Mixed Social Optima and Nash equilibrium in Linear-Quadratic-Gaussian Mean-field System

Xinwei Feng¹, Jianhui Huang² and Zhenghong Qiu³,*

Abstract—This paper investigates a class of mixed stochastic linear-quadratic-Gaussian (LQG) social optimization and Nash game in the context of a large-scale system. Two types of interactive agents are involved: a major agent and a large number of weakly-coupled minor agents. All minor agents are cooperative to minimize the social cost as the sum of their individual costs, whereas such social cost is conflictive to that of the major agent. Thus, the major agent and all minor agents are further competitive to reach some nonzero-sum Nash equilibrium. Applying the mean-field approximations and person-by-person optimality, we obtain auxiliary control problems for the major agent and minor agents, respectively. The decentralized social strategy is derived by a class of new consistency condition (CC) system, which consists of mean-field forward-backward stochastic differential equations. The well-posedness of CC system is obtained by the discounting method. The related asymptotic social optimality for minor agents and Nash equilibrium for major-minor agents are also verified.

Index Terms—Decentralized control, LQG mean-field strategy, Mean-field forward backward stochastic differential equations, Nash equilibrium, Social-optimality.

I. INTRODUCTION

The Mean-field methodology for large-population systems has been extensively studied recently. The central goal of an individual agent in the large-population system is to obtain decentralized strategy based on its own limited information. One efficient approach is the mean-field method which enables us to obtain the decentralized strategy through the limiting auxiliary control problem and the related consistency condition (CC) system. Along this direction, the interested readers are referred to [15], [17] and [18] for the derivation of mean-field games, [2], [10], [13] for linear-quadratic-Gaussian (LQG) mean-field games, [3] for probabilistic analysis in mean-field games, [21] for risk-sensitive mean-field games, [20] for discrete-time mean-field games. [24] studied the mean-field social solution to consensus problems. In the basic mean-field decision model, all agents have comparably small influence. However, in some real models, there exists an individual but centralized (non-cooperative) game in which two agents make competitive decisions based on individual but centralized information. As an extension, in our setup: one agent no longer applies such centralized decision. Instead, all its sub-units or branches will apply distributed information to optimize the original cost jointly (e.g., [1], [9] and [26]), that now is reformulated in some team-cost form. Thereby, all sub-units become “minor” agents and formalize a (cooperative) team, while another agent still applying centralized information becomes a “major” and non-cooperative player, from the viewpoint of all “minor” agents.

In our study, the problem can be solved in the following way. Firstly, for the major agent, we freeze the state average and obtain the auxiliary control problem. By the result in [29], the auxiliary control problem for the major agent can be derived. Secondly, for the minor agents, under the person-by-person optimality principle, by applying variational techniques and introducing some mean-field terms, the original minor social optimization problem can also be converted to an auxiliary LQG control problem, which can be solved using some traditional scheme in [29] as well. Thirdly, to determine the frozen mean-field terms, we construct the consistency condition (CC) system by some fixed-point analysis. Last, by using some asymptotic analysis and standard estimation of stochastic differential equations (SDEs), we show that the mean-field strategy provides an efficient approximation (i.e., the optimal loss tends to 0 as the population N tends to ∞).

Moreover, the innovative aspects of the obtained results in this paper are as follows: Firstly, in general mean-field games framework, usually the auxiliary control problem can be obtained directly by replacing the state average with some frozen mean-field term. However, this scheme will bring some...
“bad” strategy in our social optimia framework, which can not achieve the asymptotic optimality. Instead, in Section IV, variational techniques are applied to distinguish the high-order infinitesimals after the mean-filed approximation, and a new type of auxiliary control problem would be derived. Secondly, the state process and state average enter the diffusion terms. Such feature brings many difficulties when we apply the variational method to obtain the auxiliary control problem for the minor agents. In particular, \( N + 1 \) additional adjoint processes should be introduced to deal with the cross-terms in the cost function variation. Thirdly, in the estimation of optimal loss, unlike the general mean-field games framework, the asymptotic optimality is proved through investigating the Fréchet derivative of the social cost in Section VII. Last, the control process enters the diffusion terms. Because of this, the adjoint-state term will enter the drift term of the CC system. This also brings difficulties when we study the solvability of the CC system, which is a mean-field forward-backward stochastic differential equations (MF-FBSDEs) system, and the mean-field terms are represented in an embedding way. To its well-posedness, we apply some discounting methods.

The remaining of the paper is organized as follows: In Section II, we give the formulation of the mixed LQG social optimia problem. In Section III and Section IV, we find the auxiliary control problem of the major agent and minor agents respectively. The CC system is derived in Section V. Meanwhile, the well-posedness of CC system is also established. In Section VI, we compare our result with some previous literature. In Section VII, we obtain the asymptotic optimality of the decentralized strategy. Last, in Section VIII, we simulate our model with some numerical methods.

### II. PROBLEM FORMULATION

Consider a finite time horizon \( [0, T] \) for fixed \( T > 0 \). Assume that \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}) \) is a complete filtered probability space satisfying the usual conditions and \( \{W_i(t), 0 \leq i \leq N\}_{0 \leq t \leq T} \) is an \((N+1)\)-dimensional Brownian motion on this space. Let \( \mathcal{F}_t \) be the filtration generated by \( \{W_i(s), 0 \leq i \leq N\}_{0 \leq s \leq t} \) and augmented by \( \mathcal{N}_t \) (the class of all \( \mathcal{F} \)-null sets of \( \mathcal{F}_t \)). For simplicity, define \( \mathbb{E}^\mathcal{F}_w \) denotes the conditional expectation w.r.t. \( \mathcal{F}_w \).

Let \( \langle \cdot, \cdot \rangle \) denote standard Euclidean inner product and \( \| \cdot \| \) denote the norm. \( x^T \) denotes the transpose of a vector (or matrix) \( x \). \( S^n \) denotes the set of symmetric \( n \times n \) matrices with real elements. \( M > (\geq 0) \) denotes that \( M \in S^n \) which is positive (semi)definite, while \( M \geq 0 \) denotes that, for some \( \varepsilon > 0 \), \( M - \varepsilon I \geq 0 \). We introduce the following spaces for any given Euclidean space \( \mathcal{E} \). They will be used in the paper:

- \( L^2_{\mathcal{F}_t}(\Omega, \mathcal{H}) := \{ \xi : \Omega \to \mathbb{H} \text{ is } \mathcal{F}_t \text{-measurable}, \mathbb{E}^{\mathcal{F}_t}[\|\xi\|^2] < \infty \} \),
- \( L^2_{\mathcal{F}_t}(0, T; \mathcal{H}) := \{ \xi : [0, T] \times \Omega \to \mathbb{H} \text{ is } \mathcal{F}_t \text{-progressively measurable}, \mathbb{E}^{\mathcal{F}_t}[\int_0^T \|\xi(t)\|^2 dt < \infty] \}, \)
- \( L^\infty(0, T; \mathcal{H}) := \{ \xi : (0, T) \times \Omega \to \mathbb{H} \text{ is } \mathcal{F}_t \text{-adapted, continuous}, \mathbb{E}^{\mathcal{F}_t}[\sup_{0 \leq t \leq T} \|\xi(t)\| < \infty] \}, \)
- \( L^2_{\mathcal{F}_t}(\Omega; C(0, T; \mathcal{H})) := \{ \xi : [0, T] \times \Omega \to \mathbb{H} \text{ is } \mathcal{F}_t \text{-adapted, continuous, } \mathbb{E}^{\mathcal{F}_t}[\sup_{0 \leq t \leq T} \|\xi(t)\| < \infty] \}, \)

and \( \|\xi\|^2_{L^2_{\mathcal{F}_t}} := \mathbb{E}^{\mathcal{F}_t}[\int_0^T \|\xi(t)\|^2 dt] \) denotes the \( L^2 \) norm. We consider a weakly coupled large population system with a major agent \( A_0 \) and \( N \) individual minor agents denoted by \( \{A_i : 1 \leq i \leq N\} \). The dynamics of the \( N + 1 \) agents are given by a system of SDEs with mean-field coupling:

\[
\begin{align*}
\dot{x}_0(t) &= [A_0(t)x_0(t) + B_0(t)u_0(t) + F_0(t)x(N)(t)]dt + [C_0(t)x_0(t) + D_0(t)u_0(t) + \bar{F}_0(t)x(N)(t)]dW_0(t), \\
&+ [C_0(t)x_0(t) + D_0(t)u_0(t) + \bar{F}_0(t)x(N)(t)]dW_0(t), \quad (1)
\end{align*}
\]

and for \( 1 \leq i \leq N \),

\[
\begin{align*}
\dot{x}_i(t) &= [A_i(t)x_i(t) + B_i(t)u_i(t) + F_i(t)x(N)(t)]dt \\
&+ [C_i(t)x_i(t) + D_i(t)u_i(t) + \bar{F}_i(t)x(N)(t)]dW_i(t), \quad (2)
\end{align*}
\]

where \( x(N)(t) = \frac{1}{N} \sum_{i=1}^N x_i(t) \) is the average state of the minor agents.

**Remark 1:** We remark that the control process and state-average enter both the drift and diffusion terms. This makes our paper different to standard mean-field game (e.g., [28]) or social optimization (e.g., [16]) literature in which only drift terms are control-dependent.

Let \( u(\cdot) := (u_0(\cdot), u_1(\cdot), \ldots, u_N(\cdot)) \) be the set of strategies of all \( N + 1 \) agents, \( u_{-0}(\cdot) := (u_1(\cdot), \ldots, u_N(\cdot)) \) and \( u_{-i}(\cdot) := (u_0(\cdot), \ldots, u_{i-1}(\cdot), u_{i+1}(\cdot), \ldots, u_N(\cdot)) \), \( 0 \leq i \leq N \). The centralized admissible strategy set is given by

\[
\mathcal{U}_C := \{ u(\cdot) | u(t) \in \mathcal{F}_t \text{ measurable, } \mathbb{E}^{\mathcal{F}_t}[\|u(t)\|^2 dt < \infty] \}.
\]

Correspondingly, the feedback decentralized admissible strategy set for the major agent is given by

\[
\mathcal{U}_0 := \{ u_0(\cdot) | u(t) \in \mathcal{F}_{W_0}^t \text{ measurable, } \mathbb{E}^{\mathcal{F}_t}[\|u(t)\|^2 dt < \infty] \},
\]

and the feedback decentralized admissible strategy set for the \( i \)-th minor agent is given by

\[
\mathcal{U}_i := \{ u_i(\cdot) | u(t) \in \mathcal{F}_i^t \text{ measurable, } \mathbb{E}^{\mathcal{F}_t}[\|u_i(t)\|^2 dt < \infty] \}.
\]

For simplicity, define

\[
\mathcal{U}_{-0} := \{ (u_1(\cdot), \ldots, u_N(\cdot)) | u_i(t) \in \mathcal{U}_i, \quad i = 1, \ldots, N \}.
\]

The cost functional for \( A_0 \) is given by

\[
\mathcal{J}_0(u_0(\cdot), u_{-0}(\cdot)) = \frac{1}{2} \mathbb{E}^{\mathcal{F}_0}[\int_0^T (Q_0(x_0(t), x_N(t)) dt + H_0(x_0(t), x_N(t)) dt)] + (R_0(u_0(t), u_{-0}(t)))dt,
\]

and the cost functional for \( A_i, \quad 1 \leq i \leq N \), is given by

\[
\mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2} \mathbb{E}^{\mathcal{F}_i}[\int_0^T \left( \langle Q_i(x_i(t), x_N(t)) dt - H_i(x_0(t), x_N(t)) dt \right) + (R_i(u_i(t), u_{-i}(t)))dt dt
\]

**Remark 2:** It is worth pointing out that it brings no essential difficulty to introduce a terminal cost term in \([3]\) and \([4]\). This will only change the terminal value of the associated Riccati equations. Thus, for simplicity, we only consider Lagrange type cost functional here.

The aggregate team cost of \( N \) minor agents is

\[
\mathcal{J}_{N\text{-team}}(u(\cdot)) = \sum_{i=1}^N \mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)).
\]

We impose the following general assumptions, which are commonly used in LQG models, on the coefficients:

- \((H1)\) \( A_0(t), B_0(t), C_0(t), F_0(t), A_i(t), F_i(t), C_i(t), \bar{F}_i(t)) \in L^\infty(0, T; \mathbb{R}^{n \times n}) \),
- \( B_0(t), D_0(t), D_i(t) \in L^\infty(0, T; \mathbb{R}^{n \times m}) \),
- \( Q_0(t), H_0(t), Q_i(t), H_i(t), U_i(t) \in L^\infty(0, T; S^n) \),
Remark 3: Under (H1), the system (1) and (2) admits a unique strong solution \((x_0, \ldots, x_N) \in L^2_\mathbb{F}_T(\Omega; C([0, T; \mathbb{R}^n]) \times \cdots \times L^2_\mathbb{F}_T(\Omega; C([0, T; \mathbb{R}^n]))\) for any given admissible control \((u_0, \ldots, u_N) \in \mathcal{U}_c\). Under (H2), the cost functionals (3) and (4) are well defined.

Note that while the coefficients are dependent on the time variable \(t\), in what follows, the variable \(t\) will usually be suppressed if no confusion would occur. We propose the following social optimization problem:

**Problem 1:** Find a decentralized strategy set \(\bar{u}(\cdot) = (\bar{u}_0(\cdot), \bar{u}_1(\cdot), \ldots, \bar{u}_N(\cdot))\) where \(\bar{u}_i(\cdot) \in \mathcal{U}_i, 0 \leq i \leq N\), such that

\[
\begin{align*}
\mathcal{J}_0(\bar{u}_0(\cdot), u_{-0}(\cdot)) &= \inf_{u_0 \in \mathcal{U}_0} \mathcal{J}_0(u_0(\cdot), u_{-0}(\cdot)), \\
\mathcal{J}^N(\bar{u}_0(\cdot), u_{-0}(\cdot)) &= \inf_{u_{-0} \in \mathcal{U}_{-0}} \mathcal{J}^N(\bar{u}_0(\cdot), u_{-0}(\cdot)).
\end{align*}
\]

**III. AUXILIARY OPTIMAL CONTROL PROBLEM OF THE MAJOR AGENT**

Replacing \(x^{(N)}(\cdot)\) of (1) and (3) by \(\hat{x}(\cdot)\) which will be determined in Section V, the limiting major agent’s state is given by

\[
\begin{align*}
\frac{dz_0}{dt} &= (A_0z_0 + B_0v_0 + F_0\hat{x})dt + (C_0z_0 + D_0v_0 + \tilde{F}_0\hat{x})dW_0, \\
z_0(0) &= \xi_0,
\end{align*}
\]

and correspondingly the limiting cost functional is

\[
J_0(v_0(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T [(Q_0(z_0 - H_0\tilde{x}^\omega)) + (R_0v_0(\cdot), v_0(\cdot))] dt.
\]

**IV. STOCHASTIC OPTIMAL CONTROL PROBLEM FOR MINOR AGENTS**

**A. Person-by-person optimality**

Let \((\bar{u}_1(\cdot), \ldots, \bar{u}_N(\cdot))\) be centralized optimal strategies of the minor agents. We now perturb \(u_j(\cdot)\) and keep \(u_{-j}(\cdot) = (\bar{u}_0(\cdot), \bar{u}_1(\cdot), \ldots, \bar{u}_{j-1}(\cdot), \bar{u}_{j+1}(\cdot), \ldots, \bar{u}_N(\cdot))\) fixed. For \(j = 1, \ldots, N, j \neq i\), denote the perturbation \(\delta u_i(\cdot) = u_i(\cdot) - \bar{u}_i(\cdot), \delta x_i(\cdot) = x_i(\cdot) - \bar{x}_i(\cdot), \delta x_j(\cdot) = x_j(\cdot) - \bar{x}_j(\cdot), \delta x(N) = \frac{1}{N} \sum_{j=1}^{N} \delta x_j(\cdot), \) and \(\delta J_j\) is the first variation (Fréchet differential) of \(J_j\) w.r.t. \(\delta u_j\). Therefore, \(\delta x_i, \delta x_j, \delta x_0\) and \(\delta x_{-0}(\cdot) = : \sum_{j=1,j \neq i}^{N} \delta x_j(\cdot)\) are given by

\[
\begin{align*}
\delta d x_i &= (A_0 \delta x_i + B_0 \delta u_i + F_0 \delta x(N))dW_i, \\
&\quad + (C_0 \delta x_i + D_0 \delta u_i + \tilde{F}_0 \delta x(N) + \tilde{G}_0 \delta x_0)dW_i, \\
\delta d x_j &= (A_0 \delta x_j + F_0 \delta x(N))dt + (\delta x_j + \tilde{F}_0 \delta x(N))dW_j, \\
&\quad + (A_0 v_0 + F_0 \delta x(N))dt + (C_0 \delta x_0 + \tilde{F}_0 \delta x(N))dW_0, \\
\delta d x_{-0}(\cdot) &= (A_0 \delta x_{-0}(\cdot) + F_0 \delta x(N))dt, \\
&\quad + \sum (C_0 \delta x_j + \tilde{F}_0 \delta x(N) + \tilde{G}_0 \delta x_0)dW_j, \\
\delta x_i(0) &= 0, \delta x_j(0) = 0, \delta x_0(0) = 0, \delta x_{-0}(\cdot)(0) = 0.
\end{align*}
\]

By some elementary calculations, we can further obtain \(\delta J_i\) of the cost functional of \(A_i\) as follows

\[
\delta J_i = \mathbb{E} \int_0^T [(Q(x_i - H^\omega(N) - H^\omega x_0), \delta x_i - H^\omega \delta x_0 - H^\omega \delta x(N))] dt,
\]

for \(j \neq i, \delta J_j\) of the cost functional of \(A_j\) is given by

\[
\begin{align*}
\delta J_j = \mathbb{E} \int_0^T [(Q(\bar{x}_j - \bar{H} \tilde{x}^{(N)} - \bar{H} \bar{x}_0), \delta x_j - \bar{H} \delta \tilde{x}^{(N)} - \bar{H} \delta x_0)] dt, \\
&\quad + \sum (Q(\bar{x}_j - \bar{H} \tilde{x}^{(N)} - \bar{H} \bar{x}_0), \delta x_j - \bar{H} \delta \tilde{x}^{(N)} - \bar{H} \delta x_0) + (R_{\bar{u}_i}, \delta u_i)] dt.
\end{align*}
\]

We can further obtain \(\delta J^N_{soc}\), the first variation of the social cost, satisfying

\[
\begin{align*}
\delta J^N_{soc} &= \mathbb{E} \int_0^T [(Q(x_i - H^\omega(N) - H^\omega x_0), \delta x_i - H^\omega \delta x_0 - H^\omega \delta x(N))] dt, \\
&\quad + \sum (Q(x_j - H \tilde{x}^{(N)} - H \bar{x}_0), \delta x_j - H \delta \tilde{x}^{(N)} - H \delta x_0) + (R_{\bar{u}_i}, \delta u_i)] dt dt.
\end{align*}
\]

Replacing \(\tilde{x}^{(N)}(\cdot)\) in (13) by \((\tilde{x}(\cdot) - \hat{x}) + \bar{x}\),

\[
\delta J^N_{soc} = \mathbb{E} \int_0^T [(Q(x_i - H \tilde{x} - H \bar{x}_0), \delta x_i - H \delta \tilde{x} - H \delta \bar{x}_0)] dt + \sum_{j=1}^{N} (Q(x_j - H \tilde{x} - H \bar{x}_0), \delta x_j - H \delta \tilde{x} - H \delta \bar{x}_0) + (R_{\bar{u}_i}, \delta u_i)] dt dt
\]

where

\[
\begin{align*}
\varepsilon_{1,i} &= \mathbb{E} \int_0^T [(Q(H - H \tilde{x} - H \bar{x}_0), \delta \tilde{x} - H \delta x(N))] dt, \\
\varepsilon_{2,i} &= -\mathbb{E} \int_0^T [(H \tilde{x} - \tilde{x}(\cdot) - (H \tilde{x} - \tilde{x}(\cdot)), \delta \tilde{x}_i - (H \tilde{x}_i - \tilde{x}_i(N))] dt, \\
\varepsilon_{3,i} &= \mathbb{E} \int_0^T [(H \tilde{x} - \tilde{x}(\cdot) - (H \tilde{x} - \tilde{x}(\cdot)), \delta \tilde{x}_0) dt, \\
\varepsilon_{4,i} &= \mathbb{E} \int_0^T [(\tilde{H} \tilde{x} - \tilde{x}(\cdot) - (\tilde{H} \tilde{x} - \tilde{x}(\cdot)), \delta \tilde{x}(\cdot))] dt dt.
\end{align*}
\]

Introduce the limit processes \((x_0, x_0^*, x^*)\) to replace \((N \delta x_0, N \delta x_j, \delta x_{-0}(\cdot))\) by \((N \delta x_0 - x_0^*), + x_0^*, (N \delta x_j - x_j^*) + x_j^*, (\delta x_{-0}(\cdot) - x^*) + x^*)\) where

\[
\begin{align*}
\delta x_0 &= (A_0 x_0 + F_0 \delta x_0 + F_0 x^*)dt + (C_0 x_0 + \tilde{F}_0 \delta x_0 + \tilde{F}_0 x^*)dW_0, \\
\delta x_j &= (A_0 x_j + F_0 \delta x_j + F_0 x^*)dt + (C_0 x_j + \tilde{F}_0 \delta x_j + \tilde{F}_0 x^*)dW_j, \\
\delta x_{-0}(\cdot) &= (A_0 x_{-0}(\cdot) + F_0 \delta x_0 + F_0 x^*)dt, x_0^*(0) = x_0^*(0), x^*(0) = x^*(0) = 0.
\end{align*}
\]
Therefore,
\[
\delta J^{(N)}_{\text{soc}} = E \int_0^T \left[ (\langle Q \tilde{x}, \delta x \rangle - \langle Q (\dot{\tilde{x}} + H x) \rangle) + \dot{\tilde{H}} Q (\dot{\tilde{x}} + \tilde{H} x) \delta x \right] dt + \frac{1}{N} \sum_{i,j=1}^N \left( \langle Q \tilde{x}_i - \dot{\tilde{x}}_i + \tilde{H} x \rangle, \delta x_i \right) dt + \sum_{t=1}^T \varepsilon_{5,i}^T (\delta x_i),
\]
where
\[
\varepsilon_{5,i}^T = E \int_0^T \left( \dot{\tilde{H}} Q (\dot{\tilde{x}} + \tilde{H} x), \delta x_i \right) dt,
\]
\[
\varepsilon_{6,i}^T = E \int_0^T \left( H Q (\dot{\tilde{x}} + \tilde{H} x), \dot{x}_i - N \delta x_i \right) dt,
\]
\[
\varepsilon_{7,i}^T = E \int_0^T \frac{1}{N} \sum_{j=1}^N \left( \langle Q \tilde{x}_j - \dot{\tilde{x}}_j + \tilde{H} x \rangle, \delta x_j \right) dt.
\]
Replacing \( \tilde{x}_0 \) by \( (\tilde{x}_0 - z_0) + z_0 \), we have
\[
\delta J^{(N)}_{\text{soc}} = E \int_0^T \left[ (\langle Q \tilde{x}_i, \delta x_i \rangle - \langle Q (\dot{\tilde{x}} + H z) \rangle + \dot{\tilde{H}} Q (\dot{\tilde{x}} + \tilde{H} z) \delta x_i \right] dt.
\]
(17)

Applying It\'s formula to \( \langle y_1^0, x_j^* \rangle, \langle y_2, x^* \rangle \) and \( \langle y_1^0, x_0^* \rangle \), we have
\[
\begin{align*}
0 &= E \left[ \langle y_1^0(T), x_j^*(T) \rangle - E \left[ \langle y_1^0(0), x_j^*(0) \rangle \right] \right] \\
&= E \int_0^T \left[ \langle -Q (\tilde{x}_j - \dot{\tilde{x}}_j + H z_0), x_j^* \rangle + \langle F^T y_1^0 + F \tilde{y}_1^0, x^* \rangle \right. \\
&\quad + \left. \langle \tilde{G}^T \beta_1^j, x_j^* \rangle + \langle F^T y_1^1 + F \tilde{y}_1^1, \delta x_i \rangle \right] dt.
\end{align*}
\]
(19)

Therefore, considering the case when \( N \to \infty \), we introduce the first variation of the decentralized auxiliary cost functional \( \delta J_i \) as follows
\[
\delta J_i = E \int_0^T \left[ \langle Q \tilde{x}_i, \delta x_i \rangle - \langle Q (\dot{\tilde{x}} + H z_0) \rangle + \dot{\tilde{H}} Q (\dot{\tilde{x}} + \tilde{H} z) \delta x_i \right] dt.
\]
(23)

Remark 5: In (23), we ignore \( \varepsilon_{11,i} \) and (13) and introduce the first variation of the auxiliary cost functional \( \delta J_i \). Actually, \( \varepsilon_{11,i} \) and (13) have some order as \( \tilde{x}^{(N)} - \tilde{x} \), and it is sufficient to conjecture \( \| \tilde{x}^{(N)} - \tilde{x} \|_2 \to 0 \) when \( N \to \infty \) because the weakly coupled structure of our problem. The rigorous proof will be given in Section VII.

B. Decentralized strategy
Motivated by (23), one can introduce the following auxiliary problem:

Problem 3: Minimize \( J_i(u_i) \) over \( u_i \in U_i \) where
\[
\begin{align*}
dz_i &= (A z_i + B v_i + F \tilde{x}) dt \\
&\quad + (C z_i + D v_i + \tilde{G} \tilde{x} + \tilde{G} z_0) dW_i, \\
z_i(0) &= z,
\end{align*}
\]
\[
J_i(u_i) = \frac{1}{2} E \int_0^T \left[ \langle Q(z_i, z_i) - 2(S, z_i) + (R v_i, v_i) \rangle \right] dt,
\]
(24)

\[
S = Q(\dot{\tilde{x}} + H z_0) + \dot{\tilde{H}} Q(\dot{\tilde{x}} + \tilde{H} z_0) - F^T y_2 \\
- F^T \tilde{y}_1 - F \tilde{y}_1^1 - F \tilde{y}_1^0 - F \tilde{y}_1^1 - F \tilde{y}_1^1.
\]
The mean-field terms $\hat{x}$, $z_0$, $y_{21}$, $\bar{\beta}_1$, $y_0^0$, $\beta_0^0$ will be determined by the CC system in Section V. From [27], we have the following result:

**Proposition 2:** Under (H1)-(H2) and (SA), the following Riccati equation

$$\begin{aligned}
&\begin{aligned}
&\hat{P} + PA + A^T P + C^T PC + Q - (PB + C^T PD) \\
&\times (R + D^T PD)^{-1} (B^T P + D^T PC) = 0, \\
&P(T) = 0,
\end{aligned}
\end{aligned} \tag{25}$$

is strictly regularly solvable, and Problem 3 admits a feedback optimal control $\tilde{y}_1 = \Lambda_1 \hat{x} + \Lambda_2$ where

$$\begin{aligned}
&\begin{aligned}
&\Lambda_1 = -(R + D^T PD)^{-1} (B^T P + D^T PC), \\
&\Lambda_2 = - (R + D^T PD)^{-1} (B^T \varphi + D^T \eta + \varphi D (\hat{P} \hat{x} + \hat{G} z_0)),
\end{aligned}
\end{aligned} \tag{26}$$

and $(\varphi, \eta)$ satisfies

$$\begin{aligned}
&\begin{aligned}
&d\varphi = -[(A^T - (PB + C^T PD) (R + D^T PD)^{-1} B^T ) \varphi + [C^T - (PB + C^T PD) (R + D^T PD)^{-1} D^T ] \eta] \\
&+ [(PB + C^T PD) (R + D^T PD)^{-1} D^T - \hat{C} ] \\
&\times \hat{P} (\hat{x} + \hat{G} z_0) + P F \hat{x} - S dt + \eta dW_0, \\
&\varphi (T) = 0.
\end{aligned}
\end{aligned} \tag{27}$$

**V. Consistency condition**

Because of the symmetric and decentralized character, we only need a generic Brownian motion (still denoted by $W_1$) which is independent of $W_0$ to characterize the CC system.

**Proposition 3:** The undetermined parameters in Problem 3 can be determined by $(\hat{x}, z_0, y_0^0, \beta_0^0, \tilde{y}_1, y_{12}) = (E_{W_0}[z], z_0, y_0, \beta_0, E_{W_0}[\tilde{y}_1], E_{W_0}[y_{12}])$, where $(z_0, y_0, \beta_0, \tilde{y}_1, y_{12})$ is the solution of the following MF-FBSDEs:

$$\begin{aligned}
&\begin{aligned}
dz &= [(A_0 - B_0 R_0^0 - D_0) z_0 - B_0 R_0^1 - D_0] \zeta \\
&+ (F_0 - D_0) \tilde{y}_0 - D_0 \tilde{y}_1 + D_0^T \xi_0] dt + [(C_0 - D_0 R_0^0 - D_0) z_0 \\
&- D_0 R_0^1 \tilde{y}_0 - D_0 R_0^1 \tilde{y}_1 + D_0^T \xi_1] dt + (D_0 \tilde{y}_1 - D_0^T \xi_2] dt + \zeta dW_0,
\end{aligned}
\end{aligned} \tag{28}$$

Next we use discounting method to study the global solvability of FBSDEs (30). To start, we first give some results for general nonlinear forward-backward system:

$$\begin{aligned}
&\begin{aligned}
&dX(t) = b(t, X(t), E_{W_0}[X(t)]), Y(t), Z(t)] dt \\
&+ \sigma(t, X(t), E_{W_0}[X(t)], Y(t), Z(t)) dW(t),
\end{aligned}
\end{aligned} \tag{31}$$

where $W = (W_0, W_1)$, and the coefficients satisfy the following conditions:

(A1) There exist $\rho_1, \rho_2 \in \mathbb{R}$ and positive constants $k_i, i = 1, \ldots, 12$ such that for all $t, x, y, z, \tilde{z}$, a.s., left-margin:

1. $|b(t, x, y, z)| \leq \rho_1 |x - x_2|^{2}$
2. $|b(t, x, y, z)| \leq k_1 |x - x_2| + k_2 |y - y_2| + k_3 |z - z_2|$
3. $|b(t, x, y, z)| \leq k_4 |x - x_2| + k_5 |y - y_2| + k_6 |z - z_2|$
4. $|b(t, x, y, z)| \leq k_7 |x - x_2| + k_8 |y - y_2| + k_9 |z - z_2|$
5. $|b(t, x, y, z)| \leq k_{10} |x - x_2| + k_{11} |y - y_2| + k_{12} |z - z_2|$. 

with

$$\begin{aligned}
&\begin{aligned}
P := B^T P + D^T PC, \\
&\begin{aligned}
&\mathcal{R} := R + D^T PD, \\
&R_0 := R_0 + D_0 P_0 D_0.
\end{aligned}
\end{aligned}
\end{aligned} \tag{29}$$

Define $X = (z_0, \zeta, \tilde{y}_1, y_{12})^T$, $Y = (y, y_0, y_0^0, \phi, \hat{\phi})^T$, $Z = (\hat{z}_0, \beta_0^0, \beta_0^1, \eta)^T$, $Z_2 = (0^T, \beta_0^1, 0^T, 0^T)^T$, $Z = (Z_1, Z_2)$ and $W = (W_0^T, W_1^T)$. (28) take the following form:

$$\begin{aligned}
&\begin{aligned}
dX = [A_1 X + \hat{A}_1 E_{W_0}[F_{W_0}] + B_1 Y + F_1 Z_1] dt \\
&+ [C_1 X + \hat{C}_1 E_{W_0}[F_{W_0}] + D_1 Y + F_1 Z_1] dW_0 \\
&+ [C_0 X + \hat{C}_0 E_{W_0}[F_{W_0}] + D_0 Y + F_1 Z_1] dW_1,
\end{aligned}
\end{aligned} \tag{30}$$

$$\begin{aligned}
dY = [A_2 X + \hat{A}_2 E_{W_0}[F_{W_0}] + B_2 Y + B_2 E_{W_0}[Y_{W_0}] + C_2 Z_1 + \hat{C}_2 Z_2 + C_0 E_{W_0}[F_{W_0}] \hat{P} \hat{x} + \hat{G} z_0 + Z_2 dW_0 + Z_2 dW_1,
\end{aligned} \tag{31}$$

$$(z_0(0), \zeta(0), \tilde{y}_1(0), y_{12}(0)) = (0, 0, 0, 0), \quad \tilde{y}_1(T) = 0, \quad \eta(T) = 0,$$
For Remark 6: Under (H1)-(H3), there exists a constant \( \delta_1 > 0 \) depending on \( \rho_1, \rho_2, T, k_i, i = 1, 6, 7, 8, 9, 10 \) such that if \( k_i \in [0, \delta_1], i = 2, 3, 4, 5, 11, 12 \), FBSDE (31) admits a unique adapted solution \( (X, Y, Z) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m \times d}) \). Furthermore, if \( 2\rho_1 + 2\rho_2 \leq -2k_1 - 2k_2 - 2k_3^2 \), then there exists a constant \( \delta_2 > 0 \) depending on \( \rho_1, \rho_2, k_i, i = 1, 6, 7, 8, 9, 10 \) such that if \( k_i \in [0, \delta_2], i = 2, 3, 4, 5, 11, 12 \), FBSDE (31) admits a unique adapted solution \( (X, Y, Z) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m \times d}) \).

Let \( \rho_1^* \) and \( \rho_2^* \) be the largest eigenvalue of \( \frac{1}{2}(A_1 + A_1^T) \) and \( \frac{1}{2}(B_2 + B_2^T) \) respectively. Comparing (31) with (30), we can check that the parameters of (A1) can be chosen as follows:

\[
k_1 = ||A_1||, \quad k_2 = ||B_1||, \quad k_3 = ||F_1||, \quad k_4 = ||A_2||, \quad k_5 = ||A_2||, \\
k_6 = ||B_2||, \quad k_7 = ||C_2|| + ||C_2||, \quad k_8 = ||C_2||, \quad k_9 = ||C_2|| + ||C_2||, \\
k_{10} = ||C_1|| + ||C_2||, \quad k_{11} = ||D_1|| + ||D_1||, \quad k_{12} = ||F_1|| + ||F_1||.
\]

Now we introduce the following assumption:

(H3) \( 2\rho_1^* + 2\rho_2^* < -2||A_1|| - 2||B_2|| - 2(||C_2|| + ||C_2||)^2 - 2||C_2||^2 - (||C_1|| + ||C_2||)^2 - (||C_1|| + ||C_2||)^2. \)

We have the following result:

**Proposition 4:** Under (H1)-(H3), there exists a constant \( \delta_3 > 0 \) depending on \( \rho_1^* \), \( \rho_2^* \), \( k_i, i = 1, 6, 7, 8, 9, 10 \) such that if \( k_i \in [0, \delta_3], i = 2, 3, 4, 5, 11, 12 \), FBSDEs (30) admits a unique adapted solution \( (X, Y, Z) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m \times d}) \).

In what follows, we give an example to show how exactly such conditions can be applied.

**Remark 6:** For \( \varepsilon > 0 \), let \( \rho_1 = \frac{\varepsilon}{2}, \rho_2 = \frac{\varepsilon}{2}, \rho_3 = \frac{\varepsilon}{2}, \rho_4 = \frac{\varepsilon}{2}, \rho_5 = \frac{\varepsilon}{2}, \rho_6 = \frac{\varepsilon}{2}, \rho_7 = \frac{\varepsilon}{2}, \rho_8 = \frac{\varepsilon}{2}, \rho_9 = \frac{\varepsilon}{2}, \rho_10 = \frac{\varepsilon}{2}, \rho_11 = \frac{\varepsilon}{2}, \rho_12 = \frac{\varepsilon}{2} \). Then by taking expectation, the mean-field terms can be obtained by (10) and (26) respectively. Then the mean-field decentralized strategies are given by \( \bar{u}_i = \Theta_1\bar{z}_0 + \Theta_2, \bar{u}_i = \Lambda_1\bar{z}_i + \Lambda_2, \) for \( i = 1, \ldots, N \), where \( \bar{z}_0 \) and \( \bar{z}_i \) satisfy

\[
\begin{aligned}
d\bar{z}_0 &= \left((A_0 + B_0\Theta_1)\bar{z}_0 + B_0\theta_2 + F_0\mathbb{E}^{\mathbb{W}_0}[\bar{z}]\right)dt \\
&\quad + \left((C_0 + D_0\Theta_1)\bar{z}_0 + D_0\theta_2 + F_0\mathbb{E}^{\mathbb{W}_0}[\bar{z}]\right)dW_0, \\
d\bar{z}_i &= \left((A + B_1\Theta_1)\bar{z}_i + B_1\theta_2 + F_0\mathbb{E}^{\mathbb{W}_0}[\bar{z}]\right)dt \\
&\quad + \left((C + DA_1)\bar{z}_i + DA_2 + F_0\mathbb{E}^{\mathbb{W}_0}[\bar{z}] + \tilde{G}\bar{z}_0\right)dW_i, \\
\bar{z}_0(0) &= \xi_0, \quad \bar{z}_i(0) = \xi_i, \quad 1 \leq i \leq N.
\end{aligned}
\]

Through the discussion above, the mean-field decentralized strategies are characterized. In what follows, we will show some special cases and illustrate the relation between our research and the existing literature.

**VI. Special Case**

In this section, we compare our result with standard LQG control problem [27] and LQG social optima [25].

When there involves no minor agent, this problem will reduce to a standard LQG optimal control problem. By letting \( A = B = F = D = F = C = Q = H = \hat{H} = R = 0 \), we have \( x_i^{(N)} = x_i \equiv 0 \), and also by (28), we have \( z = \bar{z} = \equiv 0 \). Then by (9) and (11), we have

\[
\begin{aligned}
&\{-(P_0B_0C_0^T\Gamma_0D_0^T\Gamma_0^{-1})(R_0 + D_0^T\Gamma_0P_0D_0)\}^{-1}(B_0^T\Gamma_0P_0 + D_0^T\Gamma_0C_0^T\Gamma_0P_0C_0) + \\
&\hat{P}_0 + P_0A_0 + A_0^T\hat{P}_0 + C_0^T\Gamma_0P_0C_0 + Q_0 = 0, \quad P_0(T) = 0, \\
&\phi = 0. \quad \therefore (\Theta_1, \Theta_2) \text{ takes the following form }
\end{aligned}
\]

Such result is consistent with [27, Theorem 4.3].

On the other hand, when there involves no major agent, this problem will reduce to a problem of mean-field control in social optima. By letting \( A_0 = B_0 = F_0 = C_0 = D_0 = F = G = Q_0 = H_0 = \hat{H}_0 = R_0 = 0 \), we have \( x_0 \equiv 0 \). The CC system becomes:

\[
\begin{aligned}
dz &= \{(A - BR^C)\varphi + (F - BR^D)B^T\varphi + (F - DR^C)\Phi(\mathbb{E}^{\mathbb{W}_0}[\bar{z}]) \}dt \\
&\quad + \{(C - DR^C)\varphi + DR^D \}d\mathbb{E}^{\mathbb{W}_0}[\bar{z}]dW_1(t), \\
d\bar{g}_0 &= \{H(1 - \hat{H})\mathbb{E}[\bar{z}] - \tilde{G}\mathbb{E}[\bar{z}]\}dt, \\
d\bar{g}_1 &= \{H(1 - \hat{H})\mathbb{E}[\bar{z}] - \tilde{G}\mathbb{E}[\bar{z}]\}dt, \\
&\quad + \{(A - \hat{H})\mathbb{E}[\bar{z}] - \tilde{G}\mathbb{E}[\bar{z}]\}dt, \\
&\quad + \{(A - \hat{H})\mathbb{E}[\bar{z}] - \tilde{G}\mathbb{E}[\bar{z}]\}dt.
\end{aligned}
\]

which is consistent with the result of [25, equation (33)].

Moreover, \((\Lambda_1, \Lambda_2)\) takes the following form

\[
\begin{aligned}
\Lambda_1 &= -(R + D^T\Gamma_0P_D)^{-1}(B^T\varphi + D^T\tilde{F}), \\
\Lambda_2 &= -(R + D^T\Gamma_0P_D)^{-1}(B^T\varphi + D^T\tilde{F}),
\end{aligned}
\]

which is also consistent with [25].

Through the discussion above, we compare our result with some previous literature. For the next part, we will study the performance of the mean-field strategy. Specifically, we will prove its asymptotic optimality.
VII. ASYMPTOTIC $\varepsilon$-OPTIMALITY

Definition 1: A mixed strategy set \( \{u_i^0 \in U_i\} \) is called asymptotically $\varepsilon$-optimal if there exists $\varepsilon = \varepsilon(N) > 0$, \( \lim_{N \to \infty} \varepsilon(N) = 0 \) such that
\[
\begin{aligned}
& J_0(u_0^0, u_0^0) \leq \inf_{u_0 \in U_0} J_0(u_0, u_0^0) + \varepsilon, \\
& J_{soc}(u_0^0, u_0^0) - \inf_{u_0 \in U_0} J_{soc}(u_0, u_0^0) \leq \varepsilon,
\end{aligned}
\]
where $u_0^0 := \{u_{01}^0, \ldots, u_{0N}^0\}$. In this case, $u_0^0$, $u_0^0$ achieve an asymptotic $\varepsilon$-equilibrium, and $u_{01}^0, \ldots, u_{0N}^0$ achieve an asymptotic $\varepsilon$-social optimum.

Let $\tilde{u}$ be the mixed-field strategy given in Section V and the realized decentralized states $(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_N)$ satisfy:
\[
\begin{aligned}
& d\tilde{x}_0 = (A\tilde{x}_0 + B\tilde{u}_0 + F_0\tilde{z}(N)) dt, \\
& + (C_0\tilde{x}_0 + D_0\tilde{u}_0 + F_0\tilde{z}(N)) dW_0, \quad \tilde{x}_0(0) = \xi_0 \\
& d\tilde{x}_1 = (A\tilde{x}_1 + B\tilde{u}_1 + F_0\tilde{z}(N)) dt \\
& + [C\tilde{x}_1 + D\tilde{u}_1 + \tilde{F}\tilde{z}(N) + \tilde{G}\tilde{x}_0] dW_1, \quad \tilde{x}_1(0) = \xi_1,
\end{aligned}
\]
where $\tilde{r}(N) = \frac{1}{N} \sum_{i=1}^N \tilde{x}_i$. First, we need some estimations. In the proofs below, we will use $K$ to denote a generic constant whose value may change from line to line.

Lemma 1: [11, Lemma 5.1] Under (H1)-(H3) and (SA), there exists a constant $K_1$ independent of $N$ such that
\[
\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} \|\tilde{x}_i(t)\|^2 + \sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} \|\tilde{z}_i(t)\|^2 \leq K_1.
\]

Proof: It is easy to get that
\[
d(\tilde{x}_N(t) - E W_0[z]) = (A + F)(\tilde{x}_N(t) - E W_0[z]) dt + \frac{1}{N} \sum_{i=1}^N C\tilde{x}_i + D\tilde{u}_i + \tilde{F}\tilde{x}_0 + \tilde{G}\tilde{x}_0 \]dW_1.
\]
Therefore,
\[
\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} \|\tilde{x}_N(s) - E W_0[z]\|^2 & \leq K_1 \sup_{0 \leq s \leq t} \|\tilde{x}_N(s) - E W_0[z]\|^2 \\
& \leq C\tilde{x}_i + D\tilde{u}_i + \tilde{F}\tilde{x}_0 + \tilde{G}\tilde{x}_0 dW_1.
\end{aligned}
\]
By
\[
\begin{aligned}
& d \left( \frac{1}{N} \sum_{i=1}^N \tilde{z}_i - E W_0[z] \right) = \left( (A + B\Lambda_1) \left( \frac{1}{N} \sum_{i=1}^N \tilde{z}_i - E W_0[z] \right) \right) dt \\
& + \frac{1}{N} \sum_{i=1}^N (C + D\Lambda_1) \tilde{z}_i dW_1 + \tilde{F}E W_0[z] + \tilde{G}\tilde{z}_0 dW_1,
\end{aligned}
\]
\[
\left( \frac{1}{N} \sum_{i=1}^N \tilde{z}_i - E W_0[z] \right)(0) = 0.
\]
it is easy to get
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left\| \frac{1}{N} \sum_{i=1}^N \tilde{z}_i - E W_0[z] \right\|^2 = O \left( \frac{1}{N} \right).
\]

Then by Burkholder-Davis-Gundy inequality, we have
\[
\begin{aligned}
K & \mathbb{E} \int_0^t \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 ds \\
& = K \mathbb{E} \int_0^t \left\| \tilde{z}_N(s) - E W_0[z] \right\|^2 ds \\
& \leq K \mathbb{E} \int_0^t \left\| \tilde{z}_N(s) - E W_0[z] \right\|^2 ds + K \sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} \|\tilde{x}_i(t)\|^2.
\end{aligned}
\]
Finally, it follows from Gronwall’s inequality, and Lemma 1 that there exists a constant $K_2$ independent of $N$ such that
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 \leq K_2.
\]

Lemma 2: Under (H1)-(H3) and (SA), there exists a constant $K_2$ independent of $N$ such that
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 \leq K_2.
\]

Proof: It is easy to check that
\[
d(\tilde{x}_i - \tilde{z}_i) = \left[A(\tilde{x}_i - \tilde{z}_i) + F(\tilde{x}_N(t) - E W_0[z]) \right] dt \\
+ \left[C(\tilde{x}_i - \tilde{z}_i) + \tilde{F}(\tilde{x}_N(t) - E W_0[z]) + \tilde{G}(\tilde{x}_0 - \tilde{z}_0) \right] dW_i,
\]
and
\[
d(\tilde{x}_i - \tilde{z}_i) \leq \left[A(\tilde{x}_i - \tilde{z}_i) + F(\tilde{x}_N(t) - E W_0[z]) \right] dt \\
+ \left[C(\tilde{x}_i - \tilde{z}_i) + \tilde{F}(\tilde{x}_N(t) - E W_0[z]) + \tilde{G}(\tilde{x}_0 - \tilde{z}_0) \right] dW_i.
\]
Therefore, it follows from Burkholder-Davis-Gundy inequality that
\[
\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 & \leq K_1 \mathbb{E} \int_0^t \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 ds \\
& \leq K_1 \mathbb{E} \int_0^t \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 ds \\
& \leq K_1 \mathbb{E} \int_0^t \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2 ds + K_2 \mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - E W_0[z] \right\|^2.
\end{aligned}
\]

Lemma 3: Under (H1)-(H3) and (SA), there exists a constant $K_3$ independent of $N$ such that
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - \tilde{z}_N(s) \right\|^2 \leq K_3.
\]

Proof: It is easy to check that
\[
\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - \tilde{z}_N(s) \right\|^2 \\
& \leq K_1 \mathbb{E} \int_0^t \left\| \tilde{x}_N(s) - \tilde{z}_N(s) \right\|^2 ds \\
& \leq K_1 \mathbb{E} \int_0^t \left\| \tilde{x}_N(s) - \tilde{z}_N(s) \right\|^2 ds + K_2 \mathbb{E} \sup_{0 \leq s \leq t} \left\| \tilde{x}_N(s) - \tilde{z}_N(s) \right\|^2.
\end{aligned}
\]
Therefore, it follows from Gronwall’s inequality and Lemma 2 that
\[
\sup_{0 \leq t \leq T} \mathbb{E} \sup_{0 \leq t' \leq t} \| \tilde{x}_i(t) - \tilde{z}_i(t) \|^2 \leq \frac{K_8}{N}.
\]

A. Major agent

Lemma 4: Under (H1)-(H3) and (SA),
\[\mathcal{J}_0(\tilde{u}_0, \tilde{u}_0) - J_0(\tilde{v}_0) = O\left(\frac{1}{\sqrt{N}}\right)\cdot\]

Proof: Recall (3) and (8), it follows from
\[
\mathcal{J}_0(\tilde{u}_0, \tilde{u}_0) - J_0(\tilde{v}_0)
= \frac{1}{2} \mathbb{E} \int_0^T \langle Q_0(\tilde{x}_0 - H_0 \tilde{z}(N)), \tilde{x}_0 - H_0 \tilde{z}(N) \rangle dt
- \langle Q_0(\tilde{z}_0 - H_0 \tilde{z}(N)), \tilde{z}_0 - H_0 \tilde{z}(N) \rangle dt
\leq KE \int_0^T \left[ \| \tilde{x}_0 - \tilde{z}_0 \|^2 + \| \tilde{z}(N) - \tilde{z}(N) \|^2 \right] dt = O\left(\frac{1}{\sqrt{N}}\right),
\]
where the last equality follows from Lemmas 1-3.

B. Minor agents

1) Representation of social cost: Rewrite the large-population system (1) and (2) as follows:
\[
dx = (Ax + Bu)dt + \sum_{i=0}^N (C_i x + D_i u)dw_i,
\]
where
\[
A = \begin{pmatrix}
A_0 & \dot{P}_0 & \cdots & \dot{P}_0 \\
0 & A + \ddot{P} & \cdots & \ddot{P} \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & \ddots
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
B_0 & 0 & \cdots & 0 \\
0 & B & \cdots & 0 \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]
\[
C = \begin{pmatrix}
C_0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]
\[
D = \begin{pmatrix}
D_0 & 0 & \cdots & 0 \\
0 & D & \cdots & 0 \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & 0
\end{pmatrix},\]
\[
\Xi = \begin{pmatrix}
\tilde{z}_0 \\
\xi
\end{pmatrix},
\]

Similarly, the social cost takes the following form:
\[
\mathcal{J}_{soc}^{(N)}(u)
= \frac{1}{2} \sum_{i=1}^N \mathbb{E} \int_0^T \left[ \langle Q_i(x_i - Hx_i - \tilde{H}x^{(N)}), (x_i - Hx_i - \tilde{H}x^{(N)}) \rangle + \langle Ru_i, u_i \rangle \right] dt
\leq \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Qx, x \rangle + \langle Ru, u \rangle \right] dt,
\]
where
\[
Q = \begin{pmatrix}
Q_{00} & Q_{01} & \cdots & Q_{0N} \\
Q_{10} & Q_{11} & \cdots & Q_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{N0} & Q_{N1} & \cdots & Q_{NN}
\end{pmatrix},
\]
\[
R = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]
and for \(i = 1, \cdots, N, j \neq i,\)
\[
Q_{0j} = NQ + \tilde{H}^T \tilde{Q} \tilde{H} - \tilde{Q} \tilde{H} + \tilde{H}^T Q,
\]
\[
Q_{ij} = -\tilde{H}^T QH + QH,
\]
\[
Q_{ij} = -HQH + HQ,
\]
Next, define the following operators
\[
(Lu)(s) := \Phi(\cdot) \int_0^s \Phi(s) - 1 (B - \sum_{i=0}^N C_i D_i) u(s) ds + \sum_{i=0}^N \left\{ \Phi(s) - 1 D_i \phi \right\} \phi dt,
\]
\[
(Lu)(s) := (Lu)(\cdot)(T), \hspace{1cm} \Gamma(\Xi) := \Phi(\cdot) \phi^{-1}(0) \Xi, \hspace{1cm} \Gamma \Xi := (\Xi)(T).
\]
Correspondingly, $L^*$ is defined as the adjoint operator of $L$. Hence, we can rewrite the cost functional as follows:

$$2\mathcal{J}_{soc}^{(N)}(u) = \langle (L^*QL + R)u(\cdot), u(\cdot) \rangle + 2\langle L^*QI_{\Xi}(\cdot), u(\cdot) \rangle + \langle QI_{\Xi}(\cdot), \Gamma_{\Xi}(\cdot) \rangle$$

$$= : \langle M_2u(\cdot), u(\cdot) \rangle + 2\langle M_1, u(\cdot) \rangle + M_0. \quad (37)$$

Note that, $M_2$ is a self-adjoint positive semidefinite bounded linear operator.

2) Minor agent’s perturbation: Let us consider the case that the minor agent $A_i$ uses an alternative strategy $u_i$ while the major agent and all other minor agents $A_j, j \neq i$ use the strategies $\bar{u} - i$. The realized states with the $i^{th}$ minor agent’s perturbation are

$$d\tilde{x}_0 = \left[ A_0\tilde{x}_0 + B_0\bar{u}_0 + F_0\tilde{E}^{(N)} \right] dt + \left[ C_0\tilde{x}_0 + D_0\bar{u}_0 + F_0\tilde{E}^{(N)} \right] dW_0,$$

$$d\tilde{x}_i = \left[ A_i\tilde{x}_i + B_iu_i + F_i\tilde{E}^{(N)} \right] dt + \left[ C_i\tilde{x}_i + Du_i + F_i\tilde{E}^{(N)} + \tilde{G}_z \right] dW_i,$$

$$y_i(0) = \xi_0, \quad y_i(t) = \xi, \quad 1 \leq j \leq N, \quad j \neq i,$$

where $\tilde{E}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i$. Moreover, introduce the following system:

$$d\tilde{l}_0 = \left[ A_0\tilde{l}_0 + B_0\bar{u}_0 + F_0\tilde{E}^{(N)} \right] dt + \left[ C_0\tilde{l}_0 + D_0\bar{u}_0 + F_0\tilde{E}^{(N)} \right] dW_0,$$

$$d\tilde{l}_j = \left[ A_i\tilde{l}_j + B_ju_j + F_i\tilde{E}^{(N)} \right] dt + \left[ C_i\tilde{l}_j + Du_j + F_i\tilde{E}^{(N)} + \tilde{G}_z \right] dW_j,$$

$$\tilde{l}_0(0) = \xi_0, \quad \tilde{l}_i(0) = \xi, \quad 1 \leq j \leq N,$$

and

$$d\bar{l}_0 = \left[ A_0\bar{l}_0 + B_0\bar{u}_0 + F_0\bar{E}^{(N)} \right] dt + \left[ C_0\bar{l}_0 + D_0\bar{u}_0 + F_0\bar{E}^{(N)} \right] dW_0,$$

$$d\bar{l}_i = \left[ A_i\bar{l}_i + B_iu_i + F_i\bar{E}^{(N)} \right] dt + \left[ C_i\bar{l}_i + Du_i + F_i\bar{E}^{(N)} + \bar{G}_z \right] dW_i,$$

$$\bar{l}_0(0) = \xi_0, \quad \bar{l}_i(0) = \bar{\xi}_i(0) = \xi, \quad 1 \leq j \leq N, \quad j \neq i,$$

where $\tilde{E}^{(N)} = \sum_{i=1}^{N} \tilde{E}_i$ and $\bar{E}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \bar{E}_i$. Similar to Lemma \textbf{1} we have

$$E \sup_{0 \leq t \leq T} \|E_i(t)\|^2 + E \sup_{0 \leq t \leq T} \|\bar{E}_i(t)\|^2 \leq K \left( 1 + \|u_i\|^2_{L^2} \right),$$

$$E \sup_{0 \leq t \leq T} \|\bar{E}_i(t)\|^2 \leq K,$$

where $\bar{E}_i$ is defined by (14), (16), (18) and (22).

Similar to the computation in Section \textbf{IV.A} we have

$$\delta \mathcal{J}_{soc}^{(N)} = E \int_0^T \left[ \langle Q\bar{x}_i, \delta x_i \rangle + \langle R\bar{u}_i, \delta u_i \rangle + \langle Q(\bar{H}\bar{x} + Hz_0) + \bar{H}Q(\bar{x} - H\bar{x} - Hz_0) - F^T y_0 - F^T \bar{W}_0 [\bar{y}_0] \right. \right.$$

$$\left. - \bar{F}^T \bar{W}_0 [\bar{y}_0] - F^T y_0 - F^T \bar{W}_0 [\bar{y}_0] \rangle dW_i + \sum_{i=1}^{13} (t) \varepsilon_i(t), \right]$$

where $\varepsilon_1, \cdots, \varepsilon_{13i}$ are defined by (14), (16), (18) and (22).

Finally, we have

$$\delta \mathcal{J}_{soc}^{(N)} = E \int_0^T \left[ \langle Q\bar{x}_i, \delta x_i \rangle + \langle R\bar{u}_i, \delta u_i \rangle + \langle Q(\bar{H}\bar{x} + Hz_0) + \bar{H}Q(\bar{x} - H\bar{x} - Hz_0) - F^T y_0 - F^T \bar{W}_0 [\bar{y}_0] \right. \right.$$

$$\left. - \bar{F}^T \bar{W}_0 [\bar{y}_0] - F^T y_0 - F^T \bar{W}_0 [\bar{y}_0] \rangle dW_i + \sum_{i=1}^{15} \varepsilon_i(t), \right]$$

At the end of this subsection, we give some lemmas which will be used in subsection 3)

\textbf{Lemma 6:} Under (H1)-(H3) and (SA), there exists a constant $K_N$ independent of $N$ such that

$$E \sup_{0 \leq t \leq T} ||\bar{x}(N)(t) - \bar{E}^{(N)}(t)||^2 \leq K_N \left( \frac{1}{N} + \frac{\|u_i\|^2_{L^2}}{N^2} \right).$$

\textbf{Proof:} Note that

$$d\bar{x}(N) = \left[ A\bar{x}(N) + \frac{1}{N} BA_1 \sum_{j=1}^{N} \bar{z}_j - \frac{1}{N} BA_1 \bar{z}_i + \frac{1}{N} BA_1 - \frac{1}{N} BA_2 \right] dt + \frac{1}{N} \left[ C\bar{x}(N) + \bar{E}^{(N)} \right] dW_i,$$

and similar to [22, equation (5.7)] we have

$$d\bar{W}_0[z] = \left[ \left( A - B R^{-1} (B_1^T P + D_1^T PC_1) \right) \bar{E}^{(N)} \right] dt$$

$$- B R^{-1} D_1^T P \bar{E}^{(N)} + \bar{F}^T \bar{E}^{(N)} \right] dW_i,$$

Thus,

$$d(\bar{x}(N) - \bar{E}^{(N)}(t))$$

$$= \left[ (A + F)(\bar{x}(N) - \bar{E}^{(N)}(t)) + BA_1 \left( \frac{1}{N} \sum_{j=1}^{N} \bar{z}_j - \bar{E}^{(N)}(t) \right) \right] dt$$

$$+ \frac{1}{N} B(u_i - \Lambda_1 \bar{z}_i - \Lambda_2) \right] dW_i$$

$$+ \frac{1}{N} \left[ C\bar{x}(N) + \bar{E}^{(N)} \right] dW_i$$

$$+ \frac{1}{N} \sum_{j \neq i} \left[ C\bar{x}(N) + \bar{E}^{(N)} \right] dW_j,$$
By Burkholder-Davis-Gundy inequality and (34), we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^{(N)}(t) - E^{W_0}[z] \right\|^2 \\
\leq K \mathbb{E} \int_0^T \left[ \left\| x^{(N)}(t) - E^{W_0}[z] \right\|^2 + \frac{1}{NT^2} \left\| z_j(t) \right\|^2 + \frac{1}{N^2} \left\| u_i \right\|^2 + \frac{1}{N^2} \right] ds + O \left( \frac{1}{N} \right)
\]
\[
+ \frac{K}{N^2} \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \left[ C \delta x_j + D \delta u_i + \tilde{F}^{(N)} + \tilde{G} \right] dW_j \|^2
\]
\[
+ \frac{K}{N^2} \mathbb{E} \sup_{0 \leq t \leq T} \sum_{i \neq i} \int_0^t \left[ C \delta x_j + D \delta u_i + \tilde{F}^{(N)} + \tilde{G} \right] dW_j \|^2
\]
Finally, it follows from Gronwall’s inequality that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^{(N)}(t) - E^{W_0}[z] \right\|^2 \leq K_9 \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right)
\]

Moreover, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \delta x_0(t) \right\|^2 \leq K \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right),
\]
and
\[
\sup_{1 \leq j \leq N, j \neq i} \mathbb{E} \sup_{0 \leq t \leq T} \left\| \delta x_j(t) \right\|^2 \leq K \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right).
\]

Lemma 9: There exist constants $K_{14}, K_{15}, K_{16}$ independent of $N$ such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^{**} - \delta x_{-0,i} \right\|^2 \leq K_{14} \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right),
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^*_0 - N \delta x_0 \right\|^2 \leq K_{15} \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right),
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^*_j - N \delta x_j \right\|^2 \leq K_{16} \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right).
\]

Proof: First, we have the following dynamics
\[
\begin{aligned}
d(x^{**} - \delta x_{-0,i}) &= \left[ (A + F)(x^{**} - \delta x_{-0,i}) + \frac{1}{N} (F \delta x_i + \delta x_{-0,i}) \right] dt \\
- \sum_{i \neq i} \left[ C \delta x_j + \tilde{F}^{(N)} + \tilde{G} \right] dW_j,
\end{aligned}
\]
\[
d(x^*_0 - N \delta x_0) = \left[ A_0(x^*_0 - N \delta x_0) + F_0(x^{**} - \delta x_{-0,i}) \right] dt \\
+ \left[ C_0(x^*_0 - N \delta x_0) + \tilde{F}_0(x^{**} - \delta x_{-0,i}) \right] dW_0,
\]
\[
d(x^*_j - N \delta x_j) = \left[ A(x^*_j - N \delta x_j) + F(x^{**} - \delta x_{-0,i}) \right] dt \\
+ \left[ C(x^*_j - N \delta x_j) + \tilde{F}(x^{**} - \delta x_{-0,i}) \right] dW_j.
\]
Therefore, it follows from Burkholder-Davis-Gundy inequality that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^{**}(s) - \delta x_{-0,i}(s) \right\|^2 \\
\leq K \mathbb{E} \int_0^t \left[ \left\| x^{**}(s) - \delta x_{-0,i}(s) \right\|^2 + \frac{1}{N^2} \right] ds + K \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{i \neq i} \int_0^t \left[ \delta x_j + \delta x_{(N)} + \delta x_{0} \right] dW_j \right\|^2
\]
\[
\leq K \left( 1 + \frac{1}{N^2} \right) \left\| x^{**}(s) - \delta x_{-0,i}(s) \right\|^2 \\
+ K \left( 1 + \frac{1}{N^2} \right) \left\| \delta x_j \right\|^2 + \left\| \delta x_{(N)} \right\|^2 + \left\| \delta x_0 \right\|^2 ds.
\]
It then follows from Gronwall’s inequality and Lemma 8 that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| x^{**}(t) - \delta x_{-0,i}(t) \right\|^2 \leq K \left( \frac{1}{N^2} + \frac{\left\| u_i \right\|^2}{N^2} \right).
\]
Similarly, we have the other two estimations.
3) Asymptotic optimality: In order to prove asymptotic optimality for the minor agents, it suffices to consider the perturbations \( u_{-0} \in U_{-0} \) such that \( J_{soc}^{(N)}(\tilde{u}_0, u_{-0}) \leq J_{soc}^{(N)}(\tilde{u}_0, \bar{u}_{-0}) \). It is easy to check that \( J_{soc}^{(N)}(\tilde{u}_0, \bar{u}_{-0}) \) is a constant independent of \( N \). Therefore, in what follows, we only consider the perturbations \( u_{-0} \in U_{-0} \) satisfying \( \sum_{j=1}^{N} f_j(u_{-0}) \leq K \). Let \( \delta u_i = u_i - \bar{u}_i \), and consider a perturbation \( u = \tilde{u} + (0, \delta u_1, \ldots, \delta u_N) := \tilde{u} + \delta u \). Then by Section VII-B.1 we have

\[
2J_{soc}^{(N)}(\tilde{u} + \delta u) = (M_2(\tilde{u} + \delta u), \bar{u} + \delta u) + 2(M_1, \tilde{u} + \delta u) + M_0
\]

where \( (M_2u + M_1, \delta u_i) \) is the Fréchet differential of \( J_{soc}^{(N)} \) on \( \tilde{u} \) with variation \( \delta u_i \). Therefore, in order to prove asymptotic optimality for the minor agents, we only need to show that \( \langle M_2\tilde{u} + M_1, \delta u \rangle = o(N) \). To this end, we introduce another assumption:

(H4) There exists constants \( L_1, L_2 > 0 \) independent of \( N \) such that

\[
E \int_0^T \left\| E^{W_0}[y_1] - \frac{1}{N} \sum_{j \neq i} y_j \right\|^2 dt \leq \frac{L_1}{N},
\]

\[
E \int_0^T \left\| E^{W_0}[\beta_1]\right\|^2 dt \leq \frac{L_2}{N}.
\]

Theorem 3: Under (H1)-(H4) and (SA), \( (\tilde{u}_1, \ldots, \tilde{u}_N) \) is an asymptotically \( \varepsilon \)-optimal strategy for the minor agents whose individual cost functionals are given by (4) and the social cost is given by (5).

Proof: From Section VII-B.2 we have

\[
\langle M_2\tilde{u} + M_1, \delta u \rangle
\]

\[
= \sum_{i=1}^{N} \int_0^T \left[ (Q_i\tilde{l}_i, \delta l_i) + (S_i, \delta l_i) + (R\tilde{l}_i, \delta u_i) \right] dt + \sum_{i=1}^{N} \sum_{i=1}^{15} \xi_i l_i.
\]

From the optimality of \( \bar{u} \), we have

\[
\int_0^T \left[ (Q_i\tilde{l}_i, \delta l_i) + (S_i, \delta l_i) + (R\tilde{l}_i, \delta u_i) \right] dt = 0.
\]

Moreover, it follows from Lemma [10] and (H4) that

\[
\sum_{i=1}^{N} \sum_{i=1}^{15} \xi_i l_i = O(\sqrt{N}).
\]

Therefore, \( \langle M_2\tilde{u} + M_1, \delta u \rangle = O(\sqrt{N}) \).

Remark 7: Note that

\[
d\left( \frac{1}{N} \sum_{j \neq i} \tilde{x}_j - E^{W_0}[\tilde{x}_1] \right)
\]

\[
= \left[ A \left( \frac{1}{N} \sum_{j \neq i} \tilde{x}_j - E^{W_0}[\tilde{x}_1] \right) + F \left( \tilde{x}(N) - E^{W_0}[\tilde{x}(N)] \right) \right]
\]

\[
+ \frac{1}{N} \sum_{j \neq i} \left[ C_\tilde{x}_j + D\tilde{l}_i + \tilde{F}(N) \tilde{x}_j \right] dt
\]

Therefore, it follows from Burkholder-Davis-Gundy inequality and Gronwall’s inequality that

\[
E \sup_{0 \leq t \leq T} \left\| \frac{1}{N} \sum_{j \neq i} \tilde{x}_j(s) - E^{W_0}[\tilde{x}_1] \right\|^2 \leq \frac{K}{N}
\]

If \( C = 0 \), applying Itô’s formula to \( \frac{1}{N} \sum_{j \neq i} y_j - E^{W_0}[y_1] \), it is easy to check that (39) in (H4) holds.

Remark 8: If the state has the following form

\[
\begin{aligned}
\delta z_0 &= (A_0 x_0 + B_0 u_0 + F_0 x_0) dt + (C_0 x_0 + D_0 u_0 + \tilde{F}_0 x_0) \tilde{W}_0, \\
\delta x_i &= (A_i x_i + B_i u_i + F_i x_i + G_i x_0) dt + D_d W_i, \\
x_0(0) &= \xi_0, \quad x_i(0) = \xi, \quad 1 \leq i \leq N.
\end{aligned}
\]

Assumption (H4) is not needed to obtain the asymptotic optimality of the minor agents. However, if the state equations of the minor agents take the form (2), we need to suppose the assumption (H4) hold and we will continue to study this in the future work.

Last, by combining Theorems 2, 3 and recalling the Definition 1, we have the following result

Theorem 4: The mean-field strategies \( \tilde{u}_0, \tilde{u}_{-0} \) achieve an asymptotic \( \varepsilon \)-equilibrium between the major agent and the aggregation of minor agents, where \( \tilde{u}_0 = \Theta_1 \bar{x}_0 + \Theta_2, \tilde{u}_{-0} = (\bar{u}_1, \ldots, \bar{u}_N) \) and \( \bar{u}_i = \Lambda_i \bar{x}_0 + \Lambda_2 \). Moreover \( \tilde{u}_1, \ldots, \tilde{u}_N \) achieve an asymptotic \( \varepsilon \)-social optimum among the aggregation of minor agents. Thus, \( (\tilde{u}_0, \tilde{u}_{-0}) \) is asymptotically \( \varepsilon \)-optimal.

Proof: By Theorem 2, we have

\[
J_0(\tilde{u}_0, \tilde{u}_{-0}) \leq \inf_{u_{-0} \in U_{-0}} J_0(u_0, \tilde{u}_{-0} + O(\sqrt{N}).
\]

Moreover, by Theorem 3, we have

\[
\frac{1}{N} J_{soc}^{(N)}(\tilde{u}_0, \tilde{u}_{-0}) \leq \inf_{u_{-0} \in U_{-0}} J_{soc}^{(N)}(\tilde{u}_0, u_{-0}) \leq O(\sqrt{N}).
\]

Thus, by Definition 1, Theorem 4 holds straightforwardly.
\[Q_0 = (0.6210 \ 0.8691), Q = (0.8701 \ 0.1925), R_0 = (0.7160 \ 0.0.5984), R = (0.3885 \ 0.4182)\]. It is easy to see that such generated coefficients are constants and surely \(L^\infty\) and Lipschitz continuous. Thus, assumption (H1)-(H2) hold.

In the following simulation, we will calculate the feedback form mean-field strategies and also the corresponding state trajectories of major and minor agents. The convergence of the population average will also be simulated.

Firstly, we solve (28) by decentralizing method and decoupling method. As we mentioned above, the CC system can be rewritten as (30). Then by letting \(X_1 := E[X|F^W]\), \(Y_1 := E[Y|F^W]\), \(X_2 := (X - X_1)\), \(Y_2 := (Y - Y_1)\).

\[
\begin{align*}
&dX_1 = \left( (A_1 + \bar{A}_1)X_1 + B_1Y_1 + F_1E[Z_1|F^W] \right) dt \\
&\quad + \left[ (C^0_1 + \bar{C}_1)X_1 + D^0_1EY_1 + F^1_1E[Z_1|F^W] \right] dW_0,
\end{align*}
\]

\[
\begin{align*}
&dX_2 = \left( (\bar{A}_1X_1 - B_1Y_1 + F_1(Z_1 - E[Z_1|F^W]) \right) dt \\
&\quad + \left[ C^0_1X_2 + D_1^0Y_1 + F_0^1(E[Z_1|F^W]) \right] dW_0,
\end{align*}
\]

\[
\begin{align*}
&dY_1 = \left( (\bar{A}_2 + \bar{A}_2)X_1 + (\bar{B}_2 + \bar{B}_2)Y_1 + C_2E[Z_1|F^W] \right) dt \\
&\quad + \left[ (\bar{C}_1 + \bar{C}_2)E[Z_2|F^W] \right] dt + E[Z_1|F^W]dW_0,
\end{align*}
\]

\[
\begin{align*}
&dY_2 = \left[ (A_2X_1 - B_2Y_1 + C_2(Z_1 - E[Z_1|F^W]) \right) dt \\
&\quad + \left[ \bar{C}_2(Z_2 - E[Z_2|F^W]) \right] dW_2 + (Z_1 - E[Z_1|F^W])dW_0 + Z_2dW_1,
\end{align*}
\]

\[
X(0) = (\xi_0^T, \xi^T)^T, \quad Y(T) = (0^T, 0^T, 0^T, 0^T)^T.
\]

Further, by letting \(E[Z_1|F^W] = Z_1^1, (Z_1 - Z_1^1) = Z_1^2\), we can rewrite as

\[
\begin{align*}
&d(\xi_0^T, \xi^T)^T = \left[ (A_1 + \bar{A}_1)X_1 + (\bar{B}_1 0)Y_1 + \bar{F}_1E[Z_1|F^W] \right] dt \\
&\quad + \left[ (\bar{C}^0_1 + \bar{C}_1)X_1 + \bar{D}^0_1EY_1 + \bar{F}^1_1E[Z_1|F^W] \right] dW_0,
\end{align*}
\]

\[
\begin{align*}
&d(\xi_0^T, \xi^T)^T = \left[ (A_2 + \bar{A}_2)X_1 + (\bar{B}_2 + \bar{B}_2)Y_1 + \bar{C}_1E[Z_1|F^W] \right] dt \\
&\quad + \left[ \bar{C}_2(Z_2 - E[Z_2|F^W]) \right] dW_0 + \left[ 0^T, 0^T, 0^T, 0^T \right]^T.
\end{align*}
\]

We have

\[
\begin{align*}
&d(\xi_0^T, \xi^T)^T = \left[ A_1(\xi_0^T, \xi^T)^T + B_1(\xi_0^T, \xi^T)^T + C_1(Z_1^1, Z_1^2) \right] dt \\
&\quad + \left[ A_2(\xi_0^T, \xi^T)^T + B_2(\xi_0^T, \xi^T)^T + C_2(Z_1^1, Z_1^2) \right] dW_0
\end{align*}
\]

We can introduce the following Riccati equation

\[
\begin{align*}
\dot{K} + KA_1 + KB_1K - A_4 - B_4K + (KC_1 - C_4) \times (I - KC_2)^{-1}(KA_2 + KB_2K) = 0, \\
K(T) = 0.
\end{align*}
\]

Then it is easy to verify that

\[
\begin{align*}
&Y_1 = K(\xi_0^T, \xi^T)^T, \\
&Z_1^2 = (I - KC_2)^{-1}(KA_2 + KB_2K)(\xi_0^T, \xi^T)^T, \\
&0 = (KA_3 + KB_3K)(\xi_0^T, \xi^T)^T + KC_3(Z_1^1, Z_1^2).
\end{align*}
\]

Then (44) becomes a decoupled BSDE

\[
\begin{align*}
&d(\xi_0^T, \xi^T)^T = \left[ A_1(\xi_0^T, \xi^T)^T + C_1(I - KC_2)^{-1}(KA_2 + KB_2K) \right] (\xi_0^T, \xi^T)^T dt \\
&\quad + \left[ A_2(\xi_0^T, \xi^T)^T + C_2(I - KC_2)^{-1}(KA_2 + KB_2K) \right] (\xi_0^T, \xi^T)^T dW_0 \\
&\quad + \left[ A_3 + B_3K + C_3(I - KC_2)^{-1}(KA_2 + KB_2K) \right] (\xi_0^T, \xi^T)^T dW_1.
\end{align*}
\]

Then by using Euler–Maruyama method, Milstein method and Runge–Kutta method, \((\xi, \xi_0, \eta, \eta_0, \eta_1, \eta_2)\) can be obtained. Further, by (10) and (26), we can calculate \(\Theta_1, \Theta_2, \Lambda_1, \Lambda_2\). Then, the realized states can be obtained by (1) and (2). The following graphs are the first coordinate of the
realized states.  

Then, the corresponding feedback form mean-field controls can be obtained as well. The following graphs are the first coordinate of the mean-field controls.

Next, we simulate the convergence of the population state average \( \bar{x}(N)(t) \) to the mean field \( \bar{x} \). Specifically, we will calculate \( \mathbb{E} \sup_{0 \leq t \leq T} \| \bar{x}(N)(t) - \bar{x} \|^2 \). First, \( \sup_{0 \leq t \leq T} \| \bar{x}(N)(t) - \bar{x} \|^2 \) can be calculated directly. Second, for the expectation, we repeat such process enough times (200 times) and take the average to simulate it.

The relation between \( \mathbb{E} \sup_{0 \leq t \leq T} \| \bar{x}(N)(t) - \bar{x} \|^2 \) and \( N \) can be fitted by \( \mathbb{E} \sup_{0 \leq t \leq T} \| \bar{x}(N)(t) - \bar{x} \|^2 = \frac{441}{N} \) with R-square 0.9944.

In this sense, \( \mathbb{E} \sup_{0 \leq t \leq T} \| \bar{x}(N)(t) - \bar{x} \|^2 = O\left( \frac{1}{N} \right) \).

By the simulation above, we can see that the mean-field strategy is asymptotically optimal.

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**IX. APPENDIX**

For any given \((Y, Z) \in L^2_T (0, T; R^m) \times L^2_T (0, T; R^{m \times (d+1)})\) and \(0 \leq t \leq T\), the following SDE has a unique solution:

\[
X(t) = x + \int_0^t b(s, X(s), EW_0[X(s)], Y(s), Z(s))ds + \int_0^t \sigma(s, X(s), EW_0[X(s)], Y(s), Z(s))dW(s).
\]

Therefore, we can introduce a map \( M_1 : (Y, Z) \in L^2_T (0, T; R^m) \times L^2_T (0, T; R^{m \times (d+1)}) \to X \in L^2_T (0, T; R^n) \) by \( (Y, Z) \mapsto X \). Moreover, we have the following result:

**Lemma 10:** Let \( X_i \) be the solution of \( \text{(46)} \) corresponding to \( (Y_i, Z_i) \), \( i = 1, 2 \) respectively. Then for all \( \rho \in \mathbb{R} \) and some
constant $l_1 > 0$, we have

$$\mathbb{E}e^{-\rho t}||\hat{X}(t)||^2 + \rho \mathbb{E} \int_0^t e^{-\rho s}||\hat{X}(s)||^2 ds$$

$$\leq (k_2l_1 + k_1^2l_1) \mathbb{E} \int_0^t e^{-\rho(t-s)}||\hat{Y}(t)||^2 ds$$

$$+ (k_2l_2 + k_1^2l_2) \mathbb{E} \int_0^t e^{-\rho(t-s)}||\hat{Z}(t)||^2 ds,$$

and

$$\mathbb{E}e^{-\rho t}||\hat{X}(t)||^2 \leq (k_2l_1 + k_1^2l_1) \mathbb{E} \int_0^t e^{-\rho(t-s)}||\hat{Y}(t)||^2 ds$$

$$+ (k_2l_2 + k_1^2l_2) \mathbb{E} \int_0^t e^{-\rho(t-s)}||\hat{Z}(t)||^2 ds,$$

where $\hat{\rho}_1 = \rho - 2\rho_{12} - 2k_1 - k_2l_{12} - k_3l_{12}^2 - k_3^2 l_{10}$ and $\hat{\Phi} = \Phi_1 - \Phi_2$, $\Phi = X, Y, Z$. Moreover,

$$\mathbb{E} \int_0^T e^{-\rho t}||\hat{X}(t)||^2 dt$$

$$\leq \frac{1 - e^{-\hat{\rho}_1 T}}{\hat{\rho}_1} \left[ (k_2l_1 + k_1^2l_1) \mathbb{E} \int_0^T e^{-\rho t}||\hat{Y}(t)||^2 dt$$

$$+ (k_2l_2 + k_1^2l_2) \mathbb{E} \int_0^T e^{-\rho t}||\hat{Z}(t)||^2 dt \right],$$

and

$$e^{-\rho T} \mathbb{E} ||\hat{X}(T)||^2 \leq (1 \vee e^{-\hat{\rho}_1 T}) \left[ (k_2l_1 + k_1^2l_1) \mathbb{E} \int_0^T e^{-\rho t}||\hat{Y}(t)||^2 dt$$

$$+ (k_2l_2 + k_1^2l_2) \mathbb{E} \int_0^T e^{-\rho t}||\hat{Z}(t)||^2 dt \right].$$

Specifically, if $\hat{\rho}_1 > 0$,

$$e^{-\rho T} \mathbb{E} ||\hat{X}(T)||^2 \leq (k_2l_1 + k_1^2l_1) \mathbb{E} \int_0^T e^{-\rho t}||\hat{Y}(t)||^2 dt$$

$$+ (k_2l_2 + k_1^2l_2) \mathbb{E} \int_0^T e^{-\rho t}||\hat{Z}(t)||^2 dt.$$

**Proof:** For any $\rho > 0$, applying Itô’s formula to $e^{-\rho t}||\hat{X}(t)||^2$,

$$\mathbb{E}e^{-\rho t}||\hat{X}(t)||^2 + \rho \mathbb{E} \int_0^t e^{-\rho s}||\hat{X}(s)||^2 ds$$

$$= 2\mathbb{E} \int_0^t e^{-\rho s} \hat{X}(s) \left( b(s, X_1(s), E^{W_0}[X_1(s)], Y_1(s), Z_1(s))$$

$$- b(s, X_2(s), E^{W_0}[X_2(s)], Y_2(s), Z_2(s))) ds$$

$$+ \mathbb{E} \int_0^t e^{-\rho s} \left( \sigma(s, X_1(s), E^{W_0}[X_1(s)], Y_1(s), Z_1(s))$$

$$- \sigma(s, X_2(s), E^{W_0}[X_2(s)], Y_2(s), Z_2(s))) ds$$

$$\leq \mathbb{E} \int_0^t e^{-\rho s} \left[ (2\rho_1 + 2k_1 + k_2l_{12}^1 + k_3l_{12}^2 + k_3^2 + k_3^2 l_{10}) ||\hat{X}(s)||^2$$

$$+ (k_2l_1 + k_1^2l_1)||\hat{Y}(s)||^2 + (k_2l_2 + k_1^2l_2)||\hat{Z}(s)||^2 \right] ds.$$
\[ \mathbb{E} \int_0^T e^{-\rho t} \| Y_1(t) - Y_2(t) \|^2 dt + \mathbb{E} \int_0^T e^{-\rho t} \| Z_1(t) - Z_2(t) \|^2 dt \leq \left[ \frac{1 - e^{-\tilde{\rho}_1 T}}{\tilde{\rho}_2} + 1 \right] \left( k_4 l_3 + k_5 l_4 \right) \times \mathbb{E} \int_0^T e^{-\rho t} \| X_1(t) - X_2(t) \|^2 dt \]

\[ \leq \left[ \frac{1 - e^{-\tilde{\rho}_2 T}}{\tilde{\rho}_2} + \frac{1}{1 - k_7 l_5 - k_8 l_6} \right] \frac{1 - e^{-\tilde{\rho}_1 T}}{\tilde{\rho}_1} \left( k_4 l_3 + k_5 l_4 \right) \times \left( k_2 l_1 + k_4 l_3 \right) \int_0^T e^{-\rho t} \| U_1(t) - U_2(t) \|^2 dt \]

Choosing suitable \( \rho \), we get that \( \mathcal{M} \) is a contraction mapping.

Furthermore, if \( 2\rho_1 + 2\rho_2 < -2k_1 - 2k_6 - 2k_9^2 - 2k_8^2 - k_9^2 - k_10^2 \), we can choose \( \rho \in \mathbb{R}, 0 < k_7 l_5 < \frac{1}{2} \) and \( 0 < k_8 l_6 < \frac{1}{2} \) and sufficient large \( \tilde{l}_1, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4 \) such that \( \tilde{\rho}_1 > 0, \tilde{\rho}_2 > 0, 1 - k_7 l_5 - k_8 l_6 > 0 \).

Therefore,

\[ \mathbb{E} \int_0^T e^{-\rho t} \| Y_1(t) - Y_2(t) \|^2 dt + \mathbb{E} \int_0^T e^{-\rho t} \| Z_1(t) - Z_2(t) \|^2 dt \leq \left[ \frac{1}{\tilde{\rho}_2} + \frac{1}{1 - k_7 l_5 - k_8 l_6} \right] \frac{1}{\tilde{\rho}_1} \left( k_4 l_3 + k_5 l_4 \right) \times \left( k_2 l_1 + k_4 l_3 \right) \int_0^T e^{-\rho t} \| U_1(t) - U_2(t) \|^2 dt \]

\[ + (k_3 l_2 + k_5 l_2) \int_0^T e^{-\rho t} \| V_1(t) - V_2(t) \|^2 dt \]

The proof is complete.