In this paper, we are concerned with the nonrelativistic limit of a class of computable approximation models for radiation hydrodynamics. The models consist of the compressible Euler equations coupled with moment closure approximations to the radiative transfer equation. They are first-order partial differential equations with source terms. As hyperbolic relaxation systems, they are showed to satisfy the structural stability condition proposed by the second author. Based on this, we verify the nonrelativistic limit by combining an energy method with a formal asymptotic analysis.

**KEYWORDS**
formal asymptotic expansion, moment closure systems, nonrelativistic limit, radiation hydrodynamics, structural stability condition

**MSC CLASSIFICATION**
35Q35, 35B25, 82C40

## 1 | INTRODUCTION

Radiation hydrodynamics [1] studies interactions of radiation and matters through momentum and energy exchanges. It is modeled with the compressible Euler equations coupled with a radiation transport equation via an integral-type source [2–4]:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) + \nabla p &= -S_F, \\
\partial_t (\rho E) + \text{div}(\rho v E + pv) &= -cSE, \\
\partial_t I + c\mu \cdot \nabla I &= cS.
\end{align*}
\]

(1)

Here the unknowns \( \rho, v, \) and \( E \) denote the density, velocity, and energy of the fluid, respectively; \( I = I(x, t, \mu) \geq 0 \) is the radiative intensity depending on the \( D \)-dimensional spatial variable \( x \in \mathbb{R}^D \), the time variable \( t > 0 \), and the direction variable \( \mu \in S^{D-1} \) with \( S^{D-1} \) the unit sphere in \( \mathbb{R}^D \); the thermodynamics pressure \( p = p(\rho, \theta) \) is a smooth function of \( \rho \) and temperature \( \theta; S = S(\rho, \theta, I; c) \) is the source of radiation; \( c \) is the speed of light; the source term in (1) is taken to be [2]

\[
S = c\rho \sigma_a(\theta) \left( \frac{1}{|S^{D-1}|} b(\theta) - I \right) + \frac{1}{c} \rho \sigma_s(\theta) \left( \frac{1}{|S^{D-1}|} \int_{S^{D-1}} I d\mu - I \right).
\]

(2)
where \( \sigma_a = \sigma_a(\theta) > 0 \) is the absorption coefficient, \( \sigma_s = \sigma_s(\theta) > 0 \) is the scattering coefficient, and the Planck function \( b = b(\theta) \) is smooth and satisfies

\[
b(\theta) > 0, \quad b'(\theta) > 0;
\]

\( S_F(S_E) \) characterizes the energy (resp. impulse) exchange between the radiation and matter [5]:

\[
S_F = \int_{S^{n-1}} \mu S d\mu, \quad S_E = \int_{S^{n-1}} S d\mu.
\]

The theory of radiation hydrodynamics has a wide range of applications, including nonlinear pulsation, supernova explosions, stellar winds, and laser fusion [1, 4, 6]. However, the full set of radiation hydrodynamics equations is computationally expensive and numerically difficult to solve since the radiative equation is a high-dimensional integro-differential equation. Various solution methods have been developed. Among them, the moment method is quite attractive due to its numerous advantages such as clear physical interpretation and high efficiency in transitional regimes. It has been regarded as a successful tool to solve radiative equation [7–9].

Recently, a new moment method was proposed by Fan et al. [10, 11] for the radiative transfer equation (RTE), which is basically the last equation in Equation (1). The resultant model (called the HMP\(_N\) model) is globally hyperbolic, and some important physical properties are preserved. In this paper, we focus on Equation (1) with the last equation replaced by its HMP\(_N\) approximation (and the source terms are treated accordingly). The resultant coupling system will be called the Euler-HMP\(_N\) approximation of Equation (1). See Section 3.2 for the Euler-HMP\(_N\) approximation.

The goal of this paper is to investigate the nonrelativistic limit of Euler-HMP\(_N\) approximation that is the limit as the light speed tends to infinity. We restrict ourself to the monodimensional geometry. Under quite general assumptions, we prove that as the light speed goes to infinity, the Euler-HMP\(_N\) approximation of Equation (1) converges to

\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} b(\theta) \right) &= 0, \\
\partial_t (\rho E + b(\theta)) + \partial_x (\rho Ev + pv) &= \partial_x \left( \frac{1}{3 \rho \sigma_a(\theta)} \partial_x b(\theta) \right),
\end{aligned}
\]

with corresponding initial data. See details in Section 4.1.

Note that the nonrelativistic limit is a singular perturbation problem. Such singular limit problems have attracted much attention for many years. For instance, Marcati et al. [12] and Marcati and Milani [13] firstly analyzed the singular limit for weak solutions of hyperbolic balance laws with particular source terms. Bardos et al. [14, 15] studied the limit problem for nonsmooth solutions of the closely related nonlinear RTEs. With the well-known compensated compactness theory, Marcati and Rubino [16] studied general 2 \( \times \) 2 systems with applications to multidimensional problems and a class of one-dimensional semilinear systems. Recently, for a class of first-order symmetrizable hyperbolic systems, Lattanzio and Yong [17] and Peng and Wasiolek [18] studied the diffusion relaxation limit and derived parabolic type equations.

For the above works, the structural stability condition proposed by Yong [19, 20] is the key. It is a proper counterpart of the H-theorem for the kinetic equation. Indeed, this condition has been tacitly respected by many well-developed physical theories [21]. Recently, it was shown by Di et al. [22] to be satisfied by the hyperbolic regularization models [9, 23], which provides a basis for the first author to prove that the models well approximate the Navier–Stokes equations [24]. In contrast, the Biot/squirt (BISQ) model for wave propagation in saturated porous media violates this condition and thus allows exponentially exploding asymptotic solutions [25]. On the other hand, this condition also implies that the resultant moment system is compatible with the classical theories [24, 26].

In this paper, we verify the structural stability condition for the Euler-HMP\(_N\) system and construct formal asymptotic solutions thereof. On the basis of the stability condition, we use the energy method to prove the validity of the asymptotic approximations. Moreover, we conclude the existence of the solution to the Euler-HMP\(_N\) systems in the time interval where the approximations are well-defined.

Here, we mention some related works for the equations of radiation hydrodynamics. The system (1) was introduced by Pomraning [4] in the framework of special relativity. For the radiation hydrodynamics system with the radiation transfer equation replaced by its discrete-ordinate approximations, Rohde and Yong [27] showed the existence of
entropy solutions to the Cauchy problems in the framework of functions of bounded variation and investigated the nonrelativistic limit of the entropy solutions. Fan et al. [28] studied the nonrelativistic and low Mach number limits for the Navier–Stokes–Fourier–P1 approximation radiation model. Jiang et al. [5] studied the nonrelativistic limit problem of the compressible NSF–P1 approximation radiation hydrodynamics model arising in radiation hydrodynamics. We refer to earlier studies [29–32] for more references.

The paper is organized as follows. Section 2 presents a brief introduction of MP$_N$ and HMP$_N$ moment methods for the RTE. In Section 3, we verify the structural stability condition for the Euler-HMP$_N$ systems. Section 4 is devoted to the nonrelativistic limit. In particular, the formal asymptotic expansion is constructed in Section 4.1 and justified in Section 4.2. Finally, we conclude our paper in Section 5.

2 | HMP$_N$ MODEL

In this section, we present the HMP$_N$ model proposed by Fan et. [10] for the RTE for a gray medium in the slab geometry:

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial x} = S(I).$$

(3)

Here $I = I(x, t, \mu) \geq 0$ is the specific intensity of radiation, the variable $\mu \in [-1, 1]$ is the cosine of the angle between the photon velocity and the positive $x$-axis, the time variable $t \in \mathbb{R}$, and space variable $x \in \Omega$ with $\Omega$ a closed interval, and the right-hand side $S(I)$ is defined in (2).

Define the $k$th moment of the specific intensity as

$$E_k = \langle I \rangle_k = \frac{1}{\mu} \int_{-1}^{1} \mu^k I d\mu, \quad k \in \mathbb{N}.$$

Multiplying (3) by $\mu^k$ and integrating it with respect to $\mu$ over $[-1, 1]$ yield the moment equations

$$\frac{1}{c} \frac{\partial E_k}{\partial t} + \frac{\partial E_{k+1}}{\partial x} = \langle S(I) \rangle_k.$$

(4)

Notice that the governing equation of $E_k$ depends on the $(k + 1)$th moment $E_{k+1}$, which indicates that the full system contains an infinite number of equations, so we need to provide a so-called moment closure for the model. A common strategy is to construct an Ansatz: $I = \hat{I}(E_0, E_1, \cdots, E_N; \mu)$ with a prescribed integer $N$ such that

$$\langle \hat{I}(E_0, E_1, \cdots, E_N; \mu) \rangle_k = E_k, \quad k = 0, 1, \cdots, N.$$

Then the moment closure is given by

$$E_{N+1} = \langle \hat{I}(E_0, E_1, \cdots, E_N; \mu) \rangle_{N+1}.$$

Based on this strategy, many moment systems have been developed, such as the $P_N$ model [33], the $M_N$ model [8, 34], the positive $P_N$ model [35], the MP$_N$ model [11], and the HMP$_N$ model [10].

In this paper, we focus on the HMP$_N$ model which is based on the MP$_N$ model [11]. The latter takes the ansatz of the $M_N$ model (the first-order of the $M_N$ model) as a weight function and then constructs the ansatz by expanding the specific intensity around the weight function in terms of orthogonal polynomials in the velocity direction. Therefore, we briefly describe the MP$_N$ model.

2.1 | MP$_N$ model

The construction of the MP$_N$ model starts with the following weight function

$$\omega^{[a]}(\mu) = \frac{1}{(1 + a\mu)^\alpha}, \quad \alpha \in (-1, 1).$$

(5)
Here $\alpha$ is related to the low-order moment of radiation intensity, and its expression will be given later. Having this weight function, we use the Gram–Schmidt orthogonalization to define a series of orthogonal polynomials on the interval $[-1, 1]$:

$$\phi_0^{[\alpha]}(\mu) = 1, \quad \phi_j^{[\alpha]}(\mu) = \mu^j - \sum_{k=0}^{j-1} \kappa_{j,k} \phi_k^{[\alpha]}(\mu), \quad j \geq 1,$$

where the coefficients are

$$\kappa_{j,k} = \int_{-1}^1 \mu^j \phi_k^{[\alpha]}(\mu) \omega^{[\alpha]}(\mu) d\mu.$$

(6)

From the orthogonality, it is easy to see that

$$\kappa_{j,k} = 0, \text{ if } j < k, \quad \kappa_{k,k} = \int_{-1}^1 (\phi_k^{[\alpha]}(\mu))^2 \omega^{[\alpha]}(\mu) d\mu > 0.$$

(7)

Set $\Phi_i^{[\alpha]}(\mu) = \phi_i^{[\alpha]}(\mu) \omega^{[\alpha]}(\mu)$ for $i = 0, 1, \cdots, N$. The ansatz for the MP$_N$ model is

$$\hat{I}(E_0, E_1, \cdots, E_N; \mu) = \sum_{i=0}^N f_i \Phi_i^{[\alpha]}(\mu),$$

where $f_i$ are the expansion coefficients. Thanks to the orthogonality, the coefficients can be expressed as

$$f_i = \frac{1}{\kappa_{ii}} \left( E_i - \sum_{j=0}^{i-1} \kappa_{ij} f_j \right), \quad 0 \leq i \leq N.$$

(8)

The moment closure form is given by

$$E_{N+1} = \sum_{k=0}^N \kappa_{N+1,k} f_k.$$

For the MP$_N$ systems, the parameter $\alpha$ is taken to be

$$\alpha = -\frac{3E_1/E_0}{2 + \sqrt{4 - 3(E_1/E_0)^2}}.$$

A simple calculation shows that $f_1 = 0$.

Define the Hilbert space $\mathbb{H}_{N}^{[\alpha]}$ as

$$\mathbb{H}_{N}^{[\alpha]} := \text{span} \{ \Phi_i^{[\alpha]}(\mu), i = 0, \cdots, N \}$$

with the inner product

$$\langle \Phi, \Psi \rangle_{\mathbb{H}_{N}^{[\alpha]}} = \int_{-1}^1 \Phi(\mu) \Psi(\mu) \omega^{[\alpha]}(\mu) d\mu.$$

Let $\mathbb{H}$ be the space of all the admissible specific intensities for the RTE. Consider the map from $\mathbb{H}$ to $\mathbb{H}_{N}^{[\alpha]}$:

$$P : I \rightarrow \hat{I} = \sum_{i=0}^N f_i \Phi_i^{[\alpha]}(\mu), \quad f_i = \frac{\langle I, \Phi_i^{[\alpha]} \rangle_{\mathbb{H}_{N}^{[\alpha]}}}{\langle \Phi_i^{[\alpha]}, \Phi_i^{[\alpha]} \rangle_{\mathbb{H}_{N}^{[\alpha]}}} = \frac{\int_{-1}^1 I \phi_i^{[\alpha]} d\mu}{\kappa_{ii}},$$

where $\kappa_{ii}$ is defined in (7). Clearly, this map is an orthogonal projection.
Similar to the reduction framework in the literature [9], the MP$_{N}$ moment equation can be obtained as

$$\frac{1}{c} P \frac{\partial PI}{\partial t} + P \mu \frac{\partial PI}{\partial x} = PS(PI).$$

Note that the unknown variables are coefficients

$$w = (f_0, f_2, \ldots, f_N)^T$$

of PI in the basis space $\mathbb{H}^{[\alpha]}_N$.

The MP$_2$ moment model was showed [11] to be globally hyperbolic and perform well in numerical experiments. But it allows a non-physical characteristic velocity exceeding the speed of light. When $N \geq 3$, the global hyperbolicity fails. For these reasons, the HMP$_{N}$ moment closure model as a novel hyperbolic regularization was proposed [10].

2.2 | HMP$_{N}$ model

This class of models uses

$$\tilde{\omega}^{[\alpha]}(\mu) = \frac{1}{(1 + a \mu)^{\frac{1}{2}}}, \quad \alpha \in (-1, 1)$$

(9)

as the weight function which is different from that of MP$_N$ models. As before, we introduce the orthogonal polynomials with respect to this new weight function:

$$\tilde{\phi}_0^{[\alpha]}(\mu) = 1, \quad \tilde{\phi}_j^{[\alpha]}(\mu) = \mu^j - \sum_{k=0}^{j-1} \tilde{k}_{j,k} \tilde{\phi}_k^{[\alpha]}(\mu), \quad j \geq 1. \quad (10)$$

The coefficients are

$$\tilde{k}_{j,k} = \int_{-1}^{1} \mu^j \tilde{\phi}_k^{[\alpha]}(\mu) \tilde{\omega}^{[\alpha]}(\mu) d\mu$$

(11)

and the analog of (7) also holds the following:

$$\tilde{k}_{j,k} = 0, \text{ if } j < k, \quad \tilde{k}_{k,k} = \int_{-1}^{1} \left( \tilde{\phi}_k^{[\alpha]}(\mu) \right)^2 \tilde{\omega}^{[\alpha]}(\mu) d\mu > 0. \quad (12)$$

Similarly, we have a new Hilbert space

$$\tilde{\mathbb{H}}^{[\alpha]}_N := \text{span} \left\{ \Phi_i^{[\alpha]}(\mu) = \tilde{\phi}_i^{[\alpha]}(\mu) \tilde{\omega}^{[\alpha]}(\mu), i = 0, \ldots, N \right\}$$

with the inner product

$$\langle \Phi, \Psi \rangle_{\tilde{\mathbb{H}}^{[\alpha]}_N} = \int_{-1}^{1} \Phi(\mu) \Psi(\mu) \tilde{\omega}^{[\alpha]}(\mu) d\mu$$

(13)

and the orthogonal projection from $\mathbb{H}$ to $\tilde{\mathbb{H}}^{[\alpha]}_N$:

$$\tilde{P} : I \rightarrow \tilde{I} = \sum_{i=0}^{N} g_i \tilde{\phi}_i^{[\alpha]}(\mu), \quad g_i = \frac{\langle I, \tilde{\phi}_i^{[\alpha]}(\mu) \rangle_{\tilde{\mathbb{H}}^{[\alpha]}_N}}{\langle \tilde{\phi}_i^{[\alpha]}, \tilde{\phi}_i^{[\alpha]} \rangle_{\tilde{\mathbb{H}}^{[\alpha]}_N}} = \int_{-1}^{1} I \tilde{\phi}_i^{[\alpha]}(\mu) d\mu / \tilde{k}_{i,i}.$$
Having these preparations, the HMP$_N$ models were constructed in Fan et al. [10] as

$$\frac{1}{c} \tilde{P} \frac{\partial P_I}{\partial t} + \tilde{P} \mu \frac{\partial P_I}{\partial x} = \tilde{P} S(P_I).$$

They can be rewritten as the equations for $w = (f_0, \alpha, f_1, \cdots, f_N)^T$:

$$\frac{1}{c} \frac{\partial w}{\partial t} + \tilde{D}^{-1} \tilde{M} \frac{\partial w}{\partial x} = \tilde{D}^{-1} \tilde{S}. \quad (14)$$

Here the matrix $\tilde{D}$ is denoted as

$$\tilde{P} \frac{\partial P_I}{\partial t} = \left( \tilde{\Phi} \Phi^T \right)_0 \tilde{D} \frac{\partial w}{\partial t},$$

and

$$\tilde{M} = \tilde{\Lambda}^{-1} \left( \mu \tilde{\Phi}^T \Phi \tilde{\Phi}^T \right)_0, \quad \tilde{\Lambda} = \text{diag}(\bar{\kappa}_{0,0}, \bar{\kappa}_{1,1}, \cdots, \bar{\kappa}_{N,N}),$$

$$\tilde{S} = \left( \langle \tilde{\Phi}^T \Phi S(P_I) \rangle \tilde{\Lambda}^{-1} \right)_i = 0, \cdots, N.$$  \quad (15)

The details can be found in the literature [10].

### 3 | STABILITY ANALYSIS

#### 3.1 | Structural stability condition

Yong [36] proposed a structural stability condition for systems of first-order partial differential equations with source terms:

$$U_t + \sum_{j=1}^{D} A_j(U) U_{x_j} = Q(U),$$

where $A_j(U)$ and $Q(U)$ are $n \times n$-matrix and $n$-vector smooth functions of $U \in G \subset \mathbb{R}^n$ with state space $G$ open and convex. The subscripts $t$ and $x_j$ refer to the partial derivatives with respect to $t$ and $x_j$.

Set $Q_U = \frac{\partial Q}{\partial U}$ and define the equilibrium manifold

$$E := \{ U \in G : Q(U) = 0 \}.$$  

The structural stability condition consists of the following three items:

(i) There are an invertible $n \times n$ matrix $P(U)$ and an invertible $r \times r$ matrix $S(U)$, defined on the equilibrium manifold $E$, such that

$$P(U) Q_U(U) = \begin{bmatrix} 0 & 0 \\ 0 & S(U) \end{bmatrix} P(U), \quad \text{for } U \in E.$$

(ii) There is a symmetric positive definite matrix $A_0(U)$ such that

$$A_0(U) A_j(U) = A_j(U) A_0(U), \quad \text{for } U \in G.$$

(iii) The left-hand side and the source term are coupled in the following way:

$$A_0(U) Q_U(U) + Q_U^T(U) A_0(U) \leq -P^T(U) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} P(U), \quad \text{for } U \in E.$$

Here $I$ is the unit matrix of order $r$. 
As shown in Yong [37], this set of conditions has been tacitly respected by many well-developed physical theories. Condition (i) is classical for initial value problems of the system of ordinary differential equations (ODEs, spatially homogeneous systems), while (ii) means the symmetrizable hyperbolicity of the partial differential equation (PDE) system. Condition (iii) characterizes a kind of coupling between the ODE and PDE parts. Recently, this structural stability condition was shown [22] to be proper for certain moment closure systems. Furthermore, this condition also implies the existence and stability of the zero relaxation limit of the corresponding initial value problems [36].

3.2 Stability of the Euler-HMP\textsubscript{N} system

In this subsection, we verify the structural stability condition for the following one-dimensional Euler-HMP\textsubscript{N} system

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2 + p) &= \rho \left( c \sigma_a(\theta) + \frac{1}{c} \sigma_x(\theta) \right) \kappa_{1,0}(a) f_0, \\
\partial_t (\rho E) + \partial_x (\rho Ev + pv) &= c^2 \rho \sigma_a(\theta) \left( \kappa_{0,0}(a) f_0 - b(\theta) \right), \\
\partial_t w + c \tilde{D}^{-1} \tilde{M} \partial_x w &= c \tilde{D}^{-1} \tilde{S},
\end{align*}
\]

which is Equation (1) with its last equation replaced by the HMP\textsubscript{N} approximation (14). Here the following relations have been used:

\[
\begin{align*}
S_F &= \int_{-1}^{1} \mu s d\mu = -\rho(c \sigma_a(\theta) + \frac{1}{c} \sigma_x(\theta)) E_1, \\
S_E &= \int_{-1}^{1} s d\mu = -c \rho \sigma_a(\theta)(b(\theta) - E_0)
\end{align*}
\]

with \( E_0 = \kappa_{0,0}(a) f_0 \) and \( E_1 = \kappa_{1,0}(a) f_0 \) due to formula (8).

Let \( u = (\rho, \rho v, \rho E)^T \in \mathbb{R}^3 \) be the hydrodynamical variables and \( w = (f_0, a, f_2, \ldots, f_N)^T \in \mathbb{R}^{N+1} \) be radiation variables. Denoting \( F(u) = (\rho v, \rho v^2 + p, \rho E + pv)^T \) and \( \epsilon = 1/c \), we can rewrite (16) as

\[
\begin{align*}
\partial_t U + \frac{1}{\epsilon} A(U; \epsilon) \partial_x U &= \frac{1}{\epsilon^2} Q(U; \epsilon),
\end{align*}
\]

with

\[
U = \begin{pmatrix} u \\ w \end{pmatrix}, \quad A(U; \epsilon) = \begin{pmatrix} \epsilon F_u(u) & 0 \\ 0 & \tilde{D}^{-1} \tilde{M} \tilde{D} \end{pmatrix}, \quad Q(U; \epsilon) = \begin{pmatrix} q^{(1)}(U; \epsilon) \\ q^{(2)}(U; \epsilon) \end{pmatrix}.
\]

\[
q^{(1)}(U; \epsilon) = \begin{pmatrix} 0, \rho \epsilon \sigma_a(\theta) + \epsilon^3 \sigma_x(\theta) \kappa_{1,0}(a) f_0, \rho \sigma_a(\theta)(\kappa_{0,0}(a) f_0 - b(\theta)) \end{pmatrix}^T,
\]

\[
q^{(2)}(U; \epsilon) = \epsilon \tilde{D}^{-1} \tilde{S}(U; \epsilon).
\]

Note that \( \tilde{D} = \tilde{D}(U) \) and \( \tilde{M} = \tilde{M}(U) \) are independent of \( \epsilon \). The state space is

\[
G = \{ U = (u, w) \mid \rho > 0, \theta > 0, a \in (-1, 1) \}.
\]

Next, we write down the explicit expression of \( \tilde{S} = \tilde{S}(U; \epsilon) \). Recall that \( PI = \sum_{i=0}^{N} f_i \Phi_i^{[a]}(\mu) \). Then by its definition (2), we have

\[
S(PI) = \rho \left( \frac{1}{2} c \sigma_a b(\theta) + \frac{1}{2c} \sigma_x \int_{-1}^{1} I d\mu - \left( c \sigma_a + \frac{1}{c} \sigma_x \right) \sum_{j=0}^{N} f_j \Phi_j^{[a]} \right).
\]

From Fan et al. [10], we know that \( \Phi_j^{[a]} = \alpha \Phi_j^{[a]} + \beta_j \Phi_j^{[a]} \) with \( \beta_j = \frac{\kappa_{i,j}}{\kappa_{i,j}} \). Thus, we compute the \( i \)th component of \( \tilde{S} \) in Equation (15):
Here we have used $E_0 = \int_{-1}^{1} 1 d\mu = \kappa_{0,0} f_0$ and $\int_{-1}^{1} \tilde{\Phi}_i^{[a]} \phi_j^{[a]} d\mu = \delta_{ij} \bar{\kappa}_{li}$. Set $\tilde{S}(U; \epsilon) = \epsilon \tilde{S}(U; \epsilon)$. We have $q^{(2)}(U; \epsilon) = D^{-1} \tilde{S}(U; \epsilon)$ and

$$\tilde{S}_i(U; \epsilon) = \frac{\rho (\sigma_a b + \epsilon^2 \gamma_{0,0} f_0)}{2 \bar{\kappa}_{li}} - \frac{\rho (\sigma_a + \epsilon^2 \gamma_a) (\alpha f_{i-1} + \beta_i f_i)}{2 \bar{\kappa}_{li}}$$

with $R_i \triangleq \int_{-1}^{1} \tilde{\Phi}_i^{[a]} d\mu$. Note that $\kappa_{0,0}$, $\bar{\kappa}_{li}$, $R_i$ and $\beta_i$ depend on $a$ and $\tilde{S}_i(U; \epsilon)$ is a polynomial of $\epsilon$. Since $f_{-1} = f_1 = 0$, $\tilde{S}(U; 0)$ can be rewritten as

$$\tilde{S}_0(U; 0) = \frac{\rho \sigma_a b (b - \kappa_{0,0} f_0)}{\bar{\kappa}_{0,0}}, \quad \tilde{S}_i(U; 0) = \frac{\rho \sigma_a b R_i}{2 \bar{\kappa}_{i,1}} - \frac{\rho \sigma_a \alpha f_0}{2 \bar{\kappa}_{0,0}}, \quad \tilde{S}_2(U; 0) = \frac{\rho \sigma_a b R_2}{2 \bar{\kappa}_{2,2}} - \frac{\rho \sigma_a \beta_2 f_2}{2 \bar{\kappa}_{2,2}}$$

$$\tilde{S}_i(U; 0) = \frac{\sigma_a b}{2 \bar{\kappa}_{i,1}} R_i - \frac{\rho \sigma_a (\alpha f_{i-1} + \beta_i f_i)}{2 \bar{\kappa}_{i,1}}, \quad \text{for } i = 3, \ldots, N.$$ (19)

Here $\rho$, $\sigma_a$, $b$, $\kappa_{li}$, $\bar{\kappa}_{li} > 0$.

For $\kappa_{li}$ and $\bar{\kappa}_{li}$, we have the following explicit expressions.

$$\kappa_{0,0} = \int_{-1}^{1} w^{[a]}(\mu) d\mu = \frac{2 (3 + a^2)}{3 (1 - a^2)^2}, \quad \bar{\kappa}_{0,0} = \int_{-1}^{1} \tilde{w}^{[a]}(\mu) d\mu = \frac{2 (a^2 + 1)}{(a^2 - 1)^4},$$

$$\kappa_{1,0} = \int_{-1}^{1} \mu w^{[a]}(\mu) d\mu = \frac{8 a}{3 (a^2 - 1)^2}, \quad \bar{\kappa}_{1,0} = \int_{-1}^{1} \mu \tilde{w}^{[a]}(\mu) d\mu = \frac{-2 a (a^2 + 5)}{3 (a^2 - 1)^4},$$

(20)

$$\bar{\kappa}_{1,1}(0) = \int_{-1}^{1} (\tilde{\Phi}_1^{[a]})^2(\mu) \tilde{w}^{[a]}(\mu) d\mu = \frac{2}{3}.$$

Here $\phi_1^{[a]} = \phi_1^{[a]}(\mu) = \mu - \bar{\kappa}_{1,1}(a) / \bar{\kappa}_{0,0}$ according to Equation (10). These can be easily checked by using the expressions of $w^{[a]}(\mu)$ and $\tilde{w}^{[a]}(\mu)$ given in (5) and (9).

For $R_i = R_i(\alpha)$ in (18), we have the following.

**Lemma 1.** $R_0(\alpha) = 2$, $R_1(\alpha) = \frac{2 \alpha (a^2 + 5)}{3 (a^2 + 1)}$, $R_0(0) = 0$ and $R'_i(0) = 0$ for $i \geq 2$. 

Proof. According to Equation (10), we know that \( \tilde{\phi}_0^{[\alpha]} = 1 \) and \( \tilde{\phi}_1^{[\alpha]}(\mu) = \mu - \tilde{k}_{1,0}(\alpha)/\tilde{k}_{0,0}(\alpha) \). Thereby,

\[
R_0(\alpha) = \int_{-1}^{1} \tilde{\phi}_0^{[\alpha]}(\mu) \, d\mu = 2, \quad R_1(\alpha) = \int_{-1}^{1} \tilde{\phi}_1^{[\alpha]}(\mu) \, d\mu = \int_{-1}^{1} \mu \, d\mu - 2 \frac{\tilde{k}_{1,0}(\alpha)}{\tilde{k}_{0,0}(\alpha)} = \frac{2\alpha(\alpha^2 + 5)}{3(\alpha^2 + 1)}.
\]

Since \( \tilde{w}^{[0]}(\mu) = 1 \) in (9), we have

\[
R_i(0) = \int_{-1}^{1} \tilde{\phi}_i^{[0]}(\mu) \, d\mu = \int_{-1}^{1} \mu^i \tilde{w}^{[0]}(\mu) \, d\mu = \tilde{k}_{0,i}(0) = 0, \text{ for } i \geq 2.
\]

Here we have used the orthogonality of \( \tilde{k}_{i,j} \) in (12), that is, \( \tilde{k}_{i,j}(\alpha) = 0 \) for \( i < j \). Based on the expression of \( \tilde{k}_{i,j} \) in (11), we have

\[
\tilde{k}_{0,j}'(\alpha) = \int_{-1}^{1} \frac{\partial \tilde{\phi}_i^{[\alpha]}(\mu)}{\partial \alpha} \tilde{w}^{[\alpha]}(\mu) \, d\mu + \int_{-1}^{1} \tilde{\phi}_i^{[\alpha]}(\mu) \frac{\partial \tilde{w}^{[\alpha]}(\mu)}{\partial \alpha} \, d\mu.
\]

Note that \( \frac{\partial \tilde{\phi}_i^{[\alpha]}(\mu)}{\partial \alpha} = -5\mu/(1+\mu)^3 \); thus, \( \frac{\partial \tilde{w}^{[\alpha]}(\mu)}{\partial \alpha} = -5\mu \). Taking \( \alpha \to 0 \), we can obtain

\[
R_i'(0) = \int_{-1}^{1} \frac{\partial \tilde{\phi}_i^{[0]}(\mu)}{\partial \alpha} \tilde{w}^{[0]}(\mu) \, d\mu = \tilde{k}_{0,i}'(0) = \tilde{k}_{0,i}(0) + 5 \int_{-1}^{1} \mu \tilde{\phi}_i^{[0]}(\mu) \, d\mu
\]

\[
= \tilde{k}_{0,i}(0) + 5 \int_{-1}^{1} \mu \tilde{\phi}_i^{[0]}(\mu) \, d\mu
\]

\[
= \tilde{k}_{0,i}(0) + 5 \tilde{k}_{1,i}(0).
\]

Similarly, it follows from the orthogonality of \( \tilde{k}_{i,j} \) that

\[
R_i'(0) = 0, \text{ for } i \geq 2.
\]

The equilibrium manifold \( G_{eq} \) is defined as following

\[
G_{eq} = \{ U \in G : Q(U; 0) = 0 \}.
\]

Due to Equation (17) and the expression of \( \tilde{S} \) in (19), we know that \( U \in G_{eq} \) if and only if \( \tilde{S}(U; 0) = 0 \). We denote the equilibrium state as \( U_{eq} \). Using formulas (20) and Lemma 1, one can obtain the equilibrium state \( U_{eq} \) as

\[
f_0 = \frac{b(\theta)}{\tilde{k}_{0,0}(0)} = \frac{1}{2} b(\theta), \quad f_i = 0, \quad \text{for } i = 2, \ldots, N.
\]

It can be seen from system (17) that the source term of the fluid variable is also zero on the above-mentioned equilibrium manifold. It is worth noting that for any \( U_{eq} \in G_{eq} \) and any \( \varepsilon \), there are

\[
Q(U_{eq}; \varepsilon) = 0.
\]
Next, we verify that the Euler-HMPN system (17) satisfies the structural stability condition. Throughout this paper, we make the standard thermodynamical assumptions [38]:

\[ p_\theta(\rho, \theta), \quad p_\rho(\rho, \theta), \quad e_\rho(\rho, \theta) > 0, \quad \text{for} \quad \rho > 0, \theta > 0. \]

Assume the existence of a specific entropy function \( s = s(\rho, e) \) satisfying the classical Gibbs relationship

\[ \theta ds = de + p dv, \quad v := 1/\rho. \]

We take \( \eta(u) = -\rho s(\rho, e) \) as the classical entropy function of Euler equations. This means that \( \eta_{uu} \) is symmetrizer of Euler equations [39]. According to Equation (15), \( \bar{\Lambda}M = \langle \mu \bar{D}^{[\alpha]}(\bar{D}^{[\alpha]}(\bar{D}^{[\alpha]})^T \bar{D}^{[\alpha]} \rangle \) is symmetric. Therefore, it can be seen that the Euler-HMPN system (17) has the following symmetrizer

\[ A_0(U) = \begin{pmatrix} \eta_{uu} & 0 \\ D^T \bar{\Lambda} D & \end{pmatrix}. \quad (22) \]

That is, \( A_0(U)A(U; \epsilon) = A(U; \epsilon)^T A_0(U) \). Note that the symmetrizer \( A_0 = A_0(U) \) is independent with \( \epsilon \).

In order to verify the first and third requirements in the structural stability condition, we need to compute \( Q_U(U_{eq}; 0) \). Therefore, we now write down the source term of system (17) as follows:

\[ Q(U; 0) = \begin{pmatrix} q^{(1)}(U; 0) \\ q^{(2)}(U; 0) \end{pmatrix}, \]

\[ q^{(1)}(U; 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \rho \sigma_a(\theta)(\kappa_{0,0}(\alpha)f_0 - b(\theta))^T, \]

\[ q^{(2)}(U; 0) = D^{-1}(U)\hat{S}(U; 0). \]

Set \( S_{\rho E} = \rho \sigma_a(\theta)(\kappa_{0,0}(\alpha)f_0 - b(\theta)). \) Resorting to formulas (20) and (21), we note that, on the equilibrium manifold \( G_{eq} \),

\[ \frac{\partial S_{\rho E}}{\partial \rho} (U_{eq}; 0) = -\rho \sigma_a b' \theta \rho, \quad \frac{\partial S_{\rho E}}{\partial (\rho v)} (U_{eq}; 0) = -\rho \sigma_a b' \theta v, \]

\[ \frac{\partial S_{\rho E}}{\partial (\rho E)} (U_{eq}; 0) = -\rho \sigma_a b' \theta \rho E, \quad \frac{\partial S_{\rho E}}{\partial f_0} (U_{eq}; 0) = 2\rho \sigma_a, \]

\[ \frac{\partial S_{\rho E}}{\partial w} (U_{eq}; 0) = 0, \quad \text{for} \quad w \neq f_0. \]

For \( q^{(2)}(U; 0) \), we know that

\[ \frac{\partial}{\partial U} (D^{-1} \hat{S}) (U_{eq}; 0) = D^{-1} \frac{\partial \hat{S}}{\partial U} (U_{eq}; 0) + \frac{\partial D^{-1}}{\partial U} \hat{S} (U_{eq}; 0) = D^{-1} \frac{\partial \hat{S}}{\partial U} (U_{eq}; 0). \]

Thus, we need to compute \( \frac{\partial S}{\partial U} (U_{eq}; 0) \). Note that \( \hat{S}_0 = \frac{\rho \sigma_a(\theta)}{\kappa_{0,0}(\alpha)}(b(\theta) - \kappa_{0,0}(\alpha)f_0) \) according to Equation (19). Using formula (20), we can obtain

\[ \frac{\partial \hat{S}_0}{\partial \rho} (U_{eq}; 0) = \frac{1}{2} \rho \sigma_a b' \theta \rho, \quad \frac{\partial \hat{S}_0}{\partial (\rho v)} (U_{eq}; 0) = \frac{1}{2} \rho \sigma_a b' \theta v, \]

\[ \frac{\partial \hat{S}_0}{\partial (\rho E)} (U_{eq}; 0) = \frac{1}{2} \rho \sigma_a b' \theta \rho E, \quad \frac{\partial \hat{S}_0}{\partial f_0} (U_{eq}; 0) = -\rho \sigma_a, \]

\[ \frac{\partial \hat{S}_0}{\partial w} (U_{eq}; 0) = 0, \quad \text{for} \quad w \neq f_0. \]
Similarly, for $\tilde{S}_1 = \frac{\rho_a(\theta)(b)}{2\bar{\kappa}_1(\alpha)} R_1(\alpha) - \rho a \sigma_a(\theta)f_0$, we have

$$\frac{\partial \tilde{S}_1}{\partial u} (U_{eq};0) = 0, \quad \frac{\partial \tilde{S}_1}{\partial w} (U_{eq};0) = 0, \quad \text{for } w \neq a,$$

$$\frac{\partial \tilde{S}_1}{\partial a} (U_{eq};0) = \frac{\rho_a b R_1^r(0)\bar{\kappa}_{1,1}(0) - R_1(0)\bar{\kappa}_{1,1}'(0)}{2\bar{\kappa}_{1,1}^2(0)} - \frac{1}{2} \rho_a \sigma_a b = 2\rho_a \sigma_a b.$$

When $i \geq 2$, we know that $\tilde{S}_i(U;0) = \frac{\rho_a(\theta)(b)}{2\bar{\kappa}_{i,1}(\alpha)} R_i(\alpha) - \rho a \sigma_a(\theta)(\alpha f_{i-1} + \beta(\alpha)f_i)$. Analogously, it is easy to show that

$$\frac{\partial \tilde{S}_i}{\partial u} (U_{eq};0) = 0, \quad \frac{\partial \tilde{S}_i}{\partial f_i} (U_{eq};0) = -\rho_a \sigma_a \beta_i(0), \quad \frac{\partial \tilde{S}_i}{\partial w} (U_{eq};0) = 0, \quad \text{for } w \neq f_i.$$

Resorting to the explicit expression of $\tilde{D}$ in literature [10] and Equation (15), we can obtain

$$D^{-1}(U_{eq}) = \text{diag} \left( \beta_0^{-1}(0), (-2b(\theta))^{-1}, \beta_2^{-1}(0), \cdots, \beta_N^{-1}(0) \right),$$

$$D^T \tilde{\Lambda} D(U_{eq}) = \text{diag} \left( \beta_0^2(0)\bar{\kappa}_{0,0}(0), (-2b(\theta))^2\bar{\kappa}_{1,1}(0), \beta_2^2(0)\bar{\kappa}_{2,2}(0), \cdots, \beta_N^2(0)\bar{\kappa}_{N,N}(0) \right).$$

Here $D^{-1}(U_{eq})$ and $D^T \tilde{\Lambda} D(U_{eq})$ are matrices belonging in $\mathbb{R}^{(N+1)\times(N+1)}$. In summary, the Jacobian matrix $Q_U (U_{eq};0)$ is

$$Q_U (U_{eq};0) = \begin{pmatrix} Q_1 (U_{eq};0) & 0 \\ 0 & -\rho a \sigma_a I_{N\times N} \end{pmatrix},$$

where

$$Q_1 (U_{eq};0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} \rho_a b' \theta_{\rho} & \frac{1}{2} \rho_a b' \theta_{\varphi} & \frac{1}{2} \rho_a b' \theta_{\psi} & -\rho a \sigma_a \end{pmatrix}.$$

(25)

Obviously, the rank of $Q_1 (U_{eq};0)$ is 1, so the rank of $Q_U (U_{eq};0)$ is $N + 1$.

Using Equation (23), we can rewrite $A_0(U_{eq})$ in the same block form as

$$A_0(U_{eq}) = \begin{pmatrix} \eta_{u u} & 0 \\ 0 & D^T \tilde{\Lambda} D(U_{eq}) \end{pmatrix},$$

$$A_0(U_{eq})_{N\times N} = \text{diag} \left( (-2b(\theta))^2\bar{\kappa}_{1,1}(0), \beta_2^2(0)\bar{\kappa}_{2,2}(0), \cdots, \beta_N^2(0)\bar{\kappa}_{N,N}(0) \right).$$

Note that $\eta_{\rho E} = -\frac{1}{\theta}$ [39]; we can obtain

$$A_0(U_{eq}) Q_U (U_{eq};0) = \begin{pmatrix} H & 0 \\ 0 & -\rho a \sigma_a A_0(U_{eq})_{N\times N} \end{pmatrix},$$

where

$$H = \begin{pmatrix} \eta_{u u} & 0 \\ 0 & 2 \end{pmatrix} Q_1 (U_{eq};0) = \begin{pmatrix} -\rho a b' \theta_{\rho} & -\rho a b' \theta_{\varphi} & -\rho a b' \theta_{\psi} & 2\rho a \sigma_a \theta_{\rho} \\ 0 & -\rho a b' \theta_{\rho} & -\rho a b' \theta_{\varphi} & 2\rho a \sigma_a \theta_{\rho} \\ 0 & 0 & -\rho a b' \theta_{\rho} & 2\rho a \sigma_a \theta_{\rho} \\ \rho_a b' \theta_{\rho} & \rho_a b' \theta_{\varphi} & \rho_a b' \theta_{\psi} & -2\rho a \end{pmatrix}.$$

(27)
Take $P \in \mathbb{R}^{(N+4)\times(N+4)}$ as the following:

$$P = a \begin{pmatrix} P_1 & 0 \\ 0 & I_{N\times N} \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ -b'\theta_p & -b'\theta_p & -b'\theta_p & 2 \end{pmatrix}.$$  
(28)

Here $a$ is an undetermined nonzero constant. Obviously, $P$ is an invertible matrix since $\det(P) = 2a (1 + b'\theta_{pE}) \neq 0$. In fact, we have

$$PQ_U = a \begin{pmatrix} P_1 & 0 \\ 0 & I_{N\times N} \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & -\rho \sigma a I_{N\times N} \end{pmatrix} = a \begin{pmatrix} P_1 Q_1 & 0 \\ 0 & -\rho \sigma a I_{N\times N} \end{pmatrix}.$$  

Simple calculation shows that

$$P_1 Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\rho \sigma a (1 + b'\theta_{pE}) \end{pmatrix} P_1.$$  

Thus, the first requirement of structural stability condition is met. Moreover, the third requirement of stability condition need $P$ holds the following inequality.

$$A_0(U_{eq})Q_U (U_{eq}; 0) + Q_U^T (U_{eq}; 0) A_0(U_{eq}) + P^T \left( \begin{array}{cc} \text{diag}(0, 0, 0, 1) & 0 \\ 0 & I_{N\times N} \end{array} \right) P \leq 0.$$  

Due to expression (26), the above inequality is equivalent to

$$H + H^T + a^2 P_1^T \text{diag}(0, 0, 0, 1) P_1 \leq -2\rho \sigma a \hat{A}_0(U_{eq})_{N\times N} + a^2 I_{N\times N} \leq 0.$$  
(29)

Set

$$K = (-\rho \sigma a b'\theta_p, -\rho \sigma a b'\theta_p, -\rho \sigma a b'\theta_{pE}, 2\rho \sigma a),$$

$$L = \begin{pmatrix} \theta_p \\ \theta_p \\ \theta_{pE} \\ -1 \end{pmatrix}.$$  

Then, we have

$$H = L^T K, \quad P_1^T \text{diag}(0, 0, 0, 1) P_1 = \frac{1}{\rho^2\sigma_a^2} K^T K.$$  

Hence, the first inequality can be rewritten as

$$H + H^T + a^2 P_1^T \text{diag}(0, 0, 0, 1) P_1 = L^T K + K^T L + \frac{a^2}{\rho^2\sigma_a^2} K^T K$$

$$= \left( L^T + \frac{a^2}{2\rho^2\sigma_a^2} K^T \right) K + K^T \left( L + \frac{a^2}{2\rho^2\sigma_a^2} K \right).$$  

The above matrix is seminegative definite is equivalent to $(L + \frac{a^2}{2\rho^2\sigma_a^2} K)K^T \leq 0$, which means that

$$a^2 \leq \rho \sigma_a \frac{4 + 2b' \left( \theta_p^2 + \theta_{pE}^2 \right)}{4 + b'^2 \left( \theta_p^2 + \theta_{pE}^2 \right)} \left( \frac{\theta_p^2 + \theta_{pE}^2}{\theta_p^2 + \theta_{pE}^2} \right).$$  
(30)

According to inequalities (29) and (30), if the nonzero constant $a$ satisfies the following constraints

$$a^2 \leq \min_{2 \leq k \leq N} \left\{ \begin{array}{ll} 2 \rho \sigma a b \beta_k(0) \hat{c}_{k,k}(0), & 16 \rho \sigma a b^2, \quad \frac{4 + 2b' \left( \theta_p^2 + \theta_{pE}^2 \right)}{4 + b'^2 \left( \theta_p^2 + \theta_{pE}^2 \right)} \left( \frac{\theta_p^2 + \theta_{pE}^2}{\theta_p^2 + \theta_{pE}^2} \right) \end{array} \right\}.$$
the $P$ matrix defined in (28) satisfies the structural stability condition. And

$$\rho > 0, \quad \sigma_0 > 0, \quad b > 0, \quad b' > 0, \quad \beta_k(0) \tilde{\kappa}_{k,k}(0) > 0.$$ 

The value space of $a$ is obviously not empty. Consequently, we conclude the following theorem.

**Theorem 1.** The Euler-HMP$_N$ system (17) admits Yong's structural stability condition, which's symmetrizer $A_0$ and $P$ are defined in (22) and (28).

### 4 | NONRELATIVISTIC LIMIT

In this section, we analyze the nonrelativistic limit of the radiation hydrodynamics system (16). In other words, we focus on singular limits $\varepsilon \to 0$ of the following system

$$\partial_t U + \frac{1}{\varepsilon} A(U; \varepsilon) \partial_x U = \frac{1}{\varepsilon^2} Q(U; \varepsilon). \quad (31)$$

Here $U, A(U; \varepsilon)$ and $Q(U; \varepsilon)$ are demonstrated in (17).

As we mentioned before, Lattanzio and Yong [17] and Peng and Wasiolek [18] studied the singular limits of initial value problems for first-order quasilinear hyperbolic systems with stiff source terms. Under appropriate stability conditions and the existence of approximate solutions, they justified rigorously the validity of the asymptotic expansion on a time interval independent of the parameter. However, system (31) that the coefficient matrix and the source terms both depending on $\varepsilon$ are not considered, which introduce some additional terms.

For convenience, we rewrite the equations of hydrodynamical variables (the first three equations in (16)) as the following conservative form

$$\begin{align*}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v + \varepsilon E_1) + \partial_x (\rho v^2 + p + E_2) &= 0, \\
\partial_t (\rho E + E_0) + \partial_x (\rho Ev + pv) + \frac{1}{\varepsilon} \partial_x E_1 &= 0.
\end{align*} \quad (32)$$

Here we use the first two moment equations of RTE (4):

$$\begin{align*}
\varepsilon \partial_t E_0 + \partial_x E_1 &= S_E, \\
\varepsilon \partial_t E_1 + \partial_x E_2 &= S_F.
\end{align*} \quad (33)$$

Owing to the relation (8), we know that

$$E_0 = \kappa_{0,0} f_0, \quad E_1 = \kappa_{1,0} f_0, \quad E_2 = \kappa_{2,2} f_2 + \kappa_{2,0} f_0.$$ 

We introduce $\bar{U} = (\bar{u}, \bar{w})^T$ with

$$\begin{align*}
\bar{u} &= (\rho, \rho v + \varepsilon \kappa_{1,0} f_0, \rho E + \kappa_{0,0}(\alpha) f_0)^T, \\
\bar{w} &= (\kappa_{0,0}(\alpha) f_0 - b(\theta), \alpha, f_2, \cdots, f_N)^T
\end{align*} \quad (34)$$

and set $\bar{U}_{eq} = \bar{U}(U_{eq})$. Then, system (17) can be rewritten as

$$\partial_t \bar{U} + \frac{1}{\varepsilon} \bar{A}(U; \varepsilon) \partial_x \bar{U} = \frac{1}{\varepsilon^2} \bar{Q}(U; \varepsilon) \quad (35)$$
with

\[ \tilde{A}(U; \varepsilon) = D_U \bar{U} \begin{pmatrix} \varepsilon F_0(U) & 0 \\ D^{-1} \bar{M}D & (D_U \bar{U})^{-1} \end{pmatrix}, \]

\[ \tilde{Q}(U; \varepsilon) = D_U \bar{U} Q(U(U)) = \begin{pmatrix} 0 \\ q(U; \varepsilon) \end{pmatrix}. \]

Here, the transformation matrix of \( U \rightarrow \bar{U} \) is

\[ D_U \bar{U} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \varepsilon \kappa_{1,0} & \varepsilon \kappa'_{1,0} f_0 \\ 0 & 0 & 1 & \kappa_{0,0} & \kappa'_{0,0} f_0 \\ -b' \theta_\rho & -b' \theta_{\rho E} & -b' \theta_{\rho E} & \kappa_{0,0} & \kappa'_{0,0} f_0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{0 \times (N-1)} \begin{pmatrix} \varepsilon \kappa_{1,0} & \varepsilon \kappa'_{1,0} f_0 \\ 0 & 1 \end{pmatrix}_{(N-1) \times (N-1)}. \]

Using formula (20), a routine computation that gives rise to the determination of \( D_U \bar{U} \) is

\[ \det(D_U \bar{U}) = \kappa_{0,0} + b' \theta_{\rho E} \kappa_{0,0} + \varepsilon b' \theta_{\rho E} \kappa_{1,0} = \left( 3 + \alpha^2 \right) \left( 1 + b' \theta_{\rho E} \right) - \varepsilon 4 \alpha b' \theta_{\rho E} \frac{2}{3(1-\alpha^2)^3}. \]

Note that \( \alpha \in (-1, 1) \) and \( \theta_{\rho E} > 0 \) [39]. Then when \( \varepsilon = 0 \), we have

\[ \det(D_U \bar{U}) = \frac{2}{3(1-\alpha^2)^3} \left( 1 + b' \theta_{\rho E} \right) > 0. \]

Thus, there exists \( \varepsilon_0 > 0 \) such that \( \det(D_U \bar{U}) \neq 0 \) for \( \varepsilon \in [0, \varepsilon_0] \). Therefore, we assume that \( \varepsilon \in [0, \varepsilon_0] \) for system (35).

System (35) also satisfies Yong's structural stability condition. It is apparent from (21) that \( \bar{w} = 0 \) on equilibrium manifold. On the equilibrium manifold, we have

\[ \partial_U \tilde{Q} = \partial_U (D_U \bar{U} Q) = \partial_U (D_U \bar{U}) D_U \bar{U} Q + D_U \bar{U} Q U (D_U \bar{U})^{-1} = D_U \bar{U} Q U (D_U \bar{U})^{-1}. \]

Here we use \( Q(U_{eq}) = 0 \). Set \( \tilde{P} = P(D_U \bar{U})^{-1} (U_{eq}; 0) \), in which \( P \) expressed in (28). On the equilibrium manifold, we see that

\[ \kappa_{0,0}(0) = 2, \quad \kappa'_{0,0}(0) = 0. \]

A straightforward calculation gives rise to \( P = a D_U \bar{U} (U_{eq}; 0) \), which \( a \) is the constant in (28). Thus, \( P \) is a scalar matrix. Moreover,

\[ \tilde{P} \tilde{Q} \tilde{P}^{-1} = P(D_U \bar{U})^{-1} D_U \bar{U} Q U (D_U \bar{U})^{-1} D_U \bar{U} P^{-1} = P Q U P^{-1}. \]

This means that system (35) satisfies the first requirement of structural stability condition. For the second requirement, the symmetrizer of system (35) is \( \tilde{A}_0 = \tilde{A}_0 (U; \varepsilon) = (D_U \bar{U})^{-1} \tilde{A}_0 (D_U \bar{U})^{-1} \). A simple computation shows that the system also satisfies third requirement of structural stability condition.

For further discussions, we analyze \( \tilde{A}(U; 0) \). In Appendix A1, we show that

\[ \tilde{A}^{11}(U_{eq}; 0) = 0, \quad \partial_U \tilde{A}^{11}(U_{eq}; 0) = 0, \]

and \( \tilde{A}^{21}(U_{eq}; 0) \) is not full-rank matrix. See details in Appendix A1.

For the convenience of writing, we omit the superscript below. Then system (35) has the following form

\[ \partial_U + \frac{1}{\varepsilon} A(U; \varepsilon) \partial_U = \frac{1}{\varepsilon^2} Q(U; \varepsilon), \]

(38)
with initial conditions

\[ U(x, 0) = \bar{U}(x, \varepsilon). \]  

(39)

Here \( U = (u, w)^T \in G \subset R^{N+4} \) and

\[ u = (\rho, \rho v + \varepsilon \kappa_{1,0} f_0, \rho E + \kappa_{0,0} f_0)^T \in R^3, \quad w = (\kappa_{0,0} f_0 - b(\theta), \alpha, f_2, \cdots, f_N)^T \in R^{N+1}. \]  

(40)

Here \( A = A(U; \varepsilon) \) and \( Q = Q(U; \varepsilon) \) are the respective \( n \times n \)-matrix function and \( n \)-vector functions of \( (U; \varepsilon) \in G \times [0, \varepsilon_0]. \)

The parameter \( \varepsilon \in [0, \varepsilon_0]. \) The state space \( G \) is a open convex set, which is defined as

\[ G = \{ U = (u, w) : \rho > 0, \theta > 0, \alpha \in (-1, 1) \}. \]

The equilibrium manifold is

\[ G_{eq} = \{ U \in G : w = 0 \}. \]

**Lemma 2.** System (38) satisfies the following properties.

1. The source term has the following form:

\[ Q(U; \varepsilon) = \begin{pmatrix} 0 \\ q(U; \varepsilon) \end{pmatrix}, \quad q(U; \varepsilon) \in R^{(N+1)}, \]

and

\[ q(U; \varepsilon) = 0 \iff w = 0, \]

for all \( u \)

\[ \partial_u q(U_{eq}; 0) = 0, \quad \partial_u q(U_{eq}; 0) \text{ invertible.} \]

2. System (38) satisfies Yong’s structural stability condition and \( P \) is a scalar matrix;

3. \( A^{11}(U_{eq}; 0) = 0 \) and \( \partial_u A^{11}(U_{eq}; 0) = 0 \) for all \( U_{eq} \in G_{eq}; \)

4. \( A_0(U_{eq}; 0) = \text{diag}(A_0^{11}(U_{eq}; 0), A_0^{22}(U_{eq}; 0)) \) is a block diagonal matrix.

**Proof.** Obviously, the first two terms are clearly established. The proof of the third term is exhibited in Appendix A1. As shown in Theorem 2.2 by Yong [36], the structural stability condition implies that \( P^{-1}A_0(\varepsilon_{eq}; 0)P^{-1} \) is a block diagonal matrix. Moreover, \( P \) is a scalar matrix, so \( A_0(\varepsilon_{eq}; 0) \) is a block diagonal matrix. Through the previous discussion, we can see that the coefficient matrix \( A(U; \varepsilon) \) and the source term \( Q(U; \varepsilon) \) are smoothly dependent on \( U \) and \( \varepsilon. \) The symmetrizer \( A_0 \) is also a smooth function of \( U \) and \( \varepsilon. \)

Assuming that the initial value of the equation is periodic and smooth, according to Kato [40], for all integer \( s > \frac{1}{2}, \) there exists a maximal time \( T_\varepsilon > 0 \) such that problem (38) and (39) admits a unique local-in-time smooth solution \( U^\varepsilon \) satisfying

\[ U^\varepsilon \in C([0, T_\varepsilon), H^s) \cap C^1([0, T_\varepsilon), H^{s-1}). \]

The central problem of the study is to show that \( U^\varepsilon \) converges as \( \varepsilon \to 0 \) and \( \inf T_\varepsilon > 0. \) To do this, we study the approximate solution of (38).

We end this section with stating several calculus inequalities in Sobolev spaces [41], two elementary facts [36] related to ODEs, and the notation involved in this paper. Their proofs can be found in Yong [36] and references cited therein.

**Lemma 3** (Calculus inequalities). Let \( s, s_1, \) and \( s_2 \) be three nonnegative integers, and \( s_0 = \lfloor D/2 \rfloor + 1. \)

1. If \( s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \geq 0, \) then \( H^{s_3} \subset H^{s_1}. \) Here the inclusion symbol \( \subset \) implies the continuity of the embedding.
2. Suppose $s > s_0 + 1, A \in H^s$ and $Q \in H^{s-1}$, Then for all multi-indices $\alpha$ with $\alpha \leq s$, $[A, \partial_\tau]Q \equiv A\partial_\tau^s Q - \partial_\tau^s (AQ) \in L^2$ and

$$\|A\partial_\tau^s Q - \partial_\tau^s (AQ)\| \leq C_s\|A\|_s\|Q\|_{s-1}.$$

3. Suppose $s > s_0, A \in C_0^\infty(G)$ and $V \in H^s(R^d, G)$. Then $A(V(\cdot)) \in H^s$ and

$$\|A(V(\cdot))\|_s \leq C_s\|A\|_s(1 + \|V\|_s^2).$$

Here and below $C_s$ denotes a generic constant depending only on $s$, $n$, and $D$, and $|A|_s$ stands for $\sup_{u \in G_\alpha, |\alpha| \leq s} \|\partial_\tau^\alpha A(u)\|$.

**Lemma 4.** [36] Suppose $A(x, \tau) \in C([0, \infty), H^s)$ with $s > \frac{3}{2}$, $f(x, \tau) \in C([0, \infty), L^2)$, $\|f(\tau)\|$ decays exponentially to zero as $\tau$ goes to infinity, and $E(x), S(x) \in L^\infty$ are uniformly positive definite symmetric matrices such that for all sufficiently large $\tau$ and for all $x$,

$$E(x)A(x, \tau) + A^T(x, \tau)E(x) \leq -S(x).$$

If $V(x, \tau) \in C^1([0, \infty), L^2)$ satisfies

$$\frac{dV}{d\tau} = A(x, \tau)V + f(x, \tau),$$

then $\|V\|$ decays exponentially to zero as $\tau$ goes to infinity. Moreover, if $V(x, \tau), f(x, \tau) \in C([0, \infty), H^s)$ and $\|f(\tau)\|_s$ decays exponentially to zero as $\tau$ goes to infinity, then $\|V\|_s$ decays exponentially to zero as $\tau$ goes to infinity.

**Lemma 5.** [36] Suppose $\psi(t)$ is a positive $C^1$-function of $t \in [0, T)$ with $T \leq \infty$, $m > 1$ and $b_1(t), b_2(t)$ are integrable on $[0, T)$. If

$$\psi'(t) \leq b_1(t)\psi^m(t) + b_2(t)\psi(t),$$

then there exists $\delta > 0$, depending only on $m$, $C_{1b}$ and $C_{2b}$, such that

$$\sup_{t \in [0, T)} \psi(t) \leq e^{C_{1b}},$$

whenever $\psi(0) \in (0, \delta)$. Here

$$C_{1b} = \sup_{t \in [0, T)} \int_0^t b_1(s)ds, \quad C_{2b} = \int_0^T \max\{b_2(t), 0\}dt.$$
are found, the formal asymptotic approximation is defined as the above truncation (41). We assume that there exists an approximate solution $U^n_t$ to (38)–(39) defined on a time interval $[0, T_m]$, with $T_m > 0$ independent of $\varepsilon$.

The properties of the approximate solution strongly depend on its leading profile $(u_0, w_0)$, which is a formal limit of $U^n_t$. From the equations (41) and (38), we can obtain

$$
\varepsilon^{-2} : q(u_0, w_0; 0) = 0,
$$

$$
\varepsilon^{-1} : \left( \begin{array}{c}
A^{11}(u_0, w_0; 0) \ A^{12}(u_0, w_0; 0) \\
A^{21}(u_0, w_0; 0) \ A^{22}(u_0, w_0; 0)
\end{array} \right) \partial_x \left( \begin{array}{c}
u_0 \\
w_0
\end{array} \right) = \left( \begin{array}{c}
0 \\
w_0(u_0, w_0; 0)w_1
\end{array} \right),
$$

$$
\varepsilon^0 : \partial_t u_0 + A^{11}(u_0, w_0; 0) \partial_x u_1 + A^{12}(u_0, w_0; 0) \partial_x w_1 \\
+ (A^{11}_u(u_0, w_0; 0) u_1 + A^{11}_w(u_0, w_0; 0) w_1 + A^{11}_{xw}(u_0, w_0; 0)) \partial_x u_0 = 0.
$$

According to Lemma 2, we have

$$
w_0 = 0, \quad w_1 = q_w^{-1}(u_0, w_0; 0)A^{21}(u_0, w_0; 0) \partial_x u_0,
$$

$$
\partial_t u_0 + A^{12}(u_0, w_0; 0) \partial_x (q_w^{-1}(u_0, w_0; 0) A^{21}(u_0, w_0; 0) \partial_x u_0)
$$

$$
+ (A^{11}_u(u_0, w_0; 0) q_w^{-1}(u_0, w_0; 0) A^{21}(u_0, w_0; 0) \partial_x u_0 + A^{11}(u_0, w_0; 0)) \partial_x u_0 = 0. \tag{42}
$$

Equation (42) can be rewritten as

$$
\partial_t u_0 + A^{12}(u_0, w_0; 0) q_w^{-1}(u_0, w_0; 0) A^{21}(u_0, w_0; 0) \partial_x^2 u_0
$$

$$
+ A^{12}(u_0, w_0; 0) \partial_u (q_w^{-1}(u_0, w_0; 0) A^{21}(u_0, w_0; 0)) \partial_x u_0
$$

$$
+ (A^{11}_u(u_0, w_0; 0) q_w^{-1}(u_0, w_0; 0) A^{21}(u_0, w_0; 0) \partial_x u_0 + A^{11}(u_0, w_0; 0)) \partial_x u_0 = 0.
$$

Since $A^{21}(u_0, w_0; 0)$ is not full-rank matrix according to Appendix A1, we know that the equation of $u_0$ (42) is not strictly parabolic. Its proof is quite similar to those proved in Lattanzio and Yong [17] and Peng and Wasiolek[18].

Here we derive the specific form of the equation which satisfies $u_0$. Expanding the variables into a power series of $\varepsilon$ involved in Equation (32) yields

$$
\rho = \rho^0 + \varepsilon \rho^1 + \cdots, \quad v = v^0 + \varepsilon v^1 + \cdots,
$$

$$
E = E^0 + \varepsilon E^1 + \cdots, \quad \theta = \theta^0 + \varepsilon \theta^1 + \cdots,
$$

$$
p = p^0 + \varepsilon p^1 + \cdots, \quad f_0 = f_0^0 + \varepsilon f_1^0 + \cdots, \quad f_2 = f_2^0 + \varepsilon f_1^1 + \cdots, \tag{43}
$$

where $\theta = \theta(\rho, v, E), p = p(\rho, \theta)$. According to the definition of equilibrium state in (21), we know that $f_0^0 = \frac{1}{2} b (\theta^0), \alpha^0 = 0, f_2^0 = 0$.

Using Equation (32), we arrive at

$$
\partial_t \rho^0 + \partial_x (\rho^0 \nu^0) = 0,
$$

$$
\partial_t (\rho^0 v^0) + \partial_x \left( \rho^0 (v^0)^2 + p^0 + \kappa_2(0) f_0^0 + \kappa_2(0) f_0^0 \right) = 0,
$$

$$
\partial_t (\rho^0 E^0 + \kappa_0(0) f_0^0) + \partial_x \left( \rho^0 E^0 \nu^0 + p^0 \nu^0 + \kappa_1(0) \alpha^1 f_0^0 + \kappa_1(0) f_0^1 \right) = 0. \tag{44}
$$

Here $\kappa_2(0) = 2, \kappa_0(0) = 2, \kappa_1(0) = -\frac{8}{3}, \kappa_1(0) = 0$ due to formula (20). To get a closed system, we also need the expression of $\alpha^1$. 
In order to obtain the expression of $a^1$, we analyze equation of $E_1$ in (33):

$$\partial_t (\kappa_{1,0} f_0) + \frac{1}{\varepsilon} \partial_x \left( \kappa_{2,0} f_2 + \kappa_{2,0} f_0 \right) = -\rho \left( \frac{1}{\varepsilon^2} \sigma_a(\theta) + \sigma_s(\theta) \right) \kappa_{1,0}(\alpha) f_0.$$ 

Putting expansion (43) into above equation, the identification of $O(\varepsilon^{-1})$ yields

$$\partial_x \left( \kappa_{2,0}(0) f_0^0 \right) = -\rho^0 \sigma_a \left( \theta^0 \right) \left( \kappa_{1,0}(0) a^1 f_0^0 \right).$$

Here we used $f_2^0 = 0$ and $\kappa_{1,0}(0) = 0$. Combining $\kappa_{2,0}(0) = \frac{2}{3}$ and $f_0^0 = \frac{1}{2} b(\theta^0)$, we see that

$$\kappa_{1,0}(0) a^1 f_0^0 = -\frac{1}{3} \rho^0 \sigma_a(\theta) \partial_x \left( b(\theta^0) \right).$$

Omitting the superscript in above equation, the nonrelativistic limit equation can be obtained as

$$\partial_t \rho + \partial_x (\rho v) = 0,$$

$$\partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} b(\theta) \right) = 0,$$

$$\partial_t (\rho E + b(\theta)) + \partial_x (\rho E v + pv) = \partial_x \left( \frac{1}{3 \rho^0 \sigma_a(\theta)} \partial_x \left( b(\theta) \right) \right).$$

Previous works [2, 3, 43] also obtain the zero-order approximation of the radiation hydrodynamics system, and the system is also hyperbolic–parabolic form which is similar to the above equation. However, there is no rigorous proof of the singular limit.

Below, we derive the equations satisfied by the other coefficients in the asymptotic solution (41). To do this, we consider the residual

$$R \left( U_{\varepsilon}^m \right) = \partial_t U_{\varepsilon}^m + \frac{1}{\varepsilon} A \left( U_{\varepsilon}^m ; \varepsilon \right) + \frac{1}{\varepsilon} Q \left( U_{\varepsilon}^m ; \varepsilon \right). \quad (45)$$

Using Taylor expansion, we have

$$A \left( U_{\varepsilon}^m ; \varepsilon \right) = A \left( U_{\varepsilon}^m ; 0 \right) + \left[ \sum_{k=1}^{\infty} \varepsilon^k \partial^k_x A \left( U_{\varepsilon}^m ; 0 \right) \right],$$

$$Q \left( U_{\varepsilon}^m ; \varepsilon \right) = Q \left( U_{\varepsilon}^m ; 0 \right) + \left[ \sum_{k=1}^{\infty} \varepsilon^k \partial^k_x Q \left( U_{\varepsilon}^m ; 0 \right) \right].$$

Remark that for $W = \sum_{k=0}^{\infty} \varepsilon^k W_k$ and a sufficiently smooth function $H$, we have formally [17]

$$H(W) = H \left( \sum_{k=0}^{\infty} \varepsilon^k W_k \right) = H(W_0) + \left[ \sum_{k=1}^{\infty} \varepsilon^k \left[ \partial^k W_0 + C(H, k, W) \right] \right],$$

where coefficients $C(H, k, W)$ are completely determined by the given function $H$ and the first $k$ components $W = (W_0, W_1, W_2, \cdots, W_{k-1})$. Moreover, $C(H, 1, W) = 0$ and $C(H, k, W)$ is linear with respect to $W_{k-1}$ for $k \geq 3$. 

4.1.1 Outer expansions

As a formal solution, the outer expansion $\sum_{k=0}^{\infty} \varepsilon^k U_k(x, t)$ asymptotically satisfies system (38). Thus, we have

$$R \left( \sum_{k=0}^{\infty} \varepsilon^k U_k \right) = -\varepsilon^{-2} Q(U_0; 0) + \varepsilon^{-1} [A(U_0; 0) \partial_x U_0 - \partial_t Q(U_0; 0) - Q_U(U_0; 0) U_1]$$

$$+ \sum_{k=0}^{\infty} \varepsilon^k \partial_t U_k + \sum_{k=0}^{\infty} \varepsilon^k \frac{1}{2} \partial_x^2 A(U_0; 0) \partial_x U_{k+1-l}$$

$$+ \sum_{k=0}^{\infty} \varepsilon^k \sum_{l=0}^{k-l} \frac{1}{l!} [\partial_U (\partial^l_x A(U_0; 0)) U_{k+1-l-j} + C(\partial^l_x A(\cdot; 0), k+1-l-j, U)] \partial_x U_j$$

$$- \sum_{k=0}^{\infty} \varepsilon^k \sum_{l=0}^{k+1} \frac{1}{l!} [\partial_U (\partial^l_x Q(U_0; 0)) U_{k+2-l} + C(\partial^l_x Q(\cdot; 0); k+2-l, U)]$$

$$- \sum_{k=0}^{\infty} \varepsilon^k \frac{1}{(k+2)!} \partial_x^{k+2} Q(U_0; 0)$$

vanishes. This happens when each term of the last expansion is zero, that is,

$$\varepsilon^{-2} : Q(U_0; 0) = 0,$$

$$\varepsilon^{-1} : A(U_0; 0) \partial_x U_0 - \partial_t Q(U_0; 0) - Q_U(U_0; 0) U_1 = 0,$$

$$\varepsilon^k : \partial_t U_k + \frac{1}{k+1} \partial_x^2 A(U_0; 0) \partial_x U_{k+1-l}$$

$$+ \sum_{l=0}^{k-l} \frac{1}{l!} [\partial_U (\partial^l_x A(U_0; 0)) U_{k+1-l-j} + C(\partial^l_x A(\cdot; 0), k+1-l-j, U)] \partial_x U_j$$

$$= \sum_{l=0}^{k+1} \frac{1}{l!} [\partial_U (\partial^l_x Q(U_0; 0)) U_{k+2-l} + C(\partial^l_x Q(\cdot; 0); k+2-l, U)]$$

$$+ \frac{1}{(k+2)!} \partial_x^{k+2} Q(U_0; 0).$$

According to Lemma (2), $u_0$, $w_0$, and $w_1$ satisfy

$$Q(u_0, w_0; 0) = 0 \Rightarrow w_0 = 0, \quad \partial_t Q(u_0, 0; 0) = 0,$$

$$w_1 = q^{w-1}(u_0, 0; 0) A^{12}(u_0, 0; 0) \partial_x u_0,$$

$$\partial_t u_0 + A^{12}(u_0, 0; 0) \partial_x w_1 + A^{13}_w(u_0, 0; 0) w_1 \partial_x u_0 + A^{11}(u_0, 0; 0) \partial_x u_0 = 0. \quad (48)$$

The above equations can be rewritten with the $u$, $w$-components as

$$\partial_t u_k + \sum_{l=0}^{k+1} \frac{1}{l!} \partial_x^2 (A^{11}(U_0; 0) \partial_x u_{k+1-l} + A^{12}(U_0; 0) \partial_x w_{k+1-l})$$

$$+ \sum_{l=0}^{k-l} \frac{1}{l!} [\partial_u (\partial^l_x A^{11}(U_0; 0)) u_{k+1-l-j} \partial_x u_j + \partial_w (\partial^l_x A^{11}(U_0; 0)) w_{k+1-l-j} \partial_x u_j$$

$$+ \partial_u (\partial^l_x A^{12}(U_0; 0)) u_{k+1-l-j} \partial_x w_j + \partial_w (\partial^l_x A^{12}(U_0; 0)) w_{k+1-l-j} \partial_x w_j$$

$$+ C(\partial^l_x A^{11}(\cdot; 0), k+1-l-j, U) \partial_x u_j + C(\partial^l_x A^{12}(\cdot; 0), k+1-l-j, U) \partial_x w_j] = 0,$$
and
\[
\frac{\partial}{\partial t} w_k + \sum_{l=0}^{k+1} \frac{1}{l!} \frac{\partial^l}{\partial t^l} \left( A^{21}(U_0; 0) \partial_x u_{k+1-l} + A^{22}(U_0; 0) \partial_x w_{k+1-l} \right) \\
+ \sum_{l=0}^{k} \sum_{j=0}^{l-1} \frac{1}{l!} \left[ \partial_u \left( \frac{\partial^l}{\partial t^l} A^{21}(U_0; 0) \right) u_{k+1-l-j} \partial_x u_j + \partial_u \left( \frac{\partial^l}{\partial t^l} A^{21}(U_0; 0) \right) w_{k+1-l-j} \partial_x u_j \\
+ \partial_u \left( \frac{\partial^l}{\partial t^l} A^{22}(U_0; 0) \right) u_{k+1-l-j} \partial_x w_j + \partial_u \left( \frac{\partial^l}{\partial t^l} A^{22}(U_0; 0) \right) w_{k+1-l-j} \partial_x w_j \\
+ C \left( \frac{\partial^l}{\partial t^l} A^{21}(\cdot, 0); k+1-j \right) \partial_x u_j + C \left( \frac{\partial^l}{\partial t^l} A^{22}(\cdot, 0); k+1-j \right) \partial_x w_j \right] \\
- \sum_{l=0}^{k+1} \frac{1}{l!} \left[ \partial_u \left( \frac{\partial^l}{\partial t^l} q(U_0; 0) \right) u_{k+1-l} + \partial_u \left( \frac{\partial^l}{\partial t^l} q(U_0; 0) \right) w_{k+1-l} + C \left( \frac{\partial^l}{\partial t^l} q(\cdot; 0); k+2-l \right) \partial_x u_j \right] \\
- \frac{1}{(k+2)!} \frac{\partial^{k+2}}{\partial t^{k+2}} q(U_0; 0) = 0.
\]

Obviously, the equations in (47) need to be rewritten to determine \( U_k \) inductively. Equation (48) shows that \( U_0 \) lies on the equilibrium manifold \( G_{\epsilon_0} \). According to Equation 48, we have found the equations for \( U_0, w_0, \) and \( w_1 \). From Lemma (2), we know \( A^{13}(U_0; 0) = 0 \) and \( w_0 = 0 \). Hence, the equations of \( u_k \) (49) may depend on \( U_0, \ldots, U_k, w_{k+1} \) and their first-order derivatives but are independent of \( u_{k+1} \). From Equation 50, we can see that \( w_0 \) depend on \( U_0, \ldots, U_k, w_{k+1} \) and \( w_{k+2} \). The equations of \( w_k \) are independent of \( u_{k+2} \) due to the fact that since \( q(U_0; 0) = 0 \), we know \( \partial_u(\partial^l q(U_0; 0))u_{k+2-l} = 0 \) when \( l = 0 \). Moreover, \( \partial_u(\partial^l q(U_0; 0))w_{k+2-l} = \partial_u q(U_0; 0)w_{k+2} \) when \( l = 0 \). Therefore, (50) gives an expression of \( w_{k+2} \) as a function of \( U_0, \ldots, U_k, w_{k+1}, \) and of the known quantities and their derivatives.

Up to now, we have found the equations for \( u_0, w_0, \) and \( w_1 \). Assume inductively that we have equations for \( u_i, w_i, \) and \( w_{i+1} \) for \( i = 0, \ldots, k \). Equation 50 gives an expression of \( w_{i+2} \) of function of \( u_{k+1}, \partial_x u_{k+1} \) and of the known quantities and their derivatives. With this expression, the equation for \( u_{k+1} \) can be derived from relation (49).

Assume that \( U_0, \ldots, U_{k-1} \) are known. From Equation (49), we know that the equations of \( u_k \) can be rewritten as

\[
\partial_t u_k + A^{12}(U_0; 0) \partial_x w_{k+1} + \cdots = 0.
\]

What is omitted here and in the following equations is the derivative term of the known quantity and the known quantity \( U_0, \ldots, U_{k-1} \). Equation (50) allows to express \( w_{k+1} \) as

\[
w_{k+1} = q_{w}(U_0; 0)^{-1} A^{21}(U_0; 0) + \cdots .
\]

Hence, the coefficient of the second derivative in the equation of \( u_k \) is still \( A^{12}(U_0; 0)q_{w}(U_0; 0)^{-1} A^{21}(U_0; 0) \), which is the same as \( u_0 \), so the equation of \( u_k \) is not strictly parabolic.

From previous discussions, it remains to find initial data for the coefficients \( U_k \). For this purpose, we turn to consider the composite expansion.

4.1.2 Composite expansions
Since \( t = \epsilon^2 \tau \), we have formally

\[
\sum_{k=0}^{\infty} \epsilon^k U_k(x, t) = \sum_{k=0}^{\infty} \epsilon^k U_k(x, \epsilon^2 \tau) = \sum_{k=0}^{\infty} \epsilon^k P_k(x, \tau),
\]

where

\[
P_k(x, \tau) = \sum_{h=0}^{[k/2]} \frac{\epsilon^k}{h!} \frac{\partial^h U_k-2h}{\partial \tau^h}(x, 0)
\]

is a polynomial of degree \([k/2]\) in \( \tau \). Particularly, \( P_0(x, \tau) = U_0(x, 0) \).
The composite expansion $U^m_ε$ in (41) becomes

$$\sum_{k=0}^{m} \epsilon^k (U_k(x, t) + I_k(x, \tau)) = \sum_{k=0}^{m} \epsilon^k (P_k(x, \tau) + I_k(x, \tau)), \tag{51}$$

which is just the traditional inner expansion [42]. Now write (38) in variables $(x, \tau)$ as follows:

$$\frac{1}{\epsilon^2} \partial_\tau U + \frac{1}{\epsilon} A(U; \epsilon) \partial_\tau U = \frac{1}{\epsilon^2} Q(U; \epsilon).$$

The corrected formal solution should asymptotically satisfy Equation 38. Namely, the formal asymptotic expansion

$$R\left(\sum_{k=0}^{\infty} \epsilon^k (P_k(x, \tau) + I_k(x, \tau))\right) = -\epsilon^{-2} Q(P_0 + I_0; 0) + \epsilon^{-1} [A(P_0 + I_0; 0) \partial_\tau (P_0 + I_0) - \partial_\tau Q(P_0 + I_0; 0)$$

$$-Q_U(P_0 + I_0; 0)(P_1 + I_1)] + \sum_{k=0}^{\infty} \epsilon^k \partial_\tau (P_{k+2} + I_{k+2})$$

$$- \sum_{k=0}^{\infty} \epsilon^k \frac{1}{(k+2)!} \partial^2 Q(P_0 + I_0; 0)$$

$$+ \sum_{k=0}^{\infty} \epsilon^k \sum_{l=0}^{k+1} \frac{1}{l!} \partial^l_A(P_0 + I_0; 0) \partial_\tau (P_{k+1-l} + I_{k+1-l})$$

$$+ \sum_{k=0}^{\infty} \epsilon^k \sum_{l=0}^{k-l} \frac{1}{l!} \left[ \partial_U (\partial^l_A(P_0 + I_0; 0)) (P_{k+1-l-j} + I_{k+1-l-j}) + C(\partial^l_A(\cdot; 0), k+1-l-j, \tau; I+P) \right] \partial_\tau (P_j + I_j)$$

$$- \sum_{k=0}^{\infty} \epsilon^k \sum_{l=0}^{k+1} \frac{1}{l!} \left[ \partial^l_U (\partial^l_A(P_0 + I_0; 0)) (P_{k+2-l} + I_{k+2-l}) + C(\partial^l_A(\cdot; 0), k+2-l, \tau; I+P) \right]$$

vanishes. This happens when each term of the last expansion is zero, that is,

$$\partial_\tau (P_0 + I_0) = Q(P_0 + I_0; 0),$$

$$\partial_\tau (P_1 + I_1) = -A(P_0 + I_0; 0) \partial_\tau (P_0 + I_0) + \partial_U Q(P_0 + I_0; 0)(P_1 + I_1)$$

$$+ \partial_\tau Q(P_0 + I_0; 0)(P_0 + I_0),$$

$$\partial_\tau (P_k + I_k) = \partial_U Q(P_0 + I_0; 0)(P_k + I_k) + F(k, 1 + P), \quad k \geq 2, \tag{53}$$

where

$$F(k, 1 + P) = \frac{1}{k!} \partial^k_U Q(P_0 + I_0; 0) - \sum_{l=0}^{k-1} \frac{1}{l!} \partial^l_A(P_0 + I_0; 0) \partial_\tau (P_{k-1-l} + I_{k-1-l})$$

$$- \sum_{l=0}^{k-2} \sum_{j=0}^{k-2-l} \frac{1}{l!} \left[ \partial_U (\partial^l_A(P_0 + I_0; 0)) (P_{k-1-l-j} + I_{k-1-l-j}) + C(\partial^l_A(\cdot; 0), k-1-l-j, 1+P) \right] \partial_\tau (P_j + I_j)$$

$$+ \sum_{l=1}^{k-1} \frac{1}{l!} \left[ \partial^l_U (\partial^l_A(Q(P_0 + I_0; 0)) (P_{k-1-l} + I_{k-1-l}) + C(\partial^l_A(Q(\cdot; 0), k-l, 1+P) \right]. \tag{54}$$

Here $F(k, 1 + P)$ depends only on the first $k$ terms of the inner expansion, which is $U_0, I_0, \cdots, U_{k-1}$, and $I_{k-1}$. 
According to the definition of \( P_k \), \( \sum_{k=0}^{\infty} \varepsilon^k P_k(x, \tau) \) is also a solution of (38). Hence, we obtain as above

\[
\begin{align*}
\partial_\tau P_0 &= Q(P_0; 0), \\
\partial_\tau P_1 &= -A(P_0; 0) \partial_\tau P_0 + \partial_\tau Q(P_0; 0) P_1 + \partial_\tau Q(P_0; 0) P_0, \\
\partial_\tau P_k &= \partial_\tau Q(P_0; 0) P_k + F(k, \tilde{\lambda}).
\end{align*}
\]

(55)

Note that

\[
P_0(x, \tau) = U_0(x; 0), \quad Q(P_0; 0) = 0.
\]

We find from (53) and (55) that

\[
\begin{align*}
\partial_\tau I_0 &= Q(P_0 + I_0; 0), \\
\partial_\tau I_k &= \partial_\tau Q(P_0 + I_0; 0) I_k + [\partial_\tau Q(P_0 + I_0; 0) - \partial_\tau Q(P_0; 0)] P_k + G_k,
\end{align*}
\]

where

\[
G_k = F(k, P + I) - F(k, P)
\]

with

\[
F(I, P + I) = \partial_\tau Q(P_0 + I_0; 0)(P_0 + I_0) - A(P_0 + I_0; 0) \partial_\tau (P_0 + I_0).
\]

According to the expression of \( F(k, I + P) \) (54), \( G_k \) depends only on \( U_0, I_0, \ldots, U_{k-1}, I_{k-1} \).

4.1.3 Initial data for the outer expansion

Now we determine the initial conditions for \( \tilde{U}_k \). Assume that \( \tilde{U}(x; \varepsilon) \) has a formal asymptotic expansion as follows:

\[
\tilde{U}(x; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \tilde{U}_k(x), \quad \tilde{U}_k(x) = (\tilde{u}_k(x), \tilde{w}_k(x))^T.
\]

If the composite expansion (41) is a solution of (38) and (39), we should have

\[
\tilde{U}_k(x, 0) + I(x, 0) = \tilde{U}_k(x),
\]

or equivalently,

\[
\begin{align*}
u_k(x, 0) + I_k^I(x, 0) &= \tilde{u}_k(x), \\
w_k(x, 0) + I_k^{II}(x, 0) &= \tilde{w}_k(x).
\end{align*}
\]

(57)

From \( Q^I = 0 \) and the first equation of (56), we have \( \partial_\tau I_0^I = 0 \). Meanwhile, since \( I_0 \) satisfies \( I_0(x, +\infty) = 0 \), we know that \( I_0^I(x, \tau) = 0 \), which means that there is no zeroth-order initial layer for \( u \). Together with \( w_0 = 0 \), we obtain

\[
u_0(x, 0) = \tilde{u}_0(x), \quad I_0^{II}(x, 0) = \tilde{w}_0(x).
\]

According to (56), \( I_0^{II} \) satisfies

\[
\begin{align*}
\partial_\tau I_0^{II} &= q(\tilde{u}_0(x), I_0^{II}; 0) \\
I_0^{II}(x, 0) &= \tilde{w}_0(x).
\end{align*}
\]

(58)

Here and below, the superscript “I” (or “II”) stands for the first three (or last \( n + 1 \)) components of a vector in \( \mathbb{R}^{n+4} \).

**Lemma 6.** Let \( \tilde{w}_0 \) be sufficiently small. Then there exists a unique global smooth solution \( I_0 \) satisfying

\[
\|I_0\|_{s+m} \to 0, \text{ exponentially as } \tau \to +\infty.
\]

(59)
Proof. By Lemma 2, system (38) satisfies the structural stability condition and $P$ is a scalar matrix. Then $q_w$ satisfies

$$A_0^{22}(u, 0; 0)q_w(u, 0; 0) + q_w(u, 0; 0)^T A_0^{22}(u, 0; 0) \leq -I.$$  

Therefore, for sufficiently small data $\tilde{w}_0$, there is a unique global solution $I_0^H(x, \tau)$ (see Arnold [44]). Thanks to Lemma 2, $0 \in \mathcal{R}^{n+1}$ is a fixed point for (59). Moreover, $q_w(u, 0; 0)$ is stable due to above equation. Hence, $0 \in \mathcal{R}^{n+1}$ is locally asymptotically stable for (59). By induction, for all $\alpha$ with $\alpha \leq s + m$, $\frac{\partial^2 I^H}{\partial \alpha^2} (x, \tau)$ satisfies a linear ODE of the form

$$\partial_t Y = \partial_u q \left( \tilde{w}_0(x), I_0^H, 0 \right) Y + g_w(x, \tau).$$

Meanwhile, $g_w(x, \tau)$ decays to zero as $\tau \to +\infty$. Thanks to Lemma 4, we see the exponential decay of $\|I_0\|_{s+m} \to 0$. □

Assume that, for $k \geq 1$ and for any $i \leq k - 1$, $I_i$ exists globally in time and $\|I_i(\cdot, \tau)\|_{s+m-i}$ decays exponentially fast to zero as $\tau \to +\infty$. Then so does $\|G_k\|_{s+m-k}$ since $G_k = F(k, P + I) - F(k, P)$ is a function of $I_i, P_i (0 \leq i \leq k - 1)$ and their first-order derivatives with respect to $x$. Because the $u$-component of $Q$ is 0 for $k \geq 1$, the first three equations in (56) are

$$\partial_t I_k^I = G_k^I,$$  

(60)

Hence,

$$I_k^I(x, \tau) = I_k^I(x, 0) + \int_0^\tau G_k^I(x, \tau') d\tau',$$

which admits a limit 0 as $\tau$ goes to infinity. Therefore,

$$I_k^I(x, \tau) = -\int_\tau^{+\infty} G^I(k, x, \tau') d\tau',$$

and

$$I_k^I(x, \tau) = -\int_\tau^{+\infty} G^I(k, x, \tau') d\tau', \text{ exponentially as } \tau \to +\infty.$$  

In particular,

$$I_k^I(x, 0) = -\int_0^{+\infty} G_k^I(x, \tau') d\tau'.$$  

(61)

Together with (57), it determines the initial value of $u_k$:

$$u_k(x, 0) = \tilde{u}(x) + \int_\tau^{+\infty} G^I(k, x, \tau') d\tau'.$$  

(62)

Furthermore, we can rewrite the remaining equation in (56) as

$$\partial_t I_k^{II} = \partial_u q(P_0 + I_0; I_0^I + \int_0^{+\infty} G^I(k, x, \tau') d\tau') + [\partial_u q(P_0 + I_0; 0) - \partial_u q(P_0; 0)] P_k^{II} + [\partial_u q(P_0 + I_0; 0) - \partial_u q(P_0; 0)] I_k^{II} + G^I(k)$$  

$$\equiv \partial_u q(P_0 + I_0; 0) I_k^{II} + G'.$$  

(63)

We know that $\|G\|_{s+m-k}$ decays exponentially fast to zero as $\tau \to +\infty$ from the definition of $G$ and Lemma 6. Thanks to Lemma 4, we see the exponential decay of $\|I_k^{II}(x, \tau)\|_{s+m-k} \to 0$. Hence, the inductive process is complete.

Now we describe a procedure to determine the coefficients of the expansion (41) using Equations (46) and (53). Based on previous analysis, $I_0, U_0,$ and $w_1$ are known. Then we can solve (63) with the initial value providing in (57) to obtain $I_k^{II}$. The value of $I_k^I$ can be determined by Equation (60) and initial value (61). Hence, we can determine $U_0, U_1, I_0, I_1$ since the
equation and initial value of $u_1$ are known. Assume inductively that $U_i, I_i, w_{i+1}$ with $i \leq k$ have been obtained. Then we can solve (63) with the initial value provided in (57) to obtain $I^U_{k+1}$. And Equation (60) and the initial value (61) give the value of $I^U_{k+1}$. Thus, $I_{k+1}$ are completely determined. Moreover, (50) gives an expression of $w_{k+2}$ as a function of $u_{k+1}$, $\partial_x u_{k+1}$ and of the known quantities and their derivatives. With this expression, the equation for $u_{k+1}$ can be derived from (49) together with the initial value (62).

Therefore, we obtain $U_{k+1}, I_{k+1},$ and $w_{k+2}$. Hence, the inductive process is complete. In conclusion, we have determined all coefficients in expansion (41) and $\| I_k \|_{s+m-k}$ decays exponentially to zero as $\tau \to +\infty$.

### 4.1.4 Residual estimation

The next lemma is concerning the residual of the formal approximation $R(U^m)$.

**Theorem 2.** Let $R(U^m)$ be defined by (45). Then

$$R(U^m) = \varepsilon^{m-1} Q_U(U_0;0) U_{m+1} + \varepsilon^{m-1} F_m,$$

where $Q_U(U_0;0) U_{m+1}$ is completely determined by the first $m$ terms of the outer expansion. And $F_m$ satisfies

$$\| F_m \|_{s} \leq C \varepsilon + Ce^{-\mu \tau}, \quad \text{(64)}$$

with $\mu \geq 0$ and $C$ constants independent of $\varepsilon$.

*Proof.* The proof of this theorem mainly refers to the literature [36] and Lattanzio and Yong [17]. From the relation in (46), we have

$$R(\sum_{k=0}^{m} \varepsilon^k U_k) = \varepsilon^{m-1} Q_U(U_0;0) U_{m+1} + O(\varepsilon^m),$$

where

$$Q_U(U_0;0) U_{m+1} = \partial_t U_{m-1} + \sum_{l=0}^{m-1} \frac{1}{l!} \partial^l_t A(U_0;0) \partial_x U_{m-l}$$

$$+ \sum_{l=0}^{m-1} \sum_{j=0}^{m-1-l} \frac{1}{l!} \left[ \partial_U \left( \partial^l_t A(U_0;0) \right) U_{m-l-j} + C \left( \partial^l_t A(\cdot;0), m-l-j, U \right) \right] \partial_x U_j$$

$$- \sum_{l=1}^{m-1} \frac{1}{l!} \left[ \partial_U \left( \partial^l_t Q(U_0;0) \right) U_{m+1-l} + C \left( \partial^l_t Q(\cdot;0), m+1-l, U \right) \right]$$

$$- \frac{1}{(m+1)!} \partial^{m+1}_x Q(U_0;0).$$

Then $Q_U(U_0;0) U_{m+1}$ depend only on $U_0, \ldots, U_m$. Define $F_m$ as

$$\varepsilon^{m-1} F_m = R(U^m) - \varepsilon^{m-1} Q_U(U_0;0) U_{m+1}.$$

With this definition, we only need to prove (64).

To this end, consider the Taylor expansion with respect to $\varepsilon$ at $\varepsilon = 0$:

$$\sum_{k=0}^{m} \varepsilon^k U_k(x,t) = \sum_{k=0}^{m} \varepsilon^k U_k(x, \varepsilon^2 \tau) = \sum_{k=0}^{m} \varepsilon^k P_k(x, \tau) + \varepsilon^{m+1} P(x, t, \varepsilon),$$

where $P(x, t, \tau, \varepsilon) = O(1) \tau^{1+|m/2|}$. Thus, we can write

$$U^m_r = \sum_{k=0}^{m} \varepsilon^k (U_k(x, t) + I(x, \tau))$$

$$= \sum_{k=0}^{m} \varepsilon^k (P_k(x, \tau) + I_k(x, \tau)) + \varepsilon^{m+1} P(x, t, \varepsilon),$$
In the spirit of relation (52) for the inner expansion, we deduce from the definition of $R(U^m)$ that

$$
R(U^m) = \varepsilon^{m-1} \left[ \tilde{P}_T + C(\varepsilon, \tilde{P}; I_0 + P_0, \ldots, I_m + P_m) \right]
$$

$$
R \left( \sum_{k=0}^{m} U_k \right) = \varepsilon^{m-1} \left[ P_T + C(\varepsilon, \tilde{P}; P_0, \ldots, P_m) \right].
$$

Here $C(\varepsilon, \tilde{P}; P_0, \ldots, P_m)$ depends smoothly on the $\varepsilon, \tilde{P}; P_0, \ldots, P_m$ and their first-order derivatives with respect to the $x_i$'s.

Furthermore, it follows from the definition of $F_m$ that

$$
F_m = \varepsilon^{-(m-1)} R(U^m) - \varepsilon^{-(m-1)} \left( \sum_{k=0}^{m} U_k \right) + O(\varepsilon)
$$

$$
= C_U(\varepsilon, \tilde{P}; \cdot)I + O(\varepsilon),
$$

where $C_U(\varepsilon, \tilde{P}; \cdot)$ denotes the Fréchet derivative of the operator $C(\varepsilon, \tilde{P}; \cdot)$. Finally, the estimate in (64) follows from the decay property of the I when $\tau$ tends to infinity. □

### 4.2 Justification of formal expansions

Having constructed formal asymptotic approximations $U^m$ for the initial value problems (38) and (39), we prove here the validity of the approximations under Lemma (2) and under some regularity assumptions on the given data. For the sake of exactness, we refer to next remark and make the following assumption.

**Assumption 1.** Let $s > \frac{3}{2}$.

1. There exists a convex open set $G_0 \subset \subset G$ satisfying $G_0 \subset \subset G$ such that $\tilde{U}(x; \varepsilon) \in G_0$ for all $\varepsilon > 0$ and all $x \in \Omega$, and $\tilde{U}(\cdot; \varepsilon) \in H^s$ is periodic on $\Omega$;
2. $A(U; \varepsilon), Q(U; \varepsilon), P(U; \varepsilon), A_0(U; \varepsilon)$ are smooth function of $U \in G, \varepsilon \in [0, \varepsilon_0]$;
3. $Q_U(U_0; 0) U_{m+1} \in C([0, T_m], H^3)$;
4. $U^m$ takes value in $\tilde{G}_0$ and satisfies $U^m \in C([0, T_m], H^{\sigma+1}) \cap C([0, T_m], H^3)$. For sufficiently small $\varepsilon > 0$,

$$
\|U^m_T(0, \cdot) - \tilde{U}(\cdot, \varepsilon)\|_s \leq c \varepsilon^m,
$$

and

$$
\sup_{0 \leq t \leq T_m} \|U^m - U_0\|_s \leq C \varepsilon + C \varepsilon^2 B_{\varepsilon}(t), \quad \|\partial_t U^m\|_s \leq c + cB_{\varepsilon}(t).
$$

where $B_{\varepsilon}(t) = \varepsilon^{-2} e^{-\frac{\mu t}{\varepsilon^2}}$ and $\mu > 0$ is a constant independent of $\varepsilon$.

**Remark 1.** The first assumption is necessary to apply the existence theorem, see [40]. The second assumption is obviously. The next can be verified by using the existence theory for parabolic system in Taylor [45]. Equation (65) is a natural condition on the initial data. It stands for initial errors. In the above subsection, we have constructed $U_k$ and $I_k$. Now we show that, for any fixed $m \in N$, the approximate solution $U^m$ defined by (41) satisfies (66). Indeed, since $I_0 = 0$ and $I_0(t_0, \tau)$ decays exponentially fast to zero as $\tau \to +\infty$ with $\tau = t/\varepsilon^2$, thus $\|I_0\|_s \leq C e^{-\frac{\mu t}{\varepsilon^2}}$ with $\mu > 0$ a constant independent of $\varepsilon$. Meanwhile,

$$
\partial_t I_0(t, \tau) = \varepsilon^{-2} \partial_x I_0(t, \tau) = \varepsilon^{-2} g(\bar{a}_0(x), I^{H}_0; 0).
$$

Therefore,

$$
\|U^m - U_0\|_s = \left\| \sum_{k=1}^{m} \varepsilon^k U_k(t) + \sum_{k=0}^{m} \varepsilon^{k} I_k(t/\varepsilon^2) \right\|_s \leq C \varepsilon + C \varepsilon^2 B_{\varepsilon}(t),
$$

$$
\|\partial_t U^m\|_s = \left\| \partial_t I_0(t, \tau) + \sum_{k=0}^{m} \varepsilon^k U_k(t) + \sum_{k=1}^{m} \varepsilon^{k} I_k(t/\varepsilon^2) \right\|_s \leq c + cB_{\varepsilon}(t).
$$
Fix $\varepsilon > 0$ and recall Assumption 1. According to Theorem 2.1 in Majda [41], for any convex open set $G_1$ satisfying $G_0 \subseteq G_1 \subseteq G$, there exists $T_\varepsilon > 0$ such that that initial value problems (38) and (39) for the symmetrizable hyperbolic system have a unique $H^\infty$-solution $U^\varepsilon$ satisfying $U^\varepsilon \in C([0, T_\varepsilon], H^\infty)$ and $U^\varepsilon \in \tilde{G}_1$. Without loss of generality, we assume that $T_\varepsilon$ is the maximal time interval where the $H^\infty$-solution $U^\varepsilon$ take value in $\tilde{G}_1$. Note that $T_\varepsilon$ may shrink to zero as so does $\varepsilon$.

In order to show $T_\varepsilon \geq T_m$, we state our main result.

**Theorem 3.** Under the assumption 1 with $m > 2$, suppose $s > \frac{3}{2}$ is a integer, $[0, T_\varepsilon]$ is the maximal time interval where (38) has a solution $U^\varepsilon \in C([0, T_\varepsilon], H^\infty)$ with values in a convex set $\tilde{G}_1$, and $[0, T_m]$ a time interval where the asymptotic approximation $U^m_\varepsilon$ of the form (41).

Then there exists a constant $K$, independent of $\varepsilon$ but dependent on $T_m$, such that

$$
\|U^\varepsilon(t) - U^m_\varepsilon(t)\|_s \leq K\varepsilon^m,
$$

for sufficiently small $\varepsilon$ and $t \in [0, \min\{T_m, T_\varepsilon\})$.

Before proving this theorem, we remark that $m > 2$ is required by the following proof (see (77)) below). However, since

$$
U^m_\varepsilon(x, t) = U^m_\varepsilon(x, t) + \sum_{k=m_0+1}^m \varepsilon^k \left(U_k(x, t) + I(x, t/\varepsilon^2)\right),
$$

we have

$$
\left\|U^\varepsilon(t) - U^m_\varepsilon(t)\right\|_s \leq \left\|U^\varepsilon(t) - U^m_\varepsilon(t)\right\|_s + \sum_{k=m_0+1}^m \varepsilon^k \left\|U_k(x, t) + I(x, t/\varepsilon^2)\right\|_s.
$$

and thus,

$$
\left\|U^\varepsilon(t) - U^m_\varepsilon(t)\right\|_s = O(\varepsilon^{m_0+1})
$$

for any $m_0 \leq m$ provided that the coefficients of $\varepsilon^k$ in the sum are bounded.

In addition, on the basis of Theorem 3, we use exactly the same argument in Yong [36] to obtain the following.

**Theorem 4.** The hypotheses of Theorem 3 imply $T_\varepsilon \geq T_m$.

**Proof.** If $T_\varepsilon \leq T_m$, then Theorem 3 gives

$$
\left\|U^\varepsilon(T_\varepsilon) - U^m_\varepsilon(T_\varepsilon)\right\|_s \leq K\varepsilon^m.
$$

Thus, it follows from the embedding inequality that $U^\varepsilon(T_\varepsilon) \in G_0$ if $\varepsilon$ is small enough. Now we could apply Theorem 2.1 in Majda [41], beginning at the time $T_\varepsilon$, to continue this solution beyond $T_\varepsilon$. This is a contradiction. Therefore, we have $T_\varepsilon \geq T_m$.

Now we prove Theorem 3.

**The Proof of Theorem 3.** Let $T_* = \min\{T_\varepsilon, T_m\}$, then both the exact solution $U^\varepsilon$ and the approximate solution $U^m_\varepsilon$ that are defined on time interval $[0, T_\varepsilon)$ satisfy Equation (38) and

$$
\partial_t U^m_\varepsilon + \frac{1}{\varepsilon} A(U^m_\varepsilon; \varepsilon) \partial_x U^m_\varepsilon = \frac{1}{\varepsilon^2} Q(U^m_\varepsilon; \varepsilon) + R^m_\varepsilon.
$$

On $[0, T_*]$, we define

$$
V = U^\varepsilon - U^m_\varepsilon,
$$

then

$$
\partial_t V + \frac{1}{\varepsilon} A(U^\varepsilon; \varepsilon) \partial_x V = \frac{1}{\varepsilon} \left(A(U^m_\varepsilon; \varepsilon) - A(U^\varepsilon; \varepsilon)\right) \partial_x U^m_\varepsilon + \frac{1}{\varepsilon^2} \left(Q(U^\varepsilon; \varepsilon) - Q(U^m_\varepsilon; \varepsilon)\right) - R^m_\varepsilon.
$$

(67)
Applying $\partial_x^\alpha$ to the last equation for multi-index $\alpha$ satisfying $|\alpha| \leq s$, and setting $V_a = \partial^\alpha V$, we get

$$
\begin{align*}
\partial_t V_a + \frac{1}{\epsilon} A(U^\epsilon; \epsilon) \partial_x V_a &= \frac{1}{\epsilon} \{ (A(U^m_\epsilon; \epsilon) - A(U^\epsilon; \epsilon)) \partial_x U^m_\epsilon \}_a + \frac{1}{\epsilon^2} \{ (Q(U^\epsilon; \epsilon) - Q(U^m_\epsilon; \epsilon)) \}_a \\
&+ \frac{1}{\epsilon} [A(U^\epsilon; \epsilon) \partial_x V_a - [A(U^\epsilon; \epsilon) \partial_x V]_a] - (R^m_\epsilon)_a.
\end{align*}
$$

We consider the energy norm $e(V_a(x, t)) = V^T_a A_0(U^\epsilon; \epsilon) V_a$. Multiplying the last equation by $2V^T_a A_0(U^\epsilon; \epsilon)$ and integrating over $x \in \Omega$ yields

$$
\begin{align*}
\frac{d}{dt} \int e(V_a) &dx = \frac{2}{\epsilon} \int V^T_a A_0(U^\epsilon; \epsilon) \{ (A(U^m_\epsilon; \epsilon) - A(U^\epsilon; \epsilon)) \partial_x U^m_\epsilon \}_a \, dx \\
&+ \frac{2}{\epsilon^2} \int V^T_a A_0(U^\epsilon; \epsilon) \{ (Q(U^\epsilon; \epsilon) - Q(U^m_\epsilon; \epsilon)) \}_a \, dx \\
&+ \frac{2}{\epsilon} \int V^T_a A_0(U^\epsilon; \epsilon) (A(U^\epsilon; \epsilon) \partial_x V_a - [A(U^\epsilon; \epsilon) \partial_x V]_a) \, dx \\
&- 2 \int V^T_a A_0(U^\epsilon; \epsilon) \partial_x (R^m_\epsilon) \, dx \\
&+ \int V^T_a (\partial_t (A_0(U^\epsilon; \epsilon)) + \frac{1}{\epsilon} \partial_x (A_0(U^\epsilon; \epsilon) A(U^\epsilon; \epsilon))) \, V_a \, dx \\
&\triangleq I^1 + I^2 + I^3 + I^4 + I^5.
\end{align*}
$$

Next we estimate each term in the right-hand side of (68). Firstly, we have

$$
I^1 = \frac{2}{\epsilon} \int V^T_a A_0(U^\epsilon; \epsilon) \{ (A(U^m_\epsilon; \epsilon) - A(U^\epsilon; \epsilon)) \partial_x U^m_\epsilon \}_a \, dx
$$

$$
= \frac{2}{\epsilon} \int V^T_a (A_0(U^\epsilon; \epsilon) - A_0(U_0; 0)) \{ (A(U^m_\epsilon; \epsilon) - A(U^\epsilon; \epsilon)) \partial_x U^m_\epsilon \}_a \, dx
$$

$$
+ \frac{2}{\epsilon} \int V^T_a A_0(U_0; 0) \{ (A(U^m_\epsilon; \epsilon) - A(U^\epsilon; \epsilon)) \partial_x U^m_\epsilon \}_a \, dx.
$$

Recall that $U^\epsilon$ and $U_0$ take values in a compact subset $G_1$. By using Assumption 1, we can obtain

$$
A_0(U^\epsilon; \epsilon) - A_0(U_0; 0) = A_0(U^\epsilon; \epsilon) - A_0(U^\epsilon; 0) + A_0(U^\epsilon; 0) - A_0(U_0; 0)
$$

$$
= -\epsilon \int_{0}^{1} A_0(U^\epsilon; \epsilon) d\theta - |U^\epsilon - U_0| \int_{0}^{1} A_0(U^\epsilon + \theta(U_0 - U^\epsilon); 0) d\theta
$$

$$
\leq C\epsilon + C|U^\epsilon - U_0|
$$

$$
\leq C(\epsilon + |U^\epsilon - U^m_\epsilon| + |U^m_\epsilon - U_0|)
$$

$$
\leq C\epsilon + C\epsilon^2 \Delta + C\epsilon^2 B_\delta(t),
$$

where $\Delta \triangleq \|U^m_\epsilon - U^\epsilon\|_s / \epsilon^2$. Here and below, $C$ is a generic constant which may change from line to line. Since

$$
A(U^\epsilon; \epsilon) - A(U^m_\epsilon; \epsilon) = -\int_{0}^{1} A(U^\epsilon + \theta(U^\epsilon - U^m_\epsilon); \epsilon) V d\theta,
$$

we use the calculus inequalities in Sobolev spaces (3), to get that the first term $I^1$ can be bounded by $C(1 + \epsilon \Delta + \epsilon B_\delta(t)) \|V\|_2^2$ and the second term of $I^4$ can be rewritten as $\frac{2}{\epsilon} \int V^T_a A_0^2(U_0; 0) \{ (A(U^m_\epsilon; \epsilon) - A(U^\epsilon; \epsilon)) \partial_x U^m_\epsilon \}_a \, dx$.
plus the following formula

\[ \frac{2}{\varepsilon} \int V_a^H A_{01}^1(U_0; 0) \{ (A_{12}^1(U_0^m; \varepsilon) - A_{12}^1(U_0^r; \varepsilon)) \partial_x w_m \}_{a} dx \]
\[ + \frac{2}{\varepsilon} \int V_a^H A_{02}^2(U_0; 0) \{ (A_{21}^1(U_0^m; \varepsilon) - A_{21}^1(U_0^r; \varepsilon)) \partial_x w_m \}_{a} dx \]
\[ + \frac{2}{\varepsilon} \int V_a^H A_{02}^2(U_0; 0) \{ (A_{22}^1(U_0^m; \varepsilon) - A_{22}^1(U_0^r; \varepsilon)) \partial_x w_m \}_{a} dx. \]

The last two terms on the right-hand side are bounded by

\[ \frac{C}{\varepsilon} \| V \|_{s} \| V_a^H \| \leq \frac{\delta}{\varepsilon^2} \| V_a^H \|^2 + C \| V \|^2. \]

Since \( w_0 = 0 \), Assumption 1 yields \( \| w_m \|_{s} \leq C(\varepsilon + \varepsilon^2 B_s(t)) \). Therefore, we have

\[ \int V_a^H A_{01}^1(U_0; 0) \{ (A_{12}^1(U_0^m; \varepsilon) - A_{12}^1(U_0^r; \varepsilon)) \partial_x w_m \}_{a} dx \]
\[ \leq C(\varepsilon + \varepsilon^2 B_s(t)) \| V \|^2. \]

Moreover, we have

\[ A_{11}^1(U_0^m; \varepsilon) - A_{11}^1(U_0^r; \varepsilon) = A_{11}^1(u_0^m, \varepsilon) - A_{11}^1(u_0^r, \varepsilon) + A_{11}^1(u_0^m, \varepsilon) - A_{11}^1(u_0^r, \varepsilon) \]
\[ = - \int_0^1 \partial_u A_{11}^1(u_0^m + \theta(u_0^m - u_0^r), \varepsilon) V^I d\theta \]
\[ - \int_0^1 \partial_u A_{11}^1(u_0^r, \varepsilon) V^I d\theta. \]

The second integral above is easily estimated due to the appearance of \( \| V_a^H \| \). The first one can be treated due to condition \( \partial_u A_{11}^1(U_0; 0) = 0 \) in Lemma 2. Precisely, we write

\[ \partial_u A_{11}^1(u_0^m + \theta(u_0^m - u_0^r), \varepsilon) = \partial_u A_{11}^1(u_0^m + \theta(u_0^m - u_0^r), \varepsilon) - \partial_u A_{11}^1(u_0^m, \varepsilon) \]
\[ + \partial_u A_{11}^1(u_0^m; \varepsilon) - \partial_u A_{11}^1(u_0^m, \varepsilon) + \partial_u A_{11}^1(u_0^m, \varepsilon) - \partial_u A_{11}^1(u_0^m, \varepsilon) \]
\[ = \int_0^1 \partial_{u u} A_{11}^1(u, \varepsilon) \left( u_0^m - u_0 + \theta \left( u_0^m - u_0^r \right) \right) d\theta \]
\[ + \int_0^1 \partial_{uu} A_{11}^1(u_0^m; \varepsilon) w_0^m d\theta' + \int_0^1 \partial_{uu} A_{11}^1(u_0^m; \varepsilon) d\theta'. \]

Here \( u(\theta, \theta') = (1 - \theta')u_0 + \theta' u_0^m + \theta(u_0^m - u_0^r) \). The integral in (70) can be rewritten as

\[ \int_0^1 \partial_u A_{11}^1(u_0^m + \theta(u_0^m - u_0^r), \varepsilon) V^I d\theta = \int_0^1 \partial_{u u} A_{11}^1(u, \varepsilon) \left( \left( u_0^m - u_0 + \theta \left( u_0^m - u_0^r \right) \right) \right) V^I d\theta d\theta' \]
\[ + \int_0^1 \partial_{uu} A_{11}^1(u_0^m; \varepsilon) V^I d\theta d\theta' + \int_0^1 \partial_{uu} A_{11}^1(u_0^m; \varepsilon) \left( u, V^I \right) d\theta d\theta'. \]
Since \( ||w^m|| \) and \( \epsilon \) both can be bounded by \( C(\epsilon + \epsilon^2 B_0(t)) \), and

\[
||u^m_t - u_0||_2 \leq C\epsilon + C\epsilon^2 B_0(t), \quad ||u^r - u^m_t||_2 \leq C\epsilon^2 \Delta.
\]

then

\[
\int V^T_a I^a A^{11}(U_0; 0) \{ (A^{11}(U^m_r; \epsilon) - A^{11}(U^r_r; \epsilon)) \partial_u u^m_t \}_a dx \\
\leq \frac{\delta}{\epsilon} ||V^H_a|| + C(\epsilon + \epsilon^2 \Delta + \epsilon^2 B_0(t)) ||V^H||^2.
\]

Therefore,

\[
I^a_t \leq \frac{\delta}{\epsilon^2} ||V^H_a||^2 + C(1 + \epsilon \Delta + \epsilon B_0(t)) ||V^H||^2.
\]

(71)

The second item is

\[
I^a_2 = \frac{2}{\epsilon^2} \int V^T_a A_0(U^r_r; \epsilon)^{(Q(U^r_r; \epsilon) - Q(U^m_r; \epsilon))}_a dx.
\]

We first rewrite \( Q(U^r_r; \epsilon) - Q(U^m_r; \epsilon) \) as

\[
Q(U^r_r; \epsilon) - Q(U^m_r; \epsilon) = Q(U_0; 0)V + \epsilon \partial_u Q(U_0; 0)V + [Q(U^r_r; 0) - Q(U^m_r; 0) - Q(U_0; 0)V] \\
+ \epsilon^2 \partial_u Q(U^r_r; 0) - \partial_u Q(U^m_r; 0) - \partial_u Q(U_0; 0)V \\
+ [Q(U^r_r; \epsilon) - Q(U^r_r; 0) - \epsilon \partial_u Q(U^r_r; 0) - (Q(U^m_r; \epsilon) - Q(U^m_r; 0) - \epsilon \partial_u Q(U^m_r; 0))]
\]

which implies that

\[
I^a_2 = I^a_{21} + I^a_{22} + I^a_{23} + I^a_{24} + I^a_{25}
\]

with the natural correspondence for \( I^a_{21}, \ldots, I^a_{25} \). Now we estimate each of these terms. For \( I^a_{21} \), we write

\[
I^a_{21} = \frac{2}{\epsilon^2} \int V^T_a A_0(U^r_r; \epsilon) \{ Q(U_0; 0)V \}_a dx \\
= \frac{2}{\epsilon^2} \int V^T_a A_0(U_0; 0)Q(U_0; 0)V_a dx \\
+ \frac{2}{\epsilon^2} \int V^T_a A_0(U_0; 0)[\partial_u^\epsilon Q(U_0; 0)V] - Q(U_0; 0)V_a] dx \\
+ \frac{2}{\epsilon^2} \int V^T_a [A_0(U^r_r; \epsilon) - A_0(U_0; 0)] \{ Q(U_0; 0)V \}_a dx.
\]

From the structural stability conditions in Lemma 2, we can see

\[
2 \int V^T_a A_0(U_0; 0)Q(U_0; 0)V_a dx = V^T_a (A_0(U_0; 0)Q(U_0; 0) + Q(U_0; 0)A_0(U_0; 0)) V_a dx \\
\leq - c_0 ||V^H_a||^2
\]

with \( c_0 \) a positive constant. Since \( Q(U_0; 0) = \text{diag}(0, q_w(U_0, 0)) \), we have

\[
\int V^T_a A_0(U_0; 0)\partial_u^\epsilon(Q(U_0; 0)V) - Q(U_0; 0)V_a] dx = \int V^T_a A^{21}(U_0; 0)\partial_u^\epsilon(q_w(U_0; 0) V^H) - q_w(U_0; 0)V^H_a] dx \\
\leq C ||V^H_a|| \left\| \partial_u^\epsilon(q_w(U_0; 0) V^H) - q_w(U_0; 0)V^H_a \right\| \\
\leq C ||V^H_a|| \left\| q_w(U_0; 0) ||V^H||_{||x||-1} \right\| \\
\leq \frac{\delta}{4} ||V^H_a||^2 + C ||V^H||^2_{||x||-1}.
\]
Note that the above term vanishes when $\alpha = 0$. Here we use the calculus inequalities (Lemma (3)). And for the remaining terms, we will use the calculus inequalities in Sobolev spaces repeatedly. For the third item in $I^{s}_{21}$, we have

$$V_a^T \left[ A_0(U^r; \epsilon) - A_0(U_0; 0) \right] \partial^\alpha (Q_U(U_0; 0)V) = V_a^T \left[ A_0^{12}(U^r; \epsilon) - A_0^{12}(U_0; 0) \right] \partial^\alpha (q_U(U_0; 0)V^H)$$

$$+ V_a^T \left[ A_0^{22}(U^r; \epsilon) - A_0^{22}(U_0; 0) \right] \partial^\alpha (q_U(U_0; 0)V^H).$$

Using (69), we have

$$\int V_a^T \left[ A_0(U^r; \epsilon) - A_0(U_0; 0) \right] \partial^\alpha (Q_U(U_0; 0)V) \, dx \leq C(\epsilon + \epsilon^2 \Delta + \epsilon^2 B_\epsilon(t)) \| V^H_a \| \| V \|.$$
Using (69), we obtain
\[
\frac{2}{\varepsilon^2} \int V_a^T (A_0(U^\varepsilon; \varepsilon) - A_0(U_0; 0)) \{ [Q(U^\varepsilon; 0) - Q(U^m_\varepsilon; 0) - Q(U_\varepsilon; 0)V] \}_a \ dx
\leq C \left( \varepsilon + \varepsilon^2 \Delta + \varepsilon B_\varepsilon(t) \right) \| V \|_2^2,
\]
and
\[
\frac{2}{\varepsilon^2} \int V_a^{TT} A_0^{22}(U_0; 0) \{ [q(U^\varepsilon; 0) - q(U^m_\varepsilon; 0) - q(U_\varepsilon; 0)V] \}_a \ dx
\leq \frac{C}{\varepsilon^2} \| V \|_2 \| V \|_s
\leq \frac{\delta}{\varepsilon^2} \| V \|_a + C(1 + \varepsilon \Delta + \varepsilon B_\varepsilon(t)) \| V \|_2^2.
\]
Therefore,
\[
I_{23}^a \leq \frac{\delta}{\varepsilon^2} \| V \|_a + C(1 + \varepsilon \Delta + \varepsilon B_\varepsilon(t)) \| V \|_2^2.
\]
Similarly, \( I_{24}^a \) can be bounded by
\[
I_{24}^a = \frac{2}{\varepsilon^2} \int V_a A_0(U^\varepsilon; \varepsilon) \{ \varepsilon \left[ \partial_x Q(U^\varepsilon; 0) - \partial_y Q(U^m_\varepsilon; 0) - \partial_z Q(U_\varepsilon; 0)V \right] \}_a \ dx
\leq \frac{\delta}{\varepsilon^2} \| V \|_a + C(1 + \varepsilon \Delta + \varepsilon B_\varepsilon(t)) \| V \|_2^2.
\]
For the last term in \( I_2^a \), since
\[
Q(U^\varepsilon; \varepsilon) - Q(U^\varepsilon; 0) - \varepsilon \partial_x Q(U^\varepsilon; 0) - (Q(U^m_\varepsilon; \varepsilon) - Q(U^m_\varepsilon; 0) - \varepsilon \partial_x Q(U^m_\varepsilon; 0))
= \varepsilon^2 \int_0^1 \int_0^1 \partial_x^2 Q(U^\varepsilon; 0)(1 - \tau)U_\varepsilon^m; \theta \varepsilon) V \ dx \ v \ d\theta.
\]
We have
\[
I_{25}^a \leq \frac{2}{\varepsilon^2} C\varepsilon^2 \| V \|_2^2 \leq C \| V \|_2^2.
\]
Note that \( (1 + \varepsilon \Delta + \varepsilon B_\varepsilon(t)) \leq C(1 + \Delta^2 + B_\varepsilon(t)) \). Therefore,
\[
I_2^a \leq \frac{4\delta - C\varepsilon}{\varepsilon^2} \| V \|_2^2 + \frac{C}{\varepsilon^2} \| V \|_2^2 + C(1 + \Delta^2 + B_\varepsilon(t)) \| V \|_2^2.
\]
(72)

Next, we estimate \( I_3^a \). To this end, we observe
\[
I_3^a = \frac{2}{\varepsilon} \int V_a^T A_0(U^\varepsilon; \varepsilon) (A(U^\varepsilon; \varepsilon) \partial_x V - [A(U^\varepsilon; \varepsilon) \partial_x V]_a) \ dx
\leq \frac{2}{\varepsilon} \int V_a^T (A_0(U^\varepsilon; \varepsilon) - A_0(U_0; 0)) (A(U_0; 0) \partial_x V - [A(U_0; 0) \partial_x V]_a) \ dx
+ \frac{2}{\varepsilon} \int V_a^T A_0(U^\varepsilon; \varepsilon) (A(U^\varepsilon; \varepsilon) - A(U^m_\varepsilon; \varepsilon)) \partial_x V - [A(U^\varepsilon; \varepsilon) - A(U_\varepsilon; 0)] \partial_x V]_a) \ dx
+ \frac{2}{\varepsilon} \int V_a^T A_0(U_0; 0) (A(U_0; 0) \partial_x V - [A(U_0; 0) \partial_x V]_a) \ dx.
\]
Using the calculus inequalities and (69), the first two terms in the above equation can be bounded by

\[ C(1 + \epsilon \bigtriangledown + \epsilon B_r(t))\|V\|_s^2. \]

According to Lemma 2, we know that \( A^{11}(U_0; 0) \equiv 0 \). Thus, the last term of \( I_3^a \) can be rewritten as

\[
\frac{2}{\epsilon} \int V_a^T A_0(U_0; 0)(A(U_0; 0)\partial_a V_a - [A(U_0; 0)\partial_a V]_a)dx
\]

\[
= \frac{2}{\epsilon} \int V_a^T A_0^{11}(U_0; 0) (A^{12}(U_0; 0)\partial_a V^{11}_a - [A^{12}(U_0; 0)\partial_a V^{11}]]_a) dx
\]

\[
+ \frac{2}{\epsilon} \int V_a^T A_0^{12}(U_0; 0) (A^{21}(U_0; 0)\partial_a V^{12}_a - [A^{21}(U_0; 0)\partial_a V^{12}]]_a) dx
\]

\[
+ \frac{2}{\epsilon} \int V_a^T A_0^{22}(U_0; 0) (A^{22}(U_0; 0)\partial_a V^{22}_a - [A^{22}(U_0; 0)\partial_a V^{22}]]_a) dx,
\]

in which each term on the right-hand side contains \( V^{II} \). By the calculus inequalities, it is easy to see that

\[
\frac{2}{\epsilon} \int V_a^T A_0(U_0; 0)(A(U_0; 0)\partial_a V_a - [A(U_0; 0)\partial_a V]_a)dx \leq \frac{C}{\epsilon} \| V^{II} \|_a \| V \|_s
\]

\[
\leq \frac{\delta}{\epsilon^2} \| V^{II} \|^2_a + C \| V \|_s^2.
\]

This implies that

\[
I_3^a \leq \frac{\delta}{\epsilon^2} \| V^{II} \|^2_a + C(1 + \epsilon \bigtriangledown + \epsilon B_r(t))\|V\|_s^2. \tag{73}
\]

For \( I_4^a \), we have

\[
I_4^a = -2 \int V_a^T A_0(U^*; \epsilon)\partial_a (R^m_a) \ dx
\]

\[
= -2 \int V_a^T (A_0(U^*; \epsilon) - A_0(U_0; 0))\partial_a (R^m_a) \ dx - 2 \int V_a^T A_0(U_0; 0)\partial_a (R^m_a) \ dx.
\]

Using the \( \|F_m\|_s \leq C\epsilon + Ce^{-\mu t} = C(\epsilon + \epsilon^2 B_r(t)) \) in Theorem 2 and Assumption 1, we obtain

\[
-2 \int V_a^T (A_0(U^*; \epsilon) - A_0(U_0; 0))\partial_a (R^m_a) \ dx
\]

\[
\leq C\|U^* - U_0\|_s \| R^m_a \|_s \| V \|_s
\]

\[
\leq \epsilon^m (1 + \epsilon \bigtriangledown + \epsilon B_r(t)) \| Q^{U}(U_0; 0)\|_s \| F_m \|_s \| V \|_s
\]

\[
\leq C(1 + \epsilon \bigtriangledown + \epsilon B_r(t))^2 \| V \|_s^2 + C\epsilon^2m.
\]

The remaining term can be bounded by

\[
-2 \int V_a^T A_0(U_0; 0)\partial_a (R^m_a) \ dx
\]

\[
= -\int \epsilon^{-m-1} V_a^T A_0^{12}(U_0; 0)\partial_a (q_0(U_0; 0) w_{m+1})dx + \int \epsilon^{-m-1} V_a^T A_0(U_0; 0)\partial_a F_m dx
\]

\[
\leq \frac{\delta}{\epsilon^2} \| V^{II} \|^2_a + C(1 + \epsilon B_r(t))^2 \| V \|_s^2 + C\epsilon^{2m}.
\]

Therefore,

\[
I_4^a \leq \frac{\delta}{\epsilon^2} \| V^{II} \|^2_a + C(1 + \epsilon B_r(t))^2 \| V \|_s^2 + C\epsilon^{2m}. \tag{74}
\]
The last term is
\[ I_5^* = \int V_a^T \left( \partial_t (A_0(U^*; \varepsilon)) + \frac{1}{\varepsilon} \partial_x (A_0(U^*; \varepsilon) A(U^*; \varepsilon)) \right) V_a dx. \]

And we have
\[ V_a^T \partial_t A_0(U^*; \varepsilon) V_a \leq C |\partial_t V + \partial_t U^m_t|||V||^2. \]

The equation of \( V \) implies
\[ \partial_t V = -\frac{1}{\varepsilon} A(U^*; \varepsilon) \partial_x V + \frac{1}{\varepsilon} \left( A \left( U^m_t; \varepsilon \right) - A(U^*; \varepsilon) \right) \partial_x U^m_t + \frac{1}{\varepsilon^2} \left( Q(U^*; \varepsilon) - Q \left( U^m_t; \varepsilon \right) \right) - R^m_t. \]

Since \( ||V||_s = \varepsilon^2 \Delta \), we have
\[ | -\frac{1}{\varepsilon} A(U^*; \varepsilon) \partial_x V| \leq C \frac{1}{\varepsilon} ||V||_s = \varepsilon \Delta, \]
\[ |\frac{1}{\varepsilon} \left( A \left( U^m_t; \varepsilon \right) - A(U^*; \varepsilon) \right) \partial_x U^m_t| \leq C \frac{1}{\varepsilon} ||V||_s = \varepsilon \Delta, \]
\[ |R^m_t| \leq C \varepsilon^{m-1} \leq C. \]

Moreover,
\[ Q(U^*; \varepsilon) - Q \left( U^m_t; \varepsilon \right) = Q(U^*; \varepsilon) - Q \left( U^m_t; \varepsilon \right) - \partial_t Q \left( U^m_t; \varepsilon \right) V \]
\[ + (\partial_U Q \left( U^m_t; \varepsilon \right) - \partial_U Q(U^*; \varepsilon))V \]
\[ + (\partial_U Q(U^*; \varepsilon) - \partial_U Q(U^*; 0)V + \partial_U Q(U^*; 0)V. \]

From Lemma 2, we obtain that
\[ \partial_U Q(U^*; 0) = \begin{pmatrix} 0 & 0 \\ 0 & q_w(U^*; 0) \end{pmatrix}, \quad \partial_U Q(U^*; 0)V = \begin{pmatrix} 0 \\ q_w(U^*; 0)V \end{pmatrix}. \]

Then
\[ |Q(U^*; \varepsilon) - Q \left( U^m_t; \varepsilon \right)| \leq C \left( ||V||^2_s + \||U^m_t - U^*||_\infty \|V\|_s + \varepsilon ||V||_s + \|V^H\|_s \right) \]
\[ \leq C \left( \||U^m_t - U^*\|_s^2 + \|V\|_s^2 + \|V^H\|_s + \varepsilon^2 \right) \]
\[ \leq C \varepsilon^2 (1 + \varepsilon B_c(t))^2 + \varepsilon^4 \Delta^2 + C \||V^H\||_s. \]

Thus,
\[ |\partial_t V| \leq C(1 + \Delta^2 + B_c(t)) + \frac{C}{\varepsilon^2} \||V^H\||_s. \]

Noting \( ||\partial_t U^m^s||_s \leq C + CB_c(t) \) from Assumption 1, we obtain
\[ \int V_a^T \partial_t A_0(U^*; \varepsilon) V_a dx \leq C(1 + \Delta^2 + B_c(t)) ||V||^2_s + \frac{C}{\varepsilon^2} \||V^H||_s ||V||_s^2. \]

Setting \( \hat{A}(U^*; \varepsilon) = A_0(U^*; \varepsilon) A(U^*; \varepsilon) \), the second term of \( I_5^* \) can be treated as
\[ \frac{1}{\varepsilon} \int V_a^T \partial_x (A_0(U^*; \varepsilon) A(U^*; \varepsilon)) V_a dx = \frac{1}{\varepsilon} \int V_a^T \partial_x \hat{A}_1(U^*; \varepsilon) V_a dx + \frac{2}{\varepsilon} \int V_a^T \partial_x \hat{A}_1(U^*; \varepsilon) V_a^H dx \]
\[ + \frac{1}{\varepsilon} \int V_a^T \partial_x \hat{A}^2(U^*; \varepsilon) V_a^H dx. \]
Obviously, the last two terms are bounded by

\[ \frac{C}{\varepsilon} \|V\|_s \|V^H\| \leq \frac{\delta}{\varepsilon^2} \|V^H\|^2 + C\|V\|^2. \]

Since \( A^{11}(U_0; 0) = 0 \) and \( A_0(U_0; 0) \) is a block diagonal matrix, we have \( \hat{A}^{11}(U_0; 0) = 0 \). The first term can be bounded by

\[ \frac{1}{\varepsilon} \int V^H \partial_x \hat{A}^{11}(U^\varepsilon; \varepsilon) \, dx = \frac{1}{\varepsilon} \int V^H \partial_x (\hat{A}^{11}(U^\varepsilon; \varepsilon) - \hat{A}^{11}(U_0; 0)) \, dx \]
\[ \leq C \frac{1}{\varepsilon} \|\partial_x(U^\varepsilon - U_0)\|_\infty \|V\|^2 \]
\[ \leq C(1 + \varepsilon \Delta + \varepsilon B_t(t)) \|V\|^2. \]

Therefore,

\[ I_s^\varepsilon \leq \frac{\delta}{\varepsilon^2} \|V^H\|^2 + C(1 + \Delta^2 + B_t(t)) \|V\|^2 + \frac{C}{\varepsilon^2} \|V^H\|_s \|V\|^2. \]  \tag{75}

Substituting (71)–(75) into inequality (68) yields

\[ \frac{d}{dt} \int \sigma(V_a) \, dx + \frac{c_0 - 8\delta}{\varepsilon^2} \|V^H\|^2 \leq \frac{C}{\varepsilon^2} \|V^H\|_{|\alpha|-1} + C(1 + \Delta^2 + B_t(t)) \|V\|^2 + \frac{C}{\varepsilon^2} \|V^H\|_s \|V\|^2 + C\varepsilon^{2m}. \]

Let \( \delta \) to be sufficiently small such that \( c_1 = c_0 - 8\delta \in (0, c_0) \), then we have

\[ \frac{d}{dt} \int \sigma(V_a) \, dx + \frac{c_1}{\varepsilon^2} \|V^H\|^2 \leq \frac{C}{\varepsilon^2} \|V^H\|_{|\alpha|-1} + C(1 + \Delta^2 + B_t(t)) \|V\|^2 + \frac{C}{\varepsilon^2} \|V^H\|_s \|V\|^2 + C\varepsilon^{2m}. \]  \tag{76}

Recall \( \|V^H\|^2_{-1} = 0 \), when \( |\alpha| = 1 \), we see that \( \frac{C}{\varepsilon^2} \|V^H\|_{|\alpha|-1} \) on the right-hand side of (76) can be controlled by \( \frac{c_1}{\varepsilon^2} \|V^H\|^2 \). More generally, let \( \eta \in (0, 1) \). Multiplying (76) by \( \eta^{|\alpha|} \) and summing up the equalities for all index \( \alpha \) with \( |\alpha| \leq s \) yield

\[ \frac{d}{dt} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \int \sigma(V_a) \, dx + \frac{c_1}{\varepsilon^2} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \|V^H\|^2 \leq \frac{C}{\varepsilon^2} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \|V^H\|_{|\alpha|-1} + C(1 + \Delta^2 + B_t(t)) \|V\|^2 \]
\[ + \frac{C}{\varepsilon^2} \|V^H\|_s \|V\|^2 + C\varepsilon^{2m}. \]

in which \( C \) is independent of \( \eta \). Let \( \eta \) be suitably small. Then

\[ \frac{C}{\varepsilon^2} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \|V^H\|_{|\alpha|-1} \leq \frac{c_1}{2\varepsilon^2} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \|V^H\|^2 \]

and

\[ \frac{c_1 \eta^{|\alpha|} \|V^H\|^2}{2\varepsilon^2} \leq \frac{c_1}{2\varepsilon^2} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \|V^H\|^2. \]

Therefore,

\[ \frac{d}{dt} \sum_{|\alpha| \leq s} \eta^{|\alpha|} \int \sigma(V_a) \, dx + \frac{c_1 \eta^{|\alpha|} \|V^H\|^2}{2\varepsilon^2} \leq C(1 + \Delta^2 + B_t(t)) \|V\|^2 + \frac{C}{\varepsilon^2} \|V^H\|_s \|V\|^2 + C\varepsilon^{2m}. \]
By the Young inequality, we have
\[ C\|V^H\|_0\|V\|^2 \leq \frac{c_1\eta^4}{4}\|V^H\|_0^4 + \frac{c_2}{c_1\eta^2}\|V\|^4 \leq \frac{c_1\eta^4}{4}\|V^H\|_0^2 + \frac{c_2}{c_1\eta^2}\varepsilon^2 \Delta \|V\|^2. \]

Thus,
\[ \frac{d}{dt} \sum_{|\alpha| \leq s} \eta^{4|\alpha|} \int_{\Omega} e^{(V_a)}dx + \frac{c_1\eta^4}{4\varepsilon^2}\|V^H\|_0^2 \leq C \left( 1 + \varepsilon^2 + B_r(t) + \frac{1}{\eta^{2s}} \right) \|V\|^2 + C\varepsilon^{2m}. \]

Note that
\[ C^{-1}\|V_a\|^2 \leq e(V_a) \leq C|V_a|^2. \]

Now we fix \( \eta > 0 \). Integrating this inequality over \([0, T_m]\) and noting that \( \sum_{|\alpha| \leq s} \eta^{4|\alpha|} \int_{\Omega} e(V_a)dx \) is equivalent to \( \|V_a(T)\|^2 \), we use \( \|V(0)\|_0 = O(\varepsilon^m) \) to obtain
\[ \|V_a(T)\|^2 + \frac{1}{\varepsilon^2} \int_0^T \|V^H(t)\|_0^2 dt \leq CT\varepsilon^{2m} + \int_0^T C(1 + \varepsilon^2 + B_r(t))\|V(t)\|_0^2 dt. \]

Then
\[ \|V(T)\|_0^2 \leq CT\varepsilon^{2m} + \int_0^T C(1 + \varepsilon^2 + B_r(t))\|V(t)\|_0^2 dt. \]

We apply Gronwall’s lemma to the above equation to get
\[ \|V(T)\|_0^2 \leq CT_m\varepsilon^{2m-4} \exp \left[ \int_0^T C(1 + \varepsilon^2 + B_r(t))dt \right] \tag{77}. \]

Since \( \|V\|_0 = \varepsilon^2 \Delta \), it follows from above equation that
\[ \Delta(T)^2 \leq CT_m\varepsilon^{2m-4} \exp \left[ \int_0^T C(1 + \varepsilon^2 + B_r(t))dt \right] \equiv \Phi(T). \]

Thus,
\[ \Phi'(T) = C(1 + \varepsilon^2 + B_r(t))\Phi(T) \leq C(1 + B_r(t))\Phi(T) + C\Phi^2(T) \]

because of \( \int_0^T B_r(t) \leq \frac{1}{2\varepsilon} \). Applying the nonlinear Gronwall-type inequality in Lemma 5 to the last inequality yields
\[ \Delta(T)^2 \leq \sup_{[0,T_m]} \Phi(T) \leq C \exp \left[ \int_0^T C(1 + B_r(t))dt \right] \]

if we assume \( m > 2 \) and choose \( \varepsilon \) so small that \( \Phi(0) = CT_m\varepsilon^{2m-4} < \delta \). Then there exists a constant \( C \), independent of \( \varepsilon \), such that
\[ \Delta(T) \leq C \]

for any \( T \in [0, \min\{T_\varepsilon, T_m\}] \). Because of (77), there exists a constant \( K > 0 \), independent of \( \varepsilon \), such that
\[ \|V\|_0 \leq K\varepsilon^n. \]

This completes the proof of Theorem 3.
5 | CONCLUSIONS

In this work, we study the structural stability condition for the radiation hydrodynamics system, which is governed by Euler equation coupled with the HMP$_N$ moment model [10] of radiation transport equation. The resultant coupling system is a first-order partial differential equations with stiff source. The stability theory for hyperbolic relaxation systems [36, 46] has been verified for numerous well-known systems of PDEs in physics, and it can also be used to analyze the compatibility of hyperbolic relaxation systems [24]. This work further demonstrates the universality and the significance of the stability theory for hyperbolic relaxation systems [36, 46]. On the basis of the structural stability condition, we verify the nonrelativistic limit by combining an energy method with a formal asymptotic analysis.

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Zhiting Ma: Formal analysis, validation, and writing-original draft. Wen-An Yong: Methodology, validation, and writing-review and editing.

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In this Appendix, we prove \( \tilde{\mathcal{A}}^{11}(\tilde{U}_{eq};0) = 0 \) and \( \tilde{\mathcal{A}}_{b}^{11}(\tilde{U}_{eq};0) = 0 \) which is established in Lemma 2.

Take value on the equilibrium state and set \( \varepsilon = 0 \) in (36). We can obtain

\[
\tilde{\mathcal{A}}(\tilde{U}_{eq};0) = D_\varepsilon \tilde{U}(\tilde{U}_{eq};0) \left( \begin{array}{c}
0_{3\times 3} \\
0_{(N+1)\times 3} \\
\end{array} \right) \left( \begin{array}{c}
0_{3\times(N+1)} \\
D^{-1} M D(\tilde{U}_{eq}) \\
\end{array} \right)^{-1}(D_\varepsilon \tilde{U})(\tilde{U}_{eq};0),
\]

(A1)

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where \( \hat{\bar{D}}^{-1}\hat{\bar{M}}\hat{\bar{D}}(\hat{\bar{U}}_{eq}) \in \mathbb{R}^{(N+1)\times(N+1)} \). From the above discussion in Section 4, we know that

\[
D_U \hat{\bar{U}}(\hat{\bar{U}}_{eq}; 0) = \begin{pmatrix} P_1 & 0 \\ 0 & I_{N\times N} \end{pmatrix},
\]

in which \( P_1 \in \mathbb{R}^{4\times 4} \) defined in (28). Then \( \hat{\bar{A}}(\hat{\bar{U}}_{eq}; 0) \) can be rewritten as

\[
\hat{\bar{A}}(\hat{\bar{U}}_{eq}; 0) = \begin{pmatrix} P_1 & 0 \\ 0 & I_{N\times N} \end{pmatrix} \begin{pmatrix} 0_{3\times 3} & 0_{3\times 1} & 0_{3\times N} \\ 0_{1\times 3} & g_1 & g_2 \\ 0_{N\times 3} & g_3 & g_4 \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & I_{N\times N} \end{pmatrix},
\]

where \( g_1 \) is the first element in the upper left corner of \( \hat{\bar{D}}^{-1}\hat{\bar{M}}\hat{\bar{D}}(\hat{\bar{U}}_{eq}) \), \( g_2 \in \mathbb{R}^{1\times 1} \), \( g_3 \in \mathbb{R}^{N\times 1} \), \( g_4 \in \mathbb{R}^{N\times N} \) are corresponding block of matrix \( \hat{\bar{D}}^{-1}\hat{\bar{M}}\hat{\bar{D}}(\hat{\bar{U}}_{eq}) \).

Next, we calculate \( g_1 \) and the first component of vector \( g_3 \). Note that \( \hat{\bar{D}}(\hat{\bar{U}}_{eq}) \) is diagonal matrix which showed in (23), \( \hat{\bar{M}}(\alpha) = \hat{\bar{A}}^{-1}(\mu \hat{\bar{\Phi}}^{[\alpha]}(\mu), \hat{\bar{\Phi}}^{[\alpha]}(\mu)^T)\hat{\bar{\eta}}_N^{[\alpha]} \) and \( \hat{\bar{\lambda}}(\alpha) = \text{diag}(\hat{\bar{\kappa}}_{0,0}, \hat{\bar{\kappa}}_{1,1}, \cdots, \hat{\bar{\kappa}}_{N,N}) \) due to (15). It follows from the definition of the inner product of \( \hat{\bar{\eta}}_N^{[\alpha]} \) (13) that

\[
\hat{\bar{M}}_{11}(\alpha) = \hat{\bar{\kappa}}_{0,0}^{-1}(\alpha) \int \mu \hat{\bar{\Phi}}^{[\alpha]}(\mu)\hat{\bar{\Phi}}^{[\alpha]}(\mu)/\hat{\bar{\psi}}^{[\alpha]}(\mu)d\mu
\]

\[= \hat{\bar{\kappa}}_{0,0}^{-1}(\alpha) \int \mu \hat{\bar{\psi}}^{[\alpha]}(\mu)d\mu = \hat{\bar{\kappa}}_{0,0}^{-1}(\alpha)\hat{\bar{\kappa}}_{1,0}(\alpha), \]

\[
\hat{\bar{M}}_{21}(\alpha) = \hat{\bar{\kappa}}_{1,1}^{-1}(\alpha) \int \mu \hat{\bar{\Phi}}^{[\alpha]}(\mu)\hat{\bar{\Phi}}^{[\alpha]}(\mu)/\hat{\bar{\psi}}^{[\alpha]}(\mu)d\mu
\]

\[= \hat{\bar{\kappa}}_{1,1}^{-1}(\alpha) \int \mu \hat{\bar{\psi}}^{[\alpha]}(\mu)d\mu = \hat{\bar{\kappa}}_{1,1}^{-1}(\alpha)\hat{\bar{\kappa}}_{1,1}(\alpha) = 1. \]

Hence, \( g_1 = \hat{\bar{M}}_{11}(0) = \hat{\bar{\kappa}}_{0,0}^{-1}(0)\hat{\bar{\kappa}}_{1,0}(0) = 0 \). The first components of \( g_3 \) is \((-2b(\theta)\hat{\bar{\Sigma}}_{21} \beta_0^{-1} \neq 0 \) Thus \( g_3 \) is not zero. Therefore

\[
\hat{\bar{A}}(\hat{\bar{U}}_{eq}; 0) = \begin{pmatrix} P_1 & 0 \\ 0 & I_{N\times N} \end{pmatrix} \begin{pmatrix} 0_{3\times 3} & 0_{3\times 1} & 0_{3\times N} \\ 0_{1\times 3} & g_1 & g_2 \\ 0_{N\times 3} & g_3 & g_4 \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & I_{N\times N} \end{pmatrix}
\]

\[
= \begin{pmatrix} 0_{4\times 4} & P_1 \begin{pmatrix} 0_{3\times N} \\ g_2 \\ g_3 \end{pmatrix} P_1^{-1} \\ 0_{N\times 3} & g_3 \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & I_{N\times N} \end{pmatrix}
\]

(A2)

Since \( g_3 \) is not zero and the matrix \( P_1 \) is invertible, the rank of the matrix \( \begin{pmatrix} 0_{N\times 3} & g_3 \end{pmatrix} P_1^{-1} \) is 1. Divided the matrix \( \hat{\bar{A}} \) as follows

\[
\hat{\bar{A}}(\hat{\bar{U}}; \epsilon) \triangleq \begin{pmatrix} \hat{\bar{A}}^{11}(\hat{\bar{U}}; \epsilon) & \hat{\bar{A}}^{12}(\hat{\bar{U}}; \epsilon) \\ \hat{\bar{A}}^{21}(\hat{\bar{U}}; \epsilon) & \hat{\bar{A}}^{22}(\hat{\bar{U}}; \epsilon) \end{pmatrix},
\]

where \( \hat{\bar{A}}^{11}(\hat{\bar{U}}; \epsilon) \in \mathbb{R}^{3\times 3} \), \( \hat{\bar{A}}^{12}(\hat{\bar{U}}; \epsilon) \in \mathbb{R}^{3\times (N+1)} \), \( \hat{\bar{A}}^{21}(\hat{\bar{U}}; \epsilon) \in \mathbb{R}^{(N+1)\times 3} \), \( \hat{\bar{A}}^{22}(\hat{\bar{U}}; \epsilon) \in \mathbb{R}^{(N+1)\times (N+1)} \). Then it follows from (A2) that \( \hat{\bar{A}}^{11}(\hat{\bar{U}}_{eq}; 0) = 0 \) for all \( \hat{\bar{U}}_{eq} \in \hat{\bar{G}}_{eq} \). Meanwhile, \( \hat{\bar{A}}^{21}(\hat{\bar{U}}_{eq}; 0) \) is the matrix formed by the first three columns of the following \((N + 1) \times 4\) matrix

\[
\begin{pmatrix} 0_{1\times 4} \\ 0_{N\times 3} & g_3 \end{pmatrix} P_1^{-1}
\]

Thus, \( \hat{\bar{A}}^{21}(\hat{\bar{U}}_{eq}; 0) \) is not full-rank matrix.
Furthermore, we analyze $\tilde{A}_u^{11} (\tilde{U}_{eq}; 0)$. Firstly, we show that $\tilde{A}_u^{11} (\tilde{U}_{eq}; 0) = 0$. For any $u = (\rho, \rho v, \rho E)$, we have

$$
\tilde{A}_u (\tilde{U}_{eq}; 0) = \partial_u (D_U \tilde{U}) \left( \begin{array}{cc} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{array} \right) (D_U \tilde{U})^{-1} + D_U \tilde{U} \left( \begin{array}{cc} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{array} \right) \partial_u (D_U \tilde{U})^{-1}.
$$

(A3)

From the expression of $D_U \tilde{U}$ in (37) and $\theta = \theta(u)$, we know that

$$
\partial_u (D_U \tilde{U}) (\tilde{U}_{eq}; 0) = \begin{pmatrix} 0_{3 \times 3} \\ Y_1 \end{pmatrix},
$$

with $Y_1$ is a non-zero matrix in $\mathbb{R}^{(N+1) \times 3}$. Therefore, we have

$$
\partial_u (D_U \tilde{U}) (\tilde{U}_{eq}; 0) \begin{pmatrix} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{pmatrix} (\tilde{U}_{eq}; 0) = \begin{pmatrix} 0_{3 \times 3} & Y_1 \\ Y_1 & 0_{(N+1) \times (N+1)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{pmatrix} (\tilde{U}_{eq}; 0) = 0
$$

In the second term in (A3), since $D^{-1}M\overline{D}$ is only depend on $w$, so $\partial_u (D^{-1}M\overline{D}) = 0$, which yields the second term vanish. From above discussion, we know that

$$
D_U \tilde{U} (\tilde{U}_{eq}; 0) \begin{pmatrix} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{pmatrix} (\tilde{U}_{eq}; 0) = \begin{pmatrix} 0_{4 \times 4} \\ P_1 (0_{3 \times N} \begin{pmatrix} g_2 \\ g_4 \end{pmatrix}) \\ 0_{N \times 3} \begin{pmatrix} g_2 \\ g_4 \end{pmatrix} \end{pmatrix}. 
$$

A tedious calculation shows that the matrix of inverse transformation is

$$
(D_U \tilde{U})^{-1} (\tilde{U}; 0) = \begin{pmatrix} 0_{3 \times 3} & Y_1 \\ Y_1 & 0_{(N+1) \times (N+1)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{pmatrix} (\tilde{U}_{eq}; 0) = \begin{pmatrix} 0_{4 \times 4} \\ P_1 (0_{3 \times N} \begin{pmatrix} g_2 \\ g_4 \end{pmatrix}) \end{pmatrix}.
$$

(A4)

Thus we can obtain

$$
\partial_u (D_U \tilde{U})^{-1} (\tilde{U}_{eq}; 0) = \begin{pmatrix} Y_2 \\ 0_{N \times N} \end{pmatrix},
$$

with $Y_2 \in \mathbb{R}^{4 \times 4}$. So the third term of (A3) can be rewritten as

$$
D_U \tilde{U} (\tilde{U}_{eq}; 0) \begin{pmatrix} 0 & 0 \\ 0 & D^{-1}M\overline{D} \end{pmatrix} (\tilde{U}_{eq}; 0) \partial_u (D_U \tilde{U})^{-1} (\tilde{U}_{eq}; 0) = \begin{pmatrix} 0_{4 \times 4} \\ P_1 (0_{3 \times N} \begin{pmatrix} g_2 \\ g_4 \end{pmatrix}) \end{pmatrix} \begin{pmatrix} Y_2 \\ 0_{N \times N} \end{pmatrix} = \begin{pmatrix} 0_{4 \times 4} \\ 0_{N \times N} \end{pmatrix}
$$

with $Y_3$ is corresponding matrix. Thus that the $3 \times 3$ block in the upper left corner of matrix $\tilde{A}_u (\tilde{U}_{eq}; 0)$ is zero. This means $\tilde{A}_u^{11} (\tilde{U}_{eq}; 0) = 0$ for any $u = (\rho, \rho v, \rho E)$.

Since $\tilde{u} = \tilde{u}(u, w)$, we have

$$
\partial_{\tilde{u}} \tilde{A}_u^{11} (\tilde{U}_{eq}; 0) = \partial_{\tilde{u}} \tilde{A}_u^{11} (\tilde{U}_{eq}; 0) \frac{\partial u}{\partial \tilde{u}} + \partial_w \tilde{A}_u^{11} (\tilde{U}_{eq}; 0) \frac{\partial w}{\partial \tilde{u}} = \partial_w \tilde{A}_u^{11} (\tilde{U}_{eq}; 0) \frac{\partial w}{\partial \tilde{u}}.
$$
According to expression of $(D_U \tilde{U})^{-1}$ in (A4), we know $\frac{dw}{d\tilde{u}}$ are zero except for $\frac{df_0}{d\tilde{u}}$. Thus

$$\partial_u \tilde{A}^{11} (\tilde{U}_{eq}; 0) = \partial_u \tilde{A}^{11} (\tilde{U}_{eq}; 0) \frac{dw}{d\tilde{u}} = \partial_{f_0} \tilde{A}^{11} (\tilde{U}_{eq}; 0) \frac{df_0}{d\tilde{u}}.$$  

Thanks to the equations of hydrodynamical variables (32), we set

$$\tilde{F} (\tilde{U}; \epsilon) = (\epsilon \rho v, \epsilon (\rho v^2 + p + \kappa_{2,2} f_2 + \kappa_{2,0} f_0), \epsilon (\rho E v + pv) + \kappa_{1,0} f_0)^T,$$

with $\kappa_{2,2}, \kappa_{2,0}, \kappa_{1,0}$ are function of $\alpha$ such that $\tilde{A}^{11} (\tilde{U}; \epsilon) = \partial_u F (\tilde{U}; \epsilon)$. Thus

$$\partial_{f_0} \tilde{A}^{11} (\tilde{U}; \epsilon) = \partial_{f_0} \left( \partial_u F (\tilde{U}; \epsilon) \right) = \partial_u \partial_{f_0} F (\tilde{U}; \epsilon) = \partial_u (0, \epsilon \kappa_{2,0} (\alpha), \kappa_{1,0} (\alpha))^T.$$  

And since $\frac{dw}{d\tilde{u}}$ are zero except for $\frac{df_0}{d\tilde{u}}$, we have

$$\partial_u \kappa_{1,0} (\alpha) = \partial_u \kappa_{1,0} (\alpha) \frac{du}{d\tilde{u}} + \partial_u \kappa_{1,0} (\alpha) \frac{dw}{d\tilde{u}} = \partial_{f_0} \kappa_{1,0} (\alpha) \frac{df_0}{d\tilde{u}} = 0.$$

Similarly $\partial_u \kappa_{2,0} (\alpha) = 0$. Therefore $\partial_u \tilde{A}^{11} (\tilde{U}_{eq}; 0) = 0$.

In conclude, on all equilibrium state $\tilde{U}_{eq}$, we see that

$$\tilde{A}^{11} (\tilde{U}_{eq}; 0) = 0, \quad \partial_u \tilde{A}^{11} (\tilde{U}_{eq}; 0) = 0.$$  

Moreover, $\tilde{A}^{21} (\tilde{U}_{eq}; 0)$ is not full–rank matrix.