A 2-arc Transitive Hexavalent Nonnormal Cayley Graph on $A_{119}$

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Abstract: A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be normal if the base group $G$ is normal in $\text{Aut}\Gamma$. The concept of the normality of Cayley graphs was first proposed by M.Y. Xu in 1998 and it plays a vital role in determining the full automorphism groups of Cayley graphs. In this paper, we construct an example of a 2-arc transitive hexavalent nonnormal Cayley graph on the alternating group $A_{119}$. Furthermore, we determine the full automorphism group of this graph and show that it is isomorphic to $A_{120}$.

Keywords: simple group; nonnormal Cayley graph; arc-transitive graph; automorphism group

1. Introduction

Throughout this paper, all graphs are assumed to be finite and undirected.

For a graph $\Gamma$, we use $V(\Gamma)$, $E(\Gamma)$, $\text{Arc}\Gamma$ and $\text{Aut}\Gamma$ to denote the vertex set, edge set, arc set and full automorphism group of the graph $\Gamma$, respectively. A graph $\Gamma$ is said to be arc-transitive if the full automorphism group $\text{Aut}\Gamma$ acts transitively on $\text{Arc}\Gamma$. We use $\text{val}(\Gamma)$ to denote the valency of the $\Gamma$, and we say $\Gamma$ is a cubic, tetravalent, pentavalent or hexavalent graph, meaning $\text{val}(\Gamma) = 3, 4, 5$ or 6.

Let $G$ be a finite group with identity element 1 and $S$ (say Cayley subset) a subset of $G$ such that $1 \notin S$ and $S = S^{-1} := \{x^{-1} \mid x \in S\}$. Define the Cayley graph $\text{Cay}(G, S)$, that is, the Cayley graph of $G$ with respect to the Cayley subset $S$ as the graph with vertex set $G$ such that $g, h \in G$ are adjacent if and only if $hg^{-1} \in S$. It is easy to see that the valency of $\text{Cay}(G, S)$ is $|S|$. As we all know, $\text{Cay}(G, S)$ is connected if and only if $(S) = G$. On the other hand, letting $R(G)$ be the right regular representation of $G$ and letting $\text{Aut}\text{Cay}(G, S)$ be the full automorphism group of $\text{Cay}(G, S)$, there are clearly $R(G) \leq \text{Aut}\text{Cay}(G, S)$, and $R(G)$ acts transitively on the vertices of $\text{Cay}(G, S)$. Then, the graph $\text{Cay}(G, S)$ is vertex-transitive, and $G$ (or $R(G)$) can be viewed as a regular subgroup of $\text{Aut}\text{Cay}(G, S)$. Conversely, a connected graph $\Gamma$ is isomorphic to a Cayley graph of a group $G$ if and only if the full automorphism group $\text{Aut}\Gamma$ contains a subgroup which acts regularly on $V(\Gamma)$ and the subgroup is isomorphic to $G$ (see [1]). A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be a normal Cayley graph if the base group $G$ is normal in $\text{Aut}\Gamma$; otherwise, $\Gamma$ is said to be a nonnormal Cayley graph (see [2]).

The study about Cayley graphs on finite non-abelian simple groups has always attracted much attention because of Cayley graphs with high levels of symmetry; for example, vertex-transitivity, edge-transitivity and arc-transitivity are widely used in the design of interconnection networks. For more detailed applications, we recommend that readers refer to [3,4]. Let $G$ be a finite non-abelian simple group, and let $\Gamma = \text{Cay}(G, S)$ be a connected arc-transitive Cayley graph on $G$. The main motivation for classifying 2-arc-transitive nonnormal Cayley graphs comes from the fact that Fang, Ma and Wang [5] proved all but finitely that many locally primitive Cayley graphs of valency $d \leq 20$ or a prime number of the finite non-abelian simple groups are normal. In [5] (Problem 1.2), they proposed the following problem: classify nonnormal locally primitive Cayley graphs (note...
that 2-arc-transitive graphs must be locally primitive) of finite simple groups with valency $d \leq 20$ or a prime number. To solve this problem, we should study each valency $d \leq 20$ or a prime number. In the case where $\Gamma$ is a cubic graph (3-valent), Li [6] proved that $\Gamma$ must be normal, except for seven exceptions. On the basis of Li's result, Xu et al. [7,8] proved that $\Gamma$ must be normal, except for two exceptions on $A_{17}$. In the case where $\Gamma$ is a tetra-valent graph (4-valent), Fang et al. in [9] proved that most of such $\Gamma$ are normal, except for Cayley graphs on a list of $G$. Further, Fang et al. in [10] proved that $\Gamma$ are normal when $\Gamma$ is 2-transitive, except for two graphs on $M_{11}$. In the case where $\Gamma$ is a pentavalent graph (5-valent), Zhou and Feng [11] proved that all 1-transitive Cayley $\Gamma$ of simple groups are normal. Ling and Lou in [12] gave an example of a 2-transitive pentavalent nonnormal Cayley graph on $A_{39}$.

Therefore, the next natural problem is to study the case of the 6-valent. However, there are no known nonnormal examples of hexavalent 2-arc-transitive Cayley graphs on finite simple groups.

The aim of this paper is to construct a nonnormal example of a connected 2-arc transitive hexavalent Cayley graph on a finite non-abelian simple group. Our main result is the following theorem.

**Theorem 1.** There exists a nonnormal example of a connected 2-arc-transitive hexavalent Cayley graph on the alternating group $A_{119}$, and the full automorphism group of this graph is isomorphic to the alternating group $A_{120}$.

### 2. Preliminaries

In this section, we give some necessary preliminary results which are used in later discussions.

Let $G$ be a finite group and let $H$ be a subgroup of $G$. Then we have the following result (see [13] (Ch. I, 1.4)).

**Lemma 1.** Let $G$ be a group and let $H$ be a subgroup of $G$. Let $N_G(H)$ be the normalizer of $H$ in $G$, and let $C_G(H)$ be the centralizer of $H$ in $G$. Then, $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(H)$ of $H$.

We next introduce the definition of a Sabidussi coset graph. Let $G$ be a group, $g \in G \setminus H$ such that $g^2 \in H$, and let $H$ be a core-free subgroup of $G$. Define the *Sabidussi coset graph* $\text{Cos}(G,H,g)$ of $G$ with respect to the core-free subgroup $H$ as the graph with vertex set $|G : H|$ (the set of cosets of $H$ in $G$) such that $Hx$ and $Hy$ are adjacent if and only if $xy^{-1} \in HgH$. The following lemma follows from [14], and it can be easily proved by the definition of the coset graphs (see [15] (Theorem 3) for example).

**Lemma 2.** Let $G$ be a group, $g \in G \setminus H$ such that $g^2 \in H$, and let $H$ be a core-free subgroup of $G$. Let $\Gamma = \text{Cos}(G,H,g)$ be a Sabidussi coset graph of $G$ with respect to $H$. Then, $\Gamma$ is $G$-arc-transitive and the following holds:

1. The valency of the graph $\Gamma$ is equal to $|H : H \cap H^g|$.
2. $\Gamma$ is a connected graph if and only if $(H,g) = G$.
3. If $G$ contains a subgroup $R$ is regular on $VT$, then $\Gamma \cong \text{Cay}(R,S)$, where $S = R \cap HgH$.

Conversely, if $\Sigma$ is an $X$-arc-transitive graph, then $\Sigma$ is isomorphic to a Sabidussi coset graph $\text{Cos}(X,X_0,g)$, where $g \in N_X(X_{uv})$ is a 2-element such that $g^2 \in X_0$, and $u \in V\Sigma$, $w \in \Sigma(v)$.

**Proof.** Let $\Sigma$ be an $X$-arc-transitive graph. Let $v \in V\Sigma$ be a vertex of $\Sigma$ and $w \in \Sigma(v)$. Since $\Sigma$ is $X$-arc-transitive, there is $g$ such that $w^g = w$. For each $x \in X$, define $\varphi : Hx \rightarrow v^g$. Then we can verify that $\varphi$ is a graph isomorphic from $\Sigma$ to $\text{Cos}(X,X_{uv},g)$. Since $\Sigma$ is undirected, we have $g^2 \in X_0$. Hence, $(H \cap Hg)^g = H \cap Hg$. Thus, we can choose a 2-element $g$ satisfying $g \in N_X(X_{uv})$. \qed

Let $t_1 \geq 0$ and $t_2 \geq 0$ be two integers. We denote by the $\{2,3\}$-group the finite group of the order $2^{t_1}3^{t_2}$. Following the definition of relevant objects in [16] (Theorem 3.1), we
have the following lemma, which is about the stabilizers of arc-transitive hexavalent graphs. For the structure of the received stabilizers, see the proof in [16] (Page 926).

**Lemma 3.** Let $s$ be a positive integer, and let $\Gamma$ be a connected hexavalent $(G, s)$-transitive graph for some $G \leq \text{Aut}\Gamma$. Let $v \in V \Gamma$. Then $s \leq 4$ and one of the following statements holds:

1. For $s = 1$, the stabilizer $G_v$ is a $\{2, 3\}$-group.
2. For $s = 2$, the stabilizer $G_v \cong PSL(2, 5)$, $PGL(2, 5)$, $A_6$ or $S_6$.
3. For $s = 3$, the stabilizer $G_v \cong D_{10} \times PSL(2, 5)$, $F_{20} \times PGL(2, 5)$, $A_6 \times A_6$, $S_5 \times S_5$, $(D_{10} \times PSL(2, 5)) \cdot Z_2$ with $D_{10} \cdot Z_2 = F_{20}$ and $PSL(2, 5) \cdot Z_2 = PGL(2, 5)$, or $(A_5 \times A_6) \times Z_2$ with $A_5 \times Z_2 = S_5$ and $A_6 \times Z_2 = S_6$.
4. For $s = 4$, the stabilizer $G_v \cong Z_2^6 \times GL(2, 5) = AGL(2, 5)$.

3. A 2-arc Transitive Hexavalent Nonnormal Cayley Graph on $A_{119}$

In this section, we construct a connected 2-arc transitive hexavalent nonnormal Cayley graph on $A_{119}$ and determine its full automorphism group. In fact, if $\Gamma := \text{Cay}(G, S)$ is a Cayley graph of a non-abelian simple group $G$, then $G$ is core free in $X$, where $G \leq \text{Aut}\Gamma$. Let $v \in V \Gamma$ and $H = X_v$. Suppose that $|H| = n$. Then by Lemma 3, $n$ may be 60, 120, etc. Consider the action of $X$ on the set of $[X : G]$ by right multiplication; then, $X \leq S_n$. So, we may construct the nonnormal Cayley graph in $S_n$, where $n = 60, 120$, etc. The following example is really the case where we construct $n = 120$.

**Construction 1.** Let $G$ be the alternating group on the set $\{2, 3, \ldots, 120\}$. Then, $G \cong A_{119}$. Let $H = \langle a, b \rangle < X := A_{120}$ (the alternating group on $\{1, 2, \ldots, 120\}$), where the following holds:

$$a = (1 2 4 3)(5 13 12 17)(6 14 11 18)(7 15 10 19)(8 16 9 20)(21 61 101 81)(22 62 102 82)(23 63 103 83)(24 64 104 84)(25 65 105 85)(26 66 106 86)(27 67 107 87)(28 68 108 88)(29 69 109 89)(30 70 110 90)(31 71 111 91)(32 72 112 92)(33 73 113 93)(34 74 114 94)(35 75 115 95)(36 76 116 96)(37 77 117 97)(38 78 118 98)(39 79 119 99)(40 80 120 100)(41 56 59 46)(42 55 58 45)(43 54 57 44)(44 53 58 47)(49 51 52 50),$$

$$b = (1 21 41)(2 22 42)(3 23 43)(4 24 44)(5 25 45)(6 26 46)(7 27 47)(8 28 48)(9 29 49)(10 30 50)(11 31 51)(12 32 52)(13 33 53)(14 34 54)(15 35 55)(16 36 56)(17 37 57)(18 38 58)(19 39 59)(20 40 60)(21 81 111)(22 82 112)(23 83 113)(24 84 114)(25 85 115)(26 86 116)(27 87 117)(28 88 118)(29 89 119)(30 90 120)(31 53 60 48)(32 54 60 49)(33 55 60 45)(34 56 60 46)(35 57 60 47)(36 58 60 44)(37 59 60 43)(38 60 60 42)(39 60 60 41)(40 60 60 50),$$

Take $x \in X$ as follows:

$$x = (1 79)(2 80)(3 60)(4 58)(5 113)(6 64)(7 114)(8 63)(9 112)(10 111)(11 12)(13 47)(15 73)(16 106)(19 43)(20 41)(21 118)(22 120)(23 24)(25 50)(26 49)(27 68)(28 66)(30 62)(32 61)(33 42)(34 44)(35 103)(36 101)(37 107)(38 45)(40 75)(48 108)(53 54)(55 115)(56 116)(57 119)(59 117)(65 109)(67 110)(69 102)(70 104)(71 72)(76 105)(81 89)(82 93)(84 99)(85 98)(87 91)(88 94)(90 100)(96 97).$$

Define $\Sigma = \text{Cos}(X, H, x)$. 
Lemma 4. The graph $\Sigma = \cos(X, H, x)$ in Construction 1 is a connected 2-arc-transitive graph and isomorphic to the nonnormal hexavalent Cayley graph $\text{Cay}(G, S)$ of $G$, determined by $S = \{x_1, x_1^{-1}, x_2, x_3, x_4, x_5\}$ with the following:

$$x_1 = (2 \ 3 \ 38 \ 36 \ 95 \ 101 \ 45 \ 60 \ 80 \ 77)(4 \ 37 \ 97 \ 103 \ 46 \ 35 \ 96 \ 107 \ 58 \ 78)(5 \ 90 \ 106 \ 66 \ 51 \ 28 \ 16 \ 100 \ 113 \ 74)(6 \ 18 \ 64 \ 76 \ 98 \ 109 \ 14 \ 65 \ 85 \ 105)(7 \ 92 \ 114 \ 73 \ 68 \ 49 \ 52 \ 26 \ 27 \ 15)(8 \ 20 \ 31 \ 41 \ 63 \ 75 \ 59 \ 83 \ 117 \ 40)(9 \ 88 \ 104 \ 11 \ 69 \ 93 \ 120 \ 54 \ 21 \ 81 \ 115 \ 23 \ 56 \ 91 \ 111 \ 71)(10 \ 87 \ 116 \ 24 \ 55 \ 89 \ 118 \ 53 \ 22 \ 82 \ 102 \ 12 \ 70 \ 94 \ 112 \ 72)(13 \ 34 \ 30 \ 17 \ 62 \ 44 \ 47 \ 67 \ 86 \ 110)(19 \ 32)(29 \ 42 \ 48 \ 99 \ 119 \ 39 \ 57 \ 84 \ 108 \ 33)(43 \ 61)$;

$$x_2 = (2 \ 97)(4 \ 99)(5 \ 73)(6 \ 74)(7 \ 8)(9 \ 41)(10 \ 88)(11 \ 78)(13 \ 58)(14 \ 96)(15 \ 60)(16 \ 95)(17 \ 66)(18 \ 65)(19 \ 52)(20 \ 50)(22 \ 40)(23 \ 27)(24 \ 32)(25 \ 29)(26 \ 33)(28 \ 39)(31 \ 36)(35 \ 38)(43 \ 116)(44 \ 77)(45 \ 62)(46 \ 48)(47 \ 61)(49 \ 107)(51 \ 105)(53 \ 110)(54 \ 75)(55 \ 109)(56 \ 76)(63 \ 120)(64 \ 119)(67 \ 91)(68 \ 92)(69 \ 103)(70 \ 94)(71 \ 104)(72 \ 93)(79 \ 113)(80 \ 86)(81 \ 112)(83 \ 111)(87 \ 114)(89 \ 90)(98 \ 117)(100 \ 118)(101 \ 102)$;

$$x_3 = (3 \ 39)(4 \ 37)(5 \ 106)(6 \ 105)(7 \ 50)(8 \ 49)(9 \ 35)(10 \ 45)(11 \ 36)(12 \ 47)(13 \ 113)(14 \ 114)(15 \ 16)(18 \ 118)(19 \ 28)(20 \ 56)(21 \ 92)(23 \ 91)(26 \ 120)(27 \ 94)(29 \ 30)(31 \ 107)(32 \ 108)(33 \ 112)(34 \ 110)(38 \ 97)(40 \ 98)(41 \ 42)(43 \ 102)(44 \ 101)(51 \ 87)(52 \ 85)(53 \ 117)(54 \ 96)(57 \ 89)(58 \ 90)(59 \ 116)(60 \ 115)(62 \ 80)(63 \ 67)(64 \ 72)(65 \ 69)(66 \ 73)(68 \ 79)(71 \ 76)(75 \ 78)(81 \ 82)(83 \ 109)(84 \ 111)(93 \ 119)(99 \ 104)(100 \ 103)$;

$$x_4 = (2 \ 20)(3 \ 7)(4 \ 12)(5 \ 9)(6 \ 13)(8 \ 19)(11 \ 16)(15 \ 18)(21 \ 49)(23 \ 84)(24 \ 101)(25 \ 112)(26 \ 28)(27 \ 110)(29 \ 94)(30 \ 60)(31 \ 96)(32 \ 59)(33 \ 85)(34 \ 120)(35 \ 87)(36 \ 118)(38 \ 53)(40 \ 55)(42 \ 76)(44 \ 74)(45 \ 119)(46 \ 117)(47 \ 48)(50 \ 64)(51 \ 102)(54 \ 67)(56 \ 65)(57 \ 108)(58 \ 106)(62 \ 104)(63 \ 82)(66 \ 116)(68 \ 114)(69 \ 86)(71 \ 88)(73 \ 97)(75 \ 99)(77 \ 105)(78 \ 80)(79 \ 107)(81 \ 103)(89 \ 91)(90 \ 115)(92 \ 113)(98 \ 109)(100 \ 111)$;

$$x_5 = (2 \ 111)(3 \ 89)(4 \ 32)(5 \ 21)(6 \ 118)(7 \ 23)(8 \ 117)(9 \ 95)(11 \ 93)(13 \ 99)(14 \ 27)(15 \ 100)(16 \ 28)(17 \ 106)(18 \ 20)(19 \ 105)(22 \ 73)(24 \ 75)(29 \ 112)(30 \ 69)(33 \ 108)(34 \ 80)(35 \ 107)(36 \ 79)(37 \ 38)(39 \ 101)(40 \ 103)(41 \ 49)(42 \ 53)(44 \ 59)(45 \ 58)(47 \ 51)(48 \ 54)(50 \ 60)(56 \ 57)(61 \ 116)(62 \ 64)(63 \ 114)(65 \ 94)(66 \ 96)(67 \ 102)(68 \ 104)(71 \ 90)(72 \ 110)(77 \ 84)(78 \ 82)(85 \ 120)(86 \ 119)(87 \ 88)(91 \ 109)(97 \ 113)(98 \ 115)$.

Proof. Let $\Delta := \{1, 2, \ldots, 120\}$. Then, $X$ has a natural action on $\Delta$. By Magma [17], $\langle H, x \rangle = X$, and so the graph $\Sigma$ is connected by Lemma 2 (2). Furthermore, by Magma [17], we have that $H$ is regular on $\Delta$. However, $G$ is the stabilizer of point 1 in $X$. Hence, $X$ has a factorization $X = GH$ for $G \cap H = 1$. Therefore, $G$ is regular on $[X : H]$. By Lemma 2 (3), $\Sigma$ is isomorphic to a Cayley graph of $G = A_{119}$. Additionally, by the computation of Magma [17] (for the Magma code, see Appendix A), we have $\frac{|H|}{|H \cap X|} = 6$. Hence, Lemma 2 (1) implies that $\Sigma$ is a hexavalent graph. Since $H \cong \text{PGL}(2, 5)$, Lemma 3 implies that $\Sigma$ is 2-arc transitive. Since $X$ is a non-abelian simple group, $G$ is not normal in $X \leq \text{Aut}\Sigma$. It follows that $\Sigma$ is nonnormal. Let $x_1, x_2, x_3, x_4, x_5$ and $S$ be defined as in this lemma. By the computation of Magma [17] (for the Magma code, see Appendix B), we have $G \cap \langle HxH \rangle = S$. Thus, by Lemma 2 (3), we have that $\Sigma$ is isomorphic to $\text{Cay}(G, S)$. This completes the proof of the lemma. \qed

In the next lemma, we show that the full automorphism group $\text{Aut}\Sigma$ is isomorphic to alternating group $A_{120}$.

Lemma 5. The full automorphism group $\text{Aut}\Sigma$ of the 2-arc-transitive hexavalent graph $\Sigma = \cos(X, H, x)$ in Construction 1 is isomorphic to alternating group $A_{120}$.
Proof. Let $A = \text{Aut}\Sigma$. Assume first that the full automorphism group $A$ is quasiprimitive on $V \Sigma$. Let $N$ be a minimal normal subgroup of $A$. Then, $N$ is transitive on $V \Sigma$. It implies that $N$ is insoluble. Thus, $N$ is isomorphic to $T_1 \times T_2 \times \cdots \times T_d = T^d$, where $T_i \cong T$ for each $1 \leq i \leq d$, $T$ is a non-abelian simple group, and $d \geq 1$. Let $p$ be the largest prime factor of the order of $A_{119}$. Then, $p > 5$ and $p^2 \nmid |A_{119}|$. Since $N$ is transitive on $V \Sigma$ and $|V \Sigma| = |A_{119}|$, we have that $p$ divides $|N|$. Assume that $d \geq 2$. Then, $p^d$ divides $|N|$. However, by Lemma 3, the order of the stabilizer $A_6$, divides $2^7 \cdot 3^3 \cdot 5^3$, and so $|A|$ divides $2^7 \cdot 3^3 \cdot 5^3$, $|A_{119}|$ which is divisible by $p^d$, a contradiction. Hence, we have $d = 1$ and $N = T \unlhd A$. Let $C = C_A(T)$ be the centralizer of $T$ in $A$. Then, $C \unlhd N_A(T) = A$ and $CT = C \times T$. If $C \neq 1$, since $A$ is quasiprimitive on $V \Sigma$, this implies that $C$ is transitive on $V \Sigma$. It implies that $p$ divides $|C|$. Therefore, $p$ divides $|CT|$, which divides $|A|$, and so we have that $p^d$ divides $|A|$, a contradiction. Hence, $C = 1$, and $A \unlhd \text{Aut}(T)$ is almost simple.

Since $T \cap X = X \cong A_{120}$, it follows that $T \cap X = 1$ or $X$. If $T \cap X = 1$, then since $|A_1| = 2^4 \cdot 3^2 \cdot 5^2$, we have $|T| = 2^4 \cdot 3^2 \cdot 5^2$; note that $p > 5$, $p \nmid |T|$, a contradiction. Thus, $T \cap X = X$, and so $X \leq T$. It follows that $|T : X|$ divides $|A : X|$, which divides $2^4 \cdot 3^2 \cdot 5^2$. By [18] (pp. 135–136), we can conclude that $T = X \cong A_{120}$. Thus, $A \unlhd \text{Aut}(T) \cong S_{120}$. If $A \cong S_{120}$, then $|A_0| = |A|/|A_0| = 240$, a contradiction to Lemma 3. Hence, $A \cong A_{120}$.

Now assume that the full automorphism group $A$ is not quasiprimitive on $V \Sigma$. Then there is a minimal normal subgroup $M$ of $A$ that acts nontransitively on $V \Sigma$. Since $M \cap X \leq X$, we have $M \cap X = 1$ or $X$. For the latter case $M \cap X = X$, we have $X \leq M$, and so $M$ is transitive on $V \Sigma$, a contradiction. For the former case, $M \cap X = 1$, then we have that $|M|$ divides $|A|/|X|$, which divides $2^4 \cdot 3^2 \cdot 5^2$.

Assume that $M$ is insoluble. Since $|M|$ divides $2^4 \cdot 3^2 \cdot 5^2$, and the simple groups $A_5, A_6, \text{PSp}(4,3)$ are the only $[2,3,5]$-factor non-abelian simple groups (see [19] (Table 1), and note that the definition of the $[2,3,5]$-group is similar to $[2,3,5]$-group); by checking the orders of these groups, it is easy to figure out $M \cong A_5$ or $A_6$. Then, since $|M| \cdot |A_{120}| = |M| \cdot |X| = |L| = |V \Sigma| \cdot |L_0| = |A_{119}| \cdot |L_0|$, we have $|L_0| = 2^3 \cdot 3^2 \cdot 5^2$ or $2^7 \cdot 3^3 \cdot 5^2$, a contradiction to the description of the orders of the stabilizers in Lemma 3.

Assume that $M$ is soluble. Then $M \cong Z_2^r$ or $Z_3^s$ or $Z_5^l$, where $1 \leq r \leq 4$, $1 \leq s \leq 2$ and $1 \leq l \leq 2$. Let $L = MX$. Then $L = M \cdot X$, a split expansion of $M$ by $X$. Further, we have $L/C_L(M) \leq \text{Aut}(M) \cong GL(r,2)$ or $GL(s,3)$ or $GL(l,5)$. We note that $M$ is a subgroup of $C_L(M)$. If $M = C_L(M)$, then we have $L/C_L(M) = L/M \cong X \cong A_{120} \leq GL(r,2)$ or $GL(s,3)$ or $GL(l,5)$. However, for each $1 \leq r \leq 4$, $1 \leq s \leq 2$ and $1 \leq l \leq 2$, $GL(r,2)$, $GL(s,3)$ or $GL(l,5)$ has no subgroup isomorphic to the alternating group $A_{120}$. Hence, we have $M < C_L(M)$ and $1 \neq C_L(M)/M \leq L/M \cong A_{120}$. It implies that $A_{120} \cong C_L(M)/M$; then $|C_L(M)| = |M| \cdot |X| = |L|$ since $C_L(M) \unlhd L$, we have $C_L(M) = L = MX$, and $X$ centralizes $M$, hence, $L = M \times X$. Then $L_0/L_0 = L_0/L_0 \cap X \cong L_0X/X \cong L/X \cong M$. Thus, $L_0 \cong X_0/M$. Note that with the order of the stabilizers given in Lemma 3, we conclude $M \cong Z_3 \times Z_5$. In the case where $M \cong Z_3$, we have $|L_0| = |X_0| \cdot |M| = 360$, then $L_0 \cong A_6$, $A_6 \not\cong \text{PGL}(2,5), Z_5$, but there is no normal subgroup which is isomorphic to $\text{PGL}(2,5)$ in $A_6$, a contradiction. In the case where $M \cong Z_5$, we have $|L_0| = |X_0| \cdot |M| = 600$, then $L_0 \cong D_{10} \times \text{PSL}(2,5), D_{10} \times \text{PSL}(2,5) \cong \text{PGL}(2,5). Z_5$; by [17], there is no normal subgroup with order $120$ in $D_{10} \times \text{PSL}(2,5)$, so clearly, $\text{PGL}(2,5) \not\cong D_{10} \times \text{PSL}(2,5)$, which also leads to a contradiction. This completes the proof of the lemma.

**Proof of Theorem 1.** Now we are ready to prove our main Theorem 1. Let $\Sigma = \text{Cos}(X, H, x)$ be the graph as in Construction 1. Then, Lemma 4 shows that $G$ is a connected 2-arc-transitive graph and isomorphic to a nonnormal hexavalent Cayley graph $\text{Cay}(G, S)$, with $G \cong A_{119}$. This proves the statement of the former part of Theorem 1. The next Lemma 5 shows that the full automorphism group $\text{Aut}\Sigma$ of the graph $\Sigma$ is isomorphic to alternating group $A_{120}$. This proves the statement of the latter part of Theorem 1, and so completes the proof of Theorem 1. □
Author Contributions: B.L. (Bo Ling): formal analysis, supervision, W.L.: writing—original draft, B.L. (Bengong Lou): writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by the National Natural Science Foundation of China (12061089, 11861076, 11701503, 11761079), and the Natural Science Foundation of Yunnan Province (2019FB139, 2018FB003).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no potential conflict of interest.

Appendix A. Magma Codes Used in Computing the Valency

```magma
val:=function(H,x);
m:=Order(H)/Order(H meet H^x);
return M;
end function;
```

Appendix B. Magma Codes Used in Computing the Elements of $G \cap (H \times H)$

```magma
elt:=function(a,b,x);
X:=Alt(120);
G:=Stabilizer(X,1);
H:=sub<X|a,b>;
M:=[];
for m in H do
    for n in H do
        if 1^(m*x*n) eq 1 then
            if not m*x*n in M then
                Append(~M,m*x*n);
            end if;
        end if;
    end for;
end for;
return M;
end function;
```

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