Input-output theory for fermions in an atom cavity

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We generalize the quantum optical input-output theory developed for optical cavities to ultracold fermionic atoms confined in a trapping potential, which forms an "atom cavity". In order to account for the Pauli exclusion principle, quantum Langevin equations for all cavity modes are derived. The dissipative part of these multi-mode Langevin equations includes a coupling between cavity modes. We also derive a set of boundary conditions for the Fermi field that relate the output fields to the input fields and the field radiated by the cavity. Starting from a constant uniform current of fermions incident on one side of the cavity, we use the boundary conditions to calculate the occupation numbers and current density for the fermions that are reflected and transmitted by the cavity.

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I. INTRODUCTION

In light of the remarkable achievement of Bose-Einstein condensation in 1995 [1], there has been a growing application of ideas from quantum optics to matter waves. This new field of atom optics [2] has included both theoretical and experimental investigations of matter wave coherence [3-11], atom lasers [3,4], non-linear effects in matter waves including matter wave mixing [10,11], parametric amplification and squeezing in coupled optical and matter waves [12]. However the extension of these ideas to degenerate Fermi gases has proven difficult because of the Pauli exclusion principle, which prohibits one from developing simple theoretical models based on only a few normal modes of the Schrödinger field. In addition to this, Fermi fields do not possess a classical limit analogous to the coherent state for Bose fields thereby making it impossible to develop semiclassical mean-field theories such as the Gross-Pitaevskii equation for Bose fields.

All in all, the physical intuition obtained from quantum optics cannot be directly applied to theoretical investigations of fermions. It therefore seems necessary that in order to make progress in the theory of fermionic atom optics, fundamental model systems in quantum optics need to be reanalyzed from first principles. Recent work in this direction indicates that four-wave mixing and coherent amplification of matter waves can occur in fermionic systems as a result of cooperative many-particle quantum interference analogous toDicke superradiance [13-15].

The purpose of this paper is to consider another model system, the atomic analog of an optical cavity with two partially transmissive mirrors. A schematic of our system is illustrated in Fig. 1. It consists of an atom cavity formed by two potential barriers with a finite number of bound states. The cavity states are coupled to a continuum of free particle states on either side of the cavity via tunnelling through the barriers. Our goal is to develop an input-output theory for fermions in the atom cavity that allows one to calculate the field radiated out of the cavity in terms of the field incident on the cavity.

The input-output theory for a single mode of a lossy optical cavity was developed by Collett and Gardiner in the form of quantum Langevin equations for the cavity mode [16]. The great utility of this theory is that it allows one to incorporate the effects of quantum noise on the output field transmitted by the cavity as well as in the intracavity dynamics. Collett and Gardiner’s formalism has been extended to bosonic matter fields in order to model the output coupling of atoms from a Bose-Einstein condensate in a single mode of an atom trap [17]. As we will show below, the necessity of treating all modes of the atom cavity for fermions leads to novel features not present in the single mode bosonic theories. Most significantly, the eigenstates of the cavity become coupled due to their mutual interaction with the same external continuum states. Secondly, the coupling of the reservoir modes to all cavity modes leads to the creation of coherences between fermions occupying different single particle modes in the radiated field even if the incident field is completely incoherent.

The outline of the paper is as follows: In section II, we present our physical model for an atom cavity coupled to a continuum of reservoir states. In section III, we derive a set of quantum Langevin equations for the fermionic annihilation and creation operators of the eigenstates of the cavity in terms of both the input and output fields. These results are generalized to a two sided cavity in section IV. In section V, we consider a constant current of fermions incident on one side of the atom cavity and calculate the steady state statistics of the fermions transmitted through the other side of the cavity. In appendix A, we show that in a manner similar to the bosonic case, the presence of noise operators in the Langevin equations is necessary to preserve the anticommutation relations for the fermion operators. In appendix B we derive explicit forms for the coupling constants, which connect the intracavity modes to external continuum modes via a tunnelling Hamiltonian.
reservoir system in the subspace of states with energies greater than $V_0$ do not contain any occupied states with energies less than $V_0$. Such a restriction is valid provided the initial wave functions decay exponentially inside the potential barriers. For thick barriers, the number of bound states of the potential barriers of height $V_0$ located at cavity states by tunnelling through the potential barriers.

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\[ H = H_S + H_L + H_R + H_{SR} + H_{SL} \]  

where $H_S$, $H_L$, and $H_R$ are the free Hamiltonians for the system (i.e., the atom cavity) and the left and right reservoir states, respectively,

\[ H_S = \sum_{n=0}^{N} \hbar \Omega_n c_n^\dagger c_n \]  

\[ H_L = \sum_{k} \hbar \omega_k a_k^\dagger a_k \]  

\[ H_R = \sum_{k} \hbar \omega_k b_k^\dagger b_k. \]

Here, $c_n$ is a fermionic annihilation operator that destroys an atom in the cavity with wave function $\phi_n^{(c)}(x)$ and energy $\hbar \Omega_n = \hbar^2 K_n^2 / 2m$. Similarly, $a_k$ and $b_k$ are fermionic annihilation operators that destroy an atom in the left and right reservoirs, respectively, with the wave function

\[ \phi_k^{(l,r)}(x) = \exp(\pm ik(x \pm (a + d))/L^{1/2} \]

and energy $\hbar \omega_k = \hbar^2 k^2 / 2m$ in the regions outside the barrier.

The coupling between the system and the reservoirs, $H_{SR}$ and $H_{SL}$, is given by effective tunnelling Hamiltonians $[19, 20, 21, 22]$

\[ H_{SL} = i\hbar \sum_{n,k} \left[ \kappa_{n,k} c_n^\dagger a_k - \kappa_{n,k}^* a_k^\dagger c_n \right] \]  

\[ H_{SR} = i\hbar \sum_{n,k} \left[ \tilde{\kappa}_{n,k} c_n b_k - \tilde{\kappa}_{n,k}^* b_k^\dagger c_n \right]. \]

In all cases the summation is restricted to those states with energies below the barrier. Explicit expressions for the tunnelling matrix elements, $\kappa_{n,k}$ and $\tilde{\kappa}_{n,k}$, are given in Appendix B. We note here that in 1-D, the coupling constants depend only on the magnitude of $k$ and not its sign.

In contrast to quantum optical systems, which are often approximated as a single cavity mode with a large occupation number, a full multi-mode treatment is required for fermions even if the number of fermions in the cavity is small (~ 1). This is because of the Pauli principle, which forbids more than one atom from occupying the same cavity state, and thereby prevents one from singling out a particular state as being more important than the rest.

We conclude this section by noting that the general results presented below do not depend on the precise nature of our physical model. We use a stepwise constant potential because it leads to simple analytic results for the coupling between the reservoirs and cavity states. Our model system in Eq. (1) could be applied to any fermion system in which a finite number of discrete states are linearly coupled to a dense continuum of states.

![Fig. 1: Schematic diagram of 1D atom cavity system.](image)
III. SINGLE SIDED CAVITY

We now derive a set of integro-differential equations of motion for the cavity operators that only involve the initial or final state of the reservoir operators. We proceed by formally integrating the equations of motion for the reservoir operators and substituting these back into the equations for the cavity operators. In this section, we set $H_R = H_{SR} = 0$ so that the atom-cavity is coupled to a single reservoir. This is analogous to an optical cavity with a single partially transmittive mirror. The inclusion of the right reservoir is summarized in the following section.

It is convenient to work with slowly varying operators in the interaction representation,

$$ c_n(t) = e^{-i\Omega t} \hat{c}_n(t), \quad a_k(t) = e^{-i\omega_k t} \hat{a}_k(t). $$

By formally integrating the Heisenberg equations of motion for $\hat{a}_k(t)$ from the initial time $t_0$ to $t$,

$$ \hat{a}_k(t) = \hat{a}_k(t_0) - \sum_m \kappa_{m,k}^* \int_{t_0}^t dt' e^{i(\omega_k - \Omega_m) t'} \hat{c}_m(t') \tag{8} $$

where $\hat{a}_k(t_0)$ are the operators for the input field incident on the cavity barrier, and substituting this solution into the equations of motion for $\hat{c}_n(t)$, we obtain

$$ \frac{d}{dt} \hat{c}_n(t) = \sum_k \kappa_{n,k} e^{-i(\omega_k - \Omega_n) t} \hat{a}_k(t_0) - \sum_m e^{i(\Omega_n - \Omega_m) t} \int_0^{t-t_0} d\tau \alpha_{n,m}(\tau) \hat{c}_m(t-\tau). \tag{9} $$

Here, $\alpha_{n,m}(\tau)$ is a reservoir correlation function,

$$ \alpha_{n,m}(\tau) = \sum_k \kappa_{n,k} \kappa_{m,k}^* e^{i(\Omega_m - \omega_k) \tau}. \tag{10} $$

that decays to zero in a characteristic time $\tau_c$ due to the destructive interference between the different oscillations. Note that $\tau_c$ depends, in general, on the cavity states $n$ and $m$ that are coupled by $\alpha_{n,m}(\tau)$, and $\tau_c^{-1}$ is on the order of the bandwidth of the reservoir, $V_0/\hbar$. Furthermore, if we assume that $\hat{c}_m(t)$ only changes significantly over a time scale $T_m \gg \tau_c$, then we can make the Markov approximation by setting $\hat{c}_m(t-\tau) = \hat{c}_m(t)$ in Eq. (9).

For time intervals $t - t_0 \gg \tau_c$, we can then make the replacement

$$ \int_0^{t-t_0} d\tau \alpha_{n,m}(\tau) \approx \int_0^{\infty} d\tau \alpha_{n,m}(\tau) \tag{11} $$

where the later expression is given by

$$ \int_0^{\infty} d\tau \alpha_{n,m}(\tau) = \gamma_{n,m} + i\Delta_{n,m}, \tag{12} $$

with

$$ \gamma_{n,m} = \pi \sum_k \kappa_{n,k} \kappa_{m,k}^* \delta(\Omega_m - \omega_k) \tag{13} $$

and

$$ \Delta_{n,m} = P \sum_k \kappa_{n,k} \kappa_{m,k}^* \frac{1}{\Omega_m - \omega_k}. \tag{14} $$

These expressions are defined in the continuum limit that $\sum_k \rightarrow (L/2\pi) \int dk$.

![FIG. 2: Plot of $\alpha_{n,n}(\tau)$ for $n = 49$ and $d/a = 0.0001$(solid line), $0.001$(dotted line), $0.01$ (dashed line). Times are measured in units of $\Omega_0^{-1}$. $V_0$ and $a$ were chosen so that the cavity contains 50 bound states.](image-url)
Langevin equation for each of the cavity modes,
\[
\dot{c}_n(t) = -i\Omega_n c_n(t) - \sum_m (\gamma_{n,m} + i\Delta_{n,m}) c_m(t) + F_n^{(in)}(t).
\]
(15)

In Eq. (15), we have defined the input noise operator \( F_n^{(in)}(t) \) as
\[
F_n^{(in)}(t) = \sum_k \kappa_{n,k} e^{-i\omega_k(t-t_0)} a_k(t_0) \equiv \sum_k \kappa_{n,k} a_k^{(in)}(t)
\]
(16)
where
\[
a_k^{(in)}(t) = a_k(t_0) \exp(-i\omega_k(t - t_0))
\]
is the annihilation operator for mode \( \phi_k^{(l)}(x) \) of the input Fermi field at time \( t \), and the total input field operator is therefore
\[
\Psi^{(in)}(x, t) = \sum_k a_k^{(in)}(t) \phi_k^{(l)}(x).
\]
That is, \( \Psi^{(in)}(x, t) \) is the free Fermi field that propagates from the initial time \( t_0 \) to \( t \) in the Heisenberg picture.

Since the initial reservoir operators obey the anticommutation relations \( \{a_k(t_0), a_{k'}^\dagger(t_0)\} = \delta_{k,k'} \) and \( \{a_k(t_0), a_{k'}(t_0)\} = 0 \), it is easy to show that the noise operators obey the anticommutation relations,
\[
\{F_n^{(in)}(t), F_{m}^{(in)}(t - \tau)\} = e^{-i\Omega_m \tau} \alpha_{n,m}(\tau) \quad (17)
\]
(17)
\[
\{F_n^{(in)}(t), F_{m}^{(in)}(t - \tau)\} = 0 \quad (18)
\]
(18)

Furthermore, if we restrict ourselves to time scales much longer than the correlation time \( \tau_c \), then we can approximate the correlation function in Eq. (17) by a delta function times the area under \( \alpha_{n,m}(\tau) \), so that
\[
\{F_n^{(in)}(t), F_{m}^{(in)}(t - \tau)\} \approx 2\gamma_{n,m} \delta(\tau). \quad (19)
\]
(19)

Before proceeding, there are several features of Eq. (13) that are worth mentioning. First, the dissipative term \( \sum_m (\gamma_{n,m} + i\Delta_{n,m}) c_m(t) \) gives a damping term plus an energy shift for \( n = m \) while for \( n \neq m \) there is a non-zero coupling between cavity states. The coupling between cavity states is a result of all states coupling to the same reservoir, which leads to an indirect coupling between cavity states. Second, the noise operators couple all of the reservoir states to each of the cavity states. Moreover, the noise operators are different for each cavity state due to the \( n \)-dependence of the coupling constants.

Instead of solving for the reservoir operators in terms of the initial time \( t_0 \), one can instead solve for \( \hat{a}_k(t) \) in terms of a final time \( t_1 > t \),
\[
\hat{a}_k(t) = \hat{a}_k(t_1) + \sum_m \kappa_{m,k}^* \int_t^{t_1} dt' e^{i(\omega_k - \Omega_m)t'} \hat{c}_m(t').
\]
(20)

The operators \( \hat{a}_k(t_1) \) represent the modes of the output field that contain the field radiated by the cavity at earlier times. By substituting this expression into the equations of motion for \( \hat{c}_n(t) \), making the Markov approximation in the integrand, and transforming back to the Heisenberg representation one obtains
\[
\dot{c}_n(t) = -i\Omega_n c_n(t) + \sum_m (\gamma_{n,m} - i\Delta_{n,m}) c_m(t) + F_n^{(out)}(t).
\]
(21)

Here, \( F_n^{(out)}(t) \) is the output field noise operator for state \( n \),
\[
F_n^{(out)}(t) = \sum_k \kappa_{n,k} e^{-i\omega_k(t-t_1)} a_k(t_1) \equiv \sum_k \kappa_{n,k} a_k^{(out)}(t)
\]
(22)

where we have defined the output field annihilation operator \( a_k^{(out)}(t) \) for the mode \( \phi_k^{(l)}(x) \). The \( a_k^{(out)}(t) \) are related to the output field of the reservoir by,
\[
\Psi^{(out)}(x, t) = \sum_k a_k^{(out)}(t) \phi_k^{(l)}(x).
\]

It is clear that \( \Psi^{(out)}(x, t) \) represents the free Fermi field for the reservoir that propagates from \( t \) to the final time \( t_1 \). It is easy to see that if the \( a_k(t_1) \) obey normal fermionic anticommutation relations, then the anticommutators for the output noise operators are the same as Eqs. (17-18).

The boundary condition for the barrier separating the cavity from the reservoir is obtained by subtracting Eq. (13) from Eq. (21),
\[
F_n^{(in)}(t) - F_n^{(out)}(t) = 2 \sum_m \gamma_{n,m} c_m(t) \quad (23)
\]
(23)

Equation (23) relates the noise operator for the output field to the input noise operator reflected by the barrier and the field radiated by the cavity. It is of the same form as the boundary condition for an optical cavity except that in our case there are separate noise operators for each cavity mode since we cannot, in general, assume that the coupling constants are independent of the cavity state. In Appendix A, we use Eq. (23) and (15) to derive the anticommutation relations between the cavity operators and the noise operators at arbitrary times.

Equation (23) is not very useful since it is the mode operators of the output field, \( a_k^{(out)}(t) \), that are needed to calculate properties of the output field such as mode occupation statistics, current density, etc. Therefore, we must extract from Eq. (23) a boundary condition for the annihilation operators of the modes of the input and output fields.

First we note that in the limit of an infinite potential barrier the system-reservoir coupling vanishes, \( \kappa_{n,k} \equiv 0 \). In this limit, a fermion incident from the left is perfectly reflected by the barrier. Since \( \Psi^{(in)}(x, t) \) and \( \Psi^{(out)}(x, t) \)
are the free Schrödinger fields that propagate forward in time from \( t_0 \to -\infty \) to \( t \) and from \( t \) to \( t_1 \to \infty \), respectively, it follows that

\[ a_k(t_0)e^{i\omega_k t_0} = -a_{-k}(t_1)e^{i\omega_k t_1}. \]

The total field is the incident plus reflected fields,

\[ \Psi(x, t) = \Psi^{(\text{in})}(x, t) + \Psi^{(\text{out})}(x, t). \]

It follows from (23) and (24) that \( \Psi(-a-d, t) = 0 \) and that the eigenmodes of the reservoir are standing waves for \( V_0 \to \infty \). Note that (24) and (25) are second quantized versions of the relations for the incident and reflected wave functions from an infinite potential barrier [18].

For the reservoir, the displacement operator is given by

\[ D(x) = e^{-ipx/\hbar} \]

where \( P = \sum_k \hbar k a_k^\dagger a_k \) is the momentum operator for the reservoir [23]. Multiplying (23) by \( D(x) \) on the left and \( D^\dagger(x) \) on the right gives after some manipulation [24],

\[ \sum_m \kappa_{n,k} \left( \alpha_{m,k}^{(\text{in})}(t) - \alpha_{-m,k}^{(\text{out})}(t) \right) e^{ikx} = 2\pi \sum_{m,k} \kappa_{n,k} \kappa_{n,m,k}^* \delta(\Omega_m - \omega_k)c_m(t)e^{ikx}. \]

Multiplying Eq. (23) by \( \int_{-a-d} e^{-ik'x} \) and taking the limit \( L \to \infty \) gives the boundary condition for the modes of the input and output fields,

\[ \alpha_{m,k}^{(\text{in})}(t) - \alpha_{-m,k}^{(\text{out})}(t) = 2\pi \sum_{m} \kappa_{n,m,k}^* \delta(\Omega_m - \omega_k)c_m(t) \]

Physically, Eq. (27) says that the difference between the mode of the output field propagating away from the barrier with momentum \(-\hbar k\) is the input field with momentum \( \hbar k \) reflected by the barrier plus the field radiated by the cavity. Furthermore, only those states of the cavity that have the same energy as the reservoir mode can radiate into that mode. For a one dimensional system, there will only be a single cavity mode that contributes to the right hand side of Eq. (27). The \( \delta(\Omega_m - \omega_k) \) comes from Eq. (11) where we assumed times much longer than the width of the reservoir correlation function. Hence, it follows from the uncertainty relation \( \Delta E \Delta t \sim \hbar \) that the range of reservoir energies that couple to each cavity mode goes to zero as \( t - t_0 \to \infty \).

### IV. TWO SIDED CAVITY

The generalization of the preceding to a two sided cavity, \( H_{SR}, H_R \neq 0 \), is straightforward since the reservoirs couple independently to the cavity. For the right reservoir, we define the input and output noise operators,

\[ G_n^{(\text{in})}(t) = \sum_k \tilde{\kappa}_{n,k} \alpha_{n,k}^{(\text{in})}(t) \]

\[ G_n^{(\text{out})}(t) = \sum_k \tilde{\kappa}_{n,k} \alpha_{n,k}^{(\text{out})}(t) \]

where

\[ \alpha_{n,k}^{(\text{in})}(t) = \tilde{b}_k(t_0)e^{-i\omega_k(t-t_0)} \]

\[ \alpha_{n,k}^{(\text{out})}(t) = \tilde{b}_k(t_1)e^{-i\omega_k(t-t_1)} \]

represent the input and output annihilation operators for the free field modes of the right reservoir at time \( t \). Associated with the right reservoir noise operators are damping constants

\[ \tilde{\gamma}_{n,m} = \pi \sum_k \tilde{\kappa}_{n,k} \tilde{\kappa}_{n,m,k} \delta(\Omega_m - \omega_k), \]

and radiative energy shifts

\[ \tilde{\Delta}_{n,m} = \frac{P \sum_k \tilde{\kappa}_{n,k} \tilde{\kappa}_{n,m,k}^* \frac{1}{\Omega_m - \omega_k}}{\Omega_m - \omega_k}. \]

The quantum Langevin equations for the cavity mode operators expressed in terms of the input fields from the left and right are then

\[ \dot{\alpha}_{n}(t) = -i\Omega_n c_n(t) - \sum_m \left( (\gamma_{n,m} + i\Delta_{n,m})c_m(t) + (\tilde{\gamma}_{n,m} + i\tilde{\Delta}_{n,m})c_m(t) \right) + F_n^{(\text{in})}(t) + G_n^{(\text{in})}(t) \]

One may also derive Langevin equations analogous to Eq. (24) involving the output noise operators for the two reservoirs, which can be used along with (34) to derive the boundary conditions for the noise operators in the left and right reservoirs,

\[ G_n^{(\text{in})}(t) - G_n^{(\text{out})}(t) = 2 \sum_m \tilde{\gamma}_{n,m} c_m(t) \]

\[ F_n^{(\text{in})}(t) - F_n^{(\text{out})}(t) = 2 \sum_m \gamma_{n,m} c_m(t). \]
Using the boundary condition $b_k(t_0)e^{i\omega k t_0} = -b_{-k}(t_1)e^{i\omega k t_1}$ that corresponds to (24) for the free fields in the right reservoir, the boundary condition for $b_k^{(in)}(t)$ and $b_k^{(out)}(t)$ can be derived in the same manner as Eq. (24). One finds

$$b_k^{(in)}(t) - b_k^{(out)}(t) = 2\pi \sum_m \tilde{\kappa}_{m,k} \delta(\Omega_m - \omega_k)c_m(t), \quad (37)$$

$$a_k^{(in)}(t) - a_k^{(out)}(t) = 2\pi \sum_m \kappa_{m,k}^* \delta(\Omega_m - \omega_k)c_m(t). \quad (38)$$

Equations (34), (37), and (38) are the central result of this section. Given an initial state for the two reservoirs at $t_0$, Eq. (34) can be used to calculate the state of the cavity at some later time. The mode operators for the output field of the left and right reservoirs can then be determined from the boundary conditions, (37) and (38).

V. OUTPUT FIELD STATISTICS

We illustrate how to utilize these results for a particular initial state of the cavity plus reservoirs. Specifically, we assume that the atom cavity initially contains no atoms and that the right reservoir is likewise in the vacuum state. Furthermore, the left reservoir contains a "beam" of fermions incident on the barrier at $x = -a-d$, with a spatially uniform current density equal to $\rho b_0 q/m$ where $\rho = N/L$ is the linear atomic density and $N$ is the total number of fermions. This physical configuration is represented by the initial state vector,

$$|\Psi(t_0)\rangle = \prod_{|k-q| \leq k_F} a_k^{\dagger}(t_0)|0\rangle \quad (39)$$

where $k_F = 2\pi \rho$ is the Fermi momentum. $|\Psi(t_0)\rangle$ represents a zero temperature Fermi distribution that has been given a Galilean boost that displaces the gas by $q$ in k-space. This is analogous to the optical case where an incoherent white light source is used to drive an optical cavity.

Even though $|\Psi(t_0)\rangle$ is a state with a fixed number of atoms, it acts like a constant input flux of fermions on the cavity. This is because of the implicit use of the Born approximation in the derivation of the Langevin equations, i.e. the reservoir is assumed to be so large that the back-action of the system is negligible.

Using the results of the previous section we can calculate how the state of the left and right reservoirs are modified due to their coupling to the cavity. In particular, we focus on the occupation numbers

$$n_k^{(L)}(t) = \langle a_k^{(out)}(t) a_k^{(out)\dagger}(t) \rangle \quad (40)$$

$$n_k^{(R)}(t) = \langle b_k^{(out)}(t) b_k^{(out)\dagger}(t) \rangle \quad (41)$$

as well as the current density operators for the reservoirs

$$J^{(L,R)}(x,t) = \sum_q \tilde{j}_q^{(L,R)}(t) e^{i\tilde{q}x}$$

where

$$\tilde{j}_q^{(L)}(t) = \frac{\hbar}{2mL} \sum_k (2k + \tilde{q}) a_k^{(out)}(t) a_{k+\tilde{q}}^{(out)}(t) \quad (42)$$

$$\tilde{j}_q^{(R)}(t) = \frac{\hbar}{2mL} \sum_k (2k + \tilde{q}) b_k^{(out)}(t) b_{k+\tilde{q}}^{(out)}(t) \quad (43)$$

are the spatial Fourier components of the current. Note that $mL(\tilde{j}_q^{(L,R)}(t))$ is the average momentum in the output fields.

Equation (34) for $c_n(t)$ can be numerically integrated but we note that because of the exponential dependence of the reservoir-cavity coupling constants on the barrier height and thickness, the off-diagonal coupling is much smaller than the energy difference between the cavity modes,

$$|\Omega_n - \Omega_m| \gg |\gamma_{n,m} + \tilde{\gamma}_{n,m}|, |\Delta_{n,m} + \tilde{\Delta}_{n,m}|$$

for $n \neq m$. In fact, a numerical evaluation of $|\gamma_{n,m} + \tilde{\gamma}_{n,m}|$ and $|\Delta_{n,m} + \tilde{\Delta}_{n,m}|$ using the coupling constants of Appendix B indicate that these terms are at least two orders of magnitude smaller than the energies of the cavity modes. It is therefore an excellent approximation to neglect all off-diagonal terms in the equations of motion. Furthermore, we can perform a renormalization of the cavity mode energies by absorbing the radiative energy shifts into them, $\Omega_m + \Delta_{m,m} + \tilde{\Delta}_{m,m} \rightarrow \Omega_m$.

For times much longer than the lifetimes of the cavity modes, $(t - t_0)(\gamma_{n,m} + \tilde{\gamma}_{n,m}) \gg 1$, the system reaches a steady state with the solution,

$$c_m(t) = \sum_k \frac{\kappa_{m,k} a_k^{(in)}(t) + \tilde{\kappa}_{m,k} b_k^{(in)}(t)}{i(\Omega_m - \omega_k) + \Gamma_m} \quad (44)$$

where $\Gamma_m = \gamma_{m,m} + \tilde{\gamma}_{m,m}$. It is easy to see from Eq. (44) that the occupation numbers for the cavity modes have a Lorentzian profiles,

$$\langle c_m^\dagger c_m \rangle = \sum_k \frac{|\kappa_{m,k}|^2 \langle n_k(t_0) \rangle}{(\Omega_m - \omega_k)^2 + \Gamma_m^2} \quad (45)$$

where

$$\langle n_k(t_0) \rangle = \Theta(k_F - |k - q|)$$

are the occupation numbers for the input field. Due to the broadband nature of the input field all cavity modes with energies in the range $\omega_{k_F+q} - \omega_{k_F+q}$ will have a significant population with higher energy cavity states having larger populations due to the $|\kappa_{m,k}|^2$’s exponential dependence on energy.

Using Eqs. (34) and the boundary conditions (37) and (38), we obtain steady state expectation values of the occupation of the output field modes.
\[ \langle n^{(L)}_{-k} \rangle = \left(1 - 4\pi \sum_{m} \frac{\Gamma_{m} \delta(\Omega_{m} - \omega_{k})|\kappa_{m,k}|^{2}}{(\Omega_{m} - \omega_{k})^{2} + \Gamma_{m}^{2}} \right) \langle n_{k}(t_{0}) \rangle + 4\pi^{2} \sum_{m,n,k'} \frac{|\kappa_{m,k'}|^{2} |\kappa_{m,k}|^{2} \delta(\Omega_{m} - \Omega_{n}) \delta(\Omega_{m} - \omega_{k})}{(\Omega_{m} - \omega_{k'})^{2} + \Gamma_{m}^{2}} \langle n_{k'}(t_{0}) \rangle \] (46)

\[ \langle n^{(R)}_{k} \rangle = 4\pi^{2} \sum_{m,n,k'} \frac{|\kappa_{m,k'}|^{2} |\kappa_{m,k}|^{2} \delta(\Omega_{m} - \Omega_{n}) \delta(\Omega_{m} - \omega_{k})}{(\Omega_{m} - \omega_{k'})^{2} + \Gamma_{m}^{2}} \langle n_{k'}(t_{0}) \rangle. \] (47)

Equations (46) and (47) represent the changes in occupation numbers due to reflection and transmission through the cavity. The most significant feature of Eq. (46) is that fermions in the input beam are perfectly reflected by the barrier, \( \langle n^{(L)}_{-k} \rangle = \langle n_{k}(t_{0}) \rangle \), unless there exists a cavity mode that is degenerate in energy with state \( k \) of the reservoir. The second term represents the interference term between the fermions reflected by the barrier and the fermions that have tunneled into the barrier and subsequently tunneled back out into the left reservoir. The last term in Eq. (46) represents the tunnelling of atoms into mode \( -k \) as a result of atoms from mode \( k' \) that have tunneled into the cavity and then tunnel out of the cavity.

In order to gain additional physical insight, we simplify these expressions by noting that the denominator in Eq. (46) as well as the last term in Eq. (46) are sharply peaked around \( \omega_{k'} = \Omega_{m} = \omega_{k} \). Therefore we can replace \( k' \) with \( k \) in this term and drop the summation over \( k' \).

Using \( \Gamma_{m} = 2\gamma_{m,m} \approx 2\pi|\kappa_{m,k}|^{2} \delta(\Omega_{m} - \omega_{k}) \) we have

\[ \langle n^{(L)}_{-k} \rangle \approx \langle n_{k}(t_{0}) \rangle - \mathcal{L}(\omega_{k}) \left( \langle n_{k}(t_{0}) \rangle - \langle n_{-k}(t_{0}) \rangle \right), \] (48)

\[ \langle n^{(R)}_{k} \rangle \approx \mathcal{L}(\omega_{k}) \left( \langle n_{k}(t_{0}) \rangle + \langle n_{-k}(t_{0}) \rangle \right), \] (49)

where

\[ \mathcal{L}(\omega_{k}) = \sum_{m} \frac{\Gamma_{m}^{2}}{(\Omega_{m} - \omega_{k})^{2} + \Gamma_{m}^{2}}. \] (50)

When \( \langle n_{-k}(t_{0}) \rangle = 0 \), Eq. (48) indicates that \( \langle n^{(L)}_{-k} \rangle \approx 0 \), a result of the complete destructive interference between the fermions that are directly reflected by the barrier and the fermions that tunnel out of the cavity into the left reservoir. At the same time, Eq. (49) indicates that fermions resonant with a cavity mode tunnel through to the right side with unit probability, \( \langle n^{(R)}_{k} \rangle \approx \langle n_{k}(t_{0}) \rangle \). These transmission resonances are similar to the situation in an optical Fabry-Perot cavity.

Fig. 3 shows a plot of \( \langle n^{(L)}_{k}(t) \rangle \) and \( \langle n^{(R)}_{k}(t) \rangle \) for \( d/a = 10^{-4} \), and \( a/L = 1.2 \times 10^{-5} \), \( k \) is in units of \( 2\pi/L \).

The steady state current density in the output field of the left and right reservoirs are given by,
\[ \langle j^{(L)}_{-\bar{q}} \rangle = -\frac{\rho q}{m} \delta_{\bar{q},0} - \frac{\hbar}{2mL} \sum_{k} (2k + \bar{q}) \left(-2\pi \sum_{m} \kappa_{m,k+\bar{q}}^{*} \kappa_{m,k} \left( \frac{\delta(\Omega_{m} - \omega_{k+\bar{q}})(n_{k+\bar{q}}(t_{0}))}{-i(\Omega_{m} - \omega_{k+\bar{q}}) + \Gamma_{m}} + \frac{\delta(\Omega_{m} - \omega_{k})(n_{k}(t_{0}))}{i(\Omega_{m} - \omega_{k}) + \Gamma_{m}} \right) + 4\pi^{2} \sum_{m,n,k,k'} \kappa_{m,k}^{*} \kappa_{n,k+\bar{q}} \kappa_{n,k'} \kappa_{m,k}^{*} \delta(\Omega_{m} - \omega_{k}) \delta(\Omega_{n} - \omega_{k+\bar{q}}) \langle n_{k+\bar{q}}(t_{0}) \rangle \right) \]

and

\[ \langle j^{(R)}_{\bar{q}} \rangle = \frac{\hbar}{2mL} \sum_{k} (2k + \bar{q}) \left(4\pi^{2} \sum_{m,n,k,k'} \kappa_{m,k}^{*} \kappa_{n,k+\bar{q}} \kappa_{n,k'} \kappa_{m,k}^{*} \delta(\Omega_{m} - \omega_{k}) \delta(\Omega_{n} - \omega_{k+\bar{q}}) \langle n_{k+\bar{q}}(t_{0}) \rangle \right) \]

respectively.

The first term on the rhs of Eq. (51) is the incident current reflected by the barrier. The average momenta in the output fields are proportional to \( \langle j^{(L)}_{0} \rangle = -\hbar/mL \sum_{k} k \langle n_{k}^{(L)} \rangle \) and \( \langle j^{(R)}_{0} \rangle = \hbar/mL \sum_{k} k \langle n_{k}^{(R)} \rangle \). In the left reservoir, \( \langle j^{(L)}_{0} \rangle \approx -\rho q/m \) since most of the fermions are perfectly reflected by the barrier. In the right reservoir, the transmitted current \( \langle j^{(R)}_{0} \rangle \approx 0 \) since for those states that are resonant with the cavity \( \langle n_{k}^{(R)} \rangle = \langle n_{k}^{(L)} \rangle \) while for non-resonant reservoir states, \( \langle n_{k}^{(R)} \rangle \approx 0 \). Physically, this is due to the fact that atoms tunnel into the right reservoir from a standing wave cavity mode and therefore, they have equal probability to tunnel into states with positive and negative momentum.

The \( \bar{q} \neq 0 \) terms are the spatial modulations in the current density that build up in the reservoirs as a result of the reservoir-cavity mode coupling. This can generate a coherence between the \( k \) and \( k + \bar{q} \) modes when there is a finite amplitude for an atom initially in state \( k \) to tunnel into the cavity and then tunnel back out of the cavity into state \( k + \bar{q} \). The change in \( \langle j^{(L)}_{\bar{q}} \rangle \) is of order \( \kappa^{2} \) rather than \( \kappa^{4} \) as was the case for Eq. (46) since the current only involves generating a coherence between \( k \) and \( k + \bar{q} \) rather than the transfer of population. Furthermore, the \( \kappa^{2} \) terms in (51) are only finite for \( |\omega_{k+\bar{q}} - \omega_{k}| < \Gamma_{m} \), which implies that the coherence is only generated between reservoir states whose energies lie within the linewidth of a particular cavity mode. Consequently, decreasing the thickness and height of the barriers will make the linewidths of the cavity states larger and thereby increase the magnitude of \( \bar{q} \neq 0 \) components of the current.

Fig. 4 shows a plot of \( \langle j^{(L)}_{\bar{q}} \rangle \) for several values of \( \bar{q} \). The \( \bar{q} \neq 0 \) components of the current are about two orders of magnitude smaller than \( \langle j^{(L)}_{0} \rangle \) and they decay away with increasing \( \bar{q} \). We do not plot \( \langle j^{(R)}_{\bar{q}} \rangle \) since the current is equal to 0 to within our numerical accuracy. This is because the cavity linewidths are so narrow that it becomes nearly impossible to satisfy both the delta functions in the numerator and the Lorentzian denominators of Eq.

FIG. 4: Plot of \( \langle j^{(L)}_{0} \rangle \) for the same parameters as Fig. 3. \( \bar{q} \) are in units of \( 2\pi/L \) and the current is in units of \( \hbar/2ma^{2} \).

\( \rho q/m = 5.27 \) in these units.

\[ \text{(22)} \] for \( \bar{q} \neq 0 \).

VI. CONCLUSION

In this paper we have extended the quantum optical input-output theory to atom cavities containing fermions. This formalism can easily be applied to intra-cavity nonlinear atom optical processes such as four-wave mixing between fermions \[13\] or coherent photoassociation of fermions into molecular bosons \[26\]. In a future work we plan to extend this work to treat non-Markovian dynamics \[27\] as well as including the effects of the spatially delocalized single particle state with energies greater than \( V_{0} \).

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VII. APPENDIX A: ANTICOMMUTATION RELATIONS BETWEEN CA VITY AND NOISE OPERATORS

In this appendix, we examine some of the consequences of Eqs. (15) and (23). Correlations between the input and output fields and the cavity modes may be expressed in terms of the anticommutators for the cavity and noise operators, i.e. nonvanishing equal time anticommutators imply that the operators are not independent. From Eq. (15), one sees that the solution for \( c_n(t) \) can be expressed in terms of the initial conditions for the cavity operators, \( c_n(t_0) \), and the \( a_k(t_0) \). It follows immediately that \( \{c_n(t), F_m(t')\} = 0 \) for all \( t \) and \( t' \) since the only non-vanishing anti-commutators at \( t_0 \) are between creation and annihilation operators. In a similar manner, it immediately follows from Eq. (23) that \( \{c_n(t), F_{m(out)}(t')\} = 0 \). Formally integrating Eq. (15) from \( t_0 \) to \( t \) shows that \( c_n(t) \) depends on \( F_{m(out)}(t') \) for \( t' < t \) (note that \( c_n(t) \) will, in general, depend on all of the noise operators, \( F_{m(out)}(t') \), not just \( n = m \), due to the intra-cavity mode coupling). Using Eq. (14) it then follows that

\[
\{c_n(t - \tau), F_{m(out)}(t)\} = 0
\]

for \( \tau > 0 \). This is nothing more than a statement of causality, the system can only depend on the past input. In a similar manner, formal integration of Eq. (23) from \( t + \tau \) to \( t_1 \) implies that

\[
\{c_n(t + \tau), F_{m(out)}(t)\} = 0
\]

for \( \tau > 0 \). Again, this is nothing more than the statement that the output of the system can only depend on the state of the system in the past. Using Eqs. (15) and (23), one obtains

\[
\{c_n(t + \tau), F_{m(out)}(t)\} = 2 \sum_p \gamma^*_{m,p} \{c_n(t + \tau), c_p(t)\}
\]

while using Eq. (53) along with (23) gives,

\[
\{c_n(t - \tau), F_{m(out)}(t)\} = -2 \sum_p \gamma^*_{m,p} \{c_n(t - \tau), c_p(t)\}
\]

for \( \tau > 0 \). It follows from Eq. (23) that the equal time anticommutators can be written as \( \{c_n(t), F_{m(out)}(t)\} = \sum_p \gamma^*_{m,p} \{c_n(t), c_p(t)\} + A_{n,m} \). The operator \( A_{n,m} \) is determined from the equations of motion, (15) and (21), to be \(-i \sum_p \Delta^*_{m,n,p} \{c_n(t), c_p(t)\} \). If we define the step function \( u(\tau) \) as

\[
u(\tau) = \begin{cases} 
1 & , \quad \tau > 0 \\
1/2 & , \quad \tau = 0 \\
0 & , \quad \tau < 0 
\end{cases}
\]

then the anticommutators for arbitrary \( \tau \) are given by

\[
\{c_n(t + \tau), F_{m(out)}(t)\} = 2u(\tau) \sum_p (\gamma^*_{m,p} - i \Delta^*_{m,p}\delta_{\tau,0}) \{c_n(t + \tau), c_p(t)\}
\]

\[
\{c_n(t + \tau), F_{m(out)}(t)\} = -2u(-\tau) \sum_p (\gamma^*_{m,p} + i \Delta^*_{m,p}\delta_{\tau,0}) \{c_n(t + \tau), c_p(t)\}.
\]

Finally, we show that Eq. (15) preserve the equal time anticommutators for the system operators. Since (15) constitute an \( N \times N \) system of equations, an explicit solution will be non-trivial for \( N > 2 \). However, it is already apparent that \( \{c_n(t), c_n(t)\} = 0 \) since, as we stated before, \( c_n(t) \) can be expressed in terms of the initial states for the operators at \( t_0 \) and \( \{c_n(t_0), c_n(t_0)\} = \{c_n(t_0), a_k(t_0)\} = 0 \). We are therefore left with showing that \( \{c_n(t), c_m(t)\} = \delta_{n,m} \). Calculating the derivative of the anticommutator using (15) and (14), one obtains,

\[
\frac{d}{dt} \{c_n(t), c_m(t)\} = -i(\Omega_n - \Omega_m) \{c_n(t), c_m(t)\}.
\]

Integrating this expression and using the initial condition, \( \{c_n(t_0), c_m(t_0)\} = \delta_{n,m} \), we obtain the desired result.

VIII. APPENDIX B: TUNNELLING COUPLING CONSTANTS

In this appendix we derive explicit expressions for the tunnelling coupling constants \( \kappa_{n,k} \) and \( \tilde{k}_{n,k} \). Tunnelling in many-body systems was first treated by Bardeen [21] and later elaborated on by Prange [24] and Harrison [25] in the context of electron tunnelling across insulator junctions. Bardeen showed that if there exists a potential bar-
rier separating two regions of space, then the tunnelling matrix element, $T_{a,b}$, for the tunnelling of a particle from state $\phi_a(x)$ on one side of the barrier into the state $\phi_b(x)$ on the opposite side of the barrier is given by the off-diagonal current density in the barrier,

$$T_{a,b} = -\frac{\hbar^2}{2m} \left[ \phi_a^*(x) \frac{d}{dx} \phi_b(x) - \phi_b(x) \frac{d}{dx} \phi_a^*(x) \right] \bigg|_{x=x_1}.$$  

(57)

Here $x_1$ is a point inside the barrier and $\phi_a(x)$ and $\phi_b(x)$ are the continuations of the single particle wave functions into the barrier where they decay exponentially. $T_{a,b}$ is independent of the choice of $x_1$ provided the energy difference between the two states $\phi_a$ and $\phi_b$ is much less than the height of the barrier. The relationship between $\tilde{\kappa}_{a,b}$ and the overlap of the Hamiltonian between states localized on either side of the barrier is discussed in [20].

For the system illustrated in Fig. 1, we identify $-i\hbar \kappa_{a,b}$ with $T_{a,b}$ when $\phi_a(x) = \phi_k^{(l)}(x)$ and $\phi_b(x) = \phi_n^{(s)}(x)$. Similarly, $-i\hbar \kappa_{n,k}$ is equal to $T_{a,b}$ when $\phi_a(x) = \phi_k^{(r)}(x)$ and $\phi_b(x) = \phi_n^{(s)}(x)$.

In deriving the coupling constants, we take the $\phi_k^{(l)}(x)$ to be the eigenstates of the infinite potential well corresponding to $d \to \infty$. In the classically allowed region, $-a \leq x \leq a$, $\phi_k^{(l)}(x)$ is proportional to $\cos(Kn x)$ for $n$ even and $\sin(Kn x)$ for $n$ odd while inside the barrier $\phi_n^{(s)}(x) \sim \exp(-\sqrt{2m(V_0 - \hbar \Omega_n)/\hbar^2}|x|)$. The cavity wave numbers $K_n$ are determined by the solutions to

$$K_n a \tan(K_n a) = \sqrt{\beta^2 - (K_n a)^2}.$$  

(58)

for $n$ even and

$$K_n a \cot(K_n a) = -\sqrt{\beta^2 - (K_n a)^2}.$$  

(59)

for $n$ odd [21].

For even $n$ we find,

$$\kappa_{n,k} = \frac{-i\hbar \sigma}{m} \sqrt{\frac{2K_n}{L(1+(\sigma/k)^2)(K_n a + \cot(K_n a))}} e^{-\sigma d} \cos(K_n a)$$  

(60)

with

$$\tilde{\kappa}_{n,k} = \kappa_{n,k}$$  

(61)

In a similar manner, we find for $n$ odd,

$$\kappa_{n,k} = \frac{i\hbar \sigma}{m} \sqrt{\frac{2K_n}{L(1+(\sigma/k)^2)(K_n a - \tan(K_n a))}} e^{-\sigma d} \sin(K_n a)$$  

(62)

with

$$\tilde{\kappa}_{n,k} = -\kappa_{n,k}.$$  

(63)

In both cases, $\sigma = \sqrt{2mV_0/\hbar^2 - \beta^2}$ is the inverse of the penetration depth of the reservoir state into the barrier and $\beta^2 = 2ma^2V_0/\hbar^2$ is the dimensionless barrier height.

An interesting consequence of Eqs. (61) and (63) is that if $n$ is even and $m$ is odd or vice versa, then

$$(\gamma_{n,m} + i\Delta_{n,m}) + (\tilde{\gamma}_{n,m} + i\tilde{\Delta}_{n,m}) \equiv 0.$$  

(64)

On the other hand if $n$ and $m$ are both even or both odd, then

$$(\gamma_{n,m} + i\Delta_{n,m}) + (\tilde{\gamma}_{n,m} + i\tilde{\Delta}_{n,m}) = 2(\gamma_{n,m} + i\Delta_{n,m}).$$  

(65)

Since $\phi_n^{(s)}(x)$ is an eigenstate of the parity operator with parity $(-1)^n$, it follows from Eq. (24) that for a two-sided cavity only states of the same parity are coupled. This is a direct consequence of the our model system in Fig. 1 being invariant under spatial reflections, which implies that even when the coupling of the cavity states to the reservoirs is taken into account, parity will still be a good quantum number for the cavity states. No such result holds for the single-sided cavity since the cavity plus reservoir system is no longer invariant with respect to reflections.

Finally we note that the approximation $\omega_k \approx \Omega_n$ used in deriving $\kappa_{n,k}$ becomes exact for the evaluation of the $\gamma_{n,n}$ and the cavity boundary conditions (27) and (29) since in these expression $\kappa_{n,k}$ is always multiplied by $\delta(\Omega_n - \omega_k)$.

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