Weil-Étale Cohomology And Special Values Of L-functions At Zero

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Abstract

We construct the Weil-Étale cohomology and Euler characteristics for a subclass of the class of $\mathbb{Z}$-constructible sheaves on the spectrum of the ring of integers of a totally imaginary number field. Then we show that the special value of an Artin L-function of toric type at zero is given by the Weil-Étale Euler characteristic of an appropriate $\mathbb{Z}$-constructible sheaf up to signs. As applications of our result, we will derive some classical formulas of special values of L-functions of tori and their class numbers.

1 Introduction

Throughout this paper, we fix the following notations. $K$ is a totally imaginary number field with ring of integers $O_K$ and Galois group $G_K$. Let $X = \text{Spec}(O_K)$ and $j : \text{Spec}(K) \to X$. By a discrete $G_K$-module we mean a finitely generated abelian group with a continuous $G_K$ action. There are two main aims of this paper.

1. To construct the Weil-Étale cohomology $H^n_W(X, F)$ and Euler characteristic $\chi(F)$ for any strongly-$\mathbb{Z}$-constructible sheaf $F$ on $X$ (see definition 3.1).

2. Let $N$ be a discrete $G_K$-module and let $L(N, s)$ be the Artin L-function associated with the representation $N \otimes \mathbb{Z} \mathbb{C}$ of $G_K$. We will show that $j_*N$ is strongly-$\mathbb{Z}$-constructible and $L^*(N, 0) = \pm \chi(j_*N)$.

The fact that $L^*(N, 0)$ is related to the Euler characteristic of $j_*N$ was established by Bienenfeld and Lichtenbaum and our proof is based on the techniques they developed in [Lic75] and [BL]. However, their Euler characteristic is different from the Weil-Étale Euler characteristic constructed in this paper. The Weil-Étale cohomology in this paper is not the same as the Weil-Étale cohomology constructed by Lichtenbaum in [Lic09a] but rather is based on his ideas in [Lic09b] and [Lic14]. As applications, let $T$ be an algebraic torus over $K$ with character group $\hat{T}$, we obtain a formula for $L^*(\hat{T}, 0)$ which is similar to Ono’s Tamagawa number formula for tori [Ono63] and use it to derive some formulas of Ono [Ono871] and Katayama [Kat91] for the class numbers of tori.

The structure of the paper is as follows. We construct the Weil-Étale cohomology for $\mathbb{Z}$-constructible sheaves in section 2. In sections 3 and 4, we define the class of strongly-$\mathbb{Z}$-constructible sheaves and construct their Weil-Étale Euler characteristics. Our main result $L^*(N, 0) = \pm \chi(j_*N)$ is proved at the end of section 4. Section 5 is for applications and examples. Finally, we have an appendix containing the results about determinants of exact sequences and orders of torsion groups used in this paper.

Acknowledgment: This paper is part of my PhD thesis written at Brown University. I would like to thank my advisor Professor Stephen Lichtenbaum for his guidance and encouragement. Part of this work was written when I was a member of the SFB Higher Invariant Research Group at University of Regensburg. I would like to thank Professor Guido Kings for his support and my friend Yigeng Zhao for many helpful conversations.
2 The Weil-Étale Cohomology Of \( \mathbb{Z} \)-Constructible Sheaves

2.1 The Weil-Étale Complexes

In this section, we define the Weil-étale complex for \( \mathbb{Z} \)-constructible sheaves following the ideas of Lichtenbaum [Lic14]. First, we recall the definition of \( \mathbb{Z} \)-constructible sheaves from [Mil06 page 146].

Definition 2.1. A sheaf \( F \) on \( X \) is \( \mathbb{Z} \)-constructible if

1. there exists an open dense subscheme \( U \) of \( X \) and a finite étale covering \( U' \to U \) such that the restriction of \( F \) to \( U' \) is a constant sheaf defined by a finitely generated abelian group,

2. for any point \( p \) outside \( U \), the stalk \( F_p \) is a finitely generated abelian group.

We say that \( F \) is constructible if in the definition above the restriction of \( F \) to \( U' \) is a constant sheaf defined by a finite abelian group and for any point \( p \) outside \( U \), the stalk \( F_p \) is finite.

Example 2.2. 1. Any constant sheaf defined by a finitely-generated abelian groups.

2. Let \( M \) be a discrete \( G_K \)-module, then \( j_* M \) is a \( \mathbb{Z} \)-constructible sheaf. Furthermore, if \( M \) is finite then \( j_* M \) is constructible.

Definition 2.3. We say \( F \) is a negligible sheaf on \( X \) if there exists a finite set \( S \) of closed points of \( X \) such that \( F = \prod_{p \in S} (i_p)_* M_p \) where \( M_p \) is a finite discrete \( \hat{\mathbb{Z}} \)-module and \( i_p \) is the map \( p \to X \). Note that negligible sheaves are constructible.

Definition 2.4. Let \( F \) be \( \mathbb{Z} \)-constructible, the Weil-étale complex is defined as

\[
R\Gamma^e_w(X, F) := \tau_{\leq 1} R\Gamma^e(X, F) \oplus \tau_{\geq 2} R\text{Hom}_\mathbb{Z}(R\text{Hom}_X(F, \mathbb{G}_m), \mathbb{Z}[-2])
\]

where \( \tau_{\leq n} \) and \( \tau_{\geq n} \) are the truncation functors defined in [Wei94 1.2.7]. This is an object in the derived category of abelian groups. The Weil-étale cohomology are defined as \( H^{n}_{W}(X, F) := h^n(R\Gamma^e_w(X, F)) \).

Proposition 2.5. The Weil-étale cohomology of \( F \) satisfy

\[
H^n_w(X, F) = \begin{cases} 
H^n_{et}(X, F) & n = 0, 1 \\
\text{Hom}_X(F, \mathbb{G}_m)^D & n = 3 \\
0 & n > 3.
\end{cases}
\]

(1)

\[0 \to \text{Ext}^1_X(F, \mathbb{G}_m)^D \to H^3_w(X, F) \to \text{Hom}_X(F, \mathbb{G}_m)^* \to 0.\]

Note that \( A^D := \text{Hom}_\mathbb{Z}(A, \mathbb{Q}/\mathbb{Z}) \) and \( A^* := \text{Hom}_\mathbb{Z}(A, \mathbb{Z}) \).

Proof. From the definition of \( R\Gamma^e_w(X, F) \), we have

\[
H^n_w(X, F) = \begin{cases} 
H^n_{et}(X, F) & n = 0, 1 \\
h^n(R\text{Hom}_\mathbb{Z}(R\text{Hom}_X(F, \mathbb{G}_m), \mathbb{Z}[-2])) & n \geq 2.
\end{cases}
\]

For \( n \geq 2 \), from [Wei94 exercise 3.6.1], there is an exact sequence

\[0 \to \text{Ext}^{3-n}_X(F, \mathbb{G}_m)^D \to H^n_w(X, F) \to \text{Ext}^{3-n}_X(F, \mathbb{G}_m)^* \to 0.\]

Hence, \( H^n_w(X, F) = 0 \) for \( n \geq 4 \) and \( H^2_w(X, F) \simeq \text{Hom}_X(F, \mathbb{G}_m)^D \). \( \square \)
To compute the Weil-étale cohomology of $\mathbb{Z}$, we need the following result.

**Theorem 2.6.** Let $K$ be a totally imaginary number field. Then

$$H^n_{\text{et}}(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \text{Pic}(O_K)^D & n = 2 \\ (O_K^*)^D & n = 3 \\ 0 & n > 3 \end{cases}$$

and

$$H^n_{\text{et}}(X, \mathcal{O}_m) = \begin{cases} \mathbb{O}_K^* & n = 0 \\ \text{Pic}(O_K) & n = 1 \\ 0 & n = 2 \\ \mathbb{Q}/\mathbb{Z} & n = 3 \\ 0 & n > 3 \end{cases}$$

(2)

**Proof.** For $H^n_{\text{et}}(X, \mathbb{G}_m)$ see [Mil06 II.2.1]. For $H^n_{\text{et}}(X, \mathbb{Z})$, we can apply the Artin-Verdier duality [Mil06 II.3.1].

**Proposition 2.7.** The Weil-étale cohomology of $\mathbb{Z}$ is given by

$$H^n_{\text{W}}(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ (\mu_K)^D & n = 3 \\ 0 & n > 3 \end{cases}$$

(3)

**Proof.** This is an application of proposition 2.5 and theorem 2.6 for $F = \mathbb{Z}$.}

For each place $v$ of $K$, let $K_v$ be the completion of $K$ at $v$ and $j_v$ be the map $\text{Spec}(K_v) \to X$.

**Definition 2.8.** For a sheaf $\mathcal{F}$ on $X$, we define the Betti sheaf $\mathcal{F}_B$ to be

$$\mathcal{F}_B := \prod_{v \in S_\infty} (j_v)_*(j_v)^* \mathcal{F}.$$ Note that $\mathcal{F} \mapsto \mathcal{F}_B$ is an exact functor. Moreover, there is a natural map $\mathcal{F} \to \mathcal{F}_B$ obtained by taking the direct sum over all infinite place $v$ of $K$ of the map $\mathcal{F} \to (j_v)_*(j_v)^* \mathcal{F}$.

**Lemma 2.9.** For each infinite place $v$ of $K$, let $\mathcal{F}_{K_v}$ be the stalk of $\mathcal{F}$ at $K_v$. Then

$$H^n_{\text{et}}(X, \mathcal{F}_B) = \begin{cases} \prod_{v \in S_\infty} \mathcal{F}_{K_v} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$ 

**Proof.** This follows from the fact that $(j_v)_*$ is an exact functor and $G_{K_v} = 0$ for $v \in S_\infty$.}

### 2.2 The Regulator Pairing

We want to define a pairing for every étale sheaf on $X$ such that it generalizes the construction of the regulator of a number field when the sheaf is $\mathbb{Z}$. There is a natural map

$$\Lambda_K : \frac{H^0_{\text{et}}(X, (\mathcal{O}_m)_B)}{H^0_{\text{et}}(X, \mathcal{G}_m)} = \prod_{v \in S_\infty} \frac{K_v^*}{\mathcal{O}_K^*} \to \mathbb{R}$$

$$u = (u_1, ..., u_v) \mapsto \sum_{v \in S_\infty} \log |u_v|_v.$$ Note that by the product formula, $\Lambda_K$ is well-defined.
Definition 2.10. Let \( F \) be an étale sheaf on \( X \). The regulator pairing for \( F \) is defined as

\[
\langle \cdot, \cdot \rangle_F : \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \times \text{Hom}_X(F,\mathbb{G}_m) \to \mathbb{R}
\]

(4)

let \( \alpha \) and \( \phi \) be elements of \( \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \) and \( \text{Hom}_X(F,\mathbb{G}_m) \). By functoriality, \( \phi \) induces a map

\[
\phi_B : \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \to \frac{H^0_{et}(X,(\mathbb{G}_m)_B)}{H^0_{et}(X,(\mathbb{G}_m))} = \prod_{v \in S_{\infty}} K_v^* / \mathbb{O}^*_K.
\]

Define \( \langle \alpha, \phi \rangle_F := \Lambda_K(\phi_B(\alpha)) = \sum_{v \in S_{\infty}} \log |\phi_B(\alpha)|_v \).

Definition 2.11. Suppose the pairing \( \langle \cdot, \cdot \rangle_F \) is non-degenerate modulo torsion. Choose bases \( \{v_i\} \) and \( \{u_j\} \) for the torsion free quotient groups of \( \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \) and \( \text{Hom}_X(F,\mathbb{G}_m) \) respectively. Define \( R(F) := |\det(\langle v_i, u_j \rangle_F)| \) to be the regulator of \( F \). This definition does not depend on the choice of bases.

Example 2.12. 1. If \( F \) is constructible then \( \text{Hom}_X(F,\mathbb{G}_m) \) and \( \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \) are finite groups. Thus, the pairing is non-degenerate modulo torsion and \( R(F) = 1 \).

2. Consider the constant sheaf \( \mathbb{Z} \) on \( X \), the regulator pairing in this case is

\[
(\prod_{v \in S_{\infty}} \mathbb{Z}) / \mathbb{O}^*_K \to \mathbb{R}
\]

Let \( \{u_1,..u_{r_1+r_2-1}\} \) be a \( \mathbb{Z} \)-basis for \( O_K^*/\mu_K \). Let \( \{v_1,..v_{r_1+r_2-1}\} \) be any \( r_1+r_2-1 \) embeddings of \( K \) into \( \mathbb{C} \) considered as a basis for \( \prod_{v \in S_{\infty}} \mathbb{Z} / \mathbb{Z} \). Then \( \langle v_i, u_j \rangle = \log |u_j|_{v_i} \). The determinant of the matrix \( (\log |u_j|_{v_i}) \) is the regulator \( R \) of the number field \( K \). Hence \( R(\mathbb{Z}) = R \).

3. Let \( T \) be an algebraic torus over a totally imaginary number field \( K \). Then the regulator \( R(j_*\tilde{T}) \) is the same as the regulator \( R_T \) of the torus \( T \) defined in [Ono61].

Lemma 2.13. Let \( f : F \to G \) be a morphism of sheaves. Then the following diagram commutes

\[
\begin{array}{ccc}
\frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} & \times & \text{Hom}_X(F,\mathbb{G}_m) \\
\downarrow_{f_B} & & \downarrow_{f^*} \\
\frac{H^0_{et}(X,G_B)}{H^0_{et}(X,G)} & \times & \text{Hom}_X(G,\mathbb{G}_m)
\end{array}
\]

(5)

Proof. Let \( \alpha \) and \( \phi \) be elements of \( H^0_{et}(X,F_B) \) and \( \text{Hom}_X(G,\mathbb{G}_m) \) respectively. Then

\[
\langle f_B(\alpha), \phi \rangle = \Lambda_K(\phi_B(f_B(\alpha))) = \Lambda_K((f^*\phi)_B(\alpha)) = \langle \alpha, f^*\phi \rangle.
\]

3 Strongly-\( \mathbb{Z} \)-Constructible Sheaves

3.1 Definitions And Examples

Definition 3.1. An étale sheaf \( F \) on \( X = \text{Spec}(O_K) \) is called strongly-\( \mathbb{Z} \)-constructible if it satisfies the following conditions:

4
1. It is \( \mathbb{Z} \)-constructible.

2. The map \( H^0_{\text{et}}(X, \mathcal{F}) \to H^0_{\text{et}}(X, \mathcal{F}_B) \) has finite kernel.

3. \( H^1_{\text{et}}(X, \mathcal{F}) \) and \( H^2_{\text{et}}(X, \mathcal{F}) \) are finite abelian groups.

4. The regulator pairing \((2,10)\) is non-degenerate modulo torsion.

**Remark 3.2.** Our definition is modeled after the definition of quasi-constructible sheaves of Bie
nenfeld and Lichtenbaum (cf. [BL section 4]).

**Example 3.3.**

1. Constant sheaves defined by finitely generated abelian groups.

2. Let \( p \) be a closed point of \( X \) and \( i : p \to X \) be the natural map. Let \( M \) be a finite \( \hat{\mathbb{Z}} \)-module.

Then \( i_!M \) is strongly-\( \mathbb{Z} \)-constructible. A non-example would be \( i_* \mathbb{Z} \). Indeed, \( H^2_{\text{et}}(X, i_* \mathbb{Z}) \simeq H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z} \) which is infinite.

3. Constructible sheaves.

4. Let \( M \) be a discrete \( G_K \)-module. Then \( j_*M \) is strongly-\( \mathbb{Z} \)-constructible (see proposition 3.13).

The following proposition is a direct consequence of \((2,5)\) and the Artin-Verdier duality.

**Proposition 3.4.** Let \( \mathcal{F} \) be a strongly-\( \mathbb{Z} \)-constructible sheaf on \( X \). Then

\[
H^p_{\text{W}}(X, \mathcal{F}) = \begin{cases} 
H^2_{\text{et}}(X, \mathcal{F}) & n = 0, 1 \\
\text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^D_{\text{tor}} & n = 3 \\
0 & n \notin \{0, 1, 2, 3\}.
\end{cases}
\]

\(0 \to H^2_{\text{et}}(X, \mathcal{F}) \to H^2_{\text{W}}(X, \mathcal{F}) \to \text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^* \to 0.

In particular, if \( \mathcal{F} \) is a constructible sheaf, then \( H^n_{\text{W}}(X, \mathcal{F}) = H^n_{\text{et}}(X, \mathcal{F}) \) for all \( n \).

**Proposition 3.5.** Suppose we have an exact sequence of strongly-\( \mathbb{Z} \)-constructible sheaves

\(0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0\) \hspace{1cm} (6)

Then we have long exact sequences of Weil-\( \text{étale} \) cohomology

\(0 \to H^0_{\text{W}}(X, \mathcal{F}_1) \to H^0_{\text{W}}(X, \mathcal{F}_2) \to \ldots \to H^3_{\text{W}}(X, \mathcal{F}_1) \to H^3_{\text{W}}(X, \mathcal{F}_3) \to 0\) \hspace{1cm} (7)

**Proof.** From [We94] exercise 3.6.1, for \( i \in \{1, 2, 3\} \) we have

\(0 \to \text{Ext}^i_{\text{W}}(\mathcal{F}_i, \mathbb{G}_m)^D_{\text{tor}} \to h^1(\text{RHom}_\mathbb{Z}(\text{RHom}_X(\mathcal{F}_i, \mathbb{G}_m), \mathbb{Z}[-2])) \to \text{Ext}^i_{\text{W}}(\mathcal{F}_i, \mathbb{G}_m)^* \to 0.

As \( \mathcal{F}_i \) is strongly-\( \mathbb{Z} \)-constructible, \( \text{Ext}^i_{\text{W}}(\mathcal{F}_i, \mathbb{G}_m) \) is finite and \( \text{Ext}^2_{\text{W}}(\mathcal{F}_i, \mathbb{G}_m)^D_{\text{tor}} \simeq H^1_{\text{et}}(X, \mathcal{F}_i) \). Hence

\( h^1(\text{RHom}_\mathbb{Z}(\text{RHom}_X(\mathcal{F}_i, \mathbb{G}_m), \mathbb{Z}[-2])) \simeq H^1_{\text{et}}(X, \mathcal{F}_i) \).

As \( \text{RHom}(\cdot, \mathbb{G}_m[-1]), \text{R}^1_{\text{et}}(\cdot, \mathbb{G}_m) \) and \( \text{RHom}(\cdot, \mathbb{Z}[-3]) \) are exact functors, we have the following distinguished triangle

\(\text{RHom}_\mathbb{Z}(\text{R}^1_{\text{et}}(X, \mathcal{F}_1^D), \mathbb{Z}[-3]) \to \text{RHom}_\mathbb{Z}(\text{R}^1_{\text{et}}(X, \mathcal{F}_2^D), \mathbb{Z}[-3]) \to \text{RHom}_\mathbb{Z}(\text{R}^1_{\text{et}}(X, \mathcal{F}_3^D), \mathbb{Z}[-3]) \to \text{RHom}_\mathbb{Z}(\text{R}^1_{\text{et}}(X, \mathcal{F}_1^D), \mathbb{Z}[-3])[1]. \) \hspace{1cm} (8)

The long exact sequence of cohomology corresponding to \((8)\) yields

\(H^1_{\text{et}}(X, \mathcal{F}_1) \to H^1_{\text{et}}(X, \mathcal{F}_2) \to H^1_{\text{et}}(X, \mathcal{F}_3) \to H^2_{\text{et}}(X, \mathcal{F}_1) \to H^2_{\text{et}}(X, \mathcal{F}_2) \to \ldots \)

\( H^3_{\text{W}}(X, \mathcal{F}_1) \to H^3_{\text{W}}(X, \mathcal{F}_2) \to H^3_{\text{W}}(X, \mathcal{F}_3) \to 0 \). \hspace{1cm} (9)

Combining \((9)\) with the first 6-term of the long exact sequence of étale cohomology groups corresponding to \((6)\), we have \((7)\).
3.2 Main Properties

We study the main properties of strongly-\(\mathbb{Z}\)-constructible sheaves in this section.

**Proposition 3.6.** Suppose we have an exact sequence of étale sheaves on \(X\)
\[
0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0
\]
where \(\mathcal{F}_3\) is constructible. Then \(\mathcal{F}_1\) is strongly-\(\mathbb{Z}\)-constructible if and only if \(\mathcal{F}_2\) is strongly-\(\mathbb{Z}\)-constructible.

**Proof.** Since the category of \(\mathbb{Z}\)-constructible sheaves is an abelian category \([\text{Mil06} \text{ page 146}], \) condition 1 of \[3.1\] holds for \(\mathcal{F}_1\) if and only if it holds for \(\mathcal{F}_2\).

Since \(H^n_{et}(X, \mathcal{F}_3)\) is finite for all \(n\), \(H^n_{et}(X, \mathcal{F}_1)\) and \(H^n_{et}(X, \mathcal{F}_2)\) differ only by finite groups. As \(H^n_{et}(X, \mathcal{F}_{3,B})\) and \(H^n_{et}(X, \mathcal{F}_3)\) are finite, \((H^n_{et}(X, \mathcal{F}_{3,B})/H^n_{et}(X, \mathcal{F}_1))_R \simeq (H^n_{et}(X, \mathcal{F}_{2,B})/H^n_{et}(X, \mathcal{F}_2))_R\).
Hence, condition 2 and 3 of \[3.1\] holds for \(\mathcal{F}_1\) if and only if they hold for \(\mathcal{F}_2\).

Finally, from lemma \[2.13\] there is a commutative diagram
\[
\begin{array}{ccc}
\left( \frac{H^n_{et}(X, \mathcal{F}_{1,B})}{H^n_{et}(X, \mathcal{F}_1)} \right)_R & \times & \text{Hom}_X(\mathcal{F}_1, \mathbb{G}_m)_R \\
\downarrow \cong & & \downarrow \cong \\
\left( \frac{H^n_{et}(X, \mathcal{F}_{2,B})}{H^n_{et}(X, \mathcal{F}_2)} \right)_R & \times & \text{Hom}_X(\mathcal{F}_2, \mathbb{G}_m)_R
\end{array}
\]
As a result, condition 4 of \[3.1\] holds for \(\mathcal{F}_1\) if and only if it holds for \(\mathcal{F}_2\). \(\square\)

Next we want to show that strongly-\(\mathbb{Z}\)-constructible sheaves are stable under push-forward by a finite morphism. Let \(L/K\) be a finite Galois extension of totally imaginary number fields. Let \(\pi : \text{Spec}(L) \to \text{Spec}(K)\) and \(\pi' : Y = \text{Spec}(O_L) \to X = \text{Spec}(O_K)\) be the natural finite morphisms. We write \(S_{L,\infty}\) and \(S_{K,\infty}\) for the set of infinite places of \(L\) and \(K\) respectively.

**Lemma 3.7.** Let \(v\) be a place of \(K\). Then \(j_v^*\pi'_*\mathcal{F} \simeq \prod_{w|v} (\pi_w)_* j_w^*\mathcal{F}\) where \(j_w : \text{Spec}(L_w) \to Y\) and \(\pi_w : \text{Spec}(L_w) \to \text{Spec}(K_v)\) are the natural maps.

**Proof.** We have the commutative diagram
\[
\begin{array}{ccc}
\prod_{w|v} \text{Spec}(L_w) & \xrightarrow{\prod_{w|v} j_w} & \text{Spec}(O_L) \\
\downarrow \left( \prod_{w|v} \pi_w \right) & & \downarrow \pi' \\
\text{Spec}(K_v) & \xrightarrow{j_v} & \text{Spec}(O_K)
\end{array}
\]
Since \(\pi'_*(j_w)_* = (j_v)_*(\pi_w)_*\), for \(w|v\),
\[
\text{Hom}_{K_v}(j_v^*\pi'_*\mathcal{F}, \prod_{w|v}(\pi_w)_* j_w^*\mathcal{F}) = \text{Hom}_X(\pi'_*\mathcal{F}, \prod_{w|v}(\pi_w)_* j_w^*\mathcal{F}) = \text{Hom}_X(\pi_*\mathcal{F}, \pi'_* \prod_{w|v}(j_w)_* j_w^*\mathcal{F}).
\]
Thus, the adjoint map \(\mathcal{F} \to \prod_{w|v}(j_w)_* j_w^*\mathcal{F}\) induces a canonical map \(j_v^*\pi'_*\mathcal{F} \to \prod_{w|v}(\pi_w)_* j_w^*\mathcal{F}\). Let \(\eta_K = \text{Spec}(K)\), \(\eta_v = \text{Spec}(K_v)\) and similarly for \(\eta_L\) and \(\eta_w\). Then
\[
(j_v^*\pi'_*\mathcal{F})_{\eta_v} = \mathcal{F}_{\eta_L}^{[L:K]} = \left( \prod_{w|v}(\pi_w)_* j_w^*\mathcal{F} \right)_{\eta_v}.
\]
Therefore, \(j_v^*\pi'_*\mathcal{F} \simeq \prod_{w|v}(\pi_w)_* j_w^*\mathcal{F}\). \(\square\)
**Lemma 3.8.** Let $\mathcal{F}$ be a $\mathbb{Z}$-constructible sheaf on $Y$. Then the following hold

1. The norm map induces a natural isomorphism $Nm : \text{Ext}^0_Y(\mathcal{F}, \mathbb{G}_m) \to \text{Ext}^0_X(\pi'_*\mathcal{F}, \mathbb{G}_m)$. 
2. There is a natural isomorphism $H^n_W(X, \pi'_*\mathcal{F}) \simeq H^n_W(Y, \mathcal{F})$. 
3. The sheaf $(\pi'_*\mathcal{F})_B$ is isomorphic to $\pi'_*(\mathcal{F}_B)$. In particular, $H^n_{\text{et}}(X, (\pi'_*\mathcal{F})_B) \simeq H^n_{\text{et}}(Y, \mathcal{F}_B)$.

**Proof.**

1. We begin by describing the map $Nm$. The norm map $N_{L/K}$ induces a morphism of sheaves $N_{L/K} : \pi'_*\mathbb{G}_m,Y \to \mathbb{G}_m,X$. As $\pi'$ is a finite morphism, $\pi'_*$ is an exact functor \cite{Mil06} II.3.6]. Therefore, $\pi'_*$ induces the map $\text{Ext}^0_Y(\mathcal{F}, \mathbb{G}_m,Y) \to \text{Ext}^0_X(\pi'_*\mathcal{F}, \mathbb{G}_m,Y)$. We define $Nm$ to be the composition of the following maps

$$\text{Ext}^0_Y(\mathcal{F}, \mathbb{G}_m,Y) \xrightarrow{\pi'_*} \text{Ext}^0_X(\pi'_*\mathcal{F}, \pi'_*\mathbb{G}_m,Y) \xrightarrow{N_{L/K}} \text{Ext}^0_X(\pi'_*\mathcal{F}, \mathbb{G}_m,X).$$

The fact that $Nm$ is an isomorphism is the Norm theorem \cite{Mil06} II.3.9].

2. As $\pi'_*$ is an exact functor, $H^n_W(X, \pi'_*\mathcal{F}) \simeq H^n_W(Y, \mathcal{F})$. Therefore, $H^n_W(X, \pi'_*\mathcal{F}) \simeq H^n_W(Y, \mathcal{F})$ for $n = 0,1$. In addition, by part 1, we have

$$H^3_W(X, \pi'_*\mathcal{F}) = \text{Hom}_X(\pi'_*\mathcal{F}, \mathbb{G}_m)_\text{tor} \simeq \text{Hom}_Y(\mathcal{F}, \mathbb{G}_m)_\text{tor} = H^3_W(Y, \mathcal{F}).$$

To prove $H^2_W(Y, \mathcal{F}) \simeq H^2_W(X, \pi'_*\mathcal{F})$, we apply the 5-lemma to the following diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & H^2_{\text{et}}(X, \pi'_*\mathcal{F}) \\
& & \downarrow \simeq \downarrow \simeq \\
0 & \longrightarrow & H^2_{\text{et}}(Y, \mathcal{F}) & \longrightarrow & H^2_W(Y, \mathcal{F}) & \longrightarrow & \text{Hom}_Y(\mathcal{F}, \mathbb{G}_m)^* & \longrightarrow & 0
\end{array}
$$

3. Recall from lemma 3.7 that for $w|v$, we have $(j_v)_*(\pi_w)_* = (\pi'_w)_*(j_w)_*$. In addition, $(j_v)^*\pi'_w\mathcal{F} \simeq \prod_{w|v}(\pi_w)_*(j_w)^*\mathcal{F}$. Hence,

$$(\pi'_*\mathcal{F})_B = \prod_{v \in S_{K,\infty}} (j_v)_*(\pi_w)_*(j_w)^*\mathcal{F} \simeq \prod_{v \in S_{K,\infty}} (j_v)_*(\pi'_w)_*(j_w)^*\mathcal{F} = \prod_{w \in S_{\mathbb{L},\infty}} \pi'_w(j_w)_*(j_w)^*\mathcal{F} = \pi'_*(\mathcal{F}_B).$$

Therefore, $H^0_{\text{et}}(X, (\pi'_*\mathcal{F})_B) \simeq H^0_{\text{et}}(X, \pi'_*(\mathcal{F}_B)) \simeq H^0_{\text{et}}(Y, \mathcal{F}_B).$

\[\Box\]

**Proposition 3.9.** If $\mathcal{F}$ is strongly-$\mathbb{Z}$-constructible then so is $\pi'_*\mathcal{F}$ and $R(\pi'_*\mathcal{F}) = R(\mathcal{F})$.

**Proof.** As $\pi'_*$ preserves $\mathbb{Z}$-constructible sheaves \cite{Mil06} page 146], $\pi'_*\mathcal{F}$ is $\mathbb{Z}$-constructible. From lemma 3.8 it remains to show that the regulator pairing of $\pi'_*\mathcal{F}$ is non-degenerate and $R(\pi'_*\mathcal{F}) = R(\mathcal{F})$. They will all follow once we prove the diagram below commutes

$$
\begin{array}{ccc}
\text{Ext}^0_Y(\mathcal{F}, \mathbb{G}_m) & \longrightarrow & \text{Ext}^0_X(\pi'_*\mathcal{F}, \mathbb{G}_m) & \longrightarrow & \mathbb{R} \\
\text{Hom}_Y(\mathcal{F}, \mathbb{G}_m) & \longrightarrow & \mathbb{R}
\end{array}
$$

(11)
Let $\alpha$ and $\phi$ be elements of $H^0_c(X, (\pi'_* F)_B)$ and $\text{Hom}_Y(F, G_m)$. We need to show

$$\Lambda_L \circ \phi_B(\psi(\alpha)) = \Lambda_K \circ Nm(\phi)_B(\alpha).$$

From lemma 3.8, $Nm(\phi) = N_{L/K} \circ \pi'_* \phi$. Let us consider the following diagram

![Diagram](image)

The left square of (13) commutes by functoriality. It is not hard to see the upper triangle on the right is commutative. We shall prove that the lower triangle on the right also commutes. Let $\beta$ be an element of $H^0_c(Y, (G_m)_B) \simeq \prod_{w \in S_{L,\infty}} L_w^*$. Then

$$\Lambda_K(N_{L/K}(\beta)) = \sum_{v \in S_{K,\infty}} \log |N_{L/K}(\beta)_{v}|_v = \sum_{v \in S_{K,\infty}} \sum_{w|v} \log |N_{L_w/K_v}(\beta_w)|_w$$

$$= \sum_{v \in S_{K,\infty}} \sum_{w|v} \log |\beta_w|_w = \sum_{w \in S_{L,\infty}} \log |\beta_w|_w = \Lambda_L(\beta).$$

Therefore, diagram (13) is commutative and from this we deduce equation (12). As a result, diagram (11) commutes. Hence, the proposition is proved. \qed

**Corollary 3.10.** $\pi'_* \mathbb{Z}$ is a strongly-$\mathbb{Z}$-constructible sheaf.

We need the following theorem of Ono.

**Theorem 3.11.** Let $M$ be a discrete $G_K$-module. Then there exist finitely many Galois extensions $\{K\}_\mu$, $\{K\}_\lambda$ of $K$ and positive integers $n$, $\{m_\mu\}_\mu$, $\{m_\lambda\}_\lambda$ and a finite $G_K$-module $N$ such that

$$0 \to M^n \oplus \prod_\mu (\pi_\mu)_* \mathbb{Z}^{m_\mu} \to \prod_\lambda (\pi_\lambda)_* \mathbb{Z}^{m_\lambda} \to N \to 0$$

is an exact sequence of $G_K$-modules. Here $\pi_\mu$ is the natural map from $\text{Spec}(K_\mu)$ to $\text{Spec}(K)$. \qed

**Proof.** For a proof, see [Ono01] 1.5.1. \qed

The following proposition is a sheaf-theoretic version of theorem 3.11.

**Proposition 3.12.** Let $M$ be a discrete $G_K$-module which is also a finitely generated abelian group. Then there exist finitely many Galois extensions $\{K\}_\mu$, $\{K\}_\lambda$ of $K$ and positive integers $n$, $\{m_\mu\}_\mu$, $\{m_\lambda\}_\lambda$, a constructible sheaf $R$ and a finite $G_K$-module $N$ such that we have the following exact sequences

$$0 \to M^n \oplus \prod_\mu (\pi_\mu)_* \mathbb{Z}^{m_\mu} \to \prod_\lambda (\pi_\lambda)_* \mathbb{Z}^{m_\lambda} \to N \to 0,$$

$$0 \to (j_* M)^n \oplus \prod_\mu (\pi'_\mu)_* \mathbb{Z}^{m_\mu} \to \prod_\lambda (\pi'_\lambda)_* \mathbb{Z}^{m_\lambda} \to R \to 0,$$

where $\pi'_\mu : \text{Spec}(O_{K_\mu}) \to \text{Spec}(O_K)$ and $\pi_\mu : \text{Spec}(K_\mu) \to \text{Spec}(K)$ are the natural maps.
Proof. The existence of (14) is precisely theorem 3.11. Let \( P_1 = \prod_\mu (\pi_\mu)_* Z^{m_\mu} \) and \( P_2 = \prod_\lambda (\pi_\lambda)_* Z^{m_\lambda} \). By applying \( j_* \) to (14), we obtain the exact sequence

\[
0 \to j_* M^n \oplus j_* P_1 \to j_* P_2 \to j_* N \to R^1 j_* (M \oplus P_1)
\]

which we split into two exact sequences

\[
0 \to j_* M^n \oplus j_* P_1 \to j_* P_2 \to \mathcal{R} \to 0 \quad \text{and} \quad 0 \to \mathcal{R} \to j_* N \to \mathcal{Q} \to 0
\]

where \( \mathcal{Q} \) is a subsheaf of \( R^1 j_* (M \oplus P_1) \). As \( R^1 j_* (M \oplus P_1) \) is negligible, \( \mathcal{Q} \) is constructible. Since \( j_* P_1 = \prod_\mu (\pi_\mu)_* Z^{m_\mu} \) and \( j_* P_2 = \prod_\lambda (\pi_\lambda)_* Z^{m_\lambda} \), we have \( j_* P_1 \) and \( j_* P_2 \) are strongly-\( \mathbb{Z} \)-constructible by corollary 3.10. As \( N \) is finite, \( j_* N \) is constructible. Since \( \mathcal{Q} \) is constructible, \( \mathcal{R} \) is constructible as the category of constructible sheaves is abelian. \( \square \)

**Proposition 3.13.** Let \( M \) be a discrete \( G_K \)-module. Then \( j_* M \) is a strongly-\( \mathbb{Z} \)-constructible sheaf.

**Proof.** Consider sequence (15) of proposition 3.12. As \( \mathcal{R} \) is constructible and \( \prod_\lambda (\pi_\lambda)_* Z^{m_\lambda} \) is strongly-\( \mathbb{Z} \)-constructible, \( (j_* M)^n \oplus \prod_\mu (\pi_\mu)_* Z^{m_\mu} \) is strongly-\( \mathbb{Z} \)-constructible by proposition 3.9. Since \( \prod_\mu (\pi_\mu)_* Z^{m_\mu} \) is strongly-\( \mathbb{Z} \)-constructible, so is \( j_* M \). \( \square \)

### 4 Euler Characteristics Of Strongly-\( \mathbb{Z} \)-Constructible Sheaves

#### 4.1 Construction

Let \( \mathcal{F} \) be a strongly-\( \mathbb{Z} \)-constructible sheaf on \( X \). There are natural maps \( R\Gamma_W (X, \mathcal{F}) \to \tau_{\leq 1} R\Gamma_{et} (X, \mathcal{F}) \) and \( \tau_{\leq 1} R\Gamma_{et} (X, \mathcal{F}) \to R\Gamma_{et} (X, \mathcal{F}) \). The morphism \( \mathcal{F} \to \mathcal{F}_B \) induces a map of complexes \( R\Gamma_{et} (X, \mathcal{F}) \to R\Gamma_{et} (X, \mathcal{F}_B) \). Composing all three maps yields the map

\[
R\Gamma_W (X, \mathcal{F}) \to R\Gamma_{et} (X, \mathcal{F}_B).
\]

**Definition 4.1.** We define the complex \( D_F \) by the cone

\[
D_F := [R\Gamma_W (X, \mathcal{F}) \to R\Gamma_{et} (X, \mathcal{F}_B)][-1].
\]

**Proposition 4.2.** Let \( \mathcal{F} \) be a strongly-\( \mathbb{Z} \)-constructible sheaf on \( X \). Then \( H^n (D_F) \) satisfy

\[
0 \to H^1 (D_F) \to H^0_{et} (X, \mathcal{F}) \to H^0_{et} (X, \mathcal{F}_B) \to H^2 (D_F) \xrightarrow{\partial} H^1_{et} (X, \mathcal{F}) \to 0
\]

(17)

\[
0 \to H^2_{et} (X, \mathcal{F}) \to H^3 (D_F) \to \text{Hom}_{\mathcal{X}} (\mathcal{F}, \mathcal{G}_m)^* \to 0
\]

and \( H^4 (D_F) = \text{Hom}_{\mathcal{X}} (\mathcal{F}, \mathcal{G}_m)^{D_{tor}} \) and \( H^n (D_F) = 0 \) otherwise.

**Proof.** There is a distinguished triangle

\[
D_F [1] \to R\Gamma_W (X, \mathcal{F}) \to R\Gamma_{et} (X, \mathcal{F}_B) \to D_F [2].
\]

The long exact sequence of cohomology of this triangle yields the following exact sequence

\[
0 \to H^1 (D_F) \to H^0_{et} (X, \mathcal{F}) \to H^0_{et} (X, \mathcal{F}_B) \to H^2 (D_F) \to H^1_{et} (X, \mathcal{F}) \to 0
\]

and \( H^{n+1} (D_F) \simeq H^n_W (X, \mathcal{F}) \) for \( n \geq 2 \). The lemma then follows from proposition 2.5. \( \square \)
Lemma 4.3. Let \( \beta \) be the map \( H^2(D_F) \to H^1_{\text{et}}(X, \mathcal{F}) \) from proposition 4.2. Then there is a canonical isomorphism \( \theta : H^2(D_F)_{\mathbb{R}} \to H^3(D_F)_{\mathbb{R}} \) with \( |\det \theta| = R(\mathcal{F})/|\text{cok}(\beta_{\text{tor}})| \) with respect to integral bases.

Proof. We construct the isomorphism \( \theta : H^2(D_F)_{\mathbb{R}} \to H^3(D_F)_{\mathbb{R}} \) as follows. From the exact sequence (17) we have

\[
0 \to H^0_{\text{et}}(X, \mathcal{F}_B)/H^0_{\text{et}}(X, \mathcal{F}) \to H^2(D_F) \overset{\beta}{\to} H^1_{\text{et}}(X, \mathcal{F}) \to 0.
\]

As \( H^1_{\text{et}}(X, \mathcal{F}) \) is finite, there is an isomorphism \( \phi : (H^0_{\text{et}}(X, \mathcal{F}_B)/H^0_{\text{et}}(X, \mathcal{F}))_{\mathbb{R}} \to H^2(D_F)_{\mathbb{R}} \). By lemma 6.6, \( |\det(\phi)| = |\text{cok}(\beta_{\text{tor}})| \) with respect to integral bases.

From the exact sequence (18) and the fact that \( H^1_{\text{et}}(X, \mathcal{F}) \) is finite, there is an isomorphism \( \psi : H^3(D_F)_{\mathbb{R}} \to \text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^*_{\mathbb{R}} \). By lemma 6.6 and the fact that \( \text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^* \) is torsion free, we have \( |\det(\psi)| = 1 \) with respect to integral bases.

As \( \mathcal{F} \) is strongly-\( \mathbb{Z} \)-constructible, the regulator pairing induces the isomorphism

\[
\gamma : \left( \frac{H^0_{\text{et}}(X, \mathcal{F}_B)}{H^0_{\text{et}}(X, \mathcal{F})} \right)_{\mathbb{R}} \to \text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^*_{\mathbb{R}}.
\]

By definition 2.11, \( |\det(\gamma)| = R(\mathcal{F}) \) with respect to integral bases.

We define \( \theta \) to be \( \psi^{-1} \circ \beta \circ \phi^{-1} \). Therefore, \( \theta : H^2(D_F)_{\mathbb{R}} \to H^3(D_F)_{\mathbb{R}} \) is an isomorphism and \( |\det \theta| = R(\mathcal{F})/|\text{cok}(\beta_{\text{tor}})| \) with respect to integral bases.

The existence of \( \theta \) in lemma 4.3 enables us to make the following definition.

Definition 4.4. For a strongly-\( \mathbb{Z} \)-constructible sheaf \( \mathcal{F} \) on \( X \), we define the Euler characteristic \( \chi(\mathcal{F}) \) by

\[
\chi(\mathcal{F}) := \frac{[H^1(D_F)]/[H^3(D_F)_{\text{tor}}]}{[H^2(D_F)_{\text{tor}}]/[H^4(D_F)]} |\det(\theta)|
\]

where \( \theta \) is the isomorphism constructed in the lemma 4.3 and its determinant is computed with respect to integral bases.

Proposition 4.5. For a strongly-\( \mathbb{Z} \)-constructible sheaf \( \mathcal{F} \) on \( X \), we have

\[
\chi(\mathcal{F}) = \frac{[H^0_{\text{et}}(X, \mathcal{F})_{\text{tor}}]/[H^2_{\text{et}}(X, \mathcal{F})]R(\mathcal{F})}{[H^1_{\text{et}}(X, \mathcal{F})]/[\text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^*_{\text{tor}}]/[H^0_{\text{et}}(X, \mathcal{F}_B)_{\text{tor}}]/[\text{cok}(\delta_{\text{tor}})]}
\]

where \( \delta \) is the quotient map \( H^0_{\text{et}}(X, \mathcal{F}_B) \to H^0_{\text{et}}(X, \mathcal{F})/H^0_{\text{et}}(X, \mathcal{F})_{\text{tor}} \). In particular, if \( \mathcal{F} \) is a constructible sheaf then

\[
\chi(\mathcal{F}) = \frac{[H^0_{\text{et}}(X, \mathcal{F})]/[H^2_{\text{et}}(X, \mathcal{F})]}{[H^1_{\text{et}}(X, \mathcal{F})]/[H^3_{\text{et}}(X, \mathcal{F})]/[H^0_{\text{et}}(X, \mathcal{F}_B)]}
\]

Proof. By lemma 4.3, \( |\det \theta| = R(\mathcal{F})/|\text{cok}(\beta_{\text{tor}})| \) where \( \beta : H^2(D_F) \to H^1_{\text{et}}(X, \mathcal{F}) \). Now we compute the torsion subgroups of \( H^2(D_F) \). We have \( [H^3(D_F)_{\text{tor}}] = [H^2_{\text{et}}(X, \mathcal{F})] \) and \( [H^4(D_F)] = [\text{Hom}_X(\mathcal{F}, \mathbb{G}_m)^*_{\text{tor}}] \) from proposition 4.2. We split the exact sequence (17) into

\[
0 \to H^1(D_F) \to H^0_{\text{et}}(X, \mathcal{F}) \to H^0_{\text{et}}(X, \mathcal{F}_B) \overset{\delta}{\to} H^0_{\text{et}}(X, \mathcal{F}_B) \to 0,
\]

\[
0 \to H^0_{\text{et}}(X, \mathcal{F}_B)/H^0_{\text{et}}(X, \mathcal{F}) \to H^2(D_F) \overset{\beta}{\to} H^1_{\text{et}}(X, \mathcal{F}) \to 0.
\]
Applying lemma 6.3 to the two exact sequences above, we have

\[ [H^1(D_F)] = \frac{[H^0_{et}(X,F)_{tor}] \left[ \left( \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \right)_{tor} \right]}{[H^0_{et}(X,F_B)_{tor}] \cok(\delta_{tor})], \quad [H^2(D_F)_{tor}] = \frac{\left[ \left( \frac{H^0_{et}(X,F_B)}{H^0_{et}(X,F)} \right)_{tor} \right]}{\cok(\beta_{tor})}. \]

Putting everything together establishes the formula for \( \chi(F) \).

\[ \square \]

### 4.2 Simple Computations

**Proposition 4.6.** Let \( h, R \) and \( w \) be the class number, the regulator and the number of roots of unity of \( K \) respectively. Then \( \chi(Z) = hR/w \). In particular, \( \zeta_K(0) = -\chi(Z) \).

**Proof.** From theorem 2.6 \( H^1_{et}(X, Z) = 0 \) and \( [H^2_{et}(X, Z)] = h \). It is clear that \( H^0_{et}(X, Z_B)_{tor} = 0 \) and \( \cok(\delta_{tor}) = 0 \). In addition, \( R(Z) = R \) and \( [(O^*_K)_{tor}] = w \). As a result, \( \chi(Z) = hR/w \). Therefore, \( \zeta_K(0) = -\chi(Z) \) by the analytic class number formula.

\[ \square \]

**Proposition 4.7.** Euler characteristics of negligible sheaves are 1.

**Proof.** It suffices to prove this proposition for the constant sheaf \( Z/n \). From theorem 2.6 the étale cohomology of \( Z/n \) is given by

\[ H^r_{et}(X, Z/n) = \begin{cases} \mathbb{Z}/n & r = 0 \\ (Pic(O_K)/n)^D & r = 1 \\ \mu_n(K)^D & r = 3. \end{cases} \]

\[ 0 \to Pic(O_K)[n]^D \to H^2_{et}(X, Z/n) \to (O^*_K/(O^*_K)^n)^D \to 0. \]

Observe that \( [H^0_{et}(X, (Z/n)_B)] = n|S^\infty| \) and \( R(Z/n) = 1 \). From proposition 4.5 and Dirichlet’s Unit Theorem, \( \chi(Z/n) = 1 \).

\[ \square \]

**Proposition 4.8.** Euler characteristics of negligible sheaves are 1.

**Proof.** It is enough to prove this lemma for the sheaf \( i_*M \) where \( M \) is a finite \( \hat{Z} \)-module. We have \( H^n_{et}(X, i_*M) \simeq H^n(\hat{Z}, M) \) which is 0 for \( n \geq 2 \) [Ser95 page 189]. Moreover, \( [H^0(\hat{Z}, M)] = [H^1(\hat{Z}, M)] \) for finite \( M \). Also \( (i_*M)_B = 0 \), therefore \( \chi(i_*M) = 1 \).

\[ \square \]

**Proposition 4.9.** Let \( L \) be a finite Galois extension of \( K \), \( Y = Spec(O_L) \) and \( \pi' : Y \to X \) be the natural map. Let \( F \) be a strongly-\( Z \)-constructible sheaf on \( Y \). Then \( \pi'_*F \) is a strongly-\( Z \)-constructible sheaf on \( X \). Moreover, \( H^n(D_{\pi'_*F}) \simeq H^n(D_F) \) and \( \chi(\pi'_*F) = \chi(F) \).

**Proof.** From proposition 3.9 \( \pi'_*F \) is a strongly-\( Z \)-constructible sheaf and \( R(\pi'_*F) = R(F) \). To show \( H^n(D_{\pi'_*F}) \simeq H^n(D_F) \), let us consider the following commutative diagram

\[ \begin{array}{ccc} 0 \to H^1(D_{\pi'_*F}) & \to H^0_{et}(X, \pi'_*F) & \to H^0_{et}(X, (\pi'_*F)_B) & \to H^2(D_{\pi'_*F}) & \to H^1(D_{\pi'_*F}) & \to 0 \\\n0 \to H^1(D_F) & \to H^0_{et}(Y, F) & \to H^0_{et}(Y, F_B) & \to H^2(D_F) & \to H^1(D_F) & \to 0 \end{array} \]

where the rows are exact from proposition 4.2. Note that the map in the center is an isomorphism by proposition 3.9. Thus, the 5-lemma implies that \( H^n(D_{\pi'_*F}) \simeq H^n(D_F) \) for \( n = 1, 2 \). For \( n = 3, 4 \), again from proposition 4.2 we have \( H^n(D_{\pi'_*F}) \simeq H^n_{BZ}(X, \pi'_*F) \) and \( H^n(D_F) \simeq H^n_{BZ}(Y, F) \). Thus, \( H^n(D_{\pi'_*F}) \simeq H^n(D_F) \) for \( n = 3, 4 \) by lemma 3.8. Therefore, \( \chi(\pi'_*F) = \chi(F) \).

\[ \square \]
Corollary 4.10. The sheaf $\pi_*\mathbb{Z}$ on Spec($K$) corresponds to the induced $G_K$-module $\text{Ind}^{G_K}_{G_L}(\mathbb{Z})$. If we write $\pi_*\mathbb{Z}$ for $\text{Ind}^{G_K}_{G_L}(\mathbb{Z})$ then $L^*(K, \pi_*\mathbb{Z}, 0) = \pm \chi(\pi_*\mathbb{Z})$.

Proof. By proposition 4.39 $\chi(\pi_*\mathbb{Z}) = \chi(\mathbb{Z})$. Also by proposition 4.36 $\zeta_L^*(0) = \pm \chi(\mathbb{Z})$. Since $L(K, \pi_*\mathbb{Z}, s) = \zeta_L(s)$, $L^*(K, \pi_*\mathbb{Z}, 0) = \pm \chi(\pi_*\mathbb{Z})$. □

4.3 Multiplicative Property

Lemma 4.11. Let $\delta$ be the map $H^0_{et}(X, F_B) \to H^0_{et}(X, F_B)/H^0_{et}(X, F)$. Then the exact sequence

$$0 \to H^0_{et}(X, F)_{\mathbb{R}} \xrightarrow{\Delta} H^0_{et}(X, F_B) \xrightarrow{\delta} \frac{H^0_{et}(X, F_B)}{H^0_{et}(X, F)} \to 0$$

has determinant $[\text{cok}(\delta_{tor})]$ with respect to integral bases.

Proof. Since $F$ is strongly-$\mathbb{Z}$-constructible, the kernel $H^1(D_F)$ of the map $H^0_{et}(X, F) \to H^0_{et}(X, F_B)$ is finite. Applying lemma 6.8 to the exact sequence

$$0 \to H^1(D_F) \to H^0_{et}(X, F) \xrightarrow{\Delta} H^0_{et}(X, F_B) \xrightarrow{\delta} \frac{H^0_{et}(X, F_B)}{H^0_{et}(X, F)} \to 0$$

we deduce that (19) has determinant $[\text{cok}(\delta_{tor})]$ with respect to integral bases. □

The main result of this section is the following theorem.

Theorem 4.12. Suppose we have a short exact sequence of strongly-$\mathbb{Z}$-constructible sheaves

$$0 \to F_1 \to F_2 \to F_3 \to 0.$$  

Then $\chi(F_2) = \chi(F_1)\chi(F_3)$.

Proof. From proposition 4.5 $\chi(F_1)\chi(F_3)/\chi(F_2)$ is given by

$$\left(\prod_{i=1}^3 [H^0_{et}(X, F_i)_{tor}]^{-1+(i+1)}\right) \left(\prod_{i=1}^3 \mathcal{R}(F_i)^{-1+(i+1)}\right) \left(\prod_{i=1}^3 \left[\frac{H^2_{et}(X, F_i)}{H^1_{et}(X, F_i)}\right]^{-1+(i+1)}\right)$$

$$\left(\prod_{i=1}^3 [H^0_{et}(X, F_{i,B})_{tor}]^{-1+(i+1)}\right) \left(\prod_{i=1}^3 [\mathcal{H}om_{X}(F_i, G_{m})_{tor}]^{-1+(i+1)}\right) \left(\prod_{i=1}^3 [\text{cok}(\delta_{i,tor})]^{-1+(i+1)}\right)$$

(21)

We split the long exact sequence of cohomology associated with (20) into the following exact sequences

$$(\mathcal{H}^0): 0 \to H^0_{et}(X, F_1) \to H^0_{et}(X, F_2) \to H^0_{et}(X, F_3) \to S \to 0,$$

$$0 \to S \to H^1_{et}(X, F_1) \to H^1_{et}(X, F_2) \to \cdots \to H^2_{et}(X, F_3) \to Q \to 0,$$

(22)

(23)

where both $S$ and $Q$ are finite abelian groups. From (22), we have

$$\prod_{i=1}^3 \left[\frac{H^2_{et}(X, F_i)}{H^1_{et}(X, F_i)}\right]^{-1+(i+1)} = \frac{[Q]}{[S]}.$$  

(24)

Applying lemma 6.7 to (22), we obtain

$$\left(\prod_{i=1}^3 [H^0_{et}(X, F_i)_{tor}]^{-1+(i+1)}\right) = \nu(\mathcal{H}^0)_{\mathbb{R}}[S].$$  

(25)
As all the $F_i$ are strongly-$\mathbb{Z}$-constructible, $\text{Ext}^1_X(\mathcal{F}_i, \mathbb{G}_m)^D \simeq H^2_{\text{et}}(X, \mathcal{F}_i)$. Therefore, the following sequence is exact

$$(\mathcal{H}om) : \quad 0 \to \text{Hom}_X(\mathcal{F}_3, \mathbb{G}_m) \to \text{Hom}_X(\mathcal{F}_2, \mathbb{G}_m) \to \text{Hom}_X(\mathcal{F}_1, \mathbb{G}_m) \to Q^D \to 0. \quad (26)$$

Applying lemma 6.7 to (26) yields

$$\left( \prod_{i=1}^{3} [\text{Hom}_X(\mathcal{F}_i, \mathbb{G}_m)_{\text{tor}}]^{(-1)^{i+1}} \right) = \nu(\mathcal{H}om)_\mathbb{R}[Q^D]. \quad (27)$$

Let $(\mathcal{H}_B)$ be the exact sequence

$$(\mathcal{H}_B) : \quad 0 \to H^0_{\text{et}}(X, \mathcal{F}_{1,B}) \to H^0_{\text{et}}(X, \mathcal{F}_{2,B}) \to H^0_{\text{et}}(X, \mathcal{F}_{3,B}) \to 0. \quad (28)$$

Applying lemma 6.7 to (28), we obtain

$$\left( \prod_{i=1}^{3} [H^0_{\text{et}}(X, \mathcal{F}_{i,B})_{\text{tor}}]^{(-1)^{i+1}} \right) = \nu(\mathcal{H}_B)_\mathbb{R}. \quad (29)$$

Applying lemma 6.4 to the following diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & (H^0_{\text{et}}(X, \mathcal{F}_{1,B}))_\mathbb{R} & \longrightarrow & (H^0_{\text{et}}(X, \mathcal{F}_{2,B}))_\mathbb{R} & \longrightarrow & (H^0_{\text{et}}(X, \mathcal{F}_{3,B}))_\mathbb{R} & \longrightarrow & 0 \\
& & \downarrow{(\delta_1)_R} & & \downarrow{(\delta_2)_R} & & \downarrow{(\delta_3)_R} & & \\
0 & \longrightarrow & \text{Hom}_X(\mathcal{F}_1, \mathbb{G}_m)^*_\mathbb{R} & \longrightarrow & \text{Hom}_X(\mathcal{F}_2, \mathbb{G}_m)^*_\mathbb{R} & \longrightarrow & \text{Hom}_X(\mathcal{F}_3, \mathbb{G}_m)^*_\mathbb{R} & \longrightarrow & 0
\end{array} \quad (\mathcal{H}_B/\mathcal{H}^0)$$

yields

$$\left( \prod_{i=1}^{3} R(F_i)^{(-1)^{i+1}} \right) = \nu(\mathcal{H}_B/\mathcal{H}^0)\nu(\mathcal{H}om)_\mathbb{R}. \quad (30)$$

Applying lemma 6.2 to the diagram below where all the columns are short exact sequences

$$\begin{array}{cccccc}
0 & \longrightarrow & H^0_{\text{et}}(X, \mathcal{F}_1)_\mathbb{R} & \longrightarrow & H^0_{\text{et}}(X, \mathcal{F}_2)_\mathbb{R} & \longrightarrow & H^0_{\text{et}}(X, \mathcal{F}_3)_\mathbb{R} & \longrightarrow & 0 \\
& & \downarrow{(\delta_1)_R} & & \downarrow{(\delta_2)_R} & & \downarrow{(\delta_3)_R} & & \\
0 & \longrightarrow & H^0_{\text{et}}(X, \mathcal{F}_{1,B})_\mathbb{R} & \longrightarrow & H^0_{\text{et}}(X, \mathcal{F}_{2,B})_\mathbb{R} & \longrightarrow & H^0_{\text{et}}(X, \mathcal{F}_{3,B})_\mathbb{R} & \longrightarrow & 0
\end{array} \quad (\mathcal{H}^0)$$

and use lemma 4.13, we have

$$\left( \prod_{i=1}^{3} [\text{cok}(\delta_{i,\text{tor}})]^{(-1)^{i+1}} \right) = \frac{\nu(\mathcal{H}^0)_\mathbb{R}\nu(\mathcal{H}_B/\mathcal{H}^0)}{\nu(\mathcal{H}_B)_\mathbb{R}}. \quad (31)$$

Putting everything together, we obtain $\chi(F_2) = \chi(F_1)\chi(F_3)$. \qed

**Proposition 4.13.** Euler characteristics of constructible sheaves are 1.
Proof. Let $\mathcal{F}$ be a constructible sheaf on $X$. Then there exists an open dense subset $U$ of $X$ such that $\mathcal{F}_U := \rho^* \mathcal{F}$ is locally constant where $\rho : U \to X$ is the inclusion map. Let $\pi : V \to U$ be the finite étale morphism such that $\pi^* \mathcal{F}_U$ is a constant sheaf. Let $S$ be the set of primes of $K$ (including the infinite primes) not corresponding to a point of $U$. Let $K_S$ be the maximal subfield of $K$ that is ramified over $K$ at only primes in $S$. Let $G_S$ be $\text{Gal}(K_S/K)$. The category of locally constant sheaves with finite stalks on $U$ is equivalent to the category of discrete finite $G_S$-modules [Mil80 page 156]. Let $M$ be the $G_S$-module corresponding to $\mathcal{F}_U$. In particular, $M$ is a finite abelian group. By making $U$ smaller if necessary, we may assume $[M]$ is not divisible by the residue characteristics of any closed points of $U$. From [Mil06 II.2.9], $H^n(U, \mathcal{F}_U) \simeq H^n(G_S, M)$. Let $i_p : p \to X$ and $\mathcal{F}_p = i_p^* \mathcal{F}$. We have the canonical exact sequence

$$0 \to \rho_j^* \mathcal{F}_U \to \mathcal{F} \to \prod_{p \in X-U} (i_p)_\ast \mathcal{F}_p \to 0.$$ 

Since $\prod_{p \in X-U} (i_p)_\ast \mathcal{F}_p$ is negligible, its Euler characteristic is 1. By theorem 4.12, it suffices to show $\chi(\rho_j^* \mathcal{F}_U) = 1$. Since $(\rho_j^* \mathcal{F}_U)_B = \mathcal{F}_B$, $[H^0_\text{et}(X, (\rho_j^* \mathcal{F}_U)_B)] = \prod_{v \in S_\infty} [M]$. Let $H^n_c(U, \mathcal{F}_U)$ be the cohomology with compact support defined in [Mil06 page 165]. As $K$ is totally imaginary, $H^n_c(U, \mathcal{F}) = H^n(X, p_j^* \mathcal{F}_U)$. Then the proof is complete because by [Mil06 II.2.13],

$$\chi(\rho_j^* \mathcal{F}_U) = \frac{[H^0_c(U, \mathcal{F}_U)] [H^2_c(U, \mathcal{F}_U)]}{[H^1_c(U, \mathcal{F}_U)] [H^3_c(U, \mathcal{F}_U)] \prod_{v \in S_\infty} [M]} = 1.$$

\[\square\]

### 4.4 Special Values Of L-Functions At Zero

#### Theorem 4.14.
Let $K$ be a totally imaginary number field. Let $M$ be a discrete $G_K$-module. Then

1. $\text{ord}_{s=0} \Lambda(M, s) = \text{rank}_\mathbb{Z} \text{Hom}_X(j_* M, \mathbb{Q}_m)$.
2. $L^\ast(M, 0) = \pm \chi(j_* M)$.

Proof. 1. The ranks of $\text{Hom}_X(j_* M, \mathbb{Q}_m)$ and $\prod_{v \in S_\infty} H^0(K_v, M_v)/H^0(K, M)$ are the same because the regulator pairing for $j_* M$ is non-degenerate. Thus, from [Lat84 1.3.4],

$$\text{ord}_{s=0} \Lambda(M, s) = \sum_{v \in S_\infty} \text{rank}_\mathbb{Z} H^0(K_v, M_v) - \text{rank}_\mathbb{Z} H^0(K, M) = \text{rank}_\mathbb{Z} \text{Hom}_X(j_* M, \mathbb{Q}_m).$$

2. Consider the two exact sequences [14] and [15] from proposition 3.12. Since $\mathcal{R}$ is a constructible sheaf, by propositions 4.10 and 4.13 $\chi(\mathcal{R}) = 1$ and $\chi((\pi_i)_{m}, \mathbb{Z}_{m\lambda}) = \pm L^\ast((\pi_i^\prime)_{m}, \mathbb{Z}_{m\lambda}, 0)$. Hence, by theorem 4.12 and the fact that $N$ is a finite $G_K$-module

$$\chi(j_* M)^n = \frac{\prod_{\lambda} \chi((\pi_i^\prime)_{m}, \mathbb{Z}_{m\lambda})}{\prod_{\mu} \chi((\pi_i^\prime)_{m}, \mathbb{Z}_{m\mu})} = \frac{\prod_{\lambda} L^\ast((\pi_i^\prime)_{m}, \mathbb{Z}_{m\lambda}, 0)}{\prod_{\mu} L^\ast((\pi_i^\prime)_{m}, \mathbb{Z}_{m\mu}, 0)} = |L^\ast(M, 0)|^n.$$

Since $L^\ast(M, 0)$ is a real number, we deduce $L^\ast(M, 0) = \pm \chi(j_* M)$. \[\square\]

#### Corollary 4.15.
Let $T$ be an algebraic torus defined over a totally imaginary number field $K$ with character group $\hat{T}$. Then

$$L^\ast(\hat{T}, 0) = \pm \frac{[H^2_\text{et}(X, j_* \hat{T})] \text{R}(j_* \hat{T})}{[H^1_\text{et}(X, j_* \hat{T})] [\text{Hom}_X(j_* \hat{T}, \mathbb{Q}_m)_{\text{tor}}]} = \pm \frac{[\text{Ext}^1_X(j_* \hat{T}, \mathbb{Q}_m)] \text{R}(j_* \hat{T})}{[\text{Ext}^2_X(j_* \hat{T}, \mathbb{Q}_m)] [\text{Hom}_X(j_* \hat{T}, \mathbb{Q}_m)_{\text{tor}}]}.$$

Proof. The second equality follows from the first by the Artin-Verdier duality. The first equality follows from theorem 4.14 proposition 4.5 and the fact that $\hat{T}$ is a torsion free abelian group. \[\square\]
5 Applications: Algebraic Tori

Let \( T \) be an algebraic torus defined over a totally imaginary number field \( K \) with character group \( \hat{T} \). Corollary 4.15 gives a formula for \( L^*(\hat{T},0) \) in terms of étale cohomology. We shall derive another formula for \( L^*(\hat{T},0) \) in terms of Galois cohomology and other arithmetic invariant of \( T \).

For each finite place \( p \) of \( K \), we write \( K_p^{ur} \) for the maximal unramified extension of the completion \( K_p \) of \( K \). We denote by \( I_p \) the inertia group of \( p \). Let \( O_p \), \( O_p^{ur} \) and \( \hat{O}_p \) be the valuation rings of \( K_p \), \( K_p^{ur} \) and \( \hat{K}_p \) respectively.

5.1 Local Galois Cohomology

Lemma 5.1. Let \( N \) be a discrete \( G_K \)-module. Let \( \hat{N} = \text{Hom}_\mathbb{Z}(N, \hat{K}^*) \). Then

1. \( \text{Ext}^n_X(j_*N, j_*G_m) \simeq H^n(K, \hat{N}). \) In particular, \( \text{Ext}^n_X(j_*\hat{T}, j_*G_m) \simeq H^n(K, T). \)
2. Let \( i: p \to X \) be a closed immersion. Then \( \text{Ext}^n_X(j_*N, i_*\mathbb{Z}) \simeq \text{Ext}^n_Z(N^{I_p}, \mathbb{Z}). \)
3. For each finite prime \( p \) of \( K \), \( H^1(I_p, N) \) is finite.

Proof. 1. From [Mil06 II.1.4], \( R^qj_*G_m = 0 \) for \( q > 0 \). Thus \( \text{Ext}^p_X(j_*N, R^qj_*G_m) \Rightarrow \text{Ext}^{p+q}_G(N, \hat{K}^*) \) collapses and yields \( \text{Ext}^p_X(j_*N, j_*G_m) \simeq \text{Ext}^p_G(N, \hat{K}^*). \) From [Mil06 I.0.8], there is a spectral sequence \( H^p(G_K, \text{Ext}^q_Z(N, \hat{K}^*)) \Rightarrow \text{Ext}^{p+q}_G(N, \hat{K}^*). \) Since \( \hat{K}^* \) is divisible, \( \text{Ext}^q_Z(N, \hat{K}^*) = 0 \) for \( q > 0 \). Thus, \( H^p(G_K, \text{Hom}_Z(N, \hat{K}^*)) \simeq \text{Ext}^p_G(N, \hat{K}^*). \) Hence, \( \text{Ext}^p_X(j_*N, j_*G_m) \simeq H^n(K, \hat{N}). \) Finally, if \( N = \hat{T} \) then \( \hat{N} = \text{Hom}_\mathbb{Z}(\hat{T}, \hat{K}^*) \simeq T(\hat{K}). \) Therefore, \( \text{Ext}^n_X(j_*\hat{T}, j_*G_m) \simeq H^n(K, T). \)

2. From the spectral sequence \( \text{Ext}^p_X(j_*N, R^qi_*\mathbb{Z}) \Rightarrow \text{Ext}^{p+q}_Z(N^{I_p}, \mathbb{Z}) \) and the fact that \( i_* \) is exact, we deduce \( \text{Ext}^p_X(j_*N, i_*\mathbb{Z}) \simeq \text{Ext}^p_Z(N^{I_p}, \mathbb{Z}). \)

3. Let \( L/K \) be a finite Galois extension such that \( G_L \) acts trivially on \( N \). From the Hochschild-Serre spectral sequence \( H^i(G_{L_p^{ur}/K_p^{ur}}, H^0(I_{L_p}, N)) \Rightarrow H^{i+1}(I_p, N), \)

\[ 0 \to H^1(G_{L_p^{ur}/K_p^{ur}}, N) \to H^1(I_p, N) \to H^0(G_{L_p^{ur}/K_p^{ur}}, H^1(I_{L_p}, N)) \to H^2(G_{L_p^{ur}/K_p^{ur}}, N). \]

Note that \( G_{L_p^{ur}/K_p^{ur}} \) is isomorphic to the inertia subgroup of \( G_{L_p/K_p} \), in particular it is finite. Hence, \( H^i(G_{L_p^{ur}/K_p^{ur}}, N) \) is finite for \( i = 1, 2 \). Thus, it is enough to show \( H^1(I_{L_p}, N) \) is finite. As \( I_{L_p} \) acts trivially on \( N \), it suffices to consider only two cases namely \( N = \mathbb{Z} \) and \( N = \mathbb{Z}/n \).

Proposition 5.2. Let \( N \) be a discrete \( G_{K_p} \)-module. Let \( \hat{N} = \text{Hom}_\mathbb{Z}(N, \hat{K}_p^*) \) and \( \hat{N}^c = \text{Hom}_\mathbb{Z}(N, \hat{O}_p^*). \) Then

1. \( H^0(K_p, \hat{N}^c) = \{ f \in \text{Hom}_{G_{K_p}}(N, \hat{K}_p^*) : \text{ for all } x \in H^0(K_p, N), \text{ we have } f(x) \in O_p^* \}. \)
2. \( \text{Ext}^2_Z(N^{I_p}, \mathbb{Z}) \simeq H^2(K_p, \hat{N}). \)
3. The following sequence is exact

\[ 0 \to H^0(K_p, \hat{N}^c) \to H^0(K_p, \hat{N}) \to \text{Hom}_\mathbb{Z}(N^{I_p}, \mathbb{Z}) \to H^0(\hat{Z}, H^1(I_p, N))^D \to \]
\[ \to H^1(K_p, \hat{N}) \to \text{Ext}^1_Z(N^{I_p}, \mathbb{Z}) \to 0. \quad (32) \]
Proof. 1. Let \( f \in H^0(K_p, \hat{N}^c) \). For any \( x \in H^0(K_p, N) \), \( f(x) \in \hat{O}_p^* \) by definition. As \( f \) is \( G_{K_p} \)-invariant, \( f(x) \in O_p^* \). Thus, \( H^0(K_p, \hat{N}^c) \) is a subset of the right hand side.

Conversely, let \( f \) be an element of the right hand side. Let \( L_p \) be a finite Galois extension of \( K_p \) such that the Galois group \( G_{L_p} \) acts trivially on \( N \). For \( x \in N \), \( f(x) \in L_p^* \) as \( N = H^0(L_p, N) \).

We have \( N_{L_p/K_p}(f(x)) = f(Tr_{L_p/K_p}(x)) \). As \( Tr_{L_p/K_p}(x) \in H^0(K_p, N) \), \( f(Tr_{L_p/K_p}(x)) \in O_p^* \).

Hence, \( N_{L_p/K_p}(f(x)) \in O_p^* \). We deduce that \( f(x) \in O_p^* \subset O_p^* \). As a result, the right hand side is a subset of \( H^0(K_p, N^c) \).

2. From Tate’s local duality, \( H^2(K_p, \hat{N}) \simeq H^0(K_p, N)^D \) and \( Ext^2_Z(N^{I_p}, \mathbb{Z}) \simeq H^0(\hat{Z}, N^{I_p})^D \) which is \( H^0(K_p, N)^D \). Thus, \( H^2(K_p, N) \simeq Ext^2_Z(N^{I_p}, \mathbb{Z}) \).

3. From the spectral sequence \( H^r(\hat{Z}, H^s(I_p, N)) \Rightarrow H^{r+s}(K_p, N) \), we obtain

\[
0 \to H^1(\hat{Z}, N^{I_p}) \to H^1(K_p, N) \to H^0(\hat{Z}, H^1(I_p, N)) \to H^2(\hat{Z}, N^{I_p}) \to H^2(K_p, N)
\]

Taking Pontryagin dual and use Tate’s local duality theorem, we have

\[
\hat{H}^0(K_p, N) \xrightarrow{\hat{\Psi}} \hat{Hom}_Z(N^{I_p}, \mathbb{Z}) \to \hat{H}^0(\hat{Z}, H^1(I_p, N))^D \to \hat{H}^1(K_p, \hat{N}) \to Ext^1_Z(N^{I_p}, \mathbb{Z}) \to 0.
\]

Let \( W = \text{cok}(\hat{\Psi}) \). As \( \hat{H}^0(\hat{Z}, H^1(I_p, N)) \) is finite by lemma 5.1, \( W \) is finite. To complete the proof, we shall show the following sequence is exact.

\[
0 \to H^0(K_p, \hat{N}^c) \to H^0(K_p, \hat{N}) \xrightarrow{\hat{\Psi}} \hat{Hom}_Z(N^{I_p}, \mathbb{Z}) \to W \to 0.
\]

The map \( \Psi \) is defined as follows: for \( f \in H^0(K_p, \hat{N}) \) and \( x \in N^{I_p} \), \( \Psi(f)(x) = v(f(x)) \) where \( v \) is the normalized valuation of \( K_p \). Then \( \Psi \) is a continuous map and

\[
\ker \Psi = \{ f \in Hom_{G_{K_p}}(N, \hat{K}_p^*) : \text{ for all } x \in N^{I_p}, \text{ we have } f(x) \in (O_p^{ur})^* \}.
\]

Claim : \( \ker \Psi = H^0(K_p, \hat{N}^c) \).

Proof of claim :

- Let \( f \in H^0(K_p, \hat{N}^c) \) and \( x \in N^{I_p} \). Then \( f(x) \in H^0(I_p, \hat{K}^*) \cap \hat{O}_p^* = (O_p^{ur})^* \). Therefore, \( H^0(K_p, \hat{N}^c) \subset \ker \Psi \).

- To prove the other inclusion, we use the description of \( H^0(K_p, \hat{N}^c) \) from part 1. Let \( f \in \ker \Psi \) and \( x \in H^0(K, N) \). Then \( f(x) \in (O_p^{ur})^* \) by definition. Since \( f(x) \in K_p^* \), \( f(x) \in (O_p^{ur})^* \cap K^* = O_p^* \). Hence, \( \ker \Psi \subset H^0(K_p, \hat{N}^c) \).

Let \( W' = \text{cok}(\Psi) \). We have the following exact sequence where all the maps are strict morphisms:

\[
0 \to \hat{H}^0(K_p, \hat{N}^c) \to \hat{H}^0(K_p, \hat{N}) \xrightarrow{\hat{\Psi}} \hat{Hom}_Z(N^{I_p}, \mathbb{Z}) \to W' \to 0.
\]

As profinite completion is exact for sequences with strict morphisms, \( W' = W \). That completes the proof of the proposition. \( \square \)
Corollary 5.3. Let $N = \hat{T}$ for some torus $T$ over $K_p$. Then $H^0(K_p, \hat{N}) = T(K_p)$, $H^0(K_p, \hat{N}^c) = T^c_p$, the maximal compact subgroup of $T(K_p)$ and

$$0 \to \frac{T(K_p)}{T^c_p} \to \text{Hom}_Z(\hat{T}^I, \mathbb{Z}) \to H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))^D \to H^1(K_p, T) \to \text{Ext}_Z^1(\hat{T}^I, \mathbb{Z}) \to 0. \quad (34)$$

5.2 A Formula For $L^*(\hat{T}, 0)$

Theorem 5.4. Let $K$ be a totally imaginary number field and $T$ be an algebraic torus over $K$ with character group $\hat{T}$. Let $h_T$, $R_T$ and $w_T$ be the class number, the regulator and the number of roots of unity of $T$. \footnote{see [Ono61] for the definitions of these invariants.} Let $\mathbb{III}^n(T)$ be the Tate-Shafarevich group. Then

$$L^*(\hat{T}, 0) = \pm \frac{h_T R_T \prod_{p \notin S_{\infty}} [\mathbb{III}^1(T)] [H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))]}{w_T \prod_{p \notin S_{\infty}} [H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))]}. \quad (35)$$

Proof. From the short exact sequence of étale sheaves on $X = \text{Spec}(O_K)$

$$0 \to \mathbb{G}_m \to j_* \mathbb{G}_m \to \prod_{p \in X} i_p \mathbb{Z} \to 0$$

we obtain the long exact sequence

$$\ldots \to \text{Ext}^n_X(j_*, \mathbb{G}_m) \to H^n(K, T) \to \prod_{p \in X} \text{Ext}^n_X(j_*, i_p \mathbb{Z}) \to \ldots \quad (36)$$

By lemma 5.1 and proposition 5.2 \footnote{see [Ono61] for the definitions of these invariants.} \ (36) can be rewritten as

$$\ldots \to \text{Ext}^n_X(j_*, \mathbb{G}_m) \to H^n(K, T) \to \prod_{p \in X} \text{Ext}^n_X(\hat{T}^I, \mathbb{Z}) \to \ldots \quad (37)$$

Since $K$ is totally imaginary, for $n \geq 1$, the Tate-Shafarevich group $\mathbb{III}^n(T)$ is the kernel of the map $H^n(K, T) \to \prod_{p \in X} H^n(K_p, T)$. We split (37) into the following exact sequences

$$0 \to \text{Hom}_X(j_*, \mathbb{G}_m) \to T(K) \to \prod_{p \in X} \text{Hom}_Z(\hat{T}^I, \mathbb{Z}) \to P \to 0, \quad (38)$$

$$0 \to P \to \text{Ext}^1_X(j_*, \mathbb{G}_m) \to Q \to 0, \quad (39)$$

$$0 \to Q \to H^1(K, T) \xrightarrow{\psi} \prod_{p \in X} \text{Ext}^1_Z(\hat{T}^I, \mathbb{Z}) \to R \to 0, \quad (40)$$

$$0 \to R \to \text{Ext}^2_X(j_*, \mathbb{G}_m) \to \mathbb{III}^2(T) \to 0. \quad (41)$$

Note that $P$, $Q$ and $R$ are finite groups as $\text{Ext}^1_X(j_*, \mathbb{G}_m)$ and $\text{Ext}^2_X(j_*, \mathbb{G}_m)$ are finite. From (39) and (41), we have

$$\frac{[\text{Ext}^1_X(j_*, \mathbb{G}_m)]}{[\text{Ext}^2_X(j_*, \mathbb{G}_m)]} = \frac{[P][Q]}{[\mathbb{III}^2(T)][R]} \quad (42)$$

For each finite prime $p$ of $K$, we split the sequence (34) into

$$0 \to T(K_p)/T^c_p \to \text{Hom}_Z(\hat{T}^I, \mathbb{Z}) \to S_p \to 0, \quad (43)$$
0 \to S_p \to \text{H}^0(\hat{\mathbb{Z}}, \text{H}^1(I_p, \hat{T}))^D \to \text{H}^1(K_p, T) \to \text{Ext}^1_\mathbb{Z}(\hat{T}^\prime_p, \mathbb{Z}) \to 0. \tag{44}

Let \( U_T \) be the group of units in \( T(K) \) and \( Cl(T) \) be the class group of \( T \). We have the exact sequence

\[
0 \to U_T \to T(K) \to \prod_{p \in X} T(K_p)/T_p^c \to Cl(T) \to 0. \tag{45}
\]

From (43), we obtain the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & T(K) & \to & T(K) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \prod_{p \in X} T(K_p)/T_p^c & \to & \prod_{p \in X} \text{Hom}_{\hat{\mathbb{Z}}}(\hat{T}^\prime_p, \mathbb{Z}) & \to & \prod_{p \in X} S_p \to 0
\end{array}
\]

The Snake lemma combining with (35) and (45) yield \( U_T \simeq \text{Hom}_X(j_\ast \hat{T}, \mathbb{G}_m) \) and

\[
0 \to Cl(T) \to P \to \prod_{p \in X} S_p \to 0. \tag{46}
\]

In particular, \([\text{Hom}(j_\ast \hat{T}, \mathbb{G}_m)_{\text{tor}}] = [U_{T,\text{tor}}] = w_T\). Also as \( P \) is finite, (46) implies \( \prod_{p \in X} S_p \) is finite and \([P] = h_T \prod_{p \in X} [S_p]\). From (44), we have

\[
\begin{array}{cccccc}
0 & \to & 0 & \to & H^1(K, T) & \to & 0 \\
\downarrow & & \downarrow & & \uparrow & & \downarrow \\
0 & \to & \prod_p \text{H}^0(\hat{\mathbb{Z}}, H^1(I_p, T))^D_{S_p} & \to & \prod_p H^1(K_p, T) & \to & \prod_p \text{Ext}^1_{\hat{\mathbb{Z}}}(N^I_p, \mathbb{Z}) \to 0
\end{array}
\]

By the Snake lemma and (40), we obtain

\[
0 \to \text{III}^1(T) \to Q \to \prod_p \left( \frac{\text{H}^0(\hat{\mathbb{Z}}, H^1(I_p, T))^D_{S_p}}{\text{cok} \Delta} \right) \to 0. \tag{47}
\]

From the generalized Poitou-Tate exact sequence [Mil06 I.4.20], we have

\[
0 \to \text{III}^1(T) \to H^1(K, T) \xrightarrow{\Delta} \prod_p H^1(K_p, T) \to H^1(K, \hat{T})^D \to \text{III}^2(T) \to 0.
\]

Thus, \([\text{cok} \Delta][\text{III}^2(T)] = [H^1(K, \hat{T})]\). As \(\text{cok} \Delta\) and \(\prod_p S_p\) are finite, so is \(\prod_p H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))\). Therefore from (47),

\[
[Q] = \frac{[\text{III}^1(T)] \prod_p \frac{[\text{H}^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))^D_{S_p}]}{[\text{cok} \Delta]} [R]}{[\text{III}^1(T)][\text{III}^2(T)][R] \prod_p [\text{H}^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))^D]} = \frac{[\text{III}^1(T)][\text{III}^2(T)][R] \prod_p [\text{H}^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))^D]}{[H^1(K, T)] \prod_p [S_p]}. \tag{48}
\]

Putting together (42), (16) and (18), we have

\[
\frac{[\text{Ext}^1_X(j_\ast \hat{T}, \mathbb{G}_m)]}{[\text{Ext}^2_X(j_\ast T, \mathbb{G}_m)_{\text{tor}}]} = \frac{h_T[\text{III}^1(T)] \prod_p [\text{H}^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))^D]}{[H^1(K, T)]}. \tag{49}
\]

From corollary 113 and the fact that \(R(j_\ast \hat{T}) = R_T\), we finally obtain formula (34). \(\square\)
5.3 A Class Number Formula

We can interpret equation (49) in the proof of theorem 5.4 as a formula for the class number of a torus.

Proposition 5.5. Let $T$ be an algebraic torus over a totally imaginary number field $K$. Then

$$h_T = \frac{[\text{Ext}^1_X(j_*\hat{T}, \mathbb{G}_m)]}{[\text{Ext}^2_X(j_\ast T, \mathbb{G}_m)]} \frac{[H^1(K, \hat{T})]}{[\prod_p [H^0(\mathbb{Z}, H^1(I_p, \hat{T}))]^D]}$$

(50)

Using (5.5) we shall deduce the following theorems of Ono and Katayama.

Theorem 5.6 ([Ono87]). Let $L/K$ be a Galois extension of totally imaginary number fields with Galois group $G$. Let $L_0$ be the maximal abelian subextension of $L$ over $K$ and $I_L$ be the group of ideles of $L$. For each finite prime $v$ of $K$, choose a prime $w$ of $L$ lying over $v$ and let $D_w$ and $I_w$ be the decomposition group and the inertia group of $w$. Let $O_w$ be the ring of integers of $L_w$. Let $T = H^1_{L/K}(\mathbb{G}_m)$ be the norm torus corresponding to the extension $L/K$. Then

$$h_T = \frac{h_L[L_0 : K][H^0_I(G, O_L^*)]}{h_K[\ker(H^0_T(G, L^*) \to H^0_T(G, I_L))] \prod_p [H^0_T(D_w, O_w^*)]}.$$  

(51)

Proof. Let $\pi : \text{Spec}(L) \to \text{Spec}(K)$. We have the exact sequence of $G_K$-modules

$$0 \to \mathbb{Z} \to \pi_*\mathbb{Z} \to \hat{T} \to 0.$$  

(52)

Since $R^1 j_*\mathbb{Z} = 0$, we have the exact sequence of étale sheaves on $X = \text{Spec}(O_K)$

$$0 \to \mathbb{Z} \to \pi'_*\mathbb{Z} \to j_*\hat{T} \to 0$$  

where $\pi' : \text{Spec}(O_L) \to \text{Spec}(O_K)$. The long exact sequence of Ext-groups yields

$$0 \to H^1_I(G, O_L^*) \to \text{Ext}^1_X(j_*\hat{T}, \mathbb{G}_m) \to \text{Pic}(O_L) \to \text{Pic}(O_K) \to \text{Ext}^1_X(j_*\hat{T}, \mathbb{G}_m) \to 0.$$  

From proposition 5.5 we have

$$h_T = \frac{h_L[H^1(K, \hat{T})][H^0_I(G, O_L^*)]}{h_K[\prod_p [H^0(\mathbb{Z}, H^1(I_p, \hat{T}))]^D]}.$$

(53)

The exact sequence (52) induces

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \hat{T} \to 0.$$  

(54)

From (54) and the fact that $H^n(G, \mathbb{Z}[G]) = 0$, we deduce that

$$H^1(K, \hat{T}) \simeq H^1(G, \hat{T}) \simeq H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z}).$$

Therefore, $[H^1(K, \hat{T})] = [G^b] = [L_0 : K]$. The fact that $\prod^1(T) \simeq \ker(H^0_T(G, L^*) \to H^0_T(G, I_L))$ is proved in [PR93, 6.10]. To complete the proof we will show that if $w$ is a prime of $L$ lying above a prime $p$ of $K$ then

$$[H^0_T(D_w, O_w^*)] = [H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))].$$

(55)
Indeed, \( H^0_T(D_w, O_w^*) \simeq I^{ab}_w \) by local class field theory. On the other hand, \( H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T})) \simeq H^0(D_w/I_w, H^1(I_w, \hat{T})) \). From sequence (55) and the fact that \( \mathbb{Z}[G] \) is an induced \( I_w \)-module, we have \( H^1(I_w, \hat{T}) \simeq H^2(I_w, \hat{T}) \). Consider the spectral sequence

\[
E_2^{m,n} = H^m(D_w/I_w, H^n(I_w, \mathbb{Z})) \Rightarrow E_2^{m+n} = H^{m+n}(D_w, \mathbb{Z}).
\]

Note that \( E_2^{m,1} = 0 \) for all \( m \) and \( E_2^{m,0} = 0 \) for \( m \) odd. Therefore, we obtain the exact sequence

\[
0 \to (D_w/I_w)^D \to D_w^D \to H^0(D_w/I_w, H^2(I_w, \mathbb{Z})) \to 0.
\]

Hence, \( H^0(D_w/I_w, H^2(I_w, \mathbb{Z})) \simeq I^D_w \). Finally,

\[
[H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))] = [I^D_w] = [I^{ab}_w] = [H^0_T(D_w, O_w^*)].
\]

\[\square\]

**Corollary 5.7.** In theorem 5.6, suppose further that \( L/K \) is a cyclic extension. For each prime \( v \) of \( K \), let \( e_v(L/K) \) be the ramification index of \( v \) in \( L \). Then

\[
h_T = \frac{h_L[L : K][H^0_T(G, O^*_L)]}{h_K \prod_v e_v(L/K)}. \quad (56)
\]

**Proof.** Since \( L/K \) is cyclic, by Hasse’s theorem \( \ker(H^0_T(G, L^*) \to H^0_T(G, I_L)) = 0 \). Furthermore, \([H^0_T(D_w, O_w^*)] = [I_w] = e_v(L/K)\). Thus, the corollary follows. \[\square\]

**Theorem 5.8** [Kat91]. With notations as in theorem 5.6, let \( T' \) be the dual torus of \( R^{(1)}_{L/K}(\mathbb{G}_m) \). Then

\[
h_{T'} = \frac{h_L[H^1_T(G, O^*_L)]}{h_K \prod_v e_v(L/K)}. \quad (57)
\]

**Proof.** Let \( N \) be the character group of \( T' \). Then \( N \) satisfies the following exact sequence

\[
0 \to N \to \pi_* \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z} \to 0. \quad (58)
\]

As \( R^1 j_*(\pi_* \mathbb{Z}) = 0 \), we have the exact sequence of sheaves on \( X \)

\[
0 \to j_* N \to \pi'_* \mathbb{Z} \to \mathbb{Z} \to R^1 j_* N \to 0. \quad (59)
\]

We split (59) into

\[
0 \to j_* N \to \pi'_* \mathbb{Z} \to Q \to 0. \quad (60)
\]

\[
0 \to Q \to \mathbb{Z} \to R^1 j_* N \to 0. \quad (61)
\]

To ease notation, let \( R = R^1 j_* N \). Then \( R \) is a negligible sheaf. In particular, \( Ext^2_R(R, \mathbb{G}_m) = 0 \) for \( n = 0, 1 \). Let \( \beta \) be the map \( H^0_T(X, Q) \to H^0_T(X, \mathbb{Z}) \). From the long exact sequences of \( Ext \)-groups and cohomology groups of (61), we obtain the following:

\[
\text{Hom}_X(Q, \mathbb{G}_m) \simeq O^*_K,
\[
[Ext^1_X(Q, \mathbb{G}_m)] = h_K[Ext^1_X(R, \mathbb{G}_m)] = h_K \prod_{p \in X} [H^1(\hat{\mathbb{Z}}, H^1(I_p, N))^D],
\]

\[
[Ext^2_X(Q, \mathbb{G}_m)] = [H^1_{et}(X, Q)^D] = \frac{[H^0_{et}(X, R)]}{[cok(\beta)]} = \frac{\prod_{p \in X} [H^0(\hat{\mathbb{Z}}, H^1(I_p, N))]}{[cok(\beta)]}.
\]

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Similar argument applied to the sequence \([60]\) yields the following:

\[
[\text{Ext}_X^2(j_*N, \mathbb{G}_m)] = [H^1_{\text{et}}(X, j_*N)^D] = [\text{cok}(\alpha)],
\]
\[
[\text{Ext}_X^1(j_*N, \mathbb{G}_m)] = h_L[S][\text{Ext}_X^1(Q, \mathbb{G}_m)] = \frac{h_L[S]}{h_K[\text{cok}(\beta)]}
\]

where \(\alpha\) is the map \(H^0_{\text{et}}(X, \pi'_*\mathbb{Z}) \to H^0_{\text{et}}(X, Q)\) and \(S\) satisfies the exact sequence

\[
0 \to O^*_K \to O^*_L \to \text{Hom}_X(j_*N, \mathbb{G}_m) \to S \to 0. \tag{62}
\]

Note that \(\beta\alpha\) is the map \(H^0_{\text{et}}(X, \pi'_*\mathbb{Z}) \to H^0_{\text{et}}(X, \mathbb{Z})\) which can be identified with the map \(\epsilon : H^0(G, \mathbb{Z}[G]) \to H^0(G, \mathbb{Z})\). From the exact sequence

\[
0 \to N \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \tag{63}
\]

we deduce that \(H^1(K, N) \simeq H^1(G, N) \simeq \text{cok}(\epsilon)\). As \(\beta\) and \(\alpha\) are injective, \([\text{cok}(\alpha)][\text{cok}(\beta)] = [\text{cok}(\beta\alpha)] = [H^1(K, N)]\). From proposition \([5,5]\) we have

\[
h_T' = \frac{h_L[S]}{h_K[\text{III}^1(T') \prod_{p \in X} [H^0(\hat{\mathbb{Z}}, H^1(I, N))^D]]}.
\]

To complete the proof of the theorem, we shall prove: \(\text{III}^1(T') = 0\), \([H^0(\hat{\mathbb{Z}}, H^1(I, N))^D] = \epsilon_p(L/K)\) and \(S \simeq H^1_T(G, O^*_L)\).

- The fact that \(\text{III}^1(T') = 0\) is proved in [Kat91] page 685.
- We have \(H^0(\hat{\mathbb{Z}}, H^1(I, N))^D \simeq H^0(D_w/I_w, H^1(I_w, N))\) where \(w\) is a prime of \(L\) dividing \(p\). By \([63]\), \(H^1(I_w, N) \simeq H^0_T(I_w, \mathbb{Z})\). As \(D_w/I_w\) acts trivially on \(H^0(I_w, \mathbb{Z})\),

\[
[H^0(D_w/I_w, H^1(I_w, N))] = [H^0_T(I_w, \mathbb{Z})] = [I_w] = \epsilon_p(L/K).
\]
- From the proof of theorem \([5,4]\) \(\text{Hom}(j_*N, \mathbb{G}_m) \simeq U_T' = T'(O_K)\). Therefore, sequence \([62]\) can be identified with the sequence

\[
0 \to \mathbb{G}_m(O_K) \to R_{L/K}(\mathbb{G}_m)(O_K) \to T'(O_K) \to S \to 0 \tag{64}
\]

which is part of the long exact sequence of cohomology associated with

\[
0 \to \mathbb{G}_m(O_L) \to R_{L/K}(\mathbb{G}_m)(O_L) \to T'(O_L) \to 0. \tag{65}
\]

Note that \(R_{L/K}(\mathbb{G}_m)(O_L)\) is an induced \(G\)-module thus \(H^1_T(G, R_{L/K}(\mathbb{G}_m)(O_L)) = 0\). Consider the long exact sequence of cohomology associated with \([65]\) and compare with \([64]\), we obtain \(S \simeq H^1_T(G, O^*_L)\). \(\square\)

**Remark 5.9.** Theorems \([5,6]\) and \([5,8]\) are weaker than the original theorems of Ono and Katayama because we have to assume \(K\) is totally imaginary. Moreover, Katayama also obtained formulas for \(h_T\) and \(h_{T'}\) when \(L/K\) is not Galois, see [Kat91].
5.4 An Example $T = \text{Spec}(\mathbb{Q}(i)[x, y]/(x^2 - 3y^2 - 1))$

Let $K = \mathbb{Q}(i)$ and $L = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/12}$. Let $T = \text{Spec}(\mathbb{Q}(i)[x, y]/(x^2 - 3y^2 - 1))$. Then $T = R^{(1)}_{L/K}(\mathbb{G}_m)$. We want to illustrate the results of this section via the torus $T$.

Lemma 5.10. The unit group, class number and regulator of $K = \mathbb{Q}(i)$ are given by $O_K^* = \mu_K = \{\pm 1, \pm i\}$, $h_K = 1$, $R_K = 1$.

2. The unit group, class number and regulator of $L = \mathbb{Q}(\zeta)$ are given by $O_L^* = \mu_L \oplus (1 - \zeta^5)^\mathbb{Z}$

where $\mu_L = \langle \zeta \rangle$, $h_L = 1$, $R_L = \log(2 + \sqrt{3})$. Furthermore, the norm map $N_{L/K} : L^* \to K^*$ is given by $N_{L/K}(a + b\sqrt{3}) = a^2 - 3b^2$ for $a, b \in K$.

Proof. We only give the proof for part 2. By Dirichlet Unit Theorem, $O_L^*$ has rank 1. The torsion subgroup of $O_L^*$ is $\{\zeta^n : n = 0, \ldots, 11\}$. As $L$ is a quadratic totally imaginary extension of $\mathbb{Q}(\sqrt{3})$, $L$ is a CM field. Therefore, $[O_L^* : \mu_L \mathbb{Z}[\sqrt{3}]^*] = 1, 2$ by [Was97 4.12]. Since $\mathbb{Z}[\sqrt{3}]^* = \{\pm 1\} \times (2 + \sqrt{3})\mathbb{Z}$ and $(2 + \sqrt{3})\zeta = -(1 - \zeta^5)^2$, we conclude that $O_L^* = \mu_L \oplus (1 - \zeta^5)^\mathbb{Z}$.

The regulator of $L$ is given by $R_L = \log((1 - \zeta^5)^{1/2}) = \log(2 + \sqrt{3})$. The discriminant $\Delta_L = 12^2$. The Minkowski’s bound is $M_L = 18/\pi^2 < 2$. Therefore, $h_L = 1$. Finally as $L = K(\sqrt{3})$, $N_{L/K}(a + b\sqrt{3}) = a^2 - 3b^2$ for $a, b \in K$. \hfill \Box

Lemma 5.11. Let $G = G_{L/K}$. Then $H^n_T(G, \mu_L) \simeq \mathbb{Z}/2\mathbb{Z}$ for all $n$ and

$$H^n_T(G, O_L^*) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even}. \end{cases} \quad (66)$$

Proof. Since $G \simeq \mathbb{Z}/2\mathbb{Z}$, $H^n_T(G, O_L^*)$ only depend on the parity of $n$. We have $N_{L/K}(\zeta) = -1$ and $N_{L/K}(1 - \zeta^5) = i$. If $u \in O_L^*$ then $u = \zeta^m(1 - \zeta^5)^n$ for integers $m, n$. Thus, $N_{L/K}(u) = (-1)^{m+1}i = i^{2m+n}$. Let $I_G$ be the augmentation ideal of $G$. By direct calculation, $\text{cok}(N_{L/K}) = 0$ and

$$\ker(N_{L/K}) = (\zeta^2) \oplus (2 + \sqrt{3})\mathbb{Z} \quad \& \quad I_GO_L^* = \{\sigma(u)/u : u \in O_L^*\} = (\zeta^4) \oplus (2 + \sqrt{3})\mathbb{Z}.$$

As a result, $H^0_T(G_{L/K}, O_L^*) = O_K^*/N_{L/K}O_L^* = 0$ and

$$H^{-1}_T(G, O_L^*) = \frac{\ker(N_{L/K})}{I_GO_L^*} = \frac{(\zeta^2) \oplus (2 + \sqrt{3})\mathbb{Z}}{(\zeta^4) \oplus (2 + \sqrt{3})\mathbb{Z}} \simeq \mathbb{Z}/2\mathbb{Z}. $$

Similarly, we can show that $H^n_T(G, \mu_L) \simeq \mathbb{Z}/2\mathbb{Z}$ for all $n$. \hfill \Box

Proposition 5.12. Let $T = R^{(1)}_{L/K}(\mathbb{G}_m)$. Then

1. $\text{ord}_{s=0}L(\hat{T}, s) = 1$ and $L^*(\hat{T}, 0) = \log(2 + \sqrt{3})/3$.

2. $\prod_p[H^0(\hat{Z}, H^1(I_p, \hat{T}))] = 2$.

3. $h_T = 1$ and $\mathbb{I}^1(T) = 0$.

Proof. 1. As $T = R^{(1)}_{L/K}(\mathbb{G}_m)$, $L(\hat{T}, s) = \zeta_L(s)/\zeta_K(s)$. Thus, the first part follows from the class number formula and lemma 5.10.
2. Since $O_L = \mathbb{Z}[\mu] = O_K[\mu]$, the relative discriminant $\Delta_{L/K} = 3O_K$. Therefore, $3O_K$ is the only prime of $K$ ramified in $L$. Clearly, $e_3(L/K) = 2$. Hence, $\prod_p[H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))] = \prod_p e_p(L/K) = 2$.

3. As $L/K$ is a cyclic extension, $\mathbb{Z} \mathbb{I}^1(T) = 0$. Then using part 2), corollary 5.7 and lemma 5.11 we deduce $h_T = 1$.

\[ \text{Proposition 5.13.} \quad \text{Let } T = R_{L/K}^1(\mathbb{G}_m). \text{ Then } R_T = R(j, \hat{T}) = 2 \log(2 + \sqrt{3}) \text{ and} \]

\[
\Ext^n_X(j, \hat{T}, \mathbb{G}_m) \simeq \begin{cases} 
\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z} & n = 0 \\
0 & n = 1, 2.
\end{cases} \tag{67}
\]

\[ \text{Proof.} \quad \text{Since } \text{Pic}(O_K) = \text{Pic}(O_L) = 0, \text{ the long exact sequence of Ext-groups of (54) yields} \]

\[ \Ext^1_X(j, \hat{T}, \mathbb{G}_m) = 0 \text{ and} \]

\[ 0 \to \Hom_X(j, \hat{T}, \mathbb{G}_m) \to O_L^{N_{L/K}} \xrightarrow{\text{ker } N_{L/K}} O_K^* \to \Ext^1_X(j, \hat{T}, \mathbb{G}_m) \to 0. \]

From lemma 5.11 \[
\Ext^1_X(j, \hat{T}, \mathbb{G}_m) \simeq O_K^*/N_{L/K}O_L^* = 0 \text{ and } \Hom_X(j, \hat{T}, \mathbb{G}_m) \simeq \ker N_{L/K} \simeq \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}. \]

Finally, as the torsion free part of $\Hom_X(j, \hat{T}, \mathbb{G}_m)$ is generated by $(2 + \sqrt{3})$,

\[ R(j, \hat{T}) = \log \|(2 + \sqrt{3})^2 \mathbb{C} = 2 \log(2 + \sqrt{3}). \]

\[ \square \]

\[ \text{Corollary 5.14.} \quad \text{Let } T = R_{L/K}^1(\mathbb{G}_m). \text{ Then } \text{ord}_{s=0}L(\hat{T}, s) = \text{rank}_K \Hom_X(j, \hat{T}, \mathbb{G}_m), \]

\[ L^*(\hat{T}, 0) = \frac{\pm [\Ext^1_X(j, \hat{T}, \mathbb{G}_m)] R(j, \hat{T})}{[\Hom_X(j, \hat{T}, \mathbb{G}_m)_{\text{tor}}] [\Ext^2_X(j, \hat{T}, \mathbb{G}_m) \mathbb{Z}]} = \frac{\pm h_T R_T}{w_T} \frac{[\mathbb{Z} \mathbb{I}^1(T)]}{[H^1(K, T)]} \prod_{p \notin S_\infty} [H^0(\hat{\mathbb{Z}}, H^1(I_p, \hat{T}))]. \]

6. Appendix: Determinants And Torsions

We review some results about determinants of exact sequences and orders of torsion subgroups of finitely generated abelian groups.

6.1 Determinants Of Exact Sequences

For $n \geq 1$, consider the following exact sequence of vector spaces over $\mathbb{R}$

\[ 0 \to V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} ... \xrightarrow{T_{n-1}} V_n \to 0 \quad (\mathcal{E}). \]

Let $B_i$ be an ordered basis for $V_i$. We want to define the determinant $\nu(\mathcal{E})$ of $(\mathcal{E})$ with respect to the bases $\{B_i\}$. We shall do so inductively.

1. If $n = 1$, then $\nu(\mathcal{E}) := |\det(T_0)|$ with respect to the given bases.

2. If $n = 2$, suppose $B_0 = \{u_i\}_{i=1}^{r+s}$, $B_1 = \{v_i\}_{i=1}^{r+s}$ and $B_2 = \{w_i\}_{i=1}^s$. For $i = 1, ..., s$, let $T_1^{-1}(w_i)$ be any preimage of $w_i$ under $T_2$. We can form the following elements $\Lambda_{i=1}^{r+s} u_i$ and
Let $(\wedge_{i=1}^{r}T_0(u_i)) \wedge (\wedge_{i=1}^{s}T_1^{-1}(w_i))$ of $\wedge_{i=1}^{r+s}V_1$. Since $\wedge_{i=1}^{r+s}V_1$ is a 1 dimensional vector space over $\mathbb{R}$, there exists a unique positive real number $\delta$ such that

$$(\wedge_{i=1}^{r}T_0(u_i)) \wedge (\wedge_{i=1}^{s}T_1^{-1}(w_i)) = \pm \delta(\wedge_{i=1}^{r+s}v_i).$$

Note that the choice of the preimages of $w_i$ under $T_1$ does not affect $(\wedge_{i=1}^{r}T_0(u_i)) \wedge (\wedge_{i=1}^{s}T_1^{-1}(w_i))$. Therefore we can define $\nu(\mathcal{E}) := \delta$.

3. If $n \geq 3$, suppose we have defined $\nu(\mathcal{E})$ for $n = N$. We want to define $\nu(\mathcal{E})$ for $n = N + 1$. Let $I$ be the image of $T_{N-1}$ and choose any basis for $I$. We split $(\mathcal{E})$ into

$$0 \to V_0 \overset{T_0}{\to} V_1 \overset{T_1}{\to} \cdots \overset{T_{N-2}}{\to} V_{N-1} \overset{T_{N-1}}{\to} I \to 0 \quad (\mathcal{E}_1),$$

$$0 \to I \to V_N \overset{T_N}{\to} V_{N+1} \to 0 \quad (\mathcal{E}_2).$$

The determinant of $(\mathcal{E})$ defined to be $\nu(\mathcal{E}) := \nu(\mathcal{E}_1)\nu(\mathcal{E}_2)^{(-1)}^{N-1}$. Note that $\nu(\mathcal{E})$ is independent of the choice of basis for $I$.

**Remark 6.1.**

1. Let $(\mathcal{E})$ be an exact sequence of $\mathbb{R}$-vector spaces

$$0 \to V_0 \overset{T_0}{\to} V_1 \overset{T_1}{\to} \cdots \overset{T_{n-1}}{\to} V_n \to 0 \quad (\mathcal{E}).$$

We split $(\mathcal{E})$ into two exact sequences $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ such that $\beta \alpha = T_1$.

$$0 \to V_0 \overset{T_0}{\to} V_1 \overset{T_1}{\to} \cdots \overset{T_{n-1}}{\to} V_n \to 0 \quad (\mathcal{E}_1),$$

$$0 \to J \overset{\beta}{\to} V_i \overset{\alpha}{\to} V_i \overset{T_{n-1}}{\to} V_N \to 0 \quad (\mathcal{E}_2).$$

Then by an induction argument, we can show that $\nu(\mathcal{E}) = \nu(\mathcal{E}_1)\nu(\mathcal{E}_2)^{(-1)}^{i}$.

2. Let $(\mathcal{E}^*)$ be the dual sequence of $(\mathcal{E})$ and let $B_i^*$ be the dual basis of $B_i$. Then with respect to $\{B_i^*\}$ and $\{B_i\}$, $\nu(\mathcal{E}^*) = \nu(\mathcal{E})^{-1}$.

**Lemma 6.2.** Consider the following commutative diagram

$$\begin{array}{ccc}
0 & 0 & 0 \\
0 \to A_0 & \phi_A & A_1 \to \psi_A & A_2 \to A_3 \to 0 \quad (\mathcal{E}_A) \\
\phi_1 & \phi_2 & \phi_3 \\
0 \to B_0 & \phi_B & B_1 \to \psi_B & B_2 \to B_3 \to 0 \quad (\mathcal{E}_B) \\
\tau_1 & \tau_2 & \tau_3 \\
0 \to C_0 & \phi_C & C_1 \to \psi_C & C_2 \to C_3 \to 0 \quad (\mathcal{E}_C) \\
0 & 0 & 0 \\
\end{array}$$

Let $\{a_i, b_i, c_i\}_{i=1}^3$ be bases for $\{A_i, B_i, C_i\}_{i=1}^3$. Then with respect to these bases

$$\frac{\nu(\mathcal{E}_2)}{\nu(\mathcal{E}_1)\nu(\mathcal{E}_3)} = \frac{\nu(\mathcal{E}_B)}{\nu(\mathcal{E}_A)\nu(\mathcal{E}_C)}.$$
Proof. By the definition of $\nu(\mathcal{E}_B)$, we have
\begin{equation}
\land_i b^i_2 = \pm \nu(\mathcal{E}_B)^{-1} (\land_i \phi_B b^i_1) \land (\land_i \psi^{-1}_B b^i_3). \tag{68}
\end{equation}

Let $M := (\land_i \phi_B \theta_1(a^i_1))$ and $N := (\land_i \phi_B \tau_1^{-1}(c^i_1))$. By the definition of $\nu(\mathcal{E}_1)$,
\begin{equation}
\land_i \phi_B b^i_1 = \pm \nu(\mathcal{E}_1)^{-1} (\land_i \phi_B \theta_1(a^i_1)) \land (\land_i \phi_B \tau_1^{-1}(c^i_1)) = \pm \nu(\mathcal{E}_1)^{-1} M \land N. \tag{69}
\end{equation}

Let $P := (\land_i \psi^{-1}_B \theta_3(a^i_3))$ and $Q := (\land_i \psi^{-1}_B \tau_3^{-1}(c^i_3))$. By the definition of $\nu(\mathcal{E}_3)$,
\begin{equation}
\land_i \psi_B b^i_3 = \pm \nu(\mathcal{E}_3)^{-1} (\land_i \psi^{-1}_B \theta_3(a^i_3)) \land (\land_i \psi^{-1}_B \tau_3^{-1}(c^i_3)) = \pm \nu(\mathcal{E}_3)^{-1} P \land Q. \tag{70}
\end{equation}

Putting together (68), (69) and (70), we deduce
\begin{equation}
\land_i b^i_2 = \pm \nu(\mathcal{E}_B)^{-1} \nu(\mathcal{E}_1)^{-1} \nu(\mathcal{E}_3)^{-1} M \land N \land P \land Q. \tag{71}
\end{equation}

Let $M' := (\land_i \theta_2 \phi_A(a^i_1))$, $N' := (\land_i \tau_2^{-1} \phi_C(c^i_1))$, $P' := (\land_i \theta_2 \psi^{-1}_A(a^i_3))$, and $Q' := (\land_i \tau_2^{-1} \psi_C^{-1}(c^i_3))$. By a similar argument, we have
\begin{equation}
\land_i b^i_2 = \pm \nu(\mathcal{E}_2)^{-1} \nu(\mathcal{E}_A)^{-1} \nu(\mathcal{E}_C)^{-1} M' \land N' \land P' \land Q'. \tag{72}
\end{equation}

From (71) and (72), it is enough to show
\[ M \land N \land P \land Q = M' \land N' \land P' \land Q'. \]

Indeed, we have $M = M'$ since $\phi_B \theta_1 = \theta_2 \phi_A$. Let $x = N - N'$. As $\phi_B \tau_1^{-1}(c^i_1) - \tau_2^{-1} \phi_C(c^i_1) \in \ker \tau_2 = \text{im}(\theta_2)$, we deduce $x$ is a finite sum of wedge products such that each product has a factor which is an element of $\text{im}(\theta_2)$.

Similarly, let $y = P - P'$. As $\psi^{-1}_B \theta_3(a^i_3) - \theta_2 \psi^{-1}_A(a^i_3) \in (\ker \psi_B) = (\text{im}(\phi_B))$, $y$ is a finite sum of wedge products such that each product has a factor belonging to $\text{im}(\phi_B)$.

Since $\tau_3 \psi_B = \psi_C \tau_2$, $\psi_B^{-1} \tau_3^{-1} (c^i_3) - \tau_2^{-1} \psi_C^{-1} (c^i_3)$ is an element of $\ker(\tau_3 \psi_B)$. As a vector space, $\ker \tau_3 \psi_B$ is spanned by $\ker \psi_B = \phi_B$ and $\psi_B^{-1} \ker(\tau_3) = \psi_B^{-1} \ker(\theta_3)$. Therefore
\[ Q - Q' = (\land_i \psi^{-1}_B \tau_3^{-1}(c^i_3)) - (\land_i \tau_2^{-1} \psi_C^{-1}(c^i_3)) = z + t \]
where $z, t$ are finite sums such that each summand of $z$ (respectively $t$) has a factor belonging to $\text{im}(\phi_B)$ (respectively $\text{im}(\theta_3)$).

Claim: $M' \land x \land P' = 0$, $M \land N \land y = 0$, $M \land N \land z = 0$, $P \land t = 0$.

Proof of claim: Recall that $M' = (\land_i \theta_2 \phi_A(a^i_1))$ and $P' = (\land_i \theta_2 \psi^{-1}_A(a^i_3))$. It is clear that $\{\theta_2 \phi_A(a^i_1), \theta_2 \psi^{-1}_A(a^i_3)\}$ span $\text{im}(\theta_2)$. As each summand of $x$ has a factor belonging to $\text{im}(\theta_2)$, $M' \land x \land P' = 0$. The rest of the claim can be proved in a similar fashion. Finally,
\begin{align*}
M \land N \land P \land Q & = M \land N \land P \land (Q' + z + t) = M \land N \land P \land Q' \\
& = M \land N \land (P' + y) \land Q' = M \land N \land P' \land Q' \\
& = M' \land (N' + x) \land P' \land Q' = M' \land N' \land P' \land Q'.
\end{align*}

The following proposition can be deduced from lemma 6.2 by an induction argument (whose proof we omit).
Proposition 6.3. Consider the following commutative diagram of $\mathbb{R}$-vector spaces

\[
\begin{array}{ccccccccc}
0 & \rightarrow & V_{0,0} & \rightarrow & V_{0,1} & \rightarrow & \cdots & \rightarrow & V_{0,n} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \uparrow \\
0 & \rightarrow & V_{1,0} & \rightarrow & V_{1,1} & \rightarrow & \cdots & \rightarrow & V_{1,n} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \uparrow \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \uparrow \\
0 & \rightarrow & V_{m,0} & \rightarrow & V_{m,1} & \rightarrow & \cdots & \rightarrow & V_{m,n} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \uparrow \\
0 & & 0 & & 0 & & 0 & & 0 & \uparrow \\
\end{array}
\]

Let $B_{i,j}$ be an ordered basis for $V_{i,j}$. Then with respect to the bases $B_{i,j}$,

\[
\prod_{i=0}^{n} \nu(C_i)^{(-1)^i} = \prod_{i=0}^{m} \nu(R_i)^{(-1)^i}.
\]

Corollary 6.4. Consider the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & 0 \\
\downarrow & \phi_A & \downarrow & \psi_A & & \downarrow & \theta_1 & & \theta_2 & & \theta_3 & \downarrow & \rightarrow & 0 \\
0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & 0 \\
\end{array}
\]

Assume further that all the vertical maps are isomorphisms. Let $\{a_i, b_i\}_{i=1}^{2}$ be bases for $\{A_i, B_i\}_{i=1}^{2}$ respectively. Then with respect to these bases

\[
\frac{|\text{det } \theta_1||\text{det } \theta_3|}{|\text{det } \theta_2|} = \frac{\nu(\mathcal{E}_A)}{\nu(\mathcal{E}_B)}.
\]

6.2 Orders Of Torsion Subgroups

For a finitely generated abelian group $M$, we write $M_f$ for $M/M_{tor}$ and by an integral basis for $M$, we mean a $\mathbb{Z}$-basis for $M_f$. Moreover, if $f : M \rightarrow N$ is a group homomorphism then $f_{tor} : M_{tor} \rightarrow N_{tor}$.

Lemma 6.5. Consider the following exact sequence of finitely generated abelian groups

\[
0 \rightarrow A \rightarrow B \overset{\phi}{\rightarrow} C \overset{\psi}{\rightarrow} D \rightarrow E \rightarrow 0.
\]

Assume $A$ is finite. Then the orders of the torsion subgroups are related by

\[
\frac{[A][C_{tor}]}{[B_{tor}][D_{tor}]} = \frac{1}{[\text{cok}(\psi_{tor})]}.
\]

Proof. This is a consequence of the fact that if $A$ is finite then the map $B_{tor} \rightarrow (B/A)_{tor}$ is surjective. \(\square\)
Lemma 6.6. Let \((E)\) be an exact sequence of finitely generated abelian groups
\[0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0\]
and \((E)_{\mathbb{R}}\) be the sequence \((E)\) tensoring with \(\mathbb{R}\). Then with respect to any integral bases,
\[\nu(E)_{\mathbb{R}} = \frac{[A_{\text{tor}}][C_{\text{tor}}]}{[B_{\text{tor}}]} = [\cok(\psi_{\text{tor}})].\] (76)

Proof. From remark 6.1 for any section \(\gamma\) of \(\psi_{\mathbb{R}}\), \(\nu(E)_{\mathbb{R}} = |\det \theta_{\gamma}|\), with respect to integral bases. As \(|\det \theta_{\gamma}|\) is independent of the choice of integral bases, we only need to show that there exist a section \(\gamma\) of \(\psi_{\mathbb{R}}\) and integral bases of \(A, B\) and \(C\) such that (76) holds.

Consider the following commutative diagram
\[
\begin{array}{ccccc}
0 & \to & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \to 0 \\
& & \downarrow{\psi_{\text{tor}}} & & \downarrow{\psi} & & \downarrow{\psi_f} & \\
0 & \to & C_{\text{tor}} & \to & C & \to & C_f & \to 0 \\
\end{array}
\]
The Snake lemma yields \(\cok(\psi_f) = 0\) and \(0 \to \ker \psi_{\text{tor}} \to \ker \psi = \text{im}(\phi) \to \ker \psi_f \to \cok \psi_{\text{tor}} \to 0\).

Therefore, \([\cok(\psi_{\text{tor}})] = [\ker \psi_f/\text{im}(\phi)]\). Since \(\psi_f : B_f \to C_f\) is surjective and \(C_f\) is a free abelian group, there exists a section \(\gamma : C_f \to B_f\) of \(\psi_f\) and we have \(B_f = \ker(\psi_f) \oplus \gamma(C_f)\). Take any integral basis \(\{w_i\}_{i=1}^s\) for \(C_f\). By the Smith Normal form, there are \(\mathbb{Z}\)-bases \(\{u_i\}_{i=1}^r\) for \(A_f\) and \(\{v_i\}_{i=1}^r\) for \(\ker \psi_f\) such that \(\phi_f(u_i) = m_i v_i\) where \(m_i\) is a positive integer for \(i = 1, \ldots, r\). Then \(\{u_1, \ldots, u_r, w_1, \ldots, w_s\}\) and \(\{v_1, \ldots, v_r, \gamma(w_1), \ldots, \gamma(w_s)\}\) form integral bases for \(A_{\mathbb{R}} \oplus C_{\mathbb{R}}\) and \(B_{\mathbb{R}}\). Moreover, \([\ker \psi_f/\text{im}(\phi)] = \prod_{i=1}^r |m_i|\).

Let \(\theta : A_{\mathbb{R}} \oplus C_{\mathbb{R}} \to B_{\mathbb{R}}\) be given by \(\theta(a, c) = \psi(a) + \gamma(c)\). Then with respect to the above integral bases, \(\det(\theta_{\gamma}) = \prod_{i=1}^r |m_i|\). As a result,
\[\nu(E)_{\mathbb{R}} = |\det \theta_{\gamma}| = \prod_{i=1}^r |m_i| = \left[\frac{\ker \psi_f}{\text{im}(\phi)}\right] = [\cok(\psi_{\text{tor}})] = \frac{[A_{\text{tor}}][C_{\text{tor}}]}{[B_{\text{tor}}]} .\]

\[ \square \]

Proposition 6.7. Let \((E)\) be an exact sequence of finitely generated abelian groups
\[0 \to A_0 \to A_1 \to \ldots \to A_n \to 0.\]
Let \((E)_{\mathbb{R}}\) be the sequence \((E)\) tensoring with \(\mathbb{R}\). Let \(B_i\) be an ordered integral basis for \(A_i\). Then with respect to \(B_i\),
\[\nu(E)_{\mathbb{R}} = \prod_{i=0}^n [([A_i]_{\text{tor}})^{(-1)^i}.\]

Proof. The proof uses induction on \(n\). The base case when \(n = 2\) is lemma 6.6 \(\square\)

Corollary 6.8. Suppose we have an exact sequence of finitely generated abelian groups
\[0 \to A \to B \xrightarrow{\phi} C \xrightarrow{\psi} D \to E \to 0\]
where \(A\) and \(E\) are finite groups. Then with respect to integral bases,
\[\nu(0 \to B_{\mathbb{R}} \xrightarrow{\phi} C_{\mathbb{R}} \xrightarrow{\psi} D_{\mathbb{R}} \to 0) = \frac{[B_{\text{tor}}][D_{\text{tor}}]}{[A][C_{\text{tor}}][E]} = \frac{[\cok(\psi_{\text{tor}})]}{[\cok(\psi)]}.\]
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