TRANSVERSE HILBERT SCHEMES, BI-HAMILTONIAN SYSTEMS, AND HYPERKÄHLER GEOMETRY

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Dedicated to the memory of Sir Michael Francis Atiyah (1929-2019)

Abstract. We give a characterisation of Atiyah’s and Hitchin’s transverse Hilbert schemes of points on a symplectic surface in terms of bi-Poisson structures. Furthermore, we describe the geometry of hyperkähler manifolds arising from the transverse Hilbert scheme construction, with particular attention paid to the monopole moduli spaces.

1. Introduction

In chapter 6 of the monograph [1], Atiyah and Hitchin consider the following construction. Let $Y$ be a complex symplectic surface with a holomorphic submersion $\pi$ onto a 1-dimensional complex manifold $X$. They associate to it an open subset of the Hilbert scheme of $n$ points on $Y$ consisting of 0-dimensional complex subspaces $D$ of length $n$ such that $\pi|_D$ is an isomorphism onto its scheme-theoretic image. They observe that this transverse Hilbert scheme $Y^{[n]}$ is a symplectic manifold equipped with holomorphic submersion $\pi^{[n]}$ onto $S^nX$, the fibres of which are Lagrangian submanifolds. In particular, if $X$ is a domain in $\mathbb{C}$, then the components of $\pi^{[n]}$ define $n$ functionally independent and Poisson-commuting Hamiltonians on $Y^{[n]}$, i.e. a completely integrable system. Atiyah and Hitchin observe further that sometimes one can perform this construction on the fibres of the twistor space of a 4-dimensional hyperkähler manifold and obtain a new twistor space which then might lead to a new hyperkähler manifold. Their main example of this construction is $Y = \mathbb{C}^* \times \mathbb{C}$ with $\pi$ the projection onto the second factor. The corresponding transverse Hilbert scheme is the space of based rational maps of degree $n$ and the hyperkähler metric resulting from applying the construction to the twistor space of $S^1 \times \mathbb{R}^3$ is the $L^2$-metric on the moduli space of Euclidean monopoles of charge $n$. Further examples of this construction are given in [19, 5].

The purpose of this article is to characterise both symplectic and hyperkähler manifolds arising from this construction. Partial results in this direction have been obtained in [6] and in [17, 18]. They rely on the existence of certain endomorphism of the tangent bundle of $Y^{[n]}$. In the present work our point of view is different. We observe that $Y^{[n]}$ is equipped with a second Poisson structure, compatible with the symplectic form. Thus $Y^{[n]}$ is a completely integrable bi-Hamiltonian system. We then show that a nondegenerate bi-Poisson manifold $M^{2n}$ arises as an (open subset) of a transverse Hilbert scheme on a symplectic surface $Y$ with a submersion $Y \to \mathbb{C}$ essentially exactly then, when the coefficients of the minimal polynomial of

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the corresponding recursion operation (see [2,3] for a definition) form a submersion to \( \mathbb{C}^n \).

We then turn our attention to hyperkähler manifolds arising from the transverse Hilbert scheme construction on the fibres of the twistor space a 4-dimensional hyperkähler manifold with a tri-Hamiltonian vector field. We show that the essential feature of the geometry of a manifold \( M \) arising from this construction is the existence of a bivector \( \Pi \) on \( M \) which lies in Salamon’s component \( \Lambda^2 E \otimes S^2 H \) of \( \Lambda^2 T^CM \) and satisfies \( D \Pi = 0 \), where \( D \) is the Penrose-Ward-Salamon differential operator on \( \Lambda^2 E \otimes S^2 H \) [21]. The bivector \( \Pi \) is not Poisson, but its \( (2,0) \)-component with respect to each complex structure is a (generically log-symplectic) holomorphic Poisson bivector. Moreover, this holomorphic Poisson bivector is compatible with the parallel holomorphic symplectic form arising from the hyperkähler structure.

In the last section we identify the bivector \( \Pi \) on moduli spaces of \((2,0)\)-monopoles, i.e. hyperkähler transverse Hilbert schemes on \( S^1 \times \mathbb{R}^3 \), in terms of solutions to Nahm’s equations.

2. TRANSVERSE HILBERT SCHEMES AND BI-HAMILTONIAN SYSTEMS

2.1. Transverse Hilbert schemes. Let \( X \) be a complex manifold, \( C \) a complex manifold of dimension 1, and \( \pi : X \rightarrow C \) a holomorphic map. The transverse Hilbert scheme \( X_\pi^{[n]} \) of \( n \) points in \( X \) is an open subset of the full Hilbert scheme \( X^{[n]} \) consisting of those \( D \in X^{[n]} \) such that \( \pi_* D \) is an isomorphism onto its scheme-theoretic image [1]. Since \( C^{[n]} = S^n C \), this simply means that \( \pi(D) \) consists of \( n \) points (with multiplicities). First of all, let us observe that \( X_\pi^{[n]} \) is always smooth, unlike the full Hilbert scheme \( X^{[n]} \):

**Proposition 2.1.** Let \( \pi : X \rightarrow C \) be a holomorphic map from a complex manifold \( X \) to a 1-dimensional complex manifold \( C \). Then the transverse Hilbert scheme \( X_\pi^{[n]} \) is smooth.

**Proof.** Since \( D \in X_\pi^{[n]} \) satisfies \( D \cong \pi(D) \in S^n C \), such a \( D \) is a local complete intersection (l.c.i.). Now the claim follows from general results of deformations theory (see, e.g., [13, Theorem 1.1.(c)]).

The transverse Hilbert scheme comes equipped with a canonical map \( \pi^{[n]} : X_\pi^{[n]} \rightarrow S^n C \). If \( \pi \) is a submersion, then so is \( \pi^{[n]} \). In this case, points of \( X_\pi^{[n]} \) such that \( \pi(D) = n_1 p_1 + \cdots + n_k p_k \), with \( p_1, \ldots, p_k \) distinct points of \( C \), correspond to a choice of a section \( s_i \) of \( \pi \) in a neighbourhood of each \( p_i \), truncated to order \( n_i \) (in other words \( s_i \) is an \((n_i - 1)\)-jet of sections at \( p_i \)). Let us remark that Atiyah and Hitchin consider only the case when \( \pi \) is a submersion (they say \( \pi \) is a “complex fibration”, and the proof of the smoothness of \( X_\pi^{[n]} \), given on p. 53 in [1], makes clear that \( \pi \) must be a submersion).

Suppose now that \( X \) has a symplectic structure. If \( \dim X = 2 \), then a well-known theorem of Beauville [2] implies that \( X^{[n]} \), and hence \( X_\pi^{[n]} \), carries an induced symplectic structure. For higher dimensional \( X \), there is no induced symplectic structure on \( X^{[n]} \), not even on its smooth locus.

2.2. Log-symplectic Poisson structures. A Poisson structure on a (smooth or complex) manifold \( M^{2n} \) is given by a bivector \( \Pi \in \Gamma(\Lambda^2 TM) \) such that the Schouten bracket \([\Pi, \Pi]\) vanishes. The *symplectic locus* of the Poisson structure is the set of
points \( m \) where the induced map \( \#_\Pi : T^*_m M \to T_m M \) is an isomorphism. Its complement is called the \emph{degeneracy locus}. A Poisson structure is called \emph{log-symplectic} if \( \Pi^n \in \Gamma(\Lambda^{2n}TM) \) meets the zero section of the \( \Lambda^{2n}TM \) transversely. These structures were studied by Goto [10] in the holomorphic case, and by Guillem, Miranda and Pires in the smooth category [12] (see also [11, 8]). The name is justiﬁed by the fact that the dual 2-form \( \omega = \Pi^{-1} \) has a logarithmic singularity along the degeneracy locus. The degeneracy locus \( \Delta \) of a log-symplectic Poisson structure is a smooth Poisson hypersurface with codimension one symplectic leaves and \( M \setminus \Delta \) is a union of open symplectic leaves.

We recall from [12] that if \( f \) is a local deﬁning function for \( \Delta \), then \( \omega \) can be decomposed as

\[
\omega = \alpha \wedge \frac{df}{f} + \beta,
\]

for a 1-form \( \alpha \) and a 2-form \( \beta \). Moreover, the restrictions of \( \alpha \) and \( \beta \) to \( \Delta \) are closed, \( \alpha|_\Delta \) is intrinsically deﬁned and its kernel is the tangent space to the symplectic leaf of \( \Pi \).

2.3. Bi-Poisson structures. A \emph{bi-Poisson} structure on a (real or complex) manifold \( M \) is a pair \((\Pi_1, \Pi_2)\) of linearly independent bivectors such that every linear combination of \( \Pi_1 \) and \( \Pi_2 \) is a Poisson structure. In other words \( \Pi_1 \) and \( \Pi_2 \) satisfy \( [\Pi_1, \Pi_1] = 0, [\Pi_2, \Pi_2] = 0, [\Pi_1, \Pi_2] = 0 \), where \([, , ]\) is the Schouten bracket.

A bi-Poisson structure is called \emph{nondegenerate}, if the pencil \( t_1 \Pi_1 + t_2 \Pi_2 \) contains a symplectic structure. In what follows, we shall consider only nondegenerate bi-Poisson structures and assume that \( \Pi_1 \) is symplectic. Following Magri and Morosi [20] (see also [9]) we can deﬁne the \emph{recursion operator} \( R = \#_{\Pi_2} \circ \#_{\Pi_1}^{-1} \). It is an endomorphism of \( TM \) and Magri and Morosi show that 1) its Nijenhuis tensor vanishes; and 2) the eigenvalues of \( R \) form a commuting family with respect to both Poisson brackets.

Furthermore, \( \det R = (\mu_R)^2 \) for a well deﬁned function \( \mu_R \) on \( M \) (\( \mu_R \) is the quotient of the Pfaffians of \( \Pi_2 \) and of \( \Pi_1 \)). Thus \( \Pi_2 \) is log-symplectic if and only if 0 is a regular value of \( \mu_R \). Since \((\Pi_1, \Pi_2 - \lambda \Pi_1)\) is a nondegenerate bi-Poisson structure for each scalar \( \lambda \), the characteristic polynomial of \( R \) is of the form \( \chi_R(\lambda) = \mu_R(\lambda)^2 \).

We shall refer to \( \mu_R(\lambda) \) as the \emph{Pfaffian polynomial} of \( R \). We observe:

\textbf{Proposition 2.2.} Let \((\Pi_1, \Pi_2)\) be a real (resp. holomorphic) bi-Poisson structure on a smooth (resp. complex) manifold \( M^{2n} \) with \( \Pi_1 \) symplectic. If the coefﬁcients of the Pfaffian polynomial of the recursion operator \( R \) deﬁne a submersion \( p : M \to \mathbb{R}^n \) (resp. \( p : M \to \mathbb{C}^n \)), then the Poisson structure \( \Pi_2 - \lambda \Pi_1 \) is log-symplectic for every \( \lambda \).

We also recall the following property of bi-Poisson structures, proved by Magri and Morosi in [20]:

\textbf{Proposition 2.3 (Magri-Morosi).} Let \((M, \Pi_1, \Pi_2)\) be a bi-Poisson manifold with \( \Pi_1 \) symplectic. Then, for any polynomial \( \rho(z) \), the bivector \( \Pi_\rho \) defined by

\[
\Pi_\rho(\alpha, \cdot) = \rho(R)\Pi_1(\alpha, \cdot), \quad \alpha \in \Omega^1(M)
\]

defines a Poisson structure on \( M \), compatible with \( \Pi_1 \).
2.4. **Transverse Hilbert schemes on symplectic surfaces.** Let \( S \) be a complex surface, \( \omega \) a holomorphic symplectic form on \( S \) and \( \pi : S \to C \) a holomorphic map. As shown by Beauville [2], the Hilbert scheme \( S^{[n]} \) of \( n \) points in \( S \) has a canonically induced symplectic form \( \omega^{[n]} \). This result has been extended by Bottacin [7], who showed that any Poisson structure on \( S \) induces a Poisson structure on \( S^{[n]} \). We thus obtain two Poisson bivectors on \( S^{[n]} \): \( \Pi_1 \) induced by \( \omega^{-1} \) and \( \Pi_2 \) induced by \( \pi \cdot \omega^{-1} \), where \( \omega^{-1} = \#_{\omega} \omega \) (i.e. the bivector dual to \( \omega \)) and \( \pi \) is viewed as a function on \( S \). Since the Poisson structures on \( S \) are compatible, \( \Pi_1 \) and \( \Pi_2 \) are compatible (compatibility is trivial on the open dense subset where \( D \) consists of distinct points, and hence \([\Pi_1, \Pi_2]\) vanishes everywhere). Observe that the corresponding recursion operator \( R \) (cf. §2.3) is the endomorphism of \( TS^{[n]} \) given by the multiplication by \( \pi \) on each \( T_D S^{[n]} \cong H^0(D, N_{D/S}) \) where \( N_{D/S} \) denotes the normal sheaf of \( D \) in \( S \). This is the endomorphism considered in [6] [17] [18]. The coefficients of its Pfaffian polynomial define a map \( S^{[n]} \to S^{n} C \cong C^n \). Its restriction to the transverse Hilbert scheme \( S^{[n]}_\pi \) coincides with the canonical map \( \pi^{[n]} \) introduced in [24]. Let us prove the following properties of \( S^{[n]}_\pi \) and \( \pi^{[n]} \).

**Proposition 2.4.** Let \( S \) be a complex symplectic surface with a holomorphic map \( \pi : S \to C \). Then the transverse Hilbert scheme \( S^{[n]}_\pi \) is a nondegenerate bi-Poisson manifold with the following properties:

(i) the coefficients of the minimal polynomial of the corresponding recursion operator \( R \) coincide with the canonical map \( \pi^{[n]} : S^{[n]}_\pi \to C^n \);

(ii) at any point of its degeneracy locus, the Poisson structure \( \Pi_2 - \lambda \Pi_1 \) has rank \( 2n - 2 \) \((\lambda \in C)\);

(iii) on the subset of \( \pi^{[n]} \)-regular points, the Poisson structure \( \Pi_2 - \lambda \Pi_1 \) is log-symplectic for every \( \lambda \in C \).

**Proof.** We already know that \( S^{[n]}_\pi \) is a nondegenerate bi-Poisson manifold. Owing to the definition of the recursion operator, we know that the geometric multiplicity of each eigenvalue is even. Now observe that the multiplication by \( \pi \) defines also an endomorphism \( \tilde{R} \) of \( T_{(\pi(D))} C^n \). The geometric multiplicity of every eigenvalue of \( \tilde{R} \) is equal to 1 (since \( \pi(D) \in S^n C \) has length \( n \)). We also know that the characteristic polynomial of \( \tilde{R} \) is equal to the Pfaffian polynomial \( \mu_R(\lambda) \) of \( R \), and that the characteristic polynomial of \( R \) is \( \mu_R(\lambda)^2 \). Putting this together, we conclude that the geometric multiplicity of every eigenvalue of \( \tilde{R} \) is equal to 2 and that the minimal polynomial of \( R \) is equal to \( \mu_R(\lambda) \). This proves statements (i) and (ii). The third statement follows from Proposition 2.2. \( \square \)

**Remark 2.5.** Statement (i) has been shown in [17] Remark 2.4] under the assumption that \( \pi \) is a submersion.

**Remark 2.6.** Since, owing to the above mentioned result of Bottacin, any Poisson structure on \( S \) induces a Poisson structure on \( S^{[n]} \), we can conclude that if \( S \) is a Poisson surface with a holomorphic map \( \pi : S \to C \), then \( S^{[n]} \) is a bi-Poisson manifold. The bi-Poisson structure will, however, be degenerate if \( S \) is not symplectic.

**Remark 2.7.** Suppose that \((z, u)\) are Darboux coordinates for the symplectic form \( \omega \) on an open subset \( U \) of \( S \), i.e. \( \omega = dz \wedge du \) on \( U \). Suppose also that \( \pi(z, u) = z \) (which implies that \( \pi \) is a submersion on \( U \)). Then the corresponding open subset \( U^{[n]}_\pi \) can be described as an open subset of \((q(z), p(z))\), where \( q(z) \) is a monic
polynomial of degree \( n \) and \( p(z) \) is a polynomial of degree at most \( n - 1 \), such that, for every root \( z_i \) of \( q_i(z_i, p(z_i)) \in U \). On the open dense subset of \( U^{[n]}_\pi \), where the roots are distinct, the two Poisson structures are given by:

\[
\Pi_1 = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial u_i}, \quad \Pi_2 = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial u_i}, \quad \text{where} \quad u_i = p(z_i).
\]

We can now characterise transverse Hilbert schemes on symplectic surfaces, in the case when \( \pi \) is a submersion (i.e. the case originally considered by Atiyah and Hitchin):

**Theorem 2.8.** Let \((M^{2n}, \Pi_1, \Pi_2)\) be a holomorphic bi-Poisson manifold with \( \Pi_1 \) symplectic. Assume that the coefficients of the Pfaffian polynomial of the corresponding recursion operator \( R \) define a submersion \( p : M \to \mathbb{C}^n \) and that, for each \( \lambda \in \mathbb{C} \), if the degeneracy locus \( D_\lambda \) of \( \Pi_2 - \lambda \Pi_1 \) is nonempty, then its symplectic foliation is simple.

Then there exists a symplectic surface \( S \) with a holomorphic submersion \( \pi : S \to \mathbb{C} \) and a local bi-Poisson biholomorphism \( \Phi : (M, \Pi_1, \Pi_2) \to S^{[n]} \).

**Proof.** Let \( \mu_m(\lambda) \) denote the Pfaffian polynomial of \( R_m \). We consider the following incidence variety (cf. [1], pp.40-43, [18]):

\[
T = \{ (\lambda, m) \in \mathbb{C} \times M; m \in D_\lambda \} = \{ (\lambda, m) \in \mathbb{C} \times M; \mu_m(\lambda) = 0 \}.
\]

Due to the assumptions and to Proposition 2.2, \( T \) is smooth and the symplectic foliation on each \( D_\lambda \) is simple with codimension one leaves. We thus obtain an integrable simple foliation \( \mathcal{F} \) of \( T \), the leaf space of which is a 2-dimensional complex manifold \( S \) with a canonical holomorphic submersion \( \pi : S \to \mathbb{C} \).

On each \( D_\lambda \) there is a canonically defined (closed) 1-form \( \alpha_\lambda \) (cf. (2.1) and the following lines), the kernel of which corresponds to the symplectic foliation of \( D_\lambda \). Thus \( \alpha_\lambda \wedge d\lambda \) defines a nondegenerate, hence symplectic, 2-form on \( S \).

The scheme-theoretic inverse image of a point \( m \in M \) defines a 0-dimensional subspace \( Z_m \) of \( T \) with structure sheaf isomorphic to \( \mathcal{O}[\lambda]/(\mu_m(\lambda)) \). The projection \( T \to \mathbb{C} \) maps \( Z_m \) isomorphically onto a 0-dimensional subspace of \( \mathbb{C} \) of length \( n \). Thus \( Z_m \) descends to an element of \( S^{[n]}_\mu \), and we obtain a holomorphic map \( \Phi : M \to S^{[n]}_\mu \). It remains to show that \( \Phi \) is a local diffeomorphism. Since the coefficients of \( \mu_m \) define a submersion, the corresponding Hamiltonian vector fields do not vanish anywhere. Near any point \( p \in M \) we have therefore the “action-angle” coordinates on a neighbourhood \( U \) (given by coefficients of \( \mu \) and the local free action of \( \mathbb{C}^n \)). Let \( S_p \) be the symplectic surface obtained from \( U \) by the above procedure. On \( (S_p)_\pi^{[n]} \) there are analogous “action-angle” coordinate and therefore we obtain a holomorphic map \( \psi : (S_p)_\pi^{[n]} \to U \). Fernandes [9] shows that, on the subset where the eigenvalues are distinct, there exist local coordinates \( z_i, u_i \) such that

\[
\Pi_1 = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial u_i}, \quad \Pi_2 = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial u_i}.
\]

Thus, owing to Remark 2.7, \( \psi \) is the inverse of \( \Phi \) on the open dense subset where the roots of \( \mu_m \) are distinct (and a bi-Poisson isomorphism) and, hence, \( \psi \) is the inverse of \( \Phi\mid_U \). \( \square \)
Remark 2.9. Presumably the result remains true without the assumption that the symplectic foliations of $D_\lambda$ are simple, provided we replace “symplectic surface” with “2-dimensional symplectic stack”.

3. Hyperkähler geometry of transverse Hilbert schemes

3.1. Ward transform. Let us briefly recall the essential features of the Ward transform \cite{22, 21} in the case of hypercomplex manifolds. Let $Z$ be a complex manifold with a surjective holomorphic submersion $\pi : Z \to \mathbb{P}^1$, and let $M^C$ be the Kodaira moduli space of sections with normal bundle isomorphic to $O(1)^\otimes n$. The twistor double fibration in this case is simply

$$M^C \xrightarrow{T} M^C \times \mathbb{P}^1 \xrightarrow{\nu} Z.$$  

If $F$ is an $M^C$-uniform (i.e. $h^0(\nu(\tau^{-1}(m)))$ is constant on $M^C$) holomorphic vector bundle on $Z$, then we obtain an induced holomorphic vector bundle $\tilde{F} = \pi_* \nu^* F$ on $M^C$. In particular, if we denote by $E$ the vector bundle induced from $T_\pi Z \otimes \pi^* O(-1)$ (where $T_\pi Z = \ker d\pi$) and by $H$ the trivial vector bundle with fibre $\mathbb{C}^2$, we have $TM \simeq E \otimes H$. Furthermore, the vector bundle induced from $\pi^* O(k)$, $k \geq 0$, is simply $S^k H$, and if $F$ is $M^C$-trivial (i.e. trivial on each line $\nu(\tau^{-1}(m)))$, then the bundle induced from $F \otimes \pi^* O(k)$ is $\tilde{F} \otimes S^k H$, for any $k \geq 0$.

Recall also that an induced vector bundle comes equipped with a first-order differential operator, which arises as the pushforward of a partial connection on $\nu^* F$, which is basically the exterior derivative in the fibre directions. If $F$ is $M^C$-trivial, then this operator is a linear connection on $\tilde{F}$. We can identify this operator for bundles of the form $\tilde{F} \otimes S^k H$, which are induced from $F \otimes \pi^* O(k)$, where $F$ is $M^C$-trivial. We fix an isomorphism $H \simeq H^*$ (which corresponds to a choice of isomorphism $H^1(\mathbb{P}^1, O(-2)) \simeq \mathbb{C}$). We denote by $\alpha$ the natural projection $S^k H \otimes H \to S^{k+1} H$ (which corresponds to multiplication of sections of $O(k)$ and of $O(1)$), and use the same letter for the corresponding map on $W \otimes S^k H \otimes H \to W \otimes S^{k+1} H$ for any holomorphic vector bundle $W$. The induced differential operators $D$ are then:

1. on $S^k H$, $D = \alpha \circ d$;
2. on $\tilde{F} \otimes S^k H$, $D = \alpha \circ (\nabla \otimes \text{Id}) + \text{Id} \otimes (\alpha \circ d)$, where $F$ is $M^C$-trivial and $\nabla$ denotes the induced connection on $\tilde{F}$. The principal symbol of $D$ is $\alpha$.

The construction of $D$ as the push-forward of a partial connection shows, in particular, that $D s = 0$ if and only if $s = \tau_\ast \eta^* \bar{s}$ for a holomorphic section $\bar{s}$ of $F(k)$ on $Z$.

3.2. Hyper-Poisson bivectors. Let $M$ be a hypercomplex manifold with twistor space $Z$. As discussed above, the vector bundle $T_\pi Z \otimes \pi^* O(-1)$ is $M$-trivial, and hence the operator $D$ on the induced vector bundle $E$ is a linear connection $\nabla$. Recall that the tensor product of $\nabla$ and the flat connection on $H$ is a torsion-free linear connection on $TM$ known as the Obata connection. The induced operator $D$ on $TM$ is therefore the composition of the Obata connection and the projection $H \otimes H \to S^2 H$. Similarly, the vector bundle $\Lambda^r (T_\pi Z \otimes \pi^* O(-1))$ is $M$-trivial, and hence the vector bundle on $M$ induced from $\Lambda^r T_\pi Z$ is $\Lambda^r E \otimes S^2 H$. This is a direct summand of $\Lambda^r T^C M$, which Salamon \cite{21} identifies with the subspace of finite linear combinations of multivectors of type $(r,0)$ for different
complex structures. Salamon also shows that its $Sp(1)$-invariant complement is
\begin{equation}
B^r = \bigcap_\zeta \left(T^{(r-1)}_\xi M \oplus T^{(r-2)}_\xi M \oplus \cdots \oplus T^{(1)}_\xi M\right),
\end{equation}
where $\zeta \in \mathbb{P}^1$ labels different complex structures of the hypercomplex structure. As explained in the previous subsection, a multivector field $\Pi \in \Gamma(\Lambda^2 E \otimes S^r H)$ arises from a holomorphic section of $\Lambda^r T\pi Z$ precisely if it satisfies the equation $D\Pi = 0$. In that case, for any complex structure $I_\xi$, the $(2,0)$-component of $\Pi$ is the corresponding holomorphic multivector field on the fibre $\pi^{-1}(\xi)$ of $Z$.

**Remark 3.2.** As explained above, the condition that $\Pi_\xi^{2,0}$ is holomorphic for each $\xi$ is equivalent to $D\Pi = 0$. On the other hand, condition (ii) implies that $[\Pi, \Pi] \in \Gamma(B^3)$, where $[,]$ denotes the Schouten-Nijenhuis bracket and $B^3$ is defined in (3.2).

As usual, given a bivector field on $M$, we can define a bracket of (real- or complex-valued) functions on $M$ by
\begin{equation}
\{f, g\} = \Pi(df, dg).
\end{equation}
The name “hyper-Poisson” is justified by the following observation, which follows directly from the definition.

**Proposition 3.3.** A bivector $\Pi \in \Gamma(\Lambda^2 E \otimes S^2 H)$ is hyper-Poisson if and only if, for each complex structure $I_\xi$, the bracket (3.3) is a Poisson bracket on the sheaf $\mathcal{O}(M, I_\xi)$ of $I_\xi$-holomorphic functions. □

**Remark 3.4.** Our notion of “hyper-Poisson” is different from [16]. There, it means a triple $(\pi_1, \pi_2, \pi_3)$ of bivectors, such that $\pi_2 - i\pi_3$ is an $I_1$-holomorphic Poisson bivector etc. We do not think there is a danger of confusion, since we talk about hyper-Poisson bivectors, while [16] deals with hyper-Poisson triples.

**Definition 3.5.** Let $M$ be a hyperkähler manifold. A hyper-Poisson bivector $\Pi$ on $M$ is said to be compatible with the hyperkähler structure if, for every complex structure, the holomorphic Poisson bivector $\Pi_\xi^{2,0}$ is compatible with $\Omega^{-1}_\xi$, where $\Omega_\xi$ is the corresponding parallel holomorphic symplectic form.

**Example 3.6.** Recall that the twistor space $Z$ of a hyperkähler manifold is equipped with a fibrewise $\mathcal{O}(2)$-valued complex symplectic form $\omega$, i.e. a section of $\Lambda^2 T^*_\pi Z \otimes \mathcal{O}(2)$. It can be viewed as a (holomorphic) section of $\Lambda^2(T\pi Z(-1))^*$, i.e. it induces a symplectic form on the bundle $E$. The fibrewise bivector $\omega^{-1}$ is a section of $\Lambda^2(T\pi Z(-1))$ and so multiplying it by a real section of $\pi^*\mathcal{O}(2)$ yields a holomorphic section of $\Lambda^2 T\pi Z$ compatible with the real structure, i.e. a hyper-Poisson bivector. This bivector is simply a constant multiple of $\omega^{-1}_\alpha$, where $\omega_\alpha$ is one of the Kähler forms of the hyperkähler metric ($\alpha$ is determined by the chosen section of $\mathcal{O}(2)$). In other words, for any Kähler form $\omega_\alpha$ of the hyperkähler metric, the dual bivector $\omega^{-1}_\alpha$ is a hyper-Poisson bivector compatible with the hyperkähler structure. The corresponding Poisson bracket on $\mathcal{O}(M, I_\xi)$ is identically 0 when $I_\xi = \pm I_\alpha$. 


We can classify hyper-Poisson bivectors on 4-dimensional hyperkähler manifolds.

**Theorem 3.7.** Let \((M, g, I_1, I_2, I_3)\) be a 4-dimensional hyperkähler manifold with corresponding Kähler forms \(\omega_1, \omega_2, \omega_3\). A bivector \(\Pi\) on \(M\) is hyper-Poisson if and only if
\[
\Pi = f_1\omega_1^{-1} + f_2\omega_2^{-1} + f_3\omega_3^{-1},
\]
for smooth functions \(f_1, f_2, f_3 : M \to \mathbb{R}\) satisfying \(I_1df_1 = I_2df_2 = I_3df_3\). Such a bivector is compatible with the hyperkähler structure.

**Proof.** If \(\dim M = 4\), then \(\Lambda^2 E \otimes S^2 H\) is spanned at each point by \(\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}\). Therefore \(\Pi\) must be of the form (3.4). Its \((2, 0)\)-component with respect to \(I_1\) is equal to
\[
\frac{1}{2}(f_2 + if_3)(\omega_1^{-1} - i\omega_3^{-1}) = 2(f_2 + if_3)(\omega_2 + i\omega_3)^{-1},
\]
Hence, if \(\Pi\) is hyper-Poisson, then \(f_2 + if_3\) is \(I_1\)-holomorphic (since \(\omega_2 + i\omega_3\) is \(I_1\)-holomorphic). Similarly, \(f_3 + if_1\) must be \(I_2\)-holomorphic, and \(f_1 + if_2\) must be \(I_3\)-holomorphic. This triple of conditions is equivalent to \(I_1df_1 = I_2df_2 = I_3df_3\). Conversely, if the latter condition holds, then, for any complex structure, the \((2, 0)\)-part of \(\Pi\) is holomorphic. The \((2, 0)\)-part is also Poisson, since \(\dim_{\mathbb{C}} M = 2\). For the same reason \(\Pi\) is compatible with the hyperkähler structure. \[\square\]

**Remark 3.8.** This result can be, of course, also proved by describing holomorphic sections of the line bundle \(\Lambda^2 T_{\mathbb{C}} Z\).

**Corollary 3.9.** A 4-dimensional hyperkähler manifold \(M\) admits a hyper-Poisson bivector, other than a constant linear combination of \(\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}\), if and only if \(M\) admits a non-zero tri-Hamiltonian vector field. \[\square\]

Let \(M^{4d}\) be a hyperkähler manifold equipped with a compatible hyper-Poisson bivector \(\Pi\). For each complex structure \(I_\zeta, \zeta \in \mathbb{P}^1\), we have the compatible holomorphic Poisson bivectors \(\Pi_1 = \Omega^{-1}_{\zeta}\) and \(\Pi_2 = \Pi_{\zeta, 0}\). The Pfaffian polynomials of the corresponding recursion operators combine to define a polynomial \(p(\zeta, \lambda)\) of the form
\[
p(\zeta, \eta) = \lambda^d + p_1(\zeta)\lambda^{d-1} + \cdots + p_d(\zeta),
\]
where the degree of \(p_i\) is \(2i\). In particular \(p_1(\zeta)\) is a quadratic polynomial, compatible with the real structure of \(|O(2)|\), and hence \(p_1\) is the hyperkähler moment map for a tri-Hamiltonian vector field \(X_\Pi\). We shall call \(X_\Pi\) the *canonical Killing vector field*.

### 3.3. Hyperkähler transverse schemes.

Let \(M\) be a 4-dimensional hyperkähler manifold with a non-trivial tri-Hamiltonian Killing vector field. The moment map induces a holomorphic map \(\mu\) from the twistor space \(Z\) of \(M\) to \(|O(2)|\). Following Atiyah and Hitchin we can perform the transverse Hilbert scheme construction on fibres of \(Z \to \mathbb{P}^1\) and obtain a new twistor space \(Z_\mu^{[d]}\). Sections of \(Z_\mu^{[d]} \to \mathbb{P}^1\) are in 1–1-correspondence with 1-dimensional compact complex subspaces \(C\) of \(Z\) such that:

(i) the projection \(\pi : C \to \mathbb{P}^1\) is flat with fibres of length \(d\);

(ii) the projection \(\mu\) induces a scheme-theoretic isomorphism between \(C\) and its image in \(|O(2)|\).
Theorem 3.12. Let $M^d$ be a hyperkähler manifold, equipped with a compatible hyper-Poisson bivector $\Pi$, such that for every complex structure $I_\zeta$, the holomorphic Poisson bivectors $\Pi_1 = \Omega^{-1}_\zeta$ and $\Pi_2 = \Pi^{2,0}_\zeta$ satisfy the assumptions of Theorem 2.8. Then there exists a complex 3-fold $Z$ with properties listed in Remark 3.11 such that $M^d$ is locally isomorphic to $M_{\mu}^d$ as a hyper-Poisson hyperkähler manifold. In addition, the holomorphic map $\mu : Z \to |O(2)|$ is a submersion.
Proof. We can perform the construction in the proof of Theorem 2.8 fibrewise on the twistor space of $M^{4d}$ and obtain $Z$. Its properties follow easily. □

Remark 3.13. This theorem remains true for pseudo-hyperkähler $M^{4d}$. We do not know whether the induced metric on a hyperkähler transverse Hilbert scheme is always positive definite.

Remark 3.14. The canonical Killing field $X_H$ is transverse to the foliation defined the proof of Theorem 2.8 (on each fibre of the twistor space). Therefore the vertical vector field on $Z$, which gives the projection to $|O(2)|$ is induced by $X_H$. In particular, if $X_H$ integrates to an action of $R$ or $S^1$ on $M^{4d}$ and the resulting 3-fold $Z$ admits sections, then $Z$ is the twistor of a hyperkähler 4-manifold with a tri-Hamiltonian action of $R$ or $S^1$.

Remark 3.15. One can can consider, more generally, hyperkähler transverse Hilbert schemes on 4-manifolds $M$, the twistor space of which maps to $|O(2)|$, $r > 1$, rather than to $|O(2)|$. The corresponding object arising on $M^{4d}$ is then a section $\Pi$ of $\Lambda^2 E \otimes S^{2r} H$, satisfying $D\Pi = 0$. If we view $\Pi$ as a $\Lambda^2 E$-valued polynomial $\Pi(\zeta)$ of degree $2r$, then, for each $\zeta \in \mathbb{P}^1$, $\Pi(\zeta)$ defines an $I_{\zeta}$-holomorphic Poisson bivector on $M^{4d}$, compatible with $\Omega_{\zeta}^{-1}$. Theorem 3.12 remains true, as do the results of the next subsection (with obvious modifications).

3.4. Linear geometry of quaternionic bivectors. Let $V$ be a real vector space of dimension $4n$, equipped with the standard flat quaternionic structure $(g, I_1, I_2, I_3)$. We denote the corresponding (linear) symplectic forms by $\omega_1, \omega_2, \omega_3$. The complexification $V^C$ decomposes as $E \otimes H$, where $E$ and $H$ have complex dimensions $2n$ and $n$, respectively, and are equipped with the standard quaternionic-Hermitian structure, i.e. complex symplectic forms $\omega_E, \omega_H$ and quaternionic structures $\sigma_E, \sigma_H$, so that $\omega_E(\sigma_E(x), \sigma_E(x)) > 0$ and similarly for $H$.

Let $\Pi$ be a bivector in $\Lambda^2 V$ belonging to $\Lambda^2 E \otimes S^2 H$. We define an endomorphism $A : V \to V$ by $\#_\Pi \circ \#^{-1}_g$. Since $\Pi \in \Lambda^2 E \otimes S^2 H$, it is an eigenvector of the operator $\sum_{i=1}^3 I_i \otimes I_i$ with eigenvalue $-1$. Thus $A$ is an eigenvector of $\sum_{i=1}^3 I_i \otimes I_i$ on $V^* \otimes V$ with eigenvalue 1. Haydys [14] calls such endomorphisms \textit{aquaternionic} and shows that they are of the form $A = I_1 A_1 + I_2 A_2 + I_3 A_3$, where each $A_i$ is quaternionic, i.e. it commutes with $I_1, I_2, I_3$. Since $\Pi$ is antisymmetric and real, the $A_i$ are quaternionic matrices which are quaternion-Hermitian, i.e. $A^\dagger_i = A_i$, where $\dagger$ denotes the quaternionic adjoint. In terms of the symplectic structures (cf. Example 3.10).

$$\Pi(\alpha, \cdot) = A_1 \omega_1^{-1}(\alpha, \cdot) + A_2 \omega_2^{-1}(\alpha, \cdot) + A_3 \omega_3^{-1}(\alpha, \cdot).$$

It follows that the $(2, 0)$ component of $\Pi$ for the complex structure $I_1$ is

$$\Pi^{2,0}(\alpha, \cdot) = (A_2 + iA_3)(\omega_2 + i\omega_3)^{-1}(\alpha, \cdot),$$

and similarly for other complex structures. Viewing the $A_i$ as endomorphisms of $E$ (which corresponds to the canonical homomorphism $\mathfrak{gl}(n, \mathbb{H}) \to \mathfrak{gl}(2n, \mathbb{C})$) we obtain a quadratic endomorphism

$$A(\zeta) = (A_2 + iA_3) + 2iA_1 \zeta - (A_2 - iA_3)\zeta^2$$

of $E$, where each $A_i \in \mathfrak{gl}(2n, \mathbb{C})$ is symmetric with respect to the symplectic form $\omega_E$. We can consider the sheaf morphism $\eta = A(\zeta) : E \otimes O(-2) \to E$ on $TP^1$. Its cokernel, which we denote by $F$, is a 1-dimensional sheaf. Since each $A(\zeta)$ is
symmetric with respect to the standard symplectic form on $\mathbb{C}^{2n}$, the characteristic polynomial of $A(\zeta)$ is of the form $p(\zeta, \eta)^2$, where $p(\zeta, \eta)$ is a polynomial of degree $n$ in $\eta$. We call the scheme $C = \{(\zeta, \eta); p(\zeta, \eta) = 0\}$ as the spectral curve of the bivector $\Pi$ and view the sheaf $\mathcal{F}$ as being supported on $C$. If $\mathcal{F}$ is smooth, then $\mathcal{F}$ is a rank 2 vector bundle with $\det \mathcal{F} \cong K_C(2)$ (cf. [13], which contains more results on vector bundles arising this way). If the bivector $\Pi$ arises via the hyperkähler transverse Hilbert scheme construction from a 3-dimensional twistor space $Z$, then the spectral curve $C$ is precisely the curve in $Z$ corresponding to a point in $M^{[n]}_\mu$ and the sheaf $\mathcal{F}$ is isomorphic to $\mathcal{N}_{C/Z}(-1)$.

Remark 3.16. One can associate to $\Pi$ another spectral object. Since the $A_i$ are quaternion-Hermitian matrices, they are diagonalisable over $\mathbb{H}$ with real eigenvalues. Let us denote the product of eigenvalues of a quaternion-Hermitian matrix $X$ by $\det_X$. We can define a surface in $\mathbb{H}P^3$ as

$$S = \left\{ [x_0, x_1, x_2, x_3] \in \mathbb{H}P^3; \det_{\mathbb{H}}(x_0 - x_1A_1 - x_2A_2 - x_3A_3) = 0 \right\}.$$ 

It is a (ramified) $n$-fold of $\mathbb{R}P^2$ (via $[x_0, x_1, x_2, x_3] \to [x_1, x_2, x_3]$), and its complexification is a surface $S$ in $\mathbb{C}P^3$ which can be defined as

$$S = \left\{ [z_0, z_1, z_2, z_3] \in \mathbb{C}P^3; \det(z_0 - z_1A_1 - z_2A_2 - z_3A_3) = 0 \right\},$$

where the $A_i$ are now complex $2n \times 2n$ matrices via the homomorphism $\mathfrak{gl}(n, \mathbb{H}) \rightarrow \mathfrak{gl}(2n, \mathbb{C})$. The intersection of $S$ with the quadratic cone

$$x_1 = 2i\zeta, \quad x_2 = 1 - \zeta^2, \quad x_3 = i(1 + \zeta^2)$$

is the doubled spectral curve $C$. The sheaf $\mathcal{F}$ extends to $S$ and is defined as the cokernel of

$$z_0 - z_1A_1 - z_2A_2 - z_3A_3 : E \otimes \mathcal{O}(-2) \rightarrow E$$

on $\mathbb{P}^3$. At present we do not understand the significance of $S_R$ (as opposed to $C$) for the geometry of hyper-Poisson manifolds.

4. The hyper-Poisson bivector of the monopole moduli space

We consider the moduli space $\mathcal{M}_k$ of $SU(2)$-monopoles of charge $k$, described as the moduli space of $u(k)$-valued solutions of Nahm’s equations on $(0, 2)$, with simple poles at $t = 0, 2$ and residues defining the standard irreducible representation of $u(2)$. The Nahm equations are $\dot{T}_i = [T_i, T_0] + [T_2, T_3]$ and two further equations, obtained by cyclic permutations of indices 1, 2, 3. The tangent space at $[T_0, T_1, T_2, T_3]$ is given by quadruples of smooth maps $(t_0, t_1, t_2, t_3)$ from $[0, 2]$ to $u(k)$ satisfying the equations

$$\dot{t}_0 = [t_0, T_0] + [t_1, T_1] + [t_2, T_2] + [t_3, T_3]$$
$$\dot{t}_1 = [T_1, t_0] + [t_1, T_0] + [T_2, t_3] + [t_2, T_3]$$
$$\dot{t}_2 = [T_2, t_0] + [t_2, T_0] + [T_3, t_1] + [t_3, T_1]$$
$$\dot{t}_3 = [T_3, t_0] + [t_3, T_0] + [T_1, t_2] + [t_1, T_2].$$

The hypercomplex structure is given by the right multiplication by quaternions on $t_0 + t_1i + t_2j + t_3k$ and the Riemannian metric $g$ is

$$\| (t_0, t_1, t_2, t_3) \|^2 = -\int_0^2 \text{tr}(t_0^2 + t_1^2 + t_2^2 + t_3^2).$$
As explained by Atiyah and Hitchin in [11 Ch.6], the hyperkähler manifold $\mathcal{M}_k$ is the hyperkähler transverse Hilbert scheme associated to $S^1 \times \mathbb{R}^3$. Thus, according to [8,3] it possesses a natural hyper-Poisson bivector compatible with the hyperkähler structure. We can identify this bivector as follows:

**Theorem 4.1.** The natural hyper-Poisson bivector $\Pi$ on the moduli space $\mathcal{M}_k$ is given by

$$\#_g^{-1}\Pi = -\frac{i}{4} \int_0^2 \text{tr}\left( \sum_{i=1}^3 T_i (dT_i \wedge dT_0 - dT_0 \wedge dT_i) + \sum_{i,j,k=1}^3 \epsilon_{ijk} T_i T_j \wedge dT_k \right),$$

where $(\phi \wedge \psi)(a, b) = \phi(a)\psi(b) - \phi(b)\psi(a)$.

**Remark 4.2.** It will follow from the proof that the integral is finite.

**Proof.** We first compute the Kähler forms $\omega_2$ and $\omega_3$ corresponding to the complex structures $J$ and $K$:

$$\omega_2 = -\int_0^2 \text{tr}\left( dT_0 \wedge dT_2 + dT_1 \wedge dT_3 \right), \quad \omega_3 = -\int_0^2 \text{tr}\left( dT_0 \wedge dT_3 + dT_2 \wedge dT_1 \right).$$

The $I$-holomorphic 2-form $\omega_2 + i\omega_3$ is therefore given by

$$\omega_2 + i\omega_3 = -\int_0^2 \text{tr}(dT_0 - iT_1) \wedge d(T_2 + iT_3).$$

We can now rewrite the integrand in the formula for $\#_g^{-1}\Pi$ as follows:

$$\frac{1}{2} \text{tr}\left( (T_2 + iT_3)(\Phi_2 - i\Phi_3) + 2T_1(dT_1 \wedge dT_0 - dT_0 \wedge dT_1 + dT_2 \wedge dT_3 - dT_3 \wedge dT_2) + (T_2 - iT_3)(\Phi_2 + i\Phi_3) \right),$$

where

$$\Phi_2 = dT_2 \wedge dT_0 - dT_0 \wedge dT_2 + dT_3 \wedge dT_1 - dT_1 \wedge dT_3,$$

$$\Phi_3 = dT_3 \wedge dT_0 - dT_0 \wedge dT_3 + dT_1 \wedge dT_2 - dT_2 \wedge dT_1.$$

Observe that the first summand is of type $(0, 2)$ for the complex structure $I$, the second one of type $(1, 1)$, and the third one of type $(2, 0)$. Since $\#_g$ exchanges $(2, 0)$ and $(0, 2)$, we conclude that

$$\#_g^{-1}\Pi^{2,0} = -\frac{i}{8} \int_0^2 \text{tr}(T_2 + iT_3)(\Phi_2 - i\Phi_3) =$$

$$= -\frac{i}{8} \int_0^2 \text{tr}(T_2 + iT_3)(dT_2 - iT_3) \wedge d(T_0 + iT_1) - d(T_0 + iT_1) \wedge d(T_2 - iT_3)).$$

We now observe that for any bivector $\pi$ and $\Omega = \omega_2 + i\omega_3$:

$$\left( #_g^{-1}\pi \right)(u, v) = \left( #_g^{-1}\pi \right)(Ju + iKu, Jv + iKv).$$

Computing this for $\pi = \Pi^{2,0}$ we obtain:

$$\#_g^{-1}\Pi^{2,0} = -\frac{i}{2} \int_0^2 \text{tr}(T_2 + iT_3)(dT_2 + iT_3) \wedge d(T_0 - iT_1) - d(T_0 - iT_1) \wedge d(T_2 + iT_3)).$$
Thus, if we set $\beta = T_2 + iT_3$ and $\alpha = T_0 - iT_1$, we obtain

$$i\#^{-1}_\Omega \Pi^{2,0} = \frac{1}{2} \int_0^2 \text{tr} \, d(\beta^2) \wedge d\alpha.$$  

The complex Nahm equation is the Lax equation $\dot{\beta} = [\beta, \alpha]$. It follows that $(\beta^2, \alpha)$ also satisfies the Lax equation. Acting by a singular complex gauge transformation the formula (4.1) implies that

$$i\#^{-1}_\Omega \Pi^{2,0} = \frac{1}{2} \sum_j d(\beta_j)^2 \wedge \frac{dp_j}{\beta_j} = \sum_j \beta_j d\beta_j \wedge \frac{dp_j}{\beta_j},$$

where $\beta_j$ are the poles and $p_j$ the values of the numerator of the rational map corresponding to the given monopole (and the complex structure $I$). This means that $\Pi^{2,0}$ is precisely the holomorphic bivector obtained from the transverse Hilbert scheme construction applied to $\mathbb{C} \times \mathbb{C}^*$ with the symplectic form $d\beta \wedge \frac{dp}{\beta}$.

Observe now that, since $\#_g^{-1} \Pi^{2,0} = (\#_g^{-1} \Pi)^{0,2}$ and $Ju + iKu = (Ju) + iI(Ju)$, the formula (4.1) implies that

$$(\#_g^{-1} \Pi)^{0,2}(u, v) = \#_\Omega^{-1} \Pi^{2,0}(-Ju, -Ju).$$

Therefore the integral defining $(\#_g^{-1} \Pi)^{0,2}$ is finite. Similarly, the integral defining $(\#_g^{-1} \Pi)^{2,0}$ is finite. Since $(\#_g^{-1} \Pi)^{0,2} + (\#_g^{-1} \Pi)^{2,0}$ is the sum of all terms of the form $T_2 \cdot \phi$ and $T_3 \cdot \phi$ in the formula in the statement, repeating this decomposition for the complex structure $J$ or $K$ shows that the whole integral in the statement is finite. This also shows that

$$\Pi = \frac{1}{2} \left( \Pi^{2,0}_I + \Pi^{2,0}_{-I} + \Pi^{2,0}_J + \Pi^{2,0}_{-J} + \Pi^{2,0}_K + \Pi^{2,0}_{-K} \right),$$

and so, owing to Lemma 4.2 in [21], $\Pi \in \Gamma(\Lambda^2 E \otimes S^2 H)$. Finally, observe that $\Pi$ is real. \hfill $\Box$

**Remark 4.3.** The canonical Killing vector field $X_\Pi$ (see [32]) on the moduli space $\mathcal{M}_k$ is $(t_0, t_1, t_2, t_3) = (i, 0, 0, 0)$. We obtain

$$i(X_\Pi) \#_g^{-1} \Pi = -\frac{1}{2} \int_0^2 \text{tr} \sum_{i=1}^3 T_idT_i = -\frac{1}{4} dF,$$

where

$$F = \int_0^2 \left( \text{tr} \sum_{i=1}^3 T_i^2 + \frac{k(k^2 - 1)}{4} \left( s^{-2} + (s - 2)^{-2} \right) \right).$$

The function $F$ has been shown by Hitchin [15] to essentially give Kähler potentials of the monopole metric: for every complex structure, the sum of $F$ and some linear combination of the coefficients of the spectral curve is a Kähler potential for the corresponding Kähler form. Is this true on a general hyperkähler transverse Hilbert scheme $M_{\mu}^{[d]}$, i.e. is the 1-form $i(X_\Pi) \#_g^{-1} \Pi$ similarly related to Kähler potentials on $M_{\mu}^{[d]}$?
References

[1] M.F. Atiyah and N.J. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, Princeton (1988).

[2] A. Beauville, ‘Variétés Kähleriennes dont la première classe de Chern est nulle’, *J. Differential Geom.* 18 (1983), no. 4, 755–782.

[3] A. Beauville, ‘Determinantal hypersurfaces, *Michigan Math. J.* 48 (2000), 39–64.

[4] R. Bielawski, ‘Hyperkähler manifolds of curves in twistor spaces’, *SIGMA* 10 (2014).

[5] R. Bielawski, ‘Slices to sums of adjoint orbits, the Atiyah-Hitchin manifold, and Hilbert schemes of points’, *Complex Manifolds* 4 (2017), 16–36.

[6] R. Bielawski and L. Schwachhöfer, ‘Hypercomplex limits of pluricomplex structures and the Euclidean limit of hyperbolic monopoles’, *Ann. Global Anal. Geom.* 44 (2013), 245–256.

[7] F. Bottacin, ‘Poisson structures on Hilbert schemes of points of a surface and integrable systems’, *manuscripta math.* 97 (1998), 517–527.

[8] G.R. Cavalcanti, ‘Examples and counter-examples of log-symplectic manifolds’, *J. Topol.* 10 (2017), 1–21.

[9] R.L. Fernandes, ‘Completely integrable bi-Hamiltonian systems’, *J. Dyn. Diff. Equat.* 6 (1994), 53–69.

[10] R. Goto, ‘Rozansky-Witten invariants of log symplectic manifolds’, in *Integrable systems, topology, and physics (Tokyo, 2000)*, Contemp. Math., vol. 309, AMS, Providence, RI, 2002, 69–84.

[11] M. Guaitieri and S. Li, ‘Symplectic groupoids of log symplectic manifolds’, *IMRN* 2014 (11), 3022–3074, 2014.

[12] V. Guillemin, E. Miranda, and A. R. Pires, ‘Symplectic and Poisson geometry on b-manifolds’, *Adv. Math.* 264 (2014), 864–896.

[13] R. Hartshorne, *Deformation Theory*, Springer, New York, 2010.

[14] A. Haydys, ‘Nonlinear Dirac operator and quaternionic analysis’, *Commun. Math. Phys.* 281 (2008), 251–261.

[15] N.J. Hitchin, ‘Integrable systems in Riemannian geometry, in: *Surveys in differential geometry: integral systems*, 21–81, Int. Press, Boston, 1998.

[16] W. Hong and M. Stiénon, ‘From hypercomplex to holomorphic symplectic structures’, *J. Geom. Phys.* 96 (2015), 187–203.

[17] N. Lora Lamia Donin, ‘Transverse Hilbert schemes and completely integrable systems’, *Complex Manifolds* 4 (2017), 263–272.

[18] N. Lora Lamia Donin, ‘Hyperkähler manifolds of curves and 1-hypercomplex structures’, Ph.D. Thesis, Leibniz Universität Hannover, 2018.

[19] C. Manolescu, ‘Nilpotent slices, Hilbert schemes, and the Jones polynomial’, *Duke Math. J.* 132 (2006), 311–369.

[20] F. Magri and C. Morosi, ‘A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds’, *Quaderno S.* 19, Univ. of Milan, 1984.

[21] S.M. Salamon, ‘Differential geometry of quaternionic manifolds’, *Ann. Sci. Éc. Norm. Supér.* Serie 4, Volume 19 (1986), p. 31–55.

[22] R.S. Ward, ‘On self-dual gauge fields’, *Phys. Lett. A* 61 (1977), 81–82.

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