Unified estimation framework for unnormalized models with statistical efficiency

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Abstract

Parameter estimation of unnormalized models is a challenging problem because normalizing constants are not calculated explicitly and maximum likelihood estimation is computationally infeasible. Although some consistent estimators have been proposed earlier, the problem of statistical efficiency does remain. In this study, we propose a unified, statistically efficient estimation framework for unnormalized models and several novel efficient estimators with reasonable computational time regardless of whether the sample space is discrete or continuous. The loss functions of the proposed estimators are derived by combining the following two methods: (1) density-ratio matching using Bregman divergence, and (2) plugging-in nonparametric estimators. We also analyze the properties of the proposed estimators when the unnormalized model is misspecified. Finally, the experimental results demonstrate the advantages of our method over existing approaches.

Key Words: Unnormalized Models; Noise contrastive estimation ; Score matching; Bregman divergence; Gamma divergence; Nonparametric importance sampling; Semiparametric inference.
1 Introduction

Unnormalized models are widely used in many settings: Markov networks (Besag 1975), Boltzmann machines (Hinton 2002), models in independent component analysis (Hyvärinen 2001) and generalized gamma distributions (Stacy 1962). When the parametric model is denoted as \( p(x; \theta) \), \( p(x; \theta) \) is called an unnormalized model if its normalizing constant \( \int p(x; \theta)d\mu(x) \) cannot be calculated explicitly, or it is difficult to compute in practice. For example, when \( \mu \) is a counting measure as in the case of Markov networks and Boltzmann machines, the computational cost increases exponentially with the dimension of the sample space. When \( \mu \) is a Lebesgue measure, as in the case of models in independent component analysis or generalized gamma distributions, this cannot be calculated analytically. When we use unnormalized models, we believe that the true data generating process is approximated by the family \( \{ p(x; \theta)/\int p(x; \theta)d\mu(x), \theta \in \Theta \} \), where \( \Theta \) denotes a parameter space. Unnormalized models \( p(x; \theta) \) can be converted to normalized models by dividing their normalizing constants. However, their explicit form cannot be obtained; therefore, an exact maximum likelihood estimation (MLE) is infeasible.

Several approaches for the estimation of unnormalized models have been suggested. Two approaches are important. First, noise contrastive estimation (NCE) and contrastive divergence (CD) rely on sampling techniques, such as importance sampling and Markov Chain Monte Carlo (MCMC) (Geyer 1994; Carreira-Perpinan & Hinton 2005; Gutmann & Hyvärinen 2010; Pihlaja et al. 2010; Hyvärinen & Morioka 2016, 2017; Ceylan & Gutmann 2018; Matsuda & Hyvärinen 2019). Second, score matching uses a tractable form without the aid of a sampling technique (Hyvärinen 2005; Hyvärinen 2007; Dawid et al. 2012). Generally, the first approach is superior to the second approach in terms of statistical efficiency, whereas the second approach is superior to the first approach in terms of computational efficiency, leaving a tradeoff between computational efficiency and statistical efficiency.

In the present study, we propose a unified framework for the statistically efficient estimation of unnormalized models and several statistically efficient estimators irrespective of whether the sample space is discrete or continuous. The estimators are defined as the form of M-estimators (van der Vaart 1998) and their loss functions are derived by combining two methods: (1) density-ratio matching using Bregman divergence, and (2) plugging-in nonparametric estimators. These estimators are statistically efficient in the sense that the asymptotic variance is the same as that of the MLE; thus, the proposed estimators are su-
perior to other previously proposed estimators in terms of statistical efficiency. Moreover, the proposed estimators do not rely on any sampling techniques; therefore, they are competitive in terms of computational efficiency. Figure illustrates the superiority of our proposed estimators to the other previously proposed estimators. In summary, our contributions are as follows:

- We propose statistically efficient estimators for unnormalized models with moderate computational time (Section 3). To the best of our knowledge, proposed estimators are the first statistically efficient estimators, which works in the continuous sample space.

- We prove that the proposed estimator, associated with the Kullback-Leibler (KL) divergence, has the same asymptotic property as MLE, even when the model is misspecified (Section 5).

It should be noted that when the sample space is discrete, Takenouchi and Kanamori (2017) proposed an efficient estimator, which can be seen as a special case from our proposed framework. Importantly, it is extended to the case of continuous sample space based on the proposed framework.

2 Preliminaries

Our general setting is as follows. Let us consider a situation in which an unnormalized model \( p(x; \theta) \) is used, that is, for each \( \theta \in \Theta \), \( p(x; \theta) \) is a non-negative function and the normalizing constant defined by the integral \( \int_X p(x; \theta) d\mu(x) \), is finite. The measure \( \mu \) over the sample space \( X \) is a counting measure when the sample space is discrete, and a Lebesgue measure when the sample space is continuous. We refer to it as a baseline measure in this paper.

Our aim is to estimate \( \theta \) using a set of identically independent distributed (i.i.d) \( n \) samples \( \{x_i\}_{i=1}^n \) by assuming that these samples are obtained from the true distribution \( F_{\eta^*} \) with density \( \eta^*(x) \) with respect to the baseline measure \( \mu \). Unless otherwise noted, we assume that the unnormalized model is well-specified, that is, there exists \( \theta^* \) satisfying \( \eta^* = \exp(-c^*) p(x; \theta^*) \), \( \exp(c^*) = \int p(x; \theta^*) d\mu(x) \). The problem of unnormalized models arises because it is extremely difficult or infeasible to calculate the normalizing constant analytically. In such a case, one should avoid a direct computation of the normalizing constant; therefore, the loss function of MLE cannot be used. In this section, we review the Bregman divergence and the generalized NCE, needed to understand proposed methods.
Figure 1: Comparison of methods, where Score stands for score matching, NCE stands for a noise contrastive estimation, CD stands for a contrastive divergence method, and a self density-ratio matching estimator (SDRME) is the proposed estimator. Note that statistically efficient estimators can be constructed in the case of NCE and CD. When the ratio of the auxiliary sample size and original sample size is infinite in NCE, the estimator becomes statistically efficient. However, it is infeasible to implement it in practice. The same argument applies to CD.
We summarize frequently used notations. We denote $E^\ast \left\{ \cdot \right\}$ as an expectation under the true density $\eta^\ast(x)$. The notations $\text{Var}^\ast \left\{ \cdot \right\}$ and $\tilde{E}^\ast \left\{ \cdot \right\}$ represent variance and empirical analogues. Notations $P_n$ and $G_n$ denote an empirical distribution of $n$ samples from the true distribution $F_{\eta^\ast}$ and an empirical process $\sqrt{n}(P_n - F_{\eta^\ast})$. We denote $dP/d\mu$ as $p_n$, evaluation at $\tau$, that is, $\left| \tau = \tau^\ast \right.$ as $\left| \tau^\ast \right.$, and $\nabla_x$ as the differentiation with respect to $x$. A summary of the notation is provided in a table in the Supplementary materials.

2.1 Bregman divergence

Let $\mathbb{R}_{\geq 0}$ be a set of non-negative real numbers. We define $\mathcal{F}$ as a collection of non-negative real-valued functions on the sample space $X$, and we assume that $\mathcal{F}$ is a convex set. Given a convex function $\psi(u)$ on $\mathcal{F}$, the Bregman divergence (Bregman, 1967; Gneiting & Raftery, 2007) on $\mathcal{F} \times \mathcal{F}$ is defined as

$$B_\psi(u, v) = \psi(u) - \psi(v) - \nabla \psi(v)(u - v),$$

where $\nabla \psi(v)$ is a linear operator defined by

$$\lim_{\varepsilon \to +0} \{(\psi(v + \varepsilon h) - \psi(v))/\varepsilon\} = \nabla \psi(v)(h).$$

Here, $h$ is a function on $X$ such that $v + \varepsilon h \in \mathcal{F}$ holds for arbitrary small $\varepsilon > 0$. The convexity of $\psi(u)$ guarantees the non-negativity of the Bregman divergence. We introduce two kinds of Bregman divergences; one is separable, while the other is non-separable.

The separable Bregman divergence is defined using the function $\psi(u)$:

$$\psi(u) = E^\ast \left\{ f(u(x)) \right\},$$

where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a strictly convex function. For the differentiable $f$, the corresponding Bregman divergence $B_f(u, v; \eta^\ast)$ between $u$ and $v$ is given as

$$E^\ast \left\{ f(u(x)) - f(v(x)) - f'(v(x))(u(x) - v(x)) \right\}.$$  

For the strictly convex function $f$, the corresponding $B_f(u, v; \eta^\ast)$ vanishes if and only if $u = v$ up to a null set with respect to the measure $\eta^\ast(x)d\mu(x)$.

Example 2.1 For $f(x) = 2x \log x - 2(1 + x) \log(1 + x)$, the corresponding $B_f(u, v; \eta^\ast)$ is known as the Jensen-Shannon divergence. In other cases, for $f(x) = x \log x$, we have the Kullback-Liber (KL) divergence. For $f(x) = x^m/(m(m - 1))$, we get the $\beta$-divergence (Basu et al., 1998; Murata et al., 2004).

The Bregman divergence is non-separable if the convex function $\psi(u)$ is not expressed as (2.1). The pseudo-spherical divergence and the $\gamma$-divergence are examples of non-separable Bregman divergences, and they are commonly
The pseudo-spherical divergence is the \( \gamma \)-norm with \( \gamma > 1 \) under the density \( \eta^*(x) \), that is, \( \| u \|_\gamma = E_u [u(x)^{1+\gamma}]^{-\gamma/(1+\gamma)} \). The pseudo-spherical divergence \( B_{ps}(u; v; \eta^*) \) is defined as

\[
\| u \|_\gamma - \frac{1}{\| v \|_\gamma} E_u [v(x)^{\gamma-1}u(x)].
\] (2.3)

The pseudo-spherical divergence \( B_{ps}(u; v; \eta^*) \) vanishes if and only if \( u \) and \( v \) are linearly dependent. When we apply a log-transformation to each term in (2.3), this becomes a \( \gamma \)-divergence (Kanamori & Fujisawa, 2014), represented as

\[
B_\gamma(u; v; \eta^*) = \frac{1}{\gamma} \log E_u [u(x)^\gamma] + \frac{\gamma - 1}{\gamma} \log E_u [v(x)^\gamma] - \log E_u [v(x)^{\gamma-1}u(x)].
\]

### 2.2 Generalized noise contrastive estimation

We review an estimation method for unnormalized models focusing on generalized NCE. A generalized NCE (Pihlaja et al., 2010; Gutmann & Hirayama, 2011) is proposed by introducing a one-parameter extended model defined by \( q(x; \tau) = \exp(-c)p(x; \theta) \), \( \tau \equiv (c, \theta^\top)^\top \), where \( c \) is regarded as a parameter. Using a set of samples \( \{y_i\}_{i=1}^n \) from the auxiliary distribution with a density \( a(y) \) with respect to the baseline measure \( \mu \), the estimator \( \hat{\tau}_{NC} \) for \( \tau \) is defined as the minimizer of the following function

\[
\frac{1}{n} \sum_{i=1}^n r_{q,a}(y_i)f'(r_{q,a}(y_i)) - f(r_{q,a}(y_i)) - f'(r_{q,a}(x_i)),
\] (2.4)

where \( r_{q,a}(x) = q(x; \tau)/a(x) \), \( f(x) \) is a strictly convex function and the support of the density \( a(x) \) includes the support of \( p(x; \theta) \). This estimation is derived from a divergence perspective as follows: let the divergence between the true distribution \( \eta^*(x) \) and the one-parameter extended model \( q(x; \tau) \) be \( B_f (r_{\eta^*,a}(x), r_{q,a}(x); a(x)) \) when \( r_{\eta^*,a}(x) = \eta^*(x)/a(x) \). We have \( B_f (r_{\eta^*,a}(x), r_{q,a}(x); a(x)) \geq 0 \) and \( B_f (r_{\eta^*,a}(x), r_{q,a}(x); a(x)) = 0 \iff \eta^* = q(x; \tau^*) \). Therefore, the estimation problem of \( \tau \) is reduced to a minimization problem of \( B_f (r_{\eta^*,a}(x), r_{q,a}(x); a(x)) \) with respect to \( \tau \). By subtracting the term not associated with \( q(x; \tau) \) from \( B_f (r_{\eta^*,a}(x), r_{q,a}(x); a(x)) \), we obtain the term \(-\int f'(r_{q,a}) \eta^* d\mu + \int (f'(r_{q,a}) r_{q,a} - f(r_{q,a})) a d\mu. \) The loss function of \( \hat{\tau}_{NC} \), (2.4), is constructed using an empirical approximation of this term.

Unless otherwise noted, we hereafter assume the following properties for \( f(x) \):
Assumption 1 Function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) satisfies the following three properties: strictly convex, third-order differentiable and \( f''(1)=1 \).

Among \( f(x) \) satisfying the above conditions, the estimator when \( f = 2x \log x - 2(1 + x) \log(1 + x) \) is proven to be optimal from the perspective of asymptotic variance, irrespective of the auxiliary distribution (Uehara et al., 2018). In this case, the loss function of the estimator becomes

\[
- \frac{1}{n} \sum_{i=1}^{n} \log \frac{r_{q,a}(x_i; \tau)}{1 + r_{q,a}(x_i; \tau)} - \frac{1}{n} \sum_{i=1}^{n} \log \frac{1}{1 + r_{q,a}(y_i; \tau)}.
\]

This loss function is identical to the original NCE (Gutmann & Hyvärinen, 2010). Although it satisfies some optimality, the asymptotic variance of the estimator derived from the above loss function is larger than that of MLE. We can also use another type of \( f(x) \). For example, when \( f(x) = x \log x \), the loss function is the same as that of the Monte Carlo MLE (Geyer, 1994). When \( f(x) = 0.5x^2 \), the loss function is robust from the perspective of the influence function of the estimators (Uehara et al., 2018).

3 Estimation with self density-ratio matching

We propose two types of statistically efficient estimators with reasonable computational time. Our key idea is to match the ratio of the unnormalized model and nonparametrically estimated density using Bregman divergence. We introduce an estimator based on a separable Bregman divergence. Then, we introduce an estimator based on a non-separable Bregman divergence.

3.1 Separable case

We introduce an estimator called self density-ratio matching estimator (SDRME) for \( \tau \) as a form of M-estimators:

\[
\hat{\tau}_s = \arg \min_{\tau \in \Theta_\tau} B_f(h_1(w), h_2(w); p_n),
\]

where \( \Theta_\tau \) is a parameter space for \( \tau \), \( w(x) = q(x; \tau)/\hat{\eta}_n(x) \), \( \hat{\eta}_n(x) \) is the nonparametric estimator using an entire set of samples, \( q(x; \tau) \) is a one-parameter extended model in Section 2.2 \( p_n = d\mathbb{P}_n/d\mu \), and \( h_1(x) \) and \( h_2(x) \) are functions satisfying conditions mentioned in the next paragraph. More specifically, the loss function is written as \( 1/n \sum_{i=1}^{n} B_f(h_1(w_i), h_2(w_i)) \), where \( w_i = q(x_i; \tau)/\hat{\eta}_n(x_i) \). Importantly, it requires only sample order \( \mathcal{O}(n) \) calculation.
When the baseline measure is a counting measure, we use an empirical distribution $p_n(x)$ as $\hat{\eta}_n(x)$, whereas when the baseline measure is a Lebesgue measure, we use a kernel density estimator as $\hat{\eta}_n(x)$. Here, three conditions for $h_1(x), h_2(x)$ are assumed.

**Assumption 2** Functions $h_1 : \mathbb{R}_+ \to \mathbb{R}$ and $h_2 : \mathbb{R}_+ \to \mathbb{R}$ must be monotonically increasing functions, $h_1(x) = h_2(x) \iff x = 1$, and $h_1'(1) \neq h_2'(1)$.

The second condition is required for the identification, and the third condition comes from the asymptotic result explained, as in Section 4.

This estimator works based on the following equivalence. By replacing $p_n(x)$ and $\hat{\eta}_n(x)$ with $\eta^*(x)$ in (3.1), we obtain $B_f(h_1(w), h_2(w); \eta^*) = 0 \iff h_1(w) = h_2(w) \iff w = 1 \iff q(x; \tau) = \eta^*(x)$. This implies that the estimator is regarded as an M-estimator. As explained in Section 4, this estimator is rigorously proven to be consistent and efficient. Several specific choices can be considered as $h_1(w)$ and $h_2(w)$ as follows.

**Example 3.1 (Generalized NCE with the nonparametric estimator)**

Consider a case where $h_1(w) = 1$ and $h_2(w) = w$. The loss function becomes

$$\frac{1}{n} \sum_{i=1}^{n} \left\{ -f'(w(x_i)) + w(x_i)f'(w(x_i)) - f(w(x_i)) \right\}.$$  

This is considered to be a natural extension of generalized NCE because when we replace $a(x)$ with $\hat{\eta}_n(x)$, and $y_i$ with $x_i$ in (2.4), the loss function is the same as the one above. Especially, when $f(x) = x \log x$, the loss function is

$$-\frac{1}{n} \sum_{i=1}^{n} \log q(x_i; \tau) + \frac{1}{n} \sum_{i=1}^{n} \frac{q(x_i; \tau)}{\hat{\eta}_n(x_i)}.$$  

Note that in this case, the parameter $c$ can be profiled-out. We can also consider a broader class of estimators by setting $h_1(w) = w^\alpha$ and $h_2(w) = w^\beta$. This class includes the above as special cases.

### 3.2 Non-separable case

Similar to the separable Bregman divergence case, the pseudo-spherical divergence $B_{ps}$ and the $\gamma$-divergence $B_{\gamma}$ also provide statistically efficient estimators for unnormalized models. Following the analogy of the separable case when $h_1(w) = w^\alpha$ and $h_2(w) = w^\beta$, suppose that $B_{ps}(w^\alpha, w^\beta; \eta^*) = 0$ holds. Then, $w^\alpha$ should be proportional to $w^\beta$ because of the property of pseudo-spherical divergence. As the result, $w(x)$ is a constant function. When $w(x) = p(x; \theta)/\eta(x)$
and $\eta$ is close to $\eta^*$, $p(x; \theta)$ should be close to $\eta^*$ up to the constant factor. This implies that the parameter $\theta$ can be estimated using pseudo-spherical divergence. Replacing $\eta^*$ with an empirical distribution, SDRME with non-separable divergence $\hat{\theta}_{ns-ps}$ is obtained by

$$\arg\min_{\theta \in \Theta} B_{ps}(w^{\alpha/\gamma}, w^{\beta/\gamma}; p_n), \quad w(x) = \frac{p(x; \theta)}{\hat{\eta}_n(x)},$$

under the condition $\alpha \neq \beta$. Then, the loss function is $\left\{\sum^n_{i=1} w_i^\alpha\right\}^{1/\gamma} - \left\{\sum^n_{i=1} w_i^\beta\right\}^{(1-\gamma)/\gamma} \sum^n_{i=1} w_i^\delta$, where $\delta = (\alpha + \beta(\gamma - 1))/\gamma, w_i = p(x; \theta)/\hat{\eta}_n(x_i)$. By taking a logarithm of each term, we can construct a loss function corresponding to $\gamma$-divergence. This is equal to $B_\gamma(w^\alpha, w^\beta; p_n)$:

$$\frac{1}{\gamma} \log \sum^n_{i=1} w_i^\alpha + \frac{\gamma - 1}{\gamma} \log \sum^n_{i=1} w_i^\beta - \log \sum^n_{i=1} w_i^\delta. \quad (3.2)$$

We define estimator $\hat{\theta}_{ns-\gamma}$ as a minimizer of the above function with respect to $\theta$.

Two things should be noted. First, compared with the case of separable divergence, the unnormalized model $p(x; \theta)$ is directly used instead of a one-parameter extended model $q(x; \tau) = e^{-\tau}p(x; \theta)$. This is due to the scale-invariance property of pseudo-spherical divergence [Kanamori & Fujisawa 2014, 2015]. Second, when the baseline measure is a counting measure, Takenouchi and Kanamori (2017) have proposed an estimator that is defined as a minimizer of the following function with respect to $\theta$, $1/\gamma \log \sum_{x \in X} c_x^{1-\alpha} p(x; \theta)^\alpha + (\gamma - 1)/\gamma \log \sum_{x \in X} c_x^{1-\beta} p(x; \theta)^\beta - \log \sum_{x \in X} c_x^{1-\delta} p(x; \theta)^\delta$, where $c_x = n_x/n$, $n_x$ is a sample number taking the value of $x$. This loss function is essentially the same as (3.2) by modifying the form of summing. The case was only considered when the sample space is discrete. However, it can be generalized to the case where the sample space is continuous, using our new unified perspective. For simplicity, hereafter, we assume $\delta = 0$ to eliminate the third term in (3.2). This restriction is also reasonable to obtain the convexity as seen in Section 4.3.

4 Theoretical investigation of self density-ratio matching estimator

We prove that the asymptotic variance of estimators $\hat{\theta}_s$ and $\hat{\theta}_{ns-\gamma}$ is identical to that of MLE. We utilize the property in which our estimators take the form of Z-estimators with infinite dimensional nuisance parameters [van der Vaart 1998].
Finally, we mention the issue regarding the convexity of loss functions. For the proofs, refer to Supplementary materials.

4.1 Efficiency in the separable case

First, we discuss the case when the divergence is separable. The estimator $\hat{\tau}_s$ based on the separable divergence is defined as the minimizer of the following function $\frac{1}{n} \sum_{i=1}^{n} B_f(h_1(w_i), h_2(w_i))$, where $w_i = q(x_i; \tau_\eta(x_i))$ and $\hat{\eta}_n(x)$ is a nonparametric density estimator using an entire sample.

When $\hat{\eta}_n$ was equal to $\eta^\ast$, this estimator $\hat{\tau}_s$ would be regarded as the solution to $\tilde{E}_s[\phi(x; \tau, \eta^\ast)] = 0$, where $\phi(x; \tau, \eta) = f(h_1(w(x))) - f(h_2(w(x))) - f'(h_2(w(x))) [h_1(w(x)) - h_2(w(x))], \text{ and } w(x) = q(x; \tau)/\eta^\ast(x)$, by differentiating the loss function with respect to $\tau$. Here, the moment condition $\tilde{E}_s[\phi(x; \tau, \eta^\ast)] = 0$ holds. This condition guarantees the validity of the estimator. However, this includes the unknown term $\eta^\ast(x)$. By replacing $\eta^\ast(x)$ with the nonparametric estimator $\hat{\eta}_n$, the estimator $\hat{\tau}_s$ is still regarded as a Z-estimator. Specifically, the estimator $\hat{\tau}_s$ is constructed by solving the equation $\tilde{E}_s[\phi(x; \tau, \hat{\eta}_n)] = 0$. The consistency holds as follows.

**Theorem 1** Suppose that (1a) there exists a Glivenko–Cantelli class $F$ of functions with an integrable envelope function that contains every $\phi(x; \tau, \hat{\eta}_n)$ probability tending to 1, and (1b) for all $x$, and the estimator $\hat{\tau}_s$ satisfies $\mathbb{P}_n \phi(x; \hat{\tau}_s, \hat{\eta}_n) = o_p(1)$, (1c) $\inf_{\tau : \|\tau - \tau^\ast\| > \epsilon} \|\tilde{E}_s[\phi(x; \tau, \eta^\ast)]\| > 0$, then $\hat{\tau}_s \xrightarrow{p} \tau^\ast$.

Assumption (1a) is called a uniform convergence condition. Compared with the parametric model, it is not directly verified because there is a nonparametric component $\hat{\eta}_n(x)$. A simple way to show this is discussed in Newey & McFadden (1994). Assumption (1b) is natural because we use the knowledge $\tilde{E}_s[\phi(x; \hat{\tau}_s, \hat{\eta}_n)] = 0$. Note that $o_p(1)$ does not have to be 0. Assumption (1c) is called a well-separated mode condition. When the sample space is compact, it is equivalent to the following two conditions: (1d) $\phi(x; \tau, \eta^\ast)$ is continuous with respect to $\tau$, (1e) $E_s[\phi(x; \tau, \eta^\ast)] = 0 \iff \tau = \tau^\ast$. The condition (1d) is not strong. When the identification condition of the model $q(x; \tau_1) = q(x; \tau_2) \iff \tau_1 = \tau_2$ holds, (1e) is verified because $E_s[\phi(x; \tau, \eta^\ast)] = 0 \iff q(x; \tau) = q(x; \tau^\ast) \iff \tau = \tau^\ast$ (Uehara et al., 2018).

Next, we show the asymptotic normality of the estimator $\hat{\tau}_s$ when the sample space is discrete.
Theorem 2  When the sample space is discrete, assume that (2a) there exists a Donsker class with a square-integrable envelope function that contains every \( \phi(x; \tau, \eta) \) with probability tending to 1, (2b) \( (\tau, \eta) \rightarrow \phi(x; \tau, \eta) \) is continuous in an \( L^2 \) space \( L^2(F_\eta^\ast) \) at \( (\tau^*, \eta^*) \), (2c) \( \hat{\tau}_n \xrightarrow{p} \tau^* \), (2d) \( \frac{1}{p_n} \phi(x; \hat{\tau}_n, \hat{\eta}_n) = o_p(n^{-1/2}) \), (2e) map \( \tau \rightarrow \phi(x; \tau, \eta) \) is differentiable at \( \tau^* \) uniformly in a neighborhood of \( \eta^* \), besides, the following matrix \( \Omega = E_\ast \left[ \nabla_\tau \log q \nabla_\tau \log q \right|_{\theta^*} \) is non-singular, (2f) the second order derivative of the map \( \eta \rightarrow \phi(x; \tau, \eta) \) is uniformly bounded around in a neighborhood of \( \eta^* \), then we have

\[
\sqrt{n}(\hat{\tau}_n - \tau^*) = \Omega^{-1} G_n \left[ \nabla_\tau \log q(x; \tau) \right|_{\tau^*} + o_p(1),
\]

\[
\sqrt{n}(\hat{\tau}_n - \tau^*) \xrightarrow{d} N(0, \Omega^{-1}).
\]

These assumptions originate from Theorem 6.18. in van der Vaart (2002). Each condition is verified according to the more specific information on \( \phi(x; \tau, \eta) \), such as smoothness with respect to \( \tau \) and \( \eta \), for example, see Newey & McFadden (1994). Assumptions (2a), (2b) and (2c) are used to state \( G_n \hat{\tau}_n, \hat{\eta}_n - G_n \tau^*, \eta^*_p \rightarrow 0 \). Assumption (2a) is satisfied if simpler conditions hold like smoothness of \( \tau \rightarrow \phi(x; \tau, \eta) \). This condition can be removed by using the cross-fitting argument in Chapter 25.8. (van der Vaart, 1998). Details are therefore not discussed here. Assumption (2d) is natural because we use the information \( E_\ast[\phi(x; \tau, \eta)]_{\tau^*, \eta^*} = 0 \). The assumption (2e) states that the derivative of the map \( \tau \rightarrow E_\ast[\phi(x; \tau, \eta)] \) is differentiable at \( \tau^* \) and the derivative \( \Omega \) is non-singular. Assumption (2f) is required to control the remainder term in the proof.

The variance estimator for \( \hat{\tau}_n \) is easily constructed from Theorem 2. Finally, we prove that \( \hat{\theta}_n \) in \( \hat{\tau}_n = (\hat{c}_n, \hat{\theta}_n) \) is equivalent to MLE in terms of asymptotic variance.

Corollary 4.1  When the sample space is discrete, we have

\[
\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(0, \Sigma_{\theta}^{-1}),
\]

where \( \Sigma_{\theta} \) is the Fisher information matrix at \( \theta^* \) of the normalized model, that is, \( \text{Var}_\ast[S(x; \theta^*)] \), where \( S(x; \theta) = \nabla_\theta \left( \log p(x; \theta) - \log \int p(x; \theta)d\mu(x) \right) \).

Next, we investigate asymptotic behavior when the sample space is continuous. We use the kernel density estimator as a nonparametric estimator for \( \eta^*(x) \). Note that any nonparametric estimators can also be applied. Assume that \( \eta^*(x) \) belongs to a Hölder class of smoothness \( \nu \) (Korostelev, 2011). The kernel density estimator is constructed as \( \hat{\eta}_n(x) = (1/nh^{d_k}) \sum_{i=1}^{n} K((x_i - x)/h) \).
where \( h \) denotes a bandwidth, \( K \) denotes a \( d_x \)-dimensional kernel, and \( d_x \) denotes a dimension of \( x \) \cite{Silverman1986}. The overall error \( \| \hat{\eta}_n - \eta^* \|_\infty = O_p((\log n/n)^{1/2} h^{-d_x/2} + h^\nu) \) by choosing high-order kernel \cite{Jianqing1992}. By selecting the order of bandwidth correctly, we have \( \| \hat{\eta}_n - \eta^* \|_\infty = O_p((\log n/n)^{-1/2} h^\nu) \) \cite{Stones1982}.

From here, we analyze the asymptotic behavior of estimator \( \hat{\tau}_n \). We conclude that the estimator is still efficient.

**Theorem 3** When the sample space is continuous, under the conditions used in Theorem 3 and (2g): \( \nu/(2 \nu + d_x) > 1/4 \), (2h): \( \int \| \nabla q(x; \tau) \| \cdot d\mu(x) \) is finite, (2i): there is \( \epsilon > 0 \) such that \( \mathbb{E}_n [\sup_{|u| < \epsilon} \| \nabla q(x + u; \tau) \| \cdot d\mu(x) \] < \infty,

\[
\sqrt{n}(\hat{\tau}_n - \tau^*) \rightarrow \mathcal{N}(0, \Omega^{-1}), \quad \sqrt{n}(\hat{\theta}_n - \theta^*) \rightarrow \mathcal{N}(0, \Omega_\theta^{-1}),
\]

where \( \Omega \) is defined in Theorem 3.

Here, assumption (2g) is introduced to control a remainder term. Assumptions (2h) and (2i) are introduced following Theorem 8.11. in Newey & McFadden (1994).

### 4.2 Efficiency in the non-separable case

We consider an asymptotic analysis of estimator \( \hat{\theta}_{ns-\gamma} \) with the \( \gamma \)-divergence. When \( \mu \) is a counting measure, by differentiating \[3.2\] with respect to \( \theta \) and multiplying by \( -\gamma/\alpha \), we get \( S_{\alpha,\beta}(x; \theta) \):

\[
\int \frac{\{ \nabla \log p(x; \theta) \} w(x; \theta)^\beta}{w(x; \theta)^\beta d\mathbb{P}_n(x)} d\mathbb{P}_n(x) -
\int \frac{\{ \nabla \log p(x; \theta) \} w(x; \theta)^\alpha}{w(x; \theta)^\alpha d\mathbb{P}_n(x)} d\mathbb{P}_n(x),
\]

where \( w(x) = p(x; \theta)/\hat{\eta}_n(x) \). Importantly, compared with the case in Section 4.1, \( p(x; \theta) \) is used in \( w(x) \) instead of \( q(x; \tau) \). The estimator \( \hat{\theta}_{ns-\gamma} \) is defined as the solution to \( S_{\alpha,\beta}(x; \theta) = 0 \). The validity of the estimator is based on the relation \( 0 = T_{\alpha,\beta}(x; \theta) \), where \( T_{\alpha,\beta}(x; \theta) \) is a term replacing \( \hat{\eta}_n(x) \) with \( \eta^*(x) \) and \( \mathbb{P}_n \) with \( F_{\theta^*} \) in \( S_{\alpha,\beta}(x; \theta) \). Actually, the estimator \( \hat{\theta}_{ns-\gamma} \) can be seen as a \( Z \)-estimator with infinite and finite-dimensional nuisance parameters, that is, the solution to \( \hat{\theta}_{ns-\gamma} (U_{\alpha,\beta}(x; \theta, c_1, c_2, \hat{\eta}_n)) = 0 \), where \( U_{\alpha,\beta}(x; \theta, c_1, c_2) \):
The validity of the estimator is based on the moment condition $0 = E_u[U_{\alpha,\beta}(x;\theta,c_1,c_2,\eta)\mid \theta^*,c_1^*,c_2^*,\eta^*]$, where $\exp(c_1^*) = \exp(c^*)^\beta$ and $\exp(c_2^*) = \exp(c^*)^\alpha$. Note that $\theta$ is a parameter of interests, and $c_1$, $c_2$ and $\eta$ are nuisance parameters. We can derive the asymptotic results as in Section 4.1. We conclude that $\hat{\theta}_{\text{ns-}\gamma}$ is an efficient estimator.

**Theorem 4** When the sample space is discrete, under the conditions of Theorem 2, we have $\sqrt{n}(\hat{\theta}_{\text{ns-}\gamma} - \theta^*) \xrightarrow{d} \mathcal{N}(0, J_{\theta^*}^{-1})$. When the sample space is continuous, under conditions of Theorem 3, we have $\sqrt{n}(\hat{\theta}_{\text{ns-}\gamma} - \theta^*) \xrightarrow{d} \mathcal{N}(0, J_{\theta^*}^{-1})$.

### 4.3 Convexity

The convexity is important for optimization. Here, we consider the convexity of loss functions. Suppose that the model is expressed by unnormalized exponential models, $q(x;\tau) = \exp(\tau^T \xi(x))$, where the corresponding basis function for $c$ is $-1$. This model contains many types of unnormalized models such as Boltzmann machines and generalized gamma distributions used in Section 6. Regarding separable estimators $\hat{\tau}_s$ in Example 3.1, we can find sufficient conditions to ensure the convexity of loss functions.

**Theorem 5** Suppose that $f(z)$ satisfies the inequality $(2z - 1)f''(z) + z(z - 1)f'''(z) \geq 0$ for arbitrary $z > 0$. Then, the loss function of the estimator $\hat{\tau}_s$ in Example 3.1 is convex in $\tau$.

We see specific examples of $f(x)$, satisfying the above conditions.

**Example 4.1** For the functions $f(z) = z \log z$ and $f(z) = 2z \log z - 2(1 + z) \log(1+z)$, we can confirm the conditions in Theorem 3. However, the function $f(z) = 0.5z^2$ does not meet the above conditions. In the same way, we can find that the function $f(z) = z^m/(m(m-1))$ with a natural number $m \geq 2$ does not meet the conditions.

We have a similar result for non-separable estimators. As for the estimator with $\gamma$-divergence, the loss function is convex if the equality $\delta = 0$ holds [Takenouchi & Kanamori, 2017].

### 5 Asymptotics under misspecification

We have assumed that the model includes true density. In this section, we consider a misspecified case, showing that the behavior of the proposed estimators associated with KL divergence is asymptotically the same as that of MLE.
This implies that similarly to MLE, the proposed estimator converges to the parameter that minimizes the KL-divergence between the model and the true distribution, even when the model is misspecified.

Before analyzing the proposed estimators, we review a misspecified case where the model can be normalized properly. The MLE under the misspecified model is equivalent to finding the closest model to the true distribution regarding KL divergence (White, 1982). The MLE estimator \( \hat{\theta}_{\text{MLE}} \) converges to the value maximizing the function \( \theta \rightarrow E_{\star}[\log p(x; \theta) - \log \int p(x; \theta) d\mu(x)] \). We denote this value as \( \theta^* \). The value \( \theta^* \) satisfies the equation \( E_{\star}[S(x; \theta)] = 0 \), where \( S(x; \theta) = \nabla_{\theta} (\log p(x; \theta) - \log \int p(x; \theta) d\mu(x)) \).

Next, consider the asymptotic behavior of \( \hat{\theta}_s \) in (3.1) when the model is misspecified. We assume \( f(x) = x \log x \), as in Example 3.1. In this case, the estimator \( \hat{\theta}_s \) converges in probability to \( \theta^* \), which satisfies the equation \( E_{\star}[S(x; \theta)] = 0 \). When \( f(x) \) is not \( x \log x \), a similar result can be obtained. However, the limits of estimators no longer converge to the same \( \theta^* \). With these settings, we have the following theorem.

**Theorem 6** Under certain regularity conditions as in Theorem 2, we have

\[
\sqrt{n}(\hat{\theta}_s - \theta^*) = \Omega_{1m}^{-1} \mathcal{G}_n \left[ \nabla_{\theta} \log p(x; \theta) \right] + o_p(1),
\]

\[
\sqrt{n}(\hat{\theta}_s - \theta^*) \xrightarrow{d} N(0, \Omega_{1m}^{-1} \Omega_{2m}^{-1} \Omega_{1m}^{-1}).
\]

The specific forms of \( \Omega_{1m}^{-1} \) and \( \Omega_{2m}^{-1} \) are provided in the Supplementary materials.

Two implications are observed in this Theorem 6. First, when the model includes the true distribution, i.e., \( \eta^* = q(x; \tau^*) \), this theorem is reduced to Theorem 2. Second, the resulting form of \( \Omega_{1m}^{-1}, \Omega_{2m}^{-1} \) has a form similar to terms appeared in the asymptotic result of the MLE estimator when the model is normalized. Details are offered in the Supplementary materials.

### 6 Numerical experiments

Here we present several examples to illustrate the performance of the proposed procedure, and demonstrate that the asymptotic variance of the proposed estimators is the same as that of MLE. We ran simulations in the settings of restricted Boltzmann machines, submodular diversity models, generalized gamma
Table 1: Monte Carlo mean and standard error of the KL divergence between the true density and estimated density scaled by sample size in RBM. Parenthesis indicates a standard error.

| dim $v = 5$, dim $h = 2$, iteration: 50 | n  | s-JS     | s-KL     | s-Chi    | MLE     |
|----------------------------------------|----|----------|----------|----------|---------|
|                                        | 100| 5.91(2.66)| 5.32(2.22)| 6.73(4.07)| 5.66(2.72)|
|                                        | 500| 4.94(2.02)| 5.14(2.05)| 6.88(3.03)| 5.06(1.95)|
|                                        | 1000| 5.35(2.46)| 5.43(2.58)| 6.45(3.57)| 5.57(2.74)|

| dim $v = 8$, dim $h = 2$, iteration: 20 | n  | s-JS     | s-KL     | s-Chi    | MLE     |
|----------------------------------------|----|----------|----------|----------|---------|
|                                        | 500| 26.3(12.9)| 24.7(12.1)| 30.2(12.3)| 11.2(4.60)|
|                                        | 1000| 18.4(9.62)| 14.6(9.28)| 17.4(10.4)| 8.38(3.09)|
|                                        | 5000| 10.5(3.73)| 8.78(3.27)| 18.9(7.35)| 8.85(3.24)|

distributions, and misspecified Poisson models. We used the following package for kernel density estimation ([Hayfield & Racine, 2008]). We also used 6-th order kernel, and the bandwidth was selected by cross validation based on the likelihood. We compare the following estimators:

- **MLE**: Estimator by MLE.
- **NCE**: Estimator by NCE ([Gutmann & Hyvärinen, 2010]). The sample size of the auxiliary distribution is set as the original sample size.
- **s-KL, s-Chi, s-JS**: Proposed estimators, i.e., SDRME with a separable divergence $\hat{\theta}_s$. When $f = x \log x$, denote s-KL. When $f = 0.5x^2$, denote s-Chi. When $f = 2x \log x - 2(1 + x) \log(1 + x)$, denote s-JS.
- **ns-$\gamma$**: SDRME with the non-separable $\gamma$-divergence, $\hat{\theta}_{ns-\gamma}$ when $\alpha = 1.01, \beta = 0.01$.

We do not compare proposed estimators with a score matching type estimators because the superiority about statistical efficiency of NCE over score matching is already shown in ([Gutmann & Hyvärinen, 2010]).

### 6.1 Restricted Boltzmann Machine (RBM)

The RBM has the parameter $W \in \mathbb{R}^{d_v \times d_h}$. The joint probability of the RBM with the visible nodes $v \in \{+1, -1\}^{d_v}$ and hidden nodes $h \in \{+1, -1\}^{d_h}$ is $P(v, h; W) \propto e^{v^T W h}$ and the marginal probability of $v$ is $P(v; h; W) \propto \prod_{k=1}^{d_v} \cosh((v^T W)k)$. The unnormalized model for the RBM is expressed as $q(v; \tau) = e^{-\tau} \prod_{k=1}^{d_v} \cosh((v^T W)k)$ with the parameter $\tau = (c, W)$.  

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We compared four estimators: \textit{s-JS}, \textit{s-KL}, \textit{s-Chi} and MLE. In low dimensional models, the MLE is feasible because the normalized constant is accessible in practice. Table 1 shows Monte Carlo mean and standard error of the KL divergence between the true density and estimated density scaled by sample size. For each iteration, the new parameter \(W\) is generated as the true parameter. We confirmed the asymptotic efficiency of the proposed methods. Moreover, we observed that for high dimensional small sample case, a large sample size was required to achieve the variance of the MLE. In such case, the normalization constant in the MLE works as the regularization. For the other methods, the regularization is effective in stabilizing the behavior of the estimators as in [Takenouchi & Kanamori, 2017].

6.2 Submodular diversity model

For applications such as recommendation systems and information summary, several types of probabilistic submodular models have been developed to model the diversity of item sets. Among them, [Tschiatschek et al., 2016] proposed the FLID (Facility LocatIon Diversity) model, which is a probability distribution over subsets \(S\) of \([1, \cdots, V]\). Specifically, FLID is defined as

\[
P(S; u, w) \propto \exp \left( \sum_{i \in S} u_i + \sum_{d=1}^{L} (\max_{i \in S} w_{i,d} - \sum_{i \in S} w_{i,d}) \right),
\]

where \(u_i\) and \(w_i = (w_{i,1}, \cdots, w_{i,L})\) represent the quality and latent embedding vector of the \(i\)-th item, respectively \((i = 1, \cdots, n)\). Since the computation of the normalization constant of FLID is prohibitive, [Tschiatschek et al., 2016] proposed to estimate this model by using NCE.

We compared \textit{s-KL}, \textit{ns-\(\gamma\)} and NCE. We generated samples from the FLID model with \(L = 2\) and \(V = 12\). Here, each entry of \(u\) and \(w\) were sampled independently from the uniform distribution on \([0, 1]\). For the noise distribution in NCE, we used the product distribution following [Tschiatschek et al., 2016]. Table 2 presents the Monte Carlo mean and standard error of the KL divergence between the true density and estimated density; additionally, it presents the computation time of each estimator. These results indicate the significant superiority of \textit{s-KL} to NCE in terms of statistical efficiency with reasonable computational time. We also observe that the performance of \textit{s-KL} is more stable than that of \textit{ns-\(\gamma\)} in this case.

6.3 Generalized gamma distribution

Here, we consider a distribution with the following unnormalized density

\[
P(x; \theta_1, \theta_2) \propto \exp(-\theta_1 x^2) x^{\theta_2} I(x > 0)
\]

when the baseline measure is the Lebesgue
Table 2: Monte Carlo mean of the KL divergence between the true density and estimated density, scaled by sample size in a submodular diversity model. The computational time (seconds) is measured per each iteration when \( n = 2 \times 10^5 \).

\[
\begin{array}{cccc}
  n & \text{s-KL} & \text{ns-} \gamma & \text{NCE} \\
  5 \times 10^4 & 36.4(7.3) & 46.6(5.9) & 44.4(4.0) \\
  1 \times 10^5 & 21.5(4.9) & 46.4(4.2) & 37.5(7.8) \\
  2 \times 10^5 & 16.9(7.6) & 69.3(7.0) & 35.9(20.9) \\
  \text{Time} & 4911 & 2020 & 9827 \\
\end{array}
\]

Table 3: Monte Carlo mean of mean square errors scaled by sample size in a generalized gamma distribution. The computational time (seconds) is measured per each iteration when \( n = 2000 \).

\[
\begin{array}{ccc}
  n & \text{s-KL} & \text{ns-} \gamma & \text{NCE} \\
  500 & 68.2 & 77.6 & 250.3 \\
  1000 & 67.9 & 76.3 & 240.7 \\
  2000 & 68.3 & 75.3 & 246.1 \\
  \text{Time} & 1.3 & 1.3 & 0.5 \\
\end{array}
\]

measure, which is referred to as a generalized gamma distribution \( \text{[Stacy 1962]} \). We set the true value at \((\theta_1, \theta_2) = (1.3, 1.3)\).

We compared three estimators: \text{s-KL}, \text{ns-} \gamma and \text{NCE}. Unlike Sections 6.1 and 6.2, we used a kernel density estimator for \text{s-KL} and \text{ns-} \gamma, and a half-normal distribution for \text{NCE} as an auxiliary distribution. The Monte Carlo mean of the mean square errors is presented in Table 3. This result demonstrates the significant superiority of \text{s-KL} and \text{ns-} \gamma over \text{NCE} in terms of statistical efficiency with reasonable computational time even when the sample space is continuous.

6.4 Misspecified Poisson model

Here, we examine the behavior of each estimator when the model is misspecified. We assume unnormalized parametric models \( P(x; \theta) \propto \exp(\theta)^x / x! \), \( x \in \mathbb{N}_{\geq 0} \) based on Poisson distributions. We consider two scenarios based on the true distribution (well-specified case) \( \exp(-2.0)2.0^x / x! \) and, (misspecified case) \( 0.5 \exp(-2.0)2^{x-0.2} / (x - 0.2)! + 0.5 \exp(-1.0) / (x - 1.2)! \).

We compared five estimators: \text{s-KL}, \text{s-Chi}, \text{s-JS}, \text{ns-} \gamma and MLE. The Monte Carlo mean and the standard error of KL divergence between true density and estimated density are presented in Table 4. This experiment reveals that the performance of each estimator significantly varies in the misspecified case, but not in the well-specified case. It is indicated that \text{s-KL} is preferable in terms
Table 4: Monte Carlo mean and standard error of the KL divergence between the true density and estimated density scaled by sample size in a Poisson model. Parenthesis indicates a standard error.

\begin{tabular}{cccccccc}
| $n$ | s-KL | s-Chi | s-JS | ns-$\gamma$ | MLE |
|-----|------|------|------|-------------|-----|
| 1000 | 0.26 | 0.26 | 0.27 | 0.26 | 0.26 |
|      | (0.03) | (0.03) | (0.04) | (0.03) | (0.03) |
| 2000 | 0.25 | 0.26 | 0.25 | 0.25 | 0.25 |
|      | (0.03) | (0.04) | (0.04) | (0.03) | (0.03) |
\end{tabular}

mis specified case

\begin{tabular}{cccccccc}
| $n$ | s-KL | s-Chi | s-JS | ns-$\gamma$ | MLE |
|-----|------|------|------|-------------|-----|
| 1000 | 5.6 | 6.0 | 7.3 | 6.2 | 5.5 |
|      | (0.4) | (0.7) | (1.4) | (0.7) | (0.2) |
| 2000 | 11.1 | 11.8 | 14.6 | 12.1 | 10.9 |
|      | (0.4) | (0.9) | (1.9) | (0.7) | (0.2) |
\end{tabular}

of the KL divergence because it has a performance similar to that of MLE, even when the model is misspecified.

7 Conclusion

We have proposed self density-ratio matching estimators. Importantly, proposed estimators are as statistically efficient as MLE without calculating normalizing constants, regardless of whether the sample space is discrete or continuous. In addition, they do not rely on any sampling techniques. Among the several estimators, we recommend using s-KL in Section 6 for practical purposes because its experimental performance is stable, its loss function is convex, and it is seen as a projection regarding KL divergence, even when the model is misspecified. We can therefore apply common results obtained in MLE such as AIC and TIC (Akaike, 1974; Takeuchi, 1976), to the proposed estimator.

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A Notation

\( n \)  Total sample
\( \mu \)  Baseline measure
\( p(x; \theta) \)  Unnormalized Model
\( \tilde{p}(x; \theta) \)  Normalized model
\( c \)  Normalizing constant parameter
\( \tau = (c, \theta^\top)^\top \)  Parameter space for \( \theta \)
\( \Theta \)  Parameter space for \( \tau \)
\( \eta^*(x) \)  True density
\( \hat{\eta}_n(x) \)  Nonparametric estimator
\( F_{\eta^*} \)  True distribution
\( p_n \)  Empirical density
\( E_* \)  Expectation under true distribution
\( \hat{E} \)  Expectation under empirical distribution
\( \text{Var}_* \)  Variance under true distribution
\( \nabla_x \)  Differentiation with respect to \( x \)
\( \mathbb{P}_n \)  Empirical distribution of \( n \) samples from \( F_{\eta^*} \)
\( \mathbb{G}_n \)  Empirical process \( \sqrt{n}(\mathbb{P}_n - F_{\eta^*}) \)
\( \mathcal{I}_\theta \)  Fisher information matrix for \( \theta \)
\( |\tau| \)  the value at \( \tau = \tau^* \)
\( \mathcal{N}(A,B) \)  Normal distribution with mean \( A \), variance \( B \)
\( L^2(F_{\eta^*}) \)  \( L^2 \)-space with the underlying distribution \( F_{\eta^*} \)
\( \mathcal{X} \)  Sample space
\( B_f(u,v) \)  Bregman divergence based on \( f \) between \( u \) and \( v \)
\( K \)  Kernel
\( \hat{\tau}_s \)  Self density-ratio matching estimator with a separable divergence.
  Note that it is equal to \( (\hat{c}_s, \hat{\theta}_s) \)
\( \hat{\tau}_{ns-\gamma} \)  Self density-ratio matching estimator with a \( \gamma \)-divergence
\( \hat{\tau}_{ns-ps} \)  Self density-ratio matching estimator with a pseudo spherical divergence
\( \| \cdot \| \)  Euclidean norm
\( \| \cdot \|_\infty \)  \( l_\infty \) norm
B Proof of Theorems

Proof of Theorem 4.1. Use Theorem 5.11 directly in van der Vaart (2002).

Proof of Theorem 4.2. The estimator \( \hat{\tau}_n \) is considered as the one satisfying \( \mathbb{P}_n \phi(x; \tau, \eta)|_{\hat{\tau}_n, \hat{\eta}_n} = 0 \), where \( w(x) = q(x; \tau) \eta(x) \) and \( \phi(x; \tau, \eta) \) is

\[
\{|f'(h_1(w)) - f'(h_2(w))| h_1'(w) - f''(h_2(w)) h_2'(w) [h_1(w) - h_2(w)]\} w \nabla \tau \log q(x; \tau).
\]

From Theorem 6.17. in van der Vaart (2002) based on assumptions (2a)-(2f), we have

\[
\sqrt{n}(\hat{\tau}_n - \tau^*) = -V_{\tau^*, \eta^*, n} \sqrt{n} E_n[\phi(x)|_{\tau^*, \eta_n}] - V_{\tau^*, \eta^*, n} \mathbb{E}_n \phi(x)|_{\tau^*, \eta^*} + o_p(1 + \sqrt{n}||E_n[\phi(x)|_{\tau^*, \eta_n}]||),
\]

where \( V_{\tau^*, \eta^*} \) is a derivative of \( \tau \to \mathbb{E}_n[\phi(x; \tau, \eta^*)] \) at \( \tau^* \). First, we calculate the derivative \( V_{\tau^*, \eta^*} \). The derivative is

\[
\nabla_{\tau^*} \mathbb{E}_n[\phi(x; \tau, \eta^*)]|_{\tau^*} = \mathbb{E}_n[\nabla_{\tau^*} \phi(x; \tau, \eta^*)]|_{\tau^*}
\]

\[
= \sqrt{n} E_n[\{f''(h_1(w)) h_1'(w) - f''(h_2(w)) h_2'(w)\} h_1'(w) - f''(h_2(w)) h_2'(w)(h_1'(w) - h_2'(w))]
\]

\[
w \nabla \tau \log q(x; \tau) \{\nabla_{\tau^*} w\}(\hat{\eta}_n(x) - \eta^*(x))|_{\tau^*, \eta^*}
\]

\[
= f''(1) \{h_1'(1) - h_2'(1)\}^2 E_n[\nabla_{\tau^*} \log q \nabla_{\tau^*} \log q]|_{\tau^*}
\]

\[
= f''(1) \{h_1'(1) - h_2'(1)\}^2 \Omega.
\]

Next consider each term in (B.1). The second term in (B.1) vanishes because \( \phi(x)|_{\tau^*, \eta^*} \) is 0. Therefore, we only analyze the first term in (B.1):

\[
\sqrt{n} E_n[\phi(x)|_{\tau^*, \eta_n}] = \sqrt{n} E_n[\phi(x)|_{\tau^*, \eta_n}] - \sqrt{n} E_n[\phi(x)|_{\tau^*, \eta^*}]
\]

\[
= \sqrt{n} E_n[\nabla_{\tau^*} \phi(x)|_{\tau^*, \eta^*}(\hat{\eta}_n(x) - \eta^*(x))]
\]

\[
+ \sqrt{n} E_n[\phi(x)|_{\tau^*, \eta_n}] - \sqrt{n} E_n[\phi(x)|_{\tau^*, \eta^*}] - \sqrt{n} E_n[\nabla_{\eta^*} \phi(x)|_{\tau^*, \eta^*}(\hat{\eta}_n(x) - \eta^*(x))].
\]

We decompose \( \sqrt{n} E_n[\phi(x)|_{\tau^*, \eta_n}] \) into two terms again. The first term (B.2)
is
\[ \sqrt{n}E_s[\nabla \phi(x)]_{\tau, \eta^*} (\hat{\eta}_n(x) - \eta^*(x)) \]
\[ = \sqrt{n}E_s[F'(h_1(w))h_1'(w) - F'(h_2(w))h_2'(w)]h_1'^2(w) - F''(h_2(w))h_2'(w)(h_1'(w) - h_2'(w))\{\nabla \eta w \}
\]
\[ = -\nabla \tau \log q(x; \tau)|_{\tau^*, \eta^*} (\hat{\eta}_n(x) - \eta^*(x)) \]
\[ = -\nabla \tau (1) (h_1'^2(1) - h_2'^2(1)) \frac{\nabla \tau q(x; \tau)}{q(x; \tau)}|_{\tau^*} (\hat{\eta}_n(x) - \eta^*(x))d\mu(x) \]
\[ = -\nabla \tau (1) (h_1'^2(1) - h_2'^2(1)) \frac{\nabla \tau q(x; \tau)}{q(x; \tau)}|_{\tau^*}. \]

In addition, the second residual term (B.3) vanishes because, for some large \( C \) and \( \hat{\eta} \) is a between \( \hat{\eta}_n \), we have
\[ || \sqrt{n}E_s[\phi_{\tau^*, \hat{\eta}_n}] - \sqrt{n}E_s[\phi_{\tau^*, \eta^*}] - \sqrt{n}E_s[\nabla \phi(x; \tau, \eta)|_{\tau^*, \eta^*} (\hat{\eta}_n(x) - \eta^*(x))] || \]
\[ = \sqrt{n}|| E_s[\nabla \eta \phi(x; \tau, \eta)|_{\tau^*, \eta^*} (\hat{\eta}_n(x) - \eta^*(x))^2] || \]
\[ \leq C \sqrt{n}|| E_s[(\hat{\eta}_n(x) - \eta^*(x))^2] ||. \]

From the second line to the third line, we use an assumption (2g). The last term goes to 0 in probability. By combining all things and substituting into \([B.1]\), the statement is proved.

**Proof of Corollary 4.1** The score function \( S(x; \theta) \) can be written as
\[ \nabla \theta \log p(x; \theta) - \int \frac{p(x; \theta)}{p(x; \theta)d\mu(x)} d\mu(x). \]

Fisher information matrix \( \mathcal{I}_\theta^{-1} \) is \( \text{Var}_s[S(x; \theta)|_{\theta^*}] \), that is,
\[ E_s[\nabla \theta \log p(x; \theta) \nabla \theta^\tau \log p(x; \theta)|_{\theta^*}] = E_s[\nabla \theta \log p(x; \theta)|_{\theta^*}] E_s[\nabla \theta^\tau \log p(x; \theta)|_{\theta^*}]. \]

On the other hand, the component corresponding \( \theta^* \) in \( \Omega^{-1} \) can be also written as
\[ E_s[\nabla \theta^\tau \log p(x; \theta) \nabla \theta \log p(x; \theta)|_{\theta^*}] = E_s[\nabla \theta^\tau \log p(x; \theta)|_{\theta^*}] E_s[\nabla \theta \log p(x; \theta)|_{\theta^*}], \]
from Theorem 2 and Woodbury formula. This is the same as the Fisher information matrix. This concludes the proof.

**Proof of Theorem 3**

As we discussed in the proof of Theorem 2, the problem is a drift term. We can derive the given theorem by calculating the drift term in the same way. The drift term \( \sqrt{n}E_s[\phi_{\tau^*, \hat{\eta}_n}] \) is decomposed into two terms, the main:

\[ \sqrt{n}E_s[\nabla \phi(x)]_{\tau^*, \eta^*} (\hat{\eta}_n(x) - \eta^*(x)) \]
Next, we have
\[ p = \frac{n}{h^d} \sum_{i=1}^{n} K \left( \frac{x - x_i}{h} \right) - \eta^*(x) \] \( d\mu(x), \)
and the residual term. The main term corresponds to the term \( (B.2) \) in Theorem 2 and the residual term corresponds to the term \( (B.3) \) in Theorem 2. As revealed in the proof, the residual term is written as \( O_p(\sqrt{n}) = o_p(1) \) because \( \sqrt{n} \| \hat{\eta} - \eta^*(x) \| ^2 \). This term is equal to the order \( o_p(1) \) because \( \frac{\nu}{\nu + \tau^2} > 1/4 \) holds from the assumption (2g).

Next, we have
\[ \sqrt{n} \int \frac{\nabla_{x} q(x; \tau)}{q(x; \tau)} \bigg|_{\tau^*} \left( \frac{1}{nh^d} \sum_{i=1}^{n} K \left( \frac{x - x_i}{h} \right) - \eta^*(x) \right) d\mu(x), \]

This holds from Theorem 8.11 in Newey & McFadden (1994) using assumptions (2h) and (2i). Then, the drift term becomes
\[ \sqrt{n} \int \frac{\nabla_{\tau} q(x; \tau)}{q(x; \tau)} \bigg|_{\tau^*} \left( \frac{1}{nh^d} \sum_{i=1}^{n} K \left( \frac{x - x_i}{h} \right) d\mu(x) - dP_n(x) \right) = o_p(1). \]

Proof of Theorem 4 We redefine \( \sigma = (\theta^T, c_1, c_2)^T \). To avoid abuse of notations, we write \( U_{\alpha, \beta}(x; \sigma) \) as \( U(x) \).

As in the proof of Theorem 2, we have
\[ \sqrt{n}(\hat{\sigma}_{n, \gamma} - \sigma^*) = -V_{\sigma^*, \eta^*}^{-1} \sqrt{n}E_u[U(x)|_{\sigma^*, \eta_n}] - V_{\sigma^*, \eta^*}^{-1}G_nU(x)|_{\sigma^*, \eta^*} + o_p(1 + \sqrt{n}E_u[U(x)|_{\sigma^*, \eta_n}]). \]

where \( \hat{\sigma}_{n, \gamma} \) is a solution to \( \hat{E}[U(x; \sigma)] = 0 \) and \( V_{\sigma^*, \eta^*} \) is a derivative of the map \( \sigma \to E_u[U(x; \sigma, \eta^*)] \) at \( \sigma^* \).

First, we calculate the derivative \( V_{\sigma^*, \eta^*} \). This becomes
\[ E_u \left[ \begin{bmatrix} (\beta - \alpha)\nabla_{\theta} s(x; \theta) s(x; \theta)^T & s(x; \theta) & 0 \\ (\beta - 1) s(x; \theta)^T \exp(c_1) & \exp(c_1) & 0 \\ (\alpha - 1) s(x; \theta)^T \exp(c_2) & 0 & \exp(c_2) \end{bmatrix} \right], \]

which is evaluated at \( \sigma^* \) and \( s(x; \theta) = \nabla_{\theta} \log p(x; \theta) \). The term corresponding \( \theta \) in the above matrix \( V_{\sigma^*, \eta^*}^{-1} \) is
\[ (\beta - \alpha)^{-1}(E_u[\nabla_{\theta} \log p(x; \theta) \nabla_{\theta^T} \log p(x; \theta)] - E_u[\nabla_{\theta} \log p(x; \theta)]E_u[\nabla_{\theta^T} \log p(x; \theta)]^{-1}|_{\theta^*} \]
\[ = (\beta - \alpha)^{-1}V_{\theta^*}. \]
Then, we analyze each term in (B.4). First of all, the second term in (B.4) becomes zero because $U(x; \sigma^*, \eta^*) = 0$. Therefore, we only consider the first term in (B.4). We have

$$\sqrt{n} \mathbb{E}_* \left[ \nabla_{\theta} \left( \frac{\alpha - \beta}{\eta(x)^\beta} \frac{\alpha - 1}{\eta(x)^{\alpha - 1}} \right) (\hat{\eta}_n(x) - \eta^*(x)) \right] = \sqrt{n} \mathbb{E}_* \left[ \nabla_{\theta} \log p(x; \theta) | \hat{\eta}_n(x) - \eta^*(x) \right] + o_p(1).$$

Therefore, the first term corresponding $\theta$ in the above equation becomes

$$J_\theta^{-1} \mathbb{E}_* \left[ \nabla_{\theta} \log p(x; \theta) | \hat{\eta}_n(x) - \eta^*(x) \right] + o_p(1).$$

Finally, we get

$$\sqrt{n}(\hat{\theta}_{ns-\gamma} - \theta^*) = J_\theta^{-1} \mathbb{E}_* \left[ \nabla_{\theta} \log p(x; \theta) | \hat{\eta}_n(x) - \eta^*(x) \right] + o_p(1).$$

**Proof of Theorem 5** Let us define $\ell_i(\tau)$ as the loss for the sample $x_i$, i.e.,

$$\ell_i(\tau) = -f'(z_i) + w_i f''(z_i) - f(z_i),$$

where $z_i = q(x_i; \tau)/\hat{\eta}(x_i)$. The loss function is expressed by the total sum of $\ell_i(\tau)$ over all samples. For the unnormalized exponential model, some calculation yields the Hessian matrix of $\ell_i(\tau)$,

$$\nabla^2 \ell_i(\tau) = \left( f''(z_i) z_i^2 + (z_i - 1) (f'''(z_i) z_i^2 + f''(z_i) z_i) \right) \phi(x_i) \phi(x_i)^\top$$

$$= z_i (2z_i - 1) f'''(z_i) + z_i (z_i - 1) f''(z_i) \phi(x_i) \phi(x_i)^\top.$$

The assumption of the theorem guarantees that the coefficient above is non-negative; hence, the Hessian matrix of $\ell_i(\tau)$ is non-negative definite, so is the loss function. Eventually, the loss function is convex in the parameter $\tau$.

**C Supplement for Chapter 5**

First, we calculate the asymptotic variance of the estimator explicitly when the model is normalized. It is equal to

$$\mathbb{E}_* \left[ \nabla_{\theta} S(x; \theta) | \theta^* \right]^{-1} \text{Var}_* \left[ S(x; \theta) | \theta^* \right] \mathbb{E}_* \left[ \nabla_{\theta} S(x; \theta) | \theta^* \right]^{-1}.$$
The term $E_*[\nabla_{\theta^*} S(x; \theta)]|_{\theta^*}$ is

$$E_* \left[ \left( 1 - \frac{\tilde{p}^*(x)}{\eta^*(x)} \right) \nabla_{\theta^*} \nabla_{\theta} \log p(x; \theta)|_{\theta^*} \right] + E_* \left[ \frac{\tilde{p}^*(x)}{\eta^*(x)} \nabla_{\theta} \log p(x; \theta) \nabla_{\theta^*} \log p(x; \theta)|_{\theta^*} \right]$$

$$- E_* \left[ \frac{\tilde{p}^*(x)}{\eta^*(x)} \nabla_{\theta} \log p(x; \theta)|_{\theta^*} \right] E_* \left[ \frac{\tilde{p}^*(x)}{\eta^*(x)} \nabla_{\theta^*} \log p(x; \theta)|_{\theta^*} \right],$$

where

$$\tilde{p}(x; \theta) = p(x; \theta)/ \int p(x; \theta) d\mu(x),$$

$$\tilde{p}^*(x) = \tilde{p}(x; \theta^*).$$

We also have

$$\text{Var}_* [S(x; \theta)|_{\theta^*}] = \text{Var}_* [\nabla_{\theta} \log p(x; \theta)|_{\theta^*}].$$

Next, we prove Theorem 6. Before that, we show Lemma C.1.

**Lemma C.1** **Under certain regularity conditions, we have**

$$\sqrt{n}(\hat{r}_n - \tau^*) = \Omega_{1m}^{-1} \mathbb{G}_n [\nabla_{\tau} \log q(x; \tau)|_{\tau^*}] + o_p(1), \quad (C.1)$$

$$\sqrt{n}(\hat{r}_n - \tau^*) \overset{d}{\to} \mathcal{N}(0, \Omega_{1m}^{-1} \Omega_{2m}^{-1}), \quad (C.2)$$

where $\tau^* = (c^*, \theta^*)$ is a value such that

$$\exp(c^*) = \int p(x; \theta^*) d\mu(x), \quad 0 = E_* [S(x; \theta)],$$

and

$$\Omega_{1m} = -E_* \left[ \left( 1 - \frac{q(x; \tau)}{\eta^*(x)} \right) \nabla_{\tau^*} \nabla_{\tau} \log q(x; \tau)|_{\tau^*} \right] + E_* \left[ \frac{q(x; \tau)}{\eta^*(x)} \nabla_{\tau} \log q(x; \tau) \nabla_{\tau^*} \log q(x; \tau)|_{\tau^*} \right],$$

$$\Omega_{2m} = \text{Var}_* [\nabla_{\tau} \log q(x; \tau)|_{\tau^*}].$$

**Proof of Lemma C.1** The estimator $\hat{r}_n$ can be considered as the one satisfying $\mathbb{P}_n \phi_{r_n, \eta_n} = 0$, where

$$\phi(x; \tau, \eta) = \nabla_{\tau} \log q(x; \tau) - \left( \frac{q(x; \tau)}{\eta(x)} \right) \nabla_{\tau} \log q(x; \tau).$$

From Theorem 6.17, van der Vaart (2002), we have

$$\sqrt{n}(\hat{r}_n - \tau^*) = -V_{\tau^*, \eta^*}^{-1} \sqrt{n} E_* [\phi_{r_n, \eta_n}] - V_{\tau^*, \eta^*}^{-1} \mathbb{G}_n \phi(x_i; \tau^*, \eta^*) + o_p(1 + \sqrt{n} ||E_* [\phi_{r_n, \eta_n}]||), \quad (C.3)$$

where $V_{\tau^*, \eta^*}$ is a derivative of $\tau \to E_* [\phi(x; \tau, \eta^*)]$ at $\eta^*$.
First, we will see a more specific form of $\tau^*$. The value $\tau^*$ satisfies the equation $E_\tau[\phi(x; \tau, \eta^*)] = 0$. Noting that $\nabla_{\tau^*} \log q(x; \tau) = (1, \nabla_{\theta^*} \log p(x; \theta))$, we can get the form of $c^*$ and $\theta^*$ specified in the statement.

Next, we calculate the derivative $V_{\tau^*, \eta^*}$. The derivative is

$$\nabla_{\tau^*} E_*[\phi(x; \tau, \eta)|_{\tau^*}]$$

$$= E_*[\nabla_{\tau^*} \phi(z; \tau, \eta)|_{\tau^*}]$$

$$= E_* \left[ \left( -1 + \frac{q(x; \tau)}{q(x)} \right) \nabla_{\tau^*} \nabla_{\tau} \log q(x; \tau)|_{\tau^*} \right] - E_* \left[ \frac{q(x; \tau)}{q(x)} \nabla_{\tau} \log q(x; \tau) \nabla_{\tau^*} \log q(x; \tau) \right]|_{\tau^*}$$

$$= -\Omega_1.$$

Next, consider each term in Eq.3. The second term is

$$\Omega_1^{-1} G_n \left[ \left( 1 - \frac{q(x; \tau)}{q(x)} \right) |_{\tau^*, \eta^*} \nabla_{\tau} \log q(x; \tau)|_{\tau^*, \eta^*} \right].$$

The first term is

$$\sqrt{n} \Omega_1^{-1} E_* [\phi|_{\tau^*, \eta^*}] = \sqrt{n} \Omega_1^{-1} E_* [\phi|_{\tau^*, \eta^*}] - \sqrt{n} E_* [\phi|_{\tau^*, \eta^*}]$$

$$= \sqrt{n} \Omega_1^{-1} E_* [\nabla_{\eta^*} \phi(x)|_{\tau^*, \eta^*} (\hat{\eta}_n(x) - \eta^*(x))] + o_p(1)$$

$$= \sqrt{n} \Omega_1^{-1} E_* \left[ \frac{q(x; \tau)}{q(x)} \nabla_{\tau^*} \log q(x; \tau)|_{\tau^*, \eta^*} (\hat{\eta}_n(x) - \eta^*(x)) \right] + o_p(1)$$

$$= \sqrt{n} \Omega_1^{-1} \int q(x; \tau) \nabla_{\eta^*} \log q(x; \tau)(\hat{\eta}_n(x) - \eta^*(x)) d\mu(x) + o_p(1)$$

$$= \Omega_1^{-1} G_n \left[ \frac{q(x; \tau)}{q(x)} |_{\tau^*, \eta^*} \nabla_{\tau} \log q(x; \tau)|_{\tau^*, \eta^*} \right] + o_p(1).$$

Adding the first term and the second term, we get

$$\sqrt{n}(\hat{\tau}_n - \theta^*) = \Omega_1^{-1} G_n \left[ \nabla_{\tau^*} \log q(x; \tau)|_{\tau^*, \eta^*} \right] + o_p(1).$$

Therefore, we conclude that $\sqrt{n}(\hat{\tau}_n - \theta^*)$ converges to the normal distribution $\mathcal{N}(0, \Omega_1^{-1} \Omega_2^{-1} \Omega_1^{-1}).$

**Proof of Theorem 6.**

We calculate matrix, corresponding to $\theta$ term in Lemma C.1. The matrix $\Omega_1$ in Lemma C.1 is equal to the following block matrix:

$$ \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}, $$

where $\Omega_{11} = 1$,

$$\Omega_{21} = E_* \left[ \nabla_{\tau} \log q(x; \tau) \theta(x; \tau) |_{\tau^*, \eta^*} \right].$$

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Table 5: Median squared errors scaled by sample size

|                | Gaussian (n = 1000) | Gamma (n = 1000) | Gaussian (n = 4000) | Gamma (n = 4000) |
|----------------|---------------------|------------------|---------------------|------------------|
| MLE            | 0.24                | 14.3             | 0.26                | 14.8             |
| NCE            | 0.26                | 54.4             | 0.28                | 61.2             |
| s-KL           | 0.39                | 24.6             | 0.29                | 23.3             |
| s-Chi          | 0.35                | 15.1             | 0.43                | 19.6             |
| s-JS           | 0.48                | 16.3             | 0.26                | 18.5             |
| ns-γ           | 0.75                | 14.5             | 0.35                | 36.5             |

and

\[
\Omega_{22} = E_* \left[ \left( 1 - \frac{q(x; \tau)}{\eta^*(x)} \right) \nabla_{\theta^*} \nabla_{\theta^*} \log q(x; \tau) \big| \tau^* \right] + E_* \left[ \frac{q(x; \tau)}{\eta^*(x)} \nabla_{\theta^*} \log q(x; \tau) \nabla_{\theta^*} \log q(x; \tau) \big| \tau^* \right].
\]

From Woodbury formula, the corresponding term to \(\theta\) in \(\Omega_{1m}^{-1}\) is \(\Omega_{1m}^1\) where

\[
\Omega_{1m}^1 = E_* \left[ \left( 1 - \frac{q(x; \tau)}{\eta^*(x)} \right) \nabla_{\theta^*} \nabla_{\theta^*} \log p(x; \theta) \big| \tau^* \right] + E_* \left[ \frac{q(x; \tau)}{\eta^*(x)} \nabla_{\theta^*} \log p(x; \theta) \nabla_{\theta^*} \log p(x; \theta) \big| \tau^* \right]
- E_* \left[ \frac{q(x; \tau)}{\eta^*(x)} \nabla_{\theta^*} \log p(x; \theta) \big| \tau^* \right] E_* \left[ \frac{p(x; \theta)}{\eta^*(x)} \nabla_{\theta^*} \log p(x; \theta) \big| \tau^* \right].
\]

On the other hand, the corresponding part in \(\Omega_{2m}^1\) is \(\Omega_{2m}^1\), where

\[
\Omega_{2m}^1 = \text{Var}[\nabla_{\theta^*} \log p(x; \theta) | \theta^*],
\]

noting that \(\nabla_{\tau^*} \log q(x; \tau) = (1, \nabla_{\theta^*} \log p(x; \theta))\). This concludes the proof.

Note the difference between the normalized case and unnormalized case is that \(\tilde{p}(x; \theta^*)/\eta^*\) is used when the model is normalized; while, \(q(x; \tau^*)/\eta^*\) is used in \(\Omega_{1m}^1\) and \(\Omega_{2m}^1\) when the model is unnormalized.

## D Additional experiment

We perform toy experiments using the Gaussian distribution and gamma distribution. These experiments show that proposed estimator’s performance is almost the same as the MLE. In this section, we used median square errors rather than mean square errors.

Let us consider simple examples when the baseline measure is a Lebesgue measure. Here we define the following two unnormalized models: Gaussian distribution, gamma distribution:
\[ p(x; \theta) = \exp(-\theta x^2), \quad \bar{p}(x; \theta) = \sqrt{\frac{\theta}{\pi}} \exp(-\theta x^2), \]
\[ p(x; \theta) = x^{\theta_1-1} \exp(-\theta_2 x), \quad \bar{p}(x; \theta) = \frac{\theta^{\theta_1} x^{\theta_1-1} \exp(-\theta_2 x)}{\Gamma(\theta_1)}. \]

We write down each corresponding normalized model on the right side. Simulation is replicated for 100 times. Monte Carlo median squared errors were reported in Table D. Note that we use a half-normal distribution for the NCE in the case of the gamma distribution. It is indicated that proposed estimators have the similar performance as MLE. This supports our theoretical result. However, it seems that each proposed estimator has a slightly different performance. One reason is that our analysis does not take high-order terms into account.