Nonlinear dynamic behaviour of functionally graded circular plates resting on two-parameters foundation using differential transform method

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Abstract. Study of nonlinear dynamic behaviour of plates resting on two parameters foundation under various environmental conditions has been an area of increasing research interests in industrial and engineering applications. In this work, the dynamic behaviour of functionally graded plates resting on Winkler and Pasternak foundation is investigated. The inherent singularities in the governing equation are analyzed using differential transform method. Good relationship is established when the results of the semi - approximate analytical methods are compared with the already established research based on experimental. To assess the effect of nonlinear Pasternak foundation introduced, the governing differential equation is converted using Galerkin one-parameter approach into Duffing equation. Thereafter, the domain issue peculiar to semi analytical solutions is treated using cosine after treatment method. Analytical Solutions obtained are used to examine the material properties variation, linear Winkler, nonlinear Winkler and Pasternak foundation effects on plate vibration. It is established that natural frequency decrease with decrease in thickness of the plate, Natural frequency increases with increase in the value of Winkler and Pasternak foundation coefficients. The modal shape obtained satisfy the classical theory of the plate vibration. It is expected that the findings of this work will be relevant in tackling problems related to plate vibration in the industries.

Keywords: Non-linear Analysis; Functionally Graded Plate; Two-parameters foundation; Vibration; Differential transform method

1. Introduction

Functionally Graded Material (FGM) also known as inhomogeneous composite plate was first proposed by a materials scientist in Japan during a space aircraft construction project [1]. The intention was to design a thermal absorber material with ability to withstand high temperature over the surface and gradient area. Its higher thermal conductivity and ability to reduce high thermal pressures as compared to homogenous material have made it suitable in various fields of engineering such as space aircraft, civil, nuclear, mechanical, computer, nuclear reactor, military and aircraft industry. Consequently, there have been various studies on the thermal and dynamic behaviour of plates made of FGM in recent times.
Benyoucef et al. [2], considered simply and clamped support conditions in the analysis of thick FGM plate placed on elastic foundation. Subsequently, Fallah et al. [3] presented analysis of plate resting on Winkler foundation using Kantorovich method with infinite power series. In another work, Adel [4] adopted numerical finite method in investigating behaviour of deep beam resting on Winkler foundation. In a later study, Utkan [5] used Homotopy Perturbation Method (HPM) to examine the dynamic reaction of Euler beam on elastic foundation. Also, Ali [6] reaffirm the power of HPM in investigating the nonlinear vibration of rectangular plate. Based on literatures cited [7-11], dynamic behaviour of FGM Plate resting on nonlinear Pasternak foundation is not achievable using exact method due to the complex mathematical and computational difficulties involved. The singularities issue involved in the governing equations posed a huge challenge in analyzing the governing equation with most semi-analytical methods like Adomian decomposition, Homotopy analysis method, Variation of parameter methods and Duan-rang method. Based on preceding researchers [12-13], it is undisputed that semi-analytical method differential transform method (DTM) contrary to others, has a significant advantage over others semi-analytical method. DTM is a powerful convergence series with higher accuracy, adopted an easier algebraic approach in converting governing and boundary equations into recursive form. It has proven record of solving both linear and nonlinear engineering problems. It can easily deal-with non-trivial and singularity problems, ability to extend the small domain challenges to large domain with cosine method.

In recent time, research into plate resting on elastic foundation vibration analysis has gained more popularity as contained in many engineering and science publications. This is because of its applications into various branches of engineering like geotechnics, mechanical, piping, and bridge deck. Winkler foundation originally designed for railway track analysis [14-18] is majorly adopted as elastic foundation model in designing structures resting on elastic foundations. Winkler foundation is bad in withstanding shear stress thereby posing a different representation of soil characteristics. Hence wrong structural response is expected. To mitigate this, introduction of mechanical foundation model is adopted by researchers. Shariyat et al. [19] in the interest of taking care of this, conducted research using DTM on vibration analysis of FGM plate placed on two-parameters foundation. Subsequently, Benyoucef [2] used hyperbolic model to analyze static response of simply supported FGM plates on two-parameters foundation. Also, Ait [20] proposed higher shear deformation theory in investigating dynamic properties of FGM plate resting on elastic linear Winkler and Pasternak foundations. Thereafter, Mohammad [21] employed Rayleigh-Ritz method to determine the free vibration analysis of skewed graded plate on Winkler and Pasternak elastic foundation.

Earlier researches on dynamic behaviour of FGM plate resting on elastic foundations are restricted to linear Winkler and linear Pasternak as revealed in the above reviewed works. To the author’s best knowledge, dynamic behaviour of FGM plate resting on Winkler, and nonlinear Pasternak using DTM with sine and cosine after treatment technique has not been investigated. Therefore, this present study focuses on nonlinear dynamic behaviour of functionally graded circular plates resting on two-parameters foundation using differential transform method along with sine and cosine after treatment technique for the development of analytical solution. Subsequently, parametric investigation is carried out using the analytical solutions.

2. Problem Formulation and Mathematical Analysis

Consider a circular FGM plate with material properties classified in the radial and transverse direction as shown in figure 1. The FGM plate is resting on linear Winkler, and nonlinear Pasternak foundation.

$$E(z,r) = \left[ (E_c - E_m) \left( \frac{z}{h} + \frac{1}{2} \right)^x + E_m \right] \left[ 1 + \alpha \left( \frac{r}{b} \right)^\beta \right],$$  \hspace{1cm} (1)
The mid-plane is the reference point for beginning the $z$ coordinate with upward as positive. $g$ is the volume fractional index; $b$ is the external edge radius. Also, $\alpha$ is the coefficient of radial variation and $\beta$ is the exponent power law. $\alpha, g, \beta$ are parameters deciding the material profile variation. The relevant material properties for the constituent material adopted are presented in table 1.

![Figure 1](https://via.placeholder.com/150)

**Figure. 1** Functionally Graded Plate resting on Pasternak and Winkler foundation

\[
\rho(z) = \left[ (\rho_c - \rho_m) \left( \frac{z}{h} + \frac{1}{2} \right)^g + \rho_m \right],
\]  
\[
(2)
\]

Where $w$ is the transverse displacement, $\nabla^4$ is biharmonic operator, $h$ is the thickness of the plate, $E$ represents the modulus of elasticity of the plate, $k_w, k_s$ and $k_p$ are Winkler’s and Pasternak’s coefficient of the elastic foundation respectively. Meanwhile, $\rho$ and $D$ are mass density and flexural rigidity of plate, and $V$ is the Poisson’s ratio.

\[
D = \frac{1}{1 - \nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z, r) z^2 dz,
\]  
\[
(4)
\]

\[
\rho = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z) dz,
\]  
\[
(5)
\]

Substituting equations (1) and (2) into (4) and (5) results to the following equations:

\[
D = \frac{1}{1 - \nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ 1 + \alpha \left( \frac{r}{b} \right)^\beta \right] \left( E_c - E_m \right) \left( \frac{z}{h} + \frac{1}{2} \right)^g + E_m \right] z^2 dz,
\]  
\[
(6)
\]
$$D = \frac{h^3}{1-v^2} \left[ 1 + \alpha \left( \frac{r}{b} \right)^6 \right] \left[ (E_c - E_m) \frac{(g^2 + g + 2)}{4(g+1)(g+2)(g+3) + E_m} \right],$$  \hspace{1cm} (7)

$$\rho = \frac{1}{h} \int \left[ \rho_c - \rho_m \right] \left( \frac{z}{h} + \frac{1}{2} \right)^2 + \rho_m \left[ \left( \frac{\rho_m g + \rho_c}{\rho_m (g+1)} \right) \right] \, dz,$$  \hspace{1cm} (8)

$$D^* = \frac{h^3 E_c}{12(1-v^2)},$$  \hspace{1cm} (9)

$$A(g) = \frac{D}{D^*} = \frac{(E_m g^3 + (3E_c + 3E_m) g^2 + (3E_c + 8E_m) g + 6E_c)}{E_c (g+1)(g+2)(g+3)},$$  \hspace{1cm} (10)

In dynamic analysis of circular plate, for harmonic solution can be written as [19]:

$$w = w(r) e^{i\omega t},$$  \hspace{1cm} (11)

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}; \quad \nabla^4 = \frac{d^4}{dr^4} + \frac{2}{r} \frac{d^3}{dr^3} - \frac{1}{r^2} \frac{d^2}{dr^2} + \frac{1}{r^3} \frac{d}{dr},$$  \hspace{1cm} (12)

$$\omega$$ is the natural frequency. The governing equation turns to steady state Ordinary differential equation. Putting equations (11) and (12) into equation (3) gives:

$$D \left( \frac{d^4 w}{dr^4} + \frac{2}{r^2} \frac{d^3 w}{dr^3} - \frac{1}{r^3} \frac{d^2 w}{dr^2} + \frac{1}{r^4} \frac{dw}{dr} \right) + 2 \frac{dD}{dr} \frac{d^3 w}{dr^3} + \left[ (2 + \nu) \frac{dD}{dr} + \frac{r^2 D^2}{r^2} \frac{d^2 D}{dr^2} \right] \frac{d^2 w}{dr^2} + \left( \frac{dD}{dr} - \frac{r}{r^2} \frac{d^2 D}{dr^2} \right) \frac{dw}{dr}$$

$$- k_s \left( \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right) + k_n w + k_p w^3 = \rho \omega^3 \psi,$$  \hspace{1cm} (13)

Non-dimensionless variables are defined as:

$$R = \frac{r}{b}, \quad W = \frac{w}{W_{\max}},$$  \hspace{1cm} (14)

$$\hat{D} \left( \frac{d^4 W}{dR^4} + \frac{2}{R} \frac{d^3 W}{dR^3} - \frac{1}{R^2} \frac{d^2 W}{dR^2} + \frac{1}{R^3} \frac{dW}{dR} \right) + 2 \hat{D} \frac{d^3 W}{dR^3} + \left[ (2 + \nu) \frac{d\hat{D}}{dR} + \frac{R^2 \hat{D}^2}{R} \frac{d^2 \hat{D}}{dR^2} \right] \frac{d^2 W}{dR^2} + \left( \frac{d\hat{D}}{dR} - \frac{1}{R} \frac{d^2 \hat{D}}{dR^2} \right) \frac{dW}{dR}$$

$$- \frac{d\hat{D}}{dR} R \frac{d^2 \hat{D}}{dR^2} \frac{1}{R^2} \frac{dW}{dR} - k_s \left( \frac{d^2 W}{dR^2} + \frac{1}{R} \frac{dW}{dR} \right) b^2 \frac{b^4}{D_{\max}} + \frac{b^4}{D_{\max}} k_n W = \left( \frac{b^4 \rho h}{D_{\max}} \right) \omega^3 \psi,$$  \hspace{1cm} (15)

Further dimensionless Parameters:


\[
\Omega^2 = \frac{\rho h b^4}{D_{\text{max}}} \omega^2, \quad k_s = \frac{k_s b^2}{D_{\text{max}}}, \quad k_w = \frac{k_w W_{\text{max}} b^4}{D_{\text{max}}},
\]

\[(16)\]

Assuming \( A(g) = \frac{E_m g^3 + (3E_c + 3Em) g^2 + (3E_c + 8Em) g + 6E_c}{E_e (g+1)(g+2)(g+3)} \),

\[(17)\]

\[
Y = \frac{\rho_w g + \rho_c}{\rho_n (g+1)},
\]

\[(18)\]

\[
A(g) \left(1 + a R^\beta \right) \frac{\partial^3 W}{\partial R^3} + 2A(g) \left[1 + a(\beta + 1) R^\beta \right] \frac{1}{R} \frac{\partial^2 W}{\partial R^2} + A(g) \left[-1 + a(-1 + \beta^2 + (\nu + 1) \beta) R^\beta \right] \frac{1}{R^2} \frac{\partial^2 W}{\partial R^2} \\
+ A(g) \left(1 + a(\beta - 1)(\beta \nu - 1) R^\beta \right) \frac{1}{R^3} \frac{\partial W}{\partial R} - k_s \frac{\partial^2 W}{\partial R^2} - k_w \frac{1}{R} \frac{\partial W}{\partial R} + k_w W + k_p W^3 = \rho \omega^2 W,
\]

\[(19)\]

Boundary Conditions adopted in this research are Clamped, Free and Simply-Supported edges. As established by Classical Plate theory, these conditions can be written in dimensionless form below equations;

2.1 Boundary Condition

**Free Edge Condition:**

\[
M_r \bigg|_{r=1} = \left\{ -D \left[ \frac{d^2 W}{dR^2} + \nu \left( \frac{1}{R} \frac{dW}{dR} \right) \right] \right\}_{r=1} = 0
\]

\[(20)\]

\[
V_r \bigg|_{r=1} = Q_r \bigg|_{r=1} = \left\{ -D \left[ \frac{d^3 W}{dR^3} + \frac{1}{R} \frac{d^2 W}{dR^2} + \frac{1}{R^2} \frac{dW}{dR} \right] - Rd \left[ \frac{d^2 W}{dR^2} + \nu \frac{dW}{R dR} \right] \right\}_{r=1} = 0
\]

\[(21)\]

**Simply- Supported Edge Condition:**

\[
W(1) = 0, M_r \bigg|_{r=1} = \left\{ -D \left[ \frac{d^2 W}{dR^2} + \nu \left( \frac{1}{R} \frac{dW}{dR} \right) \right] \right\}_{r=1} = 0
\]

\[(22)\]

**Clamped Edge Condition:**

\[
W(1) = 0, \frac{dW}{dR} \bigg|_{r=1} = 0
\]

\[(23)\]

Where \( M_r \) is the radial bending moment, and \( V_r \) is the effective shear for per unit length.

**Symmetric case**

\[
\frac{dw}{dr} \bigg|_{r=0} = 0, \quad V_r \bigg|_{r=0} = \left( \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) = 0, \quad \text{for} \ (m = 0, 2, 4 \cdots),
\]

\[(24)\]
3. Method of Solution: Differential Transform Method

Differential transform method introduced by Zhou [22], is a very effective semi-analytical method especially in handling singularity issues, non-trivial challenges associated with vibration equations. The solution has a fast convergence characteristic with high accuracy and predict almost like exact method. The governing equation and boundary conditions are transformed to recursive relationship and solved algebraically. Few of the principles are stated below.

Considering a function \( w(r) \) that is analytic in the domain \( R \), then it will be differential continuously with respect to space \( r \):

\[
\frac{d^k w(r)}{dr^k} = \varphi(r, k), \quad \text{for all} \quad r \in R
\]  

(25)

For \( r = r_i \), then \( \varphi(r, k) = \varphi(r_i, k) \), where \( k \) belongs to the set of nonnegative integers, denoted as the \( k \)-domain. Therefore, equation (25) is written as

\[
W(k) = \varphi(r_i, k) = \left[ \frac{d^k w(r)}{dr^k} \right]_{r=r_i}
\]  

(26)

Where \( W_i \) is the spectrum of \( w(r) \) at \( r = r_i \), \( w(r) \) is expressed in Taylor's series, then \( w(r) \) is presented as;

\[
w(r) = \sum_{k=0}^{\infty} \left[ \frac{(r-r_i)^k}{k!} \right] W(k),
\]  

(27)

Equation (27) is the inverse of \( F(k) \) using the symbol "\( D \)" representing the differential transform process and combining equations (26) and (27), we have:

\[
w(r) = \sum_{k=0}^{\infty} \left[ \frac{(r-r_i)^k}{k!} \right] W(k) = D^{-1}W(k),
\]  

(28)

3.1 Transformation of the governing equation

Using the operational properties of DTM given in table 1, the transformation of the governing differential equation (19) yields;

\[
A(g)[(k+1)(k+2)(k+3)(k+4)W(k+4)] + A(g)a\left( \sum_{l=0}^{k} \delta(l - \beta) \right) (k-l+1)(k-l+2)(k-l+3)(k-l+4)W(k-l+4)
\]

\[
+ 2A(g) \left( \sum_{l=0}^{k} \delta(l + 1) \right) (k-l+1)(k-l+2)(k-l+3)W(k-l+3)
\]

\[
+ 2A(g) \left( \sum_{l=0}^{k} \alpha(\beta + 1) \delta(l - \beta + 1) \right) (k-l+1)(k-l+2)(k-l+3)W(k-l+3)
\]

\[- A(g) \left( \sum_{l=0}^{k} \delta(l + 2) \right) (k-l+1)(k-l+2)W(k-l+2) \]

\]
\[
A(g) \left\{ \sum_{l=0}^{k} \left[ \alpha \left( 1 + \beta^2 + (v+1)\beta \right) \delta(l-\beta +2) \right] (k-l+1)(k-l+2)W(k-l+2) \right\} \\
+ A(g) \left\{ \sum_{l=0}^{k} \left[ \delta(l+3) \right] (k-l+1)W(k-l+1) \right\} \\
+ A(g) \left\{ \sum_{l=0}^{k} \left[ \alpha \left( 1 - \beta \right) \delta(l-\beta +3) \right] (k-l+1)W(k-l+1) \right\} \\
-k_s \{ (k+1)W(k+2) \} - k_s \left\{ \sum_{l=0}^{k} \left[ \delta(l+1) \right] (k-l+1)W(k-l+1) \right\} + k_n W(k) \\
+k_p \left\{ \sum_{l=0}^{k} \sum_{p=0}^{k} W(l)W(p)W(k-l-p) \right\} = -\rho h\Omega^2 W(k) 
\]

(29)

**Table 1:** Showing Operational property of differential transform method.

| S/N | Function | Differential Transform |
|-----|----------|-----------------------|
| 1   | \(w(r) \pm f(r)\) | \(W(k) \pm F(k)\) |
| 2   | \(\alpha w(r)\) | \(\alpha W(k)\) |
| 3   | \(\frac{dw(r)}{dr}\) | \((k+1)W(k+1)\) |
| 4   | \(\frac{d^2w(r)}{dr^2}\) | \((k+1)(k+2)W(k+2)\) |
| 5   | \(w(r)f(r)\) | \(\sum_{l=0}^{k} F(l)W(k-l)\) |
| 6   | \(r^m\) | \(\delta(k-m) \Rightarrow 1 \text{ if } k = m \) \(0 \text{ if } k \neq m\) |
| 7   | \(w^3(x)\) | \(\sum_{l=0}^{k} \sum_{p=0}^{k-l} W(l)W(p)W(k-l-p)\) |
| 8   | \(w(r) \frac{dw(r)}{dr}\) | \(\sum_{l=0}^{k} (k-l+1)W(l)W(k-l+1)\) |

Simplifying equation (29) using the operational principle of DTM given in table 1, we get

\[
W(k+4) = \frac{1}{Ag(k+4)} \left\{ -k_s \{ (k+2) W(k+2) \} + \{ \Omega^2 Y + k_s \} W(k) + A_{Ag} \right\} \\
+ k_p \left\{ \sum_{l=0}^{k} \sum_{p=0}^{k} W(l)W(p)W(k-l-p) \right\} \\
= \left\{ \frac{(k-\beta+1)(k-\beta+2)(k-\beta+3)(k-\beta+4) + 2(1+\beta)(k-\beta+2)(k-\beta+3)(k-\beta+4) + \beta^2 + (v+1)\beta(k-\beta+3)(k-\beta+4) + ((\beta-1)\beta(v-1)(k-\beta+4)}{(k-\beta+1)\beta(\beta-3)(k-\beta+4) + (\beta-1)(\beta-3)(k-\beta+4)} \right\} W(k-\beta+4) 
\]

(30)
3.2 Transformation of the boundary condition

As expected, since the dimensionless equation (19) is a fourth-order governing equation then, four boundary conditions is expected for resolving the equation. Two of the conditions may be obtained from the external condition of the plate while the rest two is obtained from the condition at the center of the plate. The regularity conditions at the center are given as,

\[(k + 1)W(k + 1) = 0 \quad k = 0: W(1) = 0,\]  \hspace{1cm} (31)

\[\begin{align*}
(k + 1)(k + 2)(k + 3)W(k + 3) + \sum_{l=0}^{k} \delta(l + 1)(k - l + 1)(k - l + 2)W(k - l + 2) + \sum_{l=0}^{k} \delta(l + 2)(k - l + 1)W(k - l + 1) \\
(k + 1)(k + 2)(k + 3)W(k + 3) + [(k + 2)(k + 3)W(k + 3)] - [(k + 3)W(k + 3)] \\
\end{align*}\]

\[k = 0: W(3) = 0,\]  \hspace{1cm} (32)

Therefore, to obtain the four conditions required to analyze fourth-order differential equation as stated earlier, two unknowns will be introduced more, we have:

\[W(k) = a \Rightarrow W(0) = a,\]  \hspace{1cm} (33)

\[W(k + 1) = b \Rightarrow W(2) = \frac{b}{2},\]  \hspace{1cm} (34)

Where \(a\) and \(b\) are unknown constants to be found later. Invariably, for symmetric case regularity condition at the center the transformation is:

\[W(0) = a,\]

\[W(1) = 0,\]

\[W(2) = \frac{b}{2},\]  \hspace{1cm} (35)

\[W(3) = 0,\]

3.3 The solution Procedure

To obtain the natural frequency of the governing equation resting on elastic foundation the same principle explained earlier is adopted on the three boundary conditions considered in this study. For the symmetric case, even counters are taken into consideration.

Applying the conditions at the center equation (35) on the recursive form of the governing equation (30) varying \(k = 0, 1, 2, 3, 4, 5, \ldots\) in the above recursive relation. The following equations are developed.

\[W(4) = -\frac{\Omega^2 a}{64},\]  \hspace{1cm} (36)

\[W(5) = 0,\]  \hspace{1cm} (37)

\[W(6) = -\frac{\Omega^2 b}{1152},\]  \hspace{1cm} (38)

\[W(7) = 0,\]  \hspace{1cm} (39)
Applying the principle of DTM. The series solutions can be written in this form;

\[
f = \sum_{j=0}^{n} W[j] r^j, \tag{43}
\]

\[
f = a + \frac{1}{2} b r^2 - \frac{\Omega^2 a r^4}{64} - \frac{\Omega^4 a r^8}{1152} + \frac{\Omega^2 b r^6}{147456} + \frac{\Omega^4 b r^{10}}{7372800} - \frac{\Omega^6 a r^{12}}{2123366400} - \frac{\Omega^8 b r^{14}}{208089907200} + \frac{\Omega^{16} a r^{16}}{106542032486400} + \frac{\Omega^{18} b r^{18}}{17259809262796800}, \tag{44}
\]

To validate the obtained series solutions with already established research, values of parameters \( \beta = \alpha = k_w = k_s \to 0 \) are set as zero and \( Y \to 1 \). Using the condition equations \((20)\) to equation \((23)\) to find the unknown \( a \) and \( b \) introduced to the boundary conditions results into simultaneous equation. Which may be written in this form:

\[
\psi_{11}^{(n)}(\Omega) w_0 + \psi_{12}^{(n)}(\Omega) w_2 = 0 \tag{45}
\]

\[
\psi_{21}^{(n)}(\Omega) w_0 + \psi_{22}^{(n)}(\Omega) w_2 = 0 \tag{45}
\]

The polynomials \( \psi_{11}, \psi_{12}, \psi_{21}, \text{and} \psi_{22} \) are represented in terms of the natural frequency \( \Omega \). Meanwhile \( \psi_{11}, \psi_{12}, \psi_{21}, \text{and} \psi_{22} \) are representing a series expression obtained from equation \((44)\). Therefore, equation \((45)\) may be written in matrix form as:

\[
\begin{bmatrix}
\psi_{11}^{(n)}(\Omega) & \psi_{12}^{(n)}(\Omega) \\
\psi_{21}^{(n)}(\Omega) & \psi_{22}^{(n)}(\Omega)
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}, \tag{46}
\]

The following Characteristic determinant is obtained applying the non-trivial condition:

\[
\begin{bmatrix}
\psi_{11}^{(n)}(\Omega) & \psi_{12}^{(n)}(\Omega) \\
\psi_{21}^{(n)}(\Omega) & \psi_{22}^{(n)}(\Omega)
\end{bmatrix} = 0, \tag{47}
\]

Which can be resolved to obtain the natural frequency. The convergence criterion is used to determine where the acceptable result be chosen.

Eigenvalues obtained;
Solving the quadratic equation (48) gives the non-dimensionless natural frequency for first, second, third, fourth, fifth and sixth modes respectively.

\[ \Omega^2 + \Omega^2 + \Omega^2 + \Omega^2 + 1 = \left( \frac{6 \Omega^2 + 62224 \Omega^4 + 26213548032 \Omega^2 + 528614953058304 \Omega^2 + 19918648360112121446400 \Omega^2 + 24941611859618656419840000 \Omega^2 + 718318421557017304891392000}{191775658475520 + 245760 + 1 - 479439146188800} \right)^{\frac{1}{2}} \]  

(48)

Inserting equation (49) back to equation (46) gives respective values of constant \( a \) and \( b \) for each mode;

\[
\begin{bmatrix}
24205 \\
8948 \\
49406 \\
6939
\end{bmatrix} \begin{bmatrix}
a \\
b\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix},
\]

(50)

First mode shape obtained by solving the matrix equation (50) making \( a = 1 \), and solved for \( b = \frac{141619}{30997} \), the deflection series solution is obtained by putting values obtained after solving equation (50), into equation (44);

\[
W = \frac{64733 r^2}{28337} + \frac{29101 r^4}{17846} - \frac{18270 r^6}{116065} + \frac{8573 r^8}{379739} + \frac{2563 r^{10}}{3806552} + \frac{723 r^{12}}{38619037} - \frac{95 r^{14}}{127381433} - \frac{43 r^{16}}{127381433},
\]

(51)

Same approach was repeated for other boundary conditions.

\[
W = \left( \frac{64733 r^2}{28337} + \frac{29101 r^4}{17846} - \frac{18270 r^6}{116065} + \frac{8573 r^8}{379739} + \frac{2563 r^{10}}{3806552} + \frac{723 r^{12}}{38619037} - \frac{95 r^{14}}{127381433} - \frac{43 r^{16}}{127381433} \right) e^{\omega t},
\]

(52)

### 3.4 Convergence study

\[
\left| \frac{\Omega^{(i)} - \Omega^{(i-1)}}{\Omega^{(i)}} \right| \leq \varepsilon, \; j = 1, 2, 3, \ldots n
\]

(53)

Where the iteration counter is represented by \( i \), the estimated value of the \( j \)th dimensionless natural frequency is \( \Omega^{(i)}_j \) and small number chosen is \( \varepsilon \). For this study \( \varepsilon = 0.0001 \).

### 3.5 The nonlinear governing differential equation

Due to challenges associated with nonlinear parameters in the governing equation, conducting parametric study of Pasternak foundation is difficult, thereby resulting into conversion of the governing equation to Duffing equation by introducing of Galerkin decomposition procedure to isolate the spatial from the temporal part of the governing differential equation as:
\( \phi(u) \) is the trial function while \( u(t) \) is the generalized coordinate.

\[ W(r,t) = \phi(r)u(t), \quad (54) \]

Applying one-parameter Galerkin solution to equation (19),

\[ \int_0^l R(r,t)\phi(r)dr, \quad (55) \]

Meanwhile,

\[ R(r,t)=A(g)(1+\alpha R^\theta)R^\theta \frac{\partial^4 W}{\partial R^4} + 2A(g)(1+\alpha(\beta+1)R^\theta)R^\theta \frac{\partial^2 W}{\partial R^2} + A(g)(-1+\alpha(-1+\beta^\theta+(\nu+1)\beta)R^\theta)R^\theta \frac{\partial^2 W}{\partial R^2} + \]

\[ A(g)(1+\alpha(\beta-1)(\nu-1)R^\theta) \frac{\partial W}{\partial R} - k_R \frac{\partial^2 W}{\partial R^2} - k_R^\theta \frac{\partial^2 W}{\partial R^2} + k_R^3 W + k_R^3 W^3 + \rho h R \frac{\partial^2 W}{\partial t^2}, \quad (56) \]

We have:

\[ M \frac{\partial^2 u(t)}{\partial t^2} + Ku(t) + V(u(t))^3 = 0, \quad (57) \]

where

\[ M = \int_0^l (\rho h R^3 (\phi(r))^2)dr, \quad (58) \]

\[ V = \int_0^l (k_R R^3 (\phi(r))^3)dr, \quad (59) \]

\[ K = \left\{ \int_0^l \left( A(g)(1+\alpha R^\theta)R^\theta \frac{\partial^4 \phi(r)}{\partial R^4} \phi(r) + 2A(g)(1+\alpha(\beta+1)R^\theta)R^\theta \frac{\partial^2 \phi(r)}{\partial R^2} \phi(r) + \right) \right\} \left\{ \int_0^l \left( A(g)(1+\alpha(\beta-1)(\nu-1)R^\theta) \frac{\partial \phi(r)}{\partial R} \phi(r) - k_R \frac{\partial^2 \phi(r)}{\partial R^2} \phi(r) - k_R^\theta \frac{\partial^2 \phi(r)}{\partial R^2} \phi(r) - k_R^3 W \phi(r) - k_R^3 W^3 + \rho h R \frac{\partial^2 \phi(r)}{\partial t^2} \phi(r) \right) \right\} dr, \quad (60) \]

3.6 The initial boundary conditions.

Simply-Supported

\[ \phi(r) \Rightarrow \sin \left( \frac{n\pi}{l} \right) r \Rightarrow n = 1, l = 1 \text{ n = mode}, \quad (61) \]

\[ \phi(r) \Rightarrow \pi r, \quad (62) \]

\[ U(0,r) = A, \quad U'(0,r) = 0, \quad (63) \]

Applying DTM rule as stated earlier on equation (57) using the operational properties stated in table 1, one obtains:

\[ M \left( p+2 \right) \left( p+1 \right) (p+2) + KU(p) + V \sum_{m=0}^{n} \sum_{n=0}^{m} U(n)U(m-n)U(p-m) = 0, \quad (64) \]

\[ U(0) = A; U(1) = 0, \quad (65) \]
\[ U(p+2) = \frac{1}{M(p+2)(p+1)} \times \left\{ -KU(p) - V \sum_{m=0}^{p} \sum_{n=0}^{m} U(n)U(m-n)U(p-m) \right\}, \quad (66) \]

Varying P from 0, 1, 2, 3...10 gives;

\[ U(2) = \frac{1}{2} \frac{-VA^3 - KA}{M}; \quad (67) \]

\[ U(3) = 0, \quad (68) \]

\[ U(4) = \frac{1}{12M} \left( -1 \frac{K(-VA^3 - KA)}{2M} - \frac{3}{2} \frac{VA^2(-VA^3 - KA)}{M} \right), \quad (69) \]

\[ U(5) = 0, \quad (70) \]

\[ U(6) = \frac{1}{30M} \left( \frac{1}{M} \left( \frac{1}{2} \frac{K(-VA^3 - KA)}{M} - \frac{3}{2} \frac{VA^2(-VA^3 - KA)}{M} \right) \right)^4 \left( \frac{1}{4M} \left( \frac{1}{2} \frac{K(-VA^3 - KA)}{M} - \frac{3}{2} \frac{VA^2(-VA^3 - KA)}{M} \right) \right)^4, \quad (71) \]

3.7 Introducing Cosine after treatment rule

\[ \lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 = -2 \times \left( \frac{VA^3 - KA}{2M} \right), \quad (72) \]

\[ \lambda_1 \Omega_1^4 + \lambda_2 \Omega_2^4 = 24 \times \frac{1}{12M} \left( -1 \frac{K(-VA^3 - KA)}{2M} - \frac{3}{2} \frac{VA^2(-VA^3 - KA)}{M} \right), \quad (73) \]

\[ \lambda_1 \Omega_1^6 + \lambda_2 \Omega_2^6 = -720 \times \frac{1}{30M} \left( -1 \frac{K(-VA^3 - KA)}{2M} - \frac{3}{2} \frac{VA^2(-VA^3 - KA)}{M} \right)^2 \left( \frac{3}{4} \frac{K(-VA^3 - KA)}{M^2} \right), \quad (74) \]

\[
\begin{align*}
\lambda(t) &= A \frac{5VA^2 + 4K - \sqrt{27A^2V^2 + 42A^2KV + 16K^2}}{2} \cos \left( \frac{M(6VA^2 + 5K + \sqrt{27A^2V^2 + 42A^2KV + 16K^2})}{M} \right) \\
\sin \left( \frac{\pi r}{t} \right),
\end{align*}
\[
\begin{align*}
\omega(t) &= A \frac{-VA^2 - 4K - \sqrt{27A^2V^2 + 42A^2KV + 16K^2}}{2} \cos \left( \frac{M(-6VA^2 - 5K + \sqrt{27A^2V^2 + 42A^2KV + 16K^2})}{M} \right) \\
\end{align*}
\]

where, \( A, V, K, M \) obtained above
4. Results and Discussion

The analytical solution of governing equation of motion of the circular FGM plate resting on Winkler and Pasternak foundation with differential transform method and cosine after treatment plate under various boundary conditions is presented here. The material properties for the circular FGM plate as stated in table 2 are [19]:

Aluminum: \( E_m = 70 \, Gpa \), \( \rho_m = 2,702 \, kg / m^3 \),

Alumina: \( E_c = 380 \, Gpa \), \( \rho_c = 3,800 \, kg / m^3 \), respectively.

Due to the presence of nonlinear Winkler foundation in the model, there is need to adopt Galerkin method of separation to convert the governing equation to Duffing equation for the parametric study. The results are non-dimensionless dynamic analysis solution and its valid for all radius thickness. Table 3 shows validation of the first two natural frequency obtained with reported results in the literature and confirmed in good agreement. All modal shapes agreed with classical theory of vibration as presented in Figures 2 and 3.

| S/N | Parameters                              | Values                       |
|-----|-----------------------------------------|------------------------------|
| 1   | Winkler’s Coefficient \( k_w \)         | \( 0,50,100 \)              |
| 2   | Pasternak’s coefficient \( k_s \)       | \( 0,5,10,15 \)             |
| 3   | Dimensionless density \( Y \)           | \( 1,3386 \)                |
| 4   | Young’s Modulus of Metal \( E_m \)      | \( 70 \times 10^9 \) N/m^2  |
| 5   | Young’s Modulus of ceramic \( E_c \)    | \( 380 \times 10^9 \) N/m^2 |
| 6   | Mass density of Ceramic \( \rho_c \)    | \( 3,800 \) kg/m^3          |
| 7   | Mass density of Metal \( \rho_m \)      | \( 2,702 \) kg/m^3          |
| 8   | Thickness of FGP \( h \)                | \( 3 \times 10^{-3} \) m    |
| 9   | Poisson’s ratio \( \nu \)               | \( 0.3 \)                   |

4.1 Convergence study

Figures 2 and 3 with Table 4(a) and (4b) shows the convergence study. From the results presented, it is observed that for fundamental frequency converges at fifteen iterations while for subsequent higher modes more iterations as presented in the Tables (4a) and (4b) are required. This behaviour is peculiar to vibration problem while subsequent iterations produce further results. Figure 2 and 3 shows the graphical illustrations of the behaviour.
Figure 2. Convergence study of free

Figure 3. Convergence study of simply supported edge.

Table 4(a). Convergence study

| Iteration | free edge | Simply-supported edge |
|-----------|-----------|-----------------------|
| N         | mode1     | mode2     | mode3     | mode1     | mode2     | mode3     | mode4     |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 15        | 9.0028685 | 38.1640564| 4.9351271 | 29.1584754| 74.0544780| 135.4372504|
| 25        | 9.0032347 | 38.4391505| 4.9351358 | 29.7193067| 74.1948440| 137.3017798|
| 30        | 9.0032188 | 38.4382992| 87.7640764| 4.9351222 | 29.7201129| 74.1946931 | 137.3017798|
| 35        | 9.0032187 | 38.4387140| 87.8285588| 4.9351222 | 29.7201129| 74.1946931 | 137.3017798|

This behaviour is attributed to more complex series functions combination. Results shown in Table 4(a) and 4(b) illustrate that, first two natural frequency gives a reasonable prediction of the FGM circular plate.

Table 4(b). Convergence study

| Iteration | Clamped edge |
|-----------|--------------|
| N         | mode1     | mode2     | mode3     | mode4     |
|-----------|-----------|-----------|-----------|-----------|
| 15        | 10.2158739 | 38.0968279| 88.4600093| 152.9066917|
| 25        | 10.2158262 | 39.7711686| 89.1051165 | 158.0045689|
| 30        | 10.2158262 | 39.7711482| 89.1041501 | 158.0045689|
| 35        | 10.2158262 | 39.7711482| 89.1041501 | 158.0045689|

4.2 Effect of Material variation on natural frequencies

As earlier stated, material property variation depends on $\alpha$, $\beta$ and $g$ which is connected by equation (1), $\beta$ is a factor that depicts the compactness of material properties. Also, the FGM thickness is represented by $\alpha$, the thickness (positive) shows a direct proportional relationship with that the natural frequency. Therefore, the deflection of the plate is lower when the thickness is lower (negative). Table 5 depicts the correlation with this rule. Increase in the value of $\beta$ means more stiffness in material, thereby resulting to lower natural frequency of vibration with increase in value of $\beta$. 
### Table 5. Variation of material properties for first two mode.

| Edge Condition | \( \beta = 2 \) | \( \beta = 4 \) |
|----------------|-----------------|-----------------|
| \( \alpha = 0.5 \)  | g = 0.2 | g = 5 | g = 0.2 | g = 5 |
| Free          | \( \Omega_1 \) 6.9360 | 5.4963 | 2.1605 | 1.71207 |
|               | \( \Omega_2 \) 18+60.903*I | 14+48.262*I | 32.9205 | 26.0874 |
| \( \alpha = -0.5 \) | \( \Omega_1 \) 6.4180 | 5.0858 | 5.7808 | 4.5809 |
| Simply-Supported | \( \Omega_2 \) 27.6207 | 21.8876 | 39+69.071*I | 31+54.734*I |
| \( \alpha = 0.5 \) | \( \Omega_1 \) 4.2325 | 3.3540 | 4.8215 | 3.8207 |
| Clamped       | \( \Omega_2 \) 20.2954 | 16.0828 | 29.7426 | 23.5691 |
| \( \alpha = -0.5 \) | \( \Omega_1 \) 3.4736 | 2.7526 | 2.6685 | 2.1146 |
|               | \( \Omega_2 \) 20.6575 | 16.3698 | 17.9174 | 14.1984 |

#### 4.3 Effect of Elastic Foundation Parameters on natural frequencies

The analysis is performed on the three boundary conditions earlier discussed and one conditions at the center. In this study, Consideration is given to different values of Winkler and Pasternak foundation as stated in Tables 6-8.

Parametric effect of nonlinear Winkler foundation is carried out as earlier mentioned using Duffing equation due to difficulty encountered when analyzing with DTM. Moreover, it a peculiar character of plate to be affected by characteristic of elastic foundation, comparing Tables 6-8, it is revealed in the tables that for both plate and foundation stiffness to be comparable there is a need to properly study the foundation stiffness to be chosen.

### Table 6. Natural frequencies of combined Winkler and Pasternak Foundation using Duffing equation.

| Edge Condition | \( \beta = 2 \) | \( \beta = 4 \) |
|----------------|-----------------|-----------------|
| \( \alpha = 0.5 \)  | g = 1, \( k_w = 10, \) k_\( s \) = 5, k_\( p \) = 10 | g = 1, \( k_w = 50, \) k_\( s \) = 15, k_\( p \) = 20 | g = 1, \( k_w = 10, \) k_\( s \) = 5, k_\( p \) = 10 | g = 1, \( k_w = 50, \) k_\( s \) = 15, k_\( p \) = 20 |
| Simply-Supported | \( \Omega_1 \) 243.3867*I | 324.9961*I | 236.0982*I | 319.5743*I |
| \( \alpha = -0.5 \) | \( \Omega_1 \) 203.6599*I | 296.4164*I | 212.0662*I | 302.2539*I |
Table 7. Variation of foundation stiffness, material properties for first two modal frequencies.

| Edge Condition | β = 2 | β = 4 |
|----------------|-------|-------|
|                | g = 1, kᵦ = 10, kᵦ = 0, kₖ = 0 | g = 1, kᵦ = 100, kᵦ = 0, kₖ = 0 |
| α = 0.5        |       |       |
| Free           | Ω₁    | 2.8829 | 9.1167 | 3.4644 | 9.3169 |
|                | Ω₂    | 6.8081 | 11.0069 | 29.4151 | 30.6603 |
|                | Ω₁    | 2.8829 | 9.1166 | 2.1495 | 8.9119 |
|                | Ω₂    | 4.7409 | 9.8629 | 5.1665 | 10.0744 |
| Simply-Supported | α = -0.5 |       |       |
| Free           | Ω₁    | 18.2759 | 20.2190 | 26.6042 | 27.9747 |
|                | Ω₂    | 4.2251 | 9.6257 | 3.7338 | 9.42036 |
|                |       | 24.5170 | 25.9978 | 21.7994 | 23.4524 |

As it is expected in all cases, increasing the foundation stiffness results into higher value of natural frequencies. Moreover, it is also observed that, effect of the difference in natural frequencies is more significant for higher mode of the FGM circular plate. The results obtained ascertain the fact that elastic foundation stiffness always has a direct proportional relationship proportion to natural frequency. Table 6, 7, 8 reveals that the presence of Winkler foundation and Pasternak foundation increases with increase in the natural frequency.

Table 8: Natural frequencies of FGM Plate for various combinations of material properties, Winkler foundation under different supports conditions

| Edge Condition | β = 2 | β = 4 |
|----------------|-------|-------|
|                | g = 1, kᵦ = 0, kᵦ = 5, kₖ = 0 | g = 1, kᵦ = 0, kᵦ = 15, kₖ = 0 |
|                | g = 1, kᵦ = 0, kᵦ = 5, kₖ = 0 | g = 1, kᵦ = 0, kᵦ = 15, kₖ = 0 |
| α = 0.5 Free   | Ω₁    | 4.8226 | 11+13.581*I | 29.7431 | 30.1866 |
|                | Ω₂    | 15+53.317*I | 103.9481i | 69.2254 | 69.1329 |
|                | Ω₁    | 5.1706 | 4.2780 | 4.3448 | 33+58.309*I |
|                | Ω₂    | 22.8491 | 19.9806 | 34+60.399*I | 33-58.309*I |
|                | Ω₁    | 2.1922 | 9.5210 | 3.1149 | 16.0472 |
|                | Ω₂    | 15.5965 | 16+65.101*I | 24.2345 | 69.6912 |
| Simply-Supported | α = -0.5 |       |       |
| Free           | Ω₁    | 4.8639 | 15.0885 | 69.2254 | 69.1329 |
|                | Ω₂    | 15+53.317*I | 103.9481i | 69.2254 | 69.1329 |
|                | Ω₁    | 5.1706 | 4.2780 | 4.3448 | 33+58.309*I |
|                | Ω₂    | 22.8491 | 19.9806 | 34+60.399*I | 33-58.309*I |
|                | Ω₁    | 2.1922 | 9.5210 | 3.1149 | 16.0472 |
|                | Ω₂    | 15.5965 | 16+65.101*I | 24.2345 | 69.6912 |
| Clamped        | α = -0.5 |       |       |
|                | Ω₁    | 16.3242 | 41.0842 | 39+74.044*I | 39+71.268*I |
|                | Ω₂    | 4.8639 | 15.0885 | 69.2254 | 69.1329 |
|                | Ω₁    | 5.1706 | 4.2780 | 4.3448 | 33+58.309*I |
|                | Ω₂    | 22.8491 | 19.9806 | 34+60.399*I | 33-58.309*I |
|                | Ω₁    | 3.8427 | 16.6493 | 3.4284 | 14.7242 |
|                | Ω₂    | 21.3470 | 48.6911 | 19.5126 | 43+75.346*I |
4.4 Effect of Material variation on mode shape

The Mode shapes obtained Figures 4-6 conforms with classical theory of vibration. Figures 7-9 shows the material variation $\alpha$ along the three modes of the boundary conditions considered. According to Shariyat [19], the vanishing effect (non-trivial) of boundary according results into varying modal behaviour in the FGM plate considered.

**Figure 4.** First three mode shape of free edge condition.

**Figure 5.** First three mode shape of FGP with clamped edge

**Figure 6.** First three mode shape of FGP with simply supported edge

**Figure 7.** Modal shapes of first three displacement of FGP under free edge condition with varying material properties and thickness
5. Conclusion

The dynamic behaviour of FGM plate resting on Winkler linear, Pasternak linear and Nonlinear elastic foundation has been analyzed. The governing systems of nonlinear partial differential equation was transformed to Duffing equation using Galerkin of one-parameter. The nonlinear dynamic analysis of FGM Plate was performed using DTM and cosine after-treatment techniques, subsequently, the accuracy of the results obtained was validated with prior work done and results confirmed a better relationship with the results obtained in the previous research. Based on Parametric study conducted, it was revealed that;

- Natural frequency decrease with decrease in thickness of the plate,
- Introduction of Winkler and Pasternak foundation increases the natural frequency of the FGM plate.
- Natural frequency increases with increase value of Winkler and Pasternak foundation coefficient.
- Non-trivial nature of boundary condition affects the modal behaviour of the plate.
- Increase in $\beta$, means that the stiffness of the plate increases thereby lower the amplitude of vibration.

The study proofs the capability of DTM in handling singularity, non-trivial and nonlinear equations. It is hope that, the research will enhance the study of vibration and help structural designers in making sound judgement concerning design of plate resting on elastic foundation.

Abbreviations: Nomenclature

$\frac{d}{dr}$: Differential operator

w: Dynamic deflection

r: radius of thin plate
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