Calculation of mixed Hodge structures, Gauss-Manin connections and Picard-Fuchs equations

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Abstract

In this article we introduce algorithms which compute iterations of Gauss-Manin connections, Picard-Fuchs equations of Abelian integrals and mixed Hodge structure of affine varieties of dimension $n$ in terms of differential forms. In the case $n = 1$ such computations have many applications in differential equations and counting their limit cycles. For $n > 3$, these computations give us an explicit definition of Hodge cycles.

1 Introduction

The theory of Abelian integrals which arises in polynomial differential equations of the type $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ is one of the most fruitful areas which needs a special attention form algebraic geometry. The reader is referred to the articles [3], [6] and [1] for a history and applications of such Abelian integrals in differential equations. In this article we deal with computational aspects of such integrals. All polynomial objects which we use are defined over $\mathbb{C}$.

Let us be given a polynomial $f$ in $n+1$ variables $x_1, x_2, \ldots, x_{n+1}$, a polynomial differential $n$-form $\omega$ and a continuous family of $n$-dimensional oriented cycles $\delta_t \subset L_t := f^{-1}(t)$. The protagonist of this article is the integral $\int_{\delta_t} \omega$, called Abelian integral. Computations related to these integrals becomes easier when we put a certain kind of tameness condition on $f$ (see §2). For such a tame polynomial we can write $\int_{\delta_t} \omega$ as:

$$\sum_{\beta \in I} p_\beta(t) \int_{\delta_t} \eta_\beta$$

where $\eta_\beta, \beta \in I$ is a class of differential $n$-forms constructed from a basis of the Milnor vector space of $f$ and $p_\beta$’s are polynomials in $t$ (see §4 for the algorithm which produces $p_\beta$’s). The Guass-Manin connection $\nabla_\omega$ has the following basic property

$$\frac{\partial}{\partial t} \int_{\delta_t} \omega = \int_{\delta_t} \nabla_\omega$$

The above term can be written in the form (1) with $p_\beta$’s rational functions in $t$ with poles in the critical values of $f$ (see §5 for the algorithm which produces $p_\beta$’s). The $n$-th cohomology of a smooth fiber $L_t$ is canonically isomorphic to $\Omega^1_{L_t}/\Omega^{n-1}_{L_t}$, where $\Omega^1_{L_t}$ is the restriction of polynomial differential $i$-forms to $L_t$, and carries two natural filtrations called the weight and the Hodge filtrations (both together is called the mixed Hodge structure).
These filtrations are generalizations of classical notions of differential forms of the first, second and third type for Riemann surfaces in higher dimensional varieties. The reader who is not interested in the case $n > 1$ is invited to follow the article with $n = 1$ and with the usual notions of differential forms of the first, second and third type. How to calculate these filtrations by means of differential forms is the main theorem of [5] and related algorithms are explained in §9. Last but not the least, our protagonist satisfies a Picard-Fuchs equation $\sum_{i=0}^{k} p_i(t) \frac{\partial^{\lambda} \phi}{\partial t^{\lambda}} = 0$, where $p_i$’s are polynomials in $t$. The algorithm which produces $p_\beta$’s is explained in §8. The theory of Abelian integrals can be studied even in the case $n = 0$, i.e. $f$ is a polynomial in one variable. Since some open problems, for instance infinitesimal Hilbert Problem (see [3]), can be also stated in this case, we have included §9. All the algorithms explained in this article are implemented in the library brho.lib of SINGULAR and throughout the article we mention the related procedures of brho.lib. Nevertheless, the reader may want to implement the algorithms of this article in any other software in commutative algebra.

The main theorem of [5](see also [7]) does not give a basis of the Brieskorn module compatible with the mixed Hodge structure (see Definition 3). In §10 we obtain such bases for some examples of $f$ by modifying the one given in §7 (we do not have a general method for every $f$). Applications of our computations in differential equations and particularly in direction of the article [1] is a matter of future work.

2 Tame polynomials and Brieskorn modules

We start with a definition.

Definition 1. A polynomial $f \in \mathbb{C}[x]$ is called (weighted) tame if there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{N}$ such that $\text{Sing}(g) = \{0\}$, where $g = f_d$ is the last homogeneous piece of $f$ in the graded algebra $\mathbb{C}[x]$, $\text{deg}(x_i) = \alpha_i$.

The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the following way:

$$\lambda^* : (x_1, x_2, \ldots, x_{n+1}) \rightarrow (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_{n+1}} x_{n+1}), \lambda \in \mathbb{C}^*$$

The polynomial (resp. the polynomial form) $\omega$ in $\mathbb{C}^{n+1}$ is (weighted) homogeneous of degree $d \in \mathbb{N}$ if $\lambda^*(\omega) = \lambda^d \omega$, $\lambda \in \mathbb{C}^*$. Fix a homogeneous polynomial $g$ of degree $d$ and with an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Let $\mathcal{A}_g$ be the affine space of all tame polynomials $f = f_0 + f_1 + \cdots + f_{d-1} + g$. The space $\mathcal{A}_g$ is parameterized by the coefficients of $f_i$, $i = 0, 1, \ldots, d - 1$. The multiplicative group $\mathbb{C}^*$ acts on $\mathcal{A}_g$ by

$$\lambda \bullet f = \frac{f \circ \lambda^*}{\lambda^d} = \lambda^{-d} f_0 + \lambda^{-d+1} f_1 + \cdots + \lambda^{-1} f_d + g$$

The action of $\lambda \in \mathbb{C}^*$ takes $\lambda \bullet f = 0$ biholomorphically to $f = 0$.

Let $f \in \mathcal{A}_g$. We choose a basis $x^I := \{x^\beta \mid \beta \in I\}$ of monomials for the Milnor $\mathbb{C}$-vector space

$$V := \mathbb{C}[x]/\text{jacob}(g)$$

In SINGULAR one can get $x^I$ using kbase command. Using this command $\text{deg}(x^\beta)$ is not a decreasing sequence. In brho.lib the procedure okbase makes a permutation on the result of kbase and gives us $x^\beta$’s with $\text{deg}(x^\beta)$ decreasing. We will fix the order obtained
Each piece of the mixed Hodge structure of $\overline{\mathbb{C}}$ defined in the case the set of critical points of $f$.

Let $\omega := \sum_{i=1}^{n+1} (-1)^{i-1} w_i x_i \, dx_i$, $L_t := f^{-1}(t), t \in \mathbb{C}$.

$$A_{\beta} := \sum_{i=1}^{n+1} (\beta_i + 1) w_i, \quad \eta_{\beta} := x^\beta \eta, \quad \omega_{\beta} = x^\beta \, dx \, \beta \in I$$

where $\widehat{dx_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}$. Note that $A_{\beta} = \deg(x^{\beta+1})$. It turns out that $x^i$ is also a basis of $V_f := \mathbb{C}[x]/\text{Jac}(f)$ and so $f$ and $g$ have the same Milnor numbers (see the conclusion after Lemma 4 of [5]). We denote it by $\mu$. We denote by $P$ the set of critical points of $f$ and by $C := f(P)$ the set of critical values of $f$. We will use also $P$ for a polynomial in $\mathbb{C}[x]$. This will not make any confusion!

Let $\Omega_i, i = 1, 2, \ldots, n+1$ (resp. $\Omega_{ij}, j \in \mathbb{N} \cup \{0\}$) be the set of polynomial differential $i$-forms (resp. homogeneous polynomial differential $i$-forms) in $\mathbb{C}^{n+1}$. The Milnor vector space of $g$ can be rewritten in the form $V := \frac{\Omega^{n+1}}{f^{-1}M^n}$. The Brieskorn modules

$$H' = H'_f := \frac{\Omega^n}{df \wedge \Omega^{n-1} + d\Omega^{n-1}}, \quad H'' = H''_f := \frac{\Omega^{n+1}}{df \wedge d\Omega^{n-1}}$$

of $f$ are $\mathbb{C}[t]$-modules in a natural way: $t.\omega = [f\omega], \quad [\omega] \in H' \quad \text{resp.} \quad [\omega] \in H''$. They are defined in the case $n > 0$. The case $n = 0$ is treated separately in [9].

**Definition 2.** Let $H$ be one of $H'$ or $H''$. If $H = H''$ then by restriction of $\omega$ on $L_c$, $c \in \mathbb{C}\setminus C$ we mean the residue of $\frac{\omega}{f^c}$ in $L_c$ and by $\int_{\delta} \omega$, $\delta \in H_n(L_c, \mathbb{Z})$ we mean $\int_{\delta} \text{Res} \left( \frac{\omega}{f^c} \right)$. It is natural to define the mixed Hodge structure of $H$ as follows: $W_m H$, $m \in \mathbb{Z}$ (resp. $F^k H$, $k \in \mathbb{Z}$) consists of elements $\omega \in H$ such that the restriction of $\omega$ on all $L_c$, $c \in \mathbb{C}\setminus C$ belongs to $W_m H^n(L_c, \mathbb{C})$ (resp. $F^k H^n(L_c, \mathbb{C})$).

In the case of a tame polynomial we have

$$\{0\} = W_{n+1} \subset W_n \subset W_{n-1} = H, \quad \{0\} = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H$$

Each piece of the mixed Hodge structure of $H$ is a $\mathbb{C}[t]$-module. In the same way we define the mixed Hodge structure of the localization of $H$ over multiplicative subgroups of $\mathbb{C}[t]$. In the case $n = 1$ our definition can be simplified as follows: We have the filtrations

$$\{0\} = W_0 \subset W_1 \subset W_2 = H \quad \text{and} \quad 0 = F^2 \subset F^1 \subset F^0 = H,$$

where

$$W_1 = \{ \omega \in H \mid \text{\omega restricted to a regular fiber has no residue at infinity} \}$$

$$F^1 = \{ \omega \in H \mid \text{\omega restricted to a regular fiber has poles of maximum order 1 at infinity} \}$$

In particular we get

$$W_1 \cap F^1 = \{ \omega \in H \mid \text{\omega restricted to a regular comactified fiber is of the first kind} \}$$

For the notion of compactification of $\mathbb{C}^2$ and infinity see [1] and [4].

The projection of $F^\bullet$ in $Gr^W_m H := W_m/W_{m-1}$ gives us the filtration $F^\bullet$ in $Gr^W_m H$ and we define $Gr^k_F Gr^W_m H = F^k/F^{k+1}$.

**Definition 3.** Suppose that $H$ is freely generated module. The set $B = \cup_{m,k \in \mathbb{Z}} B^k_m \subset H$ is a basis of $H$ compatible with the mixed Hodge structure if $B^k_m$ form a basis of $Gr^k F Gr^W_m H$. 

3
3 Quasi-homogeneous singularities

Let \( f = g \) be a weighted homogeneous polynomial with an isolated singularity at origin. We explain the algorithm which writes every element of \( H'' \) of \( g \) as a \( \mathbb{C}[t] \)-linear combination of \( \omega_\beta \)'s. Recall that

\[
dg \wedge d(Pdx, dx_j) = (-1)^{i+j+\epsilon_{i,j}} \left( \frac{\partial g}{\partial x_j} \frac{\partial P}{\partial x_i} - \frac{\partial g}{\partial x_i} \frac{\partial P}{\partial x_j} \right) dx
\]

where \( \epsilon_{i,j} = 0 \) if \( i < j \) and \( = 0 \) if \( i > j \) and \( dx_i, dx_j \) is \( dx \) without \( dx_i \) and \( dx_j \) (we have not changed the order of \( dx_1, dx_2, \ldots \) in \( dx \)).

Proposition 1. For a monomial \( P = x^\beta \) we have

\[
\frac{\partial g}{\partial x_i} Pdx = \frac{d}{dA_\beta - \alpha_i} \frac{\partial P}{\partial x_i} dx + dg \wedge d \left( \sum_{j \neq i} (-1)^{i+j+\epsilon_{i,j}} \alpha_i x_j Pdx_i, dx_j \right)
\]

Proof. The proof is a straightforward calculation.

\[
\sum_{j \neq i} (-1)^{i+j+\epsilon_{i,j}} \alpha_i \frac{d}{dA_\beta - \alpha_i} dg \wedge d(x_j Pdx_i, dx_j)
\]

\[
= \frac{-1}{dA_\beta - \alpha_i} \sum_{j \neq i} (\alpha_j \frac{\partial g}{\partial x_j} \frac{\partial P}{\partial x_i} - \alpha_j \frac{\partial g}{\partial x_i} \frac{\partial P}{\partial x_j})
\]

\[
= \frac{-1}{dA_\beta - \alpha_i} ((dg - \alpha_i x_i P \frac{\partial P}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)))
\]

We use the above Proposition to write every \( Pdx \in \Omega^{n+1} \) in the form

\[
Pdx = \sum_{\beta \in I} p_\beta (g) \omega_\beta + dg \wedge d\xi, \quad p_\beta \in \mathbb{C}[t], \quad \xi \in \Omega^{n-1}, \quad \deg(p_\beta (g) \omega_\beta, dg \wedge d\xi) \leq \deg(Pdx)
\]

\* Input: The homogeneous polynomial \( g \) and \( P \in \mathbb{C}[x] \) representing \([Pdx] \in H''\)

\* Output: \( p_\beta, \beta \in I \) and \( \xi \) satisfying \([5]\)

We write

\[
Pdx = \sum_{\beta \in I} c_\beta x^\beta dx + dg \wedge \eta, \quad \deg(dg \wedge \eta) \leq \deg(Pdx)
\]

Then we apply \([4]\) to each monomial component \( \hat{P} \frac{\partial g}{\partial x_i} \) of \( dg \wedge \eta \) and then we write each \( \frac{\partial \hat{P}}{\partial x_i} dx \) in the form \([6]\). The degree of the components which make \( Pdx \) not to be of the form \([5]\) always decreases and finally we get the desired form.
This algorithm is implemented in the procedure linear1 of the library brho.lib. To find a similar algorithm for $H'$ we note that if $\eta \in \Omega^n$ is written in the form

$$\eta = \sum_{\beta \in I} p_\beta(g) \eta_\beta + dg \wedge \xi + d\xi_1, \quad p_\beta \in \mathbb{C}[t], \xi, \xi_1 \in \Omega^{n-1}$$

where each piece in the right hand side of the above equality has degree less than $\deg(\eta)$ then

$$d\eta = \sum_{\beta \in I} (p_\beta(g) A_\beta + p'_\beta(g) g) \omega_\beta - dg \wedge d\xi$$

and the inverse of the map $\mathbb{C}[t] \to \mathbb{C}[t], \ p \to A_\beta p + p'.t$ is given by $\sum_{i=0}^k a_i t^i \to \sum_{i=1}^k \frac{a_i}{A_\beta + i} t^i$. Therefore, we can recover $p_\beta$'s using linear1 for $d\eta$. The obtained algorithm is implemented in the procedure linear2 of the library brho.lib. Later we will introduce the procedure linear (resp. linearp) which works for arbitrary tame polynomial and is an extended version of linear1 (resp. linear2). For this reason, the procedures linear1, linear2 are not available to the user.

**Theorem 1.** For a weighted homogeneous polynomial $g$, the set

$$B = \bigcup_{k=1}^n B_{n+1}^k \cup \bigcup_{k=0}^n B_n^k$$

with

$$B_{n+1}^k = \{ \omega_\beta \mid A_\beta = n - k + 1 \}, B_n^k = \{ \omega_\beta \mid n - k < A_\beta < n - k + 1 \},$$

is a basis of $H''$ compatible with the mixed Hodge structure. The same is true for $H'$ replacing $\omega_\beta$ with $\eta_\beta$.

This theorem is due to J. Steenbrink [7] and its generalization for an arbitrary tame polynomial is given in [5].

4 A basis of $H'$ and $H''$

**Proposition 2.** For every tame polynomial $f \in A_g$ the forms $\omega_\beta, \beta \in I$ (resp. $\eta_\beta, \beta \in I$) form a basis of the Brieskorn module $H''$ (resp. $H'$) of $f$. More precisely, every $\omega \in \Omega^{n+1}$ (resp. $\omega \in \Omega^n$) can be written

$$\omega = \sum_{\beta \in I} p_\beta(f) \omega_\beta + df \wedge d\xi, \quad p_\beta \in \mathbb{C}[t], \xi \in \Omega^{n-1}, \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta$$

(resp.

$$\omega = \sum_{\beta \in I} p_\beta(f) \eta_\beta + df \wedge \xi + d\xi_1, \quad p_\beta \in \mathbb{C}[t], \xi \in \Omega^{n-1}, \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta$$

)

This Proposition is proved in [5] Proposition 1. The proof also gives us the following algorithm to find all the unknown data in the above equalities.
• Input: The tame polynomial \( f \) and \( P \in \mathbb{C}[x] \) representing \([Pdx] \in H''\).

Output: \( p_{\beta}, \beta \in I \) and \( \xi \) satisfying (9)

We use the algorithm of §3 and write an element \( \omega \in \Omega^{n+1}, \deg(\omega) = m \) in the form

\[
\omega = \sum_{\beta \in I} p_{\beta}(g)\omega_{\beta} + dg \wedge d\psi, \quad p_{\beta} \in \mathbb{C}[t], \quad \psi \in \Omega^{n-1}, \quad \deg(p_{\beta}(g)\omega_{\beta}), \deg(dg \wedge d\psi) \leq m
\]

This is possible because \( g \) is homogeneous. We have

\[
\omega = \sum_{\beta \in I} p_{\beta}(f)\omega_{\beta} + df \wedge d\psi + \omega', \quad \omega' = \sum_{\beta \in I} (p_{\beta}(g) - p_{\beta}(f))\omega_{\beta} + d(g - f) \wedge d\psi
\]

The degree of \( \omega' \) is strictly less than \( m \) and so we repeat what we have done at the beginning and finally we write \( \omega \) as a \( \mathbb{C}[t] \)-linear combination of \( \omega_{\beta} \)'s.

The algorithm for \( H' \) is similar. The statement about degrees is the direct consequence of the proof and [1]. The procedure \texttt{linear} takes a tame polynomial \( f \) and \( P \in \mathbb{C}[x] \) and returns a list. The first entry is a \( 1 \times \mu \) matrix \( C = [p_{\beta}(t)]_{\beta \in I} \) and a \( \mu \times \mu \) matrix representing a \((n - 1)\)-form \( \xi \). The base ring must contain at least one parameter and \( C \) is written in the first parameter. The procedure \texttt{linearp} for \( H' \) is similar to \texttt{linear} for \( H'' \).

### 5 Gauss-Manin connection

Let \( S(t) \in \mathbb{C}[t] \) such that

\[
S(f)dx = df \wedge \eta_f, \quad \eta_f = \sum_{i=1}^{n+1} (-1)^{i-1} p_i \hat{dx}_i \in \Omega^{n-1}
\]

For instance one can take \( S(t) := \text{det}(A_f - t.I) \), where \( A_f \) is the multiplication by \( f \) linear map form the Milnor vector space of \( f \) \( V := \mathbb{C}[x]/\text{Jacob}(f) \) to itself. This definition of \( S \) is implemented in the procedure \texttt{S} of \texttt{brho.lib}. The Gauss-Manin connection associated to the fibration \( f \) on \( H'' \) turns out to be a map

\[
\nabla : H'' \to H''_C, \nabla([Pdx]) = \left[\frac{(Q_P - P.S'(f))dx}{S}\right], \quad P \in \mathbb{C}[x]
\]

where

\[
Q_P = \sum_{i=1}^{n+1} \left( \frac{\partial P}{\partial x_i} p_i + P \frac{\partial p_i}{\partial x_i} \right)
\]

satisfying the Leibniz rule, where for a set \( \hat{C} \subset \mathbb{C} \) by \( H''_C \) we mean the localization of \( H'' \) on the multiplicative subgroup of \( H'' \) generated by \( t - c, \ c \in \hat{C} \). Using the Leibniz rule one can extend \( \nabla \) to a function from \( H''_C \) to itself and so the iteration \( \nabla^k = \nabla \circ \nabla \cdots \nabla \) \( k \) times, makes sense. It is given by

\[
\nabla^k = \frac{\nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0}{S(t)^k}
\]
where

\[ \nabla_k : H'' \to H'', \ \nabla_k((Pdx]) = [(Q \cdot (k+1)S'(t)P)dx] \]

To calculate \( \nabla : H' \to H'_C \), we use the fact that

\[ \nabla_k \omega = \nabla_k - 1 d\omega, \ \omega \in H' \]

where \( d : H' \to H'' \) is taking differential and is well-defined. See §3 of [5] for more details on \( \nabla \). Usually the iteration of the Gauss-Manin connection produces polynomial forms with huge number of monomials. But fortunately our Brieskorn module \( H'' \) (resp. \( H' \)) has already the canonical basis \( \omega_\beta, \beta \in I \) (resp. \( \eta_\beta, \beta \in I \)) and after writing \( \nabla \) the obtained coefficients are much more easier to read. The procedure \texttt{nabla} of \texttt{brho.lib} uses the formulas (11) and (12) and computes \( \nabla \) and its iterations. In \( H'' \) one can write

\[ S(t)\nabla(\omega_\beta) = \sum_{\beta' \in I} p_{\beta,\beta'} \omega_{\beta'}, \ p_{\beta,\beta'} \in \mathbb{C}[t], \ \deg(p_{\beta,\beta'}) \leq \deg(S) - 1 + A_\beta - A_{\beta'} \]

The bound on degrees can be obtained as follows:

\[ S\omega_\beta = df \wedge \eta, \Rightarrow ds + dA_\beta = d + \deg(\eta) \]

This is because \( f \) is tame (see the proof of Lemma 4 of [5]).

\[ \deg(p_{\beta,\beta'}) \leq \frac{\deg(d\eta)}{d} - A_{\beta'} = s - 1 + A_\beta - A_{\beta'} \]

The procedure \texttt{nablamat} in \texttt{brho.lib} calculates the matrix \( \frac{1}{S(t)}[p_{\beta,\beta'}] \). The Gauss-Manin connection \( \nabla \) has two nice properties:

1. Griffiths transversality theorem: For all \( i = 1, 2, \ldots, n+1 \) we have \( S(t)\nabla(F^i) \subset F^{i-1} \).

2. Residue killer: For all \( \omega \in H \) there exists a \( k \in \mathbb{N} \) such that \( \nabla^k \omega \in W_n \)

For the first one see [2]. The second one for \( n = 1 \) is proved in Lemma 2.3 of [4]. The proof for \( n > 1 \) is similar and uses the fact that the residue as a function in \( t \) on a cycle around infinity is a polynomial in \( t \).

6 The numbers \( d_\beta, \beta \in I \)

Let \( f \) be a tame polynomial with the last homogeneous part \( g \), \( F \) be its homogenization and

\[ V = \mathbb{C}[x, x_0]/\langle \frac{\partial F}{\partial x_i} | i = 1, 2, \ldots, n+1 \rangle \]

We consider \( V \) as a \( \mathbb{C}[x_0] \)-module and it is shown in [5] that \( V \) is freely generated by \( x^I := \{x^\beta, \beta \in I\} \). Let

\[ A_F : V \to V, \ A_F(G) = \frac{\partial F}{\partial x_0}G, \ G \in V \]

\textbf{Proposition 3.} The matrix of \( A_F \) in the basis \( x^I \) is of the form \( d_0^K \cdot \alpha_{\beta,\beta'} \), where \( K_{\beta,\beta'} := d - 1 + \deg(x^\beta) - \deg(x^\beta') \) and \( A_f := [\alpha_{\beta,\beta'}] \) is the multiplication by \( f \) in the Milnor vector space of \( f \). In particular, if \( A_{\beta'} - A_\beta \geq 1 \) then \( \alpha_{\beta,\beta'} = 0 \) and

\[ \det(A_F - t.x_0^{d-1}I) = \det(A_f - t.I)x_0^{(d-1)d} \]
Using the above Proposition, the procedure \texttt{mulDF} calculates $A_F$.

**Proof.** Since the polynomial $F$ is weighted homogeneous, we have $\sum_{i=0}^{n+1} \alpha_i x_i \frac{\partial F}{\partial x_i} = d \cdot F$ and so $x_0 \frac{\partial F}{\partial x_0} = d \cdot F$ in $V$ (Note that $\alpha_0 = 1$ by definition). Let

$$
(13) \quad F \cdot x^\beta = \sum_{\beta' \in I} x^{\beta'} c_{\beta', \beta'}(x_0) + \sum_{i=1}^{n+1} \frac{\partial F}{\partial x_i} q_i, \quad c_{\beta', \beta'}(x_0) \in \mathbb{C}[x_0], \quad q_i \in \mathbb{C}[x_0, x]
$$

Since the left hand side is homogeneous of degree $d + \deg(x^\beta)$ we can assume that the pieces of the right hand side are also homogeneous of the same degree. This can be done by taking an arbitrary equation (13) and subtracting the unnecessary parts.

Let $\tilde{C}$ be a finite subset of $\mathbb{C}$ and $\mathbb{C}[t]_{\tilde{C}}$ be the localization of $\mathbb{C}[t]$ on its multiplicative subgroup generated by $t - c, \ c \in \tilde{C}$ and $F_t = F - t \cdot x_0^d$. From now on we work with $\mathbb{C}[t]_{\tilde{C}}[x_0, x]$ instead of $\mathbb{C}[x_0, x]$ and redefine $V$ using $\mathbb{C}[t]_{\tilde{C}}[x_0, x]$. Let

$$
V_{\tilde{C}} = \mathbb{C}[t]_{\tilde{C}}[x_0, x]/ \langle \frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_i} \mid i = 1, 2, \ldots, n + 1 \rangle
$$

It is useful to reformulate $V_{\tilde{C}}$ in the following way: Let $R := \mathbb{C}[t]_{\tilde{C}}[x_0]$ be the set of polynomials in $x_0$ with coefficients in $\mathbb{C}[t]_{\tilde{C}}$ and $A_t = A_F - t \cdot d \cdot x_0^{d-1} I$. We have

$$
V_{\tilde{C}} = V/ \langle \frac{\partial F}{\partial x_0} q \mid q \in V \rangle = R^\mu / A_1 R^\mu
$$

Here $R^\mu$ is the set of $\mu \times 1$ matrices with entries in $R$. We consider the statement:

*(\tilde{C}): There is a function $\beta \in I \rightarrow d_\beta \in \mathbb{N} \cup \{0\}$ such that the $\mathbb{C}[t]_{\tilde{C}}$-module $V_{\tilde{C}}$ is freely generated by

$$
(14) \quad \{x_0^{\beta_0} x^\beta, 0 \leq \beta_0 \leq d_\beta - 1, \beta \in I\}
$$

To prove the statement *(\tilde{C}) we may introduce a kind of Gaussian elimination in $A_t$ and simplify it. For this reason we introduce the operation $GE(\beta_1, \beta_2, \beta_3)$. For $\beta \in I$ let $(A_t)_\beta$ be the $\beta$-th row of $A_t$.

- **Input:** $A_t, \beta_1, \beta_2, \beta_3 \in I$ with $A_{\beta_1} \leq A_{\beta_2}$.

- **Output:** a matrix $A_t'$ and a finite subset $B$ of $\mathbb{C}$.

We replace $(A_t)_\beta$ with

$$
-(A_{\beta_2})_{\beta_3} \cdot (A_t)_{\beta_1} + (A_t)_{\beta_2}
$$

and we set $B = \text{zero}(c(t))$, where $(A_t)_{\beta_1, \beta_2} = c(t) \cdot x_0^{K_{\beta_1, \beta_2}}$. Since for all $\beta_4 \in I$ we have

$$
K_{\beta_2, \beta_3} + K_{\beta_1, \beta_4} = K_{\beta_1, \beta_3} + K_{\beta_2, \beta_4}
$$

The obtained matrix $A_t'$ is of the form $[x_0^{K_{\beta_1, \beta_2}} c_{\beta, \beta}]$ and $c_{\beta_2, \beta_3} = 0$. If the matrix $B_t$ is obtained from $A_t$ by applying the above operation and $B \subset \tilde{C}$ then $A_t R^\mu = B_t R^\mu$.

We give an example of algorithm which calculates $d_\beta$'s for some finite set $\tilde{C} \subset \mathbb{C}$:
Proof. We have
\[ V' = R^n/A_t R^n \cong A_t^{-1} R^n / R^n = \frac{A_t^{\text{adj}} R^n}{x_0^{\mu(d-1)} / R^n} \]

The isomorphism is obtained by acting \( A_t^{-1} \) from left on \( R^n \) and \( \text{adj} \) makes the adjoint of a matrix. Now for \( \beta \in I \) let \( d_\beta \) be the pole order of \( \beta \)-th arrow of \( \frac{A_t^{\text{adj}} x_0^{\mu(d-1)}}{R^n} \).

The numbers \( d_\beta \) are the desired numbers. It is easy to see that \( \{ x_0^{\beta_0} x_0^\beta, 0 \leq \beta_0 \leq d_\beta, \beta \in I \} \) generates \( V' \).

Proposition 4. There is a function \( \beta \in I \to d_\beta \in \mathbb{N} \cup \{0\} \) such that the \( \mathbb{C}[t]_C \)-module \( V' \) is generated by \( \{ x_0^{\beta_0} x_0^\beta, 0 \leq \beta_0 \leq d_\beta, \beta \in I \} \).

Proof. We have
\[ V' = R^n/A_t R^n \cong A_t^{-1} R^n / R^n = \frac{A_t^{\text{adj}} R^n}{x_0^{\mu(d-1)} / R^n} \]

The isomorphism is obtained by acting \( A_t^{-1} \) from left on \( R^n \) and \( \text{adj} \) makes the adjoint of a matrix. Now for \( \beta \in I \) let \( d_\beta \) be the pole order of \( \beta \)-th arrow of \( \frac{A_t^{\text{adj}} x_0^{\mu(d-1)}}{R^n} \).

The numbers \( d_\beta \) are the desired numbers. It is easy to see that \( \{ x_0^{\beta_0} x_0^\beta, 0 \leq \beta_0 \leq d_\beta, \beta \in I \} \) generates \( V' \).

Proposition 5. There is a subset \( \tilde{C} \subset C \) such that the statement \( \ast(\tilde{C}) \) is true with \( d_\beta = d - 1, \beta \in I \).

Proof. We identify \( I \) with \( \{1, 2, \ldots, \mu\} \) and assume that
\[ \beta_1 \leq \beta_2 \Rightarrow A_{\beta_1} \geq A_{\beta_2} \]

By various use of operation \( GE \) on \( A_t \) we make all the entries of \( (A_t)_{\beta, \mu} = 0, \beta \in I \setminus \{\mu\} \). We repeat this for \( (A_t)_{\beta, \mu-1} = 0, \beta \in I \setminus \{\mu, \mu - 1\} \) and after \( \mu \)-times we get a lower triangular matrix. We always divide on a polynomial on \( t \) with leading coefficient one and so division by zero does not occur.

Proposition 6. Let \( \ast(\tilde{C}) \) is valid with \( d_\beta, \beta \in I \). Then
\[ A_\beta < n + 1, \quad d_\beta < d(n + 2 - A_\beta), \quad \sum_{\beta \in I} d_\beta = \mu(d-1) \]

Proof. The first one is already in Steenbrink’s Theorem. The second inequality is obtained by applying the first inequality associated to \( F - cx_0^d \) for some \( c \in \mathbb{C} \setminus \tilde{C} \):
\[ A_{(d_\beta-1, \beta)} = A_\beta + \frac{d_\beta - 1 + 1}{d} < n + 2 \]

The Milnor number of \( F - cx_0^d \) is \( \sum_{\beta \in I} d_\beta \) and equals to the Milnor number of \( g - cx_0^d \) which is \( \mu(d-1) \).
7 Main theorem of [5]

Suppose that $\ast(C)$ is valid with $d_\beta$, $\beta \in I$. Define
\[
I_{n+1}^k = \{ \beta \in I \mid A_\beta = n + 1 - k \}, \quad I_n^k = \{ \beta \in I \mid A_\beta + \frac{1}{d} \leq n + 1 - k \leq A_\beta + \frac{d_\beta}{d} \}
\]

**Theorem 2.** For a tame polynomial $f$, the set
\[
B = \cup_{k=1}^n B_{n+1}^k \cup \cup_{k=0}^n B_n^k
\]
with
\[
B_{n+1}^k = \{ \nabla^{n-k} \omega_\beta \mid \beta \in I_{n+1}^k \}, \quad B_k = \{ \nabla^{n-k} \omega_\beta \mid \beta \in I_n^k \},
\]
is a basis of $H''_C$ compatible with the mixed Hodge structure. The same is true for $H'_C$ replacing $\nabla^{n-k} \omega_\beta$ with $\nabla^{n+1-k} \eta_\beta$.

Unfortunately, this theorem gives us a basis of a localization $H$ compatible with mixed Hodge structure. In §10 we have computed such bases for the Brieskorn module itself.

To handle easier the pieces of the mixed Hodge structure of $H_C$ we make the following table.

| $n$  | $n+1$ |
|------|-------|
| $I_0$  | $I_1$  |
| $I_1$  | $I_2$  | $I_3$ |

The procedure `Imk` of `brho.lib` gives us $x^\beta$, $\beta \in I_m^n$, $m = n, n+1, k = 0, 1, \ldots n$ with the order $I_n^0, I_n^{n-1}, \ldots, I_n^0, I_{n+1}^n, I_{n+1}^{n-1}, \ldots, I_{n+1}^0$. In the case $n = 1$ we have the table

| $I_1$ |
|-------|
| $0$  |
| $1$  |
| $2$  |
| $I_1^1$  |

\[
I_1^1 = \{ \beta \in I \mid A_\beta + \frac{1}{d} \leq 1 \leq A_\beta + \frac{d_\beta}{d} \}, \quad I_1^0 = \{ \beta \in I \mid A_\beta + \frac{1}{d} \leq 2 \leq A_\beta + \frac{d_\beta}{d} \}
\]

\[
I_1^1 = \{ \beta \in I \mid A_\beta = 1 \}
\]

The forms $\omega_\beta$, $\beta \in I_1^1$ form a basis of $F^1 \cap W_1$ and the forms $\omega_\beta$, $\beta \in I_1^0$ form a basis of $H''/W_1$. Now to obtain a basis of $W_1/(F^1 \cap W_1)$ we must modify $\nabla \omega_\beta$, $\beta \in I_1^0$.

The procedure `changebase` calculates the matrix of the basis of the Brieskorn module $H''_C$ obtained in Theorem 7 in the canonical basis $\omega_\beta$, $\beta \in I$.

8 Picard-Fuchs equations

It is a well-known fact that for a polynomial $f \in \mathbb{C}[x]$ and $\omega \in H$ the integral $I(t) := \int_{t_0}^t \omega$ satisfies
\[
(\sum_{i=0}^k p_i(t) \frac{\partial^i}{\partial t^i})I_t = 0, \quad p_i(t) \in \mathbb{C}[t]
\]

called Picard-Fuchs equation, where $\delta_t \in H_n(L_t, \mathbb{Z})$ is a continuous family of topological cycles. When $f$ is tame, it is possible to calculate $p_i$ as follows:

We write
\[
\nabla^i(\omega) = \sum_{\beta \in I} p_{i,\beta} \omega_\beta
\]
and define the $k \times \mu$ matrix $A = [p_{i, \beta}]$, where $i$ runs through $1, 2, \ldots, k$ and $\beta \in I$. Let $k$ be the smallest number such that the the rows of $A_{k-1}$ are $\mathbb{C}(t)$-linear independent. Now, the rows of $A_k$ are $\mathbb{C}(t)$-linear dependent and this gives us (after multiplication by a suitable element of $\mathbb{C}[t]$)

$$\sum_{i=0}^{k} p_i(t) \nabla^i(\omega) = 0, \ p_i(t) \in \mathbb{C}[t]$$

Using the formula (2) and integrating the above equality, we get the equation (15). The procedure `PFeq` from the library `brho.lib` calculates $p_i$’s in (15).

9 Polynomials in one variable, $n = 0$

The theory developed in \S2 does not work for the case $n = 0$. For a polynomial of degree $d$ in one variable $\dim(H^0(L_t, \mathbb{C})) = d$ but $\mu = d - 1$. However, if we use the following definition of homology and cohomology for a discrete topological space $M$,

$$H_0(M, \mathbb{Z}) = \{m = \sum_i a_i m_i \mid a_i \in \mathbb{Z}, \ m_i \in M \mid \deg(m) = \sum_i a_i = 0\}$$

$$H^0(M, \mathbb{C}) = \{f : H_0(M, \mathbb{Z}) \to \mathbb{C} \text{ linear} \}/\{f \mid f \text{ is constant on } M\}$$

then

$$H' = \mathbb{C}[x]/\mathbb{C}[f], \ H'' = \mathbb{C}[x]dx/f'\mathbb{C}[f]dx, \ I = \{1, x, x^2, \ldots, x^{d-2}\}, \ \mu = d - 1$$

In this case

$$\int_{\delta} \omega = \sum_{i} a_i \omega(p_i), \ \text{where } \delta = \sum_{i} a_i p_i, \ a_i \in \mathbb{Z}, \ p_i \in f^{-1}(t), \ \omega \in H'$$

If, for instance, $f' = 0$ has $d$ distinct root then every vanishing cycle in $L_t$ is a difference of two points of $L_t$. The set $B = \{x, x^2, \ldots, x^{d-1}\}$ form a basis of $H'$ and its $\nabla$ which is $\{dx, xdx, \ldots, x^{d-2}dx\}$ (up to multiplication by some constants) form a basis of $H''$. The first fact is easy to see. We write $f = a_dx^d + f_0$ and for a polynomial $p(x) \in \mathbb{C}[x]$ whenever we find some $x^d$ we replace it with $f - f_0/a_d$ and at the end we get $p(x) = p_0(f) + \sum_{i=1}^{d-1} p_i(t)x^i$ or equivalently $p = \sum_{i=1}^{d-1} p_i(t)x^i$ in $H'$. There is no $\mathbb{C}[t]$-linear relation between the elements of $B$ because $B$ restricted to each regular fiber is of dimension $d$. We write

$$p(x)dx = \sum_{i=0}^{d-2} q_i(f)x^i dx + q_{d-1}(f)x^{d-1}dx = \sum_{i=0}^{d-2} q_i(f)x^i dx - \frac{q_{d-1}(f)f'}{d.a_d} dx + \frac{q_{d-1}(f)}{d.a_d} dx$$

and this proves the statement for $H''$.

The proposition (11) can be stated in the case $n = 0$ as follows: The only case in which $dA_\beta - \alpha_i = 0$ is when $n = 0$ and $P = 1$. In the case $n = 0$ for $P \neq 1$ we have

$$\frac{\partial g}{\partial x_i} Pdx = \frac{d}{d.A_\beta - \alpha_i} \frac{\partial P}{\partial x_i} gdx$$
and if \( P = 1 \) then \( \frac{\partial}{\partial P} Pdx \) is zero in \( H'' \). The argument in (7) and (8) can be done also in the case \( n = 0 \). In this case if

\[
\eta = \sum_{\beta \in I} p_{\beta}(g)\eta_{\beta} + p(g), \quad p, p_{\beta} \in \mathbb{C}[t]
\]

where each piece in the right hand side of the above equality has degree less than \( \deg(\eta) \) then

\[
d\eta = \sum_{\beta \in I} (p_{\beta}(g)A_{\beta} + p'_{\beta}(g)g)\omega_{\beta} + p'(g) dg
\]

Based on this observation, the procedure \texttt{linear}, \texttt{linearp}, works for the case \( n = 0 \).

In the case \( n = 0 \), we have only the set \( I_0^0 = \{ A_{\beta} + 1 \leq 1 \leq A_{\beta} + \frac{d}{d} \} \) and this is equal to \( I \). We have \( d_{\beta} < d \). (\( n + 2 - A_{\beta} = 2d - \beta - 1 = \) and \( A_{\beta} = \frac{\beta + 1}{d} \)). We conclude that

\[
d \leq d_{\beta} + \beta + 1 < 2d
\]

Now the infinitesimal Hilbert problem (see [3] Problem 7) can be stated in the case \( n = 0 \). Can one give an effective solution to this problem in this case? The positive answer to this question may give light into the the problem in the case \( n = 1 \).

### 10 Examples

For all the examples belowe we run

```
> LIB "brho.lib";
> LIB "matrix.lib";
```

#### 10.1 Examples, \( n = 0 \)

**For examples of this section we run**

```
>ring r0=(0,t),x, dp;
```

**Example 1.** \( f = x^5 - 5x, \ P = \{ e^i | i = 0, 1, 2, 3 \} \), \( C = \{ -4e^i | i = 0, 1, 2, 3 \} \), where \( x = e^{2\pi i d} \) is the \( d \)-th root of unity.

```
> int d=5; poly f=x^d-d*x; okbase(std(jacob(f)));
```

\[
\begin{array}{c}
[1] = x^3 \\
[2] = x^2 \\
[3] = x \\
[4] = 1
\end{array}
\]

> Abeta(f);

\[
\begin{array}{c}
[1,1] = 4/5 \\
[2,1] = 3/5 \\
[3,1] = 2/5 \\
[4,1] = 1/5
\end{array}
\]

```
> poly Sf=S(f); Sf;
```

\[
(t^4-256)
\]

```
> list l=nablamat(f,Sf);
```

```
1/(5t^4-1280) (-t^3), 128, (-4t^2), (16t), (-3t^2), (8t), (-4t^3), (8t^2), (-2t^4)
```

//This is the matrix of nabla in the canonical basis \( x^3, x^2, x^1, 1 \).

```
> dbeta(f,par(1));
```

\[\begin{array}{c}
0, 2, 2, 4
\end{array}\]

```
> Imk(f,par(1));
```

```
[1]:
```

The residues of \( \frac{\partial}{\partial t} \) at its poles satisfy the Picard-Fuchs equation

\[
6144 + 35625t \frac{\partial}{\partial t} + 33375t^2 \frac{\partial^2}{\partial t^2} + 8750t^3 \frac{\partial^3}{\partial t^3} + (625t^4 - 160000) \frac{\partial^4}{\partial t^4} = 0
\]

#### 10.2 Examples \( n = 1 \)

**For the examples belowe we define**

```
>ring r1=(0,t), (x,y), dp;
```

**Example 2.** \( f = xy(x + y - 1) \).

```
> poly f= x2y+xy2-xy ;
> poly g=lasthomo(f); g;
```

```
x2y+xy2
```

```
> okbase(std(jacob(g)));
```

```
[1] = y^2 \\
[2] = y \\
[3] = x \\
[4] = 1
```

```
> print(muldF(f-par(1)));
```

\[
(-3t+1/18)*x^2,-1/18*x^3, 0, 0,
\]

```
1/6*x, (-3t-1/6)*x^2,0, 0,
```

```
1/6*x, -1/6*x^2, (-3t)*x^2,0,
```

```
1/2, -1/2*x, 0, (-3t)*x^2
```

//We can take \( Sf = (t+1/27) \);

```
> poly Sf=S(f); Sf;
```

```
(t^2+1/27)
```

We take \( Sf = (t+1/27) \);

```
> list l=nablamat(f,Sf);
```

```
[1]:
```

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Example 4. \( f = x^4 + y^4 - x \).

\[
\begin{align*}
&\quad f = 2(x^3 + y^3) - 3(x^2 + y^2) \\
&\quad \text{Gr}_y \text{Gr}_x H'' = [1, x, y] \\
&\quad \text{Gr}_y \text{Gr}_x H'' = [x^2 - x - y] \\
&\quad \text{Gr}_y \text{Gr}_x H'' = [x^4, [x], [y], [y]]
\end{align*}
\]

\[
\begin{align*}
&\quad \text{Gr}_y \text{Gr}_x H'' = [x^2 - x - y] \\
&\quad \text{Gr}_y \text{Gr}_x H'' = [x^4, [x], [y], [y]]
\end{align*}
\]

We make the following remark

\[
\text{reduce}(9x^2y - 16*G(f^2)*y, \text{std}(\text{jacob}(f)));
\]
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