Abstract

We study the geometrical meaning of the Faà di Bruno polynomials in the context of KP theory. They provide a basis in a subspace $W$ of the universal Grassmannian associated to the KP hierarchy. When $W$ comes from geometrical data via the Krichever map, the Faà di Bruno recursion relation turns out to be the cocycle condition for (the Welters hypercohomology group describing) the deformations of the dynamical line bundle on the spectral curve together with the meromorphic sections which give rise to the Krichever map. Starting from this, one sees that the whole KP hierarchy has a similar cohomological meaning.
1 Introduction

The aim of this paper is to bridge between the algebro–geometrical setting of KP theory \[8, 9, 13\], whose basic tools are the Baker-Akhiezer function \(\psi\) and the Hirota \(\tau\) function, and the construction based on the generating function \(h(z) = z + \sum_{i \geq 1} h_i z^{-i}\) of the hamiltonian densities and its associated Faà di Bruno polynomials. Although dating back to the very beginning of the modern theory of soliton equations \[16, 5, 21\], up to now the second approach has been applied mostly to the algebraic study of the KP theory and its reductions (see, e.g., \[1, 6, 15\]). It has been reconsidered more recently in \[4, 14\] in the framework of the bihamiltonian theory of integrable systems.

More precisely, the motivations of our work are the following:

a) It is well known \[18, 19\] that one can associate to any formal Baker-Akhiezer function \(\psi\) of the KP theory a moving point \(W(\psi)\) in the universal Grassmannian \(Gr\) and that the KP flows linearize there. Another way of getting this result algebraically \[4, 10\] is by means of the Faà di Bruno polynomials \(h^{(k)}(z, x)\) recursively defined by eq. (2.4) below. These give rise to a basis in (another) space \(W(h)\in Gr\). The linear flows for \(W(\psi)\) correspond to slightly more general Riccati type equations for \(W(h)\).

b) For algebraic geometrical solutions, KP can be linearized on the Jacobian of the spectral curve \(C\) as well, the link between the two linearizations being the Krichever map \[13, 19\]. This map associates a point \(W \in Gr\) to the generic datum of a genus \(g\) curve \(C\), a point \(p \in C\), a local coordinate \(z^{-1}\) vanishing at \(p\), a line bundle \(L_D\) of degree \(g\) and a local trivialization \(\phi_0\) of \(L_D\) in a neighborhood \(U_0\) of \(p\).

It is then natural to ask for the geometrical meaning of the Faà di Bruno basis. The key observations to answer this question are the following:

1) Generically, \(L_D\) has a unique (up to homotheties) holomorphic section \(\sigma_D\), with divisor \(D\). If \(p \in C\) is not contained in \(D\), \(\sigma_D\) does not vanish at \(p\) and gives a trivialization of \(L\) in a neighborhood of \(p\).

2) From the cohomology sequence

\[0 \rightarrow H^0(C, \mathcal{O}) \rightarrow H^0(C, \mathcal{O}(p)) \xrightarrow{\Psi_4} \mathbb{C} \rightarrow H^1(C, \mathcal{O}) \rightarrow H^1(C, \mathcal{O}(p)) \rightarrow 0\]

associated to the injection \(\mathcal{O} \rightarrow \mathcal{O}(p)\), we get that \(\Psi_4\) is an injection. We denote by \([\alpha] := \Psi_4(z)\) the image if the generator \(z\) of \(H^0(C, \mathcal{O}(p)/\mathcal{O})\). So we have a distinguished vector field tangent to \(Pic_g(C)\).

Summing up, the generic datum of \((C, p, z, L_D)\) gives us a trivialization \(\phi_0\) of \(L_D\)
on \( U_0 \ni p \), i.e. a set of Krichever data, together with infinitesimal deformations of \( \mathcal{L}_D \), corresponding to representatives \( \alpha \) of \([\alpha]\).

3) We can then use the data above to deform \( \mathcal{L}_D \) together with \( \sigma_D \) along \( \alpha \). The cocycle condition for such an infinitesimal deformation (see eq. (3.1)) looks like the Faà di Bruno relation (see eq. (2.4)) but misses the \( x \) dependence of the Faà di Bruno polynomials. To accommodate this dependence, we exploit the existence of a universal family of line bundles over \( C \) with a section and prove that one can choose \( \alpha \) in such a way that the Faà di Bruno relation is actually the cocycle condition for the infinitesimal deformation of the elements of this family.

That the KP hierarchy was related to cohomology has been known since [17]. This set up corresponds to the projection on \( H^1(C, \mathcal{O}) \) of the cocycle conditions above, which control the infinitesimal deformations of \( \mathcal{L}_D \). The novelty of our result is that, by considering also the deformations of the sections of \( \mathcal{L}_D \) which give rise to the Krichever map, one gets as cocycle conditions the equations of the KP hierarchy as a dynamical system on \( Gr \) [18].

Section 2 quickly describes the appearance of Faà di Bruno polynomials in KP theory and the associated map to \( Gr \). The proofs of the main results are collected in Section 3. Section 4 is devoted to the explicit description of the case of elliptic curves. Some notions of deformation theory are recalled in Appendix A. In the sequel, we will use the notations of [11] while dealing with curves and their Jacobians. In particular we refer to [11], ch. 2.7 for the results recalled in Section 3.

2 Faà di Bruno polynomials, the KP theory and the Grassmannian

The KP hierarchy is an isospectral deformation of a monic operator of degree 1 in the ring of pseudodifferential operators on the circle \( S^1 \). Given such an operator

\[
Q = \partial - \sum_{l \geq 1} q_i \partial^{-i},
\]

one defines the associated linear problem for the Baker–Akhiezer function \( \psi \):

\[
\begin{aligned}
Q \psi &= z \psi \\
\frac{\partial}{\partial n} \psi &= Q^n \psi,
\end{aligned}
\]

(2.2)
where \( Q^n_+ = Q^n - Q^n_- \) is the differential part of \( Q^n \). The KP hierarchy (see, e.g., [7]) is the set of compatibility conditions of the linear system above, i.e. the hierarchy of Lax equations
\[
\frac{\partial}{\partial t_n} Q = [Q^+_+, Q] = \frac{\partial Q^+_m}{\partial t_n} - \frac{\partial Q^+_n}{\partial t_m} = [Q^+_+, Q^+_+]. \tag{2.3}
\]
A formally equivalent description starts with a monic Laurent series \( h \) of the form
\[
h = z^k + \sum_{l \geq 1} h_l(x) z^{-l}
\]
with coefficients in the space of smooth functions on the circle, \( x \) being a coordinate on \( S^1 \). Its Faà di Bruno iterates \( h^{(k)} \) are defined by the recurrence relation
\[
h^{(k+1)} = \partial_x h^{(k)} + hh^{(k)}, \quad h^{(0)} := 1. \tag{2.4}
\]
Since \( h^{(k)} \) has a Laurent expansion \( h^{(k)} = z^k + O(z^{k-2}) \), the equation \( z = h - \sum_{l \geq 1} q_l h^{(-l)} \) is meaningful in the space of formal Laurent series and sets up a 1-1 relation between the coefficients \( h_i \) and the standard KP variables \( q_i \) of Eq. (2.1). Following [21] one sets \( -(Q^j)_- = \sum_{l \geq 1} H_l^j Q^{-l} \) and constructs the Laurent series
\[
H^{(n)} = z^n + \sum_{l \geq 1} H_l^n z^{-l}. \tag{2.5}
\]
The second equation of (2.2) becomes
\[
\frac{\partial}{\partial t_n} \psi = H^{(n)} \psi. \tag{2.6}
\]
Setting \( h \equiv H^{(1)} \) and \( t_1 = x \), one gets \( h^{(k)} = (\partial_x^k \psi)/\psi \) and the obvious continuity equations \( \frac{\partial}{\partial t_j} h = \partial_x H^{(j)} \).

The facts from KP theory which are relevant for us are the following:

a) The KP equations are equivalent to the conservation laws:
\[
\frac{\partial}{\partial t_n} h = \partial_x H^{(n)} \tag{2.7}
\]
with \( H^{(n)} \) of the form (2.5).

b) \( H^{(k)} \) can be expanded as a finite Faà di Bruno “polynomial”:
\[
H^{(k)} = h^{(k)} + \sum_{l=0}^{k-1} c^k_l h^{(l)}, \tag{2.8}
\]
with \( c^k_l \) independent of \( z \).

c) The \( n \)-Gel’fand–Dickey reductions of the KP theory can be defined as the restrictions of the flows (2.7) to the invariant submanifolds defined by
\[
H^{(n)} = z^n. \tag{2.9}
\]
The key property linking to the Grassmannian picture is eq. (2.8). Let $H = L^2(S^1, \mathbb{C}) = H_+ \oplus H_-$, where $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, $H_+ = \overline{\text{span}} \{ z^j : j \geq 0 \}$, $H_- = \overline{\text{span}} \{ z^j : j < 0 \}$, where the bar means $L^2$ closure. Then the universal Grassmannian $\text{Gr}(H)$ is the set of subspaces $W \subset H$ for which the orthogonal projections $\pi_+: W \to H_+$ and $\pi_- : W \to H_-$ are respectively Fredholm and Hilbert-Schmidt operators. The virtual dimension of $W$ is by definition the index of $\pi_+$ and $\text{Gr}(H)$ is the union of connected components labeled by the virtual dimension. If $h = z + \sum_{l \geq 1} h_l/z^l$ is a smooth function of $z \in S^1$, the Faà di Bruno recurrence relations (2.4) give a point $W \in \text{Gr}$ by

$$W = \overline{\text{span}} \{ 1, h^{(1)}, h^{(2)}, \ldots \}. \quad (2.10)$$

It is not difficult to show that $H^{(k)} = \pi_+^{-1}(z^k)$.

3 Cohomology and the Faà di Bruno recursion relation

Recall how the Krichever map associates a point in the universal Grassmannian to the datum of a smooth algebraic curve $C$, a point $p \in C$, a local coordinate $z^{-1}$ centered at $p$, a line bundle $\mathcal{N}$ over $C$ and a local trivialization $\phi_0$ of $\mathcal{N}$ in a neighbourhood $U_0$ of $p$. The local coordinate $z^{-1}$ identifies $S^1 = \{ z : |z| = 1 \}$ as a submanifold of $C$, while the sections of $\mathcal{N}$ correspond to functions on $S^1$ via the local trivialization $\phi_0$. The point $W \in \text{Gr}(H)$ associated to $(C, p, z, \mathcal{N}, \phi_0)$ is the closure in $H$ of the space of sections of $\mathcal{N}$ which are holomorphic on $U_1 = C - \{ p \}$. Using the Mayer-Vietoris sequence one shows that the virtual dimension of $W$ is $\chi(\mathcal{N}) - 1$, $\chi(\mathcal{N})$ being the Euler characteristic of $\mathcal{N}$.

For completeness, we recall in the following two lemmas some standard facts. As usual, we will denote by $h^i(C, \mathcal{N}) = \dim H^i(C, \mathcal{N})$.

**Lemma 3.1** Let $p \in C$ be a non–Weierstrass point and $z^{-1}$ a local coordinate vanishing at $p$. Then the classes of $\{ z, z^2, \ldots, z^g \}$ give a basis for $H^1(C, \mathcal{O})$ while for any $k > g$ there exists a function $\lambda_k$ on $C$ with a pole of order $k$ at $p$ and without subleading poles but for those in the Weierstrass gap.

**Proof.** Let $j, l \geq 0$ and consider the exact sequence $0 \to \mathcal{O}(jp) \to \mathcal{O}(lp) \to \mathcal{S}_{l-j}(p) \to 0$ where $\mathcal{S}_{l-j}(p)$ is the skyscraper sheaf of length $l-j$ at $p$. The corresponding cohomology
sequence reads

\[ 0 \rightarrow H^0(C, \mathcal{O}(jp)) \rightarrow H^0(C, \mathcal{O}(lp)) \rightarrow \mathbb{C}^{l-j} \rightarrow H^1(C, \mathcal{O}(jp)) \rightarrow H^1(C, \mathcal{O}(lp)) \rightarrow 0. \]

One can use the covering \((U_0, U_1)\) to compute cohomology. Since \(p\) is not Weierstrass and \(h^1(C, \mathcal{O}) = g\), setting \(j = 0\) and \(l = g\) we see that \(H^1(C, \mathcal{O}(gp)) = 0\). The classes \([z^k] := \Psi_4(z^k)\) for \(k = 1, \cdots, g\) are then a basis of \(H^1(C, \mathcal{O})\). For every \(k > g\), setting \(j = k - 1, l = k\) we obtain that \(H^1(C, \mathcal{O}(kp)) = 0\). Then there is a function \(\lambda_k\) on \(C\) with a pole of order \(k\) at \(p\). Moreover, this function is defined up to sections in \(H^0(C, \mathcal{O}(k-1)p)\) (and up to homotheties) and we can use these ambiguities to fix the polar part of \(\lambda_k\) as claimed.

\[ \square \]

From now on, when we say that a function \(f\) has a "simple pole of order \(k\)" at \(p\) we mean that its Laurent expansion at \(p\) is the one stated in the lemma above.

**Lemma 3.2** Assume that \(h^0(C, \mathcal{N}) > 0\), \(h^1(C, \mathcal{N}) = 0\) and let \(s \in H^0(C, \mathcal{N})\) be a nontrivial section. For all \([\beta] \in H^1(C, \mathcal{O})\) and for every 1-cocycle \(\beta\) representing it there exists an infinitesimal deformation of the couple \((\mathcal{N}, s)\) along \(\beta\).

**Proof.** Consider the double complex \((A.3)\). On the covering \((U_0, U_1)\) we denote by \(\beta_{10}\) a 1-cocycle representing \([\beta]\) with a pole of order \(j\) at \(p\), by \(g_{10}\) the transition function of \(\mathcal{N}\) w.r.t. some trivialization and we let \((f_0, f_1)\) be the couple of functions which represent \(s\). It is clear that \(s\beta_{10}\) is a 1-cocycle and, since \(H^1(C, \mathcal{N}) = 0\), it is actually a coboundary. Hence, there are 0-cochains \((\delta f_0, \delta f_1)\) such that \(g_{10}^{-1}\delta f_1 = s\beta_{10} + \delta f_0\). Choosing \(\delta g_{10} := g_{10}\beta_{10}\), we have that

\[ g_{10}^{-1}\delta f_1 = \delta f_0 + (g_{10}^{-1}\delta g_{10})f_0. \quad (3.1) \]

The couple \((\tilde{f}_0, \tilde{f}_1) := (\delta f_0 + (g_{10}^{-1}\delta g_{10})f_0, \delta f_1)\) is a section of \(\mathcal{N}\) with a pole of order \(j\) at \(p\), therefore it is a holomorphic section of \(\mathcal{N}(jp)\). \(\square\)

Eq. \((3.1)\) looks like a pointwise version of the Faà di Bruno recursion relation with \(h = g_{10}^{-1}\delta g_{10}\). Indeed, this formal resemblance was the starting point of our work. To accommodate the \(x\)-dependence of eq. \((2.4)\) we need a family version of the construction above. Actually, the "universal version" is already at hand \([1]\). We choose in the Krichever data a line bundle \(\mathcal{N} = \mathcal{L}_D\) of degree \(g\) corresponding to a non special effective divisor \(D\), in such a way that \(h^0(C, \mathcal{L}_D) = 1\). In other words, there is a unique
(up to homotheties) non trivial section $\sigma_D \in H^0(C, \mathcal{L}_D)$ vanishing at $D$. If $p \in C$ is not in the support of $D$, $\sigma_D$ does not vanish at $p$ and can be used to trivialize $\mathcal{L}_D$ in a neighborhood $U_0$ of $p$. We get in this way the last piece of the Krichever data up to a $\mathbb{C}^\times$ action. We shall assume that $p$ is not a Weierstrass point.

Let $C^{(d)}$ be the $d$-th symmetric product of $C$, whose points are effective divisors $D = \sum_{i=1}^d q_i$. Recall [2] that on $Y = C \times C^{(d)}$ there is the universal divisor $\Delta$ of degree $d$, whose restriction to $C \times \{D\}$ is the divisor $D \subset C$ itself, and the corresponding line bundle $\mathcal{O}(\Delta)$. Let $\mu : C^{(d)} \to \mathcal{J}(C)$ be the Abel sum map $\mu(D) = \sum_i \oint_{\tilde{q}} \tilde{\omega}$ with base point $\tilde{q} \in C$ and $\tilde{\omega} = (\omega_1, \ldots, \omega_g)$ a basis of Abelian differentials on $C$. We also fix a symplectic basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $H_1(C, \mathbb{Z})$, normalize $\tilde{\omega}$ by $\oint_{a_i} \omega_j = \delta_{ij}$ and denote by $Z_{ij} = \oint_{b_i} \omega_j$ the corresponding period matrix.

The map

$$\tilde{\mu} : C \times C^{(g)} \to \mathcal{J}(C)$$

defined by $\tilde{\mu} = \mu \circ \pi_1 - \mu \circ \pi_2$ ($\pi_i, i = 1, 2$, being the projection on the $i$-th factor of $C \times C^{(g)}$) pulls back the theta bundle, translated by the Riemann constant $k$, to a line bundle $\mathcal{L}$ on $C \times C^{(g)}$ together with the section

$$\sigma(q, D) = \theta(\mu(q) - \mu(D) - k),$$

$\theta$ being the Riemann’s theta function. The restriction $\mathcal{L}|_{C \times \{D\}}$ is a line bundle of degree $g$ on $C$ with a holomorphic section $\sigma_D(q) := \sigma(q, D)$. Notice that $\sigma_D$ is actually an entire function on the universal covering $\tilde{C}$ of $C$, which transforms as

$$\sigma_D(q + \tilde{n} \cdot \tilde{a} + \tilde{m} \cdot \tilde{b}) = \sigma_D(q) \exp(-\pi i \tilde{m} Z \tilde{m} - 2\pi i \tilde{m} \cdot (\mu(q) - \mu(D) - k)),$$

under the action of the fundamental group of $C$ generated by $\tilde{a} = (a_1, \ldots, a_g)$ and $\tilde{b} = (b_1, \ldots, b_g)$. Since $D$ non special, $\sigma_D$ vanishes precisely at $D$ and hence the above restriction of $\mathcal{L}$ is isomorphic to $\mathcal{L}_D$. We denote by $C^{(g)}_0 \subset C^{(g)}$ the open subvariety given by non special effective divisors, and by $\Delta_0$ the restriction of the universal divisor $\Delta$ to $C \times C^{(g)}_0$. Summing up

**Proposition 3.3** The line bundle $\mathcal{L}$ on $C \times C^{(g)}_0$ is isomorphic to $\mathcal{O}(\Delta_0)$ and the section $\sigma$ has divisor $\Delta_0$.  

Denote by $P$ the divisor $P = \{p\} \times C^{(g)}_0 \subset C \times C^{(g)}_0$. Holomorphic sections $\sigma^{(k)}$ of $\mathcal{L}((k - 1)P)$ are the same as sections of $\mathcal{L}$ with poles of order bounded by $k - 1$ along
Lemma 3.4 For every $\omega$ where $k \geq 1$ by means of $\sigma^{(0)} := \sigma$

To link to cohomology, we work out an explicit coordinate description of $L$ on a
subvariety $B \subset C \times C_{0}^{(g)}$ described below.

0) Let $U_{0}$ be an open neighborhood of $p \in C$ with closure $\bar{U}_{0}$ and denote by $C_{0,\nu}^{(g)}$ the
open subset of $C^{(g)}$ of (non special) effective divisors $D$ with support not intersecting
$\bar{U}_{0}$. We shall work on $B = C \times C_{0,\nu}^{(g)}$. For every $D \in C_{0,\nu}^{(g)}$, $\sigma_{D}$ has divisor $D$ and it does
not vanish at $p$. Therefore, $\sigma$ gives a trivialization of $L$ on $B_{0} := U_{0} \times C_{0,\nu}^{(g)}$, which will
be denoted by $\Phi_{0}$. Possibly after shrinking, we can assume that $U_{0}$ is the domain of a
local coordinate $z^{-1}$ centered at $p$.

1) A local trivialization $\Phi_{1}$ of $L$ on $B_{1} = U_{1} \times C_{0,\nu}^{(g)}$, with $U_{1} = C - \{p\}$ is constructed
as follows. The datum of $p$ gives us the line bundle $\mathcal{O}(gp)$ on $C$, together with a
holomorphic section $\sigma_{gp} = \theta(\mu(q) - \mu(gp) - k)$ vanishing of order $g$ at $p$ and nowhere
else. As a function on the universal covering $\tilde{C}$, $\sigma_{gp}$ transforms as

$$\sigma_{gp}(q + \tilde{n} \cdot \tilde{a} + \tilde{m} \cdot \tilde{b}) = \sigma_{gp}(q) \exp(-\pi i \tilde{m} \tilde{Z} \tilde{m} - 2\pi i \tilde{m} \cdot (\mu(q) - \mu(gp) - k)).$$

We look [9, 13] for a function $\nu$ on $\tilde{C} \times C_{0,\nu}^{(g)}$ such that $\nu \sigma_{gp}$ transforms as $\sigma_{D}$ and does
not vanish on $B_{1}$. Let $\Omega^{(k)}$, $k > 0$, be the unique Abelian differential of the second kind
on $C$ with vanishing $a-$periods and $w^{-k-1}dw$ as principal part at $p$ ($w = z^{-1}$). The
$b-$periods of $\Omega^{(k)}$ are

$$\Pi_{l}^{k} := \oint_{b_{l}} \Omega^{(k)} = \frac{2\pi i}{k!} d^{k-1} \zeta_{l} (p),$$

where $\omega_{l}(w) = \zeta_{l}(w)dw$, ($l = 1, \cdots, g$) is the local form of the Abelian differentials on
$U_{0}$. Then

**Lemma 3.4** For every $D \in C_{0,\nu}^{(g)}$ there exists an Abelian differential $\Omega(D)$ of the second
kind on $C$, holomorphically depending on $D$, such that $\Phi_{1} = \sigma_{gp} \exp(\int_{q}^{p} \Omega(D))$ is a
never vanishing section of $L$ on $B_{1}$.

**Proof.** $\Phi_{1}$ has the desired property if and only if the $a-$periods of $\Omega(D)$ vanish and
the $b-$periods are $\oint_{b_{l}} \Omega(D) = 2\pi i \mu_{l}(D - gp) =: a_{l}(D)$, where $\mu_{l}$ is the $l-$th component
of the Abel map. If $\Omega(D) = \sum_{k=1}^{g} b_{k}(D) \Omega^{(k)}$, the equation above becomes

$$\sum_{k=1}^{g} b_{k}(D) \Pi_{l}^{k} = a_{l}(D).$$
Being \( p \) not Weierstrass, the matrix \( \Pi = (\Pi^k_l) \) is invertible and the solution for \( b_k(D) \) is holomorphic in \( D \). \( \square \)

**Remark 3.5** The proof uniquely defines \( \Phi_1 \). Of course there are other trivializations, given by multiplying \( \Phi_1 \) by a nowhere vanishing holomorphic function on \( B_1 \), i.e. by adding to \( \Omega(D) \) an Abelian differential of the second kind \( \tilde{\Omega}(D) \) with vanishing periods and holomorphic on \( B_1 \).

Summing up, we have that

**Lemma 3.6** In the trivialization \( \Phi_i \ (i = 0,1) \) given above, the section \( \sigma \) of \( \mathcal{L} \rightarrow \mathcal{B} \) corresponds to the couple \((1, f_1)\) of holomorphic functions on \( B_0, B_1 \), where

\[
f_1(q, D) = \frac{\theta(\mu(q) - \mu(D) - k)}{\theta(\mu(q) - \mu(gp) - k)} \exp(-\int_q^q \Omega(D)),
\]

and hence \( \mathcal{L} \) has transition function \( g_{10} = f_1 \) on \( B_1 \cap B_0 \). \( \square \)

We are now in the position of applying the computations of Appendix A. We can consider the line bundle \( \mathcal{L} \rightarrow \mathcal{B} \) as a deformation of \( \mathcal{L}_D \) for every \( D \in C^{(g)}_{0,p} \), and the section \( \sigma \) as a deformation of \( \sigma_D \). The logarithmic derivative with respect to \( D \) of the transition function \( g_{10} \) is a cocycle representing the cohomology class in \( H^1(C, \mathcal{O}) \) of the deformation of the line bundle \( \mathcal{L}_D \).

We first consider one dimensional deformations. Let \( \xi : \mathcal{X} := C \times X \rightarrow \mathcal{B} \) be the identity on \( C \) and an embedding of a disk \( X = \{ x \in \mathbb{C} : |x| < \epsilon \} \) into \( C^{(g)}_{0,p} \). For simplicity we will leave implicit the pull-back maps associated to \( \xi \). Set \( \mathcal{X}_i = U_i \times X \ (i = 0,1) \) and denote by \( D_x \) the image of \( x \in X \) on \( C^{(g)}_{0,p} \), by \( \mathcal{L}_x \) the corresponding line bundle and by \( \sigma_x \) its section.

**Proposition 3.7** There exist

1) an embedding of the disk \( X \) into \( C^{(g)}_{0,p} \),
2) a trivialization \( \Phi_1 \) of \( \mathcal{L} \) over \( \mathcal{X}_1 \),
3) sections \( \sigma^{(k)} = (f^{(k)}_0, f^{(k)}_1) \) of \( \mathcal{L} \), with a simple pole of order \( k \geq 0 \) at \( P \),

such that, for every \( x \in X \),

a) the cohomology class of the deformation of \( \mathcal{L}_x \) is \([\alpha]\) with representing cocycle \( \alpha = z + \sum_{l>0} \alpha_l z^{-l} \)

b) the cocycle condition (3.1) for the deformation \( \sigma^{(k)} \) of \( \sigma^{(k)}_x \) is the Faà di Bruno recursion relation (2.4) with \( h = h^{(1)} = \alpha \) and \( h^{(k)} = f^{(k)}_0 \).
**Proof.** Fix a divisor $D_0 \in C_{0,p}^{(g)}$. Lemma 3.4 gives us a differential $\Omega(D_0)$ and a trivialization of $\mathcal{L}_0$ on $U_1$. If $2\pi i \Pi^{(1)} \in \mathbb{C}$ are the $b$-periods of $\Omega^{(1)}$, there is $\epsilon > 0$ such that, for $|x| < \epsilon$, $D_x = D_0 + x\Pi^{(1)}$ is again in $C_{0,p}^{(g)}$. We choose for the embedding 1) the map $x \mapsto D_x$ and for the trivialization $\Phi_1$ the section of Lemma 3.4 with $\Omega(D_x) = \Omega(D_0) + x\Omega^{(1)}$. Then the transition function of $\mathcal{L}_x$ is

$$g_{10}(q,x) = \frac{\theta(\mu(q) - \mu(D_0) - x\Pi^{(1)} - k)}{\theta(\mu(q) - \mu(gp) - k)} \exp\left(-\int_q^p \Omega(D_x)\right)$$

and $\alpha = g_{10}^{-1}\partial_x g_{10}$ satisfies a). As for 3), the section $\sigma^{(0)} = \sigma$ of Lemma 3.6 is a deformation of $\sigma_x$ and the corresponding cocycle condition reads

$$g_{10}^{-1}\partial_x f_1^{(0)} = \partial_x f_0^{(0)} + \alpha f_0^{(0)},$$

showing that the couple $(f_0^{(1)}, f_1^{(1)}) := (\partial_x f_0^{(0)} + \alpha f_0^{(0)}, \partial_x f_1^{(0)})$ is a holomorphic section $\sigma_x^{(1)}$ of $\mathcal{L}_x(p)$. Since $\sigma_x^{(1)}$ depends holomorphically on $x$, we have a section $\sigma^{(1)}$ of $\mathcal{L}(P)$. Iterating this procedure one constructs all the other sections $\sigma^{(k)}$ for $k > 1$.

Setting $h^{(k)} := f_0^{(k)}$ one has that $h = h^{(1)} = \alpha$ and the cocycle condition above for the deformation $\sigma^{(k)}$ of $\sigma_x^{(k)}$ gives precisely the Faà di Bruno recursion relations. □

**Remark 3.8** There is a simple connection between our construction and the Baker-Akhiezer function of \[1, 13, 19\]. Indeed, $g_{10}(z,x) = f_1^{(0)}(z,x)|_{U_1 \cap U_0}$, and $f_1^{(0)}(z,x)$ is a holomorphic function on $U_1$ whose zeros define the effective divisor $D_x$ corresponding to the line bundle $\mathcal{L}_x$. Now, $\psi(z,x) := f_1^{(0)}(z,x)/f_1^{(0)}(z,0)$ is meromorphic on $U_1$, its poles correspond to the non-special effective divisor $D_0$ of degree $g$, and it has an essential singularity at $p$, i.e. it is a Baker-Akhiezer function. This gives another justification to our definition of $h$ as $\partial_x \log f_1^{(0)}(z,x) = \partial_x \log \psi(z,x)$.

The full KP hierarchy has a similar cohomological meaning, which will be quickly sketched below. More details will be given elsewhere. Let us go back to the universal family $\mathcal{L} \to \mathcal{B}$. If $t_1, \ldots, t_g$ are local coordinates on $C_{0,p}^{(g)}$, the classes of $g_{10}^{-1}\partial_k g_{10}$, $k = 1, \ldots, g$ are a basis of $H^1(C, \mathcal{O})$ and give vector fields on $C_{0,p}^{(g)}$. We can choose $t_1 = x$. For every $j > g$ we introduce an extra parameter $t_j$ and change the transition function to

$$\tilde{g}_{10}(z,\bar{t}) = g_{10}(z,t_1,\ldots,t_g)e^{\sum t_j \lambda_j(z)},$$

where $\lambda_j(z)$ is a meromorphic function on $C$ with a simple pole of order $j$ at $p$ and holomorphic elsewhere. The new transition function belongs to the same cohomology
class of the old one, so the family of line bundles $\mathcal{L}$ is unaffected: we have only changed the trivialization over $\mathcal{B}_1$. As a result, the image of the Krichever map (2.10) is the same as before. The motion of the point $W_T$ in the Grassmannian is easily described as follows. Let $\bar{\Gamma}(U_1, \mathcal{O})$ be the closure in $\mathcal{H}$ of the span of the holomorphic functions on $U_1$. Then
\[ W_T = g_{10}^{-1}(z, t)\bar{\Gamma}(U_1, \mathcal{O}) = \frac{g_{10}(z, 0)}{g_{10}(z, t)}W_0. \] (3.2)
Notice that $W_T$ does not depend on the choice for the trivialization of $L_t$ over $U_1$.

We apply the construction of proposition 3.7 to the section $\sigma = (1, f_1)$ along all the vector fields represented by $\alpha_k = g_{10}^{-1} \partial_k g_{10}$ getting a cocycle condition of the form
\[ H^{(k,j+1)} := g_{10}^{-1} \partial_{k+j+1} f_1 = \partial_k H^{(k,j)} + \alpha_k H^{(k,j)} \]
where $\partial_k = \partial/\partial t_k$ and the tildes have been dropped. It is clear that $H^{(k,j)} \in W_T$ (for all $k, j \in \mathbb{N}$) and, by construction, $\{H^{(k)} := H^{(k,1)} : k \in \mathbb{N}\}$ is a basis of $W_T$. To get this last identification one chooses the coordinates $t_1, \cdots, t_g$ in such a way that the $\alpha_k$ have a simple pole of order $k$ at $p$. Differentiating with respect to $t_j$ the identity $\partial_k f_1 = g_{10} H^{(k)}$ (and observing that the right hand side times $g_{10}^{-1}$ is an element of $W_T$ and can therefore be expressed as a linear combination of the basis elements $H^{(l)}$) one gets that the equation of motion (3.2) is equivalent to the set of differential equations
\[ (\partial_j + H^{(j)}) H^{(k)} = \sum_{l \in \mathbb{N}} c_{j\!k}^{\!l} H^{(l)}, \] (3.3)
found in [4]. Finally, the Faà di Bruno basis reads $\{h^{(j)} := H^{(1,j)} : j \in \mathbb{N}\}$ and (3.3) is equivalent to the conservation laws (2.7).

4 An example: elliptic curves

The simplest example is when $C$ is an elliptic curve, which we identify with its own Jacobian. Fix $p$ and $\tilde{q}$ in $C$. Then $C_{0,p}^{(1)} = C - U_0 - \{k\}$ where $k$ is the Riemann constant. Thus, $\Omega^{(1)}(w) = (\wp(w) + c)dw$ where $w$ is a uniformizing coordinate on $C$ centered at $p$, $\wp(x)$ is the Weierstrass function and $c = -\oint_a \wp(w)dw$. Since $\wp(w) = -\frac{e^2}{dw^2} \log \theta_{11}(w)$ up to a constant, it follows that
\[ \int_{\tilde{q}}^{q} \Omega^{(1)}(w) = -\frac{\theta'_{11}(w)}{\theta_{11}(w)} \]
up to a constant which we can neglect. The $b$–period of $\Omega^{(1)}$ is $2\pi i$ and we find
\[
g_{10}(z, x) = f_1^{(0)}(z, x) = \frac{\theta(z^{-1} - q_1 - x - k)}{\theta(z^{-1} - k)} \exp \left( -(q_1 + x) \frac{\theta'(z^{-1})}{\theta_{11}(z^{-1})} \right),
\]
where $z = w^{-1}$, $D_0 = q_0$ and $D_x = q_0 + x$. The Baker-Akhiezer function reads
\[
\psi(z, x) = \frac{\theta(z^{-1} - q_1 - x - k)}{\theta(z^{-1} - q_1 - k)} \exp \left( -x \frac{\theta'(z^{-1})}{\theta_{11}(z^{-1})} \right),
\]
and the Faà di Bruno generating function is
\[
h(z, x) = -\frac{\theta'(z^{-1} - q_1 - x - k)}{\theta(z^{-1} - q_1 - x - k)} - \frac{\theta'(z^{-1})}{\theta_{11}(z^{-1})}.
\]
The other elements of the Faà di Bruno basis are obtained recursively, the functions $H^{(k)}$ for $k > 1$ can be written in terms of the $\wp$–function and its derivatives and the
flows generated by $t_k$ are trivial.

Finally, we want to explain how one can recover the Jacobian of the curve $C$ from the Faà di Bruno polynomials. Since the only non–trivial flow in the KP equations (2.7) is the first, we can restrict $h$ to satisfy
\[
H^{(2)} = h^{(2)} - 2h_1 = z^2 \tag{4.1}
\]
\[
\partial_x H^{(k)} = 0 \quad \forall k \geq 3 \tag{4.2}
\]
The first condition (4.1) allows to represent the $x$–derivatives of the Laurent coefficients $h_k$ of $h$ as ordinary polynomials in the same coefficients: $h_{kx} = P_k(h_1, \ldots, h_{k+1})$, e.g.
\[
h_{1x} = -2h_2, \quad h_{2x} = -2h_3 - h_1^2, \quad h_{3x} = -2h_4 - 2h_1h_2, \quad h_{4x} = -2h_5 - 2h_1h_3 - h_2^2,
\]
and the same is valid for the Laurent coefficients of $H^{(k)}$. For $k = 3$ we have
\[
H_{-1}^{(3)} = h_3 - h_1^2, \quad H_{-2}^{(3)} = h_4 - h_1h_2, \quad H_{-3}^{(3)} = h_5 - h_1h_3.
\]
Using the condition (4.2) for $k = 3$, we infer that $h_3 = c_1 + h_1^2$ and $h_5 = c_3 + h_1h_3$, where $c_1$ and $c_3$ are the constants $c_1 = H_{-1}^{(3)}$, $c_3 = H_{-3}^{(3)}$. Thus we obtain $h_5 = h_1^2 + c_1h_1 + c_3$ which, by means of the previous relations, takes the form
\[
h_{1x}^2 = 4h_1^3 + 8c_1h_1 + 8c_3.
\]
We see immediately that this is the Weierstrass equation for the $\wp$–function after the identifications $h_1(x) = \wp(x)$, $8c_1 = -g_2$ and $8c_3 = -g_3$, so that we have an elliptic
curve with uniformizing coordinate \( x \). From our discussions it is clear that this is the Jacobian \( \mathcal{J}(C) \cong C \) of \( C \) itself, since \( x \) is the parameter for the deformations of \( \mathcal{L} \). This obviously reflects the well known result \([9, 19]\) which expresses the solution \( u = 2h_1 \) of the KdV equation as the second logarithmic derivative of the theta function of \( \mathcal{J}(C) \).

A Basic facts on deformation theory

We collect here \([12, 20]\) some notions of Kodaira–Spencer deformation theory used in the paper. Let \( C \) be a smooth algebraic curve, \( \mathcal{N} \) a line bundle over \( C \) with a non trivial holomorphic section \( s \). A deformation of the couple \( (\mathcal{N}, s) \), with parameter space a ball \( \mathcal{B} \subset \mathbb{C}^n \), is a couple \( (\mathcal{L}, \sigma) \) where

a) \( \mathcal{L} \to C \times \mathcal{B} \) is a line bundle together with an isomorphism between \( \mathcal{N} \) and \( \mathcal{L}|_{C \times \{0\}} \),

b) \( \sigma \) is a holomorphic section of \( \mathcal{L} \) such that \( \sigma_0 := \sigma|_{C \times \{0\}} = s \).

We can cover \( C \times \mathcal{B} \) with open subsets of the form \( U_j := U_j \times \mathcal{B} \) (with local coordinates \( z_j \) on \( U_j \) and \( t = (t_1, \cdots, t_n) \) on \( \mathcal{B} \)) over which \( \mathcal{L} \) trivializes with fibre coordinate \( \xi_j \in \mathbb{C} \). On the overlaps there exist transition functions \( g_{jk}(z_k, t) \) such that \( \xi_j = g_{jk}(z_k, t) \xi_k \) and satisfying the cocycle condition

\[ g_{jk}(t)g_{kl}(t) = g_{jl}(t). \]

One can assume that \( g_{jk}(t = 0) \) are the transition functions of \( \mathcal{N} \). The section \( \sigma \) of \( \mathcal{L} \) is given by local functions \( f_j(z_j, t) \) on \( U_j \) which glue as

\[ f_j(z_j, t) = g_{jk}(z_k, t)f_k(z_k, t). \]

The functions \( f_j(t = 0) \) represent the section \( s \). The infinitesimal version of the relations above at \( t = 0 \) reads

\[ g_{jk}^{-1}\partial_t g_{jk} + g_{kl}^{-1}\partial_t g_{kl} - g_{jl}^{-1}\partial_t g_{jl} = 0, \quad (A.1) \]

showing that \( g_{jk}^{-1}\partial_t g_{jk} \) is a 1–cocycle with values in \( \mathcal{O} \), and

\[ \partial_t f_j - g_{jk}\partial_t f_k = (\partial_t g_{jk}) f_k. \quad (A.2) \]

Changing the transition functions by a coboundary \((g_{jk} = g_{jk}g_k^{-1}) \) adds to the 1–cocycle above the coboundary \( g_{jk}^{-1}\partial_t g_j - g_{jk}^{-1}\partial_t g_k \). Accordingly, the isomorphism
classes of infinitesimal deformations of $\mathcal{N}$ correspond to the elements of $H^1(C, \mathcal{O})$. Of course the action of the 1–coboundaries of $\mathcal{O}$ extends to the infinitesimal deformation of $s$ by mapping $\partial_t f_j$ to $\partial_t f_j + g_j^{-1} \partial_t g_j$.

All this information can be collected in the following hypercohomology group. Consider the sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{N} \to 0,$$

where $s$· is the multiplication by $s$, and the double complex given by taking Čech cochains

$$\begin{array}{c}
C^0(C, \mathcal{O}) \xrightarrow{\delta} C^1(C, \mathcal{O}) \to \cdots \\
\downarrow s \cdot \quad \downarrow s \cdot \\
C^0(C, \mathcal{N}) \xrightarrow{\delta} C^1(C, \mathcal{N}) \to \cdots
\end{array}$$

(A.3)

where $\delta$ is the coboundary operator. Set $A^p := C^p(C, \mathcal{O}) \oplus C^{p-1}(C, \mathcal{N})$ and define the operator $\delta_s : A^p \to A^{p+1}$ by $\delta_s(u, v) = (\delta u, \delta v + (-1)^p su)$. Then $\delta_s^2 = 0$ and one defines the hypercohomology groups $H^p_s$ as the cohomology groups of $(A, \delta_s)$. By (A.1) and (A.2), $\rho := (g_j^{-1} \partial_t g_{jk}, \partial_t f_j) \in A^1$ is actually a cocycle in this hypercohomology. Zero cochains $(g_i, 0) \in A^0$ give rise to 1-coboundaries of the form $(g_i - g_j, sg_i)$ and $\rho + (g_i - g_j, sg_i)$ corresponds to an isomorphic deformation. Hence, the isomorphism classes of infinitesimal deformations of the couple $(\mathcal{N}, s)$ correspond to the elements of $H^1_s$.

Acknowledgements

We would like to thank F. Magri for discussions and B. Dubrovin for a careful reading of the manuscript.

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