Separable Symmetric Tensors and Separable Anti-symmetric Tensors

Changqing Xu1 · Kaijie Xu2

Received: 19 May 2022 / Revised: 26 August 2022 / Accepted: 2 September 2022 / Published online: 24 October 2022 © Shanghai University 2022

Abstract
In this paper, we first initialize the S-product of tensors to unify the outer product, contrac-
tive product, and the inner product of tensors. Then, we introduce the separable symmetry
tensors and separable anti-symmetry tensors, which are defined, respectively, as the sum
and the algebraic sum of rank-one tensors generated by the tensor product of some vectors.
We offer a class of tensors to achieve the upper bound for \( \|A\| \leq 6 \) for all tensors of
size \( 3 \times 3 \times 3 \). We also show that each \( 3 \times 3 \times 3 \) anti-symmetric tensor is separable.

Keywords S-product · Invertible tensor · Separable symmetric tensor · Separable anti-
symmetric tensor

Mathematics Subject Classification 53A45 · 15A69

1 Introduction

A tensor is a multi-way array with multiple indexes. It can be applied in many fields such
as the theory of relativity [24], elasticity [1], magnetics and computer vision [19, 20], and
quantum algorithm [4, 17, 22, 23]. The tensor theory also plays an important role in data
analysis recently. Most of the concepts such as the determinant, rank, eigenvalue and eigen-
vector, and the inverse in matrix theory have been introduced into the tensor theory. Nev-
ertheless, characterizations of the terminology and the relevant computations in the tensor
form are still in the exploration. For instance, it is well known that the following are equiv-
alent conditions for a square matrix \( A \in \mathbb{R}^{n \times n} \) to be nonsingular, i.e., \( \det A \neq 0 \),

Changqing Xu
cqxurichard@mail.usts.edu.cn
Kaijie Xu
kjxu@xidian.edu.cn

1 School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou 215009,
Jiangsu, China

2 School of Electronic Engineering, Xidian University, Xi’an 710071, Shaanxi, China
\[
\det A \neq 0 \iff \text{rank}(A) = n;
\]
\[
\iff 0 \notin \pi(A);
\]
\[
\iff N(A) = \{0\};
\]
\[
\iff A \text{ is invertible};
\]
\[
\iff Ax = 0 \text{ has only the zero vector as its solution};
\]
\[
\iff \text{The columns(rows) of } A \text{'s are linearly independent}.
\]

Unfortunately, we cannot confirm any of these conditions when \( A \) is replaced by a tensor \( A \in \mathbb{T}_{m,n} \). Thus, there are still many obstacles for us to move from matrices to tensors. In this paper we will tackle some of the issues mentioned above in some special cases.

There are two different definitions for the tensor rank, i.e., the CP rank and the marginal rank. The CP rank of a tensor can be traced back to 1927 [7, 8]. Given an \( m \)-order tensor \( A \) of size \( I_1 \times \cdots \times I_m \). A CP decomposition of \( A \) is the form

\[
A = \sum_{j=1}^{R} \alpha_1^j \times \alpha_2^j \times \cdots \times \alpha_m^j,
\]

where all \( \alpha_i^j \in \mathbb{R}^{I_i} \) are nonzero. The smallest number \( R \) for (1) is called the CP rank of \( A \) and is denoted by \( \text{rank}(A) \) (see [2, 5]). For a matrix \( A \in \mathbb{C}^{m \times n} \), we have \( \text{rank}(A) \leq \min \{m, n\} \). But this is not true for a tensor: a tensor \( A \) of size \( 2 \times 2 \times 2 \) can have rank 3, and it is possible that \( \text{rank}(A) = 5 \) for a tensor \( A \in \mathbb{T}_{3,3} \) (see [13]). In this paper, we will present some tensor \( A \in \mathbb{T}_{3,3} \) with \( \text{rank}(A) = 6 \).

For convenience, we denote by \([n]\) the set \{1, 2, \ldots, n\} for any positive integer \( n \) and \( \mathbb{R} \) the field of real numbers. Throughout the paper, we use the tensorial notation, i.e., tensors of order 0 (scalars) are denoted by means of italic type letters \( a, b, x, y, \) and some Greek letters \( \lambda, \mu, \) etc., tensors of order 1 (vectors) by means of boldface italic letters \( x, y, z, \) and Greek letters \( \alpha, \beta, \gamma, \) tensors of order 2 (matrices) by capital boldface letters \( A, B, X, Y, M, \) and tensors of higher orders by curlicue letters \( A, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \ldots \).

Denote by \( \mathcal{T}_{m,n} \) the set of the \( m \)-th order \( n \)-dimensional real tensors with size \( n \times n \times \cdots \times n \). A tensor \( A \in \mathcal{T}_{m,n} \) is called symmetric if each entry \( a_{i_1,i_2,\ldots,i_m} \) is invariant under any permutation on its indices. Denote by \( \mathcal{S}_{m,n} \) the set of the \( m \)-th order \( n \)-dimensional symmetric tensors. Then, each \( A \in \mathcal{S}_{m,n} \) is associated uniquely with the homogeneous polynomial

\[
f_A(x) := A x^m = \sum_{i_1,i_2,\ldots,i_m} a_{i_1,i_2,\ldots,i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.
\]

Here we use \( a_{i_1,\ldots,i_m} \) or \( a_{i_1} \) to denote an element of \( A \). A symmetric tensor \( A \in \mathcal{S}_{m,n} \) is called positive definite (positive semidefinite or PSD) if \( f_A(x) > 0 \) for all \( x \in \mathbb{R}^n \). It is easy to see that an odd-order PSD tensor must be zero-tensor. Thus, we assume that \( m \) is even in the follows. For detail on PSD tensors we refer to [3, 15, 16].

Let \( m = 2k \) where \( k \) is a positive integer. The identity tensor \( I = (\delta_{ij}) \in \mathcal{T}_{m,n} \) is defined by

\[
\delta_{i_1,i_2,\ldots,i_{j_1},\ldots,j_k} = \delta_{i_1,j_1} \delta_{i_2,j_2} \cdots \delta_{i_j,j_k}
\]

for \( \sigma := (i_1, i_2, \ldots, i_m) \in S(m,n) \) with \( m = 2k, \delta_{ij} = 0, 1, \) and \( \delta = 1 \) if \( i = j \). Recall that the \( k \)-mode multiplication of an \( m \)-order tensor \( A \) with a matrix \( M \in \mathbb{R}^{n \times q} \) is defined as
Equation (4) generalizes the matrix multiplication since we have $A \times_2 B = AB$ for $A, B \in \mathbb{R}^{m \times n}$ (note the difference between (4) and that in [11, 12]). Note that for a vector $M = v \in \mathbb{R}^n$ (4) yields a tensor of order $m - 1$, thus the product is also called a contractive product. The t-product of tensors is defined between two 3-order tensors [10].

Let $S_1 = \{s_1, s_2, \cdots, s_p\}, S_2 = \{t_1, t_2, \cdots, t_q\}$ be subsets of $[p + q]$ with $S_1 \cup S_2 = [p + q]$, where $p, q$ are positive integers. The S-product of tensors $A \in T_{p,n}$ and $B \in T_{q,n}$, denoted by $A \boxtimes_S B$, is defined by

$$
(A \boxtimes_S B)_{i_1 i_2 \cdots i_m} = \sum_{k_1, k_2, \cdots, k_t} a_{i_1 i_2 \cdots i_p} b_{j_1 j_2 \cdots j_q},
$$

(5)

where $m = p + q - 2r$, $r = |S_1 \cap S_2|$, and

$$
\{i_1, i_2, \cdots, i_p\} \subseteq S_1, \quad \{j_1, j_2, \cdots, j_q\} \subseteq S_2,
$$

and the sum in the right-hand side of (5) is taken over all subscripts in $S := \{k_1, k_2, \cdots, k_t\} = S_1 \cap S_2$. The S-product can be contractive or extensive, depending on $r$ as in the following.

(a) $r > 0$ (1 $\leq r \leq \min \{p, q\}$), i.e., $S = S_1 \cap S_2$ is nonempty. $A \boxtimes_S B$ is a tensor of order $p + q - 2r$ by (5).

(b) $r = 0$, i.e., there is no intersection between $S_1$ and $S_2$. Then, $S_1 \cup S_2 = [p + q]$, $A \boxtimes_S B$ is a tensor of order $p + q$.

If $S_2 \subseteq S_1$, then $S = S_2$, $A \boxtimes_S B$ is contractive along mode-$S_2$, yielding $AB := A \boxtimes_S B$ of order $m = p - q$. A special case is the inner product $AB = (A, B)$ when $p = q$ or equivalently $S_1 = S_2$. On the other hand, the S-product of tensors in case (b) is exactly the outer (extensive) product of tensors. The S-product can be recursively employed to yield a higher order or a lower order tensor, depending on what we need. An extreme case is a rank-1 symmetric tensor $x^n$ generated from the power of a vector $x$ in the sense of S-product of $x$. Thus, the S-product unifies all possible multiplications of tensors, including the familiar contractive and the outer (tensor) product of tensors.

The separable symmetric tensors can be used to characterize the factorizations of polynomials. In fact, it is easy to see from (2) that an $m$-order $n$-variate polynomial $f(x)$ can be decomposed into $m$ linear forms (equivalently $f$ has $m$ distinct roots) if and only if it is associated with a tensor $A \in T_{m,n}$ which is a separable symmetric tensor. On the other hand, a separable anti-symmetric tensor is closely related to the Grassmannian algebra [25]. It can also be applied in computer vision to unify and computer the multi-focal tensors (see e.g., [6]). The first author also employs the separable symmetric tensor to compute the different hyperplanes in the generalized principal component analysis (GPCA) [26]. The GPCA is an algebraic-geometric approach proposed by Vidal in 2003 [21] to model mixtures of subspaces with a unique global solution to the clustering of the data points based on the polynomial decomposition. By the homogeneous coordinate expression, the approach takes the mixture of subspaces as a projective algebraic variety which is estimated from sample data points to derive the embedding of the data analytically. The estimation of the $n$ subspaces can be transformed into that of the algebraic variety defined by a set of polynomials.
Moreover, the problem of identifying a collection of hyperplanes boils down to the estimation and the factorization of a polynomial $p_n(x)$ into a product of $n$ linear factors.

In the following sections, we present some basic properties of the identity tensors. Also introduced are the separable symmetric tensors and the separable anti-symmetric tensors. Our result on separable tensors presents a class of $3 \times 3 \times 3$ tensors that satisfy $\text{rank}(A) = 6$.

## 2 Invertibility of a Hypercubic Tensor with an Even-Order

Recall that the identity matrix $I_n$ in matrix space $\mathbb{C}^{n \times n}$ satisfies $I_n X = X I_n = X$ for all $X \in \mathbb{C}^{m \times n}$. This is also true in the tensor case.

**Lemma 1** For an even number $m = 2k$, the identity tensor $I \in T_{m,n}$ satisfies

$$I A = A I = A, \quad \forall A \in T_{m,n}. \quad (6)$$

**Proof** For any $\sigma = (i_1, i_2, \ldots, i_m) \in S(m, n)$, we have

$$(A I)_{i_1 i_2 \cdots i_m} = \sum_{j_1, j_2, \ldots, j_k} a_{i_1 i_2 \cdots j_k} \delta_{j_1 j_2 \cdots j_k i_{k+1} i_{k+2} \cdots i_m}$$

$$= \sum_{j_1, j_2, \ldots, j_k} a_{i_1 i_2 \cdots j_k} \delta_{j_1 i_{k+1}} \delta_{j_2 i_{k+2}} \cdots \delta_{j_k i_m}$$

$$= a_{i_1 i_2 \cdots i_{k+1} i_{k+2} \cdots i_m}.$$ 

Thus, $I A = A$. Similarly we can prove $A I = A$.

The identity tensor $I \in T_{m,n}$ is actually the $S$-power of the identity matrix $I_n$ in the sense of the outer-product:

$$I = I_n \boxtimes S_2 I_n \boxtimes S_3 \cdots \boxtimes S_k I_n,$$

where $S_i \equiv \{ i, k+i \} (i = 1, 2, \ldots, k)$. Therefore, we can also write $I = I_n^{[k]}$. An even-order tensor $A \in T_{m,n}$ is said to be invertible if there exists a tensor $B \in T_{m,n}$ such that

$$BA = AB = I. \quad (7)$$

$B$ is called the inverse of $A$ and is denoted by $A^{-1}$. The invertibility of an even-order tensor can be transferred to that of a square matrix by tensor matricization (or unfolding). An unfolding of a tensor $A$ is a process through which $A$ is rearranged into a matrix [12]. A tensor $A \in T_{m,n}$ can be unfolded into an $n^k \times n^k$ matrix if $m = 2k$. A normal unfolding yields a matrix $A = (a_{ij})$ whose entries are defined as $a_{ij} = a_{i_1 i_2 \cdots i_{k+r} i_{k+r+1} i_{k+r+2} \cdots i_{2k}}$ where

$$i = 1 + \sum_{r=1}^{k} (i_r - 1)n^{k-r}, \quad j = 1 + \sum_{r=1}^{k} (i_{k+r} - 1)n^{k-r}. \quad (8)$$

We call the matrix $A$ obtained by the normal unfolding a normal square or NS matrix of $A$. We have
\textbf{Theorem 1} Let $A, B \in \mathcal{T}_{m,n}$ where $m = 2k$ is an even number, and $A, B$ are, respectively, the NS matrices of $A$ and $B$. Then, $A$ is invertible if and only if $A$ is invertible. Furthermore, $B = A^{-1}$ if and only if $B = A^{-1}$.

\textbf{Proof} We take $m = 4$ for our convenience in notations. The argument in general case (i.e., $m = 2k$) follows the same route. Write $A = (a_{ij}), B = (b_{ij})$. Then, our result follows by

\[(AB)_{i_1i_2i_3i_4} = \sum_{j_1j_2} a_{i_1j_1j_2} b_{j_1j_2i_3i_4} = \sum_{j_1j_2} a_{i_1+(i_1-1)n,j_2+(i_2-1)n} b_{j_1+(i_1-1)n,j_2+(i_2-1)n} = \sum_{j=1}^{n^2} a_{i_1b_{j_1}} = (AB)_{ij},\]

where $\sigma := (i_1, i_2, i_3, i_4) \in S(4; n)$ and $i = i_2 + (i_1 - 1)n, j = i_4 + (i_3 - 1)n$.

\textbf{Corollary 1} Let $A \in \mathcal{T}_{m,n}$ where $m = 2k$ is an even number. Then, $A$ is invertible if and only if $\det A \neq 0$ where $A$ is the NS matrix of $A$.

\textbf{Corollary 2} Let $A \in \mathcal{T}_{m,n}$ where $m = 2k$ is an even number. Then, $A$ is invertible if and only if there is a tensor $B \in \mathcal{T}_{m,n}$ such that $AB = I$. Furthermore, the inverse of $A$ is unique.

The spectrum theory of tensors was independently introduced by Qi [15] and Lim [14] in 2005, and investigated by Qi [15], Lim [14], and Hu et al. [9].

Let $A = (a_{i_1i_2\cdots i_m}) \in \mathcal{T}_{m,n}$ and $0 \neq x \in \mathbb{C}^n$. Then, the product $A x^{m-1} \in \mathbb{C}^n$ is a vector. For any number $\lambda$, if there exists a nonzero vector $u \in \mathbb{C}^n$ such that

\[A u^{m-1} = \lambda u^{[m-1]}, \tag{9}\]

where $u^{k}$ is a $k$-order $n$-dimensional rank-1 tensor generated by $u$, $u^{[k]} \in \mathbb{C}^n$ is a vector whose $i$th coordinate is defined as $u^{[k]}_i$ where $u = (u_1, u_2, \cdots, u_n)^T$. We call $(\lambda, u)$ an eigenpair of $A$ in which $\lambda$ is called an eigenvalue of $A$ and $u$ is called an eigenvector associated with $\lambda$. The pair $(\lambda, u)$ is called an H-eigenpair if $u$ is a real vector, which is called an H-eigenvector. Note that $\lambda$ is also a real number in this case and is called an H-eigenvalue of $A$ if $A$ is real. It is shown by Qi [15] that a tensor $A \in \mathcal{T}_{m,n}$ (for an even $m$) is PSD if and only if all of its $H$-($Z$)-eigenvalues are nonnegative (see e.g., [18]).

### 3 Symmetric Tensors and Anti-symmetric Tensors

Let $n > 1$ be a positive integer and $m = 2k > 0$ be an even number. Denote by $\mathcal{P}_m$ the set of all permutations on set $[m]$. For any tensor $A = (a_{i_1i_2\cdots i_m}) \in \mathcal{T}_{m,n}$ and any permutation $\sigma \in \mathcal{P}_m$, we define $\sigma(A)$ as the tensor $A^{(\sigma)} = (A_{i_1i_2\cdots i_m}^{(\sigma)})$ where

\[a_{i_1i_2\cdots i_m}^{(\sigma)} = a_{i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(m)}}, \quad \forall \sigma := (i_1, i_2, \cdots, i_m) \in S(m; n). \tag{10}\]

We call $A$ $\sigma$-symmetric if it satisfies $\sigma(A) = A$. $A$ is called $\sigma$-sign symmetric if $\sigma(A) = (-1)^{i(\sigma)} A$ where $i(\sigma)$ is the inverse number of $\sigma$. A tensor $A \in \mathcal{T}_{m,n}$ is called
anti-symmetric if $A$ is $\sigma$-sign symmetric for all $\sigma \in P_m$. A symmetric tensor is $\sigma$-symmetric for all $\sigma \in P_m$. Now we denote

$$ S = \frac{1}{\sqrt{m!}} \sum_{\sigma \in P_m} \sigma, \quad (11) $$

then $S : T_{m,n} \to T_{m,n}$ is a linear operator sending each tensor in $T_{m,n}$ into $S_{m,n}$ [3]. Note that

$$ \sigma \circ S = S \circ \sigma = S, \quad \forall \sigma \in P_m. \quad (12) $$

A tensor $A \in T_{m,n}$ is symmetric if and only if $A = S(A)$. Given a tensor $A \in T_{m,n}$. The symmetrization of $A$ is defined as tensor $S(A)$. Note that the polynomial associated with $A$ is the same as that with $S(A)$, which makes reasonable for us to assume the symmetry of tensors. For our convenience, we denote the set of $m$th order $n$-dimensional symmetric tensors by $S_{m,n}$.

A symmetric tensor $A \in S_{m,n}$ is called separable if

$$ A = S(u_1 \times u_2 \times \cdots \times u_m) \quad (13) $$

for some vectors $u_1, u_2, \cdots, u_m \in \mathbb{R}^n$. Some natural questions are: when is a symmetric tensor separable? can a symmetric tensor be decomposed into the sum of some separable tensors? what is the rank of a separable tensor?

**The separable symmetric tensors** Now we let $v_1, v_2, \cdots, v_m \in \mathbb{C}^n$ where each $v_j$ is a nonzero vector, and write

$$ \mathcal{L} := \frac{1}{\sqrt{m!}} \sum_{\sigma \in P_m} (-1)^{r(\sigma)} \sigma. $$

Then, $\mathcal{L}$ is a linear operator on $T_{m,n}$. We denote

$$ v_1 \wedge \cdots \wedge v_m = \mathcal{L}(v_1 \times v_2 \times \cdots \times v_m), \quad (14) $$

and

$$ v_1 \vee v_2 \vee \cdots \vee v_m = S(v_1 \times v_2 \times \cdots \times v_m). \quad (15) $$

For $m = 2$, the operator $\wedge$ produces an $n \times n$ anti-symmetric matrix of rank 2 when $v_1, v_2 \in \mathbb{C}^n$ are linearly independent ($n \geq 2$). In the following we will show that the tensor $v_1 \wedge \cdots \wedge v_m$ must be an anti-symmetric tensor in general case.

**Theorem 2** Let $u_j, v_j \in \mathbb{R}^n, j \in [m]$, and $A = (a_{ij}) \in \mathbb{R}^{mxm}$ with $a_{ij} = \langle u_i, v_j \rangle$ for all $i, j \in [n]$. Denote $A_j = \mathcal{L}(u_1 \times \cdots \times u_m), B_j = \mathcal{L}(v_1 \times \cdots \times v_m)$, and $A_s = S(u_1 \times \cdots \times u_m), B_s = S(v_1 \times \cdots \times v_m)$. Then, we have

$$ \langle A_j, B_j \rangle = \det(A), \quad (16) $$

and

$$ \langle A_s, B_s \rangle = \text{perm}(A), \quad (17) $$

where $\text{perm}(A)$ denotes the permanent of matrix $A$. 

 Springer
Proof For convenience, we denote $U := u_1 \times \cdots \times u_m$, $V := v_1 \times \cdots \times v_m$. We come to prove (16). By the definition

$$\langle A_j, B_j \rangle = \frac{1}{m!} \sum_\theta \sum_\kappa (-1)^{\theta(\kappa)} (-1)^{\theta(\kappa)} \langle \theta(U), \kappa(V) \rangle$$

$$= \frac{1}{m!} \sum_\theta \sum_\kappa (-1)^{\theta(\kappa-\kappa)} \langle U, \theta^{-1} \kappa(V) \rangle$$

$$= \sum_{\delta \in \mathcal{P}_m} (-1)^{\delta(\theta)} \langle U, \delta(V) \rangle$$

$$= \sum_{\delta \in \mathcal{P}_m} (-1)^{\delta(\theta)} \langle u_1, \delta(v_1) \rangle \langle u_2, \delta(v_2) \rangle \cdots \langle u_m, \delta(v_m) \rangle$$

$$= \sum_{\delta \in \mathcal{P}_m} (-1)^{\delta(\theta)} a_{1j_1} a_{2j_2} \cdots a_{mj_m}$$

$$= \det(A),$$

where $\delta := (j_1, j_2, \cdots, j_m) \in \mathcal{P}_m$ is any permutation of $[m]$ in the last second equation. The proof of formula (17) can be deduced similarly.

**Corollary 3** Let $U = [u_1, u_2, \cdots, u_m] \in \mathbb{C}^{n \times m}$ and $A = u_1 \wedge \cdots \wedge u_m$. Then, we have

$$\|A\| = \sigma_1 \sigma_2 \cdots \sigma_m,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$ denote the singular values of matrix $U$, and $\|A\|$ is the Frobenius norm of a tensor.

**Proof** Denote $A = U^* U$. Then, $A = (a_{ij})$ with $a_{ij} = \langle u_i, u_j \rangle$ for all $i, j$. Let $U = QDW^*$ be the singular value decomposition of $U$ with $Q \in \mathbb{C}^{n \times n}$, $W \in \mathbb{C}^{m \times m}$ be column orthogonal and $D = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_m)$ ($\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$, here assume that $m \leq n$). Then, $\det(U^* U) = \sigma_1^2 \sigma_2^2 \cdots \sigma_m^2$. By Theorem 2 we have

$$\|A\| = \sqrt{\langle A, A \rangle} = (\det A)^{1/2} = (\det(U^* U))^{1/2} = \sigma_1 \sigma_2 \cdots \sigma_m.$$

**Corollary 4** Let $u_1, u_2, \cdots, u_m \in \mathbb{C}^n$ and $A = u_1 \wedge \cdots \wedge u_m$. Then, $A = 0$ if and only if $u_1, u_2, \cdots, u_m$ are linearly dependent.

**Proof** By Corollary 3, we get $A = 0$ if and only if $\|A\| = 0$, $\text{rank}(A) < m$, and $u_1, u_2, \cdots, u_m$ are linearly dependent.

The following lemma implies that the wedging is a multilinear operation.

**Lemma 2** Let $j \in [m]$, $u_j, w_j, v_1, v_2, \cdots, v_m \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$ be a scalar. Then

(i) $v_1 \wedge \cdots \wedge (u_j + w_j) \wedge v_m = v_1 \wedge \cdots \wedge u_j \wedge v_1 \wedge \cdots \wedge w_j \wedge v_m$;

(ii) $v_1 \wedge \cdots \wedge (\lambda v_j) \wedge v_m = \lambda (v_1 \wedge \cdots \wedge v_j \wedge v_m)$.

Now we are ready to show that $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ is an anti-symmetric tensor.
Theorem 3 Let $\mathcal{A} = \mathcal{L}(v_1 \times \cdots \times v_m)$ where $v_1, v_2, \cdots, v_m \in \mathbb{R}^n$ with each $v_j \neq 0$. Then, $\mathcal{A}$ is anti-symmetric.

Proof For any given $\phi \in \mathcal{P}_m$, we want to show that $\mathcal{A}^{(\phi)} = (-1)^{\tau(\phi)} \mathcal{A}$. For this purpose, we let $\sigma = (i_1, i_2, \cdots, i_m) \in S(m; n)$. Then, we have

$$A_{i_1i_2\cdots i_m}^{(\phi)} = a_{i_1i_2\cdots i_m} = \frac{1}{\sqrt{m!}} \sum_{\sigma \in \mathcal{P}_m} (-1)^{\tau(\sigma)} v_{\sigma(1)} \times \cdots \times v_{\sigma(m)}$$

$$= (-1)^{\tau(\phi)} \frac{1}{\sqrt{m!}} \sum_{\theta \in \mathcal{P}_m} (-1)^{\tau(\theta)} v_{\theta(1)} \times \cdots \times v_{\theta(m)}$$

$$= (-1)^{\tau(\phi)} a_{i_1i_2\cdots i_m},$$

Here we denote $\sigma \phi = \theta$. Note that $\mathcal{P}_m \phi = \mathcal{P}_m$ for any $\phi \in \mathcal{P}_m$, and that $(-1)^{\tau(\alpha \beta)} = (-1)^{\tau(\alpha)}(-1)^{\tau(\beta)}$.

A tensor $\mathcal{A} \in \mathcal{T}_{m;n}$ is called separable anti-symmetric or SAS if there exist some vectors $v_1, v_2, \cdots, v_m \in \mathbb{R}^n$ such that $\mathcal{A} = v_1 \wedge \cdots \wedge v_m$. Note that $\mathcal{A} = 0$ if $v_j = 0$ for some $j$ or $v_i = v_j$ for some distinct $i, j$. Our next theorem shows that $\mathcal{A}$ is not zero only if $v_1, v_2, \cdots, v_m$ are linearly independent. Here $O$ stands for a zero tensor of appropriate size.

Let $v_1, v_2, \cdots, v_m$ be linearly dependent. Then, there exists a vector $v_j$ which can be expressed as a linear combination of the others. We assume w.l.o.g. that $v_m = \lambda_1 v_1 + \cdots + \lambda_{m-1} v_{m-1}$ where $\lambda_j \in \mathbb{R}$. Corollary 4 can also be deduced by Lemma 2 since

$$v_1 \wedge \cdots \wedge v_{m-1} \wedge v_m = v_1 \wedge \cdots \wedge v_{m-1} \wedge \left( \sum_{j=1}^{m-1} \lambda_j v_j \right)$$

$$= \sum_{j=1}^{m-1} \lambda_j v_1 \wedge \cdots \wedge v_{m-1} \wedge v_j$$

$$= O.$$

An $m$-order $n$-dimensional anti-symmetric tensor $v_1 \wedge \cdots \wedge v_m$ can be constructed recursively from vectors $v_1, v_2, \cdots, v_m$. For this purpose, we define

$$\mathcal{A} \bowtie u = \frac{1}{p} \sum_{j=1}^{p} \mathcal{A} \bowtie_j u,$$  \hspace{1cm} (19)

where $\mathcal{A} \in \mathcal{T}_{p-1;n}$, $u \in \mathbb{R}^n$, and the $S$-product of $\mathcal{A}$ and $u$ in (19) is defined with

$$S_1 = \{1, 2, \cdots, j-1, j+1, \cdots, p\}, S_2 = \{j\}.$$

We call the multiplication defined by (19) the bowtie product of $\mathcal{A}$ and vector $u$. The bow-tie product lift a $(p-1)$-order tensor to a $p$-order tensor.

Now we are ready to state the bowtie process:
Theorem 4  Let $A(1) = v_1$, $A(2) = v_1 \wedge v_2$, and $A(k) := v_1 \wedge v_2 \wedge \cdots \wedge v_k \in T_{k,n}$, $k = 2, 3, \cdots$. Then, the sequence $A(k)$ can be constructed through formula

$$A(p+1) = A(p) \bowtie v_{p+1}, \quad p = 2, 3, \cdots, m - 1. \quad (20)$$

The following example gives an expression for an SAS tensor of order 2, i.e., an anti-symmetric matrix, in the sense of separability.

Example 1  Let $n \geq 3$ be an integer and $A = \frac{1}{2} (u \times v - v \times u)$ where $u, v \in \mathbb{R}^n$ are linearly independent. Then,

$$A = \frac{1}{2} (uv^\top - vu^\top) \in \mathbb{R}^{n \times n}$$

is an anti-symmetric matrix with $\text{ran} k(A) = 2$. Note that $A$ is not invertible since $n \geq 3$.

It is not difficult to show by Example 1 that an anti-symmetric nonzero matrix $A$ is separable if and only if $\text{ran} k(A) = 2$, which implies that not all anti-symmetric tensors of order 2 (i.e., anti-symmetric matrices) are separable.

4 Invertibility of Separable Tensors

We denote by $\pi(A)$ for the spectrum of a tensor $A$. It is known that a real symmetric matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable with all eigenvalues being real numbers, and a real anti-symmetric matrix has zero as its unique real eigenvalue. We conjecture that this phenomenon is also true in the tensor case. In the following, we first consider the spectrum of separable symmetric tensors and the spectrum of a separable anti-symmetric tensor.

Let $A \in T_{m,n}$. Then, $A$ is called a separable symmetric tensor if there exist some vectors $v_1, \cdots, v_m$ such that

$$A = S(V) = \frac{1}{\sqrt{m!}} \sum_{\sigma \in \Pi_m} v_{\sigma(1)} \times v_{\sigma(2)} \times \cdots \times v_{\sigma(m)}, \quad (21)$$

where $V = v_1 \times v_2 \times \cdots \times v_m$. A separable tensor is either a separable symmetric tensor or a separable anti-symmetric tensor. In the following we write $v_1 \diamond v_2 \diamond \cdots \diamond v_m$ for $L(V)$ or $S(V)$. Now suppose that $1 \leq m < n$ and $A$ is a separable tensor, i.e.,

$$A = v_1 \diamond v_2 \diamond \cdots \diamond v_m. \quad (22)$$

A natural question is: is $A$ invertible or not? The next result tells us that $A$ is singular (not invertible) if $m < n$.

Lemma 3  Let $1 \leq m < n$ where $m = 2k$ is an even number and $A \in T_{m,n}$ be invertible. Then $0 \notin \pi(A)$.

Proof  If $0 \notin \pi(A)$, then there is a nonzero vector $x \in \mathbb{R}^n$ which is an eigenvector of $A$ corresponding to $\lambda = 0$, i.e.,
Suppose that $A$ is invertible, then there exists a tensor $B \in T_{m,n}$ such that $AB = BA = I$. By (23) we have

$$x^{m-1} = (BA)x^{m-1} = B(Ax^{m-1}) = 0.$$  

It follows that $x = 0$, a contradiction with our assumption. Thus, $0 \notin \pi(A)$.

**Theorem 5** Let $1 \leq m < n$ where $m = 2k$ is an even number and $A \in T_{m,n}$ be a separable tensor. Then, $A$ is not invertible.

**Proof** First we assume that $A \in T_{m,n}$ is separable symmetric. Then $A = S(V) = v_1 \vee v_2 \vee \cdots \vee v_m$ for some $v_j \in \mathbb{R}^n$, $j \in [m]$, where

$$V = v_1 \times v_2 \times \cdots \times v_m.$$  

Denote $V = \text{span}\{v_1, v_2, \cdots, v_m\}$, i.e., $V$ is the subspace of $\mathbb{R}^n$ spanned by $v_1, v_2, \cdots, v_m$. If $v_1, v_2, \cdots, v_m$ are linearly independent, then $\dim(V) = m < n$. Given any $x \in V^c$ where $V^c$ is the orthogonal complementary space of $V$, we have

$$Ax^{m-1} = S(V)x^{m-1} = S(\prod_{j=2}^m (v_j^T x)v_1).$$

and it follows that $Ax^{m-1} = S(0) = 0 \in \mathbb{R}^n$ since $x \in V^c$ is orthogonal to each $v_j$. Thus, $0 \notin \pi(A)$. The result is followed by Lemma 3.

Denote by $e_j$ the $j$th column vector of the identity matrix $I_n$ for $j \in [n]$. We denote

$$Q_n = \mathcal{L}(I_n) := \mathcal{L}(e_1 \times e_2 \times \cdots \times e_n).$$

Then, $Q = (Q_{i_1i_2\cdots i_n})$ has $n!$ nonzero entries where

$$Q_{i_1i_2\cdots i_n} = \begin{cases} 1, & \text{if } (i_1, \cdots, i_n) \in E_n; \\ -1, & \text{if } (i_1, \cdots, i_n) \in O_n; \\ 0, & \text{otherwise}, \end{cases}$$

where $E_n$ and $O_n$ denote, respectively, the set of even and odd permutations on $[n]$. For example, $Q_3$ has six nonzero elements

$$Q_{123} = Q_{231} = Q_{312} = -Q_{132} = -Q_{231} = -Q_{321} = 1.$$  

We call $Q_n$ the standard separable anti-symmetric tensor or SSAS tensor. The following theorem tells us that an $n$-order $n$-dimensional real SAS tensor is a scaled SSAS tensor.

**Theorem 6** Let $A \in T_{n,n}$ be an SAS tensor. Then, $A = \lambda Q$ for some $\lambda \in \mathbb{R}$.

**Proof** Since $A$ is an SAS tensor, we may assume that $A = \mathcal{L}(A_1 \times A_2 \times \cdots \times A_n)$ where $A = (a_{ij}) = [A_1, \cdots, A_n] \in \mathbb{R}^{n \times n}$. Thus, we have
\[ \mathcal{A} = \mathcal{L} \left( \sum_{i=1}^{n} a_{i1} e_i, \sum_{i=1}^{n} a_{i2} e_i, \ldots, \sum_{i=1}^{n} a_{i3} e_i \right) \]

\[ = \sum_{i_1,i_2,\ldots,i_n} a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \mathcal{L}(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \]

\[ = Q \sum_{\sigma \in \mathcal{P}_n} (-1)^{\tau(\sigma)} a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} = \det(\mathcal{A}) Q. \]

Thus, the result holds with \( \lambda = \det(\mathcal{A}) \).

The following theorem shows that each \( 3 \times 3 \times 3 \) anti-symmetric tensor is separable.

**Theorem 7** Let \( \mathcal{A} = (a_{ijk}) \in \mathbb{R}^{3 \times 3 \times 3} \) be anti-symmetric. Then, \( \mathcal{A} \) must be separable.

**Proof** If \( \mathcal{A} = \mathcal{O} \), then the statement is true. We suppose that \( \mathcal{A} \) is a nonzero tensor. By definition of an anti-symmetric tensor, we know that \( \mathcal{A} \) satisfies

\[ a_{123} = a_{231} = a_{312} = -a_{132} = -a_{231} = -a_{321}, \]

and that all other entries shall be zero since the repetitions allowed in their subscripts. Therefore we may assume that

\[ a_{123} = a_{231} = a_{312} = a, \quad a_{132} = a_{231} = a_{321} = -a, \]  \hspace{1cm} (26)

where \( a \in \mathbb{R} \). We may assume w.l.g. that \( a > 0 \). Let \( \mathcal{A} = (a_{ijk}) = [a_1, a_2, a_3] = a^{1/3} I_3 \) be the scalar matrix of size \( 3 \times 3 \). Write \( \mathcal{E} = (E_{ijk}) = a_1 \times a_2 \times a_3 \). Then, \( \mathcal{E} \) is a rank-one tensor whose unique nonzero entry is \( E_{123} = a_{11} a_{22} a_{33} = a \). Since \( \mathcal{L}(\mathcal{E}) \) is an anti-symmetric tensor by Theorem 3, its nonzero entries coincide with that in (26). Consequently, we have \( \mathcal{A} = \mathcal{L}(\mathcal{E}) \). This shows that \( \mathcal{A} \) is a separable anti-symmetric tensor.

We shall mention that for \( n \geq 4 \) not all (anti-)symmetric tensors in \( T_{3,n} \) are separable. The following example can be used to illustrate this point.

**Example 2** Let \( \mathcal{A} \in T_{3,4} \) with its nonzero elements listed as follows:

\[ a_{123} = a_{231} = a_{312} = 1, \quad a_{132} = a_{213} = a_{321} = -1; \]  \hspace{1cm} (27)

\[ a_{124} = a_{241} = a_{412} = 2, \quad a_{142} = a_{214} = a_{421} = -2; \]  \hspace{1cm} (28)

\[ a_{134} = a_{341} = a_{413} = 3, \quad a_{143} = a_{314} = a_{431} = -3; \]  \hspace{1cm} (29)

\[ a_{234} = a_{342} = a_{423} = 1, \quad a_{243} = a_{324} = a_{432} = -1. \]  \hspace{1cm} (30)

It is easy to check that this tensor \( \mathcal{A} \) is anti-symmetric. We now show that \( \mathcal{A} \) is not separable. Let \( \mathcal{A} = \mathcal{L}(\alpha \times \beta \times \gamma) \) for some \( \mathcal{A} = [\alpha, \beta, \gamma] \in \mathbb{R}^{4 \times 3} \). Then, by Theorem 7 the \( 3 \times 3 \times 3 \) leading principal sub-tensor \( \mathcal{A}_k \) (obtained by removing the \( k \)-th layer of each mode) is separable. Furthermore, we have by Theorem 6 \( \mathcal{A}_k = \mathcal{L}(\alpha_k \times \beta_k \times \gamma_k) \) where \( \alpha_k, \beta_k, \gamma_k \in \mathbb{R}^3 \) are obtained, respectively, by removing the \( k \)-th coordinate of \( \alpha, \beta, \gamma \). Thus, by the proof of Theorem 7, we get \( \mathcal{A}[2 : 4, :] = I_3 \) by (27). Similarly, we get \( \mathcal{A}[1 : 3, :] = I_3 \), which is conflicted. Thus, \( \mathcal{A} \) cannot be separable.
5 Commutation Tensors and the Rank of Separable Tensors

In order to study the rank of a separable tensor, we introduce the definition of the commutation tensors. Recall that a commutation tensor $K_{p,q}$ is a 4-order (0,1)-tensor $K$ of size $p \times q \times q \times p$, defined in [27]

$$K_{i_1,i_2,i_3,i_4} = 1 \iff i_1 = i_4, i_2 = i_3.$$  \hspace{1cm} (31)

It is shown in [27] that

**Proposition 1** For all $x \in \mathbb{R}^q, y \in \mathbb{R}^p$, we have

$$\mathcal{K}(x \times y) = y \times x,$$  \hspace{1cm} (32)

where the multiplication $K \cdot A$ follows the rule of the contractive product.

The commutation tensor plays a role analog to that of a permutation matrix. Now we extend this definition to a general even order case. For any positive integer $m > 1$ and any given permutation $\sigma \in \mathcal{P}_m$, we define the permutation tensor $K^{(\sigma)}$ as a $2m$-order (0,1)-tensor defined by

$$K^{(\sigma)}_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = 1 \iff i_k = j_{\sigma(k)}, \forall k \in [m].$$  \hspace{1cm} (33)

For $m = 2$, there are two permutations on set {1, 2}, i.e.,

(i) the identity permutation $\sigma = (1)(2)$, in which case $K$ is exactly the identity tensor $I$ of order 4;
(ii) $\sigma = (12)$, in which case $K$ is just the commutation tensor we just mentioned.

When $m = 3$, there are $3! = 6$ permutations on [3]. For any permutation $\sigma \in \{1, 2, 3\}$, $K^{(\sigma)}$ is a 6-order (0,1)-tensor with entry

$$K^{(\sigma)}_{i_1,i_2,i_3,j_1,j_2,j_3} = 1 \iff i_k = j_{\sigma(k)}, \forall k = 1, 2, 3.$$

For example, if $\sigma = (321)$, i.e., $\sigma(1) = 3, \sigma(3) = 2, \sigma(2) = 1$, then $K_{i_1,i_2,i_3,j_1,j_2,j_3} = 1$ iff $i_1 = j_3, i_2 = j_1, i_3 = j_2$.

Similar to Proposition 1, we have

**Proposition 2** Given any permutation $\sigma \in \mathcal{P}_m$ and a group of vectors $u_1, u_2, \ldots, u_m \in \mathbb{R}^n$, we have

$$\mathcal{K}^{(\sigma)}(u_1 \times u_2 \times \cdots \times u_m) = u_{\sigma(1)} \times u_{\sigma(2)} \times \cdots \times u_{\sigma(m)},$$  \hspace{1cm} (34)

where the multiplication $K \cdot A$ follows the rule of the contractive product.

**Proof** We denote the tensor of the left-hand side and of the right-hand side of (34), respectively, by $A$ and $B$ and write $u_j = (u_{1j}, u_{2j}, \ldots, u_{nj})^T$. Then, both $A, B \in T_{m,n}$. Given $(i_1, i_2, \ldots, i_m) \in S(m, n)$, we have

---

1 A general permutation tensor can be defined without the restriction of a constant dimension in the first $m$ modes. Here we simplify it to fit our purpose.
Thus, \( A = B \). The proof is completed.

Now we are ready to prove

**Theorem 8** Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}^n \) (1 \( \leq m \leq n \)). Then, the \( m! \) vectors in set

\[
\left\{ \alpha_{\sigma(1)} \times \alpha_{\sigma(2)} \times \cdots \times \alpha_{\sigma(m)} : \sigma \in \mathcal{P}_m \right\}
\]

are linearly independent if and only if vectors \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are linearly independent.

**Proof** For sufficiency, we assume that the vectors in set (35) are linearly independent, and we want to prove that vectors \( \{ \alpha_j \}_{j=1}^m \) are linearly independent. Suppose, to the contrary, that vectors \( \{ \alpha_j \}_{j=1}^m \) are linearly dependent. Then, by Corollary 3 we have \( \alpha_1 \wedge \cdots \wedge \alpha_m = 0 \), which implies that the vectors in set (35) are linearly dependent. This is a contradiction to our hypothesis.

Now we show the necessity. We assume that vectors \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are linearly independent. To show the linear dependency of vectors in set (35), we let

\[
\sum_{\sigma \in \mathcal{P}_m} \lambda_\sigma \alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)} \otimes \cdots \otimes \alpha_{\sigma(m)} = 0,
\]

where \( 0 \) denotes the zero tensor in \( T_{m,n} \) and \( \lambda_\sigma \in \mathbb{R} \) is a scalar. It suffices to show that \( \lambda_\sigma = 0 \) for each \( \sigma \in \mathcal{P}_m \). By Proposition 2, (36) is equivalent to

\[
0 = \left( \sum_{\sigma \in \mathcal{P}_m} \lambda_\sigma K^{(\sigma)} \right) \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_m,
\]

which implies that

\[
\sum_{\sigma \in \mathcal{P}_m} \lambda_\sigma K^{(\sigma)} = 0.
\]

It is easy to see that the tensors in the set \( S := \{ K^{(\sigma)} : \sigma \in \mathcal{P}_m \} \) are linearly independent. In fact, this assertion can be easily confirmed if we consider the set of matrices \( A^\sigma \) where each \( A^\sigma \in \mathbb{R}^{m \times m} \) is obtained by matricization of tensor \( K^\sigma \) in set \( S \), i.e., \( A^\sigma = (a_{ij}) \) with

\[
a_{ij} = K_{i_1j_1 \cdots i_mj_m}.
\]

Then, \( \sum_{\sigma \in \mathcal{P}_m} \lambda_\sigma K^{(\sigma)} = 0 \) implies that \( \lambda_\sigma = 0 \) for each \( \sigma \in \mathcal{P}_m \). Thus, the proof is completed.
We note that Theorem 8 is also true if the tensor-products in set (35) are replaced by the Kronecker products. From Theorem 8, we immediately get

**Corollary 5** Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}^n \) and \( A = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \). Then, \( \text{rank}(A) \leq m! \). Furthermore, \( \text{rank}(A) = m! \) if the vectors \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are linearly independent.

Corollary 5 shows that the rank of a tensor can be very large even though its dimension \( n \) is small, a fact which is not conformal to the case when \( m = 2 \). We know that the rank of an \( n \times n \) matrix \( A \) satisfies \( \text{rank}(A) \leq n \). But a 4-order 3-dimensional tensor \( A \) can have rank \( 4! = 24 \), which is much bigger than 3 if we choose \( A \) to be a nonzero separable tensor. Our result can be used to enhance the consequences that appeared in [12].

Now we let \( A \in T_{m,n} \). We want to know when a symmetric tensor \( A \) can be separable. In the case \( m = 2 \), we see that \( A = \frac{1}{2}(\alpha\beta^T + \beta\alpha^T) \in \mathbb{R}^{n \times n} \) is a rank-2 symmetric matrix when \( \alpha, \beta \in \mathbb{R}^n \) are linearly independent. Furthermore, we can show that

**Lemma 4** Let \( \alpha, \beta \in \mathbb{R}^n (n \geq 2) \) be linearly independent and \( A = \frac{1}{2}(\alpha\beta^T + \beta\alpha^T) \). Then, \( A \) is neither PSD nor negative semidefinite.

**Proof** We may assume w.l.g. that \( ||\alpha|| = ||\beta|| = 1 \) where the norm \( || \cdot || \) denotes the Euclidean norm. Denote \( a = \langle \alpha, \beta \rangle \). Then, we have \( |a| < 1 \) since \( \langle \alpha, \beta \rangle^2 \leq \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle = 1 \) and the equality holds only if \( \alpha = \pm \beta \), which contradicts with the linear independency of \( \alpha, \beta \). We suppose that \( 0 < a < 1 \), and denote \( \eta = \frac{1-a}{2} \). Take \( \lambda = -a(1+2\eta) \) and \( u = \alpha - a(1+2\eta)\beta \). Then, we can easily check that \( u^TAu < 0 \). When \( -1 < a < 0 \), we can also find a vector \( v \) such that \( v^TAv < 0 \). Thus, \( A \) is not PSD. Similar argument also applies to show that \( A \) is not negative semidefinite.

We end the paper by conjecturing that the conclusion in Lemma 4 is also valid for a separable symmetric tensors of order \( m \geq 2 \).

**Conjecture 1** Let \( v_1, v_2, \ldots, v_m \in \mathbb{R}^n (n \geq m) \) be linearly independent vectors, and \( A = v_1 \vee v_2 \vee \cdots \vee v_m \). Then, \( A \) is neither a PSD tensor nor a negative semidefinite tensor.

**Acknowledgements** The authors would like to thank Professor Fuzhen Zhang of Nova Southeastern University for his remarks for the proof of Theorem 8. Thanks are also given to the anonymous referees for their valuable suggestions and remarks which lead to the improvement of the manuscript.

**Compliance with Ethical Standards**

**Conflict of Interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

**References**

1. Baerheim, R.: Harmonic decomposition of the anisotropic elasticity tensor. Q. J. Mech. Appl. Math. 46, 391–418 (1993)
2. Carroll, J.D., Chang, J.: Analysis of individual differences in multidimensional scaling via an n-way generalisation of “Eckart-Young” decomposition. Psychometrika 35, 283–319 (1970)
3. Comon, P., Golub, G., Lim, L.-H., Mourrain, B.: Symmetric tensors and symmetric tensor rank. SIAM J. Matrix Anal. Appl. 30, 1254–1279 (2008)
4. Gu, L., Wang, X., Zhang, G.: Quantum higher order singular value decomposition. In: 2019 IEEE International Conference on Systems, Man and Cybernetics (SMC), Bari, Italy, 6-9m October, 2019, Quantum Information Processing 20, 190 (2021)
5. Harshman, R.A.: Foundations of the PARAFAC procedure: models and conditions for an “explanatory” multi-modal factor analysis. UCLA Working Pap. Phonet. 16, 1–84 (1970)
6. Hartley, R., Schaffalitzky, F.: Reconstruction from projections using Grassmann tensors. Int. J. Comput. Vision 83, 274–293 (2009)
7. Hitchcock, F.L.: The expression of a tensor or a polyadic as a sum of products. J. Math. Phys. Camb. 6, 164–189 (1927)
8. Hitchcock, F.L.: Multiple invariants and generalized rank of a p-way matrix or tensor. J. Math. Phys. Camb. 7, 39–70 (1927)
9. Hu, S., Huang, Z., Ling, C., Qi, L.: On determinants and eigenvalue theory of tensors. J. Symbolic Comput. 50, 508–531 (2013)
10. Kilmer, M.E., Martin, C.D.: Factorization strategies for third-order tensors. Linear Algebra Appl. 435, 641–658 (2011)
11. Kolda, T.: Numerical optimization for symmetric tensor decomposition. Math. Program. 151, 225–248 (2015)
12. Kolda, T., Bader, B.W.: Tensor decompositions and applications. SIAM Rev. 51, 455–500 (2009)
13. Lavrauw, M., Pavan, A., Zanella, C.: On the rank of $3 \times 3 \times 3$-tensors. Linear Multilinear Algebra 61, 648–652 (2013)
14. Lim, L.-H.: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), pp. 129–132 (2005)
15. Qi, L.: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. 40, 1302–1324 (2005)
16. Qi, L.: Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl. 439, 228–238 (2013)
17. Qi, L., Zhang, G., Braun, D., Waldraff, F.B., Giraud, O.: Regularly decomposable tensors and classical spin states. Commun. Math. Sci. 15, 1651–1665 (2017)
18. Qi, L., Luo, Z.: Tensor Analysis: Spectral Theory and Special Tensors. SIAM, Philadelphia (2017)
19. Shashua, A., Hazan, T.: Non-negative tensor factorization with applications to statistics and computer vision. In: Proceedings of the 22nd International Conference on Machine Learning (ICML), Bonn, Germany, pp. 792–799 (2005)
20. Shashua, A., Zass, R., Hazan, T.: Multi-way Clustering Using Super-symmetric Non-negative Tensor Factorization. In: Leonardis, A., Bischof, H., Pinz, A. (eds) Computer Vision–ECCV 2006. ECCV 2006. Lecture Notes in Computer Science, vol. 3954, pp. 595–608. Springer, Berlin, Heidelberg (2006)
21. Vidal, R.: Generalized Principal Component Analysis (GPCA): an Algebraic Geometric Approach to Subspace Clustering and Motion Segmentation, Ph.D. Thesis. Electrical Engineering and Computer Sciences, University of California, Berkeley (2003)
22. Wang, X., Gu, L., Lee, H., Zhang, G.: Quantum context-aware recommendation systems based on tensor singular value decomposition. Quant. Inf. Process 20, 190 (2021)
23. Wang, X., Gu, L., Lee, H., Zhang, G.: Quantum tensor singular value decomposition. J. Phys. Commun. 5(7), 075001 (2021)
24. Weinberg, S.: Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York (1972)
25. Schulz, W.C.: Theory and Applications of Grassmann Algebra. Transgalactic Publishing Company, Flagstaff (2011)
26. Xu, C.: Tensor symmetrization and its applications in generalized principal component analysis, to appear in Pacific Journal of Optimizations, 18(3) (2022)
27. Xu, C., He, L., Lin, Z.: Commutation matrices and commutation tensors. Linear Multilinear Algebra 68, 1721–1742 (2020)