Vertex operator algebras and operads

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This paper is dedicated to the memory of Lawrence Corwin, January 20, 1943 – March 19, 1992.

Abstract

Vertex operator algebras are mathematically rigorous objects corresponding to chiral algebras in conformal field theory. Operads are mathematical devices to describe operations, that is, $n$-ary operations for all $n$ greater than or equal to 0, not just binary products. In this paper, a reformulation of the notion of vertex operator algebra in terms of operads is presented. This reformulation shows that the rich geometric structure revealed in the study of conformal field theory and the rich algebraic structure of the theory of vertex operator algebras share a precise common foundation in basic operations associated with a certain kind of (two-dimensional) “complex” geometric object, in the sense in which classical algebraic structures (groups, algebras, Lie algebras and the like) are always implicitly based on (one-dimensional) “real” geometric objects. In effect, the standard analogy between point-particle theory and string theory is being shown to manifest itself at a more fundamental mathematical level.

1 Introduction

The notion of vertex operator algebra arose in the vertex operator construction of the Monster (see [FLM1], [B1] and [FLM2]). The algebraic theory of vertex operator algebras provides deep conceptual understanding of many (but not yet all) of the phenomena of monstrous moonshine (see [CN]) and at the same time establishes a solid foundation for the algebraic aspects of conformal field theory.

Though this algebraic theory is mathematically rigorous and is very powerful in the study of issues ranging from monstrous moonshine, as in [B2], [D] and [DM], to the Batalin-Vilkovisky algebra structure and the homotopy Lie algebra structure in string theory (see [LZ]), these results still present certain mysteries. We propose that many such mysteries would dissipate with a more conceptual understanding of vertex operator algebras. Early investigations of the rich geometric structure of conformal field theory (see especially [BPZ], [FS], [Se] and [V]) already suggest that such an understanding should be related to the geometry of Riemann surfaces.
In [H1] and [H2], it has been established that the category of vertex operator algebras, in the sense of [FLM2] or [FHL], is isomorphic to a certain category — the category of geometric vertex operator algebras — defined in terms of a certain moduli space of spheres with punctures and local coordinates. Todd Trimble and Jim Stasheff commented, in connection with [H1] and [H3], respectively, that an operad-like structure appeared to be implicit in this geometric interpretation of vertex operator algebras. These comments led to the present reformulation of the notion of vertex operator algebra using the language of operads; see also our more detailed paper [HL2].

The first important example of operad-like structures occurred in Stasheff’s notion of $A_\infty$-space, in the course of a homotopy-theoretic characterization of loop spaces ([St1], [St2]). Later, May formalized the notion of operad [M]. But operads can be found “everywhere”: Classical algebraic structures (groups, algebras, Lie algebras and so on) are in fact always implicitly based on operads defined using one-dimensional geometric objects such as punctured circles and binary trees, as is pointed out in [H1] and [HL2]. The present operadic formulation of the notion of vertex operator algebra shows that vertex operator algebras can be thought of as analogues of certain classical algebraic structures, in the sense that they too can be based on certain (partial) operads. The main difference is that for vertex operator algebras the underly ing operads are defined using certain (two-dimensional) “complex” geometric objects instead of (one-dimensional) “real” ones.

Starting from this operadic formulation of the notion of vertex operator algebra, one can develop the whole theory of vertex operator algebras and related structures and concepts in parallel with traditional theories. For instance, the theory of tensor products of modules for a general class of vertex operator algebras, as developed beginning in [HL1], can be expressed using the language of operads. And it now appears that this view of the theory of vertex operator algebras can help us gain a new level of insight into the phenomena of monstrous moonshine.

In this paper we describe our operadic reformulation. For more details, see [HL2]. We begin with the definitions of vertex operator algebra and of operad. For the basic algebraic theory of vertex operator algebras, see especially [FLM2] and [FHL]. Operads and related concepts were introduced in [M]. The details of the structures and of the proofs of the results in Section 5 are given in [H1] and [H4].

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2 The notion of vertex operator algebra

In this section, we quote the definition and basic “duality” properties of vertex operator algebras from [FLM2] or [FHL]. In this definition, all the variables \( x, x_0, \ldots \) are independent commuting formal variables, and all expressions involving these variables are to be understood as formal Laurent series. We use the formal expansion \( \delta(x) = \sum_{n \in \mathbb{Z}} x^n \) in the following way:

\[
 x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m \in \mathbb{N}, n \in \mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{-m} x_2^m. \quad (1)
\]

(In general, negative powers of binomials are to be expanded in nonnegative powers of the second summand.)

**Definition 1** A **vertex operator algebra** (over \( \mathbb{C} \)) is a \( \mathbb{Z} \)-graded vector space

\[
 V = \bigoplus_{n \in \mathbb{Z}} V(n)
\]

such that

\[
 \dim V(n) < \infty \text{ for } n \in \mathbb{Z},
\]

\[
 V(n) = 0 \text{ for } n \text{ sufficiently small},
\]

equipped with a linear map \( V \otimes V \rightarrow V[[x, x^{-1}]] \), or equivalently,

\[
 v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \text{ (where } v_n \in \text{End } V),
\]

\( Y(v, x) \) denoting the **vertex operator associated with** \( v \), and equipped also with two distinguished homogeneous vectors \( 1 \in V(0) \) (the vacuum) and \( \omega \in V(2) \). The following conditions are assumed for \( u, v \in V \):

\[
 u_n v = 0 \text{ for } n \text{ sufficiently large};
\]

\[
 Y(1, x) = 1 \text{ (1 on the right being the identity operator)};
\]

the **creation property** holds:

\[
 Y(v, x)1 \in V[[x]] \text{ and } \lim_{x \to 0} Y(v, x)1 = v
\]

(that is, \( Y(v, x)1 \) involves only nonnegative integral powers of \( x \) and the constant term is \( v \)); the **Jacobi identity** (the main axiom) holds:

\[
 x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2)Y(u, x_1)
 = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)
\]

\( (9) \)
(note that when each expression in (9) is applied to any element of \( V \), the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation \( Y(\cdot, x_2) \) is understood to be extended in the obvious way to \( V[[x_0, x_0^{-1}]] \)); the Virasoro algebra relations hold:

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c
\tag{10}
\]

for \( m, n \in \mathbb{Z} \), where

\[
L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}
\tag{11}
\]

and

\[
c \in \mathbb{C};
\tag{12}
\]

\[
L(0)v = nv \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad v \in V(n);
\tag{13}
\]

\[
\frac{d}{dx} Y(v, x) = Y(L(-1), x).
\tag{14}
\]

The vertex operator algebra just defined is denoted by \((V, Y, \mathbb{1}, \omega)\). The complex number \( c \) is called the central charge or rank of \( V \) (or of \((V, Y, \mathbb{1}, \omega)\)).

Vertex operator algebras have important “rationality,” “commutativity” and “associativity” properties, collectively called “duality” properties. These properties in fact can be used as axioms replacing the Jacobi identity in the definition of vertex operator algebra, as we now explain.

In the propositions below, \( \mathbb{C}[x_1, x_2]_S \) is the ring of rational functions obtained by inverting (localizing with respect to) the products of (zero or more) elements of the set \( S \) of nonzero homogeneous linear polynomials in \( x_1 \) and \( x_2 \). Also, \( \iota_{12} \) is the operation of expanding an element of \( \mathbb{C}[x_1, x_2]_S \), that is, a polynomial in \( x_1 \) and \( x_2 \) divided by a product of linear polynomials in \( x_1 \) and \( x_2 \), as a formal series containing at most finitely many negative powers of \( x_2 \) (using binomial expansions for negative powers of linear polynomials involving both \( x_1 \) and \( x_2 \)); similarly for \( \iota_{21} \) and so on. (The distinction between rational functions and formal Laurent series is crucial.)

**Proposition 2 (a) (rationality of products)** For \( v, v_1, v_2 \in V \) and \( v' \in V' \) (the graded dual space of \( V \)), the formal series \( \langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle \), which involves only finitely many negative powers of \( x_2 \) and only finitely many positive powers of \( x_1 \), lies in the image of the map \( \iota_{12} \):

\[
\langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = \iota_{12}f(x_1, x_2),
\tag{15}
\]

where the (uniquely determined) element \( f \in \mathbb{C}[x_1, x_2]_S \) is of the form

\[
f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t}
\tag{16}
\]
for some $g \in \mathbb{C}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) (commutativity) We also have

$$\langle v', Y(v_2, x_2)Y(v_1, x_1)v \rangle = \iota_{21} f(x_1, x_2). \quad (17)$$

Proposition 3 (a) (rationality of iterates) For $v, v_1, v_2 \in V$ and $v' \in V'$, the formal series $\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle$, which involves only finitely many negative powers of $x_0$ and only finitely many positive powers of $x_2$, lies in the image of the map $\iota_{20}$:

$$\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle = \iota_{20} h(x_0, x_2), \quad (18)$$

where the (uniquely determined) element $h \in \mathbb{C}[x_0, x_2]$ is of the form

$$h(x_0, x_2) = \frac{k(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t} \quad (19)$$

for some $k \in \mathbb{C}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) The formal series $\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle$, which involves only finitely many negative powers of $x_2$ and only finitely many positive powers of $x_0$, lies in the image of $\iota_{02}$, and in fact

$$\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle = \iota_{02} h(x_0, x_2). \quad (20)$$

Proposition 4 (associativity) We have the following equality of rational functions:

$$\iota_{12}^{-1} \langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = \left( \iota_{20}^{-1} \langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle \right) \bigg|_{x_0 = x_1 - x_2} . \quad (21)$$

Proposition 5 The Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of vertex operator algebra, the Jacobi identity may be replaced by these properties.

These duality properties, when suitably interpreted, form part of the geometric interpretation of vertex operator algebras discussed in Section 5 below. This part of the geometric interpretation was initially pointed out by Igor Frenkel.

3 The notions of operads and their modules

Now we give a variant of May’s definition of operad (cf. [M]):
Definition 6 An operad $\mathcal{C}$ consists of a family of sets $\mathcal{C}(j)$, $j \in \mathbb{N}$, together with (abstract) substitution maps $\gamma$, one for each $k \in \mathbb{N}$, $j_1, \ldots, j_k \in \mathbb{N}$,

$$\gamma: \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \rightarrow \mathcal{C}(j_1 + \cdots + j_k)$$

$$(c;d_1,\ldots,d_k) \mapsto \gamma(c;d_1,\ldots,d_k),$$

an identity element $I \in \mathcal{C}(1)$ and a (left) action of the symmetric group $S_j$ on $\mathcal{C}(j)$, $j \in \mathbb{N}$ (where $S_0$ is understood to be the trivial group), satisfying the following axioms:

(i) Operad-associativity: For any $k \in \mathbb{N}$, $j_s \in \mathbb{N}$ $(s = 1,\ldots,k)$, $i_t \in \mathbb{N}$ $(t = 1,\ldots,j_1 + \cdots + j_k)$, $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$ $(s = 1,\ldots,k)$ and $e_t \in \mathcal{C}(i_t)$ $(t = 1,\ldots,j_1 + \cdots + j_k)$,

$$\gamma(\gamma(c;d_1,\ldots,d_k);e_1,\ldots,e_{j_1+\cdots+j_k}) = \gamma(c;f_1,\ldots,f_k),$$

where

$$f_s = \gamma(d_s;e_{j_1+\cdots+j_{s-1}+1},\ldots,e_{j_1+\cdots+j_s}).$$

(ii) For any $j, k \in \mathbb{N}$, $d \in \mathcal{C}(j)$ and $c \in \mathcal{C}(k)$,

$$\gamma(I;d) = d,$$

$$\gamma(c;I,\ldots,I) = c.$$ 

(In particular, for $k = 0$, $\gamma: \mathcal{C}(0) \rightarrow \mathcal{C}(0)$ is the identity map.)

(iii) For any $k \in \mathbb{N}$, $j_s \in \mathbb{N}$ $(s = 1,\ldots,k)$, $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$ $(s = 1,\ldots,k)$, $\sigma \in S_k$ and $\tau_s \in S_{j_s}$ $(s = 1,\ldots,k)$,

$$\gamma(\sigma(c);d_1,\ldots,d_k) = \sigma(j_1,\ldots,j_k)\gamma(c;d_{\sigma(1)},\ldots,d_{\sigma(k)}),$$

$$\gamma(c;\tau_1(d_1),\ldots,\tau_k(d_k)) = (\tau_1 \ominus \cdots \ominus \tau_k)\gamma(c;d_1,\ldots,d_k),$$

where $\sigma(j_1,\ldots,j_k)$ denotes the permutation of $j = \sum_{s=1}^k j_s$ letters which permutes the $k$ blocks of letters determined by the given partition of $j$ as $\sigma$ permutes $k$ letters, and $\tau_1 \ominus \cdots \ominus \tau_k$ denotes the image of $(\tau_1,\ldots,\tau_k)$ under the obvious inclusion of $S_{\sum_{s=1}^k j_s} \times \cdots \times S_{\sum_{s=1}^k j_s}$ in $S_j$; that is,

$$\sigma(j_1,\ldots,j_k)(j_{\sigma(1)}+\cdots+j_{\sigma(i-1)}+l) = j_1+\cdots+j_{\sigma(i-1)}+l,$$  

$l = 1,\ldots,j_{\sigma(i)}$,  

$i = 1,\ldots,k$

and

$$(\tau_1 \ominus \cdots \ominus \tau_k)(j_1+\cdots+j_{i-1}+l) = j_1+\cdots+j_{i-1}+\tau_i(l),$$  

$l = 1,\ldots,j_i$,  

$i = 1,\ldots,k$.  

(\text{May } \text{uses right actions, so that the description of the permutation } \sigma(j_1,\ldots,j_k) \text{ might have a different interpretation.})

This definition is the same as that in \text{M} with the following exceptions:

(i) $\mathcal{C}(0)$ need not consist of exactly one element.

(ii) The sets in the definition need not be (certain kinds of) topological spaces (and correspondingly the maps need not be continuous).
and (26) always exist. Then we call such a family of sets $C$ we define a partial pseudo-operad both (23) and (25) – (28) holds whenever both sides exist; and the left-hand sides of (25) and (26) always exist. Then we call such a family of sets $C(j)$ together with the partial maps $\gamma$, the identity $I$ and the actions of $S_j$ on $C(j)$ a partial operad. In addition, we define a partial pseudo-operad to be a family of sets $C(j)$, $j \in \mathbb{N}$, together with partially defined substitution maps $\gamma$, an identity $I$ and actions of $S_j$ on $C(j)$, $j \in \mathbb{N}$, satisfying all the axioms for partial operads except the operad-associativity. (Later, we shall typically denote partial operads by the symbol $\mathcal{P}$ rather than $C$.)

If in Definitions 6 and 7 the sets $C(j)$, $j \in \mathbb{N}$, are assumed to be objects in given categories (e.g., have certain kinds of topological, smooth or analytic structure) and the maps $\gamma$ and the actions of $S_j$, $j \in \mathbb{N}$, are morphisms in these categories (e.g., are continuous or smooth or analytic), we have the notions of operads in these categories, and we use the names of these categories plus the word “operads” to designate them (e.g., topological operads, smooth operads or analytic operads). In the case of partial operads, we also require that the domains of the substitution maps are in the category we are considering.

A morphism $\psi : \mathcal{C} \rightarrow \mathcal{C}'$ of operads $\mathcal{C}$ and $\mathcal{C}'$ is a sequence of $S_j$-equivariant maps $\psi_j : \mathcal{C}(j) \rightarrow \mathcal{C}'(j)$ such that $\psi_1(I) = I'$ and the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) & \xrightarrow{\gamma} & \mathcal{C}(j_1 + \cdots + j_k) \\
\downarrow & & \downarrow \\
\mathcal{C}'(k) \times \mathcal{C}'(j_1) \times \cdots \times \mathcal{C}'(j_k) & \xrightarrow{\gamma'} & \mathcal{C}'(j_1 + \cdots + j_k).
\end{array}
$$

(31)

For partial operads we also require that the domains of the substitution maps for $\mathcal{C}$ are mapped into the domains of the substitution maps for $\mathcal{C}'$; the diagram (31) is interpreted in the obvious way. Morphisms for partial pseudo-operads are defined in the same way as morphisms for partial operads.

Next we discuss the sense in which operads describe “operations.” In the rest of this section, all the operads will be ordinary (i.e., in the category of sets and not partial).

Let $X$ be a set and $Y$ a subset of $X$. We define the endomorphism operad $\mathcal{E}_{X,Y}$ as follows: Let $\ast$ be a one-element set (a terminal object in the category of sets) and take $X^\ast = Y^\ast = \ast$. Let $\mathcal{E}_{X,Y}(j)$, $j \in \mathbb{N}$, be the set of maps from $X^j$ to $X$ which map $Y^j$ to $Y$; then $\mathcal{E}_{X,Y}(0) = Y$. The substitution maps are defined by

$$
\gamma(f; g_1, \ldots, g_k) = f \circ (g_1 \times \cdots \times g_k)
$$

(32)

for $f \in \mathcal{E}_{X,Y}(k)$, $k \in \mathbb{N}$ and $g_s \in \mathcal{E}_{X,Y}(j_s)$, $s = 1, \ldots, k$. The identity $I_{X,Y}$ is the identity map of $X$. For $f \in \mathcal{E}_{X,Y}(j)$, $\sigma \in S_j$, $x = (x_1, \ldots, x_j) \in X^j$,

$$(\sigma(f))(x) = f(\sigma^{-1}(x))
$$

(33)
where
\[ \sigma(x) = \sigma(x_1, \ldots, x_j) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(j)}). \] (34)

It is easy to see that \( E_{X,Y} \) is an operad. The corresponding definition in [M] amounts to the case in which \( Y \) has one element. Note the special cases \( Y = \emptyset \) and \( Y = X \). Observe that operad-associativity (the associativity of substitution) is unrelated to any associativity properties that the \( j \)-ary operations might or might not have.

For vector spaces we have a more appropriate notion. Let \( V \) be a vector space and \( W \) a subspace of \( V \). We define the corresponding (multilinear) endomorphism operad \( \mathcal{M}_{V,W} \) as follows: Let \( \mathcal{M}_{V,W}(j), \ j \in \mathbb{N}, \) be the set of multilinear maps from \( V^j \) to \( V \) which map \( W^j \) to \( W \); it is understood that \( V^0 = W^0 \) is the one-element set as above, and that a “multilinear map” (a “zero-linear map”) from this set to \( V \) (or \( W \)) is a map of sets, i.e., an element of the target set. In particular, \( \mathcal{M}_{V,W}(0) = W \). The substitution maps, the identity and the actions of the symmetric groups are defined just as in the definition of endomorphism operads for sets. Then \( \mathcal{M}_{V,W} \) is an operad. Note the special cases \( W = 0 \) and \( W = V \).

**Definition 8** Let \( \mathcal{C} \) be an operad. A \( \mathcal{C} \)-space consists of a set \( X \), a subset \( Y \subset X \) and a morphism \( \psi \) of operads from \( \mathcal{C} \) to \( E_{X,Y} \) such that \( \psi_0(\mathcal{C}(0)) = \mathcal{E}_{X,Y}(0) (= Y) \). It is denoted \((X,Y,\psi)\). An element of \( \psi_0(\mathcal{C}(0)) = Y \) is called a quasi-identity element of \( X \) (for \( \mathcal{C} \) and \( \psi \)). Note that each element of \( \mathcal{C}(j), \ j \in \mathbb{N}, \) defines a \( j \)-ary operation on \( X \). A morphism from a \( \mathcal{C} \)-space \((X,Y,\psi)\) to a \( \mathcal{C} \)-space \((X',Y',\psi')\) is a map \( \eta : X \longrightarrow X' \) such that \( \eta(Y) \subset Y' \) and \( \eta \circ \psi_j(c) = \psi'_j(\eta(c)) \circ \eta^j \) for \( j \in \mathbb{N} \) and \( c \in \mathcal{C}(j) \); it follows that \( \eta(Y) = Y' \). An isomorphism of \( \mathcal{C} \)-spaces is defined in the obvious way. The definition of “\( \mathcal{C} \)-space” in [M] amounts to the case in which \( \mathcal{C}(0) \) and \( Y \) consist of one element.

**Definition 9** A \( \mathcal{C} \)-module \((V,W,\nu)\) consists of a vector space \( V \), a subspace \( W \) and a morphism \( \nu \) from \( \mathcal{C} \) to \( \mathcal{M}_{V,W} \) such that the subspace of \( V \) spanned by \( \nu_0(\mathcal{C}(0)) \) is \( \mathcal{M}_{V,W}(0) (= W) \). We call \( \nu_0(\mathcal{C}(0)) \) the set of quasi-identity elements. Each element of \( \mathcal{C}(j), \ j \in \mathbb{N}, \) defines a multilinear \( j \)-ary operation on \( V \). Morphisms and isomorphisms of \( \mathcal{C} \)-modules are defined in the obvious ways.

The notion of suboperad of an operad is defined in the obvious way. An intersection of suboperads is again a suboperad. Let \( \mathcal{C} \) be an operad and \( U \) a subset of the disjoint union of the sets \( \mathcal{C}(j), \ j \in \mathbb{N} \). The suboperad of \( \mathcal{C} \) generated by \( U \) is the smallest suboperad of \( \mathcal{C} \) such that the disjoint union of the family of sets in the suboperad contains \( U \). The operad \( \mathcal{C} \) is said to be generated by \( U \) if the suboperad generated by \( U \) is \( \mathcal{C} \) itself.

**Definition 10** Let \( \mathcal{C} \) be an operad. We call an element \( a \) of \( \mathcal{C}(2) \) associative if
\[ \gamma(a; a, I) = \gamma(a; I, a). \] (35)

We say that an operad \( \mathcal{C} \) is associative if \( \mathcal{C} \) is generated by \( \mathcal{C}(0) \) and an associative element \( a \in \mathcal{C}(2) \).
**Definition 11** Let $C$ be an associative operad with associative element $a \in C(2)$. We call a $C$-space a $C$-**monoid** and a $C$-module a $C$-**associative algebra**, with $C$-**associative binary product** given by the image of $a$ and with **quasi-identity elements** given by the image of $C(0)$.

An important example of an associative operad can be obtained from circles with punctures and local coordinates. We consider the moduli spaces $C_{j}$, $j \in \mathbb{N}$, of circles (i.e., compact connected smooth one-dimensional manifolds) with $j + 1$ ordered points (called **punctures**), the zeroth negatively oriented, the others positively oriented, and with local coordinates vanishing at these punctures. Given two such circles with punctures and local coordinates and a positively oriented puncture on the first circle, we can cut suitable intervals around the given puncture and the negatively oriented puncture on the second circle, and then identify their boundaries using the two local coordinate maps and the map $t \mapsto 1/t$ to obtain another circle with punctures and local coordinates. The ordering of the positively oriented punctures of the sewn circle is obtained by “inserting” the ordering for the second circle into that for the first. This procedure in fact induces an operation on the moduli space of circles with punctures and local coordinates. Using this operation on the moduli space we can define substitution maps. We also have a natural identity element and actions of $S_{j}$ on the $C_{j}$, $j \in \mathbb{N}$. The family $\{C_{j} \mid j \in \mathbb{N}\}$ with these structures forms an associative operad, which we denote by $C$. For more details, see [HLZ].

For this operad $C$, a $C$-monoid is a monoid in the usual sense and a $C$-associative algebra is an associative algebra in the usual sense.

## 4 Rescalable partial operads and their modules

Let $\mathcal{P}$ be a partial operad. A subset $G$ of $\mathcal{P}(1)$ is called a **rescaling group** for $\mathcal{P}$ if $G$ contains $I$; the substitution maps $\gamma$ from a subset of $\mathcal{P}(1) \times \mathcal{P}(k)$ to $\mathcal{P}(k)$ and from a subset of $\mathcal{P}(k) \times (\mathcal{P}(1))^{k}$ to $\mathcal{P}(k)$ are defined on $G \times \mathcal{P}(k)$ and on $\mathcal{P}(k) \times G$, respectively, for each $k \in \mathbb{N}$; both sides of (23) exist if $c \in G$ or $d_{1}, \ldots, d_{k} \in G$ or $c_{1}, \ldots, c_{j_{1}+\ldots+j_{k}} \in G$ and if either side of (23) exists; $\gamma$ maps $G \times G$ into $G$; and inverses of the elements of $G$ exist with respect to $\gamma$ and $I$; then $G$ is in fact a group. (Note that $G = \{I\}$ is always an example of a rescaling group for $\mathcal{P}$.) Given a rescaling group $G$ for $\mathcal{P}$, we define a corresponding equivalence relation on $\mathcal{P}$: Two elements $c_{1}$ and $c_{2}$ of $\mathcal{P}(j)$, $j \in \mathbb{N}$, are said to be $G$-equivalent if there exists $d \in G$ such that

$$c_{2} = \gamma(d; c_{1});$$

(36)

our assumptions insure that this is an equivalence relation.

**Definition 12** A (G-)**rescalable partial operad** is a partial operad $\mathcal{P}$ together with a rescaling group $G$ for $\mathcal{P}$ satisfying the following condition: For any $c \in \mathcal{P}(k)$, $k \in \mathbb{N}$, $d_{1} \in \mathcal{P}(j_{1}), \ldots, d_{k} \in \mathcal{P}(j_{k})$, $j_{i} \in \mathbb{N}$, there exist $d'_{1} \in \mathcal{P}(j'_{1}), \ldots, d'_{k} \in \mathcal{P}(j'_{k})$ which are $G$-equivalent to $d_{1}, \ldots, d_{k}$, respectively, such that $\gamma(c; d'_{1}, \ldots, d'_{k})$ exists.
Suppose that we have a set-theoretic category with a reasonable notion of “induced substructure,” such as a topological, smooth or analytic category. A **partial operad with rescaling group (or rescalable partial operad) in this category** (e.g., a topological, smooth or analytic rescalable partial operad) is a partial operad \( \mathcal{P} \) with rescaling group (or a rescalable partial operad) such that its underlying partial operad is in the category we are considering and the rescaling group is a group in this category, with the structure induced from that on \( \mathcal{P}(1) \).

Morphisms and isomorphisms of partial operads with rescaling groups are defined in the obvious ways.

The definitions of \(\mathcal{P}\)-space and \(\mathcal{P}\)-module in Section 3 also make sense when \(\mathcal{P}\) is a partial operad. But for a partial operad \(\mathcal{P}\) with a rescaling group \(G\), it is more relevant to look for some kind of “(multilinear) endomorphism partial operad” of a \(G\)-module, and then to define a “\(\mathcal{P}\)-module” to be a morphism from \(\mathcal{P}\) to such a (multilinear) endomorphism partial operad. However, we must be content with only the following “(multilinear) endomorphism partial pseudo-operads” (recall Definition 7):

Let \(G\) be a group, \(V\) a completely reducible \(G\)-module and \(W\) a \(G\)-submodule of \(V\). Then \(V = \prod_{M \in A} V(M)\), where \(A\) is the set of equivalence classes of irreducible \(G\)-modules and \(V(M)\) is the sum of the \(G\)-submodules of \(V\) in the class \(M\), and similarly for \(W\). Assume that \(\dim V(M) < \infty\) for every \(M \in A\). We define a *(multilinear)* endomorphism partial pseudo-operad \(\mathcal{H}_{V,W}^G\) as follows: For any \(j \in \mathbb{N}\) the set \(\mathcal{H}_{V,W}^G(j)\) is the set of all multilinear maps from \(V^j\) to \(V = \prod_{M \subseteq A} V(M) = V^*\) such that \(W^j\) is mapped to \(\mathcal{W} = \prod_{M \subseteq A} W(M) = W^*\), where \(\mathcal{W}\) denotes the graded dual of an \(A\)-graded vector space and \(^*\) denotes the dual space of a vector space. As in the definition of multilinear endomorphism operad above, it is understood that \(V^0 = W^0\) is the one-element set, so that the set \(\mathcal{H}_{V,W}^G(0)\) is equal to \(\mathcal{W}\). The identity \(I_{V,W}^G\) is the embedding map from \(V\) to \(\mathcal{W}\). The symmetric group \(S_j\) acts on \(\mathcal{H}_{V,W}^G(j)\) in the obvious way. To define the substitution maps, we first define a contraction operation on \(\mathcal{H}_{V,W}^G\): Given \(f \in \mathcal{H}_{V,W}^G(k)\) and \(g \in \mathcal{H}_{V,W}^G(j)\) \((k,j \in \mathbb{N})\) and a positive integer \(s \leq k\), we say that the contraction of \(f\) at the \(s\)-th argument and \(g\) at the zeroth argument exists if for any \(v_1, \ldots, v_{k+j-1} \in V\) and \(v' \in V'\), the series

\[
\sum_{M \subseteq A} \langle v', f(v_1, \ldots, v_s-1, P_M(g(v_s, \ldots, v_{s+j-1})), v_{s+j}, \ldots, v_{k+j-1}) \rangle
\]  

(37)

converges absolutely, where \(P_M : \mathcal{W} \longrightarrow V(M)\) is the projection operator. In this case the (well-defined) limits for all \(v_1, \ldots, v_{k+j-1} \in V\), \(v' \in V'\) define an element \(f \star_g g\) of \(\mathcal{H}_{V,W}^G(k+j-1)\), the contraction. More generally, given any subset of \(\{1, \ldots, k\}\) and any element of \(\bigcup_{j \in \mathbb{N}} \mathcal{H}_{V,W}^G(j)\) for each element of the subset, we have the analogous contraction, defined using the appropriate multisums, when they are absolutely convergent. The substitution map

\[
\gamma_{V,W}^G : \mathcal{H}_{V,W}^G(k) \times \mathcal{H}_{V,W}^G(j_1) \times \cdots \times \mathcal{H}_{V,W}^G(j_k) \longrightarrow \mathcal{H}_{V,W}^G(j_1 + \cdots + j_k)
\]

\[
(f; g_1, \ldots, g_k) \longmapsto \gamma_{V,W}^G(f; g_1, \ldots, g_k)
\]  

(38)
is defined by this procedure, using the whole set \( \{1, \ldots, k\} \). Of course, the cases of proper subsets of \( \{1, \ldots, k\} \) are recovered by letting some of the \( g \) be \( \eta^{G}_{V,W} \). The family \( H_{V,W}^{G} \) of sets \( H_{V,W}^{G}(j), j \in \mathbb{N} \), equipped with the substitution maps \( \gamma_{V,W}^{G} \), the identity \( I_{V,W}^{G} \) and the actions of \( S_{j} \) on \( H_{V,W}^{G}(j), j \in \mathbb{N} \), satisfies all the axioms for a partial operad except the operad-associativity and therefore is a partial pseudo-operad. The operad-associativity fails because in general we cannot expect to have the absolute convergence of the multisums corresponding to a sequence of substitutions.

Using (multilinear) endomorphism partial pseudo-operads, we define the following notions of \( P \)-pseudo-module and of \( P \)-module:

**Definition 13** Let \( P \) be a partial operad with rescaling group \( G \). A \( P \)-pseudo-module \((V,W,\nu)\) is a completely reducible \( G \)-module \( V = \coprod_{M \in A} V(M) \) with \( \dim V(M) < \infty \), together with a submodule \( W \) of \( V \) and a morphism \( \nu \) from \( P \) (viewed as a partial pseudo-operad) to the partial pseudo-operad \( H_{V,W}^{G} \), such that the submodule of \( V \) generated by the homogeneous components of the elements of \( \nu_{0}(P(0)) \) is \( W \) and the map from \( G \) to \( H_{V,W}^{G}(1) \) induced from \( \nu_{1} \) is the given representation of \( G \) on \( V \). An element of \( \nu_{0}(P(0)) \) is called a quasi-identity element of \( \nabla \) for \( P \). A morphism from a \( P \)-pseudo-module \((V,W,\nu)\) to a \( P \)-pseudo-module \((\tilde{V},\tilde{W},\tilde{\nu})\) is a \( G \)-module morphism \( \eta : V \rightarrow \tilde{V} \) such that \( \eta(W) \subset \tilde{W} \) and \( \eta(c) \circ \eta_{j}(c) = \nu_{j}(c) \circ \eta(c) \) for \( j \in \mathbb{N} \) and \( c \in P(j) \), where \( \eta \) is extended naturally to \( \eta : V \rightarrow \tilde{V} \) if \( \tilde{V} \rightarrow \tilde{V} \); it follows that \( \eta(W) = \tilde{W} \). Isomorphisms of \( P \)-pseudo-modules are defined in the obvious way. For a \( P \)-pseudo-module \((V,W,\nu)\), the image of \( P \) under \( \nu \) (where it is understood that the substitution maps are the substitution maps for \( H_{V,W}^{G} \) restricted to the images of the domains of the substitution maps for \( P \)) is a partial pseudo-operad. We define a \( P \)-module to be a \( P \)-pseudo-module \((V,W,\nu)\) such that the image of \( P \) under \( \nu \) is a partial operad, that is, such that operad-associativity holds for the image. Morphisms and isomorphisms of \( P \)-modules are defined to be morphisms and isomorphisms of the underlying \( P \)-pseudo-modules, respectively.

Though this definition of \( P \)-module is conceptually natural, it is in practice typically very difficult to determine whether a \( P \)-pseudo-module is a \( P \)-module. The issue is to insure that operad-associativity holds for certain families of multilinear maps.

The notion of partial suboperad of a partial operad is defined in the obvious way; we require that substitutions in a partial suboperad exist if and only if the corresponding substitutions in the original partial operad exist. An intersection of partial suboperads of a partial operad is a partial suboperad. We also have the notion of partial suboperad generated by a subset. If a partial operad is the partial suboperad generated by a given subset, we say that this partial operad is generated by the subset.

**Definition 14** Let \( P \) be a partial operad with rescaling group \( G \). We call an element \( a \in P(2) \) associative if there exists \( a' \in P(2) \) which is \( G \)-equivalent to \( a \) (that is, there exists \( b_{0} \in G \) such that \( a' = \gamma(b_{0};a) \)) and there exist unique \( b_{i} \in G, i = 1, \ldots, 5, \ldots, k \).
which depend on $a'$, such that $\gamma(a; a', I)$ exists and

$$\gamma(a; a', I) = \gamma(d_1; I, d_2),$$

(39)

where

$$d_1 = \gamma(b_1; \gamma(a; b_2, b_3)), \quad d_2 = \gamma(a; b_4, b_5).$$

(40)

We call a partial operad $\mathcal{P}$ with rescaling group $G$ associative if $\mathcal{P}$ is generated by $\mathcal{P}(0)$, $G$ and an associative element $a \in \mathcal{P}(2)$.

**Definition 15** Let $\mathcal{P}$ be an associative partial operad with rescaling group $G$ and associative element $a \in \mathcal{P}(2)$. We call a $\mathcal{P}$-pseudo-module a $\mathcal{P}$-associative pseudo-algebra and a $\mathcal{P}$-module a $\mathcal{P}$-associative algebra, with $\mathcal{P}$-associative binary product given by the image of $a$ and with quasi-identity elements given by the image of $\mathcal{P}(0)$.

## 5 Vertex operator algebras as modules for certain partial operads associated with spheres with tubes

Using the language developed in the previous section, we can now explain how vertex operator algebras (recall Section 2) amount to modules for the partial operads which are the complex powers of the determinant line bundle over the moduli space of spheres with tubes.

A *sphere with $n$ tubes* ($n > 0$) is a sphere (a genus-zero compact connected one-dimensional complex manifold) with $n$ distinct, ordered points (called punctures) with the zeroth puncture negatively oriented and the other punctures positively oriented, and with local analytic coordinates vanishing at these punctures. Given two spheres with tubes and given one positively oriented puncture on the first sphere, we can sew these two spheres at the given puncture on the first sphere and the negatively oriented puncture on the second sphere by first cutting disks with reciprocal radii (using the local coordinates), and containing no other punctures, around the two given punctures, and then identifying the boundaries of the remaining parts of the two spheres with tubes using the two local coordinate maps and the map $z \mapsto 1/z$. We call this procedure the *sewing operation*. Note that the conditions “with reciprocal radii” and “containing no other punctures” are not always satisfied, therefore the sewing operation is only a partial operation. But if we rescale the local coordinate map at the negatively oriented puncture on the second sphere with tubes by multiplying it by a suitable nonzero complex number, then these two conditions can always be satisfied, and thus after rescaling, any two spheres with tubes can always be sewn together at the given punctures. The ordering of the positively oriented punctures on the sewn sphere is obtained by “inserting” the ordering for the second sphere into that for the first.
Two spheres with tubes are said to be conformally equivalent if there exists an analytic diffeomorphism of the underlying one-dimensional complex manifolds preserving all the indicated structures except perhaps the local coordinate neighborhoods. The space of conformal equivalence classes of spheres with \( n \) tubes is called the moduli space of spheres with \( n \) tubes and is denoted \( K(n-1) \) (\( n - 1 \) being the number of positively oriented punctures). Given two conformal equivalence classes in \( K(m) \) and \( K(n) \), respectively, and a positive integer \( i \leq m \), if there are two spheres with tubes in the two given conformal equivalence classes which can be sewn at the \( i \)-th positively oriented puncture on the first sphere and the negatively oriented puncture on the second sphere, the conformal equivalence class of the sewn sphere with tubes, in \( K(m+n-1) \), depends only on the two given classes. This procedure for obtaining an element of \( K(m+n-1) \) from two elements of \( K(m) \) and \( K(n) \) is still called the sewing operation.

We now have a family of sets \( K(j) \), \( j \in \mathbb{N} \). Given \( Q \in K(k) \), \( Q_1 \in K(j_1) \), ..., \( Q_k \in K(j_k) \), we define the substitution \( \gamma_K(Q; Q_1, \ldots, Q_k) \) by successively sewing \( Q_i \) to \( Q \) at the \( i \)-th positively oriented puncture of \( Q \), \( 1 \leq i \leq k \). This gives us substitution maps \( \gamma_K \). The identity \( I_K \) is the conformal equivalence class of the standard sphere \( \mathbb{C} \cup \{\infty\} \) with \( \infty \) the negatively oriented puncture, 0 the only positively oriented puncture, and with standard local coordinates vanishing at \( \infty \) and 0. The symmetric groups \( S_j \) acts on \( K(j) \) by permuting the orderings of the punctures of the spheres with tubes in the conformal equivalence classes in \( K(j) \) for \( j \in \mathbb{N} \). The family \( \{K(j) \mid j \in \mathbb{N}\} \) equipped with these structures forms a partial operad which we denote by \( K \). Since after a possible rescaling of the local coordinate maps the sewing operation can always be performed, this partial operad is a \( \mathbb{C}^\times \)-rescalable partial operad. It can also be shown that \( K \) is associative, with associative element the equivalence class of the sphere \( \mathbb{C} \cup \{\infty\} \) with \( \infty \) the negatively oriented puncture, 1 and 0 the first and second positively oriented punctures, respectively, and with the standard local coordinates vanishing at these punctures. The moduli spaces \( K(j), j \in \mathbb{N} \), have natural infinite-dimensional complex manifold structures and the substitution maps are analytic maps. Thus \( K \) is an analytic associative \( \mathbb{C}^\times \)-rescalable partial operad.

The determinant line bundle over the moduli space of spheres with boundaries induces a line bundle over \( K \). We still call it the determinant line bundle and denote it by \( \bar{K}^1 \). For any complex number \( c \), the line bundle \( \bar{K}^1 \) raised to the complex power \( c \) is a well-defined line bundle over \( K \) which we denote by \( \bar{K}^c \). For any \( c \in \mathbb{C} \), \( \bar{K}^c \) is also an analytic associative \( \mathbb{C}^\times \)-rescalable partial operad.

We consider the (partial) operad \( \bar{K}^c \) for a given \( c \in \mathbb{C} \). From the properties of the determinant line bundle, we know that there is a natural connection on \( \bar{K}^c \). Moreover, this connection is flat over \( \bar{K}(j) \), the space of the conformal equivalence classes containing the sphere \( \mathbb{C} \cup \{\infty\} \) with the negatively oriented puncture \( \infty \), the positively oriented ordered punctures \( z_1, \ldots, z_{j-1}, 0 \), and with standard local coordinates vanishing at these punctures, for all \( (z_1, \ldots, z_{j-1}) \), \( z_i \neq z_k \) for \( i < k \). In
fact, \( K(j) \) can be identified with \( M^{j-1} = \{(z_1, \ldots, z_{j-1}) \mid z_i \neq z_k, i < k \} \).

Since an equivalence class of irreducible modules for \( C^\times \) is determined by an integer \( n \) such that \( a \in C^\times \) acts on modules in this class as scalar multiplication by \( a^{-n} \), any completely reducible module for \( C^\times \) is of the form \( V = \prod_{n \in Z} V(n) \) where \( V(n) \) is the sum of the \( C^\times \)-submodules in the class corresponding to the integer \( n \). In particular, the vector space of a \( K^c \)-associative pseudo-algebra is of this form. Note that for a \( K^c \)-associative pseudo-algebra we have \( \dim V(n) < \infty \) by definition.

**Definition 16** A \( K^c \)-associative pseudo-algebra \((V, W, \nu)\) is meromorphic if the following axioms are satisfied:

(i) \( V(n) = 0 \) for \( n \) sufficiently small.

(ii) For any \( v' \in V', v_1, \ldots, v_j \in V \), \( \langle v', \nu(\cdot)(v_1, \ldots, v_j) \rangle \) is analytic as a function on \( K^c(j) \).

(iii) Given any \( v_1, \ldots, v_j \in V \) and \( v' \in V' \) and any flat section \( \phi \) of the restriction of the line bundle \( K^c(j) \) to \( K(j) \), \( \langle v', \nu(\phi(\cdot))(v_1, \ldots, v_j) \rangle \) is a meromorphic function on \( K(j) = M^{j-1} \) with \( z_i = 0 \) and \( z_i = z_k, i < k \), as the only possible poles, and for fixed \( v_i, v_k \in V \) there is an upper bound for the orders of the pole \( z_i = z_k \) of the functions \( \langle v', \nu(\phi(\cdot))(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k, v_{k+1}, \ldots, v_j) \rangle \) for all \( v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_j \in V, v' \in V' \).

It can be shown that any meromorphic \( K^c \)-associative pseudo-algebra is a \( K^c \)-associative algebra.

We call a meromorphic \( K^c \)-associative (pseudo-)algebra a **vertex associative algebra with central charge** or rank \( c \). **Morphisms** (respectively, **isomorphisms**) of vertex associative algebras are morphisms (respectively, isomorphisms) of the underlying \( K^c \)-associative algebras. The main theorem in [H1] and [H2] can now be reformulated using the language we have developed as follows:

**Theorem 17** **The category of vertex operator algebras with central charge** (or rank) \( c \) **is isomorphic to the category of vertex associative algebras with central charge** (or rank) \( c \).

Here we give a brief description of the functor from the category of vertex operator algebras with central charge \( c \) to the category of vertex associative algebras with central charge \( c \). Let \((V, Y, 1, \omega)\) be a vertex operator algebra with central charge \( c \). The \( Z \)-graded vector space \( V \) is naturally a completely reducible \( C^\times \)-module. The module \( W \) for the Virasoro algebra generated by \( 1 \) is a \( Z \)-graded subspace of \( V \) and therefore is a \( C^\times \)-submodule of \( V \). In [H2] and [H3], a certain section of the line bundle \( K^c \) over \( K \) is chosen. This section restricts to a nonzero flat section \( \phi \) over \( K(2) \) as above. Any element of the fiber over \( Q \in K(2) \) is of the form \( \lambda \phi(Q) \) where \( \lambda \in C \). We define \( \nu(\lambda \phi(Q)) \) by

\[
\nu(\lambda \phi(Q))(v_1, v_2) = \lambda Y(v_1, x)v_2 \bigg|_{x=z} \tag{41}
\]
for any $v_1, v_2 \in V$. More generally, For $\bar{Q} \in \bar{K}^c(2)$, $\nu(\bar{Q})$ is obtained by modifying this expression using exponentials of certain infinite linear combinations of the $L(n)$ (recall (10), (11)) determined by the local coordinates at the three punctures (see [HL2]). For $\bar{Q} \in \bar{K}^c(j)$, $j \neq 2$, $\nu(\bar{Q})$ can be defined analogously. The triple $(V, W, \nu)$ is the vertex associative algebra corresponding to $(V, Y, 1, \omega)$.

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