Structural and analytical mesomechanics of alloys with shape memory effect in the concept of interacting levels of deformation and destruction

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Abstract. This article presents the results of the development of structural-analytical mesomechanics based on the concept of interacting multi-level (micro, meso and macro) plastic flow, the evolution of structural damage and martensitic phase transformations. It presents a system of constitutive relations, which take into account the mutual influence of plastic deformations, processes of destruction and structural phase transformations of martensitic nature for alloys with shape memory effect (SME).

1. Introduction
In this article, on the basis of [1–6], the results on the development of methods of structural-analytical mesomechanics of materials with the SME are presented. Unlike previous works on the development of structural-analytical mesomechanics, in this article, for the first time in a related and multi-level formulation, the defining relations for the macroscale level are derived that take into account the possibility of simultaneous multilevel plastic flow, reversible martensitic transformations, micro-destruction and their mutual influence.

Let us consider the problem of calculating the functional and mechanical properties of materials with an SME undergoing shear inelastic deformations, micro destruction processes, and martensitic-type structural and phase transformations. In accordance with the methodology of [1–6], the behavior of the material will be considered at three scale levels (micro-meso-and macro). At the micro level, deformations and stresses are averaged over small volumes of a crystal and are the characteristics of the movements and interactions of atoms. Here, the main acts of mass transfer are realized in the concepts of solid-state physics. At the meso-level, deformations and stresses are characterized by significant spatial gradients and they develop under essentially nonequilibrium conditions, which determines the dominant role of translational-rotational deformation models, the description of which requires the use of laws of nonequilibrium thermodynamics. And, finally, at the macro level, deformations and stresses are obtained by averaging over a large number of interacting micro and mesoregions and are interpreted in the ideas of engineering mechanics of materials. Let us consider the method of constructing the defining relations for the calculation of micro-meso-and macro-scale properties during plastic deformation of materials with an SME.
2. The effective field method

In the process of elastoplastic deformation, as a rule, the nucleation and motion of dislocations is inhomogeneous, differently in different grains, inhibition occurs at grain boundaries and other obstacles. As a result, microcracks can be opened in the most loaded structural concentrators. The deformation created in such conditions is incompatible and leads to the formation of internal or intrinsic stress fields, which in turn influence the further deformation process. The nature of the internal stress fields is such that they weaken the stresses from external forces $\sigma$ in the plastic shear zone ($\rho_{ik}$) and strengthen them where the shift is inhibited. A similar situation occurs in the crystal zone, where the process of destruction is formed, microcracks arise and appear ($\rho_{ik}$). If phase transformations are initiated, that are strongly localized under the conditions considered, then there arises an intrinsic stress field ($\rho_{ik}$), due to the heterogeneity of deformations of martensitic nature. This is equivalent to the statement that the deformation forms three regions in crystals in which the processes of plastic shears, the formation of microcracks and the formation of martensitic transformations are interconnected. To describe the processes under consideration, following the methods of structural-analytical mesomechanics [1, 6], we write out formulas for the effective stresses that act in each of the crystal regions.

In the region of the crystal, where plastic shifts occur, it is reasonable to present the effective stresses $\sigma_{ik}^*$ in the form of:

$$\sigma_{ik}^* = \sigma_{ik} - a\rho_{ik} + b\rho_{ik}^\Phi.$$  \hspace{1cm} (1)

Effective stresses $\sigma_{ik}^{**}$, acting in the zone of initiation of cracks, will look like:

$$\sigma_{ik}^{**} = \sigma_{ik} + a\rho_{ik} - b\rho_{ik}^\Phi.$$  \hspace{1cm} (2)

In the zone of initiation of phase transformations of the martensitic type, two subregions are formed: one in which the effective stresses $\sigma_{ik}^\Phi$ decrease due to plastic shears, and the other where the effective stresses $\sigma_{ik}^\Phi$ increase in comparison with $\sigma_{ik}$. Following [5, 6] you can write:

$$-\sigma_{ik}^\Phi = \sigma_{ik} - a\rho_{ik} + b\rho_{ik}^\Phi,$$

$$+\sigma_{ik}^\Phi = \sigma_{ik} + a\rho_{ik} - b\rho_{ik}^\Phi.$$  \hspace{1cm} (3)

where $a$ and $b$ are permanent; $\rho_{ik}$, $\rho_{ik}^\Phi$ are oriented stress tensors determined at the macroscale level.

3. Hereditary phenomena in reversible martensitic transformations

Plastic deformation of austenite and martensite leads to hardening associated with an increase in yield strengths in austenite $\tau^A$, martensite $\tau^M$, generation of fields of own stresses $\rho_{ik}$, $\rho_{ik}^\Phi$. Since austenite can turn into martensite (and vice versa), the natural question arises about the inheritance of strengthening factors. Within the framework of this approach, it is not possible to discuss the inheritance process in detail, therefore, we assume that the yield strengths of both phases exist and develop independently of each other. Own stress fields $\rho_{ik}$, $\rho_{ik}^\Phi$ appear during plastic deformation of any of the phases and are common to them, i.e. act in representative volumes regardless of the phase composition. Thus, inheritance is assumed without changing these fields during martensitic transformations.
4. Microscale level

Plastic deformation of austenite and martensite leads to hardening related to the micro level analysis, we introduce the volume of averaging \( V_0 \gg V_a \), where \( V_a \) is the atom volume. Following the methodology of structural-analytical mesomechanics and using the same notation as in [6], we formulate the defining relations for the description of the corresponding deformation fluxes and parameters characterizing the evolution of structure and damage at the micro level:

\[
(m) \dot{\beta}_{ik} = (m) C_{ikpq} \dot{\gamma}_{pq}, \quad (m) \dot{\rho}_{ik} = (m) \dot{\gamma}_{ik} T; \quad (4)
\]

\[
(m) \dot{\beta}_H = (m) \beta_{31}(\delta_{ij3} \delta_{k1} + \delta_{k3} \delta_{1i}), \quad (m) \dot{\rho}_H = (m) \beta_{31} + (m) \dot{\rho}_a; \quad (5)
\]

\[
(m) \dot{\rho}_3^T = A_t \exp \left[ -\frac{u_t - \gamma_{31} \text{sign} \tau_{31}}{kT} \right] (\tau_{31} \text{sign} \tau_{31})^m \text{sign} \tau_{31}; \quad (6)
\]

\[
(m) \dot{\rho}_a = A_a (\tau_{31} - \tau_0^s \text{sign} \tau_{31}) H(\tau_{31} \text{sign} \tau_{31} - (m) \tau_s^s) H(\tau_{31} \text{sign} \tau_{31} - (m) \tau_0^s); \quad (7)
\]

\[
\dot{\tau}^s = (m) A_a \dot{\beta}_{31} \text{sign} \tau_{31} - (m) \dot{\tau}^T - (m) A_s \exp \left[ -\frac{(m) u_s - (m) \dot{\tau}_s \text{sign} \tau_s}{kT} \right] \times
\]

\[
\times \left[ (m) \dot{\tau}_0^s - (m) \tau_0 + (m) \dot{\tau}_0 \right] H(\tau_{31} \text{sign} \tau_{31} - (m) \tau_0^s) H(\tau_{31} \text{sign} \tau_{31} - (m) \tau_0^s); \quad (8)
\]

\[
(m) \dot{\tau}_0^s = (m) \dot{\tau}_0^s - (m) A_a \dot{\beta}_{31} \text{sign} \tau_{31} - (m) \tau_0^s (s^*) = (m) \tau_0 + s^*; \quad \tilde{\tau}_{\text{min}} \leq s^* \leq \tilde{\tau}_{\text{max}}; \quad (9)
\]

\[
\tau_{ik} = \tau_{31}^s (\delta_{ij3} \delta_{k1} + \delta_{k3} \delta_{1i}), \quad \tau_{ik} = \tau_{31} - \alpha_{i3} \psi_{31}^{(p)} + \alpha_r \psi_{31}^{(r)} + \alpha_\Phi \psi_{31}^{(\Phi)} + r_{31},
\]

\[
\tau_{31} = \alpha_p \alpha q \sigma q \rho_{pq} \left[ 1 - \Pi_{ij} \right]^{-1}; \quad (10)
\]

\[
\psi_{ik} = \alpha_{pi} \alpha q k \rho_{pq}, \quad \psi_{ik}^{(r)} = \alpha_{pi} \alpha q k r_{pq}, \quad \psi_{ik}^{(\Phi)} = \alpha_{pi} \alpha q k \Phi_{pq}; \quad (11)
\]

\[
\dot{\pi}^0 = \left[ \tilde{\tau}_{33} - \tau_0 \right] \left[ \frac{\beta}{\beta_0} - 1 \right] \left[ \frac{\dot{\beta}}{\beta_0} + H(\beta - \beta_0) \left[ \frac{\dot{\tau}_{33}}{\tau_0} - 1 \right] \frac{\dot{\tau}_{33}}{\tau_0} \right] \times
\]

\[
\times \left[ 1 + \alpha_0 \left( \frac{\beta}{\beta_0} \right) H(1 - \pi^0) + \alpha_0 H(\beta - \beta_0) H(\tilde{\tau}_{33} - \tau_0) \frac{\dot{\beta}}{\beta_0} \right]; \quad (12)
\]
\[ \dot{\tau}^0 = \left[ H(\bar{\tau}_31 \text{sign} \bar{\tau}_31 - \tau^c) \delta \left( \frac{\beta}{\beta^c} - 1 \right) \frac{\dot{\beta}}{\beta^c} + H(\beta - \beta^c) \delta \left( \frac{\bar{\tau}_31 \text{sign} \bar{\tau}_31 - \tau^c}{\tau^c} \right) \frac{\dot{\tau}_31}{\tau^c} \right] \times \left[ 1 + \alpha^c \frac{\beta}{\beta^c} H(1 - \pi^c) + \alpha^c H(\beta - \beta^c) H(\bar{\tau}_31 \text{sign} \bar{\tau}_31 - \tau^c) \frac{\dot{\beta}}{\beta^c} \right]; \] (13)

\[ \int_0^t \exp \left[ - \frac{u_p - \gamma p \bar{\tau}_{33}^c}{kT} \right] dt = 1; \dot{\pi}^t = \delta \left( \frac{t}{t_p} - 1 \right) \frac{1}{t_p} \left( 1 + a^1 \beta \right) H(1 - \tau^t) + a^1 \dot{H} \left( \frac{t}{t_p} - 1 \right) \beta; \] (14)

\[ \dot{\pi}_1 = [\bar{\tau}_{31} \delta_{11} + \bar{\tau}_{33} H(\bar{\tau}_{33}) \delta_{33}] \pi^0 + \pi^t)^n; \pi_1 = [\bar{\tau}_{31} \delta_{11} + \bar{\tau}_{33} H(\bar{\tau}_{33}) \delta_{33}] \pi^c)^n; \] (15)

\[ \dot{\beta} = \beta^1 \text{sign} \beta^1; \beta = \int_0^t \beta^1 \text{sign} \beta^1 ds; \] (16)

\[ \check{\tau}_{ik} = \eta_{ik}^{0} \eta_{ik}^{0} \check{\tau}_{ik}^{**}; \check{\tau}_{ik} = \eta_{ik}^{c} \eta_{ik}^{c} \check{\tau}_{ik}^{**}; \check{\tau}_{ik} = \tau_{ik} + \alpha \rho \psi_{ik}^{(p)} + \alpha \tau \psi_{ik}^{(r)} + \alpha \phi \psi_{ik}^{(\Phi)} + \Sigma \check{\tau}_{ik}. \] (17)

In relations (4) – (17) and further, the point above the symbol means the time derivative. Tensors of elastic deformation \( (m) \beta_{ik}^T \), elastic coefficients \( (m) C_{ikpq} \), thermal expansion deformations \( (m) \beta_{ik}^H \) and thermal expansion coefficients \( (m) \gamma_{ik} \) are assigned to the crystallographic basis. The tensor of the total inelastic deformation rate \( (m) \dot{\beta}_{ik}^H \) (5) consists of thermally activated \( (m) \dot{\beta}_{ik}^t \) (6) and athermic \( (m) \dot{\beta}_{ik}^a \) (7) components.

The evolutionary equations for the initial \( (m) \tau_{ik}^0 \) (9) and current \( (m) \tau_{ik}^c \) (8) stresses of the flow of crystallographic shear take into account strain hardening (the first term in (8)), the temperature factor \( (T) \), and the ability to age or return (the second and third terms in (8)); the symbol \( (m) \) takes the values \( m = M \) for martensite, \( m = A \) for austenite. The effective stress \( \tau_{ik}^c \) contains, along with the components of the stresses from external loads \( \tau_{ik} \), oriented \( \psi_{ik}^{(p)} \), \( \psi_{ik}^{(r)} \), \( \psi_{ik}^{(\Phi)} \), and non-oriented \( \Sigma_{ik} \) structural stresses of various nature [1, 2]. Among non-oriented microstresses, stresses due to the anisotropy of the thermal expansion coefficients and the anisotropy of the elastic compliance of the crystals are particularly distinguished. A detailed derivation of evolutionary equations for calculating non-oriented stresses of various nature is contained in [1,2]. Note that the fields \( \Sigma_{ik} \) are mutually balanced, they are short-range, and the wavelength of the fluctuating stress field is comparable with the characteristic size.

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of the representative volume \( V_0 \). Oriented stresses \( \psi_{ik}^{(\rho)}, \psi_{ik}^{(r)} \) and \( \psi_{ik}^{(\Phi)} \) are long-range, reflect the effect on the stress state of crystals of spatial non-uniformity of nucleation and development of plastic deformation \( \psi_{ik}^{(\rho)} \), the appearance of cracks at meso and macroscale levels \( \psi_{ik}^{(r)} \) and uneven deformation of martensitic nature \( \psi_{ik}^{(\Phi)} \). The introduction of oriented structural stresses takes into account the stress relaxation factor in local volumes, where plastic shear \( \psi_{ik}^{(\rho)} \) occurs, the relaxation effect of structural concentrators in the places where the opening of cracks \( \psi_{ik}^{(r)} \) and the relaxation effect, where martensitic crystals \( \psi_{ik}^{(\Phi)} \) arise. The considered structural stresses \( \psi_{ik}^{(\rho)}, \psi_{ik}^{(r)} \) and \( \psi_{ik}^{(\Phi)} \) are formed at the meso and macroscale levels, therefore, a detailed derivation of the evolutionary equations for their calculation will be presented below when analyzing the corresponding defining relations. Equations (12) – (17) simulate the processes of cracking by a separation (12), a slice (13), opening damage of a thermal-fluctuation nature (14), (15), as well as their growth in the process of inelastic deformation.

Note that for convenience, the indicated relations are formulated in the corresponding bases: separation \( \{\hat{a}, \hat{b}, \hat{c}\} \) (12), (14), (15) and slice \( \{\hat{p}, \hat{r}, \hat{s}\} \) (13). Equations (12) and (13) take into account that microcracks arise only when two conditions are mandatory fulfilled. Firstly, when the path of microdeformations \( \beta \) (16) reaches a critical value equal to tearing cracks \( \beta^T \) and shear cracks \( \beta^S \). Secondly, when the pullout voltage \( \tau_{33}^T \) or shear stress \( \tau_{31}^T \) reaches the corresponding critical voltage pull-off \( \tau^T \) or critical voltage cut \( \tau^C \). The damage parameter \( \Pi_L \) in (10) takes into account the influence on \( \tau_{ik} \) of those damages that are revealed at the micro- and mesostructural levels.

The calculation of the parameter \( \Pi_L \) is discussed in [6], which is devoted to the formulation of the fracture criterion at the macro level. The notation \( \alpha_{ik} \) characterizes the direction cosines of the relative orientation of the local (l, m, n) and laboratory (x, y, z) bases; \( \eta_{ik}^0, \eta_{ik}^c \) are direction cosines between the local basis for the implementation of plastic shear (l, m, n), separation bases \( \{\hat{a}, \hat{b}, \hat{c}\} \) and cut \( \{\hat{p}, \hat{r}, \hat{s}\} \); \( \delta \) is Dirac delta function; \( \delta_{ik} \) is the Kronecker symbol, \( H(x) \) is the Heaviside function; \( \alpha_p, \alpha_s, \alpha_\Phi \) are the concentration coefficients of structural stresses; \( \tau_{\min}^T, \tau_{\max}^T \) are boundary values of a random field, which are estimated from the analysis of the physical content of the problem.

We note that the mere appearance of microcracks of separation or shear does not mean that they will immediately develop, it is necessary to fulfill additional criteria. In accordance with the ideas of fracture mechanics, for the growth of cracks that arise, it is necessary to fulfill the force (or energy) conditions of their propagation. In this regard, microdamage vectors are introduced [6] for tear cracks and shear cracks in the form of relations (15), where \( m \) is a constant. Formulas (15) take into account the degree of opening of microcracks in the stress field. When the values \( \tilde{x}_1, \tilde{x}_2 \) and \( \tilde{x}_3, \tilde{x}_4 \) exceed the critical level \( \pi_{kp}^2 \), the crack will be able to propagate in the crystal. The analysis and physical meaning of the constants \( A_s, A_c, \alpha, \nu_s, \gamma_s, \tau_s, m \) are given in [1]. Relations (4) - (17) describe the evolution of the loss of shear stability of the loaded material at the macroscale level, they allow us to calculate the damageability and predict the microstrains of the corresponding nature under arbitrary modes of thermomechanical action.

Let us consider the question of finding the deformation caused by martensitic inelasticity channel. If the stresses are insignificant and there are no plastic shifts in the material, then the calculation of the deformation effects associated with the martensitic mass transfer channel can be successfully carried out using well-tested relationships [1, 3–5] for environments with phase reactions that take into account the specifics of the transformation (first or second kind), statistical properties of the structure and other
features. The laws of mechanical properties are practically not investigated for the cases when plastic shears, microcracks and martensitic reactions are simultaneously initiated in the material. In this case, as already noted in the analysis of plasticity, the entire volume of the crystal is divided by the nature of the action of oriented stresses $\rho_{ik}$ and $\tau_{ik}$ into two areas. Developing the above considerations, in order to take into account the effect of plastic shears and microcracks arising on martensitic reactions, we assume that the entire representative volume where martensitic transformations are realized can be divided into two sub areas with specific volumes $a$ and $b$ containing oriented microstresses $\Phi^{\sigma+}_{ik}$ and $\Phi^{\sigma-}_{ik}$. Then, in the presence of stresses $\sigma_{ik}$, the effective stresses $\Phi^{\sigma}_{ik}$ acting in different areas during phase transformations can be represented in the first approximation in the form of relations (3). The introduction of effective stresses (3) allows us to formulate, following the method of [3], analytical relations for the calculation of deformations taking into account plastic shears, microdamages and martensitic transformations in the form

$$\dot{\Phi}^{+}_{ik} = A_0 D_{ik} \Phi^{+}, \quad \dot{\Phi}^{-}_{ik} = A_0 D_{ik} \Phi^{-}, \quad \Phi^{+}_{ik} = D_{31} (\delta_{i3} \delta_{k1} + \delta_{k3} \delta_{i1});$$

$$\Phi^{+} = -\tilde{T}^* H(1 - \Phi_{m}) H(\tilde{T}^*) H[M_{H} - \Phi^{+} (M_{H} - M_{K}) - T_{+}^* \{M_{H} - M_{K}\}^{-1} + ];$$

$$\Phi^{-} = -\tilde{T}^* H(1 - \Phi_{m}) H(\tilde{T}^*) H[M_{H} - \Phi^{-} (M_{H} - M_{K}) - T_{+}^* \{M_{H} - M_{K}\}^{-1} + ];$$

$$\tilde{T}^* = \tilde{T} - \frac{T_0}{q_0} \alpha_{ip} \alpha_{kq} D_{pq} + \sigma^{\Phi}_{ik}, \quad \tilde{T}^{*} = \tilde{T} - \frac{T_0}{q_0} \alpha_{ip} \alpha_{kq} D_{pq} - \sigma^{\Phi}_{ik};$$

$$\Phi_{m} = \int_{S_{\Gamma}} F(S_{\Gamma}) \int_{F(S_{D})} f(\Omega)(a\Phi^{+} + b\Phi^{-}) dS_{\Gamma} dS_{D} d^3 \Omega;$$

$$\int_{S_{\Gamma}} F(S_{\Gamma}) dS_{\Gamma} = 1, \quad \int_{S_{D}} F(S_{D}) dS_{D} = 1, \quad \int_{\Omega} f(\Omega) d^3 \Omega = 1.$$
take into account that the average statistical values of martensitic microstrain $\beta_{ik}$ are determined taking into account the influence of both plastic deformation and the development of damage (3) – (20).

5. Meso level-1
We introduce the volume of averaging $V_{m1}$ ($V_0 << V_{m1} << V_{m2}$), where $V_{m2}$ is the characteristic volume of meso-2. The deformation on meso-1 $\Phi_{ik}$ during martensitic reactions is identified with the distortion of martensitic transformations $B_{ik}^\Phi$ [5]. The defining equations at the scale level under consideration obey the equations of conservation of hydrodynamic type:

$$
c B_{ik}^\Phi = c B_{31}^\Phi \delta_{i3} \delta_{k1}, \quad c B_{ik}^\Phi = -\nabla_{\Omega} \cdot c \tilde{I}_\beta + c \sigma_\beta, \quad c L_{ik}^\Phi = c B_{ik}^\Phi m_{11},
$$

$$
c \tilde{I}_\beta = c A_{1\Phi} \nabla_{\Omega} \left( c \beta_{31}^\Phi \right) - \int_0^1 \left( c \beta_{31}^\Phi \right) dS,
$$

$$
\sigma_\Phi = c K_{1\Phi} \left( c \beta_{31}^\Phi \right) + l_1 (\nabla_{\Omega} \times c \tilde{a}_\Phi) \cdot \tilde{e}_n;
$$

$$
c \tilde{a}_\Phi = \left( c \beta_{31}^\Phi \right) \tilde{e}_1, \quad c l_1 = c \eta_1 \Phi \cdot \mod V_{\Omega} \left( \frac{c \beta_{31}^\Phi}{\nabla_{\Omega}^2 \left( c \beta_{31}^\Phi \right)} \right).
$$

In formulas (22), the symbol “c” takes the values “+” or “–” in accordance with formulas (3), characterizing the equations of deformation mass transfer in the sub areas of the crystal, where plastic deformation occurs “–” or in the sub areas of the crystal, where microcracks occur “+”. $c B_{ik}^\Phi$ is the transformation distortion at meso-1; the vector $c \tilde{I}_\beta$ characterizes the distortion flux density on meso-1; $c \sigma_\beta$ parameter describes the production of an internal source $c B_{ik}^\Phi$ due to the initiation in the $V_{m1}$ mesovolume of a statistical ensemble of distortions of martensitic transformations $c \beta_{31}^\Phi$ and a circulation flow of a deformation field $(l_1 (\nabla_{\Omega} \times c \tilde{a}_\Phi) \cdot \tilde{e}_n)$ component. The parameter $l_1$ reflects the size of the arising vortex, $c K_{1\Phi}$, $c A_{1\Phi}$, $c \eta_1 \Phi$ are constants of matter, $\nabla_{\Omega}$ is the "nabla" operator for orientational coordinate system; the symbol $<....>$ denotes the average value.

6. Meso level-2
We introduce the volume of averaging $V_{m2}$ ($V_{m1} << V_{m2} << V$), where $V$ is the characteristic volume of the macro level. The distortion rate $\dot{\Phi}_{ik}$ at the considered scale level will be described on the basis of the theory of large-scale fluctuations by a system of integro-differential equations, taking into account the continuity of the environment. In the framework of the model being developed, the continuity condition of the environment means the need to reconcile the slip velocities of the considered band and the shifts occurring in arbitrary deformation mesos. Using the technique developed in [5, 6], the following analytical relations can be formulated for meso-2:

$$
c \dot{\Phi}_{ik} = c \dot{\phi}_{31}^\Phi \delta_{i3} \delta_{k1},
$$
\[ c \dot{q}_{ik}^{\Phi}(\Omega_1) = \int f(\Omega'_1)R(\Omega_1,\Omega'_1)\frac{1}{2}[\alpha_{m3}(\Omega'_1)\sigma_{n1}(\Omega'_1) + \alpha_{m1}(\Omega'_1)\sigma_{n3}(\Omega'_1)]D(c^{\Phi}\epsilon_{mn})d\Omega'_1; \]

\[ c \dot{\phi}_{31}^{\Phi}(\Omega_1) = \int f(\Omega'_{am})A_{m}(\Omega_{am},\Omega'_{am})\frac{1}{2}[\alpha_{m3}(\Omega'_{am})\sigma_{n1}(\Omega'_{am}) + \alpha_{m1}(\Omega'_{am})\sigma_{n3}(\Omega'_{am})]D_{am} + \int f(\Omega'_{ma})M_{a} (\Omega_{ma},\Omega'_{ma})\frac{1}{2}[\alpha_{m3}(\Omega'_{ma})\sigma_{n1}(\Omega'_{ma}) + \alpha_{m1}(\Omega'_{ma})\sigma_{n3}(\Omega'_{ma})]D_{am} + \]

\[ c \overline{D}(\sigma_{ik}^{*}) = c^{\Phi} \sigma_{ik}^{*} - \sigma_{11}^{*}, \quad \sigma_{11}^{*} = \frac{1}{\sqrt{2}}(c^{\Phi} \sigma_{ik}^{*} - \sigma_{11}^{*})^{1/2}; \]

\[ c \overline{D}(\epsilon_{ik}^{*}) = c^{\Phi} \epsilon_{ik}^{*} - \epsilon_{11}^{*}, \quad \epsilon_{11}^{*} = \frac{1}{\sqrt{2}}(c^{\Phi} \epsilon_{ik}^{*} - \epsilon_{11}^{*})^{1/2}. \] (23)

In formulas (23), \( c \overline{D}(\sigma_{ik}^{*}), c \overline{D}(\epsilon_{ik}^{\Phi}) \) are the directors of the effective stresses and phase strain rates of the corresponding sub area; \( f(\Omega_1) \) and \( f(\Omega'_1) \) – distribution density by orientations \( \{\Omega_1\} \) and \( \{\Omega'_1\} \); \( A_{m}(\Omega_{ma},\Omega'_{ma}), M_{a} (\Omega_{ma},\Omega'_{ma}) \) and \( R(\Omega_{ma},\Omega'_{ma}) \) – influence functions that make sense of structural compliance, structural memory of form, and structural relaxation, respectively; \( \{\Omega_1\} \) – areas of orientation space where shifts occur; \( \{\Omega'_1\} \) – areas of influence [5].

7. Macro level-1

We introduce the volume of averaging \( V (V_m<<V<<R^3) \), where \( R \) is the characteristic size of the macro product. As a result of orientational averaging of parameters (23), the defining relations are obtained at the macro level in the traditional form adopted in the mechanics of a deformable solids [4]. Below is a system of defining relations for the macroscale level. The method of deriving the above equations is described in detail in [5, 6]. For areas of the crystal, where the phase deformation mechanism is initiated, the defining relations will be:

\[ -\epsilon_{ik}^{\Phi} = -A_{ikpq}^{\Phi} \sigma_{pq}^{*} + M_{ikpq}^{\Phi} - \Phi_{ik}^{*}, \quad \epsilon_{ik}^{*} = +A_{ikpq}^{\Phi} \sigma_{pq}^{*} + M_{ikpq}^{\Phi} + \Phi_{ik}^{*}, \quad \epsilon_{ik}^{\Phi} = -\epsilon_{ik}^{\Phi} + \epsilon_{ik}^{\Phi}, \] (24)

To calculate the elastic \( \epsilon_{ik}^{\gamma} \), temperature \( \epsilon_{ik}^{T} \) and plastic deformations \( \epsilon_{ik}^{H} \), following [5,6], we get:

\[ \dot{\epsilon}_{ik}^{Y} = \overline{C}_{ikmn}^{\Phi} \dot{\sigma}_{mn}, \quad \dot{\epsilon}_{ik}^{T} = \overline{C}_{ikmn}^{\Phi} \dot{\sigma}_{mn}, \quad \dot{\epsilon}_{ik}^{H} = \overline{A}_{ikmn} \dot{\sigma}_{mn}^{*} + \overline{B}_{ikmn} \dot{\sigma}_{mn}^{*}; \] (25)

\[ \dot{\rho}_{ik} = \overline{G}_{ikmn} \dot{\sigma}_{mn}^{*} - R_{ikmn}\dot{\rho}_{mn}, \quad \dot{\theta}_{ik} = \overline{P}_{ikmn} \dot{\gamma}_{mn} + \overline{Z}_{ikmn} \dot{\gamma}_{mn}, \quad \dot{\rho}_{ik}^{\Phi} = \overline{R}_{ikmn} \dot{\rho}_{mn}^{\Phi}. \] (26)

8. Macro level-2

Analytical relations (24) - (25), together with equations (4) - (17) and (1) - (3), should be considered as defining equations for a structurally inhomogeneous environment in a macropoint. To solve practical problems of engineering mechanics, it is required to formulate additional equations that take into
account the spatial arrangement of volumes $V$, dynamic and static properties of the environment, etc., as well as initial and boundary conditions for the corresponding variables. The foregoing reduces to the formulation of the following equations of the boundary value problem of mechanics:

8.1. **Dynamic Equilibrium Equations for Stresses**

$$\Delta_i \sigma_{ik} = \rho \ddot{u}_k \quad \text{(27)}$$

where $\rho$ is the density of the medium; $u_k$ is the displacement vector.

8.2. **Continuity condition for deformation**

$$e_{krs} e_{qmr} \nabla_s \nabla_t \epsilon^* = -e_{krs} e_{qmr} \nabla_s \nabla_t \epsilon^*, \quad \text{(28)}$$

where $e_{kr}$ is the Levi-Civita tensor; $\epsilon^*_i = \epsilon^*_{ik} + \delta_{ik} + H_{ik}$

8.3. **Balance condition for temperature**

$$\dot{T} = \frac{K_{ik} \nabla_j \nabla_k T}{\rho \cdot c} - \frac{q_0}{\rho \cdot c} \Phi_{ik}, \quad \text{(29)}$$

where $K_k$ is the thermal conductivity coefficient tensor; $c$ is the specific heat.

The second summand on the right-hand side (29), which contains $\Phi_{ik}$, takes into account the heat effect of the martensitic transformation reaction. In the case of isotropic heat transfer at the macro level, the tensor of thermal conductivity coefficients is equal to $K_{ik} = \rho \delta_{ik}$, where $k_0$ is the scalar coefficient of thermal conductivity.

For the final formulation of a specific boundary value problem, differential equations for stresses (27), deformations (28) and temperature (29) must be supplemented with appropriate boundary and initial conditions.

The following important aspects should be noted. In contrast to the traditional methodology of continuum mechanics, when the components of the strain tensor are introduced as derivatives of the field of displacements of the continuum points, in this approach they are specified through the corresponding micro- and meso-characteristics taking into account the physical laws of formation of martensitic crystals at the micro level and the processes of self-organization of martensitic structures at the mesoscale level.

Physically, the analytical relations (24) - (26) take into account not only the translational-rotational nature of mass transfer at the meso-level, but also the cross-effects of interaction between elements of the environment at various scale levels.

To reflect the complex functionally mechanical properties of materials with the SME, equations for calculating the kinetic coefficients of structural compliance $A_{ikpq}^{\Phi}$, structural shape memory $M_{ikpq}^{\Phi}$ and structural heterogeneity of interphase interaction $R_{ikpq}^{\Phi}$ are derived. In order to take into account the effect of structural changes in the material as a result of plastic deformation and damage accumulation, equations for calculating the kinetic coefficients of structural compliance $A_{ikpq}$, structural flowability $B_{kpq}$, structural heterogeneity $G_{ikpq}$, structural relaxation $R_{kpq}$, structural damage $\Pi_{kpq}$ and structural relaxation of damage concentrators $Z_{kpq}$ are derived [6]. The above kinetic coefficients were obtained as functionals representing fourth-rank tensor objects and along with the formulas for calculating elastic compliance $C_{ikpq}$ and thermal expansion coefficients of materials $\gamma_{ik}$, reflect nontrivial structural changes and the formation of complex deformation properties taking into account the kinetics of martensitic phase transformations and structural evolutions on micro-, meso- and macro levels.
9. Conclusion

The proposed version of the theory eliminates the main drawbacks of the previous models based on the methods of structural and analytical strength theory for materials with shape memory effect and significantly expands the possibilities of mathematical modeling of complex functional and mechanical properties of materials with a martensitic inelasticity mechanism. This result was obtained by taking into account the multilevel processes of mutual influence of the mechanisms of plastic deformation, microdestruction, and phase transformations of a martensitic nature. The results of analytical calculations showed good qualitative and quantitative agreement with experimental data in a wide range of variations in the modes of thermomechanical action [1, 3, 5, 6].

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