q)$ to $M$.

We denote the set of basic $\tau$-rigid pairs in mod $\Lambda$ by $\tau$-rigid pair$(\Lambda)$, and those having a $\tau$-rigid pair $(M, Q)$ as a direct summand by $\tau$-rigid pair$(M, Q)_\Lambda(\Lambda)$. Both of these sets contain subsets consisting of basic support $\tau$-tilting pairs, that are denoted by $\text{st}$-tilt pair$(\Lambda)$ and $\text{st}$-tilt pair$(M, Q)_\Lambda(\Lambda)$, respectively. Also, let $s\tau$-tilt$(\Lambda)$ denote the set of basic support $\tau$-tilting $\Lambda$-modules up to isomorphism. We say that a $k$-algebra is $\tau$-tilting finite if $s\tau$-tilt$(\Lambda)$ is a finite set.
The set of support $\tau$-tilting $\Lambda$-modules is inextricably linked with functorially finite torsion classes of $\text{mod} \, \Lambda$. Recall that a torsion class (resp. torsion-free class) of $\text{mod} \, \Lambda$ is a subcategory which is closed under factor modules (resp. submodules) and extensions. If $\mathcal{G}$ is a torsion class, the right perpendicular category $\mathcal{G}^\perp$ is a torsion-free class. Dually, the left perpendicular category of a torsion-free class is a torsion class. A torsion pair is a pair of subcategories $(\mathcal{G}, \mathcal{F})$ where $\mathcal{G}^\perp = \mathcal{F}$ and $^\perp \mathcal{F} = \mathcal{G}$. It is indeed the case that $\mathcal{G}$ is a torsion class and that $\mathcal{F}$ is a torsion-free class. For every $X \in \text{mod} \, \Lambda$ one has a canonical exact sequence

$$0 \rightarrow tX \rightarrow X \rightarrow fX \rightarrow 0,$$

(1.D)
in which the objects $tX$ and $fX$ are determined by functors $t : \text{mod} \, \Lambda \rightarrow \mathcal{G}$ and $f : \text{mod} \, \Lambda \rightarrow \mathcal{F}$, called the torsion radical and the torsion-free functor, respectively.

In the next theorem and throughout, given a $\Lambda$-module $M$, let $\text{gen}(M)$ denote the full subcategory of $\text{mod} \, \Lambda$ containing the $\Lambda$-modules $X$ for which there exists an epimorphism $M \oplus nX$ for some $n \geq 1$.

**Theorem 1.15 ([AIR14, Theorem 2.7]).** Let $M$ be a support $\tau$-tilting $\Lambda$-module. Then the pair $(\text{gen}(M), M^\perp)$ is a torsion pair in $\text{mod} \, \Lambda$. Moreover, the torsion class $\text{gen}(M)$ is functorially finite in $\text{mod} \, \Lambda$, and we have a bijection

$$\text{gen} : \tau\text{-tilt} (\Lambda) \rightarrow \text{f-tors} (\Lambda)$$

from the set of basic support $\tau$-tilting $\Lambda$-modules to the set of functorially finite torsion classes in $\text{mod} \, \Lambda$.

Since the set of torsion classes is partially ordered under inclusion, this bijection induces a partial order on $\tau\text{-tilt}(\Lambda)$ and $\tau\text{-tilt pair}(\Lambda)$. More explicitly, we impose that $M \geq N$ if $\text{gen}(M) \supseteq \text{gen}(N)$. If $(M, Q)$ and $(N, R)$ are basic support $\tau$-tilting pairs, we say that $(M, Q) \geq (N, R)$ if $M \geq N$ as basic support $\tau$-tilting modules.

**Theorem 1.16 ([Jan15 Proposition 4.5], [IY08 Proposition 6.2(3)]).** Let $\mathcal{C}$ be $\text{Hom}$-finite Krull–Schmidt triangulated category and $S \in \mathcal{C}$ be a silting object. Let $\text{add}(S) \ast \Sigma \text{add}(S)$ be the subcategory of $2S$-term objects in $\mathcal{C}$. The functor

$$\mathcal{C}(S, -) : \mathcal{C} \rightarrow \text{mod} \, \text{End}_{\mathcal{C}}(S)$$

induces an equivalence of categories

$$\mathcal{C}(S, -) : \frac{\text{add}(S) \ast \Sigma \text{add}(S)}{[\Sigma S]} \rightarrow \text{mod} \, \text{End}_{\mathcal{C}}(S),$$

where $[\Sigma S]$ is the ideal of $\mathcal{C}$ consisting of morphisms factoring through $\text{add}(\Sigma S)$.

The close connection between silting and support $\tau$-tilting is expressed by the following theorem.

**Theorem 1.17 ([IJY14 Theorem 4.5]).** Let $\mathcal{C}$ and $S$ be as in Theorem 1.16 and let $\Lambda = \text{End}_{\mathcal{C}}(S)$. We have a bijection

$$2S\text{-presilt}(\mathcal{C}) \xrightarrow{H_S} \tau\text{-rigid pair}(\Lambda),$$

$$X \xrightarrow{\psi} (\mathcal{C}(S, X), \mathcal{C}(S, X_1))$$

where $\Sigma X_1$ is the maximal direct summand of $X$ in $\Sigma S$. It restricts to bijections
TWO-TERM SILTING AND \( \tau \)-CLUSTER MORPHISM CATEGORIES

\[ 2_S \text{-presilt}_P(C) \xrightarrow{H_S} \ \tau \text{-rigid pair}_{H_S(P)}(\Lambda), \]

and

\[ 2_S \text{-silt}_P(C) \xrightarrow{H_S} s\tau \text{-tilt pair}_{H_S(P)}(\Lambda), \]

(1.E)

for each \( P \in 2_S \text{-presilt}(C) \). The last bijection is an isomorphism of partially ordered sets.

The isomorphism in (1.E) sends the Bongartz completion of \( P \in 2_S \text{-presilt}(C) \) to a unique maximal element in \( s\tau \text{-tilt pair}_{H_S(P)}(\Lambda) \). We refer to this as the Bongartz completion of \( \tau \)-rigid pair \( (M, Q) \) by \( (M^+, Q) \), noting that the second component \( Q \) remains unaltered.

**Proposition-Definition 1.18** ([Jas15 Proposition 3.6], [DIR+17 Theorem 4.12(a)], [BH23 Definition 3.2]). Let \( \Lambda \) be a finite dimensional \( k \)-algebra, and let \( (M, Q) \) be a \( \tau \)-rigid pair in \( \text{mod } \Lambda \). The pair \( (M, Q) \) determines a wide subcategory

\[ J(M, Q) \overset{\text{def}}{=} M^+ \cap \left( \tau M \right)^\perp \cap Q^\perp \subseteq \text{mod } \Lambda \]

called the \( \tau \)-perpendicular category of \( (M, Q) \). A wide subcategory \( W \) of \( \text{mod } \Lambda \) is a \( \tau \)-perpendicular wide subcategory if it is the \( \tau \)-perpendicular category of some \( \tau \)-rigid pair in \( \text{mod } \Lambda \).

The next result shows that \( \tau \)-perpendicular categories are module categories.

**Proposition-Definition 1.19** ([Jas15 Theorem 3.8], [DIR+17 Theorem 4.12(b)]). Let \( (M, Q) \) be a \( \tau \)-rigid pair in \( \text{mod } \Lambda \), let \( (M^+, Q) \) be the Bongartz completion of \( (M, Q) \), and let \( \Gamma = \text{End}_A(M^+)/[M] \). We then have an exact equivalence

\[ F_{(M, Q)} \overset{\text{def}}{=} \text{Hom}_A(M^+, -) : J(M, Q) \longrightarrow \text{mod } \Gamma. \]

We will refer to the algebra \( \Gamma \) as the \( \tau \)-tilting reduction of \( \Lambda \) with respect to \( (M, Q) \).

Let \( J(M, Q) \) be the \( \tau \)-perpendicular category of a \( \tau \)-rigid pair \( (M, Q) \) with Bongartz completion \( (M^+, Q) \), and let \( \Gamma = \text{End}_A(M^+)/[M] \), where \( [M] \) is the ideal of morphisms factoring through \( \text{add}(M) \). A \( \tau \)-rigid pair in \( J(M, Q) \) is a pair \( (U, R) \) in \( J(M, Q) \) such that \( F_{(M, Q)} U, F_{(M, Q)} R \) is a \( \tau \)-rigid pair in \( \text{mod } \Gamma \), where \( F_{(M, Q)} \) is the exact equivalence in Proposition-Definition 1.19. The set of basic \( \tau \)-rigid pairs in \( J(M, Q) \) will be denoted by \( \tau \)-rigid pair\( (J(M, Q)) \).

By construction, it follows that any \( \tau \)-rigid pair \( (M, Q) \) in \( \text{mod } \Lambda \) provides a bijection

\[ F_{(M, Q)} : \ \tau \text{-rigid pair}(J(M, Q)) \longrightarrow \tau \text{-rigid pair}(\Gamma), \]

(1.F)

where \( \Gamma \) is the \( \tau \)-tilting reduction of \( \Lambda \) with respect to \( (M, Q) \).

2. Silting \( t \)-structures and reduction

Silting reduction is a well-developed reduction technique, particularly for Hom-finite Krull–Schmidt triangulated categories [IY18]. The bijections in question restrict to two-term objects and are compatible with \( \tau \)-tilting reduction [Jas15 Theorem 4.12]. In this section, we study the interplay between silting reduction and \( t \)-structures. Iyama–Yang prove the following theorem, though in greater generality.
Theorem 2.1 ([IY18, Theorems 3.1, 3.6, and 3.7], [Jas15, Proposition 4.11]). Let $\mathcal{C}$ be a Hom-finite Krull–Schmidt triangulated category containing a silting object $S$. Let $P$ be a presilting object.

1. Let $\mathcal{Z}_P = \perp^0 P \cap P^{\perp^0}$. Then the composite

$$
\mathcal{Z}_P \hookrightarrow \mathcal{C} \xrightarrow{\text{loc}} \mathcal{C}/\text{thick}(P)
$$

induces a triangle equivalence $\mathcal{Z}_P/[P] \to \mathcal{C}/\text{thick}(P)$, where the triangulation on $\mathcal{Z}_P/[P]$ is described by Iyama–Yoshino [IY08, Theorem 4.2]. The inverse of the suspension of $X \in \mathcal{Z}_P/[P]$ is denoted by $X(-1)$, and is defined by forming a triangle

$$
X(-1) \to P_X \xrightarrow{\beta_X} X \to \Sigma X(-1)
$$

in which $\beta_X$ is a right $\text{add}(P)$-approximation.

2. The additive quotient $\mathcal{Z}_P \to \mathcal{Z}_P/[P]$ (and also the Verdier localisation $\mathcal{C} \to \mathcal{C}/\text{thick}(P)$) induces a bijection

$$
\varphi_P : \text{presilt}_P(\mathcal{C}) \to \text{presilt}(\mathcal{Z}_P/[P]) \cong \text{presilt}(\mathcal{C}/\text{thick}(P)).
$$

If $P$ is basic and $2S$-term, and has Bongartz completion $T^+_P$, it restricts to a bijection

$$
\varphi_P : 2S\text{-presilt}_P(\mathcal{C}) \to 2T^+_P\text{-presilt}(\mathcal{Z}_P/[P]) \cong 2T^+_P\text{-presilt}(\mathcal{C}/\text{thick}(P)).
$$

3. The bijection (2.B) restricts to an isomorphism of posets

$$
\varphi_P : \text{silt}_P(\mathcal{C}) \to \text{silt}(\mathcal{Z}_P/[P]) \cong \text{silt}(\mathcal{C}/\text{thick}(P)),
$$

which, if $P$ is basic and $2S$-term, further restricts to an isomorphism of posets

$$
\varphi_P : 2S\text{-silt}_P(\mathcal{C}) \to 2T^+_P\text{-silt}(\mathcal{Z}_P/[P]) \cong 2T^+_P\text{-silt}(\mathcal{C}/\text{thick}(P)).
$$

Since the bijection in (2.B) is induced by the ideal quotient $\mathcal{Z}_P \to \mathcal{Z}_P/[P]$, our next two lemmas are immediate consequences.

Lemma 2.2. Let $\mathcal{C}$ be as in Theorem 2.1 and let $P$ be a basic $2S$-term presilting object in $\mathcal{C}$. We define $\text{ind}_P 2S\text{-presilt}_P(\mathcal{C})$ to be the objects in $2S\text{-presilt}_P(\mathcal{C})$ of the form $P \oplus X$, where $X$ is an indecomposable object which is not contained in $\text{add}(P)$. Then the bijection in (2.B) restricts to a bijection

$$
\varphi_P : \text{ind}_P 2S\text{-presilt}_P(\mathcal{C}) \to \text{ind}_P 2T^+_P\text{-presilt}(\mathcal{Z}_P/[P]),
$$

where the codomain is the set of indecomposable objects in $2T^+_P\text{-presilt}(\mathcal{Z}_P/[P])$.

Lemma 2.3. Let $P$ and $Q$ be basic $2S$-term presilting objects in $\mathcal{C}$ such that $P \oplus Q$ also is basic $2S$-term presilting. The bijection

$$
\varphi_P : 2S\text{-presilt}_P(\mathcal{C}) \to 2T^+_P\text{-presilt}(\mathcal{Z}_P/[P])
$$

restricts to a bijection

$$
\varphi_P : 2S\text{-presilt}_{P \oplus Q}(\mathcal{C}) \to 2T^+_P\text{-presilt}_{\varphi_P(P \oplus Q)}(\mathcal{Z}_P/[P]).
$$
Lemma 2.4. Let $P \oplus Q$ be a basic $2_S$-term presilting object in $\mathcal{C}$. Then the diagram
\[
\begin{array}{ccc}
  2_S\text{-presilt}_{P \oplus Q}(\mathcal{C}) & \xrightarrow{\varphi_{P \oplus Q}} & 2_{T_P^+}\text{-presilt}_{\varphi_{P \oplus Q}}(Z_P/[P]) \\
  \downarrow \varphi_{P \oplus Q} & & \downarrow \varphi_{P \oplus Q}(\varphi) \\
  2_{T_Q^+}\text{-presilt}_{\varphi_{P \oplus Q}}(Z_Q/\{Q\}) & \xrightarrow{\varphi_{\varphi_{P \oplus Q}}(\varphi)} & 2_{T_{P \oplus Q}^+}\text{-presilt}(Z_{P \oplus Q}/[P \oplus Q])
\end{array}
\] (2.E)
commutes.

Proof. For the sake of terseness, let $[Q]$ denote the ideal of $Z_{P \oplus Q}/[P]$ consisting of morphisms factoring through $\text{add}(Q)$, and similarly for the ideal $[P]$ of $Z_{P \oplus Q}/[Q]$. By the Third Isomorphism Theorem, we have additive equivalences
\[
\frac{Z_{P \oplus Q}/[P]}{[Q]} \simeq \frac{Z_{P \oplus Q}/[P]}{[P \oplus Q]}/[P] \simeq \frac{Z_{P \oplus Q}}{[P \oplus Q]},
\] (2.F)
and similarly
\[
\frac{Z_{P \oplus Q}/[Q]}{[P]} \simeq \frac{Z_{P \oplus Q}}{[P \oplus Q]}.
\] (2.G)
The isomorphisms in (2.F) and (2.G) enable the construction of an essentially commutative diagram of ideal quotient functors
\[
\begin{array}{ccc}
  Z_{P \oplus Q} & \xrightarrow{\cong} & Z_{P \oplus Q}/[P] \\
  \downarrow \cong & & \downarrow \cong \\
  Z_{P \oplus Q}/[Q] & \xrightarrow{\cong} & Z_{P \oplus Q}/[P \oplus Q]
\end{array}
\] (2.H)
Theorem 2.1 asserts that the functors going out of $Z_{P \oplus Q}$ induce the bijections going out of $2_S\text{-presilt}_{P \oplus Q}(\mathcal{C})$ in (2.E).

We complete the proof by showing that the maps $\varphi_{\varphi_{P \oplus Q}}(\varphi)$ have the claimed codomain, and that they factorise $\varphi_{P \oplus Q}$ as shown in (2.E). To give an argument as to why $\varphi_{\varphi_{P \oplus Q}}(\varphi)$ has the claimed codomain in (2.E), it suffices to show that the functor on the bottom of (2.H) has the following property:

The ideal quotient functor $Z_{P \oplus Q}/[Q] \to Z_{P \oplus Q}/[P \oplus Q]$ sends $T_Q^+$ to $T_{P \oplus Q}^+$. (2.1)

Indeed, it follows from Theorem 2.1 and (2.G) that $\varphi_{\varphi_{P \oplus Q}}(\varphi)$ sends presilting objects to presilting objects, and the claim in (2.1) will ensure that $2_{T_Q^-}$-term objects are sent to $2_{T_{P \oplus Q}^-}$-term objects. To establish the claim in (2.1), it suffices to prove that the ideal quotient functor $Z_{P \oplus Q} \to Z_{P \oplus Q}/[P \oplus Q]$ sends $T_Q^+$ and $T_{P \oplus Q}^+$ to isomorphic objects. Using the definition of Bongartz completion (see (1.C)), it is easy to see that $T_Q^+$ and $T_{P \oplus Q}^+$ are both isomorphic to $S$ in the Verdier quotient $\mathcal{C}/\text{thick}(P \oplus Q)$. It now follows from Theorem 2.1 that $T_Q^+$ and $T_{P \oplus Q}^+$ are isomorphic in $Z_{P \oplus Q}/[P \oplus Q]$, which is enough to conclude that (2.1) holds.

By Lemma 2.3 and the previous paragraph, the maps in the (2.E) are induced by the functors in (2.H). The commutativity of the former diagram now follows from the commutativity of the latter.

We will now state our setup, for the sake of future reference.

Setup 2.5. Let $A$ be a non-positive dg $k$-algebra such that $H^iA$ is finite dimensional for all $i \in \mathbb{Z}$, and let $P$ be a basic two-term presilting object in $\text{per}(A)$, i.e. an object in $2\text{-presilt}(A)$.
We are requiring $H^iA$ to be finite dimensional for all $i \in \mathbb{Z}$ so that $\text{per}(A)$ becomes $\text{Hom}$-finite \cite{Ami99}

Proof of Proposition 2.4] (and consequently Krull–Schmidt, by idempotent completeness). Note that finite dimensional $k$-algebras are included in our setup. It would be possible to generalise Setup 2.5 to the setting of ST-triples $(\mathcal{C}, \mathcal{D}, S)$ \cite{AMY19} §4. We do not give the definition of ST-triples here, but we note that if $\mathcal{C}$ is an algebraic triangulated category (i.e. triangle equivalent to the stable category of a Frobenius category), this triple is equivalent to a triple of the form $(\text{per}(A), \mathcal{D}_{\text{id}}(A), A)$, where $A$ is as in Setup 2.5 \cite{AMY19} Proposition 6.12]. The assumption of algebricity does not reduce the scope of applications at time of writing; the use of ST-triples seems not to have occurred in a non-algebraic context.

In this section, we describe Iyama–Yang silting reduction and its interaction with silting t-structures. Silting reduction with respect to $P$ occurs in the Verdier quotient $\text{per}(A)/\text{thick}(P)$, but we will show that subcategory $P_{\text{id}}^{+1} \triangleq P_{\text{id}}^{+1} \cap \mathcal{D}_{\text{id}}(A)$ of $\mathcal{D}_{\text{id}}(A)$ also plays a role, where $P_{\text{id}}^{+1}$ is a right perpendicular subcategory of $\mathcal{D}(A)$ (see (1.3)). If $\text{per}(A)/\text{thick}(P)$ is triangle equivalent to $\text{per}(C)$ for some dg algebra $C$, then $P_{\text{id}}^{+1}$ is triangle equivalent to $\mathcal{D}_{\text{id}}(C)$ (see Corollary 2.8). Note that $P_{\text{id}}^{+1}$ is a thick subcategory of $\mathcal{D}_{\text{id}}(A)$. The t-structures defined by these silting objects will be restricted to $P_{\text{id}}^{+1}$, which is achievable since $P_{\text{id}}^{+1}$ is a t-exact subcategory of $\mathcal{D}_{\text{id}}(A)$ (see Proposition 2.8).

We first specify what it means for a t-structure to be generated by a silting object. We employ the following notational convention: Given an object $T$ in $\text{per}(A)$ and $I \subseteq \mathbb{Z}$, let $T_{I}^{+1}$ denote the right perpendicular of $\mathcal{D}(A)$, as defined in (1.3), and define $T_{\text{id}}^{+1} \triangleq T^{+1} \cap \mathcal{D}_{\text{id}}(A)$.

**Proposition-Definition 2.6** \cite{AMY19} Proposition 4.6]. Let $T$ be a silting object in $\text{per}(A)$. The pair $(T^{+1,0}, T^{+1,0})$ forms a t-structure on $\mathcal{D}(A)$. It restricts to a t-structure $(T_{\text{id}}^{+1,0}, T_{\text{id}}^{+1,0})$ on $\mathcal{D}_{\text{id}}(A)$. A t-structure of one of these forms is said to be a silting t-structure, and it is generated by $T$.

Let $\sigma_{T}^{0}$ be the cohomology functor associated to the silting t-structure $(T_{\text{id}}^{+1,0}, T_{\text{id}}^{+1,0})$. Then $\sigma_{T}^{0}$ sends $T$ to a projective generator of the heart $T_{\text{id}}^{+1,0}$, and the Hom-functor $\mathcal{D}_{\text{id}}(A)(T, -)$ restricts to an exact equivalence of abelian categories

$$
\mathcal{D}_{\text{id}}(A)(T, -): T_{\text{id}}^{+1,0} \longrightarrow \text{mod End}_{\mathcal{D}_{\text{id}}(A)}(T).
$$

**Lemma 2.7.** Let $A$ and $P$ be as in Setup 2.5 and let $(\mathcal{U}, \mathcal{U}^{+1})$ be a t-structure on $\mathcal{D}_{\text{id}}(A)$ such that $\mathcal{U} \subseteq P_{\text{id}}^{+1}$ and $\mathcal{U}^{+1} \subseteq P_{\text{id}}^{+1\geq 0}$. Then the subcategory $P_{\text{id}}^{+1}$ is t-exact with respect to $(\mathcal{U}, \mathcal{U}^{+1})$. In particular, the pair $(\mathcal{U} \cap P_{\text{id}}^{+1}, \mathcal{U}^{+1} \cap P_{\text{id}}^{+1})$ is a t-structure on $P_{\text{id}}^{+1}$.

**Proof.** Let $\sigma_{\mathcal{U}}$ and $\sigma_{\mathcal{U}^{+1}}$ be the truncation functors for $(\mathcal{U}, \mathcal{U}^{+1})$. Fixing an arbitrary object $X \in P_{\text{id}}^{+1}$, it suffices to show that $\sigma_{\mathcal{U}}X \in P_{\text{id}}^{+1}$. We have a truncation triangle

$$
\sigma_{\mathcal{U}}X \longrightarrow X \longrightarrow \sigma_{\mathcal{U}^{+1}}X \longrightarrow \Sigma \sigma_{\mathcal{U}}X.
$$

Since $\sigma_{\mathcal{U}^{+1}}X \in \mathcal{U}^{+1} \subseteq P_{\text{id}}^{+1\geq 0}$, the desuspension $\Sigma^{-1}\sigma_{\mathcal{U}^{+1}}X$ is in $P_{\text{id}}^{+1\leq 1}$. Using that $P_{\text{id}}^{+1\leq 1}$ is extension-closed in $\mathcal{D}_{\text{id}}(A)$, we deduce that $\sigma_{\mathcal{U}}X \in P_{\text{id}}^{+1\leq 1}$. Since $\sigma_{\mathcal{U}}X$ also belongs to $\mathcal{U}$, and thus to $P_{\text{id}}^{+1\geq 1}$, it follows that $\sigma_{\mathcal{U}}X \in P_{\text{id}}^{+1\leq 1} \cap P_{\text{id}}^{+1\geq 1} = P_{\text{id}}^{+1}$, as desired.

**Proposition 2.8.** Let $A$ be as in Setup 2.5.
(1) Let $T$ be a silting object in $\text{per}(A)$ of which $Q$ is a direct summand. Then $P_{\text{fd}}^{1\leq}$ is t-exact with respect to the t-structure $(T_{\text{fd}}^{1>0}, T_{\text{fd}}^{1=0})$ on $D_{\text{fd}}(A)$.

(2) Let $P \in 2\text{-presilt}(A)$ (as in Setup 2.5). Then $P_{\text{fd}}^{1\leq}$ is t-exact with respect to the standard t-structure $(A_{\text{fd}}^{1>0}, A_{\text{fd}}^{1=0})$ on $D_{\text{fd}}(A)$.

**Proof.** We will prove that both assertions are consequences of Lemma 2.7.

We first address (1). It is clear that $T_{\text{fd}}^{1>0} \subseteq Q_{\text{fd}}^{1>0} \subseteq Q_{\text{fd}}^{1>1}$ and $T_{\text{fd}}^{1=0} \subseteq Q_{\text{fd}}^{1=0}$, since $Q$ is a direct summand of $T$.

We move on to (2). Since $P$ is two-term, one can by definition find a triangle

$$A_1 \longrightarrow A_0 \longrightarrow P \longrightarrow \Sigma A_1,$$

where $A_0, A_1 \in \text{add}(A)$. It follows from a long-exact sequence argument that $A_{\text{fd}}^{1>0} \subseteq P_{\text{fd}}^{1>1}$ and that $A_{\text{fd}}^{1=0} \subseteq P_{\text{fd}}^{1=0}$. Since the conditions in Lemma 2.7 again are met, we conclude that $P_{\text{fd}}^{1\leq}$ indeed is t-exact with respect to the standard t-structure on $D_{\text{fd}}(A)$. \hfill \Box

We claimed in the introduction that the perpendicular category $P_{\text{fd}}^{1\leq}$ has a connection to the $\tau$-perpendicular category of $H_{\text{A}}(P)$. This assertion will now be made explicit.

**Proposition 2.9.** Let $A$ and $P$ be as in Setup 2.5 and let $(M, Q) = H_{\text{A}}(P)$ be the $\tau$-rigid pair in $\text{mod}(H^0A)$ corresponding to $P$.

1. The lattice isomorphism

$$− \cap A_{\text{fd}}^{1\geq 0} : \text{t-exact}_{(A^{>0}, A^{\leq 0})}(D_{\text{fd}}(A)) \longrightarrow \text{wide}(A^{1\geq 0}), \quad (2.J)$$

provided by Theorem 1.10 and Lemma 1.11 sends $P_{\text{fd}}^{1\leq}$ to $M^{1\leq} \cap ^{1\leq}(\tau M) \cap Q^{1\leq} \cap A_{\text{fd}}^{1\geq 0}$. Consequently, by Proposition-Definition 2.4, the lattice isomorphism

$$D(A, −) : \text{t-exact}_{(A^{>0}, A^{\leq 0})}(D_{\text{fd}}(A)) \longrightarrow \text{wide}(H^0A) \quad (2.K)$$

in Theorem 1.10 sends $P_{\text{fd}}^{1\leq}$ to $J(M, Q)$.

2. The isomorphism in (2.K) restricts to an isomorphism from the poset of t-exact subcategories of the form $P_{\text{fd}}^{1\leq}$, for some two-term presilting object $P$, to the poset of $\tau$-perpendicular wide subcategories of $\text{mod} H^0A$.

**Proof.** We first prove (1). By Lemma 1.11,2, the bijection $− \cap A_{\text{fd}}^{1\geq 0}$ in (2.J) is inverted by $\text{thick}_{D_{\text{fd}}(A)}(−)$. Since these are mutually inverse isomorphisms of posets, it suffices to prove that

$$P_{\text{fd}}^{1\leq} \cap A_{\text{fd}}^{1\geq 0} \subseteq I(M, Q), \quad (2.L)$$

$$\text{thick}_{D_{\text{fd}}(A)}(I(M, Q)) \subseteq P_{\text{fd}}^{1\leq}. \quad (2.M)$$

where $I(M, Q) \stackrel{\text{def}}{=} M^{1\leq} \cap ^{1\leq}(\tau M) \cap Q^{1\leq} \cap A_{\text{fd}}^{1\geq 0} \subseteq A^{1\geq 0}$.

The inclusion in (2.L) will be addressed first. Let $X \in P_{\text{fd}}^{1\leq} \cap A_{\text{fd}}^{1\geq 0}$. It is to be deduced that $X \in M^{1\leq} \cap ^{1\leq}(\tau M) \cap Q^{1\leq}$. The assertion that $X \in Q^{1\leq}$ is immediate, having assumed that $X \in P_{\text{fd}}^{1\leq} \subseteq Q_{\text{fd}}^{1\leq}$.\hfill \Box
Consider the following truncation triangle in the t-structure \( (A^{\geq 0}, A^{< 0}) \) on \( \mathcal{D}(A) \) (the once suspension of the standard one):
\[
\sigma_A^{< 0} P \longrightarrow P \longrightarrow \sigma_A^{> 0} P \longrightarrow \Sigma \sigma_A^{< 0} P \tag{2.N}
\]
Since \( P \) is two-term (so \( \sigma_A^{\geq 1} P \simeq 0 \) and \( \sigma_A^{> 1} P \simeq M \)), the third term in this triangle is indeed isomorphic to \( M \). The outer terms in (2.N) are in \( A^{\geq 0} \), so there are no non-trivial maps from these objects to \( X \in A_{\text{id}}^{\geq 0} \subset A_{\text{id}}^{< 0} \). Applying the contravariant functor \( \text{Hom}_{\mathcal{D}(A)}(-, X) \) to the triangle in (2.N) thus yields an isomorphism
\[
\text{Hom}_{\mathcal{D}(A)}(P, X) \cong \text{Hom}_{\mathcal{D}(A)}(M, X),
\]
so \( X \in M^{< 0} \). Lastly, we show that \( \text{Hom}_{H^0A}(X, \tau M) = 0 \). We decompose \( P \) into \( P_M \oplus \Sigma Q \), where the second term is the largest direct summand of \( P \) in \( \Sigma A \). Consider a triangle in \( \mathcal{D}(A) \)
\[
P_1 \xrightarrow{P_M} P_0 \xrightarrow{P_M} P_M \longrightarrow \Sigma P_1,
\]
where \( P_0, P_1 \in \text{proj}(A) \) and \( P_M \) is a right minimal morphism. By Proposition-Definition [2.6], the induced right exact sequence
\[
\text{Hom}_{\mathcal{D}(A)}(A, P_1) \xrightarrow{P_M} \text{Hom}_{\mathcal{D}(A)}(A, P_0) \longrightarrow \text{Hom}_{\mathcal{D}(A)}(A, P_M) \longrightarrow 0 \tag{2.P}
\]
is readily checked to be a minimal projective presentation of the \( H^0A \)-module \( \text{Hom}_{\mathcal{D}(A)}(A, P_M) \). By Proposition-Definition [2.6], an inverse of the exact equivalence \( \text{Hom}_{\mathcal{D}(A)}(A, -) : A^{\geq 0} \longrightarrow \text{mod} H^0A \) sends the right exact sequence in (2.P) to the right exact sequence
\[
\sigma_A^{0} P_1 \xrightarrow{\sigma_A^{0} P_M} \sigma_A^{0} P_0 \longrightarrow \sigma_A^{0} M \longrightarrow 0 \tag{2.Q}
\]
in \( A^{> 0} \), where \( \sigma_A^{0} \) is the cohomology functor for the silting t-structure \( (A^{\geq 0}, A^{< 0}) \) on \( \mathcal{D}(A) \). Using Proposition-Definition [2.6] to identify \( \text{mod} H^0A \) with \( A^{< 0} \), we may claim that \( \text{Hom}_{H^0A}(X, \tau M) = 0 \) if and only if the induced map
\[
\text{Hom}_{\mathcal{D}(A)}(\sigma_A^{0} P_1, X) \xrightarrow{-\sigma_A^{0} P_M} \text{Hom}_{\mathcal{D}(A)}(\sigma_A^{0} P_0, X)
\]
surjects [AIR14, Proposition 2.4(b)], or indeed that
\[
\text{Hom}_{\mathcal{D}(A)}(\sigma_A^{> 0} P_1, X) \xleftarrow{-\sigma_A^{> 0} P_M} \text{Hom}_{\mathcal{D}(A)}(\sigma_A^{> 0} P_0, X)
\]
surjects, since \( \sigma_A^{0} Y = \sigma_A^{> 0} Y \) for all \( Y \in A^{> 0} \). Since \( \sigma_A^{> 0} \) is left adjoint to the inclusion of \( A^{< 0} \) into \( \mathcal{D}(A) \), it is equivalent to show that
\[
\text{Hom}_{\mathcal{D}(A)}(P_1, X) \xrightarrow{-P_M} \text{Hom}_{\mathcal{D}(A)}(P_0, X) \tag{2.Q}
\]
surjects. This is shown by a long-exact sequence argument, applying \( \text{Hom}_{\mathcal{D}(A)}(-, X) \) to the triangle in (2.N). We have shown that \( \text{Hom}_{H^0A}(X, \tau M) = 0 \), and moreover that the inclusion in (2.L) holds.

To prove that the inclusion in (2.M) holds, it suffices to show that \( I(M, Q) \subseteq P_{\text{id}}^{\perp 2} \), since \( P_{\text{id}}^{\perp 2} \) is a thick subcategory of \( \mathcal{D}_{\text{id}}(A) \). Since \( I(M, Q) \subseteq A_{\text{id}}^{\geq 0} \) and \( P \) is two-term, we can immediately assert that
We conclude this section with results that will turn out useful for this purpose. To prove them, we will apply some basic results from the localisation theory of compactly generated triangulated categories, possibly infinitely generated modules. Proposition 2.10 addresses the localisation theory we need, whereas the subsequent corollaries will be applied in later sections.

We conclude the proof by addressing (2). Indeed, since the map in (2.Q) injects, so does the restricted map from the poset of τ-exact subcategories of the form $P_{\text{id}}^\perp$, for some two-term presilting object $P$, to the poset of τ-perpendicular wide subcategories of mod $H^0 A$. Surjectivity follows from (1) above, since it suffices to point out that $H^1_A(M, Q)^{\perp \perp}$ is a preimage of the arbitrary τ-perpendicular wide subcategory $J(M, Q)$. □

Section 4 is devoted to the construction of the τ-cluster morphism category in terms of two-term silting. We conclude this section with results that will turn out useful for this purpose. To prove them, we will apply some basic results from the localisation theory of compactly generated triangulated categories, especially in the “large” derived category $\mathcal{D}(A)$, where the objects are possibly unbounded complexes of possibly infinitely generated modules. Proposition 2.10 addresses the localisation theory we need, whereas the subsequent corollaries will be applied in later sections.

**Proposition 2.10.** Let $A$ and $P$ be as in Setup 2.5 and let Loc($P$) be the smallest thick subcategory of $\mathcal{D}(A)$ that is closed under set-indexed coproducts.

1. [Kra08 Theorem 5.6.1] There is a recollement of triangulated categories

\[
\begin{array}{ccc}
P_{\perp_{\text{id}}} & \xleftarrow{\pi_P} & \mathcal{D}(A) \\
\kappa_P & \downarrow & \rho_P \downarrow \lambda_P \\
P_{\text{id}} \perp_{\text{id}} P_{\text{id}} & \xrightarrow{\iota_P} & \text{Loc}(P)
\end{array}
\]

i.e. the diagram displays four adjoint pairs of triangle functors, the composite $\pi_P \lambda_P$ is zero, all functors into $\mathcal{D}(A)$ are fully faithful, and for all $X \in \mathcal{D}(A)$ we have triangles

\[
\begin{align*}
\lambda_P \rho_P X & \xrightarrow{\varepsilon} X \\
\iota_P \pi_P X & \xrightarrow{\eta} \Sigma(\lambda_P \rho_P X),
\end{align*}
\]

\[
\begin{align*}
\iota_P \kappa_P X & \xrightarrow{\varepsilon'} X \\
\mu_P \rho_P X & \xrightarrow{\eta'} \Sigma(\iota_P \kappa_P X),
\end{align*}
\]

where $\eta$ and $\eta'$ (resp. $\varepsilon$ and $\varepsilon'$) are units (resp. co-units) of the adjunctions.

2. Let $T^+_P$ be the Bongartz completion of $P$ as a two-term presilting object in per($A$). Then $\pi_P T^+_P \simeq \pi_P A$ in $P_{\perp_{\text{id}}}$.

3. [BH08 Proposition 3] We have a triangle equivalence $\mathbb{R} \text{Hom}_A(\pi_P, -) : P_{\perp_{\text{id}}} \xrightarrow{\simeq} \mathcal{D}(C_P)$, where $C_P$ is the dg endomorphism algebra $\mathcal{E}nd_A(\pi_P A)$. Moreover, the dg $k$-algebra $C_P$ is non-positive.

4. The composite of $\pi_P$ with the equivalence in (2) restricts to a functor $\pi_P : \text{per}(A) \rightarrow \text{per}(C_P)$, which is a Verdier localisation functor with respect to thick($P$). In particular, we have triangle...
equivalence

\[ \text{per}(A)/\text{thick}(P) \simeq \text{per}(C_P). \]

(5) The functor \( \mathcal{D}(C_P) \simeq P^{\perp_{\mathbb{Z}}} \to \mathcal{D}(A) \) preserves boundedness, i.e. it restricts to an embedding \( \mathcal{D}_{\text{id}}(C_P) \hookrightarrow \mathcal{D}_{\text{id}}(A). \)

Proof. The assertion in (1) is taken directly from the cited reference. We note that \( \text{Loc}(P)^{\perp_{\mathbb{Z}}} = P^{\perp_{\mathbb{Z}}}; \) the left hand side is contained in the right hand side since \( \text{Loc}(P) \supseteq P[i] \) for all \( i \in \mathbb{Z}, \) and the right hand side is contained in the left since \( \text{Loc}(P) \) can be built from \( \text{add}(P) \) using suspensions, desuspensions, extensions, and coproducts. The functors \( \iota_P \) and \( \lambda_P \) are naturally isomorphic to the inclusion functors, and we will simply treat them as inclusions for the remainder of this proof.

Assertion (2) now easily follows from (1), as we now show. Recall that the Bongartz completion \( T_P^+ \) is given by \( P \oplus Q^+ \), where \( Q^+ \) is defined by the triangle (see (1.C))

\[
A \to Q^+ \to \rho_P T_P^+ \to \Sigma A,
\]

in which \( \beta_{\Sigma A} \) is a right \( \text{add}(P) \)-approximation. Since the functor \( \varpi_P \) sends \( P \) to a zero object, applying \( \varpi_P \) to the triangle above thus gives that \( \varpi_P T_P^+ \simeq \varpi_P Q^+ \simeq \varpi_P A \) in \( P^{\perp_{\mathbb{Z}}}. \)

By (1), we may identify \( P^{\perp_{\mathbb{Z}}} \) with the Verdier quotient \( \mathcal{D}_{\text{id}}(A)/\text{Loc}(P). \) We may now point to the cited reference to prove the first claim in (3). The non-positivity of \( C_P \) will be shown by building on Brüning–Huber’s proof under our assumptions. For any \( i > 0, \) Brüning–Huber show that the \( k \)-linear map

\[
\text{Hom}_{P^{\perp_{\mathbb{Z}}}}(\varpi_P A, \Sigma^i \varpi_P A) \xrightarrow{\text{R Hom}_A(\varpi_P \cdot \cdot \cdot)} \text{Hom}_{\mathcal{D}(C_P)}(C_P, \Sigma^i C_P) \simeq H^i C_P,
\]

is an isomorphism, whence it suffices to show that \( \text{Hom}_{P^{\perp_{\mathbb{Z}}}}(\varpi_P A, \Sigma^i \varpi_P A) \) vanishes for all \( i > 0. \) By (2) above, it is equivalent to show that \( \text{Hom}_{P^{\perp_{\mathbb{Z}}}}(\varpi_P T_P^+, \Sigma^i \varpi_P T_P^+) \) vanishes for all \( i > 0. \) By (1) above, the pair \( (\text{Loc}(P), P^{\perp_{\mathbb{Z}}}) \) forms a t-structure on \( \mathcal{D}(A). \) The truncation triangle of \( T_P^+ \) in this t-structure is displayed along the first row below, and the second row is the truncation triangle of \( \Sigma^i T_P^+, \) for some \( i > 0. \)

\[
\begin{array}{ccc}
\rho_P T_P^+ & \xrightarrow{\varepsilon} & T_P^+ \\
& \xrightarrow{\eta} & \varpi_P T_P^+ \\
\Sigma^i \rho_P T_P^+ & \xrightarrow{\Sigma^i \varepsilon} & \Sigma^i T_P^+ \\
\end{array}
\]

\[
\Sigma^i \rho_P T_P^+ \xrightarrow{\Sigma^i \eta} \Sigma^i \varpi_P T_P^+ \xrightarrow{\Sigma^i \theta} \Sigma^i \rho_P T_P^+
\]

(2.S)

In this diagram, the vertical morphism \( x \) is arbitrary, and it is to be shown that \( x = 0. \)

It will be useful to show that \( \text{Hom}_{\mathcal{D}(A)}(T_P^+, \Sigma^{i+1} \rho_P T_P^+) = 0 \) for all \( i > 0. \) The object \( P \) generates a silting t-structure \( (P^{\perp_{\mathbb{Z}}} \cap \text{Loc}(P), P^{\perp_{\mathbb{Z}}} \cap \text{Loc}(P)) \) on \( \text{Loc}(P). \) The octahedral axiom gives a diagram where the long rows and columns are triangles, and the second column is a truncation triangle in
Arguing vertically, the object \( N \) is in \( P^\perp > 0 \), since it is an extension of objects in \( P^\perp \). A horizontal argument shows that \( N \) is also in \( P^\perp \leq 0 \), so \( N \) is in \( P^\perp Z \). This makes the third column a triangle where the first term is in \( \text{Loc}(P) \) and the third term is in \( P^\perp Z \). By the uniqueness of truncation triangles in the t-structure \( (\text{Loc}(P), P^\perp Z) \) (in the sense of Proposition 1.4(3)), it follows that \( \rho P T^+_p \cong \sigma^0 \rho P T^+_p \), putting \( \rho P T^+_p \) in \( P^\perp > 0 \), which shows that \( \text{Hom}_{D(A)}(T^+_p, \Sigma^i \rho P T^+_p) = 0 \) for all \( i > 0 \).

We now return to showing that \( x \) in (2.S) is the zero morphism, so that we can finish the proof of (3). Since \( \omega P \) is left adjoint to the inclusion functor \( \iota P \), the morphism \( \eta \) induces an isomorphism

\[ \text{Hom}_{D(A)}(T^+_p, \Sigma^i \omega P T^+_p) \cong \sigma^0 \text{Hom}_{D(A)}(\omega P T^+_p, \Sigma^i \omega P T^+_p). \]

By the previous paragraph, the composite \( (\Sigma^i \partial) \circ x \circ \eta \) vanishes. We can thus factor \( x \circ \eta \) through \( \Sigma^i \eta \), but since \( T^+_p \) is silting, this implies that \( x \circ \eta \) vanishes. The morphism \( x \) now factors through \( \partial \), but since there are no non-trivial morphism from \( \text{Loc}(P) \) to \( P^\perp \), we have shown that \( x \) vanishes, as desired.

We move on to (4). The assertions in (1) and (3) yield a triangle equivalence

\[ D(A)/\text{Loc}(P) \sim D(C_P). \]  

(2.T)

Since the right adjoint \( \iota P \) preserves coproducts, it is indeed the case that the left adjoint \( \omega P \) preserves compactness \[ \text{Nee96, Theorem 5.1} \]. The subcategory of compact objects in \( D(B) \) is \text{thick}(B) = \text{per}(B), for any dg algebra \( B \) \[ \text{Kel94, §5.3} \]. The composite functor

\[ \mathbb{R} \text{Hom}_{\Lambda}(\omega P \cdot, -) \circ \omega P \colon \text{per}(A) \longrightarrow D(C_P) \]

thus takes values in \( \text{per}(C_P) \), so we can define a triangle functor

\[ \pi_P \colon \text{per}(A) \longrightarrow \text{per}(C_P), \]  

(2.U)

by restriction, in turn inducing a triangle functor

\[ \text{per}(A)/\text{thick}(P) \longrightarrow \text{per}(C_P). \]

This last functor can be obtained by restricting the equivalence in (2.T) to compact objects. It is thus an equivalence, thanks to the idempotent completeness of \( D(C_P) \) \[ \text{Nee92, Theorem 2.1} \]. This proves (4).

Right adjoints between derived categories of dg algebras preserve boundedness if the left adjoint preserves compactness \[ \text{GP18, Lemma 4.2} \]. Since \( \omega P \) preserves compactness, the last assertion (5) now follows from (4).

□
Corollary 2.11. Let $A$ and $P$ be as in Setup 2.5 and let $C_P$ as in Proposition 2.10(3). Then there is a triangle equivalence $P_{\text{id}}^{1+\varepsilon} \simeq \mathcal{D}_{\text{id}}(C_P)$.

Proof. Proposition 2.10(3) provides an equivalence from the perpendicular subcategory $P^{1+\varepsilon}$ of $\mathcal{D}(A)$ to $\mathcal{D}(C_P)$, sending $\varpi_P A$ to $C_P$. In order to prove that this equivalence restricts to the bounded derived categories, it remains to show that the vector space $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(A)}(A, \Sigma^{i} \nu P X)$ is finite dimensional precisely when $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{P^{1+\varepsilon}}(\varpi_P A, \Sigma^{i} X)$ is finite dimensional, where $X \in P^{1+\varepsilon}$, but this follows directly from Proposition 2.10(1), or more specifically the fact that $(\varpi_P, \nu P)$ is an adjoint pair of triangle functors. □

Corollary 2.12. Let $A$ and $P$ be as in Setup 2.5. The functor $\pi_P : \text{per}(A) \longrightarrow \text{per}(C_P)$ from Proposition 2.10(4) induces a bijection

$$\pi_P : 2\text{-presilt}_P(A) \longrightarrow 2\text{-presilt}(C_P).$$

It fits into a commutative diagram of bijections:

$$
\begin{array}{ccc}
2\text{-presilt}_P(A) & \xrightarrow{\varphi_P} & 2_{P_P}^+\text{-presilt}(Z_P/|P|) \\
\downarrow{\pi_P} & & \downarrow{\zeta_P} \\
2\text{-presilt}(C_P) & & \\
\end{array}
$$

(2.V)

where $\zeta_P$ is induced by a triangle equivalence $Z_P/|P| \xrightarrow{\zeta_P} \text{per}(C_P)$ sending $T^+_P$ to $C_P$.

Proof. We have a triangle equivalence $\text{per}(A)/\text{thick}(P) \sim \text{per}(C_P)$, sending $A$ to $C_P$. By Proposition 2.10(2) and (3), it also sends the Bongartz completion $T^+_P$ to $C_P$. Using Theorem 2.1(1), one constructs a triangle equivalence $Z_P/|P| \xrightarrow{\zeta_P} \text{per}(C_P)$ sending $T^+_P$ to $C_P$. Since we can regard the functor $\pi_P$ as a Verdier localisation, we deduce from Theorem 2.1(2) that it indeed induces a bijection

$$2\text{-presilt}_P(A) \xrightarrow{\varphi_P} 2_{P_P}^+\text{-presilt}(Z_P/|P|) \xrightarrow{\zeta_P} 2\text{-presilt}(C_P),$$

as claimed, and the commutativity of (2.V) is immediate. □

Corollary 2.13. Let $A$ and $P$ be as in Setup 2.5, and let $Q \in 2\text{-presilt}(C_P)$. Consider the triangle functors

$$\begin{align*}
\pi_P & : \text{per}(A) \longrightarrow \text{per}(C_P) \\
\pi_Q & : \text{per}(C_P) \longrightarrow \text{per}(C_{P,Q}) \\
\pi_{\pi_P^{-1}(Q)} & : \text{per}(A) \longrightarrow \text{per}(C_{\pi_P^{-1}(Q)})
\end{align*}$$

provided by Proposition 2.10(4). There is a triangle equivalence $\text{per}(C_{P,Q}) \longrightarrow \text{per}(C_{\pi_P^{-1}(Q)})$, which we may use to identify the composite $\pi_Q \circ \pi_P$ with $\pi_{\pi_P^{-1}(Q)}$ up to natural isomorphism. Consequently, we
have that the bijections

\[ \pi_P : 2\text{-presilt}_P(A) \rightarrow 2\text{-presilt}(C_P) \]

\[ \pi_Q : 2\text{-presilt}_Q(C_P) \rightarrow 2\text{-presilt}(C_{P,Q}) \]

\[ \pi_{\pi_P^{-1}(Q)} : 2\text{-presilt}_{\pi_P^{-1}(Q)}(A) \rightarrow 2\text{-presilt}(C_{\pi_P^{-1}(Q)}) \]

provided by Corollary 2.12, satisfy \( \pi_Q \circ \pi_P = \pi_{\pi_P^{-1}(Q)} \).

**Proof.** Consider the diagram of triangle functors, where the horizontal functors are fully faithful and the vertical functors are Verdier localisations:

\[
\begin{array}{ccc}
\text{thick}(P) & \rightarrow & \text{thick}(\pi_P^{-1}(Q)) \\
\downarrow & & \downarrow \pi_P \\
\text{thick}(Q) & \rightarrow & \text{per}(C_P) \\
\downarrow \pi_Q & & \downarrow \pi_{\pi_P^{-1}(Q)} \\
\text{per}(C_{P,Q}) & & \text{per}(C_{\pi_P^{-1}(Q)})
\end{array}
\]

The universal property of Verdier localisations now induces a triangle equivalence \( \text{per}(C_{P,Q}) \rightarrow \text{per}(C_{\pi_P^{-1}(Q)}) \). It follows that \( \pi_{\pi_P^{-1}(Q)} \) can be identified with \( \pi_Q \circ \pi_P \) up to natural isomorphism, as shown on the right in our diagram. \( \square \)

3. Compatibility of the Iyama–Yang and Buan–Marsh bijections

Reduction techniques have been developed for both silting objects (known as silting reduction [IY18]) and support \( \tau \)-tilting modules (\( \tau \)-tilting reduction [Jas15]), and they are compatible [Jas15, Theorem 4.12(b)]. In this section we generalise Jasso’s compatibility theorem, by showing that the support \( \tau \)-tilting reduction of Buan–Marsh [BM21, Section 3] is compatible with silting reduction.

Buan–Marsh prove the following.

**Theorem 3.1** ([BM21, Theorem 3.6]). Let \( \Lambda \) be a finite dimensional \( k \)-algebra, and let \( (M, Q) \) be a \( \tau \)-rigid pair in \( \text{mod} \, \Lambda \). There is a bijection

\[ \mathcal{E}_{(M,Q)} : \tau\text{-rigid pair}_{(M,Q)}(\Lambda) \rightarrow \tau\text{-rigid pair}(J(M,Q)), \]

where \( J(M, Q) \) is the \( \tau \)-perpendicular category of \( (M, Q) \).

We now recall how the map \( \mathcal{E}_{(M,Q)} \) is constructed. For a \( \tau \)-rigid pair \( (M, Q) \), let \( \text{ind}_{(M,Q)} \tau\text{-rigid pair}(\Lambda) \) denote set of \( \tau \)-rigid pairs in \( \text{mod} \, \Lambda \) of the form \( (M, Q) \oplus (N, R) \), where \( (N, R) \) is indecomposable and not a direct summand of \( (M, Q) \). Buan–Marsh first address the cases where \( (M, Q) \) is either of the form \( (M, 0) \) or \( (0, Q) \). In each of the five cases below, they define a map between sets of indecomposable objects

\[ \mathcal{E}_{(M,Q)} : \text{ind}_{(M,Q)} \tau\text{-rigid pair}(\Lambda) \rightarrow \text{ind} \tau\text{-rigid pair}(J(M,Q)), \]

which is extended to a in the obvious way. We will then consider the general case, where \( (M, Q) \) is arbitrary, which will yield the bijection in Theorem 3.1.
For a \( \tau \)-rigid \( \Lambda \)-module \( X \), we denote by \( P_X \) its minimal projective presentation, considered as a two-term presilting object in \( \mathcal{K}^b(\text{proj} \Lambda) \). This is to say that one sets \( P_X \overset{\text{def}}{=} H^{-1}_\Lambda(X,0) \), where \( H_\Lambda \) is as defined in Theorem 1.17.

**Case I:** Suppose that \( Q = 0 \). We define \( E_{(M,0)}(M,0) \overset{\text{def}}{=} (0,0) \).

**Case I(a):** If \( N \) is an indecomposable \( \Lambda \)-module such that \( M \oplus N \) is \( \tau \)-rigid and \( N \not\in \text{gen}(M) \), define \( E_{(M,0)}(M \oplus N,0) \overset{\text{def}}{=} (f_M(N),0) \), where \( f_M : \text{mod} \Lambda \rightarrow M^+ \) is the torsion-free functor for the torsion pair \((\text{gen}(M),M^+)\) (see Theorem 1.15), that is, the natural functor \( \text{mod} \Lambda \rightarrow M^+ \) (cf. the paragraph surrounding (1.D)).

**Case I(b):** If \( N \) is an indecomposable \( \Lambda \)-module such that \( M \oplus N \) is \( \tau \)-rigid and \( N \in \text{gen}(M) \), define \( E_{(M,0)}(M \oplus N,0) \overset{\text{def}}{=} (0,f_M H^0 P_N(-1)) \), where

\[
P_N(-1) \xrightarrow{P_N} (P_M)_N \xrightarrow{\beta_{P_N}} P_N \xrightarrow{\Sigma P_N} (-1)
\]

is a distinguished triangle in \( \mathcal{K}^b(\text{proj} \Lambda) \) and \( \beta_{P_N} \) is a minimal right \( \text{add}(P_M) \)-approximation.

**Case I(c):** If \( R \) is an indecomposable projective \( \Lambda \)-module such that \( \text{Hom}_\Lambda(R,M) = 0 \), define

\[
E_{(M,0)}(M,R) \overset{\text{def}}{=} (0,f_M H^0((\Sigma R)(-1)));
\]

where

\[
(\Sigma R)(-1) \xrightarrow{(P_M)_{\Sigma R}} (P_M)_{\Sigma R} \xrightarrow{\beta_{\Sigma R}} \Sigma R \xrightarrow{\Sigma(\Sigma R)(-1)}
\]

is a distinguished triangle in \( \mathcal{K}^b(\text{proj} \Lambda) \) and \( \beta_{\Sigma R} \) is a minimal right \( \text{add}(P_M) \)-approximation. Note that the last term in this triangle need not be isomorphic to \( \Sigma^2 R(-1) \); we have chosen our notation to be consistent with (2.A).

**Case II:** Suppose that \( M = 0 \). We define \( E_{(0,Q)}(0,Q) \overset{\text{def}}{=} (0,0) \).

**Case II(a):** If \( N \) is an indecomposable \( \tau \)-rigid \( \Lambda \)-module such that \( \text{Hom}_\Lambda(Q,N) = 0 \), define

\[
E_{(0,Q)}(N,Q) \overset{\text{def}}{=} (N,0).
\]

**Case II(b):** If \( R \) is an indecomposable projective \( \Lambda \)-module such that \( (0,R) \in \text{ind}_{(0,Q)} \tau \)-rigid pair(\( \Lambda \)), define \( E_{(0,Q)}(0,Q \oplus R) \overset{\text{def}}{=} (0,f_Q(R)) \).

**In general:** Let \((M,Q)\) be a \( \tau \)-rigid pair in \( \text{mod} \Lambda \). Let \((M^b,0)\) be the Bongartz completion of \((M,0)\) and \((M^+,0)\) be the Bongartz completion of \((M,Q)\). Let \( \Gamma \overset{\text{def}}{=} \text{End}(M^b)/[M] \) be the \( \tau \)-tilting reduction of \( \Lambda \) with respect to \((M,0)\) and \( \Gamma = \text{End}(M^+)/[M] \) the \( \tau \)-tilting reduction of \( \Lambda \) with respect to \((M,Q)\). Case I above gives a bijection

\[
E_{(M,0)} : \tau \text{-rigid pair}_{(M,0)}(\Lambda) \rightarrow \tau \text{-rigid pair } J(M,0).
\]

It sends the pair \((0,Q)\) to \((0,Q)\), where \( Q = f_M H^0((\Sigma Q)(-1)) \) (see Case I(c)), whence we can restrict to a bijection

\[
E_{(M,0)} : \tau \text{-rigid pair}_{(M,Q)}(\Lambda) \rightarrow \tau \text{-rigid pair }_{(0,Q)} J(M,0).
\]

Restricting further to support \( \tau \)-tilting pairs gives a bijection

\[
E_{(M,0)} : s\tau \text{-tilt pair}_{(M,Q)}(\Lambda) \rightarrow s\tau \text{-tilt pair }_{(0,Q)} J(M,0). \tag{3.A}
\]
This is an isomorphism of posets; if \((N_1, R_1)\) and \((N_2, R_2)\) are elements in the domain such that \(\text{gen}(N_1) \supseteq \text{gen}(N_2)\), one checks that \(\text{gen}(\mathcal{E}(M, 0) N_1) \supseteq \text{gen}(\mathcal{E}(M, 0) N_2)\). This is a trivial matter for cases I(b) and I(c), and for I(a) this follows from the fact that the torsion-free functor \(f_M\) preserves epimorphisms.

By Proposition-Definition 1.19 we have an exact equivalence

\[
F_{(M, 0)} = \text{Hom}_\Lambda(M^b, -): J(M, 0) \longrightarrow \text{mod } \Gamma'.
\]

It induces a bijection (see (1.F))

\[
F_{(M, 0)}: \tau\text{-rigid pair}(J(M, 0)) \longrightarrow \tau\text{-rigid pair}(\Gamma'),
\]

which restricts to a bijection

\[
F_{(M, 0)}: \tau\text{-rigid pair}_{(0, \tilde{Q})}(J(M, 0)) \longrightarrow \tau\text{-rigid pair}_{(0, Q')}(\Gamma').
\]

where \(Q'\) is the projective \(\Gamma'\)-module \(\text{Hom}_\Lambda(M^b, Q)\). Applying Case II above to the \(k\)-algebra \(\Gamma'\) gives a bijection

\[
\mathcal{E}^{C'}_{(0, Q')}: \text{rigid pair}_{(0, Q')}(\Gamma') \longrightarrow \text{rigid pair } J_{\Gamma'}(0, Q'),
\]

where \(J_{\Gamma'}(0, Q')\) is the \(\tau\)-perpendicular category of \((0, Q')\) in \(\text{mod } \Gamma'\). Proposition-Definition 1.19 gives a bijection

\[
F^{\Gamma'}_{(0, Q')}: \text{rigid pair } J_{\Gamma'}(0, Q') \longrightarrow \text{rigid pair } \Delta,
\]

where \(\Delta\) is the \(k\)-algebra \(\text{End}_{\Gamma'}(B)\), and \((B, Q')\) is the Bongartz completion of \((0, Q')\). Since the bijection in (3.A) is an isomorphism of posets, it sends the maximal element to the maximal element, whence

\[
\Delta = \text{End}_{\Gamma'}(B) = \text{End}_{\text{End}_\Lambda(M^b, M^+)}(\text{Hom}_\Lambda(M^b, M^+)) \simeq \text{End}(M^+)/[M] = \Gamma.
\]

In the following, we insert \(\Gamma\) in the place of \(\Delta\) in (3.C). We define

\[
\mathcal{E}_{(M, Q)}(X, R) \overset{\text{def}}{=} F^{-1}_{(M, Q)} \circ F^{\Gamma'}_{(0, Q')} \circ \mathcal{E}^{C'}_{(0, Q')} \circ F_{(M, 0)} \circ \mathcal{E}(M, 0),
\]

as displayed on the right in (3.D) below.

This concludes our recollection of Buan–Marsh’s bijection \(\mathcal{E}_{(M, Q)}\).
Our aim is to link $E_{(M,Q)}$ to Iyama–Yang silting reduction. This is achievable in the following setup.

**Setup 3.2.** A Hom-finite Krull–Schmidt triangulated category $C$ is fixed, as well as a silting object $S \in C$. Let basic $P$ be a $2S$-term presilting object in $C$, let $\Lambda$ denote the endomorphism algebra $\text{End}_C(S)$, and let $\Gamma = \text{End}_{Z_P/[P]}(T_P^+)$, where $T_P^+$ is the Bongartz completion of $P$.

Recall that $\Gamma$ is isomorphic to the $\tau$-tilting reduction of the $\tau$-rigid pair $H_A(M,Q)$ in $\text{mod } \Lambda$ [Jas15, Theorem 4.12(a)], where $H_A$ is as defined in Theorem [1.17]. This fact will be used implicitly in our proofs below. In order to establish the compatibility of the two reduction techniques described above, we prove two important lemmas.

**Lemma 3.3.** Let $C$, $S$, $P$, and $\Lambda$ be as in Setup 3.2. For all $X \in \text{add}(S) + \Sigma \text{add}(S)$, we have a natural isomorphism $\frac{C}{[P]}(S,X) \cong f_{C(S,P)}(S,X)$, as $\Lambda$-modules, where $f_{C(S,P)} : \text{mod } \Lambda \to C(S,P)^\perp$ is the torsion-free functor for the torsion pair $(\text{gen}(C(S,P)), C(S,P)^\perp)$.

**Proof.** We have an exact sequence

$$[P](S, X) \ grp C(S, X) \ grp \frac{C}{[P]}(S, X) \quad (3.1.2)$$

of $\Lambda$-modules, where $[P]$ is the ideal of $C$ consisting of morphisms factoring through $\text{add}(P)$. It suffices to show that $[P](S, X) \in \text{gen}(C(S, P))$ and that $\frac{C}{[P]}(S, X) \in C(S, P)^\perp$.

Let $P' \ grp X$ be a right $\text{add}(P)$-approximation of $X$. By definition, the map

$$C(S,P') \ grp \ x_{\biggr|_{\mathbb{S},P'}} \ grp [P](S, X)$$

surjects, whence $[P](S, X) \in \text{gen}(C(S, P))$.

We now show that $\text{Hom}_\Lambda(C(S,P), \frac{C}{[P]}(S, X)) = 0$. This $k$-vector space appears as the third term in the following long exact sequence:

$$0 \ grp \text{Hom}_\Lambda(C(S,P), [P](S, X)) \ grp \text{Hom}_\Lambda(C(S,P), C(S,X)) \ grp \text{Hom}_\Lambda(C(S,P), \frac{C}{[P]}(S, X))$$

$$\text{Ext}^1_\Lambda(C(S,P), [P](S, X)).$$

Having already shown that $[P](S, X) \in \text{gen}(C(S, P))$, a result of Auslander–Smalø [ASS11, Proposition 5.8] asserts that the $\tau$-rigidity of $C(S,P)$ implies the vanishing of $\text{Ext}^1_\Lambda(C(S,P), [P](S, X))$. Thus it suffices to show that $i \ o \ -$ surjects. We will conclude the proof by showing that

$$\text{Hom}_\Lambda(C(S,P), C(S,X)) = \text{Hom}_\Lambda(C(S,P), [P](S, X)),$$

which would make $i \ o \ -$ an injective endomorphism of a finite dimensional $k$-vector space, and thus surjective. It follows from Theorem [1.16] that all $\Lambda$-homomorphisms from $C(S,P)$ to $C(S,X)$ are determined by an equivalence class of morphisms $P \ grp X$. Consequently, all homomorphisms in $\text{Hom}_\Lambda(C(S,P), C(S,X))$ have image in $[P](S, X)$, and $i \ o \ -$ surjects. 

$\square$
Lemma 3.4. Let $C$, $S$, $P$, $Λ$, and $Γ$ be as in Setup 3.2. Let $(M,Q) = H_S(P)$ (where $H_S$ is as defined in Theorem 1.17). Consider the equivalence $H'_S$ defined as the composite

$$
(\text{add}(T^+_P) \ast \text{add}(T^+_P)(1)) \mod Γ \rightarrow J(H_S(P)),
$$

(see Theorem 1.16 and Proposition-Definition 1.19.). Then the resulting composite bijection

$$
2_{T^+_P} \text{-presilt}(Z_P/[P]) \xrightarrow{H'_S} \tau \text{-rigid pair } Γ \xrightarrow{F^{-1}_{H_S(P)}} \tau \text{-rigid pair } J(H_S(P))
$$

(see Theorem 1.14 and (1.F)) can be expressed as follows:

$$
2_{T^+_P} \text{-presilt}(Z_P/[P]) \xrightarrow{H'_S} \tau \text{-rigid pair } J(H_S(P)) \xrightarrow{\psi} Y \xrightarrow{\psi} (f_{C(S,P)}C(S,Y), f_{C(S,P)}C(S,Y_1(-1))).
$$

where $Y_1$ is the largest direct summand of $Y$ in $T^+_P(1)$, and $(1)$ is defined dually to (2.A).

Proof. Fixing $Y \in 2_{T^+_P} \text{-presilt}(Z_P/[P])$, we first show that the $Γ$-module $F_{H_S(P)}f_{C(S,P)}C(S,Y)$ is isomorphic to $\frac{C(S,P)}{\Gamma}(T^+_P,Y)$, yielding the claimed re-expression of the first component. In order to use the definition of $F_{H_S(P)}$ provided in Proposition-Definition 1.19 we note that if $(M^+,Q)$ is the Bongartz completion of $H_S(P)$, then $M^+ \cong C(S,T^+_P)$. We get

$$
F_{H_S(P)}f_{C(S,P)}C(S,Y) = \text{Hom}_A \left( C(S,T^+_P), f_{C(S,P)}C(S,Y) \right).
$$

A result of Jasso [Jas15, Proposition 4.15] proves that the right hand side is isomorphic to $\frac{C(S,P)}{\Gamma}(T^+_P,Y)$. Since we have arrived where we desired, we conclude that the first component can be expressed as claimed.

It follows from the definition of $(-1)$ in (2.A) that $Y_1(-1) \in 2_{T^+_P} \text{-presilt}(Z_P/[P])$. One obtains an expression for the second component by repeating the reasoning in the previous paragraph, replacing $Y$ with $Y_1(-1)$. Thus, the second component also is as claimed, and the proof is complete. □

We now have all ingredients to prove the main theorem of this section.

Theorem 3.5. Let $C$, $S$, $P$, $Λ$, and $Γ$ be as in Setup 3.2. We have a commutative diagram of bijections

$$
2_S \text{-presilt}_P(C) \rightarrow H_S \rightarrow \tau \text{-rigid pair } H_S(P) \langle Λ \rangle \xrightarrow{ϕ_P} \tau \text{-rigid pair } J(H_S(P)) \xrightarrow{ψ_{H_S(P)}} \tau \text{-rigid pair } J(H_S(P))
$$

where $ψ_{H_S(P)} \overset{\text{def}}{=} F_{H_S(P)} \circ ϵ_{H_S(P)}$.

Proof. The commutativity of the lower triangle was shown in Lemma 3.4, and the right triangle (with $ψ_{H_S(P)}$ along the diagonal) commutes by definition. What remains is proving that

$$
H'_S \circ ϕ_P(X) = ϵ_{H_S(P)} \circ H_S(X),
$$

(3.G)
which should hold for all $X \in 2_S$-presilt$_P(C)$. All maps in question are constructed to distribute over direct sums, whence it suffices to consider the case where $X$ is of the form $P \oplus X'$ for some indecomposable object $X'$ which is not in add$(P)$. We treat five cases, and a general case, corresponding to the definition of $\mathcal{E}_{H_S(P)}$ we reviewed in the discussion following Theorem 3.1. We note that $\varphi_P(X) = X'$ in $\mathbb{Z}_P/[P]$.

Case I: Suppose that $P$ has no non-zero direct summands in $\Sigma S$.

Case I(a). Suppose that $X' \in 2_S$-presilt$_P(C)$ has no non-zero direct summand in $\Sigma S$ and $C(S, X') \not\in \text{gen}(C(S, P))$.

Using the definitions of $H_S$ and $\mathcal{E}_{H_S(P)}$, the right hand side of (3.4) becomes

$$\mathcal{E}_{H_S(P)} \circ H_S(X) = \mathcal{E}_{H_S(P)}(C(S, X), 0) = (f_{C(S, P)}C(S, X'), 0).$$

Using Lemma 3.4 and that $\varphi_P(X) = X'$ in $\mathbb{Z}_P/[P]$, we re-express the left hand side of (3.4) as follows:

$$H'_S \circ \varphi_P(X) = (f_{C(S, P)}C(S, X'), f_{C(S, P)}C(S, X'_1(-1))),$$

where $X'_1$ is the largest direct summand of $X'$ in $T^+_P(1)$, considered as an object in $\mathbb{Z}_P/[P]$. It now suffices to show that $f_{C(S, P)}C(S, X'_1) = 0$, or indeed that $f_{C(S, P)}C(S, T^+_P(1)) = 0$. By Lemma 3.3 one can proceed by showing that $\frac{C(S, T^+_P(1))}{C(S, T^+_P(1))} = 0$. By the definition of (1), we have a triangle

$$T^+_P \xrightarrow{\alpha} P' \xrightarrow{1} T^+_P(1) \xrightarrow{1} T^+_P[1],$$

in which $\alpha$ is a left add$(P)$-approximation. Applying the functor $C(S, -)$ yields a long-exact sequence

$$C(S, T^+_P) \xrightarrow{\alpha} C(S, P') \xrightarrow{1} C(S, T^+_P(1)) \xrightarrow{1} C(S, T^+_P[1]) = 0.$$

The vanishing of the last term proves that every morphism $S \rightarrow T^+_P(1)$ factors through add$(P)$. We conclude that $\frac{C(S, T^+_P(1))}{C(S, T^+_P(1))} = 0$, and in turn that (3.4) holds in this case.

Case I(b): When addressing the present case, as well as I(c) and II(b), we will use the equality obtained in (3.4) below. In these three cases, we assume that $C(S, X) \in \text{gen}(C(S, P))$, or equivalently that $f_{C(S, P)}C(S, X) = 0$. Under this assumption, we have $f_{C(S, P)}C(S, X') = 0$, and since $\varphi_P(X) = X'$ in $\mathbb{Z}_P/[P]$, it follows from Lemma 3.4 that

$$H'_S \circ \varphi_P(X) = (f_{C(S, P)}C(S, X'), f_{C(S, P)}C(S, X'_1(-1))) = (0, f_{C(S, P)}C(S, X'_1(-1))),$$

(3.1)

Suppose that $X' \in 2_S$-presilt$_P(C)$ has no non-zero direct summands in $\Sigma S$, but $C(S, X') \in \text{gen}(C(S, P))$.

It follows from the definition of $\mathcal{E}_{H_S(P)}$ that $\mathcal{E}_{H_S(P)}H_S(X) = (0, f_{C(S, P)}C(S, X'(-1)))$. Since $C(S, X) \in \text{gen}(C(S, P))$, we have that $f_{C(S, P)}C(S, X) = 0$. By (3.1) we have that

$$H'_S \circ \varphi_P(X) = (0, f_{C(S, P)}C(S, X'(-1))).$$

This shows that (3.4) holds.

Case I(c): Suppose that $X = P \oplus \Sigma Q$, where $Q \in \text{ind add}(S)$. Then

$$\mathcal{E}_{H_S(P)}H_S(X) = (0, f_{C(S, P)}C(S, (\Sigma Q)(-1))).$$

Since $C(S, X) = 0 \in \text{gen}(C(S, P))$, we have that (3.4) holds as a consequence of (3.1).

Case II: Suppose that $P \in \text{add}(\Sigma S)$.

Case II(a): If $X'$ has no non-zero direct summands in $\Sigma S$, one uses Lemma 3.4 to assert that

$$H'_S \circ \varphi_P(X) = (f_{C(S, P)}C(S, X'), f_{C(S, P)}C(S, X'_1(-1))).$$
There are no non-zero morphisms from $S$ to $P$, whence $f_{C(S,P)}$ is the identity functor, and we may simplify the right hand side above to $(C(S,X'), C(S, X'_1(-1)))$.

We show that the second component $C(S, X'_1(-1))$ vanishes. The Bongartz completion of $P$ is given by $T^+_P = P \oplus Q^+$, where $Q^+$ is not in $\text{add}(\Sigma S)$. Since $P \in \text{add}(\Sigma S)$, one uses the definition of (1) (see (3.H)) to show that $T^+_P(1) \cong P \oplus Q^+[1]$ in $Z_P/[P]$. It is now clear that $X'_1 \in Z_P/[P]$ does not have a direct summand in $T^+_P(1)$, so $X'_1 = 0$ and $C(S, X'_1(-1)) = 0$. We deduce that

$$H^*_S \circ \varphi_P(X) = (C(S, X'), 0).$$

Since $E_{H_S(P)}H_S(X) = (C(S, X'), 0)$ in this case, we have shown that (3.G) holds.

**Case II(b):** If $X' = \Sigma R$, where $R \in \text{ind}\text{add}(S)$, then $E_{H_S(P)}H_S(X) = (0, f_{C(S,Q)}(C(S, (\Sigma R)(-1))))$. Since $C(S, X) = 0$, we have that $f_{C(S,P)}C(S, X) = 0$. Thus (3.G) holds by (3.I).

**In general:** We can find a decomposition $P = P' \oplus \Sigma P''$, where $\Sigma P''$ is the largest direct summand of $P$ in $\Sigma S$. Let $(M, Q) = H_S(P)$. The Bongartz completion of $P$ is then of the form $T^+_P = \widehat{P'} \oplus \Sigma P''$, where $P'$ is a direct summand of $\widehat{P'}$. The goal is to show that the outer square in the following diagram commutes:

There are seven subdiagrams above, labeled 1 to 7. We complete the proof by showing that all these subdiagrams commute.

1: The upper square commutes, since one can apply Case I above to the $2_S$-term presilting object $P'$ in $C$. The map $H^*_S$ above is defined by $H^*_S$ in Lemma 3.4 but replacing $(M, Q)$ with $(M, 0)$.

The $\tau$-rigid pair $(0, \widehat{Q})$ is given by $E_{M, 0}(0, Q)$ (see Case I(c)).

2: The left triangle commutes as a result of Lemma 2.4.

3: Applying Lemma 3.4 but replacing $P$ with $P'$ and $(M, Q)$ by $(M, 0)$, shows this triangle to be commutative.

4: The hexagon on the right appears in (3.D), a diagram we have defined to be commutative. Recall that $Q'$ is the projective $\Gamma'$-module $\text{Hom}_A(M^b, Q)$. Moreover, we have an isomorphism of $k$-algebras $\Gamma'_n \cong \text{End}_{Z_{P'}}(T^+_P)$ and an isomorphism of $\Gamma'$-modules $Q' \cong \text{Hom}_{Z_{P'}}(T^+_P, P'')$. 

5: The inner square commutes, since one can apply Case II above to the 2\textsuperscript{\tau}\textsuperscript{\textit{+}}term presilting object \(\varphi P(\Sigma P')\) in \(ZP'/[P']\). We have identified the bottom left corner 2\textsuperscript{\tau}\textsuperscript{\textit{+}}presilt(\(Z\varphi P(\Sigma P')\)) with 2\textsuperscript{\tau}\textsuperscript{\textit{+}}presilt(\(Z\varphi P(\Sigma P')/\varphi P(\Sigma P')\)), where \(T_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+ \in Z\varphi P(\Sigma P') \subseteq ZP'/[P']\), under the equivalence \(Z_{\varphi P(\Sigma P')}/\varphi P(\Sigma P') ≃ ZP'/[P]\) (see (2.11) in the proof of Lemma 2.4).

6: As we just explained, the ideal quotient \(ZP'/[P]\), where \(ZP \subseteq C\), is equivalent to \(Z\varphi P(\Sigma P')/\varphi P(\Sigma P')\), where \(Z\varphi P(\Sigma P') \subseteq ZP'/[P']\). As we showed in the proof of Lemma 2.4 (see (2.11)) this equivalence sends \(T_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+\) to \(T_P^+\). Although the map \(H_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+\) formally has \(2\textsuperscript{\tau}\textsuperscript{\textit{+}}presilt(\(Z\varphi P(\Sigma P')/\varphi P(\Sigma P')\))\) as its domain, we may replace the domain with \(2\textsuperscript{\tau}\textsuperscript{\textit{+}}presilt(\(ZP'/[P]\))\) without altering the codomain or the expression of \(H_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+\). The commutativity of 6 now follows from the definition of \(H_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+\) in Lemma 3.4, applied to the presilting object \(\Sigma P'\) in \(ZP'/[P']\).

7: Before the proof can be completed, we point out that \(H_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+ = H_{T_P^+}\) as maps from \(2\textsuperscript{\tau}\textsuperscript{\textit{+}}presilt(\(ZP'/[P]\))\) to \(\tau\text{-rigid pair}(C)\). Indeed, by (2.11) in the proof of Lemma 2.4, the equivalence \(Z\varphi P(\Sigma P')/\varphi P(\Sigma P') ≃ ZP'/[P]\) sends \(T_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+\) to \(T_P^+\), so these objects can be regarded as isomorphic in \(ZP'/[P]\). The upper left map in 7 is thus \(H_{T_P^+}\), so 7 commutes thanks to the definition of \(H_{\varphi P(\Sigma P')\textsuperscript{\textit{+}}}^+\) in Lemma 3.4.

The proof is complete. □

We have accomplished the task of linking the Buan–Marsh bijection to that of Iyama–Yang.

4. \(\tau\text{-cluster morphism categories}

We are now ready to give a generalisation of \(\tau\text{-cluster morphism categories}. In this section, let \(A\) as in Setup 2.3, i.e. a non-positive dg \(k\)-algebra such that \(H^iA\) is finite dimensional for all \(i \in \mathbb{Z}\).

Given a two-term presilting object \(U \in \text{per}(A)\), we proved in Proposition 2.8(2) that the perpendicular category \(U\textsuperscript{\perp}@\) is \(t\)-exact in \(\mathcal{D}_{\text{fd}}(A)\). The \(\tau\text{-cluster morphism category should keep track of this information. To encode the information we want, there will be a morphism in the \(\tau\text{-cluster morphism category of the form}

\[
\mathcal{D}_{\text{fd}}(A) \xrightarrow{U} U\textsuperscript{\perp}@\text{fd}.
\]

By Proposition 2.10(3), there is a dg \(k\)-algebra \(C_U\) such that \(\text{per}(A)/\text{thick}(P) \simeq \text{per}(C_U)\). Given \(P \in 2\text{-presilt}(C_U)\), consider the functor

\[
\pi_U : \text{per}(A) \longrightarrow \text{per}(C_U)
\]

provided by Proposition 2.10(4), as well as the bijection

\[
\pi_U : 2\text{-presilt}(P) \longrightarrow 2\text{-presilt}(C_U)
\]

provided by Corollary 2.12. The \(\tau\text{-cluster morphism category will be constructed in such a way that it contains a morphism

\[
U\textsuperscript{\perp}@\text{fd} \xrightarrow{P} \pi_U\text{^{-1}}(P)\text{\^\perp@fd}.
\]
Moreover, it will contain a commutative diagram of the form

\[
\begin{array}{ccc}
\mathcal{D}_{\text{id}}(A) & \xrightarrow{U} & U_{\text{id}}^{1,2} \\
& {}_{\pi^{-1}_U P} \downarrow \swarrow P & \\
& (\pi^{-1}_U P)_{\text{id}}^{1,2} & \end{array}
\]

If \( U \) and \( V \) are different two-term presilting objects in \( \text{per}(A) \) such that \( U_{\text{id}}^{1,2} \) and \( V_{\text{id}}^{1,2} \) are the same t-exact subcategory of \( \mathcal{D}_{\text{id}}(A) \), it follows from Proposition 2.10(4) that there are triangle equivalences

\[
\text{per}(C_U) \leftrightarrow \text{per}(A)/\text{thick}(P) \rightarrow \text{per}(C_V),
\]

(4.B)

sending \( A \in \text{per}(A)/\text{thick}(P) \) to \( C_U \in \text{per}(C_U) \) and \( C_V \in \text{per}(C_V) \), respectively. Consequently, there is a triangle equivalence \( \beta_{V,U} : \text{per}(C_U) \rightarrow \text{per}(C_V) \) sending \( C_U \) to \( C_V \). It clearly induces a bijection

\[
\beta_{V,U} : 2_{\text{C}_U}\text{-presilt}(C_U) \rightarrow 2_{\text{C}_V}\text{-presilt}(C_V).
\]

(4.C)

**Definition 4.1.** Let \( A \) be a non-positive dg \( k \)-algebra such that \( H^i A \) is finite dimensional for all \( i \in \mathbb{Z} \).

The \( \tau \)-cluster morphism category of \( A \) is denoted by \( \mathcal{C} \), and is defined as follows: the objects of \( \mathcal{C} \) are (thick, by Lemma 1.5) t-exact subcategories \( S \) of \( \mathcal{D}_{\text{id}}(A) \) such that \( S = U_{\text{id}}^{1,2} \) for some two-term presilting object \( U \) in \( \text{per}(A) \). In this context, let \( C_U \) be as in Proposition 2.10(3). Let \( S_1 \) be an object in \( \mathcal{C} \). Fix a two-term presilting object \( U \) in \( \text{per}(A) \) such that \( S_1 = U_{\text{id}}^{1,2} \). Then for each two-term presilting object \( P \) in \( \text{per}(C_U) \), we add a morphism

\[
S_1 \xrightarrow{P} S_2
\]

(4.D)

provided that \( S_2 = (\pi^{-1}_U P)_{\text{id}}^{1,2} \), where \( \pi_U \) is the bijection

\[
\pi_U : 2\text{-presilt}_P(A) \rightarrow 2\text{-presilt}(C_U),
\]

induced by the functor \( \pi_U : \text{per}(A) \rightarrow \text{per}(C_U) \), as provided by Corollary 2.12.

Had we fixed a two-term presilting object \( V \) in \( \text{per}(A) \) such that \( V_{\text{id}}^{1,2} = U_{\text{id}}^{1,2} \), we would have transferred along \( \beta_{V,U} \) in (4.C) and added \( \beta_{V,U} P \) instead. Since the functor \( \beta_{V,U} \) is constructed to yield a natural isomorphism \( \beta_{V,U} \pi_U = \pi_V \) of functors from \( \text{per}(A) \) to \( \text{per}(C_V) \), we have that \( S_2 = (\pi^{-1}_U P)_{\text{id}}^{1,2} = (\pi^{-1}_U (\beta_{V,U} P))_{\text{id}}^{1,2} \), so we indeed get the morphism with the same domain and codomain. Keeping this transferral process in mind, the addition of the morphism \( P \) displayed in (4.D) is independent of the choice of \( U \). When working with a morphism as in (4.D) above, the choice of \( U \) will be implicit.

Given two consecutive morphisms

\[
S_1 \xrightarrow{P} S_2 \xrightarrow{Q} S_3,
\]

(4.E)

where \( P \in 2\text{-presilt}(C_U) \) for some \( U \in 2\text{-presilt}(A) \), we may, thanks to the previous paragraph, assume that \( Q \in \text{presilt}(C_V) \), where \( V \) is the element \( \pi^{-1}_U \) of \( 2\text{-presilt}(C_U) \). We define the composite morphism \( Q \circ P \) to be \( \pi^{-1}_P(Q) \), where \( \pi_P \) is the bijection

\[
2\text{-presilt}_P(C_U) \xrightarrow{\pi_P} 2\text{-presilt}(C_{U,P}),
\]

(4.F)

obtained by applying Corollary 2.12 to \( C_U \) and \( P \).
We include a lemma to show that our composition rule is well-defined.

**Lemma 4.2.** Consider the diagram in (4.E), where \( S_1 = U_{\text{id}} \) for some \( U \in 2\text{-presilt}(A) \), and \( Q \in 2_{C_U}\text{-presilt} \), where \( V \) is the element \( \pi_V^{-1}(P) \) of 2-presilt(\( C_U \)). The following assertions hold.

1. \( \pi^{-1}_U(P)(Q) \in 2\text{-presilt}(C_U) \),
2. \( S_3 = (\pi_U^{-1}(Q \circ P))_{\text{id}} \).

**Proof.** The first assertion (1) follows directly from the existence of the bijection in (4.F).

Secondly and finally, we prove the claim in (2). By definition, we have that \( S_3 = (\pi_U^{-1}(Q \circ P))_{\text{id}} \). We show that \( \pi^{-1}_U(P)(Q) \) is equal to \( S_3 \) as a t-exact subcategory of \( D_{\text{id}}(A) \). Direct calculation shows the following:

\[
\begin{align*}
(\pi_U^{-1}(Q \circ P))_{\text{id}} &= (\pi_U^{-1}(Q))_{\text{id}} = (\pi_V^{-1}(P)(Q))_{\text{id}} = (\pi_V^{-1}(Q))_{\text{id}} = S_3,
\end{align*}
\]

where we have used Corollary 2.13 to get the second equality. The proof is complete.

Having defined its objects, morphisms, and composition rule, we now check that \( \mathcal{W}_A \) is a category. It is clear that the two-term presilting object \( P = 0 \) in 2-presilt(\( C_U \)) is the identity morphism on \( U_{\text{id}} \). However, it is not obvious that the composition rule is associative. Buan–Marsh show that their \( \tau \)-cluster morphism category has an associative composition rule [BM21, Corollary 1.8] by considering several cases. Igusa–Todorov’s treatment of the representation finite hereditary case makes use of root systems [IT17, Section 1]. Our proof takes advantage of the functoriality of silting reduction.

**Theorem 4.3.** The composition rule for \( \mathcal{W}_A \) is associative. Thus \( \mathcal{W}_A \) is a category.

**Proof.** Given three composable morphisms

\[
S_1 \xrightarrow{P} S_2 \xrightarrow{Q} S_3 \xrightarrow{R} S_4
\]

we verify the identity

\[
R \circ (Q \circ P) = (R \circ Q) \circ P.
\]

The bijection in (4.F) is induced by the functor \( \pi_P \) (and similarly for \( \pi_Q \)), whence we may apply Corollary 2.13 when expanding the right hand side.

\[
(R \circ Q) \circ P = (\pi_Q^{-1}(R)) \circ P = \pi_P^{-1}(\pi_Q^{-1}(R)) = \pi_P^{-1}(\pi_Q^{-1}(Q)) = \pi_{Q \circ P}(R) = R \circ (Q \circ P).
\]

The proof is complete.
There is an equivalence of categories per((S(A))). Explicitly, this equivalence sends the object \( S \in \mathcal{A} \) to the \( \tau \)-rigid pair \( (M, R) \) in \( \text{mod} \mathcal{A} \) \cite{BH23}. There is a morphism \( W_1 \xrightarrow{(M,R)} W_2 \) if \((M, R)\) is a \( \tau \)-rigid pair in \( W_1 \) such that \( J_{W_1}(M, Q) = W_2 \), where \( J_{W_1}(M, R) \) is the \( \tau \)-perpendicular category of \((M, R)\) in \( W_1 \). If

\[
F_{(M,R)} : \text{\( \tau \)-rigid pair}(W_1) \longrightarrow \text{\( \tau \)-rigid pair}(\Gamma_1)
\]

is the bijection in \((4.I)\), and

\[
\mathcal{E}_{(M,R)} : \text{\( \tau \)-rigid pair}(F_{(M,R)}M, F_{(M,R)}R)(\Gamma_1) \longrightarrow \text{\( \tau \)-rigid pair}(W_2)
\]

is the bijection in Theorem \(5.1\), then the composite of

\[
W_1 \xrightarrow{(M,R)} W_2 \xrightarrow{(M', R')} W_3
\]

is defined by \( F_{(M,R)}^{-1} \circ \mathcal{E}_{(M,R)}^{-1}(M', R') \in \text{\( \tau \)-rigid pair}(W_1) \).

Note that our approach is independent of that of Buan–Marsh and Buan–Hanson. Hence, it gives an alternative approach to defining \( \tau \)-cluster morphism categories for all finite dimensional \( k \)-algebras, and also for non-positive \( dg \) \( k \)-algebras with finite dimensional cohomology in all degrees.

**Theorem 4.4.** Let \( A \) be a non-positive \( dg \) \( k \)-algebra such that \( H^iA \) is finite dimensional for all \( i \in \mathbb{Z} \). There is an equivalence of categories

\[
E : \mathcal{M}_A \longrightarrow \mathcal{M}^{BM}_{H^0A}.
\]

Explicitly, this equivalence sends the object \( S \in \mathcal{M}_A \) to \( S \cap \text{mod} H^0A \in \mathcal{M}^{BM}_{H^0A} \) (see Lemma \(1.17(1)\)) and the map \( S_1 \xrightarrow{P} S_2 \) to \( F_{H_0U}^{-1} \circ H_{U_1}(P) \) (see Theorem \(1.17\) and \((1.G)\)), where \( U \) is a two-term presilting object in \( \text{per}(A) \) such that \( U_{1\leq 2} = S_1 \) and \( C_U \) is the non-positive \( dg \) \( k \)-algebra defined in Proposition \(2.10(3)\). In particular, if \( A \) is a finite dimensional \( k \)-algebra, then \( \mathcal{M}_A \) is equivalent to \( \mathcal{M}^{BM}_A \).

**Proof.** The morphism set \( \mathcal{M}_A(S_1, S_2) \) is a subset of \( 2\)-presilt(\( C_U \)). On the other hand, the morphism set \( \mathcal{M}^{BM}_{H^0A}(W_1, W_2) \) is a subset of \( \tau \)-rigid pair(\( W_1 \)), where \( W_i \) is the wide subcategory \( S_i \cap \text{mod} H^0A \) of \( \text{mod} H^0A \), for \( i \in \{1, 2\} \). Using Theorem \(1.17\) and \((1.G)\), we define a map

\[
\mathcal{M}_A(S_1, S_2) \xrightarrow{H_{C_U}} \text{\( \tau \)-rigid pair}(H^0C_U) \xrightarrow{F_{(M,R)}^{-1}} \text{\( \tau \)-rigid pair}(W_1).
\]

It follows from Proposition \(2.10(1)\) that the \( \tau \)-perpendicular category of \( H_{C_U}(P) \) in \( W_1 \) is \( W_2 \). Consequently, the image of this map is a subset of \( \mathcal{M}^{BM}_{H^0A}(W_1, W_2) \), so a map of Hom-sets has thus been constructed.

It is easy to check that the functor \( E \) sends identity maps to identity maps; they are given by trivial presilting objects and trivial \( \tau \)-rigid pairs. It should also be shown that composition is respected. Let \( P \) and \( Q \) be a pair of composable morphisms in \( \mathcal{M}_A \), as shown in \((1.E)\). Their composite is then given by \( V = \pi_P^{-1}(Q) \). Let \((M, R)\) denote the \( \tau \)-rigid pair \( H_{C_U}(P) \) in \( \text{mod} H^0C_U \). Consider the following diagram
of bijections:

\[
\begin{array}{ccc}
2\text{-presilt}_P(C_U) & \overset{\varphi_P}{\longrightarrow} & H_{C_U} \\
\pi_P \downarrow & & \downarrow \zeta_P \\
2\text{-presilt}(C_V) & \overset{\zeta_P}{\longrightarrow} & \tau\text{-rigid pair}(H^0C_V) \\
\end{array}
\]

\[
\begin{array}{ccc}
& \overset{H_{C_U}}{\longrightarrow} & \\
2\text{-presilt}_P(C_U) & \overset{\varphi_P}{\longrightarrow} & H_{C_U} \circ (\pi_P^{-1}(Q)) \\
\pi_P \downarrow & & \downarrow \zeta_P \\
2\text{-presilt}(C_V) & \overset{\zeta_P}{\longrightarrow} & \tau\text{-rigid pair}(H^0C_V) \\
\end{array}
\]

\[
\begin{array}{ccc}
H_{E_{C_U}(P)} & \overset{E_{H_{C_U}(P)}}{\longrightarrow} & \tau\text{-rigid pair}_{(M,R)}(W_1) \\
\zeta_P \downarrow & & \downarrow \zeta_P \\
H_{E_{C_U}(P)} & \overset{E_{H_{C_U}(P)\circ F_{(M,R)}}}{\longrightarrow} & \tau\text{-rigid pair}_{(M,R)}(W_2) \\
\end{array}
\]

where we have referred to previous results to address the commutativity of the not-so-obviously commutative subdiagrams (recall that \(\zeta_P\) in Corollary 2.12 is induced by a triangle equivalence sending \(T_P\) to \(C_V\)). We have defined the morphism \(E(Q \circ P) = E(\pi_P^{-1}(Q))\) in \(\mathcal{M}^\text{BM}_{BM}(W_1, W_2)\) as the image of \(\pi_P^{-1}(Q)\) along the top row. By the commutativity of the diagram:

\[
E(Q \circ P) = F^{-1}_{(M,R)} \circ E_{H_{C_U}(P)}^{-1}(Q) = F^{-1}_{(M,R)} \circ H_{C_U}(Q) = F^{-1}_{(M,R)} \circ E_{H_{C_U}(P)}(E(Q)) = E(Q) \circ E(P),
\]

showing that composition is respected. We have thus defined a functor from \(\mathcal{M}_A\) to \(\mathcal{M}^\text{BM}_{BM,A}\).

Since the functor \(E\) in (4.1) is a bijection as a map of objects, as was shown in Proposition 2.9(2), it is essentially surjective. We conclude the proof by showing that \(E\) is fully faithful. It is to be shown that the maps in (4.1) inject for all \(S_1, S_2 \in \mathcal{M}_A\), and that the image is \(\mathcal{M}^\text{BM}_{BM,A}(W_1, W_2)\). Since the maps of the form (4.1) are restrictions of bijections, they inject, whence \(E\) is faithful. To show that \(E\) full, we fix an element \((M, R) \in \mathcal{M}^\text{BM}_{BM}(W_1, W_2)\) and show that it admits a preimage. Now, we have that \(H_{C_U}^{-1}(M, R) \in 2\text{-presilt}(C_U)\). Since \(W_2\) is the \(\tau\)-perpendicular category of \((M, R)\) in \(W_1\), it follows form Proposition 2.9(1) that \(E\) sends \(H_{C_U}^{-1}(M, R)\) to \(W_2\). This puts \(H_{C_U}^{-1}(M, R) \in \mathcal{M}_A(S_1, S_2)\), and this morphism in \(\mathcal{M}_A\) is sent to \((M, R) \in \mathcal{M}^\text{BM}_{BM}(W_1, W_2)\) by the map in (4.1). This shows that \(E\) is full, and in turn that it is fully faithful. \(\square\)

References

[AiI2] T. Aihara and O. Iyama, “Silting mutation in triangulated categories,” J. Lond. Math. Soc. (2), vol. 85, no. 3, pp. 633–668, 2012. [Online]. Available: https://doi.org/10.1112/jlms/jdr055

[AIR14] T. Adachi, O. Iyama, and I. Reiten, “\(\tau\)-tilting theory,” Compos. Math., vol. 150, no. 3, pp. 415–452, 2014. [Online]. Available: https://doi.org/10.1112/S0010437X13007422

[Ami09] C. Amiot, “Cluster categories for algebras of global dimension 2 and quivers with potential,” Ann. Inst. Fourier (Grenoble), vol. 59, no. 6, pp. 2525–2590, 2009. [Online]. Available: http://aif.cedram.org/item?id=AIF_2009_59_6_2525_0

[AMY19] T. Adachi, Y. Mizuno, and D. Yang, “Discreteness of silting objects and \(t\)-structures in triangulated categories,” Proc. Lond. Math. Soc. (3), vol. 118, no. 1, pp. 1–42, 2019. [Online]. Available: https://doi.org/10.1112/plms.12176

[AS81] M. Auslander and S. O. Smalø, “Almost split sequences in subcategories,” J. Algebra, vol. 69, no. 2, pp. 426–454, 1981. [Online]. Available: https://doi.org/10.1016/0021-8693(81)90214-3

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne, “Faisceaux pervers,” in Analys is and topology on singular spaces, I (Luminy, 1981), ser. Astérisque. Soc. Math. France, Paris, 1982, vol. 100, pp. 5–171.

[BBDG18] A. Beilinson, J. Bernstein, P. Deligne, and O. Gabber, Faisceaux pervers. Société mathématique de France, 2018.
K. Brüning and B. Huber, “Realising smashing localisations as morphisms of dg algebras,” Applied Categorical Structures, vol. 16, pp. 669–687, 2008.

A. Bakke Buan and E. J. Hanson, “τ-perpendicular wide subcategories,” Nagoya Mathematical Journal, vol. 252, pp. 959–984, 2023.

A. B. Buan and B. R. Marsh, “A category of wide subcategories,” Int. Math. Res. Not. IMRN, no. 13, pp. 10 278–10 338, 2021. [Online]. Available: https://doi.org/10.1093/imrn/rnz082

A. B. Buan, B. Marsh, M. Reineke, I. Reiten, and G. Todorov, “Tilting theory and cluster combinatorics,” Adv. Math., vol. 204, no. 2, pp. 572–618, 2006. [Online]. Available: https://doi.org/10.1016/j.aim.2005.06.003

A. B. Buan and Y. Zhou, “A silting theorem,” J. Pure Appl. Algebra, vol. 220, no. 7, pp. 2748–2770, 2016. [Online]. Available: https://doi.org/10.1016/j.jpaa.2015.12.009

A. B. Buan and Y. Zhou, “A silting theorem,” J. Pure Appl. Algebra, vol. 220, no. 7, pp. 2748–2770, 2016. [Online]. Available: https://doi.org/10.1016/j.jpaa.2015.12.009

L. Demonet, O. Iyama, N. Reading, I. Reiten, and H. Thomas, “Lattice theory of torsion classes,” arXiv e-prints, p. arXiv:1711.01785, Nov. 2017.

O. Iyama, P. Jørgensen, and D. Yang, “Intermediate co-t-structures, two-term silting objects, τ-tilting modules, and torsion classes,” Algebra Number Theory, vol. 8, no. 10, pp. 2413–2431, 2014. [Online]. Available: https://doi.org/10.2140/ant.2014.8.2413

K. Igusa and G. Todorov, “Signed exceptional sequences and the cluster morphism category,” arXiv e-prints, p. arXiv:1706.02041, Jun. 2017.

K. Igusa, G. Todorov, and J. Weyman, “Picture groups of finite type and cohomology in type An,” arXiv e-prints, p. arXiv:1609.02636, Sep. 2016.

O. Iyama and D. Yang, “Silting reduction and Calabi–Yau reduction of triangulated categories,” Trans. Amer. Math. Soc., vol. 370, no. 11, pp. 7861–7898, 2018. [Online]. Available: https://doi.org/10.1090/tran/7213

O. Iyama and Y. Yoshino, “Mutation in triangulated categories and rigid Cohen-Macaulay modules,” Invent. Math., vol. 172, no. 1, pp. 117–168, 2008. [Online]. Available: https://doi.org/10.1007/s00222-007-0096-4

O. Iyama and D. Yang, “Reduction of τ-tilting modules and torsion pairs,” Int. Math. Res. Not. IMRN, no. 16, pp. 7190–7237, 2015. [Online]. Available: https://doi.org/10.1093/imrn/rnu163

B. Keller, “Deriving dg categories,” in Annales scientifiques de l’Ecole normale supérieure, vol. 27, no. 1, 1994, pp. 63–102.

H. Krause, “Localization theory for triangulated categories,” arXiv preprint arXiv:0806.1324, 2008.

B. Keller and D. Vossieck, “Aisles in derived categories,” 1988, vol. 40, no. 2, pp. 239–253, deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987).

A. Neeman, “The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel,” in Annales scientifiques de l’Ecole normale supérieure, vol. 25, no. 5, 1992, pp. 547–566.

A. Neeman, “The Grothendieck duality theorem via Bousfield’s techniques and Brown representability,” Journal of the American Mathematical Society, vol. 9, no. 1, pp. 205–236, 1996.

J. Rickard, “Morita theory for derived categories,” J. Lond. Math. Soc., vol. 39, no. 3, pp. 436–456, 1989.