PRESHEAVES OF GROUPOIDS AS MODELS FOR HOMOTOPY TYPES

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Abstract. We prove that for any weak test category $\mathbf{A}$, the category of presheaves of groupoids over $\mathbf{A}$ models homotopy types in a canonical way. The approach taken is a generalization of Grothendieck’s theory of test categories.

Contents

Introduction 1
1. Preliminaries 4
2. Very brief recollection of the theory of test categories 5
3. Homotopy theory of presheaves of groupoids 10
4. Groupoidal weak test categories 13
5. Groupoidal test categories 18
6. Groupoidal strict test categories 26
7. Test categories vs. Groupoidal test categories 27
8. Weak equivalences via the nerve 33
References 35

Introduction

In his famous manuscript Pursuing Stacks from 1983 \cite{Gro22}, Grothendieck introduced the theory of test categories. Informally speaking, a test category is a small category $\mathbf{A}$ such that the category $\hat{\mathbf{A}}$ of presheaves over $\mathbf{A}$ models homotopy types in a canonical way, which was axiomatized by Grothendieck. “Models homotopy types” means here that there is a particular class of morphisms on $\hat{\mathbf{A}}$, such that the localization of $\hat{\mathbf{A}}$ with respect to this class of morphisms is equivalent to the category $\mathrm{Hot}$ of CW-complexes and homotopy classes of continuous maps between them.\footnote{The archetypal example of a test category is of course the category $\Delta$ of finite non-empty ordinals, for which it has been known since the famous result of Milnor \cite{Mil57} that the category $\hat{\Delta}$ of simplicial sets models homotopy types.} The archetypal example of a test category is of course the category $\Delta$ of finite non-empty ordinals, for which it has been known since the famous result of Milnor \cite{Mil57} that the category $\hat{\Delta}$ of simplicial sets models homotopy types.

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\footnote{In practise, this equivalence of homotopy categories will always come from an equivalence at the level of $\infty,1$-categories.}
Examples of test categories abound, such as the cubical category \cite[Corollaire 8.4.13]{Cis06}, the cubical category with connections \cite{Mal09}, Joyal’s Θ category \cite{CM11}, the dendroidal category Ω \cite{ACM19}, etc. The point of view of the theory of test categories is that the category of presheaves on any test category should be, in some sense, as good a model of homotopy types as the category of simplicial sets. But modeling homotopy types is not the only good homotopical property of the category of simplicial sets. For example, we have the following results:

(i) the category $\text{Ab}(\hat{\Delta})$ of abelian groups internal to simplicial sets models chain complexes in non-negative degree up to quasi-isomorphism \cite[Dol58, Kan58a]{Dol58},

(ii) the category $\text{Grp}(\hat{\Delta})$ of groups internal to simplicial sets models pointed connected homotopy types \cite[Kan58b]{Kan58b},

(iii) the category $\text{Grpd}(\hat{\Delta})$ of groupoids internal to simplicial sets models homotopy types \cite[Cra95, Theorem 8.3]{Cra95}, \cite[JT96, Theorem 10]{JT96}.

In this article, we only focus on property (iii) above and prove its generalization, which is very informally stated as follows:

**Theorem 1.** If $A$ is a test category, then the category $\text{Grpd}(\hat{A})$ models homotopy types in a “canonical way”.

The major contribution of the present work is perhaps not so much the conclusion of the previous result but rather the precise axiomatization of what is meant by “canonical way”. To give a slight hint of what this means, we have to recall the basic setup of the theory of test categories. For any small category $A$, Grothendieck considers the following canonical functor

$$i_A : \hat{A} \to \text{Cat},$$

which sends an object $X$ of $\hat{A}$, to its category of elements $A/X$. Now, recall the result attributed to Quillen by Illusie \cite[Ill72, Corollaire 3.3.1]{Ill72} which says that $\text{Cat}$ models homotopy types. More precisely, define the class $\mathcal{W}_\infty$ of morphisms $u : C \to D$ of $\text{Cat}$ that induce homotopy equivalences between the classifying spaces of $C$ and of $D$; then the localization of $\text{Cat}$ with respect to $\mathcal{W}_\infty$ is equivalent to $\text{Hot}$. Using this, we can define a canonical class of weak equivalences on $\hat{A}$ as:

$$\mathcal{W}_{\hat{A}} := i_A^{-1}(\mathcal{W}_\infty),$$

and ask when the induced functor after localization

$$\overline{i_A} : \hat{A}[\mathcal{W}_{\hat{A}}^{-1}] \to \text{Cat}[\mathcal{W}_\infty^{-1}] \simeq \text{Hot}$$

is an equivalence of categories.

What about $\text{Grpd}(\hat{A})$ now? There are two trivial but essential observations: (1) the category of groupoids internal to $\hat{A}$ is equivalent to the category $[A^{\text{op}}, \text{Grpd}]$ of presheaves over $A$ with values in the category of groupoids, (2) for an object $X$ of $\hat{A}$, the category of elements $A/X$ is nothing but the so-called *Grothendieck construction* of $X$, and the

\footnote{The classical “Dold–Kan” equivalence even says that there is an equivalence of categories between $\text{Ab}(\hat{\Delta})$ and the category of chain complexes in non-negative degree. However, from a homotopical point of view, it is the homotopical result stated above which is relevant.}
Grothendieck construction is also defined for presheaves with values in groupoids. In other words, there is a functor

$$I_A : \text{Grpd}(\hat{A}) \to \text{Cat},$$

whose restriction to $\hat{A}$ (when we see sets as discrete groupoids) is $i_A$. The generalization of the situation for $\hat{A}$ described earlier is then straightforward. We define a canonical class of weak equivalences on $\text{Grpd}(\hat{A})$ as:

$$W_{\text{Grpd}(\hat{A})} := I_A^{-1}(W_\infty),$$

and we can ask when the induced functor after localization

$$\mathcal{T}_A : \text{Grpd}(\hat{A})[W_{\text{Grpd}(\hat{A})}^{-1}] \to \text{Cat}[W_\infty^{-1}] \simeq \text{Hot}$$

is an equivalence of categories.

Taking this seemingly naive approach seriously, we end up defining the notion of groupoidal test category as a perfect formal analogue of the usual notion of test category. In fact, almost all of the results from the usual theory also work in the groupoidal theory, once they have been appropriately adapted. This generalization from the theory for $\text{Set}$-valued to $\text{Grpd}$-valued presheaves is exactly what this article is about. Once the new theory is well established, Theorem 1 above, which is now reformulated as “every test category is a groupoidal test category”, is obtained almost effortlessly as a natural by-product. It is to be noted that the proof of this theorem does not use the classical fact that $\text{Grpd}(\hat{A})$ models homotopy types and gives a new and more elementary proof in this particular case. Surprisingly, the converse of Theorem 1 is also true and we obtain:

**Theorem 2.** A small category $A$ is a groupoidal test category if and only if it is a test category.

As an application, we obtain a lot of new models for homotopy types: the category $\text{Grpd}(\hat{\square})$ of groupoids internal to cubical sets (with or without connections), the category $\text{Grpd}(\hat{\Theta})$ of groupoids internal to cellular sets, the category $\text{Grpd}(\hat{\Omega})$ of groupoids internal to dendroidal sets, the category $\text{Grpd}(\hat{\Delta}')$ of groupoids internal to semi-simplicial sets (this last example requires a version of Theorem 1 for weak test categories, which is also true), etc.

Let us end this introduction with a quick word on what is not treated in the present article. In his book [Cis06], Cisinski showed that the category of ($\text{Set}$-valued) presheaves on any test category admits a model structure where the weak equivalences are the ones canonically defined by Grothendieck. The generalization of this for $\text{Grpd}$-valued presheaves (known when $A = \Delta$) [Cra93, JT96]) is not addressed at all here and is left as future work.

**Organization of the paper.** The first section is a preliminary section recalling some basic homotopical algebra needed in the rest of the paper. The second section is a quick recollection, without proofs, of the classical theory of test categories. It is in the third section that we finally dive into the subject and give the basic setup of the homotopy theory of $\text{Grpd}$-valued presheaves, mimicking Grothendieck’s axiomatics for the homotopy theory of $\text{Set}$-valued presheaves. In the fourth, fifth and sixth sections,
we respectively develop the theory of groupoidal weak test categories, groupoidal test categories and groupoidal strict test categories. The seventh section is dedicated to the comparison of the classical theory with the groupoidal theory and in particular we obtain the theorems stated previously in the introduction. Finally, in the eighth and last section, we give an alternative definition of the weak equivalences of $\text{Grpd}$-valued presheaves, which makes the link with previous existing works on simplicial groupoids.

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1. Preliminaries

1.1. A *category with weak equivalences* is a pair $(\mathcal{C}, W)$, where $\mathcal{C}$ is a category and $W$ is a class of morphisms of $\mathcal{C}$, generically referred to as the *weak equivalences*, which contains all isomorphisms and satisfy the 2-out-of-3 property. We say that $W$ is *weakly saturated* if in addition it satisfies the following closure property: If $i: X \to Y$ and $r: Y \to X$ are morphisms of $W$ such that $ri = \text{id}_Y$ and $ir \in W$, then $r \in W$ (and thus so is $i$ by 2-out-of-3).

When $\mathcal{C}$ has pullbacks, we say that a morphism $f: X \to Y$ is a *universal weak equivalence*, if for every pullback square

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow^j & & \downarrow^f \\
Y' & \longrightarrow & Y,
\end{array}
$$

the morphism $f'$ is a weak equivalence (in particular $f$ is a weak equivalence).

By *homotopy category* of a category with weak equivalences $(\mathcal{C}, W)$, we mean the localization of $\mathcal{C}$ with respect to $W$ [GZ67]. It is denoted by $\text{Ho}_W(\mathcal{C})$, or simply $\text{Ho}(\mathcal{C})$ when there is no risk of confusion.

1.2. Let $(\mathcal{C}, W)$ and $(\mathcal{C}', W')$ be two categories with weak equivalences. A functor $F: \mathcal{C} \to \mathcal{C}'$ is said to *preserve weak equivalences* if $F(W) \subseteq W'$. In this case, it induces a functor at the level of homotopy categories $F: \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{C}')$.

A *homotopy inverse* of such a functor, is a functor $G: \mathcal{C}' \to \mathcal{C}$ which preserves weak equivalences and such that there exists zigzags of natural transformations $FG \rightsquigarrow \text{id}_{\mathcal{C}'}$, and $GF \rightsquigarrow \text{id}_{\mathcal{C}}$, which are pointwise weak equivalences (see (ii) below). In this case, $F$ induces an equivalence of categories $\text{Ho}(\mathcal{C}) \simeq \text{Ho}(\mathcal{C}')$ and similarly for $G$.

Finally, an adjunction $L: \mathcal{C} \leftrightarrow \mathcal{C}': R$ is a *homotopical equivalence* if:

(i) $L$ and $R$ preserve weak equivalences, $L(W) \subseteq W'$ and $R(W') \subseteq W'$.

3In fact, $F$ and $G$ are then even Dwyer–Kan equivalences [BK12], hence they induce an equivalence of $(\infty, 1)$-categories.
(ii) the unit and co–unit of the adjunction are pointwise weak equivalences, i.e. for every object \( X \) of \( C' \) and every object \( Y \) of \( C \) we have

\[
\varepsilon_X : LR(X) \to X \in \mathcal{W}' \quad \text{and} \quad \eta_Y : Y \to RL(Y) \in \mathcal{W}.
\]

Note that in this case \( L \) and \( R \) are homotopy inverses to each other.

The following lemma is a very useful criterion to detect homotopical equivalences.

**Lemma 1.3.** Let \((C, \mathcal{W})\) and \((C', \mathcal{W}')\) be two categories with weak equivalences and \(L : C \xrightarrow{\sim} C' : R\) an adjunction. If \(\mathcal{W} = L^{-1}(\mathcal{W}')\), then \(L \dashv R\) is a homotopical equivalence if and only if the co–unit of the adjunction is a pointwise equivalence.

**Remark 1.4.** The dual of the previous lemma is also true, but we won’t need it in this paper.

**Proof.** First, let’s prove that \(R\) preserves weak equivalences. Let \(f : X \to Y\) be a morphism in \(C'\) that belongs to \(\mathcal{W}'\). Since \(\mathcal{W} = L^{-1}(\mathcal{W}')\), we need to show that \(LR(f) \in \mathcal{W}'\) and this follows from the 2-out-of-3 property of \(\mathcal{W}'\), the commutativity of the square

\[
\begin{array}{ccc}
LR(X) & \xrightarrow{LR(f)} & LR(Y) \\
\varepsilon_X \downarrow & & \downarrow \varepsilon_Y \\
X & \xrightarrow{f} & Y,
\end{array}
\]

and the fact that the co–unit is a pointwise weak equivalence.

Now let’s prove that the unit of the adjunction is also a pointwise weak equivalence. Let \(Y\) be an object of \(C'\). Since \(\mathcal{W} = L^{-1}(\mathcal{W}')\), we need to show that \(L(\eta_Y)\) is a weak equivalence. By the triangle identity, the following triangle is commutative

\[
\begin{array}{ccc}
L(Y) & \xrightarrow{L(\eta_Y)} & LRL(Y) \\
& & \varepsilon_{L(Y)} \\
& \xrightarrow{id_{L(Y)}} & L(Y),
\end{array}
\]

hence the desired result follows from the 2-out-of-3 property of \(\mathcal{W}'\) and the fact the co–unit is a pointwise weak equivalence. \(\square\)

2. **Very brief recollection of the theory of test categories**

The goal of this section is only to provide a quick summary of the basic notions and results (without proofs) of the theory of test categories. For a detailed exposition, we refer the reader to Maltsiniotis’ book on the subject [Mal03].

**Notation 2.1.** For a small category \(A\), we denote by \(\hat{A}\), the category of \(\text{Set}\)-valued presheaves over \(A\), that is, the category of functors \(A^{\text{op}} \to \text{Set}\) and natural transformations between them.

We denote by \(\text{Cat}\) the category of small categories and functors between them. We use the notation \(e\) for the terminal object of \(\text{Cat}\), that is, the category with one object and no non-identity morphism.

For a small category \(A\) and an object \(a\) of \(A\), we denote by \(A/a\) the slice category of \(A\) over \(a\). Explicitly, \(A/a\) is the category whose objects are pairs \((a', p : a' \to a)\),...
where $a'$ and $p$ are respectively an object and a morphism of $A$, and whose morphisms $(a', p') \to (a'', p'')$ are morphisms $f: a' \to a''$ of $A$, such that $p'' \circ f = p'$.

More generally, if $u: A \to B$ is a morphism of $\text{Cat}$ and $b$ is an object of $B$, we denote by $A/b$ the category whose objects are pairs $(a, q): u(a) \to b$, where $a$ is an object of $A$ and $q$ is a morphism of $B$, and whose morphisms $(a, q) \to (a', q')$ are the morphisms $f: a \to a'$ of $A$ such that $q' \circ u(f) = q$. Note that we make the abuse of notation of not making $u$ appear in the notation $A/b$, but this category obviously depends on $u$.

2.2. Let $\Delta$ be the category whose objects are the ordered sets $\Delta_n := \{0 < \cdots < n\}$ for $n \geq 0$ and whose morphisms are non-decreasing functions between them. The category $\hat{\Delta}$ is referred to as the category of simplicial sets. The canonical inclusion $\Delta \hookrightarrow \text{Cat}$ induces the so-called nerve functor

$$N: \text{Cat} \to \hat{\Delta}$$

$$C \mapsto \left( \Delta_n \mapsto \text{Hom}_{\text{Cat}}(\Delta_n, C) \right).$$

Let us denote by $W_\infty$ the class of morphisms $u: A \to B$ of $\text{Cat}$ such that $N(u)$ is a weak homotopy equivalence of simplicial sets.\(^4\) Recall now the fundamental result of the homotopy theory of $\text{Cat}$ [Ill72, Corollaire 3.3.1]: the nerve functor induces an equivalence at the level of homotopy categories

$$\text{Ho}(\text{Cat}) \simeq \text{Ho}(\hat{\Delta}).$$

In other words, the category $\text{Cat}$ equipped with $W_\infty$ models homotopy types. In the theory of test categories, the point of view is reversed and $(\text{Cat}, W_\infty)$ is taken as the fundamental model of homotopy types. As it happens, the results of this theory only relies on a few formal properties of the class $W_\infty$, which are shared by others class of weak equivalences (non-necessarily modeling homotopy types), referred to as basic localizers, and whose definition is recalled below.

**Definition 2.3.** A class $W$ of morphisms of $\text{Cat}$ is called a basic localizer if it satisfies the following properties:

(i) $W$ is weakly saturated,

(ii) for every small category $A$ with a terminal object, the canonical morphism to the terminal category $A \to e$

is in $W$,

(iii) for any commutative triangle of $\text{Cat}$,

$$\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow & & \downarrow \\
C, & & \\
\end{array}$$

if for every object $c$ of $C$, the morphism induced by $u$

$$A/c \to B/c$$

is in $W$, then $u$ is also in $W$.

\(^4\)This means a weak equivalence of the Kan–Quillen model structure on simplicial sets.
Example 2.4. The class $W_\infty$ is a basic localizer. It is in fact the smallest basic localizer [Cis04]. More generally, for any $n \geq 0$, let $W_n$ be the class of morphisms $u: A \to B$ of $\text{Cat}$ such that $N(u)$ induces an equivalences on homotopy groups of simplicial sets, up to dimension $n$. Then, $W_n$ is a basic localizer [Cis06, Section 9.2] and $(\text{Cat}, W_n)$ models homotopy $n$-types.

We now fix once and for all in this section a basic localizer $W$ of $\text{Cat}$.

2.5. A small category $A$ is called $W$-aspherical, or simply aspherical, when the canonical morphism to the terminal category $A \to e$ is in $W$.

More generally, a morphism $u: A \to B$ is $W$-aspherical, or simply aspherical, if for every object $b$ of $B$, the category $A/b$ is aspherical. Note that it follows from the axioms of basic localizers that every aspherical morphism is in $W$. Practically, this is very often how we will prove that a morphism of $\text{Cat}$ is a weak equivalence.

Finally, we say that a small category $A$ is totally $W$-aspherical, or simply totally aspherical, if it is aspherical and the diagonal functor $\delta: A \to A \times A$ is aspherical.

For later reference we put here the following lemma. We refer to [Mal05, p. 1.1.15] for the definition of Grothendieck fibrations.

Lemma 2.6. Let

$$
\begin{array}{ccc}
A' & \xrightarrow{u} & A \\
\downarrow{v'} & & \downarrow{p} \\
B' & \xrightarrow{v} & B
\end{array}
$$

be a pullback square of $\text{Cat}$. If $p$ is a Grothendieck fibration and $v$ is aspherical then $u$ is also aspherical.

Proof. This is a particular case of the dual of [Mal05, Théorème 3.2.15].

The first step of the theory of test categories is to equip $\hat{A}$, for any small category $A$, with a canonical class of weak equivalences. For that, we use a canonical functor $\hat{A} \to \text{Cat}$.

2.7. Let $A$ be a small category. We write $i_A: \hat{A} \to \text{Cat}$ for the functor

$$
i_A: \hat{A} \to \text{Cat}$$

$$X \mapsto A/X,$$

where $A/X$ is the category of elements of $X$. In details, the objects of $A/X$ are pairs $(a, x)$, where $a$ is an object of $A$ and $x \in X(a)$. A morphism $(a, x) \to (a', x')$ consists of a morphism $f: a \to a'$ in $A$ such that $x = X(f)(x')$.

The functor $i_A$ has a right adjoint given by the following formula

$$i_A^*: \text{Cat} \to \hat{A}$$

$$C \mapsto \left(a \mapsto \text{Hom}_{\text{Cat}}(A/a, C)\right),$$

where $A/a$ is the slice category of $A$ over $a$.4
Definition 2.8. Let $\mathcal{A}$ be a small category. A morphism $\varphi: X \to Y$ of $\hat{\mathcal{A}}$ is a $W$-equivalence, or simply a weak equivalence, if $i_{\mathcal{A}}(\varphi)$ belongs to $W$. We denote by $W_{\hat{\mathcal{A}}}$ the class of $W$-equivalences.

An object $X$ of $\hat{\mathcal{A}}$ is $W$-aspherical, or simply aspherical, if $i_{\mathcal{A}}(X)$ is a $W$-aspherical category.

Finally, an object $X$ of $\hat{\mathcal{A}}$ is $W$-locally aspherical, or simply locally aspherical, if the canonical morphism to the terminal presheaf $X \to *$ is a universal $W$-equivalence.

Remark 2.9. If $\mathcal{A}$ is aspherical, then an object $X$ of $\hat{\mathcal{A}}$ is aspherical if and only if the canonical morphism to the terminal presheaf $X \to *$ is a weak equivalence. In particular, when $\mathcal{A}$ is aspherical, every locally aspherical presheaf is aspherical.

Example 2.10. In the case that $\mathcal{A} = \Delta$, the $W_\infty$-equivalences coincide with the usual weak homotopy equivalences, and a simplicial set is $W_\infty$-aspherical if and only if it is weakly contractible.

Definition 2.11 (Grothendieck). Let $\mathcal{A}$ be a small category. We say that:

(a) $\mathcal{A}$ is a $W$-pseudo-test category, or simply a pseudo-test category, if it satisfies both following conditions:

(i) $\mathcal{A}$ is aspherical,

(ii) $i_{\mathcal{A}}: \hat{\mathcal{A}} \to \text{Cat}$ induces an equivalence at the level of homotopy categories $\text{Ho}(\hat{\mathcal{A}}) \simeq \text{Ho}(\text{Cat})$,

(b) $\mathcal{A}$ is a $W$-weak test category, or simply a weak test category, if the adjunction $i_{\mathcal{A}} \dashv i_{\mathcal{A}}^*$ is a homotopical equivalence [1.2],

(c) $\mathcal{A}$ is a $W$-local test category, or simply a local test category, if for every object $a$ of $\mathcal{A}$, the category $\mathcal{A}/a$ is weak test,

(d) $\mathcal{A}$ is a $W$-test category, or simply a test category, if it is both a weak test category and a local test category,

(e) $\mathcal{A}$ is a $W$-strict test category, or simply a strict test category, if it is both totally aspherical and a test category.

Remark 2.12. We have the following sequence of implications

strict test $\Rightarrow$ test $\Rightarrow$ weak test $\Rightarrow$ pseudo-test,

but it can be shown that the converse of the first two implications do not hold. For the converse of the third one, it is still an open question.

Example 2.13. The archetypal example of strict test category is $\Delta$, but it is far from being the only one. The class of strict test categories also contains the cubical category with connections [Mal09], Joyal’s $\Theta$ category [CM11], etc. Examples of test categories which are not strict include the cubical category (without connections) [Cis06, Proposition 4.2.4] and the dendroidal category $\Omega$ [ACM19]. Examples of weak test categories which are not test include the subcategory $\Delta'$ of $\Delta$ with only monomorphisms as the morphisms [Mal05, Proposition 1.7.25]. All the examples above are of “shape-like” nature, but it is not always the case. For example, the monoid of non-decreasing functions $\mathbb{N} \to \mathbb{N}$, seen as a category with only one object, is a strict test category [CM11, Example 3.16].

[5] In fact, it follows from a result of Cisinski [Cis06, Proposition 4.2.4] and from Maltsiniotis [Mal05, Proposition 1.3.5] that condition (i) is implied by condition (ii).
We now sum up the classical criteria to detect weak test, test and strict test categories. For details and other characterizations, we refer to the first chapter of \[Mal05\].

Recall that we denote by $\Delta_1 = \{0 < 1\}$.

**Proposition 2.14.** Let $\mathcal{A}$ be a small category. We have the following characterizations:

(a) $\mathcal{A}$ is a weak test category if and only if for every small category $\mathcal{C}$ with a terminal object, the presheaf $i^*_\mathcal{A}(\mathcal{C})$ is aspherical,

(b) $\mathcal{A}$ is a local test category if and only if the presheaf $i^*_\mathcal{A}(\Delta_1)$ is locally aspherical,

(c) $\mathcal{A}$ is a test category if and only if it is aspherical and $i^*_\mathcal{A}(\Delta_1)$ is locally aspherical,

(d) $\mathcal{A}$ is a strict test category if and only if it is totally aspherical and $i^*_\mathcal{A}(\Delta_1)$ is aspherical.

Let us end this section with a quick word on aspherical functors and locally aspherical functors.

**Definition 2.15.** Let $\mathcal{A}$ be a small category, $i: \mathcal{A} \to \mathbf{Cat}$ a functor and let $i^*: \mathbf{Cat} \to \hat{\mathcal{A}}$ be the functor defined as

$$i^*: \mathbf{Cat} \to \hat{\mathcal{A}}$$

$$\mathcal{C} \mapsto \left(a \mapsto \text{Hom}_{\mathbf{Cat}}(i(a), \mathcal{C})\right).$$

We say that $i$ is a $W$-aspherical functor, or simply an aspherical functor, if it satisfies the two following conditions:

(a) $i(a)$ has a terminal object for every object $a$ of $\mathcal{A}$,

(b) if $\mathcal{C}$ is a small category with a terminal object, $i^*(\mathcal{C})$ is an aspherical object of $\hat{\mathcal{A}}$.

We say that $i$ is a $W$-locally aspherical functor, or simply a locally aspherical functor, if it satisfies condition (a) above and the following condition instead of (b):

(b') $i^*(\Delta_1)$ is a locally aspherical object of $\hat{\mathcal{A}}$.

**Remark 2.16.** The definition of aspherical functor given above is not the most general possible (see \[Mal05, Definition 1.7.1\]), but it will be sufficient for our purpose. (See also \[4.7\] below.)

**Remark 2.17.** The first item (resp. second item) of Proposition 2.14 can be reformulated as: $\mathcal{A}$ is a weak test category (resp. local test category) if and only if $\mathcal{A} \to \mathbf{Cat}, a \mapsto \mathcal{A}/a$ is an aspherical functor (resp. locally aspherical functor).

**Proposition 2.18.** Let $\mathcal{A}$ be a small category and $i: \mathcal{A} \to \mathbf{Cat}$ a functor. We have the following implications:

(a) if $\mathcal{A}$ is weak test and $i$ is an aspherical functor, then $i^*: \mathbf{Cat} \to \hat{\mathcal{A}}$ is a homotopy inverse of $i_\mathcal{A}$,

(b) if $i$ is a locally aspherical functor, then $\mathcal{A}$ is local test category,

(c) if $i$ is a locally aspherical functor and $\mathcal{A}$ is aspherical, then $\mathcal{A}$ is a test category and $i^*: \mathbf{Cat} \to \hat{\mathcal{A}}$ is a homotopy inverse of $i_\mathcal{A}$.

**Example 2.19.** The archetypal example of aspherical functor (which is even locally aspherical) is the inclusion functor $i: \Delta \hookrightarrow \mathbf{Cat}$. Then $i^*: \mathbf{Cat} \to \hat{\Delta}$ is nothing but the nerve functor. Hence, we recover via the above proposition that the nerve induces an equivalence of homotopy categories $\text{Ho}(\mathbf{Cat}) \simeq \text{Ho}(\hat{\Delta})$. 
Remark 2.20. The name *locally aspherical functor* is non standard, but by (b) of the previous proposition, it is equivalent to the usual notion of *local test functor*. Similarly, an aspherical functor \(i: A \to \text{Cat} \) such that \(A\) is a (weak) test category, is usually called a (weak) test functor. We will not use this terminology.

3. Homotopy theory of presheaves of groupoids

**Notation 3.1.** We denote by \(\text{Grpd}\) the category of (small) groupoids and for a small category \(A\), we denote by \(\hat{A}_{\text{Grpd}}\) the category of \(\text{Grpd}\)-valued presheaves over \(A\). That is, \(\hat{A}_{\text{Grpd}}\) is the category of functors \(A^{\text{op}} \to \text{Grpd}\) and natural transformations between them. We use the notation \(*\) for the terminal object of \(\hat{A}_{\text{Grpd}}\).

The canonical inclusion \(\text{Set} \hookrightarrow \text{Grpd}\), which identifies sets with discrete groupoids, induces a canonical inclusion \(\hat{A} \hookrightarrow \hat{A}_{\text{Grpd}}\). Hence, every \(\text{Set}\)-valued presheaf can be seen as a \(\text{Grpd}\)-valued presheaf.

3.2. Let \(A\) be a small category and \(X\) an object of \(\hat{A}_{\text{Grpd}}\). We write \(A//X\) for the category of elements of \(X\), which is defined as the following:

- an object is a pair \((a, x)\), where \(a\) is an object of \(A\) and \(x\) is an object of \(X(a)\),
- a morphism \((a, x) \to (a', x')\) is a pair \((f, k)\), where \(f: a \to a'\) is a morphism of \(A\), and \(k: x \sim X(f)(x')\) is a morphism of \(X(a)\) (which is necessarily an isomorphism).

The identity morphism of \((a, x)\) is given by \((\text{id}_a, \text{id}_x)\) and the composition of \((a, x) \xrightarrow{(f, k)} (a', x') \xrightarrow{(f', k')} (a'', x'')\) is given by \((f' \circ f, k''): (a, x) \to (a'', x'')\), where \(k''\) is the composite of \(x \xrightarrow{k} X(f)(x') \xrightarrow{X(f)(k')} X(f \circ f)(x'')\).

For every object \(X\) of \(\hat{A}_{\text{Grpd}}\), the category \(A//X\) comes equipped with a canonical morphism:

\[
\zeta_X: A//X \to A
\]

\[
(a, x) \mapsto a,
\]

which is easily checked to be a Grothendieck fibration (see also Remark 3.3 below).

Given a morphism \(\alpha: X \to X'\) of \(\hat{A}_{\text{Grpd}}\) (i.e. a natural transformation), we define a functor \(A//X \to A//X'\) in the following way:

- an object \((a, x)\) of \(A//X\) is sent to the object \((a, \alpha_a(x))\) of \(X'(a)\),
- a morphism \((f, k): (a, x) \to (a', x')\) of \(A//X\) is sent to the morphism \((f, \alpha_a(k)): (a, \alpha_a(x)) \to (a', \alpha_a(x'))\)

of \(A//X'\) (where we used the naturality of \(\alpha\) for the target of this morphism to be compatible).

---

6By that, we mean actual strict natural transformations and not pseudo natural transformations.
This makes the correspondence $X \mapsto A//X$ functorial in $X$, and yields a functor, denoted by $I_A$:

$$I_A : \hat{A}_{Grpd} \to \mathbf{Cat}$$

$$X \mapsto A//X.$$  

We will see later that $I_A$ admits a right adjoint.

**Remark 3.3.** Via the canonical inclusion $\text{Grpd} \hookrightarrow \text{Cat}$, any $\text{Grpd}$-valued presheaf $X$ can be seen as a $\text{Cat}$-valued presheaf. Then, $A//X$ is nothing but the Grothendieck construction of $X$ (see §3.1 for details) and the canonical morphism $\zeta_X : A//X \to A$ is the Grothendieck fibration associated to $X$.

**Remark 3.4.** When $X$ is an object of $\hat{A}$, which we see as an object of $\hat{A}_{Grpd}$ via the canonical inclusion $\hat{A} \hookrightarrow \hat{A}_{Grpd}$, we have

$$i_A(X) = I_A(X).$$

In other words, the following triangle is commutative

$$\begin{array}{ccc}
\hat{A} & \longrightarrow & \hat{A}_{Grpd} \\
i_A & \downarrow & \downarrow I_A \\
 & \hat{A} & \rightarrow \mathbf{Cat}.
\end{array}$$

**Remark 3.5.** By an obvious variation of the Yoneda lemma, the category $I_A(X)$ can alternatively be defined as the category whose objects are pairs $(a,p : a \to X)$ (we identify an object $a$ of $A$ with the Set-valued presheaf represented by $a$) and whose morphisms $(a,p) \to (a',p')$ are pairs $(f,\sigma)$, where $f : a \to a'$ is a morphism of $A$ and $\sigma : p \Rightarrow p' \circ f$ is a natural isomorphism.

*For the rest of this section, we fix once and for all a basic localizer $\mathcal{W}$ of $\mathbf{Cat}$. 

**Definition 3.6.** Let $A$ be a small category. A morphism $\varphi : X \to Y$ of $\hat{A}_{Grpd}$ is a $\mathcal{W}$-equivalence, or simply a weak equivalence, if $I_A(\varphi)$ is in $\mathcal{W}$. We denote by $\mathcal{W}_{\hat{A}_{Grpd}}$ the class of $\mathcal{W}$-equivalences.

An object $X$ of $\hat{A}_{Grpd}$ is $\mathcal{W}$-aspherical, or simply aspherical, if the category $I_A(X)$ is aspherical.

**Remark 3.7.** It follows from Remark 3.3 that a Set-valued presheaf is aspherical in the sense of Definition 2.8 if and only if it is aspherical in the sense of the previous definition (using the canonical inclusion $\hat{A} \hookrightarrow \hat{A}_{Grpd}$).

In the case that the category $A$ is aspherical, there is an equivalent characterization of aspherical objects of $\hat{A}_{Grpd}$ as stated in the following lemma.

**Lemma 3.8.** If $A$ is aspherical, then for every object $X$ of $\hat{A}_{Grpd}$, we have the following equivalence

$$X \to * \text{ is a weak equivalence } \iff X \text{ is aspherical.}$$

Conversely, if this equivalence is true for every $X$ in $\hat{A}_{Grpd}$, then $A$ is aspherical.
Proof. Notice \( I_A(*) \simeq A \), hence \( * \) is aspherical if and only if \( A \) is aspherical. The equivalence follows then from the obvious fact that for a morphism \( \varphi: X \rightarrow Y \) of \( \hat{\text{A}}_{\text{Grpd}} \), if \( Y \) is aspherical, then \( \varphi \) is a weak equivalence if and only if \( X \) is aspherical.

For the second part of the lemma, it suffices to notice that the identity morphism \( * \rightarrow * \) is always a weak equivalence (as all identity morphisms are) and \( I_A(*) \simeq A \). □

For later reference, we put here the following result which gives other characterizations of aspherical morphisms of \( \text{Cat} \).

**Proposition 3.9.** Let \( u: A \rightarrow B \) be a morphism of \( \text{Cat} \). The following conditions are equivalent:

(a) \( u \) is aspherical,

(b) the functor \( u^*: \hat{\text{B}}_{\text{Grpd}} \rightarrow \hat{\text{A}}_{\text{Grpd}} \) preserves and reflects aspherical objects, i.e. an object \( X \) of \( \hat{\text{B}}_{\text{Grpd}} \) is aspherical if and only if \( u^*(X) \) is aspherical,

(c) the functor \( u^*: \hat{\text{B}}_{\text{Grpd}} \rightarrow \hat{\text{A}}_{\text{Grpd}} \) preserves aspherical objects, i.e. for every aspherical object \( X \) of \( \hat{\text{B}}_{\text{Grpd}} \), \( u^*(X) \) is aspherical.

All these equivalent conditions imply the following condition:

(d) the functor \( u^*: \hat{\text{B}}_{\text{Grpd}} \rightarrow \hat{\text{A}}_{\text{Grpd}} \) preserves and reflects weak equivalences, i.e.

\[
(u^*)^{-1}(\mathcal{W}_{\hat{\text{A}}_{\text{Grpd}}}) = \mathcal{W}_{\hat{\text{B}}_{\text{Grpd}}}.
\]

If moreover \( A \) and \( B \) are aspherical, then conditions (a) to (d) are all equivalent and equivalent to the following condition:

(e) the functor \( u^* \) preserves weak equivalences, i.e.

\[
u^*(\mathcal{W}_{\hat{\text{B}}_{\text{Grpd}}}) \subseteq \mathcal{W}_{\hat{\text{A}}_{\text{Grpd}}}.
\]

**Proof.** Let us start with some preliminaries. It is easily checked that for every object \( X \) of \( \hat{\text{B}}_{\text{Grpd}} \), the following square

\[
\begin{array}{ccc}
I_A(u^*(X)) & \xrightarrow{\lambda_X} & I_B(X) \\
\zeta_{u^*(X)} \downarrow & & \downarrow \zeta_X \\
A & \xrightarrow{u} & B,
\end{array}
\]

where \( \lambda_X \) is the functor defined on objects as

\[
(a, x) \mapsto (u(a), x)
\]

and on morphisms as

\[
\left( (a, x) \xrightarrow{(f, \sigma)} (a', x') \right) \mapsto \left( (u(a), x) \xrightarrow{(u(f), \sigma)} (u(a'), x') \right)
\]

is a pullback square.

Now, using the fact the vertical morphisms of the previous pullback square are Grothendieck fibrations, it follows from Lemma [2.6] that if \( u \) is aspherical, then \( \lambda_X \) is aspherical. In particular, with these conditions, \( X \) is aspherical if and only if \( u^*(X) \) is aspherical, which proves the implication (a) \( \Rightarrow \) (b). The implication (b) \( \Rightarrow \) (c) is tautological. To prove (c) \( \Rightarrow \) (a), consider an object \( b \) of \( B \), seen as a \( \text{Set} \)-valued representable
presheaf (and as an object of \(\hat{\mathcal{A}}\text{Grpd}\) via the inclusion \(\hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}\text{Grpd}\)). It is straightforward to check that \(I_{\mathcal{A}}(u^*(b)) \simeq \mathcal{A}/b\). Hence, if condition \((c)\) is satisfied, then \(u\) is aspherical.

To prove \((c) \Rightarrow (d)\), it suffices to notice that \(\lambda_X\) is natural in \(X\), and thus, for every morphism \(f: X \rightarrow Y\) of \(\hat{\mathcal{B}}\text{Grpd}\), we have a commutative square

\[
\begin{array}{ccc}
I_{\mathcal{A}}(u^*(X)) & \xrightarrow{I_{\mathcal{A}}(u^*(f))} & I_{\mathcal{A}}(u^*(Y)) \\
\downarrow \lambda_X & & \downarrow \lambda_Y \\
I_{\mathcal{B}}(X) & \xrightarrow{I_{\mathcal{B}}(f)} & I_{\mathcal{B}}(Y).
\end{array}
\]

We have already seen that if \(u\) is aspherical, then the vertical arrows of the previous square are weak equivalences. Hence, in this case, \(I_{\mathcal{B}}(f)\) is a weak equivalence if and only if \(I_{\mathcal{A}}(u^*(f))\) is. By definition of weak equivalences of \(\text{Grpd}\)-valued presheaves, this means exactly that \(u^*\) preserves and reflects weak equivalences.

The implication \((d) \Rightarrow (e)\) is trivial. Finally, let us prove \((e) \Rightarrow (c)\). Let \(X\) be an aspherical object of \(\hat{\mathcal{B}}\text{Grpd}\). Since \(\mathcal{B}\) is aspherical, we have already seen (Lemma 3.8) that this means exactly that the canonical morphism

\[X \rightarrow \ast\]

is a weak equivalence of \(\hat{\mathcal{B}}\text{Grpd}\). If \(u^*\) preserves weak equivalences, then

\[u^*(X) \rightarrow u^*(\ast) \simeq \ast\]

is a weak equivalence of \(\hat{\mathcal{A}}\text{Grpd}\). Using that \(\mathcal{A}\) is aspherical, we deduce from Lemma 3.8 again that \(u^*(X)\) is aspherical. Hence, \(u^*\) preserves aspherical objects. \(\square\)

**Definition 3.10.** A small category \(\mathcal{A}\) is \(\mathcal{W}\)-\textit{groupoidal pseudo-test}, or simply \textit{groupoidal pseudo-test}, if

- \((a)\) \(\mathcal{A}\) is aspherical,
- \((b)\) \(I_{\mathcal{A}}\) induces an equivalence at the level of localized categories:

\[
\text{Ho}(\hat{\mathcal{A}}\text{Grpd}) \xrightarrow{\sim} \text{Ho}(\text{Cat}).
\]

4. **Groupoidal weak test categories**

**Notation 4.1.** For two (small) categories \(\mathcal{C}\) and \(\mathcal{D}\), we denote by \(\text{Hom}^{\text{iso}}_{\text{Cat}}(\mathcal{C}, \mathcal{D})\) the groupoid whose objects are functors \(\mathcal{C} \rightarrow \mathcal{D}\) and whose morphisms are natural isomorphisms between those functors.

**4.2.** Let \(i: \mathcal{A} \rightarrow \text{Cat}\) be a functor with \(\mathcal{A}\) a small category. We denote by \(I^*\) the functor

\[
I^*: \text{Cat} \rightarrow \hat{\mathcal{A}}\text{Grpd}
\]

\[
\mathcal{C} \mapsto \text{Hom}^{\text{iso}}_{\text{Cat}}(i(-), \mathcal{C}).
\]

When \(i\) is the functor \(\mathcal{A} \rightarrow \text{Cat}, a \mapsto \mathcal{A}/a\), we use the special notation \(I^*_\mathcal{A}\) for the functor \(I^*\). In other words, for a small category \(\mathcal{C}\), \(I^*_\mathcal{A}\) is the functor

\[
I^*_\mathcal{A}: \text{Cat} \rightarrow \hat{\mathcal{A}}\text{Grpd}
\]

\[
\mathcal{C} \mapsto \text{Hom}^{\text{iso}}_{\text{Cat}}(\mathcal{A}/(-), \mathcal{C}).
\]
Lemma 4.3. Let \( i : A \to \text{Cat} \) be a functor, with \( A \) a small category. The functor \( I^* : \text{Cat} \to \tilde{\text{A}}_{\text{Grpd}} \) has a left adjoint. Moreover, in the case that \( i \) is the functor \( a \mapsto A/a \) (and thus \( I^* = I_A^* \)), this left adjoint is \( I_A : \tilde{\text{A}}_{\text{Grpd}} \to \text{Cat} \).

Proof. Let us denote by \( \int_{a \in A} F(a,a) \) (resp. \( \int_{a \in A} F(a,a) \)) the co-end (resp. the end) of a functor \( F : A^{\text{op}} \times A \to \text{Cat} \). We define a functor \( I! : \tilde{\text{A}}_{\text{Grpd}} \to \text{Cat} \) as

\[
X \mapsto \int_{a \in A} X(a) \times i(a).
\]

For every small category \( C \) and every Grpd-valued presheaf \( X \), we then have the following sequence of natural isomorphisms:

\[
\text{Hom}_{\text{Cat}}(\int_{a \in A} X(a) \times i(a), C) \simeq \int_{a \in A} \text{Hom}_{\text{Cat}}(X(a) \times i(a), C) \\
\simeq \int_{a \in A} \text{Hom}_{\text{Grpd}}(X(a), I^*(C)_{\text{Cat}}(i(a))) \\
\simeq \text{Hom}_{\tilde{\text{A}}_{\text{Grpd}}}(X, I^*(C)).
\]

Hence, \( I! \) is left adjoint of \( I^* \). In the case that \( i : A \to \text{Cat} \) is the functor \( a \mapsto A/a \), we have

\[
I!(X) = \int_{a \in A} X(a) \times A/a,
\]

which is nothing but the Grothendieck construction of \( X \). To see this, recall that the Grothendieck construction of a Cat-valued functor (and, a fortiori, for a Grpd-valued functor) is its oplax colimit [Gra69], and that the oplax colimit of a contravariant functor is computed as the colimit weighted by the slices of the source [Str76]. The conclusion follows then from Remark 3.3. \( \square \)

We now fix, once and for all in this section, a basic localizer \( \mathcal{W} \) of \( \text{Cat} \).

Definition 4.4. A small category \( A \) is \( \mathcal{W} \)-groupoidal weak test, or simply groupoidal weak test, if the adjunction \( I_A \dashv I_A^* \) is a homotopical equivalence (1.2).

Remark 4.5. An immediate computation shows that \( I_A I_A^*(e) \simeq A \). Thus, if \( A \) is groupoidal weak test, the co-unit morphism \( A \to e \) is a weak equivalence and so \( A \) is aspherical. This shows that every groupoidal weak test category is a groupoidal pseudo-test category.

We would like now to find characterizations of groupoidal weak test categories. For that, we begin by studying a class of homotopically well-behaved functors \( A \to \text{Cat} \).

Definition 4.6. Let \( A \) be a small category. A functor \( i : A \to \text{Cat} \) is \( \mathcal{W} \)-groupoidal aspherical, or simply groupoidal aspherical, if:

(a) for every object \( a \) of \( A \), the category \( i(a) \) has a terminal object,
(b) for every small category \( C \) with a terminal object, the Grpd-valued presheaf \( I^*(C) \) is aspherical.

Remark 4.7. A more general notion of groupoidal aspherical functor is obtained by replacing the conditions of the previous definitions by:

(a') \( i(a) \) is aspherical for every object \( a \) of \( A \),
(b') for every small aspherical category \( C \), \( I^*(C) \) is aspherical,
which is the straightforward generalization of [Mal05, Definition 1.7.1]. (We will see in Proposition 4.1 that when (a) is satisfied, condition (b) and (b’) are equivalent). However, the author of the notes does not know if the theory fully works for this more general notion of groupoidal aspherical functor, and the restricted version we chose is sufficient for our purpose. For the interested reader, the difficulty is that it seems that [Mal05, Lemme 1.7.4 (b)] cannot by generalized to \text{Grpd}-valued presheaves. (Nevertheless, see Lemma (7,8) below for a partial generalization of this lemma.)

### 4.8

Let \( i : A \rightarrow \text{Cat} \) be a functor, with \( A \) a small category, and suppose that for every object \( a \) of \( A \), the category \( i(a) \) has a terminal object \( e_a \). We are going to define a canonical natural transformation

\[
\alpha : I_A I^* \Rightarrow \text{id}_{\text{Cat}}.
\]

Let \( C \) be a small category. Spelling out the definitions, we see that the category \( I_A I^*(C) \) has for object pairs \((a, p) : i(a) \rightarrow C\) where \( a \) is an object of \( A \) and \( p \) a morphism of \( \text{Cat} \). And a morphism \((a, p) \rightarrow (a', p')\) in \( I_A I^*(C) \) consists of a pair \((f, \sigma)\), where \( f : a \rightarrow a' \) is a morphism of \( A \) and

\[
\sigma : p \cong p' \circ i(f)
\]

is a natural isomorphism.

At the level of objects, we define the morphism \( \alpha_C : I_A I^*(C) \rightarrow C \) with the formula

\[
\alpha_C(a, p) := p(e_a).
\]

At the level of morphisms, the image of a morphism \((f, \sigma) : (a, p) \rightarrow (a', p')\) is defined as the composite

\[
\alpha_C(f, \sigma) := p(e_a) \xrightarrow{\sigma e_a} p'(i(f)(e_a)) \rightarrow p'(e_{a'})
\]

where the morphism on the right is induced by the canonical morphism \( i(f)(e_a) \rightarrow e_{a'} \).

We leave it to the reader to check that \( \alpha_C : I_A I^*(C) \rightarrow C \) does indeed define a functor and that it is natural in \( C \).

**Remark 4.9.** When \( i : A \rightarrow \text{Cat} \) is the functor \( a \mapsto A/a \), then \( \alpha : I_A I^* \Rightarrow \text{id}_{\text{Cat}} \) is nothing but the co–unit of the adjunction \( I_A \dashv I_A^* \).

**Lemma 4.10.** Let \( i : A \rightarrow \text{Cat} \) be a functor with \( A \) a small category and suppose that for every object \( a \) of \( A \), the category \( i(a) \) has a terminal object. For every small category \( C \) and \( c \) an object of \( C \), we have a canonical isomorphism

\[
I_A I^*(C)/c \simeq I_A I^*(C/c).
\]

**Proof.** Let us denote by \( e_a \) the terminal object of \( i(a) \). The category \( I_A I^*(C)/c \) is described as follows:

- an object is a triple \((a, p : i(a) \rightarrow C, g : p(e_a) \rightarrow c)\), where \( a \) is an object of \( A \), \( p \) is a morphism of \( \text{Cat} \), and \( g \) is a morphism of \( C \),
- a morphism \((a, p, g) \rightarrow (a', p', g')\) is a couple \((f, \sigma)\), where \( f : a \rightarrow a' \) is a morphism of \( A \) and

\[
\begin{array}{ccc}
i(a) & \xrightarrow{i(f)} & i(a') \\
\downarrow{\sigma} & \simeq & \downarrow{p} \\
C & \xrightarrow{\Rightarrow} & C
\end{array}
\]
is a natural isomorphism, such that the triangle
\[
p(e_a) \xrightarrow{\alpha_C(f,a)} p'(e'_a) \\
g \downarrow \downarrow \downarrow g' \quad \quad \quad c
\]
is commutative.

The category \(I^*_A(C/c)\) is described as:
- an object is a couple \((a,q): i(a) \to C/c\), where \(a\) is an object of \(A\) and \(q\) is a morphism of \(\text{Cat}\),
- a morphism \((a,q) \to (a',q')\) is a pair \((f,\sigma)\), where \(f: a \to a'\) is a morphism of \(A\) and
\[
i(a) \xrightarrow{i(f)} i(a') \\
\downarrow \sigma \Downarrow \Downarrow \Downarrow q' \\
\quad \quad \quad C/c
\]
a natural isomorphism.

Let us write \(\pi_c: C/c \to C\) for the canonical projection functor. We define a morphism of \(\text{Cat}\) as,
\[
\theta: I^*_A(C/c) \to I^*_A(C)/c \\
(a, q: i(a) \to C/c) \mapsto (a, \pi_c \circ q: i(a) \to C, q(e_a): \pi_c(q(e_a)) \to c),
\]
the definition on morphisms being the obvious one.

We now leave to the reader to verify that this morphism is indeed an isomorphism. This mainly amounts to showing the following general fact: let \(A\) and \(B\) be two categories and suppose that \(A\) has a terminal object \(t_A\). For any object \(b\) of \(B\), a functor \(f: A \to B/b\) is entirely determined by the post-composition \(A \xrightarrow{f} B/b \to B\) and by \(f(t_A)\), seen as a morphism of \(B\) whose target is \(b\). \(\square\)

**Proposition 4.11.** Let \(i: A \to \text{Cat}\) be a functor such that for every \(a\) in \(A\) the category \(i(a)\) has a terminal object. The following conditions are equivalent:

(a) \(i\) is groupoidal aspherical,
(b) \(I^*\) preserves aspherical objects, i.e. for every small aspherical category \(C\), the \(\text{Grpd}\)-valued presheaf \(I^*(C)\) is aspherical,
(c) \(I^*\) preserves and reflects aspherical objects, i.e. a small category \(C\) is aspherical if and only if the \(\text{Grpd}\)-valued presheaf \(I^*(C)\) is aspherical,
(d) for every small category \(C\), the canonical morphism
\[
\alpha_C: I^*_A(C) \to C
\]
is a weak equivalence,
(e) \(A\) is aspherical and \(I^*\) preserves and reflects weak equivalences, i.e.
\[
(I^*)^{-1}(W_{\text{Grpd}}) = W,
\]
(f) \(A\) is aspherical and \(I^*\) preserves weak equivalences, i.e.
\[
I^*(W) \subseteq W_{\text{Grpd}}.
\]
Proof. The implications \((d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)\) are trivial (the last one comes from that every category with a terminal object is aspherical). For the implication \((a) \Rightarrow (d)\), it follows from Lemma 4.10 (and the fact the slices \(C/c\) have a terminal object) that \(\alpha_C\) is aspherical hence a weak equivalence. This proves the equivalence of the first four conditions. For the implication \((d) \Rightarrow (a)\), notice first that if \(C = e\) is the terminal category, we have \(I_A I^*(e) = A\), and hence \((e)\) implies that \(A\) is aspherical. The fact that \((d)\) implies that \(I^*\) preserves and reflects weak equivalences follows from the naturality of \(\alpha\), 2-out-of-3 and the fact that \(\mathcal{W}_{\hat{\mathcal{A}}_{\text{Grpd}}} = I_A^{-1}(W)\) by definition. The implication \((e) \Rightarrow (f)\) is tautological. Finally, for the implication \((f) \Rightarrow (b)\), let \(C\) be a small aspherical category and consider the canonical morphism \(C \to e\), which is, by definition, a weak equivalence. Since \(I^*\) preserves weak equivalences by hypothesis, the induced morphism

\[
I^*(C) \to I^*(e)
\]

is a weak equivalence of \(\hat{\mathcal{A}}_{\text{Grpd}}\). But \(I_A I^*(e) \simeq A\), which is by hypothesis aspherical. Hence, \(I^*(e)\) is aspherical and then so is \(I^*(C)\). \(\square\)

We then obtain the following characterization of groupoidal weak test categories.

**Proposition 4.12.** Let \(A\) be a small category. The following conditions are equivalent:

(a) \(A\) is groupoidal weak test,
(b) for every small category \(C\) with a terminal object, the \(\text{Grpd}\)-valued presheaf \(I_A^*(C)\) is aspherical.

Proof. The second condition means exactly that the functor \(A \to \text{Cat}, a \mapsto A/a\) is groupoidal aspherical (since the slice categories \(A/a\) have a terminal object). The equivalence follows then from condition \((d)\) of Proposition 4.11 combined with Remark 4.9 and Lemma 1.3. \(\square\)

Interestingly, we also obtain the following result.

**Corollary 4.13.** Let \(A\) be a groupoidal weak test category and \(i: A \to \text{Cat}\) a groupoidal aspherical functor. Then, \(I^*: \text{Cat} \to \hat{\mathcal{A}}_{\text{Grpd}}\) is a homotopy inverse of \(I_A: \hat{\mathcal{A}}_{\text{Grpd}} \to \text{Cat}\).

Proof. We already know that if \(i\) is a groupoidal aspherical functor, we have a natural transformation \(\alpha: I_A I^* \Rightarrow \text{id}_{\text{Cat}}\) which is a weak equivalence argument by argument. Consider now the following zigzag of natural transformations

\[
I^*I_A \Rightarrow I_A^*I_A I^* \Rightarrow I_A^* I_A \Leftarrow \text{id},
\]

where the arrow on the left is induced by the unit of the adjunction \(I_A \dashv I_A^*\), the middle arrow is obtained by post-composing \(I_A^*\) to \(\alpha\) and pre-composing with \(I_A\) and the arrow on the right is again the unit of the adjunction \(I_A \dashv I_A^*\). Since \(A\) is groupoidal weak test, the unit of this adjunction is a weak equivalence argument by argument and \(I_A^*\) preserves weak equivalences. This proves that the three natural transformations of the previous zigzag are weak equivalences argument by argument. \(\square\)

**Remark 4.14.** Following the terminology from the usual test category theory, a groupoidal aspherical functor whose source is a weak test category ought to be called a **groupoidal weak test functor**.
Remark 4.15. Even if we remove the hypothesis that $A$ is groupoidal weak test from the previous corollary, it follows trivially from Proposition 4.11(d) that $I^*$ is a homotopical inverse “on one side” of $I_A$, but it does not seem to be a homotopical inverse “on both sides” in general. No counter-example, however, is known by the author of this paper.

5. Groupoidal test categories

We fix once and for all in this section a basic localizer $\mathcal{W}$ of $\text{Cat}$.

Definition 5.1. A small category $A$ is $\mathcal{W}$-groupoidal local test, or simply groupoidal local test, if for every object $a$ of $A$, the category $A/a$ is groupoidal weak test. We say that $A$ is $\mathcal{W}$-groupoidal test, or simply groupoidal test, if it is both a groupoidal weak test category and a groupoidal local test category.

We are now going to look for characterizations of groupoidal local test categories and groupoidal test categories. For that, it is useful to introduce first a variation of the notion of aspherical object of $\hat{A}_{\text{Grpd}}$.

Notation 5.2. Let $A$ be a small category and $a$ an object of $A$. The canonical projection functor $\pi_a: A/a \to A$ induces by precomposition a functor $\pi^*_a: \hat{A}_{\text{Grpd}} \to (A/a)_{\text{Grpd}}$. For an object $X$ of $\hat{A}_{\text{Grpd}}$, we use the suggestive notation $X|_{A/a} := \pi^*_a(X)$ for the image of $X$ by this functor.

Definition 5.3. Let $A$ be a small category. An object $X$ of $\hat{A}_{\text{Grpd}}$ is $\mathcal{W}$-locally aspherical, or simply locally aspherical, if for every object $a$ of $A$, the object $X|_{A/a}$ of $(A/a)_{\text{Grpd}}$ is aspherical.

We now have the following reformulation.

Proposition 5.4. Let $A$ be a small category. The following conditions are equivalent:

(a) $A$ is groupoidal local test,
(b) for every small category $C$ with a terminal object, $I^*_A(C)$ is locally aspherical.

Proof. Follows immediately from Proposition 4.12 and the fact that for every small category $C$ and every object $a$ of $A$, we have $I^*_A(a) \simeq I^*_A(C)|_{A/a}$. □

More generally, we can consider the following variation of groupoidal aspherical functor.

Definition 5.5. Let $A$ be a small category. A functor $i: A \to \text{Cat}$ is $\mathcal{W}$-groupoidal locally aspherical, or simply groupoidal locally aspherical, if the following conditions are satisfied:

(a) for every object $a$ of $A$, $i(a)$ has a terminal object,
(b) for every small category $C$ with a terminal object, $I^*(C)$ is locally aspherical.

Remark 5.6. In other words, Proposition 5.4 says that $A$ is groupoidal local test if and only if $A \to \text{Cat}, a \mapsto A/a$ is a groupoidal locally aspherical functor.
We now turn to a couple of key technical results on locally aspherical \( \text{Grpd} \)-valued presheaves.

**Lemma 5.7.** Let \( A \) be a small category, \( a \) an object of \( A \) and \( X \) an object of \( \hat{A}_{\text{Grpd}} \). We have a sequence of isomorphisms natural in \( X \)

\[
I_A(a \times X) \simeq I_{A/a}(X|_{A/a}) \simeq I_A(X)/a,
\]

where:

- on the left hand side, we abusively wrote \( a \) for the \( \text{Set} \)-valued presheaf represented by \( a \),
- on the right hand side, we implicitly used the canonical morphism \( \zeta_X : I_A(X) \to A \)

to make sense of the slice category.

**Proof.** These three categories have the same description (up to canonical isomorphism):

- an object is a triple \((a', p : a' \to a, x)\), where \( a' \) is an object of \( A \), \( p \) is a morphism of \( A \), and \( x \) is an object of \( X(a') \),
- a morphism \((a', p, x) \to (a'', p', x')\) is a pair \((f, k)\) where \( f : a' \to a'' \) is a morphism of \( A \) such that \( p' \circ f = p \) and \( k : x \to X(f)(x') \) is a morphism of \( X(a') \). \( \square \)

**Lemma 5.8.** Let \( A \) be a small category and \( X \) an object of \( \hat{A}_{\text{Grpd}} \). The following conditions are equivalent:

(a) \( X \) is locally aspherical,
(b) the image by \( I_A \) of the morphism to the terminal object \( X \to * \),

\[
I_A(X) \to A,
\]

is an aspherical morphism of \( \text{Cat} \),
(c) for every object \( a \) of \( A \), seen as a representable presheaf (and as an object of \( \hat{A}_{\text{Grpd}} \), via the canonical inclusion \( \hat{A} \hookrightarrow \hat{A}_{\text{Grpd}} \)), the product in \( \hat{A}_{\text{Grpd}} \)

\[
a \times X
\]

is aspherical,
(d) the morphism \( X \to * \) is a universal weak equivalence, i.e. for every object \( Y \) of \( \hat{A}_{\text{Grpd}} \), the canonical projection

\[
Y \times X \to Y
\]

is a weak equivalence.

**Proof.** The equivalence of (a), (b) and (c) follows immediately from Lemma 5.7. For the implication \((b) \Rightarrow (d)\), consider the following cartesian square

\[
\begin{array}{ccc}
Y \times X & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & *
\end{array}
\]
We leave it to the reader to check that the functor $I_A$ preserves cartesian squares. Hence, we obtain a cartesian square

$$
\begin{array}{ccc}
I_A(Y \times X) & \longrightarrow & I_A(X) \\
\downarrow & & \downarrow \zeta_X \\
I_A(Y) & \longrightarrow & A.
\end{array}
$$

Because $I_A(Y) \to A$ is a Grothendieck fibration, we deduce from Lemma 2.6 that the left vertical morphism of the previous square is an aspheric morphism of $\text{Cat}$, which implies that $Y \times X \to X$ is a weak equivalence. Finally, for the implication $(d) \Rightarrow (c)$, let $a$ be an object of $A$, and consider the canonical projection

$$a \times X \to a$$

(where once again we wrote $a$ for the $\text{Set}$-valued presheaf represented by $a$). Since the presheaf $a$ is aspherical (because the category $I_A(a) = A/a$ has a terminal object), it follows that $a \times X$ is aspherical.

**Remark 5.9.** Lemma 5.8 hides a subtlety between the classical theory for $\text{Set}$-valued presheaves and the theory for $\text{Grpd}$-valued presheaves. Indeed, let us call local weak equivalence a morphism $f : X \to Y$ of $\hat{A}_{\text{Grpd}}$ such that for every object $a$ of $A$, the morphism of $(\hat{A}/a)_{\text{Grpd}}$

$$f|_{\hat{A}/a} : X|_{\hat{A}/a} \to Y|_{\hat{A}/a}$$

is a weak equivalence. Then, an object $X$ of $\hat{A}_{\text{Grpd}}$ is locally aspherical if and only if $X \to \ast$ is a local weak equivalence. Now, Lemma 5.8(d) tells us that $X$ is locally aspherical if and only if $X \to \ast$ is a universal weak equivalence, and we might think that this characterization is true for all local weak equivalences (as the analogue result is true for $\text{Set}$-valued presheaves [Mal05, Proposition 1.2.5]). This does not work though. The correct generalization, which goes beyond the scope of this paper, involves what ought to be called “comma-universal weak equivalence”, whose definition is the same as universal equivalence only pullback squares are replaced by a comma squares.

**Lemma 5.10.** Let $A$ be a small category and $X$ an object of $\hat{A}_{\text{Grpd}}$. If $A$ is aspherical, then we have the following implication

$$X \text{ locally aspherical } \Rightarrow X \text{ aspherical}.$$  

**Proof.** Thanks to Lemma 5.8 (for example item (b)), we know that if an object $X$ of $\hat{A}_{\text{Grpd}}$ is locally aspherical, then $X \to \ast$ is a weak equivalence. The conclusion follows from Lemma 3.8.

From this last lemma, we immediately deduce the following two propositions.

**Proposition 5.11.** Let $A$ be a small category and $i : A \to \text{Cat}$ a groupoidal locally aspherical functor. Then $i$ is a groupoidal aspherical functor if and only if $A$ is aspherical.

**Proposition 5.12.** Let $A$ be a groupoidal local test category. Then $A$ is groupoidal test if and only if it is aspherical.
This last result means that in order to characterize groupoidal test categories, it suffices to characterize groupoidal local test categories.

5.13. Let $\mathcal{M}$ be a category with finite products (this includes a terminal object, which we denote by $e_\mathcal{M}$). An interval in $\mathcal{M}$ is a triple $(\mathcal{I}, i_0, i_1)$, where $\mathcal{I}$ is an object of $\mathcal{M}$, and $i_0$ and $i_1$ are morphisms of $\mathcal{M}$ from $e_\mathcal{M}$ to $\mathcal{I}$.

$$i_0, i_1 : e_\mathcal{M} \rightarrow \mathcal{I}.$$ 

A morphism of intervals $(\mathcal{I}, i_0, i_1) \rightarrow (\mathcal{I}', i_0', i_1')$ consists of a morphism $\varphi : \mathcal{I} \rightarrow \mathcal{I}'$ of $\mathcal{M}$ such that $i_{0,1} = \varphi \circ i_{0,1}$ for $\varepsilon = 0, 1$.

Now, let $f, g : X \rightarrow Y$ be two parallel morphisms of $\mathcal{M}$ and $(\mathcal{I}, i_0, i_1)$ an interval in $\mathcal{M}$. A $\mathcal{I}$-homotopy from $f$ to $g$ is a morphism $h : \mathcal{I} \times X \rightarrow Y$ of $\mathcal{M}$ such that the following diagram is commutative

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
i_0 \times X & \downarrow & \uparrow g \\
\mathcal{I} \times X & \xrightarrow{h} & Y \\
i_1 \times X & \downarrow & \\
X & \xrightarrow{g} & Y
\end{array}$$

We consider the smallest equivalence relation on the set $\text{Hom}_\mathcal{M}(X, Y)$ such that $f$ is equivalent to $g$ if there exists a $\mathcal{I}$-homotopy from $f$ to $g$. If two morphisms $X \rightarrow Y$ are in the same equivalence class for this relation, we say that they are $\mathcal{I}$-homotopic.

We say that an object $X$ of $\mathcal{M}$ is $\mathcal{I}$-contractible if $\text{id}_X : X \rightarrow X$ is $\mathcal{I}$-homotopic to a constant morphism (i.e. a morphism which factorizes through the terminal object $e_\mathcal{M}$).

Finally, notice that if $F : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor preserving finite products, then for any interval $(\mathcal{I}, i_0, i_1)$ of $\mathcal{M}$, $(F(\mathcal{I}), F(i_0), F(i_1))$ is an interval of $\mathcal{M}'$, and $F$ sends $\mathcal{I}$-homotopic morphisms to $F(\mathcal{I})$-homotopic morphisms. In particular, $F$ sends $\mathcal{I}$-contractible objects to $F(\mathcal{I})$-contractible objects.

Example 5.14. Let $\Delta_1$ be the poset $\{0 < 1\}$ seen as an object of $\text{Cat}$, and denote by $e_0, e_1 : e \rightarrow \Delta_1$ the canonical inclusions of $0$ and $1$ respectively. Then $(\Delta_1, e_0, e_1)$ is an interval of $\text{Cat}$. A $\Delta_1$-homotopy from a morphism $u : A \rightarrow B$ to a morphism $v : A \rightarrow B$ is nothing but a natural transformation $u \Rightarrow v$. Notice that a small category with either a terminal object or an initial object is $\Delta_1$-contractible.

Example 5.15. Let $i : A \rightarrow \text{Cat}$ be a functor where $A$ is a small category. Since $I^*$ preserves limits, $(I^*(\Delta_1), I^*(e_0), I^*(e_1))$ is an interval of $\hat{\text{Grpd}}$.

The following lemma relates the notion of $\mathcal{I}$-homotopy with the homotopy theory induced by a class of weak equivalences in the ambient category.

Lemma 5.16. Let $\mathcal{M}$ be a category with finite products, $\mathcal{W}$ a weakly saturated class of morphisms of $\mathcal{M}$ and $\mathcal{I}$ an interval of $\mathcal{M}$ such that the canonical morphism to the terminal object $\mathcal{I} \rightarrow e_\mathcal{M}$ is universally in $\mathcal{W}$. Then, for every $\mathcal{I}$-contractible object $X$ of $\mathcal{M}$, the canonical morphism $X \rightarrow e_\mathcal{M}$ is universally in $\mathcal{W}$.

Proof. This is a reformulation of [Mal02, p. 1.4.6].

We can now apply this to groupoidal locally aspherical functors.
Proposition 5.17. Let $i: A \to \text{Cat}$ be a functor, with $A$ a small category, such that for every object $a$ of $A$, the category $i(a)$ has a terminal object. The following conditions are equivalent:

(a) $i$ is groupoidal locally aspherical,
(b) $I^*(\Delta_1)$ is locally aspherical.

Proof. The implication $(a) \Rightarrow (b)$ is trivial because $\Delta_1$ has terminal object. For the converse, we need to show that for every small category $C$ with a terminal object, $I^*(C)$ is locally aspherical. By Lemma 5.8(d), this is equivalent to showing that $I^*(C) \to *$ is a universal weak equivalence. Notice that $I^*$ preserves limits and so it sends $\Delta_1$-contractible objects of $\text{Cat}$ to $I^*(\Delta_1)$-contractible objects of $\hat{\text{A}}_{\text{Grpd}}$. Since every small category with a terminal object is $\Delta_1$-contractible, the result follows from Lemma 5.16. □

We could now apply the previous proposition to the functor $A \to \text{Cat}, a \mapsto A/a$ and obtain a characterization of groupoidal local test categories. As it happens, we will soon obtain an even finer characterization, but we first need some more results on intervals.

Definition 5.18. Let $M$ be a category with finite products, and whose terminal object is denoted by $e_M$. A multiplicative interval in $M$ is an interval $(L, \lambda_0, \lambda_1)$ together with a binary operation

$$\Lambda: L \times L \to L,$$

such that $\lambda_0$ is a unit on the left and $\lambda_1$ is absorbing on the left. In other words, the following two diagrams are commutative:

$$
\begin{array}{ccc}
\epsilon_M \times L & \xrightarrow{\lambda_0 \times \text{id}_L} & L \times L \\
\downarrow{\text{id}_L} & & \downarrow{\Lambda} \\
L & \xrightarrow{\Lambda} & L
\end{array}
$$

$$
\begin{array}{ccc}
\epsilon_M \times L & \xrightarrow{\lambda_1 \times \text{id}_L} & L \times L \\
\downarrow{\text{id}_L} & & \downarrow{\lambda_1} \\
L & \xrightarrow{\Lambda} & L
\end{array}
$$

Example 5.19. The interval $(\Delta_1, e_0, e_1)$ of $\text{Cat}$ is multiplicative when equipped with the binary operation

$$\Delta_1 \times \Delta_1 \to \Delta_1$$

$$(a, b) \mapsto a + b - ab.$$ 

Since $I^*_A$ preserves limits, it follows that the interval $(I^*_A(\Delta_1), I^*_A(e_0), I^*_A(e_1))$, equipped with the image by $I^*_A$ of the above binary operation, is multiplicative.

Lemma 5.20. Let $M$ be a category with finite products, $\mathcal{W}$ a weakly saturated class of maps of $M$, $(\mathbb{L}, i_0, i_1)$ a interval in $M$ such that $\mathbb{I} \to \epsilon_M$ is universally in $\mathcal{W}$ and $(\mathbb{L}, \lambda_0, \lambda_1, \Lambda)$ a multiplicative interval in $M$. If there exists a morphism of intervals $(\mathbb{I}, i_0, i_1) \to (\mathbb{L}, \lambda_0, \lambda_1)$, then $\mathbb{I} \to \epsilon_M$ is universally in $\mathcal{W}$.

Proof. See [Mal05, Lemme 1.4.10]. □

For the next definition, recall that a $(2,1)$-category is a 2-category such that every 2-morphism is invertible (in other words, a $\text{Grpd}$-enriched category). Limits and colimits in a $(2,1)$-category are the $\text{Grpd}$-enriched ones.
Definition 5.21. Let $\mathcal{M}$ be a $(2,1)$-category with finite products (the terminal object is denoted by $e_M$) and an initial object $\varnothing$. An interval $(I, i_0, i_1)$ (of the underlying category of) $\mathcal{M}$ is said to be strongly separating if for every 2-square of $\mathcal{M}$

$$
\begin{array}{ccc}
X & \xrightarrow{\sim} & e_M \\
\downarrow & & \downarrow_{i_0} \\
e_M & \xrightarrow{i_1} & I,
\end{array}
$$

we necessarily have $X = \varnothing$ and the 2-morphism is the identity.

Example 5.22. Consider $\text{Cat}$ as a $(2,1)$-category, where the 2-morphisms are the natural isomorphisms between functors. Then $(\Delta_1, e_0, e_1)$ is strongly separating.

5.23. Let $A$ be a small category. The category $\hat{A}_{\text{Grpd}}$ has a canonical structure of a 2-category where the 2-morphisms are the strict natural 2-transformations. That is, given two parallel morphisms $\varphi, \psi: X \rightarrow Y$ of $\hat{A}_{\text{Grpd}}$, a 2-morphism $\alpha: \varphi \Rightarrow \psi$ consists of a family of natural transformations

$$
X(a) \xrightarrow{\varphi_a} Y(a),
$$

such that, for every $f: a \rightarrow a'$ in $A$, the following naturality condition is satisfied

$$
X(a') \xrightarrow{X(f)} X(a) \xrightarrow{\varphi_a} Y(a) = X(a') \xrightarrow{\varphi_{a'}} Y(a') \xrightarrow{Y(f)} Y(a).
$$

Notice that since $X$ and $Y$ take values in groupoids, every $\alpha_a$ is invertible, and it follows that every 2-morphism of $\hat{A}_{\text{Grpd}}$ is also invertible. Hence, $\hat{A}_{\text{Grpd}}$ is a $(2,1)$-category.

Lemma 5.24. Let $i: A \rightarrow \text{Cat}$ be a functor, with $A$ a small category, such that for every object $a$ of $A$, the category $i(a)$ is not empty. Then, the interval $(I^*(\Delta_1), i^*(e_0), i^*(e_1))$ of $\hat{A}_{\text{Grpd}}$ is strongly separating.

Proof. It is obvious that $I^*: \text{Cat} \rightarrow \hat{A}_{\text{Grpd}}$ can be extended to a $(2,1)$-functor, and so is its left adjoint $I_!$ which was defined in the proof of Lemma 4.3. We obtain this way a $(2,1)$-adjunction. The fact that $(\Delta_1, e_0, e_1)$ is strongly separating in $\text{Cat}$ can be expressed as the fact that the commutative square

$$
\begin{array}{ccc}
\varnothing & \xrightarrow{e} & e \\
\downarrow & & \downarrow_{e_0} \\
e & \xrightarrow{e_1} & \Delta_1,
\end{array}
$$
is a (2,1)-comma square. Since $I^*$ preserves $\text{Grpd}$-enriched limits, the following square

\[
\begin{array}{ccc}
\emptyset & \rightarrow & * \\
\downarrow & & \downarrow I^*(e_0) \\
* & \rightarrow & I^*(\Delta_1)
\end{array}
\]

is also a (2,1)-comma square (we used that $I^*(\emptyset) \simeq \emptyset$, which follows from the hypothesis on the non-emptiness of the categories $i(a)$), which means exactly that the interval $(I^*(\Delta_1), I^*(e_0), I^*(e_1))$ is strongly separating. Details are left to the reader. □

**Lemma 5.25.** Let $A$ be a small category. For every strongly separating interval $(\mathbb{I}, i_0, i_1)$ of $\widehat{A}_{\text{Grpd}}$, there exists a morphism of intervals $(\mathbb{I}, i_0, i_1) \rightarrow (I_A^*(\Delta_1), I_A^*(e_0), I_A^*(e_1))$ (non-necessarily unique).

**Proof.** By adjunction, we need to find a morphism of $\text{Cat}, u: I_A(\mathbb{I}) \rightarrow \Delta_1$, such that the following diagram is commutative

\[
\begin{array}{ccc}
A & \rightarrow & e \\
\downarrow & \downarrow & \downarrow e_0 \\
I_A(\mathbb{I}) & \rightarrow & \Delta_1 \\
\downarrow & \uparrow e_1 & \\
A & \rightarrow & e,
\end{array}
\]

where the map $A \rightarrow I_A(\mathbb{I})$ at the top is defined as $a \mapsto (a, a \rightarrow * \rightarrow i_0 \rightarrow \mathbb{I})$, and the other one similarly with $i_1$ instead of $i_0$.

Let $(a, a \rightarrow \mathbb{I})$ be an object of $I_A(\mathbb{I})$.

- If $p$ is such that there exists a natural isomorphism

\[
\begin{array}{ccc}
a & \rightarrow & * \\
\downarrow p & & \downarrow i_0 \\
\mathbb{I}
\end{array}
\]

then we define $u(a, p) = 0$,

- else we define $u(a, p) = 1$.

Given a morphism $(a', p') \rightarrow (a, p)$ of $I_A(\mathbb{I})$, notice that if $u(a, p) = 0$, then $u(a', p') = 0$ too. (In other words, the objects sent to 0 form a sieve). This allows for a unique possible way of defining $u$ on arrows.

The upper square of $(\mathbb{I})$ is commutative by definition. For the lower square, we need to prove that

\[u(a, a \rightarrow * \rightarrow i_0 \rightarrow \mathbb{I}) = 1.\]

\[7\]The definition is the same as the usual notion of comma square in a 2-category, except every 2-morphism involved is invertible.
Suppose that it is not the case: this would mean that there exists a 2-square
\[
\begin{array}{ccc}
a & \rightarrow & * \\
\downarrow & \simeq & \downarrow \\
* & \rightarrow & I_i
\end{array}
\]
which is forbidden since \((I, i_0, i_1)\) is strongly separating (and the initial presheaf \(\emptyset\) is never representable).

**Proposition 5.26.** Let \(A\) be a small category. The following are equivalent:

(a) \(A\) is groupoidal local test,
(b) \(I^*_A(\Delta_1)\) is locally aspherical,
(c) there exists a strongly separating interval \((I, i_0, i_1)\) in \(\hat{A}_{\text{Grpd}}\), such that \(I\) is locally aspherical,
(d) there exists a groupoidal locally aspherical functor \(i: A \rightarrow \text{Cat}\).

**Proof.** By Remark 5.6, \(A\) is groupoidal local test if and only if \(A \rightarrow \text{Cat}, a \mapsto A/a\) is a groupoidal locally aspherical functor. Hence, the implication \((a) \Rightarrow (d)\) is trivial and the equivalence \((a) \Leftrightarrow (b)\) follows from Proposition 5.17. From Lemma 5.24, we know that \((I^*_A(\Delta_1), I^*_A(e_0), I^*_A(e_1))\) is a strongly separating interval, hence the implication \((b) \Rightarrow (c)\).

For the implication \((c) \Rightarrow (b)\), we know from Example 5.19 that \((I^*_A(\Delta_1), I^*_A(e_0), I^*_A(e_1))\) is a multiplicative interval. Then, Lemma 5.25 implies that there exists a morphism of intervals \((I, i_0, i_1) \rightarrow (I^*_A(\Delta_1), I^*_A(e_0), I^*_A(e_1))\). Since, by hypothesis, \(I\) is locally aspherical, it follows from Lemma 5.20 that \(I^*_A(\Delta_1)\) is locally aspherical.

So far, we have shown \((c) \Leftrightarrow (b) \Leftrightarrow (a) \Rightarrow (d)\). Let us conclude with the implication \((d) \Rightarrow (c)\). If \(i: A \rightarrow \text{Cat}\) is a groupoidal locally aspherical functor, then, by definition, \(I^*(\Delta_1)\) is locally aspherical. Besides, each category \(i(a)\) has a terminal object and in particular is not empty, hence Lemma 5.24 applies and \((I^*(\Delta_1), I^*(e_0), I^*(e_1))\) is a strongly separating interval. \(\square\)

### 6. Groupoidal strict test categories

We fix once and for all in this section a basic localizer \(W\) of \(\text{Cat}\).

**6.1.** Recall that a small category \(A\) is **totally aspherical** if

(i) \(A\) is aspherical,
(ii) the diagonal functor \(\delta: A \rightarrow A \times A\)

is aspherical.

**Example 6.2.** A small category that has finite products (including the empty product) is totally aspherical [Mal05, Exemple 1.6.4].

**Definition 6.3.** A small category \(A\) is **\(W\)-groupoidal strict test**, or simply **groupoidal strict test**, if the following conditions are satisfied

(a) \(A\) is totally aspherical,
(b) \(A\) is groupoidal test.
In the following proposition, it is important to understand that “finite” includes “empty”.

**Proposition 6.4.** Let $A$ be a small category. The following are equivalent:

(a) $A$ is totally aspherical,

(b) the functor $I_A: \hat{A}_{\text{Grpd}} \to \text{Cat}$ preserves finite products up to weak equivalence, i.e. for every finite family $(X_i)_{i \in I}$ of objects of $\hat{A}_{\text{Grpd}}$, the canonical morphism

$$I_A(\prod_{i \in I} X_i) \to \prod_{i \in I} I_A(X_i)$$

is a weak equivalence,

(c) the class of aspherical objects of $\hat{A}_{\text{Grpd}}$ is stable by finite products, i.e. if $(X_i)_{i \in I}$ is a finite family of aspherical objects of $\hat{A}_{\text{Grpd}}$, then

$$\prod_{i \in I} X_i$$

is also aspherical,

(d) for every finite family of $(a_i)_{i \in I}$ of objects of $A$, seen as representable presheaves (and as objects of $\hat{A}_{\text{Grpd}}$ via the canonical inclusion $\hat{A} \to \hat{A}_{\text{Grpd}}$), the product in $\hat{A}_{\text{Grpd}}$

$$\prod_{i \in I} a_i$$

is aspherical.

**Proof.** Let us begin with $(a) \Rightarrow (b)$. For the empty product, this is simply saying that $I_A(\ast) \simeq A \to e$ is a weak equivalence, which is the case because a totally aspherical category is in particular aspherical. Now let $X$ and $Y$ be two objects of $\hat{A}_{\text{Grpd}}$, and notice that the following square

$$\begin{array}{ccc}
I_A(X \times Y) & \longrightarrow & I_A(X) \times I_A(Y) \\
\downarrow \scriptstyle{\zeta_{X \times Y}} & & \downarrow \scriptstyle{\zeta_X \times \zeta_Y} \\
A & \xrightarrow{\delta} & A \times A
\end{array}$$

is commutative and a pullback square. Since a product of Grothendieck fibrations is again a Grothendieck fibration, the right vertical arrow is a Grothendieck fibration and by Lemma 2.6 we deduce that the top horizontal arrow is aspherical. The general case follows from an immediate induction and the fact that weak equivalences in $\text{Cat}$ are stable by finite products [Mal05, Proposition 2.1.3].

The implication $(b) \Rightarrow (c)$ is immediate because a finite product of aspherical categories is aspherical. The implication $(c) \Rightarrow (d)$ is trivial.

Finally, for the implication $(d) \Rightarrow (a)$, notice first that condition $(d)$ applied to the empty product gives that the terminal $\ast$ object of $\hat{A}_{\text{Grpd}}$ is aspherical, which means exactly that $A$ is aspherical as usual. Now let $a$ and $b$ the two objects of $A$, seen as representable presheaves and thus as objects of $\hat{A}_{\text{Grpd}}$. It is straightforward to check that

$$I_A(a \times b) \simeq A/(a,b),$$
where on the right hand side, \((a,b)\) has to be understood as an object of \(A \times A\) and the slice is relative the diagonal functor \(\delta: A \to A \times A\). This slice category being aspherical for every \((a,b) \in A \times A\) means exactly that \(\delta\) is aspherical. \qed

Now, the crucial result is the following.

**Lemma 6.5.** Let \(A\) be a totally aspherical category and \(X\) an object of \(\hat{A}_{\text{Grpd}}\). The following conditions are equivalent:

(a) \(X\) is aspherical,

(b) \(X\) is locally aspherical.

**Proof.** A totally aspherical category being in particular aspherical, the implication \((b) \Rightarrow (a)\) has already been proved in Lemma 5.10. For the other implication, let \(a\) be an object of \(A\), which we see as a representable presheaf (and then as an object of \(\hat{A}_{\text{Grpd}}\) via the canonical inclusion \(\hat{A} \to \hat{A}_{\text{Grpd}}\)). Thanks to Proposition 6.4 and because representable presheaves are always aspherical, we know that

\[ a \times X \]

is an aspherical object of \(\hat{A}_{\text{Grpd}}\), which proves that \(X\) is locally aspherical by Lemma 5.8. \qed

The following results are straightforward consequences of the previous lemma.

**Proposition 6.6.** Let \(A\) be a small category and \(i: A \to \text{Cat}\) a functor. If \(A\) is totally aspherical, then the following conditions are equivalent:

(a) \(i\) is a groupoidal locally aspherical functor,

(b) \(i\) is a groupoidal aspherical functor.

**Proposition 6.7.** Let \(A\) be a small category. If \(A\) is totally aspherical, then the following are equivalent:

(a) \(A\) is groupoidal strict test,

(b) \(A\) is groupoidal test,

(c) \(A\) is groupoidal weak test,

(d) \(I_A(\Delta_1)\) is aspherical,

(e) there exists a strongly separating interval \((L, i_0, i_1)\) in \(\hat{A}_{\text{Grpd}}\) such that \(L\) is aspherical,

(f) there exists a groupoidal aspherical functor \(i: A \to \text{Cat}\).

7. **Test categories vs. Groupoidal test categories**

7.1. The comparison of the theory of groupoidal test categories and test categories relies on the following trivial but essential observation. If \(D\) is a (small) category with no non-trivial isomorphisms, then for any (small) category \(C\), the groupoid \(\text{Hom}_{\text{iso}}(C, D)\) doesn’t have any non-trivial morphisms. In other words, \(\text{Hom}_{\text{iso}}(C, D)\) is a set and we have

\[ \text{Hom}_{\text{iso}}(C, D) = \text{Hom}_{\text{Cat}}(C, D). \]

In particular, let \(i: A \to \text{Cat}\) be a functor, with \(A\) a small category. For any category \(D\) with no non-trivial isomorphisms, we have

\[ I^*(D) = i^*(D). \]
We then immediately have the following result.

**Proposition 7.2.** Let \( i: A \to \text{Cat} \) a functor such that for every object \( a \) of \( A \), the category \( i(a) \) has a terminal object. The following are equivalent:

(a) \( i \) is a groupoidal locally aspherical functor,
(b) \( i \) is a locally aspherical functor.

**Proof.** Thanks to Proposition 5.17, condition (a) is equivalent to \( I^*(\Delta_1) \) being locally aspherical. And condition (b) means that \( i^*(\Delta_1) \) is locally aspherical (see Definition 2.15). Since \( \Delta_1 \) has no non-trivial isomorphism, we have

\[ I^*(\Delta_1) = i^*(\Delta_1). \]

To conclude, let us prove that a \( \text{Set} \)-valued presheaf is locally aspherical as an object of \( \hat{\mathcal{A}} \) (Definition 2.8) if and only if it is locally aspherical as an object of \( \hat{\mathcal{A}}_{\text{Grpd}} \) (Definition 5.3). First notice that the canonical inclusion \( \hat{\mathcal{A}} \hookrightarrow \hat{\mathcal{A}}_{\text{Grpd}} \) preserves and reflects limits, and preserve and reflects weak equivalences. Hence, given an object \( X \) of \( \hat{\mathcal{A}} \), if \( X \to * \) is a universal weak equivalence of \( \hat{\mathcal{A}}_{\text{Grpd}} \), then it is also a universal weak equivalence of \( \hat{\mathcal{A}} \). The latter is the definition of locally aspherical object of \( \hat{\mathcal{A}} \) and the former is a characterisation of locally aspherical object of \( \hat{\mathcal{A}}_{\text{Grpd}} \) (Lemma 5.8(d)). This proves the “if” part. Conversely, if \( X \to * \) is a universal weak equivalence of \( \hat{\mathcal{A}} \), then for every object \( a \) of \( A \), seen as a representable presheaf, the canonical projection \( a \times X \to a \) is a weak equivalence of \( \hat{\mathcal{A}} \). Since \( a \) is an aspherical object of \( \hat{\mathcal{A}} \) (because \( i_A(a) = A/a \) has a terminal object), it follows that \( a \times X \) is an aspherical object of \( \hat{\mathcal{A}} \). By Remark 5.7, we deduce that it is also an aspherical object of \( \hat{\mathcal{A}}_{\text{Grpd}} \), which proves that \( X \) is locally aspherical as an object of \( \hat{\mathcal{A}}_{\text{Grpd}} \) by Lemma 5.8(c). \( \square \)

From this, we deduce our first comparison theorem.

**Theorem 7.3.** Let \( A \) be a small category. We have the following equivalences:

(a) \( A \) is groupoidal local test \( \iff \) \( A \) is local test,
(b) \( A \) is groupoidal test \( \iff \) \( A \) is test,
(c) \( A \) is groupoidal strict test \( \iff \) \( A \) is strict test.

**Proof.** By Remark 5.6 (resp. Remark 2.17), \( A \) is a groupoidal local test category (resp. local test category) if and only if \( A \to \text{Cat}, a \mapsto A/a \) is a locally groupoidal aspherical functor (resp. locally aspherical functor). The equivalence (a) follows then from Proposition 7.2.

By Proposition 5.12 (resp. Proposition 2.14), we know that \( A \) is groupoidal test (resp. test) if and only if it is groupoidal locally test (resp. locally test) and aspherical. Hence, the equivalence (b) follows trivially from (a).

Finally, \( A \) is groupoidal strict test (resp. strict test) if it is groupoidal test (resp. test) and totally aspherical. Hence, the equivalence (c) follows trivially from (b). \( \square \)

**Corollary 7.4.** If \( A \) is a test category (or equivalently a groupoidal test category), the canonical inclusion functor \( \hat{\mathcal{A}} \hookrightarrow \hat{\mathcal{A}}_{\text{Grpd}} \) induces an equivalence at the level of homotopy categories

\[ \text{Ho}(\hat{\mathcal{A}}) \simeq \text{Ho}(\hat{\mathcal{A}}_{\text{Grpd}}). \]
Proof. Consider the commutative triangle

\[
\hat{A} \xleftarrow{i_A} \hat{A}_{\text{Grpd}} \xrightarrow{I_A} \text{Cat}.
\]

If \( A \) is a test category (or equivalently a groupoidal test category), then both vertical arrow of the previous triangle induce equivalences at the level of homotopy categories. The result follows then by a 2-out-of-3 property for equivalences of categories. \(\square\)

Remark 7.5. In fact, the proof of the previous corollary is straightforwardly generalized to deduce that if \( A \) is a test category, then \( \hat{A} \hookrightarrow \hat{A}_{\text{Grpd}} \) induces a Dwyer–Kan equivalence \( [\text{BK12}] \) \((\hat{A}, W_{\hat{A}}) \simto (\hat{A}_{\text{Grpd}}, W_{\hat{A}_{\text{Grpd}}})\), hence an equivalence of \((\infty, 1)\)-categories.

Example 7.6. It follows from Theorem \[\text{7.3}\] that all the examples of (strict) test categories given in Example \[\text{2.13}\] are also groupoidal (strict) test categories. In particular, \( \Delta \) is a groupoidal strict test category, and we recover that the classical result that the category \( \hat{\Delta}_{\text{Grpd}} \) models homotopy types (using Corollary \[\text{7.4}\] for example). But this is also the case of \( \text{Grpd} \)-valued presheaves over: the cube category with or without connections, Joyal’s \( \Theta \) category, the dendroidal category, etc.

Let us now compare groupoidal weak test categories with weak test categories. For this, first recall the following technical result.

Lemma 7.7. Let \( u: A \rightarrow B \) be a morphism of \( \text{Cat} \), and \( i: A \rightarrow \text{Cat} \) and \( j: B \rightarrow \text{Cat} \) such that the triangle

\[
A \xrightarrow{u} B \xleftarrow{i} \text{Cat} \xrightarrow{j} \text{Cat}
\]

is commutative, and suppose that for every object \( b \) of \( B \), the category \( j(b) \) has a terminal object.

\(\text{(a)}\) if \( u \) is an aspherical morphism of \( \text{Cat} \), then \( i: A \rightarrow \text{Cat} \) is an aspherical functor if and only if \( j: B \rightarrow \text{Cat} \) is,

\(\text{(b)}\) if \( j: B \rightarrow \text{Cat} \) is fully faithful and \( i: A \rightarrow \text{Cat} \) is an aspherical functor, then \( u \) is an aspherical morphism of \( \text{Cat} \) and \( j: B \rightarrow \text{Cat} \) is an aspherical functor.

Proof. See \[\text{[Mal05, Lemma 1.7.4]}\]. \(\square\)

For \( \text{Grpd} \)-valued presheaves, we have the following partial generalization.

Lemma 7.8. Let \( u: A \rightarrow B \) be a morphism of \( \text{Cat} \), and \( i: A \rightarrow \text{Cat} \) and \( j: B \rightarrow \text{Cat} \) such that the triangle

\[
A \xrightarrow{u} B \xleftarrow{i} \text{Cat} \xrightarrow{j} \text{Cat}
\]

is commutative, and suppose that for every object \( b \) of \( B \), the category \( j(b) \) has a terminal object.
(a) if \( u \) is an aspherical morphism of \( \text{Cat} \), then \( i: A \to \text{Cat} \) is a groupoidal aspherical functor if and only if \( j: B \to \text{Cat} \) is, 

(b) if \( j: B \to \text{Cat} \) is fully faithful, \( i: A \to \text{Cat} \) is a groupoidal aspherical functor, and for every object \( b \) of \( B \), the category \( j(b) \) does not have any non-trivial isomorphism, then \( u \) is an aspherical morphism of \( \text{Cat} \) and \( j: B \to \text{Cat} \) is a groupoidal aspherical functor.

**Proof.** Notice first that the hypotheses imply that the category \( i(a) \) has a terminal object for every object \( a \) of \( A \).

Now, the given commutative triangle induces a commutative triangle 

\[
\begin{array}{ccc}
\hat{A}_{\text{Grpd}} & \xrightarrow{u^*} & \hat{B}_{\text{Grpd}} \\
\downarrow I^* & & \downarrow I^* \\
\text{Cat.} & \xrightarrow{I^*} & \text{Cat.}
\end{array}
\]

If \( u \) is aspherical, it follows from Proposition 3.9 that for a small aspherical category \( C \), \( J^*(C) \) is aspherical if and only if \( I^*(C) \) is aspherical. In particular, this proves (a).

For (b), notice that with the hypotheses, we have, for every object \( b \) of \( B \), 

\[ J^*(j(b)) = j^*(j(b)) \simeq b, \]

where the first equality comes from the fact that \( j(b) \) does not have any non-trivial isomorphism and the second from the fact that \( j \) is fully faithful (note that on the right hand side of the second equality, we abusively wrote \( b \) for the \( \text{Set} \)-valued presheaf represented by \( b \)). We then have

\[ u^*(b) \simeq u^*(J^*(j(b))) = I^*(j(b)). \]

Since \( i \) is a groupoidal aspherical functor and \( j(b) \) is an aspherical category, we have that \( I^*(j(b)) \) is aspherical and so is \( u^*(b) \). This means that the category \( i_A(u^*(b)) \) is aspherical and an immediate verification shows that we have a canonical isomorphism \( i_A(u^*(b)) \simeq A/b \), which proves by definition that \( u \) is aspherical. Hence, we can apply (a) and the conclusion follows. \( \square \)

**Remark 7.9.** As the reader might notice, it is only the (b) of Lemma 7.7 that does not generalize straightforwardly in Lemma 7.8 and for which we added the hypothesis that for every \( b \) in \( B \), \( j(b) \) does not have non-trivial isomorphisms. We do not know whether this hypothesis is necessary or not.

We can now prove the following result.

**Proposition 7.10.** Let \( i: A \to \text{Cat} \) be a functor, with \( A \) a small category, such that \( i(a) \) has a terminal object for every object \( a \) of \( A \). We have the following implication:

\( i \) is an aspherical functor \( \Rightarrow i \) is a groupoidal aspherical functor.

If we suppose moreover that for every object \( a \) of \( A \), the category \( i(a) \) does not have any non-trivial isomorphism, then we also have the converse implication:

\( i \) is a groupoidal aspherical functor \( \Rightarrow i \) is an aspherical functor.

**Proof.** We begin by the first implication. Suppose that \( i: A \to \text{Cat} \) is an aspherical functor and let \( B \) be the smallest full subcategory of \( \text{Cat} \) such that:
- every \( i(a) \), for \( a \) in \( A \), is an object of \( B \),
- \( B \) is stable by finite products.

Since a finite product of categories with terminal object has a terminal object, it follows that every object of \( B \) has a terminal object. By construction, we have a factorization,

\[
\begin{array}{ccc}
A & \xrightarrow{i_0} & B \\
\downarrow{i} & & \downarrow{j} \\
& \text{Cat}, &
\end{array}
\]

where \( j \) is the canonical inclusion and \( i_0 \) is the functor \( a \mapsto i(a) \). Since \( i \) is an aspherical functor, Lemma 7.7(b) implies that \( i_0 \) is aspherical and \( j : B \hookrightarrow \text{Cat} \) is an aspherical functor. Since \( B \) is stable by finite products, it is totally aspherical (Example 6.2), and thus \( j \) is also a locally aspherical functor (this follows easily from Lemma 6.3 applied to \( \text{Set} \)-valued presheaves). Applying Proposition 7.2, we obtain that \( j \) is a groupoidal locally aspherical functor, and then a groupoidal aspherical functor because \( B \) is aspherical. Finally, using Lemma 7.8(a), we have that \( i \) is a groupoidal aspherical functor.

For the converse implication, let \( i : A \to \text{Cat} \) be a groupoidal aspherical functor such that each \( i(a) \), for \( a \) in \( A \), does not have any non-trivial isomorphism and let \( B \) be the smallest full subcategory of \( \text{Cat} \) such that:
- every \( i(a) \), for \( a \) in \( A \), is an object of \( B \),
- every object \( \Delta_n \) of \( \Delta \), for \( n \geq 0 \), is an object of \( B \).

Notice that the objects of \( B \) do not have any non-trivial isomorphism. By construction, we have a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i_0} & B \\
\downarrow{i} & & \downarrow{j} \\
& \Delta & \text{Cat}, \\
\end{array}
\]

where:
- \( i_0 \) is the functor \( a \mapsto i(a) \),
- \( j \) is the full subcategory inclusion,
- \( k \) is the canonical fully faithful inclusion of \( \Delta \) in \( \text{Cat} \),
- \( k_0 \) is the full subcategory inclusion.

By hypothesis, \( i \) is a groupoidal aspherical functor and since \( j \) is fully faithful and the objects of \( B \) do not have any non-trivial isomorphism, it follows from Lemma 7.3(b) that \( i_0 \) is an aspherical morphism of \( \text{Cat} \). Now, since \( k : \Delta \to \text{Cat} \) is an aspherical functor [Mal05, Example 1.17.18] (the functor \( k^* : \text{Cat} \to \hat{\Delta} \) is nothing but the nerve functor), and since \( j \) is fully faithful, it follows from Lemma 7.7(b) that \( j \) is also an aspherical functor. Using that \( i_0 \) is an aspherical morphism of \( \text{Cat} \), we deduce from an application Lemma 7.7(a) that \( i \) is an aspherical functor.

\[
\square
\]

From this we deduce the following comparison theorem.

**Theorem 7.11.** Let \( A \) be a small category. We have the following implication:
A is a weak test category \( \Rightarrow \) A is a groupoidal weak test category.

If we suppose moreover that A does not have any non-trivial isomorphism, then we also have the converse implication:

A is a groupoidal weak test category \( \Rightarrow \) A is a weak test category.

**Proof.** By Proposition 4.12 (resp. Remark 2.17), A is groupoidal weak test (resp. weak test) if and only if \( A \to \text{Cat} \), \( a \mapsto A/a \) is a groupoidal aspherical functor (resp. aspherical functor). Hence, the result follows immediately from Proposition 7.10. \( \square \)

**Remark 7.12.** In light of Remark 7.9, we do not know if the hypothesis that A does not have any non-trivial isomorphism is necessary for the second implication of the previous theorem. If a counter-example exists, then it would necessarily be a small category A with non-trivial isomorphisms which is groupoidal weak test but not groupoidal test (or else Theorem 7.3 applies).

**Corollary 7.13.** If A is a weak test category, then the canonical inclusion \( \hat{A} \to \hat{A}_{\text{Grpd}} \) induces an equivalence at the level of homotopy categories

\[
\text{Ho}(\hat{A}) \simeq \text{Ho}(\hat{A}_{\text{Grpd}}).
\]

**Proof.** Similar to the proof of 7.4. \( \square \)

**Remark 7.14.** Same remark as Remark 7.5.

**Example 7.15.** It follows from Theorem 7.3 that all examples of weak test categories from Example 2.13 are also groupoidal weak test categories. For example, it is the case of the category \( \Delta' \) of finite non-empty ordinals and non-decreasing monomorphisms. In particular, the category \( (\Delta')_{\text{Grpd}} \) models homotopy types.

**7.16.** Finally, let us end this section with a quick word on the comparison of pseudo-test categories and groupoidal pseudo-test categories. Although, the following implication

(2) pseudo-test category \( \Rightarrow \) groupoidal pseudo-test category

seems reasonable to expect, it remains an open question for the author of these notes. As for the converse implication, it is not true in general. More precisely, the example below shows that there exists a basic localizer W (which is not \( W_\infty \)!) such that the class of W-groupoidal pseudo-test categories strictly contains the class of W-pseudo-test categories. (Hence, for this particular basic localizer, the implication (2) is true.) The question remains open for an arbitrary basic localizer, in particular for \( W_\infty \).

**Example 7.17.** Consider the functor \( \pi_1 : \text{Cat} \to \text{Grpd} \), left adjoint of the canonical inclusion functor \( \iota : \text{Grpd} \to \text{Cat} \), and let \( W_1 \) be the class of morphisms \( f \) of \( \text{Cat} \) such that \( \pi_1(f) \) is an equivalence of groupoids. We leave it as an exercise to the reader to show that \( W_1 \) is a basic localizer of \( \text{Cat} \). Now, since \( \iota \) is fully faithful, the co–unit of the adjunction \( \pi_1 \dashv \iota \) is an isomorphism and it follows then from Lemma 1.3 that this adjunction induces a homotopical equivalence between \( (\text{Cat}, W_1) \) and \( (\text{Grpd}, W_{\text{EqGrpd}}) \), where \( W_{\text{EqGrpd}} \) is the class of equivalences of groupoids. (This proves in particular that \( \text{Cat} \) models homotopy 1-types.) It follows that a small category A is \( W_1\)-groupoidal
pseudo-test if and only if it is $\mathcal{W}_1$-aspherical and the functor
\[
\hat{A}_{\text{Grpd}} \to \text{Grpd}
\]
\[X \mapsto \pi_1(i_A(X))\]
induces an equivalence at the level of homotopy categories.

Now, let $A = e$ be the terminal category. Then, the previous functor is nothing but the identity functor of $\text{Grpd}$ and it follows trivially that $e$ is $\mathcal{W}_1$-groupoidal pseudo-test. On the other hand, $e$ is not $\mathcal{W}_1$-pseudo-test. Indeed, $i_e: \text{Set} \to \text{Cat}$ is nothing but the canonical inclusion functor and so $i_e^{-1}(\mathcal{W}_1)$ is the class of isomorphisms of $\text{Set}$. If $e$ was $\mathcal{W}_1$ pseudo-test, this would imply that $(\text{Set}, \text{Iso}) \hookrightarrow (\text{Grpd}, \mathcal{W}_1)$ induces an equivalence at the level homotopy categories (and that $\text{Set}$ models homotopy 1-types), which is easily seen to be false.

Remark 7.18. Note that the previous example also shows that the class of $\mathcal{W}_1$-groupoidal weak test categories is strictly bigger than the class of $\mathcal{W}_1$-groupoidal pseudo-test categories. Indeed, if the terminal category $e$ were a $\mathcal{W}_1$-groupoidal weak test category, then it would be a $\mathcal{W}_1$-weak test category (since it does not have any non-trivial isomorphisms), and in particular a $\mathcal{W}_1$-pseudo-test category.

8. Weak equivalences via the nerve

The last section of this paper is devoted to giving an equivalent definition of weak equivalences of $\text{Grpd}$-valued presheaves in terms of nerve functors. In particular, in Example 8.10 below, we recover the usual definition of weak equivalences on $\hat{\Delta}_{\text{Grpd}}$ used in the literature [Cra95, section 8], [JT96].

8.1. Let $A$ be a small category. For a functor $X: A^{\text{op}} \to \text{Cat}$, we denote by $\int_A X$ the Grothendieck construction of $X$. This means that $\int_A X$ is the category such that:

- objects are pairs $(a, x)$ where $a$ is an object of $A$ and $x$ is an object of $X(a)$,
- a morphism $(a, x) \to (a', x')$ is a pair $(f, k)$ where $f: a \to a'$ is a morphism of $A$ and $k: x \to X(f)(x')$ is a morphism of $X(a)$.

(For details, we refer to [Mal05, p. 2.2.6], where the notation $\nabla_A$ for $\int_A$ is used). This construction is functorial and provides a functor
\[
\int_A: \hat{A}_{\text{Cat}} \to \text{Cat},
\]
where we write $\hat{A}_{\text{Cat}}$ for the category of functors $A^{\text{op}} \to \text{Cat}$ and natural transformations between them.

We have canonical inclusions $\hat{A} \hookrightarrow \hat{A}_{\text{Grpd}} \hookrightarrow \hat{A}_{\text{Cat}}$, and, as already observed, for $X$ an object of $\hat{A}$ (resp. $\hat{A}_{\text{Grpd}}$), we have $\int_A X = i_A(X)$ (resp. $\int_A X = I_A(X)$).

Proposition 8.2. [Mal05, Proposition 2.3.1] Let $\mathcal{W}$ be a basic localizer of $\text{Cat}$ and $A$ a small category. The functor $\int_A: \hat{A}_{\text{Cat}} \to \text{Cat}$ sends pointwise $\mathcal{W}$-equivalences to $\mathcal{W}$-equivalences.

We now fix once and for all a basic localizer $\mathcal{W}$ of $\text{Cat}$. 
8.3. Let $A$ and $B$ be two small categories and consider the presheaf category $\widehat{A \times B}$. Using the identification $\widehat{A \times B} \simeq \text{Hom}(A^{\text{op}}, B)$ and the post-composition by the functor $i_B : \widehat{B} \to \text{Cat}$ defines a functor

$$i_B : \widehat{A \times B} \to \widehat{A}_{\text{Cat}}$$

which we abusively denote by $i_B$ again. Then if we postcompose by $\int_A$, we obtain a functor

$$\int_A i_B : \widehat{A \times B} \to \text{Cat}.$$ 

The proof of the following lemma is a straightforward verification, which we leave to the reader.

Lemma 8.4. For every object $X$ of $\widehat{A \times B}$, there is a canonical isomorphism

$$i_{A \times B}(X) \simeq \int_A i_B(X),$$

which is natural in $X$.

Remark 8.5. Remember that the Grothendieck construction of a functor with values in $\text{Cat}$ is weakly equivalent to its homotopy colimit (with respect to any basic localizer on $\text{Cat}$) [Mal05, Théorème 3.1.7]. Since the functor $i_B$ is just the restriction of the Grothendieck construction to $\text{Set}$-valued presheaves, the previous lemma can be simply restated by saying that the homotopy colimit of a functor of two variables is computed by successively taking the homotopy colimit relative to each variable.

8.6. Let $A$ and $B$ be small categories and let $i : B \to \text{Cat}$ be a functor. Recall that we denote $i^* : \text{Cat} \to \widehat{B}$ the functor $C \mapsto \text{Hom}_{\text{Cat}}(i(-), C)$. By considering $\text{Grpd}$ as a subcategory of $\text{Cat}$, the functor $i^*$ induces by post-composition a functor

$$i^* : \widehat{A}_{\text{Grpd}} \to \widehat{A \times B},$$

which we abusively denote by $i^*$ as well.

Proposition 8.7. Let $A$ and $B$ be small categories and $i : B \to \text{Cat}$ a functor such that for every $b$ in $B$, the category $i(b)$ has a terminal object. Then, there exists a natural transformation

$$\begin{array}{ccc}
\widehat{A}_{\text{Grpd}} & \xrightarrow{i^*} & \widehat{A \times B} \\
\downarrow I_A & \cong & \downarrow i_{A \times B} \\
\text{Cat.} & & \\
\end{array}$$

Moreover, if $i$ is an aspherical functor, then this natural transformation is a weak equivalence argument by argument.

Proof. For every $b$ in $B$, let $e_b$ be the terminal object of $i(b)$. By an analogous construction as the one in [L8] in the case of $\text{Set}$-valued presheaves (see Cis06, p. 3.2.4] for details), for every small category $C$ we define a morphism

$$\alpha_C : i_B i^*(C) \to C$$

$$\begin{array}{c}
(b, p : i(b) \to C) \mapsto p(e_b), \\
\end{array}$$

8When the basic localizer is $\mathcal{W}_\infty$, this is a result of Thomason [Tho79].
which is natural in C. For every $X$ in $\hat{A}_{\text{Grpd}}$ and $a$ in $A$, we obtain a map
\[
\alpha_{X(a)} : i_B i^*(X(a)) \to X(a),
\]
natural in $X$ and $a$, and by applying $\int_A$, we obtain a canonical map
\[
i_{A \times B} i^*(X) \simeq \int_A i_B i^*(X) \to \int_A X = I_A(X),
\]
which is natural in $X$. Now, if $i$ is an aspherical functor, then by [Mal05, Proposition 1.7.6] (which is the analogue of our Proposition 4.11 for $\text{Set}$-valued presheaves), the map (3) is a weak equivalence. We conclude with Proposition 8.2. □

**Corollary 8.8.** Let $A$ and $B$ be small categories, and $i : B \to \text{Cat}$ an aspherical functor (such that $i(b)$ has a terminal object for every object $b$ of $B$). Then $i^* : \hat{A}_{\text{Grpd}} \to \hat{A \times B}$ preserves and reflects weak equivalences, i.e.
\[
W_{\hat{A}_{\text{Grpd}}} = i^{*-1}(W_{\hat{A \times B}}).
\]
A particular case where the previous corollary applies is the following.

**Corollary 8.9.** Let $A$ be a totally aspherical small category and $i : A \to \text{Cat}$ aspherical functor (such that $i(a)$ has a terminal object for every object $a$ of $A$). Then a morphism of $\hat{A}_{\text{Grpd}}$ is a weak equivalence if and only if its image by $i^* : \hat{A}_{\text{Grpd}} \to \hat{A \times A}$ is a diagonal weak equivalence, i.e.
\[
W_{\hat{A}_{\text{Grpd}}} = i^{*-1}(\delta^{*-1}(W_{\hat{A}})),
\]
where $\delta^* : \hat{A \times A} \to \hat{A}$ is the diagonal functor.

**Proof.** If $A$ is totally aspherical, then the diagonal functor $\delta : A \to A \times A$ is aspherical and so the induced functor $\delta^* : \hat{A \times A} \to \hat{A}$ preserves and reflects weak equivalences [Mal05, Proposition 1.2.9(d)]. □

**Example 8.10.** Let $A = B = \Delta$ and $i : \Delta \to \text{Cat}$ be the canonical inclusion, so that $i^* : \text{Cat} \to \hat{\Delta}$ is nothing but the usual nerve functor. The previous corollary implies that the weak equivalences on simplicial groupoids $\hat{\Delta}_{\text{Grpd}}$ are exactly those morphisms that induce diagonal weak equivalences of bisimplicial sets.

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