Fourier Series for Bernoulli-Type Polynomials, Euler-Type Polynomials and Genocchi-Type Polynomials of Integer Order

Cristina B. Corcino\textsuperscript{1,2}, Roberto B. Corcino\textsuperscript{1,2,*}

\textsuperscript{1} Research Institute for Computational Mathematics and Physics, Cebu Normal University, 6000 Cebu City, Philippines
\textsuperscript{2} Department of Mathematics, Cebu Normal University, 6000 Cebu City, Philippines

Abstract. Parameters $a$, $b$, $c$, and $\alpha$ are introduced to form the Bernoulli-type, Euler-type and Genocchi-type polynomials where $\alpha$ is the order of the polynomial and is a positive integer. Analytic methods are used here to obtain the Fourier series for these polynomials.

2020 Mathematics Subject Classifications: 11B68, 42A16, 11M35

Key Words and Phrases: Fourier Series, Bernoulli polynomials, Euler polynomials, Genocchi polynomials

1. Introduction

The polynomials that will be considered are given by the generating functions (1)-(3) where $B_n^{(\alpha)}(x;a,b,c)$ denotes the Bernoulli-type polynomials of order $\alpha$, $E_n^{(\alpha)}(x;a,b,c)$ denotes the Euler-type polynomials of order $\alpha$ and $G_n^{(\alpha)}(x;a,b,c)$ denotes the Genocchi-type polynomials of order $\alpha$ with $\alpha \in \mathbb{Z}^+$, $a, b, c$ are positive real numbers and $B = \ln b - \ln a > 0$.

\begin{align*}
\left( \frac{t}{b^\alpha - a^\alpha} \right)^\alpha e^{xt} &= \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;a,b,c) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{B} \tag{1} \\
\left( \frac{2}{b^\alpha + a^\alpha} \right)^\alpha e^{xt} &= \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;a,b,c) \frac{t^n}{n!}, \quad |t| < \frac{\pi}{B} \tag{2} \\
\left( \frac{2t}{b^\alpha + a^\alpha} \right)^\alpha e^{xt} &= \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;a,b,c) \frac{t^n}{n!}, \quad |t| < \frac{\pi}{B} \tag{3}
\end{align*}

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4507

Email addresses: corcinoc@cnu.edu.ph (C. Corcino), rcorcino@yahoo.com (R. Corcino)
These polynomials are generalizations of the classical Bernoulli, Euler and Genocchi polynomials, respectively. The Apostol-type of these polynomials were mentioned in [9] in the introduction of the paper. Fourier series for the tangent type of these polynomials were obtained in [7] while the Fourier series for the Apostol-Tangent polynomials were obtained in [6]. Integral representation and explicit formula at rational arguments of tangent polynomials of higher order were derived in [8]. Properties of higher order Apostol-Frobenius-type poly-Genocchi polynomials with parameters \(a, b, c\) were studied in [10]. Other interesting polynomials related to Bernoulli, Euler and Genocchi were studied in [1–4].

In this paper, the Fourier series for \(B_n^{(\alpha)}(x; a, b, c), E_n^{(\alpha)}(x; a, b, c)\) and \(G_n^{(\alpha)}(x; a, b, c)\) of positive integer order \(\alpha\) will be derived. The method used here is analytic. In particular, there will be heavy use of contour integration and residue theory. For elaborate discussion of these topics see [5].

2. The case \(\alpha = 1\)

Lemma 2.1. Let \(n \geq 2, N > 1\) and \(C_N\) be the circle about zero of radius \(R = (2N\pi - \varepsilon)/B\), where \(0 < \varepsilon < 1\) and \(B = \ln b - \ln a, b > a\). For

\[
0 < x < \left(\ln a - \frac{B}{2\pi - \varepsilon}\right) / \ln c, \quad \ln c > 0
\]

we have

\[
\lim_{N \to +\infty} \int_{C_N} \frac{e^{xt}}{b^t - a^t} \frac{dt}{t^n} = 0.
\]

Proof.

\[
\left| \int_{C_N} \frac{e^{xt}}{(b^t - a^t)^n} dt \right| \leq \int_{C_N} \left| \frac{e^{xt}}{(b^t - a^t)^n} \right| |dt|.
\]

We will show that under the conditions in the lemma, the function \(\frac{e^{xt}}{(b^t - a^t)}\) is bounded on \(C_N\).

Write \(e^{xt} = e^{x \ln c}, b^t = e^{t \ln b}, a^t = e^{t \ln a}\), where \(t \in C_N\). Let \(t = \gamma + i\rho\). Then

\[
\gamma = \frac{2N\pi - \varepsilon}{B} \cos \theta, \quad \rho = \frac{2N\pi - \varepsilon}{b} \sin \theta,
\]

where \(0 \leq \theta < 2\pi\). Then

\[
\frac{|e^{xt}|}{|b^t - a^t|} = \frac{e^{x\gamma \ln c}}{|e^{(\gamma + i\rho)\ln b} - e^{(\gamma + i\rho)\ln a}|} = \frac{e^{x\gamma \ln c}}{e^{x\gamma \ln c}} = \frac{e^{\gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1]^\frac{1}{2}}{e^{\gamma \ln c}}.
\]
\[\frac{1}{e^{\gamma[\ln a - x \ln c]}[e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1]^2}.\]

With
\[x < \frac{\ln a}{\ln c} - \frac{B}{(2\pi - \varepsilon) \ln c}\]
\[\implies x \ln c < \ln a - \frac{B}{2\pi - \varepsilon}\]
\[\implies x \ln c - \ln a < -\frac{B}{2\pi - \varepsilon}\]
\[\implies \ln a - x \ln c > \frac{B}{2\pi - \varepsilon} \geq \frac{B}{2\pi N - \varepsilon}, \forall N \geq 1.\]

Thus,
\[\frac{1}{e^{\gamma[\ln a - x \ln c]}} \leq \frac{1}{e^{\cos \theta}} \leq \frac{1}{e^{-1}} = e,\]

and
\[\frac{|e^{xt}|}{|b^t - a^t|^2} \leq \frac{e}{[e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1]^2}.\]

The denominator of the preceding expression must not be zero. With \(0 \leq \theta \leq 2\pi\), we look at 3 cases:

Case 1: \(\cos \theta < 0\)

As \(N \to +\infty, \gamma \to -\infty\) and \(e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1 \to 1\) provided \(B > 0\).

Case 2: \(\cos \theta > 0\)

As \(N \to +\infty, \gamma \to +\infty\) and \(e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1 = e^{2\gamma B} \left(1 - \frac{2 \cos \rho B}{e^{\gamma B}} + \frac{1}{e^{\gamma B}}\right) \to +\infty\), provided \(B > 0\).

Case 3: \(\cos \theta = 0\)

Then \(\gamma = 0\) and \(e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1 = 2 - 2 \cos \rho B\), which is nonzero provided that \(\cos \rho B \neq 1\). Because \(\cos \theta = 0\), we have \(\rho = \pm(2N\pi - \varepsilon)/B\). Thus,

\[\cos \rho B = \cos[(\pm 2N\pi - \varepsilon)] = 1 \text{ iff } 2N\pi - \varepsilon = 2k\pi, \text{ for some integer } k.\]

This gives
\[2(N - k)\pi = \varepsilon,\]
which is not possible because $0 < \varepsilon < 1$.

Thus, under the conditions in the lemma, in all 3 cases $c^{xt}/(b^t - a^t)$ is bounded $\forall t \in C_N$. Let $M$ be a positive integer such that

$$\left| \frac{c^{xt}}{b^t - a^t} \right| < M.$$ 

Then

$$\left| \int_{C_N} \frac{c^{xt}}{b^t - a^t} \, dt \right| < M \int_{C_N} \left| \frac{dt}{t^n} \right|$$

$$= M \cdot \frac{(2N \pi - \varepsilon)2\pi}{(2N \pi - \varepsilon)^n}$$

$$= \frac{2M \pi B^{n-1}}{(2N \pi - \varepsilon)^{n-1}} \rightarrow 0 \text{ as } N \rightarrow +\infty \text{ for } n \geq 2.$$ 

This completes the proof of the lemma.

**Theorem 2.2.** Let $a, b, c$ be positive real numbers. The Fourier series of the Bernoulli-type polynomials $B_n(x; a, b, c)$ is given by

$$B_n(x; a, b, c) = \frac{1}{B} \sum_{k \in \mathbb{Z}^+} \frac{e^{t_k(x + c - \ln a)}}{t_k^n},$$

valid for $0 < x < \left( \ln a - \frac{B}{2\pi - \varepsilon} \right) / \ln c, \quad \ln c > 0$.

where $t_k = 2k\pi i / B$, $B = \ln b - \ln a > 0$.

**Proof.** When $\alpha = 1$, the generating function (1) reduces to

$$\frac{t}{b^t - a^t}c^{xt} = \sum_{n=0}^{\infty} B_n(x; a, b, c) \frac{t^n}{n!}, |t| < \frac{2\pi}{B}.$$ 

Applying the Cauchy Integral Formula yields

$$\frac{B_n(x; a, b, c)}{n!} = \frac{1}{2\pi i} \int_{C} \frac{c^{xt}}{(b^t - a^t)t^n} \, dt,$$

where $C$ is a circle with center at 0 and radius less than $\frac{2\pi}{B}$. Let

$$f(t) = \frac{c^{xt}}{(b^t - a^t)t^n}.$$
The function \( f(t) \) has simple poles at \( t \) such that \( b^t - a^t = 0 \) and a pole at \( t = 0 \) of order \( n \). Let \( t_k \) be those values of \( t \) such that \( b^t - a^t = 0 \). These values are obtained as follows.

\[
b^t - a^t = 0 \\
e^{t \ln b} - e^{t \ln a} = 0 \\
(e^{t \ln b} = e^{t \ln a})e^{-t \ln a}
\]

\[
\log(e^{t (\ln b - \ln a)}) = 1 \\
t(\ln b - \ln a) = \log 1 = i \arg 1 + 2k\pi i \\
t = \frac{2k\pi i}{B},
\]

where \( B = \ln b - \ln a \).

Let \( t_k = \frac{2k\pi i}{B}, k \in \mathbb{Z} \). Now let \( C_N \) be the circle described in Lemma 2.1. Applying the Residue Theorem, we have

\[
\lim_{N \to +\infty} \frac{1}{2\pi i} \int_{C_N} \frac{e^{ct}}{b^t - a^t} \frac{dt}{t^n} = \text{Res}(f(t), t = 0) + \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f(t), t = t_k).
\]

By Lemma 2.1,

\[
0 = \text{Res}(f(t), t = 0) + \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f(t), t = t_k) \\
0 = \frac{B_n(x; a, b, c)}{n!} + \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f(t), t = t_k) \\
\implies \frac{B_n(x; a, b, c)}{n!} = -\sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f(t), t = t_k).
\]

Computing the residue at \( t_k \):

\[
\text{Res}(f(t), t = t_k) = \lim_{t \to t_k} \frac{2e^{ct}}{(b^t - a^t)t^n} = \frac{2e^{ct_k}t_k^{-n}}{\mu},
\]

where

\[
\mu = \frac{d}{dt}(b^t - a^t)|_{t=t_k} \\
= \frac{d}{dt}(e^{t \ln b} - e^{t \ln a})|_{t=t_k} \\
= (\ln b e^{t_k \ln b} - \ln a e^{t_k \ln a}) \frac{e^{-t_k \ln a}}{e^{-t_k \ln a}}
\]
\[= e^{t_k \ln a} (\ln b \ e^{t_k (\ln b - \ln a)} - \ln a)\]
\[= e^{t_k \ln a} (\ln b - \ln a)\]
\[= B \cdot e^{t_k \ln a}.\]
Thus,
\[\text{Res}(f(t), t = t_k) = \frac{e^{t_k} t_k^{-n}}{B \cdot e^{t_k \ln a}} \frac{e^{t_k (x \ln c - \ln a)}}{4^n \cdot t_k^n}.\]

Consequently,
\[\frac{B_n(x; a, b, c)}{n!} = -\frac{1}{B} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{e^{t_k (x \ln c - \ln a)}}{4^n \cdot t_k^n}.\]

**Lemma 2.3.** Let \(a, b, c\) be positive real numbers. Let \(n \geq 1, N > 1\) and \(C_N\) be the circle about zero of radius \(R = ((2N + 1)\pi - \varepsilon)/B\), where \(0 < \varepsilon < 1\) and \(B = \ln b - \ln a, b > a\). For \(0 < x < \left(\ln a - \frac{B}{\pi - \varepsilon}\right) / \ln c, \ln c > 0\)
we have
\[\lim_{N \to +\infty} \int_{C_N} \frac{e^{xt}}{b^t + a^t} \frac{dt}{\ln t + 1} = 0.\]

**Proof.** We will show that the function \(\frac{e^{xt}}{b^t + a^t}\) is bounded on \(C_N\) under the conditions in Lemma 2.3.

From the proof of Lemma 2.1,
\[
\frac{|e^{xt}|}{|b^t + a^t|} = \frac{e^{x\gamma \ln c}}{e^{\gamma \ln a} [e^{2\gamma B} + 2e^{\gamma B} \cos \rho B + 1]^{\frac{1}{2}}},
\]
where here,
\[\gamma = \frac{(2N + 1)\pi - \varepsilon}{B} \cos \theta,\]
\[\rho = \frac{(2N + 1)\pi - \varepsilon}{B} \sin \theta,\]
\[0 \leq \theta \leq 2\pi.\]
With
\[x < \frac{\ln a}{\ln c} - \frac{B}{(\pi - \varepsilon) \ln c}\]
\[\implies \ln a - x \ln c > \frac{B}{\pi - \varepsilon} \geq \frac{B}{(2N + 1)\pi - \varepsilon}, \quad \forall \ N \geq 0.\]
Then
\[ \frac{1}{e^{\gamma (\ln a - x \ln c)}} \leq \frac{1}{e^{\cos \theta}} \leq \frac{1}{e^{-1}} = e. \]
Thus,
\[ \left| \frac{c^{x^t}}{b^t + a^t} \right| \leq \frac{e}{[e^{2\gamma B} + 2e^{\gamma B} \cos \rho B + 1]^2}. \]

The expression \( e^{2\gamma B} + 2e^{\gamma B} \cos \rho B + 1 \) must not be zero. The results for the cases \( \cos \theta < 0 \) and \( \cos \theta > 0 \) obtained in the proof of Lemma 2.1 still hold. We reconsider here the case \( \cos \theta = 0 \).

In the case \( \theta = 0, \gamma = 0 \) and
\[ e^{2\gamma B} + 2e^{\gamma B} \cos \rho B + 1 = 2 + 2 \cos \rho B, \]
which is nonzero provided that \( \cos \rho B \neq -1 \). Since \( \cos \theta = 0 \), we have \( \rho = (\pm 1)\frac{(2N + 1)\pi - \varepsilon}{B} \).

Thus,
\[ \cos \rho B = \cos(\pm(2N + 1)\pi - \varepsilon) = -1 \quad \text{iff} \quad (2N + 1)\pi - \varepsilon = (2k + 1)\pi, \]
for some integer \( k \). Equivalently,
\[ (2N + 1)\pi - (2k + 1)\pi = \varepsilon \]
\[ 2(N - k)\pi = \varepsilon, \]
which is not possible because \( 0 < \varepsilon < 1 \). Thus, under the conditions in the Lemma, the function \( \frac{c^{x^t}}{b^t + a^t} \) is bounded on \( C_N \) as \( N \to +\infty \).

Let \( M^* \) be a positive integer such that
\[ \left| \frac{c^{x^t}}{b^t + a^t} \right| < M^*, \quad \forall t \in C_N. \]

Then
\[ \left| \int_{C_N} \frac{c^{x^t}}{b^t + a^t} \cdot \frac{dt}{t^{n+1}} \right| \leq \int_{C_N} \left| \frac{c^{x^t}}{b^t + a^t} \right| \frac{|dt|}{t^{n+1}} \]
\[ < M^* \frac{(2N + 1)\pi - \varepsilon}{B} \cdot 2\pi \]
\[ \leq \frac{2M^*\pi B^n}{(2N + 1)\pi - \varepsilon} \]
which goes to zero as \( N \to +\infty \).
Theorem 2.4. Let $a, b, c$ be positive real numbers. The Fourier series of the Euler-type polynomials $E_n(x; a, b, c)$ is given by
\[
E_n(x; a, b, c) = \frac{2}{B} \sum_{k \in \mathbb{Z}} \frac{e^{t_k(x \ln c - \ln a)}}{t_k^{n+1}},
\]
valid for
\[
0 < x < \left( \ln a - \frac{B}{\pi - \varepsilon} \right) / \ln c, \quad \ln c > 0
\]
where $t_k = (2k + 1)\pi i / B$, $B = \ln b - \ln a > 0$.

Proof. When $\alpha = 1$, the generating function (2) reduces to
\[
\left( \frac{2}{b' + a'} \right) c^x = \sum_{n=0}^{\infty} E_n(x; a, b, c) \frac{t^n}{n!}, \quad |t| < \frac{\pi}{B}.
\]
Applying the Cauchy Integral Formula,
\[
E_n(x; a, b, c) = \frac{1}{2\pi i} \int_{C} \frac{2e^{xt}}{(b' + a')t^{n+1}} dt,
\]
where $C$ is a circle about zero of radius $\pi / B$. Let
\[
g(t) = \frac{2e^{xt}}{(b' + a')t^{n+1}}.
\]
The function $g(t)$ has a pole at $t = 0$ of order $n + 1$ and simple poles at the values of $t$ such that $b' + a' = 0$. These values are $t_k = (2k + 1)\pi i / B$, $k \in \mathbb{Z}$ which are obtained similarly as those in Theorem 2.2. Let $C_N$ be the circle described in Lemma 2.3. From the Residue Theorem,
\[
\lim_{N \to +\infty} \frac{1}{2\pi i} \int_{C_N} g(t) d(t) = \text{Res}(g(t), t = 0) + \sum_{k \in \mathbb{Z}} \text{Res}(g(t), t = t_k).
\]
By Lemma 2.3, we have
\[
E_n(x; a, b, c) = -\sum_{k \in \mathbb{Z}} \text{Res}(g(t), t = t_k).
\]
Computing the residues of $g(t)$ at $t_k$:
\[
\text{Res}(g(t), t = t_k) = \lim_{t \to t_k} \frac{2e^{xt \ln c}}{b' + a' t^{n+1}}
\]
\[
= \frac{2e^{xt_k \ln c} t_k^{n-1}}{b' + a'},
\]
where
\[ \nu = \frac{d}{dt} (b^t + a^t)|_{t=t_k} \]
\[ = \left( (\ln b) e^{t_k \ln b} + (\ln a) e^{t_k \ln a} \right) e^{-t_k \ln a} \]
\[ = e^{t_k \ln a} \left( (\ln b) e^{t_k (\ln b - \ln a)} + \ln a \right) \]
\[ = e^{t_k \ln a} [-\ln b + \ln a] \]
\[ = -B \cdot e^{t_k \ln a}. \]

Thus,
\[ \text{Res}(g(t), t = t_k) = 2 \quad e^{t_k x \ln c - \ln a} - B \cdot t_n^{n+1}. \]

Consequently,
\[ \frac{E_n(x; a, b, c)}{n!} = \frac{2}{B} \sum_{k \in \mathbb{Z}} e^{t_k (x \ln c - \ln a)} t_k^{n+1}. \]

**Theorem 2.5.** Let \( a, b, c \) be positive real numbers. The Fourier series of the Genocchi-type polynomials \( G_n(x; a, b, c) \) is given by
\[ \frac{G_n(x; a, b, c)}{n!} = \frac{2}{B} \sum_{k \in \mathbb{Z}} e^{t_k (x \ln c - \ln a)} t_k^n, \]
valid for
\[ 0 < x < \left( \ln a - \frac{B}{\pi} \right) / \ln c, \quad \ln c > 0 \]
where \( t_k = (2k + 1)\pi i/B, \quad B = \ln b - \ln a > 0. \)

**Proof.** The theorem follows from Theorem 2.4.

3. The case \( \alpha \geq 2 \)

**Lemma 3.1.** Let \( a, b, c \) be positive real numbers. Let \( n \geq \alpha \geq 2, \alpha \in \mathbb{Z}^+, \quad N > 1 \) and \( C_N \) be the circle about zero of radius \( R = (2N\pi - \varepsilon)/B \), where \( 0 < \varepsilon < 1 \) and \( B = \ln b - \ln a > 0. \)

For
\[ 0 < x < \left( \alpha \ln a - \frac{B}{2\pi - \varepsilon} \right) / \ln c, \quad \ln c > 0 \]
we have
\[ \lim_{N \to +\infty} \int_{C_N} \frac{e^{xt}}{(b^t - a^t)^{\alpha}} t^{n-\alpha+1} = 0. \]
Proof. We will show that the function \( \frac{e^{\gamma t}}{(b^t - a^t)^\alpha} \) is bounded on \( C_N \). From Lemma 2.1,

\[
|b^t - a^t| = e^{\gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1]^{\frac{1}{2}}
\]

where \( t \in C_N, t = \gamma + i\rho \). That is,

\[
\gamma = \frac{2N\pi - \varepsilon}{B} \cos \theta, \quad \rho = \frac{2N\pi - \varepsilon}{B} \sin \theta,
\]

\[0 \leq \theta \leq 2\pi.\]

Then

\[
|b^t - a^t|^{\alpha} = \frac{e^{\alpha \gamma \ln a}[e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1]^{\alpha}}{B}
\]

and

\[
\left| \frac{e^{\gamma t}}{(b^t - a^t)^\alpha} \right| = \frac{e^{\gamma \ln c}}{e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1}^{\frac{\alpha}{2}}
\]

Impose that \( \alpha \ln a - x \ln c > \frac{B}{2N\pi - \varepsilon}, \forall N \geq 1. \)

This is satisfied when

\[
\alpha \ln a - x \ln c > \frac{B}{2\pi - \varepsilon}.
\]

Equivalently, impose that

\[
0 < x < \left( \alpha \ln a - \frac{B}{2\pi - \varepsilon} \right) / \ln c.
\]

Then

\[
\frac{1}{e^{\gamma(\alpha \ln a - x \ln c)}} < \frac{1}{e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1} = e.
\]

Consequently,

\[
\left| \frac{e^{\gamma t}}{(b^t - a^t)^\alpha} \right| < \frac{e}{e^{2\gamma B} - 2e^{\gamma B} \cos \rho B + 1}^{\frac{\alpha}{2}}.
\]

It follows from Lemma 2.1 that the right hand side above is bounded on \( C_N \) as \( N \to +\infty. \)

That is, there is a constant \( M \) such that

\[
\left| \frac{e^{\gamma t}}{(b^t - a^t)^\alpha} \right| < M, \quad t \in C_N \quad \text{and} \quad 0 < x < \left( \alpha \ln a - \frac{B}{2\pi - \varepsilon} \right) / \ln c.
\]

Thus,

\[
\left| \int_{C_N} \frac{e^{\gamma t}}{(b^t - a^t)^\alpha} \frac{dt}{t^{n-\alpha+1}} \right| < M \int_{C_N} \frac{|dt|}{t^{n-\alpha+1}} < \frac{M \cdot 2N\pi - \varepsilon \cdot 2\pi}{\left( \frac{2N\pi - \varepsilon}{B} \right)^{n-\alpha+1}}
\]
\[
\frac{\pi M B^{\alpha-n}}{(2N\pi - \varepsilon)^{n-\alpha}}, \quad n \geq \alpha.
\]
\[
\rightarrow 0 \text{ as } N \rightarrow +\infty.
\]

**Lemma 3.2.** For \(a, b, c \in \mathbb{R}^+, x \in \mathbb{R}, \nu, \alpha \in \mathbb{Z}^+\) with fixed \(\nu \geq \alpha,\)

\[
B^{(\alpha)}_\nu(x;a,b,c) = \sum_{l=0}^{\nu} \left(\begin{array}{c} \nu \\ l \end{array}\right) B^{(\alpha)}_l(0;a,b,c)(x \ln c)^{\nu-l}.
\]

**Proof.**

\[
\left(\frac{t}{b^l - a^l}\right)^\alpha c^{x^l} \cdot c^{yt} = \sum_{n=0}^{\infty} B^{(\alpha)}_n(x;a,b,c) \frac{t^n}{n!} \left(\sum_{n=0}^{\infty} \frac{(yt \ln c)^n}{n!}\right)
\]

\[
\left(\frac{t}{b^l - a^l}\right)^\alpha c^{(x+y)^l} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} B^{(\alpha)}_l(x;a,b,c) \frac{l! (yt \ln c)^{n-l}}{n! (n-l)!} \cdot \frac{n!}{n!}
\]

\[
\sum_{n=0}^{\infty} B^{(\alpha)}_n(x+y;a,b,c) \frac{ty}{n!} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) B^{(\alpha)}_l(x;a,b,c) (y \ln c)^{n-l} \frac{t^n}{n!}.
\]

Thus,

\[
B^{(\alpha)}_n(x+y;a,b,c) = \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) B^{(\alpha)}_l(x;a,b,c)(y \ln c)^{n-l}.
\]

Take \(y = z, x = 0\). Then

\[
B^{(\alpha)}_n(z;a,b,c) = \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) B^{(\alpha)}_l(0;a,b,c)(z \ln c)^{n-l}
\]

Now take \(n = \nu\) and \(z = x\), we have

\[
B^{(\alpha)}_\nu(x;a,b,c) = \sum_{l=0}^{\nu} \left(\begin{array}{c} \nu \\ l \end{array}\right) B^{(\alpha)}_l(0;a,b,c)(x \ln c)^{\nu-l}.
\]

**Theorem 3.3.** Let \(a, b, c\) be positive real numbers, \(N, n, \alpha \in \mathbb{Z}^+\) with \(n \geq \alpha \geq 2, N > 1\) and \(C_N\) be the circle about zero of radius \(R = (2N\pi - \varepsilon)/B\), where \(0 < \varepsilon < 1\) and \(B = \ln b - \ln a > 0\). The Fourier series of the Bernoulli-type polynomials \(B^{(\alpha)}_n(x;a,b,c)\) of order \(\alpha\) is given by
\[
B_n^{(\alpha)}(x; a, b, c) = \frac{1}{n!} \sum_{k \in \mathbb{Z}, k \neq 0} \left( \sum_{\nu=0}^{\alpha-1} \frac{(\alpha - n - 1)_{\alpha - 1 - \nu}}{\nu!} (2k\pi i)^\nu B_\nu^{(\alpha)}(x; a, b, c) \right) e^{2k\pi i(x \ln c - \alpha \ln b)} (2k\pi i)^n,
\]
valid for \(0 < x < \left( \alpha \ln a - \frac{B}{2\pi - \varepsilon} \right) / \ln c, \ln c > 0\), where \(B_\nu^{(\alpha)}(x; a, b, c)\) is given in Lemma 3.2.

**Proof.** Applying the Cauchy Integral Formula to (1),
\[
B_n^{(\alpha)}(x; a, b, c) = \frac{1}{2\pi i} \int_{C} \frac{e^{xt}}{(b^t - a^t)^\alpha t^{n-1}} dt,
\]
where \(C\) is a circle about the origin with radius less than \(\frac{2\pi}{B}\). Let
\[
f_\alpha(t) = \frac{e^{xt}}{(b^t - a^t)^\alpha t^{n-1}}, \quad n > \alpha.
\]
The function \(f_\alpha(t)\) has a pole of order \(n - \alpha + 1\) at \(t = 0\) and a pole of order \(\alpha\) at the zeros of \(b^t - a^t\) which are given by \(t_k = \frac{2k\pi i}{B}, k \in \mathbb{Z}\). Now let \(C_N, N > 1\) be the circle described in Lemma 3.1. Applying the Residue Theorem,
\[
\lim_{N \to +\infty} \frac{1}{2\pi i} \int_{C_N} \frac{e^{xt}}{(b^t - a^t)^\alpha t^{n-1}} dt = \text{Res}(f_\alpha(t), t = 0) + \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f_\alpha(t), t = t_k).
\]
By Lemma 3.1,
\[
0 = \text{Res}(f_\alpha(t), t = 0) + \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f_\alpha(t), t = t_k)
\]
\[
0 = B_n^{(\alpha)}(x; a, b, c) = \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f_\alpha(t), t = t_k)
\]
\[
\iff
B_n^{(\alpha)}(x; a, b, c) = - \sum_{k \in \mathbb{Z}, k \neq 0} \text{Res}(f_\alpha(t), t = t_k).
\]
Computing the residues at \(t_k\):
\[
\text{Res}(f_\alpha(t), t = k) = \frac{1}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \frac{(t - t_k)\alpha}{(b^t - a^t)^\alpha} \frac{1}{t^{n-1}}
\]
\[
= \frac{1}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \left[ \frac{(t - t_k)\alpha}{(b^t - a^t)^\alpha} \frac{e^{xt\ln c}}{t^{n-1}} \right].
\]
Taking \( x = 0 \) in (1) gives
\[
\left( \frac{t}{b^t - a^t} \right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{t^n}{n!}.
\]
Replacing \( t \mapsto t - t_k \) and writing \( b^t = e^{t \ln b}, \ a^t = e^{t \ln a}, \)
\[
\frac{(t - t_k)^\alpha}{(e^{(t-t_k) \ln b} - e^{(t-t_k) \ln a})^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{(t - t_k)^n}{n!}.
\] (6)

Multiplying and dividing the left hand side of (6) by \( e^{\alpha t_k \ln b} \) gives
\[
\frac{(t - t_k)^\alpha e^{\alpha t_k \ln b}}{(e^{t \ln b} - e^{t \ln a} e^{t_k B})^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{(t - t_k)^n}{n!}.
\] (7)

With \( t_k = (2k\pi i)/B \), we have \( e^{t_k B} = e^{2k\pi i} = 1 \). Thus, (7) becomes
\[
\frac{(t - t_k)^\alpha}{(b^t - a^t)^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{(t - t_k)^n}{n!}.
\] (8)

Substituting (8) to (5) gives,
\[
\text{Res}(f, t = t_k) = \frac{e^{-\alpha t_k \ln b}}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \left( \frac{e^{xt \ln c}}{t^{\alpha-a+1}} \sum_{n=0}^{\infty} B_n(0; a, b, c) \frac{(t - t_k)^n}{n!} \right).
\]

The derivatives will be obtained using Leibniz Rule. This is done as follows. Recalling the Leibniz Rule for derivatives,
\[
\frac{d^n}{dt^n} (fg) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{d^{n-k}}{dt^{n-k}} f \right) \left( \frac{d^{k}}{dt^{k}} g \right).
\]

Let \( f = t^{\alpha-n-1}, \ g = e^{xt \ln c} \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{(t - t_k)^n}{n!} \).
Then
\[
\frac{d^{\alpha-1}}{dt^{\alpha-1}} (fg) = \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} \left( \frac{d^{\nu}}{dt^{\nu}} \right) f \left( \frac{d^{\nu-1}}{dt^{\nu-1}} g \right)
\]
\[
\sum_{\nu=0}^{\alpha-1} \binom{\alpha - 1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu} t^{n-\nu} \left( \frac{d^{\nu}}{dt^{\nu}} g \right)
= \sum_{\nu=0}^{\alpha-1} \binom{\alpha - 1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu} t^{n+\nu} \left( \frac{d^{\nu}}{dt^{\nu}} g \right),
\]
where the notation \((n)_k\) is designed as
\[
(n)_k = n(n - 1)(n - 2)\ldots(n - k + 1).
\]

Also,
\[
(\alpha - n - 1)_{\alpha-1-\nu} = (-1)^{\alpha-1-\nu} (n - \alpha + 1) (n - \alpha + 2) (n - \alpha + 3) \ldots (n - \alpha + \alpha - \nu - 1).
\]

On the other hand,
\[
\frac{d^{\nu}}{dt^{\nu}} g = \frac{d^{\nu}}{dt^{\nu}} \left( \sum_{n=0}^{\infty} \binom{\alpha}{n} (0; a, b, c) \frac{(t - t_k)^n}{n!} \cdot e^{x \ln c} \right)
= \sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{\nu - l} e^{x \ln c} \sum_{n \geq l} B_n^{(a)}(0; a, b, c) \frac{(t - t_k)^n}{n!}.
\]

Now take the limit as \(t \to t_k\). Then
\[
\lim_{t \to t_k} \frac{d^{\nu}}{dt^{\nu}} g = \sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{\nu - l} e^{t_k \ln c} B_l^{(a)}(0; a, b, c).
\]
Substituting to (9) and taking the limit as \(t \to t_k\) will yield
\[
\lim_{t \to t_k} \frac{d^{\nu-1}}{dt^{\nu-1}} (fg) = \sum_{\nu=0}^{\alpha-1} \binom{\alpha - 1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu} t_k^{n+\nu} \sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{\nu - l} e^{t_k \ln c} B_l^{(a)}(0; a, b, c).
\]
\[
= \sum_{\nu=0}^{\alpha-1} \binom{\alpha - 1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu} t_k^{n+\nu} e^{t_k \ln c} \left( \sum_{l=0}^{\nu} \binom{\nu}{l} (x \ln c)^{\nu - l} B_l^{(a)}(0; a, b, c) \right). \quad (10)
\]
Applying Lemma 3.2 to (10),

$$\lim_{t \to k} \frac{d^{\alpha-1}}{dt^{\alpha-1}}(fg) = \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} (\alpha - n - 1)_{\alpha-1-\nu} t_k^{-\nu} e^{t_k \ln c}B^n_{\nu}(x; a, b, c).$$

Thus,

$$\text{Res}(f_\alpha(t), t = t_k) = e^{t_k (x \ln c - \alpha \ln b)} \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} \frac{(-n + 1 + x \ln c)(2k\pi i)^\nu}{n!},$$

(11)

The desired Fourier series is obtained by substituting (11) to (4).

Taking $\alpha = 1$, the Fourier series in Theorem 3.3 reduces to that in Theorem 2.2. For $\alpha = 2$, Theorem 3.3 gives the Fourier series of the Bernoulli-type polynomials of order 2.

This is given by

$$B_n^{(2)}(x; a, b, c) = \frac{-1}{B^2} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{(-n + 1 + x \ln c)(2k\pi i)^\nu}{n!},$$

valid under the conditions in Theorem 3.3.

**Lemma 3.4.** Let $a, b, c$ be positive real numbers with $b > a$, $n, \alpha \in \mathbb{Z}^+$ with $n \geq \alpha$, $N > 1$ and $C_N$ be the circle about zero of radius $R = ((2N + 1)\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a$. For $\ln c > 0$ and

$$0 < x < \left(\alpha \ln a - \frac{B}{\pi - \varepsilon}\right)\int_0^n \frac{c^{\gamma t}}{(b^t + a^t)^\alpha t^{n+1}} dt = 0.$$

(12)

we have

$$\lim_{N \to +\infty} \int_{C_N} \frac{c^{\gamma t}}{(b^t + a^t)^\alpha t^{n+1}} dt = 0.$$

**Proof.** From the proof of Lemma 3.2,

$$|b^t + a^t| = e^{\alpha \gamma \ln a} \left|e^{2\gamma B} + 2e^{\gamma B} \cos \rho B + 1\right|^{\frac{\gamma}{2}}, \quad t \in C_N$$

where $t = \gamma + i\rho = \frac{(2N + 1)\pi - \varepsilon}{B}(\cos \theta + i \sin \theta), 0 \leq \theta \leq 2\pi$.

Thus,

$$\gamma = \frac{(2N + 1)\pi - \varepsilon}{B} \cos \theta, \quad \rho = \frac{(2N + 1)\pi - \varepsilon}{B} \sin \theta.$$
For $x$ satisfying (12), it follows that

$$\alpha \ln a - x \ln c > \frac{B}{\pi - \varepsilon} \geq \frac{B}{(2N + 1)\pi - \varepsilon}, \quad \forall N \geq 1.$$ 

Then

$$\frac{1}{e^{|x \ln a - x \ln c|}} = \frac{1}{e^{\frac{(2N + 1)\pi - \varepsilon}{\pi - \varepsilon} \cos \theta(\alpha \ln a - x \ln c)}} < \frac{1}{e^{\cos \theta}} < 1.$$

Consequently,

$$\left| \frac{c^{\nu t}}{(b^t + a^t)^\nu} \right| = \left| \frac{c^t}{b^t + a^t} \right| = \frac{1}{e^{\nu |\alpha \ln a - x \ln c|} \left| e^{2\gamma B + 2e\gamma B \cos \rho B + 1} \right|} < \frac{1}{e^{\cos \theta}} < 1.$$

The expression $e^{2\gamma B + 2e\gamma B \cos \rho B + 1} \neq 0$ for all $t \in C_N$ as discussed in Lemma 2.3. Thus, there exists an integer $M$ such that

$$\left| \frac{c^t}{(b^t + a^t)^\nu} \right| < M, \quad \forall t \in C_N.$$

Hence,

$$\left| \int_{C_N} \frac{c^{\nu t}}{(b^t + a^t)^\nu} \frac{dt}{t^{n+1}} \right| \leq M \int_{C_N} \left| \frac{dt}{t^{n+1}} \right| \leq M \cdot \frac{(2N + 1)\pi - \varepsilon}{B} \cdot \frac{2\pi}{(2N + 1)\pi - \varepsilon} \cdot \frac{2MB^n}{((2N + 1)\pi - \varepsilon)^{n+1}} \cdot n > 1, \quad \rightarrow 0 \quad \text{as} \quad N \rightarrow +\infty.$$ 

**Lemma 3.5.** For $a, b, c \in \mathbb{R}^+$, $x \in \mathbb{R}$, $\nu, \alpha \in \mathbb{Z}^+$ with fixed $\nu \geq \alpha \geq 2$,

$$E^{(\alpha)}_{\nu}(x; a, b, c) = \sum_{l=0}^{\nu} \binom{\nu}{l} E^{(\alpha)}_{l}(0; a, b, c)(x \ln c)^{\nu-l}.$$ 

**Proof.** The proof is done similarly as that of Lemma 3.2.

**Theorem 3.6.** Let $a, b, c$ be positive real numbers with $b > a$, $N, n, \alpha \in \mathbb{Z}^+, n \geq \alpha \geq 2$, $N > 1$ and $C_N$ be the circle about zero of radius $R = ((2N + 1)\pi - \varepsilon)/B$, where $0 < \varepsilon < 1$.
and $B = \ln b - \ln a$. The Fourier series of the Euler-type polynomials $E_n^{(\alpha)}(x; a, b, c)$ of order $\alpha$ is given by

$$E_n^{(\alpha)}(x; a, b, c) = \frac{-2\alpha}{(\alpha - 1)!} \sum_{k \in \mathbb{Z}} \sum_{\nu = 0}^{\alpha - 1} \binom{\alpha - 1}{\nu} (-1)^{\nu} B_{\nu}^{(\alpha)}(x; a, b, c) \frac{e^{t_k (\ln c - \alpha \ln b)}}{t_k^{\nu + \alpha - \nu}},$$

valid for

$$0 < x < \alpha \ln a - \frac{B}{\pi - \varepsilon}/\ln c, \quad \ln c > 0.$$

**Proof.** Applying the Cauchy-Integral Formula to (2),

$$E_n^{(\alpha)}(x; a, b, c) = \frac{1}{2\pi i} \int_C \frac{2^\alpha e^{xt}}{(b^t + a^t)^{\alpha} t^{n+1}} dt,$$

where $C$ is a circle about zero of radius less than $\frac{\pi}{B}$. Let

$$g_\alpha(t) = \frac{e^{xt}}{(b^t + a^t)^{\alpha} t^{n+1}}.$$

Then

$$E_n^{(\alpha)}(x; a, b, c) = \frac{1}{2\pi i} \int_C g_\alpha(t) dt.$$

The function $g_\alpha(t)$ has a pole of order $n + 1$ at $t = 0$ and a pole of order $\alpha$ at the zeros of $b^t + a^t$ which are given by $t_k = ((2k + 1)\pi i)/B, k \in \mathbb{Z}$. Applying the Residue Theorem and taking the limit as $N \to +\infty$,

$$\lim_{N \to +\infty} \frac{1}{2\pi i} \int_C g_\alpha(t) dt = \text{Res}(g_\alpha(t), t = 0) + \sum_{k \in \mathbb{Z}} \text{Res}(g_\alpha(t), t = t_k).$$

It follows from Lemma 3.4 that

$$E_n^{(\alpha)}(x; a, b, c) = -\sum_{k \in \mathbb{Z}} \text{Res}(g_\alpha(t), t = t_k).$$

Computing the residues at $t_k$:

$$\text{Res}(g_\alpha(t), t = t_k) = \frac{1}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha - 1}}{dt^{\alpha - 1}} \left( (t - t_k)^\alpha \frac{e^{xt}}{(b^t + a^t)^t} \cdot \frac{1}{t^{n+1}} \right). \quad (13)$$

Now use (7). With $t_k = (2k + 1)\pi i/B, e^{t_k B} = e^{(2k+1)\pi i} = -1$. Thus, (7) becomes,

$$\frac{(t - t_k)^n e^{\alpha t_k \ln b}}{(e^{t \ln b} + e^{t \ln a})^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(0; a, b, c) \frac{(t - t_k)^n}{n!}.$$
\[
(t - t_k)^{\alpha} \over (b^t + a^t)^\alpha = e^{-at_k \ln b} \sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c) \frac{(t - t_k)^n}{n!}.
\]  
(14)

Substituting (14) to (13),

\[
\text{Res}(g_\alpha(t), t = t_k) = \frac{e^{-at_k \ln b}}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha - 1}}{dt^{\alpha - 1}} \left( e^{xt} t^{-n-1} \sum_{n=0}^{\infty} B_n^{(\alpha)}(0;a,b,c) \frac{(t - t_k)^n}{n!} \right).
\]

Applying the Leibniz Rule for differentiation,

\[
\text{Res}(g_\alpha(t), t = t_k) = \frac{e^{t_k(\ln c - \alpha \ln b)}}{(\alpha - 1)!} \sum_{\nu = 0}^{\alpha - 1} \binom{\alpha - 1}{\nu} (-n - 1)_{\alpha - 1 - \nu} t_k^{-n - \alpha + \nu} B_\nu^{(\alpha)}(x;a,b,c).
\]

Thus,

\[
E_n^{(\alpha)}(x;a,b,c) = \frac{2^\alpha}{(\alpha - 1)!} \sum_{k \in \mathbb{Z}} e^{t_k(\ln c - \alpha \ln b)} \sum_{\nu = 0}^{\alpha - 1} \binom{\alpha - 1}{\nu} (-n - 1)_{\alpha - 1 - \nu} t_k^{-n - \alpha + \nu} B_\nu^{(\alpha)}(x;a,b,c)
\]

\[
= \frac{2^\alpha}{(\alpha - 1)!} \sum_{k \in \mathbb{Z}} \sum_{\nu = 0}^{\alpha - 1} \binom{\alpha - 1}{\nu} (-n - 1)_{\alpha - 1 - \nu} B_\nu^{(\alpha)}(x;a,b,c) e^{t_k(\ln c - \alpha \ln b)} t_k^{n + \alpha - \nu},
\]

which is the desired Fourier series of $E_n^{(\alpha)}(x;a,b,c)$.

Taking $\alpha = 1$, the Fourier series in Theorem 3.6 reduces to that in Theorem 2.4.

For $\alpha = 2$, the Fourier series is given by

\[
E_n^{(2)}(x;a,b,c) = \frac{1}{2^2(n!)} \sum_{k \in \mathbb{Z}} (-n - 1)_{-1} B_0^{(2)}(x;a,b,c) e^{t_k(\ln c - 2\ln b)} t_k^{n+2} + B_1^{(2)}(x;a,b,c) e^{t_k(\ln c - 2\ln b)} t_k^{n+1},
\]

where

\[
B_0^{(2)}(x;a,b,c) = \frac{1}{B^2},
\]

\[
B_1^{(2)}(x;a,b,c) = \frac{x \ln c}{B^2} + \frac{\ln ab - (\ln b)^2 - \ln b \ln a - (\ln a)^2}{B^2}.
\]

Lemma 3.7. Let $a, b, c$ be positive real numbers with $b > a$. Let $N, n, \alpha \in \mathbb{Z}^+, N > 1$ and $C_N$ be the circle about zero with radius $R = \frac{(2N + 1)\pi - \varepsilon}{B}$, where $0 < \varepsilon < 1$ and $B = \ln b - \ln a$. For

\[
0 < x < \left( \frac{\alpha \ln a - B}{\pi - \varepsilon} \right) / \ln c, \quad \ln c > 0
\]

we have

\[
\lim_{N \to +\infty} \int_{C_N} \frac{e^{xt}}{(b^t + a^t)^\alpha \ln a + 1} \frac{dt}{\ln a + 1} = 0.
\]
Lemma 3.8. For \( a, b, c \in \mathbb{R}^+, x \in \mathbb{R}, \nu, \alpha \in \mathbb{Z}^+ \) with fixed \( \nu \geq \alpha \),
\[
G_{\nu}^{(\alpha)}(x; a, b, c) = \sum_{l=0}^{\nu} \binom{\nu}{l} G_{l}^{(\alpha)}(0; a, b, c)(x \ln c)^{\nu-l}.
\]

Proof. This follows from Lemma 3.4.

Theorem 3.9. Let \( a, b, c > 0 \) be positive real numbers with \( b > a \). Let \( N, n, \alpha \in \mathbb{Z}^+ \) with \( n \geq \alpha \geq 2 \), \( N > 1 \) and \( C_N \) be the circle about zero of radius \( R = ((2N + 1)\pi - \varepsilon)/B \), where \( 0 < \varepsilon < 1 \) and \( B = \ln b - \ln a \). The Fourier series of the Genocchi-type polynomials \( G_{n}^{(\alpha)}(x; a, b, c) \) of order \( \alpha \) is given by
\[
\frac{G_{n}^{(\alpha)}(x; a, b, c)}{n!} = -\frac{2^\alpha}{(\alpha - 1)!} \sum_{k \in \mathbb{Z}} \sum_{\nu=0}^{\alpha-1} \frac{(\alpha - 1 - \nu)(\alpha - n - 1)_{\alpha-1-\nu}}{\nu} B_{\nu}^{(\alpha)}(x; a, b, c) \frac{e^{t(x \ln c - \alpha \ln b)}}{t_k^{n+\nu}}.
\]

Proof. Applying the Cauchy Integral Formula to (3),
\[
\frac{G_{n}^{(\alpha)}(x; a, b, c)}{n!} = \frac{2^\alpha}{2\pi i} \int_C \frac{c^t}{(b^t + a^t)^\alpha t^{n-a+1}} \, dt,
\]
where \( C \) is a circle about zero of radius \( < \pi/B \). Let
\[
h_{\alpha}(t) = \frac{c^t}{(b^t + a^t)^\alpha t^{n-a+1}}.
\]
This function has a pole of order \( n - \alpha + 1 \) at \( t = 0 \) and a pole of order \( \alpha \) at the zeros of \( b^t + a^t \). These poles are given by \( t_k = (2k + 1)\pi i/B, k \in \mathbb{Z} \). Applying the Residue Theorem and taking the limit as \( N \to +\infty \),
\[
\lim_{N \to +\infty} \frac{1}{2\pi i} \int_C h_{\alpha}(t) \, dt = \text{Res}(h_{\alpha}(t), t = 0) + \sum_{k \in \mathbb{Z}} \text{Res}(h_{\alpha}(t), t = t_k).
\]
It follows from Lemma 3.7 that
\[
\frac{G_{n}^{(\alpha)}(x; a, b, c)}{n!} = -\sum_{k \in \mathbb{Z}} \text{Res}(h_{\alpha}(t), t = t_k),
\]
where
\[
\text{Res}(h_{\alpha}(t), t = t_k) = \frac{1}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \left( (t-t_k)^{\alpha} \frac{c^t}{(b^t + a^t)^\alpha} \cdot \frac{1}{t^{n+1-\alpha}} \right).
\]
From (14),
\[
\text{Res}(h_{\alpha}(t), t = t_k) = \frac{e^{-\alpha t_k \ln b}}{(\alpha - 1)!} \lim_{t \to t_k} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \left( c^{xt-n+\alpha-1} \sum_{n=0}^{\infty} B_{n}^{(\alpha)}(0; a, b, c) \frac{(t-t_k)^\alpha}{n!} \right).
\]
Following the computation in the Euler-type polynomials,

\[ \text{Res}(h_{\alpha}(t), t = t_k) = e^{t_k(x \ln c - \alpha \ln b)} \frac{\alpha - 1}{(\alpha - 1)!} \sum_{\nu=0}^{\alpha-1} \left( \frac{\alpha - 1}{\nu} \right) (\alpha - n - 1)^{\alpha-1} t_k^{-n+\nu} B_{\nu}^{(\alpha)}(x; a, b, c). \]  

(18)

Substituting (18) to (17) gives the desired Fourier series.

Taking \( \alpha = 1 \), the Fourier series in Theorem 3.9 reduces to that in Theorem 2.5. Taking \( \alpha = 2 \) and \( n = 4 \), the series gives

\[ \frac{G_{4}^{(2)}(x; a, b, c)}{2^4(4!)} = - \sum_{k \in \mathbb{Z}} \left\{ -3B_{0}^{(2)}(x; a, b, c) \frac{e^{(2k+1)\pi i(x \ln c - 2 \ln b)}}{(2k+1)^3} \right\} + B_{1}^{(2)}(x; a, b, c) \frac{e^{(2k+1)\pi i(x \ln c - 2 \ln b)}}{(2k+1)^3} \]

where \( B_{0}^{(2)}(x; a, b, c) \) and \( B_{1}^{(2)}(x; a, b, c) \) are given in (15) and (16), respectively.

4. Some Remarks

The Fourier series expansions obtained in this paper for \( B_{n}^{(\alpha)}(x; a, b, c) \), \( E_{n}^{(\alpha)}(x; a, b, c) \) and \( G_{n}^{(\alpha)}(x; a, b, c) \) are useful in establishing the asymptotic formulas of these polynomials. It would then be interesting to investigate the asymptotic behavior of these polynomials.

Acknowledgements

This research is funded by Cebu Normal University through its Center for Research and Development and the Research Institute for Computational Mathematics and Physics.

References

[1] Bedoya, D., Ortega, M., Ramirez, W., Urieles, A., Fourier expansion and integral representation generalized Apostol-type Frobenius-Euler polynomials. Adv. Differ. Equ. 2020 (2020), Article 534.

[2] Bedoya, D., Ortega, M., Ramirez, W., Urieles. New biparametric families of Apostol-Frobenius-Euler polynomials of level \( m \), Mat. Stud. 55 (2021), 10-23.

[3] Cesarano, C., Ramirez, W., Khan, S. A new class of degenerate Apostol?type Hermite polynomials and applications, Dolomites Res. Notes Approx. 15 (2022), 1-10.

[4] Cesarano, C., Ramirez, W., Some new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, Carpathian Math. Publ. 14(2) (2022).
[5] Churchill, R. V., Brown, J. W. *Complex Variable and Applications*, 8th ed.; McGraw-Hill Book: New York, NY, USA, 2008.

[6] Corcino, C., Castañeda, W., Corcino, R., Asymptotic Approximations of Apostol-Tangent Polynomials in terms of Hyperbolic Functions, *Computer Modeling in Engineering and Sciences*, 132(1) (2022), 133-151.

[7] Corcino, C., Corcino, R., Fourier series for the tangent polynomials, tangent-Bernoulli and tangent-Genocchi polynomials of higher order, *Axioms* 11(3) (2022), Article 86.

[8] Corcino, C., Corcino, R., Casquejo, J.; Fourier expansion,integral representation and explicit formula at rational arguments of the tangent polynomials of higher-order, *European Journal of Pure and Applied Mathematics* 14(4) (2021), 1457-1466.

[9] Khan, W., Srivastava, D., On the generalized Apostol-type Frobenius-Genocchi polynomials, *Filomat* 33(7) (2019), 1967-1977.

[10] Corcino, R., Corcino, C. Higher Order Apostol-Frobenius-Type Poly-Genocchi Polynomials With Parameters $a, b$ and $c$. *J. Inequal. Spec. Funct.* 12 (2021), 54-72.