Qualitative and quantitative analysis of stability and instability dynamics of positive lattice solitons

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We present a unified approach for qualitative and quantitative analysis of stability and instability dynamics of positive bright solitons in multi-dimensional focusing nonlinear media with a potential (lattice), which can be periodic, periodic with defects, quasiperiodic, single waveguide, etc. We show that when the soliton is unstable, the type of instability dynamic that develops depends on which of two stability conditions is violated. Specifically, violation of the slope condition leads to a focusing instability, whereas violation of the spectral condition leads to a drift instability. We also present a quantitative approach that allows to predict the stability and instability strength.

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I. INTRODUCTION

Solitons, or solitary waves, are localized nonlinear waves that maintain their shape during propagation. They are prevalent in many branches of physics, and their properties have provided deep insight into complex nonlinear systems. The stability properties of solitons are of fundamental importance. Stable solitons are both natural carriers of energy in naturally occurring systems and often the preferred carriers of energy in engineered systems. Their stability also makes them most accessible to experimental observation.

The first studies considered stability of solitons in homogeneous media. In recent years there has been a considerable interest in the study of solitons in lattice-type systems. Such solitons have been observed in optics using waveguide arrays, photo-refractive materials, photonic crystal fibers, etc., in both one-dimensional and multidimensional lattices, mostly periodic sinusoidal square lattices or single waveguide potentials, but also in discontinuous lattices (surface solitons). Radially-symmetric Bessel lattices, lattices with triangular or hexagonal symmetry, lattices with defects, with quasicrystal structures or with random potentials. Solitons have also been observed in the context of Bose-Einstein Condensates (BEC), where lattices have been induced using a variety of techniques.

Stability of lattice solitons has been studied in hundreds of papers. The majority of these papers focused on one specific physical configuration, i.e., a specific dimension (mostly in 1D), nonlinearity and lattice type. In addition, in several studies, general conditions for stability and instability were derived (see Section III). In all of these studies, the key question was whether the soliton is stable (yes) or unstable (no).

Fibich, Sivan and Weinstein went beyond this binary view by developing a qualitative and quantitative approach to stability of positive lattice solitons. This was first carried out for spatially non-homogeneous nonlinear potentials in [33, 34]. These ideas were then developed by Sivan, Fibich and coworkers in the context of linear non-homogeneous potentials in [35, 36, 37]. These studies showed that the qualitative nature of the instability dynamics is determined by the particular violated stability condition. In addition, they presented a quantitative approach for prediction of the stability or instability strength. Specifically, these papers considered the cases of a one-dimensional nonlinear lattice [33], a two-dimensional nonlinear lattice [34], a one-dimensional linear delta-function potential [36] and narrow solitons in a linear lattice [35].

In the present article, the results of [33, 34, 35, 36, 37] are combined into a unified theory for stability and instability of lattice solitons that can be summarized in a few rules (Section VI). We illustrate how these rules can be applied in a variety of examples that may be useful to experimental studies.

II. MODEL, NOTATION AND DEFINITIONS

We study the stability and instability dynamics of lattice solitons of the nonlinear Schrödinger (NLS) equation with an external potential, which in dimensionless form is given by

$$iA_{z}(x, z) + \Delta A + (1 - V_{nl}(x)) F(|A|^2) A - V_{l}(x) A = 0. \tag{1}$$

Equation (1) is also referred to as the Gross-Pitaevskii equation (GP). NLS/GP underlies many models of nonlinear wave propagation in nonlinear optics and macroscopic quantum systems (BEC). For example, in the context of laser beam propagation, $A(x, z)$ corresponds to the electric field amplitude, $z \geq 0$ is the distance along the direction of propagation, $x = (x_1, \ldots, x_d)$ is the transverse $d$-dimensional space [e.g., the $(x, y)$ plane for propagating in bulk medium] and $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_d}$ is the $d$-dimensional diffraction term. The nonlinear term models the intensity-dependence of the refractive index. For example, $F(|A|^2) = |A|^2$ corresponds to the optical Kerr effect and $F(|A|^2) = 1/ (1 + |A|^2)$ corresponds to photorefractive materials, see e.g., [38]. The potentials $V_l$ and $V_{nl}$ correspond to a modulation of the linear and nonlinear refractive indices, respectively. In BEC, $z = t$ is time, $A(x, t)$ represents the wave function of the mean-field atomic condens-
sate, \( F(\{A^2\}) = |A|^2 \) represents contact (cubic) interaction, and the potentials \( V_i(\vec{x}) \) and \( V_{nl}(\vec{x}) \) are induced by externally applied electro-magnetic fields [39].

We define a soliton to be any solution of Eq. (1) of the form \( A(\vec{x}, z) = u(\vec{x})e^{-\mu iz} \), where \( \mu \) is the propagation constant and \( u(\vec{x}) \), the soliton profile, is a real-value function that decays to zero at infinity and satisfies

\[
\Delta u + (1 - V_{nl}(\vec{x})) F(u^2) u + \mu u - V_i u = 0. \tag{2}
\]

Solitons can exist only for \( \mu \) in the gaps in the spectrum of the linear problem

\[
\Delta u + \mu u - V_i u = 0, \tag{3}
\]

i.e., for values of \( \mu \) such that the linear problem [4] does not have any non-trivial solution, see e.g., [40].

Solitons in a lattice potential, or more general non-homogeneous potential, may be understood as bounds states of an effective (self-consistent) potential, \( V_{eff} = V_i(\vec{x}) + (1 + V_{nl}(\vec{x})) F(u^2(\vec{x})) \). They arise (i) via bifurcation from the zero-amplitude state with energy at an end point of a continuous spectral band (finite or semi-infinite) or (ii) if \( V_i \) is a potential with a defect, via bifurcation from discrete eigenvalues (localized linear modes) within the spectral gaps (semi-infinite or finite), which in addition to the bare nonlinearity, can serve to nucelate a localized nonlinear bound state [47].

In this paper, we only consider positive solitons (\( u > 0 \)) of both type (i) and (ii). This is always the case for the least energy state within the semi-infinite gap, i.e., when \( -\infty < \mu < \mu_{BE}^{(V)} \), where \( \mu_{BE}^{(V)} \) is the lowest point in the spectrum of Eq. (3), at which the first band begins. Solitons whose frequencies lie in finite spectral gaps are usually referred to as gap solitons. However, gap solitons typically oscillate and change sign and are therefore not covered by the theory presented in this paper [48].

We study the dynamics of NLS/GP and its solitons in the space \( H^1 \), with norm \( \|f\|_{H^1}^2 = \int (|f|^2 + |\nabla f|^2) d\vec{x} \). The natural notion of stability is orbital stability, defined as follows:

**Definition II.1** Let \( u(\vec{x}) \) be a solution of Eq. (2) with propagation constant \( \mu \). Then, the soliton solution \( u(\vec{x})e^{-\mu iz} \) of NLS/GP eqn. (1) is orbitally stable if for all \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that for any initial condition \( A_0 \) with \( \inf_{\gamma \in \mathbb{R}} \|A_0 - we^{\gamma z}\|_{H^1} < \delta \), then for all \( z \geq 0 \) the corresponding solution \( A(\vec{x}, z) \) of Eq. (1) satisfies \( \inf_{\gamma \in \mathbb{R}} \|A(\vec{x}, z) - we^{\gamma z}\|_{H^1} < \varepsilon \).

In discussing the stability theory for NLS it is useful to refer to its Hamiltonian structure: \( i\partial_t A = \delta H / \delta A^* \), where

\[
H[A, A^*] = \int (|\nabla A|^2 + V_i |A|^2 - (1 - V_{nl})G(|A|^2)) d\vec{x},
\]

and \( G'(s) = F(s), \ G(0) = 0 \). The Hamiltonian, \( H \), and the optical power (particle number):

\[
P = \int |A|^2 d\vec{x}
\]

are conserved integrals for NLS. Eq. (2) for \( u(x; \mu) \), the soliton profile, can be written equivalently as the energy stationarity condition \( \delta E = 0 \), where \( E \equiv H - \mu P \). Soliton stability requires a study of \( \delta^2 E \), the second variational derivative of \( E \) about \( u \). For NLS in general, stable solitons need to be local energy minimizers; see e.g. [47] and also [89].

### III. SOLITON STABILITY – OVERVIEW

The first analytic result on soliton stability was obtained by Vakhitov and Kolokolov [41]. They proved, via a study of the linearized perturbation equation, that a necessary condition for stability of the soliton \( u(x; \mu) \) is

\[
\frac{dP(\mu)}{d\mu} < 0,
\]

i.e. the soliton is stable only if its power decreases with increasing propagation constant \( \mu \). This condition will be henceforth called the slope condition.

Subsequent studies of nonlinear stability analysis of solitons revealed the central role played by the number of negative eigenvalues of the operator

\[
L_+ = -\Delta + V_i - (1 - V_{nl}) (F(u^2) - 2u^2F'(u)) - \mu,
\]

which is the real part of \( \delta^2 E \). Weinstein [43] showed that for a homogeneous (translation-invariant) medium, \( (V_i \equiv 0, V_{nl} \equiv 0) \), for \( u(x; \mu) > 0 \), if the slope condition (4) is satisfied and \( L_+ \) has only one negative eigenvalue, then the solitons are nonlinearly stable. Later, in [47] (Theorem 3.1; see also Theorem 6 of [48]) it was shown that in the presence of a linear potential which is bounded below and decaying at infinity, solitons are stable if in addition, \( L_+ \) also has no zero eigenvalue(s). A related treatment was given to the narrow soliton (semi-classical limit) subcritical nonlinearity case in [33, 44, 49] and for solitons in spatially varying nonlinear potentials in [33, 34].

General sufficient conditions for instability were given by Grillakis [50] and Jones [51]. These results imply that if either the slope is positive or if \( L_+ \) has more than one negative eigenvalue, then the soliton is unstable.

A direct consequence of the arguments of Section 3 and Theorem 3.1 in [47], [48], and [50, 51], is a stability theorem (used in this paper), which applies to positive solitons of NLS (1), whose frequencies lie in the semi-infinite spectral gap of \( -\Delta + V \); see also [53].

**Theorem III.1** Let \( u(\vec{x}) \) be a positive solution of Eq. (2) with propagation constant \( \mu \) within the semi-infinite gap, i.e. \( \mu < \mu_{BE}^{(V)} \). Then, \( A = u(\vec{x})e^{-\mu iz} \) is an orbitally-stable solution of the NLS (1) if both of the following conditions hold:

1. The slope (Vakhitov-Kolokolov) condition:

\[
\frac{dP}{d\mu} < 0.
\]
2. The spectral condition: \( L_+ \) has no zero eigenvalues and
\[
n_-(L_+) = 1. \tag{7}
\]
If either \( \frac{dP}{d\mu} > 0 \) or \( n_- \geq 2 \), the soliton is unstable.

We note that Theorem III.1 does not cover two cases:
1. \( \frac{dP}{d\mu} = 0 \): For homogeneous media, \( V_l = V_\text{nl} = 0 \), solitons are unstable; see [55, 61] for power nonlinearities and [52] for general nonlinearities. There are no analytic results for inhomogeneous media.
2. \( n_-(L_+) = 1 \) and zero is an eigenvalue of multiplicity one or higher: This case will be discussed in Sections III.A and III.B.

We also note that Grillakis, Shatah and Strauss (GSS) [44, 45] gave an alternative abstract formulation of a stability theory for positive solitons Hamiltonian systems, including NLS with a general class of linear and nonlinear spatially dependent potentials. In this formulation, the spectral condition on \( n_-(L_+) \) and the slope condition are coupled, see detailed discussion in [55]. The formulation of Theorem III.1 is a more refined and stronger statement. Specifically, it decouples the slope condition and the spectral condition on \( n_-(L_+) \) as two independent necessary conditions for stability and shows that a violation of either of them would lead to instability. This decoupling is at the heart of our qualitative approach since violation of each condition leads to a different type of instability. Stability of solitons in homogeneous media has also been investigated using the Hamiltonian-Power curves, see e.g., [46].

A. Review of stability conditions in homogeneous media

Stability and instability of solitons in homogeneous media (i.e., \( V \equiv 0 \)) have been extensively investigated [54]. In this case, \( \mu_{RE} \equiv 0 \), i.e., the semi-infinite gap associated with Eq. (3) is \( (-\infty, 0) \). For every \( \mu < 0 \) and \( \vec{x}_0 \in \mathbb{R}^d \), there exists a soliton centered at \( \vec{x}_0 \) which is radially-symmetric in \( r = |\vec{x} - \vec{x}_0| \), positive, and monotonically decaying in \( r \). In the case of a power-law nonlinearity \( F(|u|) = |u|^{2\sigma} \), the slope condition (6) depends on the dimension \( d \) and nonlinearity exponent \( \sigma \) as follows [43, 55]:

1. In the subcritical case \( d < 2/\sigma \), \( \frac{dP}{d\mu} < 0 \). Hence, the slope condition is satisfied.
2. In the critical case \( d = 2/\sigma \), the soliton power does not depend on \( \mu \), i.e., \( \frac{dP}{d\mu} \equiv 0 \). By [43], the slope condition is violated.
3. In the supercritical case \( d > 2/\sigma \), \( \frac{dP}{d\mu} > 0 \). Hence, the slope condition is violated.

Thus, the slope condition is satisfied only in the subcritical case.

When \( V \equiv 0 \), the spectrum of \( L_+ \) is comprised of three (essential) parts [55], see Figure 1.

1. A simple negative eigenvalue \( \lambda_{\text{min}} < 0 \), with a corresponding positive and radially-symmetric eigenfunction \( f_{\text{min}} \). In [33], it was shown that for power nonlinearities, \( F(|u|) = |u|^{2\sigma} \), \( f_{\text{min}} = u^{\sigma+1} \) and \( \lambda_{\text{min}} = \sigma(\sigma+2)\mu \).
2. A zero eigenvalue with multiplicity \( d \), i.e., \( \lambda_{0,j} = 0 \) with eigenfunctions \( f_j = \frac{\partial u}{\partial x_j} \) for \( j = 1, \ldots, d \). These zero eigenvalues manifest the translation invariance in a homogeneous medium in all \( d \) directions.
3. A strictly positive continuous spectrum \([-\mu, \infty)\).

FIG. 1: The spectrum of \( L_+ \) in a homogeneous medium.

Theorem III.1 does not apply directly for the stability of solitons in homogeneous medium because \( \lambda_{0,j} = 0 \) and \( n_- = 1 \). Accordingly, the notion of orbital stability must be modified. Indeed, by the Galilean invariance of NLS for \( V_l = V_\text{nl} = 0 \), an arbitrarily small perturbation of a soliton can result in the soliton moving at small uniform speed to infinity. The orbit in a homogeneous medium is thus the group of all translates in phase and space, i.e., \( \{ u(\vec{x} - \vec{x}_0; \mu)e^{i\gamma} : \vec{x}_0 \in \mathbb{R}^d, \gamma \in [0, 2\pi) \} \) and orbital stability is given by Definition II.1 but where the infinums are taken over all \( \gamma \) and \( \vec{x}_0 \).

Accordingly, Weinstein showed in [43] that in the case of homogeneous media, the spectral condition can be slightly relaxed so that it is satisfied if \( L_+ \) has only one negative eigenvalue and \( d \) zero eigenvalues, associated with the translational degrees of freedom of NLS. Hence, the spectral condition is satisfied in homogeneous media and stability is determined by the slope condition alone [43]. In particular, solitons in homogeneous media with a power-law nonlinearity \( F(|u|^2) = |u|^{2\sigma} \) are stable only in the subcritical case \( \sigma < 2/d \).

B. Stability conditions in inhomogeneous media

Below we investigate how the two stability conditions are affected by a potential/lattice.

Generically, in the subcritical \( (d < 2/\sigma) \) and supercritical \( (d > 2/\sigma) \) cases, the slope has an \( O(1) \) magnitude in a homogeneous medium. Hence, a weak lattice can affect the magnitude of the slope but not its sign, see e.g., [35]. Clearly, a sufficiently strong lattice can alter the sign of the slope, see e.g., [36] for the subcritical case and [56, 57] for the supercritical case. The situation is very different in the critical case \( (d = 2/\sigma) \). Indeed, since the slope is zero in a homogeneous medium, any potential, no matter how weak, can affect the sign of the slope.

The potential can affect the spectrum of \( L_+ \) in two different ways: 1) shift the eigenvalues, and 2) open gaps (bounded-
intervals) in the continuous spectrum, see Figure 2. In general, the minimal eigenvalue of $L_+$ remains negative, i.e., $\lambda_{\min}^{(V)} < 0$, the continuous spectrum remains positive, and the zero eigenvalues can move either to the right or to the left. Hence, generically, the spectrum of $L_+$ has the following structure:

1. A simple negative eigenvalue $\lambda_{\min}^{(V)} < 0$ with a positive eigenfunction $f_{\min}^{(V)} > 0$.
2. Perturbed-zero eigenvalues $\lambda_{0,j}^{(V)}$ with eigenfunctions $f_j^{(V)}$, for $j = 1, \ldots, d$.
3. A positive continuous spectrum, sometimes with a band-gap structure, beginning at $-\mu_{BE}^{(V)} > 0$.

This structure of the spectrum was proved in [33] for solitons in the presence of a nonlinear lattice, i.e., Eq. (1) with $V_l \equiv 0$. For a linear lattice, the proof of the negativity of $\lambda_{\min}^{(V)}$ is the same as in [33]. The proof of the positivity of $-\mu_{BE}^{(V)}$ is the same as in [33] for potentials that decay to 0 as $|\vec{x}| \to \infty$.

Although generically $\lambda_{0,j}^{(V)}(\mu) \neq 0$, there are two scenarios in which $\lambda_{0,j}^{(V)}$ equals zero:

1. The potential is invariant under a subgroup of the continuous spatial-translation group. For example (see also [34]), in a one-dimensional lattice embedded in 2D, i.e., $\frac{\partial V(x,y)}{\partial y} \equiv 0$, one has $\lambda_{0,j}(\mu) \equiv 0$. In such cases, the zero eigenvalues do not lead to instability for the reasons given in Section III A. Rather, the orbit and distance function are redefined modulo the additional invariance, e.g., in the example above the orbit is $\{u(x, y - y_0; \mu)e^{i\gamma} : y_0 \in \mathbb{R}, \gamma \in [0, 2\pi]\}$.

2. In the presence of spatial inhomogeneity, $V \neq 0$, $\lambda_{0,j}^{(V)}$ can cross zero as $\mu$ is varied. See for example [81] and the examples discussed in Sections VIII D and IX. This crossing can be associated with a bifurcation and the existence of a new branch of solitons and an exchange of stability from the old to the new branch; see the symmetry breaking analysis of [81]. In such cases, stability and instability depend on the details of the potential and nonlinearity.

In some cases, there are also positive discrete eigenvalues in $(0, -\mu)$. However, these eigenvalues do not affect the orbital stability, since they are positive. They do play a role, however, in the scattering theory of solitons [77, 78].

We note that in many previous studies, only the slope condition was checked for stability. As Theorem III.1 shows, however, “ignoring” the spectral condition is justified only for solitons centered at lattice minima, since only then the spectral condition is satisfied. In all other cases, checking only the slope condition usually lead to incorrect conclusions regarding stability.

C. Instability and collapse

We recall that in a homogeneous medium with a power nonlinearity, all solutions of the subcritical NLS exist globally. For critical and supercritical NLS there are collapsing (singular) solutions [61], i.e., solutions for which $\int |\nabla A(x, z)|^2 \, dx$ tends to infinity in finite distance. Hence, in a homogeneous medium, the two phenomena of collapse and of soliton instability appear together. In fact, the two phenomena are directly related, since in the critical and supercritical cases, the instability of the solitons is manifested by the fact that they can collapse under infinitesimally small perturbations (i.e., a strong instability).

As we shall see below, the situation is different in inhomogeneous media. Indeed, the soliton can be unstable even if all solutions of the corresponding NLS exist globally. Conversely, the soliton can be stable, yet undergo collapse under a sufficiently strong perturbation. Such results on the “decoupling” of instability and collapse have already appeared in [16, 33, 34, 35, 36, 40]. In all of these cases, the “decoupling” is related to the absence of translation invariance.

IV. QUALITATIVE APPROACH – CLASSIFICATION OF INSTABILITY DYNAMICS

The dynamics of orbitally-stable solitons is relatively straightforward - the solution remains close to the unperturbed soliton. On the other hand, there are several possible ways for a soliton to become unstable: it can undergo collapse, complete diffraction, drift, breakup into separate structures, etc.

Theorem III.1 is our starting point for the classification of the instability dynamics, since it suggests that there are two independent mechanisms for (in)stability. In fact, we show below that the instability dynamics depends on which of the two conditions for stability is violated.

As noted in Section III C in a homogeneous medium with a power-law nonlinearity, when the slope condition is violated, the soliton can collapse (become singular) under an infinitesimal perturbation. If the perturbation increases the
beam power, then nonlinearity dominates over diffraction so
that the soliton amplitude becomes infinite as its width shrinks
to zero. If the perturbation is in the “opposite direction”, the
soliton diffracts to zero, i.e., its amplitude goes to zero as its
width becomes infinite, see e.g., Theorem 2 of [48]. More
generally, in other types of nonlinearities or in the presence
of inhomogeneities, there are cases where the slope condition
is violated but collapse is not possible (e.g., in the one-
dimensional NLS with a saturable nonlinearity [63]). In such
cases, a violation of the slope condition leads to a focusing
instability whereby infinitesimal changes of the soliton can
result in large changes of the beam amplitude/width, but not
in collapse or total diffraction. Accordingly, we refer to the in-
stability which is related to the violation of the slope condition
as a focusing instability (rather than as a collapse instability).

When the soliton is unstable because the spectral condition
is violated, it undergoes a drift instability whereby infinitesi-
mal shifts of the initial soliton lead to a lateral movement
of the soliton away from its initial location. The mathemati-
cal explanation for the drift instability is as follows. The
reflective case is of the soliton away from its initial location. The mathematical relation between the vi-
ability which is related to the violation of the slope condition
as a focusing instability.

According to Fermat’s Principle, light bends to-
the potential $V_{\lambda}$ at a lattice minimum, one sees that the drift instability of solitons
centered at lattice maxima is a manifestation of Fermat’s prin-
ciple.

V. QUANTITATIVE APPROACH

As noted, the soliton is drift-uneatable when $\lambda_{0,j}^{(V)} < 0$ but
drift-stable when $\lambda_{0,j}^{(V)} \geq 0$. Thus, there is a discontinuity in
the behavior as $\lambda_{0,j}^{(V)}$ passes through zero. Nevertheless, one
can expect the transition between drift instability and drift sta-
tability to be continuous, in the sense that as $\lambda_{0,j}^{(V)}$ approaches
zero from below, the rate of the drift becomes slower and slower.
Similarly, we can expect that as $\lambda_{0,j}^{(V)}$ becomes more
negative, the drift rate will increase.

The quantitative relation between the value of $\lambda_{0,j}^{(V)}$ and the
drift rate was found analytically for the first time in [5] for
narrow solitons in a Kerr medium with a linear lattice. Later,
based on the linearized NLS dynamics, it was shown in [37]
that for solitons of any width, any nonlinearity and any linear
or nonlinear potential, this quantitative relation is as follows.

Let us define the center of mass of a perturbed soliton in the
$x_j$ coordinate as

$$\langle x_j \rangle := \frac{1}{P} \int x_j |A|^2 d\vec{x}. \quad (9)$$

Then, by [37], the dynamics of $\langle x_j \rangle$ is initially governed by
the linear oscillator equation

$$\frac{d^2}{dz^2} \langle x_j \rangle - \xi_{0,j} \langle x_j \rangle = \Omega_j^2 \langle x_j \rangle - \xi_{0,j}, \quad (10)$$

with the initial conditions

$$\langle x_j \rangle \big|_{z=0} = \frac{\int x_j |A_0|^2 d\vec{x}}{P}, \quad \langle x_j \rangle \big|_{z=0} = 2d \cdot Im \int A_0^* \nabla A_0 d\vec{x}/P. \quad (11)$$

Here, $\xi_{0,j}$ is the location of the lattice critical point in the jth
direction (not to be confused with $\langle x_j \rangle \big|_{z=0}$, the value of the
center of mass at $z = 0$). The forcing is given by

$$\Omega_j^2 = -C_j \lambda_{0,j}^{(V)}, \quad C_j = \frac{\langle f_{\lambda_{0,j}^{(V)}}^v, f_{\lambda_{0,j}^{(V)}}^v \rangle}{\langle L^{-1} f_{\lambda_{0,j}^{(V)}}, f_{\lambda_{0,j}^{(V)}} \rangle}, \quad (12)$$

where $f_{\lambda_{0,j}^{(V)}}$ is the eigenmode of $L_+$ that corresponds to $\lambda_{0,j}^{(V)}$, i.e., the eigenmode along the $x_j$ direction, the operator $L_-$ is
given by

$$L_- = -\Delta - \mu - (1 - V_n(\vec{x})) F(u^2) + V_l,$$

and the inner product is defined as $\langle f, g \rangle = \int fg^* d\vec{x}$.

Since $L_-$ is non-negative for positive solitons, it follows
that $C_j > 0$. Therefore, when $\lambda_{0,j}^{(V)}$ is negative, $\Omega_j$ is real
and when $\lambda_{0,j}^{(V)}$ is positive, $\Omega_j$ is purely imaginary. Hence, by
Eqs. (10)-(12), it follows that the lateral dynamics of a general
incident beam centered near a lattice minimum is

$$\langle x_j \rangle = \langle x_j \rangle \big|_{z=0} \cos(\Omega_j |z|) + \frac{d \langle x_j \rangle \big|_{z=0}}{\Omega_j} \sin(\Omega_j |z|), \quad (13)$$

i.e., the soliton drifts along the $x_j$ coordinate at the rate $\Omega_j$.
On the other hand, the lateral dynamics of a general incident
beam centered near a lattice maximum is

$$\langle x_j \rangle = \langle x_j \rangle \big|_{z=0} \cosh(\Omega_j |z|) + \frac{d \langle x_j \rangle \big|_{z=0}}{\Omega_j} \sinh(\Omega_j |z|). \quad (14)$$

i.e., the soliton is pulled back towards $\xi_{0,j}$ by a restoring force
which is proportional to $\Omega_j^2$, so that it undergoes oscillations
around $\xi_{0,j}$ in the $x_j$ coordinate with the period $|\Omega_j|$.

As noted, the soliton is focusing-unstable when the slope
d$P/d\mu$ is non-negative, and focusing-stable when the slope
is negative. In a similar manner to the continuous transition between drift stability and instability, one can expect the transition between focusing stability and instability to be continuous. In other words, one can expect the magnitude of the slope to be related to the strength of focusing stability or instability. At present, the quantitative relation between the magnitude of the slope and the strength of the stability is not known, i.e., we do not have a relation such as (10). However, numerical evidence for this link was found in several of our earlier studies [33, 34, 35, 36]. For example, in the case of focusing-stable solitons that collapse under sufficiently large perturbations, it was observed that as the magnitude of the slope increases, the magnitude of the perturbation that is needed for the soliton to collapse also increases. Thus, the magnitude of the slope is related to the size of the basin of stability [33, 34, 35]. In cases of focusing-stable solitons where collapse is not possible, when the magnitude of the slope increases, the focusing stability is stronger in the sense that for a given perturbation, the maximal deviation of the soliton from its initial amplitude decreases [36].

A. Physical vs. Mathematical stability

The quantitative approach is especially important in the limiting cases of “weak stability/instability”, i.e., when one is near the transition between stability and instability. For example, consider a soliton for which the two conditions for stability are met, but for which $\lambda_{0,j}^{(V)}$ or the slope are very small in magnitude. Such a soliton is orbitally stable, yet it can become unstable under perturbations which are quite small compared with typical perturbations that exist in experimental setups. Hence, such a soliton is “mathematically stable” but “physically unstable”, see e.g., [33]. Conversely, consider an unstable soliton for which either $\lambda_{0,j}^{(V)}$ is negative but very small in magnitude or the slope is positive but small. In this case, the instability develops so slowly so that it can be sometimes neglected over the propagation distances of the experiment. Such a soliton is therefore “mathematically unstable” but “physically stable” [35].

VI. GENERAL RULES

We can summarize the results described so far by several general rules for stability and instability of bright positive lattice solitons.

The qualitative approach rules are:

QL1 Bright positive lattice solitons of NLS equations can become unstable in only two ways: focusing-instability or drift-instability.

QL2 Violation of the slope condition leads to an focusing-instability, i.e., either initial diffraction or initial self-focusing. In the latter case, self-focusing can lead to collapse. Note, however, that for “subcritical” nonlinearities, the self-focusing is arrested.

QL3 The spectral condition is generically satisfied when the soliton is centered at a potential minimum and violated when the soliton is centered at a potential maximum or saddle point.

QL4 Violation of the spectral condition leads to a drift-instability, i.e., an initial lateral drift of the soliton from the potential maximum/saddle point towards a nearby lattice minimum.

The quantitative theory rules are:

QN1 The strength of the focusing- and drift-stability and instability depends on the magnitude of the slope $|\frac{dP}{d\mu}|$ and the magnitude of $|\lambda_{0,j}^{(V)}|$, respectively.

QN2 The lateral dynamics of the beam is initially given by Eqs. (10)-(12).

The above rules were previously demonstrated for 1D solitons in a periodic nonlinear lattice [33], for an anisotropic 2D lattice [34] and for several specific cases of linear lattices [35, 36]. In this paper, we demonstrate that these rules apply in a general setting of dimension, nonlinearity, linear/nonlinear lattice with any structure and for any soliton width. In particular, we use these general rules to explain the dynamics of lattice solitons in a variety of examples that were not studied before.

VII. NUMERICAL METHODOLOGY

Below we present a series of numerical computations that illustrate the qualitative and quantitative approaches presented in Sections VI, VII. We present results for the 2D cubic NLS

$$iA_z(x, y, z) + \Delta A + |A|^2 A - V(x, y) A = 0,$$

with periodic lattices, lattices with a vacancy defect, and lattices with a quasicrystal structure. There are two reasons for the choice of the 2D cubic NLS. First, this equation enables us to illustrate the instability dynamics in dimensions larger than one, in particular, in cases where the dynamics in each direction is different (e.g., as for solitons centered at saddle points). Second, the 2D cubic NLS enables us to elucidate the distinction between instability and collapse. Indeed, we recall that a necessary condition for collapse in the 2D cubic NLS is that the power of the beam exceeds the critical power $P_c \approx 11.7$ [61].

We first compute the soliton profile by solving Eq. (2) using the spectral renormalization method [64]. Once the solitons are computed for a range of values of $\mu$, the slope condition (6) is straightforward to check. In order to check the spectral condition (7), the perturbed-zero eigenvalues $\lambda_{0,j}^{(V)}$ (and the corresponding eigenfunctions $f_j$) of the discrete approximation of the operator $L_+$ are computed using the numerical method presented in [35, Appendix D]. The value of $\Omega_j$ is calculated from Eq. (15) by inversion of the discrete approximation of the operator $L_+$. 
Eq. (15) is solved using an explicit Runge-Kutta four-order finite-difference scheme. Following [33, 34, 35, 36], the initial conditions are taken to be the unperturbed lattice soliton $u(x, y)$ with either

1. a small power perturbation, i.e.,
   \[ A_0(x, y) = \sqrt{1 + c u(x, y)}, \]  
   (16)

   where $c$ is a small constant that expresses the excess power of the input beam above that of the unperturbed soliton, or

2. a small lateral shift, i.e.,
   \[ A_0(x, y) = u(x - \Delta x_0, y - \Delta y_0), \]  
   (17)

   where $\Delta x_0$ and $\Delta y_0$ are small compared with the characteristic length-scale (e.g., period) of the potential.

The motivation for this choice of perturbations is that each perturbation predominantly excites only one type of instability. Indeed, by Eq. (10)-(11), it is easy to verify that under a power perturbation (16), the center of mass will remain at its initial location (cf. [33, 34, 35]), i.e., no lateral drift will occur. In this case, only an focusing instability is possible. On the other hand, the asymmetric perturbation (17) will predominantly excite a drift instability (but if the soliton is drift-stable, this perturbation can excite an focusing instability, see Figure 3).

The advantage of the perturbations (16)-(17) over adding random noise to the input soliton is that they allow us to control the type of instability that is excited. Moreover, grid convergence tests are also simpler. Once the NLS solution is computed, it is checked for focusing and drift instabilities by monitoring the evolution of the normalized peak intensity

\[ I(z) := \frac{\max_{x,y}|A(x, y, z)|^2}{|A_0(x, y)|^2}, \]  
   (18)

and of the center of mass [9], respectively.

### VIII. PERIODIC SQUARE LATTICES

We first choose the sinusoidal square lattice

\[ V(x, y) = \frac{V_0}{2} \left[ \cos^2(2\pi x) + \cos^2(2\pi y) \right], \]  
   (19)

which is depicted in Figure 3. We consider this to be the simplest 2D periodic potential, as all the local extrema are also global extrema. This lattice can be created through interference of two pairs of counter-propagating plane waves, and is standard in experimental setups, see, e.g., [65, 66]. The stability and instability dynamics are investigated below for solitons centered at the lattice maxima, minima, and saddle points, see Figure 3(b).

![Figure 3: (Color online) The sinusoidal square lattice given by Eq. (19) with $V_0 = 5$. (a) Top view. (b) Side view. The solitons investigated below are centered at the lattice maximum (0,0), lattice minimum (0.25,0.25), and saddle point (0.25,0.5).]

#### A. Solitons at lattice minima

We first investigate solitons centered at the lattice minimum $(x_0, y_0) = (0.25, 0.25)$. Figure 4(a) shows that the power of solitons at lattice minima is below the critical power for collapse, i.e., $P(\mu) < P_c \approx 11.7$ for all $\mu$. As the soliton becomes narrower ($\mu \rightarrow -\infty$), the soliton power approaches $P_c$ from below (as was shown numerically in [40] for this lattice and analytically in [35] for any linear lattice). In addition, as the soliton becomes wider ($\mu \rightarrow \mu_{BE}$, the edge of the first band), its power approaches $P_c$ from below (rather than becomes infinite, as implied in [40]), see also [92]. The minimal power is obtained at $\mu = \mu_m \approx -10$. The power curve thus has a stable branch for narrow solitons ($-\infty < \mu < \mu_m$) where the slope condition is satisfied, and an unstable branch for wide solitons ($\mu_m < \mu < \mu_{BE}$) where the slope condition is violated. Therefore, wide solitons should be focusing-unstable while narrow solitons should be focusing-stable. Figure 4(b) shows that, as expected for solitons at lattice minima, $\lambda_0^{(3)} > \lambda_0^{(2)} > 0$ for all $\mu$. Hence, the spectral condition is fulfilled. Consequently, solitons at lattice minima should not experience a drift instability.

In order to excite the focusing instability alone, we add to the soliton a small power perturbation, see Eq. (16). We contrast the dynamics in a neighborhood of stable and unstable solitons by choosing two solitons with the same power ($P \equiv 0.98P_c$), from the stable branch ($\mu = -31$) and from the unstable branch ($\mu = -3$). We perturb these solitons with the same power perturbations ($c = 0.5\%$, $1\%$, $2\%$).

When $c = 0.5\%$ and $1\%$, the input power is below the threshold for collapse ($P < P_c$). In these cases, the self-focusing process is arrested and, during further propagation, the normalized peak intensity undergoes oscillations (see Figures 5(a) and (b)). For a given perturbation, the oscillations are significantly smaller for the stable soliton compared with the unstable soliton.

When $c = 2.5\%$, the input power is above the threshold for collapse ($P > P_c$) and the solutions undergo collapse. Therefore, for such large perturbations, collapse occurs for both stable and unstable solitons, i.e., even when both the slope and spectral conditions are fulfilled. This shows yet again that in an inhomogeneous medium, collapse and instability are not
necessarily correlated.

![Diagram](image)

FIG. 4: (Color online) (a) Power, and (b) perturbed-zero eigenvalues, as functions of the propagation constant, for solitons centered at a maximum (blue, dashes) and minimum (red, dots) of the lattice [19] with \( V_0 = 5 \). Also shown are the corresponding lines for the homogeneous NLS equation (solid, green). The circles (black) correspond to the values used in Figs. [5] [20].

In order to confirm that solitons centered at a lattice minimum do not undergo a drift instability, we shift the soliton slightly upward by using the initial condition (17) with \( \mu = -31 \). Input powers are 0.5\% (red dots), 1\% (blue dashes), and 2.5\% (solid green) above the soliton power.

![Diagram](image)

FIG. 5: (Color online) Normalized peak intensity (18) of solutions of Eq. (15) with the periodic lattice (19) with \( V_0 = 5 \). Initial conditions are power-perturbed solitons [see Eq. (16)] centered at a lattice minimum: (a) Soliton from the stable branch \((\mu = -31)\); (b) Soliton from the unstable branch \((\mu = -3)\). Input powers are 0.5\% (red dots), 1\% (blue dashes), and 2.5\% (solid green) above the soliton power.

In addition, by Eq. (12), \( \Omega_y \approx 11.12i \) for \( \mu = -31 \) and \( \Omega_y \approx 2.58i \) for \( \mu = -3 \). Figure 6(a1) shows that for \( \mu = -31 \), the center of mass in the \( y \)-direction of the position-shifted soliton follows the theoretical prediction (20) accurately over several oscillations. In addition, the center of mass in the \( x \)-direction remain at \( x = 0 \) (data not shown), in agreement with Eq. (20). Thus, the soliton is indeed drift-stable.

The situation is more complex for \( \mu = -3 \). In this case, the position-shifted soliton follows the theoretical prediction (20) over more than 2 diffraction lengths (i.e., for \( z > z_0 \) where \( z_0 \approx 1 \)), but then deviates from it, see Figure 6(b1). The reason for this instability is that for \( \mu = -3 \), the slope condition is violated. Since the position-shifted initial condition can also be viewed as an asymmetric amplitude power perturbation \( \Delta A = u(x - \Delta x_0, y - \Delta y_0) - u(x, y) \), an focusing instability is excited and the soliton amplitude changes significantly, the theoretical prediction for the lateral dynamics is no longer valid. In order to be convinced that the initial instability in this case is of an focusing-type rather than drift-type, we note that for \( \mu = -31 \) for which the slope condition is satisfied, the soliton remains focusing-stable, see Figure 6(a2).

![Diagram](image)

FIG. 6: (Color online) Dynamics of solutions of Eq. (15) with the periodic lattice (19) with \( V_0 = 5 \). Initial conditions are position-shifted solitons [see Eq. (17)] centered at a lattice minimum, with \((\Delta x_0, \Delta y_0) = (0, 0.04)\). (a1) Center of mass of the soliton in the \( y \)-coordinate (blue, solid line) and analytical prediction [Eq. (20), red, dashes] for \( \mu = -31 \); (a2) Normalized peak intensity (18) for \( \mu = -31 \); (b1) and (b2) are the same as (a1) and (a2), but for \( \mu = -3 \).

B. Solitons at lattice maxima

We now investigate solitons centered at the lattice maximum \((x_0, y_0) = (0, 0)\). Figure 4 shows that in general, solitons at lattice maxima have the opposite stability characteristics compared with those of solitons centered at lattice minima: The slope condition is violated for narrow solitons and satisfied for wide solitons, the power is above \( P_c \) [92], and the perturbed-zero eigenvalues \( \lambda_{0j}^{(V)} \) are always negative. Interestingly, for the specific choice of the lattice (19), the powers and perturbed-zero eigenvalues at lattice maxima and minima are approximately, but not exactly, images of each other with respect to the case of a homogeneous medium.

The negativity of the perturbed-zero eigenvalues implies that solitons centered at a lattice maximum undergo a drift instability (see Figure 8(b)). However, if the initial condition is subject to a power perturbation, see Eq. (16), then no drift occurs. In this case, stability is determined by the slope condition. For example, Figure 7 shows the dynamics of a power-perturbed wide soliton for which the slope condition is satisfied. When the soliton’s input power is increased by

![Diagram](image)
0.5%, the solution undergoes small focusing-defocusing oscillations, as in Figure 5(a), i.e., it is stable under symmetric perturbations. When the soliton’s input power is increased by 1%, the perturbation exceeds the “basin of stability” of the soliton \([15]\) and the soliton undergoes collapse. These results again demonstrate that collapse and instability are independent phenomena.

If the initial condition is asymmetric with respect to the lattice maximum, the soliton will undergo a drift instability. In Figure 8 we excite this instability with a small upward shift, namely, Eq. (17) with \((\Delta x_0, \Delta y_0) = (0, 0.02)\). Under this perturbation, the solution of Eq. (10) is

\[
\langle x \rangle \equiv 0, \quad \langle y \rangle = \Delta y_0 \cdot \cosh(\Omega_y z). \tag{21}
\]

with \(\Omega_y \approx 3.9\). In the initial stage of the propagation \((z < 0.5)\) the soliton drifts toward the lattice minimum—precisely following the asymptotic prediction \([20]\), see Figure 8(a), but the soliton’s amplitude is almost constant), see Figure 8(b). During the second stage of the propagation \((0.5 < z < 0.99)\) the soliton drifts somewhat beyond the lattice minimum as it begins to undergo self-focusing. In the final stage \((0.99 < z < 1)\) the soliton undergoes collapse (Figure 8(b)). The global dynamics can be understood in terms of the stability conditions for solitons centered at lattice minima and maxima as follows. The initial soliton, which is centered at a lattice maximum, satisfies the slope condition but violates the spectral condition. Consistent with these traits, the soliton is focusing-stable but undergoes a drift instability. As the soliton gets closer to the lattice minimum, it can be viewed as a perturbed soliton centered at the lattice minimum, for which the spectral condition is fulfilled and the soliton power is below \(P_L\) (see Figure 8(b)). Indeed, at this stage, the drift is arrested because the beam is being attracted back towards the lattice minimum. Moreover, the beam now is a strongly power-perturbed soliton, since the beam power \((\approx 1.03 P_L)\) is \(\approx 6\%\) above the power of the soliton at a lattice minimum. Hence, in a similar manner to the results of Figure 5(a), the perturbation exceeds the “basin of stability” and the soliton undergoes collapse.

![FIG. 7: (Color online) Same as Figure 5(a) for a soliton at a lattice maximum with \(\mu = -5\) (stable branch) and input power that is 0.5% (red dots) and 1% (blue dashes) above the soliton power.](image)

![FIG. 8: (Color online) Dynamics of a soliton at a lattice maximum with \(\mu = -5\), which is position-shifted according to (17) with \((\Delta x_0, \Delta y_0) \approx (0, 0.02)\). (a) Center of mass in the \(y\) coordinate (blue, dashes) and the analytical prediction (Eq. (21)) with \(\Omega_y \approx 3.9\), solid black). Location of lattice minimum and maxima are denoted by thin magenta and black horizontal lines, respectively. (b) Normalized peak intensity.](image)
of the perturbed-zero eigenvalues that correspond to solitons centered at a lattice minimum and maximum, respectively. This can be understood by rewriting the lattice (19) as

$$V(x, y) = \frac{V_0}{2} \left[ 1 - \cos^2(2\pi(x - 0.25)) + \cos^2(2\pi y) \right].$$

Thus, apart from the constant part (i.e., the first term), the difference between the lattices is the sign before the $x$-component of the lattice. In that sense, in the $x$ direction, the saddle point is equivalent to a maximum point, hence, the similarity between the eigenvalues. Another consequence of the saddle point is the difference between the eigenvalues at lattice maxima and minima (see Figure 9(b1,b2)). As noted before, this will be no longer true if the lattice changes in the $x$ and $y$ directions will no longer be equal.

![FIG. 9: (Color online) (a) The perturbed-zero eigenvalues at the saddle point. One eigenvalue is shifted to positive values (magenta), and is indistinguishable from the eigenvalue at lattice minima (red); one eigenvalue is shifted to negative values (black), and is indistinguishable from the eigenvalue at lattice maxima (blue). (b1) Same data as in Figure 2(a), with the addition of data for solitons centered at a saddle point of the lattice (black, dash-dots). (b2) Same as (b1) showing only the data for solitons centered at a saddle point (black, dash-dots) and for the homogeneous medium soliton (green line).]

If we apply perturbations in the stable and unstable directions simultaneously $(\Delta x_0, \Delta y_0) \approx (0.0156, 0.0156)$, the dynamics in each coordinate is nearly identical to the dynamics when the perturbation was applied just in that direction. Thus, there is a "decoupling" between the (lateral dynamics in the $x$ and $y$ directions. Indeed, this decoupling follows directly from Eq. (10).

### D. Solitons at a shallow-maximum

We now consider solitons of the periodic potential

$$V(x, y) = \frac{V_0}{25} [2 \cos(2\pi x) + 2 \cos(2\pi y) + 1]^2,$$

where $V_0 = 5$ and the normalization by 25 implies that $V_0 = \max_{x,y} V(x, y)$. Unlike the lattice (19), the lattice (23) also has shallow local maxima that are not global maxima (e.g., at $(0.5, 0.5)$).

The stability and instability dynamics of solitons centered at global minima, maxima and saddle points of the lattice are similar to the case of the lattice (19), which was already studied. Hence, we focus only on the stability of solitons centered at a shallow maximum.

Since the lattice is invariant under a $90^\circ$ rotation, the perturbed-zero eigenvalues are equal, i.e., $\lambda_0^{(1)} = \lambda_0^{(2)}$. However, unlike solitons centered at a global maximum, the corresponding perturbed-zero eigenvalues are negative only for very negative values of $\mu$ (narrow beams) but become positive for values of $\mu$ near the band edge $\mu_{BE}$ (wide beams), see Figure 12(b). The reason for the positivity of $\lambda_0^{(1)} = \lambda_0^{(2)}$ despite being centered at a lattice maximum is as follows. For narrow solitons, the region where the “bulk of the beam” is located is of higher values of the potential compared with the immediate surrounding, hence, the solitons “feel” an effective lattice maximum. On the other hand, for wider solitons, the “bulk of the beam” is centered mostly at the shallow lattice maximum and the surrounding lower potential regions. Hence, although the very center of the soliton is at the shallow lattice maximum, these solitons are effectively centered at the lattice minimum with respect to the nearest global lattice maximum (see also [33], Section 4.5). The transition of the qualitative stability properties between narrow and wide solitons described above occurs when the soliton’s width is on the order of the lattice period. As noted in Section IIII B, the stability at the transition points where $\lambda_{0,j} = 0$ or $\frac{dP}{d\mu} = 0$ requires a specific study. Similarly, a comparison of Figure 12(a) and Figure 4(a) shows that the $P(\mu)$ reflects the transition between properties which are characteristic to solitons centered at lattice maxima and minima. Indeed, for narrow solitons ($\mu \rightarrow -\infty$) is similar to the power of solitons centered at a global maximum, i.e., the power is above critical and the slope is positive. On the other hand, $P(\mu)$ curve for wide solitons ($\mu \rightarrow \mu_{BE}$) is similar to the power of solitons centered at a (simple) lattice minimum, i.e., the power is below critical and the slope is positive too.

Numerical simulations (Figure 13) demonstrate this transition. For a narrow soliton ($\mu = -12$), the theoretical prediction for the dynamics of the center of mass is $x \cong 0.5 + \Delta x_0 \cosh(4.14z)$ and $y \cong 0.5 + \Delta y_0 \cosh(4.14z)$. Indeed, the narrow soliton drifts away from the shallow maximum toward the nearby (global) lattice minimum (Figure 13(a1)) and then undergoes collapse (Figure 13(a2)). This dynamics is
similar to that of solitons centered near lattice maximum or a saddle of a the lattice [19], see Sections VIII B and VIII C. On the other hand, for the wide soliton \((\mu = -2)\), the theoretical prediction for the dynamics of the center of mass is
\[
(x) \approx 0.5 + \Delta x_0 \cos(1.6z) \quad \text{and} \quad (y) \approx 0.5 + \Delta y_0 \cos(1.6z).
\]
Indeed, this soliton remains stable, undergoing small position oscillations around the shallow maximum (Figure 13(b)). This dynamics is the same as for solitons centered at a minimum of the lattice [19], see Figure 6(a). As in previous examples, the numerical results are in excellent agreement with the analytic prediction (10)-(12).

FIG. 11: (Color online) The shallow maximum periodic lattice given by Eq. (23) with \(V_0 = 5\). (a) Top view. (b) Side view. (c) Cross section along the line \(x = y\).

FIG. 12: (Color online) Same as Figure 4 for solitons centered at a shallow local maximum of the shallow-maximum periodic lattice (23).

FIG. 13: (Color online) Dynamics of a perturbed soliton at shallow-maximum periodic lattice (23) with a narrow soliton [(a1) and (a2)] with \(\mu = -12\) and a wide soliton [(b) with \(\mu = -2\)], and using \((\Delta x_0, \Delta y_0) = (0.05, 0.05)\). (a1) Center of mass \((x) = (y)\) of the narrow soliton (blue, dashes) and the analytical prediction (red dots). (a2) Normalized peak intensity of the narrow soliton. (b) Same as (a1) for the wide soliton.

IX. PERIODIC LATTICES WITH DEFECTS

Defects play a very important role in energy propagation through inhomogeneous structures. They arise due to imperfections in natural or fabricated media. They are also often specifically designed to influence the propagation.

Solitons in periodic lattices with defects have drawn much attention both experimentally and theoretically; see, for example, \([14, 67, 68, 79]\). The complexity of the lattice details offers an opportunity to demonstrate the relative ease of applying the stability/dynamics criteria to predict and decipher the soliton dynamics in them. As an example, we study lattices with a point defect. Our analysis can also extend to different types of defects such as line defects, see e.g. \([16]\).

We consider the lattice (23)
\[
V(x, y) = \frac{V_0}{25} \left| 2 \cos(2\pi x) + 2 \cos(2\pi y) + e^{i\theta(x, y)} \right|^2,
\]
where the phase function \(\theta(x, y)\) is given by
\[
\theta(x, y) = \tan^{-1} \left( \frac{y - y_0}{x} \right) - \tan^{-1} \left( \frac{y + y_0}{x} \right),
\]
see Figure 14 and also \([16]\). Compared with the shallow-maximum periodic lattice (23), here the constant (DC) component (the third term in the lattice) attains a phase distortion which creates an (effective) vacancy defect at \((0, 0)\), which is a shallow-maximum. Further, far away from the origin, the potential (24) is locally similar to the shallow-maximum periodic lattice (23). This is a generic example of a point defect, as opposed to a line defect \([69]\). In what follows, we consider solitons centered at the vacancy defect \((x_0, y_0) = (0, 0)\).

The stability properties of solitons in the shallow-maximum periodic (23) and vacancy-defect (24) lattices are strikingly similar, as can be seen from Figs. 12 and 15. In both cases, there is a marked transition between narrow and wide solitons and this transition occurs when the soliton width is of the order of the lattice period. Indeed, numerical simulations show that the dynamics of perturbed solitons is qualitatively similar in both cases – compare Figures 13 and 16. We do note that unlike the shallow-maximum periodic lattice, the perturbed-zero eigenvalues of the vacancy lattice bifurcate into different, though similar, values. The reason for this is the phase function (25) is not invariant by \(90^\circ\) rotations.

Inspecting the lattice surfaces (Figures 11 and 14), it is clearly seen that the reason for the similarity between the shallow-maximum periodic and vacancy lattices is that the vacant site is essentially a shallow local maximum itself – and only a bit shallower than those of the shallow-maximum periodic lattice (see Figure 14).

In Figure 17 we give a detailed graphical illustration of a typical instability dynamics due to a violation of the spectral condition. Figure 17(a)-(c) show contours of the soliton profiles superposed on the contour plot of the lattice. It can be seen that as a result of the initial position shift, the soliton drifts towards the lattice minimum and that it self-focuses at the same time. Figure 17(d) shows the trajectory of the beam across the lattice. In addition, Figure 17(e) shows the center of
mass dynamics as a function of the intensity $I(z)$. This shows that initially, the perturbed soliton undergoes a drift instability with little self-focusing, but that once the collapse accelerates, it is so fast so that the drift dynamics becomes negligible.

Next, we investigate solitons in quasicrystal lattices. Such lattices appear naturally in certain molecules [70, 71], have been investigated in optics [16, 23, 24, 25, 26] and in BEC [72], and can be formed optically by the far-field diffraction pattern of a mask with point-apertures that are located on the $N$ vertices of a regular polygon, or equivalently, by the sum of $N$ plane waves (cf. [16, 73]) with wavevectors $(k_x, k_y)$ whose directions are equally distributed over the unit circle. The corresponding potential is given by

$$V(x, y) = \frac{V_0}{N^2} \left| \sum_{n=0}^{N-1} e^{i(k_x^{(n)} x + k_y^{(n)} y)} \right|^2 ,$$

where $(k_x^{(n)}, k_y^{(n)}) = (K \cos(2\pi n/N), K \sin(2\pi n/N))$ [93]. The normalization by $N^2$ implies that $V_0 = \max_{x,y} V(x, y)$. The potential (26) with $N = 2, 3, 4, 6$ yields periodic lattices. All other values of $N$ correspond to quasicrystals, which have a local symmetry around the origin and long-range order, but, unlike periodic crystals, are not invariant under spatial translation [74].

We first consider the case $N = 5$ (a 5-fold symmetric “Penrose” quasicrystal) for solitons centered at the lattice maximum $(x_0, y_0) = (0, 0)$, see Figure[18]. Since the soliton profile and stability are affected mostly by the lattice landscape near its center, we can expect the stability properties of the Penrose lattice soliton at $(0, 0)$ to be qualitatively the same as for a soliton at a lattice maximum of a periodic lattice. Indeed, Figure[19] reveals the typical stability properties of solitons centered at a lattice maximum: An focusing-unstable branch for narrow solitons, an focusing-stable branch for wider solitons and negative perturbed zero-eigenvalues (compare e.g. with Figure[8]). Therefore, the Penrose soliton will drift from the lattice maximum under asymmetric perturbations and if the soliton is sufficiently narrow, it can also undergo collapse.

Figure[19] presents also the data for a perfectly periodic lattice $(N = 4)$ and for a higher-order quasicrystal $(N = 11)$.
One can see that the stability properties in these lattices is qualitatively similar to the $N = 5$ case. The only marked difference as $N$ increases is that the soliton’s power becomes larger for a given $\mu$.

These results show that in contrast to the significant effect of the quasi-periodicity on the dynamics of linear waves (compared with the effect of perfect periodicity [24]), the effect of quasi-periodicity on the dynamics of solitons is small.

The second limit is of solitons which are much narrower than the width of the potential. In this case, only the local variation of the potential affects the soliton profile and stability. Hence, the potential can be expanded as

$$V(x) = V(0) + \frac{1}{2}V''(0)x^2 + \cdots.$$ 

The qualitative and quantitative stability approaches were applied to this case in [35].

In [35, 56], the profiles, power slope and perturbed-zero eigenvalues were computed analytically (exactly or asymptotically). It was proved that the perturbed-zero eigenvalues are negative for solitons centered at lattice maxima (repulsive potential) and are positive for solitons centered at lattice minima (attractive potential). Hence, in the latter case, stability is determined by the slope condition. In those two studies, detailed numerical simulations confirmed the validity of the qualitative and quantitative approaches. Hence, we do not present a systematic stability study for localized potentials.

**XI. SINGLE WAVEGUIDE POTENTIALS**

So far we studied periodic, periodic potentials with defects and quasiperiodic potentials. However, our theory can be applied to other types of potentials. Indeed, let us consider localized potentials, such as single or multiple waveguide potentials, for which the potential decays to zero at infinity. For such potentials, there are two limits of interest. The first limit is of solitons which are much wider than the width of the potential. In this case, the potential can be approximated as a point defect in an homogeneous medium. Then, the dynamics is governed by

$$iA_x(x,z) + \Delta A + |A|^2 A - \gamma \delta(x) A = 0,$$

where $\gamma$ is a real constant. In [34], the qualitative and quantitative stability approaches were applied to Eq. (27) in one transverse dimension.

In this paper, we presented a unified approach for analyzing the stability and instability dynamics of positive bright solitons. This approach consists of a qualitative characterization of the type of instability, and a quantitative estimation of the instability growth rate and the strength of stability. This approach was summarized by several rules (Section VII) and applied to a variety of numerical examples (Sections VIII, XI), thus revealing the similarity between a variety of physical configurations which, a priori, look very different from each other. In that sense, our approach differs from most previous studies which considered a specific physical configuration.

One aspect which was emphasized in the numerical examples is the excellent agreement between direct numerical simulations of the NLS and the reduced equations for the center of mass (lateral) dynamics, Eqs. (10)-(12). Different reduced equations for the lateral dynamics were previously derived under the assumption that the beam remains close to the initial soliton profile (see e.g. [75]) or by allowing the soliton parameters to evolve with propagation distance (see e.g., [76] and references therein). These approaches, as well as ours, are valid only as long as the beam profile remains close to a soliton profile. However, unlike previous approaches, Eqs. (10)-(12) incorporate linear stability (spectral) information into the center of mass dynamics. Thus our approach shows that the beam profile evolves as a soliton perturbed by the eigenfunction $f_{0,j}^{(V)}$. The validity of this perturbation analysis is evident from the excellent comparison between the reduced Eqs. (10)-(12) and numerical simulations for a variety of lattice types. To the best of our knowledge, such an agreement was not achieved with the previous approaches.

The numerical examples in this paper were for two-dimensional Kerr media with various linear lattices. Together with our previous studies which were done for narrow solitons in any dimension [35], a linear delta-function potential [36]...
and for nonlinear lattices \cite{33, 34}, there is a strong numerical evidence that our qualitative and quantitative approaches apply to positive solitons in any dimension, any type of non-linearity of type $F(|A|^2)$ (e.g., saturable) as well as for other lattice configurations, e.g., “surface” or “corner” solitons \cite{12}. Theorem\ref{Theorem1}I has also the qualitative and quantitative approaches apply also for the $d$-dimensional discrete NLS. This equation is obtained from Eq. \eqref{eq15} by replacing $\Delta$ by the difference Laplacian operator on a discrete lattice and $V^\prime$ by a potential defined at discrete lattice sites. This model was extensively studied, mostly for periodic lattices, see e.g., for 1D and 2D discrete NLS equation with cubic nonlinearity (see e.g., \cite{11, 12, 13, 14, 15}), saturable nonlinearity (see e.g., \cite{84}), cubic-quintic nonlinearity (see e.g., \cite{85}). General results on existence and stability of solitons in $d-$dimensions with power nonlinearities appear in \cite{82, 86}. Indeed, for the discrete NLS, the operator $L_+,$ does not generally have a zero eigenvalue due to absence of continuous translation symmetry, and the continuous spectrum is a bounded interval, starting at the soliton frequency, $-\mu$ \cite{86}. However, these changes in the spectrum do not affect the stability theory, the possible types of instabilities and the analysis of their strength.

As noted, our analysis shows that for positive bright solitons, only two types of instabilities are possible - focusing instability or drift instability. Other types of instabilities may appear, but only for non-positive solitons (e.g., gap solitons or vortex solitons). A formulation of a qualitative and quantitative theories for such solitons requires further study.

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[89] Moreover, it can be shown that for gap solitons \( n_{\pm}(L_+) \equiv \infty \) (see definition in Section III) hence they are not covered by Theorem [III].
[90] Stability of solitons which are global minimizers of the Hamiltonian, \( H \), subject to fixed squared \( L^2 \) norm, \( P \), was studied by Cazenave and Lions [1]. For \( V \equiv 0 \) and power nonlinearities, \( F(|A|^2) = |A|^{2g} \), a global minimizer (and therefore stable soliton) exists in the subcritical case, \( \sigma < 2/d \). This condition on \( \sigma \) is also a consequence of the slope condition.
[91] In order to avoid confusion, we point out that the value of \( \mu \) in \( L_+ \) is fixed, so that the eigenvalues and eigenfunctions of \( L_+ \) are the solutions of

\[ L_+ (\mu; V)f(\bar{x}) = \lambda(\mu; V)f(\bar{x}). \]

[92] In fact, the soliton power approaches \( gP_c \) where \( g < 1 \), see [87].
[93] In fact, also in this case, the soliton power approaches \( gP_c \) where \( g < 1 \), i.e., the soliton power is above \( P_c \) for solitons which are not near the band edge.
[94] We note that Eq. 36 can also describe the lattices 28 and 24 for \( N = 4 \) and an additional \( k = 0 \) phase modulated plane wave.