EULER SEQUENCE FOR COMPLETE SMOOTH $K^*$-SURFACES

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Abstract. In this note we introduce exact sequences of sheaves on a complete smooth $K^*$-surface without elliptic points. These sequences are an attempt to generalize the Euler sequence for a toric variety to complexity one surfaces. As an application we show that such a surface is rigid if and only if it is Fano.

Introduction

In [Har77, V.8.13], we are presented with an exact sequence, namely

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

which allows the author to make several computations regarding differentials. This sequence is named Euler sequence and in [CLS11] it is generalized to any smooth toric variety $X$ coming from a fan $\Sigma$ whose rays span the whole ambient lattice. The exact sequence in question is

$$0 \to \Omega_X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_{\mathbb{P}^n}(-D_\rho) \to \text{Pic}(X) \otimes \mathbb{Z} \mathcal{O}_X \to 0,$$

where $D_\rho$ is the invariant divisor associated to the ray $\rho$.

This work attempts to obtain a similar result in the case of $K^*$-surfaces. We briefly recall that a $K^*$-surface $X$ without elliptic points (see Definition 1.8) is equipped with an equivariant morphism $\pi: X \to Y$, onto a smooth projective curve $Y$, which admits two distinguished sections called the source $F^+$ and the sink $F^-$ of $X$. This allows us to form the divisor $E_S = mF - \sum_{i \in I} E_i$, where $m$ is the number of reducible fibers of $\pi$ and $\{E_i : i \in I\}$ is the set of the prime components of such fibers.

Our main theorem is the following.

Theorem 1. Let $X$ be a complete smooth $K^*$-surface without elliptic points. There is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \pi^* \Omega_Y \otimes \mathcal{O}_X(E_S) \to \Omega_X \to \mathcal{G} \to \mathcal{O}_X^{n+1} \to 0,$$

where $r = |I| - m$ and $\mathcal{G}$ is the quotient of $\bigoplus_{i \in I} \mathcal{O}_X(E_i) \oplus \mathcal{O}_X(F^+) \oplus \mathcal{O}_X(F^-)$ by the subsheaf $\bigoplus_{i,j} \mathcal{O}_X(-E_i - E_j)$, where the sum is taken over all the pairs $(i, j)$ such that $E_i \cap E_j \neq \emptyset$.

The paper is organized as follows. In Section 1 we introduce the language of polyhedral divisors detailed in [AH06]. Section 2 is devoted to the construction of two exact sequences needed to prove Theorem 1. Because of the method employed to prove them, we restrict our attention to complete smooth $K^*$-surfaces without elliptic points. Finally, in Section 3, as an application, we show that the only rigid
rational $\mathbb{K}^*$-surfaces without elliptic points are Fano. This is a partial generalization of [Ilt11, Corollary 2.8] where the case of toric surfaces is considered.

**Theorem 2.** Let $X$ be a complete smooth $\mathbb{K}^*$-surface without elliptic points. Then the following are equivalent:

1. The equality $h^1(X, T_X) = 0$ holds;
2. $X$ is toric Fano.

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1. Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $X$ an algebraic variety over $\mathbb{K}$ having an action of $T := (\mathbb{K}^*)^n$ (which is called the $n$-dimensional torus). This action is called effective if the only $t \in T$, for which $t \cdot x = x$ holds for all $x \in X$, is the identity of $T$.

**Definition.** A $T$-variety is a normal algebraic variety $X$ coming with an effective $(\mathbb{K}^*)^n$-action. The **complexity** of $X$ is the difference $\dim X - n$.

These varieties admit a polyhedral description given by K. Altmann and J. Hausen for the affine case in [AH06] and later, together with H. Suß, in [AHS08] for the non-affine case. In the following section, we briefly recall this construction as well as a definition by N. Ilten and H. Suß in [IS11] that helps to simplify the notation.

1.1. Polyhedral divisors. Let $N$ be a lattice of rank $n$, and $M = \text{Hom}(N, \mathbb{Z})$ its dual. We denote by $\langle -, - \rangle : M \times N \to \mathbb{Z}$ the perfect pairing defined by $\langle u, v \rangle := u(v)$ and by $N_\mathbb{Q} := N \otimes \mathbb{Z} \mathbb{Q}$, $M_\mathbb{Q} := M \otimes \mathbb{Z} \mathbb{Q}$ the rational vector spaces. A **polyhedron** in $N_\mathbb{Q}$ is an intersection of finitely many affine half spaces in $N_\mathbb{Q}$. If we require the supporting hyperplane of any half space to be a linear subspace, the polyhedron is called a cone. If $\sigma$ is a cone in $N_\mathbb{Q}$, its **dual cone** is defined as $\sigma^\vee := \{ u \in M_\mathbb{Q} : \langle u, v \rangle \geq 0 \text{ for all } v \in N_\mathbb{Q} \}$.

Let $\Delta \subseteq N_\mathbb{Q}$ be a polyhedron. The set $\sigma := \{ v \in N_\mathbb{Q} : tv + \Delta \subseteq \Delta, \forall t \in \mathbb{Q} \}$ is a cone called the **tailcone** of $\Delta$ and $\Delta$ is called a $\sigma$-polyhedron.

**Definition 1.1.** Let $Y$ be a normal variety and $\sigma$ a cone. A **polyhedral divisor** on $Y$ is a formal sum $D := \sum_{\Delta} \Delta_P \otimes P$, where $P$ runs over all prime divisors of $Y$ and the $\Delta_P$ are all $\sigma$-polyhedrons such that $\Delta_P = \sigma$ for all but finitely many $P$. We admit the empty set as a valid $\sigma$-polyhedron too.

Let $D := \sum \Delta_P \otimes P$ be a polyhedral divisor on $Y$, with tailcone $\sigma$. For every $u \in \sigma^\vee$ we define the evaluation $D(u) := \sum_{\Delta_P} \min_{v \in \Delta_P} \langle u, v \rangle \otimes P \in \text{WDiv}_\mathbb{Q}(\text{Loc} D)$ where $\text{Loc} D := Y \setminus (\bigcup_{\Delta_P = \emptyset} P)$ is the **locus** of $D$. 

Definition 1.2. Let $Y$ be a normal variety. A proper polyhedral divisor, also called a pp-divisor, is a polyhedral divisor $\mathcal{D}$ on $Y$, such that

(i) $\mathcal{D}(u)$ is Cartier and semiample for every $u \in \sigma^e \cap M$.
(ii) $\mathcal{D}(u)$ is big for every $u \in (\text{relint } \sigma^e) \cap M$.

Now, let $\mathcal{D}$ be a pp-divisor on a semiprojective (i.e., projective over some affine variety) variety $Y$, $\mathcal{D}$ having tailcone $\sigma \subseteq N_\mathbb{Q}$. This defines an $M$-graded algebra

$$A(\mathcal{D}) := \bigoplus_{u \in \sigma^e \cap M} \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}(u))).$$

The affine scheme $X(\mathcal{D}) := \text{Spec } A(\mathcal{D})$ comes with a natural action of $\text{Spec } \mathbb{K}[M]$. Definition 1.2 is mainly motivated by the following result [AH06, Theorem 3.1 and Theorem 3.4].

Theorem 1.3. Let $\mathcal{D}$ be a pp-divisor on a normal variety $Y$. Then $X(\mathcal{D})$ is an affine $T$-variety of complexity equal to $\dim Y$. Moreover, every affine $T$-variety arises like this.

1.2. Divisorial fans. Non-affine $T$-varieties are obtained by gluing affine $T$-varieties coming from pp-divisors in a combinatorial way as specified in Definition 1.4.

Consider two polyhedral divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ on $Y$, with tailcones $\sigma$ and $\sigma'$ respectively and such that $\Delta_P \subseteq \Delta'_P$ for every $P$. We then have an inclusion

$$\bigoplus_{u \in \sigma^e \cap M} \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}(u))) \subseteq \bigoplus_{u \in \sigma'^e \cap M} \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}'(u))),$$

which induces a morphism $X(\mathcal{D}') \rightarrow X(\mathcal{D})$. We say that $\mathcal{D}'$ is a face of $\mathcal{D}$, denoted by $\mathcal{D}' \prec \mathcal{D}$, if this morphism is an open embedding.

Definition 1.4. A divisorial fan on $Y$ is a finite set $\mathcal{S}$ of pp-divisors on $Y$ such that for every pair of divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ in $\mathcal{S}$, we have $\mathcal{D} \cap \mathcal{D}' \in \mathcal{S}$ and $\mathcal{D} \succ \mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}'$, where $\mathcal{D} \cap \mathcal{D}' := \sum (\Delta_P \cap \Delta'_P) \otimes P$.

This definition allows us to glue affine $T$-varieties via

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \rightarrow X(\mathcal{D}'),$$

thus resulting in a scheme $X(\mathcal{S})$. For the following theorem see [AHS08, Theorem 5.3 and Theorem 5.6].

Theorem 1.5. The scheme $X(\mathcal{S})$ constructed above is a $T$-variety of complexity equal to $\dim Y$. Every $T$-variety can be constructed like this.

1.3. Marked fansy divisors on curves. In this subsection $Y$ will be a smooth projective curve. For a polyhedral divisor $\mathcal{D} = \sum \Delta_P \otimes P$ on $Y$, and a point $y \in Y$, set

$$\mathcal{D}_y := \sum_{P \ni y} \Delta_P.$$

Then for a divisorial fan $\mathcal{S}$ on $Y$, we define the slice of $\mathcal{S}$ at $y$ as $\{ \mathcal{D}_y : \mathcal{D} \in \mathcal{S} \}$.

Now, the slices of a divisorial fan do not give enough information about the corresponding $T$-variety. Two divisorial fans $\mathcal{S}$ and $\mathcal{S}'$ can have the same slices, yet $X(\mathcal{S}) \neq X(\mathcal{S}')$. In [IS11], this issue is fixed for the case of complete complexity-one $T$-varieties with the following definition.
Definition 1.6. A marked fansy divisor on a curve $Y$ is a formal sum $\Xi = \sum_{P \in Y} \Xi_P \otimes P$ together with a fan $\Sigma$ and a subset $C \subseteq \Sigma$, such that

(i) $\Xi_P$ is a complete polyhedral subdivision of $N_{Q_P}$, and $\text{tail}(\Xi_P) = \Sigma$ for all $P \in Y$.

(ii) For $\sigma \in C$ of full dimension, $\sum \Delta_{\sigma}$ is a pp-divisor, where $\Delta_{\sigma}$ is the only $\sigma$-polyhedron of $\Xi_P$.

(iii) For $\sigma \in C$ of full dimension and $\tau \prec \sigma$, we have $\tau \in C$ if and only if $(\sum_P \Delta_{\sigma}) \cap \tau \neq \emptyset$.

(iv) If $\tau \prec \sigma$ and $\tau \in C$, then $\sigma \in C$.

We say that the cones in $C$ are marked.

Given a complete divisorial fan $S$ on a curve $Y$, we can define the marked fansy divisor $\Xi = \sum_S \Xi_P \otimes P$ with marks on all the tailcones of divisors $D \in S$ having complete locus. We denote it by $\Xi(S)$. The following is proved in [IS11, Proposition 1.6].

Proposition 1.7. For any marked fansy divisor $\Xi$, there exists a complete divisorial fan $S$ with $\Xi(S) = \Xi$. If two divisorial fans $S$, $S'$ satisfy $\Xi(S) = \Xi(S')$, then $X(S) = X(S')$.

1.4. $\mathbb{K}^*$-surfaces. We now look at the case of a $T$-variety of dimension two and complexity one. We call this type of variety a $\mathbb{K}^*$-surface.

Definition 1.8. Let $X$ be a $\mathbb{K}^*$-surface and let $x \in X$ be a fixed point for the torus action. We say that the fixed point $x$ is:

- **elliptic** if there is an invariant open neighborhood $U$ of $x$ such that this point lies in the closure of every orbit in $U$,
- **parabolic** if $x$ lies on a curve made entirely of fixed points,
- **hyperbolic** otherwise.

Before proceeding recall that a morphism of varieties $\pi: X \to Y$ where $X$ is a $T$-variety is called a good quotient if $\pi$ is affine, constant on the orbits and the pullback $\pi^*: \mathcal{O}_Y \to \pi_* (\mathcal{O}_X)^T$ is an isomorphism.

Proposition 1.9. Let $X$ be a complete $\mathbb{K}^*$-surface. The following statements are equivalent.

(i) There exists a morphism $X \to Y$ onto a smooth projective curve $Y$ that is a good quotient for the $\mathbb{K}^*$-action.

(ii) $X$ has no elliptic fixed points.

(iii) $X$ is given by a marked fansy divisor without marks.

Proof. We prove three implications.

(i) $\Rightarrow$ (ii). Let $x \in X$ be an elliptic fixed point. There is an open neighborhood $x \in U$ such that $x$ lies in the closure of every orbit in $U$. Therefore, there cannot be a good quotient $X \to Y$ because the open set $U$ would be mapped to a single point; a contradiction.

(ii) $\Rightarrow$ (iii). Assume that the marked fansy divisor defining $X$ has a mark. There is then an open affine chart of $X$ given by a pp-divisor $D$ with complete locus $Y$. The zero-degree component of $A(D)$ is $\Gamma(Y, \mathcal{O}) \cong \mathbb{K}$, meaning that the degrees of the generators of $A(D)$ as an algebra are either all positive (if tail $D = Q_{\geq 0}$) or all negative (if tail $D = Q_{\leq 0}$). In either case, we can take local coordinates such that the origin $x_0$ lies in the closure of every orbit, i.e. $x_0$ is an elliptic fixed point.
Let $S$ be a divisorial fan on a smooth projective curve $Y$ such that the marked fansy divisor for $X$ is $\Xi(S)$. Since there are no marks, each $D \in S$ has an affine locus, so there is a morphism $X(D) \to \text{Loc}D$ coming from the inclusion $A(D)_0 := \Gamma(\text{Loc}D, \mathcal{O}) \subseteq A(D)$. These glue together to a morphism $\pi: X \to Y$ because the completeness of $X$ implies that $\{\text{Loc}D: D \in S\}$ is an affine open covering of $Y$. Thus $\pi$ is a good quotient because $A(D)_0$ is precisely the subalgebra of invariants of $A(D)$. □

Let $X$ be a complete $K^*$-surface. There exist two invariant subsets $F^- \subseteq X$ and $F^+ \subseteq X$, called sink and source respectively, such that there is an open set $U \subseteq X$ where the closure in $X$ of every orbit in $U$ intersects both $F^-$ and $F^+$. There are finitely many orbits outside of $U \cup F^- \cup F^+$, that are called the special orbits. The source can be either an elliptic point or an irreducible curve of parabolic points; the same holds true for the sink. Every fixed point outside of $F^+ \cup F^-$ is hyperbolic.

Now, consider a complete smooth $K^*$-surface having no elliptic points. Denote by $E_1, \ldots, E_r$ the closures of the special orbits. F. Orlik and P. Wagreich construct a graph having vertex set $\{E_1, \ldots, E_r, F^+, F^-\}$ and two vertices are joined by an edge if and only if the two corresponding curves intersect. Each vertex carries a weight equal to the self-intersection number of the curve that it represents. This graph takes the following form.

The $d_{i,j}$ are all positive and satisfy that the Hirzebruch-Jung continued fraction $[d_{i,1}, d_{i,2}, \ldots, d_{i,s_i}]$ equals 0 for every $1 \leq i \leq m$. On the other hand, our surface is given by a marked fansy divisor, without marks, on a smooth projective curve $Y$.

Where the smoothness of the surface implies that $b_{i,j}a_{i,j+1} - a_{i,j}b_{i,j+1} = 1$ for every $1 \leq i \leq m, 1 \leq j < s_i$, as well as $b_{i,1} = b_{i,s_i} = 1$, as shown in [Süß10, Theorem 3.3]. It turns out, as well, that each $b_{i,j}$ with $j > 1$ equals the Hirzebruch-Jung continued fraction $[d_{i,1}, d_{i,2}, \ldots, d_{i,j-1}]$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{The graph of the $K^*$-surface}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Marked fansy divisor without marks}
\end{figure}
2. Euler sequence

2.1. Euler sequence for $\mathbb{K}^*$-surfaces. Let $X$ be a complete smooth $\mathbb{K}^*$-surface having no elliptic points given by a marked fansy divisor $\Xi$ with no marks, like the one depicted in Figure 2. Each fraction $a_{i,j}/b_{i,j}$ in the figure corresponds to a divisor $E_{i,j}$ which is the closure of a special orbit of $X$.

**Definition 2.1.** The multiplicity of the divisor $E_{i,j}$ is the non-negative integer

$$\mu(E_{i,j}) := b_{i,j} - 1.$$  

According to the definition of the divisor $E_S$ given in the introduction the equality $E_S := \sum_{i,j} \mu(E_{i,j}) \cdot E_{i,j}$ holds, where $1 \leq i \leq m$ and $1 \leq j \leq s_i$.

Let $\Omega_X$ and $\Omega_Y$ be the cotangent sheaves of $X$ and $Y$ respectively. As in Section 1.4, let $F^-$ and $F^+$ denote the source and sink of $X$. Let $Z \subseteq X$ be the set of hyperbolic fixed points of $X$. In what follows we will call $F_X$ the sheaf $O_{F^-} \oplus O_{F^+} \oplus O_Z$.

**Lemma 2.2.** Let $X$ be a complete smooth $\mathbb{K}^*$-surface without elliptic points. There is an exact sequence of $O_X$-modules

$$0 \to \pi^* \Omega_Y \otimes_{O_X} O_X(E_S) \to \Omega_X \to O_X \to F_X \to 0$$

where $\alpha$ is defined by $f dz \to \deg(z) f z$, for any homogeneous local coordinate $z$ with respect to the $\mathbb{Z}$-grading of $O_X$ induced by the $\mathbb{K}^*$-action and $\iota$ is defined by $dt \otimes f \to df t$, where $t$ is the pull-back of a local coordinate on $Y$.

**Proof.** Let $\Xi$ be a fansy divisor describing $X$. Each affine chart of $X$, or an intersection of them, is given by a polyhedron $\Delta$ on some slice of $\Xi$. In other words, it is given by the pp-divisor

$$D = \Delta \otimes p + \sum_{P=(p)} \emptyset \otimes p,$$

where $P$ is the set of points of $Y$ with non-trivial slices. We analyze each possible $\Delta$ separately.

**Case 1.** $\Delta = [a_1/b_1, a_2/b_2]$ with $b_1 b_2 \neq 0$. In this case $D$ defines an open affine subset $X_D$ of $X$ which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}} \Gamma(\text{loc}(D), O(D(u))) = S^{-1}\mathbb{K}[x^{a_2} \chi^{-b_2}, x^{-a_1} \chi^{b_1}] \cong S^{-1}\mathbb{K}[z, w] =: R,$$

where $x$ is a regular function of $\text{loc}(D)$ which has a simple zero at $p$ and is non-zero at $\text{loc}(D) - \{p\}$, while $S \subseteq \mathbb{K}[x^{a_2} \chi^{-b_2}, x^{-a_1} \chi^{b_1}]$ is the multiplicative system defined by degree zero homogeneous polynomials which do not vanish on $\text{loc}(D)$. The first equality is due to our assumption on $D$. We have an exact sequence

$$Rdz \oplus Rdw \cong \Omega_R \to R \to R/I \to 0$$

where $I \subseteq R$ is the ideal $\langle \deg(z)z, \deg(w)w \rangle$. The restriction of the quotient map $\pi: X \to Y$ to the open subset $X_D$ is defined by the inclusion $\mathbb{K}[x] \subseteq R$. Since $x = z^{\deg(w)} u^{\deg(z)} = z^{b_1} u^{b_2}$, the curve $\pi^{-1}(p) \cap X_D$ has two irreducible components which are vertical curves intersecting at the fixed point $q \in Z$ of local coordinates $z = w = 0$. Thus $R/I$ defines the skyscraper sheaf $O_q$ and we get
the first exact sequence from $F|_D \cong O_q$. The sheaf $\pi^*\Omega_Y$ is locally generated by
dx = z^{b_1-1}w^{b_2-1}(b_1wdz + b_2zdw)$, thus we have the desired isomorphism

$$\pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X((b_1-1)E_1 + (b_2-1)E_2)|_{X_D} \to \ker(\alpha)|_{X_D},$$

where for $i = 1, 2$, the divisor $E_i$ is the one associated to the fraction $a_i/b_i$, as explained in the beginning of this section.

**Case 2.** $\Delta = [a_1, \infty)$. In this case $D$ defines an open affine subset $X_D$ of $X$ which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}_{\geq 0}} \Gamma(\text{loc}(D), \mathcal{O}(D(u))) = S^{-1}\mathbb{K}[x, x^{-a_1}\chi] \cong S^{-1}\mathbb{K}[z, w] =: R,$$

where $x$ and $S$ are defined in a similar way as in the first case. Since $x = z$, the curve $\pi^{-1}(p) \cap X_D$ has one irreducible component which is a vertical curve intersecting $F^+$ at one point. Again we get an exact sequence as above and observe that now $I = \langle w \rangle$. In this case $R/I$ defines the sheaf $\mathcal{O}_P|_{X_D}$ and we get the first exact sequence from $F|_{X_D} \cong \mathcal{O}_P|_{X_D}$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = dz$, thus we have an isomorphism

$$\pi^*\Omega_Y|_{X_D} \to \ker(\alpha)|_{X_D}.$$

**Case 3.** $\Delta = (-\infty, a_2]$. This is similar to the previous case and we omit the details.

**Case 4.** $\Delta = \{a/b\}$. In this case $D$ defines an open affine subset $X_D$ of $X$ which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}} \Gamma(\text{loc}(D), \mathcal{O}(D(u))) = S^{-1}\mathbb{K}[x^k\chi^{-1}, (x^{-a}\chi^b)^{\pm 1}] \cong S^{-1}\mathbb{K}[z, w^{\pm 1}] =: R,$$

where $x$ and $S$ are defined in a similar way as in the first case and $c, d$ are integers such that $bk - la = 1$. Since $x = z^{\deg(w)}w^{\deg(z)} = z^bw^d$, the curve $\pi^{-1}(p) \cap X_D$ has an irreducible component which is a vertical curve which has empty intersection with $F^+ \cup F^- \cup Z$. Again we get an exact sequence as in the first case with $I = \langle z, w^{\pm 1} \rangle = R$. In this case $R/I = 0$ and we get the first exact sequence from $F|_{X_D} = 0$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = z^{b-1}w^{d-1}(bwz + lwdw)$, thus we have an isomorphism (since $w$ is a unit in this chart)

$$\pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X((b-1)E)|_{X_D} \to \ker(\alpha)|_{X_D},$$

where $E$ is the divisor associated to the fraction $a/b$.

We define the sheaf $\mathcal{Q}_\alpha := \Omega_X/\ker(\alpha)$.

Now, maintaining the same hypothesis and notation as above, we are ready to prove part two of Theorem 2.2.

**Lemma 2.3.** There is a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{Q}_\alpha \to \mathcal{G} \to \mathcal{Z}^{r+1} \otimes \mathcal{O}_X \to 0,$$

where

$$\mathcal{G} = \mathcal{O}(-F^-) \oplus \mathcal{O}(-F^+) \oplus \left( (\oplus_{i,j} \mathcal{O}(-E_{i,j})) / (\oplus_{i=1}^m \oplus_{j=1}^n \mathcal{O}(-E_{i,j} - E_{i,j+1})) \right).$$
Proof. Let us consider the diagram

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 \\
0 & O_X & \mathcal{F}_X & 0 \\
0 & G & \mathbb{Z}^{r+2} \otimes O_X & \mathcal{F}_X & 0 \\
0 & \mathbb{Z}^{r+1} \otimes O_X & \mathbb{Z}^{r+1} \otimes O_X & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where the top row comes from Lemma 2.2. The middle columns is the direct sum of the fundamental short exact sequences

\[
0 \rightarrow O(-F^\pm) \rightarrow O_X \rightarrow O_{F^\pm} \rightarrow 0
\]

together with the following exact sequences (cf. [Bea96]) for each \( E_i \cap E_j = p \in \mathbb{Z} \)

\[
0 \rightarrow O(-E_i - E_j) \rightarrow O(-E_i) \oplus O(-E_j) \rightarrow O_X \rightarrow O_p \rightarrow 0
\]

where we replace the first two sheaves with their quotient to obtain short sequences. The middle column is simply the exact sequence of modules

\[
0 \rightarrow \mathbb{Z}^{(1,\ldots,1)} \rightarrow \mathbb{Z}^{r+2} \rightarrow \mathbb{Z}^{r+1} \rightarrow 0
\]

after tensoring by \( O_X \). The exactness of the sequences, together with the commutativity of both squares (easy to check), ensures the existence of an exact sequence on the left column. □

We can now prove the first theorem stated in the introduction.

Proof of Theorem 1. It is direct from Lemmas 2.2 and 2.3 since \( \text{im}(\alpha) \) in \( O_X \) is isomorphic to \( Q_\alpha \). □

Proposition 2.4. The following holds: \( \mathcal{E}xt^1(Q_\alpha, O_X) \cong O_Z \).

Proof. By the definition of \( Q_\alpha \), Lemma 2.2 and the long exact sequence for ext sheaves we have \( \mathcal{E}xt^1(Q_\alpha, O_X) \cong \mathcal{E}xt^2(F, O_X) \). Since the functor \( \mathcal{E}xt^i \) commutes with finite direct sums, it is enough to show that \( \mathcal{E}xt^2(O_{F^\pm}, O_X) = 0 \) and \( \mathcal{E}xt^2(O_p, O_X) \cong O_p \) for any \( p \in \mathbb{Z} \). Taking the long exact \( \mathcal{E}xt \)-sequence of the exact sequence of sheaves

\[
0 \rightarrow O_X(-F^\pm) \rightarrow O_X \rightarrow O_{F^\pm} \rightarrow 0
\]

and using \( \mathcal{E}xt^i(O_X, O_X) = 0 \) for any \( i > 0 \), by [Har77, Pro. III.6.3(b)], we get \( \mathcal{E}xt^1(O_X(-F^\pm), O_X) \cong \mathcal{E}xt^2(O_{F^\pm}, O_X) \). By [Har77, Pro. III.6.7] we conclude that these sheaves are the zero sheaf, proving the first vanishing. To prove the second isomorphism observe that for each \( p \in \mathbb{Z} \) lying in the intersection \( E_i \cap E_j \) we have the following exact sequence of sheaves [Bea96]

\[
0 \rightarrow O_X(-E_i - E_j) \rightarrow O_X(-E_i) \oplus O_X(-E_j) \rightarrow O_X \rightarrow O_p \rightarrow 0.
\]
Denoting by $\mathcal{N}$ the quotient sheaf $\mathcal{O}_X(-E_i) \oplus \mathcal{O}_X(-E_j)/\mathcal{O}_X(-E_i-E_j)$ we deduce $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_p, \mathcal{O}_X)$ and the fact that $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X)$ is the cokernel of the map $\mathcal{O}_X(E_i) \oplus \mathcal{O}_X(E_j) \to \mathcal{O}_X(E_i+E_j)$ induced by $\varphi$ taking tensor product with $\mathcal{O}_X(E_i+E_j)$. This proves the statement. 

\section{Applications}

\textbf{Lemma 3.1.} Let $\varphi: \tilde{X} \to X$ be the blow-up of a smooth projective variety at a point $p \in X$. Then $h^1(\tilde{X}, T_{\tilde{X}}) \geq h^1(X, T_X)$.

\textit{Proof.} Since $\varphi$ is a blow-up it follows that $R^i\varphi_* T_{\tilde{X}}$ vanishes for any $i > 0$. Thus the equality $h^i(\tilde{X}, T_{\tilde{X}}) = h^i(X, \varphi_* T_{\tilde{X}})$ holds for any $i$ by [Har77, Exercise III.8.1] and we conclude by the following exact sequence of sheaves

$$0 \to \varphi_* T_{\tilde{X}} \to T_X \to T_p \to 0.$$

\textit{Proof of Theorem 2.} We begin by showing (1) $\Rightarrow$ (2). Consider the good quotient map $\pi: X \to Y$. Assume first that the curve $Y$ has positive genus. If $\pi$ has only irreducible fibers, $X$ is a ruled surface so by [Sei92, Theorem 4] we have $h^1(X, T_X) > 0$. This still holds if there are reducible fibers, by Lemma 3.1, because $X$ would be a blow-up of one of such ruled surfaces. Thus, $Y$ must necessarily be rational.

We show now that $X$ contains no invariant rational curves $C$ with $C^2 = -n \leq -2$. Suppose such a curve exists. From Lemma 2.2, after tensoring by $\mathcal{O}(K_X + C)$, we have an exact sequence

$$0 \to \pi^*(\Omega^1_Y) \otimes \mathcal{O}(E_S + K_X + C) \to \Omega_X(K_X + C) \to \text{im}(\alpha) \otimes \mathcal{O}(K_X + C) \to 0.$$

Let us compute some cohomology groups for these sheaves. Assume that $K_X + C$ is linearly equivalent to an effective divisor. From the genus formula we have

$$(K_X + C) \cdot C = 2g(C) - 2 = -2 < 0,$$

so by applying [ADHL13, Proposition V.1.1.2] we see that $C$ must be in the base locus of $|K_X + C|$, meaning $K_X + C \sim C + E'$ for some effective divisor $E'$. This would imply that $K_X$ is linearly equivalent to an effective divisor, a contradiction because $X$ is rational and smooth. Thus $h^0(X, K_X + C) = 0$. Since $\text{im}(\alpha) \otimes \mathcal{O}(C + K)$ injects into $\mathcal{O}(C + K)$, then also

$$h^0(X, \text{im}(\alpha) \otimes \mathcal{O}(C + K)) = 0.$$

If $F$ is a general fiber of $\pi$, the genus formula yields $F \cdot K_X = -2$. The product $F \cdot C$ equals at most 1 (where the equality holds if $C$ is the source or sink curve), so

$$F \cdot (-2F + E_S + K_X + C) = F \cdot K_X + F \cdot C < 0.$$

Then $h^0(X, \pi^*(\Omega^1_Y) \otimes \mathcal{O}(E_S + K_X + C)) = h^0(X, -2F + E_S + K_X + C) = 0$. Going back to the exact sequence, we can now deduce that $h^0(X, \Omega_X(K_X + C)) = 0$, and due to Serre’s duality we conclude $h^2(X, T_X(-C)) = 0$.

Consider now the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$
Tensoring by $T_X$ gives a new exact sequence

$$0 \to T_X(-C) \to T_X \to T_X|_C \to 0.$$  

From the vanishing at $H^2$ shown above, there is a surjection $H^1(X, T_X) \to H^1(X, T_X|_C)$, so it suffices to show that $h^1(X, T_X|_C) \neq 0$ to prove the non-existence of this curve $C$, but this comes directly from the exact sequence

$$0 \to T_C \to T_X|_C \to N_C|_X \to 0$$

and the fact that $h^1(X, T_C) = h^2(X, T_C) = 0$ and $h^1(X, N_C|_X) = n - 1$.

We showed that any invariant rational curve of $X$ has self-intersection $\geq -1$. Since the classes of these curves generate the Mori cone of $X$ (see [ADHL13]) we conclude that $-K_X$ is ample and thus $X$ is del Pezzo. Moreover by [Hug13, Proposition 5.9] del Pezzo $K^*$-surfaces without elliptic fixed points are toric.

The proof of $(2) \Rightarrow (1)$ is a consequence of [Itl11, Corollary 2.8].

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