Constructing new $k$-uniform and absolutely maximally entangled states

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Abstract

Pure multipartite quantum states of $n$ parties and local dimension $q$ are called $k$-uniform if all reductions to $k$ parties are maximally mixed. These states are relevant for our understanding of multipartite entanglement, quantum information protocols and the construction of quantum error correcting codes. To our knowledge, the only known systematic construction of these states is based on a class of classical error correction codes known as maximum distance separable. We present a systematic method to construct other examples of $k$-uniform states and show that the states derived through our construction are not equivalent to any $k$-uniform state constructed from maximum distance separable codes. Furthermore, we used our method to construct several examples of absolutely maximally entangled states whose existence was open so far.

Introduction. Multipartite entangled states play an important role in many quantum information processing tasks, like quantum secret sharing, quantum error correcting codes, and also in the context of high energy physics [1-6]. All of these processes and applications depend on the property of the multipartite entangled states that are used as a resource. Providing a general framework for multipartite entanglement represents a highly complex problem, probably out of reach. Therefore, many efforts have focused on the study of relevant sets of states such as, for instance, graph states [7, 8] or tensor network states [9].

Recently, a special class of states have attracted the attention for a wide range of tasks. These states are called $k$-uniform states (or for simplicity $k$-uni states), and they have the property that all of their reductions to $k$ parties are maximally mixed. An $n$-qudit state $|\psi\rangle$ in $\mathcal{H}(n,q):=\mathbb{C}^{q^n}_q$ is $k$-uniform, and denoted in what follows by $k$-uni($n,q$), whenever

$$\rho_S = \text{Tr}_{S^c} |\psi\rangle\langle\psi| \propto 1 \quad \forall S \subseteq \{1,\ldots,n\}, |S| \leq k , \tag{1}$$

where $S^c$ denotes the complementary set of $S$. The Schmidt decomposition implies that a state can be at most $|n/2|-uni$, i.e., $k \leq |n/2|$. Operationally, in a $k$-uni state any subset of at most $k$ parties is maximally entangled with the rest. The $|n/2|-uni$ states are called Absolutely Maximally Entangled (AME) because they are maximally entangled along any splitting of the $n$ parties into two groups. Similarly, we denote the set of AME states in $\mathcal{H}(n,q)$ by AME($n,q$).

Despite their natural definition, little is known about the properties of $k$-uni states, such as for which values of the tuple $(k, n, q)$ they exist or systematic methods for their construction. In [10-12] these states were related to some classes of combinatorial designs known as orthogonal arrays (OA), and their quantum counterpart, quantum orthogonal arrays (QOA). To our knowledge, the most general method to construct $k$-uni states is based on a connection between them and a family of classical error correcting codes known as maximum distance separable (MDS) [13, 14]. The resulting states are called of minimal support, as they can be expressed with the minimum number of product terms needed to guarantee that the reduced states are maximally mixed.

In this work, we introduce a systematic method of constructing $k$-uni states. We call this method Cl+Q because it combines a given classical MDS code with a basis made of $k$-uni quantum states. We prove that our method is different from previous constructions as the derived states may not be of minimal support. In fact, we show that our states cannot be obtained from any state of minimal support by stochastic local operations and classical communication (SLOCC). We also use our method to construct $k$-uni states with smaller local dimension $q$ compared to the same $k$-uni state constructed from MDS codes. We then show how the $k$-uni states derived through our construction are example of graph states and provide the corresponding graph, which is different from the graphs associated to states of minimal support. Finally, we present generalizations of the the Cl+Q method and use them to construct two examples of AME states whose existence was open so far, namely AME(7, 4) and AME(11, 8)[15].

MDS codes and $k$-uni states. The first ingredient in our construction are classical MDS codes. In the language of coding theory linear error correcting codes are usually specified by the tuple of integer numbers $[n,k,d_H]_q$ and defined over a finite field $GF(q)$. Such codes encode $q^k$ many messages specified by vectors $\vec{c}_i \in [q]^k$, with $i = 1,\ldots,q^k$, into a subset of codewords $\vec{c}_i \in [q]^n$, all having Hamming distance $d_H$ [16] Chapter 1]. Here $[q] := \{0,\ldots,q-1\}$ denotes the range from 0 to $q-1$ and the Hamming distance $d_H$ between two codewords $\vec{c}_i = (c_{i}^{(1)},\ldots,c_{i}^{(n)})$ and $\vec{c}_j = (c_{j}^{(1)},\ldots,c_{j}^{(n)})$ is the number of places where they differ. The Singleton
bound [17] states that for any linear code
\[ d_H \leq n - k + 1. \]  
A code that achieves the maximum possible minimum Hamming distance for given length and dimension is called MDS code [16, Chapter 11]. Next, we specify MDS codes by the tuple \([n, k]_q\), as the Hamming distance follows from the saturation of the Singleton bound.

MDS codes have been used to derive the only known systematic construction of \(k\)-UNI states [3][13][14], which are also of minimal support, denoted by \(k\text{-UNI}_{\min}(n, q)\). For a given MDS code, consider the pure quantum state corresponding to the equally weighted superposition of all the codewords \(\vec{c}_i\) of the code, i.e.,
\[ |\psi\rangle = \sum_{i=1}^{q^k} |\vec{c}_i\rangle, \]  
(3)
It is instructive for what follows to see why 3 is a \(k\)-UNI state, that is, to show why all reductions up to \(k\) parties are maximally mixed (more details in [18]). For that we use two properties of MDS codes. First, since all codewords have a distance at least equal to the Singleton bound (2), all the off-diagonal elements of the reduced density matrices of at most \(k\) parties are zero. Second, the obtained \(k\)-UNI states are of minimal support. This refers to the minimal number of product states needed to specify the state. For \(k\)-UNI states, since the reduced state of \(k\) parties must be proportional to the identity, and hence of full rank, this number has to be at least equal to \(q^k\), which is precisely the number of terms in 3. Finally, let us recall that MDS codes have been found for the following intervals
\[ \begin{cases} n \leq q + 2 & q \text{ is even and } k = 1 \text{ or } q - 1 \smallskip \noalign{\medskip} n \leq q + 1 & \text{all other cases} \end{cases}, \]  
(4)
which in turn defines an existence interval of \(k\)-UNI_{\min} states (see [15] Chapter 11), [17].

Orthonormal basis. The second ingredient we used in our construction are orthonormal bases where all the elements are \(k\)-UNI states. In principle, this basis can be arbitrary but in what follows we show how to construct examples of such bases starting from a \(k\)-UNI_{\min} state built from an \([n, k]_q\) MDS code. Let us first introduce the unitary operators \(X\) and \(Z\) that generalize the Pauli operators to Hilbert spaces of arbitrary dimension \(q \geq 2\),
\[ X |j\rangle = |j + 1 \mod q\rangle \]  
(5)
\[ Z |j\rangle = \omega^j |j\rangle, \]  
(6)
where \(\omega := e^{i 2\pi / q}\) is the \(q\)-th root of unity. \(X\) and \(Z\) are unitary, traceless, and they satisfy the conditions \(X^n = Z^n = 1\). We now consider operators acting on \(H(n, q)\) consisting of tensor products of powers of these operators. In particular, we focus on the operators \(M(\vec{v})\) labelled by \(\vec{v} \in [q^n]\), that have the form
\[ M(\vec{v}) := Z^{v_{n-1}} \otimes \cdots \otimes Z^{v_0} \otimes X^{v_{k-1}} \otimes \cdots \otimes X^{v_0}. \]  
(7)
As we see next, these \(q^n\) unitary operators define a basis when acting on a \(k\)-UNI_{\min} state.

Lemma 1. Consider a \(k\)-UNI_{\min} state \(|\psi\rangle \in H(n, q)\) consisting of tensor products of powers of these operators. In particular, we focus on the operators \(M(\vec{v})\) labelled by \(\vec{v} \in [q^n]\), that have the form
\[ M(\vec{v}) := Z^{v_{n-1}} \otimes \cdots \otimes Z^{v_0} \otimes X^{v_{k-1}} \otimes \cdots \otimes X^{v_0}. \]  
(7)
As we see next, these \(q^n\) unitary operators define a basis when acting on a \(k\)-UNI_{\min} state.

Lemma 1. Consider a \(k\)-UNI_{\min} state \(|\psi\rangle \in H(n, q)\) and all possible vectors \(\vec{v}_i \in [q^n]\), with \(i = 1, \ldots, q^n\). Then, the states \(|\psi_i\rangle := M(\vec{v}_i) |\psi\rangle\) form a complete orthonormal basis of \(k\)-UNI_{\min} states.

In [14] this result was proven for the particular case of AME states of minimal support, leading to an AME basis. The above lemma, whose proof can be found in [20], generalizes the result to any \(k\)-UNI_{\min} states.

Constructing \(k\)-UNI states of non-minimal support. We are now ready to describe our method to construct non-minimal support \(k\)-UNI\((n, q)\) states using the previous two ingredients. The main idea is to combine the codeword of a given MDS code with the states of a complete \(k\)-UNI orthonormal basis, see figure 1(a).

Lemma 2 (Cl+Q method). Consider an \([n_{cl}, \ell]_q\) MDS code of codewords \(\vec{c}_i\) and a complete \(\ell\)-UNI\((n_q, q)\) orthonormal basis with states \(|\psi_{ij}\rangle\) such that \(n_q = \ell\). Construct the state
\[ |\phi\rangle = \sum_{i=1}^{q^n} |\vec{c}_i\rangle \otimes |\psi_{ij}\rangle \]  
(8)
This state is a $k = \min\{\ell + 1, \ell' + 1\}$-UNI state of $n = n_\text{cl} + n_\text{q}$ parties.

The condition $n_\text{q} = \ell$ is needed to ensure that the number of codewords in the code match the number of elements in the basis, as required by the construction. Note that the number of states in the $\ell'$-UNI$(n_\text{cl}, q)$ basis is $q^{n_\text{cl}}$, while the number of codewords in the MDS code is $q^k$. In fact, this condition is slightly more general and should read $n_\text{q} = \ell$ or $n_\text{q} = n_\text{cl} - \ell$, as the dual of an $[n_\text{cl}, \ell]_q$ MDS code defines an $[n_\text{cl}, n_\text{cl} - \ell]_q$ MDS code [13], which can also be used in our construction [8].

For the purpose of the proof we need to check if the reduced density matrix

$$\sigma_S = \text{Tr}_{S^c} |\phi\rangle\langle\phi| = \text{Tr}_{S^c} \left( \sum_{i,j} |\tilde{e}_i\rangle\langle\tilde{e}_j| \otimes |\psi_i\rangle\langle\psi_j| \right),$$

is proportional to the identity for every set $S$ of size $|S| = k$. In order to do so we consider the three different possibilities for $S$ when the $k$ parties are (i) all inside the classical part, (ii) all inside the quantum part (iii) split between the classical and quantum part.

**Proof of lemma 3** First, let’s consider the case (i): having a complete orthonormal basis in the quantum part ensures orthogonality, i.e., $\langle \psi_i | \psi_j \rangle = \delta_{i,j}$ and therefore the off-diagonal elements of $\sigma_S$ are zero. In addition, and similar to what happened for the construction of $k$-UNI states from MDS codes, all the diagonal elements are equal because all possible combinations of indices appear. Therefore, $\sigma_S$ is maximally mixed.

Now for the case (ii): the large Hamming distance between the terms of the classical part yields orthogonality, i.e., $\langle \tilde{e}_i | \tilde{e}_j \rangle = \delta_{i,j}$. The fact that the quantum part is a complete basis, for either choices of the classical part, implies that the reduced density matrix is a sum over all states of a basis, i.e., $\sigma_S = \sum |\psi_i\rangle\langle\psi_i| \propto 1_{n_\text{q}}$.

The case (iii) is more involved and its proof can be found in [21], together with more details about the construction.

We provide examples of $k$-UNI states for systems of smaller dimension than those obtained using MDS codes, see Table I.

| Uniformity | $n$ | CI+Q method | MDS code |
|------------|-----|-------------|----------|
| $k = 2$    |     |             |          |
| $n = 5$    | $q \geq 2$ | $q \geq 4$ |
| $n = 6$    | $q \geq 3$ | $q \geq 4$ |
| $n = 7$    | $q \geq 4$ | $q \geq 7$ |
| $n = 8$    | $q \geq 4$ | $q \geq 7$ |
| $n = 9$    | $q \geq 4$ | $q \geq 8$ |
| $n = 10$   | $q \geq 7$ | $q \geq 9$ |
| $k = 3$    |     |             |          |
| $n = 11$   | $q \geq 7$ | $q \geq 11$|
| $n = 12$   | $q \geq 7$ | $q \geq 11$|
| $n = 13$   | $q \geq 8$ | $q \geq 13$|
| $n = 14$   | $q \geq 8$ | $q \geq 13$|
| $n = 15$   | $q \geq 9$ | $q \geq 16$|
| $n = 16$   | $q \geq 11$ | $q \geq 16$|

**TABLE I:** Comparison between the local dimension $q$ of the two methods.

where $\rho_S = \text{Tr}_{S^c} |\psi\rangle\langle\psi|$. It is also well known that this number cannot be increased by SLOCC [22].

Now consider $k$-UNI state $|\phi\rangle$ in $\mathcal{H}(n,q)$ constructed from CI+Q method. All the reductions up to $k$ parties of the state $|\phi\rangle$ are maximally mixed. However, it is possible to show that there exist at least one subset of size $|S| = k + 1$ parties such that the reduced density matrix $\sigma_S = \text{Tr}_{S^c} |\phi\rangle\langle\phi| \propto 1$. This specific set contains $k$ parties of the classical part and one party from the quantum part. This implies that the state $|\phi\rangle$ is not minimal support and hence the two states $|\psi\rangle$ and $|\phi\rangle$ cannot be mapped into the other probabilistically via LOCC. Therefore, these belong to different SLOCC classes.

**Graph states.** It is also relevant to understand the construction from the point of view of graph states. A graph $G = (V, \Gamma)$ is composed of a set $V$ of $n$ vertices and a set of weighted edges specified by the adjacency matrix $\Gamma$ [7, 8, 23, 24], an $n \times n$ symmetric matrix such that $\Gamma_{ij} = 0$ if vertices $i$ and $j$ are not connected and $\Gamma_{ij} > 0$ otherwise. Graph states are pure quantum states specified by a graph. In this formalism, qudits are represented by the graph vertices $V$. The graph state associated with a given graph $G$ is the +1 eigenstate of the following set of stabilizer operators [7, 8, 23, 24]

$$S_i = X_i \sum_j (Z_j)^{\Gamma_{ij}}, \quad 1 \leq i \leq n.$$  

The $k$-UNI$_\text{min}$ states derived from MDS codes $[n, k]_q$ are examples of graph states as it is possible to connect the adjacency matrix $\Gamma$ and the code parameters [13, 14]. In particular, if one performs local Fourier transforms $F_i = \sum_{\omega} \omega^{|i\rangle\langle j|}$ on all the last $n - k$ parties of the state $|\psi\rangle$ in (3), the resulting state is a graph state corresponding to a complete bipartite graph, see Figure 2(a). This graph is partitioned into two subsets, one containing $k$ vertices and the other one $n - k$ vertices. The weights of the edges connecting the vertices in the two subsets depend on the details of the construction of the MDS code but the structure is the same for all the states $|\psi\rangle$. [4].
corresponding graphs associated to each MDS code or, equivalently, the constructed from the Cl+Q method, Eq. (8), when the states parameters \( C_q \) tributed into \( C = \begin{bmatrix} n_{cl} & k \end{bmatrix}_q \) MDS code with parameters \( \mathcal{C}_i = [n_{cl}, \left\lfloor \frac{n_{cl}}{2} \right\rfloor - 2]_q \). An AME\((n,q)\) state \(|\phi\rangle\) for \( n \) odd, with \( n = n_{cl} + 2 \), can be constructed by concatenating all the terms of each subclass with one of the Bell states of the quantum part, see also Figure 2(b).

To show that the state \(|\phi\rangle\) is an AME state we need to check all the reduced states \( \sigma_S = \text{Tr}_S\langle \phi|\phi\rangle \) on up to half of the systems. For the purpose of the proof, as we discussed before we check three different cases, and in order to do so we use two properties of the construction. One is using the fact of having large Hamming distance between every two terms of each box and terms from different boxes. The other property is the special subclasses \( \mathcal{C}_i \) that form the classical part. In [26], we present the two unknown AME\((7,4)\) and AME\((11,8)\) states.

This configuration can lead us to construct AME states for \( n \) odd when \( q \geq n - 3 \). Considering this, in some cases like AME\((n=7,q=4)\) and AME\((n=11,q=8)\), the states appear to be unknown (see [15] for table of known AME states). It is worth to note that this configuration is more general if we start with an arbitrary MDS code \([n,k]_q\) with this property that its codewords distributed into \( q^m \) many subsets of MDS codes \( \mathcal{C}_i = [n_{cl},k-m]_q \). Using this we can get a wider set of unknown AME states.

Before concluding this part, we would like to mention that the Cl+Q method can be generalized in a different way where the same quantum part is concatenated several times with the classical part. With this method, if \( r \) is the number of times that each state of the quantum part concatenates to the terms of the classical part, the \( k\)-UNI state contains \( n = n_{cl} + r n_q \) many parties. This generalization will be discussed elsewhere [25].

**Conclusion.** We have presented a method that combines a classical error correcting code with a basis of \( k\)-UNI states to derive examples of \( k\)-UNI states. We have shown that our construction is different from the only systematic construction previously known based on MDS codes: they belong to different SLOCC classes and have a different graph-state representations. Then, we have used our method to construct \( k\)-UNI states of \( n \) parties with smaller local dimensions \( q \) compared to MDS codes, and examples of AME states, AME\((7,4)\) and AME\((11,8)\), that were unknown so far. Due to the importance that \( k\)-uniform and AME states have, it is an interesting avenue to explore how to use the method presented here for quantum information tasks and, in particular, in the context of quantum error correction.

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[1] M. Hillery, V. Bužek, and A. Berthiaume, "Quantum secret sharing", Phys. Rev. A, 59, 1829, (1999).
[2] D. Gottesman, "Stabilizer codes and quantum error correction", arXiv:quant-ph/9705052 (1997).
[3] A. J. Scott, "Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions", Phys. Rev. A, 69, 052330, (2004).
[4] D. Gottesman, "An Introduction to Quantum Error Correction and Fault-Tolerant Quantum Computation", arXiv:0904.2557 (2009).
[5] J. I. Latorre, G. Sierra, "Holographic codes" arXiv:1502.06618 [quant-ph].
[6] F. Pastawski, B. Yoshida, D. Harlow, J. Preskill, "Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence" JHEP, 06, 149, (2015).
[7] M. Hein, J. Eisert, H. J. Briegel, "Multi-party entanglement in graph states", Phys. Rev. A 69, 062311 (2004).
[8] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, H. J. Briegel, "Entanglement in Graph States and its Applications", arXiv:quant-ph/0602096.
[9] R. Orús, “A practical introduction to tensor networks: Matrix product states and projected entangled pair states”, Ann. Phys. 349, 117 (2014).
[10] D. Goyeneche, K. Życzkowski, "Genuinely multipartite entangled states and orthogonal arrays" Phys. Rev. A 90, 022316, (2014).
[11] D. Goyeneche, D. Alsina, J. I Latorre, A. Riera, and K. Życzkowski, "Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices", Phys. Rev. A, 92, 032316,(2015).
[12] D. Goyeneche, Z. Raissi, S. Di Martino, K. Życzkowski, "Entanglement and quantum combinatorial designs", arXiv:1708.05946 [quant-ph]
[13] W. Helwig "Absolutely Maximally Entangled Qudit Graph States", arXiv:1306.2879 [quant-ph]
[14] Z. Raissi, C. Gogolin, A. Riera, and A. Acín, "Constructing optimal quantum error correcting codes from absolute maximally entangled states", J. Phys A: Math. and Theor. 51, 075301 (2017), arXiv:1701.03359
[15] F. Huber and N. Wyderka, "Table of AME states, perfect tensors and multi-unitary matrices", http://www.tp.nt.unisiegen.de/+fhuber/ame.html
[16] F. J. MacWilliams, N. J. A. Sloane, "The Theory of Error Correcting Codes", North-Holland, Amsterdam (1977).
[17] R. Singleton, "Maximum distance q-nary codes", IEEE Trans. Inf. Theor., 10(2), 116, (2006).
[18] See Supplemental Material Section A for details on $k$-UNI$_{min}$ states and the number of terms they have, in expanded in the computational basis.
[19] R. M. Roth, and G. Seroussi, "On generator matrix of MDS codes", IEEE Trans. Inform. Theory, vol. IT-31, pp. 826830, (1985).
[20] See Supplemental Material Section B for details of the proof of Lemma.
[21] See Supplemental Material Section C for details of the proof of Lemma.
[22] J. Eisert and H.-J. Briegel, “The Schmidt Measure as a Tool for Quantifying Multi-Particle Entanglement”, Phys. Rev. A 64, 022306 (2001).
[23] M. Van den Nest, J. Dehaene, B. De Moor, "Graphical description of the action of local Clifford transformations on graph states", Phys. Rev. A 69, 022316 (2004).
[24] M. Bahramgiri and S. Beigi, "Graph states under the action of local clifford group in non-binary case", arXiv:quant-ph/0610267, (2006).
[25] Technical version, in preparation.
[26] See Supplemental Material Section D for details of the proof of Lemma.
Supplemental Material of “Constructing new $k$-uniform and absolutely maximally entangled states”

Section A: Linear codes and dual codes

In general, an error correcting code is denoted by $(n, K, d_H)_q$, when it encodes $K$ many messages into a subset of higher dimension $[q]^n$, all having Hamming distance at least $d_H$. Linear codes are special class of codes whose set of messages is $K = [q]^k$ for some integer $k$, and the injective map from this set of messages to the $[q]^n$ set of codewords is linear. The linear codes are usually denoted as $C = [n, k, d_H]_q$, over finite field $GF(q)$ (for the reasons of using finite fields see [16, Chapter 3]). Codewords of a linear code are all possible combination of the rows of a matrix, called a generator matrix $G_{k \times n}$. For a given vector $\vec{v}_i \in [q]^k$ a codeword can be written as $\vec{c}_i = \vec{v}_i G_{k \times n}$. A generator matrix can always be written in standard formula

$$G_{k \times n} = [1_k|A],$$

where $1_k$ is a $k \times k$ identity matrix and $A \in GF(q)^{k \times (n-k)}$.

MDS codes are those linear codes that achieve maximum possible minimum Hamming distance, Eq.(2), and a $k$-UNI$_{\min}$ state $|\psi\rangle$, Eq. (1), can be constructed by taking superposition of the computational basis states corresponding to all the codewords. Considering the details of construction MDS codes we can get more general formula for the state $|\psi\rangle$, so that the equation Eq. (3) can be written as

$$|\psi\rangle = \sum_i |\vec{c}_i\rangle = \sum_i |\vec{v}_i G_{k \times n}\rangle = \sum_i |\vec{v}_i\rangle, \vec{v}_i A \rangle.$$ (13)

Given a linear code $C$ it is always possible to find dual code $C^\perp$ such that all of its codewords are orthogonal to all the codewords of the code $C$ with respect to the Euclidean inner product of the finite field $[16, Chapter 5]$. It is however obvious that two states $|\psi\rangle$ and $|\psi^\perp\rangle$ can be constructed by taking equally weighted superposition of the computational basis states corresponding to all the codewords of both codes $C$ and its dual $C^\perp$, respectively. Considering the connection between the codewords of the original code and its dual, one can check that the states $|\psi\rangle$ and $|\psi^\perp\rangle$ can be transformed into each other by transforming the basis using Fourier gates, i.e., from $Z$-eigenbasis to $X$-eigenbasis [11].

In general, the dual code $C^\perp$ of any linear MDS code $C$ is also an MDS. If $C$ is an MDS code with parameters $C = [n, k, d_H = n - k + 1]_q$, then the dual code has parameters $C^\perp = [n, n - k, d_H^\perp = k + 1]_{16}$ Chapter 1.11]. To avoid ambiguity we denote the MDS code with the message length $k \leq n/2$ by $C$ and its dual with the length of message $n - k$ by $C^\perp$. In this case, if two states $|\psi\rangle$ and $|\psi^\perp\rangle$ are constructed from the two MDS codes $C$ and its dual $C^\perp$ respectively, then they are local unitary equivalent. As acting local Fourier gates that is used to change the basis does not change the entanglement property, then the states $|\psi\rangle$ and $|\psi^\perp\rangle$ both are $k$-UNI states. It is however obvious that state $|\psi\rangle$ in the computational basis contains $q^k$ many terms while the state $|\psi^\perp\rangle$ has $q^{n-k}$ many terms.

Section B: Proof of Lemma 1

Here we discuss a family of Pauli string $M(\vec{v}_i)$, Eq. (7), that construct a complete orthonormal basis of $k$-UNI states.

Proof of lemma 1 First, note that all the $|\psi_i\rangle$ are $k$-UNI states, since local unitary operation does not change the entanglement property of the state $|\psi\rangle$. Then we should check orthonormality of the states, i.e., check that

$$\langle \psi | M(\vec{v}_i)^\dagger M(\vec{v}_{i'}) | \psi \rangle = \prod_i \delta_{i,i'}.$$ (14)

To show this we use this fact that, for any $k$-UNI state $|\psi\rangle$ constructed from an MDS code $C = [n, k, d_H = n - k + 1]_q$, the Hamming distance between all terms is at least $d_H = n - k + 1$. The large Hamming distance between each terms of the state $|\psi\rangle$ implies

$$\langle \psi | M(\vec{v}_i)^\dagger M(\vec{v}_{i'}) | \psi \rangle = \langle \psi | M(\vec{v}_i^{(1)})^\dagger M(\vec{v}_i^{(1)\prime}) | \psi \rangle \prod_{i=k+1}^n \delta_{i,i'}.$$ (15)
Now, by considering the property of having \( k \)-UNI state, we yield
\[
\langle \psi | M(\vec{v}_i) | \psi \rangle = \text{Tr}(M(\vec{v}_i) | \psi \rangle \langle \psi |) = \prod_{i=k+1}^{n} \delta_{i,i'} \delta_{i,i'} = \prod_{i=1}^{n} \delta_{i,i'} .
\]

We also used this fact that \( M(\vec{v}_i) | \psi \rangle \langle \psi | \) has weight at least \( k \).

\[ \square \]

Section C: Proof of Lemma 2

For the readers convenience we discuss proof of the Lemma 2 in more detail.

**Proof.** It is possible to use a \( \ell \)-UNI\(_{\text{min}} \) state constructed from MDS code \( C = [n_{cl}, \ell, d_H = n_{cl} + \ell + 1]_q \) or its dual \( C^\perp = [n_{cl}, n_{cl} - \ell, d_H = \ell + 1]_q \) as the classical part. As we discussed in Section A we denote these states by \( |\psi \rangle \) and \( |\psi^\perp \rangle \) respectively, both states are precisely those are given in Eq. (13). According to this, the \( k \)-UNI state of non-minimal support \( |\phi \rangle \) constructed using all of the terms of the state \( |\psi \rangle \) as the classical part can be written as
\[
|\phi \rangle = \sum_{i} \sum_{n_{cl}} \sum_{n_{cl}} |c_i^q \rangle |\psi_i \rangle = \sum_{i} |c_i G_{k \times n} \rangle |\psi_i \rangle = \sum_{i} |\vec{v}_i, \vec{v}_i A \rangle |\psi_i \rangle ,
\]
where, the above equation is the generalized form of the Eq. (8). The difference between \( |\phi \rangle \) and \( |\phi^\perp \rangle \) is the generator matrix, or alternatively the \( A \) matrix, that is used for the construction and for the case of having \( |\phi^\perp \rangle \), we have \( \vec{v}_i \in [q]^{n_{cl} - \ell} \).

The pure states \( |\phi \rangle \) or \( |\phi^\perp \rangle \), are \( k \)-UNI states iff all the reduced density matrices of all of the subsystems of size less than or equal to \( k \) are maximally mixed. Therefore, we take an arbitrary subset \( S \subseteq \{1, \ldots, n\} \) and show that the reduced density matrix of any subset \( S \), i.e., \( \sigma_S \), is maximally mixed. This subset can have various forms: (i) it may be contained entirely in the support of the classical part \( Cl = \{1, \ldots, n_{cl}\} \), (ii) or the support of the quantum part \( Q = \{1, \ldots, n_{cl}\} \), (iii) or it may be split between these two parts \( Cl \cup Q = \{1, \ldots, n\} \). We consider three different cases without loss of generality.

Case I: If the \( S \) qudits of the reduced density matrix \( \sigma_S \), come from the classical part. In this case we consider that \( S \subseteq Cl \) (there is no qudit from the quantum part.) The reduced density matrix resulting from tracing out all the quantum part of the state \( |\phi \rangle \), Eq. (17), is
\[
\sigma_S = \text{Tr}_{S^c} \text{Tr}_{Q} |\phi \rangle \langle \phi | = \sum_{i,i'} (\text{Tr}_{S^c} |\vec{v}_i, \vec{v}_i A \rangle \langle \vec{v}_{i'}, \vec{v}_{i'} A |) \langle \psi_i | \psi_{i'} \rangle = \sum_i \text{Tr}_{S^c} |\vec{v}_i, \vec{v}_i A \rangle \langle \vec{v}_i, \vec{v}_i A | ,
\]
which is a direct consequence of having complete basis as the quantum part, i.e., \( \langle \psi_i | \psi_{i'} \rangle = \delta_{i,i'} \). In case of considering the state \( |\phi^\perp \rangle \), same procedure holds when we calculate the reduced density matrix \( \sigma_S \) with the same condition for the set \( S \subseteq Cl \). We should just replace \( \vec{v}_i \in [q]^{n_{cl} - \ell} \) with \( \vec{v}_i \in [q]^{\ell} \).

The reduced density matrix \( \sigma_S \) with the size of the free indices is proportional to identity matrix. In case of considering the state \( |\phi \rangle \), the number of free indices is \( \ell \), so the state is \( \ell \)-UNI. For the case of having \( |\phi^\perp \rangle \) the same calculations holds, just in this case the state is \( n_{cl} - \ell \)-UNI, because the number of free indices is \( n_{cl} - \ell \).

Case II: If the \( S \subseteq Q \) qudits located at the quantum part. In this case, we consider the reduced density matrix \( \sigma_S \), resulting from tracing out all of the qudits of the classical part the set \( Cl = \{1, \ldots, n_{cl}\} \). The reduced density matrix simplifies to
\[
\sigma_S = \text{Tr}_{Cl} \text{Tr}_{S^c} |\phi \rangle \langle \phi | = \sum_{i,i'} (\langle \vec{v}_i | \vec{v}_i' \rangle \langle \vec{v}_i A | \vec{v}_i' A |) \langle \psi_i | \psi_{i'} \rangle ) = \sum_i |\psi_i \rangle \langle \psi_i |.
\]
where we have used that $\langle \vec{v}_i | \vec{v}_{i'} \rangle = \delta_{i,i'}$. The quantum part is a complete orthogonal basis, then the reduced density matrix in this case $\sigma_S = 1_{n_\text{cl}}$. It is discussed that in this scenario, $n_q = \ell$ or it can be $n_q = n_\text{cl} - \ell$.

Then, the reduced density matrix $\sigma_S$ is a $\ell$-UNI for the state $|\phi\rangle$ and $n_\text{cl} - \ell$-UNI for the state $|\phi^\perp\rangle$.

Case III: Now we consider the case that $S \cap \text{Cl} \neq \{\}$ and $S \cap Q \neq \{\}$. We then have the general formula

$$\sigma_S = \text{Tr}_{S \cap \text{Cl}} \text{Tr}_{S \cap Q} |\phi\rangle \langle \phi|$$

$$= \sum_{i,i'} \text{Tr}_{S \cap \text{Cl}} (|\vec{v}_i, \vec{v}_i\rangle \langle \vec{v}_{i'}, \vec{v}_{i'}|) \otimes \text{Tr}_{S \cap Q} (|\psi_i\rangle \langle \psi_{i'}|).$$

(20)

(21)

To get $\sigma_S$ maximally mixed it is enough to fulfill the following two conditions: The partial trace inside the classical part Cl must be proportional to $\delta_{i,i'}$ and the partial trace inside the quantum part Q must leave over a maximally mixed state. The second condition becomes harder to satisfy the larger $|S^c \cap Q|$ is, but because the states $|\psi_i\rangle$ in the quantum part Q are all $\ell'$-UNI, this works as long as $|S \cap Q| \leq \ell'$, which is always satisfied, because we considering only $|S| = \ell' + 1$. The problematic conditions is thus the first. As the terms $|\vec{v}_i, \vec{v}_i\rangle$ that make up the classical part of the state $|\phi\rangle$ are coming from an MDS code, they are all computational basis states. Hence, if the partial trace over $S^c \cap \text{Cl}$ is proportional to $\delta_{i,i'}$ for all $S$ with $|S \cap \text{Cl}| = \ell'$, then it will also be proportional to $\delta_{i,i'}$ for all $S$ with $|S \cap \text{Cl}| < \ell'$. This means we only need to show that

$$\text{Tr}_{S \cap \text{Cl}} (|\vec{v}_i, \vec{v}_i\rangle \langle \vec{v}_{i'}, \vec{v}_{i'}|) \propto \delta_{i,i'}$$

(22)

for all $S$ with $|S \cap \text{Cl}| = \ell'$. Fix any $S$ and let $\{ |s\rangle \}$ be the computational basis for $S \cap \text{Cl}$ and $\{ |t\rangle \}$ be that of $S^c \cap \text{Cl}$. We can then write

$$\text{Tr}_{S \cap Cl} (|\vec{v}_i, \vec{v}_i\rangle \langle \vec{v}_{i'}, \vec{v}_{i'}|) = \sum_{s,s',t} |s\rangle \langle s'| |s,t\rangle \langle s,t| \vec{v}_i, \vec{v}_i\rangle \langle \vec{v}_{i'}, \vec{v}_{i'}|s',t\rangle$$

(23)

For $\vec{v}_i \neq \vec{v}_{i'}$, the two inner products in the right hand side of the last equation can be simultaneously non-zero only if $|\vec{v}_i, \vec{v}_i\rangle$ and $|\vec{v}_{i'}, \vec{v}_{i'}\rangle$ are identical in at least $|S^c \cap \text{Cl}|$ many locations, because otherwise they cannot both be non-orthogonal to $|t\rangle$. But this means that their Hamming distance could not be larger than $n_\text{cl} - |S^c \cap \text{Cl}| \leq \ell' \leq \ell/2$, where in the second inequality we have used that $|S^c \cap \text{Cl}| \geq n_\text{cl} - \ell'$ in the third that $\ell = n_q \geq 2\ell'$. But, at the same time, we know that the Hamming distance between any two $|\vec{v}_i, \vec{v}_i\rangle$ and $|\vec{v}_{i'}, \vec{v}_{i'}\rangle$ for $\vec{v}_i \neq \vec{v}_{i'}$ is at least $d_H = n_\text{cl} - \ell + 1 \geq \ell + 1$, where the inequality follows from $n_\text{cl} \geq 2\ell$. These were only compatible if $\ell + 1 \leq \ell/2$, which is never fulfilled.

Let now consider the state $|\phi^\perp\rangle$ where the classical part is constructed from the dual code $\phi^\perp$ and the condition $n_\text{cl} - \ell = n_q$ is necessary. For the state $|\phi\rangle$, it was obvious that $\ell > \ell'$ that we used in order to show the above conditions hold true. But using the dual code as the classical part the two above conditions can be achieved only if $d_H^\perp = \ell + 1 \geq n_\text{cl} - |S^c \cap \text{Cl}|$ for all $S$ with $|S \cap \text{Cl}| = |S| - 1$. And this is only possible if we consider $|S| = \min(\ell + 1, \ell' + 1)$.

Now, considering all the three cases, we can determine the minimum size for the set $S$ for which the reduced density matrix is proportional to identity. For the state $|\phi\rangle$, we can realize that the state is $\ell' + 1$-UNI. And for the case $|\phi^\perp\rangle$, the state is $\min(\ell + 1, \ell' + 1)$-UNI, which depends on the parameters of the classical part and quantum part.

It just remains to present the necessary condition for the existence of this set of states. In order to construct the $k$-UNI non-minimal support of $n = n_\text{cl} + n_q$ qudits using the Cl+Q method, we first need to construct the $\ell$-UNI and $\ell'$-UNI states of $n_\text{cl}$ and $n_q$ qudits, respectively. Also we know that there is a direct correspondence between constructing minimal support states and the classical MDS codes. Then, in order to find the existence condition of such a state one should just check the condition of constructing the MDS codes. To show this we use that according to Eq. (11), we should find $\max\{n_q, n_\text{cl}\}$ for given local dimension $q$. Considering this, one simply can verify that $\max\{n_\text{cl}, n_q\} = n_\text{cl}$. Thus the existence of MDS code with $n_\text{cl}$ parties and local dimension $q$ is enough to guarantee that such a non-minimal support $k$-UNI state with $n > n_\text{cl}$ party and the same local dimension exist. We provide a detailed comparison in table [11].

For an example we can consider AME(5, q) with the following closed form expression [12]

$$|\phi^\perp\rangle = \sum_{l,m=0}^{q-1} |l, m, l + m\rangle \psi(l,m),$$

(24)
Case I: If the set $|S| = \lceil n/2 \rceil = \lceil nq/2 \rceil$ contains entirely in the support of the classical part $Cl = \{1, \ldots, n_{cl}\}$, and it can be split between two parts $Cl \cup Q = \{1, \ldots, n\}$. For the last case two possibilities is allowed: if one party is from the quantum part $|S \cap Q| = 1$ and the rest from the classical part, i.e., $|S \cap Cl| = \lceil n/2 \rceil - 1$, or two parties from quantum part $|S \cap Q| = |Q| = 2$ and $|S \cap Cl| = \lfloor n/2 \rfloor - 2$ from the classical part.

Case I: If the set $S$ contain entirely in the support of the classical part. This means that $S \subseteq Cl$ and $|S| = \lceil n/2 \rceil = \lceil nq/2 \rceil$.

Then, the reduced density matrix can be written as

$$
\sigma_S = \text{Tr}_{Cl} \rho = \sum_{i,i'} \sum_{j,j'} (\text{Tr}_{Cl} |\tilde{c}_{i,j} \rangle \langle \tilde{c}_{i',j'}|) \langle \psi_i | \psi_{i'} \rangle .
$$

### Table II: Comparison between local dimension $q$ of the two methods.

| Uniformity | $n$ | Cl part | Orthonormal basis for Q part | Cl+Q method | MDS code |
|------------|-----|---------|-----------------------------|-------------|---------|
|            |     |         |                             |             |         |
| $k = 2$    |     |         |                             |             |         |
| $n = 5$    | $[3,2,2]_q$ | Bell basis, $q^2$ states | $q \geq 2$ | $q \geq 4$ |
| $n = 6$    | $[4,2,3]_q$ | Bell basis, $q^2$ states | $q \geq 3$ | $q \geq 4$ |
| $n = 7$    | $[5,2,4]_q$ | Bell basis, $q^2$ states | $q \geq 4$ | $q \geq 7$ |
| $n = 8$    | $[5,3,3]_q$ | GHZ basis, $q^3$ states $q \geq 4$ | $q \geq 7$ |
| $n = 9$    | $[6,3,4]_q$ | GHZ basis, $q^3$ states | $q \geq 4$ | $q \geq 8$ |
| $n = 10$   | $[7,3,5]_q$ | GHZ basis, $q^3$ states | $q \geq 7$ | $q \geq 9$ |
| $k = 3$    |     |         |                             |             |         |
| $n = 11$   | $[7,4,4]_q$ | AME$(4,q)$ basis, $q^4$ states | $q \geq 7$ | $q \geq 11$ |
| $n = 12$   | $[8,4,5]_q$ | AME$(4,q)$ basis, $q^4$ states | $q \geq 7$ | $q \geq 11$ |
| $n = 13$   | $[9,4,6]_q$ | AME$(4,q)$ basis, $q^4$ states | $q \geq 8$ | $q \geq 13$ |
| $n = 14$   | $[9,5,5]_q$ | AME$(5,q)$ basis, $q^5$ states | $q \geq 8$ | $q \geq 13$ |
| $n = 15$   | $[10,5,6]_q$ | AME$(5,q)$ basis, $q^5$ states | $q \geq 9$ | $q \geq 16$ |
| $n = 16$   | $[11,5,7]_q$ | AME$(5,q)$ basis, $q^6$ states | $q \geq 11$ | $q \geq 16$ |

where $|\psi_i\rangle$ represent Bell basis which is considered as the quantum part,

$$
|\psi_{(l,m)}\rangle = X^l \otimes Z^m \sum_r |r, r\rangle .
$$

For the qubit case we have

$$
|\phi^+\rangle = |000\rangle|\phi^+\rangle + |011\rangle|\psi^+\rangle + |101\rangle|\phi^-\rangle + |110\rangle|\psi^-\rangle ,
$$

where $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$ are the Bell basis of the Hilbert space of 2 qubits. One can check that the reduced density matrix $\sigma_S$ up to 2 parties are all maximally mixed.

### Section D: Proof of Lemma 3

Proof. The Cl+Q method with repetition allows us to construct AME states by combining two so-called classical part and quantum part while repetition in the quantum part is allowed. In this case, for the classical part, we consider that the classification of MDS code $C = [n_{cl}, \lceil n/2 \rceil, d_H = \lceil n/2 \rceil + 1]_q$ to MDS codes with smaller parameters $C = [n_{cl}, \lceil n/2 \rceil - 2, d_H = \lceil n/2 \rceil + 3]_q$ exists. And, we consider that for a given subclass $C_i$, codewords represent by $\tilde{c}_{i,j}$ and they are $q^{\lceil n/2 \rceil - 2}$ many codewords. It is obvious that $\sum_i \sum_j |\tilde{c}_{i,j}\rangle$ is summing over all the codewords of the code $C$ with convention that $i$ is in the range $\{1, \ldots, q^2\}$ represent the number of the subclass $C_i$, and $j$ is in the range $\{1, \ldots, q^{\lceil n/2 \rceil - 2}\}$. The state

$$
|\phi\rangle = \sum_i \sum_j |\tilde{c}_{i,j}\rangle |\psi_i\rangle
$$

is a modification of Eq. (27), and it can be an AME state if the reduced density matrices $\sigma_S = \text{Tr}_{S^c} |\phi\rangle \langle \phi|$ is proportional to identity. As in the lemma, we check three different cases for any subset $S$ up to $|S| = \lceil n/2 \rceil = \lceil nq/2 \rceil$ parties: this may be contained entirely in the support of the classical part $Cl = \{1, \ldots, n_{cl}\}$, or it can be split between two parts $Cl \cup Q = \{1, \ldots, n\}$. For the last case two possibilities is allowed: if one party is from the quantum part $|S \cap Q| = 1$ and the rest from the classical part, i.e., $|S \cap Cl| = \lceil n/2 \rceil - 1$, or two parties from quantum part $|S \cap Q| = |Q| = 2$ and $|S \cap Cl| = \lfloor n/2 \rfloor - 2$ from the classical part.
To show that the partial trace over $S^c$ and the quantum part $Q$ is proportional to $\delta_{i,i'}\delta_{j,j'}$, we consider two different conditions. If the terms are belonging to one of the boxes or when they belong to different boxes. Let’s first consider the terms that belong to one of the boxes, i.e., $i = i'$. In this case Hamming distance between these terms is $d_H \geq \left[ \frac{n/2}{2} \right] + 2$ which is larger than size of the subset $S$, i.e., $d_H > |S|$. This means the partial trace over the classical part is proportional to $\delta_{i,j}$, i.e.,

$$\text{Tr}_{S^c}(|\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \propto \delta_{j,j'}.$$ (29)

For the second case, when parties are from different boxes ($i \neq i'$), we use the direct consequence of having complete basis as the quantum part, which means for different boxes we have

$$\langle \psi_i | \psi_{i'} \rangle = \delta_{i,i'}.$$ (30)

Therefore, substituting Eq. (29) and Eq. (30) into Eq. (28), we get

$$\sigma_S = \sum_{i,j}(\text{Tr}_{S^c} |\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \propto 1_{[n/2]}.$$ (31)

where we used the fact that the number of the free indices of the classical part is equal to $\left[ \frac{n/2}{2} \right] = \left[ \frac{n}{4} \right]$.

Case II: The subset $S$ split between two parts such that $|S \cap Q| = 1$ and $|S \cap Cl| = \left[ \frac{n/2}{2} \right] - 1 = \left[ \frac{n}{4} \right]$. Then the reduced density matrix $\sigma_S$ simplifies to

$$\sigma_S = \text{Tr}_{S \cap Cl} \text{Tr}_{S \cap Q} |\phi\rangle\langle \phi|$$

$$= \sum_{i,i'} \sum_{j,j'} \text{Tr}_{S \cap Cl}(|\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \otimes \text{Tr}_{S \cap Q}(|\psi_i\rangle\langle \psi_{i'}|).$$ (32)

Since the partial trace inside the classical part $|S^c \cap Cl| = \left[ \frac{n/2}{2} \right]$, is larger than $n_{cl} - d_H$ i.e., $|S^c \cap Cl| > n_{cl} - d_H = \left[ \frac{n/2}{2} \right]$ then the partial trace is proportional to $\delta_{i,j}\delta_{j,j'}$. Therefore we can obtain

$$\text{Tr}_{S^c \cap Cl}(|\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \propto \delta_{i,i'}\delta_{j,j'}.$$ (33)

The quantum part contains Bell basis that all of them are 1-UNI states, therefore

$$\text{Tr}_{S \cap Q}(|\psi_i\rangle\langle \psi_{i'}|) \propto 1.$$ (34)

These two equations lead us to the following

$$\sigma_S = \sum_i \text{Tr}_{S \cap Cl}(|\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \otimes 1.$$ (35)

Also, we know that the superposition of the all codewords in the computational basis of the MDS code $C$, i.e., $\sum_{i,j} |\tilde{c}_{i,j}\rangle$ is an AME states of $n_{cl}$ parties, the reduced density matrices up to $\left[ \frac{n/2}{2} \right]$ are all proportional to identity matrices. Considering all of theses, the reduced density matrix $\sigma_S$ in this case is proportional to $1_{[n/2]}$.

Case III: We consider subset that $|S \cap Q| = |Q| = 2$ and $|S \cap Cl| = \left[ \frac{n/2}{2} \right] - 2 = \left[ \frac{n}{4} \right] - 1$. We then have the following formula

$$\sigma_S = \text{Tr}_{S \cap Cl} |\phi\rangle\langle \phi|$$

$$= \sum_{i,i'} \sum_{j,j'} \text{Tr}_{S \cap Cl}(|\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \otimes (|\psi_i\rangle\langle \psi_{i'}|).$$ (36)

Like what we had for the case (II), since the Hamming distance between the terms of the classical part is larger than the size of the subset $|S \cap Cl|$, then the partial trace over the classical part provide $\delta_{i,i'}\delta_{j,j'}$. It is also possible to explain in this way that $|S^c \cap Cl| = \left[ \frac{n/2}{2} \right] + 2$ and since this is larger than $n_{cl} - d_H$ then Eq. (36) simplifies to

$$\sigma_S = \sum_{i,j} \sum_{j} \text{Tr}_{S \cap Cl}(|\tilde{c}_{i,j}\rangle\langle \tilde{c}_{i,j}'|) \otimes (|\psi_i\rangle\langle \psi_{i}|).$$ (37)
As we explained before the classical boxes are all MDS codes with parameters \([n_{cl}, \lceil \frac{n_{cl}}{2} \rceil - 2, d_H = \lceil \frac{n_{cl}}{2} \rceil + 2]_q\). This means that all the terms inside of the box \(i\) form a \([\frac{n_{cl}}{2}] - 2\)-UNI state such that all the reduced density matrices on up to \(|S \cap Cl| = \lceil \frac{n_{cl}}{2} \rceil - 2\) are all proportional to identity, or concretely,

\[
\sum_j \text{Tr}_{S \cap Cl}(|\vec{c}_{i,j} \rangle \langle \vec{c}_{i,j}|) \propto \mathbb{I} |\frac{n_{cl}}{2}| - 2 .
\]

(38)

Then, we get

\[
\sigma_S = \sum_{i=1}^{q^2} \mathbb{I} |\frac{n_{cl}}{2}| - 2 \otimes (|\psi_i \rangle \langle \psi_i|) .
\]

(39)

The quantum part is a complete orthonormal basis, therefore \(\sum_i |\psi_i \rangle \langle \psi_i| \propto \mathbb{I}_2\). Then, the reduced density matrix in this case \(\sigma_S \propto \mathbb{I} |\frac{n_{cl}}{2}|\).

The Cl+Q with repetition produces two new AME states, AME(7, 4) and AME(11, 8). The state AME(7, 4) can be constructed by using MDS code with parameters \([5, 3, 3]_4\) and showing that all the terms can be classified into \(4^2\) many boxes with terms forming an MDS code \([5, 1, 5]_4\). Thus, the following closed form expression is an AME(7, 4)

\[
|\phi\rangle = \sum_{i,j,l \in GF(4)} |i, j, l, i + j + l, i + xj + (1 + x)l\rangle \otimes X^{i+j} \otimes Z^{i+xl} \sum_{m \in GF(4)} |m, m\rangle ,
\]

(40)

where \(GF(4) = \{0, 1, x, 1 + x\}\) generated by \(x^2 = x + 1\).

For the state AME(11, 8) we employ MDS code \([9, 5, 5]_8\) such that it can be classified to \(8^2\) boxes of MDS codes with parameters \([9, 3, 7]_8\). And, we yield the following state that is an AME(11, 8)

\[
|\phi\rangle = \sum_{i,j,l,m,r \in GF(8)} |i, j, l, m, r, i + a_1j + a_2l + a_3m + a_4r, i + a_2j + a_3l + a_4m + a_5r, i + a_3j + a_4l + a_5m + a_6r\rangle \otimes |\psi_{i,j,l,m,r}\rangle ,
\]

(41)

with,

\[
|\psi_{i,j,l,m,r}\rangle = X^{i+a_1j+a_2l+m} \otimes Z^{a_1i+a_2j+a_4l+r} \sum_{s \in GF(4)} |s, s\rangle ,
\]

(42)

and \(GF(8) = \{0, 1, a_1, a_2, \ldots, a_6\}\) with \(a_1 = x^2, a_2 = 1 + x + x^2, a_3 = 1 + x, a_4 = x, a_5 = x + x^2, \) and \(a_6 = 1 + x^2\) generated by \(1 + x + x^3 = 0\).

[1] This can be established by checking the stabilizer formalism and graph states representation. See [25] for more details.