Linear lambda calculus with explicit substitutions as proof-search in Deep Inference

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Abstract

SBV is a deep inference system that extends the set of logical operators of multiplicative linear logic with the non commutative operator seq.

We introduce the logical system SBVr which extends SBV by adding a self-dual atom-renaming operator to it. We prove that the cut elimination holds on SBVr.

SBVr and its cut free subsystem BVr are complete and sound with respect to linear Lambda calculus with explicit substitutions. Under any strategy, a sequence of evaluation steps of any linear \( \lambda \)-term becomes a process of proof-search in SBVr (BVr) once \( M \) is mapped into a formula of SBVr.

Completeness and soundness follow from simulating linear \( \beta \)-reduction with explicit substitutions as processes. The role of the new renaming operator of SBVr is to rename channel-names on-demand. This simulates the substitution that occurs in a \( \beta \)-reduction.

Despite SBVr is a minimal extension of SBV its proof-search can compute all boolean functions, as linear lambda calculus with explicit substitutions can compute all boolean functions as well. So, proof search of SBVr and BVr is at least \( \text{ptime} \)-complete.

1 Introduction

We shall see how the functional computation that lambda calculus with explicit substitutions develops relates to proof-search inside an extension of SBV \cite{14}, the system at the core of deep inference (DI).

System SBV. Semantic motivation, intuitions, examples of its use and a cut elimination theorem of the system SBV are in \cite{14}. The cut free sub-system of SBV is BV. The idea leading to SBV is that the logical systems we may rely formal reasoning on must not necessarily exploit shallow rules, as opposed to deep ones. Rules of sequent and natural deduction systems are shallow because they build proofs with a form that mimic the structure of the formula they prove. Deep rules, instead, apply arbitrarily deep in the tree representation of a formula. Thanks to the above deepness, BV substantially extends multiplicative linear logic (MLL) \cite{12} with \( \triangleright \), the non commutative binary operator seq. Many sources of evidence about the relevance of BV exist. The deep application of rules in BV is strictly connected to its expressiveness, as compared to MLL. Any limits we might put on the application depth of BV rules would yield a strictly less expressive system \cite{14}. Moreover, under the analogy "processes-as-formulas and communication-as-proof-search", \cite{10} shows that the operator seq models the sequential behavior CCS, the system of concurrent and communicating processes \cite{23}.

Also, BV, which is \( \text{np-time-complete} \) \cite{19}, has then been extended with linear logic exponentials, in the system NEL \cite{15,16,17,32}, whose provability is undecidable \cite{30}. Finally, strong connections between BV develops and the evolution of discrete quantum systems are emerging \cite{3,2}. 


Linear lambda calculus with explicit substitutions. There is a vast literature on explicit substitutions. We just recall [1, 20, 22, 25] as pointers. We focus on the simplest version of lambda calculus endowed with the obvious notion of explicit substitutions which embodies the kernel of functional programming at its simplest level. The functions linear lambda calculus with explicit substitutions represents use their arguments exactly once in the course of the evaluation. The set of functions we can express in it are quite limited, but “large” enough to let the decision about which is the normal form of its lambda terms a polynomial time complete problem [21]. If we take the polynomial time Turing machines as computational model of reference, of course. Recall that “with explicit substitutions” means that operation substituting a lambda term for a lambda variable, in the course of a β-reduction is not meta, but a syntactical construction.

Leading motivations. Our motivation is to search how structural proof theory, based on DI methodology, can contribute to paradigmatic programming language design. The reason why we think DI can be useful to this respect is that structural proof theory of a quite vast range of logics has become very regular and modular. Proof theory of DI is now developed for classical [4, 5, 6, 8, 9], intuitionistic [33], linear [27, 28, 29, 11] and modal [7, 13, 26] logics, indeed.

We expect that much regularity and modularity at the proof-theory level can highlight useful inherent properties and new primitives, or evaluation strategies, at the level of programs. The point is to look for the computational interpretation of derivations in DI style, in the same vein as the one we are used to with shallow inference. For example, a source of new programming primitives, or evaluation strategies, can be DI deductive systems whose inference rules only manipulate atoms of formulas, and for which new notions of proof normalization exist, in addition to cut elimination.

Starting observation. A typical way to illustrate the properties of BV is to show that any derivation of the sequent ⊢_{\text{MLL}} A_1, \ldots, A_m of MLL embeds into a derivation of BV under (\cdot)^* that maps par and tensor of MLL into par and copar of BV, respectively, and whose extension to MLL sequents is:

\[(\vdash_{\text{MLL}} A_1, \ldots, A_m)^* = [A_1^* \otimes \cdots \otimes A_m^*]\]

(1)

However, alternatively to (1), intuitionistic multiplicative linear logic (IMLL) can embed into BV by mapping sequents of IMLL into formulas of SBV:

\[(\alpha_1, \ldots, \alpha_m \vdash_{\text{IMLL}} \beta)^* = \langle (\alpha_1^\otimes \cdots \otimes \alpha_m^\otimes) \cdot \beta^\otimes \rangle\]

(2)

After (2), a first step is recalling that every axiom A \vdash_{\text{IMLL}} A can give a type to a variable x of linear lambda calculus as in x : A \vdash_{\text{MLLL}} x : A. A second step is recalling the intuition behind the interpretation of any structure (R • T) of BV. The atoms of R, and T will never interact. So, the following representation \{x\}_o of x as structure in BV can make sense:

\[\{x\}_o = \langle x \cdot \overline{o} \rangle\]

(3)

In (3) x becomes the name of the input channel to the left of • that will eventually be forwarded to the output channel o, associated to x by •. Noticeably, (3) strongly resembles the base clause:

\[\{x\}_o = x @ \phi \cdot \overline{\phi}\]

(4)

of the, so called, output-based embedding of the standard lambda calculus with explicit substitutions into π-calculus [35]. In it, “@” is the sequential composition of the π-calculus and @ a generic, essentially place-holder, variable. The whole structure is a forwarder, in accordance with the terminology of [18]. We recall from [35] that output-based embedding is more liberal than the more popular input-based embeddings, inspired to the one in [24]. Output-based one simulates spine reduction of standard lambda calculus with explicit substitutions, while the input-based embedding simulates lazy β-reduction strategy.
The need to extend BV. The essential correspondence between \(3\) and \(4\) rise the question about how could we represent, at least a fragment of standard lambda calculus as a process of proof-search inside BV, in the style of the above output-based embedding. The main missing ingredient is what we can dub as on-the-fly renaming of channels able to model the substitution of a term for a bound variable.

1.1 Contributions

System \textit{SBVr}. We introduce the system \textit{SBVr} (Section 2) which extends \textit{SBV}. The extension of \textit{SBV} consists on adding a binary renaming operator \(\angle \cdot \angle\). Renaming is self-dual and binds atoms. Renaming is the inverse of \(\alpha\)-rule, its prominent defining axiom being \(R \approx \angle R\{a/b\}\angle a\). The meta-operation \(\{a/b\}\) must be a capture-free substitution of the atom \(a\) for every free occurrence of the atom \(b\) in \(R\) and of \(a\) for \(b\). The idea is that we shall rename input/output channels, i.e. atoms, in formulas that represent linear lambda terms with explicit substitutions. Renaming essentially sets the boundary where the name change can take place, without altering the set of free names of \textit{SBVr} structures.

Completeness of \textit{SBVr}. We define how to transform any linear lambda term with explicit substitutions \(M\) into a formula of \textit{SBVr} (Section 3). Then, the evaluation of \(M\) becomes a proof-search process inside \textit{SBVr}:

\[(\text{Theorem 6.7, page 12, Section 6})\] For every linear lambda term with explicit substitutions \(M\), and every atom \(o\), which plays the role of output-channel, if \(M\) reduces to \(N\), then:

\[
\begin{align*}
\langle N \rangle_o \\
\langle M \rangle_{\text{SBVr}} \\
\langle M \rangle_o
\end{align*}
\]

is a derivation of \textit{SBVr}, with \(\langle M \rangle_o\) as conclusion, and \(\langle N \rangle_o\) as premise.

Thanks to the deep application of rules, proof-search inside \textit{SBVr} is completely flexible, so it can simulate any evaluation strategy from \(M\) to \(N\).

Completeness of \textit{BVr}. In fact, we can also show that a computation from \(M\) to \(N\) in linear lambda calculus with explicit substitutions becomes a process of annihilation between the formula that represents \(M\), and the negation of the formula representing \(N\):

\[(\text{Corollary 6.8, page 12, Section 6})\] For every \(M\), and \(o\), if \(M\) reduces to \(N\), then \([\langle M \rangle_o \not= \langle N \rangle_o]\) is a theorem of \textit{BVr}.

Cut elimination for \textit{SBVr}. The completeness of \textit{BVr} follows from proving that the cut elimination holds inside \textit{SBVr} (Theorem 4.1, page 9, Section 3). The proof of cut elimination extends to \textit{SBVr} the path followed to prove the cut-elimination for \textit{SBV} [14], based on the four main steps shallow splitting, context reduction, splitting, and admissibility of the up fragment.

Soundness of \textit{SBVr}. We show that proof-search of \textit{SBVr} can be an interpreter of lambda terms with explicit substitutions. Then, the evaluation of \(M\) becomes a proof-search process inside \textit{SBVr}:

\[(\text{Theorem 6.9, page 13, Section 6})\] For every linear lambda term with explicit substitutions \(M\), and every atom \(o\), which plays the role of output-channel, if

\[
\begin{align*}
\langle N \rangle_o \\
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\langle M \rangle_o
\end{align*}
\]

is a derivation of \textit{SBVr}, with \(\langle M \rangle_o\) as conclusion, and \(\langle N \rangle_o\) as premise, then \(M\) reduces to \(N\).
Soundness of BVr. We show that proof-search of BVr can be an interpreter of lambda terms with explicit substitutions (Section 6).

(Corollary 6.10, page 13, Section 6) For every linear lambda terms with explicit substitutions \( M, N \), and every atom \( o \), which plays the role of output-channel, if \( \llbracket M \rrbracket_o \uplus \llbracket N \rrbracket_o \) is a theorem of BVr, then \( M \) reduces to \( N \).

In principle, this means that if we think \( M \) reduces to \( N \), we can check our conjecture by looking at proof of \( \llbracket M \rrbracket_o \uplus \llbracket N \rrbracket_o \) inside BVr. However, it is worth remarking that we can prove \( \llbracket M \rrbracket_o \uplus \llbracket N \rrbracket_o \) is a theorem of BVr only under a specific proof-search strategy. This might limit efficiency. Indeed, the freedom we could gain, at least in principle, thanks to the deep application of the logical rules, in the course of a proof-search might be lost by sticking to the specific strategy we are referring to and that we shall see.

Expressiveness of SBVr and BVr. Linear lambda calculus is \( \text{ptime} \)-complete, using polynomial time Turing machines as complexity model of reference [21]. The proof in [21] shows that lambda calculus computes all boolean functions. So, proof-search of SBVr and BVr can do the same. The extension of SBV to SBVr is not trivial.

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2 Systems SBVr and BVr

Structures. Let \( a, b, c, \ldots \) denote the elements of a countable set of positive propositional variables, while \( \overline{a}, \overline{b}, \overline{c}, \ldots \) denote the set of negative propositional variables, isomorphic to the set of positive ones. The set of atoms contains positive and negative propositional variables, and nothing else. Let \( \circ \) be a constant different from any atom. The grammar in Figure 1 gives the set of structures. The structures \( \text{par}, \text{copar}, \text{and} \text{ } \text{seq} \) come from SBV. Renaming \( \llbracket R \rrbracket_a \) is new and comes with the proviso that \( a \) must be a positive atom. Namely, \( \llbracket R \rrbracket_a \) is not in the syntax. Renaming implies the definition of the free names \( \text{FN}(R) \) of \( R \) as in Figure 2.

Size of the structures. The size \( |R| \) of \( R \) sums the number of occurrences of atoms in \( R \) and the number of renaming \( |T|_a \) inside \( R \) whose bound variable \( a \) belongs to \( \text{FN}(T) \). For example, \( |[b \uplus \overline{b}]| = 2 \), while \( |[a \uplus \overline{a}]| = 3 \).

Equivalence on structures. Structures are equivalent up to the smallest congruence defined by the set of axioms in Figure 3 that assigns to renaming the status of self-dual operator. The reason is intuitive. By definition, \( R[a/b] \) substitutes every (free) occurrence of the atom \( a \), and its dual \( \overline{a} \), for \( b \),
The system SBVr. It contains the set of inference rules in Figure 4 with form $\frac{T}{R}$, name $\rho$, premise $T$, and conclusion $R$. One between $R$ or $T$ may be missing, but not both. The typical use of an inference rules is $\frac{S[R]}{S{T}}$. It specifies that if a structure $U$ matches $R$ in a context $S\{\ }$, it can be rewritten to $S[T]$. Since rules apply in any context, and we use as rewriting rules $R$ is the redex of $\rho$. The following set of equivalence axioms holds as well:

\[
\begin{align*}
\rho \equiv & R \triangleright \{\} \\
\end{align*}
\]

They are $S\{\ }$, i.e. a structure with a single hole $\{\ }$ in it. If $S[R]$, then $R$ is a substructure of $S$. For example, we shall tend to shorten $S[[R \triangleright U]]$ as $S[R \triangleright U]$ when $[R \triangleright U]$ fills the hole $\{\ }$ of $S\{\ }$ exactly.
Admissible and derivable rules.

The rules in Proof.

Proposition 2.1 derive it as $T$ and $D$ or a sequence of two derivations. The topmost structure in a derivation is its conclusion. The size $|D|$ of a derivation $D$ is the number of rule instances in $D$. A derivation $D$ of a structure $R$ in $SBV^r$ is a structure or an instance of the above rules and $T$ its reduct.

The down fragment of $SBV^r$ is $\{u|, s, q|, r|\}$. Its up fragment of $SBV^r$ is $\{a|\}$. So $s$ belongs to both. Renaming is modeled by $r_1$ and $r_t$. The former can be viewed as the restriction to a self-dual quantifier of the rule $u_1$ which, in $[31]$, models the universal quantifier.

Derivation and proof. A derivation in $SBV^r$ is either a structure or an instance of the above rules or a sequence of two derivations. The topmost structure in a derivation is its conclusion. The size $|D|$ of a derivation $D$ is the number of rule instances in $D$. A derivation $D$ of a structure $R$ in $SBV^r$ is a structure or an instance of the above rules and $T$ its reduct.

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Figure 4: System $SBV^r$.

and $T$ its reduct.

The *down fragment* of $SBV^r$ is $\{u|, s, q|, r|\}$. Its *up fragment* of $SBV^r$ is $\{a|\}$. So $s$ belongs to both.

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### Derivation and proof.

A derivation in $SBV^r$ is either a structure or an instance of the above rules or a sequence of two derivations. The topmost structure in a derivation is its conclusion. The size $|D|$ of a derivation $D$ is the number of rule instances in $D$. A derivation $D$ of a structure $R$ in $SBV^r$ from a structure $T$ in $SBV^r$, only using a subset $B \subseteq SBV^r$ is $\not\exists\ B$. The equivalent space-saving form we shall tend to use is $D : T \vdash^B_{SBV^r} R$. In general, we shall drop both $B$ and $SBV^r$ when $D$ develops in full $SBV^r$. The derivation $\not\exists\ B$ is a proof whenever $T \approx \circ$. We denote it as $D \vdash^B_R \not\exists\ B$, or $D : T \vdash^B_{SBV^r} R$. When developing a derivation, we write $T \approx R$ to mean $R \approx T$. Finally, we shall write $\{\rho_1, \ldots, \rho_m\} \vdash T R$ if, together with some equivalence, we apply the set of rules $\{\rho_1, \ldots, \rho_m\}$ to derive $R$ from $T$.

The following proposition shows when two structures $R, T$ of $BV^r$ can be “moved” inside a context so that they are one aside the other and may eventually communicate going upward in a derivation.

**Proposition 2.1** (Context extrusion). $S [R \approx T] \vdash^q_{BV^r} [S][R] \approx T$, for every $S, R, T$.

**Proof.** By induction on $|S|$ (or $|T|$), proceeding by cases on the form of $S$ (or $T$). (Details in Appendix A.)

Proposition 2.1 here above, also shows how crucial it is saying that every structure is a derivation of $BV^r$. Otherwise, the statement would become meaningless in the base case.

### Equivalence of systems.

A subset $B \subseteq SBV^r$ *proves* $T$ if $\vdash^B_R T$ for some $D$. Two subsets $B$ and $B'$ of the rules in $SBV^r$ are strongly equivalent if, for every derivation $\not\exists\ B$, there exists a derivation $\not\exists\ B'$, and vice versa. Two systems are equivalent if they prove the same structures.

### Admissible and derivable rules.

A rule $\rho$ is admissible for the system $SBV^r$ if $\rho \not\in SBV^r$ and, for every derivation $D$ such that $\not\exists\ \rho| SBV^r$, there is a derivation $D'$ such that $\not\exists\|SBV^r$. A rule $\rho$ is
derivable in $B \subseteq SBVr$ if $\rho \not\in B$ and, for every instance $\rho \frac{T}{R}$, there exists a derivation $\mathcal{D}$ in $B$ such that $\mathcal{D}\models B$. Figure 5 shows a core set of rules derivable in $SBVr$. The rules $i\|$, and $i\|$ are the general interaction up and down, respectively. The rule $\text{def}_i$ uses $\overline{T}$ as a place-holder and $a$ as name for $T$. Building the derivation upward, we literally replace $T$ for $\overline{T}$. Symmetrically for $\text{def}_t$. The rules $\text{mixp}$, and $\text{pmix}$ show a hierarchy between the connectives, where $\otimes$ is the lowermost, $\ast$ lies in the middle, and $\odot$ on top.

**General interaction up is derivable in** $\{i\|, s, q\|, r\|$ **.** We can prove it by induction on $|R|$, proceeding by cases on the form of $R$. We detail out the only case new to $BVr$. Let $R \equiv [T]_a$. Then:

\[
\frac{r\|}{\frac{\langle T \circ \overline{T} \rangle_a}{\frac{\langle T \circ \overline{T} \rangle}{\frac{\langle T \circ \overline{T} \rangle}{\frac{[a]_a}{\langle a \rangle}}}}}
\]

Symmetrically, general interaction down is derivable in $\{i\|, s, q\|, r\|\}$.

**The rule $\text{def}_i$ is derivable in** $\{ai\|, s, qi\}$ **as follows**:

\[
\frac{\langle R \ast T \rangle}{\langle R \ast (a \odot T) \rangle} \quad \frac{\langle R \ast (\overline{a} \ast T) \rangle}{\langle R \ast (a \ast \overline{a} \ast T) \rangle} \quad \frac{\langle R \ast (\overline{a} \ast \overline{a} \ast T) \rangle}{\langle R \ast (a \ast \overline{a} \ast T) \rangle}
\]

Symmetrically, $\text{def}_i$ is derivable in $\{ai\|, s, qi\}$.

**The rule $\text{mixp}$ is derivable in** $\{q \|\}$ **as follows**:

\[
\frac{\langle R \odot T \rangle}{\langle (R \ast a) \odot (\circ \ast T) \rangle} \quad \frac{\langle R \ast T \rangle}{\langle (R \ast a) \ast (\circ \ast T) \rangle}
\]

Symmetrically, $\text{pmix}$ is derivable in $\{qi\}$. 

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**Figure 5:** A core-set of rules derivable in $SBVr$. 

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3 Splitting theorem of $\text{BV}_r$

The goal is to prove that $\text{SBV}_r$ and $\text{BV}_r$ are strongly equivalent. Namely, if a derivation of $T$ from $R$ exists in one of the two systems, then there is a derivation of $T$ from $R$ into the other. Proving the equivalence, amounts to proving that every up rule is admissible in $\text{BV}_r$ or, equivalently, that we can eliminate them from any derivation of $\text{SBV}_r$. Splitting theorem for $\text{BV}_r$, which extends the namesake theorem for $\text{BV}$ [14], is the effective tool we prove to exist to show that the up fragment of $\text{SBV}_r$ is admissible for $\text{BV}_r$.

**Proposition 3.1 ($\text{BV}_r$ is affine).** In every derivation $\mathcal{D} : T \vdash_{\text{BV}_r} R$, we have $|R| \geq |T|$.

**Proof.** By induction on $|\mathcal{D}|$, proceeding by cases on its last rule $\rho$. □

**Proposition 3.2 (Derivability of structures in $\text{BV}_r$).** For all structures $R, T$:

1. $\mathcal{D} : \vdash_{\text{BV}_r} (R \cdot T)$ iff $\mathcal{D}_1 : \vdash_{\text{BV}_r} R$ and $\mathcal{D}_2 : \vdash_{\text{BV}_r} T$.
2. $\mathcal{D} : \vdash_{\text{BV}_r} (R \otimes T)$ iff $\mathcal{D}_1 : \vdash_{\text{BV}_r} R$ and $\mathcal{D}_2 : \vdash_{\text{BV}_r} T$.
3. $\mathcal{D} : \vdash_{\text{BV}_r} [R]_a$ iff $\mathcal{D}' : \vdash_{\text{BV}_r} R[\{ a \}]$, for every atom $a$.

**Proof.** Both hold in $\text{BV}$ [14] while, of course, is meaningless in $\text{BV}$.

We start proving the "if implication". First, we observe that the proofs of [1] and [2] given in [14] by induction on $|\mathcal{D}|$ inside $\text{BV}$, obviously extend to the cases when the last rule of $\mathcal{D}$ is $\rho$. The reason is that the redex of $\iota$ can only be inside $R$ or $T$. Concerning the assumption implies the existence of $\mathcal{D}' : \vdash_{\text{BV}_r} R[\{ a \}]$, namely of $\mathcal{D}' : \vdash_{\text{BV}_r} R$.

For proving the "only if" direction, we use induction on $|\mathcal{D}|$, proceeding by cases on its last rule $\rho$. In all the three cases the redex of $\rho$ can only be inside $R$ or $T$. So, the statements hold by obviously applying the inductive hypotheses. □

**Proposition 3.3 (Shallow Splitting).** For all structures $R, T$ and $P$:

1. If $\mathcal{D} : \vdash \{ a \not\equiv P \}$, then $\mathcal{D}' : \overline{a} \vdash_{\text{BV}_r} P$.
2. If $\mathcal{D} : \vdash_{\text{BV}_r} [(R \cdot T) \not\equiv P]$, then $(P_1 \cdot P_2) \vdash_{\text{BV}_r} P_1, \vdash_{\text{BV}_r} [R \not\equiv P_1], \text{and} \vdash_{\text{BV}_r} [T \not\equiv P_2]$, for some $P_1, P_2$.
3. If $\mathcal{D} : \vdash_{\text{BV}_r} [(R \otimes T) \not\equiv P]$, then $[P_1 \not\equiv P_2] \vdash_{\text{BV}_r} P_1, \vdash_{\text{BV}_r} [R \not\equiv P_1], \text{and} \vdash_{\text{BV}_r} [T \not\equiv P_2]$, for some $P_1, P_2$.
4. If $\mathcal{D} : \vdash_{\text{BV}_r} [(R]_a \not\equiv P]$, then $[P]_a \vdash_{\text{BV}_r} P, \vdash_{\text{BV}_r} [R \not\equiv P']$, for some $P'$.

**Proof.** Point [1] holds by induction on $|\mathcal{D}|$, reasoning by cases on the last rule $\rho$ of $\mathcal{D}$.

From [14] we know that the statements [2] and [3] hold in $\text{BV}$ by induction on the lexicographic order of the pair $(|V|, |\mathcal{D}|)$, where $V$ is one between $(R \cdot T) \not\equiv P$ or $(R \otimes T) \not\equiv P$, proceeding by cases on the last rule $\rho$ of $\mathcal{D}$. The proof of points [2] and [3] extends to the cases where $\rho$ is $\iota$, using the same inductive measure.

Also point [4] holds by induction on the above lexicographic order of the pair $(|V|, |\mathcal{D}|)$. (Details in Appendix B.) □

**Proposition 3.4 (Context Reduction).** For all structures $R$ and contexts $S \{ \}$ such that $\mathcal{D} : \vdash_{\text{BV}_r} S[R]$, there are $U$, such that $\mathcal{D} : \vdash_{\text{BV}_r} S[U]$, and $\vdash_{\text{BV}_r} [R \not\equiv U]$.

**Proof.** The proof is by induction on $|S[\{ ]|$, proceeding by cases on the form of $S[\{ ]$. (Details in Appendix C.) □

**Remark 3.5 (Reading correctly Proposition 3.4).** The statement here above is a compressed version of the more explicit one here below:
For all structures $R$ and contexts $S \{ \}$ such that $\mathcal{D} : \vdash_{\text{BVR}} S[R]$, there are $U, a$ such that, for every structure $V$, $\mathcal{D} : [[V \vDash U]]_a \vdash_{\text{BVR}} S[V]$, and $\vdash_{\text{BVR}} [R \vDash U]$.

Namely, $S \{ \}$ supplies the “context” $U$, required for proving $R$, no matter which structure fills the hole of $S \{ \}$.

**Theorem 3.6.** *(Splitting).* For all structures $R, T$ and contexts $S \{ \}$:

1. If $\mathcal{D} : \vdash_{\text{BVR}} S(R \cdot T)$, then $[[ \{ \} \vDash (K_1 \cdot K_2)]]_a \vdash_{\text{BVR}} S \{ \}$, $\vdash_{\text{BVR}} [R \vDash K_1]$, and $\vdash_{\text{BVR}} [T \vDash K_2]$, for some $K_1, K_2, a$.

2. If $\mathcal{D} : \vdash_{\text{BVR}} S(R \vDash T)$, then $[[ \{ \} \vDash K_1 \vDash K_2]]_a \vdash S \{ \}$, $\vdash_{\text{BVR}} [R \vDash K_1]$, and $\vdash_{\text{BVR}} [T \vDash K_2]$, for some $K_1, K_2, a$.

3. If $\mathcal{D} : \vdash_{\text{BVR}} S[R \cdot a]$, then $[[ \{ \} \vDash K]]_a \vdash_{\text{BVR}} S \{ \}$, and $\vdash_{\text{BVR}} [R \vDash K]$, for some $K, a$.

**Proof.** We obtain the proof of the three statements by composing Context Reduction (Proposition 3.4), and Shallow Splitting (Proposition 3.3) in this order. (Details in Appendix 4.) \qed

## 4 Cut elimination of $\text{SBVR}$

**Theorem 4.1.** *(Admissibility of the up fragment).* The up fragment $[a_1, q_1, r_1]$ of $\text{SBVR}$ is admissible for $\text{BVR}$.

**Proof.** Using splitting (Theorem 3.6) and shallow splitting (Proposition 3.3) it is enough to show that:

(i) $a_1$ gets replaced by a derivation that contains an instance of $a_1$, (ii) $q_1$ gets replaced by a derivation that contains a couple of instances of $q_1$ and $s$ rules, (iii) $r_1$ gets replaced by a derivation that contains a couple of instances of $r_1$ and $s$ rules. (Details in Appendix 5.) \qed

**Theorem 4.1** here above directly implies:

**Corollary 4.2.** The cut elimination holds for $\text{SBVR}$.

## 5 Linear lambda calculus with explicit substitutions

It is a pair with a set of linear lambda terms, and an operational semantics on them. The operational semantics looks at substitution as explicit syntactic component and not as meta-operation.

### The linear lambda terms.

Let $\mathcal{Y}$ be a countable set of variable names we range over by $x, y, w, z$. We call $\mathcal{Y}$ the set of lambda variables. The set of linear lambda terms with explicit substitutions is $\Lambda = \bigcup_{X \in \mathcal{Y}} \Lambda_X$ we range over by $M, N, P, Q$. For every $X \in \mathcal{Y}$, the set $\Lambda_X$ contains the linear lambda terms with explicit substitutions whose free variables are in $X$, and which we define as follows: (i) $x \in \Lambda_{(a)}$; (ii) $\lambda x. M \in \Lambda_X$ if $M \in \Lambda_{X \setminus \{a\}}$; (iii) $(M) N \in \Lambda_{X \cup Y}$ if $M \in \Lambda_X, N \in \Lambda_Y$, and $X \cap Y = \emptyset$; (iv) $(M) \{x = P\} \in \Lambda_{X \setminus \{a\}}$ if $M \in \Lambda_{X \setminus \{a\}}, P \in \Lambda_Y$, and $X \cap Y = \emptyset$.

**$\beta$-reduction on linear lambda terms with explicit substitutions.** It is the relation $\rightarrow$ in Figure 6. It is the core of the very simple, indeed, computations the syntax of the terms in $\Lambda$ allow to develop. The point, however, is that the computational mechanism that replaces a terms for a variable is there, and we aim at modeling it inside $\text{BVR}$.

### Operational semantics on linear lambda terms with explicit substitutions.

It is the relation $\Rightarrow$ in Figure 7. It is the contextual and transitive closure of the above $\beta$-reduction with explicit substitution. We denote as $|M \Rightarrow N|$ the number of instances of rules in Figure 7 used to derive $M \Rightarrow N$. 

We relate functional and proof-theoretic worlds. First we map terms of LP to terms of a structure of $SBV_r$.

Figure 6: $\beta$-reduction $\Rightarrow \subseteq \Lambda \times \Lambda$ with explicit substitution.

Figure 7: Rewriting relation $\Rightarrow \subseteq \Lambda \times \Lambda$.

6 Completeness and Soundness of $SBV_r$ and $BV_r$

We relate functional and proof-theoretic worlds. First we map terms of $\Lambda$ into structures of $SBV_r$. Then, we show the completeness of $SBV_r$ and $BV_r$, i.e. that the computations of $\Lambda$ correspond to proof-search inside the two systems. Finally, we prove soundness of $SBV_r$ and $BV_r$ w.r.t. the computations of lambda calculus with explicit substitutions under a specific proof-search strategy. This means that we can use $SBV_r$ or $BV_r$ to compute any term which any given $M$ reduces to.

The map $\llbracket \cdot \rrbracket_o$. We start with the following “fake” map from $\Lambda$ to $SBV_r$:

\[
\begin{align*}
\llbracket x \rrbracket_o &= \langle x \cdot \overline{D} \rangle \\
\llbracket \lambda x . M \rrbracket_o &= \forall x. \llbracket M \rrbracket_o \otimes (p \otimes \overline{D}) \\
\llbracket (M) N \rrbracket_o &= \exists p. \llbracket M \rrbracket_o \otimes \exists q. \llbracket N \rrbracket_q \otimes (p \otimes \overline{D}) \\
\llbracket (M) \{ x = P \} \rrbracket_o &= \forall x. \llbracket \llbracket M \rrbracket_o \otimes \llbracket P \rrbracket_o \rrbracket
\end{align*}
\]

We use it only to intuitively illustrate how we shall effectively represent terms of $\Lambda$ as structures of $SBV_r$. The map here above translates $M$ into $\llbracket M \rrbracket_o$, where $o$ is a unique output channel, while the whole expression depends on a set of free input channels, each for every free variable of $M$. Clause [5] associates the input channel $x$ to the fresh output channel $\overline{D}$, under the intuition that $x$ is forwarded to $o$, using the terminology of [18]. Clause [6] assumes $\llbracket M \rrbracket_o$ has $p$ as output and (at least) $x$ as input. It renaming $p$, hidden by $\exists$, as $\overline{D}$ thanks to $(p \otimes \overline{D})$. This must work for every input $x$. For this reason we hide $x$ by means of $\forall$. Clause [7] makes the output channels of both $\llbracket M \rrbracket_o$ and $\llbracket N \rrbracket_q$ local, while renaming $p$ to $\overline{D}$ thanks to $(p \otimes \overline{D})$. If $\llbracket M \rrbracket_o$ will result in the translation of a $\lambda$-abstraction $\lambda z . P$, then the existential quantifier immediately preceding $\llbracket N \rrbracket_q$ will interact with the universal quantifier in front of $\llbracket M \rrbracket_o$. The result will be an on-the-fly channel name renaming. Clause [8] identifies the output of $\llbracket P \rrbracket_o$ with one of the existing free names of $\llbracket M \rrbracket_o$. The identification becomes local thanks to the universal quantifier.

However, in a setting where the second order quantifiers $\forall, \exists$ only operate on atoms, distinguishing between the two is meaningless. So, the renaming can be self-dual and the true map $\llbracket \cdot \rrbracket$ which adheres to the above intuition is in Figure [8].

We keep stressing that $\llbracket \cdot \rrbracket$ strongly recalls output-based embedding of standard lambda calculus with explicit substitutions into $\pi$-calculus [55]. In principle, this means that extending $SBV_r$ with the
right logical operators able to duplicate atoms, and consequently upgrading \(|\cdot|\) , we could model full 
\(\beta\)-reduction as proof-search.

\[
\begin{align*}
\langle x \rangle_o &= \langle x \cdot \overline{\gamma} \rangle \\
\langle \lambda x \cdot M \rangle_o &= \{[\langle M \rangle_p \otimes (p \otimes \overline{\gamma})]_p\}, \\
\langle (M) \cdot N \rangle_o &= \{[\langle (M) \rangle_p \otimes (p \otimes \overline{\gamma})]_p\}, \\
\langle (M) \cdot \{x = P\} \rangle_o &= \{[\langle (M) \rangle_o \otimes \{P\}]_o\},
\end{align*}
\]

Figure 8: Map \(|\cdot|\) from \(\Lambda\) to structures

6.1 Origins of the embedding \(|\cdot|\).

The very source of this work, hence of the map \(|\cdot|\) , have been:

1. An almost trivial observation on the form of the derivations of the intuitionistic and multiplicative fragment of linear logic (IMLL) \([12]\), recalled in Figure 9.

2. The internalization of the notion of IMLL sequent, usually a meta-notion, inside SBV.

The formalization of the trivial observation we mention in point (1) here above is:

**Proposition 6.1.** Every derivation \(\Pi\) of the sequent \(A_1, \ldots, A_m \vdash_{\text{MLL}} A\) starts from, at least, \(m\) instances of the rule \(\alpha\) , each proving the sequent \(\langle A_i \rangle_{\Pi} \vdash_{\text{MLL}} A\) for \(1 \leq i \leq m\). We call free axioms the set of such instances of \(\alpha\).

\[
\begin{align*}
\alpha^* &= \alpha \\
(\alpha \otimes \beta)^* &= (\alpha^* \otimes \beta^*) \\
(\alpha \rightarrow \beta)^* &= (\overline{\alpha^*} \otimes \beta^*) \\
(\alpha_1, \ldots, \alpha_m \vdash_{\text{MLL}} \beta)^* &= (\langle \alpha_1^* \otimes \cdots \otimes \alpha_m^* \rangle \cdot \beta^*)
\end{align*}
\]

Figure 10: Map \((\cdot)^*\) from formulas and sequents of IMLL to structures of SBV

**Proposition 6.2 (Internalizing sequents).** Let the map \((\cdot)^*\) from formulas and sequents of IMLL to structures of SBV be given in Figure 10. Then, we can extend it to every derivation \(\Pi\) of IMLL in a way that, if \(\Pi\) has conclusion \(A_1, \ldots, A_m \vdash_{\text{MLL}} A\), and free axioms \(A_1, \ldots, A_m \vdash_{\text{MLL}} A\), then:

\[
\begin{align*}
\langle [A_1^* \otimes \cdots \otimes A_m^*] \cdot A^* \rangle \\
\Pi^* \\
\langle (\langle A_1^* \rangle_{\Pi} \otimes \cdots \otimes \langle A_m^* \rangle_{\Pi}) \cdot A^* \rangle
\end{align*}
\]
Proof. By induction on the size $|I|$ of $I$ which counts the number of instance rules in it, proceeding by cases on its last rule. (Details in Appendix I.)

Given Proposition 6.2, it has been natural to look for the least extension of SBV where we could manage the context-sensitive mechanism of substitution of a term for a variable of linear lambda calculus with explicit substitutions. Such a least extension is the renaming operator that simply determines the scope within which we need to search the name that has to be replaced by (the representation) of a linear lambda term with explicit substitutions.

6.2 Properties of the embedding $\llbracket \cdot \rrbracket$.

Lemma 6.3 (Output names are linear). Every output name of $\llbracket M \rrbracket_o$ occurs once in it.

Proof. By structural induction on the definition of $\llbracket \cdot \rrbracket$, proceeding by cases on the form of $M$.

| Rule | Left | Right |
|-----|------|-------|
| $\alpha\text{-ren}$ | $\llbracket M \rrbracket_p \triangleright (p \triangleright \overline{r})$ | $\llbracket (l \triangleright M) \{ x = N \} \rrbracket_o$ |
| $\iota\text{-intro}$ | $\llbracket (l \triangleright M) \{ x = N \} \rrbracket_o$ | $\llbracket (\lambda x. M) \{ x = P \} \rrbracket_o$ |
| $\iota\text{-r}$ | $\llbracket \{(M) \{ x = P \} \{ N \} \rrbracket_o$ | $\llbracket \{(M) \{ N \} \{ x = P \} \rrbracket_o$ |
| $\iota\text{-l}$ | $\llbracket \{(M) \{ x = P \} \{ N \} \rrbracket_o$ | $\llbracket \{(M) \{ x = P \} \{ N \} \rrbracket_o$ |

Figure 11: Derivable rules that simulate $\beta$-reduction with explicit substitutions

Lemma 6.4 (Output renaming). For every $M$, $o$, and $p$, the rule $\alpha\text{-ren}$ in Figure 11 is derivable in the down-fragment of BVr.

Proof. By induction on the size of $M$. (Details in Appendix II.)

Lemma 6.5 (Simulating $\rightarrow$). For every $M, N, P, o, p, and q$.

1. The rules $\iota\text{-intro}, \iota\text{-l}, \iota\text{-r},$ and $\iota\text{-r}$, in Figure 11 are derivable in the down-fragment of BVr.

Proof. All the derivations require to apply the above Lemma 6.4. More specifically, $\iota\text{-var}$ requires $\upiota\text{exp}$, while $\iota\text{-intro}, \iota\text{-l}, \iota\text{-r},$ and $\iota\text{-r}$ require one instance of $\iota_l$. (Details in Appendix II.)

Remark 6.6. Were the clause $(y) \{ x = P \} \rightarrow y$ in the definition of $\rightarrow$ we could not prove Lemma 6.5 because we could not prove $\llbracket (P) \rrbracket_o$. The reason is that, given $\llbracket (x) \{ x = P \} \rrbracket_o$, it is not evident which logical tool can erase $P$ as directly as happens for $(y) \{ x = P \} \rightarrow y$. The only erasure mechanism existing in BVr is atom annihilation through the rules $\upiota_l$, and $\upiota_l$.

Theorem 6.7 (Completeness of SBVr). For every $M$, and $o$, if $M \Rightarrow N$, then $D : \llbracket N \rrbracket_o \vdash_{SBVr} \llbracket M \rrbracket_o$, where $\upiota_l$ is the unique rule of the up-fragment of BVr used in $D$.

Proof. By induction on $|M \Rightarrow N|$, proceeding by cases on the last rule used, taken among those in Figure 7 (Details in Appendix II).

Corollary 6.8 (Completeness of BVr). For every $M, N$, and $o$, if $M \Rightarrow N$, then $\vdash_{BVr} \llbracket M \rrbracket_o \vdash \llbracket N \rrbracket_o$.

Proof. Theorem 6.7 implies $D : \llbracket N \rrbracket_o \vdash_{SBVr} \llbracket M \rrbracket_o$. The rule $\upiota_l$ is derivable in BVr. So, we can plug it on top of $D$ and apply Theorem 5.1 that transforms $D$ into some $D : \vdash_{BVr} \llbracket M \rrbracket_o \vdash \llbracket N \rrbracket_o$. 

Theorem 6.9 (Soundness of SBVr). For every M, N, and o, let $\mathcal{D} : \vdash M, N, o \vdash_{SBVr}$ be derived by composing a, possibly empty, sequence of rules in Figure 11. Then $M \Rightarrow N$.

Proof. We reason by induction on $|\mathcal{D}|$, proceeding by cases on the form of $M$ and $N$.

As a first base case we assume $M$, and $N$ coincide. By definition, every structure is a derivation of SBVr. So, the statement holds.

As a second base case, let $M$, and $N$ be different with $M$ a redex of $\Rightarrow$ in Figure 6. So, $\langle M \rangle_o$ is the conclusion of one of the rules in Figure 11 a part from $o$-rename. We can derive a premise $\langle N \rangle_o$ which, by definition, translates the reduct $N$. We conclude by $\Rightarrow$ in Figure 7.

The inductive case is with different $M$, and $N$ such that $M$ contains a redex $P$. So, $\langle M \rangle_p \equiv S \langle P \rangle_p$, for some $S \{ \}$. As in the previous case, $\langle P \rangle_p$ is the conclusion of some $\rho$ among $s$-intro, $s$-simp, $s$-@$i$, $s$-@r, $\langle N \rangle_o \neq \{ \}$ and $s$-var.

So, there exists $\rho \langle Q \rangle_p$ with $\langle Q \rangle_p$ premise of $\rho$. The previous base case in this proof implies $P \Rightarrow Q$. Moreover, $\langle Q \rangle_p$ is image, through $\langle \cdot \rangle$, of some lambda term $M'$ since nothing changes in $S \{ \}$ when applying $\rho$. Specifically, $M'$ is $M$ with $Q$ in place of $P$. So, by induction on $\mathcal{D} : \vdash M, N, o \vdash_{SBVr}$ we get $M' \Rightarrow N$. The conclusion is by an instance $\Rightarrow$ in Figure 7. 

Corollary 6.10 (Soundness of BVR). For every $M, N, and o$, if $\mathcal{D} : \vdash M, N, o \vdash_{BVR}$ then $M \Rightarrow N$.

Proof. The strategy to build $\mathcal{D}$ is to start proving $\mathcal{D} : \vdash M, N, o \vdash_{SBVr}$ in SBVr, for some $N$, using Theorem 6.9. Then we plug $\Rightarrow$ which is derivable, on top of $\mathcal{D}$. Finally, we apply Corollary 6.2.

Remark 6.11 (Potential of BVR soundness). Corollary 6.10 suggests that proof-search inside BVR can be used as an interpreter of lambda calculus with explicit substitutions. The interpreter, however, has a weakness. It works under a specific strategy. Currently, we do not know if we can reformulate it so that, for example, the existence of the shortest proof of $\langle M \rangle_o \Rightarrow \langle N \rangle_o$ would always imply that $M$ evaluates to $N$. Of course, such a stronger statement could become relevant in a further extension of BVR where full lambda calculus could be simulated.

7 Conclusions and future work

We define an extension SBVr of SBV by introducing an atom renaming operator which is a self-dual limited version of universal and existential quantifiers. Renaming and $(R \ (-T)$ model the evaluation of linear lambda terms with explicit substitutions as proof-search in SBVr. So, we have not applied DI methodology to reformulate an existing logical system we already know to enjoy a Curry-Howard correspondence with the lambda calculus. Instead, we have searched to use as much as possible logical operators at the core of DI, slightly extended to get a computational behavior we could not obtain otherwise.

We conclude by listing some of the possible natural developments of the work. Concerning Remark 6.11 above, extensions of SBVr whose unconstrained proof-search strategies could be a sound interpreter of full lambda calculus (with explicit substitutions) is one natural work direction. This would really allow to implement one of the motivations leading to this work, related to the search of new programming primitives, or evaluation strategies, of (paradigmatic) programming languages. Starting point to extend SBVr could be [17][18].

Also, we can think of extending SBVr by an operator that models non-deterministic choice. One reason would be the following generalization of soundness (Theorem 6.9 page 13). Let us assume we know that $M$, applied to $P$, reduces to one among $N_1, \ldots, N_m$. Proving the statement:

If $\langle P_1 \rangle_o \cdot \cdots \cdot \langle P_{m-1} \rangle_o \cdot \langle P_m \rangle_o \vdash_{BVR} \langle M \rangle_o \Rightarrow \langle \langle N_1 \rangle_o \cdot \cdots \cdot \langle N_m \rangle_o \rangle_o$, then $M$ reduces to $N_i$ for some $P_1, \ldots, P_m$.
would represent the evaluation space of any linear lambda term with explicit substitutions as a non-deterministic process searching normal forms. Candidate rules for non-deterministic choice to extend SBVr could be:

\[
\begin{align*}
\frac{[R \otimes T] \otimes [U \otimes T]}{[R \otimes U] \otimes T} & \quad \frac{[R \otimes U] \otimes T}{([R \otimes T] \otimes (U \otimes T))} \\
\end{align*}
\]

A further reason to extend SBVr with non-deterministic choice is to keep developing the programme started in [10], aiming at a purely logical characterization of full CCS. We recall that in [10] only sequential and parallel composition of processes have been casted in logical terms.

Finally, the exploration of relations between linear lambda calculus with explicit substitutions, as we embed it in SBVr using a calculus-of-process style, and the evolution of quantum systems, as proofs of BVr [2], makes sense. Indeed, modeling a \( \lambda \)-variable \( x \) as a forwarder \( \langle x \triangleright o \rangle \) is, essentially, looking at \( x \) as a sub-case of \( \langle (x_1 \otimes \cdots \otimes x_k) \triangleright o_1 \otimes \cdots \otimes o_l \rangle \), which recalls the origins of our embedding and which can represent edges in DAGs that model quantum systems evolution [2].

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A Proof of Context extrusion (Proposition 2.1, page 6)

By induction on \(|S|\), proceeding by cases on the form of \(S\). In the base case with \(S\) is a structure, the statement holds simply because \(S[R \otimes T] = [S[R] \otimes T]\), and \(S[R \otimes T]\) being a structure, is, by definition, a derivation.

As a first case, let \(S\) is a structure, then:
\[
\begin{array}{c}
\langle S'[R \otimes T] \cdot U \rangle \equiv \langle S[R \otimes T] \rangle \\
\langle S'[R] \otimes T \cdot U \rangle \\
\end{array}
\]
\[
\begin{array}{c}
q_4 \\
[S[R] \otimes T] \equiv \langle S'[R] \cdot U \rangle \otimes T \\
\end{array}
\]

where \(\mathcal{D}\) exists by inductive hypothesis which holds thanks to \(|S'| < |S|\). If, instead \(S\) is a structure, we can proceed as here above, using \(s\) in place of \(q_4\).

As a second case, let \(S\) is a structure, then:
\[
\begin{array}{c}
[S'[R \otimes T]]_o \equiv [S[R \otimes T]] \\
\langle S'[R] \otimes T \cdot U \rangle \\
\end{array}
\]
\[
\begin{array}{c}
[(S'[R])_o \otimes T]_o \\
[S[R] \otimes T]_o \equiv [(S'[R])_o \otimes T] \\
\end{array}
\]

where \(\mathcal{D}\) exists by inductive hypothesis which holds thanks to \(|S'| < |S|\).
B Proof of Shallow Splitting (Proposition 3.3, page 8)

Point 1 holds by starting to observe that \( P \neq \circ \). Otherwise, we would contradict the assumption. Then, we proceed by induction on \( \Diamond \), reasoning by cases on the last rule \( \rho \) of \( \Diamond \). If \( \rho = \circ \), then \( P = \overline{a} \).

Otherwise, \( \rho \) rewrites \( P \) to some \( P' \), getting \( \Diamond: \vdash [a \neq P'] \), which, by inductive hypothesis, implies \( \Diamond': \forall \vdash P' \). The application of \( \rho \) gives the thesis.

From [14] we know that the statements hold in \( \text{BV} \) by induction on the lexicographic order of the pair \((|V|, |\Diamond|)\), where \( V \) is one between \([|R \times T| \neq P]\) or \([|R \times T| \neq P]\), proceeding by cases on the last rule \( \rho \) of \( \Diamond \).

We start extending the proof of points 2 and 3 to the cases where \( \rho = \circ \), hence proving that points 2 and 3 hold inside \( \text{BV} \). We focus on point 2, being analogous.

Let the last rule of \( \Diamond \) be \( \circ \). If its redex falls inside \( R, T \) or \( P \) it is enough to proceed by induction on \( |\Diamond| \). Otherwise, the redex of \( \circ \) can be the whole \([|R \times T| \neq P]\), thanks to \([|R \times T| \neq P| \approx [|R \times T|_a \equiv P_a]\), if \( a \neq \text{FN}(R 	imes T) \lor \text{FN}(P) \). So we have

\[
\begin{align*}
\forall \because & \frac{[|R \times T| \neq P_a]}{[|R \times T| \neq P]} \\
\forall \because & \frac{[|R \times T| \neq P_a]}{[|R \times T| \neq P]_{|P_a|}}
\end{align*}
\]

The derivability of structures (Proposition 3.2) applied on \( \Diamond' \) implies that, for every \( b \),

\[
\begin{align*}
\forall \because & \frac{\Diamond'}{[|R \times T| \neq P]} \\
\forall \because & \frac{[|R \times T| \neq P]}{[|R \times T| \neq P]_{|b|}}
\end{align*}
\]

The inductive hypothesis holds thanks to \(|\Diamond'| < |\Diamond|\). So, there are \( P', P'' \) such that \((P' \circ P'') \vdash P, \vdash [R \neq P'], \) and \((T \neq P''), \) which prove the statement.

Now, we prove point 2 by detailing the three relevant cases.

As a first case, let \( P = \langle [T' \times T''] \neq V \neq V'' \rangle \). So \( \Diamond \) can be:

\[
\begin{align*}
\forall \because & \frac{\langle [T' \times [R_a \neq V \neq T''] \neq V'' \rangle, \langle \circ \neq T' \rangle, \langle \circ \neq T'' \rangle, \langle \circ \neq T' \rangle \neq V'' \rangle}{[|R_a \neq V \neq T'' \neq P''] \neq V''}] \\
\forall \because & \frac{[|R_a \neq V \neq T'' \neq P''] \neq V'']}{[|R_a \neq (T' \neq T'') \neq V' \neq V'' \rangle]
\end{align*}
\]

The relations \(|\langle [T' \times [R_a \neq V \neq T''] \neq V'' \rangle, \langle \circ \neq T' \rangle, \langle \circ \neq T'' \rangle, \langle \circ \neq T' \rangle \neq V'' \rangle| < |\Diamond| \) imply the inductive hypothesis holds for point 2 on \( \Diamond' \). So, there are \( P', P'' \) such that \( \Diamond'' : (P' \circ P'') \vdash V'', \Diamond'' : \vdash [T' \neq P'], \) and \( \Diamond : \vdash [R_a \neq V \neq T'' \neq P''] \). The relation \(|P'\rangle < |V''\rangle \) implies that \(|[R_a \neq V' \neq T'' \neq P''] < |[R_a \neq (T' \times T'') \neq V' \neq V'' \rangle\)\). So, the inductive hypothesis holds for point 2 on \( \Diamond \). Hence, there is \( R' \) such that \( \Diamond' : [R_a \vdash [V \neq T \neq P'], \) and \( \Diamond' : \vdash [R \neq R'], \) where \( \Diamond'' \) is the “second half” of our thesis. Instead, the “first half” is:

\[
\begin{align*}
\forall \because & \frac{\langle [T' \times P', \langle \circ \neq V \rangle \neq V'' \rangle}{[|V' \neq (T' \times T'') \neq V'' \rangle]} \\
\forall \because & \frac{[|V' \neq (T' \times T'') \neq V'' \rangle}{[|V' \neq (\circ \neq T') \times [V \neq T'' \neq P''] \rangle]} \\
\forall \because & \frac{[|V' \neq (\circ \neq T') \times [V \neq T'' \neq P''] \rangle}{[|V' \neq (\circ \neq V) \times (T' \times T'') \rangle]} \\
\forall \because & \frac{[|V' \neq (\circ \neq V) \times (T' \times T'') \rangle}{[|V' \neq (\circ \neq V) \times (T' \times T'') \rangle}]
\end{align*}
\]
As a second case, let \( P \) be \([ (T' \otimes T'') \not\supset V' \not\supset V'' \)\. So \( \mathcal{D} \) can be:
\[
\begin{align*}
\text{\( \vdash [[T' \otimes [R]a \not\supset V' \not\supset T'']] \not\supset V'' ] \) } \\
\text{\( \vdash [[[R]a \not\supset V'] \not\supset (T' \otimes T'') \not\supset V' \not\supset V''] \) } \\
\text{\( \vdash [[R]a \not\supset (T' \otimes T'') \not\supset V' \not\supset V''] \) }
\end{align*}
\]

The relations \( \| [[(T' \otimes [R]a \not\supset V' \not\supset T'')] \not\supset V'' ] \| = \| [[R]a \not\supset (T' \otimes T'') \not\supset V' \not\supset V''] \| \), and \( | \mathcal{D}'| < | \mathcal{D} | \) imply the inductive hypothesis holds for point 4 on \( \mathcal{D}' \). So, there are \( P', P'' \) such that \( \mathcal{D}'' : [P' \not\supset P''] \not\supset V''' \), \( \mathcal{D}'' : [T' \not\supset P'] \), and \( \mathcal{D} : [ [R]a \not\supset V' \not\supset T'' \not\supset P'' ] \). The relation \( |P'| < |V''| \) implies \( \| [[R]a \not\supset V' \not\supset T'' \not\supset P''] \| < \| [[R]a \not\supset (T' \otimes T'') \not\supset V' \not\supset V''] \| \). So, the inductive hypothesis holds for point 4 on \( \mathcal{D} \). Hence, there is \( R' \) such that \( \mathcal{D}'' : [R' \not\supset [V' \not\supset T'' \not\supset P''] ] \), and \( \mathcal{D}'' : [R \not\supset R'] \), where \( \mathcal{D}'' \) is the “second half” of our thesis. Instead, the “first half” is:
\[
\begin{align*}
\text{\( \vdash [R'] a \) } \\
\text{\( \vdash [V' \not\supset T'' \not\supset P''] \) } \\
\text{\( \vdash [[(T' \otimes T'') \not\supset P'] \not\supset V' \not\supset P''] \) } \\
\text{\( \vdash [[(T' \otimes T'') \not\supset P'] \not\supset V' \not\supset P''] \) } \\
\text{\( \vdash [[T' \not\supset T''] \not\supset V' \not\supset V''] \) }
\end{align*}
\]

As a third case, let \( P \) be \([ [T'] a \not\supset T''] \). So \( \mathcal{D} \) can be:
\[
\begin{align*}
\text{\( \vdash [[[R]a \not\supset [T'] a] \not\supset T''] \) } \\
\text{\( \vdash [[[R]a \not\supset [T'] a] \not\supset T''] \) } \\
\text{\( \vdash [[[R]a \not\supset [T'] a] \not\supset T''] \) }
\end{align*}
\]

The relation \( \| [[R \not\supset [T'] a] \not\supset T''] \| \leq \| [[R]a \not\supset [T'] a \not\supset T''] \| \), and \( | \mathcal{D}'| < | \mathcal{D} | \) implies the inductive hypothesis holds for point 4 on \( \mathcal{D}' \). So, there is \( R' \) such that \( \mathcal{D} : [R' \not\supset T''] \), and \( \mathcal{D} : [R \not\supset [R' \not\supset R'] ] \), where \( \mathcal{D}'' \) is the “second half” of our thesis. Instead, the “first half” is:
\[
\begin{align*}
\text{\( \vdash [[T' \not\supset R'] a \) } \\
\text{\( \vdash [[T' a \not\supset [R'] a] \) } \\
\text{\( \vdash [[T' a \not\supset [R'] a] \) }
\end{align*}
\]

\textbf{C \ Proof of Context Reduction (Proposition 3.4, page 8)}

The proof is by induction on \(| \mathcal{S} \{ 1 \} | \), proceeding by cases on the form of \( S \{ 1 \} \).

As a first case, let \( S \{ 1 \} \equiv (S' \{ 1 \} \star P) \). So, the assumption is \( \mathcal{D} : [S' \{ 1 \} \not\supset P] \). The derivability of structures implies \( \mathcal{D} : [S' \not\supset R] \), and \( \mathcal{D}''' : [P] \). The relation \( |S' \{ 1 \} | < |S' \not\supset P] \) implies the inductive hypothesis holds on \( \mathcal{D}' \). So, there are \( U, a \) such that \( \mathcal{D} : [ [1 \not\supset U] a \not\supset S' \{ 1 \} ] \), and \( [R \not\supset U] \) which is the “second half” of the thesis. Instead, the “first half” is:
\[
\begin{align*}
\text{\( \vdash [[[1 \not\supset U] a \) } \\
\text{\( \vdash [S' \{ 1 \} \not\supset \alpha \) } \\
\text{\( \vdash [S' \not\supset P] \) }
\end{align*}
\]
As a second case, let $S \{ \} \equiv (S' \{ \} \otimes P)$. So, the assumption is $\mathcal{D} : \vdash (S'[R] \otimes P)$. The derivability of structures implies $\mathcal{D}' : \vdash S'[R]$, and $\mathcal{D}'' : \vdash P$. The relation $|S'[R]| < |S'[R] \otimes P|$ implies the inductive hypothesis holds on $\vdash$. So, there are $U, a$ such that $\mathcal{D} : [[\{ \} \equiv U]_a \vdash S'\{ \}$, and $\vdash [R \equiv U]$ which is the “second half” of the thesis. Instead, the “first half” is:

$$[[\{ \} \equiv U]_a\]$$

As a third case, let $S \{ \} \equiv [S'[\{ \} \otimes P) \equiv P)$. So, the assumption is $\mathcal{D} : \vdash ([S'[R] \otimes P') \equiv P)$. Shallow splitting implies the existence of $P_1, P_2$ such that $\mathcal{D} : (P_1 \equiv P_2) : \vdash P$, $\mathcal{D}_0 : \vdash S'[R] \equiv P_1$, and $\mathcal{D}_1 : \vdash [P' \equiv P_2]$. The relation $|S'[R] \equiv P_1| < |(S'[R] \otimes P') \equiv P|$, which holds also thanks to $|P_1| < |P|$, implies the inductive hypothesis holds on $\mathcal{D}_0$. So, there are $U, a$ such that $\mathcal{D} : [[\{ \} \equiv U]_a \vdash [S'\{ \} \equiv P_1]$, and $\vdash [R \equiv U]$ which is the “second half” of the thesis. Instead, the “first half” is:

$$[[\{ \} \equiv U]_a\]$$

As a fourth case, let $S \{ \} \equiv [(S''[\{ \} \otimes P') \equiv P]$. So, the assumption is $\mathcal{D} : \vdash [(S''[R] \otimes P') \equiv P]$.$\mathcal{D}$ Shallow splitting implies the existence of $P_1, P_2$ such that $\mathcal{D} : (P_1 \equiv P_2) : \vdash P$, $\mathcal{D}_0 : \vdash [S''[R] \equiv P_1]$, and $\mathcal{D}_1 : \vdash [P' \equiv P_2]$. The relation $|S''[R] \equiv P_1| < |(S''[R] \otimes P') \equiv P|$, which holds also thanks to $|P_1| < |P|$, implies the inductive hypothesis holds on $\mathcal{D}_0$. So, there are $U, a$ such that $\mathcal{D} : [[\{ \} \equiv U]_a \vdash [S''\{ \} \equiv P_1]$, and $\vdash [R \equiv U]$ which is the “second half” of the thesis. Instead, the “first half” is:

$$[[\{ \} \equiv U]_a\]$$

As a fifth case, let $S \{ \} \equiv [(S''[\otimes P') \equiv P]$. So, the assumption is $\mathcal{D} : \vdash [(S''[R) \otimes P') \equiv P]$. Shallow splitting implies the existence of $P_1, P_2$ such that $\mathcal{D} : [P_1 \equiv P_2] : \vdash P$, $\mathcal{D}_0 : \vdash [S''[R] \equiv P_1]$, and $\mathcal{D}_1 : \vdash [P' \equiv P_2]$. The relation $|S''[R] \equiv P_1| < |(S''[R] \otimes P') \equiv P|$, which holds also thanks to $|P_1| < |P|$, implies the inductive hypothesis holds on $\mathcal{D}_0$. So, there are $U, a$ such that $\mathcal{D} : [[\{ \} \equiv U]_a \vdash [S''\{ \} \equiv P_1]$, and $\vdash [R \equiv U]$ which is the “second half” of the thesis. Instead, the “first half” is:

$$[[\{ \} \equiv U]_a\]$$
As a sixth case, let $S \vdash [S'' \vdash P]_a \not\equiv P$ with $a \in \text{FN}(S''(R))$. Otherwise, we have to consider the case suitable to treat $S \vdash [S'' \vdash P] \not\equiv [S'' \vdash P]$. So, the assumption is $\mathcal{D} : \vdash [S''(R)]_a \not\equiv P$. Shallow splitting implies the existence of $P'$ such that $[P']_a \not\equiv P$, and $\mathcal{D} : \vdash [S''(R) \not\equiv P']$. The relation $[[S''(R) \not\equiv P']] < [[S''(R)]_a \not\equiv P]]$ implies the inductive hypothesis holds on $\mathcal{D}$. So, there are $U,a$ such that $\mathcal{D} : [S'' \vdash [P']_a \not\equiv P']$, and $\vdash [R \not\equiv U]$ which is the “second half” of the thesis. For getting to the “first half” we start observing that $P'$ implies $\mathcal{D}'' : [[[\{ \} \not\equiv U]]_a \not\equiv [P']_a]$ and that $[[[[\{ \} \not\equiv U]]_a \not\equiv [P]]_a]$. So:

\[
\begin{align*}
\mathcal{D} & : \vdash [S'' \vdash [P']_a \not\equiv P'] \\
\mathcal{D}'' & : [[[\{ \} \not\equiv U]]_a \not\equiv [P']_a] \\
\mathcal{D} & : \vdash \mathcal{D}'' \\
\mathcal{D}'' & : [[[\{ \} \not\equiv U]]_a \not\equiv [P]]_a
\end{align*}
\]

### D  Proof of Splitting (Theorem 3.6, page 9)

We obtain the proof of the three statements by composing Context Reduction (Proposition 3.4), and Shallow Splitting (Proposition 3.3) in this order.

We give the details of points 1 and 3, as point 2 is analogous to 1.

As a first case, let us focus on point 1. Context Reduction (Proposition 3.4) applies to $\mathcal{D}$. So, there are $U,a$ such that $\mathcal{D}_0 : [[[\{ \} \not\equiv U]]_a \not\equiv [K_1 \not\equiv K_2])$ and $\mathcal{D}_1 : \vdash [R \not\equiv U]$. Shallow Splitting (Proposition 3.3) applies to $\mathcal{D}_1$. So, $\mathcal{D}_0 : \vdash [K_1 \not\equiv K_2]) \not\equiv U$, $\mathcal{D}_1 : \vdash [R \not\equiv K_1]$, and $\mathcal{D}_2 : \vdash [T \not\equiv K_2]$, for some $K_1, K_2$. Both $\mathcal{D}_1$ and $\mathcal{D}_2$ are the “second half” of the proof. The “first half” is:

\[
\begin{align*}
\mathcal{D}_0 & : [[[\{ \} \not\equiv U]]_a \not\equiv [K_1 \not\equiv K_2])]_a \\
\mathcal{D}_1 & : \vdash [R \not\equiv U]]_a \\
\mathcal{D}_2 & : \vdash [T \not\equiv K_2]
\end{align*}
\]

As a second case, let us focus on point 3. Context Reduction (Proposition 3.4) applies to $\mathcal{D}$. So, there are $U,a$ such that $\mathcal{D}_0 : [[[\{ \} \not\equiv U]]_a \not\equiv [K_1 \not\equiv K_2])$ and $\mathcal{D}_1 : \vdash [R \not\equiv U]$. We notice that the existence of $\mathcal{D}_0$ means that, for every $V$, $\mathcal{D}_0 : [[V \not\equiv U]_a \not\equiv S[V]$. Shallow Splitting (Proposition 3.3) applies to $\mathcal{D}_1$. So, $\mathcal{D}_0 : [K_1]_a \not\equiv U$, $\mathcal{D}_1 : \vdash [R \not\equiv K]$, for some $K$. So, $\mathcal{D}_1$ is the “second half” of the proof. For every $V$, the “first half” is:

\[
\begin{align*}
\mathcal{D}_0 & : [[[\{ \} \not\equiv U]]_a \not\equiv [K_1 \not\equiv K_2])]_a \\
\mathcal{D}_1 & : \vdash [R \not\equiv U]]_a \\
\mathcal{D}_2 & : \vdash [T \not\equiv K_2]
\end{align*}
\]

### E  Proof of Admissibility of the up fragment (Theorem 4.1, page 9)

As a first case we show that $a \vdash$ is admissible for $BVr$. So, we start by assuming:

\[
\begin{align*}
\mathcal{D} & : \vdash S(a \not\equiv \overline{a}) \\
\mathcal{D} & : \vdash S[a \not\equiv \overline{a}]
\end{align*}
\]

Applying splitting (Theorem 3.6) to $\mathcal{D}$ we have $\mathcal{D}_0 : [[[\{ \} \not\equiv [K_1 \not\equiv K_2])_a \not\equiv S[\{ \}]$, $\mathcal{D}_1 : \vdash [a \not\equiv K_1]$, and $\mathcal{D}_2 : \vdash [\overline{a} \not\equiv K_2]$, for some $K_1, K_2,b$, where $b$, and $a$ may coincide. A basic observation is that $\mathcal{D}_0$ holds
for any structure we may plug inside { }. So, in particular, we have $\mathcal{I}_0 : [\{ ] \triangleright K_1 \triangleright K_2]_b \vdash \mathcal{S}[\{ ]$. Now, shallow splitting (Proposition 3.3) on $\mathcal{P}_1, \mathcal{P}_2$ implies $\mathcal{P}_1 : \triangleright [R \triangleright T]_b$ and $\mathcal{P}_2 : a \triangleright K_2$. So, we can build the following proof with the same conclusion as $\mathcal{D}$, but without its bottommost instance of $a$:

\[
\begin{array}{c}
\mathcal{D} \\
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right) \\
\hline
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right)
\end{array}
\]

As a second case we show that $q_1$ is admissible for $\text{Bv}_r$. So, we start by assuming:

\[
\begin{array}{c}
\mathcal{D} \\
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right) \\
\hline
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right)
\end{array}
\]

Applying splitting (Theorem 3.3) to $\mathcal{D}$ we have $\mathcal{D}_0 : [\{ ] \triangleright K_1 \triangleright K_2]_b \vdash \mathcal{S}[\{ ]$, $\mathcal{P}_1 : \vdash [R \triangleright T]_b \triangleright K_1$, and $\mathcal{P}_2 : \vdash [U \triangleright V]_b \triangleright K_2$, for some $K_1, K_2, b$, where $b$ and $a$ may coincide. A basic observation is that $\mathcal{D}_0$ holds for any structure we may plug inside { }. So, in particular, we have $\mathcal{D}_0 : [\{ ] \triangleright K_1 \triangleright K_2]_b \vdash \mathcal{S}[\{ ]$. Then, shallow splitting (Proposition 3.3) on $\mathcal{D}_1, \mathcal{D}_2$ implies $\mathcal{D}_1 : \vdash [R \triangleright K_T]_b$, $\mathcal{P}_1 : \vdash [T \triangleright K_T]_b$, $\mathcal{P}_0 : [K_U \triangleright K_V]_b \triangleright K_2$, $\mathcal{P}_1 : \vdash [U \triangleright K_U]_b$, and $\mathcal{P}_2 : \vdash [V \triangleright K_V]_b$. So, we can build the following proof with the same conclusion as $\mathcal{D}$, but without its bottommost instance of $q_1$:

\[
\begin{array}{c}
\mathcal{D} \\
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right) \\
\hline
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right)
\end{array}
\]

As a third case we show that $r_1$ is admissible for $\text{Bv}_r$. So, we start by assuming:

\[
\begin{array}{c}
\mathcal{D} \\
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right) \\
\hline
\vdash \mathcal{S} \left( (R \triangleright T) \triangleright (U \triangleright V) \right)
\end{array}
\]

Applying splitting (Theorem 3.3) to $\mathcal{D}$ we have $\mathcal{D}_0 : [\{ ] \triangleright K_1 \triangleright K_2]_b \vdash \mathcal{S}[\{ ]$, $\mathcal{D}_1 : \vdash [R \triangleright K_T]_b$, and $\mathcal{D}_2 : \vdash [T \triangleright K_T]_b$, for some $K_1, K_2, b$, where $b$ and $a$ may coincide. A basic observation is that $\mathcal{D}_0$ holds for any structure we may plug inside { }. So, in particular, we have $\mathcal{D}_0 : [\{ ] \triangleright K_1 \triangleright K_2]_b \vdash \mathcal{S}[\{ ]$. Then, shallow splitting (Proposition 3.3) on $\mathcal{D}_1, \mathcal{D}_2$ implies $\mathcal{D}_1 : \vdash [K_U \triangleright K_V]_b \triangleright K_2$, $\mathcal{P}_1 : \vdash [T \triangleright K_T]_b$. So, we can build the following proof with the same
conclusion as \( \mathcal{D} \), but without its bottommost instance of \( r_1 \):

\[
\begin{align*}
\lor \quad & \quad \Gamma, \Delta, \Theta \\
\vdash & \quad \psi \\
\equiv & \quad [[\psi]_{\Gamma, \Delta, \Theta}]
\end{align*}
\]

\[
\begin{align*}
\vdash & \quad [[(T \circ K_T) \circ [R \circ K_R]]]_{\Gamma, \Delta, \Theta} \\
\vdash & \quad [[[[R \circ K_R] \circ T] \circ K_T]]_{\Gamma, \Delta, \Theta} \\
\vdash & \quad [[[[R \circ T] \circ [K_R \circ K_T]]]_{\Gamma, \Delta, \Theta} \\
\vdash & \quad [[(R \circ T)]_{\Gamma, \Delta, \Theta} \circ [K_R \circ K_T]]_{\Gamma, \Delta, \Theta} \\
\vdash & \quad [[(R \circ T)]_{\Gamma, \Delta, \Theta} \circ [K_R \circ K_T]]_{\Gamma, \Delta, \Theta} \\
\vdash & \quad S((R \circ T)]_{\Gamma, \Delta, \Theta} \\
\end{align*}
\]

\section{Proof of Internalizing sequents (Proposition 6.2, page 11)}

By induction on the size \( |\Pi| \) of \( \Pi \) which counts the number of instance rules in it, proceeding by cases on its last rule. To avoid cluttering the derivations of SBV we are going to produce, we shall omit \( (\cdot)^* \) around the formulas.

Let the last rule of \( \Pi \) be \( \lor \). Then, the derivation we are looking for is the structure \( \langle \overline{A} \cdot A \rangle \).

Let the last rule of \( \Pi \) be \( \land \). Then:

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle \overline{A} \cdot A \rangle \circ \langle B_1 \cdot B_1 \rangle \circ \cdots \circ \langle B_m \cdot B_m \rangle \\
\end{align*}
\]

exists under the inductive hypothesis that \( \Pi_A, \Pi_B \) derive the assumptions of cut.

Let the last rule of \( \Pi \) be \( \land \). Then, there is:

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle B_1 \cdot B_1 \rangle \circ \cdots \circ \langle B_m \cdot B_m \rangle \\
\end{align*}
\]

under the inductive hypothesis that \( \Pi_A, \Pi_B \) derive the assumptions of \( \land \).

Let the last rule of \( \Pi \) be \( \rightarrow \). Then:

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle \overline{A} \cdot A \rangle \\
\end{align*}
\]

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle \overline{A} \cdot A \rangle \\
\end{align*}
\]

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle \overline{A} \cdot A \rangle \\
\end{align*}
\]

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle \overline{A} \cdot A \rangle \\
\end{align*}
\]

\[
\begin{align*}
\langle \overline{A_1} \cdot A_1 \rangle \circ \cdots \circ \langle \overline{A_m} \cdot A_m \rangle & \circ \langle \overline{A} \cdot A \rangle \\
\end{align*}
\]
exists under the inductive hypothesis that \( \Pi \) derives the assumption of \( \neg \circ \).

G  Proof of Output renaming (Lemma 6.4, page 12)

Let \( M \equiv x \). Then:

\[
\begin{align*}
\Gamma & \equiv (x \circ \overline{y}) \\
\Gamma[\circ]_{p} & \equiv (p \circ \overline{p}) \\
\end{align*}
\]

Let \( M \equiv (P) Q \). Then:

\[
\begin{align*}
\text{s,ai}_1 & \equiv \langle (P) Q \rangle_o \equiv \langle (P) p' \equiv (Q) q \equiv (p' \circ \overline{p}) \rangle_{p'}^x \\
\text{s} & \equiv \langle (P) p' \equiv (Q) q \equiv (p' \circ \overline{p}) \rangle_{p'}^x \equiv \langle (P) p \equiv (Q) q \equiv (p \circ \overline{p}) \rangle_{p'}^x \\
\text{ri} & \equiv \langle (P) p' \equiv (Q) q \equiv (p' \circ \overline{p}) \rangle_{p'}^x \equiv \langle (P) p \equiv (Q) q \equiv (p \circ \overline{p}) \rangle_{p'}^x \\
\end{align*}
\]

Let \( M \equiv \lambda x. P \). Then:

\[
\begin{align*}
\text{s,ai,pmix}_1 & \equiv \langle \lambda x. P \rangle_o \equiv \langle [\{(P) p' \equiv (Q) q \equiv (p' \circ \overline{p}) \}]_{p'}^x \rangle_x \\
\text{ri} & \equiv \langle [\{(P) p' \equiv (Q) q \equiv (p' \circ \overline{p}) \}]_{p'}^x \rangle_x \equiv \langle [\{(P) p \equiv (Q) q \equiv (p \circ \overline{p}) \}]_{p'}^x \rangle_x \\
\end{align*}
\]

Let \( M \equiv (P) \{x = Q\} \). Then:

\[
\begin{align*}
\left( (P) \{x = Q\} \right)_o & \equiv \left( (P) p \equiv (Q) q \right)_x \\
\text{ri} & \equiv \left( (P) p \equiv (Q) q \right)_x \\
\end{align*}
\]

where \( \text{s} \) exists thanks to the inductive hypothesis which holds because \( P \) is a sub-term of \( (P) Q \).

H  Proof of Simulating \( \rightarrow \) (Lemma 6.5, page 12)

Let us focus on \( s \)-var. The following derivation exists:

\[
\begin{align*}
\langle (P) \rangle_o & \equiv \langle (P) \rangle_x \\
\text{ri} & \equiv \langle [\{(P) \equiv (x \circ \overline{y})\}]_x \rangle_x \\
\end{align*}
\]

\[
\begin{align*}
\langle (x) \{x = P\} \rangle_o & \equiv \langle [(x) p \equiv (P) q \equiv (x \circ \overline{y}) \equiv (p \circ \overline{p})] \rangle_x \\
\end{align*}
\]

where \( \text{s} \) exists thanks to Lemma 6.4. The here above derivation requires \( q \) because \( \text{mexp} \) is derivable in \( \{q\} \).
Let us focus on $s$-intro. The following derivation exists:

$$
\begin{align*}
& e_1 \vdash [\{(M) \{x = N\}\}o] \\
& e_2 \vdash [\{(M) \{x = N\}\}p], \varphi \\
& e_3 \vdash [\{(M) \{x = N\}\}p, \{p \circ \overline{p}\}]_p, p \equiv [\{(M) \{x = N\}\}p, \{p \circ \overline{p}\}]_p \\
& e_4 \vdash [\{(M) \{x = N\}\}p, \{p \circ \overline{p}\}]_p, \{p_0\} \equiv [\{(M) \{x = N\}\}p, \{p \circ \overline{p}\}]_p, \{p_0\} \\
\end{align*}
$$

where:

- $\{p\}_0$ in the conclusion of $e_0$ becomes $\{p\}_q$ in its premise because $q$ only occurs as output channel name in a pair $(p'' \circ q)$, for some $p''$, and nowhereelse;

- Lemma 6.4 implies the existence of both $\mathcal{Q}, \mathcal{Q}'$;

- in the conclusion of $e_1$, $p'$ has disappeared from $\{M\}p$;

- in the conclusion of $e_2$, $p$ has disappeared from $\{(M) \{x = N\}\}o$.

Let us focus on $s$-$\omega$. The following derivation exists:

$$
\begin{align*}
& e_1 \vdash [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, p \equiv [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, \{q\} \equiv [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, \{q\} \\
& e_2 \vdash [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, \{q\} \equiv [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, \{q\} \\
& e_3 \vdash [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, \{q\} \equiv [\{(M) \{x = P\}\}x, \{p \circ \overline{p}\}]_p, \{q\} \\
\end{align*}
$$

where $p, y$ do not belong to $\{p\}_x$, and $e_1$ applies three of the axioms in Figure 3.

Let us focus on $s$-$\omega$. The following derivation exists:

$$
\begin{align*}
& \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \equiv \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \equiv \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \\
& \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \equiv \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \\
& \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \equiv \{(M) \{x = P\}\}N, \{p \circ \overline{p}\}p, \{q\} \\
\end{align*}
$$

The case relative to $s$-$\omega$ develops as for $s$-$\omega$.

## I Proof of Completeness of SBVr (Proposition 6.7, page 12)

By induction on $|M \Rightarrow N|$, proceeding by cases on the last rule used, taken among those in Figure 7.

Let the last rule be $w_m$, namely $M \Rightarrow N$ because $M \rightarrow N$. Lemma 6.3 directly implies the thesis. Let the last rule be $u_a$. The inductive hypothesis implies the existence of $\mathcal{Q}_0, \mathcal{Q}_1$:

$$
\begin{align*}
& \{N\}_0, \varphi \\
& \{P\}_o, \varphi \\
& \{M\}_o
\end{align*}
$$
Let the last rule be $\sigma r$. The inductive hypothesis implies the existence of $D$:

$$
\left\lceil [\{P\}_o \triangleright (N)_{x = 1}]_{x} \right\rceil \equiv \{((P) \{x = N\})_{x}, \\
\triangleright \right\}
$$

$$
\{((P) \{x = M\})_{x} \equiv \left\lceil [\{P\}_o \triangleright (M)_{x = 1}]_{x} \right\rceil \right\}
$$

In all the remaining cases we can proceed just as here above.