Local Differential Privacy based Federated Learning for Internet of Things

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Abstract—Internet of Vehicles (IoV) is a promising branch of the Internet of Things. IoV simulates a large variety of crowdsourcing applications such as Waze, Uber, and Amazon Mechanical Turk, etc. Users of these applications report the real-time traffic information to the cloud server which trains a machine learning model based on traffic information reported by users for intelligent traffic management. However, crowdsourcing application owners can easily infer users’ location information, which raises severe location privacy concerns of the users. In addition, as the number of vehicles increases, the frequent communication between vehicles and the cloud server incurs unexpected amount of communication cost. To avoid the privacy threat and reduce the communication cost, in this paper, we propose to integrate federated learning and local differential privacy (LDP) to facilitate the crowdsourcing applications to achieve the machine learning model. Specifically, we propose four LDP mechanisms to perturb gradients generated by vehicles. The Three-Outputs mechanism is proposed which introduces three different output possibilities to deliver a high accuracy when the privacy budget is small. The output possibilities of Three-Outputs can be encoded with two bits to reduce the communication cost. Besides, to maximize the performance when the privacy budget is large, an optimal piecewise mechanism (PM-OPT) is proposed. We further propose a suboptimal mechanism (PM-SUB) with a simple formula and comparable utility to PM-OPT. Then, we build a novel hybrid mechanism by combining Three-Outputs and PM-SUB. Finally, an LDP-FedSGD algorithm is proposed to coordinate the cloud server and vehicles to train the model collaboratively. Extensive experimental results on real-world datasets validate that our proposed algorithms are capable of protecting privacy while guaranteeing the utility.

Index Terms—Local differential privacy, Data analytic, Internet of Things (IoT), Federated learning, Internet of Vehicles (IoV).

I. INTRODUCTION

Most recently, the crowdsourcing potentiality to address the complex problems has been expediated by the Internet connected vehicles which are also referred as Internet of Vehicles (IoV). Moreover, with the development of vehicular technology, more and more such vehicles are incorporating as participants in vehicular crowdsourcing applications. Usually, vehicular crowdsourcing applications such as Waze, Uber, and Amazon Mechanical Turk gather traffic information provided by the vehicles’ owners to support intelligent traffic management using artificial intelligence techniques. However, sending the spatio-temporal data such as the real-time location information to a malicious cloud server exposes vehicle owners’ privacy to extreme vulnerabilities[1]. Because a malicious server may track and monitor vehicle owners using provided location information, which is a severe intrusion of privacy of the vehicle users. In addition, a communication efficient approach is required as the increasing vehicles incur frequent exchange of information between vehicles and the cloud server, which results in an unexpected challenge on the communication technology. To address the aforementioned privacy threat and improve communication efficiency, in this paper, we propose to integrate local differential privacy (LDP) and federated learning (FL) techniques.

Recently, LDP has been applied by some large companies, for example, Apple and Google, as a practical approach for securely sharing data. Under the LDP mechanism, users are responsible for perturbing their data themselves with LDP mechanisms and sending perturbed results to the aggregator. LDP provides plausible deniability to users; thus, an adversary cannot deduce any user’s information regardless of what prior knowledge they have[2]. LDP mechanisms are generally used in the data collecting step in the data analysis task. Data owners are encouraged to add noise to their data before releasing them. There are several options for releasing numeric values under LDP: (i) the classical Laplace mechanism[3] releases data with unbounded noise. The worst-case noise variance is large; thus, the utility of the Laplace mechanism can be greatly improved, as illustrated in Fig.1 (ii) Duchi et al.’s[4] solution perturbs and releases data in the range $[-1, 1]$ with data outside the input domain. It achieves a satisfactory utility when the privacy budget $\epsilon < 2.3$, but its utility is worse than that of the Laplace mechanism when the privacy budget $\epsilon \geq 2.3$. (iii) Wang et al. [7] propose a piecewise mechanism (PM) that releases randomized data with infinite possibilities. The range of outputs of their mechanism is continuous, and thus, it is unsuitable to be encoded by vehicles. Besides, we can reduce the worst-case noise variance through careful design.

Due to the deficiencies of existing solutions, we are motivated to explore a novel mechanism. Quan et al. [8–11] propose a series of staircase mechanisms under centralized differential privacy for optimal utility in terms of the minimum expectation of noise amplitude and power added to the query output considering privacy budgets $\epsilon \rightarrow 0$ and $\epsilon \rightarrow +\infty$. They
prove that the optimal probability density function should be staircase-shaped in the low privacy regime (the privacy budget \( \epsilon \) is large). Kairouz et al. [12] prove that two outputs are optimal if \( \epsilon \to 0 \). However, they neglect the case when the privacy budget is a middle value. Intuitively, to obtain the optimal performance, the number of output possibilities should increase as \( \epsilon \) increases. Duchi et al.’s [6] solution contains two outputs, and its performance is best in the high privacy regime (\( \epsilon \) is small). As \( \epsilon \) increases, a mechanism with three outputs may have a better performance at a specific region. Therefore, we propose Three-Outputs whose three output possibilities can be encoded using two bits. Inspired by PM of Wang et al. [7], we propose PM-OPT that considers two types of probability densities to achieve a higher utility. We also propose PM-SUB to obtain a smaller worst-case noise variance than state-of-the-art mechanisms while maintaining a simple formula. For ease of encoding, we discretize the continuous range of outputs. We prove that the variance after discretization is equal to or worse than that of the continuous case. Furthermore, we develop a novel hybrid mechanism named HM-TP by combining PM-SUB and Three-Outputs. After experimenting with real-world datasets and synthetic datasets, we confirm that our proposed mechanisms outperform existing solutions.

Additionally, FL is an emerging privacy-preserving approach for distributed learning. Users have their own datasets which are never uploaded to the cloud server [3]. Instead, each user continuously performs training based on its private data, and then it sends updates to the server to contribute to the final machine learning model. FedSGD and FedAvg are two commonly used FL algorithms: FedSGD enforces that gradient updates are averaged by the server to make a gradient descent step. While in FedAvg, users send their local model parameter updates to the server to obtain the averaged model parameters update for the global model. If all local users start from the same initialization, FedSGD is equivalent to FedAvg because averaging the gradients is strictly equivalent to averaging the weights themselves. However, FL still introduces various privacy concerns [13][14]; for example, adversaries can launch attacks to recover images from a face recognition system [15]. Thus, we propose to integrate FL with local differential privacy to enable the privacy-preserving FL training.

Contributions. Our contributions can be summarized as follows:

- We propose four local differential privacy mechanisms to defend vehicles against privacy-breaching attacks. Three-Outputs has three output possibilities and a good performance when the privacy budget is small, whereas PM-OPT achieves a better performance when the privacy budget is high. PM-SUB simplifies PM-OPT while maintaining the accuracy. HM-TP combines PM-SUB and Three-Outputs to obtain a higher accuracy.
- We discretize the continuous range of outputs in our proposed mechanisms. Through the discretization post-processing, we enable vehicles to use our proposed mechanisms. In Section VIII we confirm that the discretization post-processing algorithm maintains utility with our experiments, while reducing the communication cost.
- We propose a local differential privacy based federated learning training vehicular crowdsourcing system, and an LDP-FedSGD algorithm by taking advantage of FL and LDP.
- Experimental evaluation of our proposed mechanisms on real-world datasets and synthetic datasets demonstrates that our proposed mechanisms achieve higher accuracy in estimating the mean frequency of the data and performing empirical risk minimization tasks than existing approaches.

Organization. In the following, Section II introduces the preliminaries. Then, we introduce related works in Section III. Section IV shows the system model and the local differential privacy based FederatedSGD algorithm. Section V presents the problem formation. Section VI proposes novel solutions for the single numerical data estimation. Section VII illustrates proposed mechanisms used for multidimensional numerical data estimation. Section VIII demonstrates our experimental results. Section IX extends our proposed mechanisms to the centralized differential privacy, and it is followed by a conclusion in Section X.

II. Preliminaries

In local differential privacy, users complete the perturbation by themselves. To protect users’ privacy, each user runs a random perturbation algorithm \( \mathcal{M} \), and then he sends perturbed results to the aggregator. The privacy budget \( \epsilon \) controls the privacy-utility trade-off, and a higher privacy budget means a lower privacy protection. As a result of this, we define local differential privacy as follows:

**Definition 1.** (Local Differential Privacy.) Let \( \mathcal{M} \) be a randomized function with domain \( X \) and range \( Y \); i.e., \( \mathcal{M} \) maps each element in \( X \) to a probability distribution with sample space \( Y \). For a non-negative \( \epsilon \), the randomized mechanism \( \mathcal{M} \) satisfies \( \epsilon \)-local differential privacy if

\[
\ln \frac{P_{\mathcal{M}}[Y \in S|x]}{P_{\mathcal{M}}[Y \in S|x']} \leq \epsilon, \quad \forall x, x' \in X, \forall S \subseteq Y. \tag{1}
\]

\( P_{\mathcal{M}}[\cdot|\cdot] \) means conditional probability distribution depending on \( \mathcal{M} \). In local differential privacy, the random perturbation is performed by users instead of a centralized aggregator. Centralized aggregator only receives perturbed results, which makes sure that the aggregator is unable to distinguish whether the true tuple is \( x \) or \( x' \) with high confidence (controlled by the privacy budget \( \epsilon \)).

III. Related Work

Recently, local differential privacy has attracted much attention [16][22]. Several mechanisms for numeric data estimation have been proposed [2][5][23]. (i) Dwork et al. [5] proposed the Laplace mechanism which added the Laplace noise to real one-dimensional data directly. The Laplace mechanism was originally used in the centralized differential privacy mechanism, and it can be applied to local differential privacy directly. (ii) Duchi et al. [2] proposed an LDP framework that provided output from \( \{-C, C\} \) where \( C > 1 \). (iii) Wang et
Fig. 1: Different mechanisms’ worst-case noise variance for one-dimensional numeric data versus the privacy budget $\epsilon$.

Duchi et al. [2] proposed PM which offered an output that might contain infinite possibilities in the range of $[-A, A]$. In addition, they applied LDP mechanism to preserve the privacy of gradients generated during machine learning tasks. Both Duchi et al. [2] and Wang et al.’s [7] approaches were extended to the case of multidimensional numerical data.

**Deficiencies of existing solutions.** Fig. 1 shows that the worst-case noise variance in the noisy value returned by Duchi et al.’s [6] solution is smaller than that of Laplace mechanism’s if $\epsilon \leq 2.3$; however, the Laplace mechanism outperforms Duchi et al.’s [6] solution if $\epsilon$ is larger. The worst-case noise variance in PM is smaller than that of Laplace and Duchi et al.’s [6] solution when $\epsilon$ is large. The HM mechanism outperforms other existing solutions by taking advantage of Duchi et al.’s [6] solution when $\epsilon$ is small and PM when $\epsilon$ is large. However, PM and HM’s outputs have infinite possibilities that are hard to encode. We would like to find a mechanism that can improve the utility of existing mechanisms. In addition, we believe there may be a mechanism that retains a high utility and is easy to encode. Based on the above intuition, we propose four novel mechanisms that can be used by vehicles in Section V.

In addition, LDP has been widely used in the research of the IoT [24–28]. For example, Xu et al. [24] integrated deep learning and local differential privacy techniques and applied them to protect users’ privacy in edge computing. They developed an EdgeSanitizer framework that formed a new protection layer against sensitive inference by leveraging a deep learning model to conduct data minimization and obfuscate the learned features with noise. Choi et al. [25] explored the feasibility of applying LDP on ultra-low-power (ULP) systems. They used resampling, thresholding, and a privacy budget control algorithm to overcome the low resolution and fixed point nature of ULPs. He et al. [26] addressed two potential privacy issues induced by the wireless task offloading feature of MEC, i.e., location privacy, and usage pattern privacy, by proposing a constrained Markov decision process (CMDP) based privacy-aware task offloading scheduling algorithm. Their algorithm allowed a mobile device to achieve the best possible delay and energy consumption performance while maintaining a pre-specified level of privacy. Li et al. [27] proposed a privacy-preserving data aggregation scheme for MEC to assist IoT applications with three participants, i.e., a terminal device (TD), an edge server (ES), and a public cloud center (PCC). TDs generated and encrypted data and sent them to the ES, and then the ES submitted the aggregated data to the PCC. The PCC used its private key to recover the aggregated plaintext data. Their scheme guaranteed data privacy of the TDs and provided source authentication and integrity. In addition, their scheme could save half of the communication cost. To protect the privacy of massive data generated from the IoT platforms, Arachchige et al. [28] designed an LDP mechanism named as LATENT for deep learning. The LATENT was implemented by adding a randomization layer between the convolutional module and the fully connected module to perturb data before data left data owners for machine learning services.

Additionally, federated learning is an emerging distributed machine learning paradigm, and it is widely used to address data privacy problem in machine learning [3]. Recently, federated learning is explored extensively in the Internet of Things recently [29]–[33]. Lim et al. [29] surveyed federated learning applications in mobile edge network comprehensively, including algorithms, applications and potential research problems, etc. Hao et al. [30] proposed a differential enhanced federated learning (PEFL) scheme for industrial artificial industry. They use the differential privacy mechanism to protect the privacy of gradients, but we use a stronger privacy-preserving mechanism (LDP) to protect each vehicle’s privacy. Moreover, Lu et al. [31] proposed CLONE which is a collaborative learning framework on the edges for connected vehicles, and it reduces the training time while guaranteeing the prediction accuracy. Different from CLONE, our proposed framework inserts local differential privacy noises to protect the privacy of the uploaded data. Furthermore, Fantacci et al. [32] leverage FL to protect the privacy of mobile edge computing, while Saputra et al. [33] applied FL to predict the energy demand for electrical vehicle networks.

**IV. SYSTEM MODEL AND THE LOCAL DIFFERENTIAL PRIVACY BASED FEDERATEDSGD ALGORITHM**

In this work, we consider a scenario where a number of vehicles are connected with a cloud server as Fig. 2. Each vehicle is responsible for continuously performing training and inference locally based on data that it collects and the model initiated by the cloud server. Local training dataset is never uploaded to the cloud server. After finishing predefined epochs locally, the cloud server calculates the average of uploaded gradients from vehicles and updates the global model with the average.

In addition, we propose a local differential privacy based federated stochastic gradient descent algorithm (LDP-FedSGD) for our proposed system. Details of LDP-FedSGD are given in Algorithm 1. Unlike the FedAvg algorithm, in the FedSGD algorithm, clients (i.e., vehicles) upload updated gradients instead of model parameters to the central aggregator (i.e., cloud server) [34]. However, compared with the standard FedSGD [34], we add our proposed LDP
mechanism proposed in Section VII to prevent the privacy leakage of gradients. Each vehicle locally takes one step of gradient descent on the current model using its local data, and then it perturbs the true gradient with Algorithm 6. The server aggregates and averages the updated gradients from vehicles and then updates the model. To reduce the communication rounds, we separate vehicles into groups, so that the cloud server updates the model after gathering gradient updates from vehicles in a group. In the following sections, we will introduce how we obtain the LDP algorithm in detail.

Algorithm 1: Local Differential Privacy based FederatedSGD (LDP-FedSGD) Algorithm.

1 Server executes:
2 Server initializes the parameter as \( \theta_0 \);
3 for \( i \) from 1 to maximal iteration number do
4     Server sends \( \theta_{i-1} \) to users in group \( G_i \);
5     for each user \( i \) in Group \( G_i \) do
6         UserUpdate(\( i \), \( \Delta L \));
7     Server computes the average of the noisy gradient of group \( G_i \) and updates the parameter from \( \theta_{i-1} \) to \( \theta_i \):
8         \( \theta_i \leftarrow \theta_{i-1} - \eta \cdot \frac{1}{|G_i|} \sum_{i \in G_i} A(\Delta L(\theta_{i-1}; x_i)) \), where \( \eta \) is the learning rate;
9     if \( \theta_i \) and \( \theta_{i-1} \) are close enough or this remains no user which has not participated in the computation then
10        break;
11     end
12 end
13 VehicleUpdate (\( i \), \( \Delta L \));
14 Compute the (true) gradient \( \Delta L(\theta_{i-1}; x_i) \), where \( x_i \) is user \( i \)’s data;
15 Use local differential privacy-compliant algorithm \( A \) to compute the noisy gradient \( A(\Delta(\theta_{i-1}; x_i)) \);

V. PROBLEM FORMATION

Let \( x \) be a user’s true value, and \( Y \) be the perturbed value. Under the perturbation mechanism \( M \), we use \( \mathbb{E}_M[Y|x] \) to denote the expectation of the randomized output \( Y \) given input \( x \). \( \text{Var}_M[Y|x] \) is the variance of output \( Y \) given input \( x \). \( \text{MaxVar}(M) \) denotes the worst-case \( \text{Var}_M[Y|x] \). We are interested in finding a privatization mechanism \( M \) that minimizes \( \text{MaxVar}(M) \) by solving the following constraint minimization problem:

\[
\min_{M} \text{MaxVar}(M),
\quad \text{s.t. Eq. (1),}
\quad \mathbb{E}_M[Y|x] = x, \text{ and }
\quad P_M[Y \in \mathcal{Y}|x] = 1.
\]

The second constraint illustrates that our estimator is unbiased, and the third constraint shows the proper distribution where \( \mathcal{Y} \) is the range of randomized function \( M \). In the following sections, \( M \) is clear from the context, so that we omit the subscript \( M \) for simplicity.

VI. MECHANISMS FOR ESTIMATION OF A SINGLE NUMERIC ATTRIBUTE

To solve the problem in Section V, we propose four local differential privacy mechanisms: Three-Outputs, PM-OPT, PM-SUB, and HM-TP. Fig. 1 compares the worst-case noise variances of existing mechanisms and our proposed mechanisms. Three-Outputs has three discrete output possibilities, which incurs little communication cost because two bits are enough to encode three different outputs. Moreover, it achieves a small worst-case noise variance in the high privacy regime (small privacy budget \( \epsilon \)). However, to maintain a low worst-case noise variance in the low privacy regime (large privacy budget \( \epsilon \)), we propose PM-OPT and PM-SUB. Both of them achieve higher accuracies than Three-Outputs and other existing solutions when the privacy budget \( \epsilon \) is large. Additionally, we discretize their continuous ranges of output for vehicles to encode using a post-processing discretization algorithm. In the following sections, we will explain our proposed four mechanisms and the post-processing discretization algorithm in detail respectively.

A. Three-Outputs Mechanism

Now, we propose a mechanism with three output possibilities named as Three-Outputs which is illustrated in Algorithm 2. Three-Outputs ensures low communication cost while achieving a smaller worst-case noise variance than existing solutions in the high privacy regime (small privacy budget \( \epsilon \)). Duchi et al.’s [6] solution contains two output possibilities, and it outperforms other approaches when the privacy budget is small. However, Kairouz et al. [12] prove that two outputs are not always optimal as \( \epsilon \) increases. By outputing three values instead of two, Three-Outputs improves the performance as the privacy budget increases, which is shown in Fig. 1. When the privacy budget is small, Three-Outputs is equivalent to Duchi et al.’s [6] solution.

For notional simplicity, given a mechanism \( M \), we often write \( P_M[Y = y \mid X = x] \) as \( P_{y \leftarrow x}(M) \) below. We also sometimes omit \( M \) to obtain \( P(Y = y \mid X = x) \) and \( P_{y \leftarrow x} \).

Given a tuple \( x \in [-1, 1] \), Three-Outputs returns a perturbed value \( Y \) that equals \( -C \), 0 or \( C \) with probabilities...
Algorithm 2: Three-Outputs Mechanism for One-Dimensional Numeric Data.

Input: tuple $x \in [-1, 1]$ and privacy parameter $\epsilon$.
Output: tuple $Y \in \{-C, 0, C\}$.

1. Sampling a random variable $u$ with the probability distribution as follows:
   \[
P[u = -1] = P_{-C_{x}} - x, \\
P[u = 0] = P_{0_{x}} - x, \\
P[u = 1] = P_{C_{x}} - x,
\]
   where $P_{-C_{x}}, P_{0_{x}}$ and $P_{C_{x}}$ are given in Eq. (2), Eq. (3) and Eq. (4).
2. if $u = -1$ then
   3. \[Y = -C;\]
3. else if $u = 0$ then
   4. \[Y = 0;\]
4. else
   5. \[Y = C;\]
   6. return $Y$;

Defined by
\[
P_{-C_{x}} = \left\{ \begin{array}{ll}
\frac{1-P_{-0_{x}}}{2} + \frac{1-P_{0_{x}}}{2} - \frac{e^{\epsilon} - 3e^{\frac{\Delta_1}{2\Delta_0}}}{e^{\epsilon} + 1} & \text{if } 0 \leq x \leq 1, \\
\frac{1-P_{-0_{x}}}{2} - \frac{1-P_{0_{x}}}{2} - \frac{e^{\epsilon} - 3e^{\frac{\Delta_1}{2\Delta_0}}}{e^{\epsilon} + 1} & \text{if } -1 \leq x \leq 0,
\end{array} \right.
\]
(2)
\[
P_{C_{x}} = \left\{ \begin{array}{ll}
\frac{1-P_{-0_{x}}}{2} + \frac{e^{\epsilon} - 3e^{\frac{\Delta_1}{2\Delta_0}}}{e^{\epsilon} + 1} & \text{if } 0 \leq x \leq 1, \\
\frac{1-P_{-0_{x}}}{2} + \frac{1-P_{0_{x}}}{2} - \frac{e^{\epsilon} - 3e^{\frac{\Delta_1}{2\Delta_0}}}{e^{\epsilon} + 1} & \text{if } -1 \leq x \leq 0,
\end{array} \right.
\]
(3)
\[
P_{0_{x}} = P_{0_{-0}} + \left( \frac{P_{-0_{x}} - P_{0_{x}}}{} - P_{0_{-0}} \right)x, \quad \text{if } -1 \leq x \leq 1,
\]
(4)

where $P_{0_{-0}}$ is defined by
\[
P_{0_{-0}} := \left\{ \begin{array}{ll}
0, & \text{if } \epsilon < \ln 2, \\
-\frac{1}{6}(-4e^{\frac{\Delta_1}{2\Delta_0}} - 4e^{\frac{\Delta_1}{2\Delta_0}} - 5), & \text{if } \ln 2 \leq \epsilon \leq \epsilon', \\
+2\sqrt{\Delta_0} \cos\left(\frac{\Delta_1}{2\Delta_0}\right) + \frac{1}{2} \arccos\left(-\frac{\Delta_1}{2\Delta_0}\right)), & \text{if } \ln 2 \leq \epsilon \leq \epsilon', \\
\frac{e^{\epsilon}}{e^{\epsilon} + 2}, & \text{if } \epsilon > \epsilon',
\end{array} \right.
\]
(5)
in which
\[
\Delta_0 := e^{4\epsilon} + 14e^{3\epsilon} + 50e^{2\epsilon} - 2e^\epsilon + 25,
\]
\[
\Delta_1 := -2e^{6\epsilon} - 42e^{5\epsilon} - 270e^{4\epsilon} - 404e^{3\epsilon} - 918e^{2\epsilon} + 30e^\epsilon - 250,
\]
and $\epsilon' := \ln\left(\frac{3 + \sqrt{65}}{2}\right) \approx \ln 5.53$.

Next, we will show how we derive the above probabilities. For a mechanism which uses $x \in [-1, 1]$ as the input and only three possibilities $-C, 0, C$ for the output value, it satisfies
\[
\epsilon\text{-LDP: } \frac{P_{C_{x}} - x}{P_{C_{x}} - x}, \quad P_{0_{x}} - x, \quad P_{-C_{x}} - x \in [e^{-\epsilon}, e^{\epsilon}],
\]
(9a)

unbiased estimation:
\[
C \cdot P_{C_{x}} + 0 \cdot P_{0_{x}} + (-C) \cdot P_{-C_{x}} = x,
\]
(9b)
proper distribution:
\[
P_{y_{x}} = 0 \quad \text{and} \quad P_{C_{x}} + P_{0_{x}} + P_{-C_{x}} = 1.
\]
(9c)

To calculate values of $P_{C_{x}}, P_{0_{x}}$ and $P_{-C_{x}}$, we use Lemma 1 below to convert a mechanism $M_1$ satisfying the requirements in (9a) (9b) (9c) to a symmetric mechanism $M_2$. Then, we use Lemma 2 below to transform the symmetric mechanism further to $M_3$ whose worst-case noise variance is smaller than $M_2$'s. Next, we use $P_{0_{-0}}$ to represent other probabilities, and then we prove that we get the minimum variance when $P_{0_{-0}} = e^{-\epsilon}P_{0_{-1}}$ using Lemma 3. Finally, Lemma 4 and Lemma 5 are used to obtain values for $P_{0_{-0}}$ and the worst-case noise variance of Three-Outputs, respectively. Thus, we can obtain values of $P_{C_{x}}, P_{0_{x}}$ and $P_{-C_{x}}$ using $P_{0_{-0}}$. In the following, we will illustrate above processes in detail.

By symmetry, for any $x \in [-1, 1]$, we enforce
\[
\left\{ \begin{array}{l}
P_{C_{x}} = P_{-C_{x}} ; \\
P_{0_{x}} = P_{0_{-x}} ;
\end{array} \right.
\]
(10a)
(10b)
\[P_{0_{-x}} = P_{0_{-0}} x,\]
where Eq. (10b) can be derived from Eq. (10a). The formal justification of Eq. (10a) (10b) is given by Lemma 1 below. Since the input domain $[-1, 1]$ is symmetric, we can transform any mechanism satisfying requirements in (9a) (9b) (9c) to a symmetric mechanism while guaranteeing the worst-case noise variance will not increase in Lemma 1. Thus, we can derive probabilities when $x \in [-1, 1]$ using probabilities when $x \in [0, 1]$ based on the symmetry.

Lemma 1. For a mechanism $M_1$ satisfying the requirements in (9a) (9b) (9c), the following symmetrization process to obtain a mechanism $M_2$ will not increase (i.e., will reduce or not change) the worst-case noise variance, while mechanism $M_2$ still satisfies the requirements in (9a) (9b) (9c).

Symmetrization: For $x \in [-1, 1],$
\[
P_{C_{x}}(M_2) = P_{-C_{x}}(M_2) = \frac{P_{C_{x}}(M_1) + P_{-C_{x}}(M_1)}{2},
\]
(11)
\[
P_{0_{x}}(M_2) = P_{0_{-x}}(M_2) = \frac{P_{0_{x}}(M_1) + P_{0_{-x}}(M_1)}{2},
\]
(12)

Proof. The proof details are given in Appendix A of the online full version (i.e., this paper).

Based on Lemma 1 we define a symmetric mechanism as follows.

Symmetric Mechanism. A mechanism under (9a) (9b) (9c) is called a symmetric mechanism if it satisfies Eq. (10a) (10b). In the following, we only consider the symmetric mechanism $M_2$.

Now, we design probabilities for the symmetric mechanism $M_2$. As $M_2$ satisfies the unbiased estimation which is a linear
relationship, we set probabilities as piecewise linear functions of \( x \) as follows:

**Case 1:** For \( x \in [0, 1] \),
\[
P_{C \leftarrow x} = P_{C \leftarrow 0} + (P_{C \leftarrow 0} - P_{C \leftarrow x})x, \tag{13}
\]
\[
P_{C \leftarrow x} = P_{C \leftarrow 0} - (P_{C \leftarrow 0} - P_{C \leftarrow 1})x, \tag{14}
\]
\[
0_{x} = 1 - P_{C \leftarrow 0} + (P_{C \leftarrow 0} - P_{C \leftarrow 1})x. \tag{15}
\]

**Case 2:** For \( x \in [-1, 0] \),
\[
P_{C \leftarrow x} = P_{C \leftarrow 0} + (P_{C \leftarrow 0} - P_{C \leftarrow 1})x, \tag{16}
\]
\[
P_{C \leftarrow x} = P_{C \leftarrow 0} - (P_{C \leftarrow 0} - P_{C \leftarrow 1})x, \tag{17}
\]
\[
0_{x} = 1 - P_{C \leftarrow 0} - P_{C \leftarrow 0} + (P_{C \leftarrow 0} - P_{C \leftarrow 1})x. \tag{18}
\]

Then, we may assign values to our designed probabilities above. We find that if a symmetric mechanism satisfies Eq. (19a) and Eq. (19b), it obtains a smaller worst-case noise variance. From Lemma 2 below, we enforce
\[
\begin{cases}
  P_{C \leftarrow 0} = e'P_{C \leftarrow 0}, \\  P_{C \leftarrow 1} = e'P_{C \leftarrow 1}.
\end{cases} \tag{19a}
\]

Hence, given a symmetric mechanism \( M_2 \) satisfying Inequality (20), we can transform it to a new symmetric mechanism \( M_3 \) which satisfies Eq. (19a) and Eq. (19b) through processes of Eq. (21) (22) (23) until \( P_{C \leftarrow 0} = e'P_{C \leftarrow 0} \). After transformation, the new mechanism \( M_3 \) achieves a smaller worst-case noise variance than mechanism \( M_2 \). Therefore, we use the new symmetric mechanism to replace \( M_2 \) in the future’s discussion. Details of transformation are in the Lemma 2.

**Lemma 2.** For a symmetric mechanism \( M_2 \), if

\[
P_{C \leftarrow 1}(M_2) < e'P_{C \leftarrow 0}(M_2), \tag{20}
\]

we set a symmetric mechanism \( M_3 \) as follows: For \( x \in [-1, 1] \),
\[
P_{C \leftarrow x}(M_3) = P_{C \leftarrow x}(M_3)
\]
\[
= P_{C \leftarrow x}(M_2) - \frac{eP_{C \leftarrow 1}(M_2) - P_{C \leftarrow 1}(M_2)}{e' - 1}, \tag{21}
\]
\[
P_{C \leftarrow x}(M_3) = P_{C \leftarrow x}(M_3)
\]
\[
= P_{C \leftarrow x}(M_2) - \frac{eP_{C \leftarrow 1}(M_2) - P_{C \leftarrow 1}(M_2)}{e' - 1}, \tag{22}
\]
\[
0_{x}(M_3) = 1 - P_{C \leftarrow 0}(M_3) - P_{C \leftarrow 0}(M_3)
\]
\[
= P_{C \leftarrow 0}(M_2) + \frac{2(eP_{C \leftarrow 1}(M_2) - P_{C \leftarrow 1}(M_2))}{e' - 1}. \tag{23}
\]

Moreover, the mechanism \( M_3 \) has a worst-case noise variance smaller than that of \( M_2 \), while \( M_3 \) still satisfies the requirements in (19a) (19b) (24).

**Proof.** The proof details are given in Appendix E of the online full version [35] (i.e., this paper).

We have proved that the symmetric mechanism \( M_3 \) has a smaller worst-case noise variance than that of mechanism \( M_2 \) in Lemma 2 and then we use mechanism \( M_3 \) to obtain the relation between \( P_{0 \leftarrow 0} \) and \( P_{0 \leftarrow 0} \) to find the minimum variance. From Lemma 3 below, we enforce
\[
P_{0 \leftarrow 0} = e'P_{0 \leftarrow 1}. \tag{24}
\]

Then, we use the following Lemma 3 to obtain the relation between \( P_{0 \leftarrow 0} \) and \( P_{0 \leftarrow 0} \), so that we can obtain \( P_{C \leftarrow x}, P_{0 \leftarrow x} \) and \( P_{C \leftarrow x} \) using \( P_{0 \leftarrow 0} \).

**Lemma 3.** Given \( P_{0 \leftarrow 0} \), the variance of the output given input \( x \) is a strictly decreasing function of \( P_{0 \leftarrow 1} \) and hence is minimized when \( P_{0 \leftarrow 1} = \frac{P_{0 \leftarrow 0}}{e'} \).

**Proof.** The proof details are given in Appendix C of the online full version [35] (i.e., this paper).

Lemma 3 shows that we get the minimum variance when \( P_{0 \leftarrow 1} = \frac{P_{0 \leftarrow 0}}{e'} \). Hence, we replace \( e'P_{0 \leftarrow 1} \) with \( P_{0 \leftarrow 0} \). Then, the variance is equivalent to
\[
\text{Var}[Y | X = x] = \left( \frac{e' + 1}{(e' - 1)(1 - \frac{P_{0 \leftarrow 0}}{e'})^2} \right) \left( 1 - P_{0 \leftarrow 0} + \frac{P_{0 \leftarrow 0} - P_{0 \leftarrow 0}^2}{e'} \right)^2 - x^2. \tag{25}
\]

Complete details for obtaining Eq. (25) are in Appendix C of the online full version [35] (i.e., this paper).

Next, we use Lemma 4 to obtain the optimal \( P_{0 \leftarrow 0} \) in Three-Outputs to achieve the minimum worst-case variance as follows:

**Lemma 4.** The optimal \( P_{0 \leftarrow 0} \) to minimize the \( \max_{x \in [-1, 1]} \text{Var}[Y | x] \) is defined by Eq. (5).

**Proof.** The proof details are given in Appendix E of the online full version [35] (i.e., this paper).

**Remark 1.** Fig. 3 displays how \( P_{0 \leftarrow 0} \) changes with \( e \) in Eq. (5). When the privacy budget \( e \) is small, \( P_{0 \leftarrow 0} = 0 \). Thus, Three-Outputs is equivalent to Duchi et al.’s [6] solution when \( P_{0 \leftarrow 0} = 0 \). However, as the privacy budget \( e \) increases, \( P_{0 \leftarrow 0} \) increases, which means that the probability of outputting true value increases.

Fig. 3: Optimal \( P_{0 \leftarrow 0} \) if the privacy budget \( e \in [0, 8] \).
By summarizing above, we obtain \( P_{C \leftarrow z} \), \( P_{C \leftarrow x} \) and \( P_{0 \leftarrow x} \) from Eq. (2), Eq. (3) and Eq. (4) using \( P_{0 \leftarrow 0} \).

Then, we can calculate the optimal \( P_{0 \leftarrow 0} \) to obtain the minimum worst-case noise variance of Three-Outputs as follows:

**Lemma 5.** The minimum worst-case noise variance of Three-Outputs is obtained when \( P_{0 \leftarrow 0} \) satisfies Eq. (5).

**Proof.** The proof details are given in Appendix F of the online full version [35] (i.e., this paper). \( \square \)

### B. PM-OPT Mechanism

Now, we advocate an optimal piecewise mechanism (PM-OPT) as shown in Algorithm 5 to get a small worst-case variance when the privacy budget is large. As shown in Fig. 1, Three-Outputs's worst-case noise variance is smaller than PM's when the privacy budget \( \epsilon < 3.2 \). But it loses the advantage when the privacy budget \( \epsilon \geq 3.2 \). As the privacy budget increases, Kairouz et al. [12] suggested to send more information using more output possibilities. Besides, we observe that it is possible to improve Wang et al.'s [7] PM to achieve a smaller worst-case noise variance. Thus, inspired by them, we propose an optimal piecewise mechanism named as PM-OPT with a smaller worst-case noise variance than PM.

**Algorithm 3: PM-OPT Mechanism for One-Dimensional Numeric Data under Local Differential Privacy.**

**Input:** tuple \( x \in [-1, 1] \) and privacy parameter \( \epsilon \).

**Output:** tuple \( y \in [-A, A] \).

1. Value \( t \) is calculated in the Eq. (30);
2. Sample \( u \) uniformly at random from \([0, 1] \);
3. if \( u < \frac{e^{-0.5\epsilon}}{\sqrt{2}} \) then
4. Sample \( Y \) uniformly at random from \([L(\epsilon, x, t), R(\epsilon, x, t)] \);
5. else
6. Sample \( Y \) uniformly at random from \([ -A, L(\epsilon, x, t)] \cup (R(\epsilon, x, t), A] \);
7. return \( Y \);

For a true input \( x \in [-1, 1] \), the probability density function of the randomized output \( Y \in [-A, A] \) after applying local differential privacy is given by

\[
F[Y = y|x] = \begin{cases} 
   c, & \text{for } y \in [L(\epsilon, x, t), R(\epsilon, x, t)] \text{,} \\
   d, & \text{for } y \in [-A, L(\epsilon, x, t)) \cup (R(\epsilon, x, t), A] \text{.} 
\end{cases} 
\]

where

\[
c = \frac{e^\epsilon t(e^\epsilon - 1)}{2(1 + e^\epsilon)^2}, \\
d = \frac{e^\epsilon t(e^\epsilon - 1)}{2(1 + e^\epsilon)^2}, \\
A = \frac{(e^\epsilon + t)(t + 1)}{t(e^\epsilon - 1)}, \\
L(\epsilon, x, t) = \frac{(e^\epsilon + t)(tx - 1)}{t(e^\epsilon - 1)}, \\
R(\epsilon, x, t) = \frac{(e^\epsilon + t)(tx + 1)}{t(e^\epsilon - 1)}, \\
\]

The meaning of \( t \) can be seen from \( \frac{t}{1 + e^\epsilon} = \frac{L(\epsilon, x, t)}{R(\epsilon, x, t)} \). When the input is \( x = 1 \), the length of the higher probability density function \( F[Y = y|x] = \frac{e^\epsilon t(e^\epsilon - 1)}{2(1 + e^\epsilon)^2} \) is \( R(\epsilon, 1, t) - L(\epsilon, 1, t) \). \( R(\epsilon, 1, t) \) is the right boundary, and \( L(\epsilon, 1, t) \) is the left boundary. If \( 0 < t < \infty \), we can derive \( \lim_{t \to 0} \frac{t}{1 + e^\epsilon} = -1 \), meaning the right boundary is opposite to the left boundary if \( t \) is close to 0. Since \( \lim_{t \to \infty} \frac{t}{1 + e^\epsilon} = 1 \), it means that the right boundary is equal to the left boundary when \( t \) is close to \( \infty \).

Moreover, Fig. 4 illustrates that the probability density function of Eq. (26) contains three pieces. If \( y \in [L(\epsilon, x, t), R(\epsilon, x, t)] \), the probability density function is equal to \( c \) which is higher than other two pieces \( y \in [-A, L(\epsilon, x, t)] \) and \( y \in (R(\epsilon, x, t), A] \). We calculate the probability of a variable \( Y \) falling in the interval \([L(\epsilon, x, t), R(\epsilon, x, t)] \) as \( f_{L(\epsilon, x, t)}[L(\epsilon, x, t) \leq Y \leq R(\epsilon, x, t)] = \int_{L(\epsilon, x, t)}^{R(\epsilon, x, t)} c \, dY = \frac{e^\epsilon}{t + e^\epsilon} \).

Furthermore, we use the following lemmas to establish how we get the value \( t \) in Eq. (26).

**Lemma 6.** Algorithm 5 achieves \( \epsilon \)-local differential privacy. Given an input value \( x \), it returns a noisy value \( Y \) with
\[ \mathbb{E}[Y|x] = x \text{ and} \]
\[ \text{Var}[Y|x] = \frac{t + 1}{e^\varepsilon - 1} x^2 + \frac{(t + e^\varepsilon)((t + 1)^3 + e^\varepsilon - 1)}{3t^2(e^\varepsilon - 1)^2}. \]

Proof. The proof details are given in Appendix I of the online full version [35] (i.e., this paper).

Thus, when \( x = 1 \), we obtain the worst-case noise variance as follows:
\[ \max_{x \in [-1,1]} \text{Var}[Y|x] = \frac{t + 1}{e^\varepsilon - 1} + \frac{(t + e^\varepsilon)((t + 1)^3 + e^\varepsilon - 1)}{3t^2(e^\varepsilon - 1)^2}. \]

(32)

Then, we obtain the optimal \( t \) in Lemma 7 to minimize Eq. (32).

Lemma 7. The optimal \( t \) for \( \min_t \max_{x \in [-1,1]} \text{Var}[Y|x] \) is Eq. (30).

Proof. By computing the first-order derivative and second-order derivative of \( \min_t \max_{x \in [-1,1]} \text{Var}[Y|x] \), we get the optimal \( t \). The proof details are given in Appendix J of the online full version [35] (i.e., this paper).

C. PM-SUB Mechanism

We propose a suboptimal piecewise mechanism (PM-SUB) to simplify the sophisticated computation of \( t \) in Eq. (3) of PM-OPT, and details of PM-SUB are shown in Algorithm 4.

Algorithm 4: PM-SUB Mechanism for One-Dimensional Numeric Data under Local Differential Privacy.

Input: tuple \( x \in [-1,1] \) and privacy parameter \( \varepsilon \).

Output: tuple \( Y \in [-A, A] \).

1 Sample \( u \) uniformly at random from \([0,1]\);
2 if \( u < \frac{e^\varepsilon}{e^\varepsilon + e}\) then
3 Sample \( Y \) uniformly at random from \([(e^\varepsilon+e^\varepsilon)(xe^\varepsilon-3-1), (e^\varepsilon+e^\varepsilon)(xe^\varepsilon+3-1)]\);
4 else
5 Sample \( Y \) uniformly at random from \([-A, (e^\varepsilon+e^\varepsilon)(xe^\varepsilon-3-1) e^\varepsilon/(e^\varepsilon-1)] \cup [(e^\varepsilon+e^\varepsilon)(xe^\varepsilon+3-1) e^\varepsilon/(e^\varepsilon-1), A] \);
6 return \( Y \); Fig. 1 illustrates that PM-OPT achieves a smaller worst-case noise variance compared with PM, but the parameter \( t \) for PM-OPT in Eq. (30) is complicated to compute. Some vehicles are unable to process the complicated computation. To make \( t \) simple for vehicles to implement, we need to find a simple expression for it while ensuring the mechanism’s performance. Then, we find that Wang et al.’s [7] PM is the case when \( t = e^\varepsilon/2 \). Inspired by PM, \( ln t \) and \( \varepsilon \) can be linearly related. Then, we find that \( ln t \) is close to \( \frac{1}{2} (t \text{ for PM-OPT in Eq. (30)} \), so we can set \( e^\varepsilon/3 \) as \( t \) in Eq. (20) for a new mechanism named as PM-SUB. The probability of a variable \( Y \) falling in the interval \([L(e, x, e^\varepsilon/3), R(e, x, e^\varepsilon/3)] \) is \( \frac{e^\varepsilon}{e^\varepsilon/3+e^\varepsilon} \), and we give the detail of proof in Appendix I.

\[ \mathbb{P}[u = 1] = \left( C \cdot \left( \frac{m u}{m} + 1 \right) - y \right) \cdot \frac{m}{C} \]

2 if \( u = 1 \) then
3 \( Z = C \cdot \left[ \frac{m u}{m} \right]; \)
4 else
5 \( Z = C \cdot \left[ \frac{m u}{m} + 1 \right]; \)
6 return \( Z \);

As shown in Fig. 5, PM-SUB’s worst-case noise variance is close to PM-OPT’s, but it is smaller than PM’s, which can be observed in Fig. 1.

D. Discretization Post-Processing

Both PM-OPT and PM-SUB’s output ranges is \([-1,1] \) which is continuous, so that there are infinite output possibilities given an input \( x \). Thus, it is difficult to encode their outputs for vehicles. Hence, we consider to apply a post-processing process to discretize the continuous output range into finite output possibilities. Algorithm 5 shows our discretization post-processing steps.

Algorithm 5: Discretization Post-Processing.

Input: Perturbed data \( y \in [-C, C] \), and domain \([-C, C] \) is separated into \( 2m \) pieces, where \( m \) is a positive integer.

Output: Discrete data \( Z \).

1 Sample a Bernoulli variable \( u \) such that
2 if \( u = 1 \) then
3 \( Z = C \cdot \left[ \frac{m u}{m} \right]; \)
4 else
5 \( Z = C \cdot \left[ \frac{m u}{m} + 1 \right]; \)
6 return \( Z \);

The idea of Algorithm 5 is as follows. We discretize the range of output into \( 2m \) parts due to the symmetric range \([-C, C] \), and then we obtain \( 2m + 1 \) output possibilities.
After we get a perturbed data $y$, it will fall into one of $2m$ segments. Then, we categorize it to the left boundary or the right boundary of the segment, which resembles sampling a Bernoulli variable.

Next, we explain how we derive probabilities for the Bernoulli variable. Let the original input be $x$. A random variable $Y$ represents the intermediate output after the perturbation and a random variable $Z$ represents the output after the discretization. The range of $Y$ is $[-C, C]$. Because the range of output is symmetric with respect to 0, we discretize both $[-C,0]$ and $[0,C]$ into $m$ parts, where the value of $m$ depends on the user’ requirement. Thus, we discretize $Y$ to $Z$ to take only the following $(2m+1)$ values:

$$\left\{ i \times \frac{C}{m} : \text{integer } i \in \{-m,-m+1, \ldots, m\} \right\}.$$ (34)

When $Y$ is instantiated as $y \in [-C,C]$, we have the following two cases:

1. If $y$ is one of the above $(2m+1)$ values, we set $Z$ as $y$.
2. If $y$ is not one of the above $(2m+1)$ values, and there exist some integer $k \in \{-m,-m+1, \ldots, m-1\}$ such that $\frac{kC}{m} < y < \frac{(k+1)C}{m}$. In fact, this gives $k < \frac{ym}{C} < k+1$, so we can set $k := \lfloor \frac{ym}{C} \rfloor$. Then conditioning on that $Y$ is instantiated as $y$, we set $Z$ as $\frac{kC}{m}$ with probability $\frac{ym}{C} - k$, and as $\frac{(k+1)C}{m}$ with probability $\frac{k+1}{m} - k$, so that the expectation of $Z$ given $Y = y$ equals $y$ (as we will show in Eq. (146), this ensures that the expectation of $Z$ given the original input as $x$ equals $x$).

The following Lemma shows the probability distribution of assigning $y$ with a boundary value in the second case above when the intermediate output $y$ is not one of discrete $(2m+1)$ values.

Lemma 8. After we obtain the intermediate output $y$ after perturbation, we discretize it to a random variable $Z$ equal to $\frac{kC}{m}$ or $\frac{(k+1)C}{m}$ with the following probabilities:

$$\mathbb{P}[Z = z \mid Y = y] = \begin{cases} \frac{k+1 - \frac{ym}{C}}{C}, & \text{if } z = \frac{kC}{m}, \\ \frac{\frac{ym}{C} - k}{C}, & \text{if } z = \frac{(k+1)C}{m}. \end{cases}$$ (35)

Proof. The proof details are given in Appendix of the online full version (i.e., this paper).

After discretization, the worst-case noise variance does not change or get worse proved by Lemma 9 as follows:

Lemma 9. Let local differential privacy mechanism be Mechanism $\mathcal{M}_1$, and discretization algorithm be Mechanism $\mathcal{M}_2$. Let all of output possibilities of Mechanism $\mathcal{M}_1$ be $S_1$, and output possibilities of Mechanism $\mathcal{M}_2$ be $S_2$, $S_2 \subset S_1$. When given input $x$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are unbiased. The worst-case noise variance of Mechanism $\mathcal{M}_2$ is greater than or equal to the worst-case noise variance of Mechanism $\mathcal{M}_1$.

Proof. The proof details are given in Appendix of the online full version (i.e., this paper).

E. HM-TP Mechanism

Fig. 1 shows that Three-Outputs outperforms PM-SUB when the privacy budget $\epsilon$ is small, whereas PM-SUB achieves a smaller variance if the privacy budget $\epsilon$ is large. To fully take advantage of two mechanisms, we combine Three-Outputs and PM-SUB to create a new hybrid mechanism named as HM-TP. Fig. 1 illustrates that HM-TP obtains a lower worst-case noise variance than other solutions.

Hence, HM-TP invokes PM-SUB with probability $\beta$. Otherwise, it invokes Three-Outputs. We define the noisy variance of HM-TP as $\text{Var}_H[Y|\{x\}]$ given inputs $x$ as follows:

$$\text{Var}_H[Y|x] = \beta \cdot \text{Var}_P[Y|x] + (1 - \beta) \cdot \text{Var}_T[Y|x],$$

where $\text{Var}_P[Y|x]$ and $\text{Var}_T[Y|x]$ denote noisy outputs’ variances incurred by PM-SUB and Three-Outputs, respectively. The following lemma presents the value of $\beta$:

Lemma 10. The $\max_{x \in [-1,1]} \text{Var}_H[Y|x]$ is minimized when $\beta$ is Eq. (157). Due to the complicated equation of $\beta$, we put it in the appendix.

Proof. The proof details are given in Appendix of the online full version (i.e., this paper).

Since we have obtained the probability $\beta$, we can calculate the exact expression for the worst-case noise variance in Lemma 11 as follows:

Lemma 11. If $\beta$ satisfies Lemma 10 we obtain the worst-case noise variance of HM-TP as

$$\max_{x \in [-1,1]} \text{Var}_H[Y|x] =$$

$$\begin{cases} \text{Var}_P[Y|x^\ast], & \text{if } 0 < \beta < \frac{2(e^a - a)(e^a - a e^a (e^a + 1)^2)}{2(e^a - a)(e^a + 1)^2}, \\
\max\{\text{Var}_H[Y|0], \text{Var}_H[Y|1]\}, & \text{otherwise}, \end{cases}$$

where $x^\ast := \frac{(\beta - 1) + a (e^a + 1)^2}{2(e^a - a)(e^a + 1)^2}$ and $a = P_{0\rightarrow 0}$ which is defined in Eq. 5.

Proof. The proof details are given in Appendix of the online full version (i.e., this paper).

VII. MECHANISMS FOR ESTIMATION OF MULTIPLE NUMERIC ATTRIBUTES

Now, we consider a case in which the user’s data record contains $d > 1$ attributes. There are three existing solutions to collect multiple attributes: (i) The straightforward approach which collects each attribute with privacy budget $\epsilon/d$. Based on the composition theorem 36, it satisfies $\epsilon$-LDP after collecting of all attributes. But the added noise can be excessive if $d$ is large 5. (ii) Duchi et al.’s 6 solution, which is rather complicated, handles numeric attributes only. (iii) Wang et al.’s 7 solution is the advanced approach that deals with a data tuple containing both numeric and categorical attributes. Their algorithm requires to calculate an optimal $k < d$ based on the single dimensional attribute’s $\epsilon$-LDP mechanism, and a user submits selected $k$ dimensional attributes instead of $d$ dimensions.

Thus, we follow Wang et al.’s 7 idea to extend Section to the case of multidimensional attributes. Algorithm 6
Algorithm 6: Mechanism for Multiple-Dimensional Numeric Attributes.

Input: tuple \( x \in [-1,1]^d \) and privacy parameter \( \epsilon \).
Output: tuple \( Y \in [-A,A]^d \).

1. Let \( Y = \langle 0,0, \ldots, 0 \rangle \);
2. Let \( k = \max\{1, \min\{d, \left\lfloor \frac{\epsilon}{d} \right\rfloor \} \} \);
3. Sample \( k \) values uniformly without replacement from \( \{1,2, \ldots, d\} \);
4. for each sampled value \( j \) do
   5. Feed \( x[t_j] \) and \( \frac{\epsilon}{d} \) as input to \( \text{PM-SUB} \), \( \text{Three-Outputs} \), or \( \text{HM-TP} \), and obtain a noisy value \( y_j \);
   6. \( Y[t_j] = \frac{\epsilon}{d} y_j \);
5. return \( Y \);

shows the pseudo-code of our extension for our \( \text{PM-SUB} \), \( \text{Three-Outputs} \), and \( \text{HM-TP} \). Given a tuple \( x \in [-1,1]^d \), the algorithm returns a perturbed tuple \( Y \) that has non-zero value on \( k \) attributes, where

\[
k = \max\{1, \min\{d, \left\lfloor \frac{\epsilon}{d} \right\rfloor \} \},
\]

and Appendix S of the online full version [35] (i.e., this paper) proves our selected \( k \) is optimal after extending \( \text{PM-SUB} \), \( \text{Three-Outputs} \), and \( \text{HM-TP} \) to support \( d \) dimensional attributes.

Overall, our algorithm for collecting multiple attributes outperforms existing solutions, which is confirmed by our experiments in the Section VIII. But \( \text{Three-Outputs} \) uses only one more bit compared with Duchi et al.’s [6] solution to encode outputs. Moreover, our \( \text{Three-Outputs} \) obtains a higher accuracy in the high privacy regime (where the privacy budget is small) and saves many bits for encoding since \( \text{PM} \) and \( \text{HM} \)’s continuous output range requires infinite bits to encode, whereas \( \text{PM-SUB} \) and \( \text{HM-TP} \)’s advantages are obvious at a large privacy budget. Furthermore, because vehicles cannot encode continuous range, we discretize the continuous range of outputs to discrete outputs. Our experiments in Section VIII confirm that we can achieve similar results to algorithms before discretizing by carefully designing the number of discrete parts. Hence, our proposed algorithms are obviously more suitable for vehicles than existing solutions.

Intuitively, Algorithm 6 requires every user to submit \( k \) attributes instead of \( d \) attributes, such that the privacy budget for each attribute increases from \( \epsilon/d \) to \( \epsilon/k \), which helps to minimize the noisy variance. In addition, by setting \( k \) as Eq. (36), algorithm 6 achieves an asymptotically optimal performance while preserving privacy, which we will prove using Lemma 11 and Lemma 12. Lemma 11 and Lemma 12 are proved in the same way as that of Lemma 4 and 5 in [7].

Lemma 12. Algorithm 6 satisfies \( \epsilon \)-local differential privacy. In addition, given an input tuple \( x \), it outputs a noisy tuple \( Y \), such that for any \( j \in [1,d] \), and each \( t_j \) of those \( k \) attributes is selected uniformly at random (without replacement) from all \( d \) attributes of \( x \), and then \( E[Y[t_j]] = x[t_j] \).

Proof. Algorithm 6 composes \( k \) numbers of \( \epsilon \)-LDP perturbation algorithms; thus, based on composition theorem of differential mechanism [37], Algorithm 6 satisfies \( \epsilon \)-LDP. As we can see from Algorithm 6, each perturbed output \( Y \) equals to \( \frac{\epsilon}{d} y_j \) with probability \( \frac{\epsilon}{d} \) or equals to 0 with probability \( 1 - \frac{\epsilon}{d} \).

Thus, \( E[Y[t_j]] = \frac{\epsilon}{d} E[y_j] = E[y_j] = x[t_j] \) holds.

Lemma 13. For any \( j \in [1,d] \), let \( Z[t_j] = \frac{1}{n} \sum_{i=1}^{n} Y[t_j] \) and \( X[t_j] = \frac{1}{n} \sum_{i=1}^{n} x[t_j] \). With at least \( 1 - \beta \) probability,

\[
\left| Z[t_j] - X[t_j] \right| = O\left( \sqrt{\left( \frac{\epsilon \ln(1/\beta)}{\delta} \right)} \right).
\]

Proof. The proof details are given in Appendix R of the online full version [35] (i.e., this paper).

VIII. Experiments

We implemented both existing solutions and our proposed solutions, including \( \text{PM-SUB} \), \( \text{Three-Outputs} \), \( \text{HM-TP} \) proposed by us, \( \text{PM} \) and \( \text{HM} \) proposed by Wang et al. [7]. Duchi et al.’s [6] solution and the traditional Laplace mechanism. Our datasets include the following: (i) the WISDM Human Activity Recognition dataset [38], which is a set of accelerometer data on an Android phone from 35 subjects performing 6 activities, where the domain of the timestamps of the phone’s uptime is removed from the dataset, and the remaining 3 numeric attributes are accelerations in \( x, y \), and \( z \) directions measured by the Android phone’s accelerometer and 2 categorical attributes; and (ii) two public datasets extracted from Integrated Public Use Microdata Series [39] which contain census records from Brazil (BR) and Mexico (MX). BR contains 4M tuples and 16 attributes, of which 6 are numerical and 10 are categorical. MX contains 4M records and 19 attributes, of which 5 are numerical and 14 are categorical. We normalize the domain of each numeric attribute to \([−1,1]\). In our experiments, we report average results over 100 runs.

A. Results on the Mean Values of Numeric Attributes

We estimate the mean of each numeric attribute by collecting a noisy multidimensional tuple from each user. For comparison with Wang et al.’s [7] mechanisms, we follow their experiments and then split the total privacy budget \( \epsilon \) into two parts. Assume a tuple contains \( d \) attributes which include \( d_n \) numeric attributes and \( d_c \) categorical attributes. Then, we allocate \( d_n \epsilon/d \) budget to numeric attributes, and \( d_n \epsilon/d \) to categorical ones, respectively. We estimate the mean value for each of the numeric attributes using existing methods: (i) Duchi et al.’s [6] solution which directly handles multiple numeric attributes; (ii) the Laplace mechanism, which applies to each numeric attribute individually with \( \epsilon/d \) budget; (iii) \( \text{PM} \) and \( \text{HM} \) from Wang et al. [7]. We measure the mean square error (MSE) of the estimated mean values for numeric attributes using our proposed approaches in Section VII. Fig. 6 presents MSE results as a function of the total budget of \( \epsilon \) in the datasets (WISDM, MX and BR). Overall, our experimental evaluation shows that our proposed approaches outperform existing solutions. \( \text{HM-TP} \) outperforms existing solutions in all
settings, whereas PM-SUB’s MSE is smaller than PM’s when privacy budget $\epsilon$ is large such as 4, and Three-Outputs’s performance is better at a small privacy budget. Hence, experimental results are in accordance with our theories.

We also run a set of experiments on synthetic datasets that contain numeric attributes only. We create four synthetic datasets, including 16 numeric attributes where each attribute value is generated from a Gaussian distribution with mean value $u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and standard deviation of $\frac{1}{16}$. By evaluating the MSE in estimating mean values of numeric attributes with our proposed mechanisms, we present our experimental results in Fig. 7. Hereby, we confirm that PM-SUB, Three-Outputs and HM-TP outperform existing solutions.

**B. Results on Empirical Risk Minimization**

In this set of experiments, we evaluate the proposed algorithms’ performance using linear regression, logistic regression, and SVM classification tasks. Since both the BR and MX datasets contain the “total income” attribute, we use it as the dependent variable and consider other attributes as independent variables. We transform each categorical attribute $t_j$ with $k$ values into $k - 1$ binary attributes with a domain $\{-1, 1\}$, such that (i) the $l$-th ($l < k$) value in $t_j$ is represented by 1 on the $l$-th binary attribute and $-1$ on each of the remaining $k-2$ attributes; (ii) the $k$-th value in $t_j$ is represented by $-1$ on all binary attributes. After this transformation, the dimension of WISDN is 43, and BR (resp. MX) is 90 (resp. 94). For the BR and MX datasets, we consider the numeric attribute “total income” as the dependent variable and other attributes...
Consider each tuple of data as the dataset of a vehicle, so vehicles calculate gradients and run different LDP mechanisms to generate noisy gradients. Each mini-batch is a group of vehicles. Thus, the centralized aggregator i.e. cloud server updates the model after each group of vehicles send noisy gradients. The experiment involves 8 competitors: PM-SUB, Three-Outputs, HM-TP, PM, HM, Duchi et al.’s solution, Laplace and a non-private setting. We set the regularization factor $\lambda = 10^{-4}$ in all approaches. In each dataset, we use 10-fold cross-validation 5 times to assess the performance of each method. Fig. 8 and Fig. 9 show that the proposed mechanisms (PM-SUB, Three-Outputs, and HM-TP) have lower misclassification rates than other mechanisms. Fig. 10 shows the MSE of the linear regression model. We ignore Laplace’s result because its MSE clearly exceeds those of other mechanisms. In the selected privacy budgets, our proposed mechanisms (PM-SUB, Three-Outputs, and HM-TP) outperform existing approaches (PM, HM, Duchi et al.’s solution, and Laplace).

### Results after Discretization

In this section, we add a discretization post processing step in Algorithm 5 to the implementation of mechanisms with continuous range of outputs, including PM, PM-SUB, HM and HM-TP. To confirm that the discretization is effective, we perform the following experiments. We separate the output domain $[-C, C]$ into 2000 segments, and then we have 2001 possible outputs given an initial input $x$. We add a discretization step to the experiments in Section VIII-A. Fig. 11 displays our experimental results. We confirm that our proposed approaches outperform existing solutions in estimating the mean value using three real-world datasets: WISDM, MX, and BR after discretizing.

In addition, we use log regression and linear regression to evaluate the performance after discretization. We repeat the experiments in Section VIII-B with an additional discretization post processing step. Fig. 12 and Fig. 13 present our experimental results. Compared with other approaches, the performance is similar to that before discretizing. Furthermore, Fig. 14 illustrates how the accuracy changes as output possibilities increase. It shows that the misclassification rate of the
logistic regression task and the MSE of the linear regression task are related to the size of output possibilities. Although incurring with randomness, we find that the misclassification rate and MSE decrease as the number of output possibilities increases. When there are three output possibilities, it incurs randomness. Moreover, Fig. 15 shows that PM-SUB outperforms Three-Outputs, when the number of output possibilities is large. However, when we discretize the range of outputs into 2000 segments, the performance is satisfactory and similar to the performance with a continuous range of outputs. Hence, our proposed approaches combined with the discretization step help retain the performance while enabling the usage in vehicles.

IX. DISCUSSION

A. Apply PM-OPT and PM-SUB to centralized differential privacy

Neighboring databases. Centralized differential privacy (DP) contains bounded DP and unbounded DP [40]. In bounded DP, database $D_1$ can be obtained from database $D_2$ by changing the value of exactly one tuple [5]. In unbounded DP, we can obtain database $D_1$ and $D_2$ one from the other by adding or removing one record (i.e. one tuple) [41]. We prove that PM-OPT and PM-SUB also satisfy centralized differential privacy requirements in Theorem 1.

Theorem 1. PM-OPT satisfies centralized differential privacy constraints.

Proof. Given a query function $Q : D \rightarrow R$ performed on two neighbour datasets $D_1$ and $D_2$, let $Q_{\max}$ and $Q_{\min}$ be the maximal and minimum query result over all possible outputs respectively, we have $x := \frac{2Q(D_1) - (Q_{\max} + Q_{\min})}{Q_{\max} - Q_{\min}} \in [-1, 1]$ and
where \( x' := \frac{2 \cdot Q(D_x) - (Q_{\text{max}} + Q_{\text{min}})}{Q_{\text{max}} - Q_{\text{min}}} \in [-1, 1] \). After applying PM-OPT, we get noisy value \( Y \). Let \( Y' := \frac{2 \cdot Q(x') - (Q_{\text{max}} + Q_{\text{min}})}{Q_{\text{max}} - Q_{\text{min}}} \) and \( Y'' := \frac{2 \cdot Q'(x') - (Q_{\text{max}} + Q_{\text{min}})}{Q_{\text{max}} - Q_{\text{min}}} \), so that we have

\[
\frac{\mathbb{P} \left[ Q(x) \in S \right]}{\mathbb{P} \left[ Q(x') \in S \right]} \leq \frac{c}{d} = \exp(\epsilon),
\]

where \( S = [-A, A] \), \( c, d \) are in Eq. (27) and Eq. (28), respectively. 

**Theorem 2.** PM-SUB achieves \( \epsilon \)-differential privacy.

**Proof.** Omitted because it is similar to Theorem [1].

**B. A clarification about Three-Outputs versus Four-Outputs**

One may wonder why we consider a perturbation mechanism with three outputs (i.e., our Three-Outputs) instead of a perturbation mechanism with four outputs (referred to as Four-Outputs), since using two bits to encode the output of a perturbation mechanism can represent four outputs. The reason is as follows. The approach to design Four-Outputs is similar to that for Three-Outputs, but the detailed analysis for Four-Outputs will be even more tedious than that for Three-Outputs (which is already quite complex). Given above reasons, we elaborate Three-Outputs but not Four-Outputs in this paper.

**X. Conclusion**

In this paper, we propose PM-OPT, PM-SUB, Three-Outputs and HM-TP local differential privacy mechanisms. These mechanisms effectively preserve the privacy when collecting data records and computing accurate statistics in various data analysis tasks, including estimating the mean frequency and complex machine learning tasks such as linear regression, logistic regression, and SVM classification. Moreover, we integrate our proposed local differential privacy mechanisms with FedSGD algorithm to create an LDP-FedSGD algorithm. The LDP-FedSGD algorithm enables the vehicular crowdsourcing applications to train a machine learning model to predict the traffic status while avoiding the privacy threat and reducing the communication cost. More specifically, by leveraging LDP mechanisms, adversaries are unable to deduce the exact location information of vehicles from uploaded gradients. Then, FL enables vehicles to train their local machine learning models using collected data and then send noisy gradients instead of data to the cloud server to obtain a global model. Extensive experiments demonstrate that our proposed approaches are effective and able to perform better than existing solutions. Further, we intend to apply our proposed LDP mechanisms to deep neural network to deal with more complex data analysis tasks.

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A. Proof of Lemma 7

The mechanism $M_2$ satisfies the proper distribution in Eq. (9a) because of

$$P_{C \leftarrow x}(M_2) + P_{C \leftarrow x}(M_2) + P_{0 \leftarrow x}(M_2)$$

$$\frac{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}{2}$$

$$+ P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)$$

$$+ P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1) = 1.$$  

Besides, the mechanism $M_2$ satisfies unbiased estimation in Eq. (9b) because

$$C \cdot P_{C \leftarrow x}(M_2) + (-C) \cdot P_{C \leftarrow x}(M_2) + 0 \cdot P_{0 \leftarrow x}(M_2)$$

$$= C \cdot \frac{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}{2}$$

$$+ (-C) \cdot \frac{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}{2}$$

$$+ 0 \cdot \frac{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}{2} = x.$$  

In addition,

$$\frac{P_{C \leftarrow x}(M_2)}{P_{C \leftarrow x}(M_2)} = \frac{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}$$

and

$$\frac{P_{C \leftarrow x}(M_2)}{P_{C \leftarrow x}(M_2)} = \frac{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)}$$

and

$$\frac{P_{0 \leftarrow x}(M_2)}{P_{0 \leftarrow x}(M_2)} = \frac{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}.$$  

According to (9a), we obtain

$$\frac{e^{-x}(P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1))}{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)} \leq Eq. (38) \leq \frac{e^{x}(P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1))}{P_{C \leftarrow x}(M_1) + P_{C \leftarrow x}(M_1)},$$

which is equivalent to

$$e^{-x} \leq Eq. (38) \leq e^{x}.$$  

Similarly, we prove that Eq. (39) satisfies (9a). Hence, we conclude that $M_2$ satisfies $\epsilon$-LDP requirements.

Then, we prove that the symmetrization process does not increase the worst-case noise variance as follows: Since $M_2$ satisfies the unbiased estimation of Eq. (9b), $E[Y | X = x] = x$. Hence, the variance of mechanism $M_2$ given $x$ is

$$Var_{M_2}[Y | X = x] = E[Y^2 | X = x] - (E[Y | X = x])^2$$

$$= C^2 \cdot P_{C \leftarrow x}(M_2) + 0 \cdot P_{C \leftarrow x}(M_2)$$

$$+ (-C)^2 \cdot P_{C \leftarrow x}(M_2) - x^2$$

$$= C^2(1 - P_{0 \leftarrow x}(M_2)) - x^2$$

or it changes to

$$Var_{M_2}[Y | X = x] = E[Y^2 | X = -x] - (E[Y | X = -x])^2$$

$$= C^2 \cdot P_{C \leftarrow x}(M_2) + 0 \cdot P_{C \leftarrow x}(M_2)$$

$$+ (-C)^2 \cdot P_{C \leftarrow x}(M_2) - x^2$$

$$= C^2(1 - P_{0 \leftarrow x}(M_2)) - x^2,$$  

$$= C^2 \left(1 - \frac{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}{2}\right) - x^2,$$  

when given $-x$.

The variance of mechanism $M_1$ is

$$Var_{M_1}[Y | X = x] = E[Y^2 | X = -x] - (E[Y | X = -x])^2$$

$$= C^2 \cdot P_{C \leftarrow x}(M_1) + 0 \cdot P_{C \leftarrow x}(M_1)$$

$$+ (-C)^2 \cdot P_{C \leftarrow x}(M_1) - x^2$$

$$= C^2(1 - P_{0 \leftarrow x}(M_1)) - x^2,$$  

$$= C^2 \left(1 - \frac{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}{2}\right) - x^2.$$

Hence,

$$Var_{M_2}[Y | X = x] = Var_{M_2}[Y | X = -x]$$

$$= Var_{M_2}[Y | X = x] + Var_{M_2}[Y | X = -x]$$

$$\leq \frac{2}{2} \leq \max\{Var_{M_1}[Y | X = -x], Var_{M_1}[Y | X = x]\}.$$  

B. Proof of Lemma 2

In essence, with Eq. (21) (21), we have

$$P_{C \leftarrow 1}(M_3)$$

$$= P_{C \leftarrow 1}(M_2) - \frac{e^{x}P_{C \leftarrow 1}(M_2) - P_{C \leftarrow 1}(M_2)}{e^{-x} - 1}$$

$$= e^{x}.$$  

Similarly, we can prove that $P_{C \leftarrow 1}(M_3) \cdot P_{C \leftarrow 1}(M_3) = e^{x}$. Besides,

$$P_{C \leftarrow x} + P_{C \leftarrow x} + P_{0 \leftarrow x} = 1.$$  

In addition,

$$C \cdot P_{C \leftarrow x} + (-C) \cdot P_{C \leftarrow x} + 0 \cdot P_{0 \leftarrow x}$$

$$= C \cdot (P_{C \leftarrow x}(M_2) - P_{C \leftarrow x}(M_3)) = x.$$  

Hence, mechanism $M_3$ satisfies requirements in (9a) (9b) (9c). Because the symmetric mechanism $M_3$ satisfies requirements in (9a) (9b) (9c), the variance of $M_3$ is

$$Var_{M_3}[Y | X = x] = E[Y^2 | X = x] - (E[Y | X = x])^2$$

$$= C^2 \cdot P_{C \leftarrow x}(M_3) + 0 \cdot P_{C \leftarrow x}(M_3)$$

$$+ (-C)^2 \cdot P_{C \leftarrow x}(M_3) - x^2$$

$$= C^2 \left(1 - \frac{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}{2}\right) - x^2.$$  

or it changes to

$$Var_{M_3}[Y | X = x] = E[Y^2 | X = -x] - (E[Y | X = -x])^2$$

$$= C^2 \cdot P_{C \leftarrow x}(M_3) + 0 \cdot P_{C \leftarrow x}(M_3)$$

$$+ (-C)^2 \cdot P_{C \leftarrow x}(M_3) - x^2$$

$$= C^2 \left(1 - \frac{P_{0 \leftarrow x}(M_1) + P_{0 \left arrow x}(M_1)}{2}\right) - x^2,$$  

$$= C^2 \left(1 - \frac{P_{0 \leftarrow x}(M_1) + P_{0 \leftarrow x}(M_1)}{2}\right) - x^2.$$
Because of $P_{0\leftarrow x}(M_2) > P_{0\leftarrow x}(M_3)$, based on Eq. (23) and Inequality (20), we get $P_{0\leftarrow x}(M_2) > P_{0\leftarrow x}(M_3)$ which means that

$$\text{Var}_{M_2}[Y|X = x] > \text{Var}_{M_3}[Y|X = x].$$

Thus, the variance of $M_3$ is smaller than the variance of $M_2$ when $x \in [-1, 1]$, so we obtain that the worst-case noise variance of $M_3$ is smaller that of $M_2$ using

$$\max_{x \in [-1, 1]} \text{Var}_{M_3}[Y|X = x] > \max_{x \in [-1, 1]} \text{Var}_{M_3}[Y|X = x].$$

### C. Proof of Lemma 3

Since $P_{-C\leftarrow 0} + P_{C\leftarrow 0} + P_{0\leftarrow 0} = 1$ and unbiased estimation $-C \cdot P_{-C\leftarrow 0} + C \cdot P_{C\leftarrow 0} + 0 \cdot P_{0\leftarrow 0} = 0$, we have

$$P_{C\leftarrow 0} = P_{-C\leftarrow 0} = \frac{1 - P_{0\leftarrow 0}}{2}.$$  \hspace{1cm} (49)

Then, based on (9a) (9b) (9c) and Lemma 2 we can derive $C$ with the following steps:

$$P_{-C\leftarrow 1} + P_{C\leftarrow 1} + P_{0\leftarrow 1} = 1,$n

$$-C \cdot P_{-C\leftarrow 1} + C \cdot P_{C\leftarrow 1} + 0 \cdot P_{0\leftarrow 1} = 1.$$n

Therefore, we have

$$P_{C\leftarrow 1} = 1 - P_{0\leftarrow 1} + \frac{t}{2},$$ \hspace{1cm} (50)

$$P_{-C\leftarrow 1} = \frac{1 - P_{0\leftarrow 1} - t}{2}.$$ \hspace{1cm} (51)

From Lemma 2 we obtain

$$\frac{1 - P_{0\leftarrow 1} + t}{2} = e^t \cdot \left( \frac{1 - P_{0\leftarrow 1} - \frac{t}{2}}{2} \right),$$

which is equivalent to

$$C = \frac{e^t + 1}{(e^t - 1)(1 - P_{0\leftarrow 1})}.$$  \hspace{1cm} (52)

Hence,

$$P_{C\leftarrow 1} = P_{-C\leftarrow 1} = \frac{(1 - P_{0\leftarrow 1})e^t}{e^t + 1},$$ \hspace{1cm} (53)

$$P_{-C\leftarrow 1} = P_{C\leftarrow 1} = \frac{(1 - P_{0\leftarrow 1})}{e^t + 1}.$$ \hspace{1cm} (54)

Then, we compute the variance as follows:

**I. For $x \in [0, 1]$, we have**

$$\text{Var}[Y|X = x] = E[Y^2|X = x] - (E[Y|X = x])^2 = C^2 \cdot P_{C\leftarrow x} + 0 \cdot P_{0\leftarrow x} + (-C)^2 \cdot P_{-C\leftarrow x} - x^2$$

$$= C^2 (P_{C\leftarrow x} + P_{-C\leftarrow x}) - x^2.$$  \hspace{1cm} (55)

Substituting Eq. (13) and Eq. (14) into Eq. (52) yields

$$= C^2 (P_{C\leftarrow x} + P_{-C\leftarrow x}) - x^2$$

$$= C^2 (P_{C\leftarrow x} + P_{-C\leftarrow x}) + C^2 (P_{C\leftarrow x} + P_{C\leftarrow -x})$$

$$- C^2 (P_{C\leftarrow x} + P_{C\leftarrow -x}) x^2$$

$$= C^2 (P_{C\leftarrow x} + P_{C\leftarrow -x}) + C^2 (1 - P_{0\leftarrow x}) x^2$$

$$= C^2 (1 - P_{0\leftarrow x}) x^2.$$ \hspace{1cm} (56)

II. For $x \in [-1, 0]$, we have

$$\text{Var}[Y|X = x] = E[Y^2|X = x] - (E[Y|X = x])^2$$

$$= C^2 (1 - P_{0\leftarrow x}) + C^2 (P_{0\leftarrow x} - P_{0\leftarrow 1}) x^2.$$ \hspace{1cm} (57)

Hence, by summarizing Eq. (50), Eq. (53), and Eq. (54), we get the variance as follows:

$$\text{Var}[Y|X = x] = C^2 (1 - P_{0\leftarrow x}) + C^2 (P_{0\leftarrow x} - P_{0\leftarrow 1}) x^2.$$ \hspace{1cm} (58)

Derive the partial derivative of $\text{Var}[Y|X = x]$ to $P_{0\leftarrow 1}$, and we get

$$d(\text{Var}[Y|X = x]) = \frac{d}{dP_{0\leftarrow 1}} (\text{Var}[Y|X = x])$$

$$= \frac{(e^t + 1)^2 (2P_{0\leftarrow 0} + 2P_{0\leftarrow 1} - 1) - 2(1 - 2P_{0\leftarrow 1} - P_{0\leftarrow 1} - 2)}{(P_{0\leftarrow 1} + 1)^2 (e^t - 1)^2}. \hspace{1cm} (59)

Then, we have the following cases:

**I. If $|x| = 0$, Eq. (56) $= \frac{(e^t + 1)^2 (2P_{0\leftarrow 0} - 2)}{(P_{0\leftarrow 1} + 1)^2 (e^t - 1)^2} < 0$,**

**II. If $|x| = 1$, Eq. (56) $= \frac{(e^t + 1)^2 (P_{0\leftarrow 0} - 2)}{(P_{0\leftarrow 0} + 1)^2 (e^t - 1)^2} < 0$.**

Therefore, if given $P_{0\leftarrow 0}$, the variance of the output given input $x$ is a strictly decreasing function of $P_{0\leftarrow 1}$, Hence, we get the minimized variance when $P_{0\leftarrow 1} = \frac{P_{0\leftarrow 0}}{e^t}$. \hspace{1cm} (60)

### D. Solve Eq. (59)

To find the optimal $t$ for $\min_{t} \max_{x \in [-1, 1]} \text{Var}[Y|x]$, we calculate first-order derivative of the $\max_{x \in [-1, 1]} \text{Var}[Y|x]$ as follows:

$$\frac{2t}{3(e^t - 1)^2} + \frac{4}{3(e^t - 1)^2} + \frac{4}{3(e^t - 1)^2} - \frac{4t^2}{3(e^t - 1)^2}$$

$$= \frac{2}{3(e^t - 1)^2} [t + 2e^t - 2e^t t^{-2} - e^t t^{-3}] \hspace{1cm} (59)$$

Next, we calculate the second-order derivative of $\max_{x \in [-1, 1]} \text{Var}[Y|x]$ as follows:

$$\frac{2}{3(e^t - 1)^2} [1 + 4e^t t^{-3} + 3e^t t^{-4}] > 0. \hspace{1cm} (58)$$
Since the second-order derivative of $\max_{x \in [-1, 1]} \text{Var}[Y \mid x] > 0$, we can conclude that $\max_{x \in [-1, 1]} \text{Var}[Y \mid x]$ has a minimum point in its domain.

To find $t$ which minimizes $\max_{x \in [-1, 1]} \text{Var}[Y \mid x]$, we set $t^4 + 2e^t t^3 - 2e^t t - e^{2t} = 0$. By solving

$$t^4 + 2e^t t^3 - 2e^t t - e^{2t} = 0,$$

we obtain Eq. (59). Define Eq. (59)'s coefficients as $c_4 := 1$, $c_3 := 2e^t$, $c_2 := 0$, $c_1 := -2e^t$, $c_0 := -e^{2e}$, and we obtain

$$c_4 \cdot t^4 + c_3 \cdot t^3 + c_1 \cdot t + c_0 = 0.$$  \hspace{1cm} (60)

To change Eq. (60) into a depressed quartic form, we substitute $f := e^t$, $t := y - \frac{1}{4}c_4 = y - \frac{1}{4}$ into Eq. (60) and obtain

$$y^4 + p \cdot y^2 + q \cdot y + r = 0,$$

where

$$p = \frac{8c_2 c_4 - 3c_3^2}{8c_4^2} = -\frac{3f^2}{2},$$

$$q = \frac{c_3^2 - 4c_2 c_3 c_4 + 8c_1 c_2^2}{8c_4^2} = f^3 - 2f,$$

$$r = \frac{-3c_1^4 + 256c_0 c_4 c_3 c_4 - 64c_1 c_3^2 c_4^2 + 16c_2 c_3^2 c_4}{256c_4^4} = -\frac{3}{16}f^4.$$

Rewrite Eq. (61) to the following:

$$\left(y^2 + \frac{p}{2}\right)^2 = -qy - r + \frac{p^2}{4}.$$  \hspace{1cm} (65)

Then, we introduce a variable $m$ into the factor on the left-hand side of Eq. (65) by adding $2y^2 m + pm + m^2$ to both sides. Thus, we can change the equation to the following:

$$\left(y^2 + \frac{p}{2} + m\right) = 2my^2 - qy + m^2 + mp + \frac{p^2}{4}.$$  \hspace{1cm} (66)

Since $m$ is arbitrarily chosen, we choose the value of $m$ to get a perfect square in the right-hand side. Hence,

$$8m^3 + 8pm^2 + (2p^2 - 8r)m - q^2 = 0.$$  \hspace{1cm} (67)

To solve Eq. (67), we substitute Eq. (62), Eq. (63) and Eq. (64) into the following equations:

$$c_4' := 8,$$

$$c_3' := 8p,$$

$$c_2' := 2p^2 - 8r,$$

$$c_1' := -q^2,$$

$$\Delta_0 = (c_2')^2 - 3c_3'c_1' = (8p)^2 - 3 \cdot 8 \cdot (2p^2 - 8r) = 0,$$

$$\Delta_1 = 2(c_2')^2 - 9c_3'c_1' - 27(c_1')^2 - c_0' = 2(8p)^2 - 9 \cdot 8 \cdot 8p \cdot (2p^2 - 8r) + 27 \cdot 8^2 \cdot (-q^2) = 6912(f^4 - f^2),$$

$$C = \sqrt{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0}}{2}} = \sqrt{\Delta_1} = \sqrt{6912(f^4 - f^2)}.$$  \hspace{1cm} (68)

By solving the cubic function Eq. (68), we have roots as follows:

$$m_k = -\frac{1}{3c_3} \left(c_2' + \xi^k C + \frac{\Delta_0}{\xi^k C}\right) = \frac{f^2}{2} + \sqrt{\frac{f^2 - f^4}{2}} \xi^k, k = 0, 1, 2.$$  \hspace{1cm} (69)

We only use the real-value root; thus, we get

$$m = \frac{f^2}{2} + \sqrt{\frac{f^2 - f^4}{2}}.$$  \hspace{1cm} (70)

Thus,

$$\Delta = \frac{f^2}{2} + \sqrt{\frac{f^2 - f^4}{2}}.$$  \hspace{1cm} (71)

Then, the solutions of the original quartic equation are

$$t = -\frac{c_3}{4c_4} + \frac{\pm \sqrt{2m \pm 2 \sqrt{-(2p + 2m \pm 1) \frac{\sqrt{\Delta}}{m}}}}{2}.$$  \hspace{1cm} (72)

Since $t$ is a real number and $t > 0$, we obtain Eq. (30) after substituting $c_3, c_4, m, p, q, f$ into Eq. (72).

**E. Proof of Lemma 4**

By summarizing Lemma 1, 2, 3, our designed mechanism achieves minimum variance when it satisfies $P_{0 \rightarrow 1} = \frac{P_{0 \rightarrow 0}}{e}$. Hence, the variance is

$$\text{Var}[Y \mid X = x] = C^2 (1 - P_{0 \rightarrow 0}) + C^2 P_{0 \rightarrow 0} (1 - \frac{1}{e^x}) |x| - x^2,$$

where

$$C = \frac{e^x + 1}{(e^x - 1)(1 - \frac{P_{0 \rightarrow 0}}{e})}.$$  \hspace{1cm} (74)

For simplicity, we set

$$a = P_{0 \rightarrow 0},$$

$$b = P_{0 \rightarrow 0} (1 - \frac{1}{e^x}) = a (1 - \frac{1}{e^x}).$$  \hspace{1cm} (75)

Since $x \in [-1, 1]$, the worst-case noise variance is

$$\max_{x \in [-1, 1]} \text{Var}[Y \mid x] = \left\{(1 - a)C^2 + \frac{C^2b^2}{2}, \quad \text{if} \quad \frac{C^2b}{2} < 1, \right.$$  \hspace{1cm} (76)

$$\left.(1 - a + b)C^2 - 1, \quad \text{if} \quad \frac{C^2b}{2} \geq 1. \right\}$$  \hspace{1cm} (77)

Substituting Eq. (75), Eq. (76) and Eq. (74) into Eq. (77) yields

$$\max_{x \in [-1, 1]} \text{Var}[Y \mid x] = \left\{ \begin{array}{ll}
\frac{(e^x + 1)^2 a^2}{(e^x - 1)^2} + \frac{(1-a)^2}{4(e^x - a)^2} & \text{if} \quad \frac{C^2b}{2} < 1, \\
\frac{(e^x + 1)^2 a^2}{(e^x - 1)^2} + \frac{(1-a)^2}{4(e^x - a)^2} & \text{if} \quad \frac{C^2b}{2} \geq 1.
\end{array} \right.$$  \hspace{1cm} (78)
Substituting Eq. (76) and Eq. (74) yields
\[
\frac{C^2b}{2} = \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{2(e^\epsilon - 1)^2(e^\epsilon - a)^2} \cdot a(e^\epsilon - 1) = \frac{(e^\epsilon + 1)^2 \cdot e^\epsilon \cdot a}{2(e^\epsilon - 1)(e^\epsilon - a)^2} < 1, \tag{79}
\]
and
\[
2(e^\epsilon - 1)a^2 - [4(e^\epsilon - 1)e^\epsilon + (e^\epsilon + 1)^2 \cdot e^{2\epsilon}]a + 2(e^\epsilon - 1)e^{2\epsilon} > 0. \tag{80}
\]
To solve Eq. (80), we denote the smaller solution of the quadratic function as
\[
a^* = \frac{e^\epsilon(2e^\epsilon + 6e^\epsilon - 3) - (e^\epsilon + 1)e^\epsilon\sqrt{(e^\epsilon + 1)^2 + 8(e^\epsilon - 1)}}{4(e^\epsilon - 1)}. \tag{81}
\]
From Eq. (14), we get
\[
P_{C \to 0} \geq P_{C \to 1}. \tag{82}
\]
Then, substituting \( P_{C \to 0} \) and \( P_{C \to 1} \) with Eq. (49) and Eq. (51) in Eq. (82) yields
\[
\frac{1-P_{0 \to 0}}{2} \geq \frac{1-(1-P_{0 \to 1})}{e^\epsilon + 1}. \tag{83}
\]
Hence,
\[
a = P_{0 \to 0} \leq \frac{e^\epsilon}{e^\epsilon + 2}. \tag{84}
\]
From Eq. (84), we know that the Eq. (82) will be ensured (i) when \( 0 \leq a < a^* \) if \( a^* < \frac{e^\epsilon}{e^\epsilon + 2} \), or (ii) when \( 0 \leq a \leq \frac{e^\epsilon}{e^\epsilon + 2} \) if \( a^* \geq \frac{e^\epsilon}{e^\epsilon + 2} \). Hence, by combining with Eq. (78), we obtain
\[
\max_{x \in [-1, 1]} \text{Var}[Y|x] = \begin{cases} \left( \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) + \frac{(e^\epsilon + 1)^2 \cdot a^2}{4(e^\epsilon - a)^4} \right), & \text{for } 0 \leq a < a^*, \\ \left( \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) - 1, & \text{if } a^* < \frac{e^\epsilon}{e^\epsilon + 2}, \\ \left( \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) + \frac{(e^\epsilon + 1)^2 \cdot a^2}{4(e^\epsilon - a)^4} \right), & \text{for } 0 \leq a \leq \frac{e^\epsilon}{e^\epsilon + 2}, \end{cases} \tag{85}
\]
Substituting Eq. (81) into \( a^* = \frac{e^\epsilon}{e^\epsilon + 2} \) yields
\[
e^\epsilon(e^{2\epsilon} + 6e^\epsilon - 3) - (e^\epsilon + 1)e^\epsilon\sqrt{(e^\epsilon + 1)^2 + 8(e^\epsilon - 1)} = \frac{e^\epsilon}{e^\epsilon + 2}. \tag{86}
\]
After solving Eq. (86), we get \( \epsilon = \ln 4 \).

According to Fig. 16, we obtain that \( a^* \geq \frac{e^\epsilon}{e^\epsilon + 2} \) if \( 0 < \epsilon \leq \ln 4 \). Since \( \epsilon = \ln 4 \) is the only solution if \( \epsilon > 0 \), we conclude that \( a^* > \frac{e^\epsilon}{e^\epsilon + 2} \) if \( \epsilon > \ln 4 \). Therefore, we can replace the condition \( a^* < \frac{e^\epsilon}{e^\epsilon + 2} \) and write the variance as follows:

\[
\text{max}_{x \in [-1, 1]} \text{Var}[Y|x] = \begin{cases} \left( \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) + \frac{(e^\epsilon + 1)^2 \cdot a^2}{4(e^\epsilon - a)^4} \right), & \text{for } 0 \leq a < a^*, \\ \left( \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) - 1, & \text{if } a^* < \frac{e^\epsilon}{e^\epsilon + 2}, \\ \left( \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) + \frac{(e^\epsilon + 1)^2 \cdot a^2}{4(e^\epsilon - a)^4} \right), & \text{for } 0 \leq a \leq \frac{e^\epsilon}{e^\epsilon + 2}, \end{cases} \tag{87}
\]
To simplify the calculation of the minimum \( \text{max}_{x \in [-1, 1]} \text{Var}[Y|x] \) in Eq. (87), we define
\[
f_1(a) := \frac{(e^\epsilon + 1)^2 \cdot e^{2\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right) + \frac{(e^\epsilon + 1)^2 \cdot a^2}{4(e^\epsilon - a)^4}, \tag{88}
\]
and
\[
f_2(a) := \frac{(e^\epsilon + 1)^2 \cdot e^{\epsilon}}{(e^\epsilon - 1)^2} \cdot \left( 1 - \frac{a}{(e^\epsilon - a)^2} \right). \tag{89}
\]
First order derivative of \( f_2(a) \) in Eq. (89) is
\[
f_2'(a) = \frac{2e^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^3} > 0. \tag{90}
\]
Since \( f_2'(a) > 0 \), the worst-case noise variance monotonously increases if \( a \in [a^*, \frac{e^\epsilon}{e^\epsilon + 2}] \), we can get optimal \( a \) by analyzing \( f_1(a) \) in Eq. (88) when \( a \in [0, a^*] \) if \( \epsilon < \ln 4 \). First order derivative of Eq. (88) is
\[
f_1'(a) = \frac{2(1 - a)}{(e^\epsilon - a)^3} - \frac{1}{(e^\epsilon - a)^2} + \frac{a(e^\epsilon + 1)^2}{2(e^\epsilon - a)^4} + \frac{a^2(e^\epsilon + 1)^2}{(e^\epsilon - a)^5}. \tag{91}
\]
After simplifying \( f_1'(a) \), we have
\[
f_1'(a) = \frac{-2a^3 - a^2(-e^{2\epsilon} - 5 - 4e^\epsilon)}{2(e^\epsilon - a)^5} + \frac{-a(7e^\epsilon - 4e^{2\epsilon} - e^{3\epsilon}) - (2e^{3\epsilon} - 4e^{2\epsilon})}{2(e^\epsilon - a)^5}. \tag{92}
\]
Since $2(e^c - a)^5 > 0$, solving $f'_1(a) = 0$ is equivalent to solve the following equation
\[
2a^3 + a^2((-e^2 - 5 - 4e^c) + a(7e^c - 4e^{2c} - e^3c) + (2e^{3c} - 4e^{2c}) = 0. \tag{93}
\]
We define coefficients of Eq. (93) as follows:
\[
c_3 := 2, \tag{94}
c_2 := -e^{2c} - 5 - 4e^c, \tag{95}
c_1 := 7e^c - 4e^{2c} - e^3c, \tag{96}
c_0 := 2e^{3c} - 4e^{2c}. \tag{97}
\]
The general solution of the cubic equation involves calculation of
\[
\Delta_0 = c_2^2 - 3c_3c_1
\]
\[
= (-e^{2c} - 5 - 4e^c)^2 - 3 \times 2(7e^c - 4e^{2c} - e^3c)
\]
\[
= e^{4c} + 14e^{3c} + 50e^{2c} - 2e^c + 25 > 0,
\]
\[
\Delta_1 = 2c_2^3 - 9c_3c_1c_2 + 27c_3^2c_0
\]
\[
= 2(-e^{2c} - 5 - 4e^c)^3
\]
\[
- 9 \times 2(-e^{2c} - 5 - 4e^c)(7e^c - 4e^{2c} - e^3c)
\]
\[
+ 27 \times 2^2(2e^{3c} - 4e^{2c})
\]
\[
= -2e^{6c} - 42e^{5c} - 270e^{4c} - 404e^{3c} - 918e^{2c}
\]
\[
+ 30e^c - 250 < 0,
\]
\[
C = \sqrt[3]{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}.
\]
Substituting $\Delta_0$ and $\Delta_1$ into $C$ yields
\[
\Delta_1^2 - 4\Delta_0^3
\]
\[
= (-2c^6 - 42c^5 - 270c^4 - 404c^3 - 918c^2 + 30c - 250)^2
\]
\[
- 4(c^4 + 14c^3 + 50c^2 - 2c + 25)^3 < 0,
\]
then
\[
\sqrt[3]{\Delta_1^2 - 4\Delta_0^3} = i\sqrt{\Delta_0^3 - \Delta_1^2}, \tag{98}
\]
finally,
\[
C = \sqrt[3]{\Delta_1 - i\sqrt{\Delta_1^2 - 4\Delta_0^3}}.
\]
To eliminate the imaginary number, we change Eq. (107) using Euler's formula. Define mold of $C$ as follows:
\[
|C| = (|C|^3)^{1/3} = \left( \sqrt[3]{\Delta_1^2 + \Delta_0^3 - \Delta_1^2} \right)^{1/3} = \sqrt[3]{\Delta_0}, \tag{99}
\]
\[
C = |C|e^{i\theta}, \tag{100}
\]
\[
C^3 = |C|^3e^{3i\theta} = \sqrt[3]{\Delta_0}e^{3i\theta}. \tag{101}
\]
Therefore, we obtain $C = \sqrt[3]{\Delta_0}e^{i\theta}$. According to Euler's Formula, we have
\[
e^{i3\theta} = \cos 3\theta + i\sin 3\theta, \tag{102}
\]
\[
\cos 3\theta = \frac{\Delta_1}{2\Delta_0^2} < 0. \tag{103}
\]
\[
sin 3\theta = -\frac{\sqrt{4\Delta_0^3 - \Delta_1^2}}{2\Delta_0^2} < 0. \tag{104}
\]
Hereby,
\[
3\theta = -\pi + \arccos(-\frac{\Delta_1}{2\Delta_0^2}), \tag{105}
\]
\[
\theta = -\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_0^2}). \tag{106}
\]
The solution of the cubic function is
\[
a_k = \frac{1}{3c_3}(c_2 + \xi^kC + \frac{\Delta_0}{\xi^kC}), \quad k \in \{0, 1, 2\}. \tag{107}
\]
To solve Eq. (107), we have the following cases:

- If $k = 0$, we have
  \[
a_0 = -\frac{1}{3c_3}(c_2 + C + \frac{\Delta_0}{C}). \tag{108}
\]
  Substituting $\theta$ (106) and $c_2$ (95) into Eq. (108) yields
  \[
a_0 = -\frac{1}{6}(-e^{2c} - 4e^c - 5 + 2\sqrt{\Delta_0}\cos(-\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_0^2}))). \tag{109}
\]
- If $k = 1$, we have
  \[
a_1 = -\frac{1}{3c_3}(c_2 + \xi C + \frac{\Delta_0}{\xi C})
\]
  \[
= -\frac{1}{3c_3}(c_2 + (-\frac{1}{2} + \frac{\sqrt{3}i}{2})C + \frac{\Delta_0}{(-\frac{1}{2} + \frac{\sqrt{3}i}{2})C})
\]
  \[
= -\frac{1}{3c_3}(c_2 + C\frac{\Delta_0}{C} + \frac{\Delta_0}{C}i\frac{\Delta_0}{C})
\]
  \[
= -\frac{1}{3c_3}(c_2 + \sqrt{\Delta_0}e^{i(\theta + \frac{\pi}{3})} + \sqrt{\Delta_0}e^{i(\frac{5\pi}{4} - \theta)}). \tag{110}
\]
Simplify Eq. (110) using Eq. (102), we have
\[
a_1 = -\frac{1}{3c_3}(c_2 + 2\sqrt{\Delta_0}\cos(\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_0^2}))). \tag{111}
\]
Substituting $\theta$ (106) and $c_2$ (95) into Eq. (111) yields
\[
a_1 = -\frac{1}{6}(-e^{2c} - 4e^c - 5 + 2\sqrt{\Delta_0}\cos(\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_0^2}))). \tag{112}
\]
- If $k = 2$, we get
  \[
a_2 = -\frac{1}{3c_3}(c_2 + \xi^2 C + \frac{\Delta_0}{\xi^2 C})
\]
  \[
= -\frac{1}{3c_3}(c_2 + (-\frac{1}{2} + \frac{\sqrt{3}i}{2})^2C + \frac{\Delta_0}{(-\frac{1}{2} + \frac{\sqrt{3}i}{2})^2C})
\]
  \[
= -\frac{1}{3c_3}(c_2 + (-\frac{1}{2} + \frac{\sqrt{3}i}{2})C + \frac{\Delta_0}{(-\frac{1}{2} + \frac{\sqrt{3}i}{2})C})
\]
  \[
= -\frac{1}{3c_3}(c_2 + \sqrt{\Delta_0}e^{i(\theta + \frac{\pi}{4})} + \sqrt{\Delta_0}e^{i(\frac{5\pi}{4} - \theta)}). \tag{113}
\]
Simplify Eq. (113) using Eq. (102), we obtain
\[ a_2 = -\frac{1}{3c_3}(c_2 + 2\sqrt{\Delta_0 \cos(\theta - \frac{2\pi}{3})}). \] (114)
Substituting \( \theta \) (106) and \( c_2 \) (95) into Eq. (114) yields
\[ a_2 = -\frac{1}{6}(-e^{2\epsilon} - 4e^\epsilon - 5 + 2\sqrt{\Delta_0 \cos(-\pi + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_0^{\frac{3}{2}}}))}). \] (115)

The number of real and complex roots are determined by the discriminant of the cubic equation as follows:
\[ \Delta = 18c_3c_2c_1c_0 - 4c_3^3c_0 + c_2^2c_1^2 - 4c_3c_1^3 - 27c_3^2c_0^2. \] (116)
Substituting \( c_3 \) (94), \( c_2 \) (95), \( c_1 \) (96) and \( c_0 \) (97) into \( \Delta \) (116) yields
\[ \Delta = e^{2\epsilon}(e^{12\epsilon} + 30e^{5\epsilon} + 279e^{4\epsilon} + 580e^{3\epsilon} - 2385e^{2\epsilon} + 606e^\epsilon - 775). \] (117)
If \( \Delta = 0 \), we get \( \epsilon = \ln(\text{root of } e^{6\epsilon} + 30e^{5\epsilon} + 279e^{4\epsilon} + 580e^{3\epsilon} - 2385e^{2\epsilon} + 606e^\epsilon - 775) \approx 0.629598.

- If \( 0 < \epsilon < 0.629598 \), the equation has one real root and two non-real complex conjugate roots.
- If \( \epsilon = 0.629598 \), \( \Delta = 0 \), the equation has a multiple root all of its roots are real.
- If \( \epsilon > 0.629598 \), \( \Delta > 0 \), the equation has three distinct real roots.

From the simplified \( f_1'(a) \) Eq. (92), we know that the sign and roots of \( f_1'(a) \) are same as its numerator, defined as follows:
\[ g(a) := -2a^3 - a^2(-e^{2\epsilon} - 5 - 4e^\epsilon) - a(7e^\epsilon - 4e^{2\epsilon} - e^{3\epsilon}) - (2e^{3\epsilon} - 4e^{2\epsilon}). \] (118)
Let \( c = e^\epsilon \), we can change \( g(a) \) to the following:
\[ g(a) = -2a^3 - a^2(-e^{2\epsilon} - 5 - 4c) - a(7c - 4c^2 - e^\epsilon) - (2c^3 - 4c^2). \] (119)

Case 1: If \( 0 < \epsilon < 0.629598 \), by observing \( a_0, a_1, a_2 \), it is obvious that \( a_2 \) is a real root. Fig. 17 shows that \( a_2 > 1 \). Since \( a_2 \) is the only real value root, Eq. (91) \( f_1'(a) > 0 \) if \( 0 < \epsilon \leq 0.629598 \) and \( f_1'(a) \leq 0 \) if \( \epsilon > 0.629598 \). Therefore, we conclude that \( a = 0 \).

Case 2: If \( 0.629598 \leq \epsilon \leq \ln 2 \), \( \Delta \geq 0 \), we get real roots.
- If \( a = 0 \), \( g(0) = -(2e^{3\epsilon} - 4c^2) \).
- If \( a = 2 \), \( g(2) = 2(8c^2 + c + 2) > 0 \).
- If \( a = +\infty \), \( \lim_{a \to +\infty} g(a) = -\infty < 0 \).

If there is a root in \( (0, \frac{\epsilon' - 629598}{2}) \), which means \( g(0) \leq 0 \). By solving \( g(0) = -(2e^{3\epsilon} - 4c^2) \leq 0 \), we have \( c \geq 2 \) meaning \( \epsilon \geq \ln 2 \). In this case 0.629598 < \( \epsilon \leq \ln 2 \), we have a root in \( (2, +\infty) \).

Based on the properties of cubic function, we have
\[ a_0a_1 + a_0a_2 + a_1a_2 = \frac{c_1}{c_3} = \frac{7e^\epsilon - 4e^{2\epsilon} - e^{3\epsilon}}{2}, \] (120)
\[ a_0a_1a_2 = -\frac{c_0}{c_3} = -\frac{2e^{3\epsilon} - 4e^{2\epsilon}}{2}. \] (121)

Fig. 17: \( a_2 \) if \( \epsilon \in [0, 0.629598] \).

Fig. 18: \( a_0a_1 + a_0a_2 + a_1a_2 \) and \( a_0a_1a_2 \) if \( \epsilon \in [0.629598, \ln 2] \).

Fig. 18 shows that \( a_0a_1 + a_0a_2 + a_1a_2 < 0 \) (Eq. (120)) and \( a_0a_1a_2 > 0 \) (Eq. (121)). Therefore, we can conclude that there are one positive real root and two negative real roots or a multiple root. Since two negative roots are out of the \( a \)’s domain, we only discuss the positive root.

- If \( a \in [0, \text{root}) \), \( g(a) > 0 \) meaning \( f_1'(a) > 0 \).
- If \( a \in [\text{root}, +\infty) \), \( g(a) \leq 0 \) meaning \( f_1'(a) \leq 0 \).

From above we know that \( g(a) > 0 \), so that \( f_1'(a) \) monotonically increases if \( a \in [0, \frac{\epsilon' - 629598}{2}) \). Therefore, \( a = 0 \).

Case 3: If \( \ln 2 \leq \epsilon \leq \ln 5.53 \), \( \Delta > 0 \), there are three distinct real roots. Since \( a_0a_1 + a_0a_2 + a_1a_2 < 0 \) and \( a_0a_1a_2 < 0 \), there are one negative root or three negative roots. If there are three negative roots, \( a_0a_1 + a_0a_2 + a_1a_2 > 0 \), there is only one negative root, \( f_1'(a) \) have two positive roots and one negative root.

- If \( a = 0 \), \( g(0) = -(2c^3 - 4c^2) < 0 \).
- If \( a = 2 \), \( g(2) = 2(8c^2 + c + 2) > 0 \).
- If \( a = +\infty \), \( \lim_{a \to +\infty} g(a) = -\infty \).

From above results, we can deduce that there is one positive root in \( (0, 2) \) defined as \( \text{root}_1 \), the other positive root is in \( (2, +\infty) \) defined as \( \text{root}_2 \). Since \( \text{root}_2 > 1 \), we only discuss \( \text{root}_1 \).

- \( a \in [0, \text{root}_1) \), \( g(a) \leq 0 \).
- \( a \in (\text{root}_1, \text{root}_2) \), \( g(a) > 0 \).
Therefore, if \( g\left(\frac{c}{c+2}\right) \geq 0 \), we can conclude that \( \text{root}_1 \leq \frac{c}{c+2} \).

The exact form of \( g\left(\frac{c}{c+2}\right) \) is

\[
g\left(\frac{c}{c+2}\right) = \frac{c^2(c+1)^2(-c^2 + 3c + 14)}{(c+2)^3}. \tag{122}
\]

By solving \( g\left(\frac{c}{c+2}\right) \geq 0 \), we have \( c \leq \ln \left(\frac{3+\sqrt{53}}{2}\right) \approx 5.53 \), i.e. \( \epsilon \leq \ln 5.53 \). From Fig. 19 we can conclude that \( \text{root}_1 \) is the correct root, \( a_0 < 0 \) and \( a_2 > 1 \). \( a = a_1 = -\frac{1}{6}(-4\epsilon^2 - 4\epsilon - \epsilon + 5 + 2\sqrt{15} \cos(\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_5}))). \)

Fig. 19: \( a_0, a_1 \) and \( a_2 \) if \( \epsilon \in [\ln 2, \ln 5.53] \).

Case 4: If \( \epsilon > \ln 5.53, \Delta > 0 \), there are three distinct real roots. From analysis in Case 3, we know that if \( \epsilon > \ln 5.53, \text{root}_1 > \frac{c}{c+2} \) and \( g\left(\frac{c}{c+2}\right) < 0 \). We know that \( g(a) \leq 0 \) if \( a \in [0, \frac{c}{c+2}] \), meaning \( f_1(a) < 0 \), so that \( f_1(a) \) monotonously decreases if \( \epsilon > \ln 5.53 \). Since \( a \in [0, \frac{c}{c+2}] \), we have \( a = \frac{c^2}{\epsilon + 2} \).

Summarize above, we obtain the optimal \( a \) which is named as \( P_{0\rightarrow 0} \) in the Eq. 5. ■

F. Proof of Lemma 5

By substituting the optimal \( P_{0\rightarrow 0} \) of Eq. 5 with \( \alpha \) in the \( \max_{x \in [-1,1]} \text{Var}[Y|x] \) of Eq. 87, we obtain the worst-case noise variance of Three-Outputs as follows:

\[
\begin{align*}
\min_{0 \leq a < a_2} \max_{x \in [-1,1]} & \text{Var}[Y|x] = \\
& \begin{cases}
\frac{(\epsilon^2-1)^2}{(\epsilon^2-1)^2} & \text{for } \epsilon < \ln 2, \\
\frac{\left(\epsilon^2+1\right)^2}{(\epsilon^2-1)^2} & \left(1 - P_{0\rightarrow 0}\right) + \frac{\left(\epsilon^2+1\right)^2}{4} \left(P_{0\rightarrow 0} - P_{0\rightarrow 0}^2\right), \text{for } \ln 2 \leq \epsilon \leq \ln 5.53,
\end{cases}
\end{align*}
\]

where \( P_{0\rightarrow 0} = -\frac{1}{6}(-4\epsilon^2 - 4\epsilon - \epsilon + 5 + 2\sqrt{15} \cos(\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_5}))). \) for \( \epsilon > \ln 5.53. \)

\[
\begin{align*}
\frac{(\epsilon^2+2)^2}{4(\epsilon^2-1)^2} & \text{for } \epsilon > \ln 5.53.
\end{align*}
\tag{123}
\]

G. Proof of \( \frac{2(\epsilon^2-a)^2(\epsilon^2-1)-ace'(\epsilon^2+1)^2}{2(\epsilon^2-a)^2(\epsilon^2+1)^2+ac\epsilon^2} \leq \frac{c-1}{c+1} \) if \( t < \frac{\epsilon}{\epsilon+1} \)

Proposition 1. \( \frac{2(\epsilon^2-a)^2(\epsilon^2-1)-ace'(\epsilon^2+1)^2}{2(\epsilon^2-a)^2(\epsilon^2+1)^2+ac\epsilon^2} \leq \frac{c-1}{c+1} \) for \( \epsilon > 0 \).

Define

\[
f := \frac{2(\epsilon^2-a)^2(\epsilon^2-1)-ace'(\epsilon^2+1)^2}{2(\epsilon^2-a)^2(\epsilon^2+1)^2+ac\epsilon^2} - \frac{\epsilon^2-1}{\epsilon^2+1}.
\]

\[
h := 2(\epsilon^2-a)^2(\epsilon^2+\frac{\epsilon^2}{3}) - ace'(\epsilon^2+1)^2.
\]

Therefore, if \( \epsilon > 0 \), we have \( \epsilon^2 + \frac{\epsilon^2}{3} > 0 \) and \( ace'(\epsilon^2+1)^2 < 0 \), so that we have Eq. (124) < 0.

H. Proof of \( 2(\epsilon^2-a)^2(\epsilon^2-t) - ace'(\epsilon^2+1)^2 > 0 \).

From values of \( \epsilon \), we have the following cases:

- If \( 0 < \epsilon < \ln 2 \), we have \( a = 0 \) and \( h = -2\epsilon^2(\epsilon^2+\epsilon^3) < 0 \). Therefore, we conclude that Eq. (124) < 0.
- If \( \ln 2 \leq \epsilon \leq \ln 5.53 \), we have \( a = -\frac{1}{6}(-2\epsilon^2 - 4\epsilon - \epsilon + 5 + 2\sqrt{15} \cos(\frac{\pi}{3} + \frac{1}{3} \arccos(-\frac{\Delta_1}{2\Delta_5}))). \) Fig. 20 shows that \( h := 2(\epsilon^2-a)^2(\epsilon^2+t) - ace'(\epsilon^2+1)^2 < 0 \), so that we obtain Eq. (124) < 0.
- If \( \epsilon > \ln 5.53 \), we have \( a = \frac{\epsilon^2}{\epsilon+2} \) and \( h = \frac{\epsilon^2}{\epsilon+2} \left(\epsilon^2+\epsilon^3+2\epsilon^2\right) \). Since \( \epsilon^2 \) and \( \epsilon^3+2\epsilon^2 \) > 1, we obtain \( -2\epsilon^2 + 2(\epsilon^2+t) > 0 \) and \( h > 0 \). Hence, we conclude that Eq. (124) < 0.

Based on above analysis, we have Eq. (124) < 0 when \( \epsilon > 0 \), meaning that \( 2(\epsilon^2-a)^2(\epsilon^2-t)-ace'(\epsilon^2+1)^2 \leq \frac{\epsilon^2-1}{\epsilon^2+t} \) when \( \epsilon > 0 \). ■

I. Proving Lemma 6

From Eq. (26a) and Eq. (26b), for any \( Y \in [-A, A] \) and any two input values \( x_1, x_2 \in [-1,1] \), we have \( \frac{\text{pdf}(Y|x_1)}{\text{pdf}(Y|x_2)} \leq \frac{\epsilon}{\epsilon} = \exp(\epsilon) \). Thus, Algorithm 1 satisfies local differential privacy. For notational simplicity, with a fixed \( \epsilon \) below, we will write \( L(e,x,t) \) and \( R(e,x,t) \) as \( L_x \) and \( R_x \). Based on proper probability distribution, we have

\[
\begin{align*}
\mathbb{E}[Y = y|x] dy &= c(L_x - L_x) + d(2A - (R_x - L_x)) = 1.
\end{align*}
\tag{127}
\]
Applying Eq. (132) and Eq. (133) to Inequality (130) and $\alpha$, we define

\[ \text{by solving above Eq. (127) and Eq. (128), we have} \]

\[ \begin{align*}
L_x &= \frac{x}{1-\alpha} - \frac{1-2\alpha}{2(1-\alpha)} = x \cdot \frac{e^\epsilon - 1}{2(1-\alpha)} - \frac{e^\epsilon}{(1-\alpha)} \\
R_x &= \frac{-x}{1-2\alpha} + \frac{1-2\alpha}{2(1-\alpha)} = x \cdot \frac{e^\epsilon}{1-\alpha} + \frac{e^\epsilon}{(1-\alpha)}
\end{align*} \]

Furthermore, the variance of $Y$ is

\[
\text{Var}[Y|x] = \mathbb{E}[Y^2|x] - (\mathbb{E}[Y|x])^2
\]

Substituting Eq. (129) into Eq. (138) yields

\[
\text{Var}[Y|x] = \frac{2d}{3} A^3 + \frac{(c-d)}{3}
\]

where it is clear under privacy parameter $\epsilon$ that

\[
\xi = e^\epsilon - 1.
\]

Applying $\alpha$, $d$ and $\xi$ to Eq. (129), we have

\[
\begin{align*}
L_x &= \frac{x}{1-2\alpha} - \frac{1-2\alpha}{2(1-\alpha)} = x \cdot \frac{e^\epsilon}{1-\alpha} + \frac{e^\epsilon}{1-\alpha} \\
R_x &= \frac{-x}{1-2\alpha} + \frac{1-2\alpha}{2(1-\alpha)} = x \cdot \frac{e^\epsilon}{1-\alpha} - \frac{e^\epsilon}{1-\alpha}
\end{align*} \]

J. Calculate the probability of a variable $Y$ falling in the interval $[L(\epsilon, x, e^{\epsilon/3}), R(\epsilon, x, e^{\epsilon/3})]$. By replacing $t$ in Eq. (26) of PM-OPT with $e^{\epsilon/3}$, we obtain the probability as follows:

\[ \mathbb{P} \left[ L(\epsilon, x, e^{\epsilon/3}) \leq Y \leq R(\epsilon, x, e^{\epsilon/3}) \right] \]

\[ = \int_{L(\epsilon, x, e^{\epsilon/3})}^{R(\epsilon, x, e^{\epsilon/3})} dY \]

Furthermore, the variance of $Y$ is

\[
\text{Var}[Y|x] = \mathbb{E}[Y^2|x] - (\mathbb{E}[Y|x])^2
\]

Substituting Eq. (129) into Eq. (138) yields

\[
\text{Var}[Y|x] = \frac{2d}{3} A^3 + \frac{(c-d)}{3}
\]

where it is clear under privacy parameter $\epsilon$ that

\[
\xi = e^\epsilon - 1.
\]

Applying $\alpha$, $d$ and $\xi$ to Eq. (129), we have

\[
\begin{align*}
L_x &= \frac{x}{1-2\alpha} - \frac{1-2\alpha}{2(1-\alpha)} = x \cdot \frac{e^\epsilon}{1-\alpha} + \frac{e^\epsilon}{1-\alpha} \\
R_x &= \frac{-x}{1-2\alpha} + \frac{1-2\alpha}{2(1-\alpha)} = x \cdot \frac{e^\epsilon}{1-\alpha} - \frac{e^\epsilon}{1-\alpha}
\end{align*} \]

J. Calculate the probability of a variable $Y$ falling in the interval $[L(\epsilon, x, e^{\epsilon/3}), R(\epsilon, x, e^{\epsilon/3})]$. By replacing $t$ in Eq. (26) of PM-OPT with $e^{\epsilon/3}$, we obtain the probability as follows:

\[ \mathbb{P} \left[ L(\epsilon, x, e^{\epsilon/3}) \leq Y \leq R(\epsilon, x, e^{\epsilon/3}) \right] \]

\[ = \int_{L(\epsilon, x, e^{\epsilon/3})}^{R(\epsilon, x, e^{\epsilon/3})} dY \]

Furthermore, the variance of $Y$ is

\[
\text{Var}[Y|x] = \mathbb{E}[Y^2|x] - (\mathbb{E}[Y|x])^2
\]

Substituting Eq. (129) into Eq. (138) yields

\[
\text{Var}[Y|x] = \frac{2d}{3} A^3 + \frac{(c-d)}{3}
\]

where it is clear under privacy parameter $\epsilon$ that

\[
\xi = e^\epsilon - 1.
\]
K. Proof of Lemma 8

The expression of these two probabilities in Eq. (143) can be solved from the following:

\[
\begin{align*}
\mathbb{P}[Z = \frac{kC}{m} | Y = y] & = \frac{kC}{m}, \\
\mathbb{E}[Z | Y = y] & = y \\
\mathbb{E}[Z | Y = y] & = \mathbb{E}[Z | Y = y] = y \\
(\mathbb{P}[Z = \frac{kC}{m} | Y = y] + \mathbb{P}[Z = \frac{(k+1)C}{m} | Y = y]) & = 1, \\
(\mathbb{P}[Z = \frac{kC}{m} | Y = y] + \mathbb{P}[Z = \frac{(k+1)C}{m} | Y = y]) & = y.
\end{align*}
\]

Summarizing (1) and (2), with \(k := \left\lfloor \frac{ym}{c} \right\rfloor \), we have

\[
\mathbb{P}[Z = z | Y = y] = \begin{cases} 
\frac{k + 1 - \frac{ym}{c}}, & \text{if } z = \frac{kC}{m}, \\
\frac{ym}{c} - k, & \text{if } z = \frac{(k+1)C}{m}.
\end{cases}
\]

In the perturbation step, the distribution of \(Y\) given the input \(x\) is given by

\[
\mathbb{P}[Y = y | x] = \begin{cases} 
p_1, & \text{if } y \in [L(x), R(x)], \\
p_2, & \text{if } y \in [-C, L(x)) \cup (R(x), C].
\end{cases}
\]

Hence,

\[
\mathbb{P}[Z = z | x] = \int_y \mathbb{P}[Z = z | Y = y] \mathbb{P}[Y = y | x] dy
\]

and

\[
\mathbb{E}[Y | x] = \int_y y \times \mathbb{P}[Y = y | x] dy = x.
\]

Therefore, we obtain

\[
\mathbb{E}[Z | x] = \sum_z z \times \mathbb{P}[Z = z | x] = \int_y \left( \sum_z z \times \mathbb{P}[Z = z | Y = y] \right) \mathbb{P}[Y = y | x] dy
\]

\[
\mathbb{E}[Z^2 | x] = \int_y y \times \mathbb{P}[Y = y | x] dy = x.
\]

L. Proof of Lemma 9

To prove \(\text{Var}[Z | X = x] \geq \text{Var}[Y | X = x]\), it is equivalent to prove

\[
\mathbb{E}[Z^2 | X = x] - (\mathbb{E}[Z | X = x])^2 \geq \mathbb{E}[Y^2 | X = x] - (\mathbb{E}[Y | X = x])^2.
\]

Since \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are unbiased, we have \(\mathbb{E}[Z | X = x] = \mathbb{E}[Y | X = x] = x\). Hence, it is sufficient to prove

\[
\mathbb{E}[Z^2 | X = x] \geq \mathbb{E}[Y^2 | X = x].
\]

We can derive that

\[
\mathbb{E}[Z^2 | X = x] = \sum_z z^2 \cdot \mathbb{P}[Z = z | X = x]
\]

and

\[
\mathbb{E}[Y^2 | X = x] = \int_y y^2 \cdot \mathbb{P}[Y = y | X = x] dy.
\]

To prove Inequality (147), because of Eq. (148) and Eq. (149), it is sufficient to prove

\[
\mathbb{E}[Z^2 | Y = y] \geq y^2, \quad \forall y \in \text{Range}(Y).
\]

After getting the intermediate output \(y\) from \(\mathcal{M}_1\), we may discretize the intermediate output \(y\) into \(z_1\) with probability \(p_1\) and \(z_2\) with probability \(p_2\). Hereby,

\[
p_1 + p_2 = 1.
\]

Mechanism \(\mathcal{M}_2\) is unbiased, so that \(\mathbb{E}[Z | Y = y] = y\), and we have

\[
p_1 \cdot z_1 + p_2 \cdot z_2 = y.
\]

According to Cauchy–Schwarz inequality, we have

\[
\mathbb{E}[Z^2 | Y = y] = p_1 \cdot z_1^2 + p_2 \cdot z_2^2 \\
\leq (\mathbb{E}[z_1^2] + (\mathbb{E}[z_2^2])^2) \geq (p_1 \cdot z_1 + p_2 \cdot z_2)^2 = y^2.
\]

Thus, we get Inequality (147) and Inequality (150).

M. Proof of Lemma 10

The \(\max_{x \in [-1, 1]} \text{Var}_x[Y | x]\) is minimized when

\[
\beta = \begin{cases} 
0, & \text{if } 0 < \epsilon < \epsilon^*, \\
\beta_1 = \frac{2(\epsilon^* - a)(\epsilon^* - 1) - a \epsilon^*}{2(\epsilon^* - a)^2(\epsilon^* + 1) - a \epsilon^*}, & \text{if } \epsilon^* < \epsilon < 2, \\
\beta_2 = \frac{\sqrt{\epsilon}}{\epsilon^* - 1}, & \text{if } \epsilon \geq 2.
\end{cases}
\]

Where

\[
\epsilon^* := 3 \ln(\text{root of } 3x^5 - 2x^3 + 3x^2 - 5x - 3 \text{ near } x = 1.22588) \\approx 0.610986.
\]

Proof. If \(x = x^* = \frac{(\beta - 1) a \epsilon^* (\epsilon^* + 1)}{2(\epsilon - a)^2(\beta(\epsilon^* + 1) - \epsilon^* + 1)}\), we have variance of \(Y\) as follows:

\[
\text{Var}_x[Y^*]
\]
\[(\beta_t + 1) + \beta - 1 \cdot (2(\beta_0 + t - e^\gamma + 1)^2 - 2(\beta_0 + t - e^\gamma + 1) + 1) \]
\[+ (1 - \beta) \cdot \frac{ae^\gamma (e^\gamma + 1)^2}{(e^\gamma - 1)(e^\gamma - a)^2} \cdot \frac{(\beta - 1)ae^\gamma (e^\gamma + 1)^2}{2(e^\gamma - a)^2(\beta(e^\gamma + t) - e^\gamma + 1)}
\[+ \frac{(t + e^\gamma)(t + 1)^3 + e^\gamma + 1}{3t^2(e^\gamma - 1)^2} \beta
\[+ (1 - \beta)(1 - a) \cdot \frac{e^\gamma (e^\gamma + 1)^2}{(e^\gamma - 1)(e^\gamma - a)^2}. \] 

(153)

Let \( \gamma := (e^\gamma + t) - e^\gamma + 1 \) and \( e := e^\gamma \), we can transform \( \text{Var}_H[Y|x^*] \) Eq. (153) to the following:

\[a_2^2 c^2 (c + 1)^4 \cdot 4(c + t)^2(c - a)^4(c - 1) - 2(c + t)^2(c - a)(c - a)^4
\[+ \frac{(t + 1)^3 + a^2 c^2 (c + 1)^2}{(c + t)(c - 1)^2(c - a)^2}, \] 

(154)

Set coefficient of \( \gamma \) as \( A \):

\[A := a_2^2 c^2 (c + 1)^4 \cdot 4(c + t)^2(c - a)^4(c - 1) - 2(c + t)^2(c - a)(c - a)^4
\[+ \frac{(t + 1)^3 + c^2 (c + 1)^2}{(c + t)(c - 1)^2(c - a)^2} \] 

(155)

Set coefficient of \( \frac{1}{\gamma} \) as \( B \):

\[B := \frac{(t + 1)^2 a_2^2 c^2 (c + 1)^4}{4(c + t)^2(c - a)^4(c - 1)}. \] 

(156)

Set \( C \) as:

\[C := -\frac{(t + 1)^2 a_2^2 c^2 (c + 1)^4}{2(c + t)^2(c - a)^4(c - 1)} + \frac{(t + 1)^2 a_2^2 c^2 (c + 1)^2}{(c + t)(c - 1)^2(c - a)^2}
\[+ \frac{(t + 1)^3 + c^2 (c + 1)^2}{3t^2(c - 1)^2(c - a)^2}. \] 

(157)

Since \( \gamma \) monotonically increases with \( \beta \) in the domain \( \beta \in (0, \beta_1) \), the minimum \( \gamma \) is \( \gamma_1 := -e^\gamma + 1 \) at \( \beta = 0 \), maximum \( \gamma \) is \( \gamma_2 := 0 \) at \( \beta = \beta_1 \).

- If \( \beta \in (0, \beta_1) \), Appendix 0 proves:
  a) \( A > 0 \), if \( 0 < \epsilon < 0.610986 \).
  b) \( A = 0 \), if \( \epsilon = 0.610986 \).
  c) \( A < 0 \), if \( 0.610986 < \epsilon < \beta_1 \).

Therefore, \( \min_{\beta} \max_{x \in [-1, 1]} \text{Var}_H[Y|x] \) is at:

- \( \beta = 0 \), if \( 0 < \epsilon < 0.610986 \).
- \( \beta = \beta_1 \), if \( 0.610986 \leq \epsilon < \ln 2 \).

Therefore, \( \min_{\beta} \max_{x \in [-1, 1]} \text{Var}_H[Y|x] \) is at \( \beta = \beta_1 \) if \( \beta \in [\beta_1, \beta_2] \). Summarize above analysis, we can conclude that \( \min_{\beta} \max_{x \in [-1, 1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|x, \beta = \beta_1] \).

- \( \beta \in [\beta_1, \beta_2] \), slope_1 = \( \frac{4 + \epsilon^2}{2(\epsilon - 1)^2} + 1 + \frac{4 + \epsilon^2((t + 1)^2 + \epsilon^2 - 1)}{(\epsilon - 1)^2} - \frac{(t + 1)^2 + \epsilon^2 - 1}{(\epsilon - 1)^2} \).

Fig. 22 proves that slope_1 > 0, where \( \epsilon \in [0, \ln 2] \).
• If \( \ln 2 \leq \epsilon \leq \ln 5.53 \),
  When \( \beta \in (0, \beta_1) \), Appendix Q proves \( A < 0 \), \( B < 0 \), Appendix N proves when \( \gamma = -\sqrt{\frac{A}{B}} \), we have:
  \[
  \max_{x, \beta} \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \max_{x \in [-1,1]} \text{Var}_H[Y|x, \beta] = \beta_3. \]
  Since \( \gamma := \beta(c + t) - c + 1 \), we have \( \beta_3 := -\sqrt{\frac{A}{B}} + c - 1 \), \( \min_{x, \beta} \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \max_{x \in [-1,1]} \text{Var}_H[Y|x, \beta] = \beta_3 \).
  When \( \beta \in [\beta_1, \beta_2] \), Fig. 24 shows that slope_1 > 0 if \( \epsilon \in [\ln 2, \ln 5.53] \), Fig. 24 shows that slope_2:

  - If \( 0 < \epsilon < 1.4338 \), slope_2 < 0, \( \beta_{\text{intersection}} < \beta_1 \), see Fig. 25
    \[ \min_{x, \beta} \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \max_{x \in [-1,1]} \text{Var}_H[Y|x, \beta] = \beta_1. \]
  - If \( \epsilon \approx 1.4338 \), slope_2 = 0, \( \beta_{\text{intersection}} < \beta_1 \), see Fig. 25
    \[ \min_{x, \beta} \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \max_{x \in [-1,1]} \text{Var}_H[Y|x, \beta] = \beta_1. \]
  - If \( 1.4338 < \epsilon \leq \ln 5.53 \), slope_2 > 0, \( \min_{x, \beta} \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \max_{x \in [-1,1]} \text{Var}_H[Y|x, \beta] = \beta_1. \]
  Since \( \text{Var}_H[Y|x, \beta] = \beta_3 < \text{Var}_H[Y|x, \beta = \beta_1] \), \( \min_{x, \beta} \max_{x \in [-1,1]} \text{Var}_H[Y|x] \) is at \( \beta = \beta_3 \).

\[ N. \text{ Proof of the monotonicity of } \text{Var}_H[Y|x^*] \]

Substituting \( A \) (Eq. 155), \( B \) (Eq. 156) and \( C \) (Eq. 157) into \( \text{Var}_H[Y|x^*] \) (Eq. 154) yields:
\[
\text{Var}_H[Y|x^*] = A\gamma + \frac{B}{\gamma} + C. \quad (158)
\]
The first order derivative of (158) is:
\[
\text{Var}_H[Y|x^*]' = A - \frac{B}{\gamma^2}. \quad (159)
\]
If \( A - \frac{B}{\gamma^2} = 0 \), we get two roots:
\[
\gamma_1 = -\sqrt{\frac{B}{A}}, \quad \gamma_2 = \sqrt{\frac{B}{A}}.
\]
Since \( \gamma < 0 \), \( \gamma_1 \) is eligible. Hereby,

• If \( \gamma \in (-\infty, -\sqrt{\frac{B}{A}}] \), \( \text{Var}_H[Y|x^*]' < 0 \), \( \text{Var}_H[Y|x^*] \) monotonically decreases.
• If \( \gamma \in (-\sqrt{\frac{B}{A}}, \infty) \), \( \text{Var}_H[Y|x^*]' > 0 \), \( \text{Var}_H[Y|x^*] \) monotonically increases.

\[ O. \text{ The sign of } A \text{ to } \epsilon \]

If \( A = 0 \), we have \( \epsilon = 3 \ln(\text{root of } 3\epsilon^5 - 2\epsilon^3 + 3\epsilon^2 - 5\epsilon - 3 \text{ near } x = 1.22588) \approx 0.610986 \).
First order derivative of \( A \) is:
\[
A' = -(25\epsilon^5 - 27\epsilon^2 + 9\epsilon^5e^\epsilon - 12e^\epsilon - 6\epsilon^2e^\epsilon - 6\epsilon^4e^\epsilon + 41\epsilon^5e^\epsilon + 7\epsilon^5e^\epsilon + 5)/(9\epsilon^2e^\epsilon(2\epsilon^2e^\epsilon + 1)^2(e^\epsilon - 1)^3). \quad (160)
\]
• If \( 0 < \epsilon < \ln 2 \), Fig. 26 shows that \( A' < 0 \) and \( A \) monotonically decreases if \( \epsilon \in (0, \ln 2) \). Therefore, we have
- $A > 0$, if $0 < \epsilon < 0.610986$.
- $A = 0$, if $\epsilon = 0.610986$.
- $A < 0$, if $0.610986 < \epsilon < \ln 2$.

- If $\ln 2 \leq \epsilon \leq \ln 5.53$, Fig. 27 shows $A < 0$.

- If $\epsilon > \ln 5.53$, we obtain

$$A = -(16e^\epsilon + 21e^{2\epsilon} + 3e^{3\epsilon} + 36e^\frac{\epsilon}{2} - 12e^\frac{\epsilon}{4} - 28e^\frac{\epsilon}{2} - 8e^\frac{\epsilon}{4} - 12) \approx 0.0463914.$$  

When $A = 0$, we have three roots:

$$r_1 \approx -16.9563, r_2 \approx -1.2284, r_3 \approx 0.0463914.$$  

If the denominator

$$\lim_{\epsilon \to -\infty} -(16e^\epsilon + 21e^{2\epsilon} + 3e^{3\epsilon} + 36e^\frac{\epsilon}{2} - 12e^\frac{\epsilon}{4} - 28e^\frac{\epsilon}{2} - 8e^\frac{\epsilon}{4} - 12) = -\infty,$$

$r_3$ is the largest real value root, the sign of

$$-(16e^\epsilon + 21e^{2\epsilon} + 3e^{3\epsilon} + 36e^\frac{\epsilon}{2} - 12e^\frac{\epsilon}{4} - 28e^\frac{\epsilon}{2} - 8e^\frac{\epsilon}{4} - 12)$$

doesn’t change, so that $A < 0$ when $\epsilon > \ln 5.53$.

---

**P. The sign of slope$_1$ when $\epsilon > \ln 5.53$**

If

$$\text{slope}_1 = \frac{t + 1}{e^\epsilon - 1} + 1 - \frac{ae^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)(e^\epsilon - a)^2} + \frac{(t + e^\epsilon)((t + 1)^3 + e^\epsilon - 1)}{32((e^\epsilon - 1)^2)} - \frac{(1 - a)e^{2\epsilon}(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} = 0,$$

we have two roots:

$$e_1' \approx -0.141506, e_2' \approx 2.21598.$$  

The first order derivative of slope$_1$ is

$$\text{slope}'_1 = -\frac{e^\frac{\epsilon}{2}(10e^\epsilon + 20) + 20e^\epsilon + (-27e^\epsilon - 45)e^\frac{\epsilon}{2} + 10}{9e^\frac{\epsilon}{2}(e^\epsilon - 1)^3}.$$

The denominator of the slope$_1'$ is $> 0$. Thus,

$$\lim_{\epsilon \to -\infty} -\frac{e^\frac{\epsilon}{2}(10e^\epsilon + 20) + 20e^\epsilon + (-27e^\epsilon - 45)e^\frac{\epsilon}{2} + 10}{9e^\frac{\epsilon}{2}(e^\epsilon - 1)^3} = -\infty.$$  

If slope$_1'$ is $0$, we have three roots:

$$e_1' \approx -1.35696, e_2' \approx 0.0169067, e_3' \approx 4.22192.$$  

Since $e_3' \approx 4.22192$ is the largest real value root, the sign of slope$_1$ doesn’t change when $\epsilon > 4.22192$. Therefore, when $\epsilon > \ln 5.53$ and slope$_1' < 0$, slope$_1$ monotonically decreases. By simplifying slope$_1$, we get

$$\text{slope}_1 = -\frac{(9e^\epsilon - 5e^{2\epsilon} - 5e^{3\epsilon} + 3)}{3(e^\epsilon - 1)^2}.$$

Then, we obtain $\lim_{\epsilon \to -\infty} \text{slope}_1 = 0$. Thus, we have slope$_1 > 0$ if $\epsilon > \ln 5.53$.

---

**Q. The sign of slope$_2$ when $\epsilon > \ln 5.53$**

When

$$\text{slope}_2 = \frac{(t + e^\epsilon)((t + 1)^3 + e^\epsilon - 1)}{32((e^\epsilon - 1)^2)} - \frac{(1 - a)e^{2\epsilon}(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} = 0,$$

we have

$$\epsilon_1 \approx \ln -1.24835, \epsilon_2 \approx \ln 1.52144.$$  

The first order derivative of slope$_2$ is:

$$\text{slope}'_2 = -\frac{4e^{2\epsilon} + e^\frac{\epsilon}{2}(9e^\epsilon + 63) - 23e^\epsilon + (-20e^\epsilon - 10)e^\frac{\epsilon}{2} - 3}{9e^\frac{\epsilon}{2}(e^\epsilon - 1)^3}.$$

The denominator of the Eq. (162) is $> 0$. Besides, from nominator of Eq. (162), we obtain

$$\lim_{\epsilon \to -\infty} (-4e^{2\epsilon} + e^\frac{\epsilon}{2}(9e^\epsilon + 63) - 23e^\epsilon + (-20e^\epsilon - 10)e^\frac{\epsilon}{2} - 3) = -\infty.$$
If Eq. 162 = 0, we have two roots:
\[\epsilon_1 = 3 \ln(\text{root of } 4x^6 - 9x^5 + 20x^4 + 23x^3 - 63x^2 + 10x + 3)\]
\[\approx -3.13865,\]
\[\epsilon_2 = 3 \ln(\text{root of } 4x^6 - 9x^5 + 20x^4 + 23x^3 - 63x^2 + 10x + 3)\]
\[\approx 0.709472.\]

Since \(\epsilon \approx 0.709472\) is the largest real value root, the sign of slope2 doesn’t change if \(\epsilon > 0.709472\). Therefore, slope2 < 0 if \(\epsilon > 5.53\).

Simplify
\[
\text{slope}_2 = \frac{3e^\epsilon - 3e^\epsilon + 5e^\epsilon + 2e^\epsilon - 9}{3(e^\epsilon - 1)^2},
\]
and then we get
\[
\lim_{\epsilon \to \infty} \text{slope}_2 = 0.
\]

Thus, we have slope2 > 0 if \(\epsilon > 5.53\).

**R. Proof of Lemma 12**

For any \(i \in [1, n]\), the random variable \(Y[t_j] - x[t_j]\) has zero mean based on Lemma 12. In both PM-SUB and \(\text{HMPM-SUB, Three-Outputs}\), \(|Y[t_j] - x[t_j]| \leq \frac{d}{k} E[(e^\epsilon + e^\epsilon)(e^\epsilon + 1)]\).

By Bernstein’s inequality, we have
\[
P[|Y[t_j] - Z[t_j]| \geq \lambda] = Pr\left[\sum_{i=1}^{n} |Y[t_j] - x[t_j]| \geq n\lambda\right]
\leq 2 \cdot \exp\left(-\frac{(n\lambda)^2}{2 \cdot \sum_{i=1}^{n} \text{Var}[Y[t_j]] + d \cdot \lambda \cdot \frac{d}{k} E[(e^\epsilon + e^\epsilon)(e^\epsilon + 1)]}\right).
\]

In Algorithm 5 \(Y[t_j]\) equals \(\frac{d}{k} y_j\) with probability \(\frac{k}{n}\) and 0 with probability \(1 - \frac{k}{n}\). Moreover, we obtain \(E[Y[t_j]] = x[t_j]\) from Lemma 12 and then we get
\[
\text{Var}[Y[t_j]] = E[(Y[t_j]^2)] - E[Y[t_j]]^2
= \frac{k}{d} E[(\frac{d}{k} y_j)^2] - (x[t_j])^2
= \frac{d}{k} E[(y_j)^2] - (x[t_j])^2. \tag{163}
\]

In Algorithm 6 if Line 5 uses PM-SUB, we use the variance in Eq. 164 to compute \(E[(y_j)^2]\), the asymptotic expression involving \(\epsilon\) are in the sense of \(\epsilon \to 0\).

\[E[(y_j)^2] = \text{Var}[y_j] + (E[y_j])^2
= \frac{t(\epsilon_j^*) + 1}{e^\epsilon - 1} (x[t_j])^2 + \frac{(t(\epsilon_j^*) + e^\epsilon)(t(\epsilon_j^*) + 1 + e^\epsilon - 1)}{3(t(\epsilon_j^*)^2(e^\epsilon - 1)^2)}
+ (x[t_j])^2 = O\left(\frac{k^2}{e^\epsilon}\right). \tag{164}\]

In Algorithm 6 if Line 5 uses Three-Outputs, and then we use the variance in Eq. 172 to compute \(E[(y_j)^2]\), the asymptotic expression involving \(\epsilon\) are in the sense of \(\epsilon \to 0\).

\[E[(y_j)^2] = \text{Var}[y_j] + (E[y_j])^2
= \frac{(1 - a)e^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} + \frac{b|x[t_j]|e^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} - (x[t_j])^2
+ (x[t_j])^2
= \frac{(1 - a)e^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} + \frac{b|x[t_j]|e^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} = O\left(\frac{k^2}{e^\epsilon}\right),
\]

In Algorithm 6 if Line 5 uses HPM-TP, we have:
\[E[(y_j)^2] = \text{Var}[y_j] + (E[y_j])^2
= \left\{
\begin{array}{ll}
(\frac{e^\epsilon + 1}{(e^\epsilon - 1)^2})^2 + (x[t_j])^2, & \text{if } 0 < \epsilon < \epsilon^*, \\
\text{Var}_{\text{H}}[y_j] + (x[t_j])^2, & \text{if } \epsilon^* \leq \epsilon < \ln 2,
\end{array}
\right.
\]
\[= O\left(\frac{k^2}{e^\epsilon}\right), \tag{165}\]

where \(\epsilon^*\) is defined in the Eq. 152. Then,
\[\text{Var}[Y[t_j]] = \frac{d}{k} O\left(\frac{k^2}{e^\epsilon}\right) - (x[t_j])^2 = O\left(\frac{dk^2}{e^\epsilon}\right). \tag{166}\]

Therefore, we obtain
\[P[|Z[t_j] - X[t_j]| \geq \lambda] \leq 2 \cdot \exp\left(-\frac{n\lambda^2}{2 d \cdot (O(\ln(d/\beta)) + O(\ln(d/\epsilon)))}\right).
\]

By the union bound, there exists \(\lambda = O\left(\frac{\sqrt{\ln(\ln(d/\beta))}}{e^\epsilon}\right)\).

Therefore, \(\max_{j \in [1, d]} |Z[t_j] - X[t_j]| = \lambda = O\left(\frac{\ln(\ln(d/\beta))}{e^\epsilon}\right)\).

**S. Calculate k for PM-SUB and Three-Outputs**

We calculate the optimal \(k\) for PM-SUB and Three-Outputs.

(I) We calculate the \(k\) for \(d\) dimension PM-SUB. When \(x[t_j] = 1\), we get
\[
\max \text{Var}[Y[t_j]]
= \frac{d}{k} \left(\frac{t(\epsilon_j^*) + 1}{e^\epsilon - 1} + \frac{(t(\epsilon_j^*) + e^\epsilon)(t(\epsilon_j^*) + 1 + e^\epsilon - 1)}{3(t(\epsilon_j^*)^2(e^\epsilon - 1)^2)} + 1\right) - 1.
\]

For PM-SUB, we have
\[
\max \text{Var}[Y[t_j]]
= \frac{d}{k} \left(\frac{e^\epsilon + 1}{e^\epsilon - 1} + \frac{(e^\epsilon + e^{\epsilon*})(e^\epsilon + 1)^2 + e^\epsilon - 1}{3(e^\epsilon)^2(e^\epsilon - 1)^2} + 1\right) - 1.
\]

Let \(s = \frac{\epsilon_j^*}{e^\epsilon}\), and then
\[
\max \text{Var}[Y[t_j]]
= \frac{d}{e^\epsilon} \left(s + 1 + \frac{(s + e^{\epsilon*})(s + 1)^2 + e^{\epsilon*} - 1}{3(s)^2(s - 1)^2} + 1\right) - 1.
\]

Let \(f(s) = s + 1 + \frac{(s + e^{\epsilon*})(s + 1)^2 + e^{\epsilon*} - 1}{3(s)^2(s - 1)^2} + 1\),
and then we obtain
\[
\max \text{Var}[Y[t_j]] = \frac{d}{e^\epsilon} \cdot f(s) - 1. \tag{167}\]
From numerical experiments shown in Fig. 28, we conclude that we can get
\( \min f(s) \) and \( \max \text{Var}[Y[t_j]] \) if \( s = 2.5 \), i.e. \( k = \frac{4}{e^2} \).

(II) Calculate the \( k \) for \( d \) dimension Three-Outputs. The variance of \( Y[t_j] \) is

\[
\text{Var}[Y[t_j]] = \frac{d}{k} \left( \frac{(1-a)e^{\frac{x}{k}}(e^{\xi} + 1)^2 + b|x[t_j]|e^{\frac{x}{k}}(e^{\xi} + 1)^2}{(e^{\xi} - 1)^2(e^{\xi} - a)^2} \right) - (x[t_j])^2.
\]

where \( b \) is from Eq. (76) and \( a \) is from Eq. (75).

Let \( x[t_j]' = \frac{db}{2(k(e^{\xi} - 1)^2(e^{\xi} - a)^2)} \), if \( 0 < x[t_j]' < 1 \), the worst-case noise variance of \( Y \) is

\[
\max_{x \in [-1, 1]} \text{Var}[Y[t_j]] = \left\{ \begin{array}{ll}
\text{Var}[x[t_j]'], & \text{if } 0 < x[t_j]' < 1, \\
\max\{\text{Var}[0], \text{Var}[1]\}, & \text{otherwise}.
\end{array} \right.
\]

(168)

Let \( s = \frac{x}{k} \), and then

\[
x[t_j]' = \frac{d}{k} \cdot \frac{b(s)e^{2s}(e^x + 1)^2}{2(e^{x} - 1)^2(e^{x} - a(s))^2}.
\]

If \( 0 < \frac{d}{k} \cdot \frac{b(s)e^{2s}(e^x + 1)^2}{2(e^{x} - 1)^2(e^{x} - a(s))^2} < 1 \), and then

\[
\max \text{Var}[Y[t_j]] = \max \text{Var}[x[t_j]'] = \frac{d}{k} \left( \frac{(1-a)e^{\frac{x}{k}}(e^{\xi} + 1)^2 + b|x[t_j]|e^{\frac{x}{k}}(e^{\xi} + 1)^2}{(e^{\xi} - 1)^2(e^{\xi} - a)^2} \right) - (x[t_j]')^2.
\]

\[
= \frac{d}{k} \cdot \frac{(b(s))^2e^{4s}(e^x + 1)^4}{2(e^{x} - 1)^2(e^{x} - a(s))^4} - \frac{d^2}{e^2} \cdot \frac{(b(s))^2e^{4s}(e^x + 1)^4}{4(e^{x} - 1)^4(e^{x} - a(s))^4} + \frac{d}{e} \cdot \frac{(1-a(s))^2e^{2s}(e^x + 1)^2}{(e^{x} - 1)^4(e^{x} - a(s))^2}.
\]

(169)

Substituting \( b = a \cdot \frac{e^{-1}}{e^x} \) into Eq. (169) yields

\[
= \frac{d}{k} \cdot \frac{(b(s))^2e^{2s}(e^x + 1)^4}{4(e^{x} - 1)^2(e^{x} - a(s))^4} + \frac{d}{e} \cdot \frac{(1-a(s))^2e^{2s}(e^x + 1)^2}{(e^{x} - 1)^4(e^{x} - a(s))^2}.
\]

- If \( \epsilon < \ln 2, a = 0, b = 0 \), first-order derivative of \( \max \text{Var}[Y[t_j]] \) is

\[
\max \text{Var}[Y'[t_j]] = \frac{d}{k} \cdot \frac{(e^{x} + 1)(-4se^x + e^{2s} - 1)}{(e^{x} - 1)^3}.
\]

(170)

When \( \max \text{Var}[Y[t_j]]' = 0 \), we have root \( s \approx 2.18 \).

- If \( \ln 2 < \epsilon < \ln 5.5 \), by numerical experiments, we have optimal \( s \approx 2.5 \).

- If \( \epsilon \geq \ln 5.5 \), by numerical experiments, we have optimal \( s \approx 2.5 \).

Therefore, we pick \( s = 2.5 \) and \( k = \frac{4}{e^2} \) to simplify the experimental evaluation.

\( \Box \)

T. Extending Three-Outputs for Multiple Numeric Attributes

**Lemma 14.** For a \( d \)-dimensional numeric tuple \( x \) which is perturbed as \( Y \) under \( \epsilon \)-LDP, and for each \( t_j \) of the \( d \) attribute, the variance of \( Y[t_j] \) induced by Three-Outputs is

\[
\text{Var}[Y[t_j]] = \frac{d}{k} \left( \frac{(1-a)e^{\frac{x}{k}}(e^{\xi} + 1)^2 + b|x[t_j]|e^{\frac{x}{k}}(e^{\xi} + 1)^2}{(e^{\xi} - 1)^2(e^{\xi} - a)^2} \right) - (x[t_j])^2.
\]

**Proof of Lemma** The variance of \( Y[t_j] \) is computed as

\[
\text{Var}[Y[t_j]] = \mathbb{E}[(Y[t_j])^2] - \mathbb{E}[Y[t_j]]^2
\]

\[
= \frac{k}{d} \mathbb{E}[\left( \frac{d}{k} y_j \right)^2] - (x[t_j])^2
\]

\[
= \frac{d}{k} \mathbb{E}[y_j^2] - (x[t_j])^2.
\]

(171)

We use variance Eq. (173) to compute

\[
\mathbb{E}[y_j^2] = \text{Var}[y_j] + (\mathbb{E}[y_j])^2
\]

\[
= \left( 1-a \right) e^{\frac{x}{k}}(e^{\xi} + 1)^2 + \frac{b|x[t_j]|e^{\frac{x}{k}}(e^{\xi} + 1)^2}{(e^{\xi} - 1)^2(e^{\xi} - a)^2} - (x[t_j])^2
\]

\[
= \left( 1-a \right) e^{\frac{x}{k}}(e^{\xi} + 1)^2 + \frac{b|x[t_j]|e^{\frac{x}{k}}(e^{\xi} + 1)^2}{(e^{\xi} - 1)^2(e^{\xi} - a)^2}.
\]

Then,

\[
\text{Var}[Y[t_j]] = \frac{d}{k} \left( \frac{(1-a)e^{\frac{x}{k}}(e^{\xi} + 1)^2 + b|x[t_j]|e^{\frac{x}{k}}(e^{\xi} + 1)^2}{(e^{\xi} - 1)^2(e^{\xi} - a)^2} \right) - (x[t_j])^2.
\]

\( \Box \)
Given PM–SUB’s variance in Eq. (139), we have
\[
\text{Var}_P[Y|x] = \left(1 - \frac{1}{1 - 2Ad} \right) x^2 + \frac{2d}{3} A^3 + \frac{(1 - 2Ad)^3}{12(c-d)^2}.
\]
Substituting \(\alpha = Ad\) yields
\[
\text{Var}_P[Y|x] = \left(1 - \frac{2\alpha}{1 - 2\alpha} \right) x^2 + \frac{2\alpha}{3d^2} \left(1 - \frac{2\alpha}{1 - 2\alpha} \right)^3.
\]
In addition, substitute \(1 - 2\alpha = \frac{e^d - d}{e - d}, \text{ } d = \frac{e^d}{2(e^d + e^{-d})}; \xi := \frac{e^d - d}{e - d}\). Then, we have
\[
\text{Var}_P[Y|x] = \frac{t + 1}{e^d - 1} x^2 + \frac{(t + e^d)(t + 1)^3 + e^d - 1}{3t^2(e^d - 1)^2}.
\]
(172)

Given Three-Outputs’s variance in Eq. (173), we can simplify it as
\[
\text{Var}_T[Y|x] = (1 - a)C^2 + C^2 b|x - x^2.
\]
According to Eq. (72), we have \(C = \frac{e'}{e - a}\), so that
\[
\text{Var}_P[Y|x] = \frac{(1 - a)e^{2\alpha}(e^d + 1)^2}{(e^d - 1)^2(e^e - a)^2} + \frac{b|x|e^{2\alpha}(e^d + 1)^2}{(e^d - 1)^2(e^e - a)^2} - x^2.
\]
(173)

Based on Eq. (172) and Eq. (173), we can construct variance of hybrid mechanism as follows
\[
\text{Var}_H[Y|x] = \beta \left(\frac{t + 1}{e^d - 1} x^2 + \frac{(t + e^d)(t + 1)^3 + e^d - 1}{3t^2(e^d - 1)^2} \right) + (1 - \beta) \left(\frac{(1 - a)e^{2\alpha}(e^d + 1)^2}{(e^d - 1)^2(e^e - a)^2} + \frac{b|x|e^{2\alpha}(e^d + 1)^2}{(e^d - 1)^2(e^e - a)^2} - x^2\right),
\]
where \(t = e^{d/3}\). From Eq. (72), we set \(b = a \frac{e^d - 1}{e - a}\) to get the worst-case noise variance. Then, we have variance of the hybrid mechanism as
\[
\text{Var}_H[Y|x] = \beta \left(\frac{t + 1}{e^d - 1} x^2 + \frac{(t + e^d)(t + 1)^3 + e^d - 1}{3t^2(e^d - 1)^2} \right) + (1 - \beta) \left(\frac{(1 - a)e^{2\alpha}(e^d + 1)^2}{(e^d - 1)^2(e^e - a)^2} + \frac{b|x|e^{2\alpha}(e^d + 1)^2}{(e^d - 1)^2(e^e - a)^2} - x^2\right),
\]
where \(t = e^{d/3}\). Based on Eq. (174), we get
\[
\max_{x \in [-1,1]} \text{Var}_H[Y|x] = \begin{cases} 
\text{Var}_H[Y|x^*], & \text{if } \beta \frac{t + 1}{e^d - 1} + \beta - 1 < 0, 0 < x^* < 1, \\
\max \{\text{Var}_H[Y|0], \text{Var}_H[Y|1]\}, & \text{otherwise},
\end{cases}
\]
where \(x^* := \frac{(1 - \beta)ae^{e^d} + e^d - 1 - 2\beta}{2(e^d - a)2(e^d + 1)e^{e^d} - e^d + 1}\).
Therefore, we have the following cases to compute \(\max_{x \in [-1,1]} \text{Var}_H[Y|x]\):

(I) If \(\beta \frac{t + 1}{e^d - 1} + \beta - 1 < 0, 0 < Y < 1\), we obtain:
- \(\beta < \frac{e^{d/3} - 1}{e^{d/3}}\).
- For \(Y := \frac{(\beta-1)ae^{e^d}(e^d + 1)^2}{2(e^d - a)2(e^d + 1)e^{e^d} - e^d + 1}\), if \(0 < Y < 1\), we have \(0 < \frac{e^d - a}{2(e^d - a)2\beta(e^d + 1)e^{e^d} - e^d + 1}\), if \(Y < 1\).

If \(\frac{e^d - a}{2(e^d - a)2\beta(e^d + 1)e^{e^d} - e^d + 1}\) < 1, we have:
\[
\beta(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2) > 2(e^d - a)^2(e^d - 1) - ae(e^d + 1)^2.
\]
(176)
- If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 > 0\), we have
\[
\beta > \frac{2(e^d - a)^2(e^d - 1) - ae(e^d + 1)^2}{2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2}.
\]
- If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 = 0\) and \(2(e^d - a)^2(e^d - 1) - ae(e^d + 1)^2 < 0\), no \(\beta\) satisfies the condition.
- If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 = 0\) and \(2(e^d - a)^2(e^d - 1) - ae(e^d + 1)^2 < 0\), no \(\beta\) satisfies the condition.
- If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 < 0\), we have
\[
\beta < \frac{e^{d/3} - 1}{e^{d/3}},
\]
to get the correct domain, we compare \(\frac{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1}{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1}\) and \(\frac{e^d}{e^{d/3}}\), see Appendix [C] we have \(\frac{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1}{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1} \leq \frac{e^{d/3} - 1}{e^{d/3}}\).
Therefore, \(\beta < \frac{e^{d/3} - 1}{e^{d/3}}\).

Summarize above analysis, we have the following cases to compute \(\max_{x \in [-1,1]} \text{Var}_H[Y|x]\):

1) If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 > 0\), we have:
\[
\max_{x \in [-1,1]} \text{Var}_H[Y|x] = \begin{cases}
\text{Var}_H[Y|x^*], & \text{if } \frac{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1}{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1} < \frac{e^{d/3} - 1}{e^{d/3}}, \\
\max \{\text{Var}_H[Y|0], \text{Var}_H[Y|1]\}, & \text{otherwise}.
\end{cases}
\]
2) If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 = 0\) and \(ae(e^d + 1)^2 + 2(e^d - a)^2(1 + e^{d/3}) > 0\), we have:
\[
\max_{x \in [-1,1]} \text{Var}_H[Y|x] = \max \{\text{Var}_H[Y|0], \text{Var}_H[Y|1]\},
\]
3) If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 = 0\) and \(2(e^d - a)^2(e^d - 1) - ae(e^d + 1)^2 < 0\), we have:
\[
\max_{x \in [-1,1]} \text{Var}_H[Y|x] = \begin{cases}
\text{Var}_H[Y|x^*], & \text{if } \beta < \frac{e^{d/3} - 1}{e^{d/3}}, \\
\max \{\text{Var}_H[Y|0], \text{Var}_H[Y|1]\}, & \text{otherwise},
\end{cases}
\]
4) If \(2(e^d - a)^2(e^d + t) - ae(e^d + 1)^2 < 0\), we have:
\[
\max_{x \in [-1,1]} \text{Var}_H[Y|x] = \begin{cases}
\text{Var}_H[Y|x^*], & \text{if } \beta < \frac{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1}{2(e^d - a)^2(e^d + 1)e^{e^d} - e^d + 1}, \\
\max \{\text{Var}_H[Y|0], \text{Var}_H[Y|1]\}, & \text{otherwise}.
\end{cases}
\]
Appendix [H] proves that
\[
2(e^d - a)^2(e^d - 1) - ae(e^d + 1)^2 < \frac{e^d - 1}{e^{d/3}}.
\]
Therefore, we have
\[
\max_{x \in [-1,1]} \text{Var}_H[Y | x] = \begin{cases} 
\text{Var}_H[Y | x^*], & \text{if } 0 < \beta < \frac{2(e^\epsilon - a)^2(e^\epsilon - 1) - ae(e^\epsilon + 1)^2}{2(e^\epsilon - a)^2(e^\epsilon + t) - ae(e^\epsilon + 1)^2}, \\
\max \{ \text{Var}_H[Y | 0], \text{Var}_H[Y | 1] \}, & \text{otherwise}.
\end{cases}
\]

(II) Based on above analysis, to make \(\max_{x \in [-1,1]} \text{Var}_H[Y = y | x] = \max \{ \text{Var}_H[Y | 0], \text{Var}_H[Y | 1] \} \), \(\beta\) should satisfy constraint \(\frac{2(e^\epsilon - a)^2(e^\epsilon - 1) - ae(e^\epsilon + 1)^2}{2(e^\epsilon - a)^2(e^\epsilon + t) - ae(e^\epsilon + 1)^2} \leq \beta \leq 1\).

To get the exact value of \(\max_{x \in [-1,1]} \text{Var}_H[Y | x]\), we compare \(\text{Var}_H[Y | 1]\) and \(\text{Var}_H[Y | 0]\), values of \(\text{Var}_H[Y | 1]\) and \(\text{Var}_H[Y | 0]\) are:
\[
\begin{align*}
\text{Var}_H[Y | 1] & = \left( \frac{t + 1}{e^\epsilon - 1} + 1 - \frac{ae(e^\epsilon + 1)^2}{(e^\epsilon - 1)(e^\epsilon - a)^2} + \frac{(t + e^\epsilon)((t + 1)^3 + e^\epsilon - 1)}{3t^2(e^\epsilon - 1)^2} - \frac{(1 - a)e2^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} \right) \\
\text{Var}_H[Y | 0] & = \left( \frac{t + 1}{e^\epsilon - 1} + 1 - \frac{ae(e^\epsilon + 1)^2}{(e^\epsilon - 1)(e^\epsilon - a)^2} + \frac{(t + e^\epsilon)((t + 1)^3 + e^\epsilon - 1)}{3t^2(e^\epsilon - 1)^2} - \frac{(1 - a)e2^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2} \right).
\end{align*}
\]

Since \(\text{Var}_H[Y | 1]\) and \(\text{Var}_H[Y | 0]\) are linear equations respect to \(\beta\), we compare slopes of \(\beta\) in \(\text{Var}_H[Y | 1]\) and \(\text{Var}_H[Y | 0]\). We define the slope of \(\beta\) in \(\text{Var}_H[Y | 1]\) as
\[
slope_1 := \frac{t + 1}{e^\epsilon - 1} + 1 - \frac{ae(e^\epsilon + 1)^2}{(e^\epsilon - 1)(e^\epsilon - a)^2} + \frac{(t + e^\epsilon)((t + 1)^3 + e^\epsilon - 1)}{3t^2(e^\epsilon - 1)^2} - \frac{(1 - a)e2^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2},
\]
and the slope of \(\beta\) in \(\text{Var}_H[Y | 0]\) as
\[
slope_2 := \frac{(t + e^\epsilon)((t + 1)^3 + e^\epsilon - 1)}{3t^2(e^\epsilon - 1)^2} - \frac{(1 - a)e2^\epsilon(e^\epsilon + 1)^2}{(e^\epsilon - 1)^2(e^\epsilon - a)^2}.
\]

Then, we represent left boundary of \(\beta\) as
\[
\beta_1 := \frac{2(e^\epsilon - a)^2(e^\epsilon - 1) - ae(e^\epsilon + 1)^2}{2(e^\epsilon - a)^2(e^\epsilon + t) - ae(e^\epsilon + 1)^2},
\]
and the right boundary of \(\beta\) as
\[
\beta_2 := 1.
\]

and the value of \(\beta\) at the intersection of slope_1 and slope_2 is
\[
\beta_{\text{intersection}} := \frac{(c - 1)(c - a)^2 - ac(c + 1)^2}{(t + 1)(c - a)^2 - ac(c + 1)^2}.
\]

Then, slope_1, and slope_2 have the following possible combinations:

1) If \(\text{slope}_1 > 0\), \(\text{slope}_2 > 0\), \(\beta = \beta_1\).
2) If \(\text{slope}_1 < 0\), \(\text{slope}_2 < 0\), \(\beta = \beta_2\).
3) If \(\text{slope}_1 \cdot \text{slope}_2 < 0\), If \(\beta_1 < \beta_{\text{intersection}} < \beta_2\), \(\beta = \beta_{\text{intersection}}\).
4) If \(\text{slope}_1 \cdot \text{slope}_2 < 0\), If \(\beta_{\text{intersection}} < \beta_1\) or \(\beta_{\text{intersection}} > \beta_2\), find \(\beta\) for \(\max \{ \text{Var}[Y | 1, \beta = \beta_1], \text{Var}[Y | 0, \beta = \beta_1] \} \), \(\max \{ \text{Var}[Y | 1, \beta = \beta_2], \text{Var}[Y | 0, \beta = \beta_2] \} \).

5) If \(\beta_1 \cdot \beta_2 = 0\),
   
   Case 1. slope_1 = 0, slope_2 \neq 0,
   
   a) If \(\text{slope}_1 = 0\), \(\beta_{\text{intersection}} \in [\beta_1, \beta_2]\), \(\beta = [\beta_1, \beta_{\text{intersection}}]\).
   
   b) If \(\text{slope}_1 = 0\), \(\beta_{\text{intersection}} < \beta_1\) or \(\beta_{\text{intersection}} > \beta_2\), \(\text{Var}_H[Y | 1, \beta = [\beta_1, \beta_2]] > \text{Var}_H[Y | 0, \beta = [\beta_1, \beta_2]]\), \(\beta = [\beta_1, \beta_2]\).

   Case 2. slope_2 = 0, slope_1 \neq 0,
   
   a) If \(\text{slope}_1 > 0\), \(\text{slope}_2 = 0\), \(\beta_{\text{intersection}} \in [\beta_1, \beta_2]\), \(\beta = [\beta_1, \beta_{\text{intersection}}]\).
   
   b) If \(\text{slope}_1 < 0\), \(\beta_{\text{intersection}} \in [\beta_1, \beta_2]\), \(\beta = [\beta_1, \beta_{\text{intersection}}]\).

   Case 3. slope_1 = 0 and slope_2 = 0,
   
   a) If \(\text{Var}_H[Y | 1, \beta = [\beta_1, \beta_2]] < \text{Var}_H[Y | 0, \beta = [\beta_1, \beta_2]]\), \(\beta = [\beta_1, \beta_2]\).
   
   b) If \(\text{Var}_H[Y | 1, \beta = [\beta_1, \beta_2]] > \text{Var}_H[Y | 0, \beta = [\beta_1, \beta_2]]\), \(\beta = [\beta_1, \beta_2]\).

   c) If \(\text{Var}_H[Y | 1, \beta = [\beta_1, \beta_2]] = \text{Var}_H[Y | 0, \beta = [\beta_1, \beta_2]]\), \(\beta = [\beta_1, \beta_2]\).

Proof. 1) If \(\text{slope}_1 > 0\), \(\text{slope}_2 > 0\), \(\text{Var}_H[Y | 1]\) and \(\text{Var}_H[Y | 0]\) monotonically increase \(\beta \in [\beta_1, \beta_2]\), \(\min_{\beta} \max_{x \in [-1,1]} \text{Var}_H[Y | x]\) is at \(\beta = \beta_1\).

2) Similar to 1), we have \(\min_{\beta} \max_{x \in [-1,1]} \text{Var}_H[Y | x]\) is at \(\beta = \beta_1\).

3) If \(\beta < \beta_{\text{intersection}} < \beta_2\), we have:
   
   • If \(\text{slope}_1 > 0, \text{slope}_2 < 0\), \(\text{Var}[Y | 1]\) monotonically increases and \(\text{Var}[Y | 0]\) monotonically decreases, so when \(\beta \in [\beta_1, \beta_{\text{intersection}}]\), \(\max_{x \in [-1,1]} \text{Var}_H[Y | x] = \text{Var}_H[Y | 0]\). When \(\beta \in [\beta_{\text{intersection}}, \beta_2]\), \(\text{Var}_H[Y | x] = \text{Var}_H[Y | 1]\). Therefore,
4) If \( \beta_{\text{intersection}} < \beta_1 \) and \( \beta_2 > \beta_{\text{intersection}} \), we have:
   - If \( \text{slope}_1 > 0 \), \( \text{slope}_2 < 0 \) and \( \text{Var}_H[Y|1] > \text{Var}_H[Y|0] \), since \( \text{Var}_H[Y|1] \) monotonically increases in the domain, \( \min_\beta \text{Var}_H[Y|1] \) at \( \beta = \beta_1 \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1] = \text{Var}_H[Y|0] \) at \( \beta = \beta_{\text{intersection}} \).
   - If \( \text{slope}_1 > 0 \), \( \text{slope}_2 < 0 \) and \( \text{Var}_H[Y|1] < \text{Var}_H[Y|0] \). Since \( \text{Var}_H[Y|0] \) monotonically decreases in the domain, \( \min_\beta \text{Var}_H[Y|0] \) at \( \beta = \beta_2 \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1] = \text{Var}_H[Y|0] \) at \( \beta = \beta_{\text{intersection}} \).
   - If \( \text{slope}_1 < 0 \), \( \text{slope}_2 > 0 \) and \( \text{Var}_H[Y|1] > \text{Var}_H[Y|0] \), since \( \text{Var}_H[Y|1] \) monotonically decreases in the domain, \( \min_\beta \text{Var}_H[Y|1] \) at \( \beta = \beta_2 \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1] = \text{Var}_H[Y|0] \) at \( \beta = \beta_{\text{intersection}} \).
   - If \( \text{slope}_1 < 0 \), \( \text{slope}_2 > 0 \) and \( \text{Var}_H[Y|1] < \text{Var}_H[Y|0] \), \( \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1] \) since \( \text{Var}_H[Y|0] \) monotonically increases in the domain, \( \min_\beta \text{Var}_H[Y|0] \) at \( \beta = \beta_1 \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1] = \text{Var}_H[Y|0] \) at \( \beta = \beta_{\text{intersection}} \).

5) **Case 1:**
   - a) If \( \text{slope}_1 = 0 \), \( \text{slope}_2 > 0 \), \( \beta_{\text{intersection}} \in [\beta_1, \beta_2] \), we can conclude that \( \beta \in [\beta_1, \beta_{\text{intersection}}] \), \( \text{Var}_H[Y|1] > \text{Var}_H[Y|0] \). When \( \beta \in (\beta_{\text{intersection}}, \beta_2] \), \( \text{Var}_H[Y|0] > \text{Var}_H[Y|1] \). Since \( \text{Var}_H[Y|0] \) monotonically increases if \( \beta \in (\beta_{\text{intersection}}, \beta_2] \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1], \beta = [\beta_1, \beta_{\text{intersection}}] \).
   - b) If \( \text{slope}_1 = 0 \), \( \text{slope}_2 < 0 \), \( \beta_{\text{intersection}} \in [\beta_1, \beta_2] \), we can conclude that \( \beta \in [\beta_1, \beta_{\text{intersection}}] \), \( \text{Var}_H[Y|0] > \text{Var}_H[Y|1] \). When \( \beta \in (\beta_{\text{intersection}}, \beta_2] \), \( \text{Var}_H[Y|1] > \text{Var}_H[Y|0] \). Since \( \text{Var}_H[Y|1] \) does not change and \( \text{Var}_H[Y|0] \) monotonically decreases, \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|0], \beta = [\beta_{\text{intersection}}, \beta_2] \).
   - c) If \( \text{slope}_1 = 0 \), \( \text{slope}_2 > 0 \), \( \text{Var}_H[Y|1], \beta = [\beta_1, \beta_2] \) \( \text{Var}_H[Y|0], \beta = [\beta_1, \beta_2] \), the \( \beta_{\text{intersection}} \) > \( \beta_2 \). Since \( \text{Var}_H[Y|1] \) does not change if \( \beta \in [\beta_1, \beta_2] \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|1], \beta = [\beta_1, \beta_2] \).
   - d) If \( \text{slope}_1 = 0 \), \( \text{slope}_2 > 0 \), \( \text{Var}_H[Y|0], \beta = [\beta_1, \beta_2] \) \( \text{Var}_H[Y|1], \beta = [\beta_1, \beta_2] \), the \( \beta_{\text{intersection}} \) > \( \beta_2 \). Since \( \text{Var}_H[Y|0] \) monotonically decreases if \( \beta \in [\beta_1, \beta_2] \), \( \min_\beta \max_{x \in [-1,1]} \text{Var}_H[Y|x] = \text{Var}_H[Y|0], \beta = [\beta_1, \beta_2] \).
   - e) If \( \text{slope}_1 = 0 \), \( \text{slope}_2 < 0 \), \( \text{Var}_H[Y|1], \beta = [\beta_1, \beta_2] \) \( \text{Var}_H[Y|0], \beta = [\beta_1, \beta_2] \), the \( \beta_{\text{intersection}} \) < \( \beta_1 \). Since \( \text{Var}_H[Y|1] \) does not change if \( \beta \in [\beta_1, \beta_2] \),