A TWO-GROUP AGE OF INFECTION EPIDEMIC MODEL WITH PERIODIC BEHAVIORAL CHANGES

MAMADOU L. DIAGNE
University of Thies, Senegal

OUSMANE SEYDI* and AISSATA A. B. SY♭
*Polytechnic School of Thies, Senegal
♭University of Thies, Senegal

(Communicated by Pierre Magal)

ABSTRACT. In this paper we propose a two-group SIR age of infection epidemic model by incorporating periodical behavioral changes for both susceptible and infected individuals. Our model allows different incubation periods for the two groups. It is proved in this paper that the persistence and extinction of the disease are determined by a threshold condition given in term of the basic reproductive number $R_0$. That is, the disease is uniformly persistent if $R_0 > 1$ with the existence of a positive periodic solution, while the disease goes to extinction if $R_0 < 1$ with the global asymptotic stability of the disease free periodic solution. The model we have proposed is general and can be applied to a wide class of diseases.

1. Introduction. Mathematical modeling has become increasingly sophisticated and precise in the description of population dynamics. Several aspects are incorporated in models depending on the nature of the problem being studied. Structuring such as age of infection, chronological age, space ... have emerged as important elements in understanding population dynamics. We refer to the monographs [1, 7, 8, 16, 23, 28, 34] for an extensive discussion and numerous examples. Here we are interested not only in the age structure of infection but also in the behavior of individuals in the face of a disease. When a disease is spread by close contact between two individuals, behavior between individuals becomes a significant factor in the spread of the disease. Diseases such as Tuberculosis, HIV/AIDS, SARS, H1N1 are concrete examples of these situations. In fact it is well known that individuals are able to change their behavior to avoid the risk of infection when they are aware of the apparent signs or not of an infection [5, 20]. Here we have opted to model the behavior change by dividing the population into two groups. The first is made up of knowledgeable individuals who behave in a manner that avoids contamination, and the second group consists of individuals who behave normally. Both groups

---

2010 Mathematics Subject Classification. Primary: 92D30, 34K20; Secondary: 34K13.

Key words and phrases. Age of infection, periodic solution, uniform persistence, global stability, coexistence.

All the authors are supported by the CEA-MITIC (Senegal).

* Corresponding author: Ousmane Seydi.
are compartmentalized into Susceptible-Infected-Removed classes. The model we consider in this paper is the following

\[
\begin{aligned}
\frac{dS_1(t)}{dt} &= \lambda_1(t) - \mu S_1(t) - \epsilon_1(t)S_1(t) + \epsilon_2(t)S_2(t) - S_1(t) \int_0^{+\infty} [\beta_{11}(a)i_1(t, a) + \beta_{12}(a)i_2(t, a)]da, \\
\frac{dS_2(t)}{dt} &= \lambda_2(t) - \mu S_2(t) + \epsilon_1(t)S_1(t) - \epsilon_2(t)S_2(t) - S_2(t) \int_0^{+\infty} [\beta_{21}(a)i_1(t, a) + \beta_{22}(a)i_2(t, a)]da, \\
\frac{\partial i_1(t, a)}{\partial t} + \frac{\partial i_1(t, a)}{\partial a} &= -[\mu + d_1(a)]i_1(t, a) - \nu_1(t)i_1(t, a) + \nu_2(t)i_2(t, a), \\
i_1(t, 0) &= S_1(t) \int_0^{+\infty} [\beta_{11}(a)i_1(t, a) + \beta_{12}(a)i_2(t, a)]da, \\
\frac{\partial i_2(t, a)}{\partial t} + \frac{\partial i_2(t, a)}{\partial a} &= -[\mu + d_2(a)]i_2(t, a) - \nu_2(t)i_2(t, a) + \nu_1(t)i_1(t, a), \\
i_2(t, 0) &= S_2(t) \int_0^{+\infty} [\beta_{21}(a)i_1(t, a) + \beta_{22}(a)i_2(t, a)]da,
\end{aligned}
\]

supplemented with the initial conditions

\[S_k(0) = S_{k0} \in \mathbb{R}_+ \quad \text{and} \quad i(0, \cdot) = i_{k0} \in L^1_+(\mathbb{R}_+, \mathbb{R}), \quad \text{for } k = 1, 2.\]  

(2)

The state variables \(S_k(t)\) and \(i_k(t, a)\), \(k = 1, 2\) describe respectively the number of susceptible at time \(t\) and the density of infected individuals at time \(t\) with time since infection \(a\). The functions \(t \to \lambda_k(t)\), \(k = 1, 2\) are the time periodic rates of recruitment of individuals in the susceptible classes while the parameter \(\mu\) is the natural death rate. The age dependent function \(a \to d_k(a)\), \(k = 1, 2\) are the rates at which individuals are removed due to death, recovery or isolation. The time periodic functions \(t \to \epsilon_k(t)\) and \(t \to \nu_k(t)\) describe the rates at which individuals move from one group to another respectively for the susceptible class and the infected class. Finally the map \(a \to \beta_{ij}(a), 1 \leq i, j \leq 2\) is the rate at which susceptible individuals of group \(i\) are contaminated by infected individuals of group \(j\) with age of infection \(a\).

There have been many non-autonomous periodic coefficients epidemic models that have flourished in the literature in recent years. However most are systems of ordinary differential equations. Periodic delay differential equations have also gained many attentions and several works have been done in this subject. We refer among other to the works of [3, 11, 13, 24, 37, 32, 36]. The definition of a threshold by the basic reproduction number to know if it will be extinguished or persistence of a disease in the case of non-autonomous models has receive a lot of attentions. We can mention the works of [33, 37] respectively on the systems of differential equations and delay differential equations with periodic coefficients. Age of infection structured models with periodic coefficients are slightly less numerous in the literature. However, the work of [29] concerning the renewal theorem can be of great help in the study of these problems. The results in [2, 9, 18, 31] have also contributed a lot to these issues. One of the most common mathematical questions
in non-autonomous epidemiological models is the question of uniform persistence and the coexistence of the endemic state and the disease-free state. The works of [6, 12, 19, 26, 27, 38] allowed for a better understanding of these issues. We refer for instance to [26, 38] for further comments on this subject.

To the best of our knowledge the model (1)-(2) presented in this article has not yet been studied in the literature. However there are models that come close [14, 22, 24, 35] but remain different. In [14] the authors studied an autonomous two group age of infection epidemic model with criss-cross transmission. In [24] the author dealt with a one group model structured in age of infection with periodic coefficients whereas in [22] the author dealt with an autonomous one group age of infection model with immigration of new individuals into the susceptible, latent and infectious classes. In [35] an SIS autonomous age dependent infection model on a heterogeneous network were discussed. Let us also mention the work of [11] where an age structured model is considered by allowing the birth and death functions to be density dependent and periodic in time. In this manuscript we consider a two groups model with periodic coefficients that incorporates the criss-cross transmission, the within contamination and the change of behaviors of the individuals towards the disease. We also allow the possibility of having different incubation periods which makes our model applicable to several kind of diseases.

The paper is organized as follow. In Section 2 we deal with the existence of non negative solutions of system (1)-(2). We give a Volterra integral formulation and prove some properties of the model. The boundedness (upper and lower bound) of the solutions and the dissipativity are studied in Section 3. In Section 4 we give a description of the disease free periodic solution and give its relationship with the solutions of system (1)-(2). Section 5 is devoted to some preparatory results. In fact in Section 5 we consider a perturbed linear age structured model and study its asymptotic properties by using the renewal theorem [29]. The results obtained from the perturbed linear equation are combined with comparison principles in Section 6 in order to prove the global stability of the disease free periodic solution for system (1)-(2) when the basic reproductive is strictly less that one. In Section 7 we prove the uniform persistence of the periodic semiflow when the basic reproductive number is strictly greater that one. We also prove some extinction results of the model without any condition on the basic reproductive number whenever the initial distributions are take in some sub domain. We end up the paper in Section 8 by proving a coexistence result. More precisely we prove that when the basic reproductive is strictly greater that one then there exists an endemic periodic solution in addition to the disease free periodic solution.

2. Existence of non negative solutions. The existence of non negative globally defined solutions of (4) in $[0, +\infty)$ is classical and can be done by using either Volterra integral formulation or integrated semigroup formulation [7, 15, 30, 34]. So in this section we will only describe the the solutions by mean of Volterra integral equations. This will allow us to deal with the renewal equation as well as the basic reproductive number by using the results in [29].

The following hypothesis will be of concern.

**Assumption 2.1.** We assume that

i) The maps $\lambda_k, \nu_k, \epsilon_k : \mathbb{R} \to \mathbb{R}^+_0$, $k = 1, 2$, are non negative, continuous on $\mathbb{R}$ and $\tau$-periodic with $\tau > 0$. Furthermore there exists $\lambda_- > 0$ and $\lambda_+ > 0$ such that
\[ \lambda_\text{-} \leq \lambda_k(t) \leq \lambda_+, \quad \forall t \in \mathbb{R}, \quad k = 1, 2. \]

ii) The natural mortality satisfies \( \mu > 0 \) and the removal rates \( d_k : \mathbb{R}_+ \to \mathbb{R}_+ \), \( k = 1, 2 \) are continuous, bounded on \( \mathbb{R}_+ \) and non negative.

iii) The contact rates \( \beta_{ij} \), \( 1 \leq i, j \leq 2 \) are non negative, continuous on \( \mathbb{R}_+ \).

We denote the maximum age of infection by \( a^* \). The supports \( \text{Supp}(\beta_{ij}) \), \( 1 \leq i, j \leq 2 \) satisfy

\[
\text{Supp}(\beta_{11}) \subset \text{Supp}(\beta_{21}) \quad \text{and} \quad \text{Supp}(\beta_{22}) \subset \text{Supp}(\beta_{12})
\]

and there exists \( 0 \leq a_k < +\infty \) for \( k = 1, 2 \) such that

\[
\text{Supp}(\beta_{21}) = [a_1, a^*] \quad \text{and} \quad \text{Supp}(\beta_{12}) = [a_2, a^*].
\]

We also assume that

\[
a^* - a_* > 0
\]

with \( a_* = \max(a_1, a_2) \).

**Remark 1.** Condition iii) in Assumption 2.1 states that we may have different incubation periods for each group. The incubation periods are taken into account by \( a_1 \) for group 1 and \( a_2 \) for group 2. The condition \( \text{Supp}(\beta_{11}) \subset \text{Supp}(\beta_{21}) \)

\( \text{Supp}(\beta_{22}) \subset \text{Supp}(\beta_{12}) \) is made to encounter the possibility of criss-cross transmission that is when \( \beta_{kk} \equiv 0 \) for \( k = 1, 2 \).

We will rewrite system (1)-(2) into a more compact form by considering the following state variables

\[
S(t) = \begin{pmatrix} S_1(t) \\ S_2(t) \end{pmatrix}, \quad t \geq 0, \quad i(t, a) = \begin{pmatrix} i_1(t, a) \\ i_2(t, a) \end{pmatrix}, \quad t \geq 0, \quad a \geq 0.
\]

Next we introduce the following notations for each \( t \in \mathbb{R} \) and \( a \geq 0 \)

\[
\Lambda(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix}, \quad e(t) = \begin{pmatrix} -\varepsilon_1(t) & \varepsilon_2(t) \\ \varepsilon_1(t) & -\varepsilon_2(t) \end{pmatrix}, \quad b(a) = \begin{pmatrix} \beta_{11}(a) & \beta_{12}(a) \\ \beta_{21}(a) & \beta_{22}(a) \end{pmatrix},
\]

and

\[
d(a) = \begin{pmatrix} d_1(a) & 0 \\ 0 & d_2(a) \end{pmatrix}, \quad k(t) = \begin{pmatrix} -\nu_1(t) & \nu_2(t) \\ \nu_1(t) & -\nu_2(t) \end{pmatrix}.
\]

Because (1)-(2) is non autonomous we will consider the following system with initial time distribution \( t_0 \geq 0 \)

\[
\begin{cases}
\frac{dS(t)}{dt} = \Lambda(t) - \mu S(t) + e(t)S(t) - \text{diag}(S(t)) \int_0^{+\infty} b(a)i(t, a)da, \quad t > t_0 \\
\frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = [-\mu + d(a)]i(t, a) + k(t)i(t, a), \quad t > t_0, \quad a > 0 \\
i(t, 0) = \text{diag}(S(t)) \int_0^{+\infty} b(a)i(t, a)da, \quad t > t_0, \\
S(t_0) = S_0 \in \mathbb{R}_2^2, \quad i(t_0, \cdot) = i_0 \in L^1_+(\mathbb{R}_+, \mathbb{R})^2.
\end{cases}
\]

In the sequel the following notations will be used

\[
\begin{cases}
x \geq 0_{\mathbb{R}^2} \iff x \in \mathbb{R}_2^2 \\
x > 0_{\mathbb{R}^2} \iff x \in \mathbb{R}_2^+ \quad \text{and} \quad x \neq 0_{\mathbb{R}^2} \\
x \gg 0_{\mathbb{R}^2} \iff x \in \text{int}(\mathbb{R}_2^+).
\end{cases}
\]

Since we are considering a population of individuals, the sum norm in \( \mathbb{R}^2 \) is considered that is

\[
\|x\| = |x_1| + |x_2|, \quad \forall x \in \mathbb{R}^2.
\]
Furthermore if $M = (m_{ij})_{1 \leq i,j \leq 2}$ is a $2 \times 2$ matrix we define its norm as

$$\|M\|_{L(\mathbb{R}^2)} := \max_{1 \leq j \leq 2} \sum_{i=1}^{2} |m_{ij}|.$$  

We denote by $\mathbf{1}$ the vector with all elements equal to 1 that is

$$\mathbf{1} := \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \Leftrightarrow \mathbf{1}^T = (1, 1).$$

Note that with the above notations we have the supremum norm of $b$ and $d$ are defined by

$$\|b\|_{\infty} := \sup_{a \in \mathbb{R}^+} \|b(a)\|_{L(\mathbb{R}^2)}$$

with

$$\|b(a)\|_{L(\mathbb{R}^2)} = \max_{1 \leq j \leq 2} \sum_{i=1}^{2} \beta_{ij}(a) \leq \max_{1 \leq j \leq 2} \sum_{i=1}^{2} \|\beta_{ij}\|_{\infty}, \ \forall a \geq 0$$

where $\| \cdot \|_{\infty}$ denotes the usual supremum norm and

$$\|d\|_{\infty} := \max_{1 \leq j \leq 2} |d_j|_{\infty}.$$  

**Volterra integral formulation:** The integration along the characteristics for non-autonomous linear age structured system in general cases have been done in Inaba [10]. Inspired by [10] we will give the Volterra integral formulation for the non-linear coupled system (4). We briefly give some details for completeness and later references. The characteristic lines of system (4) are given by

$$\frac{dt}{dl} = 1 \ \text{and} \ \frac{da}{dl} = 1.$$  

Therefore in order to integrate system (4) along the characteristics we define for any $c_1 \in \mathbb{R}$ and $c_2 \geq 0$ the map $l \to L(l; c_1, c_2) \in L(\mathbb{R}^2)$ as the unique solution of the ordinary differential equation

$$\frac{d}{dl} L(l; c_1, c_2) = -[\mu - k(c_1 + l) + d(c_2 + l)] L(l; c_1, c_2), \ l > 0 \ \text{and} \ L(0; c_1, c_2) = I_{\mathbb{R}^2}.$$  

(5)

Then along the characteristic lines the $i$-equation of (4) satisfies the ordinary differential equation

$$\frac{d}{dl} i(l + c_1, l + c_2) = -[\mu - k(c_1 + l) + d(c_2 + l)] i(l + c_1, l + c_2), \ l > 0.$$  

Hence

$$i(l + c_1, l + c_2) = L(l; c_1, c_2) i(c_1, c_2), \ \forall l \geq 0.$$  

Therefore if $t - t_0 > a$ by setting $c_2 = 0$, $l = a$ and $c_1 = t - a > t_0$ we obtain

$$i(t, a) = L(a; t - a, 0) i(t - a, 0), \ 0 \leq a < t - t_0$$

and if $0 \leq t - t_0 \leq a$ by setting $c_1 = t_0$, $l = t - t_0$ and $c_2 = a - t + t_0 \geq 0$ it follows that

$$i(t, a) = L(t - t_0; t_0, a + t_0 - t) i_0(a - t + t_0), \ 0 \leq t - t_0 \leq a.$$  

Therefore by setting

$$B(t, t_0; S_0, l_0) := S(t + t_0), \ \forall t \geq 0.$$  

and
\[ B_I(t, t_0; S_0, i_0) := i(t + t_0, 0) = \text{diag}(B_S(t, t_0; S_0, i_0)) \int_0^{+\infty} b(a) i(t + t_0, a) da, \ \forall t \geq 0 \]
we deduce that
\[
 i(t, a) = \begin{cases} 
 L(a; t - a, 0) B_I(t - t_0 - a, t_0; S_0, i_0) & \text{if } 0 \leq a < t - t_0 \\
 L(t - t_0; t_0, a + t_0 - t) i_0 (a - t + t_0) & \text{if } 0 \leq t - t_0 \leq a.
\end{cases}
\]  
(6)
By combining (4) and (6) we see that the maps \( t \to B_S(t, t_0; S_0, i_0) \) and \( t \to B_I(t, t_0; S_0, i_0) \) must satisfy respectively for each \( t \geq 0 \)
\[
 B_S(t, t_0; S_0, i_0) = e^{-\mu t} S_0 + \int_0^t e^{-\mu(t-s)} [A(s + t_0) + e(s + t_0) B_S(s, t_0; S_0, i_0) - B_I(s, t_0; S_0, i_0)] ds
\]
\[
 B_I(t, t_0; S_0, i_0) = \text{diag}(B_S(t, t_0; S_0, i_0)) \int_t^{+\infty} b(a) i(t + t_0, a) da \\
+ \text{diag}(B_S(t, t_0; S_0, i_0)) \int_0^t b(a) i(t + t_0, a) da \\
+ \text{diag}(B_S(t, t_0; S_0, i_0)) G(t, t_0; i_0) \\
+ \text{diag}(B_S(t, t_0; S_0, i_0)) \int_0^t \Psi(t + t_0, a) B_I(t - a, t_0; S_0, i_0) da
\]
where we have set
\[
 \Psi(t, a) = b(a) L(a; t - a, 0), \ \forall t \in \mathbb{R}, \ \forall a \geq 0
\]  
(7)
and
\[
 G(t, t_0; i_0) = \int_t^{+\infty} b(a) L(t; t_0, a - t) i_0 (a - t) da, \ \forall t, t_0 \geq 0.
\]  
(8)
We conclude that \( t \in \mathbb{R}_+ \to (B_S(t, t_0; S_0, i_0), B_I(t, t_0; S_0, i_0)) \) is the unique continuous solution of the following Volterra integral system of equations
\[
 \begin{cases}
 B_S(t, t_0; S_0, i_0) = e^{-\mu t} S_0 + \int_0^t e^{-\mu(t-s)} [A(s + t_0) + e(s + t_0) B_S(s, t_0; S_0, i_0) - B_I(s, t_0; S_0, i_0)] ds, \ t \geq 0 \\
 B_I(t, t_0; S_0, i_0) = \text{diag}(B_S(t, t_0; S_0, i_0)) \left[ G(t, t_0; i_0) + \int_0^t \Psi(t + t_0, a) B_I(t - a, t_0; S_0, i_0) da \right], \\
 t \geq 0.
\end{cases}
\]  
(9)
with \( t \geq 0 \). Therefore we define for each \( t_0 \geq 0 \) and \( \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \in \mathbb{R}_+^2 \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \) by
\[
 U(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix}, \ \forall t \geq t_0
\]  
(10)
the unique mild solution of (4) that is to say that
\[
 \begin{pmatrix} S(t) \\ i(t, \cdot) \end{pmatrix} = U(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix}, \ \forall t \geq t_0.
\]  
(11)
Hence \( \{ U(t, t_0) \}_{t \geq t_0} \) is an evolution family on \( \mathbb{R}_+^2 \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \). More precisely
\[
 U(t, t_0) \text{ maps } \mathbb{R}_+^2 \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \text{ into itself for all } t \geq t_0 \text{ and satisfies}
\]
\[
 U(t, t) = I, \ \forall t \geq 0 \text{ and } U(t, s) U(s, t_0) = U(t, t_0), \ \forall t \geq s \geq t_0.
\]  
(12)
Furthermore if
\[
\begin{pmatrix}
    S(t)
    \\ i(t, \cdot)
\end{pmatrix} = U(t, t_0) \begin{pmatrix}
    S_0
    \\ i_{t_0}
\end{pmatrix}, \quad \forall t \geq t_0
\]
then
\[
\begin{pmatrix}
    S(t)
    \\ i(t, \cdot)
\end{pmatrix} = U(t, s) \begin{pmatrix}
    S(s)
    \\ i(s, \cdot)
\end{pmatrix}, \quad \forall t \geq s \geq t_0
\]
with \( t \to (S(t), i(t, \cdot)) \) the mild solution of (4). Note that since
\[
L(t; c_1 + \tau, c_2) = L(t; c_1, c_2), \quad \forall t \geq 0, \quad c_1 \in \mathbb{R} \quad \text{and} \quad c_2 \geq 0
\]
it is clear from (6) and (9) that \( U \) is a \( \tau \)-periodic evolution semiflow that this
\[
U(t + \tau, t_0 + \tau) = U(t, t_0), \quad \forall t \geq t_0 \geq 0.
\]
Moreover (7) and (15) also imply that
\[
\Psi(t + \tau, a) = \Psi(t, a), \quad \forall a \geq 0, \quad \forall t \geq 0
\]
and Assumption 2.1 ensures that \( \Psi \) has a compact support with respect to \( a \). More precisely we have
\[
\Psi(t, a) = 0_{\mathcal{L}(\mathbb{R}^2)}, \quad \forall a \geq a^*, \quad \forall t \geq 0.
\]
We now collect some properties on the linear operator \( L(l; c_1, c_2) \).

**Lemma 2.2.** Let Assumption 2.1 be satisfied. Then for any \( c_1 \in \mathbb{R} \) and \( c_2 \geq 0 \) we have
\[
1^T L(l; c_1, c_2) \leq e^{-\mu l} 1^T, \quad \forall l \geq 0
\]
and there exists a constant \( \gamma > 0 \) independent of \( c_1 \) and \( c_2 \) such that
\[
L(l; c_1, c_2) \geq e^{-(\mu + \gamma)l} I_{\mathbb{R}^2_+}, \quad \forall l \geq 0.
\]

**Proof.** Since the sum of the columns of \( k \) are equal to zero we have from (5) that
\[
\frac{d}{dl} 1^T L(l; c_1, c_2) = -[\mu 1^T + 1^T d(c_2 + l)] L(l; c_1, c_2), \quad l > 0 \quad \text{and} \quad 1^T L(0; c_1, c_2) = 1^T
\]

hence
\[
\frac{d}{dl} 1^T L(l; c_1, c_2) \leq -\mu 1^T L(l; c_1, c_2), \quad l > 0 \quad \text{and} \quad 1^T L(0; c_1, c_2) = 1^T.
\]

Therefore
\[
1^T L(l; c_1, c_2) \leq e^{-\mu l} 1^T, \quad \forall l \geq 0.
\]

To obtain (20) we note that since the off diagonal entries of the matrix \( l \to -d(l + c_2) + k(l + c_1) \) are non negative and the diagonal entries are uniformly bounded for \( l + c_1 \in \mathbb{R} \) and \( l + c_1 \geq 0 \) there exists \( \gamma > 0 \) such that
\[
[\gamma I_{\mathbb{R}^2} - d(l + c_2) + k(l + c_1)] x \geq 0
\]
for all \( x \in \mathbb{R}^2_+ \), \( c_1 \in \mathbb{R} \), \( c_2 \geq 0 \) and \( l \geq 0 \). Hence we have
\[
\frac{d}{dl} L(l; c_1, c_2) = -[\mu + \gamma] L(l; c_1, c_2) + [\gamma I_{\mathbb{R}^2} - d(l + c_2) + k(l + c_1)] L(l; c_1, c_2), \quad l > 0 \quad \text{and}
\]
\[
L(l; c_1, c_2) = e^{-(\mu + \gamma)l} I_{\mathbb{R}^2} + \int_0^l e^{-(\mu + \gamma)(l - s)} \gamma I_{\mathbb{R}^2} - d(s + c_2) + k(s + c_1)] L(s; c_1, c_2) ds, \quad l \geq 0
\]
\[
\geq e^{-(\mu + \gamma)l} I_{\mathbb{R}^2}, \quad l \geq 0.
\]
3. Boundedness of the solutions. In this section we will prove that the mild solutions of (4) have upper and lower bounds. Define

\[ I(t) := \int_{0}^{+\infty} i(t,a)da, \quad t \geq t_0 \quad \text{and} \quad I_0 := \int_{0}^{+\infty} i_0(a)da. \]

Before proving the boundedness properties we recall that \( \mathbb{R}^2 \) is endowed with the sum norm. More precisely we have set

\[ \|x\| = |x_1| + |x_2|, \quad \forall x \in \mathbb{R}^2. \]

Then we have

\[ \|x\| = (1, x), \quad \forall x \geq 0_{\mathbb{R}^2} \]
Upper bound: Using the initial conditions of (4) giving classical solutions we obtain that the map \( t \to (S(t), I(t)) \) satisfies
\[
\begin{cases}
\frac{dS(t)}{dt} = \Lambda(t) - \mu S(t) + e(t)S(t) - \text{diag}(S(t)) \int_0^{+\infty} b(a) i(t, a) da \\
\frac{dI(t)}{dt} = \text{diag}(S(t)) \int_0^{+\infty} b(a) i(t, a) da - \mu I(t) - \int_0^{+\infty} d(a) i(t, a) da + k(t) I(t) \\
S(t_0) = S_0 \in \mathbb{R}_+^2, \ I(t_0) = I_0 \in \mathbb{R}_+^2.
\end{cases}
\]

Since the maps \( t \to S(t) \) and \( t \to I(t) \) are non negative it follows that \( t \to ||S(t)|| \) and \( t \to ||I(t)|| \) are derivable with respect to \( t \). Furthermore we have for each \( t > t_0 \)
\[
\frac{d||S(t)||}{dt} = \frac{d\langle 1, S(t) \rangle}{dt} \quad \text{and} \quad \frac{d||I(t)||}{dt} = \frac{d\langle 1, I(t) \rangle}{dt}.
\]
Observe that for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}_+^2 \) we have
\[
\langle 1, e(t)x \rangle = 0 \quad \text{and} \quad \langle 1, k(t)x \rangle = 0.
\]
Hence taking the scalar product with \( 1 \) in the both sides of the equality of (25) we obtain
\[
\begin{cases}
\frac{d||S(t)||}{dt} = \|\Lambda(t)\| - \mu ||S(t)|| - \left\langle 1, \text{diag}(S(t)) \int_0^{+\infty} b(a) i(t, a) da \right\rangle \\
\frac{d||I(t)||}{dt} = \langle 1, \text{diag}(S(t)) \int_0^{+\infty} b(a) i(t, a) da \rangle - \mu ||I(t)|| - \left\langle 1, \int_0^{+\infty} d(a) i(t, a) da \right\rangle \\
S(t_0) = S_0 \in \mathbb{R}_+^2, \ I(t_0) = I_0 \in \mathbb{R}_+^2
\end{cases}
\]

and by summing the two equations of (27) we get
\[
(||S(t)|| + ||I(t)||)' = ||\Lambda(t)|| - \mu (||S(t)|| + ||I(t)||) - \left\langle 1, \int_0^{+\infty} d(a) i(t, a) da \right\rangle, \ t > t_0.
\]
Hence
\[
(||S(t)|| + ||I(t)||)' \leq 2\lambda_+ - \mu (||S(t)|| + ||I(t)||), \ t > t_0
\]
providing that
\[
||S(t)|| + ||I(t)|| \leq e^{-\mu(t-t_0)}(||S_0|| + ||I_0||) + 2\lambda_+ \int_{t_0}^{t} e^{-\mu(t-s)} ds,
\]
\[
\leq e^{-\mu(t-t_0)}(||S_0|| + ||I_0||) + \frac{2\lambda_+}{\mu} \left(1 - e^{-\mu(t-t_0)}\right), \forall t \geq t_0
\]
and we deduce that
\[
||I(t)|| + ||S(t)|| \leq \max \left(||I_0|| + ||S_0||, \frac{2\lambda_+}{\mu}\right), \forall t \geq t_0.
\]
Using the fact that the set of initial conditions giving classical solutions is dense in the space of the state variables [15] and (29) we obtain the following lemma.

Lemma 3.1. Let Assumption 2.1 be satisfied. Let \( S_0 \in \mathbb{R}_+^2 \) and \( I_0 \in L^1_+(\mathbb{R}_+, \mathbb{R})^2 \) be given and define
\[
\left( \begin{array}{c}
S(t) \\
I(t)\end{array} \right) = U(t, t_0) \left( \begin{array}{c}
S_0 \\
I_0\end{array} \right), \forall t \geq t_0.
\]
Then we have
\[ \|i(t, \cdot)\|_{L^1} + \|S(t)\| \leq \max \left( \|i_0\|_{L^1} + \|S_0\|, \frac{2\lambda_+}{\mu} \right), \forall t \geq t_0. \]

In order to state a uniform boundedness in bounded sets we recall that for each \( S_0 \in \mathbb{R}_+^2 \) and \( i_0 \in L^1_+(\mathbb{R}_+, \mathbb{R})^2 \) the map
\[ t \to (B_S(t, t_0; S_0, i_0), B_I(t, t_0; S_0, i_0)) \]
satisfying (9) is given by
\[ B_S(t, t_0; S_0, i_0) = S(t + t_0), \forall t \geq 0 \]
and
\[ B_I(t, t_0; S_0, i_0) = \text{diag}(S(t)) \int_0^{t_0} b(a)i(t + a, a)da, \forall t \geq 0 \]
with
\[ S(t_0) = S_0 \quad \text{and} \quad i(t_0, \cdot) = i_0. \]
Then a direct consequence of Lemma 3.1 is the following

**Corollary 1.** Let Assumption 2.1 be satisfied. Then for each \( M > 0 \) there exists \( M_+ > 0 \) such that for each \( S_0 \in \mathbb{R}_+^2 \) and \( i_0 \in L^1_+(\mathbb{R}_+, \mathbb{R})^2 \) with \( \|S_0\| + \|i_0\|_{L^1} \leq M \) if
\[ \left( \begin{array}{c} S(t) \\ i(t, \cdot) \end{array} \right) = U(t, t_0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right), \forall t \geq t_0 \]
then
\[ \|S(t)\| + \|i(t, \cdot)\| \leq M_+, \forall t \geq t_0 \quad (30) \]
and
\[ \|B_I(t, t_0; S_0, i_0)\| \leq M_+, \forall t \geq 0. \quad (31) \]

**Lower bound:** In order to obtain a lower bound for the map \( t \to S(t) \) we will also first consider the initial conditions giving classical solutions so that system (25) holds true. Next we observe that
\[ \text{diag}(S(t)) \int_0^{t_0} b(a)i(t, a)da = \text{diag} \left( \int_0^{t_0} b(a)i(t, a)da \right) S(t), \forall t \geq t_0. \]
Hence by setting
\[ \alpha_1 := \|b\|_\infty \max \left( \|i_0\|_{L^1} + \|S_0\|, \frac{2\lambda_+}{\mu} \right) > 0 \quad (32) \]
we obtain
\[ \text{diag}(S(t)) \int_0^{t_0} b(a)i(t, a)da \leq \alpha_1 S(t), \forall t \geq t_0. \quad (33) \]
Thus setting
\[ \alpha_2 := \max \left( \sup_{s \in [\tau, \tau]} \varepsilon_1(s), \sup_{s \in [\tau, \tau]} \varepsilon_2(s) \right) \geq 0, \quad (34) \]
one gets from the \( S \)-equation of (25) that
\[ \frac{dS(t)}{dt} \geq \lambda_- 1 - \mu S(t) + e(t)S(t) - \alpha_1 S(t) \geq \lambda_- 1 - |\mu + \alpha_1 + \alpha_2| S(t) + |\alpha_2 + e(t)| S(t), \forall t \geq t_0 \]
and since
\[ [\alpha_2 + e(t)] S(t) \geq 0, \forall t \geq t_0 \]
we obtain
\[
\frac{d\mathbf{S}(t)}{dt} \geq \lambda - \{\mu + \alpha_1 + \alpha_2\}\mathbf{S}(t), \quad \forall t \geq t_0.
\]

Hence
\[
\mathbf{S}(t) \geq e^{-(\mu+\alpha_1+\alpha_2)(t-t_0)}\mathbf{S}_0 + 1 - e^{-(\mu+\alpha_1+\alpha_2)(t-t_0)}, \quad \forall t \geq t_0
\]
and we deduce that
\[
S_k(t) \geq \min \left( S_{k0}, \frac{\lambda - \mu}{\mu + \alpha_1 + \alpha_2} \right), \quad \forall t \geq t_0, \quad k = 1, 2.
\]
The density of the set of initial conditions giving classical solutions \cite{15} allows us to conclude to the following lemma.

**Lemma 3.2.** Let Assumption 2.1 be satisfied. Let \( \mathbf{S}_0 \in \mathbb{R}^2 \) and \( \mathbf{i}_0 \in L^1_+(\mathbb{R}_+)^2 \) be given and define
\[
\begin{pmatrix}
\mathbf{S}(t) \\
\mathbf{i}(t, \cdot)
\end{pmatrix} = \mathbf{U}(t, t_0) \begin{pmatrix}
\mathbf{S}_0 \\
\mathbf{i}_0
\end{pmatrix}, \quad \forall t \geq t_0.
\]

Then we have
\[
S_k(t) \geq \min \left( S_{k0}, \frac{\lambda - \mu}{\mu + \alpha_1 + \alpha_2} \right), \quad \forall t \geq t_0, \quad k = 1, 2
\]
where the constants \( \alpha_k, \quad k = 1, 2 \) are defined respectively in (32) and (34). We end this section by the following lemma.

**Lemma 3.3.** Let Assumption 2.1 be satisfied. Let \( (\mathbf{S}_0, \mathbf{i}_0) \in \mathbb{R}^2 \times L^1_+(\mathbb{R}_+, \mathbb{R}^2) \) be given and define
\[
\begin{pmatrix}
\mathbf{S}(t) \\
\mathbf{i}(t, \cdot)
\end{pmatrix} = \mathbf{U}(t, t_0) \begin{pmatrix}
\mathbf{S}_0 \\
\mathbf{i}_0
\end{pmatrix}, \quad \forall t \geq t_0.
\]

Then for each \( t \geq t_0 \) the following estimate hold
\[
\left\| \int_0^{t_0} \mathbf{b}(a)\mathbf{i}(t, a) da \right\| 
\leq 2e^{-\mu(t-t_0) + \gamma} \| \mathbf{b} \|_{\infty} \int_0^{t_0} \mathbf{i}_0(a) da + \int_0^{t_0} \mathbf{i}_02(a) da
\]
with
\[
\gamma := \max \left( \| \mathbf{i}_0 \|_{L^1} + \| \mathbf{S}_0 \|, \frac{2\lambda}{\mu} \right).
\]

**Proof.** Define
\[
u(t, t_0) = \int_0^{t_0} \mathbf{b}(a)\mathbf{i}(t + t_0, a) da, \quad \forall t \geq 0 \Leftrightarrow \nu(t - t_0, t_0)
\]
and observe that
\[
\mathbf{B}_f(t, t_0; \mathbf{S}_0, \mathbf{i}_0) = \text{diag}(\mathbf{B}_S(t, t_0; \mathbf{S}_0, \mathbf{i}_0))\mathbf{u}(t, t_0), \quad \forall t \geq 0.
\]
Then by using (6) we have
\[
\mathbf{u}(t, t_0) = \mathbf{G}(t, t_0; \varphi) + \int_0^{t} \mathbf{S}(t \to a) \text{diag}(\mathbf{B}_S(t \to a, t_0; \mathbf{S}_0, \mathbf{i}_0))\mathbf{u}(t \to a, t_0) da, \quad \forall t, t_0 \geq 0.
\]
Using Lemma 3.1 we obtain
\[
\|u(t, t_0)\| \leq \|G(t, t_0; i_0)\| + \int_0^t \|b\|_\infty e^{-\mu a} \|u(t-a, t_0)\| da, \ \forall t, t_0 \geq 0
\]
with
\[
\hat{\gamma} := \max \left( \|i_0\|_{L^1} + \|s_0\|, \frac{2\lambda_+}{\mu} \right).
\]
The result follows by using Gronwall’s inequality, Lemma 2.4 and (37).

4. The disease-free periodic solution. In this section we will give a complete
description of the disease free periodic solution as well as its relationship with the
mild solutions of (4). The disease-free periodic solution is given by
\[
The foregoing ordinary differential equation (39) generates an evolution family
\[
\{E(t, t_0)\}_{t \geq t_0} \text{ on } \mathbb{R}^2.
\]
In particular for each \(t_0 \in \mathbb{R}\) and each \(x_0 \in \mathbb{R}^2\) we have
\[
x(t) = E(t, t_0)x_0, \ \forall t \geq t_0.
\]
Furthermore since the off-diagonal entries of \(E(t)\) are non negative we deduce
that \(\{E(t, t_0)\}_{t \geq t_0}\) is a non singular linear evolution family with
\[
E(t, t_0)\mathbb{R}^2_+ \subset \mathbb{R}^2_+, \ \forall t \geq t_0.
\]
Since
\[
\|E(t, t_0)x_0\| = \langle 1, E(t, t_0)x_0 \rangle, \ \forall t \geq t_0, \ \forall x_0 \in \mathbb{R}^2_+
\]
we obtain from (39) and the fact that \(\langle 1, e(t)E(t, t_0)x_0 \rangle = 0\) that
\[
\frac{d\|E(t, t_0)x_0\|}{dt} = -\mu\|E(t, t_0)x_0\|, \ \forall t \geq t_0, \ \forall x_0 \in \mathbb{R}^2_+
\]
which implies that
\[
\|E(t, t_0)x_0\| = e^{-\mu(t-t_0)}\|E(t_0, t_0)x_0\| = e^{-\mu(t-t_0)}\|x_0\|, \ \forall t \geq t_0, \ \forall x_0 \in \mathbb{R}^2_+.
\]

Lemma 4.1. Let Assumption 2.1 be satisfied. Then the disease-free periodic solution is given by
\[
\mathbf{S}(t) = \int_{-\infty}^t E(t, s)\Lambda(s)ds, \ \forall t \in \mathbb{R}
\]
and for each \(c_2 \geq 0\) we have
\[
\mathbf{S}(t) = \int_{-\infty}^t e^{-c_2(t-s)}E(t, s)\Lambda(s)ds + c_2 \int_{-\infty}^t e^{-c_2(s-t)}E(t, s)\mathbf{S}(s)ds, \ \forall t \in \mathbb{R}.
\]
Moreover we have
\[
0 \leq \frac{\lambda_-}{\mu + \alpha_2} \leq \mathbf{S}(t) \leq \frac{2\lambda_+}{\mu} \leq 1, \ \forall t \in \mathbb{R}
\]
Proof. Formulas \((41)\) and \((42)\) will be proved simultaneously since \((41)\) is a particular case of \((43)\) with \(c_2 = 0\). Let \(c_2 \geq 0\) be given. Then
\[
\frac{d\mathbf{S}(t)}{dt} = \Lambda(t) - \mu \mathbf{S}(t) + e(t)\mathbf{S}(t) - c_2 \mathbf{S}(t) + c_2 \mathbf{S}(t), \quad t \in \mathbb{R}.
\]
(44)

By using the variation of constants formula we have
\[
\mathbf{S}(t) = e^{-c_2(t-t_0)} \mathbf{E}(t, t_0)\mathbf{S}(t_0) + \int_{t_0}^{t} e^{-c_2(t-s)} \mathbf{E}(t, s) \Lambda(s) ds + c_2\int_{t_0}^{t} e^{-c_2(t-s)} \mathbf{E}(t, s) \mathbf{S}(s) ds, \quad \forall t \geq t_0.
\]
(45)

By \((40)\) and the uniform boundedness of \(t \to \mathbf{S}(t)\) it is easy to see that the integrals
\[
\int_{-\infty}^{t} e^{-c_2(t-s)} \mathbf{E}(t, s) \Lambda(s) ds = \lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{-c_2(t-s)} \mathbf{E}(t, s) \Lambda(s) ds
\]
and
\[
\int_{-\infty}^{t} e^{-c_2(t-s)} \mathbf{E}(t, s) \mathbf{S}(s) ds = \lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{-c_2(t-s)} \mathbf{E}(t, s) \mathbf{S}(s) ds
\]
exist for each \(t \in \mathbb{R}\) and
\[
\|e^{-c_2(t-t_0)} \mathbf{E}(t, t_0)\mathbf{S}(t_0)\| \leq e^{-(\mu+c_2)(t-t_0)} \sup_{t \in \mathbb{R}} \|\mathbf{S}(t)\|, \quad \forall t \geq t_0.
\]
The equality \((42)\) follows by letting \(t_0 \to -\infty\) in \((45)\). In order to prove \((43)\) we start with some observations. Let \(\alpha_2\) be defined in \((34)\). Let \(t_0 \in \mathbb{R}\) be given. Define for all \(t \geq t_0\)
\[
x_1(t) = \lambda_1 \mathbf{E}(t, t_0)1 \quad \text{and} \quad x_2(t) = e^{-(\mu+\alpha_2)(t-t_0)} \lambda_1 1.
\]
Then the maps \(t \to x_1(t)\) and \(t \to x_2(t)\) satisfy the following ordinary differential equations
\[
x_1(t) = -\mu x_1(t) + e(t)x_1(t), \quad t > t_0, \quad x_1(t_0) = \lambda_1 1
\]
and
\[
x_2(t) = -\mu x_2(t) - \alpha_2 x_2(t), \quad t > t_0, \quad x_2(t_0) = \lambda_1 1.
\]
Since by the choice of \(\alpha_2\) we have
\[
-\mu x + e(t)x \geq -\mu x - \alpha_2 x, \quad \forall x \in \mathbb{R}^2_+, \quad \forall t \in \mathbb{R}
\]
it follows that
\[
\lambda_1 \mathbf{E}(t, t_0)1 \geq e^{-(\mu+\alpha_2)(t-t_0)} \lambda_1 1, \quad \forall t \geq t_0.
\]
(46)

Let us now proceed to the proof of \((43)\). The right hand side of \((43)\) follows by simple computations. To prove the left hand side of \((43)\) note that
\[
\frac{d\mathbf{S}(t)}{dt} = \Lambda(t) - (\mu + \alpha_2) \mathbf{S}(t) + (\alpha + e(t)) \mathbf{S}(t), \quad t \in \mathbb{R}
\]
and by the variation of constants formula we have for all \(t \geq 0\)
\[
\mathbf{S}(t) = e^{-(\mu+\alpha_2) t} \mathbf{S}(0) + \int_{0}^{t} e^{-(\mu+\alpha_2)(t-s)} (\Lambda(s) + (\alpha + e(s)) \mathbf{S}(s)) ds
\]
\[
\geq e^{-(\mu+\alpha_2) t} \mathbf{S}(0) + \int_{0}^{t} e^{-(\mu+\alpha_2)(t-s)} \lambda_1 1 ds
\]
\[
\geq e^{-(\mu+\alpha_2) t} \mathbf{S}(0) + \left(1 - e^{-(\mu+\alpha_2) t}\right) \frac{1}{\mu + \alpha_2} \lambda_1 1.
\]
Next observe that
\[ S(0) = \int_{-\infty}^{0} E(0, s) \Lambda(s) ds \geq \int_{-\infty}^{0} \lambda_- E(0, s) 1 ds \]
and
\[ \frac{\lambda_-}{\mu + \alpha_2} 1 = \int_{-\infty}^{0} e^{-(\mu + \alpha_2)(0-s)} \lambda_- 1 ds \]
so we deduce from (46) that
\[ \int_{-\infty}^{0} \lambda_- E(0, s) 1 ds \geq \int_{-\infty}^{0} e^{-(\mu + \alpha_2)(0-s)} \lambda_- 1 ds = \frac{\lambda_-}{\mu + \alpha_2} 1 \]
Finally we have for each \( t \geq 0 \)
\[ S(t) \geq e^{-(\mu + \alpha_2)t} \int_{-\infty}^{0} E(0, s) \Lambda(s) ds + \left(1 - e^{-(\mu + \alpha_2)t}\right) \frac{\lambda_-}{\mu + \alpha_2} 1 \]
\[ \geq e^{-(\mu + \alpha_2)t} \frac{\lambda_-}{\mu + \alpha_2} 1 + \left(1 - e^{-(\mu + \alpha_2)t}\right) \frac{\lambda_-}{\mu + \alpha_2} 1 \]
\[ \geq \frac{\lambda_-}{\mu + \alpha_2} 1 \]
and the result follows by the periodicity of \( t \to S(t) \).

The next lemmas will play an important role in the asymptotic analysis of (4).

**Lemma 4.2.** Let Assumption 2.1 be satisfied. Let \( (S_0, i_0) \in \mathbb{R}^2 \times L^1_+(\mathbb{R}_+, \mathbb{R}^2) \) be given and define

\[ \left( \begin{array}{c} S(t) \\ i(t, \cdot) \end{array} \right) = U(t, t_0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right), \forall t \geq t_0. \]

Then for any \( \epsilon > 0 \) there exists \( t_1 := t_1(\epsilon, S_0) \geq t_0 \) such that

\[ S(t) \leq \overline{S}(t) + \epsilon 1, \forall t \geq t_1. \tag{47} \]

Moreover if
\[ S_0 = S(t_0) \leq \overline{S}(t_0) + \epsilon_0 1 \]
for some \( \epsilon_0 \geq 0 \) then
\[ S(t) \leq \overline{S}(t) + 2\epsilon_0 1, \forall t \geq t_0. \]

**Proof.** Using the variation of constants formula the \( S \)-equation of (4) solves as
\[ S(t) = E(t, t_0)S(t_0) + \int_{t_0}^{t} E(t, s)\Lambda(s) ds - \int_{t_0}^{t} E(t, s)\text{diag}(S(s)) \int_{0}^{+\infty} b(a)i(s, a) da ds \]
for all \( t \geq t_0 \). Hence
\[ S(t) \leq E(t, t_0)S(t_0) + \int_{t_0}^{t} E(t, s)\Lambda(s) ds, \forall t \geq t_0 \]
providing that
\[ S(t) \leq E(t, t_0)S_0 + \int_{-\infty}^{t} E(t, s)\Lambda(s) ds \leq E(t, t_0)S_0 + \overline{S}(t), \forall t \geq t_0. \]

Since
\[ \lim_{t \to +\infty} \|E(t, t_0)S_0\| \leq \lim_{t \to +\infty} e^{-\mu(t-t_0)}\|S_0\| = 0 \]
the inequality (47) follows for \( t \geq t_0 \) large enough. Finally if
\[
S(t_0) \leq \bar{S}(t_0) + \epsilon_0 1 = \int_{-\infty}^{t_0} E(t_0, s)\Lambda(s)ds + \epsilon_0 1
\]
for some \( \epsilon_0 \geq 0 \) then by using (48) we obtain that for each \( t \geq t_0 \)
\[
S(t) \leq E(t, t_0) \int_{-\infty}^{t_0} E(t_0, s)\Lambda(s)ds + \epsilon_0 E(t, t_0) 1 + \int_{t_0}^{t} E(t, s)\Lambda(s)ds
\]
\[
\leq -\int_{-\infty}^{t_0} E(t, s)\Lambda(s)ds + \epsilon_0 E(t, t_0) 1
\]
\[
\leq \bar{S}(t) + \epsilon_0 E(t, t_0) 1
\]
\[
\leq \bar{S}(t) + 2\epsilon_0 1.
\]
\[
\square
\]

The next lemma shows that the number of susceptible of each group becomes positive even if we start with zero susceptible in both groups. Indeed this due to the recruitment rate \( t \to \Lambda(t) \) which is assumed to satisfy \( \Lambda(t) \gg 0_{\mathbb{R}^2} \) for all \( t \in \mathbb{R} \).

**Lemma 4.3.** Let Assumption 2.1 be satisfied. Let \((S_0, i_0) \in \mathbb{R}^2 \times L^1_+ (\mathbb{R}^+, \mathbb{R}^2)\) be given and define
\[
\left( \begin{array}{c} S(t) \\ i(t, \cdot) \end{array} \right) = U(t, t_0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right), \ \forall t \geq t_0.
\]

Then there exists \( t_1 \geq t_0 \) such that
\[
S(t) \gg 0, \ \forall t \geq t_1.
\]

**Proof.** First note that Lemma 3.1 ensures that
\[
\|S(t)\| + \|i(t, \cdot)\|_{L^1} \leq \max \left( \|S(0)\| + \|i_0\|_{L^1}, \frac{2\lambda_+}{\mu} \right), \ \forall t \geq t_0.
\]

Then there exists a constant \( c > 0 \) large enough such that
\[
c1 - \int_{0}^{+\infty} b(a)i(t, a)da \geq 0_{\mathbb{R}^2}, \ \forall t \geq t_0.
\]

Since
\[
\text{diag}(S(t)) \int_{0}^{+\infty} b(a)i(t, a)da = \text{diag} \left( \int_{0}^{+\infty} b(a)i(t, a)da \right) S(t), \ \forall t \geq t_0
\]
one obtains that
\[
\frac{dS(t)}{dt} \geq \Lambda(t) - \mu S(t) + e(t)S(t) - eS(t), \ \forall t > t_0.
\]

Hence using comparison principle for ordinary differential equations and a variation of constants formula we obtain
\[
S(t) \geq e^{c(t-t_0)} E(t, t_0)S(t_0) + \int_{t_0}^{t} e^{c(t-s)} E(t, s)\Lambda(s)ds
\]
\[
\geq \lambda_- \int_{t_0}^{t} e^{c(t-s)} E(t, s)1ds \gg 0_{\mathbb{R}^2}, \ \forall t > t_0.
\]

The result follows. \[
\square
\]
Lemma 4.4. Let Assumption 2.1 be satisfied. Let \((S_0, i_0) \in \mathbb{R}^2 \times L^1_+(\mathbb{R}^+, \mathbb{R}^2)\) be given and define
\[
\begin{pmatrix}
S(t) \\
i(t, \cdot)
\end{pmatrix} = U(t, t_0) \begin{pmatrix}
S_0 \\
i_0
\end{pmatrix}, \quad \forall t \geq t_0.
\]
Assume that there exists \(c > 0\) such that
\[
0_{\mathbb{R}^2} \leq \int_0^{+\infty} b(a)i(t, a)da \leq c1, \quad \forall t \geq t_0
\]
then there exists \(t_1 := t_1(S_0, c) \geq t_0\) such that
\[
S(t) \geq \bar{S}(t) - c\frac{5\lambda_+}{\mu}1, \quad \forall t \geq t_1.
\]
Proof. Observing that
\[
\text{diag}(S(t)) \int_0^{+\infty} b(a)i(t, a)da = \text{diag} \left( \int_0^{+\infty} b(a)i(t, a)da \right) S(t), \quad \forall t \geq t_0
\]
we infer from (49) that
\[
cS(t) - \text{diag} \left( \int_0^{+\infty} b(a)i(t, a)da \right) S(t) \geq 0_{\mathbb{R}^2} \forall t \geq t_0.
\]
Hence using the \(S\)-equation of (4) one has
\[
\frac{dS(t)}{dt} \geq \Lambda(t) - \mu S(t) + e(t)S(t) - cS(t), \quad t > t_0
\]
and by using comparison principle for ordinary differential equations and Variation of constants formula we obtain
\[
S(t) \geq e^{-c(t-t_0)}E(t, t_0)S_0 + \int_{t_0}^t e^{-c(t-s)}E(t, s)\Lambda(s)ds, \quad \forall t \geq t_0.
\]
Thus noting that from (45) we have
\[
\int_{t_0}^t e^{-c(t-s)}E(t, s)\Lambda(s)ds = \bar{S}(t) - e^{-c(t-t_0)}E(t, t_0)\bar{S}(t_0) - c\int_{t_0}^t e^{-c(t-s)}E(t, s)\bar{S}(s)ds, \quad \forall t \geq t_0
\]
we obtain for each \(t \geq t_0\)
\[
\begin{align*}
S(t) & \geq e^{-c(t-t_0)}E(t, t_0)S_0 + S(t) - e^{-c(t-t_0)}E(t, t_0)S(t_0) \\
& \quad - c\int_{t_0}^t e^{-c(t-s)}E(t, s)\bar{S}(s)ds \\
& \geq e^{-c(t-t_0)}E(t, t_0)[S_0 - \bar{S}(t_0)] + \bar{S}(t) - c\int_{t_0}^t e^{-c(t-s)}E(t, s)\bar{S}(s)ds
\end{align*}
\]
Next we note that
\[
c\int_{t_0}^t e^{-c(t-s)}E(t, s)\bar{S}(s)ds = c\int_{t_0}^t e^{-(c+\mu)(t-s)}ds \sup_{t \in \mathbb{R}} \|\tilde{S}(t)\| \leq \frac{c}{c+\mu} \sup_{t \in \mathbb{R}} \|\bar{S}(t)\|
\]
for all \(t \geq t_0\) and by (43) combined with (51) we obtain
\[
c\int_{t_0}^t e^{-c(t-s)}E(t, s)\bar{S}(s)ds \leq \frac{4\lambda_+}{\mu}, \quad \forall t \geq t_0.
\]
The inequality (52) implies that
\[ S(t) \geq e^{-c(t-t_0)}E(t,t_0)[S_0 - \overline{S}(t_0)] + \overline{S}(t) - c\frac{4\lambda_1}{\mu}1, \ \forall t \geq t_0 \]
and since
\[ \lim_{t \to +\infty} \| e^{-c(t-t_0)}E(t,t_0)[S_0 - \overline{S}(t_0)] \| \leq \lim_{t \to +\infty} e^{-(c+\mu)(t-t_0)}\|S_0 - \overline{S}(t_0)\| = 0 \] (53)
the result follows. \[ \square \]

A consequence of Lemma 4.4 is the following

**Corollary 2.** Let Assumption 2.1 be satisfied. Let \((S_0, i_0) \in \mathbb{R}^2 \times L^1_+(\mathbb{R}_+\times \mathbb{R}^2)\) be given and define
\[
\begin{pmatrix}
S(t) \\
i(t, \cdot)
\end{pmatrix} = U(t, t_0) \begin{pmatrix}
S_0 \\
i_0
\end{pmatrix}, \ \forall t \geq t_0.
\]
Assume that there exists \(c_0 > 0\) such that for each \(t \geq t_0\)
\[ 0 \leq \int_0^{a^*} i_1(t, a) da \leq c_0 \quad \text{and} \quad 0 \leq \int_0^{a^*} i_2(t, a) da \leq c_0 \] (54)
then there exists \(t_1 := t_1(S_0, c_0) \geq t_0\) such that
\[ S(t) \geq \overline{S}(t) - c_0\|b\|_{\infty}\frac{10\lambda_1}{\mu}1, \ \forall t \geq t_1. \]

**Proof.** It is easy to see that
\[
\left| \int_0^{+\infty} b(a)i(t, a) da \right| \leq \|b\|_{\infty}\left( \int_0^{a^*} i_1(t, a) da + \int_0^{a^*} i_2(t, a) da \right) \leq \|b\|_{\infty}2c_0, \ \forall t \geq t_0,
\]
and we apply Lemma 4.4 with \(c = \|b\|_{\infty}2c_0. \] \[ \square \]

5. **The perturbed linear problem.** In this section we will study a perturbed linear problem that will allow us to use comparison principles in order to prove the asymptotic properties of (4). From now on we fix
\[ \eta = \frac{\lambda_-}{2(\mu + \alpha_2)} \] (56)
then from Lemma 4.1 we have
\[ \overline{S}(t) + \epsilon 1 \geq \frac{\lambda_-}{2(\mu + \alpha_2)}1 \geq 0, \ \forall t \in \mathbb{R}, \ \epsilon \geq -\eta. \] (57)
We consider for each \(\epsilon \geq -\eta\) the following linear equation
\[
\begin{cases}
\frac{\partial w_\epsilon(t, a)}{\partial t} + \frac{\partial w_\epsilon(t, a)}{\partial a} = -(\mu + d(a))w_\epsilon(t, a) + k(t)w_\epsilon(t, a), \\
w_\epsilon(t, 0) = \text{diag}(\overline{S}(t) + \epsilon 1) \int_0^{+\infty} b(a)w_\epsilon(t, a) da, \\
w_\epsilon(t_0, \cdot) = \varphi \in L^1(\mathbb{R}_+\times \mathbb{R}^2).
\end{cases}
\] (58)

**Remark 2.** Note that for \(\epsilon = 0\), system (58) coincide the linearised i-equation of (4) around the disease free periodic solution. The foregoing system (58) will play an important role in the asymptotic analysis.
Hence by using the Volterra integral equation formulation as in Section 2 one can easily obtain that system (58) generates a periodic linear evolution family \( \{ W_{\varepsilon}(t, t_0) \}_{t \geq t_0} \) defined from \( L^2_+(\mathbb{R}_+, \mathbb{R})^2 \) into itself by
\[
W_{\varepsilon}(t, t_0)(\varphi)(a) = \left\{ \begin{array}{ll}
L(a; t-a, 0)B_\varepsilon(t-t_0-a, t_0; \varphi) & \text{if } 0 \leq a < t-t_0 \\
L(t-t_0; a + t_0 - t)\varphi(a - t_0) & \text{if } 0 \leq t - t_0 \leq a.
\end{array} \right.
\]
(59)

where \( t \to B_\varepsilon(t, t_0; \varphi) \) is the unique continuous function on \( [0, +\infty) \) satisfying the following Volterra integral equation
\[
B_\varepsilon(t, t_0; \varphi) = \text{diag}(S(t + t_0) + \varepsilon 1) \left[ G(t, t_0; \varphi) + \int_0^t \Psi(t + t_0, a)B_\varepsilon(t-a, t_0; \varphi)da \right]
\]
(60)
where \( G \) and \( \Psi \) are defined respectively in (7) and (8). Let \( C_\tau(\mathbb{R}, \mathbb{R}_+^2) \) denotes the space of continuous \( \tau \)-periodic functions on \( \mathbb{R} \). Define for all \( \lambda \in \mathbb{R} \) the linear operator \( L^\lambda_\varepsilon : C_\tau(\mathbb{R}, \mathbb{R}_+^2) \to C_\tau(\mathbb{R}, \mathbb{R}_+^2) \) by
\[
(L^\lambda_\varepsilon \phi)(t) := \int_0^{+\infty} e^{-\lambda a} \tilde{\Psi}_\varepsilon(t, a)\phi(t-a)da, \quad \forall t \in \mathbb{R}, \quad \forall \phi \in C_\tau(\mathbb{R}, \mathbb{R}_+^2)
\]
(61)
with
\[
\tilde{\Psi}_\varepsilon(t, a) = \text{diag}(S(t) + \varepsilon 1)\Psi(t, a), \quad \forall t \in \mathbb{R}, \quad a \geq 0, \quad \forall \varepsilon \geq -\eta
\]
(62)
and \( \Psi \) defined in (7). It is clear from Lemma 2.3 that the family of linear operators defined in (61) are well defined for all \( \lambda \in \mathbb{R} \) and for each \( t \geq 0 \) we have
\[
\tilde{\Psi}_\varepsilon(t, a) = 0, \quad \forall a \geq a^*.
\]
(63)
In order to use the results in [29] we will verify some conditions in addition to (63). The family of kernels \( \{ \tilde{\Psi}_\varepsilon(t, a) : t \in \mathbb{R}, a \geq 0 \} \) is compact (then power compact) (See [2, 29]) for each \( \varepsilon \geq -\eta \) and satisfies
\[
\tilde{\Psi}_\varepsilon(t, a) \in \mathbb{R}_+^2, \quad \forall t \in \mathbb{R}, \quad \forall a \geq 0.
\]
Next we prove that for each fixed \( t_0 \geq 0 \) the family of periodic kernels \( \{ \tilde{\Psi}_\varepsilon(t + t_0, a) : t \in \mathbb{R}, a \geq 0 \} \) is \( \mu \)-positive (See Proposition 1) for any \( \mu \geq -\eta \). Before proceeding we prove the following lemma.

**Lemma 5.1.** Assume that \( a^* - a_* > 0 \) and \( \tau > 0 \). Then there exist two positive integers \( k_0 \geq 1 \) and \( k \geq 1 \) such that
\[
(2k + 1)a_* + \tau < 2ka^* \quad \text{and} \quad [k_0\tau, (k_0 + 2)\tau] \subset ((2k + 1)a_*, \tau, 2ka^*).
\]

**Proof.** Let \( k_1 \geq 1 \) be a fixed an integer large enough such that
\[
\tau < k_1(a^* - a_*) \iff a^* > a_* + \frac{\tau}{k_1}
\]
(64)
and \( n \geq 1 \) an integer large enough such that
\[
a_* + \tau < 2nk_1(a^* - a_*) \iff (2nk_1 + 1)a_* + \tau < 2nk_1a^*
\]
(65)
and
\[
2n\tau > a_* + 4\tau.
\]
(66)
Let \( k_2 \geq 0 \) be the unique integer such that
\[
(2nk_1 + 1)a_* + \tau = k_2\tau + r, \quad r \in [0, \tau).
\]
(67)
Then by setting \( k_0 = k_2 + 1 \) we obtain that
\[
(2nk_1 + 1)a_* + \tau < k_0\tau = (k_2 + 1)\tau.
\]
(68)
Next we prove that $2nk_1a^* > (k_0 + 2)\tau = (k_2 + 3)\tau$. Indeed by using (64) and (67) we have

$$2nk_1a^* > 2nk_1\left(a_* + \frac{\tau}{k_1}\right) = 2nk_1a_* + 2n\tau = k_2\tau + r - a_* - \tau + 2n\tau = (k_2 + 3)\tau + r - a_* - 4\tau + 2n\tau$$

and we deduce from (66) that

$$2nk_1a^* > (k_2 + 3)\tau = (k_2 + 2)\tau. \quad (69)$$

The result follows from (68) and (69) by setting $k = nk_1$. \hfill \Box

**Proposition 1.** Let Assumption 2.1 be satisfied. Let $w \in C_r(\mathbb{R}, \mathbb{R}_+^2)$ with $w \gg 0_{\mathbb{R}_2}$. Let $t_0 \geq 0$ be given. Then the following holds:

i) If $\phi, \bar{\phi} \in C([0, +\infty), \mathbb{R}_+^2)$, $\bar{\phi}(t_1) > 0_{\mathbb{R}_2}$ for some $t_1 \in [0, \tau]$ and

$$\phi(t) = \int_0^t \bar{\Psi}_e(t + t_0, a)\phi(t - a)da + \phi(t), \forall t \geq 0, \epsilon \geq -\eta$$

then there exists $\zeta > 0$ independent of $\epsilon$ and $t_0$ such that

$$\phi(t) \geq \zeta w(t), \forall t \in [k_0\tau, (k_0 + 2)\tau] \quad (70)$$

with $k_0$ given in Lemma 5.1. Moreover there exists $b > 0$ depending only on $\beta_{12}$, $\beta_{21}$ and $\bar{\phi}$ such that if $t > b + \tau$ then

$$\|\phi(t)\| > 0.$$ (71)

ii) There exists a constant $c_0 > 0$ such that such that for all $\epsilon \in (-\eta, \eta)$

$$c_0\|x\| \geq \|\bar{\Psi}_e(t + t_0, a)x\|_{w(t)} := \inf\{|c| : c \in \mathbb{R}, -cw(t) \leq \bar{\Psi}_e(t + t_0, a)x \leq cw(t)\},$$

for all $t \in \mathbb{R}$, $t_0 \geq 0$, $a \geq 0$ and $x \in \mathbb{R}_2$.

**Proof.** Proof of i): Set

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \forall t \geq 0 \text{ and } \bar{\phi}(t) = \begin{pmatrix} \bar{\phi}_1(t) \\ \bar{\phi}_2(t) \end{pmatrix}, \forall t \geq 0.$$

First of all note that by Lemma 2.2 combined with (58) we have for all $t \geq 0$

$$\int_0^t \bar{\Psi}_e(t + t_0, a)\phi(t - a)da \geq \text{diag}([\bar{S}(t + t_0) + \epsilon 1]) \int_0^t e^{-(\mu + \gamma)a}b(a)\phi(t - a)da$$

$$\geq \frac{\lambda_0}{2(\mu + \alpha_2)} \int_0^t e^{-(\mu + \gamma)a}b(a)\phi(t - a)da$$

$$\geq \frac{\lambda_0}{2(\mu + \alpha_2)} \int_0^t e^{-(\mu + \gamma)a} \begin{pmatrix} \beta_{12}(a)\phi_2(t - a) \\ \beta_{21}(a)\phi_1(t - a) \end{pmatrix} da. \quad (72)$$

Hence by setting

$$\gamma_0 = \frac{\lambda_0}{2(\mu + \alpha_2)}$$

we obtain the following system

$$\begin{cases} 
\phi_1(t) \geq \gamma_0 \int_0^t e^{-(\mu + \gamma)a}\beta_{12}(a)\phi_2(t - a)da + \bar{\phi}_1(t), \forall t \geq 0 \\
\phi_2(t) \geq \gamma_0 \int_0^t e^{-(\mu + \gamma)a}\beta_{21}(a)\phi_1(t - a)da + \bar{\phi}_2(t), \forall t \geq 0.
\end{cases} \quad (73)$$
Let \( t_1 \in [0, \tau] \) be given such that \( \bar{\phi}(t_1) > 0 \). Then either \( \bar{\phi}_1(t_1) > 0 \) or \( \bar{\phi}_2(t_1) > 0 \). The upcoming arguments for \( \bar{\phi}_1(t_1) > 0 \) or \( \bar{\phi}_2(t_1) > 0 \) are similar so we will only assume for example that \( \bar{\phi}_1(t_1) > 0 \). Then by continuity there exists \( t_3 > t_2 \) such that
\[
[t_2, t_3] \subset [0, \tau] \quad \text{and} \quad \bar{\phi}_1(t) > 0, \quad \forall t \in (t_2, t_3).
\]
Hence we have from (73) that
\[
\phi_1(t) \geq \bar{\phi}_1(t) > 0, \quad \forall t \in (t_2, t_3).
\]
Recall that \( a_* := \max(a_1, a_2) \). Note that for all \( t \in (a_* + t_2, a_* + t_3) \) the set \((t - t_3, t - t_2) \cap (a_*, a^*) \neq \emptyset \) have non zero measure and
\[
t - a \in (t_2, t_3), \quad \forall a \in (t - t_3, t - t_2).
\]
Therefore using (73) we get
\[
\phi_2(t) \geq \gamma_0 \int_{t-t_2}^{t-t_3} e^{-(\mu+\gamma) a} \beta_{21}(a) \phi_1(t-a) da > 0, \quad \forall t \in (a_* + t_2, a_* + t_3). \tag{74}
\]
Hence for any \( t \in (2a_* + t_2, 2a_* + t_3) \) we have \((t - a^* - t_3, t - a^* - t_2) \cap (a_*, a^*) \neq \emptyset \) have non zero measure and
\[
t - a \in (a_* + t_2, a_* + t_3), \quad \forall a \in (t - a^* - t_3, t - a^* - t_2)
\]
so that by using (73) and (74) we obtain
\[
\phi_1(t) \geq \gamma_0 \int_{t-a^*-t_2}^{t-a^*-t_3} e^{-(\mu+\gamma) a} \beta_{12}(a) \phi_2(t-a) da > 0, \quad \forall t \in (2a_* + t_2, 2a_* + t_3).
\]
Therefore by induction, we obtain for each \( n \geq 0 \)
\[
\phi_1(t) > 0, \quad \forall t \in (2n a_* + t_2, 2n a_* + t_3)
\]
and
\[
\phi_2(t) > 0, \quad \forall t \in ((2n + 1)a_* + t_2, (2n + 1)a_* + t_3).
\]
Using Lemma 5.1 we know that there exists \( k \geq 1 \) such that
\[
(2k + 1)a_* + \tau < 2ka^* \quad \text{and} \quad [k_0 \tau, (k_0 + 2) \tau) \subset ((2k + 1)a_* + \tau, 2ka^*).
\]
Since we have
\[
((2k + 1)a_* + \tau, 2ka^*) \subset (2ka_* + t_2, 2ka_* + t_3)
\]
and
\[
((2k + 1)a_* + \tau, 2ka^*) \subset ((2k + 1)a_* + t_2, (2k + 1)a_* + t_3)
\]
we conclude that
\[
\phi_1(t) \phi_2(t) > 0, \quad \forall t \in ((2k + 1)a_* + \tau, 2ka^*) \supset [k_0 \tau, (k_0 + 2) \tau]. \tag{75}
\]
The inequality (70) follows by setting
\[
\zeta = \max_{t \in [k_0 \tau, (k_0 + 2) \tau]} \left( \frac{1}{w_1(t) + w_2(t)} \min_{t \in [k_0 \tau, (k_0 + 2) \tau]} \phi_1(t) \right) \min_{t \in [k_0 \tau, (k_0 + 2) \tau]} \phi_2(t) > 0.
\]
Next we note that
\[
\phi_1(t) + \phi_2(t) \geq \int_0^t \min_\beta (\beta_{12}(a), \beta_{21}(a)) (\phi_1(t-a) + \phi_2(t-a)) + \bar{\phi}_1(t) + \bar{\phi}_2(t), \quad \forall t \geq 0.
\]
Since \(\min(\beta_{12}, \beta_{21})\) is not zero a.e. \([26, \text{Corollary B.6}]\) implies that there exists \(b > 0\) depending only on \(\beta_{12}, \beta_{21}\) and \(\check{\phi}\) such that \(\check{\phi}_1(t) + \check{\phi}_2(t) > 0\) for all \(t > b\) with
\[
\int_0^{t-b} (\check{\phi}_1(a) + \check{\phi}_2(a))da > 0.
\]
Since for all \(\check{\phi}(t_1) > 0\) for some \(t_1 \in [0, \tau]\) and is continuous we have
\[
\int_0^{t-b} (\check{\phi}_1(a) + \check{\phi}_2(a))da = \int_0^{t-b} \|\check{\phi}(a)\|da \geq \int_0^{\tau} \|\check{\phi}(a)\|da > 0, \ \forall t > b + \tau.
\]

**Proof of ii) :** Let
\[
\gamma_1 := \|b\|_{\infty} \sup_{t \in [0, \tau]} (|S(t)| + \eta)
\]
Let \(c_0 > 0\) be large enough such that
\[
0 < \frac{\gamma_1}{c_0} < \min \left( \min_{t \in [0, \tau]} w_1(t), \min_{t \in [0, \tau]} w_2(t) \right).
\]
Let \(x \neq 0_{\mathbb{R}^2}\) be given. Using (62) and Lemma 2.3 we have
\[
\frac{1}{c_0} \left\| \hat{\Psi}_e(t + t_0, a) x \right\| \leq \frac{\gamma_1}{c_0} \forall t \in \mathbb{R}, \ \forall t_0 \geq 0, \ \forall a \geq 0.
\]
Since we have used the sum norm it is clear that
\[
-w(t) \leq -\frac{\gamma_1}{c_0} \leq \frac{1}{c_0 \|x\|} \hat{\Psi}_e(t + t_0, a) x \leq \frac{\gamma_1}{c_0} \leq w(t), \ \forall t \in \mathbb{R}, \ \forall t_0 \geq 0, \ \forall a \geq 0
\]
and we deduce that
\[
-c_0 w(t) \leq \frac{1}{\|x\|} \hat{\Psi}_e(t + t_0, a) x \leq c_0 w(t), \ \forall t \in \mathbb{R}, \ \forall t_0 \geq 0, \ \forall a \geq 0.
\]
Hence
\[
c_0 \geq \left\| \frac{1}{\|x\|} \hat{\Psi}_e(t + t_0, a) x \right\|_{w(t)}, \ \forall t \in \mathbb{R}, \ \forall t_0 \geq 0, \ \forall a \geq 0
\]
and since \(\| \cdot \|_{w(t)}\) is a norm we obtain
\[
c_0 \|x\| \geq \left\| \hat{\Psi}_e(t + t_0, a) x \right\|_{w(t)}, \ \forall t \in \mathbb{R}, \ \forall t_0 \geq 0, \ \forall a \geq 0.
\]

\[\square\]

In order to make use of [29, Remark 2.4] we note that using (62) and Lemma 2.3 it is straightforward that
\[
\|\hat{\Psi}_e(t + t_0, a) - \hat{\Psi}_0(t + t_0, a)\| = \epsilon \|\Psi(t, a)\| \leq \epsilon \|b\|_{\infty}, \ \forall t \in \mathbb{R}, \ t_0 \geq 0, \ a \geq 0, \ \forall \epsilon \geq -\eta
\]
which provides that
\[
\lim_{\epsilon \to 0} \hat{\Psi}_e(t + t_0, a) = \hat{\Psi}_0(t + t_0, a)
\]
uniformly for \(t \in \mathbb{R}, \ t_0 \geq 0, \ a \geq 0\). Before stating the main results of this section let us introduce the metric \(d : \mathbb{R}_+^2 \to \mathbb{R}_+\) as in [29]
\[
d(x, y) := \inf \left\{ \epsilon \in \mathbb{R} : e^{-\epsilon} x \leq y \leq e^\epsilon x \right\}
\]
As a consequence of (62), (76), Proposition 1 and [29, Theorem 2.2, Theorem 2.3] we have the following result.

**Proposition 2.** Let Assumption 2.1 be satisfied. Then for each \(\epsilon \in (-\eta, \eta)\) there exists a unique pair \(\lambda_\epsilon \in \mathbb{R}\) and \(\hat{\phi}_\epsilon \in C_+ (\mathbb{R}, \mathbb{R}_+^2)\) such that the following hold.
Proposition 3. Let Assumption 2.1 be satisfied. Let \( \epsilon \in (-\eta, \eta) \) be given. If \( \lambda_\epsilon > 0 \) then

\[
\lim_{t \to +\infty} \| W_\epsilon(t, t_0)(\varphi) \|_{L^1} = 0, \forall \varphi \in L_+^1(\mathbb{R}_+, \mathbb{R})^2, \forall t_0 \geq 0
\]  

Moreover we also have

\[
\lim_{t \to +\infty} \int_{t_0}^{t} e^{-\mu(s-a)} \| W_\epsilon(s, t_0)(\varphi) \|_{L^1} ds = 0, \forall \varphi \in L_+^1(\mathbb{R}_+, \mathbb{R})^2, \forall t_0 \geq 0.
\]

Proof. Note that by (60) and Remark 3 there exists some constant \( C > 0 \) such that

\[
\| B_\epsilon(t, t_0; \varphi) \| \leq C e^{-\lambda_\epsilon t}, \forall t \geq 0.
\]

Hence

\[
\| W_\epsilon(t, t_0)(\varphi) \|_{L^1} = \int_{t_0}^{t} \| W_\epsilon(t, t_0)(\varphi)(a) \| da + \int_{t_0}^{+\infty} \| W_\epsilon(t, t_0)(\varphi)(a) \| da
\]

\[
= \int_{0}^{t} \| L(a; t + t_0 - a, 0)B_\epsilon(t - a, t_0; \varphi) \| da + \int_{t}^{+\infty} \| L(t; t_0, a - t)\varphi(a - t) \| da
\]

\[
\leq \int_{0}^{t} C e^{-\mu a} e^{-\lambda_\epsilon(t-a)} da + \int_{t}^{+\infty} e^{-\mu t} \| \varphi(a - t) \| da, \forall t \geq t_0.
\]

Therefore we obtain

\[
\| W_\epsilon(t, t_0)(\varphi) \|_{L^1} \leq \begin{cases} 
C e^{-\mu t} - e^{-\lambda_\epsilon t} \left( \frac{\lambda_\epsilon - \mu}{\lambda_\epsilon} \right) + e^{-\mu t} \| \varphi \|_{L^1}, & \forall t \geq t_0 \text{ if } \mu \neq \lambda_\epsilon \\
C t e^{-\mu t} + e^{-\mu t} \| \varphi \|_{L^1}, & \forall t \geq t_0 \text{ if } \mu = \lambda_\epsilon
\end{cases}
\]

and the result follows by simple computations.
In order to deal with the uniform persistence we will use the following sets
\[
\hat{\mathcal{M}}_0 := \left\{ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in L^1_+(\mathbb{R}_+,\mathbb{R})^2 : \int_0^{a^*} \varphi_1(a) da > 0 \text{ or } \int_0^{a^*} \varphi_2(a) da > 0 \right\},
\]
and
\[
\partial \hat{\mathcal{M}}_0 := L^1_+(\mathbb{R}_+,\mathbb{R})^2 \setminus \hat{\mathcal{M}}_0.
\]

**Lemma 5.2.** Let Assumption 2.1 be satisfied. For each \( \epsilon \in (-\eta,\eta) \), each \( \varphi \in \partial \hat{\mathcal{M}}_0 \) and each \( t_0 \geq 0 \) we have
\[
B_\epsilon(t,t_0;\varphi) = \int_0^{\infty} b(a) W_\epsilon(t + t_0,t_0)(\varphi) da = 0_{2\mathbb{R}}, \forall t \geq 0.
\]
Moreover
\[
\| W_\epsilon(t,t_0)(\varphi) \| \leq e^{-\mu(t-t_0)} \| \varphi \|_{L^1}, \forall t \geq t_0.
\]

**Proof.** Recall that for each \( \epsilon \geq -\eta \), each \( \varphi \in \partial \hat{\mathcal{M}}_0 \) and each \( t_0 \geq 0 \) we have
\[
W_\epsilon(t,t_0)(\varphi)(a) = \begin{cases} L(a; t-a,0)B_\epsilon(t-t_0-a,t_0;\varphi) & \text{if } 0 \leq a < t-t_0 \\
L(t-t_0,t_0,a+t_0-t)\varphi(a-t+t_0) & \text{if } 0 \leq t-t_0 \leq a 
\end{cases}
\]
where \( t \to B_\epsilon(t,t_0;\varphi) \) is the unique continuous function on \([0,\infty)\) satisfying the following Volterra integral equation
\[
B_\epsilon(t,t_0;\varphi) = \text{diag}(S(t+t_0)+\epsilon 1) \left[ G(t,t_0;\varphi) + \int_0^t \Psi(t+t_0,a)B_\epsilon(t-a,t_0) da \right], t \geq 0.
\]
Observe that if \( \varphi \in \partial \hat{\mathcal{M}}_0 \) then by Lemma 2.4 we have
\[
G(t,t_0;\varphi) = 0_{2\mathbb{R}}, \forall t \geq 0.
\]
Now using the fact that the matrix \( \text{diag}(S(t)+\epsilon 1) \) is invertible for all \( t \in \mathbb{R} \) and \( \epsilon \geq -\eta \) we get
\[
B_\epsilon(t,t_0;\varphi) = \text{diag}(S(t+t_0)+\epsilon 1) \int_0^t \Psi(t+t_0,a)B_\epsilon(t-a,t_0;\varphi) da, t \geq 0,
\]
and the uniqueness of the solutions provides that
\[
B_\epsilon(t,t_0;\varphi) = 0_{2\mathbb{R}}, \forall t \geq 0.
\]
Hence the result follows easily by combining (84) and (85). \( \square \)

The following proposition will be essential in the proof of our uniform persistence result.

**Proposition 4.** Let Assumption 2.1 be satisfied. For each \( \epsilon \in (-\eta,\eta) \), each \( \varphi \in \hat{\mathcal{M}}_0 \) and each \( t_0 \geq 0 \) there exists \( t_1 > t_0 \) such that
\[
B_\epsilon(t,t_0;\varphi) = \int_0^{\infty} b(a) W_\epsilon(t + t_0,t_0)(\varphi) > 0_{2\mathbb{R}}, \forall t \geq t_1
\]
and
\[
W_\epsilon(t,t_0)(\varphi) \in \hat{\mathcal{M}}_0, \forall t \geq t_0.
\]
Further if \( \lambda_\epsilon < 0 \) then
\[
\lim_{t \to +\infty} \| W_\epsilon(t,t_0)(\varphi) \|_{L^1} = +\infty, \forall \varphi \in \hat{\mathcal{M}}_0, \forall t_0 \geq 0.
\]
Proof. Let \( \epsilon \in (-\eta, \eta) \), \( \varphi \in \hat{M}_0 \) and \( t_0 \geq 0 \) be given. Since
\[
\mathbf{G}(t, t_0; \varphi) = \int_{t_0}^{+\infty} \mathbf{b}(a)\mathbf{L}(t; t_0, a - t)\varphi(a - t)da
\]
it is clear that
\[
\mathbf{G}(t, t_0; \varphi) = 0_{G_2}, \; \forall t \geq a^*. \tag{89}
\]
Also Lemma 2.2 ensures that
\[
\mathbf{G}(t, t_0; \varphi) \geq \int_{t_0}^{+\infty} e^{-(\mu + \gamma)t} \mathbf{b}(a)\varphi(a - t)da, \; \forall t \geq 0
\]
and we deduce that
\[
\mathbf{G}(t, t_0; \varphi) \geq e^{-(\mu + \gamma)t} \int_{0}^{+\infty} \left( \frac{\beta_{12}(a + t)}{\beta_{21}(a + t)} \right) da
\]
\[
\geq e^{-(\mu + \gamma)t} \left( \int_{a^*-t}^{a^*} \beta_{12}(a + t) \varphi_2(a)da \right)
\]
and since
\[
\int_{0}^{a^*} \varphi_1(a) > 0 \quad \text{or} \quad \int_{0}^{a^*} \varphi_2(a) > 0
\]
and \( \text{Supp}(\beta_{21}) = [a_1, a^*] \), \( \text{Supp}(\beta_{12}) = [a_2, a^*] \) there exists \( s \in [0, a^*) \) such that
\[
\mathbf{G}(s, t_0; \varphi) > 0 \Rightarrow \text{diag}(\mathbf{S}(s) + \epsilon\mathbf{1})\mathbf{G}(s, t_0; \varphi) > 0. \tag{90}
\]
Thus by property i) in Proposition 1 there exists \( t_1 > 0 \) large enough such that
\[
\|\mathbf{B}_e(t, t_0; \varphi)\| > 0, \; \forall t \geq t_1. \tag{91}
\]
Condition (87) is a consequence of Lemma 5.2 and (91). Assume that \( \lambda_e < 0 \). Let us now prove (88). To do so we will make use of (91) and Proposition 2. In fact (90) ensures that the parameters \( \alpha_e \), obtained by applying Proposition 2 to (60) is positive \( (\alpha_e > 0) \). Furthermore we have
\[
\lim_{t \to +\infty} d \left( e^{\lambda_e t} \mathbf{B}_e(t, t_0; \varphi), \alpha_e \hat{\varphi}_e(t) \right) = 0
\]
where \( d \) is defined in (77). Let \( \delta_0 > 0 \) be given. Then there exists \( t_2 > t_1 \) large enough such that
\[
0 \leq d \left( e^{\lambda_e t} \mathbf{B}_e(t, t_0; \varphi), \alpha_e \hat{\varphi}_e(t) \right) < \delta_0, \; \forall t \geq t_2 > t_1.
\]
Therefore using the definition of the metric \( d \) in (77) there exists \( t \in [t_2, +\infty) \to \theta(t) \) such that
\[
|\theta(t)| \leq \delta_0, \; \forall t \geq t_2 \tag{92}
\]
and
\[
e^{-\theta(t)}\alpha_e \hat{\varphi}_e(t) \leq e^{\lambda_e t} \mathbf{B}_e(t, t_0; \varphi) \leq e^{\theta(t)}\alpha_e \hat{\varphi}_e(t), \; \forall t \geq t_2 \tag{93}
\]
Now using the periodicity of \( \hat{\varphi}_e \) combined with (91) and (93) we obtain that
\[
\min_{t \geq t_2} \|\hat{\varphi}_e(t)\| > 0. \tag{94}
\]
Furthermore we have from Lemma 2.2 and (93) that for each $t \geq t_2$
\[
\|W(T, t_0)(\varphi)\|_{L^1} \geq \int_0^t \|L(a; t + t_0 - a, 0)B_\epsilon(t - a, t_0; \varphi)\| da
\]
\[
\geq \int_{t-t_2}^t \|L(a; t + t_0 - a, 0)B_\epsilon(t - a, t_0; \varphi)\| da
\]
\[
\geq \int_0^{t-t_2} e^{-(\mu + \gamma)a} \|B_\epsilon(t - a, t_0; \varphi)\| da
\]
\[
\geq \int_0^{t-t_2} e^{-(\mu + \gamma)a} \|B_\epsilon(t - a, t_0; \varphi)\|_{L^1} da
\]
\[
\geq \int_0^{t-t_2} e^{-(\mu + \gamma - \lambda_e)(t-a)} \|B_\epsilon(t - a, t_0; \varphi)\|_{L^1} e^{-\delta} da
\]
and since $-\lambda_e > 0$ we deduce that
\[
\lim_{t \to +\infty} \|W(T, t_0)(\varphi)\| = +\infty.
\]

6. **Global stability of the disease free periodic solution.** In order to study the global stability of the disease free periodic solution we will introduce the basic reproductive number $R_0$ as in [2, 24]. Define the basic reproductive number by
\[
\rho(L_0^0). = R_0 = \rho(L_0^0).
\]

**Theorem 6.1.** Let Assumption 2.1 be satisfied. If $R_0 < 1$ then the disease free periodic solution is globally asymptotically stable.

**Proof.** We will prove the theorem by applying Proposition 2 to (60) and using some comparison principles. Note that from condition iii) in Proposition 2 we have
\[
R_0 = \rho(L_0^0) < 1 = \rho(L_0^{\lambda_e}) \Rightarrow 0 < \lambda_e.
\]

Thus thanks to iv) in Proposition 2 we can fix $\epsilon \in (0, \eta)$ small enough such that $0 < \lambda_e$ and $\rho(L_0^{\lambda_e}) = 1$. Let $S_0 \in \mathbb{R}^2_+$ and $i_0 \in L_1^2(\mathbb{R}_+, \mathbb{R})$ be given and define
\[
\begin{pmatrix}
S(t) \\
i(t, \cdot)
\end{pmatrix} = U(t, t_0) \begin{pmatrix}
S_0 \\
i_0
\end{pmatrix}, \forall t \geq t_0.
\]

Using Lemma 4.2 one knows that there exists $t_1 > t_0$ such that
\[
S(t) \leq S(t) + \epsilon I, \forall t \geq t_1.
\]

Therefore we have
\[
\begin{cases}
\frac{dS(t)}{dt} = \Lambda(t) - \mu S(t) + e(t)S(t) - \text{diag}(S(t)) \int_0^{+\infty} b(a)i(t, a) da, t > t_1 \\
\frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -[\mu + d(a)]i(t, a) + k(t)i(t, a), t > t_1 \\
i(t, 0) \leq \text{diag}(S(t) + \epsilon I) \int_0^{+\infty} b(a)i(t, a) da, t > t_1
\end{cases}
\]

Then using the comparison principal in [17] we have
\[
0 \leq i(t, \cdot) \leq W_\epsilon(t, t_1)(i(t_1, \cdot)), \forall t \geq t_1.
\]

Then according to Proposition 3 we obtain
\[
0 \leq \lim_{t \to +\infty} \|i(t, \cdot)\|_{L^1} \leq \lim_{t \to +\infty} \|W_\epsilon(t, t_1)(i(t_1, \cdot))\|_{L^1} = 0.
\]
To conclude to the global asymptotic stability we will prove that
\[ \lim_{t \to +\infty} \| \mathbf{S}(t) - \mathbf{S}(t) \| = 0. \] (101)

To do so note that
\[ (\mathbf{S}(t) - \mathbf{S}(t))' = -\mu(\mathbf{S}(t) - \mathbf{S}(t)) + \mathbf{e}(t)(\mathbf{S}(t) - \mathbf{S}(t)) + \text{diag}(\mathbf{S}(t)) \int_{0}^{+\infty} \mathbf{b}(a)\mathbf{i}(t,a)\,da \]
for all \( t \geq t_1 \) and by using the variation of constants formula we have
\[ \mathbf{S}(t) - \mathbf{S}(t) = \mathbf{E}(t,t_1)(\mathbf{S}(t_1) - \mathbf{S}(t_1)) + \hat{\mathbf{E}}(t,t_1), \forall t \geq t_1 \]
with
\[ \hat{\mathbf{E}}(t,t_1) = \int_{t_1}^{t} \mathbf{E}(t,s)\text{diag}(\mathbf{S}(s)) \int_{0}^{+\infty} \mathbf{b}(a)\mathbf{i}(s,a)\,da, \forall t \geq t_1. \]

Observe that
\[ \| \mathbf{E}(t,t_1)(\mathbf{S}(t_1) - \mathbf{S}(t_1)) \| \leq e^{-\mu(t-t_1)}\| (\mathbf{S}(t_1) - \mathbf{S}(t_1)) \|, \forall t \geq t_1 \] (102)
and by using (97) and (99) we also have
\[ \| \hat{\mathbf{E}}(t,t_1) \| \leq \sup_{t \in \mathbb{R}} \| \mathbf{S}(s) + \mathbf{e}(1\| \| \mathbf{b}\| \| \int_{t_1}^{t} e^{-\mu(t-s)}\| \mathbf{W}_\epsilon(s,t_1)\mathbf{i}(t_1) \| \|_{L^1} ds \| \] (103)

We can conclude to
\[ \lim_{t \to +\infty} \| \mathbf{S}(t) - \mathbf{S}(t) \| = 0 \]
by using (102) and (103) together with Proposition 3. \( \square \)

7. Uniform persistence. In order to apply the theory of persistence for steady states we introduce the sets
\[ \mathcal{M} := \mathbb{R}_+^2 \times L^1_{+}(\mathbb{R}_+, \mathbb{R})^2, \quad \mathcal{M}_0 := \mathbb{R}_+^2 \times \hat{\mathcal{M}}_0 \] (104)
with \( \hat{\mathcal{M}}_0 \) defined in (80) and
\[ \partial \mathcal{M}_0 := \mathcal{M} \setminus \mathcal{M}_0 = \mathbb{R}_+^2 \times \partial \hat{\mathcal{M}}_0 \] (105)
with \( \partial \hat{\mathcal{M}}_0 \) defined in (81). Define the continuous map \( \xi : \mathbb{R}_+^2 \times L^1_{+}(\mathbb{R}_+, \mathbb{R})^2 \to [0, +\infty) \) by
\[ \xi \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) = \int_{0}^{a^*} i_{01}(a)\,da + \int_{0}^{a^*} i_{02}(a)\,da \] (106)
with \( i_{01}, i_{02} \) the components of \( i_0 \) and observe that we have
\[ \mathcal{M}_0 = \left\{ \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \in \mathbb{R}_+^2 \times L^1_{+}(\mathbb{R}_+, \mathbb{R})^2 : \xi \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) > 0 \right\} . \] (107)
Moreover we also have
\[ \partial \mathcal{M}_0 = \left\{ \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \in \mathbb{R}_+^2 \times L^1_{+}(\mathbb{R}_+, \mathbb{R})^2 : \xi \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) = 0 \right\} . \] (108)

Observe that the family of maps \( \{ \mathbf{U}(t,0) \}_{t \geq 0} \) is a periodic semiflow in the sense defined in [38]. More precisely it satisfies the following properties
i) \( \mathbf{U}(0,0) = I, \)
ii) \( \mathbf{U}(t + \tau, 0) = \mathbf{U}(t + \tau, \tau)\mathbf{U}(\tau, 0) = \mathbf{U}(t, 0)\mathbf{U}(\tau, 0), \) for all \( t \geq 0, \)
iii) \( \left( t, \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \right) \mapsto \mathbf{U}(t, 0) \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \) is continuous in \( \mathbb{R}_+ \times \mathbb{R}_+^2 \times L^1_{+}(\mathbb{R}_+, \mathbb{R})^2. \)

The main result of this section is the following.
Theorem 7.1. Let Assumption 2.1 be satisfied. Assume that $R_0 > 1$. Then $U$ is $\xi$-uniformly persistent with respect to the decomposition $(M_0, \partial M_0)$ in the following sense: There exists $\delta_0 > 0$ such that for each \( \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \in M_0 \)
\[
\liminf_{t \to +\infty} \xi \left( U(t, 0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \right) \geq \delta_0.
\]

The proof of Theorem 7.1 will be decomposed into several intermediate results. We will now split the nonlinear operator $U(t, t_0)$ defined by (11), (6) and (9) into two operators. Firstly we consider the non linear operator defined for all $t \geq t_0$ and \( \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \in \mathbb{R}^2_+ \times L_+^1(\mathbb{R}_+, \mathbb{R})^2 \) by
\[
U_1(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} = \begin{pmatrix} B_2(t, t_0; S_0, i_0) \\ \hat{U}_1(t, t_0)(S_0, i_0) \end{pmatrix}
\]
where we have set
\[
\hat{U}_1(t, t_0)(S_0, i_0)(a) = \begin{cases} L(a; t-a, 0)B_1(t-t_0-a, t_0; S_0, i_0) & \text{if } 0 \leq a < t-t_0 \\ 0_{\mathbb{R}^2} & \text{if } 0 \leq t-t_0 \leq a \\ 0_{\mathbb{R}^2} & \text{if } a < 0 \end{cases}
\]
and $t \in \mathbb{R}_+ \to (B_S(t, t_0; S_0, i_0), B_I(t, t_0; S_0, i_0))$ is the unique continuous solution of the Volterra integral system of equations (9). Secondly we consider the linear operator defined for each $t \geq t_0$, each \( \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \in \mathbb{R}^2_+ \times L_+^1(\mathbb{R}_+, \mathbb{R})^2 \)
\[
U_2(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \hat{U}_2(t, t_0)(S_0, i_0) \end{pmatrix}, \forall t \geq t_0
\]
with
\[
\hat{U}_2(t, t_0)(S_0, i_0)(a) = \begin{cases} 0_{\mathbb{R}^2} & \text{if } -\infty \leq a < t-t_0, \\ L(t-t_0; a, 0)S_0 + (a-t_0)i_0 & \text{if } 0 \leq t-t_0 \leq a. \end{cases}
\]
Observe that with the above notations we have for each \( \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \in \mathbb{R}^2_+ \times L_+^1(\mathbb{R}_+, \mathbb{R})^2 \)
\[
U(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} = U_1(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} + U_2(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix}, \forall t \geq t_0.
\]
The following lemma can be proved by using similar arguments in [34] together with Corollary 1, the continuity of the coefficient of (4) as well as the uniform continuity of the contact rates. The proof may be very long but require any difficulty if we follow the same ideas in [34].

Lemma 7.2. Let Assumption 2.1 be satisfied. Then for all $t > 0$ the operator $U_1(t, 0): \mathbb{R}^2_+ \times L_+^1(\mathbb{R}_+, \mathbb{R})^2 \to \mathbb{R}^2_+ \times L_+^1(\mathbb{R}_+, \mathbb{R})^2$ is compact.

Proposition 5 (Extinction). Let Assumption 2.1 be satisfied. If \( \begin{pmatrix} S_0 \\ i_0 \end{pmatrix} \in \partial M_0 \)
then for each $t_0 \geq 0$ the mild solution
\[
\begin{pmatrix} S(t) \\ i(t, \cdot) \end{pmatrix} = U(t, t_0) \begin{pmatrix} S_0 \\ i_0 \end{pmatrix}, \forall t \geq t_0
\]
satisfies
\[ B_I(t, t_0; S_0, i_0) = \int_0^{+\infty} b(a) i(t, a) da = 0_{\mathbb{R}^2}, \forall t \geq t_0. \]

Moreover the disease free periodic solution is globally asymptotically stable in \( \partial M_0 \).

More precisely we have
\[ \lim_{t \to +\infty} \left\| U(t, t_0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) - \left( \begin{array}{c} S(t) \\ 0_{L^1} \end{array} \right) \right\| = 0, \forall \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \in \partial M_0, \forall t_0 \geq 0. \]

**Proof.** Let \( t_0 \geq 0 \) be given. Define
\[ \left( \begin{array}{c} S(t) \\ i(t, \cdot) \end{array} \right) = U(t, t_0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right), \forall t \geq t_0. \]

Let \( \epsilon_0 > 0 \) be large enough such that \( S(t_0) \leq \overline{S}(t_0) + \frac{1}{2} \epsilon_0 1 \). Then by Lemma 4.2 we have
\[ S(t) \leq \overline{S}(t) + \epsilon_0 1, \forall t \geq t_0. \]

Therefore the map \( t \to i(t, \cdot) \) satisfies
\[
\begin{cases}
\frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -|\mu + d(a)| i(t, a) + k(t) i(t, a), \ t > t_0 \\
i(t, 0) = \text{diag}(S(t) + \epsilon_0 1) \int_0^{+\infty} b(a) i(t, a) da, \ t > t_0.
\end{cases}
\]

Thus by using the comparison principle in [17] we have
\[ i(t, \cdot) \leq W_{\epsilon_0}(t, t_0)(i_0), \forall t \geq t_0. \tag{114} \]

Since \( i_0 \in \partial M_0 \), Lemma 5.2 provides that
\[ \int_0^{+\infty} b(a) W_{\epsilon_0}(t_0, t_0)(i_0) da = 0_{\mathbb{R}^2}, \forall t \geq t_0 \Rightarrow \int_0^{+\infty} b(a) i(t, a) da = 0_{\mathbb{R}^2}, \forall t \geq t_0. \]

Because \( i_0 \in \partial M_0 \), Lemma 5.2 and (114) imply that
\[ \|i(t, \cdot)\|_{L^1} \leq \|W_{\epsilon_0}(t_0, t_0)(i_0)\|_{L^1} \leq e^{-\mu(t-t_0)} \|\varphi\|_{L^1}, \forall t \geq t_0 \Rightarrow \lim_{t \to +\infty} \|i(t, \cdot)\|_{L^1} = 0. \]

By using the same arguments as in the proof of Theorem 6.1 we also have
\[ \lim_{t \to +\infty} \|S(t) - \overline{S}(t)\| = 0. \]

**Remark 4.** Note that in Proposition 5 no condition is made on the basic reproductive number \( R_0 \). More precisely the disease always dies out when the initial conditions are taken into \( \partial M_0 \). In fact this not surprising because if we start with initial conditions in \( \partial M_0 \) the infected individuals in both groups are not capable to infect susceptible individuals even in the future. Therefore no new infected individual will appear in the future and the disease dies out as \( t \) goes to \( +\infty \).

**Lemma 7.3.** Let Assumption 2.1 be satisfied. If \( \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \in M_0 \) then
\[ \left( \begin{array}{c} S(t) \\ i(t, \cdot) \end{array} \right) = U(t, t_0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \in M_0, \forall t \geq t_0. \]
Proof. Let \( t_1 \geq t_0 \) be provided by Lemma 4.3 such that
\[
S(t) \gg 0_{\mathbb{R}^2}, \forall t \geq t_1
\] (115)
and define
\[
x_0 = S(t_1) \quad \text{and} \quad \varphi = i(t_1, \cdot).
\] (116)
Then we have
\[
\begin{pmatrix}
S(t) \\
i(t, \cdot)
\end{pmatrix} = U(t, t_1) \begin{pmatrix}
x_0 \\
\varphi
\end{pmatrix}, \forall t \geq t_1.
\]
Hence the map \( t \to (B_S(t, t_1; x_0, \varphi), B_I(t, t_1; x_0, \varphi)) \) satisfies
\[
B_S(t, t_1; x_0, \varphi) = S(t + t_1), \forall t \geq 0
\] (117)
and
\[
B_I(t, t_1; x_0, \varphi)
= \text{diag}(S(t + t_1)) \begin{bmatrix} G(t, t_1; \varphi) + \int_0^t \Psi(t + t_0, a)B_I(t - a, t_1; x_0, \varphi)da \end{bmatrix},
\] (118)
for all \( t \geq 0 \). Moreover by Lemma 3.2 one has
\[
S_k(t) \geq \min \left( S_k(t_1), \frac{\lambda_-}{\mu + \alpha_1 + \alpha_2} \right) > 0, \forall t \geq t_1, k = 1, 2
\] (119)
where the constants \( \alpha_k, k = 1, 2 \) are defined respectively in (32) and (34) so that
\[
S(t) \geq \gamma_1 1, \forall t \geq t_1
\] (120)
with
\[
\gamma_1 := \min \left( S_1(t_1), S_2(t_1), \frac{\lambda_-}{\mu + \alpha_1 + \alpha_2} \right).
\]
Then using (9), (120) and Lemma 2.2 it is obvious that
\[
B_I(t, t_1; x_0, \varphi) \geq G(t, t_1; \varphi) + \gamma_1 \int_0^t e^{-(\mu + \gamma)a} \begin{pmatrix} 0 & \beta_{12}(a) \\ \beta_{21}(a) & 0 \end{pmatrix} B_I(t - a, t_1; x_0, \varphi)da
\]
for all \( t \geq 0 \) and we deduce that
\[
1^T B_I(t, t_1; x_0, \varphi) \geq \gamma_1 1^T G(t, t_1; \varphi)
+ \gamma_1 \int_0^t e^{-(\mu + \gamma)a} \min(\beta_{12}(a), \beta_{21}(a)) 1^T B_I(t - a, t_1; x_0, \varphi)da
\]
for all \( t \geq 0 \). Using similar argument as in the proof of Proposition 4 we know that there exists \( s \in [0, a^*] \) such that
\[
1^T G(s, t_1; \varphi) > 0
\]
and since \( \min(\beta_{12}(a), \beta_{21}(a)) \) is not zero for almost every \( a \geq 0 \) we infer from [26, Corollary B.6] that there exists \( t_2 > t_1 \) large enough such that
\[
1^T B_I(t, t_1; x_0, \varphi) > 0, \forall t \geq t_2 \Rightarrow B_I(t, t_0; S_0, i_0)
= B_I(t, t_1; x_0, \varphi) > 0_{\mathbb{R}^2}, \forall t \geq t_2 > t_1.
\] (121)
To end the proof we will argue by contradiction. Assume that there exists some \( t_3 > t_0 \) such that
\[
\begin{pmatrix} S(t_3) \\
i(t_3, \cdot)
\end{pmatrix} = U(t_3, t_0) \begin{pmatrix} S_0 \\
i_0
\end{pmatrix} \notin \mathcal{M}_0 \iff \begin{pmatrix} S(t_3) \\
i(t_3, \cdot)
\end{pmatrix} \notin \partial \mathcal{M}_0.
Thus by Proposition 5 we have
\[ B_f(t, t_0; S_0, i_0) = B_f(t, t_3; S(t_3), i(t_3, \cdot)) = 0_{R^2}, \forall t \geq t_3 \]
which contradict (121).

**Lemma 7.4.** Let Assumption 2.1 be satisfied. Assume that there exists \( t_2 > 0 \) such that for each \( t \in (0, t_2) \) and by using the comparison principle in [17] we have
\[ \frac{dS(t)}{dt} = \Lambda(t) - \mu S(t) + e(t)S(t) - \text{diag}(S(t)) \int_0^{+\infty} b(a) i(t, a) da, \quad t > t_1 \]
\[ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -[\mu + d(a)] i(t, a) + k(t) i(t, a), \quad t > t_1 \]
\[ i(t, 0) = \text{diag}(S(t)) - e1, \quad t > t_1 \]
and by using the comparison principle in [17] we have
\[ i(t, \cdot) \geq W_{\varepsilon}(t, t_1)(i(t_1, \cdot)), \quad \forall t \geq t_1. \]
Therefore since \( i(t_1, \cdot) \in \tilde{M}_0 \) one can make use of Proposition 4 and (128) to obtain
\[
\lim_{t \to +\infty} \| W_c(t, t_1)(i(t_1, \cdot)) \|_{L^1} = +\infty \Rightarrow \lim_{t \to +\infty} \| i(t, \cdot) \|_{L^1} = +\infty.
\]

\[
\square
\]

In what follows, we consider the map \( \Theta : \mathbb{R}^2_+ \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \to \mathbb{R}_+^2 \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \) given by
\[
\Theta \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) := U(\tau, 0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \quad \text{and} \quad \Theta^0 \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) := \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \quad \text{for each } n \geq 2
\]
\[
\Theta^n = \underbrace{U(\tau, 0) \circ \cdots \circ U(\tau, 0)}_{n \text{ times}}
\]
\[
= U(n\tau, (n-1)\tau) \circ U((n-1)\tau, (n-2)\tau) \circ \cdots \circ U(\tau, 0)
\]
\[
= U(n\tau, 0).
\]

Thus by using the evolution semiflow properties of \( U \) and its \( \tau \)-periodicity we obtain
\[
\Theta^n \circ \Theta^p = U(n\tau, 0) \circ U(p\tau, 0) = U((n+p)\tau, p\tau) \circ U(p\tau, 0) = U((n+p)\tau, 0) = \Theta^{n+p}
\]
\[
\text{(131)}
\]
so that \( \{\Theta^n\}_{n \geq 0} \) is a discrete time semigroup. Let \( \delta > 0 \) be given and consider the set
\[
\mathcal{B}_\delta = \left\{ (S_0, i_0) \in \mathbb{R}^2_+ \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 : \| S_0 \| + \| i_0 \|_{L^1} \leq \frac{2\lambda_+}{\mu} + \delta \right\}. \tag{132}
\]

As a consequence of (28) and (29) we have the following

**Lemma 7.5.** Let Assumption 2.1 be satisfied. The set \( \mathcal{B}_\delta \) is forward invariant with respect to the discrete time semiflow \( \Theta \) that is to say that
\[
\Theta^n \mathcal{B}_\delta \subseteq \mathcal{B}_\delta, \quad \forall n \in \mathbb{N}.
\]
\[
\text{(133)}
\]
Moreover \( \Theta \) is point dissipative which means that for each \( \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \in \mathbb{R}^2_+ \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \) there exists \( n_0 \geq 0 \) such that
\[
\Theta^n \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \in \mathcal{B}_\delta, \quad \forall n \geq n_0.
\]

Lemma 7.2 will allow us to obtain that the discrete time semigroup \( \Theta \) is asymptotically smooth.

**Lemma 7.6.** Let Assumption 2.1 be satisfied. Then the semigroup \( \Theta \) is asymptotically smooth in \( \mathbb{R}^2_+ \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \).

**Proof.** First observe that for all \( \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \in \mathbb{R}^2_+ \times L^1_+ (\mathbb{R}_+, \mathbb{R})^2 \) we have
\[
\Theta^n \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) = U_1(n\tau, 0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) + U_2(n\tau, 0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right), \quad \forall n \geq 0.
\]
\[
\text{(134)}
\]
Since
\[
\left\| \mathbf{U}_2(n\tau, 0) \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \right\| = \int_{n\tau}^{+\infty} \| \mathbf{b}(a) \mathbf{L}(a; t - a, 0) \mathbf{i}_0(a - t) \| da \\
\leq \| \mathbf{b} \|_\infty e^{-\mu n\tau} \| \mathbf{i}_0 \|_{L^1} \\
\leq \| \mathbf{b} \|_\infty e^{-\mu n\tau} \left\| \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \right\|, \forall n \geq 1
\]
and \( \mathbf{U}_1(n\tau, 0) \) is compact for all \( n \geq 1 \) the result follows by applying Lemma 2.3.2 in [6]. \( \square \)

Thanks to Lemma 7.5 and Lemma 7.6 we can apply [6, Theorem 2.4.6] or [19, Theorem 2.6] to obtain the existence of global attractor for \( \Theta \).

**Lemma 7.7.** Let Assumption 2.1 be satisfied. Then \( \Theta \) has a compact global attractor \( \mathcal{A} \subset \mathbb{R}^2_+ \times L^1_+ \times \mathbb{R}^2 \).

**Lemma 7.8.** Let Assumption 2.1 be satisfied. Assume that \( R_0 > 1 \). Then \( \partial \mathcal{M}_0 \) is \( \xi \)-ejective for \( \Theta \) that is to say that there exists \( \varsigma_1 > 0 \) such that for all
\[
\left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \in \mathbb{R}^2_+ \times L^1_+ \times \mathbb{R}^2
\]
with
\[
0 < \xi \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) < \varsigma_1
\]
there exists \( n_0 \geq 0 \) such that
\[
\xi \left( \Theta^n \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \right) \geq \varsigma_1.
\]

**Proof.** We will argue by contradiction. Set
\[
\hat{\gamma}_1 := \frac{2\lambda_+}{\mu} + \delta.
\]
Let \( \varsigma_1 > 0 \) be given such that such that
\[
2\varsigma_1 \left( \frac{2\lambda_+}{\mu} + \delta \right) e^{\hat{\gamma}_1 \tau} \| \mathbf{b} \|_\infty < \varsigma_0
\]
where \( \varsigma_0 \) is provided by Lemma 7.4. Assume that
\[
0 < \xi \left( \Theta^n \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \right) < \varsigma_1, \forall n \geq 0.
\]
Define
\[
\left( \begin{array}{c} \mathbf{S}(t) \\ \mathbf{i}(t, \cdot) \end{array} \right) = \mathbf{U}(t, 0) \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right), \forall t \geq 0.
\]
Recall that
\[
\Theta^n \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) = \mathbf{U}(n\tau, 0) \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right), \forall n \geq 0.
\]
Since \( \Theta \) is point dissipative there exists \( n_0 \geq 0 \) large enough such that
\[
\Theta^n \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \in \mathcal{B}_\delta, \forall n \geq n_0.
\]
Thus we have
\[
\left\| \left( \begin{array}{c} \mathbf{S}(n\tau) \\ \mathbf{i}(n\tau, \cdot) \end{array} \right) \right\| = \left\| \mathbf{U}(n\tau, 0) \left( \begin{array}{c} \mathbf{S}_0 \\ \mathbf{i}_0 \end{array} \right) \right\| = \| \mathbf{S}(n\tau) \| + \| \mathbf{i}(n\tau, \cdot) \|_{L^1} \leq \frac{2\lambda_+}{\mu} + \delta, \forall n \geq n_0
\]
(138)
Next we define
\[
\left( \begin{array}{c}
\hat{S}_n(r) \\
\hat{i}_n(r, \cdot)
\end{array} \right) = U(r, 0) \left( \begin{array}{c}
S(n\tau) \\
i(n\tau, \cdot)
\end{array} \right) \quad \forall r \geq 0, \ \forall n \geq n_0
\]
(139)
and observe that by Lemma 3.3 and (139) we have for each \( r \geq 0 \) and each \( n \geq n_0 \)
\[
\left\| \int_0^{+\infty} b(a) \hat{i}_n(t, a) da \right\| \leq 2e^{-\mu r+\hat{\gamma}} \|b\|_\infty \left( \int_0^{a^*} i_1(n\tau, a) da + \int_0^{a^*} i_2(n\tau, a) da \right).
\]
with
\[
\hat{\gamma} = \max \left( \|S(n\tau)\| + \|i(n\tau, \cdot)\|_{L^1}, \frac{2\lambda_+}{\mu} \right) \leq \frac{2\lambda_+}{\mu} + \delta = \hat{\gamma}_1.
\]
Hence we have
\[
\left\| \int_0^{+\infty} b(a) \hat{i}_n(r, a) da \right\| \leq 2e^{-\mu r+\hat{\gamma} r} \|b\|_\infty \xi \left( \Theta^n \left( \begin{array}{c}
S_0 \\
i_0
\end{array} \right) \right), \quad \forall r \geq 0, \ \forall n \geq n_0
\]
(140)
Because \( M_0 \) is forward invariant with respect to \( U \) we have
\[
0 < \left\| \int_0^{+\infty} b(a) \hat{i}_n(r, a) da \right\|, \quad \forall r \geq 0, \ \forall n \geq n_0
\]
(141)
while (140) and (137) together with (141) imply that
\[
0 < \left\| \int_0^{+\infty} b(a) \hat{i}_n(r, a) da \right\| < \varsigma, \quad \forall r \in [0, \tau], \ \forall n \geq n_0.
\]
(142)
Now let \( t \geq n_0\tau \) be given. Then there exists \( n_t \geq 0 \) and \( r \in [0, \tau) \) such that \( t = r + (n_t + n_0)\tau \). Hence using the \( \tau \)-periodicity of \( U \) and the evolution semiflow properties we have
\[
U(t, 0) = U(r + (n_t + n_0)\tau, 0) = U(r + (n_t + n_0)\tau, (n_t + n_0)\tau)U((n_t + n_0)\tau, 0) = U(r, 0)U((n_t + n_0)\tau, 0)
\]
providing that
\[
\left( \begin{array}{c}
S(t) \\
i(t, \cdot)
\end{array} \right) = U(t, 0) \left( \begin{array}{c}
S_0 \\
i_0
\end{array} \right) = U(r, 0)U((n_t + n_0)\tau, 0) \left( \begin{array}{c}
S_0 \\
i_0
\end{array} \right) = \left( \begin{array}{c}
\hat{S}_{n_0+n_0}(r) \\
\hat{i}_{n_0+n_0}(r, \cdot)
\end{array} \right)
\]
(143)
and we infer from (142) and (143) that
\[
0 < \left\| \int_0^{+\infty} b(a) i(t, a) da \right\| = \left\| \int_0^{+\infty} b(a) \hat{i}_{n_0+n_0}(r, \cdot) da \right\| < \varsigma, \quad \forall t \geq n_0\tau.
\]
Hence Lemma 7.4 implies that
\[
\lim_{t \to +\infty} \|i(t, \cdot)\|_{L^1} = \lim_{n \to +\infty} \|i(n\tau, \cdot)\|_{L^1} = +\infty
\]
which is a contradiction with (138)
\[\square\]
We can now prove the main result of this section.
Proof of Theorem 7.1. Since $\Theta$ has a global attractor and $\partial M_0$ is $\xi$-ejective with respect to the decomposition $(M_0, \partial M_0)$, by using [19, Proposition 3.2] we obtain that $\Theta$ is $\xi$-uniformly persistent that is there exists $\delta_0 > 0$ such that for each

$$(S_0, i_0) \in \mathbb{R}_+^2 \times L^1_+((\mathbb{R}_+, \mathbb{R})^2
\lim_{n \to +\infty} \xi \left( \Theta^n \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \right) > \delta_0
$$

that is

$$\lim_{n \to +\infty} \xi \left( U(n\tau, 0) \left( \begin{array}{c} S_0 \\ i_0 \end{array} \right) \right) = \lim_{n \to +\infty} \left( \int_0^{a^*} i_1(n\tau, a)da + \int_0^{a^*} i_2(n\tau, a)da \right) > \delta_0.$$

The proof is completed. Indeed the uniform persistence of $\Theta$ is equivalent to the one of $\{U(t,0)\}_{t \geq 0}$ [38, Theorem 3.1.1].

8. Coexistence: Existence of a positive periodic solution. In this section we will show that system (4) admits a non trivial positive periodic solution that is different from the disease free periodic solution. This will be performed by applying [19, Theorem 4.5] to the discrete time semiflow $\Theta$ defined in (129).

Proposition 6. Let Assumption be satisfied. Assume that $R_0 > 1$. Then system (4) has a positive periodic orbit. More precisely there exists a $\tau$-periodic mild solution with

$$\left( \begin{array}{c} S^*(t) \\ i^*(t, \cdot) \end{array} \right) \in M_0, \ \forall t \geq 0.$$

In what follow we will always assume that $R_0 > 1$. We know from Lemma 7.5 and Lemma 7.6 that $\Theta$ is point dissipative and asymptotically smooth. Moreover $\Theta$ has a compact global attractor $A$ by Lemma 7.7 and is $\xi$-ejective with respect to the decomposition $(M_0, \partial M_0)$ by Lemma 7.8. Then its $\xi$-uniformly persistent with respect to the decomposition $(M_0, \partial M_0)$ [19, Proposition 3.2]. Therefore we only have to show that $\Theta$ is condensing. Before stating the exact meaning of condensing we recall that the Kuratowski measure of non compactness $\kappa$ is defined by

$$\kappa(B) = \inf \{ r > 0 : B \text{ has a finite open cover of diameter } \leq r \},$$

for all bounded subset $B \subset \mathbb{R}_+^2 \times L^1_+((\mathbb{R}_+, \mathbb{R})^2$. Therefore we say that $\Theta$ is $\kappa$-condensing if

$$\kappa(\Theta(B)) < \kappa(B),$$

for all bounded subset $B \subset \mathbb{R}_+^2 \times L^1_+((\mathbb{R}_+, \mathbb{R})^2$. We refer to [4, 21, 25] for several properties of $\kappa$.

The following lemma shows that $\Theta$ is $\kappa$-condensing.

Lemma 8.1. Let Assumption 2.1 be satisfied. Then $\Theta$ is $\kappa$-condensing.

Proof. Let $B$ be a bounded subset of $\mathbb{R}_+^2 \times L^1_+((\mathbb{R}_+, \mathbb{R})^2$. Then recalling that

$$\Theta = U(\tau, 0) = U_1(\tau, 0) + U_2(\tau, 0)$$

we have

$$\Theta(B) \subset U_1(\tau, 0)(B) + U_2(\tau, 0)(B) \Rightarrow \kappa(\Theta(B)) \leq \kappa(U_1(\tau, 0)(B)) + \kappa(U_2(\tau, 0)(B)).$$

Then since $U_1(\tau, 0)$ is compact we have $\kappa(U_1(\tau, 0)(B)) = 0$. Hence

$$\kappa(\Theta(B)) \leq \kappa(U_2(\tau, 0)(B)) \leq \| U_2(\tau, 0) \|_{L^1} \kappa(B) \leq e^{-\mu \tau} \kappa(B) < \kappa(B).$$
We can now conclude by using [19, Theorem 4.5.] that if $R_0 > 1$ then $\Theta : M_0 \to M_0$ has a compact global attractor $A_0 \subset M_0$ and has a fixed point in $A_0 \subset M_0$. Thus there exists \( \left( \begin{array}{c} S_0^* \\ i_0^* \end{array} \right) \in M_0 \) such that
\[
\left( \begin{array}{c} S_0^* \\ i_0^* \end{array} \right) = U(\tau, 0) \left( \begin{array}{c} S_0^* \\ i_0^* \end{array} \right) \in M_0.
\]

REFERENCES

[1] P. Auger, P. Magal and S. Ruan, *Structured Population Models in Biology and Epidemiology*, Lecture Notes in Mathematics, 1936. Mathematical Biosciences Subseries. Springer-Verlag, Berlin, 2008.

[2] N. Bacaër and S. Guernaoui, *The epidemic threshold of vector-borne diseases with seasonality: The case of cutaneous leishmaniasis in Chichaoua*, *Journal of Mathematical Biology*, 53 (2006), 421–436.

[3] Z. G. Bai, *Threshold dynamics of a time-delayed SEIRS model with pulse vaccination*, *Mathematical Biosciences*, 269 (2015), 178–185.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] S. Funk, M. Salathé and V. A. A. Jansen, *Modelling the influence of human behaviour on the spread of infectious diseases: A review*, *Journal of the Royal Society Interface*, 7 (2010), 1247–1256.

[6] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, 25. American Mathematical Society, Providence, RI, 1988.

[7] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics*, Applied Mathematics Monograph, Giardini editori e stampatori, 1995.

[8] H. Inaba, *Age-Structured Population Dynamics in Demography and Epidemiology*, Springer-Verlag, Singapore, 2017.

[9] H. Inaba, *The Malthusian parameter and $R_0$ for heterogeneous populations in periodic environments*, *Mathematical Biosciences and Engineering*, 9 (2012), 313–346.

[10] H. Inaba, *Weak ergodicity of population evolution processes*, *Mathematical Biosciences*, 96 (1989), 195–219.

[11] K. H. Liu, Y. J. Lou and J. H. Wu, *Analysis of an age structured model for tick populations subject to seasonal effects*, *Journal of Differential Equations*, 263 (2017), 2078–2112.

[12] P. Magal, *Perturbation of a globally stable steady state and uniform persistence*, *Journal of Dynamics and Differential Equations*, 21 (2009), 1–20.

[13] P. Magal and O. Arino, *Existence of periodic solutions for a state dependent delay differential equation*, *Journal of Differential Equations*, 165 (2000), 61–95.

[14] P. Magal and C. McCluskey, *Two group infection age model: An application to nosocomial infection*, *SIAM Journal on Applied Mathematics*, 73 (2013), 1058–1095.

[15] P. Magal and S. G. Ruan, *On semilinear Cauchy problems with non-dense domain*, *Advances in Differential Equations*, 14 (2009), 1041–1084.

[16] P. Magal and S. G. Ruan, *Theory and Applications of Abstract Semilinear Cauchy Problems*, Applied Mathematical Sciences, 201. Springer, Cham, 2018.

[17] P. Magal, O. Seydi and F.-B. Wang, *Monotone abstract non-densely defined Cauchy problems applied to age structured population dynamic models*, *J. Math. Anal. Appl.*, 479 (2019), 450–481, arXiv:math.AP/1901.01231.

[18] P. Magal and H. R. Thieme, *Eventual compactness for a semiflow generated by an age-structured models*, *Communications on Pure and Applied Analysis*, 3 (2004), 695–727.

[19] P. Magal and X.-Q. Zhao, *Global attractors and steady states for uniformly persistent dynamical systems*, *SIAM Journal on Mathematical Analysis*, 37 (2005), 251–275.

[20] P. Manfredi and A. D’Onofrio, *Modeling the Interplay Between Human Behavior and the Spread of Infectious Diseases*, Springer-Verlag, New York, 2013.

[21] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney, 1976.

[22] C. McCluskey, *Global stability for an SEI model of infectious disease with age structure and immigration of infecteds*, *Mathematical Biosciences Engineering*, 13 (2016), 381–400.
[23] J. A. J. Metz and O. Diekmann, *The Dynamics of Physiologically Structured Populations*, Lecture Notes in Biomathematics, 68. Springer-Verlag, Berlin, 1986.

[24] C. Rebelo, A. Margheri and N. Bacaër, Persistence in some periodic epidemic models with infection age or constant periods of infection, *Discrete and Continuous Dynamical Systems-Series B*, 19 (2014), 1155–1170.

[25] G. R. Sell and Y. C. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences, 143. Springer-Verlag, New York, 2002.

[26] H. L. Smith and H. R. Thieme, *Dynamical Systems and Population Persistence*, Graduate Studies in Mathematics, 118. American Mathematical Society, RI, 2011.

[27] H. L. Smith and P. Waltman, Perturbation of a globally stable steady state, *Proceedings of the American Mathematical Society*, 127 (1999), 447–453.

[28] H. R. Thieme, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press, Princeton, NJ, 2003.

[29] H. R. Thieme, Renewal theorems for linear periodic Volterra integral equations, *Journal of Integral Equations*, 7 (1984), 253–277.

[30] H. R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Differential Integral Equations*, 3 (1990), 1035–1066.

[31] H. R. Thieme and I. I. Vrabie, Relatively compact orbits and compact attractors for a class of nonlinear evolution equations, *Journal of Dynamics and Differential Equations*, 15 (2003), 731–750.

[32] H.-O. Walther, A periodic solution of a differential equation with state-dependent delay, *Journal of Differential Equations*, 244 (2008), 1910–1945.

[33] W. D. Wang and X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, *Journal of Dynamics and Differential Equations*, 20 (2008), 699–717.

[34] G. F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, Monographs and Textbooks in Pure and Applied Mathematics, 89. Marcel Dekker, Inc., New York, 1985.

[35] S. X. Zhang and H. B. Guo, Global analysis of age-structured multi-stage epidemic models for infectious diseases, *Applied Mathematics and Computation*, 337 (2018), 214–233.

[36] L. Zhao, Z.-C. Wang, L. Zhang, Threshold dynamics of a time periodic and two-group epidemic model with distributed delay, *Mathematical Biosciences and Engineering*, 14 (2017), 1535–1563.

[37] X.-Q. Zhao, Basic reproduction ratios for periodic compartmental models with time delay, *Journal of Dynamics and Differential Equations*, 29 (2017), 67–82.

[38] X.-Q. Zhao, *Dynamical Systems in Population Biology*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 16. Springer-Verlag, New York, 2003.

Received for publication January 2019.

E-mail address: mlamine.diagne@univ-thies.sn
E-mail address: oseydi@ept.sn
E-mail address: aissata.a.b.sy@aims-senegal.org