The continuum limit of the integrable open XYZ spin-1/2 chain

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Abstract
We show that the continuum limit of the integrable XYZ spin-1/2 chain on a half-line gives rise to the boundary sine-Gordon theory using the perturbation method.

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1 Introduction

The introduction of boundary interactions to integrable 1+1-dimensional quantum field theories have given rise to a series of interesting applications in various branches of physics ranging from particle physics to low-dimensional systems of condensed matter physics with dissipative forces or various types of impurities [1, 2]. In view of this, a number of studies have been made to construct the integrable extension of both solvable lattice models and integrable two-dimensional field theories to the case involving boundaries. Clarifying the connection between the former and latter models in the presence of boundary interactions is then an interesting question. In this letter we focus on the connection of the open XYZ spin-1/2 chain to the sine-Gordon theory on the semi-infinite line $[0, \infty)$.

The sine-Gordon theory with a boundary potential depending only on the boundary field $\varphi(0)$ is integrable, if the potential is of the form [3, 4]

$$V(\varphi(0)) = -M \cos \left( \frac{\beta}{2} (\varphi(0) - \phi_0) \right).$$

Here $\beta$ is the coupling constant appearing in the bulk theory, and $M$ and $\phi_0$ are arbitrary parameters. The existence of the first few integrals of motion in this theory, which we call the boundary sine-Gordon theory, can be checked by explicit computations. In the bulk theory, the continuum limit of the XYZ spin-1/2 chain is known to be described by the sine-Gordon theory [5]. In the lattice theory, the integrable extension of the XYZ model to the case with boundary interactions has been made by constructing the reflection matrix [6, 7]. The general form of the Hamiltonian for this integrable XYZ model on a semi-infinite chain is given by

$$H = H_{XYZ} + H_{b.t.},$$

$$H_{XYZ} = -J \sum_{i=0}^{\infty} \left( \frac{1}{2} \left( S^+_i S^-_{i+1} + S^-_i S^+_{i+1} \right) + \Delta S^z_i S^z_{i+1} + \frac{1}{2} \Gamma \left( S^+_i S^+_{i+1} + S^-_i S^-_{i+1} \right) \right),$$

$$H_{b.t.} = AS^z_0 + BS^-_0 + CS^+_0,$$

where the external fields $A, B, C$ are arbitrary parameters. In the following we assume that $C = B^*$ so that the Hamiltonian possesses the hermiticity.

In this letter we will show that the continuum limit of the open XYZ spin-1/2 chain (2) gives rise to the boundary sine-Gordon theory. To derive the continuum
limit rigorously, the exact evaluation of the low-lying excitation spectrum as well as the ground-state energy of the open XYZ spin chain is necessary. The diagonalization of the Hamiltonian [2] by means of, for example, the generalized Bethe ansatz method is a formidable task because of the off-diagonal boundary terms, and the calculation has not yet been accomplished. Without the rigorous results, we can still make a perturbative analysis in the limit in which the strength of zz interaction $\Delta$ and the boundary external field $A, B, C$ are small. Note that the X-Y anisotropy $\Gamma$ should be designed to approach zero as the lattice spacing $a$ goes to zero, for the continuum limit to be defined.

2 XYZ model on a line

We begin by recapitulating the perturbation method to derive the continuum limit of the closed XYZ spin chain [8], thereby explaining the tools to be used later in the extension to the case of open chain.

The XYZ spin-1/2 chain can be mapped to the interacting fermions on a lattice using the Jordan-Wigner transformation,

$$S_i^- = \psi_i \exp \left( i \pi \sum_{j=0}^{i-1} \psi_j^\dagger \psi_j \right), \quad S_i^z = \psi_i^\dagger \psi_i - \frac{1}{2}. \quad (5)$$

We deal with the continuum limit of this model by means of the perturbation to the XX model, i.e., the limit in which the parameters $\Gamma$ and $\Delta$ are small. The continuum theory of the XX model is that of free massless fermions. We express the low-energy excitations in terms of the chiral fermions defined on a pair of odd and even sites,

$$\psi_{2l} = (-)^l (\psi_{+,\nu} + \psi_{-\nu}), \quad \psi_{2l+1} = i(-)^l (\psi_{+,\nu} - \psi_{-\nu}), \quad (6)$$

where $\nu = l + 1/4$. We translate the lattice theory into the continuum theory using the prescription: $\psi_x = (2a)^{-1/2} \psi_{\pm \nu}$, $x = 2a \nu$ and $\int dx = 2a \sum_\nu$, in the limit of zero lattice spacing $a$. The resulting Lagrangian (density) of the low-energy effective theory for the XYZ model is equivalent to that for the massive Thirring model,

$$L^F = 2iv_0: \psi_+^\dagger \partial_+ \psi_+ + \psi_-^\dagger \partial_- \psi_- : + v_1 J_+ J_- + iv_2 \left( \psi_+ \psi_- - \psi_-^\dagger \psi_+^\dagger \right). \quad (7)$$

Here $J_\pm$ are the right and left moving fermion currents, $J_\pm(x^\pm) = : \psi_\pm^\dagger \psi_\pm(x^\pm) :$, and $x^\pm = t \mp x$, $\partial_\pm = (\partial_t \mp \partial_x) / 2$. The parameters appearing in (7) are related to
the coupling constants in the spin-chain Hamiltonian (2) as $v_0 = aJ (1 - \pi^{-1} \Delta)$, $v_1 = 4aJ\Delta$, $v_2 = J\Gamma$. Setting the value of the spin-wave velocity $v_0$ to the unity, we have $v_1 = 4\Delta (1 - \pi^{-1} \Delta)^{-1}$ and $v_2 = a^{-1}\Gamma (1 - \pi^{-1} \Delta)^{-1}$. To obtain the field theory limit one has to tune the lattice coupling constants appropriately as we take the limit $a \to 0$. Thus, we should make the coupling constant $\Gamma$ scale as $\Gamma = a\Gamma_r$ where $\Gamma_r$ is a renormalized coupling constant.

The fermionic field theory can be converted to the bosonic field theory through the bosonization,

$$J_\pm = 2\gamma^{-1} \partial_\pm \varphi_\pm,$$
(8)

$$\psi_\pm = \sqrt{2\mu} \gamma^{-1} e^{\pm i \varphi_\pm},$$
(9)

where $\gamma = \sqrt{4\pi}$ and $\mu$ is an infrared cutoff. The chiral boson fields $\varphi_+(x^+)$ and $\varphi_-(x^-)$ obey the commutation relations at an equal time, $[\varphi_+(x^+), \varphi_+(y^+)] = -[\varphi_-(x^-), \varphi_-(y^-)] = (i/4)\epsilon(x - y)$ and $[\varphi_+(x^+), \varphi_-(y^-)] = i/4$. Introduce the nonchiral boson fields

$$\varphi = \varphi_+ + \varphi_-, \quad \tilde{\varphi} = \varphi_+ - \varphi_-.$$ (10)

They both satisfy the canonical commutation relations and have the commutation relation $[\tilde{\varphi}(t, x), \varphi(t, y)] = i\theta(x - y)$ between them. The Lagrangian (2) then can be represented solely in terms of one of the boson fields, $\tilde{\varphi}$. After rescaling the fields as

$$\sigma = \varphi / \gamma R, \quad \tilde{\sigma} = \gamma R \tilde{\varphi},$$
(11)

we have the Lagrangian for the sine-Gordon theory

$$\mathcal{L}^B = \frac{1}{2} (\partial_\mu \tilde{\sigma})^2 + \frac{m^2}{\beta^2} \cos \beta \tilde{\sigma}.$$ 
(12)

Here the radius of the boson field $R$ and the mass scale $m$ are related to the parameters of the lattice model by $\sqrt{4\pi}R = [1 + (2\pi)^{-1}v_1]^{1/2}$ and $m^2 = 4\Gamma_r (1 + \pi^{-1} \Delta)^{-1}$, respectively. $\beta = 1/R$ is the dimensionless coupling constant of the sine-Gordon theory. In the present perturbation analysis, we have $\beta^2 = 4\pi [(1 - \pi^{-1} \Delta) / (1 + \pi^{-1} \Delta)]$. The exact renormalized value for $\beta$ is given by $\beta^2 = 8\pi [1 - \pi^{-1} \cos^{-1} (-\Delta)].$

The above perturbation result reproduces the exact result (13) to first order in $\Delta$. This is a nontrivial check of the reliability of the present perturbation method.
We note that the interaction term \((m^2/\beta^2) \cos \beta \tilde{\sigma}\) arises from the X-Y anisotropy as can be seen from the fact that \(m^2 \propto \Gamma_r\). The Lagrangian (7) contains another term which arises from the \(zz\) interaction through the Umklapp process\[9\]. However, this term is an irrelevant operator in the region of small \(|\Delta|\) and it can be neglected. It becomes marginal at \(\Delta = -1\), which is far outside the range of the present perturbation analysis.

3 XYZ model on a half-line

Now we proceed to the continuum limit of the open XYZ spin chain (2). We deal with this problem perturbatively starting from an unperturbed state in which the bulk is described with a massless boson field \(\tilde{\varphi}\) with the free boundary condition: \(\partial_x \tilde{\varphi}|_{x=0} = 0\). The \(zz\) interaction term and the X-Y anisotropy term in the bulk and the boundary external fields are treated as the perturbations on this state. Since \(\partial_t \varphi = -\partial_x \tilde{\varphi}\), the condition means \(\varphi(t,0) = \varphi_0\), where \(\varphi_0\) is a constant with respect to the time variable \(t\). In terms of the chiral bosons, it reads

\[
\varphi_+(t,0) = -\varphi_-(t,0) + \varphi_0.
\]

We first take into account the boundary perturbation only. The continuum representation of the spin operators at the boundary in terms of the fermions can be written down following the procedure described for the bulk case,

\[
S^z(0) = a^{-1} S_0^z = J(0) + G(0),
\]

\[
S^z(0) = a^{-1/2} S_0^- = \psi(0),
\]

where \(\psi = \psi_+ + \psi_-\). \(J = J_+ + J_-\) is the vector current and \(G = \psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+\) is the scalar density. The boundary Hamiltonian (4) gives the following boundary action.

\[
S_{\text{boundary}} = -\int_{-\infty}^{\infty} dt \left[ A_r (J(0) + G(0)) + B_r \psi(0) + C_r \psi^\dagger(0) \right],
\]

where \(A_r = a^{-1} A\), \(B_r = a^{-1/2} B\), \(C_r = a^{-1/2} C\) are the renormalized value of the boundary external fields. It contains the fermion linear terms as well as the fermion bilinear terms. Thus, in the bosonized representation, it is naively expected that the boundary perturbation contains interactions of the form \(\cos \gamma \tilde{\varphi}\) as well as \(\cos \gamma \tilde{\varphi}/2\)\[10\]. We show that only the latter gives rise to a relevant boundary interaction.
In the presence of the boundary, we modify the bosonization rule (9) slightly as [11]
\[ \psi_{\pm} = \sqrt{2\mu\gamma^{-1}} c_{\pm} e^{\pm i\gamma\varphi_{\pm}}, \] (18)
where \( c_{\pm} \) are the zero mode operators defined by
\[ c_{\pm} = \frac{1}{4} \exp \left[ \pm\frac{i}{2} \left( \frac{\pi}{2} - \gamma\varphi_{0} \right) \right]. \] (19)
Here \( \varphi_{0} \) is the constant operator introduced in the boundary condition (14). The relation (8) remains unchanged.

Applying the bosonization rule to the operators at the boundary, we have for the vector current
\[ J(0) = -2\gamma^{-1} \partial_{t} \tilde{\varphi}. \] (20)
This term does not contribute to the action after integrating over \( t \). For the scalar density term, we see that the two terms of \( G(0) \) add up to zero. Thus, the diagonal part of the boundary interactions, which are quadratic in the fermions, gives no contributions in the continuum limit. It remains to evaluate the off-diagonal terms. We have
\[ B_{\tau}\psi(0) + C_{\tau}\psi^\dagger(0) = \sqrt{2\mu\gamma^{-1}} |B_{\tau}| \left( \left[ c_{+} e^{i\gamma\varphi_{+}(0)} + c_{-} e^{-i\gamma\varphi_{-}(0)} \right] e^{-ib} + \left[ e^{-i\gamma\varphi_{+}(0)} c_{-} + e^{i\gamma\varphi_{-}(0)} c_{+} \right] e^{ib} \right), \] (21)
where we have put \( \text{arg} B_{\tau} = -b \). We should recall that it is the field \( \tilde{\varphi} \) that is relevant in the bulk sine-Gordon theory. The boundary interaction (21) involves both operators \( e^{\pm i\gamma\varphi(0)/2} \) and \( e^{\pm i\tilde{\varphi}(0)/2} \). The free boundary condition implies that the boundary value of the boson field \( \varphi \) is a constant in time, i.e., it is not a dynamical variable. Hence, the boundary interaction is also given by the field \( \tilde{\varphi} \).

To obtain the expression in terms of the nonchiral bosons, we first note that we have
\[ \tilde{\varphi}(0) = 2\varphi_{+}(0) - \varphi_{0}, \]
\[ = -2\varphi_{-}(0) + \varphi_{0}, \] (22)
at the boundary. Taking care of the fact that the zero mode operator \( \varphi_{0} \) does not commute with the boson field \( \tilde{\varphi} \), we can express the linear combination of the exponentials of the chiral bosons at the boundary in terms of \( \tilde{\varphi}(0) \) [11]
\[ c_{+} e^{i\gamma\varphi_{+}(0)} e^{-ib} + e^{-i\gamma\varphi_{+}(0)} c_{-} e^{ib} \]
\[ c - e^{-i\gamma \varphi_-(0)} e^{-ib} + e^{+i\gamma \varphi_-(0)} c^\prime e^{ib} = \frac{1}{2} \cos \left( \frac{\gamma}{2} \tilde{\varphi}(0) - b \right) \]  

Now we include the zz interaction term, which yields merely the rescaling (11) of the boson fields in our perturbation scheme as mentioned earlier. Substituting the relations (23) into (21) and rescaling the boson fields, we finally arrive at the boundary Lagrangian

\[ L_{b.t.} = -M \cos \frac{\beta}{2} (\tilde{\varphi}(0) - \phi_0), \] (24)

where

\[ M = \sqrt{2\mu \gamma^{-1}} |B_r|, \quad \phi_0 = \frac{2b}{\beta}. \] (25)

Thus the integrable boundary potential (1) for the sine-Gordon theory has been derived from the continuum limit of the integrable boundary interactions for the open XYZ spin-1/2 chain. We see that the free parameters \(M\) and \(\phi_0\) appearing in the boundary potential (1) are related to the strength and the phase of the off-diagonal part of the boundary external field for the open XYZ spin chain.

Finally, we check the scaling dimension of the operator (24). To this end, we apply the technique using the analytic continuation [12]. We make use of the boundary condition (14) to extend the right-moving boson \(\sigma_+\) to the negative axis, \(x < 0\), by defining it as the analytic continuation of the left-moving boson \(\sigma_-\) with

\[ \sigma_+(t, x) = -\sigma_-(t, -x) + \gamma \varphi_0/\beta \quad (x < 0). \] (26)

It is noted that we are now considering the situation in which the bulk is still described with a massless free boson \(\tilde{\varphi}\), i.e., the X-Y anisotropy is not included. This is sufficient to calculate the scaling dimension of the boundary operator in our perturbative analysis. Since the interactions exist only at the boundary, the chiral bosons \(\sigma_+\) and \(\sigma_-\) are decoupled in the bulk and the locality of the theory is not violated under this extension [13]. Now the right-moving boson \(\sigma_+\) is defined on the whole line \(-\infty < x < \infty\) and the calculation of the scaling dimensions of the operators at the boundary follows from the knowledge in the bulk theory. The scaling dimension of the operator (24) is evaluated to be \(\beta^2/8\pi\). If we exploit the exact value of \(\beta\), we see that this operator is relevant for all values of \(\Delta\) on the critical line, \(-1 \leq \Delta \leq 1\), except for the antiferromagnetic point \(\Delta = -1\) at which it becomes marginal. Thus, the interaction (24) indeed survives in the continuum limit.
To summarize, we note that our analysis has clarified that the boundary potential (1) is, in fact, the general solution for the boundary potential which is compatible with the integrability of the sine-Gordon theory with boundaries, though it has less number of free parameters than the general solution for the integrable open XYZ spin chain [3].

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