A NOTE ON A MAP FROM KNOTS TO 2-KNOTS

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1. INTRODUCTION. BASIC NOTIONS

The aim of the present paper is to construct a well defined map $\alpha$ from classical knots to (some generalization of) 2-knots, which naturally works for any classical knots in codimension 2. This map works well for braids and tangles; besides, it is natural with respect to the boundary operation. In particular, if a classical knot $K$ is slice (i.e., can be spanned by a 2-dimensional object in four-dimensional half-space bounded by the 3-space where $K$ lives) then the corresponding object $\alpha(K)$ can also be bounded by a some trivial 2-dimensional object. This leads to sliceness obstructions for classical knot.

This work naturally is closely connected with [1], where various invariants of dynamical systems valued in groups were constructed. The main example from [1], see also [?], is the construction of an invariant of classical braids valued in a certain group $G_3^n$. Braids are considered as motions of points on the plane; the generators of this group correspond to those moments where some three points are collinear (form a horizontal quadrisecant); in some sense the abstract 2-knots (diagrams modulo moves) constructed here represent the “knot” counterpart, whence group elements (words modulo relations) are the braid counterpart.

Definition 1.1. A knot $K \subset \mathbb{R}^3$ is called slice if there is a 2-manifold $M^2$ with boundary properly embedded in $\mathbb{R}^4_+$ such that the image of $\partial M^2$ coincides with $K \subset \mathbb{R}^3 = \partial \mathbb{R}^4_+$.

Now, we pass to abstract surface knot diagrams, see, e.g., [2].

Definition 1.2. Let $F$ be a 2-surface; a double decker set is a 1-complex $D \subset F$ together with a pasting rule such that:

1. the pasting is an equivalence relation for $D$ which is continuous with respect to the topology of $F$;
2. each equivalence class consists of one, two or three points from $D$; these sets are denoted by $D_1, D_2, D_3$, respectively, points from $D_2$ are called double points.
3. $\text{Card}(D \setminus D_2)$ is finite; $D_1 \sqcup D_3 \subset \overline{D}_2$;
4. elements of each equivalence class are ordered; the ordering is continuous with respect to the topology of $F$. For each two pasted points $d_1, d_2$ such that $d_1 > d_2$ we say that $d_1$ lies over $d_2$;
5. let $(d_1, d_2)$ be a double point of $D$, say, $d_1 < d_2$; then in the neighbourhoods of $d_1$ and $d_2$ the double decker set represents two open intervals; these intervals consist of double points; points from one interval are identified with points from the other interval.

1This research is supported by the Russian Science Foundation, Project No. 16-11-10291
Definition 1.3. For a surface $F$ with a double decker set we naturally get a 2-complex $F^\sim$ obtained by identifying all equivalent points.

Such a 2-complex endowed with additional over/under structure is called an abstract 2-surface diagram.

Remark 1.4. From the definition above it follows that $D_1$ consists of cusps; a neighbourhood of a cusp topologically looks like a circle $|z| < 1$ where $z$ is identified with $-z$ with $z = 0$ being the cusp point; similarly, triple points topologically look like the intersection of three hyperplanes in $\mathbb{R}^3$.

As for double points, they naturally form double lines which are 1-manifolds possibly having boundary at cusps and triple points (as $K_1 \sqcup K_3 \subset K_2$).

An abstract knot diagram is not assumed to be embedded anywhere. Nevertheless, a local neighbourhood of any point of $F$ can be embedded in $\mathbb{R}^3$. Thus, it is natural to depict local parts of an abstract 2-surface in a way similar to projections of 2-knots in 3-space. They naturally appear as an general position orthogonal projection of a 2-knot in $\mathbb{R}^4$ to some 3-space; points having more than one preimage are identified; the partial order relation is defined with respect to the projection coordinate.

Usually, we deal with 2-surface knot diagrams when $F$ is connected and 2-surface link diagrams when $F$ consists of connected components $F = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n$; here we say that $n$ is the number of components.

Roseman moves [3] are initially defined for 2-knots in 3-space.

However, they are local and can be applied to abstract knot diagrams.

Theorem 1.5 (Roseman [3]). Two 2-knot (resp., 2-link) diagrams $D, D'$ represent isotopic 2-knots (resp., 2-links) if and only if one can get from $D$ to $D'$ by a finite sequence of Roseman moves, shown in Figs. 1–7, and isotopies in $\mathbb{R}^3$.

Later on, we make no difference between combinatorially equivalent (abstract) knot diagrams.

Instead of giving a definition of a 2-knot diagram, we shall pass to a more general definition (for abstract knots).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The first Roseman move $\mathcal{R}_1$: elliptic move of type $\Omega_2$.}
\end{figure}

Definition 1.6. An abstract surface-knot is an equivalence class of abstract surface diagrams modulo Roseman moves. Namely, two diagrams $D, D'$ are called equivalent if there is a sequence of diagrams $D = D_1 \to D_2 \to \cdots \to D_n = D'$, such that each two diagrams $D_i, D_{i+1}$ are related by a Roseman move.
Figure 2. The second Roseman move $\mathcal{R}_2$: hyperbolic type $\Omega_2$ move.

Figure 3. The third Roseman move $\mathcal{R}_3$: elliptic $\Omega_1$ move.

Figure 4. The fourth Roseman move $\mathcal{R}_4$: hyperbolic $\Omega_1$ move.
2. THE CASE OF 1-KNOTS

Let $S^1_\phi$ be the angular unit sphere whose points are identified to unit vectors $e^{i\phi}$. Sometimes we also use the notation $-\phi$ for $\phi + \pi$.

Let $K$ be a knot in $\mathbb{R}^3$. Later on, we shall require some general position for $K$. We shall be especially interested in the height function for $K$; whenever mentioning “minimum” or “maximum” we assume extrema with respect to this function. Let $S(K) = T^2 = S^1 \times S^1_\phi$ be the torus where the first coordinate corresponds to the knot $K$ and the second coordinate is $\phi$ for $S^1 = S^1_\phi$.

Now, we make an abstract 2-knot diagram $D(K)$ of $S(K)$, as follows. We define the equivalence for pairs of points $(k, \phi), (k', \phi), k \in K, \phi \in S^1$ that $k$ and $k'$ are points on $K$ lying on the same horizontal plane, and $\phi$ is such that the pointing
Figure 7. The seventh Roseman move $R_7$: the tetrahedral move.

vector from $k$ to $k$ is collinear with $\phi$ (or $-\phi$); hence, if $(k, \phi) \sim (k', \phi)$ then $(k, -\phi) \sim (k', -\phi)$ and vice versa; moreover, if $(k, \phi)$ lies over $(k', \phi)$ then $(k, -\phi)$ lies under $(k', -\phi)$. We are going to identify the equivalent points;

**Definition 2.1.** We say that $K \in \mathbb{R}^3$ is in general position condition if the above identification leads to a double decker set. Namely, we require that

1. No four points $(k_1, \phi), (k_2, \phi), (k_3, \phi), (k_4, \phi)$ are equivalent. In other words, this means that $K$ has no horizontal quadrisecant.
2. The number of triples $(k_1, \phi) (k_2, \phi) (k_3, \phi)$ is finite; this means that $K$ has finitely many horizontal quadrisecants.
3. There is a finite number of cusps;
4. All other double points $X_1 = (k_1, \phi) \sim (k_2, \phi) = X_2$ are regular in the following sense. Topologically, the set of double points in the neighbourhood of $X_1$ is homeomorphic to an open interval as well as the set of double points in the neighbourhood of $X_2$, and these two intervals are identified.

**Remark 2.2.** Note that regular points can correspond to local extrema. For instance, assume $K$ locally looks like the graph of the parabola $z = x^2, y = \text{const}$ and another branch of $K$ looks just like a vertical line $x = y = 1$. Let $X = ((0, 0, 0), \pi/4)$ be origin with the vector pointing to the point $(1, 1)$. Then the neighbourhood of $X$ consists of two parts, the one with $(\varepsilon, 0, \varepsilon^2)$ for $\varepsilon > 0$ and the one for $\varepsilon < 0$; in both cases the vector pointing to $(0, 0, \varepsilon^2)$ is close to $\pi/4$. Then $D_2$ will locally consist of two double lines, one of which corresponds to the neighbourhood of $\pi/4$ and the other one corresponds to the neighbourhood of $-\pi/4$.

By $\alpha(K)$ we denote the abstract 2-knot represented by $D(K)$.

Now, we are ready to formulate the main theorem.

**Theorem 2.3.** If knots $K$ and $K'$ are isotopic in $\mathbb{R}^3$ then the abstract surface 2-knots $\alpha(K)$ and $\alpha(K')$ are equivalent.

The proof consists of case-by-case consideration of all situations where $K$ fails to be generic. As an example, we say that if we pass through a horizontal quadrisecant passing through some four points $a, b, c, d \in K$ with the same $z$-coordinate, then yields two “antipodal” seventh Roseman moves, see Fig.7. Indeed, denoting the angles of the quadrisecant by $\phi$ and $-\phi$, we can see four sheets of the surface
these four sheets have four double points, and the effect of the passing through the quadrisecant is the inversion of the tetrahedron.

The same happens for the direction \(-\phi\).

Now, the same definition and the same proof generalizes for many other objects: braids, tangles, and even 2-knots and knots in higher dimensions. For tangles (or knots in \(\mathbb{R}^2 \times [0, \infty]\)) we assume \(K\) to be a properly embedded (into \([0, 1]\) or \([0, \infty]\)) collection of closed intervals; the resulting object \(\alpha(K)\) will be an equivalence class of corresponding 2-surface diagrams with boundary; braids are a special case of tangles.

3. Dimension 2

More importantly, the same definition works in higher dimensions.

Let \(K\) be a 2-knot in 3-space. Here, we are interested in planes \(z = \text{const}, t = \text{const}\). As before, we let \(S(K) = K \times S^1_\phi\), where the first coordinate corresponds to the knot \(K\) and the second coordinate is \(\phi\) for \(S^1 = S^1_\phi\).

For each point \(x \in K\), we consider the plane \(z = \text{const}, t = \text{const}\) passing through this point. In general position, for each plane \(P = \{z = c_1, t = c_2\}\), the intersection \(K \cap P\) consists of finitely many points.

We are not going to give the formal definition of the abstract surface 3-diagram and an abstract surface 3-knot (link). They are quite similar to the 2-dimensional case. However, one should point out that the set of Roseman moves in higher dimension which originates from codimension 1 singularities for 3-knot projections to 4-space, is finite and all such moves are local (The same works in any dimension). As in the 2-dimensional case, abstract 3-knot is the equivalence class of diagrams modulo moves.

As before, we identify those points \((k, \alpha)\) and \((k', \alpha)\) if \(k\) and \(k'\) have the same third and fourth coordinate.

We can think of this map as follows. Having a 2-knot \(K\), we can consider its slices \(K_c = K \cap \{t = c\}\); for generic \(c\), these slices are just 1-links in 3-space; we can think of \(K\) as “glued” of all \(K_c\) for distinct \(c\). Thus, \(\alpha(K)\) is glued in the same way from distinct \(\alpha(K_c)\).

**Theorem 3.1.** If 2-knots \(K\) and \(K'\) are isotopic in \(\mathbb{R}^4\) then the abstract 3-knots \(\alpha(K)\) and \(\alpha(K')\) are equivalent.

The main observation is that for each neighbourhood \((\phi - \varepsilon, \phi + \varepsilon)\) the picture of \(\alpha(K)\) can be represented as a collection of sheets in the Euclidean space. Thus, whenever we perform some isotopy of \(K\) and restrict, every combinatorial diagram can be drawn in the Euclidean space and is subject to local moves in these Eucliedan space (for dimension 2 these are exactly Roseman moves).

**Theorem 3.2.** If \(K\) is slice then \(\alpha(K)\) is slice.

The above generalization for the case of 2-knots works as well for any \(n\)-manifold \(M^n\) embedded in some fixed space of \(\mathbb{R}^2 \times N^n\) for some manifold \(N^n\); the only thing we need here is the existence of the “first” two coordinates \(x, y\) so that we can take “horizontal” slices \(\mathbb{R}^2 \times \{\ast\}\) which intersect \(M^n \subset \mathbb{R}^2 \times N^n\) at finitely many points.
References

[1] V.O. Manturov, Non-Reidemeister Knot Theory and Its Applications in Dynamical Systems, Geometry, and Topology, [http://arxiv.org/abs/1501.05208](http://arxiv.org/abs/1501.05208)

[2] N. Kamada, S. Kamada, Abstract Link Diagrams and Virtual Knots, J. Knot Theory Ramifications, Vol. 9, N.1 (2000)

[3] D. Roseman, “Reidemeister-type moves for surfaces in four-dimensional space”, Knot Theory, Banach Center Publications, Vol. 42 (Polish Academy of Sciences, Warsaw, 1998), pp. 347–380

[4] D. Roseman, Elementary moves for higher-dimensional knots, Fund. Math. 184 (2004), p.291-310.

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