We propose a statistical mechanics approach to a coevolving spin system with an adaptive network of interactions. The dynamics of node states and network connections is driven by both spin configuration and network topology. We consider a Hamiltonian that merges the classical Ising model and the statistical theory of correlated random networks. As a result, we obtain rich phase diagrams with different phase transitions both in the state of nodes and in the graph topology. We argue that the coupling between the spin dynamics and the structure of the network is crucial in understanding the complex behavior of real-world systems and omitting one of the approaches renders the description incomplete.

During the last few decades, there has been a rapid development in the interdisciplinary area of network science. This may be because of the availability of vast amounts of data, much of it from such complex systems as financial markets, social and biological structures, and transportation networks. Studies of the network structure of such real-world systems as the World Wide Web [1] indicate that their topology has numerous non-trivial properties that the classical random graph model cannot explain [2]. This has produced new network models able to recreate some of these observed phenomena [3, 4]. Initially, most of these models focused on the graph evolution, often the growth in the number of nodes and edges [5]. On the other hand, a different approach has been developed that considers a statistical ensemble of graphs [6] called “exponential random graphs” [7]. This formalism, borrowed from statistical physics, has proved successful and has led to a phenomenological theory of the topological phase transitions in evolving networks [8–10].

This newly discovered concept of networks with a complex structure moved rapidly through the spin models community. Important critical properties were observed for both scale-free and small-world network versions of the canonical Ising model of ferromagnetism [11–13]. The use of complex networks became popular because they more closely resemble real-world structures than regular lattices or Poissonian graphs. This has been particularly important when modeling social and financial phenomena for which spin models are the simplest and the most common [14, 15]. Although the topology of many systems can be described using complex networks, the evolution of the model is limited to changes in the spin configuration. This implies that connection dynamics evolve more slowly than node state dynamics. Unfortunately, this assumption is not valid in most complex adaptive systems, which are describable using network tools. An Ising model with slowly evolving interactions was used as a model of a neural network [16] and as a possible tool for simulating magnetostriction in nanoscale magnetic structures [17]. A particularly interesting case involved models in which the connections and state dynamics coevolve with each other, one evolution depending on the other and resulting in non-trivial feedback. Most of these models focus on socio-economic systems and describe their dynamics [18] rather than their statistical mechanics [19, 20]. Some of them produce intriguing topological properties, mainly when the dynamics is driven by the structural characteristics of the network [21].

We here use the Hamiltonian formalism to describe Ising-like models with coevolution of spins and connections, and we want the connection dynamics to depend on both the spin configuration and network topology. Following the approach taken in Ref. [9] we use the degree as a topological variable and focus on nearest neighbor interactions. We consider undirected graphs with a fixed number $N$ of vertices and a fixed number $M$ of edges. The partition function $Z$ for our ensemble would be: 

$$Z = \sum_{\{\sigma\}} e^{-\beta H}$$

where $\{\sigma\}$ is the set of all possible spin configurations, $e$ is the base of the natural logarithm, and $\beta = 1/k_B T$ is the inverse temperature, with $k_B$ being the Boltzmann constant and $T$ the temperature.
we define to be
\[
Z = \sum_{\{c_{ij}\},\{s_i\}} e^{-\beta H(\{c_{ij}\},\{s_i\})}, \tag{1}
\]
where \(\{\cdot\}\) is all possible configurations with respect to a fixed number of links and nodes. Parameter \(\beta\) is the strength of fluctuations and is the inverse temperature. A general form of the Hamiltonian that lies within the scope of this paper is
\[
H(\{c_{ij}\},\{s_i\}) = \sum_{i<j} c_{ij} f(k_i, k_j, s_i, s_j) + \sum_i g(k_i, s_i), \tag{2}
\]
where \(c_{ij}\) is the adjacency matrix, \(k_i = \sum_j c_{ij}\) and \(s_i\) are respectively the degree and the spin of node \(i\), and \(f(\cdot)\) and \(g(\cdot)\) are functions to be determined. More specifically, we assume that the functions are such that
\[
H(\{c_{ij}\},\{s_i\}) = -\sum_{i<j} c_{ij} \left( \frac{k_i k_j}{\langle k \rangle} \right)^\phi s_i s_j - \sum_i k_i^\gamma - h \sum_i s_i, \tag{3}
\]
where \(\phi\) and \(\gamma\) are model parameters, \(\langle k \rangle = 2M/N\), and \(h\) is the external field acting on spins set to zero. This simple concrete form shows (i) that parameters \(\phi\) and \(\gamma\) allow us to continuously switch from complicated topological interactions to the classical Ising model, and (ii) that the multiplication of degrees is the simplest interaction expression. In an Ising framework, we treat it as a weight \(J_{ij}\) assigned to an edge \((i,j)\). In addition, \(J_{ij} = \left( \frac{k_i k_j}{\langle k \rangle} \right)^\phi\) is in accordance with real-world weighted network characteristics [22]. The second sum term is an external field that interacts with each local node degree and drives the preference for high or low degree nodes. In addition to the classical ferromagnetic interpretation, if we use a socio-economic model to examine the proposed Hamiltonian we find an accurate interpretation of its terms. Using an opinion model we determine the influence of a given agent by examining its connectivity. The external field term forces each agent to reach as many people as possible. In contrast, the interaction term allows the energy of the system to be strongly affected by the connections among influential high degree nodes. This works in two ways. High-degree agents with opposite spins are energetically unstable, and agents with the same spins lower the energy level.

The topological portion of the Hamiltonian changes the behavior of the Ising model. We find a variety of different effects. Some are structural, others are associated with spin configurations, and still others are a result of both. Figure 1 shows the simulation results for the phase diagram of \(\gamma\) and \(T\) with a fixed \(\phi = 0\). Here, no structural portion of the Hamiltonian describes interactions, i.e., the network structure is only locally important. Note that there are two separate topological phases and
also a continuous phase transition in magnetization, with a small impact by parameter $\gamma$.

Figure 2 shows the case $\phi = 0$ and $\gamma = 1.6$. Note that the topological transition of the highest degree is discontinuous, but also that the magnetization behaves in a way similar to a standard Ising model on a coevolving network [19]. These effects belong to different transition classes and occur at different temperatures, and we see a striking behavior in energy $E$, i.e., the value of the Hamiltonian. It exhibits multiple jumps, one of which occurs at the same temperature as the highest-degree topological phase transition. In addition, all jumps are approximately equal. This energy behavior suggests a multi-star configuration in which the maximum number of stars is restricted by a fraction $M/N$. Figure 5(a) shows that when $M = 3N$ three vertices are connected to approximately every network node. The number of stars decreases with the temperature. Eventually, the system becomes more homogeneous, and we see a sharp transition in the largest degree. Here $k_{\text{max}} \ll N$, and the degree distribution is approximately Poissonian.

When we remove the external field associated with the degree of each node and turn on the combination of structural terms in the interaction portion of the Hamiltonian, we find a different behavior. Figure 3 shows this in the phase diagram with respect to $\phi$ and $T$ when there is a neutral value of $\gamma = 1$. Figure 4 shows the same when we fix $\phi = 0.6$. When we examine the largest degree and the magnetization we see four phases. Figure 4 shows that in the one-dimensional phase diagram the topological transition is characterized by a sharp jump in the maximum degree. Thus there is an abrupt change in the magnetization. Unlike the case with a varying $\gamma$, there is also a transition that is triggered by a change in parameter $\phi$, and that is unaffected by temperature. The critical value of $\phi$ in this transition is noted $\phi_c$.

Examining the structural properties of the different phases mentioned above, we find that when $\gamma = 1$ and $\phi < \phi_c$ in a low-temperature regime there are many disconnected nodes and one big component with high-degree clustering [see Fig. 5(b)]. At the critical temperature, the network recombines into one component, and the highest-degree $k_{\text{max}}$ reaches its maximum value at the transition point. Increasing the temperature decreases the highest degree and the degree distribution to become Poissonian. When we increase $\phi$ above $\phi_c$, the system transitions into a multi-star configuration, a phase similar to the one previously observed for high $\gamma$ values.

We next analytically describe the system to produce an approximation of our numerical results. Because the structural heterogeneity disallows a simple mean-field approach, we use a semi-mean-field method and focus on the non-homogeneous elements of the system that most
FIG. 5: (Color online) Exemplary networks obtained in the simulation. Green nodes indicate +1 spin, red nodes indicate −1 spin.

strongly impact the Hamiltonian.

Figures 1 and 2 show the results when \( \phi = 0 \) and the \( \gamma \) value varies. We here assume that the most important part of the Hamiltonian is the contribution from the largest hubs, i.e., the stars connected to all other nodes. We assume non-hubs to have a degree equal to the average degree. Thus we approximate the Hamiltonian

\[
H \approx -n_h k_m^\gamma - (N - n_h) k_a^\gamma, \tag{4}
\]

where \( n_h \in \{0, 1, 2, \ldots, \left\lfloor \frac{M}{N} \right\rfloor \equiv |c| \} \) is the number of stars, \( k_m = N - 1 \) is the degree of each star, \( k_a \) is the average degree of the remaining nodes, and \( k_a \equiv k_a(n_h) = n_h + \frac{2(M - L(n_h))}{N - n_h} \), where \( L(n_h) = k_m + k_m - 1 + \ldots + k_m - (n_h - 1) = n_h \frac{2N - n_h - 2}{2} \) is the number of links required to create \( n_h \) stars. Taking into account all possible configurations, the partition function becomes

\[
Z = 2^N \sum_{n_h=0}^{\left\lfloor \frac{M}{N} \right\rfloor} \binom{N}{n_h} R(n_h)e^{\beta n_h k_m + (N - n_h) k_a^\gamma}, \tag{5}
\]

where \( 2^N \) is all spin configurations, \( \binom{N}{n_h} \) is the number of star combinations, and \( R(n_h) \) is the number of possible link configurations with \( n_h \) stars in the network. We approximate this as \( R(n_h) \approx \binom{N - n_h}{M - L(n_h)}(N - n_h - 1)/2 \). Although this is a slight over-counting when \( n_h < c \), the number of incorrect configurations is negligible when compared to the number of all other configurations.

Figures 1 and 2 show that the partition function allows us to analytically determine the energy and the highest degree. Although the estimated critical temperature diverges from the observed temperature, using this simple approach allows us to recreate the step-like behavior of the energy.

When \( \gamma = 1 \) and the \( \phi < \phi_c \) value varies, there is a shattering transition with a decreasing temperature. Some nodes disconnect from other nodes and become inactive. In contrast, when the temperature is high, and \( c = 3 \), the graph becomes random and highly connected. Thus we describe the state of the system in terms of the number of active nodes, and we assume that their degree can be approximated using the mean field approach. We denote the number of these nodes \( n_s \) and write the Hamiltonian

\[
H \approx -M \langle k_s^\gamma \rangle \phi^\gamma, \tag{6}
\]

where \( \langle k_s \rangle = 2M/n_s \) and \( \left\lfloor n_{\text{min}} \right\rfloor \leq n_s \leq N \) with \( n_{\text{min}} = (1 + \sqrt{1 + 8M})/2 \). We approximate the number of configurations for a particular \( n_s \) with \( \binom{N}{n_s} \left( \frac{n_s(n_s-1)}{M} \right) \) and derive the partition function

\[
Z = \sum_{n_s=\left\lfloor n_{\text{min}} \right\rfloor}^{N} 2^{N-n_s+1} \binom{N}{n_s} \times \left( \frac{n_s(n_s-1)}{2M} \right) e^{\beta 2^\phi N^\phi M^{\phi+1} n_s^{-2\phi}}. \tag{7}
\]

We assume that the spin direction of all active nodes is the same and that there are \( 2^{N-n_s+1} \) possible spin configurations. Figures 3 and 4 show the results when we analytically obtain the energy and the highest degree level. As in the previous case, we can use our estimation to approximate the system behavior, but not the critical temperature.

We approximate \( \phi_c \) to fully describe the phase diagrams. The critical value of \( \phi \) separates the homogeneous active node phase from the multi-star configuration phase. We assume that the energy of both phases is the same when \( \phi = \phi_c \) and define the critical value

\[
\phi_c \ln \left( \frac{(n_{\text{min}} - 1)^2}{N - 1} \right) + \ln \frac{M}{c} = \ln \left( c - 1 \right) \frac{(N - 1)^{\phi_c}}{2} + c^{\phi_c}(N - c), \tag{8}
\]

where both \( c \) and \( n_{\text{min}} \) retain the previous definitions. For the complete calculation of all cases see the Supplemental Material [23].

To statistically describe a coevolving spin system we have used a Hamiltonian that merges exponential random graphs and Ising-like models. A Hamiltonian that simultaneously depends on topological properties and node states has not been previously analyzed, and we have found complex behavior and have generated rich phase diagrams. The most striking aspect of our results is the existence, at specific temperatures, of topological phase transitions in which there are no node
state transitions. There are also transitions that influence order parameters, but this suggests that we must take into account both the topology and the state of the nodes to fully describe the system, and if we do not we miss essential aspects of systemic behavior.

Although the results presented here concern networks in which \( c = \frac{M}{N} = 3 \), we did extend our simulations to other cases and found the same qualitative results. These extended results and a detailed analysis of the asymptotic and topological properties of the transitions will be supplied in a future publication [24].

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I. ANALYTICAL APPROXIMATION

In the following we describe in detail our approach to approximating the behavior of the coevolving spin model. The general form of the Hamiltonian is

$$H(\{c_{ij}\}, \{s_i\}) = -\sum_{i<j} c_{ij} (\langle k \rangle) \phi s_i s_j - \sum_i k_i^\gamma - h \sum_i s_i,$$

where $s_i$ is the spin of node $i$ and $\langle k \rangle = \frac{2M}{N} = 2c$ is the average degree of a network with $N$ nodes and $M$ edges, described by the adjacency matrix $c_{ij}$. We set $h = 0$, and the model has two parameters $\gamma$ and $\phi$. To calculate the energy and the maximum degree, which we use as a topological order parameter, we calculate the partition function

$$Z = \sum_{\{c_{ij}\}, \{s_i\}} e^{-\beta H(\{c_{ij}\}, \{s_i\})}$$

by differentiating the partition function

$$\frac{\partial Z}{\partial \beta} = - \sum_{\{c_{ij}\}, \{s_i\}} H(\{c_{ij}\}, \{s_i\}) e^{-\beta H(\{c_{ij}\}, \{s_i\})}.$$

We obtain a formula for the energy of the system

$$-\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial \beta} = E.$$

In the main text we focus on two cases, (i) when $\phi = 0$ and $\gamma$ can have any positive value, and (ii) when $\gamma = 1$, which is equivalent to $\gamma = 0$, and a positive $\phi$ varies. In both cases we derive analytical approximations, which we describe below.

Note that when $\phi = 0$ and $\gamma = 0$ (or 1) we obtain a standard Ising-like Hamiltonian with a coevolving network (see Ref. [1]). Note also that when $\phi = 0$ and $\gamma = 2$ we obtain a form of the Hamiltonian similar to that analyzed in Refs. [2, 3], although those references do not consider node states.

A. First case: $\phi = 0$

When $\phi = 0$ we simplify the Hamiltonian to be

$$H(\{c_{ij}\}, \{s_i\}) = -\sum_{i<j} c_{ij} s_i s_j - \sum_i k_i^\gamma.$$

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Because the network can be strongly heterogeneous and the contribution of highly connected nodes non-negligible, we cannot use a simple mean-field approach, and we assume that the Hamiltonian is dominated by the largest hubs with connections to all other nodes [see Fig. 1(a)]. We also assume that all the other vertices have the same degree \( k_a \), and we neglect any effect of spins. This gives us a simplified Hamiltonian

\[
H \approx -n_h k_m^\gamma - (N - n_h)k_a^\gamma,
\]

where \( n_h \) is the number of hubs with the maximum possible degree \( k_m = N - 1 \). Note that \( n_h \leq \lfloor c \rfloor = \lfloor M/N \rfloor \). We explain the derivation of the average degree \( k_a \) for non-star vertices below. The hubs absorb a number of connections equal to

\[
L(n_h) = k_m + k_m - 1 + \ldots + k_m - (n_h - 1) = n_h \frac{k_m + k_m - (n_h - 1)}{2} = n_h \frac{2k_m + 1 - n_h}{2} = n_h \frac{2N - 1 - n_h}{2}.
\]

The remaining connections are distributed among non-star nodes, giving them an average degree \( k_a \equiv k_a(n_h) = n_h + \frac{2(M - L(n_h))}{N - n_h} \). We then approximate the number of connection configurations among non-star vertices as a function of the number of hubs using

\[
R(n_h) = \left( \frac{(N - n_h)(N - n_h - 1)/2}{M - L(n_h)} \right)^{\frac{N}{2}}.
\]

With the exception of when \( n_h = \lfloor c \rfloor \), this approximation overestimates the number of configurations for all other \( n_h \) values, but these overestimations are negligible. Merging all of these elements, the partition function is now

\[
Z = 2^N \sum_{n_h=0}^{c} \binom{N}{n_h} R(n_h)e^{\beta[n_h k_m^\gamma + (N - n_h)k_a^\gamma]},
\]

where \( 2^N \) is the number of spin configurations. Using Eq. (4) we calculate the energy,

\[
E = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{2^N}{Z} \sum_{n_h=0}^{c} \binom{N}{n_h} R(n_h) [n_h k_m^\gamma + (N - n_h)k_a^\gamma] e^{\beta[n_h k_m^\gamma + (N - n_h)k_a^\gamma]} =
\]

\[
= -\frac{2^N}{Z} \sum_{n_h=0}^{c} \binom{N}{n_h} R(n_h) [n_h(N - 1)\gamma + (N - n_h)k_a^\gamma] e^{\beta[n_h(N - 1)\gamma + (N - n_h)k_a^\gamma]} =
\]

\[
= -\frac{2^N}{Z} \left[ (N - 1)\gamma \sum_{n_h=0}^{c} \binom{N}{n_h} R(n_h) n_h e^{\beta[n_h(N - 1)\gamma + (N - n_h)k_a^\gamma]} + \sum_{n_h=0}^{c} \binom{N}{n_h} R(n_h)(N - n_h)k_a^\gamma e^{\beta[n_h(N - 1)\gamma + (N - n_h)k_a^\gamma]} \right].
\]

To determine the formula for the average maximum degree we first describe its behavior for different \( n_h \) values. When \( n_h > 0 \) there is a minimum of one star, and thus the maximum network degree is \( N - 1 \). When \( n_h = 0 \) there is no star, and we assume the network to be random. Because a random network has an approximate Poisson degree distribution, the maximum degree \( \bar{k}(n_h) \) is

\[
\bar{k}(n_h) = \begin{cases} 
N - 1 & \text{for } n_h > 0, \\
F_{\text{Poisson},\lambda}(1 - \frac{1}{N}) & \text{for } n_h = 0.
\end{cases}
\]

(11)
where \( F_{\text{Poisson}, \lambda}^{-1} \) is the inverse of a Poisson cumulative distribution function with a parameter \( \lambda = \langle k \rangle = 2M/N \). This yields the formula for the average maximum degree

\[
\langle k_{\max} \rangle = \frac{2N}{Z} \sum_{n_h=0}^{\lfloor c \rfloor} R(n_h)k(n_h)e^{\beta[n_h k_a + (N-n_h)k_\alpha]} =
\]

\[
= \frac{2N}{Z} \left\{ R(0) \cdot F_{\text{Poisson}, \lambda}^{-1} \left[ 1 - \frac{1}{N} \right] e^{\beta N(2M/N)^\gamma} + (N-1) \sum_{n_h=1}^{\lfloor c \rfloor} R(n_h)e^{\beta[n_h (N-1)^\gamma + (N-n_h)k_\alpha]} \right\},
\]

(12)

where \( k_\alpha \) is also a function of \( n_h \), as defined above.

**B. Second case: \( \gamma = 1 \)**

In the second case the Hamiltonian is equal to

\[
H(\{c_{ij}\}, \{s_i\}) = -\sum_{i<j} c_{ij} \left( \frac{k_i k_j}{\langle k \rangle} \right)^\phi s_i s_j.
\]

(13)

We here approximate two-phase diagram effects, (i) the transition between the ordered phase of a shattered network with one strongly connected subgraph that contains all the edges and the disordered phase of a random network, and (ii) the transition that occurs when \( \phi = \phi_c \) in the low-temperature regime. Here the network transitions from an ordered structure to a structure with a few hubs (stars) that absorb most of the connections, similar to the ordered phase described in subsection IA.

1. **Active component approximation**

Figure 1(b) shows an approximation that allows us to describe analytically the transition from a shattered network with one active, highly connected component to a random graph. This approach observes systemic behavior at different temperatures. Increasing the temperature above zero causes the active component to attract disconnected inactive nodes. Further increasing the temperature increases the size of the active component, which can continue to grow until it encompasses all nodes and becomes a random graph. Because we assume that every active component size can be approximated
by a random graph with a given average degree, we use a mean-field description limited to the active
nodes. This gives us the Hamiltonian

\[ H \approx -M \frac{k_s^{2\phi}}{\langle k \rangle^{\phi}}, \tag{14} \]

where \( \langle k \rangle_s \) is the average degree of the active component. When \( n_s \) is the number of component
nodes, \( n_s \in \left\{ \lceil n_{\min} \rceil, \lceil n_{\min} \rceil + 1, \ldots, N \right\} \), where \( n_{\min} = (1 + \sqrt{1 + 8M})/2 \), because all connections must
fit in the active component, and the limit is a complete subgraph with \( n_s(n_s - 1)/2 \) links. Because
\( \langle k \rangle_s = 2M/n_s \), we get

\[ H \approx -2^\phi N^\phi M^{\phi + 1} n_s^{-2\phi}. \tag{15} \]

Note that as \( n_s \) grows, the term “component” becomes misleading. In graph theory “component” is
used to describe a subset of nodes connected to each other. When \( n_s \) becomes sufficiently large the
active component can break into two or more separate components. We use the term active component
to mean a set of active nodes able to obtain connections, in contrast to inactive nodes.

The number of configurations of connections for a given size of the active component is equal to

\[ R(n_s) = \left( \frac{n_s(n_s - 1)}{2M} \right). \tag{16} \]

The partition function is then

\[ Z = \sum_{n_s = \lceil n_{\min} \rceil}^{N} 2^{N-n_s+1} \binom{N}{n_s} R(n_s) e^{\beta 2^\phi N^\phi M^{\phi + 1} n_s^{-2\phi}}. \tag{17} \]

Finally we calculate the energy using Eq. (4),

\[ E = -\frac{1}{Z} \frac{1}{n_s = \lceil n_{\min} \rceil} \sum_{n_s = \lceil n_{\min} \rceil}^{N} 2^{\phi+N-n_s+1} N^\phi M^{\phi + 1} n_s^{-2\phi} \left( \frac{n_s(n_s - 1)}{2M} \right) e^{\beta 2^\phi N^\phi M^{\phi + 1} n_s^{-2\phi}}. \tag{18} \]

Because we assume the active component is a random graph, we can approximate its maximum degree
in the same way as in subsection IA for a zero-star configuration, but note that the maximum degree
of the components is limited by \( n_s - 1 \). This is particularly important when \( n_s \) is small and the active
component dense. To avoid overestimating the maximum degree, we write its dependence on \( n_s \)

\[ \bar{k}(n_s) = \min \left\{ n_s - 1, F_{\text{ass},\lambda_s}^{-1} \left( 1 - \frac{1}{n_s} \right) \right\}, \tag{19} \]

where \( \lambda_s = \langle k \rangle_s = 2M/n_s \). We use this formula to obtain the average maximum degree

\[ \langle k_{\text{max}} \rangle = \frac{2^N}{Z} \sum_{n_s = \lceil n_{\min} \rceil}^{N} \bar{k}(n_s) 2^{N-n_s+1} \binom{N}{n_s} \left( \frac{n_s(n_s - 1)}{2M} \right) e^{\beta 2^\phi N^\phi M^{\phi + 1} n_s^{-2\phi}}. \tag{20} \]

This approximation also neglects the impact of spins. This is reasonable inside the two marginal
phases, but inappropriate between them.

2. Critical value \( \phi_c \)

We now estimate the critical line \( \phi_c \) that at low temperatures separates the model into two phases.
The phase for \( \phi < \phi_c \) has a fully connected cluster of size \( n \) and many single disconnected nodes. The
real size of \( n \) is the lower bound of \( n_s \) introduced above, and we approximate it using a continuous
number \( n \approx n_{min} \equiv (1 + \sqrt{1 + 8M})/2 \). We here assume that all nodes in the cluster are connected and all have the same degree \( n - 1 \). The energy of the system is then

\[
E_{\phi<\phi_c} = -\sum_{i=1}^{n(n-1)} \left( \frac{2M}{N} \right)^{-\phi} (n-1)^{2\phi} = -n(n-1) \left( \frac{2M}{N} \right)^{-\phi} (n-1)^{2\phi} = -M \left( \frac{2M}{N} \right)^{-\phi} (n-1)^{2\phi}. 
\]

(21)

Nodes separated from the active component are zero degree and do not contribute to the energy.

When \( \phi > \phi_c \) there are dominant hubs of maximum degree \( N - 1 \). We approximate the energy of the system by taking into account interactions between \( n_h \) stars and between stars and all other \( N - n_h \) nodes. We omit interactions between non-star nodes, where the degree multiplication is significantly smaller. We also assume non-star nodes have the same average degree. Finally we get

\[
E_{\phi>\phi_c} = -\sum_{i=1}^{n_h(n_h-1)} \left( \frac{2M}{N} \right)^{-\phi} (N-1)^{2\phi} - n_h(N-n_h) \left( \frac{2M}{N} \right)^{-\phi} (N-1)^{2\phi} n_h^\phi = \]

\[
= -n_h(N-1)^\phi \left[ \frac{n_h-1}{2} (N-1)^{\phi} + n_h^{\phi} (N-n_h) \right], 
\]

(22)

where \( n_h \) is the number of hubs. When \( T \to 0 \) we have \( n_h \approx 2M/N = c \) we assume it is continuous. At the critical line both energies (21) and (22) are equal

\[
c(N-1)^\phi_c \left( \frac{2M}{N} \right)^{-\phi_c} \left[ \frac{c-1}{2} (N-1)^{\phi_c} + c^{\phi_c} (N-c) \right] = M \left( \frac{2M}{N} \right)^{-\phi_c} (n-1)^{2\phi_c}, 
\]

\[
\frac{c-1}{2} (N-1)^{\phi_c} + c^{\phi_c} (N-c) = \frac{M}{c} \left( \frac{(N-1)^2}{N-1} \right)^{\phi_c}, 
\]

\[
\ln \left( \frac{c-1}{2} (N-1)^{\phi_c} + c^{\phi_c} (N-c) \right) = \phi_c \ln \left( \frac{(N-1)^2}{N-1} \right) + \ln \frac{M}{c}. 
\]

(23)

This allows us to numerically approximate \( \phi_c \).

II. SIMULATION

We use the Metropolis algorithm [4–6] to perform Monte Carlo simulations of the model. We begin every simulation with a random graph and a random spin configuration. Before collecting data about the system, we run the simulation for a given thermalization time determined by network size and the parameter values. We calculate the thermalization by observing the time evolution of order parameters and find it equal to the time needed to reach a stationary state with values fluctuating around the equilibrium. We then collect the order parameter values for a number of time steps.

Every time step of the simulation uses two basic mechanisms, (i) spin switching and (ii) edge rewiring. In spin switching, we randomly select one node and compute the energy difference between the current state of the system and a chosen spin in the opposite state. Note that here the energy difference is a function of the selected spin and its neighbors. Because here the network structure is constant, we take into account only the first Hamiltonian sum (1). We use the standard Metropolis rule to decide whether to flip the spin.

When we perform edge rewiring at the same time step, we randomly select one network link and two nodes to serve as possible ends of a new edge. Note that the new edge cannot overlap with any existing edges. We next calculate the energy difference between the new configuration (with the new link and without the old link) and the old configuration. Here the energy difference depends on all four (or in some rare cases three) nodes connected by the old and new connections. The difference
is determined by the neighboring nodes. Also, both the pure topological and mixed terms of the Hamiltonian are influenced. We again use the Metropolis algorithm to decide whether to rewire the edge.

We use a 1 : 1 ratio between the spin switching and the link rewiring to minimize thermalization times. We also perform simulations using other ratios. We find that only difference is in the time necessary to reach the stationary state. When ratios are extreme, the thermalization times become infinite. In pure spin switching we find a constant network structure, and in pure link rewiring the magnetization does not change. As long as the ratio is finite, the simulation trajectories cover the entire configuration space, and the equilibrium states remain the same.

To better understand and to easily reproduce and extend our results, we present a pseudo-code of the main loop of our simulation. It supplies the procedures performed in every time step.

```python
graph = random_Erdos_Rényi_graph(n, m)  # generate random graph
for i in [0, ..., n-1]:
    graph[i][spin] = random_choice(-1, +1)  # assign random spin to every node
for step in time_steps:
    v_index = rand_int(n)  # chose one of n nodes
    e_index = rand_int(m)  # chose one of m edges

    # spin switching
    neighbors = graph.get_neighbors(v_index)
    delta = graph.energy_change_spin(v_index, neighbors)  # compute energy difference
    if delta <= 0 or rand() < exp(- delta / T):
        graph[i][spin] = graph[i][spin] * -1  # flip spin according to Metropolis rule

    # edge rewiring
    old_from, old_to = graph.get_edge_ends(e_index)
    new_from, new_to = graph.draw_new_edge(exclude_existing=true)
    delta = graph.energy_change_edge(old_from, old_to, new_from, new_to)  # compute energy difference
    if delta <= 0 or rand() < exp(- delta / T):
        graph.delete_edge([v_from, v_to])  # rewire edge according to Metropolis rule
        graph.add_edge([new_from, new_to])
```

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