Generalized harmonic-fluid approach for the off-diagonal correlations of a one-dimensional interacting Bose gas

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We develop a generalized harmonic-fluid approach, based on a regularization of the effective low-energy Luttinger-liquid Hamiltonian, for a one-dimensional Bose gas with repulsive contact interactions. The method enables us to compute the complete series corresponding to the large-distance, off-diagonal behavior of the one-body density matrix for any value of the Luttinger parameter K. We compare our results with the exact ones known in the Tonks-Girardeau limit of infinitely large interactions (corresponding to K = 1) and, different from the usual harmonic-fluid approach, we recover the full structure of the series. The structure is conserved for arbitrary values of the interaction strength, with power laws fixed by the universal parameter K and a sequence of subleading corrections.

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I. INTRODUCTION

Quasi-one-dimensional quantum fluids display unique properties due to the major role played by quantum fluctuations in reduced dimensionality (see e.g., [1]). For example, in an interacting 1D Bose fluid quantum fluctuations destroy the off-diagonal long-range order of the one-body density matrix (or first-order coherence function), defined as \( \rho_1(x_1, x_2) = \langle \psi_d(x_1) \psi_d(x_2) \rangle \), where \( \psi(x) \) is the bosonic field annihilation operator at position \( x \). In contrast to its 3D counterpart, where the one-body density matrix at large distances tends to a constant which corresponds to the fraction of Bose-condensed atoms [2], in 1D the one-body density matrix at zero temperature decays as a power law [3, 4], the coefficient of the power law being fixed by the interaction strength: the decay is faster as the interaction strength increases from the Tonks-Girardeau regime [5] of impenetrable bosons.

The one-body density matrix is not only a fundamental quantity as it measures the macroscopic coherence properties of a quantum fluid with bosonic statistics, but also because it is directly related (by Fourier transformation) to the momentum distribution of the fluid. While the momentum distribution for a Bose fluid is typically narrow and peaked around wavevector \( k = 0 \), its specific form, its broadening due to interactions and its possible singularities give a wealth of information about the nature of the correlated fluid under study.

Quasi-one-dimensional Bose fluids find an experimental realization in experiments with ultra-cold atomic gases confined to the minima of a 2D optical lattice [7]. For those systems, the momentum distribution is one of the most common observables and the one-body density matrix has been measured as well [8], though not yet in the quasi-1D geometry.

From a theoretical point of view, the calculation of the one-body density matrix for the 1D interacting Bose gas has a long history. In the Tonks-Girardeau limit where the exact many-body wavefunction is known by means of a mapping onto a gas of spinless fermions [6], the problem reduces to the evaluation of a \((N - 1)\)-dimensional integral. This mathematical challenge has been addressed first by Lenard [9], and by several subsequent works (see e.g., [10, 11, 12, 13]). The main result is the evaluation of the large-distance behavior of the one-body density matrix in the form of a series expansion, (from [12])

\[
\rho_{1 TG}(z) = \frac{\rho_0}{|z|^{1/2}} \left[ 1 - \frac{1}{32} \frac{1}{z^2} - \frac{1}{8} \frac{\cos(2z)}{z^2} - \frac{3}{16} \frac{\sin(2z)}{z^3} + \frac{33}{2048} \frac{1}{z^4} + \frac{93}{256} \frac{\cos(2z)}{z^4} + \ldots \right],
\]

(1)

where the constant \( \rho_0 \) and the coefficients have been calculated exactly. In Eq. (1) and in the following we express the one-body density matrix in units of the average particle density \( \rho_0 \) and as a function of the scaled relative coordinate \( z = \pi \rho_0 (x_1 - x_2) \).

For the case of arbitrary interaction strength, the calculation of the correlation functions remains a challenge (see e.g., [14]), although the model of bosons with contact repulsive interactions is integrable by the Bethe-Ansatz technique [15]. The power-law decay at large distances can be inferred using a harmonic-fluid approach [3, 4, 16, 17, 18]. The latter is based on an effective, low-energy Hamiltonian describing the long-wavelength collective excitations of the fluid having a linear excitation spectrum \( \omega(k) = \nu_e k \). The resulting structure for the large-distance series of the one-body density matrix reads (from [4])

\[
\rho_{1 LL}(z) \sim \frac{1}{|z|^{1/2K}} \sum_{m=0}^\infty B_m \frac{\cos(2mz)}{z^{2m^2K}},
\]

(2)

\( B_m \) are coefficients fixed by the interaction strength, and the power law dependence on \( z \) is modified by the so-called effective power \( K \).
where \( K = \pi \rho_0/mv_s \) is the universal Luttinger-liquid parameter and the coefficients \( B_m \) are nonuniversal and cannot be obtained by the harmonic-fluid approach. By noticing that the harmonic-fluid approach is valid also in the Tonks-Girardeau regime and corresponds to the case of Luttinger parameter \( K = 1 \), one can directly compare the predictions of the two methods. Specifically, this comparison shows that the structure of Eq. (1) is richer than that of Eq. (2) obtained by the standard harmonic-fluid approach.

The central result of this study is by defining a properly regularized harmonic-fluid model (to be detailed below) the following general structure of the one-body density matrix can be obtained for arbitrary values of the Luttinger parameter \( K \):

\[
\rho_1(z) \sim \frac{1}{|z|^{1/2K}} \left[ 1 + \sum_{n=1}^{\infty} \frac{a'_n}{z^{2n}} + \sum_{m=1}^{\infty} b_m \cos(2mz) \left( \sum_{n=0}^{\infty} \frac{b'_n}{z^{2n}} \right) + \sum_{m=1}^{\infty} c_m \sin(2mz) \right],
\]

(3)

where all the coefficients \( a'_n, b_m, b'_m, c_m \) and \( c'_m \) are nonuniversal and need to be calculated by a fully microscopic theory (for possible methods see e.g. [19, 20]). We note that Eq. (3): (i) generalizes Eq. (2) and (ii) has the full structure of the exact result (1) in the Tonks-Girardeau limit. As a direct consequence of the series structure (3) the momentum distribution will display singularities in its derivatives for \( k = \pm 2m\pi \rho_0 \).

II. REGULARIZED HARMONIC-FLUID APPROACH FOR BOSONS

We proceed by outlining the method used. We describe the Bose gas with contact interactions (Lieb-Liniger model [13]) by a harmonic-fluid Hamiltonian, expressed in terms of the fields \( \theta(x) \) and \( \phi(x) \) which describe the density and phase fluctuations of the fluid [4]:

\[
\mathcal{H}_{LL} = \frac{\hbar v_s}{2\pi} \int_0^L dx \left[ K (\nabla \phi(x))^2 + \frac{1}{K} (\nabla \theta(x))^2 \right].
\]

(4)

The parameters \( K \) and \( v_s \) entering Eq. (4) above are related to the microscopic interaction parameters \( \hbar \omega_j b_j b^\dagger_j \), the phase field \( \phi(x) \) is related to the velocity of the fluid \( \nu(x) = \hbar \nabla \phi(x)/m \) and the field \( \theta(x) \) defines the fluctuations in the density profile. We have adopted here an effective low-energy description, which assumes that the collective excitations in the fluid are noninteracting and phonon-like. The description breaks down at a length scale \( a \) of the order of the typical inter-particle distance \( \rho_0^{-1} \), as it neglects the broadening and curvature of the Bose gas spectrum at finite momentum [15, 22]. Within the harmonic-fluid approximation, the bosonic field operator is expressed as \( \Psi^\dagger(x) = \sqrt{\rho(x)} e^{-i\phi(x)} \).

Specifically, its expression in terms of the fields \( \theta(x) \) and \( \phi(x) \) reads [4, 21]

\[
\Psi^\dagger(x) = A [\rho_0 + \Pi(x)]^{1/2} \sum_{m=-\infty}^{+\infty} e^{2mi\theta(x)} e^{-i\phi(x)},
\]

(5)

where \( A \) is a nonuniversal constant, \( \Pi(x) = \nabla \theta(x)/\pi \), and for compactness of notation we have introduced the field \( \Theta(x) = \theta_B + \pi \rho_0 x + \theta(x) \), where \( \theta_B \) is chosen in order to ensure that the average of \( \theta(x) \) vanish.

In order to calculate the one-body density matrix, we expand the fields \( \theta(x) \) and \( \phi(x) \) in terms of the normal modes (bosonic) operators \( b_j, b^\dagger_j \) which diagonalize the Hamiltonian (4) such that \( \mathcal{H}_{LL} = \sum_j \hbar \omega_j b_j b^\dagger_j \), with \( \omega_j = v_s k_j \). As we are interested in the thermodynamic limit, we have chosen for simplicity the periodic boundary conditions for a Bose gas in a uniform box of length \( L \), where \( k_j = 2\pi j/L \). In this case the mode expansion reads

\[
\phi(x) = \frac{1}{2\sqrt{K} \sum_{j \neq 0} \sqrt{\frac{\pi x}{J}} e^{-|k_j|/2}} \left( e^{ik_j x} b_j + e^{-ik_j x} b^\dagger_j \right)
\]

(6)

\[
\Theta(x) = \sqrt{\frac{K}{2} \sum_{j \neq 0} \frac{1}{\sqrt{|J|}} \left( e^{ik_j x} b_j + e^{-ik_j x} b^\dagger_j \right)} + \theta_0 + \frac{\pi x}{L} N,
\]

(7)

with \( N \) and \( J \) being the particle number and angular momentum operators, and \( \phi_0, \theta_0 \) being their conjugate fields in the phase-number representation [4]. The zero-mode fields \( \phi_0, \theta_0 \) do not enter the calculation of the one-body density matrix, as it turns out to depend only on the differences \( \theta(x_1) - \theta(x_2) \) and \( \phi(x_1) - \phi(x_2) \).

In the expressions (6) and (7) above we have introduced the short-distance cutoff \( a \sim \rho_0^{-1} \), thus regularizing the effective theory. The one-body density matrix \( \rho_1(x_1, x_2) = \langle \psi^\dagger(x_1) \psi(x_2) \rangle \) is obtained in the generalized harmonic-fluid approach from Eq. (5) as

\[
\rho_1(x_1, x_2) = |A|^2 \sum_{(m, m') \in \mathbb{Z}^2} \langle [\rho_0 + \Pi(x_1)]^{1/2} e^{im\theta(x_1)} e^{-im\theta(x_2)} \rangle e^{2im\theta(x_1)} e^{-2im\theta(x_2)},
\]

(8)

where the only nonvanishing leading terms satisfy \( m = m' \) [1].

We detail now the calculation of the quantum average appearing in Eq. (8). In order to display its dependence only on differences between fields, we re-write the central term as

\[
e^{i2m\theta(x_1)} e^{-i\phi(x_1)} e^{i\phi(x_2)} = e^{i2m(\theta(x_1) - \theta(x_2))} e^{-i(\phi(x_1) - \phi(x_2))}
\]

\[
\times e^{i(m(\theta(x_1) + \theta_0) - \phi(x_1) + \phi(x_2) - \phi_0)}
\]

(9)
where \( \varphi(x) = \phi(x) - \pi x(J)/L \) and the commutator between the \( \theta \) and \( \phi \) fields is computed in Eq. \([15]\) below. We then perform a series expansion of the square root terms \( [1 + \Pi/ho_0]^{1/2} \) in the one-body density matrix. We define \( X = \Pi(x_1)/\rho_0, Y = \Pi(x_2)/\rho_0 \), and \( Z = i2m(\theta(x_1) - \theta(x_2)) - i(\varphi(x_1) - \varphi(x_2)) \). Using the fact that the fields \( X, Y \) and \( Z \) are Gaussian with zero average, we obtain from Wick’s theorem

\[
\langle \sqrt{1 + X} e^{Z} \sqrt{1 + Y} \rangle = e^{\frac{1}{2} \langle Z^2 \rangle} \sum_{k,l=0}^{\infty} \frac{(2k)!(2l)!}{k!l!(2k-1)(2l-1)} \frac{(2k)!}{(2l)!} \frac{(i\sqrt{2})^{5(k+l)-2j}k!(k-1)!l!}{(l-j)!} H_{k-j}(\langle X \rangle) H_{l-j}(\langle Y \rangle),
\]

where \( H_n(x) \) are the Hermite polynomials. To second order in \( X \) and \( Y \) Eq. \([10]\) reads

\[
\langle \sqrt{1 + X} e^{Z} \sqrt{1 + Y} \rangle \simeq e^{\frac{1}{2} \langle Z^2 \rangle} \left[ 1 + \frac{1}{2} (\langle XZ \rangle + \langle ZY \rangle) - \frac{1}{8} (\langle X^2 \rangle + \langle Y^2 \rangle - 2\langle XY \rangle) - \frac{1}{8} (\langle XZ \rangle - \langle ZY \rangle)^2 \right].
\]

The main expression \([10]\) requires then the calculation of the various two-point correlation functions involving the three fields \( X, Y, Z \). All of them can be obtained from

\[
\langle \varphi(x_1)\varphi(x_2) \rangle = (\pi/L)^2(j_0^2)x_1x_2 - \frac{1}{4K} \ln C(x_1 - x_2),
\]

\[
\langle \theta(x_1)\theta(x_2) \rangle = \pi(\Pi_0) x_1x_2 - \frac{K}{4} \ln C(x_1 - x_2),
\]

\[
\langle \theta(x_1)\varphi(x_2) \rangle = \frac{1}{4} \ln \left[ \frac{1 - e^{-2\pi\alpha/L} + 12\pi(x_1-x_2)/L}{1 - e^{-2\pi\alpha/L} + 12\pi(x_1-x_2)/L} \right],
\]

where \( C(x) = 1 - 2\cos(2\pi x)/L \) \( e^{-2\pi\alpha} + e^{-4\pi\alpha} \), \( \Pi_0 = (N-N)/L \), and \( J_0 = J-J \). We are now in a position to calculate the correlators between the fields \( X, Y, Z \) in the thermodynamic limit \( (L \to \infty, N \to \infty) \) at fixed \( N/L = \rho_0 \). Using Eqs. \([12]-[14]\) we have

\[
\exp(\langle Z^2 \rangle/2) \simeq \left( \frac{c_0}{\sqrt{z^2 + \alpha^2}} \right)^{1 + Km^2}.
\]

\[
\langle XZ \rangle = \langle ZY \rangle \simeq \frac{z^2 + 4Km\alpha}{2\alpha z_0^2 + \alpha^2 z_0^2 + \alpha^2 z_0^2},
\]

\[
\langle XY \rangle \simeq \frac{K z^2 - z^2}{2(z^2 + \alpha^2)^2},
\]

\[
\langle X^2 \rangle = \langle Y^2 \rangle \simeq \frac{K}{2}\alpha^2,
\]

where \( \alpha = \pi \rho_0 a \). Similarly, the commutator in Eq. \([9]\) is obtained from Eq. \([15]\) as

\[
\exp(m [\theta(x_1) + \theta(x_2), \varphi(x_1) - \varphi(x_2)]) \simeq \left( \frac{a - i(x_1 - x_2)}{a + i(x_1 - x_2)} \right)^m.
\]

Notice that the effect of the zero-nodes \( \Pi_0 \) and \( J_0 \) is absent in the thermodynamic limit, because it scales as \( 1/L \). The series expansion in \( \Pi(\rho_0) \) is valid for small fluctuations of the field \( \Pi(x) \) compared to the average density \( \rho_0 \), \( \sqrt{\langle X^2 \rangle} \simeq 1 \) \( \text{i.e.} \) for \( \alpha \gtrsim \sqrt{K/2} \). By combining the previous equations we obtain the final result for the one-body density matrix in rescaled units, finding the structure displayed in Eq. \([3]\):

\[
\rho_1(z) = \frac{\rho_\infty}{|z|^{1/2K}} \left[ 1 + \frac{c_0}{2z} + \frac{c_{0,4}}{z^4} + c_{1,2} \cos(2z) \right]_z + c_{1,4} \left[ \frac{c_{2,0} \cos(2z)}{z^{2K+2}} + c_{1,5} \sin(2z) \right]_{z^{2K+1} + \ldots}.
\]

with \( \rho_\infty = |A|^2 \alpha^{1/2K} c_{0,0} \).

A few comments are in order at this point. First, by taking the limit \( a \to 0 \) in Eqs. \([12]-[15]\) we recover the results of the standard harmonic-fluid approach, \( \text{i.e.} \) Eq. \([2]\). Moreover, the generalized harmonic-fluid method produces also the corresponding coefficients of the series, namely to the order of approximation derived in this work, we have \( c_{0,0} \simeq \frac{1}{1 + 2\alpha + \frac{K}{\alpha}} \). \( c_{0,2} \simeq \frac{-a^2}{4K} \left[ c_{0,0} + \frac{2K}{\alpha} - \frac{K^2}{4\alpha^2} \right] / c_{0,0} \).

\( c_{0,4} \simeq \frac{1 + 4K}{2K} \left[ c_{0,0} + \frac{4K}{\alpha} + \frac{K^2}{2\alpha^2} \right] / c_{0,0} \). \( c_{1,2} \simeq \frac{-2K^2}{\alpha} \). \( c_{1,3} \simeq \frac{4K^2}{\alpha} \left[ c_{0,0} + \frac{2K}{\alpha} + \frac{K^2}{2\alpha^2} \right] / c_{0,0} \). \( c_{1,4} \simeq \frac{1 + 4K}{2K} \left[ c_{0,0} + \frac{4K}{\alpha} + \frac{K^2}{2\alpha^2} + \frac{K^3}{3\alpha^3} \right] / c_{0,0} \). However, it should be noted that these coefficients do not necessarily coincide \( \text{e.g.} \) in the limit \( K = 1 \) with the exact ones in Eq. \([11]\). Our approach is still effective, as it suffers from some limitations: first of all, we have just used a single-parameter regularization which neglects the details of the spectrum of the Bose fluid. Second, our approach relies on a hydrodynamic-like expression for the field operator \([\Pi]\) which implicitly assumes that the fluctuations of the field \( \Pi(x) \) are “small”, which is not always the case. On the other hand, we see from our derivation that the corrections due to the \( \Pi(x) \)
fluctuations renormalize the coefficients of the series to all orders, giving rise to the contributions in square brackets. The value chosen for the cutoff parameter is $\alpha = 1/2$. The inset shows the subleading behavior $z^{1/2}p_1(z)$ of the one-body density matrix in the same notations and units as in the main graph.

Figure 1 shows our results for the one-body density matrix (Eq. (21), solid line) for $K = 1$ in units of $\rho_0 \rho_\infty$ as a function of the scaled relative coordinate $z = \pi \rho_0 (x_1 - x_2)$ (dimensionless). The result of the generalized harmonic-fluid approximation Eq. (21) obtained without taking into account the effect of the field $\Pi(z)$ (solid line) is compared to the exact result Eq. (1) (dashed line) and to the usual harmonic-fluid approximation Eq. (2) (dotted line), with $B_0 = 1$, $B_1 = -1/2$ and $B_{m>1} = 0$. The application to fermions

It is possible to apply the above approach to the case of a 1D spinless Fermi gas with odd-wave inter-particle interaction \[24\], i.e. the 1D analog of $p$-wave interactions. For any value of the coupling strength the (attractive) Fermi gas can be mapped onto a (repulsive) Bose gas described by the Lieb-Liniger model with dimensionless coupling strength given by $\gamma_B = -1/\gamma_F$ \[24\] \[23\], the spectrum of collective excitations being the same for the two models. Hence, it is possible to describe the interacting Fermi gas by the Luttinger-liquid Hamiltonian \[1\] as well, except for the so-called Fermi-Tonks-
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