Spectra of Signless Normalized Laplace Operators for Hypergraphs

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Abstract

The spectral theory of chemical hypergraphs is further investigated, with a focus on the normalized Laplace operators. Two signless normalized Laplacians are introduced and it is shown that the spectra of such operators for classical hypergraphs coincide with the spectra of the normalized Laplacians for bipartite chemical hypergraphs. Furthermore, the spectra of special families of hypergraphs are established.

Keywords: Hypergraphs, Spectral Theory, Signless normalized Laplace Operator

1 Introduction

In this work we bring forward the study of the normalized Laplacian \( L \) that has been established for chemical hypergraphs: hypergraphs with the additional structure that each vertex in a hyperedge is either an input, an output or both (in which case we say that it is a catalyst for that hyperedge). Chemical hypergraphs have been introduced in [1] with the idea of modelling chemical reaction networks and related ones, such as metabolic networks. In this model, each vertex represents a chemical element and each hyperedge represents a chemical reaction. Furthermore, in [2], chemical hypergraphs have been used for modelling dynamical systems with high order interactions. In this model, the vertices represent oscillators while the hyperedges represent the interactions on which the dynamics depends.

The spectrum of the normalized Laplacian reflects many structural properties of the network and several theoretical results on the eigenvalues have been established in [1, 3, 4]. Furthermore, as shown in [3], by defining the vertex degree in a way that it does not take catalysts into account, studying the spectrum of \( L \) for chemical hypergraphs is equivalent to studying the spectrum of the oriented hypergraphs introduced in [5] by Reff and Rusnak, in which catalysts are not included. Therefore, without loss of generality we can work on oriented hypergraphs. Here, in particular, we focus on the bipartite case and we show that the spectra the normalized Laplacian for bipartite chemical hypergraphs coincide with the spectra of the signless normalized Laplacian that we introduce here for classical hypergraphs. Furthermore, we establish the spectra of the signless normalized Laplacian for special families of such classical hypergraphs.

Classical hypergraphs are widely used in various disciplines. For instance, they offer a valid model for transport networks [6], neural networks (in whose context they are often called neural codes) [7–14], social networks [15] and epidemiology networks [16], just to mention some examples. It is worth noting that a simplicial complex \( S \) is a particular case of hypergraph with the additional constraint that, if a hyperedge belongs to \( S \), then also all its subsets belong to \( S \). Simplicial complexes are also widely present in applications. On the one hand, their more precise structure allows for a deeper theoretical study, compared to general hypergraphs. On the other hand, the constraints of simplicial

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complexes can be translated as constraints on the model, and this is not always convenient. Consider, for instance, a collaboration network that represents coauthoring of research papers: in this case, the fact that authors \(A, B\) and \(C\) have written a paper all together does not imply that \(A, B\) and \(C\) have all written single author papers, nor that \(A\) and \(B\) have written a paper together without \(C\). In this case, a hypergraph would give a better model than a simplicial complex.

**Structure of the paper.** In Section 2 we introduce the basic definitions on oriented hypergraphs and they associated operators, and we prove some small preliminary results which are needed throughout the paper. In Section 3 we introduce and discuss twin vertices. In Section 4 we prove new properties of bipartite oriented hypergraphs and we show that, from the spectral point of view, these are equivalent to classical hypergraphs with no input/output structures. In the subsequent sections we investigate the spectra of new hypergraph structures that we introduce either with the idea of generalizing well known graph structures, or with the idea of offering a model for a specific application. Finally, in Section 11 we draw some conclusions.

2 Oriented hypergraphs

**Definition 2.1** ([1, 5]). An oriented hypergraph is a pair \(\Gamma = (V, H)\) such that \(V\) is a finite set of vertices and \(H\) is a set such that every element \(h\) in \(H\) is a pair of disjoint elements \((h_{\text{in}}, h_{\text{out}})\) (input and output) in \(\mathcal{P}(V) \setminus \{\emptyset\}\). The elements of \(H\) are called the oriented hyperedges. Changing the orientation of a hyperedge \(h\) means exchanging its input and output, leading to the pair \((h_{\text{out}}, h_{\text{in}})\).

**Definition 2.2.** Given \(h \in H\), we say that two vertices \(i\) and \(j\) are co-oriented in \(h\) if they belong to the same orientation sets of \(h\); we say that they are anti-oriented in \(h\) if they belong to different orientation sets of \(h\).

From now on, we fix such a chemical hypergraph \(\Gamma = (V, H)\) on \(N\) vertices \(v_1, \ldots, v_N\) and \(M\) hyperedges \(h_1, \ldots, h_M\).

**Definition 2.3.** We define the underlying hypergraph of \(\Gamma\) as \(\Gamma' := (V, H')\) where

\[ H' := \{ (h_{\text{in}} \cup h_{\text{out}}, \emptyset) : h = (h_{\text{in}}, h_{\text{out}}) \in H \} \]

**Remark 2.4.** The underlying hypergraph forgets about the input/output structure: all vertices are only inputs for all hyperedges in which they are contained.

**Remark 2.5.** Graphs are oriented hypergraphs such that \#\(h_{\text{in}}\) = \#\(h_{\text{out}}\) = 1 for each \(h \in H\), that is, each edge has exactly one input and one output. Therefore, the underlying hypergraph of a graph is not a graph.

**Definition 2.6** ([3]). We define the in-degree of a vertex \(v\) as

\[ \deg_{\text{in}}(v) := \# \text{ hyperedges containing } v \text{ as an input}; \]

we define its out-degree as

\[ \deg_{\text{out}}(v) := \# \text{ hyperedges containing } v \text{ as an output} \]

and we define its degree as

\[ \deg(v) := \deg_{\text{in}}(v) + \deg_{\text{out}}(v). \]

Similarly, we define the cardinality of a hyperedge \(h\) as

\[ \#h := \#\{h_{\text{in}} \cup h_{\text{out}}\}. \]
Definition 2.7 ([1, 4]). We define the normalized Laplace operator associated to \( \Gamma \) as the \( N \times N \) matrix
\[
L := \text{Id} - D^{-1}A,
\]
where \( \text{Id} \) is the \( N \times N \) identity matrix, \( D \) is the diagonal degree matrix and \( A \) is the adjacency matrix defined by \( A_{ii} := 0 \) for each \( i = 1, \ldots, n \) and
\[
A_{ij} := \# \{ \text{hyperedges in which } v_i \text{ and } v_j \text{ are anti-oriented} \} + \sum_{h: v_i \text{ input}} \frac{\gamma(h') - \sum_{h': v_i \text{ output}} \gamma(h'')}{\deg v} + \sum_{h: w \text{ output}} \frac{\gamma(h') - \sum_{h': w \text{ output}} \gamma(h'')}{\deg w},
\]
for \( i \neq j \).

Definition 2.8. We say that two vertices \( v_i \) and \( v_j \) are adjacent, denoted \( v_i \sim v_j \), if they are contained at least in one common hyperedge.

Remark 2.9. While in the graph case \( v_i \sim v_j \) if and only if \( A_{ij} = 1 \), it is clear by definition of \( A \) that this is no longer true for general hypergraphs.

Remark 2.10. Consider a graph \( \Gamma \) and let \( \Gamma' \) be its underlying hypergraph. Then, the adjacency matrix \( A \) of \( \Gamma \) and the adjacency matrix \( A' \) of \( \Gamma' \) are such that \( A' = -A \), while the degree matrices of \( \Gamma \) and \( \Gamma' \) coincide. Therefore, the normalized Laplacians of \( \Gamma \) and \( \Gamma' \) are
\[
L = \text{Id} - D^{-1}A \quad \text{and} \quad L' = \text{Id} + D^{-1}A = 2 \cdot \text{Id} - L,
\]
respectively. Hence, \( \mu \) is an eigenvalue for \( A \) if and only if \( -\mu \) is an eigenvalue for \( A' \), while \( \lambda \) is an eigenvalue for \( L \) if and only if \( 2 - \lambda \) is an eigenvalue for \( L' \).

Definition 2.11 ([1]). Let \( C(\mathcal{H}) \) be the space of functions \( \gamma : \mathcal{H} \to \mathbb{R} \), endowed by the scalar product
\[
(\gamma, \tau)_{\mathcal{H}} := \sum_{h \in \mathcal{H}} \gamma(h) \tau(h).
\]
The hyperedge-Laplacian associated to \( \Gamma \) is the operator
\[
L^H : C(\mathcal{H}) \to C(\mathcal{H})
\]
such that, given \( \gamma : \mathcal{H} \to \mathbb{R} \) and given \( h \in \mathcal{H} \),
\[
L^H \gamma(h) := \sum_{v \text{ input of } h} \frac{\sum_{h': v \text{ input}} \gamma(h') - \sum_{h': w \text{ output}} \gamma(h'')}{\deg v} + \sum_{w \text{ output of } h} \frac{\sum_{h': w \text{ input}} \gamma(h') - \sum_{h': w \text{ output}} \gamma(h'')}{\deg w}.
\]
As shown in [1], \( L^H \) has \( M = \# \mathcal{H} \) eigenvalues, counted with multiplicity, and the nonzero spectra of \( L \) and \( L^H \) coincides.

Definition 2.12. ([5]) The unnormalized Laplacian associated to \( \Gamma \) is the operator \( \Delta := D - A \).

Remark 2.13. From Remark 2.10 it follows that, if \( \Delta \) is the unnormalized Laplacian associated to a graph \( \Gamma \) and \( \Delta' \) is the unnormalized Laplacian associated to the underlying hypergraph of \( \Gamma \), then \( \Delta' = D + A \). This coincides with the well known signless Laplacian of a graph [17, 20]. Motivated by this observation, we give the following definition.

Definition 2.14. Let \( \Gamma \) be an oriented hypergraph and let \( \Gamma' \) be its underlying hypergraph. We define the signless adjacency, the signless normalized Laplacian, the signless hyperedge normalized Laplacian and the signless unnormalized Laplacian of \( \Gamma \) as the adjacency matrix, the normalized Laplacian, the hyperedge normalized Laplacian and the unnormalized Laplacian of \( \Gamma' \), respectively.
Lemma 2.15. The spectra of $L$, $L^H$, $A$ and $\Delta$ doesn’t change if we apply one of the following transformations:

1. Reverse the orientation of a hyperedge, i.e. let all its inputs become outputs and let all its outputs become inputs;

2. Reverse the role of a vertex in all the hyperedges in which it is contained, i.e. let it become an input where it is an output and let it become an output where it is an input.

Proof. The first point is clear by definition of $L$, $L^H$, $A$ and $\Delta$: none of these operators change if we reverse the orientation of a hyperedge.

The second point is shown in [1, Lemma 49] for $L$ and $L^H$. In order to see it for $A$, observe that reversing the role of a vertex $v_i$ in all hyperedges in which it is contained implies that the new adjacency matrix $A'$ obtained from this operation will be such that $A'_{ij} = -A_{ij}$ for all $j \neq i$, and $A'_{kk} = A_{kk}$ for all $k, j \neq i$. Hence, if we let $B$ be the $n \times n$ diagonal matrix such that $b_{jj} = 1$ for all $j \neq i$ and $b_{ii} = -1$, we have that $A' = BAB^{-1}$. This proves that $A$ and $A'$ are similar, therefore isospectral. Now, since the degree matrix doesn’t change when reversing all orientations of a vertex, we have that $\Delta = D - A$ and $\Delta' = D - A'$. The isospectrality of $A$ and $A'$ implies the isospectrality of $\Delta$ and $\Delta'$.

Remark 2.16. If $\Gamma$ is obtained from $\hat{\Gamma}$ by reversing the orientation of a hyperedge, both the eigenvalues and the eigenvectors of the operators associated to $\Gamma$ and $\hat{\Gamma}$ coincide.

If $\Gamma$ is obtained from $\hat{\Gamma}$ by reversing all orientations of a vertex $v$, then two eigenfunctions $f$ and $\hat{f}$ corresponding to an eigenvalue $\lambda$ for an operator associated to $\Gamma$ and $\hat{\Gamma}$, respectively, are such that $f(w) = \hat{f}(w)$ for each $w \neq v$, and $f(v) = -\hat{f}(v)$.

From here on, unless otherwise specified, we focus on the normalized Laplacian $L$ and we define the spectrum of $\Gamma$ as the spectrum of $L$ for $\Gamma$.

3 Twin vertices

In [1] it is shown that $n$ duplicate vertices produce the eigenvalue 1 with multiplicity at least $n - 1$. Similarly, in this section we discuss twin vertices.

Definition 3.1. We say that two vertices $v_i$ and $v_j$ are twins if they belong exactly to the same hyperedges, with the same orientations. In particular, $A_{ij} = -\deg(v_i) = -\deg(v_j)$ and $A_{ik} = A_{jk}$ for all $k \neq i, j$.

Remark 3.2. While duplicate vertices are known also for graphs, twin vertices cannot exist for graphs, since in this case one assumes that each edge has one input and one output.

We now generalize the notions of duplicate vertices and twin vertices by defining duplicate families of twin vertices.

Definition 3.3. Let $\Gamma = (V, H)$ be an oriented hypergraph. We say that a family of vertices $V_1 \sqcup \ldots \sqcup V_l \subseteq V$ is a $l$-duplicate family of $t$-twin vertices if

- For each $i \in \{1, \ldots, l\}$, $\#V_i = t$ and the $t$ vertices in $V_i$ are twins;

- For each $i, j \in \{1, \ldots, l\}$ with $i \neq j$, for each $v_i \in V_i$ and for each $v_j \in V_j$, we have that $A_{ij} = 0$ and $A_{ik} = A_{jk}$ for all vertices $v_k$ that are not in the $l$-family, i.e. $v_k \in V \setminus V_1 \sqcup \ldots \sqcup V_l$.

Proposition 3.4. If $\Gamma$ contains a $l$-duplicate family of $t$ twins, then:

- $t$ is an eigenvalue with multiplicity at least $l - 1$;

- $0$ is an eigenvalue with multiplicity at least $l(t - 1)$.
Proof. The fact that 0 is an eigenvalue with multiplicity at least \( t(t-1) \) follows from Proposition 3.5. In order to show that \( t \) is eigenvalue with multiplicity at least \( t-1 \), consider the following \( t-1 \) functions. For \( i = 2, \ldots, l \), let \( f_i : V \to \mathbb{R} \) such that \( f_i := 1 \) on \( V_1 \), \( f_i := -1 \) on \( V_i \) and \( f_i := 0 \) otherwise. Then,

- For each \( v_1 \in V_1 \),
  \[
  Lf(v_1) = 1 - \frac{1}{\deg v_1} \sum_{1 \neq j \in V_1} -\deg v_1 = 1 + t - 1 = t \cdot f(v_1);
  \]

- For each \( v_i \in V_i \),
  \[
  Lf(v_i) = -1 - \frac{1}{\deg v_i} \sum_{i \neq j \in V_i} \deg v_i = -1 - (t-1) = t \cdot f(v_i);
  \]

- For each \( v_k \in V \setminus V_1 \sqcup \ldots \sqcup V_l \),
  \[
  Lf(v_k) = -\frac{1}{\deg v_k} \left( \sum_{v_i \in V_1} A_{1k} - \sum_{v_i \in V_i} A_{ik} \right) = 0 = t \cdot f(v_k).
  \]

Therefore, \( f_i \) is an eigenfunction for \( t \). Furthermore, the functions \( f_2, \ldots, f_l \) are linearly independent. Therefore, \( t \) is an eigenvalue with multiplicity at least \( l-1 \).

\( \square \)

**Proposition 3.5.** If there are \( \hat{n} \) twin vertices, 0 is an eigenvalue with multiplicity at least \( \hat{n} - 1 \). Furthermore, if \( v_i \) and \( v_j \) are twin vertices and \( f \) is an eigenfunction for \( L \) with eigenvalue \( \lambda \neq 0 \), then \( f(v_i) = f(v_j) \).

Proof. The first claim follows from Proposition 3.4 by taking \( t = 1 \).

Now, assume that \( v_i \) and \( v_j \) are twin vertices and let \( f \) be an eigenfunction for \( L \) with eigenvalue \( \lambda \neq 0 \). Then,

\[
\lambda f(v_i) = Lf(v_i) = f(v_i) + f(v_j) - \frac{1}{\deg v_i} \left( \sum_{k \neq i, j} A_{ik} f(v_k) \right) = Lf(v_j) = \lambda f(v_j).
\]

Therefore, since \( \lambda \neq 0 \), this implies that \( f(v_i) = f(v_j) \). \( \square \)

### 4 Bipartite hypergraphs

**Definition 4.1.** We say that a hypergraph \( \Gamma \) is bipartite if one can decompose the vertex set as a disjoint union \( V = V_1 \sqcup V_2 \) such that, for every hyperedge \( h \) of \( \Gamma \), either \( h \) has all its inputs in \( V_1 \) and all its outputs in \( V_2 \), or vice versa (Figure 1).

We now give the definition of vertex-bipartite hypergraph that, as we shall see in Lemma 4.3 below, coincides with the definition of bipartite hypergraph.

**Definition 4.2.** We say that a hypergraph \( \Gamma \) is vertex-bipartite if one can decompose the hyperedge set as a disjoint union \( H = H_1 \sqcup H_2 \) such that, for every vertex \( v \) of \( \Gamma \), either \( v \) is an input only for hyperedges in \( H_1 \) and it is an output only for hyperedges in \( H_2 \), or vice versa.

**Lemma 4.3.** Up to changing the orientation of some hyperedges, a hypergraph is bipartite if and only if it is vertex-bipartite.
Proof. Assume that $\Gamma$ is bipartite. Up to changing the orientation of some hyperedges, we can assume that the vertex set has a decomposition $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ such that each hyperedge $h$ has all its inputs in $\mathcal{V}_1$ and all its outputs in $\mathcal{V}_2$. Therefore, every vertex in $\mathcal{V}_1$ is an input only for hyperedges in $\mathcal{H}$, and every vertex in $\mathcal{V}_2$ is only an output for hyperedges in $\mathcal{H}$. It follows that the decomposition of the hyperedge set as $\mathcal{H} = \mathcal{H} \sqcup \emptyset$ gives a vertex-bipartition.

Now, assume that $\Gamma$ is vertex-bipartite, with $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$. Assume, by contradiction, that $\Gamma$ is not bipartite. Then, up to changing the orientation of some hyperedges, there exist two vertices $v, w \in \mathcal{V}$ and two hyperedges $h_1, h_2 \in \mathcal{H}$ such that:

1. $h_1$ has both $v$ and $w$ as inputs;
2. $h_2$ has $v$ as input and $w$ as output.

The fact that $v$ is an input in both $h_1$ and $h_2$ implies that $h_1$ and $h_2$ are in the same $\mathcal{H}_i$. On the other hand, the fact that $w$ is an input for $h_1$ and an output for $h_2$ implies that $h_1$ and $h_2$ do not belong to the same $\mathcal{H}_i$. This brings to a contradiction. Therefore, $\Gamma$ is bipartite. \qed

Proposition 4.4. If $\Gamma$ is bipartite, it is isospectral (with respect to $L$, $L^H$, $A$ and $\Delta$) to its underlying hypergraph, therefore, in particular, also to every other bipartite hypergraph that has the same underlying hypergraph as $\Gamma$.

Proof. By the first point of Lemma 2.15, since $\Gamma$ is bipartite, up to switching the orientations of some hyperedges we can assume that all the inputs are in $\mathcal{V}_1$ and all the outputs are in $\mathcal{V}_2$, with $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$. Furthermore, by the second point of Lemma 2.15 we can move a vertex from $\mathcal{V}_1$ to $\mathcal{V}_2$ or vice versa, by letting it be always an output or always an input, without changing the spectrum of $L$, $L^H$, $A$ and $\Delta$. In particular, if we move all vertices to $\mathcal{V}_1$, we obtain the underlying hypergraph of $\Gamma$. \qed

Remark 4.5. As a consequence of Proposition 4.4, without loss of generality we can always assume that a bipartite hypergraph $\Gamma$ has only inputs, when studying the spectrum of the normalized Laplacian. In this case,

- $A_{ij} = -\#\{h \in \mathcal{H} : v_i, v_j \in \mathcal{H}\} \leq 0$ for each $i \neq j$;
- $\sum_j A_{ij} = -\sum_{h \ni v_i} \#h$, for each $v_i \in \mathcal{V}$;
- $L^H \gamma(h) = \sum_{v_i \in \mathcal{V}} \frac{1}{\deg v_i} \cdot (\sum_{h' \ni v_i} \gamma(h'))$, for all $\gamma : \mathcal{H} \to \mathbb{R}$ and for all $h \in \mathcal{H}$.

Remark 4.6. In view of Remark 2.10 Proposition 4.4 is a generalization of the well known fact that, for bipartite graphs, the spectrum of the adjacency matrix is symmetric.

Remark 4.7. For a hypergraph $\Gamma$ with only inputs, the quantity $\frac{A_{ij}}{\sum_j A_{ij}}$ is the transition probability of a random walker to go from a vertex $v_i$ to a vertex $v_j$.\[21\]

From now on in the paper we work on a hypergraph $\Gamma = (\mathcal{V}, \mathcal{H})$ that has only inputs. Therefore, we work on classical hypergraphs and, in view on Proposition 4.4, this is (spectrally) equivalent to working with bipartite oriented hypergraphs. Therefore, in other words, we focus on the signless normalized Laplacian of classical hypergraphs.
5 Hypertrees

In the case of graphs, trees are connected graphs with no cycles and they have been widely studied in spectral graph theory (see for instance [22–26]). Equivalently, one can define a tree as a connected graph such that the kernel of the (hyper)edge Laplacian $L^H$ is trivial. This motivates us to give the following definition.

**Definition 5.1.** We say that a hypergraph is a **hypertree** if it is connected and the kernel of $L^H$ is trivial.

**Remark 5.2.** For a hypertree on $N$ nodes and $M$ hyperedges, the multiplicity of 0 for $L$ is equal, by [1, Corollary 14], to $N - M$.

6 Hyperflowers

In this section we introduce and study **hyperflowers**: hypergraphs in which there is a set of nodes, the core, that is well connected to the other vertices, and a set of peripheral nodes such that each of them is contained in exactly one hyperedge. Hyperflowers are therefore a generalization of star graphs [27]. Before giving their formal definition, we give the preliminary definition of pendant vertex.

**Definition 6.1.** A vertex of a hypergraph is said **pendant** if it belongs to exactly one hyperedge and it is the only vertex of the hyperedge that has degree one. A vertex is said **quasi-pendant** if it belongs to the same hyperedge as a pendant vertex.

**Remark 6.2.** Definition 6.1 generalizes the definition of pendant vertex for graphs. Recall that a star of a graph is defined as a maximal subgraph formed by pendant vertices all incident with the same vertex (the center of the pendant star) and this definition includes, in particular, the well known star graph. The hyperflowers defined below are a generalization of star graphs.

**Definition 6.3.** A $(l,r)$-hyperflower with $t$ twins (Figure 2) is an hypergraph $\Gamma = (\mathcal{V}, \mathcal{H})$ whose vertex set can be written as $\mathcal{V} = U \sqcup W$, where:

- $U$ is a set of $t \cdot l$ nodes $v_{11}, \ldots, v_{1l}, \ldots, v_{tl}, \ldots, v_{tt}$ which are called **peripheral**;
- There exist $r$ disjoint sets of vertices $h_1, \ldots, h_r \in \mathcal{P}(W) \setminus \{\emptyset\}$ such that

$$\mathcal{H} = \{h | h = h_i \cup \bigcup_{z=1}^{t} v_{zj} \text{ for } i = 1, \ldots, r \text{ and } j = 1, \ldots, l\}.$$ 

If $t = 1$, we simply say that $\Gamma$ is a $(l,r)$-hyperflower. If $t = 1$ and $r = 1$, we simply say that $\Gamma$ is a $l$-hyperflower.

**Remark 6.4.** The $(l,r)$-hyperflowers in Definition 6.3 are a particular case of the hyperstars in [6], that also include weights and non-disjoint sets $h_1, \ldots, h_r$. Here we choose to study the particular structure of $(l,r)$-hyperflowers (and their generalizations with twins) because the strong symmetries of these structures allows for a deeper study of the Laplacian spectrum. Furthermore, we can give an interpretation of hyperflowers in terms of transportation networks. In fact, we can see the vertices of a hyperflower as stations, and we can see the hyperedges as public means of transport that stop in the respective stations. The core of the hyperflower then represents the most connected (central) stations, while the peripheral vertices represent the peripheral stations.

**Remark 6.5.** For $r = 1$, any $l$-hyperflower with twins is a tree with $l$ hyperedges. Therefore, by Remark 5.2, the multiplicity of 0 for $L$ is $N - l$.

**Proposition 6.6.** The spectrum of the $l$-hyperflower on $N$ nodes is given by:

- $0$, with multiplicity $N - l$;
• 1, with multiplicity \( l - 1 \);
• \( N - l + 1 \), with multiplicity 1.

**Proof.** By [4, Corollary 3.5], 1 is an eigenvalue with multiplicity at least \( l - 1 \). By [3, Theorem 3.1], \( \lambda_N = N - l + 1 \). By Remark 5.2, the multiplicity of 0 is \( N - l \).

**Proposition 6.7.** The spectrum of the \((l, 2)\)-hyperflower on \( N \) nodes is given by:
• 0, with multiplicity \( N - l - 1 \);
• 1, with multiplicity \( \geq l - 1 \);
• \( \lambda_N > 1 \);
• \( \lambda_{N-1} = N - \lambda_N - l + 1 \geq 1 \).

In the particular case in which \( \#h \) is constant for each \( h \in H \), \( \lambda_N = \frac{N-l}{2} + 1 \) and \( \lambda_{N-1} = \frac{N-l}{2} \).

**Proof.** By [4, Corollary 3.5], 1 is an eigenvalue with multiplicity at least \( l - 1 \). Now, the \( N - l \) vertices \( v_{l+1}, \ldots, v_N \) form two classes of twin vertices that generate the eigenvalue 0 with multiplicity at least \( N - l - 2 \). In particular, there exist \( N - l - 2 \) linearly independent corresponding eigenfunctions \( f_i : V \to \mathbb{R} \) such that \( f_i(v) = 1 \) for some \( v \notin \{v_1, \ldots, v_l\} \), \( f_i(w) = -1 \) for a given \( w \) twin of \( v \), and \( f_i = 0 \) otherwise. If we let \( g(v_j) := 1 \) for each \( j = 1, \ldots, l \), \( g(v'_1) := -1 \) for exactly one \( v'_1 \in h_1 \) and \( g(v'_2) := -1 \) for exactly one \( v'_2 \in h_2 \), it’s easy to see that \( g \) is also an eigenfunction of 0. Furthermore, the \( f_i \)'s and \( g \) are all linearly independent, which implies that 0 has multiplicity at least \( N - l - 1 \).

Now, by [3, Theorem 3.1], \( \lambda_N \geq \frac{\sum_{h \in H} \#h}{|H|} > 1 \). We have therefore listed already \( N - 1 \) eigenvalues and there is only one eigenvalue \( \lambda \) missing. Since \( \sum_{i=1}^{N} \lambda_i = N \), we have that \( \lambda = N - \lambda_N - l + 1 \). In particular, since by [3, Theorem 3.1] \( \lambda_N \leq \max_{h \in H} \#h \) with equality if and only if \( \#h \) is constant, and \( \max_{h \in H} \#h \leq N - l \), we have that
\[
\lambda = N - \lambda_N - l + 1 \geq 1,
\]
with equality if and only if \( \#h \) is constant and equal to \( N - l \), that is, if and only if \( \#h_1 = \#h_2 = 1 \). Hence, \( \lambda = \lambda_{N-1} \) and we have that \( \lambda_{N-1} = 1 \) if and only if \( \#h_1 = \#h_2 = 1 \).

In general, if \( \#h \) is constant for each \( h \in H \), then by [3, Theorem 3.1] \( \lambda_N = \#h = \frac{N-l}{2} + 1 \) and therefore \( \lambda_{N-1} = \frac{N-l}{2} \).
Remark 6.8. With the same argument used in the proof of Proposition 6.7 one can see that the $(l, r)$-hyperflower on $N$ nodes is such that:

- $0$ is an eigenvalue with multiplicity $\geq N - l - r + 1$;
- $1$ is an eigenvalue with multiplicity $\geq l - 1$;
- $\lambda_N > 1$.

There are $r - 1$ eigenvalues for which we cannot say anything a priori. In the particular case in which $\#h_j = 1$ for each $j = 1, \ldots, r$, the $N = l + r$ and the vertices $v_{l+1}, \ldots, v_N$ are $r$ duplicate vertices, therefore the eigenvalue $1$ in this case has multiplicity $(l - 1) + (r - 1) = N - 2$, while $\lambda_N = 2$. In this case, in particular, the $(l, r)$-hyperflower is a graph.

**Proposition 6.9.** Let $\Gamma$ be an $(l, r)$–hyperflower with pendant vertices $v_1, \ldots, v_l$. Let $\hat{\Gamma} := (\hat{V}, \hat{H})$ be the $(1, r)$–hyperflower defined by

$$\hat{V} := V \setminus \{v_2, \ldots, v_l\} \quad \text{and} \quad \hat{H} := \{h \in H : v_2, \ldots, v_l \not\in h\}.$$ 

Then, the spectrum of $\hat{\Gamma}$ is given by:

- The $N - l + 1$ eigenvalues of $\hat{\Gamma}$, with multiplicity;
- $1$, with multiplicity at least $l - 1$.

**Proof.** By [H] Corollary 3.5], adding $v_2, \ldots, v_l$ to $\hat{\Gamma}$ produces the eigenvalue $1$ with multiplicity $l - 1$. Therefore, it is left to show that, if $\lambda$ is an eigenvalue of $\hat{\Gamma}$, then $\lambda$ is also an eigenvalue of $\Gamma$. Let $L$ and $A$ be the Laplacian and the adjacency matrix on $\Gamma$, respectively, and let $\hat{L}$ and $\hat{A}$ be the Laplacian and the adjacency matrix on $\hat{\Gamma}$, respectively. Let also $f$ be an eigenfunction for $\hat{\Gamma}$ corresponding to the eigenvalue $\lambda$. Then,

$$\hat{L} \hat{f}(v_k) = \hat{f}(v_k) - \frac{1}{\deg_{\hat{\Gamma}} v_k} \sum_{v_i \in \hat{V} \setminus \{v_k\}} \hat{A}_{ik} \hat{f}(v_i) = \lambda \cdot \hat{f}(v_k), \quad \text{for all } v_k \in \hat{V}.$$ 

Now, let $f : V \to \mathbb{R}$ be such that $f := \hat{f}$ on $\hat{\Gamma}$ and $f(v_2) := \ldots := f(v_l) := \hat{f}(v_1)$. Then,

$$L f(v_1) = f(v_1) - \frac{1}{\deg v_1} \sum_{v_i \in V \setminus \{v_1\}} A_{1i} f(v_i) = \hat{f}(v_1) - \frac{1}{\deg_{\hat{\Gamma}} v_1} \sum_{v_i \in \hat{V} \setminus \{v_1\}} \hat{A}_{1i} \hat{f}(v_i) = \hat{L} \hat{f}(v_1) = \lambda \cdot \hat{f}(v_1) = \lambda \cdot f(v_1).$$

Similarly, for $j \in 2, \ldots, l$,

$$L f(v_j) = f(v_j) - \frac{1}{\deg v_j} \sum_{v_i \in V \setminus \{v_1\}} A_{ij} f(v_i) = \hat{f}(v_1) - \frac{1}{\deg_{\hat{\Gamma}} v_1} \sum_{v_i \in \hat{V} \setminus \{v_1\}} \hat{A}_{ij} \hat{f}(v_i) = \lambda \cdot \hat{f}(v_1) = \lambda \cdot f(v_j).$$

Furthermore, for each $v_k \in V \setminus \{v_1, \ldots, v_l\}$, we have that

- $\deg_{\hat{\Gamma}}(v_k) = 1$ while $\deg(v_k) = l$;
- For each $v_k' \in V \setminus \{v_1, \ldots, v_l\}$ such that $\hat{A}_{kk'} \neq 0$, $\hat{A}_{kk'} = -1$ while $A_{kk'} = -l$;
- $\hat{A}_{k1} = A_{k1} = -1$, and $A_{kj} = -1$ for each $j = 2, \ldots, l$.

Therefore, for for each $v_k \in V \setminus \{v_1, \ldots, v_l\}$,

$$L f(v_k) = f(v_k) - \frac{1}{\deg v_k} \left( \sum_{k'} A_{kk'} f(v_{k'}) + \sum_{j=1}^{l} A_{kj} f(v_j) \right).$$
Figure 3: A 5–hyperfern with 3 twins.

\[ \hat{f}(v_k) - \frac{1}{l} \left( \sum_{k'} (-l) \hat{f}(v_{k'}) + (-1) \sum_{j=1}^{l} \hat{f}(v_1) \right) \]

\[ = \hat{f}(v_k) + \sum_{k'} \hat{f}(v_{k'}) + \hat{f}(v_1) \]

\[ = \hat{L} \hat{f}(v_k) = \lambda \cdot \hat{f}(v_k) = \lambda \cdot f(v_k). \]

This proves that \( \lambda \) is an eigenvalue for \( L \), and \( f \) is a corresponding eigenfunction.

**Remark 6.10.** Proposition 6.9 tells us that, in order to know the spectrum of a \((l, r)\)–hyperflower, we can simply study the spectrum of the \((1, r)\)–hyperflower obtained by deleting \( l - 1 \) pendant vertices and the hyperedges containing them, and then add \( l - 1 \) 's to the spectrum.

**Lemma 6.11.** The \((l, r)\)-hyperflower with \( t \) twins has eigenvalue \( t \) with multiplicity at least \( l - 1 \).

**Proof.** It follows from Proposition 3.4.

**Lemma 6.12.** The spectrum of the \((l, 1)\)-hyperflower with \( t \) twins is given by:

- 0, with multiplicity \( N - l \);
- \( t \), with multiplicity \( l - 1 \);
- \( \lambda_N = N - tl + t \).

**Proof.** Since all hyperedges have cardinality \( N - tl + t \), by [3] Theorem 3.1 we have that \( \lambda_N = N - tl + t \). Furthermore, by Proposition 3.4, \( t \) is an eigenvalue with multiplicity at least \( l - 1 \). Since, clearly, \( N - tl + t > t \), we have listed \( l \) eigenvalues whose sum is \( N \). By Remark 5.2, 0 has multiplicity \( N - l \).

### 7 Hyperferns

**Definition 7.1.** We say that \( \Gamma = (\mathcal{V}, \mathcal{H}) \) is a \( l \)-hyperfern with \( t \) twins (Figure 3) if one can decompose the vertex set as \( \mathcal{V} = \mathcal{V}_1^L \sqcup \ldots \sqcup \mathcal{V}_l^L \sqcup \mathcal{V}_1^R \sqcup \ldots \sqcup \mathcal{V}_l^R \sqcup \mathcal{W} \) such that:

- \( \#\mathcal{W} = l \), \( \#\mathcal{V}_i^L = t \) and \( \#\mathcal{V}_i^R = t \), for each \( i = 1, \ldots, l \);
- \( \mathcal{W} = \{w_1, \ldots, w_l\} \) and
  \[ \mathcal{H} = \{\mathcal{W}, h_1^L, \ldots, h_l^L, h_1^R, \ldots, h_l^R\}, \]
  where \( h_i^L = \mathcal{V}_i^L \cup \{w_i\} \) and \( h_i^R = \mathcal{V}_i^R \cup \{w_i\} \), for each \( i = 1, \ldots, l \).
We say that $\mathcal{W}$ is the core and we say that all vertices in $\mathcal{V} \setminus \mathcal{W}$ are peripheral nodes.

Remark 7.2. Hyperferns can offer, for instance, a model for transportation networks in which the central hyperedge $\mathcal{W}$ represents a highway, left and right roads represent the left and right on ramps and off ramps, and the vertices are traffic junctions.

Proposition 7.3. The spectrum of the $l$–hyperfern with $t$ twins is such that:

1. There are $2tl + l$ eigenvalues whose sum is $2tl + l$;
2. $0$ is an eigenvalue with multiplicity $2tl - l - 1$;
3. $t$ is an eigenvalue with multiplicity at least $l$.

Proof. The first claim follows by construction, since the $l$–hyperfern with $t$ twins has $2tl + l$ vertices. By Remark 5.2, since hyperferns are hypertrees with $2l + 1$ hyperedges, we have that $0$ has multiplicity $N - 2l - 1 = 2tl + l - 2l - 1 = 2tl - l - 1$. Also, by Proposition 3.4, since there are $l$ 2-duplicate families of $t$ twins, $t$ is an eigenvalue with multiplicity at least $l$. $\square$

8 Complete Hypergraphs

Definition 8.1 ([$4$]). We say that $\Gamma = (\mathcal{V}, \mathcal{H})$ is the $c$-complete hypergraph, for some $c \geq 2$, if $\mathcal{V}$ has cardinality $N$ and $\mathcal{H}$ is given by all possible $\binom{N}{c}$ hyperedges of cardinality $c$.

Proposition 8.2. The spectrum of the $c$-complete hypergraph is given by:

- $\frac{N - c}{N - 1}$, with multiplicity $N - 1$;
- $c$, with multiplicity $1$.

Proof. By [3, Theorem 3.1], $\lambda_N = c$. Now, observe that each vertex $v$ has degree $d := \binom{N - 1}{c - 2}$ is constant for all $i \neq j$. Therefore, $a_{ij} = -\binom{N - 2}{c - 2}$ and

$$Lf(v) = f(v) - \frac{a}{d} \left( \sum_{w \neq v} f(w) \right) = f(v) + \frac{c - 1}{N - 1} \left( \sum_{w \neq v} f(w) \right), \quad \forall v \in \mathcal{V}.$$ 

Now, for each $i = 2, \ldots, N$, let $f(v_1) := 1$, $f(v_i) := -1$ and $f := 0$ otherwise. Then,

- $Lf(v_1) = 1 - \frac{c - 1}{N - 1} = \frac{N - c}{N - 1} \cdot f(v_1)$,
- $Lf(v_i) = -1 + \frac{c - 1}{N - 1} = \frac{N - c}{N - 1} \cdot f(v_i)$, and
- $Lf(v_j) = 0 = \frac{N - c}{N - 1} \cdot f(v_j)$ for all $j \neq 1, i$.

Therefore, the $f_i$’s are $N - 1$ linearly independent eigenfunctions for $\frac{N - c}{N - 1}$. This proves the claim. $\square$

9 Lattice Hypergraphs

Lattice graphs, which are also called grid graphs, are well known both in graph theory and in applications [28–35]. For instance, they model topologies used in transportation networks, such as Manhattan street network, and crystal structure used in crystallography. These structures and their spectra are also widely used in statistical mechanics, in the study of ASEP, TASEP and SSEP models [26–38], which have applications in the Ising model, (lattice) gas and which also describe the movement of ribosomes along the mRNA [39]. In this section we generalize the notion of lattice graph to the case of hypergraphs.
Definition 9.1. Given $l \in \mathbb{N}_{\geq 2}$, we define the $l$-lattice as the hypergraph $\Gamma = (\mathcal{V}, \mathcal{H})$ on $l^2$ nodes and $2l$ hyperedges that can be drawn so that:

- The vertices form a $l \times l$ grid, and
- The hyperedges are exactly the rows and the columns of the grid (Figure 4).

Proposition 9.2. The spectrum of the $l$-lattice is given by:

- $0$, with multiplicity $l^2 - 2l + 1$;
- $\frac{l}{2}$, with multiplicity $2(l - 1)$;
- $l$, with multiplicity $1$.

Proof. By [3, Theorem 3.1], $\lambda_\rho = l$. Furthermore, by [1, Corollay 33], since the maximum number of linearly independent hyperedges is $2l - 1$, this implies that $0$ is an eigenvalue with multiplicity $l^2 - 2l + 1$.

Now, observe that $\deg v = 2$ for each $v$ and

$$A_{ij} = \begin{cases} -1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \neq j$. Therefore,

$$L f(v) = f(v) + \frac{1}{2} \left( \sum_{w \sim v} f(w) \right), \quad \text{for all } v \in \mathcal{V}. \quad (1)$$

Fix a row of the $l$-lattice given by the vertices $w_1, \ldots, w_l$. For $i = 1, \ldots, l - 1$, let $f_i : \mathcal{V} \to \mathbb{R}$ be $1$ on the neighbors of $w_i$ with respect to the row, $-1$ on the neighbors of $w_i$ with respect to its column, and $0$ otherwise. Then, by (1), it is easy to check that $f_i$ is an eigenfunction for $\frac{l}{2}$. Since the $f_i$'s are linearly independent, this proves the claim.

\[\square\]

10 Hypercycles and hyperpaths

Definition 10.1. Fix $N$ and $l \in \{2, \ldots, \frac{N}{2}\}$. We say that $\Gamma = (\mathcal{V}, \mathcal{H})$ is the $l$-hypercycle on $N$ nodes (Figure 5) if $\mathcal{V} = \{v_1, \ldots, v_N\}$, $\mathcal{H} = \{h_1, \ldots, h_N\}$ and

$$h_i = \{v_i, \ldots, v_{i+l-1}\},$$

where we let $v_{N+i} := v_i$ for each $i = 1, \ldots, N$. 

Figure 4: A 3-lattice.
Theorem 10.2. The eigenvalues of the $l$-hypercycle are

$$\lambda_i = 1 + \frac{\sum_{r=1}^{N} m(r) \cdot \cos \left( \frac{2\pi ir}{N} \right)}{l}, \quad \text{for } i = 1, \ldots, N,$$

where $m : \{0, \ldots, N\} \to \mathbb{Z}$ is such that:

- $m(r) := l - r$ for all $r \in \{1, \ldots, l - 1\}$
- $m(N - k) := m(k) = l - k$ for all $k \in \{1, \ldots, l - 1\}$
- $m := 0$ otherwise.

Proof. By construction, all vertices have degree $l$. Therefore, by [4, Remark 2.17], proving the claim is equivalent to proving that the eigenvalues of the adjacency matrix are

$$\mu_i = -\sum_{r=1}^{N} m(r) \cdot \cos \left( \frac{2\pi ir}{N} \right), \quad \text{for } i = 1, \ldots, N.$$

Observe that the adjacency matrix can be written as

$$A = \begin{bmatrix}
0 & l-1 & l-2 & \ldots & 1 & 0 & \ldots & 0 & 1 & \ldots & l-2 & l-1 \\
l-1 & 0 & l-1 & l-2 & \ldots & 1 & 0 & \ldots & 0 & 1 & \ldots & l-2 \\
l-2 & l-1 & 0 & \ldots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
l-2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
l-1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
l-1 & l-2 & \ldots & \ldots & 0 & \ldots & 0 & 1 & \ldots & \ldots & l-1 & 0
\end{bmatrix}$$

Therefore,

$$A = -\begin{bmatrix}
m(0) & m(N-1) & m(N-2) & \ldots & m(1) \\
m(1) & m(0) & m(N-1) & \ldots & m(2) \\
m(2) & m(1) & m(0) & \ldots & m(3) \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
m(N-1) & m(N-2) & m(N-3) & \ldots & m(0)
\end{bmatrix}$$

where

- $m(r) := l - r$ for all $r \in \{1, \ldots, l - 1\}$
- $m(N - k) := m(k) = l - k$ for all $k \in \{1, \ldots, l - 1\}$
- $m := 0$ otherwise.
Hence, $A$ is a (symmetric) circulant matrix. By [40], the eigenvalues of $A$ are
\[ \mu_i = -\sum_{r=1}^{N} m(r) \cdot \cos \left( \frac{2\pi ir}{N} \right), \quad \text{for } i = 1, \ldots, N. \]

Similarly to the hypercycle, we can define a hyperpath as follows.

**Definition 10.3.** Fix $N$ and $l \in \{2, \ldots, N\}$. We say that $\Gamma = (\mathcal{V}, \mathcal{H})$ is the $l$-hyperpath on $N$ nodes (Figure 6) if $\mathcal{V} = \{v_1, \ldots, v_N\}$, $\mathcal{H} = \{h_1, \ldots, h_{N-l+1}\}$ and
\[ h_i = \{v_{i}, \ldots, v_{i+l-1}\}. \]

**Remark 10.4.** By construction, it is clear that the $N - l + 1$ hyperedges of the $l$-hyperpath are linearly independent. Therefore, every hyperpath is a hypertree and, in particular, this implies that the multiplicity of the eigenvalue 0 in this case is $l - 1$. Note that studying the entire spectrum of hyperpaths is difficult for general hypergraphs, since we cannot longer determine the spectrum of a path from the spectra of cycles as done for graphs (see for instance [31]).

**11 Conclusions**

The families of hypergraphs that we investigated are either generalizations of well known graphs or they offer practical interpretations in various fields. In future works it would be interesting, for instance, to investigate the spectral properties of the structures defined here in the more general case.
where the chemical hypergraphs are not necessarily bipartite and therefore they do not reduce to classical hypergraphs. It would be also interesting to study spectral properties of new structures, as well as to see more practical results.

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