THE ONE-LOOP QED IN NONCOMMUTATIVE SPACE

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Abstract

Returning to the old problems in ordinary QED, by an appropriate extension of the dimensional regularization method in noncommutative space we try to provide a quite coherent look into NCQED. The renormalisation of theories, the $\beta$ function, the vacuum polarisation of photon, the general structure of vertex fermion-photon, the anomalous magnetic moment (AMM) of fermions and the validity of Ward identity at the one-loop level are reinvestigated.
1 INTRODUCTION

Since 1999 the space-time non-commutativity has been realized from string theory when open string propagates in the presence of constant background antisymmetric tensor field [1]. In recent years the noncommutative field theories have generated a lot of interests in many aspects (see reference in [14]), for instance, the Lorenz invariance, the unitarity [12,16,17,20], instanton [13], Standard model on NCM space-time and then the new NCM theories [15] based on the extension of the basic relation which indicates that space-time loses its condition of continuum:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$$ (1)

In spite of that, there are some old aspects, particularly in NCMQED, which have to be reinvestigated. In the works [5],[8],[18] the structure of the vertex is considered, the Ward identity have called attention in [5], particularly [21] is an explicit investigation for its validity in the two processes $e^+e^- \rightarrow \gamma\gamma$ and $\gamma\gamma \rightarrow \gamma\gamma$. Now, in this work we try to investigate the same problems in the NCMQED at one-loop level by using the dimensional regularization and perturbative method. For the pedagogical purpose, we will organize the paper in the following way: Sec.2 is devoted to remind some principal properties of NCMQED, Sec.3 is reserved to present some results for the renormalisation of theories, Sec.4 is a brief consideration of the structure of vertex, Sec.5 for checking the validity of Ward identity, Sec.6 is the discussion of the AMM and we will finally end up with some comments and remarks.

2 PERTURBATIVE THEORY OF NCMQED

1. Let us begin with the pure U(1) NCM Yang-Mills action in space-time dimension $d$:

$$S_{YM} = -\frac{1}{4} \int d^d x F_{\mu\nu}(x) \star F^{\mu\nu}(x)$$ (2)

Here the $\star$-product is defined as:

$$f(x) \star g(x) = e^{i\theta_{\mu\nu}\partial_\mu(x)\partial_\nu(x)} f(x + \xi)g(x + \eta) \bigg|_{\xi,\eta \rightarrow 0}$$ (3)

$\theta_{\mu\nu}$ is antisymmetric matrix which has the dimension of area. To avoid problems with unitarity we will assume that only the space-space components of $\theta_{\mu\nu}$ are non zero, namely, $\theta^{0\nu} = 0$. The field strength $F_{\mu\nu}(x)$ is defined by:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)]_M$$ (4)
and the Moyal bracket is:

\[ [f, g]_M = f \star g - g \star f \]  

(5)

Note that: even in U(1) case \( A_\mu \) couples to itself since the field strength \( F_{\mu\nu} \) has the non-linear term in \( A_\mu \). Assuming that the fields decrease so promptly at infinity that the space-time integral of a Moyal bracket vanishes. It is easily to show that the action (2) is invariant under the gauge transformations following:

\[
A_\mu(x) \rightarrow A'_\mu(x) = U(x) \star A_\mu(x) \star U^{-1}(x) + \frac{i}{g} U(x) \star \partial_\mu U^{-1}(x)
\]

(6)

Where:

\[
U(x) = \left( e^{i\lambda(x)} \right)_* = 1 + i\lambda(x) + \frac{(i)^2}{2} \lambda(x) \star \lambda(x) + ...
\]

\[
U^{-1}(x) = \left( e^{-i\lambda(x)} \right)_* = 1 - i\lambda(x) + \frac{(-i)^2}{2} \lambda(x) \star \lambda(x) + ...
\]

\[
U(x) \star U^{-1}(x) = 1
\]

Now, we consider the fundamental representation of the matter in which the covariant derivative is defined as:

\[
D_\mu \psi(x) = \partial_\mu \psi(x) + ig\psi(x) \star A_\mu(x)
\]

In terms of ordinary product the action for U(1) Yang-Mills fields and the matter field can be rewritten as:

\[
S_{YM-Matter} = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{\psi}i\partial_\mu \psi - ge^{\frac{i}{2} p^\nu p' \overline{\psi} A_\mu \psi - m \overline{\psi} \psi} \right\}
\]

(7)

In order to obtain the non-singular free propagator for the gauge fields we need to introduce the gauge-fixing term. We shall do this in a consistent way by using the BRS formalism. Let us introduce the ghost fields \( c, \overline{c} \) and the auxiliary fields \( B \) and define the BRST transformations as follow:

\[
\delta_B A_\mu(x) = D_\mu c(x) = \partial_\mu c(x) - i[A_\mu(x), c(x)]_M
\]

\[
\delta_B c(x) = - (\overline{c}(x) \star c(x))
\]

\[
\delta_B \overline{c}(x) = B(x)
\]

\[
\delta_B B(x) = 0
\]

(8)

is nilpotent.

To keep track of the renormalisation of the composite transformations \( \delta_B A_\mu(x) \) and
\[ \delta_Bc(x) \text{ one also introduces the external fields } J_\mu(x) \text{ and } H(x) \text{ which couple to them.} \]
The Faddev-Popov and gauge fixing action is given as:

\[
S_{GF-ghost} = \int d^d x \left\{ -\frac{1}{2\xi} \partial_\mu A^\mu(x) \star \partial_\nu A^\nu(x) + \partial_\mu \overline{c}(x) \star (\partial_\mu c(x) - i[A_\mu(x), c(x)]_M) \right\}
\]

(9)

Where \( \xi \) is the gauge fixing parameter.

and the action for the sources is defined by:

\[
S_{sources} = \int d^d x \left\{ K^\mu \star A_\mu + J \star B + (\overline{\eta} \star \overline{c} - \overline{c} \star \eta) + H \star (c \star c) + J_\mu \star D_\mu c \right\}
\]

(10)

It is easily to show that \( S_{GF-ghost} \) is invariant under the BRST transformations. The complete action for NCQED in a general covariant gauge is given as:

\[
S_{tot} = S_{YM+Matter} + S_{gf-ghost} + S_{sources} = S_{inv} + S_{sources}
\]

(11)

where: \( S_{YM+Matter} \) and \( S_{gf-ghost} \) are defined in (7),(10) respectively. As a result, the propagator is the same as in the commutative counterpart but each vertex will accompanies with a phase factor which depends on the momenta outgoing from the vertex and the consequence is that the Feynman rules in Feynman gauge are derived and presented in figure (1).
\[
\begin{align*}
\text{i} S(p) &= \frac{\text{i}}{\not{\! p} - m + \text{i} \epsilon} \\
\text{i} D^{\mu\nu} &= \frac{-\text{i} g^{\mu\nu}}{q^2 + \text{i} \epsilon} \\
\text{i} D(p) &= \frac{\text{i}}{\not{p}^2 + \text{i} \epsilon} \\
= \text{i} e^{-\gamma^\lambda} e^\lambda (p_1 \wedge p_F) \\
&= 2 \epsilon \sin(\frac{1}{2} p_1 \wedge p_2) \\
&\quad [(p_1 - p_2)^{\mu 3} g^{\mu 1 \mu 2} + (p_2 - p_3)^{\mu 1} g^{\mu 2 \mu 3} + (p_3 - p_1)^{\mu 2} g^{\mu 3 \mu 1}] \\
&= 2 \epsilon p_1^\mu \sin(\frac{p_1 \wedge p_F}{2}) \\
&= 4 \epsilon e^2 \left[ \sin(\frac{p_1 \wedge p_2}{2}) \sin(\frac{p_1 \wedge p_4}{2}) \
\left( g^{\mu 1 \mu 3} g^{\mu 2 \mu 4} - g^{\mu 1 \mu 4} g^{\mu 2 \mu 3} \right) \\
&\quad + \sin(\frac{p_1 \wedge p_3}{2}) \sin(\frac{p_2 \wedge p_4}{2}) \
\left( g^{\mu 1 \mu 4} g^{\mu 2 \mu 3} - g^{\mu 1 \mu 2} g^{\mu 3 \mu 4} \right) \\
&\quad + \sin(\frac{p_1 \wedge p_4}{2}) \sin(\frac{p_2 \wedge p_3}{2}) \
\left( g^{\mu 1 \mu 2} g^{\mu 3 \mu 4} - g^{\mu 1 \mu 3} g^{\mu 2 \mu 4} \right) \right]
\end{align*}
\]

Fig (1): The Feynman rules for NCMQED
Fig (3): The correction to vacuum polarization of photon

(Figure 2): The correction to electron self-energy

Fig (4): The correction to $\psi\overline{\psi}A_\mu$-vertex
3 THE RENORMALISED NCMQED

It is useful to note that firstly we work in the Feynman gauge and secondly, in the following calculations we assume that the Bessel functions are finite.

1. The one-loop fermion self-energy:

The electron self-energy receives one-loop correction through only one diagram (2):

\[ \bar{\Sigma}_{(p)} = -ie^2 \mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \gamma_\nu \gamma_1 \frac{1}{(p-k)^2} \]  \hspace{1cm} (12)

Noting in this diagram that the phase factors accompanied with the two vertices cancel with each other. Thus the contribution is the same as in ordinary QED. So, not only UV divergence can be subtracted by the usual rescaling of wave function and electron’s mass but also the finite part does not change. In the dimensional renormalisation [22], the counter term for the fermion loop is defined as:

\[ Z_2 = 1 + \Delta_2 \]

where \( \Delta_2 \) is decomposed into two parts: infinite and finite part which in the on-shell condition are given by:

\[ \Delta_2^\infty = -\frac{e^2}{16\pi^2} \left( \frac{2}{\epsilon} - \gamma_E^+ \right) \]

and,

\[ \Delta_2^F = -\frac{e^2}{(4\pi)^2} \left( 2 - \ln \frac{m^2}{\mu^2} - 4 \int_0^1 dz \frac{1 - z^2}{z} \right) \]  \hspace{1cm} (13)

So, the renormalisation constant is:

\[ Z_2 = 1 - \frac{e^2}{8\pi^2\epsilon} \]  \hspace{1cm} (14)

2. The one-loop photon self-energy.

The one-loop photon self-energy receives contribution from four diagram (3a)-(3d) in figure (3).

(i) The diagram (3d) is the contribution of the electron loop. It turns out that its
contribution is the same as in ordinary QED since like the electron self-energy the two phase factors at two vertices are cancelled with each other. We get:

\[ i\Pi^{(d)}_{\mu\nu}(p) = \frac{ie^2N_f}{6\pi^2\epsilon}(p_\mu p_\nu - \eta_{\mu\nu}p^2) = \frac{ie^2}{(4\pi)^2\epsilon}(\frac{-8N_f}{3})(\eta_{\mu\nu}p^2 - p_\mu p_\nu) \] (15)

Where \( N_f \) denote the number of independent fields with charge \( \pm 1 \).

(ii) For three diagrams (3a),(3b) and (3c). In the dimensional regularization we can rewrite the contribution of these diagrams in the form:

\[ i\Pi^{(i)}_{\mu\nu} = e^2\mu^{4-d}C^{(i)} \int \frac{d^dk}{(2\pi)^d} \frac{1 - \cos \theta \wedge k}{k^2(p+k)^2} N^{(i)}_{\mu\nu} \]

where \( i=a,b,c \)

\[ C^{(a)} = \frac{1}{2}, \quad C^{(b)} = -1, \quad C^{(c)} = \frac{1}{2} \]

and:

\[ N^{(a)}_{\mu\nu} = 4\eta_{\mu\nu}(d-1)(p+k)^2 \]
\[ N^{(b)}_{\mu\nu} = 2k_\mu(p+k)_\nu \]
\[ N^{(c)}_{\mu\nu} = 2[\eta_{\mu\nu}(5p^2 + 2pk + 2k^2) + k_\mu k_\nu(4d-6) + p_\mu p_\nu(d-6) + (p_\mu k_\nu + p_\nu k_\mu)(2d-3)] \]

The contribution of three diagrams is:

\[ i\Pi^{(abc)}_{\mu\nu} = e^2\mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1 - \cos \theta \wedge k}{k^2(p+k)^2} N^{(abc)}_{\mu\nu} \] (16)

where

\[ N^{(abc)}_{\mu\nu} = \Sigma_i C^{(i)}N^{(i)}_{\mu\nu} \]
\[ = 2d\eta_{\mu\nu}k^2 + 4(d-2)k_\mu k_\nu + (2d+3)\eta_{\mu\nu}p^2 + (d-6)p_\mu p_\nu + 2(2d-1)\eta_{\mu\nu}kp + (2d-3)p_\mu k_\nu + (2d-5)p_\nu k_\mu \]

Using the dimensional regularization, we see that the photon self-energy receives the contributions from the planar part and the non-planar part:

\[ i\Pi^{(abc)}_{\mu\nu} = i\Pi^{(abc)}_{\mu\nu(\text{planar})} + i\Pi^{(abc)}_{\mu\nu(\text{non-planar})} \] (17)

where:

\[ i\Pi^{(abc)}_{\mu\nu(\text{planar})} = e^2\mu^{4-d} \int_0^1 dz \int \frac{d^dk}{(2\pi)^d} \frac{N^{(abc)}_{\mu\nu}}{(k^2 - M^2)^2} \] (18)
\[ i\Pi^{(abc)}_{\mu\nu}(\text{non-planar}) = -e^2\mu^{4-d} \int_0^1 dz \int \frac{d^dk}{(2\pi)^d} (\cos \theta k) \frac{N^{(abc)}_{\mu\nu}}{[k^2 - M^2]^2} \]  

(19)

For the planar part in the limit \( \epsilon \to 0 \) we get:

\[
\begin{align*}
  i\Pi^{(abc)}_{\mu\nu}(\text{planar}) &= \frac{ie^2}{(4\pi)^2} \left\{ \frac{20}{3\epsilon} \left( \eta_{\mu\nu}p^2 - p_{\mu}p_{\nu} \right) \right\} \\
  &- \frac{ie^2}{(4\pi)^2} \int_0^1 dz \left( \ln \frac{M^2}{4\pi\mu^2} \right) \left( 20\eta_{\mu\nu}M^2 + M_{\mu\nu} \right) - \frac{ie^2}{(4\pi)^2} \frac{10}{3} \eta_{\mu\nu}p^2
\end{align*}
\]

(20)

For the non-planar part, we obtain:

\[
\begin{align*}
  i\Pi^{(abc)}_{\mu\nu(\text{np})} &= -\frac{ie^2}{(4\pi)^2} \int_0^1 dz \left( \frac{4\pi\mu^2}{-M^2} \right)^\frac{\hat{s}}{2} \\
  &\left\{ \left( 2d^2 + 4(d-2) \right) \eta_{\mu\nu}(-M^2) \left( \frac{Z}{2} \right)^{\frac{\hat{s}}{2} - 1} K_{\frac{\hat{s}}{2} - 1}(z) \right. \\
  &\left. + \left( -M^2 \right) \left( d\eta_{\mu\nu}(\vec{p})^2 + 2(d-2)\vec{p}_\mu\vec{p}_\nu \right) \left( \frac{Z}{2} \right)^{\frac{\hat{s}}{2} - 2} K_{\frac{\hat{s}}{2} - 2}(z) \right. \\
  &\left. + 2M_{\mu\nu} \left( \frac{Z}{2} \right)^{\frac{\hat{s}}{2}} K_{\frac{\hat{s}}{2}}(z) \right\}
\end{align*}
\]

(21)

where

\[ Z = |\vec{p}|M \]

Noting that there is no divergence in Bessel functions even when \( \epsilon \to 0 \). So, the infinite terms in the contribution of three diagrams (3a)-(3c) come from the planar part. With the above result, summing (15), (20) and (21) we get the one-loop photon self-energy:

\[
\begin{align*}
  i\Pi^{(abcd)}_{\mu\nu}(p) &= i\Pi^{(d)}_{\mu\nu}(p) + i\Pi^{(abc)}_{\mu\nu}(p) \\
  i\Pi^{(abcd)}_{\mu\nu}(p) &= \frac{ie^2}{(4\pi)^2\epsilon} \left( \frac{20}{3} - \frac{8N_f}{3} \right) \left( \eta_{\mu\nu}p^2 - p_{\mu}p_{\nu} \right) + \text{finite}
\end{align*}
\]

(22)

The renormalisation constant \( Z_3 \) is:

\[ Z_3 = 1 + \frac{e^2}{16\pi^2\epsilon} \left( \frac{20}{3} - \frac{8N_f}{3} \right) \]

(23)
3. Contribution of diagram (4a)

Now, we consider the vertex presented in figure (4). In d-dimensions the proper vertex is:

$$\Lambda^{(a)}_{\mu}(p, q, p') = -ie^2 \mu^\epsilon e^{2p\wedge p'} \int \frac{d^d k}{(2\pi)^d} e^{-ik\tilde{q}} \left[ \gamma_\mu \frac{1}{p' - k - m} \gamma_\nu \frac{1}{p - k - m} \gamma_\rho \frac{1}{k^2 - m^2} \right]$$

where: $m_\gamma$ is the mass of photon, $\epsilon = 4 - d$ and $k\tilde{q} = k \wedge q = k^{\mu} \theta^{\mu\nu} q_\nu$

By standard Feynman parameterisation, we get,

$$\Lambda^{(a)}_{\mu}(p, q, p') = \frac{2e^2 \mu^\epsilon}{(4\pi)^{\frac{d}{2}}} e^{\frac{2p\wedge p'}{2}} \int_0^1 dx \int_0^{1-x} dy e^{-i(x+y)p\wedge p'}$$

$$\left\{ \gamma_\mu \left[ \frac{(2 - d)^2}{2(M^2_a)^{\frac{d}{2}}} \left( \frac{Z_a}{2} \right)^{\frac{d}{2}} K_{\frac{d}{2}}(Z_a) - \frac{(2 - d)}{4} \frac{M^2_a}{(M^2_a)^{\frac{d}{2}-1}} \left( \frac{Z_a}{2} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(Z_a) \right] \\
+ \frac{(2 - d)}{2} \frac{\bar{q}_\mu \bar{q}}{(M^2_a)^{\frac{d}{2}-1}} \left( \frac{Z_a}{2} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(Z_a) \\
- \frac{N_{2a}}{(M^2_a)^{\frac{d}{2}+1}} \left( \frac{Z_a}{2} \right)^{\frac{d}{2}+1} K_{\frac{d}{2}+1}(Z_a) \right\}$$

(25)

In the limit $d = 4$:

$$\Lambda^{(a)}_{\mu}(p, q, p') = \frac{2e^2}{(4\pi)^2} e^{\frac{2p\wedge p'}{2}} \int_0^1 dx \int_0^{1-x} dy e^{-i(x+y)p\wedge p'}$$

$$\left\{ \gamma_\mu \left[ 2K_0(Z_a) + \frac{(\bar{q})^2(M^2_a)}{Z_a} K_{-1}(Z_a) \right] \\
- (\bar{q}_\mu \bar{q}) \left( \frac{2M^2_a}{Z_a} \right) K_{-1}(Z_a) - \frac{N_{2a}}{(M^2_a)^2} \left( \frac{Z_a}{2} \right) K_{1}(Z_a) \right\}$$

(26)

Since we can safely set $\epsilon = 0$ (there is no pole) with the Bessel functions, it is easy to see that the contribution of diagram (4a) is finite.

4. Contribution of diagram (4b)

For diagram (4b). Starting with:

$$\Lambda^{(b)}_{\mu}(p, q, p') = -ie^2 \mu^\epsilon e^{2p\wedge p'} \int \frac{d^d k}{(2\pi)^d} \left( e^{ik\wedge q} e^{-ip\wedge p'} - 1 \right)$$
\[
\left[ \gamma^\rho \frac{k + m}{(k^2 - m^2)} \gamma^\nu \right] \frac{1}{(p' - k)^2 - m_\gamma^2} \frac{1}{(p - k)^2 - m_\gamma^2}
\]
\{(2p - p' - k)\gamma_\mu + (2p' - p - k)\gamma_\nu + (2k - p - p')\gamma_\mu \gamma_\nu\}
\] (27)

After performing the dimensional regularization we can separate it into two parts: planar and non-planar ones.

\[
\Lambda^{(b)}_\mu (p, q, p') = \Lambda^{(b)}_\mu (p, q, p')_{\text{planar}} + \Lambda^{(b)}_\mu (p, q, p')_{\text{nonplanar}}
\] (28)

(i) For the planar part:
The complete expression of \(\Lambda^{(b)}_{\mu(\text{planar})} (p, q, p')\) is given as:
\[
\Lambda^{(b)}_{\mu(\text{planar})} (p, q, p') = -\frac{2e^2}{(4\pi)^2} e^{i(p \wedge p')} \int_0^1 dx \int_0^{1-x} dy \gamma_\mu \left( \frac{\mu^2}{2M_b^2} \right)^{\frac{\epsilon}{2}} \Gamma\left( \frac{\epsilon}{2} \right)
\]
\[
- \frac{N_{2b}}{2M_b^2} \left( \frac{\mu^2}{-M_b^2} \right)^{\frac{\epsilon}{2}} \Gamma\left( \frac{\epsilon}{2} + 1 \right)
\}
\] (29)

In the limit \(d = 4, \epsilon \to 0\), we get:
\[
\Lambda^{(b)}_{\mu(\text{planar})} (p, q, p') = \left( \frac{3e^2}{8\pi^2} \right) \gamma_\mu e^{i(p \wedge p')} \left( \frac{1}{\epsilon} - \frac{1}{2} \gamma_E \right)
\]
\[
+ \Lambda^{(b)*}_{\mu(\text{planar})} (p, q, p')
\] (30)

and \(\Lambda^{(b)*}_{\mu(\text{planar})} (p, q, p')\) is the finite part of the vertex function for the planar diagram(b):
\[
\Lambda^{(b)*}_{\mu(\text{planar})} (p, q, p') = -\frac{e^2}{16\pi^2} e^{i(p \wedge p')} \left\{ \gamma_\mu \left( 3 \ln \frac{m^2}{\mu^2} + 6 \int_0^1 dz \ln (1 - z)^2 \right) \right.
\]
\[
- \int_0^1 dx \int_0^{1-x} dy \frac{N_{2b}}{M_b^2} \}
\] (31)

(ii) For the non-planar part:
In the same way, the contribution of the non-planar part is:
\[
\Lambda^{(b)}_{\mu(\text{nonplanar})} (p, q, p') = \frac{2e^2\mu^\epsilon}{(4\pi)^{\frac{\epsilon}{2}}} e^{i(p \wedge p')} \int_0^1 dx \int_0^{1-x} dy e^{i(x+y-1)(p \wedge p')}
\]
\[
\left\{ \gamma_\mu \left[ \frac{2(1 - d)}{M_b^2} \right]^{\frac{\epsilon}{2}} K_{\frac{\epsilon}{2}} (Z_b) - \frac{(q)^2}{2(M_b^2)^{\frac{\epsilon}{2}-1}} \left[ \frac{Z_b}{2} \right]^{\frac{\epsilon}{2}-1} K_{\frac{\epsilon}{2}-1} (Z_b) \right\}
\]

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where \( \Lambda^\mu(p, q, p') \) is the finite part of the total vertex function and defined in (25) and (34) as:

\[
\Lambda^\mu(p, q, p') = \Lambda^\mu_{\text{finite}}(p, q, p') + \Lambda^\mu_{\text{finite}}(p, q, p')
\]

From (25) and (34) the contribution of \( \psi\psi A \) vertex is given by:

\[
\Lambda^\mu(p, q, p') = \frac{3e^2}{8\pi^2} \gamma^\mu e^{\tau^\mu p'} \left( \frac{1}{\epsilon} - \frac{1}{2} \gamma_E \right) + \Lambda^\mu_{\text{finite}}(p, q, p')
\]

where \( \Lambda^\mu_{\text{finite}}(p, q, p) \) is the finite part of the total vertex function and defined in (25) and (34) as:

\[
\Lambda^\mu_{\text{finite}}(p, q, p') = \Lambda^\mu_{\text{finite}}(p, q, p') + \Lambda^\mu_{\text{finite}}(p, q, p')
\]
and imposing the infinite renormalisation constant
\[ \Delta_1^\infty = -\Lambda_{\mu}^{div} = -\frac{3e^2}{8\pi} \left( \frac{1}{\epsilon} - \frac{1}{2} \gamma_E \right) \]
we can remove the divergence part from the total vertex.
The renormalisation constants \( Z_1 \) coming from the one-loop vertex function is given by:
\[ Z_1 = 1 - \frac{3e^2}{8\pi^2\epsilon} \]
(37)
At this stage we see that at the one-loop level the NCMQED is completely renormalised.

5. The \( \beta \)-function

As in the commutative QED, the \( \beta \)-function of the theory is given by:
\[ \beta(g) = \mu \frac{\partial}{\partial \mu} g(\mu) \]
In terms of the bare coupling constant \( g_o \) and the renormalization constants \( Z_i \)
,(i=1,2,3), the renormalized coupling constant \( g(\mu) \) is defined by:
\[ g_o = \mu^{\frac{\epsilon}{2}} g(\mu) Z_1 Z_2^{-1} Z_3^{-\frac{\epsilon}{2}} \]
where \( \epsilon = 4 - D \) Now bringing together(14),(23),(37)into the above relation,we get:
\[ e_o = e \mu^{\frac{\epsilon}{2}} \left( 1 - \frac{3e^2}{8\pi^2\epsilon} \right) \left( 1 + \frac{e^2}{8\pi^2\epsilon} \right) \left[ 1 - \frac{e^2}{8\pi^2\epsilon} \left( \frac{20}{3} - \frac{8N_f}{3} \right) \right] \]
\[ = e \mu^{\frac{\epsilon}{2}} \left[ 1 + \frac{e^2}{16\pi^2\epsilon} \left( -\frac{22}{3} + \frac{4N_f}{3} \right) \right] \]
from which follows (in the limit \( \epsilon \to 0 \))
\[ \beta(\epsilon) = -\frac{e^3}{16\pi^2} \left( \frac{22}{3} - \frac{4N_f}{3} \right) \]
Remarks:
- A contribution \( \frac{22}{3} \) is due to the structure similar to non-abelian dynamics of NCM gauge fields.
- From the evaluation, the UV divergence is suggested to appear only in the planar diagrams since the non-planar part of the corresponding one-loop Feynman are assumed to be finite with finite non-commutative parameter \( \theta \)
- For \( N_f < 6 \) the NCMQED with fundamental matters is asymptotically free.

6. Vacuum polarisation of photon

Now, we determine the finite terms in the expression of \( i\Pi^{(abc)}_{\mu\nu}(p) \). Firstly, we evaluate \( i\Pi^{(abc)}_{\mu\nu(non-planar)}(p) \). In the limit \( \epsilon \to 0, \bar{p} \to 0 \) just keeping the leader terms in the expansion of Bessel function we have:

\[
i\Pi^{(abc)}_{\mu\nu(non-planar)}(p) = \frac{ie^2}{(4\pi)^2} \int_0^1 dz \left\{ 40\eta_{\mu\nu} \left\{ \frac{2}{|\bar{p}|^2} + M^2 \ln \frac{Z}{2} - \frac{1}{2} \frac{M^2}{|\bar{p}|^2} \right\} + \left(-4\eta_{\mu\nu}|\bar{p}|^2 + 4\bar{p}_\mu\bar{p}_\nu\right) \left( \frac{8}{|\bar{p}|^4} - \frac{2M^2}{|\bar{p}|^2} \right) + 2M_{\mu\nu} \ln \frac{Z}{2} \right\}
\]

Taking integration over the Feynman parameter and putting together with (20) we can rewrite the complete expression of \( i\Pi^{(abc)}_{\mu\nu}(p) \) as:

\[
i\Pi^{(abc)}_{\mu\nu}(p) = \frac{ie^2}{(4\pi)^2} \left( \frac{10}{3} \left( \eta_{\mu\nu}p^2 - p_\mu p_\nu \right) \left( \frac{2}{\epsilon} + \ln \pi |\bar{p}|^2 \mu^2 \right) + \frac{ie^2}{(4\pi)^2} \left( \frac{32}{3} \frac{\bar{p}_\mu\bar{p}_\nu}{|\bar{p}|^4} - \frac{4}{3} \frac{p^2\bar{p}_\mu\bar{p}_\nu}{|\bar{p}|^2} \right) \right)
\]

We see that the vacuum polarisation of photon can be written in the form:

\[
i\Pi^{(abc)}_{\mu\nu}(p) = A \left( \eta_{\mu\nu}p^2 - p_\mu p_\nu \right) + B\bar{p}_\mu\bar{p}_\nu
\]

where A, B, and C are the functions of the scalars \( p^2 \) and \( |\bar{p}|^2 \):

\[
A = \frac{ie^2}{(4\pi)^2} \left( \frac{2}{\epsilon} + \ln \pi |\bar{p}|^2 \mu^2 \right)
\]
\[
B = \frac{ie^2}{(4\pi)^2} \left( \frac{32}{3} \frac{1}{|\bar{p}|^4} - \frac{4}{3} \frac{p^2}{|\bar{p}|^2} \right)
\]

That is the form compatible with the Ward identity.
4 THE FERMION-PHOTON VERTEX STRUCTURE IN NCMQED

Let’s return to the electron-photon vertex contributions from two diagrams (4a),(4b). Putting (26),(34) into (36) we get the detail expression of the finite part for the proper vertex:

\[ \Lambda^{(s)}(p,q,p') = \frac{2e^2}{(4\pi)^{\frac{3}{2}}} e^{\frac{\pi}{2}p \cdot p'} \int_0^1 dx \int_0^{1-x} dy e^{-i(x+y)p \cdot p'} \]

\[ \{ \frac{\gamma_\mu}{2(M_a^2)^{\frac{3}{2}}} \left[ \left( \frac{Z_a}{2} \right)^{\frac{3}{2}} K_{\frac{3}{2}}(Z_a) - \left( \frac{2-d}{4} \right) \left( \frac{Z_a}{2} \right)^{\frac{3}{2}-1} K_{\frac{3}{2}-1}(Z_a) \right] \]

\[ + \frac{2-d}{2 \frac{\bar{q}_\mu \bar{q}}{M_a^2} \frac{2}{(M_a^2)^{\frac{3}{2}}} \left( \frac{Z_b}{2} \right)^{\frac{3}{2}-1} K_{\frac{3}{2}-1}(Z_b) - \frac{2}{(M_b^2)^{\frac{3}{2}}} \left( \frac{Z_b}{2} \right)^{\frac{3}{2}-1} K_{\frac{3}{2}-1}(Z_b) \}

\[ + \frac{2-d}{2 \frac{\bar{q}_\mu \bar{q}}{M_b^2} \frac{2}{(M_b^2)^{\frac{3}{2}}} \left( \frac{Z_b}{2} \right)^{\frac{3}{2}-1} K_{\frac{3}{2}-1}(Z_b) - \frac{2}{(M_b^2)^{\frac{3}{2}}} \left( \frac{Z_b}{2} \right)^{\frac{3}{2}-1} K_{\frac{3}{2}-1}(Z_b) \}

\[ - \frac{e^2}{16\pi^2} e^{\frac{\pi}{2}p \cdot p'} \left\{ \frac{\gamma_\mu}{2} \left[ 3 \ln \frac{m^2}{\mu^2} + 6 \int_0^1 dz (1-z)^2 \right] - \int_0^1 dx \int_0^{1-x} dy N_{2b} \right\} \] (38)

The structure of the electron-photon vertex in ordinary QED indicates that the list of vectors and scalars appearing in the vertex function was restricted to \((\gamma_\mu, q_\mu, q^2, m, e)\). In the case of NCQED, due to the presence of \(\theta_{\mu\nu}\), we have two other scalars: \((\bar{q})^2, \bar{q}\), and one other vector \((\bar{q}_\mu)\). Indeed, from the above relation we can rewrite the vertex function in the form:

\[ \Lambda^{(s)}(p,q,p') = (G\gamma_\mu + H\bar{q}_\mu + L) e^{\frac{\pi}{2}p \cdot p'} \] (39)

where \(G, H, L\) are the functions of scalars \((q^2, \bar{q}^2, \bar{q}, m, e)\). These functions can be picked out from (38) in the on-shell condition:

\[ G(e, m, |\bar{q}|^2) = \frac{2e^2}{(4\pi)^{\frac{3}{2}}} \int_0^1 dx \int_0^{1-x} dy \left\{ e^{-i(x+y)p \cdot p'} \left[ \frac{(2-d)^2}{2(M_a^2)^{\frac{3}{2}}} \left( \frac{Z_a}{2} \right)^{\frac{3}{2}} K_{\frac{3}{2}}(Z_a) \right] \right. \]

\[ \left. - \frac{2-d}{4} \left( \frac{Z_a}{2} \right)^{\frac{3}{2}-1} K_{\frac{3}{2}-1}(Z_a) \right\} \]
After sandwiching $\Gamma^*$ between $\overline{u}(p')$ and $u(p)$ we obtain

$$\overline{u}(p')\Gamma^* u(p) = i e e^{\frac{2}{\mu^2}} \overline{u}(p') \left[ F_1 \gamma_\mu + \frac{1}{2m} F_2 (i \sigma_{\mu\nu} q^\nu' + H \overline{q}_\mu) u(p) \right]$$

In the condition $q^2 = 0$ and in the limit $\epsilon \to 0$ the form factors are:

$$F_1 = 1 + \frac{2 e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{2 K_0(Z_a) e^{-i(x+y)p\cdot p'}}{2m} - 6 K_0(Z_b) e^{i(x+y-1)p\cdot p'} \right\}$$

$$+ \left( \frac{M_a^2}{Z_a} K_1(Z_a) e^{-i(x+y)p\cdot p'} - \frac{M_b^2}{Z_b} K_1(Z_b) e^{i(x+y-1)p\cdot p'} \right)$$

$$- \frac{e^2}{(4\pi)^2} \left( 3 \ln \frac{m^2}{\mu^2} + 6 \int_0^1 dz z \ln(1 - z)^2 \right)$$

$$+ \frac{m^2 e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ e^{-i(x+y)p\cdot p'} \left[ (x + y + 1)^2 - 3 \right] \frac{Z_a}{M_a^2} K_1(Z_a) \right\}$$

$$+ \frac{3m^2 e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ (\frac{Z_b}{2}) K_1(Z_b) e^{i(x+y-1)p\cdot p'} - \frac{1}{2} \right\}$$

(44)

and,

$$F_2(q^2 = 0) = -\frac{8m^2 e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy$$

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\[
\left\{ \frac{(x+y)(x+y-1)}{M_a^2} \left( \frac{Z_a}{2} \right) K_1(Z_a) e^{i(x+y)p\wedge p'} \right. \\
+ \frac{(x+y-1)(x+y-3)}{M_b^2} \left[ \left( \frac{Z_b}{2} \right) K_1(Z_b) e^{i(x+y-1)p\wedge p'} - \frac{1}{2} \right] \right\}
\]

(45)

We can see that (43) is the general form of the vertex which satisfied the gauge invariance.
As in the ordinary QED, the form factor \(F_1\) is the correction of the fermion’s charge.
At the tree level it’s natural to impose:
\[
F_1(q^2 = 0) = 1
\]

In this case, e is the electric charge of fermion in the limit \(q^2 = 0\). From (44) this choice corresponds to \((\epsilon \to 0)\):
\[
\Delta_1^F = - \frac{2e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ 2K_0(Z_a) e^{-i(x+y)p\wedge p'} - 6K_0(Z_b) e^{i(x+y-1)p\wedge p'} \right. \\
+ \left. (\bar{q})^2 \left[ \frac{(M_a^2)}{Z_a} K_1(Z_a) e^{-i(x+y)p\wedge p'} - \frac{(M_b^2)}{Z_b} K_1(Z_b) e^{i(x+y-1)p\wedge p'} \right] \right\}
\]
\[
+ \frac{e^2}{(4\pi)^2} \left( 3 \ln \frac{m^2}{\mu^2} + 6 \int_0^1 dzz \ln(1-z)^2 \right)
\]
\[
- \frac{4m^2e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ e^{-i(x+y)p\wedge p'} \left[ (x+y+1)^2 - 3 \right] \left( \frac{Z_a}{2} \right) K_1(Z_a) \right\}
\]
\[
- \frac{12m^2e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ (x+y-1)^2 \left[ \left( \frac{Z_b}{2} \right) K_1(Z_b) e^{i(x+y-1)p\wedge p'} - \frac{1}{2} \right] \right\}
\]

(46)

The form factor \(F_2(q^2 = 0)\) is used to determine the coefficient of the anomalous magnetic moment of fermion.

5 \hspace{1em} THE WARD IDENTITY

Let us first consider the total vertex \(\Gamma_\mu(p,q,p')\). By the counter terms \(Z_i = 1 + \Delta_i\) we can rewrite the total vertex in the form:
\[
\Gamma_\mu(p,q,p') = ie \left[ \gamma_\mu e^{i\pi p\wedge p'} (1 + \Delta_1) + \Lambda_\mu(p,q,p') \right]
\]

(47)

Recalling that:
\[
\Lambda_\mu(p,q,p') = \Lambda_\mu^a(p,q,p') + \Lambda_\mu^b(p,q,p')
\]
and,

\[ \Lambda^{(a)}(p, q, p') = -ie^2 \mu' e^{2p \cdot p'} \int \frac{d^d k}{(2\pi)^d} e^{-ik \bar{q}} \left[ \gamma_\nu \frac{1}{D(p' - k)} \gamma_\mu \frac{1}{D(p - k)} \gamma_\nu \frac{1}{k^2} \right] \]

\[ \Lambda^{(b)}(p, q, p') = +ie^2 \mu' e^{2p \cdot p'} \int \frac{d^d k}{(2\pi)^d} \left( 1 - e^{ik \bar{q} e^{-ip \cdot p'}} \right) \left[ \gamma_\rho \frac{1}{D(k)} \gamma_\nu \frac{1}{(p' - k)^2 (p - k)^2} \right] \]

\{ (2p - p' - k)_\nu g_{\mu \rho} + (2p' - p - k)_\rho g_{\nu \mu} + (2k - p - p')_\mu g_{\rho \nu} \} \quad (48) \]

where:

\[ D(p) \equiv \not{p} - m \]

By the simple manipulations we can show that:

\[ (p - p')_\mu \Lambda_\mu(p, q, p') = -ie^2 \mu' e^{2p \cdot p'} \int \frac{d^d k}{(2\pi)^d} \left( e^{ik \bar{q} e^{-ip \cdot p'}} - 1 \right) \frac{N_f}{D} \]

\[ -ie^2 \mu' e^{2p \cdot p'} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{k^2} \gamma_\rho \frac{1}{D(p' - k)} \gamma_\nu \frac{1}{D(p - k)} \right] \quad (49) \]

where:

\[ N_f = 2(\not{p} - \not{p'}) + 2(p - p')k \not{p}' - 2k^2(\not{p} - \not{p'}) - 2mk(p - p') \quad (50) \]

Recalling that the expression of the fermion self energy is:

\[ \sum_{(p)} = -\Delta_2 \not{p} - \Delta_0 m + \sum_{(p)} \quad (51) \]

in which \( \Delta_2 \) and \( \Delta_0 \) are the counter terms for the fermion loop while \( \sum_{(p)} \) is the correction of the fermion’s propagator whose expression is:

\[ \sum_{(p)} = -ie^2 \mu' e^{2p \cdot p'} \int \frac{d^d k}{(2\pi)^d} \left( e^{ik \bar{q} e^{-ip \cdot p'}} - 1 \right) \frac{N_f}{D} \]

\[ \sum_{(p)} = -ie^2 \mu' e^{2p \cdot p'} \int \frac{d^d k}{(2\pi)^d} \gamma_\rho \frac{1}{D(p - k)} \gamma_\nu \frac{1}{k^2} \quad (52) \]

From (49) and (52) we obtain the relation between the \( \lambda'_\mu \)s and the \( \sum_{(p)} \)s as:

\[ (p - p')_\mu \Lambda_\mu(p, q, p') = e^{2p \cdot p'} \left( \sum_{(p')} - \sum_{(p)} + \Omega \right) \quad (53) \]

The \( \Omega \)’s term in the above equation is defined as:

\[ \Omega \equiv -ie^2 \mu' \int \frac{d^d k}{(2\pi)^d} \left( e^{ik \bar{q} e^{-ip \cdot p'}} - 1 \right) \frac{N_f}{D} \quad (54) \]
where:
\[ N_f = 2pk(\not{\phi} - \not{\phi}') + 2\phi'(p - p')k - 2k^2(\not{\phi} - \not{\phi}') - 2mk(p - p') \]  
(55)

By the standard Feynman parameterisation, it is easy to show that \( \Omega \) can be decomposed into two parts: finite and infinite.

\[ \Omega = (\not{\phi} - \not{\phi}') \left( \Omega^\infty + \Omega^F \right) \]  
(56)

in which we have:
\[ \Omega^\infty(q^2 = 0) = \frac{e^2}{4\pi^2} \left( \frac{1}{\epsilon} - \frac{1}{2\gamma_E} \right) \]  
(57)

and;
\[ \Omega^F(q^2 = 0) = -\frac{e^2}{8\pi^2} \ln \frac{m^2}{\mu^2} - \frac{e^2}{4\pi^2} \int_0^1 dz z^2 \ln(1 - z)^2 + \frac{e^2}{8\pi^2} \int_0^1 \frac{z^2}{(1 - z)^2} \]
\[ - \frac{e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy e^{i(x+y-1)p\wedge p'} \]
\[ \left\{ 2K_0(Z_b) + \frac{(q^2 M_b^2)}{2} K_{-1}(Z_b) + \frac{m^2(x + y)(1 - x - y)}{M_b^2} \left( \frac{Z_b}{2} \right) K_1(Z_b) \right\} \]  
(58)

Now, returning to (47), from (51) and (53) we obtain:
\[ (p - p')^\mu \Gamma_\mu(p, q, p') = iee^{\frac{i}{2}p\wedge p'} \left\{ (\not{\phi} - \not{\phi}') \left[ \Delta_1 - \Delta_2 + \Omega^\infty + \Omega^F \right] + S^{-1}(p) - S^{-1}(p') \right\} \]  
(59)

For the divergence parts, with the result (13) and (37) in section.3:
\[ \Delta_2^\infty = -\frac{e^2}{16\pi^2} \left( \frac{2}{\epsilon} - \gamma_E \right) \]
\[ \Delta_1^\infty = -\frac{3e^2}{8\pi} \left( \frac{1}{\epsilon} - \frac{1}{2} \gamma_E \right) \]

fortunately, we see that:
\[ \Delta_1^\infty - \Delta_2^\infty + \Omega^\infty = 0 \]

and we obtain:
\[ (p - p')^\mu \Gamma_\mu(p, q, p') = iee^{\frac{i}{2}p\wedge p'} \left[ \Delta_1^F - \Delta_2^F + \Omega^F \right] + iee^{\frac{i}{2}p\wedge p'} \left[ S^{-1}(p) - S^{-1}(p') \right] \]  
(60)

For the finite part, from (13), (46) and (58) we see that:
\[ \Delta_1^F - \Delta_2^F + \Omega^F \neq 0 \]

That means we can redefine the charge of fermion e.
6 THE ANOMALOUS MAGNETIC MOMENT

As discussed in [5], we see that the coefficient $H$ in the expression of vertex function (43) will give the new contribution to the magnetic moment:

$$< \vec{\mu} > = \frac{H}{i \theta}, \quad \theta_i \equiv \epsilon_{ijk} \theta_{jk}$$  \hspace{1cm} (61)

As we see this correction of magnetic moment does not depend on spin. So far we can write the magnetic moment of fermion in NCQED as:

$$< \vec{\mu} > = g \left( \frac{e^2}{2m} \right) \vec{s} + \frac{H}{i \theta}$$  \hspace{1cm} (62)

The form factor $F_2(q^2 = 0)$ determines the coefficient of the anomalous magnetic moment:

$$g = 2[1 + F_2(q^2 = 0)]$$  \hspace{1cm} (63)

Now from (45) we can identify $F_2(q^2 = 0)$ as:

$$F_2(q^2 = 0) = -\frac{8m^2e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \left( \frac{x+y}{M_a^2} \right) \frac{Z_a}{2} K_1(Z_a) e^{-i(x+y)p \wedge p'} + \left( \frac{x+y-1}{M_b^2} \right) \frac{Z_b}{2} K_1(Z_b) e^{i(x+y-1)p \wedge p'} - \frac{1}{2} \right\}$$  \hspace{1cm} (64)

Now, we try to evaluate the magnetic moment of fermion in the rest framework such that $p \wedge p' = 0$ in which for $\theta^{0i} = 0$ we get:

$$F_2(q^2 = 0) = -\frac{8m^2e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \left( \frac{x+y}{M_a^2} \right) \frac{Z_a}{2} K_1(Z_a) + \left( \frac{x+y-1}{M_b^2} \right) \frac{Z_b}{2} K_1(Z_b) - \frac{1}{2} \right\}$$  \hspace{1cm} (65)

Now in the limit $|\vec{q}| \approx 0$ by keeping the leading terms in the expansion of the Bessel function, we obtain:

$$F_2(q^2 = 0) = -\frac{8m^2e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy$$
\begin{align*}
\{ (x+y)(1-x-y) & \left[ \frac{1}{2M_a^2} + \frac{|\tilde{q}|^2}{8} \ln \frac{m^2|\tilde{q}|^2}{4} + \frac{|\tilde{q}|^2}{8} \ln \left[ (x+y)^2 + \frac{m^2}{m^2}(1-x-y) \right] \\
& + \frac{|\tilde{q}|^2}{4} \left( -\frac{1}{2} + \gamma_E \right) \right] \\
+ (x+y-1)(x+y-3) & \left[ \frac{|\tilde{q}|^2}{8} \ln \frac{m^2|\tilde{q}|^2}{4} + \frac{|\tilde{q}|^2}{8} \ln \left[ (x+y-1)^2 + \frac{m^2}{m^2}(x+y) \right] \\
& + \frac{|\tilde{q}|^2}{4} \left( -\frac{1}{2} + \gamma_E \right) \right] \} \quad (66)
\end{align*}

After the integration over the Feynman parameters we get:

\[
F_2(q^2 = 0) = \frac{\alpha}{2\pi} - \frac{m^2e^2|\tilde{q}|^2}{48\pi^2}
\left\{ \ln \frac{m^2|\tilde{q}|^2}{4} - 1 + 2\gamma_E + 6 \int_0^1 z(1-z) \ln[z^2 + \frac{m^2}{m^2}(1-z)] \right\} \quad (67)
\]

Due to the massless photon there is an IR-divergent term in the expression of the form factor \( F_2(q^2) = 0 \). Ignoring the IR-divergent terms we receive the correction of the coefficient \( g \) as:

\[
\delta g = -\frac{m^2e^2|\tilde{q}|^2}{48\pi^2} \left\{ \ln \frac{m^2|\tilde{q}|^2}{4} - 1 + 2\gamma_E \right\} \quad (68)
\]

Now, we evaluate the contribution of the coefficient \( H \) in the rest frame. From (41) we have:

\[
H(e, m, |\tilde{q}|^2, \tilde{q}) = -\frac{4e^2\tilde{q}}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{(M_a^2)}{Z_a} K_{-1}(Z_a) + \frac{(M_b^2)}{Z_b} K_{-1}(Z_b) \right\} \quad (69)
\]

In the low momentum limit, keeping the leader terms in the expansion of Bessel’s function we get:

\[
H = -\frac{e^2(\tilde{q})}{4\pi^2|\tilde{q}|^2} \quad (70)
\]

So, the full expression of the fermion’s magnetic moment in the rest frame is:

\[
< \vec{\mu} > = \frac{e}{m} \left\{ 1 + \frac{\alpha}{2\pi} - \frac{m^2e^2|\tilde{q}|^2}{48\pi^2} \left[ \ln \frac{m^2|\tilde{q}|^2}{4} - 1 + 2\gamma_E \right] \right\} \vec{s} + \frac{ie^2(\tilde{q})}{4\pi^2|\tilde{q}|^2} \vec{\theta} \quad (71)
\]
7 CONCLUSIONS

In this paper we have studied the renormalisation of NCMQED, the vacuum polarisation of photon, the β-function, the contribution of the vertex function at one-loop level in NCQED. Based on the dimensional regularization method which have been generalized to NC-theories and by assuming Bessel functions are finite at the finite θ, the calculations show that the theory is renormalised with the counter terms. The structure of vacuum polarisation of photon satisfies Ward identity as well as the structure of vertex. It is shown that besides the normal form factors there is a new contribution to the magnetic moment comes from the parameter θ. The Ward identity is satisfied at the one-loop level of vertex and in the condition θ^{0ν} = 0.

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Appendix

We work in d-dimensional "Minkowski" space with one timelike and (d-1) spacelike dimensions. We are interested in the generic integrals:

\[ I_1 = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\tilde{q}}}{(k^2 - \Delta)^3} \]

Using Wick’s rotation: \( k^0 = ik^n \), we have:

\[ k^2 = -k_E^2 \]

and,

\[ k\tilde{q} = k^\mu \theta^{\mu\nu} q_\nu = k^0 \theta^{0\nu} q_\nu + k^i \theta^{ij} q_j \]

In the case of the spatial non-commutativity \( \theta^{0\nu} = 0 \)

\[ \Rightarrow k\tilde{q} = k^i \theta^{ij} q_j \]

So, the exponent \( e^{ik\tilde{q}} \) doesn’t change the sign and we get

\[ I_1 = (i)(-1)^3 \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\tilde{q}}}{(k^2 + \Delta)^3} = (i)(-1)^3 I_{1E} \]

In order to evaluate the generic form of this integral in Euclidean space we use the Feynman parameterisation:

\[ \frac{1}{k^2 + \Delta} = \int_0^\infty d\alpha e^{-\alpha (k^2 + \Delta)} \]

\[ \Rightarrow I_{1E} = \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \int \frac{d^D k}{(2\pi)^D} e^{-(\alpha_1 + \alpha_2 + \alpha_3)k^2 + ik\tilde{q} - \Delta(\alpha_1 + \alpha_2 + \alpha_3)} \]

Taking the integration over \( k \) we get

\[ I_{1E} = \frac{1}{(4\pi)^D} \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \frac{1}{(\alpha_1 + \alpha_2 + \alpha_3)^{D/2}} \exp\left[-\frac{(\tilde{q})^2}{4(\alpha_1 + \alpha_2 + \alpha_3)} - \Delta(\alpha_1 + \alpha_2 + \alpha_3)\right] \]

Inserting

\[ 1 = \int_0^\infty d\rho \cdot \delta(\rho - \sum_{i=1}^3 \alpha_i) \]
and rescaling $\alpha_i \rightarrow \rho \alpha_i$ we get

$$I_{1E} = \frac{1}{(4\pi)\frac{D}{2}} \frac{1}{\Gamma(\frac{3}{2})} \int_0^\infty \frac{d\rho}{\rho^{\frac{D}{2}-2}} e^{\exp[-\frac{(\tilde{q})^2}{4\rho}] - \Delta \rho]}$$

In terms of Bessel function we have:

$$I_{1E} = \frac{1}{(4\pi)\frac{D}{2}} \frac{1}{\Gamma(\frac{3}{2})} \left( \frac{Z}{2} \right)^{\frac{3-D}{2}} K_{\frac{D}{2}}(Z)$$

where $Z = |\tilde{q}|(\Delta)^{\frac{3}{2}}$ and,

$$I_1 = (i)(-1)^3 \frac{1}{(4\pi)\frac{D}{2}} \frac{1}{\Gamma(\frac{3}{2})} \left( \frac{Z}{2} \right)^{\frac{3-D}{2}} K_{\frac{D}{2}}(Z)$$

(2.2) \hspace{1cm} I_{\mu\nu} = \int \frac{d^D k}{(2\pi)^D (k^2 - \Delta)^3} k_\mu k_\nu e^{ik\tilde{q}}

After the Wick rotation the integral has the form:

$$I_{\mu\nu} = (i)(-1)^3 I_{\mu\nu(E)} = (i)(-1)^3 \int \frac{d^D k}{(2\pi)^D (k^2 + \Delta)^3} k_\mu k_\nu e^{ik\tilde{q}}$$

We evaluate $I_{\mu\nu(E)}$ from the generating function:

$$Z_E = \int \frac{d^D k}{(2\pi)^D} e^{ik(\tilde{q} - z)} (k^2 + \Delta)^{-\frac{3}{2}}$$

we see that:

$$I_{\mu\nu(E)} = \frac{1}{i^2} \frac{\partial^2}{\partial z_\mu \partial z_\nu} Z|_{z \rightarrow 0}$$

As the same way, after the parameterisation, we have:

$$Z_E = \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \frac{1}{(\alpha_1 + \alpha_2 + \alpha_3)^2} e^{-(\alpha_1 + \alpha_2 + \alpha_3)k^2 + ik(\tilde{q} - z) - \Delta(\alpha_1 + \alpha_2 + \alpha_3)}$$

with the aid of the Gaussian integral we can take integration over momentum k:

$$Z_E = \frac{1}{(4\pi)^\frac{D}{2}} \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \frac{1}{(\alpha_1 + \alpha_2 + \alpha_3)^2} e^{[exp[-\frac{(\tilde{q} - z)^2}{4(\alpha_1 + \alpha_2 + \alpha_3)} - \Delta(\alpha_1 + \alpha_2 + \alpha_3)]]}$$

Now taking derivative with respect to $z$ and then taking the limit $z \rightarrow 0$ we get:

$$I_{\mu\nu(E)} = \frac{1}{(4\pi)^\frac{D}{2}} \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \left\{ \frac{\delta_{\mu\nu}}{2(\alpha_1 + \alpha_2 + \alpha_3)^{\frac{D}{2}+1}} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{4(\alpha_1 + \alpha_2 + \alpha_3)^{\frac{D}{2}+2}} \right\}$$
\[ \exp\left[-\frac{(\tilde{q})^2}{4(\alpha_1 + \alpha_2 + \alpha_3)} - \Delta(\alpha_1 + \alpha_2 + \alpha_3)\right] \]

Inserting
\[ 1 = \int_{0}^{\infty} d\rho \cdot \delta(\rho - \sum_{i=1}^{3} \alpha_i) \]

and rescaling \( \alpha_i \rightarrow \rho \alpha_i \) we get
\[ I_{\mu\nu(E)} = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma(3)} \int_{0}^{\infty} d\rho \left\{ \frac{\delta_{\mu\nu}}{2(\rho)^{\frac{D}{2} - 1}} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{4(\rho)^{\frac{D}{2}}} \right\} \]
\[ \exp\left[-\frac{(\tilde{q})^2}{4(\rho)} - \Delta(\rho)\right] \]

In terms of the Bessel’s functions, the relation above can be rewritten in the form:
\[ I_{\mu\nu(E)} = A\delta_{\mu\nu} - B\tilde{q}_\mu \tilde{q}_\nu \]

where
\[ A = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma(3)} \frac{1}{\Delta^{2 - \frac{D}{2}}} \left( \frac{Z}{2} \right)^{-\frac{D}{2}} K_{2 - \frac{D}{2}}(Z) \]
\[ B = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma(3)} \frac{1}{2\Delta^{1 - \frac{D}{2}}} \left( \frac{Z}{2} \right)^{-\frac{D}{2}} K_{1 - \frac{D}{2}}(Z) \]

returning to the Minkowski space, the result is:
\[ I_{\mu\nu} = (-i)(-1)^{\alpha} [A g_{\mu\nu} + B\tilde{q}_\mu \tilde{q}_\nu] \]

In general, the integral:
\[ I_{\mu\nu} = \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu e^{ik\tilde{q}}}{(k^2 - \Delta)^\alpha} = (-i)(-1)^{\alpha} [A g_{\mu\nu} + B\tilde{q}_\mu \tilde{q}_\nu] \]

where:
\[ A = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma(\alpha)} \frac{1}{\Delta^{\alpha - 1 - \frac{D}{2}}} \left( \frac{Z}{2} \right)^{-\frac{D}{2}} K_{\alpha - 1 - \frac{D}{2}}(Z) \]
\[ B = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma(\alpha)} \frac{1}{2\Delta^{\alpha - 2 - \frac{D}{2}}} \left( \frac{Z}{2} \right)^{-\frac{D}{2}} K_{\alpha - 2 - \frac{D}{2}}(Z) \]

\[ I_2 = \int \frac{d^Dk}{(2\pi)^D} \frac{k^2 e^{ik\tilde{q}}}{(k^2 - \Delta)^\alpha} \]

Contracting the result is:
\[ I_2 = (-i)(-1)^{\alpha}[A \cdot D + B(\tilde{q})^2] \]
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