Non-equilibrium stationary states at the fractional quantum Hall edge

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The two-dimensional electron gas (2DEG) in the regime of quantum Hall (QH) effect is a fascinating example of a system exhibiting rich emergent physics [1, 2]. In this regime, observed in the strong magnetic field, electrons in the bulk of 2DEG form an incompressible liquid with elementary excitations having fractional charge and fractional statistics [2, 3]. At the same time, chiral one-dimensional edge states [4] are present at the boundary of the 2DEG. These collective states, that could be viewed as incompressible deformations of the electron liquid, are very similar to the optical beams. Such similarity has led to the emergence of a new subfield of condensed matter physics, the electron optics.

The QH edge states can be described in the framework the low-energy effective theory approach [5], where the effective Hamiltonian is constructed from the general considerations of locality and gauge invariance. Such effective theory has a great success in describing the very recent electron optics experiments in the regime of integer QH effect, where an integer number ν of Landau levels is filled with electrons [6, 7]. In particular, it has been fruitfully applied to strongly out-of-equilibrium situations, which take place in experiments on dephasing of the edge states [8] and on the energy relaxation at the QH edge [7]. This progress is due to the recent development of the technique of non-equilibrium bosonization [9], which allows to treat Coulomb interactions non-perturbatively in systems far from equilibrium.

Non-equilibrium edge states are typically created by bringing two edges close to each other, so that the electrons can tunnel across the formed quantum point contact (QPC), and applying a voltage bias Δμ between these edges. It has been found in Ref. [10] that there is an intermediate asymptotic in the process of relaxation of the energy distribution function of electrons at the integer QH edge after a biased QPC, see Fig. 1. This is a consequence of the integrability of dynamics of interacting one-dimensional electrons. Thus, apart from Coulomb interactions, one needs additional energy exchange mechanisms, such as, e.g., disorder, in order to equilibrate electrons to an actual Fermi distribution. Such additional mechanisms act at distances larger than the distance where the partial equilibration to the intermediate asymptotic occurs.

The intermediate distribution function has a power-law asymptotic character in startling contrast to the Fermi distribution [10]. However, an observation of the non-equilibrium states by measuring directly the distribution function might be intricate at intermediate transparency of the source QPC, where the non-equilibrium distribution is similar to the Fermi distribution. One can overcome this obstacle by measuring a quantity which has qualitatively different behaviour in this state and at equilibrium. One of such quantities is the noise power of weak backscattering currents Jbs measured by a second, detector QPC [11], see Fig. 2, upper panel:

\[ S_{bs} = \int dt \langle j_{bs}(t) j_{bs}(0) \rangle . \] (1)

These noise power has been studied thoroughly in Ref. [12] in the case of integer QH effect, where the intermediate asymptotic manifests itself in a non-analytic behaviour \( S_{bs} \propto \Delta \mu |\tau|^2 \log |\tau| \) at small values of the ampli-
tude $\tau$ of tunnelling at the QPC. In contrast, the noise power behaves as $S_{bs} \propto \Delta \mu/|\tau|$ if a true thermal equilibration occurs.

In this Letter, we consider the situation of fractional QH effect and find that the phenomenon of intermediate stationary state takes place in this situation as well, but its manifestations are even more dramatic. We investigate the behavior of the effective temperature of edge states, measured by two detectors: by a second QPC via the backscattering noise power (Fig. 2, upper panel) and by a quantum anti-dot (QD) [7, 13] via QD level broadening (Fig. 2, lower panel). Naively, one could think that these setups could be described using the perturbation theory in tunnelling at contacts. However, it has been shown in Ref. [14] that such perturbation theory is divergent and fails at zero temperature. Therefore, we resort to the non-equilibrium bosonization [9] and show that it can be used for a non-perturbative treatment of the source QPC in the weak backscattering limit, where rare scattering events of quasi-particles between edges have Poissonian statistics.

Using the non-equilibrium bosonization approach, we find the central result of this Letter: the universal linear scaling of the effective temperature with the injected current in both detection schemes,

$$\Theta_{\text{eff}} \propto \langle I \rangle,$$

which is not modified neither by interactions at the edge nor by interactions between the edge states and the QD. Such scaling is drastically different from what one expects in the case of eventual equilibration of the edge states. Indeed, at equilibrium at the temperature $\Theta$, the energy flux at the QH edge equals to $\pi^2 \Theta^2/12$. Thus, the temperature of the edge states after a QPC where power $P = \Delta \mu(I)$ is dissipated behaves as:

$$\Theta \propto \sqrt{P}.$$  (3)

Such dependance is qualitatively different from (2) in the situation of fractional QH effect, where the current-voltage characteristics of the QPC are highly non-linear. In addition, we find that the reason for the divergence mentioned above is the behavior of the noise power $S_{bs} \propto |\tau|^{4_5-2}$, which is regular, but not-analytic for the fractional values of the quasiparticle correlation function scaling dimension $\delta$.

In our analysis we focus on the $\nu = 2/m$, $m \in 2\mathbb{Z} + 1$ series of fractional QH states which are now extensively studied experimentally [13, 15, 16]. An interesting feature of these series is the presence of neutral upstream modes. Such modes have been predicted theoretically in Ref. [17] and experimentally detected recently [16]. They are interesting for us for two reasons: First, there is no average detector current, if the detector (QPC or QD) is upstream of the injection QPC, which makes the measurement of the effective temperature more simple. Second, there is a opportunity to use the two QPC setup to distinguish effective models with different values of couplings of quasi-particle excitations to the neutral mode, that are allowed by the edge effective theory at these filling factors.

**Effective theory of the edge states.**— It has been shown in Ref. [18] that general constraints of locality and gauge invariance allow one to construct an effective theory of the QH edge states at the energies much lower than the Fermi energy. Such theory, however, contains arbitrary parameters, that can be fixed experimentally or from microscopic calculations, and different realizations of these parameters are referred to as different effective models. In the minimal (i.e. simplest possible) effective models, the physics of the QH edge states at filling factors $\nu = 2/m$ are described by a set of boson fields $\phi_{\alpha}(x,t)$, where $s = L, U, R$ enumerates the three edges in the experimental setup, left, upper and right, and $\alpha = 0, 1$ enumerates the edge channels at each edge, see Fig. 2. These fields have the following canonical commutation relations:

$$[\phi_{\alpha}(x), \phi_{\alpha'}(y)] = 2\pi i(-1)^{\alpha} \delta_{\alpha \beta} \delta_{s s'} \delta(x - y),$$  (4)

where a different sign for $\alpha = 1$ reflects the opposite chirality of the corresponding channel. The charge densities at each edge in the system of units where $e = h = 1$ are given by

$$\rho_s(x) = \frac{\sqrt{\nu}}{2\pi} [\cosh \Theta \cdot \partial_x \phi_{\alpha_0}(x) + \sinh \Theta \cdot \phi_{\alpha_1}(x)],$$  (5)
where $\theta$ is the first parameter that could not be fixed at the level of effective theory and describes the strength of interaction at the edge [19, 20].

The Hamiltonian of the setup depicted in the upper panel of Fig. 2 contains several terms $H = H_0 + (A + A' + \text{h.c.})$, where the first term describes the bare chiral dynamics of the edge excitations:

$$H_0 = \sum_{s=+} \frac{v_s}{4\pi} \int dx [\partial_x \phi_{s=0}(x)]^2.$$  

(6)

The other terms describe weak backscattering at the two QPCs:

$$A = \tau \sum_{\sigma} e^{i[(\eta_{L=0} - \eta_{U=0})]}, A' = \tau' \sum_{\sigma} e^{i[(\eta_{U=0} - \eta_{R=0})]},$$  

(7)

where $\sigma = \pm$ enumerates the two types of local quasiparticle excitations with minimal charge $e^* = 1/m$ present in the model [19]:

$$\eta_{s=\sigma}(x) = \frac{1}{\sqrt{2}} \left[ \left( \frac{\cosh \theta}{\sqrt{m}} + \sigma \frac{\sinh \theta}{\sqrt{n}} \right) \phi_{s=0}(x) + \left( \frac{\sinh \theta}{\sqrt{m}} + \sigma \frac{\cosh \theta}{\sqrt{n}} \right) \phi_{s=1}(x) \right],$$  

(8)

where the number $n \in 2\mathbb{Z} + 1$ is the second parameter that could not be fixed at the level of effective theory. The question of which particular value of $n$ corresponds to actual experimental conditions could be answered either using microscopic ab initio calculations or experimentally [19]. Below we show that the experiments proposed in this Letter are also good tools for such task.

The equilibrium correlation function of quasi-particles in the effective theory described by (4) and (6) is given by

$$\left< e^{i\eta_{U=0}(x,t)} e^{-i\eta_{U=0}(x,0)} \right> \propto \left[ \Theta / \sin \pi \Theta (i t + \varepsilon) \right]^{\delta},$$  

(9a)

$$\delta = \frac{1}{2 mn} \left( m + n \right) \cosh 2\theta + 2\sigma \sqrt{mn \sinh 2\theta}. $$  

(9b)

Consequently, the injection current in the weak tunnelling limit, given by the Kubo formula $\langle I \rangle = e^* \int dt \langle [A^\dagger(t), A(0)] \rangle$, at the low temperature reads:

$$\langle I \rangle \propto |\tau|^2 \Delta \mu^{2\delta - 1}, \quad \delta = \min(\delta_+, \delta_-).$$  

(10)

We see that in the non-chiral case, scaling depends on the mixing angle $\theta$, which is related to the strength of interactions at the edge [19]. This is an obstacle in the experimental identification of the topological parameter $n$, and this also requires us to find an indication of the non-equilibrium state which is not sensitive to interactions. We show that the effective temperature measured in the two proposed schemes is indeed not sensitive to the scaling $\delta$. Therefore, without loss of generality, we focus on the situation of strong interactions $\theta = 0$, where one of the modes is completely neutral, see Eq. (5). We consider this situation to be the most relevant in view of the recent observation of neutral edge modes [16].

**Noise power.**— The backscattering currents operator at the detector (right) QPC could be found as a commutator of the total charge at the $s = R$ edge with the total Hamiltonian. The result is $j_{bs} = ie^*(A' - A'^\dagger)$, so that to the leading order in $\tau'$, the noise power (1) of these currents is given by $S_{bs} = \langle e^* \rangle^2 \int dt \langle [A'^\dagger(t), A'(0)] \rangle$, while the average current is zero. Using Eq. (7) we find that the noise power is determined by the correlation functions of the quasi-particle operators:

$$S_{bs} = 2 \langle e^* \rangle^2 |\tau'|^2 \int dt \left< e^{-i\eta_{U=0}(D,t)} e^{i\eta_{U=0}(D,0)} \right> \times \left< e^{i\eta_{R=0}(D,t)} e^{-i\eta_{R=0}(D,0)} \right>. $$  

(11)

In an equilibrium state with temperature $\Theta$, the correlation function is given by Eq. (9a) and the integral in Eq. (11) evaluates to

$$S_{bs} \propto \Theta^{2\delta - 1},$$  

(12)

where the quasi-particle’s scaling dimension at $\theta = 0$ is $\delta = 1/2m + 1/2n$. Starting from this relation, we define the effective noise temperature as $\Theta_{eff} = S_{bs}^{1/(2\delta - 1)}$.

Next, we proceed with the calculation of the noise temperature in the two QPC setup. Assuming zero base temperature, the correlation function for $s = R$ is evaluated over the ground state $\left< e^{i\eta_{U=0}(D,t)} e^{-i\eta_{R=0}(D,0)} \right> \propto \langle i t + \varepsilon \rangle^{-\delta}$. As discussed above, the perturbative theory is divergent and one needs to find a non-perturbative expression for the correlation function at the upper, $s = U$, edge in a non-equilibrium state created by the backscattering process at the injection (left) QPC. This can be done using the non-equilibrium bosonization technique proposed in Ref. [9]. In this technique, the boson fields $\eta_{U=0}$ are expressed in terms of the backscattering currents at the source QPC by solving the equations of motion generated by Hamiltonian (6) with the boundary conditions at $x = 0$:

$$j_{U=0}(t) = \partial_t \eta_{U=0}(0, t)/2\pi.$$  

(13)

In the situation we consider, the edge dynamics is given by $\eta_{U=0}(D,t) = \eta_{U=0}^{\text{eq}}(D,t) + 2\pi \sigma e^* \left[ N_+(t - D/v_1) - N_-(t - D/v_1) \right]$, where $e^* N_\sigma(t) \equiv \int_0^\infty \left[ j_{U=0}(t') - j_{U=0}^{\text{eq}}(t') \right] dt'$ are the operators of the number of quasiparticles, see Fig. 3, left panel for notations. At large times, the numbers $N_\sigma$ could be considered as classical fluctuating quantities and the correlation function factorizes as

$$\left< e^{-i\eta_{U=0}(D,t)} e^{i\eta_{U=0}(D,0)} \right> \propto \chi(t)(it + \varepsilon)^{-\delta}, $$  

(14)

where $\chi(t)$ is the purely non-equilibrium contribution expressed via the full counting statistics [21] of the quasi-
particles’ backscattering process:

\[
\chi(t) = \prod_{\sigma} \chi_{\sigma}(2\pi \sigma/m, t), \quad \chi_{\sigma}(\lambda, t) \equiv \langle e^{i\lambda N_{\sigma}(t)} e^{-i\lambda N_{\sigma}(0)} \rangle. \tag{15}
\]

Although in general the full counting statistics is an extremely complex quantity, two important simplifications arise in the limit of weak backscattering: First, the main contribution to the integral (11) comes from large times, where backscattering could be considered as a classical (Markovian) process. Second, all the cumulants of quasi-particle numbers for the rare, Poissonian, process are equal to the average number, i.e., \( \langle N^p_{\sigma}\rangle \equiv \partial^p_{\lambda} \chi_{\sigma}(\lambda, t)|_{\lambda=0} = \langle I \rangle |t|/2e^* \) for all \( p \). Thus, the non-equilibrium part of the correlation function is given by

\[
\log \chi(t) \simeq \frac{\langle I \rangle |t|}{e^*} [\cos(2\pi/m) - 1], \quad \Delta \mu |t| \gg 1. \tag{16}
\]

Substituting this expression back in Eq. (14) and then in Eq. (11) we arrive at the following result:

\[
\frac{\langle I \rangle |t|}{e^*} \left[ e^{2\pi/m} - 1 \right] \cdot \frac{\Delta \mu |t|}{1}. \tag{17}
\]

where \( s = U, R, \bar{s} = R, U \), and the dimensionless couplings \( g_s \equiv U_s/\pi v_0 \) take values between 0 and 1 and have physical meaning of charges in units of \( e^* \) accumulated at the corresponding channels due to interactions with the QD. As well, the energy \( \bar{\epsilon}_0 \) is generally renormalized [23] to \( \bar{\epsilon}_0 = \epsilon_0 + \sum_s \int dx U^2_s(x)/\epsilon_0 \).

There are two regimes in which level broadening is dominated by one of the two sources: First, quantum, source is tunnelling of the quasi-particles to and from the QD. Second, classical, source is the interaction with the non-equilibrium charge fluctuations, which “shake” the level. We focus on the situation of classical level broadening, where the QD is almost totally incoherent and can be described by a master equation [23].

The classical regime is realized if \( G' \ll G \ll 1/2\pi \), where \( G \) is the conductance of the injector QPC, \( G' \) is the conductance of the detector. Level broadening in this regime can be found by studying the rate of tunnelling of quasi-particles from the upper channel to the QD, which is given by \( \Gamma = \int dt A^*_0(t) A_0(0) \) in the leading order. Using Eq. (18) we find that the tunnelling rate is given by the following integral:

\[
\Gamma \propto \int \frac{dt}{\mu} e^{i\epsilon_0 (it + \bar{\epsilon})^{-\delta'}} , \tag{19}
\]

where \( \delta' = [g_R^2 + (1 - g_U)^2]/2m + 1/2n \), and the contour of integration goes as shown in Fig. 3, middle panel.

Evaluating the integral, we find that the scaling of the tunnelling rate is given by

\[
\Gamma(\epsilon_0) \propto \text{Re}[i(\epsilon_0 + i\Delta \epsilon_0)^{\delta'-1}], \tag{20}
\]

where \( \Delta \epsilon_0 = \langle I \rangle [\cos(2\pi/m) - 1]/e^* \). It is not universal and depends on the interaction strength via couplings \( g_s \). However, the level broadening itself has a universal linear dependence on the current so that the effective temperature in this situation is given by Eq. (2) as well.
This result serves as an additional argument in favor of the universality of relation (2) and shows that it is not a property of a particular detection scheme. The plot of the tunnelling rate (20) for the case of strong interaction is shown in Fig. 3, right panel.

To summarise, we show that the non-equilibrium state created by coupling two fractional QH edge states relaxes through an intermediate stationary state that is qualitatively different from an equilibrium one. We find non-perturbatively the noise power and the level broadening generated by such state and show that they can be described by an effective temperature which has a universal linear dependence on the injected current. The measurement of scaling of the backscattering noise with the bias in such states can be also used to efficiently distinguish effective models of edge states.

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