ON FREE RESOLUTIONS IN MULTIVARIABLE OPERATOR THEORY

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Abstract. Let $A_d$ be the complex polynomial ring in $d$ variables. A contractive $A_d$-module is Hilbert space $H$ equipped with an $A_d$ action such that for any $\xi_1, \xi_2, \ldots, \xi_d \in H$,

$$\|z_1\xi_1 + z_2\xi_2 + \cdots + z_d\xi_d\|^2 \leq \|\xi_1\|^2 + \|\xi_2\|^2 + \cdots + \|\xi_d\|^2.$$ 

Such objects have been shown to be useful for modeling $d$-tuples of mutually commuting operators acting on a Hilbert space. An example is given by $H^2_d$, which is the Hilbert space of holomorphic functions on $B_d = \{z \in \mathbb{C}^d : |z| \leq 1\}$ generated by the reproducing kernel $k(z, w) = \frac{1}{1 - \langle z, w \rangle}$, $z, w \in B_d$. For a natural subcategory of contractive $A_d$-modules, whose members are said to be pure, the module $H^2_d$ plays the role of the free module of rank one. In fact, given any pure contractive $A_d$-module $H$, there is a “free resolution”, i.e. an exact sequence of the following form:

$$\cdots \xrightarrow{\Phi_2} H^2_d \otimes C_1 \xrightarrow{\Phi_1} H^2_d \otimes C_0 \xrightarrow{\Phi_0} H \xrightarrow{} 0.$$ 

For $i \geq 1$, the map $\Phi_i$ can be viewed as a $B(C_i, C_{i-1})$-valued weakly holomorphic function on $B_d$. ($B(C_i, C_{i-1})$ is the set of bounded linear operators from $C_i$ into $C_{i-1}$.)

“Localizing” the free resolution at a point $\lambda \in B_d$, one obtains a “localized complex”:

$$\cdots \xrightarrow{\Phi_3(\lambda)} C_2 \xrightarrow{\Phi_2(\lambda)} C_1 \xrightarrow{\Phi_1(\lambda)} C_0.$$ 

We shall show that the homology of this complex is isomorphic to the homology of the Koszul complex of the $d$-tuple $(\varphi^1, \varphi^2, \ldots, \varphi^d)$, where $\varphi^i$ is the $i$th coordinate function of a M"{o}bius transform on $B_d$ such that $\varphi(\lambda) = 0$.

We shall also show that the set of M"{o}bius transforms on $B_d$ gives rise to a class of unitary operators on the Hilbert space $H^2_d$. Explicitly, if $\varphi$ is a M"{o}bius transform on $B_d$, then the map $\xi \mapsto \frac{\sqrt{1 - |\lambda|^2}}{1 - \langle \lambda, \lambda \rangle} (\xi \circ \varphi)$, where $\lambda = \varphi^{-1}(0)$, is a unitary operator on $H^2_d$.

We shall show that the set of such operators acts “ergodically” on $H^2_d$, in the sense that no non-trivial invariant subspace of $H^2_d$ is preserved by every operator of this form.
1. Introduction

Let \( A_d = \mathbb{C}[z_1, z_2, \ldots, z_d] \) be the polynomial ring in \( d \) variables. We define a \textit{contractive Hilbert} \( A_d \)-module to be a module \( \mathcal{H} \) over \( A_d \) that is a Hilbert space and that has the additional property that for all \( \xi_1, \xi_2, \ldots, \xi_d \in \mathcal{H} \),

\[
\left\| \sum_{k=1}^{d} z_k \xi_k \right\|_2 \leq \sum_{k=1}^{d} \| \xi_k \|_2.
\]

Obviously any closed submodule \( K \) of a contractive \( A \)-module \( \mathcal{H} \) is a contractive \( A \)-module. Consider the Banach space quotient \( \mathcal{H}/K \). One can identify this space with \( \mathcal{H} \oplus K \), and define a contractive \( A_d \)-module structure on it by compressing polynomials by \( P_{\mathcal{H} \oplus K} \).

The notion of a Hilbert module was used by Arveson in [4] to represent commuting \( d \)-contractions of operators. Indeed, the actions of \( z_1, z_2, \ldots, z_d \) on \( \mathcal{H} \) correspond to a mutually commuting \( d \)-tuple of linear operators \((T_1, T_2, \ldots, T_d)\) by defining \( z_i \xi = T_i \xi \) for all \( \xi \in \mathcal{H} \). The contractive condition on the module \( \mathcal{H} \) is equivalent to saying that the row operator \((T_1 T_2 \cdots T_d)\) is contractive when seen as a map from

\[
\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \rightarrow \mathcal{H}.
\]

Conversely, given a \( d \)-tuples of linear operators \((T_1, T_2, \ldots, T_d)\) acting on a Hilbert space \( \mathcal{H} \) where the components mutually commute and the row operator \((T_1 T_2 \cdots T_d)\) is contractive, one can define a contractive \( A_d \)-module structure on \( \mathcal{H} \) by defining \( z_i \xi = T_i \xi \) for each \( \xi \in \mathcal{H} \). Given a contractive \( A_d \)-module \( \mathcal{H} \), we call the \( d \)-tuple \((T_1, T_2, \ldots, T_d)\) the \textit{associated} \( d \)-tuple of \( \mathcal{H} \).

An example of a contractive \( A_d \)-module, and one which plays an important role in the theory, is the following: Let \( H^2_d \) be the Hilbert space of holomorphic functions on the unit ball \( B_d \) of \( \mathbb{C}^d \) derived from the following reproducing kernel:

\[
k(z, \lambda) = \frac{1}{1 - \langle z, \lambda \rangle}, \quad z, \lambda \in B_d.
\]

The function \( k_\lambda : z \mapsto k(z, \lambda) \) is in \( H^2_d \) and in fact for any \( \xi \in H^2_d \),

\[
\xi(\lambda) = \langle \xi, k_\lambda \rangle.
\]

A contractive \( A_d \)-module structure is defined on \( H^2_d \) as follows: For any \( \xi \in H^2_d \), \( p(z_1, z_2, \ldots, z_d) \in A_d \), \( p(z_1, z_2, \ldots, z_d)\xi \) is simply the function \( p(z_1, z_2, \ldots, z_d) \) multiplied by \( \xi \). The Hilbert space direct sum of \( n \) copies of \( H^2_d \) is also a contractive \( A_d \)-module and can be expressed by
$H_d^2 \otimes C$, where $C$ is a Hilbert space of dimension $n$. Note that we put no restrictions at this point on the cardinality of $n$. For reasons that will soon become apparent, we call $H_d^2 \otimes C$ the free contractive $A$-module of rank $\dim C$.

Another simple example of a contractive $A_d$-module arises as follows: Let $C(\partial B_d)$ be the $C^*$-algebra of continuous functions on the unit sphere in $\mathbb{C}^d$. Let $\pi : C(\partial B_d) \to B(\mathcal{K})$ be a *-representation. Then polynomials act on $\mathcal{K}$ in the natural way, and $\mathcal{K}$ becomes a contractive $A_d$-module. Such modules are called spherical modules.

Free and spherical contractive $A_d$-modules play the role of universal objects in the category of $A_d$-modules. The following result is due to W. Arveson ([2]).

**Theorem 1.1.** Let $\mathcal{H}$ be a contractive $A_d$-module. There exists a free module $F$, a spherical module $S$, and a module homomorphism $U : F \oplus S \to \mathcal{H}$ such that $UU^* = 1$, i.e. $U$ is a coisometry.

A uniqueness condition also applies. First we state the following definition:

**Definition 1.2.** Let $\mathcal{H}$ be a contractive $A_d$-module, let $F$ be a free module, let $S$ be a spherical module, and let $U : F \oplus S \to \mathcal{H}$ be a coisometry. The triple $(F, S, U)$ is said to be a minimal dilation of $\mathcal{H}$ if the closed submodule of $F \oplus S$ generated by $U^* \mathcal{H}$ is $F \oplus S$.

The following theorem, a proof of which can be found in [4], states that any two minimal dilations are naturally isomorphic.

**Theorem 1.3.** Every contractive $A_d$-module $\mathcal{H}$ has a minimal dilation. Furthermore, if $(F, S, U)$ and $(F', S', U')$ are minimal dilations, then there is a unitary module isomorphism $V : F \oplus S \to F' \oplus S'$ such that $U = U'V$. Furthermore, $V$ has the form $V = (1_{H^2_d} \otimes W) \oplus W'$.

In this paper, we will be concerned almost entirely with pure contractive $A_d$-modules.

**Definition 1.4.** A contractive $A_d$-module $\mathcal{H}$ is pure if the spherical part of any minimal dilation is trivial. In other words, $\mathcal{H}$ is unitarily isomorphic to a quotient of a free module.

There is an equivalent formulation of this definition expressed in terms of contractive $d$-tuples of commuting operators. This equivalence follows from the work of Arveson in [2].

**Theorem 1.5.** Let $\mathcal{H}$ be a contractive $A_d$-module, and let $T_1, T_2, \ldots, T_d$ be the linear operators on $\mathcal{H}$ corresponding to the actions of $z_1, z_2, \ldots, z_d$. Then $\mathcal{H}$ is pure iff
\[ SOT - \lim_{n \to \infty} \sum_{i_1,i_2,\ldots,i_n} T_{i_1}T_{i_2} \cdots T_{i_n}T_{i_n}^* \cdots T_{i_2}^*T_{i_1}^* = 0. \]

Let \( C \) be a Hilbert space. The free module \( H^2_d \otimes C \) can be viewed as a space of (weakly) holomorphic \( C \)-valued functions defined on \( B_d \). Indeed, if \( \xi \in H^2_d \otimes C \) and \( \lambda \in B_d \), then \( \xi(\lambda) \) is defined to be the unique element of \( C \) such that for all \( \eta \in C \),

\[ \langle \xi, k_\lambda \otimes \eta \rangle = \langle \xi(\lambda), \eta \rangle_C. \]

This identification of a free module with a space of vector valued holomorphic functions gives us a useful way of perceiving module homomorphisms between free modules. Indeed, let \( D \) and \( C \) be Hilbert spaces, and let \( \Phi : H^2_d \otimes D \to H^2_d \otimes C \) be a module homomorphism. For each \( \lambda \in B_d \), define the bounded linear operator \( \Phi(\lambda) : D \to C \) by \( \Phi(\lambda) \eta = \Phi(1 \otimes \eta)(\lambda) \). Since \( \Phi \) is a homomorphism, it follows that for all \( \xi \in H^2_d \otimes D \) and \( \lambda \in B_d \), \( (\Phi \xi)(\lambda) = \Phi(\lambda)\xi(\lambda) \). Hence \( \Phi \) is given by pointwise multiplication by the operator valued function \( \Phi(\lambda) \). From the definition, it is not difficult to show that the adjoint of \( \Phi \) has the following property: Let \( \lambda \in B_d \), and let \( \eta \in C \). Then

\[ \Phi^*(k_\lambda \otimes \eta) = k_\lambda \otimes \Phi(\lambda)^* \eta. \]

As a consequence of (2), we have the following boundedness condition on \( \Phi(\lambda) \).

\[ \| \Phi(\lambda) \| \leq \| \Phi \|, \forall \lambda \in B_d. \]

Thus the function \( \lambda \in B_d \mapsto \Phi(\lambda) \) is a bounded \( \mathcal{B}(D,C) \)-valued (weakly) holomorphic function on \( B_d \).

Let \( \mathcal{H} \) be any pure contractive \( \mathcal{A}_d \)-module, and let \( \mathcal{M} \) be a closed submodule. Using Theorem 1.5, it is not hard to show that \( \mathcal{M} \) is pure as a contractive \( \mathcal{A}_d \)-module. Using this, we can perform the following construction: Let \( \mathcal{H} \) be a pure contractive \( \mathcal{A}_d \)-module. Then by Theorem 1.1, there exists a free module \( \mathcal{F}_1 \) and a coisometric module homomorphism \( \Phi_1 : \mathcal{F}_1 \to \mathcal{H} \). Clearly \( \ker \Phi_1 \) is a closed submodule of \( \mathcal{F}_1 \), and by our above remarks, there exists a free module \( \mathcal{F}_2 \) and a coisometric module homomorphism \( \Phi_2 : \mathcal{F}_2 \to \ker \Phi_1 \). Extending the codomain of \( \Phi_2 \), we obtain a partially isometric module homomorphism \( \Phi_2 : \mathcal{F}_2 \to \mathcal{F}_1 \) with range space \( \ker \Phi_1 \). Continuing in this fashion, we obtain the following exact sequence:
(3) \[ \cdots \xrightarrow{\Phi_2} \mathcal{F}_1 \xrightarrow{\Phi_1} \mathcal{F}_0 \xrightarrow{\Phi_0} \mathcal{H} \longrightarrow 0, \]
where \( \mathcal{F}_i \) is free for each \( i \). We call this a free resolution of \( \mathcal{H} \), and it is analogous to the free resolution in the theory of finitely generated modules over Noetherian rings.

Each \( \mathcal{F}_i \) appearing in (3) has the form \( H^2 \otimes C_i \) for some Hilbert space \( C_i \). If we localize to a point \( \lambda \in B_d \), we obtain the following sequence of linear maps:

(4) \[ \cdots \xrightarrow{\Phi_3(\lambda)} C_2 \xrightarrow{\Phi_2(\lambda)} C_1 \xrightarrow{\Phi_1(\lambda)} C_0 \]

Since (3) is exact, it follows that (4) is a complex for each \( \lambda \).

The main result of Section 2 is that the homology of (4) at 0 is closely connected to the spectral properties of the \( d \)-tuple \((T_1, T_2, \ldots, T_d)\) associated with \( \mathcal{H} \). By “spectral properties” we are referring to the spectrum of a tuple of operators in the sense of Taylor (cf. [15]). The following discussion summarizes this more precisely: For each \( k \in \mathbb{Z} \), let \( \bigwedge^k \mathbb{C}^d \) be the \( k \)-fold alternating product of \( \mathbb{C}^d \), taken to be the trivial vector space when \( k < 0 \). For each \( k \), let \( E_k = \mathcal{H} \otimes \bigwedge^k \mathbb{C}^d \). Fix an orthonormal basis \( \{e_1, e_2, \ldots, e_d\} \) for \( \mathbb{C}^d \), and for each \( k \), define the linear map \( \partial_k : E_k \longrightarrow E_{k-1} \) as follows:

\[
\partial_k(\xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} z_{i_j} \xi \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}.
\]

A straightforward calculation shows that the following sequence is a complex.

(5) \[ 0 \longrightarrow E_d \xrightarrow{\partial_d} E_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} E_0 \longrightarrow 0 \]

What we will show in Section 2 is that

\[
\frac{\ker \partial_k}{\im \partial_{k+1}} \cong \frac{\ker \Phi_k(0)}{\im \Phi_{k+1}(0)}.
\]

Hence the homology of (5) and that of (4) when \( \lambda = 0 \) are identical.

In Section 3, we introduce the notion of a Möbius transform of a pure contractive \( \mathcal{A}_d \)-module. We recall that a Möbius transform \( \varphi \) on \( B_1 \) is a bijective holomorphism from \( B_d \) onto itself. If \( \mathcal{H} \) is a pure contractive \( \mathcal{A}_d \)-module, and \((T_1, T_2, \ldots, T_d)\) is the associated \( d \)-tuple, then we can define the Möbius transform \((\mathcal{H})_\varphi\) of \( \mathcal{H} \) to be the pure
contractive $A_d$-module where the underlying Hilbert space of $H_\lambda$ is the underlying Hilbert space of $H$, but $z_1, z_2, \ldots, z_d$ act as $\varphi^1, \varphi^2, \ldots, \varphi^d$, respectively, where $\varphi^i$ is the $i$th coordinate of $\varphi$. We will show that there exists a unitary module isomorphism $U_\varphi : H^2_d \rightarrow (H^2_d)_\lambda$, which carries the space of all functions in $H^2_d$ that vanish at 0 to the space of all functions that vanish at $\lambda$. We will also show that the $U_\lambda$'s act "ergodically" on $H^2_d$ in the sense that no proper nontrivial closed submodule of $H^2_d$ is invariant under $U_\lambda$ for all $\lambda \in B_d$.

Finally, in Section 4, we use the results on Section 3 to describe the homology of (4) for an arbitrary $\lambda \in B_d$. In particular, Theorem 4.3 states that the homology of the Koszul complex of $(H)_\varphi$ is equivalent to the localized complex (4). Hence (4) contains spectral information (in the sense of Taylor) of the Möbius transformation of $H$.

2. The Koszul complex and free resolution of a contractive $A_d$-module

In his seminal paper ([15]), J. Taylor defined the notion of invertibility of $d$-tuples of commuting operators acting on a Banach space $B$. We will briefly summarize Taylor’s construction:

Let $B$ be a Banach space, and let $(a_1, a_2, \ldots, a_d)$ be a $d$-tuple of mutually commuting bounded operators on $B$. For $k \in \mathbb{Z}$, let $\wedge^k \mathbb{C}^d$ be the $k$-fold wedge product of $\mathbb{C}^d$, where this is taken to be the trivial vector space when $k < 0$. Let $E_k = B \otimes \wedge^k \mathbb{C}^d$. Note that $E_k$ can be viewed as an $\binom{n}{k}$-fold direct sum of $E_k$'s, hence it is a Banach space itself. Fix an orthonormal basis $\{e_1, e_2, \ldots, e_d\}$ for $\mathbb{C}^d$. For each $k$, define $\partial_k : E_k \rightarrow E_{k-1}$ by

$$\partial_k \xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} = \sum_{j=1}^{d} (-1)^{j+1} a_{i_j} \xi \otimes e_{i_1} \wedge \cdots \hat{e}_{i_j} \cdots \wedge e_{i_k},$$

A simple computation shows that $\partial_{k-1} \partial_k = 0$. Hence we have the following complex of Banach spaces.

$$\cdots \rightarrow 0 \rightarrow E_d \xrightarrow{\partial_d} E_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_0} 0.$$  \hspace{1cm} (6)

The following is Taylor’s definition of invertibility:

**Definition 2.1.** A $d$-tuple of mutually commuting operators acting on a common Banach space $B$ is said to be invertible if the sequence in (6) is exact.
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One can immediately provide a definition of the spectrum of \((a_1, a_2, \ldots, a_d)\):

**Definition 2.2.** Let \(a = (a_1, a_2, \ldots, a_d)\) be a \(d\)-tuple of mutually commuting operators on a Banach space \(\mathcal{B}\). Then \(\text{spec}(a)\) is defined to be the set \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d\) where \((a_1 - \lambda_1, a_2 - \lambda_2, \ldots, a_d - \lambda_d)\) is not invertible.

We remark that in the case where \(d = 1\), both Definition 2.1 and Definition 2.2 reduce to the usual definitions of invertibility and spectrum for single operators.

If we restrict ourselves to the case where \(\mathcal{B}\) is a Hilbert space \(\mathcal{H}\), and the \(d\)-tuple \((T_1, T_2, \ldots, T_d)\) consists of mutually commuting bounded operators on \(\mathcal{H}\), then we can express homological features of (6) in terms of a single self-adjoint operator. Our presentation will follow Arveson ([1]), but this idea appeared earlier in the work of such authors as Curto ([6]) and Vasilescu ([16]).

Let \(\mathcal{H}\) and \((T_1, T_2, \ldots, T_d)\) be as above, and let \(\{e_1, e_2, \ldots, e_d\}\) be our fixed orthonormal basis for \(\mathbb{C}^d\). The space \(\bigwedge^k \mathbb{C}^d\) has a natural inner product defined by

\[
\langle z_1 \wedge z_2 \wedge \cdots \wedge z_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k \rangle = \det((\langle z_i, w_j \rangle)_{ij}), z_i, w_i \in \mathbb{C}^d.
\]

Hence the spaces \(E_k\) are tensor products of Hilbert spaces. Let \(\bigwedge \mathbb{C}^d = \bigoplus_{k \in \mathbb{Z}} \bigwedge^k \mathbb{C}^d\). Define \(E\) to be the direct sum of the \(E_k\)'s, i.e.

\[
E = \bigoplus_{k \in \mathbb{Z}} E_k = \mathcal{H} \otimes \bigwedge \mathbb{C}^d.
\]

Let \(c_1, c_2, \ldots, c_d\) be the operators on \(\bigwedge \mathbb{C}^d\) defined by

\[
c_i(z_1 \wedge z_2 \wedge \cdots \wedge z_k) = e_i \wedge z_1 \wedge \cdots \wedge z_k, z_i \in \mathbb{C}^d.
\]

We then define the linear operator \(\partial : E \longrightarrow E\) to be the sum

\[
T_1 \otimes c_1^* + T_2 \otimes c_2^* + \cdots + T_d \otimes c_d^*.
\]

One then checks that the restriction of \(\partial\) to \(E_k\) is \(\partial_k\). One can now prove the following theorem (cf. [1]):

**Theorem 2.3.** Let \((T_1, T_2, \ldots, T_d)\) be a \(d\)-tuple of mutually commuting operators on a common Hilbert space \(\mathcal{H}\). If \(\partial\) is the corresponding boundary operator, then \((T_1, T_2, \ldots, T_d)\) is invertible iff \(\partial + \partial^*\) is invertible.
The operator \( \partial + \partial^* \) is called the Dirac operator corresponding to \((T_1, T_2, \ldots, T_d)\). This idea of expressing the invertibility of a \(d\)-tuple in terms of a single operator suggests the following definition:

**Definition 2.4.** A \(d\)-tuple of operators as in Theorem 2.3 is said to be Fredholm if the corresponding Dirac operator \( \partial + \partial^* \) is Fredholm.

We note that in the case where \(d = 1\), this definition corresponds to the usual definition of Fredholmnness via an easy application of Atkinsons' equivalences (see [5], for example).

The Fredholmness of a Dirac operator has an importance consequence with respect to the homology of the Koszul complex. This is expressed in the following theorem, which follows from the definition of \(\partial\) by a straightforward argument.

**Theorem 2.5.** A \(d\)-tuple \((T_1, T_2, \ldots, T_d)\) is Fredholm iff the homology spaces \(\ker \partial_k / \text{im} \partial_{k+1}\) are finite dimensional for all \(k \in \mathbb{Z}\).

Theorem 2.5 allows us to generalize the notion of index to Fredholm \(d\)-tuples. Indeed, the index of a Fredholm \(d\)-tuple \((T_1, T_2, \ldots, T_d)\) is defined to be the alternating sum of the dimensions of the homology spaces of the Koszul complex, i.e.

\[
\text{index} (T_1, T_2, \ldots, T_d) = \sum_{k \in \mathbb{Z}} (-1)^{k+1} \frac{\ker \partial_k}{\text{im} \partial_{k+1}}.
\]

Again, we note that this definition reduces to the usual definition of index when one takes \(d = 1\).

We now show that the \(d\)-tuple associated with the free Hilbert module \(H^2_d\) is Fredholm and we compute its homology.

**Theorem 2.6.** The \(d\)-tuple \((S_1, S_2, \ldots, S_d)\) associated with \(H^2_d\) is Fredholm. Furthermore, the extended sequence of maps

\[
\cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_{1/2}} \mathbb{C} \xrightarrow{} 0,
\]

where \(\partial_{1/2}\) is defined to be the evaluation map \(\xi \mapsto \xi(0)\), is exact.

A proof of the fact the \((S_1, S_2, \ldots, S_d)\) is Fredholm can be found in [1]. We will rely on this fact to prove the remainder of the theorem.

**Proof.** Let \(k\) be an integer no less than 1. By Theorem 2.5, the space \(\text{im} \partial_{k+1}\) has finite codimension in \(\ker \partial_k\). It is a standard fact in operator theory that if the image of a bounded operator has finite codimension in a larger closed subspace, then the image is closed. Hence \(\text{im} \partial_{k+1}\) is a closed subspace of finite codimension in \(\ker \partial_k\). We now show that
im \partial_{k+1} = \ker \partial_k. To this end, let \( \xi \in \ker \partial_k \). Then \( \xi \) can be written as follows:

\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq d} \xi_{i_1, i_2, \ldots, i_k} \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}
\]

where \( \xi_{i_1, i_2, \ldots, i_k} \in H^2_d \). Supposing for the moment that these \( \xi_{i_1, i_2, \ldots, i_k} \) are homogeneous polynomials all of the same degree \( N \), then \( \partial_k \xi \) is in a similar form but with common degree \( N + 1 \). It follows in the general case that if \( \xi^n_{i_1, i_2, \ldots, i_k} \) is the \( n \)th degree homogeneous component of \( \xi_{i_1, i_2, \ldots, i_k} \), then

\[
\xi^n := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq d} \xi^n_{i_1, i_2, \ldots, i_k} \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \in \ker \partial_k.
\]

By Corollary 17.5 in [7], \( \xi^n \in \im \partial_{k+1} \) for each \( n \), hence \( \xi = \sum_{n=0}^{\infty} \xi_n \in \im \partial_{k+1} \).

We proceed to the case where \( k = 0 \). By Theorem 3.1, the following row operator is a partial isometry with image \( \{ \xi \in H^2_d : \xi(0) = 0 \} \).

\[
(S_1 S_2 \cdots S_d) : \underbrace{H^2_d \oplus \cdots \oplus H^2_d}_{d \text{ times}} \rightarrow H^2_d.
\]

A generic element \( \zeta \) of \( E_1 \) has the following form

\[
\sum_{k=1}^{d} \xi_k \otimes e_k, \ \xi_k \in H^2_d.
\]

Hence \( \partial_1 \xi = \sum_{k=1}^{d} S_k \xi_k \). It follows from (7) and the statement preceding it that \( \im \partial_1 = \ker \partial_{1/2} \).

The surjectivity of \( \partial_{1/2} \) is clear. \( \square \)

**Corollary 2.7.** Let

\[
\cdots \rightarrow \partial_2 \rightarrow E_1 \rightarrow \partial_1 \rightarrow E_0 \rightarrow 0
\]

be the Koszul complex of \( (S_1, S_2, \ldots, S_d) \). Then its homology is as follows:

\[
\frac{\ker \partial_k}{\im \partial_{k+1}} = 0, \ k \geq 1
\]
\[
\ker \partial_0 \approx \mathbb{C}.
\]

**Corollary 2.8.** Let \((S'_1, S'_2, \ldots, S'_d)\) be the d-tuple associated with a free module \(F = H^2_d \otimes \mathbb{C}\) with the following extended sequence of maps:

\[
\cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_{1/2}} \mathcal{C} \longrightarrow 0.
\]

Then for \(k \geq 1\), \(\ker \partial_k = \text{im} \partial_{k+1}\) and \(\ker \partial_0/\text{im} \partial_1 \cong \ker \partial_0 \cap (\text{im} \partial_1)^\perp \cong \mathcal{C}\). In other words \(\text{im} \partial_{k+1}\) is closed in \(\ker \partial_k\), and in the case where \(k = 1\), the codimension of \(\text{im} \partial_{k+1}\) in \(\ker \partial_1\) is \(\dim \mathcal{C}\).

Naturally, our entire discussion on Koszul complexes of d-tuples of operators can be rephrased in terms of \(\mathcal{A}_d\)-modules. Indeed, one simply takes \(\mathcal{H}\) to be the module defined by \(z_i \xi = T_i \xi\) for all \(\xi \in \mathcal{H}\). The spaces \(E_k\) are defined analogously, with \(\partial_k : E_k \longrightarrow E_{k-1}\) reexpressed as

\[
\partial_k(\xi \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} z_{i_j} \xi \otimes e_{i_1} \wedge \cdots \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k},
\]

Since \(\mathcal{A}_d\) is a commutative algebra, the maps \(\partial_k\) are module homomorphisms. Hence the sequence (6) can be viewed as a complex of \(\mathcal{A}_d\)-modules.

As summarized in Section 1, starting with a pure \(\mathcal{A}_d\)-module \(\mathcal{H}\), by means of dilation theory one may construct a free resolution of \(\mathcal{H}\):

\[
\cdots \xrightarrow{\Phi_2} F_1 \xrightarrow{\Phi_1} F_0 \xrightarrow{\Phi_0} \mathcal{H} \longrightarrow 0,
\]

which is an exact sequence where the \(\Phi_i\)'s are partial isometries and the \(F_i\)'s are free modules of the form \(H^2_d \otimes \mathcal{C}_i\). One then “localizes” (8) to a point \(\lambda \in B_d\) to obtain a complex of vector spaces:

\[
\cdots \xrightarrow{\Phi_2(\lambda)} C_2 \xrightarrow{\Phi_1(\lambda)} C_1 \xrightarrow{\Phi_0(\lambda)} C_0.
\]

The main result of this section is the following:

**Theorem 2.9.** Let \(\mathcal{H}\) be a pure contractive \(\mathcal{A}_d\)-module, and let

\[
\cdots \xrightarrow{\Phi_2} F_1 \xrightarrow{\Phi_1} F_0 \xrightarrow{\Phi_0} \mathcal{H} \longrightarrow 0.
\]

a free resolution of \(\mathcal{H}\). Localize at 0 to obtain the following complex:
Let

\[ \cdots \xrightarrow{\Phi_2(0)} C_2 \xrightarrow{\Phi_2(0)} C_1 \xrightarrow{\Phi_1(0)} C_0. \]

be the Koszul complex of \( \mathcal{H} \). Then for all \( k \geq 1 \),

\[ \ker \partial_k \cong \frac{\ker \Phi_k(0)}{\text{im} \Phi_{k+1}(0)}. \]

Proof. In the course of this proof, we will use \( \mathcal{F}_i^{k} \) and \( \mathcal{H}^k \) to denote, respectively, \( \mathcal{F}_i \otimes \wedge^k \mathbb{C}^d \) and \( \mathcal{H} \otimes \wedge^k \mathbb{C}^d \). Since for any \( k \), \( \wedge^k \mathbb{C}^d \) is finite dimensional, tensoring the components of (9) by \( \wedge^k \mathbb{C}^d \) preserves exactness. Hence the following sequence is exact:

\[ \cdots \xrightarrow{\Phi_2 \otimes 1_{\wedge^k \mathbb{C}^d}} \mathcal{F}_1^k \xrightarrow{\Phi_1 \otimes 1_{\wedge^k \mathbb{C}^d}} \mathcal{F}_0^k \xrightarrow{\Phi_0 \otimes 1_{\wedge^k \mathbb{C}^d}} \mathcal{H}^k \xrightarrow{} 0. \]

For the sake of convenience, and since it will cause no confusion in what follows, we will denote maps of the form \( \mathcal{F}_i \otimes \wedge^k \mathbb{C}^d \) by \( \mathcal{F}_i^k \) for any linear operator \( A \) on \( H^2 \). Furthermore, unless otherwise stated, we shall use \( \partial_k \) to denote any Koszul complex mapping \( \mathcal{M} \otimes \wedge^k \mathbb{C}^d \rightarrow \mathcal{M} \otimes \wedge^{k-1} \mathbb{C}^d \).

We claim that the following diagram commutes, and, with the exception of the \( \mathcal{C} \) column, is exact on rows and columns.

\[ \cdots \xrightarrow{\partial_{i+3}} \mathcal{F}_{i+2}^1 \xrightarrow{\partial_{i+3}} \mathcal{F}_{i+2}^0 \xrightarrow{\partial_{i+2}} \mathcal{F}_{i+3}^0 \xrightarrow{\partial_{i+3}(0)} \mathcal{F}_{i+3}^0 \xrightarrow{\partial_{i+3}(0)} \mathcal{F}_{i+3}^0 \xrightarrow{} \mathcal{C}_{i+2} \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{\partial_{i+3}} \mathcal{F}_{i+2}^1 \xrightarrow{\partial_{i+2}} \mathcal{F}_{i+2}^0 \xrightarrow{\partial_{i+2}(0)} \mathcal{F}_{i+3}^0 \xrightarrow{\partial_{i+3}(0)} \mathcal{F}_{i+3}^0 \xrightarrow{} \mathcal{C}_{i+2} \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{\partial_{i+1}} \mathcal{F}_{i+1}^1 \xrightarrow{\partial_{i+1}} \mathcal{F}_{i+1}^0 \xrightarrow{\partial_{i+1}(0)} \mathcal{F}_{i+1}^0 \xrightarrow{\partial_{i+1}(0)} \mathcal{F}_{i+1}^0 \xrightarrow{} \mathcal{C}_{i+1} \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{\partial_{i}} \mathcal{F}_{i}^1 \xrightarrow{\partial_{i}} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{} \mathcal{C}_{i} \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{\partial_{i}} \mathcal{F}_{i}^1 \xrightarrow{\partial_{i}} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{} \mathcal{C}_{i} \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{\partial_{i}} \mathcal{F}_{i}^1 \xrightarrow{\partial_{i}} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{} \mathcal{C}_{i} \xrightarrow{} 0 \]

\[ \cdots \xrightarrow{\partial_{i}} \mathcal{F}_{i}^1 \xrightarrow{\partial_{i}} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{\partial_{i}(0)} \mathcal{F}_{i}^0 \xrightarrow{} \mathcal{C}_{i} \xrightarrow{} 0 \]
First we check commutativity for the square

\[
\begin{array}{c}
\mathcal{F}_i \\
\Phi_i \\
\mathcal{F}_i
\end{array} \quad \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
\mathcal{F}_i^k \\
\Phi_{i+1} \\
\mathcal{F}_i^k
\end{array} \quad \begin{array}{c}
\partial_k \\
\partial_k \\
\partial_k
\end{array}
\]

(11)

where \(i, k \geq 1\). This amounts to showing that \(\partial_k \Phi_{i+1} = \Phi_{i+1} \partial_k\). Consider an element in \(\mathcal{F}_i^k\) of the form

\[
\xi \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k},
\]

(12)

where \(\xi \in \mathcal{F}_{i+1}, 1 \leq i_1 < i_2 < \cdots < i_k \leq d\). Applying \(\Phi_{i+1}\) and then \(\partial_k\) to this gives us

\[
\sum_{j=1}^{k} (-1)^{j+1} z_{i_j} \Phi_{i+1} \xi \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}.
\]

(13)

Since \(\Phi_{i+1}\) is a module homomorphism, the \(z_{i_j}\)'s commute with \(\Phi_{i+1}\), and the resulting expression

\[
\sum_{j=1}^{k} (-1)^{j+1} \Phi_{i+1} z_{i_j} \xi \otimes e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}
\]

is the result of applying \(\partial_k\) then \(\Phi_{i+1}\) to (12). Hence (11) is a commuting square. For the case of squares of the following form:

\[
\begin{array}{c}
\mathcal{F}_{i+1}^0 \\
\Phi_{i+1} \\
\mathcal{F}_{i+1}^0
\end{array} \quad \begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
\mathcal{C}_{i+1} \\
\Phi_i \\
\mathcal{C}_i
\end{array} \quad \begin{array}{c}
\partial_{1/2} \\
\partial_{1/2} \\
\partial_{1/2}
\end{array}
\]

(14)

we argue as follows. Let \(\xi \in \mathcal{F}_{i+1}^0 = \mathcal{F}_{i+1}\). The result of evaluating at 0 and then applying \(\Phi_{i+1}(0)\) results in

\[
\Phi_{i+1}(0) \xi(0)
\]

which is equivalent to \((\Phi_{i+1} \xi)(0)\), which is the result of applying \(\Phi_{i+1}\) to \(\xi\) and then evaluating at 0. Hence (14) is a commuting square.
Before proceding we make the following observation: For each \( i \geq 0 \), we denote the quotient module \( F_i/\Phi_{i+1}(F_{i+1}) \) by \( \mathcal{H}_i \). Note that by assumption \( \mathcal{H} \) is naturally isomorphic to \( \mathcal{H}_0 \). Let

\[
\cdots \xrightarrow{\partial'_{1}} \mathcal{H}_i \xrightarrow{\partial'_{0}} \mathcal{H}^0_i \xrightarrow{} 0
\]

be the Koszul complex for \( \mathcal{H}_i \). Fixing this \( i \), we let

\[
\cdots \xrightarrow{\partial''_{1}} \mathcal{H}_{i+1} \xrightarrow{\partial''_{0}} \mathcal{H}^0_{i+1} \xrightarrow{} 0
\]

be the Koszul complex for \( \mathcal{H}_{i+1} \). Note that since (10) commutes, the complexes (15) and (16) are induced by the maps of the Koszul complexes \( F^*_i \rightarrow F^*_i \) and \( F^*_{i+2} \rightarrow F^*_i \) respectively. This can be expressed by saying that the following two diagrams commute:

\[
\begin{array}{c}
\cdots \xrightarrow{\partial_2} F^1_{i+1} \xrightarrow{\partial_1} F^0_{i+1} \\
\downarrow \Phi_{i+1} \downarrow \Phi_{i+1} \\
\cdots \xrightarrow{\partial_2} F^1_i \xrightarrow{\partial_1} F^0_i \\
\downarrow \downarrow \\
\cdots \xrightarrow{\partial_2'_{1}} \mathcal{H}_i \xrightarrow{\partial'_{0}} \mathcal{H}^0_i \\
\downarrow \downarrow \\
0_0
\end{array}
\]

\[
\begin{array}{c}
\cdots \xrightarrow{\partial_2} F^1_{i+2} \xrightarrow{\partial_1} F^0_{i+2} \\
\downarrow \Phi_{i+2} \downarrow \Phi_{i+2} \\
\cdots \xrightarrow{\partial_2} F^1_{i+1} \xrightarrow{\partial_1} F^0_{i+1} \\
\downarrow \downarrow \\
\cdots \xrightarrow{\partial''_1} \mathcal{H}_{i+1} \xrightarrow{\partial''_{0}} \mathcal{H}^0_{i+1} \\
\downarrow \downarrow \\
0_0
\end{array}
\]
We will use this fact to establish the following two claims:

**Claim 2.10.** For $k \geq 2$,

\[
\frac{\ker \partial_k'}{\text{im } \partial_{k+1}'} \cong \frac{\ker \partial_{k-1}'}{\text{im } \partial_k'}.
\]

**Proof.** Consider the following portion of (10), where $k \geq 2$:

\[
\begin{array}{cccccccc}
\mathcal{F}_{i+2}^k & \overset{\partial_{i+2}}{\longrightarrow} & \mathcal{F}_{i}^k & \overset{\partial_i}{} & \mathcal{F}_{i+1}^{k-1} & \overset{\partial_{i+1}}{\longrightarrow} & \mathcal{F}_i^{k-2} & \overset{\partial_i}{} & \mathcal{F}_{i+1}^{k-3} \\
\Phi_{i+2} & \downarrow & \Phi_{i+2} & \downarrow & \Phi_{i+2} & \downarrow & \Phi_{i+2} & \downarrow & \Phi_{i+2} \\
\mathcal{F}_{i+1}^k & \overset{\partial_{i+1}}{\longrightarrow} & \mathcal{F}_{i}^k & \overset{\partial_i}{} & \mathcal{F}_{i+1}^{k-1} & \overset{\partial_{i+1}}{\longrightarrow} & \mathcal{F}_i^{k-2} & \overset{\partial_i}{} & \mathcal{F}_{i+1}^{k-3} \\
\Phi_{i+1} & \downarrow & \Phi_{i+1} & \downarrow & \Phi_{i+1} & \downarrow & \Phi_{i+1} & \downarrow & \Phi_{i+1} \\
\mathcal{F}_i^k & \overset{\partial_i}{} & \mathcal{F}_i^k & \overset{\partial_i}{} & \mathcal{F}_i^{k-1} & \overset{\partial_i}{} & \mathcal{F}_i^{k-2} & \overset{\partial_i}{} & \mathcal{F}_i^{k-3} \\
(17)
\end{array}
\]

Let $\zeta \in \ker \partial_k'/\text{im } \partial_{k+1}'$. Choose a representative $\zeta_0 \in \mathcal{F}_i^k$ for $\zeta$. By assumption, $\partial_k \zeta_0 \in \Phi_{i+1}(\mathcal{F}_{i+1}^{k-1})$. Choose $\eta_0 \in \mathcal{F}_{i+1}^{k}$ such that $\Phi_{i+1}(\partial_{i+1} \eta_0) = 0$. By exactness of the bottom row of (17), $\partial_{k-1} \Phi_{i+1} \eta_0 = 0$, hence $\partial_{k-1} \eta_0 \in \ker \Phi_{i+1}$. Hence by the exactness of the last column in (17), $\eta_0$ defines a homology class $\eta \in \ker \partial_{k-1}'/\text{im } \partial_k'$. We claim that this $\eta$ depends only on the choice of $\zeta$. Indeed, due to exactness of rows in (17), a different choice of $\eta_0$ corresponds to a perturbation by an element in the image of $\mathcal{F}_{i+1}^{k-1}$ under $\Phi_{i+2}$, which results in the new $\eta_0$ being in the same homology class. A different choice of $\zeta_0$ corresponds to a perturbation by an element of $\partial_{k+1}(\mathcal{F}_{i+1}^{k+1})$ and an element of $\Phi_{i+1}(\mathcal{F}_{i+1}^{k+1})$. The first of these is eliminated by the exactness of the bottom row of (17) and the way we defined $\eta$. Due to the commutativity of the diagram, the second perturbation corresponds to a perturbation of $\eta_0$ by an element of $\partial_k(\mathcal{F}_{i+1}^{k+1})$, which yield the same homology class $\eta$.

Conversely, if one begins with an element $\eta \in \ker \partial_{k-1}'/\text{im } \partial_k'$, for any representative $\eta_0 \in \mathcal{F}_{i+1}^{k-1}$, there is an element $\zeta_0 \in \mathcal{F}_i^k$ such that $\partial_k \zeta_0 = \Phi_{i+1} \eta_0$. This implies that $\zeta_0$ corresponds to a homology class $\zeta \in \ker \partial_k'/\text{im } \partial_{k+1}'$. We claim that this $\zeta$ depends only on $\eta$. Indeed, by the exactness of the bottom row, a different choice of $\zeta_0$ is a perturbation by an element in $\partial_{k+1}(\mathcal{F}_{i+1}^{k+1})$, which corresponds to the same homology class $\zeta$. A different choice of $\eta_0$ corresponds to a perturbation by an element in $\Phi_{i+2}(\mathcal{F}_{i+2}^{k-1})$ or in $\partial_k(\mathcal{F}_{i+1}^k)$. In the first case, the perturbation is annihilated by the time that we get to $\mathcal{F}_{i+1}^{k-1}$. In the second case, the commutativity of the diagram implies that the resulting perturbation
of \( \zeta_0 \) is an element in \( \Phi_{i+1}(\mathcal{F}_i^k) \) and hence the homology class \( \zeta \) is the same.

Clearly the above maps are inverses of each other, and hence we have the desired isomorphism.

\[ \square \]

Claim 2.11.

\[ \frac{\ker \partial'_i}{\im \partial'_2} \cong \frac{\ker \Phi_{i+1}(0)}{\im \Phi_{i+2}(0)}. \]

**Proof.** As in Claim 2.10, we focus on a piece of (10):

\[
\begin{array}{c}
\mathcal{F}_{i+2}^2 \xrightarrow{\partial_2} \mathcal{F}_{i+2}^1 \xrightarrow{\partial_1} \mathcal{F}_{i+2}^0 \xrightarrow{\partial_{i+1}^1} C_{i+2} \longrightarrow 0 \\
\Phi_{i+2} \downarrow \Phi_{i+2} \downarrow \Phi_{i+2} \downarrow \Phi_{i+2}(0) \\
\mathcal{F}_{i+1}^2 \xrightarrow{\partial_2} \mathcal{F}_{i+1}^1 \xrightarrow{\partial_1} \mathcal{F}_{i+1}^0 \xrightarrow{\partial_{i+1}^1} C_{i+1} \longrightarrow 0 \\
\Phi_{i+1} \downarrow \Phi_{i+1} \downarrow \Phi_{i+1} \downarrow \Phi_{i+1}(0) \\
\mathcal{F}_1^2 \xrightarrow{\partial_2} \mathcal{F}_1^1 \xrightarrow{\partial_1} \mathcal{F}_1^0 \xrightarrow{\partial_{i+1}^1} C_i \longrightarrow 0.
\end{array}
\]

Let \( \zeta \) be an element in \( \ker \partial'_1/\im \partial'_2 \), let \( \zeta_0 \) be a representative of \( \zeta \) in \( \mathcal{F}_i^1 \). Then \( \partial_1 \zeta_0 = \Phi_{i+1}(\mathcal{F}_i^0) \) by assumption. Choose \( \eta \in \mathcal{F}_i^0 \) such that \( \Phi_{i+1} \eta = \partial_1 \zeta_0 \), and then let \( x_0 = \eta(0) \in C_{i+1} \). By the way in which \( x_0 \) was defined and by the commutativity of (18), \( x_0 \in \ker \Phi_{i+1}(0) \).

Now let \( x \) be the homology class of \( x_0 \) in \( \ker \Phi_{i+1}(0)/\im \Phi_{i+2}(0) \). We claim that this \( x \) depends only on \( \zeta \). First, by exactness, a different choice of \( \eta \) corresponds to a perturbation by an element in \( \Phi_{i+2}(\mathcal{F}_i^0) \).

By the commutativity of (18), this corresponds to a perturbation by \( x_0 \) by an element in \( \Phi_{i+2}(0)(C_{i+2}) \), which leaves \( x \) unaltered. By the exactness of rows, any perturbation of \( \zeta_0 \) by an element of \( \partial_2(\mathcal{F}_i^2) \) or \( \Phi_{i+1}(\mathcal{F}_i^1) \) is annihilated by the time one gets to \( x_0 \in C_{i+1} \).

Conversely, suppose that one starts with an element \( x \in \ker \Phi_{i+1}(0)/\im \Phi_{i+2}(0) \). Choose a representative \( x_0 \in \ker \Phi_{i+1}(0) \) for \( x \). Let \( \eta_0 \) be a preimage of \( x_0 \) in \( \mathcal{F}_i^0 \). By the commutativity of (18), there exists \( \zeta_0 \in \mathcal{F}_i^1 \) such that \( \partial_1 \zeta_0 = \Phi_{i+1} \eta_0 \). Hence this \( \zeta_0 \) is part of a homology class \( \zeta \in \ker \partial'_1/\im \partial'_2 \). We claim that \( \zeta \) depends only on the choice of \( x_0 \). Indeed, by the exactness of the middle row in (18), a different choice of \( \eta_0 \in \mathcal{F}_i^0 \) corresponds to a perturbation by an element in \( \partial_1(\mathcal{F}_i^1) \). By the commutativity of (18), this corresponds to a perturbation of \( \zeta_0 \) by an element of \( \Phi_{i+1}(\mathcal{F}_i^1) \), yielding the same homology class \( \zeta \). A different choice of \( \zeta_0 \) so that \( \partial_1 \zeta_0 = \Phi_{i+1} \eta_0 \) corresponds, by exactness of the bottom row of (18), to a perturbation by an element
in $\partial_2(\mathcal{F}^2_1)$ which obviously yields the same homology class $\eta$. Finally, a different choice of $x_0$ corresponds to a perturbation by an element in $\Phi_{i+2}(0)(C_{i+2})$. By the exactness of the top row and second to last column of (18), one sees that this perturbation is annihilated by the time one arrives at $\mathcal{F}^0_i$.

Clearly the above two maps are inverses of each other. Hence the desired isomorphism is established.

\[ \square \]

We can now establish the statement of the theorem. Labeling more explicitly now, we let

\[ \cdots \xrightarrow{\partial_2^i} \mathcal{H}^1_i \xrightarrow{\partial_1^i} \mathcal{H}^0_i \xrightarrow{} 0 \]

be the Koszul complex of $\mathcal{H}_i$. For $k \geq 2$, Claim 2.10 yields the following sequence of isomorphisms:

\[ \begin{array}{c}
\frac{\ker \partial^0_k}{\im \partial^0_{k+1}} \cong \frac{\ker \partial^1_{k-1}}{\im \partial^1_k} \cong \cdots \frac{\ker \partial^{k-1}}{\im \partial^{k-1} 2}.
\end{array} \]  

(19)

We then use Claim 2.11 to finish off the sequence:

\[ \frac{\ker \partial^{k-1}}{\im \partial^{k-1} 2} \cong \frac{\ker \Phi_k(0)}{\im \Phi_{k+1}(0)}.
\]

(20)

Together, (19) and (20) establish the statement of the theorem.

\[ \square \]

Our goal is to describe the homology of the localized complex

\[ \cdots \xrightarrow{\Phi_1(\lambda)} C_2 \xrightarrow{\Phi_2(\lambda)} C_1 \xrightarrow{\Phi_1(\lambda)} C_0. \]

for an arbitrary $\lambda \in B_d$. We will attain this goal in Section 4, but we need some machinery first. This is the subject of the next section.

3. The Möbius Transform

In this section we define the notion of a Möbius transform of a pure contractive $A_d$-module. We begin by summarizing the main properties of Möbius transforms on the unit ball in $\mathbb{C}^d$. For a more detailed exposition, we refer the reader to [13]. Recall that a Möbius transform on the unit ball in $\mathbb{C}^d$ is a continuous bijection $\varphi : \overline{B_d} \to \overline{B_d}$ which satisfies the following (somewhat redundant) properties:

1. $\varphi$ is holomorphic in $B_d$.
2. $\varphi(B_d) = B_d$. 
3. \( \varphi(\partial B_d) = \partial B_d \).

Let \( \lambda \in B_d \), and define \( \varphi_\lambda \) as follows:

\[
\varphi_\lambda(z_1, z_2, \ldots, z_d) = \left( \frac{\sqrt{1 - |\lambda|^2} z_1}{1 - \sum_{k=1}^d \lambda_k z_k} - \lambda_1, \ldots, \frac{\sqrt{1 - |\lambda|^2} z_d}{1 - \sum_{k=1}^d \lambda_k z_k} - \lambda_d \right)
\]

or in “vector notation”,

\[
\varphi_\lambda(z) = \frac{P_{C\lambda}(z - \lambda) - (1 - |\lambda|^2) P_{C\lambda} \perp z}{1 - \langle z, \lambda \rangle}, \quad z \in B_d,
\]

where \( P_{C\lambda} \) is the orthogonal projection onto the one-dimensional subspace \( \mathbb{C}\lambda \subseteq \mathbb{C}^d \). Some calculations reveal that \( \varphi_\lambda \) is a Möbius transform and that \( \varphi_\lambda(\lambda) = 0 \). It is a striking fact that any Möbius transform \( \varphi \) can be written in the form \( u \circ \varphi_\lambda \), where \( u \) is a unitary operator on \( \mathbb{C}^d \) and \( \lambda = \varphi^{-1}(0) \). This fact allows us to find a useful formula for the expression \( \langle \varphi(w), \varphi(z) \rangle \), where \( z, w \in B_d \). Calculating this first for the case where \( \varphi = \varphi_\lambda \), we obtain the following identity:

\[
(22) \quad \langle \varphi(z), \varphi(w) \rangle = 1 - \frac{(1 - |\lambda|^2)(1 - \langle z, w \rangle)}{(1 - \langle \lambda, w \rangle)(1 - \langle z, \lambda \rangle)}, \quad z, w \in B_d.
\]

Consequently, for any Möbius transform \( \varphi \) with \( \lambda = \varphi^{-1}(0) \), \( \langle \varphi(w), \varphi(z) \rangle \) is also given by (22).

The following theorem unveils the role that is played by Möbius transforms in the theory of free contractive \( \mathcal{A}_d \)-modules.

**Theorem 3.1.** Let \( \varphi^1, \varphi^2, \ldots, \varphi^d \) be the coordinates of the Möbius transform \( \varphi \). Define the map \( \Phi : H^2_d \oplus \cdots H^2_d \to H^2_d \) to be left multiplication by the row vector \( (\varphi^1 \cdots \varphi^d) \), i.e.

\[
\Phi \left( \begin{array}{c} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_d \end{array} \right) = \sum_{k=1}^d \varphi^k \xi_k, \quad \xi_1, \xi_2, \ldots, \xi_d \in H^2_d.
\]

Then \( \Phi \) is a partially isometric module homomorphism with range \( \{ \xi \in H^2_d : \xi(\lambda) = 0 \} = \{ k_\lambda \}^\perp \).

**Proof.** It is obvious that \( \Phi \) is a module homomorphism. To show that it is partially isometric with the stated range, it suffices to show that
\[ \langle (1 - \Phi\Phi^*)k_w, k_z \rangle = \langle P_\lambda k_w, k_z \rangle, \]
for any \( w, z \in B_d \). The sufficiency of this condition follows from the fact that the set of all \( k_z \)'s forms a spanning set of \( H_d^2 \).

We compute the left side of (3). Using (1), (2), and formula (22), we have

\[
\langle (1 - \Phi\Phi^*)k_w, k_z \rangle = \frac{1}{1 - \langle z, w \rangle} - \frac{\langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle} + \frac{1 - |\lambda|^2}{(1 - \langle \lambda, w \rangle)(1 - \langle z, \lambda \rangle)}.
\]

We compute the right side of (3) as follows:

\[
\langle P_\lambda k_w, k_z \rangle = \frac{\langle k_w, k_\lambda \rangle}{\|k_\lambda\|^2} \langle k_\lambda, k_z \rangle = \frac{1 - |\lambda|^2}{(1 - \langle \lambda, w \rangle)(1 - \langle z, \lambda \rangle)},
\]
which is identical to (23), hence we have established (3).

We are now in a position to define a Möbius transform of \( H_d^2 \).

**Definition 3.2.** Let \( \varphi \) be a Möbius transform. We define \((H_d^2)_\varphi\) to be the \( \mathcal{A}_d \)-module whose underlying Hilbert space is \( H_d^2 \) and whose \( \mathcal{A}_d \) is given by \( z_i \cdot \xi = \varphi^i \xi \) for any \( \xi \in H_d^2 \) and \( i = 1, 2, \ldots, d \), where \( \varphi^i \) is the \( i \)th coordinate function of \( \varphi \).

**Theorem 3.3.** Let \( \varphi \) be a Möbius transform. Then \((H_d^2)_\varphi\) is a pure contractive \( \mathcal{A}_d \)-module. Furthermore, if we set \( \lambda = \varphi^{-1}(0) \), then the map \( U_\varphi : H_d^2 \rightarrow (H_d^2)_\varphi \) defined by \( U_\varphi \xi = (\xi \circ \varphi) \frac{k_\lambda}{\|k_\lambda\|} \) is a unitary module isomorphism.

**Proof.** Let \( (\Phi_1, \Phi_2, \ldots, \Phi_d) \) be the \( d \)-tuple such that \( \Phi_i \in \mathcal{B}(H_d^2) \) is multiplication by \( \varphi^i \). By conditions (1), (2), and (3), to prove the first part of the theorem it suffices to show that the following two conditions on \( (\Phi_1, \Phi_2, \ldots, \Phi_d) \) hold.

1. \( \sum_{k=1}^d \Phi_k \Phi_k^* \leq 1 \).
2. WOT \( \lim_{n \to \infty} \sum_{i_1, i_2, \ldots, i_n} \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_n} \Phi_{i_n}^* \cdots \Phi_{i_2}^* \Phi_{i_1}^* = 0 \).

The first condition (1) follows from Theorem 3.1, since the row operator \( (\Phi_1, \Phi_2, \ldots, \Phi_d) \) is a partial isometry. For the second condition (2), we apply the stated limit to the linear functional \( \langle (\cdot)k_z, k_w \rangle \):
where we make use of (2). By property (2) of the classical Möbius transform, \(| \langle \varphi_{\lambda}(w), \varphi_{\lambda}(z) \rangle | < 1\). Hence the limit in (25) tends to 0. Since the set of all \(k_z\) form a spanning set for \(H^2_d\), and since the sums \(\sum_{i_1, i_2, \ldots, i_n} \Phi_{i_1} \cdots \Phi_{i_n} \Phi^*_{i_n} \cdots \Phi^*_{i_1} k_z, k_w\) are uniformly bounded over \(n\), it follows that condition (2) is valid.

For the next statement in the proof, recall that by Theorem 1.3 there exists a minimal dilation \(U_\varphi : H^2_d \otimes \mathcal{D} \to (H^2_d)_\varphi\). Let \((S'_1, S'_2, \ldots, S'_d)\) be the \(d\)-tuple associated with \(H^2_d \otimes \mathcal{D}\). By Theorem 3.1 \(\sum_{k=1}^{d} S'_k \psi_k^*\) is the projection onto \((k_0 \otimes \mathcal{D})^\perp = \{\xi \in H^2_d \otimes \mathcal{D} : \xi(0) = 0\}\). Since \(U_\varphi\) is a coisometric module homomorphism, we have the following equation: Let \(P'_0\) be the orthogonal projection onto the space \(k_0 \otimes \mathcal{D}\).

\[
\lim_{n \to \infty} \sum_{i_1, i_2, \ldots, i_n} \langle \Phi_{i_1} \cdots \Phi_{i_n} \Phi^*_{i_n} \cdots \Phi^*_{i_1} k_z, k_w \rangle = \lim_{n \to \infty} \langle \varphi_{\lambda}(w), \varphi_{\lambda}(z) \rangle^n \langle k_z, k_w \rangle.
\]

(25)

Theorem 3.1. Hence \(P'_0 U_\varphi^*\) is a rank one operator, and by the minimality condition on \(U_\varphi\), the submodule generated by the image this operator must be \(H^2_d \otimes \mathcal{D}\). It follows that \(\dim \mathcal{D} = 1\), hence we may view \(U_\varphi\) as a map \(H^2_d \to (H^2_d)_\varphi\). The equation in (26) also implies that \(U_\varphi P'_0\) is a partial isometry with support \(\mathbb{C}k_0\) and image \(\mathbb{C}k_\lambda\). Hence we may set \(U_\varphi 1 = \frac{k_\lambda}{\|k_\lambda\|}\). The equation \(U_\varphi \xi = (\xi \circ \varphi) \frac{k_\lambda}{\|k_\lambda\|}\) now follows since \(U_\varphi\) is a module homomorphism.

To complete the proof, it suffices to show that \(\ker U_\varphi = \{0\}\). To this end, let \(\xi \in H^2_d\), and suppose that \((\xi \circ \varphi) \frac{k_\lambda}{\|k_\lambda\|} = 0\). Since \(k_\lambda(z)/\|k_\lambda\| = \sqrt{\frac{1-|\lambda|^2}{1-\langle z, \lambda \rangle}}\) is never 0 on \(B_d\), we must have \(\xi(\varphi(z))\) for all \(z \in B_d\). Hence \(\xi(z) = 0\) for all \(z \in B_d\) since \(\varphi\) is bijective on \(B_d\). Hence \(\xi = 0\).

The following corollary shows how a Möbius transform \(U_\varphi\) provides a means of “changing the base point” from 0 to \(\lambda \in B_d\) when considering the module \(H^2_d\).

**Corollary 3.4.** Let \(\varphi\) be a Möbius transform and let \(\lambda = \varphi^{-1}(0)\). Then \(U_\varphi \{k_0\}^\perp = \{k_\lambda\}^\perp\).

**Proof.** This is obvious from the definition of \(U_\varphi\).
To conclude this section, we demonstrate an “ergodicity” property of the set of Möbius transforms.

**Theorem 3.5.** Let $\mathcal{M}$ be a proper non-trivial closed submodule of $H_d^2$. Then there exists $\xi \in \mathcal{M}$ and a Möbius transform $\varphi$ such that $U_\varphi \xi \notin \mathcal{M}$.

**Proof.** Since $\mathcal{M}$ is a proper closed submodule, it cannot contain $k_0$. Hence

\[ M = \sup \{ |\langle \xi, k_0 \rangle| : \|\xi\| = 1, \xi \in \mathcal{M} \} < 1. \]

By Theorem 3.2 in [8], there exists $\lambda \in B_d$ such that

\[ \frac{\|P_M k_\lambda\|}{\|k_\lambda\|} > M. \]  

(27)

An explicit calculation involving (21) shows that

\[ U_{\varphi_{-\lambda}} U_{\varphi_{\lambda}} \xi = \xi \circ u, \]

where $u$ is a unitary operator on $\mathbb{C}^d$. Hence by the definition of $U_u$, $U_{\varphi_{-\lambda}} U_{\varphi_{\lambda}} = U_u$. Hence $U_{\varphi_{\lambda}}^* = U_u^* U_{\varphi_{-\lambda}}$. Therefore by our assumption that $\mathcal{M}$ is invariant under Möbius transforms, it follows that

\[ \sup \{ |\langle U_{\varphi_{\lambda}}^* \xi, k_0 \rangle| : \|\xi\| = 1, \xi \in \mathcal{M} \} \leq M. \]  

(28)

By definition, $U_{\varphi_{\lambda}} k_0 = \frac{k_\lambda}{\|k_\lambda\|}$. Hence by (28), $|\langle \xi, \frac{k_\lambda}{\|k_\lambda\|} \rangle| \leq M$ for all $\xi \in \mathcal{M}$ such that $\|\xi\| = 1$. But this implies that

\[ \frac{\|P_M k_\lambda\|}{\|k_\lambda\|} \leq M, \]

which contradicts (27). \qed

4. The homology of localized free resolutions

In this section we use the machinery developed in Section 3 to extend Theorem 2.9. We first generalize Definition 3.2.

**Definition 4.1.** Let $\mathcal{H}$ be a pure contractive $\mathcal{A}_d$-module, and let $\varphi$ be a Möbius transform. We define the Möbius transform of $\mathcal{H}$ by $\varphi$ to be the module $(\mathcal{H})_\varphi$ with underlying Hilbert space $\mathcal{H}$ and $\mathcal{A}_d$ action defined by $z_i \cdot \xi = \varphi^i(T_1, T_2, \ldots, T_d) \xi$ for all $\xi \in H_d^2$ and $i = 1, 2, \ldots, d$. 
Theorem 4.2. Let $\mathcal{H}$ be a pure contractive $A_d$-module, and let $\varphi$ be a Möbius transform. Then $(\mathcal{H})_\varphi$ is contractive and pure.

Proof. Let $V : H^2_d \otimes D \to \mathcal{H}$ be a dilation of $\mathcal{H}$. From Definition 3.2, we see that $V$ is also a module homomorphism as a map from $(H^2_d)_\varphi \otimes D$ onto $(\mathcal{H})_\varphi$. Precomposing $V$ with $U_\varphi \otimes I_D$ gives a coisometric module homomorphism $V' : H^2_d \otimes D \to (\mathcal{H})_\varphi$. Hence $(\mathcal{H})_\varphi$ is isomorphic to the quotient of a free module, whence it is pure and contractive. \hfill $\square$

The main result of this section is the following:

Theorem 4.3. Let $\mathcal{H}$ be a pure contractive $A_d$-module, and let $\varphi$ be a Möbius transform with $\lambda = \varphi^{-1}(0)$. Let

\[ \cdots \xrightarrow{\partial''_k} E_2 \xrightarrow{\partial''_1} E_1 \xrightarrow{\partial'_0} E_0 \to 0 \]

be the Koszul complex of $(\mathcal{H})_\varphi$, and let

\[ \cdots \xrightarrow{\Phi_3(\lambda)} C_2 \xrightarrow{\Phi_2(\lambda)} C_1 \xrightarrow{\Phi_1(\lambda)} C_0 \]

be the localization at $\lambda$ of a free resolution of $\mathcal{H}$. Then for $k \geq 1$,

\[ \frac{\ker \partial'_k}{\text{im } \partial'_{k+1}} \cong \frac{\ker \Phi_k(\lambda)}{\text{im } \Phi_{k+1}(\lambda)}. \]

Proof. Let

\[ \cdots \xrightarrow{\Phi_2} H^2_d \otimes C_1 \xrightarrow{\Phi_1} H^2_d \otimes C_0 \xrightarrow{\Phi_0} \mathcal{H} \to 0 \]

be the free resolution of $\mathcal{H}$ from which (29) is derived. Since $\Phi_i$ is a module homomorphism it follows that for $i \geq 0$, $\varphi^j \Phi_i = \Phi_i \varphi^j$ for $j = 1, 2, \ldots, d$. Hence we may view (30) as an exact sequence of Möbius transformed modules:

\[ \cdots \xrightarrow{\Phi_2} (H^2_d)_\varphi \otimes C_1 \xrightarrow{\Phi_1} (H^2_d)_\varphi \otimes C_0 \xrightarrow{\Phi_0} (\mathcal{H})_\varphi \to 0. \]

For $i \geq 1$ let $\Phi'_i = U_\varphi \Phi_i U_\varphi$. Then we have the following partial isomorphism of complexes:

\[ \cdots \xrightarrow{\Phi_2} (H^2_d)_\varphi \otimes C_1 \xrightarrow{\Phi_1} (H^2_d)_\varphi \otimes C_0 \xrightarrow{\Phi_0} (\mathcal{H})_\varphi \to 0 \]

\[ \begin{array}{c}
\downarrow U^*_\varphi \\
\end{array} \]

\[ \cdots \xrightarrow{\Phi'_2} H^2_d \otimes C_1 \xrightarrow{\Phi'_1} H^2_d \otimes C_0. \]
Since two isomorphic complexes have isomorphic homologies, there is a coisometric homomorphism $\Phi'_0 : H^2_d \otimes C_0 \rightarrow (\mathcal{H})_{\varphi}$ which make the following sequence exact:

$$\cdots \xrightarrow{\Phi'_2} H^2_d \otimes C_1 \xrightarrow{\Phi'_1} H^2_d \otimes C_0 \xrightarrow{\Phi'_0} (\mathcal{H})_{\varphi} \rightarrow 0.$$  

This is obviously a free resolution for $(\mathcal{H})_{\varphi}$, hence Theorem 2.9 implies that

$$\frac{\ker \partial'_k}{\text{im} \partial'_{k+1}} \cong \frac{\ker \Phi'_k(0)}{\text{im} \Phi'_{k+1}(0)}$$

for $k \geq 1$. We now compute $\Phi'_i(0)$ for $i \geq 1$. Let $\eta \in C_i$ and $\eta' \in C_{i+1}$. Then

$$\langle \Phi'_i(0)\eta, \eta' \rangle = \langle \Phi'_i(k_0 \otimes \eta), k_0 \otimes \eta' \rangle = \langle U^* \Phi_i U \varphi(k_0 \otimes \eta), k_0 \otimes \eta' \rangle$$

$$= \langle \Phi'_i(k_0 \otimes \eta), \frac{k_\lambda}{\|k_\lambda\|} \otimes \eta' \rangle = \langle \frac{k_\lambda}{\|k_\lambda\|} \otimes \eta, \Phi'_i(\frac{k_\lambda}{\|k_\lambda\|} \otimes \eta') \rangle$$

$$= \langle \frac{k_\lambda}{\|k_\lambda\|} \otimes \eta, \frac{k_\lambda}{\|k_\lambda\|} \otimes \Phi_i(\lambda)^* \eta' \rangle = \frac{\|k_\lambda\|^2 \langle \eta, \Phi_i(\lambda)^* \eta' \rangle}{\|k_\lambda\|^2} = \langle \Phi_i(\lambda)\eta, \eta' \rangle.$$  

Since $\eta$ and $\eta'$ were arbitrary, $\Phi'_i(0) = \Phi_i(\lambda)$ for each $i \geq 1$. The conclusion of the theorem now follows.  

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