Investigation of the statistical properties of a dynamic system generated by number-theoretic endomorphisms

V T Dubrovin¹, F G Gabbasov², V Yu Chebakova¹ and M S Fadeeva¹

¹Kazan Federal University, 18 Kremlin street, Kazan, 420008, Russia
²Kazan State University of Architecture and Engineering, 1 Zelenaya St., 420043, Kazan, Russia

E-mail: vchebakova@mail.ru

Abstract. Mathematical modeling uses a wide variety of mathematical methods and concepts, in particular, an ergodic theory that studies statistical properties of motions in measure spaces (dynamic systems). The article is devoted to a further study of the statistical properties of a dynamic system generated by number-theoretic endomorphisms. An estimate of the convergence rate in the central limit theorem for sums of functions from number-theoretic endomorphisms trajectories is obtained. The obtained results may be of interest to practitioners using the theory of dynamic systems and specialists in number theory.

1. Introduction

The methods of ergodic theory are widely used in the formulation of problems in mechanics and physics (see, for example, [1]). An important role in the study of problems of underground filtration [2-9] of non-Newtonian fluids, in particular, high-viscous hydrocarbons, have filtration laws, i.e. dependencies between filtration rate and fluid pressure gradient. These dependencies are determined on the basis of empirical data, so the reliability of these data is of great importance. When formulating problems of the mechanics of a solid deformable body [10-16], the dependencies between strains and stresses, which are also determined on the basis of experimental data, play a great role. Capacitive coupled radio-frequency (CCRF) discharges are widely used for treatment of organic materials because they have low gas temperature. When making numerical calculation of CCRF-discharge characteristics in approximation of continuous medium, we need apply the values of the factors of the initial-boundary value problems, which are the part of the mathematical model of the discharge (factors of diffusion, mobility of charged factors, velocity of plasma chemical reaction) [17-23]. The Monte Carlo method is used to find them [24-28]. Here we use statistical analysis of random fields. It is known that limit theorems are significantly important for statistical methods [29].

The obtained in this article results are a continuation of the dynamic systems investigation described in [30–32]. We define the dynamic system under study. Let $T$ be the interval $(0,1)$ transformation defined as follows: $T(t) = \{\varphi(t)\}$ where $\varphi = f^{-1}$, $\{\}$ is the fractional part designation, $f$ is a function satisfying one of following conditions:

- The function $f$ is defined and decreases on $[1,\infty)$, $f(1) = 1$, $f$ is strictly positive, continuous and strictly decreases on $[1,k)$ and $f = 0$ on $[1,\infty)$ where $k$ is either a natural number or $\infty$ (in
this case it means that \( \lim_{x \to \infty} f(x) = 0 \). In addition, \( |f(x_2) - f(x_1)| \leq |x_2 - x_1| \) and there is \( \lambda, 0 < \lambda < 1 \) such that \( |f(x_2) - f(x_1)| \leq \lambda(x_2 - x_1) \) if \( 1 + f(2) < x_1 < x_2 \).

- The function \( f \) is defined and increases on \([0, \infty), f(0) = 0, f \) is continuous and strictly increases on \([0, k) \) and \( f = 1 \) on \([k, \infty) \) where \( k \) is either a natural number or \( \infty \) (in this case it means that \( \lim_{x \to \infty} f(x) = 1 \)). In addition, \( f(x_2) - f(x_1) \leq x_2 - x_1 \) if \( 0 \leq x_2 < x_1 \).

- \( \text{ess sup}_{0 < c < 1} f_{E_n}(x) \leq D \leq \text{ess inf}_{0 < c < 1} f_{E_n}(x) \). Here \( E_n = (a_1, a_2, \ldots, a_n) \) where \( a_i = \varphi(T^i x) \) is the whole part designation, \( f_{E_n}(x) = f(a_1 + f(a_2 + \ldots + f(a_n + x))) \), \( D \) is a constant independent either of \( E_n \) and \( n \).

Let \( M \) be the set of points from the interval \((0,1)\) in which all powers of the \( T \) map is defined. In article [33] it is proved that if conditions a), c) or b), c) are satisfied there exists a measurable function \( p(x) \) on the interval \((0,1)\) such that \( D^{-1} \leq p(x) \leq D \int_0^1 p(x) dx = 1 \) and \( T \) is an ergodic endomorphism of the space \( M \) with measure \( \mu(x) = \int_x^1 p(x) dx \). The function \( p(x) \) is unique up to a set of measure 0.

Two special cases of transformation \( Tt = \varphi(t) \) should be noted where \( \varphi = f^{-1}, \{1\} \).

- \( f(x) = \frac{x}{\theta} \cdot \theta \geq 2 \). Here if \( \theta \) is an integer, the Lebesgue measure is an invariant measure \( \mu \).

- \( f(x) = 1/x \). Here the invariant measure \( \mu \) is approximated by the Lebesgue measure so that it can be replaced by the Lebesgue measure.

Points in which all degrees of transformation \( T \) are not defined are defined by equalities \( T^k t = 0, k = 1, 2, \ldots \). The set of such points is countable, therefore, in metric theorems the difference between \( M \) and the interval \((0,1)\) can be neglected which we do.

In [33] the so-called expansion theorem was proved. If \( f \) satisfied a) or b) the number \( t \in M \) uniquely determines by the sequence of integers, that is \( t = f(a_1 + f(a_2 + f(a_3 + \ldots))) \) where \( a_i = a_i(t) = \lfloor \varphi(T^i t) \rfloor \) \( (\lfloor \cdot \rfloor \) is the sign of the whole part). \( a_i(t), i = 1, 2, 3, \ldots \) is a stationary sequence of random variables defined in a probability space \( \{\Omega, F, \mu\} \) where \( \Omega \) is an interval \((0,1), F \) is a Borel algebra on \((0,1), \mu \) is an invariant measure under the transformation \( T \). Thus, the transformation \( T \) can be considered as a shift converting the sequence \( a_1, a_2, \ldots \) into a sequence \( a_1, a_2, a_3, \ldots \), that is \( T^k t = f(a_{k+1} + f(a_{k+2} + \ldots))k = 1, 2, \ldots \), and, therefore, the function \( g(T^k t) \) can be written in the form \( g(a_{k+1}, a_{k+2}, \ldots)k = 0, 1, 2, \ldots \). This sequence is stationary on the measure \( \mu \).

Further in [34] it was proved that if \( f_{E_n}(x) \) has bounded variation and \( \sum_{E_n} \text{var} f_{E_n} \leq K \) where \( K \) does not depend on \( n \), the sequence \( a_k(t), k = 0, 1, 2, \ldots \) satisfies the condition of uniform strong mixing with coefficient \( \alpha(k) \leq A e^{-bk} \) where \( A, b \) are positive constants.

### 2. Statement of theorems.

Let consider a sequence of real-valued and Lebesgue measurable functions \( g(T^k t), k = 0, 1, 2, \ldots \) defined on the interval \((0,1)\). Assume that conditions:

- \( \int_0^1 g(t) d\mu(t) = 0, \|g(t)\| \leq C \) where \( C \) is a constant.

- There exist constants \( B > 0, \beta > 0 \) such that \( \left( \int_0^h |g(t + h) - g(t)|^2 \right)^{1/2} \leq B \|h\|^\beta \).
• \( \lim_{n \to \infty} \frac{1}{n} n^{-1} \left( \sum_{k=0}^{n-1} g(T^k t) \right)^2 d\mu(t) = \sigma^2 > 0. \)

• \( \limsup_{n \to \infty} \frac{1}{n} \exp \left[ t \sum_{k=0}^{n-1} g(T^k t) / \sqrt{n} \right] < 1. \)

2.1. Theorem 1
In the above conditions the following relation holds:

\[
\mu \left\{ t : 0 \leq t \leq 1, \sum_{k=0}^{n-1} g(T^k t) \leq x \sigma \sqrt{n} \right\} = \Phi(x) + \mathcal{O} \left( \frac{1}{1 + |x|^\omega} \frac{\ln^{5/2} n}{\sqrt{n}} \right)
\]

Where \( \Phi(x) \) is the normal distribution function with the parameters \( 0, 1 \), \( \omega \) is an arbitrarily large fixed positive number.

In view of the foregoing the proof of the theorem reduces to solving a probabilistic problem to the formulation of which we proceed now.

Let consider a sequence of random variables formed as follows: \( \xi_k = \psi(a_k, a_{k+1}, \ldots), k = 1, 2, \ldots \) where \( a_k, a_{k+1}, \ldots \) is in the narrow sense a stationary sequence of random variables defined in probability space \( \{ \Omega, A, P \} \). \( \psi \) is measurable mappings of the numerical sequence space into a number line. Let the following conditions are satisfied:

- conditions 1) \( E \xi_1 = 0, |\xi_1| \leq C. \)
- conditions 1) 2) \( E \xi_1^2 = E \xi_1^2 [a_1, \ldots, a_{k+1}] \leq B e^{-\beta k} \) where \( E \xi_1^2[H] \) is the conditional expected value on the set of quantities \( H \) and \( B, \beta > 0 \) are constants.
- conditions 3) The sequence \( a_k, a_{k+1}, \ldots \) satisfies the condition of uniform strong mixing with a coefficient \( \alpha(k) \leq A e^{-b k} \) where \( A, b \) are positive constants.
- conditions 4) \( \lim_{n \to \infty} E \left( n^{5/2} \sum_{k=1}^{n} \xi_k \right)^2 = \sigma^2 > 0. \)

2.2. Theorem 2
When these conditions are satisfied, the following relation holds:

\[
P \left( \sum_{k=1}^{n} \xi_k \leq x \sigma \sqrt{n} \right) = \Phi(x) + \mathcal{O} \left( \frac{1}{1 + |x|^\omega} \frac{\ln^{5/2} n}{\sqrt{n}} \right).
\]

It is easy to verify that all the conditions of the theorem are also satisfied for random variables \( \xi_k = E \xi_k^2[H] \) satisfying the condition of uniform strong mixing. It is sufficient to prove Theorem 2 for the quantities \( \xi_k \) by virtue of their proximity to the quantities \( \xi_k \) which follows from condition 2).

3. Proof of Theorems.
We give the lemma used in the proof of the theorem. We denote \( \chi_v(n) \) as the semi-invariant of the \( v \)'th order of the sum \( \sum_{k=1}^{n} \xi_j \) that is \( \chi_v(n) = \frac{d^v}{dz^v} \ln E \exp(z \sum_{j=1}^{n} \xi_j) \big|_{z=0}. \)

- Lemma 1. There exists a constant \( H \) independent of \( V \) such that the estimate
\[ |\chi(r(n)| < H^n(n^{s-1}), \ 1 \leq s \leq \omega \ln n. \]

The proof of the lemma is carried out in the same way as in [35].

The theorem is proved using summation methods for weakly dependent random variables based on the idea of S. N. Bernstein. He proposed to break the sums of weakly dependent random variables into long and short partial sums as a result of which the long sums are almost independent and the contribution of short sums to the total distribution is small. In the course of the proof instead of \( \xi_j \) we write \( \hat{\xi}_j \). The sum the essence of the distribution of which we are going to study: \( S_n = \sum_{j=1}^{n} \hat{\xi}_j \).

Let \( Q \) and \( N \) are natural growing together with \( n \) numbers satisfying the condition \( |n - p(Q + N)| \leq p \). We divide the sum \( S_n \) as follows:

\[
S_n = \sqrt{Q} \sum_{j=1}^{p} y_j + \sqrt{Q} \sum_{j=1}^{p} y_j^0 = \sqrt{Q} \left( z_p + z_p^0 \right),
\]

\[
y_j = \left( \sqrt{Q} \right) \sum_{r=1}^{N} \xi_{j-(j+1)Q+N+r},
\]

\[
y_j^0 = \left( \sqrt{Q} \right) \sum_{r=1}^{N} \xi_{(j-1)(Q+N)+r},
\]

\[
y_{p+1} = \sum_{r=1}^{N} \hat{\xi}_{p(Q+N)+r}.
\]

We denote independent random variables distributed in the same way as \( y_j \) by \( \hat{y}_j \), \( j = 1, 2, \ldots, p \),

\[
\hat{z}_p = \sum_{j=1}^{p} \hat{y}_j,
\]

\[
\sigma^2(Q) = E \left( \sum_{j=1}^{p} \hat{y}_j / \sqrt{Q} \right)^2, F_p(x) = P \left( \hat{z}_p \leq x \sigma(Q) \sqrt{p} \right), \quad \hat{F}_p(x) = P \left( \hat{z}_p \leq x \sigma(Q) \sqrt{p} \right).
\]

\[
f_p(t) = E \exp(it\hat{z}_p / (\sigma(Q) \sqrt{p})), \quad \hat{f}_p(t) = E \exp(it\hat{z}_p / (\sigma(Q) \sqrt{p})).
\]

Let \( C_j, \omega_j \) denote positive constants independent of \( p, Q, N, s \). From the uniform strong mixing condition we obtain that

\[
\left| f_p(t) - \hat{f}_p(t) \right| \leq C_1 p \alpha(N - s)
\]  
(1)

Let \( N = 2[\omega_1 \ln n] \); then if \( 1 \leq s(n) \leq \omega_1 \ln n \), the estimate (1) is written so:

\[
\left| f_p(t) - \hat{f}_p(t) \right| \leq C_2 p n^{-m}
\]  
(2)

Due to condition 3) for the distribution function \( \hat{F}_p(x) \) the asymptotic expansion holds:

\[
\hat{F}_p(x) = \Phi(x) + \sum_{k=1}^{v} P_k(-\Phi) / p^{k/2} + o(1 / p^{1/2})
\]  
(3)

where \( P_k(-\Phi) = \sum_{q=1}^{k} (-1)^{k+2q} / q ! \sum_{k_1, k_2, \ldots, k_q = 2^q \ldots k_q} \ldots \Lambda_{k_q} \ldots \Lambda_{k_1} / (k_1 \ldots k_q) \Phi^{(k+2q)}(x) \)

\( \Phi^{(q)}(x) = (1 / \sqrt{2\pi}) (1 / r!) H_{r-1}(x) e^{-x^2 / 2}, \) \( H(x) \) are Chebyshev-Hermite polynomials, \( \Lambda_r = \chi_r / \sigma^{(r)}(Q) \), \( \chi_r \) is an \( r^{th} \) semi-invariant of \( y_j \). In the expansion (3) we replace \( \hat{F}_p(x) \) by \( F_p(x) \) and estimate the resulting error. We denote \( G_p(x) = \Phi(x) + \sum_{k=1}^{v} P_k(-\Phi) / p^{k/2} \). It is obvious that

\[
\left| F_p(x) - G_p(x) \right| \leq \left| F_p(x) - \hat{F}_p(x) \right| + \left| \hat{F}_p(x) - G_p(x) \right|.
\]

Further

\[
\left| F_p(x) - \hat{F}_p(x) \right| \leq 1 / \sqrt{2L(F_p, \hat{F}_p)} + \max_x \left| \hat{F}_p(x) - \hat{F}_p(x + \Delta x) \right|,
\]  
(4)
where $L(F_p, \hat{F}_p)$ is the distance between the distribution functions in the Levi metric;

$$|\Delta| = L(F_p, \hat{F}_p)/\sqrt{2}.$$ It is known that $L(F_p, \hat{F}_p) \leq 1/\pi \left| f_p(t) - \hat{f}_p(t) \right| dt + 2\epsilon \ln U/U, U > \epsilon.$

Choosing $U = n^{\alpha}$ and applying (2) we obtain

$$L(F_p, \hat{F}_p) \leq C\alpha p \ln n/n^{\alpha}\] (5)$$

It follows from (3)-(5) that $\left| F_p(x) - \hat{F}_p(x) \right| \leq C_5 \left( 1 + \rho^{v/2} + p \ln n/n^{\alpha} \right).$ Now from (3)

$$F_p(x) = \Phi(x) + \sum_{k=1}^{n} P_k(-\Phi) / p^{k/2} + O(1/p^{v/2} + p \ln n/n^{\alpha}) \] (6)$$

Next we replace the distribution function $F_p(x)$ in (6) by the distribution function $P\left( \sum_{k=1}^{n} x_k \leq x\sigma(Q)\sqrt{pQ} \right)$. The reception which is used in this case is well described in [36] and [37], so we restrict ourselves to the fact that we give the final result. We have

$$P\left( \sum_{k=1}^{n} x_k \leq x\sigma(Q)\sqrt{pQ} \right) = \Phi(x) + \sum_{k=1}^{n} P_k(-\Phi) / p^{k/2} + O(1/p^{v/2} + p \ln n/n^{\alpha} + \ln^2 n / (1 + |x|^{\alpha})).$$

We estimate the polynomials $P_k(-\Phi)$ in this expression using the estimates of the higher semi-invariants from Lemma 1: invariants from Lemma 1: $|P_k(-\Phi)| < (C_5)^k (ks)^k |x|^{3k} \varepsilon^{v/2}/Q^{k/2}$, after which we can write down

$$P\left( \sum_{k=1}^{n} x_k \leq x\sigma(Q)\sqrt{pQ} \right) = \Phi(x) + \sum_{k=1}^{n} P_k(-\Phi) / p^{k/2} + O(1/p^{v/2} + p \ln n/n^{\alpha} + \ln^2 n / \sqrt{Q}/(1 + |x|^{\alpha} + (C_5)^k (ks)^k |x|^{3k} \varepsilon^{v/2}/Q^{k/2}) \] (7)$$

Using Lemma 1 we suppose that $v = [\ln n]$. Assume that $p = [\ln n]$ and choose $Q$ from the condition $n - p(Q + N) \leq p$. After this taking into account that $\frac{1}{2} (Q) = \sigma^2 + O(1/Q), \sigma \sqrt{n/(pQ)} = 1 + O(N/Q)$ we obtain the assertion of Theorem 2 from (7) and consequently of Theorem 1.

4. Conclusion

The obtained estimates for the convergence rate in the central limit theorem require that conditions 1)-4) be fulfilled. If condition 4) is absent, then the estimate of the convergence rate in theorems can be made using the method of successive approximations set forth in [38] and [39]. However, in this case the estimate will be of the order $O(1/n^{1/2-\varepsilon})$ where $\varepsilon$ is an arbitrarily small positive number. In the future there is the possibility of improving this estimate.

Acknowledgments

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University and and the work is funded by RFBR, projects 16-01-00301.

References

[1] Arnold V I and Avez A. 1968 Ergodic problems of classical mechanics (New York, Benjamin)

[2] Badriyev I B, Zadvornov O A, Ismagilov L N and Skvortsov E V 2009 Solution of plane seepage problems for a multivalued seepage law when there is a point source Journal of Applied Mathematics and Mechanics 73 (4) 434-42 DOI: 10.1016/j.jappmathmech.2009.08.007

[3] Badriyev I B, Kalacheva N V, Shangaraeva A I and Sudakov V A 2018 Numerical solving of highly viscous fluids filtration in porous media for nonlinear filtration laws with power
growth IOP Conference Series: Earth and Environmental Science **155** (1) 012015

DOI: 10.1088/1755-1315/155/1/012015

[4] Badriev I B and Fanyuk B Y 2012 Iterative methods for solving seepage problems in multilayer beds in the presence of a point source Lobachevskii Journal of Mathematics **33** (4) 386-99

DOI: 10.1134/S1995080212040026.

[5] Badriev I B, Banderov V V, Lavrentyeva E E and Pankratova O V 2016 On the Finite Element Approximations of Mixed Variational Inequalities of Filtration Theory IOP Conference Series: Materials Science and Engineering **158** (1) 012012 DOI: 10.1088/1757-899X/158/1/012012

[6] Badriev I B 1983 Difference-schemes for linear-problems of the filtration theory with discontinuous law Izvestiya Vysshikh Uchebnykh Zavedenii Matematika **5** 3-12

[7] Badriev I B and Nechaeva L A 2013 Mathematical simulation of steady filtration with multivalued law PNRPU Mechanics Bulletin (3) 37-65

[8] Badriev I B and Karchevskii M M 1989 Convergence of the iterative Uzawa method for the solution of the stationary problem of seepage theory with a limit gradient Journal of Soviet Mathematics **45** (4) 1302-9. DOI: 10.1007/BF01095183

[9] Badriev I B 1989 Regularization of the nonlinear problem of seepage theory with a discontinuous law Journal of Soviet Mathematics **44** (5) 681-91 DOI: 10.1007/BF01095183

[10] Badriev and Shagidullin R R 1992 Study of monomeric equations of static state of soft envelope and algorithm of their solution Izvestiya vysshikh uchebnikh zavedenii. Matematika (1) 8-16

[11] Badriev I B, Banderov V V, Gnedenkova V L, Kalacheva N V, Korablev A I and Tagirov R R 2015 On the finite dimensional approximations of some mixed variational inequalities Applied Mathematical Science **9**(113-6) 5697-705 DOI: 10.12988/ams.2015.57480

[12] Davydov R L, Sultanov L U and Kharzhavina V S 2015 Elastoplastic model of deformation of three- dimensional bodies in terms of large strains Global Journal of Pure and Applied Mathematics **11**(6) 5099-108

[13] Berezhnoi D V, Balafendieva I S, Sachenkov A A and Sekaeva L R 2016 Modelling of deformation of underground tunnel lining, interacting with water-saturated soil IOP Conference Series: Materials Science and Engineering **158** (1) 012018 DOI: 10.1088/1757-899X/158/1/012018

[14] Solov'ev S I 2016 Eigenvibrations of a beam with elastically attached load Lobachevskii Journal of Mathematics **37**, 597-609 DOI: 10.1134/S1995080216050115

[15] Badriev I B, Makarov M V and Paimushin V N 2017 Numerical investigation of a physically nonlinear problem of the longitudinal bending of the sandwich plate with a transversal-soft core PNRPU Mechanics Bulletin (1) 39-51 DOI: 10.15593/perm.mech/2017.1.03

[16] Badriev I B, Makarov M V and Paimushin V N 2017 Contact statement of mechanical problems of reinforced on a contour sandwich plates with transversally-soft core Russian Mathematics **61**(1) 69-75 DOI: 10.3103/S1066369X1701008X

[17] Chebakova V Y 2017 Modeling of radio-frequency capacitive discharge under atmospheric pressure in Argon Lobachevskii Journal of Mathematics **38**(6) 1165-78 DOI: 10.1134/S1995080217060154

[18] Badriev I B, Chebakova V Yu and Zheltukhin V S 2017 Capacitive coupled RF discharge: modelling at the local statement of the problem Journal of Physics Conference Series **789** 012004 DOI: 10.1088/1742-6596/789/1/012004

[19] Chebakova V Ju 2016 Simulation of Radio-Frequency Capacitive Discharge at Atmospheric Pressure in Argon Uchenye zapiski kazanskogo universiteta-seriya fiziko-matematicheskie nauki **158** (3) 404-23 (in Russian)

[20] Zheltukhin V S, Solov'ev S I, Solov'ev P S, Chebakova V Yu and Sidorov A M 2016 Third type boundary conditions for steady state ambipolar diffusion equation Conference Series: Materials Science and Engineering **158** (1) 012102. DOI: 10.1088/1757-899X/158/1/012102

[21] Zheltukhin V S, Solovyev S I, Solovyev P S and Chebakova V Yu 2016 Existence of solutions
for electron balance problem in the stationary radio-frequency induction discharges IOP Conference Series: Materials Science and Engineering 158 (1) 012103 DOI: 10.1088/1757-899X/158/1/012103

[22] Badriev I B, Zheltukhin V.S and Ju Chebakova V 2017 Numerical solution of the initial boundary value problems of radio-frequency capacitive coupled discharge Journal of Physics: Conference Series 927 (1) 012008 DOI: 10.1088/1742-6596/927/1/012008

[23] Chebakova V J and Badriev I B 2018 Mathematical simulation of the low-temperature plasma at the interaction with oil products IOP Conference Series: Earth and Environmental Science 107 (1) 012097 DOI: 10.1088/1755-1315/107/1/012097

[24] Yousfi M Hennad A and Alkaf A 1994 Monte Carlo simulation of electron swarms at low reduced electric fields Physical Review E 49 (4) 3264-73

[25] Hans Rau 2000 Monte Carlo simulation of a microwave plasma in hydrogen J. Phys. D: Appl. Phys. 33 3214–22

[26] Askhatov R M, Badriev I B, Chebakova V Yu and Zheltukhin V S 2018 Simulation of electron moving in RF capacitively coupled discharge Journal of Physics: Conference Series 1058 (1) 012044 DOI: 10.1088/1742-6596/1058/1/012044

[27] Gabbasov F G, Dubrovin V T and Chebakova V J 2016 Vector random fields in mathematical modelling of electron motion IOP Conference Series: Materials Science and Engineering 158 (1) 012032 DOI: 10.1088/1757-899X/158/1/012032

[28] Chebakova V J, Gerasimov A V and Kirpichnikov A P 2016 On the solving of one type of problems of mathematical physics IOP Conference Series: Materials Science and Engineering 158 (1) 012023 DOI: 10.1088/1757-899X/158/1/012023

[29] Dubrovin V.T. 2014 Convergence rate in limit theorems for weakly dependent random values Lobachevskii Journal of Mathematics 35 (4) 390-6

[30] Gabbasov F G and Dubrovin V T 2013 Estimate of the convergence rate in the multidimensional limit theorem for endomorphisms of Euclidean space Uchenye Zapisiki Kazanskogo Universiteta. Serya Fiziko-Matematicheskie Nauki 155 33-43

[31] Dubrovin V T, Gabbasov F G and Chebakova V Yu 2016 Multidimensional central limit theorem for sums of functions of the trajectories of endomorphisms Lobachevskii Journal of Mathematics 37 (4) 409-17 DOI: 10.1134/S1995080216040053

[32] Dubrovin V T, Gabbasov F G and Kugurakov V S 2015 On the multidimensional limit theorem for endomorphisms of Euclidean space Uchenye Zapisiki Kazanskogo Universiteta. Serya Fiziko-Matematicheskie Nauki 157 (1) 25-34 (in Russian)

[33] Renyi A 1957 Representation for real numbers and their ergodic properties Acta Math. Acad. Sci. Hungar. 8 477-93

[34] Gordin M I 1968 On random processes generated by number-theoretic endomorphisms DAN SSSR 182 (5) 1004-6

[35] Dubrovin V T and Moskvin D A 1979 Estimation of the senior semi-invariants of sums of weakly dependent quantities Izvestiya VUZov. Matematika 5 20-8

[36] Dubrovin V T and Moskvin D A 1979 The central limit theorem for sums of functions of sequences with mixing. Teoriya veroyatnostei i ee primenenie XXIV (4) 553-64 (in Russian)

[37] Gabbasov F G 1977 A multidimensional limit theorem for sums of functions of sequences with mixing Litovskii matematicheskii sbornik XVII (4) 83-98

[38] Gabbasov F G and Dubrovin V T 2014 Multi-dimensional limit theorem on large deviations for endomorphisms of Euclidean space Uchenye Zapisiki Kazanskogo Universiteta. Serya Fiziko-Matematicheskie Nauki 156 (2) 16-24 (in Russian)

[39] Gabbasov F G and Dubrovin V T 2014 Large deviations in the multidimensional limit theorem for ergodic automorphisms of a two-dimensional torus Matematicheskiye metody v tekhnike i tekhnologiyakh-MMTT 4 242-5 (in Russian)