Analytical Description of Voids in Majumdar-Papapetrou Spacetimes

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Abstract

We discuss new Majumdar-Papapetrou solutions for the 3+1 Einstein-Maxwell equations, with charged dust acting as the external source of the fields. The solutions satisfy non-linear potential equations which are related to well-known wave equations of 1+1 soliton physics. Although the matter distributions are not localised, they present central structures which may be identified with voids.
1 Introduction

We consider solutions for the Einstein-Maxwell (EM) equations with charged dust acting as the external source of the fields. Our basic equations read

\[ G_{\mu\nu} = 8\pi T_{\mu\nu}, \]

(1)

\[ F^{\mu\nu;\nu} = 4\pi J^\mu, \]

(2)

where \( G_{\mu\nu} \) and \( F^{\mu\nu} \) denote the Einstein and Maxwell tensors, and the total energy-momentum tensor is given by

\[ T_{\mu\nu} = E_{\mu\nu} + \rho u^\mu u_\nu. \]

(3)

Here \( E_{\mu\nu} \) is the Maxwell energy-momentum tensor, and the matter term corresponds to dust with energy density \( \rho \) and four-velocity \( u^\mu \). The four-current is defined by the expression

\[ J^\mu = \sigma u^\mu, \]

(4)

where \( \sigma \) is the charge density.

We assume that the fluid is static and use the conformstatic metric

\[ ds^2 = -V^2 dt^2 + \frac{1}{V^2} h_{ij} dx^i dx^j, \]

(5)

where the background metric \( h_{ij} \) and \( V \) depend only on the space-like coordinates \( x^1, x^2, x^3 \). The electrostatic forms of \( A_\mu \) and \( J^\mu \) are given by

\[ A_\mu = A_0(x^i) \delta^\mu_0, \]

(6)

\[ J^\mu = \frac{\sigma(x^i)}{V} \delta^\mu_0, \]

(7)

with \( i = 1, 2, 3 \).

Under these conditions, Eq. (2) contains only one non-trivial equation:

\[ \frac{1}{\sqrt{h}} \partial_j \left( \sqrt{h} h^{jk} \frac{ \partial_k A_0 }{ V^2 } \right) = \frac{4\pi J^0}{V^2}, \]

(8)

where \( h \) and \( h^{ij} \) are the determinant and the inverse of \( h_{ij} \), respectively.

The trace of the Einstein equations is

\[ R = -8\pi T, \]

(9)

where \( R \) denotes the Ricci scalar and \( T = T^\mu_\mu \). We use the decomposition

\[ R = V^2 \left[ R_h + 2 \nabla_h^2 \ln V - 2 \partial_i \ln V \partial_i \ln V \right]. \]

(10)

Here \( R_h \) is the Ricci scalar associated to \( h_{ij} \), and \( \nabla_h^2 \) is the three-dimensional Laplacian operator constructed with the same metric. We assume a flat background space, with \( R_h = 0 \). Therefore, combining Eqs. (9) and (10) we obtain

\[ \nabla_h^2 \left( \frac{1}{V} \right) = \frac{4\pi T}{V^3}. \]

(11)
Following the Majumdar-Papapetrou (MP) procedure, we assume that

\[ A_0 = \alpha V, \]  

where \( \alpha = \pm 1 \). As a consequence, the Maxwell equation (8) takes the form

\[ \nabla^2_h \left( \frac{1}{V} \right) = -\frac{4\pi \alpha J^0}{V^2}, \]  

which is clearly the same as Eq. (11) whenever the condition

\[ T = -\alpha J^0 V \]  

holds. This equation can be combined with \( J^0 = \sigma V \) to obtain the alternative expression

\[ \sigma = -\alpha T. \]  

Since \( T = -\rho \) for dust, Eqs. (11) and (15) can be finally expressed as

\[ \nabla^2_h \lambda + 4\pi \rho \lambda^3 = 0, \]  

\[ \sigma = \alpha \rho, \]

where \( \lambda = \frac{1}{V} \). Due to Eqs. (3) and (12), only one Einstein equation is not trivially satisfied. Therefore, solving Eq. (16) is sufficient for finding a solution of the EM equations.

If we identify our flat background space with the Euclidean, three dimensional space and assume \( \rho = 0 \), then Eq. (14) reduces to the usual Laplace equation \( \nabla^2 \lambda = 0 \) and the electrovac, multi-black hole solution follows straightforwardly. Assuming spherical symmetry, and using spherical coordinates, we find

\[ \lambda = 1 + \frac{m}{r}. \]

In the far-asymptotic region, the behaviour of this solution is approximately given by

\[ V \approx 1 - \frac{m}{r}, \quad g_{00} \approx -1 + \frac{2m}{r}, \quad A_0 \approx \pm \left(1 - \frac{m}{r} \right). \]

The corresponding expression for the electric field is

\[ E \approx \frac{q}{r^2}, \]

where

\[ q = \pm m. \]

Equation (3) implies that the invariant area of any 2-sphere surrounding the origin is given by \( \frac{4\pi}{V(\tau) r^2} \). Therefore, the set \( r = 0, t = constant \) has a non-zero invariant area given by \( 4\pi m^2 \). In fact, a simple coordinate transform shows that the null hypersurface \( r = 0 \) is the horizon of the extremal Reissner-Nordström solution. Also, if we define the new radial coordinate \( \tilde{r} = -r \) and perform the standard analysis, then we find
that this horizon encloses a point-like, essential singularity placed at $\tilde{r} = m$. In fact, the invariant area vanishes and the scalar $J = F_{\mu\nu} F^{\mu\nu} = \lambda^{-4} \left( \frac{d\lambda}{dr} \right)^2$ blows up at that point.

Equations (16) and (17) were originally discussed by Das [6] in his study of equilibrium configurations of self-gravitating, charged dust. More recently, Gürses [3] has considered non-electrovac solutions when Eq. (16) is linear. This situation corresponds to his choice $\rho = \frac{b^2}{4\pi a^4}$ for constant $b$. In this case, Eq. (16) admits the particular solution $\lambda = \frac{a\sin br}{r}$ where $a$ is an integration constant. The oscillatory behaviour of this solution implies a geometry with a complicated radial dependence. In fact, the invariant area vanishes for a discrete, infinite set of values of $r$, and the Ricci scalar $R = \frac{2b^2 a^4}{a^2 \sin^2 br}$ blows up wherever the invariant area vanishes, except for $r = 0$. Other solutions with oscillatory behaviour have been considered by Balakrishna and Wali, [7] Braden and Varela, [8] and Ida. [4] In Section 2 we exploit the general non-linearity of Eq. (16) to obtain new solutions which are free of oscillatory singularities and allow asymptotically flat behaviour.

2 The non-linear models

The non-linear potential equation (16) takes the spherically symmetric form

$$\frac{d^2 \lambda}{dr^2} + \frac{2}{r} \frac{d\lambda}{dr} + 4\pi \rho \lambda^3 = 0. \quad (22)$$

Using the new radial coordinate $\tau = \frac{1}{r}$, the same differential equation can be written as

$$\frac{d^2 \lambda}{d\tau^2} + 4\pi \rho \frac{\tau^4 \lambda^3}{4\pi} = 0. \quad (23)$$

If $\rho$ and $\lambda$ satisfy the condition

$$\rho = \frac{b^2}{4\pi} \frac{\tau^4 \sin \lambda}{\lambda^3}, \quad (24)$$

then (23) finally reduces to the -sine-Gordon equation [11]

$$\frac{d^2 \lambda}{d\tau^2} + b^2 \sin \lambda = 0, \quad (25)$$

which has the solutions

$$\lambda^\pm (\tau) = 2 \arcsin \left[ \tanh \left( \pm b\tau + c \right) \right] + 2n\pi, \quad (26)$$

where $n$ is an arbitrary integer, $c$ is an integration constant, and $b$ is assumed to be positive. We consider only the case $n = 0$. In terms of the original radial coordinate, these solutions read

$$V^\pm (r) = \frac{1}{2 \arcsin \left[ \tanh \left( \pm \frac{b}{r} + c \right) \right]}. \quad (27)$$
We observe that $V^\pm(0)^2$ is finite, so the invariant area vanishes for $r = 0$. Therefore, the set $r = 0$, $t = constant$ is point-like with respect to both solutions. Let us deal with $V^+$ first. A preliminary numerical study of the invariants $J$, $R$, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ suggests that these quantities are bounded for non-negative $r$, whenever $c$ is positive. If we choose

$$c = \frac{1}{2} \ln \left[ \frac{1 + \sin(1/2)}{1 - \sin(1/2)} \right],$$

then the far-asymptotic behaviour of this solution is given by Eqs. (19), (20), (21) with $m = 2b\cos(1/2)$. Therefore, $V^+$ is asymptotically flat, exactly as the MP electrovac solution.

The positive definite energy density given by Eq. (24) corresponds to a non-localised matter (and charge) distribution. However, $\rho$ is negligible for $x = \frac{r}{b} \ll 0.2$. For very small $x$ the dimensionless expressions of $\rho$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ are approximately given by

$$\rho(x) \approx \frac{e^{-c}}{\pi^4} \frac{e^{-\frac{x}{b}}}{x^4}, \quad (29)$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(x) \approx 3 \left( \frac{2}{\pi} \right)^6 e^{-2e} \frac{e^{-\frac{x}{b}}}{x^8}. \quad (30)$$

These results imply a very fast decrease of $\rho(x)$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(x)$ when $x \to 0^+$, and suggest the existence of a void in the innermost region of the asymptotically flat object constructed with $V^+$. Nevertheless, this interpretation cannot be complete without a better understanding of the point-like set $r = 0$, $t = constant$. A closer look at the singularity contained in this solution is also necessary. We interpret $r = 0$, $t = constant$ as the center of symmetry and observe that the above mentioned invariants are bounded at this point. However, the coordinate transform $\tilde{r} = -r$ reveals the existence of a point-like, essential singularity at $\tilde{r} = \frac{b}{2}$. In fact, the invariant $J$ blows up at this point. The use of $V^+$ alone may imply the division of the manifold into connected parts, separated by the point-like singularity placed at $r = -\frac{b}{2}$. However, a very different situation comes out when we restrict $V^+$ to positive values of $r$ and describe the geometry for $r < 0$ with the second solution $V^-$. Then, a smooth (at least $C^1$) matching of $V^+$ and $V^-$ occurs at $r = 0$ and the arising asymptotically flat spacetime seems to be connected and singularity free, with an almost empty region near the center of symmetry. Thus, the joint use of $V^+$ and $V^-$ provides a simpler description of a MP void. The study of the global structure of these solutions is left as an open problem, which provides motivation for further research work.

Finally, we point out that other exact, non-linear solutions for this theory can be found if we impose different relationships between $\rho$ and $\lambda$. For example, the choice

$$\rho = -\frac{b^2}{4\pi} \frac{\tau^4 \sin \lambda}{\lambda^3} \quad (31)$$

leads to the sine-Gordon equation

$$\frac{d^2 \lambda}{d\tau^2} = b^2 \sin \lambda. \quad (32)$$
It has the well-known solutions

$$\lambda^\pm(\tau) = 4\arctan e^{(\pm b\tau + d)}.$$  \hspace{1cm} (33)

If we choose $$d = \ln [\tan(1/4)]$$, then both solutions have asymptotically flat behaviour. Another example is

$$\rho = \frac{b^2}{4\pi} (\lambda - \lambda^3).$$  \hspace{1cm} (34)

In this case the geometry is determined by the $$\lambda\phi^4$$ equation

$$\frac{d^2\lambda}{d\tau^2} + b^2 (\lambda - \lambda^3) = 0$$  \hspace{1cm} (35)

which admits the solutions

$$\lambda^\pm(\tau) = \tanh \left( \pm \frac{b}{\sqrt{2}} \tau + f \right).$$  \hspace{1cm} (36)

The relationship between the 3+1 EM theory and the equations of 1+1 soliton physics deserves a more detailed examination. Possible extensions of this work involve the analysis of dust models for which $$\lambda(\tau)$$ is a solution of the KdV equation, and the study of the non-linear potential equations arising in higher dimensions.\[10\]

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