Long-time existence of mean curvature flow with external force fields *

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Abstract In this paper, we study the evolution of submanifold moving by mean curvature minus a external force field. We prove that the flow has a long-time smooth solution for all time under almost optimal conditions. Those conditions are that the second fundamental form on the initial submanifolds is not too large, the external force field, with its any order derivatives, is bounded, and the field is convex with its eigenvalues satisfying a pinch inequality.

Key words parabolic equation, mean curvature flow, maximum principle (for tensor).

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1 Introduction

1.1 Background

In this paper, we study the flow

\[
\frac{dF}{dt} = -(H_\alpha - \omega_\alpha)e_\alpha \equiv -f_\alpha e_\alpha
\]  

(1.1)

where

\[F_t := F(\cdot, t) : M^n \rightarrow R^{n+k}\]

is a family of smooth immersions with \(M_t = F_t(M)\) and \(M\) is compact oriented submanifold in \(R^{n+k}\), \(H\) denotes the mean curvature vector of \(M_t\) w.r.t unit normal field \(e_\alpha, \alpha = n + 1, \ldots, n + k\), \(\omega\) is a given smooth function in \(R^{n+k}\), \(\nabla\omega\) is the standard gradient field of \(\omega\) in \(R^{n+k}\), and \(\omega_\alpha = \langle \nabla\omega, e_\alpha \rangle\).

This flow generalizes the well-known mean curvature flow, i.e., the case of \(\omega \equiv \text{const}\), and it comes directly from the study of the Ginzburg-Landau vortex. As was shown in [1,2], there are two models which are, respectively, reduced to the Ginzburg-Landau system of parabolic equations

\[
\frac{\partial V_\varepsilon}{\partial t} = \nabla V_\varepsilon + \nabla\omega \nabla V_\varepsilon + AV_\varepsilon + \frac{BV_\varepsilon}{\varepsilon^2} (1 - |V_\varepsilon|^2)
\]

(1.2)

in \(R^m \times (0, \infty)\), where \(\varepsilon\) is a small positive parameter and \(\omega, A, B\), are known functions. One is a simple equation simulating inhomogeneous type II superconducting materials [3], and the other is a three-dimensional superconducting thin films having variable thickness [4]. An important problem in Ginzburg-Landau superconductors is to study the vortex dynamics, i.e, the convergence of \(V_\varepsilon\) as well as of their zero points (which, roughly, are called vortex) as \(\varepsilon \rightarrow 0\).

When \(m = 2\) and the initial vortex consists of finite isolated points, it was proved that the vortex dynamics of the Dirichlet problem for (1.2) is described by the ODE system[1,5,6]:

\[
\frac{\partial x}{\partial t} = -\nabla\omega(x).
\]

When \(m \geq 2\) and the initial vortex consists of a filment or even a codimension k submanifold, it was proved [2] that as \(\varepsilon \rightarrow 0\), the vortex of Cauchy problem for (1.2) is approximated by the evolution of the initial vortex according to flow (1.1)
on the time internal in which the flow is smooth. Similar results were obtained for Neumann problem in [7] and for case of $\nabla \omega = 0$ in [7,8].

Therefore, it is important in physics to consider the long-time existence of the flow (1.1).

On the other hand, mean curvature flow has been strongly studied in last decades. It is well-known that the flow must blow up in finite time except that the initial submanifolds are graphic, see [9-16] for the details. Hence, it is natural to ask for what kind of functions $\omega$ (1.1) has long-time existence.

1.2 Main results

Higher co-dimension mean curvature flow, i.e., (1.1) without external force field, has been studied in [9-13], while there are a lot of studies on mean curvature flow for hypersurfaces, see [14][15][16] for example. All those papers show that mean curvature flow must blow up and so singularity happens in finite time, except that the initial surfaces are entire graphs or graphic submanifolds. In this paper, we are concentrated on the long-time existence of (1.1). Here is the main results.

**Theorem 1.1** If there exist positive constants $C, C_3, \overline{\lambda}, \underline{\lambda}$ with $\overline{\lambda} < 2\underline{\lambda}$ such that the following conditions are satisfied:

1. $\Delta |\xi|^2 \leq \nabla^2 \omega(x)\xi_i\xi_j \leq \overline{\lambda}|\xi|^2$ and $|\nabla^3 \omega(x)| \leq C_3$ for all $\xi \in \mathbb{R}^{n+1}$ and for $x \in M_t$, where $M_t$ is any solution of (1.1) on any finite time interval $[0, T]$;
2. $|A|^2 < C$ on $M_0$;
3. there exist a $\delta > 0$ such that $5a^4 + |a|C_3 + (\overline{\lambda} - 2\underline{\lambda})a^2 \leq 0$ for all $a$ satisfying $\sqrt{C} - \delta < a < \sqrt{C}$;
4. $|\nabla \omega(x)|$ is uniformly bounded for all $x \in M_t$ and for $i = 1, 2, 3, \ldots$, where $M_t$ is any solution of (1.1) on any finite time interval $[0, T]$;

then the flow (1.1) has a smooth solution for all time $t \in [0, \infty)$.

Throughout this paper, flow (1.1) is denoted by (1.1)' in the case of $k = 1$, i.e., the hypersurfaces case.

**Theorem 1.2** Suppose that the assumptions of theorem 1.1 are satisfies except that (1.1) is replaced by (1.1)' and (3) is replaced by

(3)' there exist a $\delta > 0$ such that $a^4 + |a|C_3 + (\overline{\lambda} - 2\underline{\lambda})a^2 \leq 0$ for all $a$ satisfying

3
\[ \sqrt{C} - \delta < a < \sqrt{C} + \delta; \]
then the flow \((1.1)'\) has a smooth solution for all time \(t \in [0, \infty)\).

**Remark 1.3** An easier verification of assumptions (1) and (4) is to assume that they hold for all \(x \in R^{n+k}\).

The following theorem generalizes the convexity-preserved of mean curvature flow in [14].

**Theorem 1.4** Let \(T > 0\) and \(M_t\) be a smooth solution of flow \((1.1)'\) on the time interval \([0, T]\). If \(\nabla^3 \omega \equiv 0\) and \(M_0\) is convex, then \(M_t\) is convex for all \(t \in [0, T]\).

Physically, \(\omega\) is a density function and actually has the form
\[
\omega = \frac{1}{2}(c_1 x_1^2 + \ldots + c_{n+1} x_{n+1}^2)
\]
for \(c_i > 0\), (see [3] for example), but theorems 1.1 and 1.2 cannot be applied directly to this special case, because \(|\nabla \omega|\) is not known to be bounded uniformly at this moment. However, we can give the long-time existence for this \(\omega\) under hypersurfaces case.

**Corollary 1.5** Suppose that \(\omega = \frac{1}{2}(c_1 x_1^2 + \ldots + c_{n+1} x_{n+1}^2)\) where \(c_i\) are positive constants and let \(M = \max c_i\) and \(m = \min c_i\). If \(M < 2m\) and \(|A|^2 < 2m - M\) on \(M_0\), then for any \(T > 0\) the flow \((1.1)'\) has a smooth solution for all \(t \in [0, T]\).

We would like to point out that Corollary 1.5 generalizes theorem 1.3 in [17] which studies \((1.1)'\) for the special case \(\omega = c|x|^2\). That theorem also shows that flow \((1.1)'\) must blow up in finite time either if \(c < 0\), or if \(c > 0\) and \(|A|^2 > c\) on \(M_0\), which means that both convexity of \(\omega\) (as in assumption (1)) and the small of the initial \(|A|^2\) (as in assumptions (2) and (3)) are necessary.

In section 2 we will give notations and preliminaries, and we will give the proofs of theorems 1.2, 1.4 and corollary 1.5 in section 3 and the proof of theorem 1.1 in section 4.

## 2 Preliminaries

Throughout this paper, we use the following notations. \(\langle \cdot, \cdot \rangle\) denotes the usual inner product in \(R^{n+k}\). If \(M\) is given as in section 1 and \(F\) denotes its parametrization.
in $\mathbb{R}^{n+k}$, the metric $\{g_{ij}\}$ are given by

$$g_{ij}(x) = \langle \frac{\partial F(x)}{\partial x_i}, \frac{\partial F(x)}{\partial x_j} \rangle, \quad x \in M.$$ 

Let $\nabla$ denotes the Levi-Civita connection on $M$, while $\nabla^\perp$ denotes the standard gradient in $\mathbb{R}^{n+k}$. We will use $i, j, k, \cdots$ to denote the tangent indexes and $\alpha, \beta, \gamma, \cdots$ for normal ones. Doubled indices always mean to sum from 1 to $n$ for $i, j, k, \cdots$ and from 1 to $k$ for $\alpha, \beta, \gamma, \cdots$. Indices are raised and lowered w.r.t $g^{ij}$ and $g_{ij}$. Moreover, we will identify $V \in T_xM$ with $DF(V) \in T_{F(x)}\mathbb{R}^{n+k}$. Also, we will use $\langle \cdot, \cdot \rangle$ to denote the scalar product on $M$ if there are no confusions.

The second fundamental form in direction $e_\alpha$ is denoted by

$$h_{\alpha ij}(x) = -\langle e_\alpha, \nabla_i \nabla_j F \rangle$$

and the norm of the second fundamental form by

$$|A|^2 = g^{ij} g^{kl} h_{\alpha ik} h_{\alpha lj}.$$ 

The mean curvature on $M$ in direction $e_\alpha$ is given by

$$H_\alpha = g^{ij} h_{\alpha ij}.$$ 

Let $R_{ijkl}$ denote the curvature tensor and $R^\perp_{\beta\alpha jk}$ denote the normal curvature tensor, and recall Ricci equation and Gauss equation for the submanifold of Euclid space

$$R^\perp_{\alpha\beta ij} = h_{\alpha ik} h_{\beta jk} - h_{\alpha jk} h_{\beta ik}, \quad (2.1)$$

and

$$R_{ijkl} = h_{\alpha ik} h_{\alpha jl} - h_{\alpha jk} h_{\alpha il}, \quad (2.2)$$

Of course, $R^\perp$ is zero for hypersurface. Also, we can write Weigarten equation and Codazzi equation

$$\nabla_i e_\beta = h^\perp_{\beta i} \nabla_i F + C^\gamma_{\beta\gamma} e_\gamma, \quad (2.3)$$

and

$$h_{\alpha ik, j} = h_{\alpha ij, k}, \quad (2.4)$$
where $C^\gamma_{i\beta}$ is the connection coefficient of normal connection and $C^\gamma_{i\beta} = -C^\beta_{i\gamma}$.

Besides, we will use the following basic facts.

**Proposition 2.1**[14,18]. For any hypersurface $M$ in $\mathbb{R}^{n+1}$, we have

$$\nabla_i \nabla_j F = -h_{ij} \nu, \quad (2.5)$$

$$\nabla_i \nu = h^l_i \nabla_l F, \quad (2.6)$$

$$\nabla_k h_{ij} = \nabla_j h_{ik}, \quad (2.7)$$

$$\nabla_i \nabla_j h = \Delta h_{ij} - H h^l_i h_{lj} + |A|^2 h_{ij}, \quad (2.8)$$

$$2h^{ij} \nabla_i \nabla_j H = \Delta |A|^2 - 2|\nabla A|^2 - 2Z, \quad (2.9)$$

where $\nu$ is the outer normal vector of $M$, $C = g^{ij} g^{kl} g_{st} h_{ik} h_{sj} h_{lt} = \text{tr}(A^3)$, and $Z = HC - |A|^4$.

**Proposition 2.2**[12][18]. Suppose flow (1.1) holds true for $t \in [0, T)$ with $T \leq \infty$, then we have the following equations in $[0,T)$:

$$\frac{dg_{ij}}{dt} = -2f_\alpha h_{\alpha ij}, \quad (2.10)$$

$$\frac{dg^{ij}}{dt} = 2f_\alpha h_{\alpha kl} g^{ik} g^{jl}, \quad (2.11)$$

and letting $f \equiv f_\alpha = f_1$ for a hypersurface, we have

$$\frac{dh_{ij}}{dt} = \nabla_i \nabla_j f - fh^l_i h_{lj}, \quad (2.12)$$

$$\frac{d|A|^2}{dt} = 2h^{ij} \nabla_i \nabla_j f + 2fC, \quad (2.13)$$

The following theorem for short time existence of (1.1) is well-known due to the theory of PDE and the technique of De Turk[19].

**Theorem 2.3**[19] The flow (1.1) is a system of quasilinear parabolic equations and there exists a maximal time $0 < T \leq \infty$ such that (1.1) admits a smooth solution on $[0,T)$.

### 3 Hypersurfaces case

In this section, we will prove theorems 1.2 and 1.4 and corollary 1.5. The key step is to calculate the evolution equations of $|A|^2$. 
Proposition 3.1 Suppose flow (1.1)' holds true for \( t \in [0, T) \) with \( T \leq \infty \), then we have the following equations in \([0,T)\):

\[
\frac{d|A|^2}{dt} = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2h^{ij}(\nabla_i \nabla^2 \omega)(\nabla_j F, \nu) + 2|A|^2 \nabla^2 \omega(\nu, \nu) - 4h^{ij} h^l_j \langle \nabla_i \nabla \omega, \nabla_l F \rangle - \langle \nabla \omega, \nabla |A|^2 \rangle. \tag{3.1}
\]

**Proof:** By (2.13), we have

\[
\frac{d|A|^2}{dt} = 2h^{ij} \nabla_i \nabla_j f + 2f C.
\]

By the notation \( f \equiv f_\alpha = f_1 \) and using (2.9), we have

\[
2h^{ij} \nabla_i \nabla_j f = 2h^{ij} \nabla_i \nabla_j H - 2h^{ij} \nabla_i \nabla_j \langle \nabla \omega, \nu \rangle
= \Delta |A|^2 - 2|\nabla A|^2 - 2Z - 2h^{ij} \nabla_i \nabla_j \langle \nabla \omega, \nu \rangle. \tag{3.2}
\]

It follows from (2.6) that

\[
\nabla_i \nabla_j \langle \nabla \omega, \nu \rangle = \nabla_i (\langle \nabla_j \nabla \omega, \nu \rangle + \langle \nabla \omega, \nabla_j \nu \rangle)
= \nabla_i (\langle \nabla_j \nabla \omega, \nu \rangle + h^l_j \langle \nabla \omega, \nabla_l F \rangle)
= \langle \nabla_i \nabla_j \nabla \omega, \nu \rangle + h^l_j \langle \nabla_j \nabla \omega, \nabla_l F \rangle + h^l_i \langle \nabla_j \nabla \omega, \nabla_l F \rangle + \nabla_i h^l_j \langle \nabla \omega, \nabla_l F \rangle.
\]

Using (2.5) and (2.7), we obtain

\[
\nabla_i \nabla_j \langle \nabla \omega, \nu \rangle = \langle \nabla_i \nabla_j \nabla \omega, \nu \rangle + h^l_j \langle \nabla_j \nabla \omega, \nabla_l F \rangle + h^l_i \langle \nabla_j \nabla \omega, \nabla_l F \rangle + \langle \nabla \omega, \nabla h_{ij} \rangle - h^l_j h^l_i \langle \nabla \omega, \nu \rangle, \tag{3.3}
\]

which implies

\[
2h^{ij} \nabla_i \nabla_j \langle \nabla \omega, \nu \rangle = 2h^{ij} \langle \nabla_i \nabla_j \nabla \omega, \nu \rangle + 4h^{ij} h^l_j \langle \nabla_i \nabla \omega, \nabla_l F \rangle + \langle \nabla \omega, \nabla |A|^2 \rangle - 2h^{ij} h^l_j h^l_i \langle \nabla \omega, \nu \rangle.
\]

This, together with (3.2),(2.13)and the definitions of \( f, C \) and \( Z \), gives

\[
\frac{d|A|^2}{dt} = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2h^{ij} \langle \nabla_i \nabla_j \nabla \omega, \nu \rangle
- 4h^{ij} h^l_j \langle \nabla_i \nabla \omega, \nabla_l F \rangle - \langle \nabla \omega, \nabla |A|^2 \rangle. \tag{3.4}
\]
But
\[ \langle \nabla_i \nabla_j \nabla \omega, \nu \rangle = \nabla_i (\langle \nabla_j \nabla \omega, \nu \rangle) - \langle \nabla_j \nabla \omega, \nabla_i \nu \rangle = (\nabla_i \nabla^2 \omega)(\nabla_j F, \nu) + \nabla^2 \omega (\nabla_i \nabla_j F, \nu) + \langle \nabla_j \nabla \omega, \nabla_i \nu \rangle - \langle \nabla_j \nabla \omega, \nabla_i \nu \rangle = (\nabla_i \nabla^2 \omega)(\nabla_j F, \nu) - h_{ij} \nabla^2 \omega (\nu, \nu), \tag{3.5} \]
where we have used (2.5) for the last equation. Insert this equality to (3.14) we can get the desired equality (3.1).

**Proposition 3.2** With the same assumption as in Proposition 3.1, the second fundamental form satisfies the following evolution equation for tensor in \([0,T)\): \[ \frac{dh_{ij}}{dt} = \Delta h_{ij} - 2H h_{ij} + |A|^2 h_{ij} - (\nabla_i \nabla^2 \omega)(\nabla_j F, \nu) + h_{ij} \nabla^2 \omega (\nu, \nu) - h_{ij} \nabla^2 \omega (\nabla_j F, \nabla_j F) - \langle \nabla \omega, \nabla h_{ij} \rangle + 2h_{ij} h_{il} (\nabla \omega, \nu). \tag{3.6} \]

**Proof:** It is a combination of (2.8),(2.12),(3.3) and (3.5).

**Proof of Theorem 1.4:** Applying the maximum principle for tensor \([20]\) to equation (3.6), we see that the surface \(M_t\) is always convex along the flow if \(\nabla^3 \omega \equiv 0\) and \(M_0\) is convex.

**Lemma 3.3** Suppose that \(M_t\) is the solution of (1.1)’ on \([0,T)\) and the assumptions (1), (2) and (3)’ in theorem 1.2 are satisfied. Then \(|A|^2 < C\) for all \(t \in [0,T)\).

**Proof:** Taking a local orthonormal basis \(e_i\) on \(M_t, i = 1 \ldots n\), by (3.1) we have
\[ \frac{d|A|^2}{dt} \leq \Delta |A|^2 + 2 |A|^4 - 2 h_{ij} (\nabla_i \nabla^2 \omega)(e_j, \nu) + 2 |A|^2 \nabla^2 \omega (\nu, \nu) - 4 h_{ij} h_{jl} \nabla^2 \omega (e_i, e_l) - \langle \nabla \omega, \nabla |A|^2 \rangle. \]
This, together with assumption (1), implies
\[ \frac{d|A|^2}{dt} \leq \Delta |A|^2 + 2 |A|^4 + 2 |A|C_3 + 2 |A|^2 \lambda - 4 h_{ij} h_{jl} \nabla^2 \omega (e_i, e_l) - \langle \nabla \omega, \nabla |A|^2 \rangle. \tag{3.7} \]
Next, we estimate \(-4 h_{ij} h_{jl} \nabla^2 \omega (e_i, e_l)\). Since
\[ -4 h_{ij} h_{jl} \nabla^2 \omega (e_i, e_l) = -4 h_{ij} h_{jl} (\lambda E(e_i, e_l) + \nabla^2 \omega (e_i, e_l) - \lambda E(e_i, e_l)) = -4 |A|^2 \lambda - 4 h_{ij} h_{jl} (\nabla^2 \omega - \lambda E)(e_i, e_l) \]
where $E$ is the unit matrix, we have

$$-4h_{ij}h_{jl}\nabla^2\omega(e_i,e_l) \leq -4|A|^2\bar\chi + 4|h_{ij}h_{jl}|(\nabla^2\omega - \bar\chi E)(e_i,e_l)|$$

$$\leq -4|A|^2\bar\chi + 4(\bar\chi - \Delta)|A|^2$$

$$= -4\Delta|A|^2.$$

Therefore, (3.7) becomes

$$\frac{d|A|^2}{dt} \leq \triangle|A|^2 + 2|A|^4 + 2|A|C_3 + 2(\bar\chi - 2\Delta)|A|^2 - (\nabla\omega, \nabla|A|^2).$$

Now by the assumptions and Hamilton’s maximum principle one can easily obtain that $|A|^2 < C$ for all time. Otherwise, we can choose the first time $t_0$ such that $a(t_0) = C$, where $a(t) \equiv \max_{M_t} |A|^2$. Then there exists a time $t_1 < t_0$ such that $(\sqrt{C} - \delta)^2 < a(t) < C$ for $t \in [t_1,t_0)$ and $a(t_1) < C$. Hence

$$2a^2(t) + 2\sqrt{a(t)}C_3 + 2(\bar\chi - 2\Delta)a(t) \leq 0, \forall t \in [t_1,t_0)$$

by assumption (3)’. Therefore, applying Hamilton’s maximum principle [20] to equation (3.8), we have $a(t) \leq a(t_1) < C$ for all $t \in [t_1,t_0]$. This contradicts $a(t_0) = C$.

**Remark 3.4** We would like to say that the convex condition on $\omega$ (as in assumption (1)) and the small condition of the initial $|A|^2$ (as in assumptions (2) and (3)’) are necessary. If $\nabla\omega = cx$ with either $c < 0$, or $c > 0$ and $|A|^2 > c$ on $M_0$, we have proved in [17] that $|A|^2$ must blow up in finite time and the flow exists only in finite time.

**Proof of Theorem 1.2:** From Lemma 3.3 we see that $|A|^2$ are bounded uniformly if assumptions (1)-(3) are satisfied. Thus, if we can prove that $|\nabla^m A|^2 \leq C_m$ is bounded when $t \to T$, then by a well-known theorem of partial differential equations the flow (1.1) can be extended to $[0,T+\varepsilon)$ for some small $\varepsilon > 0$, where $T < \infty$ is the maximal time interval for which (1.1)’ has a smooth solution. This concludes that the maximum time interval must be $[0,\infty)$.

To estimate $|\nabla A|^2$, the boundedness of $|\nabla^4 \omega|$ is necessary but is not enough, because we want to calculate the time derivative of $\Gamma^k_{ij}$. As we know that connection is not a tensor, but the difference of two connection is tensor, so is $\frac{d\Gamma^k_{ij}}{dt}$.
Taking normal coordinate and using (2.10), we have
\[
\frac{d\Gamma^k_{ij}}{dt} = \frac{1}{2} \frac{d}{dt} \left( g^l_k \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \right)
\]
\[
= \frac{1}{2} g^l_k \left( \frac{\partial}{\partial x^i} \left( \frac{d}{dt} g_{jl} \right) + \frac{\partial}{\partial x^j} \left( \frac{d}{dt} g_{il} \right) - \frac{\partial}{\partial x^l} \left( \frac{d}{dt} g_{ij} \right) \right)
\]
\[
= -g^l_k \left( \frac{\partial}{\partial x^i} (fh_{jl}) + \frac{\partial}{\partial x^j} (fh_{il}) - \frac{\partial}{\partial x^l} (fh_{ij}) \right).
\]

Noting that \( \partial_i f = \partial_i H - \nabla^2 \omega (\partial_i \nu) - h_{il}(\nabla \omega, \partial_l) \) and repeating the arguments of Huisken [14] we obtain the following result.

**Lemma 3.5** Suppose that \( M_t \) is the solution of (1.1)' on \([0, T)\) for \( T < \infty \). If assumptions (1), (2) and (3)' of theorem 1.2 are satisfied and \( |\nabla \omega| \) for \( i = 1, \ldots, m \) is uniformly bounded on \( M_t \), then \( |\nabla^m \omega| \) is uniformly bounded on \( M_t \).

Using lemma 3.5, we have completed the proof of theorem 1.2.

**Proof of corollary 1.5:** For the special case, \( \omega = \frac{1}{2} c_1 x_1^2 + \ldots + \frac{1}{2} c_{n+1} x_{n+1}^2 \), we have that
\[
\nabla \omega = (c_1 x_1, \ldots, c_{n+1} x_{n+1}), \quad \nabla^2 \omega = (c_i \delta_{ij}), \quad \text{and} \quad \nabla^3 \omega = 0.
\]
Let \( M = \max c_i \) and \( m = \min c_i \). Applying lemma 3.3 we get if \( M < 2m \) and \( |A|^2 < 2m - M \) on \( M_0 \), \( |A|^2 < 2m - M \) as long as flow (1.1)' exists. To get the long-time existence we have to get the higher derivative estimate of \( |A|^2 \). But lemma 3.5 can not be applied directly, because \( |\nabla \omega| \) may turn to be infinite if the surface expands to infinity. However, we can prove that the surface will not expand to infinity in finite time as follows. For this purpose, we need a theorem of [18].

**Lemma 3.6**[18] Let \( F \) be a smooth immersed solution of (1.1)' and \( \bar{F} \) be an immersed solution of this evolution equation. If \( \bar{F} \) is contained in a connected component of \( R^{n+1} \setminus F \) or in the closure of such a component at the beginning of the evolution, then this remains during the evolution. Since \( |A|^2 \leq 2m - M \) on \( M_0 \), we will prove that if the initial surface is sphere, the sphere will expand to infinity as \( t \to \infty \).

**Lemma 3.7** Suppose that \( M_0 = S^n(R) \) is the initial surface of the flow (1.1)' and \( \omega, m, M \) are as above. Let \( s(t) := \frac{1}{2} |F_t|^2 \) where \( F_t \) is the position vector of \( M_t \). If \( |A|^2 < 2m - M \) on \( M_0 \), then \( C_0 \equiv (2ms(0) - n) > 0 \) and \( s \geq \frac{n+C_0}{2m} e^{2mt} \).
for all $t > 0$.

**Proof**: Note that

$$\frac{ds}{dt} = \langle \frac{dF}{dt}, F \rangle = -(H - \langle \nabla \omega, \nu \rangle)\langle F, \nu \rangle = -n + \langle \nabla \omega, \nu \rangle\langle F, \nu \rangle.$$

Since on the spheres $\nu = \frac{1}{|F|} F$, we have

$$\langle \nabla \omega, \nu \rangle = \frac{1}{|F|}\langle \nabla \omega, F \rangle = \frac{1}{|F|}(c_1 F_1^2 + \ldots + c_{n+1} F_{n+1}^2) \geq \frac{1}{|F|} m|F|^2 = m|F|.$$

Hence,

$$\frac{ds}{dt} \geq -n + 2ms. \quad (3.9)$$

Therefore, $s \geq \frac{n+C_0}{2m} e^{2mt}$ for all $t > 0$ if $C_0 > 0$. Now by the initial condition, we have

$$2m - M > |A|^2 = \frac{1}{n} H^2 = \frac{1}{n} \frac{n^2}{|F|^2} = \frac{n}{2s(0)},$$

which implies $2s > \frac{n}{2m-M}$ and $C_0 > 0$. In this way, we have completed the proof of lemma 3.7.

Finally, Lemma 3.6 and lemma 3.7 imply $M_t$ will not expand to infinity in finite time. This, together with the above discussions, finishes the proof of corollary 1.5.

### 4 Higher co-dimension case

In this section, we will prove theorem 1.1. As the hypersurface case, the key step is to derive the evolution equation of $|A|^2$. For this purpose, we want to calculate the evolution equation of the second fundamental form tensor. In the following, for $x \in M^n$ we take orthonormal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k}$ of $R^{n+k}$ such that $\{e_1, \ldots, e_n\}$ is the basis of $T_x M^n$ and $\{e_{n+1}, \ldots, e_{n+k}\}$ (denoted by $\{e_\alpha\}$) is the
Proposition 4.1 Suppose flow (1.1) holds true for \( t \in [0, T) \) with \( T \leq \infty \), then we have the following equations in \([0, T)\):

\[
\frac{dh_{\alpha ij}}{dt} - \Delta h_{\alpha ij} = -H_\beta h_{\beta kl}h_{\alpha jl} - (\nabla_j \nabla^2 \omega) (e_i, e_\alpha) + h_{\beta ij} \nabla^2 \omega (e_\beta, e_\alpha) - h_{\alpha k j} \nabla^2 \omega (e_i, e_k) - h_{\alpha ik} \nabla^2 \omega (e_j, e_k) + h_{\beta ij} \langle e_\alpha, \frac{de_\alpha}{dt} \rangle + \langle \nabla_\omega, e_\beta \rangle (h_{\beta ik} h_{\alpha j k} + h_{\beta j k} h_{\alpha i k}) - \langle \nabla_\omega, \nabla h_{\alpha ij} \rangle - h_{\alpha im} (h_{\gamma mj} h_{\gamma kk} - h_{\gamma mk} h_{\gamma kj}) - h_{\alpha mk} (h_{\gamma mj} h_{\gamma ik} - h_{\gamma mk} h_{\gamma ij}) - h_{\beta ik} (-h_{\beta km} h_{\alpha jm} + h_{\beta jm} h_{\alpha km}). \tag{4.1}
\]

Proof: For both sides are tensors, we calculate in normal coordinate. Since \( \nabla_j \nabla_i F = -h_{\alpha ij} e_\alpha \), then by (1.1) we have

\[
\frac{dh_{\alpha ij}}{dt} = -\frac{d}{dt} \langle \nabla_j \nabla_i F, e_\alpha \rangle
= -\langle \nabla_j \nabla_i (-H_\beta e_\beta + \omega_\beta e_\beta), e_\alpha \rangle - \langle \nabla_j \nabla_i F, \frac{de_\alpha}{dt} \rangle
= \langle \nabla_j \nabla_i (H_\beta e_\beta), e_\alpha \rangle - \langle \nabla_j \nabla_i (\omega_\beta e_\beta), e_\alpha \rangle + h_{\beta ij} \langle e_\alpha, \frac{de_\alpha}{dt} \rangle. \tag{4.2}
\]

By Weigarten equation (2.3), we have

\[
\nabla_j \nabla_i e_\beta = (\nabla_j h_{\beta il}) e_l - h_{\beta il} h_{\gamma jl} e_\gamma + (\nabla_j C^\gamma_{ij}) e_\gamma + C^\gamma_{ij} \nabla_j e_\gamma
= (\nabla_j h_{\beta il}) e_l - h_{\beta il} h_{\gamma jl} e_\gamma + (\nabla_j C^\gamma_{ij}) e_\gamma + C^\gamma_{ij} h_{\gamma jl} e_l + C^\gamma_{ij} C^\eta_{jl} e_\eta
= h_{\beta il} e_l - h_{\beta il} h_{\gamma jl} e_\gamma + (\nabla_j C^\gamma_{ij}) e_\gamma + C^\gamma_{ij} C^\eta_{jl} e_\eta. \tag{4.3}
\]

This, together with (2.3), implies

\[
\nabla_j \nabla_i (H_\beta e_\beta) = (\nabla_j \nabla_i H_\beta) e_\beta + (\nabla_j H_\beta) \nabla_i e_\beta + (\nabla_i H_\beta) \nabla_j e_\beta + H_\beta \nabla_j \nabla_i e_\beta
= (\nabla_j \nabla_i H_\beta) e_\beta + (\nabla_j H_\beta) h_{\beta il} e_l + (\nabla_j H_\beta) C^\gamma_{ij} e_\gamma
+ (\nabla_i H_\beta) h_{\beta jl} e_l + (\nabla_i H_\beta C^\gamma_{j\beta}) e_\gamma + H_\beta \nabla_j \nabla_i e_\beta. \tag{4.4}
\]

Hence, \[
\langle \nabla_j \nabla_i (H_\beta e_\beta), e_\alpha \rangle = \nabla_j \nabla_i H_\alpha + \nabla_j H_\beta C^\alpha_{ij} + \nabla_i H_\beta C^\alpha_{j\beta}
- H_\beta h_{\alpha ij} h_{\beta l} + H_\beta \nabla_j C^\alpha_{ij} + H_\beta C^\alpha_{j\beta} C^\eta_{jl}, \tag{4.5}
\]
\[ \sum_k h_{akk,ij} = \nabla_j \nabla_i H_\alpha + \nabla_j H_\beta C_{ij}^\alpha + \nabla_i H_\beta C_{j\beta}^\alpha \]
\[ + H_\beta \nabla_j C_{ij}^\alpha + H_\beta C_{ij}^\alpha C_{j\gamma}^\alpha - 2h_{aki} \frac{\partial \Gamma_{ik}^l}{\partial x_j} \]  
(4.6)
and the last term of (4.6) is zero because \( \Gamma_{ik}^l = -\Gamma_{il}^k \). Then we use (4.6) to rewrite (4.5) as
\[ \langle \nabla_j \nabla_i (H_\beta e_\beta), e_\alpha \rangle = \sum_k h_{akk,ij} - H_\beta h_{\alpha ij} h_{\beta il}. \]  
(4.7)
Simon’s Identity gives
\[ \sum_k h_{akk,ij} = \Delta h_{\alpha ij} - (h_{\beta ik} R_{\beta \alpha jk} + h_{amk} R_{mijk} + h_{aim} R_{mkjk}). \]  
(4.8)
Putting (4.8) in (4.7) and using (2.1) and (2.2), we obtain that
\[ \langle \nabla_j \nabla_i (H_\beta e_\beta), e_\alpha \rangle = \Delta h_{\alpha ij} - H_\beta h_{\alpha ij} h_{\beta il} \]
\[ - h_{am}(h_{\gamma mj} h_{\gamma k} - h_{\gamma mk} h_{\gamma i}) - h_{amk}(h_{\gamma mj} h_{\gamma ik} - h_{\gamma mk} h_{\gamma ij}) \]
\[ - h_{\beta ik}(-h_{\beta jm} h_{\alpha jm} + h_{\beta jm} h_{\alpha km}). \]  
(4.9)
Next, we use (2.3) to calculate the term \( \nabla_j \nabla_i (\omega_\beta e_\beta) \) in (4.2). Since
\[ \nabla_j \nabla_i (\omega_\beta e_\beta) = \nabla_j \nabla_i (\nabla \omega - \langle \nabla \omega, e_k \rangle e_k) \]
\[ = \nabla_j \nabla_i \nabla \omega - \nabla_j (\nabla^2 \omega (e_i, e_k) e_k) - h_{\beta ik} \langle \nabla \omega, e_k \rangle e_\beta \]
\[ = \nabla_j \nabla_i \nabla i - \nabla_j (\nabla^2 \omega (e_i, e_k) e_k) + h_{\beta jk} \nabla^2 \omega (e_i, e_k) e_\beta \]
\[ + \nabla_j (h_{\beta i k} \langle \nabla \omega, e_\beta \rangle e_k - h_{\beta i k} h_{\gamma jk} \langle \nabla \omega, e_\beta \rangle e_\gamma) \]
\[ + \nabla_j (h_{\beta i k} \langle \nabla \omega, e_k \rangle e_\beta + h_{\beta i k} \nabla^2 \omega (e_j, e_k) e_\beta) \]
\[ - h_{\beta i k} h_{\gamma jk} \langle \nabla \omega, e_\gamma \rangle e_\beta + h_{\beta i k} \langle \nabla \omega, e_k \rangle \nabla_j e_\beta, \]  
(4.10)
we have
\[ \langle \nabla_j \nabla_i (\omega_\beta e_\beta), e_\alpha \rangle = \langle \nabla_j \nabla_i \nabla \omega, e_\alpha \rangle + h_{\alpha jk} \nabla^2 \omega (e_i, e_k) + h_{\alpha ik} \nabla^2 \omega (e_j, e_k) \]
\[ - \langle \nabla \omega, e_\beta \rangle (h_{\beta i k} h_{\alpha jk} + h_{\alpha ik} h_{\beta jk}) \]
\[ + (\nabla_j h_{\alpha ik} + C_{j\beta}^\alpha h_{\beta i k}) \langle \nabla \omega, e_k \rangle \]
\[ = \langle \nabla_j \nabla_i \nabla \omega, e_\alpha \rangle + h_{\alpha jk} \nabla^2 \omega (e_i, e_k) + h_{\alpha ik} \nabla^2 \omega (e_j, e_k) \]
\[ - \langle \nabla \omega, e_\beta \rangle (h_{\beta i k} h_{\alpha jk} + h_{\alpha ik} h_{\beta jk}) + h_{\alpha i k} \langle \nabla \omega, e_k \rangle. \]  
(4.11)
Due to Codazzi equation (2.4), we have

\[
\langle \nabla_j \nabla_i (\omega_\beta e_\beta), e_\alpha \rangle = \langle \nabla_j \nabla_i \omega, e_\alpha \rangle + h_{\alpha j k} \nabla^2 \omega(e_i, e_k) + h_{\alpha i k} \nabla^2 \omega(e_j, e_k) \\
- \langle \nabla \omega, e_\beta \rangle (h_{\beta i k} h_{\alpha j k} + h_{\alpha i k} h_{\beta j k}) + \langle \nabla \omega, \nabla h_{\alpha i j} \rangle \\
= \langle \nabla_j \nabla^2 \omega(e_i, e_\alpha) \rangle - h_{\beta i j} \nabla^2 \omega(e_\alpha, e_\beta) \\
+ h_{\alpha j k} \nabla^2 \omega(e_i, e_k) + h_{\alpha i k} \nabla^2 \omega(e_j, e_k) \\
- \langle \nabla \omega, e_\beta \rangle (h_{\beta i k} h_{\alpha j k} + h_{\alpha i k} h_{\beta j k}) + \langle \nabla \omega, \nabla h_{\alpha i j} \rangle.
\]

(4.12)

So (4.1) follows from (4.2),(4.9) and (4.12).

**Proposition 4.2** Suppose flow (1.1) holds true for \( t \in [0, T) \) with \( T \leq \infty \), then we have the following equation in \([0, T)\):

\[
\frac{d|A|^2}{dt} = \Delta |A|^2 - 2|\nabla A|^2 - 2h_{\alpha i j}(\nabla_j \nabla^2 \omega)(e_i, e_\alpha) \\
+ 2h_{\alpha i j} h_{\beta i j} \nabla^2 \omega(e_\alpha, e_\beta) - 4h_{\alpha i k} h_{\alpha i j} \nabla^2 \omega(e_j, e_k) - \langle \nabla \omega, \nabla |A|^2 \rangle \\
+ 2 \sum_{\alpha, \gamma, i, m} (h_{\alpha i k} h_{\gamma m k} h_{\alpha i m} h_{\gamma m k})^2 + 2 \sum_{i, j, k, m} (h_{\alpha i j} h_{\alpha m k})^2
\]

(4.13)

**Proof:** We calculate it in normal coordinate. Because \(|A|^2 = g^{ij} g^{kl} h_{\alpha i k} h_{\alpha j l} \), then

\[
\frac{d|A|^2}{dt} = 2 \frac{dg^{ik}}{dt} h_{\alpha i k} h_{\alpha j l} + 2 \frac{dh_{\alpha i j}}{dt} h_{\alpha i j}.
\]

(4.14)

Hence by (2.11) (4.1) and (4.14), we have

\[
\frac{d|A|^2}{dt} = 2h_{\alpha i j} \Delta h_{\alpha i j} + 4(H_\beta - \omega_\beta) h_{\beta i k} h_{\alpha i j} h_{\alpha k j} - 2H_\beta h_{\alpha i j} h_{\beta i l} h_{\alpha j l} \\
- 2h_{\alpha i j} (\nabla_j \nabla^2 \omega)(e_i, e_\alpha) + 2h_{\alpha i j} h_{\beta i j} \nabla^2 \omega(e_\beta, e_\alpha) \\
- 4h_{\alpha i j} h_{\alpha k j} \nabla^2 \omega(e_i, e_k) + 2h_{\alpha i j} h_{\beta i j} \langle e_\beta, \frac{de_\alpha}{dt} \rangle + 4h_{\alpha i j} \omega_\beta h_{\beta i k} h_{\alpha j k} \\
- \langle \nabla \omega, \nabla |A|^2 \rangle - 2h_{\alpha i j} h_{\alpha i m} h_{\gamma m j} H_\gamma + 2h_{\alpha i j} h_{\alpha i m} h_{\gamma m k} h_{\gamma k j} \\
- 2h_{\alpha i j} h_{\alpha m k} (h_{\gamma m j} h_{\gamma i k} - h_{\gamma m k} h_{\gamma i j}) - 2h_{\alpha i j} h_{\beta i k} (h_{\beta l j} h_{\alpha l k} - h_{\beta l k} h_{\alpha l j}).
\]

Observing that \( 2h_{\alpha i j} h_{\beta i j} \langle e_\beta, \frac{de_\alpha}{dt} \rangle \) is zero by symmetry and \( 2h_{\alpha i j} \Delta h_{\alpha i j} = \Delta |A|^2 - 2|\nabla A|^2 \), we have

\[
\frac{d|A|^2}{dt} = \Delta |A|^2 - 2|\nabla A|^2 - 2h_{\alpha i j} (\nabla_j \nabla^2 \omega)(e_i, e_\alpha) + 2h_{\alpha i j} h_{\beta i j} \nabla^2 \omega(e_\beta, e_\alpha) \\
- 4h_{\alpha i j} h_{\alpha k j} \nabla^2 \omega(e_i, e_k) - \langle \nabla \omega, \nabla |A|^2 \rangle + 2h_{\alpha i j} h_{\alpha i m} h_{\gamma m k} h_{\gamma k j} \\
- 2h_{\alpha i j} h_{\alpha m k} (h_{\gamma m j} h_{\gamma i k} - h_{\gamma m k} h_{\gamma i j}) - 2h_{\alpha i j} h_{\beta i k} (h_{\beta l j} h_{\alpha l k} - h_{\beta l k} h_{\alpha l j}).
\]
But the last three terms can be calculate as follows:

\[2h_{\alpha ij}h_{\alpha im}h_{\gamma kj} - 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ik} \]

\[+ 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ij} - 2h_{\alpha ij}h_{\beta ik}(h_{\beta ij}h_{\alpha kl} - h_{\beta ik}h_{\alpha lj}) \]

\[= 4h_{\alpha ij}h_{\alpha im}h_{\gamma mj}h_{\gamma ik} + 2h_{\alpha ij}h_{\gamma mk}h_{\alpha mk}h_{\gamma ij}. \quad (4.15)\]

Since

\[2h_{\alpha ij}h_{\alpha im}h_{\gamma mj}h_{\gamma ik} - 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ik} \]

\[= 2h_{\alpha ij}h_{\alpha ik}h_{\gamma mk}h_{\gamma mj} - 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ik} \]

\[= h_{\alpha ij}h_{\gamma mj}(h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik}) + h_{\alpha mj}h_{\gamma ij}(h_{\alpha mk}h_{\gamma ik} - h_{\alpha mk}h_{\gamma ik}) \]

\[= \sum_{\alpha,\gamma,i,m} (\sum_k h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik})^2 \quad (4.16)\]

and

\[2h_{\alpha ij}h_{\gamma mk}h_{\alpha mk}h_{\gamma ij} = 2 \sum_{i,j,k,m} (\sum_{\alpha} h_{\alpha ij}h_{\alpha mk})^2, \quad (4.17)\]

we have proved the proposition.

**Lemma 4.3** Suppose that \(M_t\) is the solution of (1.1) on \([0,T)\), and the assumptions (1), (2) and (3) of theorem 1.1 are satisfied, then \(|A|^2 \leq C\) on \(M_t\) for all \(t \in [0,T)\).

**Proof:** The proof is almost the same as that of lemma 3.3 in the case of hypersurfaces. It follows from Schwartz inequality that

\[2 \sum_{i,j,k,m} (\sum_{\alpha} h_{\alpha ij}h_{\alpha mk})^2 \leq 2 \sum_{i,j,k,m} (\sum_{\alpha} h_{\alpha ij}^2)(\sum_{\alpha} h_{\gamma mk}^2) = 2|A|^4\]

and

\[\sum_{\alpha,\gamma,i,m} (\sum_k h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik})^2 \leq 4 \sum_{\alpha,\gamma,i,m} (\sum_k h_{\alpha ik}h_{\gamma mk})^2 \leq 4|A|^4.\]

Consequently, using the same technique from (3.7) to (3.8) we obtain

\[\frac{d|A|^2}{dt} \leq \Delta |A|^2 - \langle \nabla \omega, \nabla |A|^2 \rangle + 10|A|^4 + 2|A|C_3 + 2(\overline{\lambda} - 2\underline{\lambda})|A|^2. \quad (4.18)\]

then the result follows by copying the arguments below (3.8).

**Proof of theorem 1.1:** Using Lemma 4.3 and repeating the proof of theorem 1.2, one can easily prove theorem 1.1.
References

[1] Jian, H. Y., Xu, X. W., The vortex dynamics of a Ginzburg-Landau system under pinning effect, Science in China (Ser. A), 2003, 46: 488-498.

[2] Jian, H. Y., Liu, Y. N., Ginzburg-Landau vortex and mean curvature flow with external force field, Acta Math. Sin., Engl. Ser, 2006, 22(6): 1831-1842.

[3] Chapman, S. J., Richardson, G., Vortex pinning by inhomogeneities in type-II superconductors, Phys. D., 1997, 108: 397-407.

[4] Chapman, S. J., Du, Q., Gunzburger, A model for variable thickness superconducting thin films, Z. Angew Math. Phys., 1996, 47: 410-431.

[5] Lin, F. H., Some dynamical properties of Ginzburg-Landau vortices, I, II, Commu. Pure Appl. Math., 1996, 49: 323-364.

[6] Jian, H. Y., Song, B., Vortex dynamics of Ginzburg-Landau equations in inhomogeneous superconductors, Journal of Differential Equations, 2001, 170: 123-141.

[7] Lin, F. H., Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds, Commu. Pure Appl. Math., 1998, 51: 385-441.

[8] Jerrard, R. L., Soner, H. M., Scaling limits and regularity for a class of Ginzburg-Landau systems, Ann. Inst. Henri. Poincare Analyse Nonlineaire., 1999, 16: 423-446.

[9] Jian, H. Y., Translating solitons of mean curvature flow of noncompact space-like hypersurfaces in Minkowski space, J. Differential Equations, 2006, 220: 147-162.

[10] Chen, J., Li, J., Mean curvature flow of surface in 4-manifolds, Adv. Math., 2001, 163: 287-309.

[11] Chen, J., Li, J., Singularity of mean curvature flow of Lagrangian submanifolds, Invet. Math., 2004, 156: 25-51.
[12] Wang, M. T., Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension, Invet. Math. 2002, 148: 525-543.

[13] Smoczyk, K., Wang, M. T. Mean curvature flows of Lagrangian submanifolds with convex potentials, J. Differential Geom., 2002, 62: 243-257.

[14] Huisken, G., Flow by mean curvature flow of convex surfaces into spheres, J. Differential Geom., 1984, 20: 237-266.

[15] Huisken, G., Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geom., 1990, 31: 285-299.

[16] Jian, H. Y., Liu, Q., Chen, X., Convexity of Translating Solitons of Mean Curvature Flow, Chin. Ann. Math. 2005, 26B: 413-422.

[17] Liu, Y. N., Jian, H. Y., Evolution of hypersurfaces by mean curvature minus external force field, Science in China (Ser A), (to appear)

[18] Schnürer, O., Smoczyk, K., Evolution of hypersurfaces in central force fields, J. Reine Angew Math., 2002, 550: 77-95.

[19] Hamilton, R.S., Lectures on geometric evolution equations, preprint, 1996.

[20] Hamilton, R. S., Three-manifolds with positive Ricci curvature. J. Differential Geom., 1982, 17: 255-306.