Mesons in (2+1) Dimensional Light Front QCD. II. Similarity Renormalization Approach

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Abstract

Recently we have studied the Bloch effective Hamiltonian approach to bound states in 2+1 dimensional gauge theories. Numerical calculations were carried out to investigate the vanishing energy denominator problem. In this work we study similarity renormalization approach to the same problem. By performing analytical calculations with a step function form for the similarity factor, we show that in addition to curing the vanishing energy denominator problem, similarity approach generates linear confining interaction for large transverse separations. However, for large longitudinal separations, the generated interaction grows only as the square root of the longitudinal separation and hence produces violations of rotational symmetry in the spectrum. We carry out numerical studies in the Glazek-Wilson and Wegner formalisms and present low lying eigenvalues and wavefunctions. We investigate the sensitivity of the spectra to various parameterizations of the similarity factor and other parameters of the effective Hamiltonian, especially the scale $\sigma$. Our results illustrate the need for higher order calculations of the effective Hamiltonian in the similarity renormalization scheme.

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I. INTRODUCTION

Any attempt to solve bound state problems in quantum field theory using Fock space based Hamiltonian methods immediately encounters a two-fold infinity problem. In a relativistic system there are infinitely many energy scales and there are an infinite number of particles. Typical Hamiltonians of interest couple low energy scales with high energy scales which results in ultraviolet divergences. Furthermore, Hamiltonian couples every particle

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number sector allowed by symmetries and at strong coupling brute force particle truncation can fail miserably.

Light front Hamiltonian poses special problems for the treatment of ultraviolet divergences since the counterterms for transverse divergences in this case are in general expected to be nonlocal in the longitudinal direction. In spite of the complexities due to renormalization, light front approach to QCD is appealing from several physical aspects. First and foremost is the fact that in a cutoff theory where one imposes a cutoff on the minimum longitudinal momentum of the constituents, Fock vacuum is the exact eigenstate of the Hamiltonian and there is hope that one may find a direct link between QCD on one hand and constituent quark model and quark parton model on the other hand.

In the best of all possible worlds, one can attempt to diagonalize the Hamiltonian in a single step. This is the spirit behind straightforward implementations of the Discretized Light Cone Quantization (DLCQ) method (for a review see Ref. [2]). DLCQ has been quite successful in two dimensional models, but for QCD this approach may be quite ambitious. Recall efficient numerical algorithms designed for matrix diagonalization where the matrix is first brought to a tri-diagonal form which is then diagonalized. In the same spirit, to keep the complexities under control, a two step process to the Hamiltonian diagonalization has been suggested [1]. The proposal is based on similarity renormalization scheme devised independently by Glazek and Wilson and Wegner.

In the first step, by integrating out the high energy degrees of freedom perturbatively one arrives at an effective Hamiltonian which is in a band diagonal form. At this step, one can identify the ultraviolet divergent part of the counterterms needed to be added to the Hamiltonian to remove ultraviolet divergences. It is quite advantageous to treat the ultraviolet divergences perturbatively especially in gauge theories since one can avoid pitfalls of other effective Hamiltonian approaches. For example, it is well-known that a simple truncation of the Fock space (like a Tamm-Dancoff truncation) leads to uncanceled divergences as a result of violations of gauge symmetry.

In the second step, the effective Hamiltonian is diagonalized exactly. It is important to note that the high energy states integrated out contain small \( x \) gluons, i.e., gluons having small longitudinal momentum fraction. Since the vacuum is trivial, it is hoped that, as a result of integrating out small \( x \) gluons which are sensitive to long distance physics on the light front, the effective Hamiltonian may contain interactions responsible for low energy properties of QCD. Indeed Perry [5] found a logarithmic confining interaction in the \( q\bar{q} \) sector in the lowest order effective interaction.

Initial bound state studies in the similarity renormalization approach worked in either the non-relativistic limit or in the heavy quark effective theory formalism to investigate heavy-quark systems. Only recently, work has begun in the context of glueball spectrum to address many practical problems, especially the numerical ones that one faces in this approach. Since the conceptual and technical problems one encounters in QCD are numerous, we have initiated a study of bound state problems in QCD in 2+1 dimensions [9]. Our main motivation is not the fitting of data but a critical evaluation of the strengths and weaknesses of the various assumptions and approximations made in the similarity approach.

In Ref. [9] we studied the meson sector of 2+1 dimensional light front QCD using a Bloch effective Hamiltonian in the first non-trivial order. The resulting two dimensional integral equation was converted into a matrix equation and solved numerically. The
vanishing energy denominator problem which leads to severe infrared divergences in 2+1 dimensions was investigated in detail. We defined and studied numerically a reduced model which is relativistic, free from infrared divergences, and exhibits logarithmic confinement. The manifestation and violation of rotational symmetry as a function of the coupling was studied quantitatively. Our study indicated that in the context of Fock space based effective Hamiltonian methods to tackle gauge theories in 2+1 dimensions, approaches like similarity renormalization method may be mandatory due to uncanceled infrared divergences caused by the vanishing energy denominator problem.

In this work we study similarity renormalization approach in the first non-trivial order to the same problem. The plan of this paper is as follows. In Sec. II we review the similarity renormalization approach in the Glazek-Wilson and Wegner formalisms. In Sec. III we present the effective Hamiltonian in the $q\bar{q}$ sector. Three different parameterizations of the similarity factor are discussed in Sec. IV. In Sec. V we present analytical calculations with the step function similarity factor which clearly shows the emergence of linear and square root confining interactions for large transverse and longitudinal separations respectively. In Sec. VI we perform numerical calculations and present low lying spectra and wavefunctions. Sec. VII contains discussion and conclusions.

II. SIMILARITY RENORMALIZATION THEORY FOR THE EFFECTIVE HAMILTONIAN

Realizing that a single step diagonalization of the Hamiltonian is too difficult, one may aim for the next best thing, namely, bringing the Hamiltonian to a band diagonal form. This is the method proposed by Glazek and Wilson and Wegner independently [4].

Starting from a cutoff Hamiltonian $H_B$ which includes canonical terms and counterterms we wish to arrive at an effective Hamiltonian $H_\sigma$ defined at the scale $\sigma$ via a similarity transformation

$$H_\sigma = S_\sigma \ H_B \ S_\sigma^\dagger$$

where $S_\sigma$ is chosen to be unitary.

The boundary condition is $\lim_{\sigma \to \infty} H_\sigma = H_B$.

Introduce anti-Hermitian generator of infinitesimal changes of scale $T_\sigma$ through

$$S_\sigma = T \ e^{ \int_{\infty}^\sigma \ d\sigma' \ T_{\sigma'}}$$

where $T$ puts operators in order of increasing scale.

For infinitesimal change of scale, $S_\sigma = 1 - T_\sigma \ d\sigma$ and $S_\sigma^\dagger = 1 + T_\sigma \ d\sigma$. Then we arrive at the infinitesimal form of the transformation

$$\frac{dH_\sigma}{d\sigma} = [H_\sigma, T_\sigma].$$

This equation which has been called the flow equation of the Hamiltonian is the starting point of the investigations.

The basic goal of the transformation $S_\sigma$ is that $H_\sigma$ should be band diagonal relative to the scale $\sigma$. Qualitatively this means that matrix elements of $H_\sigma$ involving energy jumps much
larger than $\sigma$ should be zero. $T_\sigma$ still remains arbitrary to a great extent. It is instructive to go through the steps of the derivation which leads to the Glazek-Wilson choice.

We write $H_B = H_{B0} + H_{BI}$ where $H_{B0}$ is the free part and $H_{BI}$ is the interaction part of the bare cutoff Hamiltonian. A brute force way of achieving our goal is to define the matrix elements $H_{I\sigma ij} = f_{\sigma ij} H_{BI ij}$ where we have introduced the function $f_{\sigma ij} = f(x_{\sigma ij})$ with $x$ a function of $\sigma^2$ and $\Delta M^2_{ij}$. The function $f(x)$ should be chosen as follows:

\begin{align*}
\text{when } \sigma^2 &>> \Delta M^2_{ij}, \quad f(x) = 1 \quad \text{(near diagonal region)}; \\
\text{when } \sigma^2 &<< \Delta M^2_{ij}, \quad f(x) = 0 \quad \text{(far off diagonal region)}; \\
\text{in between } \quad f(x) &\text{ drops from 1 to 0 } \quad \text{(transition region).} \quad (2.4)
\end{align*}

Here $\Delta M^2_{ij}(= M^2_i - M^2_j)$ denotes the difference of invariant masses of states $i$ and $j$. Because of the properties of $f$, $H_{I\sigma ij}$ is band diagonal. What is wrong with such a choice of inserting form factors by hand at the interaction vertices? First of all, we have simply discarded degrees of freedom above $\sigma$. Secondly, $H_\sigma$ will have very strong dependence on $\sigma$. Thirdly, to ensure that $H_\sigma$ has no ultraviolet cutoff dependence, $H_B$ should contain canonical and counterterms. But, in light front Hamiltonian field theory, because of the complexities due to renormalization, a priori we do not know the structure of counterterms. Note that in the definition of $H_\sigma$ given in Eq. (2.3) the form of $T_\sigma$ is still unspecified. In fact, a wide variety of choices are possible. In the following, we consider the choices made by Glazek and Wilson and Wegner. The price we have to pay for the use of flow equations is that it will generate complicated interactions even if the starting Hamiltonian has only simple interactions. For example, starting with a Hamiltonian which has only 2 particle interaction, the transformation will generate 3 particle interactions, 4 particle interactions, etc.

### A. Glazek-Wilson Formalism

Writing $H_\sigma = H_0 + H_{I\sigma}$, noting that the free Hamiltonian $H_0$ does not depend on $\sigma$ and taking matrix elements in free particle states, we have,

\begin{equation}
[H_\sigma, T_\sigma]_{ij} = (P^-_i - P^-_j) T_{\sigma ij} + [H_{I\sigma}, T_\sigma]_{ij} \quad (2.5)
\end{equation}

where $H_0 | i \rangle = P^-_i \ | i \rangle$, etc. i.e.,

\begin{equation}
\frac{1}{f_{\sigma ij}} \frac{dH_{I\sigma ij}}{d\sigma} = \frac{1}{f_{\sigma ij}} [H_{I\sigma}, T_\sigma]_{ij} + \frac{1}{f_{\sigma ij}} (P^-_i - P^-_j) T_{\sigma ij}. \quad (2.6)
\end{equation}

Since we want $H_{I\sigma ij}$ to be band diagonal, it is advantageous to trade $\frac{1}{f_{\sigma ij}} \frac{dH_{I\sigma ij}}{d\sigma}$ for $\frac{d}{d\sigma} \left[ \frac{1}{f_{\sigma ij}} H_{I\sigma ij} \right]$ which on integration has the chance to ensure that $H_{I\sigma ij}$ is band diagonal, we use

\begin{equation}
\frac{d}{d\sigma} \left[ \frac{1}{f_{\sigma ij}} H_{I\sigma ij} \right] + \frac{1}{f_{\sigma ij}^2} \frac{df_{\sigma ij}}{d\sigma} H_{I\sigma ij} = \frac{1}{f_{\sigma ij}} \frac{dH_{I\sigma ij}}{d\sigma} \quad (2.7)
\end{equation}

and arrive at
\[
\frac{d}{d\sigma} \left[ \frac{1}{f_{\sigma ij}} H_{1\sigma ij} \right] = \frac{1}{f_{\sigma ij}} (P_i^- - P_j^-) T_{\sigma ij} \\
+ \frac{1}{f_{\sigma ij}} [H_{I\sigma}, T_{\sigma}]_{ij} - \frac{1}{f_{\sigma ij}^2} \frac{df_{\sigma ij}}{d\sigma} H_{1\sigma ij}.
\] (2.8)

Still \( T_{\sigma ij} \) is not defined. We next convert this equation into two equations, one defining the flow of \( H_{1\sigma ij} \) and other defining \( T_{\sigma ij} \). Recalling the starting equation Eq. (2.3) we add and subtract \([H_{I\sigma}, T_{\sigma}]_{ij}\) to the r.h.s. and arrive at

\[
\frac{d}{d\sigma} \left[ \frac{1}{f_{\sigma ij}} H_{1\sigma ij} \right] = [H_{I\sigma}, T_{\sigma}]_{ij} - \frac{d}{d\sigma} f_{\sigma ij} [H_{I\sigma}, T_{\sigma}]_{ij}.
\] (2.9)

Glazek and Wilson choose \( T_{\sigma} \) to be

\[
T_{\sigma ij} = \frac{1}{P_j^- - P_i^-} \left[ (1 - f_{\sigma ij}) [H_{I\sigma}, T_{\sigma}]_{ij} - \frac{d}{d\sigma} (\ln f_{\sigma ij}) H_{1\sigma ij} \right].
\] (2.10)

Then from Eq. (2.9), we have,

\[
\frac{dH_{1\sigma ij}}{d\sigma} = f_{\sigma ij} [H_{I\sigma}, T_{\sigma}]_{ij}.
\] (2.11)

Integrating Eq. (2.11) from \( \sigma \) to \( \infty \), we arrive at,

\[
H_{1\sigma ij} = f_{\sigma ij} \left[ H_{1Bij} - \int_{\sigma}^{\infty} d\sigma' [H_{I\sigma'}, T_{\sigma'}]_{ij} \right].
\] (2.12)

Note that \( H_{1\sigma ij} \) is zero in the far off-diagonal region. This is clear from the solution given in Eq. (2.13) since \( f(x) \) vanishes when \( x \geq 2/3 \).

\( T_{\sigma ij} \) vanishes in the near diagonal region. When \( i \) is close to \( j \), \( f_{\sigma ij} = 1 \) and both \((1 - f_{\sigma ij})\) and \( \frac{d}{d\sigma} f_{\sigma ij} \) vanishes. It follows, then, from Eq. (2.10) that \( T_{\sigma ij} \) vanishes in the near-diagonal region. This guarantees that a perturbative solution to \( H_{1\sigma ij} \) in terms of \( H_{Bij} \) will never involve vanishing energy denominators.

Our next task is to derive the effective Hamiltonian to second order in perturbation theory. Using

\[
H_{1\sigma ik}^{(1)} \simeq f_{\sigma ik} H_{Blik}
\] (2.13)

and

\[
T_{\sigma kj} \simeq \frac{1}{P_j^- - P_k^-} \left\{ - \frac{d}{d\sigma} (\ln f_{\sigma kj}) f_{\sigma kj} H_{Blik} \right\}
\] (2.14)

in Eq. (2.12), a straightforward calculation leads to

\[
H_{1\sigma ij}^{(2)} = - \sum_k H_{Blik} H_{Blikj} \left[ \frac{g_{\sigma ik}}{P_k^- - P_j^-} + \frac{g_{\sigma jk}}{P_k^- - P_i^-} \right],
\] (2.15)
where

\[ g_{\sigma ij} = f_{\sigma ij} \int_{\sigma}^{\infty} d\sigma' f_{\sigma' ik} \frac{d}{d\sigma'} f_{\sigma' jk}, \]

\[ g_{\sigma jik} = f_{\sigma ij} \int_{\sigma}^{\infty} d\sigma' f_{\sigma' jk} \frac{d}{d\sigma'} f_{\sigma' ik}. \] (2.16)

We find that the effective Hamiltonian in similarity perturbation theory is a modification of the effective Hamiltonian in Bloch perturbation theory [10].

\[ \text{B. Wegner Formalism} \]

In the Wegner formalism [11], the flow equation is given by

\[ \frac{dH(l)}{dl} = [\tau(l), H(l)]. \] (2.17)

Wegner chooses

\[ \tau(l) = [H_d, H] = [H_d, H_r] \] (2.18)

where \( H_d \) is the diagonal part of the Hamiltonian and \( H_r \) is the rest, i.e., \( H = H_d + H_r \). Here the word diagonal is used in the particle number conserving sense. It is important to note that \( H_d \) is not the free part of the Hamiltonian and both \( H_d \) and \( H_r \) depend on the length scale \( l \).

The light front Hamiltonian has dimension of \((\text{mass})^2\) and hence \( \tau \) has the dimension of \((\text{mass})^4\), \( l \) has dimension of \((\text{mass})^{-4}\).

Expanding in powers of the coupling constant,

\[ H = H_d^{(0)} + H_r^{(1)} + H_d^{(2)} + H_r^{(2)} + \ldots \] (2.19)

where the superscript denotes the order in the coupling constant,

\[ \tau(l) = [H_d^{(0)}, H_r^{(1)}] + [H_d^{(0)}, H_r^{(2)}] + \ldots. \] (2.20)

Then, to second order,

\[ \frac{dH}{dl} = [[H_d^{(0)}, H_r^{(1)}], H_d^{(0)}] + [[H_d^{(0)}, H_r^{(1)}], H_r^{(1)}] + [[H_d^{(0)}, H_r^{(2)}], H_d^{(0)}] + \ldots. \] (2.21)

Introduce [12] the eigenstates of \( H_d^{(0)} \),

\[ H_d^{(0)} | i \rangle = P^-_i | i \rangle. \] (2.22)

Then, to second order,

\[ \frac{dH_{ij}}{dl} = -(P^-_i - P^-_j)^2 H_{r\ ij}^{(1)} + \left[ \tau_i^{(1)}, H_r^{(1)} \right]_{ij} - (P^-_i - P^-_j)^2 H_{r\ ij}^{(2)} + \ldots. \] (2.23)

To first order in the coupling,
\[
\frac{dH_{rij}}{dl} = -(P_i^- - P_j^-)^2 H_{rij}^{(1)}
\] (2.24)

which on integration yields

\[
H_{rij}^{(1)}(\sigma) = e^{-\frac{(P_i^- - P_j^-)^2}{\sigma^4}} H_{rij}^{(1)}(\Lambda)
\] (2.25)

where we have introduced the energy scale \( \sigma \) via \( l = \frac{1}{\sigma^4} \) and used the fact that \( l = 0 \) corresponds to the original bare cutoff. We notice the emergence of the similarity factor \( f_{\sigma ij} = e^{-\frac{(P_i^- - P_j^-)^2}{\sigma^4}}. \)

If we are interested only in particle number conserving (diagonal) part of the effective interaction, to second order we have,

\[
\frac{dH_{lij}}{dl} = \left[ \tau_1^{(1)}(l), H_r^{(1)} \right]_{ij}
\] (2.26)

Using

\[
\tau_1^{(1)}(l) = (P_j^- - P_i^-) H_{rij}^{(1)}
\] (2.27)

the effective interaction generated to second order in the diagonal sector is

\[
H_{lij} = \sum_k H_{lik}^B H_{kj}^B \frac{(P_i^- - P_k^-)}{(P_i^- - P_k^-)^2 + (P_j^- - P_k^-)^2} \left[ 1 - e^{-\frac{(P_i^- - P_k^-)^2 + (P_j^- - P_k^-)^2}{\sigma^4}} \right].
\] (2.28)

Even though the second order formula is very similar to the one in Glazek-Wilson formalism when an exponential form is chosen for the similarity factor (see Sec. IV), we note a slight difference. In the Glazek-Wilson formalism, since the purpose is to bring the Hamiltonian into a band diagonal form, even in the particle number conserving sectors the large jumps in energies do not appear by construction. In the version of the Wegner formalism presented here the purpose is to bring the Hamiltonian in the block diagonal form in particle number sector so that large jumps in energies are allowed by the effective Hamiltonian. Note that small energy denominators do not appear in both formalisms.

III. EFFECTIVE HAMILTONIAN IN THE Q\bar{Q} SECTOR

In similarity renormalization approach due to Glazek and Wilson, to second order, the interacting part of the effective Hamiltonian in similarity renormalization approach is given by Eq. (2.13). In this work we restrict ourselves to the \( q\bar{q} \) sector. Then the states involved in the matrix elements \( i \) and \( j \) refer to \( q\bar{q} \) states and \( k \) refer to \( q\bar{q}g \) states.

Following the steps similar to the ones outlined in Ref. [9], we arrive at the bound state equation

\[
[M^2 - \frac{m^2 + k^2}{x(1-x)}] \psi_2(x, k) = \text{SE} \, \psi_2(x, k) - 4 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \, f_{\sigma ij} \, \psi_2(y, q) \frac{1}{(x-y)^2} - \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \, \psi_2(y, q) \frac{V}{ED}.
\] (3.1)
Here $x$ and $y$ are the longitudinal momentum fractions and $k$ and $q$ are the relative transverse momenta. We introduce the cutoff $\eta$ such that $\eta \leq x, y \leq 1 - \eta$. We further introduce the regulator $\delta$ such that $|x - y| \geq \delta$. Ultraviolet divergences are regulated by the introduction of the cutoff $\Lambda$ on the relative transverse momenta $k$ and $q$. The self energy contribution

$$SE = -\frac{g^2}{2(2\pi)^2} C f \int_0^1 dy \int dq \, \theta(x - y) \left[ 1 - f_{\sigma ik}^2 \right] \frac{(q \frac{y}{x} + k \frac{y}{x} - \frac{2(k-q)}{(x-y)})^2 + m^2(x-y)^2}{(kq - qx)^2 + m^2(x-y)^2} \times k(x-y) + iV_I,$$

$$\times \frac{\theta(x-y)}{(x-y)} \left[ K(k, x, q, y) + iV_I \right]$$

The boson exchange contribution

$$V = \frac{\theta(x-y)}{(x-y)} \left[ \frac{g_{\sigma jik}}{m^2 + \frac{y}{x} + \frac{(k-q)^2}{(x-y)}} - \frac{m^2 + k^2}{m^2 + k^2} \right]$$

$$\times \left[ K(k, x, q, y) + iV_I \right]$$

where

$$K(k, x, q, y) = \left( q \frac{y}{x} + k \frac{y}{x} - \frac{2(k-q)}{(x-y)} \right) \left( \frac{q}{1-y} + k \frac{1-x}{x-y} + \frac{2(k-q)}{(x-y)} \right)$$

$$- \frac{m^2(x-y)^2}{xy(1-x)(1-y)}$$

$$V_I = -\frac{m}{xy(1-x)(1-y)} \left[ q(2y - 3x) + k(3y + x - 2) \right].$$

For all the $f$ and $g$ factors,

$$M_i^2 = \frac{k^2 + m^2}{x(1-x)} \quad \text{and} \quad M_j^2 = \frac{q^2 + m^2}{y(1-y)}.$$

For $x > y$,

$$M_k^2 = \frac{(k-q)^2}{x-y} + \frac{q^2 + m^2}{y} + \frac{k^2 + m^2}{1-x}$$

and

$$M_k^2 = \frac{(q-k)^2}{y-x} + \frac{q^2 + m^2}{1-y} + \frac{k^2 + m^2}{x}.$$
Before proceeding further, we perform the ultraviolet renormalization. The only ultraviolet divergent term arises from the factor 1 inside the square bracket in Eq. (3.2). We isolate the ultraviolet divergent term which is given by

\[
SE_{\text{divergent}} = -\frac{g^2}{2(2\pi)^2} C_f \left[ \int_{x-\delta}^{x+\delta} dy \int_{-\Lambda}^{+\Lambda} dq \frac{(x+y)^2}{xy(x-y)^2} \right. \\
+ \left. \int_{x+\delta}^{1} dy \int_{-\Lambda}^{+\Lambda} dq \frac{(2-x-y)^2}{(1-x)(1-y)(x-y)^2} \right].
\]

(3.9)

which is canceled by adding a counterterm.

IV. SIMILARITY FACTORS

A. Parameterization I

In recent numerical work [8], the following form for the similarity factor has been chosen:

\[
f_{\sigma ij} = e^{-\frac{(\Delta M^2_{ij})^2}{\sigma^4}}
\]

(4.1)

with \(\Delta M^2_{ij} = M^2_i - M^2_j\) where \(M^2_i\) denotes the invariant mass of the state \(i\), i.e., \(M^2_i = \sum_i (k^i)^2 + m^2_i\). Then

\[
g_{\sigma ik} = f_{\sigma ij} \int_{\sigma}^{\infty} d\sigma' f_{\sigma' ik} \frac{d}{d\sigma'} f_{\sigma' jk}
\]

\[
= e^{-\frac{(\Delta M^2_{ij})^2}{\sigma^4}} \frac{(\Delta M^2_{jk})^2}{(\Delta M^2_{ik})^2 + (\Delta M^2_{jk})^2} \left[ 1 - e^{-\frac{(\Delta M^2_{ij})^2 + (\Delta M^2_{jk})^2}{\sigma^4}} \right].
\]

(4.2)

For the self energy contribution, \(i = j\) and we get

\[
g_{\sigma ijk} = g_{\sigma jk} = g_{\sigma ik} = \frac{1}{2} \left[ 1 - e^{-\frac{2(\Delta M^2_{ij})^2}{\sigma^4}} \right].
\]

(4.3)

Due to the sharp fall of \(f\) with \(\sigma\), the effective Hamiltonian has a strong dependence on \(\sigma\). Note that this parameterization emerges naturally in the Wegner formalism.

B. Parameterization II

St. Glazek has proposed the following form [13] for \(f_{\sigma ij}\).

\[
f_{\sigma ij} = \frac{1}{1 + \left( \frac{u_{\sigma ij}(1-u_0)}{u_0(1-u_{\sigma ij})} \right)^{2\sigma^4}}
\]

(4.4)

with
\[ u_{\sigma ij} = \frac{\Delta M_{ij}^2}{\Sigma M_{ij}^2 + \sigma^2}, \quad (4.5) \]

\( u_0 \) a small parameter, and \( n_g \) an integer. The mass sum \( \Sigma M_{ij}^2 = M_i^2 + M_j^2 \). The derivative

\[ \frac{df_{\sigma ij}}{d\sigma} = 2^{n_g} \frac{2\sigma}{\Sigma M_{ij}^2 + \sigma^2} \left( \frac{u_{\sigma ij}}{u_0} \right)^{2n_g} \frac{(1 - u_0)^{2n_g}}{(1 - u_{\sigma ij})^{2n_g+1}} \frac{1}{\left[ 1 + \left( \frac{u_{\sigma ij}(1-u_0)}{u_0(1-u_{\sigma ij})} \right)^{2n_g} \right]^2}. \quad (4.6) \]

Note that for small \( u \), both \( 1 - f(u) \) and \( \frac{df}{d\sigma} \) vanish like \( u^{2n_g} \).

**C. Parameterization III**

For analytical calculations it is convenient to choose a step function cutoff for the similarity factor:

\[ f_{\sigma ij} = \theta(\sigma^2 - \Delta M_{ij}^2). \quad (4.7) \]

Then

\[ g_{\sigma ik} = \theta(\sigma^2 - \Delta M_{ij}^2) \theta(\Delta M_{jk}^2 - \sigma^2) \theta(\Delta M_{jk}^2 - \Delta M_{ik}^2). \quad (4.8) \]

It is the factor \( \theta(\Delta M_{jk}^2 - \sigma^2) \) in \( g_{\sigma ik} \) that prevents the energy denominator from becoming small.

**V. ANALYTICAL CALCULATIONS WITH THE STEP FUNCTION SIMILARITY FACTOR**

In this section we perform analytical calculations to understand the nature of the effective interactions generated by the similarity factor. Since there are no divergences associated with \( \eta \) and \( \Lambda \), we suppress their presence in the limits of integration in the following equations.

**A. Self energy contributions**

Consider the self energy contributions to the bound state equation Eq. (3.2). Rewriting the energy denominators to expose the most singular terms, we have,

\[
SE = -\frac{g^2}{2(2\pi)^2} C_f \int_0^1 dy \int dq \frac{\theta(x - \delta - y)}{x - y} \left\{ 1 - f_{\sigma ik}^2 \right\} \left[ \frac{\left( \frac{y}{x} + \frac{k}{x} - \frac{2(k-q)}{(x-y)} \right)^2 + \frac{m^2(x-y)^2}{x^2y^2}}{(x-y)^2 + \frac{q^2+m^2}{y} - \frac{k^2+m^2}{x}} \right]
- \frac{g^2}{2(2\pi)^2} C_f \int_0^1 dy \int dq \frac{\theta(y - x - \delta)}{y - x} \left\{ 1 - f_{\sigma ik}^2 \right\} \times \left[ \left( \frac{q}{1-y} + \frac{k}{1-x} + \frac{2(q-k)}{(y-x)} \right)^2 + \frac{m^2(y-x)^2}{(1-x)^2(1-y)^2} \right] \left( \frac{(q-k)^2}{y-x} + \frac{q^2+m^2}{y} - \frac{k^2+m^2}{x} \right). \quad (5.1)
\]
The terms associated with 1 in the curly brackets lead to ultraviolet linear divergent terms which we cancel by counterterms (see Sec. III). They also lead to an infrared divergent term which remains uncanceled. Explicitly this contribution is given by

\[
4 \frac{g^2 m^2}{2(2\pi)^2} C_f \int_{0}^{x-\delta} dy \int dq \frac{1}{(ky - qx)^2 + m^2(x - y)^2} \\
+ 4 \frac{g^2 m^2}{2(2\pi)^2} C_f \int_{x+\delta}^{1} dy \int dq \frac{1}{[k(1-y) - q(1-x)]^2 + m^2(x - y)^2}.
\]

This is simply indicative of the fact that terms associated with 1 in the curly bracket still has a vanishing energy denominator problem. We will address the resolution of this problem shortly.

Let us next consider new infrared divergences that arise as a result of the modifications due to similarity factor.

1. Leading singular terms

Keeping only the most infrared singular terms in the numerators (i.e., for \(x > y\), \(4 \frac{(k-q)^2}{(x-y)^2}\)) and denominators (i.e., for \(x > y\), \(\frac{(k-q)^2}{(x-y)^2}\)) and for \(y > x\), \(\frac{(q-k)^2}{(y-x)^2}\) and \(\frac{(q-k)^2}{(y-x)^2}\)), we have,

\[
SE_1 = \frac{g^2}{2(2\pi)^2} C_f \int_{0}^{1} dy \int dq \theta(x - \delta - y) f_{\sigma ik}^2 \frac{1}{(x - y)^2} \\
+ \frac{g^2}{2(2\pi)^2} C_f \int_{0}^{1} dy \int dq \theta(y - x - \delta) f_{\sigma ik}^2 \frac{1}{(y - x)^2}.
\]

The integral is given by

\[
\left[ \int_{0}^{x-\delta} dy \int dq \frac{1}{(x-y)^2} \theta \left( \sigma^2 - \frac{(k-q)^2}{x-y} \right) + \int_{x+\delta}^{1} dy \int dq \frac{1}{(y-x)^2} \theta \left( \sigma^2 - \frac{(k-q)^2}{y-x} \right) \right].
\]

We change the transverse momentum variable, \(p = k - q\). For \(x - \delta > y\), we set \(x - y = z\) and for \(y > x + \delta\) we set \(y - x = z\). Then, we have,

\[
4 \frac{g^2}{2(2\pi)^2} C_f \left( \int_{\delta}^{x} \frac{dz}{z^2} \int dp \theta(\sigma^2 - \frac{p^2}{z}) + \int_{\delta}^{1-x} \frac{dz}{z^2} \int dp \theta(\sigma^2 - \frac{p^2}{z}) \right) = \frac{16g^2}{2(2\pi)^2} C_f \sigma \left[ \frac{2}{\sqrt{\delta}} - \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1-x}} \right].
\]

2. Sub-leading singular terms

Next we study sub-leading singular terms containing \(\frac{1}{x-y}\) in self energy generated by the similarity transformation. They are given by
\[
SE_2 = -4 \frac{g^2}{2(2\pi)^2} C_f \left[ \int_0^{x-\delta} dy \int dq \, \theta \left( \sigma^2 - \frac{(k-q)^2}{x-y} \right) \frac{1}{x-y} \left( \frac{k^2}{x} - \frac{q^2}{y} \right) \frac{1}{(k-q)^2} \right] 
- \int_{x+\delta}^1 dy \int dq \, \theta \left( \sigma^2 - \frac{(q-k)^2}{y-x} \right) \frac{1}{y-x} \left( \frac{k^2}{1-x} - \frac{q^2}{1-y} \right) \frac{1}{(q-k)^2} \]
\]

where we have kept only \((k-q)^2\) term in the denominator since the rest vanish in the limit \(x \to y\). As before, for \(x - \delta > y\), we put \(x - y = z\), \(k - q = p\). With the symmetric integration in \(p\), terms linear in \(p\) does not contribute. Only potential source of \(\delta\) divergence is the \(p^2\) term in the integrand. Since \(p_{max} = \sigma \sqrt{z}\), after \(p\) integration \(\frac{1}{z}\) is converted into \(\frac{1}{\sqrt{z}}\) which is an integrable singularity. Same situation occurs for \(y > x\). Thus there are no terms divergent in \(\delta\) coming from sub-leading singular terms.

**B. Gluon exchange contributions**

Let us next consider the effect of similarity factors on gluon exchange terms.

1. **Instantaneous gluon exchange**

From instantaneous interaction we have,

\[
V_{\text{inst}} = -4 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \, \psi_2(y, q) f_{\sigma ij} \frac{1}{(x-y)^2}. \tag{5.7}
\]

For the sake of clarity, it is convenient to rewrite this as

\[
V_{\text{inst}} = -4 \frac{g^2}{2(2\pi)^2} \frac{1}{2} C_f \int dy \int dq \, f_{\sigma ij} \psi_2(y, q) \left[ \theta(x - y - \delta) \left\{ \frac{(k-q)^2}{x-y} + \left( \frac{q^2}{y} - \frac{k^2}{x} \right) + m^2 \left( \frac{1}{y} - \frac{1}{x} \right) \right. \right. \\
+ \left. \left. \left( \frac{k^2}{1-x} - \frac{q^2}{1-y} \right) - m^2 \left( \frac{1}{1-y} - \frac{1}{1-x} \right) \right) - \frac{(k-q)^2}{x-y} - \frac{(q^2}{1-y} - \frac{k^2}{1-x}) \right. \\
+ \left. \left. \left( \frac{k^2}{y-x} - \frac{q^2}{x} - \frac{k^2}{x} \right) - m^2 \left( \frac{1}{x} - \frac{1}{y} \right) \right) \right] \left[ \theta(y - x - \delta) \left\{ \frac{(q-k)^2}{y-x} + \left( \frac{q^2}{y} - \frac{k^2}{x} \right) + m^2 \left( \frac{1}{1-x} - \frac{1}{1-y} \right) \right. \right. \\
+ \left. \left. \left( \frac{q-k}{y-x} - \frac{k^2}{1-x} - \frac{q^2}{1-y} \right) + m^2 \left( \frac{1}{1-y} - \frac{1}{1-x} \right) \right) \right] \right] \frac{1}{(x-y)^2} \tag{5.8}
\]

We have to separately analyze the three types of terms in the numerator.

First consider terms proportional to \(m^2\) in the numerator. They are given by

\[
-4 \frac{g^2 m^2}{2(2\pi)^2} \frac{1}{2} C_f \int dy \int dq \, f_{\sigma ij} \psi_2(y, q) \times \left[ \frac{1}{(ky-qx)^2 + m^2(x-y)^2} + \frac{1}{[k(1-y) - q(1-x)]^2 + m^2(x-y)^2} \right] \tag{5.9}
\]
which leads to the logarithmic confining interaction in the nonrelativistic limit. Note, however, that Eq. (5.9) is affected by a logarithmic infrared divergence arising from the vanishing energy denominator problem. The logarithmic infrared divergence is cancelled by the self energy contribution, Eq. (5.2). Thus we have explicitly shown that the logarithmically confining Coulomb interaction survives similarity transformation but the associated infrared divergence is canceled by self energy contribution.

Next we look at the most singular term in the numerator in the limit \( x \to y \). In this limit we keep only the leading term in the denominator and we get

\[
-4 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \psi_2(y, q) f_{\sigma ij} \left\{ \frac{\theta(x - y)}{(x - y)^2} + \frac{\theta(y - x)}{(y - x)^2} \right\}.
\]

Lastly we look at the rest of the terms in the instantaneous exchange. Since we are interested only in the singularity structure, we keep only the leading term in the denominator and we get

\[
-2 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq f_{\sigma ij} \frac{1}{(k - q)^2} \times \left[ \frac{\theta(x - y)}{x - y} \left\{ \frac{g_{\sigma jk} + g_{\sigma ik}}{x(1 - x)} \right\} + \frac{\theta(y - x)}{y - x} \left\{ \frac{k^2(1 - 2x)}{x(1 - x)} \frac{g_{\sigma jk} + g_{\sigma ik}}{y(1 - y)} \right\} \right] + \frac{\theta (y - x)}{(y - x)^2} \left\{ \frac{k^2(1 - 2x)}{x(1 - x)} \right\}.
\]

\[\text{(5.10)}\]

2. Transverse gluon exchange

First, consider the most singular terms. Keeping only the most singular terms, the gluon exchange contribution is

\[
- \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \psi_2(y, q) \times
\left\{ \frac{\theta(x - \delta - y)}{(x - \delta - y)^2} \left[ \frac{g_{\sigma jk} + g_{\sigma ik}}{(k - q)^2} \right] \frac{\theta(y - x - \delta)}{y - x} \left[ \frac{g_{\sigma jk} + g_{\sigma ik}}{(q - k)^2} \right] \right\}.
\]

\[\text{(5.11)}\]

Explicitly, for \( x > y \),

\[
g_{\sigma ij} = \theta(\sigma^2 - M_{ij}^2) \theta(M_{jk}^2 - M_{ik}^2) \theta(M_{jk}^2 - \sigma^2),
\]

\[
g_{\sigma jk} = \theta(\sigma^2 - M_{ij}^2) \theta(M_{ik}^2 - M_{jk}^2) \theta(M_{ik}^2 - \sigma^2).
\]

\[\text{(5.12)}\]

We are interested in the situation \( x \) near \( y \) and \( i \) near \( j \). Then \( \theta(M_{jk}^2 - M_{ik}^2) = \frac{1}{2} = \theta(M_{ik}^2 - M_{jk}^2) \) and \( \theta(\sigma^2 - M_{ij}^2) = 1 \). Then the gluon exchange contribution is
For convenience we change variables. For $x > y$, in the infrared singular.

Thus for large $y > x$, we put $\delta x = \frac{p^+}{P^+}$ and $q - k = p^1$ where $P^+$ is the total longitudinal momentum. Thus we arrive at

$$
-4 \frac{g^2}{2(2\pi)^2} C_f \left[ \frac{\theta(x - \delta - y)}{(x - y)^2} \left( 1 - \theta \left( \frac{\sigma^2 - (k - q)^2}{x - y} \right) \right) + \frac{\theta(y - x - \delta)}{(y - x)^2} \left( 1 - \theta \left( \frac{\sigma^2 - (q - k)^2}{y - x} \right) \right) \right] \tag{5.14}
$$

where we have used $\theta(x) = 1 - \theta(-x)$. Combining with the most singular part of the instantaneous contribution given in Eq. (5.10) we arrive at

$$
-4 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \psi_2(y, q) \times

\left[ \frac{\theta(x - \delta - y)}{(x - y)^2} \theta \left( \sigma^2 - \frac{(k - q)^2}{x - y} \right) + \frac{\theta(y - x - \delta)}{(y - x)^2} \theta \left( \sigma^2 - \frac{(q - k)^2}{y - x} \right) \right]. \tag{5.15}
$$

For convenience we change variables. For $x > y$, we put $x - y = \frac{p^+}{P^+}$ and $k - q = p^1$ and for $y > x$, we put $y - x = \frac{p^+}{P^+}$ and $q - k = p^1$ where $P^+$ is the total longitudinal momentum. Thus we arrive at

$$
-4 \frac{g^2}{2(2\pi)^2} C_f P^+ \left[ \int dp \int_{P^+\delta}^{P^+} dp^+ \psi_2(x - \frac{p^+}{P^+}, k - p^1) \frac{1}{(p^+)^2} \theta \left( \sigma^2 - \frac{(p^1)^2P^+}{p^+} \right)
+ \int dp \int_{P^+\delta}^{P^+(1-x)} dp^+ \psi_2(x + \frac{p^+}{P^+}, k + p^1) \frac{1}{(p^+)^2} \theta \left( \sigma^2 - \frac{(p^1)^2P^+}{p^+} \right) \right]. \tag{5.16}
$$

Consider the Fourier transform

$$
V(x^-, x^1) = -4 \frac{g^2}{2(2\pi)^2} C_f P^+ \left[ \int_{P^+\delta}^{P^+} \frac{dp^+}{(p^+)^2} \int_{-p^1_{\text{max}}}^{+p^1_{\text{max}}} dp^1 e^{i \frac{1}{1\sigma^2} x^1 \xi^1} \theta \left( \sigma^2 - \frac{(p^1)^2P^+}{p^+} \right)
+ \int_{P^+\delta}^{P^+(1-x)} \frac{dp^+}{(p^+)^2} \int_{-p^1_{\text{max}}}^{+p^1_{\text{max}}} dp^1 e^{i \frac{1}{1\sigma^2} x^1 \xi^1} \theta \left( \sigma^2 - \frac{(p^1)^2P^+}{p^+} \right) \right] \tag{5.17}
$$

where $p^1_{\text{max}} = \sigma \sqrt{\frac{p^+}{\sigma^2}}$. We are interested in the behavior of $V(x^-, x^1)$ for large $x^-, x^1$. For large $x^-$, nonnegligible contribution to the integral comes from the region $q^+ < \frac{1}{|x^-|}$. For large $x^1$, we need $p^1_{\text{max}} x^1$ to be small, i.e., $(p^1_{\text{max}})^2 < \frac{1}{(x^1)^2}$, i.e., $p^+ < \frac{P^+}{(x^1)^2\sigma^2}$. Thus we have the requirements, $p^+ < \frac{1}{|x^-|}$, $p^+ < \frac{P^+}{(x^1)^2\sigma^2}$. We make the approximations

$$
\int_{-p^1_{\text{max}}}^{+p^1_{\text{max}}} dp^1 e^{-i p^1 x^1} \approx 2p^1_{\text{max}} \tag{5.18}
$$

and $e^{i \frac{1}{\sigma^2} x^1} \approx 1$.

For large $x^-$, we have $p^+ < \frac{1}{|x^-|} < \frac{P^+}{(x^1)^2\sigma^2}$, the upper limit of $p^+$ integral is cut off by $\frac{1}{|x^-|}$. Adding the contributions from both the integrals (which are equal), for large $x^-$, we have

$$
V(x^-, x^1) \approx 32 \frac{g^2}{2(2\pi)^2} C_f \sigma \left[ \sqrt{P^+ |x^-|} - \frac{1}{4\sqrt{\delta}} \right]. \tag{5.19}
$$

Thus for large $x^-$ the similarity factors have produced a square root potential but it is also infrared singular.
For large $x^1$ the upper limit of $p^+$ integral is cut off by $\frac{p^+}{(x^1)^2}$ and we get,

$$V(x^-,x^1) \approx 32 \frac{g^2}{2(2\pi)^2} C_f \sigma \left[ |x^1| \sigma - \frac{1}{\sqrt{\delta}} \right]. \quad (5.20)$$

For large $x^1$, similarity factors have produced a linear confining potential which is also infrared singular. We note that the rotational symmetry is violated in the finite part of the potential. In both cases, however, the infrared singular part is $-32 \frac{g^2}{2(2\pi)^2} C_f \sigma \frac{1}{\sqrt{\delta}}$ which is exactly canceled by the infrared contribution generated by similarity transformation from self energy, Eq. (5.3).

Lastly, we consider the terms that go like $\frac{1}{x-y}$. Keeping only the leading term in the energy denominator, we have,

$$-2 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \psi_2(y,q) \times$$

$$\left[ \frac{\theta(x-y)}{x-y} \frac{g_{\sigma jk} + g_{\sigma ijk}}{(k-q)^2} \left[ \frac{k^2(1-2x)}{x(1-x)} - \frac{q^2(1-2y)}{y(1-y)} \right] \right.$$  

$$+ \frac{\theta(y-x)}{y-x} \frac{g_{\sigma jk} + g_{\sigma ijk}}{(q-k)^2} \left[ \frac{q^2(1-2y)}{y(1-y)} - \frac{k^2(1-2x)}{x(1-x)} \right] \right]. \quad (5.21)$$

With the step function cut off we have

$$g_{\sigma jk} + g_{\sigma ijk} \approx f_{\sigma ij} \theta(\Delta M_{jk}^2 - \sigma^2). \quad (5.22)$$

Then, combining Eq. (5.11) and Eq. (5.21) for the sub-leading divergences, we get,

$$-2 \frac{g^2}{2(2\pi)^2} C_f \int dy \int dq \ f_{\sigma ij} \psi_2(y,q) \times$$

$$\left[ \frac{\theta(x-y)}{x-y} \frac{1}{(k-q)^2} \theta \left( \sigma^2 - \frac{(k-q)^2}{x-y} \right) \left[ \frac{k^2(1-2x)}{x(1-x)} - \frac{q^2(1-2y)}{y(1-y)} \right] \right.$$  

$$+ \frac{\theta(y-x)}{y-x} \frac{1}{(q-k)^2} \theta \left( \sigma^2 - \frac{(q-k)^2}{y-x} \right) \left[ \frac{q^2(1-2y)}{y(1-y)} - \frac{k^2(1-2x)}{x(1-x)} \right] \right]. \quad (5.23)$$

Taking the Fourier transform of this interaction, a straightforward calculation shows that no $\log \delta$ divergence arise from this term.

3. Summary of divergence analysis

The logarithmic confining Coulomb interaction of 2+1 dimensions is unaffected by similarity transformation and is still affected by a logarithmic divergence which is however cancelled by a logarithmic divergence from self energy contribution. Similarity transformation leads to a non-cancellation of the most singular ($\frac{1}{|x-y|}$) term between instantaneous and transverse gluon interaction terms. This leads to a linear confining interaction for large transverse separation and a square root confining interaction for large longitudinal separations. However, non-cancellation also leads to $\frac{1}{\sqrt{\delta}}$ divergences where $\delta$ is the cutoff on $|x-y|$. This divergence is cancelled by new contributions from self energy generated by similarity transformation. The subleading singular $\frac{1}{x-y}$ terms do not lead to any divergence in $\delta$. 

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VI. NUMERICAL STUDIES

The integral equation is converted into a matrix equation using Gaussian Quadrature. The matrix is numerically diagonalized using standard LAPACK routines [14]. (For details of numerical procedure see Ref. [9].) With the exponential form and the step function form of the similarity factor, the integral over the scale in the definition of \( g \) factors in Eq. (2.16) can be performed analytically as shown in Sec. IV. For parametrization II, we perform the integration numerically using \( ns \) quadrature points.

The first question we address is the cancellation of divergences which are of two types: (1) the \( \ln \delta \) divergence in the self energy and Coulomb interaction which has its source in the vanishing energy denominator problem that survives the similarity transformation and (2) \( \frac{1}{\sqrt{\delta}} \) divergences in the self energy and gluon exchange generated by the similarity transformation. In Table I we present the \( \delta \) independence of the first five eigenvalues of the Hamiltonian for \( g=0.6 \). Results are presented for three parametrizations of the similarity factor, namely, the exponential form, the form proposed by St. G/\( \text{a} \)lazek and the step function form used in our analytical studies. It is clear that the Gaussian Quadrature effectively achieves the cancellation of \( \delta \) divergences. Recall that in the study [9] of the same problem using Bloch approach, negative eigenvalues appeared for \( g=0.6 \) when \( \delta \) was sufficiently small (for example, .001) which was caused by the vanishing energy denominator problem. Our results in the similarity approach for the same coupling shows that this problem is absent in the latter approach.

Next we study the convergence of eigenvalues with quadrature points. In Table II we present the results for all three parametrizations of the similarity factor for \( g=0.2 \) with the transverse space discretized using \( k = \frac{1}{\kappa \tan \frac{u\pi}{2}} \) where \( u \)'s are the quadrature points [9]. The table show that for \( m=1 \), convergence is rather slow for all three choices of the similarity factor compared to the results in Bloch approach. Among the three choices, parameterization II shows better convergence.

Let us next discuss the nature of low lying levels and wavefunctions. First we show the ground state wavefunctions for all three similarity factors for a given choice of parameters in Fig. 1. As is anticipated, step function choice produces a non-smooth wavefunction. For parameterizations I and II, the wavefunctions show some structure near \( x=0.5 \). From our previous experience with calculations in the Bloch formalism, we believe that the structures indicate poor convergence with the number of grid points.

Consider now the structure of low lying levels. Recall that in the Bloch formalism, the ordering of levels was \( l = 0, 1, 0, \ldots \) corresponding to logarithmic potential in the nonrelativistic limit. In the presence of effective interactions generated by the similarity transformation, obviously the level ordering changes. Now we have additional confining interactions which, however, act differently in longitudinal and transverse directions. The wavefunctions are presented in Fig. 2a for parametrization II for the first four low lying levels.

There is extra freedom in parametrization II due to the presence of \( \Sigma M_{ij}^2 \) in the definition of \( u_{\sigma ij} \), Eq. (1.3). For zero transverse momentum of constituents, \( \Sigma M_{ij}^2 \) has the minimum value \( 8m^2 \). Thus relative insensitivity of parametrization to \( \sigma \) in parametrization II for small values of \( \sigma \) may be due to this factor. When we consider the heavy fermion mass limit, presence of \( 8m^2 \) in \( u_{\sigma ij} \) enhances the effect of similarity factor. In Fig. 2b we
present the wavefunctions corresponding to first four levels for parametrization II with $8m^2$ subtracted from $\Sigma M_{ij}^2$ in $u_{\sigma ij}$. From Figs. 2a and 2b note that the fourth level is different for parametrization II with and without $8m^2$ in $u_{\sigma ij}$.

By suitable choice of parameters we can study the interplay of rotationally symmetric logarithmically confining interaction and effective interactions generated by similarity transformation. Since for a given $g$, strength of the logarithmic interaction and similarity generated interactions are determined by $m$ and $\sigma$ respectively, for $m >> \sigma$ we should recover the Bloch spectrum [9]. Upto what levels the recovery occurs, of course depends on the exact value of $m$. As we already observed, for parametrization II this will happen only if $8m^2$ is subtracted from $\Sigma M_{ij}^2$. For this case, we present the first four levels for $m = 10$ and $\sigma = 4$ in Fig. 3 which clearly shows the level spacing corresponding to the Bloch spectrum.

Finally, we discuss the sensitivity of the spectra to the similarity scale $\sigma$. Ideally the low lying energy levels should be insensitive to $\sigma$. However we have calculated the effective Hamiltonian to only order $g^2$ and we expect significant sensitivity to $\sigma$. In Tables III and IV we present the lowest five eigenvalues for all three parametrizations of the similarity factor for $g = 0.2$ and $g = 0.6$ respectively. As expected $\sigma$ dependence is greater for larger value of $g$. Among the three parametrizations, the paramterization II is least sensitive to $\sigma$. In order to check whether this behaviour is due to the presence of $8m^2$ in $\Sigma M_{ij}^2$ in the definition of $u_{\sigma ij}$ we present the results in Table V for parametrization II with $8m^2$ subtracted from $\Sigma M_{ij}^2$. It is clear that sensitivity to $\sigma$ is still considerably less compared to the other two parametrizations. This may be due to the fact that $\Sigma M_{ij}^2$ is added to $\sigma^2$ in the definition of $\sigma$. Note that in parametrization II sensitivity to $\sigma$ is controlled also by additional parameters $u_0$ and $n_g$. The sensitivity to parameters $n_g$ and $u_0$ are presented in Tables VI and VII respectively.

VII. SUMMARY, DISCUSSION AND CONCLUSIONS

An attempt to solve 2+1 dimensional gauge theories using the Bloch effective Hamilto-

nian has revealed [9] problems due to uncancelled infrared divergences. They arise out of

vanishing energy denominators. Similarity renormalization formalism attempts to solve the

bound state problem in a two step process. At the first step, high energy degrees of freedom

are integrated out and ultraviolet renormalization carried out perturbatively. In the second

step, the effective Hamiltonian is diagonalized non-perturbatively. In this work, we have

studied the bound state problem in 2+1 dimensional gauge theories using the similarity

approach.

In order to have a better understanding of the numerical results, we have performed

analytical calculations with step function form for the similarity factor. Many interesting

results emerge from our analytical calculations. First of all, it is shown that due to the

presence of instantaneous interactions in gauge theories on the light front, the logarithmic

infrared divergence that appeared in the Bloch formalism persists in two places, namely

a part of the self energy contribution and the Coloumb interaction that gives rise to the

logarithmically confining potential in the nonrelativistic limit. However the terms that

persist are precisely those that produce a cancellation of resulting infrared divergences in the

bound state equation. The rest of the infrared problem that appeared in the Bloch formalism

due to the vanishing energy denominator problem is absent in the similarity formalism.
Similarity transformation however prevents the cancellation of the most severe $\frac{1}{(x-y)^2}$ singularity between instantaneous gluon exchange and transverse gluon exchange interactions and produces $\frac{1}{\sqrt{\delta}}$ divergences in the self energy and gluon exchange contributions which cancels between the two in the bound state equation. The resulting effective interaction between the quark and antiquark grows linearly with large transverse separation but grows only with the square root of the longitudinal separation. This produces violations of rotational symmetry in the bound state spectrum. We have also verified that no $\ln \delta$ divergence result from the $\frac{1}{x-y}$ singularity in the self energy and gluon exchange contributions.

In the Glazek-Wilson formalism the exact form of the similarity factor $f_\sigma$ is left unspecified. In the literature an exponential form has been used in numerical calculations [8]. For analytical calculations it is convenient to choose a step function even though it is well-known that it is not suitable for quantitative calculations [15]. There is also a proposal due to Stan Glazek which has two extra free parameters. We have tested all three parametrizations in our work. Our numerical results indeed show that step function choice always produces non-smooth wavefunctions. Parameterization II costs us an extra integration to be performed numerically but convergence is slightly better for small $g$ compared to exponential form. All three parametrizations produce violations of rotational symmetry even for small $g$. When an exponential form is used in the Glazek-Wilson formalism, the resulting effective Hamiltonian differs from the Wegner form only by an overall factor that restricts large energy differences between initial and final states. Numerically we have found this factor to be insignificant.

We have studied the sensitivity of the low lying eigenvalues to the similarity scale $\sigma$. Since the effective Hamiltonian is calculated only to order $g^2$ results do show sensitivity to $\sigma$. Among the three parametrizations the form II is least sensitive to $\sigma$ due to the functional form chosen. We have also studied the sensitivity of eigenvalues to the parameters $u_0$ and $n_g$.

The bound state equation has three parameters $m$, $g^2$ and $\sigma$ with dimension of mass. The strength of the logarithmically confining interaction is determined by $m$ and the strength of the rotational symmetry violating effective interactions generated by similarity transformation is determined by $\sigma$. For a given $g$ we expect the former to dominate over the latter for $m >> \sigma$. An examination of low lying eigenvalues and corresponding wavefunctions show that this is borne out by our numerical calculations.

A major problem in the calculations is the slow convergence. Compared to the Bloch formalism, in calculations with the similarity formalism various factors may contribute to this problem with the Gauss quadrature points. One important factor is the presence of linear confining interactions generated by the similarity transformation. It is well known that such interactions are highly singular in momentum space. Another factor is the presence of $\frac{1}{\sqrt{\delta}}$ divergences the cancellation of which is achieved numerically. It is of interest to carry out the same calculations with numerical procedures other than the Gauss quadrature. However, one should note that calculations in 3+1 dimensions employing basis functions and splines have also yielded [8] wavefunctions which show non-smooth structures.

An undesirable result of the similarity transformation carried out in perturbation theory is the violation of rotational symmetry. Our results show that this violation persists at all values of $g$ for $m = 1$. Such a violation was also observed in 3+1 dimensions. In that case the functional form of the logarithmic potential generated by similarity transformation is the same in longitudinal and transverse directions but the coefficients differ by a factor
of two. Same mechanism in 2+1 dimensions makes even the functional forms different. The important questions are whether the confining interactions generated by the similarity transformation are an artifact of the lowest order approximation and if they are not, then, whether the violation of rotational symmetry will diminish with higher order corrections to the effective Hamiltonian. Recall that the high energy degrees of freedom has been integrated out and the effective low energy Hamiltonian determined only to order $g^2$. A clear answer will emerge only after the determination of the effective Hamiltonian to fourth order in the coupling.
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FIGURES

FIG. 1. The ground state wavefunction for different choices of the similarity factor using the tan parametrization for transverse momentum grid and for $n_1 = 40$, $n_2 = 80$, $m=1.0$, $g = 0.2$, $\sigma = 4.0$ and $\eta = \delta = 10^{-5}$, $\kappa = 10.0$ as a function of $x$ and $k$. (A) Parametrization I, (B) Parametrization II with $n_g = 2$, $u_0 = 0.1$, $ns = 500$, (C) Parametrization III.

FIG. 2a. The wavefunctions corresponding to the lowest four eigenvalues as a function of $x$ and $k$ for parametrization II. The parameters are as in FIG. 1. (A) Lowest state, (B) first excited state, (C) second excited state, (D) third excited state.

FIG. 2b. Same as in FIG. 2a but with $8m^2$ subtracted from $\Sigma M_{ij}^2$.

FIG. 3. The wavefunctions corresponding to the lowest four eigenvalues as a function of $x$ and $k$ with parametrization III with $8m^2$ subtracted from $\Sigma M_{ij}^2$. The parameters are $g = 0.2$, $\eta = \delta = .00001$, $m = 10.0$, $\kappa = 10.0$, $n_g = 2$, $u_0 = 0.1$, $n_1 = 40$, $n_2 = 80$. (A) Lowest state, (B) first excited state, (C) second excited state, (D) third excited state.
| $\delta$ | Parametrization I |  |  |  |  |
|---|---|---|---|---|---|
| 0.1 | 4.89535 | 4.90359 | 4.90420 | 4.90420 | 4.90482 |
| 0.01 | 5.62612 | 6.38083 | 6.82963 | 7.20037 | 7.36414 |
| 0.001 | 5.68417 | 6.42147 | 6.90879 | 7.30609 | 7.64650 |
| 0.0001 | 5.68432 | 6.42148 | 6.90909 | 7.30611 | 7.64677 |
| 0.00001 | 5.68432 | 6.42148 | 6.90909 | 7.30611 | 7.64677 |
| $\delta$ | Parametrization II |  |  |  |  |
| 0.1 | 4.55364 | 4.55668 | 4.55668 | 4.55668 | 4.55669 |
| 0.01 | 4.86066 | 5.33491 | 5.49693 | 5.59838 | 5.79111 |
| 0.001 | 4.87607 | 5.35671 | 5.59226 | 5.64476 | 5.88613 |
| 0.0001 | 4.87604 | 5.35671 | 5.59236 | 5.64477 | 5.88615 |
| 0.00001 | 4.87604 | 5.35671 | 5.59236 | 5.64477 | 5.88615 |
| $\delta$ | Parametrization III |  |  |  |  |
| 0.1 | 5.12410 | 5.13039 | 5.13101 | 5.13101 | 5.13754 |
| 0.01 | 6.02600 | 6.94326 | 7.50445 | 7.98054 | 8.39927 |
| 0.001 | 6.00968 | 6.97160 | 7.55376 | 8.07749 | 8.49199 |
| 0.0001 | 5.96636 | 6.97160 | 7.51814 | 8.07751 | 8.46524 |
| 0.00001 | 5.96636 | 6.97160 | 7.51814 | 8.07751 | 8.46524 |

**TABLE I.** Variation with $\delta$ of the first five eigenvalues of the full Hamiltonian (excluding the less significant imaginary term). The parameters are $m = 1.0$, $g = 0.6$, $n_1 = 58$, $n_2 = 58$, $\eta = 0.00001$, $\Lambda = 20.0$, $\sigma = 4.0$, $(u_0 = 0.1$, $n_g = 2$ and $ns = 500$ (for $\sigma$ integration) in parametrization II)
| $n_1$ | $n_2$ | Parametrization I       |
|------|------|-------------------------|
| 10   | 10   | 4.320 4.353 4.357 4.357 4.361 |
| 20   | 20   | 4.375 4.442 4.484 4.484 4.485 |
| 20   | 30   | 4.398 4.482 4.546 4.583 4.610 |
| 20   | 40   | 4.411 4.502 4.570 4.615 4.656 |
| 30   | 40   | 4.412 4.503 4.570 4.615 4.655 |
| 40   | 50   | 4.420 4.515 4.585 4.634 4.678 |
| 40   | 60   | 4.426 4.524 4.594 4.645 4.692 |
| 40   | 80   | 4.434 4.535 4.607 4.661 4.709 |

| $n_1$ | $n_2$ | Parametrization II      |
|------|------|-------------------------|
| 10   | 10   | 4.163 4.194 4.194 4.194 4.203 |
| 20   | 20   | 4.186 4.244 4.276 4.276 4.277 |
| 20   | 30   | 4.192 4.256 4.296 4.323 4.329 |
| 20   | 40   | 4.195 4.262 4.304 4.335 4.344 |
| 30   | 40   | 4.195 4.262 4.304 4.335 4.344 |
| 40   | 50   | 4.197 4.266 4.308 4.341 4.353 |
| 40   | 60   | 4.199 4.268 4.311 4.345 4.359 |
| 40   | 80   | 4.201 4.272 4.315 4.350 4.367 |

| $n_1$ | $n_2$ | Parametrization III     |
|------|------|-------------------------|
| 10   | 10   | 4.360 4.379 4.381 4.381 4.391 |
| 20   | 20   | 4.469 4.533 4.572 4.572 4.572 |
| 20   | 30   | 4.503 4.604 4.674 4.721 4.749 |
| 20   | 40   | 4.520 4.632 4.703 4.768 4.810 |
| 30   | 40   | 4.528 4.636 4.714 4.768 4.811 |
| 40   | 50   | 4.542 4.657 4.736 4.797 4.848 |
| 40   | 60   | 4.548 4.668 4.748 4.813 4.866 |
| 40   | 80   | 4.556 4.683 4.764 4.833 4.889 |

**TABLE II.** Convergence of eigenvalues with $n_1$ and $n_2$ for the parametrization $k = \frac{1}{\kappa} \tan(q\pi/2)$. The parameters are $m=1.0$, $g=0.2$, $\eta = 0.00001$, $\delta = 0.00001$, $\kappa = 10.0$, $\sigma = 4.0$, $(u_0=0.1, n_g=2$ and the number of quadrature points $ns = 500$ for $\sigma$ integration for parametrization II).
| $\sigma$ | Parametrization I |     |     |     |     |     |
|---------|------------------|-----|-----|-----|-----|-----|
| 2.0     | 4.214            | 4.285 | 4.329 | 4.365 | 4.393 |
| 4.0     | 4.434            | 4.535 | 4.607 | 4.661 | 4.709 |
| 6.0     | 4.701            | 4.821 | 4.914 | 4.980 | 5.043 |
| $\sigma$ | Parametrization II |     |     |     |     |     |
| 2.0     | 4.157            | 4.214 | 4.248 | 4.260 | 4.276 |
| 4.0     | 4.201            | 4.272 | 4.315 | 4.350 | 4.367 |
| 6.0     | 4.266            | 4.350 | 4.404 | 4.446 | 4.483 |
| $\sigma$ | Parametrization III |     |     |     |     |     |
| 2.0     | 4.254            | 4.346 | 4.395 | 4.443 | 4.477 |
| 4.0     | 4.556            | 4.683 | 4.764 | 4.833 | 4.889 |
| 6.0     | 4.927            | 5.075 | 5.179 | 5.262 | 5.333 |

TABLE III. Variation with $\sigma$ of the full hamiltonian (excluding the imaginary term). The parameters are $m = 1.0$, $g = 0.2$, $n1 = 40$, $n2 = 80$, $\eta = 0.00001$, $\kappa = 10.0$, $(k = \frac{1}{\kappa}tan(q\pi/2))$, $\delta = 0.00001$, $(u_0 = 0.1$, $n_g = 2$, and $ns = 500$ for $\sigma$ integration for parameterization II)
| $\sigma$ | $\sigma$ | $\sigma$ |
|---------|---------|---------|
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| $\sigma$ | $\sigma$ | $\sigma$ |
| $g$ | $\sigma$ | Parametrization II |
|-----|--------|-------------------|
| 0.2 | 2.0    | 4.126 4.166 4.176 4.189 4.201 |
|     | 4.0    | 4.172 4.235 4.273 4.298 4.304 |
|     | 6.0    | 4.241 4.321 4.371 4.411 4.445 |
| 0.6 | 2.0    | 4.739 5.003 5.020 5.134 5.193 |
|     | 4.0    | 4.802 5.222 5.350 5.470 5.626 |
|     | 6.0    | 4.993 5.541 5.876 5.940 6.156 |

TABLE V. Variation with $\sigma$ of the full hamiltonian (excluding the imaginary term) after subtracting $8m^2$ from $\Sigma M^2_{ij}$ in the definition of $u_{\sigma ij}$. The parameters are $m = 1.0$, $\eta = 0.00001$, $\delta = 0.00001$, $u_0 = 0.1$, $n_g = 2$, and $n_s = 500$

1) for $g = 0.2$, $k = \frac{1}{\kappa} \tan(\frac{\kappa \pi}{2})$ with $\kappa = 10.0$ and $n_1 = 40$, $n_2 = 80$

2) for $g = 0.6$, $k = \frac{q A_m}{(1-q^2)A_m}$ with $A = 20.0$, and $n_1 = n_2 = 58$. 

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TABLE VI. Variation with $\sigma$ of the full Hamiltonian (excluding the imaginary term) after subtracting the $8m^2$ term from $\Sigma M^2_{ij}$ in the definition of $u_{\sigma ij}$. The parameters are $m = 1.0$, $\eta = 0.00001$, $\delta = 0.00001$, $u_0 = 0.1$, $n_g = 1$, and $n_s = 500$ $g = 0.6$, $k = \frac{q\Lambda m}{(1-q^2)\Lambda + m}$ with $\Lambda = 20.0$, and $n_1 = n_2 = 58$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$g$ & $\sigma$ & $M^2$ (Parameterization II) \\
\hline
0.6 & 2.0 & 4.781 & 5.049 & 5.051 & 5.161 & 5.221 \\
0.6 & 4.0 & 4.843 & 5.235 & 5.391 & 5.464 & 5.639 \\
0.6 & 6.0 & 5.030 & 5.538 & 5.848 & 5.956 & 6.106 \\
\hline
\end{tabular}
\end{table}

TABLE VII. Variation with $u_0$ of the full Hamiltonian (excluding the imaginary term) after subtracting the $8m^2$ term from $\Sigma M^2_{ij}$ in the definition of $u_{\sigma ij}$. The parameters are $m = 1.0$, $\eta = 0.00001$, $\delta = 0.00001$, $\sigma = 4.0$, $n_g = 2$, and $n_s = 500$ $g = 0.6$, $k = \frac{q\Lambda m}{(1-q^2)\Lambda + m}$ with $\Lambda = 20.0$, and $n_1 = n_2 = 58$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$g$ & $u_0$ & $M^2$ (Parameterization II) \\
\hline
0.2 & 0.2 & 4.990 & 5.548 & 5.887 & 5.969 & 6.171 \\
0.2 & 0.1 & 4.802 & 5.222 & 5.350 & 5.469 & 5.626 \\
0.6 & 0.05 & 4.746 & 5.076 & 5.091 & 5.253 & 5.313 \\
0.6 & 0.01 & 4.813 & 5.003 & 5.029 & 5.069 & 5.130 \\
\hline
\end{tabular}
\end{table}
Figure 2b
