Continuity, Topos and Infinitesimals

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Abstract

In this essay we discuss the concepts of topos (place), sunekhes (continuity) and infinitesimal as found in classical philosophy (primarily in Aristotle’s Physics) and their relationship to modern topology. The connection between the ancient and the modern was important in the later thought of René Thom, in particular in his Esquisse d’une Sémiophysique. We show how the modern (Leibnizian and Newtonian) concept of infinitesimal has its roots in classical thought and that the concepts of topos, continuity and infinitesimal are related. We show furthermore how Aristotle’s notions of topos and continuity can be given precise definitions in terms of modern topology and category theory and how they are relevant to modern problems in the foundations of mathematics and philosophy of science. (Accepted peer-reviewed article).

1 Aristotle’s Theory of the Continuum

A major preoccupation in the later thought of René Thom was bringing to light the ”primordial topological intuitions” present in Aristotle (see for instance [6, 7, 18, 19, 20, 21, 22, 23, 8]) and showing how the Peripatetic system hinges on certain fundamental intuitions and concepts which only with advent of modern topology have found a full formal development. According to Thom the key concept is that of sunekhes, a noun which is usually translated as ”continuum” or ”continuity”. The Physics can be considered as being concerned with the fundamental conditions of phusis, or nature, which include change, extension or magnitude, (megethos), place and time as well as form (morphe, eidos) and matter hule. Aristotle lays great emphasis on proving that these are all continuous quantities. Thom proposed furthermore that for Aristotle continuity plays a central role in our cognitive organisation, namely via the ”semantic spaces” of genera and species [Ch.8]: Aristotelic physics is interpreted as a semio physics, a semiotics not unlike that of Greimas or Brandt. Aristotle layed the foundation for much of early modern Western scientific and philosophical thought: continuity plays a major role in Newton, Leibniz, Kant, Hegel, Peirce and Husserl, albeit with unique developments proper to each philosopher.

What is peculiar to Aristotle’s concept of continuity which is of contemporary philosophical relevance is its affinity to recent developments in mathematics, mathematical logic and theoretical physics [33]. We will show how this connection (specially with regards to Aristotle’s concept of topos) sheds light on the present and historical role of infinitesimals in mathematics and physics even though the modern concept of infinitesimal is not in fact explicitly present in Aristotle.

To investigate the continuum in Aristotle we must consider the concepts of point (monas), infinity (apeiron), limit (peras) ( and the allied concepts of envelop, periekhomenon), divi-

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1 For convenience we do not mark vowel-length in our romanisation of Greek terms.
2 At least for those who hold that there should always be an interplay between philosophy, mathematics and science.
...ision (*diareisis*) and contact (*haphe*) and explore how we should interpret the basic distinction between the substrate (*hupokeimenon*) and its accidents/attributes (*sumbebekhos*).

The most striking characteristic of the Aristotelic *sunekhes* which sets it apart from the modern mathematical notion is that it is not conceived as an aggregate of points, as formed by the joining together of points (231a21-b10)[3]. Aristotle takes the later Euclidean definition of a point as that which has no parts. The definition given of a *sunekhes* involves contiguity: the extremities of its parts are united (227a6-16). The term "connectedness" is perhaps a more faithful translation. Since we cannot distinguish between the extremities of a point and the point itself, the coming together of two points will result in complete coalescence and thus a continuum could never be constituted from points. The extremity or limit (*peras*) of something is a kind of interface, that which allows the entity to relate to other entities or to the surrounding environment or be capable of receiving diverse predicates. These considerations have roots in Plato’s *Parmenides*. In the first hypothesis of the final "dialectical" section, the One is considered in its absolute simplicity (exemplifying an idea). It could not manifest (or exist) for then it would have to be in a certain place. But being in a place it would be *enveloped* by that place, that is, its extremities would touch, be in contact, with the surrounding environment. Plato then continues to argue (in *Parm.* 138A-B) that since the extremities are a part of the entity and the One is by definition an entity without parts, the One could not be in a place and hence could not "exist". According to Scolnikov [17][p95-100] this problem is resolved in the second hypothesis wherein the Existing One is considered as a complex unity, a composite (*sunolo*) which allows participation (or predication) whilst preserving its underlying unity. As we shall see, *sheaf semantics* offers a striking illustration of such a relativized principle of non-contradiction. Scolnikov [17][p100] makes the interesting observation that this complex is in concrete instances always finitely determined or divided (*diareton*), although it can potentially be divided infinitely - *diareisis* presupposes the possibility of making distinctions within a form— any form —and requires, in principle, that division can go on indefinitely. But potentially endless divisibility is a chief property of the Aristotelic continuum: In (200b19-20) we read: *to eis apeiron diaireton sunekhes on* - the continuum is what is infinitely divisible.

We are not proposing that in Aristotle substance *upokeimenon* is to be conceived as a continuum: in stricter Aristotelism the continuum belongs to the category of quantity. This a main point of disagreement between Aristotle and Thom. Thom recognised his interpretation of the substrate as a continuum as being "fundamentally unfaithful" [6][p.245].

However Thom is not postulating the Aristotelic continuum to be substance but rather modern Cartesian space which is not to be confused with place (*topos*). Aristotle in (208b8-22) seems to be perfectly aware of the modern concept of orientation (cartesian coordinates) and their *relativity* though he also tended to hold the view of the existence of an absolute cosmic orientation or frame of reference. In (209b11-17) he ascribes a similar view to Plato: the identification of matter and space, space being what is capable of receiving forms.

In conclusion we have seen that Aristotle gives us two fundamental characterisations of *sunekhes*: that whose parts have united extremities (and hence cannot consists of points) and that which is infinitely divisible. The modern topological definition of a continuum involves the assumption that the continuum consists of points. Yet in this framework the concepts of connectivity and local connectivity capture the first property and, for the real line, the second property is captured by that of dense linear order. But note that there are connected topological spaces which do not have anything that could correspond to the divisibility property (consider the coarse topology).

The infinite (*apeiron*) is very different from our modern conception of an infinite cardinality. Aristotle distinguishes between an infinity which "goes beyond any finite magnitude" (206b20) and a bounded infinity exemplified by the set obtained by successively adding to each other

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3 All references are to the *Physics* and we follow the text of [12]
segments of half the previous length (expressing the convergence of the geometric series). In illuminating passages (206b3-12, 207a7-25, 209b8) we see that it is that which has a limit, not in itself but in something external to itself. It is something incomplete or incompletely, aoristos(209b9), imperfect or not yet perfect or whole (207a7-15). This is also reflected in the difference between the aorist and perfect tenses in classical Greek. It can be an expression both of a completable or incompletable process (involving division or addition). It can express potentiality (dunamis) and generated being (oude menei...alla gignetai, it does not rest...but is in the process of becoming) and is likened to time (207b15).

The above passages show that Thom was quite justified in his insight that apeiron corresponds to the modern concept of open set in topology as well as in his insight that there is an intimate relationship between matter hule and the infinite and between form and limit (peras) or the envelop (periekDOMenos). It is interesting to note that distinction in (209b8) between diasteme (extension) and megethos (magnitude). In modern terms the diasteme would be the interior of the megethos considered as a closed interval (cf. [4][p.207 Note 5]). Another notable example of an open set in Aristotle is found in the discussion related to the inexistence of a first time wherein a motion started(236b 32 - 237b22). The concept of peras can correspond to the modern topological concepts boundary or limit of a sequence. A point functions as a limit and as a medium either for division or junction (Aristotle distinguishes in (226b18-24) between mere contact and the total fusion of extremities in continua). In modern topology open sets function like local homogeneous patches that generate the whole space (cf. the concept of basis of a topological space). They express the concept of the local spatial quality around a given point in which the particular size of this neighbourhood is irrelevant (cf. Hegel’s concept of indifferent quantity. See the section of [11] on the category of quantity. Hegel’s discussion of the relationship between quantity and quality in fact anticipates Thom in certain aspects as well as harkening back to Aristotle.).

In one of Thom’s topological analogies in [6] an open set correponds to matter (hule) and potentiality (dunamis) whilst the boundary (which is a closed set) corresponds to form and to actuality (energeia). The genesis of an entitiy can be seen as a separation from a matter by means of a cut along its boundary defining and giving form to the entity. But on the other hand the open set defines its own limit or boundary and the energeia of change takes place during an open interval of time.

It appears furthermore that Aristotle’s concept of the finite/whole/complete corresponds to the modern concept of compactness (which in the topology of ordinary space is equivalent to being closed and bounded).

As we mentioned, sunekhes in Aristotle is close to the modern concept of the standard (connected) topology over the line (or general euclidean space R^n) but with the difference that such a topology is always defined in terms of a sets of points. We propose that apeiron/hule should be taken as the fundamental generating element of the Aristotelic continuum (together with the endless potentiality of division or joining together) and that points and closed sets be seen as the result of taking limits of indefinite processes of division or joining. Aristotle’s sunekhes is thus characterised in terms of being built from open sets, not points, by being indefinitely divisible (which has in fact a certain relationship with standard topological ”separation” axioms) and by being connected.

We will relate this to recent advances in modern geometry and topology. In this framework it is possible to interpret rigorously not only the Aristotelic concept of topos (place) but also that of infinitesimal. In Thom’s interpretation of hupokeimenon as a continuum or closed bounded subset of R^3 (as in [6][p.153]) this continuum though not an absolute materia prima is an individuated matter which shares with it the characteristic of not yet being able to exist.

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4For a good introduction to modern general topology see for instance [10]. Some basic concepts are also defined in the appendix of this essay.
by itself. Natural beings always consist of both matter and form, and for Thom form is a continuum (or space) stratified according to different locally homogenous qualities separated by ”catastrophe sets” determined by a fiber bundle (or sheaf of local qualities). This is at the heart of Aristotelic biological division between homeomeres and anomeomeres. More generally, the passage (228a26-30) shows how Aristotle clearly recognizes that globally change is not continuous only locally continuous, a concatenation of contiguous continuous changes and is thus in this way stratified.

In the *Physics* we have also the germ of the idea of a continuous function, a quality of how a certain quantity varies in function of another quantity, such as place or some other quality, in terms of time. One property of continuous functions is that the image of a compact set is a compact set. This property relative to compact sets seems to be the intuition present in Aristotle’s proof that there can be no infinite motion in a finite time (237b23-238a11-20). Another property is that the image of a connected set is connected. A particular case of this property is Bolzano’s intermediate value theorem (taught in elementary calculus classes) which expresses *natura non datur saltus*, the fact that a continuous function has no jumps or holes, cannot skip intermediate values (more intuitively we can see that the graph is a connected curve in space). It is interesting to note that the continuity of a function can in fact be defined more intuitively in terms of the arcwise connectedness of its graph.

The basis of Thom’s Catastrophe Theory is that qualities should always be locally continuous with respect to space and time. It is quite natural to replace Thom’s use of fiber bundles with that of sheaves or their equivalent étale-bundle presentation[28][p.88] which capture exactly this property.

The sheaf-theoretic formulation lends itself more naturally to a formulation based only on the category of open sets, without any direct appeal to points, or appeal to the idea of space being made up of points. Thom in facts mentions sheaves in [6][p.209] to argue for the topological primacy of the substrate space over its fiber space. We will now show how the above considerations cast light on Aristotle’s definition of topos and on the concept of infinitesimal.

Consider a space $E$ representing the environment on which is defined a certain sheaf of qualities. Consider an individual entity given by a closed bounded subset $B$ of $E$. What is the topos of $B$ in $E$ in the Aristotelic sense? At first sight we are faced with a problem. We wish to find ”the limit of what is enveloping” (212a5). If we consider the complement of $B$, $c(B)$, then we get an open set. This open set will have its own boundary which will (for sufficiently tame kinds of set) coincide with the boundary of $B$ so that this boundary cannot be seen as the Aristotelic topos which is essentially ”separable”, being like the limit of the container (the inner surface of a bottle containing water). As Leon Robin wrote[13][p.142] the topos is ”always qualified”. If we consider a sheaf of qualities then the topos will certainly not be determined merely by the behaviour of the sheaf on the induced topology of $B$, for this ignores completely the situation of $B$ relative to $E$ (to use an analogy from elementary calculus, it is as if we were to try to calculate the lateral limit of a function by only considering the value of the function at the given point). The Aristotelic concept of topos points directly to the modern concept of infinitesimal. The topos of $B$ in $E$ is captured by a limit (in the modern category-theoretic sense) of a sequence of nested open neighbourhoods of $B$ in $E$, or rather, by the behaviour of the sections of a sheaf of qualities on such a sequence. The topos of $B$ in $E$ is determined by its infinitesimal neighbourhood. If $B$ is taken as a point $p$ then its topos is the stalk at $p$. The exact concept of topos if found remarkably in modern algebraic geometry in the concept of a formal completion of a subvariety $Y$ in a variety $X$. In [2][p. 190] Hartshorne writes:

*The formal completion of $Y$ in $X$ (...) is an object which carries information about all the infinitesimal neighbourhoods $Y_n$ of $Y$ at once. Thus it is thicker than any $Y_n$, but is contained*
inside any actual open neighborhood of $Y$ in $X$.

Aristotle was at least not always aware of this construction as in (209a7-13) where the point is stated to be identical to its topos. Waterfield [5] [p.253] also comments on the connection to the first hypothesis of the Parmenides.

An infinitesimal is something that is not a point yet is smaller than any open neighbourhood of this point. A sequence of nested neighbourhoods can be conceived as ”flowing” (cf. Newton’s *fluxiones*) and any finite number of elements of this sequence can be removed without affecting the defined infinitesimal. This can be also interpreted as an essential local homogeneity or self-similarity of the standard topology of ordinary euclidean space. All closed intervals are qualitatively the same. As we shall see in the next section, in a category-theoretic, point-free, presentation of topology, closed intervals or infinitesimals should be seen a central objects, ”generators” in some sense, of all major topological properties. In the following sections we will try to give more detailed technical illustrations of the ideas discussed in this section in what pertains to recent advances in geometry and topology.

2 Topos, Infinitesimals and Sheaves

Leaving aside for a moment the problem of a ”point-free” definition of the continuum we present some considerations on how both infinitesimals and the concept of topos can be captured using modern Sheaf Theory. An excellent introduction to Sheaf Theory can be found in [28, 15].

How can we define an infinitesimal on the line at point $a$? One possible answer is saying that it is simply the set of all open neighborhoods of $a$ - for containing the infinitesimal neighbourhood of $a$ is what they all have in common - and not simply the point $a$.

We can also consider the nested sequence $U_1 \supset U_2 \supset \ldots \supset U_i \supset \ldots$ of open subsets containing $a$ as a definition of the infinitesimal at $a$, provided that they form a countable basis of the topology. It is interesting that the infinitesimal is not any of the $U_i$’s but the very sequence of them, that is, the flow of such $U_i$’s tending to but never identifying with the point $a$. If we remove finitely many $U_i$’s we still are defining the same infinitesimal. It is all of them and none of them in particular. The classical paradox of infinite cardinalities (illustrated by the paradox of Hilbert’s hotel expressing the invariance of infinite cardinalities under finite subtraction) creates this paradox of the definition of the infinitesimal. The infinitesimal can only be defined using an infinite number of terms, any finite number of which are dispensable. The infinitesimal is simply a local (sub)basis for a topology. Aristotle’s sequence of halved lines in the *Physics* furnishes an example of such a basis for the real line. The homogeneous nature (any local open part has the same quality as the whole) of the continuum is related to the fact that it is generated by the same infinitesimal. The infinitesimal is an ideal element which can only enter into relationship to the continuum and the point $a$ through reference to an infinite sequence of subsets containing $a$, but being at the same time neither the point $a$ itself nor any infinite sequence of open subsets or indeed any single one. No finite collection of open sets or points can define an infinitesimal, nor can the point itself, for the same ”point” can be ”in” a space of any dimension.

As we have seen, the nested sequence of open sets $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$ in $\mathbb{R}^n$ all containing a certain point $x$ defines (let us just assume we have uniqueness) the infinitesimal $dx$ which we take to be the ”limit”, the infinitesimal ”ball” with center $x$ or the flowing vanishing neighbourhood of $x$, not equal to $x$ yet contained in all open sets containing $x$. Now $x$ can be replaced with any closed compact subset $K$ of $\mathbb{R}^n$ and we get the ”infinitesimal” $dK$ defined by a sequence of nested open sets all containing $K$. If we view $K$ as a ”body” within the ”body” $\mathbb{R}^n$ then $dK$ is precisely what Aristotle means by place topos: *to peras tou periekhontos somatos kath'ho sunaptei to periekhomeno* (212a5-6), the limit of the enveloping body according to where
it touches what it envelopes. Indeed $dK$ is something outside the body $K$ which is yet a limit of what is outside and that touches _sunaptei_ (without identifiyng with) $K$. In (211b10) we have _en tauto gar ta eskhata tou periekhontos kai tou periekomenon, the limits of the container and contained coincide_ (211b12-14), but without fusing and being identical.

This concept of topos make sense whenever we have an embedding of a manifold, analytic space or variety (the formal neighbourhood discussed in the previous section) within another one. Take for instance instance the "tubular neighbourhood" or normal bundle used in differential geometry and topology. But the above concept of topos has not yet any physical sense for we must consider space as a substratum of qualities, that is, we need to consider a sheaf defined over it. As we mentioned, a fundamental feature of the thought of Thom (which is later expanded by Jean Petitot in for instance [14] is that Aristotelic categories are reinterpreted in a Cartesian form, in a way characteristic of modern mathematical physics, wherein modern "space" corresponds to the Aristotelic "substrate" whilst qualities (in modern terms, fields) are captured by functions over that space, or more precisely by sections of certain bundles or, equivalently, sheaves. Sheaves allow us to glue together local information and define in a rigorous way the quality of a certain substrate at a given point. This is called taking the germ of a section at a point and is a generalisation of the original concept of derivative of a function at a point developed by Leibniz and Newton.

The concept of a _constructible sheaf_ [16][p.320] involves a stratification of the underlying space into locally homogenous qualities in a way the captures admirably the Aristotelic concept of _anomeomere_ and general scheme of Catastrophe Theory.

Consider the sheaf $C$ of continuous functions over $R^n$, which means we consider all the continuous functions defined on open sets $U$, which we denote by $C(U)$. Then if we consider a nested sequence defining the infinitesimal $dx$ there is a corresponding sequence

$$C(U_1) \to C(U_2) \to \ldots \to C(U_n) \to \ldots$$

which defines the "stalk" $C_x$ which we denote by $\frac{dC}{dx}$. What about for the _situation_ of a closed compact $K$? If we denote by $j : K \to R^n$ the inclusion, then there is a well-defined sheaf (the _inverse image sheaf_) that satisfies precisely the Aristotelic property:

$$j^{-1}C(K) = \lim_{U \supset K} C(U) = \frac{dC}{dK}$$

The meaning of this expression is that the qualities expressed by the sheaf $j^{-1}C$ all depend on the qualities of $K$ considered as enveloped by some neighbourhood $U$, in fact they depend of $K$ and its infinitesimal neighbourhood in $R^n$, its _situation_ and immersion in the larger space. It is thus a limit of the qualities on enveloping neighbourhoods of $K$. In conlusion, the concept of topos corresponds to a rigorously defined mathematical concept which is used in a wide range of areas of applied mathematics.

Aristotle considers also the situation in which both the contained and the container can change (as a boat in a river). The final definition of topos is _to tou periekhontos peras akinèton proton, tout’estin ho topos_, the first immobile limit of the enveloping entity, _that is the topos_ (212a20). This suggests that alternatively we consider the diffeomorphic action $\phi_t : T \times R^n \to R^n$ of a dynamical system and define the topos to be the intersection of all invariant open sets ($\phi_t(U) = U$) such that $K \subset U$ and hence $\phi_t(K) \subset U$.

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5Elementary algebraic topology offers us many striking embodiments of the Aristotelic topos, situations in which we are interested in the behaviour in the neighbourhood of a boundary. This is the main intuition behind the concept of relative homology of a pair $(X,A)$ and the excision theorem. According to [34][p.44]: "In particular a chain in $X$ is a cycle modulo $A$ if its boundary is contained in $A$. This reflects the structure of $X - A$ and the way that it is attached to $A$. In a sense, changes in the interior of $A$, away from its boundary with $X - A$, should not alter these homology groups". A very useful property of CW-complexes is that they are locally strong deformation retractions around the "place" in which the cells are attached.
In continuum mechanics *Cauchy's stress principle* is related to the "act" of infinitesimal homeomeric volumes of the substance on its immediate environment - its topos - through the defining region of the act, the surface. This is the concept of *surface force*. The infinitesimal volume will always be dynamic and have a surface; Cauchy's principle states that we can take the limit of the force/area ratio of the force acting on a given small area of the surface determined by a normal direction because the moments will vanish at the limit. Considering each direction we thus can define a (symmetric) tensor, the stress tensor, at each infinitesimal point.

Cauchy's principle is an example of the dynamical physical nature of the infinitesimal. Our direct physical experience of the world around us is very much like the topos of a small volume in continuum mechanics. We feel our body directly immersed and affected by the qualities of the surrounding environment (or, in J. v. Uexküll's terms, the *Umwelt*) through the medium of mechanoreceptors and sense organs.

We mentioned above the concept of stratification associated to a constructible sheaf. One of Thom's most important papers in topology\(^9\) concerns stratified sets and morphisms. The theory involved - which deals with a generalisation of smooth manifolds and algebraic and semi-algebraic spaces - dates back to the work of Hassler Whitney (1907-1989) on singularity theory\(^7\). It is perhaps the most elaborate development of the original topological intuition present in the Aristotelic concept of topos. The topos is found in the way in which in a stratification a given stratum \(X\) is related to the adjacent strata \(Y\) which are contained in its closure \(\overline{X}\). This induces a decomposition of the boundary of \(X\) into such adjacent strata which can be seen as forming the *topos* of \(X\). These strata \(Y\) in general have a very complex topological relationship to \(X\). Whitney's conditions (A) and (B) are imposed in order to maintain a tame, intuitive, non-pathological situation, ruling out what can happen with spirals or in the crossings of the stratification of the Whitney umbrella. These Whitney stratifications came closer to the intuitive notions of topos in the smooth way in which the adjacent strata envelop and contain \(X\). Many more mathematical examples that could be adduced regarding the importance of the behaviour on the "boundary" of object such as for example the Dirichlet problem in partial differential equations in which a function on a given region is completely determined by its values on the boundary of this region.

In all the previous discussion we placed ourselves in the framework of continuous, differentiable or smooth (infinitely differentiable) functions or maps. The smooth category is clearly at the heart of Thom's mathematical work and indeed of modern differential topology and differentiable dynamical systems. In a future essay we will argue for the fundamental importance of the analytic category (and some extensions thereof) both for mathematics and natural science and explain how it is distinct yet closely related to the smooth category. The analytic category and the algebraic category (of which we have already mentioned the formal completion of a subvariety) have the interesting feature that the infinitesimal and topos concepts discussed above can be given purely algebraic definitions. There is no need to appeal to intuitions of "flowing quantities" or "infinitesimally small quantities". It was O. Zariski and other who gave rigorous foundations to the geometric intuitions of the early Italian school of Algebraic Geometry. Thom recognised that his concept of "transversality" stems from their notion of "general position".

In algebraic and analytic geometry we can define many infinitesimal concepts such as infinitesimal neighbourhood (formal completion of a subvariety or formal scheme), tangent space, smooth (non-singular) variety and smooth map in purely algebraic terms involving quotients, localisations and completions of rings and modules and algebras of formal differential operators. A central role is played by *nilpotent elements* of a ring, that is, elements \(a\) for which some power \(a^n\) is zero. This algebraic approach is also at the heart of modern singularity theory and the
proofs of Thom’s theory of Elementary Catastrophes. We shall say something more about this in a further section were we discuss ”space as a quotient”.

Nilpotents and zero-divisors can be seen as giving an interpretation to the procedure of ”ignoring” infinitesimals of a higher order to obtain equalities, often excused as ”approximations”. To see how this relates to niloptency, consider Leibniz’s rule for differentiation \( (f(x)g(x))' = f(x)g'(x) + f'(x)g(x) \). But for differentials this is expressed as \( d(yz) = zdy + ydz \).

By definition \( d(yz) = (y+dy)(z+dz) - yz \). But this is \( yz + ydz + zdy + dydz - yz = ydz + zdy \) because \( dydz \) is an infinitesimal of higher order than the other terms and so can be ”neglected” or equated to zero \( dydz = 0 \). Geometrically, the rectangle \((y + dy)(z + dz)\) is divided into a constant rectangle \(yz\), two rectangles with one infinitesimal and one constant side, \(ydz\) and \(zdy\), and a rectangle with both sides infinitesimal \(dydz\), which is thus ”infinitely smaller” than the other components and may be ”neglected”.

3 Topology without Points

We have seen how the Aristotelic \textit{sunekhes} corresponds roughly to a characterisation of the standard topology on the real line (or euclidean space) in terms of (local) connectivity and divisibility. In standard terms, this \textit{divisibility property} could be described (for the line \( R \)) as follows: given any point \( p \) there are two disjoint open sets whose union is dense in \( R \) and for which \( p \) belongs to both their closures. If we consider the standard algebraic structure on the real line then we have a linear order and divisibility can be expressed by the standard \textit{density property}. For higher-dimensional spaces instead of a point we could consider hyperplanes for instance.

Following Thom we wish to capture the primordial intuition of space before it receives any further order-theoretic or algebraic determinations. We wish to find a formulation of these basic properties which does not involve an appeal to points (as in the standard definition of topological space).

In modern mathematics there are two (closely interrelated) ways in which this might be achieved. The order-theoretic one (in terms of locales) and the category-theoretic one, in terms of Topos Theory (in particular Grothendieck Toposes and Synthetic Differential Geometry). There is furthermore also an algebraic approach in which points correspond to prime (or in particular maximal ideals), used in algebraic geometry and functional analysis.

The first two approaches are essentially relational. We start from open sets or objects in a category and consider the relationships between them. In Category Theory itself we find in a sophisticated way some of the primordial Aristotelic concepts: that of limit (and colimit) and of adjoint functors where the left adjoint can be seen as expressing a vast generalisation of the idea of an open interior whilst the right adjoint is seen as a kind of closure as seen in the case of adjoint functors between posets. All the constructions we discussed to capture the notion of topos and infinitesimal in the last section involve limits.

Inspired by the characterisation above we propose the following definition of an \textit{Aristotelic Locale} \( L \). A locale is a complete distributive lattice (which hence has a top element 1 and bottom element 0) and is way of defining topology without points (see chapter II of [3] for an introduction). Since there are no points there is no obvious way to define density or closure. We might attempt to capture a point-free description of the Aristotelic properties of the standard topology of Euclidean space by the following axioms (the quantifiers are to be understood as having scope \( L \)):

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\begin{align*}
\text{(Global Connectivity)} & \quad \sim \exists u, v. 1 = u \lor v \& u \land v = 0 \\
\text{(Local Connectivity)} & \quad 1 = \bigvee \{ z : \sim \exists u, v. z = u \lor v \& u \land v = 0 \}
\end{align*}
\]
Local connectivity means that we can cover any open set with connected open sets.

Sheaves are defined in terms of the (opposite) category of open sets of a given topology. The infinitesimals we considered in the last section do not naturally belong to this category although the associated stalks belong to the target category.

Grothendieck’s philosophy\cite{26} is that the category of sheaves on a space gives a more fundamental description of a space than its topology. A Grothendieck topology is precisely a way of defining a ”topology” on an arbitrary category without any need of an underlying set of points. The underlying concept is that of ”cover”, another intuition related to continuum which can be traced to many places of Aristotle’s *Physics*, as the idea of measuring a given interval by a smaller one clearly corresponds to a cover. We can define sheaves on categories with a Grothendieck topology (called *sites*) and we obtain again infinitesimal concepts which allow us to define the germ of a sheaf.

The category of sheaves over a site is called a *Grothendieck Topos* and satisfies a remarkably small and elegant set of axioms. The more general concept of Topos was discovered independently by Lawvere where it is seen as defining an intuitionistic set theory without points. This also allows us to have temporal or spatially localised concept of truth and to give meaning to such sentences as ”The cat is black and white” and furnish a rigorous basis for the relativized form of the principle of non-contradiction. The subobject classifier object of a topos has the structure of a Heyting algebra which can be interpreted as giving the possible logical values of propositions. In fact, the poset of subobjects $Sub(A)$ of an object $A$ in a category can have diverse properties depending on the type of category. In the cases of a topos these posets are Heyting algebras. We can thus think of $Sub(A)$ as a kind of generalised space.

In order to obtain a complete formulation of the Aristotelic continuum it would be interesting to extend likewise to topoi the Divisibility Axiom above.

We noted how infinitesimals do not in general belong to the base category on which sheaves are defined. But in a topos theoretic framework we may wish for infinitesimals to actually belong to the topos. This is precisely the road taken by Lawvere’s Synthetic Differential Geometry\cite{32}. Lawvere, in his quest for the adequate categorical foundations for continuum mechanics, was influenced by Grothendieck’s project of a ”tame topology” which seeks to avoid the unintuitive paradoxes of standard topology (such as space-filling curves and the Banach-Tarski paradox\cite{6}).

In \cite{35} Lawvere gives an extraordinarily interesting category-theoretic philosophical analysis of space and allied concepts employing an interpretation of the concepts of Hegel’s Logic.

There is an interesting and promising development of category theory in relationship to philosophy and logic which has the most wide-ranging implications for understanding the Aristotelic concept of topos (and to theoretical physics in general) : *homotopy type theory* and *higher category theory*, in particular *higher topos theory*. Homotopy type theory, an extension of the constructive type theory of Martin-Löf, was first proposed by Voevodsky as an alternative foundations for mathematics suitable for computer-assisted formalisation and verification. For each type there is a whole hierarchy different levels of ”equality” analogue to the series

\[ \forall u \exists w, v : w \land v = 0 \& \forall z. w \lor v \leq z \rightarrow z = u \]
of homotopy groups $\pi_i(X)$ of a topological space. The adequate model for homotopy type theory involves higher category theory in which we have not only morphisms, but morphisms between morphisms and then morphisms between these and so forth. For equality types all such morphisms are invertible and we have what is called an $\infty$-groupoid. Models of homotopy type theory are a higher-order analogue of toposes called homotopy toposes or $(\infty, 1)$-toposes. In [30] Urs Schreiber uses higher toposes to formalise quantum field theories. It would be very interesting to analyse the crucial property of being "cohesive" in terms of the Aristotelic continuity concepts analysed in this paper.

4 Topos as a Quotient

In Aristotle’s Categories a division is made between discrete and continuous quantity. If we visualize numbers - discrete quantity - in a spatial way, then spatiality is related to divisibility, to the formation of quotients involving equivalence relations. If we take 0 and 1 then there is a space between them and we would like to name the ”points” in this space and are thus forced to ”invent” $1/2$ with the additional caution that $1/2 \cong 2/4 \cong 4/8$. ... This abstract construction is often a stumbling block for children learning mathematics. Modern algebraic geometry springs from Descartes’ theory of studying geometrical figures by means of algebraic equations defining them. These form an ideal in a commutative ring and we can consider the behavior of such functions in a small vicinity of a point - the topos of the point - and this is done by entirely analogous quotient constructions such as localisation and completion. There is also a similar concept of localization of a category. A full subcategory $i : D \to C$ is a localization if there is a left adjoint $r$ to $i$ preserving limits. If furthermore $r$ is left exact then giving a localization is the same as giving a calculus of fractions, a construction analogous to the construction above where the morphisms of the category are replaced by equivalence classes of certain pairs of morphisms $(f, g)$. We discussed in the last section how the subobjects $\text{Sub}(A)$ of an object $A$ can be seen as a generalized space. It turns out that giving a localization on a category is the same as giving a universal closure operator on $\text{Sub}(A)$ for every object $A$. This closure operator is a generalisation of the closure operator of ordinary topology. For a detailed discussion of the topics of this section see [3] [Ch.7]. We note that the inverse image functor we considered in the previous section has a powerful expression in the context of derived categories of Abelian sheaves and these are given by a localization of the homotopy category of complexes [10] [Ch.1]. In fact modern homotopy theory is centered around the interplay between different kinds of simplicial structures and model categories. The latter ”model” a homotopy theory which can be defined as a category with a class of morphisms - weak equivalences - we wish to take as isomorphisms. We form the homotopy category by localising such categories at weak equivalences. Certain classes of models form themselves a category and have a natural notion of weak equivalence. Thus they can be seen as a ”homotopy theory of homotopy theories” [1] [p.82] which can be seen as the most complex evolution of the ancient concepts of topos and space to date.

The category $\text{Set}^{\text{op}}$ of presheaves over a category $C$, which is simply the category of functors from $C^{\text{op}}$ to $\text{Set}$ which are not required to satisfy any ”gluing conditions”, can be seen as a sort

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7The linearly ordered real interval $I$ plays a central role in homotopy theory. The concept of connectivity associated to it is pathwise connectivity (which can also be extended to Topos Theory [30]). This seems particularly relevant to Thom’s Semiophysics, the attempt to described the topological architecture of semantic spaces based the Aristotelic theory of genera and species wherein two species belong to the same genus if they can be continuously deformed into each other [p.199]. The divisibility property of the continuum is what allows the division of the genus into species, the boundary of such a division corresponding to the difference which allows a definition.

8The notions of fibration and co-fibration can be seen as generalisation of what it means for a space to be respectively ”inside” and ”over” another space.
of materia prima. Given a category $C$, endowing $C$ with a Grothendieck topology is the same as defining a universal closure operator in $\text{Set}^{C^{\text{op}}}$ and the associated localization is precisely the Grothendieck topos, the category of sheaves on $C$.

Topos theory and sheaf theory are rich enough to generalise and unite most of our classical spatial intuitions. Besides quotients, associated to the rational numbers, there is also the classical concept of defining the real line by a process of ”completion”. The construction of the reals via Dedekind cuts on the rational numbers can in fact be generalised to the topos of sheaves over a topological space $X$. We can also view a topos as a model for higher-order intuitionistic logic and in this topos we formulate all the standard concepts of the differential calculus. The real numbers in this topos correspond to the sheaf of continuous functions over $X$ and this topos is a model for intuitionistic analysis in which all functions are continuous, thus furnishing a model which comes close to the Aristotelic theory of change wherein continuity is omnipresent. In view of what we discussed about local continuity in Aristotelic Physics and Catastrophe Theory it would be interesting to extend these constructions to the case of merely locally continuous functions.

5 Conclusion

We have seen how Sheaf Theory (or more generally, Topos Theory) captures the kindred concepts of topos and infinitesimal as well as furnishing a framework for defining the topology of the continuum in a point-free Aristotelian way.

This illustrates how ancient philosophy holds insights which are only recently being clarified in the light of modern knowledge.

It invites us to reconsider the essence of philosophy and its relationship to mathematics and natural science.

Until the days of Husserl it was normal for a philosopher to have a good knowledge of mathematics and physics and to not only reflect upon the foundational questions of mathematics and physics but to see these questions as fundamental to epistemology, ontology and the philosophy of mind as they are for the philosophy of nature.

The subsequent divorce of science and mathematics from philosophy has had the tragic consequence that the independent technical development of the latter has often led to it missing out on a valuable enriching interplay with the philosophical thought of the past.

Topological and geometric concepts have shown themselves to be potent factors in the progress of physics. It is true that Thom held formal logic in small regard and once wrote that the main goal of Catastrophe Theory was to ”geometrize thought and linguistic activity” and in his Esquisse he wrote that the logical was derivative of the geometric. The geometric and the topological, with their extension to spatial-temporal processes brought about by the revolutions of Poincaré and Einstein, allows us to encompass many fundamental philosophical categories such as causality and possibility (phase-space), and furnish a powerful foundation for our rationality which is mirrored equally in the structure of the mind and in natural science.

It is difficult to think of a domain in which we are given as much power at once to construct models for the natural world and to shed light on the working of our own mind and which at the same time yields so readily to formal axiomatic treatment. This can be seen in the recent developments of homotopy type theory inspired by ideas of Lawvere. Modal homotopy type theory has been proposed by Corfield as a refinement of the (multi)modal logics usually considered by philosophers and the concept of cohesive higher topos is used by Urs Schreiber to axiomatise certain aspects of theoretical physics.

Aristotle the topologist and phenomenologist is also Aristotle the formal logician and

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9 We could also mention the axiomatic treatments mereology.

10 His disciple Theophrastus developed the basis for propositional logic.
we find therein a perfect harmonisation and mutual enrichment between proof theory and semantics. We find a promising domain for unveiling "a genealogy of the understanding" with applications not only to natural science but to cognitive science and the formalisation of natural language.

Experimental data in physics has been an occasion for the both the deployment and refinement of our geometric-dynamic concepts and there are currently many active areas of investigation which propose that the classical model of space-time collapses at the Planck scale and a new concept of space is called for. Topos Theory has been applied to theoretical physics (cf. [33]). Zeno’s paradoxes of motion addressed by Aristotle in the Physics retain their relevance today: can space and time by ultimately discrete?

The movement towards abstraction and axiomatisation described in this essay is also at once a movement towards logical clarity (we might call it "logical phenomenology") and a movement towards a progressive vision and intuition of the pure essence of our primordial topological-dynamic intuitions perceived so keenly by Aristotle and Thom.

Appendix

We give here some definitions of a few important concepts in modern topology.

Let $X$ be a set. Then a topology over $X$ is a collection $\tau$ of subsets of $X$ such that $X$ and the empty set belong to $\tau$, the union of any collection of sets in $\tau$ belongs to $\tau$ and the finite intersection of sets in $\tau$ is still in $\tau$. The elements of $\tau$ are called open sets. A closed is the complement of an open set. A basis for a topology $\tau$ is a collection $\mathcal{B}$ of subsets of $X$ such that every open set can be expressed as a union of sets in $\mathcal{B}$. A basis for the standard topology of the real line is given by the open intervals $(a, b)$. Given a point $x \in X$ a neighbourhood of $x$ is an open set containing $x$. Given a subset $Y$ of $X$ the interior of $Y$ is the set of all points that have a neighbourhood contained in $Y$. The closure of $Y$ is the intersection of all open sets containing $Y$. It is always a closed set. The correspondence which associates to each subset $Y$ its closure is an example of a closure operator. For instance, the closure of the open interval $(a, b)$ is the closed interval $[a, b]$. The boundary of a set is the set of all points $x$ that satisfy the condition: any neighbourhood of $x$ will contain points both in $Y$ and not in $Y$. There is a general concept of compactness but in the case of the ordinary topology on the real line (or Euclidean space $\mathbb{R}^n$) it corresponds to being both closed and bounded (all points at a finite distance from the origin).

A category can be thought of as a collection of objects and for any two objects (possibly equal) a (possibly empty) collection of arrows (or morphisms). Each arrow has a source and a target. If the target of an arrow $f$ is equal to the source of an arrow $g$ then we can compose them to form a new arrow $g \circ f$ which will have the source of $f$ and the target of $g$. Given an object $A$ there is a unit arrow $u_A$ which goes from $A$ back to $A$. The formal definition of category requires an associative condition on composition and that the unit arrows function as units for composition, for instance $f \circ u_A = f$ when $f$ has source $A$. Given two categories $C$ and $D$ a functor is a correspondence between objects of $C$ and objects of $D$ and arrows of $C$ and arrows of $D$ which preserves composition and unit arrows. Examples of categories are the category $Set$ of sets whose objects are sets and morphisms are maps between sets and the category $\mathcal{O}(X)$ of open sets of a topology on $X$ whose objects are the open sets for any two open sets $U$ and $V$ there is either a single morphism if $U$ is contained in $V$ or no morphism at all if otherwise. In the category $Set$ we can form the cartesian product $X \times Y$ of two sets $X$ and $Y$ as well as the disjoint union $X \sqcup Y$. The properties of these constructions can be generalised to obtain the concepts of product and coproduct for any category (or more generally limit and colimit). A Topos can be understood as a category which possesses generalisations of many fundamental properties of $Set$. 
Given a category $C$ we can form the *opposite category* $C^{\text{op}}$ which is obtained by inverting the direction of all the arrows. A *presheaf* is simply a functor $F$ from $C^{\text{op}}$ to $\text{Set}$. Given an object $A$ of $C$ the elements of $F(A)$ are called *sections*. If $U \subset V$ then this arrow corresponds to an arrow $F(V) \to F(U)$ (note that source and target are switched because we are considering the opposite category) called the *restriction* map. Presheaves can be made into a category with a suitable definition of morphism (natural transformations between functors). If $C$ is the category of open sets of a topology on a space $X$ then a *sheaf* over $X$ is a presheaf on $C$ which satisfies the *gluing* condition. This says that if we express an open set $U$ as a union (cover) of open sets $U_i$ and we consider sections $s_i$ on each $U_i$ such that they agree on all the intersections $U_i \cap U_j$ then there is unique section on $U$ such that the sections $s_i$ are given by the restrictions corresponding to the inclusions $U_i \subset U$. An example of a sheaf is the correspondence which associates to each open set $U$ of a topological space $X$ the set of continuous functions defined over $X$. An example of a presheaf which is not a sheaf is given by constant functions on open sets. Considering two disjoint open sets it is easy to see why the gluing condition fails. The concept of *Grothendieck* topology is a generalisation of the concepts involved in this property to an arbitrary category. The idea of covering $U$ by the open sets $U_i$ becomes the concept of *sieve*.

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