KHOVANOV HOMOLOGY AND RASMUSSEN’S $s$-INVARIANTS
FOR PRETZEL KNOTS

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Abstract. We calculated the rational Khovanov homology of some class of pretzel knots, by using the spectral sequence constructed by P. Turner. Moreover, we determined the Rasmussen’s $s$-invariant of almost of pretzel knots with three pretzels.

1. Introduction and main results

In [3], Khovanov constructed a bi-graded homology invariant $KH(L)$ of links, whose graded Euler characteristic is the unnormalized Jones polynomial of $L$. Lee [4] modified this theory and constructed another invariant $KH_{Lee}(L)$, which is a singly-graded homology. Lee showed that the rational Khovanov homology is representable as the $E_2$ term of a spectral sequence which converges to $\mathbb{Q}^\oplus 2^n$, where $n$ is a number of components of a link $L$. In [8], Rasmussen defined a knot invariant $s(K)$ using the Khovanov homology and Lee theory. This invariant $s$ gives a lower bound for the slice genus. There are several inequalities for $s(K)$: slice-Bennequin inequality proved by Shumakovich [9] and Plamenevskaya [6] independently, sharper slice-Bennequin inequality by Kawamura [2], and crossing change inequality by Livingston [5] and Rasmussen [8].

There was no theoretical computational tool for Khovanov homology but the skein exact sequence obtained from a short exact sequence of Khovanov homology

$$0 \to C(D'') \to C(D) \to C(D') \to 0$$

(1)

where $D$ is a link diagram and $D'$ and $D''$ are obtained by resolving the same crossing of $D$ to the 0-smoothing and the 1-smoothing, respectively. We would have used this long exact sequence repeatedly before, but it requires careful bookkeeping. Turner [7] defined a spectral sequence, which is obtained by tying up in a bundle the skein exact sequences. It is simple enough to enable us fast theorical computation. In [7], Turner gives an example of an application to $(3, n)$-torus links.

In this paper, we give another application of Turner’s spectral sequence to pretzel knots. Our main results are stated below. Theorem 1.1 is a result for $KH(K)$, and Theorems 1.2 and 1.3 are for $s(K)$. In Section 2, we summarize without proofs some basic facts. In Section 3, we prove Theorem 1.1. In Section 4, we give remarks about the assumption of Theorem 1.1. In Section 5, we prove Theorems 1.2 and 1.3.

Theorem 1.1. Suppose $p$ is an odd number and $p \geq 9$, $q = p - 2$, and $r$ is a positive even number. Let $K$ be the pretzel knot $P(p, -q, -r)$. Then Rasmussen’s $s$-invariant $s(K)$ is given by

$$s(K) = 2.$$  

Furthermore, the rational Khovanov homology $KH(K)$ is computable and given explicitly as in formulae [8], [11], [20].

In the proof of Theorem 1.1 we obtain the value of $s$-invariant before we obtain the homology. Hence our approach is easily applied to calculation of $s$-invariant of...
more larger class of pretzel knots. In fact, by combinatorial use of sharper slice-
Bennequin inequality and this approach, we could obtain the value of $s$-invariant of
almost all of pretzel knots with three pretzels.

**Theorem 1.2.** Suppose $p$ and $q$ are odd numbers and $p \geq 3$, $q \geq 3$, and $r$ is a
positive even number. Let $K$ be the pretzel knot $P(p, q, -r)$. Then Rasmussen’s
$s$-invariant $s(K)$ is given by

$$s(K) = p - q. \quad (3)$$

**Theorem 1.3.** Suppose $p$, $q$ and $r$ are odd numbers and $p \geq 3$, $q \geq 3$, $r \geq 3$. Let
$K$ be the pretzel knot $P(p, q, -r)$. Then Rasmussen’s $s$-invariant $s(K)$ is given by

$$s(K) = \begin{cases} 
0 & \text{if } p > \min\{q, r\} \\
2 & \text{if } p < \min\{q, r\} 
\end{cases} \quad (4)$$

We note some properties of pretzel knot with three pretzels and its $s$-invariant.

**Proposition 1.4.** i) A pretzel knot $P(p, q, r)$ is indeed knot if and only if all of $p$, $q$, $r$
are odd numbers and one of these integers is even number and the other two
are odd numbers.

ii) For all $\sigma \in \mathcal{S}_3$, $s_1, s_2, s_3 \in \mathbb{Z}$,

$$P(s_1, s_2, s_3) \simeq P(s_{\sigma(1)}, s_{\sigma(2)}, s_{\sigma(3)}) \quad (5)$$

and

$$s(P(-s_1, -s_2, -s_3)) = s(P(s_1, s_2, s_3)) = -s(P(s_1, s_2, s_3)). \quad (6)$$

By Proposition 1.4 we do not need to consider all patterns. For example, if $p$, $q$ and $r$
satisfy the assumptions of Theorem 1.2 then $s(P(p, q, r)) = p - q$.

2. Tools and notation

2.1. **Turner’s spectral sequence.** Let $L$ be an oriented link and $D$ be an oriented
diagram of $L$. Let $\{1, \ldots, m\}$ be a subset of the set of the crossings of $D$. For
$1 \leq k \leq m$, let $D_k$ be the diagram obtained from $D$ by resolving the crossing
$1, \ldots, k$ to 1-smoothings, and $D_{k+1}$ the diagram obtained from $D$ by resolving the
crossing $1, \ldots, k - 1$ to 1-smoothings and the crossing $k$ to a 0-crossing. For $k = 0$,
let $D(0) = D(0) = D$ as an oriented diagram.

If the $k$th crossing is positive then there is the canonical orientation of the di-
gram $D_k$: since the 0-smoothing is the oriented resolution, $D_k$ inherits an
orientation from $D_{k-1}$. But for $D_k$, there is no canonical one, so we can choose
any orientation. On the contrary, if the $k$th crossing is negative then $D_k$ inherits
an orientation from $D_{k-1}$ and we can choose any orientation for $D_k$.

Turner’s spectral sequence, which is converging to $KH(D)$, is obtained by ar-
raging $KH(D)$’s and $KH(D)$ at the appropriate position represented by some
constants $\hat{a}_k$, $\hat{b}_k$, $A_k$, and $B_k$ defined from $D$’s and $D$’s.

**Proposition 2.1** (P. Turner Proposition 2.2). For every fixed $j \in \mathbb{Z}$, there is a spectral sequence
$(E^\ast, d_r : E_1^\ast \rightarrow E_r^\ast)$ converging to $KH_j(D)$ with
$E_1$-page given by

$$E_1^{s,t} = \begin{cases} 
KH_{j+B_s+\hat{a}_{s+1}}^{s+t+A_s} (\hat{D}_{s+1}) & s = 0, \ldots, m - 1 \\
KH_{j+B_m}^{m+t+A_m} (D_m) & s = m \\
0 & \text{otherwise} 
\end{cases} \quad (7)$$
where \( \tilde{a}_k, \tilde{b}_k, A_k, \) and \( B_k \) are the constants defined by the following:

\[
\begin{align*}
&\bar{n}_k^+ = \text{number of positive crossing in } D(k) \\
&\bar{n}_k^- = \text{number of negative crossing in } D(k) \\
&\tilde{n}_k^+ = \text{number of positive crossing in } \tilde{D}(k) \\
&\tilde{n}_k^- = \text{number of negative crossing in } \tilde{D}(k)
\end{align*}
\]

\[
\begin{align*}
& a_k = n_{k-1}^- - n_k^- - 1 \quad \text{and} \quad \bar{a}_k = \tilde{n}_{k-1}^- - \tilde{n}_k^- \\
& b_k = 3a_k + 1 \quad \text{and} \quad \bar{b}_k = 3\bar{a}_k - 1.
\end{align*}
\]

\[A_0 = B_0 = 0, \quad A_k = \sum_{i=1}^{k} a_i, \quad \text{and} \quad B_k = \sum_{i=1}^{k} = 3A_k + k.\]

2.2. Lee theory. To prove the theorems, we need to make use of Lee theory. Lee theory is a variant of rational Khovanov homology obtained from the same underlying vector spaces but using a different differential. We denote it by \( Kh^{\ast}_{Lee}(L) \). We summarize the results we need about Lee theory in the following proposition.

**Proposition 2.2.** For a knot \( K \),

(i) \( Kh^t_{Lee}(K) = \left\{ \begin{array}{ll} Q \oplus Q & t = 0 \\
0 & t \neq 0 \end{array} \right. \)

(ii) There is a spectral sequence \( (E_\ast^{s,t}, d_r) \) converging to \( Kh^s_{Lee}(K) \) with \( E_1^{s,t} = Kh^{s+t}_{4s+1}(K) \). The differential \( d_r \) has bi-degree \((1,4r)\) in term of index of the Khovanov homology.

2.3. Notation. Let \( P(p,q,r) \) be the standard diagram of \((p,q,r)\)-pretzel link (or link itself). By abuse of notation, if \( p \) (or \( q, r \)) is equal to 0, we consider that there is no half twist at the corresponding pretzel. For example, \( P(p,q,0) \simeq T(2,p)\#T(2,q) \) as an unoriented link.

Let \( P(p,q,r_\epsilon) \) be the diagram obtained from the diagram \( P(p,q,r + \epsilon) \) by resolving the crossing at the top of the corresponding pretzel to a \( x \)-smoothing, where \( \epsilon = 1, x = 1 \) if \( r \geq 0 \), \( \epsilon = -1, x = 0 \) if \( r < 0 \). Obviously, \( P(p,q,r_\epsilon) \simeq P(p,q,0) \) as an unoriented link. \( P(p_\epsilon, q, r), P(p,q_\epsilon, r) \) is also defined in the same manner.

\( KH(L) \) and \( Kh_{Q}(L) \) denotes the rational Khovanov homology of \( L \) and its Poincaré polynomial, respectively.

3. Proof of Theorem 1.1

We prove the theorem by two steps. Each step consists of three small steps: first, we write down the spectral sequence, secondly, we describe the possibility of \( KH(K) \), thirdly, we determine \( KH(K) \) (by using Lee theory).
Proposition 3.1 (Step 1). If \( p : odd \geq 9, q = p - 2 \),
\[
KH_Q(P(p, -q, 0)) = q^{-2p+5}t^{-p+2} + q^{-2p+9}t^{-p+3} + 2q^{-2p+9}t^{-p+4} + q^{-2p+11}t^{-p+5}
\]
\[
+ \sum_{n=\frac{p+7}{2}} q^{2n} t^n ((I_n + 1)q^{-1}t^{-2} + (I_n + 2)q^{-1}t^{-1} + I_nq^1t^0)
\]
\[
+ (I_0 + 3)q^{-1}t^{-2} + (I_0 + 3)q^{-1}t^{-1} + (I_0 + 2)q^1t^{-1} + (I_0 + 4)q^1t^0
\]
\[
+ (I_0 + 4)q^3t^0 + (I_0 + 3)q^3t^1 + (I_0 + 3)q^5t^1 + (I_0 + 4)q^5t^2
\]
\[
+ \sum_{m=\frac{p-3}{2}} q^{2m} t^m ((I_m + 1)q^3t^0 + I_mq^3t^1 + (I_m + 2)q^5t^1 + (I_m + 1)q^5t^2)
\]
\[+ q^{2p+3} t^p. \quad (8)\]
where \( I_m = \frac{p-m-3}{2}, I_0 = \frac{p-9}{2}, I_n = \frac{p+n-5}{2}. \) In other words, \( KH_Q(P(p, -q, 0)) \) is
given by Table 2 (see page 14).

Proof. We use Proposition 2.1 (in case \( m = q - 1 \)). Let \( D = D_{(0)} = \tilde{D}_{(0)} = P(p, -q, 0) \) and its orientation be as shown in Figure 1. For \( 1 \leq k \leq q - 1 \),
\( D_{(k)} := P(p, -(q - k), 0) \) (which inherits the orientation from \( D_{(k-1)} \)), \( \tilde{D}_{(k)} := P(p, -(q - k), 0) \) and its orientation be as shown in Figure 2. It is easy to see that
\( \tilde{D}_{(k)} \simeq T(2, p) \) for \( 1 \leq k \leq q - 1 \), \( D_{(q-1)} \simeq T(2, p) \), \( \tilde{a}_s = q - s + 1 \), \( \tilde{b}_s = 3q - 3s + 2 \),
\( \tilde{A}_s = 0 \), \( B_s = s \) for \( 1 \leq s \leq q - 1 \).

The spectral sequence is
\[
E_{s,t}^1 = KH_{j+q}^{i+q}(T(2, p)) \text{ if } s = 0, \ldots, q - 2 \quad (9)
\]
\[
E_{q-1,t}^1 = KH_{j+q}^{i+q-1}(T(2, p)). \quad (10)
\]
From the result of the (integral) Khovanov homology of \((2, p)\)-torus link (see [3]),
one can write down the spectral sequence for each \( j \). For \( j \geq 2p + 1 \), \( j = 2p - 3 \) or
\( j \leq 5 - 2p \), the spectral sequence is concentrated in one column and hence collapses
for dimensional reasons. For other \( j \), there exists one or several possible (nontrivial)
\( d^r \)'s \((r \geq 1)\).

If a generator of \( E_{s,t}^1 \) for some \( j \) survives to \( E_{\infty} \), it corresponds to a generator
of \( KH_{j+q}^{i+q}(D) \). So the generator of \( E_{s,t}^1 \) for some \( j \) located at the initial/terminal
end of these differential (we note that it may possibly survive to \( E_{\infty} \)) corresponds to a possible generator of \( KH_{j}^{i+q}(D) \).

Now the situation is as shown in Table 10 (see page 15). We note that there is
no possible generator outside the two diagonals: \( \deg = 2 \dim + 2 \pm 1 \). We require
the support of Lee theory. Suppose that the possible generator $z$ in bi-degree $(-p + 2, -2p + 7)$ is indeed a generator. Then it survives to $E_\infty$ of Lee’s spectral sequence, (there is nothing which kills $z$), so $KH_{\text{Lee}}^{p+2} \neq 0$, which contradicts Proposition 2.2. Hence this generator $z$ must be fake (not a generator). Moreover, by looking back to Turner’s spectral sequence, the possible generator $w$ in bi-degree $(-p + 3, -2p + 7)$ must be also fake. This is from the following reason. Since $z$ is fake, and the $E_1$ for $j = -2p + 7$ is $\begin{pmatrix} 1 & 0 \\ -p + 2 & 1 \\ 1 & 1 \end{pmatrix}$, the generator $\tilde{z} \in E_1^{0,-p+2}$ (which corresponds $z$) must be killed at $E_1$, and when $\tilde{z}$ is killed, another generator $\tilde{w} \in E_1^{1,-p+2}$ must be killed at the same time, which corresponds to $w$. Everything is going on like this. By using two spectral sequences alternately, we can show that for each even number $3 - p \leq n \leq -2$, exactly 1 pair of possible generators in bi-degree $(n-1, n, 1+2n)$ must be fake, and other possible generators must be indeed generators (details are omitted). Thus we obtain the result as shown in the proposition.

Remark 3.2. As mentioned in the proof, there is no possible generator outside the two diagonals deg $= 2\dim +2 \pm 1$. This indicates that the rational Khovanov homology of $P(p, -q, 0)$ is computable if we just know the Jones polynomial of $P(p, -q, 0)$, which is equal to $V(T(2, p))V(T(2, q))$.

Proposition 3.3 (Step 2 in case $r = 2$). If $p$ : odd $\geq 9$, $q = p - 2$, $r = 2$, $K = P(p, -q, -r)$ then

$$KH_Q(K) = q^4t^2KH_Q(P(p, -q, 0)) + q^1 + q^3 + q^3t + q^9t^3$$

$$s(K) = 2$$

and $K$ is $H$-thin.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4	extwidth}
\centering
\begin{tikzpicture}
\node at (0,0) {$P$};
\node at (2,0) {$-q$};
\node at (4,0) {$-r$};
\end{tikzpicture}
\caption{Figure 3.}
\end{subfigure}
\begin{subfigure}{0.4	extwidth}
\centering
\begin{tikzpicture}
\node at (0,0) {$P$};
\node at (2,0) {$-q$};
\node at (4,0) {-(r-k)};
\end{tikzpicture}
\caption{Figure 4.}
\end{subfigure}
\end{figure}

Proof. We use Proposition 2.1 (in case $m = 2$). Let $D = P(p, -q, -r)$ and its orientation be as shown in Figure 3, $D_{(k)} = P(p, -q, -(r - k))$ for $k = 1, 2$ and its orientation as shown in Figure 4. We note that the orientations of unmarked arcs vary with whether $k = 1$ or $k = 2$. $\tilde{D}_{(k)} = P(p, -q, -(r - k))$ (which inherits the orientation from $D_{(k-1)}$) for $k = 1, 2$. Then $\tilde{a}_1 = \tilde{a}_2 = 0$, $\tilde{b}_1 = \tilde{b}_2 = -1$, $A_1 = -3$, $B_1 = -8$, $A_2 = -2$, $B_2 = -4$.

It is easy to see $\tilde{D}_{(1)} \cong T(2, 2)$ and $\tilde{D}_{(2)}$ is equivalent to the link $L$ obtained from $T(2, 2)$ by changing the orientation of one component. In this case, we may say that $L \cong T(2, -2)$. We can easily get the homology of $L$ from the homology of $T(2, 2)$ by shifting indexes:

$$KH^i_J(L) \cong KH^{i+2}_{J+6}(T(2, 2)).$$

(13)
The spectral sequence is
\[ E^{0,t}_{k} = KH_{j-1}^{t}(T(2, 2)) \] (14)
\[ E^{1,t}_{k} = KH_{j-2}^{t}(L) = KH_{j-3}^{t}(T(2, 2)) \] (15)
\[ E^{2,t}_{k} = KH_{j-4}^{t}(P(p, -q, 0)). \] (16)

Since the homology of \( T(2, 2) \) is very simple, the spectral sequence is also simple. This fact enables us to carry out the computation. For \( j \neq 1, 3, 5, 7, 9 \), the spectral sequence is as shown in Table 1. (Let \( I_0 := (p - 9)/2 \).)

When we finish writing down the possible \( KH(K) \), we will find that the support of possible \( KH(K) \) is in the two diagonals \( \deg = 2 \dim + 2 \pm 1 \), and all possible generators are shown in Table 2. In this table, a possible generator is denoted in a bracket, like \([1], I_0 + 3\]. So we can easily see that the support of \( KH(K) \) is also in the same two diagonals. In other words, \( K \) is H-thin. From this result, we obtain \( s(K) = 2 \) immediately. In this case, there is no necessity of support of Lee’s theory to determine \( s(K) \).

We use Lee’s theory to determine \( KH(K) \). Because \( KH(K) \) is in the two diagonals mentioned above, Lee’s theory implies that
\[ \text{rank}KH_{2i+1}^{i}(K) = \text{rank}KH_{2i+2}^{i+1}(K) \quad \text{if} \quad i \neq 0, -1 \] (17)
\[ \text{rank}KH_{1}^{0}(K) - 1 = \text{rank}KH_{3}^{1}(K) \] (18)
\[ \text{rank}KH_{-1}^{-1}(K) = \text{rank}KH_{0}^{0}(K) - 1. \] (19)
Hence by (19) the possible generator at bi-degree \((0, 3)\) is indeed a generator. Then we look back Turner’s \(E_1(j = 3)\). We will find that all possible generators at bi-degree \((1, 3)\) are also generators. In this way, all possible generators at bi-degree \((*, 9)\) are indeed generators, and exactly one possible generator is fake at bi-degree \((1, 5), (2, 5), (2, 7)\) and \((3, 7)\), respectively. Then the homology turns out to be as shown in Table 3.

| deg \ dim | -1 | 0   | 1   | 2   | 3   | 4   |
|-----------|----|-----|-----|-----|-----|-----|
| 9         |     | \(I_0 + 4\) | \(I_0 + 4\) |     |     |     |
| 7         |     | \(I_0 + 3\) | \(I_0 + 4\) |     |     |     |
| 5         |     | \(I_0 + 2\) | \(I_0 + 4\) |     |     |     |
| 3         |     | \(I_0 + 4\) | \(I_0 + 4\) |     |     |     |
| 1         | \(I_0 + 1\) | \(I_0 + 3\) |     |     |     |     |
| -1        | \(I_0 + 3\) |     |     |     |     |     |

Table 3.

| deg \ dim | -1 | 0 | 1 | 2 | 3 | 4 |
|-----------|----|---|---|---|---|---|
| 9         |     | +1 | 0 |   |   |   |
| 7         |     | 0 | 0 |   |   |   |
| 5         |     | 0 | 0 |   |   |   |
| 3         |     | +1 | +1 |   |   |   |
| 1         | 0 | +1 |     |   |   |   |
| -1        | 0 |   |     |   |   |   |

Table 4.

\(KH(K)\) is almost same as \(KH(P(p, -q, 0))\) shifted by bi-degree \((2, 4)\). Table 4 shows the difference between the two homologies. From this viewpoint, we obtain \(11\).

**Proposition 3.4** (Step 2 in case \(r \geq 4\)). If \(p: \text{odd } \geq 9, q = p - 2, r: \text{even } \geq 4,\)

\[
KH_Q(K) = q^{2r}t^rKH_Q(P(p, -q, 0)) + q^{1} + q^{3} + q^{3}t^{1} + \sum_{k=1}^{r/2-1} q^{4k}t^{2k}(q^{1} + q^{3})(1 + q^{2}t) + q^{2r+5}t^{r+1}
\]

\(20\)

\[
s(K) = 2
\]

\(21\)

and \(K\) is \(H\)-thin.

**Proof.** We use Proposition 2.4 (in case \(m = r\)). Let \(D = P(p, -q, -r)\) and its orientation be as shown in Figure 6 \(D_{(k)} = P(p, -q, -(r - k))\) and its orientation as shown in Figure 6 \(\tilde{D}_{(k)} = P(p, -q, -(r - k))\) (which inherits the orientation from \(D_{(k-1)}\)) for \(1 \leq k \leq r\). Then \(a_k = 0, b_k = -1, A_{2l+1} = -2l - 3, B_{2l+1} = -4l - 8, A_{2l} = -2l, B_{2l} = -4l\) for \(1 \leq k \leq r, 0 \leq l < r/2, A_r = -r, B_r = -2r\).

It is easy to see \(\tilde{D}_{(2l+1)} \simeq T(2, 2)\) and \(\tilde{D}_{(2l)}\) is equivalent to the link \(L\) obtained from \(T(2, 2)\) by changing the orientation of one component. We can easily get its homology from the homology of \(T(2, 2)\) by shifting indexes.
The spectral sequence is
\[ E_{2}^{l,t} = KH_{j-4l-1}(T(2,2)) \quad 0 \leq l < r/2 \] (22)
\[ E_{1}^{l+1,t} = KH_{j-4l-9}(L) = KH_{j-4l-3}(T(2,2)) \quad 0 \leq l < r/2 \] (23)
\[ E_{r,t} = KH_{j-2r}(P(p,-q,0)) \] (24)
In other words,
\[ E_{s,t} = KH_{j-2s-1}(T(2,2)) \quad \text{for } 0 \leq s \leq r-1 \] (25)
This spectral sequence is also simple in some sense.

Claim 1: Suppose that \( j \in \mathbb{Z} \) fixed. Then the support of \( E_{1} \) is in two diagonals \( s+t = (j-2 \pm 1)/2 \), i.e. \( E_{s,t} = 0 \) if \( s+t \neq (j-2 \pm 1)/2 \).

Proof of Claim 1) If \( j \) is even, then \( E_{1} = 0 \) and there is nothing to prove. Suppose \( j \) is odd. Since \( KH_{a,b}(T(2,2)) \neq 0 \) if and only if \((a,b) = (0,0), (0,2), (2,4), (2,6)\), we obtain
\[ E_{s,t} = 0 \text{ if } (s,t) \neq (l-1,0), (l-3,0), (l-5,2), (l-7,2) \] (27)
for \( 0 \leq s \leq r-1 \). For \( s = r \), by Proposition 3.1 we obtain that the support of \( KH(P(p,-q,0)) \) is in two diagonals: \( \deg = 2 \dim + 2 \pm 1 \). Hence we have \( E_{r,t} = 0 \) if \( j - 2r \neq 2t + 2 \pm 1 \).

This simplicity means the support of the possible \( KH(K) \) is also in the two diagonals \( \deg = 2 \dim + 2 \pm 1 \), hence \( K \) is H-thin, and \( s(K) = 2 \).

To determine \( KH(K) \), we write down the Turner’s \( E_{1} \) and possible \( KH \) explicitly, and use two spectral sequences alternately. It seems to take long time, but the well-regulatedness and “H-thinness” (Claim 1) of \( E_{1} \) make it easier. We have to consider several cases: \( r \leq p-5, r = p-3, r = p-1, r = p+1, \) and \( r \geq p+3 \). We omit details.

\[
\begin{array}{cccccccccc}
\text{deg} & \text{dim} & 0 & 1 & \ldots & l & l+1 & \ldots & r & r+1 \\
2r+5 & & & & & & & & & +1 \\
2r+3 & & & & & & & & 0 & 0 \\
2r+5 & & & & & * & 0 & & & \\
2l+5 & & & & & +1 & * & & & \\
2l+3 & & & & & +1 & +1 & & & \\
2l+1 & & & & * & +1 & & & & \\
3 & & & & +1 & +1 & & & & \\
1 & & & & +1 & & & & & \\
\end{array}
\]

\( 0 < l < r, \quad l : \text{even} \)

Table 5.

In all cases, \( KH(K) \) is almost same as \( KH(P(p,-q,0)) \) shifted by bi-degree \((r,2r)\). Table 5 shows the difference between the two homologies. From this viewpoint, we obtain (20).

4. Some remarks about Theorem 1.1

4.1. Further calculation.

Remark 4.1. There exists the case that we can’t determine \( KH(K) \) by this technique. Even in such case, however, we may be able to determine \( s(K) \). Example 4.2 gives an example of such case.
Example 4.2. If $K = P(9, -5, -2)$, above approach is not successful. But we can determine $s(K)$.

One of the reasons of unsuccessfulness of calculation is that there is a “double knight move” type piece of possible generators. There are two possibility: all four possible generators are indeed generators, or all are fake. We can not determine which possibility holds. Hence we need another approach.

To determine $s(K)$ is, however, possible by this approach. Because possible $KH_j(K)$ vanishes if $j \neq 3, 5$, we obtain $s(K) = 4$.

| [1] | [1] |
|-----|-----|
| [1] | [1] |

Table 6. “double knight move” type piece of possible generators

4.2. The (sharper) slice-Bennequin type inequality for $s(K)$. Let us focus on $s(K)$, not on $KH(K)$. There are several slice-Bennequin type inequalities for $s(K)$.

Proposition 4.3 (Shumakovitch\[9\], Plamenevskaya\[6\]). For any knot $K$ and its diagram $D_K$, we have

$$s(K) \geq w(D_K) - O(D_K) + 1$$

(28)

where $w(D_K)$ and $O(D_K)$ are the writhe of $D_K$ and the number of Seifert circle of $D_K$, respectively.

Definition 4.4 (Kawamura\[2\]). Let $D$ be an oriented diagram and $S$ a Seifert circle of $D$. A Seifert circle $S$ is called strongly negative if it is adjacent at least two negative crossings but no positive crossings. A Seifert circle $S$ is called non-negative if it is not strongly negative.

Proposition 4.5 (\[2\]). Let $K$ be a knot and $D_K$ be a diagram of $K$. If $D_K$ has at least one non-negative Seifert circle, we have

$$s(K) \geq w(D_K) - (O_{\geq}(D_K) - O_{<}(D_K)) + 1$$

(29)

where $O_{<}(D_K)$ and $O_{\geq}(D_K)$ are the number of strongly negative and non-negative Seifert circle of $D_K$, respectively.

In our case $K = P(p, -q, -r)$ for $p : \text{odd} \geq 9$, $q = p - 2$, $r : \text{even} \geq 2$, and $D_K$ is as shown in Figure 3. $D_K$ and $D'_K$ have some non-negative Seifert circle, and $w(D_K) = r + 2$, $O(D_K) = r + 1$, $O_{<}(D_K) = 0$, $O_{\geq}(D_K) = r + 1$, $w(D'_K) = -r - 2$, $O(D'_K) = r + 1$, $O_{<}(D'_K) = r - 1$, $O_{\geq}(D'_K) = 2$. By (28), we obtain

$$s(K) \geq 2$$

(30)

$$s(K') \geq -2r - 2$$

(31)

by using the property of $s(K)$ that $s(K') = -s(K)$, we obtain

$$2 \leq s(K) \leq 2r + 2.$$ 

(32)

Similarly, by (29), we obtain

$$s(K) \geq 2$$

(33)

$$s(K') \geq -4$$

(34)

and we obtain

$$2 \leq s(K) \leq 4.$$ 

(35)
These estimations do not enable us to determine \(s(K)\). But by the above calculation, we have obtained \(s(K) = 2\).

5. **Proof of Theorem 1.2 and 1.3**

Let us survey the estimation of the value of \(s\)-invariant of pretzel knots with three pretzels before proving the theorems. We suppose all pretzel has at least two half twists. By Proposition 4.3, it is sufficient to consider the following five cases: (E) Let \(p, q\) be positive odd numbers and \(r\) be a positive even number.

(E0) \(K = P(p, q, r)\). (E1) \(K = P(p, q, -r)\). (E2) \(K = P(p, -q, r)\).

(O) Let \(p, q, r\) be positive odd numbers.

(O0) \(K = P(p, q, r)\). (O1) \(K = P(p, q, -r)\).

(E1) have a positive diagram, hence by \(\text{[5]}\), we obtain \(s(K) = p + q\). Similarly, (O0) have a negative diagram, hence \(K^1\) have a positive one, so we obtain \(s(K) = -2\). In the other three cases (E0), (E2) and (O1), Proposition 4.5 gives us estimations of \(s(K)\) and the signature \(\sigma(K)\) of \(K\). If \(K\) is alternating, then \(s(K) = -\sigma(K)\). In these table, we set \(\omega(P(i, j, k)) := ij + jk + ki\).

In case (E0), \(s(K)\) has been already determined. In case (E2), we determine \(s(K)\) in Theorem 1.2. In some case in (E3), we determine \(s(K)\) in Theorem 1.3. The next lemma plays an important part as computational shortcut.

**Lemma 5.1.** If \(p, q\) are odd numbers \(\geq 3\), then \(K = T(2, p)\#T(2, -q)\) is H-thin.

**Proof.** We use Turner’s spectral sequence with following settings. (Same as in the proof of Proposition 4.3) Let \(D = D_{(0)} = \bar{D}_{(0)} = P(p, -q, 0)\) and its orientation be as shown in Figure 1. For \(1 \leq k \leq q - 1\), \(D_{(k)} := P(p, -(q-k), 0)\) (which inherits the orientation from \(D_{(k-1)}\), \(\bar{D}_{(k)} := P(p, -(q-k), 0)\) and its orientation be as shown in Figure 2. It is easy to see that \(\bar{D}_{(k)} \simeq T(2, p)\) for \(1 \leq k \leq q - 1\), \(\bar{D}_{(q-k-1)} \simeq T(2, p)\), \(\bar{a}_s = q - s + 1\), \(\bar{b}_s = 3q - 3s + 2\), \(A_s = 0\), \(B_s = s\) for \(1 \leq s \leq q - 1\).

The spectral sequence is

\[
E_1^{s,t} = KH^{s+q}_{j+3q-2s-1}(T(2, p)) \quad \text{if } s = 0, \ldots, q - 2 \tag{36}
\]

\[
E_2^{q-1, t} = KH^{s+q-1}_{j+3q-2s-1}(T(2, p)) \tag{37}
\]
Here we note that for $p \geq 2$, support of the homology of $(2, p)$-torus knot is given by

$$\text{Supp } KH(T(2, p)) \subset \{ \deg = p - 1 \pm 2 \text{ dim}, 0 \leq \text{ dim } \leq p \}.$$  \hfill (38)

Hence the support of the spectral sequence is given by

$$\text{Supp } E_1^{s,t} \subset \left\{ t = \frac{j + q - p + 1}{2} - s, -q \leq t \leq p - q \right\} \text{ if } s = 0, \ldots, q - 2$$ \hfill (39)

$$\text{Supp } E_1^{q-1,t} \subset \left\{ t = \frac{j - q - p + 1}{2} - 1, -q + 1 \leq t \leq p - q + 1 \right\}.$$  \hfill (40)

In other words, $E_1^{s,t} = 0$ if $s + t \neq \pm \frac{j + q - p + 1}{2}$. From this we conclude that $KH_j^i(K) = 0$ if $j \neq 2i + p - q + 1$. This completes the proof. \hfill \Box

The following proposition partially proves Theorem 1.2. Note that there is an additional condition $p > q$.

**Proposition 5.2.** If $p, q$ are odd numbers $\geq 3$, $p > q$, $r$ is a positive even number and $K = P(p, -q, -r)$, then

$$s(K) = p - q.$$ \hfill (41)

**Proof.** We use Turner’s spectral sequence with following settings. (Same as in the proof of Proposition 3.4) Let $D = P(p, -q, -r)$ and its orientation be as shown in Figure 3. $D(k) = P(p, -q, -(r - k))$ and its orientation as shown in Figure 4. $D(k) = P(p, -q, -(r - k)^\ast)$ (which inherits the orientation from $D(k - 1)$) for $1 \leq k \leq r$. Then $a_k = 0, b_k = -1, A_{2l+1} = -2l - 1 - p + q, B_{2l+1} = -4l - 2 - 3p + 3q, A_2 = -2l, B_2 = -4l$ for $1 \leq k \leq r, 0 \leq i < r/2, A_r = -r, B_r = -2r$.

It is easy to see $D_{2l+1}(2p - q)$ and $D_{2l}(2p - q)$ is equivalent to the link $L$ obtained from $T(2, p - q)$ by changing the orientation of one component. We can easily get its homology from the homology of $T(2, p - q)$ by shifting indices:

$$KH_j^i(L) \simeq KH_j^{i+p-q}(T(2, p - q)).$$ \hfill (42)

The spectral sequence is

$$E_1^{s,t} = KH_j^{i - 2s - 1}(T(2, p - q)) \quad \text{for } 0 \leq s \leq r - 1$$ \hfill (43)

$$E_1^{s,t} = KH_j^{i - 2s}(P(p, -q, 0)).$$ \hfill (44)

From (38), we obtain that $E_1^{s,t}|_{s+t=0, 0 \leq s \leq r - 1} \neq 0$ only if $s = 0, j = p - q + 1$. Moreover, from Lemma 5.1 we obtain $E_1^{s,t}|_{r+t=0} \neq 0$ only if $j = p - q + 1$. Hence we conclude $E_1^{s,t}|_{s+t=0} \neq 0$ only if $j = p - q + 1$, which proves the proposition. \hfill \Box

To complete the proof of Theorem 1.2, we use the following fact.

**Lemma 5.3** (Livingston\[5\], J. Rasmussen\[8\]). Suppose $K_+$ and $K_-$ are knots that differ by a single crossing change — from a positive crossing in $K_+$ to a negative one in $K_-$. Then

$$s(K_+) \leq s(K_-) \leq s(K_-) + 2.$$  \hfill (45)

The rest of the theorem will be proved by induction. Let $K_+ = P(p, -q, -r)$, $K_- = P(p - 2, -q, -r)$. Suppose that $s(P(p, -q, -r)) = p - q$. From the lemma above, we obtain

$$p - q - 2 \leq s(K_-) \leq p - q.$$ \hfill (45)

On the other hand, from the slice-Bennequin type inequality for $s(K)$ (see Table 7, we obtain

$$p - q - 4 \leq s(K_-) \leq p - q - 2.$$ \hfill (46)

Then we obtain $s(K_-) = p - q - 2$, and we have completed the proof of Theorem 1.2.
Let us prove Theorem 1.3. We begin by proving in the case \( p = q + 2 \) and \( r \) is arbitrary.

**Proposition 5.4.** If \( p, q \) are odd numbers \( \geq 3 \), \( p = q + 2 \), \( r \) is a positive odd number and \( K = P(p, -q, -r) \), then

\[
s(K) = 0.
\]

**(Proof.** We use the spectral sequence with following settings. Let \( D = P(p, -q, -r) \), \( D_1 = P(p, -q, -(r - 1)) \) and its orientation be as shown in Figure 4 (set \( k = 0, 1 \)), \( \tilde{D} = (p, -q, -(r - 1)\mathcal{E}) \) (which inherits the orientation from \( D_0 = D \)). We note that \( r \) is an odd number. Then \( a_1 = 0, b_1 = -1, A_1 = 1, B_1 = 4 \).

In this case, our spectral sequence is

\[
E_1^{j,k} = KH^{j+k}_1(T(2, -2)) = KH^{j+k}_1(T(2, 2)) \quad (48)
\]

\[
E_1^{j,k} = KH^{j+k}_1(P(p, -q, -(r - 1))). \quad (49)
\]

It is sufficient to focus on \( E_1^{j,k}|_{j+k=0} \). It is easy to check that \( E_1^{j,k}|_{j+k=0} = 0 \) if \( j \neq \pm 1 \), which completes the proof. If \( s = 0 \), there is nothing to comment. If \( s = 1 \), it follows from Theorem 1.1 we obtain \( KH^{1+4}_1(P(p, -q, -(r - 1))) = 0 \) if \( j \neq \pm 1 \) from Theorem 1.1.

By using Lemma 5.3 inductively, the condition \( p = q + 2 \) is weakened to \( p > q \). Let us see the detail. Let \( K_+ = P(p, -q, -r) \), \( K_- = P(p + 2, -q, -r) \). Suppose that \( s(P(p, -q, -r)) = 0 \). Lemma 5.3 gives us \( -2 \leq s(K_-) \leq 0 \). On the other hand, the slice-Bennequin type inequality for \( s(K) \) gives us \( 0 \leq s(K_-) \leq 2 \). Then we obtain \( s(K_-) = 0 \). Moreover, the condition \( p > q \) is weakened to \( p > \min\{q, r\} \) because of Proposition 5.4.

It remains to consider the case \( p \leq \min\{q, r\} \). We give the partial result for the case \( p < \min\{q, r\} \), which completes the proof of Theorem 1.3.

**Proposition 5.5.** If \( p, q, r \) are odd numbers \( \geq 3 \), \( p < \min\{q, r\} \), then

\[
s(K) = 2. \quad (50)
\]

**(Proof.** Let us suppose \( p < q \) and consider Turner’s spectral sequence with following settings. Let \( D_k = P(p, -q, -(r - k)) \) and its orientation as shown in Figure 4 for \( 0 \leq k \leq r \). Let \( \tilde{D}_k = P(p, -q, -(r - k)\mathcal{E}) \) (which inherits the orientation from \( D_k \)) for \( 1 \leq k \leq r \). Then \( a_k = 0, b_k = -1, A_{2l+1} = -2l - 1 + p - q, B_{2l+1} = -4l - 2 + 3p - 3q, A_{2l} = -2l, B_{2l} = -4l \) for \( 1 \leq k \leq r, 0 \leq l < r/2 \). Especially, \( A_r = -r + p - q, B_r = -2r + 3(p - q) \). It is easy to see that \( \tilde{D}_k \approx T(2, p - q) \).

Our spectral sequence is

\[
E_1^{s,t} = KH^{s+p-q}_{j-4l+1+3(p-q)}(T(2, p - q)) \quad \text{if} \quad s = 0, \ldots, r - 1 \quad (51)
\]

\[
E_1^{s,t} = KH^{s+p-q}_{j-2r+3(p-q)}(T(2, p)\#T(2, -q)). \quad (52)
\]

We focus on \( E_1^{s,t}|_{s+t=0} \). \( E_1^{0,0} = 0 \) if \( j \neq 1, 3 \). Since \( KH^1(T(2, p - q)) = 0 \) if \( i < p - q \), \( E_1^{0,-s} = 0 \) for \( 1 \leq s \leq r - 1 \) and all \( j \). Here we suppose \( p < r \), then \( E_1^{i,-r} = 0 \) for all \( j \) because \( KH^i(T(2, p)\#T(2, -q)) = 0 \) if \( i < -q \). Hence we find that \( KH_1^0(D) = 0 \) if \( j \neq 1, 3 \). So we conclude that \( s(P(p, -q, -r)) = 2 \) if \( p < \min\{q, r\} \). \( \square \)

**Remark 5.6.** We have not yet determined \( s(K) \) in case \( p = \min\{q, r\} \).
Remark 5.7. By Lemma 5.3, the following equations are hold: if $p, q, r$ are odd numbers, then

$$P(p + 2, -q, -r) = P(p, -q, -r) \text{ or } -2$$

(53)

$$P(p, -(q + 2), -r) = P(p, -q, -r) \text{ or } +2$$

(54)

$$P(p, -q, -(r + 2)) = P(p, -q, -r) \text{ or } +2.$$  (55)

These inequalities played important role in the proof of Theorem 1.3, but they do not help to weaken the assumption of Theorem 1.3 anymore.

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Table 9. The rational Khovanov homology of $P(p, -q, 0)$ for $p : \text{odd} \geq 7, q = p - 2$.

| deg\dim | $\ldots$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $\ldots$ | $m$ | $m + 1$ | $m + 2$ | $\ldots$ | $p - 1$ | $p$ |
|----------|----------|------|------|-----|-----|-----|----------|-----|--------|--------|----------|--------|-----|
| $2p + 3$ |          |      |      |     |     |     |          |     |        |        |          |        |     |
| $2p + 1$ |          |      |      |     |     |     |          |     |        |        |          |        |     |
| $\vdots$ |          |      |      |     |     |     |          |     |        |        |          |        |     |
| $5 + 2m$ |          |      |      |     |     |     |          |     | $I_m + 2$ | $I_m + 1$ |          |        |     |
| $3 + 2m$ |          |      |      |     |     |     |          |     | $I_m + 1$ | $I_m$   |          |        |     |
| $\vdots$ |          |      |      |     |     |     |          |     |        |        |          |        |     |
| $5$      |          |      |      |     |     |     |          |     | $I_0 + 3$ | $I_0 + 4$ |          |        |     |
| $3$      |          |      |      |     |     |     |          |     | $I_0 + 4$ | $I_0 + 3$ |          |        |     |
| $1$      |          |      |      |     |     |     |          |     | $I_0 + 2$ | $I_0 + 4$ |          |        |     |
| $-1$     |          |      |      |     |     |     |          |     | $I_0 + 3$ | $I_0 + 3$ |          |        |     |
| $\vdots$ |          |      |      |     |     |     |          |     |        |        |          |        |     |

| deg\dim | $-p + 2$ | $-p + 3$ | $-p + 4$ | $-p + 5$ | $\ldots$ | $n - 2$ | $n - 1$ | $n$ | $\ldots$ | $-2$ | $-1$ | $0$ |
|----------|---------|---------|---------|---------|----------|--------|------|-----|--------|------|------|-----|
| $-1$     |         |         |         |         |          | $I_0 + 3$ | $I_0 + 3$ |     |       |       |       |     |
| $\vdots$ |         |         |         |         |          |        |      |     |       |       |       |     |
| $1 + 2n$ |         |         |         |         |          | $I_n$   | $I_n + 1$ |     |       |       |       |     |
| $-1 + 2n$|         |         |         |         |          | $I_n + 1$ | $I_n + 2$ |     |       |       |       |     |
| $\vdots$ |         |         |         |         |          |        |      |     |       |       |       |     |
| $-2p + 11$|        |         |         |         |          |        |      |     |       |       |       |     |
| $-2p + 9$|         |         |         |         |          | $1$     |       |     |       |       |       |     |
| $-2p + 7$|         |         |         |         |          | $2$     |       |     |       |       |       |     |
| $-2p + 5$|         |         |         |         |          | $1$     |       |     |       |       |       |     |

$m : \text{even}, 2 \leq m \leq p - 3, n : \text{even}, -p + 7 \leq n \leq -2, \text{and } I_m = \frac{p - m - 3}{2}, I_0 = \frac{p - 9}{2}, I_n = \frac{p + n - 5}{2}$.
$$m : \text{even, } 2 \leq m \leq p - 3, \ n : \text{even, } -p + 7 \leq n \leq -2, \text{ and } I_m = \frac{p-m-3}{2}, I_0 = \frac{p-3}{2}, I_n = \frac{p+n-5}{2}.$$

A possible generator is in a bracket, like $[1], [I_0 + 3]$.  

Table 10. The possible situation of the rational Khovanov homology of $P(p,-q,0)$ for $p : \text{odd} \geq 7, q = p - 2$