Secrecy Capacity of Two-Hop Relay Assisted Wiretap Channels

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Abstract Incorporating the physical layer characteristics to secure communications has received considerable attention in recent years. Moreover, cooperation with some nodes of network can give benefits of multiple-antenna systems, increasing the secrecy capacity of such channels. In this paper, we consider cooperative wiretap channel with the help of an Amplify and Forward (AF) relay in the middle of transmission to transmit confidential messages from source to legitimate receiver in the presence of an eavesdropper. In this regard, the secrecy capacity of AF relaying is derived, assuming the relay is subject to a peak power constraint. To this end, an achievable secrecy rate for Gaussian input assumption is derived. Then, it is proved that any rate greater than this secrecy rate is not achievable. To do this, the capacity of a genie-aided channel as an upper bound for the secrecy capacity of the underlying channel is derived, showing this upper bound is equal to the computed achievable secrecy rate with Gaussian input assumption. Moreover, the power allocation policy at the relay is formulated as a fractional quadratic problem, and the optimal solution is analytically derived. Accordingly, the corresponding secrecy capacity is compared to the Decode and Forward (DF) strategy which is served as a benchmark in the current work.

Keywords Secrecy capacity · Achievable secrecy rate · Physical layer security · Cooperative wiretap channel

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1 Introduction

Security has been regarded as one of the important issues in wireless communication networks as it may happen an illegitimate receiver to hear transmitted signal. As a result, enhancing security has attracted a great deal attentions in recent years in both of proposed by Shannon in his landmark paper [1] in which it is assumed both the legitimate receiver and eavesdropper (wiretapper) have direct access to the transmitted signal. Accordingly, using cryptographic approaches and the notion of equivocation, the level of uncertainty about the message and the key at the eavesdropper side is measured. However, this approach may not be feasible for some of wireless technologies [2]. This motivated Wyner in his pioneering work in [3] to investigate the possibility of incorporating physical layer characteristics to secure the wireless communication networks.

Wyner introduced the wiretap channel in which a source wishes to send confidential message to a legitimate receiver while keeping the eavesdropper as ignorant of this information as possible when the source-eavesdropper channel is degraded version of the source-destination channel. For more information about the degraded broadcast channel, refer to Appendix 1. Moreover, in [3] the maximum achievable secrecy rate, the rate below which the message can not be decoded at the eavesdropper, is defined as the secrecy capacity. Accordingly, the secrecy capacity of discrete memoryless wiretap channels and Gaussian wiretap channels are investigated in [3] and [4], respectively.

In Csiszár and Körner [5], generalized Wyner’s approach to broadcast channels which are not necessarily degraded. It is assumed the source wishes to transmit a common message to both legitimate receiver and eavesdropper in addition to sending a confidential message to the legitimate receiver. This channel is termed as Broadcast Channel with Confidential message (BCC). Accordingly, both the capacity-equivocation region and the secrecy capacity region of BCC are established in [5]. Moreover, it is shown that in the lack of a common message, the secrecy capacity for degraded case can be computed as

\[
C_s = \max \frac{I(X;Y)}{C_0} - \frac{I(X;Z)}{C_0}
\]

where \(X, Y\) and \(Z\) are, respectively, the source input, the channel outputs at the legitimate receiver, and the eavesdropper’s received signal where the maximization is taken over the distribution of channel input signal. Note that the secrecy capacity can be affected by channel conditions. For instance, if source-destination channel is weaker than source-eavesdropper channel the secrecy capacity will be zero meaning no confidential message can be transmitted. To overcome this issue, multiple antenna systems can be employed [6–12].

Due to the cost and size limitation, using multiple antennas at each node may not be practically feasible. Cooperative communications, however, is an effective way to get advantages of multi-antenna systems while incorporating single antenna nodes [13–18]. In cooperative communication, some nodes can act as intermediate nodes, dubbed relays, to facilitate the transmission between two nodes of network. Accordingly, there are some strategies to be employed at the relay nodes, among them, the Amplify and Forward (AF), and Decode and Forward (DF) are mostly addressed in the literature. In AF strategy, the relay sends an scaled version of its received signal to the destination without any more changes, while in DF, the relay attempts to decode the information, re-encodes again and transmits a coded version of information to the destination. As a result, the AF strategy is simpler than DF. Furthermore, in some applications, the relay nodes may have low security level, thus it is desirable that transmitted messages to be confidential for the relays. These
relays are called untrusted relays [19, 20]. In such scenarios, the AF strategy is the prominent choice as the relay nodes do not need to have an access to the information bits, hence, they are unable to eavesdrop the information bits.

More recently, a great deal of attentions are devoted to the physical layer security issues in cooperative communication networks, where it is shown that relaying can improve the achievable secrecy rate of such networks [21–24]. For instance, in [25] the secure communication for a source to destination with the help of multiple cooperating relays in the presence of one or more eavesdroppers is investigated by considering three cooperative strategies: (1) DF, (2) AF and (3) Cooperative Jamming (CJ). It should be noted that this paper fails to draw a concrete path towards finding optimal power allocation strategies at the relay nodes and merely an approximate solution is proposed. In [26], the AF beamforming under total and individual relay power constraints is studied where the goal is maximizing the secrecy rate when perfect channel state information (CSI) is available. To this end, an iterative search method as a suboptimal solution is proposed.

Moreover, the idea of relay selection for secure cooperative networks is considered in [27–29]. Also, there are some related works on this issue [30, 31].

In this paper, we derive the secrecy capacity of a simple cooperative wiretap channel in which a source wishes to send a confidential message to a legitimate destination with the help of an untrusted relay incorporating the AF strategy, where it is desirable to keep the information bits confidential from an eavesdropper. As seen in Fig. 1, this work assumes that the source is too far from the destination and eavesdropper. So, the direct links have not considerable role in data transmission and can be omitted in analysis. In this situation, a relay node is incorporated in the middle of transmission and the communication is occurred in two hops with the help of this node. In this case, the received signal at the destination is a degraded version of the relay’s received signal. Thus, the DF strategy is optimal. However, we are interested in cases in which we are dealing with an untrusted relay which is unaware of incorporated codebook at the source and the AF strategy is employed at this node. In this regard, the secrecy capacity is fully characterized.

To this end, the achievable secrecy rate for Gaussian input is derived. Then, it is demonstrated that any rate greater than this rate is not achievable. The main contribution of this paper is finding the optimum power allocation of relay node that gives the secrecy capacity. Moreover, the corresponding non-convex optimization problem is addressed and the optimal solution is analytically derived. Accordingly, the secrecy capacity of AF relaying is compared to the secrecy capacity of DF relaying to get an indication regarding the capacity loss due to the use of AF relaying. This is completely different to the works done in [25, 26] which merely investigate suboptimal solutions of AF strategy. Moreover, the secrecy capacity of AF relaying is not reported elsewhere.

The reminder of this paper is organized as follows. The system model is discussed in Sect. 2. Section 3 provides the problem formulation followed by the main results, where some of technical details are provided in the appendix. Numerical results are represented in Sect. 4. Finally, Sect. 5 summarizes findings.

The following notations are used throughout this paper: We use bold upper and lower case characters for matrices and vectors, respectively. Symbols $h$ and $I$, respectively, denote differential entropy and mutual information. $\mathbb{R}^n$ is the set of all $n$-dimensional real-valued vectors. $A \succeq 0$ means that $A$ is positive semi-definite matrix. Moreover, $E[x]$, $\text{Var}[x]$ and $\text{Cov}(x,y)$ denote the mean and variance of random variable $x$ and covariance of random variables $x$ and $y$, respectively. The notations $x^*$, $\mathbb{R}\{x\}$ and $\lvert x \lvert$ refer to complex

\footnote{Future work will investigate the effect of direct links on the studied problem.}
conjugate, real part and absolute value of complex variable $x$. Function $\{x\}^+$ is equivalent to $\max\{0, x\}$. Finally $x^n$ and $CN \sim (0, K)$, respectively, denote a sequence of length $n$ and a zero-mean circularly symmetric complex Gaussian distribution with covariance $K$.

2 System Model

We consider a wireless communication network consisting of a source node $S$, a relay node $R$, a destination node $D$, and a passive eavesdropper $E$ (see Fig. 1). Moreover, it is assumed all nodes are equipped with single antenna and operate in half duplex mode. Also, it is assumed that there is not a direct link from $S$ to $D$ and $E$, and the communication is carried out in two hops through the use of a relay in the middle of transmission. We consider a quasi-static flat fading environment where all channel coefficients are assumed to be statistically independent. Moreover, in addition to the source-to-relay channel gain, the channel gains from the relay to the destination and eavesdropper are assumed to be completely known at the relay. This is in accordance to what is assumed in some of related works including [32].

According to the model depicted in Fig. 1, the communication is occurred in two hops. During the first hop, $S$ sends the message $W$, which is uniformly taken from the index set $\mathcal{W} = \{1, 2, \ldots, 2^{nR}\}$, to the relay over a transmission interval of length $n$, where $R$ and $nR$ indicate, respectively, the transmission rate of source in units of bits per channel use and the message entropy. The mapping of each message $W$ to a codeword $x^n_s \in \chi^n_s$ is done by an encoder $f_n : \mathcal{W} \rightarrow \chi^n_s$, where $\chi^n_s$ is the transmitted vector space. Each source symbol $x_s(t)$, which appears within one time slot, has zero mean and unit power, i.e., $E[|x_s|^2] = 1$. In this case, the received signal at the relay node can be written as,

$$y_r(t) = \sqrt{P_s} h_r x_s(t) + z_r(t) \quad \text{for } t = 1, \ldots, n,$$

where $h_r$ is the channel fading coefficient from source to the relay, $z_r$ is zero-mean Additive White Gaussian Noise (AWGN) at the destination which is of unit power, i.e., $E[|z_r|^2] = 1$. Finally, $P_s$ is the transmit power per symbol.

Then, the relay depending on the incorporated strategy, broadcasts a variation of the received information to the destination as well as the eavesdropper. In the following subsections, two relaying strategies, AF and DF, are investigated.
2.1 Amplify and Forward

In AF relaying, the relay transmits a scaled version of its received signal, i.e., \( y_r \), to the destination as follows,

\[
x_r = \omega y_r,
\]

where \( x_r \) is the transmitted signal and \( \omega \) is a scaling factor, ensuring the peak power constraint at the relay is satisfied. As a result, the received signals at the nodes D and E can be written, respectively, as,

\[
y_d = h_d x_r + z_d = \sqrt{P_s} h_d \omega h_r x_r + h_d \omega z_r + z_d,
\]

\[
y_e = h_e x_r + z_e = \sqrt{P_s} h_e \omega h_r x_r + h_e \omega z_r + z_e,
\]

where \( h_d \) and \( h_e \) are channel fading coefficients from R to D and E, respectively. Also, \( z_d \sim \mathcal{CN}(0, 1) \) and \( z_e \sim \mathcal{CN}(0, 1) \) are additive white Gaussian noises at the nodes D and E, respectively.

2.2 Decode and Forward

In DF strategy, the relay attempts to decode the source message and re-encodes the estimated message \( W \) to a codeword \( x_r^n \in \mathcal{X}_r^n \) by an encoder \( g_n : \mathcal{W} \rightarrow \mathcal{X}_r^n \). For large transmission interval \( n \), invoking the channel coding theorem, the relay can correctly decode the information signal as long as the transmission rate is not greater than the capacity of source-relay link, which is given by,

\[
C_{S-R} = \log_2 \left( 1 + P_s |h_r|^2 \right).
\]

After re-encoding, the relay broadcasts a weighted version of re-encoded symbols, i.e., \( \omega x_r \), to D and E. Thus, the received signals at the nodes D and E can be respectively expressed as,

\[
y_d = h_d \omega x_r + z_d,
\]

\[
y_e = h_e \omega x_r + z_e.
\]

In both AF and DF strategies, we assume that the relay is subject to a peak power constraint, i.e. \( E[|x_r|^2] \leq P_r \) in AF and \( E[|\omega x_r|^2] \leq P_r \) in DF. Thus, the scaling factor at the relay should satisfy the following constraints,

\[
|\omega|^2 \leq \frac{P_r}{1 + |h_r|^2 P_s} \quad \text{for AF,}
\]

\[
|\omega|^2 \leq P_r \quad \text{for DF,}
\]

where \( E[|x_r|^2] = 1 \) is assumed in DF strategy.

In the sequel, we are going to compute the secrecy capacity of this network.

\[\text{2 For notational convenience, we ignore the index of symbols in the rest of paper.}\]
3 Secrecy Capacity of Channel

This section aims to address the secrecy capacity of cooperative wiretap channel when the relay makes use of AF and DF strategies which are addressed in Sects. 3.1 and 3.2, respectively.

3.1 Amplify and Forward

The Amplify and Forward cooperative wiretap single-input single-output channel can be thought as a broadcast channel. Hence, the secrecy capacity of this channel can be computed as [5],

\[
C_s(P_r) = \max_{E[u^2_s] \leq P_r} \frac{1}{2} \left[ I(u_s; y_d) - I(u_s; y_e) \right] +
\]

where \( u_s \) is an auxiliary random variable, \( p(u_s, x_s) \) is the joint Probability Density Function (PDF) of \( u_s \) and \( x_s \); and \( \rho \) is the set of all possible PDFs. Also, the factor \( \frac{1}{2} \) is due to the use of half-duplex nodes and the transmission is done during two time slots.

Evaluating the secrecy capacity of underlying channel using (10) may be computationally infeasible. This motivated us to propose the following theorem which aims at addressing this issue using an indirect approach.

**Theorem 1** The secrecy capacity of cooperative amplify and forward wiretap channel is given by,

\[
C_s(P_r) = \begin{cases} 
0 & \alpha \leq \beta \\
\frac{1}{2} \log_2 \left( \frac{\alpha \beta P_r^2 + (\alpha + \beta)P_r + \mu}{\alpha \beta P_r^2 + (\alpha + \beta)P_r + \mu} \right) & \alpha > \beta \text{ and } P_r \leq \sqrt{\frac{\mu}{\alpha \beta}} \\
\frac{1}{2} \log_2 \left( \frac{2 \sqrt{\alpha \beta \mu} + \alpha + \beta}{2 \sqrt{\alpha \beta \mu} + \alpha + \beta} \right) & \alpha > \beta \text{ and } P_r > \sqrt{\frac{\mu}{\alpha \beta}} 
\end{cases}
\]

(11)

where \( \alpha = |h_d|^2 \), \( \beta = |h_e|^2 \) and \( \mu = 1 + P_s |h_r|^2 \).

**Proof** We prove the above theorem in two steps. First, using (10), it is shown that (11) is achievable for Gaussian distribution. Next, for the converse part, we propose an upper bound and show that any transmission rate greater than (11) is not achievable.

3.1.1 The Achievability of (11)

For Gaussian input and the assumption \( x_s = u_s \), the achievable secrecy rate can be computed as,

\[
R_s(P_r) = \max_{E[|u^2_s|] \leq P_r} \frac{1}{2} \left[ I(x_s; y_d) - I(x_s; y_e) \right] +
\]

(12)

Thus, referring to (4) and noting \( x_s \sim \mathcal{N}(0, 1) \), it follows,
\[ I(x_s; y_d) = \log_2 \left( 1 + \frac{P_s |h_d|^2 |\omega|^2 |h_r|^2}{1 + |h_d|^2 |\omega|^2} \right) \]

(13)

Similarly, noting (5), one can arrive at the following,

\[ I(x_s; y_e) = \log_2 \left( 1 + \beta \mu |\omega|^2 \right) \]

(14)

Substituting (13) and (14) into (12), it turns out that the achievable secrecy rate becomes,

\[ R_s(P_r) = \max_{|\omega|^2 \leq \frac{P_r}{\mu}} \left\{ \frac{1}{2} \log_2 \left( \frac{\alpha \beta \mu |\omega|^4 + (\alpha \mu + \beta |\omega|^2 + 1)}{\alpha \beta \mu |\omega|^4 + (\alpha + \beta \mu |\omega|^2 + 1)} \right) \right\}^+ \]

(15)

To address the optimal solution of (15), the following maximization problem should be tackled,

\[ \max_{|\omega|^2 \leq \frac{P_r}{\mu}} \frac{\alpha \beta \mu |\omega|^4 + (\alpha \mu + \beta |\omega|^2 + 1)}{\alpha \beta \mu |\omega|^4 + (\alpha + \beta \mu |\omega|^2 + 1)} \]

(16)

which can be reformulated as,

\[ \max_x f(x) = \frac{\alpha \beta \mu x^2 + (\alpha \mu + \beta)x + 1}{\alpha \beta \mu x^2 + (\alpha + \beta \mu)x + 1} \]

subject to \(0 \leq x \leq X\)

(17)

where \(x = |\omega|^2\) and \(X = \frac{P_r}{\mu}\). Although the objective function of (17) is the ratio of two convex quadratic functions, this function is not convex in general [33, 34]; hence, the method of Lagrange Multipliers does not give the optimal solution. To find the optimal value of \(x\), i.e., \(\hat{x}\), we consider two possible cases of \(\alpha \leq \beta\) and \(\alpha > \beta\) as the following.

**Case \(\alpha \leq \beta\):** In this case, we show that the optimal solution of (17) is \(\hat{x} = 0\). To this end, noting the definition of \(\mu\), indicating \(\mu \geq 1\), it follows,

\[ \alpha(\mu - 1) \leq \beta(\mu - 1), \]

(18)

or equivalently,

\[ \alpha \mu + \beta \leq \alpha + \beta \mu. \]

(19)

Thus, for \(0 < x \leq X\) and noting \(f(x) = \frac{\alpha \beta \mu x^2 + (\alpha \mu + \beta)x + 1}{\alpha \beta \mu x^2 + (\alpha + \beta \mu)x + 1}\), it turns out that the denominator of \(f(x)\) is greater than the nominator. Therefore, we have \(f(x) < 1\). On the other hand, since \(f(0) = 1\), the optimal value of \(x\) becomes,

\[ \hat{x} = 0. \]

(20)

**Case \(\alpha > \beta\):** In this case, we show that the optimal value of (17) can be computed as,
\[
\hat{x} = \begin{cases} 
\frac{P_r}{\mu} & P_r \leq \sqrt{\frac{\mu}{\alpha \beta}}, \\
\frac{1}{\sqrt{\alpha \beta \mu}} & P_r > \sqrt{\frac{\mu}{\alpha \beta}}, 
\end{cases}
\] (21)

where \(\hat{x}\) is derived through using the following theorem.

Theorem 2 We consider the following optimization problem,

\[
\max_{x \in \mathbb{R}^n} f(x) = \frac{x^T Q x + q^T x + q^0}{x^T P x + p^T x + p^0},
\] (22)

where \(P\) and \(Q\) are \(n \times n\) symmetric positive semi-definite matrices. To address the optimal solution, we define the following function,

\[
F(x, \lambda) = x^T Q x + q^T x + q^0 - \lambda (x^T P x + p^T x + p^0), \quad \lambda > 0.
\] (23)

Also, we define the functions,

\[
x(\lambda) = \arg \max_{x \in \mathbb{R}^n} F(x, \lambda) \quad \forall \lambda > 0,
\] (24)

and

\[
\pi(\lambda) = \max_{x \in \mathbb{R}^n} F(x, \lambda) = F(x(\lambda), \lambda).
\] (25)

If there exists \(\hat{\lambda} > 0\) for which \(\pi(\hat{\lambda}) = 0\), then \(\hat{x} \equiv x(\hat{\lambda})\) is the optimal solution of (22).

Proof see [33].

According to the Theorem 2 and referring to (17), we define \(F(x, \lambda)\) as follows,

\[
F(x, \lambda) = x^T Q x + q^T x + q^0 - \lambda (x^T P x + p^T x + p^0), \quad \lambda > 0.
\] (23)

Also, we define the functions,

\[
x(\lambda) = \arg \max_{x \in \mathbb{R}^n} F(x, \lambda) \quad \forall \lambda > 0,
\] (24)

and

\[
\pi(\lambda) = \max_{x \in \mathbb{R}^n} F(x, \lambda) = F(x(\lambda), \lambda).
\] (25)

If there exists \(\hat{\lambda} > 0\) for which \(\pi(\hat{\lambda}) = 0\), then \(\hat{x} \equiv x(\hat{\lambda})\) is the optimal solution of (22).

Proof see Appendix 2.

Based on claim 1, it is sufficient to merely investigate \(F(x, \lambda)\) for \(1 \leq \lambda < \frac{2 \mu + \beta}{\alpha + \beta \mu}\).

It should be noted that referring to (26), since \(1 - \lambda \leq 0\), it turns out that \(F(x, \lambda)\) is a concave function of \(x\) and has two positive roots.\(^3\) As a result, depending on the value of \(\lambda\), \(F(x, \lambda)\) can be represented as one of the curves illustrated in Fig. 2. Assuming \(\hat{x}\) maximizes \(F(x, \lambda)\), if \(X\) is equal or less than \(\hat{x}\), then \(x(\lambda) = X\) gives the maximum value of \(F(x, \lambda)\) in the interval \(x \in [0, X]\) (see Fig. 2a); otherwise, \(x(\lambda)\) will be equal to \(\hat{x}\) (see Fig. 2b).

Thus, we have,

\[
x(\lambda) = \begin{cases} 
X & X \leq \hat{x} \\
\hat{x} & X > \hat{x},
\end{cases}
\] (27)

where \(\hat{x}\) is computed by taking derivation of \(F(x, \lambda)\) with respect to \(x\) and equating to zero as follows,

\(^3\) The number of positive roots of a polynomial with real coefficients ordered in terms of ascending power of the variable is either equal to the number of variations in sign of consecutive non-zero coefficients or less than this by a multiple of 2 [35].
As a result, using (28) and claim 1, (27) can be expressed as,

\[ x(\lambda) = \begin{cases} 
X & 1 \leq \lambda \leq \frac{2x\beta \mu X + x\mu + \beta}{2x\beta \mu X + x + \beta \mu} \\
\frac{\lambda(x + \beta \mu) - (x\mu + \beta)}{2x\beta \mu (1 - \lambda)} & \frac{2x\beta \mu X + x\mu + \beta}{2x\beta \mu X + x + \beta \mu} < \lambda < \frac{x\mu + \beta}{x + \beta \mu} 
\end{cases} \]  

(29)

Also, using (24) and (25), \( \pi(\lambda) \) can be obtained by,

\[ \pi(\lambda) = F(x(\lambda), \lambda) = \begin{cases} 
\pi_1(\lambda) & 1 \leq \lambda \leq \frac{2x\beta \mu X + x\mu + \beta}{2x\beta \mu X + x + \beta \mu} \\
\pi_2(\lambda) & \frac{2x\beta \mu X + x\mu + \beta}{2x\beta \mu X + x + \beta \mu} < \lambda < \frac{x\mu + \beta}{x + \beta \mu} 
\end{cases} \]  

(30)

where

\[ \pi_1(\lambda) = (x\beta \mu X^2 - (x + \beta \mu)X - 1)\lambda 
+ x\beta \mu X^2 + (x\mu + \beta)X + 1, \]  

(31)

and

\[ \pi_2(\lambda) = \frac{(\lambda(x + \beta \mu) - (x\mu + \beta))^2}{4x\beta \mu (\lambda - 1)} - \lambda + 1. \]  

(32)

Claim 2 \( \dot{\lambda} \) can be written as,

\[ \dot{\lambda} = \begin{cases} 
\dot{\lambda}_1 = \frac{x\beta \mu X^2 + (x\mu + \beta)X + 1}{x\beta \mu X^2 + (x + \beta \mu)X + 1} & X \leq \frac{1}{\sqrt{x\beta \mu}} \\
\dot{\lambda}_2 = \frac{2(x + \beta \mu)(x\mu + \beta) - 8x\beta \mu - \sqrt{\Delta}}{2(x - \beta \mu)^2} & X > \frac{1}{\sqrt{x\beta \mu}}, 
\end{cases} \]  

(33)

where

\[ \Delta = x^2 + \beta^2 \mu^2 - 4x\beta \mu X + 8x\beta \mu. \]
\[ \Delta = (8\alpha\beta\mu - 2(\alpha + \beta\mu)(\alpha\mu + \beta))^2 - 4(\alpha - \beta\mu)^2(\alpha\mu - \beta)^2. \] (34)

**Proof** See Appendix 3.

Finally, substituting (33) into (29) yields,

\[ \hat{x} = \begin{cases} 
X & X \leq \frac{1}{\sqrt{2}\beta\mu} \\
\frac{2\alpha\mu + \beta}{2\beta\mu(1 - \lambda)} & X > \frac{1}{\sqrt{2}\beta\mu}.
\end{cases} \] (35)

Moreover, comparing (35) with (27), one can arrive at the following,

\[ \hat{x} = \begin{cases} 
X & X \leq \frac{1}{\sqrt{2}\beta\mu} \\
\frac{1}{\sqrt{2}\beta\mu} & X > \frac{1}{\sqrt{2}\beta\mu},
\end{cases} \] (36)

or equivalently, we have,

\[ \hat{x} = \begin{cases} 
\frac{P_r}{\mu} & P_r \leq \sqrt{\frac{\mu}{2\beta}} \\
\frac{1}{\sqrt{2}\beta\mu} & P_r > \sqrt{\frac{\mu}{2\beta}}.
\end{cases} \] (37)

As a result, noting \( \hat{x} = |\omega_{opt}|^2 \), it turns out that if \( P_r > \sqrt{\frac{\mu}{2\beta}} \), the relay doesn’t use all of its available power. This is due to the fact that the relay sends a noisy version of \( x_s \) and additional relay’s transmit power may enhance the additive noise, thereby decreasing the secrecy rate.

Finally, using (20) and (37) and after some mathematics, one can readily observe that (11) is the achievable secrecy rate of AF relying for Gaussian input. In what follows, we are going to show that any rate greater than (11) is not achievable (the converse part), thereby (11) is actually the secrecy capacity of AF relaying.

### 3.1.2 The Converse Approach

For the converse part, we show that any rate greater than \( R_s(P_r) \) defined in (15) is not achievable. To do this, we investigate the capacity of genie-aided channel as an upper bound on the secrecy capacity of underlying channel. Then, we show that this upper bound is tight for Gaussian distribution. The following lemma establishes the capacity of corresponding genie-aided channel,

**Lemma 1** ([9]) An upper bound on the secrecy capacity of cooperative wiretap channel is,

\[ C_s(P_r) \leq \max_{P_{\omega}^{(\omega)}} \frac{1}{2} I(x_s; y_d|\omega). \] (38)

\[ \omega \in \mathbb{C}^\mathbb{C}, |\omega|^2 \leq \frac{C_s}{P}. \]
In what follows, we show that for AF relaying, the Gaussian distribution maximizes $I(x_s; y_d | y_e)$. To this end, we have,

$$I(x_s; y_d | y_e) = h(y_d | y_e) - h(y_d | x_s, y_e).$$

(39)

The second term in the right hand side of (39) can be expressed, using (4) and (5), as,

$$h(y_d | x_s, y_e) = h(h_d\omega z_r + z_d | x_s, y_e)$$

$$= h(h_d\omega z_r + z_d | h_e\omega z_r + z_e).$$

(40)

One can readily observe that (40) does not depend on the distribution of $x_s$, thus, $p(x_s)$ should be chosen such that the first term in the right hand side of (39), i.e., $h(y_d | y_e)$, is maximized. On the other hand, we have,

$$h(y_d | y_e) = a \frac{h(y_d - z_{\text{LMMSE}}y_e | y_e)}{C_0}$$

$$\leq b \frac{h(y_d - z_{\text{LMMSE}}y_e)}{C_0}$$

$$\leq \log_2(\pi e \lambda_{\text{LMMSE}}),$$

(41)

where (a) comes from the fact that adding a known value to a random variable does not change the entropy and (b) holds since we always have $h(y | x) \leq h(x)$. $\lambda_{\text{LMMSE}}$ is the corresponding coefficient of Linear Minimum Mean Square Error (LMMSE) estimation of $y_d$ by $y_e$ and $z_{\text{LMMSE}}$ is the error variance conditional on knowing $y_e$, i.e., $E[|y_d - z_{\text{LMMSE}}y_e|^2 | y_e]$. The last inequality in (41) is due to the fact that the maximum differential entropy is achieved by Gaussian distribution.

In the case that $y_d$ and $y_e$ are jointly Gaussian, the estimation error, i.e., $y_d - z_{\text{LMMSE}}y_e$ is independent of every linear function of $y_e$ [36], thus for Gaussian input we have $h(y_d - z_{\text{LMMSE}}y_e | y_e) = h(y_d - z_{\text{LMMSE}}y_e)$. Noting, the maximum differential entropy is achieved by Gaussian distribution, hence, the inequalities in (41) are held with equality for Gaussian input $x_s$ and $I(x_s; y_d | y_e)$ is maximized. So, we can rewrite (38) as,

$$C_s(P_r) \leq \max_{|\omega|^2 \leq \frac{P_r}{2}} \frac{1}{2} I(x_s; y_d | y_e)$$

$$= \max_{|\omega|^2 \leq \frac{P_r}{2}} \frac{1}{2} \log_2(\pi e \lambda_{\text{LMMSE}})$$

$$- \frac{1}{2} h(h_d\omega z_r + z_d | h_e\omega z_r + z_e),$$

(42)

where $\lambda_{\text{LMMSE}}$ can be computed as [36],

$$\lambda_{\text{LMMSE}} = \text{Var}(y_d - z_{\text{LMMSE}}y_e | y_e)$$

$$= \text{Var}(y_d) - \frac{|\text{Cov}(y_d, y_e)|^2}{\text{Var}(y_e)}.$$
\[
\begin{bmatrix}
z_d \\
z_e
\end{bmatrix} \sim \mathcal{CN}(0, K_{\phi}), \quad K_{\phi} = \begin{bmatrix} 1 & \phi^* \\ \phi & 1 \end{bmatrix}.
\] (44)

Using (44) and after some mathematics, it turns out that (43) can be computed as,
\[
\lambda_{\text{LMMSE}} = \frac{1 + (\alpha + \beta)x - |\phi|^2 - 2 \Re\{ \mu x h_d h_e^* \phi \}}{1 + \beta x}.
\] (45)

The proof is provided in Appendix 4.

Moreover, the second term in (42) can be computed as,
\[
h(h_d \omega z_r + z_d | h_e \omega z_r + z_e) \\
= h(h_d \omega z_r + z_d, h_e \omega z_r + z_e) - h(h_e \omega z_r + z_e) \\
= \pi e \left( 1 + (\alpha + \beta)x - |\phi|^2 - 2 \Re\{ x h_d h_e^* \phi \} \right) \\
= \log_2 \frac{1 + (\alpha + \beta)x - |\phi|^2 - 2 \Re\{ x h_d h_e^* \phi \}}{1 + \beta x}.
\] (46)

The proof is given in Appendix 5.

Plugging (45) and (46) into (42) and after some manipulations, it follows,
\[
C_s(P_r) \leq \max_{0 \leq x \leq \lambda} \frac{1}{2} \log_2 \left\{ \frac{1 + \beta x}{1 + \beta \mu x} \frac{1 + (\alpha + \beta)x - |\phi|^2 - 2 \Re\{ \mu x h_d h_e^* \phi \}}{1 + \beta \mu x} \right\}.
\] (47)

Note that the covariance matrix \( K_{\phi} \) should be positive semi-definite, i.e., \( K_{\phi} \succeq 0 \). This results in,
\[
|\phi| \leq 1.
\] (48)

Thus, (47) yields an upper bound just for values of \( \phi \) which satisfy (48).

Proceeding, we again consider two cases \( \alpha \leq \beta \) and \( \alpha > \beta \). For each case, an upper bound of secrecy capacity is computed with a special value of \( \phi \).

**Case \( \alpha \leq \beta \):** In this case, we choose,
\[
\phi = \frac{h_d^*}{h_e^*}.
\] (49)

Noting,
\[
|\phi|^2 = \frac{\alpha}{\beta} \leq 1,
\] (50)

thus substituting (49) into (47) gives the following upper bound,
\[
C_s(P_r) \leq \max_{0 \leq x \leq \lambda} \frac{1}{2} \log_2 \left\{ \frac{1 + \beta x}{1 + \beta \mu x} \frac{1 + (\alpha + \beta)x - |\phi|^2 - 2 \Re\{ \mu x h_d h_e^* \phi \}}{1 + \beta \mu x} \right\} \\
= \max_{0 \leq x \leq \lambda} \frac{1}{2} \log_2 \frac{\beta \mu (\beta - \alpha)x^2 + (\beta - \alpha)(\mu + 1)x + 1 - \frac{\alpha}{\beta}}{\beta \mu (\beta - \alpha)x^2 + (\beta - \alpha)(\mu + 1)x + 1 - \frac{\alpha}{\beta}} \\
= 0.
\] (51)

This results in,
\[ C_s(P_r) = 0. \]  

**Case \( \alpha > \beta \):** In this case, \( \phi \) is set to,
\[ \phi = \frac{h_e}{h_d}, \]  
where we should note the following,
\[ |\phi|^2 = \frac{\beta}{\alpha} < 1. \]  

Substituting (54) into (47), we arrive at the following,
\[ C_s(P_r) \leq \max_{0 \leq x \leq x^2} \frac{1}{2} \log_2 \frac{(1 + \beta x) \left( 1 - \frac{\beta}{2} + (\alpha - \beta) x \right)}{(1 + \beta \mu x) \left( 1 - \frac{\beta}{2} + (\alpha - \beta) x \right)} \]  
\[ = \max_{0 \leq x \leq x^2} \frac{1}{2} \log_2 \frac{1 + \beta x}{1 + \beta \mu x} \times \frac{1 + \alpha \mu x}{1 + \alpha x} \]  
\[ = \max_{0 \leq x \leq x^2} \frac{1}{2} \log_2 \frac{x \beta \mu x^2 + (\alpha \mu + \beta) x + 1}{x \beta \mu x^2 + (\alpha + \beta \mu) x + 1}, \]  
where (55) is proved in Appendix 6.

Referring to (15), it turns out that (56) is actually an achievable rate for the underlying channel. Thus, we have,
\[ C_s(P_r) = \max_{0 \leq x \leq x^2} \frac{1}{2} \log_2 \frac{x \beta \mu x^2 + (\alpha \mu + \beta) x + 1}{x \beta \mu x^2 + (\alpha + \beta \mu) x + 1}, \]  

Considering the obtained results in (52) and (57), Theorem 1 is proved.  

### 3.2 Decode and Forward

For DF relaying, using max-flow min-cut theorem, it turns out that the secrecy capacity can be computed as,
\[ C_s(P_r) = \frac{1}{2} \min \{ C_{S-R}, C_{SR-D} \}, \]  
where as mentioned earlier \( C_{S-R} \) is the capacity of source-to-relay and \( C_{SR-D} \) is the secrecy capacity of the second hop operating at full power which is given by [9],
\[ C_{SR-D} = \max_{\mu^2 \leq \mu R, \mu \in [0, \infty]} \left\{ \log_2 \left( 1 + \frac{1 + \alpha \mu R}{1 + \beta \mu R} \right) \right\}. \]  

If \( C_{SR-D} \leq C_{S-R} \), then we have \( C_s(P_r) = \frac{1}{2} C_{SR-D} \), otherwise, the minimum value of \( C_{SR-D} \) and \( C_{S-R} \) is equal to \( C_{S-R} \) and in this case, the relay does not need to use all of its available power, i.e., \( P_r \). In other words, when the secrecy capacity associated with the second hop is greater than the available information at the relay, the relay can simply adjust its power so
that not to waste any more power. In this case, we have $C_{S,R} = C_{S-R}$, where referring to (9), one can arrive at the following,\(^4\)

$$C_{S-R} = \log_2\left(\frac{1 + \alpha |\omega|^2}{1 + \beta |\omega|^2}\right). \quad (60)$$

By noting (6) and the definition of $\mu$, we get,

$$|\omega|^2 = \frac{\mu - 1}{\alpha - \beta \mu}. \quad (61)$$

As a result, the secrecy capacity of DF relaying is given by

$$C_S(P_r) = \begin{cases} 
0 & \alpha \leq \beta \\
\frac{1}{2} \log_2 \left(\frac{1 + \alpha P_r}{1 + \beta P_r}\right) & \alpha > \beta \text{ and } \frac{1 + \alpha P_r}{1 + \beta P_r} \leq \mu \\
\frac{1}{2} \log_2 \mu & \alpha > \beta \text{ and } \frac{1 + \alpha P_r}{1 + \beta P_r} > \mu,
\end{cases} \quad (62)$$

and the optimum relay’s power can be written as,

$$|\omega_{opt}|^2 = \begin{cases} 
0 & \alpha \leq \beta \\
P_r & \alpha > \beta \text{ and } \frac{1 + \alpha P_r}{1 + \beta P_r} \leq \mu \\
\frac{\mu - 1}{\alpha - \beta \mu} & \alpha > \beta \text{ and } \frac{1 + \alpha P_r}{1 + \beta P_r} > \mu.
\end{cases} \quad (63)$$

### 4 Simulation Results

This section aims to provide some numerical results to illustrate the secrecy capacity versus the power budget at the relay for cooperative wire-tap relay channel employing the AF and DF strategies. Throughout the simulations, the channel coefficients of source-relay ($h_r$), relay-destination ($h_d$) and relay-eavesdropper ($h_e$) are assumed to be Rayleigh distributed. Also, the received noises at the relay, the destination and the eavesdropper are assumed to be circularly symmetric complex Gaussian random variables with zero mean and unit variance. Moreover, the results are derived for different values of relay-destination channel strengths $\sigma_{h_d}^2 = 1, 2, 4$ and 8, while it is assumed $\sigma_{h_r}^2 = \sigma_{h_e}^2 = 1$ throughout Figs. 3, 4 and 5. Also, source transmit power is set to $P_s = 10$ dBW\(^5\) in these figures. Finally, Figs. 6 and 7 consider the scenarios in which $\sigma_{h_d}^2 = \sigma_{h_r}^2 = \sigma_{h_e}^2 = 1$ and the effect of varying $P_s$ and $P_r$ are studied.

Figures 3 and 4, respectively, show the secrecy capacity of AF and DF cooperative wiretap channels versus power budget for various relay-destination channel strengths, implying the secrecy capacity of DF is greater than that of AF strategy. This is due to the

\(^4\) It is worth mentioning that $\frac{1 + \alpha |\omega|^2}{1 + \beta |\omega|^2}$ is an increasing function with respect to $|\omega|$ for $\alpha > \beta$, thus decreasing $|\omega|$ reduces the secrecy rate of the second hop.

\(^5\) Please note that here it is assumed the transmit SNR at the source is 10 dB. Thus, noting the received noise at this node is of unit power, thus the transmit power at the source becomes 10 dBW.
fact that the received signal at the destination is a degraded version of the relay’s, thus the DF strategy is optimal. Moreover, it is demonstrated that as the relay-destination channel strength is increased, the secrecy capacity is consistently increased. Moreover, the secrecy capacity approaches to a constant value as the relay’s power tends to infinity. This is due to the fact that the capacity of the first hop acts as bottleneck. Also, Fig. 5 depicts the consumed relay’s power versus available power for AF and DF strategies, showing the AF
strategy saves more power as compared to DF strategy when the power budget at the relay increases.

In Fig. 6, the secrecy capacity of AF and DF relaying versus source transmit power, i.e., $P_s$, is represented for two power budget values. According to this figure, AF and DF relaying schemes give the same secrecy capacity for large values of $P_s$. This is due to the fact that the received signal at the relay has high SNR value, hence, the AF strategy performs like as DF strategy and the capacity loss is negligible. Moreover, the impact of
increasing the power budget at the relay on the secrecy capacity is shown for two cases of $P_r = 10, 20$ dBW.

Figure 7 represents the relay’s power consumption versus source transmit power. In this case, the power consumption at the relay is the same for both of AF and DF relaying at high SNR regime. Also, by decreasing $P_s$, the power consumption at the AF relay tends to a constant value, however, this is not the case happening for DF relay. As is mentioned in Sect. 3.2, when the capacity of source-to-relay is less than the secrecy capacity of relay-to-destination, the DF relay adjust its power so that not to waste an excess power. On the other hand, for AF relaying, since the transmit power at the AF relay is the product of incoming signal power and the square of scaling factor at the relay, i.e., $\hat{x} = |\omega_{sp}|^2$, hence, when the transmit power approaches zero, the incoming power at the relay tends to the noise power which is constant. Also, according to equation (37), the scaling factor tends to a non-zero constant value when $P_s \to 0$. As a result, the relay’s power does not fall below a certain threshold and this method is not power efficient in weak SNR regime.

5 Conclusion

This paper aims at exploring the secrecy capacity of AF relay-assisted wire-tap channel. In this regard, an achievable secrecy rate for Gaussian input assumption is derived and then it is shown that any rate greater than this is not achievable. To this end, for converse approach, the capacity of a genie-aided channel as an upper bound for the secrecy capacity is studied and it is proved that under Gaussian assumption the achievable secrecy rate meets this upper bound, thus the obtained result is actually the secrecy capacity.

Throughout the proof of the achievability, an optimization problem is being used to determine the optimum power allocation strategy at the relay. To get further steps towards finding the optimum power allocation strategy, the problem is reformulated as a fractional
quadratic problem where using recent advances in this area, a closed-from solution is derived. Finally, the secrecy capacity of DF strategy which is optimal in relay-assisted degraded BCC is being served as the benchmark to get an insight regarding the capacity loss due to using the AF strategy in untrusted relays.

Appendix 1

Definition (Degraded Broadcast Channel): In cases in which the transmitted information $x \in \mathcal{X}$ is received at two destination points with joint probability density function $p_{Y_1,Y_2|X}(y_1,y_2|x) = p_{Y_2|Y_1,X}(y_2|y_1,x) \cdot p_{Y_1}(y_1|x)$, where $y_1 \in \mathcal{Y}_1$ and $y_2 \in \mathcal{Y}_2$ are received signals at destinations 1 and 2, the channel from source-to-destination 2 is said to be a degraded version of source-to-destination 1 channel when $x \rightarrow y_1 \rightarrow y_2$ forms a Markov chain, meaning $p_{Y_2|Y_1,X}(y_2|y_1,x) = p_{Y_2|Y_1}(y_2|y_1)$.

Appendix 2: Proof of Claim 1

Note that $\hat{\lambda}$ should satisfy the equality $\pi(\hat{\lambda}) = 0$, where we have $f(\hat{x}) = \hat{\lambda}$. Thus, the value of $\hat{\lambda}$ resides between lower bound and upper bound of $f(x)$. In the case of $\alpha > \beta$, we have,

$$\alpha \mu + \beta > \alpha + \beta \mu,$$

and therefore,

$$\alpha \beta \mu^2 + (\alpha \mu + \beta) x + 1 \geq \alpha \beta \mu^2 + (\alpha + \beta \mu) x + 1,$$

where the equality is satisfied for $x = 0$. Thus, we have $f(x) \geq 1$.

On the other hand, we know that for positive values $a$, $b$ and $c$, when $b < a$, we have,

$$\frac{a + c}{b + c} < \frac{a}{b}.$$  

(66)

Based on this, by choosing $a = \alpha \mu + \beta$, $b = \alpha + \beta \mu$ and $c = \alpha \beta \mu^2 + 1$, we arrive at,

$$f(x) = \frac{\alpha \beta \mu^2 + (\alpha \mu + \beta) x + 1}{\alpha \beta \mu^2 + (\alpha + \beta \mu) x + 1} < \frac{\alpha \mu + \beta}{\alpha + \beta \mu}.$$  

(67)

Therefore, we conclude that,

$$1 \leq \hat{\lambda} < \frac{\alpha \mu + \beta}{\alpha + \beta \mu}.$$  

(68)

Appendix 3: Proof of Claim 2

Using (26) and (29), one can readily verify that $F(x, \lambda)$ and $x(\lambda)$ are continuous functions of $x$ and $\lambda$. Therefore, $\pi(\lambda)$ will also be a continuous function of $\lambda$. Furthermore, $\pi(\lambda)$ is a decreasing convex function of $\lambda$ [33]. Moreover, $\pi(\lambda)$ has positive and negative values,
respectively, at the start and end points of interval \( \lambda \in [1, \frac{\alpha + \beta}{\alpha + \beta \mu}] \), since from (31) we have the following for \( \lambda = 1 \),

\[
\pi(1) = \pi_1(1) = (\alpha \mu + \beta) - (\alpha + \beta \mu) > 0,
\]

and for \( \lambda = \frac{\alpha + \beta}{\alpha + \beta \mu} \) using (32), it follows,

\[
\pi \left( \frac{\alpha \mu + \beta}{\alpha + \beta \mu} \right) = \pi_2 \left( \frac{\alpha \mu + \beta}{\alpha + \beta \mu} \right) = 1 - \frac{\alpha \mu + \beta}{\alpha + \beta \mu} < 0.
\]

Thus, noting \( \pi(\lambda) \) is strictly decreasing function, it has one root in the interval \( \lambda \in [1, \frac{\alpha + \beta}{\alpha + \beta \mu}] \), where this root should either reside in the region in which \( \pi(\lambda) = \pi_1(\lambda) \) or \( \pi(\lambda) = \pi_2(\lambda) \) as respectively illustrated in Fig. 8a or b. To determine which of these conditions is occurred, we should compute \( \pi(\lambda) \) at the point in which these curves meet each other, i.e., at the point \( \lambda^* = \frac{\alpha \mu + \beta}{\alpha + \beta \mu \mu} \) as illustrated in Fig. 8,

\[
\bar{\pi} = \pi(\lambda^*) = \pi_1 \left( \frac{2\alpha \beta \mu X + \alpha \mu + \beta}{2\alpha \beta \mu X + \alpha + \beta \mu} \right) = \frac{(\alpha \mu + \beta - (\alpha + \beta \mu))(\alpha \beta \mu X^2 - 1)}{2\alpha \beta \mu X + (\alpha + \beta \mu)}.
\]

This implies that for the case \( X \leq \frac{1}{\sqrt{\beta \mu}} \), \( \bar{\pi} \) has negative value, hence, \( \pi(\lambda) \) can be represented as Fig. 8a. Thus, the root of \( \pi(\lambda) \) can be computed through setting \( \pi_1(\lambda) \) to zero as follows,

\[
\pi_1(\lambda) = (-\alpha \beta \mu X^2 - (\alpha + \beta \mu)X - 1)\lambda
\]

\[
+ \alpha \beta \mu X^2 + (\alpha \mu + \beta)X + 1 = 0,
\]

which gives,

\[
\lambda_1 = \frac{\alpha \beta \mu X^2 + (\alpha \mu + \beta)X + 1}{\alpha \beta \mu X^2 + (\alpha + \beta \mu)X + 1}.
\]

Alternatively, Fig. 8b corresponds to the case that we have \( X > \frac{1}{\sqrt{\beta \mu}} \). Consequently, the root of \( \pi(\lambda) \) is derived by setting \( \pi_2(\lambda) \) to zero as follows,

\[
\lambda_2 = \frac{\alpha \beta \mu X^2 + (\alpha \mu + \beta)X + 1}{\alpha \beta \mu X^2 + (\alpha + \beta \mu)X + 1}.
\]
\[
\pi_2(\lambda) = \frac{(\lambda (x + \beta \mu) - (x \mu + \beta))^2}{4x\beta \mu (\lambda^2 - 1)} - \lambda + 1 = 0, \quad (74)
\]

or equivalently, we have,
\[
(x - \beta \mu)^2 \lambda^2 + (8x\beta \mu - 2(x + \beta \mu)(x \mu + \beta))\lambda + (x \mu - \beta)^2 = 0, \quad (75)
\]

which has the following roots,
\[
\lambda_2 = \frac{2(x + \beta \mu)(x \mu + \beta) - 8x\beta \mu - \sqrt{\Delta}}{2(x - \beta \mu)^2}, \quad (76)
\]
\[
\lambda_3 = \frac{2(x + \beta \mu)(x \mu + \beta) - 8x\beta \mu + \sqrt{\Delta}}{2(x - \beta \mu)^2}. \quad (77)
\]

It is clear that \( \lambda_2 \) is the desirable root of \( \pi_2(\lambda) \). Thus, we have,
\[
\hat{\lambda} = \begin{cases} 
\frac{z\beta \mu X^2 + (x \mu + \beta)X + 1}{z\beta \mu X^2 + (x + \beta \mu)X + 1} & X \leq \frac{1}{\sqrt{z\beta \mu}} \\
\frac{2(x + \beta \mu)(x \mu + \beta) - 8x\beta \mu - \sqrt{\Delta}}{2(x - \beta \mu)^2} & X > \frac{1}{\sqrt{z\beta \mu}}.
\end{cases} \quad (78)
\]

**Appendix 4**

According to (4), we have,
\[
\text{Var}(y_d) = P_s|\omega|^2|h_d|^2|h_r|^2 + |\omega|^2|\omega|^2 + 1 \\
= 1 + |\omega|^2|\omega|^2 (1 + P_s|h_r|^2) \\
= 1 + x\mu x. \quad (79)
\]

Similarly, using (5), it follows,
\[
\text{Var}(y_e) = 1 + \beta \mu x. \quad (80)
\]

Also, the cross covariance of \( y_d \) and \( y_e \) can be computed as,
\[
\text{Cov}(y_d, y_e) = E \left[ (\sqrt{P_s}h_d|\omega| h_d h_r|\omega| |z_r| + z_d) \right] \\
\times (\sqrt{P_s}h_e|\omega| h_e h_r|\omega| |z_r| + z_e)^* \] \quad (81)
\[
= (1 + P_s|h_r|^2)|\omega|^2 h_d h_e^* + \phi^* \\
= \mu x h_d h_e^* + \phi^*.
\]

Using (43), (79), (80) and (81) can be written as,
\[
\lambda_{\text{LMMSE}} = 1 + \alpha \mu x - \frac{\alpha \beta x^2 + |\phi|^2 + 2\Re\{\mu x h_d h_e^* \phi\}}{1 + \beta \mu x} \\
= \frac{1 + (\alpha + \beta) \mu x - |\phi|^2 - 2\Re\{\mu x h_d h_e^* \phi\}}{1 + \beta \mu x}. \tag{82}
\]

**Appendix 5**

We begin with the following definition,

\[
K_x \triangleq \text{Cov}\left(\begin{bmatrix} h_d \omega z + z_d \\ h_e \omega z + z_e \end{bmatrix}\right)
= \begin{bmatrix} |\omega|^2 |h_d|^2 + 1 & |\omega|^2 h_d h_e^* + \phi^* \\ |\omega|^2 h_d h_e + \phi & |\omega|^2 |h_e|^2 + 1 \end{bmatrix}. \tag{83}
\]

We know that the following holds,

\[
h(h_d \omega z + z_d | h_e \omega z + z_e) = \log_2 \frac{\pi e |K_x|}{\text{Var}(h_e \omega z + z_e)}, \tag{84}
\]

where

\[
|K_x| = 1 + (\alpha + \beta) x - |\phi|^2 - 2\Re\{\mu x h_d h_e^* \phi\}, \tag{85}
\]

and

\[
\text{Var}(h_e \omega z + z_e) = \beta x + 1. \tag{86}
\]

So, we arrive at (46).

**Appendix 6**

To prove (55), we use the following equality,

\[
\frac{1 + \alpha x}{1 + \alpha \mu x} \times \frac{1 - \frac{\beta}{x} + (\alpha - \beta) \mu x}{1 - \frac{\beta}{x} + (\alpha - \beta) x} = \frac{\alpha \mu (\alpha - \beta) x^2 + (\alpha - \beta)(1 + \mu) x + 1 - \frac{\beta}{x}}{\alpha \mu (\alpha - \beta) x^2 + (\alpha - \beta)(1 + \mu) x + 1 - \frac{\beta}{x}} = 1. \tag{87}
\]

Thus, we have,

\[
\frac{1 + \alpha \mu x}{1 + \alpha x} = \frac{1 - \frac{\beta}{x} + (\alpha - \beta) \mu x}{1 - \frac{\beta}{x} + (\alpha - \beta) x}, \tag{88}
\]

which yields the equality of (55).
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