Double-Ended Palindromic Trees:  
A Linear-Time Data Structure and Its Applications

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Abstract

The palindromic tree (a.k.a. eertree) is a linear-size data structure that provides access to all palindromic substrings of a string. In this paper, we propose a generalized version of eertree, called double-ended eertree, which supports linear-time online double-ended queue operations on the stored string. At the heart of our construction, is a class of substrings, called surfaces, of independent interest. Namely, surfaces are neither prefixes nor suffixes of any other palindromic substrings and characterize the link structure of all palindromic substrings in the eertree.

As an application, we develop a framework for range queries involving palindromes on strings, including counting distinct palindromic substrings, and finding the longest palindromic substring, shortest unique palindromic substring and shortest absent palindrome of any substring. In particular, offline queries only use linear space. Apart from range queries, we enumerate palindromic rich strings with a given word in linear time on the length of the given word.

Keywords: Palindromes, eertrees, double-ended data structures, string algorithms.

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References
1 Introduction

Palindromes  Palindromes, which read the same backward as forward, are an interesting and important concept in many fields, e.g., linguistics [Ber73], mathematics [HS98, BHS04], physics [HKR95], biology [Lea94, LRR08, GPHD03], and computer science [Gus97]. Especially, combinatorial properties of palindrome complexity, i.e., the number of palindromes, of finite and infinite strings have been extensively studied in [Dro95, All97, DP99, Baa99, Dam00, DZ00, ABCD03, BHNR04, AAK10].

Palindromes as strings were first studied from an algorithmic perspective by Slisenko [Sli73] (cf. [Gal78]). Knuth, Morris and Pratt [KMP77] developed the well-known string matching algorithm, and applied it in recognizing concatenations of even-length palindromes. Soon after, this result was improved in [GS78] to concatenations of non-trivial palindromes. Manacher [Man75] invented a linear-time algorithm to find all palindromic prefixes of a string. Later, it was found in [ABG95] that Manacher’s algorithm can be used to find all maximal palindromic substrings, and an alternative algorithm for the same problem was proposed in [Jen94] based on suffix trees [Wei73, McC76, Ukk95, Far97]. Several parallel algorithms to find palindromic substrings were also designed [CR91b, ABG95, BG95]. Finding the longest palindromic substring was also studied in the streaming model [BEKMTSA14, GMSU19], and after single-character substitution [FNI21]. Recently, it was shown in [CPR22] that the longest palindromic substring can be found in sublinear time. In addition, a quantum algorithm for this problem was found in [LGS22] with quadratic speedup over classical algorithms. Very recently, computing maximal generalized palindromes was studied in [FMN22].

Since it was pointed out in [DJP01] that a string $s$ of length $|s|$ has at most $|s| + 1$ distinct palindromic substrings (including the empty string), many algorithms concerning palindromes have been successively proposed. Groult, Prieur and Richomme [GPR10] found a linear-time algorithm to count the number of distinct palindromic substrings of a string, and used it to check the palindromic richness of a string. Here, a string $s$ is called palindromic rich if it contains the maximum possible $|s| + 1$ distinct palindromic substrings [GJWZ09]. Palindrome pattern matching was studied in [IT13], where two strings are matched if they have the same positions of all their maximal palindromic substrings [IBT10]. Another line of research focused on concatenations of palindromes. The palindromic length of a string is the minimal $k$ such that it is a concatenation of $k$ palindromes [Rav03]. A linear-time algorithm for recognizing strings of a fixed palindromic length $k$ was given in [KRS15]. Several algorithms for computing the palindromic length were proposed [FGKK14, ISI14, RS18, BKRS17].

Recently, Rubinchik and Shur [RS18] proposed a linear-size data structure, called eertree or known as the palindromic tree, which stores all distinct palindromic substrings of a string and can be constructed online in linear time. The size of the eertree is much smaller than the length $n$ of the string on average, because the expected number of distinct palindromic substrings of a string is known to be $O(\sqrt{n})$ [AAK10, RS16]. Using this powerful data structure, they enumerated palindromic rich strings of length up to 60 (cf. sequence A216264 in OEIS [Slo64]), and reproduced a different algorithm from [FGKK14, ISI14, RS18, BKRS17] to compute the palindromic length of a string. Later, an online algorithm to count palindromes in substrings was designed in [RS17] based on eertrees. Furthermore, Mieno, Watanabe, Nakashima, et al. [MWN22] developed a type of eertree for a sliding window.

Double-ended data structures  A double-ended queue (abbreviated to deque, cf. [Tar83, Knu97]) is an abstract data structure consisting of a list of elements on which four kinds of operations can be performed:
• push_back(c): Insert an element c at the back of the deque.
• push_front(c): Insert an element c at the front of the deque.
• pop_back(): Remove an element from the back of the deque.
• pop_front(): Remove an element from the front of the deque.

If only operations allowed are push_back and pop_back (or push_front and pop_front), then the deque becomes a stack. If only push_back and pop_front (or push_front and pop_back) are allowed, then the deque becomes a queue.

As a linear data structure, deques are widely used in practical and theoretical computer science with their implementations supported in most high-level programming languages, e.g., C/C++, Java, Python, etc. Deques were investigated in different computational models, including Turing machines [Kos79], RAMs (random access machines) [Kos94] and functional programming [Hoo82, CG93, Oka95, BT95, KT99]. In addition to the basic deque operations, a natural extension of deque is to maintain more useful information related to the elements stored in it. For example, mindeque (a.k.a. deque with heap order) [GT86] is a kind of extended deque that supports the find_min operation, which finds the minimal element in the deque. Furthermore, a more powerful extension of mindeque, called catenable mindeque, was developed in [BST95], which supports catenation of two mindeque.

1.1 Main results

In this paper, we study an extension of deque, called double-ended eertree, which processes a string \( s \) (initialized to empty) with four kinds of basic operations supported:

• push_back(c): Insert a character c at the back of the string. That is, set \( s \leftarrow sc \).
• push_front(c): Insert a character c at the front of the string. That is, set \( s \leftarrow cs \).
• pop_back(): Remove a character from the back of the string. That is, set \( s \leftarrow s[1..|s| − 1] \), provided that \( s \) is not empty.
• pop_front(): Remove a character from the front of the string. That is, set \( s \leftarrow s[2..|s|] \), provided that \( s \) is not empty.

In addition to the basic operations above, we can also make online queries to the current string, including but not limited to the following:

• Find the number of distinct palindromic substrings of the current string.
• Find the longest palindromic prefix of the current string, and check whether it is unique in the current string.
• Find the longest palindromic suffix of the current string, and check whether it is unique in the current string.

The data structure adopted in our implementation is an extended eertree. This indicates that we always have access to the eertree of the current string, and therefore basic operations of the eertree are naturally inherited. Roughly speaking, we proposed a self-organizing data structure double-ended eertree which supports online deque operations on the stored string in linear time, which is stated in the following theorem.
**Theorem 1.1** (Double-ended eertrees, Theorem 5.14 restated). *Double-ended eertree can be implemented with worst-case time and space complexity $O(\log(\sigma))$ per operation, where $\sigma$ is the size of the alphabet.*

The double-ended eertree is assumed to work in the word RAM model [FW90] under the constant cost criterion (see Section 2.3 for the formal definition), which considers any operations from the C programming language as constant time, thereby a more practical computational model than RAM [CR73]. The word RAM model is arguably the most widely used computational model for practical algorithms (cf. [Hag98]). In practice, the size $\sigma$ of the alphabet is usually a constant, e.g., $\sigma = 2$ for binary strings, $\sigma = 4$ for DNA sequences, and $\sigma = 26$ for English dictionaries; in this case, the implementation of double-ended eertrees in Theorem 1.1 achieves worst-case time and space complexity $O(1)$ per operation.

For comparison, we collect known implementations of different eertrees in Table 1. If only basic operations `push_back` and `pop_back` are allowed, the double-ended eertree is called a stack eertree; and if only basic operations `push_back` and `pop_front` are allowed, the double-ended eertree is called a queue eertree. See Section 1.2 for further discussions.

| Eertree Type              | Time Complexity Per Operation |
|--------------------------|------------------------------|
| Stack Eertree [RS18]     | $O(\log(\sigma))$           |
| Queue Eertree [MWN+22]   | $O(\log(\sigma))$           |
| Double-Ended Eertree (This Paper) | $O(\log(\sigma))$ |

**Persistent double-ended eertrees** Furthermore, we study the persistent version of double-ended eertrees. They are useful in answering online range queries.

**Theorem 1.2** (Persistent double-ended eertrees, Theorem 5.15 restated). *Fully persistent double-ended eertrees can be implemented with worst-case time and space complexity $O(\log(n) + \log(\sigma))$ per operation, where $\sigma$ is the size of the alphabet and $n$ is the length of the string in the current version.*

Here, a data structure is fully persistent if every version of it can be both accessed and modified. By contrast, a data structure is partially persistent if every version of it can be accessed but only the latest version can be modified.

### 1.1.1 Applications

As an application, we apply our double-ended eertree in several computational tasks (see Table 2 for an overview).

**Range queries concerning palindromes** We studied online and offline range queries concerning palindromes on a string $s[1..n]$ of length $n$. Each query is of the form $(l, r)$ and asks problems of different types on substring $s[l..r]$:

- **Counting Distinct Palindromic Substrings**: Find the number of distinct palindromic substring of $s[l..r]$.
- **Longest Palindromic Substring**: Find the longest palindromic substring of $s[l..r]$.  

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Table 2: Applications of double-ended eertrees.

| Computational Task                                      | Our Method          | Known Methods                  |
|---------------------------------------------------------|---------------------|--------------------------------|
| **COUNTING DISTINCT**                                   | \( O(n\sqrt{q}) \) * | \( O(nq) \)†                  |
| PALINDROMIC SUBSTRINGS                                  |                     | \( O((n + q) \log(n)) \)† [RS17] |
| **LONGEST PALINDROMIC SUBSTRING**                      | \( O(n\sqrt{q}) \) * | \( O(nq) \)†                  |
|                                                          |                     | \( O(n \log^2(n) + q \log(n)) \) [ACPR20] |
| **SHORTEST UNIQUE PALINDROMIC SUBSTRING**              | \( O(n\sqrt{q}\log(n)) \) * | \( O(nq) \)†                  |
| **SHORTEST ABSENT PALINDROME**                         | \( O(n\sqrt{q} + q \log(n)) \) * | \( O(nq + q \log(n)) \)†      |
| **COUNTING RICH STRINGS with Given Word**              | \( O(n\sigma + k\sigma^k) \) | \( O(k\sigma^k(n + k)) \)†     |

* Complexity for offline queries.
† These are straightforward algorithms equipped with eertrees [RS18] or other algorithms concerning palindromes (e.g., [Man75, KMP77, GPR10]) as a subroutine.

- **Shortest Unique Palindromic Substring**: Find the shortest unique palindromic substring of \( s[l..r] \).
- **Shortest Absent Palindrome**: Find the shortest absent palindrome of \( s[l..r] \).

The trick we adopt is called Mo’s algorithm (also known as the Rectilinear Steiner Minimal Arborescence technique [KLZU06]), which is a general technique for range queries (cf. [DKPW20]). The basic operations in Mo’s algorithm are adding and deleting elements at both ends of a list, which are naturally supported in our double-ended eertree. Therefore, we can answer the aforementioned range queries efficiently.

**Corollary 1.3** (Corollary 6.1, 6.3, 6.5, 6.7 and 6.8 combined). *Online and offline range queries of type Counting Distinct Palindromic Substrings, Longest Palindromic Substring, Shortest Unique Palindromic Substring and Shortest Absent Palindrome can be answered with total time and space complexity \( \hat{O}(n\sqrt{q}) \) where \( n \) is the length of the string and \( q \leq n^2 \) is the number of queries.*

*Moreover, the space complexity of offline queries can be reduced to \( O(n) \) when \( \sigma = O(1) \).*

The reduced space complexity for offline queries in Corollary 1.3 is significant, since the size \( \sigma \) of the alphabet is usually a constant in practice as already mentioned above.

As shown in Table 2, double-ended eertrees bring speedups to all mentioned computational tasks with the only exceptions

1. It was shown in [RS17] that range queries of **Counting Distinct Palindromic Substrings** can be answered in \( O((n + q) \log(n)) \) time assuming \( \sigma = O(1) \). Our method can be still faster than their method when \( q = o(\log^2(n)) \) or \( q = \omega(\frac{n^2}{\log^2(n)}) \).

\( \hat{O}(\cdot) \) suppresses polylogarithmic factors of \( n, q \) and \( \sigma \).

2. The concurrent work of Mitani, Mieno, Seto, and Horiyama [MMSH22] appeared on arXiv on the same day of this paper. They showed that range queries of **Longest Palindromic Substring** can be solved in linear time \( O(n + q) \) assuming \( \sigma = O(1) \).
2. It was shown in [ACPR20] that range queries of **Longest Palindromic Substring** can be answered in $O(n \log^2(n) + q \log(n))$ time assuming $\sigma = O(1)$. Our method can be still faster than their method when $q = o(\log^4(n))$ or $q = \omega(n^2 / \log^2(n))$.

**Enumerating rich strings with given word** Palindromic rich strings have been extensively studied [GJWZ09, RR09, BDLGZ09, Ves14]. Recently, the number of rich strings of length $n$ was studied [RS18, GSS16, Ruk17]. Using double-ended eertree, we give an algorithm for COUNTING RICH STRINGS WITH GIVEN WORD (see Problem 5 for the formal statement).

**Corollary 1.4 (Corollary 6.9 restated).** There is an algorithm for COUNTING RICH STRINGS WITH GIVEN WORD, which computes the number of palindromic rich strings of length $n + k$ with a given word of length $n$ with time complexity $O(n \sigma + k \sigma^k)$, where $\sigma$ is the size of the alphabet.

By contrast, a naïve algorithm that enumerates all (roughly $k \sigma^k$ in total) possible candidates and then checks each of them by [GPR10] in $O(n + k)$ time would have time complexity $O(k \sigma^k(n + k))$.

The strength of our algorithm is that the parameters $n$ and $\sigma^k$ in the complexity are additive, while they are multiplicative in the naïve algorithm.

### 1.2 Our techniques

We propose a framework of double-ended eertrees (see Section 3). The difficulty to implement double-ended eertrees lies in two parts, namely, checking the uniqueness of palindromic substrings and maintaining the longest palindromic prefix and suffix. Under this framework, we propose two methods to implement double-ended eertrees, namely the occurrence recording method and the surface recording method, with their comparison in Table 3. The former is intuitive and easy to understand, and the latter is more efficient based on a new concept called surface.

| Attribute          | Occurrence Recording | Surface Recording |
|--------------------|----------------------|-------------------|
| push_back & push_front | Time Complexity | $O(\log(\sigma) + \log(n))$ | $O(\log(\sigma))$ |
|                    | Space Complexity     | $O(\log(\sigma))$ | $O(\log(\sigma))$ |
| pop_back & pop_front | Time Complexity      | $O(\log(n))$ | $O(1)$ |
|                    | Space Complexity     | $O(1)$ | $O(1)$ |
| Auxiliary Data     | $S$, prenode, sufnode | cnt, presurf, sufsurf |

† amortized complexity.

#### 1.2.1 Occurrence recording method

To overcome the difficulties mentioned above, we first propose a sub-optimal algorithm to implement double-ended eertrees with time complexity per operation $O(\log(\sigma) + \log(n))$, where $n$ is the length of the current string, based on the occurrence recording method (see Section 4).

For each palindromic substring $t$ of the current string $s$, let $v = \text{node}(t)$ denote the node in the eertree that represents string $t$. The idea is to store a reduced set $S(s, v)$, defined by Eq. (4), of some selected occurrences of $t$ in $s$, with the total size of $S(s, v)$ over every node $v$ being $O(n)$. By contrast, the number of all occurrences over all palindromic substrings of $s$ can be roughly $O(n^2)$. With the help of $S(s, v)$, we are able to check the uniqueness of a palindromic substring directly.
(see Lemma 4.1), as well as maintain the longest palindromic prefix and suffix of $s$ indirectly, under deque operations. This part of the method can be seen as a generalization of occurrence recording in queue eertrees [MWN+22], wherein the last two occurrences are recorded. By contrast, our method records a reduced set of occurrences.

Specifically, we store two sets $\text{prenode}(s, i)$ and $\text{sufnode}(s, i)$ related to the reduced sets $S(s, v)$ for every $1 \leq i \leq |s|$, defined by Eq. (6) and (7). Roughly speaking, $\text{prenode}(s, i)$ (resp. $\text{sufnode}(s, i)$) collects all nodes $v$ with reduced occurrence $i$ (resp. end position $i$), i.e., $i \in S(s, v)$ (resp. $i - \text{len}(v) + 1 \in S(s, v)$), where $\text{len}(v)$ denotes the length of the palindrome that $v$ represents. It is worth noting that the longest palindromic prefix (resp. suffix) of $s$ is indeed the palindrome represented by the node of the longest length in $\text{prenode}(s, 1)$ (resp. $\text{sufnode}(s, |s|)$) (see Lemma 4.3). Since the two series of sets $\text{prenode}(s, i)$ and $\text{sufnode}(s, i)$ can be maintained by balanced binary search trees as the reduced sets $S(s, v)$ change, the key is to maintain $S(s, v)$ efficiently.

A subtle observation yields that $S(s, v)$ can be maintained by modifying only amortized $O(1)$ elements for each deque operation (see Section 4.2 for details). With this, we design an online algorithm to implement double-ended eertrees with time complexity $O(\log(\sigma) + \log(n))$ per operation, where the $\log(n)$ term comes from operations of balanced binary search trees. The method to maintain the reduced sets $S(s, v)$ is simple, but the proof of the correctness turns out to be surprisingly complicated (see Lemma 4.4 and Lemma 4.6). To this end, we find some useful properties of palindromes and eertrees such as Lemma 2.6 and Lemma 2.7, which may be of independent interest.

1.2.2 Surface recording method

Inspired by the reduced sets in the occurrence recording method (see Lemma 5.8), we identify a class of substrings, called surfaces (see Definition 2.1), that can characterize how palindromic substrings are distributed in the string. Roughly speaking, a surface in a string $s$ is a palindromic substring of $s$ that is not a prefix or suffix of any other palindromic substrings of $s$. The name “surface” is chosen to denote not being “covered” by anyone else. Intuitively, long palindromes derive shorter ones, but a surface is such a palindrome that is not implied by any other palindromes. For better understanding, Lemma 2.2 gives an alternative definition of surfaces. For example, in the string $s = \text{abacaba}$, $s[1..7]$ is a surface but $s[1..3]$ is not because $s[1..3]$ is a palindromic proper prefix of, therefore “covered” by, $s[1..7]$. In other words, surface $s[1..7]$ naturally implies shorter palindromes $s[1..3]$ and $s[5..7]$.

The key of the surface recording method is to maintain all surfaces in the string implicitly. To achieve this, we store two lists $\text{presurf}(s, i)$ and $\text{sufsurf}(s, i)$, defined by Eq. (21) and (22), similar to the sets $\text{prenode}(s, i)$ and $\text{sufnode}(s, i)$ used in the occurrence recording method. By contrast, $\text{presurf}(s, i)$ and $\text{sufsurf}(s, i)$ only store (the pointer to) a node. Specifically, $\text{presurf}(s, i)$ (resp. $\text{sufsurf}(s, i)$) indicates the surface with occurrence $i$ (resp. with end position $i$); or the empty string $\epsilon$, if no such surface exists. It is shown in Lemma 5.9 that the longest palindromic prefix (resp. suffix) of $s$ is the leftmost surface with start position 1 (resp. the rightmost surface with end position $|s|$), whose corresponding node is $\text{presurf}(s, 1)$ (resp. $\text{sufsurf}(s, |s|)$). Therefore, the longest palindromic prefix and suffix are naturally obtained if we maintain $\text{presurf}(s, i)$ and $\text{sufsurf}(s, i)$. Looking into the relationship between surfaces, we find a way to maintain $\text{presurf}(s, i)$ and $\text{sufsurf}(s, i)$ by modifying only $O(1)$ elements with time complexity $O(1)$ per deque operation (see Section 5.3).

It remains to check the uniqueness of palindromic substrings. To achieve this, we find an efficient algorithm to maintain the number of occurrences of each existing palindromic substrings of the current string, which is rather simple. We define $\text{precnt}(s, v)$ by Eq. (17) (resp. $\text{sufcnt}(s, v)$
by Eq. (13) to denote the number of occurrences of node $v$ in string $s$ that are not a prefix (resp. suffix) of a longer palindromic substring. Surprisingly, we find that $pren(s,v) = sufcnt(s,v)$ always holds (see Lemma 5.1), thereby letting $cnt(s,v)$ denote either of them; also, the number of occurrences of every palindromic substring, whose corresponding node is $v$, is the sum of $cnt(s,u)$ over all nodes $u$ in the link tree rooted at node $v$ (see Lemma 5.6). At last, a simple update rule of $cnt(s,v)$ is established for each deque operation with time complexity $O(1)$ per operation (see Lemma 5.7). It is clear that uniqueness means only one occurrence, and the number of occurrences can be computed through $cnt(s,v)$.

1.3 Related works

**String processing** As not mentioned above, approximate palindromes are strings close to a palindrome with gaps or mismatches, which were investigated in a series of works [Gus97, Por99, PB02, HCC10, CHC12, AP14]. As in information processing, the study of palindromes aims to find efficient algorithms for strings concerning their palindromic structures. Trie [DLB59, Fre60] is one of the earliest data structures that can store and look up words in a dictionary, which is widely used in string-searching problems with its extensions developed, including Aho-Corasick automata [AC75, Mey85] and suffix trees [Wei73, McC76, Ukk95, Far97]. Suffix arrays [MM93, GBYS92] were introduced to improve the space complexity of suffix trees, and were later improved to linear-time [KSB06, NZC09]. Suffix trees with deletions on one end and insertions on the other end, namely, suffix trees that support queue operations or also called suffix trees in a sliding window, were studied in a series of work [FG89, Lar96, IS11, CHK+20, MKA+20]. However, we are not aware of any suffix trees with insertions and deletions on both ends.

**Persistent data structures** The notion of persistent data structures (cf. [Die89, Oka98, Str13]) were first introduced and investigated in [DSST89]. Since then, many basic persistent data structures have been studied, e.g., stacks [Mye83], lists [DST94], priority queues [BO96], tries [OG98, Phi01], etc. Recently, persistent techniques for suffix trees were also investigated in [HIA00, BH04, KLP11], with applications to range queries on strings [KLP11].

**Range queries** Range queries are a series of computational tasks which ask certain properties of any subset of the database, which are proved to be especially useful in computational geometry (e.g., [Wil82, AE99]). Orthogonal range queries on a $k$-dimensional space were first investigated with space-partitioning data structures, e.g., quad tree [PB74] and $k$-d tree [Ben75] with time complexity $O(n^{1/k})$ per operation. Later, new data structures for orthogonal range queries were proposed with time complexity $O(\log^k(n))$ per operation [Lue78, Ben80, Wil85]. Orthogonal range queries were further studied in [Cha90a, Cha90b]. As a special and important case, array range queries (i.e., orthogonal range queries in 1-dimensional space; cf. [Ska13]) were then extensively studied, including range sum [Fre82, Yao82, Yao85], range minimum [HT84, SV88, BFC00, BFCP+05], range medium [KMS05, HPM08, BGJS11, HNN11, JL11], range mode [KMS05, PG09, Pet08, CDL+14, GPVWX21], etc.

It turns out that offline queries can have computational advantages over online queries (e.g., [GTS05, CK91a, Cha97, CP10]). Recently, it was shown in [DKPW20] that offline range inversion queries can be solved more efficiently than using Mo’s algorithm. Here, Mo’s algorithm is a very general technique for a wide range of offline range queries and can achieve $\tilde{O}(n^{3/k})$ time for answering $q$ queries on an array of length $n$ (also known as the Rectilinear Steiner Minimal Arborescence technique [KLZU06]). It was also mentioned in [DKPW20] that Mo’s algorithm can be modified
to support online queries with persistent data structures if the number \( q \) of queries is known in advance.

Range queries concerning strings but not directly related to palindromes were also extensively studied in the literature, e.g., longest common subsequence [Tis08, Sak19, CGMW21, Sak22], longest common prefix [AAL+14, GPST18, AGH+20, MSST20], longest common substring [ACPR20], minimal and maximal suffixes [BGK+16], minimal rotation [Koc16], approximate pattern matching [CKW20], dictionary matching [CKM+21], shortest unique substring [AGPT20], and shortest absent word [BCKP22]. Counting the occurrences of a substring in another was studied in [KRRW15].

1.4 Discussion

1.4.1 Surfaces

A similar notion to surface is the maximal palindromic substring, which has been extensively studied in the literature [Man75, GS78, Jeu94, ABG95, IIBT10, IIIT13]. A maximal palindromic substring is a substring not contained in any other palindromic substrings. By contrast, a surface is a substring not contained in any other palindromic substrings as their prefix or suffix. It can be seen that any maximal palindromic substring is a surface, but not vice versa. From the perspective of eertrees, maximal palindromic substrings characterize the trie-like structure of eertrees while surfaces characterize the link tree of eertrees.

Another similar notion to surface is the border-maximal palindrome [MWN+22]. A border-maximal palindrome is a palindrome that is not a proper suffix of other palindromic substrings. The difference between surfaces and border-maximal palindromes is that the former is index sensitive but the latter is not. For example, \( bab \) is not a border-maximal palindrome in \( s = aababbaababab \) because it is a proper suffix of \( s[9..13] = babab \). By contrast, \( s[3..5] = bab \) is a surface in \( s \) because it is not contained in any other palindromic substrings as their prefix or suffix. Intuitively, border-maximal palindromes characterize global properties of a string while surfaces catch local properties.

As shown in this paper, we find that surfaces are quite useful in palindrome related problems. We are not aware that the concept of surface has been defined and used elsewhere but we believe that it will bring new insights into string processing.

1.4.2 Other possible implementations

It has been shown in [RS18] that eertree is a very efficient data structure to process palindromes. Theoretically, one might doubt whether it is possible to implement basic operations of double-ended eertrees in other ways, with or without the notion of eertrees.

By suffix trees Intuitively, one might wonder whether it is possible to adapt suffix tree tricks (cf. [Gus97]) to implementing double-ended eertrees. A common way concerning palindromic substrings of a string \( s \), for example, is to build a suffix tree of \( s\$s^R \), where \( s^R \) is the reverse of \( s \) and \$ is any character that does not appear in \( s \). Then, palindromic substrings of \( s \) can be found using longest common extension queries on \( s \) and \( s^R \). This kind of trick was already used in finding the longest palindrome [Gus97], counting palindromes [GPR10], and finding distinct palindromes online [KRS13]. However, in our case, after a \texttt{push_back}(c) (resp. \texttt{push_front}(c)) operation, the suffix tree of \( sc\$cs^R \) (resp. \( cs\$s^Rc \)) is required; also, after a \texttt{pop_back} (resp. \texttt{pop_front}) operation, the suffix tree of \( s'\$s'^R \) is required, where \( s' = s[1..|s| - 1] \) (resp. \( s' = s[2..|s|] \)). As can be seen, to support deque operations together with queries concerning palindromes on string \( s \), we probably need complicated modifications on suffix trees, such as inserting and removing characters at both
ends and certain intermediate positions of a string. Therefore, it seems difficult to implement a double-ended eertree directly using suffix trees.

**By other variants of eertrees** Since the double-ended eertree can be considered as an extension of stack eertree [RS18] or queue eertree [MWN+22], it might be possible to implement the double-ended eertree using the tricks in stack eertree and queue eertree at the first glance. Now we discuss about this issue in the following.

- The stack eertree proposed in [RS18] depends on the structure of stack. That is, a character will not change before all characters after it are removed. Based on this fact, a backup strategy is to store necessary information, e.g., the node of the current longest suffix palindromic substring, when a character is inserted; and restore the previous configuration of the eertree through the backups when the last character is removed. In the scene of double-ended eertrees, characters at the very front of the string can apparently be removed or inserted, thereby making the backup strategy no longer effective.

- The queue eertree proposed in [MWN+22] requires auxiliary data structures for removing characters, where the key technique is to check whether a palindromic substring is unique in the current string. To achieve this, they maintain the rightmost occurrence and the second rightmost occurrence of every node in the eertree in a lazy manner. When a character is inserted at the back, the occurrences of only the longest palindromic suffix are updated; when a character is removed from the front, we do not have to deal with the occurrences of the longest palindromic prefix, but its longest palindromic proper prefix (by lazy maintenance). This approach, however, is based on the monotonicity of queue operations. That is, the rightmost occurrence can become the second rightmost occurrence, but not vice versa. In the scene of double-ended eertrees, the lazy strategy will fail when characters are removed from the back, because this time the second rightmost occurrence can be the rightmost occurrence.

As discussed above, we cannot obtain a straightforward implementation of double-ended eertrees from similar or related data structures. Nevertheless, we are glad to see if there exist other efficient implementations of double-ended eertrees.

1.4.3 Potential applications in bioinformatics

As mentioned in [RS18], eertrees could have potential in Watson-Crick palindromes [KM10,MMP22] and RNA structures [MP05,Str07,MBK+11,BAI5]. In addition, gene editing is an emerging research field in biology (cf. [BGOP18]), e.g. the clustered regularly interspaced short palindrome repeats (CRISPR)-Cas9 system (cf. [BD16,LHCT18]). One might require real-time palindrome-related properties while editing DNA and RNA sequences. We believe that double-ended eertrees could have potential practical applications in such tasks.

1.5 Organization of this paper

In the rest of this paper, we first introduce preliminaries in Section 2. Then, we define a framework of implementing double-ended eertrees in Section 5. Under the framework, we propose an occurrence recording method to maintain double-ended eertrees in Section 4, inspired by which we propose a more efficient method called surface recording in Section 5. Applications of double-ended eertrees are presented in Section 6. Finally, a brief conclusion is drawn in Section 7.
2 Preliminaries

2.1 Strings

Let \( \Sigma \) be an alphabet of size \( |\Sigma| \). A string \( s \) of length \( n \) over \( \Sigma \) is an array \( s[1]s[2] \ldots s[n] \), where \( s[i] \in \Sigma \) is the \( i \)-th character of \( s \) for \( 1 \leq i \leq n \). We write \(|s|\) to denote the length of string \( s \). Let \( \Sigma^n \) denote the set of all strings of length \( n \) over \( \Sigma \), and \( \Sigma^* \) denotes the set of all (finite) strings over \( \Sigma \). Especially, \( \epsilon \) denotes the empty string, i.e., \( |\epsilon| = 0 \). \( s[i..j] \) denotes the string \( s[i]s[i+1] \ldots s[j] \) if \( 1 \leq i \leq j \leq |s| \), and \( \epsilon \) otherwise. A string \( t \) is called a substring of \( s \) if \( t = s[i..j] \) for some \( i \) and \( j \). Moreover, a substring \( t = s[i..j] \) of string \( s \) is called a prefix of \( s \) if \( i = 1 \), and is called a suffix of \( s \) if \( j = |s| \). A prefix (resp. suffix) \( t \) of string \( s \) is proper if \( t \neq s \). In particular, the empty string \( \epsilon \) is a substring, prefix and suffix of any strings. A non-empty substring \( t \) of \( s \) is unique in \( s \) if \( t \) occurs only once in \( s \), i.e., there is only one pair of indexes \( i \) and \( j \) such that \( 1 \leq i \leq j \leq |s| \) and \( s[i..j] = t \). A string \( s \) is palindromic (or a palindrome) if \( s[i] = s[|s| - i + 1] \) for all \( 1 \leq i \leq |s| \). In particular, the empty string \( \epsilon \) is a palindrome. The center of a palindromic substring \( s[i..j] \) is \( l = (i + j)/2 \). The center of \( s[i..j] \) is an integer (resp. half-integer) if its length is odd (resp. even). A non-empty palindromic substring \( s[i..j] \) is a palindrome. The center of \( s[i..j] \) is called a surface in \( s \).

Remark 2.1. In Definition 2.1, the indexes \( i \) and \( j \) of \( t = s[i..j] \) are sensitive. Consider the string \( s = abacabaxyaba \). There are three occurrences \( s[1..3], s[5..7], s[10..12] \) of \( aba \) in \( s \). According to the definition, either \( s[1..3] \) or \( s[5..7] \) is not a surface in \( s \) because \( s[1..7] = abacaba \) is a palindrome with \( s[1..3] \) and \( s[5..7] \) being its prefix and suffix, respectively. However, it can be easily verified that \( s[10..12] \) is a surface in \( s \).

The following lemma shows a useful property of the periods of palindromes.

Lemma 2.1 (Periods of palindromes, Lemma 2 and Lemma 3 in [KRS15]). Suppose \( s \) is a non-empty palindrome and \( 1 \leq p \leq |s| \) is an integer. Then \( p \) is a period of \( s \) if and only if \( s[1..|s| - p] \) is a palindrome.

Suppose \( s \) is a string and \( 1 \leq i \leq |s| \). \( \text{prepal}(s,i) \) (resp. \( \text{sufpal}(s,i) \)) denotes the longest prefix (resp. suffix) palindromic substring of \( s[i..|s|] \) (resp. \( s[1..i] \)), and \( \text{prelen}(s,i) \) (resp. \( \text{suflen}(s,i) \)) denotes its length. Formally,

\[
\text{prepal}(s,i) = s[i..i + \text{prelen}(s,i) - 1], \\
\text{sufpal}(s,i) = s[i - \text{suflen}(s,i) + 1..i],
\]

where

\[
\text{prelen}(s,i) = \max \{ 1 \leq l \leq |s| - i - 1: s[i..i + l - 1] \text{ is a palindrome} \}, \\
\text{suflen}(s,i) = \max \{ 1 \leq l \leq i: s[i - l + 1..i] \text{ is a palindrome} \}.
\]

2.2 Surfaces

We introduce a new notion of surfaced palindromic substrings, also called surfaces, which will be used in developing our algorithms.

Definition 2.1 (Surface). A non-empty palindromic substring \( t = s[i..j] \) of \( s \), where \( 1 \leq i \leq j \leq |s| \), is called surfaced (or a surface) in \( s \), if neither \( s[i..r] \) nor \( s[l..j] \) is a palindrome for any \( 1 \leq l < i \leq j < r \leq |s| \).

Remark 2.1. In Definition 2.1, the indexes \( i \) and \( j \) of \( t = s[i..j] \) are sensitive. Consider the string \( s = abacabaxyaba \). There are three occurrences \( s[1..3], s[5..7], s[10..12] \) of \( aba \) in \( s \). According to the definition, either \( s[1..3] \) or \( s[5..7] \) is not a surface in \( s \) because \( s[1..7] = abacaba \) is a palindrome with \( s[1..3] \) and \( s[5..7] \) being its prefix and suffix, respectively. However, it can be easily verified that \( s[10..12] \) is a surface in \( s \).
Intuitively, a surface is a palindromic substring that is not covered by any other palindromic substrings (that is why we call it surfaced). Lemma 2.2 gives an alternative definition of surfaces.

**Lemma 2.2.** A palindromic substring \( t = s[i..j] \) of a non-empty string \( s \), where \( 1 \leq i \leq j \leq |s| \), is a surface in \( s \) if and only if \( \text{prelen}(s, i) = \text{suflen}(s, j) = j - i + 1 \).

**Proof.** Suppose \( t = s[i..j] \) is a non-empty palindromic substring of \( s \). Then \( \text{prelen}(s, i) \geq j - i + 1 \) and \( \text{suflen}(s, j) \geq j - i + 1 \).

\( \Rightarrow \). Suppose \( t \) is a surface in \( s \). We will prove \( \text{prelen}(s, i) = j - i + 1 \) by contradiction. If \( \text{prelen}(s, i) \neq j - i + 1 \), then \( \text{prelen}(s, i) > j - i + 1 \), which implies that \( s[i..i + \text{prelen}(s, i) - 1] \) is a palindrome. That is, \( s[i..r] \) is a palindrome with

\[
\begin{align*}
    r &= i + \text{prelen}(s, i) - 1 > i + (j - i + 1) - 1 > j.
\end{align*}
\]

By Definition 2.1 \( t = s[i..j] \) is not a surface in \( s \), which is a contradiction. Therefore, we have \( \text{prelen}(s, i) = j - i + 1 \). Similarly, we can also obtain that \( \text{suflen}(s, j) = j - i + 1 \).

\( \Leftarrow \). If \( \text{prelen}(s, i) = \text{suflen}(s, j) = j - i + 1 \), by the definition of \( \text{prelen}(s, i) \), we note that \( s[i..r] \) is not a palindrome for every \( r \geq i + \text{prelen}(s, i) = j + 1 > j \). Also by the definition of \( \text{suflen}(s, j) \), \( s[l..j] \) is not a palindrome for every \( l \leq j - \text{suflen}(s, j) \leq i - 1 < i \). By Definition 2.1 we conclude that \( t \) is a surface in \( s \).

We will show in the following lemma that the longest palindromic prefix and suffix of a string are surfaces in it.

**Lemma 2.3.** Suppose \( s \) is a non-empty string. Then \( s[1..\text{prelen}(s, 1)] \) and \( s[|s| - \text{suflen}(s, |s|) + 1..|s|] \) are surfaces in \( s \).

**Proof.** We only need to show that \( s[1..\text{prelen}(s, 1)] \) is a surface in \( s \) because of symmetry. To see this, assume that \( s[1..\text{prelen}(s, 1)] \) is not a surface in \( s \). By Definition 2.1 there is an index \( \text{prelen}(s, 1) < j \leq |s| \) such that \( s[1..j] \) is a surface in \( s \). However, the existence of \( j \) conflicts with the definition of \( \text{prelen}(s, 1) \). Therefore, \( s[1..\text{prelen}(s, 1)] \) is a surface in \( s \).

### 2.3 Word RAM

The word RAM (random access machine) [FW90] is an extended computational model of RAM [CR73]. In the word RAM model, all data are stored in the memory as an array \( A \) of \( w \)-bit words. Here, a \( w \)-bit words means an integer between 0 and \( 2^w - 1 \) (inclusive), which can be represented by a binary string of length \( w \). The instruction set of the word RAM is an analog of that of the RAM, with indirect addressing the only exception. In the word RAM, the value of \( A[i] \) can only store limited addresses from 0 to \( 2^w - 1 \), therefore we only have access to \( 2^w \) addresses. The input of size \( n \) is stored in the first \( n \) elements of \( A \) initially. In our case, we are only interested in algorithms with space complexity polynomial in \( n \), thereby assuming that \( w = \Omega(\log n) \) such that \( w \) is large enough.

For our purpose, our algorithms work in the word RAM under the constant cost criterion. That is, the cost of each instruction of the word RAM is a constant, i.e., \( O(1) \). In our algorithms, we only require the instruction set of the word RAM to contain basic arithmetic operations (addition and subtraction) but not multiplication, division or bit operations.
2.4 Eertrees

The eertree of a string \( s \) is a data structure that stores the information of all palindromic substrings of \( s \) in an efficient way [RS18]. Roughly speaking, the eertree is a finite-state machine that resembles trie-like trees with additional links between internal nodes, which is similar to, for example, Aho-Corasick automata [ACT5]. Formally, the eertree of a string \( s \) is a tuple

\[
\text{eertree}(s) = (V, \text{even}, \text{odd}, \text{next}, \text{link}),
\]

where

1. \( V \) is a finite set of nodes, each of which is used to represent a palindromic substring of \( s \).
2. \( \text{even}, \text{odd} \in V \) are two special nodes, indicating the roots of palindromes of even and odd lengths, respectively.
3. \( \text{next}: V \times \Sigma \rightarrow V \cup \{\text{null}\} \) describes the tree structure of \( \text{eertree}(s) \). Specifically, for every node \( v \in V \) and character \( c \in \Sigma \), \( \text{next}(v, c) \) indicates the outgoing edge of \( v \) labeled by \( c \). In case of \( \text{next}(v, c) = \text{null} \), it means that there is no outgoing edge of \( v \) labeled by \( c \). If \( \text{next}(v, c) = u \), we write \( \text{prev}(u) = v \) to denote the node \( v \) that points to \( u \). It is guaranteed that all of the outgoing edges together form a forest consisting of two trees, whose roots are \( \text{even} \) and \( \text{odd} \), respectively.
4. \( \text{link}: V \rightarrow V \) is the suffix link.

Let \( \text{str}(v) \) denote the palindromic string that is represented by node \( v \). For every node \( v \in V \) and character \( c \in \Sigma \) such that \( \text{next}(v, c) \neq \text{null} \), define \( \text{str}(\text{next}(v, c)) = c \text{str}(v)c \). Especially, \( \text{str}(\text{even}) = \epsilon \) is the empty string and \( \text{str}(\text{odd}) = \epsilon_{-1} \) is the empty string of length \(-1\), i.e., \( |\epsilon_{-1}| = -1 \). Here, \( c\epsilon_{-1} = c \) for every character \( c \in \Sigma \). We write \( \text{len}(v) = |\text{str}(v)| \) to denote the length of the palindrome represented by node \( v \). Especially, \( \text{len}(\text{even}) = 0 \) and \( \text{len}(\text{odd}) = -1 \). For every node \( v \in V \) and character \( c \in \Sigma \) such that \( \text{next}(v, c) \neq \text{null} \), we have \( \text{len}(\text{next}(v, c)) = \text{len}(v) + 2 \). For convenience, for every palindromic substring \( t \) of \( s \), let \( \text{node}(t) \in V \) be the node in \( \text{eertree}(s) \) such that \( \text{str}(\text{node}(t)) = t \). For every node \( v \in V \setminus \{\text{even}, \text{odd}\} \), let \( \text{occur}(s, v) \) be the set of start positions of occurrences of \( \text{str}(v) \) in \( s \), i.e.,

\[
\text{occur}(s, v) = \{1 \leq i \leq |s| - \text{len}(v) + 1: s[i..i + \text{len}(v) - 1] = \text{str}(v)\}, \tag{1}
\]

and it holds that \( \text{occur}(s, v) \neq \emptyset \). It is obvious that a palindromic substring \( t \) of \( s \) is unique in \( s \) if and only if \( |\text{occur}(s, \text{node}(t))| = 1 \). For an occurrence \( i \in \text{occur}(s, v) \) of \( \text{str}(v) \) in \( s \), we also call \( i \) the start position and \( i + \text{len}(v) - 1 \) the end position of this occurrence.

For every node \( v \in V \), \( \text{link}(v) \) is the node of the longest proper palindromic suffix of \( \text{str}(v) \), i.e.,

\[
\text{link}(v) = \arg \max_{u \in V} \{\text{len}(u): \text{str}(u) \text{ is a proper palindromic suffix of } \text{str}(v)\}.
\]

In particular, \( \text{link}(\text{even}) = \text{link}(\text{odd}) = \text{odd} \). Indeed, \( \text{link}: V \rightarrow V \) defines a rooted tree \( T = (V, E) \) called the link tree with \( \text{odd} \) being the root, where

\[
E = \{ (\text{link}(v), v) : v \in V \setminus \{\text{odd}\} \}.
\]

For every \( v \in V \), let \( T_v = (V_v, E_v) \) denote the subtree of node \( v \) in the link tree \( T \), i.e.,

\[
V_v = \{ u \in V : v \in \text{link}^*(u) \},
\]

\[
E_v = \{ (a, b) \in E : \{a, b\} \subseteq V_v \},
\]
where link\(^*\)(v) = \{ link\(^k\)(v): k \in \mathbb{N} \} = \{ v, link(v), link^2(v), \ldots \}, link\(^k\)(v) = link(link\(^{k-1}\)(v)) for \( k \geq 1 \), and link\(^0\)(v) = v.

For space efficiency, we are only interested in the minimal eertree of a string \( s \). That is, there is no redundant node in the eertree with respect to string \( s \). A node \( v \) in the eertree is redundant with respect to a string \( s \) if \( \text{occur}(s, v) = \emptyset \). Throughout this paper, we use eertree\((s)\) to denote the minimal eertree of string \( s \). For every node \( v \in V \) in eertree\((s)\), let link\(cnt\)(s, \( v \)) be the number of nodes which link to \( v \), i.e.,

\[
\text{link\(cnt\)}(s, v) = |\{ u \in V: \text{link}(u) = v \}|.
\] (2)

Note that node \( u \) in Eq. (2) is not redundant with respect to \( s \).

We first show a relationship between every node and its ancestors in the link tree.

**Lemma 2.4.** Suppose \( s \) is a string and \( u \) is a node in eertree\((s)\). For every \( v \in \text{link}\(^*\)(u) \) with \( \text{len}(v) \geq 1 \), we have \( \text{occur}(s, u) \cup (\text{occur}(s, u) + \text{len}(u) - \text{len}(v)) \subseteq \text{occur}(s, v) \), where \( \text{occur}(s, u) + \text{len}(u) - \text{len}(v) = \{ i + \text{len}(u) - \text{len}(v): i \in \text{occur}(s, u) \} \).

**Proof.** Let \( i \in \text{occur}(s, u) \). By the definition, \( s[i..i + \text{len}(u) - 1] = \text{str}(u) \) is a palindrome. Note that \( \text{str}(v) \) is a palindromic prefix (and also suffix) of \( \text{str}(u) \). It follows that \( s[i..i + \text{len}(u) - 1] = s[i + \text{len}(u) - \text{len}(v)..i + \text{len}(u) - 1] = \text{str}(v) \) are two occurrences of \( v \), i.e., \( \{ i, i + \text{len}(u) - \text{len}(v) \} \subseteq \text{occur}(s, v) \).

Inspired by Lemma 2.4, we realize that \( \text{occur}(s, v) \) actually implies certain occurrences of \( \text{link}(v) \) for every node \( v \in V \), provided that \( \text{len}(\text{link}(v)) \geq 1 \). For this reason, we define the complement of \( \text{occur}(s, v) \) by

\[
\overline{\text{occur}}(s, v) = \{ i + \text{len}(v) - \text{len}(\text{link}(v)): i \in \text{occur}(s, v) \}.
\] (3)

Intuitively, the complement \( \overline{\text{occur}}(s, v) \) of \( \text{occur}(s, v) \) contains occurrences of \( \text{str}(\text{link}(v)) \) which are implied by and symmetric to those occurrences in \( \text{occur}(s, v) \) with respect to the center of \( \text{str}(v) \) as shown in the following corollary.

**Corollary 2.5.** Suppose \( s \) is a string and \( u \) is a node in eertree\((s)\). If \( \text{len}(\text{link}(u)) \geq 1 \), then \( \text{occur}(s, u) \cup \overline{\text{occur}}(s, u) \subseteq \text{occur}(s, \text{link}(u)) \).

**Proof.** It is a straightforward corollary of Lemma 2.4 by letting \( v = \text{link}(u) \).

In the following, we show that the lengths of any node and its ancestors in the link tree are convex.

**Lemma 2.6.** Suppose \( s \) is a string and \( v \) is a node in eertree\((s)\). Let \( v_1 = \text{link}(v) \) and \( v_2 = \text{link}(v_1) \). If \( \text{len}(v_2) \geq 1 \), then \( \text{len}(v) + \text{len}(v_2) \geq 2 \text{len}(v_1) \).

**Proof.** We only need to consider the case that \( \text{len}(v_1) > \text{len}(v)/2 \). Otherwise, the inequality always holds because \( 2\text{len}(v_1) - \text{len}(v) \leq 0 \leq \text{len}(v_2) \).

Note that \( \text{str}(v)[1..\text{len}(v_1)] = \text{str}(v_1) \) is the longest palindromic proper prefix of \( \text{str}(v) \). By Lemma 2.4, \( p = \text{len}(v) - \text{len}(v_1) \) is the minimal period of \( \text{str}(v) \). Similarly, \( p_1 = \text{len}(v_1) - \text{len}(v_2) \) is the minimal period of \( \text{str}(v_1) \). Since every period of \( \text{str}(v) \) implies a period of \( \text{str}(v_1) \), we have \( \text{len}(v) - \text{len}(v_1) = p \geq p_1 = \text{len}(v_1) - \text{len}(v_2) \), i.e., \( \text{len}(v) + \text{len}(v_2) \geq 2 \text{len}(v_1) \).

By Lemma 2.6, we furthermore derive properties of complements of occurrences that will be useful in analyzing our algorithms.
Lemma 2.7. Suppose $s$ is a string and $v$ is a node in eertree$(s)$. Let $i \in \text{occur}(s, v)$, $v_1 = \text{link}(v)$ and $v_2 = \text{link}(v_1)$. If $\text{len}(v_2) \geq 1$, then

$$i + \text{len}(v) - \text{len}(v_1) \in \bigcup_{\text{link}(u) = v_2} \overline{\text{occur}}(s, u).$$

Proof. We consider two cases in the following.

Case 1. $\text{len}(v_1) \leq \text{len}(v)/2$. It can be verified that

$$s' = s[i + \text{len}(v_1) - \text{len}(v_2), i + \text{len}(v) - \text{len}(v_1) - 1]$$

is a palindrome. To see this, we note that $s[i..i + \text{len}(v) - 1] = \text{str}(v)$. It remains to show that $s' \neq \epsilon$. This can be seen by

$$|s'| = \text{len}(v) - 2(\text{len}(v_1) - \text{len}(v_2)) = \text{len}(v) - 2\text{len}(v_1) + 2\text{len}(v_2) \geq 2\text{len}(v_2) > 0.$$

Let $u = \text{node}(s')$. Since $\text{len}(u) = |s'| \geq 2\text{len}(v_2) > \text{len}(v_2)$, $\text{str}(v_2)$ is a proper palindromic suffix of $\text{str}(u)$. From this, we know that there exists an integer $k \geq 1$ such that $\text{link}^k(u) = v_2$. Let $u' = \text{link}^{k-1}(u)$, then $\text{link}(u') = v_2$. Since $i + \text{len}(v_1) - \text{len}(v_2) \in \text{occur}(s, u)$, by Lemma 2.4 we have

$$i + \text{len}(v_1) - \text{len}(v_2) + \text{len}(u) - \text{len}(u') = i + \text{len}(v) - \text{len}(v_1) + \text{len}(v_2) - \text{len}(u') \in \text{occur}(s, u'),$$

which implies $i + \text{len}(v) - \text{len}(v_1) \in \overline{\text{occur}}(s, u').$

Case 2. $\text{len}(v_1) > \text{len}(v)/2$. It can still be verified (in a different way) that

$$s' = s[i + \text{len}(v_1) - \text{len}(v_2), i + \text{len}(v) - \text{len}(v_1) - 1]$$

is a palindrome. To see this, we note that $s'$ has the same center as $s[i..i + \text{len}(v) - 1] = \text{str}(v)$. It remains to show that $s' \neq \epsilon$. By Lemma 2.6 we have $\text{len}(v_2) \geq 2\text{len}(v_1) - \text{len}(v)$. Then, $|s'| = \text{len}(v) - 2\text{len}(v_1) + 2\text{len}(v_2) \geq \text{len}(v_2) \geq 1$.

Case 2.1 $|s'| = \text{len}(v_2)$. In this case, $\text{len}(v_2) = 2\text{len}(v_1) - \text{len}(v)$, and we know that $s' = \text{str}(v_2)$ because $s'$ is a proper prefix of $s[i + \text{len}(v) - \text{len}(v_1), i + \text{len}(v) - 1] = \text{str}(v_1)$. Since $i \in \text{occur}(s, v)$, by Lemma 2.4 we have $i \in \text{occur}(s, v_1)$, and thus $i + \text{len}(v) - \text{len}(v_1) = i + \text{len}(v_1) - \text{len}(v_2) \in \overline{\text{occur}}(s, v_1)$.

Case 2.2 $|s'| > \text{len}(v_2)$. Let $u = \text{node}(s')$. Note that $s[i + \text{len}(v_1) - \text{len}(v_2), i + \text{len}(v_1) - 1] = \text{str}(v_2)$ is a proper suffix of $s[i..i + \text{len}(v_1) - 1] = \text{str}(v_1)$, and also a proper prefix of $s'$. From this, we know that there exists an integer $k \geq 1$ such that $\text{link}^k(u) = v_2$. Let $u' = \text{link}^{k-1}(u)$, then $\text{link}(u') = v_2$. Since $i + \text{len}(v_1) - \text{len}(v_2) \in \text{occur}(s, u)$, by Lemma 2.4 we have

$$i + \text{len}(v_1) - \text{len}(v_2) + \text{len}(u) - \text{len}(u') \in \text{occur}(s, u').$$

Thus $i + \text{len}(v_1) - \text{len}(v_2) + \text{len}(u) - \text{len}(u') + \text{len}(u') - \text{len}(v_2) = i + \text{len}(v) - \text{len}(v_1) \in \overline{\text{occur}}(s, u')$.

The above two cases together yield that there always exists a node $u$ with $\text{link}(u) = v_2$ such that $i + \text{len}(v) - \text{len}(v_1) \in \overline{\text{occur}}(s, u)$. These complete the proof.
2.4.1 Stack eertrees

Eertrees that support stack operations, which we call stack eertrees, were studied in \[RS18\], with an efficient trick, called the direct link, proposed for adding a new character to the end of the string. The direct link is an auxiliary information that allows us to find the longest suffix palindromic substring in \(O(1)\) time. Formally, the direct link \(dlink : V \times \Sigma \rightarrow V\) is defined by

\[
dlink(v, c) = \arg \max_{u \in \text{link}^*(v) \setminus \text{odd}} \{ \text{len}(u) : \text{str}(v)[\text{len}(v) - \text{len}(u)] = c \},
\]

and \(dlink(v, c) = \text{odd}\) if no \(u\) satisfies the conditions. Proposition 2.8 shows that the longest suffix palindromic substring can be found by using direct links, and that direct links can be maintained efficiently.

**Proposition 2.8** (Longest suffix palindromes using direct links \[RS18\]). Suppose \(s\) is a string and \(c\) is a character. We have

\[
suflen(sc, |sc|) = \text{len}(dlink(\text{node}(\text{sufpal}(s, |s|)), c)) + 2.
\]

**Lemma 2.9** (Efficient direct links \[RS18\]). For every node \(v\), \(dlink(v, \cdot)\) can be created from \(dlink(\text{prev}(v), \cdot)\) with time and space complexity \(O(\log(\sigma))\), with \(dlink(v, \cdot)\) stored in a persistent binary search tree, where \(\sigma\) is the size of the alphabet.

With this efficient approach to maintain direct links, Rubinchik and Shur \[RS18\] developed an algorithm that supports efficient online stack operations on eertrees.

**Theorem 2.10** (Stack operations for eertrees \[RS18\]). Stack eertrees can be implemented with time and space complexity per operation \(O(\log(\sigma))\), where \(\sigma\) is the size of the alphabet.

2.4.2 Queue eertrees

Recently, eertrees for a sliding window were studied in \[MWN+22\]. From the perspective of data structures, they support queue operations on eertrees, which we call queue eertrees. The main difficulty met in queue operations on eertrees is to check the uniqueness of the longest palindromic prefix of a string. This is because when the front character of the string is being removed, the longest palindromic prefix of the string should be deleted from the eertree if it is unique (i.e., appears only once). This difficulty was resolved in \[MWN+22\] by maintaining the second rightmost occurrence of every palindromic substrings. To check whether a palindromic substring is unique, it is sufficient to check whether its second rightmost occurrence exists. Based on this observation, Mieno et al. \[MWN+22\] proposed an algorithm that supports efficient online queue operations on eertrees.

**Theorem 2.11** (Queue operations for eertrees \[MWN+22\]). Queue eertrees can be implemented with time and space complexity per operation \(O(\log(\sigma))\), where \(\sigma\) is the size of the alphabet.

3 Framework of Double-Ended Eertrees

For readability, we first illustrate the framework of double-ended eertrees in this section. Throughout this paper, three kinds of font types are used in our algorithms:

1. Typewriter font: Variables in typewriter font are being maintained in the algorithms, e.g., \texttt{start\_pos}, \texttt{end\_pos}, \texttt{S}, \texttt{prenode}, \texttt{sufnode}, etc. Moreover, after a variable of typewriter font, we write its index inside a squared bracket, e.g., \texttt{S[v]}, \texttt{data[start\_pos]}, etc.
2. Math italic: Variables in math italics are mathematical concepts defined independent of the execution of the algorithms, e.g., \textit{prepal}, \textit{sufpal}, etc.

3. Roman font: Variables in Roman font are simple notions clear in the context and easy to maintain (thus we need not take much attention on their maintenance), though may depend on the execution of the algorithms, e.g., \textit{len}, \textit{link}, \textit{linkcnt}, etc.

3.1 Global index system

We need some auxiliary information that helps to maintain the whole data structure. In order to indicate indexes of the strings that once occur during the operations easily and clearly, we use a range $[\text{start\_pos}, \text{end\_pos})$ to store a string $s$ of length $|s| = \text{end\_pos} - \text{start\_pos}$ with global indexes $\text{start\_pos}$ and $\text{end\_pos}$, where the $i$-th character of $s$ is associated with the index $\text{start\_pos} + i - 1$ for $1 \leq i \leq |s|$ during the execution of the algorithm. To this end, we introduce an array $\text{data}$ such that $\text{data}[\text{start\_pos} + i - 1] = s[i]$ holds during the algorithm. Every time an element is pushed into (resp. popped from) the front, we set $\text{start\_pos} \leftarrow \text{start\_pos} - 1$ (resp. $\text{start\_pos} \leftarrow \text{start\_pos} + 1$); and if it is pushed into (resp. popped from) the back, we set $\text{end\_pos} \leftarrow \text{end\_pos} + 1$ (resp. $\text{end\_pos} \leftarrow \text{end\_pos} - 1$). Initially, $s = \epsilon$ is the empty string, and we set $\text{start\_pos} = \text{end\_pos} = 0$.

3.2 \texttt{push\_back} and \texttt{push\_front}

The overall idea to push a new character $c$ into the front (resp. back) of the string $s$ is straightforward (see Algorithm 1 and Algorithm 2). Let $t$ be the longest palindromic suffix (resp. the longest palindromic prefix) of $s$. Then $\text{eertree}(sc)$ (resp. $\text{eertree}(cs)$) can be obtained from $\text{eertree}(s)$ with the help of $t$.

\begin{algorithm}[H]
\caption{The framework of \texttt{push\_back}(c)}
1: $s \leftarrow \text{data}[\text{start\_pos}..\text{end\_pos} - 1].$
2: $t \leftarrow \text{sufpal}(s, |s|).$
3: Obtain $\text{eertree}(sc)$ from $\text{eertree}(s)$ with the help of $t$.
4: $\text{data}[\text{end\_pos}] \leftarrow c.$
5: $\text{end\_pos} \leftarrow \text{end\_pos} + 1.$
\end{algorithm}

\begin{algorithm}[H]
\caption{The framework of \texttt{push\_front}(c)}
1: $s \leftarrow \text{data}[\text{start\_pos}..\text{end\_pos} - 1].$
2: $t \leftarrow \text{prepal}(s, 1).$
3: Obtain $\text{eertree}(cs)$ from $\text{eertree}(s)$ with the help of $t$.
4: $\text{start\_pos} \leftarrow \text{start\_pos} - 1.$
5: $\text{data}[\text{start\_pos}] \leftarrow c.$
\end{algorithm}

3.3 \texttt{pop\_back} and \texttt{pop\_front}

When dealing with pop operations, we need to delete the nodes in the eertree that are no longer involved (see Algorithm 3 and Algorithm 4). Let $t$ be the longest palindromic suffix (resp. prefix) of $s$. $\text{eertree}(s[1..|s| - 1])$ (resp. $\text{eertree}(s[2..|s|])$) can be obtained from $\text{eertree}(s)$, with node($t$) deleted if $t$ is unique in $s$. 

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Algorithm 3 The framework of pop\_back()
1: \( s \leftarrow \text{data[start.pos..end.pos – 1]} \).
2: \( t \leftarrow \text{sufpal}(s, |s|) \).
3: if \( t \) is unique in \( s \) then
4: Delete node\((t)\) from the eertree.
5: end if
6: end pos \( \leftarrow \) end pos – 1.

Algorithm 4 The framework of pop\_front()
1: \( s \leftarrow \text{data[start.pos..end.pos – 1]} \).
2: \( t \leftarrow \text{prepal}(s, 1) \).
3: if \( t \) is unique in \( s \) then
4: Delete node\((t)\) from the eertree.
5: end if
6: start pos \( \leftarrow \) start pos + 1.

4 Occurrence Recording Method

In order to understand the main linear-time algorithm, we first introduce a sub-optimal occurrence recording method to maintain double-ended eertrees.

The main idea of the occurrence recording method is to record the occurrences of all palindromic substrings. The idea of this method was inspired by the algorithm to implement queue eertrees [MWN+22]. However, the method used in [MWN+22] is heavily based on the monotonicity of queue operations. Here, we extend the occurrence recording strategy for our more general case.

4.1 Reduced sets of occurrences

Suppose that \( s \) is a string and \( \text{eertree}(s) = (V, \text{even}, \text{odd}, \text{next}, \text{link}) \). In fact, it is difficult to directly maintain the set \( \text{occur}(s, v) \) for every node \( v \in V \) because the total number of elements that appear in \( \text{occur}(s, v) \) for all \( v \in V \) can be roughly \( O\left(|s|^2\right) \), which is quite large for our purpose. A simple example is that \( s = a^n \), where \( |\text{occur}(s, \text{node}(a^k))| = n - k + 1 \) for every \( 1 \leq k \leq n \).

To overcome this issue, we maintain a reduced set \( S(v) \) for every \( v \in V \setminus \{\text{even, odd}\} \) instead, which represents \( \text{occur}(s, v) \) for every \( v \in V \setminus \{\text{even, odd}\} \) indirectly. Specifically, we wish to maintain a reduced set \( S(s, v) \) for every \( v \in V \setminus \{\text{even, odd}\} \) such that

\[
\text{occur}(s, v) = S(s, v) \cup \bigcup_{\text{link}(u) = v} (\text{occur}(s, u) \cup \overline{\text{occur}(s, u)}).
\] (4)

Intuitively, some of the occurrences of node \( v \) can be obtained implicitly from the occurrences of its child nodes in the link tree; and \( S(s, v) \) is used to contain those occurrences of node \( v \) that are not implied by the former.

We claim that any reduced sets \( S(s, v) \) that satisfy Eq. (4) can help to check the uniqueness of a palindromic substring of \( s \) as follows.

**Lemma 4.1** (Uniqueness checking via reduced sets of occurrences). Suppose \( s \) is a non-empty string, \( S(s, v) \) is defined by Eq. (4) for every node \( v \) in \( \text{eertree}(s) \) and \( s[i..|s|] \) is a palindrome for some \( 1 \leq i \leq |s| \). Let \( v = \text{node}(s[i..|s|]) \). Then \( s[i..|s|] \) is unique in \( s \) if and only if \( \text{linkcnt}(s, v) = 0 \) and \( |S(s, v)| = 1 \).
Proof. “⇒”. If \( s[i..|s|] \) is unique in \( s \), then \( v \) must be a leaf node of the link tree \( T \) of \( \text{eertree}(s) \). Otherwise, \( v \) has a child node \( u \) in \( T \), i.e., there exists a node \( u \) such that \( \text{link}(u) = v \). Let \( i \in \text{occur}(s, u) \), and we have \( \{i, i + \text{len}(u) - \text{len}(v)\} \subseteq \text{occur}(s, v) \). Note that \( i \neq i + \text{len}(u) - \text{len}(v) \) due to \( \text{len}(v) < \text{len}(u) \). Therefore, \( |\text{occur}(s, v)| \geq 2 \), which contradicts with the uniqueness of \( \text{str}(v) = s[i..|s|] \). Now we conclude that \( v \) is a leaf node of \( T \). According to Eq. (2), we have \( \text{linkcnt}(s, v) = 0 \), and then Eq. (4) becomes

\[
\text{occur}(s, v) = S(s, v),
\]

which implies that \( |S(s, v)| = |\text{occur}(s, v)| = 1 \).

“⇐”. If \( \text{linkcnt}(s, v) = 0 \), then Eq. (4) holds again and thus \( |\text{occur}(s, v)| = |S(s, v)| = 1 \), which implies the uniqueness of \( \text{str}(v) = s[i..|s|] \) in \( s \).

According to the reduced set \( S(s, v) \), we further define

\[
\text{prenode}(s, i) = \{ v \in V : i \in S(s, v) \}, \tag{6}
\]

\[
\text{sufnode}(s, i) = \{ v \in V : i - \text{len}(v) + 1 \in S(s, v) \}. \tag{7}
\]

As will be shown in Lemma 4.3, we can represent \( \text{prepal}(s, 1) \) and \( \text{sufpal}(s, |s|) \) by an element in \( \text{prenode}(s, 1) \) and \( \text{sufnode}(s, |s|) \), respectively. But before the use of \( \text{prenode}(s, 1) \) and \( \text{sufnode}(s, |s|) \), we should first show that they are not empty.

Lemma 4.2. Suppose \( s \) is a non-empty string, and \( S(s, v) \) is defined by Eq. (4) for every node \( v \) in \( \text{eertree}(s) \). Then \( \text{prenode}(s, 1) \neq \emptyset \) and \( \text{sufnode}(s, |s|) \neq \emptyset \).

Proof. We only need to show that \( \text{prenode}(s, 1) \neq \emptyset \) by contradiction, and \( \text{sufnode}(s, |s|) \neq \emptyset \) can be obtained similarly. Now we assume that \( \text{prenode}(s, 1) = \emptyset \). By the definition of \( \text{prenode}(s, 1) \), we have that \( 1 \notin S(s, v) \) for every node \( v \) in \( \text{eertree}(s) \). In the following, we will show that \( 1 \notin \text{occur}(s, v) \) for every node \( v \) (except \( \text{even} \) and \( \text{odd} \)) in \( \text{eertree}(s) \) by induction on the topological order of the nodes in the link tree \( T \) of \( \text{eertree}(s) \). Let \( v_1, v_2, \ldots, v_n \) be the list of all nodes (except \( \text{even} \) and \( \text{odd} \)) in \( \text{eertree}(s) \) such that \( v_j \notin T_{v_i} \) for every \( 1 \leq i < j \leq n \). That is, \( v_j \) is not in any subtree \( T_{v_i} \) of \( v_i \) in the link tree \( T \) for any \( i < j \). Now we are going to show that \( 1 \notin \text{occur}(s, v_i) \) for every \( 1 \leq i \leq n \).

Induction. Suppose \( 1 \notin \text{occur}(s, v_i) \) for every \( 1 \leq i \leq k \). It is trivial that it holds for \( k = 0 \), and our goal is to show that it holds for \( k = n \). We consider the following two cases.

1. \( v_{k+1} \) is a leaf node in the link tree \( T \) of \( \text{eertree}(s) \). By Eq. (4), we have \( \text{occur}(s, v_{k+1}) = S(s, v_{k+1}) \), and thus \( 1 \notin S(s, v_{k+1}) \in \text{occur}(s, v_{k+1}) \).

2. \( v_{k+1} \) is not a leaf node in the link tree \( T \) of \( \text{eertree}(s) \). By Eq. (4), we have

\[
\text{occur}(s, v_{k+1}) = S(s, v_{k+1}) \cup \bigcup_{\text{link}(u) = v_{k+1}} (\text{occur}(s, u) \cup \{i + \text{len}(u) - \text{len}(v_{k+1}) : i \in \text{occur}(s, u)\}).
\]

Let \( u \) be any node such that \( \text{link}(u) = v_{k+1} \). By induction, since \( u \) goes before \( v_{k+1} \) in the topological order, we have \( 1 \notin \text{occur}(s, u) \). Thus for every \( i \in \text{occur}(s, u) \), we have \( i \geq 2 \). Note that \( \text{len}(u) > \text{len}(v_{k+1}) \), thus \( i + \text{len}(u) - \text{len}(v_{k+1}) \geq 2 + 1 = 3 \). Together with \( 1 \notin S(s, v_{k+1}) \), we conclude that \( 1 \notin \text{occur}(s, v_{k+1}) \).
Combining the both cases, we have that $1 \notin \text{occur}(s, v_{i})$ for every $1 \leq i \leq k + 1$. Therefore, $1 \notin \text{occur}(s, v)$ for every $v \in V \setminus \{\text{even}, \text{odd}\}$.

Now consider the non-empty palindromic substring $t = s[1..\text{prenode}(s, 1)]$. It is obvious that $\text{node}(t) \notin V \setminus \{\text{even}, \text{odd}\}$, but we have that $1 \in \text{occur}(s, \text{node}(t))$, which contradicts with $1 \notin \text{occur}(s, v)$ for all $v \notin V \setminus \{\text{even}, \text{odd}\}$. Therefore, we have $\text{prenode}(s, 1) \neq \emptyset$, and $\text{sufnode}(s, |s|) \neq \emptyset$ similarly.

Having known from Lemma 4.2 that $\text{prenode}(s, 1)$ and $\text{sufnode}(s, |s|)$ are non-empty sets, we are able to take the maximal elements from them without undefined issues. The following lemma shows that they can be used to find the longest palindromic prefix and suffix, respectively.

**Lemma 4.3.** Suppose $s$ is a non-empty string, $S(s,v)$ is defined by Eq. (4) for every node $v$ in $\text{eertree}(s)$. We have

\[
\text{prepal}(s, 1) = \text{str} \left( \arg \max_{v \in \text{prenode}(s, 1)} \text{len}(v) \right),
\]

\[
\text{sufpal}(s, |s|) = \text{str} \left( \arg \max_{v \in \text{sufnode}(s, |s|)} \text{len}(v) \right).
\]

**Proof.** We only have to prove that

\[
\text{prelen}(s, 1) = \max_{v \in \text{prenode}(s, 1)} \text{len}(v), \quad (8)
\]

\[
\text{suflen}(s, |s|) = \max_{v \in \text{sufnode}(s, |s|)} \text{len}(v). \quad (9)
\]

We only need to prove Eq. (8), and Eq. (9) can be obtained similarly. For every node $v \in \text{prenode}(s, 1)$, we have that $1 \in S(s,v)$. By Eq. (4), we have $1 \in S(s,v) \subseteq \text{occur}(s,v)$. By the definition of $\text{prelen}(s, 1)$, we have that $\text{len}(v) \leq \text{prelen}(s, 1)$, and thus

\[
\max_{v \in \text{prenode}(s, 1)} \text{len}(v) = \max_{1 \in S(s,v)} \text{len}(v) \leq \text{prelen}(s, 1). \quad (10)
\]

On the other hand, let $t = s[1..\text{prepal}(s, 1)]$ be the longest palindromic prefix of $s$, and let $u = \text{node}(t)$. Then it holds that $1 \in \text{occur}(s,u)$ and $\text{len}(u) = \text{prepal}(s, 1)$. We will show that $1 \in S(s,u)$ by considering the following two cases.

1. $u$ is a leaf node in the link tree $T$ of $\text{eertree}(s)$. By Eq. (1), we have $S(s,u) = \text{occur}(s,u)$, and thus $1 \in S(s,u)$.

2. $u$ is not a leaf node in the link tree $T$ of $\text{eertree}(s)$. By Eq. (4), we have

\[
\text{occur}(s,u) = S(s,u) \cup \bigcup_{\text{link}(w) = u} (\text{occur}(s,w) \cup \{i + \text{len}(w) - \text{len}(u) : i \in \text{occur}(s,w)\}).
\]

Let $w$ be any node such that $\text{link}(w) = u$. We first show that $1 \notin \text{occur}(s,w)$ by contradiction. If $1 \in \text{occur}(s,w)$, then $s[1..\text{len}(w)]$ is a palindromic prefix of $s$. By the maximality of $\text{prepal}(s,1)$, we have $\text{len}(u) = \text{prepal}(s,1) \geq \text{len}(w)$. On the other hand, since $\text{link}(w) = u$, we have $\text{len}(w) > \text{len}(u)$, which is a contradiction. Therefore, we conclude that $1 \notin \text{occur}(s,w)$. Let $i \in \text{occur}(s,w)$, then $i \geq 2$ and thus $i + \text{len}(w) - \text{len}(u) \geq 2 + 1 = 3$. We conclude that $1 \notin \{i + \text{len}(w) - \text{len}(u) : i \in \text{occur}(s,w)\}$. Recall that $1 \in \text{occur}(s,u)$, which yields that $1 \in S(s,u)$.  

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Now we have $1 \in S(s, u)$, i.e., $u \in \text{prenode}(s, 1)$. Then,

$$\max_{v \in \text{prenode}(s,1)} \text{len}(v) \geq \text{len}(u) = \text{prelen}(s, 1).$$

(11)

Combining Eq. (10) and Eq. (11), we obtain Eq. (8).

4.2 The algorithm

In this subsection, we will provide an algorithm that maintains certain valid $S(s, v)$ satisfying Eq. (4) for every $v \in V$ in $O(\log(|s|))$ time for every double-ended queue operation on $s$. Especially, for each operation, there are totally $O(1)$ elements modified in all $S(s, v)$.

4.2.1 push_back and push_front

For push_back and push_front operations, we only focus on how to maintain $S(s, v)$, since other auxiliary data are defined by $S(s, v)$. Here, we use the variable $S[v]$ in our algorithm, which ought to be $S(s, v)$ for the current string $s$. The algorithms for push_back and push_front are given as follows.

- push_back($c$):
  1. Find the node $v$ of the longest palindromic suffix $\text{sufpal}(s, |s|)$ of $s$.
  2. Modify eertree($s$) to eertree($sc$) according to $v$ and $c$ by online construction of eertrees [RS18], with $v'$ being the node of the longest palindromic suffix $\text{sufpal}(sc, |sc|)$ of $sc$.
  3. Add the start position $|sc| - \text{len}(v') + 1$ of $\text{sufpal}(sc, |sc|)$ into $S[v']$.

- push_front($c$):
  1. Find the node $v$ of the longest palindromic prefix $\text{prepal}(s, 1)$ of $s$.
  2. Modify eertree($s$) to eertree($cs$) according to $v$ and $c$ by online construction of eertrees [RS18], with $v'$ being the node of the longest palindromic prefix $\text{prepal}(cs, 1)$ of $cs$.
  3. Shift right all elements in $S[u]$ by 1 for all nodes $u$.
  4. Add the start position 1 of $\text{prepal}(cs, 1)$ into $S[v']$.

It can be seen that the algorithms for push_back and push_front are almost symmetric, with the only exception that all elements in $S[u]$ are shifted right by 1 for all nodes $u$. This is because all characters in $s$ should move right in order to insert the new character $c$ at the front of $s$. A straightforward implementation of shift-right operations needs to modify all (up to $O(|s|)$) elements in $S[u]$ for all node $u$. To make it efficient, we store global indexes (see Section 3.1) rather than relative ones. In this way, a string $s[1..|s|]$ is stored and represented by a range $[\text{start pos}, \text{end pos})$ with global indexes $\text{start pos}$ and $\text{end pos}$. With the global indexes, $S[u]$ can be maintained as if no shift-rights were needed. The algorithms for push_back and push_front are now restated as follows.

- push_back($c$):
  1. Modify eertree($s$) to eertree($sc$), with $v'$ being the node of $\text{sufpal}(sc, |sc|)$.
  2. Increment $\text{end pos}$ by 1. Then, $[\text{start pos}, \text{end pos})$ indicates the range of string $sc$.
  3. Add the start global position $\text{end pos} - \text{len}(v')$ of $\text{sufpal}(sc, |sc|)$ into $S[v']$.
• push_front(c):
  1. Modify eertree(s) to eertree(cs), with v′ being the node of prepal(cs, 1).
  2. Decrement start_pos by 1. Then, [start_pos, end_pos) indicates the range of string cs.
  3. Add the start global position start_pos of prepal(cs, 1) into S[v′].

For completeness, we provide formal and detailed descriptions of the algorithms for push_back and push_front in Algorithm 5 and Algorithm 6 respectively.

Algorithm 5 push_back(c) via occurrence recording method
1: s ← data[start_pos..end_pos − 1].
2: v ← arg max_u∈sufnode[end_pos−1] len(u). ▷ This ensures that str(v) = sufpal(s, |s|).
3: Obtain eertree(sc) from eertree(s) according to v and c, and let v′ ← node(sufpal(sc, |sc|)).
4: data[end_pos] ← c.
5: end_pos ← end_pos + 1.
6: Add end_pos − len(v′) into S[v′].
7: Add v′ into prenode[end_pos − len(v′)] and sufnodesufnode[end_pos − 1].

Algorithm 6 push_front(c) via occurrence recording method
1: s ← data[start_pos..end_pos − 1].
2: v ← arg max_u∈prepal[start_pos] len(u). ▷ This ensures that str(v) = prepal(s, 1).
3: Obtain eertree(cs) from eertree(s) according to v and c, and let v′ ← node(prepal(cs, 1)).
4: start_pos ← start_pos − 1.
5: data[start_pos] ← c.
6: Add start_pos into S[v′].
7: Add v′ into prenode[start_pos] and sufnodesufnode[start_pos + len(v′) − 1].

The correctness of the algorithms for push_back and push_front is based on the following facts, Lemma 4.4 and 4.5.

Lemma 4.4 (Correctness of push_back by occurrence recording). Let s be a string and c be a character. Suppose S(s, v) defined for eertree(s) satisfies Eq. 4. Let v′ = node(sufpal(sc, |sc|)), and

\[ S(sc, v) = \begin{cases} S(s, v) \cup \{|sc| - \text{len}(v') + 1\}, & v = v', \\ S(s, v), & \text{otherwise}, \end{cases} \]

where S(s, v) = ∅ if v ∉ eertree(s). Then S(sc, v) satisfies Eq. 4) defined for eertree(sc).

Proof. Let occur(s, v) and occur(s′, v) be the set of occurrences of str(v) in s and s′ = sc, respectively. According to Eq. 11, it is straightforward that

\[ \text{occur}(s', v) = \begin{cases} \text{occur}(s, v) \cup \{|sc| - \text{len}(v') + 1\}, & v \in \text{link}^*(v'), \\ \text{occur}(s, v), & \text{otherwise}. \end{cases} \]  

(12)

Since S(s′, v) = S(s, v) and occur(s′, v) = occur(s, v) for every v ∉ link*(v′), we only need to show that S(s′, v) satisfies Eq. 4) defined for eertree(sc) for every v ∈ link*(v′) \ {even, odd}. Now we
will show that $S(s', \text{link}^k(v'))$ satisfies Eq. (11) defined for \text{eertree}(sc) for every $k \geq 0$. That is, for every $k \geq 0$ with $\text{link}^k(v') \notin \{\text{even}, \text{odd}\}$, it holds that

$$\text{occur}(s', \text{link}^k(v')) = S(s', \text{link}^k(v')) \cup \bigcup_{\text{link}(u) = \text{link}^k(v')} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u)). \quad (13)$$

The proof is split into two cases.

**Case 1.** For $k = 0$, $\text{link}^0(v') = v'$. By Eq. (12), the right hand side of Eq. (13) becomes

$$S(s', v') \cup \bigcup_{\text{link}(u) = v'} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u))$$

$$= S(s, v') \cup \{|sc| - \text{len}(v') + 1\} \cup \bigcup_{\text{link}(u) = v'} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u))$$

$$= \text{occur}(s, v') \cup \{|sc| - \text{len}(v') + 1\}$$

$$= \text{occur}(s', v').$$

**Case 2.** For $k \geq 1$ such that $\text{link}^k(v') \notin \{\text{even}, \text{odd}, v'\}$, by Eq. (12), we have

$$\text{occur}(s', \text{link}^{k-1}(v')) = \text{occur}(s, \text{link}^{k-1}(v')) \cup \{|sc| - \text{len}(\text{link}^{k-1}(v')) + 1\}.$$ 

With this, we further have

$$\overline{\text{occur}}(s', \text{link}^{k-1}(v')) = \left\{ i + \text{len}(\text{link}^{k-1}(v')) - \text{len}(\text{link}^k(v')) : i \in \text{occur}(s', \text{link}^{k-1}(v')) \right\}$$

$$= \overline{\text{occur}}(s, \text{link}^{k-1}(v')) \cup \{|sc| - \text{len}(\text{link}^k(v')) + 1\}.$$ 

The right hand side of Eq. (13) becomes

$$S(s', \text{link}^k(v')) \cup \bigcup_{\text{link}(u) = \text{link}^k(v')} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u))$$

$$= S(s, \text{link}^k(v')) \cup \bigcup_{\text{link}(u) = \text{link}^k(v')} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u))$$

$$\cup \text{occur}(s', \text{link}^{k-1}(v')) \cup \overline{\text{occur}}(s', \text{link}^{k-1}(v'))$$

$$= S(s, \text{link}^k(v')) \cup \bigcup_{\text{link}(u) = \text{link}^k(v')} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u))$$

$$\cup \{|sc| - \text{len}(\text{link}^{k-1}(v')) + 1\} \cup \{|sc| - \text{len}(\text{link}^k(v')) + 1\}$$

$$= \text{occur}(s, \text{link}^k(v')) \cup \{|sc| - \text{len}(\text{link}^{k-1}(v')) + 1\} \cup \{|sc| - \text{len}(\text{link}^k(v')) + 1\}$$

$$= \text{occur}(s', \text{link}^k(v')) \cup \{|sc| - \text{len}(\text{link}^{k-1}(v')) + 1\}.$$ 

It remains to show that $|sc| - \text{len}(\text{link}^{k-1}(v')) + 1 \in \text{occur}(s', \text{link}^k(v'))$ to complete the proof. This can be seen by Eq. (12) that $|sc| - \text{len}(\text{link}^{k-1}(v')) + 1 \in \text{occur}(s', \text{link}^{k-1}(v'))$, which means that $s'[|sc| - \text{len}(\text{link}^{k-1}(v')) + 1..|sc|] = \text{str}(\text{link}^{k-1}(v'))$. On the other hand, $\text{str}(\text{link}^k(v'))$ is a proper prefix of $\text{str}(\text{link}^{k-1}(v'))$. Therefore,

$$\text{str}(\text{link}^k(v')) = s'[|sc| - \text{len}(\text{link}^{k-1}(v')) + 1..|sc| - \text{len}(\text{link}^{k-1}(v')) + \text{len}(\text{link}^k(v'))],$$
which means that $|sc| - \text{len}(\text{link}^{k-1}(v')) + 1 \in \text{occur}(s', \text{link}^k(v'))$.

As a result, Eq. (13) is verified, which means that $S(s', v)$ satisfies Eq. (4) defined for $\text{eertree}(sc)$.

**Lemma 4.5** (Correctness of push_front by occurrence recording). Let $s$ be a string and $c$ be a character. Suppose $S(s, v)$ defined for $\text{eertree}(s)$ satisfies Eq. (4). Let $v' = \text{node}(\text{prepal}(cs, 1))$, and

$$S(cs, v) = \begin{cases} (S(s, v) + 1) \cup \{1\}, & v = v', \\ S(s, v) + 1, & \text{otherwise}, \end{cases}$$

where $S(s, v) = \emptyset$ if $v \notin \text{eertree}(s)$, and $A + a = \{ x + a : x \in A \}$ for a set $A$ and an element $a$. Then $S(cs, v)$ satisfies Eq. (4) defined for $\text{eertree}(cs)$.

**Proof.** Because of symmetry, the proof is similar to that of Lemma 4.4.

### 4.2.2 pop_back and pop_front

Now we consider how to maintain $S(s, v)$ for pop_back and pop_front operations. Here, we use the variable $S[v]$ in our algorithm, which ought to be $S(s, v)$ for the current string $s$. The algorithms for pop_back and pop_front are given as follows.

- **pop_back:**
  1. Let $v$ be the node of the longest palindromic suffix $\text{sufpal}(s, |s|)$ of $s$.
  2. If $\text{str}(v)$ is unique in $s$, then delete $v$ from the eertree. This will modify $\text{eertree}(s)$ to $\text{eertree}(s[1..|s| - 1])$.
  3. For every node $u$ in $\text{sufnode}(s, |s|)$, delete the start position $|s| - \text{len}(u) + 1$ of the suffix $\text{str}(u)$ of $s$ from $S[u]$.
  4. If $\text{str}(\text{link}(v))$ is not the empty string, add the start position $|s| - \text{len}(v) + 1$ of $\text{str}(\text{link}(v))$ into $S[\text{link}(v)]$.

- **pop_front:**
  1. Let $v$ be the node of the longest palindromic prefix $\text{prepal}(s, 1)$ of $s$.
  2. If $\text{str}(v)$ is unique in $s$, then delete $v$ from the eertree. This will modify $\text{eertree}(s)$ to $\text{eertree}(s[2..|s|])$.
  3. For every node $u$ in $\text{prenode}(s, 1)$, delete the start position 1 of the prefix $\text{str}(u)$ of $s$ from $S[u]$.
  4. If $\text{str}(\text{link}(v))$ is not the empty string, add the start position $\text{len}(v) - \text{len}(\text{link}(v)) + 1$ of $\text{str}(\text{link}(v))$ into $S[\text{link}(v)]$.
  5. Shift left all elements in $S[u]$ by 1 for all nodes $u$.

Similar to the algorithms for push_back and push_front in Section 4.2.1, the algorithms for pop_back and pop_front given here are also almost symmetric, with the only exception that all elements in $S[u]$ are shifted left by 1 for all nodes $u$. To avoid modifying every element stored in $S[u]$, we use the global indexes as in Section 4.2.1 and the algorithms for pop_back and pop_front are restated as follows.

- **pop_back:**
1. Let \( v \) be the node of the longest palindromic suffix \( \text{sufpal}(s, |s|) \) of \( s \).

2. Modify \( \text{eertree}(s) \) to \( \text{eertree}(s[1..|s| - 1]) \).

3. For every node \( u \) in \( \text{sufnode}(s, |s|) \), delete the start global position \( \text{end\_pos} - \text{len}(u) \) of the suffix \( \text{str}(u) \) of \( s \) from \( S[u] \).

4. If \( \text{str}(\text{link}(v)) \) is not the empty string, add the start global position \( \text{end\_pos} - \text{len}(v) \) of \( \text{str}(\text{link}(v)) \) into \( S[\text{link}(v)] \).

5. Decrement \( \text{end\_pos} \) by 1. Then, \([\text{start\_pos}, \text{end\_pos}]\) indicates the range of string \( s[1..|s| - 1] \).

\[\text{pop\_front:}\]

1. Let \( v \) be the node of the longest palindromic prefix \( \text{prepal}(s, 1) \) of \( s \).

2. Modify \( \text{eertree}(s) \) to \( \text{eertree}(s[2..|s|]) \).

3. For every node \( u \) in \( \text{prenode}(s, 1) \), delete the start global position \( \text{start\_pos} \) of the prefix \( \text{str}(u) \) of \( s \) from \( S[u] \).

4. If \( \text{str}(\text{link}(v)) \) is not the empty string, add the start global position \( \text{start\_pos} + \text{len}(v) - \text{len}(\text{link}(v)) \) of \( \text{str}(\text{link}(v)) \) into \( S[\text{link}(v)] \).

5. Increment \( \text{start\_pos} \) by 1. Then, \([\text{start\_pos}, \text{end\_pos}]\) indicates the range of string \( s[2..|s|] \).

For completeness, we provide formal and detailed descriptions of the algorithms for \( \text{pop\_back} \) and \( \text{pop\_front} \) in Algorithm 7 and Algorithm 8, respectively.

**Algorithm 7 pop\_back() via occurrence recording method**

\begin{enumerate}
\item \( s \leftarrow \text{data}[\text{start\_pos}..\text{end\_pos} - 1] \).
\item \( v \leftarrow \text{arg\_max}_{u \in \text{sufnode}[\text{end\_pos} - 1]} \text{len}(u) \). \hspace{1cm} \triangleright \text{This ensures that } \text{str}(v) = \text{sufpal}(s, |s|).
\item \text{if } \text{linkcnt}(s, v) = 0 \text{ and } |S[v]| = 1 \text{ then } \triangleright \text{By Lemma 4.1 it means that } \text{str}(v) \text{ is unique in } s.
\item \hspace{1cm} \text{Delete } v \text{ from the eertree.}
\item \hspace{1cm} \text{end if}
\item \hspace{1cm} \text{for every } u \in \text{sufnode}[\text{end\_pos} - 1] \text{ do}
\item \hspace{1cm} \hspace{1cm} \text{Delete } \text{end\_pos} - \text{len}(u) \text{ from } S[u].
\item \hspace{1cm} \hspace{1cm} \text{Delete } u \text{ from } \text{prenode}[\text{end\_pos} - \text{len}(u)] \text{ and } \text{sufnode}[\text{end\_pos} - 1].
\item \hspace{1cm} \text{end for}
\item \hspace{1cm} \text{if } \text{len}(\text{link}(v)) \geq 1 \text{ then}
\item \hspace{1cm} \hspace{1cm} \text{Add } \text{end\_pos} - \text{len}(v) \text{ into } S[\text{link}(v)].
\item \hspace{1cm} \hspace{1cm} \text{Add } \text{link}(v) \text{ into } \text{prenode}[\text{end\_pos} - \text{len}(v)].
\item \hspace{1cm} \hspace{1cm} \text{Add } \text{link}(v) \text{ into } \text{sufnode}[\text{end\_pos} - \text{len}(v) + \text{len}(\text{link}(v)) - 1].
\item \hspace{1cm} \text{end if}
\item \hspace{1cm} \text{end if}
\item \hspace{1cm} \text{end for}
\item \hspace{1cm} \text{end if}
\item \hspace{1cm} \text{end for}
\end{enumerate}

The correctness of the algorithms for \( \text{push\_back} \) and \( \text{push\_front} \) is based on the following facts, Lemma 4.6 and 4.9.

**Lemma 4.6** (Correctness of \( \text{pop\_back} \) by occurrence recording). Let \( s \) be a non-empty string. Suppose \( S(s, v) \) defined for \( \text{eertree}(s) \) satisfies Eq. (7). Let \( v' = \text{node}(\text{sufpal}(s, |s|)) \), and

\[
S(s', v) = \begin{cases} 
(S(s, v) \setminus \{|s| - \text{len}(v) + 1\}) \cup \{|s| - \text{len}(v') + 1\}, & \text{if } v = \text{link}(v') \text{ and } \text{len}(\text{link}(v')) \geq 1, \\
S(s, v) \setminus \{|s| - \text{len}(v) + 1\}, & \text{otherwise},
\end{cases}
\]
Algorithm 8 pop_front() via occurrence recording method

1: \( s \leftarrow \text{data[start\_pos..end\_pos - 1]} \).
2: \( v \leftarrow \text{arg max}_{u \in \text{prenode[start\_pos]}} \text{len}(u) \). \( \triangleright \) This ensures that \( \text{str}(v) = \text{prepal}(s, 1) \).
3: \textbf{if} \( \text{linkcnt}(s, v) = 0 \) and \( |S[v]| = 1 \) \textbf{then} \( \triangleright \) By Lemma 4.1 it means that \( \text{str}(v) \) is unique in \( s \).
4: \hspace{1em} Delete \( v \) from the eertree.
5: \textbf{end if}
6: \textbf{for} every \( u \in \text{prenode[start\_pos]} \) \textbf{do}
7: \hspace{1em} Delete \( \text{start\_pos} \) from \( S[u] \).
8: \hspace{1em} Delete \( u \) from \( \text{prenode[start\_pos]} \) and \( \text{sufnode[start\_pos + \text{len}(u) - 1]} \).
9: \textbf{end for}
10: \textbf{if} \( \text{len}(\text{link}(v)) \geq 1 \) \textbf{then}
11: \hspace{1em} Add \( \text{start\_pos} + \text{len}(v) - \text{len}(\text{link}(v)) \) into \( S[\text{link}(v)] \).
12: \hspace{1em} Add \( \text{link}(v) \) into \( \text{prenode[start\_pos + \text{len}(v) - \text{len}(\text{link}(v))] \).}
13: \hspace{1em} Add \( \text{link}(v) \) into \( \text{sufnode[start\_pos + \text{len}(v) - 1]} \).
14: \textbf{end if}
15: \( \text{start\_pos} \leftarrow \text{start\_pos} + 1 \).

where \( s' = [1..|s| - 1] \). Then \( S([1..|s| - 1], v) \) satisfies Eq. (4) defined for \( \text{eertree}(s[1..|s| - 1]) \).

Proof. Let \( \text{occur}(s, v) \) and \( \text{occur}(s', v) \) be the set of occurrences of \( \text{str}(v) \) in \( s \) and \( s' = [1..|s| - 1] \), respectively. According to Eq. (4), it is straightforward that

\[
\text{occur}(s', v) = \text{occur}(s, v) \setminus \{|s| - \text{len}(v) + 1\} = \begin{cases} 
\text{occur}(s, v) \setminus \{|s| - \text{len}(v) + 1\}, & v \in \text{link}^*(v'), \\
\text{occur}(s, v), & \text{otherwise}.
\end{cases}
\]  

(14)

Since \( S(s', v) = S(s, v) \) and \( \text{occur}(s', v) = \text{occur}(s, v) \) for every \( v \notin \text{link}^*(v') \), we only need to show that \( S(s', v) \) satisfies Eq. (4) defined for \( \text{eertree}(s') \) for every \( v \in \text{link}^*(v') \setminus \{\text{even}, \text{odd}\} \). Now we will show that \( S(s', \text{link}^k(v')) \) satisfies Eq. (4) defined for \( \text{eertree}(s') \) for every \( k \geq 0 \). That is, for every \( k \geq 0 \) such that \( \text{link}^k(v') \notin \{\text{even}, \text{odd}\} \), it holds that

\[
\text{occur}(s', \text{link}^k(v')) = S(s', \text{link}^k(v')) \bigcup_{\text{link}(u) = \text{link}^k(v')} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u)).
\]

(15)

The proof is split into three cases.

**Case 1.** \( k = 0 \). In this case, \( \text{link}^k(v') = \text{link}^0(v') = v' \). Let \( u \) be a node such that \( \text{link}(u) = v' \). By Eq. (14), we know that \( |s| - \text{len}(u) + 1 \notin \text{occur}(s, u) = \text{occur}(s', u) \), thus \( |s| - \text{len}(v') + 1 \notin \text{occur}(s', u) \). Therefore, the right hand side of Eq. (15) becomes

\[
\begin{align*}
S(s', v') & \cup \bigcup_{\text{link}(u) = v'} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u)) \\
& = (S(s, v') \setminus \{|s| - \text{len}(v') + 1\}) \cup \bigcup_{\text{link}(u) = v'} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u)) \\
& = \left( S(s, v') \cup \bigcup_{\text{link}(u) = v'} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u)) \right) \setminus \{|s| - \text{len}(v') + 1\} \\
& = \text{occur}(s, v') \setminus \{|s| - \text{len}(v') + 1\} \\
& = \text{occur}(s', v').
\end{align*}
\]
**Case 2.** $k = 1$. We only need to consider the case that $\text{len}(\text{link}(v')) \geq 1$. We note that

$$S(s', \text{link}(v')) = S(s, \text{link}(v')) \setminus \{|s| - \text{len}(\text{link}(v')) + 1\} \cup \{|s| - \text{len}(v') + 1\}.$$ 

Let $u$ be a node such that $\text{link}(u) = \text{link}(v')$. If $u \neq v'$, by Eq. (14), we have $|s| - \text{len}(u) + 1 \notin \text{occur}(s, u)$, thus $|s| - \text{len}(\text{link}(v')) + 1 \notin \{i + \text{len}(u) - \text{len}(\text{link}(v')) : i \in \text{occur}(s, u)\} = \overline{\text{occur}}(s, u)$. By Eq. (11), $\{s| - \text{len}(\text{link}(v')) + 1 \notin \text{occur}(s, u)$ because $|s| - \text{len}(\text{link}(v')) + 1 > |s| - \text{len}(u) + 1$. Note that these also hold for $u = v'$.

The right hand side of Eq. (15) becomes

$$S(s', \text{link}(v')) \cup \bigcup_{\text{link}(u)=\text{link}(v')} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u))$$

$$= (S(s, \text{link}(v')) \setminus \{|s| - \text{len}(\text{link}(v')) + 1\} \cup \{|s| - \text{len}(v') + 1\})$$

$$\cup \bigcup_{\text{link}(u)=\text{link}(v')} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u))$$

$$\cup (\text{occur}(s, v') \setminus \{|s| - \text{len}(v') + 1\}) \cup (\overline{\text{occur}}(s, v') \setminus \{|s| - \text{len}(\text{link}(v')) + 1\})$$

$$= \left(S(s, \text{link}(v')) \cup \bigcup_{\text{link}(u)=\text{link}(v')} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u)) \right) \setminus \{|s| - \text{len}(\text{link}(v')) + 1\}$$

$$= \text{occur}(s, \text{link}(v')) \setminus \{|s| - \text{len}(\text{link}(v')) + 1\}$$

$$= \text{occur}(s', \text{link}(v')).$$

**Case 3.** $k \geq 2$. We only need to consider the case that $\text{len}(\text{link}^k(v')) \geq 1$. Let $u$ be a node such that $\text{link}(u) = \text{link}^k(v')$. The right hand side of Eq. (15) becomes

$$S(s', \text{link}^k(v')) \cup \bigcup_{\text{link}(u)=\text{link}^k(v')} (\text{occur}(s', u) \cup \overline{\text{occur}}(s', u)) = A_1 \cup A_2 \cup A_3 \cup A_4,$$

where

$$A_1 = S(s, \text{link}^k(v')) \setminus \{|s| - \text{len}(\text{link}^k(v')) + 1\},$$

$$A_2 = \bigcup_{\text{link}(u)=\text{link}^k(v')} (\text{occur}(s, u) \cup \overline{\text{occur}}(s, u)),$$

$$A_3 = \text{occur}(s, \text{link}^{k-1}(v')) \setminus \{|s| - \text{len}(\text{link}^{k-1}(v')) + 1\},$$

$$A_4 = \overline{\text{occur}}(s, \text{link}^{k-1}(v')) \setminus \{|s| - \text{len}(\text{link}^k(v')) + 1\}.$$

In the sets $A_1$ and $A_4$, the element $|s| - \text{len}(\text{link}^k(v')) + 1$ is removed. Next, we will show that this special element is not in either $A_2$ or $A_3$.

**Lemma 4.7.** If $\text{len}(\text{link}^k(v')) \geq 1$, then $|s| - \text{len}(\text{link}^k(v')) + 1 \notin A_2 \cup A_3$.

**Proof.** Let $u$ be a node such that $\text{link}(u) = \text{link}^k(v')$, and then $\text{len}(u) > \text{len}(\text{link}^k(v'))$. Then for every $i \in \text{occur}(u)$, we have $1 \leq i \leq |s| - \text{len}(u) + 1 < |s| - \text{len}(\text{link}^k(v')) + 1$, which means $|s| - \text{len}(\text{link}^k(v')) + 1 \notin \text{occur}(u)$ and therefore $|s| - \text{len}(\text{link}^k(v')) + 1 \notin A_3$.  


If \( u \neq \text{link}^{k-1}(v') \), by Eq. (14), we know that \(|s| - \text{len}(u) + 1 \notin \text{occur}(s, u)\). This immediately leads to
\[
|s| - \text{len}\left(\text{link}^k(v')\right) + 1 \notin \left\{ i + \text{len}(u) - \text{len}\left(\text{link}^k(v')\right) : i \in \text{occur}(s, u) \right\} = \text{occur}(s, u).
\]
This implies that \(|s| - \text{len}\left(\text{link}^k(v')\right) + 1 \notin A_2\).

By Lemma 4.7, we have
\[
A_1 \cup A_2 \cup A_3 \cup A_4 = \left( S\left( s, \text{link}^k(v') \right) \cup A_2 \cup A_3 \cup A_4' \right) \setminus \left\{ |s| - \text{len}\left(\text{link}^k(v')\right) + 1 \right\}, \tag{16}
\]
where \( A_4' = \overline{\text{occur}}(s, \text{link}^{k-1}(v')). \)

There is another special element \(|s| - \text{len}(\text{link}^{k-1}(v')) + 1 \in A_2 \cup A_4'. \)

\textbf{Lemma 4.8.} If \( \text{len}(\text{link}^k(v')) \geq 1 \), then \(|s| - \text{len}(\text{link}^{k-1}(v')) + 1 \in A_2 \cup A_4'. \)

\textbf{Proof.} Let \( v = \text{link}^{k-2}(v') \), and \( i = |s| - \text{len}(v) + 1 \in \text{occur}(s, v) \). By Lemma 2.7, we have
\[
i + \text{len}(v) - \text{len}(\text{link}(v)) \in \bigcup_{\text{link}(u)=\text{link}^2(v)} \overline{\text{occur}}(s, u),
\]
which follows that
\[
|s| - \text{len}\left(\text{link}^{k-1}(v')\right) + 1 \in \bigcup_{\text{link}(u)=\text{link}^k(v')} \overline{\text{occur}}(s, u) \subseteq A_2 \cup A_4'.
\]

By Lemma 4.8 and Eq. (16), the right hand side of Eq. (15) becomes
\[
A_1 \cup A_2 \cup A_3 \cup A_4
= \left( S\left( s, \text{link}^k(v') \right) \cup A_2 \cup \text{occur}(s, \text{link}^{k-1}(v')) \cup A_4' \right) \setminus \left\{ |s| - \text{len}\left(\text{link}^k(v')\right) + 1 \right\}
= \left( S\left( s, \text{link}^k(v') \right) \cup \bigcup_{\text{link}(u)=\text{link}^k(v')} \left( \text{occur}(s, u) \cup \overline{\text{occur}}(s, u) \right) \right) \setminus \left\{ |s| - \text{len}\left(\text{link}^k(v')\right) + 1 \right\}
= \text{occur}(s, \text{link}^k(v')) \setminus \left\{ |s| - \text{len}\left(\text{link}^k(v')\right) + 1 \right\}
= \text{occur}(s', \text{link}^k(v')).
\]

\textbf{Lemma 4.9} (Correctness of \texttt{pop_front} by occurrence recording). Let \( s \) be a non-empty string. Suppose \( S(s, v) \) defined for \( \text{eertree}(s) \) satisfies Eq. (7). Let \( v' = \text{node}(\text{prepal}(s, 1)) \), and
\[
S(s', v) = \begin{cases} 
\left( (S(s, v) \setminus \{1\}) \cup \{\text{len}(v') - \text{len}(\text{link}(v')) + 1\} \right) - 1, & v = \text{link}(v') \text{ and } \text{len}(\text{link}(v')) \geq 1, \\
(S(s, v) \setminus \{1\}) - 1, & \text{otherwise},
\end{cases}
\]
where \( s' = s[2..|s|] \), and \( A - a = \{ x - a : x \in A \} \) for a set \( A \) and an element \( a \). Then \( S(s[2..|s|], v) \) satisfies Eq. (7) defined for \( \text{eertree}(s[2..|s|]) \).

\textbf{Proof.} Because of symmetry, the proof is similar to that of Lemma 4.6.
4.2.3 Complexity analysis

Theorem 4.10 (Double-ended eertree by occurrence recording). Double-ended eertrees can be implemented with amortized time complexity per operation \(O(\log(n) + \log(\sigma))\) and worst-case space complexity per operation \(O(\log(\sigma))\), where \(\sigma\) is the size of the alphabet and \(n\) is the length of the current string. More precisely,

- A `push_back` or `push_front` operation requires worst-case time complexity \(O(\log(n) + \log(\sigma))\) and worst-case space complexity \(O(\log(\sigma))\).

- A `pop_back` or `pop_front` operation requires amortized time complexity \(O(\log(n))\) and worst-case space complexity \(O(1)\).

Proof. The correctness has already been proved right after providing the algorithms in Section 4.2.1 and Section 4.2.2. Next, we analyze the time and space complexity of each of the deque operations. For convenience, we use balanced binary search trees to store \(S\), `prenode` and `sufnode`.

For one query of `push_back` and `push_front` operations, it is clear that no loop exists in their implementations (see Algorithm 4 and Algorithm 5). By the online construction of eertree in [RS18], we need \(O(\log(\sigma))\) time and space in the worst case to convert `eertree(s)` to `eertree(sc)` or `eertree(cs)`. Also, there is only one insertion operation required on each of \(S\), `prenode` and `sufnode`, which can be done in \(O(\log(n))\) time and \(O(1)\) space by binary tree insertions in the worst case. Therefore, a `push_back` or `push_front` operation requires worst-case time complexity \(O(\log(n) + \log(\sigma))\) and worst-case space complexity \(O(\log(\sigma))\).

For one query of `pop_back` (resp. `pop_front`) operations, the only loop is to delete the start position of \(\text{str}(u)\) for every node \(u\) in `sufnode[end_pos - 1]` (resp. `prenode[start_pos]`). According to the correctness of the algorithm, we conclude that \(S[u] = S(s, u), \text{sufnode[end_pos - 1]} = \text{sufnode}(s, |s|)\) and \(\text{prenode[start_pos]} = \text{prenode}(s, 1)\) at this moment. Next, we only consider the case of `pop_back` because of symmetry. Note that for every node \(u\) in `sufnode(s, |s|)`, the start position of the prefix \(\text{str}(u)\) of \(s\) must be in \(S(s, u)\). This follows that for every node \(u\) in `sufnode[end_pos - 1]`, its start position `end_pos - len(u)` is in \(S[u]\) before deleting it. Now we consider the total number of elements to be deleted. Note that every element to be deleted must have been added before. For all the four deque operations, there will be at most one element added per operation. Therefore, there will be amortized \(O(1)\) elements to be deleted per operation. So the time complexity per operation is amortized \(O(\log(n))\) and the space complexity is \(O(1)\) per operation in the worst case.

\[\square\]

5 Surface Recording Method

In the occurrence recording method, we maintain \(S(s, v)\) as a reduced set of occurrences of every palindromic substring of \(s\). Although the total size of such abstractions is \(O(|s|)\), it requires an extra logarithmic factor \(O(\log(|s|))\) to maintain `prepal(s, i)`, `sufpal(s, i)` for each \(1 \leq i \leq |s|\) and the elements in each \(S(s, v)\). Lemma 4.4 suggests a way to check the uniqueness of palindromic substrings of \(s\), which, however, does require the cardinality of \(S(s, v)\) rather than the exact elements in it. There are two aspects that can be optimized, thereby the reduced set \(S(s, v)\) no longer needed explicitly.

- To make it better, our new idea is to maintain certain number `cnt(s, v)` rather than a set \(S(s, v)\) for each node \(v\), where `cnt(s, v)` can be also used to check the uniqueness of palindromic substrings (see Lemma 5.6).
Nevertheless, we still need to maintain \textit{prenode}(s, i) and \textit{sufnode}(s, i) according to \(S(s, v)\) as in the occurrence recording method. To overcome this issue, we find a different way to maintain the longest palindromic prefix and suffix of \(s\), called the surface recording method. This section will first introduces how to check the uniqueness of palindromic substrings without the help of \(S(s, v)\), and then provide the surface recording method by clarifying the relationship between the reduced sets \(S(s, v)\) and surfaces.

5.1 Indirect occurrence counting

In this subsection, we propose an indirect occurrence counting. Let \(\text{precnt}(s, v)\) and \(\text{sufcnt}(s, v)\) denote the number of occurrences of \(\text{str}(v)\) that are the longest palindromic prefix and suffix of \(s[i..|s|]\) and \(s[1..i]\), respectively. That is,

\[
\text{precnt}(s, v) = |\{1 \leq i \leq |s|: \text{prepal}(s, i) = \text{str}(v)\}|. \quad (17)
\]

\[
\text{sufcnt}(s, v) = |\{1 \leq i \leq |s|: \text{sufpal}(s, i) = \text{str}(v)\}|. \quad (18)
\]

The following lemma shows that the definitions of \(\text{precnt}(v)\) and \(\text{sufcnt}(v)\) are equivalent.

**Lemma 5.1.** Suppose \(s\) is a string, and \(\text{eertree}(s)\) is the eertree of \(s\) with the set \(V\) of nodes. For every node \(v \in V \setminus \{\text{even, odd}\}\) in \(\text{eertree}(s)\), we have

\[
\text{precnt}(s, v) = \text{sufcnt}(s, v).
\]

Therefore, we write \(\text{cnt}(v) := \text{precnt}(v) = \text{sufcnt}(v)\) for convenience. Moreover, we find \(\text{cnt}(v)\) can be used to count the number of occurrences of palindromic substrings.

**Lemma 5.2.** Suppose \(s\) is a string and \(\text{eertree}(s)\) is the eertree of \(s\) with the set \(V\) of nodes. For every node \(v \in V \setminus \{\text{even, odd}\}\), we have

\[
|\text{occur}(s, v)| = \sum_{u \in T_v} \text{cnt}(s, u),
\]

where \(T_v\) is the subtree of node \(v\) in the link tree \(T\) of \(\text{eertree}(s)\) (see Section 2.4 for more details about the link tree).

**Proof of Lemma 5.1 and Lemma 5.2.** In this proof, we first show that

\[
|\text{occur}(s, v)| = \sum_{u \in T_v} \text{sufcnt}(s, u) = \sum_{u \in T_v} \text{precnt}(s, u),
\]

and then yield \(\text{precnt}(s, v) = \text{sufcnt}(s, v)\), which concludes Lemma 5.1 and Lemma 5.2 together.

First, we show the following proposition concerning \(\text{sufcnt}(s, v)\).

**Proposition 5.3.** For every node \(v \in V \setminus \{\text{even, odd}\}\), we have

\[
|\text{occur}(s, v)| = \sum_{u \in T_v} \text{sufcnt}(s, u). \quad (19)
\]
Proof. To see this, it is trivial to verify Proposition 5.3 for every string $s$ of length $|s| = 1$. Now we are going to prove Proposition 5.3 by induction on the length $|s|$ of $s$.

**Induction.** Suppose Proposition 5.3 holds for all strings of length $k$. That is, Eq. (19) holds for every string $s$ of length $k$ and every node $v$. Now let $s$ be a string of length $|s| = k + 1$. Let $t = \text{sufpal}(s, |s|)$ and $u_0 = \text{node}(t)$. We only need to consider nodes in link$^*(u_0)$. For every node $v \in \text{link}^*(u_0)$ with $\text{len}(v) \geq 1$, we know that $s[|s| - \text{len}(v) + 1..|s|]$ is a palindromic suffix of $s$. Let $s' = s[1..|s| - 1]$ and $\text{str}^*(u_0) = \text{str}(\text{link}^*(u_0)) = \{ \text{str}(v) : v \in \text{link}^*(u_0) \text{ and } \text{len}(v) \geq 1 \}$. Then, we have

$$|\text{occur}(s, v)| = \begin{cases} |\text{occur}(s', v)| + 1, & v \in \text{str}^*(u_0), \\ |\text{occur}(s', v)|, & \text{otherwise.} \end{cases}$$

By the definition of $\text{sufcnt}(s, v)$, we have

$$\text{sufcnt}(s, v) = \begin{cases} \text{sufcnt}(s', v) + 1, & v = u_0, \\ \text{sufcnt}(s', v), & \text{otherwise.} \end{cases}$$

For every node $v \in \text{link}^*(u_0)$, by induction, we have

$$|\text{occur}(s, v)| = |\text{occur}(s', v)| + 1 = \sum_{u \in T_v} \text{sufcnt}(s', v) + 1 = \sum_{u \in T_v \setminus \{u_0\}} \text{sufcnt}(s', v) + (\text{sufcnt}(s', u_0) + 1) = \sum_{u \in T_v \setminus \{u_0\}} \text{sufcnt}(s, v) + \text{sufcnt}(s, u_0) = \sum_{u \in T_v} \text{sufcnt}(s, v).$$

Therefore, Proposition 5.3 holds for all strings of length $k + 1$. Then, Proposition 5.3 is proved.

Similarly, we also have the following proposition symmetric to Proposition 5.3.

**Proposition 5.4.** For every node $v \in V \setminus \{\text{even, odd}\}$, we have

$$|\text{occur}(s, v)| = \sum_{u \in T_v} \text{precnt}(s, u).$$

Combining both Proposition 5.3 and Proposition 5.4, we obtain that for every node $v \in V \setminus \{\text{even, odd}\}$,

$$\sum_{u \in T_v} \text{precnt}(s, u) = \sum_{u \in T_v} \text{sufcnt}(s, u).$$

(20)

An induction on the link tree of $\text{eertree}(s)$ will lead to $\text{precnt}(s, v) = \text{sufcnt}(s, v)$ for every node $v \in V \setminus \{\text{even, odd}\}$, which is exactly Lemma 5.1. Then, Lemma 5.2 holds by Proposition 5.3, 5.4 and Lemma 5.1.

For completeness, we will show Lemma 5.1 in the following. We restate Lemma 5.1 as follows for convenience.

**Proposition 5.5.** For every node $v \in V \setminus \{\text{even, odd}\}$, we have $\text{precnt}(s, v) = \text{sufcnt}(s, v)$. 

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Proof. The basis of the proof is that if \( v \) is a leaf node in the link tree of \( \text{eertree}(s) \), i.e., \( |T_v| = 1 \), then it is straightforward that \( \text{precnt}(s, v) = \text{sufcnt}(s, v) \) by Eq. (20). Let \( l_v \) be the distance from node \( v \) to its nearest descendant leaf node. Formally,

\[
l_v = \begin{cases} 
0, & |T_v| = 1, \\
\max \{ l_u : \text{link}(u) = v \} + 1, & \text{otherwise}.
\end{cases}
\]

We will show Proposition 5.5 for every node \( v \) by induction on \( l_v \). It is clear that Proposition 5.5 holds for \( l_v = 0 \).

\textbf{Induction.} Suppose Proposition 5.5 holds for all \( 1 \leq l_v \leq k \). Now let \( v \) be a node with \( l_v = k + 1 \). By induction, for every node \( u \in T_v \{v\} \), \( \text{precnt}(s, u) = \text{sufcnt}(s, u) \) because \( 1 \leq l_u \leq k \). Then by Eq. (20), we have

\[
0 = \sum_{u \in T_v} \text{precnt}(s, u) - \sum_{u \in T_v} \text{sufcnt}(s, u) = (\text{precnt}(s, v) - \text{precnt}(s, v)) + \sum_{u \in T_v \setminus \{v\}} (\text{precnt}(s, u) - \text{precnt}(s, u)) = \text{precnt}(s, v) - \text{precnt}(s, v).
\]

Therefore, Proposition 5.5 holds for \( l_v = k + 1 \).

In conclusion, Proposition 5.5 is proved and these yield the proof.

Note that \( \text{cnt}(v) \) is not the cardinality of \( S(v) \). Indeed, both of \( \text{cnt}(v) \) and \( S(v) \) describe certain properties of occur(\( v \)) from different perspectives. Lemma 5.6 shows that \( \text{cnt}(v) \) can be used to check whether a palindromic substring is unique.

\textbf{Lemma 5.6 (Uniqueness checking via indirect occurrence counting).} Suppose \( s \) is a string, and \( s[i..|s|] \) is a palindrome for some \( 1 \leq i \leq |s| \). Let \( v = \text{node}(s[i..|s|]) \). Then \( s[i..|s|] \) is unique in \( s \) if and only if \( \text{linkcnt}(s, v) = 0 \) and \( \text{cnt}(s, v) = 1 \).

\textbf{Proof.} By the definition of occur(\( s, v \)), we know that \( s[i..|s|] \) is unique if and only if \( |\text{occur}(s, v)| = 1 \).

\( \Rightarrow \). If \( s[i..|s|] \) is unique, i.e., \( |\text{occur}(s, v)| = 1 \), by Lemma 5.2 we have

\[
|\text{occur}(s, v)| = \sum_{u \in T_v} \text{cnt}(s, u).
\]

By the definition of eertree, we have that \( \text{cnt}(s, u) \geq 1 \) for every node \( u \) in the eertree of \( s \). This implies that \( |\text{occur}(s, v)| \geq |T_v| \). Therefore, it must hold that \( |T_v| = 1 \), i.e., only one node in the subtree with root \( v \), thereby \( \text{linkcnt}(s, v) = 0 \). At last, we have \( \text{cnt}(s, v) = 1 \).

\( \Leftarrow \). If \( \text{linkcnt}(s, v) = 0 \) and \( \text{cnt}(s, v) = 1 \), by Lemma 5.2 we have that \( |\text{occur}(s, v)| = \text{cnt}(s, v) = 1 \).

To conclude this subsection, we show the relationship between \( \text{cnt}(s, v) \) and \( \text{cnt}(s', v) \), with \( s' \) being modified from \( s \) by adding or deleting a character at any of the both ends.

\textbf{Lemma 5.7.} Let \( s \) be a string and \( c \) be a character. Then

- push\_back:

\[
\text{cnt}(sc, v) = \begin{cases} 
\text{cnt}(s, v) + 1, & \text{str}(v) = \text{sufpal}(sc, |sc|), \\
\text{cnt}(s, v), & \text{otherwise}.
\end{cases}
\]
• **push_front:**

\[
\text{cnt}(cs, v) = \begin{cases} 
\text{cnt}(s, v) + 1, & \text{str}(v) = \text{prepal}(cs, 1), \\
\text{cnt}(s, v), & \text{otherwise}.
\end{cases}
\]

• **pop_back:** If \(s\) is not empty, then

\[
\text{cnt}(s[1..|s| - 1], v) = \begin{cases} 
\text{cnt}(s, v) - 1, & \text{str}(v) = \text{sufpal}(s, |s|), \\
\text{cnt}(s, v), & \text{otherwise}.
\end{cases}
\]

• **pop_front:** If \(s\) is not empty, then

\[
\text{cnt}(s[2..|s|], v) = \begin{cases} 
\text{cnt}(s, v) - 1, & \text{str}(v) = \text{prepal}(s, 1), \\
\text{cnt}(s, v), & \text{otherwise}.
\end{cases}
\]

**Proof.** We first show the identity for **push_back**. For this case, we only use the property that \(\text{cnt}(s, v) = \text{sufcnt}(s, v)\). For string \(sc\), the longest palindromic suffix of \(sc\) is the surface with end position \(|sc|\), which is \(s[|sc| - \text{suflen}(sc, |sc|) + 1..|sc|]\); and \(\text{sufpal}(sc, i)\) is not depend on \(c\) for every \(1 \leq i \leq |s|\). By the definition of \(\text{sufcnt}(sc, v)\), we have

\[
\text{sufcnt}(sc, v) = \begin{cases} 
\text{sufcnt}(s, v) + 1, & \text{str}(v) = \text{sufpal}(sc, |sc|), \\
\text{sufcnt}(s, v), & \text{otherwise}.
\end{cases}
\]

Next, we will show the identity for **pop_back**. For this case, also we only use the property that \(\text{cnt}(s, v) = \text{sufcnt}(s, v)\). Here, we only consider the case that \(|s| \geq 2\). We only need to note that only the longest palindromic suffix \(\text{sufpal}(s, |s|)\) is deleted with other \(\text{sufpal}(s, i)\) the same as \(\text{sufpal}(s[1..|s| - 1], i)\) for \(1 \leq i < |s|\). By the definition of \(\text{sufcnt}(sc, v)\), we have

\[
\text{sufcnt}(s[1..|s| - 1], v) = \begin{cases} 
\text{sufcnt}(s, v) - 1, & \text{str}(v) = \text{sufpal}(s, |s|), \\
\text{sufcnt}(s, v), & \text{otherwise}.
\end{cases}
\]

At last, the identities for **push_front** and **pop_front** are symmetric to those for **push_back** and **pop_back**. Then, these yield the proof.

**5.2 Surface recording**

For every node \(v\) in \(\text{eertree}(s)\), we define the set of surfaced occurrences of \(\text{str}(v)\) as

\[
\text{surf}(s, v) = \{ i \in \text{occur}(s, v) : s[i..i + \text{len}(v) - 1] \text{ is a surface in } s \}.
\]

Here, we give an alternative definition of the reduced set \(S(s, v)\) related to surfaces.

**Lemma 5.8.** Suppose \(s\) is a string and \(v\) is a node in \(\text{eertree}(s)\). The reduced set \(S(s, v)\) satisfies Eq. [4] if and only if \(\text{surf}(s, v) \subseteq S(s, v) \subseteq \text{occur}(s, v)\).

**Proof.** We only need to show that

\[
\text{occur}(s, v) \setminus \bigcup_{\text{link}(u) = v} (\text{occur}(s, u) \cup \text{occur}(s, u)) = \text{surf}(s, v)
\]

for every node \(v\) in \(\text{eertree}(s)\).

“\(\subseteq\)”. To show that the set of the left hand side is contained in \(\text{surf}(s, v)\), we choose any element \(i \in \text{occur}(s, v)\) such that \(i \notin \text{occur}(s, u)\) and \(i \notin \text{occur}(s, u)\) for every node \(u\) with \(\text{link}(u) = v\). That is, the following three conditions hold:
1. \( s[i..i + \text{len}(v) - 1] = \text{str}(v) \),
2. \( s[i..i + \text{len}(u) - 1] \neq \text{str}(u) \) for every node \( u \) with \( \text{link}(u) = v \),
3. \( s[i - \text{len}(u) + \text{len}(v)..i + \text{len}(v) - 1] \neq \text{str}(u) \) for every node \( u \) with \( \text{link}(u) = v \).

If \( s[i..i + \text{len}(v) - 1] \) is a proper suffix of a palindromic string \( w = s[l..i + \text{len}(v) - 1] \) for some \( 1 \leq l < i \), then there is a positive integer \( k \) such that \( \text{link}^k(\text{node}(w)) = v \), thus condition 3 does not hold by letting \( u = \text{link}^{k-1}(\text{node}(w)) \). Therefore, \( s[i..i + \text{len}(v) - 1] \) is not a proper suffix of any palindromic substrings of \( s \). Similarly, \( s[i..i + \text{len}(v) - 1] \) is not a proper prefix of any palindromic substrings of \( s \). By Definition 2.11, we have that \( s[i..i + \text{len}(v) - 1] \) is a surface in \( s \), thereby \( i \in \text{surf}(s, v) \).

As seen in Lemma 5.8, the set \( \text{surf}(s, v) \) is the minimal choice of the reduced set \( S(s, v) \). Inspired by this, we are going to maintain the set \( \text{surf}(s, v) \) implicitly. To this end, we write \( \text{presurf}(s, i) \) and \( \text{sufsurf}(s, i) \) to denote the surface in \( s \) with start and end index \( i \), respectively. That is,

\[
\text{presurf}(s, i) = \begin{cases} 
\text{prepal}(s, i), & \text{s[i..i + prelen(s, i) - 1] is a surface in } s, \\
\epsilon, & \text{otherwise.}
\end{cases}
\]  \tag{21}

\[
\text{sufsurf}(s, i) = \begin{cases} 
\text{sufpal}(s, i), & \text{s[i - suflen(s, i) + 1..i] is a surface in } s, \\
\epsilon, & \text{otherwise.}
\end{cases}
\]  \tag{22}

The following lemma shows that \( \text{prepal}(s, 1) \) and \( \text{sufpal}(s, |s|) \) can be computed by \( \text{presurf}(s, i) \) and \( \text{sufsurf}(s, i) \), respectively.

**Lemma 5.9.** Suppose \( s \) is a non-empty string. Then, we have

\[
\text{prepal}(s, 1) = \text{presurf}(s, 1),
\]

\[
\text{sufpal}(s, |s|) = \text{sufsurf}(s, |s|),
\]

**Proof.** To prove \( \text{presurf}(s, 1) = \text{prepal}(s, 1) \), we only need to show that \( s[1..\text{prelen}(s, 1)] \) is a surface in \( s \). It is clear that \( \text{suflen}(s, \text{prelen}(s, 1)) \leq \text{prelen}(s, 1) \) because \( \text{suflen}(s, i) \leq i \) for every \( 1 \leq i \leq |s| \). On the other hand, \( s[1..\text{prelen}(s, 1)] = \text{prepal}(s, 1) \) is a palindrome, so \( \text{suflen}(s, \text{prelen}(s, 1)) \geq \text{prelen}(s, 1) \). It follows that \( \text{suflen}(s, \text{prelen}(s, 1)) = \text{prelen}(s, 1) \). By Lemma 2.2 we conclude that \( s[1..\text{prelen}(s, 1)] \) is a surface in \( s \), and therefore \( \text{presurf}(s, 1) = \text{prepal}(s, 1) \).

We can show that \( \text{sufpal}(s, |s|) = \text{sufsurf}(s, |s|) \) similarly.

**5.3 The algorithm**

In this subsection, we will provide an efficient algorithm for implementing double-ended eertree based on surface recording. The algorithm is rather simple, but with a complicated correctness proof.

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5.3.1 push\_back and push\_front

The following lemma shows how \( \text{presurf}(s, i) \) and \( \text{sufsurf}(s, i) \) relate to \( \text{presurf}(sc, i) \) and \( \text{sufsurf}(sc, i) \) as a character \( c \) is added at the back of string \( s \).

**Lemma 5.10** (Surface recording for push\_back). Let \( s \) be a string and \( c \) be a character. Let \( t = \text{sufpal}(sc, |sc|) \) and \( t' = \text{str}(\text{link}(\text{node}(t))) \). Then

\[
\text{presurf}(sc, i) = \begin{cases} 
  t, & i = |sc| - |t| + 1, \\
  \epsilon, & i = |sc| \text{ and } |t| \neq 1, \\
  \text{presurf}(s, i), & \text{otherwise},
\end{cases}
\]

and

\[
\text{sufsurf}(sc, i) = \begin{cases} 
  t, & i = |sc|, \\
  \epsilon, & i = |sc| - |t| + |t'| \text{ and } \text{sufsurf}(s, |sc| - |t| + |t'|) = t' \text{ and } |t'| \geq 1, \\
  \text{sufsurf}(s, i), & \text{otherwise}.
\end{cases}
\]

**Proof.** It is trivial when \( s \) is empty. We just consider the case that \( s \) is non-empty.

We consider \( \text{presurf}(sc, i) \) first.

- If \( |t| = 1 \), by Lemma 2.21 \( \text{sc}[|sc|..|sc|] \) is a surface in \( sc \). Thus, \( \text{presurf}(sc, |sc|) = \text{sc}[|sc|..|sc|] = t \).

- If \( |t| \geq 2 \), by Lemma 2.21 \( \text{sc}[|sc| - |t| + 1..|sc|] \) is a surface in \( sc \) but \( \text{sc}[|sc|..|sc|] \) is not. Thus, \( \text{presurf}(sc, |sc| - |t| - 1) = \text{sc}[|sc| - |t| + 1..|sc|] = t \) and \( \text{presurf}(sc, |sc|) = \epsilon \).

It remains to consider the case that \( i \neq |sc| - |t| + 1 \) and \( i \neq |sc| \). Assume that \( \text{presurf}(sc, i) \neq \text{presurf}(s, i) \) for some \( 1 \leq i \leq |s| \) and \( i \neq |sc| - |t| + 1 \), let \( j = i + \text{prelen}(s, i) - 1 \leq |s| \). To yield a contradiction, we consider the following cases.

- \( s[i..j] \) is not a surface in \( s \). In this case, we have \( \text{presurf}(s, i) = \epsilon \). By Definition 2.21, there exists \( 1 \leq l < i \) or \( j < r \leq |s| \) such that \( s[l..j] \) or \( s[i..r] \) is a palindromic substring of \( s \). It can be seen that \( sc[i..j] \) is not a surface in \( sc \) because \( sc[l..j] \) or \( sc[i..r] \) is a palindromic substring of \( sc \). Therefore, \( \text{presurf}(sc, i) = \epsilon \) — a contradiction.

- \( s[i..j] \) is a surface in \( s \). If \( sc[i..j] \) is a surface in \( sc \), then \( \text{presurf}(sc, i) = \text{presurf}(s, i) \). So we only need to consider the case that \( sc[i..j] \) is not a surface in \( sc \). By Definition 2.21 this boils down into the following two cases.

1. There exists some \( 1 \leq l < i \) such that \( sc[l..j] \) is a palindromic substring of \( sc \). In this case, \( s[l..j] \) is a palindromic substring of \( s \), which implies that \( s[i..j] \) is not a surface in \( s \) — a contradiction.

2. There exists some \( j < r \leq |sc| \) such that \( sc[i..r] \) is a palindromic substring of \( sc \). If \( r \leq |s| \), then \( s[i..r] \) is a palindromic substring in \( s \), which implies that \( s[i..j] \) is not a surface in \( s \) — a contradiction; otherwise, \( r = |sc| \), and then \( sc[i..|sc|] \) is a palindrome. Furthermore, \( sc[i..|sc|] \) is a suffix of \( t \). In this case, \( |t| \geq |sc| - i + 1 \geq 2 \) and \( |sc| - |t| + 1 \leq i \leq j < |sc| \). Moreover, we have already assumed that \( i \neq |sc| - |t| + 1 \), so we only deal with the case that \( |sc| - |t| + 1 < i \leq j < |sc| \). Further, we consider two cases.

   (a) \( i + j \geq 2|sc| - |t| + 1 \). In this case, \( sc[2|sc| - |t| + 1 - j..j] = s[2|sc| - |t| + 1 - j..j] \) is a palindrome. By Definition 2.21 \( s[i..j] \) is not a surface in \( s \) — a contradiction.
(b) \(i + j < 2|sc| - |t| + 1\). In this case, \(sc[i..2|sc| - |t| + 1 - i] = s[i..2|sc| - |t| + 1 - i]\) is a palindrome. By Definition 2.1, \(s[i..j]\) is not a surface in \(s\) — a contradiction.

Next, we consider \(sufsurf(sc, i)\). By Lemma 2.3, we know that \(sc[|sc| - |t| + 1..|sc|]\) is a surface in \(sc\). Then \(sufsurf(sc, |sc|) = t\). In the following, we only need to consider the case that \(1 \leq i \leq |s|\), and to show that \(sufsurf(sc, i) = sufsurf(s, i)\) unless \(i = |sc| - |t| + |t'|\), \(sufsurf(s, |sc| - |t| + |t'|) = t'\) and \(|t'| \geq 1\). Now assume that \(sufsurf(sc, i) \neq sufsurf(s, i)\) for some \(1 \leq i \leq |s|\), and let \(j = i - suflen(s, i) + 1 \leq i\). We consider the following cases.

- \(s[j..i]\) is not a surface in \(s\). In this case, we have \(sufsurf(s, i) = \epsilon\). An argument similar to that for \(presurf(sc, i)\) will obtain that \(sc[j..i]\) is not a surface in \(sc\), and therefore \(sufsurf(sc, i) = \epsilon\) — a contradiction.

- \(s[j..i]\) is a surface in \(s\). In this case, \(sufsurf(sc, i) \neq \epsilon\). If \(sufsurf(sc, i) \neq \epsilon\) (and also \(sufsurf(sc, i) \neq sufsurf(s, i)\)), we have that \(sc[j..i]\) is a surface in \(sc\) for some \(j < j\), which implies \(s[j'..i]\) is a palindrome and thus \(s[j..i]\) is not a surface in \(s\) — a contradiction. Now we only need to consider the case that \(sufsurf(sc, i) = \epsilon\). In this case, \(sc[j..i]\) is not a surface in \(sc\). By Definition 2.1, this boils down into the following two cases.

1. There exists some \(1 \leq l < j\) such that \(sc[l..i]\) is a palindromic substring of \(sc\). In this case, \(s[l..i]\) is a palindromic substring of \(s\), which implies that \(s[j..i]\) is not a surface in \(s\) — a contradiction.

2. There exists some \(i < r \leq |sc|\) such that \(sc[j..r]\) is a palindromic substring of \(sc\). If \(r \leq |s|\), then \(s[j..r]\) is a palindromic substring of \(s\), which implies that \(s[j..i]\) is not a surface — a contradiction. Now we only need to consider the case that \(r = |sc|\). That is, \(sc[j..|sc|]\) is a palindrome. It is clear that \(j \geq |sc| - |t| + 1\). Indeed, \(j = |sc| - |t| + 1\); otherwise, \(j > |sc| - |t| + 1\), and we consider the following two cases.

   (a) \(i + j \geq 2|sc| - |t| + 1\). In this case, \(sc[2|sc| - |t| + 1 - j..j] = s[2|sc| - |t| + 1 - j..j]\) is a palindrome. By Definition 2.1, \(s[j..i]\) is not a surface in \(s\) — a contradiction.

   (b) \(i + j < 2|sc| - |t| + 1\). In this case, \(sc[i..2|sc| - |t| + 1 - i] = s[i..2|sc| - |t| + 1 - i]\) is a palindrome. By Definition 2.1, \(s[j..i]\) is not a surface in \(s\) — a contradiction.

Therefore, \(j = |sc| - |t| + 1\). Note that \(s[j..i] = sc[j..i]\) is a palindromic proper prefix of \(sc[j..|sc|]\) = \(t\), so \(node(s[j..i]) \in \text{link}^*(node(t))\). Suppose \(node(s[j..i]) = \text{link}^k(node(t))\) for some \(k \geq 1\). It can be seen that \(|t'| = \text{len(link}(node(t))) > j - i + 1 \geq 1\). If \(k \geq 2\), then \(j - i + 1 < \text{len(link}(node(t)))\), and thus \(sc[j..j + \text{len(link}(node(t))) - 1] = s[j..j + \text{len(link}(node(t))) - 1]\) is a palindrome, which implies that \(s[j..i]\) is not a surface in \(s\) — a contradiction. Now we have \(k = 1\) and \(s[j..i] = t'\), thus \(j - i + 1 = \text{len(link}(node(t))) = |t'| \geq 1\). It is clear that \(i = j - |t'| - 1 = |sc| - |t| + |t'|\). Recall that \(s[j..i]\) is a surface in \(s\), which is assumed previously. It follows that \(sufsurf(s, i) = s[j..i]\), which is \(sufsurf(s, |sc| - |t| + |t'|) = t'\).

\[\square\]

Symmetrically, we show how \(presurf(s, i)\) and \(sufsurf(s, i)\) relate to \(presurf(cs, i)\) and \(sufsurf(cs, i)\) as a character is added at the front of string \(s\).
Lemma 5.11 (Surface recording for push_front). Let $s$ be a string and $c$ be a character. Let $t = \text{prepal}(cs, 1)$ and $t' = \text{str}(\text{link}(\text{node}(t)))$. Then

$$
\text{presurf}(cs, i) = \begin{cases} 
  t, & i = 1, \\
  \epsilon, & i = |t| - |t'| + 1 \text{ and } \text{presurf}(s, |t| - |t'|) = t' \text{ and } |t'| \geq 1, \\
  \text{presurf}(s, i - 1), & \text{otherwise.}
\end{cases}
$$

and

$$
\text{sufsurf}(cs, i) = \begin{cases} 
  t, & i = |t|, \\
  \epsilon, & i = 1 \text{ and } |t| \neq 1, \\
  \text{sufsurf}(s, i - 1), & \text{otherwise.}
\end{cases}
$$

Proof. Because of symmetry, the proof is similar to that of Lemma 5.10. \qed

The algorithms for push_back and push_front are given as follows.

- **push_back($c$):**
  1. Find the node $v$ of the longest palindromic suffix $\text{sufpal}(s, |s|)$ of $s$.
  2. Modify $\text{eertree}(s)$ to $\text{eertree}(sc)$ according to $v$ and $c$ by online construction of eertrees \cite{RS18}.
  3. Maintain presurf and sufsurf according to Lemma 5.10.
  4. Maintain cnt according to Lemma 5.7.

- **push_front($c$):**
  1. Find the node $v$ of the longest palindromic prefix $\text{prepal}(s, 1)$ of $s$.
  2. Modify $\text{eertree}(s)$ to $\text{eertree}(cs)$ according to $v$ and $c$ by online construction of eertrees \cite{RS18}.
  3. Maintain presurf and sufsurf according to Lemma 5.11.
  4. Maintain cnt according to Lemma 5.7.

For completeness, we provide formal and detailed descriptions of the algorithm for push_back in Algorithm 9 and Algorithm 10.

5.3.2 pop_back and pop_front

The following lemma shows how presurf($s, i$) and sufsurf($s, i$) relate to presurf($s[1..|s| - 1], i$) and sufsurf($s[1..|s| - 1], i$) as a character is deleted from the back of string $s$.

Lemma 5.12 (Surface recording for pop_back). Let $s$ be a non-empty string. Let $t = \text{sufpal}(s, |s|)$ and $t' = \text{str}(\text{link}(\text{node}(t)))$. Then

$$
\text{presurf}(s[1..|s| - 1], i) = \begin{cases} 
  t', & i = |s| - |t| + 1 \text{ and } |\text{sufsurf}(s, |s| - |t| + |t'|)| < |t'|, \\
  \epsilon, & i = |s| - |t| + 1 \text{ and } |\text{sufsurf}(s, |s| - |t| + |t'|)| \geq |t'|, \\
  \text{presurf}(s, i), & \text{otherwise,}
\end{cases}
$$

and

$$
\text{sufsurf}(s[1..|s| - 1], i) = \begin{cases} 
  t', & i = |s| - |t| + |t'| \text{ and } |\text{sufsurf}(s, |s| - |t| + |t'|)| < |t'|, \\
  \text{sufsurf}(s, i), & \text{otherwise,}
\end{cases}
$$
Algorithm 9 \texttt{push\_back}(c) via surface recording method
\begin{algorithmic}[1]
\STATE $s \leftarrow \text{data[start\_pos..end\_pos - 1]}$. 
\STATE $v \leftarrow \text{sufsurf[end\_pos - 1]}$. \hfill $\triangleright$ This ensures that $\text{str}(v) = \text{sufpal}(s, |s|)$. 
\STATE Obtain $\text{eertree}(sc)$ from $\text{eertree}(s)$ with the help of $v$, and let $v' \leftarrow \text{node(sufpal(sc, |sc|))}$. 
\STATE $\text{data[end\_pos]} \leftarrow c$. 
\STATE $\text{end\_pos} \leftarrow \text{end\_pos} + 1$. 
\STATE $\text{presurf[end\_pos - 1]} \leftarrow \text{even}$. 
\STATE $\text{sufsurf[end\_pos - 1]} \leftarrow v'$. 
\STATE \textbf{if} $\text{len(link}(v')) \geq 1$ and $\text{sufsurf[end\_pos - len}(v') + \text{len(link}(v')) - 1 = \text{link}(v')$ \textbf{then}
\STATE $\text{sufsurf[end\_pos - len}(v') + \text{len(link}(v')) - 1] \leftarrow \text{even}$. 
\STATE $\text{cnt}[v'] \leftarrow \text{cnt}[v'] + 1$. 
\end{algorithmic}

Algorithm 10 \texttt{push\_front}(c) via surface recording method
\begin{algorithmic}[1]
\STATE $s \leftarrow \text{data[start\_pos..end\_pos - 1]}$. 
\STATE $v \leftarrow \text{presurf[start\_pos]}$. \hfill $\triangleright$ This ensures that $\text{str}(v) = \text{prepal}(s, 1)$. 
\STATE Obtain $\text{eertree}(sc)$ from $\text{eertree}(s)$ with the help of $v$, and let $v' \leftarrow \text{node(sufpal(cs, |cs|))}$. 
\STATE $\text{start\_pos} \leftarrow \text{start\_pos} - 1$. 
\STATE $\text{data[start\_pos]} \leftarrow c$. 
\STATE $\text{presurf[start\_pos]} \leftarrow v'$. 
\STATE $\text{sufsurf[start\_pos]} \leftarrow \text{even}$. 
\STATE $\text{sufsurf[start\_pos + len}(v') - 1] \leftarrow v'$. 
\STATE \textbf{if} $\text{len(link}(v')) \geq 1$ and $\text{presurf[start\_pos + len}(v') - \text{len(link}(v'))] = \text{link}(v')$ \textbf{then}
\STATE $\text{presurf[start\_pos + len}(v') - \text{len(link}(v'))] \leftarrow \text{even}$. 
\STATE $\text{cnt}[v'] \leftarrow \text{cnt}[v'] + 1$. 
\end{algorithmic}
Proof. Let \( s' = s[1..|s| - 1] \). We consider \( \text{presurf}(s', i) \) first. By Lemma 5.10, we have

\[
\text{presurf}(s, i) \begin{cases} t, & i = |s| - |t| + 1, \\
\epsilon, & i = |s| \text{ and } |t| \neq 1, \\
\text{presurf}(s', i), & \text{otherwise.}
\end{cases}
\]

This means that for every \( 1 \leq i \leq |s'| \) with \( i \neq |s| - |t| + 1 \), we have \( \text{presurf}(s', i) = \text{presurf}(s, i) \).

We only need to consider the case that \( i = |s| - |t| + 1 \leq |s'| = |s| - 1 \). In this case, \(|t| \geq 2 \) and \(|t'| \geq 1 \). We will first show that \( \text{sufsurf}(s, |s| - |t| + |t'|) \neq t' \), and thus \( \text{sufsurf}(s, |s| - |t| + |t'|) \neq t' \). Otherwise, \( \text{sufsurf}(s, |s| - |t| + |t'|) = t' \), which means that \( s[|s| - |t| + 1..|s| - |t| + |t'|] \) is a surface in \( s \). However, we know that \( s[|s| - |t| + 1..|s|] = t \) is a palindrome — a contradiction. Next, we will focus on the substring \( s'[|s| - |t| + 1..|s| - |t| + |t'|] = s[|s| - |t| + 1..|s| - |t| + |t'|] \), and consider the following two cases.

1. \(|sufsurf(s, |s| - |t| + |t'|)| > |t'| \). In this case, it is straightforward that

\[
s[|s| - |t| + |t'|] = \text{sufsurf}(s, |s| - |t| + |t'|) + 1..|s| - |t| + |t'|
\]

is a palindrome, which has a palindromic proper suffix \( s[|s| - |t| + 1..|s| - |t| + |t'|] \). It can be seen by Definition 2.1 that \( s'[|s| - |t| + 1..|s| - |t| + |t'|] \) is not a surface in \( s' \). Therefore, \( \text{presurf}(s', |s| - |t| + 1) = \epsilon \).

2. \(|sufsurf(s, |s| - |t| + |t'|)| < |t'| \). In this case, we have \( \text{sufsurf}(s, |s| - |t| + |t'|) = \epsilon \). Otherwise, \( s[|s| - |t| + |t'|] = \text{sufsurf}(s, |s| - |t| + |t'|) + 1..|s| - |t| + |t'| \) is a surface in \( s \), however, it is also a substring of a palindrome \( s[|s| - |t| + 1..|s| - |t| + |t'|] = t' \) — a contradiction.

Now we will show by contradiction that \( s'[|s| - |t| + 1..|s| - |t| + |t'|] \) is a surface in \( s' \).

- If \( s'[|s| - |t| + 1..r] \) is a palindrome for some \( |s| - |t| + |t'| < r \leq |s'| \), then \( s[|s| - |t| + 1..r] \) is a palindromic proper prefix of \( t \) with length \( > |t'| \), which contradicts the definition of \( t' \) — the longest palindromic proper prefix of \( t \).
- If \( s'[l..|s| - |t| + |t'|] \) is a palindrome for some \( 1 \leq l < |s| - |t| + 1 \), then let \( l_{\text{min}} \) be the smallest such \( l \). We can show that \( s[l_{\text{min}}..|s| - |t| + |t'|] \) is a surface in \( s \). If not, by Definition 2.1 and the minimality of \( l_{\text{min}} \), we have \( \text{prelen}(s, l_{\text{min}}) > |s| - |t| + |t'| - l_{\text{min}} + 1 \).

Let \( r = l_{\text{min}} + \text{prelen}(s, l_{\text{min}}) - 1 \leq |s| \), and then \( s[l_{\text{min}}..r] \) is a palindrome. It is clear that \( r \neq |s| \), otherwise \( s[l_{\text{min}}..r] \) is a palindromic suffix of \( s \) with length \( > |t| \), which contradicts the definition of \( t \) — the longest palindromic suffix of \( s \). Therefore, \( r \geq |s| - 1 \). Note that \( s[|s| - |t| + 1..|s| - |t| + |t'|] = t' \) is a palindromic proper suffix of \( s[l_{\text{min}}..|s| - |t| + |t'|] \), and \( s[l_{\text{min}}..|s| - |t| + |t'|] \) is a palindromic proper suffix of \( s[l_{\text{min}}..r] \). Let \( u = \text{node}(t') \). Then \( u = \text{link}^k(\text{node}(s[l_{\text{min}}..r])) \) for some \( k \geq 2 \). Let \( u_1 = \text{link}^{k-1}(\text{node}(s[l_{\text{min}}..r])) \) and \( u_2 = \text{link}^{k-2}(\text{node}(s[l_{\text{min}}..r])) \). By Lemma 2.6, we have \( \text{len}(u) \geq 2\text{len}(u_1) - \text{len}(u_2) \).

Then, we consider the following two cases.

(a) \( \text{len}(u) > 2\text{len}(u_1) - \text{len}(u_2) \). In this case, we have that the prefix

\[
s[|s| - |t| + 1..|s| - |t| + 2|t'| - 2\text{len}(u_1) + \text{len}(u_2)]
\]

of \( t \) is a palindromic substring of \( s \) because it has the same center as

\[
s[|s| - |t| + |t'| - \text{len}(u_1) + 1..|s| - |t| + |t'| - \text{len}(u_2)] = \text{str}(u_2).
\]

Note that \( s[|s| - |t| + 1..|s| - |t| + |t'|] = t' \) is a palindromic proper prefix of the palindrome \( s[|s| - |t| + 1..|s| - |t| + 2|t'| - 2\text{len}(u_1) + \text{len}(u_2)] \), which contradicts the definition of \( t' \) — the longest palindromic proper prefix of \( t \).
(b) \( \text{len}(u) = 2 \text{len}(u_1) - \text{len}(u_2) \). In this case, we have that the prefix

\[
\text{s}([\text{s} - |t| + |t| + \text{len}(u_1)]) = \text{str}(u_1)
\]

of \( t \) is a palindromic substring of \( s \). Note that \( t' \) is of course a palindromic proper prefix of \( s[[s - |t| + 1..|s| - |t| + \text{len}(u_1)] = \text{str}(u_1) \), which contradicts the definition of \( t' \) — the longest palindromic proper prefix of \( t \).

From the above arguments, we conclude that \( s[l_{\text{min}}..|s - |t| + |t'|] \) is a surface in \( s \), and therefore \( \text{sufsurf}(s, |s| - |t| + |t'|) = s[l_{\text{min}}..|s| - |t| + |t'|] \), which contradicts the condition that \( \text{sufsurf}(s, |s| - |t| + |t'|) = \epsilon \).

As each case above encounters a contradiction, we conclude that \( s'[|s| - |t| + 1..|s| - |t| + |t'|] \) is a surface in \( s' \), and thus \( \text{presurf}(s', |s| - |t| + 1) = t' \).

Next, we consider \( \text{sufsurf}(s', i) \). By Lemma 5.10, we have

\[
\text{sufsurf}(s, i) = \begin{cases} 
  t, & \text{if } i = |s|, \\
  \epsilon, & \text{if } i = |s| - |t| + |t'| \text{ and } \text{sufsurf}(s', s - |t| + |t'|) = t' \text{ and } |t'| \geq 1, \\
  \text{sufsurf}(s', i), & \text{otherwise}.
\end{cases}
\]

This means that for every \( 1 \leq i \leq |s'| \) with \( i \neq |s| - |t| + |t'| \), we have \( \text{sufsurf}(s', i) = \text{sufsurf}(s', i) \).

We only need to consider the case that \( i = |s| - |t| + |t'| \leq |s'| = |s| - 1 \). We first consider the special case that \( t' = \epsilon \), i.e., \( |t| = 1 \). In this case, \( i = |s'| = |s| - 1 \). It can be seen that \( \text{sufsurf}(s, |s| - 1) \neq \epsilon \); otherwise, \( s[|s| - |\text{sufsurf}(s, |s| - 1)|..|s| - 1] \) is not a surface in \( s \), then, by Definition 2.1, \( s[|s| - |s| - 1]|..|s| - 1] \) is a palindrome, which contradicts the definition of \( t \) — the longest palindromic suffix of \( s \). From this, we claim that \( \text{sufsurf}(s', |s| - 1) = \text{sufsurf}(s, |s| - 1); \) otherwise, \( s'[|s| - |s| - 1]|..|s| - 1] \) is not a surface in \( s' \), then, by Definition 2.1, there exists some \( 1 \leq l < |s| - |s| - 1|\) such that \( s[l|s| - 1] \) is a palindrome, which contradicts the definition of \( \text{sufsurf}(s', |s| - 1) \). On one hand, \( \text{sufsurf}(s', |s| - 1) = \text{sufsurf}(s, |s| - 1) = |s| - 1 \) (because \( \text{sufsurf}(s, |s| - 1) \neq \epsilon \). On the other hand, \( \text{sufsurf}(s', |s| - 1) \geq s[l|s| - 1] = |s| - l > |s| - 1 \). Therefore, \( \text{sufsurf}(s', |s| - 1) = \text{sufsurf}(s, |s| - 1) \) if \( t' = \epsilon \). We will consider the case that \( |t'| \geq 1 \), i.e., \( |t| \geq 2 \), by the following cases. Note that we have already shown that \( \text{sufsurf}(s, |s| - |t| + |t'|) \neq |t'| \) when considering \( \text{presurf}(s', i). \)

1. \( \text{sufsurf}(s, |s| - |t| + |t'|) < |t'| \). In this case, it can be seen that \( \text{sufsurf}(s', |s| - |t| + |t'|) = t' \), since we have already shown that \( s'[|s| - |t| + 1..|s| - |t| + |t'|] \) is a surface in \( s' \) for the same case when considering \( \text{presurf}(s', i) \).

2. \( \text{sufsurf}(s, |s| - |t| + |t'|) > |t'| \). In this case, we know that

\[
s[|s| - |t| + |t'| - \text{sufsurf}(s, |s| - |t| + |t'|)] + 1..|s| - |t| + |t'|
\]

is palindromic, and has a palindromic proper suffix \( s[|s| - |t| + 1..|s| - |t| + |t'|] = t' \). Therefore, \( s'[|s| - |t| + 1..|s| - |t| + |t'|] \) is not a surface in \( s' \), and thus \( \text{sufsurf}(s', |s| - |t| + |t'|) = \epsilon \).

Symmetrically, we show how \( \text{presurf}(s, i) \) and \( \text{sufsurf}(s, i) \) relate to \( \text{presurf}(s[2..|s|], i) \) and \( \text{sufsurf}(s[2..|s|], i) \) as a character is deleted from the front of string \( s \).
**Lemma 5.13** (Surface recording for pop_front). Let $s$ be a non-empty string. Let $t = \text{prepal}(s, 1)$ and $t' = \text{str}(\text{link}(\text{node}(t)))$. Then

$$\text{presurf}(s[2..|s|], i) = \begin{cases} t', & i = |t| - |t'| \text{ and } |\text{presurf}(s, |t| - |t'| + 1)| < |t'|, \\ \text{presurf}(s, i + 1), & \text{otherwise}, \end{cases}$$

and

$$\text{sufsurf}(s[2..|s|], i) = \begin{cases} t', & i = |t| - 1 \text{ and } |\text{presurf}(s, |s| - |t| + 1)| < |t'|, \\ \epsilon, & i = |t| - 1 \text{ and } |\text{presurf}(s, |s| - |t| + 1)| \geq |t'|, \\ \text{sufsurf}(s, i + 1), & \text{otherwise}, \end{cases}$$

*Proof.* Because of symmetry, the proof is similar to that of Lemma 5.12. □

The algorithms for pop_back and pop_front are given as follows.

- **pop_back:**
  1. Let $v$ be the node of the longest palindromic suffix $\text{sufpal}(s, |s|)$ of $s$.
  2. If $\text{str}(v)$ is unique in $s$ (checked by Lemma 5.6), then delete $v$ from the eertree. This will modify $\text{eertree}(s)$ to $\text{eertree}(s[1..|s| - 1])$.
  3. Maintain presurf and sufsurf according to Lemma 5.12.
  4. Maintain $\text{cnt}[v] \leftarrow \text{cnt}[v] - 1$ according to Lemma 5.7.

- **pop_front:**
  1. Let $v$ be the node of the longest palindromic prefix $\text{prepal}(s, 1)$ of $s$.
  2. If $\text{str}(v)$ is unique in $s$ (checked by Lemma 5.6), then delete $v$ from the eertree. This will modify $\text{eertree}(s)$ to $\text{eertree}(s[2..|s|])$.
  3. Maintain presurf and sufsurf according to Lemma 5.13.
  4. Maintain $\text{cnt}[v] \leftarrow \text{cnt}[v] - 1$ according to Lemma 5.7.

For completeness, we provide formal and detailed descriptions of the algorithm for push_back in Algorithm 11 and Algorithm 12.

### 5.3.3 Complexity analysis

**Theorem 5.14** (Double-ended eertree by surface recording). Double-ended eertrees can be implemented with worst-case time and space complexity per operation $O(\log(\sigma))$, where $\sigma$ is the size of the alphabet and $n$ is the length of the current string. More precisely,

- A push_back or push_front operation requires worst-case time and space complexity $O(\log(\sigma))$.

- A pop_back or pop_front operation requires worst-case time and space complexity $O(1)$.

*Proof.* The correctness has already been proved right after providing the algorithms in Section 5.3.1 and Section 5.3.2. We only analyze the time and space complexity of each deque operation. It is clear that no loops exist in Algorithm 9, 10, 11 and 12. The only space consumption is the online construction of eertrees in Algorithm 9 and 10 which requires $O(\log(\sigma))$ time and space per operation. □
Algorithm 11 pop\_back() via surface recording method

1: \( s \leftarrow \text{data[start..end - 1]} \).
2: \( v \leftarrow \text{sufsurf[end - 1]} \). \hspace{1cm} \triangleright \text{This ensures that str}(v) = \text{sufpal}(s, |s|).
3: if \( \text{linkcnt}(s, v) = 0 \) and \( \text{cnt}[v] = 1 \) then \hspace{1cm} \triangleright \text{By Lemma 5.6 it means that str}(v) is unique in } s.
4: \hspace{1cm} \text{Delete } v \text{ from the eertree.}
5: end if
6: \text{cnt}[v] \leftarrow \text{cnt}[v] - 1.
7: if \( \text{len}(	ext{sufsurf[end - len(v) + len(link(v)) - 1]}) < \text{len}(\text{link}(v)) \) then
8: \hspace{1cm} \text{sufsurf[end - len(v)]} \leftarrow \text{link}(v).
9: \hspace{1cm} \text{presurf[end - len(v)]} \leftarrow \text{link}(v).
10: else
11: \hspace{1cm} \text{presurf[end - len(v)]} \leftarrow \text{even}.
12: end if
13: \text{end_pos } \leftarrow \text{end_pos - 1}.

Algorithm 12 pop\_front() via surface recording method

1: \( s \leftarrow \text{data[start..end - 1]} \).
2: \( v \leftarrow \text{presurf[start_pos]} \). \hspace{1cm} \triangleright \text{This ensures that str}(v) = \text{prepal}(s, 1).
3: if \( \text{linkcnt}(s, v) = 0 \) and \( \text{cnt}[v] = 1 \) then \hspace{1cm} \triangleright \text{By Lemma 5.6 it means that str}(v) is unique in } s.
4: \hspace{1cm} \text{Delete } v \text{ from the eertree.}
5: end if
6: \text{cnt}[v] \leftarrow \text{cnt}[v] - 1.
7: if \( \text{len}([\text{presurf[start_pos + len(v) - len(link(v))]}]) < \text{len}(\text{link}(v)) \) then
8: \hspace{1cm} \text{presurf[start_pos + len(v)]} \leftarrow \text{link}(v).
9: \hspace{1cm} \text{sufsurf[start_pos + len(v) - 1]} \leftarrow \text{link}(v).
10: else
11: \hspace{1cm} \text{sufsurf[start_pos + len(v) - 1]} \leftarrow \text{even}.
12: end if
13: \text{start_pos } \leftarrow \text{start_pos + 1}.
5.4 Persistent double-ended eertrees

In this subsection, we study how to make our double-ended eertree fully persistent. A persistent data structure is a collection of data structures (of the same type), called versions, ordered by the time they are created. A data structure is called fully persistent if every version of it can be both accessed and modified.

**Theorem 5.15** (Persistent double-ended eertrees). Fully persistent double-ended eertrees can be implemented with worst-case time and space complexity per operation $O(\log(n) + \log(\sigma))$, where $\sigma$ is the size of the alphabet and $n$ is the length of the string in the current version. More precisely,

- A push_back or push_front operation requires worst-case time and space complexity $O(\log(n) + \log(\sigma))$.
- A pop_back or pop_front operation requires worst-case time and space complexity $O(\log(n))$.

**Proof.** The implementation of persistent double-ended eertree is based on Algorithm 9 to 12 wherein data, presurf and sufsurf are easy to make persistent in $O(\log(n))$ time and space per operation by binary search trees. We only need to further consider how to add and delete nodes from eertrees persistently. To this end, we maintain a persistent set nodes (by, for example, binary search trees) to contain all nodes in the eertree of the current version.

- To add a node $v$, we need to
  - add $v$ into nodes with time and space complexity $O(\log(n))$,
  - create $\text{dlink}(v, \cdot)$ from $\text{dlink}(\text{prev}(v), \cdot)$ with time and space complexity $O(\log(\sigma))$ by Lemma 2.9,
  - maintain $\text{linkcnt}(s, \text{link}(v))$ and $\text{cnt}[v]$ with time and space complexity $O(\log(n))$.

Thus adding a node requires $O(\log(n) + \log(\sigma))$ time and space.

- To delete a node $v$, we need to
  - maintain $\text{linkcnt}(s, \text{link}(v))$ and $\text{cnt}[v]$ with time and space complexity $O(\log(n))$,
  - delete $v$ from nodes with time and space complexity $O(\log(n))$.

Thus deleting a node requires $O(\log(n))$ time and space.

Therefore, double-ended eertrees can be implemented fully persistently with worst-case time and space complexity per operation $O(\log(n) + \log(\sigma))$. 

The complexity might be possible to improve using more advanced persistent data structures such as persistent arrays, lists and deques [Oka98,Str13]. We leave this issue for future work.

6 Applications

In this section, we will apply our double-ended eertrees to several computational tasks. The first part is a framework for range queries of various types concerning palindromes on a string. The second part is an efficient algorithm that enumerates palindromic rich strings with a given word.
6.1 Range queries concerning palindromes on a string

In this subsection, we aim to design a framework for range queries on a string concerning problems about palindromes.

Suppose a string \( s[1..n] \) of length \( n = |s| \) is given. We consider range queries on any substrings \( s[l..r] \) of \( s \), where \( 1 \leq l \leq r \leq n \). A query \((l, r)\) is to find

- the number of distinct palindromic substrings,
- the longest palindromic substring,
- the shortest unique palindromic substring,
- the shortest absent palindrome

of substring \( s[l..r] \) of \( s \). We formally state the four types of queries as follows.

**Problem 1 (Counting Distinct Palindromic Substrings).** Given a string \( s \) of length \( n \), for each query of the form \((l, r)\), count the number of distinct palindromes over all \( s[i..j] \) for \( l \leq i \leq j \leq r \).

**Problem 2 (Longest Palindromic Substring).** Given a string \( s \) of length \( n \), for each query of the form \((l, r)\), find the longest palindromic substring \( s[i..j] \) over \( l \leq i \leq j \leq r \). If there are multiple solutions, find any of them.

**Problem 3 (Shortest Unique Palindromic Substring).** Given a string \( s \) of length \( n \), for each query of the form \((l, r)\), find the shortest palindromic substring \( s[i..j] \) over \( l \leq i \leq j \leq r \) that occurs exactly once in \( s[l..r] \). If there are multiple solutions, find any of them.

**Problem 4 (Shortest Absent Palindrome).** Given a string \( s \) of length \( n \), for each query of the form \((l, r)\), find the shortest palindrome \( t \) that is not a substring of \( s[l..r] \). If there are multiple solutions, find any of them.

Now suppose we have \( q \leq n^2 \) queries \((l_i, r_i)\) with \( 1 \leq l_i \leq r_i \leq n \) for \( 1 \leq i \leq q \). In the following, we will consider to answer these queries in the offline and online cases, respectively.

### 6.1.1 Offline queries

To answer offline queries efficiently, we adopt the trick in Mo’s algorithm (cf. [DKPW20]). The basic idea is to maintain a double-ended ctree to iterate over the ctree of each \( s[l_i..r_i] \) for all \( 1 \leq i \leq q \). Let \( B \) be a parameter to be determined. We sort all queries by \( \lfloor (l_i - 1)/B \rfloor \) and in case of a tie by \( r_i \) (both in increasing order). Let \( T \) be a double-ended ctree of \( s[l_1..r_1] \) which can be constructed in \( O(|l_1 - r_1 + 1| \log(\sigma)) \) time. Now we will iterate all ctrees needed as follows.

- For \( 2 \leq i \leq q \) in increasing order,

1. Let \( l \leftarrow l_{i-1} \) and \( r \leftarrow r_{i-1} \) indicate that \( T \) is the ctree of \( s[l..r] \) currently.
2. Repeat the following as long as \( r < r_i \):
   - Set \( r \leftarrow r + 1 \), and then perform \texttt{push_back} \((s[r])\) on \( T \).
3. Repeat the following as long as \( l > l_i \):
   - Set \( l \leftarrow l - 1 \), and then perform \texttt{push_front} \((s[l])\) on \( T \).
4. Repeat the following as long as \( r > r_i \):
– Set \( r \leftarrow r - 1 \), and then perform \( \text{pop\_back()} \) on \( T \).

5. Repeat the following as long as \( l < l_i \):
   – Set \( l \leftarrow l - 1 \), and then perform \( \text{pop\_front()} \) on \( T \).

6. Now \( T \) stores the eertree of \( s[l_i..r_i] \).

It is clear that the time complexity of the above process is

\[
O\left( \frac{q}{2} \sum_{i=2}^{q} (|l_i - l_{i-1}| + |r_i - r_{i-1}|) \log(\sigma) \right) = O\left( \left( Bq + \frac{n^2}{B} \right) \log(\sigma) \right) = O(n\sqrt{q}\log(\sigma))
\]

by setting \( B = \lceil n/\sqrt{q} \rceil \). Now we have access to the eertree of substring \( s[l_i..r_i] \) for each \( 1 \leq i \leq q \).

In the following, we will consider different types of queries separately.

**Counting distinct palindromic substrings**

The number of distinct palindromic substrings, also known as the palindromic complexity, of a string, has been studied in the literature (e.g., [ABCD03,AAK10,GPR10,RS17]). It was noted in [RS18] that the number of distinct palindromic substrings of string \( s \) equals to the number of nodes in the eertree of \( s \) (minus 1). Immediately, we have the following result on counting distinct palindromic substrings.

**Corollary 6.1.** Offline range queries of type **Counting Distinct Palindromic Substrings** can be solved with time complexity \( O(n\sqrt{q}\log(\sigma)) \).

It was shown in [RS17] that range queries of type **Counting Distinct Palindromic Substrings** can be solved in \( O((n + q)\log(n)) \) time assuming \( \sigma = O(1) \). Our algorithm in Corollary 6.1 can be faster than the one given in [RS17] when \( q = o(\log^2(n)) \) or \( q = \omega(n^2/\log^2(n)) \).

**Longest palindromic substring**

Finding the longest palindromic substring has been extensively studied in the literature (e.g., [Man75,ABC95,Jeu94,Gus97,BEMTSA14,AB19,GMSU19,FNI21,CPR22,LGS22]). To answer range queries of type **Longest Palindromic Substring**, we use \( n \) linked lists to store all palindromic substrings of the current string. Specifically, let \( \text{list}[i] \) be the double-linked list to store palindromic substrings of length \( i \) for every \( 1 \leq i \leq n \), and let \( \text{maxlen} \) indicate the maximum length over all these palindromic substrings. Initially, all lists are empty and \( \text{maxlen} = 0 \). We can maintain these data as follows.

- When a node \( u \) is added to the double-ended eertree \( T \),
  1. Add \( u \) to \( \text{list}[\text{len}(u)] \).
  2. Set \( \text{maxlen} \leftarrow \max\{\text{maxlen}, \text{len}(u)\} \).

- When a node \( u \) is deleted from the double-ended eertree \( T \),
  1. Delete \( u \) from \( \text{list}[\text{len}(u)] \).
  2. Repeat the following until \( \text{list}[\text{maxlen}] \) is not empty:
     - Set \( \text{maxlen} \leftarrow \text{maxlen} - 1 \).

To find the longest palindromic substring of the current string in the eertree \( T \), just return any element in \( \text{list}[\text{maxlen}] \). The correctness is trivial. In the above process, there is only one loop that decrements \( \text{maxlen} \) until \( \text{list}[\text{maxlen}] \) is not empty. It can be seen that the loop repeats no more than twice by the following observation.
Proposition 6.2. If a node $u$ with $\text{len}(u) > 2$ is being deleted from a double-ended eertree due to $\text{pop\_back}$ or $\text{pop\_front}$ operations, then after node $u$ is deleted, there is a node $v$ in the eertree such that $\text{len}(v) = \text{len}(u) - 2$.

Proof. Choose node $v$ such that $\text{str}(v) = \text{str}(u)[1..\text{len}(u) - 1]$. It is trivial that $\text{len}(v) = \text{len}(u) - 2$ and $\text{str}(v)$ occurs at least once in the string after the $\text{pop\_back}$ or $\text{pop\_front}$ operation. □

Therefore, we have the following result on finding the longest palindromic substring.

Corollary 6.3. Offline range queries of type Longest Palindromic Substring can be solved with time complexity $O(n\sqrt{q}\log(\sigma))$.

It was shown in [ACPR20] that range queries of type Longest Palindromic Substring can be solved in $O(n\log^2(n) + q\log(n))$ time assuming $\sigma = O(1)$. Our algorithm in Corollary 6.3 can be faster than the one given in [ACPR20] when $q = o(\log^4(n))$ or $q = \omega(n^2/\log^2(n))$.

Shortest unique palindromic substring Motivated by molecular biology [KYK’92, YYK’92], algorithms about the shortest unique palindromic substring was investigated in a series of works [INM’18, WNI’20, FM21, MF22]. To find the shortest unique palindromic substring with respect to an interval of a string, they introduced the notion of minimal unique palindromic substrings (MUPSs). Here, a palindromic substring $s[i..j]$ of string $s$ is called a MUPS of $s$, if $s[i..j]$ occurs exactly once in $s$ and $s[i+1..j-i]$ either is empty or occurs at least twice. The set of MUPSs can be maintained after single-character substitution [FM21], and in a sliding window [MWN’22].

In our case, we are to find the shortest unique palindromic substring of a string $s$, which is actually the shortest MUPSs of $s$. To this end, we are going to maintain the set of all MUPSs of the current string. The following lemma shows that whether a palindrome is a MUPS can be reduced to uniqueness checking.

Lemma 6.4 (MUPS checking via uniqueness [MWN’22]). A palindrome $t$ is a MUPS of string $s$, if and only if $t$ is unique in $s$ and $t[1..|t| - 1]$ is not unique in $s$, where the empty string $\epsilon$ is considered to be not unique in any string.

By Lemma 5.6, whether a palindromic substring $t$ of string $s$ is a MUPS of $s$ can be checked in $O(1)$ time, given access to the node of $t$ in the eertree. Now we will maintain the set $\text{MUPS}$ to store all MUPSs of the current string. Initially, the set $\text{MUPS}$ is empty. For each deque operation on the double-ended eertree $T$, do the following.

- For each $\text{push\_back}(c)$ operation on double-ended eertree $T$,
  1. Perform $\text{push\_back}(c)$ on $T$.
  2. Let $u = \text{node}(\text{sufpal}(s, |s|))$, where $s$ is the current string.
  3. Maintain $\text{MUPS}$ according to whether $u$ as well as prev($u$) is a MUPS of $s$.
- For each $\text{push\_front}(c)$ operation on double-ended eertree $T$,
  1. Perform $\text{push\_front}(c)$ on $T$.
  2. Let $u = \text{node}(\text{prepal}(s, 1))$, where $s$ is the current string.
  3. Maintain $\text{MUPS}$ according to whether $u$ as well as prev($u$) is a MUPS of $s$.
- For each $\text{pop\_back}$ operation on double-ended eertree $T$,
1. Let \( u = \text{node}(\text{suffix}(s, |s|)) \), where \( s \) is the current string.
2. Perform \text{pop\_back}() on \( T \).
3. Maintain \text{MUPS} according to whether \( u \) as well as \text{prev}(u) \) is a \text{MUPS} of \( s \).

- Before each \text{pop\_front} operation on double-ended eertree \( T \),
  1. Let \( u = \text{node}(\text{prefix}(s, 1)) \), where \( s \) is the current string.
  2. Perform \text{pop\_front}() on \( T \).
  3. Maintain \text{MUPS} according to whether \( u \) as well as \text{prev}(u) \) is a \text{MUPS} of \( s \).

Here, step 3 of each case can be maintained as follows.
1. If \( u \) a \text{MUPS} of \( s \), add \( u \) to \text{MUPS}; otherwise, remove \( u \) from \text{MUPS}.
2. If \text{len}(u) > 2, do the same for \text{prev}(u). That is, if \text{prev}(u) \) a \text{MUPS} of \( s \), add \text{prev}(u) to \text{MUPS}; otherwise, remove \text{prev}(u) from \text{MUPS}.

To answer each query of type \text{Shortest Unique Palindromic Substring}, just return any element in \text{MUPS} with the minimum length. To achieve this, we can use the binary search tree to maintain \text{MUPS}, which introduces an extra \( O(\log(n)) \) in the complexity. Then, we have the following result.

**Corollary 6.5.** Offline range queries of type \text{Shortest Unique Palindromic Substring} can be solved with time complexity \( O(n \sqrt{q} (\log(n) + \log(\sigma))) \).

It was shown in [MWN+22] that the set of \text{MUPS}s can be maintained in the sliding window model, which is actually a special case of ours that \( l_i \leq l_{i+1} \) and \( r_i \leq r_{i+1} \) for every \( 1 \leq i < q \). In this special case, range queries of type \text{Shortest Unique Palindromic Substring} can be solved in time \( O(n(\log(n) + \log(\sigma)) + q) \) by the sliding window technique in [MWN+22]. However, their technique seems not applicable in our more general case.

**Shortest absent palindrome** Minimal absent palindromes (MAPs) is a palindromic version of the notion of minimal absent words, which was extensively studied in the literature [CMRS00, MRS02, CCT12]. The set of MAPs can be maintained in the sliding window model [MWN+22]. Here, a palindrome \( t \) is called a MAP of a string \( s \), if \( t \) does not occur in \( s \) but \( t[1..|t|−1] \) does, where the empty string \( \epsilon \) is considered to occur in any string.

In our case, we are to find the shortest absent palindrome of a string, which is actually the shortest MAP of the string. The following lemma shows an upper bound of the length of the shortest absent palindrome of a string.

**Lemma 6.6.** Suppose \( s \) is a string of length \( n \). Then, the length of the shortest absent palindrome of \( s \) is \( \leq \lceil 2\log_\sigma(n) \rceil + 1 \), where \( \sigma \) is the size of the alphabet.

**Proof.** Let \( k = \lceil 2\log_\sigma(n) \rceil + 1 \). The number of palindromes of length \( k \) is \( \sigma^{[k/2]} > n \). On the other hand, the number of distinct non-empty palindromic substrings of string \( s \) is at most \( n \) [DJP01]. We conclude that there must be a palindrome of length \( k \) that does not occur in \( s \), and these yield the proof. \( \square \)
Our main idea to find the shortest absent palindrome is to maintain the set of MAPs. In the implementation, we use linked-lists to store MAPs of each length indirectly. Specifically, let \( \text{list}[i] \) be the double-linked list to store palindromic substrings \( t \) of length \( i \) for each \( i \) with \( \text{next}(\text{node}(t), c) = \text{null} \) for at least one character \( c \). By Lemma 6.6 we can choose the range of \( i \) as \( 1 \leq i \leq \lceil 2 \log_2(n) \rceil + 1 \), and palindromic substrings of length beyond this range are ignored. We maintain these data as follows.

- When a node \( u \) is added to the double-ended ctree \( T \),
  1. Add \( u \) to \( \text{list}[	ext{len}(u)] \).
  2. If \( \text{next}(	ext{prev}(u), c) \neq \text{null} \) for every character \( c \), delete \( \text{prev}(u) \) from \( \text{list}[	ext{len}(	ext{prev}(u))] \).
- When a node \( u \) is deleted from the double-ended ctree \( T \),
  1. Delete \( u \) from \( \text{list}[	ext{len}(u)] \).
  2. If \( \text{prev}(u) \) is not in \( \text{list}[	ext{len}(\text{prev}(u))] \), add \( \text{prev}(u) \) to \( \text{list}[	ext{len}(\text{prev}(u))] \).

The above procedure can be maintained in \( O(1) \) time per operation.

We can answer each query of type Shortest Absent Palindrome as follows with the current string \( s \).

1. If there is a character \( c \) that does not occur in \( s \), i.e., \( \text{next}(	ext{odd}, c) = \text{null} \), return \( c \).
2. If there is a character \( c \) such that \( cc \) does not occur in \( s \), i.e., \( \text{next}(	ext{even}, c) = \text{null} \), return \( cc \).
3. For each \( 1 \leq i \leq \lceil 2 \log_2(n) \rceil + 1 \) in this order, if \( \text{list}[i] \) is not empty, do the following:
   - Let \( u \) be any node stored in \( \text{list}[i] \), and find any character \( c \) such that \( \text{next}(u, c) = \text{null} \).
     - Return \( c \) \( \text{str}(u) \).

Finally, we have the following result.

**Corollary 6.7.** Offline range queries of type Shortest Absent Palindrome can be solved with time complexity \( O(n \sqrt{q} \log(\sigma) + q \log(n)/\log(\sigma)) \).

**Remark 6.1.** The algorithms mentioned in Corollary\( 6.1, 6.3, 6.5 \) and 6.7 require time and space complexity roughly \( O(n \sqrt{q} \log(\sigma)) \) because each operation introduces incremental \( O(\log(\sigma)) \) space due to the online construction of ctrees by Theorem \( 5.14 \). Indeed, we have an alternative implementation with \( O(n \sqrt{q} \sigma) \) time and \( O(n \sigma) \) space. This is achieved by using a copy-based algorithm to store \( \text{dlink}(v, c) \) for every character \( c \), which requires \( O(\sigma) \) time and space per operation. Since the length of substrings of \( s \) is always \( \leq |s| = n \), the space used in the copy-based algorithm is \( O(n \sigma) \) independent of the parameter \( q \). When \( \sigma \) is small enough such that \( q = \omega(\sigma^2/\log^2(\sigma)) \), the space used in the copy-based algorithm can be much smaller than that in the \( O(n \sqrt{q} \log(\sigma)) \)-time algorithm, with the same time complexity up to a small factor. In particular, when \( \sigma = O(1) \) is a constant justified in Section 1.1, we can obtain an algorithm with \( O(n \sqrt{q}) \) time and \( O(n) \) space.
6.1.2 Online queries

Before holding each type of query, we first describe the main idea on how to prepare and store necessary information in order to obtain a time-space trade-off. We partition the string \( s \) into blocks, each of size \( B \). Specifically, for every \( 1 \leq i \leq n/B \), the start position of the \( i \)-th block is \( \ell_i = B(i - 1) + 1 \). Now for each \( 1 \leq i \leq n/B \), we are going to store the double-ended eertree of \( s[\ell_i..r] \) for every \( \ell_i \leq r \leq n \). There are \( O(n^2/B) \) double-ended eertrees to be stored. To achieve this, we build these double-ended eertrees as follows.

- For every \( 1 \leq i \leq n/B \), let \( \mathcal{T} \) be the persistent double-ended eertree of the empty string.
  - For every \( \ell_i \leq r \leq n \) in this order,
    1. perform \( \text{push-back}(s[r]) \) on \( \mathcal{T} \),
    2. store (the pointer to) \( \mathcal{T} \) as the double-ended eertree of \( s[\ell_i..r] \).

By Theorem 6.15, it is clear that it takes \( O(n^2(\log(n) + \log(\sigma))/B) \) time and space to prepare the \( O(n^2/B) \) (persistent) double-ended eertrees.

To answer a query on substring \( s[l..r] \), if \( r - l + 1 \leq B \), then build the double-ended eertree of \( s[l..r] \) directly; otherwise, let \( \mathcal{T} \) be the persistent double-ended eertree of \( s[B[(l - 1)/B] + 1..r] \), then perform \( \text{push-front} \) with characters \( s[B[(l - 1)/B]..[l]] \) in this order. It is clear that it takes \( O(B(\log(n) + \log(\sigma))) \) time and space to prepare the persistent double-ended eertree of \( s[l..r] \). If the number \( q \) of queries is known in advance, it is optimal to set \( B = \lceil n/\sqrt{\sigma} \rceil \), and then the time and space complexity becomes \( O(n\sqrt{q}(\log(n) + \log(\sigma))) \). After considering each type of query with necessary persistent auxiliary data, we have the following results.

**Corollary 6.8.** Online range queries of type Counting Distinct Palindromic Substrings, Longest Palindromic Substring and Shortest Unique Palindromic Substring can be solved with time complexity \( O(n\sqrt{q}(\log(n) + \log(\sigma))) \), and those of type Shortest Absent Palindrome can be solved with time complexity \( O(n\sqrt{q}(\log(n) + \log(\sigma)) + q\log(n)\log(\log(n))/\log(\sigma)) \), if the number \( q \) of queries is known in advance.

**Proof.** It is straightforward to obtain these results from Corollary 6.1, 6.3, 6.5 and 6.7 equipped with persistent data structures such as persistent binary search trees and persistent double-ended eertrees. Here, we especially mention that the doubly-logarithmic factor of \( n \) is introduced in Shortest Absent Palindrome due to persistent data structures used for maintaining \( \text{list}[i] \) of size \( O(\log(n)) \) in Corollary 6.7. \( \square \)

6.2 Enumerating rich strings with a given word

Palindromic rich strings have been extensively studied \[ [\text{GJWZ09,RR09,BDLGZ09,Ves14}] \]. Recently, the number of rich strings of length \( n \) was studied \[ [\text{RS18,GSS16,Ruk17}] \]. Especially, the number of binary rich strings of length \( n \) (cf. sequence A216264 in OEIS \([\text{Slo64}]\)) was efficiently computed by eertree in \[ [\text{RS18}] \], and they thus deduced an \( O(1.605^n) \) upper bound (as noted in \[ [\text{Ruk17}] \]). Shortly after, a lower bound \( \Omega\left(37.6\sqrt{n}\right) \) was given in \[ [\text{GSS16}] \].

We consider a computational task to enumerate rich strings with a given word, with a formal description in Problem 5.

**Problem 5 (Counting Rich Strings with Given Word).** Given a string \( t \) of length \( n \) and a number \( k \), count the number of palindromic rich strings \( s \) of length \( n + k \) such that \( t \) is a substring of \( s \).
Let \( \sigma \) be the size of the alphabet. There are roughly \( O(k\sigma^k) \) strings of length \( n + k \) with the given substring \( t \) of length \( n \). A simple solution to Problem 5 is to enumerate each of the \( O(k\sigma^k) \) candidates and check its richness by \([GPR10]\) in \( O(n + k) \) time, thereby with total time complexity at least \( O((n + k)k\sigma^k) \), where \( n \) and \( \sigma^k \) are multiplicative in the complexity. Using our double-ended eertree, we are able to improve the time complexity such that \( n \) and \( \sigma^k \) are additive.

**Corollary 6.9.** There is an algorithm for **Counting Rich Strings with Given Word** with time complexity \( O(n\sigma + k\sigma^k) \), where \( \sigma \) is the size of the alphabet.

**Proof.** The basic idea is to enumerate all possible characters being added at the front and the back of the string. It is clear that there are \((k + 1)\sigma^k\) ways to add \( k \) characters to both ends of a string.

Suppose we are given a string \( t \) of length \( n \) and want to enumerate all strings \( s \) of length \( n + k \) with \( t \) being its substring. To remove duplicate enumerations, we only enumerate strings of the form \( xty \) such that \( t \) does not occur in \( t[2..|t|]y \). To this end, we build the Aho-Corasick automaton \([AC75]\) of a single string \( t \). This can be done in \( O(n\sigma) \) time. Recall that an Aho-Corasick automaton is a trie-like structure with each of its node corresponding to a unique string. Especially, the root corresponds to the empty string. For our purpose, we only need the transitions between its nodes. Specifically, the transition \( \delta(u, c) \) is defined for every node \( u \) and character \( c \), which points to the node \( v \) of the largest \( \text{len}(v) \) such that \( \text{str}(v) \) is a proper suffix of \( \text{str}(u) \). Here, we follow the notations \( \text{str}(\cdot) \) and \( \text{len}(\cdot) \) as for eertrees.

With the Aho-Corasick automaton of string \( t \), we can enumerate every distinct string \( s \) with \( t \) being its substring, and simultaneously maintain the double-ended eertree of \( s \). Our algorithm consists of two parts (see Algorithm 13).

1. The first part is to enumerate all characters added at the back, with \( t \) only occurring once in the resulting string (see Algorithm 14).

2. The second part is to enumerate all characters added at the front, and then check whether the resulting string is palindromic rich (see Algorithm 15).

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### Algorithm 13: An algorithm for **Counting Rich Strings with Given Word**

**Input:** string \( t \) of length \( n \), and number \( k \).

**Output:** the number of palindromic rich strings \( s \) of length \( n + k \) such that \( t \) is a substring of \( s \).

1. Build the double-ended eertree \( T \) of \( t \).
2. Build the Aho-Corasick automaton of \( t \) with transitions \( \delta(\cdot, \cdot) \).
3. Let \( u_t \) be the node corresponding to \( t \) in the Aho-Corasick automaton of \( t \).
4. \( \text{ans} \leftarrow 0 \).
5. \( \text{enum_back}(t, u_t) \).
6. \( \text{return} \ \text{ans} \).

It is clear that the time complexity of our algorithm is \( O(n\sigma + k\sigma^k) \). In order to prove its correctness, we only need to show that the recursive function correctly enumerates every string with \( t \) being its substring exactly once. To see this, we first show that every string of length \( n + k \) with substring \( t \) will be enumerated at least once. Suppose \( s = xty \), where \( |x| + |y| = k \). If there are multiple representations of \( s \) in the form \( s = xty \), we choose the one with the shortest \( y \). Then, we can see that \( t \) is not a substring of \( t[2..|t|]y \). This implies that \( \text{enum_back}(s, u) \) in Algorithm 14.

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\(^3\)In fact, the Knuth-Morris-Pratt algorithm \([KMP77]\) for pattern matching can also achieve our goal. The Aho-Corasick automaton we use here is for its efficient state transitions.
Algorithm 14 enum_back(s, u)
1: enum_front(s).
2: if $|s| \geq n + k$ then
3: return
4: end if
5: for each character c do
6: $v \leftarrow \delta(u, c)$.
7: if $v \neq u$ then
8: push_back(c) on double-ended tree $T$.
9: enum_back(sc, v).
10: pop_back(c) on double-ended tree $T$.
11: end if
12: end for

Algorithm 15 enum_front(s)
1: if $|s| = n + k$ then
2: num $\leftarrow$ the number of distinct palindromic substrings in $s$ by the double-ended eertree $T$.
3: if num $= n + k + 1$, i.e., $s$ is palindromic rich then
4: ans $\leftarrow$ ans + 1.
5: end if
6: return
7: end if
8: for each character c do
9: push_front(c) on double-ended tree $T$.
10: enum_front(cs).
11: pop_front(c) on double-ended tree $T$.
12: end for
will eventually reach the state with $s = ty$. Upon calling $\text{enum\_front}(ty)$, it is trivial that $xty$ will be enumerated.

It remains to show that every string of length $n + k$ with substring $t$ will be enumerated at most once. We define a path $t \rightarrow ty \rightarrow xty$ to represent the recursive procedure, which means that the algorithm starts by calling $\text{enum\_back}(t)$, then calls $\text{enum\_front}(ty)$, and finally achieves $xty$ of length $n + k$. It has already been shown in the above that there is at least one such path for every string of length $n + k$ with substring $t$. If there are two different enumerations of the same string $s$, then there are two different paths $t \rightarrow ty_1 \rightarrow x_1ty_1$ and $t \rightarrow ty_2 \rightarrow x_2ty_2$ such that $s = x_1ty_1 = x_2ty_2$ with $(x_1, y_1) \neq (x_2, y_2)$. Without loss of generality, we assume that $|y_1| < |y_2|$. It can be seen that the path $t \rightarrow ty_2 \rightarrow x_2ty_2$ is impossible as follows. There is a non-empty string $w$ such that $ty_2 = wty_1$. Before reaching the state that $s = ty_2$ in $\text{enum\_back}(s, u)$, it must reach the state that $s = wt$ and $u = ut$. Since $w$ is not empty, there is no way to call $\text{enum\_back}(wt, ut)$ because of the guard $v \neq ut$ in $\text{enum\_back}(s, u)$ of Algorithm 13.

7 Conclusion

In this paper, we proposed a linear-time implementation for double-ended eertrees, and investigated its applications in range queries concerning palindromes and enumerating palindromic rich strings. Moreover, we provide a practical and efficient implementation for double-ended eertrees. There are several aspects left for future research.

- The concept of surface is useful in our algorithms. It would be interesting to find how surfaces can be used in string processing.
- Although double-ended eertrees can answer general range queries concerning palindromes, we believe that they can be addressed by more targeted algorithms to achieve better complexity, e.g., [RS17].
- Recently, palindromes in circles [Sim14] and trees [BLP15, GKRW15, GSS19, FNI+19] have been investigated. It would be interesting to generalize eertrees for palindromes in these special structures.

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