ORDERS OF QUATERNION ALGEBRAS WITH INVOLUTION

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Abstract. We introduce the notion of maximal orders over quaternion algebras with orthogonal involution and give a classification over local and global fields. Over local fields, we show that there is a correspondence between maximal and/or modular lattices and orders closed under the involution.

1. Introduction:

The study of maximal orders of quaternion algebras has a rich history, going back to the work of Hurwitz in the 1800s. Finding such orders is essentially a solved problem. Indeed, an explicit algorithm is given in [IR93] that constructs maximal orders in a semisimple algebra over an algebraic number field in polynomial time, with a simpler algorithm given in [Voi13] for the special case of quaternion algebras. It is a straightforward exercise to check that any order of a quaternion algebra is closed under the standard involution, i.e. quaternion conjugation. However, given an arbitrary involution \( \dagger \) on a quaternion algebra \( H \) and an order \( O \subset H \), it need not be the case that \( O = O^\dagger \). In the special case where \( O = O^\dagger \), we shall call \( O \) a \( \dagger \)-order. In this paper, we characterize \( \dagger \)-orders over local and global fields—in the local case, we show that either the quaternion algebra is a division algebra or there is a correspondence between maximal or modular lattices and \( \dagger \)-orders (see Theorems 4.1, 8.1, and 8.2).

The interest in this subject comes from the problem of constructing certain arithmetic sphere packings in \( \mathbb{R}^3 \). In \( \mathbb{R}^2 \), one can construct [Sta14] [Sta15] circle packings by considering the action of a Bianchi group \( \text{SL}(2, \mathcal{O}_K) \) on the real line, where \( \mathcal{O}_K \) is the ring of integers of some imaginary quadratic field \( K \). This same method can be used to produce interesting sphere packings in \( \mathbb{R}^3 \), but requires replacing the Bianchi group \( \text{SL}(2, \mathcal{O}) \) with an appropriate analog \( \text{SL}^\dagger(2, \mathcal{O}) \), where \( \mathcal{O} \) is a maximal \( \dagger \)-order (see [She]).

For this reason, we shall be primarily interested in maximal \( \dagger \)-orders—that is, \( \dagger \)-orders not properly contained inside any other \( \dagger \)-order. Since any order is contained inside a maximal order, it is easy to see that any maximal \( \dagger \)-order must be of the form \( \mathcal{O} \cap \mathcal{O}^\dagger \), where \( \mathcal{O} \) is maximal. Therefore, all maximal \( \dagger \)-orders are Eichler orders—that is, the intersection of two (not necessarily distinct) maximal orders. However, it is not the case that all Eichler orders of the form \( \mathcal{O} \cap \mathcal{O}^\dagger \) are maximal.

As a first example, consider the two maximal orders

\[
\mathcal{O}_1 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z} \frac{5 + 5i + 3j - ij}{10} \subset \left( \frac{-1, -5}{\mathbb{Q}} \right)
\]
\[
\mathcal{O}_2 = \mathbb{Z} \oplus 9\mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z} \frac{90 + 385i - 63j + ij}{90} \subset \left( \frac{-1, -5}{\mathbb{Q}} \right).
\]

Define an involution on \( \left( \frac{-1, -5}{\mathbb{Q}} \right) \) by

\[(w + xi + yj + zij)^\dagger = w + xi + yj - zij.\]

Then it is an easy computation that

\[
\mathcal{O}_1 \cap \mathcal{O}_1^\dagger = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z} \frac{1 + i + j + ij}{2}
\]
\[
\mathcal{O}_2 \cap \mathcal{O}_2^\dagger = \mathbb{Z} \oplus 9\mathbb{Z}i \oplus j \oplus 9\mathbb{Z} \frac{1 + i + j + ij}{2},
\]
which shows that $\mathcal{O}_2 \cap \mathcal{O}^\mathcal{J}$ is not a maximal $\mathcal{J}$-order. What is less evident is that $\mathcal{O}_1 \cap \mathcal{O}^\mathcal{J}$ is a maximal $\mathcal{J}$-order. As we shall describe presently, for an Eichler order $\mathcal{O} \cap \mathcal{O}^\mathcal{J}$ to be maximal, there is a necessary and sufficient restriction on the discriminant of the order.

Let $F$ be a local or global field of characteristic not 2, and $o$ its ring of integers. If $H$ is quaternion algebra over $F$ with an orthogonal involution $\mathcal{J}$, we denote by $\text{disc}(\mathcal{J}) \in F^\times / (F^\times)^2$ the discriminant of $\mathcal{J}$ (see section 2 for definitions). Unlike the discriminant $\text{disc}(H)$ of the quaternion algebra, this is not an ideal of $o$, but we can associate an ideal to $\text{disc}(\mathcal{J})$ by defining a map

$$\iota : F^\times / (F^\times)^2 \rightarrow \{\text{square-free ideals of } o\}$$

$$[\lambda] \mapsto \bigcup_{\lambda \in [\lambda] \cap o} \lambda o.$$

With these conventions, our main result is the following simple statement.

**Theorem 1.1.** Given a quaternion algebra $H$ over a local or global field $F$, the maximal $\mathcal{J}$-orders of $H$ correspond to Eichler orders of the form $\mathcal{O} \cap \mathcal{O}^\mathcal{J}$ with discriminant

$$\text{disc}(H) \cap \iota(\text{disc}(\mathcal{J})).$$

Note that it is sufficient to prove that all maximal $\mathcal{J}$-orders have discriminants of the desired form—if $\mathcal{O} \cap \mathcal{O}^\mathcal{J}$ is an Eichler order, it must be contained inside a maximal $\mathcal{J}$-order, and if their discriminants match, they must be the same.

In the process of proving Theorem 1.1, we establish a far more explicit classification of maximal $\mathcal{J}$-orders over local fields. In particular, we show that if $H = \text{End}(V)$, where $V$ is a 2-dimensional quadratic space over a local field, then there is a correspondence between either maximal or modular lattices in $V$ and maximal $\mathcal{J}$-orders of $H$.

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**2. Preliminaries:**

We recall some basic facts and conventions about involutions over central simple algebras. Let $H$ be a central simple algebra of dimension $n^2$ over a field $F$ (with $\text{char}(F) \neq 2$). Recall that an involution (of the first type) is an $F$-linear map $\mathcal{J}$ on $H$ such that

1. $(xy)^\mathcal{J} = y^\mathcal{J}x^\mathcal{J}$ $(\forall x, y \in H)$, and
2. $\mathcal{J}^2 = \text{id}$.

A homomorphism of two algebras with involution $(H_1, \mathcal{J}_1)$ and $(H_2, \mathcal{J}_2)$ is an $F$-algebra homomorphism $\phi : H_1 \rightarrow H_2$ such that

$$\phi(\sigma^{\mathcal{J}_1}) = \phi(\sigma)^{\mathcal{J}_2} \quad (\forall \sigma \in H_1).$$

This is the correct notion of homomorphism for our purposes, since isomorphisms of algebras with involutions

1. send $\mathcal{J}$-orders to $\mathcal{J}$-orders,
2. send Eichler orders of the form $\mathcal{O} \cap \mathcal{O}^\mathcal{J}$ to Eichler orders of the same form, and
3. preserve the discriminant.

Note that $H$ decomposes as a vector space as $H^+ \oplus H^-$, where

$$H^+ = \{x \in H \mid x^\mathcal{J} = x\}$$

$$H^- = \{x \in H \mid x^\mathcal{J} = -x\}.$$
It is well known that involutions on $H$ split into two basic types:

1. **Symplectic involutions**, for which $\dim H^+ = \frac{n(n-1)}{2}$ and $\dim H^- = \frac{n(n+1)}{2}$, and
2. **Orthogonal involutions**, for which $\dim H^+ = \frac{n(n+1)}{2}$ and $\dim H^- = \frac{n(n-1)}{2}$.

In the particular case where $H$ is a quaternion algebra, the only symplectic involution is the standard involution of quaternion conjugation—all other involutions are orthogonal.

For a given orthogonal involution $\dagger$, we define the group

$$GO(H, \dagger) = \{ x \in H \times | x \dagger x \in F \}.$$

There is an exact sequence

$$0 \longrightarrow \{ \pm 1 \} \longrightarrow GO(H, \dagger) \longrightarrow \left\{ \text{automorphisms of } (H, \dagger) \right\} \longrightarrow 0,$$

and so we simply identify $PGO(H, \dagger) = GO(H, \dagger)/\{ \pm 1 \}$ with the automorphism group of $(H, \dagger)$.

Finally, the discriminant of an orthogonal involution is defined as

$$\text{disc}(\dagger) = \frac{\text{nd}(h)}{2} \cdot (F^\times)^2 \in F^\times / (F^\times)^2,$$

where $h$ is any element of $H^-$, and $\text{nd}(h) = \overline{h}h$ is the standard norm map. Orthogonal involutions are classified by the discriminant—that is, given a quaternion algebra $H$ and two orthogonal involutions $\dagger_1, \dagger_2$ on $H$ such that $\text{disc}(\dagger_1) = \text{disc}(\dagger_2)$, there is an isomorphism $\phi: (H, \dagger_1) \rightarrow (H, \dagger_2)$.

### 3. Localization:

We note that to prove Theorem 1.1 it is in fact sufficient to prove it for local fields. To see this, $F$ be a global field, let $\Omega$ denote the set of places of $F$, and $\Omega_f, \Omega_\infty$ denote the finite and infinite places of $F$ respectively. Given a quaternion algebra with involution $(H, \dagger)$, we can consider the localizations

$$H_\nu = H \otimes_F F_\nu$$

$$(h \otimes t)^\dagger_\nu = h^\dagger \otimes t$$

for any $\nu \in \Omega$. We can also define

$$O_p = O \otimes_O \mathfrak{O}_p$$

for any $p \in \Omega_f$. It is easy to check that localization sends $\dagger$-orders to $\dagger_p$-orders, sends Eichler orders to Eichler orders, preserves maximality, and

$$\text{disc}(H) = \prod_{\nu \in \Omega} \text{disc}(H_\nu)$$

$$\iota(\text{disc}(\dagger)) = \prod_{\nu \in \Omega} \iota(\text{disc}(\dagger_\nu)).$$

Therefore, if Theorem 1.1 holds for all of the localizations $H_p$, it must hold for $H$ itself. Thus, henceforth we shall assume that $F$ is a local field over a place $p$ with uniformizer $\pi$.

For any $x \in F$, we shall denote by $ord_p(x)$ the smallest integer $n$ such that $x \in p^n$. 
4. THE DIVISION ALGEBRA CASE:

For any quaternion algebra over a field \( F \), it is either a division algebra, or is isomorphic to \( Mat(2, F) \) (in which case it is called split). We start by considering the division algebra case, which turns out to be especially simple.

**Theorem 4.1.** Let \( F \) be a local field, and suppose \( H \) is a division algebra. Then there is a unique maximal \( \mathfrak{h} \)-order, given by

\[
O = \{ h \in H | nrd(h) \in o \}.
\]

The order \( O \) is also maximal and furthermore,

\[
disc(O) = p = disc(H) \cap i(disc(\mathfrak{h})).
\]

**Proof.** It is well known that \( O \) is the unique maximal order of \( H \). Consequently, it must be the unique maximal \( \mathfrak{h} \)-order. It is also easy to check that

\[
disc(O) = p = disc(H).
\]

Since \( i(disc(\mathfrak{h})) \) is a square-free ideal in a local field, it is either \( o \) or \( p \). In either case, the theorem holds. \( \square \)

5. SPLIT QUATERNION ALGEBRAS:

It remains to prove Theorem 4.1 for the case where \( H \) is split. We have that \( H \cong End(V) \), where \( V \) is any 2-dimensional vector space over \( F \). We start with a basic observation that we shall use for easy computations of the discriminant.

**Lemma 5.1.** Let \( H \) be a split quaternion algebra over a local field \( F \) with involution \( \mathfrak{h} \). Choose \( \lambda \in disc(\mathfrak{h}) \) such that \( \lambda \) is a generator of \( i(disc(\mathfrak{h})) \). Then \( (H, \mathfrak{h}) \) is isomorphic to \( Mat(2, F) \) with the involution \( \mathfrak{h}_\lambda \) given by

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & c/\lambda \\ b \lambda & d \end{array} \right).
\]

**Proof.** Since \( H \cong Mat(2, F) = End(F^2) \) as algebras, and involutions on algebras are classified by their discriminant, it shall suffice to prove that \( disc(\mathfrak{h}_\lambda) = disc(\mathfrak{h}) \). But

\[
\left( \begin{array}{cc} 0 & 1 \\ -\lambda & 0 \end{array} \right) = -\left( \begin{array}{cc} 0 & 1 \\ -\lambda & 0 \end{array} \right),
\]

and the norm is

\[
det \left( \begin{array}{cc} 0 & 1 \\ -\lambda & 0 \end{array} \right) = \lambda
\]

and therefore

\[
disc(\mathfrak{h}_\lambda) = \lambda (F^\times)^2 = disc(\mathfrak{h}),
\]

as claimed. \( \square \)

With Lemma 5.1 in mind, we shall now rephrase the problem in terms of quadratic forms. Given any non-singular bilinear form \( b \) on \( V \), it is a simple computation that there is a unique involution \( \mathfrak{h}_b \) of \( End(V) \) defined by the property

\[
b(v, \sigma w) = b(\sigma^{\mathfrak{h}_b} v, w) \quad (\forall v, w \in V, \sigma \in End(V)).
\]
It is easily checked that scaling the bilinear form $b$ on $V$ does not change the involution $\dagger$. However, it is a standard result (see [KMRT98, p.1]) that $[b] \mapsto \dagger b$ defines a bijection between symmetric bilinear forms (defined up to scaling) and orthogonal involutions. Thus, we henceforth fix a symmetric bilinear form $b$ on $V$ such that the corresponding involution is the orthogonal involution of $A$. We denote the associated quadratic form $q$. To prove Theorem 1.1, we shall eventually need to consider the non-dyadic and dyadic places of $F$ separately. First, however, we give some results that hold generally.

**Lemma 5.2.**

$$GO(\text{End}(V), \dagger) = \bigcup_{l=0}^{1} \{ g \in GL(V) | q_{\dagger}(gv) = (-1)^l \det(g)q_{\dagger}(v) \forall v \in V \}.$$  

**Proof.** Let $g \in GO(\text{End}(V), \dagger)$ and $v \in V$. Then

$$q_{\dagger}(gv) = b_{\dagger}(gv, gv)$$

$$= b_{\dagger}(g^2gv, v)$$

$$= g^2gb_{\dagger}(v, v)$$

$$= \pm \det(g)q_{\dagger}(v),$$

hence

$$GO(\text{End}(V), \dagger) \subset \bigcup_{l=0}^{1} \{ g \in GL(V) | q_{\dagger}(gv) = (-1)^l \det(g)q_{\dagger}(v) \forall v \in V \}.$$  

On the other hand, if $q_{\dagger}(gv) = \pm \det(g)q_{\dagger}(v)$ for all $v \in V$, then $g$ preserves the quadratic form $q_{\dagger}$ up to scaling, hence it preserves the bilinear form $b_{\dagger}$ up to scaling. Since there is a bijective correspondence between bilinear forms (up to scaling) and orthogonal involutions on $\text{End}(V)$, we conclude that $g$ in fact preserves $\dagger$. \qed

Next, to easily describe Eichler orders $O \cap O^\dagger$ in $\text{End}(V)$, we shall need to consider $\mathfrak{o}$-lattices in the quadratic space $V$. For a given lattice $\Lambda \subset V$, we denote its dual lattice by

$$\Lambda^\sharp = \{ v \in V | b_{\dagger}(v, \Lambda) \subset \mathfrak{o} \}.$$  

Given a lattice $\Lambda$, we have a corresponding order in $\text{End}(V)$, defined by

$$\text{End}(\Lambda) = \{ \sigma \in \text{End}(V) | \sigma(\Lambda) \subset \Lambda \}.$$  

Note that scaling the lattice $\Lambda$ does not change the order.

**Lemma 5.3.** We have a map

$$\varphi : \{ \text{lattices in } V \text{ up to scaling} \} \rightarrow \{ \text{orders of } \text{End}(V) \}$$

$$[\Lambda] \mapsto \text{End}(\Lambda) \cap \text{End}(\Lambda^\sharp)$$

that surjects onto the subset of Eichler orders of the form $O \cap O^\dagger$.

**Proof.** The proof is based on two observations. First, over a local field, $\mathfrak{o}$ is a PID, and it follows that all maximal orders are of the form $\text{End}(\Lambda)$ for some lattice $\Lambda$. Secondly, $\text{End}(\Lambda)^\sharp = \text{End}(\Lambda^\sharp)$, since
We recall the definitions of modular and maximal lattices. The scale, norm, and volume of a lattice $\Lambda$ are the fractional ideals

$$s\Lambda = \bigcup_{v_1, v_2 \in \Lambda} b_t(v_1, v_2)\mathfrak{o},$$
$$n\Lambda = \bigcup_{v \in \Lambda} q_t(v)\mathfrak{o},$$
$$v\Lambda = \bigcup_{\Lambda = \mathfrak{o}v_1 \perp \mathfrak{o}v_2} \det(b_t(v_i, v_j))\mathfrak{o},$$

respectively. Let $a$ be a fractional ideal of $F$. Then $\Lambda$ is $a$-modular if $\Lambda = a\Lambda^\sharp$—equivalently, if $s\Lambda = a$ and $v\Lambda = a^2$. Also, $\Lambda$ is $a$-maximal if $n\Lambda \subset a$ and for any other lattice $\Lambda' \supset \Lambda$

$$n\Lambda' \subset a \Rightarrow \Lambda = \Lambda'.$$

Isomorphisms of algebras with involution respect maximality—that is, they send maximal lattices to maximal lattices.

**Lemma 5.4.** Let $\phi \in \text{PGO} \left(\text{End}(V), \sharp\right)$, and let $\Lambda$ be an $a$-maximal lattice. Then there exists $\lambda \in F^\times$ and a $\lambda a$-maximal lattice $\Lambda'$ such that

$$\phi \left(\text{End}(\Lambda) \cap \text{End}(\Lambda^\sharp)\right) = \text{End}(\Lambda') \cap \text{End}(\Lambda'^\sharp).$$

Conversely, if $\Lambda, \Lambda'$ are $a$-maximal lattices, then there exists $\phi \in \text{PGO} \left(\text{End}(V), \sharp\right)$ such that

$$\phi \left(\text{End}(\Lambda) \cap \text{End}(\Lambda^\sharp)\right) = \text{End}(\Lambda') \cap \text{End}(\Lambda'^\sharp).$$

**Proof.** Choose a representative $g \in \text{GO}(H, \sharp)$ such that $\phi(\sigma) = g\sigma g^{-1}$. Then

$$\phi \left(\text{End}(\Lambda) \cap \text{End}(\Lambda^\sharp)\right) = g\text{End}(\Lambda)g^{-1} \cap g\text{End}(\Lambda)^{\sharp}g^{-1}$$

$$= \text{End}(g\Lambda) \cap \text{End}(g\Lambda)^{\sharp}$$

$$= \text{End}(g\Lambda) \cap \text{End} \left((g\Lambda)^\sharp\right).$$

It is straightforward to see that if $\Lambda$ is $a$-maximal, then $g\Lambda$ is $\det(g)a$-maximal.

On the other hand, if $\Lambda, \Lambda'$ are both $a$-maximal lattices over a local field, then there is an isomorphism of quadratic spaces $g : V \rightarrow V$ such that $\Lambda = g\Lambda'$. We have the desired the isomorphism

$$\phi(\sigma) = g\sigma g^{-1}.\quad \square$$

Finally, we also need the following technical lemma.
Lemma 5.5. Let $H = \text{Mat}(2, F)$ with the involution $\frac{1}{2} \lambda$. Suppose $\Lambda$ is a lattice in $F^2$ with an orthogonal basis. Then there exists $g \in G\Omega(\text{Mat}(2, F), \frac{1}{2} \lambda)$ such that

$$\text{End}(\Lambda) \cap \text{End}(\Lambda^\perp) \subset g(\text{Mat}(2, o) \cap \text{Mat}(2, o)^\perp) g^{-1}.$$  

Proof. We are given that $\Lambda = oe_1 \perp oe_2$. Therefore, we know that the Gram matrix $(b_t(e_i, e_j))_{ij}$ can be assumed to be of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

Note that if $e_1 = (a, b), e_2 = (c, d)$, then

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} a^2 \lambda + c^2 & cd + ab \lambda \\ cd + ab \lambda & b^2 \lambda + a^2 \end{pmatrix},$$

and so

$$\lambda x/y = \frac{a^2 \lambda^2 + c^2 \lambda}{b^2 \lambda + a^2} = \frac{a^2 b^2 \lambda^2 + b^2 c^2 \lambda}{b^2 (b^2 \lambda + a^2)} = \frac{c^2 \epsilon^2 + b^2 c^2 \lambda}{b^2 (b^2 \lambda + a^2)} = \frac{c^2}{b^2} \in (F^\times)^2.$$  

Consequently, the Gram matrix must be of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \lambda \pi^{2n} z \epsilon & 0 \\ 0 & \epsilon \end{pmatrix},$$

where $z \in o, \epsilon \in o^{\times^2}$ and $n \in \mathbb{Z}$. In fact, we can assume that $\epsilon = 1$, since we are free to scale $e_2$ by an element of $D^\times$, and this scales $\sigma_t(e_2)$ by an element of $o^{\times^2}$.  

However, the Gram matrix defines the lattice $\Lambda$ up to isomorphism of quadratic spaces, which we know lift to isomorphisms of $\frac{1}{2}$-orders. Therefore, it suffices to construct a single lattice for each Gram matrix. This is done explicitly by

$$\Lambda = o\pi^n (a, -b\lambda) \oplus o\pi^n (b, a)$$

where

$$z = a^2 + b^2 \lambda = q_t((a, b)).$$

If we define a matrix

$$L = \begin{pmatrix} a & b \\ -b\lambda & a \end{pmatrix} \begin{pmatrix} \pi^n & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$\sigma \in \text{End}(\Lambda) \iff \sigma \in L\text{Mat}(2, o)L^{-1}.$$  

However,
\[
\left( \begin{array}{cc}
    a & b \\
    -b\lambda & a
\end{array} \right)^{\lambda} \left( \begin{array}{cc}
    a & b \\
    -b\lambda & a
\end{array} \right) = (a^2 + b^2\lambda)I
\]
so
\[
\left( \begin{array}{cc}
    a & b \\
    -b\lambda & a
\end{array} \right) \in GO(Mat(2,F),\mathcal{T}_\lambda).
\]

As we remarked earlier, conjugating by elements of \( GO(Mat(2,F),\mathcal{T}_\lambda) \) corresponds to isomorphism of \( \mathcal{T} \)-orders, so we may assume that
\[
L = \left( \begin{array}{cc}
    \pi^n & 0 \\
    0 & 1
\end{array} \right),
\]
and therefore
\[
End(\Lambda) = \left( \begin{array}{cc}
    \pi^n & 0 \\
    0 & 1
\end{array} \right)\cdot Mat(2,\mathcal{O}) \left( \begin{array}{cc}
    \pi^n & 0 \\
    0 & 1
\end{array} \right)^{-1},
\]
\[
End(\Lambda^\dagger) = \left( \begin{array}{cc}
    \pi^n & 0 \\
    0 & 1
\end{array} \right)^{-1} \cdot Mat(2,\mathcal{O})^\dagger \left( \begin{array}{cc}
    \pi^n & 0 \\
    0 & 1
\end{array} \right).
\]

We can then explicitly compute
\[
End(\Lambda) \cap End(\Lambda^\dagger) = \left\{ \left( \begin{array}{cc}
    a & b\pi^n \\
    c\pi^n & d
\end{array} \right) \mid a, b, c, d \in \mathcal{O} \right\} \text{ if } \iota(\text{disc}(\mathcal{T})) = p
\]
\[
\cap Mat(2,\mathcal{O}) \cap Mat(2,\mathcal{O})^\dagger.
\]

6. The Non-Dyadic Case:

We now specialize to the case where \( F \) is a local field over a non-dyadic place (that is, \( 2 \in \mathcal{O}^\times \)).

**Theorem 6.1.** Suppose that \( A \cong End(V) \) is a split quaternion algebra over a local field \( F \). Suppose \( \mathcal{O} \) is a maximal \( \mathcal{T} \)-order. Then there exists a fractional ideal \( \mathfrak{a} \subset \mathcal{O} \) and an \( \mathfrak{a} \)-maximal lattice \( \Lambda \) such that
\[
\mathcal{O} = End(\Lambda) \cap End(\Lambda^\dagger).
\]

Furthermore, all maximal \( \mathcal{T} \)-orders are isomorphic, and
\[
\text{disc}(\mathcal{O}) = \text{disc}(H) \cap \iota(\text{disc}(\mathcal{T})).
\]

**Proof.** By Lemma 5.1 we can assume that \( H = Mat(2,F) \) and \( \mathcal{T} = \mathcal{T}_\lambda \). In this case, the lattice
\[
\mathcal{O}^2 = \mathcal{O} \left( \begin{array}{c}
    1 \\
    0
\end{array} \right) + \mathcal{O} \left( \begin{array}{c}
    0 \\
    1
\end{array} \right)
\]
is \( \mathcal{O} \)-maximal. This follows from the fact (see [O'M73], p. 246) that if \( \mathfrak{n}\Lambda \subset \mathfrak{a} \) and the ideal
\[
2^2\mathfrak{a}^{-2}\mathfrak{v}\Lambda
\]
has no square factors, then \( \Lambda \) is \( \mathfrak{a} \)-maximal. It is easy to see that \( \mathfrak{n}\mathcal{O}^2 = \mathcal{O} \), and \( \mathfrak{v}\mathcal{O}^2 = \mathcal{O} \) or \( p \). In either case, the required ideal is square-free, and therefore \( \mathcal{O}^2 \) is \( \mathcal{O} \)-maximal.

We will show that all maximal \( \mathcal{T} \)-orders are isomorphic to
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\[ \text{Mat}(2, o) \cap \text{Mat}(2, o)^\dagger = \begin{cases} \{ \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix} \in \text{Mat}(2, o) \} & \text{if } \iota(\text{disc}(\dagger)) = p \\ \text{Mat}(2, o) & \text{if } \iota(\text{disc}(\dagger)) = o. \end{cases} \]

Note that this order has the desired discriminant and is the \( \dagger \)-order corresponding to the maximal lattice \( o^2 \). By Lemma 5.4, we will then have proved the theorem.

By Lemmas 5.3 and 5.2, to show that all maximal \( \dagger \)-orders are isomorphic to \( \text{Mat}(2, o) \cap \text{Mat}(2, o)^\dagger \), it is enough to show that for every lattice \( \Lambda \subset F^2 \), there exists a \( g \in \text{GO}(\text{Mat}(2, F), \dagger \lambda) \) such that

\[ \text{End}(\Lambda) \cap \text{End}(\Lambda^\dagger) \subset g \left( \text{Mat}(2, o) \cap \text{Mat}(2, o)^\dagger \right) g^{-1}. \]

Since \( F \) is non-dyadic, every lattice has an orthogonal basis, and so the result follows by Lemma 5.5. \( \square \)

7. The Dyadic Case

If \( F \) is a local field over a dyadic place, the behavior of maximal \( \dagger \)-orders is more complicated—unlike the non-dyadic case, it need not be true that \( o^2 \) is a maximal \( o \)-lattice.

**Lemma 7.1.** Let \( F \) be a local field over a dyadic place. Let \( \lambda \in F^\times \) be integral and square-free. The lattice \( o^2 \) is \( o \)-maximal for the quadratic form

\[ q_\dagger(x, y) = x^2 + \lambda y^2 \]

on \( F^2 \) if and only if \( \delta(-\lambda) = p \).

**Proof.** It is easy to check that \( \delta o^2 = o \), as desired. If \( -\lambda \) is a square (that is, \( q_\dagger \) is isotropic), then it is easy to check that \( o^2 \) is not maximal—indeed, the lattice

\[ o \left( \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right) + o \left( \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \right) \]

clearly contains \( o^2 \) and yet also has norm \( o \). Therefore, we can assume that \( q_\dagger \) is anisotropic. It follows that there is a unique maximal \( o \)-lattice, which is the set of all \( x, y \) such that

\[ q_\dagger(x, y) \in o. \]

It is evident that \( o^2 \) is a sub-lattice—it remains to show that there are non-integral \( x, y \) such that \( q_\dagger(x, y) \in o \) if and only if \( \delta(-\lambda) \neq p \).

First, suppose there is such a pair \( x, y \in F \). Let \( n \) be the smallest integer such that \( \pi^n x, \pi^n y \in o \) by assumption, \( n \geq 1 \). Therefore

\[ (\pi^n x)^2 + \lambda (\pi^n y)^2 \in p^2, \]

and since it is easily seen that \( y \neq 0 \), we arrange to find that

\[ -\lambda = (x/y)^2 + p^2, \]

which shows \( \delta(-\lambda) \neq p \). In the other direction, if \( \delta(-\lambda) \neq p \), then we can find coprime \( x, y \in D \) such that

\[ -\lambda = (x/y)^2 + p^2, \]

hence

\[ x^2 + \lambda y^2 \in p^2, \]

in which case \( (x/\pi, y/\pi) \notin o^2 \) but do belong to the maximal \( o \)-lattice. \( \square \)
Consequently, we get different behavior depending on whether \( \mathfrak{d}(\text{disc}(\mathfrak{f}) \cap \mathfrak{o}) = \mathfrak{p} \) or not.

**Theorem 7.1.** Suppose \( A \cong \text{End}(V) \) is a split quaternion algebra over \( F \), a local field over a dyadic place \( \mathfrak{p} \). Suppose that \( \mathfrak{d}(\text{disc}(\mathfrak{f}) \cap \mathfrak{o}) = \mathfrak{p} \). Let \( \mathcal{O} \) is a maximal \( \frac{1}{2} \)-order. Then there exists a fractional ideal \( \mathfrak{a} \subset F \) and an \( \mathfrak{a} \)-maximal lattice \( \Lambda \) such that

\[
\mathcal{O} = \text{End}(\Lambda) \cap \text{End}(\Lambda^\dagger).
\]

Furthermore, all maximal \( \frac{1}{2} \)-orders are isomorphic, and

\[
\text{disc}(\mathcal{O}) = \text{disc}(H) \cap \iota(\text{disc}(\mathfrak{f})).
\]

**Proof.** The proof is very similar to the proof of Theorem 6.1. We note that we can reduce to the special case \( \text{End}(V) = \text{Mat}(2, F) \), \( \frac{1}{2} = \frac{1}{2}_F \), where \( \lambda \mathfrak{o} = \mathfrak{o} \) or \( \mathfrak{p} \). By Lemma 7.1, \( \mathfrak{o}^2 \) is a maximal lattice. Ergo, if we prove that every maximal \( \frac{1}{2} \)-order \( \mathcal{O} \) of \( \text{Mat}(2, F) \) is of the form

\[
\mathcal{O} = \text{End}(\Lambda) \cap \text{End}(\Lambda^\dagger),
\]

where \( \Lambda \) is a lattice in \( F^2 \) with orthogonal base, we will be done.

Over a local field, if \( n \Lambda = s \Lambda \), there is an explicit algorithm to construct an orthogonal base of \( \Lambda \). On the other hand, if \( \Lambda = \mathfrak{o}e_1 + \mathfrak{o}e_2 \), then

\[
s \Lambda = \sum_{i,j} q_{ij}(e_i, e_j) \mathfrak{o}
\]

\[
n \Lambda = \sum_{i} q_{i}(e_i) \mathfrak{o} + 2s \Lambda.
\]

Let

\[
\Lambda = \mathfrak{o} \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

so that the Gram matrix is

\[
G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a^2 + c^2 & \lambda ab + cd \\ \lambda ab + cd & \lambda b^2 + \mathfrak{o}^2 \end{pmatrix}.
\]

Thus, the only way that \( n \Lambda \neq s \Lambda \) is if

\[
\min\{\text{ord}_p(\lambda a^2 + c^2), \text{ord}_p(\lambda b^2 + \mathfrak{o}^2)\} > \text{ord}_p(\lambda ab + cd).
\]

However,

\[
(\lambda a^2 + c^2)(\lambda b^2 + \mathfrak{o}^2) = (\lambda ab + cd)^2 + \lambda(ad - bc)^2,
\]

hence

\[
-\lambda = \lambda^2(ad - bc)^2 + \frac{\lambda \left(\lambda a^2 + c^2\right) \left(\lambda b^2 + \mathfrak{o}^2\right)}{(\lambda ab + cd)^2}
\]

\[
= \lambda^2(ad - bc)^2 + \mathfrak{p}^2,
\]

which is impossible since we are given that \( \mathfrak{d}(-\lambda) = \mathfrak{p} \). Therefore, \( \Lambda \) has an orthogonal base, and we are done. \( \square \)
If $\mathfrak{d}(-\text{disc}(\mathfrak{i})) \cap \mathfrak{o} \neq \mathfrak{p}$, we shall show in the next section that there are always at least two isomorphism classes of maximal $\frac{1}{2}$-orders. For now, we give a weaker result.

**Theorem 7.2.** Suppose $\Lambda \cong \text{End}(V)$ is a split quaternion algebra over $F$, a local field over a dyadic place $\mathfrak{p}$. Suppose that $\mathfrak{d}(-\text{disc}(\mathfrak{i})) \cap \mathfrak{o} \neq \mathfrak{p}$. Let $\mathcal{O}$ be a maximal $\frac{1}{2}$-order. Then there exists a fractional ideal $\mathfrak{a} \subset F$ and an $\mathfrak{a}$-modular lattice $\Lambda$ such that

$$\mathcal{O} = \text{End}(\Lambda) \cap \text{End}(\Lambda^{\frac{1}{2}}).$$

Additionally,

$$\text{disc}(\mathcal{O}) = \text{disc}(H) \cap \nu(\text{disc}(\mathfrak{i})) = \mathfrak{o}.$$

**Proof.** Since $\mathfrak{d}(-\text{disc}(\mathfrak{i})) \cap \mathfrak{o} \neq \mathfrak{p}$, we conclude that $\nu(\text{disc}(\mathfrak{i})) = \mathfrak{o}$—otherwise, there would be an element $\lambda \in \text{disc}(\mathfrak{i})$ such that $\text{ord}_\mathfrak{p}(-\lambda) = 1$, which would imply that $\mathfrak{d}(-\text{disc}(\mathfrak{i})) \cap \mathfrak{o} = \mathfrak{p}$.

By Lemma 5.4, we can assume that $H = \text{Mat}(2, F)$ and $\mathfrak{i} = \mathfrak{i}_\lambda$. Since $\nu(\text{disc}(\mathfrak{i})) = \mathfrak{o}$, $\mathfrak{o}^2$ is a unimodular lattice. Choose any maximal $\frac{1}{2}$-order $\mathcal{O} = \text{End}(\Lambda) \cap \text{End}(\Lambda^{\frac{1}{2}})$.

By the theory of Jordan splittings, $\Lambda$ is either modular or it has an orthogonal base. However, if it has an orthogonal base, then we know by 5.5 that $\mathcal{O}$ is isomorphic to $\text{Mat}(2, \mathfrak{o}) \cap \text{Mat}(2, \mathfrak{o})^{\frac{1}{2}}$—since $\mathfrak{o}^2$ is unimodular, we see that we can always take $\Lambda$ to be modular.

Since $\Lambda$ is $\mathfrak{a}$-modular for some fractional ideal $\mathfrak{a}$, $\Lambda = \mathfrak{a}\Lambda^{\frac{1}{2}}$. This proves that $\mathcal{O} = \text{End}(\Lambda)$, hence its discriminant is $\mathfrak{o}$. 

## 8. Counting Orders over Local Fields:

In this section, we shall give methods for precisely counting the number of maximal $\frac{1}{2}$-orders in a split quaternion algebra over a local field, as well as methods of counting the number of isomorphism classes. To start, we shall need the following lemma.

**Lemma 8.1.** Let $F$ be a local field. Then the stabilizer of $\text{Mat}(2, \mathfrak{o})$ in $\text{GL}(2, F)$ is

$$\text{GL}^0(2, D) = \ker \left( \text{GL}(2, F) \xrightarrow{\text{det}} F^\times \to F^\times / (F^\times)^2 \right) \cap \text{Mat}(2, \mathfrak{o}).$$

**Proof.** Suppose that $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{Stab}(\text{Mat}(2, \mathfrak{o}))$. Then

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} \in \text{Mat}(2, \mathfrak{o}),$$

and therefore

$$a^2, b^2, c^2, d^2 \in \det(\gamma)\mathfrak{o}.$$

If $\gamma \in \text{Stab}(\text{Mat}(2, \mathfrak{o}))$, then so is $\lambda\gamma$. Therefore, we are free to assume that either $\det(\gamma)\mathfrak{o} = \mathfrak{o}$ or $\mathfrak{p}$. Suppose first that it is $\mathfrak{p}$. Then $a, b, c, d \in \mathfrak{p}$, hence $ad, bc \in \mathfrak{p}^2$. However, this implies $ad - bc = \det(\gamma) \in \mathfrak{p}^2$, which is a contradiction.

Therefore, we are free to assume that $\det(\gamma) \in \mathfrak{o}^\times$, which proves that $a, b, c, d \in \mathfrak{o}$. What we have shown is that

$$\gamma \in \left( \begin{array}{cc} \pi^n & 0 \\ 0 & \pi^n \end{array} \right) \text{GL}(2, \mathfrak{o}) \subset \text{GL}^0(2, \mathfrak{o}).$$

It is easy to see that $\text{GL}^0(2, \mathfrak{o}) \subset \text{Stab}(\text{Mat}(2, \mathfrak{o}))$, so we are done.

**Corollary 8.1.** Given two lattices $\Lambda_1, \Lambda_2 \subset V$, $\text{End}(\Lambda_1) = \text{End}(\Lambda_2)$ if and only if $\Lambda_1 = \lambda\Lambda_2$ for some $\lambda \in F^\times$. 

Proof. We showed previously that if $L_1, L_2$ are matrices corresponding to bases of $\Lambda_1, \Lambda_2$, then
\[
\text{End}(\Lambda_1) = L_1 \text{Mat}(2, \mathfrak{o}) L_1^{-1}
\]
\[
\text{End}(\Lambda_2) = L_2 \text{Mat}(2, \mathfrak{o}) L_2^{-1},
\]
and therefore
\[
\text{End}(\Lambda_1) = \text{End}(\Lambda_2) \iff L_1^{-1} L_2 \in \text{GL}^0(2, \mathfrak{o})
\]
\[
\iff L_2 = L_1 \gamma \quad \text{(for some } \gamma \in \text{GL}^0(2, \mathfrak{o}))
\]
\[
\iff L_2 = \lambda L_1 \gamma \quad \text{(for some } \gamma \in \text{GL}(2, \mathfrak{o}), \lambda \in F^\times).
\]
However, multiplying $L_1$ by an element of $\text{GL}_2(\mathfrak{o})$ does not change the lattice $\Lambda_1$ (this amounts to a change of basis), hence the result follows. \hfill \square

**Corollary 8.2.** Given two Eichler orders
\[
\mathcal{O}_1 = \text{End}(\Lambda_1) \cap \text{End}(\Lambda_1^\sharp)
\]
\[
\mathcal{O}_2 = \text{End}(\Lambda_2) \cap \text{End}(\Lambda_2^\sharp),
\]
$\mathcal{O}_1 = \mathcal{O}_2$ if and only if $\Lambda_1 = \lambda \Lambda_2$ or $\Lambda_1 = \lambda \Lambda_2^\sharp$ (for some $\lambda \in F^\times$).

**Proof.** This follows immediately from the preceding corollary and the observation that, over a local field, the maximal orders of which an Eichler order is an intersection are uniquely determined. \hfill \square

Motivated by Corollary 8.2, we define an equivalence relation on the set of lattices in $V$ by
\[
\Lambda_1 \sim \Lambda_2 \iff \Lambda_1 = \lambda \Lambda_2 \text{ or } \Lambda_1 = \lambda \Lambda_2^\sharp \quad \text{(for some } \lambda \in F^\times).
\]
From Corollary 8.2, it follows that the set of equivalence classes of lattices is in bijection with the set of $\dagger$-orders of $\text{End}(V)$ that are Eichler orders. We can give an even stronger statement, although we need to split into two cases

**Theorem 8.1.** Let $V$ be a 2-dimensional vector space over a local field $F$ with maximal ideal $p$. Suppose either $2 \notin p$ or $\mathfrak{o}(\text{disc}(\dagger) \cap \mathfrak{o}) = p$. Then there is a well-defined bijection
\[
\varphi : \{\text{maximal lattices in } V\} / \sim \to \{\text{maximal } \dagger \text{-orders of } \text{End}(V)\}
\]
\[
\left[\Lambda\right] \mapsto \text{End}(\Lambda) \cap \text{End}(\Lambda^\dagger).
\]

**Proof.** As usual, we use Lemma 5.1 to reduce to the case where $A = \text{Mat}(2, F)$, $\dagger = \dagger\Lambda$.

To show $\varphi$ is well-defined, we need to show that if $\Lambda$ is a maximal lattice, then $\varphi([\Lambda])$ is a maximal $\dagger$-order. We can freely assume that $\Lambda$ is either $\mathfrak{o}$-maximal or $p$-maximal. Over a local field, all $\mathfrak{o}$-maximal lattices are isomorphic—since $\mathfrak{o}^2$ is maximal and $\varphi([\mathfrak{o}^2])$ is a maximal $\dagger$-order, we can conclude that any $\mathfrak{o}$-maximal lattice corresponds to a maximal $\dagger$-order. So, it remains to consider $p$-maximal orders.

If $(F^2, q_{\dagger})$ is isotropic, there is a $g \in \text{GO} (\text{Mat}(2, F), \dagger)$ such that $\det(g) = \pi$, and therefore $g \mathfrak{o}^2$ is a $p$-maximal lattice corresponding to a maximal $\dagger$-order. Ergo all $p$-maximal orders correspond to maximal $\dagger$-orders. If $(F^2, q_{\dagger})$ is anisotropic, there is a unique $p$-maximal lattice, given by
\[
\Lambda_p = \{ v \in F^2 \mid q_{\dagger}(v) \in p \}.
\]
There are two cases: either there exists $v \in F^2$ such that $q_{\dagger}(v) \in p \setminus p^2$, or there doesn’t. If there is such a $v$, then there is a $g \in \text{GO} (\text{Mat}(2, F), \dagger)$ such that $\det(g) = q_{\dagger}(v)$, and therefore $g \mathfrak{o}^2$ is a $p$-maximal lattice—the result of the argument is the same as in the isotropic case. If there is no such $v$, then $\Lambda_p$ is $p^2$-maximal, hence $\frac{1}{\pi} \Lambda_p = \mathfrak{o}^2$, and so we conclude that $\varphi([\Lambda_p])$ is a maximal $\dagger$-order.

That $\varphi$ is injective follows from Corollary 8.2. Surjectivity follows from Theorems 6.1 and 7.1. \hfill \square
Theorem 8.2. Let $V$ be a 2-dimensional vector space over a local field $F$ over a dyadic place $p$ and $\mathfrak{d}(-\text{disc}(\mathfrak{f}) \cap \mathfrak{o}) \neq p$. Then there is a well-defined bijection

$$\varphi : \{\text{modular lattices in } V \} / \sim \rightarrow \{\text{maximal } \mathfrak{f}-\text{orders of } \text{End}(V)\}$$

$$[\Lambda] \mapsto \text{End}(\Lambda) \cap \text{End}(\Lambda^\sharp) = \text{End}(\Lambda).$$

Furthermore, two maximal $\mathfrak{f}$-orders $\varphi([\Lambda_1]), \varphi([\Lambda_2])$ are isomorphic if and only if there are representatives $\Lambda_1 \in [\Lambda_1], \Lambda_2 \in [\Lambda_2]$ and an element $g \in \text{GO}(\text{End}(V), \mathfrak{f})$ such that $\Lambda_1 = g\Lambda_2$.

Finally, the number of isomorphism classes of maximal $\mathfrak{f}$-orders is

$$\left\{ \begin{array}{ll}
m + 1 & \text{if } \mathfrak{d}(-\lambda) = p^{2m+1} \\
n + 1 & \text{if } \mathfrak{d}(-\lambda) = (4) \text{ or } 0 \end{array} \right.$$  

where $n = \text{ord}_p(2)$.

Proof. If $\Lambda$ is a $\mathfrak{a}$-modular, then $\text{End}(\Lambda^\sharp) = \text{End}(\mathfrak{a}\Lambda) = \text{End}(\Lambda)$, hence $\text{End}(\Lambda)$ is a maximal $\mathfrak{f}$-order. Therefore, $\varphi$ is well-defined.

That $\varphi$ is injective follows from Corollary 8.2. Surjectivity follows from Theorem 7.2. The fact that isomorphism of maximal $\mathfrak{f}$-orders corresponds to acting on the lattices by $\text{GO}(\text{End}(V), \mathfrak{f})$ is an easy consequence of bijectivity and Lemma 5.2.

It remains to determine the number of non-isomorphic classes of maximal $\mathfrak{f}$-orders. To do this, we make use of the norm group. For a lattice $\Lambda$, its norm group is defined as

$$g\Lambda = q_1(\Lambda) + 2s\Lambda,$$

or equivalently as

$$g\Lambda = ao^{sq} + b\mathfrak{o},$$

where $a$ is a norm generator, $b$ is a weight generator, and

$$o^{sq} = \{o^2 | d \in \mathfrak{o}\}.$$

Now, notice that since $\text{GO}(\text{End}(V), \mathfrak{f})$ contains an element with determinant in $p \setminus p^2$, each isomorphism class contains at least one unimodular representative. Two unimodular lattices over a dyadic field are isomorphic if and only if the norm groups are isomorphic. Therefore, two unimodular lattices $\Lambda_1, \Lambda_2$ correspond to the same isomorphism class if and only if there is an element $\gamma \in \text{GO}(H, \mathfrak{f})$ such that

$$\det(\gamma) \in o^\times$$

$$g\Lambda_1 = \gamma^\sharp g\Lambda_2.$$

Let the norm generators of $\Lambda_1, \Lambda_2$ be $a_1, a_2$ respectively, and let the weight generators be $b_1, b_2$. Then $\Lambda_1, \Lambda_2$ are in the same isomorphism class if and only if there exist $s, t \in F$ such that $s^2 + t^2\lambda \in o^\times$ and

$$a_1 o^{sq} + b_1 \mathfrak{o} = \pm(s^2 + t^2\lambda)a_2 o^{sq} + b_2 \mathfrak{o}.$$

However, over a dyadic field, any unimodular lattice has a basis in which the Gram matrix is

$$\begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix},$$

where $\alpha, \beta \in \mathfrak{o}, -1 + \alpha\beta \in o^\times$, and $\alpha$ is a norm generator (in particular, $\text{ord}_p(\alpha) < \text{ord}_p(\beta)$). This implies that there exist $s_1, s_2, t_1, t_2 \in F$ such that
\[a_1 = s_1^2 + t_1^2 \lambda,\]
\[a_2 = s_2^2 + t_2^2 \lambda,\]
and therefore there exists \(s, t\) such that \(s^2 + t^2 \lambda \in \mathfrak{o}^\times\) and \(a_1 = (s^2 + t^2 \lambda)a_2\) if and only if
\[\text{ord}_p(a_1) = \text{ord}_p(a_2).\]

In other words, we see that the isomorphism class of \(\Lambda_1, \Lambda_2\) is wholly determined by the order of their norm and weight generators.

However, choosing a basis as above such that the Gram matrix is
\[
\begin{pmatrix}
\alpha & 1 \\
1 & \beta
\end{pmatrix},
\]
we get that the norm generator is \(\alpha\) and the weight generator is either \(\beta\) or 2 (if \(\text{ord}_p(\beta) > n\)).

There exists a unimodular lattice \(\Lambda\) with Gram matrix
\[
\begin{pmatrix}
\alpha & 1 \\
1 & \beta
\end{pmatrix}
\]
if and only if
\[\det \begin{pmatrix} -\lambda & 0 \\ 0 & 1 \end{pmatrix} = -\lambda = -1 + \alpha \beta \in \mathfrak{o}^\times / (\mathfrak{o}^\times)^2.\]

Note that \(\lambda\) is only defined up to multiplication by \((\mathfrak{o}^\times)^2\), so we can take it to be \(-1 + \pi^{2m+1} \epsilon, -1 + 4 \epsilon,\) or \(-1\) (for some \(\epsilon \in \mathfrak{o}^\times\)). Furthermore, since we can scale \(\Lambda\) by elements of \(\mathfrak{o}^\times\), we can assume that we have equality in \(\mathfrak{o}\) itself in equation (1).

First, suppose that \(\lambda = -1 + \pi^{2m+1} \epsilon\). We have that
\[-1 + \pi^{2m+1} \epsilon = -1 + \alpha \beta,
\]
showing that
\[\text{ord}_p(\alpha) + \text{ord}_p(\beta) = 2m + 1 < 2n.\]

Thus, \(\beta\) is the weight generator if \(\text{ord}_p(\alpha) \geq 2m - n + 1\). Otherwise, 2 is the weight generator. Therefore, all possible choices for the orders of the norm and weight generators are
\[
(m, m + 1), (m - 1, m + 2), \ldots (2m - n + 1, n), (2m - n, n), (2m - n - 1, n), \ldots (0, n) \quad \text{if } 2m + 1 > n
\]
\[
(m, m + 1), (m - 1, m + 2), \ldots (0, 2m + 1) \quad \text{if } 2m + 1 \leq n.
\]

Second, suppose that \(\lambda = -1 + 4 \epsilon\). We have that
\[-1 + 4 \epsilon = -1 + \alpha \beta,
\]
showing that
\[\text{ord}_p(\alpha) + \text{ord}_p(\beta) = 2n,
\]

hence \(\beta\) is the weight generator if \(\text{ord}_p(\alpha) \geq n\), and otherwise 2 is the weight generator. Therefore, all possible choices for the orders of the norm and weight generators are
\[
(n, n), (n - 1, n), (n - 2, n), \ldots (0, n).
\]

Finally, suppose that \(\lambda = -1\). We have that \(\alpha \beta = 0\). Since \(\alpha\) is taken to be norm generator, it is non-zero—therefore, \(\beta = 0\), and therefore the weight generator is simply 2. The norm generator cannot have order greater than \(n\), and therefore we see that all the possible choices for the orders of the norm and weight generators are again
This proves the claim.

\[ (n, n), (n - 1, n), (n - 2, n), \ldots (0, n). \]

\[ \square \]

REFERENCES

[IR93] Gábor Ivanyos and Lajos Rónyai. Finding maximal orders in semisimple algebras over \( \mathbb{Q} \). \textit{Computational Complexity}, 3(3):245–261, 1993.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. \textit{The Book of Involutions}. American Mathematical Society, June 1998.

[O’M73] O. Timothy O’Meara. \textit{Introduction to Quadratic Forms}. Springer, 1973.

[She] Arseniy Sheydvasser. Applications of quaternion algebras to sphere packings. in preparation.

[Sta14] Katherine E. Stange. Visualising the arithmetic of imaginary quadratic fields, 2014.

[Sta15] Katherine E. Stange. The apollonian structure of bianchi groups, 2015.

[Voi13] John Voight. Identifying the matrix ring: Algorithms for quaternion algebras and quadratic forms. \textit{Quadratic and Higher Degree Forms Developments in Mathematics}, pages 255–298, 2013.

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