SMOOTHNESS IN THE $L_p$ MINKOWSKI PROBLEM
FOR $p < 1$

GABRIELE BIANCHI, KÁROLY J. BÓRÖCZKY, AND ANDREA COLESANTI

Abstract. We discuss the smoothness and strict convexity of the solution of the $L_p$-Minkowski problem when $p < 1$ and the given measure has a positive density function.

1. Introduction

Given a convex body $K$ in the class $K^n_0$ of convex convex sets with non-empty interior in $\mathbb{R}^n$ containing the origin $o$, we write $h_K$ and $S_K$ to denote its support function and its surface area measure, respectively, and for $p \in \mathbb{R}$, $S_{K,p}$ to denote its $L_p$-area measure, where $dS_{K,p} = h_K^{1-p}dS_K$. The $L_p$-area measure defined by Lutwak [34] is a central notion in convexity, see say Barthe, Guédon, Mendelson and Naor [2], Bööröczky, Lutwak, Yang and Zhang [5], Campi and Gronchi [9], Chou [14], Cianchi, Lutwak, Yang and Zhang [16], Gage and Hamilton [18], Haberl and Parapatits [22], Haberl and Schuster [23,24], Haberl, Schuster and Xiao [25], He, Leng and Li [26], Henk and Linke [27], Ludwig [33], Lutwak, Yang and Zhang [36,37], Naor [38], Naor and Romik [41], Paouris [43], Paouris and Werner [44] and Stancu [49].

The $L_p$ Minkowski problem asks for the existence of a convex body $K \in K^n_0$ whose $L_p$ area measure is a given finite Borel measure $\nu$ on $S^{n-1}$. When $p = 1$ this is the classical Minkowski problem solved by Minkowski [39] for polytopes, and by Alexandrov [1] and Fenchel and Jessen [17] in general. The smoothness of the solution was clarified in a series of papers by Nirenberg [42], Cheng and Yau [13], Pogorelov [45] and Caffarelli [7,8]. For $p > 1$ and $p \neq n$, the $L_p$ Minkowski problem has a unique solution according to Chou and Wang [15], Guan and Lin [21] and Hug, Lutwak, Yang and Zhang [29]. The smoothness of the solution is discussed in Chou and Wang [15], Huang and Lu [28] and Lutwak and Oliker [35]. In addition, the case $p < 1$ has been intensively investigated by Bööröczky, Lutwak, Yang and Zhang [4], Bööröczky and Hai T. Trinh [6], Chen [12], Chen, Li and Zhu [10,11], Iviaki [30], Jiang [31], Lu and Wang [32], Lutwak, Yang and Zhang [38], Stancu [47,48] and Zhu [51–54].

The solution of the $L_p$-Minkowski problem may not be unique for $p < 1$ according to Chen, Li and Zhu [11] if $0 < p < 1$, according to Stancu [48] if $p = 0$, and according to Chou and Wang [15] if $p < 0$ small.

In this paper we are interested in this problem when $p < 1$ and $\nu$ is a measure with density with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ on $S^{n-1}$, i.e. in the problem

$$dS_{K,p} = f d\mathcal{H}^{n-1} \quad \text{on} \quad S^{n-1},$$

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where \( f \) is a non-negative function in \( S^{n-1} \).

According to Chou and Wang [15], if \(-n < p < 1\) and \( f \) is bounded from above and below by positive constants, then (1.1) has a solution. More general existence results are provided by the recent works Chen, Li and Zhu [10] if \( p = 0 \), Chen, Li and Zhu [11] if \( 0 < p < 1 \), and Bianchi, Böröczky and Colesanti [3] if \(-n < p < 0\) and \( 0 < p < 1\).

We observe that \( h \) is a non-negative positively \( 1 \)-homogeneous convex function in \( \mathbb{R}^n \) which solves the Monge-Ampère equation

\[
(1.2) \quad h^{1-p} \det(\nabla^2 h + hI) = nf \quad \text{on} \quad S^{n-1}
\]

in the sense of measure if and only if \( h \) is the support function of a convex body \( K \in \mathcal{K}_n^0 \) which is the solution of (1.1) (see Section 2). Naturally, if \( h \) is \( C^2 \), then (1.2) is a proper Monge-Ampère equation. The function \( h \) may vanish somewhere for certain functions \( f \), and when this happen and \( p < 1 \) the equation (1.2) is singular at the zero set of \( h \).

In this paper we study the smoothness and strict convexity of a solution \( K \in \mathcal{K}_n^0 \) of (1.1) assuming \( \tau_2 > f > \tau_1 \) for some constants \( \tau_2 > \tau_1 > 0 \). Concerning these aspects for \( p < 1 \), we summarize the known results in Theorem 1.1, and the new results in Theorem 1.2.

We say that \( x \in \partial K \) is a smooth point if there is a unique tangent hyperplane to \( K \) at \( x \) and that \( K \) is smooth if each \( x \in \partial K \) is smooth (see Section 2 for all definitions). For \( x \in \partial K \), the exterior normal cone at \( z \) is denoted by \( N(K, z) \), and for \( z \in \text{int} K \), we set \( N(K, z) = \{z\} \). Theorem 1.1 (i) and (ii) are essentially due to Caffarelli in [7] (see Theorem 5.5), and Theorem 1.1 (iii) is due to Chou, Wang [15]. If the function \( f \) in (1.1) is \( C^\alpha \) for \( \alpha > 0 \), then Caffarelli [8] proves (iv).

**Theorem 1.1 (Caffarelli, Chou, Wang).** If \( K \in \mathcal{K}_n^0 \) is a solution of (1.1) for \( n \geq 2 \) and \( p < 1 \), and \( f \) is bounded from above and below by positive constants, then the following assertions hold:

(i) The set \( X_0 \) of the points \( x \in \partial K \) with \( N(K, x) \subset N(K, o) \) is closed, any point of \( X = \partial K \setminus X_0 \) is smooth and \( X \) contains no segment.

(ii) If \( o \in \partial K \) is a smooth point, then \( K \) is smooth.

(iii) If \( p \leq 2 - n \), then \( o \in \text{int} K \), and hence \( K \) is smooth and strictly convex.

(iv) If \( o \in \text{int} K \) and the function \( f \) in (1.1) is positive and \( C^\alpha \), for some \( \alpha > 0 \), then \( \partial K \) is \( C^{2,\alpha} \).

Concerning strict convexity Claim (iii) here is optimal because Example 1.2 shows that if \( 2 - n < p < 1 \), then it is possible that \( o \) belongs to the relative interior of an \((n-1)\)-dimensional face of a solution \( K \) of (1.1) where \( f \) is a positive continuous function. Therefore the only question left open is the smoothness of the solution if \( 2 - n < p < 1 \).

We note that if \( p < 1 \) and \( K \) is a solution of (1.2) with \( f \) positive and \( o \in \partial K \), then

\[
(1.3) \quad \dim N(K, o) \leq n - 1.
\]

Therefore Theorem 1.1 (ii) yields that the solution \( K \) is smooth if \( n = 2 \). In general, we have the following partial results.

**Theorem 1.2.** If \( K \in \mathcal{K}_n^0 \) is a solution of (1.1) for \( n \geq 2 \) and \( p < 1 \), and \( f \) is bounded from above and below by positive constants, then the following assertions hold:

(i) If \( n = 2 \), \( n = 3 \) or \( n > 3 \) and \( p < 4 - n \), then \( K \) is smooth.

(ii) If \( \mathcal{H}^{n-1}(X_0) = 0 \) for the \( X_0 \) in Theorem 1.1 (i), then \( K \) is smooth.
Our results differ in some cases from the ones in Chou and Wang [15], possibly because [15] considers the equation
\[ \det(\nabla^2 h + hI) = nh^{p-1} \quad \text{on } S^{n-1} \] instead of \( (1.2) \). In the context of non-negative convex functions being a solution of this last equation is a priori more restrictive than being a solution of \( (1.2) \), even if obviously the two notions coincide where \( h \) is positive (see Section 2 for more on this point). Chou and Wang [15] proves, under our same assumptions on \( f \), the strict convexity of the solution \( h \) of \( (1.3) \), and uses this to prove that the body \( K \) is smooth. In our opinion \( (1.2) \) is the right equation to consider and using it we obtain weaker results.

To give an example of the differences of the two equations, the support function \( h \) of the body \( K \) in Example 1.2 (where \( o \) belongs to the relative interior of an \((n - 1)\)-dimensional face) is a solution of \( (1.2) \) but it is not a solution of \( (1.4) \).

According to Chou and Wang [15] (see also Lemma 3.1 below), the Monge Ampère equation \( (1.2) \) can be transferred to a Monge-Ampère equation
\[ v^{1-p} \det(\nabla^2 v) = g \] for a convex function \( v \) on \( \mathbb{R}^{n-1} \) where \( g \) is a given non-negative function.

The proofs of Claims (i) and (ii) in Theorem 1.1 use as an essential tool a result proved by Caffarelli in [7] regarding smoothness and strict convexity of convex solutions of certain Monge-Ampère equation of type \( (1.5) \) (see Theorem K.6). Proving that \( \partial K \) is \( C^1 \) is equivalent to prove that \( h_K \) is strictly convex, and [7] is the key to prove this property in \( \{ y \in S^{n-1} : h(y) > 0 \} \).

The proof of Claim (i) in Theorem 1.2 is based on the following result for the singular inequality \( v^{1-p} \det \nabla^2 v \geq g \).

**Proposition 1.3.** Let \( \Omega \subset \mathbb{R}^n \) be an open convex set, and let \( v \) be a non-negative convex function in \( \Omega \) with \( S = \{ x \in \Omega : v(x) = 0 \} \). If for \( p < 1 \) and \( \tau > 0 \), \( v \) is the solution of
\[ v^{1-p} \det \nabla^2 v \geq \tau \quad \text{in } \Omega \setminus S \]
in the sense of measure, and \( S \) is \( r \)-dimensional, for \( r \geq 1 \), then \( p \geq -n + 1 + 2r \).

The underlying idea behind the proof of this result is that on the one hand, the graph of \( v \) near \( S \) is close to be ruled, hence the total variation of the derivative is “small”, and on the other hand, the total variation of the derivative is “large” because of the Monge-Ampère inequality.

The inequality \( p \geq -n + 1 + 2r \) in this result is optimal, at least when \( r = 1 \). Indeed Example 3.2 shows that for any \( p > -n + 3 \) there exists a non-negative convex solution of \( (1.6) \) in \( \Omega \) which vanish on the intersection of \( \Omega \) with a line.

Proposition 1.3 yields actually somewhat more than Claim (i) in Theorem 1.2 namely, if \( r \geq 2 \) is an integer, \( p < \min\{1, 2r - n\} \) and \( K \in \mathcal{K}_0^n \) is a solution of \( (1.2) \) with \( o \in \partial K \), then \( \dim N(K, o) < r \). As a consequence, we have the following technical statements about \( K \), where we also use Theorem 1.2 (i) for Claim (ii).

**Corollary 1.4.** If \( p < 1 \) and \( K \in \mathcal{K}_0^n \), \( n \geq 4 \), is a solution of \( (1.1) \) with \( o \in \partial K \), then
(i) \( \dim N(K, o) < \frac{n - 1}{2} \);
(ii) if in addition \( n = 4, 5 \) and \( K \) is not smooth, then \( \dim N(K, o) = 2 \) and \( \dim F(K, u) = n - 1 \) for some \( u \in N(K, o) \).

We review the notation used in this paper in Section 2. Section 3 contains results and examples regarding Monge-Ampère equations in \( \mathbb{R}^n \), namely Proposition 1.3, Example 3.2 and Proposition 5.4. This last result is the key to prove
Theorem 1.2 [1]. In Section 4 we show, for the sake of completeness, how to prove Theorem 1.1 using ideas due to Caffarelli [7, 8] and Chou and Wang [15]. Theorem 1.2 and Corollary 1.4 are proved in Section 5.

2. Notation and preliminaries

As usual, $S^{n-1}$ denotes the unit sphere and $o$ the origin in the Euclidean $n$-space $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then $\langle x, y \rangle$ is the scalar product of $x$ and $y$, while $||x||$ is the euclidean norm of $x$. By $[x, y]$ we denote the segment with endpoint $x$ and $y$.

We write $\mathcal{H}^k$ for $k$-dimensional Hausdorff measure in $\mathbb{R}^n$.

We denote by $\partial E$, $\mathrm{int} E$, $\partial \mathrm{aff} E$, and $1_E$ the boundary, interior, closure, and characteristic function of a set $E$ in $\mathbb{R}^n$, respectively. The symbols $\mathrm{aff} E$ and $\mathrm{lin} E$ denote respectively the affine hull and the linear hull of $E$. The dimension $\text{dim} E$ is the dimension of $\mathrm{aff} E$. With the symbol $E | L$ we denote the orthogonal projection of $E$ on the linear space $L$.

For notions and facts about Monge-Ampère equations, see the survey Trudinger and Wang [50]. Given a function $v$ defined on a subset of $\mathbb{R}^n$, $\nabla v$ and $\nabla^2 v$ denote its gradient and its Hessian, respectively. When $v$ is a convex function defined in an open convex set $\Omega$, the subgradient $\partial v(x)$ of $v$ at $x \in \Omega$ is defined as

$$\partial v(x) = \{ z \in \mathbb{R}^n : v(y) \geq v(x) + \langle z, y-x \rangle \text{ for each } y \in \Omega \},$$

which is a compact convex set. If $\omega \subset \Omega$ is a Borel set, then we denote by $N_v(\omega)$ the image of $\omega$ through the gradient map of $v$, i.e.

$$N_v(\omega) = \bigcup_{x \in \omega} \partial v(x).$$

The associated Monge-Ampère measure is defined by

$$(2.1) \quad \mu_v(\omega) = \mathcal{H}^n \left( N_v(\omega) \right).$$

For $p < 1$ and non-negative $g$ on $\mathbb{R}^n$, we say that the non-negative convex function $v$ satisfies the Monge-Ampère equation

$$v^{1-p} \det(\nabla^2 v) = g$$

in the sense of measure (or in the Alexandrov sense) if

$$v^{1-p} \, d\mu_v = g \, d\mathcal{H}^n.$$

A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The treatise Gardner [19], Gruber [20], Schneider [46] are excellent general references for convex geometry. The function

$$h_K(u) = \max \{ \langle u, y \rangle : y \in K \},$$

for $u \in \mathbb{R}^n$, is the support function of $K$. When it is clear the convex body to which we refer we will drop the subscript $K$ from $h_K$ and write simply $h$. Any convex body $K$ is uniquely determined by its support function.

If $S$ is a convex set in $\mathbb{R}^n$, then a $z \in S$ is an extremal point if $z = \alpha x_1 + (1-\alpha)x_2$ for $x_1, x_2 \in S$ and $\alpha \in (0, 1)$ imply $x_1 = x_2 = z$. We note that if $S$ is compact convex, then it is the convex hull of its extremal points. Next let $C$ be a convex cone; namely, $\alpha_1 u_1 + \alpha_2 u_2 \in C$ if $u_1, u_2 \in C$ and $\alpha_1, \alpha_2 \geq 0$. For $u \in C \setminus \{o\}$, we say that $\sigma = \{ ku : k \geq 0 \}$ is an extremal ray if $\alpha_1 x_1 + \alpha_2 x_2 \in \sigma$ for $x_1, x_2 \in C$ and $\alpha_1, \alpha_2 > 0$ imply $x_1, x_2 \in \sigma$. Now if $C \not= \{o\}$ is a closed convex cone such that the origin is an extremal point of $C$, then $C$ is the convex hull of its extremal rays.

The normal cone of a convex body $K$ at $z \in K$ is defined as

$$N(K, z) = \{ u \in \mathbb{R}^n : \langle u, y \rangle \leq \langle u, z \rangle \text{ for all } y \in K \}$$
where \( N(K, z) = \{ o \} \) if \( z \in \text{int} K \) and \( \dim N(K, z) \geq 1 \) if \( z \in \partial K \). This definition can be written also as

\[
N(K, z) = \{ u \in \mathbb{R}^n : h_K(u) = \langle z, u \rangle \}.
\]

(2.2)

In particular, \( N(K, z) \) is a closed convex cone such that the origin is an extremal point, and

\[
h_K(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 h_K(u_1) + \alpha_2 h_K(u_2) \quad \text{for} \quad u_1, u_2 \in N(K, z) \quad \text{and} \quad \alpha_1, \alpha_2 > 0.
\]

(2.3)

A convex body \( K \) is smooth at \( p \in \partial K \) if \( N(K, p) \) is a ray, and \( K \) is a smooth convex body if each \( p \in \partial K \) is a smooth point. In the latter case, \( \partial K \) is \( C^1 \), which is equivalent to saying that the restriction of \( h_K \) to any hyperplane not containing \( o \) is strictly convex, by (2.3).

We say that \( K \) is strictly convex if \( \partial K \) contains no segment, or equivalently, \( h_K \) is \( C^1 \) on \( \mathbb{R}^n \setminus \{ 0 \} \) (see (2.4)).

The face of \( K \) with outer normal \( u \in \mathbb{R}^n \) is defined as

\[
F(K, u) = \{ z \in K : h_K(u) = \langle z, u \rangle \},
\]

which lies in \( \partial K \) if \( u \neq o \). Schneider [16] Thm. 1.7.4 proves that

\[
\partial h_K(u) = F(K, u).
\]

(2.4)

In particular, for any Borel \( \omega \subset S^{n-1} \), the surface area measure \( S_K \) satisfies

\[
S_K(\omega) = \mathcal{H}^{n-1}(\bigcup_{u \in \omega} F(K, u)) = \mathcal{H}^{n-1}(\bigcup_{u \in \omega} \partial h_K(u)),
\]

and hence \( S_K \) is the analogue of the Monge-Ampère measure for the restriction of \( h_K \) to \( S^{n-1} \).

Given a convex body \( K \) containing \( o \) and \( p < 1 \), let \( S_{K, p} \) denote the \( L_p \) area measure of \( K \); namely,

\[
dS_{K, p} = h_k^{1-p}dS_K.
\]

(2.5)

In particular, for a positive measurable \( f : S^{n-1} \to \mathbb{R} \), \( h_K \) is a solution of (1.2) in the sense of measure if and only if the following conditions (a) and (b) hold:

(a) \( \dim N(K, o) < n \); or equivalently,

\[
\mathcal{H}^{n-1}(\{ y \in S^{n-1} : h_K(y) = 0 \}) = 0,
\]

(2.6)

(b) for each Borel set \( \omega \subset \{ y \in S^{n-1} : h_K(y) > 0 \} \), we have

\[
S_K(\omega) = \int_\omega nf(y)h_K(y)^{p-1}d\mathcal{H}^{n-1}(y).
\]

(2.7)

Let us compare these two conditions to the conditions for \( h_K, K \in K_0^* \), being a solution of (1.4) for \( p < 1 \) and positive \( f \). On the one hand, we have (2.6) and (2.7). However, since the exponent \( p - 1 < 0 \), we have to add the condition

\[
S_K(N(K, o) \cap S^{n-1}) = \mathcal{H}^{n-1}\left( \bigcup \{ F(K, u) : u \in N(K, o) \cap S^{n-1} \} \right) = 0.
\]

(2.8)

In particular, if \( K \in K_0^* \) is a solution of (1.4) for \( p < 1 \) and \( f \) is bounded from below and above by positive constants, then combining (2.8), Theorem 1.1 (i) and Theorem 1.2 (ii) shows that \( K \) is smooth, as it was verified by Chou and Wang [15].
3. Some results on Monge-Ampère equations in Euclidean space

Lemma 3.1 is the tool to transfer the Monge-Ampère equation (1.2) on $S^{n-1}$ to a Euclidean Monge-Ampère equation on $\mathbb{R}^{n-1}$. For $e \in S^{n-1}$, we consider the restriction of a solution $h$ of (1.2) to the hyperplane tangent to $S^{n-1}$ at $e$.

**Lemma 3.1.** If $e \in S^{n-1}$, $h$ is a convex positively 1-homogeneous non-negative function on $\mathbb{R}^n$ that is a solution of (1.2) for $p < 1$ and positive $f$, and $v(y) = h(y + e)$ holds for $v : e^\perp \to \mathbb{R}$, then $v$ satisfies

\[ v^{1-p} \det(\nabla^2 v) = g \quad \text{on } e^\perp \]

where, for $y \in e^\perp$, we have

\[ g(y) = \left( 1 + \|y\|^2 \right)^{\frac{n-1-p}{2}} f \left( \frac{e + y}{\sqrt{1 + \|y\|^2}} \right). \]

**Proof.** Let $h = h_K$ for $K \in K_n^+$, and let

\[ \tilde{S} = \{ u \in S^{n-1} : h_K(u) = 0 \}, \]

which is a possibly empty spherically convex compact set whose spherical dimension is at most $n - 2$, by (2.6). According to (2.7), the Monge-Ampère equation for $h_K$ can be written in the form

\[ dS_K = h_K^{p-1} f \, d\mathcal{H}^{n-1} \quad \text{on } S^{n-1} \setminus \tilde{S}. \]

We consider $\pi : e^\perp \to S^{n-1}$ defined by

\[ \pi(x) = (1 + \|x\|^2)^{-\frac{1}{2}} (x + e), \]

which is induced by the radial projection from the tangent hyperplane $e + e^\perp$ to $S^{n-1}$. Since $\langle \pi(x), e \rangle = (1 + \|x\|^2)^{-\frac{1}{2}}$, the Jacobian of $\pi$ is

\[ \det D\pi(x) = (1 + \|x\|^2)^{-\frac{n}{2}}. \]

For $x \in e^\perp$, (2.4) and writing $h_K$ in terms of an orthonormal basis of $\mathbb{R}^n$ containing $e$, yield that $v$ satisfies

\[ \partial v(x) = \partial h_K(x + e)\mid_{e^\perp} = F(K, x + e)\mid_{e^\perp} = F(K, \pi(x))\mid_{e^\perp}. \]

Let $S = \pi^{-1}(\tilde{S})$. For a Borel set $\omega \subset e^\perp \setminus S$, we have

\[
\mathcal{H}^{n-1}(N_v(\omega)) = \mathcal{H}^{n-1}(\bigcup_{x \in \omega} \partial v(x)) = \mathcal{H}^{n-1}(\bigcup_{u \in \pi(\omega)} \langle F(K, u), e^\perp \rangle) = \int_{\pi(\omega)} \langle u, e \rangle \, dS_K(u) \\
= \int_{\pi(\omega)} \langle u, e \rangle h_K^{p-1}(u) f(u) \, d\mathcal{H}^{n-1}(u) = \int_{\omega} (1 + \|x\|^2)^{-\frac{n-1}{2}} f(\pi(x)) v(x)^{p-1} \, d\mathcal{H}^{n-1}(x)
\]

where we used at the last step that

\[ v(x) = h_K(x + e) = (1 + \|x\|^2)^{\frac{1}{2}} h_K(\pi(x)). \]

In particular, $v$ satisfies the Monge-Ampère type differential equation

\[ \det D^2 v(x) = (1 + \|x\|^2)^{-\frac{n}{2}} f(\pi(x)) v(x)^{p-1} \quad \text{on } e^\perp \setminus S. \]

Since, $\dim S \leq n - 2$ by (1.3), $v$ satisfies (3.1) on $e^\perp$. \hfill \Box

Having Lemma 3.1 at hand showing the need to understand related Monge-Ampère equations in Euclidean spaces, we prove Propositions 1.3 and 3.4 and quote Caffarelli’s Theorem 5.6.
Proof of Proposition 1.3. Up to restricting Ω and changing coordinate system, we may assume, without loss of generality, that \( \Omega = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| < s_1, \|x_2\| < s_2 \} \) and that \( S = \{(x_1, x_2) : x_2 = 0\} \), and \( v \) is continuous on \( \partial \Omega \).

Let \( \alpha = \max_{\Omega} v \) and let us consider the convex body

\[
M = \{(x_1, x_2, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R} : \|x_1\| \leq s_1, \|x_2\| \leq s_2, v(x_1, x_2) \leq y \leq \alpha \}.
\]

For \( t \in (0, s_2/2) \), let

\[
\Omega_t = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| \leq s_1/2, \|x_2\| \leq t \}.
\]

We estimate \( \mathcal{H}^n \left( N_v(\Omega_t \setminus S) \right) \). Let \((x_1, x_2) \in \Omega_t \setminus S \) and let \((z_1, z_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \) belong to \( \partial v(x_1, x_2) \). We prove that

\[
(3.4) \quad \|z_2\| \leq \frac{2\alpha}{s_2} \quad \text{and} \quad \|z_1\| \leq \frac{4\alpha}{s_1s_2} t.
\]

If \( z_2 = 0 \) the first inequality in (3.4) holds true. Assume \( z_2 \neq 0 \). The vector \((z_1, z_2, -1)\) is an exterior normal to \( M \) at \( p = (x_1, x_2, u(x_1, x_2)) \). Since

\[
q_1 = (x_1, x_2 + \frac{s_2 z_2}{\|z_2\|} \alpha) \in M
\]

(because \( \|x_2 + s_2 z_2/(\|z_2\|)\| \leq \|x_2\| + s_2/2 \leq s_2 \)) then \( \langle q_1 - p, (z_1, z_2, -1) \rangle \leq 0 \). This implies

\[
\|z_2\| \leq \frac{2}{s_2} \alpha - v(x_1, x_2)
\]

and the first inequality in (3.4). Again, if \( z_1 = 0 \) then the second inequality (3.4) holds true. Assume \( z_1 \neq 0 \). We have

\[
q_2 = (x_1 + \frac{s_1 z_1}{2\|z_1\|}, 0, u(x_1, x_2)) \in M,
\]

because \( \|x_1 + s_1 z_1/(2\|z_1\|)\| \leq s_1, (x_1 + s_1 z_1/(2\|z_1\|), 0) \in S \) and therefore \( v(x_1, x_2) \geq 0 = v(x_1 + s_1 z_1/(2\|z_1\|), 0) \). The inequality \( \langle q_2 - p, (z_1, z_2, -1) \rangle \leq 0 \) implies the second inequality (3.4).

The inequalities in (3.3) imply

\[
(3.5) \quad \mathcal{H}^n \left( N_v(\Omega_t \setminus S) \right) \leq c t^r,
\]

for a suitable constant \( c \) independent on \( t \).

Now we estimate \( \int_{\Omega_t \setminus S} v(x)^{p-1} \, dx \). The inclusion of the convex hull of \( S \times \{0\} \) and \( \{(\|x_1\| \leq s_1, \|x_2\| \leq s_2, y = \alpha) \in M \) implies that \( v(x_1, x_2) \leq \frac{\alpha}{s_1s_2} \|x_2\| \) for each \((x_1, x_2) \in \Omega_t \) by the convexity of \( v \). Using this estimate it is straightforward to compute that

\[
(3.6) \quad \int_{\Omega_t \setminus S} v(x)^{p-1} \, dx \geq d t^{n+p-r-1},
\]

for a suitable constant \( d \) independent on \( t \). The inequalities (3.5) and (3.6) and the differential inequality satisfied by \( v \) imply, as \( t \to 0^r \),

\[
\alpha t^r \geq \mathcal{H}^n \left( N_v(\Omega_t \setminus S) \right) \geq \int_{\Omega_t \setminus S} \tau v(x)^{p-1} \, dx \geq \tau d t^{n+p-r-1}.
\]

This inequality implies \( p \geq -n + 1 + 2r. \) \( \square \)

Example 3.2. Let us show that for any \( p > -n + 3 \) there exists a non-negative convex solution of (1.6) in \( \Omega = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 \in [-1, 1], \|x_2\| \leq 1 \} \) which vanish on the 1-dimensional space \( S = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_2 = 0\} \).

To prove this let

\[
v(x_1, x_2) = \|x_2\| + f(\|x_2\|) q(x_1)
\]

where \( f(r) = r^\alpha, \) with \( \alpha = (p + n - 1)/2, \) and \( q(x_1) = (1 + \beta x_1^2), \) with \( \beta > 0 \) sufficiently small. Note that \( \alpha > 1 \) exactly when \( p > -n + 3. \)
The function $v$ is invariant with respect to rotations around the line containing $S$. To compute $\det \nabla^2 v$ at an arbitrary point, it suffices to compute it at $(x_1, 0, \ldots, 0, r)$, $r \geq 0$. We get
\[
\begin{align*}
v_{x_1x_1} &= f(r)g''(x_1), \\
v_{x_1x_i} &= 0, & \text{when } 1 < i < n, \\
v_{x_1x_n} &= f'(r)g'(x_1), \\
v_{x_ix_1} &= \frac{1}{r} + \frac{f'(r)}{r}g(x_1), & \text{when } 1 < i < n, \\
v_{x_ix_j} &= 0, & \text{when } i \neq j, (i, j) \neq (1, n), (i, j) \neq (n, 1), \\
v_{x_nx_n} &= f''(r)g(x_1).
\end{align*}
\]
The function $v$ is convex if $\beta$ is sufficiently small. Indeed, the eigenvalues of $\nabla^2 v$ are $\frac{1}{r} + \frac{f'(r)}{r}g(x_1)$, with multiplicity $n - 2$, and those of the matrix
\[
\begin{pmatrix}
f g'' & f' g' \\
f' g' & f'' g
\end{pmatrix}
\]
The determinant of the latter matrix is
\[
2\alpha^2 r^{2(\alpha - 1)} \left( \alpha - 1 - 2(1 + \alpha) \beta x_1^2 \right),
\]
which is positive if $\beta > 0$ is sufficiently small. Thus all eigenvalues of $\nabla^2 v$ are positive.

We get
\[
\det \nabla^2 v = \left( f' g' - f'' g \right) \left( \frac{1}{r} + \frac{f'}{r}g \right)^{n-2}
\]
which has the same order as $r^{2\alpha - n}$ as $r \to 0^+$. Clearly $v$ has order $r$, and $v^{1-p} \det \nabla^2 v$ has order $r^{2\alpha - n + 1-p}$, which is uniformly bounded from above and below for our choice of $\alpha$.

The next statement is a slight revision of Lemmas 3.2 and 3.3 from Trudinger and Wang [50]. We remark that Lemma 3.2 in [50] proves (3.7) with $\sup_{\Omega} |v|$ instead of $|v(0)|$. The inequality (3.7) follows from that and the observation that if $u$ is any convex function in $\Omega$, which vanishes on $\partial \Omega$, and $tE \subset \Omega \subset E$ then $|u(0)| \geq t/(t+1) \sup_{\Omega} |u|$.

**Lemma 3.3.** Let $v$ be a convex function defined on the closure of an open bounded convex set $\Omega \subset \mathbb{R}^n$ satisfying the Monge-Ampere equation
\[
\det \nabla^2 v = \nu
\]
for a finite non-negative measure $\nu$ on $\Omega$, let $v \equiv 0$ on $\partial \Omega$ and let $tE \subset \Omega \subset E$ for $t > 0$ and an origin centered ellipsoid $E$.

(i) If $z \in \Omega$ satisfies $(z + sE) \cap \partial \Omega \neq \emptyset$ for $s > 0$, then
\[
|v(z)| \leq s^{1/n} \frac{1}{\alpha_0} \mathcal{H}^n(\Omega)^{1/n} \nu(\Omega)^{1/n}
\]
for some $c_0 > 0$ depending on $n, t$.

(ii) If $\nu(t\Omega) \geq b \nu(\Omega)$ for $b > 0$, then
\[
|v(0)| \geq c_1 \mathcal{H}^n(\Omega)^{1/n} \nu(\Omega)^{1/n}
\]
for some $c_1 > 0$ depending on $n$, $t$ and $b$.

The proof of Claim (ii) in Theorem 1.2 is based on the following proposition. This proposition is related to a step in the proof of Theorem E (a) in [15], however the argument that we use to prove it is substantially different from that in [15].
Proposition 3.4. Let \( \psi \) be a non-negative convex function defined on the closure of an open convex set \( \Omega \subset \mathbb{R}^n, \ n \geq 2, \) such that \( S = \{ x \in \Omega : \psi(x) = 0 \} \) is non-empty and compact, and \( \nu \) is locally strictly convex on \( \Omega \setminus S. \) Let \( \psi : (0, \infty) \to [0, \infty) \) be monotone decreasing and not identically zero; assume that \( \tau_2 > \tau_1 > 0 \) and \( \nu \) satisfy
\[
\tau_1 \psi(v) \leq \det \nabla^2 \nu \leq \tau_2 \psi(v)
\]
in the sense of measure on \( \Omega \setminus S. \) If \( \dim S \leq n-1 \) and \( \mu_D(S) = 0 \) for the associated Monge-Ampère measure \( \mu_D, \) then \( S \) is a point.

Note that (3.8) means that for each Borel set \( \omega \subset \Omega \setminus S \) we have
\[
\tau_1 \int_\omega \psi(v(x)) \, dx \leq \mu_D(\omega) \leq \tau_2 \int_\omega \psi(v(x)) \, dx,
\]
where \( \mu_D \) has been defined in (2.1).

Proof. We may assume that \( \Omega \) is bounded. We suppose that \( \dim(\text{aff } S) \geq 1, \) and seek a contradiction. We may assume that \( \omega \) is the center of mass of \( S, \) let \( L = \text{aff } S = \text{lin } S \) and let \( e = (o, 1) \in \mathbb{R}^n \times \mathbb{R}. \) Let \( \varepsilon_0 > 0 \) be the minimum of \( v \) on \( \partial \Omega \) and let us consider the convex body
\[
M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : v(x) \leq y \leq \varepsilon_0 \}.
\]
Since \( M \) is bounded and \( v \) is locally strictly convex on \( \Omega \setminus S, \) no supporting hyperplane to \( M \) intersects both \( S \) and the top facet \( F(M, e). \) There exists therefore a constant \( \theta_0 \) such that if a hyperplane \( H \) intersects both \( \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x = o, 0 \leq y \leq \varepsilon_0/2 \} \) and the top face \( \{(x, y) \in M : y = \varepsilon_0 \}, \) then both components of \( M \setminus H \) are of volume at least \( \theta_0. \) We choose \( \varepsilon_1 \in (0, \varepsilon_0/2) \) such that the volume of the cap \( \{(x, y) \in M : y \leq \varepsilon_1 \} \) is less than \( \theta_0. \)

For \( \varepsilon \in (0, \varepsilon_1), \) let \( H_\varepsilon \) be the hyperplane
(i) containing \( L + \varepsilon e \) and
(ii) cutting off the minimal volume from \( M \) (on the side containing the origin)
under condition (i).

We write \( l_\varepsilon \) to denote the linear function on \( \mathbb{R}^n \) whose graph is \( H_\varepsilon, \) and define
\[
\Omega_\varepsilon = \{ x \in \mathbb{R}^n : v(x) < l_\varepsilon(x) \}.
\]
Since \( \Omega \) is bounded, we have \( \text{cl } \Omega_\varepsilon \subset \Omega \) by the choice of \( \varepsilon_1. \)

Let us prove that for each \( w \in L^+ \cap \mathbb{R}^n \) we have
\[
\int_{\Omega_\varepsilon} \langle x, w \rangle \, dx = 0.
\]
Indeed, for \( t \in \mathbb{R} \) with \( |t| \) small, let
\[
F(t) = \int_{\{x \in \Omega_\varepsilon : l_\varepsilon(x) + t(x, w) - v(x) > 0\}} l_\varepsilon(x) + t(x, w) - v(x) \, dx.
\]
By definition of \( l_\varepsilon, \) \( F \) has a local minimum at \( t = 0. \) We have
\[
\frac{F(t) - F(0)}{t} = \int_{\{x \in \Omega_\varepsilon : l_\varepsilon(x) - v(x) > 0\}} \langle x, w \rangle \, dx
\]
\[
+ \int_{\Omega} \left( \frac{l_\varepsilon(x) - v(x)}{t} + \langle x, w \rangle \right) \left( 1_{\{x : l_\varepsilon(x) + t(x, w) - v(x) > 0\}} - 1_{\{x : l_\varepsilon(x) - v(x) > 0\}} \right) \, dx.
\]
The set where \( 1_{\{x : l_\varepsilon(x) + t(x, w) - v(x) > 0\}} - 1_{\{x : l_\varepsilon(x) - v(x) > 0\}} \) differs from 0 is contained in
\[
A_t = \{ x \in \Omega : |l_\varepsilon(x) - v(x)| < |t \langle x, w \rangle| \}.
\]
and there exists $c$ independent on $t$ such that $\mathcal{H}^n(A_t) < ct$ and $\sup_{A_t} |l_{\varepsilon}(x) - v(x)| < ct$. Therefore we have

$$\frac{dF}{dt}(0) = \int_{\Omega_\varepsilon} \langle x, w \rangle \, dx,$$

which proves (3.3). Note that (3.9) implies that the center of mass of $\Omega_\varepsilon$ is contained in $L$. Therefore (see [46, Lemma 2.3.3]) $\Omega_\varepsilon$ contains the reflection of $\Omega_\varepsilon$ with respect to this center of mass, scaled, with respect to the same center of mass, by a factor $1/n$. We deduce from this that

(3.10) \[ S \subset (\Omega_\varepsilon L^\perp) \subset n(\Omega_\varepsilon L^\perp). \]

It follows from the definition of $\Omega_\varepsilon$ that

(3.11) \[ S \subset \Omega_\varepsilon \quad \text{and} \quad \lim_{\varepsilon \to 0^+} (L \cap \Omega_\varepsilon) = S, \]

where the last limit is in the sense of the Hausdorff distance. In particular, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that if $\varepsilon \in (0, \varepsilon_2)$, then

(3.12) \[ L \cap \Omega_\varepsilon \subset 2S. \]

We observe that

(3.13) \[ v(x) - l_{\varepsilon}(x) = \begin{cases} 0 & \text{if } x \in \partial \Omega_\varepsilon \\ -\varepsilon & \text{if } x \in S. \end{cases} \]

We claim that for any $\varepsilon \in (0, \varepsilon_2)$, there exists an ellipsoid $E_\varepsilon$ centered at the origin such that

(3.14) \[ \frac{1}{8n^3} E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon. \]

According to Loewner’s or John’s theorems, there exists an ellipsoid $\tilde{E}$ centered at the origin and $z_1 \in \Omega_\varepsilon$ such that

$$z_1 + \frac{1}{n} \tilde{E} \subset \Omega_\varepsilon \subset z_1 + \tilde{E}.$$ 

It follows from (3.10) that there exists $z_2 \in \Omega_\varepsilon$ such that $z_2|L^\perp = \frac{1}{n} z_1|L^\perp.$ In particular, $y_1 = \frac{1}{1+\varepsilon} z_1 + \frac{n}{1+\varepsilon} z_2 \in \Omega_\varepsilon$ satisfies that $y_1|L^\perp = 0$, or in other words, $y_1 \in L \cap \Omega_\varepsilon$. In addition,

$$y_1 + \frac{1}{2n^2} \tilde{E} \subset \frac{1}{n+1} \left( z_1 + \frac{1}{n} \tilde{E} \right) + \frac{n}{n+1} z_2 \subset \Omega_\varepsilon.$$ 

Let $m = \dim L \leq n - 1$. Since $y_1 \in L \cap \Omega_\varepsilon$ and (3.12) imply $\frac{1}{n} y_1 \in S$, and the origin is the centroid of $S$, we deduce that $y_2 = \frac{1}{2m} y_1 \in S$. As $2m + 1 < 2n$, we have

$$\frac{1}{4n^3} \tilde{E} \subset \frac{1}{2m+1} \left( y_1 + \frac{1}{2n^2} \tilde{E} \right) + \frac{2m}{2m+1} y_2 \subset \Omega_\varepsilon.$$ 

As $\Omega_\varepsilon \subset 2\tilde{E}$ follows from $\sigma \in z_1 + \tilde{E}$, we may choose $E_\varepsilon = 2\tilde{E}$, proving (3.14).

Let us apply Lemma 5.3 to $\Omega_\varepsilon$ and to the function $v - l_\varepsilon$. Let $\nu$ denote the Monge-Ampère measure $\mu_{(v-l_\varepsilon)}$ restricted to $\Omega_\varepsilon$. If $\Omega_0$ is an open set such that $\Omega_\varepsilon \subset \Omega_0 \subset c \Omega_0 \subset \Omega$ then the set $N_\nu(\Omega_0)$ is bounded and this implies

$$\nu(\Omega_\varepsilon) \leq \mathcal{H}^n(N_{(v-l_\varepsilon)}(\Omega_\varepsilon)) \leq \mathcal{H}^n(N_\nu(\Omega_0)) < \infty.$$ 

Let $t = 1/(8n^3)$. Formula (3.14) yields that $tE_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$. 

Let us prove that \( v(\Omega_\varepsilon) \geq b v(\Omega_\varepsilon) \) if \( b = \tau_1 t^n / \tau_2 \). The function \( v \) is convex and attains its minimum at \( o \), thus \( v(x) \geq v(tx) \) for any \( x \in \Omega_\varepsilon \). By this, the monotonicity of \( \psi \), (3.13) and the assumptions on \( S \), we deduce that

\[
\nu(t \Omega_\varepsilon) = \nu(t(\Omega_\varepsilon \setminus S)) \geq \tau_1 \int_{(\Omega_\varepsilon \setminus S)} \psi(v(x)) \, dx = \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(tz)) \, dz \\
\geq \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(z)) \, dz \\
\geq \tau_1 t^n \frac{\nu(\Omega_\varepsilon \setminus S)}{\tau_2} = \frac{\tau_1 t^n}{\tau_2} \nu(\Omega_\varepsilon).
\]

Let \( c_0 \) and \( c_1 \) be the constants appearing in Lemma 3.3. It follows from (3.13) and Lemma 3.3 (i) that if \( \varepsilon \in (0, \varepsilon_2) \), then

\[
(3.15) \quad \varepsilon = |v(o) - l_\varepsilon(o)| \geq c_1 \mathcal{H}^n(\Omega_\varepsilon)^{1/n} \nu(\Omega_\varepsilon)^{1/n}.
\]

On the other hand, let \( s = c_1/(2c_0)^n \). It follows from (3.11), \( \dim S \geq 1 \) and from the fact that the origin is the centroid of \( S \) that there exists \( \varepsilon \in (0, \varepsilon_1) \) small enough, such that \( S \subset L \cap \Omega_\varepsilon \subset (1 + s) S \). In particular, there exists \( z_\varepsilon \in S \) such that \( (z_\varepsilon + sE_\varepsilon) \cap \partial \Omega_\varepsilon \neq \emptyset \). It follows from Lemma 3.3 (i) that

\[
\varepsilon = |v(z_\varepsilon) - l_\varepsilon(z_\varepsilon)| \leq c_0 s \mathcal{H}^n(\Omega_\varepsilon)^{1/n} \nu(\Omega_\varepsilon)^{1/n} = \frac{c_1}{2} \mathcal{H}^n(\Omega_\varepsilon)^{1/n} \nu(\Omega_\varepsilon)^{1/n}.
\]

This contradicts (3.15), and in turn proves Proposition 3.4. \( \square \)

We will actually use the following consequence of Proposition 3.4.

**Corollary 3.5.** Let \( \tau_2 > \tau_1 > 0 \), and let \( g \) be a function defined on an open convex set \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), such that \( \tau_2 > g(x) > \tau_1 \) for \( x \in \Omega \). For \( p < 1 \), let \( v \) be a non-negative convex solution of

\[
v^{1-p} \det \nabla^2 v = g \quad \text{in} \quad \Omega.
\]

If \( S = \{ x \in \Omega : v(x) = 0 \} \) is non-empty, compact and \( \mu_\varepsilon(S) = 0 \), and \( v \) is locally strictly convex on \( \Omega \setminus S \), then \( S \) is a point.

**Proof.** All we have to check that \( \dim S \leq n - 1 \). It follows from the fact that the left hand side of the differential equation is zero on \( S \), while the right hand side is positive. \( \square \)

The following result by Caffarelli (see Theorem 1 and Corollary 1 in [7]), handles the part of the boundary of a convex body \( K \) where the support function at some normal vector is positive.

**Theorem 3.6 (Caffarelli).** Let \( \lambda_2 > \lambda_1 > 0 \), and let \( v \) be a convex function on an open convex set \( \Omega \subset \mathbb{R}^n \) such that

\[
(3.16) \quad \lambda_1 \leq \det \nabla^2 v \leq \lambda_2
\]

in the sense of measure.

(i) If \( v \) is non-negative and \( S = \{ x \in \Omega : v(x) = 0 \} \) is not a point, then \( S \) has no extremal point in \( \Omega \).

(ii) If \( v \) is strictly convex, then \( v \) is \( C^1 \).

We recall that (3.16) is equivalent to saying that for each Borel set \( \omega \subset \Omega \) we have

\[
\lambda_1 \mathcal{H}^n(\omega) \leq \mu_\varepsilon(\omega) \leq \lambda_2 \mathcal{H}^n(\omega),
\]

where \( \mu_\varepsilon \) has been defined in (2.1).
4. Proof of Theorem 1.1

The next lemma provides a tool for the proof of Theorem 1.1 (iii). The same result is also proved in Chou and Wang [15]; we present a short argument for the sake of completeness.

**Lemma 4.1.** For $n \geq 2$ and $p \leq 2 - n$, if $K \in K^n_0$ and there exists $c > 0$ such that $S_{K,p}(\omega) \geq c \mathcal{H}^{n-1}(\omega)$ for any Borel set $\omega \subset S^{n-1}$, then $a \in \text{int } K$.

**Proof.** We suppose that $a \in \partial K$ and seek a contradiction. We choose $e \in N(K,a) \cap S^{n-1}$ such that $\{\lambda e : \lambda \geq 0\}$ is an extremal ray of $N(K,a)$. Let $H^+$ be a closed half space containing $\mathbb{R}e$ on the boundary such that $N(K,a) \cap \text{int } H^+ = \emptyset$. Let $B^n$ be the unit ball centered at the origin $o$, and let

$$V_0 = S^{n-1} \cap (e + B^n) \cap \text{int } H^+.$$ 

It follows by the condition on $S_{K,p}$ that

$$(4.1) \quad c \int_{V_0} h_K(u)^{p-1} \, d\mathcal{H}^{n-1} \leq \int_{V_0} h_K(u)^{p-1} \, dS_{K,p} = S_K(V_0) < \infty.$$

However, since $h_K$ is convex and $h_K(e) = 0$, there exists $c_0 > 0$ such that

$$h_K(x) \leq c_0 \|x - e\| \quad \text{for } x \in e + B^n.$$

We observe that the radial projection of $V_0$ onto the tangent hyperplane $e + e^\perp$ to $S^{n-1}$ at $e$ is $e + V_0'$ for

$$V_0' = e^\perp \cap (\sqrt{3} B^n) \cap \text{int } H^+.$$

If $y \in V_0'$, then $u = (e + y)/\|e + y\|$ verifies $\|u - e\| \geq \|y\|/2$. It follows that

$$\int_{V_0} h_K(u)^{p-1} \, d\mathcal{H}^{n-1} \geq c_0^{p-1} \int_{V_0} \|u - e\|^{p-1} \, d\mathcal{H}^{n-1}(u) \geq \frac{c_0^{p-1}}{2} \int_{V_0'} \frac{\|y\|^{p-1}}{(1 + \|y\|^2)^{n/2}} \, d\mathcal{H}^{n-1}(y) \geq \frac{c_0^{p-1}}{2^{n+1}} \int_{V_0'} \|y\|^{p-1} \, d\mathcal{H}^{n-1}(y) = \infty$$

as $p \leq 2 - n$. This contradicts (4.1), and hence verifies the lemma. \qed

**Proof of Theorem 1.1** Claim (i). For $u_0 \in S^{n-1} \setminus N(K,o)$, we choose a spherically convex open neighbourhood $\Omega_0$ of $u_0$ on $S^{n-1}$ such that for any $u \in \partial \Omega_0$, we have $(u, u_0) > 0$ and $u \notin N(K,o)$. Let $\Omega \subset u_0 \setminus \Omega_0$ be defined in a way such that $u_0 + \Omega$ is the radial image of $\Omega_0$ into $u_0 + u_0 \setminus \Omega_0$, and let $v$ be the function on $\Omega$ defined as in Lemma 3.1 with $h = h_K$. Since $h_K$ is positive and continuous on $\partial \Omega$, we deduce from Lemma 3.1 that there exist $\lambda_2 > \lambda_1 > 0$ depending on $K$, $u_0$ and $\Omega_0$ such that

$$(4.2) \quad \lambda_1 \leq \det \nabla^2 v \leq \lambda_2$$

on $\Omega$.

First we claim that

$$(4.3) \quad \text{if } z \in \partial K \text{ and } N(K,z) \not\subset N(K,o), \text{ then } z \text{ is a smooth point}.$$

We suppose that $\text{dim } N(K,z) \geq 2$, and seek a contradiction. Since $N(K,z)$ is a closed convex cone such that $o$ is an extremal point, the property $N(K,z) \not\subset N(K,o)$ yields an $e \in (N(K,z) \cap S^{n-1}) \setminus N(K,o)$ generating an extremal ray of $N(K,z)$. We apply the construction above for $u_0 = e$. The convexity of $h_K$ and (2.2) imply $h_K(x) \geq (z,x)$ for $x \in \mathbb{R}^n$, with equality if and only if $x \in N(K,z)$. We define $S \subset \Omega$ by $S + e = N(K,z) \cap (\Omega + e)$ and hence $o$ is an extremal point of $S$. It
follows that the function \( \tilde{v} \) defined by \( \tilde{v}(y) = v(y) - \langle z, y + \varepsilon \rangle \) is non-negative on \( \Omega \), satisfies (4.2), and

\[
S = \{ y \in \Omega : \tilde{v}(y) = 0 \}.
\]

These properties contradict Caffarelli’s Theorem 3.6 (i) as \( o \) is an extremal point of \( S \), and in turn we conclude (4.3).

Next we show that

\[
(4.4) \quad h_K \text{ is differentiable at any } u_0 \in S^{n-1} \setminus N(K,o).
\]

We apply again the construction above for \( u_0 \). If \( u \in \Omega_0 \) and \( z \in F(K,u) \) clearly \( K \) is smooth at \( z \) (i.e. \( N(K,z) \) is a ray) by (4.3). Therefore, by (2.3), \( v \) is strictly convex on \( \Omega \) and Caffarelli’s Theorem 3.6 (ii) yields that \( v \) is \( C^1 \) on \( \Omega \). In turn, we conclude (4.4).

In addition, \( F(K,u) \) is a unique smooth point for \( u \in \Omega_0 \) (see (2.3), yielding that \( \Omega_0 = \bigcup \{ F(K,u) : u \in \Omega_0 \} \) is an open subset of \( \partial K \). Therefore \( \Omega_0 \subset X \), any point of \( \Omega_0 \) is smooth (by (2.3)) and \( \Omega_0 \) contains no segment (by (2.4)), completing the proof of Claim (i).

Claim (ii). We suppose that \( o \in \partial K \) is smooth, and that there exists \( z \in \partial K \) such that \( K \) is not smooth at \( z \). Claim (i) yields that \( z \in X_0 \), and hence \( N(K,z) \subset N(K,o) \), which is a contradiction, verifying Claim (ii).

Claim (iii). This is a consequence of Lemma 3.1 and Claim (i).

Claim (iv). This is a consequence of Lemma 3.1 Claim (i) and Caffarelli [8].

**Example 4.2.** If \( n \geq 2 \) and \( p \in (-n+2,1) \), then there exists \( K \in \mathcal{K}^n_0 \) with smooth boundary such that \( o \) lies in the relative interior of a facet of \( \partial K \) and \( dS_{K,p} = f \, d\mathcal{H}^{n-1} \) for a strictly positive continuous \( f : S^{n-1} \to \mathbb{R} \).

Let \( q = (p + n - 1)/(p + n - 2) \). We have \( q > 1 \). Let

\[
g(r) = \begin{cases} 
(r - 1)^q & \text{when } r \geq 1; \\
0 & \text{when } r \in [0,1]; 
\end{cases}
\]

and \( \bar{g}(x_1, \ldots, x_{n-1}) = g(\| (x_1, \ldots, x_{n-1}) \|) \). Let \( K \in \mathcal{K}^n_0 \) be such that \( K \cap \{ x : x_n \leq 1 \} = \{ x : 1 \geq x_n \geq \bar{g}(x_1, \ldots, x_{n-1}) \} \) and \( \partial K \cap \{ x : x_n > 0 \} \) is a \( C^2 \) surface with Gauss curvature positive at every point. Clearly \( K \cap \{ x : x_n = 0 \} \) is a \( (n-1) \)-dimensional face of \( K \) which contains \( o \) in its relative interior and has unit outer normal \( (0, \ldots, 0, -1) \).

To prove that \( dS_{K,p} = f \, d\mathcal{H}^{n-1} \) for a positive continuous \( f : S^{n-1} \to \mathbb{R} \), it suffices to prove that there is a neighborhood of the South pole where \( dS_{K,p} / d\mathcal{H}^{n-1} \) is continuous and bounded from above and below by positive constants. Let \( h \) be the support function of \( K \) and, for \( y \in \mathbb{R}^{n-1} \), let \( v(y) = h(y, -1) \) be the restriction of \( h \) to the hyperplane tangent to \( S^{n-1} \) at the South pole. It suffices to prove that in a neighborhood \( U \) of \( o \), \( v \) satisfies the equation \( v^{1-p} \det \nabla^2 v = G \) with a function \( G \) which is bounded from above and below by positive constants.

If \( y \in U \setminus \{ o \} \) we have

\[
(4.5) \quad v(y) = h(y, -1) = \langle (x', \bar{g}(x')), (y, -1) \rangle \quad \text{where} \quad \nabla \bar{g}(x') = y.
\]

If \( U \) is sufficiently small then \( v(y) \) depends only on \( \| y \| \). Let \( y = (z, 0, \ldots, 0) \), with \( z > 0 \) small and let \( r = 1 + (z/q)^1/(q-1) \). We have

\[
\nabla \bar{g}(r, 0, \ldots, 0) = (z, 0, \ldots, 0)
\]
and \( \text{(4.5)} \) gives
\[
v(z,0,\ldots,0) = v(q(r - 1)^{n-1} - (r - 1)^y)
= z + \frac{q - 1}{q^{n-1} + p}z^{n-1 + p}.
\]
(Note that \( n - 1 + p > 1 \).
Clearly \( v(0,\ldots,0) = h(0,\ldots,0, -1) = 0 \). When \( z > 0 \) we have
\[
v_{y_1y_1} = \frac{q - 1}{q^{n-1} + p}(n - 1 - p)(n - 2 - p)z^{n-3+p} \quad \text{when } i \neq 1
\]
\[
v_{y_iy_i} = 1 \quad \text{when } i = 1
\]
and, as \( z \to 0^+ \)
\[
v(z,0,\ldots,0)^{1-p} \det \nabla^2 v(z,0,\ldots,0) = c + o(1),
\]
for a suitable constant \( c > 0 \). This implies the existence of a function \( G \) positive and continuous on \( U \) such that
\[
\mathcal{H}^{n-1}(N_o(\omega \cap \{ v > 0 \})) = \int_{\omega \cap \{ v > 0 \}} nG(y)v(y)^{p-1} \, dy.
\]
for any Borel set \( \omega \subset U \). To conclude the proof that \( v \) is a solution in the sense of Alexandrov of \( v^{1-p} \det \nabla^2 v = G \) in \( U \) it remains to prove that \( \mathcal{H}^{n-1}\{ y \in U : v(y) = 0 \} = 0 \), but this is obvious since \( \{ y \in U : v(y) = 0 \} = \{ o \} \).

We remark that \( h(x) \) is defined as in Theorem 1.1 (i). The equality on the left in this formula follows by (2.2) and the equality on the right follows by Theorem 1.1 (i). Thus
\[
(5.1) \quad S + e = N(K,o) \cap (e^+ + e),
\]
by \( (2.2) \). If \( K \) is not smooth at \( o \) then \( \dim S \geq 1 \) and, by Proposition 1.3 \( p \geq n - 4 \) (note that here the dimension of the ambient space is \( n - 1 \)). This proves Theorem 1.2 (i).

To prove Theorem 1.2 (ii) we observe that
\[
N_{h\cdot}(e + S) = \bigcup_{u \in N(K,o)} F(K,u) = X_0,
\]
where \( X_0 \) is defined as in Theorem 1.1 (i). The equality on the left in this formula follows by \( (2.4) \) and the equality on the right follows by Theorem 1.1 (i). Thus
\[
N_v(S) = X_0 \{ e \},
\]
and if \( \mathcal{H}^{n-1}(X_0) = 0 \) then \( \mu_v(S) = 0 \). We observe that \( S \) is compact, by \( (5.1) \), that \( v \) is locally strictly convex, by Theorem 1.1 (i), and that \( \dim S \leq n - 2 \), by \( (1.3) \). Hence Theorem 1.2 (ii) follows by Corollary 3.3 and \( (5.1) \). \( \square \)

Proof of Corollary 1.4. Claim \( (i) \) is an immediate consequence of \( (2.2) \), Proposition \( (1.3) \) and Lemma \( 3.1 \). This claim implies that when \( n = 4 \) or \( n = 5 \) and \( K \) is not smooth then \( \dim N(K,o) = 2 \). In this case \( N(K,o) \cap S^{n-1} \) is a closed arc: let \( e_1 \) and \( e_2 \) be its endpoints. If \( u \in N(K,o) \cap S^{n-1} \), \( u \neq e_1 \), \( u \neq e_2 \), then \( F(K,u) \) is contained
Therefore \( \dim F(n-1) = n-1 \) or \( \dim F(n-1) = n-1 \), because otherwise
\[
\bigcup \{ F(K, u) : u \in N(K, o) \cap S^{n-1} \} \neq 0.
\]
which coincides with \( X_0 \) by Theorem 1(i), has \((n-1)\)-dimensional Hausdorff measure equal to zero and, by Theorem 2(ii), \( K \) is smooth. 

\[\Box\]

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SMOOTHNESS IN THE $L_p$ MINKOWSKI PROBLEM

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Rétanoda u. 13-15, H-1053 Budapest, Hungary, and Department of Mathematics, Central European University, Nador u. 9, H-1051, Budapest, Hungary
E-mail address: boroczky.karoly.j@renyi.mta.hu

Dipartimento di Matematica e Informatica “U. Dini”, Università di Firenze, Viale Morgagni 67/A, Firenze, Italy I-50134
E-mail address: andrea.colesanti@unifi.it