Manifestation of nematic degrees of freedom in the Raman response function of iron pnictides

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We establish a relation between the Raman response function and the electronic contribution to the nematic susceptibility within the spin-driven approach to electron nematicity of the iron based superconductors. The spin-driven nematic phase, characterized by the broken $C_4$ symmetry, but unbroken $O(3)$ spin-rotational symmetry, is generated by the presence of magnetic fluctuations associated with the striped phase. It occurs as a separate phase between $T_N$ and $T_s$ in systems where the structural and magnetic phase transitions are separated. Detecting the presence of nematic degrees of freedom in iron-based superconductors is a difficult task, since it involves measuring higher order spin correlation functions. We show that the nematic degrees of freedom manifest themselves in the experimentally measurable Raman response function. We calculate the Raman response function in tetragonal phase in the large $N$ limit by considering Aslamazov-Larkin type of diagrams that contain a series of inserted fermionic boxes that resemble the nematic coupling constant of the theory. These diagrams effectively account for collisions between spin fluctuations. By summing an infinite number of such higher order diagrams, we demonstrate that the electronic Raman response function shows a clear maximum at the structural phase transition. Hence, the Raman response function can be used to probe nematic degrees of freedom.

I. INTRODUCTION

Iron-based superconductors show rich phase diagrams, with the high-temperature superconducting dome being in the close proximity to an antiferromagnetic striped phase that sets in at a temperature $T_N$. In addition, a structural phase transition at $T_s$, from the high-temperature tetragonal phase into an orthorhombic phase, has been shown to closely follow the magnetic transition: $T_s \geq T_N$. It was proposed that spin-fluctuations, associated with the striped phase, lead to emergent electronic nematic degrees of freedom at higher temperatures. These electronic nematic degrees of freedom then couple to the lattice and induce the structural phase transition to the orthorhombic phase.

There are several strong evidences for an electronic nematic state: resistivity-anisotropy measurements, the measurement of the elastoresistance, the observed anisotropies in thermopower, the optical conductivity, the torque magnetometry, and in STM measurements. Measurements of the elastic constants showed the shear modulus strongly softened in the high temperature tetragonal phase. A theoretical analysis based upon nematic fluctuations due to a strong magneto-elastic coupling showed that the inverse shear modulus is proportional to the susceptibility of the nematic order parameter $\chi_{\text{nem}}$, which diverges at the structural phase transition, explaining the softening of the shear modulus. The most direct evidence for the magnetic origin of nematicity so far is the scaling of the shear modulus and the NMR spin-lattice relaxation rate, seen in iron pnictides. An interesting open issue in this context is the lack of such scaling behavior in iron-chalcogenides.

A relation between nematicity and the Raman response of iron based superconductors was already studied in Ref.25, and in Ref.26 where the Kramers-Kronig transform of the Raman response was compared with the shear modulus. Here we demonstrate, based on an explicit microscopic theory, that the low frequency Raman response in $B_{1g}$ channel is given by

$$R_{B_{1g}}(\omega) = \frac{R_0(\omega)}{1 - \tilde{g} \int \chi_{\text{mag}}^2(q)}.$$  

where $R_0(\omega)$ stands for the leading order Aslamazov-Larkin diagram, $\chi_{\text{mag}}$ magnetic susceptibility, and $\tilde{g}$ the dynamic nematic coupling constant of the theory. On the other hand, the susceptibility of the nematic order parameter of our model, in the large $N$ limit is given by

$$\chi_{\text{nem}} = \frac{\int q \chi_{\text{mag}}^2(q)}{1 - g_{\text{stat}} \int q \chi_{\text{mag}}^2(q)},$$  

where in the pure electronic theory $\tilde{g} = g_{\text{stat}}$. This would lead to the divergence of the Raman response function at the structural phase transition. However, one needs to include the effect of the lattice in order to analyze this problem. We do so by introducing nemato-elastic coupling and find that, in this case, $g_{\text{stat}} = \tilde{g} + \frac{\gamma_0}{\gamma_{\text{stat}}}$. Here $\gamma_{\text{el}}$ is the elasto-nematic coupling constant, and $\gamma_{\text{stat}}$ the bare value of the orthorhombic elastic constant. We show that when magnetic and structural phase transitions are split, this leads to a maximum at the amplitude of the electronic Raman response function at the structural phase transition, in agreement with the recent experiments. Raman response function could then be used to probe the dynamic excitation spectrum of the nematic degrees of freedom, similar to inelastic neutron scattering that probes the dynamic spin excitation spectrum.
We start from the spin-driven scenario for the nematic phase, in which magnetic fluctuations stabilize a nematic phase, characterized by broken $C_4$ symmetry. The Raman response function measures electronic density-density correlator weighted by appropriate form factors. Since electrons interact with spin fluctuations, the latter will manifest themselves in the Raman response function in the form of corrections to the electron self energy and the Raman vertex, formally expressed in terms of Aslamazov-Larkin diagrams\cite{AL}. We show that the leading order Aslamazov-Larkin (AL) diagram disappears for the $B_{2g}$ symmetry of the form factor, but not for the $B_{1g}$ symmetry, in agreement with the experiment\cite{EXPERIMENT}. However, this leading order approach cannot account for the rapid increase in the amplitude of the Raman response function as one approaches the structural transition, as seen in the experiments of Refs.\cite{REF1,REF2}. Instead it would predict a similar increase only at the magnetic phase transition. Therefore, we go beyond the leading order approximation, and take into account collisions between spin fluctuations that become more and more important as one approaches the nematic / structural transition. Our approach is based on the exact same collisions between spin-fluctuations that led to the emergence of spin-induced nematicity in the first place. Formally this is accomplished by inserting a series of quartic paramagnon couplings, mediated by electronic excitations, into the Raman response function. Such quartic couplings contain a product of four fermionic Green functions and give rise to a peak of the electronic Raman response function at the structural phase transition. In summary, we have explicitly shown that the spin-nematic scenario can explain the Raman data, as was proposed in Ref.\cite{OTHERREF}.

The paper is organized as follows. In Section II we present the microscopic model for the spin-driven nematic phase. We calculate the effective action and analyze it in the large $N$ limit. We derive the condition for the susceptibility of the nematic order parameter to diverge, following reference\cite{REFERENCE}. In Section III we show how to calculate the Raman response function using a diagrammatic approach. We first calculate the leading order Aslamazov Larkin diagram, as has previously been done in Ref.\cite{PREVIOUS} and show that it vanishes in $B_{2g}$ channel, but remains finite in $B_{1g}$ channel. We then calculate higher order diagrams that take into account collisions between spin-fluctuations. Finally, after summing an infinite number of these higher-order diagrams within a controlled $1/N$ expansion, we show that the maximum of the Raman response function occurs when the nematic susceptibility diverges, i.e. at the structural phase transition. We present our conclusions in Section IV.

II. MICROSCOPIC MODEL: SPIN DRIVEN NEMATICITY

Two different approaches have been proposed in order to explain the origin of nematic phase in pnictides and its relation to the magnetic phase – the orbital scenario\cite{ORBITAL} and spin-driven nematic scenario\cite{SPIN,SPIN1}. For a discussion of these approaches see for example Ref\cite{DISCUSSION}. Here we follow approach of a spin-driven nematic state. In this scenario, the nematic phase is stabilized by magnetic fluctuations that are associated with the stripe density wave (SDW) phase. The order parameter of the SDW state\cite{SDW} can be characterized by an $O(3) \times Z_2$ manifold - $O(3)$ is the spin-rotational symmetry and $Z_2$ a discrete symmetry associated with the choice of the ordering wave-vector, $Q^X = (\pi, 0)$ or $Q^Y = (0, \pi)$. Let the two order parameters associated with these two ordering wave vectors be $\Delta^X$ and $\Delta^Y$ respectively. The SDW state is characterized by broken $O(3)$ and $Z_2$ symmetries. On the mean-field level the breaking of $Z_2$ and $O(3)$ symmetry occurs simultaneously. However, when one includes fluctuations, these transitions can be split. In case of joint transitions, they are usually both first order transitions. The criterion for breaking the discrete $Z_2$ symmetry via a second order transition is a threshold value of the magnetic correlation length $\xi$. Decreasing the temperature leads to an increase of $\xi$. Before the correlation length diverges at the magnetic phase transition temperature, the threshold value will be reached and spin-driven nematicity sets in. This naturally explains why the magnetic and structural phase boundaries are correlated and leads to an intermediate phase with $Z_2$ symmetry breaking without $O(3)$ symmetry breaking. This intermediate state is the nematic phase in the pnictides. It is characterized by unequal strength of the magnetic fluctuations associated with the ordering wave vectors $Q^X$ and $Q^Y$: $\langle \Delta^2_X - \Delta^2_Y \rangle \neq 0$, but no magnetic order, $\langle \Delta_{X,Y} \rangle = 0$.

In what follows we will outline the steps of Ref\cite{REF} and outline the mathematical model for spin-driven nematic phase. We start from a simplified itinerant model where we include the bands near the $\Gamma-$point and the $X-$ and $Y-$ points in the Brillouin zone. For our main result no explicit knowledge of the detailed parametrization of the band structure is necessary. However, in order to obtain explicit numerical results we use the simplified model of Ref.\cite{REF}. We consider parabolic dispersions with
where \( m_i \) are the band masses, \( \epsilon_0 \) is the offset energy, and \( \mu \) denotes the chemical potential. The corresponding Fermi surfaces are shown in Fig. 1.

In order to study the established stripe magnetic, we consider the Hamiltonian that contains the interactions in the spin channel with momenta near \( Q_X \) and \( Q_Y \):

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}
\]

\[
\mathcal{H}_0 = \sum_{i,k} \epsilon_{i,k} c_{i,k}^\dagger c_{i,k}
\]

\[
\mathcal{H}_{\text{int}} = - \frac{1}{2} u_s \sum_{i,q} s_{i,q} \cdot s_{i,-q}.
\]

Here, \( c_{i,k}^\dagger \) is the creation operator of an electron with momentum \( k \), spin \( \alpha \) in the band \( i \). The spin operator is given by

\[
s_{i,q} = \sum_k c_{i,k+q0}^\dagger \lambda_{\alpha\beta} c_{i,k} \beta
\]

where \( \lambda_{\alpha\beta} \) denotes the \( N^2 - 1 \) component vector of the generators of the \( SU(N) \) algebra. In the case \( N = 2 \) it holds \( \lambda_{\alpha\beta} = \frac{1}{2} \sigma_{\alpha\beta} \) with vector of the Pauli matrices \( \sigma \), \( u_s \) is the coupling in spin channel, which can be expressed in terms of density-density and pair-hopping interactions between hole and electron pockets²⁷.

The partition function is given by

\[
Z = \int dc_{i,k} dc_{i,k}^\dagger e^{-\beta \mathcal{H}}.
\]

Since, the interaction Hamiltonian is quadratic in the fermionic variables, we can decouple it using a Hubbard-Stratonovich decoupling in the spin-channel. This way, we eliminate the quartic interaction between fermions at the expense of a functional integral over two additional bosonic fields \( \Delta_X \) and \( \Delta_Y \), with \( N^2 - 1 \) components. The bosonic fields couple linearly to the corresponding fermionic spin densities. After introducing the spinor

\[
\Psi^\dagger_k = \left( c_{i,k}^\dagger \right)_i \left( c_{i,k}^\dagger \right)_k c_{i,k}^\dagger c_{i,k}^\dagger c_{i,k}^\dagger c_{i,k}^\dagger c_{i,k}^\dagger c_{i,k}^\dagger \right)
\]

we can write the partition function as:

\[
Z = \int d\Delta_i e^{-S[\Psi, \Delta_i]}.
\]

with action:

\[
S[\Psi, \Delta_i] = - \int_k \Psi_k^\dagger G^{-1}_{\Delta,k} \Psi_k + \frac{2}{u_s} \int_x \left( \Delta_X^2 + \Delta_Y^2 \right).
\]

Here, the matrix of the inverse Green’s function \( G^{-1}_{\Delta,k} \) is given by:

\[
G^{-1}_{\Delta,k} = G^{-1}_{0,k} - V_{\Delta},
\]

with the bare term:

\[
G_{0,k} = \begin{pmatrix} \hat{G}_{\Gamma,k} & 0 \\ 0 & \hat{G}_{X,k} \end{pmatrix}
\]

and the interacting term:

\[
V_{\Delta} = \begin{pmatrix} -\Delta_X \cdot \lambda & 0 & 0 \\ -\Delta_Y \cdot \lambda & 0 & 0 \\ -\Delta_Y \cdot \lambda & 0 & 0 \end{pmatrix}.
\]

\( \hat{G}_{i,k} = G_{i,k} \) with \( G_{i,k}^{-1} = i \omega_n - \epsilon_{i,k} \) and \( N \times N \) unit matrix \( \hat{1} \). We invert the matrix equation (10) by expanding the geometric series and obtain the following expression for \( G_{\Delta} \) that we will use later-on:

\[
G_{\Delta} = \sum_{n=0}^{\infty} (-1)^n \left( G_{0} V_{\Delta} \right)^n G_{0}.
\]

A. Effective action in the large-\( N \) expansion

In this section, we first show how to obtain the Ginzburg-Landau expansion of the effective action in powers of the spin fluctuation fields \( \Delta_X, \Delta_Y \) in the limit of large \( N \). Next, we re-formulate this effective action in terms of the collective nematic Ising variable \( \phi \), and analyze the equation of state for \( \phi \). We deduce the condition for the onset of the nematic phase by examining the susceptibility of the nematic order parameter. We begin by integrating out the fermionic degrees of freedom from Eq. (8). It follows:

\[
Z = \int d\Delta_i e^{-S_{\text{eff}}[\Delta_X, \Delta_Y]}
\]

with action:

\[
S_{\text{eff}}[\Delta_X, \Delta_Y] = - \text{Tr} \left[ \ln \left( 1 - G_{0} V_{\Delta} \right) \right]
\]

\[
+ \frac{2}{u_s} \int_x \left( \Delta_X^2 + \Delta_Y^2 \right).
\]

Here, \( \text{Tr} \left( \cdots \right) \) refers to sum over momentum, frequency, spin, and band indices. We further expand in powers of \( \Delta_X, \Delta_Y \) to obtain:

\[
S_{\text{eff}}[\Delta_X, \Delta_Y] = \frac{1}{2} \text{Tr} \left( G_{0,k} V_{\Delta} \right)^2 + \frac{1}{4} \text{Tr} \left( G_{0,k} V_{\Delta} \right)^4
\]

\[
+ \frac{2}{u_s} \int_x \left( \Delta_X^2 + \Delta_Y^2 \right).
\]

After using a series of identities for the generators of the \( SU(N) \) algebra, needed to evaluate the above traces (for details see ref. 12), we arrive at the following effective action in the large \( N \) limit:

\[
S_{\text{eff}}[\Delta_X, \Delta_Y] = \sum_{i} r_{0,i} \Delta_i^2 + \sum_{i,j} u_{i,j} \Delta_i^2 \Delta_j^2.
\]
with the coefficients:

\[ r_{0,i} = \frac{2}{u_s} + 2 \int_k G_{\Gamma,k} G_{i,k} \]

\[ u_{ij} = \frac{1}{8N} \int_k G^2_{\Gamma,k} G_{i,k} G_{j,k}. \]

We used the notation \( f_k = T \sum_n \int \frac{d^d k}{(2\pi)^d} \) \( k = (k, \omega_n) \) combines the momentum \( k \) and the Matsubara frequency \( \omega_n = (2n + 1) \pi T \).

Using the identities,

\[ \int_k G_{\Gamma,k} G_{X,k} = \int_k G_{\Gamma,k} G_{Y,k} \]

\[ \int_k G^2_{\Gamma,k} G_{X,k} = \int_k G^2_{\Gamma,k} G^2_{Y,k}, \]

valid because the underlying Hamiltonian obeys the full \( C_4 \) symmetry, we can write the action in the more convenient form:

\[ S_{\text{eff}}[\Delta_X, \Delta_Y] = r_0(\Delta_X^2 + \Delta_Y^2) + \frac{u}{2}(\Delta_X^2 + \Delta_Y^2)^2 \]

\[ -\frac{g}{2}(\Delta_X^2 - \Delta_Y^2)^2, \]

where we have added a field \( h_n \) conjugate to the nematic order parameter \( \Delta_Y^2 - \Delta_X^2 \). This term is needed in order to calculate the susceptibility of the nematic order parameter. A finite value of \( \frac{\phi}{\tilde{g}} \) implies non-zero expectation value of \( \frac{\phi}{\tilde{g}} = (\Delta_Y^2 - \Delta_X^2) \neq 0 \) and the system develops nematic order. The field \( \psi \) is always non-zero and describes the strength of magnetic fluctuations.

In case of split magnetic and structural phase transitions, there is no magnetic order right below the structural transition temperature, i.e. \( \Delta_{X,Y} = 0 \). Next we integrate out the \( N^2 - 1 \) component fields \( \Delta_{X,Y} \). If we further rescale the coupling constants to \( \tilde{g} = g (N^2 - 1) \) and \( \tilde{u} = u (N^2 - 1) \), required to reach a sensible large-\( N \) limit, the effective action can be written as

\[ S_{\text{eff}} = N^2 \int_q \left\{ \frac{\phi^2}{2\tilde{g}} - \frac{\psi^2}{2\tilde{u}} \right\} + \frac{N^2}{2} \int_q \left\{ \log \left( \frac{\chi_q^{-1} + \psi}{\phi} \right)^2 - (\phi + h_n)^2 \right\} \]

We note that the effective action (23) has an overall pre-factor \( N^2 - 1 \approx N^2 \). For \( N \gg 1 \) the integral over the fields \( \phi \) and \( \psi \) can be performed via the saddle-point method, i.e. by analyzing the extremum of the action. After solving for \( \partial S_{\text{eff}}[\phi, \psi] / \partial \phi = \partial S_{\text{eff}}[\phi, \psi] / \partial \psi = 0 \), we obtain the equations of state for \( \phi \) and \( \psi \):

\[ \frac{\phi}{\tilde{g}} = \int_q \left( \frac{\chi_q^{-1} + \psi}{\phi + h_n} \right)^2 \]

\[ \frac{\psi}{\tilde{u}} = \int_q \left( \frac{\chi_q^{-1} + \psi}{\phi + h_n} \right)^2 - (\phi + h_n)^2. \]

By differentiating the second equation in (24), we find that, for small \( \phi \)

\[ \frac{\partial \phi}{\partial h_n} |_{h_n=0} = -\frac{\tilde{g}}{1 - \tilde{g}} \int_k \frac{\chi^2_k}{\chi_k^2}, \]

where, from now on, we have shifted \( \chi^{-1} \to \chi^{-1} + \psi \), which simply corresponds to the re-normalisation of the mass term due to fluctuations. Similarly to the result of Ref. 29 we find that the susceptibility of the nematic order parameter \( \Delta_X^2 - \Delta_Y^2 \) is given by

\[ \chi_{\text{nem}}^2 = \frac{\int_k \chi^2_k}{1 - \frac{\tilde{g}}{\int_k \frac{\chi^2_k}{\chi_k^2}}}, \]

where \( \chi_{q}^{-1} \) is the inverse magnetic susceptibility, and

\[ \tilde{g} = -\frac{N}{16} \int_k G^2_{\Gamma,k} (G_{X,k} - G_{Y,k})^2 \]

is the nematic coupling constant of the theory. In Ref. 7 it was found that for the classical phase transition and \( u/g > 2 \) the nematic transition pre-empts the magnetic transition, i.e. the transition lines are split. Also, the nematic transition was found to be of second order. This is the regime we are interested in. What we have calculated so far is the purely electronic contribution to the nematic susceptibility. One, however needs to include the effect of the lattice, as was pointed out by...
where \( \gamma_{el} \) is the nemato-elastic coupling constant and \( \mathbf{u} \) the phonon displacement field. The phonons renormalize the nematic coupling constant to a frequency and momentum dependent coupling

\[
\tilde{g}(q, \omega) = \tilde{g} + \gamma_{el} \frac{q^2}{\omega^2}.
\]

where \( \gamma_{el} \) is the elastic constant. In particular, if one wants to determine the location of the nematic phase transition, which is dictated by the condition of divergent nematic susceptibility, one needs to look at the static limit of the coupling constant, i.e. the limit where \( \omega = 0 \) is set to zero. This leads to \( \tilde{g}_{\text{static}} = \tilde{g} + \frac{\gamma_{el}}{2} \). The full nematic susceptibility, including the effect of the coupling to the lattice, in the large \( N \) expansion is therefore given by

\[
\chi_{\text{nem}} = \frac{\int_{k} \chi_{k}^{2}}{1 - \tilde{g}_{\text{static}} \int_{k} \chi_{k}^{2}}.
\]

\[ \text{III. RAMAN RESPONSE FUNCTION} \]

Raman scattering is a valuable tool to study strongly correlated electronic systems\(^{30,42,44}\), since it probes lattice, spin and electronic degrees of freedom. It has been used to extract the information about the momentum structure and symmetry of the excitations that exist in the system, in the concept of cuprates\(^{31,42,44}\) and pnictides\(^{45,46}\).

The differential photon scattering cross section in Raman spectroscopy is directly proportional to the structure factor \( S \):

\[
S(q) = \frac{1}{\pi} [1 + n(\omega)] \text{Im} R(q).
\]

which is related to the imaginary part to the Raman response function \( R \) through the fluctuation-dissipation theorem\(^{27}\). Here, \( n(\omega) \) is the Bose-Einstein distribution function, and \( q = (q, \omega) \). Since the momentum of light is much smaller than the typical lattice momentum, one normally replaces \( q \approx 0 \) in Eq. (32).

The Raman response functions measure correlations between "effective charge density" fluctuations \( \tilde{\rho} \),

\[
R(i\omega) = \int_{0}^{1/T} d\tau e^{-i\omega\tau} \langle [\tilde{\rho}(\tau)\tilde{\rho}(0)] \rangle.
\]

The effective density, weighted by the form factors that can be changed via the geometry of the photon polarization, is defined as

\[
\tilde{\rho}_{k} = \sum_{i,k',\sigma} \gamma_{el} c_{i,k',\sigma}^{\dagger} c_{i,k',\sigma}.
\]

\[ \sigma \text{ is the spin index, } i \text{ the band index, and the operator } c_{i,k,\sigma}^{\dagger} \text{ creates an electron with spin } \sigma \text{ and momentum } k \text{ in band } i. \]

\[ \text{where } i = X, Y, Z. \]

Due to the single particle character of the source term, the generating functional Eq. (35) can be written in the form

\[
W_{h} = \frac{1}{Z} \int d\Delta_{i} d\bar{\Psi}_{\Delta_{i}} e^{-S[\Psi, \Delta_{i}]} + \bar{\Psi}^{\dagger} V_{h} \Psi
\]

\[ Z = \int d\Delta_{i} d\Psi e^{-S[\Psi, \Delta_{i}]} \]

where \( S[\Psi, \Delta_{i}] \) is given in Eq. (9). The elements of the matrix \( V_{h} \) in momentum/frequency, spin and band space are

\[
V_{h,k_{1}k_{2}\sigma_{1}\sigma_{2}ij} = h_{k_{1}} - h_{k_{2}} \gamma_{el} \delta_{\sigma_{1}\sigma_{2}} \delta_{ij},
\]

with \( h \) being the field conjugate to the effective density. The Raman response function \( \tilde{R}_{q} \) is obtained by differentiating the generating functional \( W_{h} \) with respect to the conjugate field \( h \):

\[
\tilde{R}_{q} = \frac{\delta^{2} W_{h}}{\delta h_{\bar{q}} \delta h_{q} |_{h=0}}.
\]

Due to the single particle character of the source term, the generating functional Eq. (35) can be written in the form

\[
W_{h} = \frac{1}{Z} \int d\Delta_{i} d\Psi e^{\Psi^{\dagger} G^{-1}_{\Delta_{i}} \Psi + \frac{1}{2} \int_{x} (\Delta_{X}^{2} + \Delta_{Y}^{2})} G^{-1}_{\Delta_{i},h} (\Delta_{X} + \Delta_{Y}^{2}) + \bar{\Psi}^{\dagger} V_{h} \Psi
\]

\[ S_{h}[\Delta_{i}] = \frac{2}{\upsilon_{s}} \int_{x} (\Delta_{X}^{2} + \Delta_{Y}^{2}) - \text{Tr ln} (G^{-1}_{\Delta_{i},h}) \]

Since \( W_{h} \) contains the action that is quadratic in fermions, we integrate out the fermions and obtain:

\[
W_{h} = \frac{1}{Z} \int d\Delta_{i} e^{-S_{h}[\Delta_{i}]} \]

\[ S_{h}[\Delta_{i}] = \frac{2}{\upsilon_{s}} \int_{x} (\Delta_{X}^{2} + \Delta_{Y}^{2}) - \text{Tr ln} (G^{-1}_{\Delta_{i},h}) \]

(39)
We further expand:

$$\text{Tr} \ln \left( G_{\Delta,h}^{-1} \right) = \text{Tr} \ln \left( G_{\Delta}^{-1} \right) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\text{Tr} (G_{\Delta} V_h)^n}{n}. \tag{40}$$

Then, using Eq \((39)\) and \((37)\),

$$R_q = \frac{1}{Z} \int d\Delta_i e^{-S_h[\Delta_i]} \frac{\delta^2}{\delta h_q \delta h_{-q}} \exp \left[ \text{Tr} (G_{\Delta} V_h) - \frac{1}{2} \text{Tr} (G_{\Delta} V_h)^2 \right] \bigg|_{h=0} \tag{41}$$

where \(S_h\) is given by Eq \((39)\). We define the matrix

$$\Gamma^q = \frac{\delta V_h}{\delta h_q} \tag{42}$$

$$\Gamma^q_{k,k',\sigma,\sigma',i,i'} = \delta_{k-k',q} \gamma_k \delta_{\sigma,\sigma'} \delta_{i,i'}. \tag{42}$$

**A. Self-energy and vertex correction diagrams**

Next, we analyse the leading order contributions to the Raman response function. These arise from the self-energy and vertex correction diagrams depicted in Fig \(2\). Both of these diagrams arise from differentiating the second term in the exponential \((41)\) twice with respect to \(h\).

$$R^V_q = \frac{1}{Z} \int d\Delta_i e^{-S_{\text{eff}}[\Delta_x,\Delta_y]} \text{Tr} \left( (G_0 \Gamma)^2 \right) \tag{43}$$

Here \(S_{\text{eff}}[\Delta_x,\Delta_y] = S_h[\Delta_i] \bigg|_{h=0}\) is the effective action given by \((13)\).

In order to get the vertex correction, we replace both \(G_{\Delta}\) in \((43)\) by \(G_{\Delta} \rightarrow G_0 V_{\Delta} G_0\), which comes from the perturbative expansion of Eq \((13)\).

$$R^V_q = -\frac{1}{Z} \int d\Delta_i e^{-S_{\text{eff}}[\Delta_x,\Delta_y]} \text{Tr} \left( (G_0 \Gamma V_{\Delta})^2 G_0 \Gamma \right). \tag{44}$$

Explicitly written out, these diagrams depicted in Fig \(2\) read

$$R^V(\omega) = -2T \sum_{\nu,i=x,y} \int_{k,q} \gamma_k \gamma_{k-q} \chi(q,0) + i \leftrightarrow \Gamma$$

$$R^S(\omega) = -T \sum_{\nu,i=x,y} \int_{k,q} \gamma_k \gamma_{k-q} G_i(k,\nu) G_i(k,\nu + \omega)$$

$$G_i(k-q,\nu) \chi(q,0) + i \leftrightarrow \Gamma$$

**B. Leading order Aslamazov-Larkin diagrams**

Next we analyse the Aslamazov-Larkin contributions to the Raman response function. We will show that Aslamazov-
Larkin diagrams probe different symmetries than the self-energy and vertex correction Raman diagrams. We will also show that the leading order Aslamazov-Larkin diagram scales like square of the magnetic correlation length, as opposed to the logarithm of the magnetic correlation length which was the case for the vertex and self-energy correction diagrams.

The Aslamazov-Larkin contribution to the Raman response function, analyzed in Ref. 30 arises from differentiating the first term inside the exponential in (41) twice, and from replacing $\mathcal{G}_\Delta \rightarrow (\mathcal{G}_\Delta)^2 \mathcal{G}_0$, which comes from the perturbative expansion of Eq. (13):

$$R_q = \frac{1}{Z} \int d\Delta e^{-S_{\text{eff}}[\Delta_X, \Delta_Y]} \left[ \text{Tr} \left( (\mathcal{G}_\Delta)^2 \mathcal{G}_0 \Gamma \right) \right]^2$$

Here $S_{\text{eff}}[\Delta_X, \Delta_Y] = S_\text{h}[\Delta_i]|_{h=0}$ is the effective action given by (16).

As we will see below, their key assumption of a description based on the Aslamazov-Larkin diagrams is that one neglects the interactions between spin fluctuations. In other words, one approximates the effective action in (49) by quadratic action $S_0[\Delta_i] = \frac{1}{2} \int (\Delta_X^2 + \Delta_Y^2) + \frac{1}{2} \text{Tr} (\mathcal{G}_0 \Delta_\Gamma)^2$. While this assumption is frequently justified, it is not allowed in the theory of spin-driven nematicity, as we will show below.

The leading order Aslamazov-Larkin diagram, depicted in Fig. 4 can be calculated as

$$R_0(\omega) = T \sum_{i=X,Y} \int_q \Lambda_i(q, \Omega_n, \omega) \chi(q, \Omega_n)$$

$$\chi(q, \Omega_n - i\omega) \Lambda_i(q, \Omega_n, \omega), \quad (50)$$

where

$$\Lambda_i(q, \Omega, \omega) = \Lambda_i^{(1)}(q, \Omega, \omega) + \Lambda_i^{(2)}(-q, -\Omega, -\omega)$$

$$\Lambda_i^{(1)}(q, \Omega, \omega) = T \sum_n \int_k \gamma_k G_{\Gamma}(k, \nu_n - \omega) G_{\Gamma}(k, \nu_n)$$

$$\times G_i(k - q, \nu_n - \Omega)$$

$$\Lambda_i^{(2)}(q, \Omega, \omega) = T \sum_n \int_k \gamma_k G_{\Gamma}(k, \nu_n - \omega) G_{\Gamma}(k, \nu_n)$$

$$\times G_i(k - q, \nu_n - \Omega), \quad (51)$$

similar to what was found in Ref. 49.

1. Raman response in different symmetry channels

In the concept of the pairing symmetry in high-temperature superconductors, successful theoretical models supported by experiments have been developed in order to explain the symmetry sensitivity of the Raman response function. Similarly, here, before we evaluate the leading order Aslamazov-Larkin diagram, we analyze the contribution to these diagrams in the various symmetry channels. Higher order corrections that will be discussed later do not alter this symmetry based analysis. The main contribution to the integrals in (50), comes from the regions in momentum space where all the momenta in Green’s functions are close to the Fermi surface. These are the so-called hot-spots. At a given hot spot (see Figure 4) the function $\gamma_k$ is only slowly varying (as compared to the Green’s functions in the above integral) and hence can be approximated as a constant. However, depending on the form for $\gamma_k$, it might change sign across different hot-spots. It is easy to see that $\gamma_{B_{1g}} = k_x^2 - k_y^2$ has the same sign at all different hot-spots; therefore the vertex $\Lambda(k, \Omega, \omega)$ will not be zero for this choice of $\gamma$. However, if one choses for example $\gamma_{B_{2g}} = k_x k_y$, it will have an alternating sign across different hot-spots, which will result in the vertex $\Lambda(q, \Omega, \omega) = 0$. For details see Fig 4. This is in good agreement with the experiments, where an enhancement of the Raman response function has been seen in $B_{1g}$ channel, but not in $B_{2g}$ channel.

2. Explicit calculation of leading order Aslamazov-Larkin diagram

The leading order Aslamazov-Larkin diagram has been evaluated in Ref. 30 assuming that the main contribution comes from the hot-spot regions and that the momenta of the fluctuations are peaked around $q \approx Q_{X,Y}$. After the analytic continuation to the real frequencies, it was found that the imaginary part of the Raman response function, which is
quantity of experimental interest, is given by

\[ \text{Im} R_0(\omega + i0^+) = \int_{-\infty}^{\infty} \frac{\text{d} \epsilon}{\pi} [n(\epsilon) - n(\epsilon + \omega)] \times \int \text{Im} [\chi^R(\epsilon, q)] \text{Im} [\chi^R(\epsilon + \omega, q)] \]

with the spin propagator in the tetragonal phase given by:

\[ \chi^R(q, \Omega) = \frac{1}{r + q^2 - i\Omega}. \]

In \( d = 2 \) the \( q \) integral Eq (52) can be performed exactly, which leads to the following expression

\[ \text{Im} R_0(\omega + i0^+)_{d=2} = \int_{0}^{\infty} \text{d} \epsilon [n(\epsilon_+) - n(\epsilon_-)] \epsilon_+ \epsilon_- \times [F(\epsilon_+) - F(\epsilon_-)] \]

with

\[ F(x) = \frac{1}{x} \left( \arctan \frac{r}{x} - \frac{\pi}{2} \text{sgn}(x) \right) \]

We defined \( \epsilon_\pm = \epsilon \pm \omega/2 \). The plot of the function (54) is shown in Fig 5. In particular one can deduce that, in the regime \( T \gg r \), \( R_0(\omega)_{d=2} \approx \frac{\omega^2}{T} \) for small frequencies \( \omega \), while the amplitude of the Raman response function scales as \( R_0^\text{max}(\omega)_{d=2} \approx \frac{T}{2} \) in this regime.

In principle, one could also deduce the dimensionality of the spin fluctuations by looking at the high-frequency tail of the Raman response function in the regime \( T \gg r \). We have found that \( R_0(\omega)_{d=2} \approx \frac{T}{2} \) and \( R_0(\omega)_{d=3} \approx \frac{T}{2} \). However, it might be difficult to fit the experimental data to these curves, because the high-frequency tail of the spectrum is highly sensitive to which type of particle-hole continuum has been subtracted off in the experiment.

In summary, we have shown that the leading order Aslamazov-Larkin diagram probes \( B_{1g \text{sym}} \) symmetry because the form factor doesn’t change sign across different hot-spots, while the self-energy and vertex correction Raman diagrams lead to partial cancellations in this case.

### C. Higher order Aslamazov-Larkin-like diagrams

Next, we go beyond the quadratic action approximation for \( S_{\text{eff}} \) in (49), and include full quartic action in order to evaluate Raman response function. As we will show, diagrammatically this corresponds to inserting a series of fermionic ‘boxes’ that resemble the structure of the nematic coupling constant \( \tilde{g} \) into the leading order Aslamazov-Larkin diagram. These diagrams take into account the collisions between spin fluctuations which were not accounted for in the leading order Aslamazov-Larkin diagram.

First we show how these terms arise from the diagrammatic expansion. We start from Eq (49), but this time we go beyond the quadratic approximation for the effective action, and include quartic terms

\[ R_q = \frac{1}{T} \int d\Delta_i e^{-S_{\text{eff}}[\Delta_i]} \left[ \text{Tr} \left( \left( \frac{G_0 V_\Delta}{2} \right)^2 \right) \right]^2. \]

\[ S_{\text{eff}}[\Delta_i] = S_0[\Delta_i] + \frac{1}{4} \text{Tr} \left( \frac{G_0 V_\Delta}{2} \right)^4 \]

We further expand the exponential

\[ e^{-\frac{1}{4} \text{Tr} (G_0 V_\Delta)^4} \approx \sum_{m=0}^{\infty} \frac{1}{m!} \left[ -\frac{1}{4} \text{Tr} (G_0 V_\Delta)^4 \right]^m \]

to obtain

\[ R_q = \sum_{m=0}^{\infty} \frac{1}{m!} R_q^{(m)}, \]

where we averaged the following terms with respect to the Gaussian collective spin action:

\[ R_q^{(m)} = \left\langle \left[ -\frac{1}{4} \text{Tr} (G_0 V_\Delta)^4 \right]^m \text{Tr} \left( \left( \frac{G_0 V_\Delta}{2} \right)^2 \right) \right\rangle_{S_0}. \]

In order to evaluate the expectation values one performs contractions of the \( \Delta \) fields. We obtain a series of diagrams that look like the leading order Aslamazov-Larkin diagram with an arbitrary number of inserted fermionic boxes, depicted in Fig. 5. One can, alternatively, connect the diagonally opposite corners of the box to bosonic propagators coming from the same triangular Raman vertex (rather than having them as depicted in Fig. 5). However, one can show that these types of diagrams are suppressed by a factor of \( N^2 \) as compared to the ones depicted in Fig 5 (compare Eqs B2 and B5).

The higher order diagrams effectively take into account collisions between spin fluctuations, which have been neglected in the leading order Aslamazov-Larkin diagram. As one approaches the transition line, collisions between spin fluctuations become more and more important and one would anticipate significant changes in the Raman response function due to these processes. As we will show, the re-summation of boxed Aslamazov-Larkin diagrams will lead to the maximum of the Raman response function at the structural phase transition.

The next task is to re-sum an infinite number of such diagrams. Every box can be characterized by two indices: the first one denotes the type of incoming spin fluctuations, this can be either \( X \) or \( Y \) and the second the type of exiting spin fluctuation. Let us denote this box \( B_{x, \beta} \). Summing all boxed diagrams can be most efficiently expressed as:

\[ R(\omega) = R_0(\omega) + \sum_{\Omega, \Omega', \epsilon} \int_{q, q'} \Lambda_\alpha(\omega, \Omega, q) \times \chi(q, \Omega) \chi(q, \Omega - \omega) \times B_{\alpha\beta}(q, q', \Omega, \Omega', \omega) \chi(q', \Omega') \times \chi(q', \Omega' - \omega) \Lambda_\beta(\omega, \Omega', q'). \]
The matrix $B$ was deduced from Eq (59) and Eq (B4). For details about explicit evaluation of the $SU(N)$ trace pre-factor (which arises from contractions of products of $\lambda$ matrices of boxed diagram containing arbitrary number of boxes) please see Appendix B. The matrix $B$ of irreducible boxes is then given as

$$B = -\frac{N}{2} \begin{pmatrix} g_{XX} & g_{XY} \\ g_{XY} & g_{XX} \end{pmatrix}$$

where we used the abbreviation

$$g_{XX} = \int_k G^2_{\Gamma,k} G^2_{X,k}$$
$$g_{XY} = \int_k G^2_{\Gamma,k} G^2_{Y,k},$$

and used that by symmetry: $\int_k G^2_{\Gamma,k} G^2_{X,k} = \int_k G^2_{\Gamma,k} G^2_{Y,k}$.

Next we need to determine an expression for the full box $\tilde{B}_{\alpha\beta}$, i.e. perform a sum over the leading box-diagrams within the $1/N$ expansion. This is illustrated in Fig 5 and can be written as:

$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + B_{\alpha\delta} B_{\delta\beta} \int_{q'} \chi^2(q',0) + ...$$

$$= \sum_{m=1}^{\infty} (B^m)_{\alpha\beta} \left( \int_q \chi^2(q,0) \right)^{m-1},$$

The matrix $B^m$ is given by

$$B^m = \frac{1}{2} \left( -\frac{N}{8} \right)^m \left( g^m_{+} + g^m_{-} \right) \left( g^m_{+} - g^m_{-} \right),$$

where $g_{\pm} = g_{XX} \pm g_{XY}$. From this it follows that

$$\tilde{R}(\omega) = R_0(\omega) + R_0(\omega) \frac{\tilde{g} \int_q \chi^2(q,0)}{1 - \tilde{g} \int_q \chi^2(q,0)},$$

where

$$\tilde{g} = -\frac{N}{16} \int_k G^2_{\Gamma,k} (G_{X,k} - G_{Y,k})^2$$

is precisely the nematic coupling constant in Eq (21) for the effective action. After performing the analytic continuation to real frequencies and taking the imaginary part, we get that

$$\text{Im}\tilde{R}(\omega) = \text{Im} [R_0(\omega)] + \tilde{g} \chi^\text{el}_{\text{nem}} \text{Im} [R_0(\omega)],$$

where

$$\chi^\text{el}_{\text{nem}} = \frac{\int_q \chi^2(q,0)}{1 - \tilde{g} \int_q \chi^2(q,0)}$$

is the electronic contribution to the nematic susceptibility calculated in the large $N$ limit for the model described in Section IIA. As was pointed out in Ref [27], the enhancement of static nematic coupling constant (51) does not enter the Raman response, due to the fact that the Raman response operates in the dynamical limit ($q = 0$) and finite $\omega$, and the static and dynamic limit do not commute [27]. At the nematic / structural phase transition the nematic susceptibility diverges, and

$$\left( \tilde{g} + \frac{\gamma^2}{C^2} \int_q \chi^2_{\text{mag}}(q) \right) = 1.$$ (71)

This results in the maximum of the Raman response function (69) at the structural phase transition.
IV. CONCLUSION

In summary, we have shown that the Raman scattering can be used as a tool to probe the existence of the nematic phase in pnictides. We have presented a calculation that demonstrates that, in the low-frequency limit, and large $N$ limit, the Raman response function shows a clear maximum at the structural transition temperature.

In our model, the electronic nematic phase in pnictides is stabilized by spin-fluctuations associated with the striped phase, and occurs as a thin sliver above the magnetic transition. In order to calculate the Raman response function, we have gone beyond the leading order Aslamazov-Larkin diagram, and included higher order diagrams that contain a series of quartic paramagnon couplings, mediated by electronic excitations. Such quartic couplings contain a product of four fermionic Green functions and include the effect of collisions between spin fluctuations. When re-summed these diagram lead to the maximum of the electronic Raman response function as a function of temperature, and possibly be used as a tool to probe the existence of the nematic phase in pnictides. We have presented a calculation that demonstrates this, and the Deutsche Forschungsgemeinschaft through DFG-SPP 1458 ‘Hochtemperatur-supraleitung in Eisenpniktiden’. The method that we developed analysed the Raman response function only in the regime of small frequencies. It would be desirable to extend it to the entire frequency range, such that one can analyse the entire shape of the Raman response function, and possibly be able to extract some information about the dynamical nematic susceptibility.

Further, one might expect a charge driven nematic phase to have similar signatures in the Raman response function. This could be relevant to the peculiar case of FeSe, where the nematic phase has been detected, but no magnetic phase has been seen\cite{43}. In order to do so, we would need to develop a theoretical method that goes beyond large $N$ expansion.

V. ACKNOWLEDGEMENT

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Appendix A: Effective action of the SU(N) fermionic model

1. Some useful SU(N) identities

Here, we present some useful identities for the structure constant of $SU(N)$. They have been used to determine the scaling of the boxed Aslamazov-Larkin diagrams with $N$, and to develop the Ginzburg-Landau expansion of the effective action in powers of spin fluctuation fields $\Delta$ (see Section IV). We begin by listing some standard $SU(N)$ identities for the generators $\lambda_i$, where $i = 1, \ldots N^2 - 1$. All repeated indices are summed over.

\begin{align}
\{ \lambda_j, \lambda_k \} &= \frac{1}{N} \delta_{jk} + d_{jkl} \lambda_l, \quad d_{jkl} = d_{kjl} \quad (A1) \\
[\lambda_j, \lambda_k] &= if_{jkl} \lambda_l, \quad f_{jkl} = -f_{kjl} \quad (A2) \\
\lambda_j \lambda_k &= \frac{1}{2N} \delta_{jk} + \frac{1}{2} R_{jkl} \lambda^l \quad (A3) \\
R_{jkl} &:= d_{jkl} + if_{jkl} \quad (A4)
\end{align}

Here $d_{jkl}$ is symmetric under the exchange of its indices, while $f_{jkl}$ is antisymmetric under the exchange of neighbouring indices. Further, some useful relations for the summations of structure constants can be derived\cite{50,51}, which read

\begin{align}
d_{ijkl}d_{klm} &= \frac{N^2 - 4}{N^2} \delta_{ab}, \quad (A5) \\
f_{ijkl}f_{klm} &= N \delta_{ab}, \quad (A6) \\
\sum_i d_{ij} &= 0. \quad (A7)
\end{align}

The identity (A7) follows from considering

\begin{align}
\sum_i \{ \lambda_i, \lambda_i \} &= 2 \sum_i \lambda_i^2 = \frac{1}{N} \mathbb{1} + d_{ijkl} \lambda_j, \quad (A8)
\end{align}

Since $\sum_i \lambda_i^2$ is proportional to the unit matrix $\mathbb{1}$, for the equality above to hold, we must have $\sum_i d_{ij} = 0$, which proves the identity (A7).

Useful identities that involve the traces of the $SU(N)$ matrices are

\begin{align}
\text{Tr} (\lambda_i) &= 0, \quad (A9) \\
\text{Tr} (\mathbb{1}) &= N, \quad (A10) \\
\text{Tr} (\lambda_i \lambda_j) &= \frac{1}{2} \delta_{ij} \quad (A11)
\end{align}

In order to analyse the trace of the product of four $SU(N)$ generators

\begin{align}
\text{Tr} (\lambda_i \lambda_j \lambda_k \lambda_l) &= \text{Tr} \left[ \left( \frac{1}{2N} \delta_{ij} + \frac{1}{2} R_{ijp} \lambda_p \right) \times \left( \frac{1}{2N} \delta_{kl} + \frac{1}{2} R_{klr} \lambda_r \right) \right] \\
&= \frac{1}{4N} \delta_{ij} \delta_{kl} + \frac{1}{8} R_{ijp} R_{klp} \quad (A12)
\end{align}

we used the identity (A3) in the first line, as well as Eq (A9) and Eq (A11) in the second line. These results will be of importance for the subsequent analysis of higher order diagrams.
2. Effective action from tr log expansion

First we calculate the quadratic terms in the free energy expansion. This is given by
\[
\frac{1}{2} \text{Tr} \left( G_0 V_\Delta \right)^2 = 4 \sum_\alpha \int G_{\alpha,k} G_{\Gamma,k} \sum_{i,j=1}^{N^2-1} \text{Tr} \left( \lambda_i \lambda_j \right) \Delta^i_\alpha \Delta^j_\alpha
\]
\[= 2 \sum_\alpha \int G_{\alpha,k} G_{\Gamma,k} |\Delta_\alpha|^2 \quad (A13)\]
where \( \alpha = X, Y \) and we used the identity \((A11)\).

Next we calculate the quartic term in the free energy expansion
\[
\frac{1}{4} \text{Tr} \left( G_0 V_\Delta \right)^4 = \frac{1}{2} \text{Tr} \left( \lambda_i \lambda_j \lambda_k \lambda_l \right)
\]
\[\times \sum_{\alpha = X, Y} g_{\alpha \alpha} \Delta^i_\alpha \Delta^j_\alpha \Delta^k_\alpha \Delta^l_\alpha
\]
\[+ \frac{1}{2} \text{Tr} \left( \lambda_i \lambda_j \lambda_k \lambda_l \right)
\]
\[\times \sum_{\alpha = X, Y} g_{\alpha \alpha} \Delta^i_\alpha \Delta^j_\alpha \Delta^k_\alpha \Delta^l_\alpha \quad (A14)\]
with
\[g_{XX} = g_{YY} = \int_k G_{X,k}^2 G_{\Gamma,k}^2\]
\[g_{XY} = g_{YX} = \int_k G_{X,k} G_{Y,k} G_{\Gamma,k}^2. \quad (A15)\]

We further substitute the identity \((A12)\) in \((A14)\), to write
\[
\frac{1}{4} \text{Tr} \left( G_0 V_\Delta \right)^4 = K_1 + K_2, \quad (A16)\]
where
\[K_1 = \frac{1}{8N} \sum_{\alpha = X, Y} g_{\alpha \alpha} |\Delta_\alpha|^4
\]
\[+ \frac{1}{8N} \sum_{\alpha = X, Y} g_{\alpha \alpha} |\Delta_\alpha|^2 |\Delta^\alpha|^2\]
\[K_2 = \sum_{\alpha = X, Y} \frac{g_{\alpha \alpha}}{16} R_{ijp} R_{klp} \Delta^i_\alpha \Delta^j_\alpha \Delta^k_\alpha \Delta^l_\alpha
\]
\[+ \sum_{\alpha = X, Y} \frac{g_{\alpha \alpha}}{16} R_{ijp} R_{klp} \Delta^i_\alpha \Delta^j_\alpha \Delta^k_\alpha \Delta^l_\alpha. \quad (A17)\]

Let us examine the \(K_2\) term in Eq \((A17)\) and assume that the \(R_{akl} \sim NP\), where \(p\) is the leading power dependence on \(N\) for large \(N\). We have that \(R_{akl} R_{bkl} = d_{ak} d_{bkl} - f_{akl} f_{bkl}\), which follows from \((A3)\) and the antisymmetry of \(f\). If we further use the identities \((A5)\) and \((A6)\), it follows that \(R_{akl} R_{bkl} = -\frac{1}{N} \delta_{ab}\). On the other hand \(R_{akl} R_{bkl} \sim N^{2p} (N^2 - 1)^2\) where \((N^2 - 1)^2\) comes from the two summations over \(k\) and \(l\). Therefore, it follows that \(N^4 N^{2p} \sim \frac{1}{N}\), and \(N^{2p} \sim \frac{1}{N}\). Thus, we showed that \(K_2 \sim N^{-5}\), while \(K_1 \sim N^{-1}\), and that therefore the \(K_2\) term can be omitted in the large \(N\) limit.

Combining \((A17)\) and \((A13)\), the effective action in the large \(N\) limit can be written as
\[
S_{\text{eff}} [\Delta_X, \Delta_Y] = \sum_i r_{0,i} \Delta_i^2 + \sum_{i,j} u_{ij} \Delta_i^2 \Delta_j^2
\]
with the coefficients:
\[r_{0,i} = \frac{2}{u_s} + 2 \int_k G_{\Gamma,k} G_{i,k}
\]
\[u_{ij} = \frac{1}{8N} \int_k G_{\Gamma,k} G_{i,k} G_{j,k}. \quad (A19)\]
We note that in the large \(N\) approximation there are no \(\Delta_X \cdot \Delta_Y\) terms in the action; however if one considers corrections to large \(N\) these terms might appear in the effective action.

Appendix B: Identities containing products of traces of \(SU(N)\) generators

In this appendix we derive further identities for the traces of the \(SU(N)\) generators, which have been used to deduce the dependence of the Aslamazov-Larkin boxed diagrams on \(N\). In particular, we would like to calculate
\[
T_m := \text{Tr} (\lambda_{i_1} \lambda_{i_2}) \text{Tr} (\lambda_{i_2} \lambda_{i_1}) \text{Tr} (\lambda_{i_3} \lambda_{i_4}) \text{Tr} (\lambda_{i_4} \lambda_{i_3}) \text{Tr} (\lambda_{i_5} \lambda_{i_6}) \text{Tr} (\lambda_{i_6} \lambda_{i_5}) \ldots
\]
\[\times \text{Tr} (\lambda_{i_{2m-1}} \lambda_{i_{2m-1}}) \text{Tr} (\lambda_{i_{2m-1}} \lambda_{i_{2m-1}}) \text{Tr} (\lambda_{i_{2m-2}} \lambda_{i_{2m-2}}). \quad (B1)\]

We begin by considering \(m = 1\). Written out explicitly, it follows:
\[
T_1 = \text{Tr} (\lambda_j \lambda_i) \text{Tr} (\lambda_k \lambda_l) \text{Tr} (\lambda_k \lambda_l)
\]
\[= \left( \frac{1}{4} \delta_{ij} \delta_{kl} \right) \left( \frac{1}{4N} \delta_{ij} \delta_{kl} + \frac{1}{8} f_{ijkl} f_{ijkl} \right)
\]
\[= \frac{1}{4} \frac{1}{4N} \sum_{ijkl} \delta_{ij} \delta_{kl} + \sum_{skr} \frac{1}{32} R_{sir} R_{rkr}. \quad (B2)\]

where we have used \((A11)\) and \((A12)\) to get to the second line, and the fact that \(R_{sir} = 0\) in the penultimate line, which is a consequence of \((A7)\) and the antisymmetry of \(f\). Using the
same set of identities, we find that

\[
T_2 = \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \lambda_k \\
= \left( \frac{1}{4} \delta_{ij} \delta_{kl} \right) \left( \frac{1}{4N} \delta_{ij} \delta_{sr} + \frac{1}{8} R_{ji \ell} R_{sr \ell} \right) \\
\times \left( \frac{1}{4N} \delta_{sr} \delta_{kl} + \frac{1}{8} R_{rs \ell} R_{ikz \ell} \right) \\
= \frac{1}{4} \left( \frac{1}{4N} \right)^2 \sum_{ijkl \ell} \delta_{ij} \delta_{kl} \delta_{sr} \\
= \frac{1}{4} \left( \frac{1}{4N} \right)^2 (N^2 - 1)^3. \quad \text{(B3)}
\]

Similarly, one can deduce that

\[
T_m = \frac{1}{4} \left( \frac{1}{4N} \right)^{m+1} (N^2 - 1)^{m+1} \approx \frac{N}{16} \left( \frac{N}{4} \right)^m. \quad \text{(B4)}
\]

Using the same reasoning, one can calculate another trace combination such as:

\[
T_1^{\text{diag}} = \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \text{Tr}(\lambda_2 \lambda_1) \lambda_k \\
= \left( \frac{1}{4} \delta_{ij} \delta_{kl} \right) \left( \frac{1}{4N} \delta_{ij} \delta_{kl} + \frac{1}{8} R_{ji \ell} R_{sr \ell} \right) \\
= \frac{1}{16N} \sum_{ijkl} \delta_{ij} \delta_{kl} R_{ijkl} + \frac{1}{32} \sum_{ikp} R_{ikp}^2 \\
= \frac{1}{16N} (N^2 - 1) - \frac{1}{8N} (N^2 - 1) \\
\approx N. \quad \text{(B5)}
\]

We note that \(T_1^{\text{diag}} \ll T_1\) in the large \(N\) expansion; in particular it is smaller by a factor of \(O(1/N^2)\). Using these results one can develop a controlled large-\(N\) analysis of the Raman response in spin-driven nematic systems.
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