Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem

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Abstract

For each $\beta \in (0, +\infty)$ there exists a canonical measure $\mu_\beta$ on the ring $A_f$ of finite adeles. We show that $Q^*_+$ acts ergodically on $(A_f, \mu_\beta)$ for $\beta \in (0, 1]$, and then deduce from this the uniqueness of KMS$\beta$-states for the Bost-Connes system.

Bost and Connes [BC] constructed a remarkable C$^*$-dynamical system which has a phase transition with spontaneous symmetry breaking involving a action of the Galois group $\text{Gal}(Q^{ab}/Q)$, and whose partition function is the Riemann $\zeta$ function. In their original defi-
nition the underlying algebra arises as the Hecke algebra associated with an inclusion of certain $ax + b$ groups. Recently Laca and Raeburn [LR, L2] have realized the Bost-Connes algebra as a full corner of the crossed product algebra $C_0(A_f) \times Q^*_+$. This new look at the system has allowed to simplify significantly the proof of the existence of KMS-states for all temperatures, and the proof of the phase transition theorem for $\beta > 1$ [L1]. On the other hand, for $\beta \leq 1$ the uniqueness of KMS$\beta$-states implies the ergodicity of the action of $Q^*_+$ on $A_f$ for certain measures (in particular, for the Haar measure). The aim of this note is to give a direct proof of the ergodicity, and then to show that the uniqueness of KMS$\beta$-states easily follows from it.

Though the proof of the Bost-Connes phase transition theorem (for $\beta \leq 1$) thus obtained differs from the proofs given in [BC] and [L1], it is entirely based on these papers. In particular, the key point is an application of Dirichlet’s theorem.

So let $\mathcal{P}$ be the set of prime numbers, $A_f$ the restricted product of the fields $Q_p$, $p \in \mathcal{P}$, of $p$-adic numbers, $\mathcal{R} = \prod_p \mathbb{Z}_p$ its maximal compact subring, $W = \mathcal{R}^* = \prod_p \mathbb{Z}_p^*$. The group $Q^*_+$ of positive rationals is embedded diagonally into $A_f$, and so acts by multiplications on the additive group of finite adeles. Then the Bost-Connes algebra $C_Q$ is the full corner of $C_0(A_f) \times Q^*_+$ determined by the characteristic function of $\mathcal{R}$ [L2]. The dynamics $\sigma_t$ is defined as follows [L1]; it is trivial on $C_0(A_f)$, and $\sigma_t(\lambda(q)) = q^{it}\lambda(q)$, where $\lambda(q)$ is the multiplier of $C_0(A_f) \times Q^*_+$ corresponding to $q \in Q_+^*$. Then ([L1]) there is a one-to-one correspondence between $(\beta, \sigma_t)$-KMS-states on $C_Q$ and measures $\mu$ on $A_f$ such that

$$\mu(\mathcal{R}) = 1 \text{ and } q_+^\beta \mu \text{ for all } q \in Q^*_+ \text{ (i.e., } \mu(q^{-1}X) = q^{-\beta} \mu(X)). \quad (1\beta)$$

Namely, the KMS-state corresponding to $\mu$ is the restriction of the dual weight on $C_0(A_f) \times Q^*_+$ to $C_Q$.

Note that if $\beta > 1$ and $\mu$ is a measure with the property (1$\beta$) then $\mu(W) = \frac{1}{\zeta(\beta)} > 0$, since $W = \mathcal{R}\setminus \cup_p p\mathcal{R}$. Moreover, the sets $qW, q \in Q^*_+$, are disjoint, and their union is a set of full measure (since $\sum_{n \in \mathbb{N}} \mu(nW) = 1$). Thus there exists a one-to-one correspondence between
probability measures on $W$ and measures on $A_f$ satisfying (1$\beta$). On the other hand, if $\beta \leq 1$ then $\mu(W) = 0$.

For each $\beta \in (0, +\infty)$ there is a unique $W$-invariant measure $\mu_\beta$ satisfying (1$\beta$). Explicitly, $\mu_\beta = \otimes_p \mu_{\beta,p}$, where $\mu_{\beta,p}$ is the measure on $Q_p$ such that $\mu_{1,p}$ is the Haar measure ($\mu_{1,p}(Z_p) = 1$), and

$$\frac{d\mu_{\beta,p}}{d\mu_{1,p}}(a) = \frac{1 - p^{-\beta}}{1 - p^{-1}} |a|_{p}^{\beta - 1} \text{ for } a \in Q_p.$$ 

In fact, for the proof of Proposition below we will only need to know that the restriction of $\mu_{\beta,p}$ to $Z_p^*$ is a (non-normalized) Haar measure.

**Proposition.** The action of $Q_+^*$ on $(A_f, \mu_\beta)$ is ergodic for $\beta \in (0, 1]$.

**Proof.** Consider the space $L^2(\mathcal{R}, d\mu_\beta)$ and the subspace $H$ of it consisting of the functions that are constant on $N$-orbits. In other words, $H = \{f \in L^2(\mathcal{R}, d\mu_\beta) | V_nf = f, n \in \mathbb{N}\}$, where $(V_nf)(x) = f(nx)$. Since any $Q_+^*$-invariant subset of $A_f$ is completely determined by its intersection with $\mathcal{R}$, it suffices to prove that $H$ consists of constant functions. For this we will compute the action of the projection $P : L^2(\mathcal{R}, d\mu_\beta) \to H$ on a basis of $L^2(\prod_{p \in B} Z_p, \otimes_{p \in B} \mu_{\beta,p})$ (considered as a subspace of $L^2(\mathcal{R}, d\mu_\beta)$) for each finite subset $B$ of $\mathcal{P}$.

Let $\chi$ be a character of $\prod_{p \in B} Z_p^*$. Consider $\chi$ first as a function on $\prod_{p \in B} Z_p^*$ by letting $\chi = 0$ outside of $\prod_{p \in B} Z_p^*$. Then using the projection $\mathcal{R} \to \prod_{p \in B} Z_p^*$, consider $\chi$ as a function on $\mathcal{R}$. Let $\mathbb{N}_B$ be the unital multiplicative subsemigroup of $\mathbb{N}$ generated by $p \in B$. Note that the sets $n \prod_{p \in B} Z_p^*$, $n \in \mathbb{N}_B$, are disjoint, their union is a subset of $\prod_{p \in B} Z_p^*$ of full measure, and the operator $n^{-\beta / 2}V_n^*$ maps isometrically $L^2(\prod_{p \in B} Z_p^*, \otimes_{p \in B} \mu_{\beta,p})$ onto $L^2(n \prod_{p \in B} Z_p^*, \otimes_{p \in B} \mu_{\beta,p})$ for any $n \in \mathbb{N}_B$. Hence the functions $V_n^*\chi$, $n \in \mathbb{N}_B$, $\chi \in (\prod_{p \in B} Z_p^*)^*$, form an orthogonal basis for $L^2(\prod_{p \in B} Z_p^*, \otimes_{p \in B} \mu_{\beta,p})$. So we have to compute $PV_n^*\chi$. But if $g \in H$ then $(V_n^*\chi, g) = (\chi, g)$, whence $PV_n^*\chi = P\chi$. Thus we have only to compute $P\chi$.

For a finite subset $A$ of $\mathcal{P}$, let $H_A$ be the subspace consisting of the functions that are constant on $N_A$-orbits, $P_A$ the projection onto $H_A$. Then $P_A \setminus P$ as $A \setminus \mathcal{P}$. Set

$$W_A = \prod_{p \in A} Z_p^* \times \prod_{q \in \mathcal{P} \setminus A} Z_q \subset \mathcal{R}.$$ 

Note, as above, that $\cup_{n \in N_A} nW_A$ is a subset of $\mathcal{R}$ of full measure. We state that

$$P_A f|_{N_A} \equiv \frac{1}{\zeta_A(\beta)} \sum_{n \in N_A} n^{-\beta} f(nx) \text{ for } x \in W_A,$$ 

where $\zeta_A(\beta) = \sum_{n \in N_A} n^{-\beta} = \prod_{p \in A} (1 - p^{-\beta})^{-1}$. Indeed, denoting the right hand part of (2) by $f_A$, for $g \in H_A$ we obtain

$$(f_A, g) = \sum_{n \in N_A} \int_{nW_A} f_A(x)g(x) d\mu_\beta(x) = \sum_{n \in N_A} n^{-\beta} \int_{W_A} f_A(x)g(x) d\mu_\beta(x)$$

$$= \zeta_A(\beta) \int_{W_A} f_A(x)g(x) d\mu_\beta(x) = \sum_{n \in N_A} n^{-\beta} \int_{W_A} f(nx)g(nx) d\mu_\beta(x)$$

$$= \sum_{n \in N_A} \int_{nW_A} f(x)g(x) d\mu_\beta(x) = (f, g).$$

Returning to the computation of $P\chi$, we see that

$$P_A \chi|_{N_A} \equiv \frac{\chi(x)}{\zeta_A(\beta)} \sum_{n \in N_A} n^{-\beta} \chi(n) = \chi(x) \prod_{p \in A} \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \text{ for } x \in W_A.$$
Thus if $\chi$ is trivial then $P_A \chi \equiv \prod_{p \in \mathcal{P}} (1 - p^{-\beta})$ for all $A \supset B$, hence $P \chi$ is a constant. If $\chi$ is non-trivial then since $||P_A \chi||_\infty \leq 1$ and the product $\prod_{p: \text{Re}(\chi(p)) < 0} (1 - p^{-\beta})$ diverges by Dirichlet’s theorem [S], we have $P \chi = 0$.

\[\text{Corollary, [BC]}\] For $\beta \in (0, 1]$ there exists a unique $(\beta, \sigma_t)$-KMS state on $\mathcal{C}_Q$.

\textbf{Proof.} Let $\phi_\beta$ be the KMS$_\beta$-state corresponding to $\mu_\beta$. Since $L^\infty(A_f, d\mu_\beta) \times \mathbb{Q}_+^*$ is a factor by Proposition, and $\pi_{\phi_\beta}(\mathcal{C}_Q)'$ is its reduction, $\phi_\beta$ is a factor state. This and the discussion before Proposition show that

(i) $\phi_\beta$ is an extremal KMS$_\beta$-state;

(ii) $\phi_\beta$ is a unique $W$-invariant KMS$_\beta$-state.

Now the proof is finished as in [BC, Theorem 25]:

If $\psi$ is an extremal KMS$_\beta$-state then $\int_W w_\ast \psi \, dw = \phi_\beta$. Since KMS$_\beta$-states form a simplex, we conclude that $\psi = \phi_\beta$.

\[\text{Remarks.}\]

(i) The expression for $P \chi$ in the proof of Proposition shows that the divergence of the product

$$\prod_{p \in \mathcal{P}} \left| \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \right|$$

for non-trivial $\chi$ is a necessary condition for the ergodicity (otherwise $P \chi$ would be a non-zero function, which can not be constant since $\int_R P \chi \, d\mu_\beta = \int_R \chi \, d\mu_\beta = 0$), hence for the uniqueness of KMS$_\beta$-states. So the appearance of (some form of) Dirichlet’s theorem in the proofs is not an accident.

(ii) By [BC, Theorem 5] $\pi_{\phi_\beta}(\mathcal{C}_Q)'$ is a factor of type III for $\beta \in (0, 1]$. Then the factor $L^\infty(A_f, d\mu_\beta) \times \mathbb{Q}_+^*$ is also of type III. Hence its smooth flow of weights is trivial, that means that the action of $\mathbb{Q}_+^*$ on $(\mathbb{R}_+ \times A_f, dt \otimes d\mu_\beta)$ is ergodic [CT]. In particular, the spectral subspaces of $L^\infty(A_f, d\mu_\beta)$ corresponding to the characters $q \mapsto \xi^q$ of $\mathbb{Q}_+^*$ have to be trivial for all $t \neq 0$.

But the projection $P_t$ onto the subspace $\{f \mid V_\alpha f = n^t f\}$ of $L^2(\mathcal{R}, d\mu_\beta)$ is computed with the same ease as in the proof of Proposition:

$$P_t = s - \lim_{A \not\supset \mathcal{P}} P_{t,A}, \quad (P_{t,A} f)(m,x) = \frac{m^it}{\zeta_A(\beta)} \sum_{n \in \mathbb{N}_A} n^{-\beta-it} f(nx) \quad \text{for } x \in W_A, \, m \in \mathbb{N}_A.$$

Thus the product

$$\prod_{p \in \mathcal{P}} \left| \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta-it}} \right|$$

has to be divergent for all $t \neq 0$ and all number characters $\chi$ modulo $m$. 
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