Local dynamics and gravitational collapse of a self-gravitating magnetized Fermi gas.

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We use the Bianchi-I spacetime to study the local dynamics of a magnetized self-gravitating Fermi gas. The set of Einstein-Maxwell field equations for this gas becomes a dynamical system in a 4-dimensional phase space. We consider a qualitative study and examine numeric solutions for the degenerate zero temperature case. All dynamic quantities exhibit similar qualitative behavior in the 3-dimensional sections of the phase space, with all trajectories reaching a stable attractor whenever the initial expansion scalar \(H_0\) is negative. If \(H_0\) is positive the trajectories end up in a curvature singularity that can be, depending on initial conditions, isotropic or anisotropic. In particular, if the initial magnetic field intensity is sufficiently large the collapsing singularity will always be anisotropic and pointing in the same direction of the field.

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I. INTRODUCTION

White dwarfs and neutron stars, as very dense objects found in great abundance, are astrophysical laboratories to test physical theories under intense gravity. In particular, we can consider as a theoretical model the possibility of Magnetic White Dwarfs that could be endowed with extremely large magnetic fields, which, in principle, can be stronger than those measured in earthbound laboratories. It is very interesting to study the behavior of the magnetic field lines, in its interplay with strong gravity, as such a magnetic star undergoes gravitational collapse. Various qualitative arguments show that under gravitational collapse the field lines squeeze together making the field stronger, even resisting collapse.

Several models describing magnetic white dwarfs have been constructed (see \cite{1, 2, 3}) using Landau’s well known argument setting appropriate mass limits for these objects. Ideally, a consistent model of a magnetic white dwarf would require considering numeric hydrodynamical modeling of Einstein–Maxwell equations with (at the very least) axially symmetric configurations that would have to comply with the appropriate boundary conditions and equations of state of a degenerate magnetized Fermi gas. However, we can still obtain important information on the local dynamics of the self–gravitating magnetized Fermi gas by considering the evolution of such a source in a much more simplified spacetime geometry.

The type of equation of state that we consider for a magnetized white dwarf is similar to that discussed in previous papers dealing with a magnetized electron gas \cite{4, 5, 6} (1968) and later developed for more general sources \cite{7, 8, 9, 10, 11} (2000). These articles discuss equations of state for strongly magnetized gases endowed with anisotropic pressure. Previous work in a Newtonian framework noted the possibility in which the pressure \(p_{\parallel}\) parallel to the magnetic field overtakes the pressure \(p_{\perp}\) perpendicular to the field, suggesting a magnetic collapse in this direction. This brings us to think that a general relativistic treatment of magnetized configurations will yield a singularity associated to the collapse having an extended, anisotropic, form along the direction of the magnetic field. Such type of singularities are denoted as “line” or “cigar” singularities, as opposed to isotropic “point–like” singularities.

In order to investigate the local dynamics and collapse of a magnetized Fermi gas under General Relativity, we consider one of the simplest geometric configurations compatible with the anisotropic pressure that characterizes these sources: the Bianchi I model described in terms of a Kasner metric. The present paper generalizes previous work along these lines \cite{12}.

The main justification of dealing with the simplified Bianchi I geometry is that the latter could provide a rough description of the local dynamics of a fluid element of the magnetized gas in the realistic configuration. If we consider local fluid elements far from the boundary of the configuration, so that this volume exchanges particles and energy with the rest of the system (seen as a reservoir), then we can consider it like a local volume of a great canonical distribution associated with the whole gas. While it is evident that a lot of valuable information is lost by making such simplifications, we can still get a rough description of the effects of strong gravity on local physics.

Once we consider the Bianchi I spacetime with the Kasner metric as the metric field associated with the magnetized gas, the Einstein–Maxwell system of field equations reduce to an autonomous system of four ordinary differential equations, leading to a 4–dimensional phase space that can be studied from a qualitative and numerical point of view using standard dynamical systems techniques. The physical and geometric dimensionless variables of this phase space are the single independent...
component of the shear tensor $S$, the expansion scalar $\mathcal{H}$, the normalized dimensionless magnetic field and the chemical potential, $\beta$ and $\mu$. All other quantities can be expressed in terms of these four basic quantities. The numerical examination of the system sheds light on the type of collapse singularities and their relation to specific initial conditions.

The equation of state that we are using is strictly valid for densities of the order of $10^7 g cm^{-3}$ and up to $10^{15} g cm^{-3}$ expected in compact objects from white dwarves to neutron stars. A gas of strongly magnetized and highly degenerate fermions in a neutron star is in a state that closely resembles superfluidity with near infinite conductivity (see page 291 [13]). In these conditions the role of viscosity is minor, though one can still consider the possibility of dissipative or transport phenomena, such as dissipation of rotational energy in electromagnetic and gravitational waves (see [13, 14, 15]).

However, even if viscosity is not significant (at least for neutron stars), the main reason why we are neglecting it (and other dissipative effects) is to keep a mathematically tractable problem. We feel that treating the case of thermal equilibrium is sufficient for a first approach, leaving the study of dissipative transport phenomena for a future work.

The paper is organized as follows. In section II we present and discuss the set of Einstein–Maxwell equations, the source of anisotropy and the most appropriate form of the equations of state for the magnetized Fermi gas. The qualitative dynamical analysis is carried on in Section-III, defining a set of dimensionless normalized variables, leading to a self–consistent autonomous system of four ordinary differential equations. The numerical and qualitative analysis of this system and the classification of the types of collapse singularities are dealt with in Section-IV, while the conclusion is given in section V. The main result from the numerical analysis is that once we allow for a relativistic strong gravity treatment it is always possible, for sufficiently large magnetic field initial intensity, to obtain the anisotropic “cigar” type of collapse singularity as hypothesized in previous work [9] carried on along an intuitive Newtonian framework work.

II. KASNER METRIC WITH ANISOTROPIC PRESSURE

The Kasner metric is among the simplest metrics compatible with the anisotropic pressure associated with a magnetized source. This metric is given by:

$$ds^2 = -c^2 dt^2 + A^2(t) dx^2 + B^2(t) dy^2 + C^2(t) dz^2.$$  

It is associated with a “non–tilted” Bianchi-I space time [10]. For a comoving 4-velocity $u^a = \delta_i^a$, where $u^a u_a = -1$, the 4–acceleration vanishes and the expansion scalar $\Theta$ and the shear tensor $\sigma^{\alpha\beta}$ take the forms:

$$\Theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C},$$  

$$\sigma^{\alpha\beta} = \text{diag} [\sigma_x, \sigma_y, \sigma_z, 0].$$  

where

$$\sigma_x = \frac{2\dot{A}}{3A} - \frac{\dot{B}}{3B} - \frac{\dot{C}}{3C},$$  

$$\sigma_y = \frac{2\dot{B}}{3B} - \frac{\dot{A}}{3A} - \frac{\dot{C}}{3C},$$  

$$\sigma_z = \frac{2\dot{C}}{3C} - \frac{\dot{A}}{3A} - \frac{\dot{B}}{3B}.$$  

We consider as the source for this metric the following stress–energy tensor:

$$T^a_b = (U + P)u^a u_b + P \delta^a_b + \Pi^a_b, \quad P = p - \frac{2B M}{3}.$$  

where $\Pi^a_b$ is the anisotropic pressures tensor, $U$ the energy density, $P$ the pressure, $B$ the magnetic field and $M$ the magnetization, all them are functions of the time. Notice that the anisotropy is produced by the magnetic field $B$. If this field vanishes, ie: $B = 0$, the stress-tensor reduces to that of a perfect fluid tensor with isotropic pressure $P = p$. In the general case $B \neq 0$ the tensor $\Pi^a_b$ has the form

$$\Pi^a_b = \text{diag}[\Pi, \Pi, -2\Pi, 0], \quad \Pi = -\frac{B M}{3}, \quad \Pi_a = 0.$$  

The Einstein field equations (EFE) associated with the Kasner metric and the stress–energy tensor are

$$-G^{\alpha\beta} = \frac{\dot{B}}{B} + \frac{\dot{C}}{C} = -\kappa (p - B M),$$  

$$-G^{\beta\gamma} = \frac{\dot{A}}{A} + \frac{\dot{C}}{C} = -\kappa (p - B M),$$  

$$-G^{\gamma\delta} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} = -\kappa p,$$  

$$-G^{\alpha\delta} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} = \kappa U.$$  

where a dot denotes the derivative with respect to the proper time of fundamental observers and $\kappa = 8\pi G/c^3$. From the balance equation $T^a_b;_b = 0$ and the Maxwell equations $F^{ab};_b = 0$ and $F^a_{[abc]} = 0$, we further have

$$\dot{U} + (p + U)\Theta - B M \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0,$$  

$$\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{2B} = 0.$$  

Since we need to construct a self–consistent system of first order differential equations that can be solved numerically, it is convenient to eliminate first and second derivatives of the metric functions in the Einstein–Maxwell equations in terms of the expansion scalar and the components of the shear tensor. Proceeding along
these lines, we combine equations (2), (3), (6) and (7) to eliminate the functions $A, B, C$ and their derivatives $\dot{A}, \dot{B}, \dot{C}, \dot{C}$. After some algebraic manipulation we arrive to the following constraint

$$-(\Sigma^y)^2 = (\Sigma^z)^2 - \Sigma^y \Sigma^z + \frac{\Theta^2}{3} = \kappa U,$$  \hspace{1cm} (8)

plus the following set of 5 differential equations:

$$\dot{U} + (U + p) \Theta - B M \left(\frac{2}{3} \Theta - \Sigma^z\right) = 0,$$  \hspace{1cm} (9a)

$$\Sigma^y = -\frac{\kappa}{3} B M - \Sigma^y \Theta,$$  \hspace{1cm} (9b)

$$\Sigma^z = \frac{2\kappa}{3} B M - \Sigma^z \Theta,$$  \hspace{1cm} (9c)

$$\dot{\Theta} = 2 \kappa (BM - 3/2 \rho) - \frac{\Theta^2}{2} - \frac{3}{2} \left((\Sigma^y + \Sigma^z)^2 - \Sigma^x \Sigma^z\right),$$  \hspace{1cm} (9d)

$$\dot{\beta} = \frac{2}{3} \beta (3 \Sigma^z - 2 \Theta), \quad \text{with} \quad \beta \equiv B / B_c.$$  \hspace{1cm} (9e)

where $\Sigma^z = \sigma^z$ is the independent component of the shear tensor. While the shear tensor can be fully determined by this single quantity, it will be convenient for our numerical calculations further ahead to use two components of this tensor.

The functions $U, p$ and $M$ are now given by the equation of state for the gas:

$$p = \lambda \Gamma_p (\beta, \mu), \quad B M = \lambda \beta \Gamma_M (\beta, \mu), \quad U = \lambda \Gamma_U (\beta, \mu).$$  \hspace{1cm} (10)

The $\Gamma$ functions depend on the parameter $\beta$, which is the magnetic field normalized by $B_c$, and on the chemical potential, $\mu$, normalized with the rest energy, both dimensionless quantities. The constants $B_c$ and $\lambda$ are:

$$B_c = \frac{m_e^2 c^3}{\varepsilon h}, \quad \lambda = \frac{m_e c^2}{4\pi^2 \lambda_c^2}.$$  \hspace{1cm} (11)

while $\lambda_c = \hbar / m c$ is the Compton wavelength. If we consider an electrons gas, then $m = m_e$ and $\lambda = 3.86 \times 10^{-11}$ cm. Nevertheless, $B_c = 4.414 \times 10^{13}$ G (for electrons too) which is a well known critical value of a magnetic field. In all neutron or white dwarf stars older than few second after formation, one can neglect the thermal contributions to the pressure and energy density; thus we can set $p(\beta, \mu, T) = p(\beta, \mu)$, the same for $U$ and $M$. Typical white dwarf temperatures satisfy $kT << E_{\text{Fermi}}$, where $E_{\text{Fermi}}$ is the Fermi kinetic energy. So, the thermal disorder time $t$ is not responsible for the pressure, the energy density nor the magnetization. It is now convenient to introduce the form of the $\Gamma (\beta, \mu)$ functions for the degenerate case ($T = 0$), these are [2]:

\begin{align}
\Gamma_p &= \frac{a_0}{3} \left(\mu^2 - \frac{5}{2}\right) + \frac{1}{2} \arcsinh \left(\frac{a_0}{\mu}\right) + \beta \sum_{n=0}^{s} \alpha_n (a_n - b_n - c_n), \quad (12a) \\
\Gamma_M &= \sum_{n=0}^{s} \alpha_n (a_n - b_n - 2c_n), \quad (12b) \\
\Gamma_U &= a_0 \left(\mu^2 - \frac{1}{2}\right) - \frac{1}{2} \arcsinh \left(\frac{a_0}{\mu}\right) + \beta \sum_{n=0}^{s} \alpha_n (a_n + b_n + c_n), \quad \text{where} \quad \alpha_n = 2 - \delta_{0n}, \quad n = 0, 1, (12c) \\
\text{while} \quad a_n &= \mu \sqrt{\mu^2 - 1 - 2n^2}, \quad b_n = \ln \left[\frac{(\mu + a_n \mu)}{\sqrt{1 + 2n^2}}\right], \quad c_n = 2n \beta b_n, \quad s = I \left[\frac{\mu^2 - 1}{2\beta}\right]. \quad (12d)
\end{align}

where $I[X]$ denotes the integer part of its argument $X$.

### III. DIMENSIONLESS VARIABLES

Consider now the following variables:

$$H = \frac{\Theta}{3}, \quad \frac{d}{d\tau} = \frac{1}{H_0} \frac{d}{dt}.$$  \hspace{1cm} (13)

and the dimensionless functions:

$$S^y = \frac{\Sigma^y}{H_0}, \quad S^z = \frac{\Sigma^z}{H_0}, \quad \Omega = \frac{\kappa \lambda \beta}{3 H_0^2}, \quad \mathcal{H} = \frac{H}{H_0}, \quad (14)$$

\begin{align}
(S^y)^2 - (S^z)^2 - S^y S^z + 3 \mathcal{H}^2 = 3 \Gamma_U, \quad (15)
\end{align}

where $S^y$ and $S^z$ are related to the $yy$ and $zz$ components of the shear tensor, while $\Omega$ is related to the magnetic field. The new time $\tau$ is a dimensionless time (or “logarithmic” time). The quantity $H(t)$ (because of (2)) will have dimensions of cm$^{-1}$ and the sign of $\tau$ becomes determined from the sign of $H(t)$. However, we have chosen $\Omega = \beta$ and $\kappa \lambda = 3 H_0^2$ in order to avoid the presence of annoying constants in the system of equations. Inserting the equations of state (10) and the new definitions (13), (14) into (8) and (9) we obtain the constraint:
plus the system
\begin{align}
S^y_{\tau} &= -\beta \Gamma_{\mathcal{M}} - 3S^y\mathcal{H}, \\
S^z_{\tau} &= 2\beta \Gamma_{\mathcal{M}} - 3S^z\mathcal{H},
\end{align}
(16a)
\begin{align}
\mathcal{H}_{\tau} &= \beta \Gamma_{\mathcal{M}} - \frac{3}{2}(\Gamma_p + \mathcal{H}^2 + \frac{(S^y + S^z)^2}{3} - \frac{S^yS^z}{3}), \\
\beta_{\tau} &= 2\beta(S^z - 2\mathcal{H}), \\
\mu_{\tau} &= \frac{1}{\Gamma_{U,\beta}}[(2\mathcal{H} - S^z)(\Gamma_{\mathcal{M}} + 2\Gamma_{U,\beta})\beta - 3\mathcal{H}(\Gamma_p + \Gamma_U)].
\end{align}
(16c)

Notice that, as opposed to cosmological sources and models \cite{17} where \(H_0 = 0.59 \times 10^{-28}\)cm\(^{-1}\) would play the role of the Hubble scale constant, in our magnetized Fermi gas we have \(H_0 = 0.86 \times 10^{-12}\)cm\(^{-1}\), which is a much smaller length scale. This is logical and consistent because it indicates that our simplified model is examined on local scales smaller than cosmic scales. The scale \(1/H_0 \simeq 1.15 \times 10^{12}\)cm is the order of the distance of an astronomical unit.

The results mentioned in references \cite{7, 8, 3} for the electron gas show that for an intense magnetic field, of the order of the critical field \(B_c\), all the electrons are in the ground state of the Landau level \(n = 0\), and consequently we have \(p_\perp = 0\). It is an interesting issue to study how the electron gas evolves in this case, in which the functions \(\Gamma(\beta, \mu)\) are simplified considerably, taking the following forms:
\begin{align}
\Gamma_p &= \frac{a_0}{3}(\mu^2 - \frac{5}{2}) + \frac{1}{2}\text{arcsinh}(\frac{a_0}{\mu}) + \beta(a_0 - b_0), \\
\Gamma_{\mathcal{M}} &= (a_0 - b_0), \\
\Gamma_U &= a_0(\mu^2 - \frac{1}{2}) - \frac{1}{2}\text{arcsinh}(\frac{a_0}{\mu}) + \beta(a_0 + b_0), \\
a_0 &= \mu\sqrt{\mu^2 - 1}, \quad b_0 = \ln(\mu + a_0/\mu), \quad c_0 = 0, \quad c_0 = 1.
\end{align}
(17a)
(17b)
(17c)
(17d)

Thus, substituting (17) into (16) yields a self-consistent system of five ordinary differential equations (18), with the unknown functions \(\beta, \mathcal{H}, S^y, S^z\) and \(\mu\), and the constraint (19), which only admits a numerical solution.

From the equations of state (10) it follows that the chemical potential must satisfy \(\mu \geq 1\), which is typically correct for systems with densities of the order \(\sim 10^7\)gm/cm\(^3\) or larger. For white dwarfs or neutron stars the chemical potential takes values around \(\sqrt{3} \simeq 1.732\). Since \(U > 0\), then the chemical potential \(\mu \geq 1\) and from (16) we obtain the constraint equation:
\[-(S^y)^2 - (S^z)^2 - S^yS^z + 3\mathcal{H}^2 = 3\Gamma_U \geq 0.\]
(18)
so that our physical 5-dimensional phase space is restricted by the relations:
\begin{align}
3\mathcal{H}^2 &\geq (S^y)^2 + (S^z)^2 + S^yS^z, \\
\mu^2 &\geq 1 + 2\beta, \\
\beta &\geq \frac{3\Gamma_p - \Gamma_U}{2\Gamma_{\mathcal{M}}}. 
\end{align}
(19a)
(19b)
(19c)

For non–tilted Bianchi-I models there is only one independent component of the shear tensor, which we are taking to be \(S^z\), however it is useful to use also the \(S^y\) component for several numerical calculations. Also, the form of the components of the shear tensor determines the form of the metric coefficients. From (2)–(3) we get:
\[\frac{A_{\tau}}{A} = (S^x + \mathcal{H}), \quad \frac{B_{\tau}}{B} = (S^y + \mathcal{H}), \quad \frac{C_{\tau}}{C} = (S^z + \mathcal{H}).\]
(20)

\subsection*{IV. NUMERICAL RESULTS AND DISCUSSIONS}

Using (2) and (13) we can express the local average volume \(V \equiv ABC\) in terms of \(\mathcal{H}\) and the dimensionless time \(\tau\) as
\[V(\tau) = V(0) \exp \left(3 \int_{\tau_0}^{\tau} \mathcal{H} d\tau \right).\]
(21)

Clearly illustrating why the time \(\tau\) is known as a “logarithmic” time. Notice that the sign of \(\mathcal{H}(\tau)\) denotes expanding (\(> 0\)) or collapsing (\(< 0\)) local volumes.
A. Singularities

Since the equations of state that we are considering are associated with compact objects of very high density (at least \( \sim 10^7 \text{g/cm}^3 \)), the evolution range of the models for diluting lower densities is not physically interesting and will not be pursued any further. This means that we will only consider the collapsing phase of the models, so that we will only need to examine initial conditions in which the initial expansion \( \mathcal{H}_0 \) is negative. We tested numerically the models using a wide range of different initial conditions covering the full range of physically interesting values for compact objects, from white dwarf to neutron stars, for example: \( \mu_0 = 2 \), corresponding to densities of \( \sim 10^7 \text{g/cm}^3 \), while \( \beta_0 = 10^{-5} \) represents magnetic fields of \( 10^8 \) G. Together with \( \mathcal{H}_0 < 0 \), we considered in particular: \( S_y^0 = 0, \pm 1 \) and \( S_z^0 = 0, \pm 1 \), corresponding to cases of zero initial deformation and initial deformation on the \( y \) or \( z \) directions.

As long as \( \mathcal{H}_0 < 0 \) the models exhibit a general collapsing behavior \( \mathcal{H} \to -\infty \), independent of the initial values of other functions (see an example in figure(1)). In all collapsing configurations the magnetic field intensity diverges to infinity regardless of its initial value. In figure(1) we show several numeric curve solutions for different values of other functions (see an example in figure(1)).

The shear. For example, if we have a large initial shear (deformation) in the \( x \) direction, say: \( S_x^0 \gg S_y^0, S_z^0, \beta_0, \mu_0 \), then a type cigar singularity emerges along the \( x \) direction or parallel or perpendicular to the magnetic field. An examination of all these cases reveals that the collapse state strongly depends on the magnitude of the initial shear. For example, if we have a large initial shear (deformation) in the \( x \) direction, say: \( S_x^0 \gg S_y^0, S_z^0, \beta_0, \mu_0 \), then a type cigar singularity emerges along the \( x \) direction. This is shown in figure(2), illustrating (by means of the set of equations (22)) that the metric function \( A \) tends to infinity, while the other metric functions, \( B \) and \( C \) rapidly fall to zero. In general, the initial configuration of the system and the initial values of the shear tensor determine the privileged direction of the anisotropic collapse.

The numerical trials also show that there is always a threshold value for the initial magnetic field intensity that influences the direction of the “cigar” type singularity. This is illustrated in figure(2): if we increase the initial magnetic field intensity to \( \beta_0 = 1 \) then we will obtain a “cigar” type singularity along the \( z \) direction for any \( \beta_0 \geq 1 \). However, \( \beta_0 = 1 \sim 10^{13} \text{G} \), which is not a physically realistic value in magnetic white dwarfs but could be reasonable in a primordial magnetized universe model.

An isotropic “point” singularity can always emerge for configurations with zero initial deformation, i.e. \( S_x^0 = S_y^0 = S_z^0 = 0 \) and \( \beta_0 = 0 \). However, even with zero initial deformation there is always a threshold value of initial magnetic field intensity for which the singularity becomes anisotropic along the \( z \) direction.

B. Phase Space and Critical Subspaces

As mentioned previously, the Einstein–Maxwell system can be written as an autonomous system associated with a 4-dimensional phase space in the variables \((S^2, \beta, \mu, \mathcal{H})\). Notice that \( \Sigma^y \) can always be found if we determine \( \Sigma^z = \Sigma \), which is the only independent shear component. Considering the four primordial functions the system will take the form:

\[
\begin{align*}
\dot{U} &= -(U + p - \frac{2}{3}BM)\Theta - BM\Sigma, \\
\dot{\Sigma} &= \frac{2}{3}\kappa BM - \Theta\Sigma, \\
\dot{\Theta} &= \kappa (BM + \frac{3}{2}(U - p)) - \Theta^2, \\
\dot{\beta} &= \frac{2}{3}\beta(3\Sigma - 2\Theta).
\end{align*}
\]
where $\Sigma = \sigma \bar{z}$. If we work with the dimensionless functions defined in [14] this system becomes:

\begin{align}
S^2_{\tau} &= 2\beta T_{\Lambda M} - 3H S^2, \quad \text{(23a)} \\
H_{\tau} &= \beta T_{\Lambda M} + \frac{3}{2}(\Gamma_U - \Gamma_p) - 3H^2, \quad \text{(23b)} \\
\beta_{\tau} &= 2\beta(S^2 - 2H), \quad \text{(23c)} \\
\mu_{\tau} &= \frac{1}{\Gamma_{U,\mu}}[(2H - S^2)(\Gamma_{\Lambda M} + 2\Gamma_{U,\beta})\beta - 3H(\Gamma_p + \Gamma_U)]. \\
\text{(23d)}
\end{align}

Note that only change the equation for $H(\tau)$, but we can reduce the equations (23b) or (16c) to the constraint (15), therefore both systems are equivalent. Then, (16a) is only necessary for the computation of the metric coefficients.

In figure 3 we represent a 3-dimensional section of the phase space ($S^2$, $\beta$, $\mu$) with different curves for several initial conditions. As shown in the figure, the initial value of the expansion $H_0$ determines the global evolution of the numerical solutions curves. All curves starting at $\tau = 0$ with initial expansion $H_0 = -\sqrt{\kappa \lambda / 3}$ converge into the stable attractor marked as “a”, whereas if we choose $H_0 = \sqrt{\kappa \lambda / 3}$ and start at $\tau = 0$, the curves evolve towards the anisotropic singularity. Setting the left hand sides of the equations in (23) to zero and solving the algebraic system we find the set of critical points associated with this system, including the stable attractor marked as “a”. These points are given by:

$$a = \{S^2 = 0, \beta = 0, \mu = 1, H = 0\}, \quad \text{(24)}$$

For the 4-dim phase space ($S^2$, $\beta$, $\mu$, $H$) we have 4 possible 3-dim sections of the same phase space. We have computed this numerical solutions with similar results.

V. CONCLUSION.

We have presented a model based on the dynamical description of a local volume of a magnetized, self-gravitating, Fermi gas in the basic Landau level $n = 0$. Since we are considering complicated equations of the state, we have worked with the simplified form of the Einstein-Maxwell equations that follow by assuming a Bianchi-I space time represented by a Kasner metric, whose source of anisotropy is just the magnetic field. This simplified spacetime geometry provides a convenient toy model for a rough understanding the local collapsing behavior of the type of matter found inside a magnetized star like a white dwarf or a neutron star.

The relevance of the present paper emerges from our study of the collapsing singularities, which can be isotropic point-like or anisotropic of type cigar. Point singularities emerge under very special initial conditions of zero magnetic field, zero shear deformation or both. Cigar type singularities can also be obtained in all directions, depending on the initial values of the shear deformation. However, for an initial magnetic field intensity having a sufficiently large value the end singularity always becomes of type cigar in the direction of the field. This result is important because the value of the magnetic field determines the type of collapse and this is in agreement with the non-relativistic previous paper which examined the collapse of this type of magnetized gases within a Newtonian framework [7],[8],[9].

As discussed in [18] by Collins & Ellis, orthogonal Bianchi models like the one we are considering are globally hyperbolic and only present a single singularity, so that hypersurfaces of constant time (orthogonal to the 4-velocity) are global Cauchy hypersurfaces and every point in spacetime can be causally connected to the latter. The 4-velocity is a geodesic field and the singularity is marked by a specific constant time value, so that it is non-timelike and every event in spacetime can be causally connected to this singularity (in particular by the time-like geodesics that are integral curves of the 4-velocity field).

Now, in this article we are only considering a collapsing regime from an initial hypersurface of constant time. Thus, every future directed timelike curve (geodesic or not) starting at any initial Cauchy hypersurface of constant time will terminate in the collapse singularity. Under these conditions, this singularity is obviously censored.
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