Toric codes over finite fields

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Abstract

In this note, a class of error-correcting codes is associated to a toric variety associated to a fan defined over a finite field $\mathbb{F}_q$, analogous to the class of Goppa codes associated to a curve. For such a “toric code” satisfying certain additional conditions, we present an efficient list decoding algorithm for the dual code. Many examples are given. For small $q$, many of these codes have parameters beating the Gilbert-Varshamov bound. In fact, using toric codes, we construct a $(n, k, d) = (49, 11, 28)$ code over $\mathbb{F}_8$, which is better than any other known code listed in Brouwer’s tables [B] for that $n$ and $k$. We give upper and conjectural lower bounds on the minimum distance. The upper bounds are known to be sharp in some cases. We conclude with a discussion of some decoding methods.

1 Introduction

Some of the constructions discussed here is implemented in [MAGMA] and [GAP], using the toric package1 [J].

Let $\mathbb{F}_q$ denote the finite field with $q$ elements. A $q$-ary code is, for us, a short exact sequence of vector spaces

$$0 \rightarrow C \xrightarrow{\gamma} \mathbb{F}_q^n \xrightarrow{\theta} \mathbb{F}_q^{n-k} \rightarrow 0.$$ (1)

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1The GAP version of toric requires [GUAVA].
We identify $C$ with its image under $\gamma$. If $\mathbb{F}^n$ is given the usual standard vector space basis then the matrix of $\gamma$ is the generating matrix of $C$ and the matrix of $\theta$ is the parity check matrix of $C$. The integer $n$ is called the \textbf{length} of $C$. The dimension of $C$ is $k = \dim_{\mathbb{F}_q}(C)$. A \textbf{codeword} is an element of $C$. Since the sequence (1) is exact, a vector $v \in \mathbb{F}^n$ is a codeword if and only if $\theta(v) = 0$. The \textbf{Hamming distance} between two vectors is the number of coordinates where they differ. This metric stratifies $C$ into subsets of codewords having the same Hamming distance from the origin, or “weight”. The smallest non-zero weight which occurs is called the \textbf{minimum distance}, denoted $d = d(C)$.

On one hand, it is known \cite{MS} that

$$d + k \leq n + 1.$$ 

On the other hand, it is known \cite{MS} that, as $n$ tends to infinity, there are codes for which $(d/n, k/n)$ lie above the “Gilbert-Varshamov curve”,

$$y = 1 - x \log_q(q - 1) - x \log_q(x) - (1 - x) \log_q(1 - x), \quad 0 \leq x \leq \frac{q - 1}{q}.$$ 

Many of the toric codes constructed here beat this bound.

\textbf{Example 1.1.} Let $R = \mathbb{F}_q[x_1, ..., x_m]_\ell$ denote the vector space of polynomials in $m$ variables over $\mathbb{F}_q$ having degree $\leq \ell$. Let $S = \mathbb{F}_q^m = \{p_1, ..., p_n\}$ be some indexing, where $n = q^m$. Let $C$ denote the code

$$C = \{(f(p_1), ..., f(p_n)) \mid f \in R\}.$$ 

This code is called the \textbf{Reed-Muller code}, denoted $R(\ell, m, q)$.

For instance, when $q = 2$, $m = 5$, $\ell = 1$, we have $n = 32$, $k = \dim_{\mathbb{F}_2}(C) = 6$ and $d = 16$. (This code was used in the 1972 Mariner mission to Mars.)

In general, the length $n$, dimension $k$, and minimum distance $d$ are known exactly for RM codes: If $\ell = a(q - 1) + b$, where $1 \leq b \leq q - 1$, and if $0 \leq \ell \leq m$, then $n = q^m$,

$$k = \sum_{i=0}^{\ell} \sum_{j=0}^{\lfloor i/q \rfloor} (-1)^j \binom{m}{j} \binom{m - 1 + i - qj}{m - 1},$$

and $d = (q - b)q^{m-a-1}$. 

For this, see for example [MS], ch 13, §3. It should be noted that there are easily constructed functions in R of “minimum weight” (page 920 in [HCT]).

For example, let \( m = 2 \), \( \ell = 5 \), and \( q \) is “large”. Then a basis for \( R \) is given by the monomials \( \{ x_1^{e_1} x_2^{e_2} | 0 \leq e_1 + e_2 \leq 5 \} \). We can plot the exponents \((e_1,e_2)\) of these monomials. It is a triangle with vertices \((0,0)\), \((5,0)\), \((0,5)\). The number of lattice points in this triangle is the dimension of \( C \).

All “toric codes”, defined below, are subcodes of a Reed-Muller code. They are associated to a complete non-singular \( r \)-dimensional toric variety \( X \), provided with a dense torus \( T \hookrightarrow X \), defined over \( \mathbb{F}_q \), and a \( T \)-invariant divisor \( G \). The analogy with the above example is obtained by replacing the triangle by a convex polytope in \( \mathbb{R}^r \) and replacing all the points in \( \mathbb{F}_q^m \) by a subset of the \( \mathbb{F}_q \)-rational points in \( X \) (often times but not always by those \( \mathbb{F}_q \)-rational elements in \( T, T(\mathbb{F}_q) \)). The polytope is associated to the divisor \( G \) (see [E], [F], [JV], for examples).

We shall focus here on the case of toric surfaces.

### 2 Hansen codes

There are three Danish mathematicians working in algebraic geometry with the same last name, Hansen. They are unrelated. In the title of this section, we are talking about Johan Hansen.

We recall briefly some codes associated to a toric surface, constructed by J. Hansen [H1].

Let \( \mathbb{F} = \mathbb{F}_q \) and let \( \mathbb{F} \) denote a separable algebraic closure. Let \( L \) be a lattice in \( \mathbb{Q}^2 \) generated by \( v_1, v_2 \in \mathbb{Z}^2 \), \( P \) a polytope in \( \mathbb{Q}^2 \), and \( X(P) \) the associated toric surface. Let \( P_L = P \cap \mathbb{Z}^2 \).

There is a dense embedding of \( GL(1) \times GL(1) \) into \( X(P) \) given as follows. Let \( T_L = Hom_\mathbb{Z}(L, GL(1)) \) (which is \( \cong GL(1) \times GL(1) \) by sending \( t = (t_1,t_2) \) to \( m_1 v_1 + m_2 v_2 \mapsto e(\ell)(t) = t_1^{m_1} t_2^{m_2} \)) and let \( e(\ell): T_L \to \mathbb{F} \) be defined by \( e(\ell)(t) = t(\ell) \), for \( t \in T_L \) and \( \ell \in L \).

Impose an ordering on the set \( T_L(\mathbb{F}) \) (changing the ordering leads to an equivalent code). Define the code \( C = C_P \subset \mathbb{F}^n \) to be the linear code generated by the vectors

\[
B = \{(e(\ell)(t))_{t \in T_L(\mathbb{F})} | \ell \in L \cap P_L\},
\]

where \( n = (q-1)^2 \). In some special cases, the dimension of \( C \) and an estimate
of its minimum distance can at least be conjectured explicitly (see Conjecture 4.1 below).

J. Hansen ([H1], [H3]) gives\(^2\) the minimum distance \(d\) of such codes in the cases:

(a) \(P\) is an isosceles triangle with vertices \((0,0),(a,a),(0,2a)\),
(b) \(P\) is an isosceles triangle with vertices \((0,0),(a,0),(0,a)\), or
(c) \(P\) is a rectangle with vertices \((0,0),(a,0),(0,b),(a,b)\),
(d) \(P\) is a trapazoid with vertices \((0,0), (a,0), (0,b), (a,b+am)\), where \(m>0\).

provided \(q\) is “sufficient large”. His precise result is recalled below.

Theorem 2.1. Let \(a, b\) be positive integers. Let \(P\) be the polytope defined in (a)-(d) above.

\(\text{(a)}\) Assume \(q > 2a + 1\). The code \(C = C_P\) has
\[
{n = (q-1)^2, \quad k = (a+1)^2, \quad d = n - 2a(q-1).}
\]

\(\text{(b)}\) Assume \(q > a + 1\). The code \(C = C_P\) has
\[
{n = (q-1)^2, \quad k = (a+1)(a+2)/2, \quad d = n - a(q-1).}
\]

\(\text{(c)}\) Assume \(q > \max(a,b) + 1\). The code \(C = C_P\) has
\[
{n = (q-1)^2, \quad k = (a+1)(b+1), \quad d = n - a(q-1) - b(q-1) + ab.}
\]

\(\text{(d)}\) Assume \(q > \max(a,b,b+am) + 1\). The code \(C = C_P\) has
\[
{n = (q-1)^2, \quad k = (a+1)(b+1) + m\frac{a(a+1)}{2},}
\[
d = \min((q-a-1)(q-b-1), (q-1)(q-b-am-1)).
\]

The bounds on \(q\) are best possible.

We give examples of Hansen’s theorem stated above using MAGMA (or GAP) computations [J].

Example 2.2. Part (b).

\(^2\)In fact, [H1] contains only upper bounds on \(d\). Explicit computations using the toric package suggested that these upper bounds were attained. This was proven in [H3].
3 Construction

This section gives a construction which is a little more general than Hansen’s construction recalled in [E], though it still falls in the framework of the general class of codes constructed in [Han2].

Again, we shall focus on the case of a toric surface, for simplicity.

Let \( M \cong \mathbb{Z}^2 \) be a lattice in \( V = \mathbb{R}^2 \) and let \( N \cong \mathbb{Z}^2 \) denote its dual. Let \( \Delta \) be a fan (of rational cones, with respect to \( M \)) in \( V \) and let \( X = X(\Delta) \) denote the toric variety associated to \( \Delta \). Let \( T \) denote a dense torus in \( X \).

Let \( \tau_1, \ldots, \tau_s \) denote the edges of \( \Delta \), ordered in a counterclockwise fashion, and let \( v_i \in N \) denote the lattice point closest to the origin in \( \tau_i \). We write \( v_{s+1} = v_1 \). The angle between successive vectors \( v_i, v_{i+1} \) must be in the range \((0, \pi)\), \( 1 \leq i \leq s \). In order for \( X \) to be complete, we require that \( \Delta \) contain the cones generated by each successive pair \( v_i, v_{i+1}, 1 \leq i \leq s \). In other for \( X \) to be non-singular, we require that each successive pair \( v_i, v_{i+1}, 1 \leq i \leq s \), generate \( N \). (Equivalently, we require that each matrix \((v^t_i, v^t_{i+1})\) has determinant \( \pm 1 \).)

Let \( P = P_1 + \ldots + P_n \) be a positive 1-cycle on \( X \), where the points \( P_i \in X(\mathbb{F}_q) \) are distinct. Let \( G \) be a \( T \)-invariant divisor on \( X \) which does not “meet” \( P \), in the sense that no element of the support of \( P \) intersects any element in the support of \( G \). We write this as

\[
\text{supp}(G) \cap \text{supp}(P) = \emptyset.
\]

Some additional assumptions on \( G \) and \( P \) shall be made later. Let

\[
L(G) = \{0\} \cup \{f \in \mathbb{F}_q(X)^{\times} \mid \text{div}(f) + G \geq 0\}
\]

denote the Riemann-Roch space associated to \( G \).

Over \( \mathbb{C} \), according to [E], §3.4, there is a polytope \( P_G \) in \( V \) such that \( L(G) \) is spanned by the monomials \( x^a \) (in multi-index notation), for \( a \in P_G \cap N \).
Over $\mathbb{F}_q$, we assume that the same result holds provided that, for all $a = (a_1,\ldots,a_r) \in P_G \cap N$, we have $|a_i| < q$, for all $1 \leq i \leq r$. In other words, we assume $q$ is “sufficiently large”, in this sense (depending on $G$).

Let $C_L = C_L(P,G)$ denote the code defined by

$$
C_L = \{(f(P_1),\ldots,f(P_n)) \mid f \in L(G)\}.
$$

This is the **Goppa code associated to** $X$, $P$, and $G$. The dual code is denoted

$$
C = C_L^\perp = \{(c_1,\ldots,c_n) \in \mathbb{F}_q^n \mid \sum_{i=1}^n c_i f(P_i) = 0, \forall f \in L(G)\}.
$$

It is reasonable to ask under what conditions (if any) is the map

$$
L(G) \to C_L
f \mapsto (f(P_1),\ldots,f(P_n))
$$

an injection? This question was investigated by S. Hansen in [Han3] and [Han1] for varieties, §5.3, and [H3] in the case of certain toric surfaces.

**Lemma 3.1.** (J. Hansen) If $X = X(\Delta)$ is a non-singular toric surface and $q$ is sufficiently large (i.e., $q > q_0(\Delta)$, for some $q_0(\Delta) > 1$) then (5) is injective.

This was proven in §2.3 in [H3] for certain cases (where $q_0$ was explicitly determined) but the technique applies more generally to yield the above result.

Let $\Delta$ be a fan in a lattice $L \cong \mathbb{Z}^r$. Let $\tau_i$ and $v_i$ be as above $1 \leq i \leq s$. Let $D_i$ denote the Weil divisor

$$
D_i = \text{Hom}(\tau_i^\perp \cap L^\perp, \mathbb{C}^\times),
$$

which can be regarded as the Zariski closure of an orbit of $T$.

**Example 3.2.** Let $\Delta$ be the fan generated by cones whose edges are defined by

$$
v_1 = 5e_1 - e_2, \quad v_2 = -e_1 + 5e_2, \quad v_3 = -e_1 - e_2.
$$

Let $X$ be the toric variety associated to $\Delta$. In this example, the divisor $G = 5D_3$ on $X/\mathbb{F}_8$ yields a toric code which one finds (thanks to the **toric package**) has the parameters $(49,11,28)_8$. This is a new code and beats the previous best known code (for given $n$, $k$, $q$) having parameters $(49,11,27)_8$ [B].
Example 3.3. Let $\Delta$ be the fan generated by cones whose edges are defined by
\[ v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + 2e_2, \quad v_3 = -e_1 - e_2. \]
Let $X$ be the toric variety associated to $\Delta$.

Note $v_1, v_2$ generate a cone $\sigma_1$ in the fan $\Delta$ yet do not generate the lattice $\mathbb{Z}^2$. By the criterion on page 29 of [F], $X$ is singular.

In the notation of (6), the divisor $G = d_1D_1 + d_2D_2 + d_3D_3$ is a Cartier divisor if and only if $d_1 \equiv d_2 \equiv d_3 \pmod{3}$ (this is a consequence of an exercise on page 62 of [F]). A Cartier divisor is very ample if and only if $d_1 + d_2 + d_3 > 0$ (this follows from the criterion in the proof of the proposition on page 68 in [F]). Let
\[ P_G = \{(x, y) \mid \langle(x, y), v_i \rangle \geq -d_i, \forall i\} \]
\[ = \{(x, y) \mid 2x - y \geq -d_1, -x + 2y \geq -d_2, -x - y \geq -d_3\} \]
denote the polytope associated to $G$.

We tabulate parameters of toric codes $C_L(P, G)$, where $P$ is the sum of all the $\mathbb{F}_q$-valued points in a “dense” torus $T$ in $X$. These results were obtained using the toric package. Note that not all of these divisors $G$ are Cartier.
Example 3.4. Let for now $\Delta$ be the fan generated by

$$v_1 = e_1, \quad v_2 = -e_1 + e_2, \quad v_3 = -e_1 - e_2.$$ 

Let $X$ be the toric variety associated to $\Delta$. This is singular. The divisor $G = d_1 D_1 + d_2 D_2 + d_3 D_3$ with $d_1 = 0$, $d_2 = 0$, $d_3 = a$ gives case (a) of Hansen’s code.

To create a smooth toric variety yielding Hansen’s code, let instead $\Delta$ be the refined fan generated by

$$v_1 = e_1, \quad v_2 = -e_1 + e_2, \quad v_3 = -e_1 - e_2, \quad v_4 = -e_2.$$ 

Let $X$ be the toric variety associated to $\Delta$. This is now non-singular. The divisor $G = d_1 D_1 + d_2 D_2 + d_3 D_3 + d_4 D_4$ with $d_1 = 0$, $d_2 = 0$, $d_3 = a$, $d_4 = 2a$ (for example) gives case (a) of Hansen’s code.

Example 3.5. Let $\Delta$ be the fan generated by

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = -e_1 - e_2.$$
Let $X$ be the toric variety associated to $\Delta$. This is non-singular. The divisor $G = d_1D_1 + d_2D_2 + d_3D_3$ with $d_1 = 0$, $d_2 = 0$, $d_3 = a$ gives case (b) of Hansen’s code.

**Example 3.6.** Let $\Delta$ be the fan generated by

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = -e_1, \quad v_4 = -e_2.$$ 

Let $X$ be the toric variety associated to $\Delta$. This is non-singular. The divisor $G = d_1D_1 + d_2D_2 + d_3D_3 + d_4D_4$ with $d_1 = 0$, $d_2 = 0$, $d_3 = a$, $d_4 = b$ gives case (c) of Hansen’s code.

**Example 3.7.** Let $\Delta$ be the fan generated by

$$v_1 = e_1, \quad v_2 = -e_1 + me_2 \ (m > 1) \text{ fixed}, \quad v_3 = -e_2.$$ 

Let $X$ be the toric variety associated to $\Delta$. This surface is, except for small $m$, singular.

In the notation above, the divisor $G = d_1D_1 + d_2D_2 + d_3D_3$ is a Cartier divisor if and only if $d_1 + d_2 \equiv 0 \pmod{m}$ (this is an exercise on page 64 of [F]).

Let $\Delta'$ be the refined fan generated by

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = -e_1 + me_2 \ (m > 1) \text{ fixed}, \quad v_4 = -e_2.$$ 

Let $X'$ be the toric variety associated to $\Delta'$. This surface is non-singular. The divisor $G' = d'_1D'_1 + d'_2D'_2 + d'_3D'_3 + d'_4D'_4$ (using $D'_i$ to denote the divisor in (6) defined on $X'$) is very ample if and only if $d'_2 + d'_4 \geq 0$, $d'_1 + d'_3 \geq md'_1$ (this is an exercise on page 70 of [F]). Taking

$$d'_1 = d'_2 = 0, \quad d'_3 = a, \quad d'_4 = b,$$

and swapping the roles of $x$ and $y$, gives the Hansen code of type (d) constructed above.

Let

$$P_G = \{(x, y) \mid \langle(x, y), v_i \rangle \geq -d_i, \forall i\} = \{(x, y) \mid x \geq -d_1, -x + my \geq -d_2, -y \geq -d_3\}$$

denote the polytope associated to the Weil divisor $G = d_1D_1 + d_2D_2 + d_3D_3$.

The toric package gives the following results:

$m = 3$
Indeed, the parameters arising in these case are very similar to each other.

Example 3.8. Let $\Delta$ be the fan generated by

$$v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + e_2, \quad v_3 = -e_1.$$

Let $X$ be the toric variety associated to $\Delta$. This is non-singular. All the divisors $G = d_1D_1 + d_2D_2 + d_3D_3$ are Cartier. The divisor $G = d_1D_1 + d_2D_2 + d_3D_3$ is very ample if and only if $d_1 + d_2 + d_3 > 0$.

Let

$$P_G = \{(x, y) \mid \langle(x, y), v_i \rangle \geq -d_i, \forall i\}$$

$$= \{(x, y) \mid 2x - y \geq -d_1, -x + y \geq -d_2, -x \geq -d_3\}$$

denote the polytope associated to $G$. If each $d_i > 0$ and if $d_3 > d_1 + d_2$ then the volume of this polytope is given by

$$\text{vol}(P_G) = \frac{1}{2} (2d_1 + d_3)(d_3 - d_1 - d_2).$$
We tabulate parameters of toric codes \( C_L(P,G) \), where \( P \) is the sum of all the \( \mathbb{F}_q \)-valued points in a “dense” torus \( T \) in \( X \). These parameters were obtained using the \texttt{toric} package.

| \( q \) | \( d_1 \) | \( d_2 \) | \( d_3 \) | \( n \) | \( k \) | \( d \) | Using | 
|-----|-----|-----|-----|-----|-----|-----|-----|
| 5   | 0   | 0   | 1   | 16  | 3   | 12  | \text{Best possible} |
| 5   | 0   | 0   | 2   | 16  | 6   | 8   | \( d = 9 \) is best known |
| 5   | 0   | 0   | 3   | 16  | 10  | 4   | \( d = 5 \) is best known |
| 5   | 0   | 0   | 4   | 16  | 13  | 3   | \text{Best possible} |
| 7   | 0   | 0   | 1   | 36  | 3   | 30  | \text{Best possible} |
| 7   | 0   | 0   | 2   | 36  | 6   | 24  | \text{Best known} |
| 7   | 0   | 0   | 3   | 36  | 10  | 18  | \text{Best known} |
| 7   | 0   | 0   | 4   | 36  | 15  | 12  | \( d = 14 \) is best known |
| 7   | 4   | 1   | 1   | 36  | 26  | 5   | \( d = 6 \) is best known |
| 7   | 4   | 1   | 2   | 36  | 30  | 4   | \( d = 5 \) is best possible |
| 7   | 4   | 1   | 3   | 36  | 33  | 3   | \text{Best possible} |
| 7   | 4   | 1   | 4   | 36  | 35  | 2   | \text{MDS} |
| 8   | 0   | 0   | 1   | 49  | 3   | 42  | \text{Best possible} |
| 8   | 0   | 0   | 2   | 49  | 6   | 35  | \( d = 36 \) is best known |
| 8   | 0   | 0   | 3   | 49  | 10  | 28  | \text{Best known} |
| 8   | 0   | 0   | 4   | 49  | 15  | 21  | \( d = 23 \) is best known |
| 8   | 4   | 3   | 49  | 34  | 6   | 10  | \( d = 10 \) is best known |
| 8   | 0   | 0   | 4   | 49  | 39  | 5   | \( d = 6 \) is best known |
| 8   | 2   | 4   | 49  | 46  | 3   | \text{best possible} |
| 8   | 3   | 4   | 49  | 48  | 2   | \text{MDS} |
| 8   | 4   | 1   | 49  | 43  | 4   | \( d = 5 \) is best possible |
| 9   | 0   | 0   | 1   | 64  | 3   | 56  | \text{Best possible} |
| 9   | 0   | 0   | 2   | 64  | 6   | 48  | \( d = 49 \) is best known |
| 9   | 0   | 0   | 3   | 64  | 10  | 40  | \( d = 41 \) is best known |

\section{Estimates on the parameters}

We shall now estimate the parameters \( n, k, d \) for \( C_L = C_L(G, P, X) \), conjecturally. Assume \( X \) is a complete toric variety of dimension \( r \). For many such varieties, the conjectured estimates below were verified by computer using the \texttt{toric} package.

Let \( G \) be a \( T \)-invariant Cartier divisor on \( X = X(\Delta) \), let \( P_G \) be the polytope associated to \( G \) and let \( M \) be the lattice as in \texttt{F}.
Conjecture 4.1. Let $C_L = C_L(G,P,X)$ be as in [3]. Assume

- $X$ is a non-singular, projective toric variety of dimension $r$,
- $n$ is so large that there is an integer $N > 1$ such that
  
  $$2N \cdot \text{vol}(P_G) \leq n \leq 2N^2 \cdot \text{vol}(P_G).$$

If $q$ is “large” then any $f \in L(G) = H^0(X, \mathcal{O}(G))$ has no more than $n$ zeros in $X(\mathbb{F}_q)$. Consequently,

$$d \geq n - 2N \cdot \text{vol}(P_G).$$

Here “large” make depend on $X$, $P$ and $G$ but not on $f$.

Let $G$ be a $T$-invariant Cartier divisor on $X = X(\Delta)$ and let

$$\psi_G : |\Delta| \to \mathbb{R}$$

be the linear function associated to $G$ as in [F], §3.4 (this is called a virtual support function in [E], Definition VII.4.1). If $\psi_G$ is strictly convex and $X$ is non-singular then $G$ is very ample and $\mathcal{O}(G)$ is generated by its global sections ([F], §3.4, p. 70)\(^3\). Conversely, if $\mathcal{O}(G)$ is generated by its global sections then $\psi_G$ is convex and

$$\psi_G(v) = \min_{u \in P_G \cap M} \langle u, v \rangle,$$

where $P_G$ and $M$ are as above.

Conjecture 4.2. Let $C_L = C_L(G,P,X)$ be as in [3]. Assume

- $X$ is a non-singular, projective toric variety of dimension $r$,
- $\psi_G$ is strictly convex,
- and

$$\deg(P) > \deg(G^*).$$

\(^3\)Demazure has shown that on a complete, non-singular toric variety a $T$-invariant divisor is ample if and only if it is very ample (see [E], page 71).
If \( q \) is “large” then any \( f \in L(G) = H^0(X, \mathcal{O}(G)) \) has no more than \( n = \deg(P) \) zeros in \( X(\mathbb{F}_q) \). Consequently,

\[
k \geq \dim H^0(X, \mathcal{O}(G)) = |P_G \cap M|,
\]

and

\[
d \geq n - r! \cdot |P_G \cap M|.
\]

Moreover, if \( n > r! |P_G \cap M| \) then \( \dim H^0(X, \mathcal{O}(G)) = |P_G \cap M| \).

**Remark 4.3.** This is false if \( X \) is singular.

We conclude with an upper bound on the minimum distance. Let \( M^+ = \{ m \in M \mid m_i \geq 0 \} \).

**Theorem 4.4.** Let \( C = C_L(G, P, X) \) be as in \( \text{(3)} \). Let \( H \subset P_G \cap M^+ \) be a line \( \{(e_1, ..., e_{j-1}, x, e_{j+1}, ..., e_r) \mid 0 \leq x \leq h\} \), for some integer \( h > 0 \). Then \( d \leq n - h \).

**Remark 4.5.** (1) In particular, let \( r = 2 \) and let \( h > 0 \) denote the largest integer such that \( \{(0,0), ..., (0,h)\} \subset P_G \). Then \( d \leq n - h \).

(2) This generalizes the results of J. Hansen stated above, which are known to be sharp.

**proof:** Let \( h = \max_i(d_i) \). Let \( f \in L(G) \) denote the function

\[
(x_j - a_1) ... (x_j - a_h) \prod_{i \neq j} x_i^{e_i} \in L(G),
\]

where \( a_i \in \mathbb{F}_q^\times \) are distinct. This vanishes at \( h \) points, so \( d \leq n - h \). \( \square \)

## 5 Decoding algorithm

One reasonable decoding algorithm is an algorithm for decoding more general multi-dimensional cyclic codes. We refer to J. Little’s papers [L1], [L2] and the references cited there.

In this section, we discuss a much more restrictive algorithm. The algorithm here will decode \( C = C_L(P, G) \perp \) when \( G \) is a “large” divisor and the number of errors is “small”. What it lacks in optimality it makes up in simplicity.
Let \( X = X(\Delta) \) be a toric variety associated to a fan \( \Delta \) with edges \( \tau_1, \ldots, \tau_s \). Let \( D_i \) denote the \( T \)-Weil divisors as above.

Let \( C \) be as in \( \text{[1]} \). Let \( r = c + e \) be a received vector, \( c \in C \) a code-word and \( e \) the error vector of smallest weight. Thus all \( f \in L(G) \) satisfy

\[ \sum_{i=1}^n c_i f(P_i) = 0. \]

Let

\[ I = \{ i \mid e_i \neq 0 \}. \]

We shall follow \( \text{[S]}, \text{pages 217-218} \), to (a) create an error locator function for \( r \) and \( C \), and (b) solve for \( e_i, i \in I \). The method generally only gives a list of several vectors, one of which is the true error vector \( e \) - it only yields one vector (namely, \( e \)) under a certain technical condition (see Proposition 5.2 for details).

Let \( G' \) be another divisor of \( X \). We shall make some assumptions on \( G \) and \( G' \) as we go along. Let

\[
\begin{align*}
L(G') &= \text{span}\{f_1, \ldots, f_\ell\}, \\
L(G - G') &= \text{span}\{g_1, \ldots, g_k\}, \\
L(G) &= \text{span}\{h_1, \ldots, h_m\}.
\end{align*}
\]

We assume

(A) that these spaces \( L(G), L(G'), L(G - G') \) are non-zero,

(B) that there are \( T \)-invariant divisors \( D_i \) for which (a) each \( P_i \) is contained in a \( D_i, i \in I \), (b) \( P_i \notin D_j \), for \( i \neq j \) and \( i, j \in I \), (c) \( L(G' - \sum_{i \in I} D_i) \) is non-zero, (d) for any fixed \( i_0 \in I \), \( L(G - G' - \sum_{i \in I} D_i) \) is non-empty and properly contained in \( L(G - G' - \sum_{i \in I, i \neq i_0} D_i) \).

(C) \( d > \deg(G') \).

Assumption (B) above is very strong and forces us to assume that \( |I| \leq s \).

Roughly speaking, \( G \) is a “large” divisor and the number of errors is “small”.

Let

\[ [r, \phi] = \sum_{i=1}^n r_i \phi(P_i), \]

and let \( x_1 = a_1, \ldots, x_\ell = a_\ell \) be a non-trivial solution to the system of \( k \) simultaneous equations

\[
\sum_{j=1}^\ell [r, f_j g_i] x_j = 0, \quad 1 \leq i \leq k. \quad \text{(7)}
\]
Let $f = a_1 f_1 + \ldots + a_\ell f_\ell$.

**Proposition 5.1.** Under the assumptions above, $f$ is an error locator function. In other words, $f(P_i) = 0$ for all $i \in I$.

**proof:** We have $L(G' - \sum_{i \in I} D_i)$ is non-zero, by assumption. Choose a non-zero $z \in L(G' - \sum_{i \in I} D_i) \subset L(G')$ and write $z = \gamma_1 f_1 + \ldots + \gamma_\ell f_\ell$, for some $\gamma_i \in \mathbb{F}_q$. Then $zg_j \in L(G)$, for all $1 \leq j \leq k$. If $r = (r_1, \ldots, r_\ell)$ then

$$[r, zg_j] = \sum_{i=1}^\ell [r, f_i g_j] \gamma_i.$$ 

On the other hand, $[r, zg_j] = [c + e, zg_j] = [e, zg_j]$, so

$$[r, zg_j] = [e, zg_j] = \sum_{v=1}^n e_v f_i(P_v)g_j(P_v).$$

But $e_v = 0$ for $v \notin I$ and $z(P_v) = 0$ for $v \in I$, so $[r, zg_j] = 0$. This implies that $(\gamma_1, \ldots, \gamma_\ell)$ is a non-trivial solution to (7).

Now suppose that there is an $i_0 \in I$ for which $f(P_{i_0}) \neq 0$. By assumption (B,d), there is an $h \in L(G - G')$ such that $h(P_{i_0}) \neq 0$ and $h(P_i) = 0$ for all other $i \in I$. Then

$$[r, fh] = \sum_{v=1}^n e_v f_i(P_v)h(P_v) = e_{i_0} f_i(P_{i_0})h(P_{i_0}) \neq 0.$$ 

On the other hand, $h$ is a linear combination of the $g_j$’s. So this contradicts the equation $[r, zg_j] = 0$ we obtained earlier. □

Next, we must determine the $e_i$, $i \in I$. Let

$$N(f) = \{i \mid 1 \leq i \leq n, f(P_i) = 0\}.$$ 

By the proposition, $I \subset N(f)$.

**Proposition 5.2.** The $b_i = e_i$, for $i \in N(f)$, solve

$$\sum_{i \in N(f)} b_i h_j(P_i) = [r, h_j], \quad 1 \leq j \leq m.$$ 

If $|N(f)| \leq \deg(G')$ then this solution is unique.
Remark 5.3. Naturally, it would be interesting to know if the hypothesis $|N(f)| \leq \deg(G')$ can be removed. Does the $T$-invariance of the $D_i$ and the fact that each $D_i$ is (the Zariski closure of) an orbit help?

A bound on $|N(f)|$ is known for certain $\Delta$ - see §2.3 in [H3]. A conjectural bound is given in the previous section.

proof: Since $h_j \in L(G)$ and (by the previous proposition) $e_v = 0$ for all $v \notin N(f)$, we have

$$[r, h_j] = [e, h_j] = \sum_{v=1}^{n} e_v h_j(P_v) = \sum_{v \in N(f)} e_v h_j(P_v).$$

Thus the equation displayed in Proposition 5.2 has the $e_i$'s, $i \in N(f)$, as a solution.

Now we show that this solution is unique. Suppose $b_i, i \in N(f)$, is another solution. Define $b_j = 0$ for $j \in \{1, 2, ..., n\} - N(f)$. Then for $1 \leq j \leq m$,

$$[b, h_j] = \sum_{v=1}^{n} b_v h_j(P_v) = \sum_{v \in N(f)} b_v h_j(P_v) = [r, h_j] = [e, h_j].$$

As the $h_j$'s form a basis of $L(G)$, this implies $b - e$ is a codeword. The weight of this codeword satisfies

$$wt(b - e) \leq |N(f)| \leq \deg(G') < d,$$

by our hypothesis. This forces $b_i = e_i$, so the solution is unique. □

The above Proposition tells us how to find the error vector, given $f, L(G)$, and the $P_i$'s, as desired.

We illustrate this algorithm with an example.

Example 5.4. Here is an example where we decode a received word in a toric code with $\leq 3$ errors.

Let $\Delta$ be the fan as in Example 3.3. Let $X$ be the toric variety associated to $\Delta$.

Let

$$P_D = \{(x, y) \mid \langle(x, y), v_i \rangle \geq -d_i, \forall i\}$$

$$= \{(x, y) \mid 2x - y \geq -d_1, -x + 2y \geq -d_2, -x - y \geq -d_3\}$$

denote the polytope associated to the Weil divisor $D = d_1 D_1 + d_2 D_2 + d_3 D_3$, where $D_i$ is as above.
Let 
\[ G = 10D_3, \quad D = D_1 + D_2 + D_3, \quad G' = 2D. \]

Then \( P_G \) is a triangle in the plane with vertices at \((0,0)\), \((10/3,20/3)\), and \((20/3,10/3)\). It’s area is \(\frac{1}{2}bh = 50/3\).

Let \( T \subset X \) denote the dense torus of \( X \). In this example, the patch \( U_{\sigma_1} \) is an affine variety with coordinates \( x_1, x_2, x_3 \) given by \( x_1^3 - x_2x_3 = 0 \). The torus embedding \( T \hookrightarrow U_{\sigma_1} \) is given by sending \((t_1,t_2)\) to \((x_1,x_2,x_3) = (t_1t_2, t_1t_2^2, t_1^2t_2)\). The patch \( U_{\sigma_2} \) is an affine variety with coordinates \( y_1, y_2, y_3 \) given by \( y_2^2 - y_1y_3 = 0 \). The torus embedding \( T \hookrightarrow U_{\sigma_2} \) is given by sending \((t_1,t_2)\) to \((y_1, y_2, y_3) = (t_1^2t_2^{-1}, t_1^{-1}, t_1^2t_2)\). The patch \( U_{\sigma_3} \) is an affine variety with coordinates \( z_1, z_2, z_3 \) given by \( z_2^2 - z_1z_3 = 0 \). The torus embedding \( T \hookrightarrow U_{\sigma_3} \) is given by sending \((t_1,t_2)\) to \((x_1,x_2,x_3) = (t_1^{-1}t_2^{-2}, t_2^{-1}, t_1t_2^{-1})\).

In the local coordinates of \( U_{\sigma_1} \), the space \( L(G) \) has as a basis,
\[
\{ h_1 \} = \{ 1, x_1 x_2, x_1^2 x_2, x_1 x_2^2, x_1^2 x_2^2, x_1^3 x_2^2, x_1 x_2^3, x_1^2 x_2^3, x_1^3 x_2^3, \\
x_1^4 x_2^3, x_1^5 x_2^3, x_1^6 x_2^3, x_1^7 x_2^3, x_1^8 x_2^3, x_1^9 x_2^3, x_1^{10} x_2^3, x_1^{11} x_2^3, x_1^{12} x_2^3, \\
x_1^{13} x_2^3, x_1^{14} x_2^3, x_1^{15} x_2^3, x_1^{16} x_2^3, x_1^{17} x_2^3, x_1^{18} x_2^3, x_1^{19} x_2^3, x_1^{20} x_2^3 \}.
\]

In particular, it is 22-dimensional.

The polytope \( P_{G'} \) has vertices at \((-2,-2), (2,0), (0,2)\). In the local coordinates of \( U_{\sigma_1} \), the space \( L(G') \) has as a basis,
\[
\{ f_1 \} = \{ x_1^{-2} x_2^{-2}, x_1^{-1} x_2^{-1}, x_1^{-1} x_2^{-1}, 1, x_1, x_1 x_2, x_2, x_2^2 \}.
\]

In particular, it is 10-dimensional.

The polytope \( P_{G-G'} \) has vertices at \((2,2), (10/3,14/3), (14/3,10/3)\). In the local coordinates of \( U_{\sigma_1} \), the space \( L(G-G') \) has as a basis,
\[
\{ g_1 \} = \{ x_1^2 x_2^2, x_1^3 x_2^3, x_1^4 x_2^4, x_1^5 x_2^5, x_1^6 x_2^6 \}.
\]

\( L(G-G') \) is 5-dimensional.

Choose distinct points \( P_1, \ldots, P_n \in X(\mathbb{F}_q) \) and let
\[
C = \{ (c_1, \ldots, c_n) \in \mathbb{F}_q^n \mid \sum_{i=1}^n c_i f(P_i) = 0, \forall f \in L(G) \}.
\]

Let \( P_i \in D_i \) and \( P_i \notin D_j \), with \( 1 \leq i \neq j \leq 3 \). Assume \( P_4, \ldots, P_n \notin D_i, \)
\( i = 1, 2, 3 \). (For example, let \( \{ P_4, \ldots, P_n \} \subset T(\mathbb{F}_q) \).) We must choose \( n > 50 \),
assuming Conjecture 4.2 below, to insure condition (C) above is satisfied. 

Since $\text{L}(G - 3D)$ is 1-dimensional, hence non-zero, the hypotheses and assumptions above are all satisfied.

Suppose $r = (1, 1, 1, 0, ..., 0) \in \mathbb{F}_q^n$ is the received vector (it has 3 errors in the first 3 positions). In this case,

$$[r, f] = f(P_1) + f(P_2) + f(P_3).$$

Let $a_1, a_2, ..., a_{10}$ be a solution to

$$a_1[r; f_1g_1] + a_2[r; f_2g_1] + ... + a_{10}[r; f_{10}g_1] = 0,$$
$$a_1[r; f_1g_2] + a_2[r; f_2g_2] + ... + a_{10}[r; f_{10}g_2] = 0,$$
$$\vdots$$
$$a_1[r; f_1g_5] + a_2[r; f_2g_5] + ... + a_{10}[r; f_{10}g_5] = 0.$$

This determines

$$f = \sum_{i=1}^{10} a_j f_j,$$

and hence $N(f)$. The equations

$$\sum_{i \in N(f)} b_i h_j(P_i) = [r, h_j], \quad 1 \leq j \leq 22,$$

clearly have a solution ($b_1 = b_2 = b_3 = 1$, the other $b_i = 0$).

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\[\text{For example, we know}\]

$$|X(F_2)| = 7, \quad |X(F_3)| = 13, \quad |X(F_4)| = 21,$$
$$|X(F_5)| = 31, \quad |X(F_7)| = 57, \quad |X(F_8)| = 73, ...$$

So, we may take $q \geq 7$, at least conjecturally. See also Hansen’s Lemma 3.1.
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