An efficient finite element method applied to quantum billiard systems

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An efficient finite element method (FEM) for calculating eigenvalues and eigenfunctions of quantum billiard systems is presented. We consider the FEM based on triangular \( C_1 \) continuity quartic interpolation. Various shapes of quantum billiards including an integrable unit circle are treated. The numerical results show that the applied method provides accurate set of eigenvalues exceeding a thousand levels for any shape of quantum billiards on a personal computer. Comparison with the results from the FEM based on well-known \( C_0 \) continuity quadratic interpolation proves the efficiency of the method.

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I. INTRODUCTION

There has been much interest in characterizing the quantum manifestation of classically chaotic systems \[^1,2\], since McDonald and Kaufman's pioneering investigation on the statistical characteristics of eigenvalues and eigenfunctions \[^3\]. The quantum billiard, which is represented by the two dimensional stationary Schrödinger equation of free particle with satisfying the Dirichlet boundary condition, is an intensively studied model system in the field of quantum chaos due to its simplicity. The integrability of corresponding classical billiard depends solely upon the geometry of boundary. The quantum billiard can be also expressed by the scalar Helmholtz equation, for example, which describes the electromagnetic field inside a flat microwave resonator. In that context, microwave experiments played the role of analog computation of eigenstates in quantum billiards \[^4,5\].

There are several numerical methods, which have been dominantly adopted by colleagues in this field, for calculating eigenvalues and eigenfunctions of the quantum billiards such as the boundary integral method (BIM) (reviewed in Ref. \[^6\]), the plane wave decomposition method (PWDM) \[^7,8\], the scaling method \[^9,10\], and the conformal mapping method \[^12,13,14\]. Let us briefly review the mentioned methods. The BIM, which is a rigorously established method, reduces the problem of two dimensional stationary Schrödinger equation to a one dimensional integral equation. In result, each root of the Fredholm determinant constitutes the set of eigenvalues. In practice, the determinant does not become zero due to the discretization error and the BIM approximates the minima of the lowest singular values to the eigenvalues. One of them is that the calculated results can include additional, i.e., spurious solutions which correspond to roots of outside scattering problem with Neumann boundary condition \[^15\]. For non-convex geometric billiard, the detection of spurious solutions is not a simple numerical task \[^6\] and the BIM failed in the isospectral drum introduced by Gordon, Webb, and Wolpert \[^16\] due to its strong non-convexity \[^17\]. The feasibility of missing eigenvalues is another weakness of the BIM. For higher lying eigenvalues, the spectra become more denser and the correct detection of minima is a serious problem.

The PWDM, which has been introduced by Heller \[^7\], is a rather heuristic method in the context of quantum chaos. It is appropriate for computing high lying eigenstates but incongruent for studying of spectral statistics, because only a few selected eigenstates can be calculated with many intermediate missing. Also the PWDM can fail in non-convex or multiply connected (e.g., containing a hole) billiards \[^18\]. In the literatures, there has been a considerably efficient numerical method, that is, the scaling method derived by Vergini and Saraceno \[^9\]. It represents the boundary norm as a function of energy by the use of scaling. In result, the authors of Refs. \[^9,10\] obtained all eigenvalues (without any missing) within a narrow energy range, which lie close to a chosen reference value, in a single computational step. The efficiency of scaling method is obvious from that the BIM can locate a single eigenvalue in a single computational step. For specific geometric billiard for which the conformal mapping onto the unit disk is sufficiently simple (e.g., so-called Robnik billiard \[^11\]), the conformal mapping method derived by Robnik \[^12\] have provided accurate set of eigenvalues \[^17\]. Recently, new approaches that combine each ideas of above mentioned methods have been studied, for the BIM and the PWDM \[^19\] and for the BIM and the scaling method \[^20\]. Concerning the scattering quantization method, an efficient improvement has been carried out in Ref. \[^21\].

The finite element method (FEM) is one of the most widely accepted numerical methods for partial differential equations in various fields of science and engineering \[^22,23,24\]. Compared with previously mentioned methods, the FEM has obvious advantages that it has almost no limitation on the geometric complexity of billiard (see the results in Ref. \[^25\]) and provides in a single compu-
tational step a set of all eigenvalues and eigenfunctions up to maximal level allowed by memory allocation. However, the accurate computation of high-lying eigenstates using the FEM is conventionally more difficult than the case of other mentioned methods, since the FEM discretizes not only the boundary but the whole domain of billiard (it needs more memory storage). Thereby, though its obvious advantages, the FEM has been apparently overlooked in the field of quantum chaos. As far as we know, there have been only a few studies where the FEM is used for calculating the eigenstates of quantum billiards. Among those studies, Heuveline showed an effective FEM only requiring $O(N)$ memory allocation by using the p-finite elements basis and the sparsity of matrices. Note that the FEM commonly gives rise to sparse matrices but usual FEMs do not take advantage of the sparsity (it contains quite difficult numerical tasks) and need $O(N^2)$ memory storage, where $N$ is the number of total nodes.

The aim of this paper is to show the validity of FEM as a numerical method for calculating eigenvalues and eigenfunctions of quantum billiards. For that purpose, we present an efficient FEM based on the Hermite interpolation. In each element, the wave function is interpolated by quartic polynomials involving nodal values of wave function and its first derivatives, namely, the adopted interpolation basis admits the $C_1$ continuity quartic interpolation. The results of numerically calculated eigenvalues exceeding a thousand levels for any shape of quantum billiards on a personal computer. Comparison with the results from the FEM based on well-known $C_0$ continuity quadratic interpolation proves the efficiency of the applied method. Note that, by virtue of the $C_1$ continuity, the method handles well problem that treats values of first derivatives of wave function at the boundary such as the Neumann boundary condition.

The rest of paper is organized as follows. In Sec. II, we outline numerical procedures of the FEM based on $C_1$ continuity quartic interpolation. The results of numerically calculated eigenvalues and the analysis of spectral statistics for unit disk, the Robnik billiard, and the spiral-shaped billiard are presented in Sec. III. Conclusions are given in Sec. IV.

II. NUMERICAL PROCEDURE

The quantum billiard is governed by two-dimensional stationary Schrödinger equation of free particle

$$-\nabla^2 \psi(\vec{r}) = E \psi(\vec{r}), \quad \text{for } \vec{r} \in \Omega$$

(1)

with satisfying the Dirichlet boundary condition $\psi(\vec{r}) = 0$ at the boundary of domain $\partial \Omega$. Note that we use the natural units $\hbar = 2m = 1$.

The first step of applying FEM is that the domain of billiard $\Omega$ is discretized into finite elements, i.e., mesh generation. The shape of element and the number of nodes in each element are determined according to the type of interpolation polynomials (i.e., shape functions). Here we consider the FEM based on the triangular $C_1$ continuity quartic interpolation, which has been derived by Specht and known that passes all patch test, i.e., a condition for assessing FEM convergence for arbitrary mesh configurations (see Chapter 11 of Ref. [24]). Accordingly the domain of billiard is discretized into triangular elements. In each element, there exist three nodes that locate at vertices of triangle and each node has three degrees of freedom correspond to wave function and its first derivatives ($\psi, \partial \psi/\partial x, \partial \psi/\partial y$). Thereby each element has actually nine nodes. In each element the unknown function, i.e., wave function $\psi(x, y)$ is represented as a linear combination of shape functions multiplied by as-yet-unknown nodal values of wave function and its first derivatives. The shape function is defined only over a given element and has zero value at outside of it. An explicit representation of shape functions will be postponed for a while. The numerical procedure of FEM requires that each node has three indices: local index $il$, element index $ie$, and global index $i$. The mesh generation completes the mapping from local and element index ($il_{ieh}$ node of $ie_{ih}$ element) to global index ($i_{ih}$ global node).

On numerical calculations in this paper, we use two considerable and freely available mesh generators; the DistMesh and the Triangle. The DistMesh is a Matlab based mesh generator that finds node locations settled down a equilibrium state in a truss structure. A geometry of domain is represented by the signed distance function from node to closest boundary $\partial \Omega$, negative inside the domain. It generates high quality meshes, i.e., almost equilateral triangles but can be faced with difficulty for complex geometric boundary. The Triangle is a robust Delaunay refinement code. The user-supplied data, which contains the information of nodes placing on the boundary, are employed for specifying the domain of billiard. The quality of meshes is controlled by the constraint of minimum angle (up to 34 degree) and maximum size of triangle. The Triangle has almost no limitation on the complexity of geometry.

Now we can take the next step. Eq. (1) can be obtained from the condition that the action

$$A = \int ds (\nabla \psi^*(\vec{r}) \cdot \nabla \psi(\vec{r}) - E \psi^*(\vec{r}) \psi(\vec{r}))$$

(2)

is minimized with respect to variation of $\psi^*(\vec{r})$. Here $\psi(\vec{r})$ and $\psi^*(\vec{r})$ are considered to be two independent variables. Then, the action integral of Eq. (2) is discretized into integrations over each element as

$$A = \sum_{ie} A^{(ie)}$$

(3)
where \( ne \) is the number of total elements. In Cartesian coordinates, the discretized action integral \( A^{(ic)} \) is represented by

\[
A^{(ic)} = \int \int_{\text{mesh}} dx dy \left( \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \psi^*}{\partial y} \frac{\partial \psi}{\partial y} \right) - E \int \int_{\text{shape}} dx dy (\psi^* \psi)
\]

where \( \psi = \psi(x, y) \) and \( \psi^* = \psi^*(x, y) \). For simple computation of \( A^{(ic)} \), it is advisable that the integral domain of each element is transformed into a regularized domain (called parent element) as

\[
A^{(ic)} = \int \int_{\text{shape}} dx dy f(x, y) = \int \int_{\text{parent}} d\xi d\eta g(\xi, \eta).
\]

Figure 1 shows a transformation of triangular real element into the parent element. The quantities in Eq. (4) are altered into \((\xi, \eta)\) notations as the followings. The wave function \( \psi(\xi, \eta) \) is interpolated by the \( C^1 \) continuity quartic shape functions \( \{H_{ij}(\xi, \eta)\} \) multiplied by as-yet-unknown nodal values of wave function and its first derivatives \( \{\tilde{\psi}_i\} \) as

\[
\psi(\xi, \eta) = \sum_{i} \tilde{\psi}_iH_{ij}(\xi, \eta),
\]

\[
\tilde{\psi} = \{\psi_1, \frac{\partial \psi_1}{\partial x}, \frac{\partial \psi_1}{\partial y}, \psi_2, \frac{\partial \psi_2}{\partial x}, \frac{\partial \psi_2}{\partial y}, \psi_3, \frac{\partial \psi_3}{\partial x}, \frac{\partial \psi_3}{\partial y}\}.
\]

For simple representation, we use a cyclic property of the applied shape functions and introduce the following notation \( H_a = \{H_{3a-2}(\xi, \eta), H_{3a-1}(\xi, \eta), H_{3a}(\xi, \eta)\} \) for \( a = 1, 2, 3 \). Then, the shape functions are represented by

\[
H_a^T = \left\{ \begin{array}{l} P_a - P_{a+3} + P_{c+3} + 2(P_{a+6} - P_{c+6}) \\ m_b(P_{c+6} - P_{c+3}) + m_cP_{a+6} \\ -m_b(P_{c+6} - P_{c+3}) - m_cP_{a+6} \end{array} \right\}
\]

where \( m_a = x_c - x_b, n_a = y_b - y_c, \) and \( a, b, c \) are the cyclic permutations of 1, 2, 3. The nine polynomials \( \{P_l(\xi, \eta)\} \) in Eq. (7) are expressed as

\[
P = \left\{ (1 - \xi - \eta, \xi, \eta, (1 - \xi - \eta)\xi, \xi\eta, (1 - \xi - \eta), (1 - \xi - \eta)^2 + \frac{1}{2}(1 - \xi - \eta)\eta\cdot (3(1 - \mu_3)(1 - \xi - \eta) - (1 + 3\mu_3)\xi + (1 + 3\mu_3)\eta), \xi\eta + \frac{1}{2}(1 - \xi - \eta)\eta\cdot (3(1 - \mu_1)\xi - (1 + 3\mu_1)\eta + (1 + 3\mu_1)(1 - \xi - \eta)), \right.
\]

\[
\left. \eta^2(1 - \xi - \eta) + \frac{1}{2}(1 - \xi - \eta)\eta\cdot (3(1 - \mu_2)\eta - (1 + 3\mu_2)(1 - \xi - \eta) + (1 + 3\mu_2)\xi) \right\}
\]

where \( \mu_a = (l_c - l_b)/l_a \) and \( l_a = (x_b - x_c)^2 + (y_b - y_c)^2 \) for the cyclic permutation of \( a, b, c \). The coordinate transformation between real and parent elements is given by the following \( C_9 \) continuity linear shape functions

\[
x = \sum_k^3 x_k^{(ic)} N_k(\xi, \eta), \quad y = \sum_k^3 y_k^{(ic)} N_k(\xi, \eta).
\]

The partial derivatives of wave function are transformed into \((\xi, \eta)\) notations as follows:

\[
\frac{\partial \psi(x, y)}{\partial x} = \frac{\partial}{\partial x} \left( \sum_{i} \tilde{\psi}_i^{(ic)} H_{ij}(\xi, \eta) \right),
\]

\[
= \sum_{i} \frac{\partial \tilde{\psi}_i^{(ic)}}{\partial \xi} \left( \frac{\partial H_{ij}(\xi, \eta)}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial H_{ij}(\xi, \eta)}{\partial \eta} \frac{\partial \eta}{\partial x} \right),
\]

\[
\frac{\partial \psi(x, y)}{\partial y} = \frac{\partial}{\partial y} \left( \sum_{i} \tilde{\psi}_i^{(ic)} H_{ij}(\xi, \eta) \right),
\]

\[
= \sum_{i} \frac{\partial \tilde{\psi}_i^{(ic)}}{\partial \xi} \left( \frac{\partial H_{ij}(\xi, \eta)}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial H_{ij}(\xi, \eta)}{\partial \eta} \frac{\partial \eta}{\partial y} \right).
\]

By applying Eqs. (6)-(11), the discretized action integral \( A^{(ic)} \) is represented by

\[
A^{(ic)} = \sum_{i,j,l}^{9} \tilde{\psi}_i^{(ic)*} \left( \int_0^1 d\eta \int_0^{1-n} d\xi f(\xi, \eta) \tilde{\psi}_{ij} \right) \tilde{\psi}_j^{(ic)}
\]

\[
- E \sum_{i,j,l}^{9} \tilde{\psi}_i^{(ic)*} \left( \int_0^1 d\eta \int_0^{1-n} d\xi g(\xi, \eta) \tilde{\psi}_{ij} \right) \tilde{\psi}_j^{(ic)}
\]
rule over the triangle, which has been derived by Dunavant [33]. Eq. (12) can be exactly integrated as

\[
\int_0^1 d\eta \int_0^{1-\eta} d\xi f(\xi, \eta) = \sum_{i=1}^{N} f(\xi_i, \eta_i) \cdot \omega_i 
\]  

(13)

where \((\xi_i, \eta_i)\) is a quadrature point and \(\omega_i\) is a weight. \(N\) equals 12 and 16 for sixth and eighth order polynomials, respectively (see the table in Ref. [33]). Then we obtain the following result

\[
A^{(ie)} = \sum_{i,j} \tilde{\psi}_{ij}^\dagger \cdot (L_{ij}^{(ie)} - EM_{ij}^{(ie)}) \cdot \tilde{\psi}_{ij} 
\]  

(14)

where \(L_{ij} (M_{ij})\) is a summation of all \(L_{ij}^{(ie)} (M_{ij}^{(ie)})\) for which satisfies that \(i_{kth}\) and \(j_{kth}\) nodes of \(ie_{kth}\) element are mapped into \(i_{kth}\) and \(j_{kth}\) global nodes, respectively. \(N\) is the number of total global nodes. In our case, it is equal to three times the number of physical global nodes, since each node has three degrees of freedom.

The adaptation of Dirichlet boundary condition is implemented as follows. If the \(k_{th}\) global node places at the boundary of billiard \(\partial \Omega\), the nodal value of wave function \(\tilde{\psi}_k\) equals zero. It requires that the entries of \(k_{th}\) column of \(L\) and \(M\) matrices become zero and also the entries of \(k_{th}\) row set to zero since \(\tilde{\psi}_k = 0\). In practice, this is achieved by dropping the \(k_{th}\) row and column. Then, the dimension of \(L\) and \(M\) matrices are reduced by \(N - n_k\) where \(n_k\) is the number of global nodes located at \(\partial \Omega\).

Now we vary the action \(A\) with respect to nodal values \(\tilde{\psi}\) and invoke the principle of least action. In result, we obtain the discretized version of stationary Schrödinger equation in the form of generalized eigenvalue problem

\[
L \Psi = EM \Psi 
\]  

(16)

where \(L\) and \(M\) are \(N - n_k\) dimensional real and symmetric matrices and \(\Psi\) is a column matrix, which consists of nodal values \(\tilde{\psi}\). We use the LAPACK routine DSPGV [34] for solving the generalized eigenvalue problem. Note that this routine requires about \(2(N-n_k)^2\) memory storage. Finally we obtain the set of \(N - n_k\) eigenvalues \(\{E_i\}\) and eigenfunctions \(\{\Psi_i\}\) where \(\Psi_i = \left(\tilde{\psi}_1 \tilde{\psi}_2 \cdots \tilde{\psi}_N\right)\) through restoring zero nodal values at nodes on \(\partial \Omega\).

### III. NUMERICAL RESULTS

#### A. The unit circle billiard

We firstly consider an integrable billiard whose boundary is given by the unit circle for testing an efficiency of the FEM presented in Sec. II. In this case, the eigenvalue spectra are exactly known and given by the sorted set of \(\mu^2_j\) where \(j_{th}\) is the \(k_{th}\) root of \(j_{th}\) Bessel function of the first kind with considering the degenerate case on \(j \neq 0\) for \(j = 0, 1, 2, \cdots, k \in N\).

We investigate the relative error between exact and numerically calculated eigenvalues

\[
\mathcal{E}_{rel}(i) = \frac{|\tilde{E}_i - E_i|}{E_i} 
\]  

(17)

where \(\tilde{E}_i\) is the set of exact eigenvalues. For the results obtained from the FEM based on well-known triangular \(C^0\) continuity quadratic interpolation (see Refs. [22, 23] for its shape function and isoparametric transformation), we also compute the relative error \(\mathcal{E}_{rel}\). In Fig. 2(a), we plot the relative error \(\mathcal{E}_{rel}\) for both cases of interpolation basis. For the \(C^1\) continuity quartic interpolation, we obtain 21647 (almost same number of the above) eigenvalues and depict the relative error as a orange line. Note that all calculations in this paper are performed on a personal computer possessing 2.4 GHz Quad-Core CPU and 8 GByte memory. So the available maximal number of level allowed by memory allocation is limited about 22000. In both interpolations, the relative error \(\mathcal{E}_{rel}\) increases as the number of level \(n\) increases. However, the method based on the \(C^0\) continuity quadratic interpolation seriously loses its accuracy after a few hundreds levels. The results of Fig. 2(a) proves an efficiency of the FEM based on the \(C^1\) continuity quartic interpolation.

We consider the numerical result for which the relative error \(\mathcal{E}_{rel}\) is smaller than \(3 \times 10^{-3}\) as accurate eigenvalue. In Fig. 2(b), the relative error for the \(C^1\) continuity quartic basis is depicted up to 1500 level and the result is smaller than \(3 \times 10^{-3}\) in this range. Then we obtain accurate set of 1500 eigenvalues for the unit circle billiard with this criterion. The followings will show that above conjecture is reasonable.

Another method for testing the accuracy of obtained results is checking out whether the eigenvalue spectra are complete without any intermediate missing. For the system where the analytic eigenvalues are not available, such method has no alternative. It can be performed by investigating the spectral staircase function \(N(E)\), which counts the number of energy levels below \(E\). The spectral staircase function can be divided into a smooth and a fluctuating part.
\[ N(E) = \bar{N}(E) + N_{\text{fluc}}(E). \]  
(18)

The smooth part \( \bar{N}(E) \) is represented by the generalized Weyl’s law \[ \bar{N}(E) \simeq \frac{A}{4\pi} E^{\frac{1}{3}} + \frac{L}{4\pi} \sqrt{E} + C \]  
(19)

where minus and plus sign correspond to the Dirichlet and the Neumann boundary condition, respectively. \( A \) is a area of billiard, \( L \) is a length of the perimeter, and \( C \) is a correction constant for the curvature and corners given by

\[ C = \frac{1}{12\pi} \int \kappa(s) \, ds + \frac{1}{24} \sum \left( \frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi} \right) \]  
(20)

with local curvature \( \kappa(s) \) and \( \alpha \)th corner angle \( \alpha_i \).

The so-called \( \delta_n \) quantity, which is equivalent to \( N_{\text{fluc}}(E_n) \), is a good measure for the completeness of obtained results

\[ \delta_n = n - \frac{1}{2} - \bar{N}(E_n) \]  
(21)

where \( \bar{N}(E_n) := n - \frac{1}{2} \). For complete eigenvalue spectra, it has been well known that \( \delta_n \) fluctuates around zero. In Fig. 2(c), \( \delta_n \) is drawn up to 1500 level for the \( C_1 \) continuity quartic basis. It certainly fluctuates around zero and shows that the obtained eigenvalues are complete up to 1500 level. Figure 2(c) also proves that the above criterion for the relative error is acceptable.

B. The Robnik billiard

In this section, we consider the billiard which has been introduced by Robnik [11]. The boundary of the Robnik billiard is defined by a quadratic conformal mapping from the unit circle

\[ \omega = x + iy = z + \lambda z^2 \]  
(22)

where \( z = e^{i\phi}, \phi \in [0, 2\pi] \), and \( \lambda \in [0, \frac{1}{2}] \). With increasing \( \lambda \) from zero, the boundary is continuously deformed from the unit circle. At \( \lambda = \frac{1}{2} \), the mapping of Eq. (22) is no longer conformal and the billiard has a cusp. Such limit case of the Robnik billiard is also called the cardioid billiard. It has a symmetry line at \( y = 0 \) and the desymmetrized billiard is twofold; odd and even symmetry satisfying the Dirichlet and the Neumann boundary condition at the symmetry line, respectively. In Fig. 3, we show the cardioid billiard and its desymmetrized version.

It has been proven that the cardioid billiard is ergodic, mixing, and a K system, i.e., fully chaotic system [36] and its spectral statistics have been minutely studied in Ref. [37]. We regard the odd type of desymmetrized cardioid billiard as a model of convex geometric chaotic system and test the FEM based on the \( C_1 \) continuity quartic interpolation. By applying the method, we calculate 21901 eigenvalues of the odd symmetric case. Among those we obtain 1500 accurate eigenvalue spectra. As shown in Fig. 4(a), \( \delta_n \) fluctuates around zero in this range.

![FIG. 3: The cardioid billiard, i.e., the limit case of the Robnik billiard; (a) full and (b) desymmetrized version.](image)
We investigate two spectral statistics, that is, the nearest-neighbor level spacing distribution and the spectral rigidity for 1500 accurate eigenvalues. The nearest-neighbor level spacing distribution \( P(s) \, ds \) is the probability of finding a consecutive pair of eigenstates for which the difference between their eigenvalues lies in the interval \([s, s + ds]\). It measures the short range correlation of the eigenvalue spectra. Instead of \( P(s) \), we consider the cumulative level spacing distribution

\[
I(s) = \int_0^s P(s') \, ds'
\]

(23)

to keep out of the binning problem about \( P(s) \). The spectral rigidity \( \Delta_3(L) \) is the mean square deviation of the spectral staircase function from the best fitting straight line over a length \( L \), namely

\[
\Delta_3(L) = \left\langle \min_{a,b} \frac{1}{L} \int_{-L/2}^{L/2} dc \left\{ N(E + c) - a - b c \right\}^2 \right\rangle_E .
\]

(24)

It was firstly introduced by Dyson and Mehta to describe statistics of the energy levels of many particle systems such as nuclei. It measures the long range correlation of the eigenvalue spectrum. Through studies about two mentioned spectral statistics, we rescale the eigenvalue spectra \( \{E_i\} \) into \( \{E'_i\} \) where \( E'_i = \bar{N}(E_i) \) and we omit the prime. After the rescaling, the eigenvalue spectra have a mean level spacing of unity and each billiard’s own characteristic is contained on the fluctuating part \( N_{fluc}(E) \).

It has been widely accepted that the spectral statistics of classically fully chaotic systems can be well described by the universal laws of random matrix theory (RMT). From the RMT prediction, the spectral statistics are given by the distribution of the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE) for systems with and without time reversal symmetry, respectively. Note that the time reversal invariant systems possessing specific geometric properties can show the GUE-like statistics. Concerned discussions will be addressed in next section. From Berry’s semiclassical analysis for spectral rigidity, it has been also known that the universality region where the spectral statistics follow the universal RMT prediction is finite. On paraffrasing, for fully chaotic systems, the spectral rigidity \( \Delta_3(L) \) shows a universal logarithmic increase following the prediction of RMT in the interval \( 1 \lesssim L < L_{max} \). For the case of GOE, the coefficient of logarithm is twice that of GUE. Then, in the range \( L > L_{max} \), \( \Delta_3(L) \) reaches a non-universal saturation value determined by short periodic orbits of corresponding classical billiard. \( L_{max} \) is called the outer energy scale and depends on the period of shortest periodic orbit and the mean level density.

Since the cardioid billiard has the property of time reversal invariance, it is expected that the cumulative level spacing distribution \( I(s) \) follows that of GOE and the spectral rigidity \( \Delta_3(L) \) is well described by the GOE prediction within the universality regime. In Figs. 4(b) and 4(c), we show the results of \( I(s) \) and \( \Delta_3(L) \) for the odd type of desymmetrized cardioid billiard and compare with those of the GOE and GUE (for numerical calculation of \( I(s) \) and \( \Delta_3(L) \) for the RMT predictions, see the Ref. [37]). As expected, the results show that the spectral statistics are in good agreement with the GOE predictions and the spectral rigidity saturates beyond the universality regime, which is restricted to small correlation length \( L \).

Note that we also calculate eigenvalue spectra for the even type of desymmetrized cardioid billiard by applying the Neumann boundary condition at the symmetry line. It can be easily achieved by the \( C_1 \) continuity property of the applied shape function. We obtain the same 1500 accurate eigenvalues as the odd symmetric case. The results of spectral statistics for the even symmetric case are
well described by the GOE expectation as qualitatively equivalent to the odd symmetric case of Figs. 4(b) and 4(c). We would not present these results in figure.

C. The spiral-shaped billiard

In this section we consider the spiral-shaped billiard whose boundary $\partial \Omega$ is given by

$$r(\phi) = R(1 + \epsilon \frac{\phi}{2\pi})$$

in polar coordinates $(r, \phi)$. $R$ is the radius of spiral at $\phi = 0$ and $\epsilon$ is the deformation parameter determining relative size of the notch. The spiral-shaped billiard is fully chaotic, that is, there is no stable island at all due to its peculiar asymmetric property. We fix $R = 1$ and consider two cases of deformation parameter, namely, weakly deformed case at $\epsilon = 0.1$ drawn in Fig. 5(a) and strongly deformed case at $\epsilon = 0.3$ depicted in Fig. 5(b). We take the spiral-shaped billiard as a model of non-convex geometric chaotic system and test the numerical procedure presented in Sec. II.

The spiral-shaped microcavity laser has been firstly introduced by Chern et al. for obtaining unidirectional emission [42]. Afterwards, Lee et al. have found remarkable resonance patterns of the spiral-shaped dielectric microcavity exhibiting strong localizations on a simple geometric shape [43]. It looks like a clear counter-example of the conventional scar-theory in which the localized intensity patterns are appeared only on the corresponding classical unstable periodic orbits [7], since the spiral has no simple-shaped periodic orbit, that is, all periodic orbits must bounce the notch more than once [44]. Recently, Lee et al. have shown that above strongly localized resonance patterns can be approximated by linear combinations of nearly degenerated resonance modes of the circular cavity without any support from the classical periodic orbits [45]. Such recent research interests on the spiral-shaped microcavity also motivate the studies on spectral statistics of the spiral-shaped billiard.

First we consider the weakly deformed spiral-shaped billiard at $\epsilon = 0.1$. We obtain 1500 accurate eigenvalue spectra out of 21955 results calculated from the $C_1$ continuity quartic basis FEM. The $\delta_n$ fluctuates around zero in this range as shown in Fig. 6(a). In Figs. 6(b) and 6(c), we present the results of $I(s)$ and $\Delta_3(L)$. We expect that the spectral statistics are described by the GOE prediction due to the time reversal invariance as the case of cardioid billiard. However the results follow the GUE prediction rather than the GOE in Figs. 6(b) and 6(c).

For the strongly deformed spiral-shaped billiard at $\epsilon = 0.3$, we attain the same 1500 accurate eigenvalues among 21934 calculated data (see the results of $\delta_n$ in Fig. 7(a)). In contrast to the weakly deformed case, the results of $I(s)$ and $\Delta_3(L)$ are well described by the GOE expectation as one can show in Figs. 7(b) and 7(c). In result, different degrees of deformation causes quite different spectral statistics and the unexpected GUE-like statistics are observed.

In the literatures, there have been several reports that study the spectral statistics exhibiting unexpected GUE-like behavior in time reversal invariant systems, for examples, the system with certain point symmetry [29, 46] and the so-called Monza billiard possessing the property of unidirectional motion [20]. For these systems it has been known that the GUE-like spectral statistics have their origin in the degenerated eigenstates. However the GUE-like statistics of the weakly deformed spiral-shaped billiard at $\epsilon = 0.1$ cannot be explained by this reason, since the system has no degenerated eigenstates. Note that reasonable accounts for the unexpected spectral statistics of the weakly deformed spiral-shaped billiard are not feasible at present. But we would anticipate that all of above GUE-like spectral statistics can be understood in an unified principle.

IV. CONCLUSION

We present an efficient finite element method for calculating eigenvalues and eigenfunctions of quantum billiard systems. The $C_1$ continuity quartic interpolation basis is considered. We show that the method provides accurate set of eigenvalues exceeding a thousand levels for any shape of quantum billiards on a personal computer. Comparison with the well-known $C_0$ continuity quadratic basis FEM proves the efficiency of the applied method. The spectral statistics of the Robnik and the spiral-shaped billiards are studied and the unexpected GUE-like behaviors are observed.

Note that we do not make use of the sparsity of matrices. We would expect that the generalized eigenvalue solving routine, which is optimized to sparse matrix, enhances the efficiency of presented FEM.

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FIG. 5: The spiral-shaped billiards are shown on a same scale; (a) weakly deformed case for $\epsilon = 0.1$ and (b) strongly deformed case for $\epsilon = 0.3$. 
FIG. 6: (Color online) For the weakly deformed spiral-shaped billiard at $\epsilon = 0.1$: (a) The $\delta_n$ up to 1500 level. (b) The cumulative level spacing distributions of the GOE, GUE, and the considered billiard are drawn by a black full, black dotted, and orange full line, respectively. (c) The spectral rigidities of the GOE, GUE, and the considered billiard are drawn by a black full, black dotted, and orange full line, respectively.

FIG. 7: (Color online) For the strongly deformed spiral-shaped billiard at $\epsilon = 0.3$: (a), (b), and (c) contain the same quantities of Fig. 6.

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[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, Berlin, 1990).
[2] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, 1999).
[3] S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. 42, 1189 (1979).
[4] H.-J. Stöckmann and J. Stein, Phys. Rev. Lett. 64, 2215 (1990).
[5] S. Sridhar, Phys. Rev. Lett. 67, 785 (1991).
[6] A. Bäcker, “The mathematical aspects of quantum maps” Vol. 618 in *Lecture Notes in Physics* (Springer, Berlin, 2003), pp. 91-144.
[7] E. J Heller, Phys. Rev. Lett. 53, 1515 (1984).
[8] B. Li and M. Robnik, J. Phys. A 27, 5509 (1994).
[9] E. Vergini and M. Saraceno, Phys. Rev. E 52, 2204 (1995).
[10] A. H. Barnett and T. Betcke, Chaos 17, 043125 (2007).
[11] M. Robnik, J. Phys. A 16, 3971 (1983).
[12] M. Robnik, J. Phys. A 17, 1049 (1984).
[13] M. V. Berry and M. Robnik, J. Phys. A 19, 649 (1986).
[14] T. Prosen and M. Robnik, J. Phys. A 26, 2371 (1993); 27, 8059 (1994).
[15] S. Tasaki, T. Harayama, and A. Shudo, Phys. Rev. E 56, R13 (1997).
[16] C. Gordon, D. Webb, and S. Wolpert, Bull. Am. Math. Soc. 27, 134 (1992).
[17] B. Li, M. Robnik, and B. Hu, Phys. Rev. E 57, 4095 (1998).
[18] B. Gutkin, J. Phys. A 36, 8603 (2003).
[19] D. Cohen, N. Lepore, and E. J Heller, J. Phys. A 37, 2139 (2004).
[20] G. Vele, T. Prosen, and M. Robnik, New J. Phys. 9, 15 (2007).
[21] H. E Türèci and H. G L Schwefel, J. Phys. A 40, 13869 (2007).
[22] J. N. Reddy, *An Introduction to the Finite Element
Method, 2nd edition (McGraw-Hill, New York, 1993).
[23] L. R. Ram-Mohan, Finite Element and Boundary Element Applications in Quantum Mechanics (Oxford University Press, Oxford, 2002).
[24] O. C. Zienkiewicz and R. L. Taylor, The Finite Element Method: For Solid and Structural Mechanics, 6th edition (Elsevier, Oxford, 2005).
[25] D. D. de Menezes, M. Jar e Silva, and F. M. de Aguiar, Chaos 17, 023116 (2007).
[26] F. M. de Aguiar, Phys. Rev. E 77, 036201 (2008).
[27] G. Báez, F. Leyvraz, R. A. Méndez-Sánchez, and T. H. Seligman, arXiv:nlin/0005057 (2000).
[28] V. Heuveline, J. Comput. Phys. 184, 321 (2003).
[29] B. Dietz, A. Heine, V. Heuveline, and A. Richter, Phys. Rev. E 71, 026703 (2005).
[30] B. Specht, Int. J. Numer. Methods Eng. 26, 705 (1988).
[31] P.-O. Persson and G. Strang, SIAM Review 46, 329 (2004); [http://www-math.mit.edu/~persson/mesh](http://www-math.mit.edu/~persson/mesh).
[32] J. R. Shewchuk, “Applied Computational Geometry: Towards Geometric Engineering” Vol. 1148 in Lecture Notes in Computer Science (Springer-Verlag, Berlin, 1996), pp. 203-222; [http://www.cs.cmu.edu/~quake/triangle.html](http://www.cs.cmu.edu/~quake/triangle.html).
[33] D. A. Dunavant, Int. J. Numer. Methods Eng. 21, 1129 (1985).
[34] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen, LAPACK Users’ Guide, 3rd edition (Society for Industrial and Applied Mathematics, Philadelphia, 1999).
[35] H. P. Baltes and E. R. Hill, Spectra of Finite Systems (Bibliographisches Institut, Mannheim, 1976).
[36] R. Markarian, Nonlinearity 6, 819 (1993).
[37] A. Bäcker, F. Steiner, and P. Stifter, Phys. Rev. E 52, 2463 (1995).
[38] F. J. Dyson and M. L. Mehta, J. Math. Phys. 4, 701 (1963).
[39] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Ref. Lett. 52, 1 (1984).
[40] M. L. Mehta, Random Matrices, revised and enlarged 2nd edition (Academic Press, San Diego, 1991).
[41] M. V. Berry, Proc. R. Soc. Lond. A 400, 229 (1985).
[42] G. D. Chern, H. E. Türeci, A. D. Stone, R. K. Chang, M. Kneissl, and N. M. Johnson, Appl. Phys. Lett. 83, 1710 (2003).
[43] S.-Y. Lee, S. Rim, J.-W. Ryu, T.-Y. Kwon, M. Choi, and C.-M. Kim, Phys. Rev. Lett. 93, 164102 (2004).
[44] S.-Y. Lee, S. Rim, J.-W. Ryu, T.-Y. Kwon, M. Choi, and C.-M. Kim, J. Phys. A 41, 275102 (2008).
[45] J. Lee, S. Rim, J. Cho, and C.-M. Kim, Phys. Rev. Lett. 101, 064101 (2008).
[46] F. Leyvraz, C. Schmit, and T. H. Seligman, J. Phys. A 29, L575 (1996).