Relative Subanalytic Sheaves II

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Abstract. We give a new construction of sheaves on a relative site associated to a product $X \times S$ where $S$ plays the role of a parameter space, expanding the previous construction by the same authors, where the subanalytic structure on $S$ was required. Here we let this last condition fall. In this way the construction becomes much easier to apply when the dimension of $S$ is bigger than one. We also study the functorial properties of base change with respect to the parameter space.

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1. Introduction

In a previous work [11], the authors introduced the notion of relative subanalytic sheaf associated to a sheaf on a subanalytic site of the form \((X \times S)_{sa}\), where \(X\) and \(S\) are real analytic manifolds. Here “relative” concerns the projection \(p : X \times S \to S\) so that \(S\) is regarded as a parameter space.

The background for all the constructions comes from the work of Kashiwara–Schapira [8].

The main purpose of [11] was to provide the subanalytic tools for a relative Riemann–Hilbert correspondence, now assuming that \(X\) and \(S\) are complex manifolds, generalizing the famous Kashiwara’s Riemann-Hilbert correspondence in the absolute framework (the absolute case meaning that \(S\) is a point). This correspondence was achieved in [3] under the assumption that \(S\) is a complex curve. In that case the new objects are naturally endowed with an action of \(p^{-1}O_S\)-modules, \(\mathcal{D}_X\) being replaced by the sheaf of relative differential operators \(\mathcal{D}_{X \times S/S}\).

Our new goal is to provide the subanalytic tools to prove the relative Riemann-Hilbert correspondence for arbitrary dimension of \(S\) (noted by \(d_S\)). Let us explain the motivation:

While in the absolute case the reconstruction functor was based on the notion of “tempered distribution”, “tempered holomorphic function”, giving rise to the subanalytic sheaves \(\mathcal{D}^b_X\) and \(\mathcal{O}^t_X\), in the relative case we needed to construct a “relative” version of those sheaves as sheaves on a relative site “forgetting” the growth conditions on \(S\). In [11], our idea was to construct \(X_{sa} \times S_{sa}\) respecting both subanalytic structures in \(X\) and \(S\), and there live the subanalytic sheaves \(\mathcal{D}^b_{X \times S}\) and \(\mathcal{O}^t_{X \times S}\).

However it became clear that such a construction was too strict. Let us clarify the main obstruction:

Assume that \(S\) is complex. We see that \(p^{-1}O_S\), as a sheaf on \(X_{sa} \times S_{sa}\) is not concentrated in degree zero unless \(d_S = 1\), since otherwise Stein subanalytic open subsets do not form a basis for \(S_{sa}\). So, for arbitrary \(d_S\), a new site should be constructed solving this difficulty.

The main idea was then to replace \(X_{sa} \times S_{sa}\) by \(X_{sa} \times S\) which means that we consider \(S\) as a site with the usual topology. In this new setting we are able to consider infinite coverings of open subsets in \(S\) and, in particular, we have a basis of the topology consisting of products of subanalytic open subsets of \(X\) and Stein open subsets of \(S\). Working with this construction is not trivial and requires some technical results (Propositions 3.1 and 3.5).

In order to obtain them we introduce the notion of locally weakly quasi-compact site, which provides a general setting for many examples including the relative case. Now the relative sheaves “live” on \(X_{sa} \times S\) and we note \(\mathcal{D}^b_{X \times S}\) the relative sheaf associated to \(\mathcal{D}^b_{X \times S}\) and \(\mathcal{O}^t_{X \times S}\) the relative sheaf associated to \(\mathcal{O}^t_{X \times S}\).

We study the behavior of these sheaves under morphisms of the parameter space and we end with an application to a key particular case of the relative Riemann-Hilbert functor (the \(S\)-locally constant case studied in [13]).

The contents of this paper are as follows.
In Sect. 2 we prepare preliminaries to the study of the relative site. We recall some results about the subanalytic site \((X \times S)_{sa}\) and the site \(X_{sa} \times S_{sa}\) to be used in the second part of the paper. After that we introduce the notion of locally weakly quasi compact site which will apply to the relative subanalytic site \(X_{sa} \times S\). Using this generalization it is easier to establish some basic results about sheaves (in particular working with limits) and acyclic objects.

In Sect. 3 we apply the previous framework to the case of a product \(X \times S\) where \(X\) and \(S\) are complex manifolds. In the first part we re-state and re-prove the technical tools of [11] in this new framework. In order to do that we first need to prove some results to describe sections and acyclic objects in \(X_{sa} \times S\) (Propositions 3.1 and 3.5) and relate this site with \(X \times S\) and \(X_{sa} \times S_{sa}\) (Propositions 3.3 and 3.6). After that we are able to study the relativization functor \((\bullet)^{RS}\) starting from the results of [11]. In the second part we study the functorial properties of the sheaf \(\mathcal{O}^{t.S}_{X \times S}\) in view of the Riemann-Hilbert correspondence.

2. Preliminaries

In this section we state some results needed to study the relative site of Sect. 3. First we recall the constructions of [8,11]. Then we define sheaves on locally weakly quasi-compact sites and we consider a useful family of acyclic objects. Throughout all the section, \(k\) will denote a field.

2.1. Sheaves on a Subanalytic Site

The results of this section are extracted from [8] (see also [14] for a more detailed study).

Let \(X\) be a real analytic manifold and let \(k\) be a field. Denote by \(\text{op}(X_{sa})\) the category of open subanalytic subsets of \(X\). One endows \(\text{op}(X_{sa})\) with the following topology: \(S \subset \text{op}(X_{sa})\) is a covering of \(U \in \text{op}(X_{sa})\) if for any compact \(K\) of \(X\) there exists a finite subset \(S_0 \subset S\) such that \(K \cap \bigcup_{V \in S_0} V = K \cap U\). We will call \(X_{sa}\) the subanalytic site.

Let \(\text{Mod}(k_{X_{sa}})\) denote the category of sheaves on \(X_{sa}\) and let \(\text{Mod}_{\mathbb{R}-c}(k_X)\) (resp. \(\text{Mod}_{\mathbb{R}-c}(k_X)\)) be the abelian category of \(\mathbb{R}\)-constructible sheaves on \(X\) (resp. \(\mathbb{R}\)-constructible with compact support).

We denote by \(\rho : X \to X_{sa}\) the natural morphism of sites. We have functors

\[
\text{Mod}_{\mathbb{R}-c}(k_X) \subset \text{Mod}(k_X) \xrightarrow{\rho_*} \text{Mod}(k_{X_{sa}}).
\]

The functors \(\rho^{-1}\) and \(\rho_*\) are the functors of inverse image and direct image associated to \(\rho\). The functor \(\rho^{-1}\) admits a left adjoint, denoted by \(\rho_i\). The sheaf \(\rho_! F\) is the sheaf associated to the presheaf \(\text{op}(X_{sa}) \ni U \mapsto F(U)\).

The functor \(\rho_*\) is fully faithful and exact on \(\text{Mod}_{\mathbb{R}-c}(k_X)\) and we identify \(\text{Mod}_{\mathbb{R}-c}(k_X)\) with its image in \(\text{Mod}(k_{X_{sa}})\) by \(\rho_*\).

Thanks to \(\mathbb{R}\)-constructible sheaves we have some structure results for subanalytic sheaves, namely
(i) Let \( G \in \text{Mod}_{\mathbb{K}}(k_X) \) and let \( \{F_i \} \) be a filtrant inductive system in \( \text{Mod}(k_{X_{sa}}) \). Then we have the isomorphism
\[
\lim_i \text{Hom}_{k_X}(\rho_* G, F_i) \cong \text{Hom}_{k_{X_{sa}}}(\rho_* G, \lim_i F_i).
\]
In particular, if \( U \in \text{op}(X_{sa}) \) is relatively compact
\[
\lim_i \Gamma(U; F_i) \cong \Gamma(U; \lim_i F_i).
\]

(ii) Let \( F \in \text{Mod}(k_{X_{sa}}) \). There exists a small filtrant inductive system \( \{F_i\}_{i \in I} \) in \( \text{Mod}_{\mathbb{K}}(k_X) \) such that \( F \cong \lim_i \rho_* F_i \).

In order to find suitable acyclic resolutions, let us recall the definition of quasi-injective sheaves. \( F \in \text{Mod}(k_{X_{sa}}) \) is quasi-injective if for any relatively compact \( U \in \text{op}(X_{sa}) \) the restriction morphism \( \Gamma(X; F) \to \Gamma(U; F) \) is surjective.

Let \( G \in \text{Mod}_{\mathbb{K}}(k_X) \). Then the following hold:
(i) The family of quasi-injective sheaves is injective with respect to the functor \( \text{Hom}(G, \bullet) \). In particular, it is injective with respect to the functor \( \Gamma(U; \bullet) \), for each \( U \in \text{op}(X_{sa}) \).
(ii) The family of quasi-injective sheaves is injective with respect to the functor \( \mathcal{H}om(G, \bullet) \). In particular, it is injective with respect to the functor \( \Gamma_U(\bullet) \), for each \( U \in \text{op}(X_{sa}) \).

2.2. Sheaves on a Product of Subanalytic Sites

As a motivation for the contents of this section we start by recalling the example provided by the construction of [11]. Let \( X \) and \( S \) be two real analytic manifolds. It is possible to consider the product of sites \( X_{sa} \times S_{sa} \). A basis \( T \) of this topology consists of products \( U \times V \) where \( U \in \text{op}(X_{sa}) \) and \( V \in \text{op}(S_{sa}) \) are relatively compact. Then \( W \in \text{op}(X_{sa} \times S_{sa}) \) if it is a locally finite union of elements of \( T \). A subset \( T \subset \text{op}(X_{sa} \times S_{sa}) \) is a covering of \( U \times V \in T \) if it admits a finite subcover. More generally, \( T \in \text{Cov}(W) \), where \( W \in \text{op}(X_{sa} \times S_{sa}) \), if and only if, for any \( T \supseteq U \times V \subset W \), one has \( \{W' \cap (U \times V), \ W' \in T \} \in \text{Cov}(U \times V) \) (i.e. the intersection of \( T \) with \( U \times V \) admits a finite subcover). We denote by \( \rho': X \times S \to X_{sa} \times S_{sa} \) the natural morphism of sites. There are well defined functors:
\[
\text{Mod}(k_{X \times S}) \xrightarrow{\rho'_*} \text{Mod}(k_{X_{sa} \times S_{sa}}).
\]
The functor \( \rho'_* \) is fully faithful. The functor \( \rho'^{-1} \) admits a left adjoint, denoted by \( \rho'_! \). It is fully faithful, exact and commutes with \( \otimes \). Given \( F \in \text{Mod}(k_{X_{sa} \times S_{sa}}) \), \( \rho'_! F \) is the sheaf associated to the presheaf
\[
T \ni U \times V \mapsto F(U \times V)
\]

Let us consider the category \( \text{coh}(T) \) of sheaves of \( \text{Mod}(k_{X \times S}) \), admitting a finite resolution by sums \( \oplus_{i \in I} k_{U_i \times V_i} \) with \( I \) finite and \( U_i \times V_i \in T \) for each \( i \). The category \( \text{coh}(T) \) is additive and stable by kernels and cokernels.
Moreover:
• The restriction of $\rho_i^*$ to $\text{coh}(T)$ is exact. Hence the latter can be regarded as a subcategory of $\text{Mod}(k_{X,a} \times S_{a})$.

• A sheaf $F \in \text{Mod}(k_{X,a} \times S_{a})$ can be seen as a filtrant inductive limit $\lim_i \rho_i^* F_i$ with $F_i \in \text{coh}(T)$.

• The functors $\text{Hom}(G, \bullet)$, $\mathcal{H}\text{om}(G, \bullet)$, with $G \in \text{coh}(T)$, commute with filtrant $\lim$.

Finally, recall (cf [2]) that $F \in \text{Mod}(k_{X,a} \times S_{a})$ is $T$-flabby if the restriction morphism $\Gamma(X; F) \rightarrow \Gamma(W; F)$ is surjective for each $W \in T$. $T$-flabby objects are $\text{Hom}(G, \bullet)$-acyclic and $\mathcal{H}\text{om}(G, \bullet)$-acyclic for each $G \in \text{coh}(T)$.

### 2.3. Sheaves on Locally Weakly Quasi-Compact Sites

Let $\mathcal{C}_X \subset \text{op}(X)$, where $\text{op}(X)$ is the category open subsets of a topological space $X$. Until the end of the section we will assume that $\mathcal{C}_X$ is stable under finite unions and intersections. As a category, if $U, V \in \mathcal{C}_X$ one has $U \rightarrow V$ if and only if $U \subset V$. If $U \in \mathcal{C}_X$ we denote by $\mathcal{C}_U$ the subcategory of $\mathcal{C}_X$ consisting of open subsets $V \subset U$. We suppose that a Grothendieck topology on $\mathcal{C}_X$ is given and for each $U \in \mathcal{C}_X$ the set Cov$(U)$ is given by a subfamily of topological coverings of $U$. We denote by $X$ the associated site. Given a covering $S$ of $U$, and $V \rightarrow U$, we set $S \times_U V = \{ W \cap V, \ W \in S \}$.

**Definition 2.1.** One defines the site $X^f$ as follows: $S$ is a covering of $U \in \mathcal{C}_X$ in $X^f$ if it admits a finite refinement. We set Cov$(f)(U)$ the family of coverings of $U$ in $X^f$.

**Definition 2.2.** (i) Let $U, V \in \mathcal{C}_X$ and let $V \rightarrow U$. One says that $V$ is weakly quasi-compact (= wc) in $U$ if, for any $S \in \text{Cov}(U)$, we have $S \times_U V \in \text{Cov}^f(V)$. We will write $V \xrightarrow{\text{wc}} U$ to say that $V \rightarrow U$ is wc in $U$.

(ii) Let $U, V \in \mathcal{C}_X$ with $V \rightarrow U$ and let $T \in \text{Cov}(V)$, $S \in \text{Cov}(U)$. We write $T \xrightarrow{\text{wc}} S$ if $T$ is a refinement of $S \times_U V$ and for any $V' \in T, U' \in S$ one has $V' \rightarrow U'$ iff $V' \xrightarrow{\text{wc}} U'$.

Note that when $S = U \in \mathcal{C}_U$ and $T = V \in \mathcal{C}_V$ we recover Definition 2.2(i).

**Definition 2.3.** A site $X$ is locally weakly quasi-compact if:

- LWQC1 for each $U \in \mathcal{C}_X$ $\{ V \xrightarrow{\text{wc}} U \} \in \text{Cov}(U)$,
- LWQC2 for each $U, V, W \in \mathcal{C}_X$, if $W \xrightarrow{\text{wc}} U$ and $W \xrightarrow{\text{wc}} V$ one has $W \xrightarrow{\text{wc}} U \times X V$,
- LWQC3 for each $V \xrightarrow{\text{wc}} U$ there exists $W \xrightarrow{\text{wc}} U$ with $V \xrightarrow{\text{wc}} W$.

**Lemma 2.4.** Let $V \xrightarrow{\text{wc}} U$. Then for each $S \in \text{Cov}(U)$ there exists $T^f \in \text{Cov}^f(V)$ such that $T^f \xrightarrow{\text{wc}} S$.

**Proof.** By LWQC3 there exists $V' \xrightarrow{\text{wc}} U$ with $V \xrightarrow{\text{wc}} V'$. By LWQC1 for each $U_j \in S$ we have $\{ W_{ij} \xrightarrow{\text{wc}} U_j \times_U V' \} \in \text{Cov}(U_j \times_U V')$, hence $\bigcup_j \{ W_{ij} \xrightarrow{\text{wc}} U_j \times_U V' \} \in \text{Cov}(V')$ is a refinement of $S \times_U V'$. We have $T := \bigcup_j \{ W_{ij} \xrightarrow{\text{wc}} U_j \times_U V' \rightarrow V' \} \in \text{Cov}(V')$ and $T^f := T \times_V' V \in \text{Cov}^f(V)$. This implies $T^f \xrightarrow{\text{wc}} S$. □

**Lemma 2.5.** Let $F \in \text{Psh}(k_X)$, and let $U \in \mathcal{C}_X$. If $F$ is a sheaf on $X^f$, then for any $V \xrightarrow{\text{wc}} U$ the morphism

$$F^+(U) \rightarrow F^+(V)$$

(2.1)
factors through $F(V)$.

Proof. Let $S \in \text{Cov}(U)$. There is a finite refinement $T^f \in \text{Cov}^f(V)$ of $S \times_U V$. Then the morphism (2.1) is defined by

$$F^+(U) \simeq \lim_{S \in \text{Cov}(U)} F(S) \to \lim_{S \in \text{Cov}(U)} F(S \times_U V) \to \lim_{T^f \in \text{Cov}^f(V)} F(T^f) \to \lim_{T \in \text{Cov}(V)} F(T) \simeq F^+(V).$$

$F$ is a sheaf on $X^f$, hence $F(T^f) \simeq F(V)$ and the result follows. □

Corollary 2.6. Under the hypothesis of Lemma 2.5, let $S \in \text{Cov}(U)$ and $T \in \text{Cov}(V)$. If $T \xrightarrow{\text{wc}} S$, then the morphism

$$F^+(S) \to F^+(T) \quad (2.2)$$

factors through $F(T)$. In particular, if $T \in \text{Cov}^f(V)$, then the morphism (2.2) factors through $F(V)$.

From now on we assume that $X$ is locally weakly quasi-compact.

Lemma 2.7. Let $F \in \text{Psh}(k_X)$. Then, for any $U \in \mathcal{C}_X$, we have the isomorphism

$$\lim_{V \xrightarrow{\text{wc}} U} \lim_{W \xrightarrow{\text{wc}} V} F(W) \simeq \lim_{V \xrightarrow{\text{wc}} U} F(V).$$

Proof. The result follows since, for any $U \in \mathcal{C}_X$, for each $V \xrightarrow{\text{wc}} U$, there exists $W \xrightarrow{\text{wc}} U$ such that $V \xrightarrow{\text{wc}} W$ by LWQC3. Let $V \xrightarrow{\text{wc}} U$. The restriction morphism $F(U) \to F(V)$ factors through $\lim_{W \xrightarrow{\text{wc}} V} F(W)$. Taking the projective limit we obtain the result. □

Lemma 2.8. Let $F \in \text{Psh}(k_X)$, and let $U \in \mathcal{C}_X$. If $F$ is a sheaf on $X^f$, then for any $V \xrightarrow{\text{wc}} U$ the morphism

$$F^{++}(U) \to F^{++}(V) \quad (2.3)$$

factors through $F(V)$. 
Proof. Let \( S \in \text{Cov}(U) \). Thanks to Lemma 2.4 we can construct a refinement \( T^f \in \text{Cov}^f(V) \) of \( S \times_U V \) such that \( T^f \xrightarrow{wc} S \). The morphism (2.3) is defined by

\[
F^{++}(U) \simeq \lim_{S \in \text{Cov}(U)} F^+(S) \\
\rightarrow \lim_{S \in \text{Cov}(U)} F^+(S \times_U V) \\
\rightarrow \lim_{T^f \in \text{Cov}^f(V)} F^+(T^f) \\
\rightarrow \lim_{T \in \text{Cov}(V)} F^+(T) \\
\simeq F^{++}(V).
\]

By Corollary 2.6, the morphism

\[
\lim_{S \in \text{Cov}(U)} F^+(S) \rightarrow \lim_{T^f \in \text{Cov}^f(V)} F^+(T^f)
\]

factors through \( F(V) \) and the result follows. \( \square \)

Corollary 2.9. Let \( F \in \text{Psh}(kX) \). If \( F \) is a sheaf on \( X^f \), then:

(i) for any \( V \in \mathcal{C}_X \) one has an isomorphism \( \lim_{V \xrightarrow{wc} U} F(U) \simeq \lim_{V \xrightarrow{wc} U} F^{++}(U) \).

(ii) for any \( U \in \mathcal{C}_X \) one has an isomorphism \( \lim_{U \xrightarrow{wc} V} F(V) \simeq \lim_{U \xrightarrow{wc} V} F^{++}(V) \).

Proof. By Lemma 2.8 for each \( V \xrightarrow{wc} U \) we have a commutative diagram

\[
\begin{array}{ccc}
F^{++}(U) & \rightarrow & F^{++}(V) \\
\uparrow & & \uparrow \\
F(U) & \rightarrow & F(V)
\end{array}
\]

Passing to the inductive (resp. projective) limit we obtain \( \lim_{V \xrightarrow{wc} U} F(U) \simeq \lim_{V \xrightarrow{wc} U} F^{++}(U) \) (resp. \( \lim_{U \xrightarrow{wc} V} F(V) \simeq \lim_{U \xrightarrow{wc} V} F^{++}(V) \)) as required. The isomorphism follows from the universal property and the uniqueness, up to isomorphisms, of the inductive (resp. projective limit). \( \square \)

Let \( \{F_i\}_{i \in I} \) be a filtrant inductive system in \( \text{Mod}(kX) \). One sets

\[
\text{“lim” } F_i = \text{inductive limit in the category of presheaves}, \\
\lim_{i} F_i = \text{inductive limit in the category of sheaves}.
\]

Recall that \( \lim_{i} F_i = (\text{“lim” } F_i)^{++} \).

Proposition 2.10. Let \( \{F_i\}_{i \in I} \) be a filtrant inductive system in \( \text{Mod}(kX) \). Then

\[
\text{“lim” } F_i \text{ is a sheaf on } X^f.
\]
Proof. Let \( U \in \mathcal{C}_X \) and let \( S \) be a finite covering of \( U \). Since \( \lim \) commutes with finite projective limits we obtain the isomorphism \( \left( \lim_{\rightarrow \rightarrow} F_i \right)(S) \cong \lim_{\rightarrow \rightarrow} F_i(S) \) and \( F_i(U) \cong F_i(S) \) since \( F_i \in \text{Mod}(k_T) \) for each \( i \). Moreover the family of finite coverings of \( U \) is cofinal in \( \text{Cov}(U) \). Hence \( \lim_{\rightarrow \rightarrow} F_i \cong \left( \lim_{\rightarrow \rightarrow} F_i \right)^{++} \) on \( X^f \). Applying once again the functor \(( \bullet )^+\) we get
\[
\left( \lim_{\rightarrow \rightarrow} F_i \right)^+ \cong \left( \lim_{\rightarrow \rightarrow} F_i \right)^{++} \cong \lim_{\rightarrow \rightarrow} F_i
\]
on \( X^f \) and the result follows. \( \square \)

Now, setting \( F = \lim_{\rightarrow \rightarrow} F_i \) in Lemma 2.8 and in Corollary 2.9 we obtain the following results:

**Proposition 2.11.** Let \( \{F_i\}_{i \in I} \) be a filtrant inductive system in \( \text{Mod}(k_X) \) and let \( U \in \mathcal{C}_X \). Then for any \( V \xrightarrow{w} U \) the morphism
\[
\Gamma(U; \lim_{\rightarrow \rightarrow} F_i) \to \Gamma(V; \lim_{\rightarrow \rightarrow} F_i)
\]
factors through \( \lim_{\rightarrow \rightarrow} \Gamma(V; F_i) \).

**Corollary 2.12.** Let \( \{F_i\}_{i \in I} \) be a filtrant inductive system in \( \text{Mod}(k_X) \).

(i) For any \( U \in \mathcal{C}_X \) one has the isomorphism
\[
\lim_{\rightarrow \rightarrow} V \xrightarrow{w} U \Gamma(V; F_i) \cong \lim_{\rightarrow \rightarrow} \Gamma(V; F_i) \]

(ii) For any \( U \in \mathcal{C}_X \) one has the isomorphism
\[
\lim_{\rightarrow \rightarrow} \Gamma(V; F_i) \cong \lim_{\rightarrow \rightarrow} \Gamma(V; F_i)
\]

**2.4. c-Soft Sheaves**

Let \( X \) be a locally weakly quasi-compact site.

**Definition 2.13.** We say that a sheaf \( F \) on \( X \) is c-soft if the restriction morphism
\[
\Gamma(W; F) \to \lim_{\rightarrow \rightarrow} \Gamma(U; F)
\]
is surjective for each \( V, W \in \mathcal{C}_Y \) (\( Y \in \mathcal{C}_X \)) with \( V, W \xrightarrow{w} Y \) and \( V \xrightarrow{w} W \).

**Proposition 2.14.** Let \( 0 \to F' \to F \to F'' \to 0 \) be an exact sequence in \( \text{Mod}(k_X) \), and assume that \( F' \) is c-soft. Then the sequence
\[
0 \to \lim_{\rightarrow \rightarrow} \Gamma(U; F') \to \lim_{\rightarrow \rightarrow} \Gamma(U; F) \to \lim_{\rightarrow \rightarrow} \Gamma(U; F'') \to 0
\]
is exact for any \( V \in \mathcal{C}_X \).
Proof. Let $s'' \in \lim_{\leftarrow} \Gamma(U; F'')$. Then there exists $V \xrightarrow{\text{wc}} U$ such that $s''$ is represented by $s'''_U \in \Gamma(U; F'')$. Let $\{U_i\}_{i \in I} \in \text{Cov}(U)$ such that there exists $s_i \in \Gamma(U_i; F)$ whose image is $s'''_{U_i}|_{U_i}$ for each $i$. There exists $W \xrightarrow{\text{wc}} U$ with $V \xrightarrow{\text{wc}} W$, a finite covering $\{W_j\}_{j=1}^n$ of $W$ and a map $\varepsilon : J \to I$ of the index sets such that $W_j \xrightarrow{\text{wc}} U_{\varepsilon(j)}$. We may argue by induction on $n$. If $n = 2$, set $U_i = U_{\varepsilon(i)}$, $i = 1, 2$. Then $(s_1 - s_2)|_{U_1 \cap U_2}$ belongs to $\Gamma(U_1 \cap U_2; F')$, and its restriction defines an element of $\lim_{\leftarrow} \Gamma(W'; F')$, hence it extends to $s' \in \Gamma(U; F')$. By replacing $s_1$ with $s_1 - s'$ on $W_1$ we may assume that $s_1 = s_2$ on $W_1 \cap W_2$. Then there exists $s \in \Gamma(W_1 \cup W_2; F)$ with $s|_{W_i} = s_i$. Thus the induction proceeds. □

Proposition 2.15. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\text{Mod}(k_X)$, and assume $F', F \text{-soft}$. Then $F''$ is $c$-soft.

Proof. Let $V, W, Y \in \mathcal{C}_X$ with $V \xrightarrow{\text{wc}} W \xrightarrow{\text{wc}} Y$ and let us consider the diagram below

$$
\begin{array}{ccc}
\Gamma(W; F) & \xrightarrow{\alpha} & \Gamma(W; F') \\
\downarrow & & \downarrow \\
\lim_{V \xrightarrow{\text{wc}} U} \Gamma(U; F) & \xrightarrow{\beta} & \lim_{V \xrightarrow{\text{wc}} U} \Gamma(U; F'').
\end{array}
$$

The morphism $\alpha$ is surjective since $F$ is $c$-soft and $\beta$ is surjective by Proposition 2.14. Then $\gamma$ is surjective. □

Proposition 2.16. Let $V \in \mathcal{C}_X$. The family of $c$-soft sheaves is injective with respect to the functor $\lim_{V \xrightarrow{\text{wc}} U} \Gamma(U; \bullet)$.

Proof. The family of $c$-soft sheaves contains injective sheaves, hence it is cogenerating. Then the result follows from Propositions 2.14 and 2.15. □

Remark 2.17. Remark that when $X$ is a locally compact space and we consider the category $\text{Mod}(k_X)$, the above definition of $c$-soft sheaves coincides with the classical one.

3. Relative Subanalytic Site

This section resumes the purpose of [11], allowing the dimension $d_S$ of the parameter space to be arbitrary. From now on we set $k = \mathbb{C}$.

3.1. Relative Subanalytic Sheaves

Let $X$ and $S$ be two real analytic manifolds. Let us denote by $p : X \times S \to S$ the projection. We consider the following Grothendieck topologies on $X \times S$.

- The one generated by the usual topology.
- The subanalytic topology of [8,14], defining the site $(X \times S)_{sa}$, where $\text{op}((X \times S)_{sa})$ are open subanalytic subsets of $X \times S$ and the coverings are those admitting a locally finite refinement. See Sect. 2.1 for more details.
• The product of sites $X_{sa} \times S_{sa}$ of [11]. This is the site where the open sets are locally finite unions of elements of the family $\mathcal{T}$ of products $U \times V$ with $U$ and $V$ relatively compact subanalytic open respectively in $X$ and $S$, and the coverings are the families of such open sets admitting a locally finite refinement. See Sect. 2.2 for more details.

• The product of sites $X_{sa} \times S$ for which the elements of $\text{op}(X_{sa} \times S)$ are finite unions of products $U \times V$, with $U \in \text{op}(X_{sa})$ and $V \in \text{op}(S)$. With this topology, $X_{sa} \times S$ is a locally weakly quasi-compact site. See Sect. 2.3 for more details.

Let us describe the coverings in $X_{sa} \times S$: $T \subset \text{op}(X_{sa} \times S)$ is a covering of $W = U \times V \in \text{op}(X_{sa} \times S)$ if and only if it admits a refinement $\{U_i \times V_j\}_{i \in I, j \in J}$ with $\{U_i\}_{i \in I} \in \text{Cov}(U)$ (in $X_{sa}$) and $\{V_j\}_{j \in J} \in \text{Cov}(V)$ (in $S$). In particular, when $U$ is relatively compact $I$ is finite. For a general $W \in \text{op}(X_{sa} \times S)$, $T \subset \text{op}(X_{sa} \times S)$ is a covering of $W$ if for each $U \times V \in \text{op}(X_{sa} \times S)$ with $U \times V \subset W$ one has $T' \times W (U \times V) \in \text{Cov}(U \times V)$.

Let us make explicit the notion of weakly quasi-compact in $X_{sa} \times S$: let $W, W' \in \text{op}(X_{sa} \times S)$ and suppose that $W = \bigcup_{j \in J}(U_j \times V_j)$ with $J$ finite. Then $W \overset{\text{wc}}{\rightarrow} W'$ if $U_j$ and $V_j$ are relatively compact and $U_j \times V_j \subset W'$ for each $j \in J$.

We have the following commutative diagram, where the arrows are natural morphisms of sites induced by the inclusion of families of open subsets.

\[
\begin{array}{ccc}
X_{sa} \times S & \xrightarrow{a} & X_{sa} \\
\rho_S & \downarrow & \eta
\end{array}
\]

\[
\begin{array}{ccc}
X \times S & \xrightarrow{\rho'} & X_{sa} \times S_{sa} \\
\rho & \downarrow & \\
(X \times S)_{sa}
\end{array}
\]

It follows from Theorem 3.9.2 of [16] that the functor $a_* \text{ is fully faithful and } a^{-1} \circ a_* \overset{\sim}{\rightarrow} \text{Id. Similarly, the functor } \rho_{S*} \text{ is fully faithful and } \rho_{S*}^{-1} \circ \rho_{S*} \overset{\sim}{\rightarrow} \text{Id.}

**Proposition 3.1.** Let $U \times V \in \text{op}(X_{sa} \times S)$ and let $F \in \text{Mod}((X_{sa} \times S_{sa}))$. Then

\[
\Gamma(U \times V; a^{-1}F) \simeq \lim_{V' \subset \subset V} \Gamma(U \times V'; F)
\]

with $V'$ subanalytic.

**Proof.** First suppose that $U$ is relatively compact. Remark that $\{V' \subset \subset V, V' \in \text{op}(S_{sa})\}$ is a covering of $V$ and

\[
\Gamma(U \times V; a^{-1}F) \simeq \lim_{V' \subset \subset V} \Gamma(U \times V', a^{-1}F)
\]

with $V'$ subanalytic. There exists a filtrant family $\{F_i\}$ of $\mathcal{T}$-coherent sheaves such that $F \simeq \lim_i \rho'_i F_i$. Remark that $\rho' = a \circ \rho_S$ and we have

\[
a^{-1} \lim_i \rho'_i F_i \simeq \lim_i a^{-1} \circ a_* \circ \rho_{S*} F_i \simeq \lim_i \rho_{S*} F_i.
\]
We have the chain of isomorphisms
\[
\Gamma(U \times V; a^{-1}F) \cong \lim_{V' \subset \subset V} \Gamma(U \times V'; a^{-1}F)
\]
\[
\cong \lim_{V' \subset \subset V} \Gamma(U \times V'; a^{-1} \lim_{i} \rho_{*} F_{i})
\]
\[
\cong \lim_{V' \subset \subset V} \Gamma(U \times V'; \lim_{i} \rho_{S*} F_{i})
\]
\[
\cong \lim_{V' \subset \subset V} \Gamma(U \times V'; \lim_{i} \rho'_{*} F_{i})
\]
\[
\cong \Gamma(U \times V'; F)
\]
with \(V'\) subanalytic. The fourth and sixth isomorphisms follow from Corollary 2.12 and the fifth follows since \(U\) and \(V'\) are subanalytic.

Suppose now that \(U\) is not relatively compact. Let \(\{U_{n}\}_{n \in \mathbb{N}}\) be a covering of \(X\) consisting of open subanalytic subsets such that \(U_{n} \subset \subset U_{n+1}\) for every \(n \in \mathbb{N}\). Then
\[
\Gamma(U \times V; a^{-1}F) \cong \lim_{n} \Gamma(U \cap U_{n} \times V; a^{-1}F)
\]
\[
\cong \lim_{n, V' \subset \subset V} \Gamma(U \cap U_{n} \times V'; F)
\]
\[
\cong \lim_{V' \subset \subset V} \Gamma(U \times V'; F).
\]

**Corollary 3.2.** Let \(W \in \text{op}(X_{sa} \times S)\) and let \(F \in \text{Mod}(\mathbb{C}_{X_{sa} \times S_{sa}})\). Then
\[
\Gamma(W; a^{-1}F) \cong \lim_{W' \supset W} \Gamma(W'; F)
\]
with \(W' \in \text{op}(X_{sa} \times S_{sa})\).

We shall need to introduce a left adjoint to the functor \(\rho_{S}^{-1}\).

**Proposition 3.3.** The functor \(\rho_{S}^{-1}\) has a left adjoint, denoted by \(\rho_{S!}\). The functor \(\rho_{S!}\) is exact and commutes with tensor products.

**Proof.** With the notations of (3.1) \(\rho'^{-1} \circ a_{*} \cong \rho_{S}^{-1} \circ a^{-1} \circ a_{*} \cong \rho_{S}^{-1}\), where the second isomorphism follows from Theorem 3.9.2 of [16]. By Proposition 2.4.3 of [2] the functor \(\rho'^{-1}\) has a left adjoint \(\rho'_{!}\), so the composition \(\rho'^{-1} \circ a_{*} \cong \rho_{S}^{-1}\) admits a left adjoint \(a^{-1} \circ \rho'_{!} =: \rho_{S!}\). Remark that by definition (see also Proposition 2.4.4 of [2]) \(\rho_{S!}\) is exact and commutes with tensor products. \(\square\)

**Remark 3.4.** Thanks to Proposition 3.1 and the fact that \(a^{-1} \circ \rho'_{!} =: \rho_{S!}\), one can prove that \(\rho_{S!} F\) (with \(F \in \text{Mod}(\mathbb{C}_{X})\)) is the sheaf associated to the presheaf \(U \times V \mapsto \)
lim \Gamma(U' \times V; F), with \mathcal{U} \subset U'. An alternative proof can be obtained adapting the proof of Proposition 6.6.3 of [8].

**Proposition 3.5.** The category of c-soft sheaves is injective with respect to the functor \( \Gamma(U \times V; \bullet) \), \( U \in \text{op}(X_{sa}) \), \( V \in \text{op}(S) \).

**Proof.** First remember that, if \( U' \subset X \) and \( V' \subset V \) (\( U', V' \) subanalytic), then \( U' \times V' \) is weakly quasi-compact in \( U \times V \) in the relative site \( X_{sa} \times S \). Let \( \{ U_n \}_{n \in \mathbb{N}} \) (resp. \( \{ V_n \}_{n \in \mathbb{N}} \)) be a covering of \( X \) (resp. \( V \)) consisting of open subanalytic subsets such that \( U_n \subset U_{n+1} \) (resp. \( V_n \subset V_{n+1} \)) for every \( n \in \mathbb{N} \). For any \( G \in \text{Mod}(\mathbb{C}_{X_{sa} \times S}) \) we have

\[
\Gamma(U \times V; G) \simeq \lim_{\longrightarrow} \Gamma(U \cap U_n \times V_n; G) \simeq \lim_{\longrightarrow} \lim_{\longrightarrow} \Gamma(U \cap U_n \times V_n'; G).
\]

Set for short \( W_n := U \cap U_n \times V_n', n \in \mathbb{N} \) and

\[
\lim_{\longrightarrow} \Gamma(W_n; G) := \lim_{\longrightarrow} \Gamma(U \cap U_n \times V_n'; G).
\]

Then \( U \cap U_n \times V_n \rightrightarrows W_n, W_n \) is weakly quasi-compact and we can use the results of Sect. 2.3. Remark that \( U \cap U_n \) is a relatively compact subset of \( X \) (being \( U_n \) relatively compact in \( X \)), hence every covering in \( X_{sa} \) admits a finite refinement.

Take an exact sequence \( 0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \), and suppose \( F' \) c-soft. All the sequences

\[
0 \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F') \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F) \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F'') \rightarrow 0
\]

are exact by Proposition 2.14, and the morphism

\[
\lim_{\longrightarrow} \Gamma(W_{n+1}; F') \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F')
\]

is surjective for all \( n \). Then by Proposition 1.12.3 of [6] the sequence

\[
0 \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F') \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F) \rightarrow \lim_{\longrightarrow} \Gamma(W_n; F'') \rightarrow 0
\]

is exact. \( \square \)

**Proposition 3.6.** Let \( F \in \text{Mod}(\mathbb{C}_{X_{sa} \times S_{sa}}) \) be \( T \)-flabby. Then \( a^{-1} F \) is c-soft.

**Proof.** Let us first prove that, if \( W \xrightarrow{wc} W' \) with \( W, W' \in \text{op}(X_{sa} \times S) \), then there exists \( W'' \in \text{op}(X_{sa} \times S_{sa}) \) with \( W \xrightarrow{wc} W'' \xrightarrow{wc} W' \). \( W \) is a finite union of \( U_j \times V_j \in \text{op}(X_{sa} \times S) \) with \( U_j \) and \( V_j \) relatively compact and \( U_j \times V_j \subset W' \). \( V_j \) being compact, it has a subanalytic neighborhood \( V''_j \) such that \( U_j \times V''_j \xrightarrow{wc} W' \). Then \( W'' = \bigcup_j (U_j \times V''_j) \in \text{op}(X_{sa} \times S_{sa}) \) and \( W \xrightarrow{wc} W'' \xrightarrow{wc} W' \).

This implies that, given \( W, W' \in \text{op}(X_{sa} \times S) \) with \( W \xrightarrow{wc} W' \) and \( G \in \text{Mod}(\mathbb{C}_{X_{sa} \times S}) \), when we write

\[
\lim_{\longrightarrow} \Gamma(W''; G)
\]

we may assume that \( W'' \in \text{op}(X_{sa} \times S_{sa}) \).
Let $F \in \text{Mod}(\mathbb{C}_{X_{sa} \times S_{sa}})$ be $\mathcal{T}$-flabby. Then
\[
\lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset W''}} \Gamma(W''; a^{-1}F) \simeq \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset W''}} \Gamma(W''; F)
\]
as a consequence of Proposition 3.1. Since $F$ is $\mathcal{T}$-flabby the morphism $\Gamma(X \times S; F) \to \Gamma(W''; F)$ is surjective for each $W'' \in \text{op}(X_{sa} \times S_{sa})$. So the morphism $\Gamma(X \times S; F) \to \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset W''}} \Gamma(W''; F) \simeq \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset W''}} \Gamma(W''; a^{-1}F)$ is surjective as well and factors through $\Gamma(W'; a^{-1}F)$ by Proposition 3.1. Then the result follows.

\section*{3.2. Construction of Relative Sheaves}

In the situation above, let be given a sheaf $F$ on $(X \times S)_{sa}$. As in [11], we denote by $F^{S,\sharp}$ the sheaf on $X_{sa} \times S_{sa}$ associated to the presheaf
\[
\text{op}(X_{sa} \times S_{sa})^{\text{op}} \to \text{Mod}(\mathbb{C})
\]
\[
U \times V \mapsto \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}F)
\]
\[
\simeq \text{Hom}(\mathbb{C}_{U} \boxtimes \rho(\mathbb{C}_{V}), F)
\]
\[
\simeq \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset V, W \in \text{op}(S_{sa})}} \Gamma(U \times W; F).
\]

We set
\[
F^{S} := a^{-1}F^{S,\sharp}
\]
and call it the relative sheaf associated to $F$. It is a sheaf on $X_{sa} \times S$. It is easy to check that $(\bullet)^{S}$ defines a left exact functor on $\text{Mod}(\mathbb{C}_{(X \times S)_{sa}})$. We will denote by $(\bullet)^{RS,\sharp}$ and $(\bullet)^{RS} \simeq a^{-1} \circ (\bullet)^{RS,\sharp}$ the associated right derived functors.

\textbf{Remark 3.7.} When $X$ is reduced to a point, and $F$ is a sheaf on $S_{sa}$ then $F^{S}$ is nothing more than $\rho^{-1}F$ where $\rho$ is the morphism of sites $S \to S_{sa}$.

\textbf{Lemma 3.8.} Let $U \in \text{op}(X_{sa})$, $V \in \text{op}(S)$ be relatively compact. Then
\[
\Gamma(U \times V; F^{S}) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}F)
\]
\[
\simeq \text{Hom}(\mathbb{C}_{U} \boxtimes \rho(\mathbb{C}_{V}), F).
\]

\textbf{Proof.} The second isomorphism follows by adjunction. Let us prove the first one. Let $U, V'$ be open subanalytic in $X$ and $S$ respectively. By Lemma 4.2 of [11] we have
\[
\Gamma(U \times V'; F^{S,\sharp}) \simeq \Gamma(X \times V'; \rho^{-1}\Gamma_{U \times S}F) \simeq \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset V'}} \Gamma(U \times W; F).
\]

By definition and Proposition 3.1
\[
\Gamma(U \times V; F^{S}) = \Gamma(U \times V; a^{-1}F^{S,\sharp}) \simeq \lim_{\substack{\longrightarrow \\scriptstyle V' \subset \subset V}} \Gamma(U \times V'; F^{S,\sharp})
\]
with $V'$ subanalytic. We have
\[
\lim_{\substack{\longrightarrow \\scriptstyle V' \subset \subset V}} \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset V'}} \Gamma(U \times W; F) \simeq \lim_{\substack{\longrightarrow \\scriptstyle W \subset \subset V}} \Gamma(U \times W; F) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}F).
\]
This implies that $\Gamma(U \times V; F^{S}) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}F)$ as required. \qed
Proposition 3.9. Let \( G \in D^b(\mathbb{C}_{X_{sa}}) \) and \( H \in D^b(\mathbb{C}_S) \). Let \( F \in D^b(\mathbb{C}_{(X \times S)_{sa}}) \). Then
\[
\text{RHom}(G \boxtimes H, F^{RS}) \simeq \text{RHom}(G \boxtimes \rho_! H, F) \\
\simeq \text{RHom}(\mathbb{C}_X \boxtimes H, \rho^{-1} \text{RHom}(G \boxtimes \mathbb{C}_S, F)).
\]

Proof. The second isomorphism follows by adjunction. We are going to prove the first one in several steps.

(a) Suppose that \( F \) is injective. Then \( F^{S,\sharp} \) is \( T \)-flabby by Lemma 4.4 of [11] and \( F^S = a^{-1}F^{S,\sharp} \) is c-soft by Proposition 3.6. Let \( U \in \text{op}(X_{sa}) \), let \( V \in \text{op}(S) \) and let us assume that both are relatively compact. We have
\[
R\Gamma(U \times V; F^{RS}) \simeq \Gamma(U \times V; F^S) \\
\simeq \text{Hom}(\mathbb{C}_U \boxtimes \rho_! \mathbb{C}_V, F) \\
\simeq \text{RHom}(\mathbb{C}_U \boxtimes \rho_! \mathbb{C}_V, F),
\]
where the second isomorphism follows from Lemma 3.8.

(b) Suppose that \( G \in \text{Mod}_R^c(\mathbb{C}_X) \) and \( H \in \text{Mod}_R^c(\mathbb{C}_S) \). Then \( G \) (resp. \( H \)) is quasi-isomorphic to a bounded complex \( G^\bullet \) (resp. \( H^\bullet \)) consisting of finite sums \( \oplus \mathbb{C}_W \) with \( W \) subanalytic and relatively compact in \( X \) (resp. \( S \)). Let \( F \in D^b(\mathbb{C}_{(X \times S)_{sa}}) \) and let \( F^\bullet \) be a complex of injective objects quasi-isomorphic to \( F \). We have
\[
\text{RHom}(G \boxtimes H, F^{RS}) \simeq \text{Hom}(G^\bullet \boxtimes H^\bullet, (F^\bullet)^S) \\
\simeq \text{Hom}(G^\bullet \boxtimes \rho_! H^\bullet, F^\bullet) \\
\simeq \text{RHom}(G \boxtimes \rho_! H, F),
\]
where the second isomorphism follows from (a).

(c) Suppose that \( G \in \text{Mod}(\mathbb{C}_{X_{sa}}) \) and \( H \in \text{Mod}_R^c(\mathbb{C}_S) \). Then \( G \simeq \lim \rho_* G_i \), with \( G_i \in \text{Mod}_R^c(\mathbb{C}_X) \). We have
\[
\text{RHom}(\lim \rho_* G_i \boxtimes H, F^{RS}) \simeq R\lim \text{RHom}(G_i \boxtimes H, F^{RS}) \\
\simeq R\lim \text{RHom}(G_i \boxtimes \rho_! H, F) \\
\simeq \text{RHom}(\lim \rho_* G \boxtimes \rho_! \lim H_i, F),
\]
where for the first and the third isomorphism we refer to [15] and the second one follows from (b).

(d) Suppose that \( G \in \text{Mod}(\mathbb{C}_{X_{sa}}) \) and \( H \in \text{Mod}(\mathbb{C}_S) \). Then \( H \simeq \lim \rho_* H_i \), with \( H_i \in \text{Mod}_R^c(\mathbb{C}_X) \). We have
\[
\text{RHom}(G \boxtimes \lim \rho_* H_i, F^{RS}) \simeq R\lim \text{RHom}(G \boxtimes H_i, F^{RS}) \\
\simeq R\lim \text{RHom}(G_i \boxtimes \rho_! H_i, F) \\
\simeq \text{RHom}(\lim \rho_* G \boxtimes \rho_! \lim H_i, F),
\]
where for the first and the third isomorphism we refer to [15] (we also used the fact that \( \rho_l \) commutes with \( \lim \)) and the second one follows from (c).

(e) Suppose that \( G \in D^b(\mathbb{C}_{X_{sa}}) \) and \( H \in D^b(\mathbb{C}_S) \). Then \( G \) (resp. \( H \)) is quasi-isomorphic to a bounded complex \( G^\bullet \) (resp. \( H^\bullet \)) of sheaves on \( X_{sa} \) (resp. \( S \)). By dévissage, we may reduce to the case \( G \in \text{Mod}(\mathbb{C}_{X_{sa}}) \) and \( H \in \text{Mod}(\mathbb{C}_S) \) and the result follows from (d).

\[ \square \]

**Corollary 3.10.** Let \( G \in \text{Mod}(\mathbb{C}_{X_{sa}}) \), \( H \in \text{Mod}(\mathbb{C}_S) \) and let \( F \) be an injective sheaf on \((X \times S)_{sa}\). Then \( F^S \) is \( \text{Hom}(G \boxtimes H, \bullet) \)-acyclic.

**Proof.** It follows from Proposition 3.9 that \( R\text{Hom}(G \boxtimes H, F^{RS}) \simeq R\text{Hom}(G \boxtimes \rho_l H, F) \) is concentrated in degree 0, as required. \( \square \)

**Proposition 3.11.** Let \( F \in D^b(\mathbb{C}_{(X \times S)_{sa}}) \). Let \( G \in D^b(\mathbb{C}_{X_{sa}}) \) and \( H \in D^b(\mathbb{C}_S) \).

Then

\[
\rho^{-1}_S R\text{Hom}(G \boxtimes H, F^{RS}) \simeq \rho^{-1}_S R\text{Hom}(G \boxtimes \rho_l H, F).
\]

In particular, when \( G = \mathbb{C}_X \) and \( H = \mathbb{C}_S \) we have \( \rho^{-1}_l F \simeq \rho^{-1}_S F^{RS} \simeq \rho^{-1} F^{RS} \).

**Proof.** The second isomorphism follows by adjunction. Let us prove the first one.

(a) Let \( K \in \text{Mod}(\mathbb{C}_{(X \times S)_{sa}}) \) and \( K' \in \text{Mod}(\mathbb{C}_{X_{sa} \times S}) \). A morphism

\[
\eta_* K \to a_* H
\]

defines a morphism

\[
a^{-1} \eta_* K \to K'
\]

by adjunction and, composing with \( \rho^{-1}_S \), a morphism

\[
\rho^{-1}_S a^{-1} \eta_* K \to \rho^{-1}_S K'.
\]

Remark that if \( K \in \text{Mod}(\mathbb{C}_{(X \times S)_{sa}}) \) then \( \rho^{-1}_l K \simeq \rho^{-1}_S a^{-1} \eta_* K \). Indeed, for each \( y \in X \times S \),

\[
(\rho^{-1}_l K)_y \simeq \lim_{U \times V \ni y} K(U \times V) \simeq (\rho^{-1}_S a^{-1} \eta_* K)_y
\]

with \( U \in \text{op}(X_{sa}) \), \( V \in \text{op}(S_{sa}) \).

(b) Let us first suppose that \( F, G, H \) are concentrated in degree zero. Hence, setting \( K = \mathcal{H}\text{om}(G \boxtimes \rho_l H, F) \) and \( K' = \mathcal{H}\text{om}(G \boxtimes H, F^S) \) in (a), to any morphism

\[
\eta_* \mathcal{H}\text{om}(G \boxtimes \rho_l H, F) \to a_* \mathcal{H}\text{om}(G \boxtimes H, F^S)
\]

one associates a morphism

\[
\rho^{-1}_l \mathcal{H}\text{om}(G \boxtimes \rho_l H, F) \to \rho^{-1}_S \mathcal{H}\text{om}(G \boxtimes H, F^S).
\]

Note that, for any \( U, V \in \text{op}( (X \times S)_{sa}) \), the natural morphism \( \rho_l(H_V) \to (\rho H)_V \) induces a morphism \( \text{Hom}(G_U \boxtimes (\rho H)_V, F) \to \text{Hom}(G_U \boxtimes \rho_l (H_V), F) \) hence a morphism \( \psi : \eta_* \mathcal{H}\text{om}(G \boxtimes \rho_l H, F) \to a_* \mathcal{H}\text{om}(G \boxtimes H, F^S) \), which defines a morphism \( \rho^{-1}_l \mathcal{H}\text{om}(G \boxtimes \rho_l H, F) \to \rho^{-1}_S \mathcal{H}\text{om}(G \boxtimes H, F^S) \).
Let us check on the fibers that it is an isomorphism. Let $y \in X \times S$, then

\[
(\rho^{-1}\mathcal{H}om(G \boxtimes \rho_1 H, F))_y \simeq \lim_{U \times V \ni y} \Hom(G_U \boxtimes (\rho_1 H)_V, F)
\]

\[
\simeq \lim_{U \times V \ni y} \lim_{W \subset \subset V} \Hom(G_U \boxtimes (\rho_1 H)_W, F)
\]

\[
\simeq \lim_{U \times V \ni y} \Hom(G_U \boxtimes \rho_1 (H_V), F)
\]

\[
\simeq (\rho_S^{-1}\mathcal{H}om(G \boxtimes H, F^S))_y
\]

with $U \in \text{op}((X_{sa}), V, W \in \text{op}(S_{sa})$. The second and third isomorphisms follow since the morphism $(\rho_1 H)_V \rightarrow (\rho_1 H)_W$ factors through $\lim_{W \subset \subset V} (\rho_1 H)_W$ if $V \subset \subset V (V, V', W \in \text{op}(S_{sa}))$. Moreover, one has (see [14] for more details) that

\[
\lim_{W \subset \subset V} (\rho_1 H)_W \simeq \rho_1 H \otimes \lim_{W \subset \subset V} \rho_1 C_W \simeq \rho_1 H \otimes \rho_1 C_V \simeq \rho_1 (H_V).
\]

(c) Suppose now that $F$ is injective and that $G, H$ are concentrated in degree 0. Let $U \in \text{op}(X_{sa}), V \in \text{op}(S_{sa})$. The complex

\[
R\Gamma(U \times V; R\mathcal{H}om(G \boxtimes H, F^S)) \simeq R\mathcal{H}om(G_U \boxtimes H_V, F^S)
\]

is concentrated in degree 0 by Corollary 3.10. Then $F^S$ is $\mathcal{H}om(G \boxtimes H, \bullet)$-acyclic.

(d) Let $G \in D^b(\mathcal{C}_{X_{sa}})$ and $H \in D^b(\mathcal{C}_{S})$. Let $F \in D^b(\mathcal{C}_{(X \times S)_{sa}})$ and let $F^\bullet$ be a complex of injective objects quasi-isomorphic to $F$. Then

\[
\rho^{-1}R\mathcal{H}om(G \boxtimes \rho_1 H, F) \simeq \rho^{-1}\mathcal{H}om(G \boxtimes \rho_1 H, F^\bullet)
\]

\[
\simeq \rho_S^{-1}\mathcal{H}om(G \boxtimes H, (F^\bullet)^S)
\]

\[
\simeq \rho_S^{-1}R\mathcal{H}om(G \boxtimes H, F^{RS}),
\]

where the second isomorphism follows from (b) and the third one from (c). 

\[\Box\]

**Proposition 3.12.** Suppose that $F \in \text{Mod}(\rho_1 \mathcal{C}_{X \times S}^{\infty})$ is $\Gamma(W; \bullet)$-acyclic for each $W \in \text{op}((X \times S)_{sa})$. Then $F$ is $(\bullet)^{S,\#}$-acyclic.

**Proof.** Since $(\bullet)^{RS} \simeq a^{-1} \circ (\bullet)^{RS,\#}$, it is enough to show that $H^kF^{RS,\#} = 0$ if $k \neq 0$. It is enough to prove that $F$ is $(\bullet)^{S,\#}$-acyclic. This follows from Corollary 4.9 of [11]. 

\[\Box\]

**Proposition 3.13.** Suppose that $F \in \text{Mod}(\rho_1 \mathcal{C}_{X \times S}^{\infty})$ is $\Gamma(W; \bullet)$-acyclic for each $W \in \text{op}((X \times S)_{sa})$. Then for each $U \in \text{op}(X_{sa}), V \in \text{op}(S)$ we have $R^k\Gamma(U \times V; F^{RS}) = 0$ if $k \neq 0$.

**Proof.** By Proposition 3.11 we have $R\Gamma(U \times V; F^{RS}) \simeq R\Gamma(X \times V; \rho^{-1}R\Gamma_{U \times S} F)$. Since $F$ is $\Gamma(W; \bullet)$-acyclic for each $W \in \text{op}((X \times S)_{sa})$, the complex $R\Gamma_{U \times S} F$ is concentrated in degree zero. Since $F$ is a $\rho_1 \mathcal{C}_{X \times S}^{\infty}$-module, $\rho^{-1}R\Gamma_{U \times S} F$ is a $\mathcal{C}_{X \times S}^{\infty}$-module, hence $c$-soft and $\Gamma(X \times V; \bullet)$-acyclic. This shows the result. 

\[\Box\]

### 3.3. Behaviour of $(\bullet)^{RS}$ Under Pushforward and Pull-Back Relatively to the Parameter Space

We consider a real analytic map $h : S' \rightarrow S$ and still denote by $h$ the maps $\text{Id} \times h : X \times S' \rightarrow X \times S, \text{Id} \times h : X_{sa} \times S' \rightarrow X_{sa} \times S$. 
Proposition 3.14. 1. Let $F \in D^b(C_{(X \times S)_{sa}})$. Then there exists a natural morphism
\[ h^{-1}(F^{RS}) \to (h^{-1}F)^{RS'} \]
2. Let $G \in D^b(C_{(X \times S')_{sa}})$. Then there exists a natural isomorphism
\[ (Rh_*G)^{RS} \simeq Rh_*(G^{RS'}) \]

Proof. 1. Let $U$ be a relatively compact subanalytic subset of $X$ and let $V'$ be any open set in $S'$. Let $V$ be any open subset of $S$ such that $V \supset h(V')$. By Proposition 3.11 we have
\[
R\Gamma(U \times V; F^{RS}) \simeq R\Gamma(X \times V; \rho^{-1}R\Gamma_{U \times S}F)
\]
\[
\to R\Gamma(X \times V'; \rho^{-1}h^{-1}R\Gamma_{U \times S}F)
\]
\[
\to R\Gamma(X \times V'; \rho^{-1}R\Gamma_{U \times S'}(h^{-1}F))
\]
\[
\simeq R\Gamma(U \times V'; (h^{-1}F)^{RS'})
\]
which gives the desired morphism.

2. Given $U$ and $V$ as in 1), let us set $V' = h^{-1}(V)$. Then
\[
R\Gamma(U \times V; (Rh_*G)^{RS}) \simeq R\Gamma(X \times V; \rho^{-1}R\Gamma_{U \times S}Rh_*G)
\]
\[
\simeq R\Gamma(X \times V; \rho^{-1}Rh_*R\Gamma_{U \times S'}G)
\]
\[
\simeq R\Gamma(X \times V; Rh_*\rho^{-1}R\Gamma_{U \times S'}G)
\]
\[
\simeq R\Gamma(X \times V'; \rho^{-1}R\Gamma_{U \times S'}G) \simeq R\Gamma(U \times V'; G^{RS'})
\]
\[
\simeq R\Gamma(U \times V; Rh_*G^{RS'})
\]
where all the isomorphisms commute with restrictions. This gives the desired isomorphism.

\[ \square \]

3.4. The S-Locally Constant Case

The result below is an improvement of Proposition 3.3 of [13]. The latter was a step in the construction of the relative Riemann-Hilbert functor, denoted by $RH^S_X$ where $d_S = 1$ was assumed. As already said, there the construction relied on the possibility of considering on $S_{sa}$ a basis formed by Stein open subanalytic sets. This difficulty is overcome with the new site $X_{sa} \times S$ and the proof goes then in a similar way.

Lemma 3.15. The sheaf $p^{-1}O_S$ is $\rho_{S*}$-acyclic.

Proof. Suppose that $U \in \text{op}(X_{sa})$ is connected and $V \in \text{op}(S)$ is Stein. Then, by Proposition 3.3.9 of [6], we have
\[
R\Gamma(U \times V; R\rho_{S*}p^{-1}O_S) \simeq R\Gamma(U \times V; p^{-1}O_S) \simeq R\Gamma(V; O_S)
\]
and the latter is concentrated in degree 0. The result follows since the topology of $X_{sa} \times S$ admits a basis consisting of open subsets $U \times V$ with $U \in \text{op}(X_{sa})$ contractible (as a consequence of [17]), and $V$ Stein in $S$. \[ \square \]
Proposition 3.16. If $F \in \text{Mod}(\mathbb{C}_{X \times S})$ is locally isomorphic to $p^{-1}G$, where $G$ is a coherent $\mathcal{O}_S$-module, then $R\rho_{S!}F \simeq \rho_{S!}F$.

Proof. We have to show that, if $F \in \text{Mod}(\mathbb{C}_{X \times S})$ is locally isomorphic to $p^{-1}G$, where $G$ is $\mathcal{O}_S$-coherent, the natural morphism
$$\rho_{S!}F \rightarrow R\rho_{S!}F,$$defined by the adjunction morphism $\text{Id} \rightarrow \rho_S^{-1}R\rho_S \simeq \text{Id}$ is an isomorphism. It is enough to check it on a basis for the topology of $X_{sa} \times S$. First of all, let us prove that $R\rho_{S!}F$ is concentrated in degree zero. There exists a covering $\mathcal{U}$ of $X_{sa} \times S$ such that, for each $W \in \mathcal{U}$, one has $(R\rho_{S!}F)|_W \simeq (R\rho_S p^{-1}G)|_W$, where $G$ is $\mathcal{O}_S$-coherent. Then Lemma 3.15 implies that $R\rho_{S!}F$ is concentrated in degree zero.

Up to taking a refinement, we may suppose that $W = U \times V$ with $U \in \text{op}(X_{sa})$ relatively compact and connected and $V \in \text{op}(S)$ connected. In this case $\rho_{S!}p^{-1}G$ is the sheaf associated to the presheaf
$$U \times V \mapsto \lim_{\longrightarrow} \Gamma(U' \times V; p^{-1}G)$$with $U' \subset U$. Moreover, if $U$ is connected, we can suppose that $U'$ is connected as well. Indeed, if $U$ is subanalytic, every subanalytic neighborhood of $U$ contains a subanalytic neighborhood $U'$ which is connected (as an easy consequence of the triangulation of subanalytic sets).

The result follows since, according to Proposition 3.3.9 of [6], $\Gamma(U \times V; p^{-1}G) \simeq \Gamma(V; G)$. \hfill \Box

3.5. The Sheaves $\mathcal{C}_{X \times S}^{\infty,t,S}$, $\mathcal{D}_X^{t,S}$, $\mathcal{C}_{X \times S}^{\infty,w,S}$, $\mathcal{O}_{X \times S}^{t,S}$ and $\mathcal{O}_{X \times S}^{w,S}$

Let $X$ and $S$ be real analytic manifolds. The construction given by (3.2) allows us to introduce the following sheaves:

1. $\mathcal{C}_{X \times S}^{\infty,t,S} := (\mathcal{C}_{X \times S}^{\infty,t})^S$ as the relative sheaf associated to $\mathcal{C}_{X \times S}^{\infty,t}$;
2. $\mathcal{D}_X^{t,S} := (\mathcal{D}_X^{t})^S$ as the relative sheaf associated to $\mathcal{D}_X^{t}$;
3. $\mathcal{C}_{X \times S}^{\infty,w,S} := (\mathcal{C}_{X \times S}^{\infty,w})^S$ as the relative sheaf associated to $\mathcal{C}_{X \times S}^{\infty,w}$.

Let us apply the results of the previous section to these sheaves.

Proposition 3.17. Let $U \in \text{op}(X_{sa})$, $V \in \text{op}(S_{sa})$. Then

1. $\Gamma(U \times V; \mathcal{C}_{X \times S}^{\infty,t,S}) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}\mathcal{C}_{X \times S}^{\infty,t}) \simeq \Gamma(X \times V; T\text{Hom}(\mathcal{C}_{U \times S}, \mathcal{C}_{X \times S}^{\infty,t}))$;
2. $\Gamma(U \times V; \mathcal{D}_X^{t,S}) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}\mathcal{D}_X^{t,S}) \simeq \Gamma(X \times V; T\text{Hom}(\mathcal{C}_{U \times S}, \mathcal{D}_X^{t,S}))$;
3. $\Gamma(U \times V; \mathcal{C}_{X \times S}^{\infty,w,S}) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}\mathcal{C}_{X \times S}^{\infty,w}) \simeq \Gamma(X \times V; H^0D'\mathcal{C}_U \boxtimes \mathcal{C}_S \boxtimes \mathcal{O}_{X \times S})$.

Remark 3.18. In the case of subanalytic open sets like $U \times V$ the definition of the sheaf $\mathcal{D}_X^{t,S}$ is well known. A section $s \in \Gamma(U \times V; \mathcal{D}_X^{t,S})$ belongs to $\Gamma(U \times V; \mathcal{D}_X^{t,S})$ if it extends as a section of $\Gamma(X \times V; \mathcal{D}_X^{t,S})$. It is a consequence of the definition of the functor $T\text{Hom}$ of [4].

Proposition 3.19. (i) Suppose that $\mathcal{F} = \mathcal{D}_X^{t,S}, \mathcal{C}_{X \times S}^{\infty,t,S}, \mathcal{C}_{X \times S}^{\infty,w,S}$. Then $\mathcal{F}$ is $(\bullet)^S$-acyclic. Moreover $\mathcal{D}_X^{t,S}, \mathcal{C}_{X \times S}^{\infty,t,S}$ are $\Gamma(U \times V; \bullet)$-acyclic for each $U \in \text{op}(X_{sa})$, $V \in \text{op}(S)$. 


(ii) $\mathcal{C}^{\infty,w,S}_{X \times S}$ is $\Gamma(U \times V; \bullet)$-acyclic for each $U \in \text{op}(X_{sa})$ locally cohomologically trivial and $V \in \text{op}(S)$.

**Proposition 3.20.** Let $G \in D^b(\mathbb{C}_{X_{sa}})$, $H \in D^b(C_S)$. Then

1. $\rho_S^{-1}R\text{Hom}(G \boxtimes H, \mathcal{C}^{\infty,t,S}_{X \times S}) \simeq \rho_1^{-1}R\text{Hom}(G \boxtimes \rho_H, \mathcal{C}^{\infty,t}_{X \times t}) \simeq R\text{Hom}(\mathcal{C}_X \boxtimes H, \rho_1^{-1}\text{Hom}(G \boxtimes \mathcal{C}_S, \mathcal{C}^{\infty,t}_{X \times S}))$,

2. $\rho_S^{-1}R\text{Hom}(G \boxtimes H, \mathcal{D}^{t,S}_{X \times S}) \simeq \rho^{-1}R\text{Hom}(G \boxtimes \rho_H, \mathcal{D}^{t,S}_{X \times S}) \simeq R\text{Hom}(\mathcal{C}_X \boxtimes H, \rho^{-1}\text{Hom}(G \boxtimes \mathcal{C}_S, \mathcal{D}^{t,S}_{X \times S}))$,

3. $\rho_S^{-1}R\text{Hom}(G \boxtimes H, \mathcal{C}^{\infty,w,S}_{X \times S}) \simeq \rho^{-1}R\text{Hom}(G \boxtimes \rho_H, \mathcal{C}^{\infty,w}_{X \times S}) \simeq R\text{Hom}(\mathcal{C}_X \boxtimes H, \rho^{-1}\text{Hom}(G \boxtimes \mathcal{C}_S, \mathcal{C}^{\infty,w}_{X \times S})).$

When $G \in D^b_{\text{loc-c}}(\mathbb{C}_X)$ we have

1. $\rho_S^{-1}R\text{Hom}(G \boxtimes H, \mathcal{C}^{\infty,t,S}_{X \times S}) \simeq R\text{Hom}(\mathcal{C}_X \boxtimes H, T\text{Hom}(G \boxtimes \mathcal{C}_S, \mathcal{C}^{\infty}_{X \times S}))$,

2. $\rho_S^{-1}R\text{Hom}(G \boxtimes H, \mathcal{D}^{t,S}_{X \times S}) \simeq R\text{Hom}(\mathcal{C}_X \boxtimes H, T\text{Hom}(G \boxtimes \mathcal{C}_S, \mathcal{D}^{t,S}_{X \times S}))$,

3. $\rho_S^{-1}R\text{Hom}(G \boxtimes H, \mathcal{C}^{\infty,w,S}_{X \times S}) \simeq R\text{Hom}(\mathcal{C}_X \boxtimes H, D'(G \boxtimes \mathcal{C}_S \otimes \mathcal{C}^{\infty}_{X \times S})).$

In particular, when $G = \mathcal{C}_X$ and $H = \mathcal{C}_S$ we have $\rho_S^{-1}\mathcal{C}^{\infty,t,S}_{X \times S} \simeq \mathcal{C}^{\infty,t}_{X \times S}$, $\rho_S^{-1}\mathcal{D}^{t,S}_{X \times S} \simeq \mathcal{D}^{t,S}_{X \times S}$, $\rho_S^{-1}\mathcal{C}^{\infty,w,S}_{X \times S} \simeq \mathcal{C}^{\infty,w}_{X \times S}$.

Now we are going to study the action of differential operators on these new sheaves.

**Lemma 3.21.** There is a natural action of $\rho_S^*D_{X \times S}$ on $\mathcal{C}^{\infty,t,S}_{X \times S}$ and on $\mathcal{C}^{\infty,w,S}_{X \times S}$.

**Proof.** Let $K \in \text{Mod}(\mathbb{C}_{(X \times S)_{sa}})$. In order to prove the action of $\rho_S^*D_{X \times S}$ on $K^S$ it is enough to prove the action of $\rho_1^*D_{X \times S}$ on $K^{S,t}$. When $K = \mathcal{D}^{t,S}_{X \times S}$, $\mathcal{C}^{\infty,t,S}_{X \times S}$, $\mathcal{C}^{\infty,w,S}_{X \times S}$, it is a consequence of Lemma 5.4 of [11].

Let us now assume that $X$ and $S$ are complex manifolds and consider the projection $p : X \times S \to S$. Let us denote as usual by $\overline{X} \times S$ the complex conjugate manifold. Identifying the underlying real analytic manifold $X_{\mathbb{R}} \times S_{\mathbb{R}}$ to the diagonal of $(X \times S) \times (\overline{X} \times S)$, we have:

**Lemma 3.22.** $\rho_S^*p^{-1}O_S$ acts on $\mathcal{D}^{t,S}_{X \times S}$, $\mathcal{C}^{\infty,t,S}_{X \times S}$ and on $\mathcal{C}^{\infty,w,S}_{X \times S}$.

**Proof.** Let $K = \mathcal{D}^{t,S}_{X \times S}, \mathcal{C}^{\infty,t,S}_{X \times S}, \mathcal{C}^{\infty,w,S}_{X \times S}$. By Lemma 5.5 of [11], $\rho_S^*p^{-1}O_S$ acts on $K^{S,t}$. Note that $\rho_1^* \simeq a_\ast \circ \rho_S^*$, then $a^{-1}\rho_1^*p^{-1}O_S \simeq \rho_S^*p^{-1}O_S$ acts on $a^{-1}K^{S,t} = K^S$ as required.

We also have natural actions of $\rho_S^*D_{\overline{X} \times S}$, $\rho_S^*D_{\overline{X} \times S}$ on $\mathcal{D}^{t,S}_{X \times S}$.

The construction given by (3.2) allows us to introduce the following objects of $D^b(\mathbb{C}_{X_{sa} \times S})$:

1. $\mathcal{O}^{t,S}_{X \times S} := (\mathcal{O}^t_{X \times S})^{RS}$, the relative sheaf associated to $\mathcal{O}^t_{X \times S}$; that is

   $$\mathcal{O}^{t,S}_{X \times S} \simeq (R\text{Hom}_{p^\ast \mathcal{D}_{\overline{X} \times S}}(p_\ast \mathcal{O}_{\overline{X} \times S}, \mathcal{D}^{t,S}_{X \times S}))^{RS} \simeq (R\text{Hom}_{p^\ast \mathcal{D}_{\overline{X} \times S}}(p_\ast \mathcal{O}_{\overline{X} \times S}, \mathcal{C}^{\infty,t}_{X \times S}))^{RS}.$$

2. $\mathcal{O}^{w,S}_{X \times S} := (\mathcal{O}^w_{X \times S})^{RS}$, the relative sheaf associated to $\mathcal{O}^w_{X \times S}$; that is

   $$\mathcal{O}^{w,S}_{X \times S} \simeq (R\text{Hom}_{p^\ast \mathcal{D}_{\overline{X} \times S}}(p_\ast \mathcal{O}_{\overline{X} \times S}, \mathcal{C}^{\infty,w}_{X \times S}))^{RS}.$$
The exactness of \( \rho_{S!} \) together with Proposition 3.19 allow to conclude:

**Proposition 3.23.** We have the following isomorphisms in \( D^b(C_{X \times S}) \).

\[
\mathcal{O}_{X \times S}^{l,S} \simeq R\text{Hom}_{\rho_{S!}D_{X \times S}}(\rho_{S!}\mathcal{O}_{X \times S}, D\mathcal{O}_{X \times S}^{l,S}) \\
\simeq R\text{Hom}_{\rho_{S!}D_{X \times S}}(\rho_{S!}\mathcal{O}_{X \times S}, C_{X \times S}^{\infty,l,S}) \\
\mathcal{O}_{X \times S}^{w,S} \simeq R\text{Hom}_{\rho_{S!}D_{X \times S}}(\rho_{S!}\mathcal{O}_{X \times S}, C_{X \times S}^{\infty,w,S}).
\]

Proposition 3.11 together with Proposition 7.3.2 of [8] entail:

**Proposition 3.24.** Let \( G \in D^b(C_{X \times S}) \), \( H \in D^b(C_S) \). Then

1. \( \rho^{-1}_S R\text{Hom}(G \boxtimes H, \mathcal{O}_{X \times S}^{l,S}) \simeq \rho^{-1}_S R\text{Hom}(G \boxtimes \rho(H, \mathcal{O}_{X \times S}) \simeq R\text{Hom}(C_X \boxtimes H, \rho^{-1}_S R\text{Hom}(G \boxtimes C_S, \mathcal{O}_{X \times S}^{l,S})) \)
2. \( \rho^{-1}_S R\text{Hom}(G \boxtimes H, \mathcal{O}_{X \times S}^{w,S}) \simeq \rho^{-1}_S R\text{Hom}(G \boxtimes \rho(H, \mathcal{O}_{X \times S}) \simeq R\text{Hom}(C_X \boxtimes H, \rho^{-1}_S R\text{Hom}(G \boxtimes C_S, \mathcal{O}_{X \times S}^{w,S})) \).

When \( G \in D^b_{\mathbb{R},c}(C_X) \) we have

1. \( \rho^{-1}_S R\text{Hom}(G \boxtimes H, \mathcal{O}_{X \times S}^{l,S}) \simeq R\text{Hom}(C_X \boxtimes H, T\text{Hom}(G \boxtimes C_S, \mathcal{O}_{X \times S})) \)
2. \( \rho^{-1}_S R\text{Hom}(G \boxtimes H, \mathcal{O}_{X \times S}^{w,S}) \simeq R\text{Hom}(C_X \boxtimes H, D'(G \boxtimes C_S \boxtimes \mathcal{O}_{X \times S}) \).

In particular, when \( G = C_X \) and \( H = C_S \) we have \( \rho^{-1}_S \mathcal{O}_{X \times S}^{l,S} \simeq \mathcal{O}_{X \times S} \), \( \rho^{-1}_S \mathcal{O}_{X \times S}^{w,S} \simeq \mathcal{O}_{X \times S} \).

The examples given in [11] can now be stated in a more general case (\( V \) needs no longer to be subanalytic):

**Example 3.25.** Let \( U = \{ z \in \mathbb{C}, \Im z > 0 \} \), let \( V \) be open in \( \mathbb{C}^n \) and let \( g(s) \) be a holomorphic function on \( V \). Then, after a choice of a determination of \( \log z \) on \( U \), \( z^{g(s)} \) defines a section of \( \Gamma(U \times V; \mathcal{O}_{X \times \mathbb{C}^n}^{l,S}) \).

**Example 3.26.** Let \( U = \mathbb{R}_{>0} \) with a coordinate \( x \), let \( V \) be an open set in \( \mathbb{R} \) and let \( a(s) \) be an analytic function on \( V \). Let \( f \in \Gamma(\Omega \setminus V; \mathcal{O}_\mathbb{C}) \), where \( \Omega \) is an open neighborhood of \( V \) in \( \mathbb{C} \), be such that \( a \) is the boundary value \( vb(f) \) of \( f \) as a hyperfunction. Then \( x^a \) defines a section of \( \Gamma(U \times V; \mathcal{O}_{X \times \mathbb{R}_{>0}}^{l,S}) \).

### 3.6. Functorial Properties on the Parameter Space

**Remark 3.27.** We remark that, for any \( S \), the site \( X_{sa} \times S \) is a ringed site both relatively to the sheaf \( \rho_{S!}(p^{-1}\mathcal{O}_S) \) and to the sheaf \( \rho_{S!}\mathcal{O}_{X \times S} \) (cf [9], page 449). Given a morphism of complex manifolds \( \pi : S' \to S \), we have isomorphisms of functors

\[
\rho^{-1}_{S!} \pi^{-1} \rho_{S*} \simeq \pi^{-1} \\
\rho_{S'!} \rho^{-1}_{S*} \pi^{-1} \rho_{S*} \simeq \rho_{S'!} \pi^{-1}
\]

and composing with the natural morphism \( \text{Id} \to \rho_{S'!} \rho^{-1}_{S*} \) gives a natural morphism

\[
\pi^{-1} \rho_{S*} \to \rho_{S'!} \pi^{-1}
\]

Since \( \mathcal{O}_{S'} \) is a \( \pi^{-1}\mathcal{O}_S \)-module, \( \rho_{S'!}(p^{-1}\mathcal{O}'_S) \) is a \( \rho_{S'!}(p^{-1} \pi^{-1}\mathcal{O}_S) \)-module hence a \( \pi^{-1} \rho_{S*}(p^{-1}\mathcal{O}_S) \)-module. Similarly, \( \rho_{S'!}\mathcal{O}_{X \times S'} \) is a \( \pi^{-1} \rho_{S!}\mathcal{O}_{X \times S} \)-module.
In other words, \( \pi \) induces a morphism of ringed \( S \)-sites with respect to both sheaves of rings.

Consequently, according to loc.cit Theorem 18.6.9 (i), the derived functors \( L\pi^* : D(\rho_S^*p^{-1}\mathcal{O}_S) \to D(\rho_{S'}^*p^{-1}\mathcal{O}_{S'}) \) resp. (keeping the same notation \( \pi \) for the morphism \( \text{Id} \times \pi \), \( L\pi^* : D(\rho_{S'}^*\mathcal{O}_{X \times S}) \to D(\rho_{S'}^*\mathcal{O}_{X \times S'}) \) are well defined.

If \( S' \) is a closed submanifold of \( S \), we denote by \( i_{S'} \), the closed immersion \( i_{S'} : X \times S' \hookrightarrow X \times S \). Hence \( L\pi_{S'}^* \) is the functor on \( D^b(\rho_{S'}^*(p^{-1}\mathcal{O}_S)) \) given by \( \mathcal{F} \mapsto i_{S'}^{-1}(\rho_{S'}^*(p^{-1}(\mathcal{O}_S/\mathfrak{m})) \otimes_{\rho_{S'}^*(p^{-1}\mathcal{O}_S)} \mathcal{F}) \) where \( \mathfrak{m} \) is the sheaf of ideals of \( \mathcal{O}_S \) of functions vanishing on \( S' \).

We denote by \( \mathcal{D}' \) the functor on \( D^b(p^{-1}\mathcal{O}_S) \) given by

\[
\mathcal{D}'(\mathcal{F}) = R\text{Hom}_{p^{-1}\mathcal{O}_S}(\mathcal{F}, p^{-1}\mathcal{O}_S)
\]

**Proposition 3.28.** (Action by a closed immersion) Let \( S' \) be a closed submanifold of \( S \). There are natural morphisms in \( D^b(\rho_{S'}(p^{-1}\mathcal{O}_S)) \)

1. \( Li_{S'}^*\mathcal{O}^{t,S}_{X \times S} \to \mathcal{O}^{t,S'}_{X \times S'} \)
2. \( Li_{S'}^*\mathcal{O}^{w,S}_{X \times S} \to \mathcal{O}_{X \times S} \)

which are isomorphisms.

**Proof.** Let us construct the morphism (1). (Note that for \( \text{codim} S' = 0 \) the result is trivial.)

This amounts to showing that the following morphism is an isomorphism

\[
\Gamma(V; \mathcal{O}_S/\mathfrak{m}) \otimes_{\Gamma(V; \mathcal{O}_S)}^L \Gamma(U \times V; \mathcal{O}^{t,S}_{X \times S}) \to \Gamma(U \times V; \mathcal{O}^{t,S'}_{X \times S'})
\]

for any relatively compact open subset \( U \subset X \) and any open subset \( V \subset S \) running on a basis of the topology of \( S \) consisting of Stein open relatively compact subsets and where we note \( W = V \cap S' \). We note that, according to (2) of Proposition 3.17, we have

\[
\Gamma(U \times V; \mathcal{O}^{t,S}_{X \times S}) = \Gamma(X \times V; T\text{Hom}(\mathbb{C}_{U \times S}; \mathcal{O}_{X \times S}))
\]

Recall that, for a \( \mathcal{D}_{X \times S} \)-module \( \mathcal{M} \), the pull-back by \( i_{S'} \) is given by \( \mathcal{D}_{i_{S'}^*}\mathcal{M} := \mathcal{O}_{X \times S'} \otimes_{i_{S'}^{-1}\mathcal{O}_{X \times S}} i_{S'}^{-1}\mathcal{M} \). Note that \( \mathcal{O}_{X \times S} \) is a flat \( p^{-1}\mathcal{O}_S \)-module. Hence

\[
\mathcal{O}_{X \times S'} = i_{S'}^{-1}(\mathcal{O}_{X \times S}/\mathfrak{m}\mathcal{O}_{X \times S})
= i_{S'}^{-1}(p^{-1}(\mathcal{O}_S/\mathfrak{m}) \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X \times S})
\]

Therefore Theorem 5.8 (5.16) of [7] applied to the closed embedding \( i_{S'} \) and to \( F = \mathbb{C}_{U \times S} \) becomes a natural isomorphism in \( D^b(\mathcal{D}_{X \times S'}) \)

\[
T\text{Hom}(\mathbb{C}_{U \times S'}, \mathcal{O}_{X \times S'}) \simeq Li_{S'}^*T\text{Hom}(\mathbb{C}_{U \times S}; \mathcal{O}_{X \times S}). \tag{3.3}
\]

Up to a shrinking of \( V \) so that a family of holomorphic coordinates vanishing on \( S' \) is defined in \( V \), by taking the corresponding Koszul resolution (\( \mathcal{O}_S \)-free) of \( \mathcal{O}_S/\mathfrak{m} \), we have

\[
\Gamma(V; \mathcal{O}_S/\mathfrak{m}) \otimes_{\Gamma(V; \mathcal{O}_S)}^L \Gamma(X \times V; T\text{Hom}(\mathbb{C}_{U \times S}; \mathcal{O}_{X \times S}))
\simeq R\Gamma(X \times V; p^{-1}(\mathcal{O}_S/\mathfrak{m}) \otimes_{p^{-1}\mathcal{O}_S}^L T\text{Hom}(\mathbb{C}_{U \times S}; \mathcal{O}_{X \times S}))
\]
and this last term, according to (3.3), is isomorphic to $\mathcal{R}f^!(X \times W; \mathcal{H}om(C_{U \times S'}, \mathcal{O}_{X \times S'})) \simeq \mathcal{R}f^!(U \times W; \mathcal{O}^{t,S'}_{X \times S'})$ hence the desired result follows since these isomorphisms are compatible with restrictions to open subsets.

The construction of morphism (2) is similar using (3) of Proposition 3.17 and (5.15) of Theorem 5.8 of [7].

**Proposition 3.29.** Let $S' \subset S$ be a closed submanifold of $S$ of codimension $d$. Then we have an isomorphism in $D^b(D_{X \times S'}/S')$

$$i_{S'}^{-1} R\mathcal{H}om_{\rho_S \ast p^{-1}\mathcal{O}_S}(\rho_S \ast p^{-1}(\mathcal{O}_S/\mathfrak{m}), \mathcal{O}^{t,S}_{X \times S}) \simeq \mathcal{O}^{t,S'}_{X \times S'}[-d]$$

**Proof.** We have

$$R\mathcal{H}om_{\rho_S \ast p^{-1}\mathcal{O}_S}(\rho_S \ast p^{-1}(\mathcal{O}_S/\mathfrak{m}), \mathcal{O}^{t,S}_{X \times S}) \simeq \rho_S \ast \mathcal{D}'(p^{-1}(\mathcal{O}_S/\mathfrak{m})) \otimes^L_{\rho_S \ast p^{-1}\mathcal{O}_S} \mathcal{O}^{t,S}_{X \times S}$$

$$\simeq \rho_S(p^{-1}(\mathcal{O}_S/\mathfrak{m}))[{-d}] \otimes^L_{\rho_S \ast p^{-1}\mathcal{O}_S} \mathcal{O}^{t,S}_{X \times S}$$

where the first isomorphism is given by Lemma 3.22 of [13].

Applying $i_{S'}^{-1}$ we derive un isomorphism with $\text{Li}_{S'}^* \mathcal{O}^{t,S}_{X \times S}[-d]$. The result then follows by Proposition 3.28.

As a consequence of Proposition 3.28 we conclude:

**Proposition 3.30.** (Restriction to the fibers) Let $s_0 \in S$. We simply denote by $\text{Li}_{s_0}^*$ the functor $\text{Li}^*_{\{s_0\}}$ on $D^b(\rho_S \ast (p^{-1}\mathcal{O}_S))$. There are natural morphisms

1. $\text{Li}_{s_0}^* \mathcal{O}^{t,S}_{X \times S} \rightarrow \mathcal{O}^t_X$
2. $\text{Li}_{s_0}^* \mathcal{O}^{w,S}_{X \times S} \rightarrow \mathcal{O}^w_X$

which are isomorphisms, where we identify $X_{s_0}$ with $X_{s_0} \times \{s_0\}$.

More generally we consider now a morphism $\pi : S' \rightarrow S$ of complex analytic manifolds.

**Proposition 3.31.** (Inverse image) There exists a natural morphism

$$\pi^{-1} \mathcal{O}^{t,S}_{X \times S} \rightarrow \mathcal{O}^{t,S'}_{X \times S'}$$

which coincides with the usual morphism

$$\pi^{-1} \mathcal{O}_S \rightarrow \mathcal{O}_{S'}$$

when $X = \text{pt}$. In particular we have a natural morphism $L\pi^* \mathcal{O}^{t,S}_{X \times S} \rightarrow \mathcal{O}^{t,S'}_{X \times S'}$.

**Proof.** We recall that the case of a ramification of finite degree was already treated in Lemma 2.11 of [3] in a different but equivalent framework (as said above, there one considered the relative site $X_{s_0} \times S_{s_0}$ instead of $X_{s_0} \times S$). The same argument allows us to reduce to prove the existence of a morphism in $D^b(\rho_{S'} \ast \pi^{-1} D_{X \times S}/S)$

$$\pi^{-1} \mathcal{C}^{\infty,t,S}_{X \times S} \rightarrow \mathcal{C}^{\infty,t,S'}_{X \times S'}$$

In that case the morphism is nothing more than the composition with $\pi$ which commutes with operators in $\pi^{-1} D_{X \times S}/S$ and keeps the growth conditions.

□
3.7. Application

Let $Y$ be a hypersurface of the manifold $X$ and assume that $Y$ is a normal crossing divisor. We shall note $X^* := X \setminus Y$ and denote by $j$ either the open inclusion $j : X^* \to X$ or $j : X^* \times S \to X \times S$. We also keep $j$ to denote the associated morphism $j : X^*_S \times S \to X_S \times S$. We still denote by $p$ the restriction of $p$ to $X^* \times S$.

Recall that the notion of $S$-local system on $X^* \times S$ goes back to the work of Deligne [1] and was the object of a systematic study in the Appendix of [13]. A sheaf of $p^{-1}\mathcal{O}_S$-modules is $S$-locally constant (or an $S$-local system for short) if there exists a coherent $\mathcal{O}_S$-module $G$ such that, locally on $X^* \times S$, $F \simeq p^{-1}G$.

The following result is an improvement of a statement in Lemma 3.25 of [13] where the locally free case was considered assuming $d_S = 1$. It is one important step to the relative Riemann-Hilbert correspondence for arbitrary $d_S$.

**Proposition 3.32.** Let $F$ be an $S$-local system on $X^* \times S$. Then

$$\mathcal{M} := \rho_S^{-1}Rj_*(\rhoSF \otimes_{pS^{-1}\mathcal{O}_S} \rho_{j^{-1}1O^{t,S}_{X \times S}})$$

is concentrated in degree zero and

$$\mathcal{H}^0\mathcal{M} \simeq \rho_S^{-1}j_*(\rhoSF \otimes_{pS^{-1}\mathcal{O}_S} \rho_{j^{-1}1O^{t,S}_{X \times S}})$$

**Proof.** Firstly we remark that the statements are local in $X \times S$.

We start by proving the first statement. Let $G$ be a coherent $\mathcal{O}_S$-module such that $F$ is locally isomorphic to $p^{-1}G$. We have

$$Rj_*(\rhoSF \otimes_{pS^{-1}\mathcal{O}_S} \rho_{j^{-1}1O^{t,S}_{X \times S}})$$

$$\simeq Rj_*(\rhoSF \otimes_{pS^{-1}\mathcal{O}_S} (\rho_{j^{-1}1O^{t,S}_{X \times S}}))$$

$$\simeq Rj_*(\rho_{j^{-1}1O^{t,S}_{X \times S}})$$

thanks to Proposition 3.16.

Since $\mathcal{O}_{X^* \times S}$ is $p^{-1}\mathcal{O}_S$-flat and $\rho_{j^{-1}}$ is exact and commutes with $\otimes$, we have

$$\rho_{j^{-1}} \rhoS^{-1} \mathcal{O}_{X^* \times S} \simeq \rhoS^{-1} \mathcal{O}_{X^* \times S}$$

According to the proof of Theorem 2.6 of [13] we have an isomorphism of $\mathcal{O}_{X^* \times S}$-modules

$$F \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X^* \times S} \simeq p^{-1}G \otimes_{p^{-1}\mathcal{O}_S} \mathcal{O}_{X^* \times S}$$

hence we may assume from the beginning that $F = p^{-1}G$.

Therefore we conclude

$$\mathcal{M} \simeq \rhoS^{-1}Rj_*(\rhoSF \otimes_{pS^{-1}\mathcal{O}_S} j^{-1}1O^{t,S}_{X \times S})$$

$$\simeq \rhoS^{-1}Rj_*(j^{-1}1p^{-1}G \otimes_{pS^{-1}\mathcal{O}_S} j^{-1}1O^{t,S}_{X \times S})$$

$$\simeq \rhoS^{-1}(j^{-1}1p^{-1}G \otimes_{pS^{-1}\mathcal{O}_S} Rj_*)$$

$$\simeq p^{-1}G \otimes_{p^{-1}\mathcal{O}_S} \rhoS^{-1}Rj_*j^{-1}1O^{t,S}_{X \times S}$$

where, in the last isomorphism, we used the commutation of $\rhoS^{-1}$ with $\otimes$.

The statement follows because, according to Proposition 3.9, we have $\rhoS^{-1}Rj_*j^{-1}1O^{t,S}_{X \times S} \simeq \text{THom}(\mathcal{O}_{X^* \times S}, \mathcal{O}_{X \times S}) \simeq \mathcal{O}_{X \times S}(\ast Y \times S)$
It remains to prove that, for a basis \((U \times V)\) of the topology of \(X \times S\), which can be chosen so that \(U\) and \(V\) are Stein open sets and \(U\) subanalytic relatively compact, \(R\Gamma(U \times V; Rj_*j^{-1}\mathcal{O}^{t,s}_{X \times S})\) is concentrated in degree zero.

We have

\[
R\Gamma(U \times V; Rj_*j^{-1}\mathcal{O}^{t,s}_{X \times S}) \cong R\Gamma((U \setminus Y) \times V; \mathcal{O}_{X \times S}^{t}) \cong R\Gamma((U \setminus Y) \times V; \mathcal{O}_{X \times V}^{t})
\]

and by Corollary 2.3.5 of [10] this last complex is concentrated in degree zero and thus equals \(\Gamma((U \setminus Y) \times V; \mathcal{O}_{X \times V}^{t})\) which ends the proof. □

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