Every curve is a Teichmüller curve

Jordan S. Ellenberg* and D. B. McReynolds†

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Abstract

We prove that every algebraic curve $X/\mathbb{Q}$ is birational over $\mathbb{C}$ to a Teichmüller curve.

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1 Introduction

Write $\mathcal{M}_{g,[n]}$ for the moduli space of genus $g$ Riemann surfaces with $n$ (unordered) punctures. A Teichmüller curve is a holomorphic curve $f: V \to \mathcal{M}_{g,[n]}$ such that $f$ generically one-to-one and is a local isometry of Kobayashi metrics. These special immersed curves in $\mathcal{M}_{g,[n]}$ have garnered interest for some time (especially in the unpunctured case $n = 0$) and are central objects in both Teichmüller and Grothendieck–Teichmüller theory. Additionally, these curves and the Riemann surfaces they parameterize have ties to the dynamics of polygonal billiards (see for instance [12], [15], [17], and [21]). McMullen proved [18] that every Teichmüller curve has a model as an algebraic curve over $\mathbb{Q}$ (see also [15] and [21]). The main purpose of this article is to prove the converse.

Theorem 1.1. If $X/\mathbb{Q}$ is an algebraic curve, then there exists a Teichmüller curve $V$ birational to $X_\mathbb{C}$.

In fact, this Teichmüller curve can be drawn from the rather special class of Teichmüller curves parameterizing origami or square-tiled surfaces—see [12], [15], [20], and [22] for more on these surfaces and their relationship to billiard dynamics.

We emphasize that the main ideas in the proof of Theorem 1.1 are drawn from the work of Asada [1], Thurston [19], and especially Diaz–Donagi–Harbater [8], whose title we have borrowed in order to emphasize the similarity. Part of our motivation is to cast the relevant arguments of [1] and [8], which are written in the language of algebraic geometry and field extensions, in topological terms. As this paper was being completed, we encountered the very recent article of Bux–Ershov–Rapinchuk [5], which gives a new proof of Asada’s theorem in extremely explicit group-theoretic language; the reader will note that the diagram of Riemann surfaces and fundamental groups used in our proof also appears in §6 of [5].

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One could also refine Theorem 1.1 by asking which algebraic curves over number fields are holomorphically isomorphic (not merely birational) to Teichmüller curves. The answer is not "every curve" – for example, according to Masur [16] (see also Veech [24]) Teichmüller curves cannot be compact, and so a proper algebraic curve over a number field can never be isomorphic to a Teichmüller curve. But the question of which affine algebraic curves are isomorphic over \( \mathbb{C} \) to Teichmüller curves is a long-standing open problem.

Theorem 1.1 arises as a corollary of the purely group-theoretic Theorem 1.2, which is in some sense the "real theorem" of the present paper. In order to state this result, we require some additional notation. Throughout, \( \pi_{g,n} \) will denote the fundamental group for the genus \( g \) surface \( S_{g,n} \) with \( n \) punctures, and \( \text{Mod}(S_{g,n}), \text{PMod}(S_{g,n}) \) will denote the associated mapping class group and pure mapping class group of \( S_{g,n} \). The pure mapping class group \( \text{PMod}(S_{g,n}) \) admits an outer action on \( \pi_{g,n} \), preserving each of the \( n \) conjugacy classes corresponding to the \( n \) punctures of \( S_{g,n} \). In fact, by the Dehn–Nielsen theorem, \( \text{PMod}(S_{g,n}) \) is an index two subgroup of the group of outer automorphisms of \( \pi_{g,n} \) preserving these conjugacy classes.

For a finite index subgroup \( \Delta \) of \( \pi_{1,1} \), we denote the \( \pi_{1,1} \)-conjugacy class of \( \Delta \) by \([\Delta]\). The mapping class group \( \text{Mod}(S_{1,1}) \) acts on the conjugacy classes of finite index subgroups of \( \pi_{1,1} \), and the stabilizer of \([\Delta]\) in \( \text{Mod}(S_{1,1}) \) via the outer action is called a Veech group [23, Theorem 1]. When \( \Gamma \) is a Veech subgroup of \( \text{Mod}(S_{1,1}) \cong \text{SL}(2,\mathbb{Z}) \), the corresponding quotient \( \mathbb{H}_R^2/\Gamma \) of the upper half plane has a natural description as a Teichmüller curve in \( \mathcal{M}_{g,[n]} \) parameterizing origami or square-tiled surfaces—among Teichmüller curves, these are precisely the arithmetic curves. (See [20, §21]) Teichmüller curves corresponding to non-square-tiled surfaces are substantially more difficult to construct and describe; see the recent work of Bouw–Möller [6], for instance, for a construction of Teichmüller curves corresponding to non-arithmetic triangle groups.

By Belyi’s theorem, every curve \( X/\mathbb{Q} \) is birational over \( \mathbb{C} \) to an étale cover of complex projective space minus three points \( \mathbb{P}^1 \setminus \{0,1,\infty\} \), or, equivalently, to \( \mathbb{H}_R^2/\Gamma \) where \( \Gamma \) is a finite index subgroup of the level two congruence subgroup \( \Gamma(2) \) containing the center \( \{\pm1\} \). Thus, to prove Theorem 1.1, it suffices to prove the following congruence subgroup property for \( \text{Mod}(S_{1,1}) \).

**Theorem 1.2.** Every finite index subgroup \( \Delta \) of \( \Gamma(2) \) containing \( \{\pm1\} \) is a Veech group.

The classification problem for Veech groups goes back at least to Thurston (see the Kirby problem list for more on the history of this problem; see also problem 4 of the survey by Hubert, Masur, Schmidt, and Zorich [13]). Theorem 1.2 can be seen as progress on this problem. The special case of subgroups of \( \text{SL}(2,\mathbb{Z}) \) is raised in the introduction of Schmithüsen’s thesis [23]. Theorem 1.1 also strengthens a theorem of Möller [20, Theorem 5.4], which shows that \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts faithfully on the set of isomorphism classes of Teichmüller curves. A recent paper of Hubert–Lelievre [14] provides a close study of Veech subgroups of \( \text{SL}_2(\mathbb{Z}) \) corresponding to Teichmüller curves in \( \mathcal{M}_2 \); they show that among these Veech groups are some which are not congruence.

One might ask whether every algebraic curve \( X/\mathbb{Q} \) is birational to a Teichmüller curve in \( \mathcal{M}_g \), as opposed to \( \mathcal{M}_{g,[n]} \). This statement does not quite follow from what we prove in the present paper. We construct a Teichmüller curve \( C \hookrightarrow \mathcal{M}_{g,[n]} \) for some \( g,n \), with \( C \) birational to \( X_C \). A priori, the composition \( C \hookrightarrow \mathcal{M}_{g,[n]} \to \mathcal{M}_g \) might not be generically one-to-one, in which case the image of \( C \) in \( \mathcal{M}_g \), not \( C \) itself, would be a Teichmüller curve in \( \mathcal{M}_g \). This kind of behavior can indeed occur for general origami curves—see the example after Definition 2.4 in [20]. It would be interesting to see whether such problems can be avoided in the present case, given the freedom in our construction of \( C \).
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Theorem 1.2 is also related to the congruence subgroup problem for mapping class groups. There is some variation among authors in the definition of congruence subgroups of mapping class groups. We will use Veech group for the class of subgroups of Mod(S_{1,1}) described here, and reserve principal congruence subgroup for the class of subgroups \Lambda of Mod(S_{1,1}) arising as kernels of induced maps

$$\rho_\Phi : \text{Mod}(S_{1,1}) \longrightarrow \text{Out}(\pi_{1,1}/\Phi),$$

where \Phi is a finite index, characteristic subgroup of \pi_{1,1}. In the most common terminology, a congruence subgroup is a subgroup of Mod(S_{1,1}) containing a principal congruence subgroup. We see that a Veech group is a congruence subgroup by taking \Phi to be the intersection of all subgroups of \pi_{1,1} of index \lceil \pi_{1,1} : \Phi \Delta \rceil, where

$$\text{Stab}_{\text{Mod}(S_{1,1})}(\lceil \Phi \Delta \rceil) = \Delta.$$ 

Thus, Theorem 1.2 can be thought of as a refinement of the congruence subgroup property for Mod(S_{1,1}), a theorem first established by Asada [1] (see also [4]).

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Notation For the readers’ convenience, we establish some notation that will be used throughout the remainder of this article. In a group G, the subgroup generated by the elements \{g_1, \ldots, g_n\} will be denoted by \langle g_1, \ldots, g_n \rangle. The G–conjugacy class of g will be denoted by \lbrack g \rbrack_G; we will sometimes suppress the decoration G and write simply \lbrack g \rbrack. The normal closure of a subgroup H of G will be denoted by \overline{H}. The subgroup generated by a pair of subgroups H_1, H_2 will be denoted by H_1 \cdot H_2. We denote the normalizer of H in G by N_G(H). The G–conjugacy class of a subgroup H, comprised of subgroups H' that are G–conjugate to H, will be denoted by \lbrack H \rbrack_G; as with conjugacy classes of elements, we will occasionally write simply \lbrack H \rbrack.

For a genus g surface with n punctures S_{g,n}, we denote the fundamental group of S_{g,n} by \pi_{g,n}. The mapping class group of S_{g,n} comprised of orientation preserving diffeomorphisms modulo isotopy will be denoted by PMod(S_{g,n}). The pure mapping class subgroup comprised of mapping classes fixing each of the punctures will be denoted by PMod(S_{g,n}).

2 Proof of Theorem 1.2

Before commencing the proof of Theorem 1.2, we require some additional setup. To begin, we have a homomorphism \pi_{0,4} \rightarrow \pi_{0,3} given by forgetting a puncture on S_{0,4}. We denote the distinguished conjugacy class given by the simple closed curve about the forgotten puncture by \lbrack z \rbrack_{\pi_{0,4}}. The normal closure \langle z \rangle of \langle z \rangle is the kernel of this homomorphism.

We have a regular 2–fold covering

$$F_0 : S_{1,4} \longrightarrow S_{0,4}$$

associated with the homomorphism of \pi_{0,4} to \mathbb{Z}/2\mathbb{Z} given by sending each puncture class to 1. The kernel of this homomorphism is the image (F_0)_*(\pi_{1,4}) and is the subgroup of all words in \pi_{0,4} in the free generating
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set \{x, y, z\} of even word length. In particular, \((F_0)_*(\pi_{1,4})\) is a characteristic subgroup with free generating set \(\{x^2, y^2, z^2, xy, xz\}\). When no confusion is likely, we refer to \((F_0)_*(\pi_{1,4})\) simply as \(\pi_{1,4}\). We denote the conjugacy class of \(z^2\) in \(\pi_{1,4}\) by \([\alpha]\)\(_{\pi_{1,4}}\).

We also have a 4–fold regular covering

\[
F_1: S_{1,4} \longrightarrow S_{1,1}
\]

associated with the \((\mathbb{Z}/2\mathbb{Z})\)–homology homomorphism of \(\pi_{1,1}\) to \(H_1(S_{1,1}, \mathbb{Z}/2\mathbb{Z})\). The kernel of this homomorphism is the image \((F_1)_*(\pi_{1,4})\) and is characteristic. If \(\{a, b\}\) is a free generating set for \(\pi_{1,1}\), the subgroup \((F_1)_*(\pi_{1,4})\) has free generating set \(\{a^2, b^2, (ab)^2, (ba)^2, [a, b]\}\). Again, we identify \((F_1)_*(\pi_{1,4})\) with \(\pi_{1,4}\). In total, this pair of covers yields the diagram:

\[
\begin{array}{c}
\pi_{1,1} \\
\downarrow \quad 4 \\
\pi_{1,4} \\
\downarrow \quad 2 \\
\pi_{0,4}
\end{array}
\]

The groups \(\text{PMod}(S_{0,4})\) and \(\text{Mod}(S_{1,1})\) are nearly isomorphic as abstract groups; \(\text{PMod}(S_{0,4})\) is the quotient of \(\text{Mod}(S_{1,1})\) by the center \(\text{Z}(\text{Mod}(S_{1,1}))\) of \(\text{Mod}(S_{1,1})\), a group of order 2. In fact, this abstract relationship can be realized geometrically by considering the action of both groups on \(\pi_{1,4}\), as we now explain. The material below is well-understood by the experts, but we include it here as it will play a prominent role.

We first construct a homomorphism from \(\Gamma(2)\) to \(\text{Mod}(S_{1,4})\). Let \(\Gamma(2)\) be the principal congruence subgroup of level two in \(\text{SL}(2, \mathbb{Z})\); it is well known that \(\text{SL}(2, \mathbb{Z})\) and \(\text{Mod}(S_{1,1})\) are isomorphic (see [11]). This provides us with an outer action of \(\Gamma(2)\) on \(\pi_{1,1}\). For an outer automorphism \(\theta\) representing an automorphism \(\tilde{\theta}\) of \(\pi_{1,1}\), we may restrict \(\tilde{\theta}\) to the characteristic subgroup \(\pi_{1,4}\), thus obtaining an automorphism \(\tilde{\theta}\) of \(\pi_{1,4}\). The resulting outer automorphism of \(\pi_{1,4}\) is not determined by \(\theta\) as there is an ambiguity given by the outer action of \(\pi_{1,1}/\pi_{1,4}\) on \(\pi_{1,4}\). The action of \(\pi_{1,1}/\pi_{1,4}\) by permutations of the four conjugacy classes of punctures in \(\pi_{1,4}\) is via the Klein four-group \(V_4\) in \(S_4\). Consequently, the action of an outer automorphism in \(\text{Mod}(S_{1,1})\) on these four classes can be thought of as an element of \(\text{Sym}(4)\), well-defined up to the Klein four-group \(V_4\). The resulting homomorphism \(\text{Mod}(S_{1,1})\) to \(\text{Sym}(3)\) is easily seen to be equivalent to the homomorphism induced by reduction modulo 2. In particular, when \(\theta\) is an outer automorphism of \(\pi_{1,1}\) lying in \(\Gamma(2)\), there is a unique outer automorphism of \(\pi_{1,4}\) arising by restriction of \(\theta\) that preserves the four puncture classes (i.e., lies in \(\text{PMod}(S_{1,4})\)). We have thus constructed a homomorphism \(\Gamma(2)\) to \(\text{PMod}(S_{1,4})\).

The “point push” map induces an isomorphism (see for instance [11] Theorem 4.5)

\[
\text{Push}: \pi_{0,3} \longrightarrow \text{PMod}(S_{0,4}).
\]

Let \(\theta\) be an element of \(\pi_{0,3}\), thought of as an element of \(\text{Out}(\pi_{0,4})\) via Push. As above, every automorphism \(\tilde{\theta}\) representing \(\theta\) restricts to an outer automorphism of \(\pi_{1,4}\). This outer automorphism class of \(\tilde{\theta}\) in \(\text{Out}(\pi_{1,4})\) is well-defined up to the outer action of \(\pi_{0,4}/\pi_{1,4}\). The latter is a subgroup of \(\text{Out}(\pi_{1,4})\) of order 2, whose nontrivial element we denote by \(\tau\). For future reference, \(\Gamma\) denotes the subgroup of \(\text{Out}(\pi_{1,4})\) obtained by restricting automorphisms representing elements of \(\pi_{0,3}\). Since \(\tau\) is central in \(\Gamma\), this supplies us with an isomorphism \(\pi_{0,3}\) to \(\Gamma/\langle \tau \rangle\). In addition, note that \(\Gamma\) is contained in \(\text{PMod}(S_{1,4})\), i.e. \(\Gamma\) preserves the four puncture classes.

In fact, the two outer actions we have constructed are identical.
Lemma 2.1. The image of \( \Gamma(2) \rightarrow \text{PMod}(S_{1,4}) \) lies in \( \Gamma \), and there is a surjection \( \Gamma(2) \rightarrow \pi_{0,3} \) with kernel \( \{\pm 1\} \) such that the diagram

\[
\begin{array}{ccc}
\Gamma(2) & \rightarrow & \pi_{0,3} \\
\downarrow & & \downarrow \\
\Gamma & \rightarrow & \Gamma/\langle \tau \rangle
\end{array}
\]

commutes.

Proof. Let \( H \) be the quotient of \( H_1(S_{1,4}, \mathbb{Z}) \) by the subgroup generated by the homology classes of the punctures, \( \text{Aut}_\mathbb{Z}(H) \) the associated automorphism group of \( H \), and \( \text{P Aut}_\mathbb{Z}(H) \) the projective automorphism group of \( H \). Note that \( \tau \) acts as \( -1 \) on the rank 2 free abelian group \( H \), and the action of \( \Gamma \) on \( H \) induces a homomorphism

\[
\pi_{0,3} \xrightarrow{\text{Push}} \text{PMod}(S_{0,4}) \xrightarrow{\cong} \Gamma/\langle \tau \rangle \rightarrow \text{P Aut}_\mathbb{Z}(H).
\]

If we fix a basis of \( H \), the free generators of \( \pi_{0,3} \) act as the matrices

\[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix},
\]

which are free generators for the level two congruence subgroup \( \text{PGL}(2) \) of \( \text{PSL}_2(\mathbb{Z}) \). Thus, the homomorphism \( \pi_{0,3} \rightarrow \text{P Aut}_\mathbb{Z}(H) \) induces an isomorphism between \( \pi_{0,3} \) and \( \text{P Aut}_\mathbb{Z}(H)(2) \), the latter being the subgroup consisting of automorphisms acting trivially on \( H/2H \).

The group \( \text{Mod}(S_{1,1}) \) also acts on \( H \) via its map to \( \text{PMod}(S_{1,4}) \), and once again one sees directly that the resulting homomorphism \( \Gamma(2) \rightarrow \text{Aut}_\mathbb{Z}(H)(2) \) is an isomorphism.

We are now reduced to establishing the following claim.

Claim. The image of \( \Gamma(2) \) in \( \text{PMod}(S_{1,4}) \) lies in \( \Gamma \).

This claim implies the lemma since in that case the map

\[
\Gamma(2) \rightarrow \Gamma \rightarrow \text{P Aut}_\mathbb{Z}(H)(2)
\]

provides the desired identification of \( \text{PGL}(2) \) with \( \pi_{0,3} \).

Proof of Claim. To prove the claim, we embed the groups \( \pi_{0,4}, \pi_{1,1}, \) and \( \pi_{1,4} \) in a common Fuchsian group. To this end, let \( \pi_{0,1;2^3} \) be the group generated by \( \{s, t, u, v\} \) and subject to the four relations

\[
stuv = t^2 = u^2 = v^2 = 1.
\]

The group \( \pi_{0,1;2^3} \) is the orbifold fundamental group of the surface \( S_{0,1} \) with 3 order two cone points. Define the homomorphism

\[
\varphi : \pi_{0,1;2^3} \rightarrow (\mathbb{Z}/2\mathbb{Z})^3
\]

by

\[
\varphi(s) = (1, 0, 0), \quad \varphi(t) = (1, 1, 0), \quad \varphi(u) = \varphi(1, 0, 1), \quad \varphi(v) = (1, 1, 1).
\]
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It is a simple matter to check that \( \varphi \) is a homomorphism and that the diagram (1) can be recovered by means of the identifications

\[
\pi_{1, 4} = \ker \varphi, \quad \pi_{1, 1} = \varphi^{-1}(0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}), \quad \pi_{0, 4} = \varphi^{-1}(\mathbb{Z}/2\mathbb{Z}, 0, 0).
\]

The action of \( \pi_{0, 1, 2} \) by conjugation on its normal subgroup \( \pi_{1, 1} \) yields an injection of \( \pi_{0, 1, 2} \) into \( \text{Aut}(\pi_{1, 1}) \), whose image in \( \text{Out}(\pi_{1, 1}) \) is the center \( \{ \pm 1 \} \). This assertion can be seen via the following computation, where equality is up to conjugation in \( \pi_{1, 1} = \langle tv, uv \rangle \) is denoted by \( =_{c} \):

\[
\begin{align*}
    s^{-1}(tv)s = & \\ tuv(tv)vut = tu(vt)ut =_{c} vt \\
    s^{-1}(uv)s = & \\ tuv(uv)vut = tu(vu)ut =_{c} vu \\
    t(tv)t = & \\ vt =_{c} ut(tv)tu = vu \\
    u(tv)u = & \\ _{c} tu(utvu)ut = vt, \quad u(uv)u = vu \\
    v(tv)v = & \\ vu =_{c} _{c} _{c} vuvu.
\end{align*}
\]

In particular, \( \pi_{0, 1, 2} \) embeds as a normal subgroup of \( \text{Aut}(\pi_{1, 1}) \), which yields a homomorphism of \( \text{Aut}(\pi_{1, 1}) \) to \( \text{Aut}(\pi_{0, 1, 2}) \) induced by conjugation. This embedding shows that every automorphism \( \tilde{\Theta} \) in \( \text{Aut}(\pi_{1, 1}) \) extends to an automorphism \( \overline{\Theta} \) in \( \text{Aut}(\pi_{0, 1, 2}) \). Since the restriction of \( \overline{\Theta} \) to \( \pi_{1, 4} \) factors through \( \text{Aut}(\pi_{0, 4}) \), the associated outer automorphism class \( \Theta \) of \( \overline{\Theta} \) resides in \( \Gamma \). Hence, the image of \( \Gamma(2) \) in \( \text{PMod}(S_{1, 4}) \) lies in \( \Gamma \), as claimed.

Having validated the claim, the proof the lemma is complete.

We briefly explain how Lemma 2.1 can be expressed in the language of moduli spaces (better, moduli stacks) of algebraic curves. Write \( \mathcal{M} \) for the moduli space of data \( (C, P_{1}, P_{2}, P_{3}, P_{4}) \), where \( C \) is a smooth complex genus 1 curve and the \( P_{i} \) are distinct points on \( C \) such that \( P_{1} - P_{j} \) is a 2–torsion divisor on the Jacobian of \( C \). There is a natural isomorphism from \( \mathcal{M} \) to \( X(2) \), the moduli stack of elliptic curves with level two structure, which sends \( (C, P_{1}, P_{2}, P_{3}, P_{4}) \) to the elliptic curve \( (C, P_{1}) \) with \( P_{2} - P_{1} \) and \( P_{3} - P_{1} \) as a basis of 2–torsion. On the other hand, given \( (C, P_{1}, P_{2}, P_{3}, P_{4}) \) there is a unique nontrivial involution \( t \) of \( C \) fixing all the \( P_{i} \), which affords a morphism

\[
\phi : C \longrightarrow C/t
\]

whose target is a genus 0 curve. This morphism induces a morphism \( \mathcal{M} \rightarrow \mathcal{M}_{0, 4} \) defined by

\[
(C, P_{1}, P_{2}, P_{3}, P_{4}) \mapsto (\phi(C), \phi(P_{1}), \phi(P_{2}), \phi(P_{3}), \phi(P_{4})).
\]

This map is an isomorphism on coarse moduli spaces, but the target is generically a scheme, while \( \mathcal{M} \) has a generic inertia group of \( \mathbb{Z}/2\mathbb{Z} \). The (analytic) fundamental group of \( \mathcal{M}_{0, 4} \) is precisely \( \text{PMod}(S_{1, 4}) \cong \pi_{0, 3} \). Thus, the diagram

\[
\pi_{1}(X(2)) \longrightarrow \pi_{1}(\mathcal{M}) \longrightarrow \pi_{1}(\mathcal{M}_{0, 4})
\]

can be written as

\[
\Gamma(2) \longrightarrow \pi_{1}(\mathcal{M}) \longrightarrow \pi_{0, 3}.
\]

The group \( \Gamma \) appearing in the proof of Lemma 2.1 is just \( \pi_{1}(\mathcal{M}) \), and the inclusion of \( \Gamma \) in \( \text{PMod}(S_{1, 4}) \) is induced by the inclusion of \( \mathcal{M} \) as the hyperelliptic locus in \( \mathcal{M}_{1, 4} \).
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We now continue with the prerequisites for the proof of Theorem 1.2. Let \( \Delta \) be a finite index subgroup of \( \Gamma(2) \) containing \( \{ \pm 1 \} \). Via (2), we denote the corresponding subgroup of \( \pi_{0,3} \) by \( \Delta_0 \), and let \( \tilde{\Delta} \) be the pullback of \( \Delta_0 \) to \( \pi_{1,4} \). Explicitly, \( \tilde{\Delta} \) is given by

\[
\tilde{\Delta} = \overline{(z) \cdot \Delta_0} \cap (F_0)_*(\pi_{1,4}).
\]

Note that \([\alpha]_{\pi_{1,4}}\) lies in \( \tilde{\Delta} \) and has a decomposition as a union of \( \tilde{\Delta} \)-conjugacy classes, which we write

\[
[\alpha]_{\pi_{1,4}} = \bigsqcup_j^k [\alpha_j]_{\tilde{\Delta}}.
\]

In the proof of Theorem 1.2, we need a pair of elementary lemmas. In what follows, via Lemma 2.1, we identify \( \Gamma(2) \) with the subgroup \( \Gamma \) in PMod(\( S_{1,4} \)), so that \( \Gamma(2) \) acts unambiguously on conjugacy classes of elements and subgroups of \( \pi_{1,4} \). Thus, the notation \([\Delta]\) refers to the conjugacy class \([\Delta]\) in \( \pi_{1,4} \) and not \([\Delta]_{\pi_{0,4}} \) or \([\tilde{\Delta}]_{\pi_{1,4}} \).

**Lemma 2.2.** The action of \( \Gamma(2) \) on \( \pi_{1,4} \) preserves \([\Delta]\).

**Proof.** By Lemma 2.1 it suffices to prove the same statement for the action of PMod(\( S_{0,4} \)). To this end, let

\[
\overline{\Delta} = \overline{\langle z \rangle \cdot \Delta_0}
\]

be the pullback of \( \Delta_0 \) to \( \pi_{0,4} \), \( \theta \) an element of PMod(\( S_{0,4} \)), and \( \tilde{\theta} \) a lift of \( \theta \) to Aut(\( \pi_{0,4} \)). We must show that \( \tilde{\theta}(\tilde{\Delta}) \) and \( \tilde{\Delta} \) are conjugate in \( \pi_{1,4} \). To begin, since \( \tilde{\theta} \) preserves \( \langle z \rangle \), the subgroup \( \tilde{\theta}(\tilde{\Delta}) \) contains \( \langle z \rangle \). In addition, as the projection of \( \tilde{\theta} \) to \( \pi_{0,3} \) is inner, the projection of \( \tilde{\theta}(\tilde{\Delta}) \) to \( \pi_{0,3} \) is conjugate to \( \Delta_0 \) in \( \pi_{0,3} \). Therefore, the groups \( \tilde{\theta}(\tilde{\Delta}) \) and \( \tilde{\Delta} \) are conjugate by some element \( \beta \) in \( \pi_{0,4} \). Since \( \overline{\Delta} \) contains \( z \), the homomorphism of \( \pi_{0,4} \) to \( \pi_{0,4}/\pi_{1,4} \) restricted to \( \overline{\Delta} \) is onto. In particular we may take \( \beta \) in \( \pi_{1,4} \) and thus,

\[
\tilde{\theta}(\tilde{\Delta}) = \pi_{1,4} \cap \beta^{-1}\overline{\Delta}\beta = \beta^{-1}\Delta\beta
\]

as desired. \( \square \)

It follows from Lemma 2.2 that the action of \( \Gamma(2) \) permutes the \( [\alpha_i]_{\overline{\Delta}} \). In fact, this permutation action is easy to describe.

**Lemma 2.3.** The action of \( \Gamma(2) \) on the \( [\alpha_i]_{\overline{\Delta}} \) is equivalent to the action of \( \Gamma(2) \) on the cosets of \( \Delta \). In particular, the index of \( \Delta \) in \( \Gamma(2) \) is \( k \), where \( k \) is the number of classes \( [\alpha_i]_{\overline{\Delta}} \) contained in \( [\alpha]_{\pi_{1,4}} \).

**Proof.** Keeping with the notation of Lemma 2.2 let

\[
F : S \longrightarrow S_{0,3}
\]

be the covering space associated to \( \Delta_0 \) and \( p \) a point of \( S_{0,3} \). The set \( F^{-1}(p) \) has order \( [\Gamma(2) : \Delta] \) and the action of \( \pi_{0,3} \) on \( F^{-1}(p) \) is isomorphic to the action of \( \pi_{0,3} \) on cosets of \( \Delta_0 \). Viewing \( S_{0,4} \) as \( S_{0,3} \setminus \{ p \} \), the \( \overline{\Delta} \)-conjugacy classes \( [z]_{\overline{\Delta}} \) contained in \( [z]_{\pi_{0,4}} \) are identified with \( F^{-1}(p) \). As \( z_i \) lies over the non-trivial element of \( \pi_{0,4}/\pi_{1,4} \), we can identify the \( [z_i]_{\overline{\Delta}} \) with the \( [\alpha_i]_{\overline{\Delta}} \). \( \square \)
In addition to the above elementary facts, we require a result that constructs finite index subgroups of $\Gamma(2)$ that are self-normalizing and contained in a prescribed finite index subgroup. This problem is analogous to controlling isometry groups of finite covers of a closed Riemannian manifold (see [3]). We have included a proof of using actions of finite groups on representation varieties, though it seems reasonable to expect that there is an easier proof (see the remark following the proof)—the method here can also be used to address constructive problems for isometry groups of certain negatively curved, closed Riemannian manifolds, as we will explain in a later paper ([10]).

**Proposition 2.4.** Given a finite index subgroup $\Delta$ of $\Gamma(2)$ containing $\{\pm 1\}$, there exists a finite index subgroup $\Delta'$ of $\Delta$ containing $\{\pm 1\}$ such that

$$N_{\Gamma(2)}(\Delta') = \Delta'.$$

The strategy for proving Proposition 2.4 is as follows. We reduce to the case where $\Delta$ is normal in $\Gamma(2)$; then the finite quotient group $\Theta = \Gamma(2)/\Delta$ can be viewed as a subgroup of $\text{Out}(\Delta)$. We show that $\Theta$ acts faithfully on the $\text{PSL}(2,\mathbb{C})$–character variety for $\Delta$. Using this faithful action, we produce a surjective homomorphism of $\Delta$ onto a product of finite simple groups of the form $\text{PSL}(2,\mathbb{F}_p)$. Using some elementary facts about these finite simple groups, the desired $\Delta'$ is given as the pullback of a certain subgroup in this product.

**Proof.** Replacing $\Delta$ with its normal core in $\Gamma(2)$ if necessary, we may assume $\Delta$ is normal in $\Gamma(2)$. After this modification, set $\Theta$ to be the finite quotient group $\Gamma(2)/\Delta$, which admits an injective homomorphism into $\text{Out}(\Delta)$. We denote by $X$ the variety parameterizing representations

$$\rho : \Delta \rightarrow \text{SL}(2,\mathbb{C}).$$

The variety $X$ is a closed subvariety of $(\text{SL}(2,\mathbb{C}))^N$, where $N$ is the cardinality of a fixed generating set for $\Delta$. In addition, $X$ carries an algebraic $\text{Aut}(\Delta)$–action given by pre-composition. For each element $\delta$ of $\Delta$, the function $\chi_\delta$ that sends a representation $\rho$ to the trace of $\rho(\delta)$ is an algebraic function on $X$, which is invariant under the action of subgroup $\text{Inn}(\Delta)$ of the inner automorphisms in $\text{Aut}(\Delta)$. In particular, if $\theta$ is an element of $\Theta$, the function $\chi_{\theta(\delta)}$ is well-defined.

The first step is to show that $\Theta$ acts faithfully on the associated $\text{SL}(2,\mathbb{C})$–character variety of $\Delta$. The faithfulness of this action is a consequence of [2]. However, we give a simple proof here. To this end, we first show that, for each $\theta$ in $\Theta$, there exists a $\delta$ such that $\chi_\delta^2 \neq \chi_{\theta(\delta)}^2$.

Note that the action of $\theta$ on $H^1(\Delta,\mathbb{Z})$ is not trivial; if it were, the subgroup of $\text{PGL}(2)$ generated by $\theta$ and $\theta$ would be a free group of the same rank as $\theta$, which is impossible. The action of $\theta$ is also not $-1$, since it preserves the subgroup of $H^1(\Delta,\mathbb{Z})$ pulled back from $H^1(\Gamma(2),\mathbb{Z})$. Thus, we can and do choose a homomorphism

$$\phi : \Delta \rightarrow \mathbb{Z}$$

such that $\phi \circ \theta$ is neither $\phi$ nor $-\phi$.

Let $a$ be a non-torsion point of the multiplicative torus $\mathbb{G}_m$, and define

$$\rho : \Delta \rightarrow \mathbb{G}_m \oplus \mathbb{G}_m$$

by

$$\rho(\delta) = a^{\phi(\delta)} \oplus a^{-\phi(\delta)}.$$
Every curve is a Teichmüller curve

It is easy to see that \( \delta \) is in \( \ker \rho \) if and only if \( \chi_2^2(\delta) = 4 \). Thus, if

\[
\chi_2^2 \delta = \chi_2^2(\delta),
\]

the kernels of \( \rho \) and \( \rho \circ \theta \) must be equal. In turn, this equality implies that

\[
\phi = \pm (\phi \circ \theta).
\]

However, we have ruled out this equality by our selection of \( \phi \).

For each \( \theta \in \Theta \), we choose a \( \delta_\theta \) in \( \Delta \) such that \( \chi_2^2 \delta_\theta \) and \( \chi_2^2 \theta(\delta_\theta) \) are distinct. Next, define the closed subvarieties

\[
Z_\theta = \{ x \in X : \chi_2^2(\delta_\theta)(x) = \chi_2^2(\theta(\delta_\theta))(x) \}
\]

and

\[
Z_0 = \bigcup_{\theta \in \Theta} Z_\theta.
\]

By our selection of \( \delta_\theta \), we see that \( Z_0 \) is a closed subvariety of \( X \) of positive codimension. In addition, let \( Z_1 \) be the variety parameterizing representations whose images in \( \text{PSL}(2, \mathbb{C}) \) lie in a Borel subgroup, the normalizer of a Cartan subgroup, or one of the exceptional finite subgroups of \( \text{PSL}(2, \mathbb{C}) \). Since \( \Delta \) admits a Zariski dense representation into \( \text{SL}(2, \mathbb{C}) \), the closed subvariety \( Z_1 \) of \( X \) also has positive codimension, and thus so does the union \( Z_0 \cup Z_1 \).

Write \( U \) for the complement of \( Z_0 \cup Z_1 \) in \( X \). It follows that \( U \) is a quasi-projective variety, and so \( U(\mathbb{F}_p) \) is nonempty for primes \( p \) sufficiently large. To see this assertion, note that a quasi-projective variety admits a dominant map

\[
f : U \to \mathbb{P}^1.
\]

It suffices to have the statement for an irreducible component of a non-empty fiber of this map, which is a quasi-projective variety of dimension one less than \( \dim U \). By induction, we are reduced to the case of dimension 0, where the statement follows from the Chebotarev Density Theorem.

With this said, we now choose a sufficiently large prime \( p \) such that \( U(\mathbb{F}_p) \) is non-empty and let \( x \) be a point of \( U(\mathbb{F}_p) \). Such a point corresponds to a surjective representation

\[
\rho : \Delta \longrightarrow \text{SL}(2, \mathbb{F}_p).
\]

For each \( \theta \), choose a lift \( \hat{\theta} \) of \( \theta \) to \( \text{Aut}(\Delta) \), and let \( \theta \rho \) denote the representation \( \rho \circ \hat{\theta} \) of \( \Delta \). Write \( P\rho \) for the composition of \( \rho \) with projection from \( \text{SL}(2, \mathbb{F}_p) \) to the simple group \( \text{PSL}(2, \mathbb{F}_p) \).

We claim that \( P\rho \) and \( \theta P\rho \) are not isomorphic. Since \( \rho \) does not lie in \( Z_\theta(\mathbb{F}_p) \), we know there exists a \( \delta \) such that \( \chi_2^2(\rho) \neq \chi_2^2(\theta(\delta))(\rho) \); it follows that \( P\rho \) and \( \theta P\rho \) are not conjugate by an element of \( \text{PGL}(2, \mathbb{F}_p) \).

However, conjugation by \( \text{PGL}(2, \mathbb{F}_p) \) is the full automorphism group of \( \text{PSL}(2, \mathbb{F}_p) \) [7, §12.5], and so we conclude that \( P\rho \) and \( \theta P\rho \) are non-isomorphic.

We are now ready to define the representation of \( \Delta \) that we will use to construct the asserted self-normalizing subgroup. To this end, define a homomorphism

\[
\varphi = \left( \bigoplus_{\theta \in \Theta} \theta \rho \right) : \Delta \longrightarrow \left( \prod_{\theta \in \Theta} \text{PSL}(2, \mathbb{F}_p) \right) =: Q.
\]
For notational simplicity, set \( G = \text{PSL}(2, F_p) \) and \( B = \mathcal{B}(F_p) \), where \( \mathcal{B}(F_p) \) is the standard Borel subgroup of \( \text{PSL}(2, F_p) \) comprised of upper triangular matrices. In addition, recall that \( B \) is self-normalizing in \( G \), i.e. \( N_G(B) = B \). The map \( \varphi \) projects surjectively onto each copy of \( G \), and we have shown that these projections are pairwise non-isomorphic representations of \( \Delta \); it follows from a lemma of P. Hall (see [9, Lemma 3.7]) that \( \varphi \) is surjective. Note also that \( \{ \pm 1 \} \) is contained in \( \ker \varphi \).

By construction, the kernel of \( \varphi \) is preserved by the action of \( \Theta \), so \( \Theta \) acquires a homomorphism to \( \text{Out}(Q) \); in fact, this outer action of \( \Theta \) on \( Q \) is just permutation of factors.

For a fixed \( \theta \) in \( \Theta \), define \( B_\theta \) to be the subgroup of \( Q \) to be \( \mathcal{B} \) on the \( \theta \)–factor of \( Q \) and \( G \) otherwise. Define \( \Delta_\theta \) to be the pullback by \( \varphi \) of \( B_\theta \). We now show that \( \Delta_\theta \) is the desired self-normalizing finite index subgroup.

If \( \gamma \) is in \( N_{\Gamma(2)}(\Delta_\theta) \), we denote the projection of \( \gamma \) to \( \Theta \) by \( \theta_\gamma \) and note that \( \gamma \)–conjugation on \( \Delta \) induces the outer automorphism \( \theta_\gamma \). Since \( \gamma \) is in \( N_{\Gamma(2)}(\Delta_\theta) \), the induced action on \( Q \) must fix \( B_\theta \), which immediately implies that \( \theta_\gamma \) must be trivial. Consequently, we deduce that \( \gamma \) resides in \( \Delta \), and thus

\[
N_{\Gamma(2)}(\Delta_\theta) = N_\Delta(\Delta_\theta).
\]

However, since \( N_G(B) = B \), we have \( N_Q(B_\theta) = B_\theta \), and so

\[
N_\Delta(\Delta_\theta) = \Delta_\theta
\]

as needed. \( \square \)

**Remark.** The proof of Proposition 2.4 is the only part of this paper that is in any way non-elementary. There is a more elementary, but less appealing proof of Proposition 2.4 using covering space theory. Sketch: it suffices to prove the Proposition with \( \Delta(2) \) replaced by \( \pi_{0,3} \). The finite sheeted cover of the wedge of two circles corresponding to the inclusion \( \Delta_0 \subset \pi_{0,3} \) is a 4–valent graph. Then one builds a cover of this graph that admits no symmetry.

We are now in position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \Delta \) be a finite index subgroup of \( \Gamma(2) \), which contains \( \{ \pm 1 \} \). Our goal is to produce a finite index subgroup \( \Lambda \) of \( \pi_{1,1} \) such that

\[
\text{Stab}_{\text{Mod}(S_{1,1})}([\Lambda]_{\pi_{1,1}}) = \Delta.
\]

Our strategy will be to produce a family of finite index subgroups \( \Delta_{1,\ell} \) of \( \pi_{1,4} \) parameterized by primes \( \ell \). For a co-finite set of primes \( \ell \), we will show

\[
\text{Stab}_{\text{Mod}(S_{1,1})}([\Delta_{1,\ell}]_{\pi_{1,1}}) = \text{Stab}_{\Gamma(2)}([\Delta_{1,\ell}]_{\pi_{1,4}}).
\]

We will then argue that for sufficiently large \( \ell \), we also have the equality

\[
\text{Stab}_{\Gamma(2)}([\Delta_{1,\ell}]_{\pi_{1,4}}) = \Delta.
\]

The details are as follows.
According to Proposition 2.4, there exists a finite index subgroup $\Delta'$ of $\Delta$ that contains $\{\pm 1\}$ and is self-normalizing in $\Gamma(2)$. We denote the associated subgroups of $\pi_{0,3}$ by $\Delta_0'$ and $\Delta_0$, and the pullback of these subgroups to $\pi_{1,4}$ by $\Delta'$ and $\Delta$. Note that, like $\Delta$, the subgroup $\Delta'$ contains $[\alpha]_{\pi_{1,4}}$. We claim that $\Delta'$ is self-normalizing in $\pi_{1,4}$. To verify this claim, observe that if $\eta$ in $\pi_{1,4}$ normalizes $\Delta'$, then the projection of $\eta$ to $\pi_{0,3}$ normalizes $\Delta_0'$. By our selection of $\Delta'$, the subgroup $\Delta_0'$ is self-normalizing in $\pi_{0,3}$ and so the projection of $\eta$ must reside in $\Delta_0'$, which in turn implies that $\eta$ lies in $\Delta'$.

We denote the conjugacy classes associated to the three punctures on $S_{1,4}$ distinct from the puncture associated to $[\alpha]_{\pi_{1,4}}$ by $[\beta_1]_{\pi_{1,4}}, [\beta_2]_{\pi_{1,4}}, [\beta_3]_{\pi_{1,4}}$. We select $p_1, p_2,$ and $p_3$ to be three distinct primes larger than $[\Gamma(2) : \Delta]$. Using these three primes, we obtain a surjective homomorphism

$$\pi_{1,4} \rightarrow \bigoplus_{i=1}^{3} H^1(\pi_{1,4}, \mathbb{Z}/p_i\mathbb{Z}).$$

For each $i$, set $\overline{H}_i$ to be the subgroup of $H^1(\pi_{1,4}, \mathbb{Z}/p_i\mathbb{Z})$ generated by $\{\beta_j \mid j \neq i\}$, and

$$\overline{H} = \overline{H}_1 \oplus \overline{H}_2 \oplus \overline{H}_3 < \bigoplus_{i=1}^{3} H^1(\pi_{1,4}, \mathbb{Z}/p_i\mathbb{Z}).$$

Finally, set $H$ to be the pullback of $\overline{H}$ to $\pi_{1,4}$. If $b$ is an element of a puncture class $[\beta_i]$, then $b \notin H$ but $b^{p_j} \in H$. It follows that any automorphism of $\pi_{1,4}$ preserving $[H]_{\pi_{1,4}}$ must preserve the four conjugacy classes $[\alpha]_{\pi_{1,4}}, [\beta_1]_{\pi_{1,4}}, [\beta_2]_{\pi_{1,4}},$ and $[\beta_3]_{\pi_{1,4}}$ individually. Thus, the subgroup $H$ is normal in $\pi_{1,4}$, but not in $\pi_{1,1}$; conjugation by $\pi_{1,1}$ permutes the four puncture classes via the Klein four-group action described above. Clearly, $[H]_{\pi_{1,4}}$ is preserved by the action of the subgroup $\Gamma$ of $\text{Out}(\pi_{1,4})$.

For each $j$ with $1 \leq j \leq k$, set $A_j$ to be the subgroup of $H^1(\tilde{\Delta}', \mathbb{Z})$ spanned by the images of

$$[\alpha_1]_{\tilde{\Delta}'}, \ldots, [\alpha_j - 1]_{\tilde{\Delta}'}, [\alpha_{j+1}]_{\tilde{\Delta}'}, \ldots, [\alpha_k]_{\tilde{\Delta}'}$$

in $H^1(\tilde{\Delta}', \mathbb{Z})$. Note that $[\alpha_1]_{\tilde{\Delta}'}$ is a union of conjugacy classes in $\tilde{\Delta}'$. For each prime $\ell$, let $A_{j, \ell}$ denote the image of $A_j$ in $H^1(\tilde{\Delta}', \ell \mathbb{Z})$, and define $\Delta_{j, \ell}$ to be the intersection of $H$ with the kernel of the homomorphism

$$\rho_{j, \ell} \colon \tilde{\Delta}' \rightarrow H^1(\tilde{\Delta}', \ell \mathbb{Z})/A_{j, \ell}.$$

We will prove Theorem 1.2 by proving that, for sufficiently large primes $\ell$, the stabilizer of $[\Delta_{1, \ell}]_{\pi_{1,1}}$ is $\Delta$. To establish (3), we first reduce to a problem on $[\Delta_{1, \ell}]_{\pi_{1,1}}$.

For $\theta$ in $\text{Mod}(S_{1,1})$, let $\tilde{\theta}$ be an automorphism of $\pi_{1,1}$ represented by the outer class $\theta$. We can and do choose $\tilde{\theta}$ to restrict to an element of $\text{Aut}(\pi_{1,4})$ preserving the puncture class $[\alpha]_{\pi_{1,4}}$. Suppose furthermore that

$$\tilde{\theta}(\Delta_{1, \ell}) = \eta^{-1}\Delta_{1, \ell}\eta$$

for some $\eta$ in $\pi_{1,1}$. We next assume that $\ell$ is larger than the index of $\Delta_0$ in $\pi_{0,3}$ and distinct from $p_1, p_2,$ and $p_3$. If $a$ is an element of a puncture class in $\pi_{1,4}$ such that $a \notin \Delta_{1, \ell}$ but $a^\ell \in \Delta_{1, \ell}$, then the puncture class can only be $[\alpha]_{\pi_{1,4}}$. We also have that $(\eta \tilde{\theta}(a)\eta^{-1})^\ell$ is contained in $\Delta_{1, \ell}$, while $\eta \tilde{\theta}(a)\eta^{-1}$ itself is not. This implies that $\eta \tilde{\theta}(a)\eta^{-1}$ lies in the puncture class $[\alpha]_{\pi_{1,4}}$. Since $\tilde{\theta}(a)$ lies in the same puncture class,
we conclude that $\eta$ preserves the puncture class $[\alpha]_{\pi_{1,4}}$, which implies that $\eta$ is in $\pi_{1,4}$. Thus, we need only concern ourselves with the $\pi_{1,4}$--conjugacy class of $\Delta_{1,\ell}$.

We next complete the reduction to (3) by showing that $\theta$ must be an element of $\Gamma(2)$. For this reduction, note that the puncture class $[\beta]_{\pi_{1,4}}$ can be specified as the one containing elements $b$ such that, for some integer $m$, we have $b^m \notin \Delta_{1,\ell}$ but $b^{mp} \in \Delta_{1,\ell}$. Again, this holds with $\Delta_{1,\ell}$ replaced by $\eta^{-1}\Delta_{1,\ell}\eta$, which means that the action of $\bar{\theta}$ preserves not only $[\alpha]_{\pi_{1,4}}$ but the other three puncture classes $[\beta_1]_{\pi_{1,4}}, [\beta_2]_{\pi_{1,4}}$, and $[\beta_3]_{\pi_{1,4}}$ as well. Thus, $\theta$ must reside in $\Gamma(2)$.

In total, we have shown for a co-finite set of primes $\ell$ that (3) holds. As in Lemmas 2.2 and 2.3 we are implicitly embedding $\Gamma(2)$ in PMod$(S_{1,4})$ in order to make $\Gamma(2)$ preserve $[\alpha]_{\pi_{1,4}}$, and more generally to act unambiguously on $\pi_{1,4}$--conjugacy classes.

We are now ready to show that for sufficiently large primes $\ell$ that

$$\text{Stab}_{\Gamma(2)}([\Delta_{1,\ell}]_{\pi_{1,4}}) = \Delta.$$ 

As (3) also holds, this equality suffices for proving Theorem 1.2. To this end, let $\theta$ be an element of $\Gamma(2)$ preserving $[\Delta_{1,\ell}]_{\pi_{1,4}}$. By Lemma 2.3

$$\theta([\alpha_1]_{\Delta}) = [\alpha_1]_{\Delta}$$

if and only if $\theta$ is an element of $\Delta$. Since conjugation by $\Gamma(2)$ permutes the $[\alpha_i]$, we have

$$\theta([\Delta_{1,\ell}]_{\pi_{1,4}}) = [\Delta_{j,\ell}]_{\pi_{1,4}}$$

for some $j \in \{1, \ldots, k\}$. We already know that $\theta$ preserves $[\Delta_{1,\ell}]_{\pi_{1,4}}$; we now investigate when $\Delta_{1,\ell}$ and $\Delta_{j,\ell}$ can be conjugate in $\pi_{1,4}$. For this investigation, let $\eta$ be an element of $\pi_{1,4}$ such that

$$\eta^{-1}\Delta_{1,\ell}\eta = \Delta_{j,\ell}.$$ 

According to this equality, $\Delta_{j,\ell}$ is a normal subgroup of $\hat{\Delta}'$ and of $\eta^{-1}\hat{\Delta}'\eta$, with elementary abelian $\ell$--group quotient. Thus, the chain of inclusions

$$\Delta_{j,\ell} < \eta^{-1}\hat{\Delta}'\eta \cap \hat{\Delta}' < \hat{\Delta}'$$

shows that $[\hat{\Delta}' : \eta^{-1}\hat{\Delta}'\eta \cap \hat{\Delta}']$ is a power of $\ell$. On the other hand, there are only finitely many subgroups of $\pi_{1,4}$ conjugate to $\hat{\Delta}'$. Consequently, we can choose $\ell$ to be relatively prime to $[\hat{\Delta}' : \eta^{-1}\hat{\Delta}'\eta \cap \hat{\Delta}']$ for every $\eta$ in $\pi_{1,4}$. Having done so, we have forced the equality

$$\eta^{-1}\hat{\Delta}'\eta = \hat{\Delta}'.$$ 

The self-normalization of $\hat{\Delta}'$ in $\pi_{1,4}$ mandates that $\eta$ be an element of $\hat{\Delta}'$. Given this containment, (2) implies that $\Delta_{1,\ell}$ and $\Delta_{j,\ell}$ are conjugate as subgroups of $\hat{\Delta}'$. Thus, $\Delta_{1,\ell}$ and $\Delta_{j,\ell}$ must have identical projections to $H^1(\hat{\Delta}', \mathbb{Z})$. Since the subgroups $A_1, \ldots, A_k$ of $H^1(\hat{\Delta}', \mathbb{Z})$ are all distinct, $j = 1$ and so $\theta$ is an element of $\Delta$.

We have thus shown that (2) holds for all but finitely many primes $\ell$. Choosing such an $\ell$, we now see by (3) that

$$\text{Stab}_{\text{Mod}(S_{1,1})}([\Delta_{1,\ell}]_{\pi_{1,1}}) = \Delta,$$

and so $\Delta$ is a Veech group, as advertised. \qed
Every curve is a Teichmüller curve

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Department of Mathematics
University of Wisconsin at Madison
Madison, WI 53706, USA
e-mail: ellenber@math.wisc.edu

Department of Mathematics
University of Chicago
Chicago, IL 60637, USA
e-mail: dmcreyn@math.uchicago.edu