Derivation of the Euler Equations from Many-body Quantum Mechanics∗

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(Dedicated to André Verbeure on the occasion of his sixtieth birthday)

Abstract

The Heisenberg dynamics of the energy, momentum, and particle densities for fermions with short-range pair interactions is shown to converge to the compressible Euler equations in the hydrodynamic limit. The pressure function is given by the standard formula from quantum statistical mechanics with the two-body potential under consideration. Our derivation is based on a quantum version of the entropy method and a suitable quantum virial theorem. No intermediate description, such as a Boltzmann equation or semiclassical approximation, is used in our proof. We require some technical conditions on the dynamics, which can be considered as interesting open problems in their own right.

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1. Introduction

The fundamental laws of non-relativistic microscopic physics are Newton’s and the Schrödinger equation in the classical and the quantum case respectively. These equations are impossible to solve for large systems and macroscopic dynamics is therefore modeled by phenomenological equations such as the Euler or the Navier-Stokes equations. Although the latter were derived centuries ago from continuum considerations, they are in principle consequences of the microscopic physical laws and should be viewed as secondary equations. It was first observed by Morrey ‡ in

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the fifties that the Euler equations become ‘exact’ in the Euler limit, provided that the solutions to the Newton’s equation are ‘locally’ in equilibrium. Morrey’s original work was far from rigorous and, in particular, the meaning of ‘local equilibrium’ was not clear. It is nevertheless a very original work which led to the later development of the hydrodynamical limits of interacting particle systems. In terms of a rigorous proof along the lines of Morrey’s original argument, however, there has been little progress until the recent work \cite{11}. This long delay is mostly due to a serious lack of tools for analyzing many-body dynamics, in the classical case and even more so in the quantum case.

In this lecture, we will discuss the derivation of the Euler equations from microscopic quantum dynamics. As we want to consider the genuine quantum dynamics for a system with short-range pair interactions, we cannot take a semiclassical limit. Although one-particle quantum dynamics converges to Newtonian dynamics in the limit of infinite mass, this is not the case in the thermodynamic limit, i.e., the heavy-particle limit does not commute with the infinite-number-of-particles limit. This is most clearly seen in the pressure function, for which quantum corrections survive at the macroscopic scale. In fact, one of the conclusions of our work is that under rather general conditions, the pressure function is the only place where the quantum nature of the underlying system, in particular the particle statistics, survives in the Euler limit. At the same time, our derivations also shows that it is the quantum mechanical pressure, without modification, which governs the macroscopic dynamics. A similar result should hold for all systems of macroscopic conservation laws.

The Euler equations have traditionally been derived from the Boltzmann equation both in the classical case and in the quantum case, see Kadanoff and Baym \cite{4} for the quantum case. Since the Boltzmann equation is valid only in very low density regions, these derivations are not satisfactory, especially in the quantum case where the relationship between the quantum dynamics and the Boltzmann equation is not entirely clear. There were, however, two approaches based directly on quantum dynamics. The first was due to Born and Green \cite{1}, who used an early version of what was later called the BBGKY hierarchy, together with moment methods and some truncation assumptions. A bit later, Irving and Zwanzig \cite{3} used the Wigner equation, moment methods and truncations to accomplish a similar result. These two approaches rely essentially on the moment method with the Boltzmann equation replaced by the Schrödinger equation. Unlike in the Boltzmann case, where one can do asymptotic analysis to justify this approach, it seems unlikely that this can be done for the Schrödinger dynamics. Therefore, in the present work, we follow a much more direct route to connect the micro- and macroscopic dynamics.

2. Schrödinger and Euler dynamics

We begin by considering $N$ particles on $\mathbb{R}^3$, evolving according to the Schrödinger equation

$$i\partial_t \psi_t(x_1, \cdots, x_N) = H \psi_t(x_1, \cdots, x_N)$$
where the Hamiltonian is given by

\[ H = \sum_{j=1}^{N} -\frac{\Delta_j}{2} + \sum_{1 \leq i < j \leq N} W(x_i - x_j). \tag{2.1} \]

Here, \( W \) is a two-body short-ranged super-stable isotropic pair interaction and \( \psi_t(x_1, \cdots, x_N) \) is the wave function of particles at time \( t \). We consider spinless Fermions and thus the state space \( \mathcal{H}_N \) is the subspace of antisymmetric functions in \( L^2(\mathbb{R}^3)^N \). It is convenient not to fix the total number of particles and to use the second quantization terminology. In fact, it would be extremely cumbersome to work through all arguments without the second quantization formalism. The state space of the particles, called the Fermion Fock space, is thus the direct sum of \( \mathcal{H}_N \):

\[ \mathcal{H} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N. \]

Define the annihilation and creation operators \( a_x \) and \( a_x^+ \) by

\[
(a_x^+ \Psi)^N(x_1, \cdots, x_N) = \sqrt{N+1} \Psi^{N+1}(x_1, \cdots, x_N), \\
(a_x \Psi)^N(x_1, \cdots, x_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (-1)^{j-1} \delta(x - x_j) \Psi^{N-1}(x_1, \cdots, \hat{x}_j, \cdots, x_N),
\]

where \( a_x \) and \( a_x^+ \) are operator-valued distributions and, as usual, \( \hat{\cdot} \) means “omit”.

The annihilation operator \( a_x \) is the adjoint of \( a_x^+ \) with respect to the standard inner product of the Fock space with Lebesgue measure \( dx \), and

\[ [a_x, a_y^+] := a_x a_y^+ + a_y a_x^+ = \delta(x - y), \]

where \( \delta \) is the delta distribution. The derivatives of these distributions with respect to the parameter \( x \) are denoted by \( \nabla a_x \) and \( \nabla a_x^+ \). With this notation, we can express the Hamiltonian as \( H = H_0 + V \) where \( H_0 = \frac{1}{2} \int \nabla a_x^+ \nabla a_x \, dx \) and \( V = \frac{1}{2} \int dxdy W(x - y) a_x^+ a_y^+ a_x a_x \). It is more convenient to put the Schrödinger equation into the operator form, which is sometimes called the Schrödinger-Liouville equation. Denote the density matrix of the state at time \( t \) by \( \gamma_t \). Only normal states, which can be represented by density matrices, will be considered in the time evolution. Then the Schrödinger equation is equivalent to \( i\partial_t \gamma_t = \delta_{H \gamma_t} \), with \( \delta_{H \gamma_t} := [H, \gamma_t] \). The conserved quantities of the dynamics are the number of particles, the three components of the momentum and the energy. The local densities of these quantities are denoted by \( u = (u^\mu), \mu = 0, \cdots, 4 \), and are given by the following expressions:

\[
u^0_x = n_x = a_x^+ a_x, \\
u^j_x = p^j_x = \frac{-i}{2} [\nabla_j a_x^+ a_x - a_x^+ \nabla_j a_x], \quad j = 1, 2, 3, \tag{2.2} \\
u^4_x = h_x = \frac{1}{2} \nabla a_x^+ \nabla a_x + \frac{1}{2} \int dy W(x - y) a_x^+ a_y^+ a_y a_x.
\]

The finite volumes, denoted by \( \Lambda \), will always by three-dimensional tori and, unless otherwise stated, unbounded observables on \( \Lambda \) will be defined with periodic boundary conditions. E.g., the number of particles in \( \Lambda \), the total momentum, and the
total energy of the particles in $\Lambda$, respectively, are defined by

$$N_\Lambda = \int_\Lambda dx \, n_x, \quad P^j_\Lambda = \int_\Lambda dx \, p^j_x, \quad j = 1, 2, 3, \quad H_\Lambda = \int_\Lambda dx \, h_x.$$  

We slightly generalize the definition of the grand canonical Gibbs states to include a parameter for the total momentum of the system: the Lagrange multiplier $\alpha \in \mathbb{R}^3$. We will work under the assumption that the temperature and chemical potential are in the one-phase region of the phase diagram of the system under consideration, such that the thermodynamic limit is unique. The infinite volume Gibbs states are then given by the following formula:

$$\omega_{\beta, \alpha, \mu}(X) = \lim_{\Lambda \to \mathbb{R}^d} \frac{\text{Tr} \, e^{-\beta (H_0 + V_\Lambda - \alpha \cdot P_\Lambda - \mu N_\Lambda)}}{\text{Tr} \, e^{-\beta (H_0 + V_\Lambda - \alpha \cdot P_\Lambda - \mu N_\Lambda)}}. \quad (2.3)$$

It is convenient to denote the parameters $(\beta, \alpha, \mu)$ by $\lambda = (\lambda^0, \lambda^1, \lambda^2, \lambda^3)$ with $\lambda^0 = \beta \mu$, $\lambda^1 = \beta \alpha^1$, $\lambda^2 = \beta \alpha^2$, $\lambda^3 = \beta \alpha^3$. Define (notice the sign convention)

$$\lambda \cdot u = \sum_{\mu=0}^3 \lambda^\mu \, u^\mu - \lambda^4 \, u^4 \quad \text{and} \quad \langle \lambda, u \rangle_\Lambda = |\Lambda|^{-1} \int_\Lambda dx \lambda(x) \cdot u(x).$$

These notations allow us to give a compact formula for the unique, translation invariant Gibbs state (defined with constant $\lambda$), as well as for the states describing local equilibrium (defined with $x$-dependent $\lambda$):

$$\omega_\lambda = \lim_{\Lambda \to \mathbb{R}^d} e^{\langle \lambda, u \rangle_\Lambda} / Z_\Lambda(\lambda) \quad (2.4)$$

where $Z_\Lambda(\lambda)$ is the partition function given by $Z_\Lambda(\lambda) = \text{Tr} \, e^{\langle \lambda, u \rangle_\Lambda}$. If we define the pressure, as a function of the constant vector $\lambda$, by $\psi(\lambda) = \lim_{L \to \infty} |\Lambda|^{-1} \log Z_\Lambda(\lambda)$, then

$$\frac{\partial \psi}{\partial \lambda^\mu} = \omega_\lambda(u^\mu). \quad (2.5)$$

As the states $\omega_\lambda$ are translation invariant, we have $\omega_\lambda(u^\mu) = q^\mu$. Explicitly,

$$\rho = \omega_\lambda(n_x), \quad q = \omega_\lambda(p_x), \quad e = \omega_\lambda(h_x). \quad (2.6)$$

Notice that $q$ and $e$ are momentum and energy per volume.

Again, we will work under the assumption that these parameters stay in the one-phase region, the limiting Gibbs state is unique and these definitions are unambiguous. Although momentum is preferable as a quantum observable, we also introduce the velocity in order to be able to compare with the classical case. The velocity field $v(x)$ has to be defined as a mean velocity of the particles in a neighborhood of $x$. Therefore we have $v(x) = q(x)/\rho(x)$. We also introduce the energy per particle defined by $\tilde{e} = e/\rho$. The usual Euler equations are written in terms of $\rho, v,$ and $\tilde{e}$.

In order to derive the Euler equations, we need to perform a rescaling. So we shall put all particles in a torus $\Lambda_\varepsilon$ of size $\varepsilon^{-1}$ and use $(X, T) = (\varepsilon x, \varepsilon t)$ to denote
the macroscopic coordinates. For all equations in this paper periodic boundary conditions are implicitly understood.

The Euler equations are given by

\[
\frac{\partial \rho}{\partial T} + \sum_{j=1}^{3} \frac{\partial}{\partial X_j} (\rho v_j) = 0, \\
\frac{\partial (\rho v_k)}{\partial T} + \sum_{j=1}^{3} \frac{\partial}{\partial X_j} [\rho v_j v_k] + \frac{\partial}{\partial X_k} P(e, \rho) = 0, \\
\frac{\partial (\rho \tilde{e})}{\partial T} + \sum_{j=1}^{3} \frac{\partial}{\partial X_j} [\rho \tilde{e} v_j + v_j P(e, \rho)] = 0.
\]

These equations are in form identical to the classical ones but all physical quantities are computed quantum mechanically. In particular, \( P(e, \rho) \) is the thermodynamic pressure computed from quantum statistical mechanics for the microscopic system. It is a function of \( X \) and \( T \) only through its dependence on \( e \) and \( \rho \). If no velocity dependent forces act between the molecules of the fluid under consideration (we consider only a pair potential), the pressure is independent of the velocity.

Let \( q = (q^0, \cdots, q^4) \), related to density, momenta and energy as follows:

\[
q^0 = \rho, \quad q^i = \rho v^i, \quad q^4 = e = \rho \tilde{e}.
\]

In other words \( q^1, q^2, q^3 \), and \( q^4 \) are momenta and energy per \textit{volume} instead of per \textit{particle} as in the usual Euler equation (2.6). We rewrite the Euler equations in the following form

\[
\frac{\partial q^k}{\partial T} + \sum_{i=1}^{3} \nabla_i X [A^k_i(q)] = 0, \quad k = 0, 1, 2, 3, 4.
\]

The matrix \( A \) is determined by comparison with the Euler equations:

\[
A^0_j = q^j, \quad A^i_j = \delta_{ij} P + q_0 q_j / q_0, \quad A^4_j = q^j (q^4 + P) / q_0.
\]

3. Local equilibrium

To proceed we need a microscopic description of local equilibrium. Suppose we are given macroscopic functions \( q(X) \). We wish to find a local Gibbs state with the conserved quantities given by \( q(X) \). The local Gibbs states are states locally in equilibrium. In other words, in a microscopic neighborhood of any point \( x \in T^3 \) the state is given by a Gibbs state. More precisely, we wish to find a local Gibbs state with the expected values of the energy, momentum, and particle number per unit volume at \( X \) given by \( q(X) \). To achieve this, we only have to adjust the parameter \( \lambda \) at every point \( X \). More precisely, we choose \( \lambda(X) \) such that the equation holds at every point, i.e.,

\[
\frac{\partial \Psi(\lambda(X))}{\partial \lambda^\mu(X)} = q^\mu(X).
\]
If we denote the solution to the Euler equation by \( q(X,T) \), then we can choose in a similar way a local Gibbs state with given conserved quantities at the time \( T \).

Define the local Gibbs state
\[
\omega_\epsilon^t = \frac{1}{c_\epsilon(t)} \exp \left[ \epsilon^{-3}(\lambda(\epsilon t, \epsilon \cdot), u)_{\epsilon^{-1}} \right]
\]
where \( c_\epsilon(t) \) is the normalization constant. Clearly, we have that \( \omega_\epsilon(t)(u^\mu) = q^\mu(\epsilon x, \epsilon t) \) to leading order in \( \epsilon \).

In summary, the goal is to show that, in the limit \( \epsilon \to 0 \), the following diagram commutes:

\[
\begin{array}{ccc}
q(X,0) & \xrightarrow{\text{Euler}} & q(X,T) \\
\downarrow \text{local equilibrium} & & \downarrow \text{limit } \epsilon \downarrow 0 \text{ of expectation} \\
\gamma_0 & \xrightarrow{\text{Schrödinger}} & \gamma_{\epsilon^{-1}T}
\end{array}
\]

As smooth solutions of the Euler equations are guaranteed to exist only up to a finite time \( T_0 \), say \( T_0 \), we will formulate our assumptions on the dynamics of the microscopic system for a finite time interval as well, say \( t \in [0, T_0/\epsilon] \). Note the cutoff assumptions below would hold automatically for lattice models.

4. Assumptions and the main theorem

Our main result is stated in Theorem 4.1 below. First, we state the assumptions of the theorem with some brief comments. There are three kinds of assumptions.

The first category of assumptions could be called \textit{physical} assumptions on the solution of the Euler equations that we would like to obtain as a scaling limit of the underlying dynamics, and on the pair interaction potential of this system.

I. \textbf{One-phase regime:} We assume that the pair potential, \( W \), is \( C^1 \) and with support contained in a ball of radius \( R \). Moreover, we assume that \( W \) is symmetric under reflections of each of the coordinate axes (e.g., rotation symmetric potentials automatically have this symmetry), and has the usual stability property [12]: there is a constant \( B \geq 0 \) such that, for all \( N \geq 2 \), \( x_1, \ldots, x_N \in \mathbb{R}^3 \),

\[
\sum_{1 \leq i < j \leq N} W(x_i - x_j) \geq -BN.
\]

Of the Fermion system with potential \( W \) we assume that there is an open region \( D \subset \mathbb{R}^2 \), which we will call the \textit{one-phase region}, such that the system has a unique limiting Gibbs state for all values of particle density and energy density \( (\rho, e) \in D \).

The solution of the Euler equations we consider, \( q(X,T) \), will be assumed to \( C^1 \) in \( X \) for \( T \in [0, T_0] \), and have local particle and energy density in the one-phase region for all times \( T \in [0, T_0] \). I.e., \( (\rho(X,T), e(X,T)) \in D \), for all \( X \in \Lambda_1 \) and \( T \in [0, T_0] \).
The next category of assumptions is on the local equilibrium states for the Fermion system that we construct and on their time-evolution under the Schrödinger equation. They can be considered conjectures. In fact, these assumptions have not been rigorously proved even for Gibbs states. Although one expects that these assumptions can be proved in the high-temperature and low-density region using some type of cluster expansion methods, this has only been done recently for Bosons in [7]. For the rest of this paper, we shall assume this cutoff assumptions for the solution to the Schrodinger equation that we consider, as well as for the Gibbs states in the one phase regions considered in this paper.

II. Cutoff assumptions: Suppose that \( \gamma_t \) is the solution to the Schrödinger equation with a local equilibrium state as initial condition, constructed with the parameters derived from a solution of the Euler equations (with the appropriate pressure function) that does not leave the one-phase region. We make the following two assumptions about this solution:

1. Finite velocity cutoff assumption: Let \( N_p(t) = \text{Tr} \gamma_t a_p^+ a_p \), where \( a_p^# \) is the Fourier transform of \( a_p^# \). Then there is a constant \( c > 0 \), such that for all \( t \leq T_0/\varepsilon \),

\[
\varepsilon^d \int dpe^{\varepsilon^2} N_p(t) \leq C_{T_0}.
\]

2. Non-implosion assumption: There is a constant \( C_{T_0} \), such that for all \( t \leq T_0/\varepsilon \),

\[
\text{Tr} \gamma_t \varepsilon^d \int dxn_x \left[ \int_{|x-y| \leq 2R} n_y dy \right]^2 \leq C_{T_0},
\] (4.1)

where \( R \) is the range of the interaction \( W \).

Finally, we have an assumption on the set of the time-invariant ergodic states of the Fermion system. To state this assumption we need the notion of relative entropy, of a normal state \( \gamma \) with respect to another normal state \( \omega \). Let \( \gamma \) and \( \omega \) denote the density matrices of these states. The relative entropy, \( S(\gamma \mid \omega) \), is defined by

\[
S(\gamma \mid \omega) = \begin{cases} 
\text{Tr} \gamma (\log \gamma - \log \omega) & \text{if \ ker} \omega \subset \ker \gamma, \\
+\infty & \text{else}.
\end{cases}
\]

For a pair of translation invariant locally normal states, one can show existence of the relative entropy density \([10]\), defined by the limit

\[
s(\gamma \mid \omega) = \lim_{\varepsilon \downarrow 0} \varepsilon^d S(\gamma_{x} \mid \omega_{x}),
\]

where \( \gamma_x \) and \( \omega_x \) denote the density matrices of the normal states obtained by restricting \( \gamma \) and \( \omega \) to the observables localized in \( \Lambda_x = \varepsilon^{-1} \Lambda_1 \). The existence of the limit can be proved under more general conditions on the finite volumes, but this is unimportant for us.

III. Ergodicity assumption (“Boltzmann Hypothesis”): All translation invariant ergodic stationary states to the Schrödinger equation are Gibbs states if they satisfy the following assumptions: 1) the density and energy is in one phase region. 2) The relative entropy density with respect to some Gibbs state is finite.
We expect that our assumptions hold for the solutions $\gamma_t$ of the Schrödinger equation that we employ, but it must be said that, at this moment, very little is known. We believe that these are natural conditions. To prove them under rather general conditions or even for a special class of models may be regarded as an important open problem in quantum statistical mechanics.

**Theorem 4.1** Suppose that $q(X,T)$ is a smooth solution to the Euler equation in one phase region up to time $T \leq T_0$. Let $\omega_\varepsilon^t$ be the local Gibbs state with conserved quantities given by $q(X,T)$. Suppose that the cutoff assumptions and the ergodicity assumption hold. Let $\gamma_t$ be the solution to the Schrödinger equation and $\gamma_0 = \omega_0^\varepsilon$ (Notice $\gamma_t$ depends on $\varepsilon$). Then for all $t \leq T_0/\varepsilon$ we have

$$\lim_{\varepsilon \to 0} s(\gamma_t | \omega_\varepsilon^t) = 0.$$  

In other words, $\omega_\varepsilon^t$ is a solution to the Schrödinger equation in entropy sense. In particular, for any smooth function $f$ on $\Lambda$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^3 \int_{\Lambda_\varepsilon} dx f(\varepsilon x) [\gamma_t(\varepsilon x) - q(\varepsilon t, \varepsilon x)] = 0.$$  

This theorem is a quantum analogue of the classical result by Olla, Varadhan, and Yau [11]. There are a few differences in the assumptions and the strategy followed in their paper with respect to ours. E.g., in the treatment of [11] of the classical case the cut-off assumption 1) was not needed. Instead, the usual quadratic kinetic energy was replaced by one with bounded derivatives with respect to momentum. For Fermion models on a lattice instead of in the continuum, no cut-off assumptions are required. Another difference with the treatment in [11], is that we do not add noise terms to the “native” dynamics. In [11] a weak noise term was added to the Newtonian dynamics in order to be able to prove convergence to local equilibrium. In that paper, the strength of the noise vanishes in the hydrodynamic limit. Here, we do not modify the Heisenberg dynamics in any way, but instead reduce the question of convergence to local equilibrium to the ergodicity property given in Assumption III.

5. Outline of the proof

The basic structure of our proof follows the relative entropy approach of [11] [13]. The aim is to derive a differential inequality for the relative entropy between the solution to the Schrödinger equation and a time-dependent local Gibbs state constructed to reproduce the solution of the Euler equations. The time derivative of the relative entropy can be expressed as an expectation of the local currents with respect to the solution to the Schrödinger equation: Since $\gamma_t$ is a solution to the Schrödinger equation, we have for any density matrix $\omega_\varepsilon^t$ the identity

$$\frac{d}{dt} S(\gamma_t | \omega_\varepsilon^t) = \text{Tr} \gamma_t \{ i\delta_H - \partial_t \} \log \omega_\varepsilon^t.$$  

(5.1)
Using the definition of the local equilibrium states \( \rho \), we get

\[
\frac{d}{dt}\rho(t) = \varepsilon^3 \text{Tr} \gamma \{ i\delta_H - \partial_t \} \{ \langle \lambda(t, \cdot), u \rangle - \log \varepsilon(t) \}.
\]  

(5.2)

Direct computation yields that

\[
\delta \langle \lambda(t, \cdot), u \rangle = -\varepsilon \sum_{j=1}^{3} \langle \nabla_j \lambda_j(t, \cdot), \theta_j(t) \rangle,
\]

leading order in \( \varepsilon \), and where the \( \theta_j \) are local observables for the microscopic currents of the conserved quantities. Since we do not know the solution of the Schrödinger equation, \( \gamma_t \), well enough, this expectation in the RHS of (5.2) cannot be computed or estimated explicitly. The main idea is to bound these expectations in terms of the relative entropy itself, thus obtaining a differential inequality. All these bounds are based on the following inequality, which is a consequence of the variational principle for the relative entropy \( [9, 10] \): for density matrices/states \( \omega \) and \( \gamma \), with \( \ker \omega = \{0\} \), and for all self-adjoint \( h \), and \( \delta > 0 \), one has

\[
\gamma(h) \leq \delta^{-1} \log \text{Tr} e^{\delta h + \log \omega} + \delta^{-1} S(\gamma|\omega).
\]  

(5.3)

Most of the work is then to show that the terms resulting from the first term in the RHS of this inequality are sufficiently small. This requires a number of steps. Two essential ingredients are the Euler equations (naturally), and a quantum version of Virial Theorem to relate certain terms in the currents to the thermodynamic pressure as given by quantum statistical mechanics. Next, we explain the main steps in a bit more detail.

**Step 1: Replace the local microscopic currents by macroscopic currents.** The basic idea in hydrodynamical limit is first to show that the local space time average of the solution is time invariant. From Assumption III, ergodic time invariant states are Gibbs. For Gibbs states, we can replace the local microscopic currents by macroscopic currents.

1a: Construct a commuting version of local conserved quantities. In principle, one should be able to express the macroscopic currents as functions of the conserved quantities. The local conservative quantities, however, are operators which commute only up to boundary terms. Therefore, we either need to prove that the non-commutativity does not affect the meaning of macroscopic currents or we need to construct some commuting version of the local conservative quantities. We follow the second approach and construct a commuting version of local conservative quantities with a method inspired by \( [2] \).

1b: Restriction to the one phase region. Since Assumption III can only be applied in the one phase region, we have to introduce suitable cutoff functions. As these, in general, do not commute with the local currents, this is a non-trivial step.

1c: Apply the Virial Theorem. Following the method of \( [8] \), we prove and apply a quantum virial theorem to relate the local currents to the pressure.

**Step 2: Estimate all errors by local conservative quantities.** For this step the entropy inequality (5.3) is crucial.

**Step 3: Derive a differential inequality of the entropy with error term given by a large deviation formula.** As there is no large deviation theory for non-commuting observables, it is essential that we have expressed everything by commuting objects.
The rest of the argument follows by the standard relative entropy method. The main technical difficulties, in comparison with the classical case, all stem from the non-commutativity of the algebra of observables. Simple inequalities, such as $|A + B| \leq |A| + |B|$ and $|AB| \leq |A||B|$, which are used numerous times in estimates for classical systems, are false for quantum observables. Therefore, all estimates have to be derived without taking absolute values. A full account of our work will appear shortly [14].

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