Lebesgue constants for polyhedral sets and polynomial interpolation on Lissajous-Chebyshev nodes

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Abstract

To analyze the absolute condition number of multivariate polynomial interpolation on Lissajous-Chebyshev node points, we derive upper and lower bounds for the respective Lebesgue constant. The proof is based on a relation between the Lebesgue constant for the polynomial interpolation problem and the Lebesgue constant linked to the polyhedral partial sums of Fourier series. The magnitude of the obtained bounds is determined by a product of logarithms of the side lengths of the considered polyhedral sets and shows the same behavior as the magnitude of the Lebesgue constant for polynomial interpolation on the tensor product Chebyshev grid.

Keywords: interpolation, Lissajous-Chebyshev nodes, Lebesgue constants, polyhedra

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1 Introduction

In [11, 15, 16], a multivariate polynomial interpolation scheme was developed to interpolate function values on equidistant node points along Lissajous trajectories. The consideration of such node points is motivated by applications in a novel medical imaging modality called Magnetic Particle Imaging (MPI) (see [17, 18]). In this imaging technology, the magnetic response of superparamagnetic nanoparticles is measured along particular sampling paths generated by applied magnetic fields. For a typical kind of MPI scanner, these sampling paths are Lissajous curves.

In two dimensions, the polynomial interpolation scheme given in [15] was used to recover the distribution of the magnetic particles from a reduced reconstruction on equidistant nodes along the Lissajous trajectory [17]. A particular feature of this bivariate interpolation scheme is the fact that the self-intersection and the boundary points of Lissajous curves are used as interpolation nodes and that the spectral index set of the underlying polynomial space has a triangular structure. In [11], this bivariate construction was extended to higher dimensional Lissajous curves by using polynomial spaces with a particular polygonal spectral structure that will be studied in more detail in this work. In the literature, there exist also other polynomial approximation schemes that use Lissajous trajectories as generating curves. Two such constructions for three and more dimensions for polynomial spaces of a bounded total or maximal degree can be found in [6, 7]. Note that in the choice of the Lissajous curves and the polynomial spaces these constructions differ from the approach considered in this work.

Using polynomials for interpolation, special attention has to be given to the numerical condition of the interpolation scheme. In order to exclude bad conditioning, the structure of the interpolation nodes as well as the spectral structure of the polynomial interpolants have to be studied. The goal of this article is to provide such an analysis for the absolute condition number of the polynomial interpolation schemes considered in [11, 15, 16]. The interpolation nodes under consideration have been introduced in [11] as Lissajous-Chebyshev node points $\mathbf{LC}_{\kappa}^{(\epsilon)}$ (see (3.1)). In this notation, the parameters $\kappa \in \mathbb{Z}^d$ and $\epsilon \in \{1, 2\}$ determine the underlying types of Lissajous curves, and the vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ with pairwise relatively prime entries $n_1, \ldots, n_d \in \mathbb{N}$ (1.1) describes the frequencies of the Lissajous curve with respect to the coordinate axis. The interpolation problem itself is given as follows:

For the node points $\mathbf{LC}_{\kappa}^{(\epsilon)}$ and a function $f : [-1, 1]^d \to \mathbb{R}$ with values $f(\mathbf{z})$ at the node points $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbf{LC}_{\kappa}^{(\epsilon)}$, find a $d$-variate interpolation polynomial $P_{\kappa}^{(\epsilon)} f$ such that

$$P_{\kappa}^{(\epsilon)} f(\mathbf{z}) = f(\mathbf{z}) \quad \text{for all} \quad \mathbf{z} \in \mathbf{LC}_{\kappa}^{(\epsilon)}. \quad (1.2)$$

It was shown in [11] that the interpolation problem (1.2) has a unique solution in the polynomial space $\Pi_{\kappa}^{(\epsilon)}$ that is linearly spanned by all $d$-variate Chebyshev polynomials $T_{\gamma}$, where $\gamma$ is an element of the spectral index set $\Gamma_{\kappa}^{(\epsilon)}$. The spectral index set is given by

$$\Gamma_{\kappa}^{(\epsilon)} = \left\{ \gamma \in \mathbb{N}_0^d \mid \frac{\gamma_i}{n_i} < \epsilon \quad \forall i \in \{1, \ldots, d\}, \right. \quad \frac{\gamma_i}{n_i} + \frac{\gamma_j}{n_j} \leq \epsilon \quad \forall i, j \text{ with } i \neq j, \left. \frac{\gamma_i}{n_i} + \frac{\gamma_j}{n_j} < \epsilon \quad \forall i, j \text{ with } \kappa_i \neq \kappa_j \text{ mod } 2 \right\} \cup \{(0, \ldots, 0, \epsilon n_d)\}.$$
The nodes $\mathbf{LC}^{(\epsilon n}_n$, the Chebyshev polynomials $T_\gamma$, and the interpolation problem will be recapitulated in more detail in Section 3 of this article.

The absolute condition number of the interpolation problem (1.2) with respect to the uniform norm $\|f\|_\infty = \text{ess sup}_{x \in [-1,1]^d} |f(x)|$ (see [12, p. 26]) is given by the Lebesgue constant of the interpolation problem, i.e.

$$\Lambda^{(\epsilon n}_n = \sup_{f \in C([-1,1]^d): \|f\|_\infty \leq 1} \|P^{(\epsilon n}_n f\|_\infty. \quad (1.3)$$

Besides its relation to the numerical stability of the interpolation problem (1.2), the Lebesgue constant (1.3) is also an essential tool for the investigation of the approximation error $\|f - P^{(\epsilon n}_n f\|_\infty$.

A main goal of this article is to provide for all $n$ satisfying (1.1) asymptotic upper and lower bounds for the Lebesgue constants (1.3) in the sense of (1.13). The corresponding result in Theorem 3.4 states

$$\Lambda^{(\epsilon n}_n \preceq \prod_{i=1}^d \ln(n_i + 1). \quad (1.4)$$

In particular, the upper and lower estimates have asymptotically the same magnitude as the Lebesgue constants for polynomial interpolation on the tensor product Chebyshev grid (see [8]). Therefore, the interpolation problem (1.2) in $\Pi^{(\epsilon n}_n$ is asymptotically as well-conditioned as the mentioned tensor product case. The upper estimate in (1.4) of the Lebesgue constant $\Lambda^{(\epsilon n}_n$ is further used in Corollary 3.5 to formulate a multivariate error estimate and an example of a Dini-Lipschitz-type condition for the uniform convergence of the interpolation polynomials $P^{(\epsilon n}_n f$.

In the bivariate setting, the obtained results are generalizations of the corresponding results for the Padua points in [5,9,10] and improvements of estimates given in [14].

We sketch our program for the proof of (1.4). For a finite set $\Gamma \subset \mathbb{Z}^d$, the Lebesgue constant $L(\Gamma)$ related to partial Fourier series is defined as

$$L(\Gamma) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left| \sum_{\gamma \in \mathbb{Z}^d} e^{i\gamma \cdot t} \right| dt,$$

where

$$(\gamma \cdot t) = \sum_{i=1}^d \gamma_i t_i.$$

To obtain the upper and lower bounds for (1.3), our strategy in the proof of Theorem 3.4 consists in establishing the relations

$$\Lambda^{(\epsilon n}_n \preceq L\left(\Gamma^{(\epsilon n}_n,*,\right) + \prod_{i=1}^d \ln(n_i + 1), \quad L\left(\Gamma^{(\epsilon n}_n,*,\right) \preceq \Lambda^{(\epsilon n}_n \quad (1.5)$$

between $\Lambda^{(\epsilon n}_n$ and the Lebesgue constants $L\left(\Gamma^{(\epsilon n}_n,*,\right)$ of the symmetrized sets $\Gamma^{(\epsilon n}_n,*,$. Here and in the following, for every $\Gamma \subset \mathbb{Z}^d$ its symmetrization $\Gamma^*$ is defined as

$$\Gamma^* = \{ \gamma \in \mathbb{Z}^d \mid (|\gamma_1|, \ldots, |\gamma_d|) \in \Gamma \}.$$

Using the methods developed in Section 2, Corollary 3.3 states that

$$L\left(\Gamma^{(\epsilon n}_n,*,\right) \simeq \prod_{i=1}^d \ln(n_i + 1). \quad (1.7)$$
Then, combining (1.5) and (1.7) yields (1.4).

![Figure 1.1](image)

**Figure 1.1** Illustration of the sets $\Gamma^{(m)}$ (left) and $\Sigma_1^{(m)}$ (right) for $m = (5, 10, 5)$.

The technically more sophisticated part of the sketched program is the proof of (1.7). The used methods are developed in Section 2. Therein we consider the sets

$$\Gamma^{(m)} = \left\{ \gamma \in \mathbb{N}_0^d \mid \gamma_i/m_i \leq 1 \quad \forall i \in \{1, \ldots, d\}, \quad \gamma_i/m_i + \gamma_j/m_j \leq 1 \quad \forall i, j \text{ with } i \neq j \right\}$$

and its symmetrizations $\Gamma^{(m),*}$ according to (1.6). The used methods for these sets are templates for the corresponding methods for the sets $\Gamma_{\epsilon n}^{(m)}$ and $\Gamma_{\epsilon n,\kappa}^{(m)},*$, respectively. It turns out that similar methods can be used to estimate for fixed rational $r > 0$ the Lebesgue constants of families of sets $\Sigma^{(m)}$ and its symmetrizations $\Sigma^{(m),*}$, where

$$\Sigma^{(m)} = \left\{ \gamma \in \mathbb{N}_0^d \mid \sum_{i=1}^{d} \gamma_i/m_i \leq r \right\}.$$

Note that $\Gamma^{(m_1,m_2),*} = \Sigma_1^{(m_1,m_2),*}$ in dimension $d = 2$. Sets of this kind are illustrated in Figure 1.1 and are of interest since they might be used as elementary building blocks for more complex polyhedra. Further, our results could be useful for the investigation of generalizations of the triangular partial Fourier series in [28].

Estimates of the Lebesgue constant $L(\Gamma)$ for various types of sets $\Gamma$ are extensively investigated in the literature. An overview about the state of the art can be found in the survey article [20]. Since we are dealing with sets having a polyhedral structure, estimates of the Lebesgue constants for those sets are particularly interesting for us. If $E$ is a fixed $d$-dimensional convex polyhedron containing the origin, then it is well-known (see [3, 4, 23, 27, 30, 31]) that for all real $m \geq 1$ we have

$$L\left( mE \cap \mathbb{Z}^d \right) \asymp (\ln(m+1))^d.$$

In this work, we want to refine this asymptotic result for special $d$-dimensional polyhedra in which integer-valued directional dilation parameters $m_1, \ldots, m_d \in \mathbb{N}$ are given. An example for different directional parameters is the case of rectangular sets $R^{(m)} = [0, m_1] \times \cdots \times [0, m_d]$. In this case, for all $m_1, \ldots, m_d \geq 1$, we have

$$L\left( R^{(m)} \cap \mathbb{Z}^d \right) \asymp L\left( R^{(m),*} \cap \mathbb{Z}^d \right) \asymp \prod_{i=1}^{d} \ln(m_i + 1). \quad (1.8)$$

This immediately follows from the well-known one-dimensional case (see [1]).
The starting points for our investigations of $L(\Gamma)$ are two estimates of the Lebesgue constant given in \[30\] and \[31\]. In \[30\], Theorem 2 it is stated that for all polyhedra $E \in \mathbb{R}^2$ with $n$ edges, we have the uniform upper bound
\[
L(E \cap \mathbb{Z}^2) \lesssim n (\ln \text{diam}(E))^2.
\]
(1.9)
Further, it is shown in \[31\] that for all convex sets $E \in \mathbb{R}^d$ containing a ball with radius $r \geq 1$ we have the lower bound
\[
L(E \cap \mathbb{Z}^d) \gtrsim (\ln(r+1))^d.
\]
(1.10)
Combining (1.9) and (1.10) yields that for all real $m_1, m_2 \geq 1$ we have the uniform upper and lower bound
\[
(\ln \min(m_1, m_2) + 1)^2 \lesssim L(\Gamma(m_{1, m_2}, *)) \lesssim (\ln \max(m_1, m_2) + 1)^2.
\]
(1.11)
A special case of our result (see Theorem 2.1) is that for all positive integers $m_1, m_2$ we can improve (1.11) to
\[
L(\Gamma(m_{1, m_2}, *)) \approx \frac{\ln(m_1 + 1) \ln(m_2 + 1)}{d}.
\]
(1.12)
In general, Theorem 2.1 states that for all $m \in \mathbb{N}^d$ we have
\[
L(\Gamma(m)) \asymp L(\Gamma(m, \ast)) \asymp \prod_{i=1}^d \ln(m_i + 1).
\]
Thus, the magnitude of the uniform upper and lower bounds is the same as in the rectangular case (1.8). Similarly, Theorem 2.3 states that for a fixed $r \in \mathbb{Q}$, $r > 0$, and all $m \in \mathbb{N}^d$ we have
\[
L(\Sigma(m)) \asymp L(\Sigma(m, \ast)) \asymp \prod_{i=1}^d \ln(m_i + 1).
\]
In Section 2 we consider also another type of polyhedral sets given by
\[
\Xi(m_{r, s}) = \left\{ \gamma \in \mathbb{Z}^d \mid r \leq \frac{\gamma_d}{m_d} \leq \frac{\gamma_d}{m_2} \leq \frac{\gamma_d}{m_1} \leq s \right\}.
\]
(1.12)
For fixed $r, s \in \mathbb{R}$ and all positive integers $m_1, \ldots, m_d \in \mathbb{N}$, a uniform upper bound $L(\Xi(m_{r, s})) \lesssim \prod_{i=1}^d \ln(m_i + 1)$ is established for the corresponding Lebesgue constant in Theorem 2.2. The proof of the upper bound of the Lebesgue constants for the polyhedral sets $\Gamma(m)$ and $\Gamma(m, \ast)$ uses slightly generalized versions (see (2.40)) of the polyhedral sets (1.12) as building blocks. The techniques presented in the proofs of Section 2 are interesting in their own regard and might be as well useful for the consideration of other types of polyhedral sets.

**General notation**

For $x \in \mathbb{R}$, we use $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$, $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$ and denote
\[
\lfloor x \rfloor = x - \lfloor x \rfloor, \quad \lceil x \rceil = \lceil x \rceil - x.
\]
Let $f$ and $g$ be real functions on a set $X$. The notation $f(x) \lesssim g(x)$ for all $x \in X$
has by definition the following meaning:

There exists a constant \( C > 0 \) such that \( f(x) \leq C g(x) \) for all \( x \in X \).

Furthermore, we write \( f(x) \asymp g(x) \) for all \( x \in X \), (1.13) if for all \( x \in X \) we have both \( f(x) \lesssim g(x) \) and \( g(x) \lesssim f(x) \).

We write \( x = (x_1, \ldots, x_d) \) for elements of the Euclidean space \( \mathbb{R}^d \) with fixed \( d \in \mathbb{N} \).

For \( a, b \in \mathbb{R} \), \( a < b \), \( 1 \leq p < \infty \) and Lebesgue-measurable \( f : [a, b)^d \to \mathbb{R} \), we set

\[ \|f\|_{L^p([a,b)^d)} = \left( \frac{1}{(b-a)^d} \int_{[a,b)^d} |t|^p \, dt \right)^{1/p}, \]

and for Lebesgue-measurable functions \( f : [-1,1]^d \to \mathbb{R} \), and \( 1 \leq p < \infty \), we define

\[ \|f\|_{w_d,p} = \left( \frac{1}{\pi^d} \int_{[-1,1]^d} |f(x)|^p \, w_d(x) \, dx \right)^{1/p}, \quad w_d(x) = \prod_{i=1}^d \frac{1}{\sqrt{1-x_i^2}}. \]

2 Lebesgue constants for polyhedral partial sums of Fourier series

We summarize the main results of this section.

**Theorem 2.1** For all \( m \in \mathbb{N}^d \), we have

\[ L(\Gamma^{(m)}) \asymp L(\Gamma^{(m)},^+) \asymp d \prod_{i=1}^d \ln(m_i + 1). \]

In Section 3 we will apply this theorem to obtain estimates of the Lebesgue constant for the interpolation problem on the Lissajous-Chebyshev nodes. To prove Theorem 2.1 we will use the following statement which is also interesting by itself.

**Theorem 2.2** Let \( r, s \in \mathbb{R} \), \( 0 \leq r < s \), be fixed. For all \( m \in \mathbb{N}^d \), we have

\[ L(\Xi^{(m)}_{(r,s)}) \lesssim d \prod_{i=1}^d \ln(m_i + 1). \] (2.1)

Further, let us consider the sets \( \Sigma^{(m)}_r \) and \( \Sigma^{(m)}_r^+ \). These sets can be considered as another possible generalization of the sets considered in [30] for \( m \in \mathbb{Z}^d \), and they are interesting since they may be used as building blocks for certain polyhedra.

**Theorem 2.3** Let \( r \in \mathbb{Q} \), \( r > 0 \), be fixed. For all \( m \in \mathbb{N}^d \), we have

\[ L(\Sigma^{(m)}_r) \asymp L(\Sigma^{(m)}_r^+) \asymp \prod_{i=1}^d \ln(m_i + 1). \] (2.2)

The proofs of these results are given in Subsections 2.2, 2.1, and 2.3 respectively.

### 2.1 Proof of Theorem 2.2

Let us first formulate and prove several auxiliary statements.
For $d \in \mathbb{N}$, $m \in (0, \infty)^d$, and $r, s \in \mathbb{R}$, $r < s$, we set
\[ D_{(r,s)}(t) = \sum_{\mathbf{z} \in \Xi^{(m)}_{(r,s)}} e^{i \langle \mathbf{z}, t \rangle}. \]

Let $d \geq 2$ everywhere below. For $1 \leq k \leq d$, we denote
\[ D_{k,(r,s)}(t) = D_{(r,s)}^{(m)}(t_{1}, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{d}). \]

For $2 \leq k \leq d$, we introduce
\[ D_{k,(r,s)}^{(m)}(t) = D_{(r,s)}^{(m)}(t_{1}, \ldots, t_{k-2}, t_{k-1} + t_{k} m_{k}/m_{k-1}, t_{k+1}, \ldots, t_{d}), \]
\[ \Delta_{k,(r,s)}^{(m)}(t) = D_{k,(r,s)}^{(m)}(t) - D_{k,(r,s)}^{(m)}(t), \quad \text{(2.3)} \]
and
\[ F_{k,(r,s)}^{(m)}(t) = \frac{e^{i u_k}}{e^{i u_k} - 1} \sum_{\gamma = 1}^{\gamma_d} e^{i \gamma t_1 + \cdots + \gamma_{k-2} t_{k-2} + \gamma_{k+1} t_{k+1} + \cdots + \gamma_d t_d} \times e^{i \gamma_{k-1} t_{k-1} + t_{k} m_{k}/m_{k-1}} (e^{-i \gamma_{k-1} m_{k}/m_{k-1}} t_{k} - 1). \]

Here and in the following, \( \sum_{\gamma = 1}^{\gamma_d} \) means the sum over \( (\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_d) \in \Xi^{(m)}_{(r,s)} \).

In the special case $k = d$, for simplicity, we denote
\[ D_{d,(r,s)}^{(m)}(t) = D_{d,(r,s)}^{(m)}(t), \quad D_{d,(r,s)}^{(m)}(t) = D_{d,(r,s)}^{(m)}(t), \]
\[ \Delta_{d,(r,s)}^{(m)}(t) = \Delta_{d,(r,s)}^{(m)}(t), \quad F_{d,(r,s)}^{(m)}(t) = F_{d,(r,s)}^{(m)}(t), \]
and
\[ G_{(r,s)}^{(m)}(t) = \frac{1}{e^{i u_d} - 1} \left( \Delta_{d,(r,s)}^{(m)}(t) - (e^{i |r m_d| t_d} - 1) D_{d,(r,s)}^{(m)}(t) \right). \]

**Proposition 2.4** Let $m \in (0, \infty)^d$ and $r, s \in \mathbb{R}$, $r < s$. Then
\[ D_{(r,s)}^{(m)}(t) = G_{(r,s)}^{(m)}(t) + D_{d,(r,s)}^{(m)}(t) + F_{d,(r,s)}^{(m)}(t). \]

**Proof.** First, we show that
\[ D_{(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}) = G_{(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}) + D_{d,(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}) \]
\[ + F_{d,(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}). \]

Indeed, we have that $G_{(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}) + D_{d,(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d})$ equals
\[ \frac{1}{e^{i u_d} - 1} \Delta_{d,(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}) - \frac{e^{i |r m_d| t_d} - 1}{e^{i u_d} - 1} D_{(r,s)}^{(m_d-1)}(t_{d-1}) + D_{(r,s)}^{(m_d-1)}(t_{d-1}) + t_{d} m_d/m_{d-1} \]
\[ = \frac{1}{e^{i u_d} - 1} \left( e^{i u_d} D_{(r,s)}^{(m_d-1)}(t_{d-1} + t_{d} m_d/m_{d-1}) - e^{i |r m_d| t_d} D_{(r,s)}^{(m_d-1)}(t_{d-1}) \right), \]
and
\[ F_{d,(r,s)}^{(m_d-1,m_d)}(t_{d-1}, t_{d}) = \frac{e^{i u_d}}{e^{i u_d} - 1} \sum_{\gamma_{d-1} = |r m_{d-1}|}^{\gamma_{d-1}} e^{i \gamma_{d-1} t_{d-1} + t_{d} m_d/m_{d-1}} (e^{-i |r_{d-1} m_d| m_{d-1}} t_{d-1} - 1). \]

Now, (2.6) follows from
\[ e^{i \gamma_{d-1} t_{d-1}} e^{i (|r_{d-1} m_d| m_{d-1}) t_{d-1}} = e^{i u_d} e^{i \gamma_{d-1} t_{d-1} + t_{d} m_d/m_{d-1}} e^{-i r_{d-1} m_d m_{d-1} t_{d-1}}. \]
For the functions corresponding to the symbols \( S \in \{D, D^\circ, D^t, \Delta^\circ, G, F^\circ\} \), we have the descending recursive relation

\[
S_{(r,s)}^{(m_1,\ldots,m_d)}(t_1, \ldots, t_d) = \sum_{\gamma \in [rm_d]} e^{\gamma t_d} S_{(r,\gamma/m_d)}^{(m_1,\ldots,m_d)}(t_{i+1}, \ldots, t_d), \quad 1 \leq i \leq d - 2. \tag{2.7}
\]

Equality (2.6) means that we have (2.5) with \((m_{d-1},m_d)\) in place of \(m\), and \((t_{d-1},t_d)\) in place of \(t\). Thus, induction argument using the relation (2.7) for \( S \in \{D, G, D^\circ, F^\circ\} \) implies that for \( i \in \{d-2, \ldots, 2, 1\} \) we have (2.5) with \((m_1,\ldots,m_d)\) in place of \(m\), and \((t_1, \ldots, t_d)\) in place of \(t\). In particular, for \( i = 1 \), we have (2.5). \( \square \)

Next, for \( 1 \leq k \leq d - 1 \), we introduce

\[
D_{k, (r,s)}^{(m)}(t) = D_{(r,s)}^{(m_{k-1},m_k+1,\ldots,m_d)}(t_1, \ldots, t_{k-1}, t_{k+1} + t_k m_k/m_{k+1}, t_{k+2}, \ldots, t_d),
\]

and

\[
\Delta_{k, (r,s)}^{(m)}(t) = D_{k, (r,s)}^{(m)}(t) - D_{k, (r,s)}^{(m)}(t),
\]

and, using (2.4), we set

\[
F_{k, (r,s)}^{(m)}(t) = \frac{1}{e^{bt_k}} \sum_{\gamma = 0}^{\gamma t_k - 1} e^{(\gamma t_1 + \ldots + \gamma t_{k-1} + \gamma t_{k+2} + \ldots + \gamma t_d)} \times e^{b t_{k+1} + t_k m_k/m_{k+1}} (e^{\gamma t_{k+1} m_k/m_{k+1}} t_k - 1).
\]

We also denote

\[
G_{k, (r,s)}^{(m)}(t) = \frac{1}{e^{bt_k}} \left\{ \begin{array}{ll}
(D_{1, (r,s)}^{(m)}(t) - 1) D_{k, (r,s)}^{(m)}(t) - \Delta_{k, (r,s)}^{(m)}(t) & \text{if } k = 1, \\
\Delta_{k, (r,s)}^{(m)}(t) - (e^{[r m_d]} t_d - 1) D_{k, (r,s)}^{(m)}(t) & \text{if } k = d, \\
\Delta_{k, (r,s)}^{(m)}(t) - \Delta_{k, (r,s)}^{(m)}(t) & \text{if } 2 \leq k \leq d - 1,
\end{array} \right.
\]

\[
F_{k, (r,s)}^{(m)}(t) = \left\{ \begin{array}{ll}
-F_{k, (r,s)}^{(m)}(t) & \text{if } k = 1, \\
F_{k, (r,s)}^{(m)}(t) & \text{if } k = d, \\
F_{k, (r,s)}^{(m)}(t) - F_{k, (r,s)}^{(m)}(t) & \text{if } 2 \leq k \leq d - 1,
\end{array} \right.
\]

and

\[
H_{k, (r,s)}^{(m)}(t) = \left\{ \begin{array}{ll}
0 & \text{if } k = 1, \\
D_{k, (r,s)}^{(m)}(t) & \text{if } 2 \leq k \leq d.
\end{array} \right.
\]

**Proposition 2.5** Let \( m \in (0, \infty)^d \), \( r, s \in \mathbb{R} \), \( r < s \), and \( k \in \{1, \ldots, d\} \). Then

\[
D_{(r,s)}^{(m)}(t) = G_{k, (r,s)}^{(m)}(t) + H_{k, (r,s)}^{(m)}(t) + F_{k, (r,s)}^{(m)}(t). \tag{2.8}
\]

**Proof.** In the case \( k = d \), the equality (2.8) is proved in Proposition 2.4. Let us consider the case \( 2 \leq k \leq d - 1 \). By the definitions of \( G_{k, (r,s)}^{(m)}(t) \), \( \Delta_{k, (r,s)}^{(m)}(t) \) with \( k \) instead of \( d \), \((m_1, \ldots, m_k)\) instead of \( m \), and \((t_1, \ldots, t_k)\) instead of \( t \), we have

\[
G_{k, (r,s)}^{(m_1,\ldots,m_k)}(t_1, \ldots, t_k)
= \frac{1}{e^{bt_k}} \Delta_{k, (r,s)}^{(m_1,\ldots,m_k)}(t_1, \ldots, t_k) - \frac{e^{[r m_k]} t_k - 1}{e^{bt_k} - 1} D_{k, (r,s)}^{(m_1,\ldots,m_k)}(t_1, \ldots, t_k)
= \frac{1}{e^{bt_k} - 1} D_{k, (r,s)}^{(m_1,\ldots,m_k)}(t_1, \ldots, t_k) - \frac{e^{[r m_k]} t_k}{e^{bt_k} - 1} D_{k, (r,s)}^{(m_1,\ldots,m_k)}(t_1, \ldots, t_k).
\]

At the same time, Proposition 2.4 with \((m_1, \ldots, m_k)\) instead of \( m \) and \((t_1, \ldots, t_k)\)
Thus, we get (2.12) and therefore (2.11).

\[ D_{(r,s)}^{(m_1, \ldots, m_k)}(t_1, \ldots, t_k) + \frac{e^{\frac{1}{m_1}}}{e^{t_k} - 1} D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k)}(t_1, \ldots, t_k) = \frac{e^{t_k}}{e^{t_k} - 1} D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k)}(t_1, \ldots, t_k) + F_{k(r,s)}^{\varphi,(m_1, \ldots, m_k)}(t_1, \ldots, t_k). \]  \hspace{1cm} (2.9)

For the functions corresponding to the symbols \( S \in \{ D, G, H, F \} \), we have the ascending recursion relation

\[ S_{k(r,s)}^{(m_1, \ldots, m_k)}(t_1, \ldots, t_i) = \sum_{\gamma = [r m_i]}^{[m_i]} e^{\gamma t_k} S_{k(r,s)}^{(m_1, \ldots, m_{i-1})}(t_1, \ldots, t_{i-1}), \quad k + 2 \leq i \leq d. \]  \hspace{1cm} (2.10)

Below, we will show that (2.8) is satisfied with \( (m_1, \ldots, m_k) \) and \( (t_1, \ldots, t_{k+1}) \) instead of \( t \) and \( (1, \ldots, t_k) \) instead of \( t^* \).

Thus, it remains to show (2.11). By the definitions of \( G_{k(r,s)}^{(m)}(t) \) and \( F_{k(r,s)}^{(m)}(t) \) with \( (m_1, \ldots, m_k) \) and \( (t_1, \ldots, t_{k+1}) \) instead of \( m \) and \( t \), we have

\[ G_{k(r,s)}^{(m_1, \ldots, m_k)}(t_1, \ldots, t_{k+1}) = D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k)}(t_1, \ldots, t_{k+1}) - D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}) \]  \hspace{1cm} (2.11)

and

\[ F_{k(r,s)}^{(m_1, \ldots, m_k)}(t_1, \ldots, t_{k+1}) = F_{k(r,s)}^{\varphi,(m_1, \ldots, m_k)}(t_1, \ldots, t_{k+1}) - F_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}). \]

Therefore, (2.11) is equivalent to

\[ D_{(r,s)}^{(m_1, \ldots, m_k)}(t_1, \ldots, t_{k+1}) + \frac{1}{e^{t_k} - 1} D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}) + F_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}) = \frac{e^{t_k}}{e^{t_k} - 1} D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}) + F_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}). \]  \hspace{1cm} (2.12)

Now, we observe that for \( S \in \{ D, D^*, F^* \} \) the equation in (2.10) is satisfied also for \( i = k + 1 \). Hence (2.9) implies that (2.12) and, therefore, (2.11) is equivalent to

\[ \frac{1}{e^{t_k} - 1} \sum_{\gamma_i = [r m_i]}^{[m_i]} e^{\gamma_i t_{k+1}} e^{\frac{1}{m_i} \gamma_i m_k/m_{k+1}} D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_k) = \frac{1}{e^{t_k} - 1} D_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}) + F_{k(r,s)}^{\varphi,(m_1, \ldots, m_k+1)}(t_1, \ldots, t_{k+1}). \]  \hspace{1cm} (2.13)

But (2.13) easily follows from

\[ e^{\gamma_k t_{k+1} + \frac{1}{m_k} \gamma_k m_k/m_{k+1}} e^{t_k} = e^{\gamma_k t_{k+1} + \frac{1}{m_k} \gamma_k m_k/m_{k+1}} e^{t_k}. \]  \hspace{1cm} (2.14)

Thus, we get (2.12) and therefore (2.11).

Finally, we consider the case \( k = 1 \). Equation (2.14) yields

\[ D_{(r,s)}^{(m_1, m_2)} = G_{1(r,s)}^{(m_1, m_2)} + F_{1(r,s)}^{(m_1, m_2)} = G_{1(r,s)}^{(m_1, m_2)} + H_{1(r,s)}^{(m_1, m_2)} + F_{1(r,s)}^{(m_1, m_2)}. \]
Thus, induction arguments and the relation (2.10) for $S \in \{D, G, H, F\}$ yield that for $i \in \{3, 4, \ldots, d\}$ we have (2.8) with $(m_1, \ldots, m_i)$ in place of $m,$ and $(t_1, \ldots, t_i)$ in place of $t$. In particular, for $i = d$ we have the assertion (2.8). \hfill $\square$

**Proposition 2.6** Let $r, s \in \mathbb{R}, r < s$, be fixed. Then, for all $m \in [1, \infty)^d$ and all $k \in \{1, \ldots, d\}$ we have

$$
\|G^{(m)}_{k,(r,s)}\|_{L^1([-\pi,\pi]^d)} \lesssim \ln(m_k + 1) \|D^{m_{1},\ldots,m_{k-1},m_{k+1},\ldots,m_d}_{(r,s)}\|_{L^1([-\pi,\pi]^{d-1})}.
$$

**Proof.** By using the inequality

$$
\frac{1}{e^{it} - 1} \lesssim \frac{1}{|t|}, \quad t \in [-\pi, \pi) \setminus \{0\}, \tag{2.15}
$$

it is easy to see that for all $m_1, m_d \in [1, \infty)$ we have

$$
\int_{[-\pi,\pi)} \left| \frac{e^{i(|r|+1)t} - 1}{e^{it} - 1} \right| dt \lesssim \ln(m_1 + 1) \tag{2.16}
$$

and

$$
\int_{[-\pi,\pi)} \left| \frac{e^{i|m|t} - 1}{e^{it} - 1} \right| dt \lesssim \ln(m_d + 1). \tag{2.17}
$$

Let $k \in \{2, \ldots, d\}$. Denoting

$$
A_k(m_k) = \left\{ t \in [-\pi, \pi]^d \mid |t_k| \leq \frac{1}{m_k + 1} \right\}, \quad B_k(m_k) = [-\pi, \pi)^d \setminus A_k(m_k), \tag{2.18}
$$

we have

$$
\int_{[-\pi,\pi]^d} \left| \frac{\Delta_{k,(r,s)}^{(m)}(t)}{e^{it} - 1} \right| dt = \int_{A_k(m_k)} \left| \frac{\Delta_{k,(r,s)}^{(m)}(t)}{e^{it} - 1} \right| dt + \int_{B_k(m_k)} \left| \frac{\Delta_{k,(r,s)}^{(m)}(t)}{e^{it} - 1} \right| dt = I + J. \tag{2.19}
$$

By using (2.3) and (2.15), for all $m \in [1, \infty)^d$, we obtain

$$
J \lesssim \int_{B_k(m_k)} \left| \frac{\Delta_{k,(r,s)}^{(m)}(t)}{e^{it} - 1} \right| dt \lesssim \int_{B_k(m_k)} \frac{1}{|t_k|} \left| \frac{\Delta_{k,(r,s)}^{(m)}(t)}{e^{it} - 1} \right| dt \lesssim \ln(m_k + 1) \|D^{m_{1},\ldots,m_{k-1},m_{k+1},\ldots,m_d}_{(r,s)}\|_{L^1([-\pi,\pi]^{d-1})} \tag{2.20}
$$

and

$$
I \lesssim \int_{A_k(m_k)} \frac{1}{|t_k|} \left| \frac{\Delta_{k,(r,s)}^{(m)}(t)}{e^{it} - 1} \right| dt. \tag{2.21}
$$

In the following, we will use the next two well-known statements: For all continuously differentiable $2\pi$-periodic $g : \mathbb{R} \to \mathbb{R}$ and $\delta \in \mathbb{R}$ (see [13, p. 46]):

$$
\|g(\cdot + \delta) - g\|_{L^1([-\pi,\pi])} \leq \|\delta\|_{L^1([-\pi,\pi])}. \tag{2.22}
$$

For all trigonometric polynomials $\tau_n$ of degree at most $n$, one has (see [13, p. 102]):

$$
\|\tau_n\|_{L^1([-\pi,\pi])} \leq n \|\tau_n\|_{L^1([-\pi,\pi])}. \tag{2.23}
$$

Denoting $D_{(r,s)}^\circ = D_{(r,s)}^{m_{1},\ldots,m_{k-1},m_{k+1},\ldots,m_d}$ and $\delta = t_k m_k/m_{k-1}$, we can write

$$
\Delta_{k,(r,s)}^{(m)}(t) = D_{(r,s)}(t_1, \ldots, t_{k-2}, t_{k-1} + \delta, t_{k+1}, \ldots, t_d) - D_{(r,s)}^\circ(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d).
$$

Since the degree of the trigonometric polynomial $D_{(r,s)}^\circ(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d)$ in the
variable $t_{k-1}$ is at most $\lfloor sm_{k-1} \rfloor$, using (2.22) and (2.23), we obtain
\[
\int_{[-\pi, \pi]} |\Delta^k_m(t_s(t)| dt_{k-1} \leq \left| t_k \right| \frac{m_k}{m_{k-1}} \int_{[-\pi, \pi]} \left| D^s_{(r,s)}(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_k) \right| dt_k.
\]
Thus, since (2.21), we have for all $m \in [1, \infty)^d$ that
\[
I \lesssim m_k \int_{[-\pi, \pi]} |\Delta^k_m(t_s(t)| dt_k \lesssim \left\| D^{(m)}_{(r,s)} \right\|_{L^1((-\pi, \pi)^d-1)} \leq \left\| D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_k)}_{(r,s)} \right\|_{L^1((-\pi, \pi)^d-1)}.
\]
Combining this with (2.20), (2.19) yields: For $m \in [1, \infty)^d$, $k \in \{2, \ldots, d\}$, we have
\[
\int_{[-\pi, \pi]^d} \left| \Delta^k_m(t_s(t)| \frac{1}{e^{\nu_k} - 1} \right| dt \lesssim \ln(m_k + 1) \|D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_k)}_{(r,s)}\|_{L^1((-\pi, \pi)^d-1)}.
\]  
(2.24)

In analogy to (2.24), we derive that for all $m \in [1, \infty)^d$ and all $k \in \{1, \ldots, d-1\}$
\[
\int_{[-\pi, \pi]^d} \left| \Delta^k_m(t_s(t)| \frac{1}{e^{\nu_k} - 1} \right| dt \lesssim \ln(m_k + 1) \|D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_k)}_{(r,s)}\|_{L^1((-\pi, \pi)^d-1)}.
\]  
(2.25)

Finally, having in mind the definition of $G^k_m(r,s)$, we finish the proof by combining the inequalities (2.24), (2.25), (2.16), and (2.17).

\[
\square
\]

**Proposition 2.7** Let $r, s \in \mathbb{R}$, $r < s$, be fixed. Then,

\begin{enumerate}[a)]
\item for all $m \in \mathbb{N}^d$ and all $k \in \{2, \ldots, d\}$, we have
\[
\left\| F^k_m(t_s(t)|_{L^1((-\pi, \pi)^d)} \right\| \lesssim \ln(m_k - 1) \|D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)}_{(r,s)}\|_{L^1((-\pi, \pi)^d-1)},
\]
\end{enumerate}
\[
(2.26)
\]

\begin{enumerate}[a)]
\item for all $m \in \mathbb{N}^d$ and all $k \in \{1, \ldots, d-1\}$, we have
\[
\left\| F^k_m(t_s(t)|_{L^1((-\pi, \pi)^d)} \right\| \lesssim \ln(m_k + 1) \|D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)}_{(r,s)}\|_{L^1((-\pi, \pi)^d-1)}.
\]
\end{enumerate}
\[
(2.27)
\]

**Proof.** Let $k \in \{2, \ldots, d\}$. We will show (2.26) for all $m \in \mathbb{N}^d$. Denote
\[
Q^{(m)}_{k,r}(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d)
= \sum_{\nu=0}^{\nu_k} e^{(\pi_1 + \ldots + \pi_{k-1} + \ldots + \pi_{d-1})} \gamma_k \frac{m_k}{m_{k-1}}
\]
where $\sum^{\nu_k}$ is given by (2.4). Using the equality
\[
1 \frac{1}{t_k} (e^{-1/\gamma_k m_k/m_{k-1}} t_k - 1) = \sum_{\nu=1}^{\infty} \frac{1}{\nu!} (1)^\nu \gamma_k \frac{m_k}{m_{k-1}}^\nu t_k^{\nu-1},
\]
and (2.15), we obtain that for all $m \in \mathbb{N}^d$ and all $t \in [-\pi, \pi]^d$
\[
\left| F^k_m(t_s(t)|_{L^1((-\pi, \pi)^d)} \right| \lesssim \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left| Q^{(m)}_{k,r}(t_1, \ldots, t_{k-2}, t_{k-1} + t_k m_k/m_{k-1}, t_{k+1}, \ldots, t_d) \right|.
\]
\[
(2.28)
\]

We conclude that for all $m \in \mathbb{N}^d$ the following inequality holds
\[
\left\| F^k_m(t_s(t)|_{L^1((-\pi, \pi)^d)} \right\| \lesssim \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left| Q^{(m)}_{k,r}\right|_{L^1((-\pi, \pi)^d-1)}.
\]
Thus, to prove (2.26) it is sufficient to verify that for all $\nu \geq 1$ and all $m \in \mathbb{N}^d$
\[
\|Q^{(m)}_{k,r}\|_{L^1((-\pi, \pi)^d-1)} \lesssim \ln(m_k^{1+1}) \|D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)}_{(r,s)}\|_{L^1((-\pi, \pi)^d-1)}.
\]  
(2.29)
For this we will use the 1-periodic function $h_{\nu,m}$, $m \geq 1$, determined by
\begin{equation}
\begin{aligned}
h_{\nu,m}(t) &= \begin{cases} 
\nu^t & \text{if } 0 \leq t \leq 1 - m^{-1}, \\
m (1 - m^{-1})^\nu (1 - t) & \text{if } 1 - m^{-1} \leq t < 1.
\end{cases}
\end{aligned}
\end{equation}
(2.30)

Let us abbreviate $m = m_{k-1}$. Since $\gamma_{k-1}$, $m$, and $m_k$ are integers, we have that $0 \leq \lfloor \gamma_{k-1} m_k / m \rfloor \leq 1 - m^{-1}$. Thus, taking into account that by 1-periodicity of $h_{\nu,m}$ we have $h_{\nu,m}(t) = h_{\nu,m}(\lfloor t \rfloor)$, $t \in \mathbb{R}$, we derive
\begin{equation}
h_{\nu,m}(\lfloor \gamma_{k-1} m_k / m \rfloor) = \lfloor \gamma_{k-1} m_k / m \rfloor^\nu.
\end{equation}
(2.31)

Next, by the Fourier inversion theorem, it holds
\begin{equation}
\hat{h}_{\nu,m}(\nu) = \int_{[0,1]} h_{\nu,m}(t)e^{-2\pi i \nu t} dt.
\end{equation}
(2.32)

where
\begin{equation}
\hat{h}_{\nu,m}(\nu) = \int_{[0,1]} h_{\nu,m}(t)e^{-2\pi i \nu t} dt.
\end{equation}
(2.33)

Combining (2.28), (2.32), (2.31), we get that $Q_{k,m}^m(t_1, \ldots, t_k, t_{k+1}, \ldots, t_d)$ equals
\begin{equation}
\sum_{\mu \in \mathbb{N}} e^{i(\gamma_{k-1} m_{k-1} + \cdots + \gamma_{k-1} m_k / m)} \sum_{\mu \in \mathbb{Z}} \hat{h}_{\nu,m}(\mu) D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)}(t_1, \ldots, t_{k-2}, t_k, t_{k+1}, \ldots, t_d)
\end{equation}
in $L^1([-\pi, \pi]^{d-1})$. Hence, we have for all $\nu, m \geq 1$, we have
\begin{equation}
\sum_{\mu \in \mathbb{N}} |\hat{h}_{\nu,m}(\mu)| \lesssim \ln(m \nu + 1).
\end{equation}
(2.34)

Combining this inequality and (2.33), we get (2.29) and, therefore, we have (2.26).

By analogy, we can prove (2.27).

\textbf{Proof of Theorem 2.2} Inequality (2.1) is well-known for $d = 1$, since
\begin{equation}
L^1(\bigotimes_{(r,s)} \mathcal{E}_s) = \|D^{(m)}_{(r,s)}\|_{L^1([-\pi, \pi]^d)}.
\end{equation}
(2.35)

Let $d \geq 2$. For all $k \in \{1, \ldots, d\}$, we get by Proposition 2.5 that
\begin{equation}
\|D^{(m)}_{(r,s)}\|_{L^1([-\pi, \pi]^d)} \leq \|G^{(m)}_{k,(r,s)}\|_{L^1([-\pi, \pi]^d)} + \|H^{(m)}_{k,(r,s)}\|_{L^1([-\pi, \pi]^d)} + \|F^{(m)}_{k,(r,s)}\|_{L^1([-\pi, \pi]^d)}.
\end{equation}
(2.36)

Clearly, for all $\nu, m \in \mathbb{N}^d$ and all $k \in \{1, \ldots, d\}$, we have
\begin{equation}
\|H^{(m)}_{k,(r,s)}\|_{L^1([-\pi, \pi]^d)} \leq \|D^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)}_{(r,s)}\|_{L^1([-\pi, \pi]^{d-1})}
\end{equation}
(2.37)

and, by Proposition 2.6 for all $\nu, m \in \mathbb{N}^d$ and all $k \in \{1, \ldots, d\}$ we have
\begin{equation}
\|G^{(m)}_{k,(r,s)}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_k + 1) \|L^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)}_{(r,s)}\|_{L^1([-\pi, \pi]^{d-1})}.
\end{equation}
(2.38)

Thus, we need to estimate only $\|F^{(m)}_{k,(r,s)}\|_{L^1([-\pi, \pi]^d)}$. This is done with a particular
Let \( k = k(m) \) be such that \( m_i \leq m_k \) for all \( i \in \{1, \ldots, d\} \). We consider the following three cases:

(i) If \( k = k(m) \in \{2, \ldots, d-1\} \), we have
\[
\| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} \leq \| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} + \| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)}
\]
and
\[
\ln (m_{k-1} + 1) + \ln (m_{k+1} + 1) \leq 2 \ln (m_k + 1).
\]

(ii) If \( k = k(m) = 1 \), we have
\[
\| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} = \| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} \quad \text{and} \quad \ln (m_{k+1} + 1) \leq \ln (m_{k} + 1).
\]

(iii) If \( k = k(m) = d \), we have
\[
\| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} = \| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} \quad \text{and} \quad \ln (m_{k-1} + 1) \leq \ln (m_{k} + 1).
\]
Therefore, by Proposition 2.7, we get that for all \( m \in \mathbb{N}^d \) and \( k = k(m) \) we have
\[
\| F_{k(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} \lesssim \ln (m_k + 1) \| D_{(r,s)}^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)} \|_{L^1([−\pi,\pi]^d-1)}.
\]
Combining (2.36), (2.37), (2.38), and (2.39), we get that for all \( m \in \mathbb{N}^d \) and \( k = k(m) \)
\[
\| D_{(r,s)}^{(m)} \|_{L^1([−\pi,\pi]^d)} \lesssim \ln (m_k + 1) \| D_{(r,s)}^{(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d)} \|_{L^1([−\pi,\pi]^d-1)}.
\]
Because of (2.35), we get the assertion (2.1) by a simple induction argument. \( \square \)

### 2.2 Proof of Theorem 2.1

Let \( r, s \in \mathbb{R} \), \( s \geq r > 0 \), \( m = (m_1, \ldots, m_d) \in (0, \infty)^d \), \( d \in \mathbb{N} \). Let \( S_d \) be the set of all permutations of \( \{1, \ldots, d\} \), i.e. the set of bijections from \( \{1, \ldots, d\} \) onto \( \{1, \ldots, d\} \). For \( \sigma \in S_d \) and \((<0, \ldots, <d>) \in \{<, <, =\}^{d+1} \), let
\[
\Xi_{(r,s),\sigma,(<0,\ldots,<d)}^{(m)} = \left\{ \gamma \in \mathbb{Z}^d \left| \gamma_0 \leq \frac{\gamma_{\sigma(d)}}{m_{\sigma(d)}} \leq \gamma_{d-1} \leq \cdots \leq \gamma_{d-1} \leq \frac{\gamma_{\sigma(1)}}{m_{\sigma(1)}} \leq s \right. \right\}.
\]

**Proposition 2.8** Let \( r, s \in \mathbb{R} \), \( s \geq r > 0 \), be fixed. Then, for \( m \in \mathbb{N}^d \), \( \sigma \in S_d \), and \((<0, \ldots, <d>) \in \{<, <, =\}^{d+1} \), we have
\[
L_{\Xi^{(m)}_{(r,s),\sigma,(<0,\ldots,<d)}}^d \lesssim \prod_{i=1}^d \ln (m_i + 1).
\]

**Proof.** Note that if (2.41) is proved for the identity permutation \( \sigma = \text{id} |_{\{1, \ldots, d\}} \), then (2.41) immediately follows for all \( \sigma \in S_d \). Furthermore, we can restrict the considerations to \((<0, \ldots, <d>) \in \{<, <, =\}^{d+1} \). Thus, the proof follows the lines of the proof of Theorem 2.2 in an obvious way. \( \square \)

We will use sets of the form (2.40) as building blocks in order to prove the upper estimate in Theorem 2.1 Let us formulate a technical auxiliary statement.

**Lemma 2.9** Let \( X^{(m)} \) be a set of subsets of \( \mathbb{R}^d \) and \( m \in \mathbb{N}^d \). For \( N \in \mathbb{N} \) we denote
\[
X_{0,N}^{(m)} = \left\{ \Xi_1 \cap \ldots \cap \Xi_N \left| \Xi_1, \ldots, \Xi_N \in X^{(m)} \right. \right\}, \quad \text{and} \quad X_{\cup,N}^{(m)} = \left\{ \Xi_1 \cup \ldots \cup \Xi_N \left| \Xi_1, \ldots, \Xi_N \in X^{(m)} \right. \right\}.
\]
Assume that $N \in \mathbb{N}$ is fixed and that for $m \in \mathbb{N}^d$ and all $\Xi \in \mathcal{X}^{(m)}$ we have

$$L(\Xi) \leq \prod_{i=1}^{d} \ln(m_i + 1).$$

(2.43)

Then, for the fixed $N \in \mathbb{N}$, the estimate (2.43) holds also for all $\Xi \in \mathcal{X}^{(m)}$.

**Proof.** The well-known inclusion–exclusion principle yields

$$L\left(\Xi_1 \cup \ldots \cup \Xi_j\right) \leq \sum_{k=1}^{j} \left( \sum_{1 \leq l_1 < \ldots < l_k \leq j} L\left(\Xi_{l_1} \cap \ldots \cap \Xi_{l_k}\right) \right).$$

Since $N$ is fixed, we conclude the assertion. \qed

For $m \in \mathbb{N}^d$, we consider the sets

$$\Gamma_{0}^{(m)} = \left\{ \gamma \in \mathbb{N}_0^d \mid \forall i : 2 \gamma_i \leq m_i \right\}, \quad \Gamma_{1}^{(m)} = \left\{ \gamma \in \mathbb{N}_0^d \mid \forall i : 2 \gamma_i < m_i \right\},$$

and we use the notation

$$K^{(m)}[\gamma] = \left\{ i \in \{1, \ldots, d\} \mid \gamma_i/m_i = \max^{(m)}[\gamma_i] \right\}$$

(2.44)

with $\max^{(m)}[\gamma] = \max\{\gamma_i/m_i \mid i \in \{1, \ldots, d\}\}$, and for $\emptyset \neq K \subseteq \{1, \ldots, d\}$, we denote

$$\Gamma_{1}^{(m)K} = \left\{ \gamma \in \Gamma_{1}^{(m)} \mid K^{(m)}[\gamma] = K \right\}.$$  

**Proposition 2.10** Let $d \geq 2$ and $\emptyset \neq K = \{k_1, \ldots, k_h\} \subseteq \{1, \ldots, d\}$, $k_1 < \ldots < k_h$. Then $\Gamma_{1}^{(m)K}$ is equal to

$$\bigcup_{\sigma \in S_{d,K}} \left\{ \gamma \in \mathbb{N}_0^d \mid 0 \leq \frac{\gamma(\sigma(1))}{m_{\sigma(1)}} \leq \ldots \leq \frac{\gamma(\sigma(h) + 1)}{m_{\sigma(h)}} < \frac{\gamma(\sigma(h))}{m_{\sigma(h)}} = \ldots = \frac{\gamma(\sigma(1))}{m_{\sigma(1)}} < \frac{1}{2} \right\},$$

(2.45)

where $S_{d,K} = \{ \sigma \in S_d \mid \sigma(1) = k_1, \ldots, \sigma(h) = k_h \}$.  

**Proof.** By the definition, we have

$$\Gamma_{1}^{(m)K} = \left\{ \gamma \in \mathbb{N}_0^d \mid \forall j \notin K : 0 \leq \frac{\gamma_j}{m_j} < \frac{\gamma_{k_h}}{m_{k_h}} = \ldots = \frac{\gamma_{k_1}}{m_{k_1}} < \frac{1}{2} \right\}.$$  

(2.46)

Since for $\sigma \in S_{d,K}$ we have $\sigma(1) = k_1, \ldots, \sigma(h) = k_h$ and $\sigma(h + 1), \ldots, \sigma(d) \notin K$, we conclude that (2.45) is a subset of (2.46).

Now, let $\gamma$ be an element of (2.46). Then, there exist $j_{h+1}, \ldots, j_d$ such that

$$\{j_{h+1}, \ldots, j_d\} = \{1, \ldots, d\} \setminus K \quad \text{and} \quad \frac{\gamma_{j_d}}{m_{j_d}} \leq \ldots \leq \frac{\gamma_{j_{h+1}}}{m_{j_{h+1}}}.$$  

We set $\sigma(1) = k_1, \ldots, \sigma(h) = k_h$ and $\sigma(h + 1) = j_{h+1}, \ldots, \sigma(d) = j_d$. Then, $\sigma \in S_{d,K}$ and $\gamma$ is an element of the corresponding set in the union (2.45). \qed

**Corollary 2.11** Let $\emptyset \neq K \subseteq \{1, \ldots, d\}$. Then, for all $m \in \mathbb{N}^d$, we have

$$L\left(\Gamma_{1}^{(m)K}\right) \leq \prod_{i=1}^{d} \ln(m_i + 1).$$

(2.47)

**Proof.** For $K = \{1, \ldots, d\}$, we have

$$\Gamma_{1}^{(m),(1,\ldots,d)} = \left\{ \gamma \in \mathbb{N}_0^d \mid 0 \leq \frac{\gamma_d}{m_d} = \ldots = \frac{\gamma_1}{m_1} < \frac{1}{2} \right\} = \Xi^{(m)}(0,1/2),id,(\leq,=,\ldots,\leq).$$

Thus, Proposition 2.8 implies (2.47).
Let us now consider the case \( d > 2 \) with a non-empty set \( \emptyset \neq K \subseteq \{1, \ldots, d\} \). Let \( h \in \{1, \ldots, d\} \) be the same as in Proposition 2.10. Let \( \prec_0, \prec_h \) be the relation \( \prec \). If \( h \geq 2 \), let further \( (\gamma_1, \ldots, \gamma_{h-1}) = (\ldots, =) \) and \( (\gamma_{h+1}, \ldots, \gamma_d) = (\leq, \ldots, \leq) \). With this notation Proposition 2.10 implies that

\[
\Gamma^{(m),K}_{(0/2),\prec} = \bigcup_{\sigma \in S_d K} \Xi^{(m),\sigma}_{(0/2),\prec} \cup \Xi_{(0/2),\prec}.
\]  

(2.48)

Let \( \mathcal{X}^{(m)} = \{ \Xi^{(m),\sigma}_{(0/2),\prec} : \sigma \in S_d K \} \) and \( N = (d - h) \). Then, (2.42) equals \( \mathcal{X}^{(m)} = \{ \Xi^{(m),\sigma}_{(0/2),\prec} : \sigma \in S_d K, \sigma'_j \in \{\leq, =\} \text{ if } h + 1 \leq j \leq d - 1, \sigma'_j = \prec_j \text{ else} \} \).

Since Proposition 2.8 implies (2.43) for all sets in \( \mathcal{X}^{(m)} \), Lemma 2.9 yields (2.43) for all sets in \( \mathcal{X}^{(m)} \). Now, taking into account that by (2.48) we have \( \Gamma^{(m),K}_{(0/2),\prec} \in \mathcal{X}^{(m)} \), we obtain the assertion (2.47).

For \( k \in \{1, \ldots, d\} \), we define

\[
s_k^{(m)}(\chi) = (\gamma_1, \ldots, \gamma_{k-1}, m_k - \gamma_k, \gamma_{k+1}, \ldots, \gamma_d).
\]  

(2.49)

Proposition 2.12 For \( m \in \mathbb{N}^d \), we have

\[
\Gamma^{(m)} = \Gamma^{(m)} \cup \bigcup_{\emptyset \neq K \subseteq \{1, \ldots, d\}} \bigcup_{k \in K} s_k^{(m)} (\Gamma^{(m),K}_{(0/2),\prec}).
\]  

(2.50)

and, furthermore, the right hand side of (2.50) is a union of pairwise disjoint sets.

Proof. Let \( \chi \in \Gamma^{(m)} \). We will show that \( \chi \) belongs to the right hand side of (2.50). Since this is clear if \( \chi \in \Gamma^{(m)} \), we assume that \( \chi \notin \Gamma^{(m)} \). Since \( \chi \notin \Gamma^{(m)} \), there exists \( k \) such that \( \gamma_k/m_k > 1/2 \). Therefore, by the definition of \( \Gamma^{(m)} \), we have

\[
\forall i \in \{1, \ldots, d\} \setminus \{k\} : \gamma_i/m_i < 1/2.
\]  

(2.51)

Let \( \chi' = s_k^{(m)}(\chi) \). Since \( \gamma_k/m_k > 1/2 \), we have \( \gamma'_k/m_k < 1/2 \). Thus, since for \( i \neq k \) we have \( \gamma'_i = \gamma_i \), we get from (2.51) that \( \chi' \in \Gamma^{(m)} \). By the definition of \( \Gamma^{(m)} \), we have

\[
\forall i \in \{1, \ldots, d\} \setminus \{k\} : \gamma'_i/m_i - \gamma'_k/m_k = \gamma_i/m_i + \gamma_k/m_k - 1 - 1 = 0.
\]

Thus, by the definition in (2.44), we have \( k \in K^{(m)}[\chi'] \). Obviously \( \chi' = s_k^{(m)}(\chi) \) implies \( \chi = s_k^{(m)}(\chi') \). Thus, we have \( \chi \in s_k^{(m)} (\Gamma^{(m),K}_{(0/2),\prec}) \) with \( K = K^{(m)}[\chi'] \) and \( k \in K \).

Now, let \( \chi \) belong to the right hand side of (2.50). We will show that \( \chi \in \Gamma^{(m)} \). This is clear if \( \chi \in \Gamma^{(m)} \). Suppose \( \chi \in s_k^{(m)} (\Gamma^{(m),K}_{(0/2),\prec}) \) with \( K \subseteq \{1, \ldots, d\} \) and \( k \in K \). There is \( \chi' \in \Gamma^{(m),K}_{(0/2),\prec} \) with \( \chi = s_k^{(m)}(\chi') \) and, by the definition of \( \Gamma^{(m),K}_{(0/2),\prec} \), we have \( K = K^{(m)}[\chi'] \). Since \( k \in K = K^{(m)}[\chi'] \), we have \( \gamma'_i/m_i \leq \gamma'_k/m_k \), \( i \in \{1, \ldots, d\} \), thus

\[
\forall i \in \{1, \ldots, d\} \setminus \{k\} : \gamma_i/m_i + \gamma_k/m_k = \gamma'_i/m_i - \gamma'_k/m_k + 1 \leq 1.
\]  

(2.52)

Since for \( j \neq k \) we have \( \gamma_j = \gamma'_j \), and since \( \chi' \in \Gamma^{(m)} \), we have (2.51), and therefore

\[
\forall i, j \in \{1, \ldots, d\} \setminus \{k\} : \gamma_i/m_i + \gamma_j/m_j < 1.
\]  

(2.53)

Combining (2.52) and (2.53) yields \( \chi \in \Gamma^{(m)} \).

Finally, to complete the proof, we show that the right hand side of (2.50) is the union of pairwise disjoint sets. Let \( \chi \in s_k^{(m)} (\Gamma^{(m),K}_{(0/2),\prec}) \) and \( k \in K \). Then, \( \gamma_k > m_k/2 \)
and therefore $\gamma \notin \Gamma^{(m)}_i$. Let furthermore $\gamma \in \mathcal{X}_k \left( \Gamma^{(m), K'}_i \right)$. Then, $\gamma_k > m_k / 2$. Therefore, since $\gamma \in \Gamma^{(m)}_i$, we have $k' = k$, for otherwise $1 < \gamma_k / m_k + \gamma_{k'}/m_{k'} \leq 1$. We have $\gamma = \mathcal{X}_k \left( \gamma' \right)$ for some $\gamma'$ that is uniquely determined by $\gamma' = \mathcal{X}_k \left( \gamma \right)$. Therefore, $\gamma' \in \Gamma^{(m), K'}_i$ and $\gamma \in \Gamma^{(m), K}_i$, and we conclude $K' = K^{(m)} \left[ \gamma' \right] = K$. □

**Corollary 2.13** For all $m \in \mathbb{N}^d$, we have

$$L \left( \Gamma^{(m)} \right) \lesssim \prod_{i=1}^d \ln(m_i + 1).$$

**Proof.** By Proposition 2.12, the right hand side of (2.50) is a union of pairwise disjoint sets. Therefore, (2.50) implies

$$L \left( \Gamma^{(m)}_i \right) \leq L \left( \Gamma^{(m)}_0 \right) + \sum_{\emptyset \neq K \subseteq \{1, \ldots, d\}} \sum_{k \in K} L \left( s_k^{(m)} \left( \Gamma^{(m), K}_i \right) \right).$$

(2.54)

Clearly

$$L \left( s_k^{(m)} \left( \Gamma^{(m), K}_i \right) \right) = L \left( \Gamma^{(m), K}_i \right)$$

(2.55)

and the cross product structure of $\Gamma^{(m)}_i$ implies

$$L \left( \Gamma^{(m)} \right) \lesssim \prod_{i=1}^d \ln(m_i + 1).$$

(2.56)

Combining (2.54), (2.55), Corollary 2.11 and (2.56) yields the assertion. □

**Corollary 2.14** For all $m \in \mathbb{N}^d$, we have

$$L \left( \Gamma^{(m), x} \right) \lesssim \prod_{i=1}^d \ln(m_i + 1).$$

**Proof.** For $u \in \{-1, 1\}^d$, we denote $\Gamma^{(m), x}_u = \left\{ (u_i, \gamma_1, \ldots, u_d, \gamma_d) \mid \gamma \in \Gamma^{(m)} \right\}$. Consider $X^{(m)} = \left\{ \Gamma^{(m), x}_u \mid u \in \{-1, 1\}^d \right\}$ and $N = 2^d$. Then, it is clear that

$$\Gamma^{(m), x} = \bigcup_{u \in \{-1, 1\}^d} \Gamma^{(m), x}_u \in X^{(m)}_{(1,N)}.$$  

(2.57)

Let $u^{(1)}, \ldots, u^{(j)} \in \{-1, 1\}^d$, and $M = \{ i \in \{1, \ldots, d\} \mid u^{(1)}_i = u^{(2)}_i = \ldots = u^{(j)}_i \}$. We have $\bigcap_{l=1}^j \Gamma^{(m), x}_{u^{(l)}} = \left\{ \gamma \in \Gamma^{(m), x}_{u^{(l)}} \mid \gamma_i = 0 \text{ for all } i \notin M \right\}$. If $\emptyset \neq M = \{i_1, \ldots, i_h\}$, $i_1 < \ldots < i_h$, $(m_1', \ldots, m_h') = (m_{i_1}, \ldots, m_{i_h})$, $(u_1', \ldots, u_h') = (u^{(1)}_{i_1}, \ldots, u^{(1)}_{i_h})$, then

$$L \left( \bigcap_{l=1}^j \Gamma^{(m), x}_{u^{(l)}} \right) = L \left( \Gamma^{(m_1', \ldots, m_h')}_{(u_1', \ldots, u_h')} \right) = L \left( \Gamma^{(m_1', \ldots, m_h')} \right).$$

(2.58)

At the same time, if $M = \emptyset$, then the left hand side in (2.58) is $L(\{0\}) = 1$.

Note that $\prod_{l=1}^h \ln(m_l + 1) \lesssim \prod_{l=1}^d \ln(m_l + 1)$ for $M \neq \emptyset$. Thus, using Corollary 2.13 we conclude that for all $m \in \mathbb{N}^d$ we have

$$L \left( \bigcap_{l=1}^j \Gamma^{(m), x}_{u^{(l)}} \right) \lesssim \prod_{i=1}^d \ln(m_i + 1).$$

(2.59)
Now, (2.59) implies that the assumption (2.43) is satisfied and, therefore, taking into account (2.57) and Lemma 2.9 we get the assertion. \(\square\)

**Lemma 2.15** For \(z > 0\) and \(a \in (0, 1]\), we have
\[
\max\{\ln(az), 1\} \geq \max\{a \ln z, 1\} \geq a \max\{\ln z, 1\} \geq a \ln z.
\]  
(2.60)

**Proof.** The assertion is trivial for \(a = 1\). Let \(a \in (0, 1]\). The function \(h : (0, 1] \to \mathbb{R}, \ h(u) = u(1 - \ln u), \ u \in (0, 1]\), is increasing, thus \(a(1 - \ln a) < h(1) = 1\).

We conclude \(\ln a > (a - 1)(1 - \ln a)\), i.e. \(\ln a > (a - 1)\ln(e/a)\). Thus, since \(a - 1 < 0\), for \(z > e/a\) we have \(\ln a > (a - 1)\ln z\), i.e. \(\ln(az) > a\ln z\). For \(0 < z < e/a\), we conclude \(a \ln z < a\ln(e/a) = a(1 - \ln a) < 1\). We have shown: if \(\ln(az) \geq 1\), then we have \(\ln(az) > a\ln z\), and if \(\ln(az) < 1\), then we have also \(a\ln z < 1\). \(\square\)

**Proposition 2.16** There are \(\alpha_d, \beta_d > 0\) such that for all \(\mathbf{m} \in \mathbb{N}^d\), we have
\[
L\left(\mathbf{m}\right) \geq \alpha_d \prod_{i=1}^d \ln(m_i + 1), \quad L\left(\mathbf{m}^{(m)}\right) \geq \beta_d \prod_{i=1}^d \ln(m_i + 1).
\]  
(2.61)

**Proof.** We use the following Hardy-Littlewood inequality, see [32] p. 286:
\[
\frac{1}{2} \int_{[-\pi, \pi]} \left| \sum_{\gamma=0}^{N} c_{\gamma} e^{i\gamma t} \right|^2 dt \geq \sum_{\gamma=0}^{N} \frac{|c_{\gamma}|^2}{\gamma + 1}, \quad N \in \mathbb{N}_0, \ c_0, \ldots, c_N \in \mathbb{C}.
\]  
(2.62)

By the induction argument from [24] p. 69], we get for \(N_1, \ldots, N_d \in \mathbb{N}_0, c_2 \in \mathbb{C}^d:
\[
\frac{1}{2^d} \int_{[-\pi, \pi]^d} \left| \sum_{\gamma_1=0}^{N_1} \ldots \sum_{\gamma_d=0}^{N_d} c_{\gamma_1+\ldots+\gamma_d} e^{i(\gamma_1+\ldots+\gamma_d)t} \right|^2 dt \geq \sum_{\gamma_1=0}^{N_1} \ldots \sum_{\gamma_d=0}^{N_d} \frac{|c_{\gamma_1+\ldots+\gamma_d}|^2}{(\gamma_1 + 1) \cdot \ldots \cdot (\gamma_d + 1)}.
\]  
(2.63)

Using an appropriate shifting and orthogonality, we obtain
\[
L(\mathbf{\Gamma}) \geq \frac{1}{\pi^d} \quad \text{for all finite } \emptyset \neq \mathbf{\Gamma} \subset \mathbb{Z}^d.
\]  
(2.64)

By (2.63), we get for \(L\left(\mathbf{m}\right)\) the lower bounds
\[
\frac{1}{\pi^d} \sum_{2^n \in \mathbf{m}} \frac{1}{\gamma_1 + 1} \cdot \ldots \cdot \frac{1}{\gamma_d + 1} \geq \frac{1}{\pi^d} \sum_{\gamma_1=0}^{m_1/2} \ldots \sum_{\gamma_d=0}^{m_d/2} \frac{1}{(\gamma_1 + 1) \cdot \ldots \cdot (\gamma_d + 1)}.
\]

Since
\[
\sum_{\gamma=0}^{\lfloor x \rfloor} \frac{1}{\gamma + 1} \geq \max\{\ln(x + 1), 1\}, \quad x \geq 0,
\]  
(2.65)

we have
\[
L\left(\mathbf{m}\right) \geq \frac{1}{\pi^d} \prod_{i=1}^d \max\{\ln(m_i/2 + 1), 1\} \geq \frac{1}{\pi^d} \prod_{i=1}^d \max\{\ln((m_i + 1)/2), 1\},
\]

and now Lemma 2.15 implies the first inequality in (2.61) with \(\alpha_d = (2\pi)^{-d}\).

Since \(L(\{\gamma \in \mathbb{Z} \mid |\gamma| \leq x\}) = L(\{\gamma \in \mathbb{Z} \mid 0 \leq \gamma \leq 2\lfloor x \rfloor\}), \ x \geq 0\), the Hardy-Littlewood inequality (2.62) and (2.65) imply
\[
L(\{\gamma \in \mathbb{Z} \mid |\gamma| \leq x\}) \geq \frac{1}{\pi} \sum_{\gamma=0}^{2\lfloor x \rfloor} \frac{1}{\gamma + 1} \geq \frac{1}{\pi} \max(\ln(x + 1), 1), \quad x \geq 0.
\]  
(2.66)

Thus, for \(d = 1\), (2.66) yields the second inequality in (2.61) with \(\beta_1 = \pi^{-1}\). Let us prove this inequality for \(d \geq 2\). We adapt the decomposition approach from [31].
For \( j \in \{1, \ldots, d\} \), we denote

\[
\Gamma_{-j}^{(m,*)} = \left\{ (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j+1, \ldots, \gamma_d) \mid (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j+1, \ldots, \gamma_d) \in \Gamma_{-j}^{(m,*)} \right\},
\]

and

\[
a_{j}^{(m)}(t_1, \ldots, t_j-1, t_j+1, \ldots, t_d) = \sum_{(\gamma_1, \ldots, \gamma_{j-1}, \gamma_j+1, \ldots, \gamma_d) \in \Gamma_{-j}^{(m,*)}} e^{\gamma_1 t_1 + \ldots + \gamma_{j-1} t_{j-1} + \gamma_j t_{j+1} + \ldots + \gamma_d t_d}.
\]

Using the following equality

\[
\sum_{x \in \Gamma_{-j}^{(m,*)}} e^{i(x \cdot \ell)} = e^{-im_j t} \sum_{\gamma=0}^{2m_j} e^{\gamma t} a_{j}^{(m)}(\gamma-m_j)(t_1, \ldots, t_j-1, t_j+1, \ldots, t_d)
\]

and (2.62), we get

\[
\frac{1}{2} \int_{[-\pi, \pi]} \left| \sum_{x \in \Gamma_{-j}^{(m,*)}} e^{i(x \cdot \ell)} \right| dt_j \geq \frac{2m_j}{\gamma+1} \left| a_{j}^{(m)}(\gamma-m_j)(t_1, \ldots, t_j-1, t_j+1, \ldots, t_d) \right|
\]

and, therefore, we derive

\[
L(\Gamma_{-j}^{(m,*)}) \geq \frac{1}{\pi^{d-1}} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} \right) \ln \left( \frac{m_i}{m_j} (2\gamma + 1) \right). \quad (2.67)
\]

Denote \( K = \{ i \in \{1, \ldots, d\} \mid \ln((m_i + 1)/(4e^d)) \geq 1 \} \). Having in mind (2.64), we can assume without restriction \( K \neq \emptyset \), since we can ensure \( \beta_d \in (0, \pi^{-d}) \).

Let \( j \in K \). For all \( \gamma \in \{0, \ldots, \lfloor m_j/2 \rfloor\} \), we have the cross product structure

\[
\Gamma_{-j}^{(m,*)} = \left\{ (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j+1, \ldots, \gamma_d) \mid \forall i \in \{1, \ldots, d\} \setminus \{j\} : |\gamma_i| = \frac{m_i}{m_j} \right\}.
\]

Thus, for all \( \gamma \in \{0, \ldots, \lfloor m_j/2 \rfloor\} \) the inequality (2.66) implies

\[
L(\Gamma_{-j}^{(m,*)}) \geq \frac{1}{\pi^{d-1}} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} \right) \ln \left( \frac{m_i}{m_j} (2\gamma + 1) \right). \quad (2.68)
\]

Note that the product over the empty set \( K \setminus \{j\} \) is considered as 1. In this case, by (2.64), the inequality (2.68) is satisfied. Now, (2.67) yields

\[
L(\Gamma_{-j}^{(m,*)}) \geq \frac{1}{\pi^d} \sum_{\gamma=1}^{\lfloor m_j/2 \rfloor} \frac{1}{\gamma + 1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} (2\gamma + 1) \right) + \ln \left( \frac{m_i}{m_j} (\gamma + 1) \right).
\]

For \( \gamma \geq 1 \), we have \( 2\gamma \geq \gamma + 1 \). Further, we have \( 4 \leq 2m_j \). We conclude

\[
\pi^d L(\Gamma_{-j}^{(m,*)}) + \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} + 1 \right) \geq \frac{1}{\pi^d} \sum_{\gamma=0}^{\lfloor m_j/2 \rfloor} \frac{1}{\gamma + 1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} (\gamma + 1) + 1 \right)
\]

\[
\geq \int_0^{m_j/2} \frac{1}{v + 1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} v + 1 \right) dv. \quad (2.69)
\]

Next, for \( r > 0 \), we derive

\[
\sum_{\gamma \in K} \int_0^{m_j/2} \frac{1}{v + 1} \prod_{i \in K \setminus \{j\}} \ln \left( \frac{m_i}{m_j} v + 1 \right) dv = \sum_{\gamma \in K} \int_0^{r/2} \frac{m_j}{m_j \gamma + 1} \prod_{i \in K \setminus \{j\}} \ln (m_i \gamma + 1) d\tau
\]

\[
\geq \int_0^{r/2} \sum_{\gamma \in K} \frac{m_j}{m_j \gamma + 1} \prod_{i \in K \setminus \{j\}} \ln (m_i \gamma + 1) d\tau = \prod_{i \in K} \ln \left( \frac{1}{2} rm_i + 1 \right)
\]
and, therefore, there exists $k \in K$ such that
\[
\int_0^{\frac{r m_k}{v} + 1} \prod_{i \in K \setminus \{k\}} \ln \left( \frac{m_i}{2 m_k} v + 1 \right) dv \geq \frac{1}{|K|} \prod_{i \in K} \ln \left( \frac{1}{2} r m_i + 1 \right). \tag{2.70}
\]
Using (2.70) with $r = 1/2$ and (2.69) with $j = k$ and taking into account the definition of $K$, we obtain
\[
\pi^d L \left( \frac{r(m, \Sigma)}{d} \right) \geq \frac{1}{d} \left( \ln \left( \frac{1}{4} m_k + 1 \right) - \ln \left( e^d \right) \right) \prod_{i \in K \setminus \{k\}} \ln \left( \frac{1}{4} m_i + 1 \right) \tag{2.71}
\]
\[
\geq \frac{1}{d} \prod_{i \in K} \ln \left( \frac{(m_i + 1) / (4 e^d)}{\max \left\{ \ln ((m_i + 1) / (4 e^d)), 1 \right\}} \right) = \frac{1}{d} \prod_{i=1}^{d} \max \left\{ \ln ((m_i + 1) / (4 e^d)), 1 \right\}.
\]
Now, Lemma 2.15 implies the assertion with $\beta_d = \frac{d}{d-1} \pi^{-d} (4 e^d)^{-d} \in (0, \pi^{-d}]$. \hspace{1cm} □

**Proof of Theorem 2.1.** The statement follows immediately by combining Corollary 2.13, Corollary 2.14, and Proposition 2.16. \hspace{1cm} □

### 2.3 Proof of Theorem 2.3

Let $d \geq 2$, $m \in (0, \infty)^d$, and $r > 0$. Denote $D_{\Sigma, r}^{(m)} (t) = \sum_{x \in \Sigma^{(m)}} e^{t(x)}$ and
\[
\lambda_r^{(m_1, \ldots, m_l)} (\gamma_1, \ldots, \gamma_{j-1}) = m_j \left( r - \sum_{i=1}^{j-1} \frac{\gamma_i}{m_i} \right), \quad j = 2, \ldots, d.
\]
It is easy to see that
\[
D_{\Sigma, r}^{(m)} (t) = \sum_{\gamma_1=0}^{\lfloor |r m_1| \rfloor} e^{t_1 \gamma_1} \sum_{\gamma_2=0}^{\lfloor |r m_2| \rfloor} e^{t_2 \gamma_2} \ldots \sum_{\gamma_d=0}^{\lfloor |r m_d| \rfloor} e^{t_d \gamma_d}.
\]
In what follows, we will need several auxiliary functions given by
\[
F_{\Sigma, r}^{(m)} (t) = \sum_{\gamma_1=0}^{\lfloor |r m_1| \rfloor} e^{t_1 (t_1 - m_d t_d / m_1)} \sum_{\gamma_2=0}^{\lfloor |r m_2| \rfloor} e^{t_2 (t_2 - m_d t_d / m_2)} \ldots \sum_{\gamma_d=0}^{\lfloor |r m_d| \rfloor} e^{t_d (t_d - d - 1)} f_{\Sigma, r}^{(m)} (\gamma_1, \ldots, \gamma_{d-1}, t_d), \tag{2.72}
\]
\[
f_{\Sigma, r}^{(m)} (\gamma_1, \ldots, \gamma_{d-1}, t_d) = e^{t (r m_d + 1)} t_d \sum_{t_d=0}^{\max \{0, \gamma_d\}} e^{-t (r m_d + 1)} t_d - 1 \tag{2.73}
\]
and
\[
D_{\Sigma, \Sigma, r}^{(m)} (t) = D_{\Sigma, r}^{(m_1, \ldots, m_{d-1})} (t_1, \ldots, t_{d-1})
\]
\[
D_{\Sigma, d, r}^{(m)} (t) = D_{\Sigma, r}^{(m)} (t_1 - m_d t_d / m_1, \ldots, t_{d-1} - m_d t_d / m_{d-1})
\]
\[
G_{\Sigma, r}^{(m)} (t) = \frac{1}{e^{t d} - 1} \left( e^{t (r m_d + 1)} t_d \right) D_{\Sigma, \Sigma, r}^{(m)} (t) - D_{\Sigma, d, r}^{(m)} (t).
\tag{2.74}
\]

**Proposition 2.17** We have $D_{\Sigma, r}^{(m)} (t) = G_{\Sigma, r}^{(m)} (t) + F_{\Sigma, r}^{(m)} (t)$.

**Proof.** We have for $i \in \{1, \ldots, d - 2\}$ the recursive relation
\[
S_{\Sigma, r}^{(m_1, \ldots, m_{d-2})} (t_1, \ldots, t_d) = \sum_{\gamma_i=0}^{\lfloor |r m_1| \rfloor} e^{t_1 \gamma_i} S_{\Sigma, r}^{(m_1, \ldots, m_{d-1})} (t_1, \ldots, t_d).
\tag{2.75}
\]
where $S \in \{D, F\}$. Note that
\[
G^{(m_d-1,m_d)}_{\Sigma,r}(t_{d-1}, t_d) = \frac{1}{e^{i\theta_d} - 1} \left( e^{i(rm_d+1)t_d} D^{(m_d-1)}_{\Sigma,r}(t_{d-1} - m_d t_d/m_{d-1}) - D^{(m_d-1)}_{\Sigma,r}(t_{d-1}) \right)
\]
\[
F^{(m_d-1,m_d)}_{\Sigma,r}(t_{d-1}, t_d) = \frac{e^{i(m_d+1)t_d}}{e^{i\theta_d} - 1} \sum_{\gamma_d = 0}^{m_d} e^{i\gamma_d(t_{d-1} - m_d t_d/m_{d-1})} \left( e^{-i[1][m_d(r-\gamma_d-1/m_{d-1})]t_d - 1} \right).
\]

Thus, from (2.75) for $S = D$, we immediately get the same recursive relation (2.75) for the function corresponding to the symbol $S = G$.

Next, using the equality
\[
e^{i\theta_d - i\theta_{d-1}} e^{i([m_d(r-\gamma_d-1/m_{d-1})+1])t_d} = e^{i(rm_d+1)t_d} e^{i\theta_d} e^{i\theta_{d-1}} (t_{d-1} - m_d t_d/m_{d-1}) e^{-i[1][m_d(r-\gamma_d-1/m_{d-1})]t_d - 1},
\]
we conclude that $D^{(m_d-1,m_d)}_{\Sigma,r} = G^{(m_d-1,m_d)}_{\Sigma,r} + F^{(m_d-1,m_d)}_{\Sigma,r}$. Thus, applying the relations (2.75) to $S \in \{D, G, F\}$, we obtain the assertion.

**Proposition 2.18** Let $r \in (0, \infty)$. For all $m \in [1, \infty)^d$, we have
\[
\|G^{(m)}_{\Sigma,r}\|_{L^1([-\pi, \pi]^d)} \lesssim \ln(m_d + 1) \|D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}\|_{L^1([-\pi, \pi]^{d-1})}.
\]

**Proof.** We have
\[
G^{(m)}_{\Sigma,r}(t) = \frac{\Delta^{(m)}_{\Sigma,r}(t)}{e^{i\theta_d} - 1} + L^{(m)}_{\Sigma,r}(t_d) D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}(t_1 - m_d t_d/m_{d-1}, \ldots, t_{d-1} - m_d t_d/m_{d-1}),
\]
where $L^{(m)}_{\Sigma,r}(t) = \frac{e^{i(rm_d+1)t_d}}{e^{i\theta_d} - 1}$ and $\Delta^{(m)}_{\Sigma,r}(t) = D^{(m)}_{\Sigma,d,r}(t) - D^{(m)}_{\Sigma,d,r}(t)$. 

Moreover, by the telescoping sum decomposition, we derive
\[
\Delta^{(m)}_{\Sigma,r}(t) = \sum_{i=1}^{d-1} \Delta^{(m)}_{\Sigma,r,i}(t),
\]
where
\[
\Delta^{(m)}_{\Sigma,r,i}(t) = D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}(t_1, \ldots, t_i, t_{i+1} = m_d t_d/m_{d-1}, \ldots, t_{d-1} - m_d t_d/m_{d-1} - m_d t_d/m_{d-1}) - D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}(t_1, \ldots, t_i, t_{i+1} = m_d t_d/m_{d-1}, \ldots, t_{d-1} - m_d t_d/m_{d-1} - m_d t_d/m_{d-1}).
\]

Using (2.76), (2.74), and the sets (2.18), we get
\[
\int_{[-\pi, \pi]} |G^{(m)}_{\Sigma,r}| \, dt \lesssim \sum_{i=1}^{d} I_i + J,
\]
where
\[
I_i = \int_{A_i(m_d)} \left| \Delta^{(m)}_{\Sigma,r,i}(t) \right| \, dt \lesssim \int_{A_i(m_d)} \left| \Delta^{(m)}_{\Sigma,r,i}(t) \right| \, dt,
\]
and
\[
J = \int_{B_i(m_d)} \left| D^{(m)}_{\Sigma,d,r}(t) \right| + \left| D^{(m)}_{\Sigma,d,r}(t) \right| \, dt + \int_{A_i(m_d)} \left| L^{(m)}_{\Sigma,d}(t_d) D^{(m)}_{\Sigma,d,r}(t) \right| \, dt.
\]
By (2.15), we easily get $J \lesssim \ln(m_d + 1) \|D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}\|_{L^1([-\pi, \pi]^{d-1})}$. Further, we have
\[
I_i \lesssim m_d \int_{-1/(m_d+1)}^{1/(m_d+1)} \left| \frac{D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}}{e^{i\theta_d} - 1} \right| \, dt_d \|D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}\|_{L^1([-\pi, \pi]^{d-1})} \lesssim \|D^{(m_1,\ldots,m_d-1)}_{\Sigma,r}\|_{L^1([-\pi, \pi]^{d-1})}, \quad (2.77)
\]
Indeed, using (2.22) and (2.23), we obtain for \(i \in \{1, \ldots, d - 1\}\) that
\[
\int_{[-\pi, \pi]} |\Delta_{\mu, \nu}^{(m)}(t)| \, dt_1 \leq |t_1| \frac{m_1}{m_i} \gamma m_i \int_{[-\pi, \pi]} \left| D_{\Sigma, r}^{(m_1, \ldots, m_{d-1})}(t_1, \ldots, t_{d-1}) \right| \, dt_1,
\]
and, therefore (2.77).

\[Q \square\]

**Proposition 2.19** Let \(r = p/q\) with \(p, q \in \mathbb{N}\). For all \(m \in \mathbb{N}^d\), we have
\[
\| F_{\Sigma, r}^{(m)} \|_{L^1((-\pi, \pi)^d)} \lesssim \ln(\text{lcm}(q, m_1, \ldots, m_{d-1}) + 1) \| D_{\Sigma, r}^{(m_1, \ldots, m_{d-1})} \|_{L^1((-\pi, \pi)^d)},
\]
where \(\text{lcm}(q, m_1, \ldots, m_{d-1})\) denotes the least common multiple of \(q, m_1, \ldots, m_{d-1}\).

**Proof.** The proposition can be proved by repeating the proof of Proposition 2.7. Thus, let us present the sketch of the proof.

Using (2.72), (2.73), and (2.15), we get as in the proof of Proposition 2.7 that
\[
\| F_{\Sigma, r}^{(m)} \|_{L^1((-\pi, \pi)^d)} \lesssim \sum_{\nu = 1}^{\infty} \frac{\pi^\nu}{\nu!} \| Q_{\Sigma, r}^{(m)} \|_{L^1((-\pi, \pi)^d)},
\]
where
\[
Q_{\Sigma, r}^{(m)}(t_1, \ldots, t_{d-1}) = \sum_{\gamma_1=0}^{[rm_1]} \ldots \sum_{\gamma_{d-1}=0}^{[rm_{d-1}]} e^{\mu(t_1\gamma_1 + \ldots + t_{d-1}\gamma_{d-1})} \lambda(t_1, \ldots, \gamma_{d-1})^\nu,
\]
and \(\lambda(t_1, \ldots, \gamma_{d-1}) = \Lambda_{\Sigma, r}(m_1, \ldots, m_{d-1})^{(\gamma_1, \ldots, \gamma_{d-1})}\).

Thus, to finish the proof it is sufficient to verify that for all \(\nu \geq 1\) we have
\[
\| Q_{\Sigma, r}^{(m)} \|_{L^1((-\pi, \pi)^d)} \lesssim \ln(M\nu + 1) \| D_{\Sigma, r}^{(m_1, \ldots, m_{d-1})} \|_{L^1((-\pi, \pi)^d)},
\]
where \(M = \text{lcm}(q, m_1, \ldots, m_{d-1})\).

Taking into account that \(0 \leq \| \lambda(t_1, \ldots, \gamma_{d-1}) \| \leq 1 - M^{-1}\), we get in the same way as in the proof of Proposition 2.7 that in \(L^1((-\pi, \pi)^d)\)
\[
Q_{\Sigma, r}^{(m)}(t_1, \ldots, t_{d-1}) = \sum_{\gamma_1=0}^{[rm_1]} \ldots \sum_{\gamma_{d-1}=0}^{[rm_{d-1}]} e^{\mu(t_1\gamma_1 + \ldots + t_{d-1}\gamma_{d-1})} \hat{\Lambda}_{\Sigma, r}(\mu) e^{2\pi i \mu \lambda(\gamma_1, \ldots, \gamma_{d-1})}
\]
where the function \(\hat{\Lambda}_{\Sigma, r}(\mu)\) is given by (2.30). Thus, using (2.34), we get (2.78). \(\square\)

**Proof of Theorem 2.3.** The statement of the theorem is well-known for \(d = 1\). Remark also that the case \(d = 2\) with \(r = 1\) is already considered in Theorem 2.4.

Let us prove the upper estimates for \(d \geq 2\) in (2.2). Without loss of generality we can assume that \(m_1 \leq \ldots \leq m_d\). The upper estimate for \(L(\Sigma, r)m) = \| D_{\Sigma, r}^{(m)} \|_{L^1((-\pi, \pi)^d)}\)

\[
\| D_{\Sigma, r}^{(m_1, \ldots, m_{d-1})} \|_{L^1((-\pi, \pi)^d)} \leq \| D_{\Sigma, r}^{(m_1, \ldots, m_{d-1})} \|_{L^1((-\pi, \pi)^d)},
\]

Proposition 2.19 and the induction argument. Using this, we can conclude the upper estimate for \(L(\Sigma, r)m)\) in the same way as in the proof of Corollary 2.14.

Let us consider the lower bounds. As in the proof of Proposition 2.16 we get
\[
L(\Sigma, r)m) \geq \frac{1}{\pi^d} \sum_{\gamma_1=0}^{[rm_1/d]} \ldots \sum_{\gamma_{d-1}=0}^{[rm_{d-1}/d]} \frac{1}{(\gamma_1 + 1) \ldots (\gamma_{d} + 1)} \geq \frac{1}{\pi^d} (\min\{r/d, 1\})^d \prod_{i=1}^{d} \ln(m_i + 1).
\]
To show the lower bounds for the sets $\Sigma^{(m),*}$, it is sufficient to prove that there exists $\kappa_d \in (0, \pi^{-d}]$ such that for all $r > 0$ and all $m \in \mathbb{N}^d$, we have

$$L\left(\Sigma^{(m),*}_r\right) \geq \kappa_d \prod_{i=1}^d \max \left\{ \ln \left(\gamma/m_i + 1\right), 1 \right\},$$

(2.79)
since by Lemma 2.15 we will have $L\left(\Sigma^{(m),*}_r\right) \geq \kappa_d \min\{r, 1\} \prod_{i=1}^d \ln (m_i + 1)$.

We use the induction argument. By (2.69), we can choose $\kappa_d = \pi^{-1}$. Let $d \geq 2$. By analogy with the proof of the second inequality in (2.61), we get

$$L\left(\Sigma^{(m),*}_r\right) \geq \frac{1}{\pi} \sum_{\gamma=0}^{\lfloor r m_j \rfloor} \frac{1}{\gamma + 1} L\left(\Sigma^{(m),*}_{r, j, (\gamma)}\right),$$

(2.80)

where $\Sigma^{(m),*}_{r, j, (\gamma)} = \{ (\gamma_1, \ldots, \gamma_j-1, \gamma_{j+1}, \ldots, \gamma_d) | (\gamma_1, \ldots, \gamma_j-1, \gamma, \gamma_{j+1}, \ldots, \gamma_d) \in \Sigma^{(m),*}_r \}$. Denote $K = \{ i \in \{1, \ldots, d\} | \ln((m_j + 1)/(4e^d)) \geq 1 \}$. We can assume without restriction that $K \setminus \emptyset$, since by (2.64) the number $\kappa_d$ can be chosen from $(0, \pi^{-d}]$.

Let $j \in K$. For $\gamma \in \{0, \ldots, \lfloor r m_j \rfloor \}$, we have

$$\Sigma^{(m),*}_{r, j, (\gamma)} = \Sigma^{(m_1, \ldots, m_j, \ldots, m_{j+1})}_{r - r m_j/m_j}.$$

Thus, since $r - r/m_j + \gamma/m_j \geq \gamma/m_j$, using the induction argument yields

$$L\left(\Sigma^{(m),*}_{r, j, (\gamma)}\right) \geq \kappa_{d-1} \prod_{i=1 \atop i \neq j}^d \max \left\{ \ln \left(\gamma/m_j + 1\right), 1 \right\} \geq \kappa_{d-1} \prod_{i \in K \setminus \{j\}} \ln((\gamma/m_j) m_i + 1).$$

Note again that the product over the empty set $K \setminus \{j\} = \emptyset$ is considered as 1, and that in this case by (2.64) the last inequality is satisfied with $\kappa_{d-1} \in (0, \pi^{-(d-1)})$.

We have $r \geq 1/m_j$. By analogy with (2.69), using (2.80) implies

$$L\left(\Sigma^{(m),*}_r\right) + \frac{\kappa_{d-1}}{\pi} \prod_{i \in K \setminus \{j\}} \ln \left(\frac{1}{2} r m_i + 1\right) \geq \frac{\kappa_{d-1}}{\pi} \int_0^{\lfloor r m_j \rfloor} \frac{1}{v + 1} \prod_{i \in K \setminus \{j\}} \ln \left(\frac{1}{2} m_j v + 1\right) dv.$$  

There is $k \in K$ satisfying (2.70). Using the first and second inequality in (2.60), by analogy with (2.71), we get (2.79) for $\kappa_d = d^{-1} \pi^{-1}(4e^d)^{-d} \kappa_{d-1} \in (0, \pi^{-d}]$.

\section{3 Interpolation on Lissajous-Chebyshev nodes}

We first describe the solution of the interpolation problem \[1.2\] in more detail and collect some notation from \[11\].

Let us consider for $\gamma \in \mathbb{N}_0^d$ the d-variate Chebyshev polynomials

$$T_{\gamma}(\underline{x}) = T_{\gamma_1}(x_1) \cdot \ldots \cdot T_{\gamma_d}(x_d), \quad \underline{x} \in [-1, 1]^d,$$

where $T_{\gamma}(x) = \cos(\gamma \arccos x)$. The Chebyshev polynomials form an orthogonal basis of the polynomial space $\Pi^d = \text{span}\{T_{\gamma} | \gamma \in \mathbb{N}_0^d\}$ with respect to the inner product

$$\langle f, g \rangle_{\omega_d} = \frac{1}{\pi d} \int_{[-1,1]^d} f(\underline{x}) \overline{g(\underline{x})} \omega_d(\underline{x}) d\underline{x}, \quad \omega_d(\underline{x}) = \prod_{i=1}^d \frac{1}{\sqrt{1 - x_i^2}}.$$
The corresponding norms of these basis elements are

\[ \| T_2 \|_{w,2}^2 = 2^{-\epsilon(2)}, \quad \text{where} \quad \epsilon(2) = \#\{ i \in \{1, \ldots, d\} | \gamma_i \neq 0 \}. \]

We define

\[ N_d = \{ \underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d \mid n_1, \ldots, n_d \text{ are pairwise relatively prime} \}. \]

In the following, let \( \underline{n} \in N_d, \epsilon \in \{1, 2\}, \) and \( \kappa \in \mathbb{Z}^d. \) For the index set \( \Gamma_{2}^{(\epsilon \underline{n})} \) from the introduction, we define the polynomial vector space

\[ \Pi_{2}^{(\epsilon \underline{n})} = \text{span} \{ T_2 \mid \gamma \in \Gamma_{2}^{(\epsilon \underline{n})} \}. \]

Clearly, the system \( \{ T_2 \mid \gamma \in \Gamma_{2}^{(\epsilon \underline{n})} \} \) forms an orthogonal basis of \( \Pi_{2}^{(\epsilon \underline{n})}. \)

To introduce the sets \( \mathcal{LC}_{2}^{(\epsilon \underline{n})}, \) we define

\[ \mathcal{I}_{2}^{(\epsilon \underline{n})} = \mathcal{I}_{2}^{(\epsilon \underline{n})_0} \cup \mathcal{I}_{2}^{(\epsilon \underline{n})_1}, \]

where the sets \( \mathcal{I}_{2}^{(\epsilon \underline{n})_{1}}, \) \( \epsilon \in \{0, 1\}, \) are given by

\[ \mathcal{I}_{2}^{(\epsilon \underline{n})_{1}} = \{ \underline{i} \in \mathbb{N}^d \mid 0 \leq i_1 \leq \epsilon n_1 \text{ and } i_1 \equiv \kappa_1 + \epsilon \text{ mod } 2 \text{ for all } i \in \{1, \ldots, d\} \}. \]

Then, using the notation

\[ z_{i}^{(\epsilon \underline{n})} = (z_{i1}^{(\epsilon n_1)}, \ldots, z_{id}^{(\epsilon n_d)}), \quad z_{i}^{(\epsilon \underline{n})} = \cos (i \pi / (\epsilon n)), \]

the Lissajous-Chebyshev node sets are defined as

\[ \mathcal{LC}_{2}^{(\epsilon \underline{n})} = \left\{ z_{i}^{(\epsilon \underline{n})} \mid \underline{i} \in \mathcal{I}_{2}^{(\epsilon \underline{n})} \right\}. \tag{3.1} \]

Note that the mapping \( \underline{i} \mapsto z_{i}^{(\epsilon \underline{n})} \) is a bijection from \( \mathcal{I}_{2}^{(\epsilon \underline{n})} \) onto \( \mathcal{LC}_{2}^{(\epsilon \underline{n})}. \)

Further, for \( \underline{i} \in \mathcal{I}_{2}^{(\epsilon \underline{n})}, \) we introduce the weight \( w_{\underline{i}}^{(\epsilon \underline{n})} \) by

\[ w_{\underline{i}}^{(\epsilon \underline{n})} = 2^{\#\{ i \mid 0 < i_1 < \epsilon n_1 \}} / \left( 2 \epsilon \prod_{i=1}^{d} n_i \right), \tag{3.2} \]

and for \( \gamma \in \Gamma_{2}^{(\epsilon \underline{n})}, \) we use the notation

\[ f^{(\epsilon \underline{n})}(\gamma) = \max \left\{ \#\{ i \in \{1, \ldots, d\} | \gamma_i = \epsilon n_i \} - 1, 0 \right\}. \]

Note that in the case \( \epsilon = 1, \) we have \( f^{(\epsilon \underline{n})}(\gamma) = 0 \) for all \( \gamma \in \Gamma_{2}^{(\epsilon \underline{n})}. \)

Finally, for \( \underline{i} \in \mathcal{I}_{2}^{(\epsilon \underline{n})}, \) we introduce on \([-1, 1]^d\) the polynomials

\[ L_{2}^{(\epsilon \underline{n})}(\underline{x}) = w_{\underline{i}}^{(\epsilon \underline{n})} \left( \sum_{\gamma \in \Gamma_{2}^{(\epsilon \underline{n})}} 2^{\epsilon(2)-f^{(\epsilon \underline{n})}(\gamma)} T_2(z_{i}^{(\epsilon \underline{n})}) T_2(\underline{x}) - T_{\epsilon n_d}(z_{i_d}^{(\epsilon n_d)}) T_{\epsilon n_d}(\underline{x}_d) \right) \tag{3.3} \]

that by definition belong to the space \( \Pi_{2}^{(\epsilon \underline{n})}. \)

The existence and uniqueness of a solution of the interpolation problem \([1,2]\) are guaranteed by the following theorem.

**Theorem 3.1** For \( f : [-1,1]^d \to \mathbb{R} \) the unique solution to the interpolation problem \([1,2]\) in the space \( \Pi_{2}^{(\epsilon \underline{n})} \) is given by the polynomial

\[ P_{2}^{(\epsilon \underline{n})}(\underline{x}) = \sum_{\underline{i} \in \mathcal{I}_{2}^{(\epsilon \underline{n})}} f(z_{i}^{(\epsilon \underline{n})}) L_{2}^{(\epsilon \underline{n})}(\underline{x}). \]

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The proof of this result is given in [11]. Note that for \( \epsilon = 1 \) only the case \( \kappa = 0 \) was treated. However, since the general node sets \( \mathbf{L_C}(\kappa) \) differ from \( \mathbf{L_C}(0) \) only in terms of reflections with respect to the coordinate axis, the corresponding results can be transferred immediately.

By Theorem 3.1, the discrete Lebesgue constant \( \Lambda_{\kappa}(n) \) introduced in (1.3) can be reformulated as

\[
\Lambda_{\kappa}(n) = \max_{x \in [-1,1]^d} \sum_{x \in \mathbf{I}_{\kappa}(n)} |I_{\kappa}(n)(x)|. 
\]  

(3.4)

As a first auxiliary result to estimate this constant, we prove the following Marcinkiewicz-Zygmund-type inequality.

**Proposition 3.2** Let \( \kappa \in \mathbb{Z}^d, \epsilon \in \{1,2\}, \) and \( 0 < p < \infty \) be fixed. For all \( n \in \mathcal{N}_d \) and all \( P \in \Pi_{\kappa}^{(n)} \), we have

\[
\sum_{i \in \mathbf{I}_{\kappa}(n)} w_i^{(n)} |P(z_i^{(n)})|^p \lesssim \|P\|^p_{w_d,p} = \frac{1}{\pi^d} \int_{[-1,1]^d} |P(x)|^p w_d(x) \, dx. 
\]  

(3.5)

**Proof.** The proof is based on the idea given in [29]. We proceed as in the proof of [14, Lemma 3] and use the following one-dimensional result from [21, Theorem 2]:

For all \( M \in \mathbb{N}, 0 \leq \theta_1 < \cdots < \theta_M < 2\pi, \) and for all univariate trigonometric polynomials \( q_m \) of degree at most \( m \), we have the inequality

\[
\frac{1}{m} \sum_{\nu=1}^M |q_m(\theta_\nu)|^p \leq \left( m + \frac{1}{2\eta} \right) \frac{(p+1)e}{2\pi} \int_0^{2\pi} |q_m(\theta)|^p d\theta, 
\]  

(3.6)

where \( \eta = \min(\theta_2 - \theta_1, \ldots, \theta_M - \theta_{M-1}, 2\pi - (\theta_M - \theta_1)) \).

For \( m \in \mathbb{N}, r \in \{0,1\}, \) we consider the sets \( J_r^{(m)} = \{ i \in \mathbb{N}_0 \mid i < 2m, \ i \equiv r \mod 2 \}. \)

Suppose that \( i_1, \ldots, i_m \in J_r^{(m)} \) with \( 0 \leq i_1 < \cdots < i_m < 2m. \) Setting \( M = m \) and \( \theta_\nu = i_\nu \pi/m, \) we obtain \( \eta = 2\pi/m. \) Using (3.6), we get for all univariate polynomials \( Q \) of degree at most \( m \) the inequality

\[
\frac{1}{m} \sum_{i \in J_r^{(m)}} (2 - \delta_{0,i} - \delta_{0,m}) |Q(z_i^{(m)})|^p = \frac{1}{m} \sum_{i \in J_r^{(m)}} |Q(\cos \pi_i/m)|^p = \frac{1}{m} \sum_{\nu=1}^m |Q(\cos \theta_\nu)|^p 
\leq \left( 1 + \frac{1}{4\pi} \right) \frac{(p+1)e}{2\pi} \int_0^{2\pi} |Q(\cos \theta)|^p d\theta \leq 3(p+1) \frac{1}{\pi} \int_{-1}^{1} |Q(x)|^p dx, 
\]  

(3.7)

where \( \delta_{i,j} \) denotes the Kronecker delta.

Let \( P \in \Pi_{\kappa}^{(n)} \). The degree of the univariate polynomial \( z_i \mapsto P(z_1, \ldots, z_d) \) is at most \( \epsilon n_1 \). Now, taking into account the cross product structure of \( \mathbf{I}_{\kappa}^{(n)} \), \( r \in \{0,1\}, \) the weights defined in (3.2), and applying \( d \) times inequality (3.7), we obtain

\[
\sum_{i \in \mathbf{I}_{\kappa}(n)} w_i^{(n)} |P(z_i^{(n)})|^p \lesssim \|P\|_{w_d,p}, \quad r \in \{0,1\}, 
\]  

(3.8)

for \( n \in \mathcal{N}_d \) and \( P \in \Pi_{\kappa}^{(n)} \). Since \( \mathbf{I}_{\kappa}(n) = \mathbf{I}_{\kappa|\kappa_0}^{(n)} \cup \mathbf{I}_{\kappa|\kappa_1}^{(n)} \), inequality (3.8) yields (3.5). \hspace{1cm} \Box

A slight adaption of the proof of Theorem 2.1 gives the following result.

**Corollary 3.3** Let \( \epsilon \in \{1,2\} \) and \( \kappa \in \mathbb{Z}^d \) be fixed. For all \( n \in \mathcal{N}_d \), we have

\[
L(\mathbf{I}_{\kappa}(n)) \asymp L(\mathbf{I}_{\kappa|\kappa_0}^{(n)}, \epsilon) \asymp \prod_{i=1}^d \ln(n_i + 1). 
\]

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Proof. We use the notation
\[
\Gamma^{(m)}_{\kappa,T} = \left\{ \chi \in \mathbb{N}_0^d \mid \gamma_i/n_i \leq \epsilon/2 \quad \forall i \in \{1, \ldots, d\} \text{ with } n_i \equiv r \mod 2, \right. \\
\left. \gamma_i/n_i < \epsilon/2 \quad \forall i \in \{1, \ldots, d\} \text{ with } n_i \not\equiv r \mod 2 \right\}.
\]
Further, using (2.44), let \( \Gamma^{(m)}_{\kappa,T} = \left\{ \chi \in \Gamma^{(m)}_{\kappa,T} \mid K^{(m)}(\chi) = K \right\} \) for \( \emptyset \neq K \subseteq \{1, \ldots, d\} \), and, using (2.49), define the mapping \( s^{(m)}(\gamma) = s^{(m)}_k(\gamma), k = \max K^{(m)}(\gamma) \). Employing the statements from the proof of [11, Proposition 2.6 and Proposition 3.8] we get (2.46). Using the well-known relation (2.47) and Proposition 3.3 we obtain the following estimates of the discrete Lebesgue constants.

Theorem 3.4 Let \( \epsilon \in \{1, 2\} \) and \( \kappa \in \mathbb{Z}^d \) be fixed. For all \( \mathbf{n} \in \mathcal{N}_d \), we have
\[
\Lambda^{(m)}_{\kappa} \geq \prod_{i=1}^{d} \ln(n_i + 1). 
\]

Proof. We introduce
\[
\tilde{K}^{(m)}_{\kappa}(\mathbf{x}, \mathbf{a}) = \sum_{\mathbf{z} \in \Gamma^{(m)}_{\kappa}} 2^{i(2) - j} \tilde{T}_2(\mathbf{x}) T_2(\mathbf{a}), \quad K^{(m)}_{\kappa}(\mathbf{x}, \mathbf{a}) = \sum_{\mathbf{z} \in \Gamma^{(m)}_{\kappa}} 2^{i(2)} T_2(\mathbf{x}) T_2(\mathbf{a}).
\]
From (3.3), (3.4), and Proposition 3.2 we get for all \( \mathbf{n} \in \mathcal{N}_d \) that
\[
\Lambda^{(m)}_{\kappa} = \max_{\mathbf{z} \in \{-1, 1\}^d} \sum_{i \in I_2^{(m)}} \omega_i^{(m)} \left| \tilde{K}^{(m)}_{\kappa}(\mathbf{z}, \mathbf{z}) - T_{\nu_\kappa}(\mathbf{z}) T_{\nu_\kappa}(\mathbf{a}) \right| \\
\leq \max_{\mathbf{z} \in \{-1, 1\}^d} \sum_{i \in I_2^{(m)}} \omega_i^{(m)} \left| \tilde{K}^{(m)}_{\kappa}(\mathbf{z}, \mathbf{z}) \right| + 1 \leq \max_{\mathbf{z} \in \{-1, 1\}^d} \| \tilde{K}^{(m)}_{\kappa}(\mathbf{z}, \cdot) \|_{\nu_\kappa} + 1.
\]
Using the well-known relation \( \prod_{i=1}^{r} \cos(\vartheta_i) = \frac{1}{2^r} \sum_{\mathbf{v} \in \{-1, 1\}^r} \cos(v_1 \vartheta_1 + \cdots + v_r \vartheta_r), \ r \in \mathbb{N}, \) we get
\[
\prod_{i=1}^{d} \cos(\gamma_i s_i) \cos(\gamma_i t_i) = \frac{1}{2^{2d}} \sum_{\mathbf{v} \in \{-1, 1\}^d} \left( \sum_{i=1}^{d} (v_1 \gamma_i s_i + v_i \gamma_i t_i) \right).
\]
Then, for all \( \mathbf{x} = (\cos s_1, \ldots, \cos s_d) \), we get
\[
\| K^{(m)}_{\kappa}(\mathbf{x}, \cdot) \|_{\nu_\kappa} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \sum_{\mathbf{z} \in \Gamma^{(m)}_{\kappa}} 2^{i(2)} \prod_{i=1}^{d} \cos(\gamma_i s_i) \cos(\gamma_i t_i) \right| dt \\
= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \frac{1}{2^{2d}} \sum_{\mathbf{w} \in \{-1, 1\}^d} \sum_{\mathbf{z} \in \Gamma^{(m)}_{\kappa}} 2^{i(2)} \sum_{\mathbf{v} \in \{-1, 1\}^d} \cos \left( \sum_{i=1}^{d} (v_1 \gamma_i s_i + v_i \gamma_i t_i) \right) \right| dt \\
\leq \frac{1}{2^d} \sum_{\mathbf{w} \in \{-1, 1\}^d} \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| \sum_{\mathbf{z} \in \Gamma^{(m)}_{\kappa}} 2^{i(2)-d} \sum_{\mathbf{v} \in \{-1, 1\}^d} \cos \left( \sum_{i=1}^{d} (v_1 \gamma_i s_i + v_i \gamma_i t_i) \right) \right| dt
\]
Thus, by (3.10), we obtain for all \( \mathbf{n} \in \mathcal{N}_d \) the upper estimate
\[ \Lambda_{\mathbf{n}}^{(en)}(\mathbf{r}, \cdot) \leq \| K_{\mathbf{n}}^{(en)}(1, \cdot) \|_{\mathcal{L}^1(\Gamma)} + 1. \] (3.12)

For \( \mathbf{K} \subseteq \{1, \ldots, d\} \), we denote \( \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)} = \{ \mathbf{r} \in \Gamma_{\mathbf{n}}^{(en)} \mid 2\gamma_i = \epsilon n_i \Leftrightarrow i \in \mathbf{K} \} \). Then, for all \( \mathbf{r} = (\cos t_1, \ldots, \cos t_d) \), we have
\[ \tilde{K}_{\mathbf{n}}^{(en)}(1, \mathbf{r}) = K_{\mathbf{n}}^{(en)}(1, \mathbf{r}) - \sum_{\emptyset \neq \mathbf{K} \subseteq \{1, \ldots, d\}} (1 - 2^{-\#\mathbf{K}+1}) \sum_{2^e \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)}} 2^{e(2)} \cos(\gamma_i t_i). \] (3.13)

If \( \mathbf{K} \neq \emptyset \) and \( \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)} \neq \emptyset \), then \( \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)} \subseteq \Gamma_{\mathbf{n}, \mathbf{r}}^{(en)} \) for some \( \mathbf{r} \in \{0, 1\} \). Therefore, the sets \( \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)} \) have a cross product structure and we get
\[ \left| \sum_{2^e \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)}} 2^{e(2)} \prod_{i=1}^d \cos(\gamma_i t_i) \right| \leq \prod_{i \in \{1, \ldots, d\} \setminus \mathbf{K}} \left| \sum_{\gamma_i = -\lfloor e n_i/2 \rfloor + 1}^{\lfloor e n_i/2 \rfloor - 1} e^{\gamma_i t_i} \right|. \]

Thus, for \( \mathbf{K} \neq \emptyset \) and all \( \mathbf{n} \in \mathcal{N}_d \), we have
\[ \int_{[-\pi, \pi]^d} \left| \sum_{2^e \Gamma_{\mathbf{n}, \mathbf{K}}^{(en)}} 2^{e(2)} \prod_{i=1}^d \cos(\gamma_i t_i) \right| \, dt \leq \prod_{i \in \{1, \ldots, d\} \setminus \mathbf{K}} \ln(n_i + 1). \] (3.14)

Now, combining (3.11), (3.12), (3.13), and (3.14) gives the first inequality in (1.5). Finally, Corollary 3.3 implies for \( \mathbf{n} \in \mathcal{N}_d \) the estimate from above in (3.9).

We turn to the lower bound in (3.9). Let \( \epsilon \in \{1, 2\} \), \( \mathbf{n} \in \mathbb{Z}^d \), and \( \mathbf{n} \in \mathcal{N}_d \). By [25] Theorem 1, we have
\[ \Lambda_{\mathbf{n}}^{(en)} \geq \frac{1}{3^d} \mathcal{L}_{\mathbf{n}}^{(en)}. \] (3.15)

Note that in [25] the \( L^\infty-L^\infty \)-operator norm of the partial Fourier sum operator with respect to the set \( \Gamma_{\mathbf{n}}^{(en)} \) is used to characterize the Lebesgue constant related to Fourier sums. This characterization is identical to the definition given in this article, see [20] and the references therein. The relation (3.15) and Corollary 3.3 immediately imply that for \( \mathbf{n} \in \mathcal{N}_d \) we have the estimate from below in (3.9).

To estimate the approximation error \( \| f - P_{\mathbf{n}}^{(en)} f \|_\infty \) for a continuous function \( f \in C([-1, 1]^d) \), let us consider the error of the best approximation given by
\[ E_{\mathbf{n}}^{(en)}(f) = \min_{P \in \Pi_{\mathbf{n}}^{(en)}} \| f - P \|_\infty. \]

Further, let \( P^* \in \Pi_{\mathbf{n}}^{(en)} \) be such that \( E_{\mathbf{n}}^{(en)}(f) = \| f - P^* \|_\infty \). By Theorem 3.4, we get
\[ \| f - P_{\mathbf{n}}^{(en)} f \|_\infty \leq \| P_{\mathbf{n}}^{(en)}(P^* - f) \|_\infty + \| f - P^* \|_\infty \]
\[ \leq (\Lambda_{\mathbf{n}}^{(en)} + 1) E_{\mathbf{n}}^{(en)}(f) \leq \left( \prod_{i=1}^d \ln(n_i + 1) \right) E_{\mathbf{n}}^{(en)}(f). \] (3.16)
Now, using (3.16) and a multivariate version of Jackson’s inequality (see [26, Section 5.3.2]) to estimate $E^{(m)}_{f}(f)$, we obtain the following result.

**Corollary 3.5** Let $\epsilon \in \{1, 2\}$ and $k \in \mathbb{Z}^d$ be fixed. Let also $s \in \mathbb{N}_0^d$ and

$$\frac{\partial^s f}{\partial x_j^s} \in C([-1, 1]^d), \quad j \in \{1, \ldots, d\}.$$

Then, for $n \in \mathbb{N}_a$, we have

$$\|f - P^{(m)}_{f} f\|_{\infty} \lesssim \left( \prod_{i=1}^{d} \ln(n_i + 1) \right) \sum_{j=1}^{d} \frac{1}{(n_j + 1)^{\gamma_j}} \omega \left( \frac{\partial^s f}{\partial x_j^s}; 0, \ldots, 0, \frac{2}{n_j + 1}, 0, \ldots, 0 \right),$$

where

$$\omega(f; u) = \sup_{x, x' \in [-1, 1]^d : |x - x'| \leq u} |f(x') - f(x)|$$

denotes the modulus of continuity of $f$ on $[-1, 1]^d$ (see [26, Section 6.3]).

**Proof.** In view of (3.16), we only need to give a proper estimate of the best approximation $E^{(m)}_{f}(f)$. Since $\Gamma_{k, 0}^{(m)} \subseteq \Gamma_{k}^{(m)}$, we have $E^{(m)}_{f}(f) \leq E^{(m)}_{f, 0}(f)$, where $E^{(m)}_{f, 0}(f)$ denotes the error of the best approximation in the space spanned by $T_{d, \gamma} \in \Gamma_{k, 0}^{(m)}$.

Since $\Gamma_{k, 0}^{(m)}$ has a tensor-product structure, we obtain

$$E^{(m)}_{f}(f) \leq E^{(m)}_{f, 0}(f) \leq \sum_{j=1}^{d} \frac{2^{\gamma_j} \omega \left( \frac{\partial^s f}{\partial x_j^s}; 0, \ldots, 0, \frac{2}{n_j + 1}, 0, \ldots, 0 \right)}{(n_j + 1)^{\gamma_j}} \lesssim \sum_{j=1}^{d} \omega \left( \frac{\partial^s f}{\partial x_j^s}; 0, \ldots, 0, \frac{1}{n_j + 1}, 0, \ldots, 0 \right)$$

by using the estimates from [26, Section 5.3.2]. \hfill \Box

Similar as stated in [22, Theorem 4.1] for the tensor-product case, we can also give a Dini-Lipschitz criterion for the uniform convergence of the error $\|f - P^{(m)}_{f} f\|_{\infty}$.

If $f \in C([-1, 1]^d)$ satisfies the condition

$$\omega(f; u) \prod_{i=1}^{d} \ln n_i \to 0 \quad \text{as} \quad u \to 0,$$

then the polynomials $P^{(m)}_{f} f$ converge in the $L^\infty$-norm to $f$ as $\min_{i=1, \ldots, d} n_i \to \infty$.

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