ON THE OCCUPATION TIME OF BROWNIAN EXCURSION

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Abstract
Recently, Kalvin M. Jansons derived in an elegant way the Laplace transform of the time spent by an excursion above a given level $a > 0$. This result can also be derived from previous work of the author on the occupation time of the excursion in the interval $(a, a + b)$, by sending $b \to \infty$. Several alternative derivations are included.

1 Introduction

In [5], the author derives in an elegant way the Laplace transform of the time spent by an excursion above a given level $a > 0$. This result can also be derived from the occupation time of the excursion in the interval $(a, a + b)$, by sending $b \to \infty$ (cf. [2] or [4]).

2 Occupation times

Introduce for $\alpha, \beta$ complex and $a \geq 0$,

$$\psi(\alpha, \beta, a) = \frac{\alpha \cosh(\alpha \beta) + \beta \sinh(\alpha \beta)}{\alpha \sinh(\alpha \beta) + \beta \cosh(\alpha \beta)}.$$ 

Denote by $W^+_t$, Brownian excursion with time parameter $t \in [0, 1]$, see [4], I.2 for a precise definition. According to p. 117 and p. 120 of [4], or Theorem 5.1 of [2], the Laplace transform of the occupation time $T(a, a + b) = \int_0^1 1_{(a, a+b)}(W^+_t) \, dt$, is given by:

$$E e^{-\beta T(a,a+b)} = \frac{\sqrt{2\pi}}{a^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{a^2}^{\infty} \frac{ae^x}{\sinh(a\sqrt{2}x)} \, dx$$

$$\times \left[ \sqrt{\alpha} \cosh(a\sqrt{2}\alpha) + (\alpha + \beta)^{1/2} \psi(\sqrt{\alpha}, \sqrt{\alpha + \beta}, b\sqrt{2}) \sinh(\alpha \sqrt{2\alpha}) \right]^{-1} \, d\alpha,$$

$$= \frac{\sqrt{2\pi}}{a^3} \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-\frac{1}{2a^2}} + \frac{1}{a^3} \int_{a^2}^{\infty} \frac{ae^x}{\sinh(a\sqrt{2}x)} \, dx$$

$$\times \left[ \sqrt{\alpha} \cosh(a\sqrt{2}\alpha) + (\alpha + \beta)^{1/2} \psi(\sqrt{\alpha}, \sqrt{\alpha + \beta}, b\sqrt{2}) \sinh(\alpha \sqrt{2\alpha}) \right]^{-1} \, d\alpha.$$
where the path $S$ is defined by

$$S = \{ \alpha : \alpha = iy, |y| \geq \xi \} \cup \{ \alpha : \alpha = \xi e^{i\eta}, -\pi/2 \leq \eta \leq \pi/2 \},$$

for some $\xi > 0$.

In order to write the first term on the right side of (1), which term is equal to the distribution function of the supremum of Brownian excursion, \(^1\) as a complex integral we introduce the path:

$$\Gamma = \{ \alpha : \alpha = y e^{\pm i\phi}, y \geq \xi \} \cup \{ \alpha : \alpha = \xi e^{i\eta}, -\phi \leq \eta \leq \phi \},$$

with $\pi/2 < \phi < \pi$, $\xi > 0$ and the orientation counterclockwise. We choose the angle $\phi$ in such a way that all singularities of the integrand in (1) remain on the left of the path $\Gamma$. Then

$$\frac{\sqrt{2\pi}}{a^2} \sum_{k=1}^{\infty} k^2\pi^2 e^{-k^2\pi^2/2a^2} = -\frac{1}{i\sqrt{\pi}} \int_{\Gamma} \sqrt{\alpha e^\alpha} \frac{\cosh\{a\sqrt{2\alpha}\}}{\sinh\{a\sqrt{2\alpha}\}} d\alpha,$$

(2)

since the integrand has only simple poles at $\alpha_k = -k^2\pi^2/2a^2$, $k \geq 1$. Combining (1) and (2) and deforming the path $S$ into the path $\Gamma$ (again using Cauchy’s theorem), yields

$$E e^{-\beta T(a,a+b)} = -\frac{1}{i\sqrt{\pi}} \int_{\Gamma} \sqrt{\alpha e^\alpha} d\alpha$$

$$\times \left[ \frac{\sqrt{\alpha} \sinh\{a\sqrt{2\alpha}\} + (\alpha + \beta)^{1/2} \psi(\sqrt{\alpha}, \sqrt{\alpha + \beta}, b\sqrt{2}) \cosh\{a\sqrt{2\alpha}\}}{\sqrt{\alpha} \cosh\{a\sqrt{2\alpha}\} + (\alpha + \beta)^{1/2} \psi(\sqrt{\alpha}, \sqrt{\alpha + \beta}, b\sqrt{2}) \sinh\{a\sqrt{2\alpha}\}} \right].$$

By taking the limit for $b \to \infty$, $(\psi(\cdot, \cdot, b\sqrt{2}) \to 1$, uniformly on compacta of $\Gamma$) we obtain for the Laplace transform of the occupation time $T(a) = T(a, \infty)$,

$$E e^{-\beta T(a)} = -\frac{1}{i\sqrt{\pi}} \int_{\Gamma} \sqrt{\alpha e^\alpha} \psi(\sqrt{\alpha + \beta}, \sqrt{\alpha}, a\sqrt{2}) d\alpha.$$

(4)

Alternatively, one could take the limit for $a \downarrow 0$ in (3), resulting in the transform: $E e^{-\beta (1-T(b))}$. For the occupation time $T_t(a)$ of the excursion straddling $t$, we have

$$T_t(a) \stackrel{d}{=} (L_t)^{1/2} T(a (L_t)^{-1/2}),$$

(5)

with $T(a)$ and $L_t$ independent, and where $L_t$ denotes the length of the excursion. It is readily verified from the density of $L_t$, see [1], (4.4), that for integrable $\varphi$,

$$\int_0^\infty e^{-at} E \varphi(L_t) dt = \frac{1}{2\sqrt{\pi a^3}} \int_0^\infty \varphi(y)(1 - e^{-ay}) dy.$$

(6)

\(^1\)According to the Poisson-summation formula

$$\frac{\sqrt{2\pi}}{a^2} \sum_{k=1}^{\infty} k^2\pi^2 e^{-k^2\pi^2/2a^2} = 1 + 2 \sum_{k=1}^{\infty} (1 - 4k^2a^2)e^{-2k^2a^2},$$

which is the more familiar form of this distribution function.
Hence, using (5) and (6), the Laplace transform (4) yields the double Laplace transform:

\[
\int_0^\infty e^{-\alpha t} E^{-\beta T_t(a)} \, dt = \frac{1}{\alpha} \psi(\sqrt{\alpha + \beta}, \sqrt{\alpha}, a\sqrt{2}) - \frac{1}{\alpha^{3/2}} \lim_{\alpha \to 0} \sqrt{\alpha} \psi(\sqrt{\alpha + \beta}, \sqrt{\alpha}, a\sqrt{2}) = \frac{1}{\alpha} \left[ \frac{\sqrt{\alpha} \sinh(a\sqrt{2\alpha})}{\sqrt{\alpha} \cosh(a\sqrt{2\alpha}) + \sqrt{\alpha + \beta} \cosh(a\sqrt{2\alpha})} \right] - \frac{1}{\alpha^{3/2} (1 + a\sqrt{2\beta})}.
\]

This result can also be derived starting from reflected Brownian motion \(W\) (cf. [3], p. 92, Remark (3.20)).

Perhaps the most elegant formulation of the Laplace transform of the occupation time is that for strictly positive

\[
\int_0^\infty e^{-\alpha x} \left[ 1 - E^{-\beta x T(x^{-1/2})} \right] \, dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\alpha x} \left[ 1 - E^{-\beta x T(x^{-1/2})} \right] \, dx
\]

Equation (8) can be derived as follows. On the path \(\Gamma\) we have:

\[
1 - E^{-\beta T_t(a)} = -\frac{1}{i\sqrt{\pi}} \int_\Gamma \sqrt{\alpha} e^{\alpha} \, d\alpha + \frac{1}{i\sqrt{\pi}} \int_\Gamma \sqrt{\alpha} e^{\alpha} \psi(\sqrt{\alpha + \beta}, \sqrt{\alpha}, a\sqrt{2}) \, d\alpha
\]

\[
= \frac{1}{i\sqrt{\pi}} \int_\Gamma \sqrt{\alpha} e^{\alpha} \left[ \frac{2(\sqrt{\alpha + \beta} - \sqrt{\alpha}) e^{-a\sqrt{2\alpha}}}{(\sqrt{\alpha + \sqrt{\alpha + \beta}}) e^{a\sqrt{2\alpha}} + (\sqrt{\alpha} - \sqrt{\alpha + \beta}) e^{-a\sqrt{2\alpha}}} \right] \, d\alpha.
\]

Now for \(a > 0\) the integral over the path \(\Gamma\) may be replaced by integration over the line \((c-i\infty, c+i\infty)\), where \(c > 0\) is arbitrary. Hence after the substitution \(x = az\), with \(z\) positive and replacement of the path \((c/x - i\infty, c/x + i\infty)\) by the path \((c - i\infty, c + i\infty)\), we obtain

\[
x^{-3/2} \left( 1 - E^{-\beta x T(x^{-1/2})} \right) = \frac{1}{i\sqrt{\pi}} \int_{c-i\infty}^{c+i\infty} \sqrt{z} e^{\alpha z} \left[ \frac{2(\sqrt{z + \beta} - \sqrt{z}) e^{-\sqrt{2z}}}{(\sqrt{z + \sqrt{z + \beta}}) e^{\sqrt{2z}} + (\sqrt{z} - \sqrt{z + \beta}) e^{-\sqrt{2z}}} \right] \, dz.
\]

Taking Laplace transforms on both sides gives (8).

Each of the representations (4), (7) or (8) is equivalent to Theorem 1 of [5], where the duration of the excursion was scaled with a gamma(1/2, 1/2) density. In particular, Theorem 1 of [5] can be obtained from (8) by differentiating both sides with respect to \(\alpha\) and using that

\[
\int_0^\infty x^{-1/2} e^{-ax} \, dx = \sqrt{\pi/\alpha}.
\]

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