Spectral Inclusion for $C_0$-Semigroups Drazin Invertible and Quasi-Fredholm Operators

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Abstract Let $(T(t))_{t \geq 0}$ be a $C_0$-semigroups and $A$ be its infinitesimal generator. In this work, we prove that the spectral inclusion for $(T(t))_{t \geq 0}$ remains true for the Drazin invertible and quasi-Fredholm spectra. Also, we will give conditions under which facts $A$ is quasi-Fredholm, $A$ is Drazin invertible and $A$ is B-Fredholm are equivalent.

Keywords Banach space operators · $C_0$-semigroups · Spectral inclusion · Drazin invertible operator · Quasi-Fredholm operator

Mathematics Subject Classification 47A16 · 47D06 · 47D03

1 Introduction and Preliminaries

Let $X$ a Banach space and $B(X)$ the Banach algebra of all bounded linear operators on $X$. for $T \in B(X)$, by $T^*$, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n \geq 0} R(T^n)$, $N^\infty(T) =$

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\[ \bigcup_{n \geq 0} N(T^n), \rho(T) \text{ and } \sigma(T), \] we denote, respectively, the adjoint, the null space, the range, the hyper-range, the hyper-kernel, the resolvent set and the spectrum of \( T \).

Let \( T \in B(X) \), the ascent \( a(T) \) and the descent \( d(T) \) of \( T \) are defined in [16] by \( a(T) = \inf \{ n \in \mathbb{N} : N(T^n) = N(T^{n+1}) \} \) and \( d(T) = \inf \{ n \in \mathbb{N} : R(T^n) = R(T^{n+1}) \} \), respectively.

The operator \( T \) is said to be Drazin invertible if \( d(T) < \infty \) and \( a(T) < \infty \). It is well known that \( T \) is Drazin invertible if and only if \( T = T_1 \oplus T_2 \) where \( T_1 \) is invertible and \( T_2 \) is nilpotent, see [9, Corollary 2.2]. This is equivalent to the fact that there exists an integer \( n \) such that the space \( R(T^n) \) is closed and the restriction of \( T \) of \( R(T^n) \) viewed as a map from \( R(T^n) \) into \( R(T^n) \) is invertible, see [2, Theorem 2.5]. The Drazin spectrum of \( T \) is defined by

\[ \sigma_D(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not Drazin invertible} \}. \]

Similarly, from [7], \( T \) is left (respect right) generalized Drazin invertible if and only if \( T = T_1 \oplus T_2 \) such that \( T_1 \) is left (respect right) invertible and \( T_2 \) is quasi-nilpotent.

Recall that \( T \) is said to be semi-regular or Kato operator, if \( R(T) \) is closed and \( N(T) \subset R^\infty(T) \), see for example [1]. In addition, \( T \) is said to be pseudo-Fredholm operator if there exist two closed \( T \)-invariant subspaces \( M \) and \( N \) such that \( X = M \oplus N \) and \( T = T|M \oplus T|N \) with \( T|M \) is semi-regular and \( T|N \) is nilpotent. This is equivalent to the fact that there exists an integer \( n \) such that \( R(T^n) \) is closed and the restriction of \( T \) of \( R(T^n) \) viewed as a map from \( R(T^n) \) into \( R(T^n) \) is semi-regular. The quasi-Fredholm spectrum is defined by

\[ \sigma_{QF}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not quasi-Fredholm} \}, \]

see [7,10,11] for more information.

Similarly, from [3] \( T \) is said to be a B-Fredholm operator, if there exists an integer \( n \) such that the space \( R(T^n) \) is closed and the restriction of \( T \) of \( R(T^n) \) viewed as a map from \( R(T^n) \) into \( R(T^n) \) is Fredholm.

Let \( T \in B(X) \) and \( x \in X \), the local resolvent of \( T \) at \( x \) noted \( \rho_T(x) \) is defined as the union of all open subset \( U \) of \( \mathbb{C} \) for which there is an analytic function \( f : U \to X \) such that the equation \((T - \mu I)f(\mu) = x \) holds for all \( \mu \in U \). The local spectrum \( \sigma_T(x) \) of \( T \) at \( x \) is defined by \( \sigma_T(x) = \mathbb{C} \setminus \rho_T(x) \). Evidently \( \rho_T(x) \) is an open subset of \( \mathbb{C} \) and \( \sigma_T(x) \) is closed. If \( f(z) = \sum_{i=0}^{\infty} x_i (z - \mu)^i \) (in a neighborhood of \( \mu \)), then \( \mu \in \rho_T(x) \) if and only if there exists a sequence \( (x_i)_{i \geq 0} \subseteq X \), \( x_0 = x \), \((T - \mu)x_{i+1} = x_i \), and \( \sup_i \| x_i \|^t < \infty \), see [8].

Let \( T = (T(t))_{t \geq 0} \) be a strongly continuous semigroup \((C_0\text{-semigroup in short})\) with infinitesimal generator \( A \) on \( X \). We will denote the type (growth bound) of \( T \) by \( \omega_0 \):

\[ \omega_0 = \inf \{ \omega \in \mathbb{R} : \text{there exists M such that } \| T(t) \| \leq Me^{\omega t}, t \geq 0 \}, \]

see [4,5,13] for more information. Also, in [4,5,13] the authors showed that

\[ e^{t\nu(A)} \subseteq \nu(T(t)) \subseteq e^{\nu(A)} \bigcup \{0\} \]
where \( v(.) \in \{\sigma_p(\cdot), \sigma_{ap}(\cdot), \sigma_r(\cdot)\} \) is the point spectrum, approximative spectrum or residual spectrum.

The semigroup \( T(t) \) is called differentiable for \( t > t_0 \) if for every \( x \in X, t \to T(t)x \) is differentiable for \( t > t_0 \). \( T(t) \) is called differentiable if it is differentiable for \( t > 0 \). If \( B(\lambda, t)x = \int_t^1 e^{\lambda(t-s)} T(s)x ds \), then \( B(\lambda, t)x \) is differentiable in \( t \) with \( B(\lambda, t)x = T(t)x + \lambda B(\lambda, t)x \) and \( B'_{\lambda}(t) \) is a bounded linear operator in \( X \), see [5, 13].

Spectral inclusions for various reduced spectra of a \( C_0 \)-semigroup were studied by authors in [4, 5] for point spectrum, approximative spectrum and residual spectrum. Also, the spectral equality for a \( C_0 \)-semigroup was studied by authors in [15] for semi-regular, essentially semi-regular and semi-Fredholm spectrum, respectively. In this work, we will continue in this direction, we will prove that the spectral inclusion for Drazin and quasi-Fredholm spectra. Also, we will show that if \( (T(t))_{t \geq 0} \) is a \( C_0 \)-semigroup with infinitesimal generator \( A \) such that the equality \( \lim_{t \to \infty} \frac{1}{t^2} \|T(t)\| = 0 \) holds for some \( n \in \mathbb{N} \), then the infinitesimal generator \( A \) is quasi-Fredholm if and only if it is Drazin invertible if and only if it is \( B \)-Fredholm.

## 2 Main Results

We start by given the following two lemmas which are proved in [5]. They will be used to prove our main result.

**Lemma 2.1** [5] Let \((A, D(A))\) be the infinitesimal generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) and \(B(\lambda, t) = \int_0^1 e^{\lambda(t-s)} T(s)x ds\) is a bounded operator from \(X\) to \(D(A)\). Then, for every \( \lambda \in \mathbb{C}, t > 0 \) and \( n \in \mathbb{N} \), the following statements hold:

1. \((e^{\lambda t} - T(t))^n(x) = (\lambda - A)^n B(\lambda, t)x, \quad \lambda \in \mathbb{C}, x \in X;\)
2. \((e^{\lambda t} - T(t))^n(x) = B(\lambda, t)\lambda^n(\lambda - A)^n x, \quad \lambda \in \mathbb{C}, x \in D(A);\)
3. \(R(e^{\lambda t} - T(t))^n \subseteq R(\lambda - A)^n;\)
4. \(N(\lambda - A)^n \subseteq N(e^{\lambda t} - T(t))^n.\)

Let \(Y\) be a Banach space that is continuously embedded in \(X\) (in symbols: \(Y \hookrightarrow X\)). We need the following lemma.

**Lemma 2.2** [5] Let \((A, D(A))\) be the infinitesimal generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on \(X\) and assume that the restricted semigroup \((T(t))_{t \geq 0}\) is strongly continuous on some \((T(t))_{t \geq 0}\)-invariant Banach space \(Y \hookrightarrow X\). Then the infinitesimal generator of \((T(t))_{t \geq 0}\) is the part \((A_1, D(A_1))\) of \(A\) in \(Y\).

The following two lemmas which are proved in [15] will be used in the sequel.

**Lemma 2.3** [15, Lemma 3.1] Let \(A\) the infinitesimal generator of \(C_0\)-semigroup \((T(t))_{t \geq 0}\) and \(B(\lambda, t) = \int_0^t e^{\lambda(t-s)} T(s)x ds\) is a linear bounded operator on \(X\). Then there exist \(C\) and \(D\) tow operator such that \((\lambda - A), B(\lambda, t), C, D\) are mutually commuting operators, for all \(x \in D(A)\) and \((\lambda - A) + DB(\lambda, t) = I, t > 0.\)

**Lemma 2.4** [15, Lemma 3.2] Let \((\lambda - A), B(\lambda, t), C, D\) be mutually commuting operators in \(D(A)\) such that \(C(\lambda - A) + DB(\lambda, t) = I, t > 0\). Then we have:
(1) For every positive integer \( n \) there are \( C_n, D_n \in D(A) \) such that \((\lambda - A)^n, B^n(\lambda, t), C_n, D_n \) are mutually commuting and
\[
(\lambda - A)^n C_n + B^n(\lambda, t) D_n = I.
\]

(2) For every positive integer \( n \), \( R(e^{\lambda t} - T(t))^n = R(\lambda - A)^n \cap R(B^n(\lambda, t)) \) and \( N((e^{\lambda t} - T(t))^n) = N((\lambda - A)^n) + N(B^n(\lambda, t)). \) Further \( R^\infty(e^{\lambda t} - T(t)) = R^\infty(\lambda - A) \cap R^\infty(B(\lambda, t)) \) and \( N^\infty(e^{\lambda t} - T(t)) = N^\infty(\lambda - A) + N^\infty(B(\lambda, t)). \)

(3) \( N^\infty(\lambda - A) \subseteq R^\infty(B(\lambda, t)) \) and \( N^\infty(B(\lambda, t)) \subseteq R^\infty(\lambda - A). \)

Now, we give some spectral results for differentiable \( C_0 \)-semigroup.

**Lemma 2.5** Let \((A, D(A))\) be the infinitesimal generator of a strongly continuous semigroup \( (T(t))_{t \geq 0} \), \( B(\lambda, t) = \int_0^t e^{\lambda(t-s)} T(s) xds \) is a bounded operator from \( X \) to \( D(A) \). If \( T(t) \) is differentiable for \( t > t_0 \), then for every \( \lambda \in \mathbb{C}, t > t_0 \) and \( n \in \mathbb{N} \), the following statements hold:

1. \((a) : (\lambda e^{\lambda t} - AT(t))^n(x) = (\lambda - A)^n B(\lambda, t)^n(x), \ \lambda \in \mathbb{C}, x \in X; \)
2. \((b) : (\lambda e^{\lambda t} - T(t))^n(x) = B(\lambda, t)^n(\lambda - A)^n(x) \lambda \in \mathbb{C}, x \in D(A). \)

**Proof** Assuming now that \( t > t_0 \) and differentiating \((a) \) and \((b) \) in \((1) \) of Lemma 2.1 with respect to \( t \) and \( n = 1 \), we obtain
\[
\lambda e^{\lambda t} x - AT(t)x = (\lambda I - A) B'(\lambda, t)x \quad \text{for every} \quad x \in X;
\]
\[
\lambda e^{\lambda t} x - AT(t)x = B'(\lambda, t)x(\lambda I - A) \quad \text{for every} \quad x \in D(A).
\]

This gives \((a) \) and \((b) \) for \( n = 1 \). By induction, we obtain \((a) \) and \((b) \) for all \( n \in \mathbb{N} \). The rest of Lemma follows from \((1) \). \( \square \)

**Proposition 2.1** Let \( T(t)_{t > 0} \) be a \( C_0 \)-semigroup and let \( A \) be its infinitesimal generator. If \( T(t) \) is differentiable for \( t > t_0 \) and \( \lambda \in \sigma_A(x) \) for \( x \in X \), then
\[
\lambda e^{\lambda t} \in \sigma_{AT(t)}(x).
\]

**Proof** Let \( t > t_0 \) be fixed and suppose that \( \lambda e^{\lambda t} \notin \sigma_{AT(t)}(x) \), then there exist a sequence \((x_i)_{i \in \mathbb{N}} \) of \( X \) such that \( x_0 = x, (\lambda e^{\lambda t} - AT(t))x_i = x_{i-1} \) and \( \sup_i \| x_i \|^{\frac{1}{t}} < \infty \).

We put \( y_i = B'(\lambda, t)x_i \), as \( B'(\lambda, t) \) is a bounded linear operator in \( X \), we have \( y_0 = x_0 = x, \ y_0 \in D(A), \)
\[
(\lambda - A)y_i = (\lambda - A)B'(\lambda, t)x_i B'(i-1)(\lambda, t)x_i
\]
\[
= (\lambda e^{\lambda t} - AT(t))B'(i-1)(\lambda, t)x_i
\]
\[
= B'(i-1)(\lambda, t)(\lambda e^{\lambda t} - AT(t))x_i
\]
\[
= B'(i-1)(\lambda, t)x_{i-1}
\]
\[
y_{i-1}
\]
Therefore $(\lambda - A)y_i = y_{i-1}$. On the other hand
\[ \|y_i\| = \|B^i(\lambda, t)x_i\| < \|B^i(\lambda, t)\|\|x_i\| < M^i\|x_i\|, \]
then
\[ \sup_i \|y_i\|^\frac{1}{t} < \sup_i \|x_i\|^\frac{1}{t} < \infty. \]
So that $\lambda \notin \sigma_A(x)$. \hfill \Box

Denote by $\sigma_{su}(A)$ the subjectivity spectrum of $A$. It is known that $\bigcup_{x \in X} \sigma_A(x) = \sigma_{su}(A)$ for a closed operator $A$. Hence the following corollary holds.

**Corollary 2.1** Let $T(t)_{t > 0}$ be a $C_0$-semigroup and let $A$ be its infinitesimal generator. If $T(t)$ is differentiable for $t > t_0$ and $\lambda \in \sigma_{su}(A)$, then $\lambda e^{\lambda t} \in \sigma_{su}(AT(t))$.

**Proposition 2.2** Let $T(t)_{t > 0}$ be a $C_0$-semigroup and let $A$ be its infinitesimal generator. If $T(t)$ is differentiable for $t > t_0$ and $\lambda \in \sigma_{ap}(A)$, then $\lambda e^{\lambda t} \in \sigma_{ap}(AT(t))$.

**Proof** For $t > t_0$, since $\lambda e^{\lambda t} - AT(t) = (\lambda - A)B'(\lambda, t)$ and $B'(\lambda, t)$ is bounded linear operator, then $\lambda e^{\lambda t} - AT(t)$ is a bounded linear operator. It is easy to check that for $x \in D(A)$, $B'(\lambda, t)Ax = AB'(\lambda, t)x$. If $\lambda \in \sigma_{ap}(A)$, then there exists sequence $(x_n)_{n \in \mathbb{N}} \in D(A)$ satisfying $\|x_n\| = 1$ and $\|\lambda - A\|x_n \to 0$. From (1) of Lemma 2.5, we obtain the result. \hfill \Box

By an outline of the proof of [6, Theorem 2.1], we obtain the following result.

**Proposition 2.3** Let $T(t)_{t > 0}$ be a $C_0$-semigroup and let $A$ be its infinitesimal generator. If $T(t)$ is differentiable for $t > t_0$ and $\lambda \in \nu(A)$, then $\lambda e^{\lambda t} \in \nu(AT(t))$. where $\nu(\cdot) \in \{\sigma_\gamma(\cdot), \sigma_\pi(\cdot), \sigma_{ve}(\cdot)\}$ and $\sigma_\gamma(\cdot), \sigma_\pi(\cdot), \sigma_{ve}(\cdot)$ denote the regular spectrum, essential regular spectrum and left essential spectrum.

In the next theorem, we will prove that the spectral inclusion of $C_0$-semigroups remains true for the Drazin invertible and quasi-Fredholm spectra.

**Theorem 2.1** Let $(T(t))_{t \geq 0}$ a $C_0-$semigroup, with infinitesimal generator $A$. Then
\[ e^{t\sigma_D(A)} \subseteq \sigma_D(T(t)). \]

**Proof** Let $t_0 > 0$ be fixed and suppose that $(e^{\lambda t_0} - T(t_0))$ is Drazin invertible for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $M := R(e^{\lambda t_0} - T(t_0))$ is closed and the restricted semigroup $(e^{\lambda t_0} - T(t_0)|_M)$ is invertible. We show that $(\lambda - A)$ is Drazin invertible. To this end, in the first we show that $R(\lambda - A)$ is closed. Let $x \in R((\lambda - A))$, that is, there exist $u_k \in D(A^n), k = 1, 2, ..., \text{ such that } (\lambda - A)^n u_k \to x$, hence
\[ (e^{\lambda t_0} - T(t_0))^n u_k := B(\lambda, t_0)^n(\lambda - A)^n u_k \]

\[ \Box \]
by Lemma 2.1. Also,
\[ B(\lambda, t_0)^n(\lambda - A)^nu_k \to B(\lambda, t_0)^nx. \]

Hence
\[ (e^{\lambda t_0} - T(t_0))^nu_k \to B(\lambda, t_0)^nx. \]

Since \( M := R(e^{\lambda t_0} - T(t_0))^n \) is closed, then
\[ B(\lambda, t_0)^nx \in R(e^{\lambda t_0} - T(t_0))^n. \]

Hence there exists \( u \in D(A^n) \) such that
\[ B(\lambda, t_0)^nx = (e^{\lambda t_0} - T(t_0))^nu. \]

In the other hand, From Lemma 2.1, we have that
\[ (e^{\lambda t_0} - T(t_0))^nu = B(\lambda, t_0)^n(\lambda - A)u, \]

hence
\[ B(\lambda, t_0)^nx = B(\lambda, t_0)^n(\lambda - A)u. \]

This implies that,
\[ x - (\lambda - A)^nu \in N(B(\lambda, t_0)) \subseteq R(\lambda - A)^n. \]

So \( x \in R(\lambda - A)^n \), and hence \( R(\lambda - A)^n \) is closed.

Now, let us to show that \( (\lambda - A_{\lambda R(\lambda - A)^n}) \) is invertible. For this, as \( (e^{\lambda t_0} - T(t_0)|M) \) is invertible, then \( (e^{\lambda t_0} - T(t_0)|M) \) is bounded below and \( R(e^{\lambda t_0} - T(t_0)|M) = R(e^{\lambda t_0} - T(t_0))^{n+1} \) is onto. We show that \( (\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) \) is bounded below. Since \( (e^{\lambda t_0} - T(t_0)|M) \) is bounded below, then \( (e^{\lambda t_0} - T(t_0)|M) \) is injective and \( R(e^{\lambda t_0} - T(t_0)|M) \) is closed. We show that \( (\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) \) is injective and \( (\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) \) is closed. For all \( x \in D(A) \) we have

\[ \{0\} = N(e^{\lambda t_0} - T(t_0)|M \cap D(A)) = N(\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) \]
\[ + N(B(\lambda, t_0)) \cap R(B^n(\lambda, t_0)), \]

then \( N(\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) = \{0\} \), therefore \( (\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) \) is injective. As \( R(e^{\lambda t_0} - T(t_0))^{n+1} \) is closed, then \( (\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) = R(\lambda - A)^{n+1} \) is also closed. On the other hand, as \( R(e^{\lambda t_0} - T(t_0)|M) \) is onto, we can easily verify that \( R(\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) \) is onto. In fact one can verify that
\[ R(\lambda - A_{\lambda R(\lambda - A)^n} \cap D(A)) = R(\lambda - A)^{n+1} = R(\lambda - A)^n. \]
We have $R(\lambda - A)^{n+1} \subset R(\lambda - A)^n$ and if $y \in R(\lambda - A)^n$, then there exist $x \in D(A^n)$ such that $y = R(\lambda - A)^n x$ and by (1) of Lemma 2.3, we have

$$(\lambda - A)^n x = (\lambda - A)^n C_n (\lambda - A)^n x + D_n B^n (\lambda, t) (\lambda - A)^n x$$

and as $R(e^{\lambda t_0} - T(t_0)) | (e^{\lambda t_0} - T(t_0))^n = R(e^{\lambda t_0} - T(t_0))^{n+1}$ is onto, then there exist $x' \in X$ such that $(e^{\lambda t_0} - T(t_0))^n x = (e^{\lambda t_0} - T(t_0))^{n+1} x' = (\lambda - A)^{n+1} B^{n+1} (\lambda, t) x'$, therefore $y \in R(\lambda - A)^n$ and then $R(\lambda - A | R(\lambda - A)^n \cap D(A))$ is onto.

Finally, $(\lambda - A | R(\lambda - A)^n \cap D(A))$ is Drazin invertible. □

**Corollary 2.2** For the infinitesimal generator $A$ of a strongly continuous semigroup $(T(t))_{t \geq 0}$, one has the spectral inclusion

$$e^{t \sigma_v(A)} \subseteq \sigma_v(T(t))$$

where $\sigma_v(\cdot)$ is the left Drazin and right Drazin spectra.

The following example shows that the inclusion in Theorem 2.1 is strict.

**Example 1** Let $X$ be the Banach space of continuous functions on $[0, 1]$ which are equal to zero at $x = 1$ with the supremum norm. Define

$$(T(t)f)(x) := \begin{cases} f(x + t) & \text{if } x + t \leq 1; \\ 0 & \text{if } x + t > 1. \end{cases}$$

$T(t)$ is obviously a $C_0$-semigroup on $X$. Its infinitesimal generator $A$ is given on

$$D(A) = \{ f : f \in C^1([0, 1]) \cup X, f' \in X \}$$

by

$$Af = f' \text{ for } f \in D(A).$$

One checks easily that for every $\lambda \in \mathbb{C}$ and $g \in X$ the equation $\lambda f - f' = g$ has a unique solution $f \in X$ given by

$$f(t) = \int_t^1 e^{\lambda (t - s)} g(s) ds.$$
\[ e^{t\sigma_D(A)} \subseteq \sigma_D(T(t)). \]

is strict.

**Theorem 2.2** For the infinitesimal generator \( A \) of a strongly continuous semigroup \( (T(t))_{t \geq 0} \), we have the following inclusion:

\[ e^{t\sigma_QF(A)} \subseteq \sigma_{QF}(T(t)). \]

**Proof** Let \( t_0 > 0 \) be fixed and suppose that \( (e^{\lambda t_0} - T(t_0)) \) is quasi-Fredholm, for some \( \lambda \in \mathbb{C} \setminus \{0\} \). Then \( M := R(e^{\lambda t_0} - T(t_0))^n \) is closed and the restricted semigroup \( (e^{\lambda t_0} - T(t_0)|_M) \) is semi-regular.

We show that \( (\lambda - A) \) is quasi-Fredholm, to this end we show that \( R(\lambda - A)^n \) is closed and \( (\lambda - A)|_{R(\lambda - A)^n \cap D(A)} \) is semi-regular. That is, \( (R(\lambda - A)|_{R(\lambda - A)^n \cap D(A)}) \) is closed and \( N(\lambda - A)|_{R(\lambda - A)^n \cap D(A)} \) is quasi-Fredholm.

As \( M := R(e^{\lambda t_0} - T(t_0))^n \) is closed, then by the same argument as in the proof of Theorem 2.1, we conclude that \( R(\lambda - A)^n \) is closed. Since \( (e^{\lambda t_0} - T(t_0)|_M) \) is semi-regular, then \( R(e^{\lambda t_0} - T(t_0)|_M) = R(e^{\lambda t_0} - T(t_0))^n + 1 \) is closed, this implies that \( R(\lambda - A)|_{R(\lambda - A)^n \cap D(A)} = R(\lambda - A)^n + 1 \) is closed. On the other hand, by Lemma 2.1 we have that

\[ N(\lambda - A)|_{R(\lambda - A)^n \cap D(A)} \subseteq N(e^{\lambda t_0} - T(t_0)|_M \cap D(A) \subseteq N(e^{\lambda t_0} - T(t_0)|_M) \]

Since \( (e^{\lambda t_0} - T(t_0)|_M) \) is semi-regular, then \( N(e^{\lambda t_0} - T(t_0)|_M) \subseteq R^\infty (e^{\lambda t_0} - T(t_0)|_M) = R^\infty (\lambda - A)|_{R(\lambda - A)^n \cap D(A)} \cap R^\infty B(\lambda, t_0) \subset R^\infty (\lambda - A)|_{R(\lambda - A)^n \cap D(A)}, \) hence \( (\lambda - A)|_{R(\lambda - A)^n \cap D(A)} \) is semi-regular. Finally, we conclude that \( (\lambda - A) \) is quasi-Fredholm.

\[ \square \]

**Remark 1** For an operator \( T \in B(X) \) we have \( \sigma_D(T) \subseteq \sigma_{QF}(T) \), this implies by Example 1 the inclusion spectral in Theorem 2.2 is strict.

Let \( A \) be the infinitesimal generator of a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \). In the next Theorem, we will use the concept of the weak* -integral. For more information of this integral, see [17, Appendix 1].

In the following, we will give condition on \( (T(t))_{t \geq 0} \) under which facts \( A \) is Drazin invertible, \( A \) is \( B \)-Fredholm and \( A \) is \( Q \)-Fredholm are equivalent.

**Theorem 2.3** Let \( A \) be the infinitesimal generator of a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \).

If \( \lim_{t \to \infty} \frac{1}{t^n} ||T(t)|| = 0 \), for some \( n \in \mathbb{N} \), the following assertions are equivalent:

1. \( A \) is quasi-Fredholm;
2. \( A \) is Drazin invertible;
3. \( A \) is \( B \)-Fredholm.

**Proof** (1) \( \Rightarrow \) (2):

Since \( A \) is quasi-Fredholm, then \( R(A^n) \) closed and \( A|_{R(A^n) \cap D(A)} \) is semi-regular. Let \( y \in N(A|_{R(A^n) \cap D(A)}) \), then there exists \( x \in (R(A^n) \cap D(A^n)) \) such that \( y = A^n x \). We integer by parts in the following formula:

\[ \square \]
\[ T(t)x - x = \int_0^t T(s)Ax\,ds, \]

we obtain that
\[ T(t)x = x + tA + \frac{t^2}{2!}A^2 + \int_0^t \frac{(t-s)^2}{2!}T(s)A^3x\,ds. \]

We repeat this operation for \( n \) times, we obtain that
\[ T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}T(s)A^n x\,ds. \]

Hence,
\[ T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + y \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} ds \]
\[ = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \frac{t^n}{n!}y. \]

As \( \lim_{t \to \infty} \frac{1}{t} \|T(t)\| = 0 \), then \( y = 0 \), this implies that
\[ N(A_{|(R(A^n) \cap D(A))}) = \{0\}. \]

On the other hand, let \((T(t))_{t \geq 0}\) with infinitesimal generator \( A' \) the adjoint semigroup of \((T(t))_{t \geq 0}\). Since \( A_{|(R(A^n) \cap D(A))} \) is semi-regular, then \( A'_{|(R(A^n) \cap D(A'))} \) is also semi-regular, see [12, Proposition 1.6]. Using the following formula
\[ T(t)x' - x' = \text{weak}^* \int_0^t (s)x'\,ds, \quad \forall x' \in (R(A^n) \cap D(A')), \quad \text{for all } t \geq 0 \]
which is proved in [17, Proposition 1.2.2] and by the same argument as above, we get
\[ N(A'_{|(R(A^n) \cap D(A'))}) = \{0\}. \] This is equivalent to the fact that
\[ \overline{R(A_{|(R(A^n) \cap D(A))})} = (R(A^n) \cap D(A)). \]

Hence \( R(A_{|(R(A^n) \cap D(A))}) = (R(A^n) \cap D(A)) \), since \((R(A^n) \cap D(A))\) is closed. From this it follows that \( A_{|(R(A^n) \cap D(A))} \) is surjective and hence it is invertible. Finally, \( A \) is Drazin invertible.

(2) \( \Rightarrow \) (1): is clear.

(1), (2) and (3) are equivalent, since the class of Drazin invertible operator is a subclass of B-Fredholm operator and the class of B-Fredholm operator is a subclass of quasi-Fredholm operator. \( \square \)
The following example shows that the condition $\lim_{t \to \infty} \frac{1}{t^n} \| T(t) \| = 0$, for some $n \in \mathbb{N}$ in Theorem 2.3 is needed for conclusion.

**Example 2** Let $H$ be a Hilbert space with an orthonormal basis $\{e_n\}_1^\infty$ and $T$ be unilateral weighted shift operator on $H$ defined by $Aeon = w_n e_{n+1}$, $n = 1, 2, \ldots$ [14, p. 51]. Let $T(t) = e^{tA}$ be the semigroup generated by $A$, $t \geq 0$ and $w_n = 1$. Easy to see that $\| T(t) \| = e^t$, thus, $\lim_{t \to \infty} \frac{1}{t^n} \| T(t) \| \neq 0$, for all $n \in \mathbb{N}^\times$. We know (see [1, Theorem 2.86 and Example 3.30]) that $\sigma(A) = \{ \lambda \in \mathbb{C}, |\lambda| \leq 1 \}$ and $\sigma_{ap}(A) = \{ \lambda \in \mathbb{C}, |\lambda| = 1 \}$, thus, $A$ is semi-regular then $A$ quasi-Fredholm. On the other hand $A$ is not surjective so $R(A^n) \supset R(A^{n+1})$ this implies that $A^n : R(A^n) \to R(A^n)$ is not invertible; therefore, $A$ is not Drazin invertible.

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