Possibility of realizing weak gravity in redshift space distortion measurements

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We study the possibility of realizing a growth rate of matter density perturbations lower than that in General Relativity. Using the approach of the effective field theory of modified gravity encompassing theories beyond Horndeski, we derive the effective gravitational coupling $G_{\text{eff}}$ and the gravitational slip parameter $\eta$ for perturbations deep inside the Hubble radius. In Horndeski theories we derive a necessary condition for achieving weak gravity associated with tensor perturbations, but this is not a sufficient condition due to the presence of a scalar-matter interaction that always enhances $G_{\text{eff}}$. Beyond the Horndeski domain it is possible to realize $G_{\text{eff}}$ smaller than Newton’s gravitational constant $G$, while the scalar and tensor perturbations satisfy no-ghost and stability conditions. We present a concrete dark energy scenario with varying $c_s$ and numerically study the evolution of perturbations to confront the model with the observations of redshift-space distortions and weak lensing.

I. INTRODUCTION

The observations of redshift-space distortions (RSD) and weak lensing [18], combined with Cosmic Microwave Background (CMB) measurements [11], offer the possibility of testing General Relativity (GR) on cosmological scales. In particular, the observational evidence of late-time cosmic acceleration [9] may be related to some modification of gravity at large distances. The dark energy equation of state $w$ is much smaller than the value predicted by the Lambda-Cold-Dark-Matter ($\Lambda$CDM) model. In fact, the Planck CMB measurements [11] and supernovae Ia (SN Ia) data [10], can be realized in modified gravitational theories without ghosts and instabilities [11].

If we modify gravity from GR, an extra scalar degree of freedom usually emerges due to the breaking of gauge symmetries of GR [12]. This scalar field mediates an extra gravitational force with a matter sector. In $f(R)$ gravity, for example, the effective gravitational coupling $G_{\text{eff}}$ between the gravitational scalar and matter is $4/3$ times as large as Newton’s gravitational constant $G$ in the regime where the scalar mass $M$ is much smaller than the physical momentum $k/a$ of interest [13].

The recent observations of RSD [14,10] and cluster counts [17] have measured the lower growth rate of matter density perturbations $\delta_m$ than that predicted by the $\Lambda$-Cold-Dark-Matter (LCDM) model. In fact, the Planck CMB measurements [11,18] are in tension with the RSD data and the Hubble expansion data from SN Ia. One possibility for reconciling this discrepancy is to incorporate massive neutrinos [19], but this increases the tension between the CMB and the Hubble expansion measurements [20,21].

Another possibility for realizing a lower cosmic growth rate is interacting models of vacuum energy and dark matter [22,24] (see also Ref. [27]). If there is an energy transfer from dark matter to dark energy, it is possible to reduce the tension between the CMB and RSD measurements [28]. Most of these interacting models are based on a phenomenological approach, in that the equations of motion do not follow from a concrete Lagrangian. In this case, even if a lower growth rate consistent with observations is realized, it is not generally clear whether theoretically consistent conditions such as the absence of ghosts are satisfied or not.

There exists a modified gravitational scenario—dubbed the Dvali-Gabadadze-Porrati (DGP) braneworld model [29]—which possesses an explicit Lagrangian in the five-dimensional bulk space-time. In the branch where the late-time cosmic acceleration occurs, it is known that the effective gravitational coupling $G_{\text{eff}}$ is smaller than $G$ on scales relevant to large-scale structures [30]. However, ghosts are present in this accelerating branch. Thus, in the DGP model, the lower cosmic growth rate is related to the appearance of ghosts [31].

Now, a question arises. Are there some modified gravity models with concrete Lagrangians realizing weak gravity ($G_{\text{eff}}$ smaller than $G$) on cosmological scales, while avoiding the ghosts and instabilities associated with the propagation speeds of scalar and tensor perturbations? In order to address this problem, we focus on a very general class of scalar-tensor theories dubbed Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories [32]. This also accommodates Horndeski theories [33]—the most general scalar-tensor theories with second-order equations of motion in a general space-time.

The action of GLPV theories has been derived in such a way that the Horndeski action written in the framework of the Arnowitt-Deser-Misner (ADM) formalism [34] does not obey two additional conditions. While this can generate derivatives higher than second order in a generic space-time, there is no extra propagating degree of freedom on a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background according to Hamiltonian analysis in terms of cosmological perturbations [35,36]. This conclusion also holds for odd-mode perturbations on a spherically symmetric background [37].

In GLPV theories the tensor propagation speed squared $c_t^2$ can deviate from 1 even during the radiation- and early matter-dominated epochs, while in Horndeski theories this is limited by the two extra conditions men-
tioned above [34, 38]. Even for a simple canonical scalar field $\phi$ with a potential, the deviation of $c_t^2$ from 1 can give rise to an interesting observational signature such as a large difference between two gravitational potentials $\Psi$ and $\Phi$ [39]. For constant $c_t^2$ models one has $G_{\text{eff}} < G$ in the superluminal regime ($c_t^2 > 1$), but $G_{\text{eff}}$ needs to be very close to $G$ due to the fact that the scalar propagation speed squared $c_t^2$ becomes negative as $c_t^2$ is away from 1. In this paper we show that it is possible to realize weak gravity in varying $c_t^2$ models, while satisfying the no-ghost and stability conditions associated with scalar and tensor perturbations.

In Sec. IV we derive the effective gravitational coupling $G_{\text{eff}}/G$ and the gravitational slip parameter $\eta = -\Phi/\Psi$ by employing a subhorizon approximation for the perturbations relevant to large-scale structures. In GLPV theories time derivatives of metric perturbations are left even under this approximation, so the usual quasistatic approximation is trustable only when these time derivatives are suppressed relative to other terms. In this case, we implement such time derivatives as corrections to leading-order terms.

In Sec. V we discuss the possibility of realizing weak gravity by expressing $G_{\text{eff}}/G$ and $\eta$ in terms of quantities associated with the no-ghost and stability conditions of scalar and tensor perturbations. In Horndeski theories we derive a necessary condition for realizing $G_{\text{eff}} < G$, which is related with quantities appearing in the second-order action of tensor perturbations. However, this is not a sufficient condition due to an extra scalar interaction with matter which always enhances $G_{\text{eff}}$. In GLPV theories the value of $c_t^2$ is not restricted to be close to 1 even in the early cosmological epoch. In Sec. VI we propose a simple model with a time-varying $c_t^2$ in which the realization of weak gravity is possible without ghosts and Laplacian instabilities. In Sec. VII we numerically solve the full perturbation equations of motion for the decreasing $c_t^2$ model with $0 < c_t^2 < 1$ and show that the growth rate of matter perturbations associated with RSD measurements can be lower than that predicted by the $\Lambda$CDM model.

II. MODIFIED GRAVITATIONAL THEORIES

The most general scalar-tensor theories with second-order equations of motion are known as Horndeski theories [33]. The four-dimensional action of Horndeski theories is given by $S = \int d^4x \sqrt{-g} L$ with the Lagrangian

$$L = G_2(\phi, X) + G_3(\phi, X) \Box \phi$$

$$+ G_4(\phi, X) R - 2G_4,X(\phi, X) \left[ \Box \phi - \phi_{,\mu\nu} \phi_{,\mu\nu} \right]$$

$$+ G_5(\phi, X) G_{\mu\nu} \phi_{,\mu\nu} + \frac{1}{3} G_5(X(\phi, X)) \left[ \Box \phi \right]^3$$

$$- 3 \left( \Box \phi \right) \phi_{,\mu\nu} \phi_{,\mu\nu}^2 + 2 \phi_{,\mu\mu} \phi_{,\mu\nu} \phi_{,\nu}^2 ,$$

(2.1)

where $g$ is a determinant of the four-dimensional metric $g_{\mu\nu}$, $G_{2,3,4,5}$ are functions in terms of a scalar field $\phi$ and its kinetic energy $X = \phi_{,\mu} \phi_{,\mu}$, and $R$ and $G_{\mu\nu}$ are the four-dimensional Ricci scalar and the Einstein tensor respectively, and a semicolon represents a covariant derivative with $\Box \phi \equiv (g^{\mu\nu} \phi_{,\mu\nu})_{,\mu\nu}$. GLPV theories [32] correspond to the generalization of Horndeski theories derived by reformulating the Lagrangian (2.1) in terms of the 3+1 ADM decomposition of space-time [57] with the foliation of constant-time hypersurfaces $\Sigma_t$. The ADM formalism is based upon the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$, where $N$ is the lapse, $N^i$ is the shift, and $h_{ij}$ is the three-dimensional spatial metric.

The extrinsic curvature and the intrinsic curvature are defined, respectively, by $K_{\mu\nu} = h_{ij} K_{\mu\nu}^i + R_{\mu\nu}$, where $n_\mu = (-N,0,0,0)$ is a normal vector orthogonal to $\Sigma_t$ and $R_{\mu\nu}$ is the three-dimensional Ricci tensor on $\Sigma_t$. In the following we shall focus on a flat FLRW background described by the line element $ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j$, where $a(t)$ is the scale factor. In the unitary gauge the scalar field $\phi$ depends on the time $t$ alone and hence $X = -N^{-2} \dot{\phi}^2$, where a dot represents a derivative with respect to $t$. For this gauge choice, the action of GLPV theories can be written as

$$S = \int d^4x \sqrt{-g} L(N, K, S, R, U; t)$$

$$+ \int d^4x \sqrt{-g} L_m(g_{\mu\nu}, \Psi_m) ,$$

(2.2)

where $L_m$ is the matter Lagrangian, and

$$L = A_2(N, t) + A_3(N, t) K$$

$$+ A_4(N, t) (K^2 - S) + B_4(N, t) R$$

$$+ A_5(N, t) K_3 + B_5(N, t) \left( U - \frac{1}{2} KR \right) .$$

(2.3)

Here we have defined $K \equiv K^{\mu\nu} , S \equiv K_{\mu\nu} K^{\mu\nu} , R \equiv \mathcal{R}^{\mu\nu} , U \equiv \mathcal{R}_{\mu\nu} K^{\mu\nu} , \text{ and } K_3 \equiv K^3 - 3K K_{\mu\nu} K^{\mu\nu} + 2K_{\mu\nu} K^{\mu\nu} K^{\lambda} \text{ with arbitrary functions } A_{2,3,4,5} \text{ and } B_{4,5} \text{ that depend on } N \text{ and } t \text{ [32, 34].}$

The coefficients $G_2, G_3, G_4, G_5$ in Horndeski theories are related to $A_2, A_3, A_4, A_5$ and $B_4, B_5$, as

$$A_2 = G_2 - X F_{3,\phi} ,$$

$$A_3 = 2(-X)^{3/2} F_{3,X} - 2\sqrt{-X} G_{4,X} ,$$

$$A_4 = -G_4 + 2X G_{4,X} + X G_{5,\phi}/2 ,$$

$$B_4 = G_4 + X (G_{5,\phi} - F_{5,\phi})/2 ,$$

$$A_5 = -(X)^{3/2} G_{5,X}/3 ,$$

$$B_5 = -\sqrt{-X} F_5 ,$$

(2.4)
where a comma in a lower index represents a partial derivative with respect to a given scalar quantity, and $F_3$ and $F_5$ are auxiliary functions satisfying $G_3 = F_3 + 2XF_3,X$ and $G_{5,X} = F_5/(2X) + F_{5,X}$. From these relations it follows that Horndeski theories obey the two conditions

$$A_4 = 2XB_{4,X} - B_4, \quad A_5 = \frac{1}{3}XB_{5,X}. \quad (2.5)$$

GLPV theories are described by the Lagrangian \(2.3\) without imposing the two constraints \(2.5\). Even without these restrictions, the linear perturbation equations of motion on the flat FLRW background remain of second order without having an extra propagating scalar degree of freedom \[32, 33, 34\].

For the study of growth of large-scale structures, we have taken into account the matter Lagrangian \(32, 35, 36\). We consider the perturbations and varying the first-order action with respect to \(\delta N\) and \(\delta a\), we obtain the background equations

$$\dot{L} + L_{,N} - 3HF = \rho, \quad (3.4)$$
$$\dot{L} - \dot{F} - 3HF = -P, \quad (3.5)$$

respectively, where \(F \equiv L_{,K} + 2HL_{,S}\). The matter component obeys the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0. \quad (3.6)$$

The second-order action for tensor perturbations derived from Eq. \(2.2\) reads

$$S_2^{(h)} = \int d^4x a^3q_t\delta^{ik}\delta^{jl}\left(\dot{\gamma}_{ij} \gamma_{kl} - \frac{c_s^2}{a^2} \partial \gamma_{ij} \partial \gamma_{kl}\right), \quad (3.7)$$

where

$$q_t = \frac{L_{,S}}{4}, \quad c_s^2 = \frac{E}{L_{,S}}. \quad (3.8)$$

and

$$E = L_{,R} + \frac{1}{2}L_{,tt} + \frac{3}{2}HL_{,tt}. \quad (3.9)$$

We require the two conditions

$$q_t > 0, \quad (3.10)$$
$$c_s^2 > 0, \quad (3.11)$$

to avoid the tensor ghost and small-scale Laplacian instabilities, respectively.

In the presence of matter, the second-order action \(S_2^{(\mu)}\) for scalar perturbations in GLPV theories was given in Ref. \[36, 37, 48\]. GLPV theories satisfy conditions for the absence of spatial derivatives higher than second order \[32, 51\]. Varying \(S_2^{(\mu)}\) with respect to \(\delta N\), \(\theta^2\psi\), and \(\zeta\), it follows that

$$\left(2L_{,N} + L_{,NN} - 6HW + 12L_{,S}H^2\right)\delta N$$
$$+ \left(3\dot{\zeta} - \frac{\partial^2 \psi}{a^2}\right)W - 4(D + \dot{\Sigma})a^2 2\zeta = 0, \quad (3.12)$$

$$W\delta N - 4L_{,S}\dot{\zeta} = -\delta q, \quad (3.13)$$

$$\frac{1}{a^3} \frac{d}{dt} \left(a^3 \Upsilon\right) + 4(D + \dot{\Sigma})a^2 \frac{\partial^2 \delta N}{a^2} + 4E \frac{\partial^2 \zeta}{a^2}$$
$$- 3(\rho + P)\delta N = 3\delta P, \quad (3.14)$$

respectively, where

$$W \equiv L_{,K}N + 2HL_{,SN} + 4L_{,S}H, \quad (3.15)$$
$$D \equiv L_{,NR} - \frac{1}{2}L_{,tt} + HL_{,NN}, \quad (3.16)$$
$$\Upsilon \equiv 4L_{,S}a^2 - 3\delta q. \quad (3.17)$$
The continuity equations $\delta T^\mu{}_{0\mu} = 0$ and $\delta T^\mu{}_{i\mu} = 0$ lead, respectively, to

$$\delta \rho + 3H(\delta \rho + \delta P) = -(\rho + P)
\left(3\zeta - \frac{\partial^2 \psi}{a^2}\right) - \frac{\partial^2 \delta q}{a^2} ,$$

(3.18)

$$\delta q + 3H\delta \rho = -(\rho + P)\delta N - \delta P .$$

(3.19)

Substituting Eq. (3.17) into Eq. (3.14) and employing Eq. (3.19), we obtain

$$(\dot{L}_S + HL_S)\psi + L_S\psi + (D + \mathcal{E})\delta N + \mathcal{E} = 0 .$$

(3.20)

On using Eqs. (3.12) and (3.13), the second-order scalar action $S_2$ can be expressed in terms of $\zeta$, the matter perturbations, and their derivatives [48]. Provided that the matter component does not correspond to a ghost mode, the scalar ghost is absent under the condition [34, 48, 51]

$$q_s = \frac{8q_s(16q_s w_s + 3W^2)}{W^2} > 0 ,$$

(3.21)

where $w_s = 2L_{\delta N} + L_{\delta NN} - 6HW + 12H^2L_S$. Under the condition [34, 48], Eq. (3.21) translates to $16q_s w_s + 3W^2 > 0$.

In GLPV theories the scalar propagation speed squared $c_s^2$ is affected by the presence of matter [34, 39, 48]. For nonrelativistic matter characterized by $P = 0$ and $\delta P = 0$, the value of $c_s^2$ in the small-scale limit reads

$$c_s^2 = \frac{2}{q_s} \left[ \dot{\mathcal{M}} + HM - \mathcal{E} - \frac{4L_S^2}{W^2} (1 + 2\alpha_H) \right] ,$$

(3.22)

where

$$\dot{\mathcal{M}} = \frac{4L_S(D + \mathcal{E})}{W} ,$$

(3.23)

$$\alpha_H = \frac{D + \mathcal{E}}{L_S} - 1 = c_s^2 - 1 + \frac{D}{L_S} .$$

(3.24)

The parameter $\alpha_H$ characterizes the deviation from Horndeski theories [36]. To avoid the small-scale instability of scalar perturbations, we require that

$$c_s^2 > 0 .$$

(3.25)

The four conditions [34, 48, 51, 61], and (3.25) need to be satisfied for theoretical consistency.

We define the gauge-invariant gravitational potentials [58]

$$\Psi \equiv \delta N + \psi , \quad \Phi \equiv \zeta + H \psi ,$$

(3.26)

and the gravitational slip parameter

$$\eta \equiv - \frac{\Phi}{\Psi} .$$

(3.27)

Then, Eq. (3.20) can be written as

$$\Psi + \Phi = \frac{\dot{q}_s}{Hq_s} (\zeta - \Phi) - (c_s^2 - 1) \zeta - \alpha_H \delta N .$$

(3.28)

This shows that the deviation of $\eta$ from 1 is induced by the variation of $q_s$, the deviation of $c_s^2$ from 1, and the deviation parameter $\alpha_H$ from Horndeski theories.

We also introduce the effective gravitational potential

$$\Phi_\Sigma \equiv \frac{1}{2} (\Psi - \Phi) ,$$

(3.29)

which is associated with the deviation of light rays in weak lensing and CMB observations [59].

IV. SUBHORIZON PERTURBATIONS

To study the growth of structures during the matter-dominated epoch, we shall take into account nonrelativistic matter satisfying $P = 0$ and $\delta P = 0$ for the Lagrangian $L_m$. We also define the gauge-invariant matter perturbation

$$\delta m = \delta - 3H v ,$$

(4.1)

where $\delta \equiv \delta \rho / \rho$ and $v \equiv \delta q / \rho$. Taking the time derivative of Eq. (3.18) in Fourier space and using Eq. (3.19), we obtain

$$\delta_m + 2H \delta m + \frac{k^2}{a^2} \Psi = -3\dot{B} - 6H \dot{B} ,$$

(4.2)

where $k$ is a comoving wave number and $B \equiv \zeta + Hv$. The gravitational potential $\Psi$ works as a source term for the growth of matter perturbations.

To estimate the evolution of $\Psi$, we consider the perturbations deep inside the Hubble radius, i.e., $k/a \gg H$. In Fourier space we employ the subhorizon approximation under which the dominant contributions to the perturbation equations are the terms involving $\delta \rho$ and the terms multiplied by $k^2/a^2$ [60]. In GR, the accuracy of this approximation was numerically confirmed for the subhorizon perturbations [61].

By employing the subhorizon approximation in modified gravity theories, we consider theories in which the deviation from GR is not so significant in a way that the terms other than those containing $k^2/a^2$ and $\delta \rho$ are still subdominant to the perturbation equations, as in the case of GR. For example, the orders of the terms $L_{\delta N}\delta N$ and $L_S H^2 \delta N$ in Eq. (3.12) are regarded to be at most of the orders of $M^2_{\delta N} H^2 \delta N$. Under the subhorizon approximation, Eq. (3.12) reads

$$\left(\frac{k}{aH}\right)^2 [\alpha_{wv} \chi + 4(1 + \alpha_H) \zeta] \simeq 6\Omega_m \delta ,$$

(4.3)

where

$$\alpha_{wv} \equiv \frac{W}{HL_S} , \quad \chi \equiv H \psi , \quad \Omega_m \equiv \frac{\rho}{6H^2 L_S} .$$

(4.4)

The perturbation $\chi$ is related to the gravitational potential $\Phi$, as $\Phi = \zeta + \chi$. The definition of $\Omega_m$ comes from the
fact that the Friedmann equation (5.4) can be written as
\[ 6H^2L_S = \rho - A_2 - 2H^3A_5 - A_{2,N} - 3HA_{3,N} + 6H^2L_{SN} + 12H^3A_{5,N}, \] (4.5)
where we used the relation \( A_4 = -L_S - 3HA_5 \). Alternatively, one may introduce the General-Relativistic form of the matter density parameter \( \Omega_m = \rho/(3H^2M_{pl}^2) \) by expressing Eq. (4.4) in the form \( 3H^2M_{pl}^2 = \rho + \cdots \), where \( M_{pl} \) is the reduced Planck mass. In this case, the rhs of Eq. (4.3) is replaced by \( 3M_{pl}^2\dot{\Omega}_m/L_S \).

Expressing Eq. (3.20) in terms of \( \Psi, \zeta, \chi, \) and \( \dot{\chi} \), it follows that
\[
(1 + \alpha_H)\Psi + \epsilon_s^2\zeta + (1 + \epsilon_s - \alpha_H\epsilon_H)\chi = \alpha_H\frac{\dot{\chi}}{H},
\] (4.6)
where
\[
\epsilon_s \equiv \frac{\dot{L}_S}{HL_S}, \quad \epsilon_H \equiv -\frac{\dot{H}}{H^2}.
\] (4.7)

Under the subhorizon approximation the continuity equation (3.18) reads
\[
\delta \dot{\rho} + 3H\delta \rho \simeq \frac{k^2}{a^2} \left( -\rho\dot{\psi} + 4L_s\dot{\zeta} - W\delta N \right),
\] (4.8)
where we employed Eq. (3.13) and ignored the term \( 3\dot{\zeta} \) relative to \( (k^2/a^2)\psi \). Taking the time derivative of Eq. (4.3), we can eliminate the terms \( \delta \dot{\rho} \) and \( \delta \rho \) on the lhs of Eq. (4.8). This process leads to
\[
\alpha_w\psi + (\alpha_w + \epsilon_w\alpha_w + 6\Omega_m)\chi + 4[(1 + \alpha_H)(1 + \epsilon_s) + \epsilon_{oh}]\zeta \simeq -4\alpha_H\frac{\dot{\chi}}{H},
\] (4.9)
where
\[
\epsilon_w \equiv \frac{\dot{W}}{HW}, \quad \epsilon_{oh} \equiv \frac{\dot{\alpha}_H}{H}.
\] (4.10)

In the unitary gauge \( (\delta \phi = 0) \), the mass \( M \) of the scalar degree of freedom does not explicitly appear in the perturbation equations of motion. The above subhorizon approximation is valid for \( M \) smaller than \( c_s k/a \) (6.4). In the regime \( M \gg c_s k/a \), the scalar field is nearly frozen to recover the General-Relativistic behavior, so that \( G_{eff} \) is very close to \( G \). In fact, this happens for viable \( f(R) \) dark energy models in the early stage of the matter era.

Since \( \alpha_H = 0 \) in Horndeski theories, the terms on the rhs of Eqs. (4.6) and (4.9) vanish. In this case, we can express \( \chi \) and \( \zeta \) as a function of \( \Psi \) from Eqs. (4.6) and (4.9) and then derive \( \Psi \) in terms of \( \delta \) by using Eq. (4.3).

In GLPV theories we need to deal with Eqs. (4.6) and (4.9) as the differential equations involving \( \dot{\chi} \) and \( \ddot{\zeta} \). We define the quantities
\[
\epsilon_\zeta \equiv \frac{\dot{\zeta}}{H\zeta}, \quad \epsilon_\chi \equiv \frac{\dot{\chi}}{H\chi}.
\] (4.11)
If \( \epsilon_\zeta \) and \( \epsilon_\chi \) are smaller than the order of \( 1 \), the terms on the rhs of Eqs. (4.6) and (4.9) can be regarded as the corrections to those on the lhs. Let us consider this situation and express \( \zeta \) and \( \dot{\chi} \) in terms of \( \epsilon_\zeta \) and \( \epsilon_\chi \).

Of course the evolution of \( \epsilon_\zeta \) and \( \epsilon_\chi \) is known only by solving the full perturbation equations for a given theory, so Eqs. (4.6), (4.9), and (4.11) are not closed for \( \alpha_H \neq 0 \).

We define the effective gravitational coupling \( G_{eff} \), as
\[
\frac{k^2}{a^2} \Psi = -4\pi G_{eff} \rho \delta_m,
\] (4.12)
which works as a source term for the growth of \( \delta_m \) on the lhs of Eq. (4.12). The potential (3.20) obeys
\[
\frac{k^2}{a^2} \Phi \Sigma = -4\pi G \Sigma \rho \delta_m,
\] (4.13)
where
\[
\Sigma = \frac{1 + \eta G_{eff}}{2G}.
\] (4.14)

Recall that the gravitational slip parameter is given by \( \eta = -(\zeta + \chi)/\Psi \).

It is convenient to express \( G_{eff} \), \( \eta \), and \( \Sigma \) in terms of the quantities \( q_1 \), \( q_s \), and \( c_s^2 \) because the signs of them need to be positive. Substituting the relations \( E = 4q_1c_s^2 \), \( D + E = 4q_1(1 + \alpha_H) \), \( W = 4Hq_1\alpha_w \), and \( M = 16q_1(1 + \alpha_H)/(H\alpha_w) \) into Eq. (3.22), it follows that
\[
c_s^2 = \frac{32q_1(1 + \alpha_H)}{q_s\alpha_w} \left[ 1 + 2q_s - \epsilon_w + \frac{\epsilon_{oh}}{1 + \alpha_H} \right. \\
\left. - \frac{c_s^2\alpha_w}{4(1 + \alpha_H)} - \frac{6\Omega_m 1 + 2\alpha_H}{\alpha_w 1 + \alpha_H} \right],
\] (4.15)

This relation can be employed to express \( \epsilon_w \) in terms of \( c_s^2 \). On using Eqs. (4.3), (4.13), and (4.15) with Eq. (4.11) and the relation \( \delta_m \simeq \delta \) under the subhorizon approximation, we obtain
\[
G_{eff} = \frac{M_{pl}^2 f_1}{8q_1 f_2},
\] (4.16)
\[
\eta = \frac{f_3}{f_1},
\] (4.17)
\[
\Sigma = \frac{f_1 + f_3 M_{pl}^2}{2f_2},
\] (4.18)
where \( G = (8\pi M_{pl}^2)^{-1} \) is the gravitational constant, and
\[ f_1 = 64[2(1 + \alpha_1)(1 - \alpha_H \epsilon_H - \alpha_H \epsilon_x + \epsilon_s)(1 + \epsilon_s + \alpha_H + \alpha_H \epsilon_x + \alpha_H \epsilon_s + \epsilon_{\alpha_H}) + 3 \alpha_H \epsilon_s^2 \Omega_m] q_t \]
\[ -32t^2 \alpha_w [2 + \epsilon_{\alpha_H} + 2 \epsilon_s + 2 \alpha_H (1 + \epsilon_s)] q_t + \alpha_w c_s^2 (q_s^2 - 8 \alpha_s^2), \]
\[ f_2 = (1 + \alpha_1)(1 + \alpha_1) q_s c_s^2 + 8 \alpha_1 |24(1 + \alpha_1) \Omega_m + c_s^2 \Omega_m - 4(1 + \alpha_1) \alpha_w (1 + \epsilon_H + \epsilon_s + \epsilon_x - \epsilon_s)| q_t, \]
\[ f_3 = (1 + \alpha_1)(64[2(1 + \alpha_1)(1 + \epsilon_s + \alpha_H + \alpha_H \epsilon_x + \alpha_H \epsilon_s + \epsilon_{\alpha_H}) + 3 \alpha_H \epsilon_s^2 \Omega_m] q_t \]
\[ -32 \alpha_w [1 + c_s^2 + \epsilon_{\alpha_H} + \epsilon_s + \alpha_H (2 + \epsilon_H + 2 \epsilon_s + \epsilon_x)] q_t + \alpha_w (q_s^2 + 8 \alpha_s^2). \]

Equation (4.16) shows that \( G_{\text{eff}} \) is scale independent. This comes from the fact that we have ignored the field mass \( M \) relative to \( c_s k/a \) for its derivation. In other words, the results (4.19)-(4.21) are valid in the regime \( M \ll c_s k/a \). For the scalar degree of freedom associated with dark energy, the field mass is usually smaller than \( c_s k/a \) at the late cosmological epoch. For some dark energy models in which the chameleon mechanism [62] is at work in the region of high density, there is a transition from the General-Relativistic regime \( (M \gg c_s k/a) \) with \( G_{\text{eff}} \approx G \) to the scalar-tensor regime \( (M \ll c_s k/a) \) with \( G_{\text{eff}} \) given by Eq. (4.10) [63, 64]. For the model discussed later in Sec. VI the condition \( M \ll c_s k/a \) is satisfied for subhorizon perturbations.

In Horndeski theories \((\alpha_H = 0, \epsilon_{\alpha_H} = 0)\), the above analytic solutions are closed. In GLPV theories the terms \( \alpha_H \epsilon_s \) and \( \alpha_H \epsilon_x \) do not vanish in Eqs. (4.19)-(4.21). In this case, we can check the validity of the subhorizon approximation by solving the full perturbation equations of motion numerically for a given theory and by comparing the full results with the estimations (4.16)-(4.18) derived after the substitution of numerical values of \( \epsilon_s \) and \( \epsilon_x \) into Eqs. (4.19)-(4.21). In Sec. VI we shall do so for a concrete theory in the framework of GLPV theories.

V. POSSIBILITY OF REALIZING WEAK GRAVITY IN THE COSMIC GROWTH HISTORY

In this section we discuss the possibility of realizing a gravitational interaction weaker than that in GR \((G_{\text{eff}} < G)\) on scales relevant to large-scale structures. We shall focus on GLPV theories described by the Lagrangian (2.3). Then, the tensor propagation speed squared is given by

\[ c_t^2 = \frac{B_4 + B_5}{-A_4 - 3 H A_5}. \]

For the evaluation of the scalar propagation speed squared (4.13), it is convenient to express \( \Omega_m \) by using the background equations of motion (3.4) and (3.5), i.e.,

\[ \frac{6 \Omega_m}{\alpha_w} = \frac{L_N + \dot{F}}{H W}. \]

In what follows we shall discuss the cases of Horndeski and GLPV theories separately.

A. Horndeski theories

Substituting \( \alpha_H = 0 \) and \( \epsilon_{\alpha_H} = 0 \) into Eqs. (4.19)-(4.21), we obtain

\[ f_1 = \frac{M_{\text{pl}}^2 c_s^2}{8 q_t} \left( 1 + \frac{8 Q^2}{\alpha_w c_s^2} q_t \right), \]

\[ \eta = \frac{\alpha_w q_s c_s^2 + 8(\alpha_w - 4) q_t}{\alpha_w q_s c_s^2 + 8 Q^2 q_t}, \]

\[ \Sigma = \frac{M_{\text{pl}}^2 (1 + c_t^2)}{16 q_t} \left( 1 + \frac{8 Q (Q + \alpha_w - 4) q_t}{\alpha_w c_s^2 (1 + c_t^2)} q_t \right), \]

where

\[ Q \equiv \alpha_w c_s^2 - 4 - 4 \epsilon_s. \]

Provided the rhs of Eq. (5.3) is smaller than 1, it follows that \( G_{\text{eff}} < G \). The contribution \( M_{\text{pl}}^2 c_s^2 / (8 q_t) \) in \( G_{\text{eff}} / G \) originates from the tensor part, whereas the second term in the bracket of Eq. (5.3) comes from the interaction between the scalar field and matter. Under the no-ghost and stability requirements (3.10), (3.11), (3.21), and (3.25), the latter condition is always positive. Hence the necessary condition for realizing a gravitational interaction weaker than that in GR reads

\[ \frac{M_{\text{pl}}^2 c_s^2}{8 q_t} < 1. \]

Due to the presence of the scalar-matter interaction, the condition (5.3) is not sufficient for realizing \( G_{\text{eff}} < G \). The quantities \( q_t \) and \( c_t^2 \) are given, respectively, by

\[ q_t = -\frac{1}{4} (A_4 + 3 H A_5), \]

\[ c_t^2 = \frac{B_4 + B_5}{B_4 - 2 X B_4, X + H X B_5, X}, \]

where we used the two relations (2.5). The explicit form of the quantity \( Q \) in Eq. (5.3) is given by

\[ Q = \frac{A_3, N c_s^2 + 4 H A_4, N c_s^2 + 4 A_4 + 4 H A_4 (1 - c_s^2) + 6 H^2 A_5, N c_s^2 + 12 H A_5 + 12 [H^2 (1 - c_s^2) + \dot{H}] A_5}{H (-A_4 - 3 H A_5)}. \]

\[ (5.10) \]
In GR we have \(-A_4 = B_4 = M_{pl}^2/2\), \(A_3 = 0\), and \(A_5 = B_5 = 0\), in which case \(q_t = M_{pl}^2/8\), \(c_t^2 = 1\) and \(Q = 0\). In modified gravitational theories, we generally have \(M_{pl}^2 c_t^2/(8q_t) \neq 1\) and \(Q \neq 0\).

As an example, we consider theories described by the Lagrangian

\[ L = A_2(N,t) + A_3(N,t)K + A_4(t) \left( K^2 - S - R \right), \]

(5.11)

in which case \(q_t = -A_4/4\), \(c_t^2 = 1\), and \(Q = -(A_3N + 4A_4)/H^2A_4\). The theory given by \(L = -\epsilon(\phi) X/2 - V(\phi) + M_{pl}^2 F(\phi) R/2\), where \(\epsilon\), \(V\), and \(F\) are arbitrary functions of \(\phi\), belongs to a subclass of Eq. (5.11). For the choices \(\epsilon(\phi) = \omega_B M_{pl} / \phi\) and \(F(\phi) = \phi / M_{pl}\), this Lagrangian encompasses Brans-Dicke theory \(\frac{\omega_B}{\phi}\) as a special case. On using the correspondence \(2.3\), the Lagrangian of this theory is equivalent to Eq. (5.11) with the functions

\[ A_2 = -\frac{1}{2} \epsilon(\phi) X - V(\phi), \]
\[ A_3 = -\frac{1}{N} M_{pl}^2 \phi F, \phi(\phi), \]
\[ A_4 = -B_4 = -\frac{1}{2} M_{pl}^2 F(\phi). \]

(5.12)

In this case the relation \(W = -F = M_{pl}^2 (2HF + \phi F, \phi)\) holds, so Eq. (4.11) reduces to \(c_t^2 = 1\) by using Eq. (5.12).

From Eqs. (5.13) and (5.14) it follows that

\[ G_{eff} = \frac{F}{\epsilon} \left( 1 + \frac{M_{pl}^2 F^2}{2\epsilon + 3M_{pl}^2 F^2} \right), \]
\[ \eta = \frac{F \epsilon + M_{pl}^2 F^2}{F / 2 + 2M_{pl}^2 F^2}, \]
\[ \Sigma = \frac{1}{F}. \]

(5.13)

(5.14)

(5.15)

The conditions \(q_t > 0\) and \(q_s > 0\) translate to \(F > 0\) and \(2\epsilon + 3M_{pl}^2 F^2 > 0\), respectively, so \(G_{eff}\) is larger than \(G/F\) due to the presence of the second term in the bracket of Eq. (5.13). From Eq. (5.14) the parameter \(\eta\) is smaller than 1. The enhancement of \(G_{eff}\) is compensated by the smallness of \(\eta\), so that we obtain the value (5.15). In other words, we have \(Q + \alpha_w - 4 = 0\) in Eq. (5.5) for the model discussed above.

Let us consider the full Lagrangian (2.3) with the Horndeski relations (2.2). The behavior of the quantities \(q_t\) and \(c_t^2\) is crucial for the realization of the condition (5.7). The quantity \(q_t = L_4 / 4\) is associated with the matter density parameter \(\Omega_m\) defined in Eq. (4.1), as \(\Omega_m = \rho/(2AH^2q_t)\). The matter perturbation equation (4.2) can be expressed by using \(\Omega_m\), as

\[ \delta_m'' + \left( 2 + \frac{H'}{H} \right) \delta_m' - \frac{3}{2} c_t^2 \left( 1 + \frac{8Q^2}{C_{\phi}\phi c_t^2 q_t} \right) \Omega_m \delta_m \sim 0, \]

(5.16)

where a prime represents a derivative with respect to \(N = \ln a\). Note that we neglected the contribution on the rhs of Eq. (1.2) under the subhorizon approximation.

In Eq. (5.10) the quantity \(q_t\) appearing in the denominator of Eq. (5.3) has been absorbed into the definition of \(\Omega_m\). If \(\Omega_m\) is smaller than that in GR due to large values of \(q_t\), it is possible to realize a cosmic growth rate smaller than that in GR. This is one possibility for the realization of weak gravity recently studied in Ref. [68]. For this purpose we require that the scalar-matter coupling \(Q\) is suppressed to satisfy the condition \(c_t^2(1 + 8Q^2 q_t/(\alpha_w c_t^2 q_t)) \Omega_m < 1\). In dark energy models based on \(f(R)\) theories, for example, we have that \(c_t^2 = 1\), \(8Q^2 q_t/(\alpha_w c_t^2 q_t) = 1/3\), and that the deviation of \(\Omega_m\) from the value of GR is not so significant, in which case the growth rate of \(\delta_m\) is larger than that in GR [13, 66].

Let us proceed to the discussion of \(c_t^2\) appearing in Eq. (5.10). Due to the relations (5.9), the \(X\) dependence in \(B_4\) and \(B_5\) gives rise to the functions \(A_4\) and \(A_5\) depending on \(X\) as well as \(\phi\). These terms contribute to the background equations of motion as the field energy density and pressure. To realize a viable cosmology, the derivative terms \(2XB_4\) and \(HXB_5\), and \(B_5/2\) in Eq. (5.9) should be suppressed relative to \(B_4\) in the deep matter era, in which case \(c_t^2\) is close to 1.

If the field \(\phi\) is responsible for dark energy, the derivative terms mentioned above can be comparable to \(B_4\) after the onset of cosmic acceleration. This can lead to a deviation of \(c_t^2\) from 1. Whether \(c_t^2\) decreases or not depends on the models and initial conditions. In covariant Galileons [69], for example, the entry into the region \(c_t^2 < 1\) can occur for late-time tracking solutions [70].

From Eq. (5.10) the \(t\) and \(N\) dependence in \(A_4\) and \(A_5\) as well as the \(N\) dependence in \(A_3\) lead to nonzero values of \(Q\), which enhances the rhs of Eq. (5.3). Even if \(c_t^2\) starts to decrease after the end of the matter era, the scalar-matter interaction induced by the term \(Q\) should not be large enough to violate the condition \(G_{eff} < G\) for the realization of weak gravity. In covariant Galileons, for example, this scalar-matter interaction usually gives rise to a \(G_{eff}\) larger than \(G\) even for \(c_t^2 < 1\) at the level of linear perturbations [71].

To summarize, the modification to \(G_{eff}\) coming from tensor perturbations is the first crucial factor for realizing weak gravity in Horndeski theories, but the condition (5.7) is not sufficient due to the presence of the scalar-matter interaction. In Sec. (V) we shall study what kind of difference arises in GLPV theories.

B. Theories beyond Horndeski

Since GLPV theories are not subject to the constraints (5.7), the values of \(c_t^2\) are not restricted to be close to 1 even in the early matter era. In GLPV theories, the general expressions of \(G_{eff}\), \(\eta\), and \(\Sigma\) are not as simple as those in Horndeski theories due to the extra terms \(\alpha_M\) and \(\epsilon_M\). To simplify the analysis, we focus on the theories described by the Lagrangian

\[ L = -\frac{1}{2} X - V(\phi) - \frac{M_{pl}^2}{2} (K^2 - S) + \frac{M_{pl}^2}{2} F(\phi) R, \]

(5.17)
where $F(\phi)$ is a function of $\phi$ different from 1. Unlike Ref. [39], we do not restrict the situation to the case in which $F(\phi)$ is constant. In fact, we will show that the theories with a time-varying $F(\phi)$ allow for the realization of weak gravity in the regime $0 < c^2 < 1$.

Since $\mathcal{D} = 0$ for the Lagrangian (5.17), it follows that

$$
\alpha_t = c_t^2 - 1, \quad c^2 = F(\phi) .
$$

(5.18)

We also have $q_t = M_{pl}^2 / 8$, $\epsilon_s = 0$, $W = 2HM_{pl} = -F$, $\alpha_w = 4$, and $c_{\nu}^2 = H / H^2 = -c_H$. The background equation $L_N + \mathcal{F} = \rho$, which follows from Eqs. (3.4) and (3.5), corresponds to $\dot{H} / H^2 = -3\Omega_m / 2 - 3\Omega_X$, where $\Omega_X = \phi^2 / (6H^2M_{pl}^2)$ is the density parameter of the field kinetic energy. Then, the scalar sound speed squared (4.15) reduces to

$$
c^2 = c_t^2 + 3\Omega_m(1 - c_t^2) + 2\epsilon_{\alpha} ,
$$

(5.19)

where $\epsilon_{\alpha} = 2c_t c_\alpha / H$. On using Eq. (5.19), we can express $q_t = \phi^2 / (2H^2)$ without using $\phi^2$. Then, the quantities $f_1$, $f_2$, and $f_3$ in Eqs. (4.19)-(4.21) reduce to

$$
f_1 = \frac{8M_{pl}^2c_t^2}{c_s^2 - c_t^2} \left[ 2c_t^2\epsilon_{\alpha} - 2(c^2 - c_t^2)(c_t^2 - 1)(c_t^2 + \epsilon_{\alpha})\epsilon_H - 3(c_t^2 - 1)c_t^2\Omega_m \right] - 16M_{pl}^2c_t^4(c_t^2 - 1) \left[ (c_t^2 + \epsilon_{\alpha})\epsilon_H - 3(c_t^2 - 1)(\epsilon_H + \epsilon_\chi) \right] ,
$$

(5.20)

$$
f_2 = \frac{8M_{pl}^2c_t^4}{c_s^2 - c_t^2} \left[ 2c_t^2\epsilon_H - 2c_t^2\epsilon_{\alpha} + c_t^2(2c_t^2 - 1)(2\epsilon_H - 3\Omega_m) \right] - 16M_{pl}^2c_t^4(2c_t^2 - 1)(\epsilon_H - \epsilon_\chi) ,
$$

(5.21)

$$
f_3 = \frac{8M_{pl}^2c_t^4}{c_s^2 - c_t^2} \left[ 2c_t^2[1 + c_t^2(c_t^2 - 2 - \epsilon_H + \epsilon_{\alpha})] + \epsilon_H - c_t^2(2c_t^2 - 1)(2c_t^2 - 2 - \epsilon_H + 3\Omega_m + 2\epsilon_{\alpha}) \right] + 16M_{pl}^2c_t^4(c_t^2 - 1)(c_t^2\epsilon_\chi - \epsilon_H) .
$$

(5.22)

In the deep matter-dominated epoch characterized by $\epsilon_H \simeq 3/2$ and $\Omega_m \simeq 1$, we take the quasi-static limits $\epsilon_\chi \rightarrow 0$ and $\epsilon_H \rightarrow 0$ in Eqs. (5.20)-(5.22). The validity of this approximation will be checked in Sec. VI. Then, the effective gravitational coupling (4.10) and the gravitational slip parameter (1.17) reduce, respectively, to

$$
\frac{G_{\text{eff}}}{G} \simeq 1 + \frac{1 - c_t^2}{c_t^2} + \frac{5(1 - c_t^2)(c_t^2 - c_s^2)}{c_t^2 c_t^2(3 - 3c_t^2 + 2\epsilon_{\alpha})} \epsilon_{\alpha} ,
$$

(5.23)

$$
\eta \simeq 1 + \frac{5(1 - c_t^2)(c_t^2 - c_s^2)}{3c_t^2(1 + c_t^2 - c_s^2)} + \frac{25c_t^2(c_t^2 - c_s^2)(1 - c_t^2)}{3c_t^2(1 + c_t^2 - c_s^2)} + \frac{25c_t^2(c_t^2 - c_s^2)(1 - c_t^2)}{3c_t^2(1 + c_t^2 - c_s^2)} - 3(1 + c_t^2)c_t^2(\epsilon_{\alpha} - 1) + 5c_t^2\epsilon_{\alpha} \epsilon_{\alpha} .
$$

(5.24)

When $c_s^2$ is constant, i.e., $\epsilon_{\alpha} = 0$, these results match with those derived in Ref. [39] for scaling solutions realized by the potential $V(\phi) = V_1 e^{-\lambda_1 \phi / M_p}$. If $c_t^2 > 1$, then $c_t^2$ becomes negative for $c_t^2 - 1 > 2\Omega_X / (\Omega_m - 2\Omega_X)$. Provided that $1 < c_t^2 < 1 + 2\Omega_X / (\Omega_m - 2\Omega_X)$, we have $G_{\text{eff}} < G$ from Eq. (5.23). Since $\Omega_X \ll \Omega_m$ during the matter-dominated epoch, we require that $c_t^2$ is very close to 1 to avoid the Laplacian instability of scalar perturbations. Then, the deviation of $G_{\text{eff}}$ from $G$ is restricted to be small.

If $c_t^2$ is smaller than 1 without any variation, it follows that $G_{\text{eff}} > G$. In this case, $c_t^2$ away from 1, $c_t^2$ tends to be much larger than 1 during the era. Hence $G_{\text{eff}}$ is also restricted to be close to $G$. In the regime $0 < c_t^2 \ll 1$, the parameter $\eta$ is approximately given by $\eta \simeq 1 + 5(1 - c_t^2)/(3c_t^2)$ under the condition $c_t^2 \gg 1$. This large deviation of $\eta$ from 1 is the distinguished observational signature of the model (5.17) with $c_t^2$ smaller than 1 [38].

If $c_t^2$ varies in time, both $G_{\text{eff}}$ and $\eta$ are subject to change. If $|\epsilon_{\alpha}|$ is much smaller than 1, we can ignore the contribution of the terms $\epsilon_{\alpha}$ appearing in the denominators of Eqs. (5.23) and (5.24). Let us consider the case in which $c_t^2 < 1$ and $c_t^2$ is larger than the order of 1. Then, Eqs. (5.23) and (5.24) approximate to

$$
\frac{G_{\text{eff}}}{G} \simeq 1 + \frac{1 - c_t^2}{c_t^2} + \frac{5}{3c_t^2} \epsilon_{\alpha} ,
$$

(5.25)

$$
\eta \simeq 1 + \frac{5(1 - c_t^2)}{3c_t^2} - \frac{25}{9c_t^2} \epsilon_{\alpha} .
$$

(5.26)

The decrease (increase) of $c_t^2$ leads to the negative (positive) values of $\epsilon_{\alpha} = 2c_t c_\alpha / H$. This means that, even in the subluminal regime ($c_t^2 < 1$), it is possible to have $G_{\text{eff}} < G$ for decreasing $c_t^2$. For $\epsilon_{\alpha} < 0$, the parameter $\eta$ also gets increased.
A model with varying \( c_1^2 \) can be realized for the function

\[
F(\phi) = c_1^2 e^{-2\beta \phi/M_{\text{pl}}},
\]

where \( c_1 \) and \( \beta \) are constants. For dark energy models in which \( \Omega_X \) grows in time, Eq. (5.11) shows that \( |c_1^2| \) tends to increase as we go back to the past. This behavior can be avoided for scaling dark energy models described by the field potential \([72]\)

\[
V(\phi) = V_1 e^{-\lambda_1 \phi/M_{\text{pl}}} + V_2 e^{-\lambda_2 \phi/M_{\text{pl}}},
\]

where \( V_1, V_2, \lambda_1, \lambda_2 \) are constants with \( \lambda_1 \gtrsim 10 \) and \( \lambda_2 \lesssim 1 \). During the matter era, the solution is in the scaling regime characterized by \( \Omega_X = 3/(2\lambda_1^2) \) and \( \Omega_m = 1 - 3/\lambda_1^2 \) \([73]\). After the dominance of the second potential on the rhs of Eq. (5.28), the solution finally approaches an attractor with cosmic acceleration \([74]\).

We recall that Eqs. (5.23) and (5.24) are valid only in the regime characterized by \( \Omega_m \equiv 1 - 3/\lambda_1^2 \). During the matter era, the solution is in the scaling fixed point (6.5) because it is a temporal attractor \([75]\)

\[
(6.5)
\]

whereas the late-time scalar-field fixed point is characterized by

\[
x_1 = \frac{\lambda_2}{\sqrt{6}}, \quad x_2 = \sqrt{1 - \frac{\lambda_2^2}{6}}, \quad w_\phi = -1 + \frac{\lambda_2^2}{3}, \quad \Omega_m = 0.
\]

From the big bang nucleosynthesis bound on \( \Omega_m \), the slope \( \lambda_1 \) is constrained to be \( \lambda_1 > 9.4 \) \([72]\). To realize the late-time cosmic acceleration we require that \( \lambda_2^2 < 2 \), under which the scalar-field-dominated fixed point is stable \([74]\).

In Fig. 1 we plot the evolution of \( w_\phi \), \( c_1^2 \), \( c_2^2 \) versus \( 1+z = 1/a \) for \( \beta = 0.1, \ c_1^2 = 0.9, \ \lambda_1 = 10, \ \lambda_2 = 0.5, \) and \( V_2/V_1 = 10^{-6} \) with the initial conditions \( x_1 = \sqrt{6}/(2\lambda_1), \ x_2 = \sqrt{6}/(2\lambda_1), \) and \( x_3 = 0 \) at \( z = 735.4 \). The present epoch (the redshift \( z = 0 \)) is identified by the condition \( \Omega_m = 0.3 \).

For the potential \((5.28)\) the scaling matter era corresponds to \([75]\)

\[
x_1 = x_2 = \frac{\sqrt{6}}{2\lambda_1}, \quad w_\phi = 0, \quad \Omega_m = 1 - \frac{3}{\lambda_1^2}, \quad (6.5)
\]

We consider the case in which the functions \( F(\phi) \) and \( V(\phi) \) are given, respectively, by Eqs. (5.27) and (5.28).

### A. Background equations and propagation speeds

In the presence of nonrelativistic matter the background quantities \( x_1 \equiv \phi/(\sqrt{6}HM_{\text{pl}}), \ x_2 \equiv \sqrt{V/(\sqrt{3}HM_{\text{pl}})}, \) and \( x_3 \equiv \phi/M_{\text{pl}} \) obey the equations of motion

\[
x_1' = \frac{3}{2} x_1 (x_1^2 - x_2^2 - 1) + \frac{\sqrt{6}}{2} \lambda x_2^2, \quad (6.1)
\]

\[
x_2' = \frac{3}{2} x_2 (x_1^2 - x_2^2 + 1) - \frac{\sqrt{6}}{2} \lambda x_1 x_2, \quad (6.2)
\]

\[
x_3' = \sqrt{6} x_1, \quad (6.3)
\]

where \( \lambda \equiv -M_{\text{pl}} V_\phi/V \). The field equation of state \( w_\phi = (\phi^2/2 - V(\phi))/(\phi^2/2 + V(\phi)) \) and the matter density parameter \( \Omega_m = \rho/(3H^2M_{\text{pl}}^2) \) can be expressed, respectively, as

\[
w_\phi = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}, \quad \Omega_m = 1 - x_1^2 - x_2^2. \quad (6.4)
\]

**Figure 1.** Evolution of \( w_\phi, c_1^2, c_2^2 \) versus \( 1+z = 1/a \) for \( \beta = 0.1, \ c_1^2 = 0.9, \ \lambda_1 = 10, \ \lambda_2 = 0.5, \) and \( V_2/V_1 = 10^{-6} \) with the initial conditions \( x_1 = \sqrt{6}/(2\lambda_1), \ x_2 = \sqrt{6}/(2\lambda_1), \) and \( x_3 = 0 \) at \( z = 735.4 \). The present epoch (the redshift \( z = 0 \)) is identified by the condition \( \Omega_m = 0.3 \).
subluminal regime (see Fig. 1 for the case $c^2_\text{th} = 0.9$ and $\beta = 0.1$).

Let us consider the stability condition associated with the scalar propagation speed given by Eq. (6.11). First of all, the quantity $\epsilon_{\alpha m}$ can be expressed as

$$
\epsilon_{\alpha m} = -2\sqrt{6}\beta c^2_s x_1 ,
$$

which is negative for $\beta > 0$ and $\phi > 0$. Since the deviation of $c^2_s$ from 1 gives rise to negative values of $c^2_s$ for $c^2_s > 1$, we focus on the case in which $c^2_s$ is in the subluminal regime.

During the scaling matter era characterized by Eq. (6.5), the condition $c^2_s > 0$ gives the following bound

$$
\beta < \frac{1}{4\lambda_1} \left[ 3 + (\lambda^2 - 3) \left( \frac{1}{c^2_s} - 1 \right) \right].
$$

(6.8)

When $\lambda_1 = 10$ and $c^2_s = 0.9$, for example, $\beta < 0.34$.

Substituting the values (6.6) of the scalar-field-dominated fixed point into Eq. (6.11), it follows that $c^2_s = c^2_s(1 - 4\beta/\lambda_2)$. If we demand the condition $c^2_s > 0$ for the future attractor, we obtain the bound

$$
\beta < \frac{\lambda_2}{4}.
$$

(6.9)

In the numerical simulation of Fig. 1 the parameter $\lambda_2 = 0.5$ is chosen, so the condition (6.9) translates to $\beta < 0.125$. We have numerically confirmed that, for $\beta < 0.125$, $c^2_s$ remains positive from the past to the asymptotic future. If we do not impose the condition $c^2_s > 0$ in the future, then the bound (6.9) is slightly relaxed, e.g., $\beta < 0.15$ for $\lambda_2 = 0.5$.

Figure 1 illustrates the evolution of $c^2_s$ and $c^2_t$ for $\beta = 0.1$, $c^2_\text{th} = 0.9$, $\lambda_1 = 10$, and $\lambda_2 = 0.5$. Since $c^2_t$ decreases in time, this leads to the growth of $c^2_s$ during the scaling matter era. Around the end of the scaling regime the ratio $\Omega_m/\Omega_X$ starts to evolve toward 0, so $c^2_s$ starts to the decrease. Finally, the ratio $c^2_s/c^2_t$ approaches the asymptotic value $1 - 4\beta/\lambda_2 = 0.2$.

We have thus clarified the range of $\beta$ in which the stability conditions of scalar and tensor perturbations are satisfied. Since $q_1 = M^2_p/8$ and $q_2 = \phi^2/(2H^2)$, there are no ghosts in our model.

### B. Perturbation equations and initial conditions

Let us proceed to the discussion of perturbation equations and their initial conditions. Introducing the dimensionless velocity potential

$$
V_m = Hv ,
$$

(6.10)

the perturbation equations of motion can be written as

$$
\zeta' = \delta N + \frac{3}{2} \Omega_m V_m ,
$$

(6.11)

$$
\chi' = \left( \frac{H'}{H} - 1 \right) \chi - c^2_t (\delta N + \zeta) ,
$$

(6.12)

$$
\delta' = -\frac{3}{2} \Omega_m \delta + 3(x^2 - 1)\delta N + K^2(c^2_s + V_m) ,
$$

(6.13)

$$
V'_m = -\delta N + \frac{H'}{H} V_m ,
$$

(6.14)

$$
x^2 \delta N = \frac{1}{2} \Omega_m \delta_m - \frac{1}{3} K^2(c^2_s + \chi) ,
$$

(6.15)

where $K \equiv k/(aH)$ and $H'/H = -(3/2)(1 + x^2 - x^2_2)$.

Taking the $N$ derivative of Eq. (6.14) and using other equations of motion, we obtain

$$
V''_m + \alpha_1 V'_m + \alpha_2 V_m = -K^2 \left( \chi - \frac{1}{3} \epsilon_{\alpha m} c^2_s \zeta \right) ,
$$

(6.16)

where

$$
\alpha_1 = \frac{\sqrt{6}}{4x_1} \left[ 4\lambda(1 - \Omega_m - x^2_1) - \sqrt{6}x_1(2 - \Omega_m - 2x^2_1) \right] ,
$$

(6.17)

$$
\alpha_2 = \left( 1 - c^2_t \right) \frac{\Omega_m}{2x_1} K^2 + \frac{3}{2x_1} \left[ 3x_1 \left( 4x_1^2 + 2(2\Omega_m - 3)x^2 + \Omega_m(\Omega_m - 1) \right) \right. 
$$

$$
- \sqrt{6}\lambda(\Omega_m + 4x^2_1)(\Omega_m + x^2_1 - 1) \right] .
$$

(6.18)

The general solution to Eq. (6.10) can be expressed in the following form

$$
V_m = V_m^{(h)} + V_m^{(s)} ,
$$

(6.19)

where $V_m^{(h)}$ is the homogenous solution satisfying the differential equation $V_m^{(h)'} + \alpha_1 V_m^{(h)} + \alpha_2 V_m^{(h)} = 0$, and $V_m^{(s)}$ is the special solution.

When $c^2_s < 1$ the first term on the rhs of Eq. (6.18) can be much larger than 1 for subhorizon perturbations ($K \gg 1$). During the scaling matter era characterized by Eq. (6.5), the homogenous solution obeys

$$
V_m^{(h)'} + \frac{3}{2} V_m^{(h)'} + \left[ c^2_{\text{eff}} K^2 + \frac{9(\lambda^2 - 3)}{2\lambda^2_1} \right] V_m^{(h)} = 0 ,
$$

(6.20)

where $c^2_{\text{eff}} \equiv (\lambda^2 - 3)(1 - c^2_t)/3$ and the quantity $K$ evolves as $K(N) \propto e^{N/2}$. If $c^2_\text{th}$ is constant, we obtain the following solution in the limit $x \equiv 2c_{\text{eff}}/K \gg 1$ :

$$
V_m^{(h)} \simeq a^{-3/4} \sqrt{\frac{2}{\pi x}} \left[ c_1 \cos \left( x - \frac{\pi}{4} \right) + c_2 \sin \left( x - \frac{\pi}{4} \right) \right] ,
$$

(6.21)

where $c_1$ and $c_2$ are constants. Thus, $V_m^{(h)}$ exhibits the oscillation whose frequency is related to $c_{\text{eff}}$. The amplitude of $V_m^{(h)}$ decreases in proportion to $a^{-1}$.
Provided that $c_1^2$ is a constant smaller than 1, the special solution to Eq. (6.10) is approximately given by $V_m^{(s)} \simeq 2c_1^2 \chi/|\Omega_m(c_1^2 - 1)|$ for subhorizon perturbations. In Ref. [39] it was found that the perturbations $V_m$, $\chi$, and $\zeta$ stay nearly constant along the special solution during the scaling matter era characterized by Eq. (6.23). This means that, as long as the homogeneous solution $V_m^{(h)}$ is initially suppressed relative to the special solution $V_m^{(s)}$, the latter remains the dominant contribution to $V_m$. For nonconstant $c_1^2$, there is an additional term on the rhs of Eq. (6.10). If we neglect the time derivatives of $V_m^{(s)}$ for the estimation of the special solution, we obtain

$$V_m^{(s)} = -\frac{K^2}{2\alpha^2} \left( \chi - \frac{1}{3} \frac{\epsilon_{\alpha m}}{\Omega_m} \zeta \right),$$

where, in the second line, we have picked up the first term on the rhs of Eq. (6.18). From Eq. (6.22) we find that the change of $c_1^2$ actually gives rise to a variation of $V_m^{(s)}$ of the order of $V_m^{(s)} \simeq -\epsilon_{\alpha m} V_m^{(s)}/(c_1^2 - 1)$. This implies that an oscillating mode may arise even for initial conditions of $V_m$ that are very close to the value $V_m^{(s)}$. We recall that the parameter $\beta$ is bounded from above, see Eqs. (6.39) and (6.10). In particular the slope $\beta_2$ needs to be smaller than the order of 1 in order to be consistent with observations of cosmic acceleration, so the condition (6.39) gives the bound $\beta < O(0.1)$. During the scaling matter era the parameter $\epsilon_{\alpha m}$ is given by $\epsilon_{\alpha m} = -6c_1^2 \beta/\lambda_1$, so it is smaller than the order of 0.1. Then the terms $\alpha_1 V_m^{(s)'}$ and $V_m^{(s)''}$ in Eq. (6.10) are not as large as $\alpha_2 V_m^{(s)}$ for $c_1^2$ not very close to 1.

Numerically we solve the perturbation equations of motion for the initial conditions $V_m = V_m^{(s)}$, $V_m' = 0$, and $\chi' = 0$, where $V_m^{(s)}$ is given by Eq. (6.22). For given $\delta_m$, the initial values of $\delta N$, $\zeta$, and $\chi$ are known accordingly from Eqs. (6.12), (6.14), and (6.15). The initial condition of $\delta_m$ is chosen such that its value today is equivalent to $\sigma_{8}(0) = 0.82$, where $\sigma_{8}(0)$ is the rms amplitude of overdensity at the comoving 8$h^{-1}$ Mpc scale ($h$ is the normalized Hubble parameter $H_0 = 100h \text{ km sec}^{-1}\text{Mpc}^{-1}$).

In Fig. [2] we plot the evolution of $\zeta$, $\chi$, $V_m$, and $\delta N$ for the same model parameters as those given in Fig. [1]. Unlike the constant $c_1^2$ model [39], the perturbations $V_m$, $\zeta$, and $\chi$ do not stay constant even during the scaling matter era. This variation is induced by the change of $c_1^2$ appearing in the special solution (6.23).

In Fig. [2], the perturbation $\delta N$ exhibits damped oscillations with a non-negligible initial amplitude. This comes from the fact that $\delta N$ is related to the derivative $V_m''$, as $\delta N = -V_m'' + (H'/H)V_m$. We recall that, even by choosing an initial value of $V_m^{(s)}$ that is identical to $V_m^{(s)}$ in Eq. (6.22), the contribution to $V_m$ from the homogenous solution $V_m^{(h)}$ arises due to the variation of $c_1^2$. Taking the $\mathcal{N}$ derivative of Eq. (6.21), it follows that the amplitude of $V_m^{(h)'}$ is $x/2$ times as large as that of $V_m^{(h)}$. Since we are considering the case $x \gg 1$, the oscillating mode particularly manifests itself in the perturbation $\delta N$ through the derivative $V_m^{(h)'}$.

For initial conditions of $V_m$ away from the special solution $V_m^{(s)}$, the perturbations exhibit damped oscillations with larger amplitudes than those plotted in Fig. [2]. This situation is similar to what happens for the constant $c_1^2$ model [39]. Since it is likely that such large oscillations can be severely constrained from CMB observations, we focus on the case in which $V_m$ is initially close to $V_m^{(s)}$ in the following discussion.

C. Observables

We study the evolution of observables associated with RSD, weak lensing, and CMB. In Fig. [3] we plot the gauge-invariant gravitational potentials $-\Psi$ and $\Phi$ as well as $\Phi$ versus $1 + z$ for the same model parameters and initial conditions as those used in Figs. [1] and [2]. Since $\Psi = \delta N + \chi' - (H'/H)\chi$, the oscillation of $\delta N$ seen in Fig. [2] leads to that of $-\Psi$. In Fig. [4] the oscillating amplitude of $-\Psi$ is not so large due to the choice of the initial condition $|V_m^{(h)}| \ll |V_m^{(s)}|$. With the decrease of $V_m^{(h)'}$, the oscillations of $-\Psi$ damp away. The gravitational potential $-\Psi$ exhibits an overall decrease from the matter...
Figure 3. Evolution of the gravitational potentials $\Phi$, $-\Psi$, $-\Phi_\Sigma$ for the same model parameters and initial conditions as those given in the captions of Figs.1 and 2.

era to today, so this case corresponds to weak gravity.

Figure 4 shows the evolution of $G_{\text{eff}}/G$ computed from the definition of $G_{\text{eff}}$ given in Eq. (4.12). From the scaling matter era to today we have $G_{\text{eff}}/G < 1$, while satisfying the no-ghost and stability conditions of tensor and scalar perturbations (see Fig. 1). The realization of weak gravity comes from the fact that the last term on the rhs of Eq. (5.26) is negative due to the decrease of $c_t^2$. For constant $c_t$ smaller than 1, $G_{\text{eff}}$ is larger than $G$ for subhorizon perturbations.

Since the second and third terms on the rhs of Eq. (5.26) are positive for $c_t^2 < 1$ and $\epsilon_{\text{eff}} < 0$, $\eta$ is larger than 1. In Fig. 4 we find that $\eta$ grows as the decrease of $c_t^2$ from the matter era to today and it starts to decrease in the future. This deviation of $\eta$ from 1 is one of the distinguishing features of our model. As we see in Fig. 4 the subhorizon approximation based on Eq. (4.17) with inclusion of $\epsilon_\zeta$ and $\epsilon_\chi$ is a trustable prescription for the evolution of $\eta$.

Since $\eta$ is larger than 1, the two gravitational potentials $\Phi$ and $-\Psi$ are different from each other. In Fig. 5 we find that $\Phi$ grows during the matter era, while $-\Psi$ decreases. We recall that the weak lensing gravitational potential $\Phi_\Sigma$ obeys Eq. (4.13) with $\Sigma$ given by Eq. (4.14). Since $G_{\text{eff}} < G$ and $\eta > 1$ during the matter era in the numerical simulation of Figs. 3 and 4, the small value of $G_{\text{eff}}$ appearing in $\Sigma$ is compensated by the large value of $\eta$. Hence $-\Phi_\Sigma$ does not vary much relative to $\Phi$ and $-\Psi$ (see Fig. 3).

The growth rate of matter density perturbations $\delta_m$ can be measured by peculiar velocities of galaxies in RSD surveys [76, 77]. Usually, this is quantified by the data of $f(z)\sigma_8(z)$ at redshift $z$, where $f = \delta_m/(H\delta_m)$. In Fig. 5 we plot the ten data points of $f(z)\sigma_8(z)$ with error bars derived from the measurements of 2dFGRS [1], 6dFGRS [2], WiggleZ [3], SDSSLRG [4], BOSSCMASS [5], and VIPERS [6]. The latest Planck measurement of the CMB power spectra provided the bound $\sigma_8(0) = 0.829 \pm 0.014$ at the 68% confidence level [21].

In Fig. 5 we also show the theoretical curves for $\beta = 0, 0.05, 0.1, 0.12, c_t^2 = 0.9$, and $\sigma_8(0) = 0.82$ with the initial conditions of perturbations satisfying $|V_m^{(h)}| \ll |V_m^{(s)}|$. When $\beta = 0$ we have $\epsilon_{\text{eff}} = 0$ and

Figure 4. Evolution of $G_{\text{eff}}/G$ and $\eta$ (black solid lines) for the same model parameters and initial conditions as those given in the captions of Figs. 1 and 2. The dashed red and blue lines correspond to the evolution of $G_{\text{eff}}/G$ and $\eta$, respectively, derived under the subhorizon approximation involving $\epsilon_\zeta$ and $\epsilon_\chi$, i.e., Eqs. (4.10) and (4.17).
the theoretical curves for is close to models with $\beta > RSD$ data compared to the constant $c$. For larger $\beta$ values of $f_\sigma$ show a better agreement with the recent observations of $f_\sigma$. In Sec. VI we have derived the general expressions of the effective gravitational coupling $G_{eff}$, gravitational slip parameter $\eta$, and weak lensing parameter $\Sigma$ using the quantities $q_8$, $c_2$, $q_8$, and $c_2^2$. In GLPV theories the time derivatives $\xi$ and $\chi$ do not vanish even after employing the subhorizon approximation, so we need to know the numerical values of $\xi$ and $\chi$ in Eqs. (1.19-41.21) for the computations of $G_{eff}$, $\eta$, and $\Sigma$ (unless $|\xi|$ and $|\chi|$ are much smaller than unity).

In Horndeski theories the analytic expressions of $G_{eff}$, $\eta$, and $\Sigma$ are of the simple forms (3.3)-(3.4). In this case the necessary condition for the realization of weak gravity is given by Eq. (5.7), but this is not a sufficient condition due to the presence of additional scalar-matter interactions. The scalar-matter coupling, which always enhances $G_{eff}$, should not be so large as to give rise to values of $G_{eff}$ larger than $G$.

In GLPV theories the two conditions (2.3) are absent, so $c_2^2$ can deviate from 1 even in the deep matter era. We have presented a simple time-varying $c_2^2$ model described by the Lagrangian (5.17) with the function (5.27). In this model the scalar propagation speed squared $c_2^2$, which is given by Eq. (5.19), increases to be much larger than 1 for dark energy models with decreasing $\Omega_X$ toward the past. For scaling models described by the field potential (1.28) the ratio $\Omega_m/\Omega_X$ stays constant during the matter era, so the unbound growth of $c_2^2$ in the past can be avoided.

The analytic expression of $G_{eff}/G$ given in Eq. (5.29), which is valid in the deep matter era with $c_2^2 \gg 1$ and $|\xi| < 1$, $|\chi| < 1$, shows that it is possible to realize weak gravity ($G_{eff}/G < 1$) for the decreasing $c_2^2$ model ($\kappa_m \eta < 0$) in the regime $0 < c_2^2 < 1$. In this case, the parameter $\eta$ is larger than 1 from Eq. (5.20).

In Sec. VI we have numerically solved the full perturbation equations of motion for the model (5.17) with the functions (5.27) and (5.28). The solution to the normalized velocity potential $V_m = Hc$ can be written by the sum of the homogenous solution $V_m^{(h)}$ and the special solution $V_m^{(s)}$. Provided that $V_m^{(h)}$ is initially much smaller than $V_m^{(s)}$, the amplitudes of oscillating modes of perturbations are suppressed apart from $\delta N$ and $\Psi$ that are related to the time derivative of $V_m^{(h)}$. The numerical simulations of Figs. 3 and 4 show the realization of weak gravity from the matter era to today with $\eta$ larger than 1.

In Fig. 5 we have also computed the evolution of $f_\sigma$ in the redshift range $0 \leq z \leq 1$ for several different values of $\beta$. For larger $\beta$ the theoretical values of $f_\sigma$ get.

In this paper we have studied the possibility of realizing gravitational interaction weaker than that in GR on scales relevant to large-scale structures and weak lensing. We have employed the approach of the effective field theory of modified gravity encompassing both Horndeski and GLPV theories as specific cases. The important quantities associated with the no-ghost and stability conditions of tensor and scalar perturbations are given by Eqs. (3.8), (3.21), (3.22). All of the quantities $q_8$, $c_2^2$, $q_8$, and $c_2^2$ are required to be positive for theoretical consistency.

Since our interest is the evolution of perturbations for modes deep inside the subhorizon radius ($k/a \gg H$), we have exploited the subhorizon approximation under which the dominant contributions to the perturbation equations are those involving $k^2/a^2$ and the matter density perturbation $\delta \rho$. In Sec. VI we have derived the general expressions of the effective gravitational coupling $G_{eff}$, gravitational slip parameter $\eta$, and weak lensing parameter $\Sigma$ by using the quantities $q_8$, $c_2^2$, $q_8$, and $c_2^2$. In GLPV theories the time derivatives $\xi$ and $\chi$ do not vanish even after employing the subhorizon approximation, so we need to know the numerical values of $\xi$ and $\chi$ in Eqs. (1.19-41.21) for the computations of $G_{eff}$, $\eta$, and $\Sigma$ (unless $|\xi|$ and $|\chi|$ are much smaller than unity).

In Horndeski theories the analytic expressions of $G_{eff}$, $\eta$, and $\Sigma$ are of the simple forms (3.3)-(3.4). In this case the necessary condition for the realization of weak gravity is given by Eq. (5.7), but this is not a sufficient condition due to the presence of additional scalar-matter interactions. The scalar-matter coupling, which always enhances $G_{eff}$, should not be so large as to give rise to values of $G_{eff}$ larger than $G$.

In GLPV theories the two conditions (2.3) are absent, so $c_2^2$ can deviate from 1 even in the deep matter era. We have presented a simple time-varying $c_2^2$ model described by the Lagrangian (5.17) with the function (5.27). In this model the scalar propagation speed squared $c_2^2$, which is given by Eq. (5.19), increases to be much larger than 1 for dark energy models with decreasing $\Omega_X$ toward the past. For scaling models described by the field potential (1.28) the ratio $\Omega_m/\Omega_X$ stays constant during the matter era, so the unbound growth of $c_2^2$ in the past can be avoided.

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In Sec. VI we have numerically solved the full perturbation equations of motion for the model (5.17) with the functions (5.27) and (5.28). The solution to the normalized velocity potential $V_m = Hc$ can be written by the sum of the homogenous solution $V_m^{(h)}$ and the special solution $V_m^{(s)}$. Provided that $V_m^{(h)}$ is initially much smaller than $V_m^{(s)}$, the amplitudes of oscillating modes of perturbations are suppressed apart from $\delta N$ and $\Psi$ that are related to the time derivative of $V_m^{(h)}$. The numerical simulations of Figs. 3 and 4 show the realization of weak gravity from the matter era to today with $\eta$ larger than 1.

In Fig. 5 we have also computed the evolution of $f_\sigma$ in the redshift range $0 \leq z \leq 1$ for several different values of $\beta$. For larger $\beta$ the theoretical values of $f_\sigma$ get.
smaller, so that these cases exhibit better compatibility with the recent RSD data relative to the case $\beta = 0$. It remains to be seen whether or not the future RSD data combined with other observational probes favor the lower growth rate of matter perturbations than that in the $\Lambda$CDM model.

There are several issues we have not addressed in this paper. First, under the so-called disformal transformation [78, 80], the Lagrangian (5.17) can be transformed to the one in the Einstein frame in which the tensor propagation speed is $1 \pm \frac{\alpha}{3}$. Since a nontrivial kinetic coupling with the scalar field and matter arises in the Einstein frame [36, 83, 84], the role of such an interaction should be understood in view of a coupled dark energy and dark matter scenario [85]. Moreover, it will be of interest to study the screening mechanism of the fifth force [85] in local regions of the Universe for the model [5.17]. In some modified gravity models like the quartic Galileon, it was shown that the screening mechanism can give rise to a $G_{\text{eff}}$ smaller than $G$ in the nonlinear regime of matter perturbations [57]. It will be of interest to see whether such properties persist in more general modified gravity models in the framework of GLPV theories.

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