Isotropic AdS/CFT fireball

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We study the AdS/CFT thermodynamics of the spatially isotropic counterpart of the Bjorken similarity flow in $d$-dimensional Minkowski space with $d \geq 3$, and of its generalisation to linearly expanding $d$-dimensional Friedmann-Robertson-Walker cosmologies with arbitrary values of the spatial curvature parameter $k$. The bulk solution is a nonstatic foliation of the generalised Schwarzschild-AdS black hole with a horizon of constant curvature $k$. The boundary matter is an expanding perfect fluid that satisfies the first law of thermodynamics for all values of the temperature and the spatial curvature, but it admits a description as a scale-invariant fluid in local thermal equilibrium only when the inverse Hawking temperature is negligible compared with the spatial curvature length scale. A Casimir-type term in the holographic energy-momentum tensor is identified from the threshold of black hole formation and is shown to take different forms for $k \geq 0$ and $k < 0$.

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1 Introduction

AdS/CFT duality permits one to obtain the thermodynamics of a conformal field theory (CFT) from classical solutions to higher-dimensional gravitational field equations, provided the CFT spacetime is attached to the gravitational solution as a conformal boundary under suitable asymptotic conditions. The prototype prediction is the equation of state of static $\mathcal{N} = 4$ Super-Yang-Mills matter in Minkowski space \[1, 2\], obtained from the thermodynamics of an appropriate higher-dimensional black hole solution. Currently there is a growing interest in situations where the CFT configuration is not static and the duality yields predictions for hydrodynamical transport coefficients \[3, 4, 5\]. A case of particular interest is CFT as an approximation to QCD plasma in (3+1) spacetime dimensions, in a kinematical configuration that approximates a head-on ion collision at relativistic velocities \[6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\] or otherwise invokes space-time dependence \[18, 19, 20\].

In the (3+1)-dimensional ion collision setting the kinematics distinguishes between the longitudinal spatial dimension, in which the original ion velocities are, and the two transverse spatial dimensions. This spatial anisotropy is an essential part of the physics of the situation, but it is also a source of significant mathematical subtlety, and the bulk geometry is at present known only in terms of asymptotic expansions \[16, 17\]. By contrast, in the simplified setting of (1+1)-dimensional ion collisions, where no spatial anisotropy can occur, the bulk geometry can be found explicitly \[21, 22\], and in the exactly boost-invariant special case the boundary fluid turns out to an expanding ideal gas in local thermal equilibrium \[23\].

In this paper we consider a setting in which the CFT lives on a $d$-dimensional spacetime with $d \geq 3$ and the fluid is expanding isotropically from a point-like explosion. The case of main interest is $d$-dimensional Minkowski spacetime, but it will turn out useful to consider at the same time the whole family of linearly expanding Friedmann-Robertson-Walker (FRW) cosmologies with arbitrary real values of the spatial curvature parameter $k$, with the isotropically expanding fluid in Minkowski space obtained as the special case $k = -1$. The bulk solution can be found explicitly and is the generalised Schwarzschild-AdS$_{d+1}$ black hole \[24\],

$$ds^2 = -F dt^2 + \frac{dr^2}{F} + r^2 d\Omega_{d-1}^2(k), \quad F = \left(\frac{r}{\mathcal{L}}\right)^2 + k - \frac{\mu}{(r/\mathcal{L})^{d-2}}, \quad (1.1)$$

where $\mathcal{L}$ is the length scale set by the bulk cosmological constant and the rest of the notation is explained in Section \[2\] and in the Appendix. We find:

- We first address (Sections \[3\] and \[4\]) AdS/CFT thermodynamics on the conformal boundary of the bulk solution in the static foliation \[1.1\], in which the conformal boundary is a static FRW cosmology. While some of this material is well known \[25, 26, 27, 28, 29\], the novel feature is that keeping $k$ general allows the boundary volume $V$ to be treated as a continuous thermodynamical variable, since $V \sim (\mathcal{L}/\sqrt{|k|})^{d-1}$ for $k \neq 0$. We find that the boundary fluid satisfies the first law of thermodynamics in the usual form,

$$dE = T dS - p \, dV. \quad (1.2)$$

However, the relation

$$E = TS - pV, \quad (1.3)$$
which would usually be expected to hold for a fluid in local thermal equilibrium, is satisfied only in the special case \( k = 0 \), in which the FRW cosmology is flat. For \( k \neq 0 \), \((1.3)\) is satisfied asymptotically when the bulk black hole is so large that the thermal length scale \( 1/T \) is negligible compared with the curvature length scale \( \mathcal{L}/\sqrt{|k|} \). The reason for \((1.3)\) not being satisfied for \( k \neq 0 \) is that the presence of spatial curvature breaks scale-invariance in the fundamental thermodynamic relation of the fluid.

• We then write (Sections 5 and 6) the bulk solution \((1.1)\) in a foliation that makes the boundary metric into a FRW cosmology with a linearly expanding scale factor and analyse the AdS/CFT thermodynamics of the boundary fluid. For \( k = -1 \) the fluid expands isotropically and boost-invariantly from a point-like explosion in a spacetime that is flat, but just written in a cosmological form that is adapted to the explosion; for \( k \neq -1 \), by contrast, the spacetime is a genuinely curved expanding cosmology, and the explosion occurs at a pointlike initial curvature singularity. The first law is satisfied in the usual form \((1.2)\), both when the differentials refer to the time-evolution due to the expansion, and also when the differentials refer to varying the spatial volume and the temperature at a fixed cosmological time by varying \( k \) and \( \mu \) in the bulk solution. However, the relation \((1.3)\) is again satisfied exactly only for \( k = 0 \), and for \( k \neq 0 \) it is satisfied asymptotically in the limit of large bulk black hole. The reasons are the same as in the static case.

The fact that \((1.3)\) is not satisfied for \( k \neq 0 \) should not be surprising from the bulk viewpoint. The bulk counterpart of the boundary spatial curvature is the curvature of the black hole horizon, and it has been observed both within [34] and without [35] the AdS/CFT context that horizon curvature can interfere with the description of matter as a fluid in local thermal equilibrium. A perhaps surprising feature in the nonstatic boundary case is, however, that the relation \((1.3)\) is not satisfied for \( k = -1 \), when the fluid is just expanding isotropically in Minkowski spacetime. In this case the boundary curvature that breaks the scale invariance is not that of the spacetime in which the fluid expands, but just the intrinsic curvature of the boost-invariant spacelike hyperboloids that are surfaces of constant proper time in the frame comoving with the fluid.

Owing to the spatial isotropy that we assumed at the outset, the fluid flow has a vanishing shear tensor both in the static and non-static cases. The shear viscosity does therefore not contribute to the energy-momentum tensor, and we cannot use the system to examine the usual AdS/CFT prediction \( \eta = s/(4\pi) \) for the shear viscosity \( \eta \) and the entropy density \( s \) [36]. In the non-static case the fluid flow does have a nonvanishing expansion, and the kinematics would hence allow a bulk viscosity term to appear in the energy-momentum tensor, but the coefficient of this term vanishes simply because our bulk action is pure Einstein and does not contain fields that could break conformal invariance on the boundary.

A conceptual issue with the boundary matter is whether the holographic energy-momentum tensor \( T_{\mu\nu} \) should be attributed to the boundary fluid in its entirety, or whether it should be split as \( T_{\mu\nu} = T_{\mu\nu}^{(\text{Casimir})} + T_{\mu\nu}^{(\text{fluid})} \), where \( T_{\mu\nu}^{(\text{Casimir})} \) is a Casimir-type vacuum

\footnote{For \( d = 4 \), a generalisation of this coordinate transformation to an arbitrarily-expanding FRW cosmology is given in [30]. Related discussions from the brane world perspective are given in [31, 32, 33].}
energy-momentum tensor. For example, a conformal scalar field in four-dimensional flat spacetime has a nonvanishing energy-momentum tensor in the vacuum state adapted to the isotropic expansion [37], and this tensor turns out to share the tensorial structure and time-dependence of our holographic energy-momentum tensor. Similarly, a conformal scalar field in two-dimensional flat spacetime in the analogous vacuum state has a nonvanishing energy-momentum tensor ([38], equation (7.24)), and this tensor shares the tensorial structure and time-dependence of the holographic energy-momentum tensor for the (1 + 1)-dimensional Bjorken flow [23]. When \( T^{(\text{Casimir})}_{\mu\nu} \) is chosen so that \( T^{(\text{fluid})}_{\mu\nu} \) vanishes at the threshold of a disappearing bulk black hole, the fluid in the (1+1)-dimensional Bjorken flow was found to be scale invariant [23], and we shall adopt this normalisation also in the present paper. We shall see, however, that the core conclusions of consistency of (1.2) for all \( k \) and the inconsistency of (1.3) for \( k \neq 0 \) are not sensitive to the choice of \( T^{(\text{Casimir})}_{\mu\nu} \) and hold even if one chooses for example \( T^{(\text{Casimir})}_{\mu\nu} = 0 \). Prospects of identifying a Casimir term in the case of the usual, non-isotropic Bjorken flow are discussed in [39].

In the limit \( d \to 2 \), our formulas reproduce the \( d = 2 \) results of [23]. The qualitative difference that appears in this limit is that for the fluid expanding in Minkowski space, the relation (1.3) holds for \( d = 2 \) but not for \( d \geq 3 \). From the viewpoint of this limit, the exact solvability of the case \( d = 2 \) can be attributed to the absence of spatial anisotropy in two dimensions, rather than just to the local triviality of (2 + 1)-dimensional Einstein gravity [40, 41].

The metric signature we use is mostly plus, \((-++\cdots+)\), and we use the \((+++\cdots+)\) sign convention of [42], except that in the single formula (4.12) the conventions are those of [43]. The boundary dimension is \( d \) and the bulk dimension is \( d + 1 \). For simplicity of the presentation we assume throughout \( d > 2 \), although most of the formulas remain valid also for \( d = 2 \) when interpreted in a proper limiting sense, reducing to those given in [23]. The Lorentz-signature Einstein-Hilbert-York-Gibbons-Hawking action is [35, 44]

\[
S = \frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{-g} \left( R + \frac{d(d-1)}{\mathcal{L}^2} \right) - \frac{1}{8\pi G_{d+1}} \int_{\partial M} d^{d}x \sqrt{-\gamma} K(\gamma),
\]

and the bulk field equations are

\[
R_{MN} = -\frac{d}{\mathcal{L}^2} g_{MN},
\]

where the positive constant \( \mathcal{L} \) is the length scale of the cosmological constant. To compute the holographic energy-momentum tensor [43], the bulk metric will be written in the Fefferman-Graham form

\[
ds^2 = g_{MN} dx^M dx^N = \frac{\mathcal{L}^2}{z^2} [g_{\mu\nu} dx^\mu dx^\nu + dz^2],
\]

where the boundary is at \( z \to 0 \) and \( g_{\mu\nu} \) has the small \( z \) expansion

\[
g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} z + \cdots + g^{(d)}_{\mu\nu} z^d + (\text{logarithmic term}) + \cdots.
\]

For odd \( d \), the holographic energy-momentum tensor on the conformal boundary metric \( g^{(0)}_{\mu\nu} \) is

\[
T_{\mu\nu} = \frac{d \mathcal{L}^{d-1}}{16\pi G_{d+1}} g^{(d)}_{\mu\nu}. \tag{1.8}
\]
For even $d$ there are in general additional terms \[43\]. Those for $d = 4$ are given below in \[3.12\].

## 2 AdS$_{d+1}$ black hole with constant curvature horizon

In this section we review the thermodynamics of the generalised Schwarzschild-AdS$_{d+1}$ black hole in the case of an arbitrary constant curvature horizon.

The starting point is the bulk solution \[24\]

\[
\begin{align*}
\text{d}s^2 &= - F d\text{t}^2 + \frac{d\text{r}^2}{F} + r^2 d\Omega_{d-1}^2(k), \\
F(\hat{r}) &= \hat{r}^2 + k - \frac{\mu}{\hat{r}^{d-2}},
\end{align*}
\]

where $\hat{r} = r/L$, $\mu$ is a dimensionless parameter and $d\Omega_{d-1}^2(k)$ is a $(d - 1)$-dimensional positive definite metric of constant curvature, with the Ricci scalar normalised to $(d - 1)(d - 2)k$ \[15\]. As $d\Omega_{d-1}^2(k) = |k|^{-1}d\Omega_{d-1}^2(k/|k|)$ for $k \neq 0$, it would be possible (and is indeed conventional) to normalise any nonzero $k$ to $k/|k| = \pm 1$ by a redefinition of $t$ and $r$, but we shall keep $k \in \mathbb{R}$ in order to treat the spatial volume as a thermodynamical boundary variable in Section 4.

Relevant properties of $d\Omega_{d-1}^2(k)$ are reviewed in the Appendix.

The topology of the space $M_{d-1}$, on which $d\Omega_{d-1}^2(k)$ is the metric, will not play an essential role in what follows, and the simplest choice is to take $d\Omega_{d-1}^2(1)$ to be the unit sphere, $d\Omega_{d-1}^2(-1)$ to be the unit hyperbolic space and $d\Omega_{d-1}^2(0)$ to be Euclidean $\mathbb{R}^{d-1}$, as done in the Appendix. We will however be using thermodynamical quantities that are defined with respect to unit volume of $d\Omega_{d-1}^2(k)$, and it will reduce the verbiage to regard $d\Omega_{d-1}^2(k)$ as a metric on a compact space for all values of $k$, with the finite volume denoted by $\Omega_{d-1}(k)$.

For $k \neq 0$, the scaling of the metric with $k$ implies $\Omega_{d-1}(k) = |k|^{-(d-1)/2} \Omega_{d-1}(k/|k|)$, while $\Omega_{d-1}(0)$ may take arbitrary values \[15\].

We denote the coordinates on $M_{d-1}$ by $y^i$, $i = 1, \ldots, d-1$, and write $d\Omega_{d-1}^2(k) = h_{ij} dy^i dy^j$.

The metric \[24\] is asymptotically locally AdS$_{d+1}$ at $r \to \infty$, and $\mu$ is proportional to the Arnowitt-Deser-Misner (ADM) mass \[46\]. For $\mu = 0$ the spacetime is locally AdS$_{d+1}$, in unusual coordinates. For $\mu \neq 0$ there is a scalar curvature singularity at $r \to 0$.

The global structure and the thermodynamics are analysed in \[28 \ 46\] (for certain special cases, see also \[47 \ 48 \ 49 \ 50 \ 51\]). We summarise here the properties that are relevant for the rest of the paper, taking care to write the formulas for $k \in \mathbb{R}$ instead of the conventional normalisation to $k = \pm 1$ or $k = 0$.

We consider the parameter range in which the spacetime is a nonextremal black hole, with the black (and white) hole horizon at $\hat{r} = \hat{r}_+^\pm$ and the static exterior region at $\hat{r}_+^\pm < \hat{r} < \infty$. An elementary analysis of the function $F$ \[2.11\] shows that we can adopt $\hat{r}_+^\pm$ as the independent parameter, with the range $\hat{r}_e < \hat{r}_+ < \infty$, where

\[
\hat{r}_e = \begin{cases} 
0 & \text{for } k \geq 0, \\
\sqrt{-\frac{k(d-2)}{d}} & \text{for } k < 0.
\end{cases}
\]
The mass parameter $\mu$ is then given by
\[ \mu = \hat{r}_+^d + k \hat{r}_+^{d-2}, \] (2.3)
and its range is $\mu_{\text{ex}} < \mu < \infty$, where
\[ \mu_{\text{ex}} = \hat{r}_{\text{ex}}^d + k \hat{r}_{\text{ex}}^{d-2} = \begin{cases} 
0 & \text{for } k \geq 0, \\
-(-k)^{d/2} \left( \frac{d-2}{d} \right)^{d/2} & \text{for } k < 0.
\end{cases} \] (2.4)

The black hole temperature $T$, defined with respect to the timelike Killing vector $\partial_t$, is
\[ T = \frac{1}{4\pi L} \left. \frac{dF}{dr} \right|_{r=\hat{r}_+} = \frac{1}{4\pi L} \left( d\hat{r}_+ + \frac{k(d-2)}{\hat{r}_+} \right). \] (2.5)

The Bekenstein-Hawking entropy $S$ equals $1/(4G_{d+1})$ times the horizon area,
\[ S = \frac{(L\hat{r}_+)^{d-1} \Omega_{d-1}(k)}{4G_{d+1}}. \] (2.6)

Keeping $\Omega_{d-1}(k)$ fixed and allowing $\hat{r}_+$ to vary, it is seen from (2.3), (2.5) and (2.6) that the first law of black hole thermodynamics takes the form
\[ \frac{(d-1)L^{d-2} \Omega_{d-1}(k)}{16\pi G_{d+1}} d\mu = T dS. \] (2.7)

Note that for $k \geq 0$ the limit $\hat{r}_+ \to \hat{r}_{\text{ex}}$ yields a locally AdS spacetime, but for $k < 0$ this limit yields an extremal black hole with $T = 0$ and $S > 0$. Note also that the relative sign of the two terms in (2.5) changes with the sign of $k$. $T$ has a local minimum as a function of $\hat{r}_+$ for $k > 0$, and this minimum leads to the Hawking-Page phase transition [52, 53], but no such phase transition occurs for $k \leq 0$ [51].

3 Static FRW as the conformal boundary

In this section we write out the holographic energy-momentum tensor for the bulk solution (2.1) when the infinity is approached via surfaces of constant $r$. The results agree with those found in [25, 28] in the counterterm formalism of Balasubramanian and Kraus [25], but we include here some of the computational steps as done in the counterterm formalism of de Haro, Skenderis and Solodukhin [43], since comparison with this computation will allow a concise treatment of the nonstatic foliation in Section 5.

To transform the metric (2.1) to the canonical form (1.6), we must integrate $dr/\sqrt{F} = -Ldz/z$, or equivalently
\[ \frac{d\hat{r}}{\sqrt{F}} = -\frac{d\hat{z}}{\hat{z}}, \] (3.1)
where $\hat{z} = z/L$ and the boundary condition is
\[ \hat{z}\hat{r} \to 1 \text{ as } \hat{r} \to \infty. \] (3.2)
It follows that the conformal boundary metric is
\[ g^{(0)}_{\mu\nu} = \text{diag} (-1, L^2 h_{ij}) . \] (3.3)

The boundary spacetime is thus the static \( d \)-dimensional FRW spacetime,
\[ ds^2_{\text{static}} = -dt^2 + L^2 d\Omega_d^2(k), \] (3.4)
with the spatial curvature parameter \( k \) and the constant scale factor \( L \).

In the special case \( k = 0 \), equation (3.1) integrates to
\[ \hat{r}^d/2 = \frac{1}{z^{d/2}} \left( 1 + \frac{\mu}{4} \hat{z}^d \right), \] (3.5)
and the bulk metric takes the form
\[ ds^2 = \frac{L^2}{z^2} \left\{ -\frac{\left( 1 - \frac{k \hat{z}^4}{4} \right)^2}{4 \hat{z}^{d-2}} d\tau^2 + \left( 1 + \frac{\mu \hat{z}^d}{4} \right)^4 \pi^2 L^2 d\Omega_{d-1}^2(0) + dz^2 \right\}. \] (3.6)

We shall consider general values of \( k \). We look first at arbitrary odd \( d \), where it suffices to use (1.8). The case of even \( d \) is more complicated because of the conformal anomaly \[43\], and we consider only \( d = 4 \) and \( d = 6 \).

### 3.1 Odd \( d \)

Let \( d \) be odd. The energy-momentum tensor (1.8) is then proportional to the coefficient \( g^{(d)}_{\mu\nu} \) in (1.7). To evaluate this coefficient, we observe from (3.2) that the term in \( F \) (2.1b) proportional to \( \mu \) contributes to \( g^{(d)}_{tt} \) by \( \mu / L^d \). From (3.2) it further follows that all the other contributions to \( g^{(d)}_{\mu\nu} \) are proportional to \( g^{(0)}_{\mu\nu} \), and the coefficient of these contributions is fixed by observing that \( T_{\mu\nu} \) must be traceless. We hence have

\[ T_{\mu
u}^{\cdot} = \frac{L^{d-1}}{4\pi G_{d+1} 4L^d} \frac{\mu}{4L^d} \text{diag} (1 - d, 1, 1, \ldots, 1) . \] (3.7)

### 3.2 \( d = 4 \)

Let \( d = 4 \). The transformation between \( \hat{r} \) and \( \hat{z} \) reads
\[ \hat{r}^2 = \frac{1}{z^2} \left( 1 - \frac{k \hat{z}^2}{2} + \frac{\hat{z}^4}{4} \right), \] (3.8)
where
\[ \hat{z}_+ = \frac{2}{\sqrt{2\hat{r}_+^2 + k}}, \] (3.9)
and the bulk metric becomes
\[ ds^2 = \frac{L^2}{z^2} \left[ \left( 1 - \frac{\dot{z}^4}{\dot{z}^4 + \hat{z}_+^4} \right)^2 dt^2 + \left( 1 - \frac{k\dot{z}^2}{2} + \frac{\dot{z}^4}{\dot{z}^4 + \hat{z}_+^4} \right) L^2 d\Omega_3^2(k) + dz^2 \right], \quad (3.10) \]

Note that \( \hat{z}_+ \) is always positive. The exterior region is at \( 0 < \dot{z} < \hat{z}_+ \), the infinity is at \( \dot{z} \to 0 \) and the horizon is at \( \dot{z} \to \hat{z}_+ \). If desired, we can replace \( \hat{r}_+ \) by \( \hat{z}_+ \) as the independent parameter: the range of \( \hat{z}_+ \) is then \( 0 < \hat{z}_+ < 2/\sqrt{k} \) for \( k > 0 \) and \( 0 < \hat{z}_+ < \infty \) for \( k \leq 0 \). The expression for \( \mu \) in terms of \( \hat{z}_+ \) is
\[ \mu = \frac{4}{\hat{z}_+^4} - \frac{k^2}{4}. \quad (3.11) \]

The formula for the holographic energy-momentum tensor for \( d = 4 \) reads [43]
\[ T_{\mu\nu} = \frac{L^3}{4\pi G_5} \left[ g_{\mu\nu}^{(4)} - \frac{1}{3} g_{\mu\nu}^{(0)} [(\mathrm{Tr} (g_{(2)})]^2 - \mathrm{Tr} (g_{(2)}^2)] - \frac{1}{7} (g_{(2)} g_{(0)}^{-1} g_{(2)})_{\mu\nu} + \frac{1}{4} (\mathrm{Tr} (g_{(2)}) g_{(2)})_{\mu\nu} \right]. \quad (3.12) \]

A direct computation yields
\[ T_{\mu\nu} = \frac{1}{4\pi G_5 L \hat{z}_+^4} \text{ diag } (-3, 1, 1, 1) = \frac{L^3}{4\pi G_5} \left[ \mu + \frac{1}{4} k^2 \right] \text{ diag } (-3, 1, 1, 1). \quad (3.13) \]

### 3.3 \( d = 6 \)

Let \( d = 6 \). The expansion [(3.17)] reads
\[ g_{\mu\nu} dx^\mu dx^\nu = - \left[ 1 + \frac{1}{2} k\dot{z}^2 + \frac{1}{16} k^2 \dot{z}^4 - \frac{5}{6} \mu \dot{z}^6 + O(\dot{z}^8) \right] dt^2 + \left[ 1 - \frac{1}{2} k\dot{z}^2 + \frac{1}{16} k^2 \dot{z}^4 + \frac{1}{6} \mu \dot{z}^6 + O(\dot{z}^8) \right] L^2 d\Omega_3^2(k). \quad (3.14) \]

Using the formulas in [43], we find
\[ T_{\mu\nu} = \frac{L^5}{4\pi G_7} \left[ \mu - \frac{1}{2} k^3 \right] \text{ diag } (-5, 1, 1, 1, 1). \quad (3.15) \]

### 3.4 Comparison

When the bulk is pure AdS, \( \mu = 0 \), formulas [(3.7), (3.13) and (3.15)] show that \( T_{\mu\nu} \) vanishes in odd dimensions for all \( k \) but in \( d = 4 \) and \( d = 6 \) only for \( k = 0 \).

For \( k \neq 0 \), there are three qualitative differences between \( d = 4 \) and \( d = 6 \):

- For \( k > 0 \), the signs of the correction terms in [(3.13) and (3.15)] differ. When the bulk is pure AdS, \( T_{00} \) is positive in \( d = 4 \) but negative in \( d = 6 \).
• For $k < 0$, the signs of the correction terms in (3.13) and (3.15) agree, so that for pure AdS bulk $T_{00}$ is positive both in $d = 4$ and in $d = 6$. However, in the extremal black hole limit, $\mu \rightarrow \mu_{\text{ex}}$, $T_{\mu\nu}$ vanishes in $d = 4$ but remains nonvanishing, with positive $T_{00}$, in $d = 6$.

• Let $k$ have either sign. In $d = 4$, $T_{\mu\nu}$ is nonvanishing everywhere in the (nonextremal, for $k < 0$) black hole parameter range. In $d = 6$, $T_{\mu\nu}$ changes sign at $\hat{\tau}_+ = \frac{1}{2} \sqrt{|k| \left( \sqrt{5} - \text{sgn}(k) \right)}$, which is in the black hole parameter range. Note that for $k < 0$ this value is at negative $\mu$, between pure AdS and the extremal black hole limit. We are not aware of geometrically special features of the $d = 6$ bulk at this value of $\mu$.

4 Thermodynamics on static FRW

In this section we discuss the thermodynamics of the holographic energy-momentum tensor in the metric $ds^2_{\text{static}}$ (3.4). We take $d$ to be either odd or equal to 4 or 6, with the energy-momentum tensor given respectively by (3.7), (3.13) or (3.15).

To begin, we must decide whether to attribute all of $T_{\mu\nu}$ to a fluid or whether $T_{\mu\nu}$ could contain also a Casimir-type contribution. For the reasons discussed in the Introduction, we write

$$T_{\mu\nu} = T^{(\text{Casimir})}_{\mu\nu} + T^{(\text{fluid})}_{\mu\nu},$$

(4.1)

where $T^{(\text{Casimir})}_{\mu\nu}$ is the limiting value of $T_{\mu\nu}$ when the bulk black hole vanishes for $k \geq 0$ and becomes extremal for $k < 0$,

$$
\begin{align*}
\left( T^{(\text{Casimir})} \right)_\mu^\nu & = \frac{\mathcal{L}^{d-1}}{4\pi G_{d+1}} \frac{\mu_{\text{ex}}}{4\mathcal{L}^d} \text{ diag } (1 - d, 1, 1, \ldots, 1), \quad \text{(for } d \text{ odd)} \quad (4.2a) \\
\left( T^{(\text{Casimir})} \right)_\mu^\nu & = \frac{\mathcal{L}^3}{4\pi G_5} \frac{\mu_{\text{ex}} + \frac{1}{4} k^2}{4\mathcal{L}^4} \text{ diag } (-3, 1, 1, 1), \quad \text{(for } d = 4) \quad (4.2b) \\
\left( T^{(\text{Casimir})} \right)_\mu^\nu & = \frac{\mathcal{L}^5}{4\pi G_7} \frac{\mu_{\text{ex}} - \frac{1}{8} k^3}{4\mathcal{L}^6} \text{ diag } (-5, 1, 1, 1, 1), \quad \text{(for } d = 6) \quad (4.2c)
\end{align*}

The remainder is given by

$$
\left( T^{(\text{fluid})} \right)_\mu^\nu = \frac{\mathcal{L}^{d-1}}{4\pi G_{d+1}} \frac{(\mu - \mu_{\text{ex}})}{4\mathcal{L}^d} \text{ diag } (1 - d, 1, 1, \ldots, 1).$$

(4.3)

We interpret $T^{(\text{Casimir})}_{\mu\nu}$ as the vacuum energy-momentum and $T^{(\text{fluid})}_{\mu\nu}$ as the energy-momentum due to a fluid. By construction, $T^{(\text{fluid})}_{\mu\nu}$ then vanishes when the bulk black hole vanishes for $k \geq 0$ and becomes extremal for $k < 0$.

$T^{(\text{fluid})}_{\mu\nu}$ is of the perfect fluid form,

$$T^{(\text{fluid})}_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + pg^{(0)}_{\mu\nu},$$

(4.4)
where \( u^\mu = (\partial_t)^\mu = (1, 0, \ldots, 0) \) and
\[
p = \frac{\mathcal{L}^{d-1}}{4\pi G_{d+1}} \frac{(\mu - \mu_{\text{ex}})}{4\mathcal{L}^d}, \quad \epsilon = (d - 1)p. \tag{4.5}
\]

As \( \partial_t \) is induced by the bulk Killing vector with respect to which we defined the Hawking temperature in Section 2, the bulk black hole induces for the boundary fluid the temperature \( T \) given by (2.5) and the entropy density \( s \) given from (2.6) by
\[
s = \frac{S}{V} = \frac{\hat{r}_+^{d-1}}{4G_{d+1}}, \tag{4.6}
\]
where \( V = \mathcal{L}^{d-1} \Omega_{d-1}(k) \) is the spatial volume.

Now, can the boundary fluid be interpreted as a fluid in local thermal equilibrium? The fluid certainly satisfies the first law of thermodynamics. If the spatial volume \( V \) is held constant, \( k \) is a constant, the only independent variable is \( \hat{r}_+ \), and equations (2.7), (4.5) and (4.6) show that the first law reads
\[
d\epsilon = T ds. \tag{4.7}
\]
If also \( V \) is regarded as a thermodynamical variable, \( k \) is determined in terms of \( V \) by
\[
k = \text{sgn}(k)(V_0/V)^{2/(d-1)},
\]
where
\[
V_0 = \begin{cases} 
\mathcal{L}^{d-1} \Omega_{d-1}(k/|k|) & \text{for } k \neq 0, \\
0 & \text{for } k = 0. \tag{4.8}
\end{cases}
\]
The first law can then be verified to hold in the usual form
\[
dE = TdS - pdV, \tag{4.9}
\]
where \( E = V\epsilon \) is the total energy. The thermodynamical potential \( E(S, V) \) in the microcanonical ensemble can be written in the form
\[
E(S, V) = \frac{(d - 1)V}{16\pi G_{d+1}\mathcal{L}} \left[ \left( \frac{4G_{d+1}S}{V} \right)^{d/(d-1)} + \text{sgn}(k) \left( \frac{V_0}{V} \right)^{2/(d-1)} \left( \frac{4G_{d+1}S}{V} \right)^{(d-2)/(d-1)} \right.
\]
\[
+ \Theta(-k) \frac{2}{d-2} \left( \frac{d-2}{d} \right)^{d/2} \left( \frac{V_0}{V} \right)^{d/(d-1)} \right], \tag{4.10}
\]
where \( \Theta \) is the Heaviside function. Note that as \( \Theta \) and the signum function appear in (4.10) with positive powers of \( V_0 \), the values of these functions at the origin do not affect (4.10).

Where the interpretation of the boundary matter as a fluid in local thermal equilibrium breaks down, however, is that the Helmholtz free energy density \( f \equiv \epsilon - Ts \) of such a fluid should satisfy \( f = -p \) \cite{54}, but for us this holds only when \( k = 0 \). From (2.5), (4.5) and (4.6) we find
\[
f + p = \frac{2k\hat{r}_+^{d-2} - d\mu_{\text{ex}}}{16\pi G_{d+1}\mathcal{L}}, \tag{4.11}
\]
and from (2.2) and (2.4) it follows that the right-hand side vanishes when \( k = 0 \) but cannot vanish for any value of \( \hat{r}_+ \) when \( k \neq 0 \). The physical reason is that the spatial curvature breaks scale invariance: a simultaneous scaling of \( E, S \) and \( V \), which three would conventionally be considered the extensive variables, leaves the fundamental thermodynamic relation (4.10) invariant only for \( k = 0 \). For \( k \neq 0 \), scale invariance holds asymptotically in the large volume limit, in which the second and third term in (4.10) are small compared with the first term. From the bulk point of view this is the limit of a black hole with large ADM mass.

We emphasise that this thermodynamic breakdown for \( k \neq 0 \) is not in the first law of thermodynamics but in the phenomenological description of the boundary matter as a thermalised fluid. That such a breakdown can be expected when the characteristic thermal wave length \( 1/T \) is not small compared with the curvature length scales was noted already in [35], and in the AdS/CFT context it has been observed for example in [34].

We end the section with four comments. First, if \( V \) is held constant, \( k \) is a constant, and the first law for the Helmholtz free energy density reads \( df = -s \, dT \). As a consistency check on the thermodynamics, we should therefore be able to obtain \( f \) also by evaluating the partition function of the canonical ensemble. In AdS/CFT duality, the semiclassical approximation to this partition function is \( Z_{\text{CFT}} = e^{-I} \), where \( I \) is the minimal supergravity action appropriate for fixing the induced boundary metric, in positive definite signature, and the temperature is identified as the inverse period of the imaginary time in the boundary metric. For definiteness, we specialise to \( d = 4 \). The regularised Lorentz-signature Einstein action is given in Appendix B of [43] and reads

\[
S = \frac{1}{16\pi G_5} \left\{ \int d^5x \sqrt{-g} \left( \frac{8}{\mathcal{L}^2} \right) - \int d^4x \sqrt{-\gamma} \left[ 2K - \frac{6}{\mathcal{L}} + \frac{\mathcal{L}}{2} R(\gamma) + (\log \text{term}) \right] \right\},
\]

(4.12)

where \( \gamma_{\mu\nu} \) is the induced metric on the surface \( \hat{z} = \hat{\epsilon} \), \( K \) is the extrinsic curvature of this surface, the logarithmic term is given in [43], and (unlike elsewhere in the present paper) all the sign conventions are those of [43]. Inserting the metric (3.10) in (4.12), the integration over the coordinates \( y^i \) gives the factor \( L_3 \Omega_3(k) = V \), and the integration over \( t \) gives, when continued to positive definite signature, the factor \( 1/T \). The remaining integration in the volume integral part is

\[
\int_{\hat{z}_{\hat{\epsilon}}}^{\hat{z}_+} \frac{d\hat{z}}{\mathcal{L}^{\frac{1}{2}}} \left( 1 - \frac{\hat{z}^4}{\hat{z}_{\hat{\epsilon}}^4} \right) \left( 1 - \frac{k\hat{z}^2}{2} + \frac{\hat{z}^4}{\hat{z}_{\hat{\epsilon}}^4} \right) = \frac{1}{4\hat{\epsilon}^4} - \frac{k}{4\hat{\epsilon}^2} + \frac{k\hat{z}_{\hat{\epsilon}}^2 - 1}{2\hat{z}_{\hat{\epsilon}}^4} + O(\hat{\epsilon}^2).
\]

(4.13)

The boundary term in (4.12) can be verified to cancel the divergent terms in (4.13) and to bring in no contributions that remain finite as \( \hat{\epsilon} \to 0 \). From the finite term in (4.13) we then find, after continuation to positive definite signature, that

\[
I = \frac{1}{4\pi G_5 L} \left( \frac{V}{T} \right) \frac{k\hat{z}_{\hat{\epsilon}}^2 - 1}{\hat{z}_{\hat{\epsilon}}^4} = (V/T)f_{\text{tot}},
\]

(4.14)

where \( f_{\text{tot}} \equiv \epsilon_{\text{tot}} - Ts \) and the total energy density \( \epsilon_{\text{tot}} \equiv T_{00} = T_{00}^{\text{(Casimir)}} + T_{00}^{\text{(fluid)}} \) includes both the fluid part and the Casimir part. Note that \( f_{\text{tot}} \) and \( f \) differ only for \( k > 0 \). This is the consistency check we sought for.
We note in passing that the action computed in [29] by a volume subtraction method agrees with our action (4.14) only for $k = 0$. For $k \neq 0$, the difference affects the large temperature expansion of the Helmholtz free energy so that the coefficient of the $k^2$ term in equation (43) in [29] is halved.

Second, one may ask to what extent our conclusions depend on splitting the energy-momentum tensor into the Casimir part and the fluid part as in (4.1). It can be verified that the consistency of the first law for any $k$ and the breaking of scale invariance for $k \neq 0$ hold for $d = 4$ and $d = 6$ even if we choose the Casimir term so that the fluid energy-momentum tensor vanishes for $\mu = 0$, and they hold for any $d$ even if we choose the Casimir term to vanish.

Third, for $k < 0$ the split (4.1) implies that the energy density, the pressure and the temperature all go to zero in the extremal black hole limit, but the entropy density does not. In a fluid interpretation, this would mean that the fluid becomes highly degenerate in the zero temperature limit. It has however been argued that in this limit thermal fluctuations become so large that a statistical description of the system is no longer viable [55].

Fourth, we note that for $d = 4$, the fundamental relation (4.10) with $k = 0$ has the same structure as the leading term in the fundamental relation obtained in [34] for boundary hydrodynamics in the presence R-charges in the bulk.

5 Linearly expanding FRW as the conformal boundary

In this section we write the generalised Schwarzschild-AdS$_{d+1}$ black hole (2.1) in coordinates in which the conformal boundary is a linearly expanding FRW cosmology.

Consider the metric (2.1a), and allow to begin with $F$ to be an arbitrary positive function of $r$ only. We replace $(t, r)$ by the positive-valued coordinates $(\tau, z)$ of dimension length by

\[ \frac{dr}{\mathcal{L}\sqrt{F + 1}} = -d \left[ \log \left( \frac{z}{\tau} \right) \right], \]

\[ \frac{dt}{\mathcal{L}} = \frac{d\tau}{\tau} - \frac{1}{F} d \left[ \log \left( \frac{z}{\tau} \right) \right]. \]

The metric becomes

\[ ds^2 = \frac{\mathcal{L}^2}{z^2} \left[ -(F + 1) \left( \frac{z}{\tau} \right)^2 d\tau^2 + \left( \frac{r}{\mathcal{L}\tau} \right)^2 \tau^2 d\Omega_{d-1}^2(k) + dz^2 \right], \]

where it follows from (5.1a) that $r$ and $F$ depend on the coordinates $(\tau, z)$ only through the combination $z/\tau$. The Killing vector $\partial_t$ of (2.1a) takes in (5.2) the form $\mathcal{L}^{-1}(\tau \partial_{\tau} + z \partial_z)$. Note that $z$ differs from the coordinate $z$ used in Section 3.

To proceed, suppose that $F = (r/\mathcal{L})^2 + O(1)$ at $r \to \infty$, as the case is in (2.1b), and choose the integration constant in (5.1a) so that $z/\tau = (\mathcal{L}/r) + O \left[ (\mathcal{L}/r)^3 \right]$ at $r \to \infty$. The metric (5.2) has then at $z \to 0$ the Fefferman-Graham form of (1.6) and (1.7), with the boundary metric

\[ g_{\mu\nu}^{(0)} = \text{diag} \left( -1, \tau^2 h_{ij} \right). \]
The boundary spacetime is thus the $d$-dimensional FRW cosmology with the spatial curvature parameter $k$ and the scale factor $\tau$,

$$ds^2_{\text{exp}} = -d\tau^2 + \tau^2 d\Omega^2_{d-1}(k).$$

(5.4)

As reviewed in the Appendix, $ds^2_{\text{exp}}$ has a scalar curvature singularity at $\tau \to 0$ for $k \neq -1$, but for $k = -1$ it is flat and its comoving world lines are those of an isotropic and boost-invariant explosion in Minkowski space.

We now specialise to $F$ given by (2.1b). Comparing (5.1a) and (5.2) with respectively (3.1) and (2.1) shows that the metric (5.2) is obtained from the Fefferman-Graham bulk metric of Section 3 by keeping the overall factor $L^2/z^2$ and elsewhere just making the replacements $t \to \tau$, $k \to k + 1$ and $L \to \tau$. When $d$ is arbitrary and $k = -1$, we in particular have from (3.6)

$$ds^2 = \frac{L^2}{z^2} \left\{ -\left[ 1 - \frac{\mu}{4} \left( \frac{z}{\tau} \right)^d \right]^2 \frac{d\tau^2}{d(x^2)} + \left[ 1 + \frac{\mu}{4} \left( \frac{z}{\tau} \right)^d \right]^4 \tau^2 d\Omega^2_{d-1}(-1) + dz^2 \right\},$$

(5.5)

and when $d = 4$ and $k$ is arbitrary, we have from (5.10)

$$ds^2 = \frac{L^2}{z^2} \left\{ -\left[ 1 - \left( \frac{\mu + \frac{1}{4} (k+1)^2}{4} \right) \left( \frac{z^4}{\tau^4} \right)^2 \right] \frac{d\tau^2}{2\tau^2} + \left[ 1 + \left( \frac{\mu + \frac{1}{4} (k+1)^2}{4} \right) \right] \tau^2 d\Omega^2_3(k) + dz^2 \right\}.$$

(5.6)

For $k = 0$, the metric (5.6) reduces to the case $r(\tau) = \tau/\tau_0$ of the bulk solution (17)–(18) of [56]. The loci of the coordinate singularities in these nonstatic forms of the bulk metric can be analysed as in [23], for example by adapting the Eddington-Finkelstein techniques of [17], but this will not be needed in what follows.

The holographic energy-momentum tensor can be computed as in Section 3. We find

$$T_{\mu\nu} = \frac{L^{d-1}}{4\pi G_d+1} \frac{\mu}{4^d} \frac{d^d}{d\tau^d} \text{ diag} \left( 1 - d, 1, 1, \ldots, 1 \right), \quad \text{(for } d \text{ odd)}$$

(5.7a)

$$T_{\mu\nu} = \frac{L^3}{4\pi G_5} \frac{\mu + \frac{1}{2} (k+1)^2}{4^4} \text{ diag} \left( -3, 1, 1, 1 \right), \quad \text{(for } d = 4)$$

(5.7b)

$$T_{\mu\nu} = \frac{L^5}{4\pi G_7} \frac{\mu - \frac{1}{2} (k+1)^2}{4^6} \text{ diag} \left( -5, 1, 1, 1, 1 \right). \quad \text{(for } d = 6)$$

(5.7c)
As a consistency check, recall that since both the metric (5.2) and the static metric used in Section 3 have the asymptotic form of (1.6) and (1.7), the conformal boundary metrics $ds^2_{\text{static}}$ and $ds^2_{\exp}$ must be related by a conformal transformation. From (5.1) it follows that on a surface of constant $r$ we have $\tau = (\text{const}) \times \exp(t/\mathcal{L})$, and the conformal transformation therefore reads

$$ds^2_{\exp} = (\tau^2/\mathcal{L}^2) \, ds^2_{\text{static}}.$$  

(5.8)

For odd $d$, the energy-momentum tensor in $ds^2_{\exp}$ should therefore be $(\mathcal{L}/\tau)^d$ times that in $ds^2_{\text{static}}$, and formulas (3.7) and (5.7a) show that this is indeed the case. For even $d$ the situation is more complicated since the transformation includes also the conformal anomaly. For definiteness, consider $d = 4$ with $k = -1$. In this case the anomaly term that must be added to (3.13) before scaling by $(\mathcal{L}/\tau)^4$ is given in formula (6.139) of [38] and is a linear combination of the tensors

$$(1) \quad H_{\mu}^{\nu} \equiv 2R_{\mu}^{\nu} - 2g_{\mu}^{\nu} \Box R - \frac{1}{2} R^2 g_{\mu}^{\nu} + 2RR_{\mu}^{\nu} = \frac{6}{\mathcal{L}^4} \text{diag}(-3, 1, 1, 1),$$  

(5.9a)

$$(3) \quad H_{\mu}^{\nu} \equiv \frac{1}{12} R^2 g_{\mu}^{\nu} - R_{\alpha \beta} R_{\mu}^{\alpha \beta} = \frac{1}{\mathcal{L}^4} \text{diag}(3, -1, -1, -1),$$  

(5.9b)

evaluated for $ds^2_{\text{static}}$ as shown. The anomaly contribution can thus be made to precisely cancel the term $\frac{1}{4}$ from $\mu + \frac{1}{4}$ in (3.13), and give agreement with (5.7b), provided the coefficients of the tensors (5.9) satisfy an appropriate linear relation. This linear relation is in particular satisfied by the anomaly coefficients that occur in $\mathcal{N} = 4$ SU(N) Super-Yang-Mills theory [57].

6 Thermodynamics on linearly expanding FRW: the isotropic fireball

We are now ready to address the thermodynamics of the comoving fluid in the expanding FRW cosmology (5.4). We take $d$ to be either odd or equal to 4 or 6. The treatment follows closely that of the case $d = 2$ in [23].

As in Section 4 we split the energy-momentum tensor (5.7) into a fluid part $T_{\mu \nu}^{(\text{fluid})}$ and a Casimir part $T_{\mu \nu}^{(\text{Casimir})}$, so that $T_{\mu \nu}^{(\text{fluid})}$ vanishes when the bulk black hole vanishes for $k \geq 0$ and becomes extremal for $k < 0$. The result is

$$( T_{\mu}^{(\text{fluid})} )_{\nu} = \frac{\mathcal{L}^{d-1}}{4\pi G_{d+1}} \frac{\mu - \mu_{\text{ex}}}{4\tau^d} \text{diag} \left( 1 - d, 1, 1, \ldots, 1 \right)$$  

(6.1)

and

$$( T_{\mu}^{(\text{Casimir})} )_{\nu} = \frac{\mathcal{L}^{d-1}}{4\pi G_{d+1}} \frac{\mu_{\text{ex}}}{4\tau^d} \text{diag} \left( 1 - d, 1, 1, \ldots, 1 \right), \quad \text{(for } d \text{ odd})$$  

(6.2a)

$$( T_{\mu}^{(\text{Casimir})} )_{\nu} = \frac{\mathcal{L}^3}{4\pi G_5} \frac{\mu_{\text{ex}} + \frac{1}{4}(k + 1)^2}{4\tau^4} \text{diag} \left( -3, 1, 1, 1 \right), \quad \text{(for } d = 4 \text{)}$$  

(6.2b)

$$( T_{\mu}^{(\text{Casimir})} )_{\nu} = \frac{\mathcal{L}^5}{4\pi G_7} \frac{\mu_{\text{ex}} - \frac{1}{8}(k + 1)^3}{4\tau^6} \text{diag} \left( -5, 1, 1, 1, 1 \right), \quad \text{(for } d = 6 \text{)}$$  

(6.2c)
As a consistency check on the split, we observe that for \(d = 4\) with \(k = -1\), our \(T^{(\text{Casimir})}_{\mu\nu}\) (6.2b) does have the form of the energy-momentum tensor of a conformal scalar field in the vacuum whose positive frequency modes are conformally related to those that define the static vacuum on the static universe (6.3) [37].

\(T^{(\text{fluid})}_{\mu\nu}\) (6.1) has the perfect fluid form (4.4) with a comoving fluid, \(u^\mu = (1, 0, \ldots, 0)\), and

\[
p = \frac{L^{d-1}}{4\pi G_{d+1}} \left( \frac{(d - k_{\text{ex}})}{4\tau^d} \right), \quad \epsilon = (d - 1)p.
\]  

(6.3)

The energy density and pressure (6.3) hence come from those in the static case (4.5) by scaling with the factor \(L^{d-1}/\tau^d\), for both odd and even \(d\); this follows from the conformal transformation properties of the stress-energy tensor and from our having grouped the conformal anomalies for even \(d\) in \(T^{(\text{Casimir})}_{\mu\nu}\). To define other time-dependent thermodynamical quantities, we similarly scale those of the static case by the dimensionally appropriate power of \(L/\tau\), so that \(s\) is scaled by \(L^{d-1}/\tau^d\) and \(T\) by \(L/\tau\). This gives

\[
T = \frac{1}{4\pi\tau} \left( d\hat{r}_+ + \frac{k(d - 2)}{\hat{r}_+} \right), \quad s = \frac{L^{d-1}}{4G_{d+1}} \frac{\hat{r}_+^{d-1}}{\tau^{d-1}}.
\]  

(6.4)

It is readily verified that the expanding fluid satisfies the first law of thermodynamics, in two distinct senses. The volume on the spatial hypersurface of constant \(\tau\) equals \(V = \tau^{d-1}\Omega_{d-1}(k)\), and we can define the total energy and entropy at given \(\tau\) in terms of the densities \(\epsilon\) and \(s\), given in (6.3) and (6.4), by respectively \(E = V\epsilon\) and \(S = Vs\). Now, first, if \(\tau\) is regarded fixed and \(\hat{r}_+\) and \(k\) are allowed to vary, the computations in Section 4 go through without change and show that the first law at fixed \(\tau\) holds in the standard form (4.9). Second, and perhaps more relevantly for the expanding fluid, the first law holds in the form (4.9) also when \(\hat{r}_+\) and \(k\) are regarded as fixed and we follow the expansion of the fluid in \(\tau\). Note that this second sense of the first law follows just by virtue of the inverse powers of \(\tau\) in \(s\) and \(\epsilon\) and the tracelessness of \(T^{(\text{fluid})}_{\mu\nu}\), and it does not hinge on the details of how the thermodynamical variables depend on \(\hat{r}_+\) and \(k\).

As in the static case of Section 4, the interpretation of the boundary matter as a fluid in local thermal equilibrium breaks down for \(k \neq 0\) in that the Helmholtz free energy density \(f \equiv \epsilon - Ts\) is not equal to \(-p\), except asymptotically in the limit of large \(\hat{r}_+\): the value of \(f + p\) in the expanding case is given by just multiplying the static value (4.11) by \(L^d/\tau^d\). The physical reason is again that for \(k \neq 0\) the nonvanishing spatial curvature breaks the scale invariance in \(E, V\) and \(S\). When the scale invariance does hold, however, the boundary matter is an expanding perfect fluid in local thermal equilibrium.

7 Conclusions

We have shown that the AdS/CFT dual of a linearly expanding \(d\)-dimensional FRW cosmology, with \(d \geq 3\) and an arbitrary spatial curvature parameter \(k\), is a nonstatic foliation of the generalised Schwarzschild-AdS\(_{d+1}\) black hole with a horizon of constant curvature \(k\). The boundary matter is a perfect fluid that satisfies the first law of thermodynamics, but
it admits a description as a scale-invariant fluid in local thermal equilibrium only when the inverse Hawking temperature is negligible compared with the spatial curvature length scale.

Geometrically, the absence of scale invariance for generic values of the parameters is a direct consequence of the nonvanishing spatial curvature on the boundary and should hence perhaps not be surprising. However, the absence of the scale invariance implies that the Helmholtz free energy density of the fluid is not equal to minus the pressure, which means that the fluid does not admit a conventional interpretation as a fluid in local thermal equilibrium. It may be surprising that this absence of a conventional thermalised fluid interpretation occurs even in the special case of the Milne universe, where the fluid is just an isotropically and boost-invariantly expanding fireball in Minkowski space. As a side remark, we noted that a similar absence scale invariance occurs already in the more familiar case of a static bulk foliation, where the boundary is the static FRW cosmology.

The case \( d \geq 3 \) analysed in the present paper has both similarities to and differences from the case \( d = 2 \) analysed in [23]. For \( d = 2 \) the spatial curvature necessarily vanishes, and the adiabatic expansion found in [23] for \( d = 2 \) can be regarded as a direct dimensional continuation of the adiabatic expansion found in the present paper for \( d \geq 3 \) with \( k = 0 \). However, a difference is that for \( d = 2 \) the linearly expanding cosmology describes a fluid expanding isotropically in \((1 + 1)\)-dimensional Minkowski spacetime, whereas for \( d \geq 3 \) the Minkowski space fluid description requires \( k = -1 \). The adiabatically expanding isotropic fireball in \( d = 2 \) Minkowski space does therefore not generalise into an adiabatically expanding isotropic fireball in higher-dimensional Minkowski spaces.

While the spatial isotropy has made our model exactly solvable, the model contains significantly less structure than the model with boost invariance in one spatial dimension and translational invariance in the others [16, 17]. In particular, the phenomenological interest of our model as a description of expanding gauge theory plasma, especially in the special case of a fluid expanding isotropically in Minkowski space, is limited by the absence of viscosity. One might attempt to bring in shear viscosity by giving the bulk black hole angular momentum, but the rotating \( d = 2 \) case analysed in [23] (and in which case there is no shear viscosity) raises doubt as to whether the resulting flow configuration would be phenomenologically relevant. A more likely prospect for introducing viscosity in the isotropic setting might be to include metric deformations or further bulk fields [58, 59, 60, 61, 62, 63].

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Appendix: Spaces of constant curvature and linearly expanding FRW cosmologies

In this appendix we collect relevant properties of spaces of constant curvature and linearly expanding FRW cosmologies. As in the main text, we assume $d \geq 3$.

Let $d\Omega_{d-1}^2(k)$ denote for $k > 0$ the metric on the $(d-1)$-dimensional sphere of curvature radius $1/\sqrt{k}$, for $k = 0$ the Euclidean metric on $\mathbb{R}^{d-1}$, and for $k < 0$ the metric on the $(d-1)$-dimensional hyperbolic space of curvature radius $1/\sqrt{-k}$. We denote the coordinates by $y^i$, $i = 1, \ldots, d-1$, and write the metric as $d\Omega_{d-1}^2(k) = h_{ij}dy^i dy^j$. $d\Omega_{d-1}^2(k)$ is a metric of constant curvature [45], with the Riemann tensor

$$(d-1)R_{ijmn} = k(h_{im}h_{jn} - h_{in}h_{jm}), \quad (A.1)$$

and the Ricci scalar equals $(d-1)R = k(d-1)(d-2)$. The explicit embeddings in $d$-dimensional Euclidean space for $k > 0$ and $d$-dimensional Minkowski space for $k < 0$ are given in [64]. In terms of the cosmological area-radius coordinate $r$, the metric takes the form [64]

$$d\Omega_{d-1}^2(k) = \frac{dr^2}{1 - kr^2} + r^2d\Omega_{d-2}^2(1). \quad (A.2)$$

In the main text we encounter the $d$-dimensional linearly expanding FRW cosmology (5.4),

$$ds^2_{\text{exp}} = -d\tau^2 + \tau^2 d\Omega_{d-1}^2(k), \quad (A.3)$$

where $0 < \tau < \infty$. The only nonvanishing components of the Riemann tensor of $ds^2_{\text{exp}}$ are the fully spatial components, given by

$$(d)R_{ijmn} = \tau^2(k+1)(h_{im}h_{jn} - h_{in}h_{jm}), \quad (A.4)$$

and the Ricci scalar equals $(d)R = (k+1)(d-1)(d-2)/\tau^2$. $ds^2_{\text{exp}}$ has hence a scalar curvature singularity at $\tau \to 0$ when $k \neq -1$ but is flat when $k = -1$.

In the flat special case $k = -1$, $ds^2_{\text{exp}}$ is known as (the $d$-dimensional generalisation of) the Milne universe [65],

$$ds^2_{\text{Milne}} = -d\tau^2 + \tau^2 d\Omega_{d-1}^2(-1). \quad (A.5)$$

$ds^2_{\text{Milne}}$ covers the interior of the future light cone of the origin in $d$-dimensional Minkowski space, the comoving world lines of constant $y^i$ are inertial world lines that start from the origin, and the surfaces of constant $\tau$ are spacelike hyperboloids of constant proper time $\tau$ from the origin. $ds^2_{\text{Milne}}$ is thus adapted to a spherically symmetric and boost-invariant explosion in Minkowski space, starting at one point. The transformation between (A.5) and the usual Minkowski coordinates can be found in [64] [65].

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