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Non-equilibrium behavior of large-scale axial vortex cores

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ABSTRACT

A logical basis for incorporating pressure non-equilibrium and turbulent eddy viscosity in an incompressible vortex model is presented. The infrasonic acoustic source implied in our earlier work has been examined. Finally, this non-equilibrium turbulent vortex core is shown to dissipate mechanical energy more slowly than a Burgers vortex, helping us to explain the persistence of axial vortices in nature. Recent molecular dynamics simulations replicate aspects of this non-equilibrium pressure behavior.

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I. INTRODUCTION

Zuckerwar and Ash\textsuperscript{1,2} have shown theoretically how fluids such as air and water can depart from thermodynamic equilibrium when subjected to intense local strain rates. Utilizing Hamilton’s principle and quantifiable frequency-dependent acoustic parameters,\textsuperscript{3} they were able to isolate thermodynamic non-equilibrium dilatational and shearing strain rate induced processes. We have shown previously how ambient weather conditions control dust devil and tornado core characteristics.\textsuperscript{4} In Ref. 4 (Ash, Zuckerwar, and Zardadkhan\textsuperscript{4} will be referred henceforth as AZZ), we examined strain rate driven departures from thermodynamic equilibrium in large-scale incompressible vortices, demonstrating the influence of relative humidity on the tornado core size and strength. This model evolved from the premise that extreme strain rates generated intense sound signatures while simultaneously forcing local fluid pressure to depart from thermodynamic equilibrium. Assuming quasi-reversible departures,\textsuperscript{1,2} it was possible to develop a quantifiable pressure relaxation coefficient, \( \eta_p \), controlled by ambient conditions. The pressure relaxation coefficient, \( \eta_p \) (measured in microseconds for air), is a measure of pressure gradient responsiveness.

In the 1960s, meteorologist Scorer proposed a hurricane and tornado formation process, whereby large-scale rotating air columns transformed suddenly from rotating cylinders to axial vortices.\textsuperscript{5} Assuming mean angular momentum of the rotating parent was conserved, he avoided the nonphysical potential vortex centerline velocity limit by assuming the spontaneous potential vortex domain was annular. His finite cylindrical domain is consistent with observation, but he lacked a physically plausible mechanism to describe the creation of a Rankine vortex-like core region. He sketched a viscous core evolution process similar to a Lamb–Oseen vortex evolution, but without providing details. Large-scale experimental verification of his model is lacking; only two small-scale experiments appear to support his postulated jump mechanism.\textsuperscript{7,8} If the sudden inviscid transformation of a large-scale rotating column into an inviscid axial vortex represents a natural jump instability process, the shearing strain rates imposed in the vicinity of the rotational axis most certainly reach unsustainable levels. If that is the case, away from the surface, the AZZ non-equilibrium pressure and azimuthal velocity model represents the resulting steady-state, circulation-controlled potential vortex. Furthermore, Scorer’s finite outer radius limit bounds overall angular momentum and kinetic energy.

The AZZ non-equilibrium vortex study showed how unsustainable strain rates resulted when an inviscid axial vortex is imposed on a nominally incompressible fluid. Molecular dynamics simulations have exhibited a similar depressed kinetic energy-based pressure effect resulting from altered distributions of rotational degrees of freedom.\textsuperscript{5} AZZ non-equilibrium theory predicted the following relations:
I. Far field circulation, $\Gamma_{\infty}$, is twice the core circulation, i.e., if
the maximum swirl velocity ($V_{\theta,\text{Max}}$) at core radius, $R_{\text{core}}$, is
specified, $\Gamma_{\infty} = 4\pi R_{\text{core}} V_{\theta,\text{Max}}$.

II. For specified turbulent eddy viscosity, $\nu_{turb}$ and pressure relaxation
coefficient, $\eta_{p}$, the maximum swirl velocity is $V_{\theta,\text{Max}} = \sqrt{\frac{2 \nu_{turb}}{\eta_{p}}}$.

III. If the maximum swirl velocity is known, the ratio of the turbu-
 lent eddy viscosity to the pressure relaxation coefficient can be
estimated directly ($\frac{\nu_{turb}}{\eta_{p}} = \frac{\eta_{p}}{V_{\theta,\text{Max}}}^{2}$).

IV. The core radius is controlled by the imposed circulation and
the inverse square root ratio of the turbulent eddy viscosity to
the pressure relaxation coefficient, i.e., $R_{\text{core}} = \frac{\Gamma_{\infty}}{2 \pi \sqrt{\frac{\eta_{p}}{\nu_{turb}}}}$.

V. The maximum out-of-equilibrium centerline pressure deficit is
controlled by ambient density, $\rho_{\infty}$, and the ratio of the tur-
bulent eddy viscosity to the pressure relaxation coefficient, i.e.,
$\Delta P_{\text{Min}} = -4 \rho_{\infty} \frac{\eta_{p}}{\nu_{turb}} \Rightarrow C_{p,\text{Min}} = \frac{\rho(0)-\rho_{\infty}}{\frac{1}{2} \rho_{\infty} \eta_{p} V_{\text{core}}^{2}} = -4$.

In the absence of phase change energy release, buoyancy-induced
rotating air columns become dust devil vortices rather
than tornadoes. Large-scale dust devils are turbulent, but centrifugal
forces suppress centerline fluctuations. Furthermore, Corrin and
Kistler\textsuperscript{10} proved that when atmospheric turbulence is irrotational,
mean velocity profiles are not influenced by that turbulence. The
extreme shearing strain rates imposed near an inviscid vortex axis
rule out thermodynamic equilibrium. Furthermore, near the core,
sustainable imposed shearing strain rates organize atmospheric tur-
bulence, producing local Reynolds stress gradients acting to resist
that shear. On that basis, it was logical for AZZ to utilize the simple
eddy viscosity turbulence model,

$$\sigma_{ij} = (\mu + \mu_{turb}) \left[ \frac{\partial u_{i}}{\partial x_{j}} + \frac{\partial u_{j}}{\partial x_{i}} \right] = \rho_{turb} \frac{\partial u_{i}}{\partial x_{j}}, \quad (1)$$

which only influences the steady-state vortex velocity field in the
vicinity of the core. Sinclair\textsuperscript{11} compiled the turbulent velocity, tem-
perature, and pressure surveys for three dust devils with circulation
levels between 320 m$^{2}$/s$^{-1}$ and 400 m$^{2}$/s$^{-1}$. These data were employed
in AZZ to develop the simple turbulent eddy viscosity correlation
$C_{\mu}=\mu_{turb}/\mu_{turb,0} = 2.2 \pm 1$.

In our earlier study, we identified and partitioned an acoustic
source approximation given by

$$\eta_{p} \left[ \frac{\partial}{\partial x_{i}} \left( \frac{DP}{Dt} \right) - \eta_{p} \frac{D}{Dt} \left( \frac{\partial P}{\partial x_{i}} \right) \right] \approx \eta_{p} \left[ \frac{\partial u_{i}}{\partial x_{j}} \frac{\partial P}{\partial x_{j}} - \delta_{ij} \frac{\partial}{\partial x_{i}} \left( \frac{DP}{Dt} \right) \right], \quad (2)$$
in which we found the local speed of sound, $a_{0}$, was accurate to three
significant figures. Recently, a molecular dynamics simulation has
isolated an analogous acoustic source in a small-scale hurricane-like
simulation.\textsuperscript{13} The two molecular dynamics simulations,\textsuperscript{9,12} along
with recent large-scale tornado simulator experiments reporting
centerline pressure coefficient measurements as low as $-2.8$, beneath
a boundary layer,\textsuperscript{14} supporting the AZZ minimum $C_{p,\text{Min}} = -4$, away
from any boundary layer influence, has motivated the present inves-
tigation. Unlike the compressible flow or water cavitation radial
limits assumed historically to bound large-scale vortices in nature,
we demonstrate a more fundamental incompressible vortex core
depture from thermodynamic equilibrium prior to reaching those
limits.

After validating the pressure relaxation coefficient, we have
employed the AZZ acoustic source model along with turbulent eddy
viscosity to examine the large-scale vortex core structure and associ-
ed infrasonic sound generation. Neglecting body forces, the AZZ
incompressible non-equilibrium Navier–Stokes equation is

$$\rho \frac{DN}{Dt} = -\nabla P + \eta_{p} \frac{D}{Dt} (\nabla P) + \rho_{turb} \nabla^{2} v. \quad (3)$$

The steady-state, one-dimensional incompressible solutions $P(r)$,
$v_{y}(r)$, have been utilized herein to test the usefulness of the AZZ
model.

II. VALIDITY OF PRESSURE RELAXATION COEFFICIENT

Long-accepted quasi-reversible frequency-dependent sound
attenuation models appear to violate the continuum principle of
coordinate invariance. Normal shockwaves are irreversible, but con-
tinuum theory implies that both the shockwave thickness and fre-
cency dependent sound attenuation are controlled by the second
(or bulk) coefficient of viscosity. Thermodynamic equilibrium is
assumed.

Zuckerwar and Meredith\textsuperscript{14,15} have documented the process by
which NASA reference-quality spectral acoustic absorption mea-
surements are obtained (at frequencies between 10 Hz and 2500 Hz,
in their case). Using these and similar data sources, estimates of the
bulk viscosity of air are approximately three orders of magnitude
larger than the dynamic viscosity—hardly justifying Stokes’ negative
second coefficient hypothesis. Furthermore, at moderate tempera-
tures, the inferred bulk viscosity is influenced strongly by relative
humidity. The degree to which relative humidity influences experi-
mentally measured bulk viscosity was our motivation for exploiting
Hamilton’s principle to incorporate non-equilibrium molecular
(vibrational and rotational degrees of freedom) behavior in the
conservation equations for air.\textsuperscript{1,2}

Two non-equilibrium effects were identified: (1) a thermody-
namic dilatational departure, indistinguishable from bulk viscosity,
and (2) discovery that pressure could depart from thermodynamic
equilibrium in otherwise incompressible flows.\textsuperscript{1} Subsequently,
we introduced multiple molecular degrees of freedom, showing how
they could be incorporated in the non-equilibrium dilatational (vol-
umetric) relaxation coefficient, $\eta_{V}$, and a similar pressure relax-
ation coefficient, $\eta_{p}$, defined strictly in terms of accepted acoustic
parameters.\textsuperscript{2} We showed that $\eta_{V}$, $\eta_{p}$-based shock thickness esti-
mates over the Mach number range, $1 < M < 5$, agreed with the
classical Boltzmann-equation-based estimates of Mott-Smith.\textsuperscript{16} We
also included the curious prediction that a non-equilibrium pressure
effect occurred in incompressible slow viscous (Stokes) flow past a
sphere.\textsuperscript{2}

The pressure relaxation coefficient is based solely on standard
acoustical reference parameters. Calculated pressure relaxation coeffi-
cients for dry and humid air in the $25^\circ C$–$40^\circ C$ temperature range
were extremely small (less than 0.0001 s). However, as summarized
in Table 1, the pressure relaxation coefficient for dry air (0% RH)
increases with increasing temperature, whereas the pressure relax-
ation coefficient for moisture saturated air (100% RH) decreases
with increasing temperatures. At moderate temperatures, the dry air

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TABLE I. Influence of relative humidity on the pressure relaxation coefficient, $n_p$.

| Ambient temperature (°C) | Pressure relaxation coefficient (RH = 0%) (s) | Pressure relaxation coefficient (RH = 100%) (s) | $n_{p,\text{rel}}$ |
|--------------------------|---------------------------------------------|---------------------------------------------|-----------------|
| 25                       | 6.23 × 10^{-5}                             | 2.16 × 10^{-7}                             | 288             |
| 30                       | 6.65 × 10^{-5}                             | 1.83 × 10^{-7}                             | 363             |
| 35                       | 7.08 × 10^{-5}                             | 1.55 × 10^{-7}                             | 457             |
| 40                       | 7.53 × 10^{-5}                             | 1.32 × 10^{-7}                             | 570             |

The pressure relaxation coefficient is more than two orders of magnitude larger than 100% RH air. On that basis, it was logical to isolate pressure relaxation influences from turbulence in Eq. (3), when examining our non-equilibrium vortex model. AZZ theory predicts that the maximum swirl velocity is controlled by $\sqrt{2 \frac{\rho_\infty}{\eta_p}}$. At moderate tornado–hurricane-like ambient temperatures, the table shows that variations in relative humidity can increase or decrease maximum swirl velocity estimates by a factor of 10. As the relative humidity increases, the pressure relaxation coefficient decreases, thus enabling atmospheric air to tolerate increased potential-flow-driven shearing strain rates, enhancing the circulation-driven vortex core strength.

Testing the AZZ relations for tornadoes was more difficult. Not only are tornadoes much larger than dust devils, they are hazardous. Furthermore, tornadoes evolve randomly from strong interactions between fast-moving upper-level weather fronts and rising warm, moisture-laden air. Karstens et al. have compiled in situ tornado pressure measurements for 26 tornadoes, dating from 1894. Starting in 2002, direct near-centerline pressure deficit measurements have been obtained. In six of the direct encounters, complimentary mobile mesonet data enable reasonably accurate estimates of ground level funnel width. Although actual ambient temperature and relative humidity conditions were uncertain, the six "near-direct" tornado encounters identified by Karstens et al. provided maximum pressure deficit and core radius data suitable for evaluating AZZ theory.

Without ambient and dewpoint temperatures (to determine tornado-based relative humidity), turbulence influences cannot be isolated from the pressure relaxation coefficient when employing only maximum pressure deficit measurements with associated core radius data. However, humidity variations have only minor influences on ambient density, permitting reasonably accurate estimates of $\rho_\infty$, based on ground track surface elevation and hour-by-hour weather service data. AZZ theory permits estimation of the maximum swirl velocity and far field circulation utilizing minimum pressure, core diameter, and ambient air density. The Tulia, TX tornado was of special interest because a fully instrumented SUV recorded the most extreme core pressure deficit (19 400 Pa) when it accidentally penetrated the tornado funnel. It has been difficult to explain how such an extreme pressure deficit could be produced due to violent vehicle motion within the core, but we assumed the measurement approximated the actual centerline pressure deficit. A summary of our estimates of maximum swirl velocity and far field circulation, based on the data from the work of Karstens et al. (bold headings), along with $v_{\text{Tot}}/\eta_p$ ratio estimates, is provided in Table II. Their pressure minimum and mesonet-based core radius measurements are in bold type.

III. A NON-EQUILIBRIUM SOLENOIDAL VORTEX

Saffman showed that Burgers vortices were one class of solenoidal vorticity solutions governed by

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) U + v\nabla^2 \omega$$

(4)

when the mean velocity components were

$$U_i = a_i x_j, \quad \text{with} \quad a_{ii} = 0.$$ 

(5)

This vortex stretching mean velocity field stabilizes the Burgers vortex, avoiding the transient Lamb–Oseen vortex behavior.

If a finite diameter Scorer potential vortex propels the flow, the imposed vorticity is solenoidal. The unsustainable circular non-equilibrium strain rate region around the rotational axis is the only departure from Saffman’s $\omega = \nabla \times V = 0$. Utilizing the AZZ core radius and maximum swirl velocity to define reference circulation $\Gamma = 4\pi R_{\text{core}} V_{\theta,\text{Max}}$, the inviscid, Burgers, and AZZ azimuthal velocity distributions can be represented,

$$v_{\theta,\text{inviscid}}(r) = 2V_{\theta,\text{Max}} \left( 1 - \frac{r}{R_{\text{core}}} \right),$$

(6a)

$$v_{\theta,\text{Burgers}}(r) = 2V_{\theta,\text{Max}} \left( 1 - e^{-\left( \frac{r}{R_{\text{core}}} \right)^2} \right),$$

(6b)

and

$$v_{\theta,\text{AZZ}}(r) = 2V_{\theta,\text{Max}} \left( \frac{r}{R_{\text{core}}} \right)^2,$$

(6c)

respectively. As shown in Fig. 1, the azimuthal Burgers vortex velocity profile approximates a potential vortex profile within two core radii, whereas the AZZ profile requires approximately five core radii to revert. The associated shearing rates of strain, given by

$$\dot{\varepsilon}_{\theta}(r) = \frac{1}{2} \left[ \frac{\partial}{\partial r} \left( \frac{v_{\theta}}{r} \right) + \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} \right] = \frac{r}{2} \frac{d}{dr} \left( \frac{v_{\theta}}{r} \right),$$

(7)

TABLE II. Pressure deficits and core radii from Karstens et al. along with AZZ estimates. Pressure minimum and mesonet-based core radius measurements are in bold type.

| Tornado       | $\Delta P_{\text{Min}}$ (Pa) | $R_{\text{core}}$ (m) | $V_{\theta,\text{Max}}$ (m/s) | $\Gamma_{\infty}$ (m$^2$/s$^2$) | $v_{\text{Tot}}/\eta_p$ |
|---------------|------------------------------|----------------------|-----------------------------|-------------------------------|-------------------------|
| Mullinville, KS | -2 200                      | 265                  | 32.4                        | 108 000                       | 525                     |
| Manchester, SD #1 | -9 800                 | 36                   | 68.4                        | 31 000                        | 2340                    |
| Manchester, SD #2 | -5 500                 | 15                   | 51.2                        | 9 600                         | 1310                    |
| Webb, IA       | -2 700                      | 54                   | 35.8                        | 24 200                        | 641                     |
| Tipton, KS      | -1 500                      | 205                  | 26.8                        | 69 000                        | 359                     |
| Tulia, TX       | -19 400                     | 30                   | 96.1                        | 36 200                        | 4620                    |
for Burgers and AZZ vortices are compared in Fig. 2. [The strain rate produced by a similarly scaled potential vortex is \( \dot{\varepsilon}_{\text{potential}} = -2/(r^2/R_{\text{core}}^2) \).] The AZZ strain rate relaxes more rapidly at first, and the magnitude of the most negative shearing strain rate is smaller than a Burgers vortex, approaching zero more rapidly thereafter.

Finally, the axial-vorticity-based dissipation function, \( \epsilon \), can be recast in a slightly different form in order to compare decay rates for a Burgers vortex with an AZZ vortex. That is,

\[
\zeta_{\text{Burgers}}(r) = \frac{2V_{\theta,\text{Max}}}{rR_{\text{core}}} \frac{d}{dr} \left[ 1 - \exp\left(-\frac{r^2}{R_{\text{core}}^2}\right) \right] = 4\frac{V_{\theta,\text{Max}}}{R_{\text{core}}} e^{-2\left(\frac{r}{R_{\text{core}}}\right)^2} .
\]

(8)

and

\[
\zeta_{\text{AZZ}}(r) = \frac{2V_{\theta,\text{Max}}}{rR_{\text{core}}} \frac{d}{dr} \left[ 1 - \exp\left(-\frac{r^2}{R_{\text{core}}^2}\right) \right] = 4\frac{V_{\theta,\text{Max}}}{R_{\text{core}}} e^{-2\left(\frac{r}{R_{\text{core}}}\right)^2} .
\]

Thus,

\[
\zeta_{\text{Burgers}}^2(r) = 16\left(\frac{V_{\theta,\text{Max}}}{R_{\text{core}}}\right)^2 e^{-2\left(\frac{r}{R_{\text{core}}}\right)^2} .
\]

(9)

and

\[
\zeta_{\text{AZZ}}^2(r) = 16\left(\frac{V_{\theta,\text{Max}}}{R_{\text{core}}}\right)^2 e^{-2\left(\frac{r}{R_{\text{core}}}\right)^2} .
\]

The dissipation rate assuming a slowly decreasing maximum swirl velocity is compared with

\[
\epsilon_{\text{Burgers}} = \frac{\rho \nu_{\text{turb}}}{R_{\text{core}}^3} \int_0^\infty 2\pi r \zeta_{\text{Burgers}}^2(r) dr = 8\pi \nu_{\text{turb}} V_{\theta,\text{Max}}^2 .
\]

(10)

and

\[
\epsilon_{\text{AZZ}} = \frac{\rho \nu_{\text{turb}}}{R_{\text{core}}^3} \int_0^\infty 2\pi r \zeta_{\text{AZZ}}^2(r) dr = \frac{16\pi}{3} \nu_{\text{turb}} V_{\theta,\text{Max}}^2 .
\]

(11)

The radial variation of dissipation functions \( \zeta^2(r/R_{\text{core}}) \cdot 16\left(\frac{V_{\theta,\text{Max}}}{R_{\text{core}}}\right)^2 \) for a Burgers vortex and for an AZZ vortex is compared in Fig. 3. As shown in Fig. 3 and deduced from Eqs. (10) and (11), the mechanical energy dissipation rate in an AZZ vortex is smaller than a Burgers vortex. In fact, the mechanical rate of energy loss is only 2/3 that of a Burgers vortex. On that basis, a non-equilibrium axial vortex will be longer-lived than a Burgers vortex.
The Burgers vortex model along with Saffman’s generalization and the Corrsin-Kistler turbulent mean velocity profile equivalence proof has withstood the test of time. In the absence of Saffman’s imposed three-dimensional mean velocity conditions, given by Eq. (5), the transient Burgers vortex reverts to the simple decay of a point or line vortex inserted suddenly in a viscous fluid. Scorer’s sudden transition of a large but finite diameter rotating cylindrical column with angular rotation rate, \( \omega \), into a finite circulation-imposed potential vortex (\( \Gamma = \pi \frac{D^2}{2} \omega \)) creates unsustai

\[ \frac{\omega_{\text{column}} R_{\text{max}}^2}{r} \rightarrow v_B(r) = 2V_{\theta, \text{MAX}} \left( \frac{\eta}{r_{\text{core}}} \right)^2 + 1 \]  

(12)

with a rotating core whose angular rotation rate is

\[ \omega_{\text{core}} = \frac{dv_B}{dr}(0) = 2V_{\theta, \text{MAX}} \frac{\eta}{r_{\text{core}}} \sqrt{\frac{2\pi}{\eta_{\text{turb}}}} = 16 \frac{\pi}{\eta_{\text{turb}}} \frac{\eta}{r_{\text{core}}} \]  

(13)

The associated pressure distribution is

\[ P(r) = P_{\infty} - \frac{\Delta P_{\text{CL}}}{\left( \frac{r}{r_{\text{core}}} \right)^2 + 1} = P_{\infty} - 4p v_{\text{turb}} \frac{1}{\eta r} \left( \frac{r}{r_{\text{core}}} \right)^2 + 1 \]  

(14)

Employing Eqs. (12) and (14), the acoustic source, Eq. (2), can be represented,

\[ \eta \left( \frac{\partial v_1}{\partial x_1} \frac{\partial P}{\partial x_1} + \frac{\partial v_2}{\partial x_1} \frac{\partial P}{\partial x_1} \right) \epsilon_1 + \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_2} \right) \epsilon_2 = -d_\epsilon \left( \frac{\partial}{\partial x_1} \left( \frac{Dp}{Dt} \right) \epsilon_1 - \frac{\partial}{\partial x_2} \left( \frac{Dp}{Dt} \right) \epsilon_2 \right) \]

or, in cylindrical coordinates,

\[ \left[ \frac{\partial v_r}{\partial r} \frac{\partial P}{\partial x} - \frac{1}{r} \frac{\partial v_\theta}{\partial r} \frac{\partial P}{\partial \theta} \right] \epsilon_\theta = -d_\epsilon \left[ \frac{\partial}{\partial r} \left( \frac{Dp}{Dt} \right) \epsilon_1 - \frac{\partial}{\partial \theta} \left( \frac{Dp}{Dt} \right) \epsilon_2 \right] \]

The non-equilibrium rotating cylindrical volume creates a rotating acoustic source, given by

\[ -4p v_{\theta, \text{MAX}} \frac{\eta}{r_{\text{core}}} \left( \frac{r}{r_{\text{core}}} \right)^2 + 1 \epsilon_\theta = d_\epsilon \frac{\partial}{\partial r} \left( \frac{Dp}{Dt} \right) \epsilon_\theta \]  

(15)

In the swirling or azimuthal direction,

\[ -4p v_{\theta, \text{MAX}} \frac{\eta}{r_{\text{core}}} \left( \frac{r}{r_{\text{core}}} \right)^2 + 1 \epsilon_\theta = d_\epsilon \frac{\partial}{\partial r} \left( \frac{Dp}{Dt} \right) \epsilon_\theta \]  

(16)

Hence, the density fluctuation magnitude is maximized at \( r = \frac{R_{\text{max}}}{2} \).

\[ \left( \frac{1}{\rho} \frac{\partial}{\partial r} \left( \frac{Dp}{Dt} \right) \right)_{\text{MAX}} = 4\sqrt{2} \frac{V_{\theta, \text{MAX}}}{a^2 r_{\text{core}}} \]  

(17)

A normalized plot of the dimensionless variation of \( \frac{1}{\rho} \frac{\partial}{\partial r} \left( \frac{Dp}{Dt} \right) \) with \( (r/r_{\text{core}}) \) is shown in Fig. 4. Employing the Karstens data, the minimum (largest magnitude) gradient of the material rate of change of density, normalized with respect to ambient density, varies from \(-3.4 \times 10^{-5} \) 1/s/m to a much larger \(-0.075 \) 1/s/m for the anomalous Tulia, TX tornado encounter. Additionally, a characteristic acoustic frequency produced by these rotating funnel cores should be related to the circulation-imposed centerline angular rotation rate, i.e., \( \omega_{\text{CL}} = 2V_{\theta, \text{MAX}}/r_{\text{core}} \) or

\[ f_{\text{characteristic}} = \frac{V_{\theta, \text{MAX}}}{\pi r_{\text{core}}} \]  

(18)
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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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