Initial value problem and soliton solutions of the single-cycle short pulse equation via the Riemann-Hilbert approach

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Keywords: single-cycle short pulse equation, initial value problem, Riemann-Hilbert problem

Abstract

This paper solves the initial value problem of the single-cycle short pulse equation \( u_{xt} = u + \frac{1}{6}u(u^2)_{xx} \) via the Riemann-Hilbert approach. By converting initial value problem into the Riemann–Hilbert problem, we have obtained one-soliton solutions and two-soliton solutions for the single-cycle short pulse equation.

1. Introduction

The short pulse equation (SP equation)

\[ u_{xt} = u + \frac{1}{6}(u^2)_{xx} \]

was first introduced by Schäfer and Wayne [1] as a model equation to describe the propagation of ultra-short light pulses in silica optical fibres. Different from the nonlinear Schrödinger equation (NLS equation) which is used to model the evolution of slowly varying wave trains, the SP equation is proposed to describe the pulse whose spectrum is not narrowly localized around the carrier frequency. It has been proven that as the pulse length shortens, the NLS equation approximation describing the optical pulses becomes steadily less accurate, while the SPE provides a better approximation [2]. Sakovich A and Sakovich S [3] studied the integrability of the SP equation from a zero curvature point of view and gave a transformation relating the SP equation to the sine-Gordon equation. In [4], this transformation was used to derive exact solutions of the SP equation from the known soliton solutions of the sine-Gordon equation. The recursion operator, Hamiltonian structures, conservation laws and the SP hierarchy were studied in [5] and [6]. Multisoliton solutions and periodic solutions were given in [7–9]. Generalizations of the SP equation were studied in [10–12], [13] solved the solution and studied the long-term asymptotic of the short pulse equation under the given initial value by a Riemann-Hilbert approach. Recently, [14–21] studied other nonlinear differential equations via the RH approach.

The single-cycle short pulse equation (SCP equation)

\[ u_{xt} = u + \frac{1}{2}u(u^2)_{xx} \]  \hspace{1cm} (1)

was proposed in [11], which was able to appear in physics and technology as a model equation describing the propagation of the extremely short wave packets in certain media with cubic nonlinearities. They called equation (1) single-cycle short pulse equation because the smooth envelope soliton of the SCP equation can be as short as only one cycle of its carrier frequency. Also, [11] gave the Lax pairs, bi-Hamiltonian structure of the SCP equation and derived solutions of the SCP equation from the known soliton solutions of the sine-Gordon equation. In this paper, we will solve the initial value problem of the SCP equation. By establishing the Riemann-Hilbert problem, we present the existence and uniqueness of the solution to the initial value problem of the SCP equation, and further obtain one-soliton solutions and two-soliton solutions for the SCP equation. Recently, [22] also gave one-soliton solutions of the SCP equation. Different from [22], the transformations we selected will make the results more understandable and we focus on the selection of the parameters \( C_j \) to make the solutions of the SCP equation specific.

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The novel points in this paper are as follows. First, different from \[11\] in which the two-soliton solutions of the SCP equation were obtained from the known soliton solutions of the sine-Gordon equation, in this paper, the two-soliton solutions of the SCP equation are obtained directly by the RH problem. Second, in the case when the scattering coefficient \(a(k)\) has a pair of negative conjugate zeros, unlike the two-soliton solutions of the SP equation given in \[13\], the two-soliton solutions we have obtained has the property: the partial derivative of the spatial variable \(x(y, t)\) to the variable \(y\) is always greater than or equal to 0. This means that the obtained two-soliton solutions do not appear to blow up about the space variable \(x\). Third, for the first time, we have solved the two-soliton solutions of the SCP equation for the case that the scattering coefficient \(a(k)\) has only two pure imaginary zeros. And we have proved that when one of the zeros approaches another, the limit of the two-soliton solutions is one-soliton solutions. The previous literatures \[11, 13–21\] and so on have not discussed this situation.

The paper is organized as follows. In section 2, this paper will perform a spectral analysis for Lax pairs of SCP equation with a given initial value at \(k = \infty\). Further, in section 3, we will apply the results obtained in section 2 to establish the RH problem. In section 4, we will perform a spectral analysis at \(k = 0\) and give the solution expression that satisfies the initial value problem of the SCP equation. In section 5, we will show that how to solve the \(N\)-soliton solutions for the initial value problem of SCP equation, and focus on solving the two-soliton solutions.

In this paper, \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\).

2. Spectral analysis for Lax pairs of SCP equation

In order to establish the RH problem of the initial value problem of SCP equation, in this section, we will proceed from the singularity point \(k = \infty\) for the Lax pairs of SCP equation to study the solution behavior at \(k = \infty\). Through the matrix transformation, we have obtained some properties of the transformed solution.

The initial value of the SCP equation studied in this paper is as follows:

\[
\begin{align*}
    u_{xx} &= u + \frac{1}{2}u(u^2)_{xx}, \quad t > 0, \quad -\infty < x < +\infty, \\
    u(x, 0) &= u_0(x), \quad -\infty < x < +\infty.
\end{align*}
\]

We assume that \(u_0(x)\) decays to 0 sufficiently fast:

\[
    u_0(x) \to 0, \quad x \to \pm \infty,
\]

and we seek a solution \(u(x, t)\) decaying to 0 for all \(t > 0\):

\[
    u(x, t) \to 0, \quad x \to \pm \infty.
\]

According to \[11\], the SCP equation has the following Lax pairs:

\[
\begin{align*}
    \Phi_x &= U \Phi, \\
    \Phi_t &= V \Phi,
\end{align*}
\]

where

\[
U = ik \begin{pmatrix} 1 - u_x^2 & 2u_x \\ 2u_x & -(1 - u_x^2) \end{pmatrix},
\]

\[
V = iku_x \begin{pmatrix} 1 - u_x^2 & 2u_x \\ 2u_x & -(1 - u_x^2) \end{pmatrix} + \frac{1}{4ik} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}.
\]

To know the solution behavior of the Lax pairs for the SCP equation at \(k = \infty\), we will take a transformation to Lax pairs \(6\) \(7\) of the SCP equation. First, we let

\[
m = 1 + u_x^2.
\]

Next, we define

\[
G(x, t) = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix},
\]

\[
p(x, t, k) = x - \int_x^\infty (m(\hat{x}, t) - 1) d\hat{x} - \frac{t}{4k^2}.
\]
As we can write the SCP equation (1) into the conservation law form:
\[ m_t = [u^2(1 + u_x^2)]_x, \]
then according to equations (10) (12), we can get
\[ p_x = m, p_t = u^2(1 + u_x^2) - \frac{1}{4k^2}. \]

Based on the above definition, we adopt the following transformation:
\[ \Phi(x, t, k) = G(x, t)\hat{\Phi}(x, t, k), \]
So Lax pairs (6) (7) for the SCP equation are transformed as follows:
\[ \hat{\Phi}_x - ikp_s[\sigma_3, \hat{\Phi}] = \hat{U}(x, t, k)\hat{\Phi}, \]
\[ \hat{\Phi}_t - ikp_s[\sigma_3, \hat{\Phi}] = \hat{V}(x, t, k)\hat{\Phi}, \]
where
\[ \hat{U}(x, t, k) = -\frac{u_x}{m}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{V}(x, t, k) = -\frac{u_x}{m}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \]

Here, let \( \hat{A} \) denote the operator which acts on a \( 2 \times 2 \) matrix \( X \) by \( \hat{A}X = [A, X] \), and then the Lax pairs (16) (17) can also be rewritten into the full differential form:
\[ d(e^{-\hat{A}p(s, t, k)\hat{\Phi}}) = W(x, t, k), \]
where \( W(x, t, k) \) is
\[ W(x, t, k) = e^{-\hat{A}p(s, t, k)\hat{\Phi}}(\hat{U}(x, t, k)dx + \hat{V}(x, t, k)dt). \]

Now we define the particular (Jost) solutions \( \hat{\Phi}_j(x, t, k) \) \( (j = 1, 2) \) of equations (16) (17) as the \( 2 \times 2 \) matrix-valued solutions of the associated Volterra integral equations:
\[ \hat{\Phi}_1(x, t, k) = 1 + \int_{-\infty}^{\infty} e^{ik[p(s, t, k) - p(y, t, k)]\hat{\Phi}}\hat{U}(y, t, k)\hat{\Phi}_1(y, t, k)dy, \]
\[ \hat{\Phi}_2(x, t, k) = 1 - \int_{\infty}^{\infty} e^{ik[p(s, t, k) - p(y, t, k)]\hat{\Phi}}\hat{U}(y, t, k)\hat{\Phi}_2(y, t, k)dy. \]

**Property 2.1.** The functions \( \hat{\Phi}_j(x, t, k) \) \( (j = 1, 2) \) have the following properties:

1. \( [\hat{\Phi}_1]_j(x, t, k) \) is bounded and analytic in \( D_2 \), \( [\hat{\Phi}_1]_1(x, t, k) \) is in \( D_1 \);
2. \( [\hat{\Phi}_2]_1(x, t, k) \) is bounded and analytic in \( D_1 \), \( [\hat{\Phi}_1]_2(x, t, k) \) is in \( D_2 \).

Here \( [\hat{\Phi}_j]_j(x, t, k) \) denotes the \( j \)-th column of a matrix \( \hat{\Phi}_j(x, t, k) \), \( D_1 \) and \( D_2 \) denote the upper-half and lower-half plane of the complex \( k \)-sphere, respectively.

**Proof.** Following the successive approximation method, we construct function term series for equations (21) (22) respectively:
\[ \hat{\Phi}_1^{(l)(s, t, k)} = I, \quad \hat{\Phi}_1^{(l)(s, t, k)} = \int_{-\infty}^{x} e^{ik[p(s, t, k) - p(y, t, k)]\hat{\Phi}}\hat{U}(y, t, k)\hat{\Phi}_1^{(l-1)(s, t, k)}dy, \]
\[ \hat{\Phi}_2^{(l)(s, t, k)} = I, \quad \hat{\Phi}_2^{(l)(s, t, k)} = -\int_{x}^{+\infty} e^{ik[p(s, t, k) - p(y, t, k)]\hat{\Phi}}\hat{U}(y, t, k)\hat{\Phi}_2^{(l-1)(s, t, k)}dy, \]

Then through the uniform convergence and differentiability of the sum of the function term series about \( [\hat{\Phi}_j]_j (i, j = 1, 2) \), we can prove property 2.1.

**Property 2.2.** \( \hat{\Phi}(x, t, k) \) satisfies the following asymptotic property:
\[ \hat{\Phi}(x, t, k) = 1 + \left[ D_1^{(-1)} \frac{\frac{n_m}{2m} D_2^{(-1)} \frac{1}{k} + \mathcal{O} \left( \frac{1}{k} \right)}{x \frac{n_m}{2m} D_2^{(-1)} \frac{1}{k}} \right], \quad k \rightarrow \infty. \]

**Proof.** According to property 2.1, we know that \( \hat{\Phi}(x, t, k) \) is about \( k \) parsing. So when \( k \rightarrow \infty \), \( \hat{\Phi}(x, t, k) \) can
be expanded to:
\[
\Phi(x, t, k) = D^{(0)}(x, t) + D^{(-1)}(x, t) \frac{1}{k} + O\left(\frac{1}{k^2}\right),
\]
(24)
where \( D^{(j)}(x, t) = \begin{pmatrix} D_{11}^{(j)} & D_{12}^{(j)} \\ D_{21}^{(j)} & D_{22}^{(j)} \end{pmatrix} \). Then by substituting equation (24) into the Lax pairs (16) (17) of \( \Phi(x, t, k) \) and comparing the coefficients of \( k \), we can prove property 2.2.

**Property 2.3.** The eigenfunctions \( \Phi_j(x, t, k) \) satisfy the following symmetry property:
\[
\Phi(x, t, k) = \Phi(x, t, -k) = \sigma_2 \Phi(x, t, k) \sigma_2,
\]
(25)
where \( \bar{A} \) represents the conjugate of \( A \).

Since \( \Phi_j(x, t, k) \) are the solutions to the same equation, from the properties of general solutions to differential equations, there must be a matrix unrelated with \( x \) and \( t \) between the two solutions, satisfying
\[
\Phi_j(x, t, k) = \Phi_j(x, t, k) e^{i k \phi(x, t, k)} S(k).
\]
(26)
According to the symmetry property (25), we can write the scattering matrix \( S(k) \) as follows:
\[
S(k) = \begin{pmatrix} a(k) & b(k) \\ -b(k) & a(k) \end{pmatrix},
\]
(27)
where \( a(k) = a(-k) \) and \( b(k) = b(-k) \).

From (26), we obtain
\[
a(k) = \det([\Phi_j \bar{1}], [\Phi_j \bar{1}]),
\]
(28)
where \( \det(A) \) means the determine of a matrix \( A \). We can deduce that \( a(k) \) is analytic in \( D_1 \) from property 2.1.

### 3. Riemann-Hilbert problem for the initial value problem of SCP equation

In this section, we will construct the Riemann–Hilbert problem for the initial value problem of SCP equation according to the relevant properties of \( \Phi(x, t, k) \). In the condition of the original parameter \( x \), the jump condition of the RH problem will not be determined solely by the initial value. This means that we cannot directly give solution expression for the parameters \( x \) and \( t \). For this reason, we will introduce new parameter \( y(x, t) \), and give the RH problem under the new parameter \( y \). At this time, the jump condition will not be related to \( u(x, t) \), only related to parameter \( y \) and spectral parameter \( k \).

Define \( M(x, t, k) \) as follows:
\[
M(x, t, k) = \begin{cases} [\Phi_2 \bar{1}] & k \in D_1, \\ [\Phi_1 \bar{1}] & k \in D_2. \end{cases}
\]
(29)
From (25), \( M(x, t, k) \) satisfies the following symmetry property:
\[
\bar{M}(x, t, k) = M(x, t, -k) = \sigma_2 M(x, t, k) \sigma_2.
\]
(30)

**Theorem 3.1.** \( M(x, t, k) \) satisfies the following conditions of the Riemann–Hilbert problem:

(i) Jump condition:
\[
M_+(x, t, k) = M_-(x, t, k) e^{i k \phi(x, t, k)} J_0(k),
\]
(31)
where
\[
J_0(k) = \left( \frac{1}{r(k)} \right) \left( \frac{r(k)}{1 + |r(k)|^2} \right), \quad r(k) = \frac{b(k)}{a(k)}, \quad k \in \mathbb{R};
\]
(32)
(ii) Normalization condition:
\[ M(x, t, k) \rightarrow I, k \rightarrow \infty. \]  

**Proof.** According to the definition of \( M(x, t, k) \) and combining with equation (26), we can easily get the jump condition. According to equations (23) (28), let \( k \rightarrow \infty \) and we can prove the normalization condition. 

Considering that \( e^{ik(x,t)}h_0(k) \) is not only related to \( u(x, 0) \), but also related to \( p(x, t, k) \), where \( p(x, t, k) \) is related to \( u(x, t) \). So we introduce a new parameter \( y \). Define \( y(x, t) \) as
\[ y(x, t) = x - \int_x^{+\infty} (m(\hat{x}, t) - 1) d\hat{x} = x - c_r(x, t), \]
where \( c_r(x, t) = \int_x^{+\infty} (m(\hat{x}, t) - 1) d\hat{x} \). From the definition of \( y \), we know that \( \frac{dy}{dx} = m > 0 \), so there exists the inverse function of \( x \) to \( y \), denoted as \( x(y, t) \).

Next, we define
\[ \tilde{M}(y, t, k) = M(x(y, t), t, k), \]
then \( \tilde{M}(y, t, k) \) satisfies the following property:
\[ \tilde{M}(y, t, k) = \tilde{M}(y, t, -k) = \sigma_2 \tilde{M}(y, t, k) \sigma_2. \]

**Theorem 3.2.** \( \tilde{M}(y, t, k) \) satisfies the following conditions of the Riemann-Hilbert problem:

(i) Jump condition:
\[ \tilde{M}_+(y, t, k) = \tilde{M}_-(y, t, k) \tilde{f}(y, t, k), k \in \mathbb{R}, \]
where
\[ \tilde{f}(y, t, k) = e^{i(ky - \frac{\pi}{2})} j_0(k); \]
and \( j_0(k) \) is shown in (32).

(ii) Normalization condition:
\[ \tilde{M}(y, t, k) \rightarrow I, k \rightarrow \infty. \]

**Proof.** Due to the existence of the inverse function \( x(y, t) \), then we substitute \( x(y, t) \) into theorem 3.1, and we can get theorem 3.2.

4. Expression of the solution for SCP equation with a given initial value

In this section, we will be based on the expansion of the RH problem at \( k = 0 \) to give the expression \( u(y, t) \) for parameter \( y \).

According to property 2.1, we find that if we solve the RH problem from \( k = \infty \), then solving will become very complicated. To review Lax pairs (6) (7), we observe that Lax pairs (6) (7) also have a singularity at \( k = 0 \), so we introduce the following transformation:
\[ \Phi(x, t, k) = \tilde{\Phi}^0(x, t, k) e^{i(kx + \frac{\pi}{2})^m}. \]

Under the transformation (40), Lax pairs (6) (7) can be rewritten as
\[ \tilde{\Phi}^0_x - ik[\sigma_3, \tilde{\Phi}^0] = \tilde{V}^0 \tilde{\Phi}^0, \]
\[ \tilde{\Phi}^0_t - \frac{1}{4ik} [\sigma_3, \tilde{\Phi}^0] = \tilde{V}^0 \tilde{\Phi}^0, \]
where
\[ \tilde{\Phi}^0 = ik \begin{pmatrix} -u_x^2 & 2u_x & 2u_x \\ 2u_x & u_x & 0 \\ 2u_x & u_x & 0 \end{pmatrix}, \]
\[ \tilde{V}^0 = iku^2 \begin{pmatrix} \sigma_3 + \begin{pmatrix} -u_x^2 & 2u_x \\ 2u_x & u_x \end{pmatrix} + \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix} \end{pmatrix} \]
Similar to the condition when $k \to \infty$, we give the *jost* function solutions for $k \to 0$,

\[
\Phi_i^0(x, t, k) = I + \int_{-\infty}^{x} e^{ik(x-y)} \mathcal{Q}(y, t) \Phi_i^0(y, t, k) dy,
\]

\[
\Phi_2^0(x, t, k) = I - \int_{x}^{\infty} e^{ik(x-y)} \mathcal{Q}(y, t) \Phi_2^0(y, t, k) dy.
\]

**Property 4.1.** The functions $\Phi_i^0$ ($j = 1, 2$) have the following properties:

(i) $[\Phi_i^0](x, t, k)$ is bounded and analytic in $D_2$, $[\Phi_i^0](x, t, k)$ is in $D_1$;

(ii) $[\Phi_2^0](x, t, k)$ is bounded and analytic in $D_1$, $[\Phi_2^0](x, t, k)$ is in $D_2$.

Since the proof method for property 4.1 is the same as the proof method for property 2.1, we omit the proof here.

**Property 4.2.** $\Phi_i^0(x, t, k)$ ($j = 1, 2$) has the following expansion for $k \to 0$,

\[
\Phi_i^0(x, t, k) = I + \left( \frac{ic_+}{2iu} \right) k + O(k^2),
\]

where $c_+(x, t) = \int_{x}^{\infty} (m(\tilde{x}, t) - 1) d\tilde{x}$.

**Proof.** According to property 4.1, we know that $\Phi_i^0(x, t, k)$ is about $k$ parsing. Thus, when $k \to 0$, $\Phi_i^0(x, t, k)$ can be expanded to

\[
\Phi_i^0(x, t, k) = D^{(0)}(x, t) + kD^{(1)}(x, t) + k^2D^{(2)}(x, t) + O(k^3),
\]

where

\[
D^{(i)}(x, t) = \begin{pmatrix} D^{(i)}_{11} & D^{(i)}_{12} \\ D^{(i)}_{21} & D^{(i)}_{22} \end{pmatrix}.
\]

Then bringing equation (47) to Lax pairs (41)(42) of $\Phi_i^0(x, t, k)$ and comparing with the coefficients of $k$, we will obtain the result.

Further, we notice that $\Phi_j(x, t, k)$ ($j = 1, 2$) and $\Phi_j^0(x, t, k)$ ($j = 1, 2$) are related to the same system of equation (1), from the properties of general solutions to differential equations, so we can get

\[
\Phi_j(x, t, k) = G^{-1} \Phi_j^0(x, t, k) e^{ikx + i\beta(j)\theta(k)} Q_j(k),
\]

where $Q_j(k)$ is unrelated with $x$ and $t$. As $x \to \pm \infty$, calculating (48), we know

\[
Q_1(k) = e^{-i\kappa_0 \xi}, \quad Q_2(k) = I,
\]

where $\xi = \int_{-\infty}^{\infty} (m(x, t) - 1) dx$. From the conservation law equation of the SCP equation, we know that $c$ is a conserved quantity.

**Property 4.3.** $a(k)$ has the following expansion:

\[
a(k) = 1 + ikc - \frac{c^2}{2} k^2 + O(k^3), \quad k \to 0,
\]

where $c = \int_{-\infty}^{\infty} (m(x, t) - 1) dx$.

**Proof.** According to the relation (48) between $\Phi_j$ ($j = 1, 2$) and $\Phi_j^0$ ($j = 1, 2$), we substitute (48) into (28) and let $x \to \infty$, $k \to 0$. Then combining with equation (46), we can proof it.

**Property 4.4.** $M(x, t, k)$ has the following asymptotic behavior in $k \to 0$:

\[
M(x, t, k) = G^{-1}(1 + \left( \frac{2ic_+}{2iu} \right) k + O(k^2))
\]

Here, $c_+(x, t) = \int_{\tilde{x}}^{\infty} (m(\tilde{x}, t) - 1) d\tilde{x}$. The proof method is similar to property 4.3.

**Theorem 4.1.** If the initial value satisfies condition (3) and assumptions (4)(5), then there is a unique solution to the initial value problem of the SCP equation, which can be expressed as

\[
u(x, t) = u(y(x, t), t),
\]

(52)
where
\begin{align*}
x(y, t) &= y + \lim_{k \to 0} \frac{((\bar{M}(y, t, 0))^{-1}\bar{M}(y, t, k))_{11} - 1}{2ik}, \\
u(y, t) &= \lim_{k \to 0} \frac{((\bar{M}(y, t, 0))^{-1}\bar{M}(y, t, k))_{22}}{2ik}.
\end{align*}

**Proof.** Because the jump matrix $\tilde{J}(y, t, k)$ is a Hermitian matrix, so the solution to $\bar{M}(y, t, k)$ exists. Further, according to the normalization condition, the solution for $\bar{M}(y, t, k)$ is unique. Then according to theorem 4.1 and the definition of $y(34)$, we can prove that the initial value problem of the SCP equation has a unique solution.

5. Soliton solutions for SCP equation

In this section, our main goal is to solve the initial value problem of the SCP equation and give the soliton solutions to the SCP equation. In order to solve the soliton solutions of the SCP equation, we introduce the concept of residue to solve $\bar{M}(y, t, k)$. Then according to theorem 4.1, solitary solutions and two-soliton solutions of the SCP equation are obtained.

According to property 2.1, we know that $a(k)$ is parsed in the upper half-plane, so we assume that $a(k)$ has $N$ simple zeros $\{k_j\}_{j=1}^N$ in the upper half of the complex. 

**Property 5.1.** $[M_1]$ has poles at the simple zeros $k_j$ of $a(k)$ (ke $D_1$), where $[M_1]$ has poles at the conjugates $\bar{k}_j$ (ke $D_2$), here $j = 1, 2, \ldots, N$, then the residues of $[M_1](x, t, k)$ and $[M_2](x, t, k)$ are as follows:

\begin{align*}
Res_{k=j}[M_1](x, t, k) &= C_j e^{2ik_jp(x,t,k)}[M_1](x, t, k), \\
Res_{k=j}[M_2](x, t, k) &= C_j e^{-2ik_jp(x,t,k)}[M_2](x, t, \bar{k}_j),
\end{align*}

where $C_j$ are complex constants.

**Proof.** According to assumption, we know that $a(k)$ has $N$ simple zeros $\{k_j\}_{j=1}^N$ in the upper half of the complex. From (28), we know that $[\bar{\Phi}_2](x, t, k)$ and $[\bar{\Phi}_1](x, t, k)$ are linearly related, so there exists a constant $b_j$ that satisfies

\begin{equation}
[\bar{\Phi}_1](x, t, k_j) = b_j[\bar{\Phi}_2](x, t, k_j), j = 1, 2, \ldots, N.
\end{equation}

This means

\begin{equation}
Res_{k=j}[M_1](x, t, k) = Res_{k=j}[\bar{\Phi}_1](x, t, k) = \frac{[\bar{\Phi}_2](x, t, k_j)}{\dot{a}(k_j)} = C_j e^{2ik_jp(x,t,k)}[\bar{\Phi}_2](x, t, k_j) = C_j e^{2ik_jp(x,t,k)}[M_1](x, t, k_j),
\end{equation}

where $\dot{a}(k) = \frac{da(k)}{dk}$, $C_j = \frac{b_j}{\dot{a}(k)}$. Then by the symmetry property, we can prove. 

**Property 5.2.** The residues of $[\bar{M}_1](y, t, k)$ and $[\bar{M}_2](y, t, k)$ are as follows:

\begin{align*}
Res_{k=j}[\bar{M}_1](y, t, k) &= C_j e^{2ik_jp(y,t,k)}[\bar{M}_1](y, t, k), \\
Res_{k=j}[\bar{M}_2](y, t, k) &= C_j e^{-2ik_jp(y,t,k)}[\bar{M}_2](y, t, \bar{k}_j).
\end{align*}

From the definition of $y(34)$, we know that the inverse function $x(y, t)$ exists, then substituting $x(y, t)$ into property 5.1, we can get property 5.2.

Combining symmetry property (36) with property 5.2, we convert the initial value problem of the SCP equation to the problem of solving linear algebraic equations,

\begin{equation}
[\bar{M}(y, t, k)]_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^N \frac{C_j e^{2ik_jp(y,t,k)}}{k - k_j}[\bar{M}(y, t, k_j)]_1.
\end{equation}
Taking into account the symmetry property (36), equation (59) can be rewritten as

\[
\left( \frac{-\hat{M}_{21}(y, t, k)}{\hat{M}_{11}(y, t, k)} \right) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sum_{j=1}^{N} C_{j} e^{\frac{i 2 k (y - \frac{1}{\sigma})}{k_j}} \left( \hat{M}_{11}(y, t, k_j) \right) \left( \hat{M}_{22}(y, t, k_j) \right)
\]

(60)

Substituting \( k = \overline{k} \) into equation (60), we have

\[
\left( \frac{-\hat{M}_{21}(y, t, k)}{\hat{M}_{11}(y, t, k)} \right) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sum_{j=1}^{N} C_{j} e^{\frac{i 2 k (y - \frac{1}{\sigma})}{k_j}} \left( \hat{M}_{11}(y, t, k_j) \right) \left( \hat{M}_{22}(y, t, k_j) \right)
\]

(61)

So the initial value problem of SCP equation is translated into solving \( \hat{M}_{11}(y, t, k) \) and \( \hat{M}_{22}(y, t, k) \) \((j = 1, 2, \ldots, N)\). And then bringing the results into equation (59), we can get \( [\hat{M}(y, t, k)]_{j} \). According to symmetry property (36), we can get \( [\hat{M}(y, t, k)]_{j} \). In this way, we solve the \( \hat{M}(y, t, k) \) for the RH problem of the initial value for the SCP equation. Finally, by theorem 4.1, we solve the solution \( u(y, t, x, t) \).

Next, we will solve one-soliton solutions and two-soliton solutions of the SCP equation. From section 2, we know that \( a(\overline{k}) = a(-k) \) and \( a(k) \) is parsed in the upper half complex plane. From this we know if \( a(k) \) has only a simple zero, then the zero must be a pure imaginary zero. If \( a(k) \) has only two simple zeros the upper half complex plane, then these two zeros are either zeros of negative conjugate relations or two pure imaginary zeros.

5.1. One-soliton solutions

When \( N = 1 \), \( a(k) \) has only a pure virtual null point \( k = ib \) \((b > 0)\). Then we can simplify equation (61) into the following two equations:

\[
-\hat{M}_{21}(y, t, k) = \frac{C_{1} e^{i 2 k (y - \frac{1}{\sigma})}}{k - k_1} \hat{M}_{11}(y, t, k_1),
\]

\[
\hat{M}_{11}(y, t, k_1) = 1 + \frac{C_{1} e^{i 2 k (y - \frac{1}{\sigma})}}{k - k_1} \hat{M}_{22}(y, t, k_1).
\]

(62)

If we suppose \( C_{1} = 2 i b e^{-2\psi} \), where \( c \) and \( \theta_{b} \) are real constants, after solving \( \hat{M}_{11}(y, t, k_1) \) and \( \hat{M}_{22}(y, t, k_1) \), we can obtain one-soliton solutions as follows:

\[
u (x, t) = u(y(x, t), t) = \frac{1}{2 b \cosh (2 \psi + 2 \theta_{b})},
\]

(63)

and the relationship between \( x \) and \( y \) is

\[
x = y - \frac{1}{2 b} (\tanh (2 \psi + 2 \theta_{b}) - 1).
\]

(64)

So we get

\[
\frac{\partial x}{\partial y} = 1 - \frac{1}{\cosh^2 (2 \psi + 2 \theta_{b})}.
\]

(65)

From which we can know, as \( y \to \infty, x \to y \).

5.2. Two-soliton solutions

Case 1. \( a(k) \) has a pair of complex conjugate zeros: \( a(k_1) = a(-k_1) = 0 \). Here we let \( k_1 = a + ib \) \((a > 0, b > 0)\). According to symmetry property (36) and property 5.2, we can simplify algebraic system (61) to the following two equations:

\[
-\hat{M}_{21}(y, t, k) = \frac{C_{1} e^{i 2 k (y - \frac{1}{\sigma})}}{k - k_1} \hat{M}_{11}(y, t, k_1) - \frac{C_{1} e^{i 2 k (y - \frac{1}{\sigma})}}{2 k_1} \hat{M}_{11}(y, t, k_1),
\]

\[
\hat{M}_{11}(y, t, k_1) = 1 + \frac{C_{1} e^{i 2 k (y - \frac{1}{\sigma})}}{k - k_1} \hat{M}_{22}(y, t, k_1) - \frac{C_{1} e^{i 2 k (y - \frac{1}{\sigma})}}{2 k_1} \hat{M}_{11}(y, t, k_1).
\]

(66)

Now, assume

\[
m_{11} = C_{1} e^{i 2 k (y - \frac{1}{\sigma})} \hat{M}_{11}(y, t, k_1), \quad m_{21} = C_{1} e^{i 2 k (y - \frac{1}{\sigma})} \hat{M}_{22}(y, t, k_1),
\]

\[
\varphi_1 = ay - \frac{at}{4(a^2 + b^2)} \quad \varphi_2 = by + \frac{bt}{4(a^2 + b^2)}.
\]

(66)
If we suppose \( C_1 = 2i\sqrt{\alpha^2 + \beta^{2b}e^{\phi}}, \theta_1 = \arg(C_1) - \frac{\pi}{2} \), by calculating equations (66), we obtain

\[
m_{11} = \frac{-[a^2b\sin(\phi_1)\cosh(\phi_2) + b^2\sin(\phi_2)e^{-\phi_1} + ab^2\cos(\phi_1)\cosh(\phi_2)]}{b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2)} + \frac{i[a^2b\cos(\phi_1)\cosh(\phi_2) - ab^2\sin(\phi_1)\sinh(\phi_2)]}{b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2)},
\]

(67)

\[
m_{21} = \frac{[b^3\sin(\phi_1)\cosh(\phi_2) + ab^2\left(\frac{1-e^{-2\phi_2}}{2}\right) - ab^2\cos^2(\phi_2)]}{b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2)} + \frac{i[a^2b\left(\frac{1+e^{-2\phi_2}}{2}\right) - ab^2\sin(\phi_1)\cosh(\phi_2) + b^3\sin^2(\phi_2)]}{b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2)},
\]

(68)

where \( \phi_1 = 2\varphi_1 + \theta_1 - \theta_2, \phi_2 = 2\varphi_2, \theta_2 = \arg(k_2) \).

Next, we consider the matrix \( \tilde{M}(y, t, k) \). According to the symmetry property (36), we obtain \( \tilde{M}(y, t, k) \) as follows:

\[
\tilde{M}(y, t, k) = \begin{pmatrix}
1 & \frac{-m_{11}}{k + k_1} + \frac{\omega}{k - k_1}
& m_{11} - \frac{\omega}{k + k_1} \\
\frac{-m_{11}}{k + k_1} - \frac{\omega}{k - k_1} & 1 & \frac{m_{11}}{k - k_1} - \frac{\omega}{k + k_1}
\end{pmatrix},
\]

(69)

According to theorem 4.1, we get

\[
u = -\text{Im}\left(\frac{m_{11}}{k^2}\right) + 2\text{Re}\left(\frac{m_{11}}{k^2}\right)\text{Im}\left(\frac{m_{11}}{k^2}\right) - 2\text{Re}\left(\frac{m_{11}}{k^2}\right)\text{Im}\left(\frac{m_{11}}{k^2}\right),
\]

(70)

\[
x = y + \text{Im}\left(\frac{m_{21}}{k^2}\right) - 2\text{Re}\left(\frac{m_{21}}{k^2}\right)\text{Im}\left(\frac{m_{21}}{k^2}\right) - 2\text{Re}\left(\frac{m_{21}}{k^2}\right)\text{Im}\left(\frac{m_{21}}{k^2}\right).
\]

(71)

Finally, we obtain the two-soliton solutions of the initial value problem

\[
u(x(y, t)) = -\frac{ab}{a^2 + b^2} \frac{b\sin(\phi_1)\sinh(\phi_2) + a\cos(\phi_1)\cosh(\phi_2)}{b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2)},
\]

\[
x(y, t) = y + \frac{ab}{2(a^2 + b^2)} \frac{b\sin(2\phi_1) - b\sin(2\phi_2) + 2}{b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2)}.
\]

(72)

(73)

Solving the partial derivative of \( x \) to \( y \), we get

\[
\frac{\partial x}{\partial y} = 1 - \frac{4a^2b^2\sin^2(\phi_1)\cosh^2(\phi_2)}{(b^2\sin^2(\phi_1) + a^2\cosh^2(\phi_2))^2} = \cos^2\left(2\arctan\frac{b\sin(\phi_1)}{a\cosh(\phi_2)}\right).
\]

(74)

From the above discussions, we know that the partial derivative of \( x \) to \( y \) is always greater than 0 or equal to 0, which is different from the two-soliton solutions of the SP equation given in [13]. Equation (74) means that the two-soliton solution we have obtained will not blow up about space variable \( x \). In particular, when \( a > 1 \), the derivative of \( u \) to \( t \) tends to \( \infty \) and the cuspon solution appears. For all other cases, it is a smooth breather soliton.

**Case 2.** When \( a(k) \) has a pair of pure imaginary zeros \( k_1 \) and \( k_2 \). Then letting \( k_1 = ia, k_2 = ib \) \( (a > 0, b > 0) \), \( C_1 = iae^{-2b} \) and \( C_2 = ibe^{-2b} \), where \( \theta_1 \) and \( \theta_2 \) are constants. Finally, we can simplify the algebraic system (61) into the following form:

\[
\begin{align*}
\bar{M}_{11}(y, t, k) &= \frac{C_1e^{-\phi}}{2ia} \bar{M}_{11}(y, t, k) + \frac{C_2e^{-\psi}}{i(a + b)} \bar{M}_{11}(y, t, k), \\
\bar{M}_{11}(y, t, k) &= \frac{C_1e^{-\phi}}{i(a + b)} \bar{M}_{11}(y, t, k) + \frac{C_2e^{-\psi}}{2ib} \bar{M}_{11}(y, t, k), \\
\bar{M}_{11}(y, t, k) &= 1 + \frac{C_1e^{-\phi}}{-2ia} \bar{M}_{11}(y, t, k) + \frac{C_2e^{-\psi}}{-i(a + b)} \bar{M}_{11}(y, t, k), \\
\bar{M}_{11}(y, t, k) &= 1 + \frac{C_1e^{-\phi}}{-i(a + b)} \bar{M}_{11}(y, t, k) + \frac{C_2e^{-\psi}}{-2ib} \bar{M}_{11}(y, t, k).
\end{align*}
\]

(75)

where \( \phi = 2a(y + \frac{t}{4a}) + 2\theta_1, \psi = 2b(y + \frac{t}{4b}) + 2\theta_2 \).
Assume
\[ m_{11} = C_1 e^{-\phi} \tilde{M}_1(y, t, k), \quad m_{12} = C_2 e^{-\phi} \tilde{M}_1(y, t, k), \]
\[ m_{21} = C_1 e^{-\phi} \tilde{M}_2(y, t, k), \quad m_{22} = C_2 e^{-\phi} \tilde{M}_2(y, t, k). \]

Solving the algebraic equations (75) above, we obtain
\begin{align*}
    m_{11} &= i \left[ ae^{-\phi} \left( 1 + \frac{b(a-b)}{2a+b} e^{-\phi} + \frac{a-b}{4a+b} e^{-2\phi} \right) \right], \\
    m_{21} &= i \left[ \frac{a e^{-2\phi}}{2} + \frac{b}{a+b} e^{-\phi} - \frac{(a-b)^2}{8a+b^2} e^{-2\phi} - \left( \frac{a-b}{4a+b} e^{-\phi} - 1 \right)^2 \right], \\
    m_{12} &= i \left[ be^{-\phi} \left( 1 + \frac{a(b-a)}{2a+b} e^{-\phi} + \frac{b-a}{4a+b} e^{-2\phi} \right) \right], \\
    m_{22} &= i \left[ \frac{b e^{-2\phi}}{2} + \frac{a}{a+b} e^{-\phi} + \frac{(a-b)^2}{8a+b^2} e^{-2\phi} - \left( \frac{a-b}{4a+b} e^{-\phi} - 1 \right)^2 \right].
\end{align*}
(76)

From theorem 4.1, we get the soliton solutions as follows:
\[ u = \frac{ae^{-\phi} + be^{-\phi} + (a-b)^2}{ab} \left( \frac{e^{-\phi}}{2} + \frac{e^{-2\phi}}{2} \right) + \frac{(a-b)^2}{4a+b} e^{-\phi} - 1 \left( \frac{e^{-\phi}}{2} + \frac{e^{-2\phi}}{2} \right). \]
(77)

The relation between \( x \) and \( y \) is as follows:
\[ x = y + \frac{ae^{-2\phi} + be^{-2\phi} + ab + a^2}{4} \left[ \frac{e^{-\phi}}{2} + \frac{e^{-2\phi}}{2} \right] + \frac{(a-b)^2}{16a+b^2} e^{-2\phi} - \left( \frac{e^{-\phi}}{2} + \frac{e^{-2\phi}}{2} \right). \]
(78)

Solving the derivative of \( x \) to \( y \), we get
\[ \frac{\partial x}{\partial y} = \left[ \frac{e^{-\phi}}{2} + \frac{e^{-2\phi}}{2} - \left( \frac{e^{-\phi}}{2} + \frac{e^{-2\phi}}{2} \right) \right]^2. \]
(79)

We have again come to the conclusion that the partial derivative of \( x \) to \( y \) is always greater than or equal to 0.

\textbf{Remark 5.2.1.} As \( a \to b \), equations (77)–(79) respectively become
\[ u = \frac{1}{a} \cosh(\phi), \quad x = y - \frac{1}{a} (\tanh(\phi) - 1), \quad \frac{\partial x}{\partial y} = 1 - \frac{1}{\cosh^2(\phi)}. \]

This is consistent with equations (63)–(65) obtained in solving one-soliton solutions in this section.

\section*{Acknowledgments}
This work is supported by the National Natural Science Foundation of China (No. 11471215).

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