Large Fixed-Diameter Graphs are Good Expanders

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Abstract

We revisit the classical question of the relationship between the diameter of a graph and its expansion properties. One direction is well understood: expander graphs exhibit essentially the lowest possible diameter. We focus on the reverse direction. We show that “sufficiently large” graphs of fixed diameter and degree must be “good” expanders. We prove this statement for various definitions of “sufficiently large” (multiplicative/additive factor from the largest possible size), for different forms of expansion (edge, vertex, and spectral expansion), and for both directed and undirected graphs. A recurring theme is that the lower the diameter of the graph and (more importantly) the larger its size, the better the expansion guarantees. We discuss the implications of our results for open questions in graph theory and for recent advances in computer networking, and leave the reader with many interesting open questions.

1 Introduction

Both the diameter of a graph and its expansion capture the “connectedness” of the graph, albeit in two very different senses. The diameter, i.e., the maximal distance between a pair of vertices, provides an upper bound on the length of shortest paths in the graph, whereas expansion measures the minimal ratio between a subset of vertices and its boundary. We revisit the classical question of relating these two traits. One direction is well known: good expansion implies a low diameter. Specifically, the diameter of a graph with good expansion is \( O(\log n) \) (see, e.g., [21]), which is asymptotically the lowest possible. We focus on the opposite, and largely unexplored, direction.

Generally speaking, low diameter is insufficient to guarantee good expansion. Consider, for example, a graph on \( n \) vertices that is a disjoint union of two separate cliques, each of size \( \frac{n}{2} \). Now we can remove one edge from each clique and connect the cliques via two “bridges” to obtain a \((\frac{n}{2} - 1)\)-regular graph of diameter 3 with very low expansion (which worsens as \( n \to \infty \)). We observe, however, that this “bad” graph is significantly smaller than the largest \((\frac{n}{2} - 1)\)-regular graph of diameter 3 (which is of size \( \Omega(n^3) \)). Indeed, our investigation below reveals that, in contrast to the above, when the degree and the diameter are fixed and the size of the graph is “sufficiently large”, the graph must have “good” expansion. We formalize this statement for different notions of “large”, for different forms of expansion (edge, vertex, and spectral expansion), and for undirected/directed graphs.

The Moore Bound. How large can a \( d \)-regular graph of diameter \( k \), which we shall refer to as a “\((d, k)\)-graph”, be? An upper bound on the size of such a graph is the well-studied Moore Bound [20], denoted by \( \mu_{d,k} \). Over the years, extensive research has been devoted to determining the existence of graphs whose sizes match this upper bound (a.k.a., Moore Graphs) or well-approximate

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Expansion guarantees

\((d,k)\)-graph

\[ n \geq \mu_{d,k} - O(d^{k/2}) \]

\[ \lambda(G) = O(\sqrt{d}) \]

\[ h_e(G) \geq \frac{\alpha d}{2k} \cdot \left(1 - \frac{1}{(d-1)^{k}}\right) \]

\[ \phi_V(G) \geq \frac{\alpha}{2(k-1)+\alpha} \]

\[ k=2 \]

\[ n = \alpha \cdot d^2 \]

\[ h_e(G) \geq \frac{\alpha d}{2d+1-\sqrt{4(1-\alpha)d^2+4d+1}} \]

\[ \phi_V(G) \geq \frac{\alpha}{2d+1} \]

\[ k=3 \]

\[ n = \alpha \cdot d^3 \]

\[ \phi_V(G) \geq \frac{\alpha}{2d+1} \]

\[ (d,k)\)-digraph

\[ n = \alpha \cdot \mu_{d,k} \]

\[ h_e(G) \geq \frac{\alpha}{2d+1-\sqrt{4(1-\alpha)d^2+4d+1}} \]

\[ \phi_V(G) \geq \frac{\alpha}{2d+1} \]

Table 1: Summary of Results Relating Expansion and Diameter.

Expansion vs. diameter. Our results relating the size of a \((d,k)\)-graph to its expansion are summarized in Table 1, with \(\lambda(G)\), \(h_e(G)\), and \(\phi_V(G)\) denoting spectral expansion, edge expansion, and vertex expansion, respectively.

Our analysis begins with the examination of spectral expansion. Our main result (in Section 3) establishes that if \(G\) is a \((d,k)\)-graph of size \(n \geq \mu_{d,k} - O(d^{k/2})\), then \(\lambda(G) = O(\sqrt{d})\), where \(\lambda(G)\) is the second largest absolute eigenvalue of \(G\). Since \(\lambda(G)\) is known to be at least \(\sqrt{d}\) [21], the spectral gap of such graphs is essentially optimal.

The proof of this result relies on the analysis of irreducible paths in the graph. A path is irreducible if it does not traverse an edge back and forth consecutively. We prove that the matrix that corresponds to all irreducible paths of length at most \(k\) must consist of strictly positive entries and shares all eigenvectors of the adjacency matrix \(A\). We establish the above algebraic relation between the two matrices by employing the Geronimus Polynomials [7, 37], a well-known class of orthogonal polynomials, as operators acting on the adjacency matrix. Given the spectrum of \(A\), an asymptotic estimation of the polynomials’ coefficients allows us to bound the spectrum of the irreducible paths matrix. We then subtract from the latter the all-ones matrix and use the leading eigenvalue of the remaining matrix (which can be computed directly) to bound the second eigenvalue of the adjacency matrix \(A\), thus establishing the expansion guarantee.

We next turn our attention to combinatorial notions of expansion: edge expansion and vertex expansion. We provide (in Section 4) guarantees on both the edge and the vertex expansion of \((d,k)\)-graphs in terms of their multiplicative distance from the Moore Bound. Our analysis leverages careful counting arguments to bound the ratio between the cardinality of a set of vertices and the size of its boundary. Importantly, our results along these lines provide expansion guarantees with respect to any degree and any diameter, with no hard constraints on the size of the graph (unlike the above result for spectral expansion) and pertain to both directed and undirected graphs. We also prove, through more refined analyses, improved results for the extensively studied cases of diameters 2 and 3.

“Not-so-sparse” expanders. Importantly, our research diverges from the main vein of prior research on expanders. Expanders are commonly viewed as highly-connected sparse graphs. Indeed, the bulk of literature on this topic assumes that the degree of these graphs is essentially constant.
Important examples include the design of feedback registers [17, 27], decoders [10], and network architectures [4, 5, 6, 16, 26, 35].

### Implications for the construction of (approximate) Moore Graphs.

Much research has been devoted to realizing, or at least well-approximating, the Moore Bound [33]. Unfortunately, matching the Moore Bound is, in general, impossible. Hoffman and Singleton [20] proved that a Moore Graph of diameter 2 must have degree 2, 3, 7, or 57. Subsequent studies established that no Moore Graphs exist for diameter \(k \geq 3\) [11] and that, for every \(d\) and \(k\), only finitely many \((d, k)\)-graphs are of size \(\mu_{d,k} - 1\) [2]. Consequently, various constructions for generating graphs whose sizes come “close” to the Moore Bound, which we term “approximate-Moore Graphs”, have been devised. The most general of those is perhaps the undirected de Bruijn graph, which is of size \((\frac{d}{2})^k\) and can be constructed for any diameter \(k\) and even degree \(d\) [12]. Canale and Gomez [9] presented a graph construction of size \(n \geq \left(\frac{d}{1.57}\right)^k\) for infinitely many values of \(d\) and \(k\). The state of the art construction of \((d, 2)\)-graphs are the celebrated polarity graphs, which are of size with the size of \(n = d^2 - d + 1\), and were introduced by Erdös and Rényi [15], and then independently by Brown [8].

Surprisingly, little is known about the approximability of the Moore Bound for large \((d, k)\)-graphs (our main object of study). In fact, for \(d, k \geq 3\), while the best known upper bound on the size of these graphs is essentially \(\mu_{d,k} \sim d^k\), the best universal existing construction generates graphs of size only \((\frac{d}{2})^k\). Closing this gap is an important open question. Our results can be seen as an attempt to shed light on the nature of these obscure creatures, showing that the question of well-approximating the Moore Bound is closely linked to that of generating good expanders under very restrictive constraints on their structure and size.

Beyond establishing general results for the expansion properties of approximate-Moore graphs,
our results imply new expansion guarantees for a number of well-studied graph constructions. These implications are summarized in Table 2, and constitute (to the best of our knowledge) the best-known combinatorial expansion guarantees for these constructions. The reader is referred to Appendix C for more details.

**Implications for (datacenter) network design.** Aside from inherent theoretical interest, our motivation stems from the domain of network design, where the focus on either the diameter or the expansion gave rise to two competing approaches for datacenter architectures [39, 36, 29, 6, 23, 24]. Our results show that these two approaches are, in fact, inextricably intertwined.

## 2 Preliminaries

We provide below a brief exposition of graph expansion and the Moore Bound. We refer the reader to [21] and [33] for detailed expositions of these topics.

**Graph expansion.** Let \( G = (V, E) \) be an undirected graph of size \( |V| = n \). \( G \) is said to be \( d \)-regular if each of its vertices is of degree \( d \), and of diameter \( k \) if the maximum distance between any two vertices in the graph is \( k \). \( d \)-regular graphs of diameter \( k \) are denoted throughout the paper as \((d, k)\)-graphs.

The combinatorial expansion of the graph reflects an isoperimetric view and is the minimal ratio between the boundary \( \partial S \) of a set \( S \) and its cardinality. Different interpretations of \( \partial S \) give rise to different notions of expansion.

*The edge expansion* of \( G \) is

\[
h_e(G) := \min_{|S| \leq \frac{n}{2}} \frac{|e(S, S^c)|}{|S|},
\]

where \( e(S, S^c) := \{(u, v) \in E | u \in S, v \in S^c\} \).

*The vertex expansion* of \( G \) is

\[
\phi_V(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|N(S)|}{|S|}.
\]

where \( N(S) := \{v \in S \mid \exists u \in S \text{ s.t. } (u, v) \in E\} \).

We next define the algebraic (spectral) notion of expansion. Let \( A \) be the adjacency matrix of the graph. Since \( A \) is symmetric it is diagonalizable with respect to an orthonormal basis, and the corresponding eigenvalues are real, and so can be ordered as follows:

\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.
\]

The first eigenvalue of a \( d \)-regular graph satisfies \( \lambda_1 = d \) and has the all-ones vector \( 1_n \) as the associated eigenvector. Let \( \lambda(G) := \max\{|\lambda_2|, |\lambda_n|\} \). A graph \( G \) is said to be an expander if \( \lambda(G) \) is bounded away from \( d \) by some constant [1]. The *algebraic expansion* (or spectral expansion) is then defined as \( d - \lambda(G) \), termed the *spectral gap*. The larger the gap, the better the expansion.

**The Moore Bound.** How large can a \((d, k)\)-graph be? A straightforward upper bound is obtained by summation of the vertices according to their distance from a fixed vertex \( v_0 \in V \). Let \( m_j \) denote the number of vertices at distance \( j \) from \( v_0 \). Note that \( m_0 = 1 \) and \( m_1 = d \). As vertices at distance \( j \geq 2 \) must be adjacent to some vertex at distance \( j - 1 \), we have that \( m_j \leq (d - 1)m_{j-1} \). A simple

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1The definitions of edge and vertex expansion admit several variants, based on either the size of the cut or the type of the boundary (see [21] for examples). While we tried to follow the most common of those, our results can be stated w.r.t. these variants as well.
induction implies that $m_j \leq d(d-1)^{j-1}$. Now since the diameter is $k$, all vertices have distance at most $k$ from $v_0$, and hence

$$n \leq 1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{k-1}.$$ 

We denote this expression, known as the Moore Bound of the graph, by

$$\mu_{d,k} := 1 + d \sum_{i=0}^{k-1} (d-1)^i = \begin{cases} 2k + 1 & \text{if } d = 2 \\ 1 + d \cdot \frac{(d-1)^{k-1}}{d-2} & \text{if } d > 2 \end{cases}$$

### 3 Diameter vs. Algebraic Expansion

The main result of this section is the following:

**Theorem 3.1.** If $G$ is a $d$-regular graph of diameter $k$ and size $n \geq \mu_{d,k} - O(d^{k/2})$, then $\lambda(G) = O(\sqrt{d})$.

Recall that the celebrated Alon-Boppana bound [34, 18] does not hold in this scenario (since $d$ is not a constant w.r.t. the size of the graph). Yet it is still possible to show (via the trace method) that a $d$-regular graph must satisfy $\lambda(G) \geq \sqrt{d}$ (see [21] for details). Hence, Theorem 3.1 implies that an additive approximation of $O(d^{k/2})$ to the Moore Bound implies essentially optimal spectral properties. The remainder of this section is devoted to the proof of the theorem.

Our high-level approach to proving Theorem 3.1 is the following. We aim to bound $\lambda(G)$, the second-largest eigenvalue (in absolute value) of the adjacency matrix $A$. We instead consider a different matrix $M$ that shares $A$’s eigenvectors, and whose eigenvalues can be derived from $A$’s eigenvalues. $M$ is obtained from $A$ by employing the Geronimus Polynomials. Using this class of polynomials as operators over $A$ generates a matrix whose entries correspond to the number of walks between pairs of vertices that do not traverse the same edge in consecutive hops. We prove a new bound on the coefficients of these polynomials, allowing us to show that for any eigenvector $v$ of $A$ with eigenvalue $\lambda = \Theta(d^\alpha)$, the eigenvalue of $v$ in $M$ is $\Theta(d^{k\alpha})$. By applying the Perron-Frobenius Theorem, we are able to show that the largest eigenvalue of $M$ equals $\mu_{d,k} - n$, which is $O(d^{k/2})$ by the assumption of the theorem. We thus conclude that $\alpha \leq 1/2$ and $\lambda(G) \leq O(\sqrt{d})$.

#### 3.1 Geronimus Polynomials

We begin with the definition of the following class of polynomials. Let $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - d$, and for every $t > 2$ define $P_t(x)$ by the recurrence relation

$$P_t(x) = xP_{t-1}(x) - (d-1)P_{t-2}(x).$$

These polynomials are well studied and are often referred to as the “Geronimus Polynomials” in the literature (see [7], [37]). The solution to the recurrence is usually formulated via the trigonometric expression

$$P_t(2\sqrt{d-1}; \cos \theta) = (d-1)^{t/2-1}(d-1)\sin((t+1)\theta) - \sin((t-1)\theta)$$

which holds for all $t > 0$ [38]. One can easily check that this identity applies for $t = 1, 2$ and verify that the recurrence relation holds for $t > 2$. All roots of $P_t$ are real and lie in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ [3, 28].
Applying the Geronimus Polynomials as operators over the adjacency matrix $A$ has several advantages. Algebraically, since $P_t(A)$ is a linear combination of powers of $A$, each eigenvector $v$ of $A$ is an eigenvector of $P_t(A)$ as well. It follows that the spectrum of $P_t(A)$ is given by $\text{spec}[P_t(A)] = \{P_t(\lambda) \mid \lambda$ is an eigenvalue of $A\}$. From a combinatorial perspective, this operation allows us to dismiss reducible paths from consideration. By reducible, we refer to paths that traverse an edge in both directions consecutively. Such paths can be reduced by eliminating this back and forth traversal. Note that an irreducible path need not be simple (a non-trivial cycle is a typical example of an irreducible yet non-simple path). The following claim states this observation formally. The proof is straightforward and is included for completeness.

Claim 3.2. Let $A$ be the adjacency matrix of a $d$-regular graph $G$. Then, $P_t(A)$ is the $n \times n$ matrix in which the $(u, v)$'th entry equals the number of irreducible paths of length exactly $t$ between $u$ and $v$.

Proof. We use induction on $t$. Note that $P_0(A) = I$, $P_1(A) = A$ and $P_2(A) = A^2 - dI$ satisfy the claim. For the inductive step, suppose that the claim holds for all Geronimus Polynomials of order strictly less than $t$.

Consider the term $A \cdot P_{t-1}(A)$, which corresponds to paths of length $t$ such that the first $t-1$ legs in the path are irreducible. Note that reducible paths in this term are those paths that can only be reduced by eliminating their last two arcs and so there must be exactly $(d-1)P_{t-2}(A)$ of them. Being the difference between those quantities, it follows that $P_t(A) = A \cdot P_{t-1}(A) - (d-1)P_{t-2}(A)$ corresponds to the irreducible paths\(^2\) in $A$.

As a corollary, the entries of $P_t(A)$ are non-negative for all $t \geq 0$. In addition, as $d(d-1)^{t-1}$ is the number of irreducible paths of length $t > 0$ starting from every $v \in G$, this quantity equals the sum of entries in every row of $P_t(A)$. In another words, Claim 3.2 implies that

$$P_t(A)1_n = d(d-1)^{t-1}1_n. \quad (2)$$

3.2 Estimating the Coefficients

For our purposes, it will be beneficial to use the representation $P_t(x) = \sum_{i=0}^{t} a_{t,i}x^i$, where $a_{t,i}$ is the $i$'th coefficient of the $t$'th Geronimus Polynomial. We note the following:

- $P_t$ is either odd or even\(^3\) for all $t > 0$, and the parity of $P_t$ equals the parity of $t$. This can be shown either by induction using the recurrence relation, or straightforward from the solution (1).

- A comparison of the leading coefficients in the recurrence implies that $a_{t,t} = a_{t-1,t-1}$. Applying the boundary conditions ($a_{1,1} = a_{0,0} = 1$) yields $a_{t,t} = 1$ for all $t > 0$.

- Setting $\theta = \pi/2$ in (1) yields $a_{t,0} = d(d-1)^{t/2-1}(-1)^{t/2}$ whenever $t$ is even.

The following easy-to-prove claim provides us with asymptotic estimates for the rest of the coefficients.

\(^2\)We point out that there is another approach to tackling irreducible paths in a graph, namely, Hashimoto’s non-backtracking operator [19]. This operator has been shown to be useful in a variety of scenarios (e.g., localization and centrality [30], clustering [25], and percolation [22] in networks). Claim 3.2 implies that Hashimoto’s operator is, in fact, closely related to $P_t(A)$. The main advantage of using the latter lies in its relation to the spectrum (the set of eigenvalues) of $A$.

\(^3\)A polynomial $q(x)$ is said to be even if $q(x) = q(-x)$ and odd if $q(-x) = -q(x)$.
Claim 3.3. Let $P_t(x) = \sum_{i=0}^{t} a_{t,i} x^i$ denote the Geronimous polynomial of order $t$, then

$$a_{t,i} = \begin{cases} \Theta \left( \frac{d^i}{i^2} \right) & \text{if } (t - i) \text{ is even} \\ 0 & \text{if } (t - i) \text{ is odd} \end{cases}$$

for all $0 \leq i \leq t$.

**Proof.** The parity of $P_t$ implies that $a_{t,i} = 0$ whenever $(t - i)$ is odd. Hence, it suffices to consider the case of an even difference, which we prove by induction on $(t - i)$. As $a_{t,t} = 1$, the basis of the induction holds.

Assume that the claim holds for every $t',i'$ for which $t' - i' < t - i$. A comparison of the $i'$th coefficient in the recurrence yields

$$a_{t,i} = a_{t-1,i-1} + (d-1)a_{t-2,i} = a_{t-1,i-1} + (d-1)\Theta(d^\frac{t-2-i}{2}) = a_{t-1,i-1} + \Theta(d^\frac{i}{t})$$

where the second equality is due to the induction hypothesis. Now, if $a_{t-1,i-1} = O(d^\frac{i}{t})$ then the claim holds. If this is not the case, we have $a_{t,i} = \Theta(a_{t-1,i-1})$.

We continue iteratively and obtain that $a_{t,i} = \Theta(a_{t-1,0}) = \Theta(d^{i/2})$, thus proving the claim. \( \square \)

We conclude with an asymptotic formulation of $P_t(x)$.

**Corollary 3.4.** The Geronimous Polynomial of order $t$ can be written as

$$P_t(x) = \sum_{i=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \Theta(d^i) \cdot x^{t-2i}.$$  

3.3 Putting the Pieces Together

We are now ready to apply this machinery to prove Theorem 3.1. Let $A$ be the adjacency matrix of a $(d,k)$-graph. Since all entries of the sum $\sum_{t=0}^{k} P_t(A)$ must be strictly positive (by Claim 3.2 and the diameter-$k$ restriction), this matrix can be decomposed into $\sum_{t=0}^{k} P_t(A) = J + M$, where $J$ is the all ones matrix and $M$ is non-negative. We now have

$$M 1_n = \left( \sum_{t=0}^{k} P_t(A) - J \right) 1_n = \left( 1 + d + d(d-1) + ... + d(d-1)^{k-1} - n \right) 1_n = (\mu_{d,k} - n) 1_n,$$

where the second equality is due to (2).

We now apply the Perron-Frobenius Theorem (see [32]), which states that a non-negative matrix admits a non-negative eigenvector with a non-negative eigenvalue that is larger or equal, in absolute value, to all other eigenvalues. Since $M$ is symmetric and thus diagonalizable w.r.t. an orthogonal basis, all non-negative eigenvectors must lie in $\text{span}\{1_n\}$. We conclude that all eigenvalues of $P_t(A)$ are bounded in absolute value by $\mu_{d,k} - n$.

We now turn to analyzing the non-trivial eigenvectors:

**Claim 3.5.** Let $v$ be an eigenvector of $A$ with $Av = \lambda v$. If $\lambda = \Theta(d^{\alpha})$ for some $\alpha > \frac{1}{2}$, then $P_t(A)v = \Theta(\lambda^t)v = \Theta(d^{t\alpha})v$. 

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Proof. Again, we use induction on \( t \). For \( t = 0 \) we have that \( P_0(A)v = Iv = v = \Theta(d^0)v \), and for \( t = 1 \) we have that \( P_1(A)v = Av = \lambda v = \Theta(d^1)v \). Assume that the claim holds for the Geronimus Polynomials of order less than \( t \). Using Corollary 3.4, we now have

\[
P_t(A)v = P_t(\lambda)v = \sum_{i=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \Theta(d^i) \cdot \lambda^{t-2i} = \sum_{i=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \Theta(d^i) \cdot \Theta(d^{\alpha(t-2i)})v = \sum_{i=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \Theta(d^{i+\alpha(1-2\alpha)})v.
\]

Whenever \( \alpha > \frac{1}{2} \) this equals \( \Theta(d^{\alpha})v \).

We are now ready to prove the theorem.

Proof. (of Theorem 3.1) Let \( v \neq 1_n \) be an eigenvector of \( A \) with eigenvalue \( \lambda \), and suppose that \( \lambda = \Theta(d^\alpha) \) for some \( \alpha > \frac{1}{2} \). Note that \( J \) is diagnolizable w.r.t. an orthogonal basis and that \( \text{rank}(J) = 1 \). Since \( J1_n = n1_n \) and \( v \neq 1_n \), it follows that \( Jv = 0 \). Applying Claim 3.5 yields

\[
Mv = \left( \sum_{t=0}^{k} P_t(A) - J \right) v = \sum_{t=0}^{k} P_t(A)v = \sum_{t=0}^{k} \Theta(d^{\alpha})v = \Theta(d^{k\alpha})v.
\]

Using the fact that \( M \)'s leading eigenvalue equals \( \mu_{d,k} - n \) gives

\[
|\Theta(d^{k\alpha})| \leq \mu_{d,k} - n \leq O(d^{k/2}) \quad \text{(by the assumption of the theorem)}.
\]

This is, of course, a contradiction to the assumption \( \alpha > \frac{1}{2} \). We thus conclude that \( \lambda = O(\sqrt{d}) \). \( \square \)

3.4 Expansion of Diameter-2 Graphs

We next consider the much-studied case of diameter-2 graphs (see, e.g., [15, 8, 31]). We observe that the above proof technique yields the following result for diameter-2 graphs that (unlike Theorem 3.1) does not impose any restrictions on the size of the graph.

Theorem 3.6. Suppose \( G = (V, E) \) be a \( d \)-regular graph of size \( n \) and diameter \( k = 2 \), then

\[
\lambda(G) \leq \frac{1 + \sqrt{1 + 4(d^2 + d - n)}}{2}
\]

The proof is in Appendix A.

4 Diameter vs. Combinatorial Expansion

While our insights regarding algebraic expansion easily translate to combinatorial expansion via the Cheeger inequality, analyzing combinatorial expansion directly gives rise to nontrivial expansion guarantees with respect to multiplicative approximations to the Moore Bound. Since the vast majority of known \((d, k)\)-graph constructions are within a multiplicative factor from the Moore Bound, the results in this section imply nontrivial expansion properties for these constructions, as summarized in Table 2. We discuss these implications in Appendix C.

Undirected graphs. Our main result in this section is the following:
**Theorem 4.1.** Let \( G = (V, E) \) be a \( d \)-regular graph of size \( n \) and diameter \( k \). If \( n = \alpha \cdot \mu_{d,k} \), then

\[
\begin{align*}
\ell(G) & \geq \frac{c \alpha}{2k} \cdot \left( 1 - \frac{1}{(d-1)^k} \right) \\
\phi_v(G) & \geq \frac{\alpha}{2(k-1) + \alpha}.
\end{align*}
\]

Our proof of Theorem 4.1 utilizes a counting argument. As the graph has diameter \( k \), each pair of vertices on opposite sides of a cut must be connected via a path of length at most \( k \) that traverses the boundary. However, there is an upper bound, induced by the degree and diameter of the graph, on the number of such paths that traverse a given edge/vertex. A careful examination of the implications of these two limitations provides us with a lower bound on the size of the boundary. The proof appears in Appendix B.1.

**Directed graphs.** We consider directed graphs next. We begin by introducing the relevant terminology and notation. We say that a directed graph (a.k.a. digraph) \( G = (V, E) \) is \( d \)-regular if both the out-degree and the in-degree of each vertex equals \( d \). A cut in a digraph \( e(S, S^c) = \{(u, v) \in E | u \in S, v \in S^c\} \) is asymmetric, and consists of all edges directed from \( S \) to \( S^c \). The diameter is still defined as the maximal distance between two vertices, and the corresponding Moore Bound is only slightly different (as there are potentially \( d^k \) vertices of distance \( i \) from a given vertex):

\[
\tilde{\mu}_{d,k} = \sum_{i=0}^{k} d^i = \frac{d^{k+1}-1}{d-1}.
\]

The following result is the directed analogue of Theorem 4.1, and is proved in Appendix B.2.

**Theorem 4.2.** Let \( G \) be a \( d \)-regular, \( k \)-diameter directed graph of size \( n = \alpha \cdot \tilde{\mu}_{d,k} \), then

\[
\begin{align*}
\ell(G) & \geq \frac{\alpha}{2k} \left( d - \frac{1}{d^k} \right) \\
\phi_v(G) & \geq \frac{\alpha \cdot d}{2(d+1)(k-1) + \alpha \cdot d}.
\end{align*}
\]

**Refined results for low-diameter graphs.** Much research on constructing low-diameter graphs focuses on diameters 2 and 3 (see, e.g., [33, 15, 31]). Graphs of very low diameter are particularly important from a practical perspective [6, 23, 24]. The following theorems improve upon our results for the edge expansion and vertex expansion of \((d, k)\)-graphs.

**Theorem 4.3.** Let \( G = (V, E) \) be an undirected \((d, 2)\)-graph of size \( n = \alpha \cdot d^2 \). Then

\[
\ell(G) \geq \frac{2d + 1 - \sqrt{4(1-\alpha)d^2 + 4d + 1}}{4}.
\]

**Proof.** Let \((S, S^c)\) be a cut in the graph with \(|S| = s \leq \frac{n}{2}\). Let \(x = |e(S, S^c)|\) denote the number of edges in the cut. There are \(s(n-s)\) pairs of vertices such that one vertex is in \( S \) and the other is in \( S^c \), and exactly \(x\) of these pairs are connected by a path of length 1. Hence \(s(n-s) - x\) such pairs are connected by a path of length 2.

How many such length-2 paths are there? For each vertex \( v \in V \), let \( a_v \) denote the number of edges in the cut that are incident on \( v \). Given an edge \( (u, v) \in e(S, S^c) \), the number of length-2 paths that use this edge with both ends in different sides of the cut is \((d-a_u)+(d-a_v)\). It follows that the total number of length-2 paths with both ends in different sides of the cut is exactly

\[
\sum_{(u,v) \in e(S, S^c)} ((d-a_u)+(d-a_v)) = 2dx - \sum_{(u,v) \in e(S, S^c)} (a_u + a_v).
\]

As every vertex \( u \in V \) contributes exactly \( a_u \) summands to the sum, the expression above equals

\[
2dx - \left( \sum_{u \in S} a_u^2 + \sum_{u \in S^c} a_u^2 \right).
\]
Since $\sum_{u \in S} a_u = \sum_{v \in S^c} a_v = x$, this expression is maximized (by employing, e.g., Cauchy-Schwarz inequality) whenever $a_u = \frac{x}{s}$ for all $u \in S$ and $a_v = \frac{x}{n-s}$ for all $v \in S^c$. That is, it is maximized when the edges in the cut are spread evenly between all vertices from the every side of the cut. It follows that the total number of length-2 paths which cross the cut is at most
\[
2dx - s \left( \frac{x}{s} \right)^2 - (n-s) \left( \frac{x}{n-s} \right)^2 = x \left( 2d - \frac{x}{s} - \frac{x}{n-s} \right).
\]

This number of paths should connect $s(n-s) - x$ pairs of vertices from both sides of the cut. We thus get that $s(n-s) - x \leq x \left( 2d - \frac{x}{s} - \frac{x}{n-s} \right)$. Rearranging terms yields $x^2 \cdot \frac{n}{s(n-s)} - x(2d + 1) + s(n-s) \leq 0$.

It follows that
\[
\frac{2d + 1 - \sqrt{(2d+1)^2 - 4n}}{2 \cdot \frac{n}{s(n-s)}} \leq x \leq \frac{2d + 1 + \sqrt{(2d+1)^2 - 4n}}{2 \cdot \frac{n}{s(n-s)}}.
\]

This means that the size of the cut is bounded from below and from above. In order to lower bound the edge expansion, we only need to use the inequality on the left:
\[
h_e(G) \geq \frac{\left| e(S, S^c) \right|}{\left| S \right|} = \frac{x}{s} \geq \frac{2d + 1 - \sqrt{(2d+1)^2 - 4n}}{2 \cdot \left( \frac{n}{s(n-s)} \right)} \geq \frac{2d + 1 - \sqrt{(2d+1)^2 - 4\alpha \cdot d^2}}{4}.
\]

**Theorem 4.4.** Let $G = (V, E)$ be an undirected $(d, 2)$-graph of size $n = \alpha \cdot d^2$. Then
\[
\phi_V(G) \geq \frac{2\alpha}{2\alpha + 1}.
\]

We can extend this analysis to graphs of diameter 3, giving us the following theorem.

**Theorem 4.5.** Let $G = (V, E)$ be a $d$-regular, 3-diameter graph of size $n = \alpha \cdot d^3$, then
\[
\phi_V(G) \geq \frac{\alpha}{\alpha + 1}.
\]

### 5 Conclusion and Open Questions

We revisited the classical question of relating the expansion and the diameter of graphs and showed that not only do good expanders exhibit low diameter but the converse is also, in some sense, true. We also discussed the implications of our results for constructions from the rich body literature on low-diameter graphs. We leave the reader with many interesting open questions.

**Tightening the gaps.** An obvious open question is improving upon our lower bounds and establishing upper bounds on the expansion of fixed-diameter graph constructions. In fact, our only lower bound guaranteed to be essentially tight is our result for the edge expansion of diameter-2 graphs (Theorem 4.3).

**Can our framework be leveraged to prove impossibility results for large $(d, k)$-graphs?**

A possible direction for establishing the nonexistence of large $(d, k)$-graphs might be to show that they have to be improbably good expanders (for example, their spectral gap may be so large so it would violate lower bounds for $\lambda(G)$).

**Benchmarking against the optimal (largest possible) $(d, k)$-graph.** Our analyses used the Moore Bound as a benchmark and examined the implications of approximating Moore Graphs for expansion. Another approach would be to compare against the size of the largest possible $(d, k)$-graph (which could potentially be much lower than the Moore Bound).
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Appendix

A Proof of Theorem 3.4

Proof. Let $A$ denote the adjacency matrix of $G$, and let $v \neq 1_n$ be an eigenvector of $A$ with eigenvalue $\lambda$. Using the notation in the previous subsection, we have

$$Mv = \left( \sum_{t=0}^{2} P_t(A) - J \right) v = (I + A + A^2 - dI)$$

$$= [\lambda + \lambda^2 + 0 - (d - 1)]v \leq (\mu_{d,2} - n)v = (d^2 + 1 - n)v$$

We thus have

$$\lambda + \lambda^2 - (d - 1) \leq d^2 + 1 - n,$$

or equivalently

$$\lambda^2 + \lambda - (d^2 + d - n) \leq 0.$$

Solving the quadratic inequality yields the desired result. \qed

B Proofs for Combinatorial Expansion

B.1 Undirected Graphs

Proof. (of Theorem 4.1) We use a counting argument. Let $(S, S^c)$ be a cut in the graph, and let $|S| = s \leq \frac{n}{2}$. As the diameter equals $k$, every pair of vertices that lie on both sides of the cut must be connected via a path of length at most $k$. We thus have $s(n - s)$ such paths, each of which passes through some edge in the cut.
How many paths of length at most $k$ include a given edge $e \in E$? As $G$ is $d$-regular, there are at most $(d-1)^{l-1}$ paths of length $l$ for which $e$ is in the $i$’th position in the path. It follows that no more than $l \cdot (d-1)^{l-1}$ paths of length $l$ use a specific edge, hence the number of paths of length at most $k$ that utilize a fixed edge is upper bounded by

$$f_d(k) = \sum_{l=1}^{k} l \cdot (d-1)^{l-1}.$$ 

Let us find a simpler formulation of $f_d(k)$. Integrating yields

$$F_d(k) = \sum_{l=1}^{k} (d-1)^l = \frac{(d-1)^{k+1} - 1}{(d-1) - 1}.$$ 

Differentiating brings us back to

$$f_d(k) = \frac{(k+1)(d-1)^k(d-2) - [(d-1)^{k+1} - 1]}{(d-2)^2} \leq \frac{(k+1)(d-1)^k(d-2) - (d-1)^k(d-2)}{(d-2)^2} = \frac{k(d-1)^k}{(d-2)}.$$ 

Now, $s(n-s)$ paths use the cut, and every edge in the cut can be a part of at most $f_d(k)$ paths. It follows that $|e(S, S^c)| \geq \frac{s(n-s)}{f_d(k)}$, and hence

$$h_c(G) = \frac{|e(S, S^c)|}{|S|} \geq \frac{s(n-s)(d-2)}{s \cdot k(d-1)^k} \geq \frac{n}{2} \cdot \frac{(d-2)}{k(d-1)^k} = \frac{\alpha d}{2k} \cdot \frac{1}{1 - \frac{1}{(d-1)^k}}.$$ 

A similar argument applies for the vertex expansion. Let $S \subseteq V$ be a subset of size $s \leq \frac{n}{2}$, and let $x = |N(S)|$ denote the size of its outer boundary. Then $|S| \cdot |S^c \setminus N(S)| = s(n-s-x)$ pairs of vertices are connected via a path of length $2 \leq l \leq k$ in which one of the inner vertices of the path is in $N(S)$. But how many paths of this form can there be?

A path of length $l$ consists of $(l-1)$ possible positions for an inner vertex. As there are $d(d-1)^{l-1}$ paths of length $l$ passing through a vertex in a fixed position, we conclude that the number of paths is at most

$$\sum_{l=2}^{k} x(l-1) \cdot d(d-1)^{l-1} = xd \sum_{l=2}^{k} \left[ l(d-1)^{l-1} - (d-1)^{l-1} \right]$$

$$= xd \left[ f_{d-1}(k) - 1 - \frac{(d-1)^k - (d-1)}{d-2} \right]$$

$$\leq xd \left[ \frac{k(d-1)^k}{(d-2)} - 1 - \frac{(d-1)^k - (d-1)}{d-2} \right]$$

$$\leq \frac{xd}{d-2} \left[ (k-1)(d-1)^k + 1 \right]$$

$$\leq \frac{(d+1)}{(d-2)} \cdot x(k-1)(d-1)^k.$$
As the number of paths must exceed the number of pairs connected by a path, it follows that

\[ s(n - s - x) \leq \left( \frac{d + 1}{d - 2} \right) \cdot x(k - 1)(d - 1)^k, \]

and thus

\[
\phi_V(G) \geq \frac{x}{s} \geq \frac{n - s}{0.5n} \geq \frac{\alpha \cdot \mu_{d,k}}{2(d+1)(k-1)(d-1)^k + \alpha \cdot \mu_{d,k}} \geq \frac{\alpha d \cdot \frac{(d-1)^{k-1} - 1}{d-2}}{2(d+1)(k-1)(d-1)^k + \alpha \cdot d \cdot \frac{(d-1)^{k-1} - 1}{d-2}} \geq \frac{\alpha}{2(k - 1) + \alpha}.
\]

### B.2 Directed Graphs

**Proof.** (of Theorem 4.2) We follow the steps of the analysis in the undirected case, starting with edge expansion. As before, let \((S, S^c)\) be a cut with \(|S| = s \leq n/2\). The number of paths of length at most \(k\) that utilize a specific edge is now bounded by

\[ f_d(k) = \sum_{l=1}^{k} l \cdot d^{l-1} \leq \frac{k \cdot d^k}{(d - 1)}, \]

and since \( |e(S, S^c)| \geq \frac{s(n - s)}{f_d(k)} \), we have

\[
h(G) \geq \frac{|e(S, S^c)|}{|S|} \geq \frac{s(n - s)(d - 1)}{s \cdot k \cdot d^k} \geq \frac{n}{2} \cdot \frac{(d - 1)}{kd^k} = \frac{\alpha \cdot \mu_{d,k}(d - 1)}{2k \cdot d^k} = \frac{\alpha \cdot (d^{k+1} - 1)}{2k \cdot d^k}.
\]

For the vertex expansion, we again consider a cut \((S, S^c)\) with \(|S| = s \leq n/2\) and \(x = |N(S)|\). In a digraph there are \(d\) paths of length \(l\) passing through a vertex in a fixed position. It follows that the number of paths of length \(2 \leq l \leq k\) which include a vertex of \(N(S)\) as one of its inner vertices is at most

\[
\sum_{l=2}^{k} x(l - 1) \cdot d^l = x \sum_{l=2}^{k} (ld^{l-1} - d^{l-1})
\]

\[ = xd \left( f_d(k) - 1 - \frac{d^k - d}{d - 1} \right) \leq xd \left( \frac{kd^k}{(d - 1)} - 1 - \frac{d^k - d}{d - 1} \right) \leq \left( \frac{d + 1}{d - 1} \right) \cdot x(k - 1)d^k.\]
It follows that
\[ s(n - s - x) \leq \left( \frac{d + 1}{d - 1} \right) x(k - 1)d^k. \]

Hence
\[
\phi_V(G) \geq \frac{x}{s} \geq \frac{n - s}{\left( \frac{d+1}{d-1} \right) (k-1)d^k + s}
\]
\[
\geq \frac{0.5n}{\left( \frac{d+1}{d-1} \right) (k-1)d^k + 0.5n}
\]
\[
\geq \frac{\alpha \cdot \mu_{d,k}}{2 \left( \frac{d+1}{d-1} \right) (k-1)d^k + \alpha \cdot d^{k+1}}
\]
\[
= \frac{\alpha \cdot d}{2(d+1)(k-1) + \alpha \cdot d}
\]
as claimed.

\[ \square \]

**B.3 Diameter-2 and Diameter-3 Graphs**

**Proof.** (of Theorem 4.4) Let \( S \subset V \) be a subset of size \( s \leq \frac{n}{2} \), and let \( x = |N(S)| \) denote the size of its outer boundary. Then at least \( |S| \cdot |S^c \setminus N(S)| = s(n - s - x) \) pairs of vertices are connected via length-2 paths whose middle vertex lies in \( N(S) \). However, the number of such paths through a fixed middle vertex is at most \( \left( \frac{d}{2} \right)^2 \). It follows that
\[ s(n - s - x) \leq x \left( \frac{d}{2} \right)^2, \]
and rearranging this yields the inequality
\[ s(n - s) \leq x \left( \frac{d^2}{4} + s \right). \]

We thus have
\[
\phi_V(G) \geq \frac{x}{s} \geq \frac{n - s}{0.25d^2 + s} \geq \frac{0.5n}{0.25d^2 + 0.5n} = \frac{2\alpha}{2\alpha + 1}. \]

**Proof.** (of Theorem 4.5) Again, let \( S \subset V \) be a subset of size \( s \leq \frac{n}{2} \), and let \( x = |N(S)| \) denote the size of its outer boundary. Then at least \( |S| \cdot |S^c \setminus N(S)| = s(n - s - x) \) pairs of vertices are connected via paths of length 2 or 3 which pass through \( N(S) \). Note that a length-3 path of this kind must contain a length-2 path whose middle vertex lies in \( N(S) \) and both ends in different sides of the cut. It follows that there are at most \( 2 \left( \frac{d}{2} \right)^2 (d - 1) \) such length-3 paths and \( \left( \frac{d}{2} \right)^2 \) such length-2 paths. This implies that
\[ s(n - s - x) \leq x \left( 2 \left( \frac{d}{2} \right)^2 (d - 1) + \left( \frac{d}{2} \right)^2 \right), \]
and hence
\[ s(n - s) \leq x \left( \frac{d^2}{2}(d-1) + \frac{d^2}{4} + s \right). \]
We thus have
\[
\phi_V(G) \geq \frac{x}{s} \geq \frac{n - s}{d^2(d - 1) + \frac{d^2}{4} + s} \geq \frac{0.5n}{\frac{d^2}{2} - \frac{d^2}{4} + 0.5n} = \frac{\alpha}{\alpha + 1 - \frac{1}{d}}.
\]

\[\square\]

C Implications for Known Constructions

C.1 Undirected Graphs

The results in section 4 can be directly applied to obtain expansion guarantees for well known constructions of large \((d,k)\)-graphs. The most general of these is perhaps the undirected de Bruijn graph \([12]\), which may be constructed for every diameter \(k\) and even degree \(d\) (the detailed definition may be found in \([33]\)). These graphs are of size \(n \geq \left(\frac{d}{2}\right)^k\) and have been extensively applied in various contexts, including the design of feedback registers \([17, 27]\), decoders \([10]\), and computer networks \([4, 5, 16, 26, 35]\).

As for the spectral guarantees for this construction, since the second eigenvalue of these graphs is known to be \(\lambda_2 = d \cos\left(\frac{\pi}{k+1}\right)\) \([13]\), applying the Cheeger inequality yields
\[
h_e(G) \geq \frac{d - d \cos\left(\frac{\pi}{k+1}\right)}{2} \sim \frac{d}{4} \left(\frac{\pi}{k+1}\right)^2.
\]

The vertex expansion is thus
\[
\phi_V(G) \geq \frac{1}{4} \left(\frac{\pi}{k+1}\right)^2.
\]

While applying Theorem 4.1 implies weaker guarantees, whenever \(k = 2\) this construction is a \(\frac{1}{4}\)-approximation to the Moore bound, and thus by the refined argument in Theorem 4.4, we have
\[
\phi_V(G) \geq \frac{1}{3}.
\]

This constitutes the best-known vertex-expansion guarantee for \((d,2)\)-de Bruijn graphs.

Canale and Gomez \([9]\) made considerable progress on the degree-diameter problem by giving a construction of \((d,k)\)-graphs of size \(n \geq \left(\frac{d}{2}\right)^k\) for an infinite set of values \(d\) and \(k\). In these graphs the expansion guarantees from our theorems are slightly better:
\[
h_e(G) \geq \frac{d}{2k \cdot 1.57^k} \cdot \left(1 - \frac{1}{(d - 1)^k}\right), \text{ and}
\]
\[
\phi_V(G) \geq \frac{1.57^{-k}}{2(k - 1) + 1.57^{-k}}.
\]

This, to the best of our knowledge, is the first analysis of the Canale-Gomez graphs and thus these results constitute the highest expansion guarantees for this construction.

Since the only known constructions that actually draw close to the Moore Bound are of diameter \(k = 2\), this case is of particular importance for us. The largest known such constructions are based on polarity graphs, first introduced by Erdős and Renyi \([15]\) and then independently by Brown \([8]\). The design of these graphs makes use of finite projective geometries in order to produce \(d\)-regular graphs of diameter 2 and of size \(n = d^2 - d + 1\). Another important construction, that attempts to yield large \((d,k)\)-graphs that (unlike polarity graphs) are also vertex-transitive, was introduced
by McKay, Miller and Siran in [31]. This property aims to capture some sort of symmetry by the requirement that the automorphism group of the graph acts transitively upon its vertices. This construction, known as MMS-graphs, is of size $n = \frac{5}{3}(d + \frac{1}{2})^2$ and diameter 2 and was proposed as the topology of high performance computing networks in [6] due to its good performance in simulation in terms of latency, bandwidth, resiliency, cost, and power consumption.

Applying Theorems 4.3 and 4.4 to polarity graphs imply that these graphs enjoy expansion of

$$h_e(G) \geq \frac{2d + 1 - \sqrt{4d + 1}}{4}, \text{ and}$$

$$\phi_V(G) \geq \frac{2}{3}.$$

We note that as $\frac{d}{2} - \frac{\sqrt{d-1}}{2}$ and $\frac{1}{2}$ are the best known lower bound for the edge and vertex expansion respectively (obtained by applying the Cheeger inequality for the known spectral gap of these graphs), both bounds depicted here constitute the best expansion guarantees to date for this important construction.

Applying the same theorems for MMS-graphs yields

$$h_e(G) \geq \frac{2d + 1 - \sqrt{4d^2 + 4d + 1}}{4} \approx \frac{d}{3}, \text{ and}$$

$$\phi_V(G) \geq \frac{16}{9} + 1 = \frac{16}{25}.$$

Here, the best known bounds are $\frac{2d+1}{4}$ and $\frac{1}{2}$ respectively (also derived from the Cheeger inequality). Note that while the edge expansion guarantee presented here is slightly weaker, the vertex expansion guarantee is substantially tighter.

C.2 Directed Graphs

In the case of directed graphs, the state of the art for degree $d \geq 2$ and diameter $k \geq 4$ are graphs of size $n = 25 \cdot 2^{k-4}$ obtained from the Alegre digraph (see [33]) and its iterated line digraphs. Applying Theorem 4.2 yields the bounds

$$h(G) \geq \left(\frac{2}{d}\right)^k \cdot \frac{25}{16} \cdot \frac{1}{2k} \left(d - \frac{1}{d^k}\right), \text{ and}$$

$$\phi_V(G) \geq \frac{(\frac{2}{3})^k \cdot \frac{25}{16} \cdot d}{2(d+1)(k-1) + (\frac{2}{3})^k \cdot \frac{25}{16} \cdot d}.$$

For the remaining values of degree and diameter, the iterated line digraphs of complete digraphs (known in the literature as Kautz digraphs [14]) have been proposed as the underlying topology in the design of computer networks and architectures in [5, 4]. These graphs are of size $n = d^k + d^{k-1}$, and thus by Theorem 4.2 enjoy expansion of

$$h(G) \geq \frac{1}{2k} \left(d - \frac{1}{d^k}\right), \text{ and}$$

$$\phi_V(G) \geq \frac{d}{2(d+1)(k-1) + d}.$$
Here again, these bound represent the best expansion guarantees to date. While these expressions do not demonstrate near-optimal expansion, let us recall that most applications of expander graphs only require $h(G)$ and $\phi_V(G)$ to be bounded away from zero (see [21] for a variety of examples). Hence these bounds suffice for a number of desired properties and potential applications whenever the diameter is sufficiently low.