On the construction of Bessel house-moving and its properties

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Abstract

The purpose of this paper is to introduce the construction of a new stochastic process called “δ-dimensional Bessel house-moving” and its properties. δ-dimensional Bessel house-moving is a δ-dimensional Bessel process hitting a fixed point at $t = 1$ for the first time. We have two methods for the construction of this process: characterizing it using the first hitting time of a Bessel process and obtaining it as the weak limit of conditioned Bessel bridges. We also study sample path properties of this process and give the decomposition formula for its distribution.

1 Introduction and main results

Throughout this paper, we fix $\delta > 0$ and set $\nu := \delta/2 - 1$.

For $0 \leq a < b$, let $R^a = \{R^a(t)\}_{t \geq 0}$ be a δ-dimensional Bessel process starting from $a$ and let $\tau_{a,b}$ denote the first hitting time of the point $b$ by $R^a$:

$$\tau_{a,b} := \inf\{r \geq 0 \mid R^a(r) = b\}.$$

In addition, for $0 \leq t_1 < t_2 \leq 1$, $r_{a \rightarrow b}^{t_1, t_2} = \{r_{a \rightarrow b}^{t_1, t_2}(t)\}_{t \in [t_1, t_2]}$ denotes a δ-dimensional Bessel bridge from $a$ to $b$ on $[t_1, t_2]$. We write simply $r_{a \rightarrow b} : = r_{a \rightarrow b}^{0, 1}$.

For a continuous process $X = \{X(t)\}_{t \in [0,1]}$, we denote its maximal value as

$$M_{[t_1, t_2]}(X) = \max_{t_1 \leq u \leq t_2} X(u).$$

For $\eta > 0$, $0 \leq s < t \leq 1$, and $x, y \in [0, \eta]$, we define

$$q_1^{(\eta)}(s, x, t, y) := \frac{P(R^x(t-s) \in dy)}{dy} P\left(M_{[s, t]}(r_{x \rightarrow y}^{s, t}) \leq \eta\right),$$

$$q_2^{(\eta)}(t, y) := \lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \varepsilon} q_1^{(\eta + \varepsilon)}(0, y, t, \eta) = \frac{P(R^y(t) \in d\eta)}{d\eta} \lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \varepsilon} P\left(M_{[0, t]}(r_{[0, t]}^{y, \eta}) \leq \eta + \varepsilon\right).$$

First, we construct a new stochastic process, “δ-dimensional Bessel house-moving” $H_{a \rightarrow b}^{a \rightarrow b}$ (0 ≤ $a < b$) from $a$ to $b$ on [0,1], by using the first hitting time of a δ-dimensional Bessel process.

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Theorem 1. Let $0 \leq a < b$. There exists an $\mathbb{R}$-valued continuous Markov process $H^{a\rightarrow b} = \{H^{a\rightarrow b}(t)\}_{t\in[0,1]}$ that satisfies

$$P(H^{a\rightarrow b}(t) \in dy) = P(R^a(t) \in dy \mid \tau_{a,b} = 1) = \frac{q_1^{(b)}(0,a,t,y)q_2^{(b)}(1-t,y)}{q_2^{(b)}(1,a)} dy,$$

$$P(H^{a\rightarrow b}(t) \in dy \mid H^{a\rightarrow b}(s) = x) = P(R^a(t) \in dy \mid R^a(s) = x, \tau_{a,b} = 1)$$

$$= \frac{q_1^{(b)}(s,x,t,y)q_2^{(b)}(1-t,y)}{q_2^{(b)}(1-s,x)} dy$$

for $0 < s < t < 1$ and $x, y \in (0,b)$.

Let $C([0,1],\mathbb{R})$ be a class of $\mathbb{R}$-valued continuous functions defined on $[0,1]$ and let $d_\infty(w,w') = \sup_{0\leq t \leq 1} |w(t) - w'(t)|$ ($w,w' \in C([0,1],\mathbb{R})$). $\mathcal{B}(C([0,1],\mathbb{R}))$ denotes the Borel $\sigma$-algebra with respect to the topology generated by the metric $d_\infty$. In addition, for $0 \leq s < t \leq 1$, $\pi_{[s,t]} : C([0,1],\mathbb{R}) \rightarrow C([s,t],\mathbb{R})$ denotes the restriction map.

Assume that $Y : (\Omega, \mathcal{F}, P) \rightarrow (C([0,1],\mathbb{R}), \mathcal{B}(C([0,1],\mathbb{R})))$ is a random variable and that $\Lambda \in \mathcal{B}(C([0,1],\mathbb{R}))$ satisfies $P(Y \in \Lambda) > 0$. Then, we define the probability measure $P_{Y^{-1}(\Lambda)}$ on $(Y^{-1}(\Lambda), Y^{-1}(\Lambda) \cap \mathcal{F})$ as

$$P_{Y^{-1}(\Lambda)}(A) := \frac{P(A)}{P(Y \in \Lambda)}, \quad A \in Y^{-1}(\Lambda) \cap \mathcal{F} := \{Y^{-1}(\Lambda) \cap F \mid F \in \mathcal{F}\}.$$

Throughout this paper, $P_{Y^{-1}(\Lambda)}(Y|_{\Lambda} \in \Gamma)$ is often written as $P(Y|_{\Lambda} \in \Gamma)$.

In addition, $X_n \overset{D}{\rightarrow} X$ means that $(X_n)_{n=1}^\infty$ converges to $X$ in distribution.

In [2], it was shown that 3-dimensional Bessel house-moving (i.e., Brownian house-moving) can be obtained as the weak limit of the conditioned 3-dimensional Bessel bridge. Motivated by that work, we construct the $\delta$-dimensional Bessel house-moving $H^{a\rightarrow b}$ as the weak limit of the conditioned $\delta$-dimensional Bessel bridge.

Theorem 2. Let $0 \leq a < b$. There exists an $\mathbb{R}$-valued continuous Markov process $H^{a\rightarrow b} = \{H^{a\rightarrow b}(t)\}_{t\in[0,1]}$ that satisfies

$$r^{a\rightarrow b}_{K^- \downarrow (b+\eta)} \overset{D}{\rightarrow} H^{a\rightarrow b}, \quad \eta \downarrow 0,$$

where $K^-(b+\eta) := \{w = \{w(t)\}_{t\in[0,1]} \in C([0,1],\mathbb{R}) \mid w(t) \leq b + \eta, \, 0 \leq t \leq 1\}$.

Also, we study the sample path properties of $\delta$-dimensional Bessel house-moving $H^{a\rightarrow b}$, and establish the regularity of its sample path. We show that the $\delta$-dimensional Bessel house-moving does not hit $b$ on the time interval $[0,1]$.

Proposition 1.1. For every $\gamma \in (0, \frac{1}{2})$, the path of $H^{a\rightarrow b}$ ($0 \leq a < b$) on $[0,1]$ is locally Hölder-continuous with exponent $\gamma$:

$$P \left( \bigcup_{n=1}^\infty \sup_{0 < s,t \leq 1} \frac{|H^{a\rightarrow b}(t) - H^{a\rightarrow b}(s)|}{|t-s|^{\gamma}} < \infty \right) = 1.$$
Proposition 1.2. Let $0 \leq a < b$. For $t \in (0, 1)$, it holds that

$$P \left( \max_{0 \leq u \leq t} H^{a \rightarrow b}(u) < b \right) = 1.$$  

The remainder of this paper is structured as follows: In Section 2, we introduce some basic facts related to Bessel functions, the Bessel process, and the Bessel bridge. In Sections 3 and 4, we construct the $\delta$-dimensional Bessel house-moving in two ways. In Section 3, we prove Theorem 1 which gives the construction of the Bessel house-moving by using the first hitting time of the Bessel process. In Section 4, we prove Theorem 2 which gives the construction of the Bessel house-moving as the weak limit of the conditioned $\delta$-dimensional Bessel bridge. We also show some sample path properties of the Bessel house-moving. Section 5 is devoted to proving the regularity of the sample path of the $\delta$-dimensional Bessel house-moving (Proposition 1.1). In Section 6, we prove the decomposition formula for the distribution of the Bessel house-moving (Theorem 4) and use this formula to prove some results, including Proposition 1.2.

2 Preliminaries

2.1 Bessel functions

Let $J_\alpha(z)$ and $I_\alpha(z)$ denote the Bessel function and modified Bessel function of the first kind with index $\alpha \in \mathbb{R}$, respectively defined as

$$J_\alpha(z) = \left(\frac{1}{2}z\right)^\alpha \sum_{k \in \mathbb{Z}_+} \frac{(-\frac{1}{4}z^2)^k}{k!(\alpha + k + 1)},$$

$$I_\alpha(z) = \left(\frac{1}{2}z\right)^\alpha \sum_{k \in \mathbb{Z}_+} \frac{\left(\frac{1}{4}z^2\right)^k}{k!(\alpha + k + 1)}$$

for $z \in \mathbb{C} \setminus \mathbb{R}_-$. In addition, for $z \in \mathbb{C} \setminus \mathbb{R}_-$, we define

$$K_\alpha(z) := \frac{\pi (I_{-\alpha}(z) - I_\alpha(z))}{2\sin(\alpha\pi)},$$

when $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and

$$K_\alpha(z) := \lim_{\beta \to \alpha} K_\beta(z)$$

when $\alpha \in \mathbb{Z}$. $K_\alpha(z)$ is called the modified Bessel function with index $\alpha$ of the second kind. Moreover, we define the values of $z^{-\alpha}J_\alpha(z)$ and $z^{-\alpha}I_\alpha(z)$ at zero as

$$z^{-\alpha}J_\alpha(z) |_{z=0} := \lim_{z \downarrow 0} z^{-\alpha}J_\alpha(z) = \frac{1}{2^\alpha \Gamma(\alpha + 1)},$$

$$z^{-\alpha}I_\alpha(z) |_{z=0} := \lim_{z \downarrow 0} z^{-\alpha}I_\alpha(z) = \frac{1}{2^\alpha \Gamma(\alpha + 1)}.$$
We obtain the following derivatives:
\[
\frac{d}{dz} (z^\alpha J_\alpha(z)) = z^\alpha J_{\alpha-1}(z), \quad \frac{d}{dz} (z^{-\alpha} J_\alpha(z)) = -z^{-\alpha} J_{\alpha+1}(z),
\]
\[
\frac{d}{dz} (z^\alpha I_\alpha(z)) = z^\alpha I_{\alpha-1}(z), \quad \frac{d}{dz} (z^{-\alpha} I_\alpha(z)) = z^{-\alpha} I_{\alpha+1}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}_-.
\]
Moreover, we have
\[
\frac{d}{dz} J_\alpha(z) = J_{\alpha-1}(z) - \frac{\alpha}{z} J_\alpha(z) = -J_{\alpha+1}(z) + \frac{\alpha}{z} J_\alpha(z),
\]
\[
\frac{d}{dz} I_\alpha(z) = I_{\alpha-1}(z) - \frac{\alpha}{z} I_\alpha(z) = I_{\alpha+1}(z) + \frac{\alpha}{z} I_\alpha(z).
\]
In the rest of this section, we assume that \( \alpha > -1 \). According to [6, (2.2) and (2.8)], there exists \( C_\alpha > 0 \) such that
\[
z^{-\alpha}|J_\alpha(z)| \leq C_\alpha \frac{1}{(1+z)^{\alpha+\frac{1}{2}}}, \quad z \geq 0, \tag{1}
\]
\[
z^{-\alpha}I_\alpha(z) \leq C_\alpha \frac{1}{(1+z)^{\alpha+\frac{1}{2}}}e^z, \quad z \geq 0. \tag{2}
\]
The sequence of positive zeros of the Bessel function \( J_\alpha \) is denoted by \( \{j_{\alpha,n}\}_{n=1}^{\infty} \). It is well known that for \( n = 1, 2, \ldots \)
\[
j_{\alpha,n} < j_{\alpha+1,n} < j_{\alpha,n+1}.
\]
Therefore,
\[
J_{\alpha+1}(j_{\alpha,n}) \neq 0, \quad n = 1, 2, \ldots.
\]
In addition, from [5, Appendix] we find the following asymptotics as \( n \to \infty \):
\[
j_{\alpha,n} \sim n\pi, \quad J_{\alpha+1}(j_{\alpha,n}) \sim (-1)^{n-1} \sqrt{\frac{2}{\pi j_{\alpha,n}}} \sim (-1)^{n-1} \frac{1}{\pi} \sqrt{\frac{2}{n}}. \tag{3}
\]

### 2.2 Bessel process and Bessel bridge

The \( \delta \)-dimensional Bessel process is a one-dimensional diffusion generated by \( \mathcal{L}_\delta := \frac{1}{2} \frac{d^2}{dx^2} + \frac{\delta - 1}{2x} \frac{d}{dx} \). In addition, for \( 0 \leq a < b \) the \( \delta \)-dimensional Bessel bridge from \( a \) to \( b \) on \([0, 1]\) is defined by conditioning the \( \delta \)-dimensional Bessel process from \( a \), \( R^a = \{R^a(t)\}_{t \geq 0} \), on \( R^a(1) = b \).

For \( t > 0 \) and \( x, y \in (0, \infty) \), we set
\[
n_t(x) := \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right), \quad A^{(\nu)}_t(x, y) := n_t(x)n_t(y)I_\nu \left( \frac{xy}{t} \right).
\]
Let \( a, b \geq 0 \). For \( 0 < s < t \) and \( x, y > 0 \), we have the transition densities of \( R^a \):
\[
P(R^a(t) \in dy) = 2\pi y \left( \frac{y}{a} \right)^\nu A^{(\nu)}_t(a, y)dy,
\]
\[
P(R^a(t) \in dy \mid R^a(s) = x) = 2\pi y \left( \frac{y}{x} \right)^\nu A^{(\nu)}_{t-s}(x, y)dy.
\]
For $0 < s < t < 1$ and $x, y > 0$, we have the transition densities of the $\delta$-dimensional Bessel bridge from $a$ to $b$, $r^{a \to b}$, on $[0, 1]$:

\[
P(r^{a \to b}(t) \in dy) = \frac{P(R^a(t) \in dy) P(R^b(1 - t) \in db)}{P(R^a(1) \in db)} = \frac{2\pi y A^{(\nu)}_t(a, y) A^{(\nu)}_{1-t}(y, b)}{A^{(\nu)}_1(a, b)} dy,
\]

\[
P(r^{a \to b}(t) \in dy \mid r^{a \to b}(s) = x) = \frac{P(R^a(t - s) \in dy) P(R^b(1 - t) \in db)}{P(R^a(1 - s) \in db)} = \frac{2\pi y A^{(\nu)}_{1-s}(x, y) A^{(\nu)}_{1-1-t}(y, b)}{A^{(\nu)}_{1-1-s}(x, b)} dy.
\]

In the next lemma, we express the joint densities of the Bessel bridge and the maximal value of the Bessel process by the maximal values of the Bessel bridges.

**Lemma 2.1.** Let $c \geq 0$ and $0 \leq a, b \leq c$. For $0 < s < t < 1$ and $0 \leq x, y \leq c$, we have

\[
P(r^{a \to b}(t) \in dy, M(r^{a \to b}) \leq c)
= P\left(M_{[0,t]}(r^{a \to b}_{[0,t]}) \leq c\right) P\left(M_{[t,1]}(r^{a \to b}_{[t,1]}) \leq c\right) P\left(r^{a \to b}(t) \in dy\right), \tag{4}
\]

\[
P(r^{a \to b}(t) \in dy, r^{a \to b}(s) \in dx, M(r^{a \to b}) \leq c)
= P\left(M_{[0,s]}(r^{a \to b}_{[0,s]}) \leq c\right) P\left(M_{[s,t]}(r^{a \to b}_{[s,t]}) \leq c\right) P\left(M_{[t,1]}(r^{a \to b}_{[t,1]}) \leq c\right)
\times P\left(r^{a \to b}(t) \in dy, r^{a \to b}(s) \in dx\right). \tag{5}
\]

Proof. First, we prove (4). By the Markov property of $R^a$, we have

\[
P(r^{a \to b}(t) \in dy, M(r^{a \to b}) \leq c)
= \frac{P(R^a(t) \in dy, M(R^a) \leq c, R^a(1) \in db)}{P(R^a(1) \in db)}
= \frac{P(R^a(1 - t) \in db, M_{1-t}(R^b) \leq c) \times P(R^a(t) \in dy, M(R^a) \leq c)}{P(R^a(1) \in db)}
\]

and

\[
P(r^{a \to b}(t) \in dy) = \frac{P(R^a(t) \in dy, R^a(1) \in db)}{P(R^a(1) \in db)} = \frac{P(R^a(1 - t) \in db) P(R^a(t) \in dy)}{P(R^a(1) \in db)}.
\]

Therefore, because

\[
P(R^b(1 - t) \in db, M_{1-t}(R^b) \leq c) = P\left(M_{[t,1]}(r^{a \to b}_{[t,1]}) \leq c\right) P(R^b(1 - t) \in db),
\]

\[
P(R^a(t) \in dy, M_t(R^a) \leq c) = P\left(M_{[0,t]}(r^{a \to y}_{[0,t]}) \leq c\right) P(R^a(t) \in dy),
\]

it follows that

\[
P(r^{a \to b}(t) \in dy, M(r^{a \to b}) \leq c)
= P\left(M_{[0,t]}(r^{a \to y}_{[0,t]}) \leq c\right) P\left(M_{[t,1]}(r^{a \to b}_{[t,1]}) \leq c\right) P(r^{a \to b}(t) \in dy),
\]

which completes the proof. In a similar manner to the proof of (4), we can obtain (5). \qed
3 Proof of Theorem 1

In this section, we prove Theorem 1 which gives the construction of the Bessel house-moving by using the first hitting time of the Bessel process.

Lemma 3.1. Let $b > 0$. For $t > 0$ and $y \in (0, b)$, we have

$$
\frac{P(\tau_{y,b} \in dt)}{dt} = \frac{q_2^{(b)}(t,y)}{2},
$$

(6)

$$
\frac{P(\tau_{0,b} \in dt)}{dt} = \frac{q_2^{(b)}(t,0)}{2}.
$$

(7)

Proof. First, we prove (6). It holds that

$$
P(\tau_{y,b} \in dt) = \frac{\partial}{\partial t} P(\tau_{y,b} \leq t) = -\frac{\partial}{\partial t} P(M_t(R_y) < b)
$$

$$
= -\frac{\partial}{\partial t} \int_0^b P(M_t(R_y) \leq b, R_y(t) \in dx)
$$

$$
= -\frac{\partial}{\partial t} \int_0^b P(M_{[0,t]}(r_{y,x}) \leq b) P(R_y(t) \in dx).
$$

For each $n$, we set

$$
f_n(t,x) := \frac{J_\nu(xj_{\nu,n}/b) J_\nu(yj_{\nu,n}/b)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2b^2}t\right), \quad t \geq 0, \quad x \in (0, b).
$$

Then, by Theorem 3, we have

$$
q_1^{(b)}(0,y,t,x) = 2x \left(\frac{x}{y}\right)^\nu \sum_{n=1}^\infty f_n(t,x).
$$

Let $T > 0$ be fixed. By Lemma A.1 and (3), there exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$
\left|\frac{\partial}{\partial t} f_n(t,x)\right| = \left|\frac{J_\nu(xj_{\nu,n}/b) J_\nu(yj_{\nu,n}/b)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \frac{j_{\nu,n}^2}{2b^2} \exp\left(-\frac{j_{\nu,n}^2}{2b^2}t\right)\right|
$$

$$
\leq 2\tilde{C}_\nu \sqrt{(1 + \frac{\pi x}{b})(1 + \frac{\pi y}{b})} (n\pi)^2 \exp\left(-\frac{(n\pi)^2}{8b^2}t\right)
$$

holds for $n > N_\nu$ and $t \in (T, \infty)$. Since

$$
\sum_{n=N_\nu+1}^\infty (n\pi)^2 \exp\left(-\frac{(n\pi)^2}{8b^2}T\right) < \infty
$$
holds, we have
\[
\frac{\partial}{\partial t} q_1^{(b)}(0, y, t, x) = 2x \left( \frac{x}{y} \right) \nu \sum_{n=1}^{\infty} \frac{\partial}{\partial t} f_n(t, x) \quad t \in (T, \infty),
\]
by Lebesgue’s dominated convergence theorem. Thus, we obtain
\[
\sup_{t \in (T, \infty)} \left| \frac{\partial}{\partial t} q_1^{(b)}(0, y, t, x) \right|
\leq y^{-\nu} \sum_{n=1}^{N_\nu} \frac{|x^{\nu+1} J_\nu \left( xj_{\nu,n}/b \right) J_\nu \left( yj_{\nu,n}/b \right)|}{b^2 J_{\nu+1}^2 (j_{\nu,n})} \frac{j_{\nu,n}^2}{b^2} \exp \left( -\frac{j_{\nu,n}^2}{2b^2} T \right)
+ 4 \tilde{C}_\nu \left( \frac{x}{y} \right)^\nu \sqrt{\left( 1 + \frac{x\pi}{b} \right) \left( 1 + \frac{y\pi}{b} \right)} \sum_{n=N_\nu+1}^{\infty} (n\pi)^2 \exp \left( -\frac{(n\pi)^2}{8b^2} T \right), \quad x \in (0, b). \tag{8}
\]
By (1), because there exists \( C_\nu > 0 \) such that
\[
|x^{\nu+1} J_\nu \left( xj_{\nu,n}/b \right)| \leq C_\nu \left( \frac{j_{\nu,n}}{b} \right)^\nu \frac{x^{2\nu+1}}{\left( 1 + \frac{xj_{\nu,n}}{b} \right)^{\nu+\frac{1}{2}}} \leq C_\nu \left( \frac{j_{\nu,n}}{b} \right)^{-1} x^\nu \left( 1 + \frac{xj_{\nu,n}}{b} \right), \quad x \in (0, b)
\]
holds, the functions \( x^{\nu+1} J_\nu \left( xj_{\nu,n}/b \right), n = 1, \ldots, N_\nu \) in the first term on the right-hand side of (8) are integrable with respect to \( x \) on \([0, b]\). In addition, since
\[
x^\nu \sqrt{1 + \frac{x\pi}{b}} \leq x^\nu \left( 1 + \frac{x\pi}{b} \right) \quad (x \in (0, b))
\]
holds, the function \( x^\nu \sqrt{1 + \frac{x\pi}{b}} \) in the second term on the right-hand side of (8) is integrable with respect to \( x \) on \([0, b]\). Therefore, by Lebesgue’s dominated convergence theorem,
\[
\frac{\partial}{\partial t} \int_0^b q_1^{(b)}(0, y, t, x) dx = \int_0^b \frac{\partial}{\partial t} q_1^{(b)}(0, y, t, x) dx.
\]
Let \( \tilde{L}_\nu := \frac{1}{2} \frac{d^2}{dx^2} + \frac{2\nu+1}{x} \frac{d}{dx} \) be the infinitesimal generator of \( R^a \) and let \( m(x) dx = 2x^{2\nu+1} dx \) be the speed measure of the Bessel process. Then, we obtain
\[
m(x) \frac{\partial}{\partial t} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) = m(x) \tilde{L}_\nu \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) = \frac{1}{2} \frac{\partial}{\partial x} \left( m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right).
\]
So, we get
\[
P(\tau_{y,b} \in dt) = -\frac{1}{2} \int_0^b \frac{\partial}{\partial x} \left( m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right) dx = -\frac{1}{2} \left[ m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right]_{x=0}^{x=b}.
\]
Inequality (1) and Lemma [A.1] imply the following inequality:

\[
\sum_{n=1}^{\infty} \sup_{x \in (0, \infty)} \left| \frac{j_{\nu,n} x^{-(\nu+1)} J_{\nu+1} \left( x j_{\nu,n}/b \right) J_{\nu} \left( y j_{\nu,n}/b \right)}{J_{\nu+1}^{2}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right) \right|
\leq C_{\nu+1} \sum_{n=1}^{\infty} j_{\nu,n} \left( \frac{j_{\nu,n}}{b} \right)^{\nu+1} \left| \frac{J_{\nu} (y j_{\nu,n}/b)}{J_{\nu+1} (j_{\nu,n})} \right| \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right)
\leq C_{\nu+1} \sum_{n=1}^{N_{\nu}} j_{\nu,n} \left( \frac{j_{\nu,n}}{b} \right)^{\nu+1} \left| \frac{1}{J_{\nu+1} (j_{\nu,n})} \right| \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right)
\leq C_{\nu+1} \sum_{n=1}^{N_{\nu}} j_{\nu,n} \left( \frac{j_{\nu,n}}{b} \right)^{\nu+1} \left| \frac{1}{J_{\nu+1} (j_{\nu,n})} \right| \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right)
\leq C_{\nu+1} \sum_{n=1}^{N_{\nu}} j_{\nu,n} \left( \frac{j_{\nu,n}}{b} \right)^{\nu+1} \left| \frac{1}{J_{\nu+1} (j_{\nu,n})} \right| \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right)
\]

\[\sum_{n=N_{\nu}+1}^{\infty} \sqrt{n} (2n\pi)^{\nu+2} \exp \left( -\frac{(n\pi)^2}{8b^2 t} \right).\] (9)

Then Lebesgue’s dominated convergence theorem and the inequality (9) show that

\[
m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right)
= 2x^{2\nu+1} \frac{\partial}{\partial x} \left( (xy)^{-\nu} \sum_{n=1}^{\infty} \frac{J_{\nu} (x j_{\nu,n}/b) J_{\nu} (y j_{\nu,n}/b)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right) \right)
= 2x^{2\nu+1} y^{-\nu} \sum_{n=1}^{\infty} \frac{j_{\nu,n} x^{-(\nu+1)} J_{\nu+1} (x j_{\nu,n}/b) J_{\nu} (y j_{\nu,n}/b)}{b^2 J_{\nu+1}^2(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right)
= -2x^{2\nu+2} y^{-\nu} \sum_{n=1}^{\infty} \frac{j_{\nu,n} x^{-(\nu+1)} J_{\nu+1} (x j_{\nu,n}/b) J_{\nu} (y j_{\nu,n}/b)}{b^3 J_{\nu+1}^2(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right)
\]

and

\[
\left. \left( m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right) \right|_{x=b} = -2 \frac{b}{y} \sum_{n=1}^{\infty} \frac{j_{\nu,n} J_{\nu} (y j_{\nu,n}/b)}{b^2 J_{\nu+1}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right) = -q_2^{(b)}(t, y),
\]

\[
\left. \left( m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right) \right|_{x=0}
= -2 \left[ x^{2\nu+2} y^{-\nu} \sum_{n=1}^{\infty} \frac{x j_{\nu,n}}{b} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right) \right]_{x=0}
= -2 \cdot \frac{y^{-\nu}}{2^{\nu+1} \Gamma(\nu+2)} \sum_{n=1}^{\infty} \frac{1}{J_{\nu+1}(j_{\nu,n})} \frac{J_{\nu} (y j_{\nu,n}/b)}{b^3 J_{\nu+1}^2(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 t}{2b^2} \right) = 0
\]

hold. Thus, we have

\[
-\frac{1}{2} \left. \left[ m(x) \frac{\partial}{\partial x} \left( \frac{q_1^{(b)}(0, y, t, x)}{m(x)} \right) \right] \right|_{x=b} = \frac{q_2^{(b)}(t, y)}{2}
\]

and (6) is proved. By (6), Theorem 6 and Corollary 2 we easily obtain (7).
REMARK 3.1. According to \([1](2.7)\) and \((2.8)\),

\[
P(\tau_{a,b} \leq t) = 1 - 2 \left( \frac{b}{a} \right)^\nu \sum_{n=1}^{\infty} \frac{J_\nu(a_{j,\nu}/b)}{j_{\nu,n} J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{J_{\nu,n}^2}{2b^2} t\right),
\]

\[
P(\tau_{0,b} \leq t) = 1 - \frac{1}{2^{\nu-1} \Gamma(\nu + 1)} \sum_{n=1}^{\infty} \frac{\mathcal{J}_{\nu,n}^{\nu-1}}{b^2 J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{J_{\nu,n}^2}{2b^2} t\right)
\]

for \(0 < a < b\) and \(t > 0\). By differentiating these identities, we obtain

\[
P(\tau_{a,b} \in dt) = \left( \frac{b}{a} \right)^\nu \sum_{n=1}^{\infty} \frac{J_\nu(a_{j,\nu}/b)}{b^2 J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{J_{\nu,n}^2}{2b^2} t\right) dt,
\]

\[
P(\tau_{0,b} \in dt) = \frac{1}{2^{\nu} \Gamma(\nu + 1)} \sum_{n=1}^{\infty} \frac{\mathcal{J}_{\nu,n}^{\nu+1}}{b^2 J_{\nu+1}(j_{\nu,n})} \exp\left(-\frac{J_{\nu,n}^2}{2b^2} t\right) dt.
\]

By using \((10)\) and \((11)\), we can also prove Lemma \(3.1\).

**Lemma 3.2.** Let \(b > 0\). For \(0 < t \leq 1\) and \(y \in [0, b)\), we have

\[q_2^{(b)}(t, y) > 0.\]

**Proof.** According to \([6]\) Theorem 3.3, for all \(x \in [0, 1)\) and \(t > 0\), there exists a constant \(C_\nu > 0\) such that

\[
\frac{P(\tau_{x,1} \in dt)}{dt} \geq C_\nu \frac{(1-x)(1+t)^{\nu+2}}{(x+t)^{\nu+\frac{1}{2}t^\frac{1}{2}}} \exp\left(-\frac{(1-x)^2}{2t} - \frac{1}{2} J_{\nu,1}^2 t\right).
\]

Hence, by Lemma \(3.1\) we can prove the assertion as follows:

\[
q_2^{(b)}(t, y) = 2 \frac{P(b^2 \tau_{y,1} \in dt)}{dt} \geq \frac{2}{b^2} \frac{(1-y/b)(1+\frac{t}{b^2})^{\nu+2}}{(\frac{y}{b} + \frac{t}{b^2})^{\nu+\frac{1}{2}(\frac{t}{b^2})^\frac{1}{2}}} \exp\left(-\frac{(b-y)^2}{2t} - \frac{J_{\nu,1}^2}{2b^2} t\right) > 0.
\]

\(\square\)

**Theorem 3.** Let \(0 \leq a < b\). For \(0 < s < t < 1\) and \(x, y \in (0, b)\), we have

\[
P(R^a(t) \in dy \mid \tau_{a,b} = 1) = \frac{q_2^{(b)}(0, a, t, y)}{q_2^{(b)}(1, a)} q_2^{(b)}(1-t, y) dy,
\]

\[
P(R^a(t) \in dy \mid R^a(s) = x, \tau_{a,b} = 1) = \frac{q_2^{(b)}(s, x, t, y)}{q_2^{(b)}(1-s, x)} q_2^{(b)}(1-t, y) dy.
\]

**Proof.** Using the Markov property of \(R^a\), for \(0 < t < u\), it holds that

\[
P(R^a(t) \in dy, \tau_{a,b} > u) = P(R^a(t) \in dy, M_u(R^a) < b) = P(R^a(t) \in dy, M_t(R^a) < b) P(M_a-t(R^a) < b) = P(R^a(t) \in dy, M_t(R^a) < b) P(\tau_{y,b} > u-t).
\]

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Thus, it follows that

\[ P \left( R^a(t) \in dy, \tau_{a,b} \in du \right) = -\frac{d}{du} P \left( R^a(t) \in dy, \tau_{a,b} > u \right) \]
\[ = P \left( R^a(t) \in dy, M_t(R^a) < b \right) P \left( \tau_{y,b} \in du - t \right). \quad (14) \]

We can calculate the first term of the right-hand side of (14) as

\[ P \left( R^a(t) \in dy, M_t(R^a) < b \right) = P \left( R^a(t) \in dy \right) \frac{P \left( R^a(t) \in dy, M_t(R^a) < b \right)}{P \left( R^a(t) \in dy \right)} \]
\[ = P \left( R^a(t) \in dy \right) P \left( M_{\left[0,\xi]\right]}(r_{\xi}^{\left[a\rightarrow y\right]} < b \right) \]
\[ = P \left( R^a(t) \in dy \right) P \left( M_{\left[0,\xi]\right]}(r_{\xi}^{\left[a\rightarrow y\right]} \leq b \right) \]
\[ = q_1(b)(0, a, t, y)dy. \quad (15) \]

Therefore, by (14), (15), (6), (7), and L'Hôpital's rule, we can prove (12) as follows:

\[ P \left( R^a(t) \in dy \mid \tau_{a,b} = 1 \right) = P \left( R^a(t) \in dy, M_t(R^a) < b \right) \frac{P \left( \tau_{y,b} \in du - t \right)}{P \left( \tau_{a,b} \in du \right)} \bigg|_{u=1} \]
\[ = \frac{q_1(b)(0, a, t, y)q_2(b)(1 - t, y)}{q_2(b)(1, a)}dy. \]

Next, we prove (13). Using the Markov property of \( R^a \), for \( 0 < s < t < u \), it holds that

\[ P \left( R^a(t) \in dy, R^a(s) \in dx, \tau_{a,b} > u \right) \]
\[ = P \left( R^a(t) \in dy, R^a(s) \in dx, M_u(R^a) < b \right) \]
\[ = P \left( R^a(s) \in dx, M_s(R^a) < b \right) P \left( R^a(t - s) \in dy, M_{t-s}(R^a) < b \right) P \left( M_{u-(t-s)}(R^a) < b \right) \]
\[ = P \left( R^a(s) \in dx, M_s(R^a) < b \right) P \left( R^a(t - s) \in dy, M_{t-s}(R^a) < b \right) P \left( \tau_{y,b} > u - (t - s) \right). \]

Thus, it follows that

\[ P \left( R^a(t) \in dy, R^a(s) \in dx, \tau_{a,b} \in du \right) \]
\[ = -\frac{d}{du} P \left( R^a(t) \in dy, R^a(s) \in dx, \tau_{a,b} > u \right) \]
\[ = P \left( R^a(s) \in dx, M_s(R^a) < b \right) P \left( R^a(t - s) \in dy, M_{t-s}(R^a) < b \right) P \left( \tau_{y,b} \in du - (t - s) \right). \]

On the other hand, by (14), we obtain

\[ P \left( R^a(s) \in dx, \tau_{a,b} \in du \right) = P \left( R^a(s) \in dx, M_s(R^a) < b \right) P \left( \tau_{x,b} \in du - s \right) \]
Combining this equality, (15), (6), and L'Hôpital's rule, we can prove (13) as follows:

\begin{align*}
P \left( R^a(t) \in dy \mid R^a(s) = x, \tau_{a,b} = 1 \right) &= \left. \frac{P \left( R^a(s) \in dx, M_s(R^a) < b \right) P \left( R^x(t - s) \in dy, M_{t-s}(R^x) < b \right) P \left( \tau_{y,b} \in du - (t-s) \right)}{P \left( \tau_{x,b} \in du - s \right)} \right|_{u=1} \\
&= \left. \frac{P \left( R^x(t - s) \in dy, M_{t-s}(R^x) < b \right) P \left( \tau_{y,b} \in du - (t-s) \right)}{P \left( \tau_{x,b} \in du - s \right)} \right|_{u=1} \\
&= \frac{q_1^{(b)}(s, x, t, y)q_2^{(b)}(1-t, y)}{q_2^{(b)}(1-s, x)} dy.
\end{align*}

We prepare the following inequalities:

**Lemma 3.3.** Let $0 \leq a < b$ and $\eta \in (0, 1]$. There exists some $C_{\nu,b} > 0$ such that

1. \[ q_1^{(b+\eta)}(s, x, t, y) \leq \frac{C_{\nu,b}}{(t-s)^{\nu+1}} n_{t-s}(y - x), \quad 0 \leq s < t \leq 1, x, y \in [0, b + \eta), \]

2. \[ q_1^{(b+\eta)}(r, z, 1, b) \leq \frac{C_{\nu,b}}{(1-r)^{\nu+1}} \left( 1 + \frac{2\eta(b + \eta)}{1 - r} \right) n_{1-r}(z - b), \quad 0 < r < 1, z \in (0, b + \eta). \]

**Proof.** First, we prove inequality (1). By (2), there exists some $C_\nu > 0$ such that

\[ \left( \frac{xy}{t-s} \right)^{-\nu} I_\nu \left( \frac{xy}{t-s} \right) \leq \frac{C_\nu}{(1 + \frac{xy}{t-s})^{\nu+\frac{1}{2}}} \exp \left( \frac{xy}{t-s} \right). \]

Thus, by this inequality, it follows that

\[ q_1^{(b+\eta)}(s, x, t, y) \leq \frac{P \left( R^x(t - s) \in dy \right)}{dy} \leq 2\pi y^{1+\nu} \left( \frac{y}{t-s} \right)^{\nu} n_{t-s}(x) n_{t-s}(y) \frac{C_\nu}{(1 + \frac{xy}{t-s})^{\nu+\frac{1}{2}}} \exp \left( \frac{xy}{t-s} \right) \leq \frac{\widehat{C}_{\nu,b}}{(t-s)^{\nu+1}} n_{t-s}(y - x), \]

where $\widehat{C}_{\nu,b} := \sqrt{2\pi C_\nu(b + 1)^{2\nu+1}} \sqrt{1 + (b + 1)^2}$. Next, we prove inequality (2). According to (4), there exists some $\widehat{C}_\nu > 0$ such that

\[ 2(xy)^{-\nu} \sum_{n=1}^{\infty} \frac{J_\nu(j_\nu,n,x)J_\nu(j_\nu,n,y)}{J_{\nu+1}^2(j_\nu,n)} \exp \left( -j_{\nu,n}^2 t \right) \leq \frac{\widehat{C}_\nu}{(t + xy)^{\nu+1/2}} \left( 1 + \frac{(1-x)(1-y)}{t} \right) \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{4t} - j_{\nu,1}^2 t \right), \quad x, y \in (0, 1), t > 0. \]
Using this inequality and Theorem 6, we can obtain the following estimation:

\[ q_{1(b + \eta)}(r, z, 1, b) = \frac{P(R^2(1 - r) \in [db]) P(M_{r,1}(r_{j,1}^{\omega}) \leq b + \eta)}{db} \]

\[ = \frac{b^{2\nu+1}}{(b + \eta)^{2\nu+2}} \left( \frac{z}{b + \eta} \right)^{-\nu} \sum_{n=1}^{\infty} J_{\nu}(z J_{\nu,n} / (b + \eta)) J_{\nu}(b J_{\nu,n} / (b + \eta)) \exp \left( -j_{\nu,n} \frac{1 - r}{2(b + \eta)^2} \right) \]

\[ \leq \tilde{C}_{\nu,b} \frac{b^{2\nu+1}}{(b + \eta)^{2\nu+2}} \left( 1 + \frac{1 - r}{2(b + \eta)^2} \right)^{\nu+2} \left( \frac{(1-r)^{1+2b^2 \nu}}{2(b + \eta)^2} \right)^{\nu+1} \left( 1 + 2\eta(b + \eta - z) \right) \]

\[ \times \exp \left( -j_{\nu,1} \frac{1 - r}{2(b + \eta)^2} \right) 2\sqrt{\pi}(b + \eta) n_{1-r}(z - b) \]

\[ \leq \tilde{C}_{\nu,b} \frac{2(b + 1) + 1}{2(b + \eta)^2} \left( \frac{(1+2b^2 \nu)^{\nu+5/2}}{b^4} \right)^{\nu+1} \]

\[ \times \exp \left( -j_{\nu,1} \frac{1 - r}{2(b + \eta)^2} \right) 2\sqrt{\pi}(b + \eta) n_{1-r}(z - b), \]

where \( \tilde{C}_{\nu,b} := \tilde{C}_{\nu} \sqrt{\frac{2}{b^4} (1+2b^2 \nu)^{\nu+5/2}}. \) Since we set \( C_{\nu,b} := \tilde{C}_{\nu,b} \lor \tilde{C}_{\nu,b}, \) we can obtain our assertions.

Let \( b > 0. \) For \( 0 \leq s < t \leq 1 \) and \( x, y \in [0, b], \) we define

\[ h_b(s, x, t, y) := \frac{q^{(b)}_1(s, x, t, y) q^{(b)}_2(1 - t, y)}{q^{(b)}_2(1 - s, x)}. \]  (16)

**Proposition 3.1.** Let \( b > 0. \) For \( 0 \leq s < t \leq 1 \) and \( x \in [0, b], \) we have

\[ \int_0^b h_b(s, x, t, y) dy = 1. \]

Proof. By (16), it suffices to show the following identity:

\[ q^{(b)}_2(1 - s, x) = \int_0^b q^{(b)}_1(s, x, t, y) q^{(b)}_2(1 - t, y) dy. \]

Here, using Lemma 4.1, it holds that

\[ \frac{q^{(b + \eta)}_1(s, x, 1, b)}{\eta} = \int_0^{b+\eta} q^{(b + \eta)}_1(s, x, t, y) q^{(b + \eta)}_1(t, y, 1, b) dy. \]  (17)

According to L'Hôpital's rule, we obtain

\[ \lim_{\eta \downarrow 0} \frac{q^{(b + \eta)}_1(s, x, 1, b)}{\eta} = q^{(b)}_2(1 - s, x). \]  (18)
On the other hand, by Lemma 3.3, for \( \eta \in (0, 1) \) and \( y \in (0, b + \eta) \), we have the following estimation:
\[
q_1^{(b+\eta)}(s, x, t, y) \frac{q_1^{(b+\eta)}(t, y, 1, b)}{\eta} \leq \frac{1}{\eta (t-s)^\nu+1} n_{t-s}(y-x) \frac{C_{v,b}}{(1-t)^\nu+1} \left( 1 \wedge \frac{2\eta(b+\eta)}{1-t} \right) n_{1-t}(y-b) \\
\leq \frac{C_{v,b}^2(b+1)}{\pi(t-s)^\nu+3/2(1-t)^{\nu+5/2}} < \infty.
\]
(19)

Again, using L’Hôpital’s rule, it holds that
\[
\lim_{\eta \downarrow 0} q_1^{(b+\eta)}(s, x, t, y) \frac{q_1^{(b+\eta)}(t, y, 1, b)}{\eta} = q_1^{(b)}(s, x, t, y)q_2^{(b)}(1-t, y)
\]
for \( y \in (0, b) \). Therefore, by (19), (20), and Lebesgue’s dominated convergence theorem, we obtain
\[
\lim_{\eta \downarrow 0} \frac{1}{\eta} \int_0^{b+\eta} q_1^{(b+\eta)}(s, x, t, y)q_1^{(b+\eta)}(t, y, 1, b) dy = \int_0^{b} q_1^{(b)}(s, x, t, y)q_2^{(b)}(1-t, y) dy.
\]

By this equality and (18), taking the limit \( \eta \downarrow 0 \) in (17) allows us to prove the assertion.

The following proposition implies that \( h_b(s, x, t, y) \) satisfies the Chapman–Kolmogorov identity.

**Proposition 3.2.** Let \( b > 0 \). For \( 0 < s < t < u < 1 \) and \( x, z \in (0, b) \), we have
\[
h_b(s, x, u, z) = \int_0^b h_b(s, x, t, y) h_b(t, y, u, z) dy.
\]

Proof. By the definition of \( h_b(s, x, t, y) \) (16), we have
\[
q_1^{(b)}(s, x, u, z)q_2^{(b)}(1-u, z) = \int_0^b q_1^{(b)}(s, x, t, y)q_2^{(b)}(1-t, y)q_1^{(b)}(t, y, u, z)q_2^{(b)}(1-u, z) dy.
\]
Thus, it suffices to show that
\[
q_1^{(b)}(s, x, u, z) = \int_0^b q_1^{(b)}(s, x, t, y)q_2^{(b)}(t, y, u, z) dy.
\]

According to Lemma 2.1, we can prove the assertion as follows:
\[
q_1^{(b)}(s, x, u, z) = P \left( \frac{R^x(u-s)}{dz} \right) P \left( M_{[s,u]}(r_{[s,u]}^{\rightarrow z}) \leq b \right) = \int_0^b P (r_{[s,u]}^{\rightarrow z}(t) \in dy, M_{[s,u]}(r_{[s,u]}^{\rightarrow z}) \leq b) dz \\
= \int_0^b P (M_{[s,t]}(r_{[s,t]}^{\rightarrow y}) \leq b) P \left( M_{[t,u]}(r_{[t,u]}^{\rightarrow z}) \leq b \right) P \left( r_{[t,u]}^{\rightarrow z}(t) \in dy \right) dz \\
= \int_0^b q_1^{(b)}(s, x, t, y)q_1^{(b)}(t, y, u, z) dy.
\]

By Proposition 3.1 and Proposition 3.2, \( h_b(0, a, t, y) dy \) and \( h_b(s, x, t, y) dy \) determine the continuous Markov process \( H^{a \rightarrow b} = \{ H^{a \rightarrow b}(t) \}_{t \in [0, 1]} \). Therefore, the proof of Theorem 1 is completed.
4 Proof of Theorem 2

In this section, we prove Theorem 2 which gives the construction of the Bessel house-moving as the weak limit of the conditioned $\delta$-dimensional Bessel bridge.

Lemma 4.1. Let $0 \leq a < b$ and $\eta > 0$. For $0 < s < t < 1$ and $x, y \in (0, b + \eta)$, we have

$$P \left( r^{a-b}|_{K^{-}(b+\eta)}(t) \in dy \right) = \frac{q_1^{(b+\eta)}(0, a, t, y)q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)}dy,$$

(21)

and

$$P \left( r^{a-b}|_{K^{-}(b+\eta)}(t) \in dy \mid r^{a-b}|_{K^{-}(b+\eta)}(s) = x \right) = \frac{q_1^{(b+\eta)}(s, x, t, y)q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(s, x, 1, b)}dy.$$

(22)

Proof. By Lemma 2.1 we obtain

$$P \left( r^{a-b}|_{K^{-}(b+\eta)}(t) \in dy \right)$$

$$= \frac{P \left( r^{a-b}(t) \in dy, M(r^{a-b}) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} = \frac{P \left( M_{\{0, t\}}(r^{a-b}_y) \leq b + \eta \right) P \left( M_{\{t, 1\}}(r^{y-b}_u) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} \times P \left( r^{a-b}(t) \in dy \right)$$

$$= \frac{P \left( M_{\{0, t\}}(r^{a-b}_y) \leq b + \eta \right) P \left( M_{\{t, 1\}}(r^{y-b}_u) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} \times \frac{P \left( R^{a}(t) \in dy \right) P \left( R^{y}(1-t) \in db \right)}{P \left( R^{a}(1) \in db \right)}$$

$$= \frac{q_1^{(b+\eta)}(0, a, t, y)q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)}dy.$$

Hence, (21) holds.

Next, we prove (22). By Lemma 2.1 we have

$$P \left( r^{a-b}|_{K^{-}(b+\eta)}(t) \in dy, r^{a-b}|_{K^{-}(b+\eta)}(s) \in dx \right)$$

$$= \frac{P \left( r^{a-b}(t) \in dy, r^{a-b}(s) \in dx, M(r^{a-b}) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} = \frac{P \left( M_{\{0, t\}}(r^{a-b}_y) \leq b + \eta \right) P \left( M_{\{t, 1\}}(r^{y-b}_u) \leq b + \eta \right) P \left( M_{\{0, s\}}(r^{a-b}_x) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} \times P \left( r^{a-b}(t) \in dy, r^{a-b}(s) \in dx \right)$$

(23)

and

$$P \left( r^{a-b}|_{K^{-}(b+\eta)}(s) \in dx \right)$$

$$= \frac{P \left( r^{a-b}(s) \in dx, M(r^{a-b}) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} = \frac{P \left( M_{\{0, s\}}(r^{a-b}_x) \leq b + \eta \right) P \left( M_{\{s, 1\}}(r^{x-b}_y) \leq b + \eta \right)}{P \left( M(r^{a-b}) \leq b + \eta \right)} P \left( r^{a-b}(s) \in dx \right).$$

(24)
Therefore, combining (23) and (24), we obtain

\[
\begin{align*}
P \left( r^{a\to b}|_{K-(b+\eta)}(t) \in dy \mid r^{a\to b}|_{K-(b+\eta)}(s) = x \right) & = \frac{P \left( r^{a\to b}|_{K-(b+\eta)}(t) \in dy, r^{a\to b}|_{K-(b+\eta)}(s) \in dx \right)}{P \left( r^{a\to b}|_{K-(b+\eta)}(s) \in dx \right)} \\
& = \frac{P \left( M_{s,t}(r_{s,t}^{x-y}) \leq b + \eta \right) P \left( M_{t,1}(r_{t,1}^{y-b}) \leq b + \eta \right)}{P \left( M_{s,t}(r_{s,t}^{x-y}) \leq b + \eta \right)} \times \frac{P \left( r^{a\to b}(t) \in dy \mid r^{a\to b}(s) = x \right)}{P \left( r^{a\to b}(s) \in dy \right)} \\
& = \frac{P \left( M_{s,t}(r_{s,t}^{x-y}) \leq b + \eta \right) P \left( M_{t,1}(r_{t,1}^{y-b}) \leq b + \eta \right)}{P \left( M_{s,t}(r_{s,t}^{x-y}) \leq b + \eta \right)} \times \frac{P \left( R^x(t-s) \in dy \right) P \left( R^y(1-t) \in db \right)}{P \left( R^x(1-s) \in db \right)} \\
& = \frac{q_1^{(b+\eta)}(s, x, t, y)q_2^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(s, x, 1, b)} dy.
\end{align*}
\]

Hence, (22) holds. \(\square\)

**Proposition 4.1.** Let \(0 \leq a < b\). For \(0 < s < t < 1\) and \(x, y \in (0, b)\), we have

\[
\begin{align*}
\lim_{\eta \downarrow 0} P \left( r^{a\to b}|_{K-(b+\eta)}(t) \in dy \right) &= \frac{q_1^{(b)}(0, a, t, y)q_2^{(b)}(1-t, y)}{q_2^{(b)}(1, a)} dy, & (25) \\
\lim_{\eta \downarrow 0} P \left( r^{a\to b}|_{K-(b+\eta)}(t) \in dy \mid r^{a\to b}|_{K-(b+\eta)}(s) = x \right) &= \frac{q_1^{(b)}(s, x, t, y)q_2^{(b)}(1-t, y)}{q_2^{(b)}(1-s, x)} dy. & (26)
\end{align*}
\]

**Proof.** By Lemma 4.1 and L'Hôpital’s rule, we obtain our assertion. \(\square\)

In Section 3, we proved that the right sides of (25) and (26) are the transition densities of the continuous Markov process \(H^{a\to b} = \{H^{a\to b}(t)\}_{t \in [0,1]}\). Then, by Proposition 4.1 and Lemma A.2, we obtain the convergence \(r^{a\to b}|_{K-(b+\eta)} \to H^{a\to b}\) as \(\eta \downarrow 0\) in the finite-dimensional distributional sense. Therefore, all that remains in proving Theorem 2 is the tightness of the family \(\{r^{a\to b}|_{K-(b+\eta)}\}_{0 < \eta < \eta_0}\) for some \(\eta_0 > 0\). By

\[
\lim_{\eta \downarrow 0} q_2^{(b+\eta)}(1, a) = q_2^{(b)}(1, a),
\]
we can take \(\eta_1 > 0\) so that \(q_2^{(b+\eta)}(1, a) > q_2^{(b)}(1, a)/2\) holds for \(\eta \in (0, \eta_1)\). Throughout this section, we fix \(\eta_1\) in this fashion and denote

\[
\eta_0 := \min\{\eta_1, 1\}. \quad (27)
\]

**Lemma 4.2.** Let \(0 \leq a < b\) and let \(0 < \eta < \eta_0\) be fixed. We have

\[
q_1^{(b+\eta)}(0, a, 1, b) > \eta \frac{q_2^{(b)}(1, a)}{2}.
\]
Proof. According to Taylor’s theorem, we can find \( \theta \in (0, 1) \) so that

\[
q_1^{(b+\theta)}(0, a, 1, b) = \eta q_2^{(b+\theta)}(1, a) > \eta \frac{q_2^{(b)}(1, a)}{2}.
\]

\[\square\]

Using Lemmas 3.3 and 4.2 we obtain the following moment inequalities:

**Lemma 4.3.** Let \( 0 \leq a < b \). For each \( \alpha > 0 \), we can find a constant \( C_{\alpha,\nu,a,b} > 0 \) so that

\[
\begin{align*}
(1) \quad & \sup_{0 < \eta < \eta_0} E \left[ \left| r^{a-b} \eta |K^{(b+\eta)}_-(r) - r^{a-b} |K^{(b+\eta)}_-(0)| \right|^{2\alpha} \right] \leq \frac{C_{\alpha,\nu,a,b}}{r^{\nu+1-a}(1-r)^{\nu+\frac{\alpha}{2}}}, \quad r \in (0, 1), \\
(2) \quad & \sup_{0 < \eta < \eta_0} E \left[ \left| r^{a-b} \eta \left( |K^{(b+\eta)}_-(1-r) - r^{a-b} |K^{(b+\eta)}_-(1)| \right|^{2\alpha} \right] \leq \frac{C_{\alpha,\nu,a,b}}{r^{\nu+2-a}(1-r)^{\nu+\frac{3}{2}}}, \quad r \in (0, 1), \\
(3) \quad & \sup_{0 < \eta < \eta_0} E \left[ \left| r^{a-b} \eta \left( |K^{(b+\eta)}_-(t) - r^{a-b} |K^{(b+\eta)}_-(s)| \right|^{2\alpha} \right] \leq \frac{C_{\alpha,\nu,a,b}}{(t-s)^{\nu+1-a}s^{\nu+\frac{3}{2}}(1-t)^{\nu+\frac{3}{4}}}, \quad s, t \in (0, 1).
\end{align*}
\]

Proof. By Lemmas 3.3 and 4.2 we have

\[
P \left( r^{a-b} \eta |K^{(b+\eta)}_-(u) \in dz \right) = \frac{q_1^{(b+\eta)}(0, a, u, z)q_1^{(b+\eta)}(u, z, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} dz \leq \frac{2}{\eta q_2^{(b)}(1, a)} \left( \frac{C_{\nu,b}}{u^{\nu+1}} n_u(z-a) \right) \left( \frac{C_{\nu,b}}{1-u} \left( 1 + \frac{2\eta(b+\eta)}{1-u} \right) n_{1-u}(z-b) \right) dz \leq \frac{4(b+\eta)C_{\nu,b}^2}{q_2^{(b)}(1, a)} \frac{1}{u^{\nu+1}(1-u)^{\nu+2}} n_u(z-a)n_{1-u}(z-b) dz
\]

for \( 0 < u < 1 \). On the other hand, for each \( c \in \mathbb{R} \),

\[
\int_0^{b+\eta} |z-c|^{2\alpha} n_r(z-c)dz \leq 2 \int_0^\infty w^{2\alpha} n_r(w)dw = \frac{(2\pi)^\alpha}{\sqrt{\pi}} \Gamma \left( \alpha + \frac{1}{2} \right)
\]

holds. Hence, because we have

\[
\begin{align*}
E \left[ \left| r^{a-b} \eta |K^{(b+\eta)}_-(r) - r^{a-b} |K^{(b+\eta)}_-(0)| \right|^{2\alpha} \right] & \leq \frac{4(b+\eta)C_{\nu,b}^2}{\sqrt{2\pi q_2^{(b)}(1, a)}} \frac{1}{r^{\nu+1}(1-r)^{\nu+5/2}} \int_0^{b+\eta} |z-a|^{2\alpha} n_r(z-a)dz \\
& \leq \frac{2^{\alpha+2}(b+\eta)C_{\nu,b}^2 \Gamma \left( \alpha + \frac{1}{2} \right)}{\sqrt{2\pi q_2^{(b)}(1, a)}} \frac{1}{r^{\nu+1-a}(1-r)^{\nu+5/2}}.
\end{align*}
\]
and

\[
E \left[ |r^{a-b}|_{K^{-\eta}}(1 - r) - r^{a-b}|_{K^{-\eta}}(1) \right]^{2a} \leq \frac{4(b + \eta)C_{\nu,b}^2}{\sqrt{2\pi q_2^{(b)}(1, a)}} \frac{1}{(1 - r)^{\nu + 3/2} r^{\nu + 2}} \int_0^{b+\eta} |z - b|^{2a} n_r(z - b) dz
\]

we obtain inequalities (1) and (2) as follows:

\[
\sup_{0 < \eta < \eta_0} \int E \left[ |r^{a-b}|_{K^{-\eta}}(r) - r^{a-b}|_{K^{-\eta}}(0) \right]^{2a} \leq \frac{2^{a+2}(b + 1)C_{\nu,b}^2 \Gamma(\alpha + \frac{1}{2})}{\sqrt{2\pi q_2^{(b)}(1, a)}} \frac{1}{r^{\nu+1-a}(1 - r)^{\nu+5/2}}
\]

\[
\sup_{0 < \eta < \eta_0} \int E \left[ |r^{a-b}|_{K^{-\eta}}(1 - r) - r^{a-b}|_{K^{-\eta}}(1) \right]^{2a} \leq \frac{2^{a+2}(b + 1)C_{\nu,b}^2 \Gamma(\alpha + \frac{1}{2})}{\sqrt{2\pi q_2^{(b)}(1, a)}} \frac{1}{r^{\nu+2-a}(1 - r)^{\nu+5/2}}.
\]

Next, we prove (3). We note that

\[
P \left( r^{a-b}|_{K^{-\eta}}(t) \in dy, \ r^{a-b}|_{K^{-\eta}}(s) \in dx \right) = P \left( r^{a-b}|_{K^{-\eta}}(t) \in dy \mid r^{a-b}|_{K^{-\eta}}(s) = x \right) P \left( r^{a-b}|_{K^{-\eta}}(s) \in dx \right)
\]

\[
= \frac{q_1^{(b+\eta)}(0, a, s, x) q_1^{(b+\eta)}(t, y, 1, b)}{q_1^{(b+\eta)}(0, a, 1, b)} q_1^{(b+\eta)}(s, x, t, y) dxdy.
\]

By Lemmas 3.3 and 4.2, we have

\[
P \left( r^{a-b}|_{K^{-\eta}}(t) \in dy, \ r^{a-b}|_{K^{-\eta}}(s) \in dx \right) \leq \frac{2}{\eta q_2^{(b)}(1, a)} \cdot \frac{C_{\nu,b}}{s^{\nu+1}} n_s(x - a) \cdot \frac{C_{\nu,b}}{(1 - t)^{\nu+1}} n_{1-t}(y - b) \cdot \frac{C_{\nu,b}}{(t - s)^{\nu+1}} n_{t-s}(y - x) dxdy
\]

\[
\leq \frac{2(b + \eta)C_{\nu,b}^2}{\pi q_2^{(b)}(1, a)} \cdot \frac{1}{(t - s)^{\nu+1}s^{\nu+3/2}(1 - t)^{\nu+5/2}} \cdot n_{t-s}(y - x) dxdy.
\]

On the other hand,

\[
\int_0^{b+\eta} |y - x|^{2a} n_{t-s}(y - x) dy = \int_0^{b+\eta} \left( \int_0^{b+\eta} |y - x|^{2a} n_{t-s}(y - x) dy \right) dx
\]

\[
\leq (b + \eta) \frac{(2(t - s))^\alpha}{\sqrt{\pi}} \Gamma(\alpha + \frac{1}{2})
\]
holds. Hence, we obtain

\[
E \left[ |r^{a \to b}|_{\mathcal{K}-(b+\eta)}(t) - r^{a \to b}|_{\mathcal{K}-(b+\eta)}(s) |^{2\alpha} \right]
\]

\[
= \int_{(0,b+\eta)^2} |y-x|^{2\alpha} P \left( r^{a \to b}|_{\mathcal{K}-(b+\eta)}(t) \in dy, \ r^{a \to b}|_{\mathcal{K}-(b+\eta)}(s) \in dx \right)
\]

\[
\leq \frac{2(b+\eta)^2 C_{\nu,b} \Gamma(\alpha + \frac{1}{2})}{\pi \sqrt{\pi} q_2(b)(1,a)} \frac{1}{(t-s)^{\nu+1}(1-t)^{\nu+5/2}} \int_{(0,b+\eta)^2} |y-x|^{2\alpha} n_{t-s}(y-x) dxdy
\]

\[
\leq \frac{2^{\alpha+1}(b+\eta)^2 C_{\nu,b} \Gamma(\alpha + \frac{1}{2})}{\pi \sqrt{\pi} q_2(b)(1,a)} \frac{1}{(t-s)^{\nu+1}(1-t)^{\nu+5/2}},
\]

Therefore, we have

\[
\sup_{0<\eta<\eta_0} E \left[ |r^{a \to b}|_{\mathcal{K}-(b+\eta)}(t) - r^{a \to b}|_{\mathcal{K}-(b+\eta)}(s) |^{2\alpha} \right]
\]

\[
\leq \frac{2^{\alpha+1}(b+1)^2 C_{\nu,b} \Gamma(\alpha + \frac{1}{2})}{\pi \sqrt{\pi} q_2(b)(1,a)} \frac{1}{(t-s)^{\nu+1}(1-t)^{\nu+5/2}},
\]

and inequality (3) is proved.

**Corollary 1.** Let 0 ≤ a < b. For each u ∈ (0, \frac{1}{2}), the family \( \{\pi_{u,1-u} \circ r^{a \to b}|_{\mathcal{K}-(b+\eta)}\}_\eta \in (0,\eta_0) \) is tight.

**Proof.** Using Lemma 4.3 (1) and (3) for α = \frac{1}{2} and α = ν + 3, respectively, we obtain

\[
\sup_{0<\eta<\eta_0} E \left[ |r^{a \to b}|_{\mathcal{K}-(b+\eta)}(u) \right]
\]

\[
\leq \sup_{0<\eta<\eta_0} \left( E \left[ |r^{a \to b}|_{\mathcal{K}-(b+\eta)}(u) - r^{a \to b}|_{\mathcal{K}-(b+\eta)}(0) \right] + E \left[ |r^{a \to b}|_{\mathcal{K}-(b+\eta)}(0) \right] \right)
\]

\[
\leq C_{1/2,\nu,a,b} (1-u)^{-\nu+\frac{3}{2}} u^{-\nu+\frac{1}{2}} + a < \infty
\]

and

\[
\sup_{0<\eta<\eta_0} E \left[ \left| \frac{r^{a \to b}|_{\mathcal{K}-(b+\eta)}(t) - r^{a \to b}|_{\mathcal{K}-(b+\eta)}(s)}{t-s} \right|^{2(\nu+3)} \right] \leq C_{\nu+3,\nu,a,b} t^{-\nu+\frac{3}{2}} (1-t)^{-\nu+\frac{5}{2}} (t-s)^2
\]

\[
\leq C_{\nu+3,\nu,a,b} u^{-2\nu-4}(t-s)^2
\]

for u ≤ s < t ≤ 1 − u. Hence, by Lemma A.3, we establish the assertion.

**Proposition 4.2.** Let 0 ≤ a < b. For ξ > 0, we have

\[
\lim_{u \to 0} \sup_{\eta \in (0,\eta_0)} E \left( \sup_{0 \leq t \leq u} \left| r^{a \to b}|_{\mathcal{K}-(b+\eta)}(t) - r^{a \to b}|_{\mathcal{K}-(b+\eta)}(0) \right| > \xi \right) = 0,
\]

\[
\lim_{u \to 0} \sup_{\eta \in (0,\eta_0)} E \left( \sup_{1-u \leq t \leq 1} \left| r^{a \to b}|_{\mathcal{K}-(b+\eta)}(t) - r^{a \to b}|_{\mathcal{K}-(b+\eta)}(1) \right| > \xi \right) = 0.
\]
Proof. Applying Lemma 4.3 (1)–(3) for $\alpha = 3\nu + 7$ and $t, s, r \in (0, 1)$ with $s < t$, we have

\[ \sup_{0 < \eta < \eta_0} E \left[ |r^{a-b}|_{K^-(b+\eta)}(r) - r^{a-b}|_{K^-(b+\eta)}(0) |^{2(3\nu+7)} \right] \leq C_{3\nu+7, \nu, a, b} \frac{r^{2\nu+6}}{(1 - r)^{\nu + \frac{3}{2}}}, \quad (28) \]

\[ \sup_{0 < \eta < \eta_0} E \left[ |r^{a-b}|_{K^-(b+\eta)}(t) - r^{a-b}|_{K^-(b+\eta)}(s) |^{2(3\nu+7)} \right] \leq C_{3\nu+7, \nu, a, b} \frac{|t - s|^{2\nu+6}}{s^{\nu + \frac{3}{2}} (1 - t)^{\nu + \frac{3}{2}}}, \quad (29) \]

\[ \sup_{0 < \eta < \eta_0} E \left[ |r^{a-b}|_{K^-(b+\eta)}(1-r) - r^{a-b}|_{K^-(b+\eta)}(1) |^{2(3\nu+7)} \right] \leq C_{3\nu+7, \nu, a, b} \frac{r^{2\nu+5}}{(1 - r)^{\nu + \frac{3}{2}}}. \quad (30) \]

Let $\gamma = \frac{1}{4\alpha} = \frac{1}{4(3\nu+7)}$, $0 < \eta < \eta_0$, and fix $n \in \mathbb{N}$. We define

\[ F_n^\eta = \left\{ \max_{1 \leq k \leq 2^{n-1}} |r^{a-b}|_{K^-(b+\eta)} \left( \frac{k - 1}{2^n} \right) - r^{a-b}|_{K^-(b+\eta)} \left( \frac{k}{2^n} \right) \geq 2^{-n\gamma} \right\}, \]

\[ \tilde{F}_n^\eta = \left\{ \max_{2^{n-1} \leq k \leq 2^n} |r^{a-b}|_{K^-(b+\eta)} \left( \frac{k - 1}{2^n} \right) - r^{a-b}|_{K^-(b+\eta)} \left( \frac{k}{2^n} \right) \geq 2^{-n\gamma} \right\}, \]

\[ a(n, k, \eta) = P \left( |r^{a-b}|_{K^-(b+\eta)} \left( \frac{k - 1}{2^n} \right) - r^{a-b}|_{K^-(b+\eta)} \left( \frac{k}{2^n} \right) \geq 2^{-n\gamma} \right), \quad 1 \leq k \leq 2^n. \]

Then, by Chebyshev’s inequality, we have

\[ a(n, k, \eta) \leq 2^{\frac{n}{2}} E \left[ \left| r^{a-b}|_{K^-(b+\eta)} \left( \frac{k - 1}{2^n} \right) - r^{a-b}|_{K^-(b+\eta)} \left( \frac{k}{2^n} \right) \right|^{2(3\nu+7)} \right], \quad 1 \leq k \leq 2^n. \quad (31) \]

Therefore, using \( (28), (29), (30), \) and \( (31), \) we have

\[ a(n, 1, \eta) \leq 2^{\frac{n}{2}} C_{3\nu+7, \nu, a, b} \left( \frac{2^n}{2^n - 1} \right)^{\nu - \frac{3}{2}} \left( \frac{1}{2^n} \right)^{\alpha - \nu - 1} \leq C_{3\nu+7, \nu, a, b} 2^{-n(\nu + 3)} \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{3}{2} n}, \]

\[ a(n, k, \eta) \leq 2^{\frac{n}{2}} C_{3\nu+7, \nu, a, b} \left( \frac{2^n}{k - 1} \right)^{\nu - \frac{3}{2}} \left( \frac{2^n}{k} \right)^{\frac{\nu}{2}} \left( \frac{1}{2^n} \right)^{2\nu+6} \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{3}{2} n}, \quad (2 \leq k \leq 2^n - 1), \]

\[ a(n, 2^n, \eta) \leq 2^{\frac{n}{2}} C_{3\nu+7, \nu, a, b} \left( \frac{2^n}{2^n - 1} \right)^{\nu - \frac{3}{2}} \left( \frac{1}{2^n} \right)^{2\nu+5} \leq C_{3\nu+7, \nu, a, b} 2^{-n(\nu + 3)} \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{3}{2} n}. \]

Thus, it follows that

\[ P \left( F_n^\eta \right) \leq \sum_{k=1}^{2^{n-1}} a(n, k, \eta) \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{n}{2}}, \quad P \left( \tilde{F}_n^\eta \right) \leq \sum_{k=2^{n-1}}^{2^n} a(n, k, \eta) \leq C_{3\nu+7, \nu, a, b} 2^{-\frac{n}{2}}. \]

Therefore, Lemmas A.4 and A.5 prove the desired results. \( \square \)

By Corollary 1 and Proposition 4.2, we can apply Theorem 8 to \( \{ r^{a-b}|_{K^-(b+\eta)} \}_{0 < \eta < \eta_0} \) and obtain the tightness of this family.

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5 Proof of Proposition 1.1

The proof is similar to that in Chapter 2, Theorem 2.8 in [3]. We fix \( \gamma \in (0, \frac{1}{2}) \). Then, we can find \( m_0 \in \mathbb{N} \) so that \( \gamma < \frac{m_0-3\nu-6}{2m_0} \) holds. For this \( m_0 \), combining Theorem 2, Skorohod’s theorem, Fatou’s lemma, and Lemma 4.3, we can take a positive number \( C_{m_0,\nu,a,b} \) that satisfies

\[
E \left[ \left| H^{a \to b}(r) - a \right|^{2m_0} \right] \leq \frac{C_{m_0,\nu,a,b}}{(r^{\nu+1}-m_0(1-r)^{\nu+\frac{5}{2}}),}
\]

\[
E \left[ \left| H^{a \to b}(1-r) - b \right|^{2m_0} \right] \leq \frac{C_{m_0,\nu,a,b}}{(r^{\nu+2}-m_0(1-r)^{\nu+\frac{5}{2}}),}
\]

\[
E \left[ \left| H^{a \to b}(t) - H^{a \to b}(s) \right|^{2m_0} \right] \leq \frac{C_{m_0,\nu,a,b}}{(t-s)^{\nu+1}-m_0^{\nu+\frac{5}{2}}(1-t)^{\nu+\frac{5}{2}}}
\]

for \( t, s, r \in (0, 1) \) with \( s < t \). Now, for \( n \in \mathbb{N} \) we define

\[
F_n = \left\{ \max_{1 \leq k \leq 2^n} \left| H^{a \to b} \left( \frac{k-1}{2^n} \right) - H^{a \to b} \left( \frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right\},
\]

\[
a(n, k) = P \left( \left| H^{a \to b} \left( \frac{k-1}{2^n} \right) - H^{a \to b} \left( \frac{k}{2^n} \right) \right| \geq 2^{-n\gamma} \right), \quad 1 \leq k \leq 2^n.
\]

Then, Chebyshev’s inequality yields

\[
a(n, 1) \leq 2^{2nm_0\gamma} E \left[ \left| H^{a \to b}(1/2^n) - a \right|^{2m_0} \right] \leq C_{m_0,\nu,a,b} 2^{-n(m_0-2m_0\gamma-2\nu-\frac{7}{2})},
\]

\[
a(n, 2^n) \leq 2^{2nm_0\gamma} E \left[ \left| H^{a \to b}(1-1/2^n) - b \right|^{2m_0} \right] \leq C_{m_0,\nu,a,b} 2^{-n(m_0-2m_0\gamma-2\nu-\frac{7}{2})},
\]

and, for \( 2 \leq k \leq 2^n - 1, \)

\[
a(n, k) \leq 2^{2nm_0\gamma} E \left[ \left| H^{a \to b} \left( \frac{k-1}{2^n} \right) - H^{a \to b} \left( \frac{k}{2^n} \right) \right|^{2m_0} \right] \leq C_{m_0,\nu,a,b} 2^{-n(m_0-2m_0\gamma-3\nu-5)}.
\]

Therefore, \( P(F_n) \leq C_{m_0,\nu,a,b} \times 2^{-n(m_0-2m_0\gamma-3\nu-6)} \), and because \( m_0 - 2m_0\gamma - 3\nu - 6 > 0 \), we have \( P(\lim \inf_{n \to \infty} F^c_n) = 1 \) by the first Borel–Cantelli lemma. If \( \omega \in \lim \inf_{n \to \infty} F^c_n \), then there exists \( n^*(\omega) \in \mathbb{N} \) such that \( \omega \in \bigcap_{n \geq n^*(\omega)} F^c_n \). For \( n \geq n^*(\omega) \), we can deduce that

\[
\left| H^{a \to b}(t) - H^{a \to b}(s) \right| \leq 2 \sum_{j=n+1}^{\infty} 2^{-j\gamma} = \frac{2}{1-2^{-\gamma}} 2^{-(n+1)\gamma}, \quad 0 < t-s < 2^{-n}.
\]

Now, let \( t, s \in [0, 1] \) satisfy \( 0 < t-s < 2^{-n^*(\omega)} \) and choose \( n \geq n^*(\omega) \) so that \( 2^{-(n+1)} \leq t-s < 2^{-n} \). Then, the above inequality yields

\[
\left| H^{a \to b}(t) - H^{a \to b}(s) \right| \leq \frac{2}{1-2^{-\gamma}} |t-s|^\gamma.
\]

Hence, \( H^{a \to b} \) is locally Hölder-continuous with exponent \( \gamma \) for \( \omega \in \lim \inf_{n \to \infty} F^c_n \). \[\square\]
6 Decomposition formula and sample path properties

Let \( t \in (t_1, t_2) \). For \( w_1 \in C([t_1, t], \mathbb{R}) \) and \( w_2 \in C([t, t_2], \mathbb{R}) \) that satisfy \( w_1(t) = w_2(t) \), we define \( w_1 \oplus_t w_2 \in C([t_1, t_2], \mathbb{R}) \) by

\[
(w_1 \oplus_t w_2)(s) = \begin{cases} 
    w_1(s), & s \in [t_1, t], \\
    w_2(s), & s \in [t, t_2].
\end{cases}
\]

**Theorem 4.** Let \( 0 \leq a < b \). For every bounded continuous function \( F \) on \( C([0, 1], \mathbb{R}) \), it holds that

\[
E \left[ F(H^{a\rightarrow b}) \right] = \int_{0}^{b} E \left[ F \left( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} \oplus_t H_{[t,1]}^{y\rightarrow b} \right) \right] P \left( H^{a\rightarrow b}(t) \in dy \right), \quad 0 < t < 1,
\]

where \( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} \) and \( H_{[t,1]}^{y\rightarrow b} \) are chosen to be independent.

**Proof.** By Theorem 2 because \( r^{a\rightarrow b} |_{K^{(b+\eta)(b)}} \Rightarrow H^{a\rightarrow b}, \quad \eta \downarrow 0 \) holds,

\[
E \left[ F(H^{a\rightarrow b}) \right] = \lim_{\eta \downarrow 0} E \left[ F(r^{a\rightarrow b} |_{K^{(b+\eta)(b)}}) \right]
\]

for every bounded continuous function \( F \) on \( C([0, 1], \mathbb{R}) \). We calculate the numerator

\[
E \left[ F(r^{a\rightarrow b} |_{K^{(b+\eta)}}) \right] = \frac{E \left[ F(r^{a\rightarrow b} : r^{a\rightarrow b} \in K^{(b+\eta)}(b+\eta)) \right]}{P \left( r^{a\rightarrow b} \in K^{(b+\eta)} \right)}
\]

as

\[
E \left[ F(r^{a\rightarrow b} : r^{a\rightarrow b} \in K^{(b+\eta)}) \right] = \int_{0}^{\infty} E \left[ F(r^{a\rightarrow b} : r^{a\rightarrow b} \in K^{(b+\eta)}, r^{a\rightarrow b}(t) \in dy \right]
\]

\[
= \int_{0}^{\infty} E \left[ F \left( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} + t r_{[t,1]}^{y\rightarrow b} |_{K_{[t,1]}^{(b+\eta)(b)}} \right) \right] P \left( r^{a\rightarrow b} \in K^{(b+\eta)}, r^{a\rightarrow b}(t) \in dy \right)
\]

\[
= \int_{0}^{\infty} E \left[ F \left( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} + t r_{[t,1]}^{y\rightarrow b} |_{K_{[t,1]}^{(b+\eta)(b)}} \right) \right] P \left( r^{a\rightarrow b} |_{K^{(b+\eta)}}(t) \in dy \right) P \left( r^{a\rightarrow b} \in K^{(b+\eta)} \right).
\]

Hence, we have

\[
E \left[ F(r^{a\rightarrow b} |_{K^{(b+\eta)}}) \right] = \int_{0}^{\infty} E \left[ F \left( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} + t r_{[t,1]}^{y\rightarrow b} |_{K_{[t,1]}^{(b+\eta)(b)}} \right) \right] P \left( r^{a\rightarrow b} |_{K^{(b+\eta)}}(t) \in dy \right).
\]

Then, it suffices to show that

\[
\lim_{\eta \downarrow 0} \int_{0}^{\infty} E \left[ F \left( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} + t r_{[t,1]}^{y\rightarrow b} |_{K_{[t,1]}^{(b+\eta)(b)}} \right) \right] P \left( r^{a\rightarrow b} |_{K^{(b+\eta)}}(t) \in dy \right)
\]

\[
= \int_{0}^{\infty} E \left[ F \left( r_{[0,t]}^{a\rightarrow y} |_{K_{[0,t]}^{(b+\eta)(b)}} + t H_{[t,1]}^{y\rightarrow b} \right) \right] P \left( H^{a\rightarrow b}(t) \in dy \right). \tag{32}
\]
We obtain the following estimation:

\[
\left| \int_0^\infty E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b+\eta) \oplus t \mathcal{F}^{y-b}_{[t,1]} | K_{[0,t]}^- (b+\eta)) \right] \right| = E \left[ \left( r_{[0,t]}^{a-b} | K_{[0,t]}^- (b+\eta) (t) \in dy \right) \right] \\
- \int_0^\infty E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b) \oplus t \mathcal{F}^{y-b}_{[t,1]} ) \right] P \left( H_{[0,t]}^{a-b} (t) \in dy \right) \\
\leq \int_0^\infty \left| E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b+\eta) \oplus t \mathcal{F}^{y-b}_{[t,1]} | K_{[0,t]}^- (b+\eta)) - \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b) \oplus t \mathcal{F}^{y-b}_{[t,1]} ) \right] \right| \\
\times P \left( r_{[0,t]}^{a-b} | K_{[0,t]}^- (b+\eta) (t) \in dy \right) \\
+ \int_0^\infty \left| E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b) \oplus t \mathcal{F}^{y-b}_{[t,1]} ) \right] P \left( r_{[0,t]}^{a-b} | K_{[0,t]}^- (b+\eta) (t) \in dy \right) \right| \\
=: I^{(1)}_t (\eta) + I^{(2)}_t (\eta).
\]

Then, if \( I^{(1)}_t (\eta) \), \( I^{(2)}_t (\eta) \to 0 \) as \( \eta \downarrow 0 \), we can prove (32). First, consider

\[
I^{(1)}_t (\eta) = \int_0^\infty \left| E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b+\eta) \oplus t \mathcal{F}^{y-b}_{[t,1]} | K_{[0,t]}^- (b+\eta)) - \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b) \oplus t \mathcal{F}^{y-b}_{[t,1]} ) \right] \right| \\
\times \frac{q^{(b+\eta)}_1 (0, a, t, y) q^{(b+\eta)}_1 (t, y, 1, b)}{q^{(b+\eta)}_1 (0, a, 1, b)} 1_{[0,b+\eta]} (y) dy.
\]

We have

\[
\sup_{\eta > 0, y \in (0, b+\eta)} \left| E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b+\eta) \oplus t \mathcal{F}^{y-b}_{[t,1]} | K_{[0,t]}^- (b+\eta)) - \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b) \oplus t \mathcal{F}^{y-b}_{[t,1]} ) \right] \right| \\
\leq 2 \sup_{w \in C([0,1], \mathbb{R})} |F(w)| < \infty.
\]

By Theorem 2 and Lemma A.6, it holds that

\[
\lim_{\eta \downarrow 0} E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b+\eta) \oplus t \mathcal{F}^{y-b}_{[t,1]} | K_{[0,t]}^- (b+\eta)) \right] = E \left[ \mathcal{F}(r_{[0,t]}^{a,y} | K_{[0,t]}^- (b) \oplus t \mathcal{F}^{y-b}_{[t,1]} ) \right].
\]

In addition, by Lemmas 3.3 and 4.2, and for \( \eta \in (0, \eta_0) \) and \( y \in [0, b+\eta] \), we obtain

\[
\frac{q^{(b+\eta)}_1 (0, a, t, y) q^{(b+\eta)}_1 (t, y, 1, b)}{q^{(b+\eta)}_1 (0, a, 1, b)} \\
\leq \frac{2}{\eta q^{(b)}_2 (1, a)} \left( C_{\nu,b} t^{\nu-1} \right) (\frac{C_{\nu,b}}{(1-t)^{\nu+1}}) \left( 1 + 2 \eta (b+\eta) \right) n_{1-\nu} (y - b) \\
\leq \frac{2(\nu+5/2)}{\pi t^{\nu+3/2} (1-t)^{\nu+5/2}} q^{(b)}_2 (1, a),
\]

and by Proposition 4.1, it holds that

\[
\lim_{\eta \downarrow 0} \frac{q^{(b+\eta)}_1 (0, a, t, y) q^{(b+\eta)}_1 (t, y, 1, b)}{q^{(b+\eta)}_1 (0, a, 1, b)} = \frac{q^{(b)}_1 (0, a, t, y) q^{(b)}_2 (1-t, y)}{q^{(b)}_2 (1, a)}.
\]
Therefore, according to (33), (34), (35), (36), and Lebesgue’s dominated convergence theorem,

\[ I_t^{(1)}(\eta) \to 0, \quad \eta \downarrow 0. \]

Next, we consider

\[ I_t^{(2)}(\eta) = \left\{ \int_0^\infty E \left[ F(r_{[0,t]}^{a \to y}|_{K_{[0,t]}^{b}}(t) \oplus_t H_{[t,1]}^{y \to b}) \right] \left( P \left( r_{[0,t]}^{a \to y}|_{K_{[0,t]}^{b}}(t) \in dy \right) - P \left( H_{[t,1]}^{a \to b}(t) \in dy \right) \right) \right\}. \]

We have

\[ \sup_{y>0} \left| E \left[ F(r_{[0,t]}^{a \to y}|_{K_{[0,t]}^{b}}(t) \oplus_t H_{[t,1]}^{y \to b}) \right] \right| \leq \sup_{w \in C([0,1],\mathbb{R})} |F(w)| < \infty. \]

Then, by (35), (36), and Lebesgue’s dominated convergence theorem,

\[ \lim_{\eta \downarrow 0} \int_0^\infty E \left[ F(r_{[0,t]}^{a \to y}|_{K_{[0,t]}^{b}}(t) \oplus_t H_{[t,1]}^{y \to b}) \right] P \left( r_{[0,t]}^{a \to y}|_{K_{[0,t]}^{b}}(t) \in dy \right) = \int_0^\infty E \left[ F(r_{[0,t]}^{a \to y}|_{K_{[0,t]}^{b}}(t) \oplus_t H_{[t,1]}^{y \to b}) \right] P \left( H_{[t,1]}^{a \to b}(t) \in dy \right). \]

Therefore, it follows that

\[ I_t^{(2)}(\eta) \to 0, \quad \eta \downarrow 0. \]

Thus, we prove (32) and the proof is completed.

\[ \square \]

REMARK 6.1. Let \( A \) be a closed subset of \( C([0,1],\mathbb{R}) \), \( B \in \mathcal{B}(C([0,1],\mathbb{R})) \) be a measurable subset of \( C([0,1],\mathbb{R}) \), and let

\[ \phi(t) := 1 - \int_0^1 1_{(-\infty,t]}(u)du, \quad t \in \mathbb{R}. \]

Then, we have

\[ F_n(w) := \phi(nd_{\infty}(w, A)) \downarrow 1_A(w), \quad n \to \infty. \]

Therefore, Lebesgue’s dominated convergence theorem and Dynkin’s \( \pi-\lambda \) theorem imply that Theorem 4 holds true for \( F = 1_B \) and \( F = 1_B^c = 1 - 1_B \).

Lemma 6.1. Let \( 0 \leq a < b \). For \( 0 < z \leq x \leq b \) and \( t \in (0, 1) \),

\[ P \left( \max_{u \in [0,t]} H_{[t,1]}^{a \to b}(u) = x \right) = 0, \]

\[ P \left( \max_{u \in [0,t]} H_{[t,1]}^{a \to b}(u) \leq x, H_{[t,1]}^{a \to b}(t) \leq z \right) = \int_0^z q_1^{(x)}(0, a, t, y)q_2^{(b)}(1 - t, y)dy + \frac{q_2^{(b)}(1, a)}{q_2^{(b)}(1, a)}. \]
Proof. Let $A_i (i = 1, 2)$ be closed subsets of $C([0,1], \mathbb{R})$ given by

$$A_1 := \left\{ w \in C([0,1], \mathbb{R}) : \max_{u \in [0,t]} w(u) = x \right\}, \quad A_2 := \left\{ w \in C([0,1], \mathbb{R}) : \max_{u \in [0,t]} w(u) \leq x, \ w(t) \leq z \right\}.$$ 

Remark 6.1 implies that Theorem 4 can be applied for $F = 1_{A_i} (i = 1, 2)$. Thus, we obtain

$$P \left( M_t(H^{a \to b}) = x \right) = \int_0^x P \left( r_0^{a \to y} | K_{[0,t]}^{-}(b) \in \partial K_{[0,t]}^{-}(x) \right) P \left( H^{a \to b}(t) \in dy \right), \quad (37)$$

$$P \left( M_t(H^{a \to b}) \leq x, H^{a \to b}(t) \leq z \right) = \int_0^z P \left( r_0^{a \to y} | K_{[0,t]}^{-}(b) \in K_{[0,t]}^{-}(x), r_0^{a \to y} | K_{[0,t]}^{-}(b)(t) \leq z \right) P \left( H^{a \to b}(t) \in dy \right). \quad (38)$$

By Proposition A.1 and (37), we obtain

$$P \left( M_t(H^{a \to b}) = x \right) = \int_0^x \frac{P \left( r_0^{a \to y} \in \partial K_{[0,t]}^{-}(x) \right)}{P \left( r_0^{a \to y} \in K_{[0,t]}^{-}(b) \right)} P \left( H^{a \to b}(t) \in dy \right) = 0.$$ 

Furthermore, (38) implies that

$$P \left( M_t(H^{a \to b}) \leq x, H^{a \to b}(t) \leq z \right)$$

$$= \int_0^z \frac{P \left( r_0^{a \to y} \in K_{[0,t]}^{-}(x), r_0^{a \to y}(t) \leq z \right)}{P \left( r_0^{a \to y} \in K_{[0,t]}^{-}(b) \right)} \frac{q_1^{(b)}(0, a, t, y)q_2^{(b)}(1 - t, y)}{q_2^{(b)}(1, a)} dy$$

$$= \int_0^z \frac{q_1^{(x)}(0, a, t, y)q_2^{(b)}(1 - t, y)}{q_2^{(b)}(1, a)} dy.$$ 

\[ \square \]

REMARK 6.2. Let $t \in (0,1)$. Lemma 6.1 implies that

$$P \left( M_t(H^{a \to b}) = b \right) = 0,$$

$$P \left( M_t(H^{a \to b}) \leq b \right) = P \left( M_t(H^{a \to b}) \leq b, H^{a \to b}(t) \leq b \right) = \int_0^b P \left( H^{a \to b}(t) \in dy \right) = 1.$$

Therefore, $P(M_t(H^{a \to b}) < b) = 1$ holds and Proposition 1.2 is obtained. Proposition 1.2 implies that Bessel house-moving $H^{a \to b}$ does not hit $b$ on the time interval $[0,1)$. 

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7 Numerical examples

In this section, we demonstrate numerical examples of $H_0 \to b$ for $b = 1.5$ and $\delta = 2, 3, 6, 10$. The densities of $H_0 \to b(k/10)$ ($1 \leq k \leq 9$) are shown in Figs 1, 2, 3 and 4.

Figure 1: The densities of $\{H_0 \to b(k/10)\}_{k=1}^{9}$ for $b = 1.5$ and $\delta = 2$.

Figure 2: The densities of $\{H_0 \to b(k/10)\}_{k=1}^{9}$ for $b = 1.5$ and $\delta = 3$.

Figure 3: The densities of $\{H_0 \to b(k/10)\}_{k=1}^{9}$ for $b = 1.5$ and $\delta = 6$.

Figure 4: The densities of $\{H_0 \to b(k/10)\}_{k=1}^{9}$ for $b = 1.5$ and $\delta = 10$. 

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Further, the graphs of $\{E[H^{0\to b}(t)]\}_{t\in[0,1]}$ are shown in Fig 5.

Figure 5: The graphs of $\{E[H^{0\to b}(t)]\}_{t\in[0,1]}$ for $b = 1.5$ and $\delta = 2, 3, 6, 10$.

8 Future work

Let $\mathcal{R} = \{\mathcal{R}(t)\}_{t\geq 0}$ be a regular one-dimensional diffusion on $[0, \infty)$. For an $\mathcal{R}$-bridge $r^{0\to b} = \{r^{0\to b}(s)\}_{s\in[0,1]}$, $(b > 0)$ from 0 to $b$ on $[0,1]$, we are interested in finding the weak limit of $r^{0\to b}_{|K-(b+\eta)}$ as $\eta \downarrow 0$.

A Appendix

A.1 Distribution of the maximal value of the Bessel bridge

In this subsection, we prove the results on the distribution of the maximal value of the Bessel bridge used in this paper.

Lemma A.1. Let $X > 0$. There exist some $\widetilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$\frac{n\pi}{2} < j_{\nu,n} < 2n\pi, \quad \left| \frac{1}{j_{\nu+1}(j_{\nu,n})} \right| \leq \pi\sqrt{n},$$

$$\frac{J_{\nu}(Xj_{\nu,n})}{J_{\nu+1}(j_{\nu,n})} \vee \left| \frac{J_{\nu+1}(Xj_{\nu,n})}{J_{\nu+1}(j_{\nu,n})} \right| \leq \widetilde{C}_\nu \frac{(1 + X\pi)^2}{X}, \quad (n > N_\nu).$$
Proof. According to (3), we can find the natural number \( N_\nu \geq 2 \) which satisfies

\[
\frac{n\pi}{2} < j_{\nu,n} < 2n\pi, \quad \left| \frac{1}{J_{\nu+1}(j_{\nu,n})} \right| \leq \frac{\pi}{n} \quad (n \geq N_\nu).
\]

In addition, by (1), for \( n \geq N_\nu \), the following inequalities hold:

\[
\begin{align*}
\left| \frac{J_\nu(X_{j_{\nu,n}})}{J_{\nu+1}(j_{\nu,n})} \right| \leq C_\nu \frac{(X_{j_{\nu,n}})^\nu}{(1 + X_{j_{\nu,n}})^{\nu + \frac{1}{2}}} \pi \sqrt{n} & \leq C_\nu \frac{(2Xn\pi)^\nu}{(1 + \frac{Xn\pi}{2})^{\nu + \frac{1}{2}}} \pi \sqrt{n} \leq 2^{\nu + 1/2} C_\nu \frac{(1 + X\pi)^{1/2}}{X}, \\
\left| \frac{J_{\nu+1}(X_{j_{\nu,n}})}{J_{\nu+2}(j_{\nu,n})} \right| \leq C_{\nu+1} \frac{(X_{j_{\nu,n}})^{\nu + 1}}{(1 + X_{j_{\nu,n}})^{\nu + \frac{3}{2}}} \pi \sqrt{n} & \leq C_{\nu+1} \frac{(2Xn\pi)^{\nu + 1}}{(1 + \frac{Xn\pi}{2})^{\nu + \frac{3}{2}}} \pi \sqrt{n} \leq 2^{\nu + 5/2} C_{\nu+1} \frac{(1 + X\pi)^{1/2}}{X}.
\end{align*}
\]

\( \square \)

**Theorem 5** ([5] (20)). Let \( 0 \leq x, y < c, t > 0 \), and let \( p(t; x, y) \) be the symmetric transition density of a regular one-dimensional diffusion on \( [0, \infty) \) \( R = \{ R(t) \}_{t \geq 0} \) relative to its speed measure. In addition, let \( r_{x-y}^t = r_{x-y}^t(s) \) \( s \in [0, t] \) denote an \( R \)-bridge of length \( t \) from \( x \) to \( y \). Moreover, let \( \varphi_\lambda^1 \) and \( \varphi_\lambda^1 \) denote the increasing and decreasing solutions of \( Au = \lambda u \) for \( A \) the infinitesimal generator of \( R \), normalized so that

\[
\int_0^{\infty} e^{-\lambda t} p(t; x, y) dt = \varphi_\lambda^1(x) \varphi_\lambda^1(y), \quad 0 \leq x \leq y, \lambda > 0.
\] (39)

Then, we have

\[
\int_0^{\infty} e^{-\lambda t} P \left( M_{[0,t]}(r_{x-y}^t) > c \right) p(t; x, y) dt = \varphi_\lambda^1(y) \varphi_\lambda^1(c) \frac{\varphi_\lambda^1(x)}{\varphi_\lambda^1(c)}.
\] (40)

**REMARK A.1.** In the case of the \( \delta \)-dimensional Bessel process, \( \varphi_\lambda^1 \) and \( \varphi_\lambda^1 \) in Theorem 5 are given as follows ([5] (23)):

\[
\varphi_\lambda^1(x) = I_\nu \left( \sqrt{2\lambda x} \right) x^{-\nu}, \quad \varphi_\lambda^1(x) = K_\nu \left( \sqrt{2\lambda x} \right) x^{-\nu}, \quad x \geq 0, \lambda > 0.
\] (41)

**Theorem 6.** Let \( c > 0 \) and \( 0 \leq s < t \). For \( x, y \in (0, c) \), we have

\[
P \left( M_{[s,t]}(r_{x-y}^s) \leq c \right) = \frac{1}{\pi A_{\nu}(x, y)} \sum_{n=1}^{\infty} \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2 (t - s)}{2c^2} \right).
\]

In addition, for \( y \in [0, c) \), we have

\[
P \left( M_{[s,t]}(r_{y-x}^s) \leq c \right) \quad P \left( M_{[s,t]}(r_{x-y}^s) \leq c \right)
\]
Proof. The Laplace transform for a function $f$ is denoted by $L(f)$:

$$L(f)(\lambda) := \int_0^\infty e^{-\lambda s} f(s) ds \quad \lambda > 0.$$ 

For $0 \leq x < y < c$, by (39), (40), and (41), we have

$$L \left( P \left( M_{[0,]}(r_{[0,]}^{x\rightarrow y}) \leq c \right) \rho(\cdot; x, y) \right)(\lambda)$$

$$= L \left( \left( 1 - P \left( M_{[0,]}(r_{[0,]}^{x\rightarrow y}) > c \right) \right) \rho(\cdot; x, y) \right)(\lambda)$$

$$= L(p(\cdot; x, y))(\lambda) - L \left( P \left( M_{[0,]}(r_{[0,]}^{x\rightarrow y}) > c \right) \rho(\cdot; x, y) \right)(\lambda)$$

$$= \varphi^r_\lambda(x) - \varphi^r_\lambda(c) \varphi^t_\lambda(c)$$

$$= (xy)^{-\nu} I_\nu(\sqrt{2\lambda xy}) K_\nu(\sqrt{2\lambda c}) - I_\nu(\sqrt{2\lambda y}) K_\nu(\sqrt{2\lambda c}), \quad \lambda > 0. $$

Here, note that

$$\frac{I_\nu(XC)}{I_\nu(C)} (I_\nu(C) K_\nu(YC) - I_\nu(YC) K_\nu(C)) = 2 \sum_{n=1}^{\infty} J_\nu(xj_{\nu,n}) J_\nu(yj_{\nu,n}) J_\nu(\sqrt{xj_{\nu,n}} \sqrt{yj_{\nu,n}}) \int_0^{\infty} e^{-\lambda s} ds$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \int_0^\infty e^{-\lambda r} (xy)^{-\nu} J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c) \exp \left( - \left( \lambda + \frac{j_{\nu,n}^2}{2c^2} \right) r \right) dr$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \int_0^\infty e^{-\lambda r} (xy)^{-\nu} J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c) \exp \left( - \left( \lambda + \frac{j_{\nu,n}^2}{2c^2} \right) r \right) dr, \quad \lambda > 0. \quad (42)$$

For $n$, we set

$$f_n(r) := \frac{J_\nu(xj_{\nu,n}/c) J_\nu(yj_{\nu,n}/c)}{c^2 J_{\nu+1}(j_{\nu,n})} \exp \left( - \left( \lambda + \frac{j_{\nu,n}^2}{2c^2} \right) r \right), \quad r \geq 0. $$

Then, by Lemma [A.1] and [3], there exist some $\tilde{C}_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$|f_n(r)| \leq \tilde{C}_\nu^2 \frac{\sqrt{1 + x/n}(1 + y/n)}{xy} \exp \left( - \frac{(n\pi)^2}{8c^2} \right) \leq \tilde{C}_\nu^2 \frac{1 + \pi}{xy} \exp \left( - \frac{(n\pi)^2}{8c^2} \right), \quad n > N_\nu.$$
Therefore, we can see that
\[
\sum_{n=N_0+1}^{\infty} \int_0^\infty |f_n(r)| dr \leq \tilde{C}_\nu \frac{1 + \pi}{xy} \sum_{n=N_0+1}^{\infty} \frac{8c^2}{(n\pi)^2} < \infty
\]
holds and we can integrate term by term in (43). Hence, it follows that
\[
L \left( P \left( M_{[0,t]}(r_{[0,1]}^{x \to y}) \leq c \right) \right) (\lambda) = \int_0^\infty e^{-\lambda r} (xy)^{-\nu} \sum_{n=1}^{\infty} J_\nu (x_{j\nu,n}/c) J_\nu (y_{j\nu,n}/c) \frac{c^2 J_{\nu+1}^2(j_{\nu,n}) \exp \left( -\frac{j_{\nu,n}^2}{2c^2} r \right)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{2c^2} \right) ) (\lambda) \quad (\lambda > 0).
\]

By the inverse Laplace transform of this identity, we obtain the following expression:
\[
P \left( M_{[0,t]}(r_{[0,1]}^{x \to y}) \leq c \right) = \frac{(xy)^{-\nu}}{\pi (xy)^{-\nu} A_t^{(\nu)}(x, y)} \sum_{n=1}^{\infty} J_\nu (x_{j\nu,n}/c) J_\nu (y_{j\nu,n}/c) \frac{c^2 J_{\nu+1}^2(j_{\nu,n}) \exp \left( -\frac{j_{\nu,n}^2}{2c^2} t \right)}{c^2 J_{\nu+1}^2(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{2c^2} t \right).
\]

Because the right-hand side of (42) is symmetric for X and Y, we can see that this result holds for 0 < y ≤ x < c.

Finally, for 0 ≤ y < c, we can calculate the following:
\[
P \left( M_{[0,t]}(r_{[0,t]}^{0 \to y}) \leq c \right) = \frac{2^{\nu + \frac{1}{2}}}{\sqrt{2\pi n_t(y)}} \sum_{n=1}^{\infty} \left( \frac{j_{\nu,n}}{cy} \right)^\nu J_\nu (y_{j\nu,n}/c) \exp \left( -\frac{j_{\nu,n}^2}{2c^2} t \right).
\]
Proposition A.1. Let $\eta > 0$ and $0 \leq s < t$. For $x, y \in (0, \eta)$, we have

$$
\frac{\partial}{\partial \eta} P \left( M_{[s,t]}(r_{[s,t]}^{x\to y}) \leq \eta \right)
= \frac{1}{\pi A_{t-s}^{(\nu)}(x, y)} \sum_{n=1}^{\infty} \frac{1}{J_{\nu+1}^{2}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^{2}}{2\eta^{2}}(t - s) \right)
\times \left\{ \left( -2\nu + \frac{2\nu j_{\nu,n}}{\eta^{3}}(t - s) \right) J_{\nu}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta)
+ \frac{xj_{\nu,n}}{\eta^{4}} J_{\nu+1}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta) + \frac{yj_{\nu,n}}{\eta^{4}} J_{\nu+1}(yj_{\nu,n}/\eta) J_{\nu}(xj_{\nu,n}/\eta) \right\}
$$

Proof. Let $\eta > 0$ and let $0 < x, y < \eta$ be fixed. For $n$, we set

$$
f_{n}(\eta, x, y) = \frac{1}{J_{\nu+1}^{2}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^{2}}{2\eta^{2}}(t - s) \right)
\times \left\{ \left( -2\nu + \frac{2\nu j_{\nu,n}}{\eta^{3}}(t - s) \right) J_{\nu}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta)
+ \frac{xj_{\nu,n}}{\eta^{4}} J_{\nu+1}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta) + \frac{yj_{\nu,n}}{\eta^{4}} J_{\nu+1}(yj_{\nu,n}/\eta) J_{\nu}(xj_{\nu,n}/\eta) \right\}
$$

By Lemma [A.1] there exist some $\tilde{C}_{\nu} > 0$ and $N_{\nu} \in \mathbb{N}$ such that

$$
|f_{n}(\eta, x, y)|
\leq \exp \left( -\frac{\pi^{2}(t - s)}{8\eta^{2}} n^{2} \right)
\times \left\{ \left( 2\nu + \frac{2n\pi}{\eta^{3}}(t - s) \right) J_{\nu}(xj_{\nu,n}/\eta) J_{\nu}(yj_{\nu,n}/\eta)
+ \frac{x(2n\pi)}{\eta^{4}} \eta \tilde{C}_{\nu} \left( 1 + \frac{\pi}{\eta} \right)^{1/2} \frac{1}{x}
+ \frac{y(2n\pi)}{\eta^{4}} \eta \tilde{C}_{\nu} \left( 1 + \frac{\pi}{\eta} \right)^{1/2} \frac{1}{y}
+ \frac{x(2n\pi)}{\eta^{4}} \eta \tilde{C}_{\nu} \left( 1 + \frac{\pi}{\eta} \right)^{1/2} \frac{1}{x}
\right\}
\leq \exp \left( -\frac{\pi^{2}(t - s)}{8\eta^{2}} n^{2} \right)
\times \left\{ \left( 2\nu + \frac{2n\pi}{\eta^{3}}(t - s) + \frac{2\pi(x + y)}{\eta^{2}} \right) \tilde{C}_{\nu}^{2} \left( 1 + \pi \right) \frac{1}{xy}, \ n > N_{\nu} \right\}
$$

Therefore, we can differentiate term by term the first identity of Theorem [6] in some neighborhood.
of $\eta$. Similarly, for $n$, we set

$$f_n(\eta, y) = \frac{(j_{\nu,n}/\eta)^{\nu+2}}{J_{\nu+1}(j_{\nu,n})} \frac{1}{\eta^3} \exp \left( -\frac{j_{\nu,n}^2}{2\eta^2} (t-s) \right) \times \left\{ \left( t - s - \frac{2\eta^2(\nu + 1)}{j_{\nu,n}^2} \right) J_\nu(y_j_{\nu,n}/\eta) + \frac{y\eta}{j_{\nu,n}} J_{\nu+1}(y_j_{\nu,n}/\eta) \right\}.$$ 

By Lemma A.1, there exist some $C_\nu > 0$ and $N_\nu \in \mathbb{N}$ such that

$$|f_n(\eta, y)| \leq \frac{\sqrt{\pi(1 + \pi)}}{y^n \nu^4 \sqrt{2}} C_\nu (2\pi^2)^{\nu+\frac{1}{2}} \left\{ t - s + \frac{8\eta^2(\nu + 1)}{(n\pi)^2} + \frac{2y\eta}{n\pi} \right\} \exp \left( -\frac{\pi^2(t-s)}{8\eta^2} - n^2 \right) \quad n > N_\nu.$$ 

Therefore, we can differentiate term by term the second identity of Theorem 6 in some neighborhood of $\eta$. 

According to Proposition A.1 and Lebesgue’s dominated convergence theorem, we can obtain the next corollary.

**Corollary 2.** Let $b > 0$. For $0 \leq s < t$ and $y \in (0, b)$, we have

$$\lim_{\eta \downarrow 0} \frac{\partial}{\partial \eta} P \left( M_{s,t}(r^{\nu-b}_{[s,t]}) \leq \eta \right) = \frac{1}{\pi A_{\nu-s}(y, b)} \sum_{n=1}^{\infty} \frac{j_{\nu,n} J_\nu(y_j_{\nu,n}/b)}{b^3 J_{\nu+1}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{2b^2} (t-s) \right),$$

$$\lim_{\eta \downarrow 0} \frac{\partial}{\partial \eta} P \left( M(r^{0-b}_{[s,t]}) \leq \eta \right) = \frac{2(t-s)^{\nu+\frac{1}{2}}}{\sqrt{2\pi n_{t-s}(b)}} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{\nu+1} J_{\nu+1}(\nu_j_{\nu,n})}{b^{2\nu+3} J_{\nu+1}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{2b^2} (t-s) \right).$$

**A.2 General results on continuous processes**

In this subsection, we introduce some general results used in this paper. The proofs of them are found in [2].

**Theorem 7 ([3, Chapter 2, Theorem 4.15]).** Let $\{X_n\}_{n=1}^{\infty}$ be the family of $C([0,1], \mathbb{R}^d)$-valued random variables. If the family $\{X_n\}_{n=1}^{\infty}$ is tight and the finite-dimensional distribution of $X_n$ converges to that of some $X$, then $X_n \xrightarrow{D} X$ holds.

**Lemma A.2.** Let $a, b \in \mathbb{R}^d$, and let $X_n$ and $X$ are $\mathbb{R}^d$-valued Markovian bridges from $a$ to $b$ on $[0,1]$ for $n \in \mathbb{N}$. Let $X_n$ and $X$ have the respective transition densities

$$P(X_n(t) \in dy) = q_n(t,y)dy, \quad P(X_n(t) \in dy \mid X_n(s) = x) = q_n(s,x,t,y)dy,$$

$$P(X(t) \in dy) = q(t,y)dy, \quad P(X(t) \in dy \mid X(s) = x) = q(s,x,t,y)dy$$

for $0 < s < t < 1, x, y \in \mathbb{R}^d$, and $n \in \mathbb{N}$. If

$$\lim_{n \to \infty} q_n(t,y) = q(t,y), \quad \text{a.e. } y \in \mathbb{R}^d,$$

$$\lim_{n \to \infty} q_n(s,x,t,y) = q(s,x,t,y), \quad \text{a.e. } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

for $0 < s < t < 1$, then the finite-dimensional distribution of $X_n$ converges to that of $X$ as $n \to \infty$. 

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Theorem 8. For \( \varepsilon \in \mathcal{E} \), \( X^{(e)} \) is a \((C([0, 1], \mathbb{R}^d), \mathcal{B}(C([0, 1], \mathbb{R}^d)))\)-valued random variable defined on \((\Omega^{(e)}, \mathcal{F}^{(e)}, P^{(e)})\). Assume that \( \{X^{(e)}(0)\}_{\varepsilon \in \mathcal{E}} \) is uniformly integrable and that the following conditions hold:

1. For each \( u \in (0, \frac{1}{2}) \), \( \{\pi_{[u, 1-u]} \circ X^{(e)}\}_{\varepsilon \in \mathcal{E}} \) is tight.
2. For each \( \xi > 0 \), it holds that

\[
\limsup_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(e)} \left( \sup_{0 \leq t \leq u} |X^{(e)}(t) - X^{(e)}(0)| > \xi \right) = 0,
\]

\[
\limsup_{u \downarrow 0} \sup_{\varepsilon \in \mathcal{E}} P^{(e)} \left( \sup_{1-u \leq t \leq 1} |X^{(e)}(t) - X^{(e)}(1)| > \xi \right) = 0.
\]

Then, the family \( \{X^{(e)}\}_{\varepsilon \in \mathcal{E}} \) is tight.

Lemma A.3. (Chapter 2, Problem 4.11 in [3]) For \( \varepsilon \in \mathcal{E} \), \( X^{(e)} \) is a \((C([0, 1], \mathbb{R}^d), \mathcal{B}(C([0, 1], \mathbb{R}^d)))\)-valued random variable defined on \((\Omega^{(e)}, \mathcal{F}^{(e)}, P^{(e)})\). Assume that \( \{X^{(e)}\}_{\varepsilon \in \mathcal{E}} \) satisfies the following conditions:

1. There exists some \( \nu > 0 \) that satisfies

\[
\sup_{\varepsilon \in \mathcal{E}} E^{(e)} \left[ |X^{(e)}(0)|^\nu \right] < \infty.
\]

2. There exist \( \alpha, \beta, C > 0 \) that satisfy

\[
\sup_{\varepsilon \in \mathcal{E}} E^{(e)} \left[ |X^{(e)}(t) - X^{(e)}(s)|^\alpha \right] \leq C |t - s|^{1+\beta}, \quad t, s \in [0, 1].
\]

Then \( \{X^{(e)}\}_{\varepsilon \in \mathcal{E}} \) is tight.

Lemma A.4. Let \( \gamma > 0 \). For \( \varepsilon \in \mathcal{E} \), \( X^{(e)} \) is a \((C([0, 1], \mathbb{R}^d), \mathcal{B}(C([0, 1], \mathbb{R}^d)))\)-valued random variable defined on \((\Omega^{(e)}, \mathcal{F}^{(e)}, P^{(e)})\). Assume that

\[
F^{(e)}_l := \left\{ \max_{1 \leq k \leq 2^{l-1}} \left| X^{(e)} \left( \frac{k-1}{2^l} \right) - X^{(e)} \left( \frac{k}{2^l} \right) \right| \geq 2^{-l\gamma} \right\} \in \mathcal{F}^{(e)}, \quad \varepsilon \in \mathcal{E}, \quad l = 1, 2, \ldots
\]

satisfy \( \sum_{l=1}^{\infty} \sup_{\varepsilon \in \mathcal{E}} P^{(e)}(F^{(e)}_l) < \infty \), then we have

\[
\limsup_{u \downarrow 0} P^{(e)} \left( \sup_{0 \leq t \leq u} |X^{(e)}(t) - X^{(e)}(0)| > \xi \right) = 0, \quad \xi > 0.
\]

Lemma A.5. Under the same assumption of Lemma A.4, if

\[
\tilde{F}^{(e)}_l := \left\{ \max_{2^{l-1} \leq k \leq 2^l} \left| X^{(e)} \left( \frac{k-1}{2^l} \right) - X^{(e)} \left( \frac{k}{2^l} \right) \right| \geq 2^{-l\gamma} \right\} \in \mathcal{F}^{(e)}, \quad \varepsilon \in \mathcal{E}, \quad l = 1, 2, \ldots
\]

satisfy \( \sum_{l=1}^{\infty} \sup_{\varepsilon \in \mathcal{E}} P^{(e)}(\tilde{F}^{(e)}_l) < \infty \), then we have

\[
\limsup_{u \downarrow 0} P^{(e)} \left( \sup_{0 \leq t \leq u} |X^{(e)}(1-t) - X^{(e)}(1)| > \xi \right) = 0, \quad \xi > 0.
\]
Lemma A.6. Let $S_1$ and $S_2$ be Polish spaces and let $X_n$ and $Y_n$ be random variables defined on $(\Omega_n, \mathcal{F}_n, P_n)$ that take values in $S_1$ and $S_2$, respectively. If $X_n$ and $Y_n$ are independent and $P_n \circ X_n^{-1}$ and $P_n \circ Y_n^{-1}$ converge to probability measures $Q$ on $S_1$ and $R$ on $S_2$, respectively, then $P_n \circ (X_n, Y_n)^{-1}$ converges to the product measure $Q \times R$.

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