Group gradings on upper block triangular matrices

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Abstract. It was proved by Valenti and Zaicev, in 2011, that, if \( G \) is an abelian group and \( K \) is an algebraically closed field of characteristic zero, then any \( G \)-grading on the algebra of upper block triangular matrices over \( K \) is isomorphic to a tensor product \( M_n(K) \otimes UT(n_1, n_2, \ldots, n_d) \), where \( UT(n_1, n_2, \ldots, n_d) \) is endowed with an elementary grading and \( M_n(K) \) is provided with a division grading.

In this manuscript, we prove the validity of the same result for a non necessarily commutative group and over an adequate field (characteristic either zero or large enough), not necessarily algebraically closed.

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1. Introduction

Algebras with additional structure are deeply studied nowadays, and in particular, the graded algebras was intensely investigated mainly after the works of Kemer [7], showing the importance of \( \mathbb{Z}_2 \)-graded algebras in the study of algebras with polynomial identities. These algebras constitutes a natural generalization of polynomial algebras in the commutative case. They are also related with supersymmetries in Physics. A very interesting question concerning gradings on algebras is classifying all possible gradings on a given algebra. For simple associative, Lie and Jordan algebras, the classification is essentially complete (see the book [4] for a complete reference in the subject). There exists many other algebras whose gradings was computed or partially computed.

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In this manuscript, we are interested in studying a non-simple algebra, namely the upper block triangular matrices. These algebras are defined in the following way. Let $n_1, n_2, \ldots, n_t \in \mathbb{N}$ be any integers, then set

$$UT(n_1, n_2, \ldots, n_t) = \begin{pmatrix}
  A_{11} & A_{12} & \cdots & A_{1t} \\
  0 & A_{22} & \cdots & A_{2t} \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & A_{tt}
\end{pmatrix},$$

where each $A_{ij}$, for $1 \leq i \leq j \leq t$, is a $n_i \times n_j$ matrix with entries in the field $K$. The Jacobson Radical $J$ of $UT(n_1, n_2, \ldots, n_t)$ is the set of all elements such that all $A_{ii}$ are zero, for $i = 1, 2, \ldots, t$. The upper triangular matrices is a particular case of upper block triangular matrices, if we consider $UT(1, 1, \ldots, 1)$.

The matrix algebras can also be obtained if we put $t = 1$.

In 2003, Valenti and Zaicev proved that any group grading on the algebra of upper triangular matrices over an algebraically closed field of characteristic zero, where the grading group is abelian, is elementary, up to a graded isomorphism [9]. In 2007, the same authors proved the same theorem, but for arbitrary field and any group [10]; and in the same paper the authors conjectured the classification of the group gradings over the algebra of upper block triangular matrices. But in 2011, Valenti and Zaicev solved this question, proving the validity of their conjecture for an algebraically closed field of characteristic zero and the grading group commutative and finite [11].

Following the sequence, in this manuscript, we describe the group gradings on the upper block triangular matrices, proving the conjecture of Valenti and Zaicev for arbitrary field of characteristic zero (or the characteristic greater than the dimension of the algebra) and a group not necessarily commutative, nor finite.

We recall that the upper block triangular matrices, in the ungraded sense, are related to the so called minimal varieties (see [5] and the references therein). The classification of the elementary gradings on the upper block triangular matrices was studied in [1]. The graded polynomial identities for the elementary gradings on the upper block triangular matrices was dealt in [2, 8]. Also, in [3] the authors addressed the question of when the knowledge of the graded polynomial identities for a certain grading on the upper block triangular matrices completely determines the grading.

2. Notations and preliminaries

We fix a group $G$ with multiplicative notation and an arbitrary field $K$.

Let $A$ be any algebra (associative or not) and $G$ any group. We say that $A$ is a $G$-graded algebra (or $A$ is equipped with a $G$-grading) if there exists a vector space decomposition $A = \bigoplus_{g \in G} A_g$ (where some of the $A_g$ can be zero) satisfying $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. We call the elements in $\bigcup_{g \in G} A_g$ homogeneous, and we say that $x$ has degree $g$ if $x \in A_g$, denoted $\deg x = g$. A graded division algebra is a graded algebra $A$ such that every non-zero homogeneous element of $A$ has an inverse in $A$. 
Let \( U = UT(n_1, n_2, \ldots, n_t) \). We say that a \( G \)-grading on \( U \) is elementary if there exists a sequence \( (g_1, g_2, \ldots, g_n) \in G^n \) (where \( n = n_1 + n_2 + \ldots + n_t \)) such that every matrix unit \( e_{ij} \in U \) is homogeneous of degree \( g_i g_j^{-1} \). If \( B \) is another \( G \)-graded algebra, then we can furnish a \( G \)-grading on \( U \otimes_K B \) if we put
\[
\deg e_{ij} \otimes b = g_i \deg b g_j^{-1},
\]
for all homogeneous \( b \in B \). Here, we canonically identify \( M_n \otimes M_m = M_{nm} \) via Kronecker product. In [10], Valenti and Zaicev conjectured that every grading on \( U \) is graded isomorphic to \( UT(n'_1, n'_2, \ldots, n'_t) \otimes M_n(K) \), where \( M_n(K) \) is provided with a division grading and \( UT(n'_1, n'_2, \ldots, n'_t) \) is endowed with an elementary grading. This was proved to be true, if the base field is algebraically closed of characteristic zero and the group is finite and abelian [11].

Denote by \( J \) the Jacobson Radical of \( U = UT(n_1, n_2, \ldots, n_t) \). Denote also by \( M_{ij} \) the block of matrices, so that we can write (as vector spaces) \( U = \bigoplus_{i \leq j \leq t} M_{ij} \). Thus, in this notation \( J = \bigoplus_{i < j} M_{ij} \). Note that each \( M_{ii} \) is isomorphic to the \( n_i \times n_i \) matrix algebra, and we can see \( M_{ii} \) as a subalgebra of \( U \). Let \( E_i \in M_{ii} \) be its identity matrix.

**Graded modules.** Let \( A \) be a \( G \)-graded algebra and \( V \) a vector space that is an \( A \)-module. Suppose that we have a decomposition \( V = \bigoplus_{g \in G} V_g \) into subspaces (in this case, \( V \) has a vector space grading). We say that \( V \) is a graded \( A \)-module if \( V_g A_h \subset V_{gh} \), for all \( g, h \in G \).

If \( V = \bigoplus_{g \in G} V_g \) is a graded vector space, and given \( h \in G \), we define \( V^{[h]} \) as the graded vector space with decomposition \( V^{[h]} = \bigoplus_{g \in G} V_g^{[h]} \), where \( V_{gh}^{[h]} = V_g \). Similarly we define the graded vector space \( [h]V \). Note that if \( A \) is a \( G \)-graded algebra, then \( A \) itself is a \( G \)-graded \( A \)-module.

For the special case where \( D \) is a \( G \)-graded division algebra, the structure of \( D \)-modules are well known (see, for instance, [4, Chapter 2, page 29]). If \( V \) is a \( G \)-graded \( D \)-module, then \( V = \bigoplus V_i \), where each \( V_i = [g_i]D \), for some \( g_i \in G \). In other words, every graded \( D \)-module is free.

### 3. Group gradings on the upper block triangular matrices

We start proving that some subspaces are grade:

**Lemma 1.** If \( J \) is graded, then all \( M_{ij} \) are graded subspaces.

**Proof.** Recall that it is easy to prove that the annihilator (left, right or two-sided) of a graded subset is again graded. Then \( R := \text{Ann}_U^U(J) = \bigoplus_{j=1}^t M_{1j} \) (the right annihilator of \( J \)) is graded.

It is well known that the unit of an unital associative graded algebra unit is always homogeneous. Exactly the same argument can be used to prove the following: if an associative algebra has a left unit, then there exists a homogeneous left unit in the algebra. Note that \( R \) has a left unit (the identity matrix \( E_1 \in M_{11} \)), hence it must admit a homogeneous left unit, say \( u_1 \). Clearly \( u_1^2 = u_1 \), hence \( u_1 \) is diagonalizable; moreover, the diagonal form
of \( u_1 \) is exactly \( E_1 \). So, after applying an isomorphism, we can assume \( E_1 \) homogeneous.

Now, since \( (1 - E_1)U \cong UT(n_2, n_3, \ldots, n_t) \) we can proceed by induction. Moreover, if \( i < j \) and \( E_i \) and \( E_j \) are the identity matrices of \( M_{ii} \) and \( M_{jj} \), respectively, then \( M_{ij} = E_i U E_j \) is a graded subspace. \( \square \)

So we can assume every matrix subalgebra \( M_{ii} \) graded. But gradings on matrix algebra are well known, see [4, Chapter 2] for instance. A description is given by the graded version of the Density Theorem. It follows that every \( M_{ii} \cong M_{p_i} \otimes D_i \), where \( M_{p_i} \) is a matrix algebra equipped with an elementary grading and \( D_i \) is a graded division algebra, where the grading on \( M_{p_i} \otimes D_i \) is induced by (1). Here, we use the Kronecker product to identify, as vector spaces, \( M_{p_i} \otimes D_i = M_{ii} \). Equivalently, we identify \( M_{ii} = M_{p_i}(D_i) \), the \( p_i \times p_i \) matrix algebra with coefficients in \( D_i \). Thus, we denote the elements of \( M_{ii} \) as \( m \otimes d \), or \( md \), where \( m \in M_{p_i} \) and \( d \in D_i \). As mentioned before, we assume that each \( M_{ii} \) is a natural subalgebra of \( U \). Moreover, under these identifications, each \( M_{ij} \) is a (graded) \((M_{ii}, M_{jj})\)-bimodule; and \( U \) is a (graded) \((M_{ii}, M_{jj})\)-bimodule as well.

It is well known that every automorphism of a matrix algebra is inner, hence we can find an invertible matrix \( A_i \) such that \( A_i M_{ii} A_i^{-1} = M_{p_i} \otimes D_i \), where the grading of \( M_{p_i} \otimes D_i \) is given by (1). Taking the block-diagonal matrix \( A' = \text{diag}(A_1, A_2, \ldots, A_t) \), we obtain an automorphism of \( U \) such that every \( M_{ii} = M_{p_i} \otimes D_i \).

**Lemma 2.** In the notations above, if \( J \) is graded, then \( \exists \) a graded division ring \( D \), and elements \( g_1, g_2, \ldots, g_t \in G \) such that \( D_i = [g_i]D[\overline{g_i}] \). Moreover, \( U \cong U' \otimes D \), where \( U' \) is given with an elementary \( G \)-grading.

**Proof.** For all \( i = 1, 2, \ldots, t \), denote by \( e_i \in D_i \) the unit element of the graded division algebra \( D_i \), and denote \( e_{i}^{(i)} \in M_{p_i} \) the matrix unit with 1 in the entry \((1, 1)\) of \( M_{p_i} \), and 0 elsewhere. For \( i < j \), let \( X = e_{i}^{(i)} e_i U e_{j}^{(i)} e_j \).

Note that \( X \) is a graded \( D_i \)-left module and a graded \( D_j \)-right module. If \( D_i \) consists of \( n_i' \times n_i' \) matrices and \( D_j \) is \( n_j' \times n_j' \) matrices then \( X \) is \( n_i' \times n_j' \) matrices. From the structure of graded modules over graded division algebras, we obtain \( n_i' n_j' = k_1 n_i'^2 = k_2 n_j'^2 \), for some \( k_1, k_2 \in \mathbb{N} \). This is possible only if \( n_i' = n_j' \), hence given a non-zero homogeneous \( v \in X \) of degree \( h \in G \), we have \( X = D_i v = v D_j \). As a consequence, for any \( x \in D_i \), there exists \( y \in D_j \) such that \( xy = wy \); in particular, if \( x \) is homogeneous, then \( y \) is homogeneous as well and \( \deg x = h(\deg y)h^{-1} \). Moreover, define the map \( T : x \in D_i \mapsto y \in D_j \). Clearly \( T \) is a linear map. Also, for each homogeneous \( x \in D_i \), one has \( \deg T(x) = h^{-1}(\deg x)h \). Futhermore, \( vT(x_1 x_2) = x_1 x_2 v = x_1 v T(x_2) = v T(x_1) T(x_2) \). Since \( D_i \) is a graded division algebra, one obtains \( T(x_1 x_2) = T(x_1) T(x_2) \), which means that \( T \) is a homomorphism of algebras. Thus, \( T \) is a weak isomorphism between \( D_i \) and \( D_j \). This proves the first part of the lemma.
Considering now all matrix units $e^{(r)}_{ij} \in M_{p_r}$, $e^{(s)}_{mn} \in M_{p_s}$ we can repeat the arguments for $e^{(r)}_{ij} U e^{(s)}_{mn}$, to conclude that it is a graded $(D_r, D_s)$-bimodule, with $\dim_K e^{(r)}_{ij} U e^{(s)}_{mn} = \dim_K D$. Thus, we obtain $U \simeq U' \otimes D$, for some upper block-triangular matrix algebra $U'$ endowed with an elementary grading.

A very important result is the following theorem, due to Gordienko:

**Lemma 3 (Corollary 3.3 of [6]).** Let $A$ be a finite-dimensional associative algebra over a field $F$ graded by any group $G$. Suppose that either $\text{char } F = 0$ or $\text{char } F > \dim A$. Then the Jacobson radical $J := J(A)$ is a graded ideal of $A$.

Combining Gordienko’s Theorem and Lemma 2, we obtain

**Theorem 4.** Let $G$ be any group and consider any $G$-grading on the upper block triangular matrix algebra $A = UT(n_1, n_2, \ldots, n_t)$ over a field $K$. Suppose that either $\text{char } K = 0$ or $\text{char } K > \dim A$. Then there exists a $G$-graded division algebra on $D = M_n(K)$ and an upper block triangular matrix algebra $B = UT(n'_1, n'_2, \ldots, n'_t)$ endowed with an elementary grading, such that $A \simeq B \otimes D$.

Note that for the particular case where $K$ is algebraically closed of characteristic zero and $G$ is abelian (finite or not), then $J$ is automatically graded (for instance, $J$ is graded by the duality between gradings and action). In this case, the classification of division gradings over matrix algebras is known (see, for example, [4, Chapter 1]). In this way, we re-obtain the result of Valenti and Zaicev [11]. More precisely, we have

**Corollary 5.** Let $G$ be an abelian group, and let $K$ be an algebraically closed field of characteristic zero. Let $U = UT(n_1, n_2, \ldots, n_t)$ be endowed with any $G$-grading. Then there exists a subgroup $T \subset G$, a 2-cocycle $\sigma : T \times T \to K^\times$, and a block-triangular algebra $U' = UT(n'_1, n'_2, \ldots, n'_t)$ endowed with an elementary grading (where $n_i = n'_i|T|$, for each $i$), such that $U \simeq U' \otimes K^\sigma T$.

**Proof.** In this case, a graded division algebra on a matrix algebra is $K^\sigma T$ (for instance, see Theorem 2.15 of [4]).

A very natural question is if Theorem 4 is true despite of the conditions on the characteristic of the base field.

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