NEW CANONICAL TRIPLE COVERS OF SURFACES

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(Communicated by Lev Borisov)

Abstract. We construct a surface of general type with canonical map of degree 12 which factors as a triple cover and a bidouble cover of \( \mathbb{P}^2 \). We also show the existence of a smooth surface with \( q = 0 \), \( \chi = 13 \) and \( K^2 = 9\chi \) such that its canonical map is either of degree 3 onto a surface of general type or of degree 9 onto a rational surface.

1. INTRODUCTION

Let \( S \) be a smooth minimal surface of general type. Denote by \( \phi : S \dashrightarrow \mathbb{P}^{pg-1} \) the canonical map and let \( d := \deg(\phi) \). The following Beauville’s result is well known.

**Theorem 1.1 ([3]).** If the canonical image \( \Sigma := \phi(S) \) is a surface, then either:

(i) \( pg(\Sigma) = 0 \), or

(ii) \( \Sigma \) is a canonical surface (in particular \( pg(\Sigma) = pg(S) \)).

Moreover, in case (i) \( d \leq 36 \) and in case (ii) \( d \leq 9 \).

Beauville has also constructed families of examples with \( \chi(\mathcal{O}_S) \) arbitrarily large for \( d = 2, 4, 6, 8 \) and \( pg(\Sigma) = 0 \). Although this is a classical problem, for \( d > 8 \) the number of known examples drops drastically: only Tan’s example [18, §5] with \( d = K^2 = 9 \), \( \chi = 4 \) and Persson’s example [14] with \( d = K^2 = 16 \), \( \chi = 4 \) are known. More recently, Du and Gao [7] claimed that if the canonical map is an abelian cover of \( \mathbb{P}^2 \), then these are the only possibilities for \( d > 8 \).

In this note we construct an example with \( d = K^2 = 12 \) which factors as a triple cover and a bidouble cover of \( \mathbb{P}^2 \).

Known examples for case (ii) with \( d = 3 \) date to 1991/2: Pardini’s example [13] with \( K^2 = 27 \), \( \chi = 6 \) and Tan’s examples [18] with \( K^2 \leq 6\chi \), \( 5 \leq \chi \leq 9 \). Later Barth [11, unpublished] have shown that the quintic surface constructed by van der Geer and Zagier in 1977 [20] is also the image of a triple canonical cover with \( \chi = 5 \) (see also [16]).

Nowadays the case \( d = 3 \) is still mysterious. On the one hand no one has given a bound for \( \chi \), on the other hand there are no examples for other values of the invariants.
More generally, for the case where the canonical map factors through a triple cover of a surface of general type and $d = 6$, we have only the family given in [6] Example 3.4] with invariants on the Noether’s line $K^2 = 2p_g - 4$.

Here we show the existence of a smooth regular surface $S$ with $\chi = 13$ and $K^2 = 9\chi$ such that its canonical map $\phi$ factors through a triple cover of a surface of general type. This is the first example on the border line $K^2 = 9\chi$ (recall that, for a surface of general type, one always has $2p_g - 4 \leq K^2 \leq 9\chi$).

This surface is an unramified cover of a fake projective plane. We show that if $\deg(\phi) \neq 3$, then $\phi$ is of degree 9 onto a rational surface. If this is the case, then one might expect to be able to recover the construction of the rigid surface $S$ as a covering of $\mathbb{P}^2$, which would be interesting. Since it seems very difficult to provide a geometric construction of a fake projective plane, we conjecture that $d = 3$.

**Notation.** We work over the complex numbers. All varieties are assumed to be projective algebraic. A ($-n$)-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$.

If $X$ is a normal surface and $Y$ is a smooth minimal model of $X$, then $\chi(\mathcal{O}_X)$, $q(X)$ and $p_g(X)$ mean $\chi(\mathcal{O}_Y)$, $q(Y)$ and $p_g(Y)$ (holomorphic Euler characteristic, irregularity and geometric genus, respectively).

The rest of the notation is standard in Algebraic Geometry.

## 2. Basics on Galois triple covers

Our references for triple covers are [17], [18] or [11].

Let $X$ be a smooth surface. A Galois triple cover $\pi : Y \to X$ is determined by divisors $L$, $M$, $B$, $C$ on $X$ such that $B \in |2L - M|$ and $C \in |2M - L|$. The branch locus of $\pi$ is $B + C$ and $3L \equiv 2B + C$, $3M \equiv B + 2C$. The surface $Y$ is normal iff $B + C$ is reduced. The singularities of $Y$ lie over the singularities of $B + C$.

If $B + C$ is smooth, we have

\begin{align*}
\chi(\mathcal{O}_Y) &= 3\chi(\mathcal{O}_X) + \frac{1}{2}(L^2 + K_XL) + \frac{1}{2}(M^2 + K_XM), \\
K_Y^2 &= 3K_X^2 + 4(L^2 + K_XL) + 4(M^2 + K_XM) - 4LM, \\
q(Y) &= q(X) + h^1(X, \mathcal{O}_X(K_X + L)) + h^1(X, \mathcal{O}_X(K_X + M)), \\
p_g(Y) &= p_g(X) + h^0(X, \mathcal{O}_X(K_X + L)) + h^0(X, \mathcal{O}_X(K_X + M)).
\end{align*}

Now suppose that $\sigma : X \to X'$ is the minimal resolution of a normal surface $X'$ with a set $s = \{s_1, \ldots, s_n\}$ of ordinary cusps (singularities of type $A_2$). If the ($-2$)-curves $A_i$, $A'_i$ satisfying $\sigma^{-1}(s_i) = A_i + A'_i$ can be labelled such that

$$\sum_{i=1}^n (2A_i + A'_i) \equiv 3J,$$

for some divisor $J$, then we say that $s$ is a 3-divisible set of cusps.

**Proposition 2.1.** Let $X'$ be a minimal surface of general type containing a 3-divisible set as above as only singularities. Let $\phi : Y \to X$ be a Galois triple cover with branch locus $\sum_i (A_i + A'_i)$.

If $n = 3\chi(\mathcal{O}_X)$, then $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_X)$ and $K_{Y'}^2 = 3K_X^2$, where $Y'$ is the smooth minimal model of $Y$. 

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Proof. Let $\tilde{X} \to X$ be the blow-up at the singular points of $\sum_n^1 (A_i + A'_i)$. Denote by $\widehat{A}_i, \widehat{A}'_i$ the $(-3)$-curves which are the strict transforms of $A_i, A'_i$, $i = 1, \ldots, n$. The surface $Y'$ is the minimal model of the Galois triple cover of $\tilde{X}$ with branch locus $\sum_n^1 (\widehat{A}_i + \widehat{A}'_i)$. The result follows from (2.1) and (2.2) (notice that $K^2_{\tilde{X}} = K^2_X - n$ and $K^2_{Y'} = 3K^2_X + 3n$).

Remark 2.2. Note that the cusps induce smooth points on the covering surface, i.e., the pullback of the divisor $\sum_n^1 (A_i + A'_i)$ is contracted to smooth points of $Y'$.

3. A SURFACE WITH CANONICAL MAP OF DEGREE 12

The following result has been shown by Tan [19, Thm 6.2.1], using codes. Here we give an alternative proof.

Lemma 3.1. Let $X'$ be a double cover of $\mathbb{P}^2$ ramified over a quartic curve with 3 cusps. Then the 3 cusps of $X'$ are 3-divisible.

Proof. Let $B \subset \mathbb{P}^2$ be a quartic curve with 3 cusps at points $p_1, p_2, p_3$ (it is well known that such a curve exists; it is unique up to projective equivalence). Consider the canonical resolution $X \to X'$. The strict transform of the lines through $p_1p_2, p_2p_3$ and $p_1p_3$ is a union of disjoint $(-1)$-curves $E_i, E'_i \subset X$, $i = 1, 2, 3$. These curves and the $(-2)$-curves $A_i, A'_i$, $i = 1, 2, 3$, which contract to the cusps of $X'$ can be labelled such that the intersection matrix of the curves $A_1, A'_1, A_2, A'_2, A_3, A'_3, E_1$ and $E_2$ is

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\
\end{bmatrix}
\]

This matrix has determinant zero. Since

$$b_2(X) = 12\chi(O_X) - K^2_X + 4q(X) - 2 = 8,$$

these 8 curves are dependent in $\text{Num}(X)$, and this relation has to be expressed in the nullspace of the matrix. Using computer algebra we get that this nullspace has basis

$$\begin{bmatrix} 2 & 1 & 1 & -1 & -1 & -2 & 3 & -3 \end{bmatrix}.$$

Thus $2A_1 + A'_1 + A_2 - A'_2 - A_3 - 2A'_3 + 3E_1 - 3E_2 = 0$ in $\text{NS}(X)$ (notice that $X$ has no non-trivial torsion). One has $\text{NS}(X) = \text{Pic}(X)$ for regular surfaces $X$ (Castelnuovo), hence there exists a divisor $L$ such that

$$2A_1 + A'_1 + A_2 + 2A'_2 + 2A_3 + A'_3 \equiv 3L.$$
Let $Q_1, Q_2 \subset \mathbb{P}^2$ be quartic curves with 3 cusps each such that $Q_1 + Q_2$ has 6 cusps and 16 nodes. Let $V$ be the bidouble cover of $\mathbb{P}^2$ defined by the divisors $Q_1, Q_2, Q_3 := 0$ (for information on bidouble covers see e.g. [5] or [12]). Consider the divisors $J_1, J_2, J_3$ such that $2J_1 \equiv Q_2 + Q_3, 2J_2 \equiv Q_1 + Q_3, 2J_3 \equiv Q_1 + Q_2$. Notice that since all singularities of $V$ are rational double points, its invariants do not change under the canonical resolution (see e.g. [2, V.22]). So we have

$$p_g(V) = p_g(\mathbb{P}^2) + \sum_{i=1}^{3} h^0(\mathbb{P}^2, K_{\mathbb{P}^2} + J_i) = 3,$$

$$\chi(O_V) = 4\chi(O_{\mathbb{P}^2}) + \frac{1}{2} \sum_{i=1}^{3} J_i(K_{\mathbb{P}^2} + J_i) = 4$$

and $V$ has 12 ordinary cusps and no other singularities.

Denote by $W_1, W_2, W_3$ the double covers of $\mathbb{P}^2$ with branch curves $Q_1, Q_1 + Q_2$, respectively (the intermediate surfaces of the bidouble cover). The canonical map of $V$ factors through maps $V \to W_i, i = 1, 2, 3$, hence it is of degree 4. We get from Lemma 3.1 that the 3 cusps of $W_1$ and the 3 cusps of $W_2$ are 3-divisible, therefore the 12 cusps of $V$ are also 3-divisible.

Now let $S \to V$ be the Galois triple cover ramified over the 12 cusps. We claim that $p_g(S) = p_g(V) = 3$. This implies that the canonical map of $S$ factors through the triple cover, thus it is of degree 12. From Proposition 2.1 $q(S) = 0$ and $K_S^2 = 12$.

So it remains to prove the claim. Let $\tilde{V}$ be the smooth minimal resolution of $V$. The cusps of $V$ correspond to configurations of $(-2)$-curves $A_i + A'_i \subset \tilde{V}$, $i = 1, \ldots, 12$. These can be labelled such that there exist divisors $L, M$ satisfying $2B + C \equiv 3L, B + 2C \equiv 3M$, where $B := \sum A_i$ and $C := \sum A'_i$.

Below we use the notation $D \geq 0$ for $h^0(\tilde{V}, O_{\tilde{V}}(D)) > 0$.

From (2.4), we need to show that $K_{\tilde{V}} + L \not\geq 0$ and $K_{\tilde{V}} + M \not\geq 0$. Suppose first that $K_{\tilde{V}} + L \geq 0$. Then

$$(K_{\tilde{V}} + L)A_i = -1, \forall i \implies K_{\tilde{V}} + L - B \geq 0$$

and

$$(K_{\tilde{V}} + L - B)A'_i = -1, \forall i \implies K_{\tilde{V}} + L - B - C \geq 0.$$ 

From $3K_{\tilde{V}} + 3L - 3B - 3C \geq 0$ and $3(L + M) \equiv 3(B + C)$ one gets $3K_{\tilde{V}} - 3M \geq 0$, i.e., $3K_{\tilde{V}} - B - 2C \geq 0$. This implies the existence of an element in the linear system $|3K_{\tilde{V}}|$ having multiplicity $> 1$ at each of the 12 cusps of $V$.

Let $q_1, q_2$ be the defining equations of $Q_1, Q_2$. The surface $V$ is given by equations $w^2 = q_1, t^2 = q_2$ in the weighted projective space $\mathbb{P}(x^1, y^1, z^1, w^2, t^2)$. It is easy to see that the linear system of polynomials of degree 3 has no element with multiplicity $> 1$ at the cusps of $V$ (for instance using computer algebra).

The case $K_{\tilde{V}} + M \geq 0$ is analogous.

4. A surface with $K^2 = 9\chi$

Based on the work of Prasad and Yeung [15], Cartwright and Steger [4] constructed a fake projective plane $F$ with an automorphism $j$ of order 3 such that $F/j$ is a surface of general type with $\chi = 1, p_g = 0, K^2 = 3$ and fundamental group $\mathbb{Z}_{13}$. This surface has a set of three 3-divisible cusps (cf. [9]). Denote by $B$ the unit ball in $\mathbb{C}^2$ and let $P, H$ and $G$ be the groups such that $F = B/P, F/j = B/H$ and
the universal cover of $F/j$ is $B/G$. We have the following commutative diagram, where the vertical arrows denote unramified $\mathbb{Z}_{13}$ covers and the horizontal arrows denote $\mathbb{Z}_3$ covers ramified over cusps:

$$
\begin{array}{ccc}
B/(G \cap P) & \xrightarrow{p} & B/G \\
\downarrow^{13:1} & & \downarrow^{13:1} \\
B/P & \xrightarrow{3:1} & B/H.
\end{array}
$$

Let $S := B/(G \cap P)$. Proposition 2.1 implies $\chi(O_S) = \chi(O_{B/G})$. We show that the surface $S$ is regular, hence $B/G$ is also regular and then $p_g(S) = p_g(B/G)$. This implies that the canonical map $\phi$ of $S$ factors through the triple cover $p$. Since $G \cap P$ is the fundamental group of $S$, the commutator quotient $(G \cap P)/(G \cap P, G \cap P)$ is isomorphic to $H_1(S, \mathbb{Z})$. The first Betti number $b_1(S) = 2q(S)$ is the minimal number of generators of $H_1(S, \mathbb{Z})$ modulo elements of finite order. Thus $q(S) = 0$ if $(G \cap P)/(G \cap P, G \cap P)$ is finite. This is shown in the Appendix, where we use computational GAP [8] data from Cartwright and Steger to compute $G \cap P$.

Now notice that $\phi$ is not composed with a pencil. In fact otherwise the canonical map of $B/G$ is composed with a pencil and then

$$39 = K^2_{B/G} \geq 4\chi(O_{B/G}) - 10 = 42,$$

from [21, Theorem A] (see also [10, Corollary 3.4]).

Finally we prove that if $d := \deg(\phi) \neq 3$, then $d = 9$ and the canonical image $\phi(S)$ is a rational surface. As in the proof of Proposition 4.1 of [3], we have

$$9\chi(O_S) \geq K^2_S \geq d \deg(\phi(S)) \geq nd(p_g(S) - 2)$$

where $n = 2$ if $\phi(S)$ is not ruled and $n = 1$ otherwise. This gives $d < 6$ if $\phi(S)$ is not ruled and $d < 12$ otherwise. Since $d \equiv 0 \pmod{3}$, then $d = 6$ or 9 and $\phi(S)$ is a rational surface. If $d = 6$, the canonical map of $B/G$ is of degree 2 and then $B/G$ has an involution. But, as seen in the Appendix, the structure of the automorphism group of $B/G$ is the semidirect product $\mathbb{Z}_{13} : \mathbb{Z}_3$, so there is no involution on $B/G$.

**APPENDIX: GAP code**

```gap
# We are using data from
# http://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/
# C18p3/C18p3-0-FP.gap
# namely the groups 'index9aFP', 'index3aFP' and
# the functions 'FundGp', 'AutGp'.

P:=index9aFP;
H:=index3aFP;
# B/P is a Fake projective plane.
# B/H is a quotient of the fake p.p. B/P by an order 3 automorphism.
# B/H is a surface with p_g=0 and K^2=3 having 3 cusps.

# Using the Cartwright-Steger function 'FundGp', we see that the
# fundamental group of B/H is "C13". We want to find the group G such
# that B/G is the universal cover of B/H. This is also computed by the
# function 'FundGp'.
```

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The following function is equal to 'FundGp' except that outputs G.

```plaintext
Grp:=function(G,FOList)
elocal e1,e2,e3,GFO,G0,G0FCA;
e1:=GeneratorsOfGroup(GammaBarFP);
e2:=Concatenation(e1,List(e1,Inverse));
Add(e2,One(G));
e3:=ListX(e2,e2,\*);
GFO:=Filtered(FOList,fo->fo in G);
G0:=Group(ListX(e3,GFO,function(elt,fo) return fo^elt; end));
Print(ForAll(GFO,fo->fo in G0),"\n");
Print(IsNormal(G,G0),"\n");
return G0;
end;

G:=Grp(H,FOList);
GP:=Intersection(G,P);

#The "commutative diagram":
StructureDescription(FactorGroup(P,GP)) = "C13";
StructureDescription(FactorGroup(H,G)) = "C13";
StructureDescription(FactorGroup(G,GP)) = "C3";
StructureDescription(FactorGroup(H,P)) = "C3";

#The first Betti number b_1(B/GP) = 0:
Index(GP,CommutatorSubgroup(GP,GP)) = 2916; #It is finite.

#The automorphism group of B/G:
StructureDescription(AutGp(G)) = "C13 : C3";
```

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