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Generation of Stationary Gaussian Processes and Extreme Value Distributions for High-Cycle Fatigue Models – Application to Tidal Stream Turbines

The operating environment of tidal stream turbines is random due to the variability of the sea flow (turbulence, wake, tide, streams, among others). This yields complex time-varying random loadings, making it necessary to deal with high cycle multiaxial fatigue when designing such structures. It is thus required to apprehend extreme value distributions of stress states, assuming they are stationary multivariate Gaussian processes. This work focuses on such distributions, addressing their numerical simulation with an analytical description. For that, we first focused on generating one-dimensional Gaussian processes, considering a band-limited white noise in both the narrow-band and the wide-band cases. We then fitted the resulting extreme value distributions with GEV distributions. We secondly extended the generation method to the correlated two-dimensional case, in which the joint extreme value distribution can be obtained from the associated margins. Finally, an example of application related to tidal stream turbines introduces a Bretschneider spectrum, whose shape is commonly encountered in the field of hydrology. Comparing the empirical calculations with the GEV fits for the extreme value distributions shows a very well agreement between the results.

Keywords: stochastic processes generation, correlated Gaussian processes, multivariate extreme value distribution, high cycle fatigue, tidal stream turbines

Introduction

Nowadays, it has become necessary to make the most of renewable energies. Tidal stream turbines allow this by harnessing the kinetic energy of sea streams (see Fig. ??). But in order to come through they have to be economically competitive, which implies reduced maintenance. However, they are immersed in flows having not mastered characteristics resulting from many contributions such as turbulence, wake, swell, current, etc. These structures are then submitted to random time-varying multiaxial stress states, which can be described from probability distributions associated to a time evolution. Numerically, these stress states can be estimated from a statistical description of the sea load that acts on these submerged structures; see for example Theodorsen (1934), Gmüter (1997), Guéna et al. (2006) and Thresher et al. (2009). Such stresses imply a risk of fatigue failure which must be taken into account during the design stage to ensure structural integrity. In this study, we are concerned with the infinite fatigue life, that is based on the greatest stress values reached during the service life (Lambert et al., 2010; Khalij et al., 2010). By taking into account their randomness, this leads us to study their multivariate extreme value distributions. We assume that the stress states are stationary multivariate Gaussian processes, described by their means and power spectral densities. More specifically, we focus in this paper on the way to get, from numerical simulations, bivariate extreme value distributions of components of the stress tensor, with analytical descriptions of the results. Since the mechanical stresses reach their maximum value on the skin of the structure, we shall consider the plane stress case only.

The paper is divided in three parts. The first part deals with generating one-dimensional Gaussian processes in order to evaluate its empirical extreme value distribution. Then, we perform a fit with a GEV distribution. Secondly, these concepts are adapted to multidimensional Gaussian processes and applied to a bivariate correlated case. A bivariate extreme value distribution, evaluated from the associated margins, is found to be a well-adapted model. Finally, we apply the previous considerations to a realistic example related to tidal stream turbines.

Figure 1: Projects of tidal stream turbines: Skerries Islands, Wales, 2015 (left) and Islay strait, Scotland, 2015 (right).

Nomenclature

Acronyms

| Acronym | Description |
|---------|-------------|
| AR      | Average Random |
| EVD     | Extreme Value Distribution |
| GEV     | Generalized Extreme Value |
| IFFT    | Inverse Fast Fourier Transform |
| MA      | Moving Average |
| PSD     | Power Spectrum Density |

Operators

| Operator | Description |
|----------|-------------|
| arg(●)   | argument of the complex number ● |
| E(●)     | statistical expectation of ● |
| Prob(●)  | probability of the event ● |
Let $X(t)$ be a temporal zero-mean Gaussian process. Let $S(f)$ be the associated PSD, uniformly discretized in $n_f$ points:

$$
S(f) = \begin{cases} 
S_0 & \text{if } f \in (f_1, f_2], \\
0 & \text{else} 
\end{cases}
$$

For instance:

- the narrow-band case corresponds to $[f_2 - f_1]$ “small”: in the sequel, we take $f_1 = 9.5$ Hz, $f_2 = 10.5$ Hz, $S_0 = 7$ MPa²/Hz, the frequency $f_1$ being included in the spectral definition domain;

- the wide-band case corresponds to $|f_2 - f_1|$ “large”: in the sequel, we take $f_1 = 0$ Hz, $f_2 = 20$ Hz, $S_0 = 0.35$ MPa²/Hz, the frequency $f_1$ being excluded from the spectral definition domain, since it is the zero value.

It should be noted that in either case, $S_0$ is such that $V[X(t)] = 7$ MPa², $V[X(t)]$ denoting the statistical variance of the process $X(t)$. These two extreme situations are represented in Fig. 2: the narrow-band PSD corresponds to the first case and the wide-band PSD represents the second one.

Figure 2: Narrow-band (top) and wide-band (bottom) PSDs of the investigated one-dimensional case.

As announced in the introduction, we consider the generation of these processes for the analysis of the extreme value distribution. In either case (wide and narrow) we proceed as follows: at first, $10^4$ realizations are generated. Afterwards, the extreme value distribution is constructed from these empirical data and fitted with a GEV distribution. In the following, the results regarding both cases are presented and illustrated together whenever it is possible.

**Theory of Gaussian process generation**

Let $X(t)$ be a temporal zero-mean Gaussian process. Let $S(f)$ be the associated PSD, uniformly discretized in $n_f$ points:

$$
\begin{align*}
(f_i, S(f_i)) &= (f_i, S_i) \\
\Delta f &= f_{i+1} - f_i, \quad 1 \leq i \leq n_f - 1
\end{align*}
$$

where $\Delta f$ is the frequency step and $n_f$ denotes the number of frequencies $f_i$ whose associated PSD magnitude $S_i$ is such that $S_i \neq 0$ (i.e. the number of frequencies $f_i$ such that $f_i \in [f_1, f_2]$). Two main
classes of generation methods exist in the literature (Sun, 2006). The first one includes time series models: Average Random (AR), Moving Average (MA) and ARMA processes. The other one is about spectral decomposition methods. We focus on the last category in this document. Each realization of \( X(t) \) is made of a sum of harmonic functions of time, whose amplitude and frequency are determined from the PSD:

\[
X(t) = \sum_{i=1}^{n_f} X_i(t).
\]  

(2)

One way of computing \( X(t) \) is to regard each harmonic contribution \( X_i(t) \) as a cosine function with deterministic modulus \( M_i \) and random uniformly distributed phase \( \phi_i \):

\[
X_i(t) = M_i \cos(2\pi f_i t + \phi_i)
\]  

(3)

where:

\[
\begin{align*}
M_i &\sim R(\sqrt{S_i \Delta f}) \\
\phi_i &\sim U(0, 2\pi)
\end{align*}
\]

Another possibility is to use the same expression as in Eq. (3) with \( M_i \) following a Rayleigh distribution, i.e.:

\[
\begin{align*}
M_i &\sim R(\sqrt{S_i \Delta f}) \\
\phi_i &\sim U(0, 2\pi)
\end{align*}
\]

Finally, we can write \( X_i(t) \) as a sum of a sine and a cosine functions with Gaussian amplitudes:

\[
X_i(t) = A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t)
\]  

(4)

with:

\[
\begin{align*}
A_i, B_i &\sim N(0, \sigma_i) \\
\sigma_i &\sim \sqrt{S_i \Delta f}
\end{align*}
\]

Numerical aspects

The random amplitudes \( A_i \) and \( B_i \) involved in Eq. (4) are computed using the Matlab generator of normally distributed numbers (\textit{randn} function). The expression proposed for their standard deviation arises from the statistical interpretation of the auto-PSD. As a matter of fact, the integral of \( S \) gives the variance of \( X \). The definition of \( \sigma_i \) then ensures units coherence: with this choice, \( A_i \) and \( B_i \) have the same dimensional units as \( X \). The total variance of the

Figure 3: Process samples during a short (left) and a longer (right) time, narrow-band (top) and wide-band (bottom) PSDs.
The following relation:

\[ V[X(t)] = \sum_{i=1}^{n_x} V[X_i(t)] = \sum_{i=1}^{n_x} \sigma_i^2 = \sum_{i=1}^{n_x} V[X_i] = V[X]. \]

The temporal variable vanishes due to the expected stationarity of the process. However, the reader must keep in mind that empirical processes cannot be stationary if the number of frequencies \( n_f \) is not large enough. On the contrary, they are Gaussian for any number of frequencies since they are computed as a linear combination of Gaussian random variables. In addition, it can be shown that the processes under consideration are ergodic for the mean but not for the correlation function (see Appendix). Besides, each PSD is characterized by its central frequency for the correlation function (see Appendix). Moreover, it should be noted that the computed correlation coefficient is defined as:

\[ \rho(\tau) = \frac{R(\tau)}{V[X]}. \]

Thus the knowledge of both the number of cycles and the PSD yields the sampling duration. Moreover, it should be noted that the computed processes are periodic, due to the use of harmonic functions. So we make sure to adjust the time period \( T \) to the samples length in order to avoid any duplication of the signals in the observation window. In practice, the value of \( T \) must be large enough to cover \( 10^6 \) cycles considering high cycle fatigue applications. Then a large number of frequencies are needed, which makes the frequency step \( \Delta f \) being very small and the generation time-consuming. One way to solve this problem is to take advantage of the IFFT algorithm. As matter of fact, adapting Eq. (4), we have:

\[
\begin{align*}
X(t) &= \sum_{i=1}^{n_x} A_i \cos(2\pi f_it) + B_i \sin(2\pi f_it) \\
&= \sum_{i=1}^{n_x} A_i \cos(2\pi f_it) + B_i \cos\left(-\frac{\pi}{2} + 2\pi f_it\right) \\
&= \sum_{i=1}^{n_x} A_i \text{Re} \left[ e^{i2\pi f_it} \right] + B_i \text{Re} \left[ e^{-i2\pi f_it} \right] \\
&= \text{Re} \left\{ \sum_{i=1}^{n_x} (A_i - B_i) e^{i2\pi f_it} \right\} \\
&= \sum_{i=1}^{n_x} \sum_{j=1}^{n_x} (A_i - B_j) e^{i2\pi f_j t}. 
\end{align*}
\]

To verify the adequateness of the computed samples, the empirical auto-correlation coefficient is calculated and compared to the analytical one in Fig. 5. For stationary processes, the autocorrelation coefficient is defined as:

\[ R(\tau) = \frac{E[X(t)X(t+\tau)]}{E[X^2]} \quad t \in [0, T] \]

\[ R(\tau) = \frac{S_0 \sin(2\pi f_2 \tau) - \sin(2\pi f_1 \tau)}{2\pi \tau}. \]

In order to compare the empirical and analytical auto-correlation coefficients in a proper way, both of the empirical and analytical auto-correlation functions are normalized using the theoretical value of the variance. To quantify the precision of the simulation, we shall compare in detail the theoretical and empirical values of the variances and auto-correlation coefficients. Figure 5 and Table 1 show the results, using here only \( n_f = 10^5 \) and \( 2 \times 10^5 \) frequencies for the narrow-band and the wide-band case respectively. The empirical auto-correlation coefficient and variance agree well with the analytical ones, what implies that the simulated processes actually correspond to their respective target PSDs.
Table 1: Comparison between the theoretical and the empirical variances, narrow-band (top) and wide-band (bottom) cases.

| Empirical (target) variance | Relative gap |
|-----------------------------|--------------|
| 7.0082 (7)                  | 0.12 %       |
| 7.0037 (7)                  | 0.05 %       |

Figure 5: Comparison between empirical and analytical auto-correlation coefficients, narrow-band (top) and wide-band (bottom).

The greatest value of each realization is stored in order to compute the associated distribution. In the sequel, EVD stands for “extreme value distribution”. Different verifications are made to evaluate the quality of the empirical EVDs. First, its mean and standard deviation are computed for an increasing number of samples. The values are provided in Fig. 6 with the related confidence intervals (95%), obtained from the bootstrap method. When generating $10^4$ samples of each process, the mean and standard deviation have converged, as well as the related confidence intervals. Secondly, the empirical EVD is plotted together with its 95% confidence interval, computed from the Greenwood formula. It must be underlined that this interval is extremely reduced, due to the large number of realizations. Finally, it is interesting to check (see Fig. 7) if the empirical EVDs actually “look like” usual EVDs. These are then fitted with 3 parameters Generalized Extreme Value (GEV) distributions:

$$F(x; \xi, \mu, \beta) = \exp\left[-\left(1+\frac{x-\mu}{\beta}\right)^{-\frac{1}{\xi}}\right], \quad 1+\frac{x-\mu}{\beta} > 0, \quad \xi \neq 0$$

(8)

Figure 6: Convergence of the EVDs mean (top) and standard deviation (bottom) value.

In probability theory and statistics, the GEV distribution is a family of continuous probability distributions developed under the extreme value theory in order to combine the Gumbel, Fréchet and Weibull families. The parametrization involved in Eq. (8) results from the work of von Mises (1936) and Jenkinson (1955). The GEV distribution arises from the extreme value theorem (Fisher and Tippet, 1928) as the limiting distribution of properly normalized maxima of a sequence of independent and identically distributed random variables. It has been shown that the framework can be extended to Gaussian processes; see for example De Haan and Ferreira (2006). The above-mentioned GEV distribution is computed from the simulation results by means of the maximum likelihood method (\textit{gevfit} function in Matlab). It should be noted that the resulting curve lies in the confidence interval. A particular attention is paid to the distribution tails since these may be involved in reliability calculations. We end by evaluating the relative error $err(p)$ between the empirical CDF and the related GEV distribution for any probability level $p$, $0 \leq p \leq 1$.

Let $s_{th}(p)$ be the threshold such that $\text{Prob}_{X(t) \in [0,T]}(X(t) < s_{th}) = p$ in the theoretical model (GEV, here). Let $s_{MC}(p)$ be the threshold such that $\text{Prob}_{X(t) \in [0,T]}(X(t) < s_{MC}) = p$ in the MC simulations. Then the relative error is computed as follows:

$$err(p) = \frac{s_{th}(p) - s_{MC}(p)}{s_{MC}(p)}$$

(9)

It must be underlined that the relative error between the empirical EVD and its GEV fit is low. These observations allow us to state that the amount of data (in particular the number of realizations) is large enough to yield accurate EVDs.
Extension to the Multi-Dimensional Case

Both cases introduced in the previous paragraph are now generalized to several processes. Using the PSDs relative to the \( n \) processes to be considered, we have the spectral matrix:

\[
S(\omega) = \begin{bmatrix}
S_{11}(\omega) & S_{12}(\omega) & \cdots & S_{1n}(\omega) \\
S_{21}(\omega) & S_{22}(\omega) & \cdots & S_{2n}(\omega) \\
\vdots & \vdots & \ddots & \vdots \\
S_{n1}(\omega) & S_{n2}(\omega) & \cdots & S_{nn}(\omega)
\end{bmatrix}
\]

\[= \begin{bmatrix}
S_{11}(\omega) & S_{12}(\omega) & \cdots & S_{1n}(\omega) \\
S_{21}(\omega) & S_{22}(\omega) & \cdots & S_{2n}(\omega) \\
\vdots & \vdots & \ddots & \vdots \\
S_{n1}(\omega) & S_{n2}(\omega) & \cdots & S_{nn}(\omega)
\end{bmatrix}
\]

\[S_{pp}(\omega), p = 1, \ldots, n \] being the process \( p \) auto-PSD and \( S_{pq}(\omega), p, q = 1, \ldots, n \) being the cross-PSD between processes \( p \) and \( q \). Each auto-PSD shows the variance frequential repartition of the related process. Each cross-PSD provides the covariance frequential repartition of the associated processes. In particular, we now consider the two-dimensional correlated case. Unlike the general case, cross-PSDs are real-valued in the following. As in the first section, the academic case of band-limited white noise is considered:

\[S_{11}(\omega) = \begin{cases} 
S_{01} & \text{if } \omega \in (f_1, f_2] \\
0 & \text{else}
\end{cases}
\]

\[S_{22}(\omega) = \begin{cases} 
S_{02} & \text{if } \omega \in (f_1, f_2] \\
0 & \text{else}
\end{cases}
\]

\[S_{12}(\omega) = \begin{cases} 
S_{012} & \text{if } \omega \in (f_1, f_2] \\
0 & \text{else}
\end{cases}
\]

Analogously to the one-dimensional case:

- in the narrow-band case, we take \( f_1 = 9.5 \text{ Hz}, f_2 = 10.5 \text{ Hz}, S_{01} = 3 \text{ MPa}^2/\text{Hz}, S_{02} = 5 \text{ MPa}^2/\text{Hz}, S_{012} = 0.9\sqrt{S_{01}S_{02}} \text{ MPa}^2/\text{Hz} \);

- the wide-band case, we take \( f_1 = 0 \text{ Hz}, f_2 = 20 \text{ Hz}, S_{01} = 0.15 \text{ MPa}^2/\text{Hz}, S_{02} = 0.25 \text{ MPa}^2/\text{Hz}, S_{012} = 0.9\sqrt{S_{01}S_{02}} \text{ MPa}^2/\text{Hz} \).

It should be noted that in either case, \( S_{01} \) is such that \( \text{Var}[X_1(t)] = 3 \text{ MPa}^2 \), \( S_{02} \) is such that \( \text{Var}[X_2(t)] = 5 \text{ MPa}^2 \) and \( S_{012} \) is such that \( \text{cov}[X_1(t), X_2(t)] = 0.9 \), \( \text{cov}[X_1(t), X_2(t)] \) denoting the statistical covariance between the processes \( X_1(t) \) and \( X_2(t) \). These two situations are represented in Fig. 8: the narrow-band PSD corresponds to the first case and the wide-band PSD represents the second one.
Theory of Gaussian processes generation

Here again, each realization of a Gaussian process results from a sum of harmonic functions of time with random amplitudes computed from the PSD. Compared to the one-dimensional case, the difference lies in the fact that random quantities have to be appropriately correlated such that processes are correlated according to their associated target cross-PSD. Then, Eq. (4) is generalized as follows:

$$X_p(t) = \sum_{i=1}^{n_t} \sum_{q=1}^{n_f} \left\{ A_{ipq} \cos(2 \pi f_i t + \theta_{ipq}) + B_{ipq} \sin(2 \pi f_i t + \theta_{ipq}) \right\}$$

(11)

where:

$$A_{ipq} = A_{iq}(f_i) \sigma_{ipq}$$
$$B_{ipq} = B_{iq}(f_i) \sigma_{ipq}$$
$$A_{iq}, B_{iq} \sim \mathcal{N}(0,1)$$
$$\sigma_{ipq} = |H_{pq}(f_i)| \sqrt{\Delta f}$$
$$\theta_{ipq} = \arg[H_{pq}(f_i)]$$
$$\mathbf{H}(f_i) = \mathbf{S}(f_i)$$

$i$ being the frequency index, $p$ being the current process index and $q$ being the summation index ($p$ being fixed) on all the processes (to introduce correlation between $X_p$ and $X_q$).

Results analysis

In order to verify the adequateness of the computed samples, the empirical auto and cross-correlation functions are calculated and compared to the analytical ones in Fig. 9. Analogously to Eq. (5), for a stationary process, the (auto and cross) correlation coefficients are defined as:

$$\rho_{pq}(\tau) = \frac{R_{pq}(\tau)}{\sqrt{V[X_p]} \sqrt{V[X_q]}}.$$  

(12)

$R_{pq}(\tau)$ being the related correlation function defined as:

$$R_{pq}(\tau) = \mathbb{E}[X_p(t) X_q(t+\tau)], \ t \in [0,T]$$

(13)

where $\tau$ is the time shift between the two instants $t$ and $t+\tau$ involved in Eq. (13). As mentioned earlier, $\tau \in [-T,T]$. Using Eq. (11) in (13) yields:

$$R_{pq}(\tau) = \sum_{i=1}^{n_t} \sum_{q=1}^{n_f} |H_{pq}(f_i) H_{qc}(f_i)| \Delta f \cos(2 \pi f_i t + \theta_{iqc} - \theta_{ipq})$$

(14)

Due to Eq. (10), in both the low-pass and the band-pass cases, the correlation functions are given by:

$$R_{11}(\tau) = \frac{S_{01} \sin(2 \pi f_2 \tau)}{2 \pi}$$
$$R_{22}(\tau) = \frac{S_{02} \sin(2 \pi f_2 \tau)}{2 \pi} \tau$$
$$R_{12}(\tau) = \frac{\text{Re}[S_{012}] \sin(2 \pi f_2 \tau) - \sin(2 \pi f_1 \tau)}{2 \pi}$$
$$+ \frac{\text{Im}[S_{012}] \cos(2 \pi f_2 \tau) - \cos(2 \pi f_1 \tau)}{2 \pi} \tau$$

(15)

### Table 2: Comparison between the theoretical and the empirical variances and cross-correlation coefficients, narrow-band (top) and wide-band (bottom) cases.

|                  | Empirical (target) value | Relative gap |
|------------------|--------------------------|--------------|
| $V[X_1]$         | 3.0267 (3)               | 0.89%        |
| $V[X_2]$         | 5.0462 (5)               | 0.92%        |
| $\rho_{12}$      | 0.9082 (0.9)             | 0.91%        |

|                  | Empirical (target) value | Relative gap |
|------------------|--------------------------|--------------|
| $V[X_1]$         | 2.9999 (3)               | -0.004%      |
| $V[X_2]$         | 5.0005 (5)               | 0.011%       |
| $\rho_{12}$      | 0.9901 (0.9)             | 0.008%       |

In order to compare the empirical and analytical correlation coefficients in a proper way, both the empirical and analytical correlation functions are normalized using the theoretical values of the variances. To quantify the precision of the simulation, we are interested in comparing in detail the theoretical and empirical values of the variances and cross-correlation coefficients. $n_f = 10^2$ frequencies are used in the narrow-band case and $n_f = 2 \times 10^3$ frequencies are used in the wide-band case. We can see from Fig. 9 that the empirical auto and cross-correlation coefficients agree well with the analytical ones. Moreover, Table 2 shows that the empirical variances and cross-correlation coefficients of the computed processes are in total agreement with the theoretical ones. This implies that the simulated processes actually correspond to their respective target PSDs and are accurately correlated.

Here again, storing the greatest value of each realization enables us to compute the bivariate EVD as well as the associated margins. The latter are fitted by GEV distributions in Fig. 11, as we did for...
where $F_{m1}(s_1)$ is the marginal EVD of the process $X_i(t)$, $F_{m12}(s_1, s_2)$ denotes the related bivariate EVD and $m$ is the association parameter. It is such that $m \in [1, +\infty[$, where $m = 1$ gives the independence case and $m \to +\infty$ corresponds to a complete dependence between the margins. This parameter is a function of the correlation coefficient $\rho_m$ between the marginal EVD:

$$m = (1 - \rho_m)^{-\frac{1}{2}}.$$  

Figure 9: Comparison between empirical and analytical correlation coefficients, narrow-band (left) and wide-band (right) PSDs.
In this work, $\rho_m$ is obtained from empirical data using Matlab (`corrcoef` function). Then Eq. (16) can be used with the GEV fit of the margins. The resulting distribution fits very well the empirical bivariate EVD, as shown by Fig. 10. This arises since fitting the extreme value margins with GEV distributions yields little errors (see Fig. 11).

**Application Considering an Hydrology Spectrum**

**Input data**

Here we consider mechanical stresses involved in a combination of a traction and a torsion loading, respectively $S_{xx}(t)$ and $S_{xy}(t)$. Other components of the stress tensor are considered equal to zero. In the following, instead of simulating $S_{xx}(t)$ and $S_{xy}(t)$, we generate the associated stresses processes needed for a fatigue criterion (Lambert et al., 2010):

$$\begin{cases} 
X_1(t) = \sqrt{3} S_{xx}(t) \\
X_2(t) = S_{xy}(t)
\end{cases}$$

Statistical moments are supposed to be the following:

$$\begin{align*}
E[S_{xx}(t)] &= 0 \text{MPa} \\
E[S_{xy}(t)] &= 0 \text{MPa} \\
V[S_{xx}(t)] &= 2700 \text{MPa}^2 \\
V[S_{xy}(t)] &= 675 \text{MPa}^2
\end{align*}$$

The previous considerations will now be applied to a realistic example which may be related to a tidal stream turbine. In this case, the mechanical stresses PSDs may be of the following type:

$$S_{pq}(f) = \begin{cases} 
\frac{2}{\pi} A_{pq} \left(\frac{2 \pi f}{3.11}\right)^{-\frac{5}{4}} e^{-\left(\frac{3.11}{2 \pi f}\right)^{82}} & f > 0 \\
0 & f = 0
\end{cases}$$

Equation (18) represents a Bretschneider spectrum and is illustrated in Fig. 12. This kind of PSD is often encountered in the field of hydrology (Dalrymple, 1994). The values of the constants $A_{pq}$ are set as follows: $A_{11} = 174.94$ S.I. units, $A_{22} = 131.21$ S.I. units, $A_{12} = 136.35$ S.I. units. It should be noted that in either case, $A_{11}$ is such that $V[X_1(t)] = 900 \text{MPa}^2$, $A_{22}$ is such that $V[X_2(t)] = 675 \text{MPa}^2$ and $A_{12}$ is such that $\text{cov}[X_1(t), X_2(t)] = \sqrt{V[X_1(t)]V[X_2(t)]} = 0.9$. This situation is represented in Fig. 12.
Results analysis

In order to verify the adequateness of the computed samples, the empirical auto and cross-correlation functions are compared to the analytical ones in Fig. 14. \( n_f = 10^4 \) frequencies are used. Equations (12) to (14) are still valid in this situation. To compare the empirical and analytical correlation coefficients in a proper way, both the empirical and analytical correlation functions are normalized using the theoretical values of the variances. Due to the complexity of the involved PSDs, the analytical correlation functions cannot be evaluated analytically but they are computed by means of the IFFT algorithm. To quantify the precision of the simulation, let us compare in detail the theoretical and empirical values of the variances and cross-correlation coefficients. From Fig. 14, the empirical correlation coefficients agree well with the analytical ones. Moreover, Table 3 shows that the variances and cross-correlation coefficient of the computed processes are in total agreement with the theoretical values. This implies that the simulated processes actually correspond to their respective target PSDs and are correctly correlated.

As we did in the previous subsection Results analysis, marginal EVDs are fitted by GEV distributions. Figure 15 shows that the relative error is low. Then Eq. (16) can be used with the GEV approximations of the margins. The resulting distribution fits very well the empirical bivariate EVD, as shown by Fig. 16.

Table 3: Comparison between the theoretical and the empirical variances and cross-correlation coefficients.

|                | Empirical (target) value | Relative gap  |
|----------------|--------------------------|---------------|
| \( V[X_1] \)   | 899.7903 (900)           | -0.0233 %     |
| \( V[X_2] \)   | 674.7338 (675)           | -0.0394 %     |
| \( \rho_{12} \) | 0.8997 (0.9)             | -0.0325 %     |

Figure 13: Process \( (X_i) \) sample during a short (top) and a longer (bottom) time.

Figure 14: Comparison between empirical and analytical correlation coefficients.

Figure 15: Relative error between the empirical margins and their GEV fits.
For any given realization of the process $X(t)$, the mathematical expectation of the process $X_p(t)$ is:

$$
\mu_{X_p}(t) = E[X_p(t)] = \sum_{i=1}^{n_t} \sum_{q=1}^{n_q} \sigma_{pq} \left\{ E[A_{iq} \cos(\omega_i t + \theta_{pq})] + E[B_{iq}] \sin(\omega_i t + \theta_{pq}) \right\} = 0.
$$

For any given realization of the process $X_p(t)$, the temporal mean is:

$$
\mu_{X_p}^{(r)} = \frac{1}{T} \sum_{i=1}^{n_t} \sum_{q=1}^{n_q} \sigma_{pq} \left\{ A_{iq}^{(r)} \int_0^T \cos(\omega_i t + \theta_{pq}) \, dt \right\} + B_{iq}^{(r)} \int_0^T \sin(\omega_i t + \theta_{pq}) \, dt = 0.
$$

Both the mathematical expectation and the temporal mean are equal (to zero), hence the processes are ergodic for the mean value.

### Non ergodicity for the correlation function

At any given time $t$, the correlation function between the processes $X_p(t)$ and $X_q(t)$ is:

$$
R_{X_pX_q}(t) = E[X_p(t)X_q(t + \tau)] = \sum_{i=1}^{n_t} \sum_{q=1}^{n_q} \sigma_{pq} \sigma_{iq} \left\{ E[A_{iiq} \cos(\omega_i t + \theta_{iq})] \cos(\omega_i (t + \tau) + \theta_{iq}) + E[A_{iiq}] \sin(\omega_i t + \theta_{pq}) \cos(\omega_i (t + \tau) + \theta_{iq}) \right\} + \left\{ E[B_{iq}] \sin(\omega_i t + \theta_{ipq}) \sin(\omega_i (t + \tau) + \theta_{iq}) \right\}.
$$

Since $A_{iq}$ and $B_{iq}$ are zero-mean random variables, their mean square values are equal to their (unit) variances $E[A_{iq}^2] = E[B_{iq}^2] = 1$. This yields:

$$
R_{X_pX_q}(t) = \sum_{i=1}^{n_t} \sum_{q=1}^{n_q} \sigma_{pq} \sigma_{iq} \left\{ E[A_{iiq}] \cos(\omega_i t + \theta_{iq}) \cos(\omega_i (t + \tau) + \theta_{iq}) + \sin(\omega_i t + \theta_{ipq}) \sin(\omega_i (t + \tau) + \theta_{iq}) \right\} = \sum_{i=1}^{n_t} \sum_{q=1}^{n_q} \sigma_{pq} \sigma_{iq} \cos(\omega_i t + \theta_{iq} - \theta_{ipq}).
$$

For any given realization of the processes $X_p(t)$ and $X_q(t)$, the temporal correlation function is:

$$
R_{X_pX_q}^{(r)}(t) = \mathbb{E}[X_p(t)X_q(t + \tau)] = \frac{1}{T} \int_0^T X_p(t)X_q(t + \tau) \, dt = \frac{1}{T} \sum_{i=1}^{n_t} \sum_{q=1}^{n_q} \sigma_{pq} \sigma_{iq} \left\{ A_{iiq}^{(r)} A_{iq}^{(r)} \int_0^T \cos(\omega_i t + \theta_{iq}) \cos(\omega_i (t + \tau) + \theta_{iq}) \, dt + B_{iq}^{(r)} B_{iq}^{(r)} \int_0^T \sin(\omega_i t + \theta_{ipq}) \sin(\omega_i (t + \tau) + \theta_{iq}) \, dt \right\} + \left\{ A_{iiq}^{(r)} B_{iq}^{(r)} \int_0^T \cos(\omega_i t + \theta_{iq}) \sin(\omega_i (t + \tau) + \theta_{iq}) \, dt \right\} + \left\{ B_{iq}^{(r)} A_{iiq}^{(r)} \int_0^T \sin(\omega_i t + \theta_{ipq}) \cos(\omega_i (t + \tau) + \theta_{iq}) \, dt \right\} + \left\{ B_{iq}^{(r)} B_{iq}^{(r)} \int_0^T \sin(\omega_i t + \theta_{ipq}) \sin(\omega_i (t + \tau) + \theta_{iq}) \, dt \right\}.
$$
After some calculations, we find:

\[ I_1 = \frac{1}{2} \int_0^T \cos \left( (\omega_1 - \omega_2) t - \omega_2 \tau + \theta_{1pq1} - \theta_{2q2} \right) \, dt \]

\[ I_2 = -\frac{1}{2} \int_0^T \sin \left( (\omega_1 - \omega_2) t - \omega_2 \tau + \theta_{1pq1} - \theta_{2q2} \right) \, dt \]

\[ I_3 = \frac{1}{2} \int_0^T \sin \left( (\omega_1 - \omega_2) t - \omega_2 \tau + \theta_{1pq1} - \theta_{2q2} \right) \, dt \]

\[ I_4 = \frac{1}{2} \int_0^T \cos \left( (\omega_1 - \omega_2) t - \omega_2 \tau + \theta_{1pq1} - \theta_{2q2} \right) \, dt. \]

Replacing \( I_1 \) to \( I_3 \) by their respective expressions gives:

\[ R_{pq}^{(i)}(\tau) = \sum_{i_1} \sum_{i_2} \sum_{i_3} \sum_{i_4} \sigma_{1pq1} \sigma_{2q2} \]

\[ \times \left\{ \frac{A_{1pq1}^{(i)} + B_{1pq1}^{(i)}}{2} \int_0^T \cos \left( (\omega_1 - \omega_2) t - \omega_2 \tau + \theta_{1pq1} - \theta_{2q2} \right) \, dt \right\} \]

\[ + \frac{B_{1pq1}^{(i)} - A_{1pq1}^{(i)}}{2} \int_0^T \sin \left( (\omega_1 - \omega_2) t - \omega_2 \tau + \theta_{1pq1} - \theta_{2q2} \right) \, dt \}\]

\[ = 1 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sigma_{pq1} \sigma_{q2} \]

\[ \times \int_0^T \cos \left( -\omega_2 \tau + \theta_{pq1} - \theta_{pq2} \right) \, dt \]

\[ = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sigma_{pq1} \sigma_{q2} \]

\[ \times \int_0^T \cos \left( \omega_1 \tau + \theta_{pq1} - \theta_{pq2} \right) \, dt. \]

Both the mathematical expectation and the temporal mean are not equal, hence the processes are not ergodic for the correlation function.

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