A new algebraic approach for calculating the heat kernel in quantum gravity

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It is shown that the heat kernel operator for the Laplace operator on any covariantly constant curved background, i.e. in symmetric spaces, may be presented in form of an averaging over the Lie group of isometries with some nontrivial measure. Using this representation the heat kernel diagonal, i.e. the heat kernel in coinciding points is obtained. Related topics concerning the structure of symmetric spaces and the calculation of the effective action are discussed.

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I. INTRODUCTION

The heat kernel, a very powerful tool for investigating the effective action in quantum field theory and quantum gravity, has been the subject of much investigation in recent years in physical as well as in mathematical literature (Refs. 1-22). The subject of present investigation is the low-energy limit of the one-loop contribution of a set of quantized fields $\phi$ on a $d$-dimensional Riemannian manifold $M$ of metric $g_{\mu\nu}$ with Euclidean signature to the effective action, which can best be presented using the $\zeta$-function regularization in the form

$$\Gamma(1) = -\frac{1}{2}\zeta'(0),$$

where

$$\zeta(p) = \mu^{2p}Tr F^{-p} = \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty \text{d}t \ t^{p-1}Tr U(t),$$

$$F = -\Box + Q + m^2,$$

$$U(t) = \exp(-tF),$$

with $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$, $Tr$ meaning the functional trace, $\mu$ being a renormalization parameter introduced to preserve dimensions, $Q(x)$ an arbitrary matrix-valued function (potential term), $m$ a mass parameter and $\nabla_\mu$ a covariant derivative. The covariant derivative includes, in general, not only the Levi-Civita connection but also the appropriate spin one as well as the vector gauge connection and is determined by the commutator $[\nabla_\mu, \nabla_\nu] = R_{\mu\nu}$. The Riemann curvature tensor, the curvature of background connection and the potential term completely describe the background metric and connection, at least locally. In the following we will call these quantities the background curvatures or simply curvatures and denote them symbolic by $\mathcal{R} = \{R_{\mu\nu\beta}, R_{\mu\nu}, Q\}$.

Exact evaluation of the heat kernel $U(t)$ is obviously impossible. Therefore, one should make use of various approximations. First of all, let us note the very important so called Schwinger - De Witt asymptotic expansion of the heat kernel at $t \to 0^+ - 5$

$$\text{Tr} U(t) \sim (4\pi t)^{-d/2} \exp(-tm^2) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} B_k,$$

$$B_k = \int_M \text{d}x g^{1/2} tr b_k.$$ 

This expansion is purely local and does not depend, in fact, on the global structure of the manifold. In manifolds with boundary additional terms in $B_k$ as well as new terms of order $t^{-d/2+k/2}$ in form of surface integrals over the boundary $\partial M$ appear. For details see Refs. 12,13, where all coefficients for arbitrary boundary conditions up to terms of order $t^{-d/2+1}$ are calculated. Its coefficients $b_k$ (we call them Hadamard - Minakshisundaram - De Witt - Seeley (HMDS) coefficients) are local invariants built from the curvature, the potential term and their covariant derivatives.\textsuperscript{1,5,6,14} They play a very important role both in physics and mathematics and are closely connected with various sections of mathematical physics.\textsuperscript{14,22} Therefore, the calculation of HMDS-coefficients is in itself of great importance. Various methods were used for calculating these coefficients, beginning from the direct De Witt’s method\textsuperscript{1} to modern mathematical methods, which make use of pseudodifferential operators, functorial properties of the heat kernel etc.\textsuperscript{5-13} Very good reviews of the calculation of the HMDS-coefficients are given in recent papers.\textsuperscript{14}

Nowadays, in general case only the first four coefficients are explicitly calculated. The first three coefficients were calculated in Ref. 9. An effective covariant technique for calculating HMDS-coefficients is elaborated in Refs. 10, 4, where also the first four coefficients are computed. In the case of scalar operators the fourth coefficient is also calculated in Ref. 11. Analytic approach was developed in Ref. 7, where a closed form for the intrinsic symbol of the resolvent parametrix was obtained. The leading terms in all the volume coefficients $B_k$ quadratic in the background curvatures were calculated completely independently in Refs. 15, 16.
Although the Schwinger - De Witt expansion is good for small \( t \) (viz. \( t \Re \ll 1 \)), and thereby in the case of massive quantized fields in weak background fields when \( \Re \ll m^2 \), it is absolutely inadequate for large \( t \) in strongly curved manifolds and strong background fields (\( \Re \gg m^2 \)). For investigating these cases one needs some other methods.

A possibility to exceed the limits of the Schwinger - De Witt expansion is to employ the direct partial summation. Namely, one can compare all the terms in HMDS-coefficients \( B_k \) (1.6), pick up the main (the largest in some approximation) terms and sum up the corresponding partial sum. There is always a lack of uniqueness concerned with the global structure of the manifold, when doing so. But, hopefully, fixing the topology, e.g. the trivial one, one can obtain a unique, well defined, expression that would reproduce the Schwinger -De Witt expansion, being expanded in curvature. The main advantage of such an approach is that although the result will be not exact it will be covariant and general.

Actually, the effective action is a covariant functional of the metric and depends on the geometry of the manifold as a whole, i.e. it depends on both local characteristics of the geometry like invariants of the curvature tensor and its global topological structure. However, we will not investigate in this paper the influence of the topology but concentrate our attention, as a rule, on the local effects. That means that we restrict ourselves to those physical problems where the contribution of the global effects may be neglected in comparison with local ones. Then the possible approximations for evaluating the effective action can be based on the assumptions about the local behavior of the background fields, dealing with the real physical gauge invariant variations of the local geometry, i.e. with the curvature invariants, but not with the behavior of the metric and the connection which is not invariant. Comparing the value of the curvature with that of its covariant derivatives one comes to two possible approximations: i) the short-wave (or high-energy) approximation characterized by \( \nabla \nabla \Re \gg \Re \Re \) and ii) the long-wave (or low-energy) one \( \nabla \nabla \Re \ll \Re \Re \).

The idea of partial summation was realized in short-wave approximation for investigating the nonlocal aspects of the effective action (in other words the high-energy limit of that) in Ref. 15,4, where all the terms in the HMDS-coefficients \( B_k \) with higher derivatives (quadratic in the curvature and potential term) are calculated and the corresponding asymptotic expansion is summed up. Another approach to study the high-energy limit of the effective action, so called covariant perturbation theory, is developed in Ref 17.

II. LOW ENERGY APPROXIMATION AND ITS CONSEQUENCES

The low-energy effective action, in other words, the effective potential, presents a very natural tool for investigating the vacuum of the theory, its stability and the phase structure. Here only partial success is achieved and various approaches to the problem are only outlined (see, e.g. the excellent review of Camporesi in Ref. 22 with an ample bibliography and our recent papers\textsuperscript{20,21}).

The long-wave (or low-energy) approximation is determined, as it was already stressed above, by strong slowly varying background fields. This means that the derivatives of all invariants are much smaller than the products of the invariants themselves. The zeroth order of this approximation corresponds to covariantly constant background curvatures

\[
\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu R_{\alpha\beta} = 0, \quad \nabla_\mu Q = 0. \quad (2.1)
\]

In this case the HMDS-coefficients are simply polynomials in curvature invariants and potential term of dimension \( \Re^k \) up to terms with one or more covariant derivatives of the background curvatures \( O(\nabla \Re) \)

\[
b_k = \sum_{n=0}^{k} \binom{k}{n} Q^{k-n} a_n + O(\nabla \Re), \quad (2.2)
\]
\[
a_k = b_k \big|_{Q=\nabla R=0} = \sum R^k. \quad (2.3)
\]

Note that the commutators \([Q, R_{\mu\nu}]\) are of order \( O(\nabla \nabla \Re) \) and, therefore are neglected here.
Then after summing the Schwinger-De Witt expansion (1.5) we obtain for the heat kernel, the \(\zeta\)-function and the effective action

\[
\text{Tr} U(t) = \int_M dx g^{1/2}(4\pi t)^{-d/2} \text{tr} \left\{ \exp \left( -t(m^2 + Q) \right) \left( \Omega(t) + O(\nabla^2 R) \right) \right\},
\]

\[
\zeta(p) = \int_M dx g^{1/2}(4\pi)^{-d/2} \frac{L^{2p}}{\Gamma(p)} \int_0^\infty dt \ t^{-d/2-1} \text{tr} \left\{ \exp \left( -t(m^2 + Q) \right) \left( \Omega(t) + O(\nabla^2 R) \right) \right\},
\]

\[
\Gamma(1) = \int_M dx g^{1/2} \{ V(\mathcal{R}) + O(\nabla^2 R) \},
\]

with

\[
V(\mathcal{R}) = \frac{1}{2} (4\pi)^{-d/2} \frac{1}{\Gamma(\frac{d}{2} + 1)} \int_0^\infty dt \ \left( \log (\mu^2 t) + \psi \left( \frac{d}{2} + 1 \right) \right) \times \left( \frac{\partial}{\partial t} \right)^{\frac{d}{2} + 1} \text{tr} \left\{ \exp(-t(m^2 + Q))\Omega(t) \right\}
\]

for even \(d\) and

\[
V(\mathcal{R}) = \frac{1}{2} (4\pi)^{-d/2} \frac{1}{\Gamma(\frac{d}{2} + 1)} \int_0^\infty dt t^{-1/2} \left( \frac{\partial}{\partial t} \right)^{\frac{d}{2} + 1} \text{tr} \left\{ \exp(-t(m^2 + Q))\Omega(t) \right\}
\]

for odd \(d\), where

\[
\Omega(t) \sim \sum_{k=0}^\infty \frac{(-t)^k}{k!} a_k,
\]

is a function of local invariants of the curvatures (but not of the potential).

It is naturally to call the functions \(\Omega(t)\) and \(V(\mathcal{R})\), that do not contain the covariant derivatives at all and so determine the zeroth order of the heat kernel and that of the effective action, the generating function for covariantly constant terms in HMDS-coefficients and the effective potential in quantum gravity respectively.

Let us note that such a definition of the effective potential is not conventional. It differs from the definition that is often found in the literature.\(^{24}\) What is meant usually under the notion of the effective potential is a function of the potential term only \(Q\), because it does not contain derivatives of the background fields (in contrast to Riemann curvature \(R_{\alpha\beta\gamma}\) that contains second derivatives of the metric and the curvature \(R_{\mu\nu}\) with first derivatives of the connection). So, e.g. in Ref. 24 the potential term \(Q\) is summed up exactly but an expansion is made not only in covariant derivatives but also in powers of curvatures \(R_{\mu\nu\alpha\beta}\) and \(R_{\mu\nu}\), i.e. the curvatures are treated perturbatively. Thereby the validity of this approximation for the effective action is limited to small curvatures \(R_{\mu\nu}, R_{\mu\nu\alpha\beta} \ll Q\). Such an expansion is called ‘expansion of the effective action in covariant derivatives’. Without the potential term \((Q=0)\) the effective potential in such a scheme is trivial. Hence we stress here once again, that the effective potential in our definition contains, in fact, much more information than the usual effective potential does when using the ‘expansion in covariant derivatives’. As a matter of fact, what we mean is the low-energy limit of the effective action formulated in a covariant way.

Note that the conditions (2.1) are local. They determine the geometry of the locally symmetric spaces. However, the manifold is globally symmetric one only in the case when it satisfies additionally some global topological restrictions (e.g. it is sufficient if it is simply connected and complete) and the condition (2.1) is valid everywhere, i.e. at any point of the manifold.\(^{25,26}\)

In most physical problems, the situation is radically different. The correct setting of the problem seems to be as follows. The low-energy effective action depends, in general, also essentially on the global topological
properties of the space-time manifold, i.e. on the existence of closed geodesics, boundaries or singularities that might act similarly to boundaries. But, as it was noted above, we do not investigate in this paper the influence of the topology. Therefore, consider a complete noncompact asymptotically flat manifold without boundary that is homeomorphic to $\mathbb{R}^d$. Let a finite not small, in general, domain of the manifold exists that is strongly curved and quasi-homogeneous, i.e. the invariants of the curvature in this region vary very slowly. Then the geometry of this region is locally very similar to that of a symmetric space. However one should have in mind that there are always regions in the manifold where this condition is not fulfilled. This is, first of all, the asymptotic Euclidean region that has small curvature and, therefore, the opposite short-wave approximation is valid.

The general situation in correct setting of the problem is the following. From infinity with small curvature and possibly radiation, where $\mathbb{R} \ll \nabla \mathbb{R}$, we pass on to quasi-homogeneous region where the local properties of the manifold are close to those of symmetric spaces. The size of this region can tend to zero. Then the curvature is nowhere large and the short-wave approximation is valid anywhere. If one tries to extend the limits of such region to infinity, then one has also to analyze the topological properties. The space can be compact or noncompact depending on the sign of the curvature. But first we will come across a coordinate horizon-like singularity, although no one true physical singularity really exists.

This construction can be intuitively imagined as follows. Take the flat Euclidean space $\mathbb{R}^d$, cut out from it a region $M$ with some boundary $\partial M$ and stick to it smoothly along the boundary, instead of the piece cut out, a piece of a curved symmetric space with the same boundary $\partial M$. Such a construction will be homeomorphic to the initial space and at the same time will contain a finite highly curved homogeneous region. Let us stress, that this surgery can be always done smoothly, so that in the region where the curved and the flat regions are joined no discontinuity in the curvature appears that could cause the reflected waves to produce Casimir-like effects. By the way, the exact effective action for a symmetric space differs from the effective action for built construction by a purely topological contribution. This fact seems to be useful when analyzing the effects of topology.

Thus the problem is to calculate the low-energy effective action (2.7), (2.8), i.e. the heat kernel for covariantly constant background. Although this quantity, generally speaking, depends essentially on the topology and other global aspects of the manifold, one can disengage oneself from these effects fixing the trivial topology. Since the asymptotic Schwinger - De Witt expansion does not depend on the topology, one can hold that we thereby sum up all the terms without covariant derivatives in it.

We stress here once again that our analysis is purely local. Of course, there are always special global effects (Casimir-like effects, influence of boundaries, closed geodesics etc.) that do not show up in the local expansion of the heat kernel. The aim of this paper is to study only such situations where the contribution of these effects is small in comparison with local part, i.e. the effective action is approximately given by the integral of the local formula.

In other words the problem is the following. One has to obtain a local covariant function of the invariants of the curvature $\Omega(t)$ (2.9) that would describe adequately the low-energy limit of the trace of the heat kernel and that would, being expanded in curvatures, reproduce all terms without covariant derivatives in the asymptotic expansion of heat kernel, i.e. the HMDS-coefficients $a_k$ (2.3). If one finds such an expression, then one can simply determine the $\zeta$-function (2.5) and, therefore, the low-energy limit of the effective action (2.7), (2.8).

### III. SYMMETRIC SPACES

In this paper we will get the most out of the properties of symmetric spaces. Let us list below some known ideas, facts and formulae about symmetric spaces presented in the form that is most convenient for calculating the heat kernel and the effective action.

First of all, we give some definitions (see Refs. 25 and 26 and Sect. IIILD). A Riemannian locally symmetric space which is simply connected and complete is globally symmetric space (or, simply, symmetric space)$^{26}$. A symmetric space is said to be of compact, noncompact or Euclidean type if all sectional curvatures $K(u,v) = R_{abcd}u^a v^b u^c v^d$ are positive, negative or zero. A direct product of symmetric spaces of compact and noncompact types is called semisimple symmetric space. It is well known$^{25,26}$ that a generic complete simply connected Riemannian symmetric space is a direct product of a flat space and a semisimple symmetric space...
(see also Sect. III.D). Although in the Sect. IY we will need actually only symmetric spaces of compact type the whole exposition of the Sect. III is valid for a more general case of semisimple symmetric spaces.

So, what are the direct consequences of the condition of covariant constancy of the curvature (2.1)?

A. Geometrical framework

First of all, to carry out the calculations in the curved space in a covariant way we need some auxiliary two-point geometric objects, namely the geodetic interval (or world function) $\sigma(x, x')$, defined as one half the square of the length of the geodesic connecting the points $x$ and $x'$, the tangent vectors $\sigma_\mu(x, x') = \nabla_\mu \sigma(x, x')$ and $\sigma_\mu'(x, x') = \nabla_\mu \sigma(x, x')$ to this geodesic at the points $x$ and $x'$ respectively and a frame $e^a_\alpha(x, x')$ which is covariantly constant (parallel) along the geodesic between points $x$ and $x'$, i.e. $\sigma^\mu \nabla_\mu e^a_\alpha = 0$. We denote the frame components of the tangent vector by $\sigma^a(x, x') = g^{ab} e^b_\alpha(x') \nabla_\mu \sigma(x, x')$.

Any tensor $T^a_{\ldots b}$, can be presented then in the form of covariant Taylor series

$$T^a_{\ldots b} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \sigma^{\mu_1} \ldots \sigma^{\mu_n} \left[ \nabla_{(\mu_1} \ldots \nabla_{\mu_n)} T^a_{\ldots b} \right] (x') e^a_\alpha \ldots e^b_\beta. \quad (3.1)$$

Therefrom it is clear that the frame components of a covariantly constant tensor are simply constant.

In the case of covariantly constant curvature one can express the mixed second derivatives of the geodetic interval, i.e. the matrix

$$\sigma^a_b(x, x') = e^a_\mu(x') e^b_\nu(x) \nabla^\mu \nabla_\nu \sigma(x, x'), \quad (3.2)$$

explicitly in terms of the curvature at a fixed point $x'$. Introducing a matrix $K = \{K^a_b(x, x')\}$

$$K^a_b = R^a_{\ldots c b d} \sigma^c \sigma^d, \quad (3.3)$$

one can sum up the Taylor series obtaining a closed form

$$\sigma^a_b = - \left( \frac{\sqrt{K}}{\sin \sqrt{K}} \right)^a_b. \quad (3.4)$$

This expression as well as any other similar expressions below should be always understood as a power series in the curvature.

B. Curvature

Let us consider the Riemann tensor in more detail (we follow here the sects. 3.7-3.10 of the first paper in Ref. 25). The components of the curvature tensor of any Riemannian manifold can be always presented in the form

$$R^a_{\ldots b c d} = \beta_{i k} E^i_{a b} E^k_{c d} \quad (3.5)$$

where $E^i_{a b}, \; (i = 1, \ldots, p; p \leq d(d-1)/2)$, is some set of antisymmetric matrices (2-forms) and $\beta_{i k}$ is some symmetric nondegenerate matrix.

Then define the traceless matrices $D_i = \{D^a_{i b}\}$

$$D^a_{i b} = - \beta_{i k} E^k_{a c} g^{c a} = - D^a_{b i}, \quad (3.6)$$

so that

$$R^a_{\ldots b c d} = - D^a_{a b} E^d_{i c}, \quad R^a_{\ldots b c d} = \beta_{i k} D^a_{i b} D^c_{k d}, \quad (3.7)$$

$$R^a_b = - \beta_{i k} D^a_{i c} D^c_{k b}, \quad R = - \beta_{i k} D^a_{i c} D^c_{k a} = - \beta_{i k} \text{tr}(D_i D_k) \quad (3.8)$$

where $\beta_{i k} = (\beta_{i k})^{-1}$. Because of the curvature identities we have identically

$$D^a_{j [d E^i_{c d]} = 0. \quad (3.9)$$
The matrices \( D_i \) are known to be the generators of the holonomy algebra, i.e., the Lie algebra of the restricted holonomy group, \( \mathcal{H} \) (first paper in Ref. 25, p. 97) of dimension \( \dim \mathcal{H} = p \)

\[
[D_i, D_k] = F^j_{ik} D_j, \quad \text{or} \quad D^a_{ic} D^c_{kb} - D^a_{kc} D^c_{ib} = F^j_{ik} D^a_{jb}. \tag{3.10}
\]

The structure constants \( F^j_{ik} \) of the holonomy algebra are completely determined by these commutation relations and satisfy the Jacobi identities

\[
F^i_{jk} F^j_{mn} = 0, \quad \text{or} \quad [F_i, F_k] = F^j_{ik} F^j_{mn}, \tag{3.11}
\]

where \( F_i = \{ F^k_i \} \) are the generators of the holonomy algebra in adjoint representation. Note that the restricted holonomy group \( H \) is always compact, as it is a subgroup of the orthogonal group (in Euclidean case), and connected.

Now let us rewrite the condition of integrability of the relations (2.1) given simply by the commutator

\[
[\nabla_\mu, \nabla_\nu] R_{\alpha \beta \gamma \delta} = -2 \left\{ R_{\mu \nu \lambda \delta} R^\lambda_{\beta \gamma} + R_{\mu \nu \lambda \gamma} R^\lambda_{\delta \alpha} \right\} = 0 \tag{3.12}
\]

in terms of introduced quantities. It is not difficult to show that it looks like

\[
E^i_{ac} D^c_{bk} - E^i_{bc} D^c_{ak} = E^j_{ab} F^i_{jk}. \tag{3.13}
\]

This equation takes place only in symmetric spaces and is the most important one. It is this equation that makes a Riemannian manifold the symmetric space.

From the eqs. (3.10) and (3.13) we have now

\[
\beta_{ik} F^k_{jm} + \beta_{mk} F^k_{ji} = 0, \quad \text{or} \quad F^T_i = -\beta F_i \beta^{-1}, \tag{3.14}
\]

that means that the adjoint and coadjoint representations of the restricted holonomy group are equivalent.

The eq. (3.13) leads also to some identities for the curvature tensor

\[
R^a_{i[b} R_{c]a} + R^a_{i[c} R_{e]a} = 0, \tag{3.15}
\]

\[
R^a_{i[c} D^c_{eb} = D^a_{ic} R^c_{eb}. \tag{3.16}
\]

that means, in particular, that the Ricci tensor matrix commutes with all matrices \( D_i \) and is, therefore, an invariant matrix of the holonomy algebra.

Actually, eq. (3.13) brings into existence a much wider algebra \( \mathcal{G} \) of dimension \( \dim \mathcal{G} = D = p + d \), in other words it closes this algebra. Really, let us introduce new quantities \( C^A_{BC} = -C^A_{CB}, \ (A = 1, \ldots, D) \)

\[
C^i_{ab} = E^i_{ab}, \quad C^a_{ib} = D^a_{ib}, \quad C^a_{kl} = F^i_{kl}, \tag{3.17}
\]

forming the matrices \( C_A = \{ C^B_{AC} \} = (C_a, C_i) \)

\[
C_a = \begin{pmatrix} 0 & D^b_i \\ E^j_{ac} & 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} D^b_{ia} & 0 \\ 0 & F^j_{ik} \end{pmatrix}, \tag{3.18}
\]

and symmetric nondegenerate matrix

\[
\gamma_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix}. \tag{3.19}
\]

Then one can show, first, that as a consequence of the identities (3.9)-(3.13) the quantities \( C^A_{CB} \) satisfy the Jacobi identities

\[
C^E_{D[A} C^D_{BC]} = 0, \quad \text{or} \quad [C_A, C_B] = C^C_{AB} C_C. \tag{3.20}
\]
and are, therefore, the structure constants of some Lie algebra $G$, the matrices $C_A$ being then the generators of this algebra in adjoint representation. More precisely, the commutation relations have the form

$$[C_a, C_b] = E^i_{ab}C_i, \quad [C_a, C_i] = D^b_{ai}C_b, \quad [C_i, C_k] = F^j_{ik}C_j,$$  \hspace{1cm} (3.21)

And, second, using the definition of $D$-matrices and the eq. (3.14) one can show that the structure constants satisfy also the identity

$$\gamma_{AB}C^B_{CD} + \gamma_{DB}C^B_{CA} = 0, \quad \text{or} \quad C^T_A = -\gamma C_A\gamma^{-1},$$  \hspace{1cm} (3.22)

meaning the equivalence of the adjoint and coadjoint representations of the algebra $G$.

In other words, the Jacobi identities (3.22) are equivalent to the identities (3.12) that the curvature must satisfy in the symmetric space. This means that the set of structure constants $C^A_{BC}$, satisfying the Jacobi identities, determines the curvature tensor of symmetric space $R^a_{b\cdots d}$. Vice versa the structure of the algebra $G$ is completely determined by the curvature tensor of symmetric space at a fixed point $x'$.

Now consider the curvature of background connection $R_{ab}$. One can show analogously to (3.12) that because of the integrability conditions of the eq. (2.1)

$$[\nabla_\mu, \nabla_\nu]R_{\alpha\beta} = [R_{\mu\nu}, R_{\alpha\beta}] - 2R_{\mu\nu\lambda[a}R_{\beta]\lambda] = 0$$  \hspace{1cm} (3.23)

the curvature of background connection $R_{ab}$ in semisimple symmetric spaces must have the form

$$R_{ab} = R_iE^i_{ab},$$  \hspace{1cm} (3.24)

where $E^i_{ab}$ are the same 2-forms and $R_i$ are some matrices forming a representation of the holonomy algebra

$$[R_i, R_k] = F^j_{ik}R_j.$$  \hspace{1cm} (3.25)

In a generic symmetric space with a flat subspace there are additional Abelian contributions to the curvature $R_{ab}$ (3.24) corresponding to the flat directions.

Finally, from (2.1) it follows that the potential term should commute with the curvature $R_{\mu\nu}$

$$[\nabla_\mu, \nabla_\nu]Q = [R_{\mu\nu}, Q] = 0$$  \hspace{1cm} (3.26)

and, therefore, with all the matrices $R_i$

$$[R_i, Q] = 0.$$  \hspace{1cm} (3.27)

C. Isometries

On the covariantly constant background (2.1), i.e. in symmetric spaces, one can easily solve the Killing equations

$$\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} = 0,$$  \hspace{1cm} (3.28)

where $\mathcal{L}_\xi$ means the Lie derivative. Indeed, by differentiating the equation

$$\mathcal{L}_\xi \Gamma^\lambda_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}\lambda^\lambda + R^\lambda_{(\mu[\alpha]\nu)}\xi^\alpha = 0,$$  \hspace{1cm} (3.29)

having in mind $\nabla R = 0$, and symmetrizing the derivatives we get

$$\nabla_{(\mu_1\cdots\nu_{2n})}\lambda^\lambda = (-1)^nR^\lambda_{(\mu_1[\alpha_1]\mu_2[\alpha_2]\cdots\mu_{2n-1}[\alpha_{2n}]}R^\alpha_{\mu_{2n}\mu_{2n+1}\cdots\mu_2:\mu_1]}\xi^\alpha,$$  \hspace{1cm} (3.30)

$$\nabla_{(\mu_1\cdots\nu_{2n+1})}\lambda^\lambda = (-1)^nR^\lambda_{(\mu_1[\alpha_1]\mu_2[\alpha_2]\cdots\mu_{2n}[\alpha_{2n}]}R^\alpha_{\mu_{2n}\mu_{2n+1}\cdots\mu_2:\mu_1]}\nabla_{\mu_{2n+1}}\xi^\alpha.$$  \hspace{1cm} (3.31)
Thereby we have found all the coefficients of the covariant Taylor series (3.1) for the Killing vectors of symmetric spaces. Moreover, one can now sum it up obtaining a closed form

$$\xi^\mu(x) = e^\mu_a \left\{ (\cos \sqrt{K})^a_b \xi^b(x') - \left( \frac{\sin \sqrt{K}}{\sqrt{K}} \right)^a_b \sigma^c \xi^b_{c, i}(x') \right\}, \quad (3.32)$$

where $\xi^b_{c, i} = e_{[i}^c \xi^b_{\mu, \nu} e^{\nu}_{\mu}.$

Therefore, all Killing vectors at any point $x$ are determined in terms of initial values of the vectors themselves $\xi^b(x')$ and their first derivatives $\xi^b_{c, i}(x')$ at a fixed point $x'$. The set of all Killing vectors $\mathcal{G} = \{\xi_A\}, (A = 1, \ldots, \hat{D}), \dim \hat{G} = \hat{D},$ can be split in two essentially different sets: $\mathcal{M} = \{P_a\}, \dim \mathcal{M} = d,$ with $P_a$ defined by

$$P^\mu_a(x) = e^\mu_b \left( \cos \sqrt{K} \right)^b_c P_c^a(x') \quad (3.33)$$

and $\hat{\mathcal{H}} = \{L_i\}, (i = 1, \ldots, \hat{p}),$ i.e. $\dim \hat{\mathcal{H}} = \hat{p} = \hat{D} - d,$ where

$$L^\mu_i(x) = -e^\mu_b \left( \frac{\sin \sqrt{K}}{\sqrt{K}} \right)^b_a \sigma^c L^a_{i, c}(x'), \quad (3.34)$$

according to the values of their initial parameters

$$P^\mu_a \big|_{x=x'} \neq 0, \quad L^\mu_i \big|_{x=x'} = 0. \quad (3.35)$$

Note, that for a general symmetric space $\hat{p} \neq p$ and, hence, $\hat{D} \neq D$!

The Killing vector fields $\xi_A = \xi^\mu_A \nabla_\mu$ (or $P_a = P^\mu_a \nabla_\mu$ and $L_i = L^\mu_i \nabla_\mu$) (acting on scalar fields) form the Lie algebra of isometries, $\hat{\mathcal{G}}$

$$[\xi_A, \xi_B] = \hat{C}^C_{AB} \xi_C, \quad (3.35a)$$

or, more explicitly,

$$[P_a, P_b] = \hat{E}^{\gamma}_{ab} L_\gamma, \quad [P_a, L_i] = \hat{D}^b_{ai} P_b, \quad [L_i, L_k] = \hat{F}^\mu_{ik} L_\mu, \quad (3.35b)$$

where $\{\hat{C}^C_{AB}\} = \{\hat{E}^{\gamma}_{ab}, \hat{D}^b_{ai}, \hat{F}^\gamma_{ik}\}$ are the structure constants of the algebra of isometries. One sees now that the generators $L_i$ vanishing at the point $x'$ form a subalgebra (3.35b) of the algebra of isometries $\hat{\mathcal{G}}$ (3.35a) called the isotropy algebra, $\hat{\mathcal{H}}$.

In fact, all odd symmetrized derivatives of $P^\mu_a$ and all even symmetrized derivatives of $L^\mu_i$ as well as $L^\mu_i$ themselves vanish at the point $x'$

$$\nabla_\nu P^\mu_a \big|_{x=x'} = \nabla_{(\mu_1 \cdots \nu \mu_{2n+1})} P^\mu_a \big|_{x=x'} = 0, \quad (3.36)$$

$$L^\mu_i \big|_{x=x'} = \nabla_{(\mu_1 \cdots \nu \mu_{2n})} L^\mu_i \big|_{x=x'} = 0. \quad (3.37)$$

All the parameters $P^\mu_a(x')$ are independent and, therefore, there are exactly $d$ such parameters. The maximal number of the parameters $L^\mu_{i, c}(x'')$ is $d(d-1)/2$, since they are antisymmetric, in other words $\dim \hat{\mathcal{H}} \leq d(d-1)/2$. However, they are not independent. This can be seen immediately if one recalls that the equation

$$L_{i, c} R_{\alpha \beta \gamma \delta} = 2 \{L^\sigma_{i, c} R_{\sigma \beta \alpha \gamma} + L^\sigma_{i, \alpha} R_{\beta \sigma \alpha \gamma} \} = 0 \quad (3.38)$$

holds in symmetric spaces. This equation is, actually, the integrability condition for Killing equations (3.26). It imposes strict constraints on the possible initial parameters $L^\mu_{i, c}(x')$. One can show that for the semisimple
symmetric spaces the number of independent parameters \( L^b_{i;c}(x') \) is equal to \( p \), i.e. the dimension of the isotropy algebra \( \hat{\mathcal{H}} \) (3.35b) is equal to the dimension of the holonomy algebra \( \mathcal{H} \) (3.10),
\[
\hat{p} \equiv \dim \hat{\mathcal{H}} = \dim \mathcal{H} \equiv p.
\]
Therefore, the dimension of the algebra of isometries \( \hat{\mathcal{G}} \), i.e. the total number of the Killing vectors, in semisimple symmetric spaces is equal to the dimension of the algebra \( \mathcal{G} \) (3.20) defined in previous Sect. III.B
\[
\hat{D} \equiv \dim \hat{\mathcal{G}} = \dim \mathcal{G} \equiv D.
\]
This means that there is no difference between the ordinary latin indices and the indices with hats. Hence one can omit the hats everywhere. In a symmetric space of general type having a flat subspace there are additional trivial Killing vectors corresponding to flat directions. Therefore, in general,
\[
\dim \mathcal{H} \leq \dim \hat{\mathcal{H}} \leq \frac{d(d-1)}{2}, \quad \dim \mathcal{G} \leq \dim \hat{\mathcal{G}} \leq \frac{d(d+1)}{2}.
\]
The spaces with maximal number of independent isometries, i.e. with \( p = \frac{d(d-1)}{2} \) and \( D = d + p = \frac{d(d+1)}{2} \), are the spaces of constant curvature and only those.

Thus taking into account (3.15) it is evident that one can put
\[
P^a_b(x') = \delta^a_b, \quad L^a_{i;b}(x') = -D^a_{ib}, \quad (3.39)
\]
Therefore, the generators of isometries in semisimple symmetric spaces take the form
\[
P_a = P^\mu_a \nabla_\mu = -\left( \sqrt{K} \cot \sqrt{K} \right)^b_a D_b, \quad (3.40)
\]
\[
L_i = L^\mu_i \nabla_\mu = -D^b_{ia} \sigma^a D_b, \quad (3.41)
\]
where
\[
D_a = (\sigma^a)^{-1} c^e_b \nabla_\mu = \frac{\partial}{\partial \sigma^a}. \quad (3.42)
\]
Moreover, one can show\(^{25,26}\) that for semisimple symmetric spaces the isotropy algebra \( \hat{\mathcal{H}} \) (3.35b) is isomorphic to the holonomy algebra \( \mathcal{H} \) (3.10) and the algebra of isometries \( \hat{\mathcal{G}} \) (3.35a) is isomorphic to the algebra \( \mathcal{G} \) (3.20) determined by the curvature tensor. Therefore, the commutation relations (3.35a) and (3.35b) can be rewritten in the form
\[
[\xi_A, \xi_B] = C^C_{AB} \xi_C, \quad (3.43)
\]
and
\[
[P_a, P_b] = E^j_{ab} L_j, \quad [P_a, L_i] = D^b_{ai} P_b, \quad [L_i, L_k] = F^j_{ik} L_j, \quad (3.44)
\]
with the same structure constants as in (3.20) and (3.21) defined by (3.5), (3.6), (3.10) and (3.17). Hence we conclude that the curvature tensor of the semisimple symmetric space completely determines the structure of the isotropy algebra and the algebra of isometries. For a generic symmetric space the curvature determines the algebra of isometries up to an Abelian ideal. Let us stress once again that in the case of semisimple symmetric spaces there is no need to distinguish in notation between the isotropy algebra \( \hat{\mathcal{H}} \) and the holonomy algebra \( \mathcal{H} \) and, therefore, between \( \hat{\mathcal{G}} \) and \( \mathcal{G} \) too.

D. General structure

As we already noted above the simply connected symmetric space \( M \) is isomorphic to the quotient space of the group of isometries by the isotropy subgroup \( M = G/\hat{H}.\)\(^{25}\) It is, in general, reducible, and has the following general structure\(^{25}\)
\[
M = M_0 \times M_s, \quad (3.45a)
\]
\[
M_s = M_+ \times M_-, \quad (3.45b)
\]
where $M_0$, $M_s$, $M_+$ and $M_-$ are the Euclidean, semisimple, compact and noncompact components. The corresponding algebra of isometries is a direct sum of ideals

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_s,$$

$$\mathcal{G}_s = \mathcal{G}_+ \oplus \mathcal{G}_-,$$

where $\mathcal{G}_0$ is an Abelian ideal, $\mathcal{G}_s$ is the semisimple ideal and $\mathcal{G}_+$ and $\mathcal{G}_-$ are the semi-simple compact and noncompact ones.

There is a remarkable duality relation $\ast$ between compact and noncompact objects. For any semisimple algebra of isometries $\mathcal{G} = M + H = \{P_a, L_k\}$ one defines the dual one according to $\mathcal{G}^\ast = iM + H = \{iP_a, L_k\}$, the structure constants of the dual algebra being

$$\{C^A_{\ast BC}\} = \{E^i_{\ast ab}, D^j_{\ast dk}, F^j_{\ast lm}\} = \{-E^i_{ab}, D^j_{dk}, F^j_{lm}\}.$$  

So, the star $\ast$ only changes the sign of $E^i_{ab}$ but does not act on all other structure constants. This means also that the matrix $\gamma$ (3.19) for dual algebra should have the form

$$\gamma^\ast_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix} = \begin{pmatrix} g_{ab} & 0 \\ 0 & -\beta_{ik} \end{pmatrix},$$

and, therefore, the curvature of the dual manifold has the opposite sign

$$R^\ast_{abcd} = -R_{abcd}.$$  

### IY. HEAT KERNEL

It should be noted once more that our analysis in this paper is purely local. (See the discussion in Sect. II.) We are looking for a universal local function of the curvature, $\Omega(t)$, (2.9) that describes adequately the low-energy limit of the heat kernel diagonal (up to ‘global’ nonanalytical effects that are not studied in this paper!). Our minimal requirement is that this function should reproduce all the terms without covariant derivatives of the curvature in the local Schwinger-De Witt asymptotic expansion of the heat kernel, i.e. it should give all the HMDS-coefficients $a_k$ (2.3) for any symmetric space.

It is well known that the HMDS-coefficients have a universal explicit structure, i.e. $a_k$ are scalar polynomials of the curvature of the order $k$ with universal numerical coefficients that do not depend on the particular form of the symmetric space, on the dimension etc. It is obvious that any flat subspaces do not contribute in $a_k$. Moreover, since HMDS-coefficients $a_k$ are analytic in the curvature it is evident that to find this universal structure it is sufficient to consider only symmetric spaces of compact type with positive curvature. Using the factorization property of the heat kernel and the duality between compact and noncompact symmetric spaces we can obtain then the results for the general case by analytical continuation.

That is why below in this paper we consider only the case of symmetric spaces of compact type when the matrices $\beta_{ik}$ and $\gamma_{AB}$ are positive definite. Besides, we restrict ourselves, for simplicity, to the scalar operators, i.e. $\mathcal{R}_{\alpha\beta} = 0$. The general case will be investigated in a future work.

#### A. Heat kernel operator

It is not difficult to show that the metric of the symmetric space can be presented in the form

$$g^{\mu\nu} = \gamma^{AB} \xi^\mu A^\nu = g^{ab} P^\mu_a P^\nu_b + \beta^{ik} L^\mu_i L^\nu_k.$$  

Indeed, by making use of the eqs. (3.7) and recalling the definition of the matrix $K$ (3.3) it is easy to obtain (4.1) using the explicit expressions (3.33), (3.34).
Now having the metric (4.1) we can build the Laplacian for the scalar ($R_{\alpha\beta} = 0$) case

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu = \gamma^{AB} \xi_A \xi_B,$$

where $\xi_A = \xi^\mu_A \nabla_\mu$ and the Killing equation (3.5) has been used.

It is not difficult to show that the Laplacian belongs to the center of the enveloping algebra, i.e. it commutes with all the generators of the algebra

$$[\Box, \xi_A] = 0.$$  

Let us now try to represent the heat kernel in terms of a group average, i.e. let us find a formula like

$$\exp (t \Box) = \int \frac{dk}{\sqrt{2}} \frac{1}{\sqrt{\det (\sinh (k^A C_A A / 2))}} \exp \left\{ -\frac{1}{4t} k^A \gamma_{AB} k^B + \frac{1}{6} R_G t \right\} \exp (k^A \xi_A).$$

We formulate first the answer in form of a theorem and prove it below.

**Theorem 1:**
*For any compact $D$-dimensional Lie group generated by $\xi_A$*

$$[\xi_A, \xi_B] = C^C_{AB} \xi_C$$

*it takes place the operator identity*

$$\exp (t \Box) = (4\pi t)^{-D/2} \int \frac{dk}{\sqrt{2}} \frac{1}{\sqrt{\det (\sinh (k^A C_A A / 2))}} \exp \left\{ -\frac{1}{4t} k^A \gamma_{AB} k^B + \frac{1}{6} R_G t \right\} \exp (k^A \xi_A),$$

*where $\Box = \gamma^{AB} \xi_A \xi_B$, $\gamma^{AB} = (\gamma_{AB})^{-1}$, $\gamma = \det \gamma_{AB}$, $\gamma_{AB}$ is a symmetric nondegenerate positive definite matrix connecting the generators in adjoint $C_A = (C^B_A)$ and co-adjoint $C^T_A$ representations*

$$C^T_A = -\gamma C_A \gamma^{-1},$$

$R_G$ *is the scalar curvature of the group manifold*

$$R_G = -\frac{1}{4} \gamma^{AB} C^C_{AD} C^D_{BC};$$

*and the integration is to be taken over the whole Euclidean space $\mathbb{R}^D$.*

**The proof:**

Let us consider the integral

$$\Psi(t) = \int \frac{dk}{\sqrt{2}} \frac{1}{\sqrt{\det (\sinh (k^A C_A A / 2))}} \exp (k^A \xi_A),$$

*where*

$$\Phi(t|k) = (4\pi t)^{-D/2} \det \left( \frac{\sinh (k^A C_A A / 2)}{k^A C_A A / 2} \right)^{1/2} \exp \left\{ -\frac{1}{4t} k^A \gamma_{AB} k^B + \frac{1}{6} R_G t \right\}.$$  

To prove the theorem we have to show that $\Psi(t) = \exp (t \Box)$, in other words, that it satisfies the operator equation

$$\partial_t \Psi = \Box \Psi$$

with initial condition

$$\Psi(t) \bigg|_{t=0} = 1.$$
First one can show that
\[ \xi_B \exp(k^A \xi_A) = X_B \exp(k^A \xi_A), \tag{4.13} \]
where
\[ X_A = X^M_A(k) \frac{\partial}{\partial k^M}, \tag{4.14} \]
are the left-invariant vector fields on the group that have in canonical coordinates the explicit form
\[ X^M_A(k) = \left( \frac{k^A C_A}{\exp(k^A C_A) - 1} \right)_A^M. \tag{4.15} \]
Therefore, from the definition of the Laplacian we have
\[ \Box \exp(k^A \xi_A) = X_2 \exp(k^A \xi_A), \tag{4.16} \]
\[ X_2 = \gamma^{AB} X_A X_B. \tag{4.17} \]
Then, introducing the metric on the group manifold
\[ g_{MN} = \gamma_{AB} X^{-1A}_M X^{-1B}_N, \tag{4.18} \]
and its determinant
\[ G = \det g_{MN} = \gamma \det X^{-2} = \gamma \det \left( \frac{\sinh (k^A C_A/2)}{k^A C_A/2} \right)^2, \tag{4.19} \]
one can obtain the transposition relation
\[ \left( G^{1/2} X_2 G^{-1/2} \right)^T = X_2. \tag{4.20} \]
Now, making use of (4.9), (4.16) and (4.20) and integrating by parts we obtain
\[ \Box \Psi(t) = \int dk \gamma^{1/2} \exp(k^A \xi_A) \left( G^{1/2} X_2 G^{-1/2} \Phi \right). \tag{4.21} \]
On the other hand, one has from (4.9)
\[ \partial_t \Psi(t) = \int dk \gamma^{1/2} \partial_t \Phi \exp(k^A \xi_A). \tag{4.22} \]
Thus to prove (4.11) we have to show that
\[ \partial_t \Phi = G^{1/2} X_2 G^{-1/2} \Phi. \tag{4.23} \]
Substituting the explicit expression for \( \Phi \)
\[ \Phi(t|k) = \gamma^{-1/4} G^{1/4}(k)(4\pi t)^{-D/2} \exp \left\{ -\frac{1}{4t} k^A \gamma_{AB} k^B + \frac{1}{6} R_G t \right\}, \tag{4.24} \]
and using the relations
\[ X_2 G^{-1/4} = \frac{1}{6} R_G G^{-1/4}, \tag{4.25} \]
and
\[ k^A \frac{\partial}{\partial k^A} G^{-1/4} = \frac{1}{2} (D - \text{tr} X) G^{-1/4}, \tag{4.26} \]
where
\[ \text{tr} X = X^A_A = \text{tr} (k^A C_A \coth (k^A C_A)), \tag{4.27} \]
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that hold on the group manifold, we convince ourselves that the eq. (4.23) is correct. Thereby it is shown
that Ψ(t) really satisfies the eq. (4.11).

Further, from (4.10) it follows immediately

\[ \Phi(t|k)|_{t=0} = \gamma^{-1/2} \delta(k) \]  

(4.28)

and, therefore, the initial condition (4.12). Thus we found Ψ(t) = exp(t□) that proves the theorem.

B. Heat kernel diagonal

So, we have found a very nontrivial representation (4.6) that holds on any compact Lie group. How can
we proceed now with this useful theorem?

First, we can express the scalar curvature of the group manifold in terms of the scalar curvature of the
symmetric space \( R \) and that of the isotropy subgroup \( R_H \)

\[ R_G = -\frac{1}{4} \gamma^{AB} C^C_{AD} C^D_{BC} = \frac{3}{4} R + R_H, \]

(4.29)

where

\[ R_H = -\frac{1}{4} \beta^{ik} F^m_{ul} F^l_{km}. \]

(4.30)

The representation (4.6) is valid for any generators \( \xi_A \), satisfying the commutation relations (4.5), and
so it is also valid for the infinitesimal isometries (3.40), (3.41) of the symmetric space. In this case □ is the
usual Laplacian and exp(t□) is the heat kernel operator.

For further use it is convenient to rewrite the integral (4.6) splitting the integration variables \( k^A = (q^a, \omega^i) \) in the form

\[ \exp(t□) = (4\pi t)^{-D/2} \int dq \, d\omega \eta^{1/2} \beta^{1/2} \det \left( \sinh \left( \frac{(q^a C_a + \omega^i C_i)/2}{(q^a C_a + \omega^i C_i)/2} \right) \right)^{1/2} \]

\[ \times \exp \left\{ -\frac{1}{4t} (q^a g_{ab} q^b + \omega^i \beta^{ik} \omega^k) + \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} \exp \left( q^a P_a + \omega^i L_i \right), \]

(4.31)

where \( \beta = \det \beta_{ik}, \eta = \det g_{ab} \). To get the heat kernel explicitly in coordinate representation we have to act
with the heat kernel operator exp(t□) on the delta-function on \( M \)

\[ \exp(t□)(x, x') = \exp(t□) \delta(x, x') = \int dq \, d\omega \eta^{1/2} \beta^{1/2} \Phi(t|q, \omega) \exp \left( q^a P_a + \omega^i L_i \right) \delta(x, x'). \]

(4.32)

To learn how the operator \( \exp(k^A \xi_A) \) acts on a scalar function \( f(x) \) let us introduce a new function

\[ \phi(s, k, x) = \exp(s k^A \xi_A) f(x). \]

(4.33)

This function satisfies the first order differential equation

\[ \partial_s \phi = k^A \xi_A \phi = k^A \xi^\mu_A(x) \partial_\mu \phi \]

(4.34)

with the initial condition of the form

\[ \phi \bigg|_{s=0} = f(x). \]

(4.35)

It is not difficult to prove that

\[ \phi(s, k, x) = f(x_0(s, k, x)), \]

(4.36)
where $x_0(s, k, x)$ satisfies the equation of characteristics

$$
\frac{dx_0^\mu}{ds} = k^A \xi_\mu^A(x_0)
$$

(4.37)

with initial condition

$$
x_0^\mu \bigg|_{s=0} = x^\mu.
$$

(4.38)

Therefore, we have

$$
\exp \left( k^A \xi_\mu^A \right) \delta(x, x') = \delta(x_0(1, k, x), x').
$$

(4.39)

Consider now the operator integrals of the form we need

$$
I(x, x') = \int dq d\omega \eta^{1/2} \beta^{1/2} Z(q, \omega) \exp \left( q^a P_a + \omega^i L_i \right) \delta(x, x'),
$$

(4.40)

where $Z(q, \omega)$ is some analytic function. Using the eq. (4.39) we have

$$
\exp \left( q^a P_a + \omega^i L_i \right) \delta(x, x') = \delta(x_0(1, q, \omega, x, x'), x') = \eta^{-1/2} J(\omega, x, x') \delta(q - \bar{q}),
$$

(4.41)

where $\bar{q} = \bar{q}(\omega, x, x')$ is to be determined from the equation

$$
x_0(1, \bar{q}, \omega, x, x') = x'.
$$

(4.42)

and $J(\omega, x, x')$ is the Jacobian computed at $x_0 = x'$

$$
J(\omega, x, x') = g'^{-1/2} \eta^{1/2} \det \left| \frac{\partial x_0^\mu}{\partial q^a} \right|_{q=\bar{q}, s=1}^{-1}.
$$

(4.43)

So, we can now simply integrate over $q$ in (4.40) to get

$$
I(x, x') = \int d\omega \beta^{1/2} Z(\bar{q}(\omega, x, x'), \omega) J(\omega, x, x').
$$

(4.44)

If we are interested in coincidence limit then one has to put finally $x = x'$

$$
I(x, x) = \int d\omega \beta^{1/2} Z(\bar{q}(\omega, x, x), \omega) J(\omega, x, x).
$$

(4.45)

Consider now the equation of characteristics at greater length. Making a change of variables

$$
x^\mu \rightarrow \sigma_0^a = \sigma^a(x_0, x') = e^a_{\mu'}(x') \sigma^{\mu'}(x_0, x')
$$

(4.46)

we arrive to the equation of more explicit form

$$
\frac{d\sigma_0^a}{ds} = - \left( \sqrt{K(\sigma_0)} \cot \sqrt{K(\sigma_0)} \right)^a b q^b - \omega^i D^a_{ib} \sigma_0^b.
$$

(4.47)

Let $\sigma_0^a = \sigma_0^a(s, q, \omega, \sigma^b)$ be the solution of the equation (4.47). Then $\bar{q}$ is to be determined from an equation like (4.42)

$$
\sigma_0^a(1, \bar{q}, \omega, \sigma^b) = 0
$$

(4.48)

and

$$
J(\omega, x, x') = \det \left| \frac{\partial \sigma_0^a}{\partial q^b} \right|_{q=\bar{q}, s=1}^{-1},
$$

(4.49)

where it has been taken into account $\det \left( e^a_{\mu'} \right) = g'^{-1/2} \eta^{1/2}$. 
Therefore, we have to find the solution to the equation (4.47) near the zero, i.e. assuming \( \sigma^0_a \) to be small. Moreover, we consider mostly the case when the points \( x \) and \( x' \) are close to each other that means that \( \sigma^a \) is small too. The equation (4.47) near the point \( \sigma^0_a = 0 \) looks like

\[
\frac{d\sigma^0_a}{ds} = -q^a
\]

meaning that the momentums \( q^a \) are of the same small order.

More precisely, we assume

\[
\sigma^0_a \sim \sigma^b \sim q^c \sim \varepsilon \ll 1
\]

and look for a solution of the eq. (4.47) in form of a power series in \( \varepsilon \), i.e. in form of a Taylor series in \( \sigma^a \) and \( q^a \).

In this way one simply obtains up to quadratic terms

\[
\sigma^0_a(s, q, \omega, x, x') = (\exp(-s\omega_i D_i))^a_b \sigma^b + \left( \exp(-s\omega_i D_i) - 1 \right)^a_b q^b + O(\varepsilon^2)
\]

With the same accuracy the solution of the eq. (4.48) is

\[
\bar{q}^a = \left( \frac{\omega^i D_i \exp(-s\omega_i D_i)}{1 - \exp(-s\omega_i D_i)} \right)^a_b \sigma^b + O(\sigma^a^2).
\]

Further, one finds from (4.52)

\[
\det \left| \frac{\partial \sigma^0_a}{\partial q^b} \right|_{q=\tilde{q}, s=1} = \det \left( \frac{\sinh(\omega_i D_i/2)}{\omega^i D_i/2} \right) + O(\sigma^a)
\]

and so, from (4.49)

\[
J(\omega, x, x') = \det \left( \frac{\sinh(\omega_i D_i/2)}{\omega^i D_i/2} \right)^{-1} + O(\sigma^a).
\]

Substituting (4.53) and (4.55) in (4.44) and expanding \( Z(q, \omega) \) we can calculate the integral (4.40) for near points \( x \) and \( x' \) in form of an expansion in \( \sigma^a(x, x') \).

Therefore, we have found, in particular, a useful exact result for coincidence limit (4.45).

**Lemma 2:**

For an analytical function \( Z(q, \omega) \) there holds

\[
I(x, x) = \int dq \, d\omega \eta^{1/2} \beta^{1/2} Z(q, \omega) \exp(q^a P_a + \omega^i L_i) \delta(x, x') \bigg|_{x=x'}
\]

\[
= \int d\omega \beta^{1/2} Z(0, \omega) \det \left( \frac{\sinh(\omega_i D_i/2)}{\omega^i D_i/2} \right)^{-1}
\]

with the operators \( P_a \) and \( L_i \) given by (3.40) and (3.41).

Using the obtained results (4.53), (4.55) and (4.56) and substituting the explicit form of our integral (4.31) we get the heat kernel in coordinate representation

\[
\exp(t \Box)(x, x') = (4\pi t)^{-D/2} \int d\omega \beta^{1/2} H \det \left( \frac{\sinh(\omega C_i/2)}{\omega C_i/2} \right)^{1/2} \det \left( \frac{\sinh(\omega D_i/2)}{\omega^i D_i/2} \right)^{-1}
\]

\[
\times \exp \left\{ -\frac{1}{4t}(\omega^i \beta_{ik} \omega^k + \sigma^a g_{ab} B^c_b(\omega) \sigma^b) + \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} + O(\sigma^a),
\]

(4.57)
where \( B(\omega) = \{ B_\omega^a(\omega) \} \) is a matrix of the form

\[
B(\omega) = \left( \frac{\sinh (\omega^i D_i/2)}{\omega^i D_i/2} \right)^{-2}.
\] (4.58)

Now, from (3.18) it is not difficult to find that

\[
det \left( \frac{\sinh (\omega^i C_i/2)}{\omega^i C_i/2} \right) = \det \left( \frac{\sinh (\omega^i D_i/2)}{\omega^i D_i/2} \right) \det \left( \frac{\sinh (\omega^i F_i/2)}{\omega^i F_i/2} \right).
\] (4.59)

Therefore, the final result after taking into account (4.59) looks like

\[
\exp(t\Box)(x, x') = (4\pi t)^{-D/2} \int d\omega \beta^{1/2} \det \left( \frac{\sinh (\omega^i F_i/2)}{\omega^i F_i/2} \right)^{1/2} \det \left( \frac{\sinh (\omega^i D_i/2)}{\omega^i D_i/2} \right)^{-1/2} \times \exp \left\{ -\frac{1}{4t}(\omega^i \beta_{ik}\omega^k + \sigma^a g_{ab} B_\omega^a(\omega) \sigma^b) + \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} + O(\sigma^a).
\] (4.60)

The coincidence limit of this heat kernel is then simply derived by putting \( x = x' \), i.e. \( \sigma^a = 0 \),

\[
\exp(t\Box)(x, x) = (4\pi t)^{-D/2} \int d\omega \beta^{1/2} \det \left( \frac{\sinh (\omega^i F_i/2)}{\omega^i F_i/2} \right)^{1/2} \det \left( \frac{\sinh (\omega^i D_i/2)}{\omega^i D_i/2} \right)^{-1/2} \times \exp \left\{ -\frac{1}{4t}(\omega^i \beta_{ik}\omega^k) + \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\}
\] (4.61)

Note, that this formula is exact (up to possible nonanalytic topological contributions, see the discussion in sect. 11). This gives a nontrivial example how the heat kernel can be constructed using only the algebraic properties of the isometries of the symmetric space.

One can derive an alternative nontrivial formal representation of this result. Substituting the equation

\[
(2\pi)^{-p/2} \beta^{1/2} \exp \left( -\frac{1}{4t} \omega^i \beta_{ik}\omega^k \right) = (2\pi)^{-p} \int dp \exp \left( ip_k \omega^k - tp_k \beta^{nk} p_n \right)
\] (4.62)

into the integral (4.61) and integrating over \( \omega \) we obtain

\[
\exp(t\Box)(x, x) = (4\pi t)^{-d/2} \exp \left\{ \left( \frac{1}{8} R + \frac{1}{6} R_H \right) \right\} \times \int dp \exp \left( -tp_n \beta^{nk} p_k \right) \det \left( \frac{\sinh (\omega^i F_k/2)}{\omega^i F_k/2} \right)^{1/2} \det \left( \frac{\sinh (\omega^i D_k/2)}{\omega^i D_k/2} \right)^{-1/2} \delta(p),
\] (4.63)

where \( \partial^k = \partial/\partial p_k \). Therefrom integrating by parts and changing the integration variables \( p_k \rightarrow \imath t^{-1/2} p_k \) we get finally an expression without any integration

\[
\exp(t\Box)(x, x) = (4\pi t)^{-d/2} \exp \left\{ \left( \frac{1}{8} R + \frac{1}{6} R_H \right) \right\} \times \det \left( \frac{\sinh (\imath \partial^k F_k/2)}{\sqrt{\imath \partial^k F_k/2}} \right)^{1/2} \det \left( \frac{\sinh (\imath \partial^k D_k/2)}{\sqrt{\imath \partial^k D_k/2}} \right)^{-1/2} \exp \left( p_n \beta^{nk} p_k \right) \bigg|_{p=0}.
\] (4.64)

This formal solution should be understood as a power series in the derivatives \( \partial^i \) that is well defined and determines the heat kernel asymptotic expansion at \( t \rightarrow 0 \).
C. Heat kernel asymptotics

Using obtained result one can get easily the explicit form of the generating function for HMDS-coefficients (2.9)

\[
\Omega(t|x, x) = (4\pi t)^{-p/2} \int d\omega \beta^{1/2} \det \left( \frac{\sinh (\omega F_i/2)}{\omega F_i/2} \right)^{1/2} \det \left( \frac{\sinh (\omega D_i/2)}{\omega D_i/2} \right)^{-1/2} 
\times \exp \left\{ -\frac{1}{4t} \omega^i \beta_{ik} \omega^k + \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\}
\]

(4.65)

This formula can be used now to generate all HMDS-coefficients \( a_k \) for any symmetric space, i.e. for any space with covariantly constant curvature, simply by expanding it in a power series in \( t \).

Changing the integration variables \( \omega \rightarrow \sqrt{t} \omega \) and introducing a Gaussian averaging over \( \omega \)

\[
\langle f(\omega) \rangle = (4\pi)^{-p/2} \int d\omega \beta^{1/2} \exp \left\{ -\frac{1}{4} \omega^i \beta_{ik} \omega^k \right\} f(\omega)
\]

(4.66)

we get

\[
\Omega(t|x, x) = \exp \left\{ \left( \frac{1}{8} R + \frac{1}{6} R_H \right) t \right\} \left\langle \det \left( \frac{\sinh (\sqrt{t} \omega^i F_i/2)}{\sqrt{t} \omega^i F_i/2} \right)^{1/2} \det \left( \frac{\sinh (\sqrt{t} \omega^i D_i/2)}{\sqrt{t} \omega^i D_i/2} \right)^{-1/2} \right\}
\]

(4.67)

Using the standard Gaussian averages

\[
<1> = 1 \quad , \quad <\omega^i> = 0 \quad , \quad <\omega^i \omega^k> = \frac{1}{2} \beta^{ik}
\]

\[
<\omega^i_1 \cdots \omega^{i_{2n+1}}> = 0, \quad <\omega^i_1 \cdots \omega^{i_{2n}}> = \left( \frac{2n!}{2^{2n+1} n!} \right) \beta^{i_1 i_2} \cdots \beta^{i_{2n-1} i_{2n}}
\]

(4.68)

one can obtain now all HMDS-coefficients in terms of various foldings of the quantities \( D^a_{\alpha\beta} \) and \( F^j_{ik} \) with the help of matrix \( \beta^{ik} \). All these quantities are curvature invariants and can be expressed directly in terms of Riemann tensor. Thereby one finds all covariantly constant terms in all HMDS-coefficients in manifestly covariant way. We are going to obtain the explicit formulae in a further work.

Y. CONCLUDING REMARKS

In present paper we continued the study of the heat kernel that we conducted in our papers (Ref. 4,10,15,21). Here we have discussed some ideas connected with the point that was left aside in previous papers, namely, the problem of calculating the low-energy limit of the effective action in quantum gravity. We have analyzed in detail the status of the low-energy limit in quantum gravity and stressed the central role playing by the Lie group of isometries that naturally appears when generalizing consistently the low-energy limit to curved space.

We have proposed a promising, to our mind, approach for calculating the low-energy heat kernel and realized, thereby, the idea of partial summation of the terms without covariant derivatives in local asymptotic expansion for computing the effective action that was suggested in Ref. 2,4.

Of course, there are left many unsolved problems. First of all, one has to obtain explicitly the covariantly constant terms in HMDS-coefficients. This would be the opposite case to the high-derivative approximation\(^\text{15,16}\) and can be of certain interest in mathematical physics. Then, we still do not know how to calculate the low-energy heat kernel in general case of covariantly constant curvatures, i.e. when all background curvatures ( \( \mathcal{R} = \{ R_{\mu\nu\alpha\beta}, \mathcal{R}_{\mu\nu}, Q \} \) ) are present. Besides, it is not perfectly clear how to do
the analytical continuation of Euclidean low-energy effective action to the space of Lorentzian signature for obtaining physical results.

Let us make a final remark concerning the relation of our work to that of previous authors who seems to treat almost the same problem (see the review paper of Camporesi in Ref. 22 and references therein and Ref. 27). What we have been trying to do in present paper is rather different from what the other authors did. These are the global topological problems and effects that are of prime interest in those papers. The authors of those papers make use of the techniques of geometric analysis on homogenous spaces with emphasis on exact results. That is why only very special concrete examples of symmetric spaces (group manifolds, spheres, rank-one symmetric spaces, split-rank symmetric spaces etc.) which allow to obtain closed formulas were considered. The results obtained in this way are presented in terms of the root vectors and their multiplicities. The complexity of the method depends critically on the rank of symmetric space. As far as we know the explicit results for the heat kernel are obt ained for rank-one and for some rank-two symmetric spaces.

We are interested, in contrary, first of all in local effects of strongly curved approximately homogeneous manifolds. Therefore, our approach is thought of only as a framework for a perturbation theory in non-homogeneity. In typical physical problems we need rather general approximation scheme instead of exact exceptional results. The point is we need the effective action as a functional of a generic metric which could be varied to obtain the physical currents.

There is, of course, the difficult question, whether the global effects might be neglected in comparison with local ones. This question is open. We can only say that if it is the case, i.e. if the local effects are dominant, then the heat kernel is given by explicit covariant formulas obtained in sect. IV.

Y. NOTE ADDED

After this paper was completed we became aware of the similar results on heat kernel in symmetric spaces by Fegan (Ref. 27). Though they were obtained in completely different rather geometrical setting incorporating the nontrivial global topology, one would, due to intrinsic locality of the heat kernel expansion, expect that the two expressions, i.e ours and that of Ref. 27, should coincide under an appropriate representation of the special functions obtained in Ref. 27.

Another comment concerns the meaning of the effective potential. If the symmetry in question is that of Euclidean space (that is not the case, in general!) our expansion should reduce to the quasi-local expansion of Brown and Duff$^{28}$, which was extended to curved spaces by Hu and O’Connor.$^{29}$ Therefore, one might consider our work as an extension of the quasi-local expansion to a symmetric space and quasi-homogenous setting.

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