Cohomological rank functions and Syzygies of Abelian varieties

Pour Olivier Debarre, avec admiration et gratitude

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1 Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$. Given a polarized abelian variety $(A, L)$ of dimension $g$, it is well-known that $L^\otimes 2$ is basepoint free and $L^\otimes 3$ is very ample. However it is a quite subtle question when $L$ is basepoint free or very ample. We may assume that the polarization type of $L$ is $(\delta_1, \ldots, \delta_g)$ where $\delta_1 | \delta_2 | \cdots | \delta_g$. We are interested in the projective geometry of $(A, L)$ when $\delta_1 = 1$ in this article. Such polarizations are called primitive polarizations.

Debarre, Hulek, and Spandaw studied primitive polarizations of type $(1, \ldots, 1, \delta_g)$ by a degeneration method in [5] and proved that when $(A, L)$ is generic in the corresponding moduli space, then

- $L$ is basepoint free iff $\delta_g > g$;
- when $\delta_g > g + 1$, the linear system $|L|$ induces a birational morphism from $A$ to its image;
- when $\delta_g > 2^g$, $L$ is very ample.

On the other hand, let’s recall the following famous conjecture which generalizes Fujita’s conjecture (see [17, Conjecture 5.4]).

**Conjecture 1.1** Let $X$ be a smooth projective variety, $x \in X$ a point, $L$ be a nef and big line bundle on $X$. Assume that

$$(L^\dim X) > (\dim X)^\dim X,$$

and for any positive dimensional irreducible subvariety $x \in Z$,

$$(L^\dim Z \cdot Z) \geq (\dim X)^\dim Z,$$

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then $K_X + L$ is basepoint free at $x$.

Since $L$ is big and nef, it is easy to see by the Kawamata-Viehweg vanishing theorem that $K_X + L$ is basepoint free at $x \in X$ is equivalent to the fact that $H^1(X, \mathcal{I} \otimes \mathcal{O}_X(K_X + L)) = 0$. Hence the basic strategy to attack this conjecture is the following: construct a singular $\mathbb{Q}$-divisor $D$ numerically equivalent to $cL$ for some rational number $0 < c < 1$ such that $x$ is an isolated component of the non-klt locus $N_{\operatorname{klt}}(X, D)$ of $(X, D)$. Recall that $N_{\operatorname{klt}}(X, D)$ is simply the subscheme defined by the multiplier ideal $\mathcal{J}(X, D)$. Then one can conclude by the Nadel vanishing theorem.

There are other positive properties we like to know about ample divisors. Let $L$ be an ample line bundle on a projective variety $X$. We say that $L$ satisfies property $(N_p)$ if the first $p$ steps of the minimal graded free resolution of the section algebra $R_L := \bigoplus_m H^0(X, L^m)$ over the polynomial ring $S_L = \operatorname{Sym}^* H^0(X, L)$ are linear (see for instance [19]). For instance, $(N_0)$ means that $L$ is projectively normal and $(N_1)$ is equivalent to say that the homogeneous ideal of $X$ in $\mathbb{P}(H^0(X, L)^*)$ is generated by quadrics. We will also regard basepoint-freeness as condition $(N_{-1})$.

When $(A, L)$ is a polarized abelian variety, Lazarsfeld, Pareschi, and Popa [21] provided a very nice criterion to tell when $L$ satisfies property $(N_p)$. This method is similar to the general approach for Fujita’s conjecture. More explicitly, let $o$ be the origin of $A$ and define

$$r(L) := \min\{c \in \mathbb{Q} \mid \text{there exists an effective } \mathbb{Q}-\text{divisor } D = cL$$
$$\text{such that } \mathcal{J}(D) = \mathcal{J}_o\}$$

Then the main result of [21] essentially states that if $r(L) < \frac{1}{p+2}$, then $L$ satisfies property $(N_p)$.

Ito [10] and Lozovanu [22] asked the following conjecture on abelian varieties.

**Conjecture 1.2** Let $(A, L)$ be a polarized abelian variety of dimension $g$. Let $p \geq -1$ be an integer. Denote by $D = \frac{1}{p+2} L$ an ample $\mathbb{Q}$-divisor. Assume that $$\langle D^{\dim B} \cdot B \rangle > (\dim B)^{\dim B}$$ for any positive-dimensional abelian subvariety $B$ of $A$. Then $L$ satisfies property $(N_p)$.

Note that the condition in Conjecture 1.2 is much weaker than Conjecture 1.1 restricted on abelian varieties: we just need to bound the intersection numbers with abelian subvarieties and the bounds are also much smaller.

Caucci applied cohomological rank functions defined in [13] to study higher syzygies of ample line bundles on abelian varieties [2]. He considered the basepoint freeness threshold of $L$ defined in [13]:

$$\beta(L) := \inf\{ t \in \mathbb{Q} \mid h^1_{\mathcal{J}_o, L}(t) = 0\},$$

where $h^1_{\mathcal{J}_o, L}(t)$ is the cohomological rank function of the ideal sheaf of the origin $\mathcal{J}_o$ associated to $L$. Then Caucci proved that if $\beta(L) < \frac{1}{p+2}$, then $L$ satisfies property $(N_p)$ ([2, Theorem 1.1]). He then asked a variant of Conjecture 1.2.

**Conjecture 1.3** Let $(A, L)$ be a polarized abelian variety of dimension $g$. Let $p \geq -1$ be an integer. Denote by $D = \frac{1}{p+2} L$ an ample $\mathbb{Q}$-divisor. Assume that $$\langle D^{\dim B} \cdot B \rangle > (\dim B)^{\dim B}$$
for any positive-dimensional abelian subvariety $B$ of $A$. Then
\[ \beta(L) < \frac{1}{p+2}. \]

This conjecture was confirmed by Ito in dimension $\leq 3$ (see [9]). Define $r'(L)$ to be
\[ \min\{c \in \mathbb{Q} \mid \text{there exists an } \mathbb{Q}-\text{divisor } D \equiv cL \text{ such that } o \text{ is an isolated component of Nklt}(A,D)\}. \]

It is clear that $r'(L) \leq r(L)$. Ito’s important observation is that $\beta(L) \leq r'(L)$ (see [9, Proposition 1.10]).

The main results of this article are the following.

**Theorem 1.4** Assume that $(A, L)$ is a polarized abelian variety of dimension $g$. Let $p \geq -1$ be an integer. Assume that
\[ (L^{\dim B} \cdot B) > (2(p + 2)(\dim B))^\dim B \]
for any positive-dimensional abelian subvariety $B$ of $A$. Then $\beta(L) < \frac{1}{p+2}$. In particular, $L$ satisfies property $(N_p)$.

For generic polarized abelian varieties with special polarizations, we can prove Conjecture 1.3.

**Theorem 1.5** Let $(A, L)$ be a very general polarized abelian variety with polarization type $(1, \ldots, 1, \delta_g)$. For some integer $p \geq -1$, assume that $(L^g) > ((p+2)g)^g$, then $\beta(L) < \frac{1}{p+2}$ and hence $L$ satisfies property $(N_p)$.

Here the condition that $(A, L)$ is very general is explicit. We just require the space of Hodge classes is of dimension 1 in each degree. See Theorem 3.6 for the more precise version of this result.

**Remark 1.6** We recall some known results in this direction.

For $p = -1$ i.e. basepoint-freeness, the condition $(L^g) > (2g)^g$, which implies that $h^0(A, L) > 2^g g^{p+1} 10^{g-p-1} > 2^g g$, is much stronger than the condition in Proposition 2 of [5]. However the proof in [5] says nothing about the locus in the moduli space of $(A, L)$ when $|L|$ is basepoint free.

For $p = 0$ i.e. projective normality, Iyer proved in [11] that if $(L^g) > 2^g (g!)^2$ and $A$ is simple, then $L$ satisfies property $(N_0)$. The bound in Theorem 1.4 is already better than Iyer’s bound when $A$ is simple and $g \geq 10$.

Lazarsfeld, Pareschi, and Popa proved that if $(L^g) > \frac{1}{2}(4(p+2)g)^g$ and $(A, L)$ is very general, then $L$ satisfies property $(N_p)$ for any $p \geq -1$ ([21, Corollary B]).

Ito [9,10] proved Conjecture 1.3 for abelian surfaces and abelian threefolds. He also proved that [9, Proposition 3.3] if $(L^{\dim B} \cdot B) > (g(p+2)(\dim B))^\dim B$ for any positive-dimensional abelian subvariety $B$ of $A$, then $\beta(L) < \frac{1}{p+2}$.

### 2 Preliminaries

#### 2.1 Minimal log canonical centers

Let $A$ be an abelian variety and $D$ an ample $\mathbb{Q}$-divisor on $A$. Assume that there exists an effective $\mathbb{Q}$-divisor $D' \equiv cD$ with $0 < c < 1$ a rational number such that $(A, D')$ is not klt. Then we have the following
Lemma 2.1 There exists $c_1$ such that $0 < |c_1 - c| < < 1$ and an effective $\mathbb{Q}$-divisor $D'_1 = c_1 D$ such that $(A, D'_1)$ is a log canonical pair satisfying:

1. $\text{Nklt}(A, D'_1)$ is a normal subvariety $Z_1$ of $A$ containing the origin $o \in A$;
2. $Z_1$ is the minimal lc center through $o$ and is smooth at $o$.

Note that in this case, the multiplier ideal sheaf $\mathcal{J}(A, D'_1)$ (see [20, Chapter 9] for the definition) is the ideal sheaf $\mathcal{J}_{Z_1}$ of $Z_1$.

Proof We first take

$$c'_1 = \text{lct}(D') := \inf\{ t > 0 \mid \mathcal{J}(t \cdot D') \text{ is non-trivial} \}.$$

Then $c'_1 \leq 1$ is a rational number and $(A, c'_1 D')$ is log canonical. We now consider the set $\text{CLC}(A, c'_1 D')$ of log canonical centers of $(A, c_1 D)$ (see [14, Definition 1.3]). Let $W$ be a minimal lc center of $(A, c'_1 D')$ of minimal dimension. Recall that the minimal lc centers are normal (see [14, Proposition 1.5]). We also know that [14, Proposition 1.5] the intersection of two lc centers of $(A, c'_1 D')$ is the union of lc centers $(A, c'_1 D')$. Then for any subvariety $Y \subset \text{CLC}(A, c'_1 D')$, either $W \subset Y$ or $W \cap Y = \emptyset$.

Since $D$ is ample, we can then take $D_2 \subset |kD|$ for some $k > 0$ sufficiently divisible such that $D_2$ contains $W$, $D_2$ does not contain any other subvarieties in $\text{CLC}(A, c'_1 D')$, and $D_2$ is smooth away from $W$. By Tie-breaking (see [18, Proposition 8.7.1] and its proof), for some $0 < \epsilon_1, \epsilon_2 < < 1$, the pair $(A, (1 - \epsilon_1)c'_1 D' + \epsilon_2 D_2)$ is log canonical and $W$ is the only lc center.

Finally, let $z \in W$ be a smooth point of $W$ and considering the translation morphism $t_z : A \rightarrow A, x \rightarrow x + z$. Let $D'_1 = t_z^*((1 - \epsilon_1)c'_1 D' + \epsilon_2 D_2)$. We conclude that the pair $(A, D'_1)$ satisfies both 1) and 2), where $Z_1 = W - z$ is a translation of $W$. \qed

We also recall Helmke’s induction [8, Proposition 3.2 and Corollary 4.6].

Proposition 2.2 Let $A$ be an abelian variety of dimension $g > 0$. Let $D$ be an effective ample $\mathbb{Q}$-divisor on $A$ such that $\text{m}_D(D) > g$. Let $D_1 = c_1 D$ be an effective $\mathbb{Q}$-divisor for some $0 < c_1 < 1$ such that $(A, D_1)$ is log canonical at $o \in A$ and $Z_1$ is a minimal lc center of $(A, D_1)$ through $o$ of dimension $d > 0$. If $(D^d \cdot Z_1) \geq g^d \left( \frac{g}{d^2} \right)$, then there exists an effective $\mathbb{Q}$-divisor $D_2 \equiv \epsilon D$ such that $c_2 := c_1 + \epsilon < 1$ and for $D_2 = D_1 + D_2$, we have $(A, D_2)$ is log canonical at $o$ with the minimal lc center $Z_2$ through $o$ properly contained in $Z_1$.

Proof It suffices to note that in [8, Proposition 3.2], $g > \frac{b(A, D_1)}{1 - c_1}$ (using Helmke’s terminology). \qed

The subadjunction formula (see [7,15,16]) will be repeatedly applied.

Theorem 2.3 Assume that $X$ is a smooth projective variety and $(X, D)$ is a log canonical pair. Assume that $Z$ is a minimal lc center of $(X, D)$. There exists an effective $\mathbb{Q}$-divisor $D_Z$ on $Z$ such that $(Z, D_Z)$ is klt and

$$(K_X + D)|_Z \sim_{\mathbb{Q}} K_Z + D_Z.$$

2.2 Some results on generic vanishing

For an abelian variety $A$ of dimension $g$, we denote by $\text{Pic}^0(A)$ the dual abelian variety of $A$. For any coherent sheaf $\mathcal{F}$ on $A$, we denote by

$$V^i(\mathcal{F}) := \{ P \in \text{Pic}^0(A) \mid h^i(A, \mathcal{F} \otimes P) \neq 0 \}$$

In fact $c_1 = \min\{ c \mid (A, cD) \text{ is log canonical at } o \}$ is the log-canonical threshold of $(A, D)$ at $o$.\[Springer]
the \(i\)-th cohomological support loci of \(\mathcal{F}\). We say that \(\mathcal{F}\) is IT\(0\) if \(V^i(\mathcal{F}) = \emptyset\) for \(i \geq 1\). We say that \(\mathcal{F}\) is M-regular (resp. GV) if \(\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) > i\) (resp. \(\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) \geq i\)) for \(i \geq 1\). Another way to define M-regular or GV sheaves is through the Fourier-Mukai functor. Let \(\mathcal{P}\) be the normalized Poincaré line bundle on \(A \times \text{Pic}^0(A)\). We denote by

\[
\Phi_{\mathcal{P}} : D^b(A) \to D^b(\text{Pic}^0(A))
\]

the Fourier-Mukai functor induced by \(\mathcal{P}\) between the derived categories of bounded complexes of coherent sheaves on \(A\) and \(\text{Pic}^0(A)\). Then we know that a coherent sheaf \(\mathcal{F}\) is M-regular (resp. GV) if \(\text{codim}_{\text{Pic}^0(A)} \text{Supp}^R \Phi_{\mathcal{P}}(\mathcal{F}) > i\) (resp. \(\text{codim}_{\text{Pic}^0(A)} \text{Supp}^R \Phi_{\mathcal{P}}(\mathcal{F}) \geq i\)) for \(i \geq 1\) (see for instance [26, Lemma 3.6]).

Let \(a : X \to A\) be a morphism from a smooth projective variety to \(A\) and \(a\) is generically finite over its image. Then by [6], we know that \(a_*\omega_X\) is GV. Moreover, if \(a(X)\) is not fibred by abelian subvarieties of \(A\), then \(a_*\omega_X\) is M-regular (see for instance [12, Lemma 2.1]).

We recall the definition of cohomological rank functions over an abelian variety of dimension \(g\). Let \(\mathcal{F}\) be a coherent sheaf on \(A\) and let \(L\) be an ample line bundle on \(A\), for a rational number \(t \in \mathbb{Q}\),

\[
h^i_{\mathcal{F},L}(t) := \frac{1}{M^{2g}} h^i(A, \pi_M^* \mathcal{F} \otimes L^{M^2 t} \otimes Q),
\]

where \(M\) is a sufficiently divisible integer such that \(M^2 t \in \mathbb{Z}\), \(\pi_M : A \to A\) is the multiplication-by-\(M\) map, and \(Q \in \text{Pic}^0(A)\) is general. It is easy to check that \(h^i_{\mathcal{F},L} : \mathbb{Q} \to \mathbb{Q}\) is a well-defined map. In [13], it has been proved that \(h^i_{\mathcal{F},L}\) can be extended to a continuous function from \(\mathbb{R}\) to \(\mathbb{R}\).

If \(D \equiv cL\) for some rational number \(c > 0\), we can similarly define \(h^i_{\mathcal{F},D}(t) := h^i_{\mathcal{F},L}(ct)\) for any \(t \in \mathbb{Q}\). For any ample \(\mathbb{Q}\)-divisor \(D\), we define

\[
\beta(D) := \inf \{ t \in \mathbb{Q} \mid h^i_{\mathcal{F},D}(t) = 0 \},
\]

where \(\mathcal{I}_o \subset \mathcal{O}_A\) is the ideal sheaf of the origin \(o\) of \(A\). It is clear that if an ample \(\mathbb{Q}\)-divisor \(D\) is numerically equivalent to \(cL\), then \(\beta(L) = c\beta(D)\).

We will use the following facts:

1. [2, Theorem 1.1] if \(\beta(L) < \frac{1}{p+2}\), \(L\) satisfies property \((N_p)\);
2. [25, Proposition 3.1 and Theorem 3.2] Assume that \(\mathcal{F}\) is a GV sheaf on \(A\) and \(V\) is a locally free IT\(0\) (resp. GV) sheaf on \(A\), then \(\mathcal{F} \otimes V\) is still IT\(0\) (resp. GV);
3. [13, Theorem 5.2] Let \(\mathcal{F}\) be a GV sheaf on \(A\), then \(h^i_{\mathcal{F},L}(t) = 0\) for \(i \geq 1\) and \(t > 0\);
4. Given a short exact sequence of coherent sheaves on \(A\)

\[
0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,
\]

then if \(h^i_{\mathcal{F}_1,L}(t) = h^i_{\mathcal{F}_3,L}(t) = 0\), we have \(h^i_{\mathcal{F}_2,L}(t) = 0\) for each \(i \geq 0\).

Given an irreducible subvariety \(Z\) of \(A\) containing the origin \(o\) as a smooth point, we will denote by \(\mathcal{I}_o,Z\) the ideal sheaf of \(o\) in \(Z\). Thus we have

\[
0 \to \mathcal{I}_Z \to \mathcal{I}_o \to \mathcal{I}_{o,Z} \to 0.
\]

Combining (4) above and Nadel vanishing, we have the following criterion.

**Lemma 2.4** Let \(D\) be an ample \(\mathbb{Q}\)-divisor. Assume that there exists an effective \(\mathbb{Q}\)-divisor \(D' \equiv cD\) for some rational number \(0 < c < 1\) such that \(\mathcal{I}(D') = \mathcal{I}_Z\) for some irreducible normal subvariety \(Z\) containing \(o\) as a smooth point, and \(h^i_{\mathcal{I}_{o,Z},D}(1 - \epsilon) = 0\) for some \(0 < \epsilon < 1 - c\), then \(\beta(D) < 1\).
Corollary 2.5 Assume that Conjecture 1.3 holds in dimension \( g - 1 \).

Let \((A, L)\) be a polarized abelian variety of dimension \( g \) and \( p \geq -1 \) an integer. Let \( D = \frac{1}{p+2}L \). Assume that \((D^{\dim B} \cdot B) > (\dim B)^{\dim B}\) for any positive-dimensional proper abelian subvariety \( B \) of \( A \). If there exists an effective divisor \( D' \equiv cD \) for some rational number \( 0 < c < 1 \) such that \( f(D') = f_B \), where \( B \) is a proper abelian subvariety of \( A \), then \( \beta(L) < \frac{1}{p+2} \).

Proof We just need to apply Lemma 2.4 and the hypothesis that Conjecture 1.3 holds in dimension \( g - 1 \).

The following lemma is simple but is quite crucial.

Lemma 2.6 Let \( Z \) be a subvariety of \( A \) containing \( o \) as a smooth point and let \( \rho : Z' \to Z \) be a desingularization of \( Z \). Assume that \( Z' \) is of general type. Then for any \( Q \in \Pic^0(Z') \), \( \rho_*(\omega_{Z'} \otimes Q) \) is \( M \)-regular and \( \rho_*(\omega_{Z'} \otimes Q) \otimes f_{o,Z} \) is GV.

Proof It is well-known that \( \rho_*(\omega_{Z'} \otimes Q) \) is \( M \)-regular when \( Z' \) is of general type (see for instance [12, Lemma 2.1] or the [27, Section 16, Note]). Note that \( \rho_*(\omega_{Z'} \otimes Q) \) is locally free of rank one around \( o \), we have

\[
0 \to \rho_*(\omega_{Z'} \otimes Q) \otimes f_{o,Z} \to \rho_*(\omega_{Z'} \otimes Q) \to \mathbb{C}_o \to 0.
\]

Hence \( V^j(\rho_*(\omega_{Z'} \otimes Q) \otimes f_{o,Z}) \) has codimension at least \( j + 1 \) in \( \Pic^0(A) \) for \( j \geq 2 \). Since \( \rho_*(\omega_{Z'} \otimes Q) \) is \( M \)-regular, it is continuously globally generated by [23, Proposition 2.13]. Thus for \( P \in \Pic^0(A) \) general, \( H^0(\rho_*(\omega_{Z'} \otimes Q) \otimes P) \to \mathbb{C}_o \) is surjective. Hence \( V^1(\rho_*(\omega_{Z'} \otimes Q) \otimes f_{o,Z}) \) is a proper subset of \( \Pic^0(A) \). Thus \( \rho_*(\omega_{Z'} \otimes Q) \otimes f_{o,Z} \) is GV.

Remark 2.7 Assume that \( f \) is a \( M \)-regular rank 1 sheaf supported on \( Z \), which is locally free around \( o \), then the same argument also shows that \( f \otimes f_{o,Z} \) is GV.

3 Intersection numbers and the proof of Theorem 1.5

Assume that \( (D^g) > g^g \). Then we know that there exists an effective \( \mathbb{Q} \)-divisor \( D'_1 \sim_{\mathbb{Q}} cD \) with \( 0 < c < 1 \) such that \((A, D'_1) \) is not klt at \( o \in A \) (see for instance [20, Lemma 10.4.12 and Proposition 9.3.2]). We then apply Lemma 2.1. For some \( 0 < c_1 < 1 \), there exists \( D_1 \equiv c_1D \) such that an irreducible normal subvariety \( Z_1 \) of dimension \( d_1 \) is the unique lc center of \((A, D_1) \) and \( o \in Z_1 \) is a smooth point.

In general it seems difficult to estimate \((D^{d_i} \cdot Z_1) \), except when \( Z_1 \) is of dimension 1 or codimension 1.

Let \( H \) be an effective divisor on \( A \), we know that there exists a quotient of abelian varieties with connected fibers \( \varphi_h : A \to A_h \) with an ample divisor \( H' \) on \( A_h \) such that \( H \) is algebraically equivalent to \( \varphi^*_h H' \). We denote by \( K_h \) the kernel of \( \varphi_h \). Then the numerical dimension of \( H \) is equal to \( \dim A_h \).

Lemma 3.1 Let \( D \) be an ample \( \mathbb{Q} \)-divisor on \( A \).

1. For any effective divisor \( H \) on \( A \) of numerical dimension \( k > 0 \), we have

\[
(D^{g-1} \cdot H) \geq (k!(D^{g-k} \cdot K_h))^\frac{1}{k} (D^g)^{\frac{k-1}{k}}.
\]

In particular, if \( H \) is ample, we have \((D^{g-1} \cdot H) \geq (D^g)^{\frac{g-1}{g}} \).

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2. Assume that \( C \) is a curve generating \( A \), then
\[
(D \cdot C) \geq g \left\lfloor \frac{(D^g)}{g!} \right\rfloor > (D^g)^{\frac{1}{g}}.
\]

**Proof** By Hodge type inequalities (see [19, Corollary 1.6.3]), we have
\[
(D^{g-1} \cdot H) \geq (D^{g-k} \cdot H^k)^{\frac{1}{g}} (D^g)^{\frac{k-1}{g}}.
\]

Since the cohomology class of the cycle \( H^k \) is equal to the cohomology class of the cycle \( (H^k)_{Ah} \cdot K_h \), we have
\[
(D^{g-k} \cdot H^k) = (D^{g-k} \cdot K_h) \cdot (H^k)_{Ah} \geq k!(D^{g-k} \cdot K_h).
\]

This result leads us to ask the following question.

**Question 3.2** Let \((A, L)\) be a polarized abelian variety of dimension \( g \geq 4 \). Let \( Z \subset A \) be an irreducible (geometrically) non-degenerate subvariety of dimension \( 2 \leq d \leq g - 2 \). Then we have
\[
(Z \cdot L^d) \geq (L^g)^{\frac{d}{g}}.
\]

If the polarized abelian variety \((A, L)\) is very general, then the answer is affirmative.

**Definition 3.3** We say that a polarized abelian variety \((A, L)\) is Hodge theoretically very general if \( \dim_Q H^{k,k}(A, \mathbb{Q}) = 1 \), where
\[
H^{k,k}(A, \mathbb{Q}) := H^{k,k}(A) \cap H^{2k}(A, \mathbb{Q})
\]
is the vector space of Hodge classes of codimension \( k \).

**Remark 3.4** We know by Mattuck (see [1, Theorem 17.4.1]) that if \((A, L)\) is very general in the moduli space of polarized abelian varieties with polarization type \((\delta_1, \ldots, \delta_g)\), then \( \dim_Q H^{k,k}(A, \mathbb{Q}) = 1 \) for \( 0 \leq k \leq g \).

**Lemma 3.5** Assume that \((A, L)\) is a polarized abelian variety of dimension \( g > 1 \) of polarization type \((\delta_1, \ldots, \delta_g)\) and \((A, L)\) is Hodge theoretically very general. Then \((L^k \cdot Z) > (L^g)^{\frac{k}{g}}\) for any irreducible subvariety \( Z \) of codimension \( 1 \leq k \leq g - 1 \).

**Proof** We know that \( \frac{1}{(g-k)!\delta_1 \cdots \delta_{g-k}} [L^{g-k}] \) is integral and is a minimal cohomology class. Since \( \dim_Q H^{g-k,g-k}(A, \mathbb{Q}) = 1 \), for any irreducible subvariety \( Z \) of dimension \( k \), its cohomology class \([Z]\) is an integral multiple of \( \frac{1}{(g-k)!\delta_1 \cdots \delta_{g-k}} [L^{g-k}] \). Thus
\[
(L^k \cdot Z) \geq \frac{(L^g)}{(g-k)!\delta_1 \cdots \delta_{g-k}} = \frac{g!\delta_{g-k+1} \cdots \delta_g}{(g-k)!}.
\]

It is easy to see that \( \frac{g!}{(g-k)!} > (g!)^{\frac{1}{g}} \) and \( \delta_{g-k+1} \cdots \delta_g \geq (\delta_1 \cdots \delta_g)^{\frac{k}{g}} \). Hence \((L^k \cdot Z) > (L^g)^{\frac{k}{g}}\).

By a similar computation, we have the following result for generic polarized abelian varieties.
Theorem 3.6 Fix $p \geq -1$ an integer. Assume that $(A, L)$ is a polarized abelian variety of dimension $g \geq 2$ of polarization type $(\delta_1, \ldots, \delta_g)$. Assume that

1. $(L^g) > ((p+2)g)^g$;
2. $(p+2)g \geq (g-i)\delta_i$ for $1 \leq i \leq g$;
3. $\dim_{\mathbb{Q}} H^{k,k}(A, \mathbb{Q}) = 1$ for all $1 \leq k \leq g$,

then $r'(L) < \frac{1}{p+2}$ and hence $L$ satisfies property \((N_p)\).

Proof Let $D = \frac{1}{p+2} L$. Since $(D^g) > (g)^g$, we may assume that $D$ is an effective $\mathbb{Q}$-divisor with $m_o(D) > g$. Then there exists an effective $\mathbb{Q}$-divisor $D_1 = c_1 D$ for some $0 < c_1 < 1$ such that $(A, D_1)$ is log canonical at $o$ and the minimal lc center through $o$ is a normal subvariety $Z_1$ of dimension $k \geq 0$. If $k = 0$, we know that $r'(L) = \frac{1}{p+2} r'(D) < \frac{1}{p+2}$ and we conclude by [9, Proposition 1.10] and [2, Theorem 1.1].

If $k > 0$, by assumption (1), (2), and (3), we have

$$(D^k \cdot Z_1) \geq \frac{1}{(p+2)^k} \frac{(L^g)}{(g-k)! \delta_1 \cdots \delta_{g-k}}$$

$$\geq (p+2)^{g-k} \frac{g^g}{(g-k)! \delta_1 \cdots \delta_{g-k}} = g^k (p+2)^{g-k}$$

$$\geq g^k \frac{(g-1) \cdots \delta g}{(g-k)!} = g^k \frac{(g-1) \cdots k}{k-1}.$$ 

Then by Proposition 2.2, there exists $D_2 \equiv c_2 D$ with $c_1 < c_2 < 1$ such that $(A, D_2)$ is lc at $o$ and the minimal lc center through $o$ is $Z_2 \subsetneq Z_1$. We continue this process to yield that $r'(D) < 1$ and conclude by [9, Proposition 1.10].

\[ \square \]

Corollary 3.7 Assume that $(A, L)$ is a very general polarized abelian variety of type $(1, \ldots, 1, \delta_g)$, then if $(L^g) > ((p+2)g)^g$, $L$ satisfies property \((N_p)\).

4 The case that the minimal lc center is a divisor

The following result deals the cases when the minimal lc center is a divisor or is a divisor in an abelian subvariety of $A$.

Proposition 4.1 Assume that Conjecture 1.3 holds in dimension $\leq g - 1$.

Let $(A, L)$ be a polarized abelian variety of dimension $g$ and $p \geq -1$ an integer. Let $D = \frac{1}{p+2} L$. Assume that $(D^{\dim B} \cdot B) > (\dim B)^{\dim B}$ for any positive-dimensional proper abelian subvarieties $B$ of $A$ and there exists an effective divisor $D' \equiv cD$ for some rational number $0 < c < 1$ such that $\mathcal{F}(D') = \mathcal{F}_Z$. Then in the following cases, we have $\beta(L) < \frac{1}{p+2}$:

1. $Z$ is a divisor on $A$;
2. $Z$ generates a positive-dimensional abelian subvariety $B$, and there exists an integral divisor $H_B$ on $B$ such that $D'|_B - H_B$ is a nef $\mathbb{Q}$-divisor on $B$ and $H_B|_Z \sim K_Z$ (thus $Z$ is Gorenstein).
4.1 The proof of Proposition 4.1 (1)

We first assume that $Z$ is an ample divisor. After translation, we may assume that $o \in Z$ is a smooth point and it suffices to show that $h^{1}_{\mathcal{O}_{o,Z},D}(1-\epsilon) = 0$ for some $0 < \epsilon < < 1$ by Lemma 2.4.

By adjunction, we know that $\mathcal{O}_{Z}(Z) = \omega_{Z}$ is the dualizing sheaf on $Z$. From

$$0 \to \mathcal{O}_{A} \to \mathcal{O}_{A}(Z) \to \mathcal{O}_{Z}(Z) \to 0,$$

we see easily that $\mathcal{O}_{Z}(Z)$ is M-regular. Thus $\mathcal{O}_{Z}(Z) \otimes \mathcal{I}_{o,Z}$ is GV by Remark 2.7. Hence $h^{1}_{\mathcal{O}_{Z}(Z) \otimes \mathcal{I}_{o,Z},D}(\delta) = 0$ for $0 < \delta < < 1$ by Sect. 2.2(3).

We then take $M > 0$ sufficiently large and divisible such that $M^{2}D'$ is an integral divisor.

Since $D' = Z + Z'$ where $Z'$ is an effective $\mathbb{Q}$-divisor. We see that $M^{2}D' - M^{2}Z$ is an effective divisor on $A$, thus we see from [1, Theorem 3.5.5] that $\mathcal{O}_{A}(M^{2}D' - M^{2}Z)$ is GV. Let $\pi_{M} : A \to A$ be the multiplication-by-$M$ map.

By Sect. 2.2 (2), we know that $\pi_{M}(\mathcal{O}_{Z}(Z) \otimes \mathcal{I}_{o,Z}) \otimes \mathcal{O}_{A}(M^{2}D')$ is GV. Hence $\pi_{M}(\mathcal{I}_{o,Z}) \otimes \mathcal{O}_{A}(M^{2}D')$ is GV. Thus $h^{1}_{\mathcal{O}_{o,Z},D}(t) = 0$ for $t \geq c$ by Sect. 2.2 (2) and (3).

If $Z$ is not ample, we can again assume that $Z$ is fibred by abelian subvariety $K$ and we consider the commutative diagram

$$
\begin{array}{ccc}
Z & \to & A \\
\downarrow h & & \downarrow \\
Z_{K} & \to & A/K
\end{array}
$$

We may assume that $o_{K} := h(o)$ is a smooth point of $Z_{K}$ and let $K_{o} := h^{-1}(o)$. Consider

$$0 \to \mathcal{I}_{K_{o},Z} \to \mathcal{I}_{o,Z} \to \mathcal{I}_{o,K_{o}} \to 0,$$

hence as before, it suffices to show that $h^{1}_{\mathcal{I}_{K_{o},Z},D}(1-\epsilon) = 0$ and $h^{1}_{\mathcal{I}_{o,K_{o},D}}(1-\epsilon) = 0$ for $0 < \epsilon < < 1$.

The latter is a consequence of the assumption that Conjecture 1.3 holds in dimension $\leq g - 1$.

We note that $\mathcal{I}_{K_{o},Z} = h^{*}\mathcal{I}_{o,K_{o},Z_{K}}$ and

$$\mathcal{O}_{Z}(Z) \otimes \mathcal{I}_{K_{o},Z} = h^{*}(\omega_{Z_{K}} \otimes \mathcal{I}_{o,K_{o},Z_{K}})$$

is again GV. Then by exactly the same argument as the ample case, we conclude the proof.

4.2 The proof of Proposition 4.1 (2)

The proof is quite similar to the previous case. After translation, we may assume that $o \in Z$ is a smooth point. By Theorem 2.3, $D'|_{Z} \sim_{\mathbb{Q}} K_{Z} + D_{Z}$ such that $(Z, D_{Z})$ is klt. Moreover, by assumption, $Z$ is Gorenstein. Then $Z$ has canonical singularities. For a resolution $\rho : Z' \to Z$ we have $\rho_{*}K_{Z'} = K_{Z}$. Thus by Lemma 2.6, we know that $\mathcal{I}_{o,Z} \otimes H_{B}$ is GV.

Take $D'' = D' + \epsilon D$ such that $c < c'' := c + \epsilon < 1$. Then $D'' - H_{B}$ is an ample $\mathbb{Q}$-divisor on $B$. 
Let $M$ be an integer sufficiently large and divisible so that $M^2D''$ is an integral divisor and let $\pi_M : A \to A$ be the multiplication-by-$M$ map. We have the commutative diagram

$$
\begin{array}{ccc}
Z^{(M)} & \longrightarrow & B \\
\mu_M & \downarrow & \pi_M \\
Z & \longrightarrow & A,
\end{array}
$$

where $Z^{(M)} := \pi_M^{-1}(Z)$.

We take a general $s \in H^0(B, \mathcal{O}_B(M^2D'' - M^2H))$ such that $o$ is not contained in the corresponding divisor of $s$. Let $W$ be the zero locus of $s|_{Z^{(M)}}$. We have the following two exact sequences:

$$
0 \to \mathcal{O}_{Z^{(M)}}(M^2H) \to \mathcal{O}_{Z^{(M)}}(M^2D'') \to \mathcal{O}_W(M^2D'') \to 0
$$

and

$$
0 \to \mu_M^*(J(E_Z(H))) \to \pi_M^*(J(E_Z))(M^2D'') \to \mathcal{O}_W(M^2D'') \to 0.
$$

In the first exact sequence, we know that $M^2H|_{Z^{(M)}} \equiv K_{Z^{(M)}}$, hence $\mathcal{O}_{Z^{(M)}}(M^2H)$ is GV. Moreover, since $J(D) = J_E$, we have $J(\pi_M^*D') = J(E_Z)$, we then see from Nadel vanishing that $\mathcal{O}_{Z^{(M)}}(M^2D'')$ is $\Gamma^0$. Thus $\mathcal{O}_W(M^2D'')$ is M-regular.

In the second exact sequence, we already know that $\mu_M^*(J(E_Z(H)))$ is GV, thus $\pi_M^*(J(E_Z))(M^2D'')$ is also GV and we then conclude that $\beta(L) < \frac{1}{p+2}$.

5 The proof of Theorem 1.4

Theorem 5.1 Let $(A, L)$ be a polarized abelian variety of dimension $g$ and $p \geq -1$ be an integer. Let $D = \frac{1}{2(p+2)} L$ be a $\mathbb{Q}$-Cartier divisor. Assume that

$$(D^\dim A \cdot B) > (\dim B)^\dim A$$

for any positive-dimensional abelian subvariety of $A$, then $\beta(L) < \frac{1}{p+2}$ and hence $L$ satisfies property $(N_p)$.

Proof Since $(D^g) > g^g$, by Lemma 2.1, we may take $D_1 \equiv c_1 D$ for some $0 < c_1 < 1$ such that an irreducible normal variety $Z$ is the unique lc center of $(A, D_1)$ containing $o$ as a smooth point. Hence $J(A, D_1) = J_Z$. By Lemma 2.4, it suffices to show that $h^1_{J_Z, ZD}(1 - \eta) = 0$ for $0 < \delta << 1$.

5.1 When $Z$ is not fibred by abelian subvarieties

We first treat the case $Z$ is not fibred by abelian subvarieties, or in other words, any desingularization of $Z$ is of general type.

By Theorem 2.3, we know that there exists an effective $\mathbb{Q}$-divisor $D_Z$ on $Z$ such that $(Z, D_Z)$ is a klt pair and

$$D_1|_Z \sim_{\mathbb{Q}} K_Z + D_Z.$$
We then take a log resolution \( \mu : \tilde{Z} \to Z \), which is an isomorphism over the smooth locus of \( Z \), and write

\[
K_{\tilde{Z}} + \tilde{D}_Z + E_2 = \mu^*(K_Z + D_Z) + E_1,
\]

where \( \tilde{D}_Z \) is the strict transform of \( D_Z \), \( E_1, E_2 \) are exceptional divisors, and \( \tilde{D}_Z + E_1 + E_2 \) has SNC support.

Note that \( 2\mu^*(D_1) + 2E_1 \sim_{\mathbb{Q}} 2K_{\tilde{Z}} + 2\tilde{D}_Z + 2E_2 \). Thus \( 2\mu^*(D_1) + 2E_1 - K_{\tilde{Z}} \sim_{\mathbb{Q}} K_{\tilde{Z}} + 2\tilde{D}_Z + 2E_2 \). We then take \( D_2 = 2D_1 + \epsilon L \equiv (\frac{c_1}{p+2} + \epsilon)L \) for some rational number \( 0 < \epsilon << 1 \) such that \( \frac{c_1}{p+2} + \epsilon < \frac{1}{p+2} \). We denote \( c_2 = \frac{c_1}{p+2} + \epsilon \). Then

\[
\mu^*(D_2) + 2E_1 - K_{\tilde{Z}} \sim_{\mathbb{Q}} K_{\tilde{Z}} + D',
\]

where \( D' \) is a big \( \mathbb{Q} \)-divisor on \( \tilde{Z} \).

We then take \( M > 0 \) an integer sufficiently large and divisible such that \( M^2\epsilon \) and \( M^2\frac{c_1}{p+2} \) are integers so that \( M^2D_2 \) is \( \mathbb{Q} \)-equivalent to an integral divisor \( L^{\otimes M^2c_2} \). Let \( \pi_M : A \to A \) be the multiplication-by-\( M \) map and we consider the Cartesian

\[
\begin{array}{ccc}
\tilde{Z}(M) & \overset{\mu_M}{\longrightarrow} & Z(M) \\
\pi_Z & \downarrow & \downarrow \pi_Z \\
\tilde{Z} & \overset{\mu}{\longrightarrow} & Z.
\end{array}
\]

We will also denote by \( \tilde{E}_i = \pi_Z^*E_i \) on \( \tilde{Z}(M) \) for \( i = 1, 2 \). They are also \( \mu_M \)-exceptional divisors. Let \( \tilde{E} := [2\tilde{E}_1] \) be an integral effective divisor which is again \( \mu_M \)-exceptional.

We claim that

\[
H^0(\tilde{Z}(M), \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)})(\tilde{E} - K_{\tilde{Z}(M)})) \neq 0.
\]

Indeed, by (1), \( \mu_M^* \left( (L^{\otimes M^2c_2}|_{Z(M)})(2\tilde{E}_1 - K_{\tilde{Z}(M)}) \sim_{\mathbb{Q}} K_{\tilde{Z}(M)} + \tilde{E} - K_{\tilde{Z}(M)} \right) \sim_{\mathbb{Q}} K_{\tilde{Z}(M)} + \tilde{E} - K_{\tilde{Z}(M)} \sim_{\mathbb{Q}} \tilde{E} \) and \( \tilde{E} - K_{\tilde{Z}(M)} \) is a big divisor on \( \tilde{Z}(M) \). By [20, Theorem 11.2.12], for any nef divisor \( H \) on \( A \),

\[
H^i(\tilde{Z}(M), \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)})(\tilde{E} - K_{\tilde{Z}(M)}) \otimes \mathcal{F} (||\pi_Z^*D'||) \otimes \mu_M^*H) = 0
\]

for \( i > 0 \). From this, we deduce that

\[
L^{\otimes M^2c_2}|_{Z(M)} \otimes \mu_M^* \left( \mathcal{O}_{\tilde{Z}(M)}(\tilde{E} - K_{\tilde{Z}(M)}) \otimes \mathcal{F} (||\pi_Z^*D'||) \right)
\]

is a non-zero \( \mathcal{O}_0 \) sheaf on \( A \) supported on \( Z(M) \). Hence it has a non-zero global section and so does \( \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)})(\tilde{E} - K_{\tilde{Z}(M)}) \).

We then take a global section

\[
0 \neq s \in H^0(\tilde{Z}(M), \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)})(\tilde{E} - K_{\tilde{Z}(M)}))
\]

and let \( \tilde{D} \) be the corresponding divisor on \( \tilde{Z}(M) \). We then have

\[
0 \to \mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)}) \overset{s}{\to} \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)})(\tilde{E}) \to \mathcal{O}_{\tilde{D}}(K_{\tilde{D}}) \to 0,
\]

where \( K_{\tilde{D}} = (K_{\tilde{Z}(M)} + \tilde{D})|_{\tilde{D}} \) is the canonical bundle of \( \tilde{D} \). Note that \( \mu_M^* \left( \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)})(\tilde{E}) \right) = \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)}) \) since \( \tilde{E} \) on \( \tilde{Z}(M) \) is \( \mu_M \)-exceptional. Since we have \( R^1\mu_M^*\mathcal{O}_{\tilde{Z}(M)} \),
\( (K_{\tilde{Z}(M)}) = 0 \) by Grauert-Riemenschneider, we can apply \( \mu_{M*} \) to the above short exact sequence to get
\[
0 \rightarrow \mu_{M*}(\mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)})) \rightarrow L^{\otimes M^2c_2}|_{Z(M)} \rightarrow \mu_{M*}\mathcal{O}_{\tilde{D}}(K_{\tilde{D}}) \rightarrow 0.
\]

We claim that \( \mu_{M*}\mathcal{O}_{\tilde{D}}(K_{\tilde{D}}) \) is M-regular on \( A \). We already know that \( \mu_{M*} \mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)}) \) is GV. On the other hand, \( \mathcal{J}_Z = \mathcal{J}(D_1) \). Since \( \pi_M \) is étale, we have \( \mathcal{J}_{Z(M)} = \mathcal{J}(\pi_M^*D_1) \).

Note that \( \pi_M^*D_1 \equiv M^2D_1 \) and \( L^{\otimes M^2c_2} - M^2D_1 \equiv (M^2c_2 - M^2c_1\epsilon/p+2)L \) is an ample \( \mathbb{Q} \)-divisor. Thus by Nadel vanishing (see [20, Theorem 9.4.8]), \( L^{\otimes M^2c_2} \otimes \mathcal{J}_{Z(M)} \) is IT\( Z \), thus so is \( L^{\otimes M^2c_2}|_{Z(M)} \). Thus, \( V^i(\mu_{M*}\mathcal{O}_{\tilde{D}}(K_{\tilde{D}})) \subset V^{i+1}(\mu_{M*}\mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)})) \) for all \( i > 0 \). Hence \( \mu_{M*}\mathcal{O}_{\tilde{D}}(K_{\tilde{D}}) \) is M-regular.

By the construction, \( \mu \) is an isomorphism around \( o \) and we thus still denote by \( o \) its preimage in \( \tilde{Z} \). Then \( \mu_M \) is an isomorphism around \( \tilde{\pi}_Z^{-1}(o) \). After a translation, we can also assume that \( \tilde{\pi}_Z^{-1}(o) \cap \tilde{D} = \emptyset \). We then have
\[
0 \rightarrow \mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)}) \otimes \mathcal{O}_{\tilde{Z}}(o) \rightarrow \mu_M^*(L^{\otimes M^2c_2}|_{Z(M)}) \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o) \rightarrow \mathcal{O}_{\tilde{D}}(K_{\tilde{D}}) \rightarrow 0.
\]

We again pushforward this short exact sequence to \( Z(M) \) and get
\[
0 \rightarrow \mu_{M*}(\mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)})) \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o) \rightarrow L^{\otimes M^2c_2}|_{Z(M)} \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o) \rightarrow \mu_{M*}\mathcal{O}_{\tilde{D}}(K_{\tilde{D}}) \rightarrow 0. \tag{2}
\]

Note that \( \mu_{M*}(\mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)})) \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o) = \mu_{M*}(\tilde{\pi}_Z^*(\mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}}) \otimes \mathcal{J}_o)) \), by étale base change,
\[
\mu_{M*}(\mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)})) \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o) = \pi_Z^*(\mu_{*}\mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}}) \otimes \mathcal{J}_o).
\]

By Lemma 2.6, \( \mu_{*}(\mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}})) \otimes \mathcal{J}_o \) is a GV sheaf on \( A \). Hence
\[
\mu_{M*}(\mathcal{O}_{\tilde{Z}(M)}(K_{\tilde{Z}(M)})) \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o)
\]

is also a GV sheaf on \( A \). From the short exact sequence (2), we see that \( L^{\otimes M^2c_2}|_{Z(M)} \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o) \) is again GV. Hence by Sect. 2.2 (4), we know that
\[
h^1_{L^{\otimes M^2c_2}|_{Z(M)} \otimes \mathcal{O}_{\tilde{Z}}^{-1}(o),L}(t) = 0
\]
for \( t > 0 \). Note that \( \mathcal{O}_{\tilde{Z}}^{-1}(o) = \pi_Z^*\mathcal{J}_o, Z \), thus we have
\[
h^1_{\mathcal{J}_o,Z,L}(t) = 0
\]
for \( t \geq c_2 \). Because \( c_2 = 2c_1 + \epsilon < \frac{1}{p+2} \). We finish the proof by Lemma 2.4 when \( \tilde{Z} \) is of general type,

5.2 The general case

We shall apply induction on dimensions of \( A \). We now assume that \( \beta(L|_B) < \frac{1}{p+2} \) for any proper abelian subvariety \( B \) of \( A \).
We assume that $Z$ is fibred by an abelian subvariety $K$ of $A$ and $Z \hookrightarrow A/K$ is not fibred by any abelian subvarieties of $A/K$. We have

$$
\begin{array}{c}
Z \hookrightarrow A \\
\quad \downarrow \quad p_Z \\
\quad \downarrow \\
\quad \overline{Z} \hookrightarrow A/K
\end{array}
$$

Note that $p_Z$ is smooth, $\overline{o} = p(o)$ is a smooth point of $\overline{Z}$. Then $K$ is exactly the fiber of $p_Z$ over $\overline{o}$. Then it suffices to show that

$$
h^1_{p_Z^*(\mathcal{I}_o, Z), 2D}(1 - \epsilon) = 0
$$

for some $0 < \epsilon << 1$, because we have

$$
0 \to p_Z^*(\mathcal{I}_o, Z) \to \mathcal{I}_o, Z \to \mathcal{I}_o, K \to 0,
$$

hence combining (3) with the assumption that $\beta(L|_K) < \frac{1}{p+2}$, we have $h^1_{\mathcal{I}_o, Z, 2D}(1 - \epsilon) = 0$, which implies Theorem 5.1.

We do not have a natural line bundle on $A/K$. Hence we need to apply Poincaré’s reducibility theorem [1, Theorem 5.3.5], there exists an abelian subvariety $K'$, which is complementary to $K$, i.e. $K' + K = A$ and $K' \cap K = G$ is an abelian finite group, such that for the natural addition map $\mu : K' \times K \to A$ we have

$$
\mu^* L \simeq (L|_{K'}) \boxtimes (L|_K).
$$

We will write $L_{K'} := L|_{K'}$ and $L_K := L|_K$. We also note that $\deg \mu = |G|$.

We also note $\mu^{-1}(Z) = Z' \times K$, where $Z' \hookrightarrow K'$ is indeed isomorphic to the base change $\overline{Z} \times_{A/K} K'$. Since $\mathcal{I}(D_1) = \mathcal{I}_Z$, we have $\mathcal{I}(\mu^* D_1) = \mathcal{I}_{\mu^{-1}(Z)}$. By [20, Theorem 9.5.35], we know that for $x \in K$ general, $\mathcal{I}(\mu^* D_1|_{K' \times \{x\}}) = \mathcal{I}_{\mu^{-1}(Z)|_{K' \times \{x\}}}$. Hence there exists $D_1' \equiv \frac{c_1}{2(p+2)} L_{K'}$ such that $\mathcal{I}(D_1') = \mathcal{I}Z'$ and $Z'$ is indeed the unique minimal lc center of $(K', D_1')$.

We now put everything in one commutative diagram:

$$
\begin{array}{c}
K' \times K \\
\quad \downarrow \quad \mu \\
\quad \downarrow \\
Z' \times K \\
\quad \downarrow p_Z \\
\quad \downarrow \\
Z' \\
\quad \tau_Z \\
\quad \downarrow \\
K' \\
\quad \tau \\
\downarrow \\
A/K
\end{array}
$$

Note that $\mu^{-1}_Z(p_Z^*(\mathcal{I}_o, Z)) = \mathcal{I}_{\tau^{-1}_Z(o), Z'} \boxtimes \mathcal{O}_K$. We have

$$
h^1_{p_Z^*(\mathcal{I}_o, Z), 2D}(t) = \frac{1}{|G|} h^1_{\mathcal{I}_{\tau^{-1}_Z(o), Z'}, \boxtimes \mathcal{O}_K, 2\mu^* D(t)}.
$$
Hence by Küneth formula, in order to prove (3), we just need to show that
\[ h^1_{\mathcal{H}_Z^{-1}(\text{Z'}, \text{Z'})} \cdot \frac{1}{p+2} L_{K'} (1 - \epsilon) = 0 \]
for \( 0 < \epsilon << 1 \).

We now apply exactly the same argument as in the previous case. We will not go through again the whole argument but rather point out several crucial points. Let \( \rho : \tilde{Z} \to Z \) be a desingularization and as before, we may assume that \( \rho \) is an isomorphism over an open neighborhood of \( \tilde{o} \). We then consider the Cartesian
\[
\begin{array}{ccc}
\tilde{Z}' & \xrightarrow{\tau_{\tilde{Z}'}} & \tilde{Z} \\
\downarrow \rho' & & \downarrow \rho \\
Z' & \xrightarrow{\tau_Z} & Z.
\end{array}
\]
Then since \( Z \) is of general type, \( \rho_* \mathcal{O}_{\tilde{Z}'}(K_{\tilde{Z}'}) \otimes \mathcal{J}_{\tilde{Z}', Z} \) is GV by Lemma 2.6. Hence
\[
\rho'_* \mathcal{O}_{Z'}(K_{Z'}) \otimes \mathcal{J}_{Z', Z} = \tau^*_{Z}(\rho_* \mathcal{O}_{\tilde{Z}'}(K_{\tilde{Z}'}) \otimes \mathcal{J}_{\tilde{Z}', Z})
\]
is also GV.

Secondly, since \( Z' \) is the minimal lc center of \( (K', D'_1) \), \( D'_1 \equiv \frac{c_1}{2(p+2)} L_{K'} \), and \( 0 < c_1 < 1 \), we have \( \frac{1}{p+2} L_{K'} | Z' = 2 \rho'_*(K_{Z'}) \) is an effective big \( \mathbb{Q} \)-divisor on \( Z' \).

We can then proceed as the previous case. \( \square \)

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