Characterization of Carleson measures by the Hausdorff-Young property

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Abstract

It is shown that the Laplace transform of an \(L^p\) \((1 < p \leq 2)\) function defined on the positive semiaxis satisfies the Hausdorff-Young type inequality with a positive weight in the right complex half-plane if and only if the weight is a Carleson measure.

1 Main theorem

The classical Hausdorff-Young inequality for the (one-dimensional) Fourier transform

\[
u(t) \to \mathcal{F}u(\xi) = \int_{\mathbb{R}} u(t)e^{-it\xi} \, dt
\]
says

\[
\|\mathcal{F}u\|_{L^{p'}(\mathbb{R})} \leq B(p) \|u\|_{L^p(\mathbb{R})}.
\]

Here \(1 \leq p \leq 2\), \(p' = (1 - 1/p)^{-1}\) is the conjugate exponent, with \(p = 1\) corresponding to \(p' = \infty\). Titchmarsh’s now-textbook estimate \(B(p) \leq (2\pi)^{1/p'}\) follows from the Parceval theorem and the Riesz-Thorin interpolation theorem. The sharp constant, which will be of some importance here,

\[
B(p) = (2\pi)^{1/p'} \left( \frac{p^{1/p}}{p'^{1/p'}} \right)^{1/2}
\]

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has been determined by Babenko [1] for even integer \( p' \) and Beckner [2] in the general case.

We will be dealing with functions \( u(t) \) defined on the positive half-line \( \mathbb{R}_+ = (0, +\infty) \), in which case the Fourier transform is analytic in the upper half-plane \( \mathbb{H} = \{ z \mid \text{Im} \, z > 0 \} \) and belongs to the Hardy class \( H^{p'}(\mathbb{H}) \) whenever \( u \in L^p(\mathbb{R}_+) \), \( 1 \leq p \leq 2 \).

In an equivalent setting, which we prefer for the reason of complex-conjugate symmetry, the upper half-plane \( \mathbb{H} \) is replaced by the right half-plane \( \mathbb{C}_+ = \{ z \mid \text{Re} \, z > 0 \} \) and the Fourier transform is replaced by the Laplace transform

\[
\mathcal{L}u(z) = \int_0^\infty u(t) e^{-zt} \, dt. \tag{3}
\]

Let \( H^s(\mathbb{C}_+) \) denote the Hardy class for the right half-plane. For \( s \geq 1 \), the norm in \( H^s(\mathbb{C}_+) \) of a function \( v(z) \) is

\[
\|v\|_{H^s(\mathbb{C}_+)} = \sup_{x > 0} \left( \int_{-\infty}^\infty |v(x + iy)|^s \, dy \right)^{1/s}.
\]

As a consequence of (3), we have

\[
\|\mathcal{L}u\|_{H^{p'}(\mathbb{C}_+, d\mu)} \leq B(p)\|u\|_{L^p(\mathbb{R}_+)}. \tag{4}
\]

Our main theorem asserts equivalence of the two classes of measures: those with Hausdorff-Young property of order \( p > 1 \), and Carleson measures.

**Definition 1.** Let \( \mu \) be a non-negative Borel measure supported on the closed right half-plane \( \overline{\mathbb{C}_+} = \{ z \mid \text{Re} \, z \geq 0 \} \). We say that \( \mu \) has the Hausdorff-Young property of order \( p \) or, in short, that \( \mu \) is \( \text{HY}(p) \), and write \( \mu \in \text{HY}(p) \) if there exists a constant \( C \) such that

\[
\|\mathcal{L}u(z)\|_{L^{p'}(\overline{\mathbb{C}_+}, d\mu)} \leq C\|u\|_{L^p(\mathbb{R}_+)}
\]

for any \( u \in L^p(\mathbb{R}_+) \). If \( C \) is the smallest such constant and \( p > 1 \), we denote \( N_{\text{HY}, p}(\mu) = C^{p'} \), the “Hausdorff-Young norm of order \( p \)” of \( \mu \). (We leave \( N_{\text{HY}, 1}(\mu) \) undefined.)
The Lebesgue measure along the $y$-axis, $\delta(x) \otimes dy$, is $HY(p)$ for all $p \in [1, 2]$ according to [1]. The same is true for the Lebesgue measure along the positive $x$-semiaxis, $dx \otimes \delta(y)$ ($x > 0$). The corresponding inequality

$$\|\mathcal{L}\|_{L^p(\mathbb{R}_+) \to L^{p'}(\mathbb{R}_+)} \leq \left(\frac{2\pi}{p'}\right)^{1/p'}$$

(5)

belongs to Hardy [3].

Note that any positive Borel measure on $\mathbb{C}_+$ is $HY(1)$, due to the trivial pointwise estimate

$$|\mathcal{L}u(z)| \leq \|u\|_{L^1(\mathbb{R}_+)}.$$

Our definition of Carleson measures will be slightly unconventional (cf. e.g. [3 § I.5]), to include a possible nontrivial mass at the boundary. Consider a family of squares adjacent to the boundary of $\mathbb{C}_+$,

$$Q_{a,h} = \{z \mid \text{Im } z \in [a, a + h], \text{ Re } z \in [0, h]\} \quad (a \in \mathbb{R}, h > 0).$$

(6)

**Definition 2.** A nonnegative Borel measure $\mu$ on $\mathbb{C}_+$ is a Carleson measure and $N_C(\mu)$ its Carleson norm if

$$N_C(\mu) = \sup_{a,h} \frac{\mu(Q_{a,h})}{h}$$

is finite.

**Remark.** The measure $\mu$ in Definition 2 can be split as $\mu = \mu_1 + \mu_2$, where $\mu_1 = \mu|_{x=0}$ is the boundary part of $\mu$ and $\mu_2$ is a Carleson measure in the conventional sense (except that we work in the right half-plane $\mathbb{C}_+$ instead of upper half-plane $\mathbb{H}$). The definition implies that $\mu_1$ is absolutely continuous relative to the Lebesgue measure $dy$. More precisely, the Radon-Nikodym derivative is bounded:

$$\frac{d\mu_1}{dy} \leq N_C(\mu),$$

(7)

hence $L^q(d\mu_1) \subset L^q(dy)$ for all $q \geq 1$. 

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**Theorem 1** (HY-characterization of Carleson measures). The following are equivalent:

(a) $\mu$ is a Carleson measure on $\mathbb{C}_+$;
(b) $\mu \in HY(p)$ for some $p \in (1, 2]$;
(c) $\mu \in HY(p)$ for all $p \in [1, 2]$.

Moreover, for any $p \in (1, 2]$

$$A_1(p)^{-1} N_C(\mu) \leq N_{HY,p}(\mu) \leq A_2(p) N_C(\mu),$$

where $A_1(p) \leq 2^{3/2} p'$ and $A_2(p) \leq 160 \pi \sqrt{e/p'}$. These estimates are order-sharp as $p' \to \infty$, that is, $A_1(p) \neq o(p')$ and $A_2(p) \neq o(p'^{-1/2})$.

**Proof.** We will first show that the qualitative statement (a) $\Rightarrow$ (c) is a simple corollary of Carleson’s theorem [3, Th. II.3.9]. The implication (b) $\Rightarrow$ (a) and the left inequality in (8) will be derived similarly to the proof of the converse part of that same theorem. The proof of the right inequality in (8) is put over to Section 2.

(a) $\Rightarrow$ (c): The case $p = 1$ is trivial, so assume that $1 < p \leq 2$. If $u \in L^p(\mathbb{R}_+)$, then $Lu \in H^{p'}(\mathbb{C}_+)$ and (4) holds. Writing $\mu = \mu_1 + \mu_2$ as in Remark after Definition 2, we have: $\|Lu\|_{L^{p'}(d\mu_1)} < \infty$ by that Remark, and $\|Lu\|_{L^{p'}(d\mu_2)} < \infty$ by Carleson’s theorem [3, Th. II.3.9, part (a) $\Rightarrow$ (b)].

(b) $\Rightarrow$ (a): We mimic the proof of part (c) $\Rightarrow$ (a) of the Carleson theorem cited above. Let $u \in L^p(\mathbb{R}_+)$ be a function with $\|u\|_p = 1$ whose Laplace transform $v = Lu$ satisfies $b = \inf_{z \in Q_{0,1}} |v(z)| > 0$. For $h > 0$, define $u_h(t) = h^{1/p} u(ht)$. Then $\|u_h\|_p = 1$ and $v_h(z) = Lu_h(z) = h^{-1/p'} v(z/h)$. If $z \in Q_{0,h}$, then $|v_h(z)| \geq h^{-1/p'} b$. Therefore

$$\|v_h\|_{L^{p'}(d\mu)}^{p'} \geq \int_{Q_h} |v_h(z)|^{p'} d\mu(z) \geq \mu(Q_h) h^{-1/p'} b^{p'}.$$

On the other hand, by condition (b) of the Theorem,

$$\|v_h\|_{L^{p'}(d\mu)}^{p'} \leq N_{HY,p}(\mu) \|u_h\|_p^{p'} = N_{HY,p}(\mu).$$

It follows that

$$\mu(Q_{0,h}) \leq h b^{-p'} N_{HY,p}(\mu).$$
Estimates for $\mu(Q_{a,h})$ for $a \neq 0$ are obtained similarly by considering the test functions $u_h(t)e^{iat}$. We conclude that $\mu$ is Carleson, with Carleson norm estimate $N_C(\mu) \leq b^{-p'}N_{HY,p}(\mu)$. The left part of the inequality (8) follows with $A_1(p) \leq b^{-p'}$.

The final part of the argument is aimed at obtaining an estimate for $b^{-p'}$ that grows linearly with $p'$. Consider the test function

$$u(t) = \begin{cases} \varepsilon^{-1/p}, & 0 < t < \varepsilon, \\ 0, & t > \varepsilon \end{cases},$$

Then $v(z) = \varepsilon^{1/p'}(1 - e^{-z\varepsilon})/(\varepsilon z)$. The inequality $|e^{-w} - w + 1| \leq |w|^2/2$ is valid whenever $\Re w \geq 0$ (say, by Taylor’s formula with remainder in the integral form). Thus $|(e^{-w} - 1)/w| \geq 1 - |w|/2$. Consequently,

$$|v(z)|^{p'} \geq \varepsilon \left(1 - \frac{\varepsilon|z|}{2}\right)^{p'}.$$

For $z \in Q_{0,1}$, the minimum occurs at the corner $z = 1 + i$ where $|z| = \sqrt{2}$. Finally, setting $\varepsilon = \sqrt{2}/p'$, we get

$$b^{-p'} \leq \frac{p'}{\sqrt{2}} \left(1 - \frac{1}{p'}\right)^{-p'} \leq \frac{4p'}{\sqrt{2}},$$

as claimed.

2 Evaluation of constants in the inequalities

To prove the upper bound for the constant $A_2(p)$ in the right inequality (8), we need Carleson’s estimate for the Poisson integral with a numeric constant. An independent derivation of it is given below. (We make no claim that the obtained constant is better than what can reconstructed from proofs in existing literature.) In the interpolatory part of Carleson’s theorem, we use a combination of Marcinkiewicz and Riesz-Thorin theorems to ensure the desired behaviour of the constant as a function of $p$.

In this theorem, the conventional Carleson measures in $\mathbb{H}$ (no mass at the boundary) are used. The squares $Q_{a,h}$ in Definition 2 are to be substituted
by the squares $R_{a,h} = \{ x + iy | a < x < a + h, \ 0 < y < h \}$ and the formula for $N_C(\mu)$ is to be modified accordingly.

In addition, the following notation will be needed:

- $P_a(t) = \pi^{-1}a/(a^2 + t^2)$, the Poisson kernel;
- $\mathcal{P}$, the Poisson convolution operator for the upper half-plane:

\[
\mathcal{P}f(x + iy) = \int_{-\infty}^{\infty} P_y(x - t) f(t) \, dt;
\]
- $E_g(\lambda)$, the “large value set”: for a function $g(z)$ and $\lambda > 0$,

\[
E_g(\lambda) = \{ z : |g(z)| > \lambda \}.
\]

**Theorem 2 (Carleson’s theorem with numeric constant).** Let $\mu$ be a Carleson measure in $\mathbb{H}$. Suppose that $f(x) \in L^p(\mathbb{R})$ and $g(x + iy) = \mathcal{P}f(x + iy)$.

(a) If $p = 1$, then

\[
\mu(E_g(\lambda)) \leq 10 N_C(\mu) \| f \|_1.
\]  

(b) If $1 < p < \infty$, then

\[
\| g \|_{L^p(d\mu)} \leq (M(p) N_C(\mu))^{1/p} \| f \|_p,
\]  

where $M(p) \leq 40 p'$ when $1 < p < 2$, and $M(p) \leq 79$ when $p \geq 2$.

We will finish the proof of Theorem 1 and prove Theorem 2 afterwards.

Given a Carleson measure $\mu$ on $\mathbb{C}_+$, write $\mu = \mu_1 + \mu_2$ as in the Remark after Definition 2. For a function $v \in H^{p'}(\mathbb{C}_+)$, we have

\[
\| v \|_{L^{p'}(d\mu)} = \| v \|_{L^{p'}(d\mu_1)} + \| v \|_{L^{p'}(d\mu_2)} \leq N_C(\mu) (1 + M(p')) \| v \|_{H^{p'}}.
\]

by (7) and (10). It follows by (4) that for $v = \mathcal{L}u$

\[
\| v \|_{L^{p'}(d\mu)} \leq (1 + M(p')) \cdot (B(p))^{p'}.
\]

(The appearance of $M + 1$ explains why we have favored the mingy constant 79 over a generous 80 in Theorem 2.) Substituting the evaluation of $B(p)$ from (2) and the estimate for $M(p')$ from Theorem 2, we get

\[
N_{HY,p}(\mu) \leq 80 \cdot \pi^{p'/2p} p'^{-1/2}.
\]
To obtain the upper bound for $A_2(p)$ as claimed in Theorem 1, it remains to notice that $\sup_{1 < p \leq 2} (p' p^{-1} \ln p) = \lim_{p \to 1} (\ldots) = 1$.

Finally, let us show that the obtained upper bounds for $A_1(p)$ and $A_2(p)$ in (8) are order-sharp.

1. Take $\mu = \delta(z - 1/p')$, that is, $\int v(z) \, d\mu = v(1/p')$. Clearly, $N_C(\mu) = p'$, while Hölder’s inequality implies that $N_{H^p}(\mu) \leq 1$. Hence $A_1(p) \geq p'$.

2. Take $\mu = dy$, the Lebesgue measure along the imaginary axis. Then $N_C(\mu) = 1$, while $N_{H^p}(\mu) = B(p)p'$, hence $A_2(p) \geq B(p)p' \sim cp' - 1/2$ as $p' \to \infty$.

The proof of Theorem 1 is complete.

Proof of Theorem 2. Let us first derive part (b) from part (a). The inequality [3], the trivial inequality $\|g\|_\infty \leq \|f\|_\infty$, and the Marcinkiewicz interpolation theorem [3, Th. I.4.5] yield

$$\|g\|_{L^p(d\mu)} \leq 2 \left(10p' N_C(\mu)^p\right)^{1/p} \|f\|_p.$$ 

We wish to obtain a constant that behaves like $O(1)^{1/p}$ as $p \to \infty$, while the constant in the above inequality tends to the limit 2. To optimize the upper bound, note that by the Riesz-Thorin theorem

$$\left(\frac{\|g\|_{L^p(d\mu)}}{\|f\|_p}\right)^p \leq \inf_{1 < r \leq p} \left(\frac{\|g\|_{L^r(d\mu)}}{\|f\|_r}\right)^r \leq \inf_{1 < r \leq p} \left(2^r \cdot 10r' N_C(\mu)\right).$$

The function $r \to 2^r r' = 2^r r/(r - 1)$ attains its minimum $m$ at the root $r_0$ of the equation $r_0(r_0 - 1) \ln 2 = 1$. Calculation gives $r_0 \approx 1.80104$ and $m \approx 7.83495$. The upper bounds for $M(p)$ as stated in Theorem 2 are obtained by rounding up.

Our proof of part (a) is a shortcut of a standard proof [3]. The three underlying steps are: Calderon-Zygmun decomposition $\Rightarrow$ Hardy-Littlewood maximal theorem $\Rightarrow$ Estimate for nontangent maximal function $\Rightarrow$ Carleson’s theorem. These steps will be implicit in our calculation.

Fix $\lambda > 0$ and consider the Calderón-Zygmun decomposition for $f$ at height $\alpha = \lambda/7$ [3, Lemma VI.2.2]: $\mathbb{R} = B \cup G$, $B \cap G = \emptyset$, $f(x) \leq \alpha$
for almost every $x \in G$, $B = \bigcup I_j$, a finite or countable union of disjoint intervals, and

$$\alpha \leq M_j \leq 2\alpha,$$

where

$$M_j = \frac{1}{|I_j|} \int_{I_j} |f| dx.$$

We may assume for simplicity that the number of intervals $I_j$ is finite: functions $f$ for which this is true are dense in $L^1$.

If $I_j = [a_j, b_j]$, let $\tilde{I}_j = [a_j - |I_j|, b_j + |I_j|]$, so that $|\tilde{I}_j| = 3|I_j|$. Let $R_j$ be the square in $\mathbb{H}$ with base $\tilde{I}_j$, i.e. $x + iy \in R_j \iff x \in \tilde{I}_j, 0 < y \leq 3|I_j|$. Our goal is to show that $E_g(\lambda) \subset \bigcup R_j$.

Fix $z = x_0 + iy \notin \bigcup R_j$ and consider separately contributions to $g(x_0 + iy)$ of $f$ restricted to the intervals $I_j$ according to whether $\tilde{I}_j$ contains $x_0$ or lies to the right, resp. to the left of $x_0$. Formally: let $x_j$ be the point in $I_j$ closest to $x_0$. Define the mutually disjoint sets of indices $S_0$, $S_+$, $S_-$ as follows: $j \in S_0$ (resp. $S_+$ or $S_-$) if $x_j - x_0 = 0$ (resp. $> 0$ or $< 0$).

In case $S_0 \neq \emptyset$, let $L = \max_{j \in S_0} |I_j|$ occur for $j = j_0$. Then $y > 3L$, as otherwise we would have $z \in R_{j_0}$. Clearly, $I_j \subset [x_0 - 2L, x_0 + 2L]$ for any $j \in S_0$, hence $\sum_{j \in S_0} |I_j| \leq 4L$. Since $P_y(t) < (\pi y)^{-1}$, we get

$$\sum_{j \in S_0} \int P_y(x_0 - t)|f(t)| dt \leq \frac{1}{\pi y} \sum_{j \in S_0} M_j |I_j| \leq \frac{2\alpha \cdot 4L}{\pi y} < \frac{8}{3\pi} \alpha. \quad (11)$$

Let us now evaluate contribution of the intervals $I_j$ with $j \in S_+$. Define the counting function for the total length of such intervals:

$$F(t) = \sum |I_j| \quad \text{over } j \text{ such that } x_0 + |I_j| < a_j \leq x_0 + t.$$

The function $F$ defined in $(0, +\infty)$ is nondecreasing, upper-semicontinuous, and $F(x) = 0$ in the right neighborhood of $x_0$. In addition, we have the important inequalities

$$F(b_j - x_0) \leq b_j - x_0 \quad (j \in S_+) \quad \text{and} \quad F(x) < 2x. \quad (12)$$
The first inequality is obvious; moreover, if $b_j \leq x < a_{j+1}$, then $F(x-x_0) < x-x_0$. And if $a_j \leq x < b_j$, then $F(x-x_0) \leq (a_j - x_0) + |I_j| < 2(a_j - x_0)$.

Let $J = \max j$. We have

$$
\sum_{j \in S_+} \int_{I_j} P_y(x_0 - t)|f(t)| \, dt
\leq \sum_{j \in S_+} P_y(a_j - x_0) \int_{I_j} |f(t)| \, dt
\leq 2\alpha \sum_{j \in S_+} P_y(a_j - x_0)|I_j|
= 2\alpha \int_{b_j-x_0}^0 P_y(t) \, dF(t).
$$

Integrating by parts, using (12) and the elementary inequality $tP_y(t) \leq (2\pi)^{-1}$, we get

$$
\sum_{j \in S_+} \int_{I_j} P_y(x - t)|f(t)| \, dt
\leq 2\alpha \left( \frac{1}{2\pi} - \int_0^\infty \frac{d}{dt} P_y(t) 2t \, dt \right)
\leq 2\alpha \left( \frac{1}{2\pi} + 1 \right).
$$

The contribution of the intervals $I_j$ with $j \in S_-$ has the same upper bound. Combining with (11), we obtain

$$
\int_B P_y(x_0 - t)|f(t)| \, dt \leq \alpha \left( \frac{2}{\pi} + 4 + \frac{8}{3\pi} \right) < \frac{11}{2}\alpha.
$$

Finally, $\sup_{G} |f| \leq \alpha$, and we conclude:

$$
|g(x_0 + iy)| \leq \int_{B \cup G} P_y(x_0 - t)|f(t)| \, dt < \frac{13}{2}\alpha.
$$

Summarizing, we can cover the set $E_g(\lambda)$ by the union of squares $R_j$. The total of their sidelengths is $\sum 3|I_j| \leq 3\|f\|_1/\alpha < 10\|f\|_1/\lambda$. The inequality (9) follows. \qed
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