Blow-up criteria for Boussinesq system and MHD system and Landau-Lifshitz equations in a bounded domain

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Abstract
In this paper, we prove some blow-up criteria for the 3D Boussinesq system with zero heat conductivity and MHD system and Landau-Lifshitz equations in a bounded domain.

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Running Title: Boussinesq-MHD-Landau-Lifshitz

1 Introduction
Let Ω be a bounded, simply connected domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \), and \( \nu \) be the unit outward normal vector to \( \partial \Omega \). First, we consider the regularity criterion of the
Boussinesq system with zero heat conductivity:

\[
\begin{align*}
\text{div } u &= 0, \\
\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u &= \theta e_3, \\
\partial_t \theta + u \cdot \nabla \theta &= 0 \quad \text{in } \Omega \times (0, \infty), \\
u \cdot \nu &= 0, \text{curl } u \times \nu = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
(u, \theta)(\cdot, 0) &= (u_0, \theta_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3,
\end{align*}
\]

where \( u, \pi, \) and \( \theta \) denote unknown velocity vector field, pressure scalar, and temperature scalar of the fluid, respectively. \( \omega := \text{curl } u \) is the vorticity and \( e_3 := (0, 0, 1) \).

When \( \theta = 0 \), (1.1) and (1.2) are the well-known Navier-Stokes system. Giga [1], Kim [2], Kang and Kim [3] have proved some Serrin type regularity criteria.

The first aim of this paper is to prove a new regularity criterion for the problem (1.1)-(1.5), we will prove \( u \) satisfies

\[
\nabla u \in L^1(0, T; BMO(\Omega))
\]

with \( 0 < T < \infty \), then the solution \( u, \theta \) can be extended beyond \( T > 0 \). Here \( BMO \) denotes the space of bounded mean oscillation.

Secondly, we consider the blow-up criterion of the 3D MHD system

\[
\begin{align*}
\text{div } u &= \text{div } b = 0, \\
\partial_t u + u \cdot \nabla u + \nabla \left( \pi + \frac{1}{2}|b|^2 \right) - \Delta u &= b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b - b \cdot \nabla u &= \Delta b \quad \text{in } \Omega \times (0, \infty), \\
u \cdot \nu &= 0, \text{curl } u \times \nu = 0, b \cdot \nu = 0, \text{curl } b \times \nu = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
(u, b)(\cdot, 0) &= (u_0, b_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3.
\end{align*}
\]

Here \( b \) is the magnetic field of the fluid.

It is well-known that the problem (1.7)-(1.11) has a unique local strong solution [4]. But whether this local solution can exist globally is an outstanding problem. Kang and Kim [3] prove some Serrin type regularity criteria.

The second aim of this paper is to prove a new regularity criterion for the problem (1.7)-(1.11), we will prove

\[
\text{Theorem 1.2. Let } u_0, b_0 \in H^3 \text{ with } \text{div } u_0 = \text{div } b_0 = 0 \text{ in } \Omega \text{ and } u_0 \cdot \nu = b_0 \cdot \nu = 0, \text{curl } u_0 \times \nu = \text{curl } b_0 \times \nu = 0 \text{ on } \partial \Omega. \text{ Let } (u, b) \text{ be a strong solution to the problem (1.7)-(1.11). If (1.6) holds true, then the solution } (u, b) \text{ can be extended beyond } T > 0.
\]
Remark 1.1. When $\Omega := \mathbb{R}^3$, our result gives the following well-known regularity criterion

$$\omega := \text{curl } u \in L^1(0, T; \dot{B}^0_{\infty, \infty}),$$

but the method of proof we used is different from that in [14, 15]. Here $\dot{B}^0_{\infty, \infty}$ denotes the homogeneous Besov space [13].

Next, we consider the following 3D density-dependent MHD equations:

$$\begin{align*}
\text{div } u &= \text{div } b = 0, \\
\partial_t \rho + \text{div } (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div } (\rho u \otimes u) + \nabla \left( \pi + \frac{1}{2} |b|^2 \right) - \Delta u &= b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b - b \cdot \nabla u &= \Delta b \text{ in } \Omega \times (0, \infty), \\
\rho, \rho u, b &\cdot \nu = 0, \text{ curl } b \times \nu = 0 \text{ on } \partial \Omega \times (0, \infty), \\
(\rho, \rho u, b)(\cdot, 0) &= (\rho_0, \rho_0 u_0, b_0) \text{ in } \Omega \subset \mathbb{R}^3.
\end{align*}$$

For this problem, in [5], Wu proved that if the initial data $\rho_0, u_0, b_0$ satisfy

$$0 \leq \rho_0 \in H^2, \ u_0 \in H^1 \cap H^2, \ b_0 \in H^2, \ -\Delta u_0 + \nabla \left( \pi_0 + \frac{1}{2} |b_0|^2 \right) = b_0 \cdot \nabla b_0 + \sqrt{\rho_0} g$$

for some $(\pi_0, g) \in H^1 \times L^2$, then there exists a positive time $T_*$ and a unique strong solution $(\rho, u, b)$ to the problem (1.12)-(1.17) such that

$$\begin{align*}
\rho &\in C([0, T_*]; H^2), \ u \in C([0, T_*]; H^1) \cap L^2(0, T_*; H^2), \\
u_0 &\in L^2(0, T_*; H^1), \sqrt{\rho_0} u_t \in L^\infty(0, T_*; L^2), \\
b &\in L^\infty(0, T_*; H^2) \cap L^2(0, T_*; H^3), \ b_t \in L^\infty(0, T_*; L^2) \cap L^2(0, T_*; H^1).
\end{align*}$$

And when $b = 0$, Kim [2] proved the following regularity criterion:

$$u \in L^{\frac{2s}{s-3}}(0, T; L^s_w(\Omega)) \text{ with } 3 < s \leq \infty.$$  (1.20)

Here $L^s_w$ denotes the weak-$L^s$ space and $L^\infty_w = L^\infty$.

The aim of this paper is to refine (1.20), we will prove

**Theorem 1.3.** Let $\rho_0, u_0,$ and $b_0$ satisfy (1.18). Let $(\rho, u, b)$ be a strong solution of the problem (1.12)-(1.17) in the class (1.19). If $u$ satisfies one of the following two conditions:

$$\begin{align*}
(i) \quad &\int_0^T \frac{\|u(t)\|_{L^{\frac{2s}{s-3}}}}{1 + \log(e + \|u(t)\|_{L^s_w})} \, dt < \infty \text{ with } 3 < s \leq \infty, \\
(ii) \quad &u \in L^2(0, T; BMO(\Omega)).
\end{align*}$$

with $0 < T < \infty$, then the solution $(\rho, u, b)$ can be extended beyond $T > 0$.  (1.21)
Finally, we consider the 3D Landau-Lifshitz system:

$$\partial_t d - \Delta d = d|\nabla d|^2 + d \times \Delta d, |d| = 1 \text{ in } \Omega \times (0, \infty),$$  \hfill (1.23)

$$\partial_n d = 0 \text{ on } \partial \Omega \times (0, \infty),$$  \hfill (1.24)

$$d(\cdot, 0) = d_0, |d_0| = 1 \text{ in } \Omega \subseteq \mathbb{R}^3.$$  \hfill (1.25)

Carbou and Fabrie \[6\] showed the existence and uniqueness of local smooth solutions. When $\Omega := \mathbb{R}^n \ (n = 2, 3, 4)$, Fan and Ozawa \[7\] proved some regularity criteria. The aim of this paper is to prove a logarithmic blow-up criterion for the problem (1.23)-(1.25) when $\Omega$ is a bounded domain. We will prove

**Theorem 1.4.** Let $d_0 \in H^3(\Omega)$ with $|d_0| = 1$ in $\Omega$ and $\partial_n d_0 = 0$ on $\partial \Omega$. Let $d$ be a local smooth solution to the problem (1.23) - (1.25). If $d$ satisfies

$$\int_0^T \frac{\|\nabla d\|_{L^q}^2}{1 + \log(e + \|\nabla d\|_{L^q})} \, dt < \infty \text{ with } 3 < q \leq \infty,$$  \hfill (1.26)

and $0 < T < \infty$, then the solution can be extended beyond $T > 0$.

In the following section 2, we give some preliminary Lemmas which will be used in the following sections. The proof of Theorem 1.1 of problem (1.1) - (1.5) will be given in section 3. The new regularly criterion of Theorem 1.2 for the 3D MHD problem (1.7) - (1.11) will be proved in section 4. In section 5 is the proof of the Theorem 1.3, and in the next section 6 we give the main proof of final Theorem 1.4.

## 2 Preliminary Lemmas

In the following proofs, we will use the following logarithmic Sobolev inequality \[8\]:

$$\|\nabla u\|_{L^\infty} \leq C(1 + \|\nabla u\|_{BMO} \log(e + \|u\|_{W^{s,p}})) \text{ with } s > 1 + \frac{3}{p},$$  \hfill (2.1)

and the following three lemmas.

**Lemma 2.1.** (\[9\]). Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain, let $b : \Omega \rightarrow \mathbb{R}^3$ be a smooth vector field, and let $1 < p < \infty$. Then

$$-\int_\Omega \Delta b \cdot b|b|^{p-2} \, dx = \frac{1}{2} \int_\Omega |b|^{p-2}|\nabla b|^2 \, dx + 4\frac{p-2}{p^2} \int_\Omega |\nabla |b|^\frac{2}{p}|^2 \, dx$$

$$- \int_{\partial \Omega} |b|^{p-2}(b \cdot \nabla) b \cdot \nu d\sigma - \int_{\partial \Omega} |b|^{p-2}(\text{curl} b \times \nu) \cdot b d\sigma.$$

**(2.2)**

**Lemma 2.2.** (\[10, 11\]). Let $\Omega$ be a smooth and bounded open set and let $1 < p < \infty$. Then the following estimate:

$$\|b\|_{L^p(\partial \Omega)} \leq C\|b\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|b\|_{W^{1,p}(\Omega)}^\frac{1}{p}$$

**(2.3)** holds for any $b \in W^{1,p}(\Omega)$.  

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Lemma 2.3. There holds
\[ \|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \log^\frac{1}{2}(1 + \|f\|_{W^{1,4}(\Omega)})) \] (2.4)
for any \( f \in W^{1,4}_0(\Omega) \).

**Proof.** When \( \Omega := \mathbb{R}^3 \), (2.4) has been proved in Ogawa [12]. When \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \). We can define
\[ \tilde{f} := \begin{cases} f & \text{in } \Omega, \\ 0 & \text{in } \Omega^c := \mathbb{R}^3 \setminus \Omega. \end{cases} \]
Then we have [10, Page 71]
\[ \|\tilde{f}\|_{W^{1,4}(\mathbb{R}^3)} = \|f\|_{W^{1,4}(\Omega)} \]
and it is obvious that
\[ \|\tilde{f}\|_{L^\infty(\mathbb{R}^3)} = \|f\|_{L^\infty(\Omega)}, \|\tilde{f}\|_{BMO(\mathbb{R}^3)} = \|f\|_{BMO(\Omega)}. \]
Thus (2.4) is proved. \( \square \)

Finally, when \( b \) satisfies \( b \cdot \nu = 0 \) on \( \partial \Omega \), we will also use the identity
\[ (b \cdot \nabla)b \cdot \nu = -(b \cdot \nabla)\nu \cdot b \quad \text{on } \partial \Omega \] (2.5)
for any sufficiently smooth vector field \( b \).

### 3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to prove that the problem (1.1) - (1.5) has a unique local-in-time strong solution, we omit the details here. We only need to establish a priori estimates.

First, thanks to the maximum principle, it follows from (1.1) and (1.3) that
\[ \|\theta\|_{L^\infty(0,T;L^\infty)} \leq C. \] (3.1)

Testing (1.2) by \( u \) and using (1.1) and (3.1), we see that
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + \int_\Omega |\text{curl } u|^2 dx \leq \int_\Omega \theta e_3 \cdot u dx \leq \frac{1}{2} \int_\Omega \theta^2 dx + \frac{1}{2} \int_\Omega u^2 dx, \]
which gives
\[ \|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C. \] (3.2)

Applying curl to (1.2) and setting \( \omega := \text{curl } u \), we find that
\[ \partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u + \text{curl } (\theta e_3). \] (3.3)
Testing (3.3) by \( \omega \) and using (1.1) and (3.1), we infer that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\omega|^2 dx + \int_\Omega |\text{curl} \omega|^2 dx = \int_\Omega (\omega \cdot \nabla)u \cdot \omega dx + \int_\Omega \theta e_3 \text{curl} \omega dx
\]

\[
\leq \|\nabla u\|_{L^\infty} \int_\Omega \omega^2 dx + \frac{1}{2} \int_\Omega |\text{curl} \omega|^2 dx + C,
\]

which implies

\[
\frac{d}{dt} \int_\Omega |\omega|^2 dx + \int_\Omega |\text{curl} \omega|^2 dx \leq C \|\nabla u\|_{L^\infty} \int_\Omega |\omega|^2 dx + C
\]

\[
\leq C(1 + \|\nabla u\|_{BMO}) \log(e + \|u\|_{H^3}) \int_\Omega |\omega|^2 dx + C,
\]

and therefore

\[
\int_\Omega |\omega|^2 dx + \int_{t_0}^t \|\text{curl} \omega\|^2_{L^2} d\tau \leq C(e + y)^{C_0 \epsilon} \quad (3.4)
\]

provided that

\[
\int_{t_0}^t \|\nabla u\|_{BMO} d\tau \leq \epsilon << 1 \quad (3.5)
\]

and \( y(t) := \sup_{[t_0, t]} \|u\|_{H^3} \) for any \( 0 < t_0 \leq t \leq T \) and \( C_0 \) is an absolute constant.

Applying \( \partial_t \) to (1.2), we deduce that

\[
\partial_t^2 u + u \cdot \nabla u_t + \nabla \pi_t - \Delta u_t = -u_t \cdot \nabla u + \theta_t e_3. \quad (3.6)
\]

Testing (3.6) by \( u_t \), using (1.1), (1.3), (3.1) and (3.2), we derive

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 dx + \int_\Omega |\text{curl} u_t|^2 dx
\]

\[
= - \int_\Omega u_t \cdot \nabla u \cdot u_t dx + \int_\Omega \theta e_3 u_t dx
\]

\[
= - \int_\Omega u_t \cdot \nabla u \cdot u_t dx - \int_\Omega \text{div}(\theta e_3) e_3 u_t dx
\]

\[
= - \int_\Omega u_t \cdot \nabla u \cdot u_t dx + \int_\Omega \theta \nabla \cdot (e_3 u_t) dx
\]

\[
\leq \|\nabla u\|_{L^\infty} \int_\Omega |u_t|^2 dx + \frac{1}{2} \int_\Omega |\text{curl} u_t|^2 dx + C
\]

\[
\leq C(1 + \|\nabla u\|_{BMO}) \log(e + \|u\|_{H^3}) \int_\Omega |u_t|^2 dx + \frac{1}{2} \int_\Omega |\text{curl} u_t|^2 dx + C,
\]

which yields

\[
\int_\Omega |u_t|^2 dx + \int_{t_0}^t \int_\Omega |\text{curl} u_t|^2 dx d\tau \leq C(e + y)^{C_0 \epsilon}. \quad (3.7)
\]
On the other hand, thanks to the $H^2$-theory of the Stokes system, if follows from (1.2), (3.1), (3.4) and (3.7) that
\[
\|u\|_{H^2} \leq C\| - \Delta u + \nabla \pi \|_{L^2} \\
\leq C\|\partial_t u + u \cdot \nabla u - \theta e_3\|_{L^2} \\
\leq C\|u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\theta\|_{L^2} \\
\leq C\|u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|u\|_{H^2}^2 + C,
\]
which implies
\[
\|u\|_{H^2} \leq C\|u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C \leq C(e + y)^{C\epsilon}.
\] (3.8)
Applying $\nabla$ to (1.3), testing by $|\nabla \theta|^{p-2}\nabla \theta$ $(2 \leq p < \infty)$ and using (1.1), we get
\[
\frac{d}{dt}\|\nabla \theta\|_{L^p} \leq C\|\nabla u\|_{L^{\infty}}\|\nabla \theta\|_{L^p} \\
\leq C(1 + \|\nabla u\|_{BMO}) \log(e + y)\|\nabla \theta\|_{L^p},
\]
which leads to
\[
\|\nabla \theta\|_{L^{\infty}(t_0,t;L^p)} \leq C(e + y)^{C\epsilon} \text{ with } 2 \leq p < \infty.
\] (3.9)
Testing (3.6) by $-\Delta u_t + \nabla \pi_t$, using (1.1), (1.3), (3.7), (3.8) and (3.9), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\text{curl } u_t|^2 \, dx + \int_{\Omega} \| - \Delta u_t + \nabla \pi_t \|^2 \, dx \\
= \int_{\Omega} (-u_t \cdot \nabla u + \theta_t e_3 - u \cdot \nabla u_t)(-\Delta u_t + \nabla \pi_t) \, dx \\
\leq (\|\nabla u\|_{L^6}\|u_t\|_{L^3} + \|u\|_{L^{\infty}}\|\nabla \theta\|_{L^2} + \|u\|_{L^{\infty}}\|\nabla u_t\|_{L^2}) \|\Delta u_t + \nabla \pi_t\|_{L^2} \\
\leq \|u\|_{H^2}(\|u_t\|_{H^1} + \|\nabla \theta\|_{L^2}) \|\Delta u_t + \nabla \pi_t\|_{L^2} \\
\leq \frac{1}{2}\|\Delta u_t + \nabla \pi_t\|_{L^2}^2 + C\|u\|_{H^2}^2(\|u_t\|_{H^1} + \|\nabla \theta\|_{L^2}^2),
\]
which leads to
\[
\int_{\Omega} |\text{curl } u_t|^2 \, dx + \int_{t_0}^t \|u\|_{H^2}^2 \, d\tau \leq C(e + y)^{C\epsilon}.
\] (3.10)
On the other hand, if follows from (3.3), (3.10), (3.9) and (3.8) that
\[
\|u\|_{H^3} \leq C(1 + \|\Delta \omega\|_{H^2}) \\
\leq C(1 + \|\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u - \text{curl } (\theta e_3)\|_{L^2}) \\
\leq C(1 + \|\partial_t \omega\|_{L^2} + \|u\|_{L^6}\|\nabla \omega\|_{L^3} + \|\omega\|_{L^4}\|\nabla u\|_{L^4} + \|\nabla \theta\|_{L^2}) \\
\leq C(e + y)^{C\epsilon},
\]
which gives
\[
\|u\|_{L^{\infty}(0,T;H^3)} \leq C,
\] (3.11)
and
\[
\|\theta\|_{L^{\infty}(0,T;W^{1,p})} \leq C \text{ with } 3 \leq p \leq 6.
\] (3.12)
This completes the proof of Theorem 1.1.
\[
\Box
\]
4 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2, we only need to prove a priori estimates. First, testing (1.8) by \( u \) and using (1.7), we see that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} |\text{curl} \, u|^2 dx = \int_{\Omega} (b \cdot \nabla) b \cdot \nabla u dx. \tag{4.1}
\]

Testing (1.9) by \( b \) and using (1.7), we find that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} b^2 dx + \int_{\Omega} |\text{curl} \, b|^2 dx = \int_{\Omega} (b \cdot \nabla) u \cdot b dx. \tag{4.2}
\]

Summing up (4.1) and (4.2), we get the well-known energy inequality

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^2 + b^2) dx + \int_{\Omega} (|\text{curl} \, u|^2 + |\text{curl} \, b|^2) dx \leq 0. \tag{4.3}
\]

Testing (1.9) by \( |b|^{p-2}b \) (2 \( \leq p \leq 6 \)), using (1.7), (2.2), (2.3) and (2.5), we derive

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} |b|^p dx + \frac{1}{2} \int_{\Omega} |b|^{p-2} |\nabla b|^2 dx + 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^\frac{p}{2}|^2 dx
\]

\[
= - \int_{\partial \Omega} |b|^{p-2} (b \cdot \nabla) \nu \cdot b d\sigma + \int_{\Omega} b \cdot \nabla u \cdot |b|^{p-2} b dx
\]

\[
\leq C \int_{\Omega} |b|^p dx + \|\nabla u\|_{L^\infty} \int_{\Omega} |b|^p dx
\]

\[
\leq 2 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^\frac{p}{2}|^2 dx + C(1 + \|\nabla u\|_{L^\infty}) \int_{\Omega} |b|^p dx
\]

\[
\leq 2 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^\frac{p}{2}|^2 dx + C(1 + \|\nabla u\|_{BMO}) \int_{\Omega} |b|^p dx \log(e + y),
\]

which implies

\[
\|b\|_{L^\infty(t_0, t; L^p)} + \int_{t_0}^t \int_{\Omega} |b|^{p} |\nabla b|^2 dx dt \leq C(e + y)^{C_0\epsilon} \quad \text{with} \quad 2 \leq p \leq 6, \tag{4.4}
\]

with the same \( y \) and \( \epsilon \) as that in (3.5).

Taking curl to (1.8) and (1.9), respectively, and setting \( \omega := \text{curl} \, u \) and \( j := \text{curl} \, b \), we infer that

\[
\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u + b \cdot \nabla j + \sum_i \nabla b_i \times \partial_i b,
\]

\[
\partial_t j + u \cdot \nabla j - \Delta j = b \cdot \nabla \omega + \sum_i \nabla b_i \times \partial_i u - \sum_i \nabla u_i \times \partial_i b. \tag{4.6}
\]
Testing (1.5) and (1.6) by $\omega$ and $j$, respectively, summing up the result and using (1.7), we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\omega^2 + j^2) dx + \int_{\Omega} (|\text{curl}\,\omega|^2 + |\text{curl}\,j|^2) dx \]

\[ = \int_{\Omega} (\omega \cdot \nabla - \partial_i b_i) + \sum_i \int_{\Omega} (\nabla b_i \times \partial_i u) \cdot j dx - \sum_i \int_{\Omega} (\nabla u_i \times \partial_i b) \cdot j dx \]

\[ \leq C \|\nabla u\|_{L^\infty} \int_{\Omega} (\omega^2 + j^2) dx \]

\[ \leq C (1 + \|\nabla u\|_{BMO}) \int_{\Omega} (\omega^2 + j^2) dx \log(e + y), \]

which implies

\[ \int_{\Omega} (\omega^2 + j^2) dx + \int_{t_0}^t \int_{\Omega} (|\text{curl}\,\omega|^2 + |\text{curl}\,j|^2) dx d\tau \leq C (e + y)^{C_0}. \quad (4.7) \]

Thus, it follows from (1.8), (1.9) and (4.7) that

\[ \int_{t_0}^t \int_{\Omega} (|u_t|^2 + |b_t|^2) dx d\tau \leq C (e + y)^{C_0}. \quad (4.8) \]

Applying $\partial_t$ to (1.8), we have

\[ \partial_t^2 u + u \cdot \nabla u_t + \nabla \pi_t - \Delta u_t = \text{div} (b \otimes b)_t - u_t \cdot \nabla u. \quad (4.9) \]

Testing (1.9) by $u$ and using (1.7), we get

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t|^2 dx + \int_{\Omega} |\text{curl}\,u_t|^2 dx \]

\[ = - \sum_{i,j} \int_{\Omega} (b^i b^j) \partial_j u_i dx - \int_{\Omega} u_t \cdot \nabla u \cdot u_t dx \]

\[ \leq C \|b_t\|_{L^2} \|b\|_{L^6} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|u_t\|_{L^4}^2 \]

\[ \leq C \|b_t\|_{L^2} \|\text{curl}\,b_t\|_{L^2} \|u_t\|_{L^2} \|b\|_{L^6} + C \|\nabla u\|_{L^2} \|u_t\|_{L^2} \|\text{curl}\,u_t\|_{L^2} \]

\[ \leq \delta \|\text{curl}\,u_t\|_{L^2}^2 + \delta \|\text{curl}\,b_t\|_{L^2}^2 + C \|b_t\|_{L^2}^2 \|b\|_{L^5}^2 + C \|\nabla u\|_{L^2}^2 \|u_t\|_{L^2}^2 \quad (4.10) \]

for any $\delta \in (0, 1)$. Applying $\partial_t$ to (1.9), we have

\[ \partial_t^2 b + u \cdot \nabla b_t - \Delta b_t = b_t \cdot \nabla u + b \cdot \nabla u_t - u_t \cdot \nabla b. \quad (4.11) \]
Testing (4.11) by $b_t$ and using (4.7), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |b_t|^2 dx + \int_{\Omega} |\nabla b_t|^2 dx
\]
\[
= \int_{\Omega} (b_t \cdot \nabla u + b \cdot \nabla u_t - u_t \cdot \nabla b)b_t dx
\]
\[
\leq \| \nabla u \|_{L^2} \| b_t \|_{L^4}^2 + \| b \|_{L^6} \| \nabla u_t \|_{L^2} \| b_t \|_{L^3} + \| \nabla b \|_{L^2} \| u_t \|_{L^4} \| b_t \|_{L^4}
\]
\[
\leq \delta \| \nabla b_t \|_{L^2}^2 + \delta \| \nabla u_t \|_{L^2}^2
\]
\[
+ C \| \nabla u \|_{L^2} \| b_t \|_{L^2}^2 + C \| b \|_{L^6} \| b_t \|_{L^2}^2 + C \| \nabla b \|_{L^2} \| u_t \|_{L^2}^2 + \| b_t \|_{L^2}^2
\]
\[
(4.12)
\]
for any $\delta \in (0, 1)$.

Combining (4.10) and (4.12) and taking $\delta$ small enough and using (4.7) and (4.8), we have
\[
\int_{\Omega} \left( |u_t|^2 + |b_t|^2 \right) dx + \int_{t_0}^t \int_{\Omega} \left( |\nabla u_t|^2 + |\nabla b_t|^2 \right) dx d\tau \leq C(e + y)^{C_{\text{loc}}}. \tag{4.13}
\]

It follows from (4.8), (4.9), (4.10) and (4.13) that
\[
\| u \|_{L^\infty(0; t; H^2)} + \| b \|_{L^\infty(0; t; H^2)} \leq C(e + y)^{C_{\text{loc}}}. \tag{4.14}
\]

Testing (4.9) by $\nabla \left( \pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t$, and using (1.7), we find that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_t|^2 dx + \int_{\Omega} \left| \nabla \left( \pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right|^2 dx
\]
\[
= \int_{\Omega} \left( (b \cdot \nabla b)_t - u_t \cdot \nabla u - u \cdot \nabla u_t \right) \left( \nabla \left( \pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right) dx
\]
\[
\leq C \| b \|_{L^\infty} \| \nabla b_t \|_{L^2} + \| b_t \|_{L^6} \| \nabla b \|_{L^3} + \| u_t \|_{L^6} \| \nabla u \|_{L^3}
\]
\[
+ \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} \left\| \nabla \left( \pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right\|_{L^2}
\]
\[
\leq \frac{1}{4} \left\| \nabla \left( \pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right\|_{L^2}^2 + C(\| u \|_{L^\infty}^2 + \| \nabla u \|_{L^3}^2) \| \nabla u_t \|_{L^2}^2
\]
\[
+ C(\| b \|_{L^\infty}^2 + \| \nabla b \|_{L^3}^2) \| \nabla b_t \|_{L^2}^2. \tag{4.15}
\]

Similarly, testing (4.11) by $-\Delta b_t$, we infer that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla b_t|^2 dx + \int_{\Omega} |\Delta b_t|^2 dx
\]
\[
= \int_{\Omega} (u_t \cdot \nabla b + u \cdot \nabla b_t - b_t \cdot \nabla u - b \cdot \nabla u_t) \Delta b_t dx
\]
\[
\leq \left( \| u_t \|_{L^6} \| \nabla b \|_{L^3} + \| u \|_{L^\infty} \| \nabla b_t \|_{L^2} + \| \nabla u \|_{L^3} \| b_t \|_{L^6} + \| b \|_{L^\infty} \| \nabla u_t \|_{L^2} \right) \| \Delta b_t \|_{L^2}
\]
\[
\leq \frac{1}{4} \| \Delta b_t \|_{L^2}^2 + C(\| u \|_{L^\infty}^2 + \| \nabla u \|_{L^3}^2) \| \nabla b_t \|_{L^2}^2 + C(\| b \|_{L^\infty}^2 + \| \nabla b \|_{L^3}^2) \| \nabla u_t \|_{L^2}^2. \tag{4.16}
\]
Combining (4.15) and (4.16) and using (4.14) and (4.13), we have

\[ \int_{\Omega} (|\text{curl } u|^2 + |\text{curl } b|^2) \, dx + \int_{t_0}^{t} \int_{\Omega} (|\Delta u|^2 + |\Delta b|^2) \, dxd\tau \leq C(e + y)^{C_0}. \]  

(4.17)

On the other hand, it follows from (4.5), (4.6), (4.3), (4.17) and (4.14) that

\[ \| u(t) \|_{H^3} + \| b(t) \|_{H^3} \leq C \left( 1 + \| \Delta \omega \|_{L^2} + \| \Delta j \|_{L^2} \right) \]
\[ \leq C(1 + \| \partial_t \omega + u \cdot \nabla - \omega \cdot \nabla u - b \cdot \nabla j - \sum_i \nabla b_i \times \partial_i b \|_{L^2} \]
\[ + \| \partial_t j + u \cdot \nabla j - b \cdot \nabla \omega + \sum_i \nabla u_i \times \partial_i b - \sum_i \nabla b_i \times \partial_i u \|_{L^2} \]
\[ \leq C(e + y(t))^{C_0}, \]

which yields

\[ \| u \|_{L^\infty(0,T;H^3)} + \| b \|_{L^\infty(0,T;H^3)} \leq C, \]

This completes the proof of Theorem 1.2.

\[ \square \]

5 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3; we only need to establish a priori estimates.

First, it follows from (1.12) and (1.13) that

\[ \| \rho \|_{L^\infty(0,T;L^\infty)} \leq C. \]  

(5.1)

Testing (1.14) by u and using (1.12) and (1.13), we see that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho u^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} (b \cdot \nabla) u \cdot u \, dx. \]  

(5.2)

And testing (1.15) by b and using (1.12) and (1.16), we find that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |b|^2 \, dx + \int_{\Omega} |\text{curl } b|^2 \, dx = \int_{\Omega} (b \cdot \nabla) u \cdot b \, dx. \]  

(5.3)

Summing up (5.2) and (5.3), we get the well-known energy inequality

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho |u|^2 + |b|^2) \, dx + \int_{\Omega} (|\nabla u|^2 + |\text{curl } b|^2) \, dx \leq 0. \]  

(5.4)

(I) Let (1.21) hold true.
Testing \([1.15]\) by \(|b|^{p-2}b\) \((2 \leq p < \infty)\), using \([1.12], [2.2], [2.3] \) and \([2.5]\), and setting \(\phi = |b|^\frac{p}{2}\), and using the Gagliardo-Nirenberg inequality \([3]\):

\[
\|\phi\|^s_{L^\frac{2s}{s+2}} \leq C\|\phi\|^\frac{1}{r} L^{r} \|\phi\|^\frac{3}{2}, \quad \text{with} \quad 3 < s \leq \infty, \tag{5.5}
\]

and the generalized Hölder inequality \([13]\):

\[
\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \tag{5.6}
\]

with \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\), we derive

\[
\frac{1}{p}\frac{d}{dt} \int_\Omega |b|^p dx + \frac{1}{2} \int_\Omega |b|^{p-2} \nabla b|^2 dx + \frac{p-2}{p^2} \int_\Omega |\nabla |b|^{\frac{p}{2}}|^2 dx
\]

\[
= -\int_\partial\Omega |b|^{p-2} (b \cdot \nabla)\nu \cdot b d\sigma + \int_\Omega (b \cdot \nabla)u \cdot |b|^{p-2} b dx
\]

\[
\leq \|\nabla u\|^\infty \int_\partial\Omega |b|^p d\sigma - \sum_i \int_\Omega b_i \partial_i (|b|^{p-2} b) dx
\]

\[
\leq C \int_\partial\Omega \phi^2 d\sigma + C \int_\Omega \phi^2 dx
\]

\[
\leq C \int_\partial\Omega \phi^2 d\sigma + C \|u\|_{L^\infty} \|\phi\|_{L^\frac{2s}{s+2}} \|\nabla \phi\|_{L^2}
\]

\[
\leq C \|\phi\|_{L^2} \|\phi\|_{H^1} + C \|u\|_{L^\infty} \|\phi\|_{L^\frac{2s}{s+2}} \|\nabla \phi\|_{L^2}^{1+\frac{3}{2}}
\]

\[
\leq 2\frac{p-2}{p^2} \int_\Omega |\nabla \phi|^2 dx + C\|\phi\|_{L^2}^2 + C \|u\|_{L^\frac{2s}{s+2}}^2 \|\phi\|_{L^2}^2,
\]

which yields

\[
\frac{d}{dt} \int_\Omega \phi^2 dx + C \int_\Omega |\nabla \phi|^2 dx \leq C(1 + \|u\|_{L^\frac{2s}{s+2}}^\frac{2s}{s+2})\|\phi\|_{L^2}^2
\]

\[
\leq C \left(1 + \frac{\|u\|_{L^\frac{2s}{s+2}}^\frac{2s}{s+2}}{1 + \log(e + \|u\|_{L^\infty})}\right)\|\phi\|_{L^2}^2 (1 + \log(e + \|u\|_{L^\infty}))
\]

\[
\leq C \left(1 + \frac{\|u\|_{L^\frac{2s}{s+2}}^\frac{2s}{s+2}}{1 + \log(e + \|u\|_{L^\infty})}\right) (1 + \log(e + y))\|\phi\|_{L^2}^2,
\]

from which it follows that

\[
\|b\|_{L^\infty(t_0,t;L^p)} + \int_{t_0}^t \int_\Omega |b|^2 |\nabla b|^2 dxd\tau \leq C(e + y(t))^{C_0e} \tag{5.7}
\]

with

\[
y(t) := \sup_{[t_0,t]} \|u\|_{W^{1,4}}
\]
for any $0 < t_0 \leq t \leq T$ and $C_0$ is an absolute constant, provided that
\begin{equation}
\int_{t_0}^{T} \left( \frac{\|u\|^{2}_{L_\infty^\alpha}}{1 + \log(e + \|u\|_{L^\infty})} \right) dt \leq \epsilon << 1.
\end{equation} (5.8)

Testing (1.14) by $u_t$, using (1.12) and (1.13), we infer that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \rho |u_t|^2 dx = - \int_{\Omega} \rho u \cdot \nabla u \cdot u_t dx + \int_{\Omega} b \cdot \nabla b \cdot u_t dx
= : I_1 + I_2.
\end{equation} (5.9)

We first compute $I_2$.
\begin{equation}
I_2 = \int_{\Omega} \text{div} \ (b \otimes b) \cdot u_t dx = - \int_{\Omega} b \otimes b : \nabla u_t dx
= - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + 2 \int_{\Omega} b \otimes b_t : \nabla u dx
\leq - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + C \|b_t\|_{L^2} \|b\|_{L^6} \|\nabla u\|_{L^3}
\leq - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + C \|b_t\|_{L^2} \|b\|_{L^6} \|\nabla u\|_{L^3} \|u\|_{H^2}^{\frac{1}{2}}
\leq - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + \delta \|b_t\|_{L^2}^2 + \delta \|u\|_{H^2}^2 + C \|b\|_{L^6} \|\nabla u\|_{L^2}^2
\end{equation} (5.10)

for any $0 < \delta < 1$.

We use (5.11), (5.5) and (5.6) to bound $I_1$ as follows.
\begin{align*}
I_1 &\leq \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho}\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L_\infty^{\frac{2}{3}}}
\leq C \|\sqrt{\rho} u_t\|_{L^2} \|u\|_{L^6} \|\nabla u\|_{L^2} \|u\|_{H^2}^{\frac{1}{2}}
\leq \delta \|u_t\|_{L^2}^2 + \delta \|u\|_{H^2}^2 + C \|u\|_{L^6} \|\nabla u\|_{L^2}^2
\end{align*} (5.11)

for any $0 < \delta < 1$.

On the other hand, by the $H^2$-theory of the Stokes system, and using (5.11), (5.5) and (5.6), we obtain
\begin{align*}
\|u\|_{H^2} &\leq C \left\| - \Delta u + \nabla \left( \pi + \frac{1}{2} |b|^2 \right) \right\|_{L^2}
\leq C \|\rho \partial_t u + \rho u \cdot \nabla u - b \cdot \nabla b\|_{L^2}
\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L_\infty^{\frac{2}{3}}} + C \|b \cdot \nabla b\|_{L^2}
\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^2} \|u\|_{H^2}^{\frac{1}{2}} + C \|b \cdot \nabla b\|_{L^2},
\end{align*}
which gives
\begin{equation}
\|u\|_{H^2} \leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|b \cdot \nabla b\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^2}.
\end{equation} (5.12)
Testing (1.15) by \( b_t - \Delta b \), using (5.3) and (5.6), we deduce that
\[
\frac{d}{dt} \int_\Omega |\text{curl } b|^2 \, dx + \int_\Omega (|b_t|^2 + |\Delta b|^2) \, dx
\]
\[
= \int_{\Omega} (b \cdot \nabla u - u \cdot \nabla b)(b_t - \Delta b) \, dx
\]
\[
\leq (\|u\|_{L^\infty} \|\nabla b\|_{L^2}^2 + \|b_t\|_{L^6} \|\nabla u\|_{L^3})(\|b_t\|_{L^2} + \|\Delta b\|_{L^2})
\]
\[
\leq C(\|u\|_{L^\infty} \|\nabla b\|_{L^2}^{1/2} \|b_t\|_{H^1}^{1/2} + C\|b\|_{L^6} \|\nabla u\|_{L^2}^{1/2} \|u\|_{H^2}^{1/2})(\|b_t\|_{L^2} + \|\Delta b\|_{L^2})
\]
\[
\leq \frac{1}{2}(\|b_t\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \delta \|u\|_{H^2}^2 + C\|b\|_{L^6}^2 \|\nabla u\|_{L^2}^2 + C\|u\|_{L^6} \|\nabla b\|_{L^2}^2 + C \quad (5.13)
\]
for any \( 0 < \delta < 1 \).

It is easy to compute that
\[
\frac{d}{dt} \int_{\Omega} |b|^4 \, dx \leq \int_{\Omega} |b|^3 |b_t| \, dx
\]
\[
\leq C\|b_t\|_{L^6}^2 \|b_t\|_{L^2} \leq \delta \|b_t\|_{L^2}^2 + C\|b\|_{L^6}^6 \quad (5.14)
\]
for any \( 0 < \delta < 1 \).

Combining (5.9), (5.10), (5.11), (5.12), (5.13) and (5.14), and taking \( \delta \) small enough, we obtain
\[
\frac{d}{dt} \int_{\Omega} (|\nabla u|^2 + |\text{curl } b|^2 + b \otimes b : \nabla u + C_0|b|^4) \, dx + \int_{\Omega} (\rho|u_t|^2 + |b_t|^2 + |\Delta b|^2) \, dx + \|u\|_{H^2}^2
\]
\[
\leq C\|b\|_{L^6}^4 \|\nabla u\|_{L^2}^2 + C\|u\|_{L^\infty} \|\nabla u\|_{L^2}^2 + \|\text{curl } b\|_{L^2}^2 + C\|b \cdot \nabla b\|_{L^2}^2 + C \quad (5.15)
\]
Using (5.4), (5.7), (5.8) and the Gronwall inequality, we have
\[
\int_{\Omega} (|\nabla u|^2 + |\text{curl } b|^2 + b \otimes b : \nabla u + C_0|b|^4) \, dx
\]
\[
\leq \int_{\Omega} (|\nabla u_0|^2 + |\text{curl } b_0|^2 + b_0 \otimes b_0 : \nabla u_0 + C_0|b_0|^4) \, dx
\]
\[
+ C\|b\|_{L^\infty(t_0,t;L^p)}^4 \int_{t_0}^t \|\nabla u\|_{L^2}^2 \, d\tau + C(t - t_0) + C \int_{t_0}^t \|b \cdot \nabla b\|_{L^2}^2 \, d\tau \exp \left( \int_{t_0}^t \|u\|_{L^6}^2 \, d\tau \right)
\]
\[
\leq C(e + y)^{C_0} \exp \left[ \int_{t_0}^t \frac{\|u\|_{L^6}^2}{1 + \log(e + \|u\|_{L^6})} \, d\tau (1 + \log(e + y)) \right]
\]
\[
\leq C(e + y)^{C_0} \quad (5.16)
\]
Plugging (5.16) into (5.13) and integrating over \([t_0, t]\), we have
\[
\int_{t_0}^t \int_{\Omega} (\rho|u_t|^2 + |b_t|^2 + |\Delta b|^2) \, dx \, d\tau + \int_{t_0}^t \|u\|_{H^2}^2 \, d\tau \leq C(e + y)^{C_0} \quad (5.17)
\]
Applying $\partial_t$ to (1.15), testing by $u_t$, using (1.12) and (1.13), we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_t|^2 dx + \int_\Omega |\nabla u_t|^2 dx
= - \int_\Omega \rho u_t \cdot \nabla |u_t|^2 dx - \int_\Omega \rho u_t \cdot \nabla (u \cdot \nabla u \cdot u_t) dx
- \int_\Omega \rho u_t \cdot \nabla u \cdot u_t dx + \int_\Omega b \cdot \nabla u_t dx + \int_\Omega b_t \nabla : \nabla u_t dx
\leq C \|u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2}
+ C \|u\|_{L^6}^2 \|\Delta u\|_{L^2} \|u_t\|_{L^6} + C \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2}
+ C \|\sqrt{\rho} u_t\|^4_{L^4} \|\nabla u\|_{L^2} + C \|b\|_{L^6} \|b_t\|_{L^3} \|\nabla u_t\|_{L^2}
\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|^2_{L^2} \|\nabla u_t\|_{L^2}
+ C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|^2_{L^2} \|\nabla u_t\|_{L^2}
+ C \|b\|_{L^6} \|b_t\|_{L^3} \|\nabla u_t\|_{L^2}
\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|^2_{L^2} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|^2_{L^2} \|\nabla u_t\|_{L^2}
+ C \|b\|_{L^6} \|b_t\|_{L^3} \|\nabla u_t\|_{L^2}
\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|^2_{L^2} \|\sqrt{\rho} u_t\|^2_{L^2} + \|u\|^2_{H^2} + C \|b\|^2_{L^6} \|b_t\|^2_{L^3}
\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{4} \|\nabla u\|^2_{L^2} \|\sqrt{\rho} u_t\|^4_{L^2} + \|u\|^2_{H^2} + C \|\nabla u\|^4_{L^2} \|b_t\|^2_{L^2}. \quad (5.18)
$$

Applying $\partial_t$ to (1.15), testing by $b_t$ and using (1.12), we get

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega |b_t|^2 dx + \int_\Omega |\text{curl} \ b_t|^2 dx
= - \int_\Omega (u_t \cdot \nabla b - b_t \nabla u - b \cdot \nabla u_t) b_t dx
\leq \|u_t\|_{L^6} \|\nabla b\|_{L^2} \|b_t\|_{L^3} + \|\nabla u_t\|_{L^2} \|b_t\|^2_{L^4} + \|\nabla u_t\|_{L^2} \|b\|_{L^6} \|b_t\|_{L^3}
\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{4} \|\text{curl} \ b_t\|^2_{L^2} + C \|\text{curl} \ b_t\|^4_{L^2} \|b_t\|^2_{L^2} + C \|\nabla u\|^4_{L^2} \|b_t\|^2_{L^2}. \quad (5.19)
$$

Combining (5.18) and (5.19) and integrating over $[t_0, t]$, we have

$$
\int_\Omega (|\rho u_t|^2 + |b_t|^2) dx + \int_{t_0}^t \int_\Omega (|\nabla u_t|^2 + |\text{curl} \ b_t|^2) dx dt \leq C(e + y)^{C_{0e}}. \quad (5.20)
$$

Similarly to (5.12), we deduce that

$$
\|u\|_{H^2} \leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^3} + C \|b\|_{L^6} \|\nabla b\|_{L^3}
\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^2} \|u\|^\frac{1}{2}_{H^2} + C \|b\|_{L^6} \|\nabla b\|_{L^2} \|b\|^\frac{1}{2}_{H^2},
$$

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which leads to
\[
\|u\|_{H^2}^2 \leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|
abla u\|_{L^2}^6 + C\|
abla b\|_{L^2}^6 + \frac{1}{2}\|b\|_{H^2}^2. \tag{5.21}
\]
Similarly, we have
\[
\|b\|_{H^2}^2 \leq C\|b_t\|_{L^2}^2 + C\|u\|_{L^6}\|
abla b\|_{L^3} + C\|u\|_{L^6}\|
abla u\|_{L^3} \leq C\|b_t\|_{L^2}^2 + C\|u\|_{L^6}^2\|
abla b\|_{L^2}^6 + C\|b\|_{L^6}^2\|
abla u\|_{L^2}^6 + \frac{1}{2}\|b\|_{H^2}^2, \tag{5.22}
\]
which implies
\[
\|b\|_{H^2}^2 \leq C\|b_t\|_{L^2}^2 + C\|u\|_{L^6}^2\|
abla u\|_{L^2}^6 + \frac{1}{2}\|b\|_{H^2}^2. \tag{5.22}
\]
Combining (5.21) and (5.22), using (5.20) and (5.16), we conclude that
\[
\|u\|_{H^2}^2 + \|b\|_{H^2}^2 \leq C(e+y)^{C_0}, \tag{5.23}
\]
and thus
\[
\|u\|_{L^\infty(0,T;H^2)} + \|b\|_{L^\infty(0,T;H^2)} \leq C. \tag{5.24}
\]
Now it is standard to prove that
\[
\|u\|_{L^2(0,T;H^3)} + \|b\|_{L^2(0,T;H^3)} \leq C, \tag{5.25}
\]
\[
\|\rho\|_{L^\infty(0,T;H^2)} \leq C. \tag{5.26}
\]
(II) Let (1.22) hold true.
Similarly to (5.7), we take \(s = \infty\) and using (2.4), we still get (5.7) provided that
\[
\int_0^T \|u(t)\|_{BMO}^2 dt \leq \epsilon << 1. \tag{5.27}
\]
We still have (5.9), (5.10), (5.11) with \(s = \infty\), (5.12) with \(s = \infty\), (5.13) with \(s = \infty\), (5.14), (5.15) with \(s = \infty\), and then using (5.27) and (2.4), we arrive at (5.16) and (5.17). Then by the same calculations as that in (5.18)-(5.26), we conclude that (5.18)-(5.26) hold true.
This completes the proof of Theorem 1.3.

\[\square\]

6 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4, we only need to establish a priori estimates.
First, using the formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, and the fact that $|d| = 1$ implies $d\Delta d = -|\nabla d|^2$, we have the following equivalent equation

$$\frac{1}{2} \frac{d}{dt} - \frac{1}{2} d \times d_t = \Delta d + d|\nabla d|^2. \quad (6.1)$$

Testing (6.1) by $d_t$ and using $(a \times b) \cdot b = 0$ and $d \cdot d_t = 0$, we get

$$\frac{d}{dt} \int_\Omega |\nabla d|^2 dx + \int_\Omega |d_t|^2 dx \leq 0. \quad (6.2)$$

Testing (1.23) by $-\Delta d_t$ and using $|d| = 1$, we find that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta d|^2 dx + \int_\Omega |\nabla d_t|^2 dx = -\int_\Omega (d|\nabla d|^2 + d \times \Delta d) \cdot \Delta d_t dx$$

$$= \int_\Omega \nabla d(\nabla d)|\nabla d|^2 + d \times \Delta d \cdot \nabla d_t dx$$

$$\leq C(\nabla d|_{L^4} \nabla d|_{L^8}^2 \Delta d|_{L^2} + ||\nabla d||_{L^8} \Delta d|_{L^2} + ||\Delta \Delta d||_{L^2}) ||\nabla d_t||_{L^2}$$

$$\leq C(\nabla d|_{L^4} \Delta d|_{L^2} \Delta d|_{L^2} + ||\nabla d||_{L^8} \Delta d|_{L^2}) ||\nabla d_t||_{L^2}$$

$$\leq C(\nabla d|_{L^4} \Delta d|_{L^2} \delta ||d||_{H^3}^2 + C||\nabla d||_{L^8} \Delta d|_{L^2}) ||\nabla d_t||_{L^2}$$

$$\leq \frac{1}{4} ||\nabla d_t||_{L^2}^2 + \delta ||d||_{H^3}^2 + C||\nabla d||_{L^8} \Delta d|_{L^2}^2 \quad (6.3)$$

for any $0 < \delta < 1$. Here we have used the Gagliardo-Nirenberg inequalities:

$$||\nabla d||_{L^4} \Delta d|_{L^2} \leq C ||d||_{L^8} ||\Delta d||_{L^2} \quad (6.4)$$

$$||\Delta d||_{L^2} \Delta d|_{L^2} \leq C ||\Delta d||_{L^2} \delta ||d||_{H^3}^2 \quad (6.5)$$

Applying $\partial_t$ to (1.23), we get

$$\partial_t d_t - \Delta \partial_t d = \partial_t (d|\nabla d|^2) + \partial_t d \times \Delta d + d \times \Delta \partial_t d.$$

Testing the above equation by $\Delta \partial_t d$, summing over $i$, and using (6.4) and (6.5) and $|d| = 1$, we obtain

$$||d||_{H^3} \leq C(||d||_{L^2} + ||\nabla \Delta d||_{L^2})$$

$$\leq C + C||\nabla d_t||_{L^2} + C||\nabla (d|\nabla d|^2)||_{L^2} + \sum_i C||\partial_i d \times \Delta d||_{L^2}$$

$$\leq C + C||\nabla d_t||_{L^2} + C||\nabla d||_{L^8} ||\nabla d||_{L^8} \Delta d|_{L^2} + \sum_i C||\nabla (d|\nabla d|^2)||_{L^2}$$

$$\leq C + C||\nabla d_t||_{L^2} + C||\nabla d||_{L^8} ||\Delta d||_{L^2}^2$$

$$\leq C + C||\nabla d_t||_{L^2} + C||\nabla d||_{L^8} ||\Delta d||_{L^2}^2 \quad (6.6)$$

$$\leq C + C||\nabla d_t||_{L^2} + C||\nabla d||_{L^8} ||\Delta d||_{L^2}^2 ||d||_{H^3}^2,$$
which yields
\[ ||d||_{H^3} \leq C + C||\nabla d_t||_{L^2} + C||\nabla d||_{L^2}^{\frac{3}{2}}||\Delta d||_{L^2}. \] (6.6)

Plugging (6.6) into (6.3) and taking \( \delta \) small enough, we have
\[
\frac{d}{dt} \int_{\Omega} |\Delta d|^2 dx + \int_{\Omega} |\nabla d_t|^2 dx \\
\leq C + C||\nabla d||_{L^2}^{\frac{3}{2}}||\Delta d||_{L^2}^{\frac{3}{2}} \\
\leq C + C \frac{||\nabla d||_{L^2}^{\frac{3}{2}}}{1 + \log(e + ||\nabla d||_{L^2})} ||\Delta d||_{L^2}^2 \log(e + ||\nabla d||_{L^2}) \\
\leq C + C \frac{||\nabla d||_{L^2}^{\frac{3}{2}}}{1 + \log(e + ||\nabla d||_{L^2})} ||\Delta d||_{L^2}^2 (e + y),
\]
which implies
\[
\int_{\Omega} |\Delta d|^2 dx + \int_{t_0}^{t} \int_{\Omega} |\nabla d_t|^2 dx d\tau \leq C(e + y)^{C_0}. \] (6.7)

Provided that
\[
\int_{t_0}^{T} \frac{||\nabla d||_{L^2}^{\frac{3}{2}}}{1 + \log(e + ||\nabla d||_{L^2})} d\tau \leq \epsilon << 1,
\]
with \( y(t) := \sup_{[t_0, t]} ||d||_{H^3} \) for any \( 0 < t_0 \leq t \leq T \) and \( C_0 \) is an absolute constant.

It follows from (1.23), (6.6) and (6.7) that
\[
\int_{\Omega} |d_t|^2 dx + \int_{t_0}^{t} ||d||_{H^3}^2 d\tau \leq C(e + y)^{C_0}. \] (6.8)

Applying \( \partial_t \) to (1.23), testing by \(-\Delta d_t\), and using \( |d| = 1 \), (6.7) and (6.8), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d_t|^2 dx + \int_{\Omega} |\Delta d_t|^2 dx = - \int_{\Omega} [\partial_t(\nabla d_t^2) + d_t \times \Delta d_t] dx \\
\leq C(||\nabla d||_{L^6}||d_t||_{L^6} + ||\nabla d||_{L^6}||\nabla d_t||_{L^3} + ||d_t||_{L^\infty}||\Delta d||_{L^2}) ||\Delta d_t||_{L^2} \\
\leq C(||\nabla d||_{L^6}^2||d_t||_{L^6} + ||\Delta d||_{L^2}||\nabla d_t||_{L^2}^2 + ||\Delta d||_{L^2}||d_t||_{L^6}) ||\Delta d_t||_{L^2} \\
\leq \frac{1}{2} ||\Delta d_t||_{L^2}^2 + C||d||_{H^2}^4||d_t||_{H^1}^2 + C||d||_{H^2}^2||d_t||_{L^2}^2,
\]
which implies
\[
\int_{\Omega} |\nabla d_t|^2 dx + \int_{t_0}^{t} ||\Delta d_t||_{L^2}^2 d\tau \leq C(e + y)^{C_0}. \] (6.9)

It follows from (6.6), (6.7), (6.8) and (6.9) that
\[
||d||_{H^3} \leq C + C||\nabla d_t||_{L^2} + C||\nabla d||_{L^6}^2||\Delta d||_{L^2} \leq C(e + y)^{C_0}. 
\]
which leads to
\[
\|d\|_{L^\infty(0,T;H^3)} \leq C.
\]

This completes the proof of Theorem 1.4. □

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References

[1] Y. Giga, Solutions for semilinear parabolic equations in \( L^p \) and regularity of weak solutions of the Navier-Stokes system. J. Diff. Equ. 62(1986) 186-212.

[2] H. Kim, A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations. SIAM J. Math. Anal. 37(2006) 1417-1434.

[3] K. Kang, J. Kim, Regularity criteria of the magnetohydrodynamic equations in bounded domains or a half space. J. Diff. Equ. 253(2)(2012) 764-794.

[4] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations. Comm. Pure Appl. Math. 36(5)(1983) 635-664.

[5] H. Wu, Strong solution to the incompressible MHD equations with vacuum. Computers and Mathematics with Applications 61(2011) 2742-2753.

[6] G. Carbou, P. Fabrie, Regular solutions for Landau-Lifshitz equation in a bounded domain. Differential Integral Equations 14(2001) 213-229.

[7] J. Fan, T. Ozawa, Logarithmically improved regularity criteria for Navier-Stokes and related equations. Math. Meth. Appl. Sci. 32(2009) 2309-2318.

[8] T. Ogawa, Y. Taniuchi, A note on blow-up criterion to the 3D Euler equations in a bounded domain. J. Diff. Equ. 190(2003) 39-63.

[9] H. Beirão da Veiga, F. Cripo, Sharp inviscid limit results under Navier type boundary conditions. An \( L^p \) theory. J. Math. Fluid Mech. 12(2010) 397-411.

[10] R. A. Adams, J. F. Fournier, Sobolev Spaces. 2nd ed., Pure and Appl. Math. (Amsterdam), vol. 140, Amsterdam: Elsevier/Academic Press, 2003.
[11] A. Lunardi, Interpolation Theory. 2nd ed., Lecture Notes. Scuola Normale Superiore di Pisa (New Series), Edizioni della Normale, Pisa, 2009.

[12] T. Ogawa, Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow. SIAM J. Math. Anal. 34(2003) 1318-1330.

[13] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[14] Q. Chen, C. Miao, Z. Zhang, The Beale-Kato-Majda criterion to the 3D magnetohydrodynamics equations. Comm. Math. Phys. 275(2007) 861-872.

[15] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear evolution equations. Math. Z. 242(2002) 251-278.