COMPLETELY MULTIPLICATIVE FUNCTIONS TAKING
VALUES IN \(-1, 1\)

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Abstract. Define the Liouville function for \(A\), a subset of the primes \(P\), by \(\lambda_A(n) = (-1)^{\Omega_A(n)}\) where \(\Omega_A(n)\) is the number of prime factors of \(n\) coming from \(A\) counting multiplicity. For the traditional Liouville function, \(A\) is the set of all primes. Denote \(L_A(n) := \sum_{k \leq n} \lambda_A(k)\) and \(R_A := \lim_{n \to \infty} \frac{L_A(n)}{n}\).

We show that for every \(\alpha \in [0, 1]\) there is an \(A \subset P\) such that \(R_A = \alpha\). Given certain restrictions on \(A\), asymptotic estimates for \(\sum_{k \leq n} \lambda_A(k)\) are also given. With further restrictions, more can be said. For character–like functions \(\lambda_p\) (\(\lambda_p\) agrees with a Dirichlet character \(\chi\) when \(\chi(n) \neq 0\)) exact values and asymptotics are given; in particular \(\sum_{k \leq n} \lambda_p(k) \ll \log n\).

Within the course of discussion, the ratio \(\phi(n)/\sigma(n)\) is considered.

1. Introduction

Let \(\Omega(n)\) be the number of distinct prime factors in \(n\) (with multiple factors counted multiply). The Liouville \(\lambda\)–function is defined by \(\lambda(n) := (-1)^{\Omega(n)}\).

So \(\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1\) and \(\lambda(2) = \lambda(5) = \lambda(7) = \lambda(8) = -1\). In particular, \(\lambda(p) = -1\) for any prime \(p\). It is well-known (e.g. See §22.10 of [7]) that \(\Omega\) is completely additive, i.e., \(\Omega(mn) = \Omega(m) + \Omega(n)\) for any \(m, n\) and hence \(\lambda\) is completely multiplicative, i.e., \(\lambda(mn) = \lambda(m)\lambda(n)\) for all \(m, n \in \mathbb{N}\). It is interesting to note that on the set of square-free positive integers \(\lambda(n) = \mu(n)\), where \(\mu\) is the Möbius function. In this respect, the Liouville \(\lambda\)–function can be thought of as an extension of the Möbius function.

Similar to the Möbius function, many investigations surrounding the \(\lambda\)–function concern the summatory function of initial values of \(\lambda\); that is, the sum \(L(x) := \sum_{n \leq x} \lambda(n)\).

Historically, this function has been studied by many mathematicians, including Liouville, Landau, Pólya, and Turán. Recent attention to the summatory function...
of the Möbius function has been given by Ng \[13, 14\]. Larger classes of completely multiplicative functions have been studied by Granville and Soundararajan \[4, 5, 6\].

One of the most important questions is that of the asymptotic order of $L(x)$; more formally, the question is to determine the smallest value of $\vartheta$ for which

$$\lim_{x \to \infty} \frac{L(x)}{x^{\vartheta}} = 0.$$ 

It is known that the value of $\vartheta = 1$ is equivalent to the prime number theorem \[11, 12\] and that $\vartheta = \frac{1}{2} + \varepsilon$ for any arbitrarily small positive constant $\varepsilon$ is equivalent to the Riemann hypothesis \[2\] (The value of $\frac{1}{2} + \varepsilon$ is best possible, as $\limsup_{x \to \infty} L(x)/\sqrt{x} > .061867$, see Borwein, Ferguson, and Mossinghoff \[3\]). Indeed, any result asserting a fixed $\vartheta \in (\frac{1}{2}, 1)$ would give an expansion of the zero-free region of the Riemann zeta function, $\zeta(s)$, to $\Re(s) \geq \vartheta$.

Unfortunately, a closed form for determining $L(x)$ is unknown. This brings us to the motivating question behind this investigation: are there functions similar to $\lambda$, so that the corresponding summatory function does yield a closed form?

Throughout this investigation $P$ will denote the set of all primes. As an analogue to the traditional $\lambda$ and $\Omega$, define the Liouville function for $A \subset P$ by

$$\lambda_A(n) = (-1)^{\Omega_A(n)}$$

where $\Omega_A(n)$ is the number of prime factors of $n$ coming from $A$ counting multiplicity. Alternatively, one can define $\lambda_A$ as the completely multiplicative function with $\lambda_A(p) = -1$ for each prime $p \in A$ and $\lambda_A(p) = 1$ for all $p \not\in A$. Every completely multiplicative function taking only $\pm 1$ values is built this way. The class of functions from $\mathbb{N}$ to $\{-1, 1\}$ is denoted $\mathcal{F}(\{-1, 1\})$ (as in \[5\]). Also, define

$$L_A := \sum_{n \leq x} \lambda_A(n) \quad \text{and} \quad R_A := \lim_{n \to \infty} \frac{L_A(x)}{n}.$$ 

In this paper, we first consider questions regarding the properties of the function $\lambda_A$ by studying the function $R_A$. The structure of $R_A$ is determined and it is shown that for each $\alpha \in [0, 1]$ there is a subset $A$ of primes such that $R_A = \alpha$. The rest of this paper considers an extended investigation on those functions in $\mathcal{F}(\{-1, 1\})$ which are character-like in nature (meaning that they agree with a real Dirichlet character $\chi$ at nonzero values). Within the course of discussion, the ratio $\phi(n)/\sigma(n)$ is considered.

2. Properties of $L_A(x)$

Define the generalized Liouville sequence as

$$\mathcal{L}_A := \{\lambda_A(1), \lambda_A(2), \ldots\}.$$ 

Theorem 1. The sequence $\mathcal{L}_A$ is not eventually periodic.

Proof. Towards a contradiction, suppose that $\mathcal{L}_A$ is eventually periodic, say the sequence is periodic after the $M$–th term and has period $k$. Now there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $nk > M$. Since $A \neq \emptyset$, pick $p \in A$. Then

$$\lambda_A(pnk) = \lambda_A(p) \cdot \lambda_A(nk) = -\lambda_A(nk).$$

But $pnk \equiv nk (\text{mod } k)$, a contradiction to the eventual $k$–periodicity of $\mathcal{L}_A$. \[\Box\]

Corollary 1. If $A \subset P$ is nonempty, then $\lambda_A$ is not a Dirichlet character.
Proof. This is a direct consequence of the non–periodicity of $\mathcal{L}_A$. □

To get more acquainted with the sequence $\mathcal{L}_A$, we study the partial sums $L_A(x)$ of $\mathcal{L}_A$, and to study these, we consider the Dirichlet series with coefficients $\lambda_A(n)$. Starting with singleton sets $\{p\}$ of the primes, a nice relation becomes apparent; for $\Re(s) > 1$

\begin{equation}
\frac{(1 - p^{-s})(1 - q^{-s})}{(1 + p^{-s})(1 + q^{-s})} \zeta(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\{p,q\}}(n)}{n^s},
\end{equation}

and for sets $\{p,q\}$,

\begin{equation}
(1 - p^{-s})(1 - q^{-s}) = \sum_{n=1}^{\infty} \frac{\lambda_{\{p,q\}}(n)}{n^s}.
\end{equation}

For any subset $A$ of primes, since $\lambda_A$ is completely multiplicative, for $\Re(s) > 1$ we have

\begin{equation}
\begin{split}
\mathcal{L}_A(s) := \sum_{n=1}^{\infty} \frac{\lambda_A(n)}{n^s} &= \prod_p \left( \sum_{l=0}^{\infty} \frac{\lambda_A(p^l)}{p^ls} \right) \\
&= \prod_{p \in A} \left( \sum_{l=0}^{\infty} \frac{(-1)^l}{p^{ls}} \right) \prod_{p \in A} \left( \sum_{l=0}^{\infty} \frac{1}{p^{ls}} \right) = \prod_{p \in A} \left( \frac{1}{1 + \frac{1}{p}} \right) \prod_{p \in A} \left( \frac{1}{1 - \frac{1}{p}} \right) \\
&= \zeta(s) \prod_{p \in A} \left( \frac{1 - p^{-s}}{1 + p^{-s}} \right).
\end{split}
\end{equation}

This relation leads us to our next theorem, but first let us recall a vital piece of notation from the introduction.

Definition 1. For $A \subset P$ denote

\[ R_A := \lim_{n \to \infty} \frac{\lambda_A(1) + \lambda_A(2) + \ldots + \lambda_A(n)}{n}. \]

The existence of the limit $R_A$ is guaranteed by Wirsing’s Theorem. In fact, Wirsing in [17] showed more generally that every real multiplicative function $f$ with $|f(n)| \leq 1$ has a mean value, i.e, the limit

\[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) \]

exists. Furthermore, in [10] Wintner showed that

\[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \prod_p \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots \right) \left( 1 - \frac{1}{p} \right) \neq 0 \]

if and only if $\sum_{p \mid 1 - f(p)}|1 - f(p)|/p$ converges; otherwise the mean value is zero. This gives the following theorem.

Theorem 2. For the completely multiplicative function $\lambda_A(n)$, the limit $R_A$ exists and

\begin{equation}
R_A = \begin{cases} 
\prod_{p \in A} \frac{p^{-1}}{p+1} & \text{if } \sum_{p \in A} p^{-1} < \infty, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Example 1. For any prime $p$, $R(p) = \frac{p-1}{p+1}$. 

To be a little more descriptive, let us make some notational comments. Denote by \( \mathcal{P}(P) \) the power set of the set of primes. Note that
\[
\frac{p - 1}{p + 1} = 1 - \frac{2}{p + 1}.
\]
Recall from above that \( R : \mathcal{P}(P) \to \mathbb{R} \), is defined by
\[
R_A := \prod_{p \in A} \left( 1 - \frac{2}{p + 1} \right).
\]
It is immediate that \( R \) is bounded above by 1 and below by 0, so that we need only consider that \( R : \mathcal{P}(P) \to [0, 1] \). It is also immediate that \( R_\emptyset = 1 \) and \( R_P = 0 \).

**Remark 1.** For an example of a subset of primes with mean value in \((0, 1)\), consider the set \( K \) of primes defined by
\[
K := \left\{ p_n \in P : p_n = \min_{q > n^3} \{ q \in P \} \text{ for } n \in \mathbb{N} \right\}.
\]
Since there is always a prime in the interval \((x, x + x^{5/8})\) (see Ingham [9]), these primes are well defined; that is, \( p_{n+1} > p_n \) for all \( n \in \mathbb{N} \). The first few values give
\[
K = \{ 11, 29, 67, 127, 223, 347, 521, 733, 1009, 1361, \ldots \}.
\]
Note that
\[
\frac{p_n - 1}{p_n + 1} > \frac{n^3 - 1}{n^3 + 1},
\]
so that
\[
R_k = \prod_{p \in K} \left( \frac{p - 1}{p + 1} \right) \geq \prod_{n=2}^{\infty} \left( \frac{n^3 - 1}{n^3 + 1} \right) = \frac{2}{3}.
\]
Also \( R_K < (11 - 1)/(11 + 1) = 5/6 \), so that
\[
\frac{2}{3} \leq R_K < \frac{5}{6},
\]
and \( R_K \in (0, 1) \).

There are some very interesting and important examples of sets of primes \( A \) for which \( R_A = 0 \). Indeed, results of von Mangoldt [15] and Landau [11, 12] give the following equivalence.

**Theorem 3.** The prime number theorem is equivalent to \( R_P = 0 \).

We may be a bit more specific regarding the values of \( R_A \), for \( A \in \mathcal{P}(P) \). We will show that for each \( \alpha \in (0, 1) \), there is a set of primes \( A \) such that
\[
R_A = \prod_{p \in A} \left( \frac{p - 1}{p + 1} \right) = \alpha.
\]

**Lemma 1.** Let \( p_n \) denote the \( n \)th prime. For all \( k \in \mathbb{N} \), \( R_{[k, \infty)} = 0 \).

**Proof.** Let \( A = P \cap [k, \infty) \). For any \( x \geq k \), we have
\[
\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{1}{p} = \log \log x + O_k(1).
\]
Since this series diverges, so \( R_A = 0 \) by [4]. \( \square \)
Theorem 4. The function \( R : \mathcal{P}(P) \rightarrow [0, 1] \) is surjective. That is, for each \( \alpha \in [0, 1] \) there is a set of primes \( A \) such that \( R_A = \alpha \).

Proof. Note first that \( R_P = 0 \) and \( R_\emptyset = 1 \). To prove the statement for the remainder of the values, let \( \alpha \in (0, 1) \). Then since
\[
\lim_{p \to \infty} R_{\{p\}} = \lim_{p \to \infty} \left(1 - \frac{2}{p} \right) = 1,
\]
there is a minimal prime \( q_1 \) such that
\[
R_{\{q_1\}} = \left(1 - \frac{2}{q_1 + 1} \right) > \alpha
\]
i.e.,
\[
\frac{1}{\alpha} \cdot R_{\{q_1\}} = \frac{1}{\alpha} \left(1 - \frac{2}{q_1 + 1} \right) > 1.
\]
Similarly, for each \( N \in \mathbb{N} \), we may continue in the same fashion, choosing \( q_i > q_{i-1} \) (for \( i = 2 \ldots N \)) minimally, we have
\[
\frac{1}{\alpha} \cdot R_{\{q_1, q_2, \ldots, q_N\}} = \frac{1}{\alpha} \prod_{i=1}^{N} \left(1 - \frac{2}{q_i + 1} \right) > 1.
\]
Now consider
\[
\lim_{N \to \infty} \frac{1}{\alpha} \cdot R_{\{q_1, q_2, \ldots, q_N\}} = \frac{1}{\alpha} \prod_{i=1}^{\infty} \left(1 - \frac{2}{q_i + 1} \right),
\]
where the \( q_i \) are chosen as before. Denote \( A = \{q_i\}_{i=1}^{\infty} \). We know that
\[
\frac{1}{\alpha} \cdot R_A = \frac{1}{\alpha} \prod_{i=1}^{\infty} \left(1 - \frac{2}{q_i + 1} \right) \geq 1.
\]
We claim that \( R_A = \alpha \). To this end, let us suppose to the contrary that
\[
\frac{1}{\alpha} \cdot R_A = \frac{1}{\alpha} \prod_{i=1}^{\infty} \left(1 - \frac{2}{q_i + 1} \right) > 1.
\]
Applying Lemma \( \zeta \) we see that \( P \setminus A \) is infinite (here \( P \) is the set of all primes).

As earlier, since
\[
\lim_{p \to \infty} R_{\{p\}} = \lim_{p \to \infty} \left(1 - \frac{2}{p} \right) = 1,
\]
there is a minimal prime \( q \in A \setminus P \) such that
\[
\frac{1}{\alpha} \cdot R_A \cdot R_{\{q\}} = \frac{1}{\alpha} \left[ \prod_{i=1}^{\infty} \left(1 - \frac{2}{q_i + 1} \right) \right] \cdot \left(1 - \frac{2}{q + 1} \right) > 1.
\]
Since \( q \) is a prime and \( q \notin A \), there is an \( i \in \mathbb{N} \) with \( q_i < q < q_{i+1} \). This contradicts that \( q_{i+1} \) was a minimal choice. Hence
\[
\frac{1}{\alpha} \cdot R_A = \frac{1}{\alpha} \prod_{i=1}^{\infty} \left(1 - \frac{2}{q_i + 1} \right) = 1,
\]
and there is a set \( A \) of primes such that \( R_A = \alpha \). \( \square \)

The following theorem gives asymptotic formulas for the mean value of \( \lambda_A \) if certain condition on the density of \( A \) in \( P \) is assumed.
Theorem 5. Suppose $A$ be a subset of primes with density

\begin{equation}
\sum_{\substack{p \leq x \\ p \in A}} \log p \frac{1}{p} = \frac{1 - \kappa}{2} \log x + O(1)
\end{equation}

and $-1 \leq \kappa \leq 1$.

If $0 < \kappa \leq 1$, then we have

\[ \sum_{n \leq x} \frac{\lambda_A(n)}{n} = c_\kappa (\log x)^\kappa + O(1) \]

and

\[ \sum_{n \leq x} \lambda_A(n) = (1 + o(1)) c_\kappa x (\log x)^{\kappa - 1}, \]

where

\begin{equation}
c_\kappa = \frac{1}{\Gamma(\kappa + 1)} \prod_p \left(1 - \frac{1}{p}\right)^\kappa \left(1 - \frac{\lambda_A(p)}{p}\right)^{-1}.
\end{equation}

In particular,

\[ R_A = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = \begin{cases} c_1 = \prod_{p \in A} \left(\frac{p-1}{p+1}\right) & \text{if } \kappa = 1, \\ 0 & \text{if } 0 < \kappa < 1. \end{cases} \]

Furthermore, $\mathcal{L}_A(s)$ has a pole at $s = 1$ of order $\kappa$ with residue $c_\kappa \Gamma(\kappa + 1)$, i.e.,

\[ \mathcal{L}_A(s) = \frac{c_\kappa \Gamma(\kappa + 1)}{(s - 1)^\kappa} + \psi(s), \quad \Re(s) > 1, \]

for some function $\psi(s)$ analytic on the region $\Re(s) \geq 1$. If $-1 \leq \kappa < 0$, then $\mathcal{L}_A(s)$ has zero at $s = 1$ of order $-\kappa$, i.e.,

\[ \mathcal{L}_A(s) = \frac{\zeta(2s)}{c_\kappa \Gamma(-\kappa + 1)} (s - 1)^{-\kappa} (1 + \varphi(s)) \]

for some function $\varphi(s)$ analytic on the region $\Re(s) \geq 1$ and hence

\[ \mathcal{L}_A(1) = \sum_{n=1}^\infty \frac{\lambda_A(n)}{n} = 0 \]

and

\[ R_A = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = 0. \]

If $\kappa = 0$, then $\mathcal{L}_A(s)$ has no pole nor zero at $s = 1$. In particular, we have

\[ \sum_{n=1}^\infty \frac{\lambda_A(n)}{n} = \alpha \neq 0 \]

and

\[ R_A = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = 0. \]

The proof of Theorem 5 will require the following result.
Theorem 6 (Wirsing). Suppose $f$ is a completely multiplicative function which satisfies

\begin{equation}
\sum_{n \leq x} \Lambda(n)f(n) = \kappa \log x + O(1)
\end{equation}

and

\begin{equation}
\sum_{n \leq x} |f(n)| \ll \log x
\end{equation}

with $0 \leq \kappa \leq 1$ where $\Lambda(n)$ is the von Mangoldt function. Then we have

\begin{equation}
\sum_{n \leq x} f(n) = c_f (\log x)^\kappa + O(1)
\end{equation}

where

\begin{equation}
c_f := \frac{1}{\Gamma(\kappa + 1)} \prod_p \left(1 - \frac{1}{p}\right)^{\kappa} \left(1 - \frac{1}{1 - f(p)}\right)
\end{equation}

where $\Gamma(\kappa)$ is the Gamma function.

Proof. This can be found in Theorem 1.1 at P.27 of [10] by replacing condition (1.89) by (8). \qed

Proof of Theorem 2. Suppose first that $0 < \kappa \leq 1$. We choose $f(n) = \frac{\lambda_A(n)}{n}$ in Wirsing Theorem. Since

\begin{equation}
\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p \leq x} \log \frac{p}{p} + O(1) = \log x + O(1),
\end{equation}

so

\begin{equation}
\sum_{n \leq x} \frac{\Lambda(n)}{n} \lambda_A(n) = \sum_{p \leq x} \frac{\log p}{p} \lambda_A(p) + O \left( \sum_{p' \leq x, l \geq 2} \frac{\log p}{p'} \right)
\end{equation}

\begin{equation}
= \sum_{p \leq x} \frac{\log p}{p} \lambda_A(p) + O \left( \sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{p \leq x} \frac{\log p}{p} \right)
\end{equation}

\begin{equation}
= \sum_{p \leq x} \frac{\log p}{p} \lambda_A(p) + O(1).
\end{equation}

On the other hand, from (5) we have

\begin{equation}
\sum_{p \leq x} \frac{\log p}{p} \lambda_A(p) = \sum_{p \leq x} \frac{\log p}{p} - 2 \sum_{p \leq x, p \notin A} \frac{\log p}{p}
\end{equation}

\begin{equation}
= \kappa \log x + O(1).
\end{equation}

Hence we have

\begin{equation}
\sum_{n \leq x} \frac{\Lambda(n)}{n} \lambda_A(n) = \kappa \log x + O(1)
\end{equation}

and condition (7) is satisfied.
It then follows from (9) and (6) that
\[
\sum_{n \leq x} \frac{\lambda_A(n)}{n} = c_\kappa (\log x)^\kappa + O(1).
\]

From (5), we have
\[
L_A(s + 1) = \sum_{n=1}^{\infty} \frac{\lambda_A(n)}{n^{s+1}} = \int_1^{\infty} y^{-s} d \sum_{n \leq y} \frac{\lambda_A(n)}{n}
\]
\[
= \int_1^{\infty} y^{-s} d \left( c_\kappa (\log y)^\kappa + O(1) \right)
\]
\[
= c_\kappa \kappa \int_1^{\infty} \frac{(\log y)^{\kappa-1}}{y^{s+1}} dy + \int_1^{\infty} y^{-s} dO(1)
\]
\[
= c_\kappa \Gamma(\kappa + 1) s^{-\kappa} + \psi(s)
\]
for \(\Re(s) > 0\) because
\[
\int_1^{\infty} \frac{(\log y)^{\kappa-1}}{y^{s+1}} dy = \Gamma(\kappa) s^{-\kappa}.
\]

Here \(\psi(s)\) is an analytic function on \(\Re(s) \geq 0\).

Therefore, \(L_A(s)\) has a pole at \(s = 1\) of order 0 < \(\kappa \leq 1\). Now from a generalization of the Wiener-Ikehara theorem (e.g. Theorem 7.7 of [1]), we have
\[
\sum_{n \leq x} \lambda_A(n) = (1 + o(1)) c_\kappa \kappa x (\log x)^\kappa - 1
\]
and hence
\[
R_A = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = \begin{cases} c_1 & \text{if } \kappa = 1, \\ 0 & \text{if } 0 < \kappa < 1. \end{cases}
\]

However,
\[
c_1 = \prod_p \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{\lambda_A(p)}{p} \right)^{-1} = \prod_{p \in A} \left( 1 - \frac{1}{1 + p^{-1}} \right).
\]

If \(-1 \leq \kappa < 0\), we denote the complement of \(A\) by \(\overline{A}\). Then we have
\[
L_{\overline{\pi}}(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\overline{\pi}}(n)}{n^s} = \zeta(s) \prod_{p \not\in \overline{A}} \left( 1 - \frac{1}{1 + p^{-s}} \right)
\]
\[
= \frac{\zeta(2s)}{\zeta(s)} \prod_{p \in \overline{A}} \left( 1 + \frac{1}{p^{-s}} \right) = \frac{\zeta(2s)}{L_A(s)}
\]
for \(\Re(s) > 1\). Hence, for \(\Re(s) > 0\), we have
\[
(11) \quad L_{\overline{\pi}}(s)L_A(s) = \zeta(2s).
\]

From (5), we have
\[
\sum_{p \leq x} \frac{\log p}{p} = \sum_{p \in \overline{A}} \frac{\log p}{p} + \sum_{p \leq x} \frac{\log p}{p} = \frac{1 + \kappa}{2} \log x + O(1)
\]
and
\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} \lambda_A(n) = -\kappa \log x + O(1).
\]
We then apply the above case to \( L_A(s) \) and deduce that \( L_A(s) \) has a pole at \( s = 1 \) of order \(-\kappa\), then in view of (11), \( L_A(s) \) has a zero at \( s = 1 \) of order \(-\kappa\), i.e.,

\[
L_A(s) = \frac{\zeta(2s)}{c_{-\kappa} \Gamma(-\kappa + 1)}(s - 1)^{-\kappa}(1 + \varphi(s))
\]

for some function \( \varphi(s) \) analytic on the region \( \mathbb{R}(s) \geq 1 \). In particular, we have

\[
L_A(1) = \sum_{n=1}^{\infty} \frac{\lambda_A(n)}{n} = 0.
\]

This completes the proof of Theorem 5. \( \square \)

Recall that Theorem 4 tells us that any \( \alpha \in [0, 1] \) is a mean value of a function in \( \mathcal{F}([-1, 1]) \). The functions in \( \mathcal{F}([-1, 1]) \) can be put into two natural classes: those with mean value 0 and those with positive mean value.

Asymptotically, those functions with mean value zero are more interesting, and it is in this class which the Liouville \( \lambda \)-function resides, and in that which concerns the prime number theorem and the Riemann hypothesis. We consider an extended example of such functions in Section 4. Before this consideration, we ask some questions about those functions \( f \in \mathcal{F}([-1, 1]) \) with positive mean value.

3. ONE QUESTION TWICE

It is obvious that if \( \alpha \notin \mathbb{Q} \), then \( R_A \neq \alpha \) for any finite set \( A \subset \mathbb{P} \). We also know that if \( A \subset \mathbb{P} \) is finite, then \( R_A \in \mathbb{Q} \).

**Question 1.** Is there a converse to this; that is, for \( \alpha \in \mathbb{Q} \) is there a finite subset \( A \) of \( \mathbb{P} \), such that \( R_A = \alpha \)?

The above question can be posed in a more interesting fashion. Indeed, note that for any finite set of primes \( A \), we have that

\[
R_A = \prod_{p \in A} \frac{p - 1}{p + 1} = \prod_{p \in A} \frac{\phi(p)}{\sigma(p)} = \frac{\phi(z)}{\sigma(z)}
\]

where \( z = \prod_{p \in A} p \), \( \phi \) is Euler’s totient function and \( \sigma \) is the sum of divisors function. Alternatively, we may view the finite set of primes \( A \) as determined by the square–free integer \( z \). In fact, the function \( f \) from the set of square–free integers to the set of finite subsets of primes, defined by

\[
f(z) = f(p_1p_2\cdots p_r) = \{p_1, p_2, \ldots, p_r\}, \quad (z = p_1p_2\cdots p_r)
\]

is bijective, giving a one–to–one correspondence between these two sets.

In this terminology, we ask the question as:

**Question 2.** Is the image of \( \phi(z)/\sigma(z) : \{\text{square–free integers}\} \to \mathbb{Q} \cap (0, 1) \) a surjection?

That is, for every rational \( q \in (0, 1) \), is there a square–free integer \( z \) such that \( \frac{\phi(z)}{\sigma(z)} = q \)? As a start, we have Theorem 4 which gives a nice corollary.

**Corollary 2.** If \( S \) is the set of square–free integers, then

\[
x \in \mathbb{R} : x = \lim_{k \to \infty} \frac{\phi(n_k)}{\sigma(n_k)} \in [0, 1].
\]
Proof. Let \( \alpha \in [0, 1] \) and \( A \) be a subset of primes for which \( R_A = \alpha \). If \( A \) is finite we are done, so suppose \( A \) is infinite. Write
\[
A = \{a_1, a_2, a_3, \ldots \}
\]
where \( a_i < a_{i+1} \) for \( i = 1, 2, 3, \ldots \) and define \( n_k = \prod_{i=1}^{k} a_i \). The sequence \((n_k)\) satisfies the needed limit. \( \square \)

4. The functions \( \lambda_p(n) \)

We now turn our attention to those functions \( F(\{-1, 1\}) \) with mean value 0; in particular, we wish to examine functions for which a sort of Riemann hypothesis holds: functions for which \( L_A(s) = \sum_{n \in \mathbb{N}} \frac{\lambda_A(n)}{n^s} \) has a large zero–free region; that is, functions for which \( \sum_{n \leq x} \lambda_A(n) \) grows slowly.

To this end, let \( p \) be a prime number. Recall that the Legendre symbol modulo \( p \) is defined as
\[
\left( \frac{q}{p} \right) = \begin{cases} 
1 & \text{if } q \text{ is a quadratic residue modulo } p, \\
-1 & \text{if } q \text{ is a quadratic non-residue modulo } p, \\
0 & \text{if } q \equiv 0 \pmod{p}.
\end{cases}
\]

Here \( q \) is a quadratic residue modulo \( p \) provided \( q \equiv x^2 \pmod{p} \) for some \( x \neq 0 \pmod{p} \).

Define the function \( \Omega_p(n) \) to be the number of prime factors, \( q \), of \( n \) with \( \left( \frac{q}{p} \right) = -1 \); that is,
\[
\Omega_p(n) = \# \{ q : q \text{ is a prime, } q | n, \text{ and } \left( \frac{q}{p} \right) = -1 \}.
\]

Definition 2. The modified Liouville function for quadratic non-residues modulo \( p \) is defined as
\[
\lambda_p(n) := (-1)^{\Omega_p(n)}.
\]

Analogous to \( \Omega(n) \), since \( \Omega_p(n) \) counts primes with multiplicities, \( \Omega_p(n) \) is completely additive, and so \( \lambda_p(n) \) is completely multiplicative. This being the case, we may define \( \lambda_p(n) \) uniquely by its values at primes.

Lemma 2. The function \( \lambda_p(n) \) is the unique completely multiplicative function defined by \( \lambda_p(p) = 1 \), and for primes \( q \neq p \) by
\[
\lambda_p(q) = \left( \frac{q}{p} \right).
\]

Proof. Let \( q \) be a prime with \( q | n \). Now \( \Omega_p(q) = 0 \) or 1 depending on whether \( \left( \frac{q}{p} \right) = 1 \) or \(-1 \), respectively. If \( \left( \frac{q}{p} \right) = 1 \), then \( \Omega_p(q) = 0 \), and so \( \lambda_p(q) = 1 \).

On the other hand, if \( \left( \frac{q}{p} \right) = -1 \), then \( \Omega_p(q) = 1 \), and so \( \lambda_p(q) = -1 \). In either case, we have
\[
\lambda_p(q) = \left( \frac{q}{p} \right).
\]

\( \square \)

\( ^{1} \)Note that using the given definition \( \lambda_p(p) = \left( \frac{p}{p} \right) = 1. \)
Hence if \( n = p^k m \) with \( p \nmid m \), then we have

\[
\lambda_p(p^k m) = \left( \frac{m}{p} \right).
\]

Similarly, we may define the function \( \Omega'_p(n) \) to be the number of prime factors \( q \) of \( n \) with \( \left( \frac{q}{p} \right) = 1 \); that is,

\[
\Omega'_p(n) = \# \left\{ q : q \text{ is a prime, } q|n, \text{ and } \left( \frac{q}{p} \right) = 1 \right\}.
\]

Analogous to Lemma 2 we have the following lemma for \( \lambda'_p(n) \) and theorem relating these two functions to the traditional Liouville \( \lambda \)-function.

**Lemma 3.** The function \( \lambda'_p(n) \) is the unique completely multiplicative function defined by \( \lambda'_p(p) = 1 \) and for primes \( q \neq p \), as

\[
\lambda'_p(q) = -\left( \frac{q}{p} \right).
\]

**Theorem 7.** If \( \lambda(n) \) is the standard Liouville \( \lambda \)-function, then

\[
\lambda(n) = (-1)^k \cdot \lambda_p(n) \cdot \lambda'_p(n)
\]

where \( p^k \| n \), i.e., \( p^k|n \) and \( p^{k+1} \nmid n \).

**Proof.** It is clear that the theorem is true for \( n = 1 \). Since all functions involved are completely multiplicative, it suffices to show the equivalence for all primes. Note that \( \lambda(q) = -1 \) for any prime \( q \). Now if \( n = p \), then \( k = 1 \) and

\[
(-1)^1 \cdot \lambda_p(p) \cdot \lambda'_p(p) = (-1) \cdot (1) \cdot (1) = -1 = \lambda(p).
\]

If \( n = q \neq p \), then

\[
(-1)^0 \cdot \lambda_p(q) \cdot \lambda'_p(q) = \left( \frac{q}{p} \right) \cdot \left( -\left( \frac{q}{p} \right) \right) = -\left( \frac{q^2}{p} \right) = -1 = \lambda(q),
\]

and so the theorem is proved. \( \square \)

To mirror the relationship between \( L \) and \( \lambda \), denote by \( L_p(n) \), the summatory function of \( \lambda_p(n) \); that is, define

\[
L_p(n) := \sum_{k=1}^{n} \lambda_p(k).
\]

It is quite immediate that \( L_p(n) \) is not positive\(^2\) for all \( n \) and \( p \). To find an example we need only look at the first few primes. For \( p = 5 \) and \( n = 3 \), we have

\[
L_5(3) = \lambda_5(1) + \lambda_5(2) + \lambda_5(3) = 1 - 1 - 1 = -1 < 0.
\]

Indeed, the next few theorems are sufficient to show that there is a positive proportion (at least 1/2) of the primes for which \( L_p(n) < 0 \) for some \( n \in \mathbb{N} \).

\(^2\)For the traditional \( L(n) \), it was conjectured by Pólya that \( L(n) \geq 0 \) for all \( n \), though this was proven to be a non-trivial statement and ultimately false (See Haselgrove [8]).
Theorem 8. Let  
\[ n = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k \]
be the base \( p \) expansion of \( n \), where \( a_j \in \{0, 1, 2, \ldots, p - 1\} \). Then we have
\[
L_p(n) := \sum_{l=1}^{n} \lambda_p(l) = \sum_{l=1}^{a_0} \lambda_p(l) + \sum_{l=1}^{a_1} \lambda_p(l) + \ldots + \sum_{l=1}^{a_k} \lambda_p(l).
\]
Here the sum over \( l \) is regarded as empty if \( a_j = 0 \).

Instead of giving a proof of Theorem 8 in this specific form, we will prove a more general result for which Theorem 8 is a direct corollary. To this end, let \( \chi \) be a non-principal Dirichlet character modulo \( p \) and for any prime \( q \) let
\[
f(q) := \begin{cases} 1 & \text{if } p = q, \\ \chi(q) & \text{if } p \neq q. \end{cases}
\]
We extend \( f \) to be a completely multiplicative function and get
\[
f(p^lm) = \chi(m)
\]
for \( l \geq 0 \) and \( p \nmid m \).

Theorem 9. Let \( N(n, l) \) be the number of digits \( l \) in the base \( p \) expansion of \( n \). Then
\[
\sum_{j=1}^{n} f(j) = \sum_{l=0}^{p-1} N(n, l) \left( \sum_{m \leq l} \chi(m) \right).
\]

Proof. We write the base \( p \) expansion of \( n \) as
\[
n = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k
\]
where \( 0 \leq a_j \leq p - 1 \). We then observe that, by writing \( j = p^lm \) with \( p \nmid m \),
\[
\sum_{j=1}^{n} f(j) = \sum_{l=0}^{k} \sum_{j=1}^{n} f(j) = \sum_{l=0}^{k} \sum_{m \leq n/p^l} \sum_{(m,p)=1} f(p^lm).
\]
For simplicity, we write
\[ A := a_0 + a_1 p + \ldots + a_l p^l \quad \text{and} \quad B := a_{l+1} + a_{l+2} p + \ldots + a_k p^{k-l-1} \]
so that \( n = A + B p^{l+1} \) in (17). It now follows from (16) and (17) that
\[
\sum_{j=1}^{n} f(j) = \sum_{l=0}^{k-1} \sum_{m \leq a_l + 1} \chi(m) = \sum_{l=0}^{k} \sum_{m \leq a_l} \chi(m) = \sum_{l=0}^{k} \sum_{m \leq A/p^l} \chi(m)
\]
because \( \chi(p) = 0 \) and \( \sum_{m=a+1}^{a+p} \chi(m) = 0 \) for any \( a \). Now since
\[ a_l \leq A/p^l = (a_0 + a_1 p + \ldots + a_l p^l)/p^l < a_l + 1 \]
so we have
\[
\sum_{j=1}^{n} f(j) = \sum_{l=0}^{k} \sum_{m \leq a_l} \chi(m) = \sum_{l=0}^{p-1} N(n, l) \left( \sum_{m \leq l} \chi(m) \right).
\]
This proves the theorem. \( \square \)
In this language, Theorem 8 [8] can be stated as follows.

Corollary 3. If \( N(n, l) \) is the number of digits \( l \) in the base \( p \) expansion of \( n \), then

\[
L_p(n) = \sum_{j=1}^{n} \lambda_p(j) = \sum_{l=0}^{p-1} N(n, l) \left( \sum_{m \leq l} \left( \frac{m}{p} \right) \right).
\]

As an application of this theorem consider \( p = 3 \).

Application 1. The value of \( L_3(n) \) is equal to the number of 1's in the base 3 expansion of \( n \).

Proof. Since \( \left( \frac{1}{3} \right) = 1 \) and \( \left( \frac{4}{3} \right) = 0 \), so if \( n = a_0 + a_13 + a_23^2 + \ldots + a_k3^k \) is the base 3 expansion of \( n \), then the right-hand side of (14) (or equivalently, the right-hand side of (13)) is equal to \( D_3(n) \). The result then follows from Theorem 8 (or equivalently Corollary 3).

Note that \( L_3(n) = k \) for the first time when \( n = 3^0 + 3^1 + 3^2 + \ldots + 3^k \) and is never negative. This is in stark contrast to the traditional \( L(n) \), which is negative more often than not. Indeed, we may classify all \( p \) for which \( L_p(n) \geq 0 \) for all \( n \in \mathbb{N} \).

Theorem 10. The function \( L_p(n) \geq 0 \) for all \( n \) exactly for those odd primes \( p \) for which

\[
\left( \frac{1}{p} \right) + \left( \frac{2}{p} \right) + \ldots + \left( \frac{k}{p} \right) \geq 0
\]

for all \( 1 \leq k \leq p \).

Proof. We first observe from (13) that if \( 0 \leq r < p \), then

\[
\sum_{l=1}^{r} \frac{l}{p} = \sum_{l=1}^{r} \left( \frac{l}{p} \right).
\]

From theorem 8

\[
\sum_{l=1}^{n} \lambda_p(l) = \sum_{l=1}^{a_0} \lambda_p(l) + \sum_{l=1}^{a_1} \lambda_p(l) + \ldots + \sum_{l=1}^{a_k} \lambda_p(l) = \sum_{l=1}^{a_0} \left( \frac{l}{p} \right) + \sum_{l=1}^{a_1} \left( \frac{l}{p} \right) + \ldots + \sum_{l=1}^{a_k} \left( \frac{l}{p} \right)
\]

because all \( a_j \) are between 0 and \( p-1 \). The result then follows.

Corollary 4. For \( n \in \mathbb{N} \), we have

\[
0 \leq L_3(n) \leq \lceil \log_3 n \rceil + 1.
\]

Proof. This follows from Theorem 10, Application 1, and the fact that the number of 1's in the base three expansion of \( n \) is \( \leq \lceil \log_3 n \rceil + 1 \).

As a further example, let \( p = 5 \).

Corollary 5. The value of \( L_5(n) \) is equal to the number of 1's in the base 5 expansion of \( n \) minus the number of 3's in the base 5 expansion of \( n \). Also for \( n \geq 1 \),

\[
|L_5(n)| \leq \lceil \log_5 n \rceil + 1.
\]
Recall from above, that $L_3(n)$ is always nonnegative, but $L_5(n)$ isn’t. Also $L_5(n) = k$ for the first time when $n = 5^0 + 5^1 + 5^2 + \ldots + 5^k$ and $L_5(n) = -k$ for the first time when $n = 3 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + \ldots + 3 \cdot 5^k$.

**Remark 2.** The reason for specific $p$ values in the proceeding two corollaries is that, in general, it’s not always the case that $|L_p(n)| \leq \lfloor \log_p n \rfloor + 1$.

We now return to our classification of primes for which $L_p(n) \geq 0$ for all $n \geq 1$.

**Definition 3.** Denote by $L^+$, the set of primes $p$ for which $L_p(n) \geq 0$ for all $n \in \mathbb{N}$.

We have found, by computation, that the first few values in $L^+$ are

$L^+ = \{3, 7, 11, 23, 31, 47, 59, 71, 79, 83, 103, 131, 151, 167, 191, 199, 239, 251 \ldots \}$.

By inspection, $L^+$ doesn’t seem to contain any primes $p$, with $p \equiv 1 \pmod{4}$. This is not a coincidence, as demonstrated by the following theorem.

**Theorem 11.** If $p \in L^+$, then $p \equiv 3 \pmod{4}$.

**Proof.** Note that if $p \equiv 1 \pmod{4}$, then

$$\left( \frac{a}{p} \right) = \left( \frac{-a}{p} \right)$$

for all $1 \leq a \leq p - 1$, so that

$$\sum_{a=1}^{p-1} \left( \frac{a}{p} \right) = 0.$$

Consider the case that $\left( \frac{(p-1)/2}{p} \right) = 1$. Then

$$\sum_{a=1}^{p-1} \left( \frac{a}{p} \right) = \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) + \left( \frac{(p-1)/2}{p} \right) = \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) + 1,$$

so that

$$\sum_{a=1}^{p-1} \left( \frac{a}{p} \right) = -1 < 0.$$

On the other hand, if $\left( \frac{(p-1)/2}{p} \right) = -1$, then since $\left( \frac{(p-1)/2}{p} \right) = \left( \frac{(p-1)/2+1}{p} \right)$, we have

$$\sum_{a=1}^{p-1} \left( \frac{a}{p} \right) = \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) - \left( \frac{(p-1)/2+1}{p} \right) = \sum_{a=1}^{p-1+1} \left( \frac{a}{p} \right) + 1,$$

so that

$$\sum_{a=1}^{p-1+1} \left( \frac{a}{p} \right) = -1 < 0. \quad \Box$$
5. A Bound for $|L_p(n)|$

Above we were able to give exact bounds on the function $|L_p(n)|$. As explained in Remark 2 this is not always possible, though an asymptotic bound is easily attained with a few preliminary results.

**Lemma 4.** For all $r,n \in \mathbb{N}$ we have $L_p(p^r n) = L_p(n)$.

**Proof.** For $i = 1, \ldots, p-1$ and $k \in \mathbb{N}$, $\lambda_p(kp+i) = \lambda_p(i)$. This relation immediately gives for $k \in \mathbb{N}$ that $L_p(p(k+1) - 1) - L_p(pk) = 0$, since $L_p(p-1) = 0$. Thus

$$L_p(p^r n) = \sum_{k=1}^{p^r-1} \lambda_p(k) = \sum_{k=1}^{p^r-1} \lambda_p(pk) = \sum_{k=1}^{p^{r-1}-1} \lambda_p(p) \lambda_p(k) = \sum_{k=1}^{p^{r-1}-1} \lambda_p(k) = L_p(p^{r-1} n).$$

The lemma follows immediately. □

**Theorem 12.** The maximum value of $|L_p(n)|$ for $n < p^i$ occurs at $n = k \cdot \sigma(p^{i-1})$ with value

$$\max_{n < p^i} |L_p(n)| = i \cdot \max_{n < p} |L_p(n)|,$$

where $\sigma(n)$ is the sum of the divisors of $n$.

**Proof.** This follows directly from Lemma 4. □

**Corollary 6.** If $p$ is an odd prime, then $|L_p(n)| \ll \log n$; furthermore,

$$\max_{n \leq x} |L_p(x)| \asymp \log x.$$
14. Nathan Ng, *The distribution of the summatory function of the Möbius function*, Proc. London Math. Soc. (3) **89** (2004), no. 2, 361–389.

15. Hans Carl Friedrich von Mangoldt, *Beweis der Gleichung* \( \sum_{k=0}^{\infty} \frac{\mu(k)}{k} = 0 \), Proc. Royal Pruss. Acad. of Sci. of Berlin (1897), 835–852.

16. Aurel Wintner, *The Theory of Measure in Arithmetical Semi-Groups*, publisher unknown, Baltimore, Md., 1944.

17. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen. II*, Acta Math. Acad. Sci. Hungar. **18** (1967), 411–467.

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