Nonlinear vibration mechanism of elastic beam in large overall rotation with rotor eccentricity effect

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Abstract. The eccentric effect of rotor has seriously affected on the stability and precision of the mechanism at high speed for light and flexible mechanism in industrial application. The analysis of nonlinear vibration mechanism must be completed to realize precision and stable control of elastic beam in a large overall rotation. Firstly, the slender flexible rod is assumed as Euler-Bernoulli beam. We apply the deformation theory of elastic beam to establish its kinematic constraint relationship. Its dynamic model is built with Galerkin method and Hamilton principle. Then the Lagrange method is used to establish the rotor dynamics model based on the Jeffcott rotor. The coupling part between the motor and the elastic beam is assumed to be rigidly connected, and the dynamic model of the motor-elastic beam system can be obtained. Lastly, we analyze the nonlinear vibration mechanism of the elastic beam with rotor eccentricity effect through applying the fourth-order Runge-Kutta method and the superposition principle of traveling wave. It provides an important theoretical basis for making an effective vibration suppression strategy for elastic beam.

1. Introduction
There are two kinds of problems with the development of the system in many engineering fields. The first is the structural strength analysis of the system. In recent years, the methods to solve the problem have been basically mature with the continuous deepening of research. The second is the development of the system to solve complex kinematics and dynamics problems. Its force included not only the interaction between external forces and various components of the system but also the complex coupling effects of electromagnetic excitation and mechanical systems. Therefore, the problem of electromechanical coupling resonance at high speed becomes more complicated and prominent [1].

In recent years, rigid-flexible coupling dynamics of large overall high speed motion and large size flexible structures has gradually a hotspot in the field of nonlinear dynamics [2]. To suppress this kind of vibration, it is necessary to analyze the dynamics of the rigid-flexible coupling and the sensitive parameters of the system [3]. In-depth study of the nonlinear vibration characteristics of elastic beams, and determine the coupling relationship between the sensitive parameters of the system and its stability. It provides an important theoretical basis and data support for the subsequent research on the design of chaos suppression controllers of high-dimensional multi-body flexible systems.
2. Dynamics model of rotor

Establish the rotor kinematic equation based on the Jeffcott rotor model and study vibration characteristics of large overall rotation beam with the effect of rotor eccentricity.

Due to factors such as material unevenness and machining errors, the center of mass \( P(x_p, y_p) \) of the rotor deviates from the axis, \( d \) is the eccentricity, \( C(x_c, y_c) \) is the geometric center of the disc, \( r \) is the deflection of the shaft at the mounting position, as shown in Figure 1 below.

As shown in Figure 1(a), the motor drives the elastic beam to a large overall rotation. The flexible link rod is a slender rod, and the coupling system is assumed rigid. Based on the relation of rotor eccentricity in Figure 1(b), the centrifugal force on the rotor can be derived as

\[
F = (d + r)m\Omega^2
\]

In equation (1), \( m \) is the quality of the disc.

This centrifugal force is balanced by the elastic restoring force of the rotor \(^4\), and the stiffness of the shaft at the disc placement is \( k \). Then equation is

\[
k r = (d + r)m\Omega^2
\]

Suppose \( \omega_j = \sqrt{k/m} \) and \( \omega_j \) is the natural vibration frequency, it can be derived from equation (2)

\[
r = \frac{d\Omega^2}{\omega_j^2 - \Omega^2}
\]

Equation (3) shows that the deflection increases with the increase of the speed. When the speed \( \Omega = \omega_j \), resonance occurs \(^5\). Deflection \( r \) is proportional to eccentricity \( d \).

When \( \varphi = \omega \), \( \Omega = \varphi \). In the case of the rotor steady-state operation, the kinematic equation of \( C \) is \(^6\)

\[
\ddot{x}_c + \omega_j^2 x_c = d\Omega^2 \cos \Omega t
\]

\[
\ddot{y}_c + \omega_j^2 y_c = d\Omega^2 \sin \Omega t
\]

Form equation (4), the trajectory of \( C \) is circular. Let \( \sigma = \Omega/\omega_j \), its trajectory radius is

\[
R_c = \frac{d\sigma^2}{1 - \sigma^2}
\]

Based on geometric constraint relationship of eccentricity, the trajectory radius of \( P \) is

\[
R_p = \frac{d^2}{1 - \sigma^2}
\]
According to equations (4)-(6), the rotor experiences simple harmonic vibration of the same frequency and amplitude in the x and y directions, with a phase difference of 90°.

3. Nonlinear dynamic modeling of elastic beam

3.1. Kinematic constraint relationship of elastic beam

Rigid-flexible coupling motion exists when the flexible slender link rod rotates in a large overall. The displacement constraint relationship before and after deformation at any point on the centerline of the elastic beam is shown in Figure 2 below.

![Fig.2 The kinematic coordinate system of elastic beam](image)

In Figure 2, $e^i$ is the inertial basis, $e^f$ is the floating basis consolidated on the centerline of the undeformed elastic beam, $r_0$ is the vector diameter of the floating basis under the inertial basis, $\rho_0$ is the vector diameter of any point $S$ on the non-center line of the undeformed beam under the floating basis, and $\kappa$ is the deformation displacement. According to the space vector method can obtain

$$ r = r_0 + \rho + \kappa $$

(7)

The coordinate matrix of the velocity of any point on the beam respect to $e^i$ is

$$ \frac{dr}{dt} = \frac{dr}{d\tau} + \omega r $$

(8)

In equation (8), $\omega = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}$, $\omega$ is the angular velocity of rotation around the $Z'$-axis.

According to the theory of elastic beam dynamics, the relationship between the displacement $\kappa$ of any point on the beam and the displacement $\lambda$ of the corresponding centerline point is

$$ \kappa_1 = \lambda_1 + \lambda_2 $$

$$ \kappa_2 = \lambda_3 $$

(9)

In equation (9), $\lambda_1$ is the longitudinal displacement on the x-axis, $\lambda_2$ is the lateral displacement on the y-axis and $\lambda_3$ is the coupling deformation of the centerline deformation displacement.

3.2. Nonlinear dynamic modeling of elastic beams

The virtual work done by the inertial force can be represented as

$$ \delta W = \int_0^{l} \gamma A \left( \frac{dr}{dt} \right)^T \delta c \, dx $$

(10)

In equation (10), $\gamma$ is the density of the elastic beam, $A$ is the area of the uniform beam, and $l$ is the length of the elastic beam.

$$ \delta W = \delta W_o + \delta W_i $$

(11)

In equation (11), $\delta W_o$ is the virtual work done without considering the centerline coupling deformation, and $\delta W_i$ is the virtual work done by the central line coupling deformation. Among them,
\[ \delta W_o = \int_{\Omega} ^{} y A A \left[ k_1 - 2 \alpha \kappa - \alpha ^2 (x + \kappa) \right] \delta \kappa \, d\Omega + \int_{\Gamma} ^{} y A A \left[ k_2 + 2 \alpha \kappa - \alpha ^2 \kappa \right] \delta \kappa \, d\Gamma \]  
\[ (12) \]

\[ \delta W_\ell = \int_{\Omega} ^{} y A A \left( \dot{\alpha}_x - \alpha \dot{\alpha}_x \right) \delta \dot{\alpha}_x \, d\Omega + \int_{\Gamma} ^{} 2 y A A \dot{\alpha}_x \delta \dot{\alpha}_x \, d\Gamma + \int_{\Gamma} ^{} y A A \left( \dot{\alpha}_x + \dot{\alpha}_x - 2 \alpha \dot{\alpha} - \alpha ^2 (x + \lambda_1) \right) \delta \dot{\alpha}_x \, d\Gamma \]  
\[ (13) \]

Without considering the strain energy generated by the shear strain, suppose the elastic beam is composed of orthogonal isotropic linear materials \([10]\) . Then equation is,
\[ \Pi = \frac{1}{2} \int_0 ^{\ell} E A c^2 \dot{\theta} \, dx \]  
\[ (14) \]

In equation (14), \( E \) is the elastic Young’s modulus of the material, \( \epsilon_\alpha \) is the positive strain in the \( x \)-direction. Considering the influence of geometric nonlinearity, the relationship between strain and geometric nonlinearity is
\[ \epsilon_\alpha = \frac{\partial \delta}{\partial \alpha} \cdot \epsilon_\alpha \cdot \frac{1}{2} \left( \frac{\partial \delta}{\partial \alpha} \right) \]  
\[ (15) \]

\[ \Pi_1 = \frac{1}{2} \int_0 ^{\ell} E A \left( \frac{\partial \delta}{\partial \alpha} \right)^2 \dot{x} \, dx + \frac{1}{2} \int_0 ^{\ell} E I \left( \frac{\partial \delta}{\partial \alpha} \right)^2 \dot{x} \, dx \]  
\[ (16) \]

In equation (16), \( I \) is the moment of inertia of the section, \( \Pi_1 \) respectively represent the variable properties generated by linear elastic term and geometric nonlinearity when geometric nonlinearity is not considered.
\[ \Pi_2 = \frac{1}{2} \int_0 ^{\ell} E A \left( \frac{\partial \delta}{\partial \alpha} \right)^2 \dot{x} \, dx + \frac{1}{2} \int_0 ^{\ell} E I \left( \frac{\partial \delta}{\partial \alpha} \right)^2 \dot{x} \, dx \]  
\[ (17) \]

Since \( \Pi_2 \) appears in the form of elastic force in the dynamic equation, it is no directly related to the rotation of large overall \([11]\) . Only the linear elastic model is considered, so its deformation energy under high speed is
\[ \Pi_2 = \frac{1}{2} \int_0 ^{\ell} E A \left( \frac{\partial \delta}{\partial \alpha} \right)^2 \dot{x} \, dx + \frac{1}{2} \int_0 ^{\ell} E I \left( \frac{\partial \delta}{\partial \alpha} \right)^2 \dot{x} \, dx \]  
\[ (18) \]

Using Galerkin method, assume that the deformation displacement of the elastic beam is
\[ \sigma = \phi q \]  
\[ (20) \]

In equation (20), \( q = [ q_1 \ q_2 \ldots] \cdot \phi = [ \phi_1 \ \phi_2 \ldots \ \phi_n ] \cdot \phi_i = [ \phi_{i1} \ \phi_{i2} \ldots \ \phi_{in} ] \cdot \phi = \left[ \begin{array}{c} \phi_1 \\ 0 \\ \phi_2 \\ \vdots \\ 0 \\ \phi_n \\ \end{array} \right] \).

Since the longitudinal deformation caused by the rotation of the cross-section is small which is ignored \([12]\) . Based on Hamilton principle,
\[ \delta H_\ell = \int_0 ^{\ell} (\delta W - \Pi + \delta \sigma) = 0 \]  
\[ (21) \]

In equation (21), \( H_\ell \) is the Hamilton function, \( \delta \sigma \) is the virtual work done by the external force, substituting equations (11), (19) and (20) into equation (21) can obtain the nonlinear dynamic model.

The motor drives the beam to rotate in a large overall on the \( xoy \) plane, ignoring the influence of gravity on the rotor. \( F_x \) and \( F_y \) are the external parametric forces experienced by the elastic beam in the \( x \) and \( y \) directions. The coupling part between motor and elastic beam is assumed rigid, and the kinematic equation of the motor-elastic beam system can be obtained as follows:
\[ \begin{cases} 
\dot{x} + \eta \dot{x} + \alpha \dot{\alpha} \cos \alpha = d \omega \dot{\alpha} \cos \theta \\
\dot{y} + \eta \dot{y} + \alpha \dot{\alpha} \sin \alpha = d \omega \dot{\alpha} \sin \theta \\
x = 2 \omega \dot{y} + \frac{E A \pi^2 - 4 m \omega^2}{4 m l} x + \frac{4 \pi - 7}{12l} (y - 2 \pi - 7 \frac{x}{24l})^2 + \frac{4 \pi - 2 \pi l}{\pi^2} \omega^2 = F_x \\
y = 2 \omega \dot{x} + \frac{E I \pi^4}{16 m l^7} \left( \frac{\pi^2 - 3}{6} \alpha \dot{\alpha} \right) y + \frac{3 \pi^2 + 16 \pi^2}{128 l^7} y (y - 2 \pi - 7 \frac{x}{12l} - 3 \pi^2 + 16 \pi^2 \frac{x}{128 l^7})^2 = F_y 
\end{cases} \]  
\[ (22) \]
4. Nonlinear vibration characteristics of elastic beam
Taking the first-order assumed mode of the lateral vibration of the Euler-Bernoulli beam satisfying the boundary conditions [13]. To systematically analyze the coupling mechanism of the rotor eccentricity effect and the nonlinear dynamic behavior of the elastic beam [14], this article compares the nonlinear vibration characteristics of the elastic beam with and without rotor eccentricity. The structural parameters of the elastic beam are shown in Table 1.

| Parameter | Length/m | Young's modulus N/m² | Cross-sectional area/m² | Moment of inertia/kg·m² | Density kg/m³ |
|-----------|----------|----------------------|-------------------------|-------------------------|--------------|
| 1         | 0.5      | 6.895×10⁹            | 7.0×10⁻⁶                | 8.22×10⁻⁷              | 2.77×10³     |

When the rotor eccentricity distance \( d \) tends to 0, it can be regarded as without rotor eccentricity. The vibration curve of the elastic beam without rotor eccentricity effect at different speeds through applying the fourth-order Runge-Kutta method is shown in Figure 3 below.

Figure 3 shows that the vibration characteristic of the beam is dominated by lateral vibration when the rotor eccentricity is zero. The simulation results are consistent with the actual situation. The longitudinal vibration frequency of the elastic beam is significantly higher than the lateral vibration frequency in the same amount of time, which proves that the natural frequency of the longitudinal vibration is higher than the lateral vibration. The elastic beam will appear "dynamic stiffness" phenomenon [15] as the speed increases.

In order to study the coupling relationship [16] between rotor eccentricity and beam vibration, the vibration characteristics of the system with different rotor eccentricity at high speed were tested.
It can be seen from Figure 4(a) that the longitudinal vibration presents aperiodic vibration with the increase of rotor eccentricity at high speed. Figure 4(b) shows that the longitudinal vibration has certain periodic vibration characteristics, but it is presenting an unstable periodic vibration as time increase.

Figure 5(a) shows that the lateral vibration of the rotating beam with the change of the rotor eccentricity produces obvious non-periodic vibration at high speed. Its amplitude of lateral vibration increases sharply with the increase of rotor eccentricity. Literature [6] proves that the elastic beam is dominated by periodic vibration at low speed. Figure 5(b) shows that the lateral vibration presents aperiodic and irregular motion. It can be seen that the strong coupling relationship between the external force amplitude of the system and the vibration of the elastic beam changes the essential characteristics of the elastic beam vibration.

The simulation results show that the rotor eccentricity effect is mainly determined by the rotor eccentricity and the speed of the rotating beam. The sensitive parameter is the amplitude of the exciting
force generated by the motor-elastic beam system. Large amplitude and aperiodic vibration of the rotating beam is easily caused by small changes of sensitive parameters.

5. Conclusions
(1) The vibration of the elastic beam tends to stable with the change of time when without rotor eccentricity. The greater the speed, the faster the vibration tends to converge to the equilibrium position. The flexible problem of elastic beam must be considered in large overall rotation at high speeds with the effect of rotor eccentricity. Due to the phenomenon of "dynamic negative stiffness" when entering high speeds, the traditional rigid model is only suitable for describing the dynamic behavior of rotating beams at low speeds.

(2) There is a strong coupling relationship between the rotor eccentricity effect and the nonlinear vibration characteristics of the rotating beam. The eccentricity and rotation speed determines the strength of the rotor eccentricity effect. With the slight change of sensitive parameters, the elastic nonlinear vibration characteristics change from periodic vibration at low speed to aperiodic vibration.

(3) At high speed, the increase of rotor eccentricity will cause more and more obvious beam vibration, and its amplitude will increase rapidly. Simulation data show that when the eccentricity reaches 1mm, the longitudinal amplitude increases to 7.12mm, indicating that the elastic beam system is unstable at this time. When the eccentricity is 0.1mm, the maximum amplitude of lateral vibration reaches 240mm. Eliminating or reducing the rotor eccentricity effect is beneficial to greatly improve the stability of the system at high speed and restrain the sudden or intermittent excitation of the elastic rotating beam.

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