A uniqueness result for propagation-based phase contrast imaging from a single measurement

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Abstract. Phase contrast imaging seeks to reconstruct the complex refractive index of an unknown sample from scattering intensities, measured for example under illumination with coherent X-rays. By incorporating refraction, this method yields improved contrast compared to purely absorption-based radiography but involves a phase retrieval problem which, in general, allows for ambiguous reconstructions. In this paper, we show uniqueness of propagation-based phase contrast imaging for compactly supported objects in the near field regime, based on a description by the projection- and paraxial approximations. In this setting, propagation is governed by the Fresnel propagator and the unscattered part of the illumination function provides a known reference wave at the detector which facilitates phase reconstruction. The uniqueness theorem is derived using the theory of entire functions. Unlike previous results based on exact solution formulae, it is valid for arbitrary complex objects and requires intensity measurements only at a single detector distance and illumination wave length. We also deduce a uniqueness criterion for phase contrast tomography, which may be applied to resolve the three-dimensional structure of micro- and nano-scale samples. Moreover, our results may have some significance to electronic imaging methods due to the equivalence of paraxial wave propagation and Schrödinger’s equation.

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1. Introduction

The advent of coherent X-ray sources, such as synchrotrons and - more recently - free-electron-lasers, has allowed to extend the scope of radiography to quasi-transparent specimen by phase-sensitive imaging techniques [32]. Examples include micro- or nano-scale objects composed mainly of light elements, most prominently biological cells [4, 42] but also organic or ceramic foams [5, 11]. Phase-contrast imaging seeks to reconstruct the spatially varying complex refractive index $n = 1 - \delta + i\beta$ of such samples from measurements of scattered wave fields. In particular, the approach takes into account the real component $\delta$ governing the refractive phase shifts that are imprinted to transmitted radiation. For the specimens and wavelengths in question, $\delta$ is typically up to three orders of magnitude larger than the absorptive part $\beta$ [11, 25]. Consequently, solely absorption-based approaches are bound to result in poor contrast. On the other hand, refractive information is encoded entirely in the phase of the transmitted wave fields which cannot be observed directly by common CCD detectors due to their physical limitation to wave intensities [35]. The required phase sensitivity can be achieved either by interferometric techniques [8, 28, 45], or by measuring the propagated wave field on a distant detector rather than close to the exit-surface of the sample [16, 30, 34, 37]. In the latter case, diffraction encodes the phase information into observable intensities. This raises the question whether the phase can be uniquely recovered from the data.

In the far field limit of large distances where the propagation essentially reduces to a Fourier transform [35], this problem has been subject to extensive analytical studies based on the ideas of Akutowicz and Walther [1, 2, 44]. See for instance [23, 27] for reviews. The principal result for such a reconstruction of a compactly supported function from Fourier intensity data is that solutions to the phase retrieval problem are in general highly non-unique in $\mathbb{R}$. On the contrary, non-trivial ambiguities are “pathologically rare” in higher dimensions [3, 6], occurring only for objects within a set of measure zero. Non-uniqueness can even be ruled out completely by restriction to non-absorbing, symmetric objects [22] for example. However, ab initio reconstructions require iteratively updated support estimates [13, 14] in order to overcome the “trivial” ambiguities induced by the invariance of the data under translations and reflections of the object. Far field phase contrast has been successfully applied to 2D- and 3D-imaging of quasi-transparent specimen ($\beta = 0$) [26] and single-material objects ($\beta \propto \delta$) [10, 24].

Concerning uniqueness, the main advantage of measurements in the near field regime, also called the holographic- or Fresnel regime, lies in the presence of the unscattered probe beam in the measured intensities. This provides a supposedly known reference signal for the superimposed scattered part that encodes the object information. Propagation in this regime is described by the Fresnel propagator. The corresponding phase retrieval problem has been proven to be uniquely solvable for general compactly supported objects, given at least two independent intensity profiles recorded at different detector distances or incident wavelengths [20]. To the best of our knowledge, no equally general analogue has ever been derived for a single measurement setting. On the other
hand, we emphasize that the uniqueness problem may be addressed using the same tools as in the far field case, namely the theory of entire functions.

Exact solution formulae for near field phase contrast imaging via a single intensity profile are restricted to single-material samples. Additionally, these approaches assume small propagation distances to approximate the transport-of-intensity equation \[33, 38\] or weak absorption and slowly varying phase shifts \[43\]. For general objects, \(\delta\) and \(\beta\) have to be reconstructed independently. Recent numerical results \[39\] suggest that a single distance may be sufficient for this in the case of phase contrast tomography. Nevertheless, referring to the singular "phase vortex" counter-example \[31\], it is commonly argued \[9\] that two real-valued intensity patterns are required to uniquely determine the complex refractive index. However intuitive, this work aims to disprove this widespread believe at least in the case of compactly supported objects illuminated by plane waves or Gaussian beams. Our central uniqueness result reads as follows:

**Theorem 1.** Let \(\mathcal{S}'(\mathbb{R}^n) \supset \mathcal{S}'_c(\mathbb{R}^n)\) denote the tempered (and compactly supported) distributions and \(\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)\) the Fourier transform. For \(w \in \mathcal{C}_\infty(\mathbb{R}^n)\) everywhere nonzero, \(\alpha \in \mathbb{C} \setminus \mathbb{R}\) and \(P_0 \in \mathcal{S}'_c(\mathbb{R}^n) \setminus \{0\}\) define

\[
F : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{C}_\infty(\mathbb{R}^n); \quad F(h) = |\mathcal{F}(P_0) \exp(\alpha \|\cdot\|^2) + \mathcal{F}(wh)|^2
\]

Then \(F\) is well-defined and injective. Moreover, any \(h \in \mathcal{S}'(\mathbb{R}^n)\) is uniquely determined by data \(F(h)|_U\) restricted to an arbitrary open set \(U \subset \mathbb{R}^n\).

Physically, the first summand in (1) is associated with the unscattered probe beam, whereas the second one relates to the scattered wave field encoding the object transmission function, which is parametrized by \(h\). Details on the physical problem of near field phase contrast imaging and how it matches the framework of Theorem 1 are discussed in section 2. Section 3 gives a recap of the theory of entire functions as a preparation for the proofs of the main results in section 4.

2. Physical problem

Figure 1 shows an idealized experimental setup for propagation-based phase contrast imaging: incident coherent electromagnetic waves of wavenumber \(k\) interact with an unknown object of thickness \(L\) in the beam line, leading to a slightly perturbed wave field at the exit-surface \(E_0\). We parametrize the object by its refractive index \(n = 1 - \delta + i\beta\), where \(\delta\) and \(\beta\) are real-valued and compactly supported. The intensity of the scattered radiation is measured in the detector plane \(E_d\) at finite distance \(d > 0\) to the object. Although the physical setting in Figure 1 is three-dimensional, we consider the more general case of \(n \in \mathbb{N}\) lateral dimensions, denoted by \(\mathbf{x}\), plus the axial \(z\)-direction.

It is well known that the cartesian components of a monochromatic electromagnetic wave in a medium of refractive index \(n\) can be described by a single complex-valued time-independent field \(\Psi\), governed by the *Helmholtz equation* \[35\]

\[
\Delta \Psi + k^2 n^2 \Psi = 0.
\]
We assume that the object is sufficiently weak and thin for the paraxial- and the projection approximation \[35\] to hold. We discuss these briefly here, referring to \[20\] for detailed error estimates. The former requires that $\Psi$ is of the form $\Psi(\mathbf{x}, z) = e^{ikz} \tilde{\Psi}(\mathbf{x}, z)$, where the envelope $\tilde{\Psi}$ is slowly varying on length scales $1/k$, so that $\partial_z^2 \tilde{\Psi}$ may be neglected in (2). This leads to the paraxial Helmholtz equation \[35\]

$$(2i k \partial_z + \Delta_\perp - 2k^2 (\delta - i \beta)) \tilde{\Psi} = 0,$$

where $\Delta_\perp$ denotes the lateral Laplacian and quadratic terms in $\delta, \beta$ have been neglected since these are typically $\leq O(10^{-6})$ for X-rays. The projection approximation describes the interaction with the object by ray optics, neglecting diffraction by additionally suppressing the term $\Delta_\perp \tilde{\Psi}$ in (3) within the sample domain $z \in [-L; 0]$ in Figure 1.

Under these assumptions, it follows from (3) that the wave field at $E_0$ is given by

$$\Psi_0(\mathbf{x}) = P(\mathbf{x}) O(\mathbf{x}) = P(\mathbf{x}) \exp \left( -ik \int_{-L}^{0} (\delta(\mathbf{x}, z) - i \beta(\mathbf{x}, z)) \, dz \right),$$

(4)

where $P$ denotes the probe function that describes the illumination wave field at $E_0$ and $O$ the object transmission function encoding the sample structure \[41\]. The vacuum between the planes $E_0$ and $E_d$ in Figure 1 is characterized by $n = 1$, i.e. $\delta = \beta = 0$. In this setting, (3) can be solved analytically, yielding a relation between the wave field $\Psi_d$ at the detector and $\Psi_0$ described by the Fresnel propagator $D^F_d$ \[35\]:

$$\Psi_d(\mathbf{x}) = D^F_d(\Psi_0)(\mathbf{x}) = -\frac{i k e^{i k d}}{(2\pi)^{n/2} d} \exp \left( \frac{i k |\mathbf{x}|^2}{2d} \right) \int \Psi_0(\mathbf{x}') \exp \left( \frac{i k |\mathbf{x}'|^2}{2d} \right) \exp \left( -\frac{i k \mathbf{x} \cdot \mathbf{x}'}{d} \right) \, d\mathbf{x}'.$$

(5)

Combining (4) and (5) and taking into account the physical limitation of the detector measurements at $E_d$ to wave intensities, i.e. to the squared modulus of $\Psi_d$, the observable data in propagation-based phase contrast imaging is thus given by

$$I_d = |\Psi_d|^2 = |D^F_d(PO)|^2.$$

(6)
One seeks to reconstruct the object transmission function $O$ from the intensities $I_d$, solving the phase retrieval problem (6). We show that Theorem 1 guarantees uniqueness of such a reconstruction for a certain class of known probe functions $P$.

By introducing new coordinates $\xi := (k/d)^{1/2}x$, $\xi' := (k/d)^{1/2}x'$ and defining $w(\xi) := n_F(\xi) := \exp(i\|\xi\|^2/2)$, (5) can be restated as

$$D_F^d(\Psi_0) = -i e^{i k d} n_F \cdot F(w \cdot \Psi_0),$$

(7)

where $F$ denotes the $n$-dimensional Fourier transform. Note that the far field limit of large propagation distances corresponds to the approximation $w = 1$ in (7), whereas we retain the exact factor here, focussing on the near field case. Setting $h := -i P(O - 1)$, the resulting intensity profile at $E_d$ according to (6) and (7) is of the form

$$I_d = |D_F^d(P) + D_F^d(h)|^2 = \left| \frac{e^{-i k d}}{n_F} D_F^d(P) + F(wh) \right|^2.$$

(8)

By (4), $h$ is compactly supported if the object, parametrized by $\delta$ and $\beta$, is compactly supported. Moreover, $O$ can be recovered uniquely from $h$ if $P$ is known and everywhere nonzero. Hence, feasibility of the considered phase contrast imaging problem reduces to the question whether any compactly supported $h$ is uniquely determined by data of the form (8).

For constant probe functions $P = e^\gamma$, i.e. illumination by incident plane waves, we have $e^{-i k d} D_F^d(P) = P$, so that the near field intensity data is given by

$$I_d(\xi) = \left| \exp \left( \gamma - \frac{i}{2} \|\xi\|^2 \right) + F(w h)(\xi) \right|^2$$

(9)

and thus matches the setting considered in Theorem 1 with $P_0 = (2\pi)^{n/2} e^{\gamma \delta_0}$ and $\alpha = -i/2$. Here, $\delta_0$ denotes the Dirac-delta distribution centered at 0.

A second, more general class of probe functions is given by Gaussian beams that propagate along the $z$-direction. These constitute analytical solutions to the paraxial Helmholtz equation (3) in vacuum for $\delta = \beta = 0$, such that the lateral intensity profile is everywhere of Gaussian shape [40]. More precisely, the propagated illumination wave field at $E_d$ is of the form

$$D_F^d(P)(\xi) = \exp(\gamma_0 + \alpha_0 \|\xi\|^2), \quad \Re(\alpha_0) < 0$$

(10)

If the axial focus of the beam is located left of $E_d$, i.e. if the wave field is divergent at the detector, we further have $\Im(\alpha_0) \leq 0$ so that the resulting probe term

$$\frac{e^{-i k d}}{n_F} D_F^d(P) = \exp(\gamma + \alpha \|\xi\|^2) \quad \text{with} \quad \gamma = \gamma_0 - i k d, \quad \alpha = \gamma_0 - i k d,$$

(11)

to be substituted into (8), is in accordance with the assumptions of Theorem 1.

All in all, our analysis shows that Theorem 1 is applicable to the considered setting of phase contrast imaging, i.e. we have proven the following corollary:
Corollary 1 (Uniqueness of propagation-based phase contrast imaging). Any object transmission function $O \in 1 + \mathcal{S}′(\mathbb{R}^n)$, arising from a compactly supported sample illuminated by a known probe function $P$ of Gaussian- or plane wave shape via (4), is uniquely determined by intensity data $(I_d)|_U$ of the form (6) measured at a single distance $d > 0$ on any open set $U \subset \mathbb{R}^n$.

Proof. Apply Theorem 1 to the operators defined by (8), (9), (11). Use $O = 1 + i h/P$. □

$O$ provides merely a shadow image of the actual sample, as indicated by the line integrals in (4). In order to resolve the spatially varying refractive index $n = 1 - \delta + i \beta$ itself, the object in Figure 1 can be rotated by angles $\theta \in S^1$ in a plane containing the $z$-direction, yielding $\theta$-dependent intensity profiles $\{I_d, \theta\}_{\theta \in S^1}$ by the influence of the radiation’s incident direction. This is the setup of phase contrast tomography. Mathematically, the resulting $\theta$-dependence of the line integrals in (4) can be expressed as a two-dimensional Radon transform $\mathcal{R}$ in the chosen plane [29], i.e. the object transmission functions $O_\theta$ for the different incident angles are of the form

$$O_\theta = \exp(-i k \mathcal{R}_\theta(\delta - i \beta)),$$

where $\mathcal{R}_\theta f := \mathcal{R}(f)(\theta, \cdot)$. (12)

According to Corollary 1, the $\{O_\theta\}_{\theta \in S^1}$ are uniquely determined by the measurements $\{I_d, \theta\}_{\theta \in S^1}$. Hence, by (12), the object $\delta - i \beta$ can be recovered from the tomographic data if the exponential and the Radon transform in (12) are invertible. This leads to the following corollary which is proven in section 4:

Corollary 2 (Uniqueness of phase contrast tomography modulo phase-wrapping). Any compactly supported object $\delta - i \beta \in \mathcal{S}′(\mathbb{R}^{n+1})$ s.t. $0 \leq k \mathcal{R}(\delta) < 2\pi$, illuminated by a known Gaussian beam- or plane wave probe function $P$, is uniquely determined by tomographic intensity data $\{(I_d, \theta)|_U\}_{\theta \in V}$ of the form (6), (12) measured at a single distance $d > 0$ on any open sets $U \subset \mathbb{R}^n, V \subset S^1$.

We conclude this sections with some remarks concerning possible generalizations of the considered idealized setup. Firstly, the setting of Theorem 1 allows for much more general probe functions than the above examples, although this generality is difficult to translate into a particular set of admissible choices. Moreover, if the illumination is unknown, an additional flat field measurement of the intensity profile without an object in the beam line may be used to normalize the scattering intensities by division through the empty beam data. This procedure yields a good approximation of the hypothetical data in the idealized plane wave illumination case if the probe field varies only on coarse length scales compared to the object structure [17]. Finally, note that the above derivations assume a parallel beam geometry, i.e. non-divergent incident radiation. However, we emphasize that measurements obtained with cone beam illumination as generated by point sources can be associated with an approximate parallel beam setup via the Fresnel scaling theorem [36, 37] or incorporated explicitly into the model [21].
3. Mathematical preliminaries

In the following, we review elements of the theory of entire functions. The given overview is based on the more detailed treatment in [7, 12, 15]. The relevance of entire functions to phase retrieval is due to the well-known Paley-Wiener-Schwartz theorem:

**Theorem 2** (Paley-Wiener-Schwartz [19]). Let $K \subset \mathbb{R}^n$ compact and convex. Then, if $u$ is a distribution of order $N \in \mathbb{N}_0$ with support contained in $K$ and $\hat{u} := \mathcal{F}(u)$, $\hat{u}$ has an extension to an entire function and there exists a constant $C > 0$ such that

$$|\hat{u}(\xi)| \leq C(1 + \|\xi\|)^N \exp\left(\sup_{x \in K} \Re(\xi) \cdot x\right) \quad \forall \xi \in \mathbb{C}^n$$

Conversely, any entire function $\hat{u}$ satisfying (13) is the complex extension of the Fourier transform of a distribution $u$ of order $\leq N$ and support in $K$.

The essence of the result is that the compactly supported objects can be identified with entire functions of limited growth via the Fourier transform. Owing to the relatedness of the transforms by (7), the same holds true for the Fresnel propagator.

For simplicity, we restrict to entire functions $f : \mathbb{C} \to \mathbb{C}$ in one dimension. This will enable us to prove Theorem 1 for $n = 1$ and then deduce the general statement by reducing the $n$-dimensional phase retrieval problem to family of one-dimensional problems. In order to characterize the growth behavior, we define

$$M_f(r) := \max_{\xi \in \mathbb{C} : |\xi| = r} |f(\xi)| \quad \text{and} \quad m_f(r) := \min_{\xi \in \mathbb{C} : |\xi| = r} |f(\xi)|.$$

Asymptotic bounds for $f$ give rise to the definition of its **order** $\lambda_f$ and **type** $\tau_f$:

$$\lambda_f := \begin{cases} 
0 & \text{for } f \text{ constant} \\
\limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r} & \text{else} \end{cases} \quad (15a)$$

$$\tau_f := \begin{cases} 
0 & \text{if } \lambda_f = 0 \\
\limsup_{r \to \infty} r^{-\lambda_f} \log M_f(r) & \text{else} \end{cases} \quad (15b)$$

If $f$ is of finite order and type $0 < \lambda_f, \tau_f < \infty$, we write $f(\xi) = O(\exp(\tau_f|\xi|^{\lambda_f}))$. Moreover, we say that order 1 entire functions are of exponential order. According to Theorem 2, the Fourier transform of any compactly supported function is an entire function of at most exponential order of finite type.

Alternatively, an entire function $f : \mathbb{C} \to \mathbb{C}$ may be characterized by its zeros in the complex plane. If $f$ is not identically zero, its roots counted by their multiplicity form an at most countably infinite sequence $Z_f := \{a_j\}_{j \in J} \subset \mathbb{C} \setminus \{0\}, J \subset \mathbb{N}$ with no accumulation point in $\mathbb{C}$. Note that we exclude a possible zero in the origin from this definition and that $Z_f$ is assumed to be monotonically increasing in modulus. Classifying the divergence behavior of $Z_f$, we define the **convergence exponent** $\rho_f \in [0; \infty]$ and **rank**
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Let $f$ be an entire function of finite order $\lambda_f$ and not identically zero. Then $f$ has rank $p_f \leq \lambda_f$ and it admits a factorization

$$f(\xi) = \xi^m \exp(q_f(\xi)) \prod_{j \in J} E_{p_f} \left( \frac{\xi}{a_j} \right) \forall \xi \in \mathbb{C},$$

where $m \in \mathbb{N}_0$ is the order of the zero at $\xi = 0$, $q_f$ a polynomial of degree $\leq \lambda_f$ and

$$E_n(z) = (1 - z) \exp \left( \sum_{j=1}^{n} \frac{z^j}{j} \right).$$

The product in (17) converges uniformly on any compact subset $K \subset \mathbb{C}$.

Conversely, for any sequence of zeros of finite convergence exponent, canonical products of the form (17) define entire functions of finite order:

**Theorem 4** (Borel [15]). Let $\{a_j\}_{j \in J} \subset \mathbb{C} \setminus \{0\}, J \subset \mathbb{N}$ be a possibly finite sequence of monotonically increasing modulus, finite rank $p$ and convergence exponent $\rho$. Then a product of the form (17) defines an entire function $f$ for any polynomial $q_f$ and $m \in \mathbb{N}_0$. Moreover, the order of $f$ is

$$\lambda_f = \max\{\deg(q_f), \rho\}.$$
Lemma 1. Let \( f, \tilde{f} \) be entire functions of finite order \( \lambda_f = \lambda_{\tilde{f}} \) such that for some \( U \subset \mathbb{R} \) open

\[
|f|^2_U = |	ilde{f}|^2_U.
\]

Then there exist entire functions \( f_1, f_2 \) of order \( \leq \lambda_f \) such that

\[
f = f_1 \cdot f_2 \quad \text{and} \quad \tilde{f} = f_1 \cdot f_2^*.
\]

Conversely, if \( f_1 \) and \( f_2 \) are entire functions of order \( \lambda \), then \( f \) and \( \tilde{f} \) defined by (19) are entire functions of order \( \leq \lambda \) satisfying \( |f|^2_U = |	ilde{f}|^2_U \).

Proof. It is sufficient to consider the case \( f \neq 0 \). By Theorem 3, \( f \) admits a factorization of the form (17). Noting that \( |f^2|^U = (f \cdot f^*)|^U \) and that \( f^* \) is entire, we find that \( |f|^2 \) has an extension to an entire function \( F = f \cdot f^* \) of order \( \leq \lambda_f \). As such, \( F \) is uniquely determined by its values on \( U \) (e.g. by Taylor expansion) and thus coincides with the respective entire extension \( \tilde{F} \) of \( |\tilde{f}|^2_U \). By the factorization of \( f \), we have for all \( \xi \in \mathbb{C} \)

\[
\tilde{F}(\xi) = F(\xi) = f(\xi)\overline{f(\xi)} = \xi^{2m} \exp(2\Re(q_f)(\xi)) \prod_{j \in J} E_{p_f} \left( \frac{\xi}{a_j} \right) \cdot E_{p_f} \left( \frac{\xi}{\overline{a_j}} \right).
\]

In particular, we find that \( F \) uniquely determines all zeros of \( f \) modulo complex conjugation, as well as the real parts of the coefficients of \( q_f \) and \( m \in \mathbb{N}_0 \). Consequently, \( \tilde{f} \) may differ from \( f \) at most by a subset \( I \subset J \) of “flipped” zeros and a multiplicative factor \( e^{Q} \), where \( Q \) is a polynomial of degree \( \leq \lambda_f \) with purely imaginary coefficients. Thus, the Hadamard factorization of \( \tilde{f} \) is given by

\[
\tilde{f}(\xi) = \xi^m \exp(q_f(\xi) + Q(\xi)) \prod_{j \in J \setminus I} E_{p_f} \left( \frac{\xi}{a_j} \right) \cdot \prod_{j \in I} E_{p_f} \left( \frac{\xi}{\overline{a_j}} \right) = \xi^m \exp \left( q_f(\xi) + \frac{Q(\xi)}{2} \right) \prod_{j \in J \setminus I} E_{p_f} \left( \frac{\xi}{a_j} \right) \cdot \exp \left( \frac{Q(\xi)}{2} \right) \prod_{j \in I} E_{p_f} \left( \frac{\xi}{\overline{a_j}} \right).
\]

Since the convergence exponents of the subsequences \( \{a_j\}_{j \in J \setminus I} \) and \( \{a_j\}_{j \in I} \) are at most as large as that of the total one, \( f_1 \) and \( f_2 \) are entire functions of order \( \leq \lambda_f \) according to Theorem 4. Noting that \( Q^* = -Q \), we further obtain for all \( \xi \in \mathbb{C} \)

\[
f_1(\xi)f_2(\xi) = \xi^m \exp \left( q_f(\xi) + \frac{(Q + Q^*)(\xi)}{2} \right) \prod_{j \in J \setminus I} E_{p_f} \left( \frac{\xi}{a_j} \right) \cdot \prod_{j \in I} E_{p_f} \left( \frac{\xi}{\overline{a_j}} \right) = \xi^m \exp(q_f(\xi)) \prod_{j \in J} E_{p_f} \left( \frac{\xi}{a_j} \right) = f(\xi).
\]

This proves the first claim. For the converse, we simply note that for all \( x \in \mathbb{R} \)

\[
|f(x)|^2 = (f \cdot f^*)(x) = (f_1 \cdot f_2 \cdot f_1^* \cdot f_2^*)(x) = (\tilde{f} \cdot \tilde{f}^*)(x) = |	ilde{f}(x)|^2.
\]
The principal idea for the proof of Theorem 1 in the case \( n = 1 \) is to show that the factorization procedure outlined by Lemma 1 never yields \( h \neq \tilde{h} \in \mathcal{S}_c'(\mathbb{R}) \) such that \( F(h) = F(\tilde{h}) \) due to the specific order 2 growth of the first summand in (1). By definition, adding a lower order function \( g, \lambda_g < \lambda_f \to f \) does not change its order, nor its type. Likewise, it is clear that multiplication with \( g \) cannot increase any of this properties. For the growth estimates to be made, we further need that they may neither decrease if \( g \) is of at most exponential order and not identically zero:

**Lemma 2** (Decay bounds for low order entire functions [7]). Let \( f \) be an entire function of order \( 0 \leq \lambda_f \leq 1 \) that is not identically zero and let \( \varepsilon > 0 \). Then

\[
\limsup_{r \to \infty} m_f(r) M_f(r)^{1+\varepsilon} > 0
\]

In particular, if \( f \) is at most of exponential type \( \tau_f \), then \( \limsup_{r \to \infty} m_f(r) e^{(\tau_f+\varepsilon)r} = \infty \).

The essential message of Lemma 2 is that non-vanishing factors of at most exponential order may never weaken super-exponential growth.

4. Proof of the main results

With the preparations of section 3, we are now in a position to prove Theorem 1. For completeness, we start by proving well-definedness:

**Lemma 3.** The operator \( F \) in Theorem 1 is well-defined. If \( \Re(\alpha) \leq 0 \), then furthermore \( F(\mathcal{S}_c'(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{R}^n) \).

**Proof.** For \( h \in \mathcal{S}_c'(\mathbb{R}^n) \), we have \( wh \in \mathcal{S}_c'(\mathbb{R}^n) \) so that \( \mathcal{F}(P_0) \) and \( \mathcal{F}(wh) \) have extensions to entire functions in \( \mathbb{C}^n \) by Theorem 2. In particular, this implies \( \mathcal{F}(P_0), \mathcal{F}(wh) \in \mathcal{C}^\infty(\mathbb{R}^n) \). Hence, the same holds true for \( F(h) \) defined by

\[
F(h)(\xi) = |\mathcal{F}(P_0)(\xi)\exp(\alpha\|\xi\|^2) + \mathcal{F}(wh)(\xi)|^2.
\]

Additionally for \( \Re(\alpha) \leq 0 \), all terms inside the modulus are of at most algebraical growth in \( \mathbb{R}^n \) so that \( F(h) \) defines a tempered distribution, i.e. \( F(h) \in \mathcal{S}'(\mathbb{R}^n) \).

The next step is to prove Theorem 1 in the one-dimensional case \( n = 1 \). This permits to apply the theory of univariate entire functions outlined in section 3:

**Lemma 4.** Theorem 1 holds true for \( n = 1 \).
Proof. Let \( h, \tilde{h} \in \mathcal{S}'(\mathbb{R}) \) such that \( F(h)_U = F(\tilde{h})_U \). Define

\[
f(\xi) := \mathcal{F}(P_0)(\xi) \exp(\alpha \xi^2) + \mathcal{F}(wh)(\xi)
\]

for all \( \xi \in \mathbb{C} \) and \( \tilde{f} \) analogously, so that \( F(h) = |f|_\mathbb{R}^2 \) and \( \tilde{F}(h) = |\tilde{f}|_\mathbb{R}^2 \). Since \( \alpha \neq 0 \) and \( P_0 \in \mathcal{S}'(\mathbb{R}) \setminus \{0\} \), \( f \) and \( \tilde{f} \) are entire functions of order 2 by Theorem 2 and Lemma 2, matching the setting of Lemma 1. Accordingly, we have

\[
f = f_1 \cdot f_2 \quad \text{and} \quad \tilde{f} = f_1 \cdot f_2^*
\]

for some entire functions \( f_1, f_2 \) of order \( \leq 2 \). Moreover,

\[
\mathcal{F}(w \cdot (h - \tilde{h})) = f - \tilde{f} = f_1 \cdot (f_2 - f_2^*) =: g.
\]

where \( h - \tilde{h} \in \mathcal{S}'(\mathbb{R}) \) is of compact support. Thus, \( g \) is an entire function of at most exponential order according to Theorem 2 and therefore of rank \( p_g \leq 1 \) by Theorem 3.

We show that \( h = \tilde{h} \) by contradiction. Accordingly, assume \( h - \tilde{h} \neq 0 \). Then \( f_1 \) and \( f_2 - f_2^* \) are nonzero factors of order \( \leq 2 \) of \( g \). Consequently, the rank of \( f_1 \) must be smaller or equal \( p_{g} \leq 1 \) because its zeros \( \{a_j\}_{j \in I} \subset \mathbb{C} \setminus \{0\} \) form a subset of those of \( g \), \( \{a_j\}_{j \in J} \), where \( I \subset J \subset \mathbb{N} \). This implies that the Hadamard factorization of \( f_1 \) can be written in the form

\[
f_1(\xi) = \xi^m \exp(\mu_0 + \mu_1 \xi + \mu_2 \xi^2) \prod_{j \in I} E_1 \left( \frac{\xi}{a_j} \right)
\]

\[
= \exp(\mu_2 \xi^2) \left( \xi^m \exp(\mu_0 + \mu_1 \xi) \prod_{j \in I} E_1 \left( \frac{\xi}{a_j} \right) \right)
\]

\[
=: f_0(\xi)
\]

for some \( \mu_0, \mu_1, \mu \in \mathbb{C} \). By the same argument as with the rank of \( f_1 \), the convergence exponent \( \rho_{f_0} \) of \( f_0 \), determined by \( \{a_j\}_{j \in I} \), can be at most as large as \( \rho_g \). On the other hand, an application of Theorem 4 to the Hadamard factorization of \( g \) yields \( \rho_g \leq \lambda_g \leq 1 \). By Theorem 4, this implies that \( f_0 \), as defined in (20), is of at most exponential order.

Substituting (20) into the factorizations of \( f \) and \( \tilde{f} \) setting \( \beta := \Re(\mu), \gamma := -\Im(\mu) \), we find that for all \( \xi \in \mathbb{C} \)

\[
f(\xi) = \exp(\mu_2 \xi^2) f_0(\xi) f_2(\xi) = \exp(-i \gamma \xi^2) f_0(\xi) \exp(\beta \xi^2) f_2(\xi)
\]

\[
\tilde{f}(\xi) = \exp(\mu_2 \xi^2) f_0(\xi) \overline{f_2(\xi)} = \exp(-i \gamma \xi^2) f_0(\xi) \left( \overline{\exp(\beta \xi^2) f_2(\xi)} \right)
\]

These equalities show that the factor \( \exp(\beta \xi^2) \) may be absorbed in \( f_2 \), as it is invariant under formation of the Schwarz reflection \( ^* \). Thus, we may assume \( \beta = 0 \) without loss of generality. This implies for all \( \xi \in \mathbb{C} \)

\[
g(\xi) = \exp(-i \gamma \xi^2) f_0(\xi) (f_2(\xi) - f_2^*(\xi))
\]

and by multiplication with \( f_0^* \) and application of \( ^* \)

\[
f_0(\xi) g^*(\xi) = -\exp(2i \gamma \xi^2) f_0^*(\xi) g(\xi).
\]
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$f_0 \cdot g^*$ and $f_0^* \cdot g$ are both nonzero entire functions of order $\leq 1$, whereas $\xi \mapsto \exp(2i\gamma \xi^2)$ is of order 2 for any $\gamma \neq 0$. According to Lemma 2, this super-exponential growth could not be compensated by the remaining at most exponential order factors on the right hand side of (21), so that the only possibility for (21) to hold for all $\xi \in \mathbb{C}$ is $\gamma = 0$.

Recalling the definition of $f$ and $\tilde{f}$ and setting $a := F(P_0)$, $b := F(wh)$, $\tilde{b} := F(w\tilde{h})$ and $e(\xi) := \exp(\alpha \xi^2)$ for all $\xi \in \mathbb{C}$, this yields combined with the preceding results

\[ f_0 \cdot f_2 = f = a \cdot e + b \tag{22a} \]
\[ f_0^* \cdot f_2 = \tilde{f}^* = a^* \cdot e^* + \tilde{b}^*. \tag{22b} \]

By multiplication of (22a) and (22b) with $f_0^*$ and $f_0$, respectively, we obtain

\[ f_0^* \cdot (a \cdot e + b) = f_0^* \cdot f_0 \cdot f_2 = f_0 \cdot (a^* \cdot e^* + \tilde{b}^*) \tag{23} \]

For $c \in \{-1, 1\}$, consider the diagonals

\[ D_c := \{z \in \mathbb{C} : \Re(z) = c \Im(z)\} \]

in the complex plane and let $s$ denote the sign of $\Im(\alpha)$. Then we have

\[ |e(\xi)| = \begin{cases} 
\exp(-|\Im(\alpha)||\xi|^2) & \text{for } \xi \in D_s \\
\exp(|\Im(\alpha)||\xi|^2) & \text{for } \xi \in D_{-s}
\end{cases} \]
\[ |e^*(\xi)| = \begin{cases} 
\exp(|\Im(\alpha)||\xi|^2) & \text{for } \xi \in D_s \\
\exp(-|\Im(\alpha)||\xi|^2) & \text{for } \xi \in D_{-s}
\end{cases} \]

Since all of the remaining factors in (23) are non-vanishing entire functions of at most exponential order, this implies that the right hand side of (23) is $O(\exp(|\Im(\alpha)||\xi|^2))$ in $D_s$, whereas the left hand side grows at most exponentially along this diagonal. Contradiction!

Accordingly, the initial assumption $h \neq \tilde{h}$ must be wrong. By generality of $h, \tilde{h} \in \mathcal{S}_c'(\mathbb{R})$, this proves injectivity of the operator $F$ in the case $n = 1$. \hfill \Box

Using the 1D result, we are finally in a position to prove Theorem 1:

**Proof of Theorem 1.** Well-definedness of $F$ has been shown in Lemma 3. Moreover, we have proven injectivity in the case $n = 1$, so that we may restrict ourselves to $n \geq 2$.

Let $h, \tilde{h} \in \mathcal{S}_c'(\mathbb{R}^n)$ such that $F(h)|_U = F(\tilde{h})|_U$ for some $U \subset \mathbb{R}^n$ open. Like in the 1D case, $F(h)$ and $F(\tilde{h})$ have extensions to entire analytic functions in $\mathbb{C}^n$ by Theorem 2, so that $F(h) = F(\tilde{h})$ everywhere. Let $\mathcal{F}_{n-1} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ denote the Fourier transform in all but the first variable. For $\xi_y \in \mathbb{R}^{n-1}$, we set

\[ P_{0, \xi_y} := \mathcal{F}_{n-1}(P_0)(\cdot, \xi_y) \exp(\alpha||\xi_y||^2), \]
\[ h_{\xi_y} := \mathcal{F}_{n-1}(wh)(\cdot, \xi_y), \]
\[ \tilde{h}_{\xi_y} := \mathcal{F}_{n-1}(w\tilde{h})(\cdot, \xi_y). \tag{24} \]
Then \( P_0, \xi_y, \tilde{h}_{\xi_y} \in \mathcal{S}'(\mathbb{R}) \) and there exists an open set \( V \subset \mathbb{R}^n \) such that \( P_0, \xi_y \neq 0 \) for all \( \xi_y \in V \). By construction, we have for all \( \xi_y \in \mathbb{R}^{n-1}, \xi_z \in \mathbb{R}, \xi = (\xi_z, \xi_y) \)

\[
\mathcal{F}(P_0, \xi_y)(\xi_z) \exp(\alpha \xi_z^2) + \mathcal{F}(h_{\xi_y})(\xi_z) = \mathcal{F}(P_0)(\xi) \exp(\alpha \|\xi\|^2) + \mathcal{F}(wh)(\xi)
\]

and an analogous equality for \( h \) and \( \tilde{h}_{\xi_y} \). This implies by assumption

\[
|\mathcal{F}(P_0, \xi_y)(\xi_z) \exp(\alpha \xi_z^2) + \mathcal{F}(h_{\xi_y})(\xi_z)|^2 = \mathcal{F}(h)(\xi)
\]

\[
= \mathcal{F}(\tilde{h})(\xi) = |\mathcal{F}(P_0, \xi_y)(\xi_z) \exp(\alpha \xi_z^2) + \mathcal{F}(\tilde{h}_{\xi_y})(\xi_z)|^2
\]

(25)

for all \( \xi_y \in \mathbb{R}^{n-1}, \xi_z \in \mathbb{R} \).

The leftmost and rightmost expressions in (25) are exactly images of \( h_{\xi_y} \) and \( \tilde{h}_{\xi_y} \), respectively, under the operator \( F \) in the one-dimensional setting \( n = 1 \). By application of Lemma 4, (25) thus implies

\[
h_{\xi_y} = \tilde{h}_{\xi_y} \quad \text{for all} \quad \xi_y \in V
\]

(26)

According to Theorem 2, \( \xi_y \mapsto h_{\xi_y}(x) \) and \( \xi_y \mapsto \tilde{h}_{\xi_y}(x) \) are entire analytic functions for all \( x \in \mathbb{R} \) so that (26) holds even for \( \xi_y \in \mathbb{R}^{n-1} \). By bijectivity of \( \mathcal{F}_{n-1} \) and invertibility of \( w \) in a multiplicative sense, \( h \) and \( \tilde{h} \) can be recovered uniquely from \( \{h_{\xi_y}\}_{\xi_y \in \mathbb{R}^{n-1}} \) and \( \{\tilde{h}_{\xi_y}\}_{\xi_y \in \mathbb{R}^{n-1}} \), respectively, by inversion of (24).

Since these function families coincide by the 1D uniqueness result, we obtain \( h = \tilde{h} \), which proves injectivity of \( F \).

We conclude this section with the proof of Corollary 2, showing uniqueness of phase contrast tomography in the absence of phase-wrapping:

**Proof of Corollary 2.** By Corollary 1, the object transmission functions \( \{O_\theta\}_{\theta \in V} \) can be uniquely reconstructed from the data. As \( \exp : \mathbb{C} \to \mathbb{C} \) is injective on \( \{z \in \mathbb{C} : -2\pi < \Im(z) \leq 0\} \), the same holds true for \( \{\mathcal{R}_\theta(\delta - i\beta)\}_{\theta \in V} \) due to the relations

\[
O_\theta = \exp(-ik\mathcal{R}_\theta(\delta - i\beta)) \quad \text{and} \quad 0 \leq k\mathcal{R}(\delta) < 2\pi.
\]

The Radon transform relates to the Fourier transform via the Fourier-Slice-Theorem. See [29] for details. In particular, \( \{\mathcal{R}_\theta(\delta - i\beta)\}_{\theta \in V} \) uniquely determines \( \mathcal{F}(\delta - i\beta) \) on some wedge-shaped open subset \( W_V \subset \mathbb{R}^{n+1} \) which is defined by the angles in \( V \). On the other hand, \( \mathcal{F}(\delta - i\beta) \) is entire analytic according to Theorem 2 since \( \delta - i\beta \) is compactly supported. Hence, the values on \( W_V \) uniquely determine \( \mathcal{F}(\delta - i\beta) \) in \( \mathbb{R}^{n+1} \) and are thus sufficient for the reconstruction of the unknown object. \( \square \)
5. Conclusions

In this paper, we have proven Theorem 1 showing uniqueness of phase retrieval of compactly supported objects superimposed upon a certain class of known reference signals. Notably, this uniqueness is deterministic and absolute - unlike previous results for plain Fourier data [3, 6], which hold only modulo “trivial” ambiguities or sets of measure zero. Yet, the proof is non-constructive, i.e. it does not yield any particular reconstruction method. An essential ingredient to it lies in the specific super-exponential growth of the reference term in (1) in $\mathbb{C}^n$. This behavior is never retained by the usual “zero-flipping” construction [1, 2, 44] of alternate solutions to the phase problem. We have applied our result to propagation-based phase contrast imaging with coherent X-rays in the near field regime, where wave propagation is governed by the Fresnel propagator and the unscattered part of the known illumination profile provides the required reference. Physically, our work implies that the complex wave field transmitted by an arbitrary compactly supported object, i.e. its contact image or transmission function, can be reconstructed from intensities recorded at a single detector distance and incident wavenumber - at least in theory. Adopted to phase contrast tomography, this allows for the reconstruction of the complete spatial structure of a compact sample, including both refractive- and absorptive components of its complex refractive index. Thus, we disprove the common believe [9, 31] that phase contrast imaging of general objects requires intensity data from at least two independent setups.

However, we emphasize that ab initio phase retrieval remains severely ill-posed as can be seen for example from the zeros of the contrast transfer function in the weak object limit, see for instance [37]. On the other hand, numerical results for phase contrast tomography, obtained via alternating projection algorithms [39] or iteratively regularized Newton-type methods as in [18], suggest that simultaneous reconstructions of refractive phase shifts and absorption are numerically feasible. For these applications, it will be of interest whether our uniqueness theorem can be adopted to the discretization of near field phase contrast imaging or whether the so far purely algebraic statement may be supplemented with stability results. Another question is in how far features of the illumination wave field, that we have assumed to be known, can be recovered simultaneously with the unknown object. These topics may be subject to future research. Finally, we note that our results may also be relevant to electronic imaging methods owing to the equivalence of the dynamics of paraxial electromagnetic waves applied herein, described by (3), and Schrödinger’s equation in quantum mechanics.

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References

[1] Akutowicz E J. On the determination of the phase of a fourier integral, i. *Transactions of the American Mathematical Society*, pages 179–192, 1956.
[2] Akutowicz E J. On the determination of the phase of a fourier integral, ii. *Proceedings of the American Mathematical Society*, 8(2):234–238, 1957.
[3] Barakat R and Newsam G. Necessary conditions for a unique solution to two-dimensional phase recovery. *Journal of mathematical physics*, 25(11):3190–3193, 1984.
[4] Bartels M, Priebe M, Wilke R N, Krüger S P, Giewekemeyer K, Kalbfleisch S, Oelendrowitz C, Sprung M, and Salditt T. Low-dose three-dimensional hard x-ray imaging of bacterial cells. *Optical Nanoscopy*, 1(1):1–7, 2012.
[5] Barty A, Marchesini S, Chapman H, Cui C, Howells M, Shapiro D, Minor A, Spence J, Weierstall U, Ilavsky J, et al. Three-dimensional coherent x-ray diffraction imaging of a ceramic nanofoam: Determination of structural deformation mechanisms. *Physical review letters*, 101(5):055501, 2008.
[6] Bates R. Uniqueness of solutions to two-dimensional fourier phase problems for localized and positive images. *Computer vision, graphics, and image processing*, 25(2):205–217, 1984.
[7] Boas R P. *Entire functions*, volume 5. Academic Press, 2011.
[8] Bonse U and Hart M. An x-ray interferometer. *Applied Physics Letters*, 6(8):155–156, 1965.
[9] Burvall A, Lundström U, Takman P A, Larsson D H, and Hertz H M. Phase retrieval in x-ray phase-contrast imaging suitable for tomography. *Optics express*, 19(11):10359–10376, 2011.
[10] Chapman H N, Barty A, Marchesini S, Noy A, Hau-Riege S P, Cui C, Howells M R, Rosen R, He H, Spence J C, et al. High-resolution ab initio three-dimensional x-ray diffraction microscopy. *JOSA A*, 23(5):1179–1200, 2006.
[11] Cloetens P, Ludwig W, Baruchel J, Van Dyck D, Van Landuyt J, Guigay J, and Schlenker M. Holotomography: Quantitative phase tomography with micrometer resolution using hard synchrotron radiation x rays. *Applied Physics Letters*, 75(19):2912–2914, 1999.
[12] Conway J B and Conway J B. *Functions of one complex variable*, volume 2. Springer, 1973.
[13] Fienup J and Wackerman C. Phase-retrieval stagnation problems and solutions. *JOSA A*, 3(11):1897–1907, 1986.
[14] Fienup J R. Reconstruction of a complex-valued object from the modulus of its fourier transform using a support constraint. *JOSA A*, 4(1):118–123, 1987.
[15] Freiling G and Yurko V. *Introduction to the theory of entire functions*. Schriftenreihe des Instituts für Mathematik der Universität-Duisburg-Essen, 2003.
[16] Gureyev T, Raven C, Snigirev A, Snigireva I, and Wilkins S. Hard x-ray quantitative non-interferometric phase-contrast microscopy. *Journal of Physics D: Applied Physics*, 32(5):563, 1999.
[17] Hagemann J, Robisch A L, Luke D, Homann C, Hohage T, Cloetens P, Suhonen H, and Salditt T. Reconstruction of wave front and object for inline holography from a set of detection planes. *Optics Express*, 22(10):11552–11569, 2014.
[18] Hohage T and Werner F. Iteratively regularized Newton-type methods for general data misfit functionals and applications to Poisson data. *Numerische Mathematik*, 123(4):745–779, 2013.
[19] Hörmander L. *The analysis of linear partial differential operators I*. Springer, Berlin, 2003.
[20] Jonas P and Louis A. Phase contrast tomography using holographic measurements. *Inverse Problems*, 20(1):75, 2004.
[21] Jonas P and Louis A. Cone beam geometry for small objects in phase contrast tomography. *Inverse Problems*, 29(9):095013, 2013.
[22] Klibanov M V. On the recovery of a 2-d function from the modulus of its fourier transform. *Journal of mathematical analysis and applications*, 323(2):818–843, 2006.
[23] Klibanov M V, Sacks P E, and Tikhonravov A V. The phase retrieval problem. *Inverse problems*, 11(1):1, 1995.
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[24] Marchesini S, Chapman H, Hau-Riege S, London R, Szoke A, He H, Howells M, Padmore H, Rosen R, Spence J, et al. Coherent x-ray diffractive imaging: applications and limitations. *Optics Express*, 11(19):2344–2353, 2003.

[25] Mayo S, Miller P, Wilkins S, Davis T, Gao D, Gureyev T, Paganin D, Parry D, Pogany A, and Stevenson A. Quantitative x-ray projection microscopy: phase-contrast and multi-spectral imaging. *Journal of Microscopy*, 207(2):79–96, 2002.

[26] Miao J, Hodgson K O, Ishikawa T, Larabell C A, LeGros M A, and Nishino Y. Imaging whole escherichia coli bacteria by using single-particle x-ray diffraction. *Proceedings of the National Academy of Sciences*, 100(1):110–112, 2003.

[27] Millane R. Phase retrieval in crystallography and optics. *JOSA A*, 7(3):394–411, 1990.

[28] Monose A, Takeda T, and Itai Y. Phase-contrast x-ray computed tomography for observing biological specimens and organic materials. *Review of Scientific Instruments*, 66(2):1434–1436, 1995.

[29] Natterer F. *The mathematics of computerized tomography*, volume 32 of *Classics of Applied Mathematics*. Society for Industrial and Applied Mathematics, 2001.

[30] Nugent K, Gureyev T, Cookson D, Paganin D, and Barnea Z. Quantitative phase imaging using hard x rays. *Physical Review Letters*, 77(14):2964, 1996.

[31] Nugent K A. X-ray noninterferometric phase imaging: a unified picture. *JOSA A*, 24(2):536–547, 2007.

[32] Nugent K A. Coherent methods in the x-ray sciences. *Advances in Physics*, 59(1):1–99, 2010.

[33] Paganin D, Mayo S, Gureyev T E, Miller P R, and Wilkins S W. Simultaneous phase and amplitude extraction from a single defocused image of a homogeneous object. *Journal of Microscopy*, 206(1):33–40, 2002.

[34] Paganin D and Nugent K A. Noninterferometric phase imaging with partially coherent light. *Physical Review Letters*, 80(12):2586, 1998.

[35] Paganin D M. *Coherent X-ray optics*, volume 1. Oxford University Press Oxford, 2006.

[36] Papoulis A. Systems and transforms with applications in optics. *McGraw-Hill Series in System Science, Malabar: Krieger*, 1968.

[37] Pogany A, Gao D, and Wilkins S. Contrast and resolution in imaging with a microfocus x-ray source. *Review of Scientific Instruments*, 68(7):2774–2782, 1997.

[38] Reed Teague M. Deterministic phase retrieval: a green’s function solution. *JOSA A*, 73(11):1434–1441, 1983.

[39] Ruhlandt A, Kreukel M, Bartels M, and Sadlitt T. Three-dimensional phase retrieval in propagation-based phase-contrast imaging. *Physical Review A*, 89(3):033847, 2014.

[40] Teich M C and Saleh B. Fundamentals of photonics. *Canada, Wiley InterScience*, page 3, 1991.

[41] Thibault P, Dierolf M, Kewish C M, Menzel A, Bunk O, and Pfeiffer F. Contrast mechanisms in scanning transmission x-ray microscopy. *Physical Review A*, 80(4):043813, 2009.

[42] Thibault P, Elser V, Jacobsen C, Shapiro D, and Sayre D. Reconstruction of a yeast cell from x-ray diffraction data. *Acta Crystallographica Section A: Foundations of Crystallography*, 62(4):248–261, 2006.

[43] Turner L, Dhal B, Hayes J, Mancuso A, Nugent K, Paterson D, Scholten R, Tran C, and Peele A. X-ray phase imaging: Demonstration of extended conditions for homogeneous objects. *Optics Express*, 12(13):2960–2965, 2004.

[44] Walther A. The question of phase retrieval in optics. *Journal of Modern Optics*, 10(1):41–49, 1963.

[45] Wilkins S, Gureyev T, Gao D, Pogany A, and Stevenson A. Phase-contrast imaging using polychromatic hard x-rays. *Nature*, 384(6607):335–338, 1996.