Strong Subgraph Connectivity of Digraphs: A Survey

Yuefang Sun\(^1\) and Gregory Gutin\(^2\)

\(^1\) School of Mathematics and Statistics, Ningbo University, Zhejiang 315211, P. R. China, sunyuefang@nbu.edu.cn
\(^2\) School of Computer Science and Mathematics, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, UK, g.gutin@rhul.ac.uk

Abstract

In this survey we overview known results on the strong subgraph \(k\)-connectivity and strong subgraph \(k\)-arc-connectivity of digraphs. After an introductory section, the paper is divided into four sections: basic results, algorithms and complexity, sharp bounds for strong subgraph \(k\)-(arc-)connectivity, minimally strong subgraph \((k,\ell)\)-(arc-) connected digraphs. This survey contains several conjectures and open problems for further study.

Keywords: Strong subgraph \(k\)-connectivity; Strong subgraph \(k\)-arc-connectivity; Digraph packing; Directed \(q\)-linkage; Directed weak \(q\)-linkage; Semicomplete digraphs; Eulerian digraphs; Symmetric digraphs; Generalized \(k\)-connectivity; Generalized \(k\)-edge-connectivity.

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1 Introduction

The generalized \(k\)-connectivity \(\kappa_k(G)\) of a graph \(G = (V,E)\) was introduced by Hager \([17]\) in 1985 \((2 \leq k \leq |V|)\). For a graph \(G = (V,E)\) and a set \(S \subseteq V\) of at least two vertices, an \(S\)-Steiner tree or, simply, an \(S\)-tree is a subgraph \(T\) of \(G\) which is a tree with \(S \subseteq V(T)\). Two \(S\)-trees \(T_1\) and \(T_2\) are said to be edge-disjoint if \(E(T_1) \cap E(T_2) = \emptyset\). Two edge-disjoint \(S\)-trees \(T_1\) and \(T_2\) are said to be internally disjoint if \(V(T_1) \cap V(T_2) = S\). The generalized local connectivity \(\kappa_S(G)\) is the maximum number of internally disjoint \(S\)-trees in \(G\). For an integer \(k\) with \(2 \leq k \leq n\), the generalized \(k\)-connectivity is defined as

\[
\kappa_k(G) = \min \{ \kappa_S(G) \mid S \subseteq V(G), |S| = k \}. 
\]

Observe that \(\kappa_2(G) = \kappa(G)\). Li, Mao and Sun \([20]\) introduced the following concept of generalized \(k\)-edge-connectivity. The generalized local edge-connectivity \(\lambda_S(G)\) is the maximum number of edge-disjoint \(S\)-trees in \(G\).
For an integer $k$ with $2 \leq k \leq n$, the \textit{generalized $k$-edge-connectivity} is defined as
\[ \lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}. \]

Observe that $\lambda_2(G) = \lambda(G)$. Generalized connectivity of graphs has become an established area in graph theory, see a monograph [19] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized $k$-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [25] observed that in the definition of $\kappa_V$, one can replace “an $S$-tree” by “a connected subgraph of $G$ containing $S$.” Therefore, Sun et al. [25] defined \textit{strong subgraph $k$-connectivity} by replacing “connected” with “strongly connected” (or, simply, “strong”) as follows. Let $D = (V, A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. A subdigraph $H$ of $D$ is called an \textit{S-strong subgraph} if $H$ is strong and $S \subseteq V(H)$. Two $S$-strong subgraphs $D_1$ and $D_2$ are said to be \textit{arc-disjoint} if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint $S$-strong subgraphs $D_1$ and $D_2$ are said to be \textit{internally disjoint} if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ be the maximum number of internally disjoint $S$-strong subgraphs in $D$. The \textit{strong subgraph $k$-connectivity} of $D$ is defined as
\[ \kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}. \]

By definition, $\kappa_k(D) = 0$ if $D$ is not strong.

As a natural counterpart of the strong subgraph $k$-connectivity, Sun and Gutin [23] introduced the concept of strong subgraph $k$-arc-connectivity. Let $D = (V(D), A(D))$ be a digraph of order $n$, $S \subseteq V$ a $k$-subset of $V(D)$ and $2 \leq k \leq n$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint $S$-strong digraphs in $D$. The \textit{strong subgraph $k$-arc-connectivity} of $D$ is defined as
\[ \lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}. \]

By definition, $\lambda_k(D) = 0$ if $D$ is not strong.

The strong subgraph $k$-(arc-)connectivity is not only a natural extension of the concept of generalized $k$-(edge-)connectivity, but also relates to important problems in graph theory. For $k = 2$, $\kappa_2(\overrightarrow{G}) = \kappa(G)$ [25] and $\lambda_2(\overrightarrow{G}) = \lambda(G)$ [23]. Hence, $\kappa_k(D)$ and $\lambda_k(D)$ could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For $k = n$, $\kappa_n(D) = \lambda_n(D)$ is the maximum number of arc-disjoint spanning strong subgraphs of $D$. Moreover, since $\kappa_S(D)$ and $\lambda_S(D)$ are the number of internally disjoint and arc-disjoint strong subgraphs containing a given set $S$, respectively, these parameters are relevant to the subdigraph packing problem, see [5–9, 13, 24].

Some basic results are introduced in Section 2. In Section 3, we sum up the results on algorithms and computational complexity for $\kappa_S(D)$, $\kappa_k(D)$, $\lambda_S(D)$ and $\lambda_k(D)$. We present many upper and lower bounds for the parameters $\kappa_k(D)$ and $\lambda_k(D)$ in Section 4. Finally, in Section 5, results on minimally strong subgraph $(k, \ell)$-(arc-)connected digraphs are surveyed.
Additional Terminology and Notation. For a digraph $D$, its reverse $D^{rev}$ is a digraph with same vertex set and such that $xy \in A(D^{rev})$ if and only if $yx \in A(D)$. A digraph $D$ is symmetric if $D^{rev} = D$. In other words, a symmetric digraph $D$ can be obtained from its underlying undirected graph $G$ by replacing each edge of $G$ with the corresponding arcs of both directions, that is, $D = \overrightarrow{G}$. A 2-cycle $xyx$ of a strong digraph $D$ is called a bridge if $D - \{xy, yx\}$ is disconnected. Thus, a bridge corresponds to a bridge in the underlying undirected graph of $D$. An orientation of a digraph $D$ is a digraph obtained from $D$ by deleting an arc in each 2-cycle of $D$. A digraph $D$ is semicomplete if for every distinct $x, y \in V(D)$ at least one of the arcs $xy, yx$ is in $D$. We refer the readers to [3,4,11] for graph theoretical notation and terminology not given here.

2 Basic Results

The following propositions can be easily verified using definitions of $\lambda_k(D)$ and $\kappa_k(D)$.

Proposition 2.1 [23, 25] Let $D$ be a digraph of order $n$, and let $k \geq 2$ be an integer. Then

$$\lambda_{k+1}(D) \leq \lambda_k(D) \text{ for every } k \leq n - 1 \quad (1)$$

$$\kappa_k(D') \leq \kappa_k(D), \lambda_k(D') \leq \lambda_k(D) \text{ where } D' \text{ is a spanning subdigraph of } D \quad (2)$$

$$\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\} \quad (3)$$

By Tillson’s decomposition theorem [30], we can determine the exact values for $\kappa_k(\overrightarrow{K}_n)$ and $\lambda_k(\overrightarrow{K}_n)$.

Proposition 2.2 [25] For $2 \leq k \leq n$, we have

$$\kappa_k(\overrightarrow{K}_n) = \begin{cases} n - 1, & \text{if } k \not\in \{4, 6\}; \\ n - 2, & \text{otherwise}. \end{cases}$$

Proposition 2.3 [23] For $2 \leq k \leq n$, we have

$$\lambda_k(\overrightarrow{K}_n) = \begin{cases} n - 1, & \text{if } k \not\in \{4, 6\}, \text{ or, } k \in \{4, 6\} \text{ and } k < n; \\ n - 2, & \text{if } k = n \in \{4, 6\}. \end{cases}$$

Proposition 2.4 [23] For every fixed $k \geq 2$, a digraph $D$ is strong if and only if $\lambda_k(D) \geq 1$.

Sun and Zhang determined the precise value for the strong subgraph $k$-arc-connectivity of a complete bipartite digraph.

Proposition 2.5 [28] For two positive integers $a$ and $b$ with $a \leq b$, we have

$$\lambda_k(\overrightarrow{K}_{a,b}) = a$$

for $2 \leq k \leq a + b$. 

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3 Algorithms and Complexity

3.1 Results for $\kappa_S(D)$ and $\kappa_k(D)$

3.1.1 General digraphs

For a fixed $k \geq 2$, it is easy to decide whether $\kappa_k(D) \geq 1$ for a digraph $D$: it holds if and only if $D$ is strong. Unfortunately, deciding whether $\kappa_S(D) \geq 2$ is already NP-complete for $S \subseteq V(D)$ with $|S| = k$, where $k \geq 2$ is a fixed integer.

The well-known Directed $q$-Linkage problem was proved to be NP-complete even for the case that $q = 2$ [16]. The problem is formulated as follows: for a fixed integer $q \geq 2$, given a digraph $D$ and a (terminal) sequence $((s_1, t_1), \ldots, (s_q, t_q))$ of distinct vertices of $D$, decide whether $D$ has $q$ vertex-disjoint paths $P_1, \ldots, P_q$, where $P_i$ starts at $s_i$ and ends at $t_i$ for all $i \in [q]$.

By using the reduction from the Directed $q$-Linkage problem, we can prove the following intractability result.

**Theorem 3.1** [25] Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let $D$ be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\kappa_S(D) \geq \ell$ is NP-complete.

In the above theorem, Sun et al. obtained the complexity result of the parameter $\kappa_S(D)$ for an arbitrary digraph $D$. For $\kappa_k(D)$, they made the following conjecture.

**Conjecture 1** [25] It is NP-complete to decide for fixed integers $k \geq 2$ and $\ell \geq 2$ and a given digraph $D$ whether $\kappa_k(D) \geq \ell$.

3.1.2 Semicomplete digraphs

Recently, Chudnovsky, Scott and Seymour [14] proved the following powerful result.

**Theorem 3.2** [14] Let $D$ be a digraph and let $q$ and $c$ be fixed positive integers. Given a partition of the vertices of $D$ into $c$ sets each inducing a semicomplete digraph and a terminal sequence $((s_1, t_1), \ldots, (s_q, t_q))$ of distinct vertices of $D$, the Directed $q$-Linkage for $D$ and $((s_1, t_1), \ldots, (s_q, t_q))$ can be solved in polynomial time.

The following nontrivial lemma can be deduced from Theorem 3.2.

**Lemma 3.3** [25] Let $k$ and $\ell$ be fixed positive integers. Let $D$ be a digraph and let $X_1, X_2, \ldots, X_\ell$ be $\ell$ vertex disjoint subsets of $V(D)$, such that $|X_i| \leq k$ for all $i \in [\ell]$. Let $X = \bigcup_{i=1}^{\ell} X_i$ and assume that for every $v \in V(D) \setminus X$ and every $w \in V(D)$, there is an arc from $v$ to $w$ or an arc from $w$ to $v$. Then we can in polynomial time decide if there exist vertex disjoint subsets $Z_1, Z_2, \ldots, Z_\ell$ of $V(D)$, such that $X_i \subseteq Z_i$ and $D[Z_i]$ is strongly connected for each $i \in [\ell]$. 

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Using Lemma 3.3, Sun, Gutin, Yeo and Zhang proved the following result for semicomplete digraphs.

**Theorem 3.4** [25] For any fixed integers \( k, \ell \geq 2 \), we can decide whether \( \kappa_k(D) \geq \ell \) for a semicomplete digraph \( D \) in polynomial time.

### 3.1.3 Eulerian digraphs

Sun and Yeo determined the complexity of the directed 2-linkage problem on Eulerian digraphs.

**Theorem 3.5** [27] The Directed 2-Linkage problem restricted to Eulerian digraphs is NP-complete.

Using Theorem 3.5, Sun and Zhang proved the following:

**Theorem 3.6** [28] Let \( k, \ell \geq 2 \) be fixed. For any Eulerian digraph \( D \) and \( S \subseteq V(D) \) with \( |S| = k \), deciding whether \( \kappa_S(D) \geq \ell \) is NP-complete.

### 3.1.4 Symmetric digraphs

Now we turn our attention to symmetric digraphs. We start with the following structural result.

**Theorem 3.7** [25] For every graph \( G \) we have \( \kappa_2(\overrightarrow{G}) = \kappa(G) \).

Theorem 3.7 immediately implies the following positive result, which follows from the fact that \( \kappa(G) \) can be computed in polynomial time.

**Corollary 3.8** [25] For a graph \( G \), \( \kappa_2(\overrightarrow{G}) \) can be computed in polynomial time.

Theorem 3.7 states that \( \kappa_k(\overrightarrow{G}) = \kappa_k(G) \) when \( k = 2 \). However when \( k \geq 3 \), then \( \kappa_k(\overrightarrow{G}) \) is not always equal to \( \kappa_k(G) \), as can be seen from \( \kappa_3(K_3) = 2 \neq 1 = \kappa_3(K_3) \). Chen, Li, Liu and Mao [12] introduced the following problem, which turned out to be NP-complete.

**CLLM Problem:** Given a tripartite graph \( G = (V,E) \) with a 3-partition \((U, V, W)\) such that \(|U| = |V| = |W| = q\), decide whether there is a partition of \( V \) into \( q \) disjoint 3-sets \( V_1, \ldots, V_q \) such that for every \( V_i = \{v_{i1}, v_{i2}, v_{i3}\} \) \( v_{i1} \in U, v_{i2} \in V, v_{i3} \in W \) and \( G[V_i] \) is connected.

**Lemma 3.9** [12] The CLLM Problem is NP-complete.

Now restricted to symmetric digraphs \( D \), for any fixed integer \( k \geq 3 \), by Lemma 3.9, the problem of deciding whether \( \kappa_S(D) \geq \ell \) (\( \ell \geq 1 \)) is NP-complete for \( S \subseteq V(D) \) with \( |S| = k \).

**Theorem 3.10** [25] For any fixed integer \( k \geq 3 \), given a symmetric digraph \( D \), a \( k \)-subset \( S \) of \( V(D) \) and an integer \( \ell \) (\( \ell \geq 1 \)), deciding whether \( \kappa_S(D) \geq \ell \), is NP-complete.
The last theorem assumes that $k$ is fixed but $\ell$ is a part of input. When $\ell$ is fixed but $k$ is a part of input, the problem is still NP-complete.

**Theorem 3.11** [28] For any fixed integer $\ell \geq 2$, given a symmetric digraph $D$, a $k$-subset $S$ of $V(D)$ and an integer $k$ ($k \geq 2$), deciding whether $\kappa_S(D) \geq \ell$, is NP-complete.

Now for the remaining case that both $k$ and $\ell$ are fixed, the problem of deciding whether $\kappa_S(D) \geq \ell$ for a symmetric digraph $D$, is polynomial-time solvable. We will start with the following technical lemma.

**Lemma 3.12** [25] Let $k, \ell \geq 2$ be fixed. Let $G$ be a graph and let $S \subseteq V(G)$ be an independent set in $G$ with $|S| = k$. For $i \in [\ell]$, let $D_i$ be any set of arcs with both end-vertices in $S$. Let a forest $F_i$ in $G$ be called $(S, D_i)$-acceptable if the digraph $\overrightarrow{F_i} + D_i$ is strong and contains $S$. In polynomial time, we can decide whether there exists edge-disjoint forests $F_1, F_2, \ldots, F_\ell$ such that $F_i$ is $(S, D_i)$-acceptable for all $i \in [\ell]$ and $V(F_i) \cap V(F_j) \subseteq S$ for all $1 \leq i < j \leq \ell$.

Now we can prove the following result by Lemma 3.12:

**Theorem 3.13** [25] Let $k, \ell \geq 2$ be fixed. For any symmetric digraph $D$ and $S \subseteq V(D)$ with $|S| = k$ we can in polynomial time decide whether $\kappa_S(D) \geq \ell$.

### 3.1.5 Open problems

The **Directed $q$-Linkage** problem is polynomial-time solvable for planar digraphs [21] and digraphs of bounded directed treewidth [18]. However, it seems that we cannot use the approach in proving Theorem 3.4 directly as the structure of minimum-size strong subgraphs in these two classes of digraphs is more complicated than in semicomplete digraphs. Certainly, we cannot exclude the possibility that computing strong subgraph $k$-connectivity in planar digraphs and/or in digraphs of bounded directed treewidth is NP-complete.

**Problem 3.14** [25] What is the complexity of deciding whether $\kappa_k(D) \geq \ell$ for fixed integers $k \geq 2$, and $\ell \geq 2$ and a given planar digraph $D$?

**Problem 3.15** [25] What is the complexity of deciding whether $\kappa_k(D) \geq \ell$ for fixed integers $k \geq 2$, and $\ell \geq 2$ and a digraph $D$ of bounded directed treewidth?

It would be interesting to identify large classes of digraphs for which the $\kappa_k(D) \geq \ell$ problem can be decided in polynomial time.
3.2 Results for $\lambda_S(D)$ and $\lambda_k(D)$

3.2.1 General digraphs

Yeo proved that it is an NP-complete problem to decide whether a 2-regular digraph has two arc-disjoint hamiltonian cycles (see, e.g., Theorem 6.6 in [8]). (A digraph is 2-regular if the out-degree and in-degree of every vertex equals 2.) Thus, the problem of deciding whether $\lambda_n(D) \geq 2$ is NP-complete, where $n$ is the order of $D$. Sun and Gutin [23] extended this result in Theorem 3.16.

Let $D$ be a digraph and let $s_1, s_2, \ldots, s_q, t_1, t_2, \ldots, t_q$ be a collection of not necessarily distinct vertices of $D$. A weak $q$-linkage from $(s_1, s_2, \ldots, s_q)$ to $(t_1, t_2, \ldots, t_q)$ is a collection of $q$ arc-disjoint paths $P_1, \ldots, P_q$ such that $P_i$ is an $(s_i, t_i)$-path for each $i \in [q]$. A digraph $D = (V, A)$ is weakly $q$-linked if it contains a weak $q$-linkage from $(s_1, s_2, \ldots, s_q)$ to $(t_1, t_2, \ldots, t_q)$ for every choice of (not necessarily distinct) vertices $s_1, \ldots, s_q, t_1, \ldots, t_q$. The DIRECTED WEAK $q$-LINKAGE problem is the following. Given a digraph $D = (V, A)$ and distinct vertices $v_1, v_2, \ldots, v_q, y_1, y_2, \ldots, y_q$, decide whether $D$ contains $q$ arc-disjoint paths $P_1, \ldots, P_q$ such that $P_i$ is an $(v_i, y_i)$-path. The problem is well-known to be NP-complete already for $q = 2$ [16]. By using the reduction from the DIRECTED WEAK $q$-LINKAGE problem, we can prove the following intractability result.

**Theorem 3.16** [23] Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let $D$ be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\lambda_S(D) \geq \ell$ is NP-complete.

3.2.2 Semicomplete digraphs, semicomplete compositions and symmetric digraphs

Bang-Jensen and Yeo [8] conjectured the following:

**Conjecture 2** For every $\lambda \geq 2$ there is a finite set $S_\lambda$ of digraphs such that $\lambda$-arc-strong semicomplete digraph $D$ contains $\lambda$ arc-disjoint spanning strong subgraphs unless $D \in S_\lambda$.

Bang-Jensen and Yeo [8] proved the conjecture for $\lambda = 2$ by showing that $|S_2| = 1$ and describing the unique digraph $S_1$ of $S_2$ of order 4. This result and Theorem 4.4 imply the following:

**Theorem 3.17** [23] For a semicomplete digraph $D$, of order $n$ and an integer $k$ such that $2 \leq k \leq n$, $\lambda_k(D) \geq 2$ if and only if $D$ is 2-arc-strong and the following does not hold: $D \subseteq S_1$ and $k = 4$.

Now consider a much larger class of digraphs called semicomplete compositions. Let $T$ be a digraph with $t$ vertices $u_1, \ldots, u_t$ and let $H_1, \ldots, H_t$ be digraphs such that $H_i$ has vertices $u_i, j_i$, $1 \leq j_i \leq n_i$. Then the composition $Q = T[H_1, \ldots, H_t]$ is a digraph with vertex set $\bigcup_{i=1}^t V(H_i) = \{u_i, j_i \mid 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$\left( \bigcup_{i=1}^t A(H_i) \right) \cup \left( \bigcup_{u_i u_p \in A(T)} \{u_i, j_i u_p, q_p \mid 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\} \right).$$
When $T$ is semicomplete, $Q$ is called a *semicomplete composition*. To see that semicomplete compositions form a much larger class of digraphs than semicomplete digraphs, note that the Hamiltonian cycle problem is polynomial time solvable for the latter but NP-complete for the former [1] (see also [5]).

The next result is a characterization of Bang-Jensen, Gutin and Yeo [5], which we reformulate using the notion of $\lambda_n(D)$. In this theorem, we will use the following notation: $K_2$ and $K_3$ denote digraphs with no arcs of order 2 and 3, respectively, $\vec{P}_2$ a directed path with two vertices, and $\overrightarrow{C}_3$ a directed cycle with three vertices.

**Theorem 3.18** Let $D$ be a semicomplete composition with $n \geq 2$ vertices. Then $\lambda_n(D) \geq 2$ if and only if $D$ is 2-arc-strong and $D$ is not isomorphic to one of the following four digraphs: $S_4$, $\overrightarrow{C}_3[K_2, K_2, K_2]$, $\overrightarrow{C}_3[\vec{P}_2, K_2, K_2]$, $\overrightarrow{C}_3[K_2, K_2, K_3]$.

It is easy to see that the following holds.

**Corollary 3.19** Let $D$ be a semicomplete composition with $n \geq 2$ vertices. In polynomial time, we can decide whether $\lambda_n(D) \geq 2$.

Now we turn our attention to symmetric digraphs. We start from characterizing symmetric digraphs $D$ with $\lambda_k(D) \geq 2$, an analog of Theorem 3.17. To prove it we need the following result of Boesch and Tindell [10] translated from the language of mixed graphs to that of digraphs.

**Theorem 3.20** A strong digraph $D$ has a strong orientation if and only if $D$ has no bridge.

Here is the characterization by Sun and Gutin.

**Theorem 3.21** [23] For a strong symmetric digraph $D$ of order $n$ and an integer $k$ such that $2 \leq k \leq n$, $\lambda_k(D) \geq 2$ if and only if $D$ has no bridge.

Theorems 3.17 and 3.21 imply the following complexity result, which we believe to be extendable from $\ell = 2$ to any natural $\ell \geq 2$.

**Corollary 3.22** [23] The problem of deciding whether $\lambda_k(D) \geq 2$ is polynomial-time solvable if $D$ is either semicomplete or symmetric digraph of order $n$ and $2 \leq k \leq n$.

Sun and Gutin gave a lower bound on $\lambda_k(D)$ for symmetric digraphs $D$.

**Theorem 3.23** [23] For every graph $G$, we have

$$\lambda_k(\vec{G}) \geq \lambda_k(G).$$

Moreover, this bound is sharp. In particular, we have $\lambda_2(\vec{G}) = \lambda_2(G)$.

Theorem 3.23 immediately implies the next result, which follows from the fact that $\lambda(G)$ can be computed in polynomial time.

**Corollary 3.24** [23] For a symmetric digraph $D$, $\lambda_2(D)$ can be computed in polynomial time.
3.2.3 Open problems

Corollaries 3.22 and 3.24 shed some light on the complexity of deciding, for fixed $k, \ell \geq 2$, whether $\lambda_k(D) \geq \ell$ for semicomplete and symmetric digraphs $D$. However, it is unclear what is the complexity above for every fixed $k, \ell \geq 2$. If Conjecture 2 is correct, then the $\lambda_k(D) \geq \ell$ problem can be solved in polynomial time for semicomplete digraphs. However, Conjecture 2 seems to be very difficult. It was proved in [25] that for fixed $k, \ell \geq 2$ the problem of deciding whether $\kappa_k(D) \geq \ell$ is polynomial-time solvable for both semicomplete and symmetric digraphs, but it appears that the approaches to prove the two results cannot be used for $\lambda_k(D)$. Some well-known results such as the fact that the hamiltonicity problem is NP-complete for undirected 3-regular graphs, indicate that the $\lambda_k(D) \geq \ell$ problem for symmetric digraphs may be NP-complete, too.

**Problem 3.25** [23] What is the complexity of deciding whether $\lambda_k(D) \geq \ell$ for fixed integers $k \geq 2$ and $\ell \geq 2$, and a semicomplete digraph $D$?

**Problem 3.26** [23] What is the complexity of deciding whether $\lambda_k(D) \geq \ell$ for fixed integers $k \geq 2$ and $\ell \geq 2$, and a symmetric digraph $D$?

It would be interesting to identify large classes of digraphs for which the $\lambda_k(D) \geq \ell$ problem can be decided in polynomial time.

3.3 Inapproximability results

The famous problem of **Steiner Tree Packing** is to find a largest collection of edge-disjoint $S$-Steiner trees in a given undirected graph $G$. Besides this classical version, researchers have also studied some variations or extensions. For example, Cheriyan and Salavatipour [13], and Sun and Yeo [27] studied the directed Steiner tree packing problem.

Sun and Zhang [28] introduced the following two types of strong subgraph packing problems in digraphs which are analogs of **Steiner Tree Packing** problem and are related to $\kappa_S(D)$ and $\lambda_S(D)$. The input of **Arc-disjoint Strong Subgraph Packing (ASSP)** consists of a digraph $D$ and a subset of vertices $S \subseteq V(D)$, and the goal is to find a largest collection of arc-disjoint $S$-strong subgraphs. Similarly, the input of **Internally-disjoint Strong Subgraph Packing (ISSP)** consists of a digraph $D$ and a subset of vertices $S \subseteq V(D)$, and the goal is to find a largest collection of internally disjoint $S$-strong subgraphs.

In the **Set Cover Packing** problem, the input consists of a bipartite graph $G = (C \cup B, E)$, and the goal is to find a largest collection of pairwise disjoint set covers of $B$, where a set cover of $B$ is a subset $S \subseteq C$ such that each vertex of $B$ has a neighbor in $S$. Feige et al. [15] proved the following inapproximability result on the **Set Cover Packing** problem.

**Theorem 3.27** [15] Unless $P=NP$, there is no $o(\log n)$-approximation algorithm for **Set Cover Packing**, where $n$ is the order of $G$.

Sun and Zhang obtained two inapproximability results on ISSP and ASSP by reductions from the **Set Cover Packing** problem.
Theorem 3.28 [28] The following assertions hold:

(i) Unless \( P=NP \), there is no \( o(\log n) \)-approximation algorithm for ISSP, even restricted to the case that \( D \) is a symmetric digraph and \( S \) is independent in \( D \), where \( n \) is the order of \( D \).

(ii) Unless \( P=NP \), there is no \( o(\log n) \)-approximation algorithm for ASSP, even restricted to the case that \( S \) is independent in \( D \), where \( n \) is the order of \( D \).

4 Bounds for Strong Subgraph \( k \)-(Arc-)Connectivity

4.1 Results for \( \kappa_k(D) \)

By Propositions 2.1 and 2.2, Sun, Gutin, Yeo and Zhang obtained a sharp lower bound and a sharp upper bound for \( \kappa_k(D) \), where \( 2 \leq k \leq n \).

Theorem 4.1 [25] Let \( 2 \leq k \leq n \). For a strong digraph \( D \) of order \( n \), we have

\[ 1 \leq \kappa_k(D) \leq n - 1. \]

Moreover, both bounds are sharp, and the upper bound holds if and only if \( D \cong \overrightarrow{K}_n \), \( 2 \leq k \leq n \) and \( k \not\in \{4, 6\} \).

Sun and Gutin gave the following sharp upper bound for \( \kappa_k(D) \) which improves (3) of Proposition 2.1.

Theorem 4.2 [23] For \( k \in \{2, \ldots, n\} \) and \( n \geq \kappa(D) + k \), we have

\[ \kappa_k(D) \leq \kappa(D). \]

Moreover, the bound is sharp.

4.2 Results for \( \lambda_k(D) \)

By Propositions 2.1 and 2.2, Sun and Gutin obtained a sharp lower bound and a sharp upper bound for \( \lambda_k(D) \), where \( 2 \leq k \leq n \).

Theorem 4.3 [23] Let \( 2 \leq k \leq n \). For a strong digraph \( D \) of order \( n \), we have

\[ 1 \leq \lambda_k(D) \leq n - 1. \]

Moreover, both bounds are sharp, and the upper bound holds if and only if \( D \cong \overrightarrow{K}_n \), where \( k \not\in \{4, 6\} \), or, \( k \in \{4, 6\} \) and \( k < n \).

They also gave the following sharp upper bound for \( \lambda_k(D) \) which improves (3) of Proposition 2.1.

Theorem 4.4 [23] For \( 2 \leq k \leq n \), we have

\[ \lambda_k(D) \leq \lambda(D). \]

Moreover, the bound is sharp.
Shiloach [22] proved the following:

**Theorem 4.5** [22] A digraph D is weakly k-linked if and only if D is k-arc-strong.

Using Shiloach’s Theorem, Sun and Gutin [23] proved the following lower bound for \( \lambda_k(D) \). Such a bound does not hold for \( \kappa_k(D) \) since it was shown in [25] using Thomassen’s result in [29] that for every \( \ell \) there are digraphs \( D \) with \( \kappa(D) = \ell \) and \( \kappa_2(D) = 1 \).

**Proposition 4.6** [23] Let \( k \leq \ell = \lambda(D) \). We have \( \lambda_k(D) \geq \lceil \ell/k \rceil \).

For a digraph \( D = (V(D), A(D)) \), the complement digraph, denoted by \( D^c \), is a digraph with vertex set \( V(D^c) = V(D) \) such that \( xy \in A(D^c) \) if and only if \( xy \notin A(D) \).

Given a graph parameter \( f(G) \), the Nordhaus-Gaddum Problem is to determine sharp bounds for (1) \( f(G) + f(G^c) \) and (2) \( f(G)f(G^c) \), and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a survey paper [2] by Aouchiche and Hansen.

By using Proposition 2.4, the following Theorem 4.7 concerning such type of a problem for the parameter \( \lambda_k \) can be obtained.

**Theorem 4.7** [23] For a digraph \( D \) with order \( n \), the following assertions hold:

(i) \( 0 \leq \lambda_k(D) + \lambda_k(D^c) \leq n - 1 \). Moreover, both bounds are sharp. In particular, the lower bound holds if and only if \( \lambda(D) = \lambda(D^c) = 0 \).

(ii) \( 0 \leq \lambda_k(D)\lambda_k(D^c) \leq (\frac{n+1}{2})^2 \). Moreover, both bounds are sharp. In particular, the lower bound holds if and only if \( \lambda(D) = 0 \) or \( \lambda(D^c) = 0 \).

5 Minimally Strong Subgraph \((k, \ell)-(Arc-)Connected Digraphs\)

5.1 Results for minimally strong subgraph \((k, \ell)\)-connected digraphs

A digraph \( D = (V(D), A(D)) \) is called minimally strong subgraph \((k, \ell)\)-connected if \( \kappa_k(D) \geq \ell \) but for any arc \( e \in A(D) \), \( \kappa_k(D - e) \leq \ell - 1 \) [23]. By the definition of \( \kappa_k(D) \) and Theorem 4.1, we know \( 2 \leq k \leq n, 1 \leq \ell \leq n - 1 \). Let \( \mathcal{F}(n, k, \ell) \) be the set of all minimally strong subgraph \((k, \ell)\)-connected digraphs with order \( n \). We define

\[
F(n, k, \ell) = \max \{|A(D)| \mid D \in \mathcal{F}(n, k, \ell)\}
\]

and

\[
f(n, k, \ell) = \min \{|A(D)| \mid D \in \mathcal{F}(n, k, \ell)\}.
\]

We further define

\[
Ex(n, k, \ell) = \{D \mid D \in \mathcal{F}(n, k, \ell), |A(D)| = F(n, k, \ell)\}
\]
and
\[\text{ex}(n, k, \ell) = \{D \mid D \in \mathcal{F}(n, k, \ell), |A(D)| = f(n, k, \ell)\}.\]

By the definition of a minimally strong subgraph \((k, \ell)\)-connected digraph, we can get the following observation.

**Proposition 5.1** [23] A digraph \(D\) is minimally strong subgraph \((k, \ell)\)-connected if and only if \(\kappa_k(D) = \ell\) and \(\kappa_k(D-e) = \ell - 1\) for any arc \(e \in A(D)\).

A digraph \(D\) is minimally strong if \(D\) is strong but \(D-e\) is not for every arc \(e\) of \(D\).

**Proposition 5.2** [23] The following assertions hold:
(i) A digraph \(D\) is minimally strong subgraph \((k, 1)\)-connected if and only if \(D\) is minimally strong digraph;
(ii) For \(k \neq 4, 6\), a digraph \(D\) is minimally strong subgraph \((k, n-1)\)-connected if and only if \(D \cong \overrightarrow{K}_n\).

The following result characterizes minimally strong subgraph \((2, n-2)\)-connected digraphs.

**Theorem 5.3** [23] A digraph \(D\) is minimally strong subgraph \((2, n-2)\)-connected if and only if \(D\) is a digraph obtained from the complete digraph \(\overrightarrow{K}_n\) by deleting an arc set \(M\) such that \(\overrightarrow{K}_n[M]\) is a directed 3-cycle or a union of \([n/2]\) vertex-disjoint directed 2-cycles. In particular, we have \(f(n, 2, n-2) = n(n-1) - 2[n/2]\), \(F(n, 2, n-2) = n(n-1) - 3\).

Note that Theorem 5.3 implies that \(\text{Ex}(n, 2, n-2) = \{\overrightarrow{K}_n - M\}\) where \(M\) is an arc set such that \(\overrightarrow{K}_n[M]\) is a directed 3-cycle, and \(\text{ex}(n, 2, n-1) = \{\overrightarrow{K}_n - M\}\) where \(M\) is an arc set such that \(\overrightarrow{K}_n[M]\) is a union of \([n/2]\) vertex-disjoint directed 2-cycles.

The following result concerns a sharp lower bound for the parameter \(f(n, k, \ell)\).

**Theorem 5.4** [23] For \(2 \leq k \leq n\), we have
\[f(n, k, \ell) \geq n\ell.\]
Moreover, the following assertions hold:
(i) If \(\ell = 1\), then \(f(n, k, \ell) = n\); (ii) If \(2 \leq \ell \leq n-1\), then \(f(n, n, \ell) = n\ell\) for \(k = n \notin \{4, 6\}\); (iii) If \(n\) is even and \(\ell = n-2\), then \(f(n, 2, \ell) = n\ell\).

To prove two upper bounds on the number of arcs in a minimally strong subgraph \((k, \ell)\)-connected digraph, Sun and Gutin used the following result, see e.g. [3].

**Theorem 5.5** Every strong digraph \(D\) on \(n\) vertices has a strong spanning subgraph \(H\) with at most \(2n-2\) arcs and equality holds only if \(H\) is a symmetric digraph whose underlying undirected graph is a tree.
Proposition 5.6 [23] We have (i) \( F(n, n, \ell) \leq 2\ell(n - 1) \); (ii) For every \( 2 \leq k \leq n \), \( F(n, k, 1) = 2(n - 1) \) and \( Ex(n, k, 1) \) consists of symmetric digraphs whose underlying undirected graphs are trees.

The minimally strong subgraph \((2, n - 2)\)-connected digraphs was characterized in Theorem 5.3. As a simple consequence of the characterization, we can determine the values of \( f(n, 2, n - 2) \) and \( F(n, 2, n - 2) \). It would be interesting to determine \( f(n, k, n - 2) \) and \( F(n, k, n - 2) \) for every value of \( k \geq 3 \) since obtaining characterizations of all \((k, n - 2)\)-connected digraphs for \( k \geq 3 \) seems a very difficult problem.

Problem 5.7 [23] Determine \( f(n, k, n - 2) \) and \( F(n, k, n - 2) \) for every value of \( k \geq 3 \). It would also be interesting to find a sharp upper bound for \( F(n, k, \ell) \) for all \( k \geq 2 \) and \( \ell \geq 2 \).

Problem 5.8 [23] Find a sharp upper bound for \( F(n, k, \ell) \) for all \( k \geq 2 \) and \( \ell \geq 2 \).

5.2 Results for minimally strong subgraph \((k, \ell)\)-arc-connected digraphs

A digraph \( D = (V(D), A(D)) \) is called minimally strong subgraph \((k, \ell)\)-arc-connected if \( \lambda_k(D) \geq \ell \) but for any arc \( e \in A(D) \), \( \lambda_k(D - e) \leq \ell - 1 \). By the definition of \( \lambda_k(D) \) and Theorem 4.3, we know \( 2 \leq k \leq n, 1 \leq \ell \leq n - 1 \). Let \( \mathfrak{G}(n, k, \ell) \) be the set of all minimally strong subgraph \((k, \ell)\)-arc-connected digraphs with order \( n \). We define

\[
G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}
\]

and

\[
g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}.
\]

We further define

\[
Ex'(n, k, \ell) = \{D \mid D \in \mathfrak{G}(n, k, \ell), |A(D)| = G(n, k, \ell)\}
\]

and

\[
ex'(n, k, \ell) = \{D \mid D \in \mathfrak{G}(n, k, \ell), |A(D)| = g(n, k, \ell)\}.
\]

Sun and Gutin [23] gave the following characterizations.

Proposition 5.9 [23] The following assertions hold:

(i) A digraph \( D \) is minimally strong subgraph \((k, 1)\)-arc-connected if and only if \( D \) is minimally strong digraph;

(ii) Let \( 2 \leq k \leq n \). If \( k \notin \{4, 6\} \), or, \( k \in \{4, 6\} \) and \( k < n \), then a digraph \( D \) is minimally strong subgraph \((k, n - 1)\)-arc-connected if and only if \( D \cong \overrightarrow{K}_n \).

Theorem 5.10 [23] A digraph \( D \) is minimally strong subgraph \((2, n - 2)\)-arc-connected if and only if \( D \) is a digraph obtained from the complete digraph \( \overrightarrow{K}_n \) by deleting an arc set \( M \) such that \( \overrightarrow{K}_n[M] \) is a union of vertex-disjoint cycles which cover all but at most one vertex of \( \overrightarrow{K}_n \).
Sun and Jin characterized the minimally strong subgraph $(3, n - 2)$-arc-connected digraphs.

**Theorem 5.11** [26] A digraph $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $\vec{K}_n$ by deleting an arc set $M$ such that $\vec{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\vec{K}_n$.

Theorems 5.10 and 5.11 imply that the following assertions hold: (i) For $k \in \{2, 3\}$, $Ex'(n, k, n - 2) = \{\vec{K}_n - M\}$ where $M$ is an arc set such that $\vec{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but exactly one vertex of $\vec{K}_n$. (ii) For $k \in \{2, 3\}$, $ex'(n, k, n - 2) = \{\vec{K}_n - M\}$ where $M$ is an arc set such that $\vec{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\vec{K}_n$.

Sun and Jin completely determined the precise value for $g(n, k, \ell)$. Note that $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$ by Theorem 4.3 and the definition of $g(n, k, \ell)$.

**Theorem 5.12** [26] For any triple $(n, k, \ell)$ with $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ such that $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$, we have

$$g(n, k, \ell) = n\ell.$$

Some results for $G(n, k, \ell)$ were obtained as well.

**Proposition 5.13** [26] We have (i) $G(n, n, \ell) \leq 2\ell(n - 1)$; (ii) For every $k$ ($2 \leq k \leq n$), $G(n, k, 1) = 2(n - 1)$ and $Ex'(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees; (iii) $G(n, k, n - 2) = (n - 1)^2$ for $k \in \{2, 3\}$.

Note that the precise values of $g(n, k, \ell)$ for each pair of $k$ and $\ell$ and the precise values of $G(n, k, n - 2)$ for $k \in \{2, 3\}$ were determined. Hence, similar to problems 5.7 and 5.8, the following problems are also interesting.

**Problem 5.14** [26] Determine $G(n, k, n - 2)$ for every value of $k \geq 2$.

**Problem 5.15** [26] Find a sharp upper bound for $G(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.

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