SUPER SELF-DUALITY AS ANALYTICITY IN HARMONIC SUPERSPACE

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Abstract

A twistor correspondence for the self-duality equations for supersymmetric Yang-Mills theories is developed. Their solutions are shown to be encoded in analytic harmonic superfields satisfying appropriate generalised Cauchy-Riemann conditions. An action principle yielding these conditions is presented.

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1. There has recently been a revival of interest in self-duality equations, arising from numerous confirmations of a remarkable suggestion [1] that all integrable systems are obtainable by dimensional reduction from 4D self-dual theories. The purpose of this letter is to show that for supersymmetric Yang-Mills theories [2] the self-duality equations can be written in a form in which their integrability becomes manifest and their solutions can be constructed in terms of superfields which are in a definite sense holomorphic. In other words, we shall establish the so-called “twistor correspondence” for the supersymmetric self-duality equations analogous to that for the ordinary \((N = 0)\) self-duality equations [3], which in the harmonic space language [4-6] involves a splitting of the coordinate \(x^{\alpha \dot{\alpha}}\) into \(x^{+\alpha} = u^{\dot{\beta}} x^{\alpha \dot{\beta}}\) and \(x^{-\alpha} = u_{\beta} x^{\alpha \beta}\), where \(u^{\pm}_{\dot{\beta}}\) are harmonics on the two-sphere, which appears in the harmonisation of the rotation group [4-6], and \(\alpha\) and \(\dot{\beta}\) are two-spinor indices. The gist of ordinary self-duality is the Cauchy-Riemann-like equation

\[
\nabla^{+}_{\alpha} \phi = 0,
\]

where \(\nabla^{+}_{\alpha}\) is the covariant derivative in \(x^{-\alpha}\). We shall show that this construction can be extended naturally to supersymmetric gauge theories. In the \(N = 1\) case we shall use the harmonic superspace with coordinates

\[
x^{\pm \alpha} \equiv u^{\dot{\beta}} x^{\alpha \dot{\beta}}, \quad \bar{\vartheta}^{\pm} \equiv u^{\dot{\alpha}} \bar{\vartheta}^{\alpha}, \quad u^{\pm}_{\dot{\alpha}}.
\]

Now super-self-duality is the condition for the integrability of the equation

\[
\mathcal{D}^{+} \phi = 0,
\]

where \(\mathcal{D}^{+}\) is the gauge-covariant spinorial derivative with respect to the variable \(\bar{\vartheta}^{-}\). \(N = 1\) supersymmetric gauge theories respect chirality, so we may, without loss of generality, take the field \(\phi\) in (3) to be a chiral superfield \(\phi(x^{+\alpha}, x^{-\alpha}, \bar{\vartheta}^{+}, \bar{\vartheta}^{-})\) independent of \(\vartheta^{\alpha}\), i.e.

\[
D_{\alpha} \phi = 0,
\]

where \(D_{\alpha}\) is the covariant spinorial derivative with respect to the variables \(\vartheta^{\alpha}\), which may be made flat by definition; and we shall set, throughout this letter, the corresponding connections to zero,

\[
A_{\alpha} = 0.
\]

Of importance is the fact that consistency of (3) and (4) implies (1) in virtue of the algebra of spinorial derivatives, so \(N = 1\) self-duality implies the usual \(N = 0\) self-duality. We shall demonstrate that all solutions of the supersymmetric self-duality conditions are encoded in a “holomorphic” chiral superfield which satisfies generalised Cauchy-Riemann equations. By holomorphicity we mean that there is a basis in which this superfield is independent of \(x^{-\alpha}, \bar{\vartheta}^{-}\) and \(\vartheta^{\alpha}\). The resulting formulation greatly simplifies the problem of constructing gauge superpotentials \(A_{\alpha \dot{\beta}}, A_{\dot{\beta}}\) solving the
super-self-duality equations. It also helps in the search for an action principle for super-self-duality.

Although our considerations are rigorous only for 4D Euclidean space, we shall remain in the complexified picture with Lorentz group $SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_R$, with $\alpha$ and $\dot{\alpha}$ labelling fundamental representations of $SL(2, \mathbb{C})_L$ and $SL(2, \mathbb{C})_R$, respectively. In [6] it was shown that consideration of the complexified picture is required by conformal invariance of the self-duality equations. Reality conditions appropriate for the required signature of four dimensional real space may be imposed; e.g. by identifying undotted and dotted spinors as representations of the two different $SU(2)'s$ in the 4D Lorentz group $SU(2)_L \times SU(2)_R$ corresponding to a Euclidean signature; or by identifying them as representations of two different $SL(2, \mathbb{R})'$s, with Lorentz group $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ corresponding to a $(2,2)$ signature. In the latter case we expect intriguing peculiarities due to the non-compactness of $SL(2, \mathbb{R})$ and an appropriate harmonic space needs to be considered (see e.g. [7]).

2. In complexified superspace $\mathcal{M}_{4|4}$ of complex dimension $(4|4)$ with coordinates $(x^\alpha, \vartheta^\alpha, \bar{\vartheta}^{\dot{\alpha}})$, the $N = 1$ super Yang-Mills theory is conventionally described in terms of two spinorial field strengths $w_\alpha$, $\bar{w}_{\dot{\alpha}}$ defined by

$$[\bar{D}_{\dot{\beta}}, D_\alpha] = \epsilon_{\dot{\beta}\dot{\alpha}} w_\alpha$$

$$[D_\beta, \bar{D}_{\dot{\alpha}}] = \epsilon_{\beta\dot{\alpha}} \bar{w}_{\dot{\alpha}},$$

(6)

where the gauge-covariant derivatives $D_A \equiv \partial_A + A_A = (D_\alpha, \bar{D}_{\dot{\alpha}}, D_\beta, \bar{D}_{\dot{\beta}})$ satisfy the familiar constraints

$$\{D_\alpha, D_\beta\} = 0 \ (7a)$$

$$\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \ (7b)$$

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2 D_\alpha \bar{D}_{\dot{\beta}} \ (7c)$$

and the supertranslations

$$\partial_A = (\partial_{\alpha\dot{\beta}}, D_\alpha, \bar{D}_{\dot{\beta}}) \equiv (\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \vartheta^\alpha}, \frac{\partial}{\partial \bar{\vartheta}^{\dot{\beta}}} + 2\vartheta^\alpha \frac{\partial}{\partial x^\alpha},)$$

realise the free superalgebra

$$[D_\beta, \partial_{\alpha\dot{\beta}}] = [D_\dot{\alpha}, \partial_{\alpha\dot{\beta}}] = [D_\alpha, D_\beta] = \{D_\alpha, D_\beta\} = \{D_\dot{\alpha}, D_{\dot{\beta}}\} = 0$$

$$[D_\alpha, D_{\dot{\beta}}] = 2 \partial_{\alpha\dot{\beta}}.$$ 

The self-duality equations for the superconnection $A_A$ take the form of the following further (in comparison with (6)) constraints

$$[D_\beta, D_\alpha] = 0,$$

(8)

which say that $\bar{w}_{\dot{\alpha}}$ vanishes. That this is the supersymmetrisation of the usual ($N = 0$) self-duality condition is evident from the dimension $(1|1)$ Jacobi identity which yields the following superfield equations

$$f_{\dot{\alpha}\dot{\beta}} = 0 \ (a), \quad D_\alpha w_\beta = 2 f_{\alpha\beta} \ (b)$$

(9)
for the self- and anti-dual vector field strengths \( f_{\dot{\alpha}\dot{\beta}}, f_{\alpha\beta} \) appearing in the definition

\[
[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}.
\]

The dimension (\( \frac{3}{2}|1 \) and (\( \frac{1}{2}|\frac{3}{2} \)) Jacobi identities are then satisfied, respectively, if \( w_{\alpha} \) and \( f_{\alpha\beta} \) satisfy the (anti-) chirality equations

\[
D_\gamma f_{\alpha\beta} = 0 \ (c), \quad \bar{D}_\dot{\beta} w_{\alpha} = 0 \ (d).
\]

All other Jacobi identities are then automatic, requiring the introduction of no further superfield strengths. Equations (8) are the superfield self-duality equations. They indeed imply the equations of motion \( \epsilon^{\gamma\alpha} D_\gamma \epsilon_{\alpha\beta} f_{\alpha\beta} = 0 \) (c), \( \epsilon^{\dot{\gamma}\dot{\alpha}} \bar{D}_{\dot{\gamma}} w_{\alpha} = 0 \) (d).

We have therefore shown that the constraints (7), (8) for \( A_A \) imply the superfield equations (9) for the superfield-strength \( w_{\alpha} \) and superfield vector-potential \( A_{\alpha\dot{\beta}} \). These in turn imply the ordinary space supersymmetric self-duality equations for the component fields (which we denote by the same symbols as the superfields of which they are the leading components)

\[
f_{\dot{\alpha}\dot{\beta}} = 0, \quad \epsilon^{\gamma\alpha} D_\gamma w_{\alpha} = \bar{w}_{\dot{\alpha}} = 0,
\]
on eliminating all gauge degrees of freedom depending on the anticommuting superspace coordinates. The converse, that given a set of component fields satisfying these component equations, one can reconstruct superfields satisfying (9), and in turn the superconnection \( A_A \) satisfying (7), (8) is also true. The proof closely follows the methods of [8].

3. Our main purpose here, however, is to introduce yet another piece of data corresponding to the above three: a ‘holomorphic’ prepotential in harmonic superspace. We shall show that the constraints (7), (8) imply generalised Cauchy-Riemann (CR) conditions for a prepotential in harmonic superspace and that any superconnection satisfying (7), (8) may be expressed in terms of such holomorphic prepotentials. (We shall use ‘holomorphic’ in this generalised sense; to describe solutions of these generalised CR conditions). This construction is a realisation of the twistor construction [3] for supersymmetric self-dual systems in the harmonic superspace framework.

In the present complex setting, the harmonics \( u_\pm^\dot{\alpha} \) remain, as usual, \( S^2 \) harmonics, the 2-sphere being a coset of the \( SL(2, \mathbb{C})_R \) part of the Lorentz group with its maximal parabolic subgroup. In this setting, \( u^+ \) and \( u^- \) are not complex conjugates of each other, and are defined up to parabolic subgroup transformations. They obey the usual constraints \( u^+\dot{\alpha} u^-_{\dot{\alpha}} = 1 \). Details of this construction, the role of conformal invariance, as well as appropriate reality conditions may be found in [6]. For our \( N = 1 \) harmonic superspace the derivatives \( \partial_\pm^\dot{\alpha} \equiv u^\pm\dot{\alpha} \partial_{\alpha\dot{\alpha}}, \quad \bar{D}^+ = u^\pm\dot{\alpha} \bar{D}_{\dot{\alpha}}, \quad D_\alpha, \) together with harmonic ones

\[
D^{\pm} = u^{\pm\dot{\alpha}} \frac{\partial}{\partial u^{\pm\alpha}}, \quad D^0 = u^{+\dot{\alpha}} \frac{\partial}{\partial u^{+\alpha}} - u^{-\dot{\alpha}} \frac{\partial}{\partial u^{-\alpha}},
\]
realise the free superalgebra

\[
\{ D_\alpha, \bar{D}^\pm \} = 2 \partial_\pm^\alpha, \quad \{ D_\alpha, \bar{D}^\pm \} = 2 \partial_\pm^\alpha,
\]

\[
[D^{++}, D^{--}] = D^0, \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}
\]
\[ [D^{\pm \pm}, D^\mp] = D^\pm, \quad [D^{\pm \pm}, \partial_\alpha^\pm] = \partial_\alpha^\pm, \quad (10c) \]

with all other commutators vanishing. Now, in terms of the gauge-covariant derivatives

\[ \nabla^\pm_\alpha = \partial^\pm_\alpha + A^+_\alpha = u^{+\bar{\alpha}} D_{\alpha \bar{\alpha}}, \quad \bar{D}^+ = \bar{D}^+ + \bar{A}^+ = u^{+\bar{\alpha}} \bar{D}_{\alpha}, \quad D_\alpha = D_\alpha, \]

(recall that \( A_\alpha = 0 \)) obeying the commutation relations

\[ [D^{++}, D^+] = 0, \quad (11a) \]
\[ [D^{\pm \pm}, D_\alpha] = 0, \quad (11b) \]
\[ [D^{++}, \nabla^+_\alpha] = 0, \quad (11c) \]

the constraints (7), (8) are equivalent to the Cauchy-Riemann system

\[ [\bar{D}^+, \nabla^+_\alpha] = 0, \quad (12a) \]
\[ [D_\beta, \nabla^+_\alpha] = 0, \quad (12b) \]
\[ [\bar{D}^+, D_\alpha] = 2 \nabla^+_\alpha. \quad (12c) \]

Remarkably, these are precisely the integrability conditions for equations (1), (3) and (4). We therefore have the following pure-gauge-like expressions for \( A^+_\alpha \) and \( \bar{A}^+ \)

\[ A^+_\alpha = -\partial^+_\alpha \phi \phi^{-1}, \quad \bar{A}^+ = -\bar{D}^+ \phi \phi^{-1}. \quad (13) \]

4. Equations (11), (12) are therefore equivalent to the constraints (7), (8). Let us now choose a coordinate basis, which we shall call the analytic frame, in which the derivatives take the forms

\[ \widehat{D}_\alpha = \phi^{-1} [D_\alpha] \phi = \frac{\partial}{\partial \vartheta^\alpha}; \]
\[ \widehat{\nabla}^+_\alpha = \phi^{-1} [\nabla^+_\alpha] \phi = \frac{\partial}{\partial x^\alpha}; \]
\[ \widehat{D}^+ = \phi^{-1} [D^+] \phi = \frac{\partial}{\partial \vartheta^\alpha} + 2 \partial^\alpha \frac{\partial}{\partial x^\alpha}, \quad (14) \]
\[ \widehat{D}^{++} = \phi^{-1} [D^{++}] \phi = D^{++} + V^{++} \]
\[ \widehat{D}^{--} = \phi^{-1} [D^{--}] \phi = D^{--} + V^{--}. \]

In this basis the covariant derivatives \( \widehat{D}^+ \) and the \( \widehat{\nabla}^+_\alpha \) become flat, losing their connections (\( \widehat{D}_\alpha \) remains flat), while the harmonic derivatives \( D^{\pm \pm} \) clearly acquire the connections

\[ V^{++} = \phi^{-1} D^{++} \phi, \quad (15a) \]
\[ V^{--} = \phi^{-1} D^{--} \phi. \quad (15b) \]
In order to preserve the operator $D^0$ as a charge counting operator, we have used (as usual, see e.g. [3]) the conventional gauge in which it does not acquire a connection:

$$[\hat{D}^{++}, \hat{D}^{--}] = D^0.$$  \hfill (16)

In this basis the dynamical content is contained entirely in $(11)$, the rest of the equations being kinematical. Using the identity

$$A(Bff^{-1}) \equiv f(B(f^{-1}Af))f^{-1} + [A, B]ff^{-1},$$

for arbitrary differential operators $A$ and $B$, eqs. $(11)$ take the form of generalised CR conditions

$$\frac{\partial}{\partial \vartheta^-} V^{++} = 0,$$
$$\frac{\partial}{\partial \vartheta^\alpha} V^{\mp\mp} = 0,$$
$$\frac{\partial}{\partial x^{-\alpha}} V^{++} = 0.$$  \hfill (17)

$V^{++}$ is therefore holomorphic: it depends on $x^{+\alpha}, \vartheta^+,$ and $u^\pm$, being independent of $x^{-\alpha}, \vartheta^-$ and $\vartheta^\alpha$; both $V^{++}$ and $V^{--}$ are chiral. Note that the third equation is a consequence of the other two. We have shown that to any solution of the super self-duality constraints $(8)$, there corresponds a chiral holomorphic superfield $V^{++}$ taking values in the gauge algebra and having component expansion

$$V^{++}(x^{+\alpha}, \vartheta^+, u^\pm_\alpha) = v^{++}(x^{+\alpha}, u^\pm_\alpha) + \bar{\vartheta}^+ \chi^+(x^{+\alpha}, u^\pm_\alpha).$$  \hfill (18)

The superfield $V^{++}$ is defined modulo gauge transformations

$$\delta V^{++} = [\hat{D}^{++}, \lambda],$$

where $\lambda$ is an arbitrary holomorphic superfield. Note that due to the presence of the fermion mode there is an important difference with the $N = 0$ case: whereas the $N = 0$ connection was encoded in one function on the 2-sphere, in the present case we have two functions, $v^{++}$ and $\chi^+$, instead.

5. The converse statement, that any chiral analytic superfield prepotential $V^{++}$ encodes a superconnection $A_A$ satisfying $(7), (8)$ also holds. To reconstruct the superconnection $A_A$ there are two options.

a) We can start with the chiral superfield $\phi$. In this case we need to solve $(15a)$ for $\phi$; $V^{++}$ being given. Equations on the 2-sphere of this kind are not too easy to solve and they appear in many applications of the harmonic-twistor approach, see, e.g. [4]. In order to determine the corresponding superconnection solving $(7), (8)$, we need to insert the $\phi$ thus obtained into the following formulae

$$A_{\alpha\dot{\alpha}} = 2 \int d^2 u \ u^-_{\dot{\alpha}} \phi \partial^+_{\dot{\alpha}} \phi^{-1}, \quad A_{\dot{\alpha}} = 2 \int d^2 u \ u^-_{\alpha} \phi \bar{D}^+ \phi^{-1}.$$  \hfill (19)
which follow immediately from (13); and $A_\alpha$ is of course zero.

b) Instead of $\phi$, we can start with the harmonic connection $V^{--}$. It follows from (16) that

$$Z \equiv D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0.$$  \hspace{1cm} (20)

For a given holomorphic $V^{++}$ taking values in the gauge algebra, it contains a set of coupled first-order linear equations for the gauge algebra components of $V^{--}$. These may be solved [9] somewhat more easily than (15a). Now, as further consequences of gauge-covariantising the harmonic derivatives (14), we have, from (10),

$$[\hat{D}^{--}, \bar{D}^+] = \bar{D}^-, \quad [\hat{D}^{--}, \bar{D}^-] = 0, \quad [\hat{D}^{--}, \partial^+_{\alpha}] = \nabla^-_{\alpha}, \quad [\hat{D}^{--}, \nabla^-_{\alpha}] = 0$$ \hspace{1cm} (21)

from which we obtain superconnections

$$A^- = -\partial^+ V^{--}, \quad A^+_{\beta} = -\partial^+_{\beta} V^{--},$$ \hspace{1cm} (22)

in terms of solutions of (20). Now, superconnections satisfying (7),(8) may be recovered by harmonic integration similar to (19). In fact, from $V^{--}$ we may also directly construct the superfield strength $w_{\alpha}$ satisfying the superfield equations (9); namely, from (6),

$$w_{\alpha} = -[\bar{D}^+, \nabla^-_{\alpha}] = \bar{D}^+ \partial^+_{\alpha} V^{--}.$$ \hspace{1cm} (23)

An alternative to equation (20) follows from the following commutators contained in the superalgebra (10) (in the analytic frame)

$$[\hat{D}^-, \nabla^-_{\alpha}] = 0 = [\nabla^-_{\alpha}, \nabla^-_{\beta}].$$

These yield the alternative equations for $V^{--}$:

$$L^{-\alpha} \equiv -\partial^+_{\alpha} \bar{D}^- V^{--} + \partial^-_{\alpha} \bar{D}^+ V^{--} + [\bar{D}^+ V^{--}, \partial^+_{\alpha} V^{--}] = 0,$$ \hspace{1cm} (24)

$$L^{--} \equiv \partial^{+\alpha} \partial^-_{\alpha} V^{--} + [\partial^{+\alpha} V^{--}, \partial^+_{\alpha} V^{--}] = 0.$$ \hspace{1cm} (25)

The latter equation is in fact the one introduced for the $N = 0$ case in [10]. We note the following interesting interrelations amongst the left-hand-sides of equations (20), (24) and (25):

$$\bar{D}^+ L^{--} = \partial^{+\alpha} L^{-\alpha}, \quad \nabla^-_{\alpha} \partial^+_{\alpha} Z = \hat{D}^{++} L^{--}, \quad \nabla^-_{\alpha} \bar{D}^+ Z = \hat{D}^{++} L^{-\alpha}.$$  

6. We now present an action for super self-duality. Since all we need to have is the generalised CR conditions (17), with $V^{++}$ expressed in terms of the field $\phi$, as a variational equation, we plug this condition into an action functional with the help of a Lagrange multiplier-type auxiliary field. The latter does not propagate if it contains only gauge degrees of freedom. For the chiral superfield $\phi$, the action functional

$$S = \int d^4x \, d^2\bar{\theta} \, du \, tr \, (\bar{D}^+ \zeta^{-3} \phi^{-1} D^{++} \phi)$$ \hspace{1cm} (26)
yields, on varying the auxiliary field $\zeta^{-3}$, the CR condition $\bar{D}^+V^{++} = 0$, $V^{++}$ is chiral by definition; and the final condition $\frac{\partial}{\partial x^-}V^{++} = 0$ is a consequence. Now, on varying $\phi$, we obtain

$$-\phi^{-1}D^{++}[\phi\bar{D}^+\zeta^{-3}\phi^{-1}] = 0. \quad (27a)$$

It follows (cf. [11]) that

$$\bar{D}^{+}\zeta^{-3} = 0. \quad (27b)$$

All solutions of this equation have the form

$$\zeta^{-3} = \bar{D}^+y^{-4}. \quad (28)$$

However $\zeta^{-3}$ enters the action via $\bar{D}^{+}\zeta^{-3}$, so it is only defined modulo the addition of $\bar{D}^+y^{-4}$. $\zeta^{-3}$ therefore does not represent any additional physical degree of freedom. For $N = 0$ an analogous action was discussed in [11,12].

Alternatively, we may choose $V^{--}$ (instead of $\phi$) as the dynamical field, express $V^{++}$ in terms $V^{--}$ with the help of (20), and construct an action having analyticity conditions for the functional $V^{++}[V^{--}]$ as variational equations.

There also exists the possibility (analogous to the $N = 0$ action considered in [10]) of writing an action for eq.(25) trilinear in $V^{--}$. It explicitly contains a constant harmonic factor of charge +4, say $(u_1^+u_2^+)^2$, and is consequently not Lorentz invariant.

7. To conclude we generalise our construction to $N$-extended harmonic-superspace with coordinates $x^{\pm \alpha}$, $\theta^{\alpha i}$, $\bar{\theta}^{\pm j} \equiv u^{\pm \dot{\alpha}}_i \bar{\theta}^{\dot{\alpha}}_j$, $u^{\pm \dot{\alpha}}_i$, where $i,j = 1,...,N$. Solutions of the generalised CR conditions

$$\frac{\partial}{\partial \theta^j}(\phi^{-1}D^{++}\phi) = 0,$$

$$\frac{\partial}{\partial \theta^{\alpha i}}(\phi^{-1}D^{\pm \pm}\phi) = 0,$$

$$\frac{\partial}{\partial x^-}(\phi^{-1}D^{++}\phi) = 0 \quad (29)$$

encode $N$-extended self-dual superconnections. It may be verified that the superconnection components

$$A_{\alpha \dot{\alpha}} = \int d^2u \; u^-_{\dot{\alpha}} \phi \phi^+ \phi^{-1},$$

$$A^{ij}_{\dot{\alpha}} = \int d^2u \; u^-_{\dot{\alpha}} \phi \bar{D}^{ij}\phi^{-1},$$

$$A_{\alpha i} = 0,$$

with $\phi$ satisfying (29), automatically satisfy the self-dual restrictions of the conventional extended superconnection constraints

$$F_{\alpha \dot{\alpha} j \beta} = 0 = F^{ij}_{\dot{\alpha} \beta} + F^{ij}_{\dot{\beta} \dot{\alpha}},$$

$$F^i_{\alpha \beta j} = 0.$$
That integrability conditions for these equations yield (29) follows from reasoning parallel to that for the N=1 case above.

For the full (non-self-dual) N = 2 and 3 theories, for which harmonic superspace formulations (harmonising the internal automorphism group) exist [4,6], the self-duality conditions are also equivalent to “double” analyticity conditions arising from Lorentz as well as internal automorphism group harmonisation. We intend to return to these theories elsewhere.

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