Variational symmetries and superintegrability in multifield cosmology

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We consider a spatially flat Friedmann–Lemaitre–Robertson–Walker background space with an ideal gas and a multifield Lagrangian consisting of two minimally coupled scalar fields which evolve in a field space of constant curvature. For this cosmological model we classify the potential function for the scalar fields such that variational point symmetries exist. The corresponding conservation laws are calculated. Finally, analytic solutions are presented for specific functional forms of the scalar field potential in which the cosmological field equations are characterized as a Liouville integrable system by point symmetries. The free parameters of the cosmological model are constrained in order to describe analytic solutions for an inflationary epoch. Finally, stability properties of exact closed-form solutions are investigated. These solutions are scaling solutions with important physical properties for the cosmological model.

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1. INTRODUCTION

Scalar fields play an important role in the theoretical description of two accelerated expansion phases of the observed universe [1–7]. The early acceleration phase of the universe known as inflation is attributed to domination for a finite time of the inflaton [8, 9]. The inflaton is a single scalar field model that drives the dynamics; providing a matter source in the universe which can display antigravitating behavior. Because of the latter antigravitating property, scalar fields have been used in the literature as dark energy models [10–15] to overpass the problems of Λ-cosmology [16]. Moreover, because of the additional degrees of freedom provided by scalar fields, the scalar field models can be used as unified models for the description of the dark matter and dark energy [17–20], while scalar fields can attribute degrees of freedom for higher-order theories of gravity [21] providing an equivalent description of modified theories of gravity.

The simplest scalar field model proposed in the literature is the quintessence model consisting of a minimally coupled single scalar field with positive energy density and equation of state parameter \( w_Q \) bounded in the range \( |w_Q| \leq 1 \) [22]. The phantom field is an alternative to quintessence model in which the scalar fields have a negative kinetic energy, such that the equation of state parameter \( w_P \) is bounded as \( w_P \leq -1 \), which means that \( w_P \) can cross the phantom divide line \( w_P = -1 \) [23]. Scalar fields nonminimally coupled to gravity have been also studied in detail for example in Brans-Dicke theory [24], O’Hanlon gravity [25], Hordenski theory [26] and others [27–29]. Multifield models have been widely studied in literature; the additional degrees of freedom that multifield models provide overpass various problems of the single scalar field models, providing a richer cosmological evolution [30–33].

In this work we are interested in multifield cosmological models in a spatially flat Friedmann–Lemaitre–Robertson–Walker (FLRW) background space consisting of two scalar fields minimally coupled to gravity which are defined in a hyperbolic field space. Additionally, we consider the contribution to the cosmic fluid of a perfect fluid of constant equation of state parameter, that is, an ideal gas. This specific multifield model has been widely studied to describe the inflationary epoch as alternative to the inflation mechanism [34–39]. In addition, it can provide a cosmological history which explains the transition from the inflationary epoch to the matter era and then to the late-time acceleration phase of the universe [40, 41]. For this cosmological model we investigate the existence of conservation laws and the
integrability properties of the field equations. The equations of motion for the scale factor and the scalar fields are of second-order and form a Hamiltonian system described by a point-like Lagrangian. We use that property in order to find all the functional forms for the potential functions of the scalar fields such that variational point symmetries exist \cite{42}. Specifically, we apply Noether’s first and second theorem to calculate the variational point symmetries and write the corresponding conservation laws \cite{43}. Point symmetries have been widely applied in gravitational physics and in cosmology \cite{44–46}.

The main idea of this work is to use the variational symmetries to perform a classification scheme of the potential function. This classification of a given set of differential equations according to the admitted symmetry vectors has been proposed by Ovsiannikov \cite{47}. The requirement of the unknown potential function to be constrained by the symmetry conditions is also a geometric selection rule. Variational symmetries of the field equations are related to geometric symmetries of the kinematic metric which define the point-like Lagrangian \cite{48, 49}. Consequently, the requirement that the two scalar fields are defined in a hyperbolic field space is directly related to the existence of variational symmetries. Previous studies for the determination of exact solutions for the two-scalar field model are presented in \cite{50–54}.

The plan of the paper is as follows: in Section 2 we present the cosmological model of our considerations and we derive the field equations. The main properties and definitions of the variational symmetries are given in Section 3. The classification problem of this work is performed in Section 4 where we present all the functional forms of the scalar field potentials in which the field equations admit variational point symmetries and conservation laws linear in the momentum. In Section 5 we construct analytic and exact solutions to the cosmological models that are Liouville integrable which are obtained from the classification scheme. In Section 6 we perform a stability analysis of the scaling solutions using a similar procedure in order to obtain the scaling solutions in the previous Section. Finally, in Section 7 we discuss our results and our conclusions are drawn.

2. FIELD EQUATIONS

According to the cosmological principle in large scales the universe is isotropic and homogeneous and its geometry is described by the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime

\[ ds^2 = -N(t)^2 dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2), \]

where \( N(t) \) is the lapse function and \( a(t) \) is the scale factor which is the radius of a three-dimensional hypersurface.

For the gravitational model we assume the Action Integral

\[ S = S_{GR} + S_{multifield} + S_m, \]

where \( S_{GR} \) is the action integral of General Relativity

\[ S = \int \sqrt{-g} dx^4 R, \]

in which \( R \) is the Ricci scalar of the background metric tensor \( g_{\mu \nu} \).

The Action Integral \( S_{multifield} \) is given by \cite{50}

\[ S_{multifield} = -\int \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{\varepsilon}{2} g^{\mu \nu} e^{\kappa \phi} \nabla_\mu \psi \nabla_\nu \psi + V(\phi, \psi) \right), \]

where the two scalar fields are defined in a hyperbolic field space, \( \kappa \) is the coupling parameter and it is related to the curvature of the hyperbolic space \( R_h \) as \( R_h = -\kappa^2/2 \). Parameter \( \varepsilon = \pm 1 \) indicates if the second field \( \psi(x^\mu) \) is phantom \( (\varepsilon = -1) \) or not \( (\varepsilon = +1) \). In the special case where \( \kappa = 0 \) and \( \varepsilon = -1 \), the quintom model is recovered, however in this work we assume \( \kappa \neq 0 \). Finally, \( S_m \) corresponds to the action integral for a perfect fluid with energy density \( \rho \), pressure \( p \) and constant equation of state parameter \( p = w_m \rho \) in which \( w_m \in (-1, 1) \) \cite{55, 56}.

For the background space (1) and by assuming that scalar fields inherit the spacetime symmetries we ended with the gravitational field equation

\[ 3H^2 = \frac{1}{2N^2} \dot{\phi}^2 + \frac{\varepsilon}{2N^2} e^{\kappa \phi} \dot{\psi}^2 + V(\phi) + \rho, \]

\[ -\left( 2\dot{H} + 3H^2 \right) = \frac{1}{2N^2} \dot{\psi}^2 + \frac{\varepsilon}{2N^2} e^{\kappa \phi} \dot{\psi} - V(\phi) + w_m \rho, \]
where from this last one, the conservation law follows

\[ (\ddot{\phi} + \frac{\dot{N}}{N} \phi + 3H \phi) - \frac{\varepsilon}{2} \kappa e^{\kappa\phi} \dot{\psi}^2 + N^2 V_{,\phi} = 0 , \]  

(7)

\[ \ddot{\psi} - \frac{\dot{N}}{N} \dot{\psi} + 3H \dot{\psi} + \kappa \dot{\phi} \dot{\psi} + \varepsilon e^{-\kappa\phi} V_{,\psi} = 0 , \]  

(8)

\[ \dot{\rho} + 3NH(1 + w_m)\rho = 0 \]  

(9)

where from this last one, the conservation law follows \( \rho(t) = \rho_m a(t)^{-3(1+w_m)} \).

By replacing in (5)-(8) and for arbitrary \( N(t) \) we end with a Hamiltonian dynamical system which is described by the singular point-like Lagrangian

\[ L\left(N, a, \dot{a}, \phi, \dot{\phi}, \psi, \dot{\psi}\right) = \frac{1}{2N(t)} \left(-6a\dot{a}^2 + a^3 \left(\dot{\phi}^2 + \varepsilon e^{\kappa\phi} \dot{\psi}^2\right)\right) - N(t) a^3 V(\phi, \psi) + \rho_m a^{-3w_m}\right) . \]  

(10)

In the following we assume the lapse function \( N(t) = a^{3w_m} \). Then, the point-like Lagrangian (10) describes the trajectory \( U(t) = U(a(t), \phi(t), \psi(t)) \) of a point particle in the three dimensional geometry

\[ ds^2_{(3)} = a^{-3w_m} \left(-6a\dot{a}^2 + a^3 \left(\dot{\phi}^2 + \varepsilon e^{\kappa\phi} \dot{\psi}^2\right)\right) , \]  

(11)

under the action of the effective potential \( V_{eff} = a^{3(1+w_m)} V(\phi, \psi) \). The integration constant \( \rho_m \) is related to the "energy \( h \)" of the point-particle, that is, \( h = -\rho_m \), in which \( h \) is a conservation law for the equation of motions.

We continue with the presentation of the basic properties for the variational symmetries.

3. VARIATIONAL SYMMETRIES

Variational symmetries can be defined for differential equations of any order which are deduced from a variational principle. Thus in this article we work with second-order differential equations with a Lagrangian function \( L = L(t, q, \dot{q}) \).

Considering the infinitesimal transformation

\[ \bar{t} = t + \varepsilon \tau(t, q) , \quad \bar{q} = q + \varepsilon \eta(t, q), \]  

(12)

generated by the differential operator \( \Gamma = \tau \partial_t + \eta \partial_q \), where \( \varepsilon \) is an infinitesimal parameter. Under this transformation, the Action Integral \( A = \int_{t_0}^{t_1} L(t, q, \dot{q}) \, dt \) becomes \( \bar{A} = \int_{\bar{t}_0}^{\bar{t}_1} L(\bar{t}, \bar{q}, \dot{\bar{q}}) \, d\bar{t} \), which up to first order in the infinitesimal \( \varepsilon \) is written as follows

\[ \bar{A} = A + \varepsilon \int_{t_0}^{t_1} \left[ L + \varepsilon \left(\tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \tau \dot{L}\right)\right] \, dt + \varepsilon F, \]  

\[ F := \tau t_1 L(t_1, q_1, \dot{q}_1) - \tau t_0 L(t_0, q_0, \dot{q}_0) , \]  

(13)

where now \( \zeta = \dot{\eta} - \dot{\dot{\eta}} \) and \( L(t_0, q_0, \dot{q}_0) \) and \( L(t_1, q_1, \dot{q}_1) \) are the values of \( L \) at the endpoints \( t_0 \) and \( t_1 \), respectively. Therefore, it follows

\[ \bar{A} = A + \varepsilon \int_{t_0}^{t_1} \left(\tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \tau \dot{L}\right) \, dt + \varepsilon F. \]  

(14)

As \( F \) depends only upon the endpoints, we may write it as

\[ F = -\int_{t_0}^{t_1} \int_{\bar{t}_0}^{\bar{t}_1} \bar{f} \, d\bar{t} \, dt. \]  

(15)

We have to say that the generator \( \Gamma \) of the infinitesimal transformation will be a variational symmetry, that is, a Noether symmetry if \( A = \bar{A} \), i.e.

\[ \int_{t_0}^{t_1} \left(\tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \tau \dot{L} - \bar{f}\right) \, dt = 0, \]  

(16)
TABLE I: Commutator of the four-dimensional Homothetic algebra

|      | K^1  | K^2  | K^3  | H     |
|------|------|------|------|-------|
| [·]  |      |      |      |       |
| K^1  | 0    | -\frac{1}{2}K^1 - K^2 0 |       |       |
| K^2  | -\frac{1}{2}K^1       | 0    | -\frac{1}{2}K^3 0 |       |
| K^3  | K^2  | -\frac{1}{2}K^3 0       | 0    |       |
| H    | 0    | 0    | 0    | 0     |

from which it follows the symmetry condition (Noether’s first theorem):

\[ \dot{f} = \tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\tau}L. \]  

(17)

Hence, according to Noether’s second theorem if there exists a vector field \( \Gamma \) and a function \( f(t, q, \dot{q}) \) such that condition (17) is true, then the quantity

\[ I(t, q, \dot{q}) = f - \left[ \tau L + (\eta - \dot{q}\tau) \frac{\partial L}{\partial \dot{q}} \right], \]

(18)

is a conservation law for the Euler-Lagrange equations with Lagrangian function \( L(t, q, \dot{q}) \). For a recent discussion on Noether’s theorem and for extensions and generalizations we refer the reader to [42].

4. SYMMETRY CLASSIFICATION

For Lagrangian functions of the form

\[ L(t, q, \dot{q}) = \frac{1}{2} \gamma_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta - V(q), \]

(19)

variational symmetries are constructed by the elements of the Homothetic algebra of the metric tensor \( \gamma_{\alpha\beta} \). We omit the presentation of the details for the derivation of the variational symmetry vectors and of the symmetry conditions which constraint the effective potential function for Lagrangian density of the form (19). They are presented into a theorem in [57] with applications which demonstrate the main theorem.

The minisuperspace line element (11) is conformally flat and admits the vector field as elements of the Homothetic algebra

\[ K^1 = \psi \partial_\phi + \left( \frac{e^{-\kappa \phi}}{\kappa \psi} - \frac{\kappa}{4} \psi^2 \right) \partial_\psi, \]

\[ K^2 = \partial_\psi - \frac{\kappa}{2} \psi \partial_\phi, \]

\[ K^3 = \partial_\phi, \quad H = \frac{2}{3(1 - w_m)} a \partial_a, \]

in which \( K^1 \), \( K^2 \) and \( K^3 \) are Killing symmetries and \( H \) is the proper Homothetic vector field. Because variational symmetries are generated by the vector fields \( \{K^1, K^2, K^3, H\} \) or linear combination of them, we should derive the one-dimensional optimal system of the Lie algebra.
The one-dimensional optimal system provides all the independent dynamical systems which do not communicate under the adjoint representation. Indeed, we consider the two generic vector fields
\[ Z = \alpha_1 K^1 + \alpha_2 K^2 + \alpha_3 K^3 + \alpha_0 H, \]
\[ W = \beta_1 K^1 + \beta_2 K^2 + \beta_3 K^3 + \beta_0 H, \]
where \( \alpha \) and \( \beta \) are constants. The two vector fields \( Z, W \) are equivalent if and only if
\[ W = \sum_{j=1}^{n} Ad (\exp (\mu_j X^j)) Z \]
or
\[ W = cZ, \quad c = \text{const}. \]
where the operator \( Ad (\exp (\mu X)) \) is called the Adjoint representation [58]. In Tables I and II the commutators and the Adjoint representation for the Homothetic algebra are presented in Tables III and IV.

Therefore, from the results of the Tables we calculate the one-dimensional system given by the one-dimensional Lie algebra \( \{ K^1, K^2, K^3, H \} \) are presented. Therefore, we present the conservation law.

However, in a special case in which \( \kappa = \pm \frac{\sqrt{2}}{2} (w_m - 1) \) the admitted Homothetic algebra is of higher dimension. Let us assume \( \kappa = + \frac{\sqrt{2}}{2} (w_m - 1) \), the case with \( \kappa = - \frac{\sqrt{2}}{2} (w_m - 1) \) is recovered with the change of variables \( \phi \rightarrow -\phi \). Therefore, for \( \kappa = \pm \frac{\sqrt{2}}{2} (w_m - 1) \), the admitted Homothetic vector fields by the minisuperspace are
\[ \bar{K}^1 = \left( a^2 e^{\frac{\sqrt{2}}{2} \phi} (w_m - 1) \right) \left( a \psi \partial_a + \sqrt{6} \psi \partial_\phi + \frac{4e^{-\sqrt{2}(w_m - 1)\phi}}{\varepsilon (w_m - 1)} \right), \]
\[ \bar{K}^2 = \left( a^2 e^{\frac{\sqrt{2}}{2} \phi} (w_m - 1) \right) \left( a \partial_a + \sqrt{6} \partial_\phi \right), \]
\[ \bar{K}^3 = \frac{1}{8} a^2 (w_m - 1) \left( 8 \left( w_m - 1 \right)^2 \psi^2 e^{\frac{\sqrt{2}}{2}(w_m - 1)\phi} + 8e^{-\sqrt{2}(w_m - 1)\phi} \right) \partial_a + \left( 3 \left( w_m - 1 \right)^2 \psi^2 e^{\frac{\sqrt{2}}{2}(w_m - 1)\phi} - 8e^{-\sqrt{2}(w_m - 1)\phi} \right) \partial_\phi + 3 \left( w_m - 1 \right) a^2 (w_m - 1) \psi e^{-\sqrt{2}(w_m - 1)\phi} \partial_\psi, \]
\[ \bar{K}^4 = \psi \partial_\phi + \left( \frac{8 \sqrt{6}}{\varepsilon (w_m - 1)} e^{-\sqrt{2}(w_m - 1)\phi} - 3 (w_m - 1) \sqrt{6} \psi^2 \right) \partial_\phi, \]
\[ \bar{K}^5 = \partial_\phi - \frac{\sqrt{6}}{4} (w_m - 1) \psi \partial_\psi, \quad \bar{K}^6 = \partial_\psi, \quad H = \frac{2}{3 (1 - w_m)} a \partial_a, \]
where \( H \) is the proper Homothetic vector field and \( \{ \bar{K}^1, \bar{K}^2, \bar{K}^3, \bar{K}^4, \bar{K}^5, \bar{K}^6 \} \) form a six dimensional Killing algebra. Consequently, when \( \kappa = \pm \frac{\sqrt{2}}{2} (w_m - 1) \) the minisuperspace is maximally symmetric. Moreover, we observe that the vector fields \( \{ \bar{K}^1, \bar{K}^2, \bar{K}^3 \} \) are gradient from which we infer that the minisuperspace is the flat space. Therefore, \( \{ \bar{K}^1, \bar{K}^2, \bar{K}^3 \} \) corresponding to the three translation symmetries of the three dimensional flat space and \( \{ \bar{K}^4, \bar{K}^5, \bar{K}^6 \} \) form the \( SO(3) \). The commutators and the Adjoint representation of the seven-dimensional Homothetic algebra are presented in Tables III and IV.

Consequently, from Tables III and IV we calculate the one-dimensional system given by the one-dimensional Lie algebras \( \{ \bar{K}^1 \}, \{ \bar{K}^1 + \beta \bar{K}^2 \}, \{ \bar{K}^1 + \alpha \bar{K}^2 + \beta \bar{K}^3 \}, \{ \bar{K}^1 \}, \{ \bar{K}^1 + \alpha \bar{K}^1 \}, \{ H \}, \{ \bar{K}^J + \gamma H \} \); in which \( A, B = 1, 2, 3 \) and \( J = 4, 5, 6 \).

Until now we have assumed that \( w_m - 1 \neq 0 \). The case where \( w_m = 1 \) will be studied elsewhere.

4.1. Classification of \( V(\phi, \psi) \)

We continue with the presentation of the special potential forms \( V(\phi, \psi) \) where the cosmological field equations (5)-(9) admit conservation laws. We omit the calculations and for each potential function which admit a variational symmetry we present the conservation law.
the cosmological field equations admit the conservation law
\[ p_I \psi + \kappa \mp \kappa \psi + \kappa \phi \]
with the corresponding conservation laws
\[ K^5 \psi + \kappa \mp \kappa \psi + \kappa \phi \]
Moreover, for the potential function \[ V = \psi - \psi \phi \]
the system admits the variational symmetry
\[ \frac{\delta L}{\delta q} \psi = \psi \mp \psi \phi + \kappa \psi \phi \]
and the corresponding conservation laws
\[ K^5 \psi + \kappa \mp \kappa \psi + \kappa \phi \]
1.4.1. Arbitrary \( \kappa \)

For potential \( V_A(\phi, \psi) = V \left(G_A(\phi, \psi)\right)\), \( G_A(\phi, \psi) = e^{-\frac{\phi}{\kappa}} (4 + \kappa^2 \psi^2 e^{\kappa \psi}) \), the vector field \( K^1 \) is a variational symmetry with conservation law function \( I^1(q, p) = K^1(q, p) \), where \( p = (\mathbf{p}, \mathbf{p}_\phi, \mathbf{p}_\psi) \); that is \( I^1(a, \phi, \psi, p_\phi, p_\psi) = \psi \mathbf{p}_\phi + \left(e^{\frac{\phi}{\kappa \psi}} - \frac{\psi}{\kappa \psi} \right) p_\phi \). Functions \( \mathbf{p} \) are the momentum defined as \( p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \), i.e. \( p_\alpha = -6a^{1-3w_m} \dot{a} + \mathbf{p}_\phi = a^{3(1-w_m)} \frac{\phi}{\psi} \) and \( p_\psi = a^{3(1-w_m)} \kappa \psi e^{\kappa \psi} \).

For \( V_B(\phi, \psi) = V \left(G_B(\phi, \psi)\right)\), \( G_B(\phi, \psi) = \psi e^{\frac{\phi}{\kappa \psi}} \), the vector field \( K^2 \) is a variational symmetry and the corresponding Noetherian conservation law is \( I^2(q, p) = K^2(q, p) \).

When \( V_C(\phi, \psi) = V \left(G_C(\phi, \psi)\right)\), with \( G_C(\phi, \psi) = e^{\frac{\phi}{\kappa \psi}} \mp \psi \dot{\psi} + \kappa \dot{\psi} \), the field equations admit the variational symmetry \( K^3 \) and the corresponding conservation law \( I^3(q, p) = K^3(q, p) \).

Finally, when \( V_D(\phi, \psi) = V \) the field equations admit three variational symmetries, the fields \( K^1, K^2 \) and \( K^3 \) with the corresponding conservation laws \( I^1(q, p), I^2(q, p) \) and \( I^3(q, p) \).

4.1.2. Case \( \kappa = \frac{\sqrt{2}}{2} (w_m - 1) \)

For the potential function \( \tilde{V}_A(\phi, \psi) = V \left(\tilde{G}_A(\phi, \psi)\right)\) with \( \tilde{G}_A(\phi, \psi) = \frac{8}{3} e^{-\frac{\sqrt{2}(w_m+1) \phi}{\kappa} + \kappa \psi} \), the cosmological field equations admit the conservation law \( \tilde{I}^1(q, p) = \tilde{K}^1(q, p) \) where the corresponding symmetry vector is the isometry \( \tilde{K}^1 \).

For \( \tilde{V}_B(\phi, \psi) = V \left(\tilde{G}_B(\phi, \psi)\right)\), \( \tilde{G}_B(\phi, \psi) = \psi e^{\frac{\phi}{\kappa \psi}} \), the vector field \( \tilde{K}^2 \) is a variational symmetry and the corresponding conservation law is \( \tilde{I}^2(q, p) = \tilde{K}^2(q, p) \).

Similarly, for \( \tilde{V}_C(\phi, \psi) = V \left(\tilde{G}_C(\phi, \psi)\right)\) with \( \tilde{G}_C(\phi, \psi) = \left(\frac{8}{3} e^{-\frac{\sqrt{2}}{2} \phi} + \kappa \psi \right) e^{\frac{\sqrt{2}}{2} \phi} \), the symmetry vector is \( \tilde{K}^3 \) with \( \tilde{I}^3(q, p) = \tilde{K}^3(q, p) \).

When potential \( \tilde{V}_D(\phi, \psi) = V \left(\tilde{G}_D(\phi, \psi)\right)\), \( \tilde{G}_D(\phi, \psi) = \psi e^{\frac{\phi}{\kappa \psi}} \) the system admits the variational symmetry \( \tilde{K}^3 \) with conservation law \( \tilde{I}^3(q, p) = \tilde{K}^3(q, p) \).

For \( \tilde{V}_E(\phi, \psi) = V \left(\tilde{G}_E(\phi, \psi)\right)\), \( \tilde{G}_E(\phi, \psi) = \frac{8}{3} e^{-\frac{\sqrt{2}(w_m-1) \psi}{\kappa} + \kappa \psi} \), the vector field \( \tilde{K}^6 \) is a variational symmetry and the corresponding Noetherian conservation law is \( \tilde{I}^6(q, p) = \tilde{K}^6(q, p) \) generated by the symmetry vector \( \tilde{K}^6 \).

Moreover, for the potential function \( \tilde{V}_F(\phi, \psi) = V \left(\tilde{G}_F(\phi, \psi)\right)\) with \( \tilde{G}_F(\phi, \psi) = \frac{8}{3} e^{-\frac{\sqrt{2}(w_m-1) \phi}{\kappa} + \kappa \psi} \), the vector field \( \tilde{K}^1 + \tilde{K}^2 \) is a variational symmetry and the corresponding Noetherian conservation law is \( \tilde{I}^6(q, p) = \tilde{K}^6(q, p) \).
vation law is \( \mathcal{I}_3^2 (q, p) = (\bar{K}^1 (q) + \beta \bar{K}^3 (q)) p \).

The vector field \( K^3 \) is a variational symmetry for the field equations with conservation law \( \mathcal{I}_3 (q, p) = \bar{K}^3 (q) p \) if the potential function \( V_G (\phi, \psi) = V (\phi, \psi) \) satisfies the differential equation \( F^3 = 0 \) in which

\[
F^3 \equiv -e^{-\frac{\sqrt{6} (w_m + 1)}{2}} \left( \sqrt{6} V_{\phi \phi} - 3 (w_m - 1) \psi V_{\phi \psi} - 3 (w_m + 1) V \right) + \\
+ 3 (w_m - 1)^2 e^{\frac{\sqrt{6} (w_m + 1)}{2}} \left( 3 (w_m + 1) V + \sqrt{6} V_{\phi \phi} \right).
\] (24)

Similarly for the linear combinations \( K^1 + \beta K^3, K^2 + \beta K^3 \) and \( K^1 + \beta K^2 + \gamma K^3 \) the symmetry conditions are

\[
0 = \psi e^{\frac{\sqrt{6} (w_m - 1)}{2}} \left( 3 (w_m + 1) V + \sqrt{6} V_{\phi \phi} \right) + 4 e^{\frac{\sqrt{6} (w_m - 1)}{2}} V_{\phi \phi} + \\
+ \beta (w_m - 1) e^{\frac{\sqrt{6} (w_m - 1)}{2}} \left( 3 (w_m + 1) V + \sqrt{6} V_{\phi \phi} \right) + \epsilon (w_m - 1) \frac{\gamma}{8} F^3,
\] (25)

respectively, while the corresponding conservation laws are \( \mathcal{I}_{13}^2 (q, p) = (\bar{K}^1 (q) + \beta \bar{K}^3 (q)) p \), \( \mathcal{I}_{23}^2 (q, p) = (\bar{K}^2 (q) + \beta \bar{K}^3 (q)) p \), and \( \mathcal{I}_{25}^2 (q, p) = (\bar{K}^1 (q) + \beta \bar{K}^2 (q) + \gamma \bar{K}^3 (q)) p \). The rest of the linear combinations do not provide any other potential function which satisfies a conservation law.

We have found all the possible cases for the scalar field potential where the cosmological field equations admit at least a variational symmetry. The field equations form a dynamical system of three dimensions, therefore in order to infer about the integrability by point symmetries we need at least three conservation laws. By definition an autonomous Hamiltonian system \( \mathcal{H} \) of dimension \( n \) is characterized as Liouville integrable [59] if there exist at least \( n - 1 \) independent conservation laws \( I_J, \ J = 1, 2, 3, ..., n - 1 \), i.e. \( \{ I_J, \mathcal{H} \} = 0 \), which are in involution, that is \( \{ I_J, I_J \} = 0 \).

For the scalar field potential

\[
V_I (\phi, \psi) = V_0 e^{-\frac{\sqrt{6} (w_m + 1)}{2} \phi},
\] (26)

we find that the cosmological field equations admit the conservation laws \( \mathcal{I}_1^1 (q, p) \), \( \mathcal{I}_2^1 (q, p) \) and \( \mathcal{I}_6^1 (q, p) = \bar{K}^6 (q) p \). Potential \( V_I (\phi, \psi) \) admits asymptotic solution the so-called hyperinflation model where the two scalar fields drive the dynamics. We refer the reader to [54] and references therein for more details.

The potential function

\[
V_{II} (\phi, \psi) = V_0 e^{\frac{\sqrt{6} (w_m + 1)^2}{2} \phi} \left( 3 \frac{e^{\frac{\sqrt{6} (w_m + 1)}{2}}}{\psi^2 (w_m - 1)^2} e^{\frac{\sqrt{6} \psi \psi_0 \phi}{2}} + 8 e^{\frac{\sqrt{6} \phi}{2}} \right)^{-\frac{2 w_m + 3}{2 w_m - 1}},
\] (27)

admits the variational symmetries \( \bar{K}^1 \), \( \bar{K}^3 \) and \( \bar{K}^4 \) with conservation laws \( \mathcal{I}_1^1 (q, p) \), \( \mathcal{I}_3^1 (q, p) \) and \( \mathcal{I}_6^1 (q, p) = \bar{K}^6 (q) p \).

For the scalar field potential

\[
V_{III} (\phi, \psi) = V_0 e^{-\frac{\sqrt{6} w_m + 4}{2} \phi} \psi^{-\frac{2 w_m + 4}{w_m - 1}}
\] (28)

the Hamiltonian system admits the conservation laws \( \mathcal{I}_2^2 (q, p) \), \( \mathcal{I}_3^3 (q, p) \) and \( \mathcal{I}_6^1 (q, p) = \bar{K}^4 (q) p \) generated by the Noether symmetries \( \bar{K}^2, \bar{K}^3 \) and \( \bar{K}^5 \).

Furthermore, when

\[
V_{IV} (\phi, \psi) = V_0 e^{\frac{\sqrt{6} (w_m + 1)^2}{2} \phi} \left( 3 \psi (2 \beta + \psi) (w_m - 1)^2 e^{\frac{\sqrt{6} \psi \psi_0 \phi}{2}} + 8 e^{\frac{\sqrt{6} \phi}{2}} \right)^{-\frac{2 w_m + 3}{2 w_m - 1}},
\] (29)

the conservation laws are \( \mathcal{I}_2^2 (q, p) \), \( \mathcal{I}_3^3 (q, p) \) and \( \mathcal{I}_{15}^4 (q, p) = (\bar{K}^4 (q) + \beta \bar{K}^5 (q)) p \).
The potential

\[ V_V (\phi, \psi) = V_0 e^{-\frac{\sqrt{\pi (w_m+1)}}{2 \sqrt{w_m-1}} \phi \left( 3 \beta \psi (w_m - 1)^2 \varepsilon + 4 \right) - 2 \frac{\sqrt{w_m + 1}}{w_m - 1} \psi}, \]  

admits the conservation laws \( \bar{I}^{13} (q, p) , \bar{I}^{2} (q, p) \) and \( \bar{I}^{56} (q, p) = (\bar{K}^5 (q) - \frac{\sqrt{6}}{3 \beta \varepsilon (w_m - 1)} \bar{K}^6 (q)) p. \)

For the potential function

\[ V_{VI} (\phi, \psi) = V_0 e^{\frac{\sqrt{\pi (w_m+1)}}{2 \sqrt{w_m-1}} \phi \left( 3 \gamma \psi (2 \beta + \psi) (w_m - 1)^2 + 8 \beta \right) e^{\frac{\sqrt{\pi (w_m + 1)}}{2} + 8 \gamma e^{\frac{\sqrt{w_m + 1}}{2}}}, \]

the Noetherian conservation laws are \( \bar{I}^{12} (q, p) = (\bar{K}^1 (q) + \beta \bar{K}^2 (q)) p \), \( \bar{I}^{12} (q, p) = (\bar{K}^1 (q) + \gamma \bar{K}^2 (q)) p \) and \( \bar{I}^{56} (q, p) = (\bar{K}^5 (q) - \frac{\sqrt{6}}{3 \beta \varepsilon (w_m - 1)} \bar{K}^6 (q)) p, \) with \( \beta \neq \gamma. \)

For the special case where \( w_m = 0 \), there exist additional scalar field potentials which admit conservation laws. These potentials are presented in [60]. The field equations for the six scalar field potentials \( V_{I-VI} (\phi, \psi) \) are Liouville integrable and specifically they are superintegrable because they admit more than three independent conservation laws. In the following, we construct the analytic solutions for these integrable dynamical systems.

### 5. ANALYTIC SOLUTIONS

In order to calculate the analytic solutions for the superintegrable scalar field potentials \( V_{I-VI} (\phi, \psi) \) we prefer to work with normal coordinates. We consider the change of variables

\[ a (x, y, z) = \left( \frac{2}{w_m - 1} \right)^{\frac{2}{w_m - 1}} \left( 4yz - \frac{3}{2} \varepsilon x^2 \right), \]

\[ \phi (x, y, z) = -\frac{\sqrt{6}}{3 (w_m - 1)} \ln \left( \frac{4yz - \frac{3}{2} \varepsilon x^2}{y^2} (w_m - 1)^2 \right), \]

\[ \psi (x, y, z) = \frac{x}{y}, \]

in which the point-like Lagrangian for the cosmological field equations becomes

\[ L (x, \dot{x}, y, \dot{y}, z, \dot{z}) = \frac{1}{2} \varepsilon \dot{x}^2 - \frac{4}{3} \dot{y} \dot{z} - U_{eff} (x, y, z) - \rho_{m0}, \]

where \( U_{eff} (x, y, z) = a^{3(1+w_m)} V (\phi, \psi). \) In the new variables the second-order field equations become

\[ \varepsilon \ddot{x} + (U_{eff})_x = 0, \]

\[ \ddot{y} - \frac{3}{4} (U_{eff})_z = 0, \]

\[ \ddot{z} - \frac{3}{4} (U_{eff})_y = 0, \]

while the constraint equation becomes

\[ \frac{1}{2} \varepsilon \dot{x}^2 - \frac{4}{3} \dot{y} \dot{z} + U_{eff} (x, y, z) + \rho_{m0} = 0. \]

#### 5.1. Scalar field potential \( V_I (\phi, \psi) \)

The exact solution for potential \( V_I (\phi, \psi) \) was derived for the first time in [54]. However, we present the analytic solution for completeness. For the scalar field potential \( V_I (\phi, \psi) \) the effective potential in the variables \( \{x, y, z\} \) of point-like Lagrangian \(35\) becomes \( U_{eff}^I (x, y, z) = V_0 \dot{y} - 2 \frac{\sqrt{w_m + 1}}{w_m - 1}. \)

Thus, the field equations \(36)\)-(38) become

\[ \varepsilon \ddot{x} = 0 , \dot{y} = 0, \]
\[ \ddot{z} + \frac{3}{2} \frac{w_m + 1}{w_m - 1} V_0 y - 3 \frac{w_m + 1}{w_m - 1} = 0. \]  

Therefore, the analytic solution is

\[
x(t) = x_0(t - t_1), \quad y(t) = y_0(t - t_2),
\]

\[
z(t) = -3V_0(w_m - 1)y_0 \frac{3w_m + 1}{w_m - 1} (t - t_2)^{\frac{w_m + 3}{w_m - 1}} + z_0(t - t_3),
\]

with constraint condition

\[
\rho_{m0} = \left( \frac{4}{3} y_0 z_0 - \frac{\varepsilon}{2} (x_0)^2 \right).
\]

In the special case where \( w_m = -\frac{1}{3} \), the closed-form solution is

\[
x(t) = x_0(t - t_1), \quad y(t) = y_0(t - t_2),
\]

\[
z(t) = \frac{3}{8} V_0(t - t_2)^2 + z_0(t - t_3),
\]

with the same constraint condition. However, when \( w_m = -\frac{1}{3} \) the additional matter source can play the role of the spatial curvature \( k \), where \( \rho_{m0} = k \), and in such case we recover the closed-form solution for the multifield model in a nonflat FLRW background space and spatial curvature \( k \).

Thus, the scale factor is

\[
(a(t))^{-3(w_m - 1)} \approx \left( 4 \frac{3V_0 (1 - w_m)}{4(w_m + 3)} (y_0(t - t_2))^{\frac{w_m + 3}{w_m - 1} + 1} + z_0y_0(t - t_2)(t - t_3) \right) - \frac{3}{2} \varepsilon x_0(t - t_1),
\]

when the term \((t - t_2)^{\frac{w_m + 3}{w_m - 1} + 1}\) dominates, the scale factor is approximated as \( a(t) \approx (t - t_2)^{\frac{4}{8(w_m - 1)^2}} \).

Thus, the line element of the background FLRW space is approximated as

\[
ds^2 = -(t - t_2)^{\frac{8w_m}{w_m + 1}} dt^2 + (t - t_2)^{\frac{8}{3(w_m - 1)^2}} (dx^2 + dy^2 + dz^2),
\]

or, equivalently

\[
ds^2 = -d\tau^2 + \tau^{\frac{8}{3(w_m - 1)^2 + w_m}} \left( dx^2 + dy^2 + dz^2 \right).
\]

For \( w_m \) the closed-form solution can be found in [40].

### 5.2. Scalar field potential \( V_I(\phi, \psi) \)

For the scalar field potential \( V_I(\phi, \psi) \) we derive \( U_{eff}^I(x, y, z) = V_0 z^{-\frac{3w_m + 1}{w_m - 1}} \). We observe that changing variables \((y, z) \rightarrow (z, y)\) we end with the dynamical system of potential \( V_I(\phi, \psi) \). The field equations (36)-(38) become

\[
\varepsilon \ddot{x} = 0, \quad \ddot{z} = 0,
\]

\[
\ddot{y} + \frac{3}{2} \frac{w_m + 1}{w_m - 1} V_0 z^{-\frac{3w_m + 1}{w_m - 1}} = 0.
\]

Hence, the analytic solution is

\[
x(t) = x_0(t - t_1), \quad z(t) = z_0(t - t_2), \quad y(t) = -3V_0(w_m - 1)z_0 \frac{3w_m + 1}{w_m - 1} (t - t_2)^{\frac{w_m + 3}{w_m - 1}} + y_0(t - t_3),
\]

with \( \rho_{m0} = \left( \frac{4}{3} y_0 z_0 - \frac{\varepsilon}{2} (x_0)^2 \right) \). At the limit where \( t \rightarrow t_2 \) we have a similar behaviour as for the previous potential.
5.3.  Scalar field potential $V_{III}(\phi, \psi)$

For the third potential of our analysis $V_{III}(\phi, \psi)$ in the new variables it follows $U_{eff}^{III}(x, y, z) = V_0 x^{-\frac{2m+4}{w_m-1}}$.  The field equations (36)-(38) become

\[ \ddot{x} - 2V_0 x^{\frac{w_m+1}{w_m-1}} x^{-\frac{2w_m+4}{w_m-1}} = 0, \]  \hspace{1cm} \text{(53)}

\[ \dot{y} = 0, \quad \ddot{z} = 0. \]  \hspace{1cm} \text{(54)}

Furthermore, from the Friedmann equation it follows

\[ \frac{1}{2} \dot{\varepsilon} x^{2} + V_0 x^{-\frac{2w_m+4}{w_m-1}} - \frac{4}{3} y_0 z_0 + \rho_{m0} = 0, \]  \hspace{1cm} \text{(55)}

that is

\[ \int \frac{dx}{\sqrt{2\varepsilon \left(\frac{4}{3} y_0 z_0 - \rho_{m0} - V_0 x^{-\frac{2w_m+4}{w_m-1}}\right)}} = t - t_0. \]  \hspace{1cm} \text{(56)}

Firstly, we solve (53). The general solution is given in terms of a hypergeometric function $\text{$_2F_1$(}a, b; c; u)$:

\[ x^2 \left(1 - 2V_0 x^{\frac{2(w_m+1)}{w_m-1}} \right) 2F_1 \left(\frac{1}{2}, \frac{1-w_m}{2w_m+2}, \frac{w_m+3}{2w_m+2}, \frac{2V_0 x^{\frac{2(w_m+1)}{w_m-1}}}{c_1}\right) \right)^2 = (c_2 + t)^2. \]  \hspace{1cm} \text{(57)}

Given the complexity in solving this implicit equation, we investigate powerlaw solutions of the system.

In the special case where $\frac{4}{3} y_0 z_0 - \rho_{m0} = 0$, the closed form solution is $x(t) \simeq (t - t_0)^{\frac{w_m-1}{2w_m}}$, where for $w_m = -\frac{1}{3}$, $x(t) \simeq (t - t_0)^2$.

For $w_m \neq 0$ we can obtain a power-law solution as follows. We propose the ansatz $x(t) = x_0 (t - t_1)^p$. Then we obtain the equations:

\[ (p-1)px_0 (t - t_1)^{p-2} - 2V_0 (w_m+1)x_0^{\frac{4}{w_m-1}} \left( t - t_1 \right)^{\frac{3w_m+4p}{w_m-1}} = 0, \]  \hspace{1cm} \text{(58)}

\[ \frac{1}{2} p^2 x_0^2 (t - t_1)^{2p-2} + V_0 x_0^{\frac{2(w_m+1)}{w_m-1}} \left( t - t_1 \right)^{-\frac{2w_m+4}{w_m-1}} + \rho_{m0} - \frac{4y_0 z_0}{3} = 0. \]  \hspace{1cm} \text{(59)}

The first equation must be valid for all $t$. Then, balancing the powers we obtain $p = \frac{w_m-1}{2w_m}$. Substituting back in the second equation we obtain

\[ \rho_{m0} + \frac{x_0^2 (t - t_1)^{-\frac{w_m+4}{w_m}} \left( 8V_0 w_m x_0^{\frac{4w_m}{w_m-1}} + (w_m - 1)^2 \varepsilon \right) - 4y_0 z_0}{8w_m^2} = 0. \]  \hspace{1cm} \text{(60)}

This expression must be valid for all $t$. Therefore, we have the additional restrictions in the parameters:

\[ V_0 = -\frac{(w_m - 1)^2 \varepsilon x_0^{\frac{4w_m}{w_m-1}}}{8w_m^2}, \quad \rho_{m0} - \frac{4y_0 z_0}{3} = 0. \]  \hspace{1cm} \text{(61)}

Finally, we have the powerlaw analytic solution

\[ x(t) = x_0 (t - t_1)^{\frac{w_m-1}{2w_m}}, \quad y(t) = y_0 (t - t_2), \quad z(t) = z_0 (t - t_3). \]  \hspace{1cm} \text{(62)}

Consider now the limit where $x(t) \simeq (t - t_1)^{\frac{w_m-1}{2w_m}}$ then for $w_m > 0$, because $\frac{w_m-1}{2w_m} < 0$, the scale factor at the limit $t \to t_1$ is approximated as $a(t) \simeq (t - t_1)^{-\frac{1}{w_m}}$. Therefore, the line element for the background space becomes

\[ ds^2 = -(t - t_1)^{-1} dt^2 + (t - t_1)^{\frac{1}{w_m}} (dx^2 + dy^2 + dz^2), \]  \hspace{1cm} \text{(63)}
or
\[ ds^2 = -d\tau^2 + e^{-\frac{1}{3w_m} \tau} \left( dx^2 + dy^2 + dz^2 \right), \quad (64) \]
which describes a de Sitter universe.

On the other hand when \( w_m < 0 \), when \( t \to t_1 \) the scale factor is approximated as \( a(t) \simeq t^{\frac{-3}{2w_m - 1}} \). Therefore, the line element for background space is simplified as
\[ ds^2 = -t^{-\frac{4}{2w_m - 1}} dt^2 + t^{-\frac{1}{2w_m - 1}} \left( dx^2 + dy^2 + dz^2 \right), \quad (65) \]
that is,
\[ ds^2 = -d\tau + \tau^{\frac{1}{2w_m - 1}} \left( dx^2 + dy^2 + dz^2 \right), \quad (66) \]
which describes an accelerated universe for \( w_m < -\frac{1}{3} \).

### 5.4 Scalar field potential \( V_{IV} (\phi, \psi) \)

The effective potential which corresponds to \( V_{IV} (\phi, \psi) \) is derived \( U^{IV} (x, y, z) = \bar{V}_0 \left( 3\beta \varepsilon x + 4z \right)^{-\frac{w_m + 1}{2w_m - 1}} \) with \( \bar{V}_0 = V_0 \left( 2 \left( w - 1 \right)^2 \right)^{-\frac{2w_m + 1}{2w_m - 1}} \). The field equations become
\[ \ddot{x} - 6\bar{V}_0 \frac{w_m + 1}{w_m - 1} \beta \varepsilon Z^{-\frac{2w_m + 1}{2w_m - 1}} = 0, \quad (67) \]
\[ \ddot{y} + 6\bar{V}_0 \frac{w_m + 1}{w_m - 1} Z^{-\frac{2w_m + 1}{2w_m - 1}} = 0, \quad (68) \]
\[ \ddot{Z} - 18\bar{V}_0 \frac{w_m + 1}{w_m - 1} \beta^2 \varepsilon Z^{-\frac{2w_m + 1}{2w_m - 1}} = 0, \quad (69) \]
where \( Z = 3\beta \varepsilon x + 4z \). The analytic solution of (69) is expressed in terms of a hypergeometric function
\[ Z^2 \left( 1 - 18\beta^3 \varepsilon \bar{V}_0 Z^{-\frac{2w_m + 1}{w_m - 1}} \right) \frac{\Gamma (\frac{1}{3}, \frac{1}{2w_m + 1}, \frac{w_m + 3}{2w_m + 2}, \frac{18\bar{V}_0 \beta^3 \varepsilon Z^{-\frac{2w_m + 1}{w_m - 1}}}{c_1})}{c_1 - 18\beta^3 \varepsilon \bar{V}_0 Z^{-\frac{2w_m + 1}{w_m - 1}}} = (c + t)^2. \quad (70) \]

For \( w_m \neq 0 \) there exists the exact solution \( Z (t) = Z_0 (t - t_1)^{\frac{w_m - 1}{2w_m - 1}} \), where \( Z_0 \) is a solution of \( (w_m - 1)^2 + 72Z_0 \frac{4}{w_m - 1} \bar{V}_0 w^2 \beta^2 \varepsilon = 0 \). Hence, we have the powerlaw solution
\[ x (t) = \frac{Z_0 (t - t_1)^{\frac{w_m - 1}{2w_m - 1}}}{3\beta} + x_0 (t - t_2), \quad (71) \]
\[ y (t) = -\frac{Z_0 (t - t_1)^{\frac{w_m - 1}{2w_m - 1}}}{3\beta^2} \varepsilon + y_0 (t - t_3), \quad (72) \]
\[ Z (t) = Z_0 (t - t_1)^{\frac{w_m - 1}{2w_m - 1}}. \quad (73) \]

The restriction \( \rho_{m0} + \bar{V}_0 Z^{-\frac{2w_m + 1}{w_m - 1}} - \frac{1}{3} \beta \varepsilon \dot{Z} + \frac{1}{2} \dot{\varepsilon} \dot{x}^2 = 0 \) becomes \( \rho_{m0} + \frac{\varepsilon}{2} \beta^2 x y = 0 \).

Thus for \( w_m > 0 \) and in the limit \( t \to t_1 \) the scale factor becomes \( a(t) \simeq (t - t_1)^{-\frac{3}{2w_m - 1}} \) for \( \frac{w_m - 1}{2w_m} + 1 < 0 \). Hence the background space becomes
\[ ds^2 = -(t - t_1)^{-\frac{3w_m - 1}{w_m - 1}} dt^2 + (t - t_1)^{-\frac{3w_m - 1}{w_m - 1}} \left( dx^2 + dy^2 + dz^2 \right), \quad (74) \]
or equivalently
\[ ds^2 = -d\tau^2 + \tau^{\frac{1}{2w_m}} \left( dx^2 + dy^2 + dz^2 \right), \quad (75) \]
which describes an accelerated universe for \( w_m \in (0, \frac{1}{3}) \).

On the other hand, for \( w_m < 0 \) for large values of \( t \), the scale factor is approximated as \( a(t) \simeq t^{-\frac{3}{2w_m - 1}} \), which leads to the line element (65).
5.5. Scalar field potential \( V_\phi (\phi, \psi) \)

From the potential function \( V_\phi (\phi, \psi) \) we calculate \( U_{\phi,\psi}^V (x, y, z) = V_0 \left( (w - 1)^2 \beta \varepsilon x + 4y \right)^{-\frac{2 w_m + 1}{w_m - 1}} \). We define the new variable \( Y = (w - 1)^2 \beta \varepsilon x + 4y \) where the field equations are written as

\[
\begin{align*}
\ddot{Y} - 2V_0 (w_m - 1)^3 (w_m + 1) \varepsilon \beta^2 Y^{-\frac{3 w_m + 1}{w_m - 1}} &= 0, \\
\ddot{x} - 2V_0 (w_m^2 - 1) \beta Y^{-\frac{3 w_m + 1}{w_m - 1}} &= 0, \\
\ddot{z} + \frac{6V_0 (w_m + 1)}{w_m - 1} Y^{-\frac{3 w_m + 1}{w_m - 1}} &= 0,
\end{align*}
\]

with constraint

\[
\rho_{m0} + V_0 Y^{-\frac{2 (w_m + 1)}{w_m - 1}} - \frac{1}{3} \dot{Y} \left( \dot{Y} - \beta (w_m - 1)^2 \varepsilon \dot{x} \right) + \frac{1}{2} \varepsilon \dot{x}^2 = 0.
\]

The closed-form solution of equation (76) is expressed in terms of hypergeometric function

\[
Y^2 \left( 1 - \frac{2 \beta^2 V_0 (w_m - 1)^3 \varepsilon Y^{-\frac{2 (w_m + 1)}{w_m - 1}}}{c_1} \right) 2F1 \left( \frac{1}{2}, \frac{1 - w_m}{2 w_m + 2}, \frac{w_m + 3}{2 w_m + 2}; \frac{2V_0 (w_m - 1)^3 \beta \varepsilon Y^{-\frac{2 (w_m + 1)}{w_m - 1}}}{c_1} \right) = (c_2 + t)^2.
\]

For \( w_m = -\frac{1}{3} \) we recover the closed-form solution

\[
\begin{align*}
\dot{x}(t) &= -\frac{8}{9} \beta V_0 (t - t_1)^2 + x_0 (t - t_2), \\
Y(t) &= Y_0 (t - t_1)^2, \\
\dot{z}(t) &= \frac{3}{2} V_0 (t - t_1)^2 + z_0 (t - t_3),
\end{align*}
\]

with \( V_0 = -\frac{81Y_0}{128 \beta \varepsilon}, \quad 6 \rho_{m0} + x_0 \varepsilon \left( 3x_0 + \frac{32 \beta z_0}{9} \right) = 0. \)

The special exact solutions for arbitrary value of \( w_m \neq 0 \) exist and for this potential in a similar way as we calculated them for potential \( V_{IV} (\phi, \psi) \) where we have to perform the change of variables \((y, z) \rightarrow (z, y)\). We can obtain a power-law solution

\[
\begin{align*}
Y(t) &= Y_0 (t - t_1)^{\frac{w_m - 1}{2 w_m - 1}}, \\
x(t) &= \varepsilon Y_0 (t - t_1)^{\frac{w_m - 1}{2 w_m - 1}} + x_0 (t - t_2), \\
z(t) &= -\frac{3}{\beta^2} Y_0 (t - t_1)^{\frac{w_m - 1}{2 w_m - 1}} + z_0 (t - t_3),
\end{align*}
\]

where \( 8 \beta^2 V_0 (w_m - 1)^3 w_m(w_m + 1) \varepsilon Y_0^{-\frac{4 (w_m - 1)}{w_m - 1}} + w_m^2 + 1 = 0 \) gives \( V_0 \) and \( 6 \rho_{m0} + x_0 \varepsilon \left( 3x_0 + 2 \beta (w_m - 1)^2 z_0 \right) = 0. \)

5.6. Scalar field potential \( V_{II} (\phi, \psi) \)

Finally, from \( V_{II} (\phi, \psi) \) it follows \( U_{\phi,\psi}^{V_{II}} (x, y, z) = V_0 \left( 6 (w_m - 1)^2 \beta \varepsilon x + 8 \left( \beta y + (w - 1)^2 \right) z \right)^{-\frac{2 w_m + 1}{w_m - 1}} \). With the use of the new variable \( U = 6 (w_m - 1)^2 \beta \varepsilon x \gamma + 8 \left( \beta y + (w - 1)^2 \gamma z \right) \), the field equations become

\[
\begin{align*}
\ddot{U} - 24 \gamma V_0 (w_m^2 - 1) \beta \left( \varepsilon \beta (w_m - 1)^2 - 8 \right) U^{-\frac{2 w_m + 1}{w_m - 1}} &= 0, \\
\ddot{x} - 12V_0 \gamma (w_m^2 - 1) \beta U^{-\frac{2 w_m + 1}{w_m - 1}} &= 0, \\
\ddot{z} + \frac{12V_0 (w_m + 1)}{w_m - 1} \beta U^{-\frac{2 w_m + 1}{w_m - 1}} &= 0.
\end{align*}
\]
with constraint
\[ \rho_{m0} - \frac{\dot{U} \ddot{z}}{6\beta} + V_0 U^{\frac{2(w+1)}{w-1}} + \gamma (w - 1)^2 \varepsilon \dot{x} \ddot{z} + \frac{4\gamma (w - 1)^2 \varepsilon^2}{3\beta} + \frac{1}{2} \varepsilon \dot{x}^2 = 0. \] (88)

The results from the previous two potentials follow: the behaviour of the solution at the limits has similar properties as before.

Firstly, we solve (85). The general solution is given in terms of a hypergeometric function:

\[ U^2 \left( 1 - \frac{24\beta \gamma V_0 (w_m-1)^2 U^{-\frac{2(w_m-1)}{w_m-1}} (\beta (w_m-1)^2 \varepsilon - 8)}{c_1} \right)^2 F_1 \left( \frac{1}{2}, 1 - \frac{w_m}{2w_m+2}, \frac{w_m+3}{2w_m+2}, \frac{24\beta \gamma V_0 (w_m-1)^2 \beta (w_m-1)^2 \varepsilon - 8}{w_m+1} U^{-\frac{2(w_m-1)}{w_m-1}} \right) = (c_2 + t)^2. \] (89)

For \( w_m \neq 0 \) the system (85), (86), (87) admits the following power-law solution

\[ U(t) = U_0(t - t_1)^{\frac{w_m-1}{w_m}}, \] (90)

\[ x(t) = U_0(t - t_1)^{\frac{w_m-1}{w_m}} \frac{6\beta \gamma (w_m-1)^2 \varepsilon - 16}{6\beta \gamma (w_m-1)^2 \varepsilon - 16} + x_0(t - t_2), \] (91)

\[ z(t) = -\frac{U_0(t - t_1)^{\frac{w_m-1}{w_m}}}{2\gamma (w_m-1)^2 (3\beta \gamma (w_m-1)^2 \varepsilon - 8)} + z_0(t - t_3), \] (92)

where \( U_0^{\frac{w_m}{w_m-1}} + 96\beta \gamma V_0 w_m^2 (3\beta \gamma (w_m - 1)^2 \varepsilon - 8) = 0 \) gives \( V_0 \) and \( 6\beta \rho_{m0} + 2\gamma (w - 1)^2 z_0 (3\beta x_0 \varepsilon + 4z_0) + 3\beta x_0^2 \varepsilon = 0 \) gives \( \rho_{m0} \).

6. STABILITY ANALYSIS OF THE SCALING SOLUTIONS

According to the methods in [61–63] let be

\[ F(\phi, \dot{\phi}, \phi) = 0, \] (93)

a second order differential equation in one dimension which admits a singular powerlaw solution

\[ \phi_c(t) = \phi_0 t^{\beta}. \] (94)

To examine the stability of the solution \( \phi_c \), the logarithmic time \( \tau \) through \( t = e^\tau \) is introduced, such that \( t \to 0 \) as \( \tau \to -\infty \) and \( t \to +\infty \) as \( \tau \to +\infty \). We use \( \phi' = \frac{\phi}{4\tau} \) in the following discussion.

The following dimensionless function is introduced

\[ u(\tau) = \frac{\phi(\tau)}{\phi_c(\tau)}, \] (95)

and the stability analysis in translated into the analysis of the stability of \( u = 1 \) of a transformed dynamical system. To construct the aforementioned system the following relations are useful:

\[ \dot{\phi} = e^{-\tau} \phi', \quad \ddot{\phi} = e^{-2\tau} (\phi'' - \phi'), \quad \frac{\phi'}{\phi_c} = \beta \text{ if } \phi_c(t) = \phi_0 t^{\beta}. \] (96)

In this section we use a similar procedure for analyzing stability of the scaling solutions that are obtained in Section 5.

Due to the complexity of the dynamics, we have supported our analytical results of this section in numerical simulations by using initial conditions within the constraint surfaces: \( (\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = 1 \) and \( \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 1 \). The auxiliary variables \( \phi_i, \Phi_i \) are dummy variables that are defined for each potential, and they are used to classify the stability of the powerlaw solutions obtained for each potential. The forward integration is based in two simple basis: i) if the point is an attractor, the boundary surfaces act as trapping surfaces due to all the orbits are attracted by the basing of attraction of the \((1,0,1,0,1,0); \) ii) if the point \((1,0,1,0,1,0)\) have saddle behavior, some orbits would abandon this region passing through the boundary surfaces to higher radii, say, having to the past and to the future that \((\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 > 1 \) and \( \Phi_1^2 + \Phi_2^2 + \Phi_3^2 > 1 \).
6.1. Scalar field potential \( V_1 (\phi, \psi) \)

In this section we analyze the stability of the analytic solution (42), (43) of equations (40) (41). We set for simplicity the integration constants \( t_1, t_2, t_3 \) to zero because they are not relevant as \( t \to \infty \).

With the time variable \( \tau = \ln(t) \), and defining the new variables \( Y = V_0 e^{2\tau} y^{\frac{3w_m+1}{w_m-1}} \) and \( Z = z(\tau) - z_0 e^\tau \) the equations (40) and (41) become

\[
x''(\tau) - x'(\tau) = 0, \quad Z''(\tau) - Z'(\tau) = \frac{3}{2} w_m + 1 \frac{Y(\tau)}{w_m - 1},
-4w_m Y'(\tau)^2 + Y(\tau) (3w_m + 1) Y''(\tau) + (w_m - 5) Y'(\tau) + 2(w_m + 3) Y(\tau)^2 = 0.
\]

The analytic solution (42), (43) becomes:

\[
x_c(\tau) = x_0 e^\tau, \quad Y_c(\tau) = Y_0 e^{-\frac{\tau(w_m+3)}{w_m-1}}, \quad Z_c(\tau) = z_1 e^{-\frac{\tau(w_m+3)}{w_m-1}}, \quad z_1 = -\frac{3(w-1)Y_0}{4(w+3)}.
\]

Defining the dimensionless variables

\[
\phi_1 = \frac{x}{x_c}, \quad \phi_2 = \frac{Y}{Y_c}, \quad \phi_3 = \frac{Z}{Z_c},
\]

we obtain the dynamical system

\[
\phi'_1 = \Phi_1, \quad \Phi'_1 = -\Phi_1, \\
\phi'_2 = \Phi_2, \quad \Phi'_2 = -\Phi_2 + \frac{4w_m\Phi_2^2}{\phi_2(1 + 3w_m)}, \\
\phi'_3 = \Phi_3, \quad \Phi'_3 = \frac{3w_m + 5}\phi_3 + \frac{4w_m + 1(w_m + 3)(\phi_2 - \phi_3)}{(w_m - 1)^2}.
\]

Now we analyze the stability of the solution \( P := (\phi_1, \Phi_1, \phi_2, \Phi_2, \phi_3, \Phi_3) = (1, 0, 1, 0, 1, 0) \). The subsystems for \((\phi_1, \Phi_1), (\phi_2, \Phi_2)\) and \((\phi_3, \Phi_3)\) are decoupled. The Jacobian matrix of the full system is

\[
J := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{4w_m\Phi_2^2}{(3w_m+1)\phi_2^2} & \frac{8w_m\phi_3}{3w_m\phi_2 + \phi_2^2} - 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2(w_m+1)(w_m+3) & 0 & -\frac{2(w_m+1)(w_m+3)}{(w_m-1)^2} & 3 + \frac{8}{w_m-1} \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

Evaluating \( J \) at the fixed point \( P \) the eigenvalues \( \{0, 0, -1, -1, -\frac{2(w_m+1)}{1-w_m}, -\frac{w_m+3}{1-w_m}\} \) are obtained. The stable manifold is 4D if \(-1 < w_m < 1\). If the analysis is restricted to the subspace \((\phi_2, \Phi_2, \phi_3, \Phi_3)\) the eigenvalues are \( \{0, -1, -\frac{2(w_m+1)}{1-w_m}, -\frac{w_m+3}{1-w_m}\} \). The stable manifold in this subspace is 3D if \(-1 < w_m < 1\).

Introducing the new variables

\[
u_1 = \Phi_2 + \phi_2 - 1, \quad u_2 = \Phi_1 + \phi_1 - 1, \quad v_1 = \frac{\Phi_2(w_m + 3)}{3w_m + 1}, \quad v_2 = \Phi_1, \\
v_3 = \frac{2(w_m+1)(\Phi_3(w_m-1)(3w_m+1) + (w_m+3)(\Phi_2(w_m-1) + 3w_m\phi_2 - (3w_m + 1)\phi_3 + \phi_2))}{(w_m - 1)^2 (3w_m + 1)}, \\
v_4 = \frac{-(w_m + 3)(\Phi_2(w_m - 1) + \Phi_3(w_m - 1) + 2(w_m + 1)\phi_2 - 2(w_m + 1)\phi_3)}{(w_m - 1)^2},
\]

(102) (103) (104)
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right). We have represented the spheres 
\[(\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2 \quad \text{and} \quad \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2, \quad r \in \{1, \sqrt{2}\}.
\]

(b) From left to right, from top to bottom: Projections in the planes \((u_1, u_2), (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (u_1, v_1)\) and \((u_2, v_2)\). The red line in fig 8 (resp. in fig 9) of the bottom array 
denotes the invariant set \(v_1 = 0\) (resp. \(v_2 = 0\)) in the projection \(u_1\) vs \(v_1\) (resp. \(u_2\) vs \(v_2\)).

FIG. 1: Some solutions of (a) the system (98)-(100) and (b) the system (105) for the potential \(V_I(\phi, \psi)\) when \(\omega_m = -1\). Notice in the projection \(u_1\) vs \(u_2\) (which contains the center manifold) the origin behaves as a saddle point. The orbits for the left along the \(u_1\)-axis tend to the origin, but for the right the orbits depart from the origin.
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right). We have represented the spheres 
\[ (\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2 \text{ and } \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2, \quad r \in \{1, \sqrt{2}\}. \]

(b) From left to right, from top to bottom: Projections in the planes \((u_1, u_2)\), \((v_1, v_2)\), \((v_1, v_3)\), \((v_1, v_4)\), \((v_2, v_3)\), \((v_2, v_4)\), \((v_3, v_4)\), \((u_1, v_1)\) and \((u_2, v_2)\). The red line in fig 8 (resp. in fig 9) of the bottom array 
denotes the invariant set \(v_1 = 0\) (resp. \(v_2 = 0\)) in the projection \(u_1 \text{ vs } v_1\) (resp. \(u_2 \text{ vs } v_2\)).

FIG. 2: Some solutions of (a) the system (98)-(100) and (b) the system (105) for the potential \(V_I(\phi, \psi)\) when \(\omega_m = 0\).

Notice in the projection \(u_1 \text{ vs } u_2\) (which contains the center manifold) the origin is stable (but not asymptotically stable) 
since any \(\epsilon\)-neighborhood of the origin will contain a \(\delta\)-neighborhood of origin with other points apart of the origin with 
\((u_1', u_2')\) \(u_1 = u_1^*, u_2 = u_2^* = (0, 0)\). Therefore, they remain in \(\delta\)-neighborhood of origin.
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right). We have represented the spheres \((\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2\) and \(\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2, r \in \{1, \sqrt{2}\}\).

(b) From left to right, from top to bottom: Projections in the planes \((u_1, u_2), (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (u_1, v_1)\) and \((u_2, v_2)\). The red line in fig 8 (resp. in fig 9) of the bottom array denotes the invariant set \(v_1 = 0\) (resp. \(v_2 = 0\)) in the projection \(u_1\) vs \(v_1\) (resp. \(u_2\) vs \(v_2\)).

FIG. 3: Some solutions of (a) the system (98)-(100) and (b) the system (105) for the potential \(V_I(\phi, \psi)\) when \(\omega_m = \frac{1}{4}\). Notice in the projection \(u_1\) vs \(u_2\) (which contains the center manifold) the origin behaves as a saddle point. The orbits for the left along the \(u_1\)-axis tend to the origin, but for the right the orbits depart from the origin.
we obtain the system
\[
\begin{align*}
u'_1 &= \frac{4v_1^2w_m(3w_m + 1)}{(w_m + 3)(w_m + 3) - v_1(3w_m + 1) + w_m + 3}, \\
u'_2 &= 0, \\
u'_3 &= \frac{v_1(-u_1(w_m + 3) + 7v_1w_m + v_1 - w_m - 3)}{u_1(w_m + 3) - v_1(3w_m + 1) + w_m + 3}, \\
u'_4 &= -v_2, \\
u'_5 &= \frac{2(w_m + 1)(v_3(u_1(w_m + 3) - 3v_1w_m - v_1 + w_m + 3) + 4v_3^2w_m)}{(w_m - 1)(u_1(w_m + 3) - v_1(3w_m + 1) + w_m + 3)}.
\end{align*}
\]

(105a)
(105b)
(105c)
(105d)
(105e)

The center manifold of the origin is given by
\[
\left\{ (u_1, u_2, v_1, v_2, v_3, v_4) \in \mathbb{R}^6 : v_i = h_i(u_1, u_2), \frac{\partial h_i}{\partial u_1}(0, 0) = 0, \frac{\partial h_i}{\partial u_2}(0, 0) = 0, h_i(0, 0) = 0, i = 1 \ldots 4 \right\},
\]

(106)

where \( h_i(u_1, u_2) \) satisfies the system of quasi-linear partial differential equations
\[
\begin{align*}
h_1 &\left( 4w_m(3w_m + 1) + \frac{\partial h_1}{\partial u_1} + (-w_m - 3)(7w_m + 1) + (u_1 + 1)(w_m + 3)^2 \right) = 0, \\
4w_m(3w_m + 1) &\frac{h_2}{h_1} \frac{\partial h_2}{\partial u_1} + (w_m + 3)h_2(3w_m + 1)h_1 - (3w_m + 1)h_1 = 0, \\
2w_m((2 - 3w_m)w_m + 1) &\frac{h_3}{h_1} \frac{\partial h_3}{\partial u_1} + (w_m + 1)(w_m + 3)(h_3((u_1 + 1)(w_m + 3) - (3w_m + 1)h_1) + 4w_mh_2^2) = 0, \\
- (u_1 + 1)(w_m + 3)^3 &h_4 + (3w_m + 1)h_1 \left( 4w_mh_1 \left( (w_m - 1) \frac{\partial h_4}{\partial u_4} + w_m + 3 \right) + (w_m + 3)^2h_4 \right) = 0.
\end{align*}
\]

(107)
(108)
(109)
(110)

Using the expansion in series
\[
h_i(u_1, u_2) = \sum_{n=0}^{N} \sum_{k=0}^{n} a_n^{[i]} u_1^{n-k} u_2^k + \mathcal{O}((\|u_1, u_2\|)^{N+1}), i = 1, \ldots 4
\]

(111)

the zero solution for any given accuracy is found.

In figures 1-3 some solutions of the systems (98)-(100) and (105) for the potential \( V_I(\phi, \psi) \) when \( \omega_m = -1, 0 \) and \( \frac{1}{4} \) are represented. More specific, in the upper panel the solutions are projected in the space \( (\phi_1, \phi_2, \phi_3) \) and \( (\Phi_1, \Phi_2, \Phi_3) \), where we have represented the spheres \( (\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2 \) and \( \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2 \) with \( r \in \{1, \sqrt{2}\} \). In the lower panel projections in the spaces \( (u_1, u_2), (v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4), (u_1, v_1) \) and \( (u_2, v_2) \) are represented. In these figures we have depicted a red line in the projections \( (u_1, v_1) \) and \( (u_2, v_2) \) which denotes the invariant set \( v_1 = 0 \) and \( v_2 = 0 \), respectively. Both lines are stable in these projections. Notice that in figures 1 and 3, the projection \( u_1 \) vs \( u_2 \), which contains the center manifold; the orbit behaves as a saddle point. The orbits from the left along the \( u_1 \)-axis tend to the origin, but from the right the orbits depart from the origin. Then, the solution is unstable (saddle behavior). This behavior is also represented in the 3D projection \( (\phi_1, \phi_2, \phi_3) \) where some orbits abandon the inner spheres backward and forward in time. On the other hand, the projection \( u_1 \) vs \( u_2 \) in figure 2; the orbit is stable (but not asymptotically stable) since any \( \epsilon \)-neighborhood of the origin will contain a \( \delta \)-neighborhood of origin with other points apart of the origin with \( (u'_1, u'_2)|_{u_1=u_1^*, u_2=u_2^*} = (0, 0) \). Therefore, they remain in \( \delta \)-neighborhood of origin.

6.2. Scalar field potential \( V_{II}(\phi, \psi) \)

In this section we study the stability of the solution (52) of the system (50), (51). We set for simplicity the integration constants \( t_1, t_2, t_3 \) to zero because they are not relevant as \( t \to \infty \). Noticing for the scalar field potential \( V_{II}(\phi, \psi) \) we derive \( U_{II}^{eff}(x, y, z) = V_0 z^{-\frac{2m+1}{2m-3}} \). We observe that doing the change of variables \( (y, z) \to (z, y) \) we
end with the dynamical system of potential \(V_f(\phi, \psi)\). Hence, with the time variable \(\tau = \ln(t)\), and defining the new variables \(Z = V_0 e^{2\tau} z^{\frac{3w_m+1}{w_m-1}}\) and \(Y = y(\tau) - y_0 e^\tau\) the field equations (50), (51) become

\[
x''(\tau) - x'(\tau) = 0, \quad Y''(\tau) - Y'(\tau) + \frac{3}{2} \frac{w_m+1}{w_m-1} Z(\tau) = 0,
-4w_m Z'(\tau)^2 + Z(\tau) \left( (3w_m+1)Z''(\tau) + (w_m - 5)Z'(\tau) \right) + 2(w_m + 3)Z(\tau)^2 = 0.
\]

The analytical solution of the original system becomes

\[
x_c(\tau) = x_0 e^\tau, \quad Y_c(\tau) = y_1 e^{-\frac{\tau(w_m+3)}{w_m-1}}, \quad y_1 = -\frac{3(w-1)Z_0}{4(w+3)}, \quad Z_c(\tau) = Z_0 e^{-\frac{\tau(w_m+3)}{w_m-1}}, \quad Z_0 = V_0 z_0^{\frac{3w_m+1}{w_m-1}}.
\]

Defining the dimensionless variables

\[
\phi_1 = \frac{x}{x_c}, \quad \phi_2 = \frac{Y}{Y_c}, \quad \phi_3 = \frac{Z}{Z_c},
\]

we obtain the dynamical system

\[
\phi_1' = \Phi_1, \quad \Phi_1' = -\Phi_1, \quad \phi_1 = \phi_1(\Phi_1, \phi_2, \phi_3, \phi_3), (112)
\]

\[
\phi_2' = \Phi_2, \quad \Phi_2' = \frac{(3w_m+5)\Phi_2}{w_m-1} + \frac{2(w_m+1)(w_m+3)(\phi_3 - \phi_2)}{(w_m-1)^2}, \quad \phi_2 = \phi_2(\Phi_2, \phi_3, \phi_3), (113)
\]

\[
\phi_3' = \Phi_3, \quad \Phi_3' = -\Phi_3 + \frac{4w_m\Phi_3^2}{\phi_3(1+3w_m)}, \quad \phi_3 = \phi_3(\Phi_3, \phi_3, \phi_3), (114)
\]

Now we analyze the stability of the fixed point \(P := (\phi_1, \phi_1, \phi_2, \phi_3, \phi_3) = (1, 0, 1, 0, 1, 0)\). The subsystems for \((\phi_1, \phi_1, \phi_2, \phi_3)\) and \((\phi_2, \phi_2, \phi_3, \phi_3)\) are decoupled.

The Jacobian matrix of the full system is

\[
J := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -\frac{4w_m\Phi_3^2}{(3w_m+1)\phi_3} \\
0 & 0 & 0 & 0 & 0 & \frac{8w_m\Phi_3}{(3w_m+1)\phi_3} \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
\]

Evaluating \(J\) at the fixed point \(P\) the eigenvalues \(\{0, 0, -1, -1, -\frac{2(w_m+1)}{1-w_m}, -\frac{w_m+3}{1-w_m}\}\) are found. The stable manifold is 4D if \(-1 < w_m < 1\). If the analysis is restricted to the subspace \((\phi_2, \phi_2, \phi_3, \phi_3)\) the eigenvalues are \(\{0, -1, -\frac{2(w_m+1)}{1-w_m}, -\frac{w_m+3}{1-w_m}\}\). The stable manifold in this subspace is 3D if \(-1 < w_m < 1\).

Introducing the new variables

\[
u_1 = \Phi_3 + \phi_3 - 1, \quad u_2 = \Phi_1 + \phi_1 - 1, \quad v_1 = \Phi_3, \quad v_2 = \Phi_1,
\]

\[
v_3 = \frac{2(w_m+1)(w_m-1)(w_m+3) + (-3w_m-1)(\Phi_2 - w_m(\Phi_2 + \phi_3) + (w_m + 3)\phi_2 - 3\phi_3))}{(w_m-1)^2(3w_m+1)},
\]

\[
v_4 = \frac{(w_m+3)(\Phi_2 + \phi_3 - w_m(\Phi_2 + \phi_3 + 2\phi_3) + 2(w_m+1)\phi_2 - 2\phi_3)}{(w_m-1)^2}.
\]
(a) Projections in the space $\left(\phi_1, \phi_2, \phi_3\right)$ (left) and $\left(\Phi_1, \Phi_2, \Phi_3\right)$ (right). We have represented the spheres $\left(\phi_1 - 1\right)^2 + \left(\phi_2 - 1\right)^2 + \left(\phi_3 - 1\right)^2 = r^2$ and $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2$, $r \in \{1, \sqrt{2}\}$.

(b) From left to right, from top to bottom: Projections in the planes $\left(u_1, u_2\right)$, $\left(v_1, v_2\right)$, $\left(v_1, v_3\right)$, $\left(v_1, v_4\right)$, $\left(v_2, v_3\right)$, $\left(v_2, v_4\right)$, $\left(v_3, v_4\right)$, $\left(u_1, v_1\right)$ and $\left(u_2, v_2\right)$. The red line in fig 8 (resp. fig 9) of the bottom array denotes the invariant set $v_1 = 0$ (resp. $v_2 = 0$) in the projection $u_1$ vs $v_1$ (resp. $u_2$ vs $v_2$).

FIG. 4: Some solutions of (a) the system (112)-(114) and (b) the system (116) for the potential $V_{II}(\phi, \psi)$ when $\omega_m = -1$. Notice in the projection $u_1$ vs $u_2$ (which contains the center manifold) the origin behaves as a saddle point. The orbits for the left along the $u_1$-axis tend to the origin, but for the right the orbits depart from the origin.
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right). We have represented the spheres \((\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2\) and \(\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2\), \(r \in \{1, \sqrt{2}\}\).

(b) From left to right, from top to bottom: Projections in the planes \((u_1, u_2)\), \((v_1, v_2)\), \((v_1, v_3)\), \((v_1, v_4)\), \((v_2, v_3)\), \((v_2, v_4)\), \((v_3, v_4)\), \((u_1, v_1)\) and \((u_2, v_1)\). The red line in fig 8 (resp. fig 9) of the bottom array denotes the invariant set \(v_1 = 0\) (resp. \(v_2 = 0\)) in the projection \(u_1\) vs \(v_1\) (resp. \(u_2\) vs \(v_2\)).

FIG. 5: Some solutions of (a) the system (112)-(114) and (b) the system (116) for the potential \(V_{II}(\phi, \psi)\) when \(\omega_m = 0\). Notice in the projection \(u_1\) vs \(u_2\) (which contains the center manifold) the origin is stable (but not asymptotically stable) since any \(\epsilon\)-neighborhood of the origin will contain a \(\delta\)-neighborhood of origin with other points apart of the origin with \((u_1', u_2')|_{u_1 = u_1^*, u_2 = u_2^*} = (0, 0)\). Therefore, they remain in \(\delta\)-neighborhood of origin.
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right). We have represented the spheres 
\((\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2\) and \(\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2\), \(r \in \{1, \sqrt{2}\}\).

(b) From left to right, from top to bottom: Projections in the planes \((u_1, u_2)\), \((v_1, v_2)\), \((v_1, v_3)\), \((v_1, v_4)\), \((v_2, v_3)\), \((v_2, v_4)\), \((v_3, v_4)\), \((u_1, v_1)\) and \((u_2, v_2)\). The red line in fig 8 (resp. fig 9) of the bottom array 
denotes the invariant set \(v_1 = 0\) (resp. \(v_2 = 0\)) in the projection \(u_1 \text{ vs } v_1\) (resp. \(u_2 \text{ vs } v_2\)).

FIG. 6: Some solutions of (a) the system (112)-(114) and (b) the system (116) for the potential \(V\{\phi, \psi\}\) when \(\omega_m = \frac{1}{2}\). Notice 
in the projection \(u_1 \text{ vs } u_2\) (which contains the center manifold) the origin behaves as a saddle point. The orbits for the left 
along the \(u_1\)-axis tend to the origin, but for the right the orbits depart from the origin.
The center manifold of the origin is given by

\[ u_1' = \frac{4v_1^2 w_m}{(3w_m + 1)(u_1 - v_1 + 1)}, \]  

\[ u_2' = 0, \]  

\[ v_1' = v_1 \left( \frac{4v_1 w_m}{(3w_m + 1)(u_1 - v_1 + 1)} - 1 \right), \]  

\[ v_2' = -v_2, \]  

\[ v_3' = \frac{2(w_m + 1) \left(v_3(3w_m + 1)^2(u_1 - v_1 + 1) + 4v_1^2 w_m(w_m + 3)\right)}{(w_m - 1)(3w_m + 1)^2(u_1 - v_1 + 1)}, \]  

\[ v_4' = \frac{(w_m + 3) \left(v_4(3w_m + 1)(u_1 - v_1 + 1) - 4v_1^2 w_m\right)}{(w_m - 1)(3w_m + 1)(u_1 - v_1 + 1)}. \]

The center manifold of the origin is given by

\[ \{ (u_1, u_2, v_1, v_2, v_3, v_4) \in \mathbb{R}^6 : v_i = h_i(u_1, u_2), \frac{\partial h_i}{\partial u_1}(0, 0) = 0, \frac{\partial h_i}{\partial u_2}(0, 0) = 0, h_i(0, 0) = 0, i = 1 \ldots 4 \} \]  

where \( h_i(u_1, u_2) \) satisfies the system of quasi-linear partial differential equations

\[ h_1((7w_m + 1)h_1 + (-u_1 - 1)(3w_m + 1)) - 4w_m h_1^2 \frac{\partial h_1}{\partial u_1} = 0, \]  

\[ 4w_m h_1^2 \frac{\partial h_2}{\partial u_1} + (3w_m + 1)(-h_1 + u_1 + 1)h_2 = 0, \]  

\[ 2w_m (2 - 3w_m)w_m + 1)h_1^2 \frac{\partial h_3}{\partial u_1} + (w_m + 1) \left((3w_m + 1)^2(-h_1 + u_1 + 1)h_3 + 4w_m(w_m + 3)h_1^2\right) = 0, \]  

\[ (w_m + 3) \left((3w_m + 1)(-h_1 + u_1 + 1)h_4 - 4w_m h_1^2\right) - 4(w_m - 1)w_m h_1^2 \frac{\partial h_4}{\partial u_1} = 0. \]

Using the expansion in series

\[ h_i(u_1, u_2) = \sum_{\substack{n=0 \ldots N \times \infty}} \sum_{k=0}^n a_{nk}^i u_1^{n-k} u_2^k + O(||(u_1, u_2)||^{N+1}), i = 1 \ldots 4 \]

the zero solution is found for any given accuracy.

In figures 4-6 some solutions of the systems (112)-(114) and (116) for the potential \( V_{II}(\phi, \psi) \) when \( \omega_m = -1, 0 \) and \( \frac{1}{4} \) are represented. More specific, in the upper panel the solutions are projected in the space \( (\phi_1, \phi_2, \phi_3) \) and \( (\Phi_1, \Phi_2, \Phi_3) \), where we have represented the spheres \( (\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2 \) and \( \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2 \) with \( r \in \{1, \sqrt{2}\} \). In the lower panel projections in the spaces \( (u_1, u_2), (v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4), (u_1, v_1) \) and \( (u_2, v_2) \) are represented. In these figures we have depicted a red line in the projections \( (u_1, v_1) \) and \( (v_2, v_2) \) which denotes the invariant set \( v_1 = 0 \) and \( v_2 = 0 \), respectively. Both lines are stable in these projections. Notice that in figures 4 and 6, the projection \( u_1 \) vs \( u_2 \) the origin behaves as a saddle point. The orbits from the left along the \( u_1 \)-axis tend to the origin, but from the right the orbits depart from the origin. Then, the solution is unstable (saddle behavior). This behavior is also represented in the 3D projection \( (\phi_1, \phi_2, \phi_3) \) where some orbits abandon the inner spheres backward and forward in time. On the other hand, the projection \( u_1 \) vs \( u_2 \) in figure 5 the origin is stable (but not asymptotically stable) since any \( \epsilon \)-neighborhood of the origin will contain a \( \delta \)-neighborhood of origin with other points apart of the origin with \( (u_1', u_2')|_{u_1 = u_2'} = (0, 0) \). Therefore, they remain in \( \delta \)-neighborhood of origin.

### 6.3. Scalar field potential \( V_{II}(\phi, \psi) \)

In this section we analyze the stability of the analytic solution (62) of equations (53), (54). We set for simplicity the integration constants \( t_1, t_2, t_3 \) to zero because they are not relevant as \( t \to \infty \) and we assume \( w_m \neq 0, 1 \).

With the time variable \( \tau = \ln(t) \) and defining the new variable \( X = V_0 \varepsilon e^{2\tau} x^r \), with \( r = -\frac{4w_m}{w_m - 1} \) to balance the powers of \( t \), the equations (53), (54) become

\[ -32w_m^2 (w_m + 1)X(\tau)^3 - (w_m - 1)^2 ((1 - 5w_m)X'(\tau)^2 - 4X(\tau) (X'(\tau) - w_mX''(\tau)) + 4(w_m + 1)X(\tau)^2) = 0, \]

\[ y''(\tau) - y'(\tau) = 0, \quad z''(\tau) - z'(\tau) = 0. \]
The analytical solution of the original system becomes
\[ X_c(t) = -\frac{(w_m - 1)^2}{8w_m^2}, \quad y_c(\tau) = y_0 e^{\tau}, \quad z_c(\tau) = z_0 e^{\tau}. \]

Defining the dimensionless variables
\[
\phi_1 = \frac{X}{X_c}, \quad \phi_2 = \frac{y}{y_c}, \quad \phi_3 = \frac{z}{z_c},
\]
we obtain for \( w_m \neq 0, 1 \) the dynamical system
\[
\begin{align*}
\phi'_1 &= \Phi_1, \quad \phi'_1 = \frac{\Phi_1}{w_m} + \frac{\Phi_1^2(5w_m - 1)}{4w_m \phi_1} + \frac{(w_m + 1)(\phi_1 - \phi_1)}{w_m}, \\
\phi'_2 &= \Phi_2, \quad \phi'_2 = -\Phi_2, \\
\phi'_3 &= \Phi_3, \quad \phi'_3 = -\Phi_3. 
\end{align*}
\]

Now we analyze the stability of the fixed point \( P := (\phi_1, \Phi_1, \phi_2, \Phi_2, \phi_3, \Phi_3) = (1, 0, 1, 0, 1, 0) \). The subsystems for \((\phi_1, \Phi_1), (\phi_2, \Phi_2)\) and \((\phi_3, \Phi_3)\) are decoupled.

The Jacobian matrix of the full system is
\[
J := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4(w_m + 1)(5w_m - 1)}{4w_m \phi_1^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

Evaluating \( J \) at the fixed point \( P \) the eigenvalues \( \{0, 0, -1, -1, -1, \frac{1}{w_m + 1}\} \) are obtained. Therefore, if \(-1 < w_m < 0\) the stable manifold of \( P \) for the full system is 4D. If the analysis is restricted to the subspace \((\phi_2, \Phi_2, \phi_3, \Phi_3)\) the eigenvalues are \( \{0, 0, -1, -1\} \). The stable manifold in this subspace is 2D for all \( w_m \neq 0, 1 \).

Defining the new variables
\[
\begin{align*}
u_1 &= \Phi_3 + \phi_3 - 1, \
u_2 &= \Phi_2 + \phi_2 - 1, \quad v_1 = \Phi_3, \quad v_2 = \Phi_2, \\
v_3 &= \frac{\Phi_1 w_m + (-w_m - 1)(\phi_1 - 1)}{2w_m + 1}, \quad v_4 = \frac{(w_m + 1)(\phi_1 + \phi_1 - 1)}{2w_m + 1}.
\end{align*}
\]

Using the previous variables we obtain the decoupled equations:
\[
\begin{align*}
u'_1 &= 0, \\
u'_2 &= 0, \\
u'_3 &= -v_1, \\
u'_4 &= -v_2, \\
v'_3 &= \frac{(5w_m - 1)(v_3 + v_4)^2}{w_m + 1} - 4(w_m + 1)(v_3 + v_4) + \frac{4(w_m(v_3 - v_4 - 1) + 1(v_3 w_m + v_3 - v_4 w_m))}{w_m + 1} + 4(v_3 + v_4), \\
v'_4 &= \frac{(5w_m - 1)(v_3 + v_4)^2}{w_m + 1} + \frac{v_3 + v_4}{w_m} + \frac{(w_m(v_3 - v_4 - 1) + 1(v_3 w_m + v_3 - v_4 w_m))}{w_m + 1} + v_3 + v_4.
\end{align*}
\]

In figures 7-9 some solutions of the systems (123)-(125) and (127) for the potential \( V_{II}(\phi, \psi) \) when \( \omega_m = -1, -\frac{1}{3} \) and \( \frac{1}{2} \) are represented. More specific, projections in the spaces \((\phi_1, \Phi_1), (\phi_2, \Phi_2), (\phi_3, \Phi_3), (u_1, u_2), (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (u_1, v_1) \) and \((u_2, v_2)\) are represented. In these figures we have depicted a red line in the projections \((u_1, v_1)\) and \((u_2, v_2)\) that denotes the invariant set \( v_1 = 0 \) and \( v_2 = 0 \), respectively. Both lines are stable in these projections. Notice that in figures 7 and 8 in the projection \( u_1 \) vs \( u_2 \) (which contains the center manifold) the origin is stable (but not asymptotically stable) since any \( \epsilon \)-neighborhood of the origin will contain a \( \delta \)-neighborhood of origin with other points apart of the origin with \((u_1', u_2')|u_1 = u_1^*, u_2 = u_2^* = (0, 0)\). Therefore, they remain in \( \delta \)-neighborhood of origin if \(-1 < w_m < 0\). On the other hand, in figure 9 the origin behaves as a saddle point since \( w_m > 0 \).
FIG. 7: Some solutions of (a) the system (123)-(125) and (b) the system (127) for the potential $V_{III}(\phi, \psi)$ when $\omega_m = -1$. From left to right, from top to bottom: Projections in the planes $(\phi_1, \Phi_1)$, $(\phi_2, \Phi_2)$, $(\phi_3, \Phi_3)$, $(u_1, u_2)$, $(v_1, v_2)$, $(v_1, v_3)$, $(v_1, v_4)$, $(v_2, v_3)$, $(v_2, v_4)$, $(v_3, v_4)$, $(u_1, v_1)$ and $(u_2, v_2)$. The red line in fig 11 (resp. fig 12) of the bottom array denotes the invariant set $v_1 = 0$ (resp. $v_2 = 0$) in the projections $u_1$ vs $v_1$ (resp. $u_2$ vs $v_2$). Notice in the projection $u_1$ vs $u_2$ (which contains the center manifold) the origin is stable (but not asymptotically stable) since any $\epsilon$-neighborhood of the origin will contain a $\delta$-neighborhood of origin with other points apart of the origin with $(u_1', u_2')|_{u_1'=u_1^*, u_2'=u_2^*} = (0, 0)$. Therefore, they remain in $\delta$-neighborhood of origin.
FIG. 8: Some solutions of (a) the system (123)-(125) and (b) the system (127) for the potential $V_{III}(\phi, \psi)$ when $\omega_m = -\frac{1}{3}$.

From left to right, from top to bottom: Projections in the planes $(\phi_1, \Phi_1)$, $(\phi_2, \Phi_2)$, $(\phi_3, \Phi_3)$, $(u_1, v_1)$, $(v_1, v_2)$, $(v_1, v_3)$, $(v_1, v_4)$, $(v_2, v_3)$, $(v_2, v_4)$, $(u_1, v_1)$ and $(u_2, v_2)$. The red line in fig 11 (resp. fig 12) of the bottom array denotes the invariant set $v_1 = 0$ (resp. $v_2 = 0$) in the projections $u_1$ vs $v_1$ (resp. $u_2$ vs $v_2$). Notice in the projection $u_1$ vs $u_2$ (which contains the center manifold) the origin is stable (but not asymptotically stable) since any $\epsilon$-neighborhood of the origin will contain a $\delta$-neighborhood of origin with other points apart of the origin with $(u'_1, u'_2)|_{u_1=u'_1, u_2=u'_2} = (0, 0)$. Therefore, they remain in $\delta$-neighborhood of origin.
FIG. 9: Some solutions of (a) the system (123)-(125) and (b) the system (127) for the potential $V_{III}(\phi, \psi)$ when $\omega_m = \frac{1}{3}$.

From left to right, from top to bottom: Projections in the planes $(\phi_1, \Phi_1)$, $(\phi_2, \Phi_2)$, $(\phi_3, \Phi_3)$, $(u_1, u_2)$, $(v_1, v_2)$, $(v_1, v_3)$, $(v_1, v_4)$, $(v_2, v_3)$, $(v_2, v_4)$, $(v_3, v_4)$, $(u_1, v_1)$ and $(u_2, v_2)$. The red line in fig 11 (resp. fig 12) of the bottom array denotes the invariant set $v_1 = 0$ (resp. $v_2 = 0$) in the projections $u_1$ vs $v_1$ (resp. $u_2$ vs $v_2$). This plot shows that the equilibrium point is a saddle for $w_m > 0$.

6.4. Scalar field potential $V_{IV}(\phi, \psi)$

In this section we study the stability of the solution (71), (72), (73) of (67), (68), (69). We set for simplicity the integration constants $t_1, t_2, t_3$ to zero because they are not relevant as $t \to \infty$.

With the time variable $\tau = \ln(t)$, and defining the new variables $X(\tau) = x(\tau) - x_0 e^\tau$, $Y(\tau) = y(\tau) - y_0 e^\tau$ and
\[ Z(\tau) = 6V_0 e^{2\tau} Z' \] with \( r = \frac{-3w_m + 1}{w_m - 1} \) the equations (67), (68), (69) become

\[
X''(\tau) - X'(\tau) - \frac{w_m + 1}{w_m - 1} \beta Z(\tau) = 0, \quad Y''(\tau) - Y'(\tau) + \frac{w_m + 1}{w_m - 1} \beta Z(\tau) = 0,
\]

\[
Z''(\tau) = -\left( w_m - 5 \right) Z(\tau) Z'(\tau) + 4w_m Z'(\tau)^2 - 2(w_m + 3) Z(\tau)^2 - \left( w_m + 1 \right) \left( 3w_m + 1 \right) Z_1 \varepsilon e^{\frac{2\tau(w_m - 1)}{3w_m + 1} Z(\tau)} \frac{7w_m + 1}{(w_m - 1)^2},
\]

where \( Z_1 = \beta^2 \frac{1 - w_m}{2w_m + 1} \frac{2(w_m + 1)}{3w_m + 1} V_0^{\frac{2}{3w_m + 1}}. \) The solution (71), (72), (73) becomes

\[
X_c(\tau) = \frac{Z_0 e^{\frac{2\tau(w_m - 1)}{3w_m + 1} Z(\tau)}}{3\beta}, \quad Y_c(\tau) = -\frac{Z_0 e^{\frac{2\tau(w_m - 1)}{3w_m + 1} Z(\tau)}}{3\beta} \varepsilon, \quad Z_c(\tau) = -\frac{(w_m - 1)^2 Z_0 e^{\frac{2\tau(w_m - 1)}{3w_m + 1} Z(\tau)}}{12\beta^2 w_m^2} \varepsilon.
\]

Defining the dimensionless variables

\[
\phi_1 = \frac{X}{X_c}, \quad \phi_2 = \frac{Y}{Y_c}, \quad \phi_3 = \frac{Z}{Z_c},
\]

we obtain for \( w_m \neq 0, -1/3 \) the dynamical system

\[
\phi'_1 = \Phi_1, \quad \Phi_1 = \frac{\Phi_1}{w_m} + \frac{\left( w_m^2 - 1 \right) \phi_1}{4w_m^2} + \frac{1}{4} \left( \frac{1}{w_m^2} - 1 \right) \phi_3,
\]

\[
\phi'_2 = \Phi_2, \quad \Phi_2 = \frac{\Phi_2}{w_m} + \frac{\left( w_m^2 - 1 \right) \phi_2}{4w_m^2} + \frac{1}{4} \left( \frac{1}{w_m^2} - 1 \right) \phi_3,
\]

\[
\phi'_3 = \Phi_3, \quad \Phi_3 = \frac{\Phi_3}{w_m} + \frac{\left( 4\Phi_3 w_m \right)}{(3w_m + 1) \phi_3} + \frac{(w_m + 1)(3w_m + 1) \phi_3}{4w_m^2} + \frac{(w_m + 1)(3w_m + 1) \phi_3}{4w_m^2}.
\]

Now we analyze the stability of the fixed point \( P := (\phi_1, \Phi_1, \phi_2, \Phi_2, \phi_3, \Phi_3) = (1, 0, 1, 0, 0, 0). \) The Jacobian matrix of the full system is

\[
\mathbf{J} := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{4} - \frac{w_m - 1}{2w_m} & \frac{1}{w_m} & 0 & 0 & \frac{1}{4} \left( \frac{1}{w_m} - 1 \right) & 0 \\
0 & 0 & \frac{1}{w_m} & 0 & \frac{1}{4} \left( \frac{1}{w_m} - 1 \right) & 0 \\
0 & 0 & 0 & \frac{1}{w_m} & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{(w_m + 1)(3w_m + 1) \phi_3}{4w_m^2} + \frac{8\Phi_3 w_m}{3w_m \phi_3 + \phi_3} + \frac{1}{w_m} & 0 \\
\end{pmatrix}
\]

Evaluating \( \mathbf{J} \) at the fixed point \( P \) the eigenvalues \( \left\{-1, -\frac{w_m - 1}{2w_m}, -\frac{w_m - 1}{2w_m}, -\frac{w_m + 1}{2w_m}, \frac{w_m + 1}{2w_m}, \frac{w_m + 1}{w_m} \right\} \) are obtained. For \(-1 < w_m < 0 \) \( P \) is a sink, or a saddle for \( w_m > 0 \).

Defining the variables

\[
\begin{align*}
v_1 &= \frac{(w_m + 1) \left( w_m^2 (6\Phi_2 + 2\Phi_3 + 3\phi_2 + \phi_3 - 4) - 2w_m (-\Phi_2 + \Phi_3 + \phi_2 + \phi_3 - 2) - \phi_2 + \phi_3 \right)}{4w_m^2 (3w_m + 1)}, \\
v_2 &= \frac{(w_m - 1) \left( 2w_m w_m (3\Phi_2 + \Phi_3 + 2) + \Phi_2 - \Phi_3 + 2 \right) - (w_m + 1)(3w_m + 1) \phi_2 + w_m^2 (-\phi_3 + \phi_3)}{4w_m^2 (3w_m + 1)}, \\
v_3 &= \frac{\Phi_3 w_m + w_m - (w_m + 1) \phi_3 + 1}{2w_m + 1}, \\
v_4 &= \frac{(w_m + 1) \left( w_m^2 (6\Phi_1 + 2\Phi_3 + 3\phi_1 + \phi_3 - 4) - 2w_m (-\Phi_1 + \Phi_3 + \phi_1 + \phi_3 - 2) - \phi_1 + \phi_3 \right)}{4w_m^2 (3w_m + 1)}, \\
v_5 &= \frac{(w_m - 1) \left( 2w_m w_m (3\Phi_1 + \Phi_3 + 2) + \Phi_1 - \Phi_3 + 2 \right) - (w_m + 1)(3w_m + 1) \phi_1 + w_m^2 (-\phi_3 + \phi_3)}{4w_m^2 (3w_m + 1)}, \\
v_6 &= \frac{(w_m + 1) (\Phi_3 + \phi_3 - 1)}{2w_m + 1}.
\end{align*}
\]
(a) Projections in the space $(\phi_1, \phi_2, \phi_3)$ (left) and $(\Phi_1, \Phi_2, \Phi_3)$ (right). We have represented the spheres $(\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2$ and $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2$, $r \in \{1, \sqrt{2}\}$.

(b) From left to right, from top to bottom: Projections in the planes $(v_1, v_2)$, $(v_1, v_4)$, $(v_1, v_5)$, $(v_2, v_3)$, $(v_2, v_4)$, $(v_2, v_5)$, $(v_3, v_6)$, $(v_4, v_5)$ and $(v_5, v_6)$.

FIG. 10: Some solutions of (a) the system (129)-(131) and (b) the system (133) for the potential $V_{IV}(\phi, \psi)$ when $\omega_m = -\frac{1}{4}$. The point $P$ is asymptotically stable.
(a) Projections in the space $(\phi_1, \phi_2, \phi_3)$ (left) and $(\Phi_1, \Phi_2, \Phi_3)$ (right). We have represented the spheres $(\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2$ and $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2$, $r \in \{1, \sqrt{2}\}$.

(b) From left to right, from top to bottom: Projections in the planes $(v_1, v_2)$, $(v_1, v_4)$, $(v_1, v_5)$, $(v_2, v_6)$, $(v_3, v_6)$, $(v_4, v_5)$, $(v_4, v_6)$, $(v_5, v_6)$.

FIG. 11: Some solutions of (a) the system (129)-(131) and (b) the system (133) for the potential $V_{IV}(\phi, \psi)$ when $\omega_m = -\frac{1}{2}$. The point $P$ is asymptotically stable.
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right). We have represented the spheres 
\[(\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2\] and 
\[\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2; \ r \in \{1, \sqrt{2}\}\].

(b) From left to right, from top to bottom: Projections in the planes \((v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5),\) 
\((v_1, v_6), (v_3, v_4), (v_5, v_6), (v_4, v_6)\) and \((v_5, v_6)\).

FIG. 12: Some solutions of (a) the system (129)-(131) and (b) the system (133) for the potential \(V_{IV}(\phi, \psi)\) when \(\omega_m = 1\). The point \(P\) is a saddle.
we obtain the system

\[ v'_{1} = \frac{(w_{m} - 1)(w_{m} + 1)^{2}(-v_{3} + v_{6} - \frac{v_{6}}{w_{m} + 1} + 1)^{\frac{w_{m} + 1}{w_{m} + 1}}}{8w_{m}^{3}} \]

\[ + \frac{1}{8w_{m}^{3}(3w_{m} + 1)^{2}(v_{3} + (v_{3} - v_{6} - 1)w_{m} - 1)} \left( v_{3}^{2}(w_{m} - 1)(5w_{m} + 1)(w_{m}(w_{m} + 6) + 1)(w_{m} + 1)^{2} \right. \]

\[ - 2v_{3}(w_{m}(-2v_{1}(3w_{m} + 1)^{2} + w_{m}(15w_{m} + 8) - 14) + v_{6}(w_{m}(-1)(w_{m}(37w_{m} + 10) + 1) - 8) - 1)(w_{m} + 1)^{2} \]

\[ + w_{m}(v_{6}^{2}w_{m}(w_{m} + 1)(5w_{m} - 1)(w_{m} - 1)^{2} - w_{m}(4v_{1}(w_{m} + 1)^{2}(3w_{m} + 1)^{2} - w_{m}(w_{m}(3w_{m}(3w_{m} + 8) + 13) - 16) + 21) \]

\[ - 2v_{6}(w_{m} + 1)(3w_{m} + 1)(w_{m}(w_{m}(-5w_{m} + v_{3}(6w_{m} + 2) - 1) + 5) + 1) - 8 - 1) \right) \right) . \] (133a)

\[ v'_{2} = \frac{(w_{m} - 1)^{2}(w_{m} + 1)^{2}(-v_{3} + v_{6} - \frac{v_{6}}{w_{m} + 1} + 1)^{\frac{w_{m} + 1}{w_{m} + 1}}}{8w_{m}^{3}} \]

\[ + \frac{1}{8w_{m}^{3}(3w_{m} + 1)^{2}(v_{3} + (v_{3} - v_{6} - 1)w_{m} - 1)} \left( v_{3}^{2}(w_{m} + 1)(5w_{m} + 1)(w_{m}(w_{m} + 6) + 1)(w_{m} - 1)^{2} \right. \]

\[ + w_{m}(v_{6}^{2}w_{m}(5w_{m} - 1)(w_{m} - 1)^{3} + 2v_{6}(3w_{m} + 1)(w_{m}((6v_{6} + 5)w_{m}^{2} + 2v_{6}w_{m} + w_{m} - 5) - 1)(w_{m} - 1) \]

\[ + w_{m}(4v_{1}(w_{m} - 1)(3w_{m} + 1)^{2} + w_{m}(3w_{m} + 4)(w_{m}(3w_{m} - 2) - 3) + 7) + 6) \]

\[ - 2v_{3}(w_{m}^{2} - 1)(w_{m}(w_{m}(2v_{2}(3w_{m} + 1)^{2} + w_{m}(15w_{m} + 8) - 14) + v_{6}(w_{m} - 1)(w_{m}(37w_{m} + 10) + 1) - 8) - 1) \right) \right) , \] (133b)

\[ v'_{3} = \frac{(3w_{m}^{2} + 4w_{m} + 1)(-v_{3} + v_{6} - \frac{v_{6}}{w_{m} + 1} + 1)^{\frac{w_{m} + 1}{w_{m} + 1}}}{8w_{m}^{3} + 4w_{m}} \]

\[ + \frac{1}{4w_{m}(2w_{m} + 1)(3w_{m} + 1)(v_{3} + (v_{3} - v_{6} - 1)w_{m} - 1)} \left( - (w_{m} - 1)(w_{m} + 1)(w_{m}(19w_{m} + 8) + 1)v_{3}^{2} \right. \]

\[ - 2(w_{m}(w_{m}(w_{m} + 2)(3w_{m} + 10) + v_{6}(w_{m}(25w_{m} + 37) + 9) + 1) + 8) + 1)v_{3} \]

\[ + w_{m}((5v_{6}(v_{5} + 6) + 9w_{m}^{3} + 2((23 - 3v_{6})v_{6} + 12)w_{m} + v_{6}(v_{6} + 18)w_{m} + 22w_{m} + 2v_{6} + 8) + 1) \right) , \] (133c)

\[ v'_{4} = \frac{(w_{m} - 1)(w_{m} + 1)^{2}(-v_{3} + v_{6} - \frac{v_{6}}{w_{m} + 1} + 1)^{\frac{w_{m} + 1}{w_{m} + 1}}}{8w_{m}^{3}} \]

\[ + \frac{1}{8w_{m}^{3}(3w_{m} + 1)^{2}(v_{3} + (v_{3} - v_{6} - 1)w_{m} - 1)} \left( v_{3}^{2}(w_{m} - 1)(5w_{m} + 1)(w_{m}(w_{m} + 6) + 1)(w_{m} + 1)^{2} \right. \]

\[ - 2v_{3}(w_{m}(w_{m}(-2v_{1}(3w_{m} + 1)^{2} + w_{m}(15w_{m} + 8) - 14) + v_{6}(w_{m} - 1)(w_{m}(37w_{m} + 10) + 1) - 8) - 1)(w_{m} + 1)^{2} \]

\[ + w_{m}(v_{6}^{2}w_{m}(w_{m} + 1)(5w_{m} - 1)(w_{m} - 1)^{2} - w_{m}(4v_{1}(w_{m} + 1)^{2}(3w_{m} + 1)^{2} - w_{m}(w_{m}(3w_{m}(3w_{m} + 8) + 13) - 16) + 21) \]

\[ - 2v_{6}(w_{m} + 1)(3w_{m} + 1)(w_{m}(w_{m}(-5w_{m} + v_{3}(6w_{m} + 2) - 1) + 5) + 1) - 8 - 1) \right) \right) \right) . \] (133d)

\[ v'_{5} = \frac{(w_{m} - 1)^{2}(w_{m} + 1)^{2}(-v_{3} + v_{6} - \frac{v_{6}}{w_{m} + 1} + 1)^{\frac{w_{m} + 1}{w_{m} + 1}}}{8w_{m}^{3}} \]

\[ + \frac{1}{8w_{m}^{3}(3w_{m} + 1)^{2}(v_{3} + (v_{3} - v_{6} - 1)w_{m} - 1)} \left( v_{3}^{2}(w_{m} + 1)(5w_{m} + 1)(w_{m}(w_{m} + 6) + 1)(w_{m} - 1)^{2} \right. \]

\[ + w_{m}(v_{6}^{2}w_{m}(5w_{m} - 1)(w_{m} - 1)^{3} + 2v_{6}(3w_{m} + 1)(w_{m}((6v_{6} + 5)w_{m}^{2} + 2v_{3}w_{m} + w_{m} - 5) - 1)(w_{m} - 1) \]

\[ + w_{m}(4v_{1}(w_{m} - 1)(3w_{m} + 1)^{2} + w_{m}(3w_{m} + 4)(w_{m}(3w_{m} - 2) - 3) + 7) + 6) \]

\[ - 2v_{3}(w_{m}^{2} - 1)(w_{m}(w_{m}(2v_{2}(3w_{m} + 1)^{2} + w_{m}(15w_{m} + 8) - 14) + v_{6}(w_{m} - 1)(w_{m}(37w_{m} + 10) + 1) - 8) - 1) + 1) , \] (133e)
In the upper panel the solutions are projected in the space $v_1(v_2), v_1(v_4), v_1(v_5), v_2(v_3), (v_2,v_4), (v_3,v_6), (v_4,v_5)$ and $(v_5,v_6)$ are represented. From these figures, it is confirmed that the point $(1,0,1,0,1,0)$ is a saddle when $\omega_m > 0$ while for $-1 < \omega_m < 0$ is an attractor of the system, i.e., it is asymptotically stable.

### 6.5. Scalar field potential $V_{\phi}(\phi, \bar{\psi})$

In this section we study the stability of the solution (82), (83), (84) of system (76), (77), (78). We set for simplicity the integration constants $t_1, t_2, t_3$ to zero because they are not relevant as $t \to \infty$.

With the time variable $\tau = \ln(t)$, and defining the new variables $X(\tau) = x(\tau) - x_0 e^\tau, \gamma(\tau) = 6 \varepsilon V_0 e^{2\tau} Y'$ with $r = -\frac{3w_{m+1}}{w_m-1}$ and $Z(\tau) = z(\tau) - z_0 e^\tau$ the equations (76), (77), (78) become

$$
Y''(\tau) = \frac{4w_m Y'(\tau)^2}{3w_m Y(\tau) + Y'(\tau)} + \frac{(5 - w_m) Y'(\tau)}{3w_m + 1} - \frac{2(w_m + 3) Y(\tau)}{3w_m + 1} - Y_1(w_m - 1)^2 Y'(\tau) + 3w_{m+1}(1) Y(\tau) - \beta Y(\tau) = 0, \quad Z''(\tau) - Z'(\tau) + \frac{(w_m + 1)}{w_m - 1} \varepsilon Y(\tau) = 0,
$$

where $Y_1 = \beta^2 \frac{1 - w_m + \varepsilon}{3w_m + 1} \frac{w_m - 1}{\varepsilon} \varepsilon V_0 e^{3w_m + 1}$. The solution (82), (83), (84) becomes

$$
X_c(\tau) = \frac{3Y_0 e^{3w_m - 1}}{\beta (w_m - 1)^2}, \quad Y_c(\tau) = -\frac{3Y_0 e^{3w_m - 1}}{4\beta^2 (w_m - 1)^2 w_m}, \quad Z_c(\tau) = -\frac{3Y_0 e^{3w_m - 1}}{\beta^2 (w_m - 1)^4}.
$$

Defining the dimensionless variables

$$
\phi_1 = \frac{X}{X_c}, \quad \phi_2 = \frac{Y}{Y_c}, \quad \phi_3 = \frac{Z}{Z_c},
$$

we obtain the dynamical system

$$
\phi'_1 = \Phi_1, \quad \Phi_1 = \frac{\phi_1}{w_m} + \frac{1}{4} \frac{1}{w_m^2} \phi_1 + \frac{1}{4} \left( \frac{1}{w_m^2} - 1 \right) \phi_2,
$$

$$
\phi'_2 = \Phi_2, \quad \Phi_2 = \frac{\phi_2}{w_m} + \frac{4\Phi_2 w_m}{3w_m \phi_2 + \phi_2} + \frac{(w_m + 1)(3w_m + 1) \phi_2}{4w_m^2} - \frac{(w_m + 1)(3w_m + 1) \phi_1}{4w_m^2},
$$

$$
\phi'_3 = \Phi_3, \quad \Phi_3 = \frac{\phi_3}{w_m} + \frac{1}{4} \left( \frac{1}{w_m^2} - 1 \right) \phi_2 + \frac{1}{4} \left( \frac{1}{w_m^2} - 1 \right) \phi_3.
$$

Now we analyze the stability of the fixed point $P := (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (1,0,1,0,1,0)$. The Jacobian matrix of
the full system is

\[
J := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{\eta} - \frac{1}{4w_m} & 1 & \frac{1}{m} \left( \frac{1}{w_m} - 1 \right) & 0 & 0 & 0 \\
0 & 0 & \frac{10 w_m + 2}{3 w_m} & 0 & 1 & 0 \\
0 & \frac{(w_m + 1)(3 w_m + 1)(7 w_m + 1) \phi_2 \phi_1}{4 w_m^2} & -16 \phi_1^2 \phi_2^2 - (w_m + 1)(3 w_m + 1) \phi_2^2 & \frac{8 \phi_2 w_m}{3 w_m} \phi_2 \phi_1 + \frac{1}{w_m} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \frac{1}{w_m} - 1 & 0 & \frac{1}{4 w_m} - \frac{1}{1 - w_m} \\
\end{pmatrix}
\]

Evaluating \( J \) at the fixed point \( P \) the eigenvalues \( \{-1, -\frac{w_m - 1}{2 w_m}, -\frac{w_m - 1}{2 w_m}, \frac{w_m + 1}{2 w_m}, \frac{w_m + 1}{2 w_m}, \frac{w_m + 1}{w_m} \} \) are obtained. For \(-1 < w_m < 0 \) \( P \) is a sink, or a saddle for \( w_m > 0 \).

In figures 13(a)-13(c) some solutions of the system (136a)-(136c) are represented for the potential \( V_V(\phi, \psi) \) when \( \omega_m = -\frac{1}{2}, -\frac{1}{3} \) and \( \frac{1}{4} \), respectively. More specific, in each sub-figure the solutions are projected in the space \((\phi_1, \phi_2, \phi_3) \) (left) and \((\Phi_1, \Phi_2, \Phi_3) \) (right), where additionally we have represented the spheres \((\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2\) and \(\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2\) with \( r \in \{1, \sqrt{2}\} \). From these figures, it is confirmed that the point \((1, 0, 1, 0, 0)\) is a saddle when \( \omega_m > 0 \) while for \(-1 < \omega_m < 0 \) is an attractor of the system, i. e., it is asymptotically stable.

### 6.6. Scalar field potential \( V_{VI}(\phi, \psi) \)

In this section we study the stability of the powerlaw solution (90), (91), (92) of the system (85), (86), (87). We set for simplicity the integration constants \( x, v \) for \( \tau \) and \( \phi \) for \( \eta \). We have considered a cosmological model consisting of two-scalar fields minimally coupled to gravity and an ideal gas. The two scalar fields interact in the kinetic and in the potential term. In particular the kinematics of the two scalar fields lie on a space of constant curvature. This kind of scalar field models are known also as Chiral models. The Chiral model is the main mechanism for the description of the hyperbolic inflation.

### 7. CONCLUSIONS

We have considered a cosmological model consisting of two-scalar fields minimally coupled to gravity and an ideal gas. The two scalar fields interact in the kinetic and in the potential term. In particular the kinematics of the two scalar fields lie on a space of constant curvature. This kind of scalar field models are known also as Chiral models. The Chiral model is the main mechanism for the description of the hyperbolic inflation.
(a) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right) for \(\omega_m = -\frac{1}{4}\). The point \(P\) is asymptotically stable.

(b) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right) for \(\omega_m = -\frac{1}{7}\). The point \(P\) is asymptotically stable.

(c) Projections in the space \((\phi_1, \phi_2, \phi_3)\) (left) and \((\Phi_1, \Phi_2, \Phi_3)\) (right) for \(\omega_m = \frac{1}{3}\). The point \(P\) is a saddle.

**FIG. 13:** Some solutions of the system (136a)-(136c) for the potential \(V_V(\phi, \psi)\) when \(\omega_m = -\frac{1}{4}, -\frac{1}{7}\) and \(\frac{1}{3}\). In these figures we have represented the spheres \((\phi_1 - 1)^2 + (\phi_2 - 1)^2 + (\phi_3 - 1)^2 = r^2\) and \(\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = r^2\), \(r \in \{1, \sqrt{2}\}\).
In the case of a spatially flat FLRW universe, the field equations form an autonomous dynamical system which is described by a point-like Lagrangian. The Lagrangian function depends on an unknown potential function which drives the dynamics of the two scalar fields. In this work we focused on the integrability properties for the field equations and we applied geometrical selection rules in order to constraint the functional forms of the potential.

We applied the theory of symmetries of differential equations to constraint the scalar field potential such that field equations admit conservation laws. Specifically, we investigate the existence of variational symmetries where the corresponding conservation laws are constructed with the use of Noether’s theorem. Because the point-like dynamical system is defined on a three dimensional space with dependent variables the scale factor and two scalar fields \( \{ a, \phi, \psi \} \), and because the system is autonomous, the Hamiltonian function is one conservation law. Thus, two additional conservation laws should be found such that the system is Liouville integrable.

Therefore, we performed a classification of the variational symmetries for the point-like Lagrangian which describes the field equations and we found six potential functions \( V(\phi, \psi) \) where the field equations are Liouville integrable. For these six potential functions we define the normal coordinates, which is used to construct the analytic solution for the field equations. The free parameters of the cosmological model are constrained in order to describe analytic and exact solutions for the scale factor which describe the hyperbolic inflation era.

Finally, the stability properties of exact closed-form solutions were investigated using a dynamical systems formulation and numerical tools for these potential functions. In particular, for potentials \( V_I(\phi, \psi) \) and \( V_{II}(\phi, \psi) \) the scaling solution is a saddle for \( -1 < w_m < 1, w_m \neq 0 \) and stable, but not asymptotically stable for \( w_m = 0 \); and for potential \( V_{III}(\phi, \psi) \) the scaling solution is a saddle when \( w_m > 0 \) while for \( -1 < w_m < 0 \) the scaling solution is stable, but not asymptotically stable. For potentials \( V_{IV}(\phi, \psi) \), \( V_{V}(\phi, \psi) \) and \( V_{VI}(\phi, \psi) \) the scaling solutions are saddle when \( w_m > 0 \) while for \( -1 < w_m < 0 \) are asymptotically stable.

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