Number representation using generalized \((-\beta)-\text{transformation}\)

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Abstract

We study non-standard number systems with negative base \(-\beta\). Instead of the Ito-Sadahiro definition, based on the transformation \(T_{-\beta}\) of the interval \([-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})\) into itself, we suggest a generalization using an interval \([l, l+1)\) with \(l \in (-1, 0]\). Such generalization may eliminate certain disadvantages of the Ito-Sadahiro system. We focus on the description of admissible digit strings and their periodicity.

Keywords: numeration system, negative base

1. Introduction

In 2008, Ito and Sadahiro [3], introduced a numeration system with negative base \(-\beta\), where \(\beta > 1\). It has been shown that every number \(x \in I = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right)\) can be written in the form

\[
x = \sum_{i=1}^{\infty} \frac{x_i}{(-\beta)^i}, \quad x_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\}.
\]

The string of digits \(d(x) = x_1x_2x_3\cdots\), which we call here Ito-Sadahiro expansion of \(x\), can be obtained by the transformation \(T : I \mapsto I\) defined by

\[
T(x) = -\beta x - \left[ -\beta x + \frac{\beta}{\beta+1} \right].
\]

The digits of the string \(d(x)\) are then given by

\[
x_i = \left[ -\beta T^{i-1}(x) + \frac{\beta}{\beta+1} \right],
\]

i.e. \(x_i\) lie in the alphabet \(A = \{0, 1, \ldots, \lfloor \beta \rfloor\}\).

The natural ordering of reals in the interval \(I\) is preserved by Ito-Sadahiro expansion, when considering the alternate order \(\preceq_{\text{alt}}\) on the set of digit strings over an ordered alphabet \(A\). We say that \(x_1x_2x_3\cdots \preceq_{\text{alt}} y_1y_2y_3\cdots\) if the first non-zero element of the sequence \((-1)^i(y_i - x_i), i \geq 1\), is positive.

The Ito-Sadahiro numeration system shares many properties of numeration with positive (in general non-integer) base considered by Rényi [11] and then by many others from different points of view. The main advantage of considering negative base is the possibility of representing every real number with the same set of digits without...
the use of sign. On the other hand, many aspects of number system with negative base are different and essentially more complicated, as an example, let us mention the structure of \((-\beta)\)-integers which is studied in [1] and [13].

Similarly as in case of number representation using a positive base, not every string of digits from the alphabet \(A = \{0, 1, \ldots, [\beta]\}\) appears as the expansion \(d(x)\) of some \(x \in I\). Ito and Sadahiro [3] have shown that a necessary and sufficient condition so that \(x_1x_2x_3 \cdots\) be equal to \(d(x)\) for some \(x\) is that

\[
d\left(\frac{-\beta}{\beta+1}\right) \leq_{\text{alt}} x_ix_{i+1}x_{i+2} \cdots \leq_{\text{alt}} d^*\left(\frac{1}{\beta+1}\right) \quad \text{for all } i = 1, 2, 3, \ldots, \tag{1}
\]

where

\[
d^*\left(\frac{1}{\beta+1}\right) = \lim_{\varepsilon \to 0+} d\left(\frac{1}{\beta+1} - \varepsilon\right).
\]

The upper and lower bounds deciding about admissibility of a digit string \(x_1x_2x_3 \cdots\) as the Ito-Sadahiro expansion are closely related. If the lower bound is a purely periodic digit string with odd period-length, i.e. \(d\left(\frac{-\beta}{\beta+1}\right) = (d_1d_2 \cdots d_{2q+1})^\omega\), where \(w^\omega\) stands for infinite repetition of the string \(w\), then \(d^*\left(\frac{1}{\beta+1}\right) = (0d_1d_2 \cdots d_{2q}(d_{2q+1} - 1))^\omega\). Otherwise, \(d^*\left(\frac{1}{\beta+1}\right) = 0d\left(\frac{-\beta}{\beta+1}\right)\) such a close relation of \(d\left(\frac{-\beta}{\beta+1}\right)\) and \(d^*\left(\frac{1}{\beta+1}\right)\) allowed to show that the \((-\beta)\)-shift is sofic if and only if the lower bound is eventually periodic [3].

Frougny and Lai [4] have shown that for \(\beta\) Pisot number every element of \(\mathbb{Q}(\beta) \cap I\) has eventually periodic Ito-Sadahiro expansion, and so for such \(\beta\), the \((-\beta)\)-shift is sofic.

Using base \(-\beta\), Ito and Sadahiro have represented only numbers within the interval \(I\). Since for every \(x \in \mathbb{R}\) there exists \(k \in \mathbb{Z}\) such that \(\frac{x}{(-\beta)^k} \in I\), every \(x \in \mathbb{R}\) can be written in the form

\[
x = x_k(-\beta)^k + x_{k-1}(-\beta)^{k-1} + x_{k-2}(-\beta)^{k-2} + \cdots,
\]

where any suffix of the digit string \(x_kx_{k-1}x_{k-2} \cdots\) verifies the condition (1). The use of negative base thus allows one to represent every real number using the same set of digits \(\{0, 1, \ldots, [\beta]\}\). A disadvantage is that the representation of \(x\) has no obvious relation to the representation of \(-x\). More serious from the practical point of view is, however, the fact that the digit strings for representation of \(x\) and \(\frac{x}{(-\beta)^k}\) can substantially differ (see discussion in Example 5). Recall that in a number system with positive base \(\alpha\), the representation of \(\frac{x}{\alpha}\) is always obtained from the representation of \(x\) by shifting the fractional point. Among the disadvantages from the arithmetical point of view, we can mention that for some bases \(-\beta\), the number zero is the only element of \(I\) for which the Ito-Sadahiro expansion has only finitely many non-zero digits. In the positive base number systems such phenomenon does not occur.

In their paper, Ito and Sadahiro do not explain the reasons for choosing for representation the transformation \(T\) with domain \(I = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right]\). The aim of this paper is to discuss the influence of the choice of the domain on the properties of the resulting number system. Similar study for the positive base with focus on the dynamical aspects and tilings has been performed in [7].

We consider the following generalization of the Ito-Sadahiro expansion.
Definition 1. For a given real number $\beta > 1$ and $l \in (-1,0]$ we define the mapping $T : I \mapsto I$ with domain $I := [l,l+1)$, by the prescription

$$T(x) := -\beta x - \lfloor -\beta x - l \rfloor, \quad \text{for } x \in I. \quad (2)$$

The mapping $T$ will be called the $(-\beta,l)$-transformation. To every $x \in I$ we associate an infinite string of integer digits $d(x) = x_1x_2x_3\cdots$ by

$$x_i := \lfloor -\beta T^{i-1}(x) - l \rfloor, \quad \text{for any } i = 1, 2, 3, \ldots \quad (3)$$

The mapping $d : I \mapsto \mathbb{Z}^\mathbb{N}$ will be called the $(-\beta,l)$-expansion. The string $d(x)$ is the $(-\beta,l)$-expansion of $x \in I$.

The paper is organized in the following way. In Section 2 we recall the properties of number systems with positive real base. In Section 3 we study number representation in a system with negative base from a general point of view, based on theorem of Thurston [14], see Theorem 3. We explain in what sense our choice of transformation $T$ is general. In section 4 we focus on the properties of the $(-\beta,l)$-transformation and study how the choice of its domain influences certain aspects of the corresponding number system. In Section 5 we characterize for fixed $\beta$ and $l$ the family of strings in $\mathbb{Z}^\mathbb{N}$ which are $(-\beta,l)$-expansions of some $x$. Analogously to the case of Ito-Sadahiro expansions, we use the alternate order on $\mathbb{Z}^\mathbb{N}$ and two reference digit strings denoted by $d(l)$ and $d^*(r)$. In Section 6 we show which property of the transformation $T$ connects these two strings. We also describe some of their properties and discuss the question which digit strings may play the role of the reference strings $d(l)$ and $d^*(r)$ for some $\beta$ and $l$. Finally, in Section 7 we prove that periodicity of $(-\beta,l)$-expansions relates to the notion of Pisot and Salem numbers, as it is the case of positive base number systems.

2. Representing reals using a positive base

Let us briefly recall several facts about numeration in positive base number systems for which we try to find analogues in the case of systems with negative base associated with $(-\beta,l)$-transformation. For a real number $\beta > 1$, Rényi in [11] has considered the transformation $T_\beta(x) := \beta x - \lfloor \beta x \rfloor$ of the interval $[0,1)$. One can represent every number $x \in [0,1)$ in the form

$$x = \sum_{i=1}^\infty \frac{x_i}{\beta^i}, \quad x_i \in \{0,1,\ldots,[\beta] - 1\}.$$

where the digits $x_i$, $i \geq 1$ are obtained by

$$x_i = \left\lfloor \beta T_\beta^{i-1}(x) \right\rfloor.$$

The string of these digits is denoted $d_\beta(x) = x_1x_2x_3\cdots$ and called the $\beta$-expansion of $x$. One can define the $\beta$-expansion of every positive real number $x$ by dividing $x$ by a suitable power of $\beta$ so that $\frac{x}{\beta^k} \in [0,1)$, finding the $\beta$-expansion $d_\beta\left(\frac{x}{\beta^k}\right)$ and shifting the fractional point.
Not every infinite string with digits in \( \{0, 1, \ldots, \lfloor \beta \rfloor \} \) is admissible as a \( \beta \)-expansion of some \( x \in [0, 1) \). The characterization of the admissible digit strings uses the lexicographic order \( \leq_{\text{lex}} \) and the infinite Rényi \( \beta \)-expansion of 1, namely the string

\[
d^{\varepsilon}_{\beta}(1) = \lim_{\varepsilon \to 0^+} d_{\beta}(1 - \varepsilon).
\]

Here we make use of the fact that the space \( A^\mathbb{N} \) of infinite words over any finite alphabet \( A \) is compact with respect to the product topology. The limit of a sequence of digit strings which have longer and longer common prefixes thus can be defined. The necessary and sufficient condition for admissibility of digit strings was formulated by Parry [10].

**Theorem 2** (Parry). Let \( \beta > 1 \). The string \( x_1 x_2 x_3 \cdots \) of integer digits is a \( \beta \)-expansion of some \( x \in [0, 1) \) if and only if every suffix \( x_i x_{i+1} x_{i+2} \cdots \) of \( x_1 x_2 x_3 \cdots \) satisfies

\[
0^\omega \preceq_{\text{lex}} x_i x_{i+1} x_{i+2} \cdots \preceq_{\text{lex}} d^{\varepsilon}_{\beta}(1).
\]

(4)

As a consequence of the above theorem, Parry has also given a criterion for sequences of digit strings which can play role of the Rényi \( \beta \)-expansion of 1 for some \( \beta \). In fact, a sequence of integers \( x_1 x_2 x_3 \cdots \) is equal to \( d^{\varepsilon}_{\beta}(1) \) for some \( \beta > 1 \) if and only if every suffix \( x_i x_{i+1} x_{i+2} \cdots \) satisfies

\[
0^\omega \preceq_{\text{lex}} x_i x_{i+1} x_{i+2} \cdots \preceq_{\text{lex}} x_1 x_2 x_3 \cdots.
\]

For a fixed \( \beta > 1 \), the \( \beta \)-shift is the closure of the set of infinite sequences appearing as \( \beta \)-expansions of numbers in \([0, 1)\), which is a shift-invariant subspace of the space of all infinite sequences over the same alphabet. Such a dynamical system is sofic, if the set of its finite factors is recognized by a finite automaton. This is equivalent to saying that the reference string \( d^{\varepsilon}_{\beta}(1) \), which is plays a crucial role in Theorem 2, is eventually periodic. The base \( \beta \) leading to a sofic \( \beta \)-shift is called a Parry number. Here, the number 1 is expressed as a power series \( 1 = \sum_{i=1}^{\infty} \frac{x_i}{\beta^i} \), where the integer coefficients \( x_i \) form an eventually periodic sequence, then \( \beta \) is a root of a monic polynomial in \( \mathbb{Z}[X] \), and so every Parry number is an algebraic integer.

When studying the positive base number systems from the arithmetical point of view, one is mainly interested in the set \( \text{Fin}(\beta) \) of numbers with finite \( \beta \)-expansions, in particular, in the set of numbers \( x \in [0, 1) \) such that \( d_{\beta}(x) \) ends in \( 0^\omega \). This set is always infinite. One can also study which numbers have eventually periodic expansion \( d_{\beta}(x) \). Two classes of algebraic numbers prove to be of particular interest, namely Pisot and Salem numbers. Pisot numbers are algebraic integers \( > 1 \) whose all conjugates are in modulus \( < 1 \), Salem numbers are algebraic integers whose all conjugates are in modulus \( \leq 1 \), with at least one unimodular conjugate. Being a Pisot number turns out to be a necessary condition for the base so that the set \( \text{Fin}(\beta) \) has a ring structure [5], it allows to construct fractal tilings [2], etc. It is also known [10] that all Pisot numbers are Parry numbers. The question whether Salem numbers are Parry numbers has been only solved for Salem numbers of small degree. Schmidt [12] shows that Pisot numbers share in some sense the properties of natural numbers, since for Pisot bases \( \beta \) the set of numbers in \([0, 1)\) with eventually periodic \( \beta \)-expansions is equal to \( \mathbb{Q}(\beta) \cap [0, 1) \). On the other hand, if every \( x \in \mathbb{Q}(\beta) \cap [0, 1) \) has eventually periodic \( d_{\beta}(x) \) then \( \beta \) is either Pisot or Salem.
3. Representing reals using a negative base

Let us approach the question of representing numbers using powers of a basis from a more general point of view. Thurston [14] stated the following simple theorem.

**Theorem 3 (Thurston).** Given \( \alpha \in \mathbb{C}, \ |\alpha| > 1 \), a finite alphabet \( \mathcal{A} \subset \mathbb{C} \) and a bounded set \( V \subset \mathbb{C} \) such that

\[
\alpha V \subset \bigcup_{a \in \mathcal{A}} (V + a).
\] (5)

Then for every \( z \in V \) there exists a sequence \( a_1a_2a_3\cdots \in \mathcal{A}^\mathbb{N} \) such that

\[
z = \sum_{i=1}^{\infty} \frac{a_i}{\alpha^i}.
\] (6)

If, moreover, the point 0 lies in the interior of \( V \), then every \( z \in \mathbb{C} \) can be written in the form

\[
z = b_k\alpha^k + b_{k-1}\alpha^{k-1} + b_{k-2}\alpha^{k-2} + \cdots \quad \text{for some } k \in \mathbb{Z} \text{ and } b_i \in \mathcal{A}.
\]

The proof of this theorem is simple and constructive, we give it here for the purpose of discussion of uniqueness of the sequence \( a_1a_2a_3\cdots \) dependently on the choice of the set \( V \) and the alphabet \( \mathcal{A} \).

**Proof.** Consider \( z \in V \). According to (5), there exist \( z_1 \in V \) and \( a_1 \in \mathcal{A} \) such that

\[
\alpha z = z_1 + a_1, \quad \text{i.e. } z = \frac{a_1}{\alpha} + \frac{z_1}{\alpha}.
\]

Again, by (5), there exist \( z_2 \in V \) and \( a_2 \in \mathcal{A} \) such that

\[
\alpha z_1 = z_2 + a_2, \quad \text{i.e. } z = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \frac{z_2}{\alpha^2}.
\]

This procedure can be repeated, so that after the \( n \)-th step we obtain

\[
z = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \cdots + \frac{a_n}{\alpha^n} + \frac{z_n}{\alpha^n}, \quad \text{where } z_n \in V \text{ and } a_1,\ldots,a_n \in \mathcal{A}.
\]

As \( V \) is bounded and \( |\alpha| > 1 \), we have easily \( \lim_{n \to \infty} \frac{z_n}{\alpha^n} = 0 \), which proves the first statement of the theorem. The second statement follows easily from the following fact. Since 0 belongs to the interior of \( V \), one can find for every \( z \in \mathbb{C} \) an exponent \( k \in \mathbb{Z} \) such that \( \frac{z}{\alpha^k} \in V \). The first statement ensures existence of a representation

\[
\frac{z}{\alpha^k} = \sum_{i=1}^{\infty} \frac{a_i}{\alpha^i}.
\]

We shall use the above method for representing real numbers \( x \) by powers of a negative base \( \alpha = -\beta, \beta > 1 \). We will, moreover, require the following:

1. The region \( V \) is a bounded interval \( I \subset \mathbb{R} \);
2. the expression \( x = \sum_{i=1}^{\infty} \frac{a_i}{(-\beta)^i} \) from the theorem is unique for every \( x \in V \);
3. the alphabet \( \mathcal{A} \subset \mathbb{R} \) is minimal in the sense that for every \( a \in \mathcal{A} \) there is an \( x \in \mathbb{R} \) in whose representation \( a_1a_2a_3\cdots \) the letter \( a \) appears at least once.

Requirement 2. implies that \( -\beta I \subset \bigcup_{a \in \mathcal{A}} (I + a) \), where the union is disjoint. By requirement 1. the set \( -\beta I \) is an interval, which together with the disjointness means that \( I \) is a semi-closed interval and, moreover, the intervals \( I + a, a \in \mathcal{A} \), concur without gaps and overlaps. Denoting \( m = \min \mathcal{A} \), we obtain using requirement 3. that \( \mathcal{A} = \{m, m + |I|, m + 2|I|, \ldots, m + (\#\mathcal{A} - 1)|I|\} \), where \( |I| \) denotes the length of the
interval \( I \). Since (5) remains valid when scaling both \( V \) and \( \mathcal{A} \) by the same fixed factor, we can, without loss of generality, set \( |I| = 1 \) and \( I = [l, r) \) for some \( l \in \mathbb{R}, r = l + 1 \). In such a case, the alphabet is of the form \( \mathcal{A} = \{m + k \mid k = 0, 1, \ldots, \#\mathcal{A} - 1\} \). Imposing another requirement,

4. \( \mathcal{A} \subset \mathbb{Z} \),

the alphabet \( \mathcal{A} \) becomes a finite set of consecutive integers.

Now one can provide a simple prescription which to a given \( x \) assigns the first digit \( a_1 \) and the remainder \( z_1 \),

\[-\beta x \in [l, r] + a_1 \iff l + a_1 \leq -\beta x < r + a_1 \iff -\beta x - r < a_1 \leq -\beta x - l.\]

Since \( a_1 \in \mathbb{Z} \), we obtain \( a_1 = \lfloor -\beta x - l \rfloor \) and \( z_1 = -\beta x - a_1 = -\beta x - \lfloor -\beta x - l \rfloor \). Note that \( z_1 = T(x) \), where \( T \) is the transformation defined in (2) and \( a_1 = x_1 \) from (3). The digits take values in the alphabet

\[
\mathcal{A}_{-\beta,l} := \left\{ \lfloor -l(\beta + 1) - \beta \rfloor, \ldots, \lfloor -l(\beta + 1) \rfloor \right\}. \tag{7}
\]

which depends on \( l \) and \( \beta \). Thus for any \( \beta \) and \( l \), the digits in the numeration system form a bounded set of \( P \) consecutive integers where \( P = \lfloor \beta \rfloor + 1 \) or \( P = \lfloor \beta \rfloor + 2 \).

One may impose some further natural requirements on the number system:

- That the digits can take the value 0. For that, we need that \(-l(\beta + 1) - \beta < 1\) and \(-l(\beta + 1) \geq 0\), which is equivalent to \(-1 < l \leq 0\), i.e. \( 0 \in [l, r) \).

- That the expansion of 0 is equal to \( d(0) = 000 \cdots = 0^\omega \). This is satisfied if 0 is a fixed point of the transformation \( T \), for which we need \( \lfloor -l \rfloor = 0 = T(0) \), which again gives the condition \( 0 \in [l, r) \).

- That the numeration system can be extended to represent not only numbers from \([l, r)\) but all reals. Here we need that every real number can be multiplied by an integer power of \(-\beta\) so that it falls within the interval \([l, r)\). Again, one gets the condition \( 0 \in [l, r) \).

The above items justify our choice of considering \(-1 < l \leq 0\), which we impose throughout the paper. Note that one can derive by simple algebraic manipulation that the digits in the alphabet (7) are then bounded by \( \lfloor \beta \rfloor \) in modulus.

4. The \((-\beta,l)\)-transformation \( T \)

Let us now see how the choice of the domain \([l, l+1) = [l, r)\) of the transformation \( T \) influences the properties of the resulting numeration system, or, on the other hand, how further requirements on the numeration system limit the choice of \( l \).

One may prefer that the digits are all non-negative, as it is the case of Rényi expansions with positive base. Such a condition is equivalent to the requirement \(-l(\beta + 1) - \beta \geq 0\), which implies \( l \geq -\frac{\beta}{\beta + 1} \). When, moreover, one requires the digits to be smaller or equal to \( \beta \), we must ask for \(-l(\beta + 1) < \lfloor \beta \rfloor + 1\), i.e. \( l < -\frac{\beta + 1}{\beta + 1} \). Putting together, we have the following remark.
Remark 4. The numeration system with negative base $-\beta$ given by the transformation (2) gives representation of numbers over the digit set \{0, 1, \ldots, \lfloor \beta \rfloor \} if and only if the left end-point of the domain $I = [l, r)$ of $T$ satisfies
\[ \frac{-\beta}{\beta + 1} \leq l < \frac{-\lfloor \beta \rfloor + 1}{\beta + 1}. \] (8)

Example 5. Putting $l = \frac{-\beta}{\beta + 1}$ ensures that (8) is satisfied universally for all $\beta$. Such choice of $l$ corresponds to the number system introduced in [3] by Ito and Sadahiro. In this case, we have the transformation
\[ T : \left[ -\frac{\beta}{\beta + 1}, \frac{1}{\beta + 1} \right) \mapsto \left[ -\frac{\beta}{\beta + 1}, \frac{1}{\beta + 1} \right), \quad T(x) = -\beta x - \left\lfloor -\beta x + \frac{\beta}{\beta + 1} \right\rfloor. \] (9)

Note that the end-points of the interval $[l, r)$ now satisfy
\[ \frac{l}{-\beta} = l + 1 = r. \]

This fact is of certain help when deriving the condition on the admissibility of digit strings. On the other hand, it brings one non-desirable phenomena. Just as in the usual positional number systems, we would expect that multiplying by a power of the base only shifts the digit string representing the number. So it would be natural to ask that
\[ \text{if } d(x) = x_1 x_2 x_3 \cdots, \text{ then } d \left( \frac{x}{(-\beta)^k} \right) = 0^k x_1 x_2 x_3 \cdots. \]

This is however not true for all $x \in \left[ -\frac{\beta}{\beta + 1}, \frac{1}{\beta + 1} \right)$ in the Ito-Sadahiro numeration system. Namely, taking for $x$ the boundary point $l = \frac{-\beta}{\beta + 1}$ and denoting $d(l) = l_1 l_2 l_3 \cdots$, we have $(-\beta)^{-2} l \in [l, r)$ but
\[ d \left( \frac{l}{(-\beta)^2} \right) = 1 l_1 l_2 l_3 \cdots, \]
as can be verified easily using (9).

The non-desirable phenomena occurring in the Ito-Sadahiro numeration system is caused by the fact that in the interval $[l, r)$ there is an element, namely $l$ itself, which divided by $-\beta$ falls outside of $[l, r)$. Such phenomena can be avoided choosing such $l$ that for given $\beta$ the following implication holds
\[ x \in [l, r) \implies \frac{x}{-\beta} \in [l, r). \] (10)

For, the transformation $T$ from (2) satisfies $T \left( \frac{x}{-\beta} \right) = x$ for all $x \in [l, r)$. Thus, the implication is a necessary and sufficient condition so that the expansion of every real number $x$ is unique. One verifies easily that implication (10) is ensured by the following condition.

Remark 6. The numeration system with negative base $-\beta$ given by the transformation (2) gives a unique representation $(-\beta, l)$-expansion of a non-zero $x \in \mathbb{R}$ in the form $x = \sum_{i=k}^{\infty} x_i (-\beta)^i$, $x_k \neq 0$, if and only if the left end-point of the domain $I = [l, r)$ of $T$ satisfies
\[ \frac{-\beta}{\beta + 1} < l \leq \frac{-1}{\beta + 1}. \] (11)
Example 7. A suitable choice of \( l \) satisfying (11) universally for all \( \beta > 1 \), is \( l = -\frac{1}{2} \).
In this case, we obtain the digit set \( \left\{ \left\lfloor \frac{-\beta+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{\beta+1}{2} \right\rfloor \right\} \) which, for \( \beta + 1 \notin 2\mathbb{Z} \), results in the symmetric alphabet
\[
\left\{ -\left\lfloor \frac{\beta+1}{2} \right\rfloor, \ldots, \left\lfloor \frac{\beta+1}{2} \right\rfloor \right\}.
\]
Consider the transformation \( T(x) = -\beta x - \left\lfloor -\beta x + \frac{1}{2} \right\rfloor \). For all but finitely many \( x \) from the interval \( [-\frac{1}{2}, \frac{1}{2}) \) we can write
\[
T(-x) = -\beta(-x) - \left\lfloor -\beta(-x) + \frac{1}{2} \right\rfloor = -\left( -\beta x - \left\lfloor -\beta x - \frac{1}{2} \right\rfloor \right) = -\left( -\beta x - \left\lfloor -\beta x + \frac{1}{2} \right\rfloor \right) = -T(x).
\]
Here we have used that \( \lfloor y \rfloor = -\lceil -y \rceil \) for all \( y \in \mathbb{R} \) and \( \lceil -y \rceil = \lfloor y \rfloor + 1 \) for all \( y \notin \mathbb{Z} \).
For \( l = -\frac{1}{2} \), we have
\[
d(-x) = d(x), \quad \text{for all } x \in [-\frac{1}{2}, \frac{1}{2}) \text{ up to a countably many exceptions.}
\]

Let us specify that the notation \( \overline{a} \) stands for the digit \(-a\); similarly for a string of digits, we write \( \overline{a_1a_2 \ldots} \) meaning \( \overline{a_1a_2 \ldots} = (-a_1)(-a_2) \cdots \). Due to the latter property, we may call the numeration system with \( l = -\frac{1}{2} \) the balanced system.

A very important practical requirement for the numeration system is that the set of numbers with finite expansions is non-trivial. A number is said to have finite \((-\beta, l)\)-expansion if \( d(x) \) ends in \( 0^\omega \). We denote the set of such numbers by \( \text{Fin}(-\beta, l) \). As we have explained above, \( 0 \in \text{Fin}(-\beta, l) \) for any \( \beta > 1 \) and any \( l \) such that \( 0 \in [l, r) \). We can then write
\[
\text{Fin}(-\beta, l) = \left\{ x \in [l, r) \mid T^n(x) = 0 \text{ for some } n \in \mathbb{N} \right\}.
\]

However, for some choices of \( l \) there are no other elements with finite \((-\beta, l)\)-expansion besides zero. The following proposition characterizes the numeration systems for which this does not happen.

Proposition 8. The necessary and sufficient condition for \( \text{Fin}(-\beta, l) \) to be different from \( \{0\} \) is that
\[
\frac{-1}{\beta} \text{ or } \frac{1}{\beta} \in [l, r).
\] 

Proof. If a non-zero number belongs to \( \text{Fin}(-\beta, l) = \{ x \in [l, r) \mid T^n(x) = 0 \text{ for some } n \in \mathbb{N} \} \), then necessarily, there exists a non-zero number \( y \in [l, r) \) such that \( T(y) = 0 \). Since \( T(y) = -\beta y - \left\lfloor -\beta y - l \right\rfloor \), the pre-images of 0 in the interval \([l, r)\) are of the form \( \frac{a}{\beta} \) for some integer \( a \). On the other hand, \( T\left( \frac{a}{\beta} \right) = a - \lfloor a - l \rfloor = 0 \) for all \( a \) such that \( \frac{a}{\beta} \in [r, l) \). Obviously, \( \frac{a}{\beta} \in [r, l) \) for some non-zero integer \( a \) if and only if \( \frac{1}{\beta} \) or \( -\frac{1}{\beta} \) belongs to \([l, r)\). \( \square \)
Consider the Ito-Sadahiro numeration system. Here the condition (13) gives \( \text{Fin}(−\beta, −\beta+1) \neq \{0\} \) if and only if \( \beta \geq \tau \), where \( \tau = \frac{1}{2}(1+\sqrt{5}) \) is the golden ratio, as already mentioned in [9]. In the balanced numeration system we have \( \text{Fin}(−\beta, −\frac{1}{2}) \neq \{0\} \) if and only if \( \beta \geq 2 \). On the other hand, we can choose a system which for any \( \beta \) provides a non-trivial set of finite expansions. Such a system is obtained for example if \( l = −\frac{1}{2} \). Unfortunatelly, now the condition (11) for uniqueness of the representation of numbers is not universally satisfied. In particular, it is valid if and only if \( \beta > \tau \).

Let us now study the properties of the \((-\beta, l)\)-transformation \( T \) of the interval \([l, r)\) in more depth. These properties will mainly be used in the proof of Theorem 15. From the prescription \( T(x) = −\beta x − [−\beta x − l] \) using integer part we derive that, on its domain, \( T \) is always continuous from the left, but there are discontinuity points in which \( T \) is discontinuous from the right, namely
\[
\mathcal{D} = \left\{ \frac{a+l}{\beta} \left| a \in \mathcal{A}_{−\beta,l} \right. \right\}.
\] (14)

For illustration of the graph of \( T \) see Figure 1. It is obvious that for a discontinuity point \( x = \frac{a+l}{\beta} \in \mathcal{D} \), we have
\[
T(x) = −\beta \frac{a+l}{\beta} − [−\beta \frac{a+l}{\beta} − l] = l.
\]
Moreover, we easily realize that
\[
T(x) = l \iff x \in \mathcal{D}.
\] (15)

The domain \([l, r)\) of the transformation \( T \) is divided by the discontinuity points \( \mathcal{D} \) into disjoint intervals \( I_a \), \( a \in \mathcal{A}_{−\beta,l} \), in such a way that for every \( x \in I_a \), the first digit \( x_1 \) in the \((-\beta, l)\)-expansion of \( x \) is equal to \( a \). Note that for the maximal digit \( a = d_1 \), the interval \( I_{d_1} \) is closed, degenerate or non-degenerate. On the other hand, for the minimal digit \( a \), the interval \( I_a \) is open. For the other digits, we always have the interval \( I_a \) of the form \((\cdot, \cdot]\).

From the prescription (2) we easily derive that fixed points of the \((-\beta, l)\)-transformation are precisely the points \( f_a := −\frac{a}{\beta+1} \), where \( a \) is a digit in \( \mathcal{A}_{−\beta,l} \). We thus have \( d(f_a) = a^\omega \) for every \( a \in \mathcal{A}_{−\beta,l} \). Obviously, we have \( f_a \in I_a \) and, moreover, \( f_a \) is in most cases an inner point of \( I_a \). In fact, it can lie on the boundary of \( I_a \) only if
\[
[−\beta(f_a + \varepsilon) − l] < [−\beta f_a − l] = a \quad \text{for any} \ \varepsilon > 0,
\]
and this happens only if \( −\beta f_a − l = a \), i.e. \( l = f_a \). In such a case, \( a \) is the maximal digit and \( I_a \) is a degenerate interval, and thus the digit \( a \) occurs in the \((-\beta, l)\)-expansion of only countably many numbers in the domain \([l, r)\). It follows that always at least two among the intervals \( I_a \), \( a \in \mathcal{A}_{−\beta,l} \) are non-degenerate.

Consider \( a \in \mathcal{A}_{−\beta,l} \) such that \( I_a \) is non-degenerate. The corresponding fixed point \( f_a \) of the transformation \( T \) is an inner point of \( I_a \), i.e. \( T \) is continuous on some neighborhood of \( f_a \). We can derive that for every \( n \in \mathbb{N} \) there exists a positive \( \delta_n \) satisfying \( T^n(f_a − \delta_n, f_a + \delta_n) \subset I_a \) for \( i = 0, 1, \ldots, n−1 \). Necessarily, for all \( x \in (f_a − \delta_n, f_a + \delta_n) \), the \((-\beta, l)\)-expansion of \( x \) starts with the prefix \( a^n \). We have thus proved the following lemma which will be of use in Section 7.
Figure 1: \((-\beta)\)-transformations corresponding to \(\beta\) minimal Pisot number (the real root of \(x^3 - x - 1\)) with different choices of domain \([l, r]\). For \(l = -\frac{\beta}{\beta+1}\) it is the Ito-Sadahiro system, for \(l = -\frac{1}{\beta+1}\) it is the balanced system. In the third case, the interval \(I_1 = \{-\frac{1}{\beta+1}\}\) is degenerate, thus the corresponding digit 1 appears in \((-\beta, l)\)-expansions of only countably many numbers. In all the cases, the graph of the transformation intersects the line \(y = 0\) only at the origin, and thus the only element of \(\text{Fin}(-\beta, l)\) is 0.

Lemma 9. For every \(\beta > 1\) and \(l \in (-1, 0]\) there exists a non-zero digit \(a\) such that for all \(n \in \mathbb{N}\) we can find an interval \(J \subset [l, r)\) such that the \((-\beta, l)\)-expansions of every \(x \in J\) is of the form
\[
d(x) = a^n x_1 x_2 x_3 \cdots.
\] (16)

5. Alternate order and admissible \((-\beta, l)\)-expansions

Let us show that, just as in the case of Ito-Sadahiro expansions, the alternate order on digit strings \(d(x)\) corresponds to the natural order of real numbers in the interval \([l, r)\).

Theorem 10. Alternate order on \((-\beta, l)\)-expansions preserves the natural order on the interval \([l, r)\), i.e. for \(x, y \in [l, r)\) with \((-\beta, l)\)-expansions \(d(x) = x_1 x_2 x_3 \cdots\) and \(d(y) = y_1 y_2 y_3 \cdots\) we have
\[
x < y \iff d(x) \prec_{\text{alt}} d(y).
\]

Proof. For \(x, y \in [l, r)\) with \((-\beta, l)\)-expansions \(d(x) = x_1 x_2 x_3 \cdots\) and \(d(y) = y_1 y_2 y_3 \cdots\) we will show that
\[
x < y \iff x_m (-1)^m < y_m (-1)^m,
\]
where \(m := \min\{k \in \mathbb{N} \mid x_k \neq y_k\}\). From the definition of \(m\) and the construction of \((-\beta, l)\)-expansions it follows that there exist \(\tilde{x}, \tilde{y} \in [l, r)\) and \(z \in \mathbb{R}\) such that
\[
x = z + \frac{1}{(-\beta)^{m-1}} \tilde{x} \quad \text{and} \quad y = z + \frac{1}{(-\beta)^{m-1}} \tilde{y},
\]
where \(x_m x_{m+1} x_{m+2} \cdots, y_m y_{m+1} y_{m+2} \cdots\) are the \((-\beta, l)\)-expansions of \(\tilde{x}, \tilde{y}\), respectively. We thus have
\[
x < y \iff \frac{\tilde{x}}{(-\beta)^{m-1}} < \frac{\tilde{y}}{(-\beta)^{m-1}} \iff \tilde{x} (-1)^{m-1} < \tilde{y} (-1)^{m-1}.
\] (17)
Since \( x_m = \lfloor -\beta \bar{x} - l \rfloor \), \( y_m = \lfloor -\beta \bar{y} - l \rfloor \), we have the following implications
\[
\begin{align*}
x_m < y_m & \quad \Rightarrow \quad -\beta \bar{x} - l < -\beta \bar{y} - l \quad \Rightarrow \quad \bar{x} > \bar{y}, \\
x_m > y_m & \quad \Rightarrow \quad -\beta \bar{x} - l > -\beta \bar{y} - l \quad \Rightarrow \quad \bar{x} < \bar{y},
\end{align*}
\]
and thus
\[
\bar{x} > \bar{y} \quad \iff \quad x_m < y_m \tag{18}
\]
If \( m \) is odd, equivalences (17) and (18) imply \( x < y \iff \bar{x} < \bar{y} \iff x_m > y_m \iff x_m(-1)^m < y_m(-1)^m \). If \( m \) is even, we have \( x < y \iff \bar{x} > \bar{y} \iff x_m < y_m \iff x_m(-1)^m < y_m(-1)^m \).

**Corollary 11.** The \((-\beta, l)\)-expansion as a function \( d : [l, r) \mapsto \mathbb{Z}^\mathbb{N} \) is strictly increasing on the interval \([l, r)\) in the alternate order, and therefore the limits
\[
d^*(l) := \lim_{\varepsilon \to 0^+} d(l + \varepsilon), \quad d^*(r) := \lim_{\varepsilon \to 0^+} d(r - \varepsilon)
\]
exist and \( d(x) \preceq_{\text{alt}} d^*(l) \).

The notation from Corollary 11 will be used throughout the paper.

Theorem 10 will now be used to prove a necessary and sufficient condition for admissibility of digit strings.

**Definition 12.** An infinite string \( x_1x_2x_3 \cdots \) of integers is called \((-\beta, l)\)-admissible (or just admissible), if there exists an \( x \in [l, r) \) such that \( x_1x_2x_3 \cdots \) is its \((-\beta, l)\)-expansion, i.e. \( x_1x_2x_3 \cdots = d(x) \).

**Theorem 13.** Let \( d \) be the \((-\beta, l)\)-expansion and denote \( d(l) = l_1l_2l_3 \cdots \) and \( d^*(r) = r_1r_2r_3 \cdots \). An infinite string \( x_1x_2x_3 \cdots \) of integers is \((-\beta, l)\)-admissible, if and only if
\[
l_1l_2l_3 \cdots \preceq_{\text{alt}} x_i x_{i+1} x_{i+2} \cdots \preceq_{\text{alt}} r_1r_2r_3 \cdots, \quad \text{for all } i \geq 1. \tag{19}
\]

**Proof.** The fact that the condition (19) is necessary follows as a consequence of Theorem 10. It remains to show that it is sufficient, i.e. that a digit string \( x_1x_2x_3 \cdots \) satisfying the condition (19) is a \((-\beta, l)\)-expansion of an \( x \in [l, r) \). It is sufficient to show that for all \( n = 1, 2, 3, \ldots \) we have
\[
\bullet x_n x_{n+1} x_{n+2} \cdots = \sum_{i=1}^{\infty} \frac{x_{n-1+i}}{(-\beta)^i} \in [l, r). \tag{20}
\]
where we use the notation \( \bullet y_1y_2y_3 \cdots \) for the number represented by the digit string \( y_1y_2y_3 \cdots \), i.e. for the value
\[
\bullet y_1y_2y_3 \cdots = \frac{y_1}{(-\beta)} + \frac{y_2}{(-\beta)^2} + \frac{y_3}{(-\beta)^3} + \cdots = \sum_{i=1}^{\infty} \frac{y_i}{(-\beta)^i}.
\]
Let us fix an \( n \in \mathbb{N} \). Since \( x_n x_{n+1} x_{n+2} \cdots \preceq_{\text{alt}} r_1r_2r_3 \cdots = d^*(r) \), there exists an \( \varepsilon > 0 \) such that \( x_n x_{n+1} x_{n+2} \cdots \preceq_{\text{alt}} d(r - \varepsilon) \). Choose an \( \varepsilon > 0 \) such that also
\[
x_{n+1} x_{n+2} x_{n+3} \cdots \preceq_{\text{alt}} d(r - \varepsilon) = \tilde{r}_1 \tilde{r}_2 \tilde{r}_3 \cdots \tag{21}
\]
We first show by induction on \( s \) the following statement:
Claim 14. Let \( z \in [l, r) \) have the \((-\beta, l)\)-expansion of the form \( d(z) = z_1z_2z_3 \cdots \) and let \( s, m \in \mathbb{N} \).

If \( x_nx_{n+1} \cdots x_{n+s}0^\omega \preceq_{\text{alt}} z_mz_{m+1} \cdots z_{m+s}0^\omega \), then \( \bullet x_nx_{n+1} \cdots x_{n+s} \geq \bullet z_mz_{m+1} \cdots - \frac{1}{\beta^{s+1}} \).

If \( x_nx_{n+1} \cdots x_{n+s}0^\omega \succeq_{\text{alt}} z_mz_{m+1} \cdots z_{m+s}0^\omega \), then \( \bullet x_nx_{n+1} \cdots x_{n+s} \leq \bullet z_mz_{m+1} \cdots + \frac{1}{\beta^{s+1}} \).

Proof. For \( s = 0 \), the inequality \( x_n0^\omega \preceq_{\text{alt}} z_m0^\omega \) is equivalent to \( x_n \leq z_m \). We thus have
\[
\frac{x_n}{-\beta} = \frac{z_m}{-\beta} = \bullet z_mz_{m+1} \cdots + \frac{1}{\beta^s (z_{m+1}z_{m+2} \cdots )} \geq \bullet z_mz_{m+1} \cdots - \frac{1}{\beta^s+1}.
\]

For \( s \geq 1 \), the inequality \( x_nx_{n+1} \cdots x_{n+s}0^\omega \preceq_{\text{alt}} z_mz_{m+1} \cdots z_{m+s}0^\omega \) implies that one of the following happens,

(i) either \( x_n = z_m \) and \( x_{n+1} \cdots x_{n+s}0^\omega \preceq_{\text{alt}} z_{m+1} \cdots z_{m+s}0^\omega \),

(ii) or \( x_n \leq z_m - 1 \).

In case (i) that \( x_n = z_m \), we use induction hypothesis to derive from \( x_{n+1} \cdots x_{n+s}0^\omega \preceq_{\text{alt}} z_{m+1} \cdots z_{m+s}0^\omega \) that
\[
\bullet x_{n+1} \cdots x_{n+s} \leq \bullet z_{m+1} \cdots + \frac{1}{\beta^s}.
\]

Dividing by \(-\beta\) and adding \( x_n(-\beta)^{-1} = z_m(-\beta)^{-1} \), we have
\[
\bullet x_nx_{n+1} \cdots x_{n+s} \geq \bullet z_mz_{m+1} \cdots - \frac{1}{\beta^{s+1}}.
\]

In case (ii), we have \( z_m - x_n \geq 1 \) and thus
\[
\frac{z_m - x_n}{-\beta} \leq -\frac{1}{\beta}.
\]

(22)

For the string \( x_{n+1} \cdots x_{n+s} \) we can derive using (21) that for prefixes \( l_1l_2 \cdots l_s \) of \( d(l) \) and \( \tilde{r}_1 \tilde{r}_2 \cdots \tilde{r}_s \) of \( d(r - \varepsilon) \) it holds that
\[
l_1l_2 \cdots l_s0^\omega \preceq_{\text{alt}} x_{n+1} \cdots x_{n+s}0^\omega \preceq_{\text{alt}} \tilde{r}_1 \tilde{r}_2 \cdots \tilde{r}_s0^\omega.
\]

By induction hypothesis, we have
\[
\frac{l - \frac{1}{\beta^s}}{l} \leq \bullet x_{n+1} \cdots x_{n+s} \leq r - \varepsilon + \frac{1}{\beta^s}.
\]

(23)

We therefore have
\[
\bullet x_nx_{n+1} \cdots x_{n+s} - \bullet z_mz_{m+1} \cdots = \frac{x_n - z_m}{-\beta} - \frac{1}{\beta} \left( \bullet x_{n+1} \cdots x_{n+s} - \bullet z_{m+1}z_{m+2} \cdots \right) \geq \frac{1}{\beta^s+1} \geq \frac{1}{\beta^s+1} \geq \frac{1}{\beta^s+1},
\]

where we have used (22) and (23). By this, the claim is proved. \( \square \)
Let us now use Claim 14 for proving the statement of Theorem 13. For \( z \in [l, r) \) of
the claim we put \( z = l \) and \( z = r - \varepsilon \). For all \( s \in \mathbb{N} \), we have
\[
l_1 l_2 \cdots l_s 0^\omega \preceq_{\text{alt}} x_n x_{n+1} \cdots x_{n+s} 0^\omega \preceq_{\text{alt}} \tilde{r}_1 \tilde{r}_2 \cdots \tilde{r}_s 0^\omega ,
\]
and therefore by claim,
\[
l - \frac{1}{\beta^s} \leq x_n x_{n+1} \cdots x_{n+s-1} \leq r - \varepsilon + \frac{1}{\beta^s} .
\]
As \( s \) tends to infinity, we derive
\[
l \leq x_n x_{n+1} x_{n+2} \cdots \leq r - \varepsilon < r ,
\]
which is equivalent to (20) that we were to show. \( \square \)

6. Reference strings \( d(l) \) and \( d^*(r) \)

Let us now study the digit strings which appear in the \((-\beta, l)\)-admissibility criteria.
The following theorem states the relation between \( d(l) \) and the limit \( d^*(l) = \lim_{\varepsilon \to 0+} d(l + \varepsilon) \). Although \( d^*(l) \) itself does not appear in the admissibility condition, it may, as we shall see on examples below, be used to derive the form of \( d^*(r) \).

**Theorem 15.** Let \( T \) be the \((-\beta, l)\)-transformation. If \( T^q(l) \neq l \) for all \( q \in \mathbb{N} \), or the equality \( T^q(l) = l \) occurs only for even \( q \in \mathbb{N} \), then
\[
d^*(l) = d(l) .
\]
If on the other hand \( T^q(l) = l \) for some \( q \in \mathbb{N} \), \( q \) odd, i.e. \( d(l) = (l_1 l_2 \cdots l_{q-1}l_q)^\omega \), then
\[
d^*(l) = l_1 l_2 \cdots l_{q-1}(l_q - 1)d^*(r) .
\]

**Proof.** The proof uses the properties of the \((-\beta, l)\)-transformation stated in Section 4. Recall that the \((-\beta, l)\)-transformation is continuous on \([l, r)\) except in the discontinuity points forming the set \( D \) given by (14). The discontinuity points divide the interval \([l, r)\) into disjoint union \([l, r) = \bigcup_{a \in A_{-\beta, l}} I_a \), where \( I_a = \{ x \in [l, r) \mid a \text{ is a prefix of } d(x) \} \).

By (15) we have that \( T(x) = l \) if and only if \( x \in D \). Suppose that for all \( i \leq k \) one has \( T^i(l) \neq l \). Then \( T^{k-1}(l) \notin D \). Denote by \( M \) the set
\[
M = \{ T^i(l) \mid i = 0, 1, \ldots, k - 1 \} .
\]
Choose \( \delta > 0 \) such that
\[
\beta^k \delta < \min\{ |x - y| \mid x \in M, y \in D \} . \tag{24}
\]
For such \( \delta \), we have \([l, l + \delta) \subset I_{l_1} , \)
\[
T([l, l + \delta)) = (T(l) - \beta \delta, T(l)] \subset I_{l_2} ,
\]
and more generally, we have for $2i - 1 \leq k$ that

$$T^{2i-1}([l, l + \delta]) = (T^{2i-1}(l) - \beta^{2i-1}\delta, T^{2i-1}(l)] \subset I_{2i},$$

(25)

and for $2i \leq k$ that

$$T^{2i}([l, l + \delta]) = [T^{2i}(l), T^{2i}(l) + \beta^{2i}\delta] \subset I_{2i+1}.$$  

(26)

Since $T^j([l, l + \delta]) \subset I_{j+1}$ for every $j \leq k$, the $(-\beta, l)$-expansion $d(x)$ of every $x \in [l, l + \delta]$ has the prefix $l_1l_2 \cdots l_{k+1}$.

Suppose now that $T^i(l) \neq l$ for any $i \in \mathbb{N}$. Then obviously

$$d^*(l) = \lim_{\varepsilon \to 0^+} d(l + \varepsilon) = d(l).$$

Suppose on the other hand that one can find $q$ such that $T^q(l) = l$. Let $q$ be the minimal exponent with this property. We can find $\delta$ so that (24) is satisfied with $k = q - 1$. One derives that $T^{q-1}([l, l + \delta]) \subset I_q$ and $T^{q-1}(l) \in D$.

First let $q - 1$ be odd. By (25) we have

$$T^{q-1}([l, l + \delta]) = (T^{q-1}(l) - \beta^{q-1}\delta, T^{q-1}(l)] \subset I_q.$$  

Although $T^{q-1}(l) \in D$, the transformation $T$ is continuous from the left at any discontinuity point, and so

$$T^q([l, l + \delta]) = [T^q(l), T^q(l) + \beta^q\delta] = [l, l + \beta^q\delta].$$

This implies that for an arbitrarily long prefix $w$ of $d(l)$ one finds an $\varepsilon > 0$ such that for all $x \in [l, l + \varepsilon)$, the $(-\beta, l)$-expansion of $x$ has the prefix $w$. In other words,

$$d^*(l) = \lim_{\varepsilon \to 0^+} d(l + \varepsilon) = d(l).$$

Let now $q - 1$ be even. By (26) we have

$$T^{q-1}([l, l + \delta]) = [T^{q-1}(l), T^{q-1}(l) + \beta^{q-1}\delta] \subset I_q.$$  

Now $T^{q-1}(l) \in D$ and $T$ is not continuous from the right, and thus

$$T^q([l, l + \delta]) = \{l\} \cup (r - \beta^q\delta, r).$$

This implies that for all $0 < \varepsilon < \delta$, the $(-\beta, l)$-expansion of $x = l + \varepsilon$ is of the form

$$d(x) = l_1l_2 \cdots l_{q-1}(l_q - 1)x_{q+1}x_{q+2} \cdots,$$

where

$$x_{q+1}x_{q+2} \cdots = d(r - \beta^q\varepsilon).$$

Therefore

$$d^*(l) = \lim_{\varepsilon \to 0^+} d(l + \varepsilon) = l_1l_2 \cdots l_{q-1}(l_q - 1)d^*(r).$$

\[\square\]
We now illustrate the use of Theorem 15 for deriving the exact prescription for the relation of the reference strings \(d(l)\) and \(d^*(r)\) for three choices of \(l\). First, let us take the example of the Ito-Sadahiro numeration system.

**Example 16.** Let \(l = -\frac{\beta}{\beta+1}\). We can compute

\[
T(r-\varepsilon) = -\beta \left( -\frac{\beta}{\beta+1} + 1 \right) + \beta \varepsilon - \left\lfloor \frac{\beta^2 - \beta + \beta \varepsilon}{\beta+1} \right\rfloor = -\frac{\beta}{\beta+1} + \varepsilon = l + \varepsilon.
\]

Therefore

\[
d^*(r) = 0 d^*(l) .
\]

If, moreover, \(d(l) = (l_1 \cdots l_q)^\omega\) is purely periodic with odd period length \(q\), then using (27) and Theorem 15 we have

\[
d^*(r) = 0 l_1 l_2 \cdots l_{q-1} (l_q - 1) d^*(r) ,
\]

which implies

\[
d^*(r) = \left( 0 l_1 l_2 \cdots l_{q-1} (l_q - 1) \right)^\omega .
\]

If, on the other hand, \(d(l)\) is not purely periodic with odd period, then (27) implies that simply \(d^*(r) = 0 d(l)\). This together corresponds to the result of Ito and Sadahiro cited in the introduction.

**Example 17.** Let \(l = -\frac{1}{\beta}\). As we have shown in (12), we have \(T(-x) = -T(x)\) for almost all \(x \in [l, r)\). It follows that

\[
\overline{d^*(l)} = d^*(r) .
\]

In case that \(d(l)\) is not purely periodic with odd period length, then

\[
d^*(r) = \overline{d^*(l)} = \overline{d(l)} .
\]

On the other hand, if \(d(l) = (l_1 \cdots l_q)^\omega\) is purely periodic with odd period length \(q\), then using Theorem 15 we obtain

\[
d^*(r) = \overline{l_1 \cdots l_{q-1} (l_q - 1) d^*(r)} ,
\]

and by iteration

\[
d^*(r) = \overline{l_1 \cdots l_{q-1} (l_q - 1) l_1 \cdots l_{q-1} (l_q - 1) d^*(r)} ,
\]

which implies

\[
d^*(r) = \left( \overline{l_1 \cdots l_{q-1} (l_q - 1) l_1 \cdots l_{q-1} (l_q - 1)} \right)^\omega
\]

**Example 18.** Let \(l = -\frac{1}{\beta}\). In that case \(l_1 = \left\lfloor -\beta (-\frac{1}{\beta}) + \frac{1}{\beta} \right\rfloor = 1\) and \(T(l) = -\beta (-\frac{1}{\beta}) - 1 = 0\). Consequently, the \((-\beta, -\frac{1}{\beta})\)-expansion of \(l\) is \(d(l) = 10^\omega\) for every base \(-\beta\). Since it is not purely periodic, we have for the left limit that \(d^*(l) = d(l)\). In this case, the right limit \(d^*(r)\) has no relation to \(d(l)\).
In the remaining part of this section we focus on the question what sequences may play role of the left and right reference strings in the admissibility condition. We describe some properties of these digit sequences, but, as we shall see, the question about a necessary a sufficient condition is far from being solved even for the case of Ito-Sadahiro numeration system. We explain the phenomenon on two examples which, in fact, represent an impeachment to Theorem 25 of Góra [6] who approaches this problem in a more general setting.

We start by a simple consequence of Theorem 13.

**Corollary 19.** Let \( d \) be the \((-\beta, l)\)-expansion. Every suffix of \( d(l) = l_1l_2l_3 \cdots \) satisfies the admissibility condition (19), i.e.

\[
l_1l_2l_3 \cdots \preceq_{alt} l_1l_{i+1}l_{i+2} \cdots \preceq_{alt} r_1r_2r_3 \cdots , \quad \text{for all } i \geq 1. \tag{28}\n\]

Every suffix of \( d^*(r) = \lim_{\varepsilon \to 0^+} d(r - \varepsilon) = r_1r_2r_3 \cdots \) satisfies

\[
l_1l_2l_3 \cdots \preceq_{alt} r_i r_{i+1} r_{i+2} \cdots \preceq_{alt} r_1r_2r_3 \cdots , \quad \text{for all } i \geq 1. \tag{29}\n\]

**Proof.** The admissibility condition for \( d(l) \) is an obvious consequence of Theorem 13.

In order to show the inequalities for \( d^*(r) \), realize that every suffix \( r_i r_{i+1} r_{i+2} \cdots \) of \( d^*(r) \) is a limit of \((-\beta, l)\)-expansions of numbers \( T^{i-1}(r - \varepsilon) \), as \( \varepsilon \) tends to 0 from the right. Since by Theorem 13 we have

\[
d\left(T^{i-1}(r - \varepsilon)\right) \prec_{alt} d^*(r).\n\]

Applying the limit, the strict inequality \( \prec_{alt} \) becomes \( \preceq_{alt} \).

The admissibility condition of Corollary 19 has interesting implications for repetition of blocks in prefixes of the reference digit strings \( d(l) \) and \( d^*(r) \).

**Corollary 20.** If \( w \) is a block of digits in the alphabet \( A \) of odd length such that \( ww \) is a prefix of \( d(l) \), then \( d(l) = w^\omega \).

If \( w \) is a block of digits in the alphabet \( A \) of odd length such that \( ww \) is a prefix of \( d^*(r) \), then \( d^*(r) = w^\omega \).

**Proof.** Let us write

\[
d(l) = \underbrace{ww \cdots w}_{k \text{ times}} vs , \tag{30}\n\]

where \( v \) is a string over \( A \) of the same length as \( w \), \( k \geq 2 \) and \( s \) is a suffix of \( d(l) \). Admissibility condition in Corollary 19 implies

\[
\underbrace{ww \cdots w}_{k \text{ times}} vs \preceq_{alt} wvs
\]

Now we use the simple fact that the alternate order satisfies

\[
x \preceq_{alt} y \iff wx \succeq_{alt} wy
\]

for a string \( w \) of odd length and any strings \( x, y \). For our case this implies

\[
\underbrace{ww \cdots w}_{k-1 \text{ times}} vs \succeq_{alt} vs
\]
and thus \(w0^\omega \succeq \text{alt} v0^\omega\). On the other hand, \(w\) is the smallest among prefixes of the same length of digit strings \(d(x)\) for all \(x \in [l, r]\), and therefore \(w0^\omega \preceq \text{alt} v0^\omega\). This implies that \(w = v\) and we may deduce that \(d(l)\) has prefix \(w^{k+1}\). The procedure may be repeated and hence \(d(l) = w^\omega\).

The same argument is used for the right reference string \(d^*(r)\).

**Remark 21.** As a result, we can observe that whenever \(d(l)\) starts with \(aa\) for some digit \(a\), then necessarily \(d(l) = a^\omega\) and consequently \(l = -\frac{a}{\beta + 1}\). Similarly, prefix \(bb\) in the right reference string implies that \(d^*(r) = b^\omega\).

Corollary 19 above shows that the bounds used in the admissibility condition satisfy the condition by themselves, i.e. the admissibility conditions (28) and (29) are a necessary condition so that strings \(l_1l_2l_3 \cdots\) and \(r_1r_2r_3 \cdots\) coincide with the reference strings \(d(l)\) and \(d^*(r)\) corresponding to some \(\beta > 1\) and \(l \in (-1, 0]\). This is analogous to the fact that in the case of positive base \(\beta\) the Rényi expansion of 1 satisfied the Parry condition. Let us recall that for base \(\beta > 1\) one uses the transformation \(T(x) = \beta x - \lfloor \beta x \rfloor\) of the interval \([0, 1)\). The string of integers \(d_\beta^r(1) = t_1t_2t_3 \cdots\) must satisfy

\[
0^\omega \preceq \text{lex} t_it_{i+1}t_{i+2} \cdots \preceq \text{lex} t_1t_2t_3 \cdots \quad \text{for every } i = 1, 2, 3, \ldots .
\]

Parry [10] has shown that this condition is also sufficient, in order that a digit string \(t_1t_2t_3 \cdots\) is equal to \(d_\beta^r(1)\) for some \(\beta > 1\).

A natural question arises, whether satisfying conditions

\[
\begin{align*}
&l_1l_2l_3 \cdots \preceq_{\text{alt}} l_i l_{i+1} l_{i+2} \cdots \preceq_{\text{alt}} r_1r_2r_3 \cdots & \text{for every } i = 1, 2, 3, \ldots, \\
&l_1l_2l_3 \cdots \preceq_{\text{alt}} r_i r_{i+1} r_{i+2} \cdots \preceq_{\text{alt}} r_1r_2r_3 \cdots
\end{align*}
\]

for a pair of integer sequences \((l_i)_{i=1}^{\infty}\), \((r_i)_{i=1}^{\infty}\) already ensures the existence of \(l\) and \(\beta\), so that the sequences are equal to the left and right reference strings \(d(l)\) and \(d^*(r)\) corresponding to some \(\beta > 1\) and \(l \in (-1, 0]\).

Such question was addressed by Góra. In [6], he studies a more general number system. However, a special case of his is equivalent to representation with negative base. Theorem 25 in [6] can be transformed into the statement that conditions (31) are sufficient for \((l_i)_{i=1}^{\infty}\), \((r_i)_{i=1}^{\infty}\) being the left and right reference strings for some \(\beta\) and \(l\). Here we show that his statement cannot be true.

In order to clarify the problem, let us focus on the Ito-Sadahiro numeration system, where \(l = -\frac{\beta}{\beta + 1}\) and the sequences \(d(l)\) and \(d^*(r)\) are connected as recalled in Example 16, i.e. when \(d(l)\) is non-periodic, then \(d^*(r) = 0d(l)\).

**Example 22.** Consider the sequence \((x_i)_{i=1}^{\infty} = a(a - 1)0a^\omega\) for some positive integer \(a > 1\). It is easily verified that every suffix \(x_i x_{i+1} x_{i+2} \cdots\) of this digit string satisfies

\[
a(a - 1)0a^\omega \preceq_{\text{alt}} x_ix_{i+1}x_{i+2} \cdots \preceq_{\text{alt}} 0a(a - 1)0a^\omega .
\]

Now if it holds that \(a(a - 1)0a^\omega\) is the left reference string of the Ito-Sadahiro system, then

\[
-\frac{\beta}{\beta + 1} = \frac{a}{-\beta} + \frac{a - 1}{(-\beta)^2} + \frac{a}{(-\beta)^4} + \frac{a}{(-\beta)^5} + \cdots,
\]

which implies that \(\beta = a\). However, applying the Ito-Sadahiro transformation \(T\) with the domain \([-\frac{\beta}{\beta + 1}, \frac{1}{\beta + 1}]\) to the point \(-\frac{\beta}{\beta + 1} = -\frac{a}{a+1}\) yields \(d(-\frac{\beta}{\beta + 1}) = a^\omega\). Thus, \(a(a - 1)0a^\omega\) is not an admissible left reference string, although it satisfies (32).
The following example shows that such a case is not isolated and need not correspond to integer base $-\beta$.

**Example 23.** Consider in the Ito-Sadahiro system the digit string $200(21)^\omega$. One easily verifies that every suffix $x_i x_{i+1} x_{i+2} \cdots$ of $200(21)^\omega$ satisfies

$$200(21)^\omega \preceq_{\text{alt}} x_i x_{i+1} x_{i+2} \cdots \prec_{\text{alt}} 0200(21)^\omega.$$ 

Comparing

$$\frac{-\beta}{\beta+1} = \frac{2}{-\beta} + \frac{2}{(-\beta)^4} + \frac{1}{(-\beta)^5} + \frac{2}{(-\beta)^6} + \frac{1}{(-\beta)^7} + \cdots,$$

leads to $\beta = \frac{1}{2} (3 + \sqrt{5})$. However, for such a base, the left reference string in the Ito-Sadahiro number system is equal to $(21)^\omega$.

### 7. Periodic $(-\beta,l)$-expansions

Representations of numbers in Ito-Sadahiro numeration system (the case $l = \beta$) from the point of view of dynamical systems have been studied by Frougny and Lai [4]. They have shown that if $\beta$ is a Pisot number, then $d(x)$ is eventually periodic for any $x \in [\beta, \beta+1) \cap \mathbb{Q}(\beta)$. In particular, their result implies that for every Pisot number the reference strings $d(l)$, $d^*(r)$ in the admissibility condition are eventually periodic. Bases $\beta$, for which in the Ito-Sadahiro numeration system the reference string $d(l)$ is eventually periodic have been called Ito-Sadahiro numbers and by [3] they are exactly the bases for which the Ito-Sadahiro $(-\beta)$-shift is sofic. The result of Frougny and Lai thus implies that Pisot numbers are Ito-Sadahiro numbers. This is an analogy to the statement for positive bases, namely that all Pisot numbers are Parry numbers. Liao and Steiner [8] show that surprisingly the notion of Ito-Sadahiro numbers and Parry numbers do not coincide.

In this section, we study the question of periodic $(-\beta,l)$-expansions. We show that the statement on Ito-Sadahiro expansions can be simply extended to the case of generalized $(-\beta)$-numeration defined on $[l,r)$, with $l \in (-1,0]$ and $r = l+1$. Note that the proof, just as the one in [4], is only a slight modification of the proof for positive bases given by Schmidt in [12].

**Theorem 24.** If $\beta$ is a Pisot number and $d$ is the $(-\beta,l)$-expansion for some $l \in (-1,0]$, then $d(x)$ is eventually periodic for any $x \in [l,r) \cap \mathbb{Q}(\beta)$.

**Proof.** Let $\beta_2, \ldots, \beta_d$ be the algebraic conjugates of $\beta = \beta_1$. Let $x \in [l,r) \cap \mathbb{Q}(\beta)$ be fixed. Such a number obviously belongs to the set $\frac{1}{q}(\mathbb{Z} + \beta \mathbb{Z} + \cdots + \beta^{d-1} \mathbb{Z})$ for some integer $q$. Denote $d(x) = x_1 x_2 x_3 \cdots$ and write

$$r_n = \frac{x_{n+1}}{-\beta} + \frac{x_{n+2}}{(-\beta)^2} + \cdots = (-\beta)^n \left( x - \sum_{k=1}^{n} x_k (-\beta)^{-k} \right).$$

Note that since $\beta$ is an algebraic integer, we have $\beta^k \in \mathbb{Z} + \beta \mathbb{Z} + \cdots + \beta^{d-1} \mathbb{Z}$, for every non-negative integer $k$. This implies that

$$r_n \in \frac{1}{q}(\mathbb{Z} + \beta \mathbb{Z} + \cdots + \beta^{d-1} \mathbb{Z}) \quad \text{for every } n \in \mathbb{N}. \quad (33)$$
For $2 \leq j \leq d$, we apply the isomorphism between $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta_j)$ defined by

$$z = c_0 + c_1 \beta + \cdots + c_{d-1} \beta^{d-1} \mapsto z^{(j)} = c_0 + c_1 \beta^{(j)} + \cdots + c_{d-1} (\beta^{(j)})^{d-1}$$

to obtain

$$r_n^{(j)}(x) = (-\beta_j)^n \left( x^{(j)} - \sum_{k=1}^{n} x_k (-\beta_j)^{-k} \right).$$

Denoting $\eta = \max\{|\beta_j|, 2 \leq j \leq d\}$, and since the digits $x_i$ of $d(x)$ are bounded by $[\beta]$, we find that

$$|r_n^{(j)}| \leq \eta^n x^{(j)} + [\beta] \sum_{k=0}^{n-1} \eta^k.$$

Since $\beta$ is a Pisot number, we have $\eta < 1$, which implies that $|r_n^{(2)}|, \ldots, |r_n^{(d)}|$ are bounded by a constant, independent of $n \in \mathbb{N}$. Also $r_n^{(1)} = r_n$ is bounded, since $r_n = T^n(x)$, and thus $r_n \in [l, r)$, which implies $|r_n| < 1$.

Consider the lattice $L$ in $\mathbb{R}^d$ spanned by vectors

$$\frac{1}{q} (1, 1, \ldots, 1), \quad \frac{1}{q} (\beta_1, \beta_2, \ldots, \beta_d), \quad \ldots, \quad \frac{1}{q} (\beta_1^{d-1}, \beta_2^{d-1}, \ldots, \beta_d^{d-1}).$$

From (33), it is obvious that the $d$-tuples

$$R_1 = (r_1^{(2)}, \ldots, r_1^{(d-1)}), \quad R_2 = (r_2^{(2)}, \ldots, r_2^{(d-1)}), \quad \ldots, \quad R_n = (r_n^{(2)}, \ldots, r_n^{(d-1)}), \quad \ldots$$

belong to the lattice $L$ for all $n \in \mathbb{N}$, and we have derived that they are found within a bounded region of $\mathbb{R}^d$. There are only finitely many lattice points in a bounded region, and hence one finds vectors $R_k, R_m$ for $k < m$ that coincide. It follows that $r_m = r_k$ and $d(x)$ is eventually periodic. \hfill \Box

**Remark 25.** Note that the above theorem does not speak about periodicity of $d(l)$ if $l \notin \mathbb{Q}(\beta)$. In fact, if $l \notin \mathbb{Q}(\beta)$, then the expansion $d(l) = l_1 l_2 l_3 \cdots$ is necessarily non-periodic, even if $\beta$ is a Pisot number.

On the other hand, the fact that $d(l)$ is eventually periodic does not predicate about the algebraic character of the base $\beta$. It only says that $l \in \mathbb{Q}(\beta)$. However, assuming that both reference strings appearing in the admissibility condition are eventually periodic already implies that $\beta$ is an algebraic integer. For, if $d(l)$ and $\mathbf{d}^*(r)$ are eventually periodic, we can derive an expression for the number $1 = \mathbf{d}^*(r) - \mathbf{d}(l)$ as a power series in $(-\beta)^{-1}$ whose integer coefficients are arranged in an eventually periodic order. Such an expression gives rise to a monic polynomial with integer coefficients having $\beta$ as a root.

The following theorem is an almost reverse to Theorem 24. Assuming eventually periodic $(-\beta, l)$-expansion of all rationals in the interval $[l, r)$, we derive that $\beta$ must be Pisot or Salem. Here we would like to point out that the original proof of Schmidt could not be adapted so easily because of the phenomena of degenerated intervals mentioned in Section 6.
Theorem 26. Let $d$ be the $(-\beta, l)$-expansion. If any rational $x \in [l, r)$ has eventually periodic $(-\beta, l)$-expansion, then $\beta$ is either Pisot or Salem number.

Proof. First realize that by assumption there exists a rational number from $[l, r)$ of the form $\frac{1}{m}$, $m \in \Z$ with eventually periodic expansion. It is easy to see that this gives us a monic polynomial from $\Z[x]$ with $\beta$ as its root, hence $\beta$ is an algebraic integer. It remains to show that all conjugates of $\beta$ are in modulus smaller than or equal to 1.

From Lemma 9 we derive that there exists a non-zero digit $a$ such that for all $N \in \N$ we can find a rational number $y$ satisfying

$$d(y) = a^N y_1 y_2 y_3 \cdots .$$

Expression (34) can be rewritten

$$y = \frac{a}{(-\beta)} + \cdots + \frac{a}{(-\beta)^N} + \sum_{i=N+1}^{\infty} \frac{y_{i-N}}{(-\beta)^i} = a \frac{1 - (-\beta)^{-N}}{\beta + 1} + \sum_{i=1}^{\infty} \frac{y_i}{(-\beta)^{i+N}}$$

and so

$$y = \frac{a}{\beta + 1} + \frac{1}{(-\beta)^N} \left( - \frac{a}{\gamma + 1} + \sum_{i=1}^{\infty} \frac{y_i}{(-\gamma)^i} \right).$$

(35)

As $y \in \Q$, by assumption, the infinite word $y_1 y_2 y_3 \cdots$ is eventually periodic and by summing a geometric series, $\sum_{i=1}^{\infty} \frac{y_i}{(-\beta)^i}$ can be rewritten as

$$\sum_{i=1}^{\infty} \frac{y_i}{(-\beta)^i} = c_0 + c_1 \beta + \cdots + c_{d-1} \beta^{d-1} \in \Q(\beta),$$

where $d$ is the degree of the algebraic integer $\beta$. In order to prove the theorem by contradiction, assume that a conjugate $\gamma \neq \beta$ is in modulus greater than 1. By application of the isomorphism between $\Q(\beta)$ and $\Q(\gamma)$, we get

$$c_0 + c_1 \gamma + \cdots + c_{d-1} \gamma^{d-1} = \sum_{i=1}^{\infty} \frac{y_i}{(-\gamma)^i},$$

and thus from (35) we get

$$y = \frac{a}{\gamma + 1} + \frac{1}{(-\gamma)^N} \left( - \frac{a}{\gamma + 1} + \sum_{i=1}^{\infty} \frac{y_i}{(-\gamma)^i} \right).$$

(36)

Now denote $\eta = \max\{|\beta|^{-1}, |\gamma|^{-1}\} < 1$ and realize that $[\beta]$ estimates the greatest digit in modulus. Comparing (35) and (36), we obtain

$$0 < \left| \frac{a}{\beta + 1} - \frac{a}{\gamma + 1} \right| \leq \eta^N \left( \left| \frac{a}{\beta + 1} - \frac{a}{\gamma + 1} \right| + 2 \sum_{i=1}^{\infty} \eta^i \right).$$

(37)

Obviously, the value in the bracket on the right hand side of (37) is a finite constant independent on $N$, and thus the right hand side tends to zero with $N$ increasing to infinity. This leads to contradiction, since on the left hand side of (37) we have a fixed positive number.

$\Box$
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