On the resolving strong domination number of graphs: a new notion

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Abstract. The study of metric dimension of graph $G$ has widely given some results and contribution of graph research of interest, including the domination set theory. The dominating set theory has been quickly growing and there are a lot of natural extension of this study, such as vertex domination, edge domination, total domination, power domination as well as the strong domination. In this study, we initiate to combine the two above concepts, namely metric dimension and strong domination set. Thus we have a resolving strong domination set. We have obtained the resolving strong domination number, denoted by $\gamma_{rst}(G)$, of some graphs.

1. Introduction
Let $G = (V,E)$ be a finite, simple and undirected graph, no loops and multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. The sets $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph $G$. For a further reference please see Chartrand et. al \cite{1}.

A vertex $u \in V(G)$ resolves a pair $x, y \subset V(G)$ if $d(u,x) \neq d(u,y)$. A set of vertices $S \subseteq V(G)$ is a resolved set of $G$ if every pair of vertices of $G$ is resolved by some vertex in $S$. The metric dimension of $G$, denoted by $\text{dim}(G)$, is the minimum cardinality of a resolving set of $G$. The resolving number, written as $\text{res}(G)$, is the minimum $k$ such that every $k$-subset of $V(G)$ is a resolving set of $G$. Obviously, every set $S \subseteq V(G)$ with $|S| \geq \text{res}(G)$ is a resolving set of $G$. A strong dominating set of a graph $G$ is a set $D \subseteq V(G)$ with the property that for all vertices $x \in V(G)$ there is a vertex $y \in N(x) \cap D$ with $d(x) \leq d(y)$, i.e. every vertex not in $D$ is dominated by a vertex in $D$ having at least the same degree. In this case we say that $y$ strongly dominates $x$. The strong domination number $\gamma_{st}(G)$ of a graph $G$ is defined as the minimum cardinality of a strong dominating set of $G$ \cite{2}.

A resolving strong dominating set is a set $R_D \subset V(G)$ which $R_D$ satisfies the definition of strong dominating set and resolving set. The strong domination number of graph $G$, denoted by $\gamma_{rst}(G)$ is the minimum cardinality of resolving strong dominating set of $G$. This study is working on a new notion of the combination of two concept, thus the research results are mostly referred to the strong dominating set of $G$ \cite{3} and the new results on the domination number study in \cite{4, 5, 6, 7}. The study on resolving domination number of graphs can be seen in \cite{8}.
In this study, we obtained the $\gamma_{rst}(G)$ on some graphs, namely path graph $P_n$, cycle graph $C_n$, complete graph $K_n$, fan graph $F_n$, and wheel graph $W_n$.

2. Resolving Strong Domination Number of Graphs
The some results on resolving strong domination number are showed in the following Theorems.

**Definition 2.1** A resolving strong dominating set is a set $R_D \subset V(G)$ which $R_D$ satisfies the definition of strong dominating set and resolving set. The strong domination number of graph $G$, denoted by $\gamma_{rst}(G)$ is the minimum cardinality of resolving strong dominating set of $G$.

**Lemma 2.1** Let $G$ be a connected graph with $\gamma_{st}$ is strong domination of $G$, $\gamma_{rst}$ is resolving strong domination of $G$, and $\dim(G)$ is the resolving number of $G$,

$$\max\{\gamma_{st}(G), \dim(G)\} \leq \gamma_{rst}(G) \leq \min\{\gamma_{st}(G) + \dim(G), |V(G)| - 1\}.$$ 

**Proof.** Let $R_D$ be the minimum resolving strong dominating set with $|R_D| = \gamma_{rst}(G)$. Since $R_D$ is resolving set and strong dominating set, thus $\dim(G) \leq |R_D|$ and $\gamma_{st}(G) \leq |R_D|$, so $\max\{\gamma_{st}(G), \dim(G)\} \leq \gamma_{rst}(G)$.

Let $D$ be a minimum strong dominating set with $|D| = \gamma_{st}(G)$ and $W$ be a minimum resolving set with $|W| = \dim(G)$. Therefore $D \cup W$ is resolving strong dominating set, thus $\gamma_{rst}(G) \leq |D \cup W| \leq \gamma_{st}(G) + \dim(G)$ Since every subset of $V(G)$ with the cardinality $|V(G)| - 1$ is a resolving set and strong dominating set, thus $\gamma_{rst}(G) \leq |V(G)| - 1$. Therefore $\gamma_{rst}(G) \leq \min\{\gamma_{st}(G) + \dim(G), |V(G)| - 1\}$. Hence $\max\{\gamma_{st}(G), \dim(G)\} \leq \gamma_{rst}(G) \leq \min\{\gamma_{st}(G) + \dim(G), |V(G)| - 1\}$. \hfill $\blacksquare$

**Theorem 2.1** Let $P_n$ be a path graph. For every positive integer $n \geq 2$,

$$\gamma_{rst}(P_n) = \begin{cases} 2, & \text{if } n = 3 \\ \lceil \frac{n}{3} \rceil, & \text{otherwise} \end{cases}$$

**Proof.** Let $P_n$ be a path graph with vertex set $V(P_n) = \{v_i; i \in [1, n]\}, |V(P_n)| = n$ and edge set $E(P_n) = \{v_iv_{i+1}; i \in [1, n-1]\}, |E(P_n)| = n - 1$. Based on Maryam et al [11] $\dim(P_n) = 1$ and [9] $\gamma_{st}(P_n) = \lceil \frac{n}{3} \rceil$. Based on Lemma 2.1, $\gamma_{st}(P_n) \leq \gamma_{rst}(P_n) \leq \gamma_{st}(P_n) + 1$.

For $n = 3$ $V(P_n) = \{v_1, v_2, v_3\}$ and $E(P_n) = \{v_1v_2, v_2v_3\}$. There is one strong dominating set of $P_3$, that is $\{v_2\}$ and $\{v_2\}$ is not resolving set of $P_3$ but every resolving set $|W(P_3)| \geq 2$ is resolving set then $|R_D| \geq 2$ is resolving strong dominating set. Therefore $\gamma_{st}(P_3) \geq 2$, so $\gamma_{rst}(P_3) = 2$.

For $n \neq 3$ we have $R_D = \{v_i: \equiv 2 \mod 3\}$ so based on Vaidya et al [9] $|D| = \lceil \frac{n}{3} \rceil$ and the superset of resolving set is resolving set. Thus $\gamma_{rst}(P_n) = \lceil \frac{n}{3} \rceil$.

For illustration, we can depict in the following picture.

![Figure 1. The illustration of resolving strong dominating set on $P_{10}$](image)

(1,4,7,8) (1,2,5,6) (3,0,3,4) (5,2,1,2) (7,4,1,0)
(0,3,6,7) (2,1,4,5) (4,1,2,3) (6,3,0,1) (8,4,2,1)

$v_1$ $v_2$ $v_3$ $v_4$ $v_5$ $v_6$ $v_7$ $v_8$ $v_9$ $v_{10}$
Theorem 2.2 Let $C_n$ be a cycle graph. For every positive integer $n \geq 3$,

$$\gamma_{rst}(C_n) = \begin{cases} 2, & \text{if } n = 3 \\ \lceil \frac{n}{3} \rceil, & \text{otherwise} \end{cases}$$

Proof. Let $C_n$ be a cycle graph with vertex set $V(C_n) = \{v_i : i \in [1, n]\}, |V(C_n)| = n$ and edge set $E(C_n) = \{v_i v_{i+1} : i \in [1, n-1] \} \cup \{v_1 v_n\}, |E(C_n)| = n$. Based on Murtaza et al [12] $\dim(C_n) = 2$ and $\lceil \frac{n}{3} \rceil$. Based on Lemma 2.1, $\gamma_{rst}(C_n) \leq \gamma_{rst}(C_n) \leq \gamma_{rst}(C_n) + 2$.

For $n = 3$ $V(C_n) = \{v_1, v_2, v_3\}$ and $E(C_n) = \{v_1 v_2, v_2 v_3, v_3 v_1\}$. There is one strong dominating set of $C_3$, that is $\{v_1\}$ and $\{v_2\}$ is not resolving set of $C_3$ but every resolving set $|W(C_3)| \geq 2$ is resolving set then $|R_D| \geq 2$ is resolving strong dominating set. Therefore $\gamma_{rst}(C_3) \geq 2$, so $\gamma_{rst}(C_3) = 2$.

For $n \neq 3$ we have $R_D = \{v_i : i \equiv 2 \mod 3\}$ so based on Dieter $[3] |D| = \lceil \frac{n}{3} \rceil$ and the superset of resolving set is resolving set. Thus $\gamma_{rst}(C_n) = \lceil \frac{n}{3} \rceil$.

To have a good understanding of this concept, we give an illustration in Figure 2.

![Figure 2](image)

Figure 2. The illustration of resolving strong dominating set on $C_{12}$

Theorem 2.3 Let $K_n$ be a complete graph. For every positive integer $n \geq 3$, $\gamma_{rst}(K_n) = n - 1$.

Proof. Let $K_n$ be a complete graph with vertex set $V(K_n) = \{v_i : i \in [1, n]\}, |V(K_n)| = n$ and edge set $E(K_n) = \{v_i v_j : i \neq j\}, |E(K_n)| = \frac{n(n-1)}{2}$. Based on Saputro et al [13] $\dim(K_n) = n - 1$ and $\gamma_{st}(K_n) = 1$. Based on Lemma 2.1, $\dim(K_n) \leq \gamma_{rst}(K_n) \leq n - 1$. Since $\dim(K_n) = n - 1$ thus $n - 1 \leq \gamma_{rst}(K_n) \leq n - 1$. Therefore $\gamma_{rst}(K_n) = n - 1$. See the Figure 3 for an illustration.

Theorem 2.4 Let $F_n$ be a friendship graph. For every positive integer $n \geq 2$, $\gamma_{rst}(F_n) = n + 1$.

Proof. Let $F_n$ be a friendship graph with vertex set $V(F_n) = \{a, x_i, y_i : i \in [1, n]\}, |V(F_n)| = 2n + 1$ and edge set $E(F_n) = \{ax_i, ay_i, x_i y_i : i \in [1, n]\}, |E(F_n)| = 3n$. Based on $\dim(F_n) = n$ and $\gamma_{st}(F_n) = 1$. Based on Lemma 2.1, $\dim(F_n) \leq \gamma_{rst}(F_n) \leq \dim(F_n) + \gamma_{st}(F_n)$. We have $R_D = \{v_i : 1 \leq i \leq n\}$ so $R_D$ is a resolving set but not strong dominating set because $a$ as a vertex with maximum degree is not a dominating set, that is not satisfied the definition of strong dominating set. Thus, $R_D = \{x_i : 1 \leq i \leq n\} \cup \{a\}$ and $R_D$ is resolving strong dominating set with cardinality $|R_D| = n + 1$. Therefore $\gamma_{rst}(F_n) = n + 1$.

For illustration, we can draw a friendship graph of order 8 with their representation of resolving set in Figure 4.
Theorem 2.5 Let $F_n$ be a fan graph. For every positive integer $n \notin \{2, 3, 6\}$, $\gamma_{rst}(F_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1$.

Proof. Let $F_n$ be a fan graph with vertex set $V(F_n) = \{a, x_i : i \in [1, n]\}$, $|V(F_n)| = n + 1$ and edge set $E(F_n) = \{ax_i : i \in [1, n]\} \cup \{x_ix_{i+1} : i \in [1, n-1]\}$, $|E(F_n)| = 2n - 1$. Based on Yero et al [14] $\dim(F_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ and [10] $\gamma_{st}(F_n) = 1$. Based on Lemma 2.1, $\dim(F_n) \leq \gamma_{rst}(F_n) \leq \dim(F_n) + \gamma_{st}(F_n)$. We have $R_D = \{x_i : 2 \leq i \leq n - 1 \text{ and } i \text{ is even}\}$ so $R_D$ is a resolving set but not strong dominating set because $a$ as a vertex with maximum degree is not a strong dominating set, that is not satisfied the definition of strong dominating set. Thus $R_D = \{x_i : 2 \leq i \leq n - 1 \text{ and } i \text{ is even}\} \cup \{a\}$ and $R_D$ is a resolving strong dominating set with cardinality $|R_D| = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1$. Therefore $\gamma_{rst}(F_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1$. See Figure 5 for an illustration. □

![Figure 3](image3.png)

**Figure 3.** The illustration of resolving strong dominating set on $K_{14}$

![Figure 4](image4.png)

**Figure 4.** The illustration of resolving strong dominating set on $F_8$

For illustration, we can draw a wheel graph of order 16 with their representation of resolving set in Figure 6.
Figure 5. The illustration of Resolving Strong Dominating Set on $F_{14}$

Theorem 2.6 Let $W_n$ be a wheel graph. For every positive integer $n \notin \{3, 6\}$, $\gamma_{rst}(W_n) = \left\lceil \frac{2n+2}{5} \right\rceil + 1$.

Proof. Let $W_n$ be a wheel graph with vertex set $V(W_n) = \{a, x_i : i \in [1, n]\}$, $|V(W_n)| = n + 1$ and edge set $E(W_n) = \{ax_i : i \in [1, n]\} \cup \{x_ix_{i+1} : i \in [1, n-1]\} \cup \{x_nx_1\}$, $|E(F_n)| = 2n$. $dim(W_n) = \left\lceil \frac{2n+2}{5} \right\rceil$ and $\gamma_{st}(W_n) = 1$. Based on Lemma 2.1, $dim(W_n) \leq \gamma_{rst}(W_n) \leq dim(W_n) + \gamma_{st}(W_n)$. We have $R_D = \{x_i : 2 \leq i \leq n-1 \text{ and } i \text{ is even} \}$ so $R_D$ is a resolving set but not strong dominating set because $a$ as a vertex with maximum degree is not a strong dominating set, that is not satisfied the definition of strong dominating set. Thus $R_D = \{x_i : 2 \leq i \leq n-1 \text{ and } i \text{ is even} \} \cup \{a\}$ and $R_D$ is resolving strong dominating set with cardinality $|R_D| = \left\lceil \frac{2n+2}{5} \right\rceil + 1$. Therefore $\gamma_{rst}(W_n) = \left\lceil \frac{2n+2}{5} \right\rceil + 1$. □

3. Concluding Remark
We have determined the exact values of resolving strong domination number of some graphs, namely path, cycle, complete, friendship, fan, and wheel. Since this study is a new notion,
namely the combination of two concepts of dominating and metric dimension, thus there exists many problems. Therefore, we propose the following open problems.

Open problems are given in the following:

(i) Determine the exact value of resolving strong domination number of any family graphs apart from graphs in this paper.

(ii) Determine the exact value of resolving strong domination number of any operation graphs.

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