SLOW ENTROPY FOR ABELIAN ACTIONS

CHANGGUANG DONG

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Abstract

We calculate slow entropy type invariant introduced by A. Katok and J.-P. Thouvenot in [5] for higher rank smooth abelian actions for two leading cases: when the invariant measure is absolutely continuous and when it is hyperbolic. As a by-product, we generalize Brin-Katok local entropy Theorem to the abelian action for the above two cases. We also prove that, for abelian actions, the transversal Hausdorff dimensions are universal, i.e. dependent on the action but not on any individual element of the action.

1 Introduction and Main Results

Metric entropy is an important numerical invariant in dynamical systems. It reflects exponential orbit growth rate of a system in measure theoretic sense, which is well studied in smooth ergodic theory for $\mathbb{Z}$- and $\mathbb{R}$- actions. However, if we consider higher rank abelian actions, and want to measure the complexity of such system, the direct extension of metric entropy fails to be useful. In most cases, it is equal to zero unless some or all transformations have infinite metric entropy, see [1], [2] and [3]. So, there is a need to find some other entropy type invariants.

One natural way is to change the normalization and measure exponential growth rate against the radius of the ball in the acting group instead of the volume of the ball. Very similar to Katok’s definition in [4], slow entropy type invariants for abelian actions have been defined in [5], and further studied by A. Katok, S. Katok and F. R. Hertz in [1]. In the latter paper, they consider the case of Cartan actions on the torus and find some connection with Fried average entropy (see [1] and the references therein). From now on, we will speak of the slow entropy for abelian actions defined in [5] as simply the slow entropy.

In this paper, we consider this slow entropy for abelian actions of more general type. An explicit formula is given for that, which is our main result. Before that, let’s make some basic settings throughout this paper. Let $(M, d)$ be a compact smooth manifold with a metric $d$, $m = \dim M$, and $\alpha : \mathbb{R}^k \to \text{Diff}^{1+r}(M)(r > 0)$ be a locally free $\mathbb{R}^k$-action on $M$; $\mu$ is an invariant Borel probability measure for $\alpha$, and also assume it is ergodic; let $p$ be an arbitrary norm on $\mathbb{R}^k$. We say, an invariant measure $\mu$ is hyperbolic if there exists $m - k$ nontrivial exponents, equivalently there exists a $t$ such that $\alpha(t)$ has $m - k$ nonzero exponents. Let $\{\chi_i\}_{1 \leq i \leq D}$ be the Lyapunov exponents in Lyapunov
decomposition, $\gamma_i(t)$ be the corresponding transversal Hausdorff dimension (THD) for $\chi_i(t)$ (see sections 2.1 and 2.2 for the detailed definition). Note that, by definition, $\gamma_i(t)$ is defined to be $\gamma_i(-t)$ when $\chi_i(t) < 0$. Hence the domain of $\gamma_i(t)$ is \{t : $\chi_i(t) \neq 0$\}. As a preparation for the slow entropy formula, we first give the following general result on the universality of THDs, which can be used independently. Let’s mention here that, it is known by certain amount of dynamists, however there is no proof yet.

**Theorem 1.1.** As a function of t, $\gamma_i(t)$ is a nonnegative constant in \{t : $\chi_i(t) \neq 0$\}. Moreover, if we do not assume $\mu$ to be ergodic, then $\gamma_i(t)$ is a nonnegative constant in each ergodic component of $\mu$.

Now we are ready to state our main result:

**Theorem 1.2 (Slow Entropy Formula).** For abelian action $\alpha$, assume $\mu$ is either hyperbolic or absolutely continuous with respect to a volume form on $M$, then

$$sh_{\mu}(\alpha, p) = \sum_{i=1}^{D} \gamma_i \max_{t \cdot p(t) \leq 1} \chi_i(t).$$

For the detailed definition of slow entropy, see section 2.3. Here, one can easily see that, slow entropy is always finite if every element has finite metric entropy; and it does not vanish unless every element has zero metric entropy. Careful reader may find the similarity between the above formula and Ledrappier-Young formula for metric entropy (Theorem C’ in [6], and when $k = 1$, $p$ is the standard norm, it reduces to the usual metric entropy case. So here we will call it slow entropy type Ledrappier-Young formula, though we can only prove it under some restrictions on the measure.

Here, it is also important to note that, measure rigidity results for higher rank abelian group actions, especially those from [12] and [13], indicate that the case of absolutely continuous measure is indeed the central one. In this case, $\gamma_i$ will be the multiplicity of the corresponding exponent $\chi_i$, and the formula then becomes the slow entropy version of Pesin entropy formula.

As a by-product, we also prove the following generalized Brin-Katok local entropy Theorem:

**Theorem 1.3.** Under the same assumptions as in Theorem 1.2 for $\mu$ a.e. $x$,

$$\lim_{\epsilon \to 0} \liminf_{s \to \infty} \frac{-\log(\mu(B(\alpha, F_s^p, x, \epsilon)))}{s} = \lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{-\log(\mu(B(\alpha, F_s^p, x, \epsilon)))}{s},$$

and this limit is equal to

$$\sum_{i=1}^{D} \gamma_i \max_{t \cdot p(t) \leq 1} \chi_i(t).$$

Here, $B(\alpha, F_s^p, x, \epsilon) = \{y \in M : d(\alpha(t)x, \alpha(t)y) \leq \epsilon, \forall t \text{ s.t. } p(t) \leq s\}$ is the so called Bowen Ball. In fact, most of our work goes into proving this theorem, and then Theorem 1.2 is an easy consequence.
Let us point out the main difficulties in proving Theorem 1.2. Recall that, for metric entropy of diffeomorphisms, Brin-Katok Theorem on local entropy, Shannon-McMillan-Breiman(SMB) Theorem and partition theory (Sinai partition) are highly used, see [6]. However, for abelian actions, SMB Theorem is not that useful, because the extension of SMB Theorem for actions includes faster growth of the denominator than what is needed in our case. Another difficulty is that, we heavily use a local entropy type theorem (Theorem 1.3) to prove slow entropy formula, but we can not prove it in the general case, because we can neither generalize the proof of Brin-Katok Theorem to our case (which highly uses SMB Theorem) nor come out with a new proof. As a result, we have to put extra assumptions on the measure into our main result. In addition, unfortunately there is no way to construct an increasing partition for the action, hence we lose many powerful tools from partition theory.

In contrast to the metric entropy, another huge problem we can not avoid is the existence of zero Lyapunov exponents, which, equivalently speaking, considering the case of non-hyperbolic measure for actions. Hyperbolic measure of a $C^{1+r}$ diffeomorphism locally has so called asymptotically almost local product structure. Namely, such kind of measure is exact dimensional, see [8] for details. The proof in [8] essentially exploits results from [6], and uses a combinatorial argument based on a special partition constructed in [6]. If we just consider hyperbolic measure for abelian actions, then similar method allows us to handle the problem. However, due to the existence of zero Lyapunov exponents, it is difficult to control the behavior in the neutral directions. This is a very subtle issue in dimension theory and smooth ergodic theory.

A similar problem is to give a close enough lower bound of the lower pointwise dimension not only for hyperbolic measure but for arbitrary Borel probability invariant measures, which should be similar to Theorem F for upper pointwise dimension in [6]. For example, in [6] the following quantities (whenever they are well defined) are considered, which are called stable and unstable pointwise dimensions of measure $\mu$,

$$d^s(x) = \lim_{r \to 0} \frac{\log \mu^s_s(B^s(x, r))}{\log r};$$

$$d^u(x) = \lim_{r \to 0} \frac{\log \mu^u_u(B^u(x, r))}{\log r};$$

here see [6] or [8] for more details. Now the question is, can one get, for $\mu$ a.e. $x$

$$d^s(x) + d^u(x) \leq d(x) := \lim inf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Finally, let us emphasize here, slow entropy type invariant may have some applications to the study of Kakutani (or orbit) equivalence and rigidity problems of actions of higher rank abelian groups, which is our subsequent study in the future.

In this paper, we will heavily use results and methods from [6]. We also use an important technique from H. Hu’s paper [7] to prove Theorem 1.1. For the proof of Theorem 1.3, a combinatorial argument from [6] and scaling trick from [9] are applied.

\footnote{For details, see [10] and further [6].}
This paper is organized as follows. In section 2, some definitions and settings are presented. Proof of Theorem 1.1 appears in section 3. The principal and essential part is section 4, 5 and 6, where we prove Theorem 1.3. In section 7, we discuss slow entropy for higher rank abelian actions and finally prove Theorem 1.2. In the last section, some open questions and possible characterization for slow entropy are discussed.

2 Preliminaries

2.1 Lyapunov Exponents, Suspension, Charts

Let $T_x M$ be the tangent space of $M$ at $x$, and for $t \in \mathbb{R}^k$, $\alpha(t)$ induces a map $D_x \alpha(t) : T_x M \to T_{\alpha(t)x} M$. One may always assume that $k \geq 2$, otherwise it will reduce to the usual case (flow). For simplicity, we will use $t$ as the diffeomorphism instead of $\alpha(t)$ in some cases.

Let’s first consider a $\mathbb{Z}^k$ action. According to the Multiplicative Ergodic Theorem, there exists a measurable set $\Gamma$ with $\mu(\Gamma) = 1$, such that for all $x \in \Gamma$, nonzero $u \in T_x M$, such that for every $t \in \mathbb{Z}^k$ the limit

$$\chi(x, u, \alpha(t)) = \lim_{n \to \infty} \frac{\log |D_x \alpha(nt)u|}{n}$$

exists and we call it the Lyapunov exponent of $u$ at $x$ for $\alpha(t)$. One can easily see that, for each $t$, the Lyapunov exponent can only take finite numbers. Since $\alpha$ is an abelian action, we can get a common splitting for the tangent space $T M = \bigoplus E_\chi$. And also, since $\mu$ is ergodic, $\chi$ is independent on $x$. Thus we will only denote $\chi_i(t)$ to be the $i$-th Lyapunov exponent for $\alpha(t)$. And the common refinement $T M = \bigoplus E_\chi$ is called the Lyapunov decomposition for $\alpha$.

For each $\chi_i$, viewed as a function of $t$, is a linear functional from $\mathbb{Z}^k$ to $\mathbb{R}$. It can be linearly extended to a functional on $\mathbb{R}^k$. The hyperplanes $\ker \chi_i \subset \mathbb{R}^k$ are called the Lyapunov hyperplanes and the connected components of $\mathbb{R}^k \setminus \bigcup_i \ker \chi_i$ are called the Weyl chambers of $\alpha$. The elements in the union of the Lyapunov hyperplanes are called singular, and elements in the union of Weyl Chambers are called regular. For more details on the general theory, see [16].

Now given a $\mathbb{Z}^k$ action on $M$, let $\mathbb{Z}^k$ act on $\mathbb{R}^k \times M$ by

$$t(s, m) = (s - t, tm)$$

and form the quotient space

$$S = \mathbb{R}^k \times M / \mathbb{Z}^k \cong \mathbb{T}^k \times M.$$ 

Note that the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times M$ by $s(t, m) = (s + t, m)$ commutes with the $\mathbb{Z}^k$ action and therefore we can get a $\mathbb{R}^k$ action on $S$. This action is closely related to the original action, and we call it the suspension of $\mathbb{Z}^k$ action. In fact, when $k = 1$, it is the usual suspension for one diffeomorphism. We can build a natural correspondence between invariant measures, nonzero Lyapunov exponents and stable/unstable distributions etc. between the suspension and original $\mathbb{Z}^k$ action. For example, if the $\mathbb{Z}^k$ action preserves $\mu$, then $\mathbb{R}^k$ action preserves $\lambda \times \mu$, here $\lambda$ is the...
Let the Lebesgue measure on $\mathbb{T}^k$. And this is why we mostly only need to deal with $\mathbb{R}^k$ actions in this paper.

The following is a result directly quoted from [12].

**Proposition 2.1** (Proposition 2.1. [12]). Let $\alpha$ be a locally free $C^{1+r}$, action of $\mathbb{R}^k$ on a manifold $M$ preserving an ergodic invariant measure $\mu$. There are linear functionals $\chi_i$, $i = 1, \cdots, D$, on $\mathbb{R}^k$ and an $\alpha$-invariant measurable splitting called the Lyapunov decomposition, of the tangent bundle of $M$

$$TM = T\mathcal{O} \oplus \bigoplus_{i=1}^{D} E_i$$

over a set of full measure $\Gamma$, where $T\mathcal{O}$ is the distribution tangent to the $\mathbb{R}^k$ orbits, such that for any $t \in \mathbb{R}^k$ and any nonzero vector $v \in E_i$ the Lyapunov exponent of $v$ is equal to $\chi_i(t)$, i.e.

$$\lim_{n \to \pm \infty} \frac{1}{n} \log ||D_x(\alpha(nt))v|| = \chi_i(t),$$

where $|| \cdot ||$ is any continuous norm on $TM$. Any point $x \in \Gamma$ is called a regular point.

Furthermore, for any $\epsilon > 0$ there exist positive measurable functions $C_\epsilon(x)$ and $K_\epsilon(x)$ such that for all $x \in \Gamma$, $v \in E_i(x)$, $t \in \mathbb{R}^k$, and $i = 1, \cdots, D$,

1. $C^{-1}_\epsilon(x)e^{\chi_i(t) - \frac{1}{2}\epsilon||\alpha(t)||||v||} \leq ||D_x(\alpha(t))v|| \leq C_\epsilon(x)e^{\chi_i(t) + \frac{1}{2}\epsilon||\alpha(t)||||v||}$;

2. Angles $\angle(E_i(x), T\mathcal{O}) \geq K_\epsilon(x)$ and $\angle(E_i(x), E_j(x)) \geq K_\epsilon(x)$, $i \neq j$;

3. $C_\epsilon(\alpha(t)x) \leq C_\epsilon(x)e^{\epsilon||\alpha(t)||}$ and $K_\epsilon(\alpha(t)x) \geq K_\epsilon(x)e^{-\epsilon||\alpha(t)||}$.

Finally, let’s now construct Lyapunov charts for the action $\alpha$. The following are a generalized proposition from [7] with some modification of notations. We include here for further use, and for simplicity omit the proof because it is similar to Proposition 4.1. in [7].

Let $|| \cdot ||$ be the standard norm on $\mathbb{R}^k$, and $| \cdot |$ be the usual norm on $\mathbb{R}^m$, here $m = \dim M$. Also let $B(\rho)$ $(\rho > 0)$ be the ball in $\mathbb{R}^m$ centered at the origin with radius $\rho$. We also assume the action is ergodic.

Denote $\{t_1, \cdots, t_k\}$ as the standard basis for $\mathbb{Z}^k$ w.r.t. the norm $|| \cdot ||$ on $\mathbb{R}^k$, i.e. it will span $\mathbb{Z}^k$ via coefficients in $\mathbb{Z}$. For $t_1$, we denote its exponents correspondingly as $\chi_1(t_1) > \cdots > \chi_m(t_1)(t_1)$. Let $\chi_+(t_1) = \min\{\chi_1(t_1), \chi_i(t_1) > 0\}$, $\chi_-(t_1) = \max\{\chi_i(t_1), \chi_i(t_1) < 0\}$ and $\Delta(t_1) = \min\{\chi_i(t_1) - \chi_{i+1}(t_1), i = 1, \cdots, m(t_1) - 1\}$. Define $\chi_\pm(t_1), \Delta(x, t_1)$ similarly. Then take $\epsilon > 0$, and

$$0 < \epsilon \leq \frac{1}{200mk} \min\{\Delta(t_i), \chi_\pm(t_i), i = 1, \cdots, k\}.$$

**Proposition 2.2.** For the $\epsilon$ defined above, there exists a measurable function $l : \Gamma \to [0, \infty)$ with $l(\alpha(t)x) \leq l(x)e^{\epsilon||t||}$, and a set of embeddings $\Phi_x : B(l(x)^{-1}) \to M$ at each point $x \in \Gamma$ such that the following holds:
i) \( \Phi_x(0) = x \), and the preimages \( D\Phi_x(0)^{-1}(T\mathcal{O}(x)) \) and \( R_i(x) = D\Phi_x(0)^{-1}(E_i(x)) \) of \( E_i(x) \) are mutually orthogonal in \( \mathbb{R}^m \), where \( E_i(x) \) is the Lyapunov subspace for some exponent in Lyapunov decomposition.

ii) Let \( \tilde{t}_x = \Phi_x^{-1} \circ t \circ \Phi_x \) be the connecting map between the chart at \( x \) and the chart at \( tx \). Then \( (t + s)_x = \tilde{t}(s)\tilde{s}_x = \tilde{s}(t)_x \tilde{t}_x \) and \( (t + s)^{-1}_x = \tilde{t}^{-1}(s)\tilde{s}^{-1}_x = \tilde{s}^{-1}(t)_x \tilde{t}^{-1}_x \) for any \( s, t \).

iii) For any \( q, 1 \leq q \leq m \), a nonzero vector \( u \in E_q(x), v \in T\mathcal{O}(x) \),

\[
|u|e^{\chi_0(x,t) - \epsilon \|t\|} \leq |D_x \tilde{t}(0)u| \leq |u|e^{\chi_0(x,t) + \epsilon \|t\|}, \quad i = 1, \cdots, k,
\]

\[
|v|e^{-\epsilon \|t\|} \leq |D_x \tilde{t}(0)v| \leq 2|v|,
\]

\[
|u|e^{\chi_0(x,t) + \chi_0(x,s) - \epsilon \|t\| + \|s\|} \leq |D_x (t + s)(0)u| \leq |u|e^{\chi_0(x,t) + \chi_0(x,s) + \epsilon \|t\| + \|s\|}, \quad \forall \ s, t.
\]

iv) Let \( L(\Psi) \) be the Lipschitz constant of the function \( \Psi \). Then for any \( t \),

\[
L(\tilde{t}_x - D\tilde{t}_x(0)) \leq e \|t\|, \quad L(D\tilde{t}_x) \leq l(x)e^{\|\|t\\|}.
\]

v) There exists a number \( \epsilon > 0 \) depending on \( \epsilon \) and the exponents such that \( \forall x \in \Gamma \),

\[
|\tilde{t}_x u| \leq e^{\epsilon \|t\|}|u|, \quad \forall u \in B(e^{-\epsilon - \epsilon} l(x)^{-1}).
\]

vi) For all \( u, v \in B(l(x)^{-1}) \), we have

\[
K^{-1} d(\Phi_x u, \Phi_x v) \leq |u - v| \leq l(x) d(\Phi_x u, \Phi_x v),
\]

for some universal constant \( K \).

We'll call such local charts \( \{ \Phi_x : x \in \Gamma \} \) \((\epsilon, l)\)-charts. Let \( \epsilon \) (small enough) in Propositions 2.1 and 2.2 be the same, and \( \Gamma_\epsilon := \{ x \in \Gamma : l(x) \leq l, C_\epsilon(x) \leq l, K_\epsilon(x) \geq \frac{1}{\tau} \} \), it is easy to see that when \( l \) large enough, \( \mu(\Gamma_\epsilon) > 0 \).

### 2.2 Transversal Hausdorff Dimension (THD)

Now we just consider one \( C^{1+r} \) diffeomorphism \( f := \alpha(t) \) on \( M \) for some \( t \in \mathbb{R}^k \). It is a well-known fact that we can choose \( t \) properly such that \( f \) is ergodic with respect to \( \mu \). The following are some definitions and results from section 7 in [6].

For \( f \), let

\[
\chi_1 > \chi_2 > \cdots > \chi_q
\]

be the distinct Lyapunov exponents, and

\[
TM = E_1 \oplus \cdots \oplus E_q
\]
be the corresponding decomposition of its tangent space. Note that these are all defined \( \mu \) a.e. Let \( u = \max\{i : \chi_i > 0\} \), and for \( 1 \leq i \leq u \), define

\[
W^i(x) = \left\{ y \in M : \limsup_{n \to \infty} \frac{\log d(f^{-n}x, f^{-n}y)}{n} \leq -\chi_i \right\},
\]

here \( d(\cdot, \cdot) \) is a Riemannian metric on \( M \). We call \( W^i(x) \) (a \( C^{1+r} \) immersed submanifold) the \( i \)-th unstable manifold (leaf of a foliation\(^2\)) of \( f \) at \( x \). We then have a nested family of a.e. foliations

\[ W^1 \subset W^2 \subset \cdots \subset W^u. \]

Each \( W^i(x) \) inherits a Riemannian structure from \( M \), and hence gives a metric on each leaf of \( W^i \), which is denoted by \( d^i \). The measure \( \mu \) also induces conditional measures on \( W^i(x) \). More precisely, given a measurable partition \( \xi \) of \( M \) which is subordinate to the \( W^i \)-foliation, there is a system of conditional measures induced from \( \mu \) associated to each atom of \( \xi \). In fact, these measures are defined up to a scalar multiple. Let \( \{\mu^i_{x}\} \) be the conditional measures for \( \xi \).

Let \( \epsilon > 0 \), for \( x \in \Gamma \) and \( n \in \mathbb{N} \), define

\[ V^i(x, n, \epsilon) = \{ y \in W^i(x) : d^i(f^kx, f^ky) < \epsilon \text{ for } 0 \leq k < n \}. \]

Then define

\[
\underline{h}_i(x, \epsilon, \xi) = \liminf_{n \to \infty} \frac{\log \mu^i_{x}(V^i(x, n, \epsilon))}{n}
\]

and

\[
\bar{h}_i(x, \epsilon, \xi) = \limsup_{n \to \infty} \frac{\log \mu^i_{x}(V^i(x, n, \epsilon))}{n}.
\]

It is proved (Proposition 7.2.1. in [6]) that

\[
\lim_{\epsilon \to 0} \underline{h}_i(x, \epsilon, \xi) = \lim_{\epsilon \to 0} \bar{h}_i(x, \epsilon, \xi) \quad \mu \text{ a.e. } x,
\]

and this is independent of the choice of \( \xi \) or \( \{\mu^i_{x}\} \). As a function, this quantity is measurable, and hence by ergodicity of \( \mu \), it is constant almost everywhere. We denote this constant by \( h_i \), and it is called the entropy along the \( i \)-th unstable manifold.

Let \( B^i(x, \epsilon) \) be the \( d^i \)-ball in \( W^i(x) \) centered at \( x \) of radius \( \epsilon \). For \( x \in \Gamma \), define

\[
\underline{\delta}_i(x, \xi) = \liminf_{\epsilon \to 0} \frac{\log \mu^i_{x}(B^i(x, \epsilon))}{\log \epsilon}
\]

and

\[
\bar{\delta}_i(x, \xi) = \limsup_{\epsilon \to 0} \frac{\log \mu^i_{x}(B^i(x, \epsilon))}{\log \epsilon}.
\]

\(^2\)See sections 1.3 and 1.4 in [6] for details.
Again, it is proved (Proposition 7.3.1. in [6]) that
\[ \delta_i(x, \xi) = \bar{\delta}_i(x, \xi) \quad \mu \text{ a.e. } x, \]
and this is independent on \( \xi \). By ergodicity, it is constant a.e. We denote the constant by \( \delta_i \), and it is called the dimension of \( \mu \) on \( W^s \)-manifolds. Note that by definition \( \delta_{u} = d^u \) for \( f \).

It is a celebrated result (Theorem C′ in [6]) that:

(i) \( h_1 = \chi_1 \delta_1 \),
(ii) \( h_i - h_{i-1} = \chi_i (\delta_i - \delta_{i-1}) \) for \( 2 \leq i \leq u \),
(iii) \( h_u = h_\mu(f) \).

Let \( \gamma_1 = \delta_1 \) and \( \gamma_i = \delta_i - \delta_{i-1} \) for \( i = 2, \ldots, u \). Replacing \( f \) by \( f^{-1} \), we can get \( \delta_{q+1-s}, \ldots, \delta_q \), where \( s = \# \{ i : \chi_i < 0 \} \) is the number of distinct negative Lyapunov exponents for \( f \). And let \( \gamma_q = \delta_q \) and \( \gamma_i = \delta_i - \delta_{i+1} \) for \( i = q + 1 - s, \ldots, q-1 \). For \( i \) that \( \chi_i = 0 \), simply define \( \gamma_i = \dim E_{\chi_i} \). Hence
\[ \sum_i \gamma_i \chi_i = 0, \]
and
\[ \sum_i \gamma_i |\chi_i| = 2 h_\mu(f). \]

Here, \( \gamma_i \) is called the transversal Hausdorff dimension of \( \mu \) with respect to \( \chi_i \). Those numbers depend on the diffeomorphism \( f \) and the measure \( \mu \). However, in the abelian action case, as we’ll prove in next section, they do not depend on the choice of an element of the action.

The essential fact behind the above definitions and results is that all intermediate stable and unstable distribution are integrable. Namely, \( \bigoplus_{1 \leq i \leq j} E^i \) are integrable for \( 1 \leq j \leq u \). One should be careful extending the definitions to abelian actions, because sometimes the \( \gamma_i \) for \( \alpha(t) \) will split to two or more THDs for some other \( \alpha(s) \).

### 2.3 Slow Entropy Type Invariants

There are two approaches to slow entropy for \( \mathbb{Z}^k \) action. One is based on an idea of coding. First, fix a finite measurable partition \( \xi \) of \( M \), and then define a metric \( d_F \) to be the pull-back of the Hamming metric in the space of codes, see section 1.1. in [5] for details. Denote \( S^H_\xi(\alpha, F, \epsilon, \delta) \) as the minimal number of \( \epsilon-d_F \) balls whose union has measure \( \geq 1 - \delta \). Given a norm \( p \) on \( \mathbb{R}^k \), let \( F^p_s \) be the set of points in \( \mathbb{Z}^k \) which is also contained in the ball centered at 0 with radius \( s \). We define the slow entropy of \( \alpha \) with respect to the norm \( p \) and the partition \( \xi \) as
\[
sh_\mu(\alpha, p, \xi) = \lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{\log S^H_\xi(\alpha, F^p_s, \epsilon, \delta)}{s}.
\]
Then we define
\[
sh_\mu(\alpha, p) = \sup_\xi sh_\mu(\alpha, p, \xi).
\]

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The other approach is to start with a metric $d$ on $M$, and define $d_F = \max_{t \in F} d \circ \alpha(t)$. Denote $S_d(\alpha, F, \epsilon, \delta)$ as the minimal number of $\epsilon$-$d_F$ balls whose union has measure $\geq 1 - \delta$. And for the same $F^p_\delta$, define

$$sh_\mu(\alpha, p) = \lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{\log S_d(\alpha, F^p_\delta, \epsilon, \delta)}{s}.$$ 

These two definitions coincide (see [5]), and hence for the latter it does not depend on the choice of $d$. Finally, slow entropy for $\alpha$ is defined as

$$sh_\mu(\alpha) = \inf_{p; \text{vol}(p) = 1} sh_\mu(\alpha, p),$$

here $\text{vol}(p)$ is the volume of the unit ball in the norm $p$. In this paper, we will mostly consider the quantity $sh_\mu(\alpha, p)$ instead of $sh_\mu(\alpha)$.

For $\mathbb{R}^k$ action, we will use the second definition.

In the case of a non-ergodic invariant measure, the procedure follows the convention, decompose the measure into ergodic components, and then integrate the slow entropy of all ergodic components. For arbitrary actions, we can not use this convention, but in the smooth case, we can. Let’s emphasize a little here the norm $p$, which can be seen as a time change, namely, changing the norm $p$ means a time change of the abelian action!

For more details and discussions about slow entropy, see section 1 in [5] and section 3 in [11].

3 Transversal Hausdorff Dimensions

In this section, we consider $\mathbb{R}^k$ action $\alpha$ on $M$ by $C^{1+r}$ diffeomorphisms. Our goal is to prove Theorem [14]. It is worth to mention that, in [19], a similar but stronger result is also obtained.

At first, associated to $\alpha$, let $TM = T\mathcal{O} \oplus \bigoplus_{i=1}^D E_i$, where $E_i$ is the Lyapunov subspace with respect to $\chi_i$. And for each $t$, there is an order for the positive exponents $\chi_i(t) 1 \leq i \leq u(t)$, and the corresponding $E_i$, such that for every $1 \leq j \leq u(t)$, the distribution $\bigoplus_{1 \leq i \leq j} E_i$ is integrable; similarly for the negative exponents. Below in this section, we will neglect $k$ zero exponents from the direction of flow.

Next, let’s prove the following slightly generalized proposition of Proposition 8.1. in [7]:

**Proposition 3.1.** Let $f$ and $g$ be commuting $C^{1+r}$ diffeomorphisms on $M$, that preserve a measure $\mu$. Also assume $\mu$ is ergodic. Let $\chi_1(f) > \chi_2(f) > \cdots > \chi_{u(f)}(f) > 0 > \cdots > \chi_{m(f)}(f)$ be all distinct Lyapunov exponents of $f$, possibly there are extra zero exponents. Similarly for $g$, $\chi_1(g) > \chi_2(g) > \cdots > \chi_{u(g)}(g) > 0 > \cdots > \chi_{m(g)}(g)$. Assume for some $i$, $1 \leq i \leq \min\{u(f), u(g)\}$, $\bigoplus_{1 \leq j \leq i} E_{\chi_j(f)} = \bigoplus_{1 \leq j \leq i} E_{\chi_j(g)}$, and there exists $\lambda(f), \lambda(g) > 0$ such that $\chi_{i+1}(f) < \lambda(f) < \chi_i(f), \chi_{i+1}(g) < \lambda(g) < \chi_i(g)$. Then there exists a measurable partition $\xi$ on $M$, such that

1) $\xi$ is subordinate to $W^i$ foliation, here $W^i$ is integrated by $\bigoplus_{1 \leq j \leq i} E_{\chi_j(f)} = \bigoplus_{1 \leq j \leq i} E_{\chi_j(g)}$;
2) $\xi$ is an increasing partition for both $f$ and $g$;
3) Both $\bigvee_{n=0}^{\infty} f^{-n}\xi$ and $\bigvee_{n=0}^{\infty} g^{-n}\xi$ are the partition into points (mod 0).
4) The biggest $\sigma$-algebra contained in $\bigcap_{n=0}^{\infty} \bigcap_{m=0}^{\infty} f^{n}g^{m}\xi$ is $\mathcal{B}^i$.

Here, we say $\xi$ is subordinate to $W^i$ foliation if for $\mu$ a.e. $x$, $\xi(x) \subset W^i(x)$ and $\xi(x)$ contains a neighborhood of $x$ open in the submanifold topology of $W^i(x)$. Partition $\xi_1$ refines $\xi_2$ ($\xi_1 > \xi_2$) if at $\mu$ a.e. $x \in M$, $\xi_1(x) \subset \xi_2(x)$. A partition is said to be increasing if $\xi > f\xi$. $\mathcal{B}^i$ is the sub-$\sigma$-algebra of Borel $\sigma$-algebra on $M$ whose elements are unions of entire $W^i$-leaf. For more details, see [6].

**Proof.** The proof is similar to the proof of Proposition 8.1. in [7]. There are several modifications. One is for Lemma 8.3. there, considering details, see [6].

For Lemma 8.4, prove

$$d^{i}(f^{-n}g^{-k}y, \partial B(x, \rho))e^{n(\lambda(f)-2\epsilon)+k(\lambda(g)-2\epsilon)} < b^{-1}.$$ 

For Lemma 8.4, prove

$$d^{i}(f^{-n}g^{-k}y, f^{-n}g^{-k}z) \leq 2Kl(z)d^{i}(y, z)e^{n(\lambda(f)-2\epsilon)-k(\lambda(g)-2\epsilon)}$$

instead.

For the proof there, replace $W^{i}_\alpha$ by $W^{i}_\alpha$, $d_\alpha$ by $d^{i}$, the same argument would work in our case. We omit the detailed proof here for simplicity.

In fact, the above proposition, can be applied to the splitting that appears in the Lyapunov decomposition for $\alpha$, because the splitting in the proposition (or for two diffeomorphisms) is coarser than this. And this is what we really need!

Considering the partition $\xi$ above, we have

**Proposition 3.2.** $H_\mu(\xi|fg\xi) = H_\mu(\xi|f\xi) + H_\mu(\xi|g\xi)$.

**Proof.** $H_\mu(\xi|fg\xi) = H_\mu(\xi \vee g\xi|fg\xi) = H_\mu(\xi|fg\xi) + H_\mu(\xi|g\xi \vee fg\xi) = H_\mu(\xi|f\xi) + H_\mu(\xi|g\xi)$.

Note also that we have $H_\mu(\xi|f\xi) = h_\xi(f)$ for any such partition (a result of section 9 of [6]). Combining $h_\xi(f) = \sum_{j=1}^{i} \gamma_j(f)\chi_j(f)$, we have $H_\mu(\xi|f\xi) = \sum_{j=1}^{i} \gamma_j(f)\chi_j(f)$. This also applies to $g$ and $fg$, then Proposition 3.2 implies,

$$\sum_{j=1}^{i} \gamma_j(fg)\chi_j(fg) = \sum_{j=1}^{i} \gamma_j(f)\chi_j(f) + \sum_{j=1}^{i} \gamma_j(g)\chi_j(g).$$  \hspace{1cm} (3)

Now, we are ready to prove Theorem 1.1. The proof is divided into the following four parts. Below $n \in \mathbb{N}^+$, $r \in \mathbb{R}^+$, and $t \neq 0$.

1) $\gamma_i(rt) = \gamma_i(t)$. Considering a partition $\xi$ (depend on $i$) built in Lemma 9.1.1. in [6] for $t$, then it is also a partition for $nt$ satisfying the same conditions. Combine $H_\mu(\xi|nt\xi) = nH_\mu(\xi|t\xi)$ and $H_\mu(\xi|t\xi) = h_\xi(t)$, for $i = 1, \cdots, u(t)$, then we get that $\gamma_i(nt) = \gamma_i(t)$ for $\chi_i(t) > 0$. This
also gives us that for any positive rational numbers \( u, \gamma_i(u) = \gamma_i(t) \) for \( \chi_i(t) > 0 \). Now pick arbitrary \( s, t \) with \( s = rt \) for some \( r \), then

\[
\gamma_1(s + t)\chi_1(s + t) = \gamma_1(s)\chi_1(s) + \gamma_1(t)\chi_1(t)
\]

hence

\[
\gamma_1(t) - \gamma_1(s + t) = (\gamma_1(s + t) - \gamma_1(s))r.
\]

If at least one of \((\gamma_1(t) - \gamma_1(s + t))\) and \((\gamma_1(s + t) - \gamma_1(s))\) is not 0, then we can replace \( r \) by \( nr \) (arbitrary \( n > 0 \)), then we will get a contradiction, because all \( \gamma_1 \) are bounded by \( \dim E_1 \). Hence

\[
\gamma_1(t) = \gamma_1(s + t) = \gamma_1(s),
\]

and this finished the first step. The same argument works for the subsequent \( \gamma_i \)s.

(2) We consider in one Weyl Chamber \( C \). Assume there are \( u \) positive exponents. Dividing \( C \) by hyperplanes \( L_{i,j} := \{ t : \chi_i(t) - \chi_j(t) = 0 \} \) into some small sub-chambers. Now in each sub-chamber, we can have an order of the positive exponents, and this order will not change when we change \( t \). So we can apply Proposition \([3,2]\) and use induction on \( i \). From Equation \( (3) \), when \( i = 1 \), for every \( s, t \) in that sub-chamber, we have

\[
\gamma_1(t + s)\chi_1(t + s) = \gamma_1(t)\chi_1(t) + \gamma_1(s)\chi_1(s),
\]

hence

\[
(\gamma_1(t) - \gamma_1(s + t))\chi_1(t) = (\gamma_1(s + t) - \gamma_1(s))\chi_1(s).
\]

If at least one of \((\gamma_1(t) - \gamma_1(s + t))\) and \((\gamma_1(s + t) - \gamma_1(s))\) is not 0, then we can let \( s \) or \( t \) go to \( \infty \), then we will get a contradiction due to the same reason in part (1). Hence

\[
\gamma_1(t) = \gamma_1(s + t) = \gamma_1(s),
\]

and this finished the first step. Suppose for \( i < u \), we have \( \gamma_j(t), j \leq i \) are all constant for all \( t \) in the sub-chamber. Then consider Equation \( (3) \) for \( i + 1 \), since the first \( i \) THDs are equal, this will leave us

\[
\gamma_{i+1}(t + s)\chi_{i+1}(t + s) = \gamma_{i+1}(t)\chi_{i+1}(t) + \gamma_{i+1}(s)\chi_{i+1}(s).
\]

Use the argument in the first step, we can get our result. Hence for all positive exponents, we have THDs are constant. The same is true for the negative exponents if we just consider the negative of the sub-chamber.

The argument also works when we consider points in one hyperplane not crossing any other hyperplanes or Lyapunov hyperplane.

(3) We still consider in one Weyl Chamber \( C \). we consider two adjacent (means separated by only one hyperplane) sub-chambers, \( C_1 \) and \( C_2 \). Note that, maybe there are two or more hyperplanes coincide (If not, we can skip the following and go to next paragraph!). Let’s assume \( L_{i,j} \) and \( L_{p,q} \) are two of them, here \( i, j, p, q \) are four different numbers, one can easily get that, on the hyperplane

\[
\chi_i = \chi_j > 0, \quad \chi_p = \chi_q > 0, \quad \chi_i \neq \chi_p.
\]
So the paired exponents will take different values on the hyperplane, and hence we can always consider them one by another, ordered from the paired exponents that take the greatest value to the paired exponents take the least value. Of course, we also need to take other positive exponents into account, which can be tackled by argument from part (2).

Without loss of generosity, suppose the first hyperplane is $L_{i,j} = \{ t : \chi_i(t) - \chi_j(t) = 0 \}$. Consider $s \in C_1$ and $t \in C_2$, they are very close to $L_{i,j}$ and comparably far away from other hyperplanes or Lyapunov hyperplanes. For such $s, t$, $\chi_i$ and $\chi_j$ are two closed exponents in both sub-chambers, and only these two exponents will change order. On hyperplane $L_{i,j}$, they coincide. Suppose $\chi_i, \chi_j$ locate at $k, k + 1$ in the order, then for the first $k - 1$ exponents, the THDs are constant by argument in part (2). Apply Proposition 3.2 for $k + 1$, we cancel the first $k - 1$ exponents, then get

$$
\gamma_i(s + t)\chi_i(s + t) + \gamma_j(s + t)\chi_j(s + t) = \gamma_i(s)\chi_i(s) + \gamma_j(s)\chi_j(s) + \gamma_i(t)\chi_i(t) + \gamma_j(t)\chi_j(t). \quad (4)
$$

When one of $s, t$ lies in $L_{i,j}$, Equation (4) still holds. Suppose $t \in L_{i,j}$, this will give us

$$
\gamma_i(s) + \gamma_j(s) = \gamma_i(t) + \gamma_j(t)
$$

for $s$ in either $C_1$ or $C_2$. Hence from this, when $s, t$ in different sub-chambers,

$$
\gamma_i(s) + \gamma_j(s) = \gamma_i(t) + \gamma_j(t).
$$

Now, suppose $s \in C_1$, $t \in C_2$ and $s + t \in C_1$, Equation (4) is

$$
(\gamma_i(s) - \gamma_i(t))\chi_i(t) = (\gamma_j(t) - \gamma_j(s))\chi_j(t).
$$

Since $\chi_i(t) \neq \chi_j(t)$, hence we have

$$
\gamma_i(s) = \gamma_i(t), \; \gamma_j(s) = \gamma_j(t).
$$

For other positive exponents, arguments in part (2) and the above work similarly. Hence the constantness of THDs can be proved when crossing the hyperplanes. All the above arguments can be also applied to one hyperplane when crossing some other hyperplane. And these show that the THDs are all constant in one Weyl Chamber.

(4) We consider the case when crossing the Lyapunov hyperplane. There may be several exponents changing their sign. However, we do not need to consider these exponents, instead we only consider those exponents remain to be positive. The argument in (3) works in this case. We omit the details here.

Hence we complete the proof of Theorem 1.1. For future use, we denote $\gamma_i(t)$ by $\gamma_i$.

**Remark 3.1.** One may easily figure out that, for maximal rank actions, say Cartan actions on tori, $E_\chi_i$ is integrable to some $W^i$ for each $i$, and the corresponding THD $\gamma_i$ is, in fact the pointwise dimension of the conditional measure of $\mu$ restricted to $W^i$. In this case, we would rather call $\gamma_i$ conditional dimension instead of transversal dimension! However, in the more general cases, especially when there are positive proportional exponents, some $\gamma_i$ really represents the dimension of the transversal direction rather than conditional dimension.
4 Main Reduction

The following three sections are dedicated to the proof of Theorem 1.3. In this section, we restate the theorem to Proposition 4.1 and give a reduction from abelian action to one diffeomorphism case. The complete proof is separated into the following two sections, each of them deals with one case.

Proposition 4.1. Under the same assumptions as in Theorem 1.2, for \( \mu \) a.e. \( x \),

\[
\lim_{\epsilon \to 0} \lim_{s \to \infty} \frac{-\log(\mu(B(\alpha, F^p_s, x, \epsilon)))}{s} = \lim_{\epsilon \to 0} \lim_{s \to \infty} \frac{-\log(\mu(B(\alpha, F^p_s, x, \epsilon)))}{s},
\]

(5)

and this limit is equal to

\[
\sum_{i=1}^{D} \gamma_i \max_{t: p(t) \leq 1} \chi_i(t).
\]

Since \( \mu \) is ergodic, it is easy to see that,

\[
\lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{-\log(\mu(B(\alpha, F^p_s, x, \epsilon)))}{s}
\]

and

\[
\lim_{\epsilon \to 0} \liminf_{s \to \infty} \frac{-\log(\mu(B(\alpha, F^p_s, x, \epsilon)))}{s}
\]

are constant a.e. We will denote the two constant by \( D_\mu \) and \( E_\mu \) respectively. Basically, we are going to prove the following two inequalities:

\[
D_\mu \leq \sum_{i=1}^{D} \gamma_i \max_{t: p(t) \leq 1} \chi_i(t),
\]

(6)

and

\[
E_\mu \geq \sum_{i=1}^{D} \gamma_i \max_{t: p(t) \leq 1} \chi_i(t).
\]

(7)

Before moving forward, let’s first pick a diffeomorphism out of the abelian action.

Choose a \( t \in \mathbb{R}^k \), \( f := \alpha(t) \), such that: (1) \( p(t) \leq 1 \); (2) \( \mu \) is ergodic with respect to \( f \); (3) there is no extra zero exponent, i.e. no nontrivial exponents for \( \alpha \) vanishes for \( f \); (4) No two different exponents in the Lyapunov decomposition coincide for \( f \). Denote \( u \) be the dimension of unstable Lyapunov subspace, and \( s \) for the stable one, and the exponents \( \chi_1 > \cdots > \chi_u > \chi_{u+1} = 0 \) (possible!) > \( \chi_{u+2} > \cdots > \chi_D \), corresponding Lyapunov space \( E_i, d_i = \dim E_i \). We fix the order of exponents as this once and for all. Let \( W^i \) be the \( i \)-foliation integrated by \( \bigoplus_{1 \leq j \leq i} E_j \) when \( i \leq u \), and \( \bigoplus_{D+1-i \leq j \leq D} E_j \) when \( i \geq u + 2 \); \( \xi_i \) be a measurable (Sinai) partition\(^3\) subordinate to \( W^i \), and \( \{ \mu_{x}^i \} \) be a system of the induced conditional measures.

The following result, to the best of the author’s knowledge, is first noticed by J. Schmeling and S. Troubetzkoy in [9], though it is in fact proved in [6] (in the proof of Theorem C there).

\(^3\)See [10] or further [6].
Proposition 4.2. Let $a_1, \ldots, a_i, i \leq u$ be negative numbers and $R_i^j = [-e^{a_i t}, e^{a_i t}]^{d_i} \times \cdots \times [-e^{a_i t}, e^{a_i t}]^{d_i} \times \{0\} \times \cdots \times \{0\}$. Then for $\mu$ a.e. $x$ the following limit exists

$$\rho_i^j(a_1, \ldots, a_i) := \rho_i^j = \lim_{t \to \infty} \frac{-\log \mu_i^j(\Phi_x R_i^j)}{t} = -\sum_{j \leq i} a_j \gamma_j.$$

The same is true by replacing $[-e^{a_i t}, e^{a_i t}]^{d_i}$ by $d_i$ dimensional balls of radius $e^{a_i t}$.

Proof. The proof is almost the same with the proof of Ledrappier-Young Formula [6], the only main change is replacing the balls by the new scaling rectangles (or balls) with proper scales at each time. See [9] or [6] for details. \qed

The above proposition is used to prove the following Proposition.

Proposition 4.3. For $i \leq u$, $\mu$ a.e. $x$,

$$h^i := \lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{-\log(\mu_i^j(B(\alpha, F_p^s, x, \epsilon)))}{s} = \lim_{\epsilon \to 0} \liminf_{s \to \infty} \frac{-\log(\mu_i^j(B(\alpha, F_p^s, x, \epsilon)))}{s},$$

and equals to

$$\sum_{j \leq i} \gamma_j \max_{t: p(t) \leq 1} \chi_j(t).$$

The above two results are true when considering the stable case. Particularly, from them we have

$$h^u = \sum_{j \leq u} \gamma_j \max_{t: p(t) \leq 1} \chi_j(t), \quad h^s = \sum_{j \geq u + 2} \gamma_j \max_{t: p(t) \leq 1} \chi_j(t).$$

Lemma 4.1. Let $a_i = \max_{t: p(t) \leq 1} \chi_i(t)$ for $1 \leq i \leq D$, take $\epsilon \leq \frac{1}{100m} \min \{1, a_1, \ldots, a_D\}$, then for any $x \in \Gamma_i$, there exists $s(x) > 0$, such that when $s \geq s(x),$

$$K^{-1} \left( \prod_{i \leq u} B_i \left( 0, \frac{e^{-(a_i + 2\epsilon)s}}{m + 1} \right) \times B_{u+1} \left( 0, \frac{e^{-2\epsilon s}}{m + 1} \right) \prod_{i \geq u + 2} B_i \left( 0, \frac{e^{-(a_i + 2\epsilon)s}}{m + 1} \right) \right) \subset \Phi_{x}^{-1} \left( B(\alpha, F_p^s, x, \epsilon) \right) \subset \prod_{i \leq u} B_i \left( 0, (m + 1)e^{-(a_i + 2\epsilon)s} \right) \times B_{u+1} \left( 0, (m + 1)e^{-2\epsilon s} \right) \prod_{i \geq u + 2} B_i \left( 0, (m + 1)e^{-(a_i + 2\epsilon)s} \right). \tag{8}$$

Here, $B_i$ is the ball centered at origin in $\mathbb{R}^{d_i}$, and $\prod$ means the usual direct product.
Proof of Lemma 4.1. Note that, for the neutral direction, it will neither contract more than sub-exponentially nor expand more than \((m + 1)\epsilon\). So we only need to prove the inclusion for the other directions. First, we prove the left hand side inclusion. It is enough to show that for any \(u \in \) and any \(t\) with \(p(t) \leq s\), we have \(d(\alpha(t)x, \alpha(t)\Phi_x(u)) \leq \epsilon\).

Now we know that
\[
|\Phi^{-1}_{\alpha(t)x} \alpha(t)\Phi_x(u)| \leq K^{-1}\left( se \sum_{i=1}^{D} e^{-(a_i+2\epsilon)s} \frac{m + 1}{m + 1} + \sum_{i=1}^{D} e^{sa_i + \epsilon e^{-(a_i+2\epsilon)s}} \right).
\]
Hence when \(s\) is great enough, we have
\[
|\Phi^{-1}_{\alpha(t)x} \alpha(t)\Phi_x(u)| \leq K^{-1}\epsilon
\]
and then
\[
d(\alpha(t)x, \alpha(t)\Phi_x(u)) \leq K|\Phi^{-1}_{\alpha(t)x} \alpha(t)\Phi_x(u)| \leq \epsilon.
\]

Now we come to the proof of the other side. Assume
\[
u := (u_1, u_2, \cdots, u_d) \in \Phi^{-1}_x B(\alpha, F^p, x, \epsilon)
\]
here, \(u_i \in \mathbb{R}^{d_i}\). It is enough to show that, \(u_i \in lB_i\left(0, (m + 1)\epsilon e^{-(a_i+2\epsilon)s}\right)\) for every \(i \leq u\), and similar for the rest is. First, choose \(t\) such that \(\chi_i(t) = sa_i\), then
\[
|\Phi^{-1}_{tx} \alpha(t)\Phi_x u| \geq (-\epsilon \epsilon + e^{(a_i-\epsilon)s}) |u_i|.
\]
And
\[
|\Phi^{-1}_{tx} \alpha(t)\Phi_x u| \leq ld(\Phi^{-1}_{tx} \Phi^{-1}_{tx} \alpha(t)\Phi_x u), \alpha(t)x) \leq l\epsilon.
\]
Hence, we get
\[
|u_i| \leq l\epsilon e^{-(a_i-2\epsilon)s}\left(\frac{e^{(a_i-2\epsilon)s}}{-\epsilon \epsilon + e^{(a_i-\epsilon)s}}\right),
\]
and when \(s\) is great enough, we get \(|u_i| \leq l\epsilon e^{-(a_i-2\epsilon)s} \). □

Proof of Proposition 4.3. It is a direct consequence of Proposition 4.2 and Lemma 4.1. □
5 Absolutely Continuous Case

Now we consider $\mu$ is absolutely continuous. Then $\gamma_i = d_i$. Note here, we do not really need $\alpha$ to preserve $\mu$, because all volume forms are equivalent, and the limit what we consider will not change in use of the following well-known result.

**Lemma 5.1.** Suppose $X$ is a Euclidean space, $\mu$ is a Borel measure on $X$, and let $\lambda$ be the Lebesgue measure, then $\mu$ is absolutely continuous with respect to $\lambda$ if and only if

$$\liminf_{r \to 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} < \infty, \mu \text{ a.e. } x \in X.$$  

For $x \in \Gamma_1$, we have an embedding $\Phi_x : B(l(x)^{-1}) \to M$, then the pullback of $\mu$ restricted to the image of $\Phi_x$, $\Phi_x^* \mu(\cdot) := \mu(\Phi_x(\cdot))$ is also absolutely continuous, because $\Phi_x$ is smooth and has bounded derivative. Noting that, for the Bowen ball, due to Lemma 4.1, it can be controlled both sides by the images of corresponding rectangles in the tangent space. So we need and only need to evaluate the limit of these rectangles on tangent space. By Lemma 5.1, we do need to do that for the standard volume form $\lambda$. By direct calculation,

$$\lambda\left(\prod_{i \leq u} B_i\left(0, \frac{e^{-(a_i+2)c}s}{(m+1)K}\right) \times B_{u+1}\left(0, \frac{e}{(m+1)K}e^{-2c}s\right) \prod_{i \geq u+2} B_i\left(0, \frac{e}{(m+1)K}e^{-(a_i+2)c}s\right)\right)$$

$$= C_1(K, m, e)e^{-s\left(\sum_{i=1}^D d_i a_i\right)-2m\epsilon s};$$

$$\lambda\left(\prod_{i \leq u} B_i\left(0, l(m+1) e^{-(a_i-2)c}s\right) \times B_{u+1}\left(0, l(m+1)e\right) \prod_{i \geq u+2} B_i\left(0, l(m+1)e^{-(a_i-2)c}s\right)\right)$$

$$= C_2(l, m, e)e^{-s\left(\sum_{i=1}^D d_i a_i\right)+2m\epsilon s}.$$  

Take limit on $s$, and let $\epsilon \to 0$, we get

$$\lim \liminf_{\epsilon \to 0} \liminf_{s \to \infty} \frac{-\log(\mu(B(\alpha, F^p_s, x, \epsilon)))}{s} = \lim \limsup_{\epsilon \to 0} \sup_{s \to \infty} \frac{-\log(\mu(B(\alpha, F^p_s, x, \epsilon)))}{s} \mu \text{ a.e. }, \quad (9)$$

and it equals to

$$\sum_{i=1}^D d_i a_i.$$  

We now give another proof of inequality (7) which uses the idea from [17]. It is well known that $M$ can be smoothly embedded into $\mathbb{R}^{2m+1}$. We denote the embedding map by $\iota$. And hence $\iota(M)$ is a smooth submanifold of $\mathbb{R}^{2m+1}$, we then pick a bounded tubular neighborhood $N$ of $\iota(M)$, which we can regard as a normal bundle of $\iota(M)$. For any $f \in \text{Diff}^{1+r}(M)$ preserving $\mu$, we can define $F \in \text{Diff}^{1+r}(N)$ such that $F \circ \iota = \iota \circ f$, and $\iota(M)$ is a closed invariant set of $F$, $F|\iota(M)$ preserve $\iota_*\mu$. Then the dynamics of $f$ on $M$ is the same (in the smooth sense) with the
Here, we use the same scale of volume form on $\iota(M)$. The idea to define $F$ is through local charts, and let $F$ preserve (as $f$) the base $\iota(M)$, but contract in all the normal directions. Therefore, by this way, we can identify $\iota(M)$ with $M$, and without confusion still use the same notations, for example, $\alpha$ is action, $d(\cdot, \cdot)$ is the metric. Below, by Lemma 5.1 we can always use $\mu$ as a volume form on $M$, or induced volume form on any submanifolds of $M$.

**Definition 5.1.** $E$ is a normed space with the splitting $E = E_1 \oplus E_2$. We call a subset $G \subset E$ is a $(E_1, E_2)$-graph if there exists an open $U \subset E_2$ and a $C^1$ map $\Psi : U \to E_1$ satisfying $G = \{ x + \Psi(x) | x \in U \}$. The dispersion of $G$ is the number $\sup \{ ||\Psi(x) - \Psi(y)|| / ||x - y|| | \forall x, y \in U \}$.

For the specific $f$, we have splitting $TM = E^u \oplus E^{cs}$, where $E^{cs} := E^c \oplus E^s$. Fix $\epsilon > 0$, by Egorov’s theorem, we can choose a compact set $L \subset M$ with $\mu(L) \geq 1 - \epsilon$ such that the splitting is continuous with the change of $x \in L$. We do need also to require $K$ to meet that all holonomy maps from unstable manifold to unstable manifold in the local charts are continuous with respect to the base points, and $L \subset \Gamma_l$ (for $l$ great enough).

**Lemma 5.2.** For $w$ small enough, there exists $v > 0$, such that $\forall x \in K$, $\mu$ a.e. $y \in M$ and $d(x, y) < v$, then $y + E^u(x)$ is a $(E^u(x), E^{cs}(x))$-graph with dispersion $\leq w$. Moreover, when $y \in E^{cs}(x)$, $\epsilon < l(y)^{-1}$ small, $s$ is great, $\mu((y + E^u(x)) \cap B(\alpha, F^p_s, x, \epsilon)) < \mu(E^u(x) \cap B(\alpha, F^p_s, x, 3\epsilon))$.

*Here, we use the same scale of volume form on $E^u$.***

**Proof.** The first one is easy, we only need to prove the second result. To that end, note that

$$(y + E^u(x)) \cap B(\alpha, F^p_s, x, \epsilon) \subset (y + E^u(x)) \cap B(\alpha, F^p_s, y, 2\epsilon),$$

because $y + E^u(x)$ is a $(E^u(x), E^{cs}(x))$-graph with dispersion $c$ small, and then translate them back to $x$ and get the desired result. \qed

By Proposition 4.3 we can easily see the following lemma:

**Lemma 5.3.** For $\mu$ a.e. $x \in M$,

$$\lim_{\epsilon \to 0} \liminf_{s \to \infty} \frac{-\log(\mu(E^u(x) \cap B(\alpha, F^p_s, x, \epsilon)))}{s} = \lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{-\log(\mu(E^u(x) \cap B(\alpha, F^p_s, x, \epsilon)))}{s},$$

and equals to

$$\sum_{i=1}^{n} d_i \max_{t \in \Gamma_p(t) \leq 1} \chi_i(t).$$

*Similar result is true for stable one.*
Proposition 5.1. For μ a.e. x ∈ M,
\[ \lim_{\epsilon \to 0} \lim_{s \to \infty} \inf_{s} -\log(\mu(E^{cs}(x) \cap B(\alpha, F_{s}^{p}, x, \epsilon))) = \lim_{\epsilon \to 0} \lim_{s \to \infty} \sup_{s} -\log(\mu(E^{cs}(x) \cap B(\alpha, F_{s}^{p}, x, \epsilon))) \]
and equals to
\[ \sum_{i=u+2}^{D} d_{i} \max_{t:|p(t)| \leq 1} \chi_{i}(t). \]

Proof. We note that in this case, the Bowen ball in the $E^{c}$ direction will not expand more than $\epsilon$ or contract more than subexponentially $e^{-s\epsilon}$, hence we can use integration on $E^{c}$ and Lemma 5.3, then get the result.

Now we begin our proof of inequality (7). For $x \in L$, there exists $C_{0} > 0$ such that
\[ \mu(B(\alpha, F_{s}^{p}, x, \epsilon)) = C_{0} \int_{E^{cs}(x)} \mu((y + E^{u}(x)) \cap B(\alpha, F_{s}^{p}, x, \epsilon))d(\mu(y)) \]
for all $s$ great enough. By Lemma 5.2, we have that
\[ \mu(B(\alpha, F_{s}^{p}, x, \epsilon)) \leq C_{1}\mu(E^{cs}(x) \cap B(\alpha, F_{s}^{p}, x, \epsilon))\mu((E^{u}(x)) \cap B(\alpha, F_{s}^{p}, x, 3\epsilon)) \]
for some constant $C_{1} > 0$. Take log both sides, and use Lemma 5.3 and Proposition 5.1, then we get the inequality (7).

6 Hyperbolic Case

In this section, we give a proof of Proposition 4.1 in the case of hyperbolic measure. Below without loss of generality, we consider $\mathbb{Z}^{k}$-action with one ergodic element $f$. Note that in the following $\mathbb{Z}^{k}$-action here is not important but the assumption on $f$ is crucial. Indeed for general actions, we can pick one element and consider one of its nontrivial ergodic measures induced to its ergodic component from $\mu$. Let’s remind some settings and results first, $\xi^{u}$ and $\xi^{s}$ are the Sinai partition subordinate to unstable and stable foliations, $\{\mu_{k}^{u}\}$ and $\{\mu_{k}^{s}\}$ are corresponding conditional measures, and $h$ is the metric entropy of $f$. Note that there is no zero exponent in this case. We have two proofs of the inequality (6). Then we exploit the methods from [8] and [9] to deal with the general $\mathbb{R}^{k}$-actions. Interested reader may skip the first two subsections.

6.1 A Proof of (6) in $\mathbb{Z}^{k}$-action case

In [6], they constructed a special countable partition $P$ of $M$ of finite entropy, and satisfies the following properties. Given $0 < \epsilon < \frac{1}{100} c$ (here $c > 0$ is a constant such that for any nontrivial exponent $\chi_{i}$ of $f$, $|\chi_{i}| > c$), let $a = \frac{\max_{i} \{a_{i}\}}{c - \epsilon}$. There exists a set $\Gamma_{1} \subset \Gamma$ of measure $\mu(\Gamma_{1}) > 1 - \epsilon/2$, an integer $N_{0}$ and a number $C_{0} > 1$ such that for $x \in \Gamma_{1}$ and $n > N_{0}$, then
Lemma 12.4.2 from [6]. We include the details here for completeness.

Proof of (6.1).

where

Hence,

and

Now for any

with

Here \( \{\mu_x\} \) is a system of conditional measures associated with \( \eta = \xi^u \vee \mathcal{P}_0^{\infty} \). And for this conditional measures, we have for \( \mu \) a.e. \( x \),

\[
\lim_n -\log \mu_x(\mathcal{P}_0^n(x)) = h_\mu(f, \mathcal{P}), \quad (\text{Lemma 12.4.1 in [6]})
\]

and

\[
\lim_{\epsilon \to 0} \lim_{s \to \infty} \sup_{s} -\log(\mu_x(\mathcal{B}(\alpha, F_\xi^p, x, \epsilon))) \leq h^\mu. \quad (12)
\]

Proof of (6.7). The basic idea is taking \( \eta \) as a partition of each atom of \( \xi^u \), and then going through Lemma 12.4.2 from [6]. We include the details here for completeness.

We have, from Proposition 4.3 that

\[
\lim_{\epsilon \to 0} \lim_{s \to \infty} \sup_{s} -\log(\mu_x(\mathcal{B}(\alpha, F_\xi^p, x, \epsilon))) = h^\mu.
\]

Now fix arbitrary small \( \delta \), such that \( \exists \epsilon_0 \) for any \( \epsilon \leq \epsilon_0 \),

\[
\lim_{s \to \infty} \sup_{s} -\log(\mu_x(\mathcal{B}(\alpha, F_\xi^p, x, \epsilon))) \leq h^\mu + \delta.
\]

Let

\[
B_{s,\epsilon} = \{ y \in \xi^u(x) : \mu_x(\mathcal{B}(\alpha, F_\xi^p, y, \epsilon)) \geq e^{-s(h^\mu+2\delta)} \}
\]

with \( \liminf_s \mu_x(B_{s,\epsilon}) = 1 \) for \( \epsilon \leq \epsilon_0 \) for \( \mu \) a.e. \( x \). Consider now

\[
A_{s,\epsilon} = \{ y \in \xi^u(x) : \mu_x(\mathcal{B}(\alpha, F_\xi^p, y, \epsilon)) \leq e^{-s(h^\mu+4\delta)} \}.
\]

Now for any \( z \in B_{s,\epsilon}, \)

\[
\mu_x(A_{s,\epsilon} \cap B(\alpha, F_\xi^p, z, \epsilon/2)) = \int y \mu_x(A_{s,\epsilon} \cap B(\alpha, F_\xi^p, z, \epsilon/2)) d\mu(y).
\]

If \( \mu_y(A_{s,\epsilon} \cap B(\alpha, F_\xi^p, z, \epsilon/2)) > 0 \), then there exists \( y' \in \eta(y) \cap A_{s,\epsilon} \cap B(\alpha, F_\xi^p, z, \epsilon/2) \) and

\[
\mu_y(A_{s,\epsilon} \cap B(\alpha, F_\xi^p, z, \epsilon/2)) \leq \mu_y'(B(\alpha, F_\xi^p, y', \epsilon)) \leq e^{-s(h^\mu+4\delta)}.
\]

Hence,

\[
\mu_x(A_{s,\epsilon} \cap B(\alpha, F_\xi^p, z, \epsilon/2)) \leq e^{-2s\delta} \mu_x(B(\alpha, F_\xi^p, z, \epsilon)).
\]

Then, pick a maximal \( \epsilon/2 \) separated set in \( B_{s,\epsilon} \), they will cover \( B_{s,\epsilon} \) no more than \( C_u \) times, where \( C_u \) is a constant depend on \( u \). Then \( \mu_x(A_{s,\epsilon} \cap B_{s,\epsilon}) \leq e^{-2s\delta} C_u \), hence we get the result. 

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Before moving forward, we need a similar result for $B(\alpha, F^p_{x}, x, \epsilon)$ as the ball in Borel density theorem. Of course, now we can not use Borel density theorem directly. The following idea comes from [9]. Let’s briefly go through it. We can lift $B(\alpha, F^p_{x}, x, \epsilon)$ to the tangent space, and then take $\Phi_{x}^{-1}B(\alpha, F^p_{s}, x, \epsilon)$ as somewhat rectangles centered at $x$. To actually do this, it is better to deal with $R_t = [-e^{-at}, e^{-at}]^{d_1} \times \cdots \times [-e^{-at}, e^{-at}]^{d_k}$ which is centered at $x$. Pick $x \in \Gamma_1$ be a density point, namely, for ball $B(0, r)$ with radius $r < l(x)^{-1}$, $\mu(\Gamma_1 \cap \Phi_{x}B(0, r)) > \frac{1}{2}\mu(\Phi_{x}(B(0, r))) > 0$. Then for $y \in B(0, r) \cap \Phi_{x}^{-1}\Gamma_1$, consider the collection of rectangles in the Lyapunov chart of $x$ defined by $R(y, t) := y + R_t$. For these defined rectangles, the Besicovitch covering lemma applies and we then have for any $f \in L^1(\mu \circ \Phi_{x})$

$$\lim_{t \to \infty} \frac{1}{\mu(\Phi_{x}(R(y, t)))} \int_{R(y, t)} f d(\mu \circ \Phi_{x}) = f(y) \text{ for } \mu \text{ a.e. } y.$$  

One may also easily see that in our case, two rectangles centered at different point in the chart of $x$, will not affect the limits $\lim_{t \to \infty} \frac{-\log \mu(\Phi_{x}R_t)}{t}$ for $z \in B(0, r)$ and $\tilde{\mu} = \mu_{x}^{\alpha}, \mu_{x}^{u}$ or $\mu$. However, for the sake of simplicity, in the following, we mainly consider $B(\alpha, F^p_{y}, y, \epsilon)$ instead of $\Phi_{x}(R(y, t))$, although the latter is more accurate. This is guaranteed by Lemma 4.1 that makes the two quantities asymptotically the same.

By Lemma 4.1 and the above argument, we have $\Gamma_2 \subset \Gamma_1$ with $\mu(\Gamma_2) > 0$, constants $C_1 < 1$ small and $N_1 \geq N_0$ such that for $y \in \Gamma_2$, $s \geq N_1$, then

(d) $\mu_{x}^{s}(B(\alpha, F^p_{s}, y, \epsilon)) \geq e^{-s(h^u+\epsilon)}, \mu_{x}^{s}(B(\alpha, F^p_{s}, y, \epsilon)) \geq e^{-s(h^u+\epsilon)},$

(e) $\mu_{x}^{s}(\Gamma_2 \cap B(\alpha, F^p_{s}, y, \epsilon)) \geq C_1 e^{-s(h^u+\epsilon)}, \mu_{x}(\Gamma_2 \cap B(\alpha, F^p_{s}, y, \epsilon)) \geq C_1 e^{-s(h^u+\epsilon)}.$

Fix $x \in \Gamma_2$ and let $H = \lim_{t \to 0} \log \mu(B(\alpha, F^p_{s}, x, \epsilon_1)) = -\log(\mu(B(\alpha, F^p_{s}, x, \epsilon_1))).$ Then for $\epsilon_1$ small enough, there exist infinitely many integers $s$ such that $\mu(B(\alpha, F^p_{s}, x, \epsilon_1)) < e^{-s(H-\epsilon)}$.

We can take $s$ or $N_1$ great enough, to make sure the diameter of atoms of $\mathcal{P}_{sa}$ is very small compare to $B(\alpha, F^p_{s}, x, \epsilon_1)$. Now we consider the number

$$N = \# \left\{ \text{atoms of } \mathcal{P}_{sa} \text{ intersecting } \Gamma_2 \cap B(\alpha, F^p_{s}, x, \frac{\epsilon_1}{2}) \right\}.$$  

On the one hand, we can easily get the upper bound on $N$:

$$N \leq C_0 e^{2sa(h+2\epsilon)} e^{-s(H-\epsilon)}.$$  

On the other hand, for $x \in \Gamma_2$, we have (e). Then for any $y \in \xi^s \cap \Gamma_2 \cap B(\alpha, F^p_{s}, x, \frac{\epsilon_1}{4})$, we have

$$\mu_{x}^{s}(\mathcal{P}_{sa}^{0}(y)) = \mu_{y}^{s}(\mathcal{P}_{sa}^{0}(y)) \leq C_0 e^{-sa(h-2\epsilon)}.$$  

Then

$$\# \left\{ \text{atoms of } \mathcal{P}_{sa}^{0} \text{ intersecting } \Gamma_2 \cap B(\alpha, F^p_{s}, x, \frac{\epsilon_1}{4}) \right\} \geq C_0^{-1} C_1 e^{sa(h-2\epsilon)} e^{-s(h+\epsilon)}.$$  

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Now let's fix one of these atoms, say $P_u$, and choose $y \in P_u \cap \Gamma_2 \cap B(\alpha, F^p_s, y, \frac{\epsilon_1}{4})$. By (e) and (b), we have for any $z \in \eta(y) \cap \Gamma_2 \cap B(\alpha, F^p_s, y, \frac{\epsilon_1}{4})$,
\[
\mu_y(P_{sa}^0(z)) = \mu_z(P_{sa}^0(z)) \leq C_0 e^{-sa(h-2\epsilon)}.
\]
Thus
\[
\# \left\{ \text{atoms of } P_{sa}^0 \text{ intersecting } \eta(y) \cap \Gamma_2 \cap B(\alpha, F^p_s, y, \frac{\epsilon_1}{4}) \right\} \geq C^{-1}_0 e^{sa(h-2\epsilon)} e^{-s(h^u+\epsilon)}.
\]

Also, we know that $\eta(y) \subset P_\infty \subset P_u$, we then have
\[
\# \left\{ \text{atoms of } P_{sa}^0 \text{ intersecting } P_u \cap \Gamma_2 \cap B(\alpha, F^p_s, y, \frac{\epsilon_1}{4}) \right\} \geq C^{-1}_0 e^{sa(h-2\epsilon)} e^{-s(h^u+\epsilon)}.
\]

Let $P_s$ be one atom, then $P_u \cap P_s$ is an atom of $P_{sa}^0$, which intersect $\Gamma_2 \cap B(\alpha, F^p_s, x, \frac{\epsilon_1}{4})$ for some $y \in P_u \cap \Gamma_2 \cap B(\alpha, F^p_s, y, \frac{\epsilon_1}{4})$. Hence $P_{sa}^{-1}P_{sa}^0$ is an atom in $\Gamma_2 \cap B(\alpha, F^p_s, x, \frac{\epsilon_1}{4})$. Thus
\[
N \geq C_0^{-2}C_1 e^{sa(h-2\epsilon)} e^{-s(h^u+h^s+2\epsilon)}.
\]

Combining the upper bound and lower bound of $N$, we get $H \leq h^u + h^s$. And this completes the proof of inequality (6).

**Remark 6.1.** The scale $a$ is important in this argument because we need the atom of the partition to be small compared to the Bowen ball. That is also why this argument can not be extended to the general case unless we have a good control in the neutral direction (for example, $f$ is an isometry in these directions). Actually, we can do such a control by dividing the atom $P_{sa}^{-1}P_{sa}^0(x)$ into smaller atoms, but it will generate one nontrivial term on the right hand side of the inequality (6).

### 6.2 Proof of Proposition 4.1 in $\mathbb{Z}^k$-action case

The following is the main result of [9], which can be applied easily to our case.

**Theorem 6.1** (Schmeling and Troubetzkoy). Suppose $\mu$ is a regular hyperbolic measure. Let $a_1, \ldots, a_D$ be negative numbers and $R_t = [-e^{a_1 t}, e^{a_1 t}] d_1 \times \cdots \times [-e^{a_D t}, e^{a_D t}] d_D$. Then for $\mu$ a.e. $z$ the following limit exists
\[
\lim_{t \to \infty} \frac{-\log \mu(\Phi_z R_t)}{t} = -\sum_{j \leq D} a_j \gamma_j.
\]

**Remark 6.2.** The proof uses the same combinatorial idea of [8]. The only modification is for rectangles, the Borel density Theorem will not be true. However, one can still use rectangles to get a somewhat density result in a set of positive measure. More about this, see [9].

**Proof of Proposition 4.1** This is a direct result of Lemma 4.1 and Theorem 6.1. In the proof of the latter theorem, it also prove the same kind of inequalities like (6) and (7). We should also note that in our case all Borel probability measure are regular. \qed
6.3 $\mathbb{R}^k$-action Case

We are now in the position to prove Proposition 4.1 in the $\mathbb{R}^k$-action case. Let’s first remind the setting in this case, $\alpha$ is a $C^{1+r}$ $\mathbb{R}^k$-action on $M$ with an ergodic hyperbolic invariant measure $\mu$; $a_i = \max_{p(t) \leq 1} \chi_i(t)$; $f$ is an element satisfying certain conditions (See section 4, after Proposition 4.1); $R_t(x, \epsilon) := \{e^{-(a_1+\delta)}t, \ldots \times [e^{-(a_1+\epsilon)}t, e^{-(a_1+\epsilon)}t]\}$ is a rectangle defined in the tangent space $T_x M$, where $\mathbb{R}^k(e^{-\epsilon t})$ denotes the $k$ dimensional rectangle with equal sides of length $e^{-\epsilon t}$ in the subspace $T_x \mathcal{O}$; similarly, $\bar{R}_t(x, \epsilon) := \{e^{-(a_1-\epsilon)}t, \ldots \times [e^{-(a_1-\delta)}t, e^{-(a_1-\delta)}t]\}$. Lemma 4.1 tells us that for a.e. $x \in M$, and $s$ small and $\epsilon < 10^{-10}$, we have $R_s(x, \epsilon) \subset B(\alpha, \mathbb{R}^k \delta, x, \epsilon) \subset \bar{R}_s(x, \epsilon)$. It is easy to see that the central and stable bundles of $f$ is integrable, the same is true for central and unstable, while it is not for stable and unstable bundles.

As in subsection 6.1, and also due to Rudolph’s result about rectangular tiling for $\mathbb{R}^k$-actions (20), there is a special countable partition $\mathcal{P}$ of $M$ of finite entropy, whose elements are rectangles consists of pieces of orbits of the action, and satisfies the following properties. Given $0 < \epsilon < \frac{\epsilon}{10}$ (here $\epsilon > 0$ is a constant such that for any nontrivial exponent $\chi_i$ of $f$, $|\chi_i| > \epsilon$), let $a = 1 + \left[ \frac{\max \{a_i\}}{\epsilon - \epsilon} \right]$, where $\left[ \right]$ means the integer part. There exists a set $X \subset \mathcal{P}$ of measure $\mu(X) > 1 - \epsilon/2$, an integer $N_0$ and a number $C_0 > 1$ such that for $x \in \Gamma_1$ and $n > N_0$, then

(a)

$$C_0^{1 - \epsilon} e^{-2na(h+2\epsilon)} \leq \mu(P^{-na}(x)) \leq C_0 e^{-2na(h-2\epsilon)}.$$ 

(b)

$$\xi^s(x) \cap \bigcap_{n \geq 0} P^0_n(x) \supset B^s(x, e^{-N_0}),$$

$$\xi^u(x) \cap \bigcap_{n \geq 0} P^{-n}_n(x) \supset B^u(x, e^{-N_0}).$$

Let $\delta > 0$ with $\delta < \frac{\epsilon}{10}$, there exist $x$ and $\Gamma_2 \subset \Gamma_1 \cap \Phi_y(B(l(x)^{-1})) (x \in \Gamma_2)$ with $\mu(\Gamma_2) > 0$, constants $C_1 < 1$ small and $N_1 \geq N_0$ such that for $y \in \Gamma_2$, $s \geq N_1$, then

(c)

$$e^{-s(h+s+\epsilon)} \leq \mu_y(\Phi_y R_s(y, \delta)) \leq e^{-s(h-s+\epsilon)} \leq \mu_y(\Phi_y R_s(y, \delta)) \leq e^{-s(h-s+\epsilon)}.$$ 

(d)

$$\mu_y(\Gamma_2 \cap \Phi_y R_s(y, \delta)) \geq C_1 e^{-s(h+s+\epsilon)} , \mu_y(\Gamma_2 \cap \Phi_y R_s(y, \delta)) \geq C_1 e^{-s(h+s+\epsilon)}.$$ 

Let $\eta^{cu} = P^0_{-\infty}$, and $\eta^{cs} = P^0_{0}$, denote the conditional measure $\mu^{cu} \mu^{cs}$ respectively. Locally, $\mu^{cu}_{\alpha} = \mu^{cs}_{\alpha} \times \lambda$ up to a positive scalar multiple depending on $x$, and similarly for $\mu^{cs}_{\alpha}$. Slightly modify the argument of (6.1), one can get from SMB Theorem the following: for a.e. $x$

$$h - \epsilon \leq \lim_{n \to \infty} \frac{-\log(\mu^{cs}_{\alpha}(P^{-n}_n(x)))}{n} \leq h,$$

(13)
\[ h - \epsilon \leq \lim_{n \to \infty} \frac{-\log(\mu_x^c(P^n_0(x)))}{n} \leq h, \quad (14) \]

\[ e^{-n(h^*+\epsilon)} \leq \mu_x^c(\Phi y R_n(y, \delta)) \leq e^{-n(h^*-\epsilon)} \]
\[ e^{-n(h^*+\epsilon)} \leq \mu_x^c(\Phi y R_n(y, \delta)) \leq e^{-n(h^*-\epsilon)}. \]

Note that we can assume, not taking the limit but when \( n \geq N_1 \), the above inequalities hold for all \( x \in \Gamma_1 \).

Let \( b > 0 \) small compared to the radius of maximal open balls inside \( \Phi y^{-1}\xi u(x) \) and \( \Phi y^{-1}\xi s(x) \), \( S_b(x) := [-b, b]^{d_1} \times \cdots \times [-b, b]^{d_0} \times \{0\} \subset T_x M \) be a section, choose \( c > 0 \), such that

\[ \Phi y^{-1} : \bigcup_{t \in [-2c, 2c]^k} \alpha(t)\Phi y S_b(x) \to B(l(x)^{-1}) \text{ is injective}, \quad \bigcup_{t \in [-2c, 2c]^k} \alpha(t)\Phi y S_b(x) \subset \Phi y B(l(x)^{-1}). \]

Let \( X = \bigcup_{t \in [-c, c]^k} \alpha(t)\Phi y S_b(x) \), note that we can assume \( X \subset \mathcal{P}(x) \). Define a transversal measure \( \mu^T \) on \( S_b(x) \):

\[ \mu^T(A) = \lim_{\kappa \to \infty} \frac{\mu(\bigcup_{t \in [-\kappa, \kappa]^k} \alpha(t)\Phi y A)}{\mu(\bigcup_{t \in [-\kappa, \kappa]^k} \alpha(t)\Phi y S_b(x))}, \quad A \subset S_b(x). \]

We can further assume that \( \mu^T(S_b(x) \cap \Phi y^{-1}\Gamma_2) > 0, \mu^T(S_b(x) \cap \Gamma_2) > 0 \) and \( \mu^T(S_b(x) \cap \Gamma_2) > 0 \). Define a projection map \( \pi : X \to S_b(x), X \ni z \mapsto S_b(x) \cap \Phi y^{-1} \bigcup_{t \in [-c, c]^k} \alpha(t)\Phi y z \).

Note that, \( \mu|_X \) has a local product structure, namely \( X \approx S_b(x) \times [-c, c]^k \) and \( \mu|_X \approx (\Phi y)_* \mu^T \times \lambda \) (up to a scalar multiple). Assume \( \epsilon = \frac{C}{\log n} \) for further use.

Define \( Q_n(y, \delta) = \bigcup_{z} \mathcal{P}^0_{-\text{an}}(z) \), where the union is taken over \( y \in \Gamma_2 \cap \Phi y 2 R_n(y, \delta) \) for which \( \mathcal{P}^0_{-\text{an}}(z) \cap \xi u(y) \cap \Phi y R_n(y, \delta) \neq \emptyset \) and \( \mathcal{P}^0_{-\text{an}}(z) \cap \xi s(y) \cap \Phi y R_n(y, \delta) \neq \emptyset \). Then when \( n \) is great enough, \( \Phi y R_n(y, \delta) \cap \Gamma_2 \subset Q_n(y, \delta) \cap \Gamma_2 \subset \Phi y 4 R_n(y, \delta) \) and \( \mathcal{P}^0_{-\text{an}}(z) \subset Q_n(y, \delta) \) if \( z \in Q_n(y, \delta) \). Define \( \bar{Q}_n(y, \delta) \) similarly by replacing \( R_n(y, \delta) \) by \( \bar{R}_n(y, \delta) \). Then define two classes \( \mathcal{R}(n) \) and \( \mathcal{F}(n) \) as in [8]: \( \mathcal{R}(n) = \{ \mathcal{P}^-_{-\text{an}}(y) \subset X : \mathcal{P}^-_{-\text{an}}(y) \cap \Gamma_2 \neq \emptyset \}, \mathcal{F}(n) = \{ \mathcal{P}^0_{-\text{an}}(y) \subset X : \mathcal{P}^-_{-\text{an}}(y) \cap \Gamma_2 \neq \emptyset \}. \)

Our idea is simple, we use a combinatorial argument from [8] to obtain our result. Basically, we consider the number of atoms in \( \mathcal{F}(n) \) intersecting \( Q_n(y, \delta) \). We find that this number is greater than the number of atoms in \( \mathcal{F}(n) \) intersecting \( \Phi y 4 R_n(y, \delta) \), and hence we can get the desired result. In the rest of this section, the argument works when taking into account \( \bar{Q}_n(y, \delta) \) instead of \( Q_n(y, \delta) \).

Let \( A \subset X \), using the same notations as in [8] define

\[ N(n, A) = \# \{ R \in \mathcal{R}(n) : R \cap A \neq \emptyset \}, \]
\[ N^*(n, y, A) = \# \{ R \in \mathcal{R}(n) : R \cap \xi u(y) \cap \Gamma_1 \cap A \neq \emptyset \}, \]
\[ N^{cs}(n, y, A) = \# \{ R \in \mathcal{R}(n) : R \cap \xi s(y) \cap \Gamma_1 \cap A \neq \emptyset \}, \]
\[ N^{cu}(n, y, A) = \# \{ R \in \mathcal{R}(n) : R \cap \xi u(y) \cap \Gamma_1 \cap A \neq \emptyset \}, \]
\[ N^{cu}(n, y, A) = \# \{ R \in \mathcal{R}(n) : R \cap \xi s(y) \cap \Gamma_1 \cap A \neq \emptyset \}. \]
\[ \hat{N}^s(n, y, A) = \# \{ R \in \mathcal{F}(n) : R \cap \xi^s(y) \cap A \neq \emptyset \}, \]
\[ \hat{N}^{cs}(n, y, A) = \# \{ R \in \mathcal{F}(n) : R \cap \xi^{cs}(y) \cap A \neq \emptyset \}, \]
\[ \hat{N}^u(n, y, A) = \# \{ R \in \mathcal{F}(n) : R \cap \xi^u(y) \cap A \neq \emptyset \}, \]
\[ \hat{N}^{cu}(n, y, A) = \# \{ R \in \mathcal{F}(n) : R \cap \xi^{cu}(y) \cap A \neq \emptyset \}. \]

The lemmata in the sequel are preparatory results, which are similar to those in [8] except some replacement of balls by rectangles. By this reason, we omit the proof here for simplicity. We will use $C$ to denote any constant afterwards.

**Lemma 6.1** (Lemma 1, [8]). For each $y \in X \cap \Gamma_1$ and integer $n \geq N_1$, then
\[
\mu^s_y(\Phi_x R_n(y, \delta)) \cdot C^{-1} e^{anh+ane} \leq N^s(n, y, Q_n(y, \delta)) \leq \mu^s_y(\Phi_x 4R_n(y, \delta)) \cdot Ce^{anh+ane},
\]
\[
\mu^u_y(\Phi_x R_n(y, \delta)) \cdot C^{-1} e^{anh+ane} \leq N^u(n, y, Q_n(y, \delta)) \leq \mu^u_y(\Phi_x 4R_n(y, \delta)) \cdot Ce^{anh+ane}.
\]

**Lemma 6.2.** For almost every $y \in X \cap \Gamma_1$ and integer $n \geq N_1$, then
\[
N^s(n, y, Q_n(y, \delta)) \leq N^{cs}(n, y, Q_n(y, \delta)) \leq N^s(n, y, Q_n(y, \delta)) \cdot C e^{3ane},
\]
\[
N^u(n, y, Q_n(y, \delta)) \leq N^{cu}(n, y, Q_n(y, \delta)) \leq N^u(n, y, Q_n(y, \delta)) \cdot C e^{3ane}.
\]

**Proof.** Combine inequalities (c), (13) and (14) with Lemma 6.1. \qed

**Lemma 6.3** (Lemma 2, [8]). For each $y \in X \cap \Gamma_2$ and integer $n \geq N_1$, then $\mu(\Phi_x R_n(y, \delta)) \leq N(n, Q_n(y, \delta)) \cdot C e^{-2anh+2ane}$.

**Lemma 6.4** (Lemma 3, [8]). For each $x \in X \cap \Gamma_2$ and integer $n \geq N_1$, we have
\[
N(n, Q_n(y, \delta)) \leq \hat{N}^{cs}(n, x, Q_n(y, \delta)) \cdot \hat{N}^{cu}(n, x, Q_n(y, \delta))Ce^{4ane}.
\]

**Lemma 6.5** (Lemma 4, [8]). For each $y \in X \cap \Gamma_2$ and integer $n \geq N_1$, we have
\[
\hat{N}^s(n, y, P(x)) \leq Ce^{anh+3ane},
\]
\[
\hat{N}^u(n, y, P(x)) \leq Ce^{anh+3ane},
\]
\[
\hat{N}^{cs}(n, y, P(x)) \leq Ce^{anh+6ane},
\]
\[
\hat{N}^{cu}(n, y, P(x)) \leq Ce^{anh+6ane}.
\]
Lemma 6.6. For $\mu$-almost every $y \in X \cap \Gamma_2$ we have
\[
\limsup_{n \to \infty} \frac{\tilde{N}^s(n, y, Q_n(y, \delta))}{N^s(n, y, Q_n(y, \delta))} e^{-7an^e} < 1,
\]
\[
\limsup_{n \to \infty} \frac{\tilde{N}^u(n, y, Q_n(y, \delta))}{N^u(n, y, Q_n(y, \delta))} e^{-7an^e} < 1,
\]
\[
\limsup_{n \to \infty} \frac{\tilde{N}^{cs}(n, y, Q_n(y, \delta))}{N^{cs}(n, y, Q_n(y, \delta))} e^{-13an^e} < 1,
\]
\[
\limsup_{n \to \infty} \frac{\tilde{N}^{cu}(n, y, Q_n(y, \delta))}{N^{cu}(n, y, Q_n(y, \delta))} e^{-13an^e} < 1.
\]

Proof. We use a similar argument of Lemma 5 of [8]. Let’s consider the set
\[
F = \left\{ y \in \Gamma_2 : \limsup_{n \to \infty} \frac{\tilde{N}^s(n, y, Q_n(y, \delta))}{N^s(n, y, Q_n(y, \delta))} e^{-7an^e} \geq 1 \right\}.
\]

For each $z \in F$, there exists an increasing sequence $\{m_j(z)\}_{j=1}^\infty$ of positive increasing integers such that, for $m_j \geq N_1$,
\[
\tilde{N}^s(m_j, z, Q_{m_j}(z, \delta)) \geq C N^s(m_j, z, Q_{m_j}(z, \delta)) e^{7am_j^e} \geq C e^{-m_j h^e + am_j h + 5am_j^e}.
\]

Assume $\mu(F) > 0$, hence we may choose $y \in F$ and small $\tau > 0$ such that $\mu^y_\tau(F \cap \xi^s(y) \cap B^s(y, \tau)) \geq o(\tau) > 0$. Since $B := \{\Phi_x R_n(z, \delta) : z \in F\}$ satisfies Besicovitch covering lemma (21), one can find a sub cover $Q \subset B$ of $F \cap \xi^s(y) \cap B^s(y, \tau)$ of arbitrarily small diameter and finite multiplicity $\rho(\dim \xi^s)$. That is, for any $L > 0$, one can choose a sequence of points $\{z_i \in F \cap \xi^s(y)\}_{i=1}^\infty$ and a sequence of integers $\{t_i\}_{i=1}^\infty$, where $t_i \in \{m_j(z_i)\}_{j=1}^\infty$ and $t_i > L$ for each $i$, such that the collection of rectangles
\[
Q_L = \{\Phi_x 4R_{t_i}(z_i, \delta) : i = 1, 2, \cdots \}
\]
comprises a cover of $F \cap \xi^s(y) \cap B^s(y, \tau)$ with multiplicity not greater than $\rho$. We have
\[
0 < o(\tau) \leq \\
\sum_{\Phi_x R_{t_i}(z_i, \delta) \in Q_L} \mu^y_\tau(\Phi_x R_{t_i}(z_i, \delta)) \\
\leq \\
C \cdot \sum_{\Phi_x R_{t_i}(z_i, \delta) \in Q_L} e^{-t_i(h^e - \epsilon)} \\
\leq \\
C \cdot \sum_{\Phi_x R_{t_i}(z_i, \delta) \in Q_L} \tilde{N}^s(t_i, z, Q_{t_i}(z_i, \delta)) e^{-at_i h - 4at_i \epsilon} \\
\leq \\
C \cdot \sum_{q=L}^{\infty} e^{-aq^h - 4aq^e} \sum_{i : t_i = q} \tilde{N}^s(t_i, z, Q_{t_i}(z_i, \delta)).
\]
Since the multiplicity of the subcover $Q_L$ is at most $\rho$, and $Q_{t_i}(z_i, \delta) \subset \Phi_x R_{t_i}(z_i, \delta)$, each set $Q_{t_i}(z_i, \delta)$ appears in the sum $\sum_{i : t_i = q} \hat{N}^s(t_i, z, Q_{t_i}(z_i, \delta))$ at most $\rho$ times, hence
\[
\sum_{i : t_i = q} \hat{N}^s(t_i, z, Q_{t_i}(z_i, \delta)) \leq \rho \hat{N}^s(q, y, \mathcal{P}(y)).
\]
Therefore, using Lemma 6.5, we have
\[
0 < o(\tau) \leq \mu^s_y(F \cap \xi^s(y) \cap B^s(y, \tau)) \leq C \cdot \sum_{q = L}^{\infty} e^{-aqh - 4aqe} \sum_{i : t_i = q} \hat{N}^s(t_i, z, Q_{t_i}(z_i, \delta)) \leq C \rho \cdot \sum_{q = L}^{\infty} e^{-aqh - 4aqe + \frac{3}{2}h + 3a} \hat{N}^s(q, y, \mathcal{P}(y)) \leq C e^{-La e}.
\]
Since $L$ can be chosen to be arbitrarily large, we get a contradiction. Hence $\mu(F) = 0$. The proof of the others are similar. \[\Box\]

Now let’s finish the proof. Let $\Gamma_3 = \{y \in \Gamma_2| \text{the inequalities in Lemma 6.6 hold for all } n \geq N'\}$. Note that we can always assume $N'$ is great enough that $\mu(\Gamma_3) \geq \mu(\Gamma_2) - \epsilon > 0$. We first prove inequality (6). This argument is almost the same to the proof of Lemma 6 in [8]. For $y \in \Gamma_3$ and $z \in \xi^u(y)$, when $n \geq N'$ then
\[
N^u(n, y, Q_n(y, \delta)) \leq \hat{N}^u(n, y, Q_n(y, \delta)) = \hat{N}^u(n, z, Q_n(y, \delta)) \leq \inf_z \{N^u(n, z, Q_n(y, \delta)) \} e^{7ane}.
\]
We also have
\[
\hat{N}^s(n, y, Q_n(y, \delta)) \times \inf_{z \in \xi^u(y)} \{N^u(n, z, Q_n(y, \delta)) \} \leq N(n, Q_n(y, \delta)),
\]
hence we have
\[
N^s(n, y, Q_n(y, \delta)) \times N^u(n, y, Q_n(y, \delta)) \leq N(n, Q_n(y, \delta)) e^{14ane}.
\]
Combining the above and Lemma 6.1, we get that
\[
\mu^s_y(\Phi_x R_n(y, \delta)) \mu^u_y(\Phi_x R_n(y, \delta)) e^{2anh - 6ane} \leq N(n, Q_n(y, \delta)) e^{14ane}.
\]
Together with
\[
N(n, Q_n(y, \delta)) \leq \frac{\mu(Q_n(t, \delta))}{\min\{\mu(P_{\hat{\Pi}_n}^{-1}n)(z) : z \in Q_n(y, \delta) \cap \Gamma_1\}} \leq \mu(\Phi_x AR_n(y, \delta)) \cdot \mu^s_y(\Phi_x R_n(y, \delta)) \cdot C e^{2anh + 2ane},
\]

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we have

\[ \mu_y^n(\Phi_x R_n(y, \delta)) \mu_y^n(\Phi_x R_n(y, \delta)) \leq \mu(\Phi_x A R_n(y, \delta)) \cdot C e^{20 \alpha n}. \]

This completes the proof of inequality (5).

To prove inequality (7), we consider \( \tilde{Q}_n(y, \delta) \). We first note that we can have a version of similar results (from Lemma 6.1 to Lemma 6.6) when replacing \( Q_n(y, \delta) \) by \( \tilde{Q}_n(y, \delta) \). Hence we have

\[
\mu(\tilde{Q}_n(y, \delta) \cap \Gamma_3) \leq C N^{cu}(n, x, Q_n(y, \delta)) \cdot \tilde{N}^{cu}(n, x, Q_n(y, \delta)) e^{-2 \alpha n h + 10 \alpha n e} \\
\leq C N^{cu}(n, x, Q_n(y, \delta)) \cdot N^{cu}(n, x, Q_n(y, \delta)) e^{-2 \alpha n h + 36 \alpha n e} \\
\leq C N^{u}(n, x, Q_n(y, \delta)) \cdot N^{u}(n, x, Q_n(y, \delta)) e^{-2 \alpha n h + 42 \alpha n e} \\
\leq C \mu_y^n(\Phi_x A R_n(y, \delta)) \cdot \mu_y^n(\Phi_x A R_n(y, \delta)) e^{2 \alpha n h + 2 \alpha n e} e^{-2 \alpha n h + 42 \alpha n e} \\
= C \mu_y^n(\Phi_x A R_n(y, \delta)) \cdot \mu_y^n(\Phi_x A R_n(y, \delta)) e^{44 \alpha n e}.
\]

Combine the above with \( \mu(\Phi_x R_n(y, \delta)) \leq C \mu(\tilde{Q}_n(y, \delta) \cap \Gamma_3) \), we obtain the desired result (7).

Remark 6.3. The argument may be modified to prove the exactness of hyperbolic measure of a \( \mathbb{R}^k \)-action, i.e. the lower and upper pointwise dimensions coincide almost everywhere.

7 Slow Entropy Formula

We now prove slow entropy formula. Let \( \Delta = \sum_{t=1}^D \gamma_t \max_{t \leq t \leq 1} \chi_t(t) \). By Proposition 4.1, there exists \( M' \) with full measure, such that the following holds: for any \( \theta > 0 \), we have \( \forall x \in M' \), \( \exists \epsilon(x) > 0 \), \( s(x) > 0 \), such that \( \forall \epsilon < \epsilon(x), s > s(x) \),

\[ e^{-s(\Delta + \theta)} \leq \mu(B(\alpha, F^p_x, x, \epsilon)) \leq e^{-s(\Delta - \theta)}. \]

Define \( M'_n := \{ x \in M' : \epsilon(x) > \frac{1}{n}, s(x) \leq n \} \). Then \( M' = \bigcup_{n} M'_n \). For \( \delta > 0 \), there exists \( n_0 > 0 \), such that \( \mu(M'_n) > 1 - \frac{\delta}{2} \). And then inside \( M'_n \), there exists a compact subset \( L \) such that \( \mu(L) > 1 - \delta \).

Consider the minimal number \( \# B_s \) of Bowen balls covering \( L \). On one hand, it is easy to see

\[ \# B_s \geq \frac{1 - \delta}{\max_{x \in L} \mu(B(\alpha, F^p_x, x, \epsilon))} \geq (1 - \delta) e^{s(\Delta - \theta)} \]

On the other hand, pick a set of points \( \{ x_i \} \) in \( L \) such that \( d_{F^p_x}(x_i, x_j) \geq \epsilon, \forall i \neq j \). We can choose such set with the maximal number of elements, denote the number by \( \# M_s \) and the set by \( \Xi \). Then \( \{ B(\alpha, F^p_x, x, \epsilon) \}_{x \in \Xi} \) covers \( L \). Furthermore, we can easily see that \( \forall x \in M, \) it is covered by at most \( C_d \) Bowen balls, where \( C_d \) is a constant only depend on dimension \( d \). So

\[ \# B_s \leq \# M_s \leq \frac{C_d}{\min_{x \in \Xi} \mu(B(\alpha, F^p_x, x, \epsilon))} \leq C_d e^{s(\Delta + \theta)}. \]
Combine the above two inequalities together, and note that we can let $\theta \to 0$ when $\epsilon \to 0$, then
\[
sh_\mu(\alpha, p) = \lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{\log \#B_s}{s} = \Delta = \sum_{i=1}^{D} \gamma_i \max_{t : p(t) \leq 1} \chi_i(t).
\]
This completes the proof of Theorem 1.2.

**Remark 7.1.** By the above argument, we can easily see,
\[
\lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{\log \#B_s}{s} = \lim_{\epsilon, \delta \to 0} \liminf_{s \to \infty} \frac{\log \#B_s}{s}.
\]
Hence in the definition of slow entropy,
\[
sh_\mu(\alpha, p) = \lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{\log S_d(\alpha, F_s^p, \epsilon, \delta)}{s} = \lim_{\epsilon, \delta \to 0} \liminf_{s \to \infty} \frac{\log S_d(\alpha, F_s^p, \epsilon, \delta)}{s}.
\]

8 Open Questions and Possible Characterization

We point out some open questions and possible characterization for slow entropy for further study. Of course, the first one is what we left:

**Question 8.1.** Prove slow entropy formula for general invariant measure.

It is not so easy to deal with this problem, as we already mentioned a little bit in the introduction. However, the next one may be a little more interesting.

**Question 8.2.** Can we generalize the definition of slow entropy to more general group actions, say free or amenable group actions, and prove an entropy formula?

Slow entropy is used to determine whether an action has a smooth realization ([5]), so one may encounter problems of general group actions. Then this question is meaningful. However, there are some difficulties for this, for example, Theorem 1.1 is no longer true.

Next, we point out a direction to generalize a slow entropy version SMB theorem. In contrast to the main result in [2] and [3], it seems not possible to have an analogy of SMB Theorem for slow entropy. But, when restricting to one Weyl Chamber or an open cone, the following question arises:

**Question 8.3.** Can one obtain a SMB type theorem for slow entropy when restrict the action to some Weyl Chamber or open cone? Namely: Considering a \(\mathbb{Z}^k\) action \(\alpha\) preserving an ergodic measure \(\mu\), and pick one Weyl Chamber \(C\), let \(\xi\) be a measurable partition on \(M\). Define
\[
\xi_s^{\alpha, C} := \bigvee_{p(t) \leq s, t \in \mathbb{Z}^k \cap C} \alpha(-t)\xi.
\]
For \(\mu\) a.e. \(x\),
\[
\lim_{s \to \infty} \frac{-\log \mu(\xi_s^{\alpha, C}(x))}{s} = \sum_{i} \gamma_i \max_{p(t) \leq 1, t \in C} \chi_i^+(t).
\]
Conjecturally, it is possible to get such type result on any open convex cone, and it will have
the form as slow entropy formula but with one side. This question may be useful to answer the first
question.

There are many characterizations for metric entropy. We wish to make some good analogies
to those. Here, we only consider an extension via Poincaré recurrence, see for example [18].
Similarly, define

$$R_s(x, \varepsilon) = \inf \{ p(t) \mid \alpha(t)x \in B(\alpha, F^p_s, x, \varepsilon), t \in \mathbb{Z}^k \}. $$

**Question 8.4.** Suppose $\alpha$ is a free abelian action, $\mu$ is ergodic, will the following two limits
$$\lim_{\varepsilon \to 0} \liminf_{s \to \infty} \frac{\log(R_s(x, \varepsilon))}{s} \text{ and } \lim_{\varepsilon \to 0} \limsup_{s \to \infty} \frac{\log(R_s(x, \varepsilon))}{s}$$
exist and equal for $\mu$ a.e. $x$? Furthermore, the limit is equal to

$$c(p) \sum \gamma_i \max_{t:p(t) \leq 1} \chi_i(t),$$

here $c(p)$ is a constant depend only on the norm $p$, which may have form $\text{vol}(p)^{-\frac{1}{k}}$.

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