INEQUALITIES FOR THE SCHMIDT NUMBER OF BIPARTITE STATES

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Abstract. In this short note we show two completely opposite methods of constructing entangled states. Given a bipartite state \( \gamma \in M_k \otimes M_k \), define \( \gamma_S = (Id + F)\gamma(Id+F) \), \( \gamma_A = (Id-F)\gamma(Id-F) \), where \( F \in M_k \otimes M_k \) is the flip operator. In the first method, entanglement is a consequence of the inequality \( \text{rank}(\gamma_S) < \sqrt{\text{rank}(\gamma_A)} \). In the second method, there is no correlation between \( \gamma_S \) and \( \gamma_A \). These two methods show how diverse is quantum entanglement.

We prove that any bipartite state \( \gamma \in M_k \otimes M_k \) satisfies

\[
SN(\gamma) \geq \max \left\{ \frac{GR(\gamma)}{\text{rank}(\gamma)}, \frac{SN(\gamma_S)}{2}, \frac{SN(\gamma_A)}{2} \right\},
\]

where \( GR(\gamma) \) stands for the largest Schmidt rank of a tensor within the image of \( \gamma \) and \( SN(\gamma) \) stands for its Schmidt number.

We also present a family of PPT states in \( M_k \otimes M_k \), whose members have Schmidt number equal to \( n \), for any given \( 1 \leq n \leq \left\lceil \frac{k-1}{2} \right\rceil \). This is a new contribution to the open problem of finding the best possible Schmidt number for PPT states.

1. Introduction

The separability problem in Quantum Information Theory asks for a deterministic criterion to distinguish the entangled states from the separable states [3]. This problem is known to be a hard problem even for bipartite mixed states [4,5].

The Schmidt number of a state \( SN(\gamma) \) - Definition 2.1 is a measure of how entangled a state is [10][11]. If its Schmidt number is 1 then the state is separable. If its Schmidt number is greater than 1 then the state is entangled. A method to compute the Schmidt Number is unknown.

Denote by \( M_k \) the set of complex matrices of order \( k \). The separability problem has been completely solved in \( M_2 \otimes M_2 \). A state in \( M_2 \otimes M_2 \) is separable if and only if it is positive under partial transposition or simply PPT (Definition 2.1) [6,9]. Therefore, the Schmidt number of a PPT state in \( M_2 \otimes M_2 \) is equal to 1. Recently, the Schmidt number of every PPT state of \( M_3 \otimes M_3 \) has been proved to be less or equal to 2 [2][12].

The authors of [8] left an open problem to determine the best possible Schmidt number for PPT states. They also presented a construction of PPT states in \( M_k \otimes M_k \) whose Schmidt numbers are greater or equal to \( \left\lceil \frac{k+1}{4} \right\rceil \). This was the first explicit example of a family of PPT states achieving a Schmidt number that scales linearly in the local dimension.

We investigate this matter. We present an explicit construction of PPT states in \( M_k \otimes M_k \), whose Schmidt numbers are equal to \( n \), for any given \( 1 \leq n \leq \left\lceil \frac{k-1}{2} \right\rceil \). This is a new contribution to their open problem.

We manage to compute the Schmidt number of these PPT states using the following inequality

\[
SN(\gamma) \geq \max \left\{ \frac{SN(\gamma_S)}{2}, \frac{SN(\gamma_A)}{2} \right\}, \tag{1.1}
\]

where \( \gamma_S = (Id + F)\gamma(Id+F) \), \( \gamma_A = (Id-F)\gamma(Id-F) \) and \( F \in M_k \otimes M_k \) is the flip operator (i.e., \( F(a \otimes b) = b \otimes a \), for every \( a, b \in \mathbb{C}^k \)).

We believe this is one of the simplest constructions of an entangled PPT state made so far.
Another inequality that we present here connects some unexpected quantities with the Schmidt number of an arbitrary state. Define the generic rank of $\gamma \in M_k \otimes M_k$ as the largest Schmidt rank of a tensor within the image of $\gamma$ and denote it by $GR(\gamma)$.

We show that every state $\gamma$ of $M_k \otimes M_k$ satisfies

$$\text{rank}(\gamma)SN(\gamma) \geq GR(\gamma). \quad (1.2)$$

Notice that if $\text{rank}(\gamma) = 1$ then $SN(\gamma) = GR(\gamma)$. Hence, this inequality is sharp. We can use this inequality to obtain a lower bound for the Schmidt number of states with low rank.

Next, through a series of very technical results, the author of [1] obtained the following lower bounds for the rank($\gamma_S$) of any separable state $\gamma \in M_k \otimes M_k$

$$\text{rank}(\gamma_S) \geq \max \left\{ \frac{r}{2}, \frac{2}{r} \text{rank}(\gamma_A) \right\},$$

where $r$ is the marginal rank of $\gamma + F\gamma F$.

These inequalities can be combined into one inequality:

$$\text{rank}(\gamma_S) \geq \frac{2}{r} \text{rank}(\gamma_A) \geq \frac{\text{rank}(\gamma_A)}{\text{rank}(\gamma_S)}.$$

Hence, $\text{rank}(\gamma_S) \geq \sqrt{\text{rank}(\gamma_A)}$ for every separable state $\gamma \in M_k \otimes M_k$. Therefore, if $\text{rank}(\gamma_S) < \sqrt{\text{rank}(\gamma_A)}$ then $\gamma$ is entangled.

Now, combining equations (1.1) and (1.2) we get

- $\text{rank}(\gamma_S)SN(\gamma) \geq \frac{GR(\gamma_S)}{2}$,
- $\text{rank}(\gamma_A)SN(\gamma) \geq \frac{GR(\gamma_A)}{2}$.

Notice that we can easily create entangled states by satisfying $\frac{GR(\gamma_S)}{\text{rank}(\gamma_S)} > 2$ or $\frac{GR(\gamma_A)}{\text{rank}(\gamma_A)} > 2$ and no correlation between $\gamma_S$ and $\gamma_A$ is required.

These two methods of creating entangled states are completely opposite. One depends on a correlation between $\gamma_S$, $\gamma_A$ and the other does not. They show how diverse is quantum entanglement.

This paper is organized as follows.

- In Section II, we prove that $SN(\gamma) \geq \max \left\{ \frac{SN(\gamma_S)}{2}, \frac{SN(\gamma_A)}{2} \right\}$ (Proposition 2.2) and we construct a PPT state whose Schmidt number is equal to $n$, for any given $n \in \{1, \ldots, \left\lceil \frac{k-1}{2} \right\rceil \}$ (Proposition 2.3).
- In Section III, we prove our main inequality $\text{rank}(\gamma)SN(\gamma) \geq GR(\gamma)$ (Theorem 3.1) and two corollaries $\text{rank}(\gamma_S)SN(\gamma) \geq \frac{1}{2}GR(\gamma_S)$ and $\text{rank}(\gamma_A)SN(\gamma) \geq \frac{1}{2}GR(\gamma_A)$ (Corollaries 3.3 and 3.4).

Notation: Given $x \in \mathbb{R}$, define $[x] = \min\{n \in \mathbb{Z}, n \geq x\}$. Identify $M_k \otimes M_k \simeq M_{k^2}$ and $\mathbb{C}^k \otimes \mathbb{C}^k \simeq \mathbb{C}^{k^2}$ via Kronecker product. Let us call a positive semidefinite Hermitian matrix of $M_{k^2}$ a (non-normalized bipartite) state of $M_k \otimes M_k$. Let $\mathcal{S}(\delta)$ denote the image of $\delta \in M_k \otimes M_k$ within $\mathbb{C}^k \otimes \mathbb{C}^k$. Given $w \in \mathbb{C}^k \otimes \mathbb{C}^k$ denote by $SR(w)$ its Schmidt rank (or tensor rank).
2. Preliminary Inequalities

In this section we present two preliminary inequalities (Proposition 2.2). They have independent interest as we can see in Proposition 2.3. There we construct a family of PPT states in $M_k \otimes M_k$ whose members have Schmidt number equal to $n$, for any given $1 \leq n \leq \left\lceil \frac{k-1}{2} \right\rceil$.

**Definition 2.1.** Given a state $\delta = \sum_{i=1}^{n} A_i \otimes B_i \in M_k \otimes M_k$, define

- the generic rank of $\delta$ as $\text{GR}(\delta) = \max\{\text{SR}(w), w \in \text{Image of } \delta\}$.
- the Schmidt number of $\delta$ as $\text{SN}(\delta) = \min \left\{ \max_j \{\text{SR}(w_j)\}, \delta = \sum_{j=1}^{m} w_j \overline{w}_j^t \right\}$. (This minimum is taken over all decompositions of $\delta$ as $\sum_{j=1}^{m} w_j \overline{w}_j^t$)
- $\delta^T = \sum_{i=1}^{n} A_i \otimes B_i^t$ as its partial transposition on the right side. Moreover, let us say that $\delta$ is positive under partial transposition or simply a PPT state if and only if $\delta$ and $\delta^T$ are states.

**Proposition 2.2.** Every state $\gamma \in M_k \otimes M_k$ satisfies $\text{SN}(\gamma) \geq \max \left\{ \frac{\text{SN}(\gamma_S)}{2}, \frac{\text{SN}(\gamma_A)}{2} \right\}$.

*Proof.* By definition 2.1 there is a subset $\{w_1, \ldots, w_n\} \subset \mathbb{C}^k \otimes \mathbb{C}^k$ such that $\gamma = \sum_{i=1}^{n} w_i \overline{w}_i^t$ and $\text{SR}(w_i) \leq \text{SN}(\gamma)$, for every $i$.

Therefore, $(\text{Id} \pm F)\gamma(I \pm F) = \sum_{i=1}^{n} v_i \overline{v}_i^t$, where $v_i = (\text{Id} \pm F)w_i$. Notice that, for every $i$,

$$\text{SR}(v_i) = \text{SR}(w_i \pm Fw_i) \leq 2\text{SR}(w_i) \leq 2\text{SN}(\gamma).$$

Hence, $\text{SN}((\text{Id} \pm F)\gamma(I \pm F)) \leq 2\text{SN}(\gamma)$.

**Proposition 2.3.** Let $a = \sum_{i=1}^{n} a_i \otimes b_i - b_i \otimes a_i$ and $v = \sum_{i=1}^{n} a_i \otimes b_i + b_i \otimes a_i$, where $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is a linearly independent subset of $\mathbb{C}^k$. Define

$$\gamma = \text{Id} + F + \epsilon(vv^t + a\overline{a}^t) \in M_k \otimes M_k.$$

1. For every $\epsilon > 0$, $\text{SN}(\gamma) = n$. Notice that $1 \leq n \leq \left\lceil \frac{k-1}{2} \right\rceil$.
2. There is $\epsilon > 0$ such that $\gamma$ is positive under partial transposition.

*Proof.* (1) Notice that $\gamma_A = 4\epsilon(a\overline{a}^t)$. Therefore, $\text{SN}(\gamma_A) = \text{SR}(a) = 2n$. Thus, $\text{SN}(\gamma) \geq \frac{\text{SN}(\gamma_A)}{2} = n$, by Proposition 2.2.

Next, notice that

$$vv^t + a\overline{a}^t = \frac{1}{2}((v + a)(v + a)^t + (v - a)(v - a)^t).$$

Since $\text{SR}(v + a) = \text{SR}(v - a) = n$ then $\text{SN}(vv^t + a\overline{a}^t) \leq n$.

The separability of $\text{Id} + F \in M_k \otimes M_k$ is a well known fact, therefore $\text{SN}(\text{Id} + F) = 1$.

Finally, $\text{SN}(\gamma) \leq \max\{\text{SN}(\text{Id} + F), \text{SN}(vv^t + a\overline{a}^t)\} \leq \max\{1, n\} = n$. Therefore, $\text{SN}(\gamma) = n$.

(2) Notice that $(\text{Id} + F)^T = \text{Id} + uu^t$, where $u = \sum_{i=1}^{k} e_i \otimes e_i$ and $\{e_1, \ldots, e_k\}$ is the canonical basis of $\mathbb{C}^k$. So $(\text{Id} + F)^T$ is positive definite and, for a small $\epsilon$, $(\text{Id} + F)^T + \epsilon(vv^t + a\overline{a}^t)^T$ is positive definite too.
3. Main Inequality

In this section, we present our main result (Theorem 3.1) and two corollaries (Corollaries 3.3 and 3.4).

**Theorem 3.1.** If $\gamma \in M_k \otimes M_k$ is a state then $\text{rank}(\gamma)SN(\gamma) \geq GR(\gamma)$.

*Proof.* The proof is an induction on $\text{rank}(\gamma)$. The cases $\text{rank}(\gamma) = 0$ and $\text{rank}(\gamma) = 1$ are trivial.

Let $v \in \mathfrak{Z}(\gamma) \setminus \{0\}$ be such that $SR(v) = GR(\gamma) = n$.

Let $R,S \in M_k$ be invertible matrices such that 
\[
u = (R \otimes S)v = \sum_{i=1}^{n} e_i \otimes e_i,
\]
where $\{e_1, \ldots, e_n, \ldots, e_k\}$ is the canonical basis of $\mathbb{C}^k$.

Define $\delta = (R \otimes S)\gamma(R \otimes S)^*$. Notice that 
- $\text{rank}(\delta) = \text{rank}(\gamma)$ (Since $(R \otimes S)$ is invertible),
- $u \in \mathfrak{Z}(\delta)$, since $\mathfrak{Z}(\delta) \subset (R \otimes S)(\mathfrak{Z}(\gamma))$ and
- $SN(\delta) = SN(\gamma)$.

Let $V = \sum_{i=1}^{n} e_i e_i^*$ and define $\zeta = (V \otimes V)\delta(V \otimes V)$. Notice that 
- $\text{rank}(\zeta) \leq \text{rank}(\delta)$, since $\mathfrak{Z}(\zeta) \subset (V \otimes V)(\mathfrak{Z}(\delta))$.
- $u \in \mathfrak{Z}(\zeta)$,
- $SN(\zeta) \leq SN(\delta)$.

Now, we can assume without loss of generality that $\zeta \in M_n \otimes M_n$.

Recall that $\mathfrak{Z}(\zeta) \neq \{0\}$ and let $w \in \mathfrak{Z}(\zeta)$ be such that $SR(w) = SN(\zeta)$. Next, let $U \in M_n$ be such that $\text{rank}(U) = n - SN(\zeta)$ and $(U \otimes Id)w = 0$.

Define $\beta = (U \otimes Id)\zeta(U \otimes Id)^* \in M_n \otimes M_n$. Notice that 
- $\text{rank}(\beta) \leq \text{rank}(\zeta) - 1$, since $\mathfrak{Z}(\beta) \subset (U \otimes Id)(\mathfrak{Z}(\zeta))$ and $(U \otimes Id)w = 0$.
- $(U \otimes Id)u \in \mathfrak{Z}(\beta)$,
- $SN(\beta) \leq SN(\zeta)$.

Since $\text{rank}(\beta) \leq \text{rank}(\zeta) \leq \text{rank}(\delta) = \text{rank}(\gamma)$ then, by induction hypothesis, $\text{rank}(\beta)SN(\beta) \geq GR(\beta)$.

Since $(U \otimes Id)u \in \mathfrak{Z}(\beta)$ then 
\[
\text{rank}(\beta)SN(\beta) \geq SR((U \otimes Id)u) = \text{rank}(U).
\]

Recall that $\text{rank}(\beta) \leq \text{rank}(\zeta) - 1$, $SN(\beta) \leq SN(\zeta)$ and $\text{rank}(U) = n - SN(\zeta) = GR(\gamma) - SN(\zeta)$.

Therefore, $(\text{rank}(\zeta) - 1)SN(\zeta) \geq GR(\gamma) - SN(\zeta)$. So 
\[
\text{rank}(\zeta)SN(\zeta) \geq GR(\gamma).
\]

Finally, since $\text{rank}(\zeta)SN(\zeta) \leq \text{rank}(\gamma)SN(\gamma)$ then the induction is complete. \qed
Remark 3.2. For a separable state $\gamma$, it is known that its rank is greater or equal to both marginal ranks \cite[Theorem 1]{7}, which are greater or equal to its generic rank. So for this particular case this inequality was previously known.

Corollary 3.3. If $\gamma \in M_k \otimes M_k$ is a state then $\text{rank}(\gamma_A)SN(\gamma) \geq \frac{GR(\gamma_A)}{2}$.

Proof. Since $SN(\gamma_A) \leq 2SN(\gamma)$, by Proposition 2.2 then $\text{rank}(\gamma_A)SN(\gamma) \geq \frac{1}{2} \text{rank}(\gamma_A)SN(\gamma_A) \geq \frac{GR(\gamma_A)}{2}$, by Theorem 3.1. \hfill \Box

Corollary 3.4. If $\gamma \in M_k \otimes M_k$ is a state then $\text{rank}(\gamma_S)SN(\gamma) \geq \frac{GR(\gamma_S)}{2}$.

Proof. Since $SN(\gamma_S) \leq 2SN(\gamma)$, by Proposition 2.2 then $\text{rank}(\gamma_S)SN(\gamma) \geq \frac{1}{2} \text{rank}(\gamma_S)SN(\gamma_S) \geq \frac{GR(\gamma_S)}{2}$, by Theorem 3.1. \hfill \Box

**Summary and Conclusion**

We presented an inequality that relates the Schmidt number of any bipartite state of $M_k \otimes M_k$, the generic rank of a tensor within the image of the state and its rank. Using this inequality, we described a method of constructing entangled states which is not based on any correlation between $\text{rank}(\gamma_A)$ and $\text{rank}(\gamma_S)$. This form of entanglement differs completely from the entanglement derived from the inequality $\text{rank}(\gamma_S) < \sqrt{\text{rank}(\gamma_A)}$. We also constructed a family of PPT states whose members have Schmidt number equal to $n$, for any given $1 \leq n \leq \lceil \frac{k-1}{2} \rceil$. This is a new contribution to the open problem of finding the best possible Schmidt number for PPT states.

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