A YAMABE-TYPE PROBLEM ON SMOOTH METRIC MEASURE SPACES

JEFFREY S. CASE

Abstract. We describe and partially solve a natural Yamabe-type problem on smooth metric measure spaces which interpolates between the Yamabe problem and the problem of finding minimizers for Perelman’s $\nu$-entropy. This problem reduces in all dimensions on Euclidean space to the characterization of the minimizers of the family of Gagliardo–Nirenberg–Sobolev inequalities studied by Del Pino and Dolbeault. We show that minimizers always exist on a compact manifold provided the so-called weighted Yamabe constant is strictly less than its value on Euclidean space. We also show that strict inequality holds for a large class of smooth metric measure spaces, but we will also give an example which shows that minimizers of the weighted Yamabe constant do not always exist.

1. Introduction

The Yamabe constant and Perelman’s $\nu$-entropy are two important geometric invariants in Riemannian geometry which are closely related in a number of ways. For example, both constants are intimately related to sharp Sobolev-type inequalities on Euclidean space, with the Yamabe constant recovering the best constant for the Sobolev inequality and the $\nu$-entropy recovering the best constant for the logarithmic Sobolev inequality. In the curved setting, these constants are defined as the infima of certain Sobolev-type ratios which are modified by scalar curvature terms, and it turns out that one can show that the infima are achieved by positive smooth functions through a two-step process. First, one shows that minimizing sequences cannot concentrate provided the Yamabe constant (resp. $\nu$-entropy) is strictly less than the best constant for the Sobolev inequality (resp. logarithmic Sobolev inequality) on Euclidean space. Second one shows that strict inequality always holds on a compact manifold, except in the case of the Yamabe constant on the standard conformal sphere.

It turns out that there is a natural one-parameter family of geometric invariants which interpolate between the Yamabe constant and the $\nu$-entropy in a natural way. These invariants, which we call weighted Yamabe constants, were introduced by the author [14] as curved analogues of the best constants in the family of Gagliardo–Nirenberg–Sobolev inequalities whose minimizers were computed by Del Pino and Dolbeault [18]. The main purpose of this article is to study to what extent these invariants interpolate between the Yamabe constant and the $\nu$-entropy, focusing

2000 Mathematics Subject Classification. Primary 53C21; Secondary 53A30, 58E11.
Key words and phrases. smooth metric measure space, Yamabe problem, Gagliardo-Nirenberg inequality.
Partially supported by NSF-DMS Grant No. 1004394.
primarily on issues related to the problem of finding functions which realize the infima defining the invariants.

In order to explain how these geometric invariants interpolate between the Yamabe constant and the $\nu$-entropy, let us first recall the aforementioned result of Del Pino and Dolbeault [15].

**Theorem 1.1** (Del Pino–Dolbeault). Fix $m \in [0, \infty)$. Given any $w \in W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ it holds that

$$
\Lambda_{m,n} \left( \int_{\mathbb{R}^n} w^{2(m+n-2)} \mathrm{dvol} \right)^{\frac{2m}{m+n-2}} \leq \left( \int_{\mathbb{R}^n} |\nabla w|^2 \right) \left( \int_{\mathbb{R}^n} w^{2(m+n-1)} \mathrm{dvol} \right)^{\frac{2m}{m+n-2}},
$$

where the constant $\Lambda_{m,n}$ is given by

$$
\Lambda_{m,n} = \frac{n\pi(m+n-2)^2}{2m+n-2} \left( \frac{2(m+n-1)}{2m+n-2} \right)^{\frac{2m}{2m+n-2}} \left( \frac{\Gamma\left(\frac{2m+n}{2}\right)}{\Gamma(m+n)} \right)^{\frac{2}{n}}.
$$

Moreover, equality holds in (1.1) if and only if there is a constant $\varepsilon > 0$ and a point $x_0 \in \mathbb{R}^n$ such that $w$ is a constant multiple of the function $w_{\varepsilon,x_0} \in C^\infty(\mathbb{R}^n)$ defined by

$$
w_{\varepsilon,x_0}(x) := \left( \frac{2\varepsilon}{\varepsilon^2 + |x-x_0|^2} \right)^{\frac{m+n}{2m+n-2}}.
$$

There are four features of Theorem 1.1 which we wish to emphasize. First, in the case $m = 0$, the inequality (1.1) together with the identification of the extremal functions is precisely the sharp Sobolev inequality [5, 29], while in the case $m = \infty$, the inequality (1.1) together with the identification of the extremal functions is precisely the sharp logarithmic Sobolev inequality [21]. Second, the extremal functions (1.3) are all the same, except for the dependence of the exponent on the parameter $m$. Third, in the limit $\varepsilon \to 0$ the functions $w_{\varepsilon,x_0}$ concentrate at $x_0$. Together, these three observations already provide a fairly convincing argument that the family (1.1) provides a particularly “good” interpolation between the sharp Sobolev inequality and the sharp logarithmic Sobolev inequality. The fourth feature of Theorem 1.1 we wish to emphasize, which requires some more explanation, is that the family (1.1) of Gagliardo–Nirenberg–Sobolev (GNS) inequalities is in a certain sense the only such family with geometrically significant extremal functions.

Given constants $2 \leq p \leq q \leq \frac{2n}{n-2}$, it is a straightforward consequence of the Sobolev inequality and Hölder’s inequality that there is some positive constant $C_{p,q}$ such that the GNS inequality

$$
\|w\|_q \leq C_{p,q} \|\nabla w\|_p \|w\|_1^{-\theta}
$$

holds for all $w \in C^\infty_0(\mathbb{R}^n)$. At present, only in the case $2p = q + 2$, corresponding to the family (1.1), is the best constant $C_{p,q}$ known (there are other cases known in the range $1 \leq p \leq q \leq \frac{2n}{n-2}$, e.g. [11] [18]). This leads one to wonder if there is some geometric reason distinguishing this family. One possible explanation was given by the author in a previous article [14], where it was observed that the formalism of smooth metric measure spaces allows one to define conformal invariants which give a curved analogue of the sharp constant $C_{p,q}$ in (1.4) on an arbitrary smooth metric measure space in a manner akin to the Yamabe constant and Perelman’s
\(\nu\)-entropy. In particular, these constants can be regarded as the infima of the so-called total weighted scalar curvature subject to certain volume constraints, and the family (1.1) has the property that it is the only family of GNS inequalities (1.4) for which the extremal functions on Euclidean space are also critical points of the constrained total weighted scalar curvature functional through variations of the metric or the measure. This generalizes the fact that extremal functions of the Sobolev inequality (resp. logarithmic Sobolev inequality) give rise to conformally flat Einstein metrics on \(\mathbb{R}^n\) (resp. Gaussian measures on \(\mathbb{R}^n\)).

To make the above more precise, let us introduce some terminology. A smooth metric measure space is a four-tuple \((M^n, g, e^{-\phi} \, d\operatorname{vol}, m)\) of a Riemannian manifold \((M^n, g)\), a smooth measure \(e^{-\phi} \, d\operatorname{vol}\) determined by a function \(\phi \in C^\infty(M)\) and the Riemannian volume element of \(g\), and a dimensional parameter \(m \in [0, \infty]\).

The weighted scalar curvature \(R^m_\phi\) of a smooth metric measure space is \(R^m_\phi := R + 2\Delta \phi - \frac{m+1}{m} |\nabla \phi|^2\), where \(R\) and \(\Delta\) are the scalar curvature and Laplacian associated to the metric \(g\) (we use the convention \(-\Delta \geq 0\)). The weighted Yamabe quotient is the functional

\[
Q(w) := \left( \frac{\int |\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R^m_\phi w^2}{\left( \int |w|^{2\frac{m+n-1}{m+n-2}} \right)^{\frac{2m}{m+n-2}}} \right)^{\frac{2m}{n}},
\]

where all integrals are taken with respect to \(e^{-\phi} \, d\operatorname{vol}\); when \(m = \infty\), this is

\[
Q(w) := \frac{\int |\nabla w|^2 + \frac{1}{4} R^\infty \, w^2}{\int w^2} \exp \left( -\frac{2}{n} \int_M \frac{w^2}{\|w\|^2} \log \frac{w^2 e^{-\phi}}{\|w\|^2} \right),
\]

which is just the limit of (1.5m) as \(m \to \infty\). The weighted Yamabe quotient is conformally invariant in the sense that if one defines

\[
\left( M^n, \hat{g}, e^{-\hat{\phi}} \, d\operatorname{vol}, m \right) := \left( M^n, e^\sigma g, e^{\frac{\sigma}{m+n}} \, e^{-\frac{\sigma}{m+n}} e^{-\phi} \, d\operatorname{vol} \right)
\]

for some \(\sigma \in C^\infty(M)\), then \(\hat{Q}(w) = Q(we^{\sigma/2})\); this is how one regards the weighted Yamabe quotient as a volume-normalized total weighted scalar curvature functional. There exist similar conformally invariant functionals on smooth metric measure spaces for which the exponents in second factor of the numerator are chosen to depend on \(2 \leq p \leq \frac{2(m+n)}{m+n-2} = q\) as in (1.4), and it is through these functionals that one characterizes the family (1.1) as the only GNS inequalities for which the extremal functions are critical points through variations of the metric or the measure (14).

Given a compact smooth metric measure space \((M^n, g, e^{-\phi} \, d\operatorname{vol}, m)\), one defines the weighted Yamabe constant by

\[
\Lambda[g, e^{-\phi} \, d\operatorname{vol}, m] := \inf \{ Q(w) : 0 < w \in C^\infty(M) \}.
\]

In particular, the weighted Yamabe constant of a smooth metric measure space with \(m = 0\) is the Yamabe constant, while Theorem 1.1 states that Euclidean space \((\mathbb{R}^n, dx^2, d\operatorname{vol}, m)\) has weighted Yamabe constant \(\Lambda_{m, n}\), where \(dx^2\) denotes the usual flat metric on Euclidean space. With a little more work, one sees that in the case \(m = \infty\) the weighted Yamabe constant, when positive, is equivalent to Perelman’s \(\nu\)-entropy; see Section 3 for details. This shows that the weighted Yamabe constant interpolates between the Yamabe constant and Perelman’s \(\nu\)-entropy. In this paper we will study what we term the weighted Yamabe problem,
which asks for the existence of functions which minimize the weighted Yamabe quotient, and to a lesser extent will also consider the uniqueness of these functions in a geometrically significant setting. Our results will illustrate the interpolatory nature of the weighted Yamabe constants, though, as we describe below, there are some surprising difficulties which are unique to the case $m \in (0, \infty)$.

Through the work of Yamabe [32], Trudinger [30], Aubin [4], and Schoen [27] (see also [24]), it is known that minimizers of the Yamabe constant exist for all compact Riemannian manifolds $(M^n, g)$. The proof consists of two steps. First, one shows that if the Yamabe constant is strictly less than the Yamabe constant of Euclidean space — which, by stereographic projection, is the Yamabe constant of the standard sphere — then Yamabe’s variational approach works. This step is necessary to rule out concentration for minimizing sequences, which we have already seen in Theorem 1.1 does occur in Euclidean space (and hence on the sphere). Second, one shows that the Yamabe constant is bounded above by the Yamabe constant of Euclidean space, and that equality holds if and only if $(M^n, g)$ is conformally equivalent to the standard $n$-sphere. Since Theorem 1.1 gives the characterization of minimizers on the sphere, this solves the Yamabe Problem.

Perelman [26] showed that for compact Riemannian manifolds with finite $\nu$-entropy — defined to be the infimum of his $W$-functional over all suitably-normalized pairs $(w, \tau) \in C^\infty(M) \times (0, \infty)$ — there always exists a pair $(w, \tau)$ which minimizes the $\nu$-entropy. The proof again consists of two steps. First, he showed that if the $\nu$-entropy is strictly less than the $\nu$-entropy of Euclidean space, then there is a $\tau \in (0, \infty)$ such that the $\nu$-entropy is equal to the infimum of the map $w \mapsto W(w, \tau)$ over suitably-normalized $w$. Since the parameter $\tau$ reflects the “scale” of the metric, and $\tau \to 0$ corresponds to “blow-up,” this is the analogue of the first step of the resolution of the Yamabe Problem. Second, he showed that the $\nu$-entropy of every compact Riemannian manifold is strictly less than the $\nu$-entropy of Euclidean space. This is of course different from the situation for the Yamabe Problem, and one aspect of our work will shed some light on this dichotomy.

We shall study the existence problem for minimizers of the weighted Yamabe constant by following the same outline as above. As a first step, we show that concentration for minimizing sequences can be ruled out provided the weighted Yamabe constant is strictly less than the weighted Yamabe constant of Euclidean space. Indeed, we have the following analogue of Aubin’s observation for the Yamabe Problem [4] and Perelman’s observation for the $\nu$-entropy [26].

**Theorem 1.2.** Let $(M^n, g, e^{-\phi} \, d\text{vol}, m)$ be a compact smooth metric measure space. Then

\begin{equation}
\Lambda[g, e^{-\phi} \, d\text{vol}, m] \leq \Lambda[\mathbb{R}^n, dx^2, d\text{vol}, m].
\end{equation}

Moreover, if the inequality in (1.6) is strict, then there exists a positive function $w \in C^\infty(M)$ such that

\[ Q(w) = \Lambda[g, e^{-\phi} \, d\text{vol}, m]. \]

In contrast to the Yamabe Problem (the case $m = 0$) and the existence problem for minimizers of the $\nu$-entropy (the case $m = \infty$), we cannot give a complete characterization of when equality holds in (1.6). However, we do have the following necessary conditions for equality to hold when $m \in \mathbb{N} \cup \{0, \infty\}$. Moreover, if Conjecture 1.6 below is true, then this result characterizes the equality case for these values of $m$. 

\[ \text{(1.6)} \]
Theorem 1.3. Let \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) be a compact smooth metric measure space such that \(m \in \mathbb{N} \cup \{0, \infty\}\). If additionally
\[
\Lambda[g, e^{-\phi} \, d\text{vol}, m] = \Lambda[\mathbb{R}^n, dx^2, d\text{vol}, m],
\]
then \(m \in \{0, 1\}\) and \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) is conformally equivalent to \((S^n, g_0, d\text{vol}, m)\) for \(g_0\) the standard metric of constant sectional curvature one on \(S^n\).

In particular, this yields the following partial solution to the weighted Yamabe problem.

Corollary 1.4. Let \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) be a compact smooth metric measure space with \(m \in \mathbb{N} \cup \{0, \infty\}\). Then there exists a positive function \(w \in C^\infty(M)\) such that
\[
\mathcal{Q}(w) = \Lambda[g, e^{-\phi} \, d\text{vol}, m].
\]

One reason we have been unable to give satisfactory necessary conditions for equality to hold in (1.10) for all \(m \in [0, \infty]\), and in particular solve the weighted Yamabe problem, is that minimizers of the weighted Yamabe constant do not always exist. As a consequence, there is no hope to replicate either the Aubin–Schoen argument \([4, 24, 27]\) showing that the inequality (1.10) is strict in the case \(m = 0\) for manifolds not conformally equivalent to the sphere, or Perelman’s heat flow argument \([26]\) in the case \(m = \infty\). To prove Theorem 1.3 we instead use minimizers of the weighted Yamabe constant of \((M^n, g, d\text{vol}, m)\) as test functions to estimate the weighted Yamabe constant of \((M^n, g, d\text{vol}, m + 1)\). Indeed, if \(w\) is a minimizer of \(\Lambda[g, d\text{vol}, m] \geq 0\), then using \(w^{m+n-2} \) as a test function for \(\Lambda[g, d\text{vol}, m + 1]\) yields the estimate
\[
\frac{\Lambda[g, d\text{vol}, m + 1]}{\Lambda[g, d\text{vol}, m]} \leq \frac{\Lambda[\mathbb{R}^n, dx^2, d\text{vol}, m + 1]}{\Lambda[\mathbb{R}^n, dx^2, d\text{vol}, m]};
\]
for a precise statement which includes a necessary condition for equality, see Theorem 7.1. In particular, this will allow us to iterate the Aubin–Schoen characterization of equality when \(m = 0\) in (1.10) to all integers \(m\).

The statement of Theorem 7.1 is particularly interesting in the case \(m = 0\). Given a compact Riemannian manifold \((M^n, g)\) such that the Yamabe constant \(\Lambda[g] = \mathcal{Q}(1)\) — that is, \(g\) is a Yamabe metric — it holds for any positive function \(v \in C^\infty(M)\) that
\[
(1.7) \quad \Lambda[g, v \, d\text{vol}, 1] \leq \frac{(n-1)^2}{n(n-2)} \Lambda[g].
\]
Moreover, if equality holds, then \(-\Delta \bar{v} = \frac{R}{n-1} \bar{v}\) for \(\bar{v}\) the average of \(v\) with respect to \(d\text{vol}\). In particular, equality holds on Euclidean space with \(v = 1\), a fact which was exploited by Carlen, Carrillo and Loss \([10]\) to give an interesting relationship between the Yamabe constant, its dual Hardy–Littlewood–Sobolev inequality, and a particular fast diffusion equation. It is tempting to speculate that there should be a curved analogue of their observation when equality holds in (1.7).

While Theorem 7.1 is the main step in proving Theorem 1.3, it does not complete the proof; there remains the possibility that the weighted Yamabe constant \(\Lambda[g, d\text{vol}, m]\) becomes nonnegative as \(m\) increases. Generalizing an observation of Akutagawa, Ishida and LeBrun \([3, Proposition 1]\) relating the sign of the Yamabe constant and the finiteness of Perelman’s \(\nu\)-entropy, we show that this is not the case: in particular, there are topological obstructions to the positivity (resp. nonnegativity) of the weighted Yamabe constant (see \([20]\) and references therein).
Theorem 1.5. Let \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) be a compact smooth metric measure space with positive (resp. nonnegative) weighted Yamabe constant. Then the Yamabe constant of \((M^n, g)\) is positive (resp. nonnegative).

As mentioned above, the weighted Yamabe problem is not always solvable; in Section 8 we will show that there does not exist a positive smooth minimizer of the weighted Yamabe constant of \((S^n, g_0, \text{vol}, 1/2)\). Our proof is based on a more general relationship that exists between the weighted Yamabe constant of \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) and the Yamabe constant of the warped product \((M^n \times \mathbb{R}^{2m}, g \oplus e^{-2\phi/m} \, dy^2)\). Indeed, this observation illustrates a further geometric significance to the family \((1.1)\) of GNS inequalities. By exploiting the special form of the extremal functions for the Sobolev inequality, Bakry gave an alternative proof (cf. [6, 9]) of Theorem 1.1 by giving an explicit relationship between the best constant in \((1.1)\) for \((\mathbb{R}^n, dx^2, d\text{vol}, m)\) and the best constant for the Sobolev inequality of \(\mathbb{R}^{n+2m}\). As we will show in Section 8, a weaker form of this facts persists in the curved setting, in that one can give a lower bound for the weighted Yamabe constant of \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) in terms of the Yamabe constant of \((M^n \times \mathbb{R}^{2m}, g \oplus e^{-2\phi/m} \, dy^2)\).

We expect that the weighted Yamabe problem is always solvable for \(m \in \{0\} \cup [1, \infty)\), but not for \(m \in (0, 1)\). If Conjecture 1.6 below holds, we can verify the latter expectation by proving that positive smooth minimizers of the weighted Yamabe constant of \((S^n, g_0, \text{vol}, m)\) do not exist for any \(m \in (0, 1)\). On the other hand, one can show that for \(m > 1\), the weighted Yamabe constant of \((S^n, g_0, \text{vol}, m)\) is strictly less than that of Euclidean space, and it is for this reason that we expect the weighted Yamabe problem to be solvable for \(m \in \{0\} \cup [1, \infty)\).

We conclude this article with a discussion of arguably the most interesting uniqueness result for minimizers of the Yamabe constant and the \(\nu\)-entropy — namely Obata’s characterization [25] of critical points of the Yamabe quotient on a compact Einstein manifold and Perelman’s characterization [26] of critical points of the \(\mathcal{W}\)-functional on a compact gradient Ricci soliton — and the possibility of generalizing them to the weighted setting. Obata showed that if \(w\) is a critical point of the Yamabe quotient of an Einstein manifold \((M^n, g)\), then either \(w\) is constant or \((M^n, g)\) is conformally equivalent to the standard \(n\)-sphere, while Perelman showed that if \((w, \tau)\) is a critical point of the volume-constrained \(\mathcal{W}\)-functional on a compact shrinking gradient Ricci soliton, then \(w\) is constant and \(\tau\) is determined by the soliton. Motivated by these results, we conjecture that the following “weighted Obata theorem” is true.

Conjecture 1.6. Let \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) be a compact smooth metric measure space, and suppose that there exists a constant \(\lambda \in \mathbb{R}\) such that

\[
\text{Ric}_\phi^m - \frac{R^m}{2(m+n-1)} g = \frac{m+n-2}{2(m+n-1)} \lambda g,
\]

where \(\text{Ric}_\phi^m\) is the \((m-)\)Bakry-Émery Ricci tensor. If \(w \in C^\infty(M)\) is a positive critical point of the map \(w \mapsto \mathcal{Q}(w)\), then either

1. \(w\) is constant, or
2. \(m \in \{0, 1\},\)

\[
\left( M^n, g, e^{-\phi} \, d\text{vol}_g, m \right) := \left( M^n, w^{\frac{4}{n-2}} g, w^{\frac{2(m+n-1)}{n-2}} e^{-\phi} \, d\text{vol}_g, m \right)
\]
satisfies (1.8) for some constant $\hat{\lambda} > 0$, and both $(M^n, g)$ and $(M^n, \hat{g})$ are homothetic to the standard $n$-sphere.

Before discussing what we do know about the validity of Conjecture 1.6 let us comment on the hypothesis (1.8). The tensor appearing on the left-hand side is the so-called weighted Schouten tensor (cf. [13]), and thus (1.8) is the analogue of the assumption that the Schouten tensor of a Riemannian manifold is a constant multiple of the metric. In the latter case, this implies that the Ricci tensor is constant. However, when $m \in (0, \infty)$ it is not the case that (1.8) implies that $\text{Ric}_\phi^n$ is a constant multiple of the metric; i.e. that $(M^n, g, e^{-\phi} \text{dvol}, m)$ is a so-called quasi-Einstein manifold. The reason that the weighted Schouten tensor, rather than the Bakry-Émery Ricci tensor itself, appears in (1.8) is due to the choice of the second factor in the numerator of (1.5) (cf. [14]).

Unfortunately, the obvious modification of the proofs of Conjecture 1.6 in the cases $m = 0$ [25] and $m = \infty$ [26] does not yield a proof of Conjecture 1.6 with the difficulty coming from the uncertainty as to the signs of certain lower order terms which appear. This is not to say that this approach cannot work, but only that some additional idea must be used which we have been unable to find. Nevertheless, the Obata–Perelman argument does prove Conjecture 1.6 under one of the following two sets of additional assumptions:

1. $(w, \phi)$ is a critical point of the map $(\xi, \psi) \mapsto Q[g, e^{-\psi} \text{dvol}, m](\xi)$; or
2. $(M^n, g, e^{-\phi} \text{dvol}, m)$ is isometric to Euclidean space and $w$ is such that the function

$$\tilde{w}(x) := |x|^{2-m-n}w\left(\frac{x}{|x|^2}\right)$$

can be extended to a positive function in $C^2(\mathbb{R}^n)$.

For precise statements, see Theorem 9.6 and Theorem 9.8 respectively. As we will make precise in Section 9 imposing either of these assumptions allows us to show that the aforementioned lower order terms vanish (at least in an integral sense). In particular, the second assumption nearly yields an alternative proof of the characterization of the extremal functions (1.3) in Theorem 1.1. However, since we feel like the greater interest of Conjecture 1.6 is as a statement about (curved) smooth metric measure spaces, we do not here try to show that critical points of the weighted Yamabe functional on Euclidean space necessarily satisfy the decay estimate above.

This article is organized as follows.

In Section 2 we give a more detailed introduction to smooth metric measure spaces and the ways in which they will be studied in this article.

In Section 3 we discuss basic properties of the weighted Yamabe quotient and introduce an auxiliary functional which is the analogue for $m \in (0, \infty)$ of Perelman’s $W$-functional [26].

In Section 4 we compute the first variation of the weighted Yamabe functional and our $W$-functionals, which in particular will make clear the significance of the assumption (1.8) in Conjecture 1.6.

In Section 5 we give a geometric description of Theorem 1.1 as a solution to the weighted Yamabe problem on Euclidean space, and use this discussion to establish some estimates necessary to study concentration for minimizing sequences of the weighted Yamabe constant of arbitrary compact smooth metric measure spaces.
In Section 6 we study the analytic aspects of the weighted Yamabe problem, culminating in the proof of Theorem 1.2.

In Section 7 we establish the aforementioned relationship between the weighted Yamabe constants of \((M^n, g, dv_0, m)\) and \((M^n, g, dv_0, m + 1)\), and then use it to prove Theorem 1.3 and Corollary 1.4.

In Section 8 we establish the aforementioned relationship between the weighted Yamabe constant of \((M^n, g, e^{-\phi}dv_0, m)\) and the Yamabe constant of \((S^n, g_0, dv_0, m)\times\mathbb{R}^2\), and then use it to show that there does not exist a smooth minimizer for the weighted Yamabe constant of \((S^n, g_0, dv_0, 1/2)\).

In Section 9 we present our uniqueness results, and also explain why Conjecture 1.6 would imply that there does not exist a smooth minimizer for the weighted Yamabe constant of \((S^n, g_0, dv_0, m)\) for any \(m \in (0, 1)\).

2. Smooth metric measure spaces

To begin, we review those notions used in the study of smooth metric measure spaces as will be necessarily in this article, following the presentation found in [12].

**Definition 2.1.** A smooth metric measure space is a four-tuple \((M^n, g, e^{-\phi}dv_0, m)\) of a Riemannian manifold \((M^n, g)\), a smooth measure \(e^{-\phi}dv_0\) determined by \(\phi \in C^\infty(M)\) and the Riemannian volume element \(dv_0\) determined by \(g\), and a dimensional parameter \(m \in [0, \infty)\). In the case \(m = 0\), we require \(\phi = 0\).

We will frequently denote abstract smooth metric measure spaces as triples \((M^n, g, v_mdvol)\), where the measure is \(v_mdvol\) and the dimensional parameter is encoded as the exponent of \(v\). In accordance with this convention, \(v\) and \(\phi\) will denote throughout this article functions which are related by \(v_m = e^{-\phi}\); when \(m = \infty\), this is to be interpreted as the formal definition of the symbol \(v^\infty\).

**Definition 2.2.** The weighted divergence \(\delta_\phi\) of a smooth metric measure space \((M^n, g, v_mdvol)\) is the operator defined on tensor fields \(T\) by

\[
(\delta_\phi T)(X, Y, \ldots) = \sum_{i=1}^n \nabla_{E_i} T(E_i, X, Y, \ldots) - T(\nabla_\phi, X, Y, \ldots)
\]

for all \(p \in M\) and all vector fields \(X, Y\) in a neighborhood of \(p\), where \(\{E_i\}\) is a parallel orthonormal frame in a neighborhood of \(p\) and we use the metric \(g\) to change the type of \(T\) if necessary.

In this article, we will only apply this definition to sections of \(TM, T^*M,\) and \(T^*M \otimes T^*M\). For example, this definition leads to the natural definition of the weighted Laplacian.

**Definition 2.3.** Let \((M^n, g, v_mdvol)\) be a smooth metric measure space. The weighted Laplacian \(\Delta_\phi: C^\infty(M) \to C^\infty(M)\) is the operator

\[
\Delta_\phi = \delta_\phi d.
\]

Equivalently, \(\Delta_\phi = \Delta - \nabla_\phi\).

**Definition 2.4.** Let \((M^n, g, v_mdvol)\) be a smooth metric measure space. The Bakry-Émery Ricci curvature \(\text{Ric}_\phi^m\) and the weighted scalar curvature \(R_\phi^m\) are the
Definition 2.5. Two smooth metric measure spaces \((M^n, g, e^{-\phi} \, d\text{vol}_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}} \, d\text{vol}_{\hat{g}}, m)\) are pointwise conformally equivalent if there is a function \(\sigma \in C^\infty(M)\) such that
\begin{align*}
(M^n, \hat{g}, e^{-\hat{\phi}} \, d\text{vol}_{\hat{g}}, m) = (M^n, e^{\frac{2}{m+n-2} \sigma} g, e^{\frac{m+n}{m+n-2} \sigma} e^{-\phi} \, d\text{vol}_g, m) .
\end{align*}

Put another way, two smooth metric measure spaces are pointwise conformally equivalent if their metrics are pointwise conformally equivalent in the usual sense, and the measures are related through this equivalence as would be the measures of two pointwise conformally equivalent \((m+n)\)-dimensional manifolds. When \(m = 0\), Definition 2.5 is of course the definition that two Riemannian manifolds be pointwise conformally equivalent. A key point of this definition is that it also makes sense when \(m = \infty\): Two smooth metric measure spaces with \(m = \infty\) are pointwise conformally equivalent if the underlying Riemannian manifolds are the same. To say that \((M^n, g, e^{-\phi} \, d\text{vol}_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}} \, d\text{vol}_{\hat{g}}, m)\) are conformally equivalent is to say that there is a diffeomorphism \(F: \hat{M} \rightarrow M\) such that \((\hat{M}^n, F^* g, F^*(e^{-\phi} \, d\text{vol}_g), m)\) is pointwise conformally equivalent to \((M^n, \hat{g}, e^{-\hat{\phi}} \, d\text{vol}_{\hat{g}}, m)\).

Definition 2.6. A geometric invariant \(T[g, e^{-\phi} \, d\text{vol}, m]\) defined on a smooth metric measure space \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) is a conformal invariant of weight \(w \in \mathbb{R}\) if for all \(\sigma \in C^\infty(M)\),
\begin{align*}
T[e^{\frac{2}{m+n-2} \sigma} g, e^{\frac{m+n}{m+n-2} \sigma} e^{-\phi} \, d\text{vol}_g, m] = e^{\frac{(m+n)w}{m+n-2} \sigma} T[g, e^{-\phi} \, d\text{vol}_g, m] .
\end{align*}

Note in particular that our convention in defining the weight of a conformal invariant is different than the usual convention of conformal geometry. The reason for this is to ensure that we get a meaningful definition of the weight of a conformal invariant in the limiting case \(m = \infty\).

Definition 2.7. An operator \(T[g, e^{-\phi} \, d\text{vol}, m] : C^\infty(M) \rightarrow C^\infty(M)\) defined on a smooth metric measure space \((M^n, g, e^{-\phi} \, d\text{vol}, m)\) is conformally covariant of bidegree \((a, b)\) if for all \(\sigma \in C^\infty(M)\),
\begin{align*}
T[e^{\frac{2}{m+n-2} \sigma} g, e^{\frac{m+n}{m+n-2} \sigma} e^{-\phi} \, d\text{vol}_g, m] = e^{-\frac{(m+n)k}{m+n-2} \sigma} \circ T[g, e^{-\phi} \, d\text{vol}_g, m] \circ e^{\frac{(m+n)\sigma}{m+n-2}} ,
\end{align*}
where the right hand side denotes a pre- and post-composition of \(T\) with two multiplication operators.
Again we adopt a different convention than usual in conformal geometry so that we obtain a meaningful definition in the limit \( m = \infty \) (cf. [16]).

The simplest nontrivial example of a conformally covariant operator, and the one which we shall be concerned with in this article, is the weighted conformal Laplacian.

**Definition 2.8.** Let \((M^n, g, v^m \, \text{dvol})\) be a smooth metric measure space. The weighted conformal Laplacian \( L_{\phi}^m : C^\infty(M) \to C^\infty(M)\) is the operator

\[
L_{\phi}^m := -\Delta_{\phi} + \frac{m + n - 2}{4(m + n - 1)} R_{\phi}^m.
\]

When \( m = 0 \), this is just the conformal Laplacian, and on the smooth metric measure space \((M^n, g, \text{dvol}, \infty)\), it is precisely the operator \(-\Delta + \frac{R}{4}\) appearing in Perelman’s definition of the \( F \)-functional [26]. As stated above, a key property of the weighted conformal Laplacian is that it is conformally covariant as an operator on smooth metric measure spaces.

**Proposition 2.9** ([15, Lemma 3.4]). The weighted conformal Laplacian is a conformally covariant operator of bidegree \( \left( \frac{m + n - 2}{2(m + n)}, \frac{m + n + 2}{2(m + n)} \right) \).

### 3. The Weighted Yamabe Constant

Let us now turn our attention to the weighted Yamabe problem. To that end, in this section we describe some basic properties of the weighted Yamabe quotient and the weighted Yamabe constant, and also introduce a general version of Perelman’s \( W \)-functional and its associated entropy (which we shall call an energy), which will be needed to deal with the possibility of concentration for minimizing sequences of the weighted Yamabe constant.

#### 3.1. The \( Q \)-functional

To begin, let us recall the definition of the weighted Yamabe quotient.

**Definition 3.1.** Let \((M^n, g, v^m \, \text{dvol})\) be a compact smooth metric measure space. The weighted Yamabe quotient \( Q : C^\infty(M) \to \mathbb{R} \) is defined by

\[
Q[g, v^m \, \text{dvol}] (w) = \frac{(L_{\phi}^m w, w) \left( \int |w|^{2(m + n - 1)} v^{-1} \right)^{\frac{2m}{m + n}}}{\left( \int |w|^{2(m + n)} v^m \, \text{dvol} \right)^{\frac{2m + n - 2}{m + n}}} \tag{3.1a}
\]

when \( m < \infty \) and by

\[
Q[g, e^{-\phi} \, \text{dvol}] (w) = \frac{(L_{\phi}^\infty w, w)}{\|w\|_2} \exp \left( -\frac{2}{n} \int_M \frac{w^2}{\|w\|_2^2} \log \frac{w^2 e^{-\phi}}{\|w\|_2^2} \right) \tag{3.1b}
\]

when \( m = \infty \).

The weighted Yamabe constant \( \Lambda[g, v^m \, \text{dvol}] \) of \((M^n, g, v^m \, \text{dvol})\) is

\[
\Lambda[g, v^m \, \text{dvol}] = \inf \{ Q[g, v^m \, \text{dvol}] (w) : 0 \neq w \in W^{1,2}(M, v^m \, \text{dvol}) \}.
\]

As usual, all integrals are taken with respect to the measure \( v^m \, \text{dvol} \). Here \( W^{1,2}(M, v^m \, \text{dvol}) \) denotes the closure of \( C^\infty(M) \) with respect to the norm

\[
\|w\|_{W^{1,2}(M, v^m \, \text{dvol})} := \int_M (|\nabla w|^2 + w^2).
\]
which, since $M$ is compact and we require $v$ to be positive, is equivalent to the usual $W^{1,2}(M)$-norm. Since $C^\infty(M)$ is dense in $W^{1,2}(M)$ and $Q(w) = Q(|w|)$ for $w$ smooth, we see that we may equivalently define the weighted Yamabe constant by minimizing over the space of positive smooth functions on $M$, as we shall often do without further comment.

Like the realization of the logarithmic Sobolev inequality as the limiting case $m = \infty$ of (3.1), it is the case that (3.1b) is the limiting case of (3.1a) when $m = \infty$:

**Proposition 3.2.** Let $(M^n, g)$ be a compact Riemannian manifold and fix $\phi \in C^\infty(M)$ and $m \in [0, \infty]$. Given any $w \in C^\infty(M)$, it holds that

$$\lim_{k \to m} Q[g, e^{-\phi} \text{dvol}, k](w) = Q[g, e^{-\phi} \text{dvol}, m](w).$$

**Proof.** This is clear in the case $m < \infty$, while in the case $m = \infty$ it follows easily from the expansion

$$\frac{\int |w|^\frac{2(m+n-1)}{m+n-2} e^{-\phi} \text{dvol}}{\int |w|^{\frac{2(m+n)}{m+n-2}} e^{-\phi} \text{dvol}} = 1 - \frac{1}{m} \left( \int_M \frac{|w|^2}{\|w\|^2} \log(w^2 e^{-\phi}) e^{-\phi} \text{dvol} \right) + O(m^{-2})$$

for $m$ large (cf. [18]).

The reason for the particular choices of exponents in the definition of the weighted Yamabe constant is to ensure that it is a curved analogue of (1.1), which is conformally invariant under constant rescalings of the metric, the measure, and the function $w$. The latter properties are easily checked, and the former is a consequence of the conformal covariance of the weighted conformal Laplacian.

**Proposition 3.3.** Let $(M^n, g, v^m \text{dvol})$ be a compact smooth metric measure space. For any $\sigma, w \in C^\infty(M)$ it holds that

$$Q\left[ e^{-\frac{2\sigma}{m+n-2}} g, e^{\frac{m+n-2\sigma}{m+n-2}} v^m \text{dvol}_g \right](w) = Q[g, v^m \text{dvol}_g]\left(e^{\frac{\sigma}{m} w}\right).$$

In particular,

$$\Lambda \left[ e^{-\frac{2\sigma}{m+n-2}} g, e^{\frac{m+n-2\sigma}{m+n-2}} v^m \text{dvol}_g \right] = \Lambda[g, v^m \text{dvol}_g].$$

**Proof.** It is clear that the integrals

$$\int_M |w|^\frac{2(m+n-1)}{m+n-2} v^{m-1} \text{dvol} \quad \text{and} \quad \int_M |w|^{\frac{2(m+n)}{m+n-2}} v^m \text{dvol}$$

are invariant under the conformal transformation

$$g, v^m \text{dvol}, w) \mapsto \left( e^{-\frac{2\sigma}{m+n-2}} g, e^{\frac{m+n-2\sigma}{m+n-2}} v^m \text{dvol}_g, e^{-\frac{\sigma}{m} w} \right).$$

That $(L^m, g, v^m \text{dvol})$ is also invariant under the transformation (3.2) follows immediately from Proposition 3.3.

The above observation that $\int |w|^\frac{2(m+n)}{m+n-2}$ is conformally invariant in particular implies that this integral measures the weighted volume of

$$(M^n, \hat{g}, \hat{v}^m \text{dvol}_{\hat{g}}) := \left( M^n, w^{\frac{4\sigma}{m+n-2}} g, w^{\frac{2(m+n)}{m+n-2}} v^m \text{dvol}_g \right).$$

In order to remove the trivial noncompactness in the weighted Yamabe problem coming from the freedom to multiply $w$ by a constant, we will typically normalize...
so that this integral is one. It is convenient to introduce terminology for such functions; our choice is motivated by the geometric interpretation of this integral.

**Definition 3.4.** Let \((M^n,g,v^m\,dvol)\) be a smooth metric measure space. We say that \(w \in C^\infty(M)\) is **volume-normalized** if

\[
\int_M w^{2(m+n)} v^m \, dvol = 1.
\]

Let us conclude this subsection with some simple, but useful, observations. First, the weighted Yamabe constant of a compact smooth metric measure space is always finite. Indeed, finiteness follows easily in the following slight more general situation.

**Lemma 3.5.** Let \((M^n,g,v^m\,dvol)\) be a smooth metric measure space and suppose that

\[
\int_M v^m \, dvol, \quad \int_M v^{-n} \, dvol, \quad \text{and} \quad \sup_{x \in M} -R^m_\phi(x)
\]

are all finite. Then there is a (finite) constant \(C\) depending only on \(3.4\) such that

\[
\Lambda[ g, v^m \, dvol ] > C.
\]

**Proof.** Let \(w \in C^\infty(M)\) be positive. By the scale-invariance of the weighted Yamabe quotient, we may suppose that \(w\) is volume-normalized. It is clear that

\[
(L^m_\phi w, w) \geq \frac{m+n-2}{4(m+n-1)} \left( - \sup_{x \in M} R^m_\phi \right) \int_M w^2 v^m \, dvol.
\]

When \(m < \infty\), Hölder’s inequality and the normalization \(3.3\) of \(w\) give

\[
\int_M w^2 \leq \left( \int_M v^m \, dvol \right)^{\frac{2}{m+n}} \quad \text{and} \quad \int_M w^{2(m+n-1)} v^{-1} \leq \left( \int_M v^{-n} \, dvol \right)^{\frac{m-1}{m+n}},
\]

while when \(m = \infty\), the basic estimate \(x \log x \geq -e^{-1}\) gives

\[
- \int_M \log \left( \frac{w^2 e^{-\phi}}{\|w\|_2^2} \right) \frac{w^2 e^{-\phi}}{\|w\|_2^2} \, dvol \leq e^{-1} \int_M \, dvol.
\]

These estimates and \(3.3\) together yield the result. \(\square\)

Second, the sign of the weighted Yamabe constant is the same as the sign of the weighted conformal Laplacian.

**Proposition 3.6.** Let \((M^n,g,v^m\,dvol)\) be a compact smooth metric measure space and denote by \(\lambda_1(L^m_\phi)\) the bottom of the spectrum of the weighted conformal Laplacian; i.e.

\[
\lambda_1(L^m_\phi) = \inf \left\{ \frac{(L^m_\phi w, w)}{\|w\|_2^2} : 0 \neq w \in W^{1,2}(M,v^m \, dvol) \right\}.
\]

The exactly one of the three statements is true:

- (1) \(\lambda_1(L^m_\phi)\) and \(\Lambda[ g, v^m \, dvol ]\) are both positive.
- (2) \(\lambda_1(L^m_\phi)\) and \(\Lambda[ g, v^m \, dvol ]\) are both zero.
- (3) \(\lambda_1(L^m_\phi)\) and \(\Lambda[ g, v^m \, dvol ]\) are both negative.
Except for the assertion that $\Lambda[g, v^m \, dvol] = 0$ implies $\lambda_1(L^m_\phi) = 0$, the proof is trivial. The proof of this assertion requires a few words about the existence of minimizers for related problems, and will be given in Section 6.

Third, a useful consequence of Proposition 3.6 is the monotonicity in $m$ of the sign of the weighted Yamabe constant.

**Proposition 3.7.** Let $(M^n, g, v^m \, dvol)$ be a compact smooth metric measure space with negative (resp. nonpositive) weighted Yamabe constant. Then for any $k \geq 0$, the weighted Yamabe constant of $(M^n, g, v^{m+k} \, dvol)$ is negative (resp. nonpositive).

In particular, this implies Theorem 1.5.

**Proof.** First observe that it suffices to show that if $\Lambda[g, 1^{m+k} \, dvol]$ is negative (resp. nonpositive), then so too is $\Lambda[g, 1^m \, dvol]$. Indeed, by the conformal invariance of the weighted Yamabe constant, we have that

$$\Lambda[g, v^{m+k} \, dvol_g] = \Lambda[v^{-2}g, 1^{m+k} \, dvol_{v^{-2}g}],$$

$$\Lambda[g, v^m \, dvol_g] = \Lambda[v^{-2}g, 1^m \, dvol_{v^{-2}g}].$$

Second, by Proposition 3.6 it suffices to show that if the first eigenvalue of the weighted conformal Laplacian $L^m_\phi$ of $(M^n, g, 1^m \, dvol)$ is negative (resp. nonpositive), then so too is the first eigenvalue of the weighted conformal Laplacian $L^{m+k}_\phi$ of $(M^n, g, 1^{m+k} \, dvol)$.

Now, a straightforward computation shows that

$$\frac{(m+k+n-1)(m+n-2)}{(m+k+n-2)(m+n-1)}(L^{m+k}_\phi w, w) \leq (L^m_\phi w, w)$$

for all $w \in W^{1,2}(M)$, with the left and right sides computed with respect to $(M^n, g, 1^{m+k} \, dvol)$ and $(M^n, g, 1^m \, dvol)$, respectively. It is a straightforward exercise in elliptic PDE to show that there exists a nonzero function $w \in C^\infty(M)$ such that $L^m_\phi w = \lambda_1(L^m_\phi)w$. Inserting this function into the above display yields

$$\frac{(m+k+n-1)(m+n-2)}{(m+k+n-2)(m+n-1)}\lambda_1(L^{m+k}_\phi) \leq \lambda_1(L^m_\phi),$$

from which the conclusion immediately follows. □

Finally, the weighted Yamabe constant is upper semicontinuous in $m$, and continuous from the left (resp. the right) near values of $m$ for which the weighted Yamabe constant is nonpositive (resp. nonnegative).

**Lemma 3.8.** Let $(M^n, g)$ be a compact Riemannian manifold and fix $m \in [0, \infty]$. Then

$$\limsup_{k \to m} \Lambda[g, 1^k \, dvol] \leq \Lambda[g, 1^m \, dvol].$$

Moreover,

1. if also there exists an $m' > m$ such that $\Lambda[g, 1^{m'} \, dvol] \geq 0$, then

$$\lim_{k \to m^+} \Lambda[g, 1^k \, dvol] = \Lambda[g, 1^m \, dvol];$$

2. if also there exists an $m' < m$ such that $\Lambda[g, 1^{m'} \, dvol] \leq 0$, then

$$\lim_{k \to m^-} \Lambda[g, 1^k \, dvol] = \Lambda[g, 1^m \, dvol];$$
Proof. By Proposition 3.2 we have that for any fixed positive function \( w \in C^\infty(M) \),
\[
\Lambda [g, 1^k \operatorname{dvol}] \leq \mathcal{Q}[g, 1^k \operatorname{dvol}](w) \to \mathcal{Q}[g, 1^m \operatorname{dvol}](w)
\]
as \( k \to m \). Applying this inequality to a minimizing sequence \( \{ w_i \} \) for \( \Lambda [g, 1^m \operatorname{dvol}] \) yields (3.6).

Suppose now that there exists an \( m' > m \) such that \( \Lambda [g, 1^{m'} \operatorname{dvol}] \geq 0 \). By Proposition 3.7 \( \Lambda [g, 1^k \operatorname{dvol}] \geq 0 \) for all \( k \leq m' \). Hölder’s inequality thus gives
\[
\mathcal{Q}[g, 1^m \operatorname{dvol}](w) = \mathcal{Q}[g, 1^k \operatorname{dvol}](w) \cdot \left( \frac{\int w^{2(m+n-1)/m+n-2} \operatorname{vol}^m}{\int w^{2(m+n-2)/n} \operatorname{vol}^n} \frac{\int w^{2(k+n-1)/k+n-2} \operatorname{vol}^m}{\int w^{2(k+n-2)/n} \operatorname{vol}^n} \right)^{\frac{2m}{2m+k-n}}
\]
\[
\leq \mathcal{Q}[g, 1^k \operatorname{dvol}](w)
\]
for \( m \leq k \leq m' \). Therefore
\[
\liminf_{k \to m^+} \Lambda [g, 1^k \operatorname{dvol}] \geq \Lambda [g, 1^m \operatorname{dvol}],
\]
yielding (3.7). A similar argument yields (3.8) in the case \( \Lambda [g, 1^{m'} \operatorname{dvol}] \leq 0 \) for some \( m' < m \). \( \Box \)

3.2. The \( \mathcal{W} \)-functional. One difficulty which arises when trying to minimize the weighted Yamabe functional directly is that it cannot be used to show that minimizing sequences are \textit{a priori} bounded in \( W^{1,2}(M, v^m \operatorname{dvol}) \). More precisely, there is no reason that a minimizing sequence \( \{ w_i \} \) of volume-normalized functions must also have \( \| w_i v^{-1} \|_2 \) uniformly bounded away from zero, or equivalently, \( (L^m_\phi w_i, w_i) \) uniformly bounded above. Indeed, the explicit minimizers \( 1.3 \) in Theorem 1.1 exhibit this behavior as \( \varepsilon \to 0 \).

To overcome this difficulty, it is useful to recast the weighted Yamabe problem as an optimization problem in \( w \) and an additional “scale” parameter which reflects the norm \( \| w_i \|_2 \). This is accomplished through the introduction of the following generalization of Perelman’s \( \mathcal{W} \)-functional \( 20 \), and parallels the approach of Del Pino and Dolbeault \( 18 \) for proving the existence of extremal functions for (1.1).

\textbf{Definition 3.9.} Let \( (M^n, g, v^m \operatorname{dvol}) \) be a compact smooth metric measure space. The \textit{GNS energy functional} \( \mathcal{W} : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R} \) is defined by

\[
\mathcal{W}(w, \tau) = \tau^m \int_M \left( \tau^{-\frac{m+m-n}{2m+n-2}} w^{2(m+n-1)/m+n-2} v^{-1} - w^{2m+n-2} \right)
\]
when \( m < \infty \) and by

\[
\mathcal{W}(w, \tau) = \tau \left( L^\infty_\phi w, w \right) - \int_M w^2 \log \left( \tau^2 w^2 e^{-\phi} \right)
\]
when \( m = \infty \).

When \( m = 0 \) the GNS energy functional is the unnormalized Yamabe quotient, and when \( m = \infty \) the GNS energy functional is, up to dimensional constants, precisely the \( \mathcal{W} \)-functional; see below for further details.

The \( \mathcal{W} \)-functional will allow us to better understand the issue of concentration of minimizing sequences of the weighted Yamabe quotient when the weighted Yamabe constant is positive. Analogous to the expanding entropy functional \( \mathcal{W}_e \) introduced by Feldman, Ilmanen and Ni \( 19 \), there also exists an “expanding GNS energy...
functional” for smooth metric measure spaces which is related to the GNS energy functional by changing a sign.

**Definition 3.10.** Let \((M^n, g, v^m \, d\nu)\) be a compact smooth metric measure space. The expanding GNS energy functional \(\mathcal{W}_+ : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}\) is defined by

\[
\mathcal{W}_+ (w, \tau) = \tau^{\frac{m}{m+n}} \left( L^m_\phi w, w \right) - m \int_M \left( \tau^{- \frac{2(m+n)}{2m+n-2}} w^{\frac{2(m+n-1)}{m+n-2}} \right) - w^{- \frac{2(m+n)}{2m+n-2}}
\]

when \(m < \infty\) and by

\[
\mathcal{W}_+ (w, \tau) = \tau \left( L^\infty_\phi w, w \right) + \int_M w^2 \log \left( \tau^{\frac{n}{2}} w^2 e^{-\phi} \right)
\]

when \(m = \infty\), where all integrals are taken with respect to the measure \(v^m \, d\nu\).

This functional should be useful when studying smooth metric measure spaces with negative weighted Yamabe constant. However, it is not important for establishing the existence of minimizers of the weighted Yamabe functional in this case. For this reason, we shall focus in the remainder of this article on the GNS energy functional, though we note that most facts we prove below about \(\mathcal{W}\) are easily modified to also apply to \(\mathcal{W}_+\).

Similar to Proposition 3.2, the GNS energy functional is continuous for \(m \in [0, \infty]\).

**Lemma 3.11.** Let \((M^n, g)\) be a compact Riemannian manifold and fix \(\phi \in C^\infty(M)\) and \(m \in [0, \infty]\). Given any \(w \in C^\infty(M)\), it holds that

\[
\lim_{k \to m} \mathcal{W}[g, e^{-\phi} \, d\nu, k](w) = \mathcal{W}[g, e^{-\phi} \, d\nu, m](w).
\]

The proof is a simple calculus exercise, and will be omitted.

One way to regard the role of the parameter \(\tau\) is as a mechanism to break the freedom to rescale the measure \(v^m \, d\nu\) in the weighted Yamabe quotient. From this standpoint, the following symmetries of the GNS energy functional are expected.

**Proposition 3.12.** Let \((M^n, g, v^m \, d\nu)\) be a compact smooth metric measure space. The GNS energy functional is conformally invariant in its first component, in that

\[
\mathcal{W}[e^{2\sigma} g, e^{(m+n)\sigma} v^m \, d\nu](w, \tau) = \mathcal{W}[g, v^m \, d\nu] \left( e^{\frac{m+n-2}{2} \sigma w}, \tau \right)
\]

for all \(\sigma, w \in C^\infty(M)\) and all \(\tau > 0\), and it is scale invariant in its second component, in that

\[
\mathcal{W}[cg, v^m \, d\nu](w, \tau) = \mathcal{W}[g, v^m \, d\nu] \left( e^{-\frac{m+n-2}{4} \tau w}, e^{-\tau} \right)
\]

for all \(w \in C^\infty(M)\) and all \(c, \tau > 0\).

**Proof.** (3.11) follows as in Proposition 3.3. (3.12) follows by direct computation. \(\Box\)

We define the energy of a smooth metric measure space by extremizing the GNS functional in the natural way.

**Definition 3.13.** Let \((M^n, g, v^m \, d\nu)\) be a compact smooth metric measure space. Given \(\tau > 0\), the \(\tau\)-energy \(\nu[g, v^m \, d\nu](\tau) \in \mathbb{R}\) is defined by

\[
\nu[g, v^m \, d\nu](\tau) = \inf \left\{ \mathcal{W}(w, \tau) : w \in W^{1,2}(M^n, v^m \, d\nu), \int_M w^{\frac{2(m+n)}{m+n-2}} = 1 \right\}.
\]
The energy \( \nu[g, v^m \text{dvol}] \in \mathbb{R} \cup \{-\infty\} \) is defined by
\[
\nu[g, v^m \text{dvol}] = \inf_{\tau > 0} \nu[g, v^m \text{dvol}](\tau).
\]

Proposition 3.12 implies that if one defines a new functional \( W \) by
\[
W[g, v^m \text{dvol}](w, \tau) = W[g, v^m \text{dvol}]
\left(\frac{n(m+n-2)}{4(m+n)} w, \tau\right),
\]
then \( W[cg, v^m \text{dvol}](w, \tau) = W[g, v^m \text{dvol}](w, c\tau) \); i.e. \( W \) has the same symmetries as Perelman’s \( W \)-functional. Indeed, in the case \( m = \infty \), this functional \( W \) agrees with Perelman’s \( W \)-functional up to the addition of a constant multiple of the volume \( \int w^2 \tau^{-n/2} \text{dvol} \). Thus the energy of a smooth metric measure space with \( m = \infty \) is precisely Perelman’s \( \nu \)-entropy [26], up to the addition of a dimensional constant.

Proposition 3.14 also implies that the \((\tau-)\)energy is conformally invariant in \( w \) (and scale-invariant in \( \tau \)).

**Proposition 3.14.** Let \((M^n, g, v^m \text{dvol})\) be a compact smooth metric measure space. Then
\[
\nu \left[ c e^{2\sigma} g, e^{(m+n)\sigma} v^m \text{dvol}_g \right] (c\tau) = \nu[g, v^m \text{dvol}] (\tau),
\]
\[
\nu \left[ c e^{2\sigma} g, e^{(m+n)\sigma} v^m \text{dvol}_g \right] = \nu[g, v^m \text{dvol}]
\]
for all \( \sigma \in C^\infty(M) \) and for all \( c > 0 \).

A key fact about the energy is that it is equivalent to the weighted Yamabe constant when the latter is positive. Indeed, we have the following explicit relationship between the two constants.

**Proposition 3.15.** Let \((M^n, g, v^m \text{dvol})\) be a compact smooth metric measure space and denote by \( \Lambda \) and \( \nu \) the weighted Yamabe constant and the energy of \( M \), respectively.

1. \( \Lambda < 0 \) if and only if \( \nu = -\infty \),
2. \( \Lambda = 0 \) if and only if \( \nu = -m \), and
3. \( \Lambda > 0 \) if and only if \( \nu > -m \). Moreover, in this case we have
\[
\nu = \frac{2m + n}{2} \left( \frac{2\Lambda}{n} \right)^{\frac{n}{m+n}} - m,
\]
and \( w \) is a volume-normalized minimizer of \( \Lambda \) if and only if \((w, \tau)\) is a volume-normalized minimizer of \( \nu \) for
\[
\tau = \left( \frac{n \int w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \text{dvol}}{2(L_j^p w, w)} \right)^{\frac{2(m+n)}{2m+n}}.
\]

**Remark 3.16.** In the limiting case \( m = \infty \), Proposition 3.15 states that \( \Lambda \leq 0 \) if and only if \( \nu = -\infty \), and moreover, if \( \Lambda > 0 \) then
\[
\nu = \frac{n}{2} \log \left( \frac{2\Lambda e}{n} \right).
\]

**Remark 3.17.** By defining \( \nu_+ \) in terms of \( W_+ \) as in [19], the proof of Proposition 3.15 given below is easily adapted to establish that

1. \( \Lambda > 0 \) if and only if \( \nu_+ = \infty \),

(2) $\Lambda = 0$ if and only if $\nu_+ = m$, and

(3) $\Lambda < 0$ if and only if $\nu_+ < m$. Moreover, in this case

$$\nu_+ = -\frac{2m+n}{2} \left( -\frac{2\Lambda}{n} \right)^{\frac{n}{m+n}} + m$$

with the corresponding relationship between the minimizers of $\Lambda$ and the extremals of $\nu_+$.

**Proof of Proposition 3.15.** If $\Lambda < 0$, then there is a $w \in C^\infty(M)$ such that $L_m^m(w, w) < 0$ and $\|w\|_{2^*} = 1$. It is then clear that

$\mathcal{W}(w, \tau) \to -\infty$

as $\tau \to \infty$.

Thus suppose $\Lambda \geq 0$. A straightforward calculus exercise shows that if $A, B \geq 0$, then

$$\inf_{x > 0} \left\{ Ax^{2m} + mBx^{-n} \right\} = \frac{2m+n}{2} \left( \frac{2AB^{2m}}{n} \right)^{\frac{n}{m+n}}$$

for all $x > 0$, with equality if and only if

$$x = \left( \frac{nB}{2A} \right)^{\frac{1}{2m+n}}.$$  

It then follows immediately from the definitions of $\Lambda$ and $\nu$ that (3.13) holds. \qed

4. **Variational formulae for the weighted energy functionals**

This section is devoted to computing the first variation of the weighted Yamabe functional and the weighted energy functional, which play an important role in establishing our main results. We will use these formulae in two ways. First, the Euler–Lagrange equation for the weighted Yamabe functional will play an important role both in establishing the regularity in Theorem 1.2 as well as facilitating the comparison between the weighted Yamabe constant of smooth metric measure spaces with different dimensional parameters $m$ as needed in the proof of Theorem 1.3. Second, the first variation of the weighted Yamabe functional for variations of the metric and the measure will yield a divergence structure for the weighted scalar curvature, making it possible to generalize in some ways Obata’s argument [25] characterizing conformally Einstein constant scalar curvature metrics on compact manifolds, which is the main ingredient in our attack on Conjecture 1.6.

To begin, we investigate the critical points of the $Q$- and $W$-functional. From the standpoint of studying the weighted Yamabe problem, the following computation of the Euler–Lagrange equation for minimizers of the weighted Yamabe quotient will be quite useful.

**Proposition 4.1.** Let $(M^n, g, v^m \text{dvol})$ be a compact smooth metric measure space and suppose that $0 \leq w \in W^{1,2}(M)$ is a volume-normalized minimizer of the weighted Yamabe constant; that is, suppose that

$$\int_M w^{\frac{2(m+n)}{m+n+2}} = 1, \quad Q(w) = \Lambda[g, v^m \text{dvol}] := \Lambda.$$  

Then $w$ is a weak solution of

$$L_{\varphi}^m w + c_1 w^{\frac{m+n}{m+n+2}} v^{-1} = c_2 w^{\frac{m+n+2}{m+n+2}},$$
where
\[
c_1 = \frac{2m(m + n - 1)\Lambda}{n(m + n - 2)} \left( \int_M w \frac{2(m+n-1)}{m+n-2} v^{-1} \right)^{-\frac{2m+n}{n}}
\]
\[
c_2 = \frac{(2m + n - 2)(m + n)\Lambda}{n(m + n - 2)} \left( \int_M w \frac{2(m+n-1)}{m+n-2} v^{-1} \right)^{-\frac{2m+n}{n}}.
\]

Proof. Since the weighted conformal Laplacian is self-adjoint, it follows that, as a critical point of the weighted Yamabe functional, \( w \) satisfies
\[
L_\phi^m w + \frac{2m(m + n - 1)}{n(m + n - 2)} \int w \frac{2(m+n-1)}{m+n-2} v^{-1}
\]
\[
= \frac{(2m + n - 2)(m + n)}{n(m + n - 2)} \int w \frac{2(m+n-1)}{m+n-2} v^{-1}.
\]

Using the assumptions yields (4.4). \( \square \)

When \( m > 0 \), the nonlinearities in (1.2) have subcritical exponents, and thus it is straightforward to conclude that weak solutions of (1.2) are in fact smooth. Unfortunately, this does not answer the weighted Yamabe problem because, as remarked in Section 3, there is no a priori reason why volume-normalized minimizing sequences for the weighted Yamabe functional should converge. It is for this reason we introduced the GNS energy functional, and thus the following computation of the Euler–Lagrange equations for its critical points will also be of use to us.

Lemma 4.2. Let \( (M^n, g, v^m \text{dvol}) \) be a compact smooth metric measure space, fix \( \tau > 0 \), and suppose that \( 0 \leq w \in W^{1,2}(M) \) is a critical point of the map \( \xi \mapsto \mathcal{W}(\xi, \tau) \) acting on the space of volume-normalized elements of \( W^{1,2}(M, v^m \text{dvol}) \). Then \( w \) is a weak solution of
\[
\tau \tau^m \phi^m w + \frac{m(m + n - 1)}{m+n-2} \tau^{-\frac{m+n}{m+n-2}} w^{\frac{m+n}{m+n-2}} v^{-1} = c_1 w^{\frac{m+n+2}{m+n}}
\]
for some constant \( c_1 \). If additionally \( (w, \tau) \) is a minimizer of the energy, then
\[
c_1 = \frac{(2m + n - 2)(m + n)}{(2m + n)(m + n - 2)} v[g, v^m \text{dvol}].
\]

Proof. (4.3) follows immediately from the definition of \( \mathcal{W} \). Under the additional assumption that \( (w, \tau) \) is a critical point of the map \( (w, \tau) \mapsto \mathcal{W}(w, \tau) \), we compute that
\[
\tau^2 \frac{2m+n}{2} \left( L_\phi^m w, w \right) = \frac{n}{2} \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}.
\]

Using this identity we can express \( v[g, v^m \text{dvol}] \) in terms of \( (L_\phi^m w, w) \), while integrating against \( w^m v^m \text{dvol} \) yields, via another application of the above display, an expression for \( c_1 \) in terms of \( (L_\phi^m w, w) \). Combining these two facts then yields (4.4). \( \square \)

Next, in order to work towards a weighted analogue of Obata’s theorem [25], we consider the first variation of the \( \mathcal{W} \)-functional through changes in the metric and the measure. In order to perform this computation, it is convenient to introduce the total weighted scalar curvature functional.
Definition 4.3. Let $M^n$ be a compact manifold, fix $m \in [0, \infty]$ and $\mu \in \mathbb{R}$, and denote by $\text{Met}(M)$ and $\mathfrak{M}_+$ the spaces of smooth Riemannian metrics and smooth measures, respectively, on $M$. The total weighted scalar curvature functional

$$W[\mu] : \text{Met}(M) \times \mathfrak{M}_+ \to \mathbb{R}$$

is the functional

$$W[\mu](g, v^m \text{dvol}) := \int_M \left( R^m_\phi + m \mu v^{-1} \right) ,$$

where $R^m_\phi$ is the weighted scalar curvature of $(M^n, g, v^m \text{dvol})$ and integration is performed with respect to $v^m \text{dvol}$.

It is straightforward to check that, depending on whether $\mu$ is zero, positive, or negative, the total weighted scalar curvature functional is effectively the same as $(L^m_w, w)$, the GNS energy functional, or the expanding GNS energy functional, respectively. Indeed,

$$W[g, v^m \text{dvol}](1, r) = \tau \int_M \left[ \frac{m + n - 2}{4(m + n - 1)} R^m_\phi + m \tau^{-\frac{2m+n}{2m+n}} v^{-1} \right] ,$$

so that the explicit form of the claimed equivalence can be readily formulated using the conformal invariance of the weighted conformal Laplacian and the GNS energy functional.

With this notation in hand, computations carried out in [12] yield the first variation of the total weighted scalar curvature functional for any fixed $\mu$. In fact, it will be convenient to write this result in two equivalent ways.

Theorem 4.4. Let $M^n$ be a compact Riemannian manifold, fix $m \in [0, \infty]$ and $\mu \in \mathbb{R}$, and let $g(s), \phi(s)$ be a smooth one parameter family of metrics and functions, respectively. Set $g = g(0), h = h(0), \phi = \phi(0)$, and $\psi = \phi'(0)$. Then the first variation

$$\delta W := \left. \frac{d}{ds} \right|_{s=0} W[\mu] \left( g(s), e^{\phi(s)} \text{dvol}_{g(s)}, m \right)$$

of the total weighted scalar curvature functional is

$$\delta W = -\int_M \left( \text{Ric}_\phi^m - \frac{R^m_\phi}{2(m+n-1)} g, h + \frac{2}{m} \psi g \right)$$

$$- \frac{m + n - 2}{m + n - 1} \int_M \left( R^m_\phi + \frac{m(m + n - 1) \mu}{m + n - 2} v^{-1} \right) \left( \frac{m - 1}{m} \psi - \frac{1}{2} \text{tr } h \right) ,$$

or equivalently,

$$\delta W = -\int_M \left( \text{Ric}_\phi^m - \frac{R^m_\phi}{m + n} g - \frac{m \mu v^{-1}}{2(m + n)} g, h + \frac{2}{m} \psi g \right) v^m \text{dvol}$$

$$- \frac{m + n - 2}{m + n} \int_M \left( R^m_\phi + \frac{m(m + n - 1) \mu}{m + n - 2} v^{-1} \right) \left( \psi - \frac{1}{2} \text{tr } h \right) v^m \text{dvol} ,$$

where all geometric invariants are computed using $(M^n, g, v^m \text{dvol})$.

Proof. In [12] Proposition 4.19 it is shown that

$$\left. \frac{d}{ds} \right|_{s=0} R^m_\phi = -\int_M \left( \text{Ric}_\phi^m - \frac{1}{2} R^m_\phi g, h \right) + \left( R^m_\phi - \frac{2}{m} \Delta \phi \right) \psi .$$

The result then follows from the trivial computation of the first variation of $\int v^{-1}$ and straightforward algebraic manipulations. \(\square\)
There are a few comments to make about Theorem 4.4. First, using the correspondence between the total weighted scalar curvature functional and the GNS energy functional, it recovers Lemma 4.2 when restricted to variations within a conformal class of smooth metric measure spaces. This is because for such a variation, the metric $v^{-2}g$ is always fixed, and hence $h + \frac{2}{m} \psi g = 0$. Second, the variational formulae (4.6) and (4.7) reveal that it is natural to consider a smooth metric measure space $(M^n, g, v^m dvol)$ to be a weighted Einstein manifold if there are constants $\lambda$ and $\mu$ such that

\begin{align}
Ric^m - \frac{R_m^m}{2(m+n-1)} g &= \frac{(m+n-2)\lambda}{2(m+n-1)} g \\
Ric^m - \frac{R^m}{m+n} g &= \frac{m\mu v - 1}{2(m+n)} g.
\end{align}

(4.8)

(4.9)

Third, applying (4.7) to variations generated by a one-parameter family of diffeomorphisms yields the following useful Bianchi-type formula involving the weighted scalar curvature. Indeed, as will be made precise in Section 9, this result will establish that (4.8) implies (4.9) and vice versa.

Proposition 4.5. Let $(M^n, g, v^m dvol)$ be a smooth metric measure space, fix $\mu \in \mathbb{R}$, and denote

\begin{align}
E^m_\phi &:= Ric^m - \frac{R_m^m}{m+n} g, \\
\tilde{E}^m_\phi &:= E^m - \frac{m\mu v - 1}{2(m+n)} g.
\end{align}

(4.10a)

(4.10b)

It holds that

\begin{equation}
\delta \phi \left( \tilde{E}^m_\phi \right) = \frac{1}{m} \text{tr} \left( \tilde{E}^m_\phi \right) d\phi + \frac{m+n-2}{2(m+n)} d \left( \frac{R^m_\phi + \frac{m(m+n-1)\mu v - 1}{m+n-2}}{m+n-2} \right).
\end{equation}

(4.11)

Proof. Let $X$ be a compactly-supported vector field on $M$, and let $\{f_s : M \to M\}$, $s \in (-\varepsilon, \varepsilon)$ be the one-parameter family of diffeomorphisms such that $f_0 = \text{id}$ and $\frac{\partial}{\partial s} |_{s=0} f_s = X$. Set $g(s) = f_s^* g$, $v(s) = f_s^* v$, so that $\delta g = L_X g$ and $\delta \phi = L_X \phi$. Since the total weighted scalar curvature functional is a geometric invariant, $\delta V = 0$. Hence (4.7) implies that

\begin{equation}
\int_M \left< \tilde{E}^m_\phi, L_X g + \frac{2}{m} X \phi g \right> = \frac{m+n-2}{m+n} \int_M \left( \frac{R^m_\phi + \frac{m(m+n-1)\mu v - 1}{m+n-2}}{m+n-2} \right) \delta \phi X.
\end{equation}

The result then follows by integration by parts. \hfill \square

5. Euclidean space as the model space

As described in the introduction, Del Pino and Dolbeault [18] have already completely solved the weighted Yamabe problem on Euclidean space by giving a complete classification of the minimizers — and indeed, the positive extremal functions — on $(\mathbb{R}^n, dx^2, 1^m dvol)$. In this section, we discuss their result in terms of the GNS energy functional of Euclidean space. In particular, this will allow us to clearly illustrate the relationship between the parameter $\tau$ and concentration for minimizers, and will lead to Proposition 5.1 which gives an important estimate needed in Section 6.

In keeping with the theme of this article, we shall present these examples in a way which is continuous in $m \in [0, \infty]$; in particular, we will normalize our functions...
different from Theorem 1.1 To that end, fix \( n \geq 3 \) and \( m \in [0, \infty] \). Given any \( x_0 \in \mathbb{R}^n, \tau > 0 \) define the function \( w_{x_0, \tau} \in C^\infty(\mathbb{R}^n) \) by

\[
w_{x_0, \tau}(x) = \tau^{\frac{n(m+n-2)}{4(m+n)}} \left( 1 + \frac{(m+n-1)|x-x_0|^2}{(m+n-2)^2 \tau} \right)^{-\frac{m+n-2}{2}}.
\]

Note that in the limiting case \( m = \infty \), this declares that \( w_{x_0, \tau}(x) = \tau^{-\frac{n}{4}} \exp \left( -\frac{|x-x_0|^2}{2 \tau} \right) \).

A straightforward computation shows that

\[
\int_{\mathbb{R}^n} w_{x_0, \tau}^{2(m+n-1)/m} \, 1^m \, d\text{vol} = \int_{\mathbb{R}^n} \frac{2(m+n)}{m} \, w_{x_0, \tau}^{2(m+n-1)/m} \, 1^m \, d\text{vol} =: V
\]

and

\[
-\tau \frac{m+n}{m+n} \Delta w_{x_0, \tau} + \frac{m(m+n-1)}{m+n-2} \tau^{-\frac{2(m+n)}{m+n-2}} w_{x_0, \tau}^{2(m+n-1)/m} = \frac{(m+n)(m+n-1)}{m+n-2} w_{x_0, \tau}^{m+n+2/m+n-2}.
\]

Since the weighted scalar curvature of \( (\mathbb{R}^n, dx^2, 1^m \, d\text{vol}) \) vanishes identically, Lemma 4.2 together with (5.2) and (5.3) imply that \( w_{x_0, \tau} \) is a critical point of the map \( w \mapsto \mathcal{W}(w, \tau) \) subject to the constraint \( \int w^{2(m+n)} = V \). Indeed, it was shown by Del Pino and Dolbeault that the only nonnegative critical points in \( W^{1,2}(\mathbb{R}^n) \) subject to this constraint are the functions \( w_{x_0, \tau} \); see [18, Theorem 4].

We point out that the explicit dependence of \( w_{x_0, \tau} \) on \( \tau \) is a consequence of the scale-invariance of the GNS energy functional and the fact that the Euclidean metric is homothetic to itself. Indeed, (4.12) implies that

\[
\mathcal{W}[g, v^m \, d\text{vol}](w, 1) = \mathcal{W}[\tau g, v^m \, d\text{vol}](\tau^{-\frac{2(m+n)}{m+n-2}} w, \tau)
\]

for any smooth metric measure space \((M^n, g, v^m \, d\text{vol})\) and any \( w \in C^\infty(M) \), while it is easily seen that

\[
\int_M w_{x_0, \tau}^{2(m+n)} \, v^m \, d\text{vol}_g = \int_M (\tau^{-\frac{n(m+n-2)}{4(m+n)}} w)^{2(m+n)} \, v^m \, d\text{vol}_{\tau g}.
\]

Specializing to Euclidean space with \( w = w_{x_0, 1} \), we see that \( w_{x_0, \tau} \) arises from \( w_{x_0, 1} \) via this rescaling and the fact that \( w_{x_0, \tau} \) satisfies (5.2) and (5.3) follows from the scale-invariance of the \( L^{2(m+n)} \)-norm and the energy. We shall apply a similar rescaling argument in Section 6 as a part of our proof of Theorem 1.2.

For our purposes, the most interesting aspect of the solutions (5.2) is the way in which they concentrate. It is straightforward to check that

\[
\sup_{x \in \mathbb{R}^n} w_{x_0, \tau}(x) = w_{x_0, \tau}(x_0) = \tau^{-\frac{n(m+n-2)}{4(m+n)}} w_{x_0, 1}^{m+n+2/m+n-2}.
\]

and also that, for any \( y \neq x_0 \),

\[
\lim_{\tau \to 0^+} w_{x_0, \tau}(y) = 0.
\]

These observations tell us two things. First, concentration does indeed occur for the solutions to the weighted Yamabe problem on Euclidean space, with concentration at the points \( x_0 \) occurring for the sequence of solutions \( w_{x_0, \tau} \) as \( \tau \to 0^+ \). Second, when \( m > 0 \), the critical points of the volume-constrained functional \( w \mapsto \mathcal{W}(w, \tau) \) with fixed \( \tau \) do not concentrate. These phenomena persist in the curved case.
Proposition 5.1. Let $(M^n, g, v^m \, \text{dvol})$ be a compact smooth metric measure space with $m > 0$, fix $\tau_0$, and suppose that $w \in C^\infty(M)$ is a positive function such that

$$W(w, \tau) = \nu[g, \, v^m \, \text{dvol}](\tau) \quad \text{and} \quad \int_M w^{2(m+n-1)} = 1$$

for some $0 < \tau \leq \tau_0$. Then there exist constants $C_1, C_2 > 0$ depending only on $(M^n, g, v^m \, \text{dvol})$ and $\tau_0$ such that

$$C_1 \leq \sup_{x \in M} \tau^{\frac{n(m+n-2)}{(m+n)}} w(x) \leq C_2. \quad (5.6)$$

Proof. We will denote throughout the course of this proof by $C, C_1, C_2 > 0$ constants depending only on $(M^n, g, v^m \, \text{dvol})$ and $\tau_0$ whose values may change from line to line.

To begin, we observe that, by Hölder’s inequality and the fact that $w$ is volume-normalized,

$$\tau^{\frac{m+n}{m+n}} (L^m_\phi w, w) \geq \tau_0^{\frac{m}{m+n}} \text{Vol}_\phi(M)^{\frac{2}{m+n}} \min \{0, R_\phi^m\}. \quad (5.7)$$

In particular, the right hand side depends only on $(M^n, g, v^m \, \text{dvol})$ and $\tau_0$. By Lemma 4.2 there exists a constant $c \in \mathbb{R}$ such that

$$\tau^{\frac{m+n}{m+n}} \left( L^m_\phi w, w \right) + \frac{m(m+n-1)}{m+n-2} \tau^{-\frac{m}{2(m+n)}} w^{m+n} v^{-1} = c w^{\frac{m+n+2}{2}}, \quad (5.8)$$

while the fact that $w$ realizes $\nu[g, \, v^m \, \text{dvol}](\tau)$ implies that

$$\tau^{\frac{m+n}{m+n}} (L^m_\phi w, w) + m \tau^{-\frac{m+n}{m+n}} \int_M w^{2(m+n-1)} v^{-1} = \nu(\tau). \quad (5.9)$$

As a consequence, we have that

$$c - \nu = \frac{m}{m+n-2} \tau^{-\frac{m}{2(m+n)}} \int_M w^{2(m+n-1)} v^{-1} \geq 0$$

$$\nu - \frac{m+n-2}{m+n-1} c = \frac{1}{m+n-1} \tau^{\frac{m+n}{m+n}} (L^m_\phi w, w).$$

In particular, it follows from $(5.7)$ that $C_1 \leq e^c \leq C_2$. Note also that $(5.7)$, $(5.9)$, and the assumption $m > 0$ together imply that the lower bound in $(5.6)$ holds.

The upper bound for $w$ is a consequence of the above estimates and the fact that the nonlinearity in $(5.8)$ is subcritical when $m > 0$. More precisely, Proposition 3.12 implies that the rescaling

$$(\hat{g}, \hat{w}) := \left( \tau^{-1} g, \tau^{\frac{n(m+n-2)}{(m+n)}} w \right)$$

is such that $\hat{w}$ satisfies

$$\hat{L}^m_\phi \hat{w} + \frac{m(m+n-1)}{m+n-2} \hat{w}^{\frac{m+n}{m+n-2}} v^{-1} = c \hat{w}^{\frac{m+n+2}{2}}, \quad \int_M \hat{w}^{2(m+n-1)} v^{-1} = \nu(\tau),$$

with respect to $(M^n, \hat{g}, v^m \, \text{dvol})$. Since $\tau \leq \tau_0$, it follows that

$$\sup_{x \in M} \left| R^m_\phi(x) \right| \leq \tau_0 \sup_{x \in M} \left| P^m_\phi \right|,$$

and hence $\hat{w}$ satisfies

$$- \hat{\Delta} \hat{w} \leq c \hat{w}^{\frac{m+n+2}{2}} - C \hat{w}. \quad (5.10)$$
On the other hand, it is well-known (e.g. [22]) that there are constants $A, B > 0$ depending only on $(\mathcal{M}^n, g)$ such that
\begin{equation}
\left( \int_M f^{\frac{2n}{n-2}} \operatorname{dvol}_g \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla f|^2 \operatorname{dvol}_g + B \int_M f^2 \operatorname{dvol}_g
\end{equation}
for all $f \in C^\infty(M)$, which implies that
\begin{equation}
(5.11) \quad \left( \int_M f^{\frac{2n}{n-2}} \operatorname{dvol}_g \right)^{\frac{n-2}{n}} \leq C_1 \int_M |\nabla f|^2 \operatorname{dvol}_g + C_2 \tau_0 \int_M f^2 \operatorname{dvol}_g.
\end{equation}
Since $m > 0$, the nonlinearity of (5.10) is subcritical, and in particular it follows readily from Moser iteration and (5.11) that $\sup \tilde{w} \leq C$, which is equivalent to the upper bound in (5.6).

6. The Existence of Minimizers

We are now in a position to establish the existence of minimizers of the weighted Yamabe quotient as asserted in the introduction. Like the proof of the corresponding result for the Yamabe Problem (cf. [4, 30]), it is convenient to separate the proof of Theorem 1.2 into two cases. First, in Proposition 6.1 we will show by a direct compactness argument that if the weighted Yamabe constant is negative, then there exists a smooth, positive minimizer. Second, in Proposition 6.3 we will show that as $\tau \to 0$, the $\tau$-energy of a smooth metric measure space with nonnegative weighted Yamabe constant tends to the weighted Yamabe constant of Euclidean space with the same dimensional parameter $m$. In contrast to Perelman’s proof in the case $m = \infty$, the proof of Proposition 6.3 presented here uses a direct blow-up argument. As will be made precise below, these two results together yield Theorem 1.2.

Proposition 6.1. Let $(\mathcal{M}^n, g, v^m \operatorname{dvol})$ be a compact smooth metric measure space with $m > 0$ and negative weighted Yamabe constant. Then there exists a positive function $w \in C^\infty(M)$ such that $Q(w) = \Lambda[g, v^m \operatorname{dvol}]$.

Proof. Let $\{w_k\} \subset C^\infty(M)$ be a volume-normalized minimizing sequence of the weighted Yamabe constant. Since the weighted Yamabe constant is negative, it follows that for $k$ sufficiently large
\begin{equation}
0 > (L^m_{\phi} w_k, w_k) \geq ||\nabla w_k||^2_2 - C ||w_k||^2_2
\end{equation}
for $C$ a constant depending only on $(\mathcal{M}^n, g, v^m \operatorname{dvol})$. Since the functions $w_k$ are volume-normalized, H"{o}lder’s inequality and the above display imply that $\{w_k\}$ is uniformly bounded in $W^{1,2}(M)$. This in particular implies that there is a constant $C > 0$ independent of $k$ such that
\begin{equation}
\int_M w_k^{\frac{2(m+n)}{m+n-1}} w^{-1} \geq C.
\end{equation}
Since $m > 0$, the embedding $W^{1,2}(M) \subset L^{\frac{2(m+n)}{m+n-1}}$ is compact, and hence there is a $w \in W^{1,2}(M)$ such that $w_k$ converges weakly in $W^{1,2}$ to $w$ and strongly in $L^{\frac{2(m+n)}{m+n-1}}$ to $w$. By construction, $w$ minimizes the weighted Yamabe constant, and
also satisfies the lower bound (6.1). It then follows from Proposition 4.1 that \( w \) is a weak solution to

\[
L^m \phi w + c_1 v^{-1} w^{\frac{m+n-2}{m+n}} = c_2 w^{\frac{m+n+2}{2m+n}}
\]

for explicit constants \( c_1, c_2 \). Since \( 1 \leq \frac{m+n}{m+n-2} \leq \frac{m+n+2}{m+n-2} < \frac{m+n}{2} \) when \( m > 0 \), the usual elliptic regularity argument for subcritical equations allows us to conclude that \( w \) is in fact smooth and positive. \( \square \)

Before stating Proposition 6.3 we first make a simple observation about the existence of minimizers of the \( \tau \)-energy.

**Lemma 6.2.** Let \((M^n, g, v^m \text{dvol})\) be a compact smooth metric measure space with \( m > 0 \) and fix \( \tau \in (0, \infty) \). Then there exists a positive function \( w \in C^\infty(M) \) such that

\[
W[g, v^m \text{dvol}](w, \tau) = \nu(\tau) \quad \text{and} \quad \int_M w^{\frac{2(m+n)}{m+n-2}} = 1.
\]

**Proof.** First observe that, as a consequence of Hölder’s inequality, there is a constant \( C \) depending only on \((M^n, g, v^m \text{dvol})\) and \( \tau \) such that

\[
W[g, v^m \text{dvol}](w, \tau) \geq -C \tau^{\frac{m}{m+n}} - m
\]

for all volume-normalized \( w \in W^{1,2}(M, v^m \text{dvol}) \). In particular, \( \nu(\tau) > -\infty \) and any volume-normalized minimizing sequence \( \{w_k\} \) of \( \nu(\tau) \) is uniformly bounded in \( W^{1,2}(M, v^m \text{dvol}) \). Since \( m > 0 \), the embedding \( W^{1,2} \subset L^{\frac{2(m+n)}{m+n-2}} \) is compact. Hence, taking a subsequence if necessary, we see that \( w_k \) converges to a volume-normalized \( w \in W^{1,2}(M) \) such that

\[
W[g, v^m \text{dvol}](w, \tau) = \nu(\tau).
\]

By Lemma 4.2, \( w \) is a weak solution to the subcritical elliptic PDE (4.3), and hence \( w \) is smooth and positive. \( \square \)

**Proposition 6.3.** Let \((M^n, g, v^m \text{dvol})\) be a compact smooth metric measure space with \( m > 0 \). Then

\[
\lim_{\tau \to 0} \nu[g, v^m \text{dvol}](\tau) = \nu[\mathbb{R}^n, dx^2, 1^m \text{dvol}]\text{.}
\]

**Proof.** We begin by showing that \( \liminf_{\tau \to 0} \nu(\tau) \geq \Lambda[\mathbb{R}^n, dx^2, 1^m \text{dvol}] \). To that end, let \( \{\tau_i\} \) be a decreasing sequence of positive numbers tending to zero and suppose that \( \nu(\tau_i) \) converges, say to \( \tilde{\nu} \). By Lemma 6.2 there are positive functions \( w_i \in C^\infty(M) \) such that \( W(w_i, \tau_i) = \nu(\tau_i) \) and \( \int_{\mathbb{R}^n} w_i^{\frac{2(m+n)}{m+n-2}} = 1 \). It follows from Lemma 4.2 that there are constants \( c_i \) such that

\[
\tau_i \int_{\mathbb{R}^n} L^m w_i \geq \frac{m(m+n-1)}{m+n-2} \tau_i - \frac{2m+n}{m+n-2} w_i^{\frac{m+n}{m+n-2}} - c_i w_i^{\frac{m+n+2}{2m+n}}.
\]

The proof of Proposition 5.1 shows that the constants \( c_i \) are uniformly bounded above and below. Using the choice of \( w_i \) and (6.2) together with the formula for the derivative of the map \( \tau \mapsto W(w, \tau) \) yields

\[
\nu(\tau_i) - \frac{(2m+n)(m+n-2)}{(2m+n-2)(m+n)} c_i = \frac{2\tau_i - d}{2m+n-2 dt} W(w_i, t)|_{t=\tau_i}.
\]

In particular, we have that

\[
\lim_{i \to \infty} \nu(\tau_i) - \frac{(2m+n)(m+n-2)}{(2m+n-2)(m+n)} c_i = 0.
\]
Next set $\tilde{g}_i = \tau_i^{-1} g$ and $\tilde{w}_i = \tau_i^{\frac{n(m+n-2)}{4(m+n)-2}} w_i$, in terms of which (6.2) becomes

$$
(6.4) \quad \tilde{L}_0^m \tilde{w}_i + \frac{m(m+n-1)}{m+n-2} \tilde{w}_i^{\frac{m+n}{m+n-2}} v^{-1} = c_i \tilde{w}_i^{\frac{m+n+2}{m+n-2}};
$$

observe also that the normalization of $w_i$ persists,

$$
\int_M \tilde{w}_i^{\frac{2(m+n)}{m+n-2}} v^m \text{dvol}_{\tilde{g}_i} = 1.
$$

By Proposition 5.1 there are constants $C_1, C_2 > 0$ independent of $i$ such that $C_1 \leq \sup \tilde{w}_i \leq C_2$. Let $x_i \in M$ be such that $\tilde{w}_i(x_i) = \sup \tilde{w}_i$. By taking a subsequence if necessary, we may suppose that the points $x_i$ converge as $i \to \infty$, say to $x_0$, and that $\sup \tilde{w}_i$ and $c_i$ converge as $i \to \infty$. Since $v(x_0)$ is positive and finite, we may assume without loss of generality that $v(x_0) = 1$; this is a straightforward consequence of (3.11).

Now, given any fixed normal coordinate chart $U$ of $x_0$, it follows that $C_1 \leq \sup_U \tilde{w}_i \leq C_2$. Indeed, it follows from (5.8) and elliptic regularity theory that the $C^2,\alpha$-norms of $\tilde{w}_i|_U$ are uniformly bounded. We may thus extract a subsequence such that $\tilde{w}_i|_U$ converges in $C^2,\alpha$ to a nonnegative function $\bar{w} \in C^\infty(\mathbb{R}^n)$ which satisfies $\bar{w}(0) > 0$ and

$$
-\Delta \bar{w} + \frac{m(m+n-1)}{m+n-2} \bar{w}^{\frac{m+n}{m+n-2}} = c \bar{w}^{\frac{m+n+2}{m+n-2}}
$$

for $c = \lim_{i \to \infty} c_i$. It follows from Theorem 1.1 that

$$
\frac{(2m+n)(m+n-2)}{(2m+n-2)((m+n)} c = \nu[\mathbb{R}^n, dx^2, \text{dvol}],
$$

and hence (6.3) implies that $\lim \inf_{\tau \to 0} \nu(\tau) \geq \Lambda[\mathbb{R}^n, dx^2, 1^m \text{dvol}]$.

Let us now show that $\lim \sup_{\tau \to 0} \nu(\tau) \leq \Lambda[\mathbb{R}^n, dx^2, 1^m \text{dvol}]$. Fix a point $p \in M$ and let $\{x_i\}$ be normal coordinates in some fixed neighborhood $U$ of $p = (0, \ldots, 0)$. Let $\varepsilon > 0$ be sufficiently small so that $B(p, 2\varepsilon) \subset U$. Let $\eta: M \to [0, 1]$ be a cutoff function such that $\eta \equiv 1$ on $B(p, \varepsilon)$ and $\text{supp} \eta \subset B(p, 2\varepsilon)$. For each $0 < \tau < 1$, define $f_\tau: M \to \mathbb{R}$ by $f_\tau(x_1, \ldots, x_n) = \eta w_0,\tau(x_1, \ldots, x_n)$, and set $\tilde{f}_\tau = V_\tau^{-\frac{n+2}{m+n}} f_\tau$ for

$$
V_\tau = \int_M f_\tau^{\frac{2(m+n)}{m+n-2}};
$$

this ensures $\|\tilde{f}_\tau\|_{\frac{2(m+n)}{m+n-2}} = 1$. Note that, by choice of $f_\tau$, the constants $V_\tau$ will be uniformly bounded away from zero. With $V$ as in (5.2), we have that

$$
\tau^{\frac{n+2}{m+n}} \int_{\mathbb{R}^n} |\nabla w_0,\tau|^2 + m V^{-\frac{1}{m+n}} \tau^{-\frac{m+n}{m+n-2}} \int_{\mathbb{R}^n} w_0,\tau^{\frac{2(m+n-1)}{m+n-2}} = V^{\frac{m+n+2}{m+n-2}} \nu[\mathbb{R}^n, dx^2, 1^m \text{dvol}].
$$

Computing as in [24, Lemma 3.4], it is easy to see that

$$
\mathcal{W}[g, v^m \text{dvol}](\tilde{f}_\tau, \tau) \leq \nu[\mathbb{R}^n, dx^2, 1^m \text{dvol}] (1 + C_1 \varepsilon) \left(1 + C_2 \tau^{\frac{n-2}{m+n-2}}\right),
$$

where the constant $C_1 > 0$ depends only on $(M^n, g, v_0^m \text{dvol})$ and the constant $C_2 > 0$ depends only on $(M^n, g, v^m \text{dvol})$ and $\varepsilon$. Taking $\tau \to 0$ and then $\varepsilon \to 0$ yields the desired result. \qed
Proof of Proposition 3.6. It is clear from the definitions of the weighted Yamabe constant and the weighted conformal Laplacian that $$\Lambda[g, v^m \, d\text{vol}] < 0$$ if and only if $$\lambda_1(L^m_w) < 0$$. If $$\lambda_1(L^m_w) = 0$$, a standard argument shows that there exists a positive $$w \in C^\infty(M)$$ such that $$L^m_w w = 0$$; in particular, it follows that $$\Lambda[g, v^m \, d\text{vol}] = 0$$. It thus remains to show that if $$\lambda_1(L^m_w) > 0$$ then $$\Lambda[g, v^m \, d\text{vol}] > 0$$.

To that end, set

\[\kappa = \inf \left\{ (L^m_w, w) : w \in W^{1,2}(M, v^m \, d\text{vol}), \int_M w^{2(m+n) \over 2m+n-2} = 1 \right\}.\]

We claim that if $$\lambda_1(L^m_w) > 0$$, then $$\kappa > 0$$. Indeed, it is clear that if $$\lambda_1(L^m_w) > 0$$ then $$\kappa \geq 0$$, so it suffices to show that if $$\kappa = 0$$, then $$\lambda_1(L^m_w) = 0$$. An argument similar to the proof of Lemma 6.2 shows that positive smooth minimizers of $$\kappa$$ always exist. In particular, if $$\kappa = 0$$ then there exists a positive $$w \in C^\infty(M)$$ such that $$L^m_w w = 0$$, completing the proof of the claim.

Finally, we observe that if $$\kappa > 0$$, then for all positive volume-normalized $$w \in C^\infty(M)$$ and all $$\tau > 0$$,

\begin{equation}
W(w, \tau) \geq \kappa \tau^{m \over m+n} - m
\end{equation}

provided $$m < \infty$$, while when $$m = \infty$$ we can proceed as in [33] p. 11] to establish the estimate

\begin{equation}
W(w, \tau) \geq (\tau - \varepsilon)\kappa - {n \over 2} \log \tau - C
\end{equation}

for any $$\varepsilon > 0$$, where the constant $$C > 0$$ depends only on $$(M^n, g, e^{-\phi} \, d\text{vol}, \infty)$$ and $$\varepsilon$$. More precisely, Jensen’s inequality yields

\[\int_M w^2 \log(w^2 e^{-\phi}) \leq {n - 2 \over 2} \log \int_M w^{2m \over n} e^{-m \phi} \, d\text{vol},\]

so that the Sobolev inequality for $$(M^n, g)$$ yields

\[W(w, \tau) = \tau (L^m_w w, w) - {n \over 2} \log \tau - \int_M w^2 \log(w^2 e^{-\phi}) \geq (\tau - \varepsilon)(L^m_w w, w) - {n \over 2} \log \tau - C\]

for any $$\varepsilon > 0$$, where $$C$$ depends on $$(M^n, g, e^{-\phi} \, d\text{vol}, \infty)$$ and $$\varepsilon$$.

In either case, it follows from (6.6) that $$\nu(\tau) > -m$$ for all $$\tau \in (0, \infty)$$, and moreover, $$\nu(\tau) \rightarrow \infty$$ as $$\tau \rightarrow \infty$$. It thus follows from Proposition 6.3 that $$\nu > -m$$, and hence, by Proposition 3.15 that $$\Lambda[g, v^m \, d\text{vol}] > 0$$.

Proof of Theorem 1.2. First note that in the case $$m = 0$$, Theorem 1.2 is already contained in Aubin’s work [4] on the Yamabe Problem, so we may assume that $$m > 0$$. It then follows immediately from Proposition 3.15 and Proposition 6.3 that (1.1) holds.

Suppose now that strict inequality holds in (1.1). If the weighted Yamabe constant is negative, the result follows from Proposition 6.1. If the weighted Yamabe constant is zero, then Proposition 3.6 implies that $$\lambda_1(L^m_w) = 0$$. As already observed in the proof of Proposition 3.6, this yields the result. We thus suppose that the weighted Yamabe constant is positive. It follows from the proof of Proposition 3.6 that $$\nu(\tau) \rightarrow \infty$$ as $$\tau \rightarrow \infty$$, so that Proposition 6.3 implies that there is a $$\tau \in (0, \infty)$$ such that $$\nu(\tau) = \nu$$. The result is then an immediate consequence of Proposition 3.15 and Lemma 6.2.
Let us now turn our attention to the proofs of Theorem 1.3, which provides a necessary condition for a smooth metric measure space with dimensional parameter \( m \in \mathbb{N} \cup \{0, \infty\} \) to have weighted Yamabe constant equal to that of Euclidean space, and of Corollary 1.4, which proves the existence of smooth minimizers of the weighted Yamabe constant for smooth metric measure spaces with \( m \in \mathbb{N} \cup \{0, \infty\} \). As stated in the introduction, these results follow from the following more general result relating the weighted Yamabe constants of \((M^n, g, v^m \text{ dvol})\) and \((M^n, g, v^{m+1} \text{ dvol})\).

**Theorem 7.1.** Let \((M^n, g, v^m \text{ dvol})\) be a compact smooth metric measure space with \(\Lambda[g, v^m \text{ dvol}] \geq 0\), and suppose that there exists a smooth, positive minimizer of the weighted Yamabe constant. Then

\[
\Lambda[g, v^{m+1} \text{ dvol}] \leq \frac{\Lambda[\mathbb{R}^n, dx^2, 1^{m+1} \text{ dvol}]}{\Lambda[\mathbb{R}^n, dx^2, 1^m \text{ dvol}]} \Lambda[g, v^m \text{ dvol}].
\]

Moreover, if equality holds in (7.1), then

\[
-\Delta \hat{v}^{m+1} = C \Lambda[g, v^m \text{ dvol}] \left( \hat{v}^{m+1} - \hat{v}^m \int_M \hat{v}^{m+1} \text{ dvol}_g \right)
\]

for

\[
C = \frac{4(m + 1)(m + n)(2m + n - 2)}{n(m + n - 2)^2} \left( \int_M \hat{v}^{m-1} \text{ dvol}_g \right)^{-\frac{2m}{n}}
\]

and, if also \( m > 0 \),

\[
\left( \int_M \hat{v}^{m} \text{ dvol}_g \right) \left( \int_M \hat{v}^{m+1} \text{ dvol}_g \right) = \frac{(2m + n)(m + n - 1)}{(2m + n - 2)(m + n)},
\]

where

\[
\hat{g} := w^{-\frac{4}{m+n}} g \quad \text{and} \quad \hat{v} = w^{-\frac{2}{m+n}} v
\]
determine the conformal rescaling of \((M^n, g, v^m \text{ dvol})\) induced by a positive volume-normalized minimizer \( w \in C^\infty(M) \) of its weighted Yamabe constant.

The basic idea underlying the proof of Theorem 7.1 is that, by weight considerations (cf. Theorem 1.1), if \( w \) is a minimizer of the weighted Yamabe constant of \((M^n, g, v^m \text{ dvol})\), then \( w^{\frac{m+n-1}{m+n}} \) is a natural test function for estimating the weighted Yamabe constant of \((M^n, g, v^{m+1} \text{ dvol})\). The key step in the realization of this idea is the following computation.

**Proposition 7.2.** Let \((M^n, g)\) be a compact Riemannian manifold and fix \( m \in [0, \infty) \) and a positive function \( v \in C^\infty(M) \). Given any \( w \in C^\infty(M) \), it holds that

\[
\left( L^{m+1}_\phi w^{\frac{m+n-1}{m+n}}, w^{\frac{m+n-1}{m+n}} \right) = \frac{(m + n - 1)^2}{(m + n)(m + n - 2)} \left( L^m_\phi w, w^{\frac{m+n}{m+n-2}} v \right),
\]

where the left and the right hand side are defined relative to the smooth metric measure spaces \((M^n, g, v^m \text{ dvol})\) and \((M^n, g, v^{m+1} \text{ dvol})\), respectively.

**Proof.** First note that both sides of (7.4) are conformally invariant in the sense of smooth metric measure spaces, so that, as in the proof of Proposition 3.7, we may assume \( v = 1 \). The result then follows immediately from the identity

\[
\left| \nabla w^{\frac{m+n-1}{m+n}} \right|^2 = \frac{(m + n - 1)^2}{(m + n)(m + n - 2)} \left( \nabla w, \nabla w^{\frac{m+n}{m+n-2}} \right).
\]
and the definition of the weighted conformal Laplacian.

Remark 7.3. One might hope that a similar relationship exists between the weighted conformal Laplacians of \((M^n, g, v^m \text{dvol})\) and \((M^n, g, v^{m+k} \text{dvol})\). It turns out that when \(k \neq 1\) the analogue of (7.4) involves an extra integral involving the scalar curvature of the metric \(v^{-2}g\); indeed, one readily computes that

\[
\left\langle L_{\phi}^{m+k} w, w \right\rangle = \frac{(m + n + k - 2)^2}{(m + n - 2)(m + n + 2k - 2)} \left( L_{\phi}^{m} w, w \right) - \frac{k(k - 1)(m + n + k - 2)}{4(m + n - 1)(m + n + k - 1)(m + n + 2k - 2)} \int_M R v^{-2} g w^2 v^{m-2}
\]

for all \(w \in C^\infty(M)\). Unfortunately, we have not been able to find a way to use this formula to (partially) remove the assumption \(m \in \mathbb{N} \cup \{0, \infty\}\) from Theorem 7.1 or Corollary 1.4.

Proof of Theorem 7.1. By Proposition 7.2 we have that for all positive \(w \in C^\infty(M)\) which are volume-normalized with respect to \((M^n, g, v^m \text{dvol})\),

\[
Q[g, v^{m+1} \text{dvol}] \left( w, w \right) = \frac{(m + n - 1)^2}{(m + n)(m + n - 2)} \left( L_{\phi}^{m} w, w \right) - \frac{2m(m + n - 1) \Lambda}{n(m + n - 2)} \left( \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \right)^{-\frac{2m+n}{n}} \int_M w^{\frac{2(m+n+1)}{m+n-2}} v
\]

(7.5)

where all quantities on the right hand side are defined in terms of \((M^n, g, v^m \text{dvol})\). On the other hand, Proposition 4.1 implies that, if we additionally take \(w\) to be a minimizer of \(\Lambda := \Lambda[g, v^m \text{dvol}]\), then

\[
\left( L_{\phi}^{m} w, w \right) = -\frac{2m(m + n - 1) \Lambda}{n(m + n - 2)} \left( \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \right)^{-\frac{2m+n}{n}} \int_M w^{\frac{2(m+n+1)}{m+n-2}} v
\]

(7.6)

Combining (7.5) and (7.6) yields

\[
Q[g, v^{m+1} \text{dvol}] \left( w, w \right) = \frac{(m + n - 1)^2 \Lambda}{n(m + n)(m + n - 2)^2} \Phi(x)
\]

(7.7)

for

\[
\Phi(x) = (2m + n - 2)(m + n)x^{-\frac{2m+n}{n}} - 2m(m + n - 1)x^{-\frac{2m+n}{n}},
\]

\[
x = \left( \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \right) \left( \int_M w^{\frac{2(m+n+1)}{m+n-2}} v \right).
\]

By Hölder’s inequality, the volume-normalization of \(w\) implies that \(x \geq 1\). On the other hand, a straightforward calculus exercise reveals that

\[
\Phi(x) \leq \frac{(2m + n - 2)(m + n)n}{2m + n} \left( \frac{(2m + n - 2)(m + n)}{(2m + n)(m + n - 1)} \right)^{\frac{2m}{n}}
\]

(7.8)

with equality if and only if \(m = 0\) or

\[
x = \frac{(2m + n)(m + n - 1)}{(2m + n - 2)(m + n)}.
\]

(7.9)
Since the right hand side of (7.9) is at least one, it follows from (7.7) and the explicit values for $\Lambda[\mathbb{R}^n, dx^2, 1^m \text{vol}]$ and $\Lambda[\mathbb{R}^n, dx^2, 1^m \text{vol}]$ given in Theorem 1.1 that

$$\Lambda[g, v^{m+1} \text{vol}] \leq \frac{\Lambda[\mathbb{R}^n, dx^2, 1^m \text{vol}]}{\Lambda[\mathbb{R}^n, dx^2, 1^m \text{vol}]} \Lambda[g, v^m \text{vol}],$$

establishing (7.1).

Suppose now that equality holds in (7.1). Then $w^{m+n-2}$ is a minimizer of $\Lambda[g, v^{m+1} \text{vol}]$. In particular, equality holds in (7.8), and hence $m = 0$ or (7.9) holds. By conformal invariance, since $w$ is a volume-normalized minimizer of $\Lambda[g, v^m \text{vol}]$ and $w^{m+n-2}$ is a minimizer of $\Lambda[g, v^{m+1} \text{vol}]$, it follows that the constant function $1$ is a volume-normalized minimizer for the weighted Yamabe constants of $(M^n, \hat{g}, \hat{v}^m \text{vol})$ and $(M^n, \hat{g}, \hat{v}^{m+1} \text{vol})$. Since the integrands appearing in the definition of $x$ are conformally invariant, it follows that if $m > 0$, then (7.3) holds. On the other hand, Proposition 4.1 applied to the minimizer $1$ yields

$$\frac{m+n-2}{4(m+n-1)} R^m_{\phi} = \frac{(2m+n-2)(m+n)\Lambda_m}{n(m+n-2)} \left( \int_M \hat{v}^{m-1} \text{vol}_{\hat{g}} \right)^{- \frac{2m}{n}} - \frac{2m(m+n-1)\Lambda_m}{n(m+n-2)} \left( \int_M \hat{v}^{m-1} \text{vol}_{\hat{g}} \right)^{- \frac{2m+n}{n} - 1} \hat{v}^{-1}$$

$$\frac{m+n-1}{4(m+n)} R^{m+1}_{\phi} = \frac{(2m+n)(m+n+1)\Lambda_{m+1}}{n(m+n-1)} \left( \int_M \hat{v}^{m+1} \text{vol}_{\hat{g}} \right)^{- \frac{2m+n}{n} - 1} - \frac{2(m+1)(m+n)\Lambda_{m+1}}{n(m+n-1)} \left( \int_M \hat{v}^{m+1} \text{vol}_{\hat{g}} \right)^{- \frac{2m+n}{n} - 1} \hat{v}^{-1}$$

for $\Lambda_m := \Lambda[g, v^m \text{vol}]$ and $\Lambda_{m+1} := \Lambda[g, v^{m+1} \text{vol}]$. The identity (7.2) then follows from the general identity

$$\hat{R}^{m+1}_{\phi} = \hat{R}^m - 2\hat{\Delta} \hat{v} - 2m\hat{v}^{-2} |\nabla \hat{v}|_{\hat{g}}^2 = \hat{R}^m - \frac{2}{m+1} \hat{v}^{-m-1} \hat{\Delta} \hat{v}^{m+1},$$

the relationship between $\Lambda_m$ and $\Lambda_{m+1}$ given by the assumption that equality holds in (7.1), and the identity (7.3) when $m > 0$.

As a consequence of Theorem 7.1, we can now prove Theorem 1.3 and Corollary 1.4. It is convenient to prove the two theorems simultaneously.

**Proof of Theorem 1.3 and Corollary 1.4.** To begin, we recall that the case $m = 0$ has already been proven through the work of Yamabe, Trudinger, Aubin, and Schoen (cf. 23), while the case $m = \infty$ has already been proven by Perelman.

Consider now the case $m = 1$. By the resolution of the Yamabe Problem, we know that there exists a smooth positive minimizer of $\Lambda[g]$. If $\Lambda[g] \leq 0$, then Proposition 6.7 implies that $\Lambda[g, v^1 \text{vol}] \leq 0$, and hence, by Theorem 1.2 Corollary 1.4 holds in this case. If instead $\Lambda[g] > 0$, we may apply Theorem 7.1 to conclude that

$$\Lambda[g, v^1 \text{vol}] \leq \Lambda[\mathbb{R}^n, dx^2, 1^1 \text{vol}]$$

with equality if and only if $\Lambda[g] = \Lambda[\mathbb{R}^n, dx^2]$ and equality holds in (7.1) with $m = 0$. In particular, either $\Lambda[g, v^1 \text{vol}] < \Lambda[\mathbb{R}^n, dx^2, 1^1 \text{vol}]$ — and hence Theorem 1.2 yields the desired minimizer — or a positive minimizer $w \in C^\infty(M)$ of $\Lambda[g]$ yields a positive minimizer $w^{\frac{n-1}{n-2}}$ of $\Lambda[g, v^1 \text{vol}]$; in either case, Corollary 1.4 holds. To
prove Theorem 1.3 when \( m = 1 \), note that if equality holds in (7.10), then the work of Aubin [4] and Schoen [27] implies that \((M^n, g)\) is conformally equivalent to the standard \( n \)-sphere, while Theorem 7.1 implies that \((M^n, g, v^1 \text{dvol})\) is conformally equivalent to \((S^n, g_0, v_0^1 \text{dvol})\) for \( v_0 \in C^\infty(M) \) a positive function such that

\[
- \Delta v_0 = n \left( v_0 - \int_{S^n} v_0 \text{dvol} \right),
\]

where we choose \( g_0 \) to have constant sectional curvature one and \( \int v_0 \text{dvol} \) denotes the average of \( v_0 \). However, these are exactly the functions for which \( v_0^{-2} g_0 \) is an Einstein metric on \( S^n \), finishing the proof of Theorem 1.3.

Next, consider the case \( m = 2 \). From the previous paragraph, we know that a positive volume-normalized minimizer \( w \in C^\infty(M) \) of \( \Lambda[g, v^1 \text{dvol}] \) exists. If \( \Lambda[g, v^1 \text{dvol}] \leq 0 \), then Proposition 3.7 implies that \( \Lambda[g, v^2 \text{dvol}] \leq 0 \), and hence, by Theorem 1.2, Corollary 1.4 holds in this case. If instead \( \Lambda[g, v^1 \text{dvol}] > 0 \), we may apply Theorem 7.1 to conclude that

\[
(7.12) \quad \Lambda[g, v^2 \text{dvol}] \leq \Lambda[R^n, dx^2, 1^m \text{dvol}];
\]

moreover, if equality holds, then \( \Lambda[g, v^1 \text{dvol}] = \Lambda[R^n, dx^2, 1^1 \text{dvol}] \) and

\[
(7.13) \quad \left( \int_M w \frac{\Delta w}{n+1} \text{dvol} \right) \left( \int_M w^\frac{n+2}{n-1} v^2 \text{dvol} \right) = \frac{n+2}{n+1}.
\]

From the discussion in the previous paragraph, it follows that if equality holds in (7.12), then \((M^n, g, v^1 \text{dvol})\) is conformally equivalent to \((S^n, g_0, v_0^1 \text{dvol})\) for \( v_0 \in C^\infty(M) \) a positive function such that (7.11) holds. The conformal invariance of the integrals in (7.13) implies that \( v_0 \) satisfies additionally

\[
(7.14) \quad \left( \int_{S^n} \text{dvol}_{g_0} \right) \left( \int_{S^n} v_0^2 \text{dvol}_{g_0} \right) = \frac{n+2}{n+1} \left( \int_{S^n} v_0 \text{dvol}_{g_0} \right)^2.
\]

As a solution to (7.11), we know that \( v_0 = ax + b \) for \( a \) a solution to \(- \Delta x = nx \) with \( \text{sup} x = 1 \). Since \( \int x = 0 \) and \( \int x^2 = 1 \), it follows that (7.11) holds if and only if \( a^2 = b^2 \), which is impossible under the positivity assumption on \( v_0 \). Thus the inequality in (7.12) is strict, yielding Theorem 1.3. By Theorem 1.2 Corollary 1.4 also holds.

Finally, we may apply Theorem 1.2, Proposition 3.7, and Theorem 7.1 inductively as above to deduce that for all \( 3 \leq m \in \mathbb{N} \),

\[
(7.15) \quad \Lambda[g, v^m \text{dvol}] \leq \frac{\Lambda[R^n, dx^2, 1^m \text{dvol}]}{\Lambda[R^n, dx^2, 1^1 \text{dvol}]} \Lambda[g, v^2 \text{dvol}] \leq \Lambda[R^n, dx^2, 1^m \text{dvol}]
\]

and hence Theorem 1.3 and Corollary 1.4 hold. \( \square \)

8. Another interpretation of the weighted Yamabe constant

There are, to the best of the author’s knowledge, now three different proofs of Theorem 1.1. The first proof is the original proof by Del Pino and Dolbeault [18], and takes the PDE approach. The second proof is due to Cordero-Erausquin, Nazaret and Villani [17], and takes an optimal transport approach to the problem. The third proof, the full details of which are not yet published, is due to Bakry (cf.
and is based upon the observation that if \( w \in C^\infty(\mathbb{R}^n) \) is a minimizer \((\ref{1.3})\) of the GNS inequality \((\ref{1.1})\), then there is a constant \( \tau \) such that

\[
 f(x, y) := \left( w^{-\frac{2}{m+n}}(x) + |y|^2/\tau \right)^{\frac{2m+n-2}{2}} \in C^\infty(\mathbb{R}^{n+2m})
\]

is a minimizer for the usual \( L^2 \)-Sobolev inequality.

In this section we consider the curved analogue of Bakry’s approach. Indeed, we will exhibit an explicit relationship between the weighted Yamabe constant of \((M^n, g, v^m\text{dvol})\) and the Yamabe constant of \((M^n \times \mathbb{R}^{2m}, g \oplus v^2\text{d}y^2)\) when \(2m \in \mathbb{N} \cup \{0\}\); for the precise statement, see Theorem \((\ref{8.3})\) below. This gives further evidence for the geometric significance of the family \((\ref{1.1})\) of GNS inequalities (cf. [17], p. 321).

The main part of the computation is contained in the next two lemmas.

**Lemma 8.1.** Fix \( k, l \geq 0, 2m \in \mathbb{N} \), and constants \( a, \tau > 0 \). Then

\[
 \int_{\mathbb{R}^{2m}} \frac{|y|^2}{(a + |y|^2/\tau)^{2m+k}} \text{d}y = \frac{\pi^m \Gamma(m + l) \Gamma(m + k - l) \tau^{m+l}}{\Gamma(2m + k) \Gamma(m + l) a^m \tau^k}
\]

where \( \Gamma(x) \) is Euler’s gamma function.

**Proof.** Both formulae follow immediately by writing the integrals in spherical coordinates and using the facts (see, for example, [1])

\[
 \text{Vol}(S^{2m-1}) = \frac{2\pi^m}{\Gamma(m)} \quad \text{and} \quad \int_0^\infty \frac{t^{\alpha-1}}{(1 + t)^{\alpha+\beta}} dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad \Box
\]

**Lemma 8.2.** Let \((M^n, g, \text{dvol}^m)\) be a compact smooth metric measure space with \(2m \in \mathbb{N} \cup \{0\}\) and let \( w \in C^\infty(M) \) be a positive function. Given any \( \tau > 0 \), define \( f \in C^\infty(M \times \mathbb{R}^{2m}) \) by

\[
 f(x, y) = \left( w^{-\frac{2}{m+n}}(x) + |y|^2/\tau \right)^{\frac{2m+n-2}{2}}
\]

Denote by \( L \) the conformal Laplacian of \((M^n \times \mathbb{R}^{2m}, g \oplus \text{d}y^2)\) and by \( L^m_\phi \) the weighted conformal Laplacian of \((M^n, g, \text{dvol}^m)\), and suppose additionally that \( L^m_\phi \geq 0 \); that is, the first eigenvalue of \( L^m_\phi \) is nonnegative. Then it holds that

\[
 Q(f) \geq C \left( \frac{2m + n - 2}{n(m + n - 2)} Q(w) \right)^{\frac{n}{2m+n}}
\]

for

\[
 C := (2m + n)(2m + n - 2) \left( \frac{\pi^m \Gamma(m + n)}{\Gamma(2m + n)} \right)^{\frac{2}{2m+n}} \left( \frac{2m + n - 2}{(2m + n - 1)} \right)^{\frac{2m}{2m+n}}
\]

Moreover, equality holds in \((\ref{8.3})\) if and only if

\[
 \tau = \frac{n \int w^{\frac{2m+n-1}{2m+n-2}}}{2(L^m_\phi w, w)}
\]

**Proof.** In what follows, all integrals are computed with respect to the Riemannian measure on the specified (product) manifold.

First we observe that, as an immediate consequence of \((\ref{8.1})\)

\[
 \int_{M^n \times \mathbb{R}^{2m}} \frac{\pi^m \tau^n \Gamma(m + n) }{\Gamma(2m + n)} \int_M w^{\frac{2(m+n)}{m+n-2}}
\]
and
\[ \int_{M^n \times \mathbb{R}^{2m}} Rf^2 = \frac{(2m + n - 1)(2m + n - 2)}{(m + n - 1)(m + n - 2)\Gamma(2m + n)} \int_M Rw^2. \]

Next, direct computation shows that
\[ |\nabla f|^2 = \left( \frac{2m + n - 2}{2} \right)^2 (w^{-\frac{2m}{m+n}} + |y|^2 / \tau) \frac{2m+n}{m+n} \left( |\nabla w^{-\frac{m}{m+n}}|^2 + 4|y|^2 / \tau^2 \right). \]

Using (8.1) again, we see that
\[ \int_{M^n \times \mathbb{R}^{2m}} |\nabla f|^2 = \left( \frac{2m + n - 2}{m + n - 2} \right)^2 \frac{\pi^m \tau \Gamma(m + n)}{\Gamma(2m + n)} \int_M |\nabla w|^2 + \frac{m(2m + n - 2)^2 \pi^m \tau^{m-1} \Gamma(m + n)}{(m + n - 1)\Gamma(2m + n)} \int_M w^{\frac{2(m+n-1)}{m+n}}. \]

Combining these equations yields
\[ (L f, f) = \left( \frac{2m + n - 2}{m + n - 2} \right)^2 \frac{\pi^m \tau \Gamma(m + n)}{\Gamma(2m + n)} \tilde{\mathcal{W}}(w, \tau) \]
for
\[ \tilde{\mathcal{W}}(w, \tau) := \tau^{\frac{m}{m+n}} \left( L^m_0 w, w \right) + \frac{m(2m + n - 2)^2}{m + n - 1} \tau^{-\frac{m}{m+n}} \int_M w^{\frac{2(m+n-1)}{m+n}}. \]

Since \( (L^m_0 w, w) \geq 0 \), (8.1), and (8.11) imply that
\[ \tilde{\mathcal{W}}(w, \tau) \geq \frac{2m + n}{2} \left( \frac{2(L^m_0 w, w)}{n} \right)^{\frac{m}{m+n}} \left( \frac{(m + n - 2)^2}{m + n - 1} \int_M w^{\frac{2(m+n-1)}{m+n}} \right)^{\frac{m}{m+n}} \]
with equality if and only if
\[ \tau = \frac{n \int w^{\frac{2(m+n-1)}{m+n}}}{2(L^m_0 w, w)}. \]

Combining (8.5), (8.6), and the above lower bound for \( \tilde{\mathcal{W}}(w, \tau) \) yields the result. □

Using the above computational facts we can establish the following relationship between the weighted Yamabe constant of \((M^n, g, v^m \text{ dvol})\) and the Yamabe constant of \((M^n \times \mathbb{R}^{2m}, g \oplus v^2 \text{ d}x^2)\).

**Theorem 8.3.** Let \((M^n, g, v^m \text{ dvol})\) be a compact smooth metric measure space with nonnegative weighted Yamabe constant \(\Lambda[g, v^m \text{ dvol}]\) and \(2m \in \mathbb{N} \cup \{0\}\). Then
\[ \frac{\Lambda[g, v^m \text{ dvol}]}{\Lambda[\mathbb{R}^n, dx^2, 1^m \text{ dvol}]} \geq \left( \frac{\Lambda[M^n \times \mathbb{R}^{2m}, g \oplus v^2 \text{ d}y^2]}{\Lambda[\mathbb{R}^{n+2m}, dx^2]} \right)^{\frac{2m+n}{m+n}}, \]
where \(dy^2\) denotes the Euclidean metric on \(\mathbb{R}^{2m}\).

In fact, Theorem 8.3 is an immediate consequence of the following more precise result.

**Theorem 8.4.** Let \((M^n, g, v^m \text{ dvol})\) be a compact smooth metric measure space with nonnegative weighted Yamabe constant \(\Lambda[g, v^m \text{ dvol}]\) and \(2m \in \mathbb{N} \cup \{0\}\). Denote by \(\Lambda_p\) the constant
\[ \Lambda_p = \inf \left\{ Q[M^n \times \mathbb{R}^{2m}, v^{-2} g \oplus dy^2, \text{ dvol}, 0](f) : f \text{ is of the form } f_2 \right\}. \]
Then
\[ \frac{\Lambda[g, v^m \text{dvol}]}{\Lambda[\mathbb{R}^n, dx^2, 1^m \text{dvol}]} \geq \left( \frac{\hat{\Lambda}_p}{\Lambda[\mathbb{R}^{n+2m}, dx^2]} \right)^{\frac{2m+n}{n}}. \]
Moreover, if \( w \) is a minimizer of \( \Lambda[g, v^m \text{dvol}] \) and equality holds in the above display, then the function \( f \) defined by (8.2) for \( \tau \) defined by (8.4) is a minimizer of \( \tilde{\Lambda}_p \).

**Proof.** Since both the weighted Yamabe constant and the metric \( v^{-2}g \) are conformal invariants of \( (M^n, g, v^m \text{dvol}) \), it suffices to consider the case \( v = 1 \). Now, let \( w \in C^\infty(M) \), define \( \tau \) by (8.4), and define \( f \in C^\infty(M \times \mathbb{R}^2m) \) by (8.2). By the definition of \( \tilde{\Lambda}_p \) we have that \( Q(f) \geq \tilde{\Lambda}_p \). It then follows from Lemma 8.2 that
\[ Q(w) = C \left( \frac{Q(f)}{(2m+n)(2m+n-2)} \right)^{\frac{2m+n}{n}} \geq C \left( \frac{\hat{\Lambda}_p}{(2m+n)(2m+n-2)} \right)^{\frac{2m+n}{n}} \]
for
\[ C = \frac{n(m+n-2)^2}{2m+n-2} \left( \frac{\Gamma(2m+n)}{\pi^m \Gamma(m+n)} \right)^{\frac{m}{2}} \left( \frac{2(m+n-1)}{2m+n-2} \right)^{\frac{2m}{n}}. \]
The result then follows from the formula (1.2). \( \square \)

**Corollary 8.5.** Minimizers for the weighted Yamabe constant of \( (S^n, g, 1^{1/2} \text{dvol}) \) do not exist.

**Proof.** Using a result of Caffarelli, Gidas and Spruck [8], Schoen [28] proved that the Yamabe constant of \( S^n \times \mathbb{R} \) is equal to the Yamabe constant of \( S^{n+1} \), and the minimizers are precisely the constant multiples of the function \((\cosh t)^{-\frac{n+1}{2}}\) for \( t \) a choice of affine parameter of \( \mathbb{R} \). Theorem 1.2 and Theorem 8.3 thus imply that
\[ \Lambda[g, 1^{1/2} \text{dvol}] = \Lambda[\mathbb{R}^n, dx^2, 1^{1/2} \text{dvol}]. \]
By Theorem 8.3, if there exists a positive minimizer \( w \in C^\infty(S^n) \) of the weighted Yamabe constant of \( (S^n, g, 1^{1/2} \text{dvol}) \), then the function
\[ f(x, t) = \left( w^{-\frac{1}{n-1}}(x) + t^2 / \tau \right)^{-\frac{n-1}{2}} \]
for \( \tau \) as in (8.4) is a minimizer of the Yamabe constant of \( S^n \times \mathbb{R} \), contradicting Schoen’s result. \( \square \)

**Remark 8.6.** Akutagawa, Florit and Petean [2] studied the Yamabe constant of Riemannian products of a compact Riemannian manifold \( (M^n, g) \) with Euclidean space, showing in particular that \( \Lambda[M^n \times \mathbb{R}^{2m}, g \oplus v^2 dy^2] < \Lambda[\mathbb{R}^{n+2m}, dx^2] \) when \( 2m > 1 \) (see [2, Theorem 1.3]). Thus Theorem 8.3 cannot be used to compute the weighted Yamabe constant of any smooth metric measure space except for those which are conformally equivalent to \( (S^n, g, 1^{1/2} \text{dvol}) \).

9. **On the uniqueness of minimizers**

We conclude this article with a discussion of Conjecture 1.6, giving both a possible outline for its proof as well as explaining how its verification would show that minimizers of the weighted Yamabe constant of \( (S^n, g_0, 1^m \text{dvol}) \) do not exist for any \( m \in (0, 1) \).
As stated in the introduction, Conjecture 1.6 arises as a weighted analogue of Obata’s theorem [25], and is also known by the work of Perelman [26] to be true in the case $m = \infty$. Since Perelman’s proof can be realized as a weighted analogue of Obata’s proof (cf. [15]), we expect that a proof of Conjecture 1.6 can be given in that same spirit.

To begin, let us observe that the assumption (1.8) is equivalent to the assumption that $(M^n, g, v^m \text{dvol})$ is a critical point of the total GNS energy functional — that is, that (4.10b) vanishes for suitable choice of $\mu$ — and thus Conjecture 1.6 is indeed a weighted analogue of Obata’s theorem. This fact is a consequence of the following analogue of a result of D.-S. Kim and Y. H. Kim [23].

**Lemma 9.1.** Let $(M^n, g, v^m \text{dvol})$ be a connected smooth metric measure space such that $m > 0$ and

\[
\text{Ric}^m_\phi - \frac{R^m_\phi}{2(m+n-1)} g = \frac{m + n - 2}{2(m+n-1)} \lambda g
\]

for some constant $\lambda \in \mathbb{R}$. Then there is a constant $\mu \in \mathbb{R}$ such that

\[
R^m_\phi + \frac{m(m+n-1)}{m+n-2} \mu v^{-1} = (m + n)\lambda,
\]

holds. In particular, the tensor $\tilde{E}^m_\phi$ defined by (4.10b) vanishes.

Conversely, suppose that there is a constant $\mu \in \mathbb{R}$ such that $\tilde{E}^m_\phi = 0$. Then there is a constant $\lambda \in \mathbb{R}$ such that (9.2), and hence (9.1), holds.

**Proof.** It follows from Proposition 4.5 (see also [13, (5.3a) and (5.6)]) that

\[
\delta_\phi \left( \text{Ric}^m_\phi - \frac{R^m_\phi}{2(m+n-1)} g \right) - \frac{1}{m} \text{tr} \left( \text{Ric}^m_\phi - \frac{R^m_\phi}{2(m+n-1)} g \right) d\phi
\]

\[
= \frac{m + n - 2}{2(m+n-1)} \mu v^{-1} d\left( vR^m_\phi \right).
\]

It follows immediately from (9.3) that if (9.1) holds, then there is a constant $\mu$ such that (9.2) holds. Combining (9.1) and (9.2) then yields $E^m_\phi = \frac{mu^{-1}}{2(m+n)} g$. In the same way, if $E^m_\phi = \frac{mu^{-1}}{2(m+n)} g$ for some constant $\mu$, then (4.11) implies that there exists a constant $\lambda$ such that (9.2), and hence (9.1), holds.

One of the main ingredients in Obata’s proof of Conjecture 1.6 in the case $m = 0$ is the observation that if $(M^n, g)$ is conformally Einstein, then there exists a positive function $u \in C^\infty(M)$ such that the tracefree part of the Hessian of $u$ is proportional to the tracefree part of the Ricci curvature of $g$. The weighted analogue of this fact is the following observation.

**Proposition 9.2.** Let $(M^n, g, v^m \text{dvol})$ be a compact smooth metric measure space with $m > 0$, fix a constant $\mu \in \mathbb{R}$, and let $E^m_\phi$, $\tilde{E}^m_\phi$ be as in (4.10). Suppose additionally that there exists a positive function $u \in C^\infty(M)$ such that the smooth metric measure space

\[
(M^n, \hat{g}, \hat{v}^m \text{dvol}_{\hat{g}}) := \left( M^n, u^{-2}g, u^{-m-n}v^m \text{dvol}_{\hat{g}} \right)
\]

satisfies

\[
\tilde{E}^m_\phi = \frac{m\hat{\mu}^{-1}}{2(m+n)} \hat{g}
\]
for some constant $\hat{\mu} \in \mathbb{R}$. Then the vector field $X = -\frac{m+n-2}{2}\nabla u$ is such that

$$
\frac{1}{2} L_X g + \frac{1}{m} d\phi(X) g = u \left( \tilde{E}_\phi^m + \frac{1}{m} \text{tr} \left( \tilde{E}_\phi^m \right) g \right) + \frac{\mu u - \hat{\mu}}{2v} g.
$$

Remark 9.3. In the case $m = 0$, the correct conclusion is that (9.6) holds after taking the projection onto the tracefree part of both sides of the equality. The case $m = \infty$ is understood by writing $u^m + n - 2 = e^f$ and observing that all quantities make sense in the limit $m = \infty$ when written in terms of $f$.

Proof. Using the formulae in [12, Proposition 4.4], we compute that

$$
\tilde{E}_\phi^m = E_\phi^m + \frac{m+n-2}{u} \left( \nabla^2 u - \frac{1}{m+n} \Delta \phi u g \right).
$$

Since $m > 0$, we compute that

$$
\nabla^2 u + \frac{1}{m} (\nabla u, \nabla \phi) g = \nabla^2 u - \frac{1}{m+n} \Delta \phi u g + \frac{1}{m} \text{tr} \left( \nabla^2 u - \frac{1}{m+n} \Delta \phi u g \right) g.
$$

The result then follows from (9.5). \qed

Corollary 9.4. Let $(M^n, g, v^n \text{dvol})$ be a smooth metric measure space, suppose that there are constants $\lambda, \mu \in \mathbb{R}$ such that (9.2) holds, and denote by $\tilde{E}_\phi^m$ the tensor determined by $\mu$ as in (4.10). Suppose additionally that there exists a positive function $u \in C^\infty(M)$ such that

$$(M^n, \hat{g}, \tilde{v}^n \text{dvol}) := (M^n, u^{-2} g, u^{-m-n} v^m \text{dvol})$$

satisfies

$$
\tilde{E}_\phi^m = \frac{m \hat{\mu} v^{-1}}{2(m+n)} \hat{g}
$$

for some constant $\hat{\mu} \in \mathbb{R}$. Then

$$
\delta_\phi \left( \tilde{E}_\phi^m(X) \right) = u \left[ \left| \tilde{E}_\phi^m \right|^2 + \frac{1}{m} \left( \text{tr} \tilde{E}_\phi^m \right)^2 \right] + \frac{\mu u - \hat{\mu}}{2v} \text{tr} \tilde{E}_\phi^m.
$$

for $X = -\frac{m+n-2}{2}\nabla u$.

Proof. Proposition 4.5 and the assumption (9.2) together imply that

$$
\delta_\phi \tilde{E}_\phi^m = \frac{1}{m} \left( \text{tr} \tilde{E}_\phi^m \right) d\phi.
$$

In particular, it follows that

$$
\delta_\phi \left( \tilde{E}_\phi^m(X) \right) = \left< \tilde{E}_\phi^m, \frac{1}{2} L_X g + \frac{1}{m} X \phi g \right>
$$

for all vector fields $X$ on $M$. Taking $X = -\frac{m+n-2}{2}\nabla u$ and applying Proposition 9.2 thus yields the desired result. \qed

The difficulty in applying Corollary 9.4 to verify Conjecture 1.6 is that there does not seem to be any a priori reason why (the integral of) the last summand of (9.8) should be nonnegative. One way to overcome this difficulty, which will be used in Theorem 9.6 below, is to assume that $\text{tr} \tilde{E}_\phi^m = 0$. Another way, which will be used in Theorem 9.8 below, is to assume $u = v$ and $\hat{\mu} = 0$. However, to get the full statement of Conjecture 1.6 even in these special cases, we need the following characterizations of smooth metric measure spaces admitting at least two solutions to (9.1) in a given conformal class.
Proposition 9.5. Let \((M^n, g, v^m \, \text{dvol}_g)\) and \((M^n, \hat{g}, \hat{v}^m \, \text{dvol}_{\hat{g}})\) be two pointwise conformally equivalent smooth metric measure spaces satisfying (1.8) for constants \(\lambda, \hat{\lambda} \in \mathbb{R}\), respectively. Define \(0 < u \in C^\infty(M)\) such that \(\hat{g} = u^{-2} g\). If \(u\) is nonconstant, then any connected component \(U\) of the set \(\{ p \in M : |\nabla u|(p) \neq 0 \}\) splits isometrically as a warped product \((I \times \Sigma^{n-1}, dt^2 \oplus \psi^2(t)h)\) for \(I\) an open interval and \((\Sigma, h)\) isometric to some level set of \(u\) in \(U\), and moreover there is a constant \(c \in \mathbb{R}\) such that \(u = u(t)\) satisfies

\[
(u')^2 = -\frac{1}{m+n-1} \left( \lambda u^2 - 2cu + \hat{\lambda} \right)
\]

in \(U\). Moreover,

1. if we assume additionally that \(M\) is compact, then \((M^n, g, v^m \, \text{dvol}_g)\) is conformally equivalent to \((S^n, g, 1^m \, \text{dvol}_g)\) and \(m \in \{0, 1\}\);

2. if we assume additionally that \((M^n, g)\) is complete, \(\lambda = 0\), and \(\hat{\lambda} > 0\), then \(c > 0\) and there exists a point \(x_0 \in \mathbb{R}^n\) such that \(u(x) = \frac{c}{2(m+n-1)}|x - x_0|^2 + \frac{\hat{\lambda}}{2c}\).

Proof. For convenience, denote by \(P^m_\phi\) the tensor

\[P^m_\phi := \text{Ric}^m_\phi - \frac{R^m_\phi}{2(m+n-1)} g.\]

From [12] Proposition 4.4] we compute that

\[\hat{P}^m_\phi = P^m_\phi + (m+n-2)u^{-1} \left( \nabla^2 u - \frac{1}{2} u^{-1} |\nabla u|^2 \right) g.\]

Since \((M^n, g, v^m \, \text{dvol}_g)\) and \((M^n, \hat{g}, \hat{v}^m \, \text{dvol}_{\hat{g}})\) both satisfy (1.8), it follows that

\[
u^{-1} \nabla^2 u - \frac{1}{2} u^{-2} |\nabla u|^2 g = \frac{1}{2(m+n-1)} \left( \hat{\lambda} u^{-2} - \lambda \right) g.
\]

Suppose that \(u\) is nonconstant. Since \(\nabla^2 u\) is a multiple of the metric, this implies that the set \(U = \{ p \in M : |\nabla u|(p) \neq 0 \}\) splits isometrically as a warped product \((I \times \Sigma^{n-1}, dt^2 \oplus \psi^2(t)h)\) for \(I\) an open interval and \((\Sigma, h)\) some level set of \(u\) in \(U\). Moreover, necessarily \(u = u(t)\) and \(\psi = ku'(t)\) for some constant \(k > 0\) (cf. [7]). Inserting this into (9.10) yields

\[
u^{-1} u'' - \frac{1}{2} u^{-2} (u')^2 = \frac{1}{2(m+n-1)} \left( \hat{\lambda} u^{-2} - \lambda \right).
\]

This can be integrated, yielding a constant \(c \in \mathbb{R}\) such that (9.9) holds.

Now suppose that \(M\) is compact and \(u\) is nonconstant. It follows from (1.8) and symmetry considerations that \(\lambda, \hat{\lambda} > 0\) and \(c^2 > \lambda \hat{\lambda}\), and moreover that, after adding a constant to \(t\) if necessary,

\[
u(t) = \frac{\sqrt{c^2 - \lambda \hat{\lambda}}}{\hat{\lambda}} \cos \left( \sqrt{\frac{\lambda}{m+n-1}} t \right) + \frac{c}{\hat{\lambda}}.
\]

It thus follows from the splitting of \(U\) that \((M^n, g)\) is isometric to \((S^n, g_0)\). A straightforward computation shows that \((S^n, g_0, v^m \, \text{dvol})\) satisfies (1.8) with \(\lambda > 0\) and \(v > 0\) if and only if either \(m = 0\) or \(m = 1\) and \(v(p) = a + b \cos (d(p, q))\) for some fixed point \(q \in S^n\) and constants \(a > b > 0\). But then \(v^{-2}g_0\) is again an Einstein metric on \(S^n\), as desired.
Theorem 9.6. Let \((M^n, g)\) be a compact smooth metric measure space such that (9.2) holds for constants \(\lambda, \mu \in \mathbb{R}\) and such that \((1, v)\) is a critical point of the map \((\xi, \nu) \mapsto Q[g, v^m \text{dvol}] \phi\). Suppose additionally that there exists a positive function \(u \in C^\infty(M)\) such that

\[
\left( M^n, \tilde{g}, \tilde{v}^m \text{dvol} \right) := \left( M^n, u^{-2}g, u^{-m-n}v^m \text{dvol} \right)
\]

satisfies (1.8) for some constant \(\tilde{\lambda} \in \mathbb{R}\). Then \((M^n, g, v^m \text{dvol})\) satisfies (1.8).

Moreover, either \(u\) is constant or \(m \in \{0, 1\}\) and \((M^n, g, v^m \text{dvol})\) is conformally equivalent to \((S^n, g_0, 1^m \text{dvol})\).

Remark 9.7. Theorem 9.6 equivalently states that Conjecture 1.6 is true under the additional assumption that \((w, v)\) is a critical point of the map \((\xi, \nu) \mapsto Q[g, v^m \text{dvol}] \phi\).

Proof. Theorem 1.4 and the assumption that \((1, v)\) is a critical point of the map \((\xi, \nu) \mapsto Q[g, v^m \text{dvol}] \phi\) together imply that the tensor \(\tilde{E}^m_\phi\) defined in terms of \((M^n, g, v^m \text{dvol})\) and \(\mu\) is tracefree. By Corollary 9.4 it follows that

\[
\int_M u \left[ \tilde{E}^m_\phi \right]^2 + \frac{1}{m} \left( \text{tr} \tilde{E}^m_\phi \right)^2 = 0.
\]

Hence \(\tilde{E}^m_\phi\) vanishes identically, and the result then follows from Lemma 9.1 and Proposition 9.5.

Theorem 9.8. Fix \(m \in [0, \infty)\) and consider \((\mathbb{R}^n, dx^2, 1^m \text{dvol})\) as a smooth metric measure space. Let \(w \in C^\infty(M)\) be a positive critical point of the weighted Yamabe quotient, and suppose additionally that the functional

\[
\tilde{w}(x) := |x|^{2-m-n}w \left( \frac{x}{|x|^2} \right)
\]

admits an extension to a positive element of \(C^2(\mathbb{R}^n)\). Then there exist constants \(a, b > 0\) and a point \(x_0 \in \mathbb{R}^n\) such that

\[
\tilde{w}(x) = (a + b|x - x_0|^2)^{-\frac{m+n-2}{2}}.
\]

Proof. By Theorem 1.4, the smooth metric measure space

\[
(S^n \setminus \{p\}, g, v^m \text{dvol}_g) := \left( \mathbb{R}^n, \frac{1}{|x|^{m+n}} \text{d}x^2, \frac{1}{|x|^{m+n}} 1^m \text{dvol}_{dx^2} \right)
\]

is such that (9.2) holds for constants \(\mu, \lambda \in \mathbb{R}\), and moreover that the tensor \(\tilde{E}^m_\phi\) defined in terms of \(\mu\) by (4.10) satisfies

\[
\int_{S^n \setminus \{p\}} \text{tr} \left( \tilde{E}^m_\phi \right) = 0.
\]

Since \(\tilde{\mu} = 0\) and \(v = u\), Corollary 9.4 yields

\[
\delta_\phi \left( \tilde{E}^m_\phi(X) \right) = u \left[ \tilde{E}^m_\phi \right]^2 + \frac{1}{m} \left( \text{tr} \tilde{E}^m_\phi \right)^2 + \frac{\mu}{2} \text{tr} \tilde{E}^m_\phi.
\]
In particular, (9.12) and (9.13) imply that

\[
\int_{S^n \setminus \{p\}} \delta_\phi \left( \overline{E^m}_\phi(X) \right) = \int_{S^n \setminus \{p\}} \left[ \left| \overline{E^m}_\phi \right|^2 + \frac{1}{m} \left( \text{tr} \overline{E^m}_\phi \right)^2 \right] u.
\]

We claim that the left hand side vanishes. Indeed, suppose that the metric \( g \) and the function \( v \) defined by (9.11) can be extended to a \( C^2 \) metric and function, respectively, on \( S^n \); note that \( v(p) = 0 \). In particular, \( |\overline{E^m}_\phi| \leq C v^{-2} \) for some constant \( C > 0 \). Since

\[
\int_{\{v > \varepsilon\}} \delta_\phi \left( \overline{E^m}_\phi(X) \right) = \int_{v^{-1}(\varepsilon)} \overline{E^m}_\phi(\nabla v, \nabla v)|\nabla v|^{-1} v^m \text{dvol}
\]

for any regular value \( \varepsilon > 0 \) of \( v \), taking \( \varepsilon \to 0 \) yields the claim.

Finally, let us verify that the metric \( g \) and the function \( v \) defined by (9.11) admit \( C^2 \) extensions to \( S^n \). To that end, observe that \( u := w^{-\frac{n-1}{2}} \) has the property that \( \hat{u}(x) := |x|^2 u\left(\frac{x}{|x|^2}\right) \) can be extended to a positive \( C^2 \) function on \( \mathbb{R}^n \) if and only if \( \hat{w} \) can be extended to a positive \( C^2 \) function on \( \mathbb{R}^n \). Fix \( p \in S^n \) and denote by \( r(\cdot) = d(p, \cdot) \) the distance from \( p \) in \( S^n \). Stereographic projection from \( p \) yields the isometry

\[
(S^n \setminus \{p\}, g_0, (1 - \cos r)^m \text{dvol}_{g_0}) = (\mathbb{R}^n, u_0^{-2} dx^2, u_0^{-m-n} m_{\text{dvol}}dx^2).
\]

for \( u_0(x) := \frac{1+|x|^2}{2} \). Using this observation, it is well-known (see, for example, [31]) that \( g \) admits a \( C^2 \) extension to \( S^n \) if and only if \( \hat{u} \) admits a \( C^2 \) extension to \( \mathbb{R}^n \), and the same argument implies that \( v \) admits a \( C^2 \) extension to \( S^n \) if and only if \( \hat{w} \) admits a \( C^2 \) extension to \( S^n \).

Finally, we observe the following role of Conjecture 1.6 in the question of the existence of minimizers of the weighted Yamabe functional.

**Theorem 9.9.** Suppose that Conjecture 1.6 is true. Then given any \( m \in (0, 1) \), there does not exist a positive smooth minimizer of the weighted Yamabe quotient of \( (S^n, g_0, 1^m \text{dvol}) \).

**Proof.** Suppose to the contrary that there does exist a positive smooth minimizer \( w \) of \( \Lambda[S^n, g_0, 1^m \text{dvol}] \). Under the assumption that Conjecture 1.6 holds, it follows that \( w \) is constant. Thus

\[
\Lambda[S^n, g_0, 1^m \text{dvol}] = \mathcal{Q}(1) = \frac{n(n-1)(m+n-2)\pi}{m+n-1} \left( \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right)^\frac{2m}{n}.
\]

In particular, if we denote \( \Lambda = \Lambda[S^n, g_0, 1^m \text{dvol}] \) and \( \Lambda_0 = \Lambda[\mathbb{R}^n, dx^2, 1^m \text{dvol}] \), we see that

\[
\frac{\Lambda}{\Lambda_0} = \frac{\frac{(n-1)(2m+n-2)}{(m+n-1)(m+n-2)} \left( \frac{2m+n-2}{m+n-1} \right)^\frac{2m}{n} \left( \frac{\Gamma(m+n)\Gamma\left(\frac{n}{2}\right)}{\Gamma(2m+n)\Gamma(n)} \right)^\frac{2m}{n}}.
\]

It follows from Lemma 9.10 below that the right hand side is strictly greater than one when \( m \in (0, 1) \), contradicting Theorem 1.2.

**Lemma 9.10.** Define \( F \colon [0, \infty) \times (2, \infty) \to \mathbb{R} \) by

\[
F(m, n) = \frac{(n-1)(2m+n-2)}{(m+n-1)(m+n-2)} \left( \frac{2m+n-2}{2m+n-1} \right)^\frac{2m}{n} \left( \frac{\Gamma(m+n)\Gamma\left(\frac{n}{2}\right)}{\Gamma(2m+n)\Gamma(n)} \right)^\frac{2m}{n}.
\]

Then
(1) \( F(0, n) = F(1, n) = 1 \) for all \( n > 2 \),
(2) \( F(m, n) > 1 \) for all \( m \in (0, 1), \ n > 2 \), and
(3) \( F(m, n) < 1 \) for all \( m > 1, \ n > 2 \).

**Proof.** It is clear by direct computation that \( F(0, n) = F(1, n) = 1 \) for all \( n \). We first show that the conclusion follows for all \( n \) sufficiently large. To that end, recall Stirling’s approximation

\[
\log \Gamma(n) = x \log x - x - \frac{1}{2} \log \frac{x}{2\pi} + \frac{1}{12x} + O(x^{-3})
\]

for \( x \) large \([1]\). Fix \( m \in [0, \infty) \). From (9.15) we see that

\[
\log \frac{\Gamma(m + n)\Gamma(\frac{n}{2})}{\Gamma(\frac{m+n}{2}+\frac{n}{2})\Gamma(n)} = m \log \frac{2(m + n - 1)}{2m + n - 2} + \frac{n}{2} \log \frac{(m + n - 1)^2}{n(2m + n - 2)}
\]

\[
- \frac{1}{2} \log \frac{m + n - 1}{2m + n - 2} + \frac{m - 1}{4(m + n - 1)(2m + n - 2)} + O(n^{-3}),
\]

while it follows from the Maclaurin expansion of \( \log(1 + x) \) that

\[
\log \frac{(n - 1)(2m + n - 2)}{(m + n - 1)(m + n - 2)} = - \frac{m(m - 1)}{(m + n - 1)(m + n - 2)} + O(n^{-4}).
\]

Combining these two expressions yields

\[
\log F(m, n) = - \frac{m(m - 1)}{2n^3} + O(n^{-4}),
\]

thus establishing the claim for \( n \) sufficiently large.

Now set \( H(m, n) = F(m, n)^{n/2} \). Clearly \( H(m, n) \) is greater than (resp. less than) one if and only if \( F(m, n) \) is greater than (resp. less than) one. A straightforward computation shows that

\[
\log \frac{H(m, n + 2)}{H(m, n)} = \frac{n}{2} \log \left( \frac{(n + 1)(2m + n)(m + n - 1)(m + n - 2)}{(m + n + 1)(m + n)(n - 1)(2m + n - 2)} \right)
\]

\[
+ m \log \left( \frac{2m + n}{m + n + 1} \right).
\]

Using the estimate \( \log(1 + x) \leq x \) for all \( x > 0 \), we compute from (9.16) that

\[
\log \frac{H(m, n + 2)}{H(m, n)} \leq \frac{m(m - 1)(2m + n)}{(m + n)(m + n + 1)(n - 1)(2m + n - 2)}.
\]

In particular, if \( m \in (0, 1) \) we have that \( H(m, n) \geq H(m, n + 2) \). Iterating this shows that \( H(m, n) \geq H(m, n + 2k) \) for \( k \) sufficiently large, so that the previous paragraph yields \( H(m, n) > 1 \) for all \( m \in (0, 1), \ n > 2 \).

Using instead the estimate \( \log(x) \geq \frac{3x}{4x^2} \) for all \( x > 0 \), we compute from (9.16) that

\[
\log \frac{H(m, n + 2)}{H(m, n)} \geq \frac{m(m - 1)(-2m + n + 4)}{(2m + n)(m + n - 1)(n + 1)(m + n - 2)}.
\]

In particular, if \( m \in (1, 2) \) we have that \( H(m, n) \leq H(m, n + 2) \). Iterating this and using the conclusion of the first paragraph yields \( H(m, n) < 1 \) for all \( m \in (1, 2], \ n > 2 \).

Finally, we compute that

\[
\frac{H(m + 1, n)}{H(m, n)} = \left( \frac{2m + n}{2m + n - 2} \right)^{\frac{m}{2}} \left( \frac{2m + n}{m + n - 1} \right)^m.
\]
Taking the logarithm of both sides and using again the estimate \( \log(1+x) \leq x \) yields \( H(m+1,n) \leq H(m,n) \) for all \( m > 0, n > 2 \). In particular, \( H(m,n) \geq H(x(m),n) \) for all \( m > 1 \), where \( x(m) \) is the unique element of \( [1,2] \) such that \( m - x(m) \in \mathbb{Z} \).

The result then follows from the previous paragraph. \( \square \)

**Remark 9.11.** Lemma 9.10 and the computation from the proof of Theorem 9.9 show that \( \Lambda[S^n,\gamma_0,1^m \mathrm{dvol}] < \Lambda[\mathbb{R}^n,dx^2,1^m \mathrm{dvol}] \) for all \( m > 1 \), which is the reason we expect that the weighted Yamabe problem is solvable for all \( m \geq 1 \).

**References**

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of National Bureau of Standards Applied Mathematics Series. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] K. Akutagawa, L. A. Florit, and J. Petean. On Yamabe constants of Riemannian products. *Comm. Anal. Geom.*, 15(5):947–969, 2007.

[3] K. Akutagawa, M. Ishida, and C. LeBrun. Perelman’s invariant, Ricci flow, and the Yamabe invariants of smooth manifolds. *Arch. Math. (Basel)*, 88(1):71–76, 2007.

[4] T. Aubin. Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)*, 55(3):269–296, 1976.

[5] T. Aubin. Problèmes isopérimétroiques et espaces de Sobolev. *J. Differential Geometry*, 11(4):573–598, 1976.

[6] D. Bakry, Y. Gentil, and M. Ledoux. *Analysis and geometry of diffusion semigroups*. Monograph in preparation.

[7] H. W. Brinkmann. Einstein spaces which are mapped conformally on each other. *Math. Ann.*, 94(1):119–145, 1925.

[8] L. A. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3):271–297, 1989.

[9] E. Carlen and A. Figalli. Stability for a GNS inequality and the Log-HLS inequality, with application to the critical mass Keller-Segel equation. arXiv:1107.5976, preprint.

[10] E. A. Carlen, J. A. Carrillo, and M. Loss. Hardy-Littlewood-Sobolev inequalities via fast diffusion flows. *Proc. Natl. Acad. Sci. USA*, 107(46):19696–19701, 2010.

[11] E. A. Carlen and M. Loss. Sharp constant in Nash’s inequality. *Internat. Math. Res. Notices*, (7):213–215, 1993.

[12] J. S. Case. Smooth metric measure spaces and quasi-Einstein metrics. *Internat. J. Math.*, 23(10):1250110, 36 pp., 2012.

[13] J. S. Case. Smooth metric measure spaces, quasi-Einstein metrics, and tractors. *Cent. Eur. J. Math.*, 10(5):1733–1762, 2012.

[14] J. S. Case. Conformal invariants measuring the best constants for Gagliardo-Nirenberg-Sobolev inequalities. *Calc. Var. Partial Differential Equations*, to appear.

[15] J. S. Case. The energy of a smooth metric measure space and applications. arXiv:1011.2728, preprint.

[16] S.-Y. A. Chang. Conformal invariants and partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 42(3):365–393, 2005.

[17] D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.*, 182(2):307–332, 2004.

[18] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875, 2002.

[19] M. Feldman, T. Ilmanen, and L. Ni. Entropy and reduced distance for Ricci expanders. *J. Geom. Anal.*, 15(1):49–62, 2005.

[20] M. Gromov and H. B. Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

[21] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4):1061–1083, 1975.
[22] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999.

[23] D.-S. Kim and Y. H. Kim. Compact Einstein warped product spaces with nonpositive scalar curvature. *Proc. Amer. Math. Soc.*, 131(8):2573–2576 (electronic), 2003.

[24] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[25] M. Obata. The conjectures on conformal transformations of Riemannian manifolds. *J. Differential Geometry*, 6:247–258, 1971/72.

[26] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:0211159, preprint.

[27] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

[28] R. M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.

[29] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

[30] N. S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:265–274, 1968.

[31] J. A. Viaclovsky. Conformally invariant Monge-Ampère equations: global solutions. *Trans. Amer. Math. Soc.*, 352(9):4371–4379, 2000.

[32] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

[33] Q. S. Zhang. Extremal of log Sobolev inequality and W entropy on noncompact manifolds. *J. Funct. Anal.*, 263(7):2051–2101, 2012.

Department of Mathematics, Princeton University, Princeton, NJ 08540

E-mail address: jscase@math.princeton.edu