Resolving the issue of branched Hamiltonian in Lanczos-Lovelock gravity

Soumendranath Ruz\textsuperscript{1} and Ranajit Mandal\textsuperscript{2}
\textit{Dept.of Physics, University of Kalyani, Nadia, India - 741235}

Subhra Debnath\textsuperscript{3} and Abhik Kumar Sanyal\textsuperscript{4}
\textit{Dept.of Physics, Jangipur College, Murshidabad, India - 742213}

The Hamiltonian constraint \( H = NH = 0 \), defines a diffeomorphic structure on spatial manifolds by the lapse function \( N \) in general theory of relativity. However, it is not manifest in Lanczos-Lovelock gravity since the expression for velocity in terms of the momentum is multivalued and thus the Hamiltonian is a branch function of momentum. Here we propose an extended theory of Lanczos-Lovelock gravity to construct a unique Hamiltonian which results in manifest diffeomorphic invariance and canonical quantization.

I. INTRODUCTION

It is well known that gravity is non-renormalizable in 4 dimension. However, in dimension \( D \geq 5 \) a renormalized and unitary theory of gravity has been realized following Lanczos-Lovelock model \textsuperscript{[1]} at least at the one-loop level. Lovelock invariants are second-rank symmetric tensors with vanishing covariant derivatives and depend only on the metric and its first and second derivatives. Thus, they have the same general properties as the Einstein tensor itself, but are higher order in curvature. The Lovelock invariants are constructed in such a way that under variation, no higher than second derivatives appear in the field equations avoiding the problem of unitarity. However, in \( D = 4 \), Lovelock invariant is simply the Ricci scalar (R) admitting nothing other than ordinary general relativity. Nevertheless, in \( D \geq 5 \), it admits Gauss-Bonnet term in addition and the corresponding theory is known as Lanczos-Lovelock gravity. In dimensions \( D \geq 6 \) two more Lovelock invariants are admissible in ADD \textsuperscript{[2]} formalism containing 8 and 25 terms respectively, which are not our present concern. Interestingly, the Lovelock invariants are found to be just the forms admissible by the higher order curvature terms generated in perturbative critical string theory \textsuperscript{[3]}. This is what one should expect if string theories are to avoid all these troublesome ghost and unitarity issues. The Lanczos-Lovelock action in \( D \) dimension reads,

\begin{equation}
A_1 = \int \sqrt{-g} \, d^D x \left[ \frac{R - 2\Lambda_0}{16\pi G_N} + \gamma \mathcal{G}_D \right] + \Sigma_{R_D} + \Sigma_{\mathcal{G}_D},
\end{equation}

where, \( G_N \) is the Newton’s gravitational constant, \( \mathcal{G}_D(= R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}) \) is the Gauss-Bonnet term and

\begin{equation}
\Sigma_{R_D} = \frac{1}{8\pi G_N} \int_{\partial V} d^{D-1} x \sqrt{h} K,
\end{equation}

\begin{equation}
\Sigma_{\mathcal{G}_D} = 4\gamma \int_{\partial V} \sqrt{h} d^{D-1} x \left( 2G_{ij}K^{ij} + \frac{K}{3} \right)
\end{equation}

are the boundary terms corresponding to Ricci scalar term \( R \) and Gauss-Bonnet term \( \mathcal{G}_D \) respectively in \( D \) dimension. In \( D = 5 \) the symbol \( K \) stands for \( K = (K^3 - 3KK^{ij}K_{ij} + 2K^{ij}K_{ik}K^k_j) \) where, \( K \) is the trace of the extrinsic curvature tensor \( K_{ij} \) and \( G_{ij} \) is the Einstein tensor built out of the induced metric \( h_{ij} \) on the boundary. Very importance of such an action in the context of astrophysics and cosmology has already been established. For example, asymptotically dS/AdS together with flat solutions \textsuperscript{[5]} and conformal anomaly from higher derivative gravity in AdS/CFT correspondence \textsuperscript{[6]} have been realized. For the zero-temperature background, solutions both in pure Gauss-Bonnet gravity and that with non-trivial matter have been found in asymptotically Lifshitz spacetimes in five dimensions \textsuperscript{[7]}. Static solution corresponding to such an action in vacuum has been presented \textsuperscript{[8]} and Birkhoffs theorem has been established \textsuperscript{[9]}. It has also been studied in the context of steep inflationary scenario \textsuperscript{[10,11]} and Friedmann-like solutions have been realized \textsuperscript{[10,11]}.

However, while performing canonical analysis of Lovelock action in 5 dimension, Deser and Franklin noticed that \textsuperscript{[10]} the presence of cubic kinetic terms and quadratic constraints make the theory intrinsically nonlinear. Even its linearized version is cubic rather than quadratic. Such a pronounced exotic behaviour of the action does not allow Hamiltonian formulation of Lanczos-Lovelock gravity following conventional Legendre transformation. As a result diffeomorphic invariance is not manifest and standard canonical formulation of the theory is not possible. Such a situation arises as the Lagrangian is quartic in velocities and as a result the expression for velocities are multivalued functions of momentum, resulting in the so called multiply branched Hamiltonian with cusps. This makes classical solution unpredictable as at any time one can jump from one branch of the Hamiltonian to the other. Further, the momentum does not provide a complete set of commut-
ing observables resulting in non-unitary time evolution of quantum states. Thus the main aim of constructing Lanczos-Lovelock gravity falls short perturbatively. Although, it shows unitary time evolution of quantum states, when expanded perturbatively about the flat Minkowski background, non-perturbatively, the situation is awesome. Our aim is to explore the associated problem in the present Robertson-Walker (R-W) minisuperspace and to show that an additional scalar curvature squared term \( R^2 \) term alleviates the problem, removing all the pathological behaviour of Lanczos-Lovelock gravity.

II. PROBLEM ASSOCIATED WITH CANONIZATION

In the \( D \) dimensional Robertson-Walker minisuperspace

\[
ds^2 = -N^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 + dX_3^2 \right], \quad (3)
\]

where \( \delta = (D-4) \) is the extra dimensions and \( d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2) \), the expressions for the Ricci scalar, \( R \) and the Gauss-Bonnet term, \( \mathcal{G}_D \) are

\[
R = \frac{D-1}{N^2} \left[ \frac{2a}{a} + \frac{(D-2)\dot{a}^2}{a^2} - \frac{2aN}{aN} \right] + \frac{6k}{a^2} \quad (4a)
\]

\[
\mathcal{G}_D = \frac{(D-3)}{a^3 N^2} \left[ (D-4) \frac{\dot{a}^2}{a} \left( \Delta_1 \frac{\dot{a}^2}{a^2} + 12k \right) + 4\Delta_1 \frac{\dot{a}^2}{N} \right] \quad (4b)
\]

where \( \Delta_1 = (D-1)(D-2) \). Plugging in the above expressions in action (1) and after cancelling the total derivative terms appearing under integration by parts with the supplementary boundary terms as usual, it reads

\[
A_1 = \int \left[ \frac{\alpha(D-3)}{\kappa} \left( 3kN - \Delta_1 \frac{\dot{a}^2}{2N} - Na^2 \Lambda_0 \right) - \gamma \Delta_2 \frac{\dot{a}^2}{N} \left( \Delta_1 \frac{\dot{a}^2}{3N^2} + 12k \right) \right] dt \quad (5)
\]

where, \( \Delta_2 = (D-3)(D-4) \) and \( \kappa = 8\pi G_N \). Therefore the canonical momenta are

\[
p_N = 0 \quad (6a)
\]

\[
p_a = -\frac{\Delta_1 \alpha(D-3)}{\kappa N} - \frac{4\gamma \Delta_2 \dot{a}}{N a^{(5-D)}} \left( \Delta_1 \frac{\dot{a}^2}{3N^2} + 6k \right) \quad (6b)
\]

Corresponding Hamiltonian, obtained from \( N \) variation equation is

\[
H_c = N \left[ \frac{\alpha(D-3)}{\kappa} \left( \frac{\Delta_1}{2} \frac{\dot{a}^2}{N^2} + 3k - a^2 \Lambda_0 \right) + \gamma \Delta_2 \frac{\dot{a}^2}{N^2} \left( \Delta_1 \frac{\dot{a}^2}{N^2} + 12k \right) \right] \quad (7)
\]

Although it is unique in terms of the velocities, there indeed exists three different Hamiltonians in terms of the phase space variables, since the momentum (6b) appears in third degree algebraic equation in \( \dot{a} \) and one can have either one or three real values of \( \dot{a} \) corresponding to a given value of momentum \( p_a \). This problem was noticed by Deser and Franklin, who stated it as “a most un-Hamiltonian system” [9]. Such multivalued Hamiltonian with cusps, usually called the branched Hamiltonian, makes the classical theory unpredictable and does not allow standard canonical formulation of the theory. Further, as energy has to be an observable, the momentum does not ensure a complete set of commuting observable and finally, the theory also suffers from the disease of having non-unitary time evolution of the quantum state. Last point is most important in the context of Lanczos-Lovelock gravity. Expanding the Lanczos-Lovelock action in the perturbative series about the linearized theory reveals that it is free from ghosts and thus is unitary. However, non-perturbatively the theory lacks unitary time evolution, as mentioned. Such unpleasant issue arising out of branched Hamiltonian was addressed long ago [12]. Starting from a toy model, the authors [12] had shown that in the path integral formalism one can associate a perfectly smooth quantum theory which possesses a clear operator interpretation and a smooth, deterministic, classical limit. Nevertheless, it puts up question on the standard classical variational principle and of course on the canonical quantization scheme. The same issue has also been addressed by several authors in the recent years [13, 14]. However, in order to solve the problem they had to tinker with some fundamental aspects like Legendre transformation and the usual Heisenberg commutation relations. Therefore none of these techniques are fully developed or rigorous. In this connection, we adopt a completely different technique to alleviate the problem by associating an additional scalar curvature invariant term \( R^2 \) in the Lanczos-Lovelock action. Note that such a term plays a crucial role in the early universe. For example, the dominance of such term leads to inflation without invoking phase transition [15] and canonical quantization yields a Schrödinger like equation, leading to quantum mechanical probability interpretation in a straightforward manner [16, 17]. Additionally, semiclassical wave-function obtained under WKB approximation has been found to be oscillatory, indicating that the region is classically allowed and it is strongly peaked about a set of solutions to the classical field
equations 18, 19. In the following we show how such a modified Lanczos-Lovelock action becomes free from pathological behaviour.

III. CONSTRUCTING A UNIQUE PHASE-SPACE HAMILTONIAN

Let us therefore supplement action (1) by $R^2$ term and express the complete action as,

$$A_2 = \int \sqrt{-g} d^Dx \left( R - \frac{2\Lambda_0}{2\kappa} + \beta R^2 + \gamma G_D \right) + \Sigma_{R^2} + \Sigma_{R^2_D} + \Sigma_{G_D},$$

where, $\Sigma_{R^2_D} = 4\beta \frac{\delta}{\delta y} RK \sqrt{h} d^{(D-1)}x$ is the boundary term required to supplement the gravitational action carrying a scalar curvature squared term $R^2$ in $D$-dimension. Before we proceed, let us mention that for a viable quantum description, canonical formulation of $R^2$ gravity was developed by Sanyal and his co-workers [13, 14] unperturbatively in the Robertson-Walker minisuperspace model, along the following prescription,

- Split the boundary term for $R^2$ into two parts, viz.

$$\Sigma_{R^2} = \Sigma_{R^2_D} + \Sigma_{R^2_{D^2}},$$

$$\Sigma_{R^2_{D^2}} = \int \left[ (\bar{R}^3) + (4R - 3\bar{R}) \right] K \sqrt{h} d^{(D-1)}x,$$

where $(\bar{R})$ is the Ricci scalar built out of $h_{ij}$ and $(4R)$ is the usual Ricci scalar in $D$ dimension.

- Express the action in terms of $h_{ij}$ and remove possible total derivative term which cancels $\Sigma_{R^2_{D^2}}$.

- Introduce auxiliary variable in the action following Horowitz’s proposal [20]. Integration by parts then takes care of the boundary term $\Sigma_{R^2_{D^2}}$. The action is then automatically expressed in canonical form.

- Find Hamiltonian constraint equation from $N$ variation equation which guarantees its diffeomorphic invariance. Now expressing it in terms of the basic variables, standard prescription for canonical quantization is followed.

Here, we follow the same route. It has been noticed that the boundary terms corresponding to $R^2$ part appearing in $D$-dimension can be taken care of, under the choice $z = h_{ij} \frac{\delta}{\delta y} = a^\frac{1}{2}$. Therefore in the present case we recast the action in view of the above variable, integrate it by parts so that all the total derivative terms get cancelled with the boundary terms retaining only $\Sigma_{R^2_{D^2}}$ and in the process the action (8) reads

$$A_2 = \int \left[ \frac{1}{\kappa} \left( 3kNz \frac{2(D-3)}{D} - \frac{2\Delta_1 z^2}{D^2 Nz^2} - \Lambda_0 Nz^2 \frac{2(D-1)}{D} \right) \right. \right.$$

$$\left. + 4\beta \frac{16(D-1)^2}{D^2 N^3 z^2} \left( \frac{\dot{\bar{z}}^2}{N} + \frac{\ddot{\bar{z}}^2 N^2}{z^2} \right) \right] dt + \Sigma_{R^2_{D^2}},$$

where, $\Delta_3 = (D-1)(D-6)$. Now introducing the following auxiliary variable

$$Q = \frac{\partial A}{\partial \dot{\bar{z}}} = 32\beta \frac{(D-1)^2}{D^2 N^3} \left( \frac{\dot{\bar{z}}}{N} - \frac{\ddot{\bar{z}} N^2}{z^2} \right),$$

the action (10) may be judiciously expressed as,

$$A_2 = \int \left[ \frac{N}{\kappa} \left( 3kz \frac{2(D-3)}{D} - \frac{2\Delta_1 z^2}{D^2 N^2 z^2} - \Lambda_0 z \frac{2(D-1)}{D} \right) \right. \right.$$

$$\left. + Q \dot{\bar{z}} \right. \left. + \Theta - \frac{16\bar{\gamma}_2}{D^2} \left( \frac{\Delta_1 z^4}{3D^2 N^3 z^2 \frac{2(D-1)}{D}} + \frac{3k z^2}{N z^2} \right) \right] dt + \Sigma_{R^2_{D^2}},$$

where, $\Theta = -\frac{z Q \dot{N}}{N} - \frac{4\beta k \Delta_3 z^2}{D^2 N^2 z^2} + \frac{36\beta k N z^2 \frac{2(D-1)}{D}}{D^2 N z^2}$. Now after removing rest of the total derivative terms under integration by parts once again, which gets cancelled with the remaining boundary term, the action in its final canonical form is expressed as,

$$A_2 = \int \left[ \frac{N}{\kappa} \left( 3kz \frac{2(D-3)}{D} - \frac{2\Delta_1 z^2}{D^2 N^2 z^2} - \Lambda_0 z \frac{2(D-1)}{D} \right) \right. \right.$$

$$\left. + Q \dot{\bar{z}} \right. \left. + \Theta - \frac{16\gamma z^4}{D^2} \left( \frac{\Delta_1 z^4}{3D^2 N^3 z^2 \frac{2(D-1)}{D}} + \frac{3k z^2}{N z^2} \right) \right] dt.$$

The canonical momenta are,

$$p_N = -\frac{Q z}{N}$$

$$p_Q = -\dot{z},$$

$$p_z = -\frac{z}{D^2 N} \left( \frac{\Delta_1}{2\pi G z^\frac{3}{2}} + \frac{96k (\beta \Delta_3 + \gamma \Delta_2)}{z^\frac{3}{2}} - \dot{Q} - \frac{Q N}{N} - \frac{64\bar{\gamma} \Delta_1 \Delta_2 \dot{z}^3}{3D^4 N^2 z^2 \frac{2(D-1)}{D}} \right).$$

Now removing $\dot{z}$ term in view of the definition of $Q$ given in (11), one can easily check that the $N$ variation
equation gives the Hamiltonian of the system,

\[
H_c = N \left[ -\frac{1}{\kappa} \left( \frac{2\Delta_1 \dot{z}^2}{D^2 z^2} + 3kz^{2(\nu-3)} - \Lambda_0 z^{2(\nu-1)} \right) \right]
\]

\[-\frac{\dot{Q}}{N} + N\Theta - \frac{16\gamma \Delta_2}{D^2} \left( \frac{\Delta_1 \dot{z}^4}{D^2 N^2 z^{2(\nu-1)}} + \frac{3kz^2}{N^2} \right),
\]

(15)

which is constrained to vanish. Now, in view of the definitions of momenta (14b) and (14c), it is possible to construct the phase-space formulation of the Hamiltonian constraint equation as,

\[
H_c = N \left( \frac{2\Delta_1 p Q^2}{D^2 N^2 z^2} - 3kz^{2(\nu-3)} + \Lambda_0 z^{2(\nu-1)} \right) - p Q \dot{z}
\]

\[+ \frac{D^2 N^2 z^2}{64\beta(D-1)^2} \frac{Q^2}{48\beta k \Delta_1 p Q^2} - 36\beta k^2 N^2 z^{2(\nu-1)}
\]

\[+ 16\gamma \frac{\Delta_2}{D^2} \left( \frac{\Delta_1 \dot{z}^4}{3D^2 N^2 z^{2(\nu-1)}} + \frac{3kz^2}{N^2} \right) = 0
\]

(16)

The definition of momentum \( p_z \) (14c) still indicates that it is multivalued in \( \dot{z} \), but \( p_Q \) (14b) being a single valued function of \( z \), \( p_z \) turns out to be single valued in \( Q \). Thus standard Legendre transformation is admissible and it is not required to consider Legendre-Fenchel transformation considered by Chi and He [14]. In the process, the presence of higher order theory alleviates the problem with branched Hamiltonian, presenting a unique Hamiltonian (16) in phase space quite naturally. This is a common feature of curvature squared gravity theory and here such an additional \( R^2 \) term cures the disease of the lack in Hamiltonian structure of the Lovelock action. Finally, to expatiate diffeomorphic invariance of the action \( (H_c = N\mathcal{H}) \), it is required to translate the equation (16) in terms of the basic variables \( h_{ij} \) and \( K_{ij} \), instead of the auxiliary one. For this purpose we choose \( x = \frac{z}{N} \) and replace \( Q \) and \( p_Q \) by,

\[
Q = \frac{\partial A}{\partial z} = \frac{p_z}{N}, \quad \text{and} \quad p_Q = -\dot{z} = -N x,
\]

(17)
as was done by Horowitz [20]. Therefore, the Hamiltonian constraint equation (16) now takes the form,

\[
H_c = N \left[ x p_x + \frac{1}{\kappa} \left( \frac{2\Delta_1 x^2}{D^2 z^2} - 3kz^{2(\nu-3)} + \Lambda_0 z^{2(\nu-1)} \right) \right]
\]

\[+ \frac{D^2 N^2 z^2}{64\beta(D-1)^2} \frac{p_x^2}{48\beta k \Delta_1 p_x^2} - 36\beta k^2 N^2 z^{2(\nu-1)}
\]

\[+ 16\gamma \frac{\Delta_2}{D^2} \left( \frac{\Delta_1 x^4}{3D^2 z^{2(\nu-1)}} + \frac{3kx^2}{z^2} \right) \right] = N\mathcal{H} = 0
\]

(18)

Hence, diffeomorphic invariance of the action is also manifest. Thus we observe that incorporating higher order \( (R^2) \) term, the modified Lanczos-Lovelock gravity [8] is free from all pathological behaviour.

IV. CANONICAL QUANTIZATION

Canonical quantization of the above Hamiltonian constraint equation (18) reads

\[
i\hbar^{-1} \frac{\partial^2}{\partial z^2} = -\frac{\hbar^2 D^2}{64\beta x(D-1)^2} \left( \frac{\partial^2}{\partial x^2} + \frac{n \partial}{x \partial x} \right) \Psi
\]

\[+ \left[ \frac{1}{\kappa} \left( \frac{2\Delta_1 x}{D^2 z^2} - 3kz^{2(\nu-4)} + \Lambda_0 z^{2(\nu-2)} \right) \right]
\]

\[+ 12\beta k \left( \frac{4\Delta_3 x}{D^2 z^2} - 3kz^{2(\nu-6)} \right)
\]

\[+ 16\gamma \frac{\Delta_2}{D^2} \left( \frac{\Delta_1 x^3}{3D^2 z^{2(\nu-2)}} + \frac{3kx^2}{z^2} \right) \right] \Psi\]

(19)

Under a further change of variable \( (\alpha = z^{\frac{\nu-2}{2}}) \) equation (19) takes the look of the Schrödinger equation,

\[
i\hbar^{-1} \frac{\partial \Psi}{\partial \alpha} = \Xi \left( \frac{\partial^2}{\partial x^2} + \frac{n \partial}{x \partial x} \right) \Psi + V_c(x, \alpha) \Psi = \hat{H}_c \Psi
\]

(20)

where, \( \Xi = \frac{\hbar^2 D^2}{64\beta x(D-1)^2(D+2)} \). Here \( \hat{H}_c \) is the effective Hamiltonian and \( \alpha \) plays the role of internal time parameter. The effective potential \( V_c \) is given by,

\[
V_c(x, \alpha) = \frac{\Delta_4}{\kappa} \left( \frac{2x \Delta_4}{D^2 \alpha^{1/2}} - 3k \alpha^{2(\nu-4)} \right) \left( + \frac{N_0}{x \alpha^{(\nu-2)}} \right)
\]

\[+ 12\beta k \Delta_4 \left( \frac{4x \Delta_3}{D^2 \alpha^{1/2}} - 3k \alpha^{2(\nu-6)} \right)
\]

\[+ 16\gamma \frac{\Delta_2 \Delta_4}{D^2} \left( \frac{3kx}{\alpha^{1/2}} + \frac{\Delta_1 x^3}{3D^2 \alpha^{1/2}} \right) \]

(21)

where, \( \Delta_4 = \frac{D^2}{N^2} \). Clearly \( \hat{H}_c \) is hermitian. Thus unitarity of the modified Lanczos-Lovelock action is established non-perturbatively. The hermiticity of \( \hat{H}_c \) allows one to write the continuity equation for a particular choice of operator ordering index \( n = -1 \), as,

\[
\frac{\partial \rho}{\partial z} + \nabla \cdot \mathbf{J} = 0,
\]

(22)

where, \( \rho = \Psi^* \Psi \) and \( \mathbf{J} = (\mathbf{J}_x, 0, 0) \) are the probability density and the current density respectively, with,

\[
\mathbf{J}_x = \frac{i\hbar D^3}{64\beta x(D-1)^2(D+2)} (\Psi^* \Psi_x - \Psi^* \Psi_x).
\]

(23)

In the process, factor ordering index here too has been fixed as \( n = -1 \) from physical argument. Note that under the choice \( \gamma = 0 \), the Gauss-Bonnet term disappears and the results obtained earlier in 19 are recovered. Now a viable quantum theory should exhibit unitary time evolution and reproduce the original classical
scenario in appropriate limits under certain semiclassical approximation. Thus, it is now left to be shown if
the present quantum prescription admits a viable semiclassical approximation. For this purpose we first find
a classical solution to the field equations under consideration. Clearly, the Hamiltonian constraint equations
(16) admits de-Sitter solution (for \(k = 0\)) in the form
\[
a = a_0 \exp (\text{H}t),
\]
provided,
\[
(D - 2) + 16\pi G_N H^2 (D - 4) \left[ \frac{D(D - 1)}{D - 1} \beta \right] = \frac{2\Lambda_0}{(D - 1) H^2},
\]
where, \(H\) is a constant. Finally, to present semiclassical solution in the standard WKB approximation, let us,
for the sake of simplicity, take up the time-independent equation (11) instead of considering the time-dependent
Schrödinger equation (19) and express it as
\[
\psi = \psi_0 e^{\frac{2}{D} S(x, z)} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi - i \hbar \frac{\partial \Psi}{\partial z} + V \psi = 0,
\]
where
\[
V = \frac{1}{8\pi G} \left( 2 \frac{\Delta_{11} x^2}{D^2 z^2} - 3k z^{2(D - 3)} + \Lambda_0 z^{2(D - 1)} \right)
+ 12\beta k \frac{\Delta_{11} x^4}{3D^2 z^2} - 3k \beta z^{2(D - 1)}
+ 16\gamma \frac{\Delta_{21} x^4}{3D^2 z^2} - 3k \beta z^{2(D - 1)}
\]
The above equation may be treated as time-independent Schrödinger equation with two variables \(x\) and \(z\) and therefore, as usual, let us sought the solution of equation (26) as,
\[
\psi = \psi_0 e^{\frac{2}{D} S(x, z)}
\]
and expand \(S\) in power series of \(h\) as,
\[
S = S_0(x, z) + \hbar S_1(x, z) + \hbar^2 S_2(x, z) + \ldots.
\]
Now following our earlier work [18] the semiclassical wavefunction around the classical solution reads (up to
first order approximation)
\[
\psi = A_0 e^{\frac{1}{\hbar} \left[ \frac{D(D - 4)}{8} + 4\beta (D - 1) H^2 - 4\gamma (D - 2) \beta H \right] \left[ \frac{D(D - 1)}{2} \right]}
\]
where, \(A_0 = \psi_0 \left( \frac{D^2 \psi}{16\beta (D - 1)^2 \hbar} \right)^{\frac{1}{D}}\). The wavefunction clearly is oscillatory and is peaked around the classical solution (23). Thus we have administered all the fundamental features of a viable quantized theory corresponding to the modified Lanczos-Lovelock action.

V. SUMMARY

Lanczos-Lovelock gravity suffers from the disease of having branched Hamiltonian. We have modified the
said action adding a scalar curvature squared term to get round the difficulty. Although unitarity of Lanczos-
Lovelock gravity has been established perturbatively, finding its propagators which arise from \(h^2\) expansion
of the theory around a fixed Minkowskian (flat) background \(\eta_{\mu\nu} = \eta_{\mu\nu}\) (\(\eta_{\mu\nu}\) is the Minkowski metric,
\(\eta_{\mu\nu}\) being the perturbation with \(h = h^\mu_{\mu}\)), and also around other fixed background, its fate was not known
non-perturbatively in the context of quantum cosmology, since it lacks a Hamiltonian structure in canonical form. Here we have also resolved this problem. In this connection one may note that in homogeneous and isotropic space-time the most general form of an action viz.,
\[
A = \int \sqrt{-g} d^D x \left[ R - \frac{2\Lambda_0}{2\kappa} + \beta_1 R^2 + \beta_2 I_1 + \beta_3 I_2 \right]
\]
reduces to
\[
A = \int \sqrt{-g} d^D x \left[ R - \frac{2\Lambda_0}{2\kappa} + \beta R^2 + \gamma G_D \right]
\]
since, \(R_{\mu\nu} R^{\mu\nu} - \frac{D}{4(D - 1)} R^2\) is topologically invariant (where \(I_1 = R_{\mu\nu} R^{\mu\nu}\) and \(I_2 = R_{\mu\nu\delta\sigma} R^{\mu\nu\delta\sigma}\)). Unitarity of the above action has been established [21] in the fixed curved dS/AdS background, under the constraints
\[
\frac{1}{\kappa_c} = \frac{1}{2\kappa} + \frac{4\Lambda D}{D - 2} \beta + \frac{4\Lambda \Delta_2}{\Delta_1} \gamma
\]
for \(D \geq 3\). In the above \(\Lambda\) is the effective cosmological constant which is related to the bare cosmological constant \(\Lambda_0\), by the following relation,
\[
\frac{\Lambda - \Lambda_0}{4\kappa} + \left[ \frac{D(D - 4)}{(D - 2)^2} \beta + \frac{\Delta_2}{\Delta_1} \gamma \right] \Lambda^2 = 0,
\]
while, the mass of the scalar mode in dS and AdS backgrounds should satisfy following conditions respectively,
\[
m^2 \geq \frac{D - 2}{4(D - 1) \beta \kappa_c} - 2 \frac{\Lambda D}{\Delta_1} \geq 0
\]
\[
m^2 \geq \frac{D - 1}{\Delta_1 \Lambda}, \quad \text{with} \quad \kappa_c > 0.
\]
Therefore, despite the usual thought that (due to the absence of fourth derivative terms in the field equations) if
the propagator of a higher order theory reduces to that corresponding to Lanczos-Lovelock gravity in
generic \(D\)-dimension, then only the theory is unitary, the analysis in [21] clearly admits \(R^2\) term in addition. Therefore the extended Lanczos-Lovelock action is free from all pathologies. We therefore conclude that higher order theory cures the problem associated with branched Hamiltonian.
[1] C. Lanczos, *A Remarkable Property of the Riemann-Cartoffel Tensor in Four Dimensions*, Ann. Math. 39 (1938) 842.

[2] D. Lovelock, *The Einstein Tensor and Its Generalizations*, J. Math. Phys. 12 (1971) 498.

[3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phenomenology, Astrophysics and Cosmology of Theories with Sub-Millimeter Dimensions and TeV Scale Quantum Gravity*, Phys. Rev. D59 (1999) 086004 and *The Hierarchy Problem and New Dimensions at a Millimeter*, Phys. Lett. B429 (1998) 263.

[4] B. Zwiebach, *Curvature squared terms and string theories*, Phys. Lett. B156 (1985) 315.

[5] B. Zumino, *Gravity theories in more than four dimensions*, Phys. Rept. 137 (1986) 109.

[6] N. E. Mavromatos and J. Rizos, *String-inspired higher-curvature terms and the Randall-Sundrum scenario*, Phys. Rev. D62 (2000) 124004.

[7] T. Padmanabhan, *Gravitation: Foundations and Frontiers*, Cambridge University Press, Cambridge U.K. (2004), pg. 668.

[8] S. Nojiri and S. D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*, Int. J. Mod. Phys. A15 (2000) 413.

[9] Da-Wei Pang, *On charged Lifshitz black holes*, JHEP10 (2009) 031.

[10] G. Dotti, J. Oliva and R. Troncoso, *Exact solutions for the Einstein-Gauss-Bonnet theory in five dimensions: Black holes, wormholes, and spacetime horns*, Phys. Rev. D70 (2004) 064019.

[11] S. Nojiri and S. D. Odintsov, *Brane-world cosmology in higher derivative gravity or warped compactification in the next-to-leading order of AdS/CFT correspondence*, JHEP 0007 (2000) 049.

[12] M. Henneaux, C. Teitelboim and J. Zanelli, *Quantum mechanics for multivalued Hamiltonians*, Phys. Rev. A36 (1987) 4417.

[13] A. Shapere and F. Wilczek, *Branched Quantization*, Phys. Rev. Lett. 109 (2012) 200402.

[14] Huan-Hang Chi and Hong-Jian He, *Single-valued Hamiltonian via Legendre-Fenchel transformation and time translation symmetry*, Nuclear Physics B 885 (2014) 448.

[15] A. A. Starobinsky, *A new type of isotropic cosmological models without singularity*, Phys. lett. B91 99 (1980).

[16] K. I. Maeda, *Inflation as a transient attractor in R^2 cosmology*, Phys. Rev. D37 (1988) 858.

[17] A. K. Sanyal and B. Modak, *Quantum cosmology with a curvature squared action*, Phys. Rev. D63 (2001) 064021.

[18] A. K. Sanyal, *Noether symmetry in the higher order gravity theory*, Gen. Rel. Grav.37 (2005) 1957.

[19] A. K. Sanyal, S. Debnath and S. Ruz, *Canonical formulation of the curvature-squared action in the presence of a lapse function*, Class. Quant. Grav.29 (2012) 215007.

[20] G. T. Horowitz, *Quantum cosmology with a positive-definite action*, Phys. Rev. D31 (1985) 1169.

[21] I. Güllü and B. Tekin, *Massive higher derivative gravity in D-dimensional anti-de Sitter spacetimes*, Phys. Rev. D80 (2009) 064033.