LIFTING INVOLUTIONS IN A WEYL GROUP
TO THE NORMALIZER OF THE TORUS

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Abstract. Let $N$ be the normalizer of a maximal torus $T$ in a split reductive group over $F_q$ and let $w$ be an involution in the Weyl group $N/T$. We construct a section of $W$ satisfying the braid relations, such that the image of the lift $n$ of $w$ under the Frobenius map is equal to the inverse of $n$.

1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field $F$. Let $T$ be a maximal torus in $G$, and let $W = N/T$ denote the associated Weyl group, where $N$ denotes the normalizer of $T$ in $G$, and let $X_*(T)$ denote the cocharacter lattice of $T$. Fix a realization of the root system $\Phi$ in $G$ (see [2]), a set of positive roots $\Phi^+$, and let $\Delta$ be the associated set of simple roots. We obtain the Tits section $w \mapsto \dot{\omega}$ of the natural map $N \to W$ [Tit66].

Let us recall some setup from a recent work of Lusztig [Lus18i]. If $F$ is an algebraic closure of $F_q$, we define $\phi : F \to F$ by $\phi(c) = c^q$, and if $F = \mathbb{C}$, we define $\phi : F \to F$ by $\phi(c) = \overline{c}$ (complex conjugation). In the first case, we assume that $G$ has a fixed $F_q$-rational structure with Frobenius map $\phi : G \to G$ such that $\phi(t) = t^q$ for all $t \in T$. In the second case, we assume that $G$ has a fixed $\mathbb{R}$-structure so that $G(\mathbb{R})$ is the fixed point set of an antiholomorphic involution $\phi : G \to G$ such that $\phi(y(c)) = y(\phi(c))$ for all $y \in X_*(T), c \in F^\times$. We may also assume that $\phi(\dot{\omega}) = \dot{\omega}$ for any $\omega \in W$.

Now let $w$ be an involution in $W$. In [Lus18i], a lift $n$ of $w$ was constructed such that $\phi(n) = n^{-1}$. The construction was quite complicated, and involved reduction arguments and case by case computations. In this paper, we construct a natural section $S$ of the entire Weyl group $W$ that satisfies the braid relations, which accomplishes the same result (namely, that $\phi(S(w)) = S(w)^{-1}$ for any involution $w$ in $W$). Our methodology in one sense illustrates the power of the braid relations, allowing us to prove the main result quickly. We moreover note that in a recent work [Adr22], all sections of the Weyl group that satisfy the braid relations were computed, for an almost-simple connected reductive group over an algebraically closed field.

We explain our method. Let $S$ be any section of $W$ (by a section of $W$, we mean a section of the map $N \to W$). For $\alpha \in \Delta$, we may write $S(s_\alpha t_\alpha) = t_\alpha s_\alpha$ for some $t_\alpha \in T$, where $s_\alpha \in W$ is the simple reflection associated to $\alpha$. Let $z_\alpha \in F^\times$ be arbitrary, with $\alpha \in \Delta$, and consider now the specific torus elements $t_\alpha = \alpha^\alpha(z_\alpha)$.
We show that the map $s_\alpha \mapsto t_\alpha s_\alpha$ extends to a section, denoted $S$, of $W$ that satisfies the braid relations. We then prove that the condition $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ for all $\alpha \in \Delta$ implies that $\phi(S(w)) = S(w)^{-1}$ for any involution $w$ in $W$. We conclude that if we define a section of $W$ by the property $s_\alpha \mapsto \alpha^\vee(z_\alpha)s_\alpha$ where $z_\alpha \in F^\times$ for all $\alpha \in \Delta$, with the property $\alpha^\vee(\phi(z_\alpha)z_\alpha^{-1}) = \alpha^\vee(-1)$, then we accomplish the goal of the paper. In the finite field case, this equality can be accomplished by setting $z_\alpha$ to be a $q-1$ root of $-1$ for every $\alpha \in \Delta$, and in the real case the equality can be accomplished by setting $z_\alpha$ to be a primitive fourth root of unity for every $\alpha \in \Delta$. Our main result, therefore, is:

**Theorem 1.1.** If $F$ is an algebraic closure of $\mathbb{F}_q$, set $\zeta$ to be a $q-1$ root of $-1$. If $F = \mathbb{C}$, set $\zeta$ to be a primitive fourth root of unity.

For each $\alpha \in \Delta$, define the map $s_\alpha \mapsto \alpha^\vee(\zeta)s_\alpha$. This maps extends to a section $S : W \to N$ that satisfies the braid relations. Moreover, for any involution $w$ in $W$, $\phi(S(w)) = S(w)^{-1}$.

One utility of finding a lifting of the involutions in the Weyl group, with the property that Frobenius acts by inversion, can be found in an ensuing paper of Lusztig [Lus18ii]. A key role is played by such a lifting in proving one of the main results (Theorem 0.4) of the paper.

### 2. Preliminaries

We first remind the reader of the definition of Tits’ section from [Tit66], as well as some generalities about general sections. We follow [Spr98] §8.1, §9.3 closely. Let $G$ be a connected reductive group over an algebraically closed field $F$, let $T$ be a maximal torus in $G$, and let $\Phi$ be the associated set of roots. For each $\alpha \in \Phi$, let $s_\alpha$ denote the associated reflection in the Weyl group $W = N/T$.

**Proposition 2.1** ([Spr98 Proposition 8.1.1]).

1. For $\alpha \in \Phi$ there exists an isomorphism $u_\alpha$ of $G_\alpha$ onto a unique closed subgroup $U_\alpha$ of $G$ such that $t_\alpha(x)t_\alpha^{-1} = u_\alpha(\alpha(x)t) \ (t \in T, x \in F)$.
2. $T$ and the $U_\alpha$ ($\alpha \in \Phi$) generate $G$.

Tits then defines a representative $\sigma_\alpha$, of $s_\alpha$, in $N$:

**Lemma 2.2** ([Spr98 Lemma 8.1.4]).

1. The $u_\alpha$ may be chosen such that for all $\alpha \in \Phi$,

\[ \sigma_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1) \]

lies in $N$ and has image $s_\alpha$ in $W$. For $x \in F^\times$, we have

\[ u_\alpha(x)u_{-\alpha}(-x^{-1})u_\alpha(x) = \alpha^\vee(x)\sigma_\alpha; \]

2. $\sigma_\alpha^2 = \alpha^\vee(-1)$ and $\sigma_{-\alpha} = \sigma_\alpha^{-1}$;
3. If $u \in U_\alpha - \{1\}$ there is a unique $u' \in U_{-\alpha} - \{1\}$ such that $uu'u \in N$;
4. If $(u'_\alpha)_{\alpha \in \Phi}$ is a second family with the property (1) of Proposition 2.1 and property (1) of Lemma 2.2 there exist $c_\alpha \in F^\times$ such that

\[ u'_\alpha(x) = u_\alpha(c_\alpha x), \ c_\alpha c_{-\alpha} = 1 \ (\alpha \in \Phi, x \in F). \]

A family $(u_\alpha)_{\alpha \in \Phi}$ with the properties (1) of Proposition 2.1 and Lemma 2.2 is called a realization of the root system $\Phi$ in $G$ (see [Spr98] §8.1).
Proposition 2.3 ([Spr98, Proposition 8.3.3]). Let \( \mu \) be a map of \( S \) into a multiplicative monoid with the property: if \( s, t \in S, \ s \neq t, \) then
\[
\mu(s)\mu(t)\mu(s)\cdots = \mu(t)\mu(s)\mu(t)\cdots,
\]
where in both sides the number of factors is \( m(s,t) \). Then there exists a unique extension of \( \mu \) to \( W \) such that if \( s_1\cdots s_h \) is a reduced decomposition for \( w \in W \), we have
\[
\mu(w) = \mu(s_1)\cdots \mu(s_h).
\]

We now fix a realization \( (u_\alpha)_{\alpha \in \Phi} \) of \( \Phi \) in \( G \). Let \( \alpha, \beta \in \Phi \) be linearly independent. We denote \( m(\alpha, \beta) \) the order of \( s_\alpha s_\beta \). Then \( m(\alpha, \beta) \) equals one of the integers 2, 3, 4, 6.

Proposition 2.4 ([Spr98, Proposition 9.3.2]). Assume that \( \alpha \) and \( \beta \) are simple roots, relative to some system of positive roots. Then
\[
\sigma_\alpha \sigma_\beta \sigma_\alpha \cdots = \sigma_\beta \sigma_\alpha \sigma_\beta \cdots,
\]
the number of factors on either side being \( m(\alpha, \beta) \).

Following [Spr98, §9.3.3], let \( w = s_{\alpha_1}\cdots s_{\alpha_h} \) be a reduced expression for \( w \in W \), with \( \alpha_1, \ldots, \alpha_h \in \Delta \). The element \( N_0(w) := \sigma_{\alpha_1}\cdots \sigma_{\alpha_h} \) is independent of the choice of reduced expression of \( w \). We therefore obtain a section \( N_0 : W \to N \) of the homomorphism \( N \to W \). This is the section of Tits [Tit66].

3. The section \( S \)

By Proposition 2.3 any section \( S \) of \( W \) satisfying the braid relations is determined by its values on a set of simple reflections. Let us write \( S(s_\alpha) = t_\alpha \sigma_\alpha \) for some \( t_\alpha \in T \).

Let \( \alpha, \beta \in \Delta \). In order that \( t_\alpha \sigma_\alpha \) and \( t_\beta \sigma_\beta \) satisfy the braid relations, it is necessary and sufficient that
\[
t_\alpha \sigma_\alpha t_\beta \sigma_\beta t_\alpha \sigma_\alpha \cdots = t_\beta \sigma_\beta t_\alpha \sigma_\alpha t_\beta \sigma_\beta \cdots,
\]
where in both sides the number of factors is \( m(\alpha, \beta) \). As \( \sigma_\alpha t_\beta \sigma_\alpha^{-1} = s_\alpha(t_\beta) \), and since the \( \sigma \) satisfy the braid relations, \((1)\) is equivalent to
\[
t_\alpha s_\alpha(t_\beta)s_\beta s_\alpha(t_\beta) \cdots = t_\beta s_\beta(t_\alpha)s_\beta s_\alpha(t_\beta) \cdots.
\]

Proposition 3.1. Choose any \( z_\alpha \in F^\times \), for \( \alpha \in \Delta \). Then the map \( s_\alpha \mapsto \alpha^\vee(z_\alpha) \sigma_\alpha \) extends to a section \( S : W \to N \) which satisfies the braid relations.

Proof. As in the proof of [Spr98 Proposition 9.3.2], we need only check the cases \( A_1 \times A_1, A_2, B_2, \) and \( G_2 \).

\( A_1 \times A_1 \): We compute
\[
\alpha^\vee(z_\alpha)s_\alpha(\beta^\vee(z_\beta)) = \alpha^\vee(z_\alpha)\beta^\vee(z_\beta) = \beta^\vee(z_\beta)s_\beta(\alpha^\vee(z_\alpha))
\]
since \( s_\alpha, s_\beta \) commute.

\( A_2 \): We compute
\[
t_\alpha s_\alpha(t_\beta)s_\alpha s_\beta(t_\alpha)
= \alpha^\vee(z_\alpha)s_\alpha(\beta^\vee(z_\beta))s_\alpha s_\beta(\alpha^\vee(z_\alpha)) = \alpha^\vee(z_\alpha)(\alpha + \beta)^\vee(z_\beta)s_\alpha(\alpha + \beta)^\vee(z_\alpha)
= \alpha^\vee(z_\alpha)(\alpha + \beta)^\vee(z_\beta)\beta^\vee(z_\alpha) = \alpha^\vee(z_\alpha)\alpha^\vee(z_\beta)\beta^\vee(z_\alpha)\beta^\vee(z_\beta)
\]
Proposition 3.3. Let $\beta$ be the short root, $\alpha$ the long root. We have $s_\alpha(\beta^\vee) = (\alpha + \beta)^\vee = \alpha^\vee + \beta^\vee$. Thus,

$$t_\alpha s_\alpha(t_\beta)s_\alpha s_\alpha(t_\beta) = s_\alpha(\beta^\vee) s_\alpha(\alpha^\vee) s_\alpha s_\alpha(t_\beta) = s_\alpha(\beta^\vee) s_\alpha(\alpha^\vee) s_\alpha s_\alpha(t_\beta).$$

One may now compute the same result for $t_\beta s_\beta(t_\alpha)s_\beta s_\alpha(t_\beta)$. □

We need the following result about involutions, see [Deo82] Theorem 5.4].

Proposition 3.2. Any involution $w \in W$ can be obtained starting from the involution $e$ by a sequence of length-increasing operations that are either multiplication of an involution by a simple reflection $s_\alpha$ with which it commutes, or conjugation by a simple reflection $s_\alpha$ with which it does not commute.

Proof. By induction on the length $\ell(w)$. If $\ell(w) = 0$, then $w = e$. So suppose that $\ell(w) > 0$, and let $\alpha$ be a simple root such that $\ell(ws_\alpha) < \ell(w)$. Distinguish the cases on whether or not $s_\alpha$ commutes with $w$. If it commutes, then $ws_\alpha$ is an involution, and $w$ is obtained from it by commuting multiplication by $s_\alpha$. If they don’t commute, then we can see as follows that $\ell(s_\alpha ws_\alpha) = \ell(w) - 2$. From $\ell(ws_\alpha) < \ell(w)$, there is a reduced expression for $w$ that ends with $s_\alpha$, and reversing it we obtain a reduced expression for $w^{-1} = w$ starting with $s_\alpha$, say $s_\alpha s_{\alpha_2} \cdots s_{\alpha_{(w)}}$. Now the exchange condition says that an expression for $ws_\alpha$ can be obtained by striking out one of the generators in the latter reduced expression, and it cannot be the initial $s_\alpha$ as that would give $s_\alpha w$ which is supposed to differ from $ws_\alpha$. Now left-multiplying by $s_\alpha$ gives an expression for $s_\alpha ws_\alpha$ obtained by striking out a generator in $s_{\alpha_2} \cdots s_{\alpha_{(w)}}$, and therefore of length $\ell(w) - 2$. This $s_\alpha ws_\alpha$ is an involution of shorter length than $w$, from which $w$ can be obtained by conjugation by $s_\alpha$ with which it does not commute. □

Proposition 3.3. Choose any $z_\alpha \in F^\times$, for $\alpha \in \Delta$. Suppose that $t_\alpha = \alpha^\vee(z_\alpha)$, for $\alpha \in \Delta$. If $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ for all $\alpha \in \Delta$, then $\phi(S(w)) = S(w)^{-1}$ for any involution $w$ in $W$.  

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Proof. We induct using Proposition 3.2. First we let \( w = s_\alpha \), a simple reflection. Then \( \phi(S(w))S(w) = \phi(t_\alpha s_\alpha) t_\alpha s_\alpha = \phi(t_\alpha) s_\alpha t_\alpha s_\alpha = \phi(t_\alpha) t_\alpha^{-1} \alpha^\vee(-1) = 1 \), since \( \phi(w) = \bar{w} \) for all \( w \in W \) and since \( \sigma_\alpha t_\alpha \sigma_\alpha^{-1} = t_\alpha^{-1} \).

Now suppose \( w \in W \) is an involution satisfying \( \phi(S(w)) = S(w)^{-1} \). We need to show firstly that \( \phi(S(s_\alpha w)) = S(s_\alpha w)^{-1} \) for any \( \alpha \in \Delta \) such that \( s_\alpha \) commutes with \( w \) and \( \ell(s_\alpha w) > \ell(w) \), and secondly that \( \phi(S(s_\alpha ws_\alpha)) = S(s_\alpha ws_\alpha)^{-1} \) for any \( \alpha \in \Delta \) with \( \ell(s_\alpha ws_\alpha) > \ell(w) \), such that \( s_\alpha \) and \( w \) do not commute.

Suppose that \( \alpha \in \Delta \), \( s_\alpha \) commutes with \( w \), and \( \ell(s_\alpha w) > \ell(w) \). Then

\[
\phi(S(s_\alpha w))S(s_\alpha w) = \phi(t_\alpha s_\alpha S(w))S(ws_\alpha) = \phi(t_\alpha) s_\alpha S(w)^{-1} S(w) t_\alpha s_\alpha
\]

where in the above we are using that \( S \) satisfies the braid relations and that \( s_\alpha \) commutes with \( w \).

Moreover, for any \( \alpha \in \Delta \) with \( \ell(s_\alpha ws_\alpha) > \ell(w) \), we also have

\[
\phi(S(s_\alpha ws_\alpha))S(s_\alpha ws_\alpha) = \phi(t_\alpha s_\alpha S(w) t_\alpha s_\alpha) t_\alpha s_\alpha S(w) t_\alpha s_\alpha
\]

\[
= \phi(t_\alpha) s_\alpha S(w)^{-1} \phi(t_\alpha) s_\alpha t_\alpha s_\alpha S(w) t_\alpha s_\alpha = \phi(t_\alpha) s_\alpha S(w)^{-1} \phi(t_\alpha) t_\alpha^{-1} \alpha^\vee(-1) S(w) t_\alpha s_\alpha
\]

\[
= \phi(t_\alpha) s_\alpha t_\alpha s_\alpha = \phi(t_\alpha) t_\alpha^{-1} \alpha^\vee(-1) = 1,
\]

where again we are using that \( S \) satisfies the braid relations, and that \( s_\alpha \) and \( w \) do not commute. \( \square \)

Proposition 3.4.

(1) Let \( F \) be an algebraic closure of \( \mathbb{F}_q \). Set \( \zeta \) to be a \( q - 1 \) root of \(-1 \). If \( t_\alpha = \alpha^\vee(\zeta) \) for all \( \alpha \in \Delta \), then \( \phi(t_\alpha) = \alpha^\vee(-1) t_\alpha \) for all \( \alpha \in \Delta \).

(2) Let \( F = \mathbb{C} \). Set \( \zeta \) to be a primitive fourth root of unity. If \( t_\alpha = \alpha^\vee(\zeta) \) for all \( \alpha \in \Delta \), then \( \phi(t_\alpha) = \alpha^\vee(-1) t_\alpha \) for all \( \alpha \in \Delta \).

Proof. For (1): Note that \( \phi(t_\alpha) = t_\alpha^q \), so if we set \( t_\alpha = \alpha^\vee(z_\alpha) \), then the desired equality \( \phi(t_\alpha) = \alpha^\vee(-1) t_\alpha \) is equivalent to \( \alpha^\vee(z_\alpha^{-1}) = \alpha^\vee(-1) \). Setting \( z_\alpha \) to be a \( q - 1 \) root of \(-1 \) gives us our result.

For (2): Recall that \( \phi(y(c)) = y(\phi(c)) \) for all \( y \in X_*(T), c \in F^x \). Setting \( t_\alpha = \alpha^\vee(z_\alpha) \), then the desired equality \( \phi(t_\alpha) = \alpha^\vee(-1) t_\alpha \) is equivalent to \( \alpha^\vee(z_\alpha^{-1}) = \alpha^\vee(-1) \). Setting \( z_\alpha \) to be a primitive fourth root of unity gives us our result. \( \square \)

By Proposition 3.1, Proposition 3.3 and Proposition 3.4, we have now proven Theorem 1.1.

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