Transverse conformal Killing forms on Kähler foliations

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Abstract. On a closed, connected Riemannian manifold with a Kähler foliation of codimension $q = 2m$, any transverse Killing $r$-form is parallel (Jung and Jung, 2012). In this paper, we study transverse conformal Killing forms on Kähler foliations. In fact, if the foliation is minimal, then for any transverse conformal Killing $r$-form $\phi$ ($r \neq m, \ 2 \leq r \leq q - 2$), $J\phi$ is parallel. Here $J$ is defined in section 4.

1 Introduction

On Riemannian manifolds, conformal Killing forms are generalizations of conformal Killing fields, which were introduced by K. Yano [20] and T. Kashiwada [10,11]. Many researchers have studied the conformal Killing forms [13,16,17,18]. On a foliated Riemannian manifold, we can study the analogous problems. Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a compact oriented Riemannian manifold $M$ with codimension $q$. A transversal conformal Killing field is a normal field with a flow preserving the conformal class of the transverse metric. As a generalization of a transversal conformal Killing field, we define the transverse conformal Killing $r$-forms $\phi$ as follows: for any vector field $X$ normal to the foliation,

$$\nabla_X \phi = \frac{1}{r+1} i(X)d\phi + \frac{1}{q-r+1} X^b \wedge \delta_T \phi = 0,$$

where $r$ is the degree of the form $\phi$ and $X^b$ is the dual 1-form of $X$. For the definition of $\delta_T$, see Section 3. The transverse conformal Killing forms $\phi$ with $\delta_T \phi = 0$ are called transverse Killing forms. Recently, S. D. Jung and K. Richardson [6] studied the transverse Killing and conformal Killing forms on Riemannian foliations. And S. D. Jung and M. J. Jung [4] studied some properties of the transverse

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Killing forms on Kähler foliations. That is, on a closed, connected Riemannian manifold with a Kähler foliation of codimension $q = 2m$, any transverse Killing $r(\geq 2)$-form is parallel. In this paper, we study the transverse conformal Killing forms on Kähler foliations. In section 2, we review the basic facts on a Riemannian foliation. In section 3, we study the transverse conformal Killing forms and curvature properties on Riemannian foliations. In section 4, we study the curvatures and several operators on Kähler foliations. In section 5, we prove the following: on a Kähler foliation with $q = 2m$, if $\phi$ is a transverse conformal Killing $m$-form, then $J\phi$ is parallel. In particular, when $(\mathcal{F}, J)$ is minimal, for any transverse conformal Killing $r$ ($2 \leq r \leq q - 2$)-forms $\phi$, $J\phi$ is also parallel. Here $J$ is an extension of the complex structure to the basic forms.

2 Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p + q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Then there exists an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \overset{\pi}{\longrightarrow} Q \longrightarrow 0,$$

(2.1)

where $L$ is the tangent bundle and $Q = TM/L$ is the normal bundle of $\mathcal{F}$. The metric $g_M$ determines an orthogonal decomposition $TM = L \oplus L^\perp$, identifying $Q$ with $L^\perp$ and inducing a metric $g_Q$ on $Q$. Let $\nabla$ be the transverse Levi-Civita connection on $Q$, which is torsion-free and metric with respect to $g_Q$ [7]. Let $R^\nabla, K^\nabla, \rho^\nabla$ and $\sigma^\nabla$ be the transversal curvature tensor, transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to $\nabla$, respectively. Let $\Omega_B^*(\mathcal{F})$ be the space of all basic forms on $M$, i.e.,

$$\Omega_B^*(\mathcal{F}) = \{ \phi \in \Omega^*(M) \mid i(X)\phi = 0, \ i(X)d\phi = 0, \ \forall X \in \Gamma L\}. \quad (2.2)$$

Then $L^2\Omega^*(M)$ is decomposed as [1]

$$L^2\Omega(M) = L^2\Omega_B(\mathcal{F}) \oplus L^2\Omega_B(\mathcal{F})^\perp. \quad (2.3)$$

Now we define the connection $\nabla$ on $\Omega_B^*(\mathcal{F})$, which is induced from the connection $\nabla$ on $Q$ and Riemannian connection $\nabla^M$ of $g_M$. This connection $\nabla$ extends the partial Bott connection $\tilde{\nabla}$ given by $\tilde{\nabla}_X \phi = \theta(X)\phi$ for any $X \in \Gamma L$ [9], where $\theta(X)$ is the transversal Lie derivative. Then the basic forms are characterized by $\Omega_B^*(\mathcal{F}) = \text{Ker} \tilde{\nabla} \subset \Gamma(\wedge Q^*(\mathcal{F}))$. By a direct calculation, we have the following lemma.
Lemma 2.1 Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric $g_M$. Then for any $X, Y, Z \in \Gamma Q$,

$$[R^\nabla (X, Y), i(Z)] = i(R^\nabla (X, Y)Z).$$

The exterior differential $d$ on the de Rham complex $\Omega^*(M)$ restricts a differential $d_B : \Omega^r_B(\mathcal{F}) \to \Omega^{r+1}_B(\mathcal{F})$. Let $\kappa \in Q^*$ be the mean curvature form of $\mathcal{F}$. Then it is well known that the basic part $\kappa_B$ of $\kappa$ is closed \cite{1}. We now recall the star operator $\ast : \Omega^r(M) \to \Omega^{n-r}(M)$ given by \cite{15, 19}

$$\ast \phi = (-1)^{p(q-r)}(\phi \wedge \chi_{\mathcal{F}}), \quad \forall \phi \in \Omega^r(M),$$

(2.4)

where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator associated to $g_M$. The operator $\ast$ maps basic forms to basic forms. For any $\phi, \psi \in \Omega^2_B(\mathcal{F})$, $\phi \wedge \ast \psi = \psi \wedge \ast \phi$ and also $\ast^2 \phi = (-1)^{r(q-r)}\phi$ \cite{15}. Let $\nu$ be the transversal volume form, i.e., $\ast \nu = \chi_{\mathcal{F}}$. The pointwise inner product $\langle \ , \ \rangle$ on $\Lambda^r Q^*$ is defined uniquely by

$$\langle \phi, \psi \rangle_\nu = \phi \wedge \ast \psi.$$

(2.5)

The global inner product $(\cdot, \cdot)_B$ on $L^2 \Omega^r_B(\mathcal{F})$ is defined by

$$(\phi, \psi)_B = \int_M \langle \phi, \psi \rangle \mu_M, \quad \forall \phi, \psi \in \Omega^r_B(\mathcal{F}),$$

(2.6)

where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form with respect to $g_M$. With respect to this scalar product, the formal adjoint $\delta_B : \Omega^r_B(\mathcal{F}) \to \Omega^{r-1}_B(\mathcal{F})$ of $d_B$ is given by \cite{15}

$$\delta_B \phi = (-1)^{q(r+1)+1} \ast d_B \ast \phi = \delta_T \phi + i(\kappa_B^2) \phi,$$

(2.7)

where $d_T = d - \kappa_B \wedge$ and $\delta_T = (-1)^{q(r+1)+1} \ast d_B \ast$ is the formal adjoint operator of $d_T$. Here $\langle \cdot, \cdot \rangle$ is a $g_Q$-dual vector to $(\cdot, \cdot)$. The basic Laplacian $\Delta_B$ is given by $\Delta_B = d_B \delta_B + \delta_B d_B$. Let $\{E_a\} (a = 1, \cdots, q)$ be a local orthonormal basic frame on $Q$. We define $\nabla^*_\nabla^* : \Omega^r_B(\mathcal{F}) \to \Omega^r_B(\mathcal{F})$ by

$$\nabla^*_\nabla^* \phi = \sum_a \nabla^2_{E_a, E_a} \phi + \nabla_{\kappa_B} \phi, \quad \phi \in \Omega^r_B(\mathcal{F}),$$

(2.8)

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for any $X, Y \in TM$. Then the operator $\nabla^*\nabla^*$ is positive definite and formally self adjoint on the space of basic forms \cite{2}. We define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in TM$ \cite{8} by

$$A_Y \phi = \theta(Y) \phi - \nabla_Y \phi.$$

(2.9)
For any $X \in \Gamma L$, $\theta(X)\phi = \nabla_X \phi$ \textsuperscript{9} and so $A_X\phi = 0$. Now we define the curvature endomorphism $F : \Omega^r_B(F) \to \Omega^r_B(F)$ by

$$F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b)R^\nabla(E_b,E_a)\phi,$$

where $\theta^a$ is a $g_Q$-dual 1-form to $E_a$. Then we have the generalized Weitzenböck formula.

**Theorem 2.2** \textsuperscript{3} On a Riemannian foliation $\mathcal{F}$, we have that for any $\phi \in \Omega^r_B(\mathcal{F})$,

$$\Delta_B \phi = \nabla^*_u \nabla^*_x \phi + F(\phi) + A_{\kappa_B}^r \phi.$$  

In particular, if $\phi$ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

**Corollary 2.3** On a Riemannian foliation $\mathcal{F}$, we have that for any $\phi \in \Omega^r_B(\mathcal{F})$,

$$\frac{1}{2} \Delta_B |\phi|^2 = \langle \Delta_B \phi, \phi \rangle - |\nabla^*_x \phi|^2 - \langle F(\phi), \phi \rangle - \langle A_{\kappa_B}^r \phi, \phi \rangle.$$

Now, we recall the following generalized maximum principle.

**Theorem 2.4** \textsuperscript{12} Let $\mathcal{F}$ be a Riemannian foliation on a closed, connected Riemannian manifold $(M,g_M)$. If $(\Delta_B - \kappa_B^r)f \geq 0$ (or $\leq 0$) for any basic function $f$, then $f$ is constant.

### 3 The transverse conformal Killing forms

Let $(M,g_M,\mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$.

**Definition 3.1** A basic $r$-form $\phi \in \Omega^r_B(\mathcal{F})$ is called a transverse conformal Killing $r$-form if for any vector field $X \in \Gamma Q$,

$$\nabla_X \phi = \frac{1}{r+1} i(X)d_B\phi - \frac{1}{r^* + 1} X^b \wedge \delta_T \phi,$$

where $r^* = q - r$ and $X^b$ is the $g_Q$-dual 1-form of $X$. In addition, if the basic $r$-form $\phi$ satisfies $\delta_T \phi = 0$, it is called a transverse Killing $r$-form.
Note that a transverse conformal Killing 1-form (resp. transverse Killing 1-form) is a $g_Q$-dual form of a transversal conformal Killing field (resp. transversal Killing field).

**Proposition 3.2**  
Let $\phi$ be a transverse conformal Killing $r$-form. Then

\[
F(\phi) = \frac{r}{r+1} \delta_T d_B \phi + \frac{r^*}{r^*+1} d_B \delta_T \phi, \tag{3.1}
\]

\[
\nabla^*_u \nabla_u \phi = \frac{1}{r+1} \delta_B d_B \phi + \frac{1}{r^*+1} d_T \delta_T \phi. \tag{3.2}
\]

**Lemma 3.3**  
Let $\phi$ be a transverse conformal Killing $r$-form. Then

\[
\nabla_X \nabla_Y \phi = \frac{1}{r+1} \{ i(\nabla_X Y) d_B \phi + i(Y) \nabla_X d_B \phi \}
\]

\[
- \frac{1}{r^*+1} \{ \nabla_X Y^b \wedge \delta_T \phi + Y^b \wedge \nabla_X \delta_T \phi \}
\]

for any $X, Y \in \Gamma Q$.

We define the operators $R^\nabla_\pm(X) : \wedge^r Q^* \to \wedge^{r \pm 1} Q^*$ for any $X \in TM$ by

\[
R^\nabla_+(X) \phi = \sum_a \theta^a \wedge R^\nabla(X, E_a) \phi, \tag{3.3}
\]

\[
R^\nabla_-(X) \phi = \sum_a i(E_a) R^\nabla(X, E_a) \phi. \tag{3.4}
\]

Then we have the following lemma.

**Lemma 3.4**  
Let $\phi$ be a transverse conformal Killing $r$-form. Then for all $X \in \Gamma Q$,

\[
\nabla_X d_B \phi = \frac{r+1}{r} \{ R^\nabla_+(X) \phi + \frac{1}{r^*+1} X^b \wedge d_B \delta_T \phi \}, \tag{3.5}
\]

\[
\nabla_X \delta_T \phi = - \frac{r^*+1}{r^*} \{ R^\nabla(X) \phi + \frac{1}{r+1} i(X) \delta_T d_B \phi \}. \tag{3.6}
\]

**Proof.** Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$. Since $\sum_a \theta^a \wedge i(E_a) \phi = r \phi$ for any $\phi \in \Omega_B^r(F)$, from Lemma 3.3

\[
R^\nabla_+(X) \phi = \frac{r}{r+1} \nabla_X d_B \phi - \frac{1}{r^*+1} X^b \wedge d_B \delta_T \phi,
\]

which proves (3.5). The proof of (3.6) is similar. \hfill $\Box$
Proposition 3.5 Let $\phi$ be a transverse conformal Killing $r$-form. Then for any $X,Y \in \Gamma Q$,

$$R^\nabla(X,Y)\phi$$

$$= \frac{1}{rr^*}\left(Y^b \wedge i(X) - X^b \wedge i(Y)\right)F(\phi)$$

$$+ \frac{1}{r}\left(i(Y)R^\nabla_+(X) - i(X)R^\nabla_+(Y)\right)\phi + \frac{1}{r^*}\left(Y^b \wedge R^\nabla(X) - X^b \wedge R^\nabla(Y)\right)\phi.$$

Proof. Let $\phi$ be the transverse conformal Killing $r$-form. From Lemma 3.3,

$$R^\nabla(X,Y)\phi = \frac{1}{r+1}\{i(Y)\nabla_Xd_B\phi - i(X)\nabla_Yd_B\phi\} - \frac{1}{r^*+1}\{Y^b \wedge \nabla_X\delta_T\phi - X^b \wedge \nabla_Y\delta_T\phi\}.$$

From Lemma 3.4, we have

$$R^\nabla(X,Y)\phi = \frac{1}{r}\{i(Y)R^\nabla_+(X) - i(X)R^\nabla_+(Y)\}\phi$$

$$- \left(X^b \wedge i(Y) - Y^b \wedge i(X)\right)\left\{\frac{1}{r(r^* + 1)}d_B\delta_T\phi + \frac{1}{r^*+1}\delta_Td_B\phi\right\}.$$

Hence the proof follows from (3.1). □

Lemma 3.6 Let $\phi$ be a transverse conformal Killing $r$-form. Then

$$\sum_a i(E_a)R^\nabla_+(E_a)\phi = \sum_a \theta^a \wedge R^\nabla_+(E_a)\phi = 0.$$

Proof. Since $\phi$ is a transverse conformal Killing $r$-form, from Proposition 3.5,

$$\sum_a i(E_a)R^\nabla_+(E_a)\phi = \frac{2}{r^*}\sum_{a,b} i(E_a)i(E_b)\{\theta^b \wedge R^\nabla_+(E_a)\phi\}$$

$$= \frac{2(r+1)}{r^*}\sum_a i(E_a)R^\nabla_+(E_a)\phi,$$

which means that $\sum_a i(E_a)R^\nabla_+(E_a)\phi = 0$. Similarly, we have

$$\sum_a \theta^a \wedge R^\nabla_+(E_a)\phi = \frac{2}{r}\sum_{a,b} \theta^a \wedge \theta^b \wedge \{i(E_b)R^\nabla_+(E_a)\phi\}$$

$$= \frac{2(r+1)}{r}\sum_a \theta^a \wedge R^\nabla_+(E_a)\phi,$$
which proves the second equality. □

4 Curvatures on a Kähler foliation

Let \((M, g_M, J, \mathcal{F})\) be a compact Riemannian manifold with a Kähler foliation \(\mathcal{F}\) of codimension \(q = 2m\) and a bundle-like metric \(g_M\) [14]. Namely, there is a holonomy invariant almost complex structure \(J : Q \rightarrow Q\) with respect to which \(g_Q\) is Hermitian, i.e., \(g_Q(JX, JY) = g_Q(X, Y)\) for \(X, Y \in Q\) and \(\nabla J = 0\). Note that for any \(X, Y \in \Gamma Q\),

\[
\Omega(X, Y) = g_Q(X, JY) \tag{4.1}
\]
defines a basic 2-form \(\Omega\), which is closed as consequence of \(\nabla g_Q = 0\) and \(\nabla J = 0\). Then

\[
\Omega = -\frac{1}{2} \sum_{a=1}^{2m} \theta^a \wedge J\theta^a. \tag{4.2}
\]

Moreover, we have the following identities: for any \(X, Y \in \Gamma Q\),

\[
R^\nabla(X, Y)J = JR^\nabla(X, Y), \quad R^\nabla(JX, JY) = R^\nabla(X, Y). \tag{4.3}
\]

Trivially, we have the following lemma.

**Lemma 4.1** On a Kähler foliation \((\mathcal{F}, J)\), the following holds:

\[
\sum_a \theta^a \wedge \rho^\nabla(E_a)^b = 0.
\]

**Proof.** By a direct calculation, we have

\[
\sum_a \theta^a \wedge \rho^\nabla(E_a)^b = \sum_{a, b} \theta^a \wedge R^\nabla(E_a, J E_b) J\theta^b
\]

\[
= \sum_{a, b, c} \theta^a \wedge g_Q(R^\nabla(E_a, J E_b) J E_b, E_c) \theta^c
\]

\[
= \sum_{a, b} R^\nabla(E_b, J E_a) J\theta^b \wedge \theta^a
\]

\[
= \sum_a \rho^\nabla(E_a)^b \wedge \theta^a,
\]

which completes the proof. □
Lemma 4.2 On a Kähler foliation \((F, J)\), we have that for any \(\phi \in \Omega_B^r(F)\),
\[
\sum_a i(E_a)R_+^\nabla(E_a)\phi = \sum_a \theta^a \wedge R_-^\nabla(E_a)\phi = -F(\phi), 
\tag{4.4}
\]
\[
\sum_a i(E_a)R_+^\nabla(JE_a)\phi = \sum_a \theta^a \wedge R_+^\nabla(JE_a)\phi = 0. 
\tag{4.5}
\]

**Proof.** The proof of (4.4) is trivial. Note that for any \(X, Y \in \Gamma Q\),
\[
R_+^\nabla(JX, Y) = R_+^\nabla(JY, X). 
\tag{4.6}
\]
From (4.6), the proof of (4.5) is trivial. \(\square\)

Lemma 4.3 On a Kähler foliation \((F, J)\), we have that for any \(\phi \in \Omega_B^r(F)\),
\[
\sum_a R_+^\nabla(JE_a)i(E_a)\phi = 0.
\]

**Proof.** Let \(\phi = \frac{1}{r!} \sum_{i_1, \ldots, i_r} \phi_{i_1 \ldots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}\) be a basic \(r\)-form. Then by a long calculation, we have
\[
\sum_{a,b} \theta^a \wedge R_+^\nabla(JE_a, E_b)i(E_b)\phi \\
= \frac{1}{r!} \sum_{i_1, \ldots, i_r, a<k<l} (-1)^{k+l-1} \phi_{i_1 \ldots i_r} \theta^a \wedge \{R_+^\nabla(JE_a, E_{i_k})\theta^{i_k} - R_+^\nabla(JE_a, E_{i_l})\theta^{i_l}\} \wedge \psi_{k,l} \\
= \frac{2}{r!} \sum_{i_1, \ldots, i_r, a<k<l} (-1)^{k+l-1} \phi_{i_1 \ldots i_r} \theta^a \wedge R_+^\nabla(JE_a, E_{i_k})\theta^{i_k} \wedge \psi_{k,l},
\]
where \(\psi_{k,l} = \theta^{i_1} \wedge \cdots \wedge \hat{\theta}^{i_k} \wedge \cdots \wedge \hat{\theta}^{i_l} \wedge \cdots \wedge \theta^{i_r}\). From (4.6),
\[
\sum_{i_k,i_l} \phi_{i_1 \ldots i_r} R_+^\nabla(JE_{i_k}, E_{i_l}) = 0.
\]
Hence, by the first Bianchi identity, we have
\[
\sum_{a,i_k,i_l} \phi_{i_1 \ldots i_r} \theta^a \wedge R_+^\nabla(JE_a, E_{i_k})\theta^{i_k} = \sum_{a,b,i_k,i_l} \phi_{i_1 \ldots i_r} g_Q(R_+^\nabla(JE_a, E_{i_k})E_{i_l}, E_b)\theta^a \wedge \theta^b \\
= \sum_{a,i_k,i_l} \phi_{i_1 \ldots i_r} R_+^\nabla(E_{i_k}, E_a)\theta^{i_k} \wedge \theta^a \\
= \sum_{a,i_k,i_l} \phi_{i_1 \ldots i_r} R_+^\nabla(JE_a, E_{i_k})\theta^{i_k} \wedge \theta^a \\
= \sum_{a,i_k,i_l} \phi_{i_1 \ldots i_r} R_+^\nabla(JE_a, E_{i_k})\theta^{i_k} \wedge \theta^a,
\]
which means
\[ \sum_{a,i} \phi_{i}^{1} \wedge R^{\nabla}(JE_{a},E_{i})\theta^{i} = 0. \]

Hence the proof is completed. \( \square \)

Let \( L: \Omega_{B}^{r}(\mathcal{F}) \to \Omega_{B}^{r+2}(\mathcal{F}) \) and \( \Lambda: \Omega_{B}^{r}(\mathcal{F}) \to \Omega_{B}^{r-2}(\mathcal{F}) \) be given respectively by
\[
L(\phi) = \epsilon(\phi), \quad \Lambda(\phi) = \iota(\phi),
\]
where \( \epsilon(\phi) = \Omega \wedge \phi \) and \( \iota(\phi) = -\frac{1}{2} \sum_{a=1}^{2m} i(JE_{a})i(E_{a}). \) Trivially, for any basic forms \( \phi \in \Omega_{B}^{r}(\mathcal{F}) \) and \( \psi \in \Omega_{B}^{r+2}(\mathcal{F}), \langle L(\phi), \psi \rangle = \langle \phi, \Lambda(\psi) \rangle. \) Moreover, for any basic r-form \( \phi, [\Lambda, L] \phi = -\frac{1}{2}(q - 2r)\phi. \) Also, we have the following lemma.

**Lemma 4.4** [5] On a Kähler foliation \( (\mathcal{F}, J) \), we have that for any \( X \in Q \),
\[
[L, i(X)] = \epsilon(JX^{b}), \quad [L, \epsilon(X^{b})] = [\Lambda, i(X)] = 0, \quad [\Lambda, \epsilon(X^{b})] = -i(JX).
\]

Now, we define the operators \( \tilde{J}: \Omega_{B}^{r}(\mathcal{F}) \to \Omega_{B}^{r}(\mathcal{F}) \) and \( S: \Omega_{B}^{r}(\mathcal{F}) \to \Omega_{B}^{r}(\mathcal{F}) \) respectively by
\[
\tilde{J}(\phi) = \sum_{a=1}^{2m} J\theta^{a} \wedge i(E_{a})\phi, \quad (4.8)
\]
\[
S(\phi) = \sum_{a=1}^{2m} J\theta^{a} \wedge i(\rho^{\nabla}(E_{a}))\phi. \quad (4.9)
\]

Trivially, if \( \phi \in \Omega_{B}^{1}(\mathcal{F}), \) then \( \tilde{J}\phi = J\phi. \) From now on, if we have no confusion, we write \( \tilde{J} \equiv J. \)

**Lemma 4.5** On a Kähler foliation \( (\mathcal{F}, J) \), we have that for any \( X, Y \in Q \),
\[
[J, i(X)] = i(JX), \quad [J, \epsilon(X^{b})] = \epsilon(JX^{b}), \quad [R^{\nabla}(X,Y), J] = 0.
\]

**Proof.** The first two equations are trivial. Since \( \sum_{a} R^{\nabla}(X,Y)J\theta^{a} \wedge i(E_{a}) + J\theta^{a} \wedge i(R^{\nabla}(X,Y)E_{a}) = 0, \) for any \( X, Y \in Q, \)
\[
R^{\nabla}(X,Y)J\phi = \sum_{a} J\theta^{a} \wedge i(E_{a})R^{\nabla}(X,Y)\phi
\]
\[
= JR^{\nabla}(X,Y)\phi,
\]
which proves the third equation. \( \square \)
Lemma 4.6 On a Kähler foliation \((\mathcal{F}, J)\), we have that for any \(\phi \in \Omega^r_B(\mathcal{F})\),
\[
\sum_a R^\nabla(E_a, JE_a)\phi = -2S(\phi), \tag{4.10}
\]
\[
\sum_a \theta^a \wedge R^\nabla(JE_a)\phi = \sum_a i(E_a)R^\nabla(JE_a)\phi = S(\phi). \tag{4.11}
\]

Proof. Note that for any \(X \in \Gamma Q\),
\[
\sum_a R^\nabla(E_a, JE_a)X^b = -2\rho^\nabla(JX)^b. \tag{4.12}
\]

Let \(\phi = \frac{1}{r!} \sum_{i_1, \ldots, i_r} \phi_{i_1 \cdots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}\). From (4.12), we have
\[
\sum_a R^\nabla(E_a, JE_a)\phi = \sum_{k, i_1, \ldots, i_r} \phi_{i_1 \cdots i_r} \theta^{i_1} \wedge \cdots \wedge \rho^\nabla(JE_{i_k})^b \wedge \cdots \wedge \theta^{i_r}
\]
\[
= 2 \sum \theta^a \wedge i(\rho^\nabla(JE_a))\phi = -2S(\phi),
\]
which proves (4.10). From Lemma 2.1, we have
\[
\sum_a R^\nabla(JE_a)i(E_a)\phi = \sum_a \theta^a \wedge R^\nabla(JE_a)\phi + \sum_a \theta^a \wedge i(\rho^\nabla(JE_a))\phi. \tag{4.13}
\]

From Lemma 4.3 and (4.13), we have
\[
\sum_a \theta^a \wedge R^\nabla(JE_a)\phi = S(\phi).
\]
Moreover, since \(\sum_a R^\nabla(JE_a, E_a)\phi = \sum_a \theta^a \wedge R^\nabla(JE_a)\phi + \sum_a i(E_a)R^\nabla(JE_a)\phi\), the proof of (4.11) follows. \(\square\)

Lemma 4.7 On a Kähler foliation \((\mathcal{F}, J)\), we have that for any \(\phi \in \Omega^r_B(\mathcal{F})\),
\[
\sum_a \theta^a \wedge JR^\nabla(E_a)\phi = \sum_a i(E_a)JR^\nabla(E_a)\phi = S(\phi) - F(J\phi). \tag{4.14}
\]

Proof. From Lemma 4.2, Lemma 4.5 and Lemma 4.6, we have
\[
\sum_a \theta^a \wedge JR^\nabla(E_a)\phi = \sum_a J\{\theta^a \wedge R^\nabla(E_a)\phi\} - \sum_a J\theta^a \wedge R^\nabla(E_a)\phi
\]
\[
= S(\phi) - JF(\phi) = S(\phi) - F(J\phi).
\]

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The last equality in the above follows from $[J, F] = 0$. On the other hand, from Lemma 4.5 and Lemma 4.6, we have

\[
\sum_a i(E_a) J R^\varphi_+(E_a) \phi = \sum_a i(E_a) J \{ \theta^b \wedge R^\varphi(E_a, E_b) \phi \} \\
= \sum_{a,b} i(E_a) \{ \theta^b \wedge J R^\varphi(E_a, E_b) \phi + J \theta^b \wedge R^\varphi(E_a, E_b) \phi \} \\
= \sum_{a,b} i(E_a) \{ \theta^b \wedge R^\varphi(E_a, E_b) J \phi + J \theta^b \wedge R^\varphi(E_a, E_b) \phi \} \\
= -F(J \phi) + \sum_a R^\varphi(J E_a, E_a) \phi + \sum_a J \theta^a \wedge R^\varphi(E_a) \phi \\
= S(\phi) - F(J \phi). \quad \blacksquare
\]

**Lemma 4.8** On a Kähler foliation $(F, J)$, we have

\[
[J, L] = [J, \Lambda] = [F, J] = [S, J] = [S, \Lambda] = [S, L] = 0.
\]

**Proof.** From Lemma 4.5, we have

\[
[F, J] \phi = - \sum_{a,b} J \theta^b \wedge i(E_a) R^\varphi(E_a, E_b) \phi - \sum_{a,b} \theta^b \wedge i(J E_a) R^\varphi(E_a, E_b) \phi \\
= 0.
\]

Others are easily proved. \(\blacksquare\)

Now, we recall the operators $d^c_B : \Omega^r_B(F) \to \Omega^{r+1}_B(F)$ and $\delta^c_B : \Omega^r_B(F) \to \Omega^{r-1}_B(F)$, which are given by

\[
d^c_B \phi = \sum_{a=1}^{2m} J \theta^a \wedge \nabla_{E_a} \phi, \quad (4.15)
\]

\[
\delta^c_B \phi = - \sum_{a=1}^{2m} i(J E_a) \nabla_{E_a} \phi + i(J \kappa_B^a) \phi. \quad (4.16)
\]

Trivially, $\delta^c_B$ is a formal adjoint of $d^c_B$ and $\delta^c_B \circ \delta^c_B = d^c_B \circ d^c_B = 0 \quad [5]$. Also, we define two operators $d^c_T$ and $\delta^c_T$ by

\[
d^c_T = d^c_B - \epsilon(J \kappa_B), \quad \delta^c_T = \delta^c_B - i(J \kappa_B^a). \quad (4.17)
\]

Then we have the following lemma.
Lemma 4.9. On a Kähler foliation \((\mathcal{F}, J)\), we have that
\[
[L, d_B] = [L, d^c_B] = 0, \quad [L, \delta_B] = -d^c_T, \quad [L, \delta^c_B] = d_T, \quad (4.18)
\]
\[
[\Lambda, \delta_B] = [\Lambda, d^c_B] = 0, \quad [\Lambda, d_B] = \delta^c_T, \quad [\Lambda, d^c_B] = -\delta_T, \quad (4.19)
\]
\[
[J, d_B] = d^c_B, \quad [J, d_B] = \delta^c_B, \quad [J, d_B] = -d_B, \quad [J, d^c_B] = -\delta_B. \quad (4.20)
\]

Proof. Note that on Kähler foliations, \(\nabla J = 0\) and then \(\nabla \bar{J} = 0\). Hence by Lemma 4.5, the proof follows. \(\square\)

Proposition 4.10. On a Kähler foliation \((\mathcal{F}, J)\), we have
\[
d_B \delta^c T + \delta^c T d_B = -[L, \delta_B] \delta^c B - \delta^c B [L, \delta_B] = 0. \quad (4.21)
\]
\[
d_B \delta^c_\Lambda + \delta^c_\Lambda d_B = d_B d^c B + d^c B d_B = 0. \quad (4.22)
\]

Proof. From Lemma 4.9, we have
\[
d^c_T \delta_B + \delta_B d^c_T = d_B \delta^c_T + \delta^c_T d_B = 0.
\]
Others are similarly proved. \(\square\)

Now, we put that for any \(X \in TM\),
\[
e(X) \phi = \delta_B i(X) \phi + i(X) \delta_B \phi. \quad (4.24)
\]

Then we have the following.

Lemma 4.11. On a Kähler foliation \((\mathcal{F}, J)\), we have that
\[
[J, \Delta_B] = \theta(J \kappa^2_B) + \theta(J \kappa^2_B)^t, \quad [\Lambda, \Delta_B] = e(J \kappa^2_B),
\]
where \(\theta(X)^t\) is a formal adjoint of \(\theta(X)\) for any \(X \in Q\).

Now, we recall that \(\mathcal{F}\) is minimal if \(\kappa = 0\). Then we have the following corollary.

Corollary 4.12. On a minimal Kähler foliation \((\mathcal{F}, J)\), we have
\[
[J, \Delta_B] = [\Lambda, \Delta_B] = 0. \quad (4.25)
\]
5 Transverse conformal Killing forms on Kähler foliations

Let \((M, g_M, J, \mathcal{F})\) be a compact Riemannian manifold with a Kähler foliation \(\mathcal{F}\) of codimension \(q = 2m\) and a bundle-like metric \(g_M\) with respect to \(\mathcal{F}\).

**Proposition 5.1** On a Kähler foliation \((\mathcal{F}, J)\), if \(\phi\) is a transverse conformal Killing \(r\)-form, then

\[(q + r^2 - qr)S(\phi) = F(J\phi). \tag{5.1}\]

**Proof.** Let \(\phi\) be a transverse conformal Killing \(r\)-form. From Proposition 3.5,

\[
\sum_a R^\nabla(E_a, JE_a)\phi = \frac{2}{rr^*}JF(\phi) + \frac{2}{r} \sum_a i(JE_a) R^\nabla_+(E_a)\phi
+ \frac{2}{r^*} \sum_a J\theta^a \wedge R^\nabla_-(E_a)\phi.
\]

Hence the proof follows from Lemma 4.6. \(\square\)

From Proposition 5.1, we have the following corollary.

**Corollary 5.2** On a Kähler foliation \((\mathcal{F}, J)\) of codimension \(q = 4\), if \(\phi\) is a transverse conformal Killing 2-form, then

\[F(J\phi) = 0.\]

**Lemma 5.3** Let \(\phi\) be a transverse conformal Killing \(r\)-form on a Kähler foliation. Then

\[
(rr^* - r - 2)d_B^c\phi = (r^* + 1)d_B J\phi - 2(r + 1)\delta_T L\phi, \tag{5.2}
\]

\[
(rr^* - r^* - 2)\delta_T^c\phi = (r + 1)\delta_T J\phi + 2(r^* + 1)d_B \Lambda\phi. \tag{5.3}
\]

**Proof.** Since \(\phi\) is a transverse conformal Killing \(r\)-form, from (4.7), (4.8) and (4.15), we have

\[
d_B^c\phi = \sum_a J\theta^a \wedge \nabla_{E_a} \phi = \frac{1}{r + 1} Jd_B \phi - \frac{2}{r^* + 1} L\delta_T \phi.
\]

From the second equation in (4.18), it is trivial that \([L, \delta_T] = -d_B^c\). Hence from Lemma 4.9, we obtain

\[
\frac{rr^* - r - 2}{(r + 1)(r^* + 1)} d_B^c\phi = \frac{1}{r + 1} d_B J\phi - \frac{2}{r^* + 1} \delta_T L\phi,
\]

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which proves (5.2). The proof of (5.3) is similar. \(\Box\)

Since \(\delta_T^2 \phi = -e(\kappa_B^2) \phi\) for any \(\phi\), from Lemma 5.3, we have that for any transverse conformal Killing \(r\)-form \(\phi\),

\[
(\mathfrak{r} \mathfrak{r}^* - r^* - 2) \delta_T d_B^* \phi = (r^* + 1) \delta_T d_B J \phi + 2(r + 1) e(\kappa_B^2) L \phi, \tag{5.4}
\]

\[
(\mathfrak{r} \mathfrak{r}^* - r^* - 2) d_B \delta_T^* \phi = (r + 1) d_B \delta_T J \phi, \tag{5.5}
\]

\[
(\mathfrak{r} \mathfrak{r}^* - r^* - 2) \delta_T \delta_T^* \phi = 2(r^* + 1) \delta_T d_B \Lambda \phi - (r + 1) e(\kappa_B^2) J \phi. \tag{5.6}
\]

Hence we have the following lemma.

**Theorem 5.4** Let \((M, g_M, \mathcal{F}, J)\) be a closed, connected Riemannian manifold with a Kähler foliation of codimension \(q = 4\). Then for any transverse conformal Killing \(2\)-form, \(J \phi\) is parallel.

**Proof.** Let \(\phi\) be a transverse conformal Killing \(2\)-form. Since \(F(J \phi) = 0\) by Corollary 5.2, we have

\[
d_B \delta_T J \phi + \delta_T d_B J \phi = 0.
\]

Therefore, we have

\[
\Delta_B J \phi = \theta(\kappa_B^2) J \phi.
\]

Hence, by the generalized Weitzenböck formula (Corollary 2.3),

\[
\frac{1}{2}(\Delta_B - \kappa_B^2)|J \phi|^2 = -|\nabla_{\text{tr}} J \phi|^2 \leq 0. \tag{5.7}
\]

From the generalized maximum principle (Theorem 2.4), \(|J \phi|\) is constant. Again, from (5.7), we have

\[
\nabla_{\text{tr}} J \phi = 0,
\]

which implies that \(J \phi \in \Omega_B^2(F)\) is parallel. \(\Box\)

**Corollary 5.5** (cf. [13]) Let \((M, g_M, J)\) be a closed Kähler manifold of dimension 4. Then for any conformal Killing \(2\)-form \(\phi\), \(J \phi\) is parallel.

On the other hand, for any basic \(r\)-form \(\phi\), Lemma 4.9 implies that

\[
J \Lambda d_B \delta_B \phi = d_B \delta_B J \Lambda \phi + d_B^* \delta_B \Lambda \phi + d_B \delta_B^* \Lambda \phi + J \delta_T^* \delta_B \phi, \tag{5.8}
\]

\[
J \Lambda \delta_B d_B \phi = \delta_B d_B J \Lambda \phi + \delta_B^* d_B^* \Lambda \phi + \delta_B^* d_B \Lambda \phi + J \delta_B \delta_T^* \phi. \tag{5.9}
\]

Hence we have the following lemma.
Lemma 5.6 Let \( \phi \) be a transverse conformal Killing \( r(\neq q) \)-form. Then

\[
J\Lambda d_B\delta_B\phi = d_B\delta_BJ\Lambda \phi + d_B\delta_B^c\Lambda \phi + d_B^c\delta_B\Lambda \phi - \frac{2(r^* + 1)}{r^*(r + 1)} J\Lambda \delta_Bd_B\phi
\]  
(5.10)

\[
- Je(J\kappa_B^* )\phi + \frac{1}{r^*} J\delta_Bi(\kappa_B^* )J\phi,
\]

\[
J\Lambda \delta_Bd_B\phi = \delta_Bd_BJ\Lambda \phi + \delta_Bd_B^c\Lambda \phi + \delta_B^c\delta_Bd_B\Lambda \phi + \frac{2(r^* + 1)}{r^*(r + 1)} J\Lambda \delta_Bd_B\phi
\]  
(5.11)

\[
- \frac{1}{r^*} J\delta_Bi(\kappa_B^* )J\phi.
\]

Proof. Let \( \phi \) be a transverse conformal Killing \( r \)-form. From (4.23), \( \delta_T \delta_B \phi = -\delta_B \delta_T \phi - e(J\kappa_B^*) \phi \). Hence from Lemma 4.9 and Lemma 5.3, we have

\[
(rr^* - r^* - 2)J\delta_T \delta_B \phi = -2(r^* + 1) J\delta_Bd_B\Lambda \phi + (r + 1) J\delta_Bi(\kappa_B^*) J\phi
\]

\[
- (rr^* - r^* - 2) Je(J\kappa_B^*) \phi
\]

\[
= -2(r^* + 1) J\Lambda \delta_Bd_B\phi + 2(r^* + 1) J\delta_B \delta_T \phi
\]

\[
- (rr^* - r^* - 2) Je(J\kappa_B^*) \phi + (r + 1) J\delta_Bi(\kappa_B^*) J\phi.
\]

Therefore, we have

\[
r^*(r + 1) J\delta_T \delta_B \phi = -2(r^* + 1) J\Lambda \delta_Bd_B\phi - r^*(r + 1) Je(J\kappa_B^*) \phi
\]

\[
+ (r + 1) J\delta_Bi(\kappa_B^*) J\phi.
\]

From (5.8), the proof of (5.10) follows. The proof of (5.11) is similar from (5.9).

\[\Box\]

Lemma 5.7 Let \((\mathcal{F}, J) \) be a minimal Kähler foliation. Then for a transverse conformal Killing \( r \) \( (2 \leq r \leq q - 2) \)-form \( \phi \),

\[
\delta_Bd_B\Lambda \phi = -\frac{r + 1}{(r - 1)(r^* + 1)} \{ J\Lambda d_B\delta_B\phi + \delta_Bd_B^c\Lambda \phi \},
\]  
(5.12)

\[
\delta_Bd_B^c\Lambda \phi = \frac{r^* + 1}{(r + 1)(r^* - 1)} \{ J\Lambda \delta_Bd_B\phi - \delta_B^c d_B\Lambda \phi \}.
\]  
(5.13)

Proof. From Lemma 4.9 and Proposition 4.10, we have

\[
\delta_B^c d_B\Lambda \phi = -\Lambda d_B\delta_T^c \phi + \Lambda i(J\kappa_B^*)d_B^c \phi - \delta_B^c d_T^c \phi,
\]  
(5.14)

\[
\delta_B^c d_B\Lambda \phi = -\Lambda d_B d_T^c \phi + \Lambda i(\kappa_B^*)d_B^c \phi + \delta_B d_T \phi.
\]  
(5.15)
From (5.6) and (5.14), we have

\[ (r^*-r^*-2)\delta_B^c d_B \Lambda \phi = -(r+1)\Lambda d_B \delta_T J \phi + (rr^*-r^*-2)\Lambda i(J\kappa_B^\sharp) d_B \phi \]

\[ - (r^*-r^*-2)\delta_B^T \delta_T^c \phi = - (r+1) \{ J\Lambda d_B \delta_T \phi - \Lambda d_B \delta_T^c \phi \} \]

\[ + (rr^*-r^*-2)\Lambda i(J\kappa_B^\sharp) d_B \phi - (rr^*-r^*-2)\delta_B^T \delta_T^c \phi. \]

By using (5.14) and (5.15), the above equation gives

\[ \delta_B^c d_B \Lambda \phi = - \frac{r+1}{(r^*+1)(r-1)} \{ J\Lambda d_B \delta_T \phi + \delta_B^c d_B \Lambda \phi \} \]

\[ + \frac{r+1}{(r^*+1)(r-1)} \{ \Lambda i(J\kappa_B^\sharp) d_B \phi + \Lambda i(\kappa_B^\sharp) d_B \phi - \delta_B^c \delta_T^c \phi + \delta_B \delta_T \phi \}. \]

Since \( F \) is minimal, \( \delta_T^c \delta_B^c \phi = \delta_B^c \delta_T^c = 0 \) and \( \delta_T \delta_B \phi = 0 \). Hence the above equation proves (5.12). From (5.15), (5.13) is similarly proved. \( \square \)

Now, we put

\[ x = J\Lambda(d_B \delta_B \phi), \quad y = J\Lambda(d_B d_B \phi), \quad \alpha = \delta_B^c d_B \Lambda \phi, \quad \beta = \delta_B^c d_B \Lambda \phi, \quad a = d_B \delta_B J\Lambda \phi, \quad b = \delta_B d_B J\Lambda \phi. \]  

(5.16)

(5.17)

From now on, we assume that \( F \) is minimal. From Lemma 5.6, we have

\[ x = a - \alpha - \beta - \frac{2(r^*+1)}{r^*(r+1)} y, \]  

(5.18)

\[ y = b + \alpha + \beta + \frac{2(r^*+1)}{r^*(r+1)} y. \]  

(5.19)

Hence from (5.18) and (5.19), we have

\[ (rr^*-r^*-2)y = r^*(r+1)(b + \alpha + \beta), \]  

(5.20)

\[ (rr^*-r^*-2)x = (rr^*-r^*-2)a - 2(r^*+1)b - r^*(r+1)(\alpha + \beta). \]  

(5.21)

On the other hand, from Lemma 5.7, we have

\[ \alpha = - \frac{r+1}{(r^*+1)(r-1)} (x + \beta), \]  

(5.22)

\[ \beta = \frac{r^*+1}{(r+1)(r^*-1)} (y - \alpha). \]  

(5.23)
Note that \( r^r_* - r^* - 2 = 0 \) if and only if \( q = 4 \). Hence from (5.20), (5.21), (5.22) and (5.23), if \( q \neq 4 \), then

\[
\begin{align*}
\lambda_1 \lambda_3 b &= (1 - \lambda_1 \lambda_3) \beta + \lambda_3 (1 - \lambda_1) \alpha, \\
\lambda_2 a + \lambda_2 (1 - \lambda_1) b &= (\lambda_1 \lambda_2 - 1) \alpha + \lambda_2 (\lambda_1 - 1) \beta,
\end{align*}
\]

where \( \lambda_1 = \frac{r^r(r+1)}{r^r - r - 2} \), \( \lambda_2 = \frac{r+1}{(r+1)(r-1)} \) and \( \lambda_3 = \frac{r^r+1}{(r+1)(r^r - 2)} \). Hence we have the following theorem.

**Theorem 5.8** Let \((M,g_M,J,\mathcal{F})\) be a closed Riemannian manifold with a minimal Kähler foliation \(\mathcal{F}\) of codimension \(q = 2m\) and a bundle-like metric \(g_M\). Then for any transverse conformal Killing \((2 \leq r \leq q - 2)\)-form \(\phi\), \(J\Lambda\phi\) is basic-harmonic.

**Proof.** From Lemma 4.9, \(d_B\delta_B^c + \delta_B^c d_B = \theta(J\kappa^c_B)\phi\). Hence we have

\[
\int_M \langle b, \alpha \rangle \mu_M = \int_M \langle d_B J\Lambda \phi, \theta(J\kappa^c_B) d_B \Lambda \phi \rangle \mu_M.
\]

Since \(\mathcal{F}\) is minimal, we have

\[
\int_M \langle b, \alpha \rangle \mu_M = 0. \tag{5.26}
\]

Similarly, we have

\[
\int_M \langle \beta, \alpha \rangle \mu_M = 0 \tag{5.27}
\]

and

\[
\int_M \langle a, b \rangle \mu_M = \int_M \langle a, \beta \rangle \mu_M = 0. \tag{5.28}
\]

(i) In case of \(q \neq 4\). From (5.24), (5.26) and (5.27), we have

\[
\lambda_3 (1 - \lambda_1) \int_M |\alpha|^2 \mu_M = 0.
\]

Since \(\lambda_3 \neq 0\) and \(\lambda_1 \neq 1\), \(\alpha = 0\). From (5.20) and (5.28), \(a = 0\). Therefore, from (5.24) and (5.25), since \(\lambda_2 (1 - \lambda_1) \neq 0\), we have

\[
\lambda_1 \lambda_3 b = (1 - \lambda_1 \lambda_3) \beta, \quad b = \beta.
\]

Hence \(b = \beta = 0\). Therefore, \(x = y = 0\). So from Corollary 4.12, \(\Delta_B J\Lambda \phi = J\Lambda \Delta_B \phi = x + y = 0\). That is, \(J\Lambda \phi\) is basic-harmonic. (ii) In case of \(q = 4\). From Theorem 5.4, \(J\phi \in \Omega^2_B(\mathcal{F})\) is parallel and so basic-harmonic, i.e., \(\Delta_B J\phi = 0\). Hence from Corollary 4.12, \(\Delta_B J\Lambda \phi = 0\), i.e., \(J\Lambda \phi\) is basic-harmonic. \(\square\)
Corollary 5.9 Let \((M,g_M,J,F)\) be as in Theorem 5.8. Then for a transverse conformal Killing \(r\) \((2 \leq r \leq q-2)\)-form \(\phi\), \(J\Lambda\phi\) is parallel.

Proof. Let \(\phi\) be a transverse conformal Killing form. Since \(F\) is minimal, from Theorem 5.8, \(\Delta_B(J\Lambda\phi) = 0\). Hence from the generalized Weitzenböck formula (Theorem 2.2), we have

\[ F(J\Lambda\phi) + \nabla^*_\text{tr} \nabla_{\text{tr}}J\Lambda\phi = 0. \]

On the other hand, from Proposition 3.2, we have

\[ F(J\Lambda\phi) = \frac{r}{r+1}y + \frac{r^*}{r^*+1}x. \]  

(5.29)

In the proof of Theorem 5.8, \(x = y = 0\). Hence \(F(J\Lambda\phi) = 0\), which means that \(J\Lambda\phi\) is parallel. \(\Box\)

Theorem 5.10 Let \((M,g_M,J,F)\) be a closed Riemannian manifold with a minimal Kähler foliation \(F\) of codimension \(q = 2m(\neq 4)\) and a bundle-like metric \(g_M\). Then for a transverse conformal Killing \(r(r \neq m, 2 \leq r \leq q-2)\)-form, \(J\phi\) is parallel.

Proof. Let \(\phi\) be a transverse conformal Killing \(r\)-form. Then \(\bar{\phi}\) is also a transverse conformal Killing \((q-r)\)-form \([6]\). Hence by Corollary 5.9, \(J\Lambda\bar{\phi}\) is parallel. Since \([\nabla_{\text{tr}},\bar{\phi}] = 0, [J,\bar{\phi}] = 0\) and \(L\bar{\phi} = \Lambda\Lambda\bar{\phi} = \pm LJ\phi\) is parallel. Note that \((m-r)J\phi = [\Lambda, L]J\phi\). Since \([L,\nabla_{\text{tr}}] = [\Lambda, \nabla_{\text{tr}}] = [J,\Lambda] = 0\), from Corollary 5.9, we get

\[ (m-r)\nabla_{\text{tr}}J\phi = \nabla_{\text{tr}}\Lambda L\phi - \nabla_{\text{tr}}L\Lambda J\phi = \Lambda L\nabla_{\text{tr}}L\phi - L\nabla_{\text{tr}}J\Lambda\phi = 0. \]

Hence if \(r \neq m\), then \(J\phi\) is parallel. \(\Box\)

Remark. (1) When \(q = 4\), \(J\phi\) is parallel for any transverse conformal 2-form \(\phi\) (Theorem 5.4).

(2) For the point foliation, Theorem 5.10 has been proved in \([13]\).

Question. When \(F\) is not minimal, is Theorem 5.10 true?

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