Superminimal Surfaces in the 6-Sphere

Martins, J.K.*

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Abstract

In this article, we use the harmonic sequence associated to a weakly conformal harmonic map \( f : S \to S^6 \) in order to determine explicit examples of linearly full almost complex 2-spheres of \( S^6 \) with at most two singularities. We prove that the singularity type of these almost complex 2-spheres has an extra symmetry and this allows us to determine the moduli space of such curves with suitably small area. We also characterize projectively equivalent almost complex curves of \( S^6 \) in terms of \( G_2 \)-equivalence of their directrix curves.

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Email: akay333@gmail.com

1 Preliminaries

The use of harmonic sequence is a very well known technique since, in recent times, it has been used by several authors (\cite{10}, \cite{12}, \cite{3}) and we shall give here the same treatment as in \cite{3}, which was suitably specialized for the case of the spheres. We start by establishing some background results about almost complex curves of the 6-sphere.

A map \( \psi : S \to W \) between Riemannian manifolds is harmonic if it satisfies the Euler-Lagrange equation

\[
\text{tr}(\nabla d\psi) = 0.
\]

Throughout this article, we shall use \( S \) to denote a Riemann surface and \( z = x + iy \) shall denote a local complex coordinate \( z \) on \( S \). In this case, the harmonicity condition of \( \psi \) is simplified to

\[
(\nabla_{\partial z} d\psi)(\frac{\partial}{\partial z}) = 0.
\]

Let \( V \to S \) be a complex vector bundle over the Riemann surface \( S \) and assume that \( \nabla \) is a connection on \( V \). By the Koszul-Malgrange Theorem, \( V \) admits the structure of a holomorphic vector bundle. Here a section \( s \) is a holomorphic section if and only if

\[
\nabla_{\partial z} s = 0.
\]

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Given a harmonic map $\psi_0 := \psi : S \to \mathbb{C}P^n$, several authors ([16], [12], [10], [3]) have dealt with the sequence of harmonic maps $\psi_k : S \to \mathbb{C}P^n$ obtained from $\psi$ via an inductive construction of a sequence of complex line bundles over $S$.

In the sequel, we outline this construction and give some of the main features of this sequence.

Let $\mathcal{L}$ be the tautological line bundle over $\mathbb{C}P^n$. Let $L_0$ and $L_1^\perp$ be the pull-backs via $\psi_0$ of $\mathcal{L}$ and $\mathcal{L}^\perp$ respectively. $L_0$ and $L_1^\perp$ are endowed with naturally induced connections for they are vector subbundles of the trivial $\mathbb{C}^{n+1}$-bundle over $S$.

Explicitly, if $s$ is a section of a subbundle $L$ of the trivial bundle $S \times \mathbb{C}^{n+1}$, then $s$ may be regarded as a map $S \to \mathbb{C}^{n+1}$. Given $X \in T_s S$, we can define a connection $\nabla_X s$ by the orthogonal projection $(Xs)^\perp$ of $Xs$ onto $L$. Similarly, we also define a connection in $L^\perp$. Thus the line bundles $L_0$ and $L_1^\perp$ have structures of holomorphic vector bundles over $S$.

The map $\psi_0$ determines a bundle map $\partial_0 : L_0 \to L_1^\perp$. Indeed, if we consider a holomorphic local section $f_0 : S \to \mathbb{C}^{n+1} \setminus \{0\}$ of $L_0$, then we define $\partial_0 f_0 = (\frac{\partial f_0}{\partial z})_{L_0^\perp}$, and $\overline{\partial_0 f_0} = (\frac{\partial f_0}{\partial \overline{z}})_{L_0^\perp}$.

It follows from (1) and (2) that $\overline{\partial_0 (\overline{\partial_0})}$ is a holomorphic (anti-holomorphic) bundle map if and only if $\psi_0 = [f_0]$ is a harmonic map. Therefore if $f_1 := \partial_0 f_0$ is not identically zero then it is a holomorphic section of the bundle $L_1^\perp$ and hence its zeros (if any) are isolated. Let $z_0$ be such a zero, then for some holomorphic local section $f_1$ we have $f_1(z) = (z-z_0)^m f_1(z)$ with $f_1(z_0) \neq 0$. This latter map will then yield a well-defined map $\psi_1(z) := [f_1(z)]$ from $S$ into $\mathbb{C}P^n$ and $f_1$ is a meromorphic local section for a complex line bundle $L_1 \subset L_1^\perp$.

By defining $\partial_1$ in a similar way, and verifying that $f_2 := \partial_1 f_1$ is a holomorphic local section of $L_1^\perp$, it follows that $\psi_1$ is also harmonic.

Therefore, as long as the bundle section $f_k$ is not identically zero, that is, $\psi_{k-1}$ is not anti-holomorphic (or $\psi_{k+1}$ is not holomorphic, when considering the descending sequence given by $\psi_{-i+1} = \overline{\psi_{-i}}(\psi_{-1})$ where $i = 0, 1, 2, \ldots$), we can carry on with this process, defining a sequence $\psi_k = [f_k(z)]$ of harmonic maps such that the local sections $f_k$ are characterized by the following properties:

\[
\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{1}{|f_p|^2} \frac{\partial}{\partial z} f_p f_p \nonumber \\
= f_{p+1} + \frac{1}{|f_p|^2} \partial_z (\log |f_p|^2) f_p \nonumber \\
= f_{p+1} + \alpha_p f_p \quad \text{where} \quad \alpha_p := \partial_z (\log |f_p|^2) \tag{3} \\
\frac{\partial f_{p+1}}{\partial \overline{z}} = -\gamma_p f_p \quad \text{where} \quad \gamma_p := \frac{|f_{p+1}|^2}{|f_p|^2}. \tag{4}
\]

Here $\langle , \rangle$ denotes the standard Hermitian product on $\mathbb{C}^{n+1}$ and $| |$ denotes the associated norm. It is known [12] that the harmonic sequence $\psi_i$ where $i \in \mathbb{Z}$, terminates at one end if and only if it terminates at both ends. If this happens, we say that each element of the sequence is superminimal and it is customary to consider the range for the indices starting at the holomorphic element of the sequence, that is, $\{\psi_j\}_{j=0}^n$ denotes the harmonic sequence generated by $\psi_0$.

This holomorphic map is usually named in the literature as the directrix curve associated to any of the harmonic maps $\psi_j$ referring to the terminology adopted when dealing with harmonic 2-spheres in $S^{2m}$ (cf. [1]).
If \( \psi_m = [f_m] \) for some harmonic map \( f_m : S \to S^n \), then we can consider \( f_m \) as a nowhere vanishing global holomorphic section of \( L_0 \) so that the sequence of meromorphic sections \( f_j \) will also satisfy the condition:

\[
\mathcal{F}_{m+k} = (-1)^k |f_{m+k}|^2 f_{m-k}. \tag{5}
\]

In particular,

\[
|f_{m+k}|/|f_{m-k}| \equiv 1. \tag{6}
\]

Note that in this situation, the element \( \psi_0 \) will necessarily be in the middle of the sequence, that is \( n = 2m \), because \( f_{m+k} \equiv 0 \) if and only if \( f_{m-k} \equiv 0 \). Moreover, \( k \) and \( |f_m| \equiv 1 \) implies

\[
\partial_z f_m = f_{m+1}. \tag{7}
\]

**Definition 1** We say that a map from a Riemannian manifold \( N \) into \( \mathbb{C}P^n \) is linearly full, when its image is not contained in any complex space form \( \mathbb{C}P^k \) for \( k < n \).

If the Riemann surface is homeomorphic to the sphere \( S^2 \) then Wolfson \[16\] shows that the corresponding complex line bundles are mutually orthogonal and consequently the harmonic sequence terminates, that is, all the harmonic 2-spheres of \( \mathbb{C}P^n \) are superminimal. Moreover, in this case, the length of the sequence achieves its maximum, \( n + 1 \), if and only if \( \psi \) is linearly full.

A detailed discussion of the holomorphic curves of a complex projective space can be found in [13] (pages 263-268), but here we describe some material on this topic to be used in this article.

Let \( \psi(z) = [f(z)] : S \to \mathbb{C}P^n \) be a holomorphic curve from the Riemann surface \( S \) into \( \mathbb{C}P^n \), where \( f : S \to \mathbb{C}^{n+1} \setminus \{0\} \) is a local holomorphic lift of \( \psi \). Then the \( j^{th} \)-osculating curve of \( \psi \) is the holomorphic curve \( \sigma_j : S \to \mathbb{C}P^n \) (where \( n_j := \lceil n+1 \rceil - 1 \)) defined by

\[
\sigma_j(z) = [f \wedge \ldots \wedge f^{(j)}](z),
\]

where \( f^{(j)} = \frac{\partial^j f}{\partial z^j} \) and \( j = 0, \ldots, n-1 \).

A higher order singularity of \( \psi \) is a point \( p \in S \) which is a singularity for some \( j^{th} \)-osculating curve \( (j = 0, 1, \ldots, n-1) \). The ramification index of \( \sigma_j \) at a point \( p \) is the order \( r_{j+1} \) of this point as a zero of the derivative of the curve \( \sigma_j \).

Thus, the holomorphic curve \( \psi \) is said to have singularity type \( (r_1, r_2, \ldots, r_n) \) at the point \( p \).

If \( S \) is a compact Riemann surface, then the curve \( \psi \) has a finite set \( \mathcal{Z}_\psi = \{p_1, \ldots, p_k\} \) of higher order singularities. We shall denote by \( R_{j+1} \) the sum of the ramification indices of \( \sigma_j \) at each singularity, that is,

\[
R_{j+1} = \sum_{i=1}^k r_{j+1}(p_i) \tag{8}
\]
and we shall refer to $R_{i+1}$ just as the ramification index of $\sigma_j$. Moreover, we can define the **total ramification index** of $\psi$ as the sum $\sum_{j=1}^{n} R_j$.

Then, we shall say that the holomorphic curve $\psi$ or any element of its corresponding harmonic sequence has **singularity type** $(r_1(p), \ldots, r_n(p))$ at the point $p$ and has **total singularity type** $(R_1, \ldots, R_n)$.

The curve $\psi$ is **totally unramified** when its total ramification index is zero. Otherwise, $\psi$ is said to be **$k$-point ramified** if the set $Z_\psi$ has cardinality $k$.

In terms of a local complex coordinate $z$ for $S$ centred on $p$, that is $z(p) = 0$, it is possible to determine a basis $\{v_0, \ldots, v_n\}$ for $\mathbb{C}^{n+1}$ in such a way that the holomorphic map $f$ can be written in the normal form

$$f(z) = \sum_{i=0}^{n} z^{k_0 + \ldots + k_i} h_i(z) v_i,$$

where $k_0 = 0$, $k_i = r_i(p) + 1$ ($j = 1, \ldots, n$) and $h_i(z)$ denotes a holomorphic function satisfying $h_i(0) \neq 0$.

If $\psi : S^2 \to \mathbb{C}P^n$ is 2-point ramified, then we can find a local complex coordinate for $S^2$ so that the higher order singularities of $\psi$ (if any) occur at $z = 0$ and $z = \infty$. Indeed, this follows from the fact that the Möbius group of conformal transformations of the 2-sphere acts triply transitively on $S^2$.

We observe that $\psi$ is a holomorphic map between algebraic varieties, since $S^2 = \mathbb{C}P^1$, and so $\psi$ is an algebraic map (c.f. [13]), that is $f$ is a rational function. Without loss of generality, we may assume $f$ to be a $\mathbb{C}^{n+1}$-valued polynomial function.

**Definition 2** We say that a harmonic map $\psi : S^2 \to \mathbb{C}P^n$ has **$S^1$-symmetry** if there exists non-trivial $S^1$-actions on $S^2$ and $\mathbb{C}P^n$, where the action on $\mathbb{C}P^n$ is by holomorphic isometries, such that for all $z \in S^2$,

$$\psi(e^{i\theta} z) = e^{i\theta} \psi(z).$$

We are now able to state the main results ([4],[7]) to be used in the following sections, which are concerned with the characterization of the $k$-point ramified harmonic 2-spheres of $\mathbb{C}P^n$ for $k \leq 2$.

**Theorem 1** Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full harmonic map with $S^1$-symmetry. Then there exists a holomorphic coordinate $z$ on $S^2$ such that the directrix curve $\psi_0 = [f_0]$ of $\psi$ can be expressed up to holomorphic isometries of $\mathbb{C}P^n$ by

$$f_0(z) = \sum_{p=0}^{n} z^{k_1 + \ldots + k_p} v_p,$$

where $\{v_0, \ldots, v_n\}$ is an orthogonal basis of $\mathbb{C}^{n+1}$ and the scalars $k_j$ are positive integers. Furthermore, $\psi$ is either totally unramified or 2-point ramified. In the latter case $\psi$ has singularities at $z = 0$ and $z = \infty$ with corresponding singularity type at $z = 0$ and $z = \infty$ given respectively by $(k_1 - 1, \ldots, k_n - 1)$ and $(k_n - 1, \ldots, k_1 - 1)$.

**Theorem 2** Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full harmonic map which is $k$-point ramified for $k \leq 2$. Let $z$ be a complex coordinate on $S^2$, then
(i) The higher order singularities of \( \psi \) (if any) occur at \( z = 0 \) and \( z = \infty \) if and only if its directrix curve \( \psi_0(z) = [f_0(z)] \) can be written in the form

\[
f_0(z) = \sum_{p=0}^{n} z^{k_1+\ldots+k_p}v_p,
\]

where the scalars \( k_i \) are positive integers and the vectors \( v_j \) constitute a basis of \( \mathbb{C}^{n+1} \). Furthermore, \( \psi \) has \( S^1 \)-symmetry (with fixed points of the \( S^1 \)-action at \( z = 0 \) and \( z = \infty \)) if and only if the basis \( \{v_0, \ldots, v_n\} \) is orthogonal.

(ii) \( \psi(S^2) \subset \mathbb{R}P^n \) and one of the conditions (and so both) in the first equivalence stated in (i) occurs if and only if \( n = 2m \) for some integer \( m \) and the following properties are satisfied.

\[
k_j = k_{n-j+1} \quad \text{for } i, j \in \{1, \ldots, n\}, \quad \langle v_j, \overline{v_i} \rangle = (-1)^j \delta_{(j,n-\cdot)} \mu \lambda_j, \quad \text{for } i, j \in \{0, \ldots, n\},
\]

where \( \mu \) is a constant and \( \lambda_j := \prod_{1 \leq r < s \leq n} (k_r + \ldots + k_s) \prod_{r=1}^{n-j} (k_{j+1} + \ldots + k_{n-r+1}) \).

The constant \( \mu \) in the theorem can be chosen to be 1 by rescaling the homogeneous coordinates of \( \mathbb{C}P^n \). However, we will avoid this in order to facilitate our calculations later on when determining examples of superminimal almost complex curves.

**Remark 1** The theorem above shows in particular that there does not exist a 1-point ramified linearly full harmonic 2-sphere in \( \mathbb{C}P^n \).

**Definition 3** Two maps \( \psi, \tilde{\psi} : S \to \mathbb{C}P^n \) are said to be projectively equivalent if there exists \( [A] \in \text{PGL}(n+1, \mathbb{C}) \) so that \( \tilde{\psi} = [A](\psi) \).

**Corollary 1** Any two \( k \)-point ramified (\( k \leq 2 \)) linearly full harmonic maps \( \psi, \tilde{\psi} : S^2 \to \mathbb{C}P^n \) with the same singularity type are projectively equivalent up to a conformal transformation of \( S^2 \).

**Proof:** According to Theorem (2) these curves are uniquely determined by their singularity type and a choice of basis for \( \mathbb{C}^{n+1} \). Thus, item (i) of that theorem shows that the corresponding directrix curves differ by an element of \( \text{GL}(n+1, \mathbb{C}) \).

\[\square\]

2 The twistor fibration \( \pi : Q^5 \to S^6 \)

Let \( Q^5 \) denote the quadric of \( \mathbb{C}P^6 \), which is the Kähler submanifold defined

\[
Q^5 = \{[x] \in \mathbb{C}P^6 \text{ such that } (x, x) = 0\},
\]

where \( (, ) \) denotes the symmetric bilinear Euclidean inner product on \( \mathbb{C}^7 \) given by \( (x, y) = \langle x, \overline{y} \rangle \).

The twistor fibration \( \pi : Q^5 \to S^6 \) is defined by the map

\[
\pi([x]) = \frac{i}{|x|^2} \bar{x} \times x.
\]
Where the cross product \((\mathbb{C}^7, \times)\) we are considering here is given in [6] and we shall also be using its main properties presented in that reference.

We shall see in the next section that the superminimal almost complex curves of the 6-sphere can be characterized as the projections of a special type of holomorphic curves of this quadric. This led us to investigate what is the group of holomorphic transformations of the quadric which preserves the superhorizontal distribution to be defined ahead.

Although \(\pi\) is not a Riemannian submersion, it is quite close to that as we shall see below. Moreover, the fact that \(\pi\) can be easily expressed in terms of the cross product \(\times\) on \(\mathbb{R}^7\) (extended \(\mathbb{C}\)-linearly to \(\mathbb{C}^7\)) yields some good methods to investigate properties of any lifting.

The exceptional Lie group \(G_2\) acts transitively on the manifolds \(Q^5\) and \(S^6\) in such a way that these manifolds can be realized also as the homogeneous spaces \(G_2/U(2)\) and \(G_2/SU(3)\) respectively. By considering these homogeneous spaces, it is possible to show that the twistor fibration just defined is nothing but the canonical projection of the first space onto the second one.

We can write any element of \(Q^5\) as \([x] = [a - ib]\) where \(a\) and \(b\) are orthonormal vectors of \(\mathbb{R}^7\). In this case \(\pi\) reduces to

\[
\pi[x] = a \times b.
\]

**Remark 2** It follows from the characterization of \(G_2\) as the group of automorphisms of \((\mathbb{R}^7, \times)\) that the map \(\pi\) is \(G_2\)-equivariant, that is, \(\pi[\gamma x] = g(\pi[x])\) for every \(g \in G_2\).

Using the Hopf fibration \(H : S^{13}_x \rightarrow CP^6\), we have for each \(x \in S^{13} \subset \mathbb{C}^7\), an isomorphism \(dH_x : T^*_xS^{13} \rightarrow T_{[x]}CP^6\). Where the horizontal space:

\[
T^*_xS^{13} = \{v \in \mathbb{C}^7 \text{ such that } (v, \overline{v}) = 0 = \{ix\}\}
\]

induces the natural decomposition \(T_xS^{13} = \text{span}_{\mathbb{R}}\{ix\} \oplus T^*_xS^{13}\) and it yields also the isomorphism:

\[
T_{[x]}Q^5 \cong \{v \in \mathbb{C}^7 \text{ such that } (v, \overline{v}) = 0 \text{ and } (v, x) = 0.\}
\]

From now on, we shall be identifying these tangent spaces with the linear subspaces of \(\mathbb{C}^7\) mentioned above with no further reference. Using the definition of \(\pi\) we have

\[
\pi^*_x(v) = \frac{i}{|x|^2} (\overline{v} \times v - x \times \overline{v}).
\]

Thus, the vertical distribution on \(Q^5\) defined as the kernel of \(\pi^*_x\) is given by the set of those tangent vectors \(v \in T_{[x]}Q^5\) such that the imaginary part of \(x \times \overline{v}\) is zero. However, writing \(v = c + id\), using the notation \(\Im\) and \(\Re\) to denote the imaginary vector and real vector parts in \(\mathbb{R}^7\) of a vector in \(\mathbb{C}^7\), we have

\[
\Im(x \times \overline{v}) \times d = (d, d)a - (b \times c) \times d = -\Re(x \times \overline{v}) \times c
\]

\[
\Im(x \times \overline{v}) \times c = (c, c)b - (a \times d) \times c = \Re(x \times \overline{v}) \times d.
\]

In order to obtain the equations above, we have used the ordinary property:

\[
u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - w(v, u).
\]
Thus, \(\Im(x \times \overline{y}) = 0\) if and only if \(\Re(x \times \overline{y}) = 0\) which implies that the vertical distribution is characterized by

\[ V_{[x]} = \text{Ker}\pi_\ast = \{ v \in T_{[x]}Q^5 \text{ such that } x \times \overline{v} = 0 \}. \tag{15} \]

We should also note that \(V_{[x]}\) is an isotropic subspace of \(T_{[x]}Q^5\), since we have for any \(v \in V_{[x]}\):

\[ 0 = \overline{v} \times (x \times \overline{v}) = 2(\overline{v} \times v)x. \]

This yields a distribution of isotropic subspaces \(H' := \nabla\) of the horizontal spaces \(H = V^\perp\), which we henceforth will name as the superhorizontal distribution.

It follows then from (15) that this vector space is characterized at \([x]\) by:

\[ H' = \{ v \in T_{[x]}Q^5 \text{ such that } x \times v = 0 \}. \tag{16} \]

Incidentally, looking at the point \(v_0 = \pi[x] \in S^6\) as a real vector of \(\mathbb{C}^7\), it is clear that \(v_0 \in T_{[x]}Q^5\). Moreover, \(v_0\) is a horizontal vector because we have for any vertical vector \(v \in V\)

\[ \langle \pi[x], v \rangle = (\overline{x} \times x, \overline{x}) = -(\overline{x} \times x, \overline{x}) = 0. \quad \forall \ v \in V. \]

Furthermore, the equation above also shows that \(v_0\) is orthogonal to \(H'\). Thus, using the 1-dimensional complex space \(D\) spanned by \(v_0\), we can split the horizontal distribution as follows.

\[ H = D \oplus H'. \tag{17} \]

We shall investigate now how far the map \(\pi\) is prevented from being a holomorphic Riemannian submersion, by looking at its behaviour concerning to length-preservation and \(C\)-linearity of its differential. We split these properties into two lemmas.

**Lemma 1** \(\pi_\ast\) is length-preserving in \(H'\) and it reduces the length by a \(\sqrt{2}\)-factor in \(D\).

**Lemma 2** \(\pi_\ast\) is \(C\)-linear in \(H'\) and it is \(C\)-anti-linear in \(D\).

We have that \(G_2\) acts on \(Q^5\) as a transitive group of isometries which preserve horizontal subspaces. Hence it is enough to show both Lemmas at the point \(x = e_1 + ie_5\). In this case, we have \(\pi([x]) = e_4\) and the vertical subspace is given by \(V_{[x]} = \text{span}_C\{e_2 + ie_6, e_3 + ie_7\}\) and \(H' = \nabla_{[x]}\).

**Proof of Lemma (1):** The restriction \(d\pi|_H'\) is length preserving as for \(v = e_2 - ie_6\) or \(v = e_3 - ie_7\) we have \(|v| = |x| = \sqrt{2}\) giving \(\|v\| = 2\) in the Fubini-Study metric and \(d\pi(v) = 2e_7\) or \(2e_6\), giving \(d\pi(v) = 2 = \|v\|\).

The restriction \(d\pi|_D\) is reducing length since \(|e_4| = 1\), \(|x| = \sqrt{2}\) so \(\|v\| = \frac{2}{\sqrt{2}} = \sqrt{2}\) in the Fubini-Study metric and \(d\pi(e_4) = -e_1\) has length \(|d\pi(e_4)| = |e_1| = 1 = \frac{1}{\sqrt{2}}\|e_4\|\). So \(d\pi\) reduces lengths by the factor \(\sqrt{2}\). \(\bigcirc\)

**Proof of Lemma (2):** The map \(d\pi|_{H'}\) is complex linear, since for \(v = e_2 - ie_6\) or \(v = e_3 - ie_7\) we have \(d\pi(iv) = Jd\pi(v) = e_4 \times d\pi(v)\).

The map \(d\pi|_D\) is complex anti-linear, since \(d\pi(ie_4) = -Jd\pi(e_4) = -e_4 \times d\pi(e_4)\). \(\bigcirc\)

Hopf hypersurfaces of space forms have been studied in [15] and the twistor fibration presented here, can provide a way to study this type of hypersurface
in the quadric \( Q^5 \). The results above show that although \( \pi \) is not a Riemannian submersion it is not so far from this. Consequently, we can ask whether the lift \( M = \pi^{-1}(M) \) of a Hopf hypersurface \( M \) of \( S^6 \) is also a Hopf hypersurface in \( Q^5 \) or not. However, it is not hard to see that the horizontal lift of a normal vector field on \( M \) cannot lie either in the distribution \( D \) or in \( H' \). This fact makes clear that \( M \) cannot be a tubular hypersurface. Furthermore, the decomposition (17) of the horizontal distribution makes it rather complicated to work with the Riemannian connection of \( Q^5 \).

We shall see later on in this article that the superminimal almost complex curves of \( S^6 \) are in 1-1 correspondence with the holomorphic curves of \( Q^5 \) which are tangential to the superhorizontal distribution. This motivates us to determine what is the group of holomorphic transformations of \( Q^5 \) which preserves the superhorizontal distribution.

Let us consider the Lie group \( H_1 = \{ M \in GL(n + 1, \mathbb{C}) : \lambda \in \mathbb{C}^* \} \) and its Lie subgroup \( H_2 = \{ M \in SO(n + 1, \mathbb{C}) : \lambda \in \mathbb{C} \} \) and \( \lambda^{n+1} = 1 \). It is well known (for instance, [13] page 65) that \( PGL(n + 1, \mathbb{C}) = GL(n + 1, \mathbb{C})/H_1 \) is the group of holomorphic transformations of \( \mathbb{C}P^n \). The following lemma is easy to prove.

**Lemma 3** \( SO(n + 1, \mathbb{C})/H_2 \) is the group of holomorphic transformations of the quadric \( Q^{n-1} \).

In the next proposition we will need the following elementary properties of the distributions \( D, H' \) and \( V \).

\[
\begin{align*}
D_{[x]} & = D_{[\pi]} \\
H'_{[x]} & = V'_{[\pi]} \\
V_{[x]} & = H'_{[\pi]}. 
\end{align*}
\]

**Remark 3** Let \( G_2^7 \) be the group of automorphisms of \( (\mathbb{C}^7, \times) \). It is clear from the definition of \( G_2 \) and the characterization of the superhorizontal distribution given in (16), that \( G_2^7 \) is a Lie subgroup of \( SO(7, \mathbb{C}) \) and it preserves the superhorizontal distribution.

A \( G_2 \)-basis (or canonical basis) of \( \mathbb{R}^7 \) is an orthonormal basis \( \{ f_1, \ldots, f_7 \} \) of \( \mathbb{R}^7 \) satisfying the relations: \( f_3 = f_1 \times f_2, f_5 = f_1 \times f_4, f_6 = f_2 \times f_4, f_7 = f_3 \times f_4 \). Hence, if \( f_1, f_2, f_4 \) are orthogonal unit vectors such that \( f_4 \perp f_1 \times f_2 \) then \( f_1, f_2, f_4 \) determine a unique \( G_2 \)-basis subject to the relations \( f_i \times (f_j \times f_k) = 2\delta_{jk}f_j - \delta_{ij}f_k - \delta_{ik}f_j \). Every \( G_2 \)-basis will have the following multiplication table

\[
\begin{array}{c|ccccccc}
\times & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & f_3 & -f_2 & f_5 & -f_4 & -f_7 & f_6 \\
2 & -f_3 & 0 & f_4 & f_6 & f_7 & -f_4 & -f_5 \\
3 & f_2 & -f_1 & 0 & f_7 & -f_6 & f_5 & -f_4 \\
4 & -f_5 & -f_6 & -f_7 & 0 & f_1 & f_2 & f_3 \\
5 & f_4 & f_7 & f_6 & -f_1 & 0 & -f_5 & f_2 \\
6 & f_7 & f_4 & -f_5 & -f_2 & f_3 & 0 & -f_1 \\
7 & -f_6 & f_5 & f_4 & f_3 & -f_2 & f_1 & 0 \\
\end{array}
\]

For any two \( G_2 \)-bases \( \{ f_1, \ldots, f_7 \} \) and \( \{ \tilde{f}_1, \ldots, \tilde{f}_7 \} \) there exists a unique \( g \in G_2 \) such that \( gf_i = \tilde{f}_i \) (we simply define \( g \) by \( gf_1 = f_1 \) and check that it is in \( G_2 \)).
Also \( g \in G_2 \) mapping \( e_1, \ldots, e_7 \) to \( f_1, \ldots, f_7 \) can be represented by the matrix
\[
(f_1 \ldots | f_7) \in SO(7)
\]
(with respect to the standard basis \( e_1, \ldots, e_7 \)). Further details on \( G_2 \)-basis can be found in [3].

**Proposition 1** The group \( \tilde{G} \) of holomorphic transformations of \( Q^5 \) which preserves the superhorizontal distribution is \( G_2^5 \).

**Proof:**
In accordance with the remark [3] above, we have \( G_2^5 \subset \tilde{G} \). Let \([g]\) be an arbitrary element of \( \tilde{G} \) that is, \([g] \in SO(7, \mathbb{C})/H_2 \). Thus \(+g\) or \(-g\) lies in \( SO(7, \mathbb{C}) \). By assumption we have for every \([x] \in Q^5 \) and \( v \in H'_{[x]} \) that
\[
gx \times gv = 0. \tag{18}
\]
We shall first observe that \( g \) also lies in \( \tilde{G} \). Indeed, given \( v \in H'_{[x]} \) we have \( \overline{v} \in V_{[x]} = H_{[x]}' \) and hence
\[
\overline{g}x \times \overline{gv} = \overline{gx \times gv} = 0. \tag{19}
\]
The superhorizontal subspaces at the points \([x_0] = [e_1 - ie_5] \) and \([x_1] = [e_1 - ie_4] \) are \( H'_{[x_0]} = \text{span}_C(e_2 + ie_6, e_3 + ie_7) \) and \( H'_{[x_1]} = \text{span}_C(e_7 + ie_2, e_3 + ie_6) \) respectively. If we then apply (18) and (19) to these vectors at their corresponding points, we obtain
\[
ge_1 \times ge_2 = ge_4 \times ge_7 = ge_6 \times ge_5
\]
\[
ge_1 \times ge_3 = ge_6 \times ge_4 = ge_7 \times ge_5
\]
\[
ge_1 \times ge_6 = ge_5 \times ge_2 = ge_4 \times ge_3
\]
\[
ge_1 \times ge_7 = ge_5 \times ge_3 = ge_2 \times ge_4
\]
The vectors \( \{ge_1, ..., ge_7\} \) are orthonormal with respect to the Euclidean product \((, ) \) in \( \mathbb{C}^7 \) since
\[
(ge_i, ge_j) = (ge_i, \overline{ge_j}) = (g^\dagger ge_i, e_j) = \delta_{ij}.
\]
Recalling that \((*, *, \cdot)\) is skew-symmetric, we see that
\[
(ge_i, \overline{ge_j}) = 0.
\]
Thus it follows from (20) that \( ge_1 \times ge_2 = \pm ge_3 \).

If \( ge_1 \times ge_2 = ge_3 \) then we can use directly (21), (22) and (23) to show that \( \{ge_1, ..., ge_7\} \) is a \( G_2 \)-basis for \( \mathbb{C}^7 \) and hence \( g \in G_2^5 \).

Similarly, if \( ge_1 \times ge_2 = -ge_3 \) then we can repeat the same process above to deduce that \( -g \in G_2^5 \). \( \bigcirc \)

### 3 Superminimal Surfaces in \( S^6 \)

**Definition 4** Let \( S \) be a Riemann surface. We say that a smooth map \( f: S \to S^6 \) is an almost complex curve of the nearly Kähler \( S^6 \) if \( f_* \) is complex linear.
Therefore, using a local complex coordinate $z = x + iy$ for $S$ we can characterize these curves by

$$
\partial_y f = f \times \partial_x f.
$$

(24)

It follows that almost complex curves of $S^6$ are weakly conformal maps which are also harmonic maps because if we differentiate again (24), we obtain $f \times (\partial_{xx} f + \partial_{yy} f) = 0$, yielding that $\partial_{xx} f + \partial_{yy} f$ is normal to $S^6$, and hence $f$ is harmonic in accordance with [11].

Therefore an almost complex curve $f : S \to S^6$ determines a harmonic sequence of maps $\psi_k : S \to \mathbb{C}P^6$ so that $\psi_0 = [f]$. Using this sequence and some invariants associated to their elements, a full classification of these curves was obtained in [6] according to the following four types:

(I ) Linearly full in $S^6$ and superminimal,

(II ) Linearly full in $S^6$ but not superminimal,

(III ) Linearly full in a totally geodesic $S^5$ in $S^6$,

(IV ) Totally geodesic.

A result of Bryant [9] highlights the importance of the Type-I almost complex curves of $S^6$. Bryant has shown that every compact Riemann surface of any genus can be realised as such an almost complex curve of the 6-sphere.

In this section we shall obtain explicitly all the 0-point and 2-point ramified linearly full almost complex 2-spheres of the 6-sphere. This is done by using the normal form for such surfaces as given by Theorem [2].

In particular, we will also prove that these surfaces are uniquely determined by their singularity type up to $G_2^7$-equivalence of their directrix curves. It is worthwhile mentioning here that a similar result in the more general situation of harmonic 2-spheres of $S^n$ and $\mathbb{C}P^n$ has been obtained in [5] but replacing, of course, the group $G_2^7$ by $SO(n + 1, \mathbb{C})$.

**Proposition 2** Let $f : S \to S^6$ be a linearly full superminimal almost complex curve. If $(\psi_j = [f_j])_{j=0}^6$ is the harmonic sequence corresponding to the harmonic map $\psi_3 = [f] = [f]$ then the meromorphic local sections $f_k : \mathbb{C} \to \mathbb{C}^7$ have the following multiplication table for $f_i \times f_j$, where the cross product $\times$ is extended $\mathbb{C}$-linearly to $\mathbb{C}^7$:

| $i\setminus j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|---|---|---|---|---|---|---|
| 0              | 0 | 0 | 0 | $-i f_0$ | $-2i f_1$ | $-2i f_2$ | $-i f_3$ |
| 1              | 0 | $i f_0$ | $i f_1$ | 0 | $-i f_3$ | $-i f_4$ |
| 2              | 0 | $-i f_0$ | 0 | $i f_3$ | 0 | $-i f_5$ |
| 3              | $i f_0$ | $-i f_1$ | $-i f_2$ | 0 | $i f_4$ | $i f_5$ | $-i f_6$ |
| 4              | 2$i f_1$ | 0 | $-i f_3$ | $-i f_4$ | 0 | 2$i f_6$ | 0 |
| 5              | 2$i f_2$ | $i f_3$ | 0 | $-i f_5$ | $-2i f_6$ | 0 | 0 |
| 6              | $i f_3$ | $i f_4$ | $i f_5$ | $-i f_6$ | 0 | 0 | 0 |

(25)

Furthermore, the following relation holds

$$
|f_4|^2 |f_5|^2 = 2|f_6|^2.
$$

(26)
The linearly fullness and superminimality conditions can easily be used to calculate as in [6] both the condition (26) and the following products:

\[ f_3 \times f_4 = \text{if}_{\bar{f}_4}, \]
\[ f_3 \times f_5 = \text{if}_{f_5} \]
\[ f_3 \times f_6 = -\text{if}_{\bar{f}_6}. \]

Then all the other products can be obtained by taking derivatives \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \bar{z}} \) and the properties given by formulae (3), (4) and (6).

**Remark 4** It is worth mentioning that the condition (26) characterizes the linearly full superminimal almost complex curves of the 6-sphere (cf. [6]) in the sense that a weakly conformal harmonic map \( f : S \to S^6 \) is \( O(7) \)-congruent to a linearly full superminimal almost complex curve if and only if (26) holds.

We say that a map \( \psi \) from a Riemann surface \( S \) into \( Q^5 \) is superhorizontal if at each point of \( S \), \( \psi^* \) takes values in the superhorizontal distribution. We shall recall now the 1-1 correspondence (cf. [2]) between superminimal almost complex curves in \( S^6 \) and holomorphic superhorizontal curves in \( Q^5 \). We intend to make use of this correspondence later on in this article in order to work out explicit examples of superminimal 2-spheres of \( S^6 \).

By using (16), the superhorizontal condition of \( \psi = [f] \) can be described analytically as follows.

\[ \psi \text{ is superhorizontal} \iff f_*|_{T_pS} \subseteq H^\psi_{(p)} = \{ v \in T_{[f(p)]}Q^5 : f \times v = 0 \} \]
\[ \iff f \times f_*(a \frac{\partial}{\partial z} + b \frac{\partial}{\partial \bar{z}}) = 0 \quad \forall a, b \in \mathbb{R} \]
\[ \iff f \times f_\bar{z} = 0 \quad \text{and} \quad f \times f_z = 0. \]

thus a holomorphic map \( \psi : S \to Q^5 \) is superhorizontal if and only if

\[ f \times \frac{df}{dz} = 0. \quad (27) \]

As a consequence of this characterization, Theorem below is proved by Bryant in [9] and Bolton et al have given a tidy treatment in [2].

**Theorem 3** A map \( g : S \to Q^5 \) is linearly full, holomorphic and superhorizontal if and only if \( \psi = \pi(g) : S \to S^6 \) is a linearly full superminimal almost complex curve in \( S^6 \) with directrix curve \( g \) where \( \pi \) denotes the twistor map from \( Q^5 \) onto \( S^6 \).

Here we can prove the following characterization for almost complex curves in \( S^6 \) in terms of their directrix curves.

**Theorem 4** Let \( f \) and \( \tilde{f} \) be linearly full almost complex 2-spheres of \( S^6 \). Then their directrix curves are projectively equivalent if and only if they are also \( G^\mathbb{C}_2 \)-equivalent.

**Proof:**

\((\Leftarrow)\)

The converse in the theorem is obvious since \( G^\mathbb{C}_2 \) is a subgroup of \( SO(7, \mathbb{C}) \).

\((\Rightarrow)\)

Let \( \{ \psi_j = [f_j] \}_{j=0}^9 \) denote the harmonic sequence corresponding to the harmonic map \( [f] \) and let \( \tilde{\psi}_0 \) denote the directrix curve of the harmonic map \( [\tilde{f}] \). By assumption there exists an element \( [A] \in PGL(7, \mathbb{C}) \) such that \( \tilde{\psi}_0 = [Af_0] \).
It is shown in [5] (Theorem 3.3) that two linearly full harmonic 2-spheres of $S^{2m}$ are projectively equivalent if and only if they are $SO(2m+1,\mathbb{C})$-equivalent. Thus we can assume in our particular case here that $A$ lies in $SO(7,\mathbb{C})$.

According to theorem [30] the map $Af_0 : S^2 \to \mathbb{C}^7$ is holomorphic and superhorizontal and hence

$$Af_0 \times Af_0' = 0. \quad (f_0' = \frac{df_0}{dz}) \quad (28)$$

We shall make use in the sequel of the following properties satisfied by the functions $(\alpha_j)^6_{j=0}$ defined by equation (3).

$$\alpha_3 = 0. \quad \text{Follows from (7)}.$$  

$$\alpha_j = -\alpha_{6-j}. \quad \text{Follows from (6), (29)}.$$  

$$\alpha_6 = \alpha_5 + \alpha_4. \quad \text{Follows from (26), (30)}.$$  

We differentiate (28) with respect to $z$ and use (3), obtaining in this way the cross product between different vectors $Af_j$. By repeating this process we can derive some relations among the cross product of the vectors $Af_j$, namely

$$Af_0 \times Af_1 = 0 \quad (31)$$

$$Af_0 \times Af_2 = 0 \quad (32)$$

$$Af_0 \times Af_3 = -Af_1 \times Af_2 \quad (33)$$

$$Af_0 \times Af_4 = -2Af_1 \times Af_3 \quad (34)$$

$$Af_0 \times Af_5 = -2Af_2 \times Af_3 - 3Af_1 \times Af_4 \quad (35)$$

$$Af_0 \times Af_6 = -5Af_2 \times Af_4 - 4Af_1 \times Af_5 + 3\alpha_5 Af_1 \times Af_4. \quad (36)$$

As $A \in SO(7,\mathbb{C})$, it follows from (5) that

$$(Af_i, Af_j) = \langle Af_i, \overline{Af_j} \rangle = \langle Af_i, Af_j \rangle = \langle f_i, f_j \rangle = (-1)^i \delta_{i,6-j}. \quad (37)$$

The vectors $\{Af_0, \ldots, Af_6\}$ form a basis for $\mathbb{C}^7$ since $A \in SO(7,\mathbb{C})$. Thus, from (37) we see immediately that $Af_0 \times Af_3$ can be written as the linear combination:

$$Af_0 \times Af_3 = aAf_0 + bAf_1 + cAf_2.$$  

But if we take the cross product of this equation with $Af_2$ and $Af_1$, then it follows that $b = c = 0$. Indeed,

$$0 = Af_2 \times (Af_0 \times Af_3) \quad \text{use (37)}$$

$$= b Af_2 \times Af_1 \quad \text{use (32)}$$

$$= b(Af_0 \times Af_3), \quad \text{use (33)}$$

and

$$0 = Af_1 \times (Af_0 \times Af_3) \quad \text{use (37)}$$

$$= c Af_1 \times Af_2 \quad \text{use (31)}$$

$$= -c(Af_0 \times Af_3). \quad \text{use (39)}$$

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On the other hand, if we also take the cross product with $Af_6$, we see that $a = \pm i$. Indeed,

\[
a(Af_0 \times Af_6) = -Af_0 \times (Af_0 \times Af_3) = (Af_0 \times Af_0) \times Af_3 - Af_3,
\]

and the Euclidean product of this with $Af_0$ gives:

\[
-1 = a(Af_0 \times Af_0, Af_3) = -a(Af_0 \times Af_3, Af_6) = -a^2(Af_0, Af_0)
\]

Let us first assume the case $Af_0 \times Af_3 = iAf_0$. Then (34) yields $Af_1 \times Af_3 = iAf_1$, indeed

\[
-2Af_1 \times Af_3 = Af_0 \times Af_4 = i(Af_0 \times Af_3) \times Af_4 = -i(Af_0 \times Af_4) \times Af_3 = 2i(Af_1 \times Af_3) \times Af_3 = -2iAf_1.
\]

Thus from (34) we get

\[
Af_0 \times Af_4 = -2iAf_1.
\]

And this yields:

\[
Af_1 \times Af_4 = \frac{i}{2}(Af_0 \times Af_4) \times Af_4 = 0. \quad \text{(Using (24)).}
\]

Now, by using the equations (25),..,(37), we can carry on with this process to determine all the cross products of the vectors \{Af_0,..,Af_6\} and to verify that they satisfy the multiplication table (25) in the following sense

\[
A(f_i \times f_j) = Af_i \times Af_j.
\]

Therefore, $A$ is an element of $G^7_C$ since \{f_0,..,f_6\} is a basis for $\mathbb{C}^7$.

In the other case to be considered, that is, when $Af_0 \times Af_3 = iAf_0$, we can use the same procedure as above to prove that $-A \in G^7_C$. However, this contradicts our assumption that $A \in SO(7, \mathbb{C})$. ☐

Let $f : S^2 \to S^6$ be a k-point ramified ($k \leq 2$) linearly full almost complex curve and let $\psi_j = [f_j]$ ($j = 0, \ldots, 6$) denote the harmonic sequence corresponding to the harmonic map $\psi_3 = [f_3] = [f]$. Then according to Theorem (2), we can find a local complex coordinate $z$ for $S^2$ and a basis \{v_0,..,v_6\} for $\mathbb{C}^7$, so that the directrix curve $\psi_0 = [f_0]$ can be expressed by:

\[
f_0 = v_0 + z^{k_1}v_1 + z^{k_1+k_2}v_2 + z^{k_1+k_2+k_3}v_3 + z^{k_1+k_2+k_3+k_4}v_4 + z^{k_1+k_2+k_3+k_4+k_5}v_5 + z^{2k_1+2k_2+2k_3+k_4+k_5+k_6}.
\]

Using the meromorphic sections $f_j$ we can choose a particular orthonormal basis \{u_0,..,u_6\} for $\mathbb{C}^7$ so that each vector $u_j$ spans the same complex line bundle as $f_j$. Indeed, as for $z \neq 0$ the function $\frac{f_j}{f_j}$ takes values in the sphere $S^{13}$
then for each \( j \in \{0, \ldots, 6\} \), by compactness, there exists a sequence \( w_{jk} \rightarrow 0 \) so that we have

\[
\lim_{k \to \infty} \frac{f_j}{|f_j|}(w_{jk}) = u_j
\]  

(39)

Now, we can notice that the vectors \( u_j \) are not obtained by orthonormalization but the triangular display below appears naturally from manipulation of equations (38) and (39). For example, the second equation is obtained in the following way:

\[
f_1 + \alpha_0 f_0 = \frac{\partial f_0}{\partial z} = k_1 z^{k_1} v_1 + \ldots \]

(40)

Consequently, we obtain a vector function \( \rho(z) \) satisfying \( \lim_{z \to 0} \rho(z) = 0 \) and

\[
|f_1(w_{jk})|u_1 + \alpha_0 |f_0(w_{jk})|u_0 = k_1 (w_{jk})^{k_1} [v_1 + \rho(w_{jk})]
\]

and hence \( v_1 \) is a linear combination of the vectors \( u_1 \) and \( u_0 \). Similarly, by differentiating successively equation (10), we obtain the following triangular shape:

\[
v_0 = a_{(0,0)} u_0 \\
v_1 = a_{(1,0)} u_0 + a_{(1,1)} u_1 \\
v_2 = a_{(2,0)} u_0 + a_{(2,1)} u_1 + a_{(2,2)} u_2 \\
v_3 = a_{(3,0)} u_0 + a_{(3,1)} u_1 + a_{(3,2)} u_2 + a_{(3,3)} u_3 \\
v_4 = a_{(4,0)} u_0 + a_{(4,1)} u_1 + a_{(4,2)} u_2 + a_{(4,3)} u_3 + a_{(4,4)} u_4 \\
v_5 = a_{(5,0)} u_0 + a_{(5,1)} u_1 + a_{(5,2)} u_2 + a_{(5,3)} u_3 + a_{(5,4)} u_4 + a_{(5,5)} u_5 \\
v_6 = a_{(6,0)} u_0 + a_{(6,1)} u_1 + a_{(6,2)} u_2 + a_{(6,3)} u_3 + a_{(6,4)} u_4 + a_{(6,5)} u_5 + a_{(6,6)} u_6
\]

where the scalars \( a_{(i,j)} \) appearing in the linear combinations are complex numbers.

From (39) and Proposition (2) we can easily determine the cross product of the vectors \( u_j \) and consequently also of the vectors \( v_j \). Namely, the vectors \( u_j \) have the following multiplication table for \( u_i \times u_j \):

\[
\begin{array}{cccccccc}
 i \backslash j & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & -i u_0 & -i \sqrt{2} u_1 & -i u_3 & -i u_4 \\
1 & 0 & 0 & i \sqrt{2} u_0 & i u_1 & 0 & -i u_3 & -i \sqrt{2} u_4 \\
2 & 0 & -i \sqrt{2} u_0 & 0 & i u_2 & i u_3 & 0 & -i \sqrt{2} u_5 \\
3 & i u_0 & -i u_1 & -i u_2 & 0 & i u_4 & i u_5 & -i u_6 \\
4 & i \sqrt{2} u_1 & 0 & -i u_3 & -i u_4 & 0 & i \sqrt{2} u_6 & 0 \\
5 & i \sqrt{2} u_2 & i u_3 & 0 & -i u_5 & -i \sqrt{2} u_6 & 0 & 0 \\
6 & i u_3 & i \sqrt{2} u_4 & i u_6 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Now, we notice that the coefficient \( a_{(0,0)} \) must be non-zero since \( f \) is linearly full and also \( f_0 \) is a holomorphic superhorizontal curve because of the characterization given in Theorem (3). These facts together with equation (27), give us a cumbersome but straightforward calculation to determine the following example.

**Example 1** Let \( \{e_1, \ldots, e_7\} \) denote a \( G_2 \)-basis for \( \mathbb{R}^7 \). Let \( k_1, k_2 \) denote positive integers. Then the holomorphic map \( \psi_0 = [f_0] : S^2 \to \mathbb{CP}^6 \), determined by the polynomial \( f_0(z) = \sum_{j=1}^7 a_j(z) e_j \) where the \( a_j(z) \) are given by
\[ a_1(z) = \frac{\sqrt[3]{3} k_2 k_1 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2} + \sqrt{\frac{3}{2}} z^{3k_1 + 2k_2}, \]
\[ a_2(z) = \frac{12 \sqrt{3} k_2 k_1 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2} + \sqrt{\frac{3}{2}} z^{3k_1 + 2k_2}, \]
\[ a_3(z) = \frac{i 4 \sqrt{3} k_1 k_2^2 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2} + i \sqrt{\frac{3}{2}} z^{3k_1 + 2k_2}, \]
\[ a_4(z) = \frac{6 \sqrt{3} k_1 k_2 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2}, \]
\[ a_5(z) = \frac{i 4 \sqrt{3} k_1 k_2^2 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2} + i \sqrt{\frac{3}{2}} z^{3k_1 + 2k_2}, \]
\[ a_6(z) = -\frac{i 4 \sqrt{3} k_1 k_2 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2} + i \sqrt{\frac{3}{2}} z^{3k_1 + 2k_2}, \]
\[ a_7(z) = -\frac{6 \sqrt{3} k_1 k_2 (k_1 + k_2)}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} z^{k_1 + k_2} + \sqrt{\frac{3}{2}} z^{3k_1 + 2k_2}, \]

is the directrix curve of a linearly full \( S^1 \)-symmetric almost complex 2-sphere in \( S^6 \) with the same singularity type \((k_1 - 1, k_2 - 1, k_1 - 1, k_1 - 1, k_2 - 1, k_1 - 1)\) at \( z = 0 \) and \( z = \infty \).

**Theorem 5** Let \( f : S^2 \to S^6 \) be a \( k \)-point ramified (\( k \leq 2 \)) linearly full almost complex curve with any singularities at \( z = 0 \) and \( z = \infty \). Then for a suitable choice of complex coordinate on \( S^2 \), the harmonic map \([f(z)] : S^2 \to CP^6\) has the same singularity type \((k_1 - 1, k_2 - 1, k_1 - 1, k_1 - 1, k_2 - 1, k_1 - 1)\) at \( z = 0 \) and \( z = \infty \). Moreover, the directrix curve of \( f \) is \( C_2^s \)-equivalent to the \( S^1 \)-symmetric curve given in the Example 1.

**Proof:**
Let \( \psi_0 : S^2 \to Q^3 \) given by \( \psi_0(z) = [f_0(z)] \) be the directrix curve of the map \([f_0(z)] = [f(z)]\). The first part of the statement follows from item (ii) of Theorem 2 and the following observation.

By comparing the exponents of the variable \( z \) appearing in the polynomial \( f \times f_0 = 0 \), we obtain the symmetry \( k_3 = k_1 \) in the singularity type.

Now, we can use the condition given by the equation 27 with the multiplication table for the products \( u_i \times u_j \) in order to determine the vectors \( (u_j) \) in a more simplified way in terms of the vectors \( u_j \), and expressed only in terms of the 8 complex parameters \( r_1 = a_{1,4}, r_2 = a_{5,4}, r_3 = a_{6,3}, r_4 = a_{6,4}, r_5 = a_{6,5}, r_6 = a_{6,2}, r_7 = a_{6,1}, r_8 = a_{5,5} \), in the following way:

\[ r_0 = \frac{k_1 k_2 r_1^{r_2^2}}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} u_0, \]
\[ r_1 = \frac{k_1 k_2 r_1^{r_2^2}}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} (r_5 u_0 + u_1), \]
\[ r_2 = \frac{k_1 k_2 r_1^{r_2^2}}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} (r_2 r_5 - r_4 r_8) u_0 + r_2 u_1 + r_8 u_2, \]
\[ r_3 = \frac{k_1 k_2 r_1^{r_2^2}}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} (\sqrt{2} r_3 u_0 + 2 r_4 u_1 + 2 r_5 u_2 + 2 r_7 u_3 + 2 u_4), \]
\[ r_4 = \frac{k_1 k_2 r_1^{r_2^2}}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} ((2r_3 r_5 - 2r_8) u_0 + (2r_4 r_5 - \sqrt{2} r_3) u_1 + 2 r_5^2 u_2 + 2 \sqrt{2} r_5 u_3 + 2 u_4), \]
\[ r_5 = \frac{k_1 k_2 r_1^{r_2^2}}{(3k_1^2 + 2k_2^2)(2k_1 + k_2)^2} (r_2 r_5 r_5 - \sqrt{2} r_3 r_4 r_8 + r_7 r_8 - r_2 r_6) u_0 + (r_2 r_4 r_5 - \sqrt{2} r_2 r_3 - r_4 r_8 u_2), \]
\[ + (r_2 r_5 - \sqrt{2} r_3 r_8) u_2 + \sqrt{2} (r_2 r_5 - r_4 r_8) u_3 + r_2 u_4 + r_8 u_5, \]
\[ r_6 = (r_5 r_7 + \frac{1}{2} r_2^2 - r_4 r_6) u_0 + r_7 u_1 + r_6 u_2 + r_3 u_3 + r_4 u_4 + r_5 u_5 + u_6. \]

By Corollary 1, we can assume \( f \) to be \( S^1 \)-symmetric. Theorem 2 shows that the \( S^1 \)-symmetric linearly full almost complex 2-spheres are characterized by the orthogonality of the vectors \( (u_j) \) and hence according to the formulae above we must have \( r_2 = \ldots = r_7 = 0 \). Thus, the directrix curve is described
by the 2-parameter family

\[f_0(z) = \frac{k_1k_2^2(k_1 + k_2)r_1^2r_8^2}{(3k_1 + 2k_2)(3k_1 + k_2)(2k_1 + k_2)}z^{k_1}u_1 + \frac{k_1k_2r_1^2r_8}{(3k_1 + 2k_2)(2k_1 + k_2)}z^{k_1}u_1\]

\[+ \frac{k_2(k_1 + k_2)r_1r_8}{(2k_1 + k_2)(3k_1 + k_2)}z^{k_1+k_2}u_2 + \sqrt{2} \frac{k_2r_1r_8}{(2k_1 + k_2)}z^{2k_1+k_2}u_3\]

\[+ r_1z^{3k_1+k_2}u_4 + r_8z^{3k_1+2k_2}u_5 + z^{4k_1+2k_2}u_6.\]

Now, we shall apply a suitable conformal transformation to the domain and also apply an appropriate element of \(G_2\) to the co-domain in order to prove that \(f\) is indeed equivalent to the curve given in the example above.

Let \(r\) be a complex root for the equation

\[r^{2k_1+k_2}r_1r_8 = \sqrt{90}. \quad (41)\]

Then we shall consider the conformal transformation \(z \mapsto rz\), and the element \(A \in G_2\) defined by

\[Au_0 := \left(\frac{90}{r_8}\right)u_0, \quad Au_1 := \left(\frac{15\sqrt{6}}{r_1r_8}\right)u_1, \quad Au_2 := \left(\frac{6\sqrt{7}}{r_1r_8}\right)u_2, \quad Au_3 := \left(\frac{\sqrt{6}}{(4r_1r_8)}\right)u_3, \quad Au_4 := \left(\frac{\sqrt{3}}{(4r_1r_8)}\right)u_4, \quad Au_5 := \left(\frac{\sqrt{3}}{(4r_1r_8)}\right)u_5, \quad Au_6 := \left(\frac{1}{(4r_1r_8)}\right)u_6.\]

Using (41) and the multiplication table (11) for the vectors \(u_j\) we deduce that

\[A(u_i \times u_j) = Au_i \times Au_j,\]

which implies that \(A \in G_2\). Thus, the holomorphic curve \(Af_0\) is reduced to

\[Af_0(z) = \frac{90k_1k_2^2(k_1 + k_2)}{(3k_1 + 2k_2)(3k_1 + k_2)(2k_1 + k_2)}z^{k_1}u_1 + \frac{15\sqrt{6}k_1k_2}{(3k_1 + 2k_2)(2k_1 + k_2)}z^{k_1}u_1\]

\[+ \frac{6\sqrt{15}k_2(k_1 + k_2)}{(2k_1 + k_2)(3k_1 + k_2)}z^{k_1+k_2}u_2 + \sqrt{2} \frac{6\sqrt{7}k_2}{(2k_1 + k_2)}z^{2k_1+k_2}u_3\]

\[+ \sqrt{15}z^{3k_1+k_2}u_4 + \sqrt{6}z^{3k_1+2k_2}u_5 + z^{4k_1+2k_2}u_6.\]

Using again that multiplication table we can also deduce by straightforward calculations that the vectors \(e_j \in \mathbb{R}^7\) (\(j = 1, \ldots, 7\)) defined by

\[u_0 = \frac{1}{\sqrt{2}}(-e_7 + ie_3), \quad u_1 = \frac{1}{\sqrt{2}}(e_2 - ie_6), \quad u_2 = \frac{1}{\sqrt{2}}(-e_1 + ie_5), \quad u_3 = e_4, \quad u_4 = \frac{1}{\sqrt{2}}(e_1 + ie_5), \quad u_5 = \frac{1}{\sqrt{2}}(e_2 + ie_6), \quad u_6 = \frac{1}{\sqrt{2}}(e_7 + ie_3),\]

form a \(G_2\)-basis for \(\mathbb{R}^7\) and the holomorphic curve \(Af_0(z)\) is written in terms of this basis exactly as the one we gave in the example.

**Corollary 2** If \(H^{o,0}\) is the space of linearly full totally unramified almost complex maps of \(S^2\) into \(S^6\) then \(H^{o,0} = G_2^c\).

**Proof:**

Indeed, this follows from the theorem above and the fact that the harmonic sequence corresponding to a harmonic map \([f]\), where \(f \in H^{o,0}\), is uniquely determined by its directrix curve. Some care is required here since the composition of \(f\) with the antipodal map of \(S^6\) gives also a harmonic map with that same
directrix curve. However, the map $-f$ is fortunately an almost anticomplex curve as we can see from (24).

Let $M$ denote the quotient set of the manifold $N = \{(p, q) \in S^2 \times S^2 / p \neq q\}$ by the equivalence relation: $(p, q) \cong (a, b)$ if and only if $p = b$ and $q = a$.

**Corollary 3** Let $H^{r_1, r_2}$ denote the space of linearly full almost complex maps of $S^2$ into $S^6$ with 2 higher singularities each of type $(r_1, r_2, r_1, r_1, r_2, r_1)$. Then $H^{r_1, r_2} = M \times G^2$.

Let $\psi_0, \ldots, \psi_n : S^2 \to \mathbb{C}P^n$ be a harmonic sequence with corresponding local lifts $f_0, \ldots, f_n : S^2 \setminus W \to \mathbb{C}P^n$ given in accordance with (3) and (4), where $W$ is the set of all singularities of the harmonic maps $\psi_p$. Bolton et al have proved in [8] that when $\psi_p$ is an immersion, the area $A(\psi_p)$ of $S^2$ with the metric induced by $\psi_p$ is given by

$$A(\psi_p) = \pi(\delta_{p-1} + \delta_p),$$

where $\delta_{-1} = 0$ and $\delta_p$ is the degree of the $(p - 1)$-st osculating curve $\sigma_{p-1}$. Moreover, they calculate this degree in terms of the $\gamma_p$ invariants. Namely,

$$\delta_p = \frac{1}{2\pi i} \int_{S^2} \gamma_p d\bar{\sigma} \wedge dz.$$  \hspace{1cm} (43)

Bolton et al carry on working out the following global Plücker formula, relating the ramification indices $R_p$ of the curves $\sigma_{p-1}$ to the degrees $\delta_p$ by

$$R_p = -2 - \delta_{p-2} + 2\delta_{p-1} - \delta_p.$$  \hspace{1cm} (44)

Finally, they also write down the degrees $\delta_p$ in terms of the ramification indices $R_p$ as follows

$$\delta_p = (p + 1)(n - p) + \frac{n - 1}{n + 1} \sum_{k=0}^{p-1} (k + 1)R_k + \frac{\delta_1 + 1}{n + 1} \sum_{k=p}^{n-1} (n - k)R_k.$$  \hspace{1cm} (45)

Using these results for the case $n = 6$ we can now produce the following consequence

**Lemma 4** Let $f : S^2 \to S^6$ be a linearly full almost complex curve. Let $(\psi_p)_{p=0}^6$ be the harmonic sequence determined by $f$. Then the ramification indices $R_p$ of the associated osculating curves $\sigma_{p-1}$ satisfy

(i) $R_j = R_{7-j}$ for $j = 1, \ldots, 6$.

(ii) $R_3 = R_2$.

**Proof:** Considering that $f$ is superminimal (see paragraph after Definition (1)) item (i) follows from direct application of (44), (43), (6) and (26), while item (ii) follows from (44), (43) and (26).  \hspace{1cm} (∗)

**Proposition 3** Let $f : S^2 \to S^6$ be a linearly full almost complex curve with total singularity type $(R_1, \ldots, R_6)$. Then the area $A(\psi)$ of the harmonic map $\psi = [f] : S^2 \to \mathbb{C}P^6$ is given by

$$A(\psi) = 4\pi(6 + 2R_1 + R_2).$$  \hspace{1cm} (46)
Proof:
Using the lemma above and (45) we have
\[ \delta_3 = 12 + 4R_1 + 2R_2 = \delta_4. \]
Thus, the Corollary follows from (42).

\[ \Box \]

Theorem 6 Let \( \mathcal{H}^d \) be the space of linearly full almost complex maps of \( S^2 \) into \( S^6 \) of area \( 4\pi d \). Then \( d \geq 6 \) and
(i ) \( \mathcal{H}^6 = \mathcal{H}^{0,0} = G_2^C \),
(ii ) \( \mathcal{H}^7 \) is empty,
(iii ) \( \mathcal{H}^8 = \mathcal{H}^{(0,1)} = M \times G_2^C \).
Furthermore, every element of \( \mathcal{H}^8 \) has directrix curve \( G_2^C \)-equivalent to the following \( S^1 \)-symmetric case
\[ f(z) = (70\sqrt{15}z^5 - 126\sqrt{15}z^3)e_1 + (70\sqrt{6}z^7 + 75\sqrt{6})e_2 \
+ (i135 + i70z^8)e_3 + 210\sqrt{10}z^4e_4 + (i70\sqrt{15}z^3 + i126\sqrt{15}z^3)e_5 \
+ (70\sqrt{6}z^7 - i75\sqrt{6}z)e_6 + (-135 + 70z^8)e_7. \quad (47) \]

Proof:
Item (i) and \( d \geq 6 \) follow are consequence of (46), whilst item (ii) follows from the fact already noted in the Remark that there does not exist a 1-point ramified harmonic 2-sphere in \( \mathbb{C}P^n \).

Now to prove (iii) we start noting that according to (46) the first possibility for the singularity type is \( R_0 = 0 \) and \( R_1 = 2 \). This case implies that \( [f] \) is 2-point ramified with singularity type at each point given by \( (0, 1, 0, 0, 1, 0) \) and hence (iii) and (47) follow from Theorem (5).

\[ \Box \]

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