A new Bound for the Maker-Breaker Triangle Game

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Abstract

The triangle game introduced by Chvátal and Erdős (1978) is one of the most famous combinatorial games. For $n, q \in \mathbb{N}$, the $(n, q)$-triangle game is played by two players, called Maker and Breaker, on the complete graph $K_n$. Alternately Maker claims one edge and thereafter Breaker claims $q$ edges of the graph. Maker wins the game if he can claim all three edges of a triangle, otherwise Breaker wins. Chvátal and Erdős (1978) proved that for $q < \sqrt{2n + 2} - 5/2 \approx 1.144\sqrt{n}$ Maker has a winning strategy, and for $q \geq 2\sqrt{n}$ Breaker has a winning strategy. Since then, the problem of finding the exact leading constant for the threshold bias of the triangle game has been one of the famous open problems in combinatorial game theory. In fact, the constant is not known for any graph with a cycle and we do not even know if such a constant exists. Balogh and Samotij (2011) slightly improved the Chvátal-Erdős constant for Breaker’s winning strategy from 2 to 1.935 with a randomized approach. Since then no progress was made. In this work, we present a new deterministic strategy for Breaker’s win whenever $n$ is sufficiently large and $q \geq \sqrt{(8/3 + o(1))n} \approx 1.633\sqrt{n}$, significantly reducing the gap towards the lower bound. In previous strategies Breaker chooses his edges such that one node is part of the last edge chosen by Maker, whereas the remaining node is chosen more or less arbitrarily. In contrast, we introduce a suitable potential function on the set of nodes. This allows Breaker to pick edges that connect the most ‘dangerous’ nodes. The total potential of the game may still increase, even for several turns, but finally Breaker’s strategy prevents the total potential of the game from exceeding a critical level and leads to Breaker’s win.
1 Introduction

For \(n, q \in \mathbb{N}\) the \((n, q)\)-triangle game is played on the complete graph \(K_n\). In every turn Maker claims an unclaimed edge from \(K_n\), followed by Breaker claiming \(q\) edges. The game ends when all edges are claimed either by Maker or Breaker. If Maker manages to build a triangle, he wins the game, otherwise Breaker wins. This game is one of the most prominent examples of Maker-Breaker games, in which Maker tries to build a certain structure while Breaker tries to prevent this. These are games of perfect information without chance moves, so either Maker or Breaker has a winning strategy, in which case we say the game is Maker’s win or Breaker’s win, respectively. For more information on Maker-Breaker games we refer to the paper by Krivelevich [7].

Figure 1: The \((7, 2)\)-triangle game is a Maker’s win. Maker-edges are red, Breaker-edges blue.

1.1 Previous work

Maker-Breaker games have been extensively studied by Beck [2, 3, 4], concerning, e.g., games in which Maker tries to build a Hamiltonian cycle, a spanning tree or a big star. Beck [3] also presented very general sufficient conditions for Maker’s win and Breaker’s win. In his work he generalized the Erdös-Selfridge Theorem [9], which gives a winning criterion for Breaker for the case \(q = 1\). A direct application of these criteria to specific games as the triangle game often does not lead to strong results. However, it turned out to be a powerful tool, e.g. Bednarska and Łuczak [6] used it to prove the following fundamental result: For a fixed graph \(G\) consider the \((G, n, q)\)-game, in which Maker has to build a copy of \(G\). There exist constants \(c_0, C_0\) such that the game is a Maker’s win for \(q \leq c_0 n^{1/m(G)}\) and a Breaker’s win for \(q \geq C_0 n^{1/m(G)}\), where \(m(G) := \max \left\{ \frac{e(H) - 1}{v(H) - 2} : H \subseteq G, v(H) \geq 3 \right\}\). This result recently was further generalized for hypergraphs by Kusch et al. [8]. Bednarska and Łuczak also conjectured that \(c_0\) and \(C_0\) can be chosen arbitrarily close to each other. Until today this conjecture couldn’t be proved or disproved for any fixed graph \(G\) that contains a cycle.

The special case of the triangle game was proposed by Chvátal and Erdős [5], who presented a winning strategy for Maker if \(q < \sqrt{2n^2 + 2} - 5/2\), and a winning strategy for Breaker if \(q \geq 2\sqrt{n}\). Both strategies are rather simple: If \(q < \sqrt{2n^2 + 2} - 5/2\), Maker can win by fixing a node \(v\) and then simply claiming all his edges incident to \(v\). At some point of time Breaker will not be able to close all Maker paths of length 2, so Maker can complete such a path to build a triangle. If \(q \geq 2\sqrt{n}\), Breaker can always close all paths of length 2 created by Maker and at the same time prevent Maker from building a star of size \(q/2\). To achieve this he first closes all new Maker paths of length 2 and then claims arbitrary
edges such that at the end of the turn he claimed exactly $\sqrt{n}$ edges incident in $u$ and the remaining $\sqrt{n}$ edges incident in $v$, where $\{u, v\}$ is the edge recently claimed by Maker.

Chvátal and Erdős also asked for the threshold bias of this game, i.e., the value $q_0(n)$ such that the game is Maker’s win for $q \leq q_0(n)$ and Breaker’s win for $q > q_0(n)$. For the triangle game, the asymptotic order of $\Theta(\sqrt{n})$ follows directly from the two strategies given above and also occurs as special case of the result by Bednarska and Luczak, but the gap between $\sqrt{2n}$ and $2\sqrt{n}$ could not be narrowed for many years. In 2011, Balogh and Samotij \cite{BalSamo11} presented a randomized Breaker strategy, improving the lower bound for Breaker’s win to about $(2 - 1/24)\sqrt{n} \approx 1.958 \sqrt{n}$.

1.2 Our Contribution

In our work we present a new deterministic strategy for Breaker that further improves the recent lower bound for Breaker’s win to $q = \sqrt{(8/3 + o(1))n} \approx 1.633 \sqrt{n}$, assuming $n$ to be sufficiently large. The global idea of our strategy is as follows. Instead of claiming arbitrary edges incident in the nodes of the last edge claimed by Maker, as done in the strategy of Chvátal and Erdős, Breaker claims only edges that connect the ‘most dangerous’ nodes, i.e., nodes that already have many incident Maker edges and rather few Breaker edges. Proceeding this way, Breaker needs fewer edges to prevent Maker from building a $q/2$-star. For the realization of this idea we use an (efficiently computable) potential function to decide which edges are most dangerous and should be claimed next to prevent Maker from building any triangle or big star. In contrast to Beck \cite{Beck87}, instead of assigning a potential to every winning set, our potential function is defined directly on the set of nodes. However, the most significantly difference to Beck and other previous potential-based approaches is that our potential function is not necessarily decreasing in every single turn. Some critical turns may occur in which the potential increases, so the challenge is to bound the number and the impact of these critical rounds. This new approach requires plenty of analytic work but turns out to be a more powerful technique than classic potential-based approaches and also might be of interest for other kinds of Maker-Breaker games.

2 Breaker’s strategy

We start by introducing the potential function which forms the basis for Breaker’s strategy. During the game, denote by $M$ the Maker graph consisting of all edges claimed by Maker so far and let $B$ denote the corresponding Breaker graph. For $v \in V$ and $H \in \{M, B\}$ let $\deg_H(v)$ denote the degree of $v$ in $H$. For a turn $t$, $\deg_{H,t}(v)$ denotes the degree of $v$ in $H$ directly after turn $t$.

2.1 The potential function

Let $\epsilon^* > 0$ and $\beta = \frac{8}{3} + \epsilon^*$. In this chapter we consider the $(n, q)$-Triangle game with $q = \sqrt{\beta n}$. As mentioned in the introduction, for $\beta \geq 4$ there exist known winning strategies for Maker, so we will assume $\beta \leq 4$ if necessary. Fix $\delta \in (0, 1 - \frac{8}{3\beta})$.

Definition 1. For every $v \in V$ define the balance of $v$ as

$$\text{bal}(v) := \frac{8(n - \deg_B(v))}{q^2(1 - \delta)(3 + \delta) - 4\deg_M(v)(2q - \deg_M(v))}.$$
Moreover we define $p_0$ as the balance of a node in the very beginning of the game, i.e.

$$p_0 := \frac{8n}{q^2(1-\delta)(3+\delta)} = \frac{8}{\beta(1-\delta)(3+\delta)}.$$ 

The balance of a node is a measure of the ratio of Maker- and Breaker-edges incident in this node: The more Maker-edges and the fewer Breaker-edges incident in $v$, the bigger the balance value gets. A detailed interpretation of the balance value can be found in Section 6.2. This is assured by the next remark.

**Remark 2.** It holds $\frac{8}{3\delta} < p_0 < \frac{8}{\delta(1-\delta)} < 1$.

**Proof.** The second and third inequality follow directly from $\delta \in (0,1-\frac{8}{3\delta})$. For the first inequality, note that $(1-\delta)(3+\delta) = 3 - 2\delta - \delta^2 < 3$, so we get $p_0 = \frac{8}{\beta(1-\delta)(3+\delta)} > \frac{8}{3\delta}$. 

During the game, Breaker will not be able to keep all nodes at their start balance. Some nodes will get more Breaker-edges than needed, others less. This *deficit* of a node will be used to define its potential.

**Definition 3.** Consider the game at an arbitrary point of time. For a node $v \in V$ let $\deg^*(v) \in \mathbb{R}$ be the balanced Breaker-degree of this node, i.e. the Breaker-degree that would be necessary, so that $\text{bal}(v) = p_0$. Formally we define

$$\deg^*(v) := n - p_0 \left( \frac{q^2(1-\delta)(3+\delta)}{8} - \deg_M(v) \left( q - \frac{\deg_M(v)}{2} \right) \right).$$

The deficit of $v$ is defined by

$$d(v) := \deg^*(v) - \deg_B(v).$$

Finally, let $\mu := 1 + \frac{6\beta \ln(n)}{4q}$. Define the potential of $v$ as

$$\text{pot}(v) := \begin{cases} 0 & \text{if } \deg_M(v) + \deg_B(v) = n - 1 \\ \mu^{d(v)/q} & \text{else} \end{cases}$$

and for an unclaimed edge $e = \{u,w\}$ define the potential of $e$ as $\text{pot}(e) := \text{pot}(u) + \text{pot}(v)$. For every turn $t$ we define $\text{pot}_t(v)$ (pot$_t(e)$, resp.) as the potential of $v$ (e, resp.) directly after turn $t$ and pot$_0(v)$ as the potential of $v$ at the beginning of the game. Analogously we define $\deg^*_t(v)$ and $d_t(v)$. The total potential of a turn $t$ is defined as $\text{POT}_t := \sum_{v \in V} \text{pot}_t(v)$. The total starting potential is defined as $\text{POT}_0 := \sum_{v \in V} \text{pot}_0(v)$.

**Lemma 4.** The total starting potential fulfills $\text{POT}_0 = n$.

**Proof.** Let $v \in V$ with $\deg_M(v) = \deg_B(v) = 0$. Then,

$$\deg^*(v) = n - p_0 \left( \frac{q^2(1-\delta)(3+\delta)}{8} \right) = n - p_0 \cdot n \cdot p_0^{-1} = 0.$$

This implies $\text{pot}(v) = \mu^{d(v)/q} = \mu^{(\deg^*(v) - \deg_B(v))/q} = \mu^0 = 1$, so

$$\text{POT}_0 = \sum_{v \in V} \text{pot}_0(v) = \sum_{v \in V} 1 = n.$$
Breaker’s aim is to keep the total potential as low as possible. The next lemma ensures that if Breaker can keep the potential of every single node below \(2^{n}\), he can prevent Maker from raising the Maker-degree of a node above \(q/2\). We will later show (Theorem 8) that Breaker is even able to keep the total potential of the game below \(2^{n}\).

**Lemma 5.** If \(n\) is sufficiently big, for every turn \(t\) and every node \(v \in V\) the following holds:

\[
0 < \text{pot}_t(v) \leq 2n \Rightarrow \deg_{M,t}(v) \neq \lceil q/2 \rceil - 1.
\]

**Proof.** Let \(t\) be a turn and \(v \in V\) with \(\text{pot}_t(v) > 0\) and \(\deg_{M,t}(v) = \lceil q/2 \rceil - 1\). We show that \(\text{pot}_t(v) > 2n\). Because \(\text{pot}_t(v) \neq 0\), we have \(\text{pot}_t(v) = \mu^{d_t(v)/q}\). We claim (and later prove) that \(d_t(v) \geq \frac{2\delta n}{3}\).

This implies

\[
\text{pot}_t(v) = \mu^{d_t(v)/q} \geq \mu^{2\delta n/3q} = \left(1 + \frac{6\beta \ln(n)}{\delta q}\right)^{2\delta n/3q} \\
= \left(1 + \frac{6\beta \ln(n)}{\delta q}\right)^{(n/\mu q + 1)(1/\mu q + 1)^{-1} 2\delta n/3q} \geq e^\alpha,
\]

where

\[
\alpha = \left(\frac{\delta q}{6\beta \ln(n)} + 1\right)^{-1} \frac{2\delta n}{3q} = \left(\frac{\delta q \mu}{6\beta \ln(n)}\right)^{-1} \frac{2\delta n}{3q} = \frac{4\beta n \ln(n)}{q^2 \mu} > 2\ln(n),
\]

where for the last inequality we used that \(\mu < 2\) if \(n\) is big enough. Finally we get \(\text{pot}_t(v) \geq e^\alpha > n^2\) and for \(n \geq 2\) this is at least \(2n\).

We still have to prove claim (1). Recall that \(d_t(v) = \deg^*_t(v) - \deg_{B,t}(v)\). We estimate \(\deg^*_t(v)\) as

\[
\deg^*_t(v) = n - p_0 \left(\frac{q^2(1-\delta)(3+\delta)}{8} - \deg_{M,t}(v) \left(q - \frac{\deg_{M,t}(v)}{2}\right)\right) \\
\geq n - p_0 \left(\frac{q^2(1-\delta)(3+\delta)}{8} - \left(q - \frac{q}{2} - 1\right) \left(q - \frac{q}{4}\right)\right) \\
= n + p_0 \left(\frac{q^2}{4} + q \left(\frac{q^2 - 3}{8}\right)\right) \\
\geq n + \frac{p_0 \beta 2n}{4}\quad\text{and}\quad n \geq \frac{2\delta n}{3} \quad\text{by Remark 2},
\]

Therefore,

\[
d_t(v) = \deg^*_t(v) - \deg_{B,t}(v) \geq n + \frac{2\delta n}{3} - n = \frac{2\delta n}{3}.
\]
2.2 The detailed strategy

The basic idea is that Maker’s task to build a triangle is closely related to the task of connecting big stars. Assume that at any time during the game Maker manages to build a path \((u,v,w)\) of length 2. Then Breaker is forced to immediately close this path by claiming the edge \(\{u,w\}\) if he doesn’t already own this edge. So every sensible Breaker-strategy will follow the simple rule of immediately closing all Maker-paths of length 2. Hence, the only chance for Maker to win the game is to construct more than \(q\) paths of length 2 in a single turn, so that Breaker can’t claim enough edges to close all of them immediately. By claiming an edge \(\{u,v\}\), Maker is building \(\deg_M(u) + \deg_M(v)\) new paths of length 2. This implies that if Breaker at each turn closes all Maker-paths of length 2 and simultaneously manages to prevent Maker from building a \(q/2\)-star, he will win the game.

**Strategy 1.** Consider an arbitrary turn \(t\). Let \(e_M = \{u,v\}\) be the edge claimed by Maker in this turn. Breaker’s moves for this turn are split into two parts.

**Part 1: closing paths.** Breaker claims \(\deg_{M,t-1}(v)\) edges incident in \(u\) and \(\deg_{M,t-1}(u)\) edges incident in \(v\) to close all new Maker-paths of length 2. If such a path is already closed, he claims an arbitrary edge incident in \(u\) (\(v\), resp.) instead. If all edges incident in \(u\) (\(v\), resp.) are already claimed, we call the turn \(t\) an isolation turn. In this case, Breaker claims arbitrary unclaimed edges instead. We call the edges claimed during Part 1 closing edges. \(u\) (\(v\), resp.) is called the head of the closing edge, whereas the corresponding second node of the edge is called its tail.

**Part 2: free edges.** If after part 1 Breaker still has edges left to claim (we will later show that this is always the case), he iteratively claims an edge \(e\) with \(\pot(e) \geq \pot(e')\) for all unclaimed edges \(e'\), until he claimed all of his \(q\) edges. We call the edges claimed in Part 2 free edges. The number of free edges claimed in turn \(t\) is denoted by \(f(t)\). Note that

\[
f(t) = q - \deg_{M,t-1}(u) - \deg_{M,t-1}(v).
\]

Part 1 of the strategy is more or less obligatory, because a Maker-path of length 2 that is not closed by Breaker can be completed to a triangle in the next turn. Part 2 is more interesting. Our aim in the following sections is to prove Theorem 9 where we show that part 2 of the strategy prevents Maker from building a \(q/2\)-star, so that Breaker wins the game.

**Observation 6.** We can assume that the game contains no isolation turns.

**Proof.** Consider an arbitrary isolation \(t\) turn in the game, i.e., a turn, after which one of the nodes of the edge \(e_M\) claimed in this turn by Maker has no unclaimed incident edges left. Right after the turn, every triangle \(e_M\) belongs to is already blocked by Breaker, so the edge \(e_M\) is of no use for Maker from this time on. Breaker even could pretend that the edge \(e_M\) belongs to his own edges, so that in the turn \(t\) Breaker claimed \(q + 1\) edges and Maker didn’t claim any edge. Hence, a perfectly playing Maker will always try to avoid isolation turns. If he can’t, he will definitely loose the game, since he can only claim useless edges until the end of the game.

The following observation states that, as long as Breaker can keep the total potential below \(2n\), he will have at least 2 free edges in every turn.

**Observation 7.** For every turn \(t\) with \(f(t) \leq 1\) there exists a turn \(t' < t\) with \(\text{POT}_{t'} > 2n\).
Proof. Let $t$ be a turn with $f(t) \leq 1$ and let $\{u,v\}$ be the Maker-edge of this turn. Because $f(t) = q - \deg_{M,t-1}(u) - \deg_{M,t-1}(v)$, we get $\deg_{M,t-1}(u) + \deg_{M,t-1}(v) \geq q - 1$, so there exists $w \in \{u,v\}$ with $\deg_{M,t-1}(w) \geq \left\lceil \frac{q-1}{2} \right\rceil \geq \left\lceil \frac{q}{2} \right\rceil - 1$. Hence, there exists a turn $t' \leq t-1$ with $\deg_{M,t'}(w) = \left\lceil \frac{q}{2} \right\rceil - 1$ and $\pot_{t'}(w) > 0$. We apply Lemma 5 and get $\pot_{t'}(w) > 2n$, so especially $\POT_{t'} > 2n$. 

2.3 Main results

In this subsection we prove that Strategy 1 works correctly and is a winning strategy. For both theorems in this subsection we assume that Breaker plays according to Strategy 1.

We further assume that $q = \sqrt{(\frac{8}{3} + \epsilon^*) n}$ for some $\epsilon^* > 0$ as stated above and that $n$ is sufficiently large. For Breaker’s strategy it is crucial that the potential of every node is kept below a certain level. This is ensured by the following theorem.

Theorem 8. For every turn $s$ it holds $\POT_s < 2n$.

The proof of this theorem is the mathematical core of this paper and is given in the next section. The main result of our work is:

Theorem 9. At the end of the game there exists no node with Maker-degree of at least $q/2$ and Breaker wins the game.

Proof. Assume that there exists a node $v$ with $\deg_{M}(v) \geq q/2$ at the end of the game. Then, $\deg_{M}(v) \geq \lceil q/2 \rceil$. Let $t$ denote the turn in which Maker claimed his $\lceil q/2 \rceil$-th edge incident in $v$, so $\deg_{M,t-1}(v) = \lceil q/2 \rceil - 1$. Due to Theorem 8 we know that $\pot_{t-1}(v) \leq \POT_{t-1} < 2n$. Note that after turn $t - 1$ there are still unclaimed edges incident in $v$, so $\pot_{t-1}(v) > 0$. We apply Lemma 5 and get $\deg_{M,t-1}(v) \neq \lceil q/2 \rceil - 1$, a contradiction.

So with every edge $\{u,v\}$ that Maker chooses he creates less than $\deg_{M}(u) + \deg_{M}(v) < q$ new Maker-paths of length 2. Hence, Breaker always has enough edges to close all Maker-paths of length 2 and finally wins the game. 

3 Analysis

3.1 Outline of the proof

We proceed to prove Theorem 8. As it is depending on a series of lemmas, for the reader’s convenience we first outline the argumentation in an informal way. We distinguish two types of turns. A turn is called non-critical, if a certain fraction of the Breaker-edges in this turn suffices to compensate the total potential increase caused by Maker in this turn. Otherwise, we call it critical. We start with an arbitrary critical turn $t_0$ in which the potential exceeds $n$. Lemma 15 gives us a useful characterization of critical turns. This enables us to prove Theorem 16, where we state that before a constant number of additional critical turns is played, the total potential will sink below $n$ again. Because a constant number of critical turns cannot increase the total potential considerably much (Lemma 19), we can prove that the total potential of the game never exceeds $2n$. 

3.2 Potential change in a single turn

To analyze the potential change of a single turn, we first present a few tools for estimation of potential change caused by single Maker- and Breaker-edges. The next lemma shows how the addition of a single Maker-edge changes the deficit of a node.

Lemma 10. Consider an arbitrary point of time in the game. Let $u \in V$ and let $\deg^*(u), \deg_M^*(u)$ and $d'(u)$ be the balanced Breaker-degree, Maker-degree and deficit of $u$ after an additional edge incident in $u$ was claimed by Maker. Then,

$$d'(u) - d(u) = \deg^*(u) - \deg^*(u) \leq p_0(q - \deg_M(u)).$$

Proof. The equation follows from the fact that an additional Maker-edge does not change $\deg_B(u)$. Using that $\deg^*_M(u) = \deg_M(u) + 1$ we continue

$$\deg^*_M(u) - \deg^*_u$$

$$= p_0 \deg^*_M(u) \left( q - \frac{\deg^*_M(u)}{2} \right) - p_0 \deg_M(u) \left( q - \frac{\deg_M(u)}{2} \right)$$

$$= p_0 \left( \deg_M(u) \left( q - \frac{\deg_M(u) + 1}{2} \right) + \left( q - \frac{\deg_M(u) + 1}{2} \right) \right)$$

$$- p_0 \deg_M(u) \left( q - \frac{\deg_M(u)}{2} \right)$$

$$= p_0 \left( q - \deg_M(u) - 1/2 \right) \leq p_0 \left( q - \deg_M(u) \right).$$

Lemma 11. (i) A single edge $e_M$ claimed by Maker increases the potential of a node by at most a factor of $\mu$ and causes a total potential increase of at most $(\mu - 1)\text{pot}(e_M)$ (where $\text{pot}(e_M)$ denotes the potential of $e_M$ when claimed by Maker).

(ii) A single edge $e_B$ claimed by Breaker causes a total potential decrease of at most $(1 - \mu^{-1/q})\text{pot}(e_B)$ (where $\text{pot}(e_B)$ denotes the potential of $e_B$ when claimed by Breaker).

Proof. (i). Let $e_M = \{u, v\}$. For $w \in V$ let $\text{pot}(w)$ denote the potential of $w$ before Maker claimed $e_M$ and $\text{pot}'(w)$ denote the potential of $w$ directly after Maker claimed $e_M$. If $e_M$ is not incident in $w$, the potential of $w$ remains unchanged. If $e_M$ is the last unclaimed edge incident in $w$, $\text{pot}'(w) = 0$ and we are done. Otherwise we can apply Lemma 10 and Remark 2 and get

$$\frac{\text{pot}'(w)}{\text{pot}(w)} = \mu^{\deg(w) - \deg(u)/q} \leq \mu^{p_0(q - \deg_M(w))/q} \leq \mu.$$

Because $e_M$ only changes the potential of $u$ and $v$, the total potential increase is $\text{pot}'(v) - \text{pot}(v) + \text{pot}'(u) - \text{pot}(u) \leq (\mu - 1)\text{pot}(e_M)$.

(ii). Let $e_B = \{u, v\}$. Because $e_B$ only changes the potential of $u$ and $v$, the total potential decrease caused by $e_B$ is $\text{pot}(v) - \text{pot}'(v) + \text{pot}(u) - \text{pot}'(u),$ where $\text{pot}(w)$ denotes the potential of $w$ before Breaker claimed $e_B$ and $\text{pot}'(w)$ denote the potential of $w$ directly after Breaker claimed $e_B$. We show that

$$\text{pot}(v) - \text{pot}'(v) \geq (1 - \mu^{-1/q})\text{pot}(v).$$
Because the same holds for $u$, the claim (ii) follows. If $e_B$ is the last unclaimed edge in $v$, $\text{pot}'(v) = 0$. Otherwise, \[
\frac{\text{pot}'(v)}{\text{pot}(v)} = \mu^{(d'(u)-d(u))/q} = \mu^{-1/q}, \]
where the last equation follows from the fact that a Breaker-edge does not change $\text{deg}^*(v)$ and increases $\text{deg}_B(v)$ by 1. 

Every turn $t$ starts with a Maker move, i.e. an edge $\{u, v\}$ being claimed by Maker followed by $q$ Breaker moves. While the Maker move causes a potential increase, Breaker’s moves cause a decrease. For every node $w \in V$, we denote its potential increase by $I_t(w)$ and its potential decrease by $D_t(w)$. Note that every claimed edge only changes the potential of its two incident nodes. When following Breaker’s strategy, there are four possible ways of potential decrease for the node $w$: decrease caused by free edges, denoted by $D^\text{free}_t(w)$ and decrease caused by closing edges, either $w$ being their head, denoted by $D^\text{heads}_t(w)$, or their tail, denoted by $D^\text{tails}_t(w)$. In the special case in which Maker or Breaker claim the last unclaimed edge incident in $w$, the potential of $w$ is set to 0, which causes an additional potential decrease. For technical reasons, this additional decrease is considered separately and denoted by $D^0_t(w)$. If for example Breaker claims a free edge that is the last unclaimed edge incident in $w$, this edge contributes both to $D^\text{free}_t(w)$ and $D^0_t(w)$. For the contribution to $D^\text{free}_t(w)$ we only compute the potential change caused by the change of the balance value and for the contribution to $D^0_t(w)$ we take the real potential decrease caused by the edge and subtract the computed contribution to $D^\text{free}_t(w)$. Moreover, we further split $D^\text{heads}_t(w)$ into two parts $D^\text{heads}_t(w) = D^-_t(w) + D^+_t(w)$, where

\[
D^-_t(w) := \min\{I_t(w), D^\text{heads}_t(w)\} \quad \text{and} \quad D^+_t(w) := \max\{D^\text{heads}_t(w) - I_t(w), 0\}. 
\]

If Maker claims an edge that connects two nodes with a very high Maker-degree, it might happen that $D^\text{heads}_t(w) > I_t(w)$ for one or both of the newly connected nodes. Otherwise, $D^-_t = 0$ and $D^-_t(w) = D^\text{heads}_t(w)$.

If for one of these values we omit the argument, we always mean the total potential increase (decrease) added up over all nodes. For example, $I_t := \sum_{v \in V} I_t(v)$. For every turn $t$ we have

$$\text{POT}_t - \text{POT}_{t-1} = I_t - D_t = I_t - (D^\text{free}_t + D^-_t + D^+_t + D^\text{tails}_t + D^0_t).$$

**Lemma 12.** Let $t$ be an arbitrary turn. Let $e_M$ be the Maker-edge of this turn. Then,

(i) for every $w \in V$ it holds $I_t(w) - D^-_t(w) \leq (\mu^{p_M(t)/q} - 1)\text{pot}_{t-1}(w)$.

(ii) $I_t - D_t \leq (\mu^{p_M(t)/q} - 1)\text{pot}_{t-1}(e_M)$.

**Proof.** (i). Let $e_M = \{u, v\}$. First note that if $D^-_t(w) \neq D^\text{heads}_t(w)$, it follows that $D^-_t(w) = I_t(w)$, so there is nothing more to show. Otherwise, the term $I_t(w) - D^-_t(w)$ describes the change of the potential of $w$ from the beginning of the turn $t$ to the end of part 1 of Breaker’s moves in the same turn, where we ignore the changes caused by tails of closing edges. For $w \notin \{u, v\}$ this is 0 and we are done. So let $w \in \{u, v\}$ and let $\text{deg}_{M, t}^{(1)}(w), \text{deg}_{B, t}^{(1)}(w), \text{deg}^s_t(w)$ and $d^-_t(w), \text{pot}^-_t(w)$ be the Maker-degree, Breaker-degree, balanced degree, deficit and potential of $w$ after part 1 of Breaker’s moves (i.e. after all closing edges have been claimed).
To compute the change of the potential of \( w \), we start by computing the change of its deficit. We have
\[
d_t^{(1)}(w) - d_{t-1}(w) = \deg^*(t)(w) - \deg^{(1)}_{B,t}(w) - \deg^*_{t-1}(w) + \deg_{B,t-1}(w)
\]
\[
= (\deg^*(t)(w) - \deg^*_{t-1}(w)) - (\deg^{(1)}_{B,t-1}(w) - \deg_{B,t}(w)).
\]
The first term describes the change of \( \deg^*(w) \). Since Breaker-edges do not influence this value, this change is caused solely by \( e_M \). Due to Lemma 10 this is at most \( p_0(b - \deg_{M,t-1}(w)) \). The second term simply describes the number of closing edges claimed incident to \( w \). Due to Observation 6 \( t \) is no isolation turn, so in case of \( w = u \), this is \( \deg_{M,t-1}(v) \) and in case of \( w = v \) this is \( \deg_{M,t-1}(u) \). Together with (2) and Remark 2 this gives
\[
d_t^{(1)}(u) - d_{t-1}(u) = p_0(q - \deg_{M,t-1}(u)) - \deg_{M,t-1}(v) \leq p_0 f(t)
\]
and
\[
d_t^{(1)}(v) - d_{t-1}(v) = p_0(q - \deg_{M,t-1}(v)) - \deg_{M,t-1}(u) \leq p_0 f(t).
\]
This implies
\[
I_t(w) - D_t(w) = \text{pot}_t^{(1)}(w) - \text{pot}_{t-1}(w) = \mu d_t^{(1)}(w)/q - \text{pot}_{t-1}(w)
\]
\[
= (\mu d_t^{(1)}(w)/q - \text{pot}_{t-1}(w) - 1)\text{pot}_{t-1}(w)
\]
\[
\leq (\mu p_0 f(t)/q - 1)\text{pot}_{t-1}(w).
\]
(ii). Note that \( I_t = I_t(u) + I_t(v) \) and \( D_t = D_t(u) + D_t(v) \), so we have
\[
I_t - D_t = I_t(u) + I_t(v) - (D_t(u) + D_t(v))
\]
\[
= I_t(u) - D_t(u) + I_t(v) - D_t(v)
\]
\[
\leq (\mu p_0 f(t)/q - 1)\text{pot}_{t-1}(u) + (\mu p_0 f(t)/q - 1)\text{pot}_{t-1}(v)
\]
\[
= (\mu p_0 f(t)/q - 1)\text{pot}_{t-1}(e_M).
\]

3.3 Critical turns

Since \( \mu \xrightarrow{n \to \infty} 1 \), with Remark 2 and \( n \) big enough we get \( \mu p_0 < 1 \). Fix \( \eta \in (0, 1 - \mu p_0) \) and define the following parts of potential change.

**Definition 13.** For every turn \( t \) let
\[
\Delta_t := I_t - D_t - (1 - \eta)D_t^{\text{free}}
\]
and
\[
r_t := D_t^{\text{free}} + D_t^{\text{tails}} + \eta D_t^{\text{free}} + D_t^0.
\]
We call \( t \) critical, if \( \Delta_t > 0 \) and non-critical otherwise.

Note that \( \text{POT}_t - \text{POT}_{t-1} = \Delta_t - r_t \). Since \( r_t \geq 0 \), every turn \( t \) with \( \text{POT}_t > \text{POT}_{t-1} \) is critical.
Lemma 14. For all $x \in \mathbb{R}$ with $x \geq 1$ it holds $x(1 - \mu^{-1/q}) \geq 1 - \mu^{-x/q}$.

Proof. We define $g(x) := x(1 - \mu^{-1/q})$ and $h(x) := 1 - \mu^{-x/q}$, so we have to show $g(x) \geq h(x)$ for all $x \geq 1$. First note that $g(1) = h(1)$, so it suffices to show that $g'(x) \geq h'(x)$ for all $x \geq 1$. We have $g'(x) = 1 - \mu^{-1/q}$ and $h'(x) = \mu^{-x/q} \ln(\mu)/q$. Because for all $x > 0$ we have

$$h''(x) = -\mu^{-x/q} \left(\frac{\ln(\mu)}{q}\right)^2 < 0 = g''(x),$$

it suffices to show that $g''(1) \geq h''(1)$. To see this, we use the fact that $e^y - 1 \geq y$ for all $y \geq 0$, so especially $\mu^{1/q} - 1 \geq \frac{\ln(\mu)}{q}$. If we multiply both sides with $\mu^{-1/q}$, we get

$$1 - \mu^{-1/q} \geq \mu^{-1/q} \frac{\ln(\mu)}{q}.$$

Because the left hand side is $g'(1)$ and the right hand side is $h'(1)$, we are done. \qed

The following lemma provides an important characterization of critical turns by an upper bound for the potential of all edges still unclaimed after the turn.

Lemma 15. Let $t$ be a critical turn with $f(t) \geq 2$ and let $e_M$ be the edge chosen by Maker in this turn. For every edge $e$ that is still unclaimed after $t$ it holds

$$\text{pot}_t(e) < \frac{\mu p_0}{(1 - \eta)} \text{pot}_{t-1}(e_M).$$

Proof. Let $e_M = \{u, v\}$. By Lemma 12 (ii) we have

$$I_t - D_t \leq (\mu p_0 f(t)/q - 1) \text{pot}_{t-1}(e_M)$$

$$= \mu p_0 f(t)/q(1 - \mu^{-p_0 f(t)/q}) \text{pot}_{t-1}(e_M)$$

$$\leq \mu(1 - \mu^{-p_0 f(t)/q}) \text{pot}_{t-1}(e_M)$$

We apply Lemma 14 with $x := p_0 f(t)$ (note that due to Remark 2 we have $x > \frac{8}{12} f(t) \geq \frac{16}{12} > 1$) and get

$$I_t - D_t \leq \mu p_0 f(t)(1 - \mu^{-1/q}) \text{pot}_{t-1}(e_M).$$

Because $t$ is a critical turn, we get

$$0 < \Delta_t = I_t - D_t - (1 - \eta) D_t^{\text{free}}$$

$$\leq \mu p_0 f(t)(1 - \mu^{-1/q}) \text{pot}_{t-1}(e_M) - (1 - \eta) D_t^{\text{free}},$$

implying

$$(1 - \eta) D_t^{\text{free}} < \mu p_0 f(t)(1 - \mu^{-1/q}) \text{pot}_{t-1}(e_M). \quad (5)$$

Now let $e$ be an edge that after turn $t$ still is unclaimed. Then every free edge claimed by Breaker in turn $t$ has at least a potential of $\text{pot}_t(e)$ because Breaker iteratively chooses the edge with maximum potential and every edge claimed by Breaker only decreases potential. Due to Lemma 11 (ii) every free edge causes a total potential decrease of at least $\text{pot}_t(e)(1 - \mu^{-1/q})$ and hence we get

$$D_t^{\text{free}} \geq f(t) \text{pot}_t(e)(1 - \mu^{-1/q}).$$

Together with (5) this implies $\text{pot}_t(e) < \frac{\mu p_0}{(1 - \eta)} \text{pot}_{t-1}(e_M)$. \qed
3.4 Increase of total potential

With our strategy we cannot guarantee that \( \text{POT}_t \leq \text{POT}_{t-1} \) for all turns \( t \). But we will show that each turn \( t_0 \) at which the potential exceeds \( n \) is followed closely by a turn at which the total potential is at most as big as it was before \( t_0 \). So in the long run we obtain a decrease of the total potential, which will ensure Breaker’s win.

Fix constant parameters \( \gamma \in (0, 1) \), and \( \epsilon > 0 \) with

\[
\frac{1 - \eta}{(1 + \epsilon)\mu p_0} > 1.
\]

Recall that this possible, because \( \eta < 1 - \mu p_0 \) by the choice of \( \eta \). Define

\[
c := \left[ \frac{1 - \log(1 - \gamma)}{\log(1 - \eta) - \log(1 + \epsilon) - \log(\mu p_0)} \right]
\]

and note that \( c > 0 \) due to (6). Although \( c \) depends on \( n \), it is bounded by constants because \( 1 < \mu < 2 \) for \( n \) sufficiently big. Let \( t_0 \) be a turn with \( \text{POT}_{t_0} > n, \text{POT}_{t_0-1} \leq n \) and \( \text{POT}_t < 2n \) for all \( t < t_0 \). Then, \( t_0 \) is a critical turn and due to Observation 7 it holds \( f(t_0) \geq 2 \). Let \( e_0 = \{u, v\} \) be the edge claimed by Maker in this turn and w.l.o.g. let \( \text{pot}_{t_0-1}(u) \geq \text{pot}_{t_0-1}(v) \). We consider three points of time:

- Let \( t_1 \) be the first turn after \( t_0 - 1 \) with \( \text{pot}_{t_1}(u) \leq (1 - \gamma)\text{pot}_{t_0-1}(u) \).
- Let \( t_2 \) be the first turn after \( t_0 \) with \( \text{pot}_{t_2}(w) \geq (1 + \epsilon)\text{pot}_s(w) \) for some \( w \in V \) and some turn \( s \) with \( t_0 \leq s < t_2 \).
- Let \( t_3 \) be the \( c \)-th critical turn after \( t_0 - 1 \).

If the game ends before the turn \( t_i \) is reached, let \( t_i := \infty \). We set \( t^* := \min(t_1, t_2, t_3) \) (note that \( t^* = \infty \) is possible) and aim to prove the following theorem

**Theorem 16.** Let \( n \) sufficiently big. If the game is not ended before turn \( t^* \), then \( \text{POT}_{t^*} \leq \text{POT}_{t_0-1} \).

Since the proof is quite involved, it is split into several parts. We start with an observation, that between the turns \( t_0 \) and \( t_2 \) the total potential will not exceed \( 2n \).

**Observation 17.** If \( n \) is sufficiently large, for every turn \( t \) with \( t_0 \leq t < t_2 \) it holds \( \text{POT}_t < 2n \).

**Proof.** Because \( t < t_2 \), for every \( v \in V \) it holds \( \text{pot}_t(v) \leq (1 + \epsilon)\text{pot}_{t_0}(v) \) by definition of \( t_2 \). This implies

\[
\text{POT}_t = \sum_{v \in V} \text{pot}_t(v) \leq \sum_{v \in V} (1 + \epsilon)\text{pot}_{t_0}(v) = (1 + \epsilon)\text{POT}_{t_0}.
\]

By Lemma 12 (ii) we have

\[
\text{POT}_{t_0} = \text{POT}_{t_0} - \text{POT}_{t_0-1} + \text{POT}_{t_0-1} \leq I_{t_0} - D_{t_0} + \text{POT}_{t_0-1} \\
\leq \mu p_0 f(t_0)/q \text{POT}_{t_0-1} \leq \mu \text{POT}_{t_0-1},
\]

so finally,

\[
\text{POT}_t \leq (1 + \epsilon)\text{POT}_{t_0} \leq (1 + \epsilon)\mu (1 + \epsilon)\text{POT}_{t_0-1} \leq \mu(1 + \epsilon)n < \frac{3}{2} \mu n.
\]

For sufficiently large \( n \) we have \( \mu < \frac{4}{3} \) and the proof is complete. \( \square \)
In the following we assume that the game is not ended before turn \( t^* \) is reached. In the next lemma we further refine the characterization of critical turns from Lemma [15]. We only consider turns between \( t_0 \) and \( t_2 \) and prove that the number of critical turns in this interval affects the maximum possible potential of unclaimed edges exponentially.

**Lemma 18.** Let \( s \) be a turn with \( t_0 \leq s \leq t^* \) and \( s < t_2 \). Let \( \text{crit}(s) \in [c] \) be the number of critical turns between \( t_0 \) and \( s \) (including \( t_0 \) and \( s \)). Then, for every edge \( e \) unclaimed after turn \( s \) it holds

\[
\text{pot}_s(e) < \left( \frac{(1 + \epsilon)\mu p_0}{(1 - \eta)} \right)^{\text{crit}(s)} 2\text{pot}_{t_0-1}(u).
\]

**Proof.** Via induction over \( \text{crit}(s) \).

Let \( \text{crit}(s) = 1 \). Recall that \( e_0 = \{u, v\} \) is the edge claimed by Maker in turn \( t_0 \) and that \( \text{pot}_{t_0-1}(u) \geq \text{pot}_{t_0-1}(v) \). Let \( e = \{x, y\} \) be an edge unclaimed after turn \( s \). Because \( s < t_2 \), we know that

\[
\text{pot}_s(e) = \text{pot}_s(x) + \text{pot}_s(y) \leq (1 + \epsilon)\text{pot}_{t_0}(x) + (1 + \epsilon)\text{pot}_{t_0}(y) = (1 + \epsilon)\text{pot}_{t_0}(e)
\]

and because \( f(t_0) \geq 2 \), by Lemma [15]

\[
(1 + \epsilon)\text{pot}_{t_0}(e) < (1 + \epsilon)\frac{\mu p_0}{(1 - \eta)}\text{pot}_{t_0-1}(e_0) \leq \left( \frac{(1 + \epsilon)\mu p_0}{(1 - \eta)} \right) 2\text{pot}_{t_0-1}(u).
\]

Now let the claim be true for all \( s' \) with \( \text{crit}(s') = i, i \in [c - 1] \). Let \( s \) be a turn with \( \text{crit}(s) = i + 1 \). Let \( s' \) be the last critical turn before \( s \) (if \( s \) is critical, let \( s' = s \)). Then \( \text{crit}(s' - 1) = i \). Let \( e_M \) be the edge claimed by Maker in turn \( s' \). We get

\[
\text{pot}_s(e) \leq (1 + \epsilon)\text{pot}_{s'}(e) \quad (t < t_2)
\]

\[
\leq (1 + \epsilon)\frac{\mu p_0}{(1 - \eta)}\text{pot}_{s'-1}(e_M) \quad \text{(Lemma [15])}
\]

\[
\leq (1 + \epsilon)\frac{\mu p_0}{(1 - \eta)} \left( \frac{(1 + \epsilon)\mu p_0}{1 - \eta} \right) i \text{pot}_{t_0-1}(u) \quad \text{(IH)}
\]

\[
= \left( \frac{(1 + \epsilon)\mu p_0}{1 - \eta} \right)^{i + 1} 2\text{pot}_{t_0-1}(u).
\]

Note that for the above application of Lemma [15] we need to ensure that \( f(s') \geq 2 \). Due to Observation [7] it suffices to show that \( \text{POT}_t < 2n \) for all \( t < s' \). By choice of \( t_0 \), we already know that \( \text{POT}_t < 2n \) for all \( t < t_0 \) and because \( s' \leq s < t_2 \), for all \( t_0 \leq t < s' \) we can apply Observation [17] and get \( \text{POT}_t < 2n \).

**Lemma 19.** For every \( \xi > 0 \), if \( n \) is sufficiently big, we have

\[
\sum_{\substack{t_0 \leq s \leq t^* \\text{s critical}}} I_s \leq 2\epsilon(\mu - 1)\text{pot}_{t_0-1}(u) < \xi\text{pot}_{t_0-1}(u).
\]

**Proof.** Let \( \xi > 0 \). First note that due to Lemma [11] (i)

\[
I_{t_0} \leq (\mu - 1)\text{pot}_{t_0-1}(e_0) \leq 2(\mu - 1)\text{pot}_{t_0-1}(u).
\]

(7)
Now let $s$ be a critical turn with $t_0 < s \leq t^*$. Let $e_M$ be the edge claimed by Maker in this turn. We get

$$I_s \leq (\mu - 1)\text{pot}_{s-1}(e_M) \quad \text{(Lemma 11 (i))}$$

$$< (\mu - 1) \left( \frac{(1 + \epsilon)p_0}{(1 - \eta)} \right)^{\text{crit}(s-1)} 2\text{pot}_{t_0-1}(u) \quad \text{(Lemma 18)}$$

$$\leq (\mu - 1)2\text{pot}_{t_0-1}(u).$$

So for every critical turn $s$ with $t_0 \leq s \leq t^*$ we have

$$I_s \leq 2(\mu - 1)\text{pot}_{t_0-1}(u). \quad (8)$$

Because $t \leq t_3$, there are at most $c$ critical turns between $t_0$ and $t^*$, so finally we get

$$\sum_{t_0 \leq s \leq t^*} I_s \leq \sum_{t_0 \leq s \leq t^*} 2(\mu - 1)\text{pot}_{t_0-1}(u) \leq 2c(\mu - 1)\text{pot}_{t_0-1}(u).$$

Recall that $(\mu - 1) = 6 \ln(u)\beta/\delta q = 6 \ln(u)\sqrt{\beta/\delta} \sqrt{n} \xrightarrow{n \to \infty} 0$, whereas $c$ is bounded by a constant. So for $n$ sufficiently big, the whole term is smaller than $\xi \text{pot}_{t_0-1}(u)$.

By definition, $t^*$ always has one of the three values $t_1, t_2, t_3$. In the following three lemmas we consider all possible cases. These lemmas combined directly imply Theorem 16.

We always assume $n$ to be sufficiently big if needed.

**Lemma 20.** If $t_1 \leq t_2$ and $t_1 \leq t_3$, then $\text{POT}_{t_0-1} \geq \text{POT}_{t^*}$.

*Proof.* Let $\xi \in (0, \eta \gamma)$. By assumption $t^* = \min(t_1, t_2, t_3) = t_1$ and hence, by definition of $t_1$ we have $\text{pot}_{t_1}(u) \leq (1 - \gamma)\text{pot}_{t_0-1}(u)$. Let $R := \sum_{t_0 \leq s \leq t^*} r_s$. Then,

$$\text{POT}_{t^*} - \text{POT}_{t_0-1} = \sum_{t_0 \leq s \leq t^*} \text{POT}_s - \text{POT}_{s-1} = \sum_{t_0 \leq s \leq t^*} \Delta_s - r_s$$

$$= \sum_{t_0 \leq s \leq t^*} \Delta_s + \sum_{s \text{ critical}} \Delta_s - R$$

$$\leq \sum_{t_0 \leq s \leq t^*} \Delta_s - R$$

$$\leq \xi \text{pot}_{t_0-1}(u) - R, \quad \text{(Lemma 19)}$$
hence it suffices to show that \( R \geq \xi \text{pot}_{t_0-1}(u) \). We have

\[
\begin{align*}
\xi \text{pot}_{t_0-1}(u) \\
\leq \eta \gamma \text{pot}_{t_0-1}(u) \\
\leq \eta (\text{pot}_{t_0-1}(u) - \text{pot}_t(u)) \\
= \eta \left( \sum_{t_0 \leq s \leq t^*} D_s(u) - I_s(u) \right) \\
= \eta \left( \sum_{t_0 \leq s \leq t^*} D_s^+(u) + D_s^-(u) + D_s^{\text{tails}}(u) + D_s^{\text{free}}(u) + D_s^0(u) - I_s(u) \right) \\
\leq \eta \left( \sum_{t_0 \leq s \leq t^*} D_s^+(u) + D_s^{\text{tails}}(u) + D_s^{\text{free}}(u) + D_s^0(u) \right) \\
\leq \sum_{t_0 \leq s \leq t^*} r_s = R.
\end{align*}
\]

\( \square \)

**Lemma 21.** If \( t_2 < t_1 \) and \( t_2 \leq t_3 \), then \( \text{POT}_{t_0-1} \geq \text{POT}_{t^*} \).

**Proof.** Let \( \xi > 0 \) with \( \xi \leq \eta (1 - \gamma)(1 - (1 + \epsilon)^{-1/p_0}) \). We have \( t^* = t_2 \), so there exists a turn \( s_0 \) with \( t_0 \leq s_0 < t^* \) and a vertex \( w \in V \), such that \( \text{pot}_t(w) \geq (1 + \epsilon)\text{pot}_s(w) \). Because \( t^* < t_1 \), the potential of \( u \) was not set to 0 and as in the proof of Lemma 20 it suffices to show that \( R \geq \xi \text{pot}_{t_0-1}(u) \). We start by showing that for all turns \( t \) with \( s_0 \leq t \leq t^* \) it holds

\[
\text{pot}_t(w) \leq \text{pot}_{s_0}(w) \prod_{s_0 < s \leq t} \mu^{p_0f(s)/q}, \tag{9}
\]

We prove (9) via induction over \( t \). For \( t = s_0 \) the claim obviously holds. Now let \( t > s_0 \). Then, \( t - 1 \geq s_0 \) and by Lemma 12 (i) we have

\[
\text{pot}_t(w) - \text{pot}_{t-1}(w) \leq I_t(w) - D_t(w) \leq \text{pot}_{t-1}(w)(\mu^{p_0f(t)/q} - 1),
\]

so

\[
\text{pot}_t(w) \leq \text{pot}_{t-1}(w)\mu^{p_0f(t)/q}.
\]

By applying the induction hypothesis we finish the proof of (9):

\[
\text{pot}_t(w) \leq \left( \text{pot}_{s_0}(w) \prod_{s_0 < s \leq t-1} \mu^{p_0f(s)/q} \right) \mu^{p_0f(t)/q} = \text{pot}_{s_0}(w) \prod_{s_0 < s \leq t} \mu^{p_0f(s)/q}.
\]

Using (9), we get

\[
(1 + \epsilon)\text{pot}_{s_0}(w) \leq \text{pot}_{t^*}(w) \leq \text{pot}_{s_0}(w) \prod_{s_0 < s \leq t^*} \mu^{p_0f(s)/q},
\]

15
so

$$(1 + \varepsilon) \leq \prod_{s_0 < s \leq t^*} \mu^{p_0 f(s)/q} = \mu^{(p_0 \sum_{s_0 < s \leq t^*} f(s))/q}$$

which, taking the logarithm gives

\[ \sum_{s_0 < s \leq t^*} f(s) \geq \frac{q \ln(1 + \varepsilon)}{p_0 \ln(\mu)} =: x, \]

so at least $x$ free edges were claimed by Breaker between the turns $s_0$ and $t^*$. Because $t^* < t_1$, at the whole time from $t_0$ to $t^*$ the potential of $u$ is at least $(1 - \gamma)\text{pot}_{t_0-1}(u)$. Hence, during this time every unclaimed edge incident in $u$ has a potential of at least $(1 - \gamma)\text{pot}_{t_0-1}(u)$, so especially every free edge claimed by Breaker has at least this potential and, due to Lemma 11 (ii), causes a decrease of the total potential of at least $(1 - \gamma)\text{pot}_{t_0-1}(u)(1 - \mu^{-\frac{1}{3}})$. Therefore, we get

\[ R \geq \eta \sum_{s_0 < s \leq t^*} D_s^{\text{free}} \]

\[ \geq \eta x (1 - \gamma)\text{pot}_{t_0-1}(u) \left(1 - \mu^{-\frac{1}{3}}\right) \]

\[ \geq \eta (1 - \gamma)\text{pot}_{t_0-1}(u) \left(1 - \mu^{-\frac{1}{3}}\right) \]

\[ \geq \eta (1 - \gamma)\text{pot}_{t_0-1}(u) \left(1 - (1 + \varepsilon)^{-\frac{1}{p_0}}\right) \]

\[ \geq \xi \text{pot}_{t_0-1}(u). \]

Lemma 22. $t_3 \geq \min(t_1, t_2)$.

Proof. Let us assume that $t_3 < \min(t_1, t_2)$. Then $t^* = t_3$, so $t^*$ is the $c$-th critical turn after $t_0 - 1$. We apply Lemma 18 to $s = t^* < t_2$ and obtain that for every unclaimed edge $e$ after turn $t^*$ it holds

\[ \text{pot}_{t^*}(e) < \left(\frac{(1 + \varepsilon)\mu p_0}{1 - \eta}\right)^c 2\text{pot}_{t_0-1}(u) \leq (1 - \gamma)\text{pot}_{t_0-1}(u) \]

by the choice of $c$. Since $t^* < t_1$, we have $\text{pot}_{t_0}(u) \geq (1 - \gamma)\text{pot}_{t_0-1}(u)$, so directly after turn $t^*$, every unclaimed edge incident in $u$ has a potential of at least $(1 - \gamma)\text{pot}_{t_0-1}(u)$. Hence, after turn $t^*$ there exists no unclaimed edge incident in $u$ and this implies that the potential of $u$ must have been set to 0 at some turn $s$ with $t_0 \leq s \leq t^*$. But then $t_1 \leq s \leq t^* = t_3$, a contradiction.

Proof of Theorem 8. Let $s$ be some turn with $\text{POT}_t < 2n$ for all $t < s$. We show that this already implies $\text{POT}_s < 2n$.

If $\text{POT}_s < n$, there is nothing to show, so let $\text{POT}_s > n$. Let $t_0$ be maximal satisfying $t_0 \leq s$ and $\text{POT}_{t_0-1} \leq n$ ($t_0$ exists due to Lemma 4). Define $t^*$ as in Section 3.3. If $s = t^*$, we can apply Theorem 16 and get $\text{POT}_s = \text{POT}_{t^*} \leq \text{POT}_{t_0-1} \leq n$, so we may assume $s < t^*$. But then, $s < t_2$, so we can apply Observation 17 and obtain that $\text{POT}_s < 2n$. \(\square\)
4 Open Questions

We have narrowed the gap for the threshold bias to $[1.414\sqrt{n}, 1.633\sqrt{n}]$. Of course, the question about the exact threshold value remains. At first sight our strategy still has some unused potential for improvement, since the secondary goal of preventing Maker from building a $q/2$-star is very restricting. Breaker could allow Maker to build a few bigger stars, if at the same time he is able to claim all edges connecting these stars. For $q \leq \sqrt{8n/3}$ the strategy still could be used to prevent Maker from building an $\alpha q$-star for some $\alpha > 1/2$. But it certainly needs some additional variations of the strategy to prevent Maker from connecting stars of size at least $q/2$.

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6 Appendix

6.1 List of variables

- \( n \): number of nodes in the game graph
- \( q \): number of Breaker-edges per turn
- \( \beta \): defined as \( \beta := \frac{2^2}{n} \); the strategy in this paper works for \( \beta > \frac{8}{3} \).
- \( \epsilon^* \): a strictly positive constant.
- \( \deg_{M,t}(v) \): Maker-degree of \( v \); number of Maker-edges incident in \( v \) after turn \( t \)
- \( \deg_{B,t}(v) \): Breaker-degree of \( v \); number of Breaker-edges incident in \( v \) after turn \( t \)
- \( \delta \): a constant with \( 0 < \delta < 1 - \frac{8}{37} \); chosen in Section 2.1
- \( \text{bal}(v) \): the balance of \( v \); a measure of the ratio of Maker and Breaker-edges incident in \( v \); introduced in Definition 1
- \( p_0 \): the balance of a node without incident Maker or Breaker-edges; introduced in Definition 1
- \( \deg^*(v) \): the balanced Breaker-degree of \( v \); introduced in Definition 3
- \( d(v) \): the deficit of \( v \), exponent in the potential function; introduced in Definition 3
- \( \mu \): base in the potential function; introduced in Definition 3
- \( \text{pot}(v) \): the potential of \( v \), in part 2 of the strategy Breaker always claims edges \( \{u, v\} \) maximizing \( \text{pot}(u) + \text{pot}(v) \); introduced in Definition 3
- \( \text{POT}_t \): the total potential of a turn \( t \); introduced in Definition 3
- \( f(t) \): number of free edges claimed by Breaker in turn \( t \); introduced in Section 2.2
- \( I_t(v) \): potential increase of \( v \) in turn \( t \)
- \( D_t(v) \): potential decrease of \( v \) in turn \( t \)
- \( D^\text{free}_t(v) \): potential decrease of \( v \) in turn \( t \) caused by free edges
- \( D^\text{heads}_t(v) \): potential decrease of \( v \) in turn \( t \) caused by closing edges with \( v \) as head
We will show that \( \text{bal}(v) \) : the fraction of available incident in \( v \) by showing that \( \text{bal}(v) \) is defined as the fraction of edges that Breaker has to claim in order to keep \( \text{deg}_v \) as long as there are unclaimed edges incident in \( v \). Because Maker claims one edge per turn and concentrates on \( v \), we get \( T = \frac{v(1-\delta)}{2} - \text{deg}_M(v) \)

- \( D_t^{\text{tails}}(v) \) : potential decrease of \( v \) in turn \( t \) caused by closing edges with \( v \) as tail
- \( D_t^{\text{heads}}(v) \) : potential decrease of \( v \) in turn \( t \) caused by claiming the last unclaimed edge of \( v \)
- \( D_t(v) = \min\{I_t(v), D_t^{\text{heads}}(v)\} \); it holds \( D_t(v) + D_t^+(v) = D_t^{\text{heads}}(v) \)
- \( D_t^+(v) = \max\{D_t^{\text{heads}}(v) - I_t(v), 0\} \); it holds \( D_t(v) + D_t^+(v) = D_t^{\text{heads}}(v) \)
- \( \eta : a \) constant with \( 0 < \eta < 1 - \mu_0 \); introduced in Section 3.3
- \( \Delta_t : \) main part of the total potential change in turn \( t \) with \( \Delta_t + r_t = \text{POT}_t - \text{POT}_{t-1} \); introduced in Definition 13
- \( r_t : \) rest part of the total potential change in turn \( t \), with \( \Delta_t + r_t = \text{POT}_t - \text{POT}_{t-1} \); introduced in Definition 13
- \( \gamma : \) a strictly positive constant; introduced in Section 3.4
- \( \epsilon : \) a strictly positive constant; introduced in Section 3.4
- \( c : \) a strictly positive value bounded by a constant; introduced in Section 3.4
- \( t_i, i = 0, 1, 2, 3 : \) certain turns considered in Section 3.4
- \( t^* = \min(t_1, t_2, t_3) \); introduced in Section 3.4

### 6.2 Interpretation of the balance value

In the following we motivate the definition of the balance value of a node by giving an ‘in-game’-example. Let \( v \in V \) with \( \text{deg}_M(v) < \frac{v(1-\delta)}{2} \) and suppose that Maker decides to concentrate on the node \( v \), i.e., from this moment on he will claim all of his edges incident in \( v \) as long as there are unclaimed edges incident in \( v \). Moreover suppose that Breaker’s aim, besides closing all Maker-paths of length 2, is to keep \( \text{deg}_M(v) \) below \( \frac{v(1-\delta)}{2} \). To achieve this, he must claim a certain amount of edges incident in \( v \) himself. Denote this amount by \( B_v \). Let \( T \) denote the number of turns that Maker needs to raise \( \text{deg}_M(v) \) above \( \left\lfloor \frac{v(1-\delta)}{2} \right\rfloor \). Then \( B_{\text{total}} := Tb \) is the number of edges that Breaker can claim before \( \text{deg}_M(v) \) above \( \left\lfloor \frac{v(1-\delta)}{2} \right\rfloor \). But there is a certain number \( C \) of edges that Breaker has to claim at different places, not incident in \( v \), to close new Maker-paths.

Setting \( A := B_{\text{total}} - C \) as the amount of available Breaker-edges, the term \( \frac{A}{A} \) represents the fraction of \( \text{available} \) Breaker-edges necessary to prevent Maker from building a \( \frac{v}{2} \)-star.

We will show that bal(v) is an approximation of \( \frac{A}{A} \), hence it is a measure for the ‘danger’ of \( v \). The smaller \( \frac{A}{A} \) is, the less attention Breaker has to spend to the node \( v \). If \( \frac{A}{A} > 1 \), this means that Breaker cannot achieve his goal of keeping \( \text{deg}_M(v) \) below \( \frac{v}{2} \).

For \( f, g : \mathbb{N} \rightarrow \mathbb{R} \) we write \( f \sim g \) if and only if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \). We will close this subsection by showing that bal(v) \( \sim \frac{B_v}{A} \) for some \( A' \leq A \). To prevent Maker from building a \( \frac{v(1-\delta)}{2} \)-star at \( v \), at the end of the game Breaker must possess at least \( n - \frac{v(1-\delta)}{2} \) edges incident in \( v \). Hence, the number of edges still to claim is \( B_v = n - \frac{v(1-\delta)}{2} - \text{deg}_B(v) \sim n - \text{deg}_B(v) \).
and $B_{\text{total}} = \frac{q^2(1-\delta)}{2} - q\deg_M(v)$. The exact value of $C$ depends on the choices of Maker and on how many closing edges are already owned by Breaker. If we assume that all closing edges are previously unclaimed, we can upper bound $C$ by

$$C' := \sum_{i=\deg_M(v)}^{\lfloor q(1-\delta)/2 \rfloor - 1} i = \frac{(\lfloor q(1-\delta)/2 \rfloor - 1) \cdot \lfloor q(1-\delta)/2 \rfloor - (\deg_M(v) + 1)\deg_M(v)}{2} - \frac{\deg_M(v)^2}{2}.$$  

Finally, for $A' := B_{\text{total}} - C' \leq A$ we get

$$\frac{B_v}{A'} = \frac{B_v}{B_{\text{total}} - C'} \sim \frac{n - \deg_B(v)}{q^2(1-\delta) - b\deg_M(v) - \left(\frac{q^2(1-\delta)^2}{8} - \frac{\deg_M(v)^2}{2}\right)} = \frac{8(n - \deg_B(v))}{q^2(1-\delta)(3 + \delta) - 4\deg_M(v)(2q - \deg_M(v))} = \text{bal}(v).$$

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