LETTER TO THE EDITOR

Renormalization group method and canonical perturbation theory

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Abstract. Renormalization group method is one of the most powerful tool to obtain approximate solutions to differential equations. We apply the renormalization group method to Hamiltonian systems whose integrable parts linearly depend on action variables. We show that the renormalization group method gives the same approximate solutions as canonical perturbation theory up to the second order of a small parameter with action-angle coordinates.

PACS numbers: 03.20.+i, 05.45.+b

Submitted to: J. Phys. A: Math. Gen.

1. Introduction

Dynamical systems written by differential equations are useful to understand temporal evolutions of the nature. Exact solutions to the equations are not always obtained because of non-integrability of systems, and naive perturbation often yields secular terms which prevent us from getting approximate but global solutions. Singular perturbation techniques [1], eg averaging methods, multi scale methods, matched asymptotic expansions and WKB methods, are available to construct global solutions. However, they provide no systematic procedures for general systems because we must select a suitable assumption about the structure of a perturbation series.

Recently, renormalization group method is proposed [2, 3] as a tool for global asymptotic analysis of the solutions to differential equations. It unifies the techniques listed above, and can treat many systems irrespective of their features. We apply the renormalization group method to Hamiltonian systems, and compare it with canonical perturbation theory [4, 5], which is one of the most developed perturbation theory for Hamiltonian systems. In this letter, we show that the renormalization group method

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also unifies the canonical perturbation theory. That is, the former and the latter give the same solutions to equations of motion up to the second order of a small parameter.

We use action-angle coordinates as they are suitable for perturbed Hamiltonian systems, and Hamiltonians are

\[ H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta), \]

where both \( I \) and \( \theta \) are \( N \)-dimensional vectors, the integrable part \( H_0 \) is

\[ H_0(I) = \omega \cdot I, \]

and \( H_1(I, \theta) \) is periodic with respect to each element of \( \theta \).

We derive an approximate solution with naive perturbation in section 2, and then we renormalize secular terms to constants of integration in section 3. Finally, in section 4 we compare the renormalized solutions with solutions obtained by canonical perturbation theory.

2. Naive Solution

The equation of motion for the system (1) is

\[ \frac{d}{dt} \{ x, H_0(x) + \epsilon H_1(x) \} = \{ x, \omega \cdot I \}, \]

where \( x = (I, \theta) \) is a \( 2N \)-dimensional vector and the symbol \( \{ \cdot, \cdot \} \) is Poisson bracket with respect to the subscript, in this case, \( x \). We expand \( x \) as a series of powers of \( \epsilon \),

\[ x = x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \ldots, \]

and then equation (3) becomes

\[ \frac{d}{dt} \left( x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \ldots \right) = \left\{ x, \omega \cdot I \right\} (x^{(0)}) + \ldots, \]

where, in the right-hand-side, we substitute \( x^{(0)} \) to \( x \) after the Poisson bracket has been operated.

The solution to \( O(\epsilon^0) \) is

\[ I^{(0)} = \alpha_0, \quad \theta^{(0)} = \omega t + \beta_0, \]

where \( N \)-dimensional vectors \( \alpha_0 \) and \( \beta_0 \) are constants of integration.

The equation of motion for \( O(\epsilon^1) \) is

\[ \frac{d}{dt} \{ x^{(1)} \} = \left\{ x, \omega \cdot I \right\} (x^{(0)}), \]

and hence the solution to \( O(\epsilon^1) \) is

\[ x^{(1)} = \left\{ x, S_1(x) \right\} + t \{ x, \langle H_1(x) \rangle_t \}, \]

where we introduced the symbols

\[ \chi = (\alpha, \beta), \quad \alpha = \alpha_0, \quad \beta = \omega t + \beta_0, \]

\[ x^{(2)} = \left\{ x, S_2(x) \right\} + \left\{ x, S_2(x^{(1)}) \right\}, \]

where we introduced the symbol
\( \langle \cdot \rangle_t \) represents average over \( t \), and

\[
S_1(\chi) \equiv \int dt \ (H_1(\chi) - \langle H_1(\chi) \rangle_t).
\] (10)

The following relation was also used

\[
\{ f(x) , \ g(x) \}_x (x^{(0)}) = \{ f(x) , \ g(x) \}_x,
\] (11)
which is satisfied by arbitrary functions \( f \) and \( g \) that are periodic for \( \theta \) and \( \beta \).

The equation to \( O(\epsilon^2) \) is

\[
\frac{d\mathbf{x}^{(2)}}{dt} = \left\{ \chi , \ \{ H_1(\chi) , S_1(\chi) \}_x \right\}_x + \left\{ \chi , \ S_1(\chi) \right\}_x , H_1(\chi) \right\}_x
\]
\[
+ t \left\{ \chi , H_1(\chi) \right\}_x , \langle H_1(\chi) \rangle_t \right\}_x,
\] (12)

and the solution to \( O(\epsilon^2) \) is

\[
\mathbf{x}^{(2)} = \{ \chi , S_2 \}_x + \frac{1}{2} \left\{ \chi , S_1 \right\}_x , S_1 \right\}_x + t \left\{ \chi , \langle F_2 \rangle_t \right\}_x
\]
\[
+ (t^2\text{-secular terms}) + (t\text{-secular terms with non-constants}).
\] (13)

Here

\[
S_2(\chi) \equiv \int dt \ (F_2(\chi) - \langle F_2(\chi) \rangle_t),
\]
\[
F_2(\chi) \equiv \{ H_1(\chi) , S_1(\chi) \}_x + \frac{1}{2} \left\{ \{ H_0(\chi) , S_1(\chi) \}_x , S_1 \right\}_x , \]
\[
\] (14)

and

\[
\left\{ \chi , S_1 \right\}_x , H_1 \right\}_x = \left\{ \chi , \frac{1}{2} \left\{ \{ H_0 , S_1 \}_x , S_1 \right\}_x \right\}_x + A(\chi) + B(\chi),
\] (15)

\[
\int dt \ A(\chi) = \frac{1}{2} \left\{ \chi , S_1 \right\}_x , S_1 \right\}_x,
\]
\[
\int dt \ t \ \left\{ \chi , H_1 \right\}_x , \langle H_1 \rangle_t \right\}_x = - \int dt \ B(\chi)
\]
\[
+ (t^2\text{-secular terms}) + (t\text{-secular terms with non-constants}),
\] (16)

which are proven by using Fourier expressions of \( H_1(\chi) , \langle H_1(\chi) \rangle_t \) and \( S_1(\chi) \).

The concrete forms of the Fourier expressions, \( A(\chi) \) and \( B(\chi) \) are shown in appendix.

Consequently, the naive solution to equation (3) is, up to \( O(\epsilon^2) \),

\[
\mathbf{x} = \chi + \epsilon \left\{ \chi , S_1 \right\}_x + \epsilon^2 \left[ \chi , S_2 \right\}_x + \frac{1}{2} \left\{ \chi , S_1 \right\}_x , S_1 \right\}_x \right] \] (18)
3. Renormalization of Secular Terms

We renormalize the secular terms of the naive solution \( I^8 \) to the constants of integration. First we regard \( \alpha_0 \) and \( \beta_0 \) as functions of \( t \) which are

\[
\begin{align*}
\alpha_0(t) &= \alpha_0 + t \left[ \epsilon \{ \alpha , \langle H_1 \rangle_t \} \chi + \epsilon^2 \{ \alpha , \langle F_2 \rangle_t \} \chi \right], \\
\beta_0(t) &= \beta_0 + t \left[ \epsilon \{ \beta , \langle H_1 \rangle_t \} \chi + \epsilon^2 \{ \beta , \langle F_2 \rangle_t \} \chi \right],
\end{align*}
\]

Second we introduce assumptions with which the renormalization transformations \( I^9 \) becomes to be a Lie group. In this case, we assume that equation \( I^9 \) is a truncated Taylor series of \( \alpha_0(t) \) and \( \beta_0(t) \) around the initial time \( t = 0 \) \( I^9 \). From time-evolutional symmetry of the system \( I^1 \), the renormalization group equation becomes

\[
\begin{align*}
\frac{d\alpha_0}{dt} &= \epsilon \{ \alpha , \langle H_1 \rangle_t \} \chi + \epsilon^2 \{ \alpha , \langle F_2 \rangle_t \} \chi + O(\epsilon^3), \\
\frac{d\beta_0}{dt} &= \epsilon \{ \beta , \langle H_1 \rangle_t \} \chi + \epsilon^2 \{ \beta , \langle F_2 \rangle_t \} \chi + O(\epsilon^3),
\end{align*}
\]

in other words,

\[
\frac{d\chi}{dt} = \{ \chi , H_0(\chi) \} \chi + \epsilon \{ \chi , \langle H_1 \rangle_t \} \chi + \epsilon^2 \{ \chi , \langle F_2 \rangle_t \} \chi + O(\epsilon^3). \tag{21}
\]

The renormalized solution is therefore

\[
\chi = \chi + \epsilon \{ \chi , S_1 \} \chi + \epsilon^2 \left[ \{ \chi , S_2 \} \chi + \frac{1}{2} \left\{ \{ \chi , S_1 \} \chi , S_1 \right\} \chi \right] + O(\epsilon^3), \tag{22}
\]

where \( \chi \) is governed by equation \( I^{21} \). Here, \( t^2 \)-secular terms and \( t \)-secular terms with non-constants in \( O(\epsilon^2) \) of equation \( I^8 \) are renormalized to coefficients of \( t \)-secular terms and coefficients of non-constant terms, respectively.

4. Comparison with Canonical Perturbation Theory

Finally we compare the renormalized solution \( I^{22} \) and solution obtained by canonical perturbation theory. The strategy of the theory is to canonically transform coordinates \( \chi = (I, \theta) \) to \( \chi^* = (I^*, \theta^*) \) with the generator \( S(\chi^*) \)

\[
\chi = \exp(\epsilon D_S)\chi^*, \quad D_S f(\chi^*) \equiv \{ f(\chi^*) , S(\chi^*) \} \chi^*, \tag{23}
\]

such that secular terms do not appear in the coordinates \( \chi^* \). What we must calculate are the generator \( S \) and the transformed Hamiltonian \( H^* \). Canonical perturbation theory \( I^1, I^3 \) states that the required generator \( S = S_1 + \epsilon S_2 + \ldots \) is expressed as

\[
\begin{align*}
S_1(\chi^*) &= \int d\tau \ (H_1(\chi^*) - \langle H_1(\chi^*) \rangle_\tau), \\
S_2(\chi^*) &= \int d\tau \ (F_2(\chi^*) - \langle F_2(\chi^*) \rangle_\tau),
\end{align*}
\]

and the transformed Hamiltonian \( H^* = H_0^* + \epsilon H_1^* + \epsilon^2 H_2^* + \ldots \) as

\[
\begin{align*}
H_0^*(\chi^*) &= H_0(\chi^*), \\
H_1^*(\chi^*) &= \langle H_1(\chi^*) \rangle_\tau, \\
H_2^*(\chi^*) &= \langle F_2(\chi^*) \rangle_\tau, \tag{26}
\end{align*}
\]
where
\[ F_2(x^*) = \{ H_1(x^*), S_1(x^*) \}_{x^*} + \frac{1}{2} \{ \{ H_0(x^*) , S_1(x^*) \}_{x^*} , S_1(x^*) \}_{x^*}. \] (27)

The symbol \( \langle \cdot \rangle_\tau \) represents the average over \( \tau \), the time of \( x^* \) following \( H_0^* \), that is,
\[ \frac{dx^*}{d\tau} = \{ x^* , H_0^*(x^*) \}_{x^*}. \] (28)

Consequently, this theory gives an approximate solution determined by
\[ \frac{dx^*}{dt} = \{ x^* , H_0(x^*) \}_{x^*} + \epsilon \{ x^* , \langle H_1 \rangle_\tau \}_{x^*} + \epsilon^2 \{ x^* , \langle F_2 \rangle_\tau \}_{x^*} + O(\epsilon^3). \] (29)

and the canonical transformation (23)
\[ x = x^* + \epsilon \{ x^* , S_1 \}_{x^*} + \epsilon^2 \left[ \{ x^* , S_2 \}_{x^*} + \frac{1}{2} \{ \{ x^* , S_1 \}_{x^*} , S_1 \}_{x^*} \right] + O(\epsilon^3). \] (30)

The approximate solution (30) is the same as the renormalized solution (22) since temporal evolutions of \( x^* \) and \( \chi \) are governed by equations (29) and (21) respectively, and the two equations have the same structure.

5. Summary and Discussions

We showed that renormalization group method gives the same approximate solutions as canonical perturbation theory up to the second order of a small parameter to the Hamiltonian systems whose integrable parts linearly depend on action variables. That is, renormalization group method unifies not only averaging methods, multi scale methods, matched asymptotic expansions and WKB methods, but canonical perturbation theory. We suppose that the unification holds even in higher orders of the small parameter.

In systems whose integrable parts are not linear, secular terms are not always proportional to time \( t \), and may be proportional to \( t^n \) \((n \neq 1)\). Canonical perturbation theory cannot remove the latter secular terms since subtracting time-averages of perturbative part of Hamiltonian is effective only for the \( t \)-linear secular terms. On the contrary, renormalization group method gives global solutions by introducing assumptions with which renormalization transformations become to a Lie group \[ \text{[7]} \] and can treat \( t^n \) type secular terms.

In the previous paper \[ \text{[8]} \], we discussed relation between integrability of original systems and symplectic properties of renormalization group equations in Cartesian coordinates. From equation (21), we clarified that renormalization group equations are always Hamiltonian systems in action-angle coordinates whose Hamiltonian is
\[ H_{RG}(\chi) = H_0 + \epsilon \langle H_1 \rangle_t + \epsilon^2 \langle F_2 \rangle_t. \] (31)

Symplectic properties are recovered even in Cartesian coordinates by using “gauge freedom” which is homogeneous terms of \( O(\epsilon^1) \). Details will show in the next paper \[ \text{[7]} \].

YN is supported in part by the Grand-In-Aid for Scientific Research of the Ministry of Education, Science, Sports and Culture of Japan(09740196).
Appendix

Let us introduce Fourier series of $H_1, \langle H_1 \rangle_t$ and $S_1$ as

$$H_1(\chi) = \sum_m \hat{H}_{1,m}(\alpha)e^{im\cdot\beta},$$

$$\langle H_1 \rangle_t(\chi) = \sum_{m: \text{Res}} \hat{H}_{1,m}(\alpha)e^{im\cdot\beta}.$$ 

$$S_1(\chi) = \sum_{m: \text{Non}} \frac{1}{im \cdot \omega} \hat{H}_{1,m}(\alpha)e^{im\cdot\beta}.$$

where the symbols $\sum_{m: \text{Res}}$ and $\sum_{m: \text{Non}}$ represent to take summations over $m$ such that $m \cdot \omega = 0$ and $m \cdot \omega \neq 0$, respectively. By using these expressions and $\eta = (-\theta, I)$, the concrete forms of $A(\chi)$ and $B(\chi)$ are

$$A(\chi) = \frac{1}{2} \sum_{m: \text{Non}} \sum_{n: \text{Non}} \left( \frac{1}{im \cdot \omega} + \frac{1}{in \cdot \omega} \right) C(\chi)$$

and

$$B(\chi) = \sum_{m: \text{Res}} \sum_{n: \text{Non}} \frac{1}{in \cdot \omega} C(\chi)$$

respectively, where

$$C(\chi) = \left[ in_k \left( \frac{\partial \hat{H}_{1,n}}{\partial \eta} + \frac{\partial (in \cdot \beta)}{\partial \eta} \hat{H}_{1,n} \right) \frac{\partial \hat{H}_{1,m}}{\partial \alpha_k} - im_k \left( \frac{\partial^2 \hat{H}_{1,n}}{\partial \alpha_k \partial \eta} + \frac{\partial (in \cdot \beta)}{\partial \eta} \frac{\partial \hat{H}_{1,n}}{\partial \alpha_k} \right) \hat{H}_{1,m} \right] e^{i(m+n)\cdot\beta}.$$