THE ZERO TENSION LIMIT
OF STRINGS AND SUPERSTRINGS

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ABSTRACT

The string equivalent of a massless particle \((m = 0)\) is the tensionless string \((T = 0)\). The study of such strings is of interest when trying to understand the high energy limit of ordinary strings. I discuss the classical \(T \to 0\) limit of the bosonic string, the spinning string and the superstring. A common feature is the appearance of a space-time (super-) conformal symmetry replacing the world-sheet Weyl invariance. The question of whether this symmetry may survive quantization is addressed. A light-cone analysis of the quantized bosonic tensionless string leads to severe constraints on the physical states: they are space-time diffeomorphism singlets characterized by their topological properties only.

1. Introduction

Both the point particle and the string have actions of the type

\[ S = T \int \mathcal{L} \]

where \(T\) is a dimensionful parameter. In the particle case we have good reasons to be interested in the limit \(T \to 0\), since this describes massless particles. These we want to study in their own right or as models of particle behaviour at high energies. Similarly it is interesting to consider strings in the limit \(T \to 0\), which presumably would be a useful description of strings at high energies \([1]\). The study of such strings turns out to be quite rewarding: as I will mention at the end of this talk, they may provide a link to the ”topological phase” of string theory.

This review describes work that I have done in various collaborations (see the acknowledgement). A number of other authors have studied tensionless strings \([2]-[5]\), originating with the work of Shild \([1]\).

2. The Bosonic String

The way to construct the \(T \to 0\) limit of the action \([1]\) used in the point particle case is to write

\[ S \to \int \Phi \mathcal{L}^2 + \Phi^{-1} T^2. \]

Integrating out the auxiliary field \(\Phi\) one recovers the previous action, but in \([2]\) the limit \(T \to 0\) is immediate. This procedure has been used to find a bosonic tensionless
string too \cite{7}, but it is only for the particle that this procedure has any relation to the world-volume geometry. (There $\Phi$ is identified with the ”einbein” e.) It has proven useful to keep a relation to the geometry and I will describe a procedure for doing so. To keep things general I start from the action for a $p$-brane \cite{1}

$$S = T \int d^{p+1} \xi \sqrt{-\gamma}$$

(3)

where $\gamma = \det \gamma_{ab}$ and the line element is

$$ds^2 = dX^\mu dX^\nu \eta_{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} d\xi^a d\xi^b \equiv \gamma_{ab} d\xi^a d\xi^b.$$ 

(4)

($\gamma_{ab}$ is the induced metric on the world-volume.) We pass to the Hamiltonian formulation via the canonical momenta derived from (3):

$$P_\mu = T \sqrt{-\gamma} \gamma^{a0} \partial_\mu \eta.$$ 

(5)

They obey the constraints

$$P^2 + T^2 \gamma^{00} = 0, \quad P_\mu \partial_\mu X^\mu = 0, \quad i = 1, ..., p.$$ 

(6)

The Hamiltonian is just these constraints multiplied by Lagrange multipliers that I call $\lambda$ and $\rho^i$,

$$\mathcal{H} = \lambda (P^2 + T^2 \gamma^{00}) + \rho^i P \cdot \partial_i X,$$ 

(7)

where $\cdot$ denotes contraction using the space-time metric $\eta$. The corresponding phase space action is

$$S_{PS} = \int d^{p+1} \xi \left\{ P \cdot \dot{X} - \lambda (P^2 + T^2 \gamma^{00}) - \rho^i P \cdot \partial_i X \right\}.$$ 

(8)

To obtain a new configuration space action we integrate out the momenta $P_\mu$ to find

$$S_{CS}^1 = \frac{1}{2} \int d^{p+1} \xi \left( \frac{1}{\lambda} \right) \left\{ \dot{X}^2 - 2 \rho^i \dot{X} \cdot \partial_i X + \rho^i \rho^j \partial_i X \cdot \partial_j X - 4 \lambda T^2 \gamma^{00} \right\}.$$ 

(9)

As an aside we note that integrating out also the $\rho^i$’s we recover the form (2) after a suitable identification between $\lambda$ and $\Phi$.

For $p = 1$ we have only one $\rho^i$ and may now identify the Lagrange multipliers with components of the $D = 2$ metric directly, thus arriving at the Brink-Howe-DiVecchia-Deser-Zumino \cite{8,9} form of the bosonic string. For general $p$ we have to go through one more step. We rewrite (3) using a $p$-dimensional auxiliary metric $G_{ij}$ and a rank 1 matrix $h^{ab}$:

$$S_{CS}^1 = \frac{1}{2} \int d^{p+1} \xi \left\{ \frac{h_{ab} \gamma^{ab}}{2 \lambda} - 2 \lambda T^2 G(p-1) + 2 \lambda GG^{ij} \gamma_{ij} \right\}.$$ 

(10)

where

$$h_{ab} = \begin{pmatrix} 1 & -\rho^i & -\rho^j \\ -\rho^i & \rho^j & \rho^j \\ -\rho^j & \rho^j & \rho^j \end{pmatrix}.$$ 

(11)

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\footnote{This section is based on material derived in collaboration with R.v.Unge}
and \( G \equiv \text{det}G_{ij} \). The equivalence between (9) and (10) is seen by integrating out \( G_{ij} \).

If we now keep \( T \neq 0 \), we may make the identification
\[
g^{ab} = \frac{1}{4} T^{-2}\lambda^{-2} G^{-1} \begin{pmatrix} -1 & \rho^i \\
\rho^i & -\rho^i \rho^j + 4\lambda^2 T^2 G G^i j \end{pmatrix}
\]
(12)
to find the usual \( p \)-brane action \([10]\) in terms of the auxiliary world-volume metric \( g_{ab} \):
\[
S_g = -\frac{T}{2} \int d^{p+1}\xi \sqrt{-g} \{ g^{ab} \gamma_{ab} - (p - 1) \}.
\]
(13)

In the limit \( T \to 0 \) we instead make the identification
\[
V^a = \frac{1}{\sqrt{2}\lambda} (1, -\rho^i).
\]
(14)

This yields the action for the zero tension theory. For the string it reads
\[
S_V = \int d^2\xi V^a V^b \gamma_{ab}
\]
(15)
with \( V^a \) a world-sheet vector density. This form of the action has proven very useful. For example it is readily supersymmetrized to give the zero tension limit of the superstring and of the spinning string. The ”geometrical” structure also made it possible to write down a number of models with world-sheet supersymmetry by constructing a new type of \( D = 2 \)” degenerate” supergravity \([11]\).

In taking the \( T \to 0 \) limit the Weyl-invariance of the string theory has been replaced by space-time conformal invariance. This is clear from the fact that a space-time conformal transformation will scale the induced metric, a scaling that may be absorbed by a compensating scaling of \( V^a \). This was not possible for the tensionful action
\[
S_g = T \int d^2\xi \sqrt{-g g^{ab} \gamma_{ab}},
\]
(16)
since \( \sqrt{-g g^{ab} \gamma_{ab}} \) is invariant by itself under scalings. That there should be such a symmetry is natural from the description of the theory in the 2D-diffeomorphism gauge \( V^a = (v, 0) \) where the field equations read
\[
\ddot{X}^\mu(\xi) = 0, \quad \dot{X}^2(\xi) = X^\mu(\xi) X^\nu_{\mu}(\xi) = 0,
\]
(17)
i.e., for each \( \sigma \) we have a massless particle moving on a null trajectory (and satisfying an orthogonality constraint). Conformal symmetry is precisely the symmetry that preserves the light-cone (causal) structure.

There are two topics based on the classical formulation \([13]\) of the tensionless string that I want to introduce: The question of supersymmetrization and the quantization of the bosonic string.

3. The Superstring

The zero tension limit of the superstring \([12]\) is achieved by a (space-time) supersymmetrization of the bosonic model \([13]\):
\[
\partial X^\mu \to \Pi^\mu_a = \partial X^\mu - i\partial \Gamma^\mu \partial_a \theta
\]
(18)
with \( \theta(\xi) \) are Majorana (or Weyl) space-time spinors. The action becomes

\[
S_V \rightarrow \int d^2 \xi V^a V^b \Pi_a^\mu \Pi_b^\nu
\] (19)

The symmetries of this action are (global) Space-time super symmetry, (in certain dimensions it is even superconformally invariant), and (local) Siegel-invariance.

The global supersymmetry is evident from the definition (18). Superconformal invariance follows as in the bosonic case, since a superconformal transformation will scale the super-line element

\[
ds^2_S = \eta_{\mu\nu} \Pi^\mu_a \Pi^\nu_b d\xi^a d\xi^b.
\] (20)

The superconformal group exists in \( D = 2 - 6 \). The Siegel invariance is given by

\[
\delta^\kappa = i\frac{\kappa}{\Pi} \kappa^a, \quad \delta^\kappa X^\mu = i\bar{\theta}^\mu \delta^\kappa \theta,
\] (21)

where \( \kappa^a \) is a world-sheet vector space-time spinor. For \( T \neq 0 \) one has to add a Wess-Zumino term to the supersymmetrization of the bosonic action to achieve \( \kappa \)-invariance. This is not the necessary in the \( T = 0 \) case at hand. With

\[
\kappa^a \equiv V^a \kappa,
\] (22)

where \( \kappa \) is a density of opposite weight to \( V^a \), and

\[
\delta_a V^a = 2V^a V^b (\partial_b \bar{\theta}) \kappa,
\] (23)

the model is Siegel-invariant as it stands. (In fact, the WZW-term is separately invariant under the transformations (21,23) in \( D = 2 \mod 8 \).) On shell the \( \kappa \)-dependence is through the combination \( V^a \Pi_a \kappa \) only, and \( V^a \Pi_a \) is nilpotent there. Hence \( \kappa \) carries half the degrees of freedom of a spinor in complete analogy to the \( T \neq 0 \) case. For closure of the Siegel symmetry one finds a version of the usual local bosonic symmetry

\[
\delta_\lambda V^a = 0, \quad \delta_\lambda \theta = \lambda V^a \partial_a \theta, \quad \delta_\lambda X^\mu = i\bar{\theta}^\mu \delta_\lambda \theta.
\] (24)

In the diffeomorphism gauge \( V^a = (v,0) \) we again find that the equations of motion describe, for each \( \sigma \), a massless superparticle moving on a superspace equivalent of a null hypersurface:

\[
\bar{\Pi}_0 = 0, \quad \partial \Pi_0 = 0, \quad (\Pi_0)^2 = 0, \quad \Pi_0 \cdot \Pi_1 = 0.
\] (25)

4. The Spinning String

The zero tension limit of the spinning string can likewise be constructed starting from the bosonic action (13). A world-sheet supersymmetrization leads to

\[
S = \int d^2 \xi \left\{ (V^a \partial_a X^\mu + i\Psi^\mu \chi)(V^b \partial_b X_\mu + i\Psi_\mu \chi) + i\bar{\Psi}^\mu V^a \partial_a \Psi_\mu \right\},
\] (26)
where $\chi$ is the fermionic partner of $V^a$ and $\Psi^\mu$ is that of $X^\mu$. (See further [13]). The world-sheet "spinors" are really Grassmann numbers here. The world-sheet supersymmetry transformations that leave (27) invariant are

$$
\delta X^\mu = i\varepsilon \Psi^\mu, \quad \delta \Psi^\mu = -\varepsilon \partial X^\mu - \frac{1}{2} i\varepsilon (\Psi^\mu \chi)
$$

$$
\delta V^a = i V^a (\varepsilon \chi), \quad \delta \chi = \nabla \varepsilon,
$$

(27)

where $\varepsilon$ is a spinor density, $\partial \equiv V^a \partial_a$ and $\nabla \equiv V^a \nabla_a$. The covariant derivative involves a connection about which it is sufficient to assume $\nabla_a V^a = 0$, which thus is the "metricity condition" of our theory.

In the gauge $V^a = (v, 0)$, we find that the model describes the motion of a massless spinning particle for each $\sigma$.

The action (26) of this theory closely resembles that of a massless spinning particle as studied by Howe, Pernati, Pernici and Townsend [14]. Just like that model it can be extended to arbitrary $N$ number of supersymmetries and carry a gauged $O(N)$ symmetry. The $(1,1)$ model is the zero tension limit of the spinning string.

Further, introducing a degenerate local superspace supergravity theory corresponding to the $(V^a, \chi)$ multiplet [11], the action (26) takes the form

$$
S = \int d^2 \xi d\theta \nabla X^\mu \nabla^2 X^\mu,
$$

(28)

where the superfields and superspace covariant derivatives are

$$
\nabla = E \partial_b + E^a \partial_a, \quad E^a \mid = V^a, \quad \nabla E \mid = \left(\frac{3}{4}\right) \chi, \quad \chi^\mu = X^\mu + \theta \Psi^\mu.
$$

(29)

The superspace formulation leads to the introduction of a number of new models. E.g., the zero tension limit of the $(2,2)$ string is introduced via a complexification of the $\chi^\mu$ field.

5. The Quantum Theory

To investigate the quantum properties of the bosonic theory one may of course proceed in several ways, as for the tensionful string. The only avenue which is not open is to demand Weyl-invariance of the quantum theory.

Both the hamiltonian and the lagrangian BRST versions with diffeomorphism ghosts lead to nilpotent BRST charges independent of the dimension of space-time. Likewise for closure of the Lorentz algebra in the light-cone gauge: no obstruction, no critical dimension.

However, as was mentioned previously, the Weyl-invariance of the $T \neq 0$ string is replaced by global conformal symmetry of the ambient space-time. This brings up the question of survival of this symmetry in the quantum theory, a question addressed in [15].

After going to light-cone gauge and solving the constraints we are left with highly non-linear expressions for the conformal generators in terms of the transversal coordinates $X^i(\sigma)$ and momenta $\Pi_i(\sigma)$. Skipping the details of how to ensure
hermiticity of the operators, choosing a reference operator ordering, regularization et.c., canonical quantization is achieved by prescribing the operator commutation relations

\[ [X^i(\sigma), \Pi^j(\tilde{\sigma})] = i\delta^{ij}\delta(\sigma - \tilde{\sigma}), \quad [X^-, \pi^+] = -i, \] (30)

where + and – refer to the two lightlike directions. We then find a systematic and controlled way of calculating the conformal algebra. The Lorentz subalgebra closes indeed. However, leaving that subalgebra we immediately encounter obstructions: operator anomalies. E.g., the commutator between the generators of special conformal and Lorentz transformations, \(K\) and \(M\), should close to \(K\). Instead we find

\[ [K^i, M^j^-] - \delta^{ij} K^- \propto \int d\sigma \left( \frac{1}{\varepsilon \pi^+} \right) \{X^i\Pi^j - x^i\pi^j\} \equiv \left( \frac{1}{\varepsilon \pi^+} \right) L^{ij} \neq 0. \] (31)

Lower case letters denote zero-modes, i.e. \(\sigma\) independent pieces of the operators. (Note, in passing, that for the particle theory the zero-modes are the whole story, and the obstructions vanish. The conformal symmetry is a good quantum symmetry for the massless particle.)

The route we choose from here is to impose

\[ L^{ij} |\text{PHYS} > = 0 \] (32)

and see if we can carry on this procedure to find the state space of the quantum theory. We now have to commute \(L^{ij}\) with the conformal generators. Eventually this procedure will terminate and leave us with some algebra. Invoking a result by Ogievetsky \[16\], this closure is the algebra of general coordinate transformations. \[2\] Thus, for the space-time conformal group to survive as a symmetry in the quantized theory the physical states must be singlets under general coordinate transformations! The states of the theory should hence correspond to equivalence classes of string configurations differing only in their topological properties.

6. Discussion

Admittedly, the route taking in imposing the constraints (32) is unorthodox. Normally one would perhaps have concluded that there are anomalies and that the space-time symmetry is not a good quantum symmetry. However, our goal was precisely to investigate under what circumstances the conformal symmetry survives as a quantum symmetry. One interpretation is then that we modify the theory so as to include the constraints and that this defines our quantum theory.

One should also bear in mind that we are trying to find the large symmetry that Gross finds indicatinhs of in studying high energy scattering of strings. It would be nice to be able to say something conclusive about that topic, but all we can say is that our results do not contradict those of Gross. The selection rules he finds say

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\(^2\)In fact, the closure yields only the analytic diffeomorphisms. I thank M.B. Green and G. Papandopolous for this comment.
that in the $T \to 0$ limit amplitudes are given by polarizations (spin) in the scattering plane, i.e., the plane defined by the relative momenta. Other polarization directions do not affect the amplitudes. The constraint (32) imply that no spin is allowed for a single tensionless string, but on the other hand there is no relative momentum in this single string Hilbert space.

Finally I emphasize that we have only found restrictions on the Hilbert space, we have no explicit construction of the spectrum. The result seems to tie in nicely with the ideas of unbroken general covariance and few short-distance degrees of freedom, though.

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1. D.J. Gross, *Phys. Rev. Lett.*, **60** (1988) 1229.
2. A. A. Zheltukhin, *Sov. J. Nucl. Phys.* **48** (1988) 375.
3. F.Lizzi, B. Rai, G. Sparano and A. Srivastava, *Phys.Lett.* **B 182** (1986) 326.
4. J. Gamboa, C. Ramires, and M. Ruiz-Altaba, *Nucl. Phys B** **338** (1990) 143.
5. A. Barcelos-Neto and M. Ruiz-Altaba, *Phys. Lett.* **B 228** (1989) 193.
6. A. Shild, *Phys. Rev. D** **16** (1977) 1722.
7. A. Karlhede and U. Lindström, *Class. Quantum Grav.* **3** (1986) L73.
8. L. Brink, P.di Vecchia and P.S. Howe, *Phys. Lett., B65* (1976) 435.
9. S. Deser and B. Zumino, *Phys. Lett. B** **65** (1976) 369.
10. P.S. Howe and R.W. Tucker, *J.Math.Phys.* **19** (1978) 981.
11. U. Lindström and M.Roček, *Phys. Lett. B271* (1991) 79.
12. U. Lindström, B. Sundborg and G. Theodoridis, *Phys. Lett. B** **253** (1991) 319.
13. U. Lindström, B. Sundborg and G. Theodoridis, *Phys. Lett. B** **215** (1988) 555.
14. P. S. Howe, S. Penati, M. Pernici and P. Townsend, *Phys. Lett. B** **258** (1991) 331.
15. J. Isberg, U. Lindström and B. Sundborg, *Phys. Lett. B** **293** (1992) 321.
16. V. I. Ogievetsky, *Lett. Nuovo Cimento B** **8** (1973) 988.