Abstract

Frequency control in power networks is designed to maintain power balance by adjusting generation, what allows to keep frequency at its nominal value (i.e. 50 Hz). If power disturbance occurs, it leads to frequency oscillations and deviation from nominal value, that are suppressed by the control. Behavior of the existing control depends on a number of parameters, that are currently chosen form the set, that guarantees control stability. But they can be adjusted within that set to increase control efficiency. The two main factors, that define efficiency are maximal frequency deviations and frequency convergence rate to equilibrium point. However frequencies functions on buses are highly oscillatory and have infinite amount of extremum. The aim of this work is firstly to present analytical conservative estimations of absolute values of frequencies deviations, in order to approximate dependence of frequency behavior on the control parameters and secondly present zero order optimization algorithm that would improve system dynamics by adjusting control parameters.

1 Introduction

Power networks are susceptible to power imbalances due to changes in power demand. Additionally generator or line failure may result in significant disturbance in power balance and power flows. As a result, frequency of the power network oscillates [1, 2, 3], and large oscillations may result in equipment damage or emergency shutdown or lines overload. In order to counter this effects frequency control is used. It adjusts power generation in order to restore power balance and deliver frequency to its nominal value (e.g. 50 Hz).

Currently used traditional version of frequency control consists of three different parts. Its first part called Droop control (or Primary Frequency control), which is aimed to reduce initial frequency drop after power disturbance appearance and works at timescale of tens of seconds. During this period the system is most vulnerable due to frequency oscillations. There exist various other versions of
frequency control schemes [7]-[9] however they are not implemented in power systems, therefore they are not considered within this paper.

Droop control has a set of control parameters, which are currently chosen to ensure system’s stability. However it is possible to adjust these parameters to reduce frequency oscillations without loosing stability of the system. There is no agreed way to describe system’s behavior with a set of particular parameters without simulations. In particular maximal frequency deviations (nadir) from the nominal value are the key factors, that influence system’s reliability. Aim of the paper is development of approach, that would minimize maximal absolute value of frequency oscillations without loss of system’s stability.

Dynamics of power network are described by system of linear differential equations. Frequency deviations on each bus of the system is highly oscillatory and has infinite number of extremums, therefore calculation of maximum of their absolute values among all buses is computationally difficult.

Additionally, as a function of control parameters, this maximum is defined by eigenvectors and eigenvalues of the system’s matrix, which cannot be calculated analytically. As a result maximum of frequency deviations is a complicated function with many extremum. Applying zero order method for its minimization results in obtaining local minimum.

In order to counter this effects majorants for absolute values of frequency deviations are derived. These majorants still depend on eigenvalues and eigenvectors of the system’s matrix, but have simpler structure. As a result, zero order method often finds not local, but global optimum and does that in a much faster time, since there is no need in maximization of oscillatory function for every bus.

Classical linearized model of transmission power network is considered. It is assumed that several buses (both loads and generators) suffer from a step change of power generation or consumption, what results in frequency oscillations.Aim of this work is derivation of analytical estimations nadir and convergence rate of frequency to the nominal value.

Since control and system’s model are described by linear system of differential equations, eigenvalue analysis is applied to obtain necessary estimations.

The article is organized in the following way. In section 2 network model and optimization aims are described, in section 3 matrix representation of power network dynamics and description of vector variables are introduced, in section 4 axillary lemmas, necessary for derivation of majorants are given, in section 5 approach for search of absolute values of maximal frequency deviations on each bus is described, in section 3 majorants of absolute values of frequency deviations are introduced, in section 7 zero order optimization method is presented, in section 8 results of numerical experiments are presented, in section 9 final observations and directions of the future work are discussed.

2 Problem Statement

2.1 Notations

Let \( \mathbb{R} \) be set of real numbers, cardinality of a finite set \( S \) is defined as \( |S| \). Imaginary one is defined as \( i \). For an arbitrary matrix (vector) \( X \) its transpose is denoted by \( X^T \). For an arbitrary vector \( x \in \mathbb{R}^n \) we define subvector \( x_{l:j} = (x_l, \ldots, x_j)^T, \ i \leq j \leq n \). \( 0 \) is zero matrix of the corresponding size, \( I \) is identity matrix of the corresponding size, \( 0 \) is zero vector of the corresponding size, \( \rho \) is vector of ones of the corresponding size. For vector \( x = (x_1, \ldots, x_n) \) we denote by \( \text{diag}(x) = \text{diag}(x_1, \ldots, x_2) \) diagonal matrix with elements \( x_l, \ l = 1, n \). Operations \( \text{Re} \ X, \text{Im} \ X, |X| \) are considered to be elementwise, if \( X \) is a vector or a matrix.
2.2 Model description

Classical generator model [4], [5] is used. The power transmission network [1] is described by a directed graph \((V, E)\), where \(V\) is the set of \(n\) buses, \(E\) is set of \(m\) lines. It is assumed, that the network is connected.

Dynamics of the power transmission network is defined by the system of linear differential equations [7]-[9]. Kron reduction [13]-[15] is applied, therefore system has the following form:

\[
\dot{\theta}_l = \omega_l, \quad l = \overline{1, n} \quad (1a)
\]

\[
M_l \dot{\omega}_l = -d_l \omega_l + \sum_{j: (i, j) \in E} p_{ji} - \sum_{j: (j, i) \in E} p_{ij} - p^M_l + p^d_l, \quad \omega_l(0) = 0, \quad l = \overline{1, n}, \quad (1b)
\]

\[
p_{ij} = b_{ij}(\theta_l - \theta_j), \quad p_{ij}(0) = 0, \quad (l, j) \in E, \quad (1c)
\]

\[
T^G_l p^M_l = -p^M_l + p^C_l, \quad p^M_l(0) = 0, \quad l = \overline{1, n}, \quad (1d)
\]

\[
T^B_l \dot{\psi}_l = -\psi_l + \psi^C_l, \quad \psi_l(0) = 0, \quad l = \overline{1, n}. \quad (1e)
\]

Variables of the system have the following meanings:

- \(\theta_l, \quad l = \overline{1, n}\) are deviations of bus voltage angles from their nominal values,
- \(\omega_l, \quad l = \overline{1, n}\) are deviations of bus frequencies from nominal value,
- \(p_{ij}, \quad (l, j) \in E\) are deviations of line active power flows from their reference values,
- \(p^M_l, \quad l = \overline{1, n}\) are mechanic power injections at generators,
- \(\psi_l, \quad l = \overline{1, n}\) are positions of valves,
- \(p^C_l, \quad l = \overline{1, n}\) are control values.

Parameters of the system:

- \(M_l > 0, \quad l = \overline{1, n}\) are generators inertia constants,
- \(d_l > 0, \quad l = \overline{1, n}\) are steam and mechanical damping of generators and frequency-dependent loads,
- \(p^d_l, \quad l = \overline{1, n}\) are unknown disturbances (assumed constant),
- \(b_{ij} > 0, \quad (i, j) \in E\) are line parameters that depend on line susceptances, voltage magnitudes and reference phase angles,
- \(T^G_l \quad l = \overline{1, n}\) are time constants that characterize time delay in fluid dynamics in the turbine,
- \(T^B_l \quad l = \overline{1, n}\) are time constants that characterize time delay in governor response.

Equations of the system:

- Equations (1b) are generator swing equations,
• Equations (1c) are equations of direct current linearized power flows,
• Equations (1d) are turbine dynamics,
• Equations (1e) are governor dynamics.

Equation (1c) can be differentiated. Variables \( \dot{\theta}_l, l = 1, n \) can be substituted from (1a). As a result, system can be described purely by differential equations:

\[
M_l \ddot{\omega}_l = -d_l \omega_l + \sum_{j:(j,l) \in E} p_{jl} - \sum_{j:(l,j) \in E} p_{lj} - p_l^M + p_l^d, \quad \omega_l(0) = 0, \ l = 1, n, \tag{2a}
\]

\[
\dot{p}_{lj} = b_{lj} (\omega_l - \omega_j), \quad p_{lj}(0) = 0, \ (l, j) \in E, \tag{2b}
\]

\[
T_l \dot{p}_l^M = -p_l^M + p_l^C, \quad p_l^M(0) = 0, \ l = 1, n, \tag{2c}
\]

\[
T_l^B \dot{\psi}_l = -\psi_l + p_l^C, \quad \psi_l(0) = 0, \ l = 1, n. \tag{2d}
\]

2.3 Control description

Droop control is applied to counter frequency drop, that occurs in a case of power deficit. In practice it reduces maximal frequency deviations from the nominal value (e.g. 50Hz) of the system. Droop control is given buy the following formula

\[
p_l^C(t) = -r_l \omega_l(t), \ l = 1, n, \tag{3}
\]

here

\[
r_l \geq r_l > 0, \ l = 1, n, \tag{4}
\]

is control parameter. These parameters are chosen with respect to (4) in order to ensure system’s stability. It is possible to adjust this parameters in order to reduce maximal frequency deviations even further and still keep system stable.

2.4 Optimization aims

Problem statement has the following form:

\[
\max_{l=1, n} \max_{t \in [0, t_1]} |\omega_l(t)| \rightarrow \min_r, \ \bar{r} \geq r > 0,
\]

where \( \omega(t) \) is part of the solution of the system (2), that depends on \( r \). Primary frequency control, considered in this paper stabilizes frequency at equilibrium during the first several tens of seconds [6]. Therefore in the experiments \( t_1 \) is taken equal to 100 seconds.

Main purpose of the work is to minimize maximal frequency deviations among all buses by adjusting control parameters \( r_l, l = 1, n \). Firstly majorants for the absolute values of frequency deviations are derived based on the eigenvectors and eigenvalues of the system’s matrix. In practice they have unique maxima therefore golden section method can be applied to find them. Then Hooke and Jeeves method is used to minimize majorants, thus reducing maximal absolute value of frequency oscillations among all buses of the system.
3 Matrix Representation

The system (2) with control $p_C^l(t)$, $l = 1, n$ substituted from (3) can be presented in the following form:

$$\dot{x} = Ax + P^D, \quad x(0) = 0,$$

$$A = ZQ$$

$$Z = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & T^G & 0 \\ 0 & 0 & 0 & T^B \end{pmatrix}, \quad Q = \begin{pmatrix} -D & -C & I & 0 \\ C^T & 0 & 0 & 0 \\ 0 & 0 & -I & I \\ -R & 0 & 0 & -I \end{pmatrix},$$

$$x(t) = \begin{pmatrix} \omega(t) \\ p(t) \\ p^M(t) \\ \psi(t) \end{pmatrix}, \quad P^D(t) = \begin{pmatrix} p^D \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$  (5)

Here $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))^T$, $p(t) = (p_1(t), \ldots, p_m(t))^T$, $p^M(t) = (p_1^M(t), \ldots, p^M_n(t))^T$, $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$, $D = \text{diag}(d_1, \ldots, d_n) > 0$, $M = \text{diag}(1/M_1, \ldots, 1/M_n) > 0$, $B = \text{diag}(1/b_1, \ldots, 1/b_m) > 0$, $T^G = \text{diag}(1/T^G_1, \ldots, 1/T^G_n) > 0$, $T^B = \text{diag}(1/T^B_1, \ldots, 1/T^B_n) > 0$, $R = \text{diag}(r_1, \ldots, r_n) > 0$, $C \in \mathbb{R}^{n \times m}$ is the incidence matrix [10] of the system graph $G$, $P^D = (p^D_1, \ldots, p^D_n)^T$ is the inhomogeneity vector of disturbances.

4 Auxiliary lemmas

In the models, describing energy systems, matrix $A$ is diagonalizable [11]-[12]. Let $A = S\Lambda S^{-1}$ be eigenvalues decomposition, where $\Lambda$ is diagonal matrix of eigenvalues with eigenvalues $\lambda_1, \ldots, \lambda_{3n+m}$ ordered with respect to real parts increase: $\text{Re} \lambda_l \leq \text{Re} \lambda_{l+1}$, $l = 1, 3n + m - 1$. Further for each column $s$ (eigenvector or generalized eigenvector) of transition matrix $S$ we will use the following notation:

$$s = \begin{pmatrix} \omega^s \\ p^s \\ p^M_s \\ \psi^s \end{pmatrix}.$$  (6)

Lemma 4.1. Kernel of $A$ has the following form:

$$\ker(A) = \begin{pmatrix} 0 \\ p^s \\ 0 \\ 0 \end{pmatrix},$$

where $p^s \in \ker(C)$.  (7)
Proof. Eigenvectors $x^s$ from $\ker(A)$ have to satisfy the following system:

$$
- D\omega^s + Cp^s + p^M s = 0, 
$$

$$
- C^T\omega^s = 0, 
$$

$$
- p^M s + \psi^s = 0, 
$$

$$
- R\omega^s - \psi^s = 0. 
$$

From (1d), (1e) $p^M s = \psi^s = -R\omega$. Substituting the latter into (7a) we get

$$
-(D + R)\omega + Cp = 0.
$$

Sum of the rows of this equation gives

$$
\sum_{l=1}^{n}(d_l + r_l)\omega_l = 0.
$$

But form (7b) $\omega_l = \omega_j$ for all $l = 1, n$, $j = 1, n$, therefore $\omega = 0$ and $Cp^s = 0$.

Lemma 4.2. There exists diagonal matrix $R \in \mathbb{R}^{n \times n}$ such, that for all $R$ for which $r_l \geq r_1 > 0$ and for any corresponding matrix $A$ the following properties are satisfied:

1. Matrix $A$ is negative semi-definite.

2. Matrix $A$ does not have purely imaginary eigenvalues.

Proof. (1) Consider the following matrices:

$$
G = \begin{pmatrix}
-I & -D^{-\frac{1}{2}}C & I & 0 \\
C^T D^{-\frac{1}{2}} & 0 & 0 & 0 \\
0 & 0 & -I & I \\
-RC D^{-\frac{1}{2}} & 0 & 0 & -I
\end{pmatrix}, 
$$

$$
P = \begin{pmatrix}
D^{-\frac{1}{2}} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{pmatrix}.
$$

Then $G = PAP$. Let us consider quadratic for with $G$:

$$
x^T G x = -\omega^T \omega + \omega^T p^M - (p^M)^T p^M + \psi^T p^M - \psi^T \psi - \omega^T RD^{-1}\omega = 
$$

$$
= -\frac{1}{2}(\omega - p^M)^T (\omega - p^M) - \frac{1}{2}(p^M - \psi)^T (p^M - \psi) - \frac{1}{2}\omega^T \omega - \frac{1}{2}\psi^T \psi - \omega^T RD^{-1}\psi.
$$

Therefore $x^T G x \geq 0$ if $r_l \leq d_l$, $l = 1, n$ and, as a result [16], matrices $G$ and $A$ are negative semi-definite. We will take

$$
R^1 = D.
$$

(2) From the contradiction. Let us assume, that $ik$ is a purely imaginary eigenvalue. Than we have the following system:

$$
- MD\omega^s + MCp^s + Mp^M s = ik\omega^s, 
$$

$$
- BC^T \omega^s = ikp^s, 
$$

$$
- T^G p^M s + T^G \psi^s = ikp^M s, 
$$

$$
- T^B R\omega^s - T^B \psi^s = ik\psi^s.
$$
Excluding all variables except $\omega^s$ we have

$$H \omega^s = 0,$$

(9)

where

$$H = -MD + \frac{i}{k} MCBC^T -$$

$$-MR(I - k^2(T^G)^{-1}(T^B)^{-1} - ik((T^G)^{-1} + (T^B)^{-1})) ((I + k^2(T^G)^{-2})(I + k^2(T^B)^{-2})) -$$

$$-ikI.$$

Here

$$\text{Re } H = -MD - MR(I - k^2(T^G)^{-1}(T^B)^{-1}) ((I + k^2(T^G)^{-2})(I + k^2(T^B)^{-2})).$$

Matrix

$$V(k) = (I - k^2(T^G)^{-1}(T^B)^{-1}) ((I + k^2(T^G)^{-2})(I + k^2(T^B)^{-2}))$$

is diagonal with diagonal elements equal

$$V_l(k) = \frac{1 - \frac{k^2(T^G)^{-1}(T^B)^{-1}}{1 + k^2(T^G)^{-2}I + k^2(T^B)^{-2}}}{1 + \frac{k^2(T^G)^{-1}(T^B)^{-1}}{1 + k^2(T^G)^{-2}I + k^2(T^B)^{-2}}}, \quad l = 1, n.$$

It can be seen, that

$$\lim_{k \to -\infty} V_l(k) = \lim_{k \to \infty} V_l(k) = 0, \quad l = 1, n.$$

Functions $V_l(k)$ are continuous, therefore they have infima $V_l$. Let us choose $R^2$ such that

$$-MD - MR^2 V \succ 0.$$

Than for all $R \leq R^2$ element wise we have

$$\text{Re } H \succ 0.$$

Complex system of linear equations (9) is equivalent to

$$\begin{pmatrix} \text{Re } H & -\text{Im } H \\ \text{Im } H & \text{Re } H \end{pmatrix} \begin{pmatrix} \text{Re } \omega^s \\ \text{Im } \omega^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

(10)

Real and imaginary parts of $H$ are symmetrical matrices, and have real eigenvalues [17]. $\text{Re } H$ is diagonal nonsingular matrix. Determinant of the matrix from (10) can be found using Schur Complement [18].

$$\det \begin{pmatrix} \text{Re } H & -\text{Im } H \\ \text{Im } H & \text{Re } H \end{pmatrix} = \det \text{Re } H \det (\text{Re } H + \text{Im } H(\text{Re } H)^{-1} \text{Im } H) =$$

$$= \det \text{Re } H \det (I + \text{Im } H(\text{Re } H)^{-1} \text{Im } H(\text{Re } H)^{-1}) \det \text{Re } H =$$

$$= \det \text{Re } H \det \left(I + (\text{Im } H(\text{Re } H)^{-1})^2\right) \det \text{Re } H.$$
Here $\text{Re} \, H \succ 0$, therefore $\det \text{Re} \, H \neq 0$, $(\text{Im} \, H (\text{Re} \, H)^{-1})^2 \succeq 0$, hence $I + (\text{Im} \, H (\text{Re} \, H)^{-1}) \succ 0$ and $\det (I + (\text{Im} \, H (\text{Re} \, H)^{-1})) \neq 0$. Matrix of the system (10) is not singular and $\omega^* = 0$. As a result $x^* = 0$.

Taking
\[
\overline{R} = \min\{\overline{R}_1, \overline{R}_2\}
\]
we ensure that statements of the lemma will be fulfilled. \qed

**Lemma 4.3.** System
\[
Ax + p^D = 0 \tag{11}
\]
is consistent.

**Proof.** System (11) can be represented as
\[
-MD\omega + MCp + M(p^G) + Mp^D = 0, \tag{12a}
\]
\[-BC^T\omega = 0, \tag{12b}
\]
\[-(p^G) + \psi = 0 \tag{12c}
\]
\[-R\omega - \psi = 0. \tag{12d}
\]

Similarly to lemma 4.1 we have $\sum_{l=1}^n (d_l + r_l)\omega_l = \sum_{l=1}^n p_l^D$, and $\omega_j = \omega_l$. Now we need to show, that solution in $p$ exists. It is given by the first set of equations (12a). From it we have
\[
MCp = -(D + R)\omega - p^D. \tag{13}
\]

Here sum of the elements of the left hand side vector is equal 0. By the Fredholm theorem of alternative (13) has solution if for any $y$ such that
\[
C^Ty = 0 \tag{14}
\]
we have $y^T((D + R)\omega - p^D) = 0$. But $C^Ty = 0$ has only solutions of the form $y = k\rho$, $k \in \mathbb{R}$, therefore $y^T(-(D + R)\omega - p^D) = \sum_{l=1}^n(-(d_l + r_l)\omega_l - p_l^D) = 0$. \qed

**Corollary 4.4.** Every stationary solution
\[
x^* = \begin{pmatrix}
\omega^*
p^*
(p^G)^*
\psi^*
\end{pmatrix}
\]
of the system (11) satisfies
\[
\omega_l^* = \frac{\sum_{l=1}^n p_l^D}{\sum_{l=1}^n (d_l + r_l)}, \quad l = 1, n. \tag{15}
\]

If system’s graph $G$ is not a tree, then $A$ is singular and eigenvalues matrix $\Lambda$ can be presented in the following form
\[
\Lambda = \begin{pmatrix}
\Lambda_1 & 0 \\
0 & 0
\end{pmatrix}, \tag{16}
\]
where \( \Lambda \) is diagonal nonsingular matrix, corresponding eigenvector matrix is given by

\[
S = \begin{pmatrix} \Omega^1_s & 0 \\ P^s_1 & P^s_2 \\ p^s_1 & 0 \\ \Psi^s_{1} & 0 \end{pmatrix},
\]

here \( S^1 \) is the submatrix of eigenvectors corresponding to nonzero eigenvalues, \( S^2 \) is the submatrix of vectors, corresponding to 0 eigenvalue. Let us introduce the following matrix

\[
\Xi = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},
\]

where \( \Lambda_2 \) is an arbitrary matrix and matrix

\[
\Xi = S\Xi S^{-1}.
\]

Then the following lemma is correct.

**Lemma 4.5.** Let

\[
x(t) = \begin{pmatrix} \omega(t) \\ p(t) \\ p^G(t) \\ \alpha(t) \end{pmatrix}
\]

be solution of the system (5) and

\[
\bar{x}(t) = \begin{pmatrix} \bar{\omega}(t) \\ \bar{p}(t) \\ \bar{p}^G(t) \\ \bar{\alpha}(t) \end{pmatrix}
\]

be solution of the system

\[
\dot{x} = \Xi x + p^D, \quad \bar{x}(0) = 0.
\]  

(17)

then

\[
\omega(t) = \bar{\omega}(t).
\]

**Proof.** Solution of (5) is given by

\[
x(t) = \int_0^t e^{A(t-\tau)} p^D d\tau = S e^{\Lambda t} \int_0^t e^{-\Lambda \tau} d\tau S^{-1} p^D =
\]

\[
= S e^{\Lambda t} \begin{pmatrix} \Lambda_1^{-1}(I - e^{-\Lambda_1 t}) & 0 \\ 0 & lt \end{pmatrix} S^{-1} p^D = S \begin{pmatrix} \Lambda_1^{-1}(e^{\Lambda_1 t} - I) & 0 \\ 0 & lt \end{pmatrix} S^{-1} p^D.
\]

Similarly

\[
\bar{x}(t) = S \begin{pmatrix} \Lambda_1^{-1}(e^{\Lambda_1 t} - I) & 0 \\ 0 & \int_0^t e^{\Lambda_2(t-\tau)} d\tau \end{pmatrix} S^{-1} p^D.
\]

Let us consider their difference

\[
x(t) - \bar{x}(t) = S \begin{pmatrix} 0 & 0 \\ 0 & \int_0^t e^{\Lambda_2(t-\tau)} d\tau \end{pmatrix} S^{-1} p^D =
\]
\[
\begin{pmatrix}
S_1 & S_2
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & \int_0^t e^{A_2(t-\tau)}d\tau
\end{pmatrix}
S^{-1}p^D =
\]
\[
\begin{pmatrix}
\Omega_s^1 & 0 \\
0 & \int_0^t e^{A_2(t-\tau)}d\tau
\end{pmatrix}
S^{-1}p^D =
\]
\[
\begin{pmatrix}
P_1^s & P_2^s \\
P_1^{Gs} & 0 \\
\Psi & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & \int_0^t e^{A_2(t-\tau)}d\tau
\end{pmatrix}
S^{-1}p^D =
\]

First \( n \) elements of this vector, corresponding to \( \omega(t) - \bar{\omega}(t) \) are zero.

Further it is assumed, that \( \Lambda_2 \) is negative definite diagonal matrix.

**Corollary 4.6.** Solution of (1) always converges to some stationary pointy.

**Proof.** It is a consequence of the fact, that solution of (2) is the same as solution of asymptotically stable system (17).

**Lemma 4.7.** The solution of the (1) can be presented in the following form:

\[
\bar{x}(t) = S\Lambda^{-1} \text{diag}(S^{-1}p^D)e^{\bar{x}_0t} + x^*,
\]

where

\[
x^* = \begin{pmatrix}
\omega^* \\
p^{*M} \\
\psi^*
\end{pmatrix} = -A^{-1}p^D.
\]

**Proof.**

\[
\bar{x}(t) = \int_0^t e^{\bar{A}(t-\tau)}p^D d\tau = S\Lambda^{-1} e^{\bar{x}_0t} S^{-1}p^D - A^{-1}p^D =
\]
\[
S\Lambda^{-1} \text{diag}(S^{-1}p^D)e^{\bar{x}_0t} - A^{-1}p^D.
\]

**Corollary 4.8.** For diagonal matrix \( \hat{\Lambda} \) with diagonal entries

\[
\hat{\lambda}_l = \begin{cases}
\frac{1}{\xi_l} & \lambda_l \neq 0 \\
0 & \lambda_l = 0
\end{cases}
\]

and

\[
\hat{x}(t) = Y e^{\Lambda_0t} - A^{-1}p^D,
\]

where

\[
Y = S\hat{\Lambda} \text{diag}(S^{-1}p^D),
\]

we have

\[
\hat{\omega}(t) = \bar{\omega}(t) = \omega(t).
\]
Corollary 4.9. For $Y_\omega = Y_{1:n}$ the following equality is correct

$$\omega(t) = Y_\omega e^{\Lambda t} + \omega^* = \text{Re } Y_\omega e^{\text{Re } \Lambda t}\cos \text{Im } \Lambda t - \text{Im } Y_\omega e^{\text{Re } \Lambda t}\sin \text{Im } \Lambda t + \omega^*$$

or in element-wise representation

$$\omega_l(t) = \sum_{j=1}^{3n+m} e^{\text{Re } \lambda_j t} \left( \text{Re } y_{lj}\cos \lambda_j t - \text{Im } y_{lj}\sin \lambda_j t \right) + \omega_l^*$$.  

5 D.c. approximation of frequency deviations

Functions $\omega_l(t)$ can be represented in the following way:

$$\omega_l(t) = h_l(t) - q_l(t), \ l = 1, n,$$

where

$$h_l(t) = \omega_l(t) + \frac{1}{2} k_l t^2, \ q_l(t) = \frac{1}{2} k_l t^2.$$

Here $k_l$ is obtained as follows.

$$\frac{d^2}{dt^2} \omega_l(t) = \sum_{j=1}^{3n+m} e^{\text{Re } \lambda_j t} \left( ((\text{Re } \lambda_j)^2 \text{Re } y_{lj} - 2 \text{Re } \lambda_j \text{Im } \lambda_j \text{Im } y_{lj} - (\text{Im } \lambda_j)^2 \text{Re } y_{lj}) \cos \lambda_j t +

((\text{Im } \lambda_j)^2 \text{Im } y_{lj} - 2 \text{Re } \lambda_j \text{Im } \lambda_j \text{Re } y_{lj} - (\text{Re } \lambda_j)^2 \text{Im } y_{lj}) \sin \lambda_j t \right) \leq

\leq \sum_{j=1}^{3n+m} ((\text{Re } \lambda_j)^2 \text{Re } y_{lj} - 2 \text{Re } \lambda_j \text{Im } \lambda_j \text{Re } y_{lj} - (\text{Im } \lambda_j)^2 \text{Re } y_{lj})^2 +

+((\text{Im } \lambda_j)^2 \text{Im } y_{lj} - 2 \text{Re } \lambda_j \text{Im } \lambda_j \text{Re } y_{lj} - (\text{Re } \lambda_j)^2 \text{Im } y_{lj})^{\frac{1}{2}} = k_l.$$

Global maximum of each function $|\omega_l(t)|, \ l = 1, n$ is obtained using branch and bound method with concave overestimators and d.c. approximation. Convergence of the method is given in [19].

We denote obtained maximum of absolute value of frequency deviation as

$$\omega_l^{\text{max}} = \max_{t \in [0, t_1]} |\omega_l(t)|, \ l = 1, n.$$

6 Majorants of frequency deviations

Theorem 6.1. For the frequency deviation $\omega(t)$ in (5) the following estimation exists:

$$|\omega(t) - \omega^*| \leq M^1(t) = |Y_\omega| e^{\text{Re } \Lambda pt}.$$

Proof.

$$|\omega(t) - \omega^*| = |Y_\omega e^{\Lambda pt} + \omega^* - \omega^*|_{1:n} = |Y_\omega e^{\Lambda pt}| \leq |Y_\omega| e^{\text{Re } \Lambda pt}.\qed$$
Corollary 6.2.

\[ |\omega(t)| \leq |Y_\omega| e^{\text{Re} \Lambda t} + |\omega^*| \]

Theorem 6.3. For the frequency deviation \( \omega(t) \) in (5) we have the following estimation:

\[ |\omega(t)| \leq M^2(t) = \sum_{j=1}^{3n+m} |\text{Im } y_j| e^{\text{Re} \lambda_j t} \min \{|\text{Im } \lambda_j t|, 1\} + \]

\[ + \sum_{j=1}^{3n+m} |\text{Re } y_j| \min \left\{ \left| \frac{d}{dt} f_j(t_j^0) \right| t, |f_j(t_j^0) - 1| \right\}, \]

where

\[ f_j(t) = e^{\text{Re} \lambda_j t} \cos(\text{Im } \lambda_j t), \]

\[ t_j^0 = \begin{cases} \left( \pi + \arctan \frac{\text{Re } \lambda_j}{\text{Im } \lambda_j} \right) / \text{Im } \lambda_j, & \lambda_j \neq 0, \\ 0, & \text{otherwise}. \end{cases} \]

\[ t_j^1 = \begin{cases} \arctan \left( \frac{(\text{Re } \lambda_j)^2 - (\text{Im } \lambda_j)^2}{2 \text{Re } \lambda_j \text{ Im } \lambda_j} \right), & \text{Re } \lambda_j \neq 0 \text{ and } \text{Im } \lambda_j \neq 0, \\ \text{Re } \lambda_j, & \text{Re } \lambda_j \neq 0 \text{ and } \text{Im } \lambda = 0, \\ 0, & \text{otherwise}. \end{cases} \]

Proof. Since \( \text{Im } \omega(t) = 0 \), we can use form

\[ \omega(t) = \omega_s(t) + \omega_c(t), \]

where

\[ \omega_s(t) = \text{Im } (Y_\omega) e^{\text{Re} \Lambda t} \sin(\text{Im } \Lambda t) \rho, \quad \omega_c(t) = \text{Re } (Y_\omega) e^{\text{Re} \Lambda t} \cos(\text{Im } \Lambda t) \rho + \omega^*. \]

We will approximate each of this function separately.

\[ |(\omega_s(t))_t| = \sum_{j=1}^{3n+m} |\text{Im } y_j e^{\text{Re} \lambda_j t} \sin(\text{Im } \lambda_j t)| \leq \sum_{j=1}^{3n+m} |\text{Im } y_j e^{\text{Re} \lambda_j t} \sin(\text{Im } \lambda_j t)| \leq \]

\[ \leq \sum_{j=1}^{3n+m} |y_j e^{\text{Re} \lambda_j t} \min \{|\text{Im } \lambda_j t|, 1\}. \]

To approximate \( \omega_c(t) \) we will use the following expression:

\[ |(\omega_c(t))_t| = \left| \sum_{j=1}^{3n+m} \text{Re } y_j f_j(t) \right|, \]

Since \( \omega(0) = 0 \), and \( \omega_s(0) = 0 \), we have \( \omega_c(0) = 0 \).

Let \( t_j^0 \) be a solution of the problem

\[ \max_{\tau > 0} |f_j(\tau) - 1|, \]
then

\[
t^0_j = \begin{cases} \left( \pi + \arctan \frac{\text{Re} \lambda_j}{\text{Im} \lambda_j} \right) / \text{Im} \lambda_j, & \lambda_j \neq 0, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( t^1_j \) be the solution of the problem

\[
\max_{\tau \geq 0} |f'_j(\tau)|,
\]

\[
t^1_j = \begin{cases} \arctan \left( \frac{(\text{Re} \lambda_j)^2 - (\text{Im} \lambda_j)^2}{2\text{Re} \lambda_j \text{Im} \lambda_j} \right) & \text{Re} \lambda_j \neq 0 \text{ and } \text{Im} \lambda_j \neq 0, \\ \text{Re} \lambda_j & \text{Re} \lambda_j \neq 0 \text{ and } \text{Im} \lambda = 0, \\ 0 & \text{otherwise}, \end{cases}
\]

Then we have the following estimation

\[
|\omega_c(t)| = \int_0^t \frac{d}{dt} \left( \sum_{j=1}^{3n+m} \text{Re} y_{ij} f_j(\eta) d\eta \right) \leq \sum_{j=1}^{3n+m} \left| \text{Re} y_{ij} \right| \left| \int_0^t f'_j(\eta) d\eta \right| =
\]

\[
= \sum_{j=1}^{3n+m} \left| \text{Re} y_{ij} \right| \min \left\{ \int_0^t \left| f'_j(\eta) \right| d\eta, \left| \int_0^t f'_j(\eta) d\eta \right| \right\} =
\]

\[
= \sum_{j=1}^{3n+m} \left| \text{Re} y_{ij} \right| \min \left\{ \int_0^t \max_{\tau \geq 0} \left| f'_j(\tau) \right| d\eta, \left| f_j(t) - 1 \right| \right\} \leq
\]

\[
\leq \sum_{j=1}^{3n+m} \left| \text{Re} y_{ij} \right| \min \left\{ \max_{\tau \geq 0} \left| f'_j(\tau) \right| t, \left| f_j(t^0_j) - 1 \right| \right\} \leq
\]

\[
\leq \sum_{j=1}^{3n+m} \left| \text{Re} y_{ij} \right| \min \left\{ \left| f'_j(t^1_j) \right| t, \left| f_j(t^0_j) - 1 \right| \right\}.
\]

\[
\square
\]

**Corollary 6.4.**

\[
|\omega_l(t)| \leq M^3(t) = \min\{M^1(t), M^2(t)\},
\]

where

\[
M^2(t) = (M^2_1(t), \ldots, M^2_n(t))^T, \quad M^3(t) = (M^3_1(t), \ldots, M^3_n(t))^T,
\]

In practice each functions \( M^3_l(t) \), \( l = 1, n \) have unique maximum. Taking this observation into consideration, we assume, that majorant maximum can be found using golden section algorithm. Maximum is denoted

\[
M^l_{\text{max}} = \max_{t \in [0, t_1]} M^3_l(t), \quad l = 1, n.
\]
7 Optimization algorithm

Frequency deviations as well as majorants, presented in the previous chapter depend on the vector of control parameters \( r = (r_1, \ldots, r_n)^T \). Hence \( \omega_{l \text{max}} \) and \( M_{l \text{max}} \) can be represented as functions of \( r \):

\[
\omega_{l \text{max}} = \omega_{l \text{max}}(r), \quad M_{l \text{max}} = M_{l \text{max}}(r).
\]

Therefore, we can introduce the following functions, describing absolute value of maximal frequency deviations among all buses and maximum among all majorants as following:

\[
F(r) = \max_{l=1,n} \omega_{l \text{max}}(r),
\]

\[
G(r) = \max_{l=1,n} M_{l \text{max}}(r).
\]

Here \( r \in \Gamma \). And set \( \Gamma \) is obtained based out the following two constraints.

1. Parameters \( r \) have to be chosen so, that system of differential equations will remain stable. For all \( r \in \Gamma \) we have

\[
\max_{l=1,n} \lambda_l = 0. \quad (20)
\]

2. Oscillation of the frequency oscillations must decrease at reasonable speed. There is no standardized constraints on the frequency oscillations decreases, therefore here the following constraint is used:

\[
\max_{l=1,n} \left| \frac{\text{Re} \lambda_l}{\text{Im} \lambda_l} \right| \geq \gamma, \quad \gamma > 0. \quad (21)
\]

If \( r \notin \Gamma \), we take

\[
F(r) = G(r) = \infty.
\]

Although it is known \([21]\), that eigenvalues and eigenvectors are continuous functions of matrix entries, functions \( F \) and \( G \) are require knowledge of eigenvectors and eigenvalues of the matrix \( A \), which cannot be calculated analytically. Therefore Hooke and Jeeves algorithm \([20]\) is used optimize both function \( F(r) \) and \( G(r) \). Points found by this algorithm are denoted by

\[
F^* = F(r_F^*), \quad G^* = G(r_G^*).
\]

8 Numerical results

The algorithms were coded in Matlab. Computations were made in PC with Intel Core i7 /2.4GHz / 16GB. Numerical results are presented in the table \([1]\). Here column System shows for which power system experiments were held, \( \gamma \) is taken 0.01. Two systems are considered \([12]\): 3 generators 9 buses system and system of New England, that consists of 10 generators and 39 buses. For each system 100 test with different vectors of disturbances. Averaged results are presented in the table. Column Optimization time represents time in seconds, required for the Hooke and Jeeves method, applied to functions \( F \) and \( G \). Next column contains number of function (\( F \) or \( G \)) calculations required for the method. Last column contains values of function \( F \) (maximal absolute value of frequency deviations) at the starting point, after optimization of \( F \) and after optimization of \( G \). Although in the last case
Table 1: Results of numerical experiments for 3 generators system and New England bus system.

| System       | Optimization Time (seconds) | Function calculations | $F$ values | Starting point |
|--------------|-----------------------------|-----------------------|------------|----------------|
|              | $F$            | $G$                  | $F$            | $G$            |                |
| New England  | 20.7           | 1.5                  | 225         | 142            | 2.67           | 1.55 | 1.45 |
| 3 generators | 3.63           | 0.15                 | 225         | 142            | 1.18           | 0.55 | 0.55 |

function $G$ is optimized, aim of the algorithm is to minimize maximal frequency deviations, therefore values of $F$ are provided. Parameters in starting point $r$ are taken from [12].

Time, required for optimization of $F$ is bigger than for $G$, due to the fact that, during every calculation of $F$ algorithm has to solve n d.c. optimization problems, while during calculation of $F$ golden section computations are required. As can be seen from the table, optimization of $G$ might give better set of parameters, than optimization of $F$. Moreover less calculations of the function are required. This effect can be seen in figures 1 and 2. Here New England system is considered, all parameters $r_l$ are frozen with the exception of the first two. Red points on figures 1 and 2 represent $G^*$ and $F(r_G^*)$ respectively, green point on figure 2 represents $G^*$. As can be seen, $F(r_G^*)$ is global optimum of $F$, so optimization of majorants gives better result, than optimization of maximal values of frequency deviations directly. This happens due to the fact, that function $G$ is smoother. Consider subregion, containing both points $G^*$ and $F^*$ on figures 3 and 4. Function $F$ in this case have unique minimum $F^*$, while $G$ has local minimums, and $G^*$ is one of them.

Dynamics of the frequencies and majorant for New England System are given in figures 5, 6, 7 and 8. Figure 5 represents behavior of the system with starting values of the parameters. Figure 6 represents behavior of the system with starting values of the parameters. Figure 7 represents behavior of the system with starting values of the parameters. Figure 8 represents behavior of the system with starting values of the parameters. Figure 8 system dynamics, after optimization of $F$ without suppression of oscillations (21). As can be seen, here maximal frequency deviations are smaller, than in 6 however they do not decay, during the observation time.

9 Conclusion

The paper proposes method, that allows to minimize maximum of absolute values of frequency deviations in power network under droop control. Control parameters are obtained by this method in a way, that ensures system’s stability and good decay rate of the oscillations.

Functions, that describe frequencies behavior on every bus are oscillatory and have infinite number of extremums, therefore finding nadir requires solving global optimization problem. Due to the fact, that system can contain thousands of buses, search for the nadir is computationally complicated problem. In order to avoid this difficulty the following approach is proposed in this paper. Firstly conservative estimates (majorants) for absolute values of frequency deviations are derived. In practice this majorants have unique maxima, which can be found by golden section method. Then Hooke and Jeeves method [20] is applied to optimize parameters values.

Numerical results represent advantage of the majorants in comparison with usage of the frequency deviations functions directly.

Aims of the future work are extension of the proposed method on systems with secondary fre-
Figure 1: Function $G(r)$ for the New England system with $r_l$, $l = 3, 10$ fixed. Red point represent value, found by the Hooke and Jeeves algorithm $G^*$. 
Figure 2: Function $F(r)$ for the New England system with $r_t$, $l = 3, 10$ fixed. Green point represents value, found by the Hooke and Jeeves algorithm $F^*$. Red point represents value of $F$ in the point $r^*_G$. 
Figure 3: Function $G(r)$ for the New England system with $r_l$, $l = 3, 10$ fixed. Subregion, containing optimal point. Red point represent value, found by the Hooke and Jeeves algorithm $G^*$. 
Figure 4: Function $F(r)$ for the New England system with $r_l$, $l = 3, 10$ fixed. Subregion, containing optimal point. Green point represent value, found by the Hooke and Jeeves algorithm $F^*$. Red point represents value of $F$ in the point $r^*_G$. 
Figure 5: New England system. Absolute values of frequency deviations and corresponding majorants for starting values of $r$. 
Figure 6: New England system. Absolute values of frequency deviations and corresponding majorants for $r^*_G$. 
Figure 7: New England system. Absolute values of frequency deviations and corresponding majorants for $\phi_F$. 
Figure 8: New England system. Absolute values of frequency deviations and corresponding majorants for $r^*_G$ without reduction of oscillations.
quency control and increase of the number of optimization methods applied for parameters optimization.

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