Note on a sign-dependent regularity for the polyharmonic
Dirichlet problem

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Abstract

A priori estimates for semilinear higher order elliptic equations usually have to deal with
the absence of a maximum principle. This note presents some regularity estimates for the
polyharmonic Dirichlet problem that will make a distinction between the influence on the
solution of the positive and the negative part of the right-hand side.

Keywords: polyharmonic, m-laplace problem, higher order, sign-dependent regularity

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1 Introduction and main result

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega \in C^{2m,\gamma}$ with $m \in \mathbb{N}^+$ and $\gamma \in (0, 1)$, and consider
the Dirichlet problem for the poly-laplace operator:

$$
\begin{cases}
(-\Delta)^m u = f \quad &\text{in } \Omega, \\
u = \frac{\partial}{\partial n} u = \cdots = \left(\frac{\partial}{\partial n}\right)^{m-1} u = 0 \quad &\text{on } \partial \Omega.
\end{cases}
$$

Suppose that

$$
f^+ := \max(0, f) \quad \text{and} \quad f^- := \max(0, -f)
$$

is such that $f^+ \in L^{p_+}(\Omega)$ and $f^- \in L^{p_-}(\Omega)$ with $p_+, p_- \in (1, \infty)$. For the second order case, that is $m = 1$, one may use the maximum principle and solve

$$
\begin{cases}
-\Delta u^+ = f^+ \quad &\text{in } \Omega, \\
0 \leq u^+ \in W^{2,p_+}(\Omega) \cap W_0^{1,p_+}(\Omega) \quad &\text{on } \partial \Omega,
\end{cases}
\quad \begin{cases}
-\Delta u^- = f^- \quad &\text{in } \Omega, \\
0 \leq u^- \in W^{2,p_-}(\Omega) \cap W_0^{1,p_-}(\Omega) \quad &\text{on } \partial \Omega,
\end{cases}
$$

separately to find $u = u^+ - u^-$ for

$$
0 \leq u^+ \in W^{2,p_+}(\Omega) \cap W_0^{1,p_+}(\Omega) \quad \text{and} \quad 0 \leq u^- \in W^{2,p_-}(\Omega) \cap W_0^{1,p_-}(\Omega),
$$

with the usual regularity estimates (2):

$$
\|u^+\|_{W^{2m,p_+}(\Omega)} \leq c_{p_+} \|f^+\|_{L^{p_+}(\Omega)} \quad \text{and} \quad \|u^-\|_{W^{2m,p_-}(\Omega)} \leq c_{p_-} \|f^-\|_{L^{p_-}(\Omega)}.
$$

The constants will depend on $\Omega$, but that dependence we will suppress in our notation.

Whenever $m \geq 2$ there is no maximum principle or, unless we have a special domain like a
ball [3], a positivity preserving property in the sense that $f \geq 0$ in (1) would result in $u \geq 0$.
Nevertheless, it is possible to find a result quite similar to (3) for the solution of (1). Such a
separation of the regularity for the positive and negative part is something we need for a higher
order semilinear problem that we consider in [4]. Since we believe it has some interest in itself,
we present this sign-dependent regularity in this separate note.

Our main result for (1) with $m \in \mathbb{N}^{++} := \{2, 3, \ldots\}$ is as follows:

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Theorem 1 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega \in C^{2m,\gamma}$ and let $p_+ \in (1, \infty)$. Suppose that $f = f^+ - f^-$ as in [3] with $f^+ \in L^{p^+}(\Omega)$ and $f^- \in L^{p^-}(\Omega)$. Then there exist constants $c_{p_+,m}, c_{p_-,m} > 0$, independent of $f^+, f^-$, such that the following holds. The unique solution $u$ of (1) can be written as $u = u^\oplus - u^\ominus$, with

\[ 0 \leq u^\ominus \in W^{2m,p_+}(\Omega) \cap W_0^{m,p^+}(\Omega), \]
\[ 0 \leq u^\oplus \in W^{2m,p_-}(\Omega) \cap W_0^{m,p^-}(\Omega), \]
and

\[ \| u^\ominus \|_{W^{2m,p_+}(\Omega)} \leq c_{p_+,m} \left( \| f^+ \|_{L^{p^+}(\Omega)} + \| f^- \|_{L^1(\Omega)} \right), \]
\[ \| u^\oplus \|_{W^{2m,p_-}(\Omega)} \leq c_{p_-,m} \left( \| f^- \|_{L^{p^-}(\Omega)} + \| f^+ \|_{L^1(\Omega)} \right). \]

Although we will construct $u^\ominus, u^\oplus$ in a way such that $u^\ominus, u^\oplus$ is unique, the statement in the theorem does not give uniqueness of this decomposition $u^\ominus, u^\oplus$. Since $f \in L^p(\Omega)$ with $p = \min \{ p_-, p_+ \} > 1$ and $\partial \Omega \in C^{2m,\gamma}$, the solution is unique in $W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$.

Generically $u^\ominus \neq u^+$, but since $u^+ = (u^\ominus - u^\oplus)^+ \leq (u^\ominus)^+ = u^\ominus$, we find that

\[ -u^\ominus \leq -u^- \leq 0 \leq u^+ \leq u^\oplus. \quad (4) \]

With this estimate one also finds a signed Sobolev inequality. Setting

\[ q_{n,m,p} := \frac{np}{n - 2mp} \quad (5) \]
we may combine with the Sobolev imbedding theorem, see [1, Theorem 4.12], to obtain the following:

Corollary 2 Let $\Omega \subset \mathbb{R}^n$ be bounded with $\partial \Omega \in C^{2m,\gamma}$ and let $p_+ \in (1, \infty)$. Suppose that $f = f^+ - f^-$ with $f^+ \in L^{p^+}(\Omega)$ and $f^- \in L^{p^-}(\Omega)$. Let $u$ be the solution of (1) as in Theorem 1. Then the following holds:

1. If moreover $p_+ \leq \frac{n}{2m}$ (so $n > 2m$) and $q \in \left[1, q_{n,m,p_+}\right]$ with $q < \infty$, then there is $c_{p_+,q,m} > 0$ such that

\[ \| u^+ \|_{L^q(\Omega)} \leq c_{p_+,q,m} \left( \| f^+ \|_{L^{p^+}(\Omega)} + \| f^- \|_{L^1(\Omega)} \right). \]

2. If moreover $p_+ > \frac{n}{2m}$, then there is $c'_{p_+,m} > 0$ such that

\[ \sup u \leq c'_{p_+,m} \left( \| f^+ \|_{L^{p^+}(\Omega)} + \| f^- \|_{L^1(\Omega)} \right). \]

Similar results depending on $p_-$ hold for $u^-$ and $\sup (-u)$.

2 Relation to previous results

Since the fundamental contributions by Agmon, Douglis and Nirenberg [2] it is known, assuming that $\Omega$ is bounded with a smooth enough boundary, that for each $p \in (1, \infty)$ and $f \in L^p(\Omega)$ a solution of (1) satisfies $u \in W^{2m,p}(\Omega)$. Whenever the solution is unique, and with the $C^{2m,\gamma}$ boundary the solution for (1) is unique for any $p \in (1, \infty)$, there exist $C_{m,p} > 0$, independent of $f$, such that

\[ \| u \|_{W^{2m,p}(\Omega)} \leq C_{m,p} \| f \|_{L^p(\Omega)}. \quad (6) \]

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Whenever \( p \in (1, \frac{n}{2m}) \), and such \( p \) exist when \( n > 2m \), the Sobolev imbedding shows that for 
\( q \leq q_{n,m,p} \), as in (1), a constant \( C_{m,p,q}^\prime > 0 \) exists such that 
\[
\|u\|_{L^q(\Omega)} \leq C_{m,p,q}^\prime \|u\|_{W^{2m,p}(\Omega)}.
\]
Combining both estimates will lead to an estimate as in Corollary (2) but then without the sign. However, since \( 0 \leq u^+ \leq u^- \) holds, one finds \( \|u^+\|_{L^q(\Omega)} \leq \|u^-\|_{L^q(\Omega)} \) and one is left with proving the result in Theorem 1.

For \( \Omega = \mathbb{R}^n \) signed estimates as in the corollary will follow directly from the Riesz potential \( I_{2m}, [3] \), for the Riesz potential solution of \((−\Delta)^m u = f \), when \( f \) goes to zero at \( \infty \) in an appropriate sense. Indeed, see [10, Chapter V], that solution is given by 
\[
u(x) = (I_{2m}f)(x) := \frac{\Gamma(\frac{4n-m}{2})}{\pi^{\frac{4n}{2}}n^{\Gamma(m)}} \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{2m-n}}.
\] (7)
Since the kernel in (7) is positive, it allows one to consider separately the influence of \( f^+ \in L^{p_+}(\mathbb{R}^n) \) and \( f^- \in L^{p_-}(\mathbb{R}^n) \) with \( p_+, p_- \in (1, \frac{2m}{n}) \). Indeed, on \( \mathbb{R}^n \) the function \( u = u^+ - u^- \) with 
\[
u^\pm(x) := (I_{2m}f^\pm)(x) \text{ and } u^\pm(x) := (I_{2m}f^\pm)(x)
\]
is such that \( u^\pm \in L^{q\pm}(\mathbb{R}^n) \) and \( u^\pm \in L^{q\mp}(\mathbb{R}^n) \) with \( q_\pm := q_{n,m,p_\pm} \).

On a bounded domain with homogeneous Dirichlet boundary conditions the crucial ingredient that allows us to consider \( f^+ \) and \( f^- \) separately, comes from [4]. There one finds that the Green function \( G_{\Omega,m} \) for (1), that is 
\[
u(x) = (G f)(x) := \int_{\Omega} G_{\Omega,m}(x,y) f(y) dy
\] (8)
solves (1), is such that the following estimate holds for some \( \tilde{c}_{1,\Omega}, \tilde{c}_{2,\Omega}, \tilde{c}_{3,\Omega} > 0 \): 
\[
\tilde{c}_{1,\Omega} H(x,y) \leq G_{\Omega,m}(x,y) + \tilde{c}_{2,\Omega} d(x)^m d(y)^m \leq \tilde{c}_{3,\Omega} H(x,y)
\] (9)
with \( H : \overline{\Omega} \times \overline{\Omega} \to [0, \infty] \) defined by 
\[
H(x,y) = \begin{cases} 
|y-x|^{2m-n} \min \left( 1, \left( \frac{d(x)d(y)}{|y-x|^2} \right)^n \right) & \text{for } n > 2m, \\
\log \left( 1 + \left( \frac{d(x)d(y)}{|y-x|^2} \right)^n \right) & \text{for } n = 2m, \\
(d(x)d(y))^{m-n/2} \min \left( 1, \left( \frac{d(x)d(y)}{|y-x|^2} \right)^{n/2} \right) & \text{for } n < 2m.
\end{cases}
\] (10)
Here \( d \) is the distance to the boundary \( \partial \Omega \): 
\[
d(x) = d(x, \partial \Omega) := \inf \left\{ |x-x^\ast| ; x^\ast \in \partial \Omega \right\}.
\]
The estimate in (10) allows us to separate the solution operator in a signed singular part and a smooth bounded part. The singular part will have the same regularity properties as in (10) from [2], but the fixed sign allows us to separate \( f^+ \) and \( f^- \).

One may wonder how general such signed regularity estimates may hold for higher order elliptic boundary value problems. Such estimates are known for pure powers of the negative laplacian \( -\Delta \). Pure powers of second order elliptic operators with constant coefficients may be allowed and they may even be perturbed by small lower order terms. See [7]. However, a recent paper [5] shows examples of higher order elliptic operators, even with constant coefficients, for which the singularity at \( x = y \) is sign-changing. Obviously for such a problem there is no estimate like (10) possible.
3 The proof

The distance function $d$ is at most Lipschitz, even on $C^\infty$-domains. So as a first step we will replace $d(x)^m d(y)^m$ in (9) by a smoother function, namely by $w(x)w(y)$ where

$$w := e_1^m \quad (11)$$

and $e_1$ is the solution of

$$\begin{cases} -\Delta e_1 = 1 & \text{in } \Omega, \\ e_1 = 0 & \text{on } \partial \Omega. \end{cases}$$

Lemma 3 This function $w$ from (11) inherits the smoothness of the boundary $\partial \Omega$, in the sense that $\partial \Omega \in C^{2m,\gamma}$ implies $w \in C^{2m,\gamma}(\Omega)$. Moreover, there exist $c_1, c_2 > 0$ such that

$$c_1 \ d(x)^m \leq w(x) \leq c_2 \ d(x)^m \text{ for all } x \in \Omega. \quad (12)$$

Proof. By the maximum principle and more precisely a uniform Hopf’s boundary point lemma, which holds for $\partial \Omega \in C^{1,\gamma}$, one finds that a constant $C_H > 0$ exists with $e_1(x) \geq C_H d(x)$ for all $x \in \Omega$. Since $e_1 \in C^1(\Omega) \cap C_0(\Omega)$, one finds another constant $C_c > 0$ such that $e_1(x) \leq C_c d(x)$ for all $x \in \Omega$. By [10, Theorem 6.19] $e_1 \in C^{2m,\gamma}(\Omega)$ and hence $w = e_1^m \in C^{2m,\gamma}(\Omega)$ and satisfies (12).

Since the function $H(\cdot,\cdot)$ from (10) satisfies for some $C_{GRS} > 0$

$$H(x, y) \geq C_{GRS} \ (d(x)d(y))^m \text{ for all } x, y \in \overline{\Omega}$$

there exists $\hat{c}_1, \hat{c}_2, \hat{c}_3 > 0$ such that the following variant of (9) holds:

$$\hat{c}_1 H(x, y) \leq G_{\Omega, m}(x, y) + \hat{c}_2 \ w(x) \ w(y) \leq \hat{c}_3 H(x, y). \quad (13)$$

We do not directly replace $d$ in $H$ by $e_1$, but instead define the function $H_{\Omega, m} : \overline{\Omega} \times \overline{\Omega} \to [0, \infty]$ by

$$H_{\Omega, m}(x, y) := G_{\Omega, m}(x, y) + \hat{c}_2 \ w(x) \ w(y). \quad (14)$$

In the next theorem we will state some properties of the operator $\mathcal{H} : C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by

$$(\mathcal{H}f)(x) := \int_{\Omega} H_{\Omega, m}(x, y) f(y) dy. \quad (15)$$

For later use we also set $\mathcal{D} = \mathcal{H} - \mathcal{G}$, i.e.

$$(\mathcal{D}f)(x) := \hat{c}_2 \ w(x) \int_{\Omega} w(y) f(y) dy, \quad (16)$$

which is well-defined for $f \in L^1(\Omega)$ and bounded as operator from $L^1(\Omega)$ to $C^{2m,\gamma}(\Omega)$. Note that $w \in W_0^{0,q}(\Omega)$ for any $q \in (1, \infty)$ and hence $\mathcal{D}$ can even be extended to $W^{-m,p}(\Omega) := W_0^{m,p/(p-1)}(\Omega)^\prime$.

For $n > 2m$ one finds from (13) and (10) that there is $C_{2,\Omega, m} > 0$ such that

$$0 \leq H_{\Omega, m}(x, y) \leq C_{2, m} \ |x - y|^{2m-n} \text{ for all } (x, y) \in \overline{\Omega} \times \overline{\Omega}, \quad (17)$$

Hence from the Hardy-Littlewood-Sobolev Theorem of fractional integration (see [10, Theorem 1, p. 119]) one finds the first two statements of:

Lemma 4 Let $\Omega$ be a bounded domain and $\partial \Omega \in C^{2m,\gamma}$ for some $\gamma \in (0, 1)$, and let $p \in [1, \infty)$. Then $\mathcal{H}f$ in (14)- (12) is well-defined for all $f \in L^p(\Omega)$:

1. For all $p \in [1, \infty)$ and $f \in L^p(\Omega)$ the integral in (15) is absolute convergent for almost every $x \in \Omega$. 

2. Suppose \( n > 2m \). For all \( p \in (1, \frac{n}{2m}) \) and \( q \in \left[1, \frac{np}{n-2mp}\right] \) there exist constants \( C_{p,q} > 0 \) independent of \( f \in L^p(\Omega) \), such that

\[
\|\mathcal{H}f\|_{L^q(\Omega)} \leq C_{p,q} \|f\|_{L^p(\Omega)}.
\]

(18)

3. For all \( p \geq 1 \) with \( p > \frac{n}{2m} \) there exist constants \( C_p > 0 \) independent of \( f \in L^p(\Omega) \), such that

\[
\|\mathcal{H}f\|_{L^\infty(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}.
\]

(19)

Proof. By [17] we find for \( f \geq 0 \) and \( f \) extended by 0 outside of \( \Omega \), that

\[
0 \leq \mathcal{H}f \leq c_n I_{2m}f
\]

with \( I_{2m}f \) a Riesz potential of \( f \) defined in [7]. By [10] Theorem 1, p. 119] the first item holds. Moreover, [10] Theorem 1, p. 119] also states that for \( p \in \left(1, \frac{n}{2m}\right) \) there exists \( C_{\text{HLS},p,n,m} > 0 \) such that

\[
\|I_{2m}f\|_{L^{\frac{np}{n-2mp}}(\Omega)} \leq C_{\text{HLS},p,n,m} \|f\|_{L^p(\Omega)}.
\]

(20)

With (20) one finds (18) for positive \( f \) and \( q = \frac{np}{n-2mp} \). Since \( \Omega \) is bounded, the estimate holds for all \( q \in \left[1, \frac{np}{n-2mp}\right] \). For general \( f \) one splits by \( f = f^+ - f^- \), uses linearity, \( \|f\|_{L^p(\Omega)} \leq \|f^\pm\|_{L^p(\Omega)} \), and finds (18) with twice the constant for \( f \) with fixed sign.

If \( n > 2m \) and \( p > \frac{n}{2m} \), then the third item follows from (17) and the usual estimate by Hölder’s inequality applied to the Riesz potential:

\[
\|\mathcal{H}f\|_{L^\infty(\Omega)} \leq c'_{m,n} \sup_{x \in \Omega} \|x - \cdot\|_{L^{p/(p-1)}(\Omega)} \|f\|_{L^p(\Omega)}
\]

For \( n = 2m \) the logarithmic singularity lies in \( L^q \) for any \( q < \infty \), which gives the estimate by Hölder for any \( p > 1 \). For \( n < 2m \) the kernel of \( \mathcal{H} \) is uniformly bounded, which yields the estimate for \( p = 1 \) and hence for any \( p \geq 1 \).

Proposition 5 Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \in C^{2m,\gamma} \) for some \( \gamma \in (0,1) \). Let \( \mathcal{H} \) be defined by (1A and 1D). Then for any \( p \in (1, \infty) \) there exists \( C_{m,p} > 0 \) such that for all \( f \in L^p(\Omega) \), it holds that \( \mathcal{H}f \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \) and

\[
\|\mathcal{H}f\|_{W^{2m,p}(\Omega)} \leq C_{m,p} \|f\|_{L^p(\Omega)}.
\]

(21)

Proof. For \( 0 \leq f \in L^p(\Omega) \) let \( f_\varepsilon := \varphi_\varepsilon \ast f \in C(\overline{\Omega}) \) denote the usual mollification, with \( \varphi_\varepsilon \) the mollifier from Friedrichs and \( f \) extended by 0 outside of \( \Omega \). By Lemma [1] and suitable \( p,q \), that is \( \frac{1}{q} \geq \frac{1}{p} - \frac{2m}{n} \), one finds

\[
\|\mathcal{H}f - \mathcal{H}f_\varepsilon\|_{L^q(\Omega)} \leq C_{p,q} \|f - f_\varepsilon\|_{L^p(\Omega)}.
\]

(22)

Also \( D \) is well-defined for \( f \in L^1(\Omega) \), which contains \( L^p(\Omega) \), and since \( w \in C^{2m,\gamma}(\overline{\Omega}) \) holds by Lemma [2] one even finds for some \( c_m \) that

\[
\|Df\|_{C^{2m,\gamma}(\overline{\Omega})} \leq c_m \|f\|_{L^1(\Omega)}.
\]

(23)

and hence also that

\[
\|Df - Df_\varepsilon\|_{L^q(\Omega)} \leq c'_{m,p,q} \|f - f_\varepsilon\|_{L^p(\Omega)}.
\]

(24)

Replacing \( f \) on the right-hand side of (11) by \( f_\varepsilon \), we find as solution

\[
Gf_\varepsilon = \mathcal{H}f_\varepsilon - Df_\varepsilon.
\]

(25)
Letting \(u \in W^{2m,p}(\Omega) \cap W^{m,p}_0(\Omega)\) denote the solution of (1) for \(f\) on the right-hand side, we find by [2] that
\[
\|u - Gf\|_{W^{2m,p}(\Omega)} \leq C_{\text{ADN}, 2m,p} \|f - f\|_{L^p(\Omega)} \rightarrow 0 \text{ for } \varepsilon \downarrow 0.
\]
From (22), (24) and (25) one finds
\[
\|(H - D)f - Gf\|_{L^q(\Omega)} \leq c_{m,p,q} \|f - f\|_{L^p(\Omega)} \rightarrow 0 \text{ for } \varepsilon \downarrow 0.
\]
Hence \((H - D)f = u \in W^{2m,p}(\Omega) \cap W^{m,p}_0(\Omega)\). With \(Df \in C^{2m,\gamma}(\Omega) \cap C^{m-1}_0(\Omega)\) one also finds \(Hf \in W^{2m,p}(\Omega) \cap W^{m,p}_0(\Omega)\).

Moreover, there exists \(C_{4,m,p} > 0\) such that
\[
\|Hf\|_{W^{2m,p}(\Omega)} = \|u + Df\|_{W^{2m,p}(\Omega)} \leq \|u\|_{W^{2m,p}(\Omega)} + \|Df\|_{W^{2m,p}(\Omega)}
\]
\[
\leq C_{\text{ADN}, 2m,p} \|f\|_{L^p(\Omega)} + c_m' \|f\|_{L^1(\Omega)} \leq C_{4,m,p} \|f\|_{L^p(\Omega)}.
\]

It remains to combine these results to the statement of Theorem 1.

**Proof of Theorem 1.** Since \(H\) and \(D\) preserve the sign, we may consider separately the solutions \(u_+\) and \(u_-\) of
\[
\begin{cases}
(-\Delta)^m u_\pm = f^\pm & \text{in } \Omega, \\
D^\alpha u_\pm = 0 & \text{for } |\alpha| < m \text{ on } \partial\Omega.
\end{cases}
\]
Here \(u_\pm\) does not have a sign but just denotes the solution parts depending on the signed splitting of the right-hand side \(f^\pm\). One finds
\[
u_+ (x) = (Hf^+) (x) - (Df^+) (x),
\]
\[
u_- (x) = (Hf^-) (x) - (Df^-) (x).
\]
So
\[
u (x) = (Hf^+) (x) - (Df^+) (x) - (Hf^-) (x) + (Df^-) (x).
\]
We split this expression into two parts:
\[
u^\oplus (x) = (Hf^+) (x) + (Df^-) (x),
\]
\[
u^\ominus (x) = (Hf^-) (x) + (Df^+) (x),
\]
and \(u (x) = u^\oplus (x) - u^\ominus (x)\) with both parts \(u^\oplus, u^\ominus\) being nonnegative. For the \(H\)-part of \(u^\oplus, u^\ominus\) we use the results of Proposition 5. The estimate in (23) takes care of the \(D\)-part.

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