Kalb-Ramond fields in the Petiau-Duffin-Kemmer formalism and scale invariance

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Abstract

Kalb-Ramond equations for massive and massless particles are considered in the framework of the Petiau-Duffin-Kemmer formalism. We obtain $10 \times 10$ matrices of the relativistic wave equation of the first-order and solutions in the form of density matrix. The canonical and Belinfante energy-momentum tensors are found. We investigate the scale invariance and obtain the conserved dilatation current. It was demonstrated that the conformal symmetry is broken even for massless fields.

1 Introduction

Antisymmetric tensor massless fields introduced in [1], [2] give dual description of the massless scalar fields (associated with zero helicity). Such fields appear naturally in the (super)string theories and D-branes [3], [4], and supergravity [5]. The anomaly cancelation in the superstring theories are due to antisymmetric tensor fields. These fields also play an important role in the noncommutative geometry [6], and cosmology [7], [8]. The specific coupling term with antisymmetric tensor fields can generate an effective topological mass for the massless vector fields [9], [10]. Different aspects of the theory of antisymmetric tensor fields were considered in [11], [12], [13].

We investigate the Kalb-Ramond fields for massive and massless particles in the framework of the first-order relativistic wave equation (RWE) (Petiau-Duffin-Kemmer [14], [15], [16] formalism).

The paper is organized as follows. In Sec.2, we consider massive and massless fields in the form of RWE of the first-order with two mass parameters. Solutions of the wave equation for free fields are obtained in the form of projection matrices in Sec.3. We obtain the canonical and symmetrical
Belinfante energy momentum tensors and the dilatation currents in Sec. 4. It is shown that the scale invariance is broken in the case of massive fields but in the massless case the modified dilatation current is conserved. We make a conclusion in Sec. 5.

The Euclidean metric is explored and the system of units $\hbar = c = 1$ is used.

2 The first-order RWE

Let us introduce equations with two mass parameters for the third rank antisymmetric tensor field

$$\partial_\mu \psi_{[\mu\nu\alpha]} + m_1 \psi_{[\nu\alpha]} = 0,$$

$$\partial_\mu \psi_{[\nu\alpha]} + \partial_\alpha \psi_{[\mu\nu]} + \partial_\nu \psi_{[\alpha\mu]} + m_2 \psi_{[\mu\nu\alpha]} = 0.$$

that are the generalization of [1], [2]. Putting $m_1 = 0$, and renormalizing fields $m_2 \psi_{[\mu\nu\alpha]} \rightarrow -\phi_{\mu\nu\alpha}$, we arrive at the Kalb-Ramond massless field equations. At $m_1 = m_2 = m$, one has the fields with the mass $m$. In the general case, fields possess the mass $m = \sqrt{m_1 m_2}$. The fields $\psi_{[\mu\nu]}, \psi_{[\mu\nu\alpha]}$ have the same dimension. Consider the wave function

$$\Psi(x) = \{\psi_A(x)\} = \left( \begin{array}{c} \psi_{[\mu\nu]}(x) \\ \psi_{[\mu\nu\alpha]}(x) \end{array} \right) \quad (A = [\mu\nu], [\mu\nu\alpha]).$$

(2)

With the help of the elements of the entire matrix algebra $\varepsilon^{A,B}$, with matrix elements and products [17]:

$$\varepsilon^{M,N}_{AB} = \delta_{MA} \delta_{NB}, \varepsilon^{M,A} \varepsilon^{B,N} = \delta_{AB} \varepsilon^{M,N},$$

where $A, B, M, N = [\mu\nu], [\mu\nu\alpha]$, and generalized Kronecker symbols

$$\delta_{[\mu\nu][\alpha\beta]} = \delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha},$$

$$\delta_{[\mu\nu\alpha][\nu\beta\gamma]} = \delta_{\mu\alpha} \delta_{\nu\beta} \delta_{\nu\gamma} + \delta_{\mu\beta} \delta_{\nu\gamma} \delta_{\alpha\nu} + \delta_{\mu\gamma} \delta_{\nu\beta} \delta_{\alpha\nu}$$

$$- \delta_{\mu\alpha} \delta_{\nu\beta} \delta_{\nu\gamma} - \delta_{\mu\beta} \delta_{\nu\gamma} \delta_{\alpha\nu} - \delta_{\mu\gamma} \delta_{\nu\beta} \delta_{\alpha\nu},$$

(3)

the system of equations (1) becomes the first-order RWE

$$(\beta_\mu \partial_\mu + m_1 P_1 + m_2 P_2) \Psi(x) = 0,$$

(4)

where we have introduced $10 \times 10$ matrices

$$\beta_\mu = \frac{1}{2} (\varepsilon^{[\nu\alpha][\mu\nu\alpha]} + \varepsilon^{[\mu\nu\alpha][\nu\alpha]}), \quad P_1 = \frac{1}{2} \varepsilon^{[\mu\nu][\mu\nu]}, \quad P_2 = \frac{1}{6} \varepsilon^{[\mu\nu\alpha][\mu\nu\alpha]}.$$
A summation over all repeated indices is implied. The Hermitian matrices $\beta^\mu_\mu = \beta_\mu$ obey the PDK algebra

$$\beta_\mu \beta_\nu \beta_\alpha + \beta_\alpha \beta_\nu \beta_\mu = \delta_\mu \nu \beta_\alpha + \delta_\alpha \nu \beta_\mu,$$

and (Hermitian) projection matrices $P_1, P_2$ satisfy the relations as follows:

$$P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 + P_2 = 1, \quad P_1 P_2 = P_2 P_1 = 0,$$

$$\beta_\mu P_1 + P_1 \beta_\mu = \beta_\mu, \quad \beta_\mu P_2 + P_2 \beta_\mu = \beta_\mu.$$

(6)

The operator $P_1$ extracts the six-dimensional subspace $(\psi_{[\mu \nu \alpha]})$ of the wave function $\Psi$, and the projection operator $P_2$ extracts the four-dimensional tensor subspace corresponding to the $\psi_{[\mu \nu \alpha]}$. At $m_1 = m_2 = m$, Eq.(4), becomes the PDK equation for massive fields

$$(\beta_\mu \partial_\mu + m) \Psi(x) = 0.$$

(7)

The massless PDK equation at $m_1 = 0$ reads

$$(\beta_\mu \partial_\mu + m_2 P_2) \Psi(x) = 0.$$

(8)

The generators of the Lorentz group are given by

$$J_{\mu \nu} = \beta_\mu \beta_\nu - \beta_\nu \beta_\mu = \varepsilon_{[\mu \nu], [\mu \nu]} - \varepsilon_{[\nu \mu], [\nu \gamma]} + \frac{1}{2} \left( \varepsilon_{[\mu \gamma \sigma], [\nu \gamma \sigma]} - \varepsilon_{[\nu \gamma \sigma], [\mu \gamma \sigma]} \right),$$

(10)

and obey the commutation relations

$$[J_{\rho \sigma}, J_{\mu \nu}] = \delta_{\sigma \mu} J_{\rho \nu} - \delta_{\rho \nu} J_{\sigma \mu} - \delta_{\mu \rho} J_{\sigma \nu} - \delta_{\sigma \nu} J_{\mu \rho},$$

$$[\beta_\lambda, J_{\mu \nu}] = \delta_{\lambda \mu} \beta_\nu - \delta_{\lambda \nu} \beta_\mu, \quad [P_1, J_{\mu \nu}] = 0, \quad [P_2, J_{\mu \nu}] = 0.$$  

(11)

The Hermitianizing matrix, $\eta$, is defined as

$$\eta = 2\beta_4^2 - 1 = \frac{1}{2} \varepsilon^{[4bc],[4bc]} - \frac{1}{6} \varepsilon_{[abc],[abc]} + \frac{1}{2} \varepsilon_{[ab],[ab]} - \varepsilon_{[4b],[4b]},$$

and the Lorentz-invariant is \(\Psi \Psi = \Psi^+ \eta \Psi \) [18], where $\Psi^+$ is the Hermitian-conjugated wave function. The Hermitian matrix $\eta^+ = \eta$ obeys the relations: $\eta \beta_\mu = -\beta_\mu \eta$ (m=1,2,3), $\eta \beta_4 = \beta_4 \eta$. Using these relations, we obtain the conjugated equation

$$\overline{\Psi}(x) \left( \beta_\mu \partial_\mu - m_1 P_1 - m_2 P_2 \right) = 0.$$

(13)
The Lagrangian can be chosen in the form

\[ L = -\frac{1}{2} \bar{\Psi}(x) \left( \beta_{\mu} \partial_{\mu} + m_1 P_1 + m_2 P_2 \right) \Psi(x) + \frac{1}{2} \bar{\Psi}(x) \left( \beta_{\mu} \partial_{\mu} - m_1 P_1 - m_2 P_2 \right) \Psi(x). \]  

Eq. (14) is obtained by varying the action corresponding to Lagrangian (13). Let us consider for simplicity the neutral fields. With the help of Eq. (2), (12), one finds the conjugated function

\[ \bar{\Psi}(x) = \left( \psi_{[\mu\nu]}(x), -\psi_{[\mu\nu\alpha]}(x) \right). \]  

The Lagrangian (14) with the help of Eq. (2), (5), (15) becomes

\[ L = \frac{1}{2} \psi_{[\mu\nu\alpha]} \partial_{\mu} \psi_{[\nu\alpha]} - \frac{1}{2} \psi_{[\mu\alpha]} \partial_{\mu} \psi_{[\nu\alpha]} - \frac{1}{2} m_1 \psi_{[\mu\nu]}^2 + \frac{1}{6} m_2 \psi_{[\mu\nu\alpha]}^2. \]  

Euler-Lagrange equations give the equations of motion (1). Lagrangians (14), and (16) become zero for fields \( \psi_A \) obeying Eq. (1) similar to the Dirac theory.

### 3 Solutions to the matrix equation

The solution to Eq. (8) for massive fields, in the form of the projection operator in the momentum space, for the positive \((+p)\) and negative \((-p)\) energies, is given by (see, for example, [17])

\[ \Pi_{\pm} = \frac{\pm i \hat{p} (\pm i \hat{p} - m)}{2m^2}, \]  

where \( \hat{p} = \beta_{\mu} p_{\mu} \), and the four-momentum is \( p_{\mu} = (p, ip_0) \) \((p^2 = p^2 - p_0^2)\).

Every column of the matrix (17) is the solution to Eq. (8) in the momentum space, on-shell, when \( p^2 = -m^2 \). The Matrix \( \Pi_{\mp} \) is the projection operator, \( \Pi_{\pm} = \Pi_{\mp}^2 \), and represents the density matrix.

For the case of massless particles, \( m_1 = 0 \), the matrix of the equation (9), in the momentum space (for the positive energy), is \( \Lambda(0) = i \hat{p} + m_2 P_2 \) and satisfies the minimal matrix equation for any \( p^2 \):

\[ \Lambda(0) \left( \Lambda(0) - m_2 \right) \left[ \Lambda(0) \left( \Lambda(0) - m_2 \right) + p^2 \right] = 0. \]  

(18)
The solution to Eq.(9), in the form the density matrix (off-shell) is given by

\[ \Pi^{(0)} = -\frac{1}{m_2 p^2} \left( \Lambda^{(0)} - m_2 \right) \left[ \Lambda^{(0)} \left( \Lambda^{(0)} - m_2 \right) + p^2 \right] = \frac{p_\mu p_\nu \varepsilon^{[\mu\gamma], [\nu\gamma]}}{2p^2}, \] (19)

so that the \( \Pi^{(0)} \) is the projection operator, \( \Pi^{(0)2} = \Pi^{(0)} \). It should be noted that Eq.(19) is valid only off-shell, \( p^2 \neq 0 \). If \( p^2 = 0 \), Eq.(18) becomes

\[ \Lambda^{(0)2} \left( \Lambda^{(0)} - m_2 \right)^2 = 0, \] (20)

and zero eigenvalues of the operator \( \Lambda^{(0)} \) are degenerated. In this case it is impossible to obtain solutions in the form of projection matrix \( [19] \). As Kalb-Ramond fields are longitudinal fields (with the helicity \( h = 0 \)), we do not need the spin operators. The density matrix (17) can be used for different quantum calculations with Kalb-Ramond fields. These fields also give the dual description of scalar fields (because the helicity is zero).

4 The energy-momentum tensor and scale invariance

From Lagrangian (14), using the general procedure \( [18] \), we obtain the canonical energy-momentum tensor

\[ T^c_{\mu\nu} = \frac{1}{2} \left( \partial_\nu \overline{\Psi}(x) \right) \beta_\mu \Psi(x) - \frac{1}{2} \overline{\Psi}(x) \beta_\mu \partial_\nu \Psi(x). \] (21)

We took into account that the Lagrangian (14) vanishes for fields obeying the equations of motion. With the help of Eq.(2),(5),(15), the canonical energy-momentum tensor (21) becomes

\[ T^c_{\mu\nu} = \frac{1}{2} \left[ \psi_{[\mu\beta\alpha]} \partial_\nu \psi_{[\beta\alpha]} - \psi_{[\beta\alpha]} \partial_\nu \psi_{[\mu\beta\alpha]} \right]. \] (22)

One can verify, with the aid of Eq.(1), that the energy-momentum tensor (22) is conserved tensor, \( \partial_\mu T^c_{\mu\nu} = 0 \). The canonical energy-momentum tensor is not the symmetric tensor, and its trace does not equal zero:

\[ T^c_{\mu\mu} = \frac{1}{2} m_1 \psi^2_{[\mu\nu]} - \frac{1}{6} m_2 \psi^2_{[\mu\nu\alpha]}. \] (23)
To investigate the dilatation symmetry, we explore the method of [20] adopted to first-order formalism [21]. The canonical dilatation current is given by [20]

\[ D^c_\mu = x_\alpha T^c_{\mu \alpha} + \Pi_\mu d \Psi, \quad (24) \]

where

\[ \Pi_\mu = \frac{\partial L}{\partial (\partial_\mu \Psi)} = -\overline{\Psi} \beta_\mu, \quad (25) \]

and the matrix \( d \) is defined by the field dimension. In the case of Bose fields the \( d \) is the unit matrix. From Eq. (24), (25), one finds

\[ \partial_\mu D^c_\mu = T^c_{\mu \mu}, \quad (26) \]

where we use the conservation of the current \( \partial_\mu j_\mu = \partial_\mu (i\overline{\Psi} \beta_\mu \Psi) = 0 \). It should be noted that for neutral fields the electric current vanishes, \( j_\mu = 0 \). One can verify with the help of Eq. (2), (5), (15) that \( \overline{\Psi} \beta_\mu \Psi = 0 \). The relation (26) also follows from [20]

\[ \partial_\mu D^c_\mu = \Pi_\mu (d + 1) \partial_\mu \Psi + \overline{\Psi} d \frac{\partial L}{\partial \Psi} + \overline{\Psi} d \left( \overline{\Psi} \beta_\mu \Psi \right) = 1 \]

(27)

The dilatation symmetry is broken due to the presence of massive parameters \( m_1 \) and \( m_2 \). For the massless fields, \( m_1 = 0 \), the dilatation current \( D^c_\mu \) is also not conserved, but we will introduce new conserved current.

Let us consider the Belinfante tensor [20]

\[ T^B_{\mu \alpha} = T^c_{\mu \alpha} + \partial_\beta X_{\beta \mu \alpha}, \quad (28) \]

where

\[ X_{\beta \mu \alpha} \equiv \frac{1}{2} \left[ \Pi_\beta J_{\mu \alpha} \Psi - \Pi_\beta J_{\mu \alpha} \Psi - \Pi_\alpha J_{\beta \mu} \Psi \right]. \quad (29) \]

From Eq. (2), (5), (10), (15), (25), we find the tensor \( X_{\beta \mu \alpha} \):

\[ X_{\beta \mu \alpha} = \overline{\Psi} \beta_\beta \alpha \beta_\mu \Psi - \delta_{\mu \alpha} \overline{\Psi} \beta_\beta \Psi \]

(30)
\[ \frac{1}{2} \left( \delta_{\beta\alpha} \psi[\mu\lambda\sigma] \psi[\lambda\sigma] - \delta_{\alpha\mu} \psi[\beta\lambda\sigma] \psi[\lambda\sigma] \right) - 2 \psi[\beta\lambda\mu] \psi[\alpha\lambda]. \]

Using expressions (28),(30), and equations of motion, one obtains the Belinfante energy-momentum tensor

\[ T_{\rho\nu}^B = -2m_1 \psi[\rho\alpha] \psi[\nu\alpha] + \delta_{\rho\nu} \left( \frac{1}{2} m_1 \psi^2[\alpha\beta] + \frac{1}{6} m_2 \psi^2[\mu\alpha\beta] \right) \]
\[ + \psi[\rho\alpha\beta] \partial_\nu \psi[\alpha\beta] - 2 \psi[\mu\alpha\rho] \partial_\mu \psi[\nu\alpha]. \]  

From Eq.(31), we find the trace of the Belinfante energy-momentum tensor

\[ T_{\mu\mu}^B = -\frac{1}{3} m_2 \psi^2[\mu\alpha\beta]. \]  

The modified Belinfante dilatation current is defined as [20]

\[ D_{\mu}^B = x_\alpha T_{\mu\alpha}^B + V_\mu, \]  

where the field-virial \( V_\mu \) is

\[ V_\mu = \Pi_\mu \psi - \Pi_\alpha J_{\alpha\mu} \psi = -\nabla_\beta \psi + \nabla_\alpha J_{\alpha\mu} \psi. \]  

With the help of (2),(5),(15), one obtains

\[ V_\mu = -\frac{1}{2} \psi[\mu\alpha\beta] \psi[\alpha\beta], \quad \partial_\mu V_\mu = \frac{1}{2} m_1 \psi^2[\mu\nu] + \frac{1}{6} m_2 \psi^2[\mu\nu\alpha]. \]  

As a result, we find

\[ \partial_\mu D_{\mu}^B = T_{\mu\mu}^B + \partial_\mu V_\mu = \frac{1}{2} m_1 \psi^2[\mu\nu] - \frac{1}{6} m_2 \psi^2[\mu\nu\alpha] = \partial_\mu D_\mu^c. \]  

Thus, the divergence of the modified Belinfante dilatation current equals the divergence of the canonical dilatation current. For the massive fields, \( m_1 = m_2 \), the dilatation symmetry is broken. In the case of massless fields \( (m_1 = 0) \), the currents \( D_{\mu}^c, D_{\mu}^B \) are not conserved. We introduce the new conserved current

\[ D_\mu = D_{\mu}^B + V_\mu = x_\alpha T_{\alpha\mu}^B + 2V_\mu, \]

and \( \partial_\mu D_\mu = 0 \). The massless fields possess the dilatation symmetry with the new dilatation current (37). However, the field-virial \( V_\mu \) (35) is not the total divergence of some local quantity \( \sigma_{\alpha\beta} \), and, therefore, the conformal symmetry is broken [20] even for the massless Kalb-Ramond fields.

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5 Conclusion

We have considered the massive and massless Kalb-Ramond fields in the PDK formalism. Solutions in the form of density matrix obtained allow us to make calculations of quantum processes with particles associated with Kalb-Ramond fields in the covariant form \[19\]. The canonical and symmetric Belinfante energy-momentum tensors found possess their nonzero traces. We have investigated the scale invariance in the first-order formalism. It was shown that the dilatation symmetry is broken for massive fields but in the massless case the new dilatation current is conserved. The conformal symmetry is broken as for massive as for the massless Kalb-Ramond fields.

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