Causality and Sound Speed for General Scalar Field Models, including \( w < -1 \),
tachyonic, phantom, k-essence and curvature corrections

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The result from the SN1a projects suggest that the dark energy can be represented by a fluid with \( w < -1 \). However, it is commonly argued that a fluid with \( |w| > 1 \) contradicts causality.

Here, we will show that a fluid with \( |w| > 1 \) does not contradict causality if \( w \) is not constant. Scalar field are the most promising candidates for describing the dark energy and they do not have a constant equation of state parameter \( w \). Scalar potentials may lead to regions where \( w \) is larger or smaller than one and even regions where the group velocity \( dp/d\rho \) diverges.

We study the evolution of scalar field perturbations and we show that the "sound speed" is always smaller than the speed of light independently of the value of \( w = p/\rho \) or \( dp/d\rho \). In general, it is neither the phase velocity nor the group velocity that gives the "sound speed". In the analysis we include the special cases of \( |w| > 1 \), tachyonic, phantom, k-essence scalar fields and curvature corrections. Our results show that scalar fields do not contradict causality as long as there is dispersion.

The existence of a dark energy, an energy density \( \rho \) with negative pressure \( p \) and equation of state \( w = p/\rho \), has been determined by the SN1a \( ^{2} \) and the CMBR observations \( ^{1} \). These observations show that we are living in a flat universe with a matter contribution today \( \Omega_{m} \simeq 0.3 \) and a dark energy \( \Omega_{de} \simeq 0.7 \) with \( |w| = 2/3 \). The cosmological results do not rule out an energy density with \( w < -1 \) \( ^{3} \).

Perhaps the most economic solution to the dark energy is that of a scalar field. Not only because they are widely predicted by particle physics but also because their dynamics may lead in a natural way to an accelerated universe. These scalar fields are called quintessence and they are homogeneous in space and have only gravitational interaction with all other fields. Other interesting possibilities have been consider such as phantom fields \( ^{2} \), k-essence \( ^{8} \) or tachyonic scalar fields and curvature corrections \( ^{10-12} \).

The possibility of having a fluid with \( w < -1 \) is problematic. There are objections against such a fluid from vacuum instability, causality and sound speed among other considerations. For a fluid with a constant equation of state parameter \( w \) it cannot exceed the value of \( |w| > 1 \). One of the many problems appears as the propagation of sound, since for \( w \) constant it gives the sound speed and clearly \( w \) must be smaller than the speed of light \( c = 1 \) \( ^{4} \). Another related problem is the vacuum instability for fluids with a constant \( w < -1 \) \( ^{5} \). Here, we are interested in studying the problem of causality and sound speed for fluids with \( |w| > 1 \) or \( |dp/d\rho| > 1 \). As we will see in the next section a scalar field with \( |w| > 1 \) and/or \( |dp/d\rho| > 1 \) can be easily obtained if the field has negative potential \( ^{20} \) (e.g. a tachyon, \( m^2 < 0 \)) or for a phantom field with negative kinetic energy and positive potential \( ^{7} \). Concerning the evolution of the perturbations of the scalar field we will show that in the absence of gravity a tachyonic field is equivalent to a phantom field.

We will show that the information speed for a scalar field does not contradict special relativity (see \( ^{17} \)) even in the cases where \( |w| > 1 \) and \( |dp/d\rho| > 1 \). However, in these cases the amplitude of the perturbations grows rapidly and therefore the first order approximation used to study the sound speed fails. This signals that the space (universe) is no longer homogenous and non-perturbative mathematical methods must be used to analyze the behavior of the universe. Still, we want to emphasis that scalar fields with \( |w| > 1 \) and \( |dp/d\rho| > 1 \), tachyons, phantom or k-essence models do not violate causality and the sound speed is smaller or equal to the speed of light.

1. MOTIVATION

It is usually stated that the "speed of sound" for a fluid is given either by \( w_{\phi} \equiv p/\rho \) if it is constant or by \( dp_{\phi}/d\rho_{\phi} \) if \( w_{\phi} \) is time dependent, where \( p_{\phi} \) is the pressure and \( \rho_{\phi} \) the energy density. In the case that \( w \equiv p/\rho \) is constant then the two quantities \( dp/d\rho = w \) coincide. The requirement that the propagation of information should not exceed the speed of light implies that \( |w_{\phi}| < 1 \) and/or \( |dp_{\phi}/d\rho_{\phi}| < 1 \). However, as we will show, neither of these two conditions are necessarily satisfied for a scalar field.

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Considering an homogenous scalar field, it is well known that its evolution leads naturally to a non-constant \( w_\phi \), so \( w_\phi \) should not be interpreted as the ”speed of sound”. The equation of state parameter for a scalar field is

\[
w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\lambda \frac{1}{2} \dot{\phi}^2 - V(\phi)}{\lambda \frac{1}{2} \dot{\phi}^2 + V(\phi)} \tag{1}\]

where the \( o \) denotes that we are considering an homogenous scalar with \( \phi_o(t) \) and \( \dot{\phi} \equiv d\phi/dt \) and we have included the parameter \( \lambda = \pm 1 \) to take into account a canonical scalar field \( \lambda = 1 \) and a phantom field \( \lambda = -1 \). It is easy to see from the above equation that for any scalar field with positive kinetic term \( E_k = \dot{\phi}^2 / 2 \geq 0 \) that the magnitude and sign \( w_\phi \) depends on the value of the potential \( V(\phi) \) and eq. \( 1 \) gives

\[
|w_\phi| \leq 1 \iff V(\phi_o) \geq 0 \tag{2}
\]

\[
w_\phi > 1 \iff E_k > -V > 0 \tag{3}
\]

\[
w_\phi < -1 \iff E_k + V < 0.
\]

As we see from eqs. \( 3 \) we can obtain \( w_\phi < -1 \) for a scalar field with a negative potential (e.g. for a tachyon field with \( V = m^2 \phi_o^2, \ m^2 < 0 \)). We can also have \( w_\phi < -1 \) for a phantom field with negative kinetic energy \( (E_k < 0) \) with positive potential \( (V(\phi_o) > 0) \). It was shown in \( 20 \) that this kind of behavior is reached naturally for scalar field with a negative minimum in the presence of a barotropic fluid that can be, for example, matter or radiation. Therefore, we do not need exotic fluids to have regions with \( |w_\phi| > 1 \).

Now, let us analyze the quantity \( dp_\phi/d\rho_\phi \) for a scalar field with arbitrary potential. The homogenous part of the scalar field depends only on time and using eq. \( 1 \) we have the adiabatic quantity

\[
\frac{dp_\phi}{d\rho_\phi} = \frac{\dot{p}_\phi}{\dot{\rho}_\phi} = \frac{1 - \frac{2V'}{\lambda \phi + V'}}{1 + \frac{2V'}{3 \lambda H \phi}} \tag{4}
\]

where we have used

\[
\dot{p}_\phi = \dot{\phi}(\lambda \ddot{\phi} - V') \quad \dot{\rho}_\phi = \dot{\phi}(\lambda \ddot{\phi} + V') \tag{5}
\]

and the equation of motion of the scalar field

\[
\lambda(\ddot{\phi} + 3H \dot{\phi}) + V' = 0 \tag{6}
\]

for the second equality of eq. \( 5 \) with \( V' \equiv dV/d\phi \). Notice that in general eq. \( 4 \) has not only regions where \( |dp_\phi/d\rho_\phi| \) is larger than one but regions where it diverges. This will happen at the turning points of the field oscillations around the minimum of the potential when \( \dot{\phi} = 0 \), if the point is reached at a non-extremum of the potential \( V \), i.e. \( \lambda \ddot{\phi} = -V' \neq 0 \), e.g. for any potential with a minimum at finite \( \phi \) as for a massive scalar field with \( V = m^2 \phi^2 \).

Clearly, we see that under normal conditions both (homogenous) quantities \( w_\phi, dp_\phi/d\rho_\phi \) can be larger than the one (the speed of light). Does this imply that the sound speed of scalar fields travel faster then the speed of light? The answer is no, as we will show later, and the reason is that neither \( w_\phi \) nor \( dp_\phi/d\rho_\phi \) gives the correct interpretation of the sound speed.

### II. SUMMARY OF DIFFERENT VELOCITIES

In order to clarify the notion of causality for fluid or a scalar field it will be useful to define the different velocities involved in determining the transportation of a signal in a medium.

For simplicity in presentation purposes we will consider only one space dimension. The wave equation for a fluid is given by

\[
\ddot{\rho} - \frac{v^2}{c^2} \nabla^2 \rho + d \dot{\rho} + b \rho = 0 \tag{7}
\]

where \( c \) is the speed of light and the parameters \( v, b, d \) may depend, in general, on the coordinates \( t, x \) and they define the type of medium the fluid is in. If we take a Fourier transformation

\[
\rho(t, x) = \int_{-\infty}^{\infty} dk \rho_k(t)e^{ikx} \tag{8}
\]
The parameter \(v\) defines the phase velocity while the friction term is given by \(d\). The simplest wave equation is given by \(d = b = 0\) and \(v\) constant. In this case all monochromatic wave functions will have the same speed and \(v\) can be interpreted as the sound speed. For \(d = 0\) and \((\nu^2 k^2/c^2 + b)/k^2\) is \(k\)-dependent, the medium is said to be dispersive and the monochromatic wave functions will travel with different phase velocities. In this case it is the group velocity that determines the speed of sound. If we allow \(d\) to be different than zero then we have a dissipative medium and the wave amplitude will suffer a damping or growth depending on the sign of \(d\) and neither the phase nor the group velocity would give the "sound speed".

For \(d = b = 0\) and \(v = cte\) the solution to eq.\((9)\) involves a sum of monochromatic waves

\[
\rho(t, x) = \int_{-\infty}^{\infty} \left( A_k \ e^{-i(kx - \omega t)} + B_k \ e^{i(kx - \omega t)} \right) dk
\]

with \(\omega, k\) constants and the amplitudes \(A_k, B_k\) also constant. From eq.\((9)\), with \(b = d = 0\), the frequency \(\omega\) (it should not be confused with \(w\)) the equation of state parameter defined in the previous section) is a function of \(k\)

\[
\omega^2 = \frac{\nu^2}{c^2} k^2.
\]

The wave number is given by \(k = 2\pi/\lambda\) and gives the inverse of the wave length. From now on we will set the speed of light back to one \((c = 1)\). The speed for a monochromatic wave, e.g. the speed of the maximum of the wave, is
given by the constrain \(k x - \omega t = cte\) from which we have the phase velocity

\[
v_p = \frac{dx}{dt} = \frac{\omega}{k} = v
\]

if \(k, \omega\) are independent of \(x, t\). A velocity smaller than \(c\) requires \(v_p = \omega/k < 1\). Notice that in this case \(v_p\) is the same for all monochromatic wave functions since \(w/k\) is \(k\)-independent.

If we are in a dispersive medium \(b \neq 0\) and the wave frequency depends on the value of \(k\). In this case one needs to consider the evolution of wave packets since their shape will be time dependent. A wave packet occupies a limited region in space. It is common to assume that the monochromatic frequencies that form the packet are concentrated around a central value \(\omega\) with corresponding wave number \(k\). For a wave function with a slightly different wave number \(k' = k + \Delta k\) the corresponding frequency is \(\omega(k + \Delta k)\) and for \(\Delta k\) small we can write \(\omega(k + \Delta k) \simeq \omega(k) + (d\omega/dk)\Delta k\).

The solution to the wave equation for a wave packet is of the form

\[
\rho(t, x) = A \ e^{i(kx - \omega t)} f((\Delta k x - t \Delta \omega))
\]

and the amplitude of the wave is modulated by the function \(f\). The group velocity is defined by requiring the argument of \(f\) to be constant, i.e. by the solution of \(\Delta k x - t \Delta \omega = cte\), giving the group velocity

\[
v_g = \frac{\Delta \omega}{\Delta k} = \frac{d\omega}{dk}
\]

if \(d\omega/dk\) is \(t, x\) independent. The group velocity \(v_g\) is valid for any dependence of \(\omega(k)\) but only as long as \(\Delta k\) is small. If the medium is non dispersive \(\omega\) is constant then the group velocity and the phase velocity coincide \(v_g = v_p = \omega/k\).

In general the form of the packet will not remain constant and it will be smoothed out during its propagation and the condition of having a small \(\Delta k\) will no be maintained.

Lastly let as consider a dissipative medium in the simple case of constant \(b, d, v\). In this case the solution to eq.\((9)\), with the ansatz \(\rho_k = Ae^{i\omega t} + Be^{-i\omega t}\) give the equation

\[
-\omega^2 + i\omega + \frac{\nu^2}{c^2} k^2 + b = 0
\]

with

\[
\omega = \frac{id \pm \sqrt{4(\nu^2/c^2) k^2 + b} - d^2}{2}
\]
and we see that \( \omega \) takes imaginary values if \( d \neq 0 \) or \( 4(k^2v^2/c^2 + b) - d^2 < 0 \). This will give damping or a growing wave function solution. If \( 4(k^2v^2/c^2 + b) - d^2 > 0 \) the damping term will be the same for all monochromatic modes.

The short explanation we have given above also shows conditions for the validity of the approximation: the packet must be quasi-monochromatic and the frequency a slowly varying real function of \( k \). If either of these approximations breaks down, the group velocity loses its physical meaning. In the general case the "speed of sound" is neither the group nor the phase velocity and one must use the more complex and subtle notion of signal velocity [25, 26].

The signal velocity is defined as the velocity with which a given standard amplitude of the wave packet moves, for instance half that of the maximum amplitude. The standard Sommerfeld-Brillouin definition amounts to this. The condition of retarded wave guarantees then that the signal velocity will be always smaller than \( c \), even if the wave packet deforms very much and the group velocity becomes \( v_g > c \). That this may happen is shown in the classical Sommerfeld-Brillouin diagram [22] for the three velocities in the neighborhood of an absorption line (Fig. 1). In the case of a scalar field the group velocity diverges when \( \dot{\phi} = 0 \) with \( V' \neq 0 \). This will happen in most cases and even for a potential of the form \( V = m^2\phi^2 \) around the turning points of the scalar field.

### III. PERFECT FLUID AND SCALAR FIELD

#### A. Perfect Fluid

A perfect fluid has an energy momentum tensor \( T^{\mu\nu} \equiv (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \) where \( \rho \) is the energy density and \( p \) the pressure of the fluid, \( g^{\mu\nu} \) the metric tensor while \( u^\mu \) the 4-velocity. In the rest frame \( u^0 = 1, u^i = 0 \) and \( T^{\mu\nu} \) gives the usual energy momentum tensor \( T^\mu_\nu = diag[\rho, -p, -p, -p] \). For an arbitrary velocity \( v \) the four velocity is
motion, with a Minkowski metric, the equations are

determined by the conservation equation \( \partial_\mu T^{\mu\nu} = 0 \) and
to solved them one needs to add an equation of state \( p = w\rho \). Matter and radiation have a constant \( w \) equal to zero,
1/3 respectively.

In order to study the propagation of sound it is common to take perturbations around a central value \( p_o, \rho_o \)
\[ p(t, x) = p_o(t) + p_1(t, x), \quad \rho(t, x) = \rho_o(t) + \rho_1(t, x) \] (17)

where the central values do not depend on the space coordinates while the perturbations \( p_1, \rho_1 \) depend on \( t \) and \( x \).

Keeping first order terms in the pressure and energy density perturbations one gets by solving the equation of motion, with a Minkowski metric, the equations
\[ \frac{\partial p_1}{\partial t} = -(\rho + p)\nabla v \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho + p} \nabla p_1. \] (18)

Using the relationship \( p = w\rho \) (or equivalently \( p_1 = w\rho_1 \)) one has
\[ \nabla p_1 = w\nabla \rho_1 \] (19)

which is valid if \( w \) is not a function of \( x \). The restriction on \( w = p_1/\rho_1 = \partial p/\partial \rho \) to not depend on the spatial variations
follows from the hypothesis of adiabatic perturbations. Eliminating \( v \) from eqs. (18) and using (19) we get the usual
wave equation
\[ \ddot{\rho}_1 - w^2\nabla^2 \rho_1 = 0 \] (20)

Comparing eq. (20) with eq. (9) we see that the phase velocity is given by \( v_p = |w| = v \) and it is the same for all
monochromatic wave functions of the fluid perturbations \( \rho_1 \), since \( b = d = 0 \). Eq. (20) is valid for a static (Minkowski)
space in the absence of gravitational interactions.

If we consider a flat FRW space, with metric \( ds^2 = dt^2 - a(t)^2dx^2 \), then the perturbation equation for radiation
with \( \delta_r = \delta \rho_r/\rho_r \) is given (in the transverse gauge) by
\[ \dddot{\delta}_r + \frac{1}{3}k^2\delta_r = \frac{4}{3}\dot{a} \] (21)

where \( \delta_r = \delta \rho_c/\rho_c \) is the perturbation of (cold) matter and the r.h.s. term in eq. (21) is a source term for \( \delta_r \). Comparing
eq. (21) and eq. (9) we see that the phase velocity is given by \( v^2 = 1/3 \) and in the absence of the source term, the group
and the phase velocity would coincide. Since there is no dissipative term \( (d = 0) \) the sound velocity for radiation
would be given be \( v_p = v_v = v_p = 1/\sqrt{3} \).

Notice that even in an expanding universe the perfect fluid (in this case for radiation) has no dispersion.

### B. Scalar Field

A scalar field is defined by the action
\[ S = \int dx^4 \sqrt{-g} \left[ \frac{\lambda}{2} (\partial\phi)^2 - V(\phi) \right] \] (22)

where \( g \equiv det[g_{\mu\nu}] \) and we have introduced the parameter \( \lambda = \pm 1 \) in order to incorporate phantom fields in the
analysis (+1 for a canonical scalar fields and -1 for phantom fields). From this eq. we can extract the energy
momentum tensor \( T_{\mu\nu}(\phi) = \lambda \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{\lambda}{2} \phi \partial_\phi \phi \partial_\phi \phi - V(\phi) \right] \) and the equation of motion is
\[ \lambda (\partial^\mu \partial_\mu \phi + 3H \dot{\phi}) + V' = 0. \] (23)

where we have set \( \partial_\mu (g^{\mu\nu} \sqrt{-g})/\sqrt{-g} \partial_\nu \phi = 3H \dot{\phi} \), valid in a flat FRW metric, with \( \sqrt{-g} = a(t)^3, H = \dot{a}/a \) the
Hubble parameter and \( a(t) \) the scale factor. The \( T^\phi_{\phi} = \rho \) component defines the energy density while \( T^i_i = -p_\phi \) the
pressure, for simplicity and presentation purposes we will only consider perturbations in one direction \( i = x \), giving
\[ \rho_\phi = \lambda \left( \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \partial^i \phi \partial_i \phi \right) + V(\phi) \quad p_\phi = \lambda \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \partial^i \phi \partial_i \phi \right) - V(\phi) \] (24)
and the equation of state parameter is

$$w_φ = \frac{p_φ}{ρ_φ} = \frac{\lambda(\frac{V}{2} - \frac{1}{2}\partial^i φ \partial_i φ)}{\lambda(\frac{V}{2} - \frac{1}{2}\partial^i φ \partial_i φ) + V(φ)}, \quad (25)$$

In cosmology it is usual to assume the scalar field to be homogenous in space and therefore $φ(t, x) = φ_o(t)$ is only a function of time and the space derivatives vanish, $\partial_i φ_o = 0$. In such a case eq. (25) reduces to eq. (1).

However, contrary to a perfect fluid, in the case of a scalar field the equation of state cannot be imposed and it is dynamically determined by the evolution of $φ$ given by eq. (25). It is easy to see that $w$ in eq. (25) does depend on the spatial coordinates. Taking a perturbations in the $x$-direction around the average solution $φ_o$ one has

$$φ(t, x) = φ_o(t) + φ(x, t) \quad (26)$$

and to first order in $φ$ we have $V(φ(t, x)) \approx V_o(φ_o(t)) + V'_o(φ) + O(φ^2)$ with $V'_o = \partial V/∂φ|_{φ_o}$ and $(\dot{φ}^2 - \partial^i φ \partial_i φ)/2 = \dot{φ}_o^2/2 + \dot{φ}_o φ + O(φ^2)$. Eq. (25) up to first order in $φ$ gives

$$w_φ = w_{φ_o} + \frac{2λ}{p_{φ_o}^2} \left[ V'_o(φ_φ(t, x)) - 2V_0 φ_o(φ(t, x)) \right] + O(φ^2) \quad (27)$$

where $w_{φ_o}$ is given by eq. (1) and $p_{φ_o} = λ\dot{φ}_o^2/2 - V_o(φ_o)$. The perturbed equation (28) up to first order in $φ$ is

$$λ(\ddot{φ} - \nabla^2 φ + 3H \dot{φ}) + V(φ_o)'' φ = -\frac{1}{2} \frac{\dot{H}}{H} \dot{φ}_o \quad (28)$$

where the r.h.s in eq. (28) is due to the variation of the metric tensor $g_{μν} = η_{μν} + h_{μν}$ and $\nabla^2 \equiv a^{-2} \nabla^2$ and $V'' = \partial^2 V/∂φ^2$. In terms of a Fourier transformation $φ(t, x) = (2π)^{-3} \int d^3k φ_k e^{i k \cdot x} / a$ we can write eq. (25) as

$$\ddot{φ}_k + 3H \dot{φ}_k + (k^2 + \frac{V(φ_o(t))''}{λ}) φ_k = -\frac{1}{2λ} \frac{\dot{H}}{H} φ_o h^* \quad (29)$$

with $k^2 = k \cdot k$. Eq. (29) gives the wave equation for the scalar field perturbations and we can see that the medium is dissipative ($H \neq 0$) and dispersive ($(k^2/a^2 + V''/λ)/k^2$ is $k$ dependent). In conformal time $dτ = dt$ eq. (29) becomes

$$\ddot{φ}_k^* + 2H φ_k^* + (k^2 + \frac{a^2 V(φ_o(t))''}{λ}) φ_k = -\frac{1}{2λ} \frac{\dot{H}}{H} φ_o h^* \quad (30)$$

where $*$ means derivative w.r.t. $dτ$ and $H ≡ a^*/a$. Comparing eq. (30) with eq. (9) we see that $v_p = 1$, i.e. the "phase velocity" is the speed of light, $b = a^2 V(φ_o(t))''/λ$ and $d = 2H$ are non zero and time dependent. Therefore, the wave function will have a non constant amplitude and the velocity of each monochromatic wave will differ thus giving a group velocity different than the phase velocity. We also remark that the left hand side of eq. (30) is the same for a canonical scalar field with negative mass square (i.e. a tachyon field with $V''/λ = V''(φ_o) < 0$) to a phantom field with positive $m^2$ (i.e. $V''/λ = -V'' < 0$). On the other hand, the gravitational contribution given by the r.h.s of eq. (30) has opposite sign for a canonical field and a phantom field since $λ$ changes sign.

C. K-essence

Let us now consider the following (k-essence) action for a scalar field $S$

$$S = \int dx^4 \sqrt{-g} K(φ) F(X) \quad (31)$$

with $K(φ)$ an arbitrary function of the scalar field $φ$ and $F(X)$ a function of the kinetic term $X = \partial_μ φ \partial^μ φ/2$. The equation of motion of eq.(31), in a flat FRW metric, is given by

$$\partial_μ \left( a(t)^3 F'(X) K(φ) \partial^μ φ \right) - a(t)^3 F'(X) K'(φ) = 0 \quad (32)$$

where we have taken $\sqrt{-g} = a^3(t)$ and a prime denotes derivative w.r.t. the argument and we used $δX/δ∂^μ φ = δ_μ φ$. Eq. (32) is then

$$\partial_μ \partial^μ φ + \frac{K'}{K} \partial_μ φ \partial^μ φ + \frac{F''}{F'} a(t)^3 X \partial^μ φ + 3H \dot{φ} - \frac{F}{F'} K' \frac{K'}{K} = 0 \quad (33)$$
The Lagrangian in eq. (31) can be reduced to a canonical kinetic term scalar Lagrangian for \( F = X, K = 1 \) and eq. (33) reduces to \( \partial_\mu \partial^\mu \phi + 3H \dot{\phi} = 0 \), i.e. a scalar field without potential. For an homogenous scalar field \( \phi = \phi_o(t) \) we have \( \partial_\mu X = \dot{\phi}_o \phi_o \) and eq. (33) becomes

\[
\partial_\mu \partial^\mu \phi_o + 3H \dot{\phi}_o + \frac{F'}{F'} \frac{F'' - F'}{F''} - \frac{K'}{K} \left( \frac{\phi_o^2 F - F'}{F''} \right) = 0 \tag{34}
\]

In order to analyze the evolution of the perturbations we will expand eq. (33) to first order in \( \varphi(t, x) = \phi_o(t) + \varphi(t, x) \). Let us defined the functions \( D(\phi) \equiv K'(\phi)/K(\phi), \ E(X) \equiv F''(X)/F'(X) \) and \( S(X) \equiv F(X)/F'(X) \) which appear in eq. (33). We expand the functions \( D, E, S \) to first order in \( \varphi(t, x) \), i.e. \( D = D_o + D' \varphi, \ E = E_o + E' \delta X, \ S = S_o + S' \delta X \) with \( \delta X = \dot{\phi}_o \phi_o \) where the prime denotes derivative w.r.t. the argument. The expansion of \( X \) and \( \partial_\mu X \) are given by

\[
X = X_o + \delta X = X_o + \dot{\phi}_o \phi_o + O(\varphi^2)
\]

\[
\partial_\mu X = \partial_\mu X_o + \dot{\phi}_o \phi_o + \phi_o \partial_\mu \dot{\phi}_o + O(\varphi^2)
\tag{35}
\]

with \( X_o = \frac{\dot{\phi}_o^2}{2}, \dot{X}_o = \dot{\phi}_o \phi_o, \partial_\alpha X_o = \partial_\alpha (\partial_\mu \phi_o \partial^\mu \phi_o/2) = \dot{\phi}_o \phi_o \delta_{\alpha \alpha} \) and \( \partial_\mu X \partial^\mu \phi = \dot{\phi}_o \phi_o + 2 \phi_o \dot{\phi}_o \dot{\phi}_o + \dot{\phi}_o^2 \dot{\phi}_o \). To first order in \( \varphi(t, x) \) eq. (33) becomes

\[
\dot{\phi} - B \nabla^2 \varphi + 3BH \dot{\phi} + A \dot{\phi} + m_{eff}^2 \varphi = -\frac{1}{2} \dot{\phi}_o \dot{h}
\tag{36}
\]

where

\[
B = 1/(1 + E_o \dot{\phi}_o^2) = F'/F' + F'' \dot{\phi}_o^2
\]

\[
A = B(2D_o \dot{\phi}_o + E' \dot{\phi}_o^2) - S_D \dot{\phi}_o + 2E_o \dot{\phi}_o \dot{\phi}_o \tag{37}
\]

and the effective mass is given by

\[
m_{eff}^2 = B(\dot{\phi}_o^2 - S_o)D' = \frac{(\dot{\phi}_o^2 - S_o)D'}{1 + E_o \dot{\phi}_o^2}
\tag{38}
\]

Making a fourier transformation of eq. (33), as in eq. (29), we get

\[
\tilde{\varphi}_k + (3BH + A) \tilde{\varphi}_k + (k^2 B + m_{eff}^2) \tilde{\varphi}_k = -\frac{1}{2} \dot{\phi}_o \dot{h}
\tag{39}
\]

The r.h.s. in eq. (39) is due to the variation of the metric as in eq. (25) and the factor of \( B \) in eqs. (38) and (39) arises because \( \partial_\mu X \partial^\mu \phi \) involves a second order time derivative of \( \varphi \). Comparing equation (38) to eq. (29) we see that they are similar, eq. (38) has a second time derivative term, a friction term and an effective mass \( m_{eff}^2 \). The friction term, the effective mass and the phase velocity \( v^2 = B \), as seen from eq. (9), are model and time dependent. The phase velocity will, in general, be different than the speed of light contrary to a canonical scalar field. The effective mass square can in principle be positive or negative and the effect of the friction term is to damp or enhance the amplitude of the fluctuations, depending on its sign.

Let us now consider the particular case presented in (8). In this special case \( K(\phi) = 1/\phi^2 \) and \( F(X) = g(y)/y \) with \( y \) defined as \( y = 1/\sqrt{X} \) and the pressure, energy density, equation of state and group velocity are given by

\[
p = \frac{g(y)}{\phi^2 y} \quad \rho = -\frac{g'(y)}{\phi^2} \tag{40}
\]

\[
w = -\frac{g(y)}{yg'(y)} \quad v_g = \frac{g(y) - yg'(y)}{yg''(y)} \tag{41}
\]

As seen from eqs. (11), the equation of state parameter (i.e. the phase velocity) diverges when \( g'(y) = 0 \) while the group velocity diverges at \( g''(y) = 0 \). In this class of models we have \( D(\phi) = K'(\phi)/K(\phi) = -2/\phi, \ D' = dD/d\phi = 2/\phi^2, \ E(X) = F''(X)/F'(X) = y^2(y'g + yg' - g)/(2(g - yg')) \) and \( S(X) = F(X)/F'(X) = 2g/(y^2(g - yg')) \) and the effective mass of eq. (33) is given by

\[
m_{eff}^2 = \frac{4}{y^2 \phi_o^2} \frac{[\dot{\phi}_o^2 + 2g(y - yg') - 2g]}{[2(g - yg') + (yg + yg' - g)y^2 \phi_o^2]} \tag{42}
\]

The coefficients \( A, B \) of eq. (37) can be easily obtained. However, even in this particular case the effective mass, \( A \) and \( B \) have not a simple form and are still model and time dependent but the main conclusions remain valid. The sound speed for k-essence differs from a canonical scalar field, its phase velocity is no longer equal to the speed of light but and its sound velocity is smaller than the speed of light.
D. Curvature Correction

In the case where we consider corrections to the Einstein general relativity we have a lagrangian \( S = \int dx^4 \sqrt{-g} (f(R) + L_m) \) (43) with \( f(R) \) an arbitrary function of the Ricci scalar \( R \) and \( L_m(\phi) \) the matter lagrangian. The equation of motion of eq.(43) in a flat FRW metric, is given by

\[
f'(R)R_{\alpha\beta} - \frac{1}{2} f(R)g_{\alpha\beta} = f'(R)^{\mu\nu} (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) + \bar{T}^M_{\alpha\beta}
\]

(44)

which can be written in a suggestive form \(13\)

\[
R_{\alpha\beta} - \frac{1}{2} Rg_{\alpha\beta} = T^C_{\alpha\beta} + T^M_{\alpha\beta}
\]

(45)

where

\[
T^C_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2} \left[ f(R) - Rf'(R) \right] + f'(R)^{\mu\nu} (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) \right\}
\]

(46)

\[
T^M_{\alpha\beta} = \frac{1}{f'(R)\bar{T}^M_{\alpha\beta}}
\]

(47)

In some interesting cases, such as

\[
f(R) = R^2, \quad f(R) = R - \frac{\mu^2}{R}
\]

(48)

these fourth order equations can be transformed to the Einstein frame, where the gravitational equations take the Einstein form, coupled to a scalar field. This is done through a conformal mapping \(13\)

\[
\tilde{g}_{\alpha\beta} = F(R)g_{\alpha\beta}, \quad \phi = G(R)
\]

(49)

with suitable chosen functions \(F(R)\) and \(G(R)\). The resulting equations are

\[
\tilde{R}_{\alpha\beta} - \frac{1}{2} \tilde{R}g_{\alpha\beta} = T^C_{\alpha\beta} + T^M_{\alpha\beta}
\]

(50)

\[
\nabla^\mu \nabla_\mu \phi - V'(\phi) = \sigma
\]

(51)

for a suitable source term \(\sigma\) \(13\). Thus, in many interesting cases, curvature corrections model of dark energy can be mapped on the scalar field quintessence paradigm. Therefore, the small disturbances equations will have the form \(29\). It should be noted that if the action \(43\) is treated with a Palatini variation \(10\) the resulting field equations are of the second order type. The resulting field equations have an effective cosmological constant which, for a suitable choice of parameters, can be fitted to reproduce the present deceleration parameter. Since these equations have the Einstein form, small perturbations will behave much the same way as in a cosmological model with a varying cosmological constant. In particular, adiabatic scalar modes will obey a scalar wave equation, formally similar to \(29\). However, for lagrangians singular in \(R = 0\) the newtonian limit cannot be recovered \(15\).

IV. SOLUTION

We shall study now the propagation speeds of a scalar field. For the sake of simplicity, let us consider Eq. \(28\) in Minkowski space-time, with \(\sqrt{-g} = 1\) in a 1+1 dimensional space-time. Let us consider the Lagrangian density \(\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 - (\partial_\phi \phi)^2] - V(\phi)\), where the constant \(\lambda\) allow us to include phantom fields in the following discussion. The equation of motion as given by eq.(29) and the perturbed equation of motion for \(\phi(z,t)\), eq.(29) \(\phi(z,t) = \phi_0(t) + \varphi(z,t)\), in the absence of the source term, is

\[
\lambda \left( \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial z^2} \right) + m^2 \varphi = 0.
\]

(52)
where we have taken $V(\phi_o)^2 = m^2$. In terms of the Fourier transformation we have
\[
\frac{\lambda}{c} \dddot{\varphi}_k + (\lambda k^2 + m^2) \varphi_k = 0. \tag{53}
\]
We have not incorporated a friction term because it only enhances or damps the amplitude of the fluctuations but does not interfere with the sound speed. Eqs. (52) or (53) encompasses several particular cases commonly found in the cosmological literature: a standard scalar field $\lambda > 0$, $m^2 > 0$ with $1 > w_{\phi_o} > -1$, a negative potential field $\lambda > 0$, $m^2 < 0$ and a phantom field $\lambda < 0$, $m^2 > 0$. The last two examples have regions with $|w_{\phi_o}| > 1$ and/or $|dp_o/dp_o| > 1$. The second case corresponds to $\phi$ near a maximum of the potential, which means the existence of an effective tachyon. It would seem that the sound velocity becomes greater than $c$ in this case, but this is not so: the fluid model breaks down when the propagation of perturbations in the system is considered, even in the “normal” case $|w| < 1$. The same thing happens for the third case, i.e. phantom field.

We are interested in finding the general solution with the initial conditions
\[
\varphi(z, 0) = f(z), \quad \left(\frac{\partial \varphi}{\partial t}\right)_{t=0} = g(z). \tag{54}
\]
It is enough to find the fundamental solution of eq. (52) \(\Delta(z, t)\), satisfying
\[
\Delta(z, 0) = 0, \quad \left(\frac{\partial \Delta}{\partial t}\right)_{t=0} = \delta(z) \tag{55}
\]
since for any set of initial conditions (54) we have
\[
\varphi(z, t) = \int_{-\infty}^{\infty} d\zeta \left[ \Delta(z - \zeta, t)f(\zeta) + \Delta(z - \zeta, t)g(\zeta) \right]. \tag{56}
\]
Indeed
\[
\varphi(z, 0) = f(z), \quad \lim_{t \to 0} \frac{\partial \varphi(z, t)}{\partial t} = \lim_{t \to 0} \int_{-\infty}^{\infty} d\zeta \left[ \Delta(z - \zeta, t)f(\zeta) + \Delta(z - \zeta, t)g(\zeta) \right] \tag{57}
\]
\[
= \lim_{t \to 0} \int_{-\infty}^{\infty} d\zeta \left[ \left( \frac{\partial^2 \Delta}{\partial z^2} - m^2 \Delta \right) f(\zeta) + \delta(z - \zeta)g(\zeta) \right] \tag{58}
\]
\[
= g(z). \tag{59}
\]
Using a Fourier decomposition for \(\Delta\)
\[
\Delta(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikz} \Delta(k)e^{i\omega t} \tag{60}
\]
we find the dispersion relation
\[
\omega = \pm c\sqrt{k^2 + \mu^2} \tag{62}
\]
with $\mu^2 \equiv m^2/\lambda$. Observe that negative kinetic energy, $\lambda < 0$, behaves as a negative $m^2$ with respect to propagation. Using the initial conditions eqs. (55) we find
\[
\Delta(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{\sin \omega(k) t}{\omega(k)} e^{ikz}. \tag{63}
\]
Now we use the integral representation \[24\] (Integral 6.677.6)]
\[
\frac{\sin ct\sqrt{k^2 + \mu^2}}{c \sqrt{k^2 + \mu^2}} = \frac{1}{2c} \int_{-ct}^{ct} d\xi J_0 \left( \mu \sqrt{(ct)^2 - \xi^2} \right) e^{-ik\xi} \tag{64}
\]
\[
= \frac{1}{2c} \int_{-\infty}^{\infty} d\xi J_0 \left( \mu \sqrt{(ct)^2 - \xi^2} \right) \Theta[(ct)^2 - \xi^2] e^{-ik\xi} \tag{65}
\]
where $\Theta[x]$ is the Heaveside function ($\Theta[x] = 1$ for $x \geq 0$ and $\Theta[x] = 0$, $x < 0$) and $J_0(x)$ the Bessel function of zero order (notice that the change in integration from eq. (51) to eq. (53) is taken care by the Heaveside function), to obtain

$$
\Delta(z, t) = \frac{1}{4\pi c} \int_{-\infty}^{\infty} dk e^{ikz} \int_{-ct}^{ct} J_0 \left( \frac{\mu \sqrt{(ct)^2 - \xi^2}}{ct} \right) e^{-ik\xi} d\xi
$$

(66)

$$
= \frac{1}{4\pi c} \int_{-\infty}^{\infty} dk d\xi J_0 \left( \frac{\mu \sqrt{(ct)^2 - \xi^2}}{ct} \right) \Theta[(ct)^2 - \xi^2] e^{ik(z-\xi)}
$$

(67)

$$
= \frac{1}{2c} J_0 \left( \frac{\mu \sqrt{(ct)^2 - z^2}}{ct} \right) \Theta[(ct)^2 - z^2].
$$

(68)

The last equation (68) is valid for $t \neq 0$ and it gives $\Delta(z, t = 0) = 0$, as can be seen from eq. (67). Taking the derivative of eq. (68) we have

$$
\dot{\Delta}(z, t) = - \frac{\mu ct}{2} \frac{J_1 \left( \frac{\mu \sqrt{(ct)^2 - z^2}}{ct} \right)}{\sqrt{(ct)^2 - z^2}} \Theta[(ct)^2 - z^2]
$$

(69)

and for $t = 0$ we get $\dot{\Delta}(z, 0) = \delta(z)$. Eqs. (68) and (69) are the main results of this section. They show that a signal propagates always with speed smaller or equal than $c$, since $\Delta(z, t)$ and $\dot{\Delta}(z, t)$ vanish for $|z| > ct$. This ensures that a signal generated at $t = 0, z_0$ will necessarily arrive at a point $|z - z_0|$ at a time larger or equal than $ct$, as can be seen from the vanishing of the wave function $\phi(z, t)$ for $|z - z_0| > ct$ from eqs. (50), (53) and (55).

On the other hand, the result for $m^2 < 0$ (or $\lambda < 0$) can be obtained easily from Eq. (65) or (68) using analytic continuation in the parameter $\mu = m/\sqrt{\lambda}$. In both cases one gets

$$
\Delta_{I}(z, t) = \frac{1}{2c} I_0 \left( |\mu| \sqrt{(ct)^2 - z^2} \right) \Theta[(ct)^2 - z^2]
$$

(70)

where $I_0$ is the modified Bessel function of zero order. The amplitude of eq. (70) grows exponentially with time, showing explicitly the instability of a field with $m^2 < 0$ (or $\lambda < 0$). However, it also shows that the speed of the wave packet is smaller or equal to the speed of light, as for the standard scalar field case ($m^2 > 0, \lambda > 0$). The instability growth of the perturbation is ameliorated by the expansion of the universe since instead of growing exponentially fast it grows power like. The increase of the amplitude implies that the first order approximation in the perturbed differential equation will cease to be valid. This implies that we can no longer use eq. (52) to study the evolution of $\phi$ and non-perturbative methods must be used. In this case the universe will be become inhomogeneous. However, we would like to emphasis that the sound speed never becomes larger than the speed of light.

It should be stressed that the above results are general. Indeed, the Principle of Equivalence guarantees that in a freely falling reference systems, the equations of motion for the fluid or scalar field will take their Minkowskian form.

This behavior can be simply illustrated solving eq. (72) in 1+1 dimensional space-time. We assume that a rectangular semi-infinite wave train of unit amplitude is incident at the spatial point $z = 0$ and look its subsequent behavior. The initial conditions for such a pulse are

$$
f(z) = \sin(kx) \Theta(-x)
$$

$$
g(z) = -\frac{d f(z)}{dz}
$$

Fig.2a shows the normal case with $\mu^2 > 0$ and $E = \omega = 2\mu$. The front of the train moves towards the right with a speed equal to $c$, but the first crest diminishes and settles slowly into the group velocity. This process is just beginning in the figure. The field is stable and there is no increase of the amplitude with time.

Fig.2b shows the propagation of the same wave train in the case $\mu^2 < 0$, i.e. when the field is “tachyonic”. In spite of that character, the signal velocity is equal to $c$ and the amplitude of the wave vanishes outside the lightcone (i.e. for $z > t$). However, the system is unstable and shows exponential growth of the signal as can be seen from the height of the amplitude around $z = 0, t = 8$.

V. CONCLUSIONS

We have shown that scalar fields have naturally regions where $|w_{\omega}| > 1$ and/or $|dp_\omega/dp_\phi| > 1$ and these properties seem to contradict causality. In some of these regions the scalar field can be interpreted as an effective tachyon (i.e.
FIG. 2: Propagation of a semi-infinite wave train of frequency $\omega = 2\mu$, moving to the right. Fig.a shows the amplitude of the pulse as a function of $(z, t)$ in the normal $\mu^2 > 0$ case. Fig.b displays the propagation of the same initial train in the anomalous (tachyonic) case $\mu^2 < 0$. In this latter case, the signal shows exponential growth but in both cases the amplitude vanishes outside the lightcone, i.e. for $z > t$.

$m^2 < 0$) or phantom (i.e. $\lambda \dot{\phi}^2 < 0$) scalar field, i.e. $\mu^2 < 0$ in both cases. We studied the evolution of the scalar field perturbations for models with $|w_{\phi\phi}| > 1$ and/or $|d\rho_o/d\rho_o| > 1$ including tachyon, phantom, k-essence and curvature models. We have shown that even in these extreme cases the scalar field is consistent with causality and Lorentz invariance provided dispersion is taken into account and the sound speed is never larger than the speed of light. Dispersion, however, appears naturally for scalar fields having nontrivial potentials (e.g. mass term) such as those used to model dark energy. However, when $\mu^2$ becomes negative the amplitude of the perturbations grows rapidly and the first order approximation breaks down. The universe becomes, therefore, inhomogeneous and non-perturbative
mathematical methods must be used to study the evolution of the perturbations.

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