The Pandora’s Box problem asks to find a search strategy over $n$ alternatives given stochastic information about their values, aiming to minimize the sum of the search cost and the value of the chosen alternative. Even though the case of independently distributed values is well understood, our algorithmic understanding of the problem is very limited once the independence assumption is dropped.

Our work aims to characterize the complexity of approximating the Pandora’s Box problem under correlated value distributions. To that end, we present a general reduction to a simpler version of Pandora’s Box, that only asks to find a value below a certain threshold, and eliminates the need to reason about future values that will arise during the search. Using this general tool, we study two cases of correlation; the case of explicitly given distributions of support $m$ and the case of mixtures of $m$ product distributions.

- In the first case, we connect Pandora’s Box to the well studied problem of Optimal Decision Tree, obtaining an $O(\log m)$ approximation but also showing that the problem is strictly easier as it is equivalent (up to constant factors) to the Uniform Decision Tree problem.

- In the case of mixtures of product distributions, the problem is again related to the noisy variant of Optimal Decision Tree which is significantly more challenging. We give a constant-factor approximation that runs in time $n^{O(m^2/\varepsilon^2)}$ for $m$ mixture components whose marginals on every alternative are either identical or separated in TV distance by $\varepsilon$. 

1 Introduction

Many everyday tasks involve making decisions under uncertainty; for example driving to work using the fastest route or buying a house at the best price. Despite not knowing how our current decisions will turn out, or how they affect future outcomes, there is usually some prior information which we can use to facilitate the decision making process. For example, having driven on the possible routes to work before, we know which is more frequently the busiest one. It is also common in such cases that we can remove part of the uncertainty by paying some additional cost. This type of problem is modeled by Pandora’s Box, first formalized by Weitzman in [Wei79]. In this problem, the algorithm is given $n$ alternatives called boxes, each containing a value from a known distribution. The exact value is not known, but can be revealed for a known opening cost specific to the box. The goal is for the algorithm to decide which is the next box to open and whether to select a value and stop, such that the total opening cost plus the minimum value revealed is minimized. In the case of independent distributions on the boxes’ values, this problem has a very elegant and simple solution, as described by Weitzman [Wei79] which obtains the optimal cost; calculate an index for each box, open the boxes in decreasing index, and stop when the expected gain is worse than the value already obtained.

Weitzman’s model makes the crucial assumption that the distributions on the values are independent. This, however, is not always the case in practice and, as it turns out, the simple algorithm of the independent case fails to find the optimal solution under correlated distributions. Generally, the complexity of the Pandora’s Box with correlations is not yet well understood. The first step towards this direction was made by [CGT+20], who considered competing against a simpler benchmark, namely the optimal performance achievable using a strategy that cannot adapt the order in which it opens boxes to the values revealed. In general, optimal strategies can decide both the ordering of the boxes and the stopping time based on the values revealed, but such strategies can be hard to learn using samples.

In this work we study the complexity of Pandora’s Box problem with correlated value distributions against the most general benchmark and provide the first non-trivial approximations. We start by presenting a reduction to a simpler version of Pandora’s Box, in which we optimize the search cost until a value less than a threshold is found. The goal of this reduction is to serve as a tool to remove the need to account for values altogether, making the problem easier to approach. The generality of this tool allows us to use it under any correlated setting. We specifically study two cases of succinctly representable correlated distributions: the case of explicitly given distributions over a small support of size $m$ and the case of mixtures of $m$ product distributions.

In the case of correlated distributions with small (explicitly given) support, we show that Pandora’s Box is tightly connected to another well known problem in decision making, the Optimal Decision Tree (ODT). In ODT, we are asked to identify an unknown hypothesis, out of $m$ possible ones, by performing a sequence of tests. Each test has a cost and, if chosen, reveals a result, which depends on which hypothesis is realized. The goal of the algorithm is to minimize the total cost of tests performed in order to learn which is the correct hypothesis. This problem has been studied for many years, and has various applications in medical diagnosis (e.g. [PKSR02]), fault diagnosis (e.g. [PD92]), and active learning ([GB09, GKR10]). Currently there is a log $m$-approximation for this problem [GB09] which is also shown to be the best possible in [CPR+11]. By connecting Pandora’s Box to Optimal Decision tree, we immediately obtain a log $m$-approximation algorithm. However, going one step further, we show that Pandora’s Box is in fact equivalent, up to constant

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This is a special case of Gittins index [GJ74].
factors, to Uniform Decision Tree; a special case of ODT where the distribution over hypotheses is uniform. The UDT problem was recently shown to be strictly easier than the general version [LLM20], which also makes Pandora’s Box strictly easier.

In the mixture of $m$ distributions case, we observe that Pandora’s Box is related to the noisy version of ODT, where the result of every test and every hypothesis is not deterministic. Previous work in this area obtained algorithms whose approximation and runtime depend on the amount of noise. In our case, by only requiring that the marginals of the mixtures differ enough in TV distance for each box, we obtain a constant approximation that only depends on the number of boxes and the number of the mixtures.

We give a more detailed overview of our results in the next section.

1.1 Our Results

A tool for removing values: in section 3 we present a reduction to the threshold version of Pandora’s Box ($PB \leq T$), where the objective is to find all values below a threshold $T$, instead of the minimum value. Given an $O(\alpha)$ approximation for $PB \leq T$ our reduction produces an algorithm that is $O(\alpha)$ approximate for the original instance with values, while making no assumptions on the type of correlation.

Explicitly given distributions: in this case we show that Pandora’s Box is closely related to the optimal decision tree problem (Theorem 3.1), which implies an $O(\log m)$ approximation for Pandora’s Box. Subsequently we show that, in fact, Pandora’s Box is equivalent up to constant factors to the Uniform Decision Tree problem (Theorems 4.2, 4.4), which is known to be strictly easier than the general Optimal Decision Tree (as shown in [LLM20]).

Mixture of $m$ product distributions: in this case Pandora’s Box is related to the noisy version of the optimal decision tree. The noise can be arbitrary and we only require the mixtures to satisfy a separability condition; the marginals should differ by at least $\varepsilon$ in TV distance. Using this property, we design a constant-factor approximation for $PB \leq T$ that runs in $n^{O(m^2/\varepsilon^2)}$ (Theorem 5.1), where $n$ is the number of boxes, which also implies a constant-factor for the initial Pandora’s Box problem with values (Corollary 5.1.1), when using the tool of Theorem 3.2.

1.2 Related work

The Pandora’s Box problem was first introduced by Weitzman in the Economics literature [Wei79]. Since then, there has been a long line of research studying Pandora’s Box and its variants [Dov18, BK19, EHL19, CGT+20, BFL20], the generalized setting where more information can be obtained for a price [CFG+00, GK01, CJK+15, CHKK15] and in settings with more complex combinatorial constraints [Sin18, GGM06, GN13, ASW16, GNS16, GNS17, GJSS19].

Optimal decision tree is an old problem studied in a variety of settings, while its most notable application is in active learning settings. It was proven to be NP-Hard by Hyafil and Rivest [HR76]. Since then the problem of finding the best algorithm was an active one [GG74, Lov85, KP98, Das04, CPR+11, CPRS09, GB09, GNR17, CJLM10, AH12], where finally a greedy $\log m$ for the general case was given by [GB09]. This approximation ratio is proven to be the best possible [CPR+11]. For the case of Uniform decision tree less is known, until recently the best algorithm was the same as the optimal decision tree, and the lower bound was 4 [CPR+11]. The recent work of Li et al. [LLM20] showed that there is an algorithm strictly better than $\log m$ for the uniform decision tree.
The noisy version of optimal decision tree was first studied in [GKR10]\(^2\), which gave an algorithm with runtime that depends exponentially on the number of noisy outcomes. Subsequently, Jia et al. in [JNNR19] gave an \((\min(r, h) + \log m)\)-approximation algorithm, where \(r\) (resp. \(h\)) is the maximum number of different test results per test (resp. scenario) using a reduction to Adaptive Submodular Ranking problem [KNN17]. In the case of large number of noisy outcome they obtain a \(\log m\) approximation exploiting the connection to Stochastic Set Cover [LPRY08, INvdZ16].

### 2 Preliminaries

We formally define the problems used in the following sections. We distinguish them in two families; the variants of Decision Tree and the variants of Pandora’s Box.

2.1 Decision Tree-like problems

In Decision Tree problems we are given a set \(S\) of \(m\) scenarios \(s \in S\) each occurring with (known) probability \(p_s\) and \(n\) tests \(T_i\) each with non-negative cost \(c_i\) for \(i \in [n]\). Each test \(T_i\) gives a specific result \(r_i(s)\in \mathbb{R}\) for every scenario \(s\). Nature picks a scenario \(s \in S\) from the distribution \(p\). The goal of the algorithm is to run a series of tests to determine which scenario is realized.

The output of the algorithm is a decision tree where at each node there is a test that is performed, and the branches are the outcomes of the test. In each of the leaves there is an individual scenario that is the only one consistent with the results of the test in the branch from the root to this leaf. We can think of this tree as an *adaptive policy* that, given the set of outcomes so far, decides the next test to perform.

The objective is to find a decision tree that minimizes the cost \(\sum_{s \in S} p_s d(s)\) i.e. the average cost for all scenarios to be identified where \(d(s)\) is the total cost of the tests used to reach scenario \(s \in S\). We denote this general version as \(\text{ODT}:\) Optimal Decision Tree. In the case where all \(p_s = 1/m\), the problem is called *Uniform Decision Tree* or \(\text{UDT}\).

**Definition 2.1 (Policy \(\pi\)).** Let \(T\) be the set of possible tests and \(R\) the set of possible results for each test. A policy \(\pi : (2^T, 2^R) \rightarrow T\) is a function that given a set of tests done so far and their results, it returns the next test to be performed.

2.2 Pandora’s Box-like problems

In the original Pandora’s Box problem we are given \(n\) boxes, each with cost \(c_i \geq 0\) and value \(v_i \sim D_i\), where the distributions are known and independent. To learn the exact value inside box \(i\) we need to pay the respective probing cost \(c_i\). The objective is to minimize the sum of the total probing cost and the minimum value obtained. In the correlated version of Pandora’s Box problem, denoted by \(\text{PB}_{\min,v}\), the distributions on the values can be arbitrary correlated we use \(D\) to denote the joint distribution over the values and \(D_i\) its marginal for the value in box \(i\). The objective is to find a policy \(\pi\) of opening boxes such that the expected cost of the boxes opened plus the minimum value discovered is minimized, where the expectation is taken over all possible realizations of the values in each box. Formally we want to minimize

\[
\mathbb{E}_s \left[ \min_{j \in \pi} v_{js} + \sum_{i \in \pi} c_i \right],
\]

\(^2\)This result is based on a result from [GK11] which turned out to be wrong [NS17]. The correct results are presented in [GK17].
where we slightly abused notation and denoted by \( \pi \) the set of boxes opened.

We also introduce Pandora’s Box with costly outside option and parameters \( T \) (called the threshold) and \( \ell \) (called the outside option). In this version the objective is to minimize the cost of finding a value \( \leq T \), while we have the extra option to quit searching by opening a box of cost \( \ell \). We say that a scenario is covered in a given run of the algorithm if it does not choose the outside option box \( \ell \). For the remainder of this paper, we always have \( \ell = T \), and denote the problem by \( \mathcal{PB}_{\leq T} \).

Finally, we define by \( \pi^I \) and \( \pi^*_I \) the policy and optimal policy for an instance of \( \mathcal{PB}_{\min v} \) or \( \mathcal{PB}_{\leq T} \). Additionally, the policy for a specific scenario \( s \in \mathcal{S} \) is denoted by \( \pi^I(s) \).

### 2.2.1 Modeling Correlation

In this work we study two general ways of modeling the correlation between the values in the boxes.

**Explicit Distributions:** in this case, \( D \) is a distribution over \( m \) scenarios where the \( j \)'th scenario is realized with probability \( p_j \), for \( j \in [m] \). Every scenario corresponds to a fixed and known vector of values contained in each box. Specifically, box \( i \) has value \( v_{ij} \in \mathbb{R}^+ \cup \{\infty\} \) for scenario \( j \).

**Mixture of Distributions:** We also consider a more general setting, where \( D \) is a mixture of \( m \) product distributions. Specifically, each scenario \( j \) is a product distribution; instead of giving a deterministic value for every box \( i \), the result is drawn from distribution \( D_{ij} \). This setting is a generalization of the explicit distributions setting described before.

### 3 A Tool Removing Values from Pandora’s Box

#### 3.1 Warmup: A naive reduction

A solution to Pandora’s Box involves two components: the order in which to open boxes and a stopping rule. As a warm-up, we present a simple reduction from \( \mathcal{PB}_{\min v} \) to \( \mathcal{PB}_{\leq T} \) that is computationally efficient for the explicit distribution setting. This result essentially simplifies the stopping rule of the problem allowing us to focus on the order in which boxes are opened.

**Theorem 3.1.** Given an efficient \( \alpha \)-approximation for \( \mathcal{PB}_{\leq T} \) for arbitrary \( T \), there exits a \( 2\alpha \)-approximation for \( \mathcal{PB}_{\min v} \) that runs in polynomial time in the number of scenarios, number of boxes, and the number of values.

The main idea is we can move the value information contained in the boxes into the cost of the boxes by creating one new box for every box and realized value pair. We still need to use the original boxes to obtain information about which scenario is realized. We do so by replacing values in the original boxes by high values while maintaining correlation.

**\( \mathcal{PB}_{\leq T} \) Instance:** given an instance \( \mathcal{I} = (\{p_i\}_{i=1}^m, \{c_j\}_{j=1}^n, V) \) of \( \mathcal{PB}_{\min v} \), we construct an instance \( \mathcal{I}' = (\{p_i'\}_{i=1}^m, \{c'_j\}_{j=1}^{n+m}, V') \) of \( \mathcal{PB}_{\leq T} \). We need \( T \) to be sufficiently large so that the outside option is never chosen and so we can easily get a policy for \( \mathcal{PB}_{\min v} \) from a policy for \( \mathcal{PB}_{\leq T} \). This is just a technical nuance. In particular, choosing \( T \) to be larger than the sum of all the boxes plus the largest value that ever could be achieved will ensure the outside option is never better than even opening all boxes. Now, define \( p' = p \). Next, we define \( V'_{ij} = V_{ij} + T + 1 \). Note all of these values will be larger than \( T \) and so we cannot stop after receiving such a value. However, they will cause the same branching behaviour as before since each distinct value is mapped in a bijectively to a
new distinct value. Also, we add additional “final” boxes for each pair \((j, v)\) of a box and a value that this box could give. Each “final” box \((j, v)\) has cost \(c_j + v\) and value 0 for the scenarios where box \(j\) gives exactly value \(v\) and values \(T + 1\) for all other scenarios. Formally,

\[
V'_{i,(j,v)} = \begin{cases} 
0 & \text{if } V_{i,j} = v \\
T + 1 & \text{else}
\end{cases}
\]

Intuitively, these “final” boxes indicate to a policy that this will be the last box opened, and so its values, which is at least that of the best values of the boxes chosen, should now be taken into account in the cost of the solution. The proof of the theorem is deferred to section A.1 of the Appendix.

### 3.2 Main Tool

The reduction presented in the previous section, even though simple to describe, cannot be generally applied to any correlation setting. Specifically, observe that the number of boxes is increased to a number proportional to the size of the support, which could even be exponential. In this section we introduce a more sophisticated reduction, that is able to overcome these issues, and is still preserving the approximation up to logarithmic factors.

**Theorem 3.2.** If there exists an \(\alpha\)-approximation for \(PB_{\leq T}\), then there exists an \(\tilde{O}(\alpha)\)-approximation for \(PB_{\min v}\).

On a high level, in this reduction we repeatedly run the algorithm for \(PB_{\leq T}\) with increasingly large value of \(T\) with the goal of capturing some mass of scenarios at every step. The thresholds for every run have to be cleverly chosen to guarantee that enough mass is captured at every run. The distributions on the boxes remain the same, and this reduction does not increase the number of boxes, therefore avoiding the issues the simple reduction of section 3.1 faced.

**Algorithm 1:** Reduction from \(PB_{\min v}\) to \(PB_{\leq T}\).

1. \(i \leftarrow 0\) // Number of current Phase
2. haveSolution \(\leftarrow\) False
3. while haveSolution == False do
   4. \(T_i \leftarrow\) Binary Search on \(T_i\)’s s.t. \(\Pr[\text{accepting } T_i] \leq 0.2\)
   5. Run \(PB_{\leq T}\) with \(T = T_i\)
   6. if Did not accept \(T_i\) then
      7. haveSolution \(\leftarrow\) True // Stop if found value below \(T_i\)
   8. end
   9. Rescale probabilities by 0.2
10. end

In our theorem proof we are using the following quantity called \(p\)-threshold. A \(p\)-threshold is the minimum possible threshold \(T\) such that at most \(p\) mass of the scenarios has cost more than \(T\) in \(PB_{\min v}\). Formally

**Definition 3.3** (\(p\)-Threshold). Let \(I\) be an instance of \(PB_{\min v}\) and \(c_s\) be the cost of scenario \(s \in S\) in \(\pi^*_I\), we define the \(p\)-threshold as

\[
t_p = \min\{T : \Pr[c_s > T] \leq p\}
\]
Before continuing to the proof of the theorem, we show two key lemmas that guarantee that enough probability mass is covered at every phase we run the \( \mathcal{PB}_{\leq T} \) algorithm (Lemma 3.5) and show a lower bound on the optimal policy for the initial problem in every phase of the reduction (Lemma 3.6). Their proofs are deferred to section A.2 of the Appendix. We first formally define what is a sub-instance of Pandora’s Box which is used both in our reduction and lemma.

**Definition 3.4** (Sub-instance). Let \( \mathcal{I} \) be an instance of \( \{ \mathcal{PB}_{\leq T}, \mathcal{PB}_{\min v}\} \) with set of scenarios \( S_T \) each with probability \( p_s^T \). For any \( q \in [0, 1] \), we call \( \mathcal{I}_q \) a sub-instance of \( \mathcal{I} \) if \( S_{\mathcal{I}_q} \subseteq S_T \) and \( \sum_{s \in S_{\mathcal{I}_q}} p_s^T = q \).

**Lemma 3.5** (Threshold Bound). Given an instance \( \mathcal{I} \) of \( \mathcal{PB}_{\leq T} \), an \( \alpha \)-approximation algorithm \( \pi_T \) to \( \mathcal{PB}_{\leq T} \), and let \( \mathcal{I}_q \) be a sub-instance of \( \mathcal{I} \). If the threshold \( T \) satisfies
\[
T \leq t_q/(10\alpha) + 10\alpha \sum_{c_s \in [t_q, t_q/(10\alpha)]} \frac{c_s p_s}{q},
\]
then when running \( \pi_T \), at most 0.2q scenarios pick the outside option box \( T \).

**Lemma 3.6.** (Optimal Lower Bound) In the reduction of Theorem 3.2, let \( \mathcal{I} \) be the instance of \( \text{pbv} \). For the optimal policy for \( \mathcal{PB}_{\min v} \) for every phase holds that
\[
\pi^*_T \geq \sum_{i=1}^{\infty} \frac{1}{10\alpha} \cdot (0.2)^i (0.2)^i / 10\alpha.
\]

**Proof of Theorem 3.2.** Given an instance \( \mathcal{I} \) of \( \mathcal{PB}_{\min v} \), we repeatedly run \( \mathcal{PB}_{\leq T} \) in phases. Phase \( i \) consists of running \( \mathcal{PB}_{\leq T} \) with threshold \( T_i \) on a sub-instance of the original problem, denoted by \( \mathcal{I}_{(0.2)^i} \). After every run \( i \), we remove the probability mass that was covered\(^5\), and run \( \mathcal{PB}_{\leq T} \) on this new instance with a new threshold \( T_{i+1} \). In each phase, the boxes, costs and values remain the same, but in this case the objective is different; we are seeking values less than \( T_i \). The thresholds are chosen such that at the end of each phase, 0.8 of the remaining probability mass is covered. The reduction process is formally shown in Algorithm 1.

**Calculating the thresholds:** for every phase \( i \) we choose a threshold \( T_i \) such that \( T_i = \min\{ T : \Pr[c_s > T_i] \leq 0.2 \} \) i.e. at most 0.2 of the probability mass of the scenarios are not covered. In order to select this threshold, we do binary search starting from \( T = 1 \), running every time the \( \alpha \)-approximation algorithm for \( \mathcal{PB}_{\leq T} \) with outside option box \( B \) and checking how many scenarios select it. We denote by \( \text{Int}_i = [t_{(0.2)^i}, t_{(0.2)^i}/(10\alpha)] \) the relevant interval of costs at every run of the algorithm, then by Lemma 3.5, we know that for remaining total probability mass \( (0.2)^i \), a threshold which satisfies
\[
T_i \leq t_{(0.2)^i}/(10\alpha) + 10\alpha \sum_{c_s \in \text{Int}_i} \frac{c_s p_s}{(0.2)^i}, \tag{1}
\]
also satisfies the desired covering property; at least 0.8 mass of the current scenarios is covered. Therefore the threshold \( T_i \) found by our binary search satisfies inequality (1).

**Constructing the final policy:** by running \( \mathcal{PB}_{\leq T} \) in phases, we get a different policy \( \pi^*_T (0.2)^i \), which we denote by \( \pi^i \) for brevity, for every phase \( i \). We construct policy \( \pi_T \) for the original instance, by following \( \pi^i \) in each phase until the total probing cost exceeds \( T_i \), at which point \( \pi_T \) starts following \( \pi^{i+1} \) or stops if a value below \( T_i \) is found.

\(^5\)Recall, a scenario is covered if it does not choose the outside option box.
Accounting for the values: in the initial $\mathcal{PB}_{\min,v}$ problem, the value chosen for every scenario is part of the cost, while in the transformed $\mathcal{PB}_{\leq T}$ problem, only the probing cost is part of the cost. Let $j$ be one of these scenarios, and $c_{\min,v}(j)$ and $c_{\leq T}(j)$ be its cost in $\mathcal{PB}_{\min,v}$ and $\mathcal{PB}_{\leq T}$ respectively. We claim that $c_{\min,v}(j) \leq 2c_{\leq T}(j)$. Observe that in every run with threshold $T_i$, only the scenarios $j$ with $v_{xj} \leq T_i$ for some box $x$ can be covered and removed from the instance. The way we constructed the final policy, is essentially the same as running ski-rental for every scenario; in the phase the scenario is covered, its value is at most $T_i$, and we stop when the probing cost exceeds this amount. Therefore, the total cost without the value will be within 2 the cost with the value included.

Bounding the final cost: using the guarantee that at the end of every phase we cover 0.8 of the scenarios, observe that the algorithm for $\mathcal{PB}_{\leq T}$ is run in an interval of the form $\text{Int}_i = [t(0.2)\cdot i, t(0.2)\cdot i/(10\alpha)]$. Note also that these intervals are overlapping. Bounding the cost of the final policy $\pi_T$ for all intervals we get

$$\pi_T \leq \sum_{i=0}^{\infty} (0.2)^i T_i$$

$$\leq \sum_{i=0}^{\infty} \left( (0.2)^i t(0.2)^i - 1/10\alpha + 10\alpha \sum_{s \in S} c_s p_s \right)$$

From inequality (1)

$$\leq 2 \cdot 10\alpha \pi^*_T + 10\alpha \sum_{i=0}^{\infty} \sum_{s \in S} c_s p_s$$

Using Lemma 3.6

$$\leq 20\alpha \log \alpha \cdot \pi^*_T.$$

Where the last inequality follows since each scenario with cost $c_s$ can belong to at most $\log \alpha$ intervals, therefore we get the theorem. The extra factor of 2 accounts for the values in $\mathcal{PB}_{\min,v}$.

Notice the generality of this reduction; the distributions on the values are preserved, and we did not make any more assumptions on the scenarios or values throughout the proof. Therefore we can apply this tool regardless of the type of correlation or the way it is given to us, e.g. we could be given a parametric distribution, or an explicitly given distribution, as we see in the next section.

4 Explicit Distributions

In this section we assume we study the case where the distributions are explicitly given, in the form of scenarios. We show that in this case, our problem is directly related to the optimal decision tree literature. We first describe a straightforward reduction from $\mathcal{PB}_{\leq T}$ to Optimal Decision Tree. Then we show that the problem is actually easier and reduces to Uniform Decision Tree. The full picture of our reductions is shown in figure 1. Using the $\mathcal{A} \leq \mathcal{B}$ notation means that problem $\mathcal{A}$ reduces to $\mathcal{B}$ and the theorem number where this is shown is mentioned above the $\leq$. Similarly $\equiv$ is the same as having both $\leq, \geq$ directions.
\[ \mathcal{P}_B \leq T_{3.1, 3.2} \leq \mathcal{P}_B \leq T_{4.1} \leq ODT \]

| 4.2, 4.4 |

Figure 1: Summary of reductions for the explicit distributions case

4.1 Reduction to ODT

We first show that \( \mathcal{P}_B \) can be reduced to the more general Optimal decision tree. This reduction implies that any known algorithm for ODT can be applied to \( \mathcal{P}_B \) and give the same guarantees. Since the best possible algorithm for ODT is a log \( m \)-approximation, this implies a log \( m \) approximation for \( \mathcal{P}_B \).

Theorem 4.1. If there exists an \( \alpha \)-approximation algorithm for ODT then there exists a \( \alpha \)-approximation for \( \mathcal{P}_B \).

Proof. We show how to convert an input given for \( \mathcal{P}_B \) to one for ODT and use this solution to get one for \( \mathcal{P}_B \) in polynomial time with the claimed approximation factor.

ODT Instance: Let \( \mathcal{I} \) be the initial instance of \( \mathcal{P}_B \). To construct the instance of ODT, \( \mathcal{I} \), we subdivide the \( m \) scenarios and their probabilities. To be precise, for any scenario \( j \) we construct two subdivisions \( j_1, j_2 \) both having probability \( p_j / 2 \). For every box \( i \) with cost \( c_i \) we create a test \( T_i \) costing \( c_i \) with possible outcomes from \( [2m] \cup \{ v + m + 1 | v \text{ is a possible value of box } i \} \). The result of this test, when scenario \( j_k \) is realized is

\[ T_i(j_k) = \begin{cases} j \ast k & \text{if } v_{ij} < T \\ v_{ij} + m + 1 & \text{o.w} \end{cases} \]

In other words, if a box \( i \) gave less than \( T \) reward for a scenario \( j \), the test corresponding to the box will isolate the subdivisions of that scenario, so that ODT can stop.

We also have a single test \( T'' \) that is a multi-way test that distinguishes all scenarios instantly at a cost of \( T \). This simulates the outside option.

Constructing the policy: We construct a policy \( \pi' \) given a policy \( \pi \) for ODT. We start at the root of \( \pi \), and open boxes in \( \mathcal{I} \) as suggested by our current location in \( \pi \). At every step, the outcome of the test suggests a step to take in the tree \( \pi' \). To be precise, there are two cases to consider.

1. If we encounter a test \( T_i \), we open the corresponding box \( i \) in \( \pi' \).
   
   (a) If the value of this box is less than \( T \), \( \pi' \) terminates.
   
   (b) If the value of this box is at least \( T \), then \( \pi' \) will continue following the corresponding branch in \( \pi \).

2. If we encounter the test \( T'' \), then \( \pi' \) takes the outside option and terminates.
We argue that $\pi'$ is in fact a feasible policy for the instance $I'$. This follows since the only ways that $\pi$ could isolate a scenario subdivision $j_k$, is by either running a test $T_i$ satisfying $T_i(j_k) = j \ast k$ or by running $T''$ which isolates all scenario subdivisions by definition. In the latter case, $\pi'$ will take the outside option in step 2 and so $\pi'$ will give a valid solution to scenario $j$. In the former case, we know that $T_i(j_k) = j \ast k$ only if $v_{ij} < T$. Thus, box $i$ is opened on $j$’s branch in step 1 and a value less than $T$ is achieved. Hence, $\pi'$ is a feasible policy for the instance.

**Approximation ratio:** First, note that $\pi'$ always open boxes (or takes the outside option) with same cost as the corresponding test run by $\pi$ by construction. Also, we have subdivisions will always go down the same branches and are always isolated at the same time. This follows since either a test corresponding to a box isolated a subdivision and so would isolate both or running $T''$ isolated them. Hence, the branch for scenario $j$ has exactly the same cost in $\pi'$ as its subdivisions in $\pi$ and their total probabilities are both the same. So, $c(\pi'(j)) \leq c(\pi(j))$.

For the optimal solutions we show that $c(\pi^*_T(j)) \leq c(\pi^*_T)$. This follows almost identically to the argument above just by swapping boxes with tests. In particular, for any branch of $\pi^*_T$, we either reach a box with a value less than $T$ or take the outside option. If we do the sequence of tests corresponding to this sequence of box openings and end $T''$ whenever the outside option is taken, we end up with a solution to the instance $I$ of cost at most $c(\pi^*_T)$. Putting it altogether, we get that

$$c(\pi') \leq c(\pi) \leq \alpha c(\pi^*_T)$$

Clearly, the reduction can be done in polynomial time and so this yields a $4\alpha$-approximation for $\mathcal{PB}_{\leq T}$.

**4.2 A Stronger Result: Equivalence with $UDT$**

The previous reduction to the optimal decision tree problem highlights the similarity of these two problems. However, optimal decision tree is a very general and powerful problem. In this section we show that the Pandora’s Box problem is actually strictly easier than the optimal decision tree. Specifically, we reduce to the uniform decision tree, which Li et al. [LLM20] proved that admits a strictly better approximation than the best possible for optimal decision tree.

**4.2.1 Reducing $\mathcal{PB}_{\leq T}$ to $UDT$**

In this section we show a reduction from $\mathcal{PB}_{\leq T}$ to $UDT$, formally stated in Theorem 4.2. The currently known results for $UDT$ all assume uniform cost tests but this reduction, even though simple, introduces a test with cost $T$. If the initial instance had non-uniform cost boxes, introducing costs is unavoidable. However, if the initial $\mathcal{PB}_{\leq T}$ instance had uniform costs, except the outside option box, we show that it is possible to avoid introducing costs (Theorem 4.3), and therefore all the known results for $UDT$ apply to $\mathcal{PB}_{\leq T}$.

**Theorem 4.2.** If there exists an $\alpha(m)$ approximation for $UDT$ where $m$ is the number of scenarios, then there exists an $O(\alpha(Tm))$-approximation for $\mathcal{PB}_{\leq T}$.

**Theorem 4.3.** If there exists an $\alpha(m)$ approximation for $UDT$ with uniform costs where $m$ is the number of scenarios, then there exists an $O(\alpha(m^2))$-approximation for $\mathcal{PB}_{\leq T}$ with uniform costs.

This theorem combined with the result of Li et al. [LLM20] that gives a $\log m / \log c(\pi^*)$-approximation for $UDT$ with uniform test costs and with our reduction from section 3.2 we get the following corollary.
Corollary 4.3.1. For an instance $\mathcal{I}$ of $\mathcal{PB}_{\min v}$ with uniform costs and $m$ scenarios, there exists a $\tilde{O}\left(\frac{\log m}{c(\pi_T)}\right)$-approximation algorithm for $\mathcal{PB}_{\min v}$. Additionally if the tests only have a constant number of different results, the approximation ratio is $\tilde{O}\left(\frac{\log m}{\log \log m}\right)$.

The proof of Theorem 4.3 follows similarly to that for Theorem 4.2, and is deferred to section B.1 of the Appendix.

Proof of Theorem 4.2. Given an instance $\mathcal{I}$ of $\mathcal{PB}_{\leq T}$ with outside cost box $b_T$, we construct the instance $\mathcal{I}'$ of UDT as follows.

Constructing the instance: we first remove all scenarios with $p_i \leq c_{\min}/(Tm)$. Let $\ell$ be the number of scenarios remaining so that $p_\ell$ is the smallest probability remaining. All probabilities are scaled by $c$ such that $\sum_{i=1}^{\ell} c \cdot p_i = 1$. We then create $\frac{p_i}{p_\ell}$ copies of scenario $i$, called sub-scenarios and denoted by $s_{ij}$ where $1 \leq j \leq \frac{p_i}{p_\ell}$ each having equal probability $p_{ij} = c p_i$. For simplicity, we assume the number of copies is integral. For any box $B$ of $\mathcal{I}$, we create a test $T_B$ called a box test such that

$$T_B(i,j) = \begin{cases} j, & \text{if } v_B(i) < T \\ \infty, & \text{else} \end{cases}$$

In other words, if a box $B$ had a value less than $T$ for scenario $i$, the test corresponding to the box isolates each sub-scenario by sending them to their own branch and send all other scenario copies to the same $\infty$ branch. We also add an outside option test $T_{\text{out}}$ that costs $T$ and isolates all copies of all scenarios by giving a different result for each.

Constructing the policy: given a policy $\pi_{\mathcal{I}'}$, we construct a policy $\pi_{\mathcal{I}}$. Starting from the root of $\pi_{\mathcal{I}'}$, whenever $\pi_{\mathcal{I}'}$ chooses to run a box test $T_B$ on a branch with a copy of scenario $i$, $\pi_{\mathcal{I}}$ opens box $B$ on the branch for scenario $i$. If $\pi_{\mathcal{I}'}$ chooses the outside option test, then $\pi_{\mathcal{I}}$ also chooses the outside option box $b_T$. If at some point the policy $\pi_{\mathcal{I}'}$ has spent more than $T$, we stop and take the outside option box $b_T$, incurring at most twice the cost of $\pi_{\mathcal{I}'}$.

For the constructed policy $\pi_{\mathcal{I}}$ to be feasible we show that for any scenario $i$ either (1) $\pi_{\mathcal{I}}$ opens a box giving value less than $T$ or (2) $\pi_{\mathcal{I}}$ takes $b_T$. Feasibility is immediate since the isolating tests correspond one-to-one to the boxes with value less than $T$, and the low probability scenarios take $b_T$.

Approximation ratio: the low probability scenarios at worst take the outside option incurring cost

$$2 \frac{c_{\min}}{Tm} \cdot T \cdot m \leq 2c_{\min} \leq 2c(\pi_{\mathcal{I}'})$$

since there is at most $m$ and their probability is at most $c_{\min}/(Tm)$. Let $s_i$ be any scenario with $p_i > p_\ell$. Then for every test run, the corresponding box is opened, and whenever $\pi_{\mathcal{I}'}$ isolates, at the same box $\pi_{\mathcal{I}}$ also finds a value less than $T$. Therefore, scenario $s_i$ contributes the same cost, but with scaled probability $c p_i$ and since $c > 1$ it holds that $c(\pi_{\mathcal{I}}(s_i)) \leq c(\pi_{\mathcal{I}'}(s_i))$. Summing up, we get $c(\pi_{\mathcal{I}}) \leq 2c(\pi_{\mathcal{I}'})$. Putting it all together we get

$$c(\pi_{\mathcal{I}}) \leq 2c(\pi_{\mathcal{I}'}) \leq 2\alpha c(\pi_{\mathcal{I}'}^*) \leq 4\alpha c(\pi_{\mathcal{I}}^*),$$

---

4This holds since $c(\pi_{\mathcal{I}}) \geq \log_K m$ where $K$ is number of different test results.
where the second inequality follows since we are given an $\alpha$ approximation and the last inequality since if we are given an optimal policy for $PB_{\leq T}$, the exact same policy is also feasible for any $T'$ instance, which has cost at least $c(\pi^*_{T'})$.

4.2.2 Reducing $UDT$ to $PB_{\leq T}$

We proceed to show the reverse reduction, from $UDT$ to $PB_{\leq T}$, showing thus that the problems are equivalent up to constant factors.

Theorem 4.4. If there exists an $\alpha$-approximation for $PB_{\leq T}$ then there exists an $O(\alpha)$-approximation for $UDT$.

Given an instance $I'$ of $UDT$ we construct an instance $I$ of $PB_{\leq T}$ by keeping the scenarios and probabilities the same and choosing $T = \sum_{i=1}^{n} c_i + 1$, where $c_i$ is the cost of test $i$. For every test $T_j$, we construct a test box $B_j$ and we define $B_j(i) = \infty$ so that each value for a standard box is an infinity chosen to match the same branching as $T_j$. The costs of these boxes is the same as the corresponding test.

Next, we introduce isolating boxes for each scenario $i$, we define isolating box $B^i$ satisfying $B^i(i) = 0$ and $B^i(k) = \infty$ for all other scenarios $k$. The cost of an isolating box is the minimum cost test needed to isolate $i$ from scenario $k$ where $k$ is the scenario that maximizes this quantity. Formally, if $c(i, k) = \min \{ c_j | T_j(i) \neq T_j(k) \}$, then $c(B^i) = \max_{k \in [m]} c(i, k)$. Overall, the instance will have $n + m$ boxes and $m$ scenarios.

The policy for $UDT$ is constructed by following the policy given by $PB_{\leq T}$, and ensuring that every time there are at most two scenarios that are not distinguished at every leaf of the policy tree. The full proof of the theorem is deferred to section B.2 of the Appendix.

5 Mixture of Product Distributions

In this section we switch gears and consider the case where we are given a mixture of $m$ product distributions. Observe that using the tool described in section 3.2, we can reduce this problem to $PB_{\leq T}$. This now is equivalent to the noisy version of $ODT$ [GK17, JNNR19] where for a specific scenario, the result of each test is not deterministic and can get different values with different probabilities.

Comparison with previous work: previous work on noisy decision tree, considers limited noise models or the runtime and approximation ratio depends on the type of noise. For example in the main result of [JNNR19], the noise outcomes are binary with equal probability. The authors mention that it is possible to extend the following ways:

- to probabilities within $\delta, 1 - \delta$, incurring an extra $1/\delta$ factor in the approximation
- to non-binary noise outcomes, incurring an extra at most $m$ factor in the approximation

Additionally, their algorithm works by expanding the scenarios for every possible noise outcome (e.g. to $2^m$ for binary noise). In our work the number of noisy outcomes does not affect the number of scenarios whatsoever.

In our work, we obtain a constant approximation factor, that does not depend in any way on the type of the noise. Additionally, the outcomes of the noisy tests can be arbitrary, and do not affect either the approximation factor or the runtime. We only require a separability condition to hold; the distributions either differ enough or are exactly the same. Formally, we require
that for any two scenarios \( s_1, s_2 \in S \) and for every box \( i \), the distributions \( D_{is_1} \) and \( D_{is_2} \) satisfy 
\[ |D_{is_1} - D_{is_2}| \in \mathbb{R}_{\geq \varepsilon} \cup \{0\}, \]
where \( |A - B| \) is the total variation distance of distributions \( A \) and \( B \).

### 5.1 A DP Algorithm for noisy \( PB_{\leq T} \)

We move on to designing a dynamic programming algorithm to solve the \( PB_{\leq T} \) problem, in the case of a mixtures of product distributions. The guarantees of our dynamic programming algorithm are given in the following theorem.

**Theorem 5.1.** For any \( \beta > 0 \), let \( \pi_{\text{DP}} \) and \( \pi^* \) be the policies produced by Algorithm \( \text{DP}(\beta) \) described by Equation (2) and the optimal policy respectively and \( UB = \frac{m^2}{\varepsilon^2} \log \frac{m^2 T}{c_{\text{min}} / \beta} \). Then it holds that
\[
c(\pi_{\text{DP}}) \leq (1 + \beta)c(\pi^*),
\]
and the DP runs in time \( n^{UB} \), where \( n \) is the number of boxes and \( c_{\text{min}} \) is the minimum cost box.

Using the reduction described in section 3.2 and the previous theorem we can get a constant-approximation algorithm for the initial \( PB_{\text{min} v} \) problem given a mixture of product distributions. Observe that in the reduction, for every instance of \( PB_{\leq T} \) it runs, the chosen threshold \( T \) satisfies \( T \leq (\beta + 1)\frac{c(\pi^*_T)}{0.2} \) where \( \pi^*_T \) is the optimal policy for the threshold \( T \). The inequality holds since the algorithm for the threshold \( T \) is a \((\beta + 1)\) approximation and it covers 80% of the scenarios left (i.e. pays 0.2\( T \) for the rest). This is formalized in the following corollary.

**Corollary 5.1.1.** Given an instance of \( PB_{\text{min} v} \) on \( m \) scenarios, and the DP algorithm described in Equation (2), then using Algorithm 1 we obtain an \( O(1) \)-approximation algorithm for \( PB_{\text{min} v} \) than runs in \( n^{O(m^2/\varepsilon^2)} \).

Observe that the naive DP, that keeps track of all the boxes and possible outcomes, has space exponential in the number of boxes, which can be very large. In our DP, we exploit the separability property of the distributions by distinguishing the boxes in two different types based on a given set of scenarios. Informally, the *informative* boxes help us distinguish between two scenarios, by giving us enough TV distance, while the *non-informative* always have zero TV distance. The formal definition follows.

**Definition 5.2 (Informative and non-informative boxes).** Let \( S \subseteq S \) be a set of scenarios. Then we call a box \( k \) *informative* if there exist \( s_i, s_j \in S \) such that
\[
|D_{ks_i} - D_{ks_j}| \geq \varepsilon.
\]
We denote the set of all *informative* boxes by \( \text{IB}(S) \). Similarly, the boxes for which the above does not hold are called *non-informative* and the set of these boxes is denoted by \( \text{NIB}(S) \).

**Recursive calls of the DP:** Our dynamic program chooses at every step one of the following options:

1. open an informative box: this step contributes towards eliminating improbable scenarios. From the definition of informative boxes, every time such a box is opened, it gives TV distance at least \( \varepsilon \) between at least two scenarios, making one of them more probable than the other. We show (Lemma 5.3) that it takes a finite amount of these boxes to decide, with high probability, which scenario is the one realized (i.e. eliminating all but one scenarios).
2. open a non-informative box: this is a greedy step; the best non-informative box to open next is the one that maximizes the probability of finding a value smaller than $T$. Given a set $S$ of scenarios that are not yet eliminated, there is a unique next non-informative box which is best. We denote by $\text{NIB}^+(S)$ the function that returns this next best non-informative box. Observe that the non-informative boxes do not affect the greedy ordering of which is the next best, since they do not affect which scenarios are eliminated.

**State space of the DP:** the DP keeps track of the following three quantities:

1. a list $M$ which consists of sets of informative boxes opened and numbers of non-informative ones opened in between the sets of informative ones. Specifically, $M$ has the following form: $M = S_1|x_1|S_2|x_2|\ldots|S_L|x_L^5$ where $S_i$ is a set of informative boxes, and $x_i \in \mathbb{N}$ is the number of non-informative boxes opened exactly after the boxes in set $S_i$. We also denote by $\text{IB}(M)$ the informative boxes in the list $M$.

In order to update $M$ at every recursive call, we either append a new informative box $b_i$ opened (denoted by $M|b_i$) or, when a non-informative box is opened, we add 1 at the end, denoted by $M + 1$.

2. a list $E$ of $m^2$ tuples of integers $(z_{ij}, t_{ij})$, one for each pair of distinct scenarios $(s_i, s_j)$ with $i, j \in [m]$. The number $z_{ij}$ keeps track of the number of informative boxes between $s_i$ and $s_j$ that the value discovered had higher probability for scenario $s_i$, and the number $t_{ij}$ is the total number of informative for scenarios $s_i$ and $s_j$ opened. Every time an informative box is opened, we increase the $t_{ij}$ variables for the scenarios the box was informative and add 1 to the $z_{ij}$ if the value discovered had higher probability in $s_i$. When a non-informative box is opened, the list remains the same. We denote this update by $E++$.

3. a list $S$ of the scenarios not yet eliminated. Every time an informative test is performed, and the list $E$ updated, if for some scenario $s_i$ there exists another scenario $s_j$ such that $t_{ij} > 1/\varepsilon^2 \log(1/\delta)$ and $|z_{ij} - \mathbb{E}[z_{ij}|s_i]| \leq \varepsilon/2$ then $s_j$ is removed from $S$, otherwise $s_i$ is removed$^6$. This update is denoted by $S^++$.

**Base cases:** if a value below $T$ is found, the algorithm stops. The other base case is when $|S| = 1$, which means that the scenario realized is identified, we either take the outside option $T$ or search the boxes for a value below $T$, whichever is cheapest. If the scenario is identified correctly, the DP finds the expected optimal for this scenario. We later show that we make a mistake only with low probability, thus increasing the cost only by a constant factor. We denote by $\text{Nat}(\cdot, \cdot, \cdot)$ the "nature’s" move, where the value in the box we chose is realized, and $\text{Sol}(\cdot, \cdot, \cdot)$ is the minimum value obtained by opening boxes. The recursive formula is shown below.

$$
\text{Sol}(M, E, S) = \begin{cases} 
\min(T, c_{\text{NIB}^+(S)} + \text{Nat}(M+1, E, S)) & \text{if } |S| = 1 \\
\min(T, \min_{i \in \text{IB}(M)} (t_{ij} + \text{Nat}(M|i, E, S))) \\
\text{Nat}(M, E++, S^{++}) 
\end{cases} \quad (2)
$$

$$
\text{Nat}(M, E, S) = \begin{cases} 
0 & \text{if } v_{\text{last box opened}} \leq T \\
\text{Sol}(M, E++, S^{++}) & \text{else}
\end{cases}
$$

$^5$If $b_i$ for $i \in [n]$ are boxes, the list $M$ looks like this: $b_3b_6b_{13}|5|b_{42}b_1|6|b_2$

$^6$This is the process of elimination in the proof of Lemma 5.3
The final solution is \( \text{DP}(\beta) = \text{Sol}(\emptyset, E^0, S) \), where \( E^0 \) is a list of tuples of the form \((0, 0)\), and in order to update \( S \) we set \( \delta = \beta c_{\min}/(m^2T) \).

**Lemma 5.3.** Let \( s_1, s_2 \in S \) be any two scenarios. Then after opening \( \frac{\log(1/\delta)}{\varepsilon^2} \) informative boxes, we can eliminate one scenario with probability at least \( 1 - \delta \).

We defer the proof of this lemma and Theorem 5.1 to section C of the Appendix.

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A Proofs from section 3

A.1 Proofs from subsection 3.1

In order to prove the result, we use the following two key lemmas. In Lemma A.1 we show that the optimal solution for the transformed instance of $\mathcal{PB}_{\leq T}$ is not much higher than the optimal for initial problem. In Lemma A.2 we show how to obtain a policy for the initial problem with values, given a policy for the problem with a threshold.

Lemma A.1. Given the instance $I$ of $\mathcal{PB}_{\min \cdot v}$ and the instance $I'$ of $\mathcal{PB}_{\leq T}$, it holds that

$$c(\pi_{I'}^*) \leq 2c(\pi_I^*).$$

Proof of Lemma A.1. We show that given an optimal policy for $\mathcal{PB}_{\min \cdot v}$, we can construct a feasible policy $\pi'$ for $I'$ such that $c(\pi_0) \leq 2c(\pi_I^*)$. We construct the policy $\pi'$ by opening the same boxes as $\pi$ and finally opening the corresponding “values” box, in order to find the 0 and stop.

Fix any scenario $i$, and let the smallest values box opened for $i$ be box $j$ which gives value $V_{i,j}$. Since $j$ is opened, in the instance $I'$ we open box $(j, V_{i,j})$, and from the construction of $I'$ we have that $V_{i,(j, V_{i,j})} = 0$. Since on every branch we open a box with values $0^7$, we get that $\pi'$ is a feasible policy for $I'$. For scenario $i$, we have that the cost of $\pi(i)$ is

$$c(\pi(i)) = \min_{k \in \pi(i)} V_{i,k} + \sum_{k \in \pi(i)} c_k$$

while in $\pi'(i)$ the minimum values is 0 and there is the additional cost of the “values” box. Formally, the cost of $\pi'(i)$ is

$$c(\pi'(i)) = 0 + \sum_{k \in \pi(i)} c_k + c(j, V_{i,j}) = \min_{k \in \pi(i)} V_{i,k} + \sum_{k \in \pi(i)} c_k + c_j = c(\pi(i)) + c_j$$

Since $c_j$ appears in the cost of $\pi(i)$, we know that $c(\pi(i)) \geq c_j$. Thus, $c(\pi'(i)) = c(\pi(i)) + c_j \leq 2c(\pi(i))$, which implies that $c(\pi') \leq 2c(\pi_I^*)$ for our feasible policy $\pi'$. Observing that $c(\pi') \geq c(\pi_I^*)$ for any policy, gives the lemma.

Lemma A.2. Given a policy $\pi'$ for the instance $I'$ of $\mathcal{PB}_{\leq T}$, there exists a feasible policy $\pi$ for the instance $I$ of $\mathcal{PB}_{\min \cdot v}$ of no larger expected cost. Furthermore, any branch of $\pi$ can be constructed from $\pi'$ in polynomial time.

Proof of Lemma A.2. We construct a policy $\pi$ for $I$ using the policy $\pi'$. Fix some branch of $\pi'$, if $\pi'$ opens a box $j$, our policy $\pi$ opens the same box. When $\pi'$ opens a “final” box $(j, v)$, our policy $\pi$ opens box $j$ if it has not been opened already. There are two cases to consider depending on where the “final” box $(j, v)$ is opened.

1. “Final” box $(j, v)$ is at a leaf of $\pi'$: since $\pi'$ has finite expected cost and this is the first “final” box we encountered, the result must be 0, therefore for policy $\pi$ the values will be $v$, by definition of $I'$. Observe that in this case, $c(\pi) \leq c(\pi')$ since the (at most) extra $v$ paid by $\pi$ for the value term, has already been paid by the box cost in $\pi'$ when box $(j, v)$ was opened.

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7 $\pi$ opens at least one box.
2. “Final” box \((j, v)\) is at an intermediate node of \(\pi'\): after \(\pi\) opens box \(j\), we copy the subtree of \(\pi'\) that follows the 0 branch into the branch of \(\pi\) that follows the \(v\) branch, and the subtree of \(\pi'\) that follows the \(\infty_1\) branch into each branch that has a value different from \(v\) (the non-\(v\) branches). The cost of this new subtree is improved as the root of the original subtree had cost \(c_j + v\) and now it has cost \(c_j\). The \(v\) branch may accrue an additional cost of \(v\) or smaller if \(j\) was not the smallest values box on this branch of the tree, so in total the \(v\) branch has cost at most that originally. However, the non-\(v\) branches have a \(v\) term removed going down the tree. Specifically, since the feedback of \((j, v)\) down the non-\(v\) branch was \(\infty_1\), some other box with 0 values had to be opened at some point, and this box is still available to be used as the final values for this branch later on (since if this branch already had a 0, it would have stopped). Thus, the cost of this subtree is at most that originally, and has one fewer “final” box opened.

Thus, we construct a policy \(\pi\) for \(I\) with cost at most that of \(\pi'\). Also, computing a branch for \(\pi\) is as simple as following the corresponding branch of \(\pi'\), opening box \(j\) instead of box \((j, v)\) and remembering the feedback to know which boxes of \(\pi'\) to open in the future. Hence, we can compute a branch of \(\pi\) from \(\pi'\) in polynomial time.

**Proof of Theorem 3.1.** Suppose we have an \(\alpha\)-approximation for \(PB_{\leq T}\). Given an instance \(I\) to \(PB_{\min v}\), we construct the instance \(I'\) for \(PB_{\leq T}\) as described before and then run the approximation algorithm on \(I'\) to get a policy \(\pi_{T'}\). Next, we prune the tree as described in Lemma A.2 to get a policy, \(\pi_I\) of no worse cost. Also, our policy will use time at most polynomially more than the policy for \(PB_{\leq T}\) since each branch of \(\pi_T\) can be computed in polynomial time from \(\pi_{T'}\). Hence, the runtime is polynomial in the size of \(I'\). We also note that we added at most \(mn\) total “final” boxes to construct our new instance \(I'\), and so this algorithm will run in polynomial time in \(m\) and \(n\).

Then, by Lemma A.2 and Lemma A.1 we know the cost of the constructed policy is

\[
c(\pi) \leq c(\pi') \leq \alpha c(\pi_{T'}^*) \leq 2\alpha c(\pi_{T'}^*)
\]

Hence, this algorithm is be a \(2\alpha\)-approximation for \(PB_{\min v}\).

**A.2 Proofs from subsection 3.2**

**Proof of Lemma 3.5.** Consider a policy \(\pi_{I_q^*}\) that for \(1/(10\alpha)\) of the scenarios chooses the outside option \(T\) and for the rest of the scenarios it runs the policy \(\pi_{I_q^*}\), then we get

\[
c(\pi_{I_q^*}) \leq c(\pi_{I_q}) = \frac{T}{10\alpha} + \sum_{s \in [t_q, t_q/(10\alpha)]} c_s \frac{p_s}{q},
\]

(3)

On the other hand since \(A_T\) is an \(\alpha\)-approximation to the optimal we have that

\[
c(\pi_{I_q}) \leq \alpha c(\pi_{I_q}^*) \leq \frac{T}{5}
\]

where for the last inequality we put together the assumption from our lemma, (3) and that \(\pi_T^*\) pays at least for the scenarios in \([t_q, t_q/(10\alpha)]\). Since the expected cost of \(A_T\) is at most \(T/5\) using Markov’s inequality, we get that \(\Pr[c_s \geq T] \leq (T/5)/T = 1/5\). Therefore, \(A_T\) covers at least 0.8 mass every time.

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Proof of Lemma 3.6. In every interval of the form \( \mathcal{I}_i = [t_{(0.2)^i}, t_{(0.2)^i}/(10\alpha)] \) the optimal policy for \( \mathcal{PB}_{\text{min}} \) covers at least \( 1/(10\alpha) \) of the probability mass that remains. Since the values belong in the interval \( \mathcal{I}_i \) in phase \( i \), it follows that the minimum possible value that the optimal policy might pay is \( t_{(0.2)^i} \), i.e. the lower end of the interval. Summing up for all intervals, we get the lemma.

B Proofs from section 4

B.1 Proofs from section 4.2.1

In order to show Theorem 4.3 we first need the following lemma that bounds the cost of a scenario that takes at least one outside option test in the transformed instance of \( \mathcal{PB}_{\leq T} \).

Lemma B.1. Let \( \mathcal{I} \) be an instance of \( \mathcal{PB}_{\leq T} \), and \( \mathcal{I}' \) the instance of UDT constructed by the reduction of Theorem 4.3. For a scenario \( s_i \) with probability \( p_i > 1/m^2 \), if there is at least one outside option test run in \( \pi_{\mathcal{I}} \), then \( c(\pi_{\mathcal{I}}(s_i)) \leq 3c(\pi_{\mathcal{I}'}(s_i)) \).

Proof of Lemma B.1. Denote by \( M \) the box tests ran before there were \( T/2 \) total outside option tests ran on the branch of \( \pi \) corresponding to scenario \( i \), and similarly denote by \( N \) the total number of outside option tests on \( i \)'s branch. In \( \pi_{\mathcal{I}'} \), the smallest cost such a combination of tests could achieve is if all the outside option tests are run first since the outside option tests cause some copies to be isolated and so can reduce their cost. Then, the copies of \( i \) can be split into two groups; those that were isolated before \( T/2 \) outside option tests were run, and those that were isolated after. We distinguish between the following cases, based on the value of \( N \).

1. \( N \geq T/2 \). Observe that in this case both groups are not empty. For those isolated after, we know each pays at least \( M + T/2 \) for the initial box tests and the initial sequence of outside option tests. On the other hand, we can lower bound the cost of those isolated before by imagining they were all run in sequence immediately as in the last paragraph. The cost of all the copies in \( \pi \) then is at least

\[
\sum_{j=1}^{K_i} \sum_{k=1}^{T/2} \frac{c_{T/2}^j}{T} k + \sum_{j=1}^{K_i} \sum_{k=T/2+1}^{T} \frac{c_{T/2}^j}{T} (T/2 + M) \geq c_{T/2}^i (3T/8 + M/2) \geq \frac{3}{8} p_i (T + M)
\]

Since \( N \geq T/2 \) by assumption, we know that \( \pi_0 \) will take the outside option for \( i \). By construction, we have that \( \pi_0 \) will take the outside option immediately after running the \( M \) initial box tests. So, the total contribution \( i \) has on the expected cost of \( \pi_0 \) is at most \( p_i (M + T) \) in this case. Hence, we have that \( i \)'s contribution in \( \pi_0 \) is at most \( \frac{3}{8} \leq 3 \) times \( i \)'s contribution in \( \pi \).

2. \( N < T/2 \). By construction, we have that \( \pi_0 \) will only run the \( M \) box tests and this was sufficient for finding a value less than \( T \). The total contribution \( i \) has on the expected cost of \( \pi_0 \) is exactly \( p_i M \) in this case. On the other hand, since \( N < T/2 \) we know that at least half

\[The final box test can also isolate but if it were done sooner, then \( i \)'s copies would be isolated and so \( \pi \) could not have run at least \( M + N \) tests, which contradicts our definition of \( M \) and \( N \).]
of each copy’s copies will have to pay $M$ for all of the box tests. In symbols, the cost of all the copies is at least

$$\sum_{j=1}^{K_i} \sum_{k=N}^{T} \frac{cp_\ell}{T} M = cp_\ell \frac{T - N}{T} M \geq cp_\ell M/2$$

This means that the contribution $i$ has on $\pi$’s expected cost is at least $cp_\ell M/2$. Hence, we have $c(\pi_0) \leq 3c(\pi)$ if ignoring the low-probability scenarios.

\[\square\]

We now show the theorem. Since the proof follows almost identically to Theorem 4.2, we just highlight the key differences.

**Proof of Theorem 4.3.**  We are again given an instance $I$ of $\mathcal{PB}_{\leq T}$ and transform it to an instance $I'$ of $\mathcal{UDT}$.

**Construction of the instance:** we remove all scenarios with $p_i \leq \frac{1}{m^2}$, and for every scenario $s_i$ create $\frac{K_i}{p_i} \cdot T$ sub-scenarios, denoted by $s_{ijk}$ where $j \in \left[\frac{K_i}{p_i}\right]$ and $k \in [T]$. We also denote $K_i = \frac{K_i}{p_i}$. Each subscenario each has equal probability $p_{ijk} = \frac{cp_\ell}{T}$. The box tests $T^B$ boxes are created as

$$T^B(i, j, k) = \begin{cases} j \cdot k, & \text{if } v_{B_i} < T \\ \infty, & \text{else} \end{cases}$$

In order to simulate box $b_T$ without introducing a test with non-unit cost, we use a sequence of tests. We define $T$ total outside option tests:

$$T_h(i, j, k) = \begin{cases} \sum_{f=1}^{i-1} K_f + j, & \text{if } k = h \\ \infty, & \text{else} \end{cases}$$

The $h$’th test isolates all scenarios with third subscript being $h$.

**Construction of the policy:** the only difference in constructing the policy $\pi_I$ is that it ignores any outside option tests that $\pi_I'$ runs until $\pi_I'$ has run at least $T/2$ such tests, at which point $\pi_I$ takes the outside option box $b_T$. In order to show the feasibility of $\pi_I$, assume that scenario $i$ does not take $b_T$ in $\pi_I$. Then, the policy $\pi_I'$ must have ran less than $T/2$ outside option tests on any sub-scenario of $i$’s branch of $\pi_I'$ and let outside option test $h$ be an outside option test not ran. Then, scenarios $s_{i1h}, \ldots, s_{iK_ih}$ are on the same branch absent any box tests. However, since $\pi_I'$ is feasible for $\mathcal{UDT}$, it must distinguish these scenarios and let $T^B$ be the box test that has been ran to distinguish them. Observe that $T^B$ only gives non-infinite value for $s_{ijh}$ if the corresponding box $B$ gave less than $T$ value for scenario $i$, and since $T^B$ is ran, we have that box $B$ is opened, thus giving a value less than $T$.

\[\text{Observe that there are exactly } T \text{ possible options for } k \text{ for any scenario. Choosing all these tests costs } T \text{ and isolates all scenarios thus simulating } b_T.\]
Approximation ratio: we show that \( c(\pi_I) \leq 3c(\pi_{I'}) \), when not considering the low-probability scenarios. Let \( s_i \) be any scenario in \( I \). We distinguish between the following cases, depending on whether there are outside option tests on \( s_i \)'s branch.

1. **No outside option tests** on \( s_i \)'s branch: since box tests always give the same value for all copies of a scenario, we know that all copies of \( s_i \) will have run the same tests and will have paid the same total cost. Let the total number of tests run for any copy be \( M \). Then, we have the total contribution of copies of scenario \( s_i \) to \( c(\pi_{I'}) \) is \( c(\pi_{i})M \). By definition of \( \pi_I \), the branches for \( s_i \) in \( \pi_I \) and its copies in \( \pi_{I'} \) are the same, since \( \pi_I \) opens boxes that branch the same as the corresponding tests in \( \pi_{I'} \). Hence, \( M \) boxes are opened on the \( i \) branch of \( \pi_I \) and so \( i \)'s contribution to \( c(\pi_I) \) is exactly \( p_iM \). Using that \( c \geq 1 \), we have \( c(\pi_I(s_i)) \leq c(\pi_{I'}(s_i)) \) for this case.

2. **Some outside option tests** on \( i \)'s branch: for this case, from Lemma B.1 we get that \( c(\pi_0(s_i)) \leq 3c(\pi(s_i)) \).

The low probability scenarios in the worst case take the outside option incurring cost

\[
2 \frac{1}{m^2} \cdot T \cdot m \leq 2c(\pi_{I'})
\]

since there are at most \( m \) scenarios, their probability is at most \( 1/(m^2) \) and the maximum cost these can pay is by opening all \( m \) unit-cost boxes. The last inequality follows since the optimal solution opens at least one box. Putting it all together we have

\[
c(\pi_I) \leq 4c(\pi_{I'}) \leq 4\alpha c(\pi_{I'}^*) \leq 6\alpha c(\pi_I^*).  
\]

Proof of Lemma B.2. It suffices to show that every scenario is isolated. Fix scenario \( s_i \). Observe that \( s_i \)'s branch has opened the isolating box \( B_i \) at some point, since that is the only box giving a less than \( T \) value for \( s_i \). Also, by construction no branch will ever take the outside option since it costs more than opening all boxes. Let \( S \) be the set of scenarios just before \( B_i \) is opened and note that by definition \( s_i \in S \).

If \( |S| = 1 \), then since \( \pi_{I'} \) runs tests giving the same branching behavior by definition of \( \pi_{I'} \), and \( s_i \) is the only scenario left, we have that the branch of \( \pi_{I'} \) isolates scenario \( s_i \).

If \( |S| > 1 \) then all scenarios in \( S \setminus \{s_i\} \) do not receive small values from opening box \( B_i \), therefore they receive their small values at strictly deeper leaves in the tree. By induction on the depth of the tree, we can assume that for each scenario \( s_j \in S \setminus \{s_i\} \) scenario \( s_j \) is isolated in \( \pi_{I'} \). We distinguish the following cases based on when we encounter \( B_i \) among the isolating boxes on \( s_i \)'s branch.
1. **B\textsuperscript{i} was the first isolating box opened on the branch:** by definition, the policy π\textsubscript{I} ignores box B\textsuperscript{i}. Since every leaf holds a unique scenario in S \ {s\textsubscript{i}}, if we simply ignore s\textsubscript{i} it will follow some path of tests and either be isolated or end up in a node that originally would have had only one scenario, as shown in figure 2. Hence, there are only two scenarios at that node and π\textsubscript{I} simply run the cheapest test distinguishing s\textsubscript{i} from that scenario.

![Figure 2: Case 1: S is the set of scenarios remaining when B\textsuperscript{i} is opened, s\textsubscript{leaf} is the scenario that s\textsubscript{i} ends up with.](image)

2. **A different box B\textsuperscript{j} was opened before B\textsuperscript{i}:** by our construction, instead of ignoring B\textsuperscript{i} we now run the cheapest test that distinguishes s\textsubscript{i} from s\textsubscript{j}, causing i and j to go down separate branches, as shown in figure 3. Then, we apply the induction hypothesis again to the scenarios in these sub-branches. The same argument as before applies so that for both s\textsubscript{i} and s\textsubscript{j} they either are isolated or end up in a node with a single scenario and then get distinguished by the last case of π\textsubscript{I}'s construction.

Hence, π\textsubscript{I} is isolating for scenario i and so π\textsubscript{I} is a feasible solution to the instance of UDT. Also, notice that any two scenarios that have isolating boxes on the same branch will end up in distinct subtrees of the lower node.

**Lemma B.3.** Given an instance I of PB\textsubscript{\textleq}T and an instance I’ of UDT, in the reduction of Theorem 4.4 it holds that

\[ c(\pi_{I'}) \leq 2c(\pi_I). \]

**Proof of Lemma B.3.** Let s\textsubscript{i} be any scenario in S. We use induction on the number of isolating boxes along s\textsubscript{i}'s branch in I'. Initially observe that B\textsuperscript{i} will always exist in s\textsubscript{i}'s branch, in any feasible solution to I. We use c(B\textsuperscript{j}) and c(T\textsubscript{k}) to denote the costs of box B\textsuperscript{j} and test T\textsubscript{k}, for any k \in [n] and j \in [n + m].

1. **Only B\textsuperscript{i} is on the branch:** since B\textsuperscript{i} will be ignored, we end up with s\textsubscript{i} and some other scenario, not yet isolated, let s\textsubscript{leaf} be that scenario. To isolate s\textsubscript{i} and s\textsubscript{leaf} we run the cheapest
Figure 3: Case 2: run test $T_i$ vs $j$ to distinguish $s_i$ and $s_j$. Sets $S_1$ and $S_2$ partition $S$

test that distinguishes between these. From the definition of the cost of $B^j$ we know that $c(T_{s_i \text{ vs } s_\text{leaf}}) \leq c(B^j)$. Additionally, since $c(s_i) \leq c(s_\text{leaf})$, overall we have $c(\pi_I) \leq 2c(\pi_{I'}).

2. More than one isolating boxes are on the branch: using a similar argument, observe additionally that for any extra isolating box $B^j$ we encounter, we substitute it with a test that distinguishes between $s_i$ and $s_j$ and costs at most $c(B^j)$. Given that $c(s_i) \leq c(s_\text{leaf})$ we again have $c(\pi_I) \leq 2c(\pi_{I'}).$

Proof of Theorem 4.4.

Constructing the policy: given a policy $\pi_I$ for the instance of $\mathcal{PB}_{\leq T}$, we construct a policy $\pi_{I'}$. For any test box $B_j$ that $\pi_I$ opens, $\pi_{I'}$ runs the equivalent test $T_j$. Then, there are a few special cases to consider.

1. If $\pi_I$ opens an isolating box $B^j$ for the first time on the current branch, then $\pi_{I'}$ ignores this box but remembers the scenario $i$, which $B^i$ would have value 0 for.

2. If $\pi_I$ opens another isolating box $B^j$ after some $B^i$ on the branch, $\pi_{I'}$ runs the minimum cost test that distinguishes scenario $j$ from $k$ where $B^k$ was the most recent isolating box opened on this branch prior to $B^j$.

3. If we are at the end of $\pi_I$, there can be at most 2 scenarios remaining on the branch, so $\pi_{I'}$ just runs the minimum cost test that distinguishes these two scenarios.

By Lemma B.2, we get that the above policy is feasible for $\mathcal{UDT}$.

Approximation ratio: from Lemma B.3 we have that $c(\pi_{I'}) \leq 2c(\pi_I)$. For the optimal policy, we have that $c(\pi_{I'}^*) \leq 3c(\pi_I^*)$. This holds since if we have an optimal solution to $\mathcal{UDT}$, we can add an isolating box at every leaf to make it feasible for $\mathcal{PB}_{\leq T}$, by only increasing the cost by a factor
of \(3^{10}\), which means that \(c(\pi_T^*)T\) will be less than this transformed \(\mathcal{PB}_{\leq T}\) solution. Overall, if \(\pi_T\) is computed from an \(\alpha\)-approximation for \(\mathcal{PB}_{\leq T}\), we have
\[
c(\pi_T) \leq 2c(\pi_T) \leq 2\alpha c(\pi_T^*) \leq 6\alpha c(\pi_T^*)
\]
So, our construction is an \(6\alpha\)-approximation for \(\mathcal{UDT}\).

\[\square\]

C Proofs from section 5

Proof of Lemma 5.3. Let \(s_1, s_2 \in S\) be any two scenarios in the instance of \(\mathcal{PB}_{\min v}\) and let \(v_i\) be the value returned by opening the \(i\)'th informative box, which has distributions \(D_{i,s_1}\) and \(D_{i,s_2}\) for scenarios \(s_1\) and \(s_2\) respectively. Then by the definition of informative boxes for every such box opened, there is a set of values \(v\) for which \(\Pr_{D_{i,s_1}}[v] \geq \Pr_{D_{i,s_2}}[v]\) and a set for which the reverse holds. Denote these sets by \(M_{s_1}^i\) and \(M_{s_2}^i\) respectively. We also define the indicator variables \(X_{s_i}^i = 1[v_i \in M_{s_i}^i]\). Define \(\overline{X} = \sum_{i \in [k]} X_{s_i}^i/k\) and observe that \(\mathbb{E}[\overline{X}|s_1] = \sum_{i \in [k]} \Pr[M_{s_1}^i]/k\).

Since for every box we have an \(\varepsilon\) gap in TV distance between the scenarios \(s_1, s_2\) we have that
\[
|\mathbb{E}[\overline{X}|s_1] - \mathbb{E}[\overline{X}|s_2]| \geq \varepsilon,
\]
therefore if \(|\overline{X} - \mathbb{E}[\overline{X}|s_1]| \leq \varepsilon/2\) we conclude that scenario \(s_2\) is eliminated, otherwise we eliminate scenario \(s_1\). The probability of error is \(\Pr_{D_{i,s_1}}[\overline{X} - \mathbb{E}[\overline{X}|s_1] > \varepsilon/2] \leq e^{-2k(\varepsilon/2)^2}\), where we used Hoeffding’s inequality since \(X_i \in \{0, 1\}\). Since we want the probability of error to be less than \(\delta\), we need to open \(O\left(\frac{\log 1/\delta}{\varepsilon^2}\right)\) informative boxes.

\[\square\]

Proof of Theorem 5.1. We describe how to bound the final cost, and calculate the runtime of the DP. Denote by \(L = m^2/\varepsilon^2 \log 1/\delta\) where we show that in order to get \((1 + \beta)\)-approximation we set \(\delta = \frac{\beta c_{\min}}{m^2\varepsilon^2}\).

Cost of the final solution: observe that the only case where the DP limits the search space is when \(|S| = 1\). If the scenario is identified correctly, the DP finds the optimal solution by running the greedy order; every time choosing the box with the highest probability of a value below \(T\).

In order to eliminate all scenarios but one, we should eliminate all but one of the \(m^2\) pairs in the list \(E\). From Lemma 5.3, and a union bound on all \(m^2\) pairs, the probability of the last scenario being the wrong one is at most \(m^2\delta\). By setting \(\delta = \frac{\beta c_{\min}}{(m^2T)}\), we get that the probability of error is at most \(\beta c_{\min}/T\), in which case we pay at most \(T\), therefore getting an extra \(\beta c_{\min} \leq \beta c(\pi^*)\) factor.

Runtime: the DP maintains a list \(M\) of sets of informative boxes opened, and numbers of non informative ones. Recall that \(M\) has the following form \(M = S_1|x_1|S_2|x_2|\ldots|S_k|x_k\), where \(k \leq L\) from Lemma 5.3 and the fact that there are \(m^2\) pairs in \(E\). There are in total \(n\) boxes, and \(L\) “positions” for them, therefore the size of the state space is \(\binom{L}{n} = O(n^L)\). There is also an extra \(n\) factor for searching in the list of informative boxes at every step of the recursion. Observe that the numbers of non-informative boxes also add a factor of at most \(n\) in the state space. The list \(E\) adds another factor at most \(nm^2\), and the list \(S\) a factor of \(2^n\) making the total runtime to be \(n^\tilde{O}(m^2/\varepsilon^2)\).

\[\square\]

\[\text{footnote text}^{10}\text{This is because for every two scenarios, the } UDT \text{ solution must distinguish between them, but one of these scenarios is the max scenario from the definition of } T_j, \text{ for which we pay less than } T_j.\]

\[\text{footnote text}^{11}\text{When there is only one scenario, this is exactly Weitzman’s algorithm.}\]