Are there metric theories of gravity other than General Relativity?

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Summary: Current generalizations of the classical Einstein–Hilbert Lagrangian formulation of General Relativity are reviewed. Some alternative variational principles, based on different choices of the gravitational field variable (metric tensor, affine connection, or both) are known to reproduce – more or less directly – Einstein’s gravitational equations, and should therefore be regarded as equivalent descriptions of the same physical model, while other variational principles (“Scalar–tensor theories” and “Higher–derivative theories”) yield pictures of the gravitational interaction which appear to be, *a priori*, physically distinct from GR. Such theories, however, are also known to admit a reformulation (in a different set of variables) which is formally identical to General Relativity (with auxiliary fields having nonlinear self–interaction). The physical significance of this change of variables has been questioned by several authors in recent years. Here, we investigate to which extent purely affine, metric–affine, scalar–tensor and purely metric theories can be regarded as *physically* equivalent to GR. Focusing on the so–called “nonlinear theories of gravity” (NLG theories), which presently enjoy a renewed attention as possible models for inflationary cosmology, we show that if the metric tensor occurring in a nonlinear Lagrangian is identified by assumption with the physical spacetime metric, relevant physical properties (positivity of energy and stability of the vacuum) cannot be assessed. On the other hand, using an alternative set of variables (the “Einstein frame”) one can prove that for a wide class of NLG theories positivity and stability properties do hold. This leads one to regard the rescaled metric (Einstein frame) as the true physical one. As a direct consequence, the physical content of such “alternative” models is reset to coincide with General Relativity, and the “Nonlinear Gravity Theories” become nothing but exotic reformulations of General Relativity in terms of unphysical variables.
I. Lagrangians for classical gravity

Among the various classical theories of gravity which have been proposed after the formulation of General Relativity, it is not always easy to single out those which are truly “alternative” theories from those which turn out to be mere reformulations of General Relativity itself. Even for the models which are known to be mathematically equivalent to GR, i.e., for which there exists a well-defined rule transforming the gravitational field equations into Einstein equations, the problem of the physical interpretation of such an equivalence often remains open.

We shall first comment on different cases of “equivalence” with General Relativity. The considerations below, which are restricted to classical (i.e. non-quantum) aspects, refer to the four-dimensional framework, although some of the results quoted hold also in higher dimensions (the two-dimensional case would instead require a separate discussion).

In a relativistic theory of gravity, according to the common wisdom, any configuration of the gravitational field should correspond to a unique metric (and therefore causal) structure and to a unique geodesic structure on spacetime. Hence, any (physical) solution of the field equations should allow one to define on the spacetime manifold a metric tensorfield and an affine connection. In the purely metric variational principles, only the metric tensor acts as the dynamical variable, while in the metric-affine Lagrangians the metric and the connection are both present as independent variables, their relationship being determined \textit{a posteriori} by the field equations. It is also possible to introduce purely affine Lagrangians, which depend only on the connection. In other models, now very popular, the gravitational interaction involves other fields (typically, a scalar field) in addition to the “geometrical” degrees of freedom. We are interested here in discussing to what extent the various action principles lead to theories which physically differ from General Relativity.

Two distinctive features of General Relativity, as far as the gravitational field alone is considered, are the following:

\begin{itemize}
  \item[(GR\textsubscript{1})]: in the absence of matter, the metric tensor providing the physical notion of spacetime distance obeys the vacuum Einstein equation;
  \item[(GR\textsubscript{2})]: independently of external matter sources, the affine connection which singles out the worldlines of free falling test particles is the Levi-Civita connection of the physical metric tensor.
\end{itemize}

To determine completely the physical content of any theory of the gravitational interaction one should, moreover, specify how gravity is coupled to the other fields. As a matter of fact, the various theories of gravity are usually classified according to their purely gravitational part, the determination of matter coupling being viewed as a secondary problem. It is commonly believed that a universal recipe to include matter interaction in General
Relativity is provided by the “minimal coupling” prescription, but there is no definitive evidence (neither theoretical nor experimental) that it should be so (for instance, applying the minimal coupling prescription to the Fierz–Pauli Lagrangian for a spin–2 field leads to inconsistencies [1]). On the other hand, criteria to accept or reject possible models of interaction are provided by energy conditions on the stress tensor (see [2]).

The interaction of matter with gravity remains however crucial while discussing the equivalence of different theories. We say that two theories are physically equivalent iff (A) the gravitational field has the same vacuum dynamics, and (B) the coupling between the gravitational metric (or the gravitational connection) and any kind of matter is the same for both theories.

Let us now apply these criteria to the most common types of gravitational Lagrangians.

**PURELY AFFINE THEORIES** — The gravitational field variable is a symmetric linear connection $\Gamma^\alpha_{\mu\nu}$. The first example of a purely affine action principle has been introduced by Einstein and Eddington:

$$L_{EE}(\Gamma, \partial \Gamma) = \sqrt{|\det R(\mu\nu)|}$$

(1.1)

The metric structure associated to each solution is obtained using the prescription:

$$g^{\mu\nu} \sqrt{|g|} = \frac{\partial L_{EE}}{\partial R(\mu\nu)}$$

(1.2)

(where $|g|$ denotes the absolute value of the inverse of the determinant of $g^{\mu\nu}$). This prescription is the prototype of a method that we shall extensively use in the sequel; in dimension four and for the vacuum Einstein–Eddington Lagrangian (1.1), the metric so defined turns out to be proportional to the (symmetrized) Ricci tensor of $\Gamma^\alpha_{\mu\nu}$. The second–order Euler–Lagrange equation generated by (1.1), upon insertion of the definition (1.2), becomes equivalent to

$$\nabla_\nu (g^{\alpha\beta} \sqrt{|g|}) = 0$$

(1.3)

where $\nabla$ denotes the covariant derivative w.r. to $\Gamma$. Eq. (1.3) can be satisfied only if $\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu}\}_g$; hence, the symmetric connection $\Gamma$ is forced by the field equations to coincide with the Levi–Civita connection of $g$. By (1.2) $g^{\mu\nu}$ is then necessarily proportional to its own Ricci tensor, and the solutions of the vacuum gravitational equations generated by the purely affine Lagrangian (1.1) are the same as the solutions of a vacuum Einstein equation with cosmological constant. Hence, there is no doubt that the Einstein–Eddington purely affine theory is physically equivalent to vacuum General Relativity.

The Einstein–Eddington Lagrangian (1.1) is the only covariant scalar density (of the appropriate weight) which can be built out of the symmetric part of the Ricci tensor,
up to a constant factor. Moreover, a covariant Lagrangian cannot depend explicitly on the components of the connection $\Gamma$, unless other fields are present. In this sense, the Einstein–Eddington Lagrangian provides the only possible vacuum purely affine model of gravity.

When matter interaction is implemented in this model, the situation becomes more complicated. We investigate this problem using a Legendre–transformation technique, first introduced in this context by Ferraris and Kijowski [4]. Let

$$L_{PA} = L_{PA}(R_{(\mu \nu)}, \Gamma^\alpha_{\mu \nu}, \Psi^A, \Psi^A_{\mu})$$

be any scalar density costructed using the (symmetrized) Ricci tensor of $\Gamma$, $\Gamma$ itself, and other (unspecified) fields $\Psi$ with their first derivatives $\Psi^A_{\mu} \equiv \partial_\mu \Psi^A$. One could assume, for instance, that $L_{PA}$ is the sum of the vacuum EE Lagrangian and some interaction Lagrangian including covariant derivatives of $\Psi$ w.r. to the connection $\Gamma$. In particular cases (scalar field, electromagnetic field) only ordinary derivatives will occur, and the Lagrangian will not depend on the connection components $\Gamma^\alpha_{\mu \nu}$.

Let us introduce a “conjugate momentum” to the connection, that we denote by $\pi^{\alpha \beta}$. The Legendre map relating $\pi^{\alpha \beta}$ to the “configuration and velocity variables” $(\Gamma, R)$ is defined as follows:

$$\pi^{\mu \nu} = \frac{\partial L_{PA}}{\partial R_{(\mu \nu)}}.$$  

(1.5)

The Legendre transformation allows one to recast the original action principle into a dynamically equivalent one, where the fields $(\Gamma, \pi)$ are regarded as independent variables. One should first compute the inverse Legendre map $r_{\mu \nu} = r_{\mu \nu}(\pi^{\alpha \beta}, \Gamma^\alpha_{\mu \nu}, \Psi^A, \Psi^A_{\mu})$, which is implicitly defined by the following relation:

$$\left.\frac{\partial L_{PA}}{\partial R_{(\mu \nu)}}\right|_{R_{(\mu \nu)}=r_{\mu \nu}(\pi,...)} \equiv \pi^{\mu \nu}.$$  

(1.6)

Then, one is able to introduce a new Lagrangian, the “Helmholtz Lagrangian”:

$$L_H = [R_{(\mu \nu)} - r_{\mu \nu}] \pi^{\mu \nu} + L_{PA}(r_{\mu \nu}, \Gamma^\alpha_{\mu \nu}, \Psi^A, \Psi^A_{\mu});$$  

(1.7)

the variation of $L_H$ w.r. to the independent variables $\Gamma$, $\pi$ and $\Psi$ is equivalent to the variation of $L_{PA}$ w.r. to $\Gamma$ and $\Psi$ only. In fact,

$$\delta L_H = \left[ R_{(\mu \nu)} - r_{\mu \nu} - \frac{\partial r_{\rho \sigma}}{\partial \pi^{\rho \sigma}} \pi^{\rho \sigma} + \frac{\partial L_{PA}}{\partial r_{\rho \sigma}} \frac{\partial r_{\rho \sigma}}{\partial \pi^{\rho \sigma}} \right] \delta \pi^{\mu \nu} +$$

$$\pi^{\mu \nu} \delta R_{(\mu \nu)} + \left[ -\frac{\partial r_{\rho \sigma}}{\partial \Gamma^\alpha_{\mu \nu}} \pi^{\rho \sigma} + \frac{\partial L_{PA}}{\partial r_{\rho \sigma}} \frac{\partial r_{\rho \sigma}}{\partial \Gamma^\alpha_{\mu \nu}} \right] \delta \Gamma^\alpha_{\mu \nu} +$$

$$+ \left[ -\frac{\partial r_{\rho \sigma}}{\partial \Psi^A} \pi^{\rho \sigma} + \frac{\partial L_{PA}}{\partial r_{\rho \sigma}} \frac{\partial r_{\rho \sigma}}{\partial \Psi^A} \right] \delta \Psi^A +$$

$$+ \left[ -\frac{\partial r_{\rho \sigma}}{\partial \Psi^A_{\mu}} \pi^{\rho \sigma} + \frac{\partial L_{PA}}{\partial r_{\rho \sigma}} \frac{\partial r_{\rho \sigma}}{\partial \Psi^A_{\mu}} \right] \delta \Psi^A_{\mu}.$$
On account of (1.6), the coefficient of $\delta\pi$ vanishes iff $R_{(\mu\nu)} = r_{\mu\nu}$: hence, the remaining part of the variation,

$$\frac{\partial L_{PA}}{\partial R_{(\mu\nu)}} \delta R_{(\mu\nu)} + \frac{\partial L_{PA}}{\partial \Gamma_{\mu\nu}^\alpha} \delta \Gamma_{\mu\nu}^\alpha + \frac{\partial L_{PA}}{\partial \Psi^A} \delta \Psi^A + \frac{\partial L_{PA}}{\partial \Psi^A_{\mu}} \delta \Psi^A_{\mu},$$

coincides with the total variation of the original purely affine Lagrangian. On the other hand, using the well–known Palatini formula for the variation of the Ricci tensor,

$$\delta R_{(\mu\nu)} = \nabla_\alpha \delta \Gamma_{\mu\nu}^\alpha - \nabla_{(\mu} \delta \Gamma_{\nu)}^\alpha,$$

(1.8)

and subtracting a full divergence, one finds that the variation of $L_H$ w.r. to $\Gamma$ gives the equation

$$\nabla_\alpha \pi_{\mu\nu} - \nabla_\lambda \pi^{\lambda(\mu} \delta_{\nu)} = \frac{\partial L_{PA}}{\partial \Gamma_{\mu\nu}^\alpha}.$$

(1.9)

Whenever the tensor density $\pi$ (which is symmetric by construction) is non–degenerate, it can be re–expressed in terms of a symmetric tensor: $\pi_{\mu\nu} = g_{\mu\nu} \sqrt{|g|}$; writing the Legendre map in terms of the new variable $g$, one gets exactly the Einstein–Eddington prescription (1.2). Equation (1.9) implies that the covariant derivative of the metric tensor $g$ w.r. to the connection $\Gamma$ vanishes identically if $L_{PA}$ does not depend explicitly on $\Gamma_{\mu\nu}^\alpha$. This holds for the vacuum theory and for particular types of matter coupling, as mentioned above. In such cases, one has $\Gamma_{\mu\nu}^\alpha = \{\alpha_{\mu\nu}\}_g$ and the model is fully equivalent to General Relativity. Otherwise, the connection $\Gamma$ is not the metric connection. Then, if we assume that $\Gamma$ is the gravitational connection, i.e. determines the worldlines of free falling particles, postulate $\text{GR}_2$ above is violated and the theory is not physically equivalent to General Relativity.

If $\Gamma$ is not the metric connection, one could try to restore the equivalence with GR by assuming that $\text{GR}_2$ holds by definiton, i.e., the physical geodesic structure is determined by $\{\alpha_{\mu\nu}\}_g$, while $\Gamma_{\mu\nu}^\alpha$ has a different physical interpretation. In that case, what about postulate $\text{GR}_1$? As we have seen, the dynamical equation generated by the variation of the Helmholtz Lagrangian with respect to a variation of the metric $g$ (through its associate tensor density $\pi$) is

$$R_{\mu\nu}(\Gamma) = r_{\mu\nu}(\pi^{\alpha\beta}, \Gamma_{\alpha\beta}^\lambda, \Psi^A, \Psi^A_{\alpha}).$$

(1.10)

To recast it in a more familiar form, recall that the difference between the Ricci tensor of $\Gamma$ and the Ricci tensor of $g$ can be expressed in terms of the first and second covariant derivatives of $g$ with respect to $\Gamma$; namely, one has

$$R_{(\mu\nu)}(\Gamma) - R_{\mu\nu}(g) = \nabla_{(\mu} Q_{\nu)}^\alpha - \nabla_\alpha Q_{\mu\nu}^\alpha + Q_{\beta(\mu}^\beta Q_{\nu)\alpha}^\alpha - Q_{\mu\nu}^\alpha Q_{\alpha\beta}^\beta$$

with $Q_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \{\alpha_{\mu\nu}\}_g = \frac{1}{2} g^{\alpha\beta} (\nabla_{\mu} g_{\nu\beta} + \nabla_{\nu} g_{\beta\mu} - \nabla_{\beta} g_{\mu\nu})$.

(1.11)
Equation (1.9) allows one to replace, after some manipulations, the covariant derivatives $\nabla g$ with terms containing $\Gamma$ and the covariant derivatives of the matterfields $\Psi$ with respect to the Levi–Civita connection of $g$. The final result is that the dynamical equation for $g$ can indeed be put in Einstein form, with an “effective stress–energy tensor” $T_{\mu\nu}$ which is not of variational origin but still allows one to obtain consistent conservation laws.

Let us summarize the results concerning purely affine models as follows:

(i) the affine theory is fully equivalent to GR whenever the right-hand side of (1.9) vanishes identically;

(ii) if the r.h.s. of (1.9) does not vanish, and therefore the connection $\Gamma$ is not the metric connection of $g$, the theory would still be equivalent to GR provided the Levi–Civita connection of $g$ is assumed to be the gravitational connection, while the tensor $Q_{\alpha\mu\nu}$ (1.11) is regarded as an external field representing other than gravitation.

In the latter case, it should be stressed that minimal coupling of matter fields with the connection $\Gamma$ does not entail, in general, that the same matter fields are minimally coupled to $g$: on the contrary, such a coupling would determine not only a non–standard interaction of matter with gravity, but also a direct interaction between $\Psi$ and the tensorfield $Q$, which in general has no reasonable physical interpretation. To construct a purely affine model being physically equivalent to GR in the presence of matter, according to the criterion (B) above, one should instead find a suitable (possibly non–minimal) coupling between $\Psi$ and $\Gamma$ in the affine Lagrangian (1.4) such that, after Legendre transformation, the matter coupling with $g$ which is assumed to hold in General Relativity is recovered. Such interaction terms can be explicitly computed, at least in some cases, using the inverse Legendre transformation; we discuss below an explicit example of the analogous situation occurring in the case of nonlinear metric theories.

**METRIC–AFFINE THEORIES** — The difference between metric–affine and purely metric variational principles for gravity consists in regarding the metric and the connection as mutually independent variables while taking the variation of the action integral\(^1\). Metric–affine models are quite popular, and the well–known “Einstein–Cartan” theory of gravity is based on a suitable generalization of the metric–affine framework.

The equivalence between a particular class of metric–affine action principles and GR has been described in full detail in [5]; in analogy with the purely affine case, it holds for

\(^1\) Some authors seem to take for granted that independent variations of metric and connection (according to what is improperly called “Palatini method”) lead to the same gravitational equations which are obtained in the purely metric variational scheme. As it has been pointed out in [5], this is true for Lagrangians which depend linearly on the curvature scalar $R$, but is otherwise false.
the vacuum theory but is broken by generic matter couplings. Consider the “nonlinear metric–affine Lagrangian”

\[ L_{MA} = f(R) \sqrt{|g|} + L_{\text{mat}}(g, \Gamma, \Psi, \partial \Psi), \tag{1.12} \]

where \( R = R_{\mu\nu}^{(\Gamma)} g^{\mu\nu} \) is the “metric–affine curvature scalar” obtained by taking the trace of the Ricci tensor of \( \Gamma \) with the metric \( g \). It seems reasonable to assume that covariant derivatives are everywhere defined by the connection \( \Gamma \), and hence that \( L_{\text{mat}} \) does not contain derivatives of the metric \( g \) (we shall see below, however, that this assumption has strong physical implications).

Suppose first that there is no matter interaction at all, \( L_{\text{mat}} \equiv 0 \). Then, the Euler–Lagrange equations generated by (1.12), after few manipulations, become

\[
\begin{cases}
  f''(R) R_{(\mu\nu)}^{(\Gamma)} - \frac{1}{2} f(R) g_{\mu\nu} = 0 \\
  \nabla_\alpha (f'(R) \sqrt{g} g^{\mu\nu}) = 0,
\end{cases}
\tag{1.13}
\]

where \( \nabla_\alpha \) still denotes the covariant derivative with respect to \( \Gamma \). Taking the trace of eq. (1.13a) one obtains (in dimension four)

\[ f'(R) R - 2 f(R) = 0. \tag{1.14} \]

Now, (1.14) is an algebraic (or transcendental) equation for \( R \): assuming that the function \( f \) is analytic, it can have only a discrete set of roots \( \{\rho_i\}_{i=1,2,...} \) (unless it is identically satisfied, which happens in \( d = 4 \) if \( f(R) = R^2 \)). Any solution of (1.13a) must then have constant metric–affine curvature:

\[ R = \rho_i \tag{1.15} \]

One substitutes this value of \( R \) into (1.13b) and, provided \( f'(\rho_i) \neq 0 \), the resulting equation is

\[ \nabla_\alpha (g^{\mu\nu} \sqrt{g}) = 0; \tag{1.16} \]

which is nothing but the metricity condition for \( \Gamma \). Consequently, the Ricci tensor of \( \Gamma \) should coincide with the Ricci tensor of \( g \), and the system (1.13) finally becomes

\[
\begin{cases}
  R_{\mu\nu}^{(g)} - \frac{1}{4} \rho_i g_{\mu\nu} = 0 \\
  \Gamma^\alpha_{\mu\nu} = \{\alpha_{\mu\nu}\} g.
\end{cases}
\tag{1.17}
\]

The only dynamical equation left is thus the Einstein equation in the vacuum, with cosmological constant \( \Lambda_i = \frac{\rho_i}{4} \). The function \( f(R) \) occurring in the original nonlinear Lagrangian (1.12) is only reflected in the spectrum of possible values \( \{\Lambda_i\}_{i=1,2,...} \) of the cosmological constant.
In other words, each solution of the system (1.13) is completely represented by the metric $g$ of an Einstein space; each solution corresponds to a definite root $\rho_i$, so that one would actually observe the same value of the cosmological constant at all points of space–time. The cosmological constant, however, can be different for different solutions of the same system of equations, in contrast to the case of General Relativity. In this sense, a single vacuum metric–affine Lagrangian (1.12) is equivalent to a whole set of Einstein–Hilbert Lagrangians, spanned by the allowed values of $\Lambda$ determined by the function $f(R)$ through (1.14).

We now investigate the consequences of matter coupling, using the Legendre transformation. For Lagrangians depending on the Ricci tensor only through the metric–affine curvature scalar, the conjugate momentum is a scalar density $\pi$, and the Legendre map is defined as follows\(^2\):

$$\pi = \frac{\partial L_{MA}}{\partial R} = f'(R) \sqrt{|g|} .$$

(1.18)

It is more convenient to express $\pi$ in terms of the associate scalar field $p = \pi |g|^{-1/2}$; with this definition, and provided

$$\frac{\partial^2 L}{\partial R^2} \neq 0 \quad \Rightarrow \quad f''(R) \neq 0 .$$

(1.19)

we can find a local inverse $r(p)$ of the Legendre map, which fulfills the identity

$$f'[r(p)] \equiv p .$$

(1.20)

The Helmholtz Lagrangian dynamically equivalent to (1.12) is then

$$L_H(g, \Gamma, \partial \Gamma, p, \Psi, \partial \Psi) = p[R_{\mu\nu}(\Gamma)g^{\nu u} - r(p)] \sqrt{|g|} + f[r(p)] \sqrt{|g|} + L_{\text{mat}} .$$

(1.21)

The total variation of $L_H$ with respect to the four independent fields $g$, $p$, $\Gamma$ and $\Psi$ yields the system

\begin{align*}
\left\{ \begin{array}{l}
pR_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}[p(R - r) + f(r)] - [p - f'(r)] \frac{\partial r}{\partial g^{\mu\nu}} = -\frac{\partial L_{\text{mat}}}{\partial g^{\mu\nu}} \\
R - r - [p - f'(r)] \frac{\partial r}{\partial p} = 0 \\
\nabla_\alpha [p \sqrt{g} g^{\mu\nu}] - \nabla_\lambda [p \sqrt{g} g^{\lambda(\mu} \delta^{\nu)\alpha}] = \frac{\partial L_{\text{mat}}}{\partial \Gamma^\alpha_{\mu\nu}} \\
\end{array} \right.
\end{align*}

(1.22)

On account of (1.20), the first two equations simplify to

\begin{align*}
\left\{ \begin{array}{l}
p[R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R] + \frac{1}{2}g_{\mu\nu}[p \cdot r - f(r)] = -\frac{\partial L_{\text{mat}}}{\partial g^{\mu\nu}} \\
R = r(p)
\end{array} \right.
\end{align*}

(1.23)

\(^2\) This definition is based on a general mathematical approach described in [6].
Inserting (1.23b) into the trace of (1.23a) one finds on the left–hand side a function of \( p \) alone:

\[
p \cdot r - 2f(r) = -g^{\mu\nu} \partial L_{\text{mat}} / \partial g^{\mu\nu}.
\]

(1.24)

If the trace of the "matter stress tensor" relative to \( g \) (which coincides with the partial derivative \( \partial L_{\text{mat}} / \partial g^{\mu\nu} \), since \( L_{\text{mat}} \) does not contain derivatives of the metric) vanishes identically, we recover the result already shown: equation (1.24) is in fact equivalent to (1.14). Otherwise, the field equations do not force the scalar field \( p \) to be constant, but rather to be a prescribed function of the arguments of \( L_{\text{mat}} \). It is thus evident that the nonmetricity of \( \Gamma \), described by (1.22c), vanishes only if the matter Lagrangian is independent of \( \Gamma \) and the stress tensor on the r.h.s. of (1.22a) is traceless.

Thus, for particular kinds of matter fields (including the relevant case of the electromagnetic field), the metric–affine nonlinear Lagrangians (1.12) are directly equivalent to GR. For other types of matter couplings, \( \Gamma^\alpha_{\mu\nu} \neq \{^\alpha_{\mu\nu}\}_g \); hence, to establish a physical equivalence one should first assume that the Levi–Civita connection of \( g \), rather than \( \Gamma \), should be regarded as the gravitational field. To recast (1.22a) into a genuine Einstein equation for \( g \), the difference between the Ricci tensors of \( g \) and \( \Gamma \) should then be included in a suitably defined “effective stress–energy tensor”; this would entail derivative couplings between the various fields and the tensor \( Q^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \{^\alpha_{\mu\nu}\}_g \), which would be hardly acceptable on physical grounds. Moreover, as for the purely affine case, one should be aware that minimal coupling to \( \Gamma \) would produce unphysical results, and minimal coupling to \( g \) should be considered instead (a dependence of \( L_{\text{mat}} \) on \( \{^\alpha_{\mu\nu}\}_g \) would contradict the assumption made in (1.12), but would only marginally affect the computations).

As a side remark, let us suggest that the peculiar (and troublesome) consequences of matter interaction in the nonlinear metric–affine framework might yet offer some unexpected resources to the cosmologist. According to our previous discussion, in a vacuum region of space–time the physics described by such models is the same as for General Relativity: \( p \) is constant and determines the value of \( \Lambda \) in that region. A layer of matter separating two vacuum regions may cause a transition between two different values of \( p \), and in that case one would observe different values of the cosmological constant in distinct vacuum regions of the same (connected) universe.

**SCALAR–TENSOR THEORIES** — From the formal viewpoint, scalar–tensor models are purely metric theories including a nonminimal coupling between a (positive–valued) scalar field and the curvature scalar \( R \) of the metric \( g \). Their prototype is the (Jordan–Fierz–)Brans–Dicke Lagrangian:

\[
L_{\text{BD}} = \left[ \varphi R - \frac{\omega}{\varphi} g^{\mu\nu} \varphi,_{\mu} \varphi,_{\nu} \right] \sqrt{|g|}.
\]

(1.25)
The action can be generalized by allowing $\omega$ to depend on the scalar field $\varphi$ and introducing a “cosmological function” $\lambda(\varphi)$ (see [7], [8]):

$$L_{\text{ST}} = \left[ \varphi R - \frac{\omega(\varphi)}{\varphi} g^{\mu\nu} \varphi,_{\mu} \varphi,_{\nu} + 2\varphi \lambda(\varphi) \right] \sqrt{|g|}. \tag{1.26}$$

In such theories, however, the scalar field $\varphi$ is not supposed to describe gravitating matter, but rather an additional degree of freedom of the gravitational field. The gravitational field becomes thus a doublet consisting of a spin–two field (the metric $g$) and a spin–zero field: the latter has no geometric significance but influences the coupling between space–time geometry and matter sources. From the physical viewpoint, therefore, the Lagrangian (1.26) should be regarded as a vacuum Lagrangian. The cosmological function seldom occurs in the current literature and in the sequel we neglect it, assuming $\lambda(\varphi) \equiv 0$.

Gravitating matter is represented by adding to (1.26) a standard interaction Lagrangian, with minimal coupling to $g$. No direct coupling is assumed between matter and the spin–zero gravity field $\varphi$, since there is no physical evidence at all for such an interaction. The full Lagrangian thus becomes

$$L = \left[ \varphi R - \frac{\omega(\varphi)}{\varphi} g^{\mu\nu} \varphi,_{\mu} \varphi,_{\nu} + \ell_{\text{mat}}(\Psi, g) \right] \sqrt{|g|}. \tag{1.27}$$

A procedure known since 1962 as Dicke transformation [9] allows to recast a scalar–tensor Lagrangian into a standard Einstein–Hilbert one, by means of a conformal rescaling. One defines a new metric

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}; \tag{1.28}$$

in terms of the new variables ($\tilde{g}_{\mu\nu}, \varphi$) the action becomes (up to a full divergence term)

$$L = \left[ \tilde{R} - \left( \frac{3}{2} \omega(\varphi) \right) \varphi^{-2} \tilde{g}^{\mu\nu} \varphi,_{\mu} \varphi,_{\nu} + \varphi^{-2} \ell_{\text{mat}}(\Psi, \varphi^{-1} \tilde{g}) \right] \sqrt{|\tilde{g}|}, \tag{1.29}$$

($\tilde{R}$ being the curvature scalar of the metric $\tilde{g}$) and after a redefinition of the Brans–Dicke scalar,

$$d\phi \equiv \left( \frac{3}{2} \omega(\varphi) + \frac{3}{2} \right) \frac{d\varphi}{\varphi}, \quad \omega > -\frac{3}{2} \tag{1.30}$$

it takes the standard form of the action for a linear massless scalar field minimally coupled to the metric, at the price of introducing a coupling between the external matter and the scalar field $\phi$:

$$L = \left[ \tilde{R} - \tilde{g}^{\mu\nu} \phi,_{\mu} \phi,_{\nu} + \tilde{\ell}_{\text{mat}}(\Psi, \phi, \tilde{g}) \right] \sqrt{|\tilde{g}|}, \tag{1.31}$$

where $\tilde{\ell}_{\text{mat}} = \varphi^{-2} \ell_{\text{mat}}(\Psi, \varphi^{-1} \tilde{g})$, with $\varphi$ replaced by the function of $\phi$ defined by (1.30). The possible insertion of a cosmological function in the scalar–tensor Lagrangian (1.27) would merely end up in a potential term for the scalar field $\phi$. 

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According to a common terminology, the original metric $g$ is referred to as the *Jordan frame* while the rescaled metric $\tilde{g}$ is said to provide the *Einstein frame*. Apart from the use of the word “frame”, which seems objectionable, this terminology is somehow misleading as it suggests that the rescaled metric obeys the Einstein equation, while the original one does not. As a matter of fact, also the dynamics of the Jordan frame metric $g$ can be represented by an Einstein equation. After some manipulations described in full detail in [10], the field equations for the metric and the Brans–Dicke scalar field can be rewritten as follows ($\varphi > 0$ everywhere by assumption):

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{1}{\varphi} (\nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \Box \varphi) + \frac{\omega}{\varphi^2} (\varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} \varphi_{,\alpha} \varphi_{,\alpha}) + \frac{1}{\varphi} T_{\mu\nu} (\Psi, g),$$ (1.32)

$$\Box \varphi = \frac{1}{2\omega + 3} \left( T_\alpha^\alpha \frac{d\omega}{d\varphi} \varphi_{,\alpha} \varphi_{,\alpha} \right), \quad \text{with} \quad T_{\mu\nu} \equiv - \frac{1}{\sqrt{|g|}} \delta \left( \ell_{\text{mat}} \sqrt{|g|} \right) \delta g_{\mu\nu}. \quad (1.33)$$

Let us compare these equations with the Einstein–frame field equations, generated by (1.31). For simplicity we restrict ourselves to the case of Brans–Dicke theory, i.e. we set $\omega \equiv \text{const}$. Having defined $\gamma = (\omega + \frac{3}{2})^{-\frac{1}{3}}$, the equations for $\tilde{g}_{\mu\nu}$ and for the scalar field $\phi = \frac{1}{\gamma} \ln \varphi$ become:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R} \tilde{g}_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + e^{-\gamma \phi} T_{\mu\nu},$$ (1.34)

$$\tilde{\Box} \phi = \gamma \frac{2}{2} T_{\mu\nu} \tilde{g}^{\mu\nu}. \quad (1.35)$$

In the latter equations, $T_{\mu\nu}$ is still the stress tensor defined in (1.33), now depending on the fields $(\Psi, \phi, \tilde{g})$ through the original variables $(\Psi, g)$; the stress tensor corresponding to $\ell_{\text{mat}}$ in (1.31) is instead $\tilde{T}_{\mu\nu} = e^{-\gamma \phi} T_{\mu\nu}$. The symbol $\tilde{\Box}$ denotes the d’Alembert wave operator associated to the new metric $\tilde{g}$.

One learns by comparing (1.32) and (1.34) that the structural difference between the Jordan frame and the Einstein frame representation of the field dynamics lies only in the different properties of the total stress–energy tensor for matter and scalar gravity.

The contribution of the Brans–Dicke scalar to the effective stress–energy tensor in (1.32) has unphysical features: it is linear in the second derivatives of $\varphi$, and therefore it does not fulfill the Weak Energy Condition. For this reason, in the Jordan frame it is impossible to prove that the total ADM energy is bounded from below and the ground state vacuum solution is stable against matter perturbation. In fact, the only available criterion for this purpose is the Positive Energy Theorem [11], which holds provided the stress tensor satisfies the Dominant Energy Condition. Notice that the failure of the Dominant Energy Condition does not necessarily entail that the energy is unbounded from below and
no stable ground state exists: in fact, the vacuum scalar–tensor Lagrangian (1.26), being equivalent to (1.31), has the same stability properies as GR. The stability of the theory, however, can be proved only after rescaling the metric from the Jordan to the Einstein frame, since in the Jordan frame the total stress–energy tensor is always indefinite.

Hence, the equivalence between scalar–tensor models and General Relativity holds in a sense which is quite different from what we have encountered in the previous examples of purely affine or metric–affine gravity. In the original variables, the model is equivalent to a general–relativistic scalar field with an unphysical coupling to the metric, while in the rescaled Einstein frame the additional degree of freedom is instead represented by a linear massless scalar field, minimally coupled. The scalar field is assumed to be positive everywhere for physical solutions, therefore for such solutions the conformal rescaling can be always performed globally on space–time.

As long as the scalar field is a priori interpreted as representing part of the gravitational interaction in the scalar–tensor theory, the equivalence with GR should apparently be regarded as purely mathematical. The fact that physical spacetime distances are assumed to be measured by the Jordan frame metric, rather than by the Einstein frame metric, is however the only real difference between the Brans–Dicke model and General Relativity with an external scalar field, since the scalar–tensor Lagrangian (1.27) is nothing but the ordinary Einstein–Hilbert Lagrangian (1.31) written in an alternative set of variables.

PURELY METRIC GRAVITY THEORIES — In the most popular versions of gravity theory, the only independent variable representing the gravitational field is the metric tensor. It is impossible to construct a covariant Lagrangian containing only first derivatives of the metric (unless a fixed background connection is introduced [12]), therefore a purely metric Lagrangian is necessarily of second order, and depends on the Riemann tensor components. Thus, a generic purely metric Lagrangian (different from the Einstein–Hilbert Lagrangian) generates fourth–order equations: such models are thus called higher–derivative gravity theories. A typical higher–derivative Lagrangian, including matter interaction, has the following form:

\[ L_{PM} = f(g_{\mu\nu}, R^\alpha_{\beta\mu\nu}, \Psi^A, \nabla_\mu \Psi^A) \sqrt{|g|}, \quad (1.36) \]

where the covariant derivatives and the curvature tensor are those relative to the metric \( g \). In most of the current literature, quadratic dependence of the Lagrangian on the curvature tensor is generally assumed (possibly including a linear term of the Einstein–Hilbert type), while higher powers are seldom considered\(^3\).

\( ^3 \) Quite recently, some authors ([13], [14]) have investigated Lagrangians including

Among all higher-derivative models, a subclass deserves special consideration, namely the Lagrangians depending on the derivatives of the metric only through a (polynomial or analytic) function of the curvature scalar $R$. These are called nonlinear gravitational Lagrangians, and are commonly written in the form

$$ L_{NL} = f(R) \sqrt{|g|} + L_{\text{mat}}(g_{\mu\nu}, \Psi^A, \nabla_\mu \Psi^A) \quad (1.37) $$

(as we shall see in the sequel, however, a more general form should be considered). The analogy with (1.12) is evident, but the dynamics is quite different in the purely metric variational framework. The gravitational field equation is in fact

$$ f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) = T_{\mu\nu}, \quad (1.38) $$

where, as usual, $T_{\mu\nu} = -\frac{1}{\sqrt{|g|}} \frac{\delta L_{\text{mat}}}{\delta g^{\mu\nu}}$.

It is well known that equation (1.38) can be formally recast into the equations of General Relativity with a scalar field. Many authors describe the relationship between the two theories purely in terms of a conformal transformation of the metric, as in the case of scalar–tensor theories (the history of the introduction of the conformal rescaling and the appropriate references can be found in [10], [15]). This is probably the easiest way to obtain this result as far as only the field equations are considered, but the comparison of the whole lagrangian structure of the two theories requires a deeper analysis.

Similarly to what has been done above while dealing with the metric–affine case, let us introduce a scalar conjugate momentum $p$, with the Legendre map

$$ p = \frac{1}{\sqrt{|g|}} \frac{\partial L_{NL}}{\partial R} = f'(R). \quad (1.39) $$

Also, let $r(p)$ be a function such that $f'(R)\big|_{R=r(p)} \equiv p$; such a function (possibly not unique) exists if $f''(R) \neq 0$. The Helmholtz Lagrangian equivalent to (1.37) is

$$ L_h = p[R(g) - r(p)] \sqrt{|g|} + f[r(p)] \sqrt{|g|} + L_{\text{mat}}. \quad (1.40) $$

The difference between (1.40) and (1.21) is that the independent fields are now $g$, $p$ and $\Psi$ only; the variation of the action corresponding to (1.40) yields the equations

$$ \begin{cases} p R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} [p(R - r) + f(r)] + g_{\mu\nu} \Box p - \nabla_\mu \nabla_\nu p = T_{\mu\nu} \\ R(g) = r(p) \\ \frac{\delta L_{\text{mat}}}{\delta \Psi^A} = 0, \end{cases} \quad (1.41) $$

derivatives of the Riemann tensor (in particular, scalar terms of the form $\Box^k R$), in view of possible generalizations of the order–lowering technique already known for fourth–order gravity, but without evident physical motivation.
where the stress tensor $T_{\mu\nu}$ is the same as in (1.38). Due to the second derivatives of $p$ occurring in (1.41), which were absent in the metric–affine case (1.22), the trace of the first equation does no longer produce an algebraic equation for $p$, but rather a differential equation which can be used to recast the system in the following form (excluding the points where $p = 0$):

\[
\begin{cases}
G_{\mu\nu} = p^{-1} \nabla_\mu \nabla_\nu p - \frac{1}{6} \{ p^{-1} f[r(p)] + r(p) \} g_{\mu\nu} + p^{-1} T_{\mu\nu} \equiv \theta_{\mu\nu} \\
\Box p = \frac{2}{3} f[r(p)] - \frac{1}{3} p \cdot r(p) ;
\end{cases}
\]

(1.42)

It is instructive to compare (1.42) with the equations (1.32) of scalar–tensor theory. In fact, the “Legendre transform” of the nonlinear Lagrangian (1.37) is (formally) a particular case of scalar–tensor theory, and the resulting field equations can be put in Einstein form, introducing the “effective stress–energy” tensor $\theta_{\mu\nu}$ which is not a variational derivative, but is still covariantly conserved due to Bianchi identities. Let us stress that the Einstein equation (1.42), as well as (1.32), has been obtained without any conformal transformation.

Mathematically, each purely metric (PM) nonlinear theory can be regarded as a metric–affine (MA) theory where the additional constraint $\Gamma_{\mu\nu}^{\alpha} = \{ \alpha_{\mu\nu} \}_g$ has been imposed while taking the variation of the action. From this viewpoint, the relationship between purely metric and metric–affine nonlinear gravity theories can be described as follows:

(i) When the metricity constraint is present (PM theories), the degrees of freedom in the second–order picture of the dynamics include a metric tensor and a scalar field (besides possible external matter fields). The metric tensor always obeys Einstein equations, while the self–interaction potential of the scalar field depends on the choice of the function $f(R)$ in the Lagrangian (but does not depend on the matter Lagrangian).

(ii) If the metricity constraint is removed (MA theories) the scalar field is “frozen down”: its dynamical equation is substituted by an algebraic equation, including a term depending on the presence of matter interaction (r.h.s. of (1.24)). In the vacuum case, $p$ acts as a cosmological constant, and the nonmetricity of the connection vanishes as a consequence of the field equations\(^4\). If matter is present, instead, the connection is not metric and the field $p$ is nonconstant; yet $p$ does not represent an independent degree of freedom, since it is completely determined by the matter distribution.

Let us mention the fact that the existence of a second–order picture of the dynamics, which (at least formally) coincides with General Relativity, is not only a feature of the

\(^4\) Therefore, in the vacuum case each solution of the field equations of the MA theory is also a solution of the corresponding PM model: this can be easily seen by imposing $p = \text{const.}$ in the system (1.42). The classical solutions of a MA model are thus a proper subset of the (much larger) set of classical solutions of the PM model with the same $f(R)$.
nonlinear metric theories (1.37), but holds for most higher–derivative models (1.36). If
the purely metric Lagrangian depends on the full Ricci tensor, as is the case for the most
general quadratic Lagrangian in dimension four\(^5\), the conjugate momentum is a rank–two
tensor, and the Legendre transform of the theory is a sort of “bimetric” theory rather than
a scalar–tensor model. For a complete discussion, we refer the reader to [6], [15], [16].

II. Determination of the physical variables in nonlinear gravity theories and
physical equivalence with General Relativity

We shall now present a physical motivation to introduce a conformal rescaling in the
case of nonlinear metric gravity theories. As we have just recalled, a GR–like formulation
can be obtained without any redefinition of the metric: for this purpose, it is enough to
isolate the additional spin–0 degree of freedom due to the occurrence of nonlinear second-
order terms in the Lagrangian, and encode it into an auxiliary scalar field by means
of the Legendre transformation. However, while the dynamical terms for the metric in
equation (1.42a) are exactly the standard ones, the source terms containing the field \(p\)
are substantially different from those expected for a gravitating scalar field, according
to General Relativity: the effective stress-energy tensor \(\theta_{\mu \nu}\) is plagued by the same unphysical
features as the r.h.s. of (1.32). As we have already seen for the case of scalar–
tensor theories, the Dominant Energy Condition can be restored by a redefinition of the field variables. A
redefinition of the scalar field alone is useless to this purpose, while the following conformal
rescaling of the metric by the field \(p\) yields the correct result:

\[
\tilde{g}_{\mu \nu} = pg_{\mu \nu}.
\]

(2.1)

It can be checked by direct computation that this conformal transformation is the only one
which deletes the linear second–order term \(\nabla_\mu \nabla_\nu p\). To get exactly the standard coupling
terms, one redefines also the scalar field, \(p \mapsto \phi\), by

\[
p = e^{\sqrt{\frac{2}{3}} \phi};
\]

(2.2)

both definitions make sense only if \(p > 0\), which is not always satisfied (for quadratic
Lagrangians including a linear Einstein–Hilbert term, \(p\) is everywhere positive for solutions
close to the Minkowski vacuum; for further details we refer the reader to [10]). The
Lagrangian then becomes (up to a full divergence)

\[
\tilde{L} = \left[ \tilde{R} - \tilde{g}^{\mu \nu} \phi,_{\mu} \phi,_{\nu} - V(\phi) \right] \sqrt{\left| g \right|}
\]

(2.3)

\(^5\) For \(d = 4\), the Lagrangian density \((R^2 - 4R^{\mu \nu}R_{\mu \nu} + R^{\alpha \beta \mu \nu}R_{\alpha \beta \mu \nu})\sqrt{|g|}\) generates trivial
field equations and can be freely added to any action integral, allowing one to remove any
quadratic term containing the Weyl tensor; such terms play therefore an effective role only
in higher–dimensional theories.
with the potential
\[
V(\phi) = e^{-\sqrt{2\phi} r[p(\phi)]} - e^{-2\sqrt{2\phi} f(r[p(\phi)])}
\]  (2.4)

In the “rescaled” Lagrangian (2.3), only the self–interaction potential \( V(\phi) \) keeps trace of the original nonlinear Lagrangian (1.37), while the dynamical terms are “universal” and are exactly those required by General Relativity. Borrowing the terminology from scalar–tensor theories, we say that the set of variables \((g, p)\) provides the “Jordan frame”, while the pair \((\tilde{g}, \phi)\) defines the “Einstein frame” for nonlinear theories.

Now, long–posed questions such as “are nonlinear metric theories of gravity physically equivalent to General Relativity?” or “which is the physical significance of the conformal rescaling?” can be phrased in rigorous and unambiguous terms. The problem is actually reduced to the following: “which metric is the true gravitational metric?” In fact, both metrics obey Einstein equations; therefore, according to our initial remarks, whether the theory is physically equivalent to General Relativity or not is a property which should be assessed by considering the coupling of the other fields with the gravitational field. Since the coupling with the metric is strongly affected by the conformal transformation, it is evident that an external field cannot be coupled in a physically satisfactory way to both metrics at the same time.

Previous attempts to determine the physical metric on theoretical grounds were based on two distinct kinds of arguments: a first group of authors (e.g. [17], [18]) tried to show that only one picture of the theory (i.e., either the Jordan frame or the Einstein frame) leads to a consistent formulation, while the other entails a breakdown of the expected conservation laws. The second group of authors (see [10] for refs.) observed that the matter fields are minimally coupled to the original metric \( g \) in the Lagrangian (1.37), and regarding the rescaled metric as the gravitational field would produce unwanted effects such as nonconstant masses depending on the scalar field \( \phi \).

The consistency argument, however, typically rests on calculations in which the stress–energy tensor is computed by taking the variational derivative of \( L_{\text{mat}} \) with respect to one metric, while covariant derivatives are taken using the Levi–Civita connection of the other metric. A careful analysis shows that such calculations are ill–grounded, and appropriate conservation laws do hold in both frames [10]. The argument based on matter coupling, on the other hand, is a sort of \textit{petitio principii}. In fact, it is evident that the Lagrangian (1.37) is constructed by assuming \textit{a priori} that matter should be minimally coupled to \( g \). To deal with a concrete example, suppose that one wishes to couple a (complex) charged scalar field to higher–derivative gravity (with a quadratic Lagrangian) and to the electromagnetic field. According to (1.37), one would be lead to define the Lagrangian as follows:

\[
L_{\text{NL}} = \left[ aR^2 + R - g^{\mu\nu} D_\mu \psi (D_\nu \psi)^* - m^2 \psi \psi^* - \frac{1}{8\pi} F_{\alpha\mu} F_{\beta\nu} g^{\mu\nu} g^{\alpha\beta} \right] \sqrt{|g|},
\]  (2.5)
where \( D_\mu \psi \equiv \partial_\mu \psi - ie A_\mu \psi \). In the corresponding Einstein frame Lagrangian, the coupling of \( \psi \) with \( \tilde{g} \) becomes unphysical, and the mass of \( \psi \) is rescaled by a nonconstant factor depending on \( \phi \). However, unless there are independent motivations to assume that the Lagrangian should necessarily be as in (2.5), one might consider instead the following Lagrangian:

\[
L_{NL} = \left\{ \frac{[R - g^{\mu\nu} D_\mu \psi (D_\nu \psi)^* + \frac{1}{2a}]}{4m^2 \psi \psi^* + \frac{1}{a}} - \frac{1}{8\pi} F_{\alpha\mu} F_{\beta\nu} g^{\mu\nu} g^{\alpha\beta} - \frac{1}{4a} \right\} \sqrt{|g|}, \tag{2.6}
\]

which, in spite of its exotic appearance, reduces to the same vacuum Lagrangian \((aR^2 + R)\) when the field \( \psi \) is “switched off”. The Einstein frame Lagrangian corresponding to (2.6) turns out to be

\[
\tilde{L} = \left[ \tilde{R} - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \left( e^{-\sqrt{\frac{3}{2}} \phi} - 1 \right)^2 \frac{1}{4ae^{-\sqrt{\frac{3}{2}} \phi}} - \tilde{g}^{\mu\nu} D_\mu \psi (D_\nu \psi)^* - m^2 \psi \psi^* - \frac{1}{8\pi} F_{\alpha\mu} F_{\beta\nu} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} \right] \sqrt{|\tilde{g}|}, \tag{2.7}
\]

and in this case one would easily accept the idea that the physical metric is \( \tilde{g} \). From this example one learns that the coupling of a given metric to matter fields is in fact determined by the physical significance ascribed to it, i.e., by its relation to the physical metric. Thus, the physical metric should be singled out \textit{a priori}, i.e., already in the vacuum theory.

One might ask whether nonlinear Lagrangians such as (2.6), yielding minimal coupling with the Einstein frame metric, can be systematically produced. Let us warn the reader that the naive procedure consisting in taking a vacuum nonlinear Lagrangian, performing a conformal rescaling, then adding to the Einstein frame Lagrangian the appropriate minimal interaction term and finally rescaling back the metric by the same conformal factor, leads to incorrect results. The inverse rescaling has to be done in a more subtle way, and once again the fundamental role played by the Legendre transformation becomes apparent. In [10] the reader can find a detailed description of the correct method.

It could seem at this point that the two frames are equally satisfactory: both pictures are consistent, and both allow minimal coupling with external matter. However, we have seen that a vacuum nonlinear Lagrangian is equivalent to a scalar–tensor theory, which in turn reduces to General Relativity with an unphysical effective stress–energy tensor in the Jordan frame, or with a well–behaved stress–energy tensor in the Einstein frame. In the Einstein frame, the Positive Energy Theorem can be applied and one can prove the existence of a stable ground state; thus, one knows that the vacuum theory is stable (for suitable choices of the function \( f(R) \), see [10]; the stability of a quadratic Lagrangian was first proved in [19]). What about the stability of the interacting model?

Let us revert to the example of the two Lagrangians (2.5) and (2.6). In both cases, the Positive Energy Theorem cannot be applied directly to the fourth–order picture (to
our present knowledge), and we should rely on the second–order picture, introducing the scalar field $p$ (or $\phi$). For the Lagrangian (2.5), although the matter Lagrangian fulfills the appropriate energy condition, the total stress–energy tensor in the second–order picture has indefinite signature, due to the unavoidable contribution of the scalar field $p$. Thus, in the Jordan frame one loses all control on the positivity of the total energy and on the stability of the theory, as soon as ordinary matter is coupled to gravity. On the contrary, inserting the same matter Lagrangian (with the rescaled metric) in the Einstein frame Lagrangian, as was done in (2.7), is perfectly safe, since the scalar field $\phi$ gives a positive contribution to the total stress–energy tensor.

The stability of the model in the presence of matter coupling with the Jordan metric can indeed be checked, upon transformation to the Einstein frame: since the stress–energy tensor of external matter is simply rescaled by the conformal factor (which is assumed to be positive), it turns out that a standard coupling in the Jordan frame does not break the Dominant Energy Condition. However, even in this case, the conserved quantity which attains a minimum at the stable vacuum, and thus can be physically identified with the total energy of the system, is the ADM energy defined in the Einstein frame, not in the Jordan frame. For this reason the Einstein frame is the most natural candidate for the role of physical frame in nonlinear gravity theories. The Jordan frame formulation may instead be regarded as a useful tool to circumvent the problem of the physical nature of the scalar field: a nonlinear Lagrangian, which does not contain any scalar field, can be introduced as the primitive object of the theory, then the scalar field is generated as a by–product of the transformation to physical variables. Such a mechanism could be compared to the standard procedure of “spontaneous symmetry breaking” in gauge theories.

In conclusion, according to our analysis, the correct way to formulate a nonlinear metric theory as a viable model of gravity (including matter) consists in assuming that the Einstein metric is the gravitational metric, and therefore that the theory is physically equivalent to General Relativity. This suggests a negative answer to the question raised in the title of this talk.

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References
[1] C. Aragone, S. Deser, *Nuovo Cim.* **57B** (1980) 33
[2] S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space-time*, Cambridge Univ. Press (Cambridge 1973)
[3] A.S. Eddington, *The Mathematical Theory of Relativity*, Cambridge Univ. Press (Cambridge 1924)
[4] M. Ferraris, J. Kijowski, *Gen. Rel. Grav.* **14** (1982) 165
[5] M. Ferraris, M. Francaviglia, I. Volovich, *Class. Quantum Grav.* **11** (1994) 1505
[6] G. Magnano, M. Ferraris, M. Francaviglia, *J. Math. Phys.* **31** (1990) 378
[7] T. Damour, G. Esposito–Farèse, *Class. Quantum Grav.* **9** (1992) 2093
[8] C.M. Will, *Theory and Experiment in Gravitational Physics*, Cambridge Univ. Press (Cambridge 1981)
[9] R.H. Dicke, *Phys. Rev.* **125** (1962) 2163
[10] G. Magnano, L.M. Sokolowski, *Phys. Rev.* **D50** (1994) 5039
[11] G.T. Horowitz, in: *Asymptotic Behavior of Mass and Spacetime Geometry*, Lect. Notes in Physics Vol. 202, ed. F. Flaherty, Springer (Berlin 1984)
[12] M. Ferraris, M. Francaviglia, *Gen. Rel. Grav.* **22** (1990) 965
[13] D. Wands, *Class. Quantum Grav.* **11** (1994) 269
[14] F.J. de Urries, J. Julve, *preprint IMAFF 95/36* (1995)
[15] G. Magnano, M. Ferraris, M. Francaviglia, *Class. Quantum Grav.* **7** (1990) 557
[16] J. Alonso, F. Barbero, J. Julve, A. Tiemblo, *Class. Quantum Grav.* **11** (1994) 865
[17] C.H. Brans, *Class. Quantum Grav.* **5** (1988) L197
[18] S. Cotsakis, *Phys. Rev.* **D47** (1993) 1437
[19] A. Strominger, *Phys. Rev.* **D30** (1984) 2257