The Leading-Order Dressing Phase in ABJM Theory

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Abstract: We study the planar asymptotic dilatation operator of ABJM theory in the $SU(2) \times SU(2)$ sector up to eight loops. Combining Bethe Ansatz techniques and $\mathcal{N} = 2$ superspace methods, we are able to fix all the coefficients appearing in the maximal-reshuffling terms. In particular, we can directly compute from Feynman diagrams the leading order coefficient $\beta^{(6)}_{2,3}$ of the dressing phase and find an agreement with the relation conjectured by Gromov and Vieira between the ABJM and $\mathcal{N} = 4$ SYM phase factor.

Keywords: AdS/CFT, Chern–Simons matter theories, integrability, dressing phase.
1. Introduction

In the last few years remarkable mathematical structures were shown to emerge in the analysis of supersymmetric Chern-Simons-matter theories in three dimensions. In this class of theories a distinguished role is played by the $\mathcal{N} = 6$ ABJM model [1] which is a $U(N)_k \times U(N)_{-k}$ superconformal gauge theory with Chern-Simons level $k$. Indeed, in the large $N$ limit the ABJM theory has been conjectured to be the AdS/CFT dual description of M-theory on an $AdS_4 \times S_7/Z_k$ background and, for $k \ll N \ll k^5$, of a type IIA string theory on $AdS_4 \times CP_3$. For this reason, soon after its discovery the ABJM model has quickly become the ideal three-dimensional playground to study AdS/CFT as much as $\mathcal{N} = 4$ SYM has been in the four-dimensional case.

Quite surprisingly, the ABJM model seems to share a number of remarkable properties with $\mathcal{N} = 4$ SYM theory even if the two theories are a priori different in nature. One of the common features is provided by the fact that also in the ABJM case, in the
limit of large $N$ and $k$ with $\lambda = N/k$ kept fixed, integrable structures naturally show up (for a review see [2]).

At first it was found in [3, 4] that, at the two-loop order and in the $SU(4)$ flavor sector, the anomalous dimensions of composite operators could be mapped to the energy spectrum of an integrable Hamiltonian acting on an alternating fundamental-antifundamental spin-chain.

The two-loop analysis was then extended to the full theory in [5, 6] by the introduction of an $OSp(2, 2|6)$ chain. The parity breaking ABJ model [7] was also studied at two loops in [6, 8], where it was found to be integrable at the given order.

Soon afterwards, paralleling the progresses done in the four-dimensional case, a set of all-loop Bethe equations for the asymptotic spectrum of the full ABJM theory was proposed in [9]. The Bethe equations nicely interpolated between the weak coupling results and the coset string construction at strong coupling [10, 11], together with the algebraic curve approach developed in [12].

One of the salient features of the Bethe equations introduced by Gromov and Vieira is that they are strongly constrained by the symmetries of the theory. In fact only a pair of undetermined functions of the coupling $\lambda$ are left open in the description of the spectrum. One is the interpolating function $h(\lambda)$ which relates the weak and strong coupling regimes of the single magnon dispersion relation. The other one is the dressing function $\theta(\lambda)$, which had to be introduced also in the $\mathcal{N} = 4$ SYM case [13, 14, 15, 16, 17, 18] in order to have a proper matching between the weak and strong coupling descriptions. In the ABJM case, the dressing phase plays an even more fundamental role since its presence has also been conjectured in [9] to give rise to the coupling between even- and odd-site excitations on the chain at high-loop orders.

The description of the asymptotic spectrum depicted in [9] has been subsequently checked at the perturbative level beyond two-loops by direct Feynman diagrammatic computations in [19, 20, 21, 22]. The four-loop dilatation operator has been fully computed for both the ABJM and ABJ models by using the component formulation in [19, 20] and the $\mathcal{N} = 2$ superspace formalism in [21]. As a result, the form of dilatation operator was found to be compatible with the spectrum predicted by the Bethe equations and moreover it was possible to fix the next-to-leading order coefficient in the weak coupling expansion of the function $h(\lambda)$.

In [23] the analysis was pushed up to six loops. At this order, a full Feynman diagrammatic analysis looks very complicated, even using superspace techniques. Nevertheless, in [23] the expression of the dilatation operator could be derived by computing a suitable set of Feynman diagrams. For further perturbative checks on the spectrum see [24].

Meanwhile, the internal S-matrix approach to integrability was developed in a
number of papers [25, 26, 27, 28, 29, 30]. An all loop S-matrix has been found which is compatible with the all-loop Bethe equations of [9]. Moreover, the thermodynamic Bethe Ansatz and Y-system framework has also been applied to the three dimensional case [31, 32, 33, 34].

In the present paper we would like to provide further evidence for the exactness of the conjectured integrable scenario by performing a direct computation of the first non-vanishing coefficient of the Bethe Ansatz dressing phase. We restrict ourselves to the $SU(2) \times SU(2)$ scalar sector of the theory, where the spin chain is simply given by a pair of coupled $SU(2)$ spin chains. In [9] it was mentioned that the dressing phase starts contributing at eight loops, leading to the result that the two $SU(2)$ chains at odd and even sites are decoupled up to six loops. Indeed, it was explicitly shown in [19, 20] and [21] by direct computations, that this is true at four-loop order: the contributions to the dilatation operator of the diagrams that could lead to interactions between the two types of magnons cancel. Moreover, the results of [23] imply that this is true also at six-loop order. This latter fact isn’t so trivial and has the consequence that the order $\lambda^4$ coefficient of the dressing factor which can be a priori present at six loops, actually vanishes. Therefore, the first non-trivial effects of the presence of the dressing phase are to be found at eight-loop order. With the aim to check the above picture, we analyze the form of the eight-loop dilatation operator and extract the value of the leading order coefficient of the dressing factor. We make use of a procedure inspired to the one used in [35] for the computation of the analogous coefficient in $\mathcal{N} = 4$ SYM.

The plan of the paper is the following. After reviewing in Section 2 the aspects of integrability in the ABJM model we shall need, in Section 3 we introduce a procedure that is useful to constrain the form of the dilatation operator as much as possible. This procedure is based on symmetry arguments and on a matching between the entries of the diagonalized dilatation matrix with the Bethe equations predictions for the energies of one and two-impurity states. We then use this procedure to write the maximal reshuffling part of the dilatation operator in terms of a small number of unknown parameters, one of which is the dressing phase. In Section 4, we perform a direct Feynman diagram computation of the maximal reshuffling diagrams to fix the unknown coefficients including the dressing phase. We end up with a result which is in complete agreement with the integrable picture conjectured in [9]. As we shall discuss, our computation, besides providing the explicit value for the leading coefficient of the dressing factor, represents a non-trivial consistency check of the Bethe equations, since, as mentioned before, the dressing phase is also shown to play the structural role of coupling even and odd excitations in the ABJM model. We give further comments on our results in Section 5 while several technical aspects are collected in the Appendices.
2. The Bethe Ansatz in the $SU(2) \times SU(2)$ Sector

In this section, we collect the integrability tools we need for the construction of the dilatation operator. We refer to Appendix A for the formulation of ABJM theory in terms of $\mathcal{N} = 2$ superfields. Throughout the entire work, we restrict ourselves to the $SU(2) \times SU(2)$ sector, where operators are made out only of the chiral superfields $Z^A, W_B$:

$$O_{B_1\cdots B_L}^{A_1\cdots A_L} = \text{Tr}[Z^{A_1}W_{B_1} \cdots Z^{A_L}W_{B_L}],$$  \hspace{1cm} (2.1)

with $A_j, B_j = 1, 2$. This sector is closed under renormalization at all loop-order. In the spin-chain picture, the operators (2.1) are mapped to states of a circular alternating spin-chain, with the fields $Z^A$ being interpreted as spins lying on odd sites and the $W_A$ as spins lying on even sites. We will refer to fields $Z^2$ and $W_2$ as impurities and they correspond to “spin down” states. The ground state of length $2L$ is chosen to be

$$|0\rangle = \text{Tr}[Z^1W_1 \cdots Z^1W_1],$$  \hspace{1cm} (2.2)

while the operators

$$|2k\rangle = \text{Tr}[Z^1W_1 \cdots Z^1W_2 \cdots Z^1W_1],$$

$$|2k+1\rangle = \text{Tr}[Z^1W_1 \cdots Z^2W_1 \cdots Z^1W_1]$$  \hspace{1cm} (2.3)

represent states with a single excitation on the site $2k$ and $2k+1$ respectively. There are two kinds of magnon states, depending on their momentum $p$ being excited on even or odd sites\(^1\):

$$|p\rangle_e = \sum_{k=1}^{L} e^{i p k} |2k\rangle,$$

$$|p\rangle_o = \sum_{k=1}^{L} e^{i p k} |2k+1\rangle.$$  \hspace{1cm} (2.4)

The magnon dispersion relation is given by

$$E(p) = \frac{1}{2} \left( \sqrt{1 + 16 h^2(\lambda) \sin^2 \frac{p}{2} - 1} \right),$$  \hspace{1cm} (2.5)

where $h(\lambda)$ is the interpolating function, which has the weak-coupling expansion \(^2\)

$$h^2(\lambda) = \sum_{k=1}^{\infty} h_{2k} \lambda^{2k}$$  \hspace{1cm} (2.6)

\(^1\)In the sum, the identification $2L + 1 \sim 1$ is understood.

\(^2\)For discussions on the strong coupling expansion of $h(\lambda)$ see [36]-[45].
with $h_2 = 1$ and $h_4 = -4 \zeta(2)$ the only known coefficients [19, 20, 21].

The all-loop asymptotic Bethe equations for the length $2L$ spin chain can be obtained from [9] by restricting to the $SU(2) \times SU(2)$ flavor sector

$$
\left[ \frac{x(u_j + i/2)}{x(u_j - i/2)} \right]_L^L = \prod_{\substack{j=1 \, \text{or} \, j+1 \neq k \neq j}}^{M_u} \frac{u_j - u_k + i}{u_j - u_k - i} e^{i\theta(u_j,u_k)} \prod_{k=1}^{M_u} e^{i\theta(u_j,u_k)},
$$

$$
\left[ \frac{x(v_j + i/2)}{x(v_j - i/2)} \right]_L^L = \prod_{\substack{j=1 \, \text{or} \, j+1 \neq k \neq j}}^{M_v} \frac{v_j - v_k + i}{v_j - v_k - i} e^{i\theta(v_j,v_k)} \prod_{k=1}^{M_v} e^{i\theta(v_j,v_k)},
$$

and must be supplemented by the momentum constraint

$$
\prod_{j=1}^{M_u} \frac{x(u_j + i/2)}{x(u_j - i/2)} \prod_{j=1}^{M_v} \frac{x(v_j + i/2)}{x(v_j - i/2)} = 1. \tag{2.8}
$$

We have called $u_i$ and $v_i$ the Bethe roots of each $SU(2)$ factor. We also denoted by $M_u$ and $M_v$ the number of $u$ and $v$ roots. Moreover, we introduced the function

$$
x(w) = \frac{w}{2} \left( 1 + \sqrt{1 - 4 \frac{h^2(\lambda)}{w^2}} \right), \tag{2.9}
$$

where $w = u, v$. As we see from (2.7), the dressing factor $e^{i\theta}$ introduces extra self-interactions for the roots and also couples the two $SU(2)$ spin chains at higher loops. Following [9] we define the dressing phase as in [14, 17]

$$
\theta(w_i, w_j) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{r,s}(\lambda) [q_r(w_i)q_s(w_j) - q_s(w_i)q_r(w_j)], \tag{2.10}
$$

where the coefficient functions $\beta_{r,s}(\lambda)$ can be expanded in $\lambda$ as

$$
\beta_{r,s}(\lambda) = \sum_{k=s-1}^{\infty} \beta_{r,s}^{(2k)} \lambda^{2k} \tag{2.11}
$$

and the quantities

$$
q_r(w) = \frac{1}{r - 1} \left( \frac{i}{x(w + i/2)^{r-1}} - \frac{i}{x(w - i/2)^{r-1}} \right) \tag{2.12}
$$

are related to the eigenvalues of the conserved charges of the theory, whose existence is ensured by integrability. In particular, the second charge is the hamiltonian of the
integrable model and is identified with the dilatation operator of the gauge theory. The energy eigenvalues are given by

\[ E = h^2(\lambda) \left( \sum_{j=1}^{M_u} q_2(u_j) + \sum_{j=1}^{M_v} q_2(v_j) \right). \]  

(2.13)

Let’s observe that, for two-impurity states, the Bethe Ansatz is simplified. The momentum constraint requires

\[ w_1 = -w_2 \equiv w, \]

while the Bethe equations reduce to

\[ \left( \frac{w + i/2}{w - i/2} \right)^L = \left( \frac{1 + \sqrt{1 - \frac{4h^2(\lambda)}{(w-i/2)^2}}}{1 + \sqrt{1 - \frac{4h^2(\lambda)}{(w+i/2)^2}}} \right)^L e^{i\theta(w,-w)} \]

for \( M_u = M_v = 1 \), and

\[ \left( \frac{w + i/2}{w - i/2} \right)^{L-1} = \left( \frac{1 + \sqrt{1 - \frac{4h^2(\lambda)}{(w-i/2)^2}}}{1 + \sqrt{1 - \frac{4h^2(\lambda)}{(w+i/2)^2}}} \right)^L e^{i\theta(w,-w)} \]

for \( M_u = 2, M_v = 0 \) or \( M_u = 0, M_v = 2 \). For any fixed \( L \), these equations can be solved order by order in perturbation theory. This is the basic tool for the computation of anomalous dimensions of long operators.

As can be seen from the Bethe equations, the anomalous dimensions of operators will depend, in general, on the coefficients of the function \( h(\lambda) \) and of the dressing phase. It seems that such coefficients cannot be determined using the integrability of the theory only. However, some general considerations on their values can be made, as we now explain. In the case of \( \mathcal{N} = 4 \) SYM theory it was shown [17] that the coefficients of the dressing phase should have a well defined degree of transcendentality in order to preserve the Kotikov-Lipatov transcendentality principle on the scaling function of the theory [46]. Such principle can be generalized to the ABJM case: the scaling function of ABJM theory is prescribed to be [9]

\[ f_{\text{ABJM}}(\lambda) = \frac{1}{2} f_{\text{SYM}}(g) |_{g \to h(\lambda)}, \]

(2.17)
where \( g = \sqrt{\frac{\lambda}{4\pi}} \) in \( \mathcal{N} = 4 \) SYM. Taking the weak coupling expansion for \( f_{\text{SYM}}(g) \) obtained in [17] and inserting the expansion of \( h(\lambda) \), we thus get, up to eight loops,

\[
f_{\text{ABJM}}(\lambda) = 4 \lambda^2 - \left( \frac{4}{3} \pi^2 - 4 h_4 \right) \lambda^4 + \left( \frac{44}{45} \pi^4 - \frac{8}{3} h_4 \pi^2 + 4 h_6 \right) \lambda^6 - 4 \left( \frac{73}{315} \pi^6 - 8 \zeta(3)^2 - \frac{11}{15} h_4 \pi^4 + \frac{1}{3} h_4^2 \pi^2 + \frac{2}{3} h_6 \pi^2 - h_8 + 4 \beta_{2,3}^{(6)} \zeta(3) \right) \lambda^8 + \mathcal{O}(\lambda^{10}).
\]

(2.18)

In deriving (2.18) we have used the fact that \( \beta_{r,s} = 0 \) for \( r + s \) even and that, in our case, the coefficients \( \beta_{r,s}(\lambda) \) should contribute starting at order \( \mathcal{O}(\lambda^{2(r+s-2)}) \) rather than \( \mathcal{O}(\lambda^{2(r+s)-1}) \), as expected from (2.11). See [14, 17] for an explanation. The latter property will be verified, for the \( \beta_{2,3} \) coefficient, in Section 3. The transcendentality principle, extended to ABJM theory, states that:

Assigning degree of transcendentality \( k \) to constants \( \pi^k \) and \( \zeta(k) \), the \( \ell \)-loop contribution to the scaling function \( f_{\text{ABJM}} \) has uniform degree of transcendentality \( \ell - 2 \).

Recalling that \( h_4 = -4 \zeta(2) \), we see that this principle is satisfied up to four loops. Moreover, from the six-loop contribution, we immediately see that \( h_6 \) should have degree four. Starting from eight loops also the unknown dressing phase starts contributing to the scaling function. One very natural way\(^3\) to preserve the transcendentality principle is to conjecture that:

- \( h_{2k} \) has degree of transcendentality \( 2k - 2 \),
- \( \beta_{r,s}^{(2k)} \) has degree of transcendentality \( 2k + 2 - r - s \).

In particular, we notice that the transcendentality principle implies that the leading order coefficient of the dressing phase should be of the form

\[
\beta_{2,3}^{(6)} = a \pi^3 + b \zeta(3),
\]

(2.19)

where \( a \) and \( b \) are some rational numbers. The main goal of this paper is to compute such constants.

\(^3\)We’re neglecting the possibility that \( h_8 \) and \( \beta_{2,3}^{(6)} \) can have higher (or lower) degrees of transcendentality which cancel out between them.
3. Construction of the Dilatation Operator

The dilatation operator is the generator of scaling transformations and measures the scaling dimensions of composite operators. The perturbative expansion of the dilatation operator in the $SU(2) \times SU(2)$ sector is

$$D(\lambda) = L + \sum_{k=1}^{\infty} D_{2k} \lambda^{2k},$$

(3.1)

where the 0-loop term $L$ is proportional to the identity and yields the classical dimension of an operator made up of $2L$ chiral superfields, while the second term is the quantum contribution and yields its anomalous dimension. The latter can be extracted from the perturbative renormalization of the composite operators $O_a$

$$O_a^{\text{ren}} = Z_a^b O_b^{\text{bare}}, \quad Z = 1 + \lambda^2 Z_2 + \lambda^4 Z_4 + \cdots.$$  

(3.2)

The renormalization factor $Z$ is introduced in order to remove UV divergences from correlation functions of operators and can be computed in perturbation theory by means of standard methods. Using dimensional regularization in spacetime dimension $D = 3 - 2\varepsilon$, quantum divergences show up as inverse powers of $\varepsilon$ in the limit $\varepsilon \to 0$. By introducing the 't Hooft mass $\mu$ and the dimensionful combination $\lambda \mu^{2\varepsilon}$ the dilatation operator is then extracted from $Z$ as

$$D = L + \mu \frac{d}{d\mu} \ln Z(\lambda \mu^{2\varepsilon}, \varepsilon) = L + \lim_{\varepsilon \to 0} \left[ 2\varepsilon \lambda \frac{d}{d\lambda} \ln Z(\lambda, \varepsilon) \right].$$

(3.3)

This definition effectively extracts the $1/\varepsilon$ pole of $\ln Z$; the higher order poles must be absent. A full-fledged quantum field theory computation of the dilatation operator can be an extremely difficult task, especially for high loop orders in perturbation theory. Fortunately, things get simplified if we make use of the integrability results.

Here we discuss a general procedure to construct the asymptotic dilatation operator of ABJM theory. It is similar to the procedure used in the case of $\mathcal{N} = 4$ SYM and described in [47, 48, 49, 14, 50]. The idea is the following: starting from an exactly integrable hamiltonian (which is next-to-nearest neighbor for ABJM theory), we deform it with local interactions of range linearly increasing with the perturbative order, obtaining a long-ranged spin chain. At $\ell$ loops the maximum range is $\ell + 1$. When the range of an interaction exceeds the length $2L$ of the spin chain, wrapping interactions typically appear and the Bethe Ansatz breaks down. So, we work with asymptotic states, i.e. states of length $2L > \ell$.

The two-loop dilatation operator [3, 4] is the sum of two XXX+$\frac{1}{4}$ Heisenberg hamiltonians, one living on odd and the other on even sites: this hamiltonian is deformed with interactions of longer range at higher perturbative orders.
The dilatation operator is conveniently written in terms of the basis of permutation structures

\[
\{a_1, \ldots, a_n\} = \sum_{j=1}^{L} P_{2j+a_1,2j+a_1+2} \cdots P_{2j+a_n,2j+a_n+2},
\]  

(3.4)

where \(P_{i,j}\) are permutation operators. Permutation structures represent local interactions of spins summed over all different positions on the spin chain: their properties are collected in Appendix B. Here, we just need to define the number

\[
\mathcal{R} = \max(a_1, \ldots, a_n) - \min(a_1, \ldots, a_n) + 3
\]  

(3.5)

which is the range of the interaction.

In Table 1 we list all the independent permutation structures that appear in the dilatation operator up to eight loops. In order to read a basis for the dilatation operator at \(\ell\) loops, one has to include all the permutation structures up to range \(\mathcal{R} = \ell + 1\), neglecting the ones in parenthesis, because they appear at the next loop order.

In order to construct the \(\ell\)-loop dilatation operator, we first write the most general combination of permutation structures up to range \(\ell + 1\). Then, we fix a large part of the unknown coefficients as follows:

1. Impose hermiticity and parity invariance:

\[
\mathcal{D}_\ell^\dagger = \mathcal{D}_\ell, \quad \mathcal{P} \mathcal{D}_\ell \mathcal{P}^{-1} = \mathcal{D}_\ell;
\]

2. Require that the vacuum state is protected (i.e., it has zero energy):

\[
\mathcal{D}_\ell |0\rangle = 0;
\]

3. Impose the magnon dispersion relation on one-impurity states on odd and even sites (let \(E_\ell\) be the \(\ell\)-loop coefficient of (2.5) in the small \(\lambda\) expansion):

\[
\mathcal{D}_\ell |p\rangle_{o,e} = E_\ell(p) |p\rangle_{o,e};
\]

4. Use the asymptotic Bethe ansatz on two-impurity states in order to fix the remaining parameters.

Point 4 works in the following manner:

• Fix a length of the spin chain, with \(L > \ell/2\), so that wrapping interactions are absent;
Table 1: Permutation structures needed up to eight loops, grouped according to their range \( \mathcal{R} \). Permutation structures of odd (even) range \( \mathcal{R} \) appear at \( \mathcal{R} - 1 \) (\( \mathcal{R} \)) loops, apart from the ones in parenthesis, which appear at the next loop order. Permutation structures contributing at ten loops or beyond are not written. Note that, in order to find the complete basis, one has to add \{\ldots, a + 1, \ldots\} to each permutation structure \{\ldots, a, \ldots\} listed here.

- Find a basis of two-impurity operators of length \( 2L \). Note that there are two types of such bases, as the impurities can be both on even (or odd) sites or one

| \( \mathcal{R} \) | Basis of permutation structures |
|----------------|--------------------------------|
| 1              | \{\}                          |
| 2              | -                             |
| 3              | \{0\}                         |
| 4              | \{0,1\}                       |
| 5              | \{0,2\}, \{2,0\} \(\text{\{0,1,2\}, \{2,1,0\}\})\) |
| 6              | \{0,3\} \{0,3,1\} \{0,2,3\}, \{2,0,3\} \(\text{\{0,1,2,3\}, \{2,1,0,3\}, \{0,3,2,1\}, \{2,3,0,1\}\})\) |
| 7              | \{0,4\} \{0,2,4\}, \{4,2,0\}, \{0,4,2\}, \{2,0,4\} \(\text{\{0,1,4\}, \{0,3,4\}\{(2,0,4,2)\}}\) \(\text{\{0,1,2,4\}, \{2,1,0,4\}, \{4,1,2,0\}, \{4,1,0,2\}\{(0,1,3,4\), \{0,3,1,4\}\}}\) \(\text{\{0,2,3,4\}, \{2,0,3,4\}, \{0,4,3,2\}, \{4,3,2,0\}\}}\) |
| 8              | \{0,5\}, \{0,1,5\}, \{0,4,5\} \{0,2,5\}, \{2,0,5\}, \{0,3,5\}, \{0,5,3\} \{0,1,3,5\}, \{0,3,1,5\}, \{0,1,5,3\}, \{5,3,1,0\} \{0,2,3,5\}, \{2,0,3,5\}, \{0,2,5,3\}, \{2,0,5,3\} \{0,2,4,5\}, \{2,0,4,5\}, \{0,4,2,5\}, \{4,2,0,5\} |
| 9              | \{0,6\} \{0,2,6\}, \{2,0,6\}, \{0,4,6\}, \{0,6,4\} \{0,3,6\} \{0,2,4,6\}, \{0,2,6,4\}, \{0,4,2,6\}, \{2,0,4,6\} \{2,0,6,4\}, \{0,6,4,2\}, \{4,2,0,6\}, \{6,4,2,0\} |
on even and the other on odd sites\(^4\);

- Explicitly diagonalize the matrix representation of \( \mathcal{D}(\lambda) = \sum_{k=1}^{\ell/2} \mathcal{D}_{2k} \lambda^{2k} \) on the basis of length \( 2L \) states;

- Compute the eigenvalues from the asymptotic Bethe equations (2.7) and compare them with those found in the previous step, containing the unknown parameters.

The explicit diagonalization of the dilatation operator and the solution of the Bethe equations can be performed with the help of Mathematica. At the end of the procedure, some parameters may remain unfixed. Some of them are related to similarity transformations of the dilatation operator:

\[
\mathcal{D}' = e^{-i\chi} \mathcal{D} e^{i\chi}.
\] (3.6)

They don’t appear in the spectrum but only affect the eigenvectors. Therefore, they can be regarded as unphysical and their values depend on the renormalization scheme.

Another way to fix the unknown parameters is through conservation of the first higher charge \( Q_3 \), if we assume perturbative integrability. This requires the construction of \( Q_3 \) up to order \( \ell \) with a procedure similar to that described for the dilatation operator, with the difference that \( Q_3^{(\ell)} \) has odd parity and is anti-hermitian. Since \( Q_3 \) has higher maximal range, \( \ell + 2 \), at high loops it can be quite complicated to compute. Some other parameters may depend on the dressing phase and cannot be fixed through the Bethe Ansatz.

We now apply the procedure described above to constrain the form of the dilatation operator up to eighth loops. We immediately get

\[
\mathcal{D}_2 = 2 \{ \} - \{ 0 \} - \{ 1 \}
\] (3.7)

for the two-loop dilatation operator and

\[
\mathcal{D}_4 = 2 (-4 + h_4) \{ \} + (6 - h_4) (\{ 0 \} + \{ 1 \}) - ((\{ 0, 2 \} + \{ 2, 0 \} + \{ 1, 3 \} + \{ 3, 1 \})
\] (3.8)

for the four-loop dilatation operator. We see that (3.7) reproduces the result of [3, 4], while (3.8) reproduces the results\(^5\) of [19] and [22].

\(^4\)For the same reason, we have seen in section 2 that there are two types of Bethe equations for two-impurity states.

\(^5\)The computation of [19] followed a procedure similar to ours, up to point 3. This fixes all but one coefficient in the dilatation operator, namely the coefficient of \( \{ 0, 1 \} + \{ 1, 2 \} \). We simply use point 4 of our procedure on length six two-impurity states to prove that it vanishes, without computing any Feynman diagram.
The six-loop dilatation operator was computed in [23] through Feynman diagrams. We apply our technique to rederive it. Assuming that the dressing phase is absent at this loop order, following [14, 17], we can completely fix the dilatation operator with our procedure. If we don’t assume that the dressing phase is absent, a single input from the results of [23] is sufficient to determine the full dilatation operator. We follow this latter approach. First of all we write the six-loop dilatation operator in the basis of permutation structures. After having imposed hermiticity and parity invariance, it is given by

\[
D_6 = a \{ \} + D_{6,\text{even}} + D_{6,\text{odd}} + D_{6,\text{mixed}},
\]

where

\[
D_{6,\text{even}} = b \{0\} + d (\{0, 2\} + \{2, 0\}) + m \{0, 4\} + l (\{0, 2, 4\} + \{4, 2, 0\}) + \bar{l} (\{2, 0, 4\} + \{0, 4, 2\})
\]

\[
+ i c_1 (\{2, 0, 4\} - \{0, 4, 2\}),
\]

\[
D_{6,\text{odd}} = D_{6,\text{even}} (\{\ldots, a, \ldots\} \leftrightarrow \{\ldots, a+1, \ldots\}),
\]

\[
D_{6,\text{mixed}} = c (\{0, 1\} + \{1, 2\}) + e (\{0, 3\} + \{1, 4\}) + f (\{0, 1, 2\} + \{2, 1, 0\} + \{1, 2, 3\} + \{3, 2, 1\}) + g (\{0, 1, 3\} + \{0, 3, 1\} + \{1, 3, 4\} + \{3, 1, 4\}) + \bar{h}_4 (\{0, 2, 3\} + \{2, 0, 3\} + \{1, 4, 2\} + \{1, 2, 4\})
\]

\[
+ i c_2 (\{0, 1, 3\} - \{0, 3, 1\} - \{1, 3, 4\} + \{3, 1, 4\} - \{0, 2, 3\} + \{2, 0, 3\} - \{1, 4, 2\} + \{1, 2, 4\}),
\]

where all parameters are real. Zero-energy of the vacuum gives the equation

\[
a + 2b + 4d + 4l + 4\bar{l} + 2m + 2c + 2e + 4f + 8g = 0.
\]

We then compute the action of \(D_6\) on \(|p\rangle_s\) and \(|p\rangle_o\) and compare the result with the six loop coefficient of the weak-coupling expansion of (2.5) obtaining the equations

\[
l = -2,
\]

\[
b = -48 - 2c - 2e - 2f - 4g + 12h_4 - h_6 + 2\bar{l} - 2m,
\]

\[
d = 12 - f - 2g - 2h_4 - 2\bar{l}.
\]
Now we consider two-impurity states. In order to avoid wrapping, we have to start from $L = 4$. For $L = 4$ (length 8 states), comparing the eigenvalues of the dilatation operator with the solutions of the Bethe equations we get

$$\tilde{l} = 0, \quad 2m = 4 - 3\beta_{2,3}^{(4)}, \quad f = 0, \quad g = 0, \quad c + e = 0. \quad (3.15)$$

The results of [23] (obtained with an independent approach) imply, in our notations, that $m = 2$. From (3.15) we thus see that $\beta_{2,3}^{(4)}$ must vanish. For $L = 5$ (length 10 states) we get

$$c = 0, \quad e = 0. \quad (3.16)$$

We therefore have explicitly shown that, up to irrelevant parameters, $D_{6,mixed} = 0$. It means that the two types of magnons don’t interact at six loops, i.e. the two $SU(2)$ factors are decoupled, as claimed in [9].

The six-loop dilatation operator is then given by

$$D_6 = 2 (30 - 8 h_4 + h_6) \{ \} + (-52 + 12 h_4 - h_6) (\{ 0 \} + \{ 1 \})$$
$$+ 2 (\{ 0, 4 \} + \{ 1, 5 \}) + (12 - 2 h_4) (\{ 0, 2 \} + \{ 2, 0 \} + \{ 1, 3 \} + \{ 3, 1 \})$$
$$- 2 (\{ 0, 2, 4 \} + \{ 4, 2, 0 \} + \{ 1, 3, 5 \} + \{ 5, 3, 1 \})$$
$$+ i \epsilon_1 (\{ 2, 0, 4 \} - \{ 0, 4, 2 \} - \{ 1, 5, 3 \} + \{ 3, 1, 5 \})$$
$$+ i \epsilon_2 (\{ 0, 1, 3 \} - \{ 0, 3, 1 \} - \{ 1, 3, 4 \} + \{ 3, 1, 4 \})$$
$$- \{ 0, 2, 3 \} + \{ 2, 0, 3 \} - \{ 1, 4, 2 \} + \{ 1, 2, 4 \} \). \quad (3.17)$$

Let’s observe that, setting the unphysical coefficients $\epsilon_1$ and $\epsilon_2$ to zero, $D_6$ coincides with the operator presented in [23]. The coefficient $\beta_{2,3}^{(4)}$, which can be a priori present according to (2.11), actually vanishes. Therefore, the first contribution to the dressing phase should be searched at eight loops. We are thus led to consider the eight-loop dilatation operator: since it isn’t known at present, we now use our procedure to constrain it as much as possible.

The permutation structures contributing to $D_8$ are listed in Table 1. After imposing hermiticity and parity invariance, the eight-loop dilatation operator can be written (with real coefficients) as

$$D_8 = a \{ \} + D_{8,even} + D_{8,odd} + D_{8,mixed} . \quad (3.18)$$
where

\[ D_{8,\text{even}} = b \{0\} + c (\{0,2\} + \{2,0\}) + d \{0,4\} + e_1 (\{0,2,4\} + \{4,2,0\}) + e_2 (\{0,4,2\} + \{2,0,4\}) + i \epsilon_3 (\{0,4,2\} - \{2,0,4\}) + f \{0,6\} + g (\{0,2,6\} + \{2,0,6\} + \{0,4,6\} + \{0,6,4\}) + i e_1 (\{0,2,6\} - \{2,0,6\} - \{0,4,6\} + \{0,6,4\}) + l_1 (\{0,2,4,6\} + \{6,4,2,0\}) + l_2 (\{0,2,6,4\} + \{2,0,4,6\} + \{0,6,4,2\} + \{4,2,0,6\}) + i e_2 (\{0,2,6,4\} - \{2,0,4,6\} + \{0,6,4,2\} - \{4,2,0,6\}) + l_3 (\{0,4,2,6\} + \{2,0,6,4\}) + m \{2,0,4,2\} \]

(3.19)

\[ D_{8,\text{mixed}} = \alpha_1 \{0,1\} + \alpha_2 (\{0,1,2\} + \{2,1,0\}) + \alpha_3 \{0,3\} + \alpha_4 (\{0,1,3\} + \{0,3,1\} + \{0,2,3\} + \{2,0,3\}) + i \epsilon_{m_1} (\{0,1,3\} - \{0,3,1\} - \{0,2,3\} + \{2,0,3\}) + \alpha_5 (\{0,1,2,3\} + \{2,3,0,1\}) + \alpha_6 (\{2,1,0,3\} + \{0,3,2,1\}) + i \epsilon_{m_2} (\{2,1,0,3\} - \{0,3,2,1\}) + \alpha_7 (\{0,1,4\} + \{0,3,4\}) + \alpha_8 (\{0,1,2,4\} + \{4,1,2,0\} + \{0,2,3,4\} + \{4,3,2,0\}) + i \epsilon_{m_3} (\{0,1,2,4\} - \{4,1,2,0\} - \{0,2,3,4\} + \{4,3,2,0\}) + \alpha_9 (\{2,1,0,4\} + \{4,1,0,2\} + \{2,0,3,4\} + \{0,4,3,2\}) + i \epsilon_{m_4} (\{2,1,0,4\} - \{4,1,0,2\} + \{2,0,3,4\} - \{0,4,3,2\}) + \alpha_{10} (\{0,1,3,4\} + \{0,3,1,4\}) + \alpha_{11} \{0,5\} + \alpha_{12} \{0,1,5\} + \{0,4,5\} + \alpha_{13} \{0,2,5\} + \{2,0,5\} + \{0,3,5\} + \{0,5,3\} + i \epsilon_{m_5} (\{0,2,5\} - \{2,0,5\} - \{0,3,5\} + \{0,5,3\}) + \alpha_{14} (\{0,1,3,5\} + \{4,2,0,5\} + \{5,3,1,0\} + \{0,2,4,5\}) + i \epsilon_{m_6} (\{0,1,3,5\} + \{4,2,0,5\} - \{5,3,1,0\} - \{0,2,4,5\}) + \alpha_{15} (\{0,3,1,5\} + \{0,1,5,3\} + \{2,0,4,5\} + \{0,4,2,5\}) + i \epsilon_{m_7} (\{0,3,1,5\} - \{0,1,5,3\} + \{2,0,4,5\} - \{0,4,2,5\}) + \alpha_{16} (\{0,2,3,5\} + \{2,0,5,3\}) + \alpha_{17} (\{2,0,3,5\} + \{0,2,5,3\}) + i \epsilon_{m_8} (\{2,0,3,5\} - \{0,2,5,3\}) + \alpha_{18} \{0,3,6\} + (\ldots, a \leftrightarrow a + 1, \ldots) \]
Requiring that the vacuum state is protected we have one condition on the coefficients. If we then impose the eight-loop dispersion relation on one-magnon states we obtain four more conditions, which in particular fix the coefficient \( l_1 = -5 \). In order to fix other parameters we consider two-impurity states: for sufficiently long states in order to avoid wrapping effects, with the help of Mathematica, we explicitly diagonalize the dilatation operator and compare its eigenvalues with those obtained from the perturbative solutions of the asymptotic Bethe equations. For \( L = 5, 6, 7 \) (length 10, 12, 14 states) we find 12 more conditions, while higher length states give no more equations. So the point 4 of the procedure described above isn’t sufficient to completely fix the dilatation operator. Nevertheless, we stress that an important result already follows from the equations we could derive from point 4: a non vanishing\(^6\) coefficient \( \beta_{2,3}^{(6)} \) for the dressing phase would imply that the \( \alpha_i \) coefficients, appearing in \( D_{8,mixed} \), cannot be all zero. This proves that \( D_{8,mixed} \neq 0 \). In principle, the undetermined coefficients could be fixed imposing the commutation of \( D \) with the third charge \( Q_3 \); this is laborious and we won’t do it, since it isn’t necessary for the computations of the dressing phase, as we will see in the next chapter. We quote here the part of the dilatation operator which acts on even sites only and exchanges the fields inside the operator in a maximal way:

\[
D_{8,mr} = + m \{2, 0, 4, 2\} \\
+ l_3 (\{0, 4, 2, 6\} + \{2, 0, 6, 4\}) \\
+ \left( -1 + \frac{1}{2}m + i \epsilon_{2a} + \frac{1}{4} \beta \right) (\{0, 2, 6, 4\} + \{0, 6, 4, 2\}) \\
+ \left( -1 + \frac{1}{2}m - i \epsilon_{2a} + \frac{1}{4} \beta \right) (\{2, 0, 4, 6\} + \{4, 2, 0, 6\}) \\
- 5 (\{0, 2, 4, 6\} + \{6, 4, 2, 0\})
\]

where the coefficient \( \epsilon_{2a} \) corresponds to similarity transformations and depends on the particular renormalization scheme used. The dressing phase enters also \( D_{8,mr} \) through the parameter \( \beta = \beta_{2,3}^{(6)} \). The form of (3.21) is the same of the \( \mathcal{N} = 4 \) SYM counterpart in the four-loop dilatation operator [35]. It is interesting to note that the coefficients \( \alpha_i \) and the coefficients of \( h(\lambda) \) don’t appear in (3.21). However, the rest of the dilatation operator contains also these parameters: in general, the “mixed” hamiltonian acts coupling the two types of magnons; this is a completely new feature of ABJM theory which appears at eight loops, making this theory substantially different from the \( \mathcal{N} = 4 \) SYM theory at four loops.

\(^6\)We will see in Section 4 that this is, indeed, the case.
4. Computation of the Dressing Phase

In this Section we proceed to the computation of the leading-order coefficient of the dressing phase, \( \text{i.e.} \) the parameter \( \beta_{2,3}^{(6)} \) which appears in the eight-loop dilatation operator. We determine it by direct field-theory calculations, in a similar fashion to the \( \mathcal{N} = 4 \) SYM case [35]. Here, we use \( \mathcal{N} = 2 \) superspace methods for the evaluation of Feynman diagrams. We refer to Appendix A for a very brief review of such methods.

We first rewrite the dilatation operator in the basis of chiral functions introduced in Appendix B, which is directly related to the chiral structure of the supergraphs. The part of the dilatation operator leading to maximal reshuffling of spins\(^7\) preserves the form of (3.21):

\[
\mathcal{D}_{8,\text{mr}} = m \chi(2, 0, 4, 2) \\
+ l_3 \left[ \chi(0, 4, 2, 6) + \chi(2, 0, 6, 4) \right] \\
+ \left( -1 + \frac{1}{2} m + i \epsilon_{2a} + \frac{1}{4} \beta \right) \left[ \chi(0, 2, 6, 4) + \chi(0, 6, 4, 2) \right] \\
+ \left( -1 + \frac{1}{2} m - i \epsilon_{2a} + \frac{1}{4} \beta \right) \left[ \chi(2, 0, 4, 6) + \chi(4, 2, 0, 6) \right] \\
- 5 \left( \chi(0, 2, 4, 6) + \chi(6, 4, 2, 0) \right).
\]

This part of the dilatation operator is sufficient to compute the parameter \( \beta_{2,3}^{(6)} \): \( \mathcal{D}_{8,\text{mr}} \) has just the same form as the maximal reshuffling hamiltonian of \( \mathcal{N} = 4 \) SYM [35]. The coefficients of the five terms in (4.1) can be computed from the supergraphs contributing to \( \chi(2, 0, 4, 2) \), \( \chi(0, 4, 2, 6) \), \( \chi(0, 2, 6, 4) \), \( \chi(2, 0, 4, 6) \), \( \chi(0, 2, 4, 6) \). Only one supergraph contributes to each term and it contains only scalar interactions. The coefficient of the last term was already fixed by imposing the dispersion relation on one-magnon states. The first four terms contain precisely four undetermined parameters, namely \( m \), \( l_3 \), \( \epsilon_{2a} \) and \( \beta \): in order to compute such parameters we need to evaluate the supergraphs in Figure 1 and isolate the overall UV divergence by subtracting all their UV subdivergences.

First of all, we perform standard D-algebra manipulations (see Appendix A) to reduce the supergraphs to ordinary momentum-space integrals \( I_{8x} \). Such integrals are displayed in Figure 2. The Feynman rules imply that the color factor is always \( N^8 \), which combines with \( \left( \frac{4\pi}{k} \right)^8 \) to give the right power of the 't Hooft coupling, \( \text{i.e.} \) \( \lambda^8 \) (the \( (4\pi)^8 \) factor will be simplified by the factor \( \frac{1}{(4\pi)^8} \) present in each integral \( I_{8x} \)). The pole parts of the integrals with the subdivergences subtracted are denoted by \( \bar{I}_{8x} \). The maximal reshuffling supergraphs, thus, evaluate to:

\(^7\)We give here only the terms which act non-trivially on even sites.
Figure 1: Supergraphs corresponding to the maximal reshuffling terms of the eight-loop dilatation operator of ABJM theory. From left to right, and together with their reflected diagrams, they are associated to each line in (4.1). The horizontal bar represents the operator itself, or analogously, the spin chain.

\[
S_a = \begin{array}{c}
\text{Diagram}
\end{array} = \lambda^8 (4\pi)^8 \bar{I}_{8a} |_{\frac{1}{2}} \chi(2, 0, 4, 2), \quad (4.2)
\]

\[
S_b = \begin{array}{c}
\text{Diagram}
\end{array} = \lambda^8 (4\pi)^8 \bar{I}_{8b} |_{\frac{1}{2}} \chi(0, 4, 2, 6), \quad (4.3)
\]

\[
S_c = \begin{array}{c}
\text{Diagram}
\end{array} = \lambda^8 (4\pi)^8 \bar{I}_{8c} |_{\frac{1}{2}} \chi(0, 2, 6, 4), \quad (4.4)
\]

\[
S_d = \begin{array}{c}
\text{Diagram}
\end{array} = \lambda^8 (4\pi)^8 \bar{I}_{8d} |_{\frac{1}{2}} \chi(2, 0, 4, 6), \quad (4.5)
\]

\[
S_e = \begin{array}{c}
\text{Diagram}
\end{array} = \lambda^8 (4\pi)^8 \bar{I}_{8e} |_{\frac{1}{2}} \chi(0, 2, 4, 6). \quad (4.6)
\]

The subtracted integrals are computed in Appendix D and are given by:

\[
\bar{I}_{8a} = \begin{array}{c}
\text{Diagram}
\end{array} = \frac{1}{(4\pi)^8} \left(-\frac{1}{3072 \varepsilon^4} + \frac{1}{192 \varepsilon^3} - \frac{1}{48 \varepsilon^2} + \frac{y}{\varepsilon}\right), \quad (4.7)
\]

\[
\bar{I}_{8b} = \begin{array}{c}
\text{Diagram}
\end{array} = \frac{1}{(4\pi)^8} \left(-\frac{5}{6144 \varepsilon^4} + \frac{5}{768 \varepsilon^3} - \frac{1}{384 \varepsilon^2} - \frac{1}{32 \varepsilon}\right),
\]

\[
\bar{I}_{8c} = \begin{array}{c}
\text{Diagram}
\end{array} = \frac{1}{(4\pi)^8} \left(-\frac{1}{2048 \varepsilon^4} + \frac{1}{128 \varepsilon^3} - \frac{3}{64 \varepsilon^2} + \frac{5}{192 \varepsilon}\right),
\]
\[
\tilde{I}_{8d} = \left( -\frac{1}{2048} + \frac{1}{192} \varepsilon^2 - \frac{11}{192} \right),
\]
\[
\tilde{I}_{8e} = \left( -\frac{1}{6144} + \frac{1}{256} \varepsilon^3 - \frac{19}{384} \varepsilon^2 + \frac{5}{16} \varepsilon \right).
\]
For their evaluation, we used the \textit{Gegenbauer polynomials x-space technique} (GPXT). We briefly review this technique in Appendix C.

We could analytically compute all the integrals in (4.7) apart from the $\frac{1}{\varepsilon^4}$ pole of $I_{8a}$, denoted by $y$ in the (4.7): in this case GPXT seems to be quite inefficient, so we followed a different strategy. We evaluated the integral numerically through Mellin-Barnes techniques. Remarkably, guided by the transcendentality principle, it was possible to extract the corresponding analytical result from the numerical one by means of the PSLQ algorithm [51]. It is given by
\[
y = -\frac{5}{32} + \frac{1}{8} \zeta(3).
\]
All the details are given in Appendix D.

We can write now the final results for the supergraphs contributing to maximal reshuffling terms of the dilatation operator:
\[
S_a = \lambda^8 \left( -\frac{5}{32} + \frac{1}{8} \zeta(3) \right) \chi(2, 0, 4, 2),
S_b = -\lambda^8 \frac{1}{32} \chi(0, 4, 2, 6),
S_c = \lambda^8 \frac{5}{192} \chi(0, 2, 6, 4),
S_d = -\lambda^8 \frac{11}{192} \chi(2, 0, 4, 6),
S_e = \lambda^8 \frac{5}{16} \chi(0, 2, 4, 6),
\]
According to the definition (3.3) of the dilatation operator, the corresponding contributions to the renormalization factor are obtained multiplying these values by -16. At this point, we can find the unknown parameters appearing in (4.1). Our results are 

\[ m = \frac{5}{2} - 2\zeta(3), \quad l_3 = \frac{1}{2}, \quad \beta = 4\zeta(3), \quad \epsilon_{2a} = \frac{2}{3}i. \] (4.10)

In particular we have computed the value of the leading order coefficient of the dressing phase, which is \[ \beta_{2,3}^{(6)} = 4\zeta(3). \] It is the same value found in \( \mathcal{N} = 4 \) SYM for the \( \beta_{2,3}^{(3)} \) coefficient. In the Conclusions we comment on this result.

5. Conclusions

In this paper we have investigated the dressing phase of ABJM theory at weak coupling. It appears in the Bethe equations, as well as in the dilatation operator, starting at eight loops. A simple procedure based on the Bethe Ansatz allowed us to construct the asymptotic dilatation operator in the \( SU(2) \times SU(2) \) sector up to six loops. We verified that the dressing phase is not present up to this order. Furthermore, we could largely constrain the eight-loop dilatation operator and show that the two \( SU(2) \) sectors are coupled starting at this order. In this case we focused on the terms corresponding to maximal interactions: they contain the unknown parameter \( \beta_{2,3}^{(6)} \), which is the leading order coefficient of the dressing phase. Thanks to superspace techniques and with the help of the transcendentality principle, we could directly compute its value from Feynman supergraphs. A great simplification follows from the fact that we considered maximal interactions: we needed to compute a very small number of Feynman diagrams (in fact, just three are necessary to completely fix \( \beta_{2,3}^{(6)} \)) which contain only scalar interactions. The result we found is \( \beta_{2,3}^{(6)} = 4\zeta(3) \) and coincides with the value of the leading coefficient of the dressing phase of \( \mathcal{N} = 4 \) SYM. In that situation, it was argued [17] that the weak-coupling expansion of \( \beta_{r,s}(\lambda) \) can be extrapolated by analytical continuation of the all-loop strong-coupling coefficient functions [52, 16]

\[ \beta_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{r+s-n-1}, \]

with

\[ c_{r,s}^{(n)} = \frac{1 - (-1)^{r+s}\zeta(n)}{2(-2\pi)^n \Gamma(n-1)} \left( r - 1 \right)(s-1) \frac{\Gamma\left[\left(\frac{1}{2}(s+r+n-3)\right)\right] \Gamma\left[\left(\frac{1}{2}(s-r+n-1)\right)\right]}{\Gamma\left[\left(\frac{1}{2}(s+r-n+1)\right)\right] \Gamma\left[\left(\frac{1}{2}(s-r-n+3)\right)\right]}, \] (5.1)

The presence of the factor \( i \) in the unphysical \( \epsilon_{2a} \) seems to spoil hermiticity of the dilatation operator, see [35] for further comments on this. Moreover, the coefficient of the chiral function \( \chi(0,2,4,6) \) turns out to be -5, as it was already known from the previous section: it, thus, constitutes a non trivial check of the whole procedure.
to negative values of $n$:

$$\beta_{r,s}(g) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)} g^{r+s+n-1}, \quad (5.2)$$

where $g = \sqrt{\lambda}$. So, the dressing phase coefficients were conjectured to be

$$\beta_{r,s}^{(r)} = -c_{r,s}^{(r+s-2r-1)}. \quad (5.3)$$

In particular, $c_{2,3}^{(-2)} = -4\zeta(3)$, so that the leading order coefficient of the SYM dressing phase was $\beta_{2,3}^{(3)} = 4\zeta(3)$. It was shown that this guess is indeed correct [35].

Since in the Bethe Ansatz proposal of [9] the strong coupling limit of the dressing phase is the same of the $\mathcal{N} = 4$ SYM case, we can suppose, by analogy of the $\mathcal{N} = 4$ SYM case and with the usual replacement $g \rightarrow h(\lambda)$, that in ABJM we have

$$\beta_{r,s}(\lambda) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)} h(\lambda)^{r+s+n-1}. \quad (5.4)$$

For $r = 2$, $s = 3$, we see that we should obtain (since $c_{2,3}^{(-1)} = c_{2,3}^{(-3)} = 0$)

$$\beta_{2,3}(\lambda) = -h_{2}^{3}c_{2,3}^{(-2)} \lambda^{6} - \left(h_{2}^{4}c_{2,3}^{(-4)} + 3h_{4}h_{2}^{2}c_{2,3}^{(-2)}\right) \lambda^{8} + \cdots \quad (5.5)$$

We observe that, since the first weak coupling coefficient of $h(\lambda)$ is $h_{2} = 1$, the leading order contribution to ABJM dressing phase is just $\beta_{2,3}^{(6)} = 4\zeta(3)$, which is precisely the same value obtained in $\mathcal{N} = 4$ SYM. At higher orders, however, the $c_{r,s}^{(-n)}$ coefficients mix with the non trivial coefficients of $h(\lambda)$ yielding dressing phase coefficients which differ from the $\mathcal{N} = 4$ SYM ones. We stress that, in our context, this is merely a conjecture: anyway, we computed the coefficient $\beta_{2,3}^{(6)}$ by independent field theory techniques and we found that the above prediction is indeed correct. It would be interesting to further test the validity of (5.4) beyond the leading order, through direct computations: hopefully, restricting to the case of maximal interactions, the calculations will be simplified and have a chance of being performed with standard field theory techniques.

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A. ABJM Theory in $\mathcal{N} = 2$ superspace

In this Appendix we briefly review the $\mathcal{N} = 2$ superspace formulation of ABJM theory. This was first given in [53], but we follow the notations used in [21] which are adapted from the ones of [54]. ABJM theory has two $\mathcal{N} = 2$ vector supermultiplets, $V$ and $\hat{V}$, with $V$ transforming in the adjoint of the first $U(N)$ factor and $\hat{V}$ in the adjoint of the second $U(N)$ factor of the gauge group. In order to extend the supersymmetry to $\mathcal{N} = 6$, the ABJM action also contains two sets of chiral matter superfields, $Z_A$ and $W_A$ with $A = 1, 2$. $Z_A$ and $W_A$ transform respectively in the $(2, 1)$ and $(1, 2)$ of the global $SU(2) \times SU(2)$ flavour group. Moreover, they transform in the bifundamental representations $(N, \bar{N})$ and $(\bar{N}, N)$ of $U(N) \times U(N)$. The gauge fixed ABJM action in $\mathcal{N} = 2$ superspace reads

$$
S = \frac{k}{4\pi} \left\{ \int d^3x \, d^4\theta \, \int_0^1 dt \, \text{Tr} \left( V \bar{D}^\alpha e^{-tV} D_\alpha e^{tV} - \hat{V} \bar{D}^\alpha e^{-t\hat{V}} D_\alpha e^{t\hat{V}} \right) 
+ \int d^3x \, d^4\theta \, \text{Tr} \left( Z_A e^{\hat{V}} Z^A e^{-\hat{V}} + W^B e^{\hat{V}} W_B e^{-\hat{V}} \right) 
+ i \frac{1}{2} \left[ \int d^3x \, d^2\theta \, \epsilon_{AC} \epsilon^{BD} \text{Tr} Z_A W_B Z^C W_D + \int d^3x \, d^2\theta \, \epsilon^AC \epsilon^{BD} \text{Tr} \bar{Z}_A \bar{W}_B \bar{Z}^C \bar{W}^D \right] 
+ \text{gauge fixing and ghost terms} \right\}. 
$$

(A.1)

The first line contains the Chern-Simons action, the second line contains the kinetic term of the matter fields and their coupling with gauge fields, while the third line is the superpotential.

The three-dimensional, $\mathcal{N} = 2$ superspace spinor covariant derivatives $D_\alpha, \bar{D}_\alpha$ satisfy the algebra

$$
\{D_\alpha, D_\beta\} = \{\bar{D}_\alpha, \bar{D}_\beta\} = 0, \quad \{D_\alpha, \bar{D}_\beta\} = p_{\alpha\beta}. 
$$

(A.2)

The metric $\epsilon_{AB}$ for the $SU(2)$ flavor indices is given by

$$
\epsilon_{12} = 1, \quad \epsilon^{12} = 1, \quad \epsilon^{AC} \epsilon_{BD} = \delta^A_C \delta^B_D - \delta^A_D \delta^B_C. 
$$

(A.3)

For the integration over the superspace our conventions are $\int d^2\theta = \frac{1}{2} \partial^\alpha \partial_\alpha$, $\int d^2\bar{\theta} = \frac{1}{2} \bar{\partial}^\alpha \bar{\partial}_\alpha$ and $\int d^4\theta = \int d^2\theta d^2\bar{\theta}$, such that

$$
\int d^3x \, d^2\theta = \int d^3x \, D^2|_{\theta = \bar{\theta} = 0}, \quad \int d^3x \, d^2\bar{\theta} = \int d^3x \, \bar{D}^2|_{\theta = \bar{\theta} = 0}, \quad \int d^3x \, d^4\theta = \int d^3x \, D^2 \bar{D}^2|_{\theta = \bar{\theta} = 0}. 
$$

(A.4)
The $\theta$-space $\delta$-function is given by

$$\delta^4(\theta - \theta') = (\theta - \theta')^2(\bar{\theta} - \bar{\theta}')^2.$$  \hspace{1cm} (A.5)

We now give the Euclidean Feynman rules  \textit{(i.e.} we have Wick-rotated to $e^S$ in the path integral\textit{)} of the theory, relevant for the computations of the diagrams in Figure 1. The chiral field propagators are

$$\langle \bar{Z}_B(p) Z_A(-p) \rangle = \langle \bar{W}_B(p) W_A(-p) \rangle = \frac{4\pi}{k}\frac{\delta^B_A}{p^2}\delta^4(\theta_1 - \theta_2), \hspace{1cm} (A.6)$$

where diagonality in the gauge indices have been suppressed. The vertices are obtained by taking the functional derivatives of the Wick rotated action w.r.t. the corresponding superfields. When a functional derivative w.r.t. the (anti)-chiral superfields is taken, factors of $(D^2)\bar{D}^2$ are generated in the vertices. We need only the quartic superpotential vertices:

$$\begin{align*}
\begin{array}{c}
\bar{Z}_A
\end{array}
\begin{array}{c}
\bar{Z}_A
\end{array}
\begin{array}{c}
\bar{Z}_A
\end{array}
\begin{array}{c}
\bar{Z}_A
\end{array} & = i\epsilon^{AC}\epsilon_{BD} \frac{k}{4\pi} \left[ \text{Tr}(\bar{B}^a B_a B^a B_a^\perp) - \text{Tr}(\bar{B}^a B_a B^a B_a^\perp) \right] \hspace{1cm} (A.7) \\
\begin{array}{c}
\bar{W}_A
\end{array}
\begin{array}{c}
\bar{W}_A
\end{array}
\begin{array}{c}
\bar{W}_A
\end{array}
\begin{array}{c}
\bar{W}_A
\end{array} & = i\epsilon_{AC}\epsilon^{BD} \frac{k}{4\pi} \left[ \text{Tr}(\bar{B}_a B_a \bar{B}_a B^d) - \text{Tr}(\bar{B}_a B_a \bar{B}_a B^d) \right]. \hspace{1cm} (A.8)
\end{align*}$$

where the color indices are labeled $(a, b, c, d)$ counter-clockwise starting with the leg in the upper left corner. Note also that, in a standard way, one of the $(D^2)\bar{D}^2$ factors has been absorbed into the (anti)chiral integration such that the integration measure of the (anti)chiral vertex is promoted to the full superspace measure. We have introduced matrices $\bar{B}_a^a$ and $B_a$, with underlined $a = 1, \cdots, N^2$ indices that transform in the $(N, \bar{N})$ of the gauge group $U(N) \times U(N)$.

\section*{B. Permutation structures}

The spin-chains which arise in the study of ABJM theory at $\ell$ loops in the $SU(2)\times SU(2)$ sector are long-range deformations of the alternating Heisenberg spin chain with next-to-nearest neighbor interactions. The interactions among spins are represented by products of permutations of next-to-neighbor spins, therefore we introduce the permutation structures as \cite{19}

$$\{a_1, \cdots, a_n\} = \sum_{j=1}^{L} P_{2j+a_1, 2j+a_1+2} \cdots P_{2j+a_n, 2j+a_n+2}, \hspace{1cm} (B.1)$$
where the permutation operators $P_{i,j}$ exchange spins at sites $i$ and $j$. Indices are understood modulo $2L$. If $n = 0$, we denote $\{\} = L$. The range $\mathcal{R}$ of a permutation structure is

$$\mathcal{R} = \max(a_1, \ldots, a_n) - \min(a_1, \ldots, a_n) + 3. \quad (B.2)$$

The integer $n$ is called length of the permutation structure and, in terms of the associated Feynman diagrams, it coincides with the number of chiral and antichiral vertices. The permutation structures satisfy the following relations:

\[
\begin{align*}
\{ \ldots, a, a, \ldots \} &= \{ \ldots, \ldots \}; \\
\{ \ldots, a, b, \ldots \} &= \{ \ldots, b, a, \ldots \}, \quad \text{if } |b - a| \neq 2; \\
\{a, b, \ldots\} &= \{a + 2m, b + 2m, \ldots\}, \quad m = 0, 1, 2, \ldots \quad (B.3) \\
\{ \ldots, a + 2, a, \ldots \} &= \{ \ldots, a, a + 2, \ldots \} + \{ \ldots, a + 2, a, \ldots \} - \{ \ldots, a, \ldots \} \\
&\quad - \{ \ldots, a + 2, \ldots \} + \{ \ldots, \ldots \}.
\end{align*}
\]

Under hermitian conjugation and parity, they transform as

\[
\begin{align*}
\{a_1, \ldots, a_n\}^\dagger &= \{a_n, \ldots, a_1\}; \\
\mathcal{P}\{a_1, \ldots, a_n\}\mathcal{P}^{-1} &= \{-a_1, \ldots, -a_n\}. \quad (B.4)
\end{align*}
\]

The permutation structures represent a convenient basis for the dilatation operator. Nevertheless, when dealing with Feynman diagrams, it is useful to use a different basis, namely the basis of chiral functions, defined in terms of permutation structures as [55, 56, 50]

\[
\begin{align*}
\chi() &= \{ \}; \\
\chi(a) &= \{ a \} - \{ \}; \\
\chi(a, b) &= \{ a, b \} - \{ a \} - \{ b \} + \{ \}; \\
\chi(a, b, c) &= \{ a, b, c \} - \{ a, b \} - \{ a, c \} - \{ b, c \} + \{ a \} + \{ b \} + \{ c \} - \{ \}; \\
\chi(a, b, c, d) &= \{ a, b, c, d \} - \{ a, b, c \} - \{ a, b, d \} - \{ a, c, d \} - \{ b, c, d \} \\
&\quad + \{ a, b \} + \{ a, c \} + \{ a, d \} + \{ b, c \} + \{ b, d \} + \{ c, d \} \\
&\quad - \{ a \} - \{ b \} - \{ c \} - \{ d \} + \{ \}.
\end{align*}
\]

These functions are directly related to the chiral structure of a supergraph and precisely describe the flavor flow of fields inside (2.1) under the action of the interaction. All diagrams contributing to the same chiral function share the same chiral structure. For the maximal interactions encountered in this work, no vector interactions can be present and the chiral structure alone determines the unique diagram which contributes. The chiral structure of the function $\chi(a_1, \ldots, a_n)$ contains $n$ chiral vertices and $n$ antichiral
vertices, connected by $\langle Z\bar{Z} \rangle$ or $\langle W\bar{W} \rangle$ propagators. Each pair of chiral and antichiral vertices (plus the propagator connecting them) describes an elementary permutation of next-to-nearest neighbors.

C. The Gegenbauer Polynomial $x$-Space Technique

In this Appendix we review the Gegenbauer Polynomial $x$-Space Technique (GPXT), which has been applied to find the UV divergent part of the Feynman integrals needed in the present work. The technique was introduced in [57] and developed in [58, 59]. See also [60, 50] for useful reviews. All the integrals are computed using dimensional regularization in Euclidean space of dimension

$$D = 2(\lambda + 1),$$

where

$$\lambda = \frac{1}{2} - \varepsilon$$

in order to get three-dimensional space in the $\varepsilon \to 0$ limit.

In our computation, a generic Wick-rotated $\ell$-loop integral in momentum space has always the form

$$I_\ell = \frac{1}{(2\pi)^\ell D} \int \frac{d^Dk_1 \cdots d^Dk_\ell}{\Pi_1 \cdots \Pi_P},$$

where $P$ is the number of propagators $\Pi_i$, which, in general, depend on the loop momenta $k_1, \ldots, k_\ell$ and the external momenta $p_1, \ldots, p_e$. We always consider massless propagators.

A propagator with weight $\alpha$ in momentum space is Fourier-transformed to coordinate space according to the following formula

$$\frac{1}{k^{2\alpha}} = \frac{\Gamma(\lambda + 1 - \alpha)}{\Gamma(\alpha)\pi^{\lambda+1}} \int \frac{d^Dx e^{2ikx}}{x^{2(\lambda+1-\alpha)}},$$

where $\Gamma(z)$ is the Euler gamma function. The weight $\alpha$ is a generic complex number. We stress the presence of the unconventional factor 2 in the exponential. According to this definition, we have

$$\int d^Dk e^{2ikx} = \pi^{2(\lambda+1)}\delta(x).$$

According to GPXT technique, the computations are made directly in coordinate space rather than in momentum space. The technique is grounded on the observation that, in $x$-space, the scalar propagator always depends on the difference of two points,

$$\Delta(x_i, x_j) = \frac{1}{(x_i - x_j)^{2\lambda}},$$
and can thus be expanded in terms of the Gegenbauer polynomials, which form an orthogonal set on the unit sphere in $\mathbb{R}^D$. For the moment, $D$ is an arbitrary integer dimension and $\lambda = D/2 - 1$. The analytic continuation to complex $D$ will be done in a second step.

The Gegenbauer polynomials $C_n^\lambda$ are defined in terms of a generating function,

$$
\frac{1}{(1 - 2xt + t^2)^\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) t^n, \tag{C.7}
$$

where $x \in [-1, 1]$. We refer to the quantity $\lambda$ as the weight of the polynomial, while $n$ is its index. The Gegenbauer polynomials are orthogonal with respect to the weight function $(1 - x^2)^{\lambda - 1/2}$:

$$
\int_{-1}^{1} dx \ (1 - x^2)^{\lambda - \frac{1}{2}} C_n^\lambda(x) C_m^\lambda(x) = \frac{\pi^{2\lambda - 2} \Gamma(n + 2\lambda) \Gamma(\lambda)}{n!(n + \lambda) \Gamma(\lambda)^2} \delta_{nm}. \tag{C.8}
$$

The following particular values are often needed:

$$
C_n^\alpha(1) = \frac{\Gamma(n + 2\alpha)}{n! \Gamma(2\alpha)}, \tag{C.9}
$$

$$
C_0^\alpha(x) = 1.
$$

Let’s now turn to the description of the technique. We will consider here only diagrams with a single external momentum $p$, entering and leaving the graph at points $x_{in}$ and $x_{out}$ in $x$-space. The generic $\ell$-loop integral in (C.3) becomes, in $x$-space,

$$
I_\ell = \frac{\Gamma(\lambda)^P}{(4\ell^P \pi^P)^{\lambda + 1}} \int d^D x_1 \cdots d^D x_P \frac{e^{2ip(x_{out} - x_{in})}}{\Delta_1 \cdots \Delta_P} \tag{C.10}
$$

where $\Delta_i$ are the propagators in coordinate space.

It is now convenient to move to spherical coordinates in $D$ dimensions: to this purpose we define

$$
r = x^2, \quad \hat{x} = \frac{x}{\sqrt{r}}. \tag{C.11}
$$

So $r$ is the (squared) radial coordinate and $\hat{x}$ is the unit vector pointing in the same direction as $x$. The integration measure changes to

$$
d^D x = \frac{1}{2} S_{D-1} r^{\lambda} dr \hat{x}, \tag{C.12}
$$

where

$$
S_{D-1} = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda + 1)}. \tag{C.13}
$$
is the surface of the unit sphere in $\mathbb{R}^D$. The integral (C.10), therefore, transforms to

$$I_\ell = N_\lambda(\ell, P) \int \frac{d\tau_1 \cdots d\tau_{D-\ell} d\hat{x}_1 \cdots d\hat{x}_{D-\ell} r_1^\lambda \cdots r_{D-\ell}^\lambda e^{2ip(x_{\text{out}} - x_{\text{in}})}}{\Delta_1 \cdots \Delta_P} \quad (C.14)$$

with the normalization factor

$$N_\lambda(\ell, P) = \frac{\Gamma(\lambda + 1)^\ell}{(4\pi)^{\ell(\lambda+1)}\lambda P}. \quad (C.15)$$

At this point we can expand the propagators in terms of the Gegenbauer polynomials: from (C.7) we have

$$\Delta(x_i, x_j) = \frac{1}{(x_i - x_j)^{2\lambda}} = \frac{1}{M_{i,j}^\lambda} \sum_{n=0}^{\infty} C_n^\lambda(\hat{x}_i \cdot \hat{x}_j) \left(\frac{m_{i,j}}{M_{i,j}}\right)^n, \quad (C.16)$$

where we have introduced the notation

$$m_{i,j} = \min(r_i, r_j), \quad M_{i,j} = \max(r_i, r_j). \quad (C.17)$$

If we are interested only in the UV divergent part of the loop integral, as in all the cases in this work, a great simplification occurs: in $x$-space, UV divergences appear when all coordinates are small, so we can approximate the exponential factor with unit, neglecting it. Since it is equivalent to set to zero the external momentum $p$, dropping the exponential will introduce IR divergences, in the region where some coordinates are large, which mix to UV ones altering the final result. We thus regulate them by introducing an infrared cutoff $R$ in the radial integrations. At the end of the computation, when all the subdivergences have been subtracted, the principal part of the integral must be independent of the regulator $R$, while the finite part in general depends on $R$ and should be discarded from the result.

The expansion of the propagators in series of Gegenbauer polynomials allows us to separate the integral in radial and angular parts and introduces as many infinite sums as the number of the propagators themselves. However, we can minimize the number of series in this way: thanks to translational invariance of the integral (C.10), we can choose one of the vertices of the diagram as the origin of $D$-dimensional space. We call this vertex the root vertex. All the propagators directly connected to the root vertex, which we call root propagators, are simply given by $1/r_i^\lambda$, where $r_i$ is the radial coordinate of the vertex which connects the propagator to the root vertex, and so they don’t produce any series expansion in the Gegenbauer polynomials. Obviously, the best choice of the root vertex is usually such that the number of propagators attached to it is maximized. Therefore, in most cases, the root vertex will coincide with the composite operator.
Once we have chosen the root vertex and we have expanded the propagators depending on differences of coordinates in terms of Gegenbauer polynomials, the angular and radial integrations are performed separately.

The angular integration can be performed by repeated use of the orthogonality relation (C.8) of the Gegenbauer polynomials, which can be rewritten in terms of the angular variables as

$$\int \hat{d}x \, C_\lambda^n(\hat{x}_i \cdot \hat{x}) C_\lambda^m(\hat{x} \cdot \hat{x}_j) = \frac{\lambda}{n + \lambda} \delta_{nm} C_\lambda^n(\hat{x}_i \cdot \hat{x}_j). \quad \text{(C.18)}$$

The angular integration is normalized as $\int \hat{d}x = 1$. In particular, since $C_\lambda^0(x) = 1$, we have

$$\int \hat{d}x_i \hat{d}x_j C_\lambda^n(\hat{x}_i \cdot \hat{x}_j) = \delta_{n0}. \quad \text{(C.19)}$$

Kronecker deltas can be used to decrease the number of summations. Angular integration is usually the hardest part in multiloop computations and can be extremely simplified if the root vertex is chosen so as to minimize the angular loop number and the number of infinite summations. In fact, GPXT is at its best when the number of infinite summations can be reduced at most to one.

Let’s now turn to the radial integration. Having dropped the exponential and introduced the infrared cutoff, radial integrands consist of simple powers. The only difficulty is that, because of the presence of the min and max functions, the domain of integration (which is an hypercube of length $R$) has to be split into $(P-\ell)!$ subdomains, defined by the different orderings of the radial variables. This number can be large. Of course this is not a problem if the procedure is automated with the help of a computer$^9$. Anyway, it is useful to find all the possible symmetries of the integrand in order to reduce the independent domains of integrations.

At this point, when the angular and radial integrations have been performed, the next point is to promote the dimension $D$, or equivalently $\lambda$, to a complex parameter through the formula (C.2) and then perform the Laurent expansion of the result around $\varepsilon = 0$. If multiple poles are present, we proceed to the subtraction of subdivergences. These must be computed within the same renormalization scheme as the original integral, $i.e.$ using GPXT and introducing the same cutoff procedure for infrared regularization.

The last step is to perform the summations that possibly survived after the angular integrations. As stated before, finding analytical results can be very hard, especially when multiple series are present, and sometimes only a numerical analysis is possible.

$^9$In particular, we used the Mathematica routine for radial integrals described in [60].
D. Integrals

In the following, we list the results of the integrals relevant for the computation of maximal reshuffling diagrams of ABJM theory in the $SU(2) \times SU(2)$ sector (see Figure 1). They are computed via GPXT as described in Appendix C. We denote by $I_j$ the value of the integral. We give here the $\varepsilon$ expansions up to the order needed for the computation of the UV divergences of the diagrams in Figure 1. We denote by $R$ the infrared regulator. For convenience, we leave here the normalization factor (C.15) unexpanded: this will be easily reintroduced in the subtraction of subdivergences.

\[ I_2 = \begin{array}{c}
\end{array} = N_\lambda(2, 3) \left( \frac{1}{2\varepsilon} + \log R + \varepsilon \log^2 R + \frac{2}{3} \varepsilon^2 \log^3 R \right), \quad (D.1) \]

\[ I_4 = \begin{array}{c}
\end{array} = N_\lambda(4, 6) \left\{ \frac{1}{8\varepsilon^2} + \frac{1 + \log R}{2\varepsilon} - 1 + 2 \log R + \log^2 R \\
+ \frac{2}{3} \varepsilon (3 - 6 \log R + 6 \log^2 R + 2 \log^3 R) \right\}, \quad (D.2) \]

\[ I_{4a} = \begin{array}{c}
\end{array} = N_\lambda(4, 6) \left( \frac{1}{4\varepsilon^2} + \frac{\log R}{\varepsilon} + 2 \log^2 R + \frac{8}{3} \varepsilon \log^3 R \right), \quad (D.3) \]

\[ I_6 = \begin{array}{c}
\end{array} = N_\lambda(6, 9) \left\{ \frac{1}{48\varepsilon^3} + \frac{2 + \log R}{8\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{5}{6} + \frac{3}{2} \log R + \frac{3}{8} \log^2 R \right) \\
- \frac{29}{3} + 5 \log R + \frac{9}{2} \log^2 R + \frac{3}{4} \log^3 R \right\}, \quad (D.4) \]

\[ I_{6a} = \begin{array}{c}
\end{array} = N_\lambda(6, 9) \left\{ \frac{1}{24\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{1}{3} + \frac{1}{4} \log R \right) + \frac{1}{\varepsilon} \left( -\frac{4}{3} + 2 \log R + \frac{3}{4} \log^2 R \right) \\
+ 8 - 8 \log R + 6 \log^2 R + \frac{3}{2} \log^3 R \right\} \quad (D.5) \]

\[ I_{6b} = \begin{array}{c}
\end{array} = N_\lambda(6, 9) \left( \frac{1}{16\varepsilon^3} + \frac{2 + 3 \log R}{8\varepsilon^2} + \frac{-4 + 12 \log R + 9 \log^2 R}{8\varepsilon} \\
+ 1 - 3 \log R + \frac{9}{2} \log^2 R + \frac{9}{4} \log^3 R \right) \quad (D.6) \]

\[ I_{6c} = \begin{array}{c}
\end{array} = N_\lambda(6, 9) \left\{ \frac{1}{24\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{1}{6} + \frac{1}{4} \log R \right) + \frac{1}{\varepsilon} \left( \frac{1}{3} + \log R + \frac{3}{4} \log^2 R \right) \\
- 2 + 2 \log R + 3 \log^2 R \right\} \quad (D.7) \]
\[ I_{8a} = N_\lambda(8, 12) \left\{ \frac{1}{192 \varepsilon^4} + \frac{1}{12 \varepsilon^3} \left( 1 + \frac{1}{2} \log R \right) \right. \\
+ \frac{1}{3 \varepsilon^2} \left( -\frac{1}{4} + 2 \log R + \frac{1}{2} \log^2 R \right) \right. \\
+ \frac{1}{3 \varepsilon} \left( x - 2 \log R + 8 \log^2 R + \frac{4}{3} \log^3 R \right) \left. \right\}, \quad (D.8) \]

\[ I_{8b} = N_\lambda(8, 12) \left\{ \frac{5}{384 \varepsilon^4} + \frac{5}{48 \varepsilon^3} \left( 1 + \log R \right) \right. \\
+ \frac{1}{6 \varepsilon^2} \left( \frac{1}{2} + 5 \log R + \frac{5}{2} \log^2 R \right) \right. \\
+ \frac{1}{\varepsilon} \left( -\frac{7}{3} + \frac{2}{3} \log R + \frac{10}{3} \log^2 R + \frac{10}{9} \log^3 R \right) \left. \right\} \quad (D.9) \]

\[ I_{8c} = N_\lambda(8, 12) \left\{ \frac{1}{128 \varepsilon^4} + \frac{1}{4 \varepsilon^3} \left( \frac{1}{3} + \frac{1}{4} \log R \right) \right. \\
+ \frac{1}{\varepsilon^2} \left( \frac{1}{24} + \frac{2}{3} \log R + \frac{1}{4} \log^2 R \right) \right. \\
+ \frac{1}{3 \varepsilon} \left( \frac{1}{2} + 5 \log R + 8 \log^2 R + 2 \log^3 R \right) \left. \right\} \quad (D.10) \]

\[ I_{8d} = N_\lambda(8, 12) \left\{ \frac{1}{128 \varepsilon^4} + \frac{1}{8 \varepsilon^3} \left( 1 + \frac{1}{2} \log R \right) \right. \\
+ \frac{1}{\varepsilon^2} \left( \frac{1}{8} + \log R + \frac{1}{4} \log^2 R \right) \right. \\
+ \frac{1}{\varepsilon} \left( -3 + \log R + 4 \log^2 R + \frac{2}{3} \log^3 R \right) \left. \right\} \quad (D.11) \]

\[ I_{8e} = N_\lambda(8, 12) \left\{ \frac{1}{384 \varepsilon^4} + \frac{1}{8 \varepsilon^3} \left( \frac{1}{2} + \frac{1}{6} \log R \right) \right. \\
+ \frac{1}{\varepsilon^2} \left( \frac{2}{3} + \frac{1}{2} \log R + \frac{1}{12} \log^2 R \right) \right. \\
+ \frac{1}{\varepsilon} \left( -\frac{7}{12} + \frac{16}{3} \log R + 2 \log^2 R + \frac{2}{9} \log^3 R \right) \left. \right\} \quad (D.12) \]

where, in \( I_{8a} \), \( x \) is reduced to a multiple series which we don’t show here (see below for a more effective method to compute the \( I_{8a} \) integral). The subtracted integrals are denoted by \( \overline{I}_j \) and are graphically represented by a box around the corresponding integral. We conveniently factor out \( 1/(4\pi)^\ell \) from the results. The pole parts of such integrals are given by:

\[ \overline{I}_2 = \frac{1}{(4\pi)^2} \frac{1}{4 \varepsilon} \quad (D.13) \]
\[ \bar{I}_4 = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = \frac{1}{(4\pi)^4} \left( -\frac{1}{32\varepsilon^2} + \frac{1}{8\varepsilon} \right) \quad (D.14) \]

\[ \bar{I}_{4a} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} - 2 \begin{array}{c}
\end{array} = \frac{1}{(4\pi)^4} \left( -\frac{1}{16\varepsilon^2} \right) \quad (D.15) \]

\[ \bar{I}_6 = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = \frac{1}{(4\pi)^6} \left( \frac{1}{384\varepsilon^3} - \frac{1}{32\varepsilon^2} + \frac{1}{6 \varepsilon} \right) \quad (D.16) \]

\[ \bar{I}_{6a} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} - 2 \begin{array}{c}
\end{array} = \frac{1}{(4\pi)^6} \left( \frac{1}{192\varepsilon^3} - \frac{1}{48\varepsilon^2} - \frac{1}{24 \varepsilon} \right) \quad (D.17) \]

\[ \bar{I}_{6b} = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \quad (D.18) \]

\[ - \frac{1}{128\varepsilon^3} - \frac{1}{32\varepsilon^2} \]
\[ I_{6c} = - 2 - 2 \] 
\[ = \frac{1}{(4\pi)^6} \left( \frac{1}{192\varepsilon^3} - \frac{1}{24\varepsilon^2} + \frac{1}{24\varepsilon} \right) \] 

(D.19)

\[ I_{8a} = - 2 \] 
\[ = \frac{1}{(4\pi)^8} \left\{ - \frac{1}{3072\varepsilon^4} + \frac{1}{192\varepsilon^3} - \frac{1}{48\varepsilon^2} + \frac{y}{\varepsilon} \right\} , \] 

(D.20)

\[ I_{8b} = - \] 
\[ = \frac{1}{(4\pi)^8} \left( - \frac{5}{6144\varepsilon^4} + \frac{5}{768\varepsilon^3} - \frac{1}{384\varepsilon^2} - \frac{1}{32\varepsilon} \right) \] 

(D.21)
\[
\bar{I}_{sc} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} = \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} - \begin{array}{c}
\text{Diagram 7} \\
\text{Diagram 8}
\end{array} - \begin{array}{c}
\text{Diagram 9}
\end{array}
\]

\[
= \frac{1}{(4\pi)^8} \left( -\frac{1}{2048\varepsilon^4} + \frac{1}{128\varepsilon^3} - \frac{3}{64\varepsilon^2} + \frac{5}{192\varepsilon} \right)
\]

(D.22)

\[
\bar{I}_{sd} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} = \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} - \begin{array}{c}
\text{Diagram 7} \\
\text{Diagram 8}
\end{array} - \begin{array}{c}
\text{Diagram 9} \\
\text{Diagram 10}
\end{array} - \begin{array}{c}
\text{Diagram 11} \\
\text{Diagram 12}
\end{array}
\]

\[
= \frac{1}{(4\pi)^8} \left( -\frac{1}{2048\varepsilon^4} + \frac{1}{192\varepsilon^3} - \frac{1}{64\varepsilon^2} - \frac{11}{192\varepsilon} \right)
\]

(D.23)

\[
\bar{I}_{se} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} = \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} - \begin{array}{c}
\text{Diagram 7}
\end{array}
\]

\[
= \frac{1}{(4\pi)^8} \left( -\frac{1}{6144\varepsilon^4} + \frac{1}{256\varepsilon^3} - \frac{19}{384\varepsilon^2} + \frac{5}{16\varepsilon} \right).
\]

(D.24)
Regarding \( I_{8a} \), GPXT allowed us to analytically compute the higher order poles in \( \varepsilon \) and to reduce the first order pole \( y \) to a multiple series. Such a series is not easy to sum and we choose to compute the integral \( I_{8a} \) through Mellin-Barnes technique \([61, 62]\) in order to have a reliable numerical estimate of its pole \( I_{8a} \mid_{1} \).

First of all, we contract the bubbles to reduce to the evaluation of a four-loop integral:

\[
I_{8a} = G(1, 1)^3 G(1, 1/2 + \varepsilon) ,
\]

where the “G-functions” are defined by

\[
G(\alpha, \beta) = \frac{\Gamma(\lambda + 1 - \alpha)\Gamma(\lambda + 1 - \beta)\Gamma(\alpha + \beta - \lambda - 1)}{(4\pi)^{\lambda+1}\Gamma(\alpha)\Gamma(\beta)\Gamma(2\lambda + 2 - \alpha - \beta)} .
\]

(D.26)

Since the G-functions in (D.25) present a simple pole in \( \varepsilon \), the four-loop integral

\[
I_{4b} =
\]

has to be computed up to order zero in \( \varepsilon \). Let’s introduce the corresponding four-loop master integral with generic powers of the propagators:

\[
J_{4b}(\alpha_1, \ldots, \alpha_8) =
\]

\[
= \frac{1}{(2\pi)^D} \int d^Dk_1 d^Dk_2 d^Dk_3 d^Dk_4 \frac{1}{(k_1^2)^{\alpha_1}(k_2^2)^{\alpha_2}(k_3^2)^{\alpha_3}(k_4^2)^{\alpha_4}}
\]

\[
\times \frac{1}{[(k_1 - k_4 - p)^2]^{\alpha_5}[ (k_2 - k_3 - p)^2]^{\alpha_6}[(k_1 - k_2)^2]^{\alpha_7}[(k_3 - k_4)^2]^{\alpha_8}} .
\]

(D.28)

The Mellin-Barnes representation of this integral and its analytical continuation in \( \varepsilon \) can be obtained with the help of the Mathematica packages AMBRE \([63]\) and MB \([64]\) respectively.

The MB package also contains a routine suitable for numerical integration of MB representations. Besides the built-in Mathematica function NIntegrate, it uses the
CUBA library [65] of numerical integration routines and the CERN libraries [66] for the implementation of gamma and psi functions, in order to prepare Fortran programs, which are more efficient, in terms of computational time, for high dimension MB integrals.

A MB representation for the integral $J_{4b}$ is

\[
J_{4b} = \frac{1}{(4\pi)^{\lambda+1}} \int_{i\infty}^{i\infty} \frac{dz_1}{2\pi i} \cdots \int_{i\infty}^{i\infty} \frac{dz_6}{2\pi i} \ \frac{\Gamma(\lambda + 1 - \alpha_{17} - z_1)\Gamma(\lambda + 1 - \alpha_{15} - z_2)}{\Gamma(\alpha_1)\Gamma(\alpha_3)\Gamma(\alpha_5)\Gamma(\alpha_6)}
\]

\[
\times \frac{\Gamma(-z_2)\Gamma(\alpha_1 + z_{12})\Gamma(\lambda + 1 - \alpha_{26} + z_1 - z_3)\Gamma(2\lambda + 2 - \alpha_{1257} - z_{24})\Gamma(-z_4)}{\Gamma(2\lambda + 2 - \alpha_{157})\Gamma(\alpha_7)\Gamma(\alpha_2 - z_1)\Gamma(3\lambda + 3 - \alpha_{12567} - z_2)}
\]

\[
\times \frac{\Gamma(\alpha_2 - z_1 + z_{34})\Gamma(-2\lambda - 2 + \alpha_{12567} + z_{234})\Gamma(3\lambda + 3 - \alpha_{1235678} - z_{2345})}{\Gamma(4\lambda + 4 - \alpha_{1235678} - z_{24}}\Gamma(-2\lambda - 2 + \alpha_{12567} + z_{234})\Gamma(\alpha_4 - z_5)
\]

\[
\times \frac{\Gamma(-z_5)\Gamma(\lambda + 1 - \alpha_4 + z_5)\Gamma(\lambda + 1 - \alpha_3 + z_3 - z_6)\Gamma(4\lambda + 4 - \alpha_{1235678} - z_6)}{\Gamma(5\lambda + 5 - \alpha_{12345678} - z_6)}
\]

\[
\times \frac{\Gamma(-z_6)\Gamma(-4\lambda - 4 + \alpha_{12345678} + z_6)\Gamma(\alpha_3 + z_56)\Gamma(-3\lambda - 3 + \alpha_{1235678} + z_{2456})}{\Gamma(-3\lambda - 3 + \alpha_{1235678} + z_6)}
\]

(D.29)

where we have denoted $\alpha_{ijk\cdots} = \alpha_i + \alpha_j + \alpha_k + \cdots$ and similarly for $z_k$.

We have to compute

\[
I_{4b} = J_{4b}(1/2 + \varepsilon, 1, 1, 1/2 + \varepsilon, 2\varepsilon, 1, 1, 1).
\]

(D.30)

Putting this integral into the MB package we can make the $\varepsilon$-expansion and find the following numerical result:

\[
I_{4b} = 2.116213934935895 \times 10^{-8} + (8.26225 \pm 0.00003) \times 10^{-7}
\]

\[
+ (0.0000177432 \pm 0.0000000003) \times \varepsilon + (0.0002705914 \pm 0.0000000003) \times \varepsilon^2 + (0.0000000003) \times \varepsilon^3.
\]

(D.31)

We then insert this result in (D.25), make the $\varepsilon$-expansion and subtract the subdivergences, to obtain the following numerical result:

\[
\bar{I}_{8a} = -5.2348 \ldots 10^{-13} + 8.37567 \ldots 10^{-12} - 3.35028 \ldots 10^{-11} - 9.63613 \ldots 10^{-12}.
\]

(D.32)
Comparing this result with (D.20) we see that the poles of second, third and fourth order perfectly agree. We recall that these poles were computed analytically via GPXT. In addition, we now have the numerical result for the first order pole:

\[
I_{8a}|_{1/\varepsilon} = \frac{1}{(4\pi)^8} (-0.0059921 \pm 0.0000008). \tag{D.33}
\]

We now want to extract the analytical result from this number. In section 2 we argued that the parameter \(\beta\) of (4.1) is purely transcendental and that it can be a rational combination of just two transcendental constants, namely \(\pi^3\) and \(\zeta(3)\). Since the only transcendental contribution to \(\beta\) can come from the \(\frac{1}{\varepsilon}\) pole of \(\bar{I}_{8a}\), such pole should be a rational combination of the constants 1, \(\pi^3\), \(\zeta(3)\). In order to extract the coefficients of this generic linear combination from the numerical value (D.33) we use the Mathematica implementation [67] of the PSLQ algorithm\(^\text{10}\) [51] (see also [68]). This is a powerful integer relation detection algorithm: given a vector \(x = (x_1, \ldots, x_n)\) of real or complex numbers and a precision \(10^{-p}\), PSLQ looks for a vector \(a = (a_1, \ldots, a_n)\), if exists, of integers \(a_i\), not all zero, such that

\[
a_1 x_1 + \cdots + a_n x_n = 0. \tag{D.34}
\]

In our case, the vector \(x\) contains the numerical value in \(\bar{I}_{8a}|_{1/\varepsilon}\) and the transcendental constants it should be fitted with:

\[
x = \left( -0.0059921, 1, \pi^3, \zeta(3) \right). \tag{D.35}
\]

Application of PSLQ algorithm with precision \(10^{-7}\), which is the same within which the numerical value of \(\bar{I}_{8a}|_{1/\varepsilon}\) has been computed, gives the solution

\[
a = (-32, -5, 0, 4). \tag{D.36}
\]

It means that the exact value of the \(\frac{1}{\varepsilon}\) pole of the \(\bar{I}_{8a}\) integral should be

\[
\bar{I}_{8a}|_{1/\varepsilon} = \frac{1}{(4\pi)^8} \left( -\frac{5}{32} + \frac{1}{8} \zeta(3) \right). \tag{D.37}
\]

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