Bertotti–Robinson solutions in five-dimensional quadratic gravity

Gérard Clément

LAPTh, Université de Savoie, CNRS, 9 chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux cedex, France

E-mail: gerard.clement@lapth.cnrs.fr

Received 14 December 2013, revised 30 January 2014
Accepted for publication 4 February 2014
Published 5 March 2014

Abstract
We construct new solutions of five-dimensional quadratic gravity as direct products of a constant curvature two-surface with a solution of three-dimensional new massive gravity with constant scalar curvature. These solutions could represent near-horizon limits of five-dimensional asymptotically flat black strings or black rings. A number of these non-asymptotically flat solutions are themselves black strings or rings. A by-product of our analysis is the construction of new solutions of four-dimensional quadratic gravity obtained by toroidal reduction of the five-dimensional solutions with flat transverse space. These again include black strings or rings, and an AdS$_2 \times T^2$ solution of $f(R)$ gravity for a specific relation between the model parameters.

Keywords: exact solutions, quadratic gravity, new massive gravity
PACS numbers: 04.20.Jb, 04.50.Gh, 04.50.Kd, 11.25.Mj

1. Introduction
It is well known that Einstein’s theory of gravity does not lead upon quantization to a perturbatively renormalizable theory. The addition to the linear Einstein–Hilbert action of curvature squared terms leads to a renormalizable theory [1]. This theory is generically non-unitary due to the occurrence in the linearized theory of massive spin-2 ghost modes, which are absent only if the quadratic action is of the Einstein–Gauss–Bonnet form [2]. However, it has recently been shown that in four [3] and in higher dimensions [4] the massive spin-2 mode may be rendered massless by a (more general) suitable choice of the model parameters.

Few exact solutions of the generic theory of $D$-dimensional quadratic gravity are known, with the exception of classes of algebraically special solutions, such as AdS waves [5], and
type III and type N solutions [6] (for a review and references, see [7]). This paper is devoted to the search for exact solutions of five- (and four-) dimensional quadratic gravity describing spacetimes which are the direct product of two manifolds, one of which at least has constant curvature.

The first example of such a solution is due to Nariai [8], who showed that the four-dimensional Einstein equations with a positive cosmological constant admit a solution with the geometry $dS_2 \times S^2$. This has been shown to be the near-extreme, near-horizon limit of the de Sitter–Schwarzschild solution [9]. The anti-Nariai solution of the Einstein equations with a negative cosmological constant, with the geometry $AdS_2 \times H^2$, can be generated from the Nariai solution by an appropriate duality transformation [10]. Better known is the Bertotti–Robinson solution [11] of four-dimensional Einstein–Maxwell theory, representing a spacetime with the geometry $AdS_2 \times S^2$ supported by a monopole electric or magnetic flux. This is the near-horizon limit of the extreme Reissner–Nordström black hole, and can be generalized to a solution of the Einstein–Maxwell equations with cosmological constant [11] (for a review, see [12]).

The anti-Nariai and Bertotti–Robinson solutions have been generalized to solutions of the $D$-dimensional cosmological Einstein equations with the geometry $AdS_2 \times \Sigma_{D-2}$, with $\Sigma_{D-2}$ a constant curvature Riemannian manifold or a product of such manifolds [13–16]. These are the near-horizon limits of $D$-dimensional black holes. Near-horizon geometries of extremal black hole solutions to higher-dimensional supergravities were classified in [17]. It has been shown that such product spacetimes also solve the $D$-dimensional Einstein–Gauss–Bonnet equations [18] and, more generally, the $D$-dimensional Einstein–Maxwell–Dilaton equations with all possible higher order corrections [19]. In this paper, we shall consider a different generalization to $D = 5$ of the Bertotti–Robinson solution, namely product spacetimes with the geometry $M_3 \times \Sigma_2$, where $M_3$ is a (not necessarily constant curvature) Lorentzian manifold, and $\Sigma_2 = S^2$, $T^2$ or $H^2$ is constant curvature. These could represent near-horizon limits of five-dimensional black strings or black rings.

To construct such solutions to five-dimensional quadratic gravity, we shall follow the strategy initiated in [20]. The direct product ansatz splits the five-dimensional field equations into a longitudinal set of three-dimensional field equations (for the $M_3$), and a transverse set of two-dimensional field equations (for the $\Sigma_2$), which in the case of a constant curvature $\Sigma_2$ amount to a constraint on the curvature invariants of the $M_3$. This reduction is carried out in section 2. It turns out that in the present case the three-dimensional field equations are those of new massive gravity (NMG) [21], while the constraint equation can be solved for a constant scalar curvature. In section 3 we review the various known solutions of NMG with constant scalar curvature, and the resulting uplifted solutions to five-dimensional quadratic gravity. Depending on the values of the coupling constants, these include in particular black strings and black rings. In the special case of a flat transverse space $\Sigma_2$, these five-dimensional solutions can be toroidally reduced to a class of non-asymptotically flat cylindrical solutions of four-dimensional quadratic gravity with the geometry $M_3 \times S^1$, again including some black strings or black rings. Our results are summarized in the last section.

2. Reduction of five-dimensional quadratic gravity

The generic action for $D$-dimensional quadratic gravity is [22]

$$I_{(D)} = \int d^Dx \sqrt{|g_{(D)}|} \left\{ \frac{\mathcal{R}}{\kappa} - \frac{2\Lambda}{\kappa} + \alpha \mathcal{R}^2 + \beta \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \gamma (\mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} - 4 \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \mathcal{R}^2) \right\}$$

(2.1)
where $\kappa$ is the $D$-dimensional Einstein constant, $\Lambda$ the cosmological constant and $\alpha, \beta, \gamma$ the quadratic coupling constants.

The equations of motion that follow from (2.1) are
\[
\frac{1}{\kappa} \left( R_{\mu\nu} - \frac{1}{2} g_{(D)\mu\nu} R + \Lambda g_{(D)\mu\nu} \right) + 2\alpha R \left( R_{\mu\nu} - \frac{1}{4} g_{(D)\mu\nu} R \right) + 2\beta \left( R_{\sigma\rho\mu\nu} - \frac{1}{4} g_{(5)\sigma\rho\mu\nu} R_{\sigma\rho} \right) R_{\sigma\rho} + 2\gamma \left( R_{\sigma\rho\sigma\rho \mu\nu} - 2R_{\sigma\rho\sigma\rho \mu\nu} R_{\sigma\rho} - 2R_{\sigma\rho\sigma\rho \mu\nu} R_{\sigma\rho} \right) = 0,
\]
where $R_{\sigma\rho\mu\nu}$ and $\nabla_{\mu}$ are the Riemann tensor and covariant derivative for the $D$-dimensional metric $g_{(D)\mu\nu}$.

Taking $D = 5$, let us carry out dimensional reduction of the field equation (2.2) relative to a constant curvature two-surface $\Sigma_2$, assuming the direct product ansatz
\[
\text{d}x_5^{(3)} = g_{\alpha\beta}(x') \text{d}x^3 + \alpha^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),
\]
where $\alpha, \beta, \gamma = 1, 2, 3$, and $k = 1, 0$ or $-1$, with
\[
s_1 = \sin \theta, \quad s_0 = \theta, \quad s_{-1} = \sinh \theta
\]
($\theta \in [0, \pi]$ for $k = 1$ and $\theta \in [0, \infty]$ for $k = 0, -1$). The non-vanishing components of the Riemann tensor are
\[
\begin{align*}
R^\alpha_{\beta\gamma\delta} &= \delta^\alpha_{\beta} R_{\gamma\delta} - \delta^\alpha_{\gamma} R_{\beta\delta} + g_{\beta\gamma} R^\alpha_{\delta} - g_{\beta\delta} R^\alpha_{\gamma} - \frac{1}{2} \left( \delta^\alpha_{\beta} R_{\gamma\delta} - \delta^\alpha_{\gamma} R_{\beta\delta} \right) R, \\
R^\alpha_{\beta\gamma\delta} &= k, \quad R^0_{\phi\phi\phi} = k s_1^2.
\end{align*}
\]

Therefore, the five-dimensional Lovelock–Gauss–Bonnet equations ($\alpha = \beta = 0$ in (2.2)) reduce to the three-dimensional cosmological Einstein equations with renormalized coupling constants.

The $(\alpha\beta)$ components of the full equation (2.2) reduce to
\[
\frac{\beta}{2} K_{\alpha\beta} + \left( 2\alpha + \frac{3\beta}{4} \right) (g_{\alpha\beta} R^2 - D_{\alpha} D_{\beta} R) + \left[ \frac{1}{\kappa} + \frac{4k\alpha + \gamma}{a^2} \right] + \left[ 2\alpha - \frac{\beta}{4} \right] R \right] \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) + \left[ \frac{\Lambda - k\alpha^2}{\kappa} - \frac{2k^2\alpha}{a^2} + \left( \frac{\alpha}{2} + \frac{11\beta}{16} \right) \frac{R^2}{a^2} \right] g_{\alpha\beta} = 0,
\]
where
\[
K_{\alpha\beta} = 2D^2 R_{\alpha\beta} - \frac{1}{2} (D_{\alpha} D_{\beta} + g_{\alpha\beta} D^2) R - 8R_{\gamma\beta} R_{\gamma\delta} + \frac{9}{2} R R_{\alpha\beta} + \left( 3R^3 \gamma_{\gamma\delta} - \frac{13}{8} R^2 \right) g_{\alpha\beta},
\]
and $D_{\alpha}$ is the three-dimensional covariant derivative. The solutions of the three-dimensional field equation (2.7) are restricted by the constraint, resulting from the $\theta\theta$ equation (2.2)
\[
\left( 2\alpha + \frac{\beta}{2} \right) D^2 R + \left( \frac{\Lambda}{\kappa} + \frac{k^2\alpha + \beta}{a^4} \right) - \frac{1}{2\kappa} R - \frac{\alpha}{2} R^2 - \frac{\beta}{2} R^3 R_{\gamma\delta} = 0.
\]
Let us make the further assumption $R = \text{constant}$. In this case, the field equation (2.7) reduce to the equations of NMG

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \lambda g_{\alpha\beta} - \frac{1}{2m^2}K_{\alpha\beta} = 0,$$  
(2.10)

with the additional constraint

$$\frac{\beta}{2}R^{\gamma\delta}R_{\gamma\delta} + \frac{\alpha}{2}R^2 + \frac{1}{2\kappa}R - \left(\frac{k^2(2\alpha + \beta)}{a^4} + \frac{\Lambda}{\kappa}\right) = 0.$$  
(2.11)

The parameters of the effective three-dimensional theory (2.10) are related to the parameters of the original five-dimensional theory, the compactification scale $a^2$ and the three-dimensional Ricci scalar by

$$\lambda = c^{-1}\left[\frac{\Lambda - ka^{-2}}{\kappa} - \frac{2k^2\alpha}{a^4} + \left(\frac{\alpha}{2} + \frac{11\beta}{16}\right)R^2\right],$$

$$m^{-2} = -c^{-1}\beta, \quad c = \frac{1}{\kappa} + \frac{4k(\alpha + \gamma)}{a^2} + \left(2\alpha - \frac{\beta}{4}\right)R$$  
(2.12)

(the real constant $m^2$ is not necessarily positive). When $\beta = 0$ the reduced field equation (2.10) reduce to the three-dimensional cosmological Einstein equations, so it is the five-dimensional quadratic Ricci coupling $\beta R^{\mu\nu}R_{\mu\nu}$ which is responsible for the occurrence of the three-dimensional quadratic couplings in equation (2.10).

Tracing the NMG field equation (2.10) yields the equation

$$-\frac{1}{2}R + 3\lambda - \frac{1}{2m^2}\left(R^{\gamma\delta}R_{\gamma\delta} - \frac{3}{8}R^2\right) = 0,$$  
(2.13)

which can be rewritten as

$$\frac{\beta}{2}R^{\gamma\delta}R_{\gamma\delta} - \frac{3\beta}{16}R^2 - \frac{c}{2}R + 3c\lambda = 0.$$  
(2.14)

This can be combined with equation (2.11) to eliminate the Ricci square term. Taking equations (2.12) into account, we finally arrive at the quadratic constraint on the Ricci scalar, equivalent to (2.11),

$$2\beta\bar{R}^2 - [1 + 2(\alpha + \gamma)x]\bar{R} + 4\bar{\Lambda} - 3x - (4\alpha - \beta)x^2 = 0,$$  
(2.15)

where we have put

$$\bar{R} = \kappa R, \quad \bar{\Lambda} = \kappa \Lambda, \quad x = kx a^{-2}.$$  

The special case of solutions with $k = 0$ admits a simple four-dimensional interpretation. In this case the constant curvature two-surface $\Sigma_2$ is simply $T^2$, so that the five-dimensional metric can be trivially reduced to four dimensions by

$$\text{d}s^2_{(5)} = \text{d}s^2_{(4)} + \text{d}y^2,$$  
(2.16)

with

$$\text{d}s^2_{(4)} = g_{\alpha\beta}(x^\gamma)\text{d}x^\alpha\text{d}x^\beta + \text{d}z^2.$$  

So solutions of equations (2.10) and (2.11) with $k = 0$ (if they exist) will lead to cylindrical solutions of four-dimensional quadratic gravity. Because the Gauss–Bonnet term is a topological invariant in four dimensions, we do not expect the existence of such solutions to depend on the value of the Gauss–Bonnet coupling constant $\gamma$.  


3. Solutions

A number of exact solutions to the equation (2.10) of NMG are known [21, 23–28]. Several of these have a constant Ricci scalar. In the following we shall concentrate on solutions of NMG with constant Ricci scalar, leading to non-asymptotically flat black ring solutions of five- (or possibly four-) dimensional quadratic gravity.

3.1. BTZ

Equation (2.10) are trivially solved by Einstein metrics, leading in the case of a negative Ricci scalar to BTZ black holes [21, 23]. The resulting five-dimensional solutions are

\[ ds^2(5) = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 (dz + N^2 dr)^2 + a^2 (d\theta^2 + s^2 k d\phi^2), \]

(3.1)

where

\[ N^2 = \frac{l^2}{l^2} - M + \frac{J^2}{4l^2}, \quad N^2 = \frac{J}{2l^2}. \]

(3.2)

These solutions depend on two dynamical parameters \( M \) and \( J \) (integration constants), and two scale parameters \( a^2 (\Sigma_1^2 \text{ scale}) \) and \( l^2 (\text{AdS}_3 \text{ scale}) \). The AdS3 scale is related to the NMG parameters and to the Ricci scalar by [21, 23]

\[ R = -6l^{-2}, \quad l^{-2} = 2m^2[-1 \pm \sqrt{1 - \lambda/m^2}]. \]

(3.3)

Inserting these in equations (2.12) and (2.15), we obtain a system of two quadratic equations for the rescaled parameters

\[ x = k\kappa a^{-2}, \quad y = \kappa l^{-2}, \]

in terms of the input parameters \( \alpha, \beta, \gamma \) and \( \Lambda \):

\[ 2\alpha x^2 - 4(\alpha + \gamma)xy - 2(3\alpha + 13\beta)y^2 + x - y - \Lambda = 0, \]

\[ (4\alpha - \beta)x^2 - 12(\alpha + \gamma)xy - 72\beta y^2 + 3x - 6y - 4\Lambda = 0. \]

(3.4)

Eliminating \( x \) between these equations generically yields an equation of the sixth degree for the variable \( y \). We consider here only some special cases:

(a) \( \alpha = \beta = 0 \) (Gauss–Bonnet). In this case the equations are solved by

\[ x = \frac{2\Lambda}{3 + 4\gamma\Lambda}, \quad y = \frac{-\Lambda}{3}. \]

(3.5)

Assuming \( \kappa > 0 \), and comparing the signs of \( x \) (\( k \)) and \( y \) (\( l^{-2} \)), we find that for \( \gamma > 0 \), the five-dimensional geometry is \( \text{BTZ} \times S^2 \) for \( \Lambda < -3/4\gamma \), \( \text{BTZ} \times H^2 \) for \( -3/4\gamma < \Lambda < 0 \), and \( dS_3 \times S^2 \) for \( \Lambda > 0 \), while for \( \gamma < 0 \) it is \( \text{BTZ} \times H^2 \) for \( \Lambda < 0 \), \( dS_3 \times S^2 \) for \( 0 < \Lambda < -3/4\gamma \), and \( dS_3 \times H^2 \) for \( \Lambda > -3/4\gamma \). For \( \gamma = 0 \) (pure Einstein case), the solution reduces to the five-dimensional Nariai or anti-Nariai solution, with the geometry \( dS_3 \times S^2 \) for \( \Lambda > 0 \), or \( \text{BTZ} \times H^2 \) for \( \Lambda < 0 \).

(b) \( \Lambda = -(2\alpha + \beta)/(4\alpha + \beta)^2 \): In this case,

\[ x = \frac{1}{4\alpha + \beta}, \quad y = 0, \]

(3.6)

and the geometry is \( \text{Minkowski}_3 \times S^2 \) or \( \text{Minkowski}_3 \times H^2 \), depending on the sign of \( 4\alpha + \beta \). This can be toroidally reduced to a solution of four-dimensional quadratic gravity with the geometry \( \text{Minkowski}_2 \times \Sigma_1 \).
\[
\Lambda = -3(3\alpha + 7\beta)/8(3\alpha + 4\beta)^2.
\]

This leads to
\[
x = 0, \quad y = \frac{1}{3\alpha + 4\beta},
\]
with the geometry BTZ \( \times T^2 \) or dS \( \times T^2 \) according to the sign of \( 3\alpha + 4\beta \). This can also be toroidally reduced to a solution of four-dimensional quadratic gravity, with the geometry \( \text{BTZ} \times S^1 \) or dS \( \times S^1 \).

### 3.2. AdS wave

As shown in [24, 25], the field equations of NMG also admit AdS wave solutions which generalize extreme BTZ black holes, and have the same constant Ricci scalar. From the stationary solutions given in [25], we obtain the solutions of five-dimensional quadratic gravity

\[
d s_{(5)}^2 = \begin{bmatrix} -2l^2 \rho + F(\rho) \end{bmatrix} d\tau^2 - 2l F(\rho) d\tau d\zeta + \begin{bmatrix} 2 \rho + l^2 F(\rho) \end{bmatrix} d\zeta^2 + \begin{bmatrix} \frac{\rho^2}{4\rho^2} \end{bmatrix} d\rho^2
\]

\[+ a^2 (d\theta^2 + s_5^2 d\phi^2),
\]

with
\[
F(\rho) = a_+ \rho^{\alpha_+} + a_- \rho^{\alpha_-} + M/2, \quad p_k = \frac{1 \pm \sqrt{m^2 l^2 + 1/2}}{2},
\]

depending on three integration constants \( a_+, a_- \) and the BTZ mass parameter \( M \). The AdS \( \times S^1 \) scale \( l^2 \) is again given by the BTZ relation (3.3) (the sign \( \pm \) in (3.9) is independent from that in (3.3)). For \( m^2 l^2 < -1/2 \), the two constants \( a_+ \) are complex conjugate, while for \( m^2 l^2 = -1/2 \) and \( m^2 l^2 > 1/2 \), the form (3.9) of \( F(\rho) \) degenerates and must be replaced by forms involving logarithms, which are given in [25]. For \( a_+ = a_- = 0 \), the solution (3.8) reduces, after the coordinate transformation \( \rho = r^2/2 \), to the extreme BTZ solution (3.1) with \( J = Ml \). For \( a_+ = 0 \) but \( a_- \neq 0 \), the solution (3.8) is asymptotic (for \( \rho \to \infty \)) to the extreme BTZ solution for \( m^2 l^2 > 1/2 \), and weakly asymptotic (in the sense of log gravity) to extreme BTZ for \( m^2 l^2 = 1/2 \).

The analysis of [25] shows that the three-dimensional AdS wave solutions describe regular black holes with a null Killing vector, leading to regular five-dimensional black rings, only for the discrete values \( m^2 l^2 = 17/2, m^2 l^2 = 7/2, m^2 l^2 = 1/2 \). The black rings for \( m^2 l^2 = 17/2 \) are actually a special case of the warped AdS black rings discussed in the next subsection, so we only consider the two other possibilities, focusing on the subcase \( k = 0 \). In this case, equation (3.7) gives \( l^2 = \kappa (3\alpha + 4\beta) \), leading on account of (2.12) to

\[
m^2 l^2 = \frac{18\alpha - 11\beta}{2\beta}.
\]

The massless \( m^2 l^2 = 7/2 \) black rings are obtained for \( \alpha = \beta, \quad \Lambda = -15/196\beta \). The five-dimensional metric is given by (3.8), (3.9) with \( k = 0, a_+ = M = 0, a_- > 0, p_\pm = 3/2, \) with a double horizon at \( x \equiv \rho^{1/2} = 0 \) hiding a timelike causal singularity (for details, see [25]).

If the coupling constants are related by \( \alpha = 2\beta/3, \quad \Lambda = -3/32\beta \), the choice \( k = 0 \) leads to \( m^2 l^2 = 1/2 \). The logarithmic solution which replaces (3.9) in this case leads to three possible kinds of regular five-dimensional black rings (or four-dimensional black strings). Only one of which is massive. with the four-dimensional black string metric (after an appropriate coordinate transformation)

\[
d s_{(4)}^2 = \frac{4\rho^2}{l^2 F} d\tau^2 + r^2 \left[ d\phi - \frac{b l \ln |\rho/\rho_0|}{r^2} d\tau \right]^2 + \frac{l^2 \rho^2}{4\rho^2} d\phi^2 + d\zeta^2
\]

\[+ (r^2 = 2\rho + bl^2 \ln |\rho/\rho_0|),
\]
depending on two parameters \( b < 0 \) and \( \rho_0 > 0 \). Again, the double horizon at \( \rho = 0 \) shields a timelike causal singularity. This spacetime is asymptotically \( \text{AdS}_3 \times S_1 \) in the sense of log gravity, and its mass and angular momentum satisfy the extremality condition \( J = Ml \). That this mass, given by
\[
M = \frac{2\mu \tau}{G_3}
\]
(with \( G_3 \) the effective three-dimensional Newton constant and \( \tau \) the period of the coordinate \( z \)), is positive is nontrivial. As discussed in [25], \( b \) must be negative in order to avoid naked CTC, so that for a positive \( G_3 \), \( M \) is negative. However, comparing (2.7) and (2.12), we see that the effective three-dimensional Newton constant
\[
G_3 \propto e^{-1} = -\frac{1}{\beta m^2} = -\frac{\kappa}{\beta y^2 t^2} = -12\kappa
\]
(using (3.7) with \( \alpha = 2\beta/3 \) and \( m^2 t^2 = 1/2 \)) is negative definite, ensuring a positive mass (3.12).

The stationary \( \text{AdS} \) wave solutions of [25] are actually special cases of the more general \( \text{AdS}_3 \) wave solutions of NMG [24], which similarly lead to solutions of five-dimensional quadratic gravity
\[
\begin{align*}
\text{d}s^2 &= 2t^{-2} \rho \, du \, dv + F(\rho, u) \, du^2 + \frac{r^2}{4\rho^2} \left[ d\theta^2 + \frac{\rho^2 + (1-b^2)\omega}{r^2} \, dt \right]^2 + \frac{1}{b^2 \zeta^2} \frac{d\rho^2}{\rho^2 - \rho_0^2} + a^2 (d\phi^2 + s^2 \, d\psi^2),
\end{align*}
\]
where \( u = lz - t, \) \( v = lz + t, \) and \( F(\rho, u) \) is given by (3.9) with the constants \( a_\pm \) replaced by arbitrary functions of \( u \). These are different from the \( \text{AdS}_5 \) wave solutions of [5]. Another special case of \( \text{AdS} \) wave solutions of NMG is the dynamical black hole metric of [28], which can similarly be uplifted to five dimensions.

### 3.3. Warped \( \text{AdS} \)

The warped \( \text{AdS}_3 \) solutions of NMG (previously discussed as solutions of topologically massive gravity [29] (TMG) [30–32]) given in [23] lead to the following solutions of five-dimensional quadratic gravity:
\[
\text{d}s^2 = -b^2 \frac{\rho^2 - \rho_0^2}{r^2} \, du^2 + r^2 \left[ dz - \frac{\rho + (1-b^2)\omega}{r^2} \, dt \right]^2 + \frac{1}{b^2 \zeta^2} \frac{d\rho^2}{\rho^2 - \rho_0^2} + a^2 (d\phi^2 + s^2 \, d\psi^2),
\]
where
\[
r^2 = \rho^2 + 2\omega \rho + \omega^2 (1-b^2) + \frac{b^2 \rho_0^2}{1-b^2},
\]
and the constants \( b^2 \) and \( \zeta \) are given by
\[
b^2 = \frac{9 - 21\lambda/m^2 \mp 2 \sqrt{3(5 + 7\lambda/m^2)}}{4(1-\lambda/m^2)}, \quad \zeta^{-2} = \frac{21 - 4b^2}{8m^2}.
\]
As shown in [32], these metrics correspond to regular black holes provided
\[
\zeta^2 > 0, \quad 0 < b^2 < 1, \quad \omega > 0.
\]
In the limiting cases \( b^2 = 1 \) and \( b^2 = 0 \), the three-dimensional reduced metric should be replaced by the special solutions given in [32] (where \( \mu_E \) should be replaced by \( \zeta \)).

We recall that the NMG parameters \( \lambda \) and \( m^2 \) in (3.17) are related to the original five-dimensional parameters and to the three-dimensional curvature associated with (3.15),
\[
R = \frac{\zeta^2}{2} (1-4b^2)
\]
by (2.12) and the constraint (2.15). The general case is intricate and unlightening, so we concentrate on the special case $\alpha = \gamma = 0$. To compute the values of the solution parameters corresponding to a given set of values of the model parameters, we proceed in the following fashion. Equation (3.17) can be inverted to yield, for each value of $b^2$, two values of the ratio $\lambda/m^2$:

$$\frac{\lambda}{m^2} = \frac{16b^4 - 72b^2 + 21}{(4b^2 - 21)^2}, \quad (b) \quad \frac{\lambda}{m^2} = 1$$

(3.20)

(for $\lambda/m^2 = 1$ the first equation (3.17) with the lower sign is indeterminate$^1$). From (3.19) and the second and third equations (2.12), we obtain

$$\beta \mathcal{R} = \frac{4b^2 - 1}{5}, \quad \xi^2 = -\frac{5}{2} \beta \kappa.$$  

(3.21)

Using this together with the equations (2.12), we can compute another value of the ratio $\lambda/m^2$ in terms of the solution parameters $b^2$ and $x$:

$$\frac{\lambda}{m^2} = \frac{400\beta(x - \overline{\kappa}) - 11(4b^2 - 1)^2}{(4b^2 - 21)^2}.$$  

(3.22)

Comparing this with (3.20), we obtain for each value of $b^2$ two possible values for $\beta(x - \overline{\kappa})$:

$$\beta(x - \overline{\kappa}) = \frac{2}{25}(2b^2 - 1)(3b^2 - 1), \quad (b) \quad \beta(x - \overline{\kappa}) = \frac{48b^4 - 64b^2 + 113}{100}.$$  

(3.23)

Finally these results are combined with the constraint (2.15), written as

$$(\beta x + 1/2)^2 = 4\beta(x - \overline{\kappa}) + 3/8 - 2(\beta \mathcal{R} - 1/4)^2,$$  

(3.24)

to obtain the two possible relations between the solution parameters $x = k\kappa a^{-2}$ and $b^2$:

$$\beta x = -\frac{1}{2} \pm \frac{\sqrt{64b^4 - 16b^2 + 29}}{10}, \quad (b) \quad \beta x = -\frac{1}{2} \pm \frac{\sqrt{64b^4 - 112b^2 + 449}}{10}.$$  

(3.25)

The corresponding values of the model parameter $\beta \overline{\kappa}$ are given in terms of $b^2$ by (3.23) and (3.25). The sign $k$ of the two-dimensional curvature is determined from the sign of $\beta x$, taking into account that $\beta \kappa < 0$ from the second equation (3.21). The outcome is that for both cases (a) and (b) in (3.25), $k = \mp 1$, so that there is no solution of this kind in four-dimensional quadratic gravity ($k = 0$). The mass of the five-dimensional warped AdS black rings is proportional to the three-dimensional mass of warped AdS black holes in NMG, which was computed in [23] to be

$$M = \frac{\xi^3 b^2 (1 - b^2)}{2G_5 m^2} \propto b^2 (1 - b^2) \omega,$$  

(3.26)

where we have used (2.12) and (3.21). This mass is positive by virtue of the regularity constraints (3.18).

### 3.4. AdS$_2 \times S^1 \times \Sigma_2$

In the special case $\lambda = m^2$, another solution of NMG is AdS$_2 \times S^1$ [23], leading to the five-dimensional solution

$$ds^2 = -(\rho^2 - \rho_0^2) \, dr^2 - 2m^2 \, dz^2 - \frac{d\rho^2}{2m^2(\rho^2 - \rho_0^2)} + a^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).$$  

(3.27)

---

$^1$ This fact was overlooked in [23].
This solution—which leads to a four-dimensional solution after integrating out the cyclic coordinate $z$—has the Minkowkian signature in the range $\rho^2 > \rho_0^2$ provided $m^2 < 0$ and $\alpha^2 > 0$. The three-dimensional curvature is $R = 4m^2$. Inserting this into the second and third equations (2.12), we obtain the linear relation

$$1 + 4(\alpha + \gamma)x + 4\gamma = 0$$

(3.28)

between the rescaled parameters

$$x = \kappa \alpha^{-2}, \quad y = 2\kappa m^2.$$  

(3.29)

This relation shows that there is no solution of this kind for $\alpha = \gamma = 0$.

One can show that there is also no solution for $\alpha = 0$, even if $\gamma \neq 0$. For $\alpha \neq 0$, equation (3.28) can be used to eliminate $y$ from the first equations (2.12) with $\lambda = m^2$ and the constraint (2.15), leading to the system

$$[\alpha^2(3\beta + 4\gamma) + 2\alpha\gamma(3\beta + \gamma + 3\beta\gamma^2)x^2 + \left(\frac{\alpha}{2}(3\beta + 2\gamma) + \frac{3\beta\gamma}{2}\right)x + \frac{\alpha}{8} + \frac{3\beta}{16} + \alpha^2\Lambda = 0,$$

$$[\alpha^2(9\beta + 8\gamma) + 4\alpha\gamma(4\beta + \gamma + 8\beta\gamma^2)x^2 + \left[\alpha(4\beta + 3\gamma) + 4\beta\gamma\right]x + \frac{\alpha + \beta}{2} + 4\alpha^2\Lambda = 0,$$

(3.30)

which is overdetermined, meaning that there must be a specific relation between the model parameters $\alpha$, $\beta$, $\gamma$, and $\Lambda$.

Again the general case is intricate, so we focus on the example $\gamma = 0$. In this case the system (3.30) has a solution provided the model parameters are related by

$$3\xi^2 + 2(6\alpha - 11\beta)\xi + 12\alpha^2 + 20\alpha\beta + 3\beta^2 = 0,$$

where $\xi = 16\alpha^2\Lambda$. The further assumption $\Lambda = 0$ leads to the two possibilities

(a) $2\alpha + 3\beta = 0, \quad y = \frac{-x}{2} = \frac{-1}{6\beta}$

(3.32)

(b) $6\alpha + \beta = 0, \quad y = \frac{x}{2} = \frac{1}{2\beta}$

(3.33)

Remembering that $m^2 < 0$, we find that in the first case, $k = 1$ and the five-dimensional geometry is $\text{AdS}_2 \times S^2 \times S^1$, while in the second case, $k = -1$ and the geometry is $\text{AdS}_2 \times H^2 \times S^1$.

Returning to the general case, let us point out that, contrary to appearance, these five-dimensional solutions with the geometry $\text{AdS}_2 \times S^1 \times \Sigma_2$ will not generically lead, upon integration along the $S^1$, to Bertotti–Robinson-like solutions of four-dimensional quadratic gravity. The reason is that the five-dimensional equation (2.2) contain, besides the equations of four-dimensional quadratic gravity, the $(zz)$ component which leads to the additional constraint $L_4 = 0$, where $L_4$ is the integrand of the quadratic action (2.1) for $D = 4$. However, as already mentioned, the case $k = 0$ does lead upon toroidal reduction to a solution of four-dimensional quadratic gravity. Assuming $x = 0$, we obtain from (3.28) $y = -1/4\alpha$, which satisfies the system (3.30) provided

$$\beta = 0, \quad \Lambda = -1/8\alpha.$$

(3.34)

Note that $\beta = 0$ with $m^2$ finite means from (2.12) that $c = 0$, contrary to the assumption made in deriving (2.10) from (2.7). However it can be checked directly that equation (2.7) are satisfied for these relations between the coupling constants if $R = -1/2\kappa \alpha$. It is tempting to speculate that this $\text{AdS}_2 \times S^1 \times S^1$ solution could be the near-horizon limit of some four-dimensional black ring solution to $f(R)$ gravity.
4. Conclusion

In this paper, we have constructed new solutions of five-dimensional quadratic gravity as direct products $M_3 \times \Sigma_2$, where $\Sigma_2 = S^2$, $T^2$ or $H^2$ is a constant curvature two-surface, and $M_3$ is a solution of three-dimensional new massive gravity with constant scalar curvature. These non-asymptotically flat solutions could represent near-horizon limits of five-dimensional asymptotically flat black strings or black rings. A number of these solutions are themselves black strings or rings (topological if $\Sigma_2 = T^2$ or $H^2$) of the BTZ (3.1), null Killing vector (3.8) or warped $\text{AdS}_3$ (3.15) type.

A by-product of our analysis is the construction of new solutions of four-dimensional quadratic gravity with the geometry $M_3 \times S^1$, obtained by toroidal reduction of the five-dimensional solutions with flat transverse space. These again include black strings or rings of the BTZ or null Killing vector type (among which the log black string (3.14)), and an $\text{AdS}_2 \times T^2$ solution of $f(R)$ gravity for a specific relation between the model parameters.

We close by commenting on the relation of the present work with recent work on higher-dimensional supergravities with curvature squared invariants [33–36]. In [33], a six-dimensional supergravity with quadratic couplings was reduced on a three-sphere, yielding a massive three-dimensional supergravity, which is closely related to the general massive supergravity of [26]. Freezing out the scalars in this theory leads to the general massive gravity (NMG + TMG) of [21]. Similarly to what has been done here, we expect solutions of the latter theory [37, 38] to lead to Bertotti–Robinson-like solutions of the original six-dimensional theory. The five-dimensional supergravity of [34] with a Riemann squared invariant is expected to compactify over a two-sphere to a supersymmetric extension of topologically massive gravity. Likewise, known solutions of TMG should lift to solutions of the five-dimensional theory. In [35] a higher derivative extension of a six-dimensional gauged supergravity was shown to admit solutions given by direct products $M_4 \times \Sigma_2$ or $M_3 \times \Sigma_3$, with $M_p$ and $\Sigma_q$ both constant curvature. In [36] a supersymmetric completion of the Einstein–Gauss–Bonnet theory was similarly shown to admit solutions with $\text{AdS}_3 \times S^2$ and $\text{AdS}_2 \times S^3$ structures. Presumably these theories should also admit more general product solutions similar to those presented here.

Acknowledgment

I wish to thank Dmitry Gal’tsov for stimulating discussions.

References

[1] Stelle K S 1977 Phys. Rev. D 16 953
[2] Stelle K S 1978 Gen. Rel. Grav. 9 353
[3] Zwiebach B 1985 Phys. Lett. B 156 315
[4] Lü H and Pope C N 2011 Phys. Rev. Lett. 106 181302 (arXiv:1101.1971)
[5] deser S, liu H, Lü H, Pope C N, Sisman T C and Tekin B 2011 Phys. Rev. D 83 061502 (arXiv:1101.4009)
[6] Gullu I, Gurses M, Sisman T C and Tekin B 2011 Phys. Rev. D 83 084015 (arXiv:1102.1921)
[7] Malek T and Pravda V 2011 Phys. Rev. D 84 024047 (arXiv:1106.0331)
[8] Malek T 2012 Exact solutions of general relativity and quadratic gravity in arbitrary dimension Doctoral PhD Thesis Prague arXiv:1204.0291
[9] Nariai H 1950 Sci. Rep. Tohoku Univ. 34 160
[10] Nariai H 1951 Sci. Rep. Tohoku Univ. 35 62
[11] Ginsparg P and Perry M J 1983 Nucl. Phys. B 222 245
[12] Dadhich N On product space-time with 2-sphere of constant curvature arXiv:gr-qc/0003026
[11] Bertotti B 1959 Phys. Rev. 116 1331
[12] Dias O J C and Lemos J P 2003 Phys. Rev. D 68 104010 (arXiv:hep-th/0306194)
[13] Caldarelli M, Vanzo L and Zerbini Z 2001 The extremal limit of D-dimensional black holes
Geometrical Aspects of Quantum Fields ed A A Bytsenko, A E Goncalves and B M Pimentel
(Singapore: World Scientific) (arXiv:hep-th/0008136)
[14] Cardoso V, Dias O J C and Lemos J P 2004 Phys. Rev. D 70 024002 (arXiv:hep-th/0401192)
[15] Figueras P, Kunduri H K, Lucietti J and Rangamani M 2008 Phys. Rev. D 78 044042
(arXiv:0803.2998)
[16] Kunduri H K and Lucietti J 2009 J. Math. Phys. 50 082502 (arXiv:0806.2051)
Kunduri H K and Lucietti J 2011 Commun. Math. Phys. 303 31 (arXiv:1002.4656)
[17] Kunduri H K and Lucietti J 2007 J. High Energy Phys. JHEP12(2007)015 (arXiv:0708.3695)
Kunduri H K and Lucietti J 2013 Living Rev. Rel. 16 8 (arXiv:1306.2517)
[18] Dadhich N and Pons J M 2013 J. Math. Phys. 54 102501 (arXiv:1210.1109)
[19] Gürses M 1992 Phys. Rev. D 46 2522
Gürses M and Sermutlu E 1995 Class. Quantum Grav. 12 2799 (arXiv:hep-th/9509076)
[20] Bouchareb A, Chen C M, Clément G and Gal’tsov D V 2013 Phys. Rev. D 88 084048
(arXiv:1308.6461)
[21] Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. Lett. 102 201301 (arXiv:0901.1766)
[22] Deser S and Tekin B 2003 Phys. Rev. D 67 084009 (arXiv:hep-th/0212292)
[23] Clément G 2009 Class. Quantum Grav. 26 105015 (arXiv:0902.4634)
[24] Ayón-Beato E, Giribet G and Hassaine M 2009 J. High Energy Phys. JHEP05(2009)029
(arXiv:0904.0668)
[25] Clément G 2009 Class. Quantum Grav. 26 165002 (arXiv:0905.0553)
Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. D 79 124042 (arXiv:0905.1259)
[26] Ayón-Beato E, Garbazi A, Giribet G and Hassaine M 2009 Phys. Rev. D 80 104029
(arXiv:0909.1347)
[27] Flory M and Sachs I 2013 Phys. Rev. D 88 044034 (arXiv:1304.1704)
[28] Deser S, Jackiw R and Templeton S 1982 Phys. Rev. Lett. 48 975
Deser S, Jackiw R and Templeton S 1982 Ann. Phys., NY 140 372
[29] Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. D 79 124042 (arXiv:0905.1259)
[30] Anninos D, Li W, Padi M, Song W and Strominger A 2009 J. High Energy Phys. JHEP03(2009)130
(arXiv:0807.3040)
[31] Moussa K A, Clément G, Guennoune H and Leygnac C 2008 Phys. Rev. D 78 064065
(arXiv:0807.4241)
[32] Lü H, Pope C N and Sezgin E 2010 J. High Energy Phys. JHEP10(2010)016 (arXiv:1007.0173)
Bergshoeff E A, Rosseel J and Sezgin E 2011 Class. Quantum Grav. 28 225016 (arXiv:1107.2825)
[33] Bergshoeff E, Coomans F, Sezgin E and Van Proeyen A 2012 J. High Energy Phys. JHEP07(2012)011
(arXiv:1203.2975)
[34] Ozkan M and Pang Y 2013 J. High Energy Phys. JHEP03(2013)158
Ozkan M and Pang Y 2013 J. High Energy Phys. JHEP07(2013)152 (arXiv:1301.6622) (erratum)
[35] Nam S, Park J D and Yi S H 2010 J. High Energy Phys. JHEP07(2010)088 (arXiv:1005.1619)
[36] Tonni E 2010 J. High Energy Phys. JHEP08(2010)070 arXiv:1006.3489