The Geometrical Modelling of Fluids

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Abstract

The paper considers the nonlinear electrodynamics type model and its relation with relativistic hydrodynamics with no dissipation (including string and membrane hydrodynamics). We are able to convert arbitrary flux of fluid to the family of geodesics by the conformal transformation of metric. The conditions of transformation of nonlinear electrodynamics solution to linear electrodynamics solution by changing of metric are presented.

1 Nonlinear electrodynamics-type model

Let us consider the following action for the model of non-linear electrodynamics type with \((D - n - 1)\)-form potential \(I (I_{M_1...M_{D-n-1}} = I_{[M_1...M_{D-n-1}]} )\) in \(D\)-dimensional space-time with metric \(g_{MN}\) of signature \((- , +, +, \ldots , +)\)

\[
S_1[I] = \int d^D x \sqrt{|g|} L(\|J\|),
\]

(1)

where \(J = dI\).

We use the following notation:

\[
(dA)_{M_0M_1...M_k} = (k + 1) \partial_{[M_0} A_{M_1...M_k]},
\]

\[
(A, B) = \frac{1}{k!} A_{M_1...M_k} B^{M_1...M_k}, \quad \|A\|^2 = (A, A),
\]

\[
(\delta A)^{M_1...M_{k-1}} = \frac{1}{\sqrt{|g|}} \partial_{M_k} \left( \sqrt{|g|} A^{M_1...M_{k-1}M_k} \right).
\]

The "first pair of Maxwell equations" \(dJ = 0\) is the consequence of the definition \(J = dI\).

The variation of action with respect to \(I\) provides the field equations ("second pair of Maxwell equations")

\[
\delta \left( J \frac{L'(\|J\|)}{\|J\|} \right) = 0, \quad \Leftrightarrow \quad \partial_{M_{D-n}} \left( \sqrt{|g|} J^{M_1...M_{D-n}} \frac{L'(\|J\|)}{\|J\|} \right) = 0.
\]

(2)

The models becomes linear if \(L(\|J\|) \sim \|J\|^2\). In this case field equations are \(\delta J = 0\). It is the regular Maxwell electrodynamics in 4-dimensional space-time if \(D = 4, n = 2\).

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2 Relativistic hydrodynamics

The non-linear electrodynamics-type model [1] could also describe the potential nondissipative flux of fluid, consists of \((n - 1)\)-dimensional membranes. The fluid is continuous distribution of membranes, where each point of \(D\)-dimensional space-time belongs to the single membrane world-surface \(V\). If \(n = 1\) the membrane is particle, i.e. we have a regular fluid. If \(n = 2\) the membrane is a string, i.e. we have a string-fluid.

\((D - n)\)-form \(J\) is dual density of membrane flux. For regular fluid of particles \((n = 1)\) \(J_{M_2...M_D} = j^{M_1} \sqrt{|g|} \varepsilon_{M_1...M_D}\), where \(j^M\) is particle flux density, and \(\varepsilon_{M_1...M_D}\) \((\varepsilon_{M_1...M_D} = \varepsilon_{[M_1...M_D]}\), \(\varepsilon_{01...D-1} = +1\) \) is totally antisymmetric symbol.

\(J\) specifies \((D - n)\) directions orthogonal to world-surface of dimension \(n\), i.e. every vector \(v^M\), tangent to world-surface \(V\), satisfies the condition \(v^{M_1} J_{M_1...M_{D-n}} = 0\). \(\|J\|\) is the density of particiles/strins/membranes, and \(-L(\|J\|)\) is energy density in attendant frame.

Statement 1. To describe the flux of particle/strings/membranes by field \(J\) one specifies the world-surfaces as a level-surfaces for set of scalars \(\varphi^\alpha, \alpha = 1, \ldots, D - n\) (see [1] and references therein). The field \(J\) represented in terms of \(\varphi^\alpha\) by the following equation

\[
J = J_\varphi = f(\varphi) \, d\varphi^1 \wedge \cdots \wedge d\varphi^{D-n} \iff J_{M_1...M_{D-n}} = f(\varphi) \, (D - n) \left( \frac{1}{D - n - 1} \right)！ \partial_{[M_1} \varphi^1 ... \partial_{M_{D-n}]\varphi^{D-n}},
\]

where \(f(\varphi)\) is a function of \(\varphi^\alpha\). \(f(\varphi)\) could be set to unit by redefinition of fields \(\varphi^\alpha\), so we assume \(f(\varphi) \equiv 1\).

The equation (2) becomes the equation of potential flux of membrane fluid (regular fluid if \(n = 1\)), consist of non-intersecting membranes with world-surfaces \(\varphi = \text{const}\). Pressure acting on membrane in membrane in attendant frame is

\[
p_\perp = L - \|J\| \cdot L' .
\]

E.g. the Lagrangian \(L(\|J\|) \sim \|J\|\) describes zero pressure membrane fluid ("membrane dust").

Similar to regular fluid, membrane fluid equations of motion could be derived from energy-momentum tensor continuity conditions \(\nabla_M T^{MN} = 0\). So, to prove the correspondence one has to check the energy-momentum tensor

\[
T_{MN} = L(\|J\|) \, P_{MN} + p_\perp \left( g_{MN} - P_{MN} \right),
\]

where

\[
P_{MN} = g_{MN} - J_{M_2...M_{D-n}} J_{N_2...N_{D-n}} \left( \frac{1}{D - n - 1} \right) ! \|J\|^2 .
\]

If \(J\) is defined by equation (3), then the tensor \(P_{MN}\) is a projector to tangent subspace at world-surface \(V\) in the point considered.

Statement 2. One can skip the potentiality condition for the flux (3) by the modification of action by \((D - n)\)-form Lagrange factor \(K\):

\[
S_2[I, \varphi, K] = \int d^D x \sqrt{|g[|} [L(\|J\|) + (K, J - J_\varphi)] .
\]

After the removing of Lagrange factor \(K\) new equations of motion could be represented in the following form (forms \(J\) and \(J_\varphi\) are identified):

\[
 J_{M_1...M_{D-n-1}} \partial_{M_{D-n}} \left( \frac{1}{\sqrt{|g|}} J^{M_1...M_{D-n}1M_{D-n}} L'(\|J\|) \right) = 0 .
\]

2
Equations (4) could be derived from equations (2) by projection to directions orthogonal to membrane world-surface. So, for the field (3) equation (4) is more weak, than equation (2). It corresponds to skipping of flux potentiality condition.

3 Metric transformations and nonlinearity

3.1 Conformal transformations of metric

One can remove the extra factor (in comparison to linear theory) $L'(\|J\|)/\|J\|$ in field equations (2), by modification of space-time metric. Let us consider the conformal transformation of metric:

$$g_{MN} \rightarrow \tilde{g}_{MN} = F^2 g_{MN},$$
$$g^{MN} \rightarrow \tilde{g}^{MN} = F^{-2} g^{MN},$$
$$\sqrt{|g|} \rightarrow \sqrt{|\tilde{g}|} = F^D \sqrt{|g|},$$
$$J_{M_1 \ldots M_{D-n}} \rightarrow J_{M_1 \ldots M_{D-n}},$$
$$\tilde{J}^{M_1 \ldots M_{D-n}} = F^{-2(D-n)} J^{M_1 \ldots M_{D-n}},$$
$$\|J\| \rightarrow \|\tilde{J}\| = F^{-(D-n)} \|J\|.$$

**Statement 3.** If the conformal factor $F$, satisfies the condition

$$F^{2n-D} = \frac{L'\|J\|}{\|J\|},$$

then using metric $\tilde{g}_{MN}$ field equations could be represented in the following form

$$\delta \tilde{J} = 0.$$

I.e. in new metric one has standard linear equations for free (with no sources) closed $(D-n)$-form $J$. For the regular non-linear electrodynamics in 4-dimensional space-time $(D = 4, n = 2)$ the transformation (5) is not possible.

In the general case (if $D \neq 2n$) one can use the conformal transformations of metric in both directions, connecting the solutions of different non-linearity in conformally-equivalent spaces.

For linear electrodynamics-type model in space with any fixed metric superposition principle holds. According the principle sum of two field equation solutions is a new solution of the same equations. One could spread the superposition principle to nonlinear models using the following procedure.

**Statement 4.** Nonlinear superposition principle. The field is described by action (1) with $D \neq 2n$. Let $J^{(1)}$ and $J^{(2)}$ are solutions of field equations at some conformally-equivalent spaces with metrics $g_{MN}^{(1)}$ and $g_{MN}^{(2)}$, such that the metrics become the same after the transfer to linear model, i.e.

$$F^{(1)} g_{MN}^{(1)} = F^{(2)} g_{MN}^{(2)} = g_{MN}^{(0)},$$

where

$$F^{(i)} = \left( \frac{L'(\|J^{(i)}\|^{(i)})}{\|J^{(i)}\|^{(i)}} \right)^{\frac{1}{2n-D}},$$

3
here $\| \cdot \|_{(i)}$ is calculated using the metric $g^{(i)}_{MN}$. One can transform both solutions to solutions in the space with the same metric $g^{(0)}_{MN}$ and build a linear combination of transformed solutions:

$$J^{(\alpha+\beta)} = \alpha J^{(1)} + \beta J^{(2)};$$

(6) $\alpha$ and $\beta$ are constants. The new field $J^{(\alpha+\beta)}$ is a solution of the same nonlinear equations at the space with new metric

$$g^{(\alpha+\beta)}_{MN} = F^{(-2)}_{(\alpha+\beta)} g^{(0)}_{MN},$$

(7) where $F^{(-2)}_{(\alpha+\beta)}$ is specified by the relation

$$F^{(-2)}_{(\alpha+\beta)} = \left( \frac{L'}{\| J^{(\alpha+\beta)} \|_{(\alpha+\beta)}} \right)^{\frac{1}{2n-D}} = \left( \frac{L' \left( \frac{F^{(-2)}_{(\alpha+\beta)}}{\| J^{(\alpha+\beta)} \|_{(\alpha+\beta)}} \right)}{\sqrt{g^{(0)}} \left( \frac{F^{(-2)}_{(\alpha+\beta)}}{\| J^{(\alpha+\beta)} \|_{(\alpha+\beta)}} \right)} \right)^{\frac{1}{2n-D}}.$$

3.2 Metric transform for ”electrostatics” and ”magnetostatics”

In special case $D = 2n$ one could not remove nonlinearity by conformal metric transform. Nevertheless, if there is no time dependence, then the problem could be treated in the important cases of static fields in static space-time.

Projection of $(D-n)$-form $J$ to surface $t = const$ produce $(D-n)$-form $J^H$ (components are numerated by indices with no time) and $(D-n-1)$-form $J^E$ (components are numerated by indices with time):

$$J = J^H + dt \wedge J^E.$$

If $D = 2n = 4$, then form $J^E$ is covector of electric field, and $J^H$ is 2-form of magnetic field.

If $J^H = 0$ (”electrostatic” case) or $J^E = 0$ (”magnetostatic” case), Lagrangian is a function of $\| dt \wedge J^E \|$ or $\| J^H \|$.

Let $g_{\alpha\beta} = g^{\alpha\beta} = 0$, for $\alpha \neq 0$. In this case $\| dt \wedge J^E \| = \sqrt{g^{(0)}} \| J^E \|_h$, where $\| \cdot \|_h$ is calculated using the metric $h_{\alpha\beta} = g_{\alpha\beta} (\alpha, \beta = 1, \ldots, D-1)$, restricted to $(D-1)$-dimensional space $t = const$.

Now one can introduce two $(D-1)$-dimensional actions.

$$S_H[I_H] = \int d^{D-1}x \sqrt{|h|} L_H(\| J^H \|_h), \quad L_H(\| J^H \|_h) = \sqrt{g^{(0)}} L(\| J^H \|_h),$$

(8)

$$S_E[I_E] = \int d^{D-1}x \sqrt{|h|} L_E(\| J^E \|_h), \quad L_E(\| J^E \|_h) = \sqrt{g^{(0)}} L(\sqrt{g^{(0)}} \| J^E \|_h),$$

(9)

where $J^E = dI_E$, $J^H = dI_H$. Here $I_H$ is $(D-n-1)$-form (for regular electromagnetic field it is vector-potential), and $I_E$ is $(D-n-2)$-form (for regular electromagnetic field it is scalar potential).

So, instead of one theory we study two static theories at space $t = const$. Each of the theories admits the conformal transform described above (5), which makes theory linear. Now the transform is not space-time transform, but pure space transform. After the transformations one has linear ”electrostatics” and linear ”magnetostatics”.

Statement 5. For ”electrostatics” and ”magnetostatics” nonlinear superposition principle similar to (6), (7) is applicable with replacement of parameters of action (11) to appropriate parameters of action (9) or action (8).
3.3 Other metric transformations

In this section we consider the useful examples of metric transformations, which are not conformal.

**Example 1.** Metric of non-rotating black hole could be replaced by flat metric of Minkowski space-time, if linear electrostatic field is considered. The metric has the form

\[ ds^2 = k(r) \, dt^2 - \frac{dr^2}{k(r)} - r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right). \]  

(10)

Electrostatic potential has the only component \( I_r \), i.e. \( I = a(r) \, dt \). Electric field also has the only component \( J_{rt} \), i.e. \( J = a'(r) \, dr \wedge dt \). Omitting of \( k(r) \) in \( g_{tt} \) and \( g_{rr} \) does not change \( g = \det(g_{MN}) \) and the only contravariant field component \( J^t \). So, one can replace in the electromagnetic field equations black hole metric (10) by Minkowski space-time metric. In the example considered we transform linear electrodynamics to linear electrodynamics again in different space-time. The transformation considered is based upon the specific form of metric tensor. The conformal transformations we discussed above were applicable to arbitrary metric of certain dimension.

**Example 2.** Similarly, spherically-symmetric solution of nonlinear electrodynamics equations \( J = a'(r) \, dr \wedge dt \) in spherically-symmetric space-time

\[ ds^2 = k(r) \, dt^2 - q(r) \, dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \]

could not be transformed to linear theory by conformal transformation \((D = 2n = 4)\). Nevertheless, we can transform radial and time components of metric to transform the particle-like solution into the vacuum solution of linear theory:

\[ g_{tt} = k(r) \rightarrow F^2 g_{tt}, \quad g_{rr} = -q(r) \rightarrow F^{-c} g_{rr}. \]

4 Metric transformations in hydrodynamics

Let field \( J \) specifies the solution of (membrane) fluid. The model with zero pressure \( p_\perp = 0 \) is (membrane) dust. In dust model particles (membranes) do not interact with each other. This model could be considered to be preferable. For (membrane) dust lines/surfaces of flow \( \varphi = \text{const} \) in space-time are geodesic lines/surfaces.

**Statement 6.** If we choose the factor \( F \) according to the following condition (instead of condition (5))

\[ F^n = L'(\|J\|), \]  

(11)

then in the metric \( \tilde{g}_{MN} \) the equation of motion has the form

\[ \tilde{\delta} \left( \frac{\tilde{J}}{\|\tilde{J}\|} \right) = 0. \]  

(12)

The equation describes (membrane) dust with the dual density of particles/strings/membranes flow \( \tilde{J} \).

If one applies the transformation (11) to electrostatics or magnetostatics, then the world surfaces of electric field-lines or of magnetic field-surfaces are geodesic surface of a metric conformally-equivalent to initial one. For electrostatics in Minkowski space-time

\[ F^4 \sim E^2. \]
It transforms the point charge into infinite tube.

**Example 3.** For single particle in Minkowski space-time

\[
ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2) \quad \rightarrow \quad ds^2 = \frac{dt^2}{r^2} - \left(\frac{dr}{r}\right)^2 - (d\theta^2 + \sin^2 \theta \, d\varphi^2).
\]

Electrostatic fields does not depend upon time. So, it is natural to represent field-lines as geodesics of 3-dimensional geometry (not as sections \(t = \text{const}\) of geodesics surfaces of 4-dimensional geometry).

If one removes time coordinate, it produces the theory with \(D = 3, n = 1\) (electrostatics) and the theory with \(D = 3, n = 2\) (magnetostatics).

So, we get the other (3-dimensional) conformal factor, which makes electric field-lines geodesics:

\[F^2 \sim E^2.\]

This transformation revert charges "upside down" (see the next example).

**Example 4.** For the single charge Euclidean space undergo the inversion transformation:

\[
dl^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2) \quad \rightarrow \quad d\tilde{l}^2 = \left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2).
\]

In the magnetostatic case \(D = 3, n = 1\) the transformation is generated by factor

\[F^4 \sim H^2.\]

Here, since the field-surfaces belongs to subspace \(t = \text{const}\), the conformal factor is the same in 3-dimensional and 4-dimensional cases.

### 5 Conclusion

Many authors, working in general relativity, postulate, that the theory combines two styles: "high" geometrical style of Einstein tensor and "low" field style of energy-momentum tensor. The geometrical interpretation of matter field allows to make field style "higher", i.e. more geometrical.

The main result of the paper is the demonstration of possibility to modify matter action by metric transformations. In many cases one can transform nonlinear theory into liner one and introduce analogue of superposition principle. Nevertheless nonlinearity does not disappear, it is hidden in the metric transformation.

Similarly to many other problems, the most physical case (electromagnetic field in 4-dimensional space-time) is exceptional case, linear and nonlinear electrodynamics could not be transformed to each other by conformal metric transformation due to conformal invariance.

The reduction of relativistic dynamics of (membrane) fluid to description of family of geodesics is the other interesting application of conformal metric transformation. So, the pressure of membranes/strings/particles is modified (it could be set to zero) by conformal transformations. The pressure provides the interaction between the close membranes. If the pressure is switched off, then one has just free motion of nonintersecting membranes/strings/particles. It could be interesting for string and brane theories (see review [2] and references therein) and for relativistic elasticity theory [1, 3].
Description of dynamics through the introduction of the additional metric is also considered in *acoustic geometry*, i.e. it describes *acoustic black holes* [4] (large number of publications could be found by the key words). In this approach the wave equation in non-relativistic fluid is written in terms of some pseudo-Riemannian metric. For fluid in rest it is the Minkowski metric with speed of light replaced by speed of sound.

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