Sini R · Nijo Varghese · V C Kuriakose

Quasi-normal modes of spherically symmetric black hole spacetimes with cosmic string in a Dirac field

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Abstract Dirac equation for a general black hole metric having a cosmic string is derived. The quasi-normal mode frequencies for Schwarzschild, RN extremal, SdS and near extremal SdS black hole space-times with cosmic string perturbed by a massless Dirac field are obtained using WKB approximation and found that in all these cases, decay is less in black holes having cosmic string compared to black holes without string.

Keywords Dirac field · cosmic string · quasi-normal modes

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1 Introduction

Though black holes are natural solutions of Einstein’s general theory of relativity, they are yet to be discovered. Various theoretical models to account for their discoveries have been proposed. The study of quasi-normal modes of black holes is one among them. It is proposed that topological defects, such as strings, monopoles etc might have been created in the very early universe [1]. Among the topological defects, cosmic string has been proved to be the most potential one for cosmic structure formation. The possibility of having strings in the early universe has been suggested by Kibble in 1976 [2]. Cosmic strings seem to be of particular interest because they provide a unique tool to learn the physics of the very early universe and are considered as a possible “seed” for galaxy formation [3, 4] and as a possible gravitational lens [5].
In this paper we consider black holes with the strings[6]. A black hole is distinguished by the fact that no information can escape from within the event horizon and hence the presence of black holes can be inferred only through indirect methods. The question of stability of black hole was first treated by Regge and Wheeler [7] who investigated linear perturbations of the exterior Schwarzschild space-time. Further work on this problem [8] led to the existence of quasi-normal modes(QNMs) and to the studies of the response of a black hole to external perturbations. Studies on perturbations of black holes by gravitational and matter fields have an important place in black hole physics. Since QNMs depend on black hole properties such as mass, angular momentum and charge, they allow a direct way of identifying the space-time parameters.

The QNMs of scalar perturbations around a Schwarzschild black hole pierced by a cosmic string was done earlier [9]. In the present work we study the influence of cosmic string on the QNMs of various black hole background space-times which are perturbed by a massless Dirac field. In section 2, we study the Dirac equation in a general spherically symmetric space-time with a cosmic string and its deduction into a set of second order differential equations. In Section 3 we evaluate the Dirac quasi-normal frequencies for the massless case using WKB scheme for Schwarzschild, RN extremal, SdS and near extremal SdS black hole space-times.

2 General metric for a black hole with cosmic string space-time perturbed by a Dirac field

The metric describing a spherically symmetric black hole with a cosmic string can be written as[9],

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\theta^2 + b^2r^2\sin^2\theta d\phi^2. \]  

(1)

It can be constructed by removing a wedge, which is done by requiring that the azimuthal angle around the axis runs over the range \(0 < \phi' < 2\pi b\), with \(\phi' = b\phi\) where \(\phi\) runs over zero to \(2\pi\). Here \(b = 1 - 4\hat{\mu}\) with \(\hat{\mu}\) being the linear mass density of the string. Following the procedure adopted in reference [10], we develop the Dirac equation in a general background space-time. We start with the Dirac equation;

\((\gamma^\mu(\partial_\mu - \Gamma_\mu) + m)\Psi = 0,\)  

(2)

where \(m\) is the the mass of the Dirac field. Here

\(\gamma^\mu = g^{\mu\nu}\gamma_\nu,\)  

(3)

and

\(\gamma_\nu = e^a_\nu\gamma_a,\)  

(4)

where \(\gamma^a\) are the Dirac matrices,

\(\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix},\)  

(5)
and $\sigma_i$ are the Pauli matrices. $e^a_\nu$ is the tetrad given by,

$$e^t_\nu = f^2, \quad e^r_\nu = \frac{1}{f^2}, \quad e^0_\nu = r, \quad e^\phi_\nu = b r \sin \theta.$$ (6)

The inverse of the tetrad $e^a_\nu$ is defined by,

$$g^{\mu\nu} = \eta^{ab} e^a_\mu e^b_\nu,$$ (7)

with $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$, the Minkowski metric. The spin connection $\Gamma_\mu$ is given by

$$\Gamma_\mu = -\frac{1}{2} [\gamma_a, \gamma_b] e^a_\nu e^b_\nu,$$ (8)

where $e^b_\nu = \partial_\nu e^b_\mu + \Gamma_\nu^\kappa_\mu e^b_\kappa$ is the covariant derivative of $e^b_\nu$. The spin connections for the above metric are obtained as,

$$\Gamma_t = \frac{1}{4} \gamma_1 \gamma_0 \frac{\partial f(r)}{\partial r},$$ (9)

$$\Gamma_r = 0,$$ (10)

$$\Gamma_\theta = \frac{1}{2} \gamma_1 \gamma_2 f^\frac{1}{2}(r),$$ (11)

$$\Gamma_\phi = \frac{1}{2} \gamma_2 \gamma_3 b \cos \theta + \frac{1}{2} \gamma_1 \gamma_3 b \sin \theta f^\frac{1}{2}(r).$$ (12)

Substituting the spin connections in Eq. (2) we will get,

$$\left[ -\gamma_0 \frac{\partial}{\partial t} + \gamma_1 f^\frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial f}{\partial r} \right) + \frac{\gamma_2}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2 \cot \theta} \right) + \frac{\gamma_3}{b r \sin \theta} \frac{\partial}{\partial \phi} + m \right] \psi(t, r, \theta, \phi) = 0,$$ (13)

Using the transformation $\psi(t, r, \theta, \phi) = \exp(-i E t) r_f^\frac{1}{2} \sin \theta \frac{1}{2} \chi(r, \theta, \phi)$, Eq. (13) becomes,

$$\left[ -E \frac{\partial}{\partial t} + \frac{1}{4} \gamma_0 \gamma_1 f^\frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial f}{\partial r} \right) + \frac{\gamma_1}{r} \gamma_1 \gamma_0 (\gamma_2 \frac{\partial}{\partial \theta} + \frac{\gamma_3}{b \sin \theta} \frac{\partial}{\partial \phi} ) - \frac{1}{r} \frac{\partial}{\partial \phi} - \frac{\gamma_0}{r} \right] \chi(r, \theta, \phi) = 0.$$ (14)

Dirac equation can be separated out into radial and angular parts by the following substitution,

$$\chi(r, \theta, \phi) = R(r) \Omega(\theta, \phi).$$ (15)

The angular momentum operator is introduced as, [10]

$$K_{(b)} = -i \gamma_1 \gamma_0 \left( \gamma_2 \frac{\partial}{\partial \theta} + \frac{\gamma_3}{b \sin \theta} \right) (b \sin \theta)^{-1} \frac{\partial}{\partial \phi},$$ (16)

such that,

$$K_{(b)} \Omega(\theta, \phi) = k_b \Omega(\theta, \phi),$$ (17)

where $k_b = \frac{k}{b}$ are the eigenvalues of $K_{(b)}$. Here $k$ is a positive or a negative nonzero integer with $l = \left| k + \frac{1}{2} \right| - \frac{1}{2}$, where $l$ is the total orbital angular
momentum. The cosmic string presence is codified in the eigenvalues of the 
an angular momentum operator [11]. Substituting Eqs. (15) and (17) in Eq. (14), 
we will get radial equation which contains $\gamma_0$ and $\gamma_1$. As $\gamma_0$ and $\gamma_1$ can 
be represented by 2 $\times$ 2 matrices, we write the radial factor $R(r)$ by a two 
component spinor notation,

$$R(r) = \begin{bmatrix} F \\ G \end{bmatrix}. \tag{18}$$

Then the radial equation in F an G are given by,

$$\frac{f}{dr} \frac{dG}{dr} + \frac{k}{r}G + \frac{m}{r}EF = EF, \tag{19}$$

$$\frac{f}{dr} \frac{dF}{dr} - \frac{k}{r}F + \frac{m}{r}EG = -EG. \tag{20}$$

Introducing a co-ordinate change as,

$$dr_* = \frac{dr}{f}, \tag{21}$$

Eq. (19) and Eq. (20) can be combined into a single equation as,

$$\frac{\partial}{\partial r_*} \begin{bmatrix} G \\ F \end{bmatrix} + f^* \begin{bmatrix} \frac{k}{m} & m \\ m & -k \end{bmatrix} \begin{bmatrix} G \\ F \end{bmatrix} = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix} \begin{bmatrix} G \\ F \end{bmatrix}. \tag{22}$$

Defining,

$$\begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} G \\ F \end{bmatrix}, \tag{23}$$

where for positive value of k,

$$\theta = \tan^{-1}(\frac{mr}{kb}). \tag{24}$$

Eq. (22) now becomes,

$$\frac{\partial}{\partial r_*} \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix} + f^* \sqrt\left(\frac{kb}{r} \right)^2 + m^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix} = -E \begin{bmatrix} 1 + \frac{1}{2E} \frac{fkm}{k^2m^2 + r^2} \\ \frac{fkm}{k^2m^2 + r^2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix}. \tag{25}$$

By making another change of the variable;

$$dr_* = \frac{dr}{\sqrt{\frac{fkm}{k^2m^2 + r^2}}} \tag{26}$$

Eq. (25) can be simplified to,

$$\frac{\partial}{\partial r_*} \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix} + f^* \sqrt\left(\frac{kb}{r} \right)^2 + m^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix} = E \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix}. \tag{27}$$
\[
\frac{\partial}{\partial \hat{r}^*} \left[ \hat{F} \right] + W \left[ \hat{F} \right] = E \left[ \hat{F} \right],
\]
(28)

where
\[
W = f \frac{1}{\left[ \frac{k}{b} \right]^2 + m^2} \left[ \left( \frac{\sqrt{b^2 r^2 - m^2}}{k} \right)^2 + 1 \right].
\]
(29)

Thus from Eq. (28), we will get two coupled equations for \( \hat{G} \) and \( \hat{F} \) which are given bellow,
\[
- \frac{\partial^2 \hat{F}}{\partial \hat{r}^*^2} + V_1 \hat{F} = E^2 \hat{F},
\]
(30)
\[
- \frac{\partial^2 \hat{G}}{\partial \hat{r}^*^2} + V_2 \hat{G} = E^2 \hat{G},
\]
(31)

where
\[
V_{1,2} = \pm \frac{\partial W}{\partial \hat{r}^*} + W^2.
\]
(32)

From Eq. (30) and Eq. (31), we can evaluate the quasi-normal mode frequencies for various black hole space-times. Here \( V_1 \) and \( V_2 \) are the super symmetric partners derived from the same super potential \( W \) and these potentials give same spectra of quasi-normal mode frequencies.

3 Quasi-normal mode frequencies

We shall now evaluate the quasi-normal frequencies for various black hole space-times perturbed by a massless Dirac field using WKB approximation. As \( V_1 \) and \( V_2 \) give same spectra of quasi-normal mode frequencies we avoid the subscripts and write \( V \) for the potential function. Thus, for massless case the equation for the potential given by Eq.(32) becomes,
\[
V = f \frac{\partial \left( f^{1/2} \hat{r} \right)}{\partial r} + f \left( \frac{k_b}{r} \right)^2.
\]
(33)

3.1 Schwarzschild black hole

We first consider the most simple black hole, viz., the Schwarzschild black hole for which,
\[
f(r) = \left( 1 - \frac{2M}{r} \right).
\]
(34)

Substituting the above \( f \) in Eq.(33) we get,
\[
V = \left( 1 - \frac{2M}{r} \right) \frac{\partial}{\partial r} \left( \left( 1 - \frac{2M}{r} \right)^{1/2} \frac{k_b}{r} \right) + \left( 1 - \frac{2M}{r} \right) \left( \frac{k_b}{r} \right)^2.
\]
(35)
where $M$ is the mass of the Schwarzschild black hole. The effective potential $V$ which depends on the absolute value of $k_b$, is in the form of a barrier. The peak of the barrier gets higher and higher as $|k|$ increases for fixed $b$ values. We repeat the calculation for different $b$ values ($b = 1, 0.5, 0.1$), and find that the height of the potential increases with $b$ values decreasing, i.e., the presence of the cosmic string, causes an increase in the height of the potential (Fig. 1).

To evaluate the quasi-normal mode frequencies, we use the WKB approximation [13,14,15]. This method can be accurate for both the real and imaginary parts of the frequencies for low lying modes with $n < k$, where $n$ is the mode number and $k$ is the angular momentum quantum number. The formula for complex quasi-normal mode frequencies $E$ in the WKB approximation, carried out to third order is,

$$E^2 = \left[ V_0 + \left(-2V_0''\right)^{\frac{1}{2}} \Lambda \right] - i \left(n + \frac{1}{2}\right) \left(-2V_0''\right)^{\frac{3}{2}} (1 + \Omega), \tag{36}$$

where

$$\Lambda = \frac{1}{(-2V_0'')^{1/2}} \left[ \frac{1}{8} \left( \frac{V_0^{(4)}}{V_0} \right) \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V_0''}{V_0} \right)^2 \left( 7 + 60\alpha^2 \right) \right], \tag{37}$$
\begin{align*}
\Omega &= \frac{1}{(-2V_0')} \left\{ \frac{5}{6912} \left( \frac{V_0''}{V_0} \right)^4 (77 + 188\alpha^2) - \frac{1}{384} \left( \frac{V_0'''}{V_0} \right) (51 + 100\alpha^2) \right. \\
&\quad + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0} \right)^2 (67 + 68\alpha^2) + \frac{1}{288} \left( \frac{V_0''''/V_0^{(5)}}{V_0^{(3)}} \right) (19 + 28\alpha^2) \\
&\quad \left. - \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0} \right) (5 + 4\alpha^2) \right\}. 
\end{align*}

(38)

Here

\[ \alpha = n + \frac{1}{2}, n = \begin{cases} 0, 1, 2, \ldots, & \text{Re}(E) > 0 \\
-1, -2, -3, \ldots, & \text{Re}(E) < 0, \end{cases} \]

(39)

\[ V_0^{(n)} = \frac{d^n V}{dr_*^n} |_{r_* = r_* (r_{\text{max}})}, \]

(40)

where the values of \( n \) lie in the range: \( 0 \leq n < k \). Plugging the effective potential in Eq. (35) into the formula given above, we obtain the complex quasi-normal mode frequencies for Schwarzschild black hole having cosmic string perturbed by a massless Dirac field. The values of Re\( (E) \) and Im\( (E) \) calculated for different values of \( b \) are given in Table 1.

For a fixed \( b \) value, Re\( (E) \) decreases as the mode number \( n \) increases for the same angular momentum quantum number \( k \) and |Im\( (E) \)| increases with \( n \). This indicates that quasi-normal modes with higher mode numbers decay faster than the low-lying one. The variation of mode frequencies for a fixed \( k \) and changing the \( b \) values are shown in Fig. 2. When the cosmic string effect is large, i.e. when \( b \) is small, Re\( (E) \) increases and |Im\( (E) \)| decreases for a fixed \( k \) [Table 1]. This implies that the decay is less in the case of Schwarzschild black hole having cosmic string compared to the case of black hole without string. For \( b = 1 \) case, we obtain the same results given in Ref. [10] where the quasi-normal modes of Schwarzschild black hole perturbed by a massless Dirac field was calculated.

3.2 RN extremal black hole

Now we will consider RN extremal black hole for which,

\[ f(r) = \left( 1 - \frac{r_0}{r} \right)^2. \]

(41)

Substituting the above \( f \) in Eq. (33) we will get,

\[ V = \left( 1 - \frac{r_0}{r} \right)^2 \frac{\partial}{\partial r} \left( \left( 1 - \frac{r_0}{r} \right)^2 \frac{k_b}{r} \right) + \left( 1 - \frac{r_0}{r} \right)^2 \left( \frac{k_b}{r} \right)^2. \]

(42)

Here the barrier potential \( V \) depends on the absolute value of \( k_b \) and for a fixed \( b \) value, the peak of the barrier gets higher and higher as |\( k \)| increases. We now take 3 different values for \( b (b = 1, 0.5, 0.1) \) and from Fig. 3 it is found that the presence of the cosmic string causes an increase in the peak of the potential.
Table 1 Quasi-normal frequencies of Schwarzschild black hole with cosmic string.

| $b$ | $k$ | $n$ | $ReE$  | $ImE$  |
|-----|-----|-----|--------|--------|
| 0.1 | 1   | 0   | 1.92351| -0.0962324|
|     | 2   | 0   | 3.84851| -0.0962269|
|     | 1   | 3.84584| -0.288768|
|     | 3   | 5.77297| -0.0960667|
|     |     | 1   | 5.77431| -0.290031|
|     |     | 2   | 5.768| -0.434385|
|     | 4   | 7.97476| -0.0962255|
|     |     | 1   | 7.96642| -0.286698|
|     |     | 2   | 7.96376| -0.412976|
|     |     | 3   | 7.88076| -0.73882|
|     | 5   | 9.62231| -0.0962273|
|     |     | 1   | 9.62124| -0.28869|
|     |     | 2   | 9.619| -0.481196|
|     |     | 3   | 9.6159| -0.673772|
|     | 4   | 9.6165| -0.866445|
| 0.5 | 1   | 0   | 0.378627| -0.0965424|
|     | 2   | 0   | 0.767194| -0.096276|
|     |     | 1   | 0.753657| 0.291848|
|     |     | 2   | 0.736303| -0.0962463|
|     |     | 1   | 0.73316| -0.28984|
|     |     | 2   | 0.72741| -0.485759|
|     | 4   | 1.53836| -0.0962367|
|     |     | 1   | 1.5317| -0.289257|
|     |     | 2   | 1.51881| -0.483803|
|     |     | 3   | 1.50043| -0.680554|
|     | 5   | 0   | 1.92351| -0.0962324|
|     |     | 1   | 1.91818| -0.289147|
|     |     | 2   | 1.91013| -0.482862|
|     |     | 3   | 1.89259| -0.68198|
|     |     | 4   | 1.87327| -0.87639|
| 1   | 1   | 0   | 0.176452| -0.106109|
|     | 2   | 0   | 0.378627| -0.0965424|
|     |     | 1   | 0.353604| -0.298746|
|     |     | 3   | 0.573685| -0.0963242|
|     |     | 1   | 0.556186| -0.292861|
|     |     | 2   | 0.527289| -0.497187|
|     |     | 4   | 0.767194| -0.096276|
|     |     | 1   | 0.753657| -0.291848|
|     |     | 2   | 0.736303| -0.485759|
|     |     | 3   | 0.72741| -0.68198|
|     | 5   | 0   | 0.960215| -0.0962564|
|     |     | 1   | 0.949593| -0.290179|
|     |     | 2   | 0.929979| -0.487634|
|     |     | 3   | 0.903578| -0.689241|
|     | 4   | 0.872052| -0.894412|

When the effective potential given in Eq. (42) is substituted in Eq. (36), we will get the complex quasi-normal mode frequencies of RN extremal black hole perturbed by a massless Dirac field [Table 1].

Here also we find, for a fixed $b$ value, $Re(E)$ decreases while $|Im(E)|$ increases with the mode number $n$ increasing for the same angular momentum.
Fig. 2 Quasi-normal modes of Schwarzschild black hole with cosmic string perturbed by a massless Dirac field is plotted for different values of $b$ ($b = 0.1, 0.5,$ and $1$) for $k = 5$.

Fig. 3 Potential versus $r$ for RN extremal black hole with cosmic string in Dirac field is plotted for different values of $b$ ($b = 1, 0.5, 0.1$) for $k = 0$ to $k = 5$.

eigenvalue $k$. This means that quasi-normal modes with higher mode numbers decay faster than the low-lying ones. For a fixed $k$, the variation of mode frequencies with $b$ values are shown in Fig. 4. When $b < 1$, i.e., when the cosmic string is present, $Re(E)$ increases and $|Im(E)|$ decreases for a fixed $k$, compared to the $b = 1$ case [Table 2]. Thus compared to RN extremal black hole, the decay is less in the case of RN extremal black hole having cosmic string.
Table 2 Quasi-normal modes of RN extremal black hole having Cosmic string

| b  | k  | n  | ReE  | ImE   |
|----|----|----|------|-------|
| 0.1| 1  | 0  | 1.24941 | -0.0441915 |
| 2  | 2  | 3.7498 -0.132939 |
| 3  | 1  | 3.74902 -0.132939 |
| 2  | 3.74746 -0.221018 |
| 4  | 4.99985 -0.044194 |
| 1  | 4.99927 -0.132588 |
| 2  | 4.9981 -0.220998 |
| 3  | 4.99634 -0.309835 |
| 5  | 6.24988 -0.0441941 |
| 1  | 6.24941 -0.132986 |
| 2  | 6.24848 -0.220988 |
| 3  | 6.24707 -0.309408 |
| 4  | 6.2452 -0.397852 |
| 0.5| 1  | 0  | 0.246679 -0.044139 |
| 2  | 0  | 0.398474 -0.0441761 |
| 1  | 0.492516 -0.131115 |
| 3  | 0.740004 -0.0441865 |
| 1  | 0.745065 -0.132812 |
| 2  | 0.742055 -0.221430 |
| 4  | 0.745299 -0.309835 |
| 1  | 0.986315 -0.13271 |
| 2  | 0.990498 -0.221638 |
| 3  | 0.981942 -0.31209 |
| 5  | 1.24941 -0.0441915 |
| 1  | 1.24706 -0.132664 |
| 2  | 1.24399 -0.221398 |
| 3  | 1.23848 -0.305554 |
| 4  | 1.22643 -0.402669 |
| 1  | 0.177481 -0.0441736 |
| 2  | 0.246679 -0.0441391 |
| 1  | 0.234639 -0.135043 |
| 3  | 0.37291 -0.0441622 |
| 1  | 0.36905 -0.133523 |
| 2  | 0.350195 -0.225851 |
| 4  | 0.498474 -0.0441761 |
| 1  | 0.492516 -0.131115 |
| 2  | 0.481162 -0.130162 |
| 3  | 0.469287 -0.316232 |
| 5  | 0.329396 -0.044132 |
| 1  | 0.490633 -0.132918 |
| 2  | 0.60859 -0.22276 |
| 3  | 0.596707 -0.313914 |
| 4  | 0.580161 -0.306822 |

3.3 Schwarzschild-de Sitter black hole

We will now take the case of SdS black hole and see how the quasi-normal modes are affected when the cosmic string pierces a SdS black hole. The
The space-time metric is,

\[ f(r) = 1 - \frac{2M}{r} - \frac{r^2}{a^2}, \quad (43) \]

where \( M \) denotes the black hole mass and \( a^2 = \frac{3}{\Lambda} \), \( \Lambda \) being the cosmological constant. The space-time possesses two horizons: the black-hole horizon at \( r = r_b \) and the cosmological horizon at \( r = r_c \). The function \( f \) has zeros at \( r_b, r_c \) and \( r_0 = -(r_b + r_c) \). Substituting the above \( f \) in Eq.\( (33) \), we will get

\[ V = \left(1 - \frac{2M}{r} - \frac{r^2}{a^2}\right) \frac{\partial}{\partial r} \left(\left(1 - \frac{2M}{r} - \frac{r^2}{a^2}\right)^{\frac{1}{2}} k_b\right) + \left(1 - \frac{2M}{r} - \frac{r^2}{a^2}\right) \left(\frac{k_b}{r}\right)^2. \quad (44) \]

We consider three values of \( b (b = 1, 0.5, 0.1) \) and from Fig.\( 5 \) we can see that the height of the potential increases with \( b \) decreasing.

By substituting Eq.\( (44) \) in Eq.\( (36) \), we obtain the complex quasi-normal modes for SdS black hole with cosmic string perturbed by a massless Dirac field. Quasi-normal modes of SdS black hole with cosmic string for various \( b \) values are shown in Table 3.

Here for a fixed \( b \) value, \( Re(E) \) and \( |Im(E)| \) show similar behavior as those of Schwarzschild and RN extremal black holes having cosmic string. The variation of mode frequencies for a fixed \( k \) with different \( b \) values are shown in Fig. 6. From Table 3 we can see that the behavior of \( |Im(E)| \) for \( n = 0 \) mode is different from the behavior of \( |Im(E)| \) for non zero values of \( n \). From this table we can see that the decay is less in the case of SdS black hole having cosmic string.
3.4 Near extremal Schwarzschild-de Sitter black hole.

As a last example, we consider near extremal SdS black hole, which is defined as the space-time for which the cosmological horizon $r_c$ is very close to black hole horizon $r_b$, i.e., \( \frac{r_c}{r_b} \ll 1 \). For this space-time, one can make the following approximations,

\[
r_0 \sim 2r_b; \quad a^2 \sim 3r_b^2; \quad M \sim \frac{r_b}{3}.
\]
Table 3 Quasi-normal frequencies of SdS black hole with cosmic string

| b  | k | n  | Re E | Im E |
|----|---|----|------|------|
| 0.1| 1 | 0  | 1.32881 | -0.193788 |
| 2  | 1 | 3.20515 | -0.193823 |
| 3  | 1 | 4.91537 | -0.192659 |
| 2  | 1 | 4.89017 | -0.196325 |
| 4  | 1 | 6.56408 | -0.192473 |
| 2  | 1 | 6.5452 | -0.196232 |
| 3  | 1 | 0.184451 | -0.199973 |
| 2  | 0 | 0.611882 | -0.191711 |
| 3  | 0 | 0.506702 | -0.602245 |
| 1  | 0 | 0.118178 | -0.329652 |
| 3  | 0 | 0.18445 | -0.179973 |
| 2  | 0 | 0.0172463 | -0.686957 |
| 3  | 0 | 0.424791 | -0.187411 |
| 2  | 0 | 0.18445 | -0.179973 |
| 1  | 0 | 0.283273 | -0.623849 |
| 2  | 0 | 0.0808308 | -1.10559 |
| 3  | 0 | 0.611882 | -1.683538 |
| 2  | 1 | 0.506702 | -0.602245 |
| 1  | 0 | 0.118178 | -0.329652 |
| 3  | 0 | 0.18445 | -0.179973 |
| 2  | 0 | 0.0172463 | -0.686957 |
| 3  | 0 | 0.424791 | -0.187411 |
| 2  | 0 | 0.18445 | -0.179973 |
| 1  | 0 | 0.283273 | -0.623849 |
| 2  | 0 | 0.0808308 | -1.10559 |
| 3  | 0 | 0.611882 | -1.683538 |
| 2  | 1 | 0.506702 | -0.602245 |
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| 3  | 0 | 0.18445 | -0.179973 |
| 2  | 0 | 0.0172463 | -0.686957 |
| 3  | 0 | 0.424791 | -0.187411 |
| 2  | 0 | 0.18445 | -0.179973 |
| 1  | 0 | 0.283273 | -0.623849 |
| 2  | 0 | 0.0808308 | -1.10559 |
| 3  | 0 | 0.611882 | -1.683538 |

Furthermore, since \( r \) is constrained to vary between \( r_b \) and \( r_c \), we get \( r - r_0 \sim r_b - r_0 \sim 3r_b \) and thus

\[
\mathcal{f} \sim \frac{(r - r_b)(r_c - r)}{r_b^2}.
\]  
(46)
Substituting Eq. (46) in Eq. (33), we get,

\[ V = \frac{(r - r_b)(r_c - r)}{r_b^2} \partial_r \left( \frac{(r - r_b)(r_c - r)}{r_b^2} \right)^{\frac{1}{2}} k_b r + \left( \frac{(r - r_b)(r_c - r)}{r_b^2} \right) \left( \frac{k_b}{r} \right)^2. \]  

(47)

From Fig. 7 we can see that the height of the potential increases when the effect of cosmic string increases.

By substituting Eq. (47) in Eq. (36) we obtain the quasi-normal modes of near extremal SdS black hole with cosmic string perturbed by a massless Dirac field. For a near extremal SdS black hole also the quasi-normal frequencies are obtained for different values of \( b \) [Table 4].

We can see that for a fixed \( b \) value, \( \text{Re}(E) \) remains same and \( |\text{Im}(E)| \) increases as the mode number \( n \) increases for the same \( k \) value. The behavior of mode frequencies by changing the \( b \) values for a fixed \( k \) are shown in Fig. 8. When \( b \) is small, i.e., when the effect of cosmic string is high, \( \text{Re}(E) \) increases while \( |\text{Im}(E)| \) have almost same but with very small decreases for a mode of fixed \( k \) value [Table 4]. The decay is less in the case of near extremal SdS black hole having cosmic string.

4 Conclusion

We have evaluated the quasi-normal mode frequencies for Schwarzschild, RN extremal, SdS and near extremal SdS black hole space-times having cosmic string perturbed by a massless Dirac field. In all these cases, we have found
Table 4 Quasi-normal frequencies of SdS near extremal black hole with cosmic string

| b  | k  | n  | ReE       | ImE       |
|----|----|----|-----------|-----------|
| 0.1| 1  | 0  | 0.0349689 | -0.001875 |
|    | 2  | 0  | 0.0699379 | -0.001875 |
|    |    | 1  | 0.0699379 | -0.005625 |
|    | 3  | 0  | 0.104901  | -0.001875 |
|    |    | 1  | 0.104901  | -0.005625 |
|    |    | 2  | 0.104901  | -0.009375 |
|    | 4  | 0  | 0.139876  | -0.001875 |
|    |    | 1  | 0.139876  | -0.005625 |
|    |    | 2  | 0.139876  | -0.009375 |
|    |    | 3  | 0.139876  | -0.013125 |
|    | 5  | 0  | 0.174845  | -0.001875 |
|    |    | 1  | 0.174845  | -0.005625 |
|    |    | 2  | 0.174845  | -0.009375 |
|    |    | 3  | 0.174845  | -0.013125 |
|    |    | 4  | 0.174845  | -0.016875 |
| 0.5| 1  | 0  | 0.0069951 | 0.00187539 |
|    | 2  | 0  | 0.0139876 | -0.00187594 |
|    |    | 1  | 0.0139876 | -0.00562519 |
|    |    | 3  | 0.0209814 | -0.00187591 |
|    |    |    | 0.0209814 | -0.00562504 |
|    |    | 2  | 0.0209814 | -0.00937506 |
|    | 4  | 0  | 0.027951  | -0.001875 |
|    |    | 1  | 0.027951  | -0.00562501 |
|    |    | 2  | 0.027951  | -0.00937502 |
|    |    | 3  | 0.027951  | -0.013125 |
|    | 5  | 0  | 0.0349689 | -0.001875 |
|    |    | 1  | 0.0349689 | -0.00562501 |
|    |    | 2  | 0.0349689 | -0.00937501 |
|    |    | 3  | 0.0349689 | -0.013125 |
|    |    | 4  | 0.0349689 | -0.016875 |
| 1  | 1  | 0  | 0.0034864 | -0.00188365 |
|    | 2  | 0  | 0.0069934 | -0.00187594 |
|    |    | 1  | 0.0069934 | -0.00562583 |
|    |    | 3  | 0.0104906 | -0.00187521 |
|    |    |    | 0.0104906 | -0.00562531 |
|    |    | 2  | 0.0104911 | -0.00937551 |
|    |    | 4  | 0.0139876 | -0.00187504 |
|    |    |    | 0.0139876 | -0.00562519 |
|    |    | 2  | 0.0139877 | -0.00937525 |
|    |    | 3  | 0.0139878 | -0.0131253 |
|    | 5  | 0  | 0.0174845 | -0.00187503 |
|    |    | 1  | 0.0174845 | -0.00562508 |
|    |    | 2  | 0.0174845 | -0.00937512 |
|    |    | 3  | 0.0174846 | -0.0131251 |
|    |    | 4  | 0.0174846 | -0.0168751 |
that quasi-normal modes with higher mode numbers decay faster than the low-lying ones. We have also found that, when the effect of cosmic string is high, |Im(E)| decreases while Re(E) increases for fixed $k$ implying that the decay is less when cosmic string is present.

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