Renormalization group theory of percolation on pseudo-fractal simplicial and cell complexes

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Simplicial complexes are gaining increasing scientific attention as they are generalized network structures that can represent the many-body interactions existing in complex systems ranging from the brain to high-order social networks. Simplicial complexes are formed by simplicies, such as nodes, links, triangles and so on. Cell complexes further extend these generalized network structures as they are formed by regular polytopes such as squares, pentagons etc. Pseudo-fractal simplicial and cell complexes are a major example of generalized network structures and they can be obtained by gluing 2-dimensional \( m \)-polygons \((m = 2 \text{ triangles}, m = 4 \text{ squares}, m = 5 \text{ pentagons}, \text{etc.})\) along their links according to a simple iterative rule. Here we investigate the interplay between the topology of pseudo-fractal simplicial and cell complexes and their dynamics by characterizing the critical properties of percolation on pseudo-fractal cell complexes. Indeed our work shows that the pseudo-fractal simplicial and cell complexes have a continuous percolation threshold at \( p_c = 0 \). When the pseudo-fractal structure is formed by polygons of the same size \( m \), the transition is characterized by an exponential suppression of the order parameter \( P_\infty \) that depends on the number of sides \( m \) of the polygons forming the pseudo-fractal cell complex, i.e., \( P_\infty \propto \exp(-\alpha/m^2) \). Here these results are also generalized to random pseudo-fractal cell complexes formed by polygons of different number of sides \( m \).

I. INTRODUCTION

Simplicial and cell complexes \([1,2]\) are generalized network structures capturing the many-body interactions existing in complex systems such as brain networks \([3-5]\), social networks \([6,8]\), and complex materials \([9,10]\). Simplicial and cell complexes are not only formed by nodes and links like networks, but they are also formed by higher dimensional simplexes and polytopes such as triangles, squares, pentagons, etc. Being formed by geometrical and topological building blocks simplicial complexes are ideal structures to study network geometry \([11,13]\). Moreover, simplicial complexes are key to investigate the role that network geometry and many-body interactions have on dynamics. Among the vast variety of dynamical processes that are starting to be investigated on simplicial complexes we mention percolation \([14,16,31,36]\), spin (Ising and Potts) models \([37-40]\), and Gaussian models \([25,26]\) in these structures. In particular in Refs. \([15,16]\) the robustness of the result obtained by Boettcher, Singh and Ziff in Ref. \([31]\) has been investigated by considering more general simplicial and cell complexes. It has been found that two-dimensional simplicial and cell complexes, i.e., simplicial and cell complexes build by gluing two-dimensional polygons along their links, can display a large variety of critical behaviors for the order parameter of link percolation. Here we extend this line of research and we characterize the link percolation transition to random pseudo-fractal simplicial and cell complexes. Pseudo-fractal simplicial complexes have been originally proposed as deterministic model for complex networks in Ref. \([41]\). Link percolation on these deterministic pseudo-fractal networks has been discussed previously in Ref. \([42]\). Here however we provide a more extensive treatment of the problem and are able to show that the critical percolation properties of the deterministic pseudo-fractal simplicial complex differs from the percolation properties of the deterministic pseudo-fractal cell complex and the random pseudo-fractal simplicial complexes. Indeed our work shows that for the deterministic pseudo-fractal simplicial complexes formed by \( m \)-polygons, the phase transition is at \( p_c = 0 \.

(RG) calculations. Using RG theory there has been very important progress in characterizing the critical properties of percolation \([14,16,31,36]\), spin (Ising and Potts) models \([37-40]\), and Gaussian models \([25,26]\) in these structures. In particular in Refs. \([15,16]\) the robustness of the result obtained by Boettcher, Singh and Ziff in Ref. \([31]\) has been investigated by considering more general simplicial and cell complexes. It has been found that two-dimensional simplicial and cell complexes, i.e., simplicial and cell complexes build by gluing two-dimensional polygons along their links, can display a large variety of critical behaviors for the order parameter of link percolation. Here we extend this line of research and we characterize the link percolation transition to random pseudo-fractal simplicial and cell complexes. Pseudo-fractal simplicial complexes have been originally proposed as deterministic model for complex networks in Ref. \([41]\). Link percolation on these deterministic pseudo-fractal networks has been discussed previously in Ref. \([42]\). Here however we provide a more extensive treatment of the problem and are able to show that the critical percolation properties of the deterministic pseudo-fractal simplicial complex differs from the percolation properties of the deterministic pseudo-fractal cell complex and the random pseudo-fractal simplicial complexes. Indeed our work shows that for the deterministic pseudo-fractal simplicial complexes formed by \( m \)-polygons, the phase transition is at \( p_c = 0 \.

\[ P_\infty \propto \exp(-\alpha/m^2) \]
and the order parameter behaves as

$$P_\infty \propto p \exp(-\alpha/p^{m-2})$$  \hspace{2cm} (1)$$

where $\alpha$ is a constant. Therefore the exponential suppression goes like $1/p$ for $m = 3$ as obtained by Ref. [12] but goes like $1/p^{m-2}$ for $m > 3$. On a side note we mention also that our derivation also captures the factor $p$ in Eq. (1) not discussed in Ref. [41]. Finally for random pseudo-fractal simplicial complexes we show that the critical behavior is dictated by the smaller value of $m$ of the polygons of the cell complex.

The paper is structured as follows: in Sec. II we describe the main properties of the random pseudo-fractal cell complexes studied in this work; in Sec. III we introduce link percolation on pseudo-fractal cell complexes, we derive the iterative equations for the linking probability defining the RG equations, and we derive the expression for the order parameter in terms of the linking probability, in Sec. IV we discuss the RG flow, in Sec. V we derive the critical behavior of the order parameter, finally in Sec. VI we provide the conclusions.

II. RANDOM PSEUDO-FRACTAL SIMPLICIAL AND CELL COMPLEXES

The pseudo-fractal simplicial complex [41] is constructed iteratively starting at iteration $n = 0$ from a single link. At each time $n \geq 1$ we attach a triangle to every link introduced at iteration $0 \leq n' < n$. This construction can be generalized by considering a random cell complex formed by regular $m$-polygons with different $m \geq 3$. We start at iteration $n = 0$ from an initial link. At each iteration $n \geq 1$ we glue a $m$-polygon to every link of the cell complex introduced at iteration $0 \leq n' < n$ with $m \geq 3$ drawn from a $q_m$ distribution. It is easy to show that at iteration $n$ the expected number of nodes $N_n$ and links $L_n$ are given by

$$N_n = 2 + \langle m \rangle - 2 \langle m \rangle^n - 1, \quad L_n = \langle m \rangle^n,$$  \hspace{2cm} (2)$$

where $\langle m \rangle = \sum_{m \geq 3} m q_m$. In the following we will refer to these generalized network structures as random cell complexes. However for $q_m = \delta_{m,3}$ the random pseudo-fractal cell complex reduces to the pseudo-fractal simplicial complex (see Fig. 1a). Moreover for $q_{m'} = \delta_{m',m}$ and $m > 3$ we obtain a deterministic cell complex formed by gluing only $m$-polygons. (see Fig. 1b for an example of a deterministic cell complex with $m = 4$). Only if the distribution $q_m$ is not a Kronecker delta, the model reduces to a genuine random cell complex (see Fig. 1b for an example of a random cell complex with $q_3 = q_4 = 1/2$).

III. LINK PERCOLATION ON PSEUDO-FRACTAL SIMPLICIAL AND CELL COMPLEXES

A. Link probability

In this paper we investigate the critical properties of link percolation on pseudo-fractal cell complexes. We assume that each link is retained with probability $p$. It follows that each link is removed with probability $q = 1-p$. In order to study link percolation on pseudo-fractal cell complexes we first derive the RG equations for the linking probability $T_n$ that the two initial nodes of the pseudo-fractal cell complex are linked at iteration $n$. At iteration $n = 0$ the two initial nodes are connected if the link between them is present, therefore $T_0 = p$. At iteration $n \geq 0$ the two initial nodes are connected by a path except if the initial link is not present and the two nodes are not connected by any path passing through any of the $m$-polygons glued to initial link at different iterations. Therefore for a deterministic pseudo-fractal cell complex with $q_m = \delta_{m',m}$ we obtain

$$T_{n+1} = 1 - (1 - p) \prod_{j=0}^{n} (1 - T_{j}^{m-1})$$  \hspace{2cm} (3)$$

with initial condition $T_0 = p$. For the random pseudo-fractal cell complexes the iterative equations determining $\{T_n\}_{n \geq n}$ needs to take into account the randomness of $m$. It is therefore immediate to show that we have

$$T_{n+1} = 1 - (1 - p) \prod_{j=0}^{n} (1 - Q(T_j)),$$  \hspace{2cm} (4)$$

where $Q(T)$ is given by

$$Q(T) = \sum_{m \geq 3} q_m T^{m-1},$$  \hspace{2cm} (5)$$

with $T_0 = p$. This recursive set of equations can be also written as

$$T_{n+1} = 1 - (1 - T_n)(1 - Q(T_n)), \quad T_0 = p.$$  \hspace{2cm} (6)$$

with the fixed point solutions only

$$T^* = 0, \quad T^* = 1.$$  \hspace{2cm} (7)$$

For any $p > 0$ the recursive equations go to the fixed point $T^* = 1$. Indeed exactly at $p = 0$ the steady state solution is $T^* = 0$. Therefore for any link probability $p > 0$ the percolation probability of an infinite network is $T^* = 1$. Indeed the RG flow described by Eq. (6) starts with $T_0 = p$ and in the limit $n \rightarrow \infty$ reaches

$$\lim_{n \rightarrow \infty} T_n = T^* = \begin{cases} 1 & \text{if } p > 0, \\ 0 & \text{if } p = 0. \end{cases}$$  \hspace{2cm} (8)$$
Therefore the (upper) percolation threshold is

\[ p_c = 0. \]

(9)

At \( p = p_c = 0 \) the percolation probability is

\[ T_c = 0. \]

(10)

**B. Generating function**

In this paragraph we derive the expression for the generating function \( T_n(x) \) and \( S_n(x, y) \) which are key to determine the properties of the link percolation in the pseudo-fractal cell complexes. The function \( T_n(x) \) is the generating function of the number of nodes in the connected component linked to both initial nodes of the considered random branching network. The function \( S_n(x, y) \) is the generating function of the sizes of the two connected components linked exclusively to one of the two initial nodes of the same network. These generating functions are defined as

\[
\hat{T}_n(x) = \sum_{\ell=0}^{\infty} t_n(\ell)x^\ell, \\
\hat{S}_n(x, y) = \sum_{\ell, \bar{\ell}} s_n(\ell, \bar{\ell})x^\ell y^{\bar{\ell}},
\]

(11)

where \( t_n(\ell) \) indicates the distribution of the number of nodes \( \ell \) connected to to the two initial nodes and \( s_n(\ell, \bar{\ell}) \) indicates the joint distribution of the number of nodes \( \ell \) connected exclusively to one initial node and the number of nodes \( \bar{\ell} \) connected exclusively to other initial node. By following the diagrammatic representation of these quantities, as described in detail in Refs. 15, 16, 81, we obtain the recursive equations for \( T_n(x) \) and \( S_n(x, y) \) given by

\[
\hat{S}_{n+1}(x, y) = (1 - p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ \sum_{r=0}^{m-2} x^r y^{m-2-r} \hat{T}_j(x) \hat{S}_j(y, y) \hat{T}_j^{m-2-r}(y) + \sum_{s=0}^{m-3} \sum_{r=0}^{s} x^r y^{s-r} \hat{T}_j(x) \hat{S}_j(y, 1) \hat{S}_j(y, 1) \hat{T}_j^{s-r}(y) \right] \right\},
\]

\[
\hat{T}_{n+1}(x) = \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ x^{m-2} \hat{T}_j^{m-1}(x) + (m - 1)x^{m-2} \hat{T}_j^{m-2}(x) \hat{S}_j(x, x) + \sum_{s=0}^{m-3} (s + 1)x^s \hat{T}_j(x) \hat{S}_j(x, 1) \hat{S}_n(1, x) \right] \right\}
\]

\[
- (1 - p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ (m - 1)x^{m-2} \hat{T}_j^{m-2}(x) \hat{S}_j(x, x) + \sum_{s=0}^{m-3} (s + 1)x^s \hat{T}_j(x) \hat{S}_j(x, 1) \hat{S}_j(1, x) \right] \right\},
\]

with initial conditions \( T_0(x) = 1 - \hat{S}_0(x, y) = p \).

We are particularly interested in the generating func-
tion $\hat{T}_n(x)$ whose derivative calculated for $x = 1$ gives the expected size of the giant component. The generating function $\hat{T}_{n+1}(x)$ depends on the generating functions $\hat{T}_j(x)$ and the functions $\hat{\Sigma}_j(x) = \hat{S}_j(x, x)$, and $\hat{S}_j(x) = \hat{S}_j(1, x)$ at iterations $0 \leq j \leq n$. From the above equations for $\hat{T}_{n+1}(x)$ and $S_{n+1}(x, y)$ we can deduce directly the set of recursive equations for $\hat{T}_{n+1}(x)$, $\hat{\Sigma}_{n+1}(x)$, and $\hat{S}_{n+1}(x)$ which read

\[
\hat{T}_{n+1}(x) = \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ x^{m-2}\hat{T}_j^{m-1}(x) + (m-1)x^{m-2}\hat{T}_j^{m-2}(x)\hat{\Sigma}_j(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i\hat{T}_j^i(x) \right) S_j^2(x) \right] \right\} - (1-p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ (m-1)x^{m-2}\hat{T}_j^{m-2}(x)\hat{\Sigma}_j(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i\hat{T}_j^i(x) \right) S_j^2(x) \right] \right\},
\]

\[
\hat{\Sigma}_{n+1}(x) = (1-p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ (m-1)x^{m-2}\hat{T}_j^{m-2}(x)\hat{\Sigma}_j(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i\hat{T}_j^i(x) \right) S_j^2(x) \right] \right\},
\]

\[
\hat{S}_{n+1}(x) = (1-p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ \sum_{i=0}^{m-2} x^i\hat{T}_j^i(x) S_j(x) \right] \right\}.
\]

(12)

These equations differ significantly from the corresponding equations valid for two-dimensional hyperbolic manifolds [13, 31] and for branched simplicial complexes [10]. In fact these equations for $\hat{T}_{n+1}(x)$, $\hat{\Sigma}_{n+1}(x)$ and $\hat{S}_{n+1}(x)$ depend on the entire RG flow of the process, i.e., their left hand side if a function of all $\hat{T}_j(x)$, $\hat{\Sigma}_j(x)$ and $\hat{S}_j(x)$ all previous iterations $j$ with $0 \leq j \leq n$. This apparent complication of the obtained equations can be removed by introducing an auxiliary function $K_{n+1}(x)$ (see for instance a similar trick used for the Gaussian model in Refs. [23, 26]). In order to show this let us rewrite the Eqs. (12) as

\[
\hat{T}_{n+1}(x) = \hat{K}_{n+1}(x) - \hat{\Sigma}_{n+1}(x)
\]

\[
\hat{K}_{n+1}(x) = \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ x^{m-2}\hat{T}_j^{m-1}(x) + (m-1)x^{m-2}\hat{T}_j^{m-2}(x)\hat{\Sigma}_j(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i\hat{T}_j^i(x) \right) S_j^2(x) \right] \right\}
\]

\[
\hat{\Sigma}_{n+1}(x) = (1-p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ (m-1)x^{m-2}\hat{T}_j^{m-2}(x)\hat{\Sigma}_j(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i\hat{T}_j^i(x) \right) S_j^2(x) \right] \right\},
\]

\[
\hat{S}_{n+1}(x) = (1-p) \prod_{j=0}^{n} \left\{ \sum_{m \geq 3} q_m \left[ \sum_{i=0}^{m-2} x^i\hat{T}_j^i(x) S_j(x) \right] \right\}.
\]

(13)

with initial conditions $\hat{T}_0(x) = p, \hat{\Sigma}_0(x) = \hat{S}_0(x) = 1 - p, \hat{K}_0(x) = 1$. This latter system of equations can be expressed by a set of iterative equations between the variables at iteration $n$ and the variable at iteration $n+1$, i.e.,
\[ \dot{T}_{n+1}(x) = \dot{K}_{n+1}(x) - \dot{\Sigma}_{n+1}(x) \]
\[ \dot{K}_{n+1}(x) = \dot{K}(x) \left\{ \sum_{m \geq 3} q_m \left[ x^{m-2} \dot{T}^{m-1}(x) + (m-1)x^{m-2} \dot{T}^{m-2}(x) \Sigma_n(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i \dot{T}^i_n(x) \right) S_n^2(x) \right] \right\} \]
\[ \dot{\Sigma}_{n+1}(x) = \Sigma_n(x) \left\{ \sum_{m \geq 3} q_m \left[ (m-1)x^{m-2} \dot{T}^{m-2}(x) \Sigma_n(x) + \left( \sum_{i=0}^{m-3} (i+1)x^i \dot{T}^i_n(x) \right) S_n^2(x) \right] \right\}, \]
\[ \dot{S}_{n+1}(x) = S_n(x) \left\{ \sum_{m \geq 3} q_m \left[ \left( \sum_{i=0}^{m-2} x^i \dot{T}^i_n(x) \right) S_n(x) \right] \right\}. \]  

(14)

As we will see in the next section, this recursive set of equations will turn out to be particular useful for evaluating the expected size of the giant component.

C. Order parameter

The order parameter of link percolation is the fraction of nodes \(P_\infty\) that in thermodynamic limit belongs to the giant component, i.e.,

\[ P_\infty = \lim_{n \to \infty} \frac{M_n}{N_n}. \]  

where \(M_n\) is the expected size of the giant component connected to the two initial nodes of the cell complex. The value of \(M_n\) can be derived from the generating function \(\tilde{T}_n(x)\) by derivation, i.e.,

\[ M_n = \frac{d\tilde{T}_n(x)}{dx} \bigg|_{x=1}. \]  

(16)

In order to obtain \(M_n\) we rewrite Eqs. (14) in terms of the vector

\[ V_n(x) = (V_1^n(x), V_2^n(x), V_3^n(x), V_4^n(x))^\top \]
\[ = (\dot{T}_n(x), \dot{K}_n(x), \dot{\Sigma}_n(x), \dot{S}_n(x))^\top, \]  

(17)

as

\[ V_n^\ast(x) = F_n(V_n(x), x). \]  

(18)

By using this notation, we note that the derivative of \(V_n(x)\) calculated at \(x=1\) follows

\[ \frac{dV_{n+1}(x)}{dx} \bigg|_{x=1} = J_n \frac{dV_n^p(x)}{dx} \bigg|_{x=1} + \frac{\partial F_n^p}{\partial x} \bigg|_{x=1}, \]  

(19)

where \(J_n\) indicates the Jacobian matrix of the system of Eqs. (18). The initial condition of Eq. (19) is \(V_0 = 0\) obtained by taking into consideration that the initial nodes are not counted. In order to evaluate Eqs. (18) we need to provide an explicit expression of the Jacobian matrix \(J_n\) whose elements are given by

\[ [J_n]_{ij} = \frac{\partial F^i_{n+1}}{\partial V^j_n(x)} \bigg|_{x=1}. \]  

(20)

Let us we indicate with \(T_n = \tilde{T}_n(1), \Sigma_n = \Sigma(1), S_n = \dot{S}_n(1)\) and \(K_n = \dot{K}_n(1)\) that by definition satisfy \(T_n = 1 - S_n = 1 - \Sigma_n\) and \(K_n = 1\).

By direct calculation of the Jacobian \(J_n\) we notice that \(J_n\) can be expressed as a function of \(Q(T_n)\) and \(H(T_n)\) with \(Q(T)\) given by Eq. (15) and \(H(T)\) given by

\[ H(T) = \sum_{m \geq 3} q_m \sum_{i=0}^{m-2} T^i. \]  

(21)

In fact, by using the following two relations

\[ (1 - T_n) \sum_{i=0}^{m-3} i(i+1)T^{i-1}_n = 2 \sum_{i=0}^{m-3} (i+1)T^n_i - (m-1)(m-2)T^{m-3}_n, \]  

(22)

and

\[ (1 - T_n) \sum_{i=0}^{m-3} i(i+1)T^n_i = \sum_{i=0}^{m-3} T^n_i - (m-1)T^{m-2}_n, \]

and using \(T_n = 1 - S_n = 1 - \Sigma_n, K_n = 1\), a direct calculation show that \(J_n\) is given by
Similarly the inhomogeneous term can be expressed as
\[
\frac{\partial F_n}{\partial x} = \begin{pmatrix}
(m - 2)Q(T_n) + 2T_n^2[H(T_n) - Q'(T_n)] & (m - 2)Q(T_n) + 2T_n[H(T_n) - Q'(T_n)] \\
2T_nS_n[H(T_n) - Q'(T_n)] & T_nS_n[H(T_n) - Q'(T_n)]
\end{pmatrix}.
\]

Since we have now an explicit expression for both \(J_n\) and \(\frac{\partial F_n}{\partial x}\), we can numerically integrate Eq. (18) finding the number of nodes \(M_n\) in the giant component of pseudo-fractal cell complexes for any value of \(n\) (numerical precision permitting). However we also want to have some analytical predictions of the critical properties of link percolation. To this end we notice that for \(n > 0\) and \(T_n < 1\) the non-homogeneous term \(\frac{\partial F_n}{\partial x}\) is sub-leading with respect to the homogeneous one in Eq. (18). However for \(n = 0\) the homogeneous term vanishes due to the initial condition \(\dot{V}_0 = 0\) so therefore the non-homogeneous term cannot be neglected. Therefore we can express \(\dot{V}_{n+1}\) as
\[
\dot{V}_{n+1} \simeq A_n \prod_{n'=1}^{n} \delta_{n'} u_{n'},
\]
where \(\lambda_n\) and \(u_n\) are the largest eigenvalue and the corresponding left eigenvector of the Jacobian matrix \(J_n\) and

\[\mathcal{A}_n\] is given by
\[
\mathcal{A}_n = \left( \prod_{n'=2}^{n} (v_{n'} | u_{n'-1}) \right) (v_1 | V_1),
\]
with \(\dot{V}_0 = \frac{\partial F_0}{\partial x}\) and \(v_n\), indicating the right eigenvector corresponding to the largest eigenvalue of the Jacobian \(J_n\).

Using Eq. (23) we can directly calculate the largest eigenvalue \(\lambda_n\) of the Jacobian matrix \(J_n\) which is given by
\[
\lambda_n = \frac{1}{2} \left( \hat{\kappa}(T_n) + \sqrt{\hat{\Delta}(T_n)} \right),
\]
where \(\hat{\Delta}(T_n)\) and \(\hat{\kappa}(T_n)\) are given by
\[
\hat{\kappa}(T_n) = (1 - 2T_n)Q'(T_n) + 2H(T_n) + 1,
\]
\[
\hat{\Delta}(T_n) = \left[ \hat{\kappa}(T_n) \right]^2 + 8(T - 1) [H^2(T_n) + Q'(T_n)].
\]

Note that for \(T_n \to 1\) then \(\lambda_n \to \langle m \rangle\).

The right eigenvector \(v_n\) corresponding to the largest eigenvalue of \(J_n\) is given by:

\[v_n = \frac{1}{C^R} \begin{pmatrix}
K(T_n) - 4H(T_n)(1 - T_n) + \sqrt{\hat{\Delta}(T_n)}2(H(T_n) - Q'(T_n))(T_n - 1) \\
K(T_n) - 4Q'(T_n)(1 - T_n) + \sqrt{\hat{\Delta}(T_n)}
\end{pmatrix},
\]

and the corresponding left eigenvector \(u_n\) is given by:

\[u_n = \frac{1}{C^L} \begin{pmatrix}
2H^3(T_n) + 4H(T_n)Q'(T_n) - 2Q^2(T_n) + H^2(T_n) \left[ -1 - 3Q'(T_n) + 2T_nQ'(T_n) + \sqrt{\hat{\Delta}(T_n)} \right] \\
2H^2(T_n) - 2H(T_n)Q'(T_n) + Q'(T_n) \left[ 1 - Q'(T_n) + 2T_nQ'(T_n) + \sqrt{\hat{\Delta}(T_n)} \right]
\end{pmatrix}.
\]
where $C^R$ and $C^L$ are normalization constants which guarantee that the right and left eigenvectors have absolute value one. Note that the right and left eigenvectors of $v_n$ and $u_n$ satisfy by definition
\begin{equation}
(v_n | u_n) = 1.
\end{equation}
From Eqs. (24) and (16) it follows that the expected number of nodes $M_{n+1}$ in the giant component can be expressed as
\begin{equation}
M_{n+1} \simeq A_n \frac{\prod_{n'=1}^{n} \lambda_n u_n^1}{},
\end{equation}
where $u_n^1$ indicates the first element of the vector $u_n$.

In Sec. V we will use Eq. (31) to derive the critical properties of link percolation on the pseudo-fractal cell complexes.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{(Color online) The RG flow is represented by plotting $\ln(y/y_0)$ (where $y_0 = -\ln(1-p)$) versus $n$ for $p = 5 \times 10^{-5}$ with $m = 3$ (blue solid line) $m = 4$ (orange dashed line) and $m = 5$ (green dot-dashed line).}
\end{figure}

\section{IV. RG FLOW}

In this section we study the RG flow described by Eq. (6) that we rewrite here for convenience
\begin{equation}
T_{n+1} = 1 - (1 - T_n)(1 - Q(T_n)),
\end{equation}
with initial condition $T_0 = p$. By defining the auxiliary variable
\begin{equation}
y_n = -\ln(1 - T_n)
\end{equation}
the RG flow described by Eq. (32) can be written as
\begin{equation}
y_{n+1} = G(y_n) = y_n - \ln(1 - Q(1 - e^{-y_n})).
\end{equation}

For $p \ll 1$, i.e., close to $p_c = 0$ we can develop Eq. (34) close to $T = T_n = 0$, $y_c = 0$. Stopping at the first relevant term in the expansion of $y_{n+1} - y_n$ we obtain
\begin{equation}
y_{n+1} - y_n = q_{n/2} y^{n-1}_n,
\end{equation}
with initial condition $y_0 = -\ln(1-p)$. Note that in Eq. (35), $\bar{m}$ indicates the minimum value of $m$ for which $q_m > 0$. By going in the continuous limit and substituting $y_n$ with a function $y(n)$, Eq. (35) can be written as
\begin{equation}
\frac{dy}{dn} = q_{\bar{m}} y^{\bar{m} - 1}.
\end{equation}
By integrating this equation from 0 up to $n$ we get
\begin{equation}
y = y_0 \left[1 - n/n_c\right]^{-1/(\bar{m} - 2)},
\end{equation}
with
\begin{equation}
n_c = \left[\left(\bar{m} - 2\right)|\ln(1-p)|^{\bar{m} - 2} q_{\bar{m}}\right]^{-1}.
\end{equation}
In particular $y$ diverges at a finite value of $n = n_c$. In Fig. 2 we show the very good agreement between the numerically integrated value of $y_n$ and the expression given by Eq. (37).

From Eq. (37) using
\begin{equation}
1 - T_n \simeq e^{-y_n}
\end{equation}
we get the the asymptotic scaling valid for $y \ll 1$ and $p \ll 1$
\begin{equation}
1 - T_n = (1 - p)^{\theta_n}
\end{equation}
with
\begin{equation}
\theta_n = \left[1 - (\bar{m} - 2)|\ln(1-p)|^{\bar{m} - 2} q_{\bar{m}}\right]^{-1/(\bar{m} - 2)}.
\end{equation}
For $n \ll n_c$ we can made a further approximation and express $\theta_n$ as
\begin{equation}
\theta_n \simeq \exp \left[q_{\bar{m}}p^{\bar{m} - 2}n\right].
\end{equation}
Therefore for $n \ll n_c$

$$y_n = y_0 \exp \left( q \frac{m}{p} - 2n \right),$$

(43)

with $y_0 = |\ln(1 - p)|$.

We notice that although Eq. (37) is obtained in the limit $y \ll 1$ we can see from numerical integration of the RG flow that $y$ retains the structure

$$y = y_0 f(n/n_c).$$

(44)

Although the functional form of $f(n/n_c)$ obtained in the expansion for $0 < y \ll 1$ (which can be deduced from Eq. (37)) is not exact close to $n \approx n_c$, from this expansion we can deduce that $y$ diverges for a finite value of $n$ of the order of $n_c$. In correspondence of this divergence the linking probability $T_n$ jumps to $T_n = 1$ (see Fig. 3).

V. CRITICAL PROPERTIES OF THE ORDER PARAMETER

A. Critical region

We are interested in characterizing the properties of the order parameter

$$P_\infty = \lim_{n \to \infty} \frac{M_n}{N_n},$$

(45)

in the critical region, i.e., close to the percolation threshold $p_c = 0$ taking $0 < p \ll 1$. To this end we first discuss the properties of the expected number of nodes $M_n$ in the giant component when the pseudo-fractal cell complex has evolved up to iteration $n$. According to the derivation obtained in Sec. III C, using Eq. (31), $M_n$ can be approximated as

$$M_n \approx A_n^{-1} \prod_{n' = 1}^{n-1} \lambda_{n'} u_{n'}^1,$$

(46)

where $A_n$ is given by Eq. (25), which can be written also as

$$A_n = D_n \langle v_1 | \tilde{V}_1 \rangle,$$

(47)

where $D_n$ is given by

$$D_n = \prod_{n' = 2}^{n} \langle v_{n'} | u_{n'-1} \rangle$$

(48)

For $p \approx p_c$, $D_n$ is in first approximation independent of $n$ and approximately equal to one, as the right and left eigenvectors will change slowly with $n$ and by definition Eq. (30) is satisfied. Therefore Eq. (46) can be written as

$$M_n \approx \langle v_1 | \tilde{V}_1 \rangle \prod_{n' = 1}^{n-1} \lambda_{n'} u_{n'}^1,$$

(49)

B. Critical expansions

Our major goal is to study the critical behavior of the order parameter $P_\infty$ (given by Eq. (45)) depending on the scaling of the expected number of of nodes $M_n$ (whose leading behavior is given by Eq. (49)) in the pseudo-fractal simplicial complex with the number of iterations $n$.

To this end in this paragraph we will investigate the scaling of $\lambda_n$ with $n$ for $0 < p \ll 1$ and we will investigate the scaling of the other factors $\langle v_1 | \tilde{V}_1 \rangle$ and $u_1^1$ present in Eq. (49) with $p$.

The leading eigenvalues $\lambda_n$ of the Jacobian matrix $J_n$ is expressed according to Eq. (49) as a function of $H(T_n)$ and $Q(T_n)$. For $y_n = -\ln(1 - T_n) \ll 1$ we can expand both $H(T_n) = H(1 - e^{-y_n})$ and $Q(T_n) = Q(1 - e^{-y_n})$

$$Q'(1 - e^{-y_n}) = q \frac{m}{p} - 1 y_n^{m-2} + O(y_n^{m-1}),$$

$$H(1 - e^{-y_n}) = 1 + y_n + O(y_n^2).$$

(50)

where $m$ indicates the smaller value of $m$ for which $q_n > 0$. Using this expansion in the Eq. (26) for the maximum eigenvalue $\lambda_n$ of the Jacobian matrix, we get

$$\lambda_n = 2(1 + y_n) + O(y_n^2)$$

(51)

For $y_n \ll 1$ also the inhomogeneous term $\partial F_n / \partial x$ can be expanded to give

$$\frac{\partial F_n}{\partial x} \sim \begin{pmatrix} 2(3)^{p_3} y_n^2 \\ 2n \\ 2n \\ y_n \end{pmatrix},$$

(52)

For $n = 0$ where the homogeneous term vanishes due to the trivial initial condition $\tilde{V}_0 = 0$ and the inhomogeneous term has the leading behavior

$$\tilde{V}_1 = \frac{\partial F_0}{\partial x} \sim \begin{pmatrix} 2(3)^{p_3} p^2 \\ 2p \\ 2p \\ p \end{pmatrix}.$$  

(53)

Moreover the leading term of $\tilde{v}_1$ is

$$\tilde{v}_1 \simeq \frac{1}{6 - 4p/3} ((1 + p), (1 - p), (-1 + p), 2).$$

(54)

Therefore for $p \ll 1$ we have that $\langle v_1 | \tilde{V}_1 \rangle$ scales linearly with $p$. In particular

$$\langle v_1 | \tilde{V}_1 \rangle \simeq \frac{p}{3}.$$  

(55)

Finally we observe that for $n \gg 1$ we have $T_n \simeq 1$ and the right eigenvector corresponding to the largest eigenvalue scales like

$$u_n = (1, 1, 0, 0)^\top.$$  

(56)
By considering the scaling relations determined by Eq. \(55\) and Eq. \(56\) in Eq. \(49\) we obtain that for \(n \gg 1\) the fraction of nodes \(M_n\) in the giant component obeys
\[
M_n \propto p \prod_{n'=1}^{n-1} \lambda_{n'}.
\] (57)
with \(\lambda_n\) following Eq. \(51\) for \(y_n \ll 1\).

C. Critical scaling of the order parameter

In this paragraph we derive the asymptotic behavior of the order parameter \(P_\infty\) given by Eq. \(45\) close to the percolation threshold \(p_c = 0\). By approximating \(M_n\) with Eq. \(57\) the order parameter \(P_\infty\) given by Eq. \(45\) can be easily shown to obey for \(0 < p \ll 1\)
\[
P_\infty \propto p \lim_{n \to \infty} \frac{1}{N_n} \prod_{n'=1}^{n} \lambda_{n'}
\]
\[
= p \exp \left[ -\ln(m) \int_0^\infty dn (1 - \psi_n) \right]
\] (58)
where \(\psi_n\) is defined as
\[
\psi_n = \frac{\ln \lambda_n}{\ln(m)}.
\] (59)

Using for \(\lambda_n\) the expansion given by Eq. \(51\) \(\psi_n\) can be expanded to give
\[
\psi_n = \frac{\ln \lambda_n}{\ln(m)} = \frac{\ln 2}{\ln(m)} + \frac{y_n}{\ln(m)} + O(y_n^2).
\] (60)

Therefore in the continuous limit for \(n\) we get
\[
1 - \psi \simeq 1 - \frac{\ln 2}{\ln(m)} - \frac{y}{\ln m}
\] (61)
with the function \(y(n)\) given by the scaling function Eq. \(44\) and diverging for \(n = n_c\). At a value of \(n \sim n_c, T_n\) jumps to \(T_n = 1, \lambda_n = \langle m \rangle\). Consequently we have that \(1 - \psi_n\) will also have a discontinuity at \(n_c\), i.e.,
\[
1 - \psi_n = \begin{cases} 
  f_\psi(\hat{n}/n_c) & \text{for } n < n_c \\
  0 & \text{for } n > n_c.
\end{cases}
\] (62)

where \(f_\psi(x)\) is a scaling function. Using this expression in Eq. \(58\) we obtain
\[
P_\infty \propto p \exp \left[ -\ln(m) \int_0^{n_c} dn (1 - \psi_n) \right]
\]
\[
= p \exp \left[ -\ln(m) \int_0^{n_c} d\hat{n} f_\psi(\hat{n}/n_c) \right]
\] (63)

By changing the variable of integration from \(n\) to \(x = n/n_c\) we obtain
\[
P_\infty \propto p \exp \left[ -n_c \ln\langle m \rangle \int_0^1 dx f_\psi(x) \right]
\] (64)

FIG. 4: (Color online) The scaling of the order parameter \(P_n\) is shown as a function of \(p\) for the deterministic pseudo-fractal simplicial and cell complexes with \(m = 3\) (top panel), \(m = 4\) (central panel) and \(m = 5\) (bottom panel). The order parameter \(P_n\) is shown for different values of \(n = 20000, 10000, 5000, 2500\) indicated with solid (red), dashed (green) dot-dashed (orange) and dotted (blue) thick lines. The predicted scaling of the order parameter in the infinite network limit is indicated with the thin dashed (black) line.
Finally using the expression for $n_c$ given by Eq. 38, by indicating with $\alpha$ the constant
\[
\alpha = \frac{\ln(m)}{(m-2)q_m} \int_0^1 f_\psi(x) dx,
\]
we obtain
\[
P_\infty \propto p \exp \left( -\frac{\alpha}{\ln(1-p)^{m-2}} \right).
\]
Because in the critical region $p \ll 1$, it follows that $P_\infty$ follows the asymptotic scaling
\[
P_\infty \propto p \exp(-\alpha/p^{m-2}).
\]
This scaling can be validated by numerically integrating Eq. 19 and using the finite size scaling of $P_n$ defined as the fraction of nodes in the giant component of a pseudo-fractal cell complexes evolved up to iteration $n$, i.e.,
\[
P_n = \frac{M_n}{N_n}.
\]
Our numerical results shown in Fig. 4 clearly demonstrates that if $n > n_c$ (where $n_c$ is a function of $p$ defined by Eq. 38) then $P_n$ follows the asymptotic scaling defined in Eq. 67. However if $n < n_c$, then $P_n$ saturates to a constant value. This phenomenology is in perfect agreement with our theoretical understanding of the critical properties of link percolation on pseudo-fractal cell complexes.

VI. CONCLUSIONS

In this work we have studied the nature of the link percolation transition in pseudo-fractal simplicial and cell complexes. The pseudo-fractal generalized networks under study include deterministic and random cell complexes, made by gluing together $m$-polygons with the same number of sides $m$ or with random number of sides $m$ drawn from a $q_m$ distribution. All these generalized network topologies display a link percolation transition at $p_c = 0$. However the critical behavior of the order parameter depends on the topology of the generalized network structure. For deterministic pseudo-fractal simplicial complexes ($m = 3$) we confirm the results of Ref. 12 showing that the order parameter is exponentially suppressed by a term $1/p$ and we predict an additional modulation of the order parameter by a factor $p$. For deterministic pseudo-fractal cell complexes with $m > 3$ we show that the exponential suppression is more severe than for simplicial complexes and decays as $1/p^{m-2}$. Finally for random cell complexes we show that the critical behavior is dominated by the smallest value of $m$, $\bar{m}$ for which $q_m > 0$. This work shows clearly that the dynamical processes defined on simplicial complexes and their cell complex counterpart might be significantly different, emphasizing the important role that network topology and geometry have on dynamical processes.
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