Quantifying entanglement of arbitrary-dimensional multipartite pure states in terms of the singular values of coefficient matrices

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The entanglement quantification and classification of multipartite quantum states are two important research fields in quantum information. In this work, we study the entanglement of arbitrary-dimensional multipartite pure states by looking at the averaged partial entropies of various bipartite partitions of the system, namely, the so-called Manhattan distance (l1 norm) of averaged partial entropies (MAPE), and it is proved to be an entanglement measure for pure states. We connected the MAPE with the coefficient matrices, which are important tools in entanglement classification and reexpressed the MAPE for arbitrary-dimensional multipartite pure states by the nonzero singular values of the coefficient matrices. The entanglement properties of the n-qubit Dicke states, arbitrary-dimensional Greenberger-Horne-Zeilinger states, and \( D_n^l \) states are investigated in terms of the MAPE, and the relation between the rank of the coefficient matrix and the degree of entanglement is demonstrated for symmetric states by two examples.

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I. INTRODUCTION

The entanglement of quantum systems was pointed out by Einstein, Podolsky, and Rosen (EPR) [1] and Schrödinger [2]. Then the concept of entanglement was brought into the physical world with extraordinary properties and applications [3], and it plays vital roles in quantum information theoretically and experimentally. Currently, entanglement is an essential resource for quantum information, which includes quantum teleportation, quantum cryptography, quantum computation, etc. [4–6]. The study of quantum entanglement has become more and more popular with the explosive development of quantum information, two of the most important studies are the classification and the quantification of entanglement.

The main approach of entanglement classification under stochastic local operations and classical communication (SLOCC) is to find an invariant which is preserved under SLOCC, and considerable research has been conducted since the beginning of this century [7–17]. Recently, Li and Li have proposed the coefficient matrices as important tools in entanglement classification under SLOCC [18, 19]. If we have an \( n \)-qubit pure state \( |\psi\rangle \), we can always expand \( |\psi\rangle \) as \( |\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle \), where \( a_i \) are the coefficients and \( |i\rangle \) are the binary basis states. The coefficient matrices corresponding to \( |\psi\rangle \) can be constructed as

\[
C_{1−l, l+1−n}(|\psi\rangle_{1−n}) = \begin{pmatrix}
    a_{00 \ldots 01 \ldots 1} & \cdots & a_{00 \ldots 10 \ldots 0} \\
    a_{01 \ldots 00 \ldots 0} & \cdots & a_{01 \ldots 10 \ldots 0} \\
    \vdots & \ddots & \vdots \\
    a_{10 \ldots 00 \ldots 0} & \cdots & a_{10 \ldots 10 \ldots 0}
\end{pmatrix},
\]

where \( 1 \leq l \leq n−1 \). As a natural generalization, an arbitrary-dimensional multipartite pure state \( |\psi\rangle \) in the \( n \)-partite Hilbert space \( \mathcal{H}^n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \), where \( \mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n \) have the dimensions \( d_1, d_2, \ldots, d_n \), respectively, can be expanded as

\[
|\psi\rangle = \sum_{i=0}^{\prod_{i=1}^{n-1} d_i - 1} a_i |s_1 s_2 \cdots s_n\rangle,
\]

where \( a_i \) are the coefficients and \( |s_1 s_2 \cdots s_n\rangle \) are the basis states,

\[
|s_1 s_2 \cdots s_n\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle,
\]


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with \( s_k \in \{0, 1, \cdots, d_k - 1\}, k = 1, \cdots, n \). We can construct the coefficient matrices by arranging the coefficients in a lexicographic ascending order \([20]\):

\[
C_{1-l, l+1-n}(\psi)_{1-n} = \begin{pmatrix}
\begin{array}{c}
a_0 \cdots 00 \cdots 0 \\
\cdots \\
\cdots \\
\cdots \\
a_{d_1-1} \cdots d_l-1 \cdots 0 \cdots 0
\end{array}
\end{pmatrix}_{n-1-l} \begin{pmatrix}
\begin{array}{c}
\cdots \begin{array}{c}
a_0 \cdots 0 \\
\cdots \\
\cdots \\
\cdots \\
a_{d_1-1} \cdots d_l-1 \cdots 0 \cdots 0
\end{array}
\end{array}
\end{pmatrix}_{n-1-l},
\end{equation}

Each permutation of qubits (or qudits) gives a permutation \( q_1, q_2, \cdots, q_n \) of \( \{1, 2, \cdots, n\} \). So in this case, the coefficient matrices \( C_{q_1, q_2, \cdots, q_n}(\psi) \) for short, omitting the column qudits can be constructed by taking the corresponding permutation. The coefficient matrices \( C_{q_1, q_2, \cdots, q_n}(\psi) \) have been proved to be invariant under SLOCC \([18, 20]\), which provides us with an approach of entanglement classification for arbitrary-dimensional multipartite pure states.

Despite the classification of entanglement, the quantification of entanglement is also an important research area in quantum information. Much effort has been put into it in recent years \([21-29]\). However, the situation becomes much more complicated when faced with many particles. Actually, only the simplest case, where states have two particles, can be completely described by current theories. There are a variety of methods of entanglement quantification of multipartite states \([30-35]\). In a quantum system with many particles, no particle is superior to the others; thus the calculation treat all the particles equally. Our entanglement measure, defined in this context, accounts for all the particles.

In this paper, we propose an entanglement measure named the Manhattan distance of averaged partial entropies (MAPE). The connection between the MAPE and the coefficient matrices is established. By means of the MAPE, we discover many noble entanglement properties of several arbitrary-dimensional multipartite pure states. With two examples, we show that the rank of \( C_{q_1, \cdots, q_{n/2}} \) and the degree of entanglement are closely linked.

This paper is organized as follows: in Sec. \( \text{III} \) we introduce an entanglement measure named the MAPE. The mathematical connection between the MAPE and the coefficient matrices is established. We prove that the MAPE is an entanglement measure for pure states. In Sec. \( \text{III} \) we investigate entanglement properties of the \( n \)-qubit Dicke states, arbitrary-dimensional Greenberger-Horne-Zeilinger (GHZ) states, and \( D^*_j \) states in terms of the MAPE. The relation between the rank of \( C_{q_1, \cdots, q_{n/2}} \) and the MAPE, i.e., the degree of entanglement, is investigated for symmetric states using examples. In Sec. \( \text{IV} \) we give a short summary and prospects.

II. THE MAPE AND COEFFICIENT MATRICES

Different from the bipartite partial entropy or its modified versions, the averaged partial entropies (APE) take into account all the partitions for a multipartite pure state. The complete entanglement measure in terms of the APE was pointed out in Ref. \([36]\), where they named the entanglement measure multiple entropy measures (MEMS). Suppose \( q_1, q_2, \cdots, q_n \) is a permutation of \( \{1, 2, \cdots, n\} \); the MEMS for multipartite pure quantum states is defined as a vector,

\[
S = (S_1, S_2, \cdots, S_{[n/2]}),
\]

the elements of which are the APE,

\[
S_l = \left\lfloor \frac{n}{[q_l, \cdots, q_1]} \right\rfloor \frac{1}{C_l},
\]

where \( 1 \leq l \leq [n/2] \), \( E_{q_l, \cdots, q_1} = -\text{Tr}(\rho_{q_l, \cdots, q_1} \log_2 \rho_{q_l, \cdots, q_1}) \) is the reduced von Neumann entropy with the other \( n - l \) particles being traced out, and

\[
C_l = \frac{n!}{(n-l)!l!}.
\]

Our entanglement measure is defined as the Manhattan distance \( (l_1 \text{ norm}) \) of APE (MAPE), namely,

\[
M = |S_1| + |S_2| + \cdots + |S_{[n/2]}| = S_1 + S_2 + \cdots + S_{[n/2]},
\]

where we have considered that \( S_l \geq 0 \). It needs to be noted that the \( l_2 \) norm of the APE cannot be used to define a measure since it is not an entanglement monotone, the proof of which is given in the Appendix. We show that the MAPE is closely connected to the coefficient matrices; the relationship directly links entanglement quantification with entanglement classification.

Theorem 1. The MAPE of an arbitrary-dimensional multipartite pure state can be reexpressed by the nonzero singular values of the coefficient matrices, namely,

\[
\sum_{l=1}^{[n/2]} \left| \prod_{q_l, \cdots, q_1}^{n} - \sum_{i} \lambda_{q_l, \cdots, q_1}^{2} \log_2 \lambda_{q_l, \cdots, q_1}^{2} \right| ^{1/\lambda_{q_l, \cdots, q_1}},
\]

where \( \lambda_{q_l, \cdots, q_1} \) are the nonzero singular values of the coefficient matrices.
where \( \lambda_{q_i-q_i,i} \) are the nonzero singular values of \( C_{q_i-q_i}((\psi)_1 \cdots n) \).

**Proof.** The relation between all the reduced density matrices and the coefficient matrices is given by [19]

\[
\rho_{q_i-q_i}((\psi)_1 \cdots n) = C_{q_i-q_i}((\psi)_1 \cdots n)C^\dagger_{q_i-q_i}((\psi)_1 \cdots n),
\]

where \( C^\dagger_{q_i-q_i}((\psi)_1 \cdots n) \) is the conjugate transpose of \( C_{q_i-q_i}((\psi)_1 \cdots n) \).

In the following proof, under the premise of no confusion, we use \( \rho \) and \( C \) to represent the reduced density matrix \( \rho_{q_i-q_i}((\psi)_1 \cdots n) \) and the coefficient matrix \( C_{q_i-q_i}((\psi)_1 \cdots n) \), respectively, for convenience.

The singular value decomposition of \( C \) is given by

\[
C = V \Sigma U^\dagger.
\]

According to Eq. (10), the reduced density matrix can be expressed as

\[
\rho = CC^\dagger = U \Sigma V^\dagger \Sigma V U^\dagger.
\]

Since \( V \) is unitary, namely, \( V^\dagger V = I \), we have

\[
\rho = U \Sigma \Sigma U^\dagger,
\]

which represents the diagonalization of \( \rho \). The columns of \( U \) are the eigenvectors of \( \rho \). Suppose the nonzero eigenvalues of \( \rho \) are \( \Lambda_i \); then

\[
\Lambda_i = \lambda_i \Lambda_i^* = \lambda_i^2,
\]

where \( \lambda_i \) are the corresponding nonzero singular values of \( C \).

The von Neumann entropy of \( \rho \) is defined by

\[
S(\rho) = -\text{Tr}(\rho \log_2 \rho).
\]

Equation (15) can be reexpressed by the nonzero eigenvalues of \( \rho \):

\[
S(\rho) = -\sum_i \Lambda_i \log_2 \Lambda_i.
\]

Thus the von Neumann entropy of \( \rho_{q_i-q_i}((\psi)_1 \cdots n) \) can be expressed as

\[
E(\rho_{q_i-q_i}((\psi)_1 \cdots n)) = -\sum_i \lambda_i^2 \log_2 \Lambda_i,
\]

where \( \lambda_{q_i-q_i,i} \) are the nonzero singular values of \( C_{q_i-q_i}((\psi)_1 \cdots n) \). Then we have

\[
S_I = \left[ \prod_{q_i-q_i=1}^n -\sum_i \lambda_i^2 \log_2 \Lambda_i \right]^{1/2}.
\]

Therefore we get Eq. (9).

**Theorem 2.** The MAPE is an entanglement measure for pure states.

**Proof.** We first prove that the MAPE is an entanglement monotone; namely, it does not increase, on average, under local operations and classical communication (LOCC). By using LOCC, a pure state \( |\psi\rangle \) can be transformed into the state

\[
|\phi_k\rangle = \frac{L_k|\psi\rangle}{\sqrt{\text{Tr}(L_k^2|\psi\rangle\langle\psi|)}}.
\]

with a probability \( p_k \). Here \( L_k = A_1^k \otimes \cdots \otimes A_n^k \) satisfies

\[
\sum_k L_k^2 L_k = I,
\]

where \( A_1^k, \cdots, A_n^k \) are local operators on each particle, with \( A_1^k A_i^k \leq I \) (\( i = 1, \cdots, n \)), and \( p_k = \text{Tr}(L_k^2|\psi\rangle\langle\psi|) \) represents the probability of obtaining \( |\phi_k\rangle \) after LOCC [4, 7]. It can be easily verified that \( \sum p_k = 1 \) according to Eq. (20).

Noting that the von Neumann entropy does not increase on average under LOCC; thus the APE do not, on average, increase under LOCC, namely,

\[
\sum_k p_k S(|\phi_k\rangle) \leq S(|\psi\rangle),
\]

it can be verified that

\[
\sum_k p_k M(|\phi_k\rangle) = \sum_k p_k [S_1(|\phi_2\rangle) + S_2(|\phi_2\rangle) + \cdots + S_{|\rho|/2}(|\phi_2\rangle)] \leq M(|\psi\rangle).
\]

Therefore the MAPE does not increase, on average, under LOCC.

It is easy to see that the MAPE is non-negative. Next, we prove the MAPE is zero for fully separable pure states. For fully separable pure states, the ranks of all the coefficient matrices are 1 [12, 37]. Since

\[
\text{Tr}[\rho_{q_i-q_i}((\psi)_1 \cdots n)] = 1,
\]

we have

\[
\sum_i \lambda_{q_i-q_i,i}^2 = 1.
\]

In the case where \( r(C_{q_i-q_i}((\psi)_1 \cdots n)) = 1 \), there exists only one nonzero singular value 1. According to Eq. (9), \( M = 0 \).

Therefore the MAPE satisfies the requirements of an entanglement measure for pure states.

**Theorem 3.** For a genuinely entangled multipartite pure state, the MAPE is not zero.

**Proof.** It has been proved that a multipartite pure state is genuinely entangled if and only if the ranks of all the coefficient matrices are greater than 1 [19, 37], which indicates that all \( \lambda_{q_i-q_i,i} \)'s are not 1. Thus \( M \neq 0 \) for genuinely entangled pure states.

**III. APPLICATIONS**

In terms of the MAPE, we discuss the entanglement properties of the \( n \)-qubit Dicke states, arbitrary-dimensional GHZ states, and \( D^n_s \) states.

**A. The \( n \)-qubit Dicke states**

The \( n \)-qubit Dicke states are defined as

\[
|l_1, n\rangle = \left( \frac{n!}{l_1! l_0!} \right)^{-1/2} \sum_k P_k [1, \cdots, 1, 0, \cdots, 0]_{l_1, l_0}.
\]
where $|1\rangle$ represents the excitation with respect to the ground state $|0\rangle$, $l_1$ is the number of excitations $|1\rangle$, which satisfy $0 \leq l_1 \leq n$ and $l_0 = n - l_1$. $\{P_k\}$ is the set of all permutations. The MAPE of three-, six-, and nine-qubit Dicke states is shown in Fig. 1, which indicates that the states are maximally entangled when the energy levels $|0\rangle$ and $|1\rangle$ are equally occupied.

![Graph showing the MAPE of three, six, and nine-qubit Dicke states.](image)

FIG. 1. (Color online) The MAPE of three, six and nine-qubit Dicke states.

### B. The arbitrary-dimensional GHZ states

The $n$-partite and $d$-dimensional GHZ state has a simple expression;

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \frac{i}{n} \cdots \frac{i}{n} \cdots \frac{i}{n}. \quad (26)$$

It can be calculated that all the coefficient matrices have the form

$$C = \begin{pmatrix} \frac{1}{\sqrt{d}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\sqrt{d}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{d}} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{d}} \end{pmatrix}. \quad (27)$$

where the coefficient matrices are usually not square matrices. They have $d$ diagonal elements that are nonzero, and the nondiagonal elements are all zero. Thus, for an $n$-partite and $d$-dimensional GHZ state, the coefficient matrices have $d$ nonzero singular values which equal to $\frac{1}{\sqrt{d}}$. Therefore

$$S_l = -\log_2 \frac{1}{d} = \log_2 d, \quad (28)$$

which obviously leads to

$$\mathcal{M} = [n/2] \log_2 d. \quad (29)$$

### C. The $D_n^3$ states

The $D_n^3$ states are defined as

$$|l_1, l_2, n\rangle = \left(\frac{-n!}{l_1!l_2!l_0!}\right)^{1/2} \sum_k P_k \left|1, \ldots, 1, 2, \ldots, 2, 0, \ldots, 0\right\rangle, \quad (30)$$

where $|1\rangle$ and $|2\rangle$ are the excitations, $|0\rangle$ represents the ground state, and $l_0, l_1, l_2$ are the numbers of states $|0\rangle, |1\rangle, |2\rangle$, respectively, which satisfy $0 \leq l_1 + l_2 \leq n$ and $l_0 = n - l_1 - l_2$. $\{P_k\}$ is the set that contains all permutations. The MAPE for $D_n^3$ states are shown in Fig. 2. The result shows that $D_n^3$ states are maximally entangled when $l_0 = l_1 = l_2 = 3$, namely, when the energy levels are equally occupied.

![Graph showing the MAPE of $D_n^3$ states.](image)

FIG. 2. (Color online) The MAPE of $D_n^3$ states.

### D. Relation between the rank of $C_{q_1\ldots q_{\lfloor n/2 \rfloor}}$ and degree of entanglement

The relation between the ranks of the coefficient matrices and degree of entanglement is of great interest [20]. The question is demonstrated for symmetric states by two examples, namely, the eight-qubit Dicke state and $D_3^3$ states.

It has been shown that the rank of $C_{q_1\ldots q_{\lfloor n/2 \rfloor}}(|\psi\rangle_1\ldots n)$ corresponding to $n$-qubit Dicke states is $k + 1$ (when $0 \leq k \leq [n/2]$) [21]. The rank of $C_{q_1\ldots q_{\lfloor n/2 \rfloor}}(|\psi\rangle_1\ldots n)$ and $M$ of eight-qubit Dicke states are shown in Fig. 3. It can be seen that the rank of $C_{q_1\ldots q_{\lfloor n/2 \rfloor}}(|\psi\rangle_1\ldots n)$ and $M$ of eight-qubit Dicke states have the same trend, and the case where the rank of $C_{q_1\ldots q_{\lfloor n/2 \rfloor}}(|\psi\rangle_1\ldots n)$ of eight-qubit Dicke states is maximized corresponds to the maximum degree of entanglement.

Next, we study the $D_n^3$ states. Numerical results have shown that the rank of $C_{q_1\ldots q_{\lfloor n/2 \rfloor}}(|\psi\rangle_1\ldots n)$ and $M$ for $D_n^3$ are maximized simultaneously when $l_1, l_2, l_0$ are all 3. The results for $l_1$ fixed to 3 are shown in Fig. 4, which shows that the rank of
the MAPE and the coefficient matrices was established, which indicates that entanglement classification and quantification are closely linked to the number and value of the nonzero singular values of the coefficient matrices. Examples were discussed to show that the MAPE is capable of dealing with quantum pure states with arbitrary dimensions. The rank of the coefficient matrix $C_{q_{1}, \ldots, q_{n/2}}$ and the degree of entanglement for eight-qubit Dicke and $D^{q}_{3}$ states are proved to have positive correlations.

It needs to be noted that Eq. [17] provides us with a way of calculating the von Neumann entropy in terms of the coefficient matrix, which is also a useful tool in analyzing other problems in a simpler manner. For instance, by means of the coefficient matrix and the criteria shown in Refs. [30, 31], it can be easily proved that for $n$-qubit Schmidt decomposable pure states, the ranks, i.e., the number of nonzero singular values, of the coefficient matrices $C_{q_{1}}(|\psi\rangle_{1-n})$ are equal (being either 1 or 2), and in the case where the ranks $C_{q_{1}}(|\psi\rangle_{1-n})$ are 2, two nonzero singular values are one-to-one correspondent. In the meantime, it can be proved that the $(n-1)$-partite reduced states $\rho'_{q_{1}, \ldots, q_{i}}(|\psi\rangle_{1-n})$ of an $n$-qubit Schmidt decomposable state are all pure or mixed.

We expect that our work could come up with further theoretical and experimental results.

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APPENDIX

We prove the monotonicity of $S_{l}$ ($1 \leq l \leq [n/2]$) cannot guarantee the monotonicity of the $l_{2}$ norm (module) of the APE, namely,

$$M = |\tilde{S}| = \sqrt{S_{1}^{2} + S_{2}^{2} + \cdots + S_{[n/2]}^{2}}.$$  \hspace{1cm} (A.1)

A mathematical counterexample can be given. Consider a pure state with $n = 6$; without loss of generality, suppose the LOCC gives $|\psi\rangle \rightarrow p_{1}|\phi_{1}\rangle \langle \phi_{1}| + p_{2}|\phi_{2}\rangle \langle \phi_{2}|$. The monotonicity of the APE implies that

$$p_{1} S_{1}(|\phi_{1}\rangle) + p_{2} S_{1}(|\phi_{2}\rangle) \leq S_{1}(|\psi\rangle), \quad p_{1} S_{2}(|\phi_{1}\rangle) + p_{2} S_{2}(|\phi_{2}\rangle) \leq S_{2}(|\psi\rangle), \quad p_{1} S_{3}(|\phi_{1}\rangle) + p_{2} S_{3}(|\phi_{2}\rangle) \leq S_{3}(|\psi\rangle).$$  \hspace{1cm} (A.2)

Since both sides of the inequalities are non-negative, further calculation yields

$$[p_{1} S_{1}(|\phi_{1}\rangle) + p_{2} S_{1}(|\phi_{2}\rangle)]^{2} \leq S_{1}^{2}(|\psi\rangle), \quad [p_{1} S_{2}(|\phi_{1}\rangle) + p_{2} S_{2}(|\phi_{2}\rangle)]^{2} \leq S_{2}^{2}(|\psi\rangle), \quad [p_{1} S_{3}(|\phi_{1}\rangle) + p_{2} S_{3}(|\phi_{2}\rangle)]^{2} \leq S_{3}^{2}(|\psi\rangle).$$  \hspace{1cm} (A.3)

It can be calculated that

$$[p_{1} \mathcal{M}(|\phi_{1}\rangle) + p_{2} \mathcal{M}(|\phi_{2}\rangle)]^{2} = (p_{1} \sqrt{S_{1}^{2}(|\phi_{1}\rangle)} + S_{2}^{2}(|\phi_{1}\rangle) + S_{3}^{2}(|\phi_{1}\rangle) + p_{2} \sqrt{S_{1}^{2}(|\phi_{2}\rangle)} + S_{2}^{2}(|\phi_{2}\rangle) + S_{3}^{2}(|\phi_{2}\rangle)).$$
we further get
\[
\sqrt{S_1^2(\langle \phi_1 \rangle) + S_2^2(\langle \phi_1 \rangle) + S_3^2(\langle \phi_1 \rangle)} + 2p_1p_2 \sqrt{S_1^2(\langle \phi_2 \rangle) + S_2^2(\langle \phi_2 \rangle) + S_3^2(\langle \phi_2 \rangle)}.
\] (A.4)

Note that
\[
\sqrt{S_1^2(\langle \phi_1 \rangle) + S_2^2(\langle \phi_1 \rangle) + S_3^2(\langle \phi_1 \rangle)} \geq S_1(\langle \phi_1 \rangle)S_1(\langle \phi_1 \rangle) + S_2(\langle \phi_1 \rangle)S_2(\langle \phi_1 \rangle) + S_3(\langle \phi_1 \rangle)S_3(\langle \phi_1 \rangle);
\] (A.5)

we further get
\[
[p_1 \mathcal{M}(\langle \phi_1 \rangle) + p_2 \mathcal{M}(\langle \phi_2 \rangle)]^2 \geq [p_1 S_1(\langle \phi_1 \rangle) + p_2 S_1(\langle \phi_2 \rangle)]^2 + [p_1 S_2(\langle \phi_1 \rangle) + p_2 S_2(\langle \phi_2 \rangle)]^2 + [p_1 S_3(\langle \phi_1 \rangle) + p_2 S_3(\langle \phi_2 \rangle)]^2.
\] (A.6)

Recall that
\[
\mathcal{M}^2(\langle \psi \rangle) = S_1^2(\langle \psi \rangle) + S_2^2(\langle \psi \rangle) + S_3^2(\langle \psi \rangle).
\] (A.7)

Therefore, according to Eq. (A.3), the monotonicity of \( \mathcal{M} \) given in Eq. (A.1) is not guaranteed by the monotonicity of \( S_1, S_2 \) and \( S_3 \).