Interaction of $d=2$ $c=1$ Discrete States from String Field Theory

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Abstract

Starting from string field theory for 2d gravity coupled to $c=1$ matter we analyze the off-shell tree amplitudes of discrete states. The amplitudes exhibit the pole structure and we use the off-shell calculus to extract the residues and prove that just the residues are constrained by the Ward Identities. The residues generate a simple effective action.

1 Introduction

Two-dimensional gravity coupled to a $c = 1$ matter is still a subject of intense study. The reason is that it admits non-perturbative investigations as well as it has a reach symmetry structure. There are two ways to describe the model. In the first approach one deals with a matrix model and in the second one with the continuum Liouville theory.

The characteristic property of the model is an appearance of discrete states (DS) which together with tachyon states exhaust the spectrum of physical states. This fact was firstly recognized in the matrix model description [1]. The appearance of these states is rather transparent from the Liouville point of view when the model looks like a 2d string [2, 11]. Since we deal with a string one can expect an infinite number of higher level states. However, since we are dealing with a 2-dimensional gauge invariant theory we also can expect that all these states can be eliminated by a gauge transformation. In fact due to a nontrivial background not all the states are purely gauge. Namely, the states with special fixed values of momentum survive in the physical spectrum [2, 4]. Or in other words, there exist non-trivial cohomologies of the corresponding BRST charge [3]-[5].

This specific nature of the spectrum provides an enormous symmetry group [6, 7]. The discrete states are spin one currents and they generate a $W_\infty$ current algebra. There

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are also spin zero states generating a ground ring \( \mathbb{R} \). It is natural to conjecture that the existence of this algebra leads to some Ward Identities (WI) for scattering amplitudes. This problem was addressed in [10] where these identities were derived for amplitudes defined by formal CFT expressions. However, one can convince that some of the amplitudes are ill defined [11]. The origin of the divergences is the specific kinematics of the ”particles” living with fixed momenta only. Two-vectors of energy-momentum form the two-dimensional lattice and when summing up a pair of discrete momenta (according to energy conservation law with the background charge) one gets the momentum of a DS again. So, all the internal lines of tree-level amplitudes describe a propagation of real ”particles”. Hence, to obtain the amplitude one has to deal with actual poles and a meaning of the WI is rather problematic.

A natural way to expose the pole structure of a scattering amplitude is to go off-shell. A consistent off-shell description is provided by string field theory (SFT).

SFT for 2d string in a nontrivial background has been proposed in [12]. As usual, the interest in SFT was motivated by the fact that SFT is supposed to be background independent and to give a framework for discussing nonperturbative effects. Now we observe that SFT appears to be a suitable tool to investigate the symmetries of correlation functions.

In this paper we are going to use the machinery of SFT to analyse the specific singularities of Feynman string diagrams of 2d string. The main advantage of SFT for the problem in question is the fact that it brings regular way to obtain off-shell amplitudes. A given Feynman tree diagram of SFT defines a meromorphic function of kinematical invariants. In the usual case to obtain on-shell tree amplitudes one has to restrict this off-shell function on some hyperplane. In our case, since the energies and momenta of DS are components of fixed 2d vectors, to go on-shell one has to sit on a fixed point and some of these points are nothing but the actual poles. Just these poles lead to the above mentioned divergences in the amplitudes of CFT.

Our strategy will be the following one. We start with the general expression for off-shell amplitudes obtained by slightly modifying the off-shell calculation for the critical string field theory [13]. We present the explicit formula for the off-shell four-point Feynman diagrams for arbitrary discrete states. We investigate the behaviour of this formula near the mass-shell. For any two given external states this point is uniquely defined by the kinematics. As one expects, this point manifests the potential pole of the amplitude and we give the rule to calculate the residue. In contrast to the usual case, these residues will be constants since the value of \( t \) is also fixed. In accordance with the unitarity condition this residue must be a product of a pair of two three-point amplitudes. The set of these three-point constants can be used to write down an effective action such that the residues can be obtained from it. To reflect the fact that we deal with \(+\) or \(−\) states numbered by pairs of integers (half-integers) \( m, n \) the effective action is a function of two sets of variables \( \Phi \) and \( \Phi^\dagger \) both of which are numbered by the pairs \( (n, m) \). This effective action also seems to give the explicit expressions for the residues of the leading poles of \( n \)-point off-shell tree amplitudes. This conjecture seems to be natural as the leading singularity is fulfilled by real ”particles” in intermediate states, that brings the expression for the residue as a product of corresponding three-point amplitudes. Turning back to the issue of the WI we realize that the proper objects to be constrained by the WI are just the residues and not the correlation functions. These identities are the origin of the symmetries of the effective action.
The paper is organised as follows. In section 2 we summarize the main points of the model. In section 3 we specify the scheme of SFT off-shell calculations. In section 4 we calculate the off-shell four-point amplitudes. In section 5 we speculate on a problem of effective action and derive the WI.

2 The model

In this section we shall recall the model and briefly discuss some specific features which are essential for our investigation.

The classical action of the model looks like an action of $d = 2$ string in the Euclidean space-time with a non-zero background charge for the Liouville mode $\phi$

$$S = \frac{1}{8\pi} \int d^2 \xi \sqrt{\hat{g}}(\hat{g}^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\mu - Q \hat{R} \phi + \text{ghosts})$$  \(1\)

The value of the background charge $Q$ is fixed by the usual requirement of vanishing of the conformal anomaly for the total matter $X_\mu = (x, \phi)$ and ghost $(b, c)$ system:

$$1 + c\phi = 26, \quad c\phi = 1 + 3Q^2, \quad \rightarrow \quad Q = 2\sqrt{2}.$$  \(2\)

The non-zero $Q$ together with two-dimensional kinematics make the spectrum of the model somewhat unusual, as was discovered in [2, 3]. The straightforward way to see it is to analyze the standard Virasoro constraints

$$L_0 \mid \text{phys. state} \rangle = 0, \quad L_m \mid \text{phys. state} \rangle = 0$$

for $m > 0$ in the light-cone parametrization

$$L_m = P^+(m)\alpha^-_m + P^-(m)\alpha^+_m + \sum_{n \neq 0, m} \alpha^+_m \alpha^-_n,$$

$$L_0 = k^+ k^- + k^+ - k^- + \hat{N};$$  \(3\)

where

$$P^\pm(m) = k^\pm + (m + 1), \quad k^\pm = \frac{1}{\sqrt{2}}(k_1 \pm ik_2), \quad \alpha^\pm_m = \frac{1}{\sqrt{2}}(\alpha_{m1} \pm i\alpha_{m2}).$$  \(4\)

The constraints $L_m = 0$ becomes drastically simplified if one analyze the states created only by $\alpha^+_m (\alpha^-_m)$ modes. Let us take a state with $\hat{N} = m_0$

$$\mid \text{phys. state} \rangle = (\alpha^+_{m_0} + \sum_{m_1 + m_2 = m_0} C_{m_1 m_2} \alpha^+_{m_1} \alpha^-_{m_2} + ...) \mid k^+, k^- \rangle,$$  \(5\)

then

$$L_{m_0} \mid \text{phys. state} \rangle = [P^+(m_0)\alpha^-_{m_0} + \sum_{n > m_0} \alpha^+_m \alpha^-_n] \mid \text{phys. state} \rangle =$$

$$= [P^+(m_0) + \text{oscillator terms}] \mid k^+, k^- \rangle = 0$$  \(6\)

and the necessary condition for a state to be physical for some specific $m = m_0$ is $P^+(m_0) = 0$ which fixes the value of $k^+$ to be

$$k^+ = m_0 + 1$$  \(7\)
The value of $k^-$ is fixed by the mass-shell condition 
\[(L_0 - 1)|\text{phys. state}\rangle = 0: \quad k^- = -2\]

The example of such a physical state, the "vector" one, is

\[|\text{phys. state}\rangle = \alpha^+_1|2, -2\rangle. \quad (8)\]

In a more general setting, the discrete states appear as a non-trivial cohomology of BRST charge $Q_{BRST}$: 
\[Q_{BRST}|\psi\rangle = 0, \quad |\psi\rangle \neq Q_{BRST}|\lambda\rangle, \quad \text{and they can be classified according to their ghost number. The nontrivial cohomology can be found for ghost number 0, ..., 3 and it can be proved that not only the states with fixed momenta are in the spectrum but the whole spectrum is exhausted by these states plus the tachyon with non-discrete momentum.}

The physical states in the total (matter + ghosts) Fock space can be explicitly described in two ways: by using Shur polynomials \cite{9} or in terms of $SU(2)$ raising and lowering operators \cite{3, 7}. For our purposes the second description is more convenient.

The universal formula for on-shell physical states in terms of conformal fields $Y_r$ reads

\[Y_{J,n}^\pm = c W_{J,n}^\pm, \quad (9)\]

where

\[W_{J,n}^\pm = \sqrt{\frac{(2J - n)!}{n!(2J)!}} [H^-, \ldots [H^-, W_{J,J}^\pm], \ldots], \quad (10)\]

and

\[H^- = \oint \frac{dz}{2\pi i} e^{-\sqrt{2}X(z)}, \quad W_{J,J}^\pm = e^{\sqrt{2}iJX} e^{\sqrt{2}(1 \mp J)\phi}. \quad (11)\]

$J$ is a positive integer or a half integer. For the sake of simplicity we shall assume the "space" dimension $x$ to be compactified, in this case the tachyon state will have a discrete momentum also. The momenta of discrete states read \cite{4} - \cite{6}:

\[k_\mu = (p, -i\varepsilon) = \sqrt{2}(n, -i(1 \mp J)) \quad (12)\]

where

\[J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots; \quad n = -J, -J + 1, \ldots, J.\]

3 The Scheme of the Off-shell Calculations

The strategy and all the necessary tools for off-shell calculations in SFT for usual critical strings were developed in a series of papers by Samuel et al. The interested reader can find the details in the original manuscripts \cite{13}. Here we adopt this calculus for the case of the critical string in a non-trivial background.

The action for 2d SFT is taken to be of the usual Witten type and the amplitudes are computed using perturbation theory. With each Feynman graph is associated a string configuration $R_\tau$. External strings are semi-infinite rectangular strips of width $\pi$. The $i$-th internal string propagator is a strip of length $\tau_i$ and width $\pi$. The interaction glues the strips in pairwise manner.

The external states located at points $w_i$ are represented by vertex operators $Y_r = c W_r$. For any fixed set of the off-shell states $Y_r$-s it is possible to obtain the the amplitude but, to treat the states of a general form by using off-shell conformal methods one has to
guarantee that $Y_r$-s are conformal fields of definite conformal dimension $\Delta_r$. So we have to specify the proper rule for going off-shell. Taking $Y_{J,n}$ in the form

$$Y_{J,n}^\pm = cV_{J,n}e^{\sqrt{2}(1 \mp J)\phi}$$

we can relax the mass-shell condition by simply substituting: $\sqrt{2}(1 \mp J)\phi \rightarrow \varepsilon\phi$ for some parameter $\varepsilon$. The fields $Y_{J,n}^\varepsilon = cV_{J,n}e^{\varepsilon\phi}$ will be conformal fields with conformal dimension

$$\Delta(\varepsilon) = -1 + J^2 - \frac{1}{2}\varepsilon^2 + \sqrt{2}\varepsilon$$

which goes to zero on-shell ($\varepsilon = \sqrt{2}(1 \mp J)$)

The contribution to a particular Feynman graph is

$$A_N = (\prod_{i=1}^{N-3} \int_0^\infty d\tau_i < Y_1(w_1)Y_2(w_2) \int dw'_1 b(w'_1) \ldots Y_N(w_N) >_{R_r}, \quad (15)$$

where the correlation function is taken on string configuration.

Although the world-sheet action for the first-quantized string is quadratic the complicated geometry of the string configuration makes the computation non-trivial. The key idea is to conformally map the string configuration to the upper half-plane. The map $\rho(z)$ which takes the half-plane into the string configuration for the $N$-point tree diagram was given by Giddings and Martinec [14].

The correlation function is a product of two factors $\langle \ldots \rangle_{R_{bc}} = \langle \ldots \rangle_{R_{x-\phi}}^{x-\phi} \langle \ldots \rangle_{R_{bc}}$ which can be calculated separately. The on-shell condition is irrelevant for the $b-c$ factor, hence one obtains the standard result

$$< \ldots >_{R_{bc}}^{x-\phi} \prod_{i=1}^{N-3} d\tau_i = (z_1 - z_2)(z_1 - z_3)\ldots (z_2 - z_N) \prod_{r=1}^{N} \left( \frac{d\rho}{dz} e^{-\rho} \right)_{z_r}^{-1} \prod_{i=1}^{N-3} dz_i, \quad (16)$$

where $z_r$ are the asymptotic positions on the real axis of $N$ external states. The $x-\phi$ factor transforms under the mapping $\rho$ into:

$$< \ldots >_{R_{x-\phi}}^{x-\phi} = \prod_{r=1}^{N} \left( \frac{d\rho}{dz} e^{-\rho} \right)_{z_r}^{-\Delta_r+1} \prod_{r=1}^{N} \langle W_r(z_r) \rangle. \quad (17)$$

Note, that for on-shell states all the $\Delta$-s are equal to zero. Collecting the factors (16) and (17) together one achieves for the amplitude (15)

$$A_N = \prod_{i=1}^{N-3} \int d\tau_i < Y_1(z_1)Y_2(z_2)W_3(z_3) \ldots W_{N-1}(z_{N-1})Y_N(z_N) >_{z-plane} \prod_{r=1}^{N} e^{N_{rr}^{00}\Delta_r}. \quad (18)$$

where $N_{rr}^{00}$ are the coefficients of the Neuman functions.

### 4 Four-Point Off-Shell Amplitude.

As it has been mentioned in the Introduction, because of the specific nature of allowed momenta one falls into a puzzling situation as soon as the calculation of amplitudes for
DS is performed. Namely, let us consider, for example, an s-channel four point amplitude with the following values of particle’s momenta $k_1 = (\sqrt{2},0)$, $k_2 = (-\sqrt{2},0)$, $k_3 = (\sqrt{2},0)$, $k_4 = (-\sqrt{2},-i2\sqrt{2})$ which satisfy the shifted energy-momentum conservation law in presence of the background charge: $\sum p_i = 0, \sum \varepsilon_i = Q$. It appears to be natural to define the invariant variable $s$ in the presence of the background charge by the following formula

$$s = (k_1 + k_2)^2 - (\varepsilon_1 + \varepsilon_2)^2 + 2\sqrt{2}(\varepsilon_1 + \varepsilon_2)$$  \hspace{1cm} (19)

Hence for this configuration the value of $s$ is also fixed to be $s = 0$. We are sitting on the pole, the amplitude is obviously divergent and this is the typical case for the theory.

In this section we derive the expression for the residues of the intermediate state poles for general s-channel four-point amplitude:

$$A_4 = \int dz_3 \langle Y_1(z_1)Y_2(z_2)W_3(z_3)Y_4(z_4)\rangle \left(\prod_{r=1}^{4} e^{N_{r0}^{0\alpha} \Delta_r}\right).$$ \hspace{1cm} (20)

Here we have omitted the $r, n$ indices for $Y-s$ and labelled them by the numbers of states. The positions of the points $z_i$ on the real axis are fixed by the parameter $\alpha$ of the Giddings mapping $\rho(z)$ as follows $z_2 = -z_3 = \alpha$, $z_1 = -z_4 = 1/\alpha$.

To convert the variable $\alpha$ to the Koba-Nielsen variable $x$ one maps the upper half-plane into itself by using $SL(2, R)$ mapping

$$x = \frac{\alpha^2 - 1}{\alpha^2 + 1 \alpha^2 - 1}$$ \hspace{1cm} (21)

The three points $z_1$, $z_2$, $z_4$ are mapped to $\infty$, 1, 0 as it must be and the final result of straightforward but tedious calculations, reads

$$A_4^s = \lim_{x_1 \rightarrow \infty} \int_{1/2}^{1} dx \langle Y_1(x_1)Y_2(1)W_3(x)Y_4(0)\rangle \left(\prod_{r=1}^{4} \langle N_{r0}^{0\alpha} \Delta_r \rangle\right).$$ \hspace{1cm} (22)

x^{2\Delta_1 + \frac{1}{2}(\Delta_3+\Delta_4-\Delta_1-\Delta_2)}(1-x)^{\Delta_2+\Delta_3}\left[\frac{\alpha(x)}{2}\right]^{\sum_{r=1}^{4} \Delta_r}.

Here we exploited the fact that $N_{00}^{11} = N_{00}^{44} = \ln \kappa/\alpha$, $N_{00}^{22} = N_{00}^{33} = \ln \kappa \alpha$, where $\kappa$ is a regular function in the region $1/2 \leq x \leq 1$, $(\sqrt{2} - 1 \geq \alpha \geq 0)$, nonzero at the point $x = 1$.

To analyze the pole structure of (22) note that the poles correspond to the divergences of the integral on the upper limit or, in other words, to terms in the integrand having factors of $1-x$ to negative powers. Such factors come from the explicit factor of $(1-x)^{\Delta_2+\Delta_3}$ and from possible contractions of the pair $Y_2(1)W_3(x)$ . One can present the OPE for these two operators in the form

$$Y_2(1)W_3(x) \sim (1-x)^{2n_2n_3-\varepsilon_2\varepsilon_3}(1-x)^R,$$ \hspace{1cm} (23)

where the first factor originates from the product of the exponents and the second one - from the contraction of the Shur polynomials. It is important to stress that $R$ is an integer. Hence, we can present the amplitude in the form

$$A_4^s = \int_{1/2}^{1} dx (1-x)^{2n_2n_3-\varepsilon_2\varepsilon_3+\Delta_2+\Delta_3-R}F(x),$$ \hspace{1cm} (24)

where $F(x) = F(x; s_r, n_r, \varepsilon_r)$ is a regular function at $x \sim 1$. 

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Moreover, according to the adopted definition of the invariant variable \( s \) we have
\[
2n_2n_3 - \varepsilon_2\varepsilon_3 + \Delta_2 + \Delta_3 = \frac{1}{2}s - 2, \tag{25}
\]
so, expanding \( F(x) \) in the Taylor series
\[
F(x) = \sum_{i=1}^{\infty} F_i(1 - x)^i
\]
we get the final expression for \( A_4^{(s)} \)
\[
A_4^{(s)} = \sum_{i=1}^{\infty} \frac{1}{2}s - R - 1 + i \left( \frac{1}{2} \right)^{s-n-1+i} F_i. \tag{26}
\]

Now, we turn to extract some information from eq. (26). Note, that for every given on-shell states 2 and 3 \( s \) is an even number:
\[
s = 2[(n_2 + n_3)^2 - (J_2 + J_3)^2 + 2(J_2 + J_3) + 4] = s_0 \tag{27}
\]
Hence, there is one and only one pole for \( i_0 = R + 1 - \frac{1}{2}s_0 \) in \( A_4^{(s)} \) and the \( F^{\text{on-shell}}_{i_0} \) is the residue (maybe equal to zero) in this pole. Moreover, to find the residue there is no need to develop the off-shell calculations, it is sufficient to extract from the on-shell correlation function
\[
\langle Y_1(\infty)Y_2(1)W_3(x)Y_4(0) \rangle \tag{28}
\]
the coefficient of the term \( \sim (1 - x)^{-1} \). It will be precisely the \( F^{\text{on-shell}}_{i_0} \).

As it is known the total s,t-channel amplitude is a sum of two Feynman diagrams of SFT. The first diagram is presented below and the second one is obtained by the cyclic permutation of the external states: (1234) \( \rightarrow \) (4123). The residue of the corresponding t-pole obviously will be given by the corresponding term in the expansion of eq. (28) with permuted labels of vertex operators.

5 The Effective Lagrangian

As it was proposed by Klebanov and Polyakov \[7\] the model can be described by its effective action. In this section we are going to connect the concept of effective action with the scattering amplitudes.

The above discussion of the four-point amplitude (see also \[11\]) shows that it is natural to associate an effective action with the leading singularities of the amplitudes. Indeed, using the OPE for \( Y_2(1) \) and \( W_3(x) \)
\[
Y_{J_2,n_2}^{+}(1)W_{J_3,n_3}^{+}(x) = \frac{1}{1-x} f_{J_2+n_2,J_3+n_3}^{J_2+J_3-1,n_2+n_3} Y_{J_2+J_3-1,n_2+n_3}^{+}(1) + \text{regular terms} \tag{29}
\]
the expression for the residue can be presented in the form
\[
\tilde{A}_4 = \text{Res} A_4 \big|_{s=s_0} = f_{J_2,n_2,J_3,n_3}^{J_1,n_1,J_2+J_3,J_3+J_2} f_{J_1+J_2,n_1+n_2}^{J_2,n_2,J_1,n_1} \tag{30}
\]
the structure constants of OPE were calculated in \[8, 7\]
\[
f_{J_1,n_1,J_2,n_2}^{J_3,n_3} = \delta_{J_3,J_1+J_2-1} \delta_{n_3,n_1+n_2} f_{J_1,n_1,J_4,n_4} =
\]
Here $\tilde{N}(Jm)$ is a normalisation factor.

This result (eq. $30$) can be reproduced by a simple effective field theory which describes an interaction of two independent fields $\Phi_{J,n}$ and $\bar{\Phi}_{J,n}$ with indices $J \geq 0$, $-J \leq n \leq J$ having a trivial propagator. The Lagrangian has the form

$$\mathcal{L}(\Phi, \bar{\Phi}) = \sum_a \bar{\Phi}_a \Phi^a + g \sum_{a,b,c} \Phi^a \Phi^b \bar{\Phi}_c f^c_{ab}.$$  

(32)

where $\Phi^a = \Phi_{s,n}$ and $\bar{\Phi}_a = \bar{\Phi}_{s,-n}$ and

$$f^c_{ab} = <Y^a_x Y^b_y Y^c_z>.$$  

(33)

It seems natural that the Lagrangian (32) correctly reproduces the coefficients of the leading singularities for all the n-point tree amplitudes by a simple one to one correspondence between graphs of the SFT and the ones of the effective theory. There is no rigorous proof at our disposal but there are some indirect arguments in favour of this hypothesis. In the previous paper [11] it was shown that if a special regularization is adopted then the residues of the leading poles (with respect to the regularization parameter) are given by the effective Lagrangian (32). Following this interpretation of the effective theory we conclude that there are no reasons to add to the effective action the quartic and higher order terms if we deal with the tree amplitudes.

Now, let us discuss the issue of Ward Identities. The symmetry properties of on-shell amplitudes were considered in [10]. The identities were derived by using the usual contour deformation trick. Namely, inserting the zero-dimensional charge

$$Q^\pm_{s,m} = \oint \frac{dz}{2\pi i} W^\pm_{s,m}(z),$$  

(34)

in the correlation function, changing the contour of integration and using the OPE’s of $W^+_{s,m}$ and $Y^+_{s,m}$ one gets the WI. However this consideration is rather formal, since some of amplitudes are ill defined, as it was explained above. The well defined objects are off-shell amplitudes and it seems natural to search for some WI for these amplitudes. However there is an essential obstacle to do this as it is impossible to apply the contour deformation argument to off-shell amplitudes. This originates from the fact that it is impossible to define an action of the non-local operator (34) on conformal fields of non-integer or non-half-integer dimension. So, WI for off-shell amplitudes of SFT cannot be derived in this way.

To solve this puzzle let us recall that we have the expression for residues of amplitudes in terms of on-shell correlations functions in eq. (28). Inserting the operator (34) in eq. (28) we get the relation between corresponding correlations functions

$$\tilde{f}_{sm,s_1m_1} (Y^+_{s_1+s-1,m_1+m} (\infty) Y^+_{s_2,m_2} (1) W^+_{s_3,m_3} (x) Y^-_{s_4,m_4} (0)) \pm$$

$$\tilde{f}_{sm,s_2m_2} (Y^+_{s_1,m_1} (\infty) Y^+_{s_2+s-1,m_2+m} (1) W^+_{s_3,m_3} (x) Y^-_{s_4,m_4} (0)) \pm$$

$$\tilde{f}_{sm,s_3m_3} (Y^+_{s_1,m_1} (\infty) Y^+_{s_2,m_2} (1) W^+_{s_3+s-1,m_3+m} (x) Y^-_{s_4,m_4} (0)) \pm$$

$$\tilde{f}_{sm,s_4m_4} (Y^+_{s_1,m_1} (\infty) Y^+_{s_2,m_2} (1) W^+_{s_3,m_3} (x) Y^-_{s_4+s-1,m_4+m} (0)) = 0$$  

(35)
It is evident that this relation is also true for the terms proportional to \((1 - x)^{-1}\) in expansion of eq.(33) which define the corresponding residues. So, the proper objects to be constrained by the WI are the residues and the identities read

\[
\tilde{f}_{sm,s_1 m_1} \tilde{A}((s_1 + s + 1, m_1 + m)^+, (s_2, m_2)^+, (s_3, m_3)^+, (s_4, m_4)^-) \pm \\
\pm \tilde{f}_{sm,s_2 m_2} \tilde{A}((s_1, m_1)^+, (s_2 + s + 1, m_2 + m)^+, (s_3, m_3)^+, (s_4, m_4)^-) \pm \\
\pm \tilde{f}_{sm,s_3 m_3} \tilde{A}((s_1, m_1)^+, (s_2, m_2)^+, (s_3 + s + 1, m_3 + m)^+, (s_4, m_4)^-) \pm \\
\pm \tilde{f}_{sm,s_4 m_4} \tilde{A}((s_1, m_1)^+, (s_2, m_2)^+, (s_3, m_3)^+, (s_4 - s + 1, m_4 + m)^-) = 0. \tag{36}
\]

Coming back to the effective action (32) it is evident that relations (36) should represent a symmetry of the effective Lagrangian. This symmetry does exist. Namely, the Lagrangian (32) is invariant under infinite number of infinitesimal transformations

\[
\delta_c \Phi^a = f^a_{cb} \Phi^b, \quad \delta_c \bar{\Phi}_a = f^{b}_{ac} \bar{\Phi}_b. \tag{37}
\]

6 Concluding Remarks

In summary, we have argued that off-shell scattering amplitudes of 2d open string discrete states exhibit the pole structure and the residues are described by the effective Lagrangian with rich symmetry structure. We have presented the rigorous proof of these statements for four-point open string tree amplitudes. The N-point case needs more detailed analysis.

Our effective action contains only cubic terms. This does not agree with the Klebanov and Polyakov conjecture \cite{7} that the effective action is nonpolynomial. However, it is possible that loops will destroy its cubic character.

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