On a Zeta-Barnes type function associated to graded modules
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Abstract
Let $K$ be a field and let $S = \bigoplus_{n \geq 0} S_n$ be a positively graded $K$-algebra. Given $M = \bigoplus_{n \geq 0} M_n$, a finitely generated graded $S$-module, and $w > 0$, we introduce the function $\zeta_{M}(z, w) := \sum_{n=0}^{\infty} \frac{H(M,n)}{(n+w)^z},$ where $H(M,n) := \dim_K M_n$, $n \geq 0$, is the Hilbert function of $M$, and we study the relations between the algebraic properties of $M$ and the analytic properties of $\zeta_{M}(z, w)$. In particular, in the standard graded case, we prove that the multiplicity of $M$, $e(M) = (m-1)! \lim_{w \searrow 0} \text{Res}_{z=m} \zeta_{M}(z, w)$.

Keywords: Graded modules, quasi-polynomials, Zeta-Barnes function.

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Introduction
Let $K$ be a field and let $S$ be a positively graded $K$-algebra. Let $M$ be a finitely generated $S$-module of dimension $m \geq 0$. Given a real number $w > 0$, we consider the function

$$\zeta_{M}(z, w) := \sum_{n=0}^{\infty} \frac{H(M,n)}{(n+w)^z},$$

where $H(M,n) := \dim_K M_n$, $n \geq 0$, is the Hilbert function of $M$. According to a Theorem of Serre, see for instance [5, Theorem 4.4.3], there exists a positive integer $D$ such that

$$H(M,n) = d_{M,m-1}(n)n^m + \cdots + d_{M,1}(n)n + d_{M,0}(n), \ (\forall)n \gg 0,$$

where $d_{M,j}(n+D) = d_{M,j}(n), \ (\forall)n \geq 0$. We introduce the zeta-Barnes [3] type function $\zeta_{M}(z, w) := \sum_{n=0}^{\infty} \frac{H(M,n)}{(n+w)^z}$, $z \in \mathbb{C}$, and we study its properties. In Theorem 1.1 we show that $\zeta_{M}(z, w)$ can be written as a linear combination of Hurwitz-Zeta functions and it is a meromorphic function on the whole complex plane with poles at most in the set $\{1, 2, \ldots, m\}$. We compute the residues of $\zeta_{M}(z, w)$ in terms of the quasi-polynomial $q_{M}(n)$. Other properties of $\zeta_{M}(z, w)$ are given in Proposition 1.2, 1.3 and Corollary 1.4, 1.5.

We also consider the function $\zeta_{M}(z) := \lim_{w \searrow 0} (\zeta_{M}(z, w) - H(M,0)w^{-z})$. In Proposition 1.6 we compute $\zeta_{M}(z)$ and its residues. In the second section, we apply the results obtained in the first section in the case when $S = K[x_1, \ldots, x_r]$ is the ring of polynomials with $\deg(x_i) = a_i$, $1 \leq i \leq r$. Given a graded $S$-module $M$, we compute the residues of $\zeta_{M}(z, w)$ and $\zeta_{M}(z)$ in terms of the graded Betti numbers of $M$ and the Bernoulli-Barnes polynomial associated to $(a_1, \ldots, a_r)$, see Corollary 2.2.

In the third section, we consider the standard graded case and we prove that the multiplicity of $M$, is

$$e(M) = (m-1)! \lim_{w \searrow 0} \text{Res}_{z=m} \zeta_{M}(z, w),$$

see Corollary 3.3. In the fourth case, we outline the non-graded case and we give a formula for the multiplicity of the module with respect to an ideal, see Proposition 4.1.
1 Graded modules over positively graded $K$-algebras

Let $K$ be a field and let $S$ be a positively graded $K$-algebra, that is

$$ S := \bigoplus_{n \geq 0} S_n, S_0 = K,$$

and $S$ is finitely generated over $K$. Assume $S = K[u_1, \ldots, u_r]$, where $u_i \in S$ are homogeneous elements of $\text{deg}(u_i) = a_i$. Let

$$ M = \bigoplus_{n \in \mathbb{N}} M_n$$

be a finitely generated graded $S$-module with the Krull dimension $m := \dim(M)$. The Hilbert function of $M$ is

$$ H(M, n) : \mathbb{N} \to \mathbb{N}, \quad H(M, n) := \dim_K(M_n), \quad n \in \mathbb{N}. $$

The Hilbert series of $M$ is

$$ H_M(t) := \sum_{n=0}^{\infty} H(M, n) t^n \in \mathbb{Z}[[t]].$$

According to the Hibert-Serre’s Theorem [1, Theorem 11.1] and [5, Exercise 4.4.11]

$$ H_M(t) = \frac{h_M(t)}{(1 - t^{a_1}) \cdots (1 - t^{a_r})},$$

where $h_M(t) \in \mathbb{Z}[t]$. According to Serre’s Theorem [5, Theorem 4.4.3] and [5, Exercise 4.4.11] there exists a quasi-polynomial $q_M(n)$ of degree $m - 1$ with the period $D := \text{lcm}(a_1, \ldots, a_r)$ such that

$$ H(M, n) = q_M(n) = d_{M,m-1}(n)n^{m-1} + \cdots + d_{M,1}(n)n + d_{M,0}(n), \quad (\forall)n \gg 0, \quad (1.1)$$

where $d_{M,k}(n + D) = d_{M,k}(n)$ for any $n \geq 0$ and $0 \leq k \leq m - 1$. We denote

$$ \alpha(M) := \min\{n_0 : H(M, n) = q_M(n), \quad (\forall)n \geq n_0\}. \quad (1.2)$$

Let $w > 0$ be a real number. We denote

$$ \zeta_M(z, w) := \sum_{n \geq 0} \frac{H(M, n)}{(n + w)^z}, \quad z \in \mathbb{C}, \quad (1.3)$$

and we call the Zeta-Barnes type function associated to $M$ and $w$. We also denote

$$ \theta_M(z, w) := \sum_{n=0}^{\alpha(M)-1} \frac{H(M, n)}{(n + w)^z}, \quad z \in \mathbb{C}. \quad (1.4)$$

The function $\theta_M(z, w)$ is entire. Moreover, $M$ is Artinian if and only if $\zeta_M(z, w) = \theta_M(z, w)$. Also, $\alpha(M) = 0$ if and only if $\theta_M(z, w) = 0.$

Theorem 1.1. We have that

\[ \zeta_M(z, w) = \theta_M(z, w) + D^{-n} \sum_{k=0}^{m-1} \sum_{j=0}^{D-1} d_{M,k}(j+\alpha(M)) \sum_{\ell=0}^{k} \binom{k}{\ell} (-w)^\ell D^{k-\ell} \zeta(z-k+\ell, \frac{j+\alpha(M) + w}{D}), \]

where \( \zeta(z, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^z} \) is the Hurwitz-zeta function.

Moreover, \( \zeta_M(z, w) \) is a meromorphic function on \( \mathbb{C} \) with the poles in the set \( \{1, 2, \ldots, m\} \) which are simple with residues

\[ R_M(w, k+1) := \text{Res}_{z=k+1} \zeta_M(z, w) = \frac{1}{D} \sum_{\ell=k}^{m-1} \binom{\ell}{k} (-w)^{\ell-k} \sum_{j=0}^{D-1} d_{M,k}(j), \ 0 \leq k \leq m-1. \]

Proof. The proof follows the line of the proof of [6 Proposition 3.2]. According to (1.1), (1.2), (1.3) and (1.4), we have

\[ \zeta_M(z, w) = \theta_M(z, w) + \sum_{n=\alpha(M)}^{\infty} \frac{q_{M}(n)}{(n+w)^z} = \theta_M(z, w) + \sum_{k=0}^{m-1} \sum_{n=\alpha(M)}^{\infty} \frac{d_{M,k}(n) n^k}{(n+w)^z}. \tag{1.5} \]

For any \( 0 \leq k \leq m-1 \), we write

\[ n^k = (n+w-w)^k = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (n+w)^{k-\ell} w^\ell. \tag{1.6} \]

By (1.5) and (1.6) and the fact that \( d_{M,k}(n+D) = d_{M,k}(n), \ (\forall)n, k \), it follows that

\[ \zeta_M(z, w) = \theta_M(z, w) + \sum_{k=0}^{m-1} \sum_{n=\alpha(M)}^{\infty} d_{M,k}(n) \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} w^\ell \frac{1}{(n+w)^{z-k+\ell}} = \theta_M(z, w) + \]

\[ + \sum_{k=0}^{m-1} \sum_{j=0}^{D-1} d_{M,k}(j+\alpha(M)) \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} w^\ell \sum_{\ell=0}^{\infty} \frac{1}{(j+tD+\alpha(M)+w)^{z-k+\ell}}. \tag{1.7} \]

On the other hand,

\[ \sum_{\ell=0}^{\infty} \frac{1}{(j+tD+\alpha(M)+w)^{z-k+\ell}} = \sum_{\ell=0}^{\infty} \frac{D^{-z+k-\ell}}{(t+\frac{j+\alpha(M) + w}{D})^{z-k+\ell}} = D^{-z+k-\ell} \zeta(z-k+\ell, \frac{j+\alpha(M) + w}{D}). \tag{1.8} \]

Replacing (1.8) in (1.7) we get the required result.

The last assertion is a consequence of the fact that the Hurwitz-zeta function \( \zeta(z-k, w) \) is a meromorphic function and has a simple pole at \( k+1 \) with the residue 1 and, also, \( \theta_M(z, w) \) is an entire function.
Proposition 1.2. Let $0 \to U \to M \to N \to 0$ be a graded short exact sequence of $S$-modules. Then 
\[ \zeta_M(z, w) = \zeta_U(z, w) + \zeta_N(z, w). \]

Proof. It follows from $H(M, n) = H(U, n) + H(N, n)$, $n \geq 0$, and (1.3). □

Proposition 1.3. For any $k \geq 0$, it holds that 
\[ \zeta_{M}(-k)(z, w) = \zeta_M(z, w + k). \]

Proof. Since $M(-k)_n = M_{n-k}$, it follows that $H(M(-k), n) = 0$ for all $0 \leq n < k$ and $H(M(-k), n) = H(M, n - k)$, for all $n \geq k$. Consequently, by (1.3), we get 
\[ \zeta_{M}(-k)(z, w) = \sum_{n=0}^{\infty} \frac{H(M(-k), n)}{(n+w)^z} = \sum_{n=k}^{\infty} \frac{H(M, n-k)}{(n+k+w)^z} = \zeta_M(z, w + k). \]

□

Corollary 1.4. If $f \in S_k$ is regular on $M$, then 
\[ \zeta_{fM}(z, w) = \zeta_M(z, w) - \zeta_M(z, w + k). \]

Proof. We consider the short exact sequence 
\[ 0 \to M(-k) \xrightarrow{f} M \to M/fM \to 0. \]

The conclusion follows from Proposition 1.2 and Proposition 1.3. □

Corollary 1.5. If $f_1, \ldots, f_p \in S$ is a regular sequence on $M$, consisting of homogeneous elements with $\deg(f_i) = k_i$, then 
\[ \zeta_{(f_1,\ldots,f_p)M}(z) = \zeta_M(z, w) + \sum_{\ell=1}^{p} (-1)^{\ell} \sum_{1 \leq i_1 < \cdots < i_\ell \leq p} \zeta_M(z, w + k_{i_1} + \ldots + k_{i_\ell}). \]

Proof. It follows from Corollary 1.4, using induction on $k \geq 1$. □

Let 
\[ \zeta_M(z) := \lim_{w \to 0} (\zeta_M(z, w) - H(M, 0)w^{-z}) = \sum_{n=1}^{\infty} \frac{H(M, n)}{n^z}. \tag{1.9} \]

Note that $\zeta_M(z)$ codify all the information about the Hilbert function of $M$ with the exception of $H(M, 0)$. Let 
\[ \theta_M(z) := \sum_{n=1}^{\alpha(M)-1} \frac{H(M, n)}{n^z}, \tag{1.10} \]

Note that $\theta_M(z)$ is an entire function. Also, if $\alpha(M) \leq 1$ then $\theta_M(z)$ is identically zero.
Proposition 1.6. We have that
\[ \zeta_M(z) = \theta_M(z) + \sum_{k=0}^{m-1} \frac{1}{Dz-k} \sum_{j=0}^{D-1} d_{M,k}(j + \alpha(M))\zeta(z - k, \frac{j + \alpha(M) + 1}{D}). \]

The function \( \zeta_M(z) \) is meromorphic with poles at most in the set \( \{1, \ldots, m\} \) which are all simple with residues
\[ R_M(k+1) := \text{Res}_{z=k+1} \zeta_M(z) = \frac{1}{D} \sum_{j=0}^{D} d_{M,k}(j), \quad 0 \leq k \leq m - 1. \]

**Proof.** The proof is similar to the proof of Theorem 1.1, therefore we will omit it. Also, the result could be derived from the proof of [6, Proposition 3.4(i)]. \( \square \)

Let \( k \geq 1 \) be an integer and let
\[ M(k) := \bigoplus_{n=-k}^{\infty} M_{n+k}. \]

Given a real number \( w > k \), we consider the function
\[ \zeta_{M(k)}(z, w) := \sum_{n=-k}^{\infty} \frac{H(M, n + k)}{(n + w)^{z}} = \sum_{n=0}^{\infty} \frac{H(M, n)}{(n + w - k)^{z}} = \zeta_M(z, w - k). \quad (1.11) \]

Let \( a(S) := \deg(H_S(t)) \) be the \( a \)-invariant of \( S \). Assume \( S \) is Gorenstein. Then, according to [5, Proposition 3.6.11], the canonical module of \( S \), \( \omega_S \) is isomorphic to \( S(a(S)) \). Consequently, we get \( \zeta_{\omega_S}(z, w) = \zeta_S(z, w - a(S)) \), where \( w > \max\{0, a(s)\} \).

**Proposition 1.7.** Let \( S \) be a Cohen-Macaulay domain with the canonical module \( \omega_S \). Then \( S \) is Gorenstein if and only if \( \zeta_{\omega_S}(z, w) = \zeta_S(z, w - a(S)) \).

**Proof.** Note that \( \zeta_{\omega_S}(z, w) = \zeta_S(z, w - a(S)) \) is equivalent to \( H_{\omega_S}(t) = t^{a(S)}H_S(t) \). Hence, according to [5, Theorem 4.4.5(2)], this is equivalent to \( S \) is Gorenstein. \( \square \)

**Remark 1.8.** Assume that \( S = K[x_1, \ldots, x_r] \) is the ring of polynomials with \( \deg(x_i) = a_i \), \( 1 \leq i \leq r \). The Hilbert series of \( S \) is
\[ H_S(t) = \frac{1}{(1 - t^{a_1}) \cdots (1 - t^{a_r})}, \]

hence \( a(S) = -(a_1 + \cdots + a_r) \). It is well known that \( S \) is Gorenstein, therefore
\[ \omega_S \cong S(a(S)) = S(-a_1 - \cdots - a_r). \]

It follows that
\[ \zeta_{\omega_S}(z, w) = \zeta_S(z, w + a_1 + \cdots + a_r), \quad (\forall)w > 0. \]

In the next section we will discuss the case of graded modules over \( S \).
2 Graded modules over the ring of polynomials.

Let \( a = (a_1, \ldots, a_r) \) be a sequence of positive integers. In the following, \( S = K[x_1, \ldots, x_r] \) is the ring of polynomials in \( r \) indeterminates, with \( \text{deg}(x_i) = a_i, 1 \leq i \leq r \). The restricted partition function associated to \( a \) is \( p_a : \mathbb{N} \to \mathbb{N} \),

\[ p_a(n) := \text{the number of integer solutions } (x_1, \ldots, x_r) \text{ of } \sum_{i=1}^{r} a_i x_i = n \text{ with } x_i \geq 0. \]

For a kindly introduction on the restricted partition function we refer to [2]. One can easily see that \( p_a(n) = H(S, n), (\forall)n \geq 1 \), hence

\[ \zeta_S(z, w) = \zeta_a(z, w) := \sum_{n=0}^{\infty} \frac{p_a(n)}{(n+w)^2} \tag{2.1} \]

is the Zeta-Barnes function associated to the sequence \( a \). We also have

\[ \zeta_S(z) = \zeta_a(z) := \lim_{w \to 0} (\zeta_a(z, w) - w^2) = \sum_{n=1}^{\infty} \frac{p_a(n)}{n^2}. \tag{2.2} \]

See [6] for further details on the properties of the function \( \zeta_a(z) \).

**Proposition 2.1.** Let \( M \) be a finitely generated graded \( S \)-module. Then:

1. \( \zeta_M(z, w) := \sum_{i=0}^{p} (-1)^i \sum_{j \geq i} \beta_{ij}(M) \zeta_a(z, w+j), \) where \( \beta_{ij}(M) := \dim_K(Tor_i(M, K))_j \) are the graded Betti numbers of \( M \) and \( p \) is the projective dimension of \( M \).
2. \( \zeta_M(z) = \sum_{i=0}^{p} (-1)^i \sum_{j \geq \max(i, 1)} \beta_{ij}(M) \zeta_a(z, j) + \beta_{00}(M) \zeta_a(z). \)

**Proof.** (1) Let

\[ F : 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0, \tag{2.3} \]

be the minimal free resolution of \( M \). We have that \( F_i = \bigoplus_{j \geq 0} S(-j)^{\beta_{ij}} \). By (2.1), Proposition 1.2 and Proposition 1.3, it follows that

\[ \zeta_{F_i}(z, w) = \sum_{j \geq 0} \beta_{ij} \zeta_a(z, w+j). \]

The result follows from Proposition 1.2 applied several times to the exact sequence (2.3).

(2) By (2.1), it follows that

\[ \lim_{w \to 0} \zeta_a(z, j + w) = \zeta_a(z, j), (\forall) j \geq 1. \tag{2.4} \]

Using (2.2), (2.4) and (1) we get the required result. \( \square \)
The Bernoulli numbers $B_\ell$ are defined by
\[
\frac{z}{e^z - 1} = \sum_{\ell=0}^{\infty} B_\ell \frac{z^\ell}{\ell!},
\]
\[B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30} \quad \text{and} \quad B_n = 0 \text{ if } n \geq 3 \text{ is odd.}
\]
For $k > 0$ we have the Faulhaber’s identity
\[
1^k + 2^k + \cdots + n^k = \frac{1}{k+1} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell n^{1+k-\ell}.
\]
The Bernoulli-Barnes polynomials $B_\ell(x; a_1, \ldots, a_r)$ are defined by
\[
\frac{z^r e^z x}{(e^{az} - 1) \cdots (e^{az} - r)} = \sum_{\ell=0}^{\infty} B_\ell(x; a_1, \ldots, a_r) \frac{z^\ell}{\ell!}.
\]
According to formula (3.9) in Ruijsenaars [8],
\[
\Res_{z=\ell} \zeta_a(z, w) = \frac{(-1)^{r-\ell}}{(\ell-1)!(r-\ell)!} B_{r-\ell}(w; a_1, \ldots, a_r), \quad 1 \leq \ell \leq r.
\]
(2.5)
The Bernoulli-Barnes numbers are defined by
\[
B_\ell(a_1, \ldots, a_r) := B_\ell(0; a_1, \ldots, a_r).
\]
The Bernoulli-Barnes numbers and the Bernoulli numbers are related by
\[
B_\ell(a_1, \ldots, a_r) = \sum_{i_1 + \cdots + i_r = \ell} \binom{\ell}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r} a_1^{i_1-1} \cdots a_r^{i_r-1},
\]
see Bayad and Beck [4, Page 2] for further details. According to [6, Theorem 3.10],
\[
\Res_{z=\ell} \zeta_a(z) = \frac{(-1)^{r-\ell}}{(\ell-1)!(r-\ell)!} B_{r-\ell}(a_1, \ldots, a_r), \quad 1 \leq \ell \leq r.
\]
(2.6)
Note that (2.6) can be deduced from (2.5).

**Corollary 2.2.** Let $M$ be a finitely generated graded $S$-module and $w > 0$. Then
\begin{enumerate}
\item $R_{M}(w, \ell) = \sum_{i=0}^{p} \sum_{j \geq 0} \beta_{ij}(M) \frac{(-1)^{i+j-r-\ell}}{(\ell-1)!(r-\ell)!} B_{r-\ell}(w+j; a_1, \ldots, a_r), \quad 1 \leq \ell \leq r.$
\item $R_{M}(\ell) = \sum_{i=0}^{p} \sum_{j \geq 0} \beta_{ij}(M) \frac{(-1)^{i+j-r-\ell}}{(\ell-1)!(r-\ell)!} B_{r-\ell}(j; a_1, \ldots, a_r), \quad 1 \leq \ell \leq r.$
\end{enumerate}

**Proof.** The results follow from Proposition 2.1 and the formulas (2.5) and (2.6).
Example 2.3. Let $a = (a_1, \ldots, a_r)$ be a sequence of positive integers, $D = \text{lcm}(a_1, \ldots, a_r)$. We consider the ideal $I = (x_1^{a_1}, \ldots, x_r^{a_r}) \subset S$. Note that $I$ is an Artinian complete intersection monomial ideal generated in degree $D$, w.r.t. the $a$-grading. According to (2.2) and Corollary 1.6, we have
\[
\zeta_{S/I}(z, w) = \theta_{S/I}(z, w) = \sum_{j=0}^r (-1)^j \binom{r}{j} \zeta_a(z, w + Dj).
\]

On the other hand, one can easily check that
\[
H_{S/I}(t) = \frac{(1-t_D)^r}{(1-t^{a_1}) \cdots (1-t^{a_r})} = (1 + t^{a_1} + \cdots + t^{a_1(\frac{D}{a_1}-1)}) \cdots (1 + t^{a_r} + \cdots + t^{a_r(\frac{D}{a_r}-1)})
\]
is a reciprocal polynomial of degree $Dr - a_1 - \cdots - a_r$. The coefficient of $t^n$ in $H_{S/I}(t)$ equals to
\[
f_a(n) = \# \{(x_1, \ldots, x_r) \in \mathbb{Z}^r : a_1 x_1 + \cdots + a_r x_r = n, \ 0 \leq x_1 < \frac{D}{a_1}-1, \ldots, 0 \leq x_r < \frac{D}{a_r}-1\}.
\]
By (2.7) it follows that
\[
\sum_{n=0}^{Dr-a_1-\cdots-a_r} f_a(n)(n+w)^{-z} = \sum_{j=0}^r (-1)^j \binom{r}{j} \zeta_a(z, w + Dj).
\]
See Rødseth and Sellers [7] for further details on the coefficients $f_a(n)$.

Example 2.4. Let $S = K[x_1, x_2]$ with $\deg(x_1) = 2$, $\deg(x_2) = 3$. Let $a = (2, 3)$. The polynomial $f = x_1^2 - x_2^3 \in S$ is homogeneous of degree 6. Let $R = S/(f)$. $R$ has the minimal graded free resolution
\[
0 \to S(-6) \xrightarrow{\partial} S \to R \to 0
\]
It follows that the non-zero Betti numbers of $R$ are $\beta_{00}(R) = 1$ and $\beta_{16}(R) = 1$. Let $w > 0$. According to (2.1) and Corollary 1.4 (or (2.8) and Proposition 2.1(1)) we have
\[
\zeta_R(z, w) = \zeta_a(z, w) - \zeta_a(z, w + 6) = \sum_{n=0}^{\infty} \frac{p_a(n)}{(n+w)^z} - \sum_{n=0}^{\infty} \frac{p_a(n)}{(n+w+6)^z} = \]
\[
= \sum_{n=0}^5 \frac{p_a(n)}{(n+w)^z} + \sum_{n=6}^{\infty} \frac{p_a(n) - p_a(n-6)}{(n+w)^z} = \frac{1}{w^z} + \sum_{n=2}^{\infty} \frac{1}{(n+w)^z} = \frac{1}{w^z} + \zeta(z, w + 2).
\]
In particular, the Hilbert series of $R$ is
\[
H_R(t) = 1 + \sum_{n=2}^{\infty} t^n = 1 + \frac{t^2}{1-t} = \frac{t^2 - t + 1}{1 - t},
\]
hence $\alpha(R) = a(R) = 1$. It follows that $\theta_R(z, w) = \frac{1}{w^z}$. Also,
\[
\zeta_R(z) = \lim_{w \to 0} (\zeta_R(z, w) - \frac{1}{w^z}) = \zeta(z, 2) \quad \text{and} \quad \theta_R(z) = 0.
\]
3 The standard graded case

Let $S$ be a standard graded $K$-algebra, that is $S = \bigoplus_{n \geq 0} S_n$, $S_0 = K$ and $S = K[S_1]$. Let $M$ be a finitely generated graded $S$-module. According to the Hilbert-Serre’s Theorem, it holds that

$$H_M(t) = \frac{h_M(t)}{(t-1)^m},$$

(3.1)

where $h_M \in \mathbb{Z}[t]$, $m = \dim(M)$ and $h_M(1) \neq 0$. Also, there exists a polynomial $P_M(t) \in \mathbb{Z}[t]$ of degree $m - 1$, such that

$$H(M, n) = P_M(n), \ (\forall) n \gg 0,$$

which is called the Hilbert polynomial of $M$.

The number $e(M) := h_M(1)$ is called the multiplicity of the module $M$.

**Proposition 3.1.** If $P_M(t) = d_{M,m-1}t^{m-1} + \cdots + d_{M,1}t + d_{M,0}$ is the Hilbert polynomial of $M$, then

$$\zeta_M(z,w) = \theta_M(z,w) + \sum_{k=0}^{m-1} d_{M,k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-w)^{\ell} \zeta(z-k+\ell, \alpha(M) + w)$$

is a meromorphic function on $\mathbb{C}$ with the poles in the set $\{1, 2, \ldots, m\}$ which are simple with residues

$$R_M(w, k+1) := \text{Res}_{z=k+1} \zeta_M(z,w) = \sum_{\ell=k}^{m-1} \binom{\ell}{k} (-w)^{\ell-k} d_{M,\ell}, \ 0 \leq k \leq m - 1.$$

**Proof.** It is the particular case of Theorem 1.1 for $a = (1, \ldots, 1)$.

**Proposition 3.2.** We have that

$$\zeta_M(z) = \theta_M(z) + \sum_{k=0}^{m-1} d_{M,k} \zeta(z-k+\ell, \alpha(M) + 1)$$

is a meromorphic function on $\mathbb{C}$ with the poles in the set $\{1, 2, \ldots, m\}$ which are simple with residues

$$R_M(\ell+1) := \text{Res}_{z=\ell+1} \zeta_M(z) = d_{M,\ell}.$$

**Proof.** It is the particular case of Proposition 1.6 for $a = (1, \ldots, 1)$.

If $\dim M \geq 1$, then we can write

$$P_M(t) = \sum_{k=0}^{m-1} (-1)^k e_k(M) \binom{t+m-1-k}{m-1-k}.$$  

(3.2)

According to [5, Proposition 4.1.9], we have

$$e_k(M) = \frac{h_M^{(k)}(t)}{k!}, \ (\forall) 0 \leq k \leq m - 1.$$  

(3.3)
Corollary 3.3. We have that
\[ e(M) = e_0(M) = (m - 1)!d_{M,m-1} = (m - 1)!R_M(m). \]

Proof. It follows from (3.2), (3.3) and Proposition 3.2. \(\square\)

The higher iterated Hilbert functions \(H_i(M,n), i \in \mathbb{N}\), of a finitely generated \(S\)-module \(M\) are defined recursively as follows:

\[ H_0(M,n) := H(M,n), \text{ and } H_i(M,n) = \sum_{j=0}^{n} H_{i-1}(M,n), i \geq 1. \] (3.4)

The functions \(H_i(M,n)\) are of polynomial type of degree \(m + i - 1\), hence

\[ H_i(M,n) = P_i(M,n) := d_{M,m+i-1}n^{m+i-1} + \cdots + d_{M,1}n + d_{M,0}, (\forall)n \gg 0. \] (3.5)

We define the higher Zeta-Barnes type functions associated to \(M\) as follows:

\[ \zeta_M^{i}(z,w) := \sum_{n=0}^{\infty} \frac{H_i(M,n)}{(n + w)^z}, i \geq 0. \] (3.6)

and

\[ \zeta_M^{i}(z) = \lim_{w \to 0} (\zeta_M^{i}(z,w) - H(M,0)w^{-z}), i \geq 0. \] (3.7)

Let \(\alpha^i(M) := \min\{n_0 \in \mathbb{N} : H_i(M,n) = P_i(M,n), (\forall)n \geq n_0\}\).

We define

\[ \theta_M^{i}(z,w) = \sum_{n=0}^{\alpha^i(M)-1} \frac{H_i(M,n)}{(n + w)^z} \text{ and } \theta_M^{i}(z) = \sum_{n=1}^{\alpha^i(M)-1} \frac{H_i(M,n)}{n^z}. \]

Similar to Proposition 2.1 and Proposition 2.2 we have the following result.

Proposition 3.4. With the above notations:

1. \(\zeta_M^{i}(z,w) = \theta_M^{i}(z,w) + \sum_{k=0}^{m+i-1} d_{M,k} \sum_{\ell=0}^{k} \binom{k}{\ell}(-w)^\ell \zeta(z-k+\ell, \alpha^i(M)+w)\) is a meromorphic function on \(\mathbb{C}\) with the poles in the set \(\{1,2,\ldots,m+i\}\) which are simple with residues

\[ R_M^{i}(w,k + 1) := \text{Res}_{z=k+1}\zeta_M^{i}(z,w) = \sum_{\ell=k}^{m+i-1} \binom{\ell}{k}(-w)^{\ell-k}d_{M,\ell}, 0 \leq k \leq m + i - 1. \]

2. \(\zeta_M^{i}(z) = \theta_M^{i}(z) + \sum_{k=0}^{m+i-1} d_{M,k} \zeta(z-k+\ell, \alpha^i(M)+1)\) is a meromorphic function on \(\mathbb{C}\) with the poles in the set \(\{1,2,\ldots,m+i\}\) which are simple with residues

\[ R_M^{i}(k + 1) := \text{Res}_{z=k+1}\zeta_M^{i}(z) = d_{M,k}, 0 \leq k \leq m + i - 1. \]
Corollary 3.5. We have that $e(M) = m!R^1_M(m + 1)$.

Proof. According to [5, Remark 4.1.6], $H_1(M, n) = d^1_{M,n}n^m + \cdots + d^1_{n,1}n + d^1_{0,0}$, $(\forall)n \gg 0$, and $e(M) = m!d^1_{M,n}$. Now, apply Proposition 3.4(2). \[ \square \]

Remark 3.6. Let $S = K[x_1, \ldots, x_r]$ and $I \subset S$ a graded ideal. We say that $S/I$ has a pure resolution of type $(d_1, \ldots, d_p)$ if its minimal resolution is

$$0 \to S(-d_p) \beta_p \to \cdots \to S(-d_1) \beta_1 \to S \to S/I \to 0,$$

where $p$ is the projective dimension of $S/I$, $d_1 < d_2 < \cdots < d_p$ and $\beta_i = \sum_{j \geq 0} \beta_{ij}(S/I)$, $1 \leq i \leq p$, are the Betti numbers of $S/I$. According to Corollary 3.3, $e(S/I) = R_{S/I}(m)$, where $m = \dim(S/I)$. On the other hand, according to Corollary 2.2(2), we have

$$R_{S/I}(m) = \sum_{i=0}^{p} \beta_i \frac{(-1)^{r-m}}{(m-1)!(r-m)!} B_{r-m}(d_i; 1, 1, \ldots, 1). \quad (3.8)$$

Suppose $S/I$ is Cohen-Macaulay and has a pure resolution of type $(d_1, \ldots, d_p)$. According to [5, Theorem 4.1.15],

$$\beta_i = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{d_j - d_i} \quad \text{and} \quad e(S/I) = \frac{d_1d_2\cdots d_p}{p!}. \quad (3.9)$$

The Ausländer-Buchsbaum formula [5, Theorem 1.3.3] implies $p = r - m$, hence (3.8) and (3.9) give the identity:

$$\sum_{i=0}^{p} (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{d_j - d_i} B_p(d_i; 1, 1, \ldots, 1) = (m - 1)!(-1)^{p}d_1d_2\cdots d_p.$$

4 The non-graded case

Let $(S, m, K)$ be a Noetherian local ring, where $m$ is the maximal ideal of $S$ and $K = S/m$ is the residue field. Let $M$ be a finitely generated $S$-module, with $m = \dim(M)$, and let $I \subset S$ be an ideal such that $m^n M \subset IM$ for some $n \geq 1$. The associated graded ring is

$$\text{gr}_I(S) = \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}} = \frac{S}{I} \oplus \frac{I}{I^2} \oplus \cdots.$$

The associated graded module of $M$, with respect to $I$, is

$$\text{gr}_I(M) := \bigoplus_{n \geq 0} \frac{I^n M}{I^{n+1} M},$$

which has a structure of a $\text{gr}_I(S)$-module. According to [5, Theorem 4.5.6], it holds that

$$\dim(\text{gr}_I(M)) = \dim(M) = m.$$
The Hilbert-Samuel function of $M$, w.r.t. $I$, is

$$\chi_M(n) := H_1(\text{gr}_I(M), n) = \sum_{i=0}^{n} H(\text{gr}_I(M), i) = \dim_K \frac{M}{I^{n+1}M}, \quad (\forall)n \geq 0.$$  

The multiplicity of $M$ with respect to $I$ is $e(M, I) := e(\text{gr}_I(M))$. For $n \gg 0$, according to [5, Remark 4.1.6], we have that

$$\chi_M(n) = \frac{e(M, I)}{m!} n^d + \text{terms in lower powers of } n. \quad (4.1)$$

We consider the functions

$$\zeta_{M, I}^i(z, w) := \zeta_{\text{gr}_I(M)}^i(z, w) \quad \text{and} \quad \zeta_{M, I}^i(z) := \zeta_{\text{gr}_I(M)}^i(z), \quad i \geq 0. \quad (4.2)$$

**Proposition 4.1.** It holds that

$$e(M, I) = m! \text{Res}_{z=m+1} \zeta_{M, I}^1(z).$$

**Proof.** This follows from (4.1), (4.2) and Corollary 3.5. □

**References**

[1] M. F. Atiyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, (1969).

[2] J. L. Ramirez Alfonsin, *The Diophantine Frobenius Problem*, Oxford Lecture Series in Mathematics and its Applications 30, (2005).

[3] E. W. Barnes, *On the theory of the multiple gamma function*, Trans. Camb. Philos. Soc. 19 (1904), 374-425.

[4] A. Bayad, M. Beck, *Relations for Bernoulli-Barnes Numbers and Barnes Zeta Functions*, International Journal of Number Theory 10 , (2014), 1321-1335.

[5] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics 39, revised edition (1998).

[6] M. Cimpoeaş, F. Nicolae, *On the restricted partition function*, Ramanujan J. (2018), https://doi.org/10.1007/s11139-017-9985-3.

[7] Ø. J. Rødseth, J. A. Sellers, *Partitions with parts in a finite set*, International Journal of Number Theory, Vol. 02, No. 03, (2006), 455-468.

[8] S. N. M. Ruijsenaars, *On Barnes Multiple Zeta and Gamma Functions*, Advances in Mathematics 156, (2000), 107-132.

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