On convolution powers of $1/x$

Andreas B.G. Blobel
andreas.blobel@kabelmail.de

March 21, 2022

Abstract

Convolution powers of $1/x$ are transformed into functions $f_n$, which satisfy a simple recurrence relation. Solutions are characterized and analyzed.

Contents

0. Overview . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
1. Convolution powers of $1/x$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
2. Some definitions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
3. Decomposition of $f_n$ in terms of $J^{n-k[1]}$ . . . . . . . . . . . . . . . . . . . 5
4. Decomposition of $J^{n[1]}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

List of Tables

1 Matrices $A^o$ through $A^6$ computed from $(34a)/(34b)$ . . . . . . . . . . . . . . 9

0. Overview

In section 1 it is shown, that convolution powers of $1/x$ may be transformed into functions $f_n$, as given in $(5a)/(5b)$. It is proven by induction, that the $f_n$ solve the recurrence relation $(8a)/(8b)/(8c)$ and satisfy the reflection property $(14)$.

Section 2 contains some definitions used in sections 3 and 4.

In section 3 it is shown, that the functions $f_n$ can be expressed as linear combinations $(24)$ of functions $J^{n-k[1]}$, with coefficients $\beta_k$ being determined by the recurrence relation $(26a)/(26b)$. 

1
Section 4 is dedicated to the further analysis of the functions $J^n[1]$, $n$-fold powers of the $J$ operator applied to the 1-function. They can be decomposed into expressions of the type $Q_j(x) \cdot \ln(x)^{n-j}$, with functions $Q_j$ being generated by the recurrence relation. The functions $Q_j$, in turn, are power series in $1/x$, whose coefficients are determined by some matrices $A_{m,j}$, which may be computed from. Example data are listed in Table 1.

1. Convolution powers of $1/x$

Let $\lambda$ and $a$ be real parameters fulfilling the condition

$$\lambda + a > 0$$

(1)

Define real functions $\varphi_{\lambda,a} : \mathbb{R} \to \mathbb{R}$

$$\varphi_{\lambda,a}(x) = \begin{cases} 0 & x < \lambda \\ \frac{1}{x + a} & x \geq \lambda \end{cases}$$

(2a)

(2b)

Convolution powers of $\varphi_{\lambda,a}$ are defined by the recurrence relation

$$\varphi_{\lambda,a}^{*1} = \varphi_{\lambda,a}$$

(3a)

$$\varphi_{\lambda,a}^{*(n+1)}(x) = \int_{-\infty}^{\infty} \varphi_{\lambda,a}(x-t) \cdot \varphi_{\lambda,a}^{*n}(t) \, dt \quad x \in \mathbb{R} \quad n \geq 1$$

(3b)

Cut-off condition (2a) translates into the general property

$$\varphi_{\lambda,a}^{*n}(x) = 0 \quad x < n \cdot \lambda \quad n \geq 1$$

(4)

Regarding the non-zero part of $\varphi_{\lambda,a}^{*n}(x)$, consider the transformation

$$\varphi_{\lambda,a}^{*n}(x) = \frac{n!}{x + n \cdot a} \cdot f_{n-1} \left( \frac{x - n \cdot \lambda}{\lambda + a} \right) \quad x \geq n \cdot \lambda \quad n \geq 1$$

(5a)

$$f_{n-1}(y) = \frac{y + n}{n!} \cdot (\lambda + a) \cdot \varphi_{\lambda,a}^{*n}((\lambda + a) \cdot y + n \cdot \lambda) \quad y \geq 0 \quad n \geq 1$$

(5b)

The functions $f_{n}$ are defined on non-negative real numbers, and their index $n$ starts from zero.
Rewriting (3a)/(3b) in terms of \( f_n \) yields

\[
fo = 1 \quad (6a)
\]

\[
(n + 1) \cdot f_n(y) = (y + n + 1) \cdot \int_o^y \frac{1}{y - s + 1} \cdot \frac{1}{s + n} \cdot f_{n-1}(s) \; ds
\]

\[
= \int_o^y \frac{1}{y - s + 1} \cdot f_{n-1}(s) \; ds + \int_o^y \frac{1}{s + n} \cdot f_{n-1}(s) \; ds
\]

\[
y \geq 0 \quad n \geq 1 \quad (6b)
\]

This makes clear, that through the transformation (5a)/(5b), the parameters \( \lambda \) and \( a \) have been eliminated. (6b) implies

\[
f_n(0) = 0 \quad n \geq 1 \quad (7)
\]

Consider the set of conditions

\[
f_o = 1 \quad (8a)
\]

\[
f_n(0) = 0 \quad n \geq 1 \quad (8b)
\]

\[
f'_n(y) = \frac{1}{y + n} \cdot f_{n-1}(y) \quad y \geq 0 \quad n \geq 1 \quad (8c)
\]

(8a) and (8b) just repeat (6a) and (7). In the recurrence relation (8c), \( f'_n \) denotes the derivative of \( f_n \) with respect to its argument.

Assertion (8c) will be proved by induction. Replacing \( n = 1 \) in (6b), while observing (6a), gives

\[
2 \cdot f_1(y) = \int_o^y \frac{1}{y - s + 1} \; ds + \int_o^y \frac{1}{s + 1} \; ds
\]

\[
= \left[ - \ln(y - s + 1) + \ln(s + 1) \right]_{s = y}^{s = 0}
\]

\[
= 2 \cdot \ln(y + 1) \quad y \geq 0 \quad (9)
\]

which verifies the induction hypothesis (8c) for \( n = 1 \).

When deriving (6b) with respect to \( y \), while applying the Leibniz integral rule, one gets

\[
(n + 1) \cdot f'_n(y) = \left( 1 + \frac{1}{y + n} \right) \cdot f_{n-1}(y)
\]

\[
- \int_o^y \frac{1}{(y - s + 1)^2} \cdot f_{n-1}(s) \; ds \quad y \geq 0 \quad n \geq 1 \quad (10)
\]
Substituting $n \to n + 1$, and applying partial integration to the second term with respect to $s$ yields

$$
(n + 2) \cdot f'_{n+1}(y) = \frac{1}{y + n + 1} \cdot f_n(y) + \frac{1}{y + 1} \cdot f_n(0)
$$

$$
+ \int_y^\infty \frac{1}{y - s + 1} \cdot f'_n(s) \, ds \quad y \geq 0 \quad n \geq 0
$$

(11)

On the other hand, using the induction hypothesis (8c) and (6b), we have

$$
\int_y^\infty \frac{1}{y - s + 1} \cdot f'_n(s) \, ds = \int_y^\infty \frac{1}{y - s + 1} \cdot \frac{1}{s + n} \cdot f_{n-1}(s) \, ds
$$

$$
= \frac{n + 1}{y + n + 1} \cdot f_n(y) \quad y \geq 0 \quad n \geq 1
$$

(12)

Finally, inserting (12) into (11) gives

$$
f'_{n+1}(y) = \frac{1}{n + 2} \cdot \left( \frac{n + 2}{y + n + 1} \cdot f_n(y) + \frac{1}{y + 1} \cdot f_n(0) \right) \quad y \geq 0 \quad n \geq 1
$$

(13)

from which the induction step follows, if one observes (8b). □

The (non-recursive) reflection property

$$
\int_y^\infty \frac{1}{y - s + 1} \cdot f_n(s) \, ds = (n + 1) \cdot \int_y^\infty \frac{1}{s + n + 1} \cdot f_n(s) \, ds \quad y \geq 0 \quad n \geq 0
$$

(14)

follows as a corollary, if one replaces

$$
f_n(y) = \int_y^\infty f'_n(s) \, ds \quad (8c) \quad \int_y^\infty \frac{1}{s + n} \cdot f_{n-1}(s) \, ds
$$

in the left-hand side of (6b).

2. Some definitions

Given a smooth, real function $f$, declare the ‘harmonic’ integration operator $H$ as the antiderivative

$$
H[f] := \int \frac{f(x)}{x} \, dx
$$

(15)

In particular, if $g$ denotes a power series in $1/x$ with radius of convergence $C > 0$

$$
g(x) = \sum_{k=0}^\infty a_k \cdot x^{-k} \quad x > 1/C > 0
$$

(16)
one gets
\[ H[g](x) = a_o \cdot \ln(x) - \sum_{k=1}^{\infty} \frac{a_k}{k} \cdot x^{-k} \quad x > 1/C \] (17)

Here, \( \ln \) denotes the natural logarithm. Clearly, if \( g(x) \) converges for \( x > 1/C \), so does the second term in (17).

More generally, applying \( H \) multiple times, we get
\[ H^n[g](x) = a_o \cdot \ln(x)^m + (-1)^m \cdot \sum_{k=1}^{\infty} \frac{a_k}{k^m} \cdot x^{-k} \quad m \geq 0 \quad x > 1/C \] (18)

Let \( \nabla \) denote the backward difference operator \[ \nabla[f](x) := f(x) - f(x - 1) \] (19)

Applying \( \nabla \) to (16) and to the natural logarithm gives
\[ \nabla[g](x) = - \sum_{k=2}^{\infty} \left( \sum_{r=1}^{k-1} \frac{(k-1)!}{r!} \cdot a_{k-r} \right) \cdot x^{-k} \quad x > 1/C + 1 \] (20a)
\[ \nabla[\ln](x) = \text{Li}_1(1/x) \quad x > 1 \] (20b)

In (20b), the notation \( \text{Li}_s \) refers to the polylogarithm. Finally, declare the operators
\[ S := I - \nabla \] (21a)
\[ J := H \circ S \] (21b)

In (21a), \( I \) denotes the identity operator.

3. Decomposition of \( f_n \) in terms of \( J^{n-k}[1] \)

It is shown in this section, that solutions \( f_n \) of the recurrence relation (8a)/(8b)/(8c) are linear combinations of \( J^{n-k}[1] \). Applying \( x \cdot \frac{d}{dx} \) to \( J^m[1] \), one gets
\[ x \cdot \frac{d}{dx} J^m[1](x) = x \cdot \frac{d}{dx} H \left[ S \left[ J^{m-1}[1](x) \right] \right] = S \left[ J^{m-1}[1](x) \right] = J^{m-1}[1](x - 1) \quad x \geq m \quad m \geq 1 \] (22)

Hence, replacing \( x \rightarrow y + n \) in (22), for \( n \geq m \):
\[ (y + n) \cdot \frac{d}{dy} J^m[1](y + n) = J^{m-1}[1](y + n - 1) \quad y \geq 0 \quad n \geq m \geq 1 \] (23)
Therefore, if one writes
\[ f_n(y) = \sum_{k=0}^{n} \beta_k \cdot J^{n-k}[1] (y + n) \quad y \geq 0 \quad n \geq 0 \] (24)
and applies \((y + n) \cdot \frac{d}{dy}\) to (24), while making use of (23), one gets
\[ (y + n) \cdot \frac{d}{dy} f_n(y) = \sum_{k=0}^{n} \beta_k \cdot (y + n) \cdot \frac{d}{dy} J^{n-k}[1] (y + n) \]
\[ = \sum_{k=0}^{n-1} \beta_k \cdot J^{n-1-k}[1] (y + n - 1) \]
\[ = f_{n-1}(y) \quad y \geq 0 \quad n \geq 1 \] (25)
Here, the fact has been exploited, that \(\frac{d}{dy} J^{0}[1] = \frac{d}{dy} 1 = 0\).
(25) clearly reproduces (24c), while (23a)/(23b) impose the following recurrence relation on the coefficients:
\[ \beta_0 = 1 \] (26a)
\[ \beta_n = - \sum_{k=0}^{n-1} \beta_k \cdot J^{n-k}[1] (n) \quad n \geq 1 \] (26b)
The first few instances of \(\beta_n\) are listed here:
\[ \beta_0 = 1 \]
\[ \beta_1 = 0 \]
\[ \beta_2 = - J^2[1] (2) \]
\[ \beta_3 = - J^3[1] (3) + J^2[1] (2) \cdot J[1] (3) \]
\[ \beta_4 = - J^4[1] (4) + J^2[1] (2) \cdot J^2[1] (4) + \left( J^3[1] (3) - J^2[1] (2) \cdot J[1] (3) \right) \cdot J[1] (4) \] (27)
Here, use has been made of the property \(J[1] (1) = \ln(1) = 0\).

4. Decomposition of \(J^n[1]\)

Results in this section are stated without giving detailed proofs.
\(J^n[1]\), \(n\)-fold powers of the \(J\) operator (21b) applied to the 1-function, may be expanded in terms of powers of the natural logarithm:
\[ J^n[1](x) = \sum_{j=0}^{n} (-1)^j \cdot Q_j(x) \cdot \frac{\ln(x)^{n-j}}{(n-j)!} \quad x \geq n \quad n \geq 0 \] (28)
with coefficient functions $Q_j$ fulfilling the recurrence relation

\[ Q_o = 1 \]

\[ Q_{n+1}(x) = -H \left[ \frac{1}{n!} \cdot \text{Li}_1(1/x)^n - \nabla[Q_n](x) \right] \quad x \geq n + 1 \quad n \geq 0 \]  

(29a)

(29b)

Operators $H$ and $\nabla$ are declared in (15) and (19) respectively. The notation $\text{Li}_s$ refers to the polylogarithm \[Wiki\].

The first instances are listed here explicitly:

\[ Q_o(x) = 1 \]  

(30a)

\[ Q_1(x) = 0 \]  

(30b)

\[ Q_2(x) = -H \left[ \text{Li}_1(1/x) \right] = \text{Li}_2(1/x) \quad x \geq 2 \]  

(30c)

\[ Q_3(x) = -H \left[ \frac{1}{2} \cdot \text{Li}_1(1/x)^2 - \nabla[\text{Li}_2(1/x)] \right] \]

\[ = \sum_{k=2}^{\infty} \frac{1}{k} \cdot \left( \frac{1}{k!} \cdot \left[ \frac{k}{2} \right] + \sum_{r=1}^{k-1} \left( \frac{1}{r} - \frac{1}{(k-r)^2} \right) \right) \cdot x^{-k} \quad x \geq 3 \]  

(30d)

In (30d), use has been made of (20a) and the identity \[Sed09\]

\[ \frac{1}{n!} \cdot \text{Li}_1(z^n) = \sum_{k=n}^{\infty} \frac{1}{k!} \left[ \frac{k}{n} \right] \cdot z^k \quad |z| < 1 \quad n \geq 0 \]  

(31)

Here, the notation $\left[ \frac{k}{n} \right]$ refers to Stirling numbers of the 1\textsuperscript{st} kind \[Wiki\].

For $n \geq 2$, the functions $Q_n(x)$ are power series in $1/x$:

\[ Q_n(x) = \sum_{s=0}^{\infty} q_{n,s} \cdot x^{-s} \quad x \geq n \geq 2 \]  

(32)

The first non-zero coefficient of $Q_n(x)$ is $q_{n,n-1}$. In particular, it turns out that

\[ q_{n+1,s} = 0 \quad 0 \leq s < n \quad n \geq 1 \]  

(33a)

\[ q_{n+1,s} = \frac{1}{s} \cdot \frac{1}{s} \cdot \sum_{\nu=0}^{n-1} \sum_{\sigma=0}^{\nu+s-n} \left[ \frac{s-\sigma}{n-\nu} \right] \cdot \left( \frac{s}{\sigma} \right) \cdot A^s_{\sigma,\nu} \quad s \geq n \quad n \geq 1 \]  

(33b)

(33b) involves Stirling numbers 1\textsuperscript{st}, binomials, and the quantities $A^s_{m,j}$, which are lower triangular integer matrices of dimension $s + 1$, whose elements obey the recurrence relation

\[ A^s_{m,o} = \delta_{m,o} \quad s \geq m \geq 0 \]  

(34a)

\[ A^s_{m,j} = \sum_{\mu=0}^{m-j} A^s_{m-\mu-1,j-1} \cdot \left( \frac{m}{\mu+1} \right) \cdot (s - m + 1)^{\mu} \quad s \geq m \geq j \geq 1 \]  

(34b)
In (34a), $\delta_{i,j}$ denotes the Kronecker delta. (34b) involves binomials and rising factorials. Special values of $A_{m,j}^s$ are

$$A_{m,o}^s = \delta_{m,o} \quad s \geq m \geq 0 \quad (35a)$$

$$A_{m,1}^s = (s - 1)^{m-1} \quad s \geq m \geq 1 \quad (35b)$$

$$A_{m,m}^s = m! \quad s \geq m \geq 0 \quad (35c)$$

The notation in (35b) refers to falling factorials. From the triangle shape of $A^s$ and (35c) it follows [Intb]:

$$\det (A^s) = \prod_{k=0}^{s} k! \quad s \geq 0 \quad (36)$$

Integers in the last row of $A^s$

$$A_{s,j}^s \quad 0 \leq j \leq s \quad (37)$$

are related to the Bell matrix with generator $1/j$ for $j \geq 1$ [Lus] and match [Inta]. Example data are listed in Table I.
\begin{table}
\centering
\begin{tabular}{c|ccccccc}
  \( j \) & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
  \( m \) & \\
  0 & 1 & & & & & & \\
  1 & 0 & 1 & & & & & \\
  0 & 1 & 0 & 0 & & & & \\
  1 & 0 & 1 & 0 & & & & \\
  2 & 0 & 1 & 2 & & & & \\
  0 & 1 & 0 & 0 & 0 & & & \\
  1 & 0 & 1 & 0 & 0 & & & \\
  2 & 0 & 2 & 2 & 0 & & & \\
  3 & 0 & 2 & 9 & 6 & & & \\
  0 & 1 & 0 & 0 & 0 & 0 & & \\
  1 & 0 & 1 & 0 & 0 & 0 & & \\
  2 & 0 & 3 & 2 & 0 & 0 & & \\
  3 & 0 & 6 & 15 & 6 & 0 & & \\
  4 & 0 & 6 & 50 & 72 & 24 & & \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
  1 & 0 & 1 & 0 & 0 & 0 & 0 & \\
  2 & 0 & 4 & 2 & 0 & 0 & 0 & \\
  3 & 0 & 12 & 21 & 6 & 0 & 0 & \\
  4 & 0 & 24 & 120 & 108 & 24 & 0 & \\
  5 & 0 & 24 & 350 & 850 & 600 & 120 & \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
  1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
  2 & 0 & 5 & 2 & 0 & 0 & 0 & 0 & & \\
  3 & 0 & 20 & 27 & 6 & 0 & 0 & 0 & \\
  4 & 0 & 60 & 218 & 144 & 24 & 0 & 0 & \\
  5 & 0 & 120 & 1120 & 1750 & 840 & 120 & 0 & \\
  6 & 0 & 120 & 3014 & 11250 & 12900 & 5400 & 720 & \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{tabular}
\caption{Matrices \( A^0 \) through \( A^6 \) computed from (34a)/(34b)}
\end{table}
References

[Sed09] Philippe Flajolet; Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009, p. 736.

[Inta] Online Encyclopedia of Integer Sequences. *Bell matrix with generator 1/j*. URL: https://oeis.org/A265607.

[Intb] Online Encyclopedia of Integer Sequences. *Superfactorials: product of first n factorials*. URL: https://oeis.org/A000178.

[Lus] Peter Luschny. *The Bell matrix*. URL: https://oeis.org/wiki/User:Peter_Luschny/BellTransform.

[Wika] Wikipedia. *Antiderivative*. URL: https://en.wikipedia.org/wiki/Antiderivative.

[Wikb] Wikipedia. *Convolution power*. URL: https://en.wikipedia.org/wiki/Convolution_power.

[Wikc] Wikipedia. *Finite difference*. URL: https://en.wikipedia.org/wiki/Finite_difference.

[Wikd] Wikipedia. *Leibniz integral rule*. URL: https://en.wikipedia.org/wiki/Leibniz_integral_rule.

[Wike] Wikipedia. *Polylogarithm*. URL: https://en.wikipedia.org/wiki/Polylogarithm.

[Wikf] Wikipedia. *Stirling numbers of the first kind*. URL: https://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind.