Wave patterns generated by a flow of a two-component Bose-Einstein condensate with spin-orbit interaction past a localized obstacle

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Abstract – It is shown that a spin-orbit interaction leads to drastic changes in wave patterns generated by a flow of a two-component Bose-Einstein condensate (BEC) past an obstacle. The combined Rashba and Dresselhaus spin-orbit interaction affects in different ways two types of excitations —density and polarization waves— which can propagate in a two-component BEC. We show that the density and polarization “ship wave” patterns rotate in opposite directions around the axis located at the obstacle position and the angle of rotation depends on the strength of the spin-orbit interaction. This rotation is accompanied by a narrowing of the Mach cone. The influence of the spin-orbit coupling on density solitons and polarization breathers is studied numerically.

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Introduction. – Flows of BECs past obstacles play an important role in the superfluidity physics. At the initial stage of the development of the superfluidity theory such kind of problems permitted Landau to formulate his famous superfluidity criterion and the two-fluid hydrodynamics model explaining different viscosity properties of the liquid HeII in different experimental setups [1]. Physically, Landau criterion corresponds to the threshold value of the flow velocity for the generation of linear quasiparticles —phonons and rotons in HeII. In reality, the loss of superfluidity can occur at smaller flow velocity which corresponds to the threshold of generation of nonlinear excitations, e.g., vortices or vortex rings in HeII [2], and this effect has been intensely studied theoretically in the model of a weakly nonlinear Bose gas and realized experimentally in experiments with cold atoms (see, e.g., [3,4] and references therein). It was found that the critical velocity at which the superfluid behavior breaks down is about $V_c \approx 0.35c_s$, where $c_s$ denotes the value of the sound velocity in a uniform condensate located far enough from the obstacle. For the flow velocity above the sound velocity, $V > c_s$, a new channel of dissipation opens —Cherenkov radiation of Bogoliubov waves whose interference results in the formation of a specific “ship wave” pattern located, due to the properties of the Bogoliubov dispersion relation, outside the Mach cone [5–8]. Emission of vortices dominates in the interval of velocities $c_s < V < 1.43c_s$ where vortices form the so-called “vortex streets” located inside the Mach cone. For velocities $V > 1.43c_s$ the vortex streets transform into oblique dark solitons [9] which become effectively stable with respect to the decay into vortices due to the transition from the absolute instability of dark solitons to their convective instability in the reference frame related with the obstacle [10–12]. Such oblique solitons have been realized in experiments with polariton condensates [13,14].

The variety of wave patterns generated by the flow of a condensate past an obstacle becomes even richer in the case of two-component condensates in which two different kinds of motion become possible —density modes with in-phase motion of the components and polarization modes with their counter-phase motion (see, e.g., [15,16]). Correspondingly, in the linear regime there are two types of sound waves with different sound velocities $c_d$ and $c_p$ and different Mach cones. Consequently, two partially overlapping ship wave patterns are generated by the flow [16]. In the nonlinear regime, the density and polarization modes demonstrate a quite different behavior: there exist one type of nonlinear density excitations (density solitons) and a number of nonlinear polarization excitations (solitons, breathers, kinks); see, e.g., [17]. Not all of them can be generated by the flow of the two-component BEC past an
SO coupling is a mixture of Rashba and Dresselhaus variables depending on the coordinates of atomic levels coupled with laser fields. The spin-orbit interaction depends on the component $\psi_x$, assume that the flow is directed along the $x$-axis and the spin-orbit interaction depends on the configuration of atomic levels coupled with laser fields.

The model. – We shall consider the 2D geometry when all variables depend on the coordinates $r = (x, y)$ only. Such a geometry appears naturally in experiments with polariton condensates and can be realized in experiments with atomic condensates confined in pancake traps. We assume that the flow is directed along the $x$-axis and the SO coupling is a mixture of Rashba and Dresselhaus interactions which leads to the anisotropic Hamiltonian

$$H_{SO} = \frac{1}{2} \left[ (p_x - \gamma_x \sigma_y)^2 + (p_y - \gamma_y \sigma_x)^2 \right],$$

written here in standard non-dimensional units; $\sigma_y$ and $\sigma_x$ are Pauli matrices. In what follows we assume that the SO coupling is tuned in such a way that $|\gamma_x| \ll |\gamma_y|$ and, hence, the $x$-component of the SO coupling can be neglected in the main approximation. As a result, an incident flow does not “feel” this interaction. However, when the flow is disturbed by the obstacle, then the $y$-component of the flow velocity in the wave motion interacts with the gauge field that results in a drastic modification of the entire wave pattern as we shall show below.

Taking standard Gross-Pitaevskii (GP) equations for the two-component BEC, we modify them by account of the SO interaction and obtain the system ($\gamma_y \equiv \gamma$)

$$i\partial_t \psi_1 + \frac{1}{2} \Delta \psi_1 - (g_{11} |\psi_1|^2 + g_{12} |\psi_2|^2) \psi_1
- i\gamma \partial_y \psi_1 = \kappa_1 V_{\text{obs}}(r) \psi_1,$$

$$i\partial_t \psi_2 + \frac{1}{2} \Delta \psi_2 - (g_{12} |\psi_1|^2 + g_{22} |\psi_2|^2) \psi_2
- i\gamma \partial_y \psi_1 = \kappa_2 V_{\text{obs}}(r) \psi_2,$$

where $\kappa_1, \kappa_2 V_{\text{obs}}(r)$ denotes the obstacle potential which is repulsive for both components if $\kappa_1 = \kappa_2 = 1$ and it is repulsive for the $\psi_1$-component and attractive for the $\psi_2$-component if $\kappa_1 = -\kappa_2 = 1$. The nonlinear interaction constants are positive and we suppose for simplicity that $g_{11} = g_{22} \equiv g_1 > g_{12} \equiv g_2$. To avoid any confusion, it should be stressed that while linear terms describing the SO coupling can be excluded from eqs. (2) by a simple gauge transformation, this transformation does not influence the nonlinear terms only in the case $g_1 = g_2$. If $g_1 \neq g_2$, then such a transformation introduces complicated non-linear terms with spatially periodic coefficients (see appendix) thereby qualitatively changing the structure of equations, which shows that the suggested form of SO coupling is physically non-trivial. We shall consider here some of its consequences.

It is easier to study the excitations with the use of such variables in which the corresponding motions are separated. To this end we introduce a spinor representation of the field variables [20]

$$\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = \sqrt{\rho} e^{i \phi/2} \begin{pmatrix}
\cos \frac{\phi}{2} e^{-i \varphi/2} \\
\sin \frac{\phi}{2} e^{i \varphi/2}
\end{pmatrix},$$

where $\Phi$ stands for the velocity potential of the in-phase motion ($\mathbf{U} = \nabla \Phi$) related with variations of the total density $\rho = |\psi_1|^2 + |\psi_2|^2$, the angle $\theta$ is the variable describing the relative density of the two components ($\cos \theta = (\rho_1 - \rho_2)/\rho, \rho_1 = |\psi_1|^2 = \rho \cos^2(\theta/2), \rho_2 = |\psi_2|^2 = \rho \sin^2(\theta/2)$) and $\phi$ is the potential of the relative counter-phase motion ($\mathbf{v} = \nabla \phi$). In these variables the GP equations (2) take the form

$$\begin{align*}
\rho_t + \frac{1}{2} \nabla [\rho (\nabla \Phi - \cos \theta \nabla \phi)] - \gamma (\rho \sin \theta \cos \phi) \partial_y = 0,
\Phi_t + \frac{1}{2} \nabla [\rho (\nabla \Phi - \cos \theta \nabla \phi)] - \gamma (\rho \sin \theta \cos \phi) \rho = 0,
\end{align*}$$

$$\begin{align*}
\rho \Phi_t + \frac{1}{2} (\nabla (\rho \sin \theta \nabla \phi) + \rho \nabla \theta \cdot \nabla \Phi)
\end{align*}$$

where we omitted the terms with the obstacle potential.

If this system is solved, then the velocities of the BEC components are equal to

$$\begin{align*}
v_1 &= \frac{1}{2} \nabla (\Phi - \phi) = \frac{1}{2} (\mathbf{U} - \mathbf{v}),
v_2 &= \frac{1}{2} \nabla (\Phi + \phi) = \frac{1}{2} (\mathbf{U} + \mathbf{v}).
\end{align*}$$

We suppose that far enough from the obstacle the BEC components have constant densities and the incident flow is uniform and directed along the $x$-axis ($\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{V} = (V, 0)$). Their overall density is equal here to $\rho = \rho_0$ and the ratio of the densities of the components is determined by the angle $\theta_0 (\rho_1^0/\rho_2^0 = \cot^2(\theta_0/2))$. Then the state of
the BEC flow disturbed by the obstacle is described by the variables
\[ \rho = \rho_0 + \rho', \quad \Phi = -[V^2 + \rho_0(g_1 + g_2)]t + \Phi', \quad \theta = \theta_0 + \theta', \quad \phi = \phi', \]
where the primed variables correspond to small deviations from the incident flow parameters and describe the wave generated by the flow past an obstacle.

We are interested in finding the overall transformation of the wave pattern caused by the SO coupling between the BEC components. It is known that in the situations without the SO coupling the wave patterns consist of different regions separated by the Mach cone lines (see [5–9,16,18]). Therefore, the general features of the wave patterns are determined by the positions of the Mach cone lines and here we restrict ourselves to an analytical treatment of these most important wave characteristics.

To this end, we linearize the system (4) with respect to the small primed variables and neglect the dispersion effects (note that we have to keep the terms with the second-order space derivatives of the potentials \( \Phi' \) and \( \phi' \) since they correspond to the first-order derivatives of the physical variables —velocities \( v_1 \) and \( v_2 \)). This approximation corresponds to the fact that the position of the Mach cones separating the smooth part of the wave packet from its oscillating part is determined by the long-wave (dispersionless) limit of linearized equations. We shall see below that the uncertainty introduced by the dispersion effects into the definition of the Mach cone location is not essential from the practical point of view. Looking for the plane-wave solution \( \rho', \Phi', \theta', \phi' \propto \exp[i(k_xx + k_yy - \omega t)] \) of the linearized equations, we obtain in the usual way the dispersion relation
\[ \tilde{\omega}^4 - (\rho_0 g_1 k^2 + 2\gamma^2 k_y^2)\omega^2 + 2\rho_0 g_2 \gamma \sin \theta_0 k^2 k_y \tilde{\omega} \\
+ \frac{\rho_0}{4}(g_1^2 - g_2^2) \sin^2 \theta_0 k^4 - \gamma^2 (\rho_0 g_1 k^2 - \gamma^2 k_y^2) k_y^2 = 0, \]
where \( \tilde{\omega} = \omega - V k_x \), \( k^2 = k_x^2 + k_y^2 \). If \( \gamma = 0 \), then eq. (7) gives well-known expressions for the density \( c_d \) and polarization \( c_p \) sound velocities which in our notation take the form
\[ c_d^2 = \frac{\rho_0}{2} \left[ g_1 \pm \sqrt{g_1^2 \cos^2 \theta_0 + g_2^2 \sin^2 \theta_0} \right], \]
(8)
The positions of the Mach cone lines can be found in the following way. Such a line represents a stationary plane wave inclined with respect to the \( x \)-axis by the angle \( \chi_M \). Hence, along it we have \( \omega = 0 \) and \( k_y = -k \sin \chi_M \), \( k_x = k \cos \chi_M \). The substitution of these values of the parameters into eq. (7) yields the equation for \( \chi_M \) whose roots determine the positions of the Mach cone lines in the \( (x, y) \)-plane:
\[ V^4 \sin^4 \chi_M - (\rho_0 g_1 + 2\gamma^2 \cos^2 \chi_M) V^2 \sin^2 \chi_M \\
+ 2V \rho_0 g_2 \gamma \sin \theta_0 \cos \chi_M \sin \chi_M + \frac{\rho_0}{4}(g_1^2 - g_2^2) \sin^2 \theta_0 \\
- \gamma^2 (\rho_0 g_1 - \gamma^2 \cos^2 \chi_M) \cos^2 \chi_M = 0. \]
(9)

To get a general view on the structure of these roots, it is convenient to introduce the variables
\[ v = V \sin \chi_M, \quad \zeta = \gamma \cos \chi_M \]
(10)
In terms of which eq. (9) takes the form
\[ v^4 + \zeta^4 - 2v^2 \zeta^2 - g_1 (v^2 + \zeta^2) \\
+ 2g_2 \sin \theta_0 v \zeta + \frac{1}{4} (g_1^2 - g_2^2) \sin^2 \theta_0 = 0. \]
(11)
In the \((v, \zeta)\)-plane this equation determines a quartic curve. On the other hand, the \((v, \zeta)\)-points lie on the ellipse
\[ \frac{v^2}{V^2} + \frac{\zeta^2}{\gamma^2} = 1. \]
(12)

Thus, the solutions of eq. (9) are represented geometrically by the points of intersection of the curve (11) with the ellipse (12). It is important to notice that the coefficients of the curve (11) are fixed by the parameters of the condensate (the nonlinear coupling constants and the ratio of densities of the components), whereas the coefficients of the ellipse play the role of the control parameters —the flow velocity \( V \) and the SO coupling constant \( \gamma \).

We have plotted these curves in fig. 1 for two situations —small \( \gamma = 0.2 \) and large \( \gamma = 2.0 \) in the case of supersonic flow velocity \( V = 2.6 \). Since the curve (11) has several branches, we observe a complicated dependence of the intersection points on the parameters \( V \) and \( \gamma \). For \( \gamma \sim c_{d,p} \), that is in the vicinity of the parameter values when the ellipse touches one of the branches of the curve (11).

If coordinates \((v, \zeta)\) of the intersection points are known, the values of the Mach cone angles can be found as \( \chi_M = \arcsin(v/V) \). The results are illustrated in fig. 2 for the case of parameters \( \rho_0 = 2, \theta_0 = 1.2 \) and \( V = 2.6 \). In the case of non-equal densities \( \theta_0 \neq \pi/2 \) this dependence shown in fig. 2 by dots is quite complicated in the vicinity of the value \( \gamma \sim c_{d,p} \) (for example, for \( \theta_0 = 1.2 \) one has \( c_{d,p} = 1.2909 \) and \( c_{p} = 0.5776 \)). However, it greatly simplifies if \( \theta_0 = \pi/2 \) (equal densities of the components)
The relations (14) can be interpreted as the relations corresponding to the spherical (cylindrical in 2D geometry) waves propagating with the sound velocities \( c_{d,p} \) and convected by the flows with effective velocities \((V,\mp\gamma)\). Hence, the first important conclusion is that the density and polarization patterns are rotated due to SO coupling in opposite directions by the angles
\[
\alpha_r = \mp \arctan \left( \frac{\gamma}{V} \right),
\]
where the upper sign corresponds to the density wave and the lower one to the polarization wave. Besides that, the effective Mach numbers are changed to
\[
M_{d,p} = \frac{\sqrt{V^2 + \gamma^2}}{c_{d,p}},
\]
and the corresponding Mach angle \( \alpha_c \) between the Mach lines and their bisectrix is determined now by the equation
\[
\sin \alpha_c = \frac{1}{M_{d,p}},
\]
and it is different again for the density and polarization waves. The angles \( \chi_M \) introduced above are therefore given by \( \chi_M = \alpha_r \pm \alpha_c \). These expressions completely determine the unusual transformations of the linear wave patterns under the action of the SO interaction between BEC components. To extend these predictions to the nonlinear waves, we shall resort to the numerical solutions of the GP equations (2).

**Numerical simulations.** We have studied exact numerical solutions of the GP equations (2) with a twofold aim: i) to determine the obstacle potentials corresponding to the generation of the density or polarization waves in SO-coupled BEC; ii) to find out what happens with nonlinear excitations (oblique solitons and breathers) under the action of the SO interaction.

First, we consider the flow past a non-polarized obstacle with \( \kappa_1 = \kappa_2 = 1 \) and typical results are shown in fig. 3 for different values of \( \gamma \). As one can see, in all cases only density waves are generated and the entire wave pattern rotates in the direction predicted by eq. (15) with negative \( y \)-component of the effective “flow velocity” equal to \( \gamma \). The Mach cone narrows down with the growth of \( \gamma \). The absence of the polarization mode follows from the identity of the wave patterns for different components shown in fig. 3(e) and their coincidence with the wave pattern for the overall density shown in fig. 3(f).

To perform a quantitative comparison of numerical results with analytical formulae, we have plotted the dependence of the rotation angle \( \alpha_r \) (defined geometrically in fig. 3(c)) on \( \gamma \) in fig. 4(a). The numerical dependence perfectly agrees with the analytical formula (15). The position of the Mach cone cannot be defined unambiguously in numerical plots because the analytical formulas refer to the dispersionless limit of a very long wavelength but in reality the wavelength of typical excitations is finite and the transition from the region “inside” the Mach cone to
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Fig. 3: (Color online) Transverse density distributions at $t = 112$ generated by the flow with velocity $V = 2.6$ past a non-polarized obstacle with $\kappa_1 = \kappa_2 = 1$ at $\gamma = 0$ (a), $\gamma = 1$ (b), $\gamma = 2$ (c), $\gamma = 3$ (d), and $\gamma = 5$ (e), (f). The red arrows indicate the direction of the flow. The rotation angle $\alpha_r$ of the entire pattern and the angle between two oblique dark solitons $\alpha_s$ are indicated on the panel (c).

Fig. 4: (Color online) (a) Mach cone angle $\alpha_c$ and cone rotation angle $\alpha_r$ (solid red circles), as well as half of the angle between oblique dark solitons $\alpha_s$ (open blue circles) vs. spin-orbit coupling strength $\gamma$ in the case of a non-polarized obstacle with $\kappa_1 = \kappa_2 = 1$. The solid lines show the analytical prediction, while the circles show the numerical results. (b) Depth of oblique dark solitons generated in the flow past a non-polarized obstacle vs. $\gamma$. In all cases $V = 2.6$.

The region “outside” it is quite smooth. Therefore, we use a practical definition of the position of the Mach cone line as a line of maxima of the total density close to the region of the ship wave oscillations. The corresponding numerical values of $\alpha_s$ are shown in fig. 4(a) and they slightly exceed the analytically predicted Mach cone angle (17). Besides that, we have determined the angle $\alpha_s$ between the oblique soliton and the bisectrix of the Mach cone (see fig. 3(c)). Its dependence on $\gamma$ is also shown in fig. 4(a). As one can see, the angle $\alpha_s$ approaches the analytically predicted Mach cone angle with the increase of $\gamma$. This means that the oblique solitons become shallower with the increase of $\gamma$ and such a behavior of the soliton’s depth is corroborated by the numerical data plotted in fig. 4(b). It should be stressed that our choice of the SO coupling in eqs. (2) does not change the profile of the soliton contrarily to some other forms of the SO interactions discussed in the literature; see, e.g., [21]. Therefore, we can use the standard analytical theory (see, e.g., [16]) according to which the soliton depth is a function of its squared total velocity, that is, in our case, of the combination $V^2 + \gamma^2$. Consequently, for small $\gamma$, $|\gamma| \ll V$, we get the parabolic dependence of the soliton depth on $\gamma$ in agreement with the plot in fig. 4(b). Our numerical results show that oblique solitons are quite robust with respect to the influence of the SO interaction.

Finally, we have studied the dependence of the wave patterns generated by the flow past a polarized obstacle on the SO coupling constant. The results are shown in fig. 5. Although both density and polarization ship waves are generated, only the polarization oblique breather is formed now by the polarized obstacle. Since the density and polarization wave patterns have different structures and they are rotated by the SO interaction in opposite directions, the entire wave pattern becomes asymmetric. For small $\gamma$, these two types of waves overlap, which leads to considerable deformations of oblique breathers (see fig. 5(b)). For large $\gamma$, the density and polarization waves are separated due to large values of the rotation angles $\alpha_r$ that agree...
crease of the SO coupling strength is also demonstrated. Considerable narrowing of the Mach cones upon the interaction by the SO coupling (1) in opposite directions. The flows of two-component condensates past obstacles are rotationally symmetric and polarizations wave structures generated by the SO coupling and the nonlinear interaction with non-equal nonlinearity constants. The oblique breathers are only for larger values of \( \gamma \) of the order of \( \gamma \approx 4 \) the polarization and density cones are very well resolvable in the wave pattern (fig. 5(d)). Thus, the oblique breathers are more fragile structures than the oblique solitons.

**Conclusion.** — Summarizing, we predicted that density and polarizations wave structures generated by the flows of two-component condensates past obstacles are rotated by the SO coupling (1) in opposite directions. The considerable narrowing of the Mach cones upon the increase of the SO coupling strength is also demonstrated.

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**Appendix**

To exclude the SO coupling by means of a gauge transformation, we rewrite the linear part of the system (2) in spinor form

\[
\begin{align*}
\iota \partial_t \tilde{\psi}_1 + \frac{1}{2} \left[ \partial_x^2 + (\partial_y - i \gamma \sigma_z)^2 + \gamma^2 \right] \psi + \text{nonlinear terms} &= 0, \\
\psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

Equation (A.1) suggests that the SO coupling terms can be eliminated by means of the gauge transformation

\[
\Psi = \exp \left( i \gamma y \sigma_z \right) \tilde{\Psi}.
\]

Substitution into (A.1) with account of nonlinear terms yields

\[
\begin{align*}
\iota \partial_t \tilde{\psi}_1 + \frac{1}{2} \Delta \tilde{\psi}_1 + \frac{1}{2} \gamma^2 \tilde{\psi}_1 \\
- \left[ (g_1 \cos 2 \gamma y + g_2 \sin 2 \gamma y) |\tilde{\psi}_1|^2 + (g_1 \sin 2 \gamma y + g_2 \cos 2 \gamma y) |\tilde{\psi}_2^2|^2 \right. \\
+ \sin(2 \gamma y) (g_1 - g_2) \text{Im}(\tilde{\psi}_1 \tilde{\psi}_2^*) ] \psi_1 &= 0, \\
\iota \partial_t \tilde{\psi}_2 + \frac{1}{2} \Delta \tilde{\psi}_2 + \frac{1}{2} \gamma^2 \tilde{\psi}_2 \\
- \left[ (g_1 \sin 2 \gamma y + g_2 \cos 2 \gamma y) |\tilde{\psi}_1|^2 + (g_1 \cos 2 \gamma y + g_2 \sin 2 \gamma y) |\tilde{\psi}_2|^2 \right. \\
+ \sin(2 \gamma y) (g_1 - g_2) \text{Im}(\tilde{\psi}_1 \tilde{\psi}_2^*) ] \psi_2 &= 0.
\end{align*}
\]

As one can see, the nonlinear terms are invariant with respect to the transformation (A.3) if \( g_1 = g_2 \) only. Hence, the effects considered here are the result of interplay of the SO coupling and the nonlinear interaction with non-equal nonlinearity constants.

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