How far does small chemotactic interaction perturb the Fisher–KPP dynamics?

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Abstract
This paper deals with nonnegative solutions of the Neumann initial-boundary value problem for the fully parabolic chemotaxis-growth system,

\[
\begin{aligned}
(\u)_{t} &= \Delta \u - \varepsilon \nabla \cdot (\u \nabla \v) + \mu \u (1 - \u), & x \in \Omega, \ t > 0, \\
(\v)_{t} &= \Delta \v - \v + \u, & x \in \Omega, \ t > 0,
\end{aligned}
\]

with positive small parameter \( \varepsilon > 0 \) in a bounded convex domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) with smooth boundary. The solutions converge to the solution \( \u \) to the Fisher–KPP equation as \( \varepsilon \to 0 \). It is shown that for all \( \mu > 0 \) and any suitably regular nonnegative initial data \((\u_{\text{init}}, \v_{\text{init}})\) there are some constants \( \varepsilon_0 > 0 \) and \( C > 0 \) such that

\[
\sup_{t > 0} \| \u (\cdot, t) - \u (\cdot, t) \|_{L^\infty (\Omega)} \leq C \varepsilon \text{ for all } \varepsilon \in (0, \varepsilon_0). 
\]

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1 Introduction

The Fisher-KPP equation ([7, 16])

\[ u_t = \Delta u + \mu u(1 - u) \]  

(1.1)

modelling the spread and growth of a biological population - or, in the original setting, of the prevalence of an advantageous gene within the population ([7]) - is well-studied and clearly of interest on its own, and there is a large corpus of literature bearing witness to this, ranging from articles on existence and speed ([7, 16]) or stability ([14]) of travelling waves, long-term behaviour of solutions and a 'hair-trigger effect' (i.e. instability of the rest state $u \equiv 0$) ([2]) to treatments of system variants in a heterogeneous environment ([4]), with nonlinear ([19]) or fractional ([26]) diffusion or nonlocal interaction ([9]), spatio-temporal delays ([1]), or investigations of the spreading as free boundary problem ([5]), to name but a few.

At the same time owing to the rather simple structure of the equation, it is no wonder that (1.1) makes an appearance as constituent of more complex models.

In the present article, we shall view chemotaxis systems with logistic growth terms as perturbation of (1.1) and ask to what extent the behaviour of solutions to (1.1) is altered in the presence of weak chemotactic effects.

Chemotaxis models with growth terms. The Keller-Segel model ([15], see also the surveys [13, 3]) has arisen from the ambition to understand chemotaxis, i.e. the partially directed movement of cells (bacteria, slime mould, etc.; with density denoted by $u$) in the direction indicated by concentration gradient of a signal substance (concentration $v$) they themselves produce:

\[ \begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), \\
    \tau v_t &= \Delta v - v + u,
\end{align*} \]  

(1.2)

Herein, the constant $\chi$ stands for the chemotactic sensitivity and $f$ is used to denote growth terms, one of the most natural forms (apart from $f \equiv 0$) being $f(u) = u - u^2$.

The model plays an important role in the mathematical study of emergence of pattern and structure in many different biological contexts (see [12]), e.g. slime mould formation, bacterial patterning, embryonic development, progression of cancer, and has spawned an abundance of mathematical literature over the past decades (see [13, 3] and the references therein).

In particular in the presence of logistic source terms like $f(u) = \kappa u - \mu u^2$ (cf. also [12, Sec. 2.8]) structure formation can be observed in ([12]), as witnessed by the numerical experiments in [25] or [6], attractor results in [24] and transient growth phenomena demonstrated in [31, 17, 29]. Let us recall some known results about this system:

In the simplified parabolic-elliptic case (i.e. $\tau = 0$), with $f$ generalizing $f(u) = \kappa u - \mu u^2$, Tello and Winkler proved the existence of global weak solutions and global classical solutions if $\mu > \frac{n-2}{n}$. They also showed convergence of solutions to the constant steady state under a stricter condition on the source terms. Very weak solutions have been constructed for sources of the form $f(u) = \kappa u - \mu u^\alpha$ for $\alpha > 2 - \frac{1}{n}$ in [30].

As to the fully parabolic case of (1.2) ($\tau = 1$, again with $f(u) = \kappa u - \mu u^2$) it is known that globally bounded classical solutions exist in two-dimensional domains ([24]) or if $\mu$ is sufficiently large ([32]); these solutions converge, provided a further largeness requirement is satisfied by $\mu$ ([33]). For any positive $\mu$, global weak solutions exist ([15]), which moreover in three-dimensional settings are known to become classical after some waiting time and enter an absorbing ball in $C^{2+\alpha}$ if $\kappa$ is sufficiently small ([15]).
Extensive studies regarding the interplay of exponents $\alpha$ and $\beta$ with respect to global existence of bounded solutions to the system obtained from (1.2) upon replacing the production term $+u$ in the second equation by, roughly speaking, $+u^{\beta}$ and with $f(u) = u - u^\alpha$ have been conducted by Nakaguchi and Osaki [22, 23]. The existence of very weak solutions to (1.2) with $f(u) = u - u^\alpha$, $\alpha > 2 - \frac{1}{n}$, has been established by Viglialoro [28] for bounded domains of arbitrary dimension.

The convergence rate of solutions for both the parabolic-elliptic and the parabolic-parabolic variant of (1.2) has recently been studied by He and Zheng [10].

Limiting cases as parameters tend to zero. Letting different parameters in (1.2) tend to zero can help uncover dynamical properties in (1.2) and the relation to affiliated models. In [34, 17], considering $\varepsilon \to 0$ in

$$
\begin{align*}
\left\{ \begin{array}{l}
    u_t = \varepsilon \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\
    0 = \Delta v - v + u
\end{array} \right.
\end{align*}
$$

was used to obtain insight into some transient growth phenomenon of solutions from a blow-up result in the hyperbolic-elliptic limit system (1.3) with $\varepsilon = 0$, both in the one-dimensional (1.4) and in the higher-dimensional radially symmetric case (1.7).

For quite general choices of $f$, system (1.2) had been suggested and investigated by Mimura and Tsuchikawa [21]. Inter alia, they considered the limit $\varepsilon \to 0$ of the time-rescaled system with Allee effect

$$
\begin{align*}
\left\{ \begin{array}{l}
    u_t = \varepsilon^2 \Delta u - \varepsilon \nabla \cdot (u \nabla v) + u(1 - u)(u - a), \\
    v_t = \Delta v + u - v,
\end{array} \right.
\end{align*}
$$

$a \in (0, \frac{1}{2})$, thus showing the existence of localized aggregating patterns.

In the present paper, we want to investigate the disturbances to Fisher-KPP dynamics caused by weak chemotactic effects and hence consider the system

$$
\begin{align*}
\left\{ \begin{array}{l}
    (u_{\varepsilon})_t = \Delta u_{\varepsilon} - \varepsilon \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) + \mu u_{\varepsilon}(1 - u_{\varepsilon}), \\
    (v_{\varepsilon})_t = \Delta v_{\varepsilon} - v_{\varepsilon} + u_{\varepsilon}, \\
    \frac{\partial u_{\varepsilon}}{\partial \nu} = \frac{\partial v_{\varepsilon}}{\partial \nu} = 0, \\
    u_{\varepsilon}(x,0) = u_{\text{init}}(x), v_{\varepsilon}(x,0) = v_{\text{init}}(x),
\end{array} \right. \\
\end{align*}
$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary, where $\mu > 0$ and $u_{\text{init}}, v_{\text{init}} \in C^0(\bar{\Omega})$ and $v_{\text{init}} \in W^{1,\infty}(\Omega)$ are nonnegative (1.4) and where $\varepsilon > 0$ is to be small. We will compare its solutions to those of

$$
\begin{align*}
\left\{ \begin{array}{l}
    u_t = \Delta u + \mu u(1 - u), \\
    \frac{\partial u}{\partial \nu} = 0, \\
    u(x,0) = u_{\text{init}}(x),
\end{array} \right. \\
\end{align*}
$$

Main result. Whereas large chemotaxis terms can cause significantly altered solution behaviour (cf. e.g. [27, Thm. 4.3]), intuition leads to surmise that in presence of weak chemotactic effects, solutions to (1.3) should be close to solutions of (1.5). For example, as $\varepsilon \to 0$, one might expect convergence in some $L^p(\Omega)$ on each finite time interval. We will prove that the solutions converge uniformly in $\Omega \times (0, \infty)$ and moreover show that this convergence is linear in the chemotactic strength $\varepsilon$: 3
Theorem 1.1. Let \( n \in \mathbb{N} \) and let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain with smooth boundary. Let \( \mu > 0 \) and suppose that \((u_{\text{init}}, v_{\text{init}})\) satisfies (1.4). Then there are \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), (1.3) has a global classical solution and that for all \( \varepsilon \in (0, \varepsilon_0) \) the solution \( u_{\varepsilon} \) to (1.3) and the solution \( u \) to (1.4) satisfy
\[
\sup_{t > 0} \| u_{\varepsilon}(\cdot, t) - u(\cdot, t) \|_{L^\infty(\Omega)} \leq C \varepsilon. \tag{1.6}
\]

Strategy and plan of the paper. Having ensured global existence and some uniform bounds for \( u_{\varepsilon} \) and \( \nabla v_{\varepsilon} \) in Section 2 and recalling the well-known result about local existence of solutions to (1.3) (see [32, Lemma 1.1], [33,Lemma 2.1]). In this section we shall show global existence and uniform boundedness of solutions to (1.3). Firstly we will recall the well-known result about local existence of solutions to (1.3) (see [32, Lemma 1.1], [33, Lemma 2.1]). We also denote the solution of (1.5) by \( u \), also denote the solution of (1.5) by \( u \) and suppose that \( \varepsilon > 0 \) that for all \( \varepsilon \in [0, \varepsilon_0] \), (1.3) has a global classical solution and that for all \( \varepsilon \in (0, \varepsilon_0) \) the solution \( u_{\varepsilon} \) to (1.3) and the solution \( u \) to (1.4) satisfy
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\sup_{t > 0} \| u_{\varepsilon}(\cdot, t) - u(\cdot, t) \|_{L^\infty(\Omega)} \leq C \varepsilon. \tag{1.6}
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Lemma 2.1. Let \( \varepsilon \in [0, \infty) \), \( \mu > 0 \), and suppose that \((u_{\text{init}}, v_{\text{init}})\) satisfies (1.4). Then there exist \( T_{\max, \varepsilon} \in (0, \infty] \) and a classical solution \((u_{\varepsilon}, v_{\varepsilon})\) of (1.3) in \( \Omega \times (0, T_{\max, \varepsilon}) \), which satisfy
\[
\text{either } T_{\max, \varepsilon} = \infty \text{ or } \limsup_{t \nearrow T_{\max, \varepsilon}} \| u_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} = \infty. \tag{2.1}
\]
Furthermore, this solution is uniquely determined in the class of function couples such that
\[
\begin{align*}
u \in C^0(\Omega \times [0, T_{\max, \varepsilon}]) & \cap C^{2,1}(\Omega \times (0, T_{\max, \varepsilon})) \quad \text{and} \\
v \in C^0(\Omega \times [0, T_{\max, \varepsilon}]) & \cap C^{2,1}(\Omega \times (0, T_{\max, \varepsilon})) \cap L^\infty([0, T_{\max, \varepsilon}); W^{1,\infty}(\Omega)).
\end{align*}
\]

Throughout the sequel, we keep \( n \in \mathbb{N}, \Omega \subset \mathbb{R}^n, \mu > 0 \) and initial data \( u_{\text{init}} \) and \( v_{\text{init}} \) satisfying (1.4) fixed and, without loss of generality, assume \( u_{\text{init}} \neq 0 \). (If \( u_{\text{init}} \equiv 0 \), also \( u_{\text{init}} \equiv 0 \) for any \( \varepsilon > 0 \), and (1.4) trivially holds true.) Moreover, we let \( T_{\max, \varepsilon} \) and \((u_{\varepsilon}, v_{\varepsilon})\) be as given by Lemma 2.1. We also denote the solution of (1.3) by \( u = u_0 \).

To simplify notation we shall abbreviate the deviations from the nonzero homogeneous steady state by introducing
\[
U_{\varepsilon}(x, t) := u_{\varepsilon}(x, t) - 1 \quad \text{and} \quad V_{\varepsilon}(x, t) := v_{\varepsilon}(x, t) - 1. \tag{2.2}
\]
for \( x \in \overline{\Omega} \) and \( t > 0 \). Then by straightforward computation it follows that \((U_{\varepsilon}, V_{\varepsilon})\) solves
\[
\begin{aligned}
(U_{\varepsilon})_t & = \Delta U_{\varepsilon} - \varepsilon \nabla \cdot (u_{\varepsilon} \nabla V_{\varepsilon}) - \mu U_{\varepsilon} - \mu V_{\varepsilon}^2, & x \in \Omega, \ t > 0, \\
(V_{\varepsilon})_t & = \Delta V_{\varepsilon} - V_{\varepsilon} + U_{\varepsilon}, & x \in \Omega, \ t > 0, \\
\frac{\partial u_{\varepsilon}}{\partial \nu} & = \frac{\partial u_{\varepsilon}}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
U_{\varepsilon}(x, 0) & = u_{\text{init}}(x) - 1, & x \in \Omega.
\end{aligned} \tag{2.3}
We will prove global existence and boundedness of solutions to (1.3). For the pointwise comparison argument (cf. [33, Lemma 3.1]) used in this proof, convexity of the domain is essential.

**Lemma 2.2.** For any \( \varepsilon \in [0, \frac{4\mu}{n}) \), the solution of (1.3) exists globally. Moreover, there is \( c_1 > 0 \) such that

\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 e^{-t} + 1 + \frac{(\mu - 1)^2 + \varepsilon}{4\mu - \varepsilon}
\]

for all \( t > 0 \) and for all \( \varepsilon \in [0, \frac{4\mu}{n}) \).

**Proof.** With \( U_\varepsilon \) and \( V_\varepsilon \) as defined in (2.2), we let

\[
z_\varepsilon(x, t) := U_\varepsilon(x, t) + \frac{\varepsilon}{2} \|\nabla V_\varepsilon(x, t)\|^2
\]

for \( x \in \Omega \) and \( t \in (0, T_{\text{max}, \varepsilon}) \).

Then \( z_\varepsilon \) satisfies

\[
(z_\varepsilon)_t - \Delta z_\varepsilon + z_\varepsilon = -\varepsilon |D^2 V_\varepsilon|^2 - \varepsilon u_\varepsilon \Delta V_\varepsilon - (\mu - 1)U_\varepsilon - \mu U_\varepsilon^2 - \frac{\varepsilon}{2} \|\nabla V_\varepsilon\|^2
\]

\[
\leq \frac{n\varepsilon}{4} U_\varepsilon^2 + \frac{n\varepsilon}{2} U_\varepsilon + \frac{n\varepsilon}{4} - (\mu - 1)U_\varepsilon - \mu U_\varepsilon^2
\]

\[
\leq \frac{(\mu - 1)^2 + n\varepsilon}{4(\mu - \frac{n\varepsilon}{4})} + \frac{n\varepsilon}{4}
\]

\[
= \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon}
\]

for all \( x \in \Omega \) and \( t \in (0, T_{\text{max}, \varepsilon}) \), where we have used the condition \( 4\mu > n\varepsilon \) (for more detail, see [33, Lemma 3.1]). In order to derive an estimate for \( z_\varepsilon \) itself from this, we note that since \( \Omega \) is convex and \( \frac{\partial \varepsilon}{\partial \nu} = 0 \) on \( \partial \Omega \), we have \( \frac{\partial |\nabla u_\varepsilon|^2}{\partial \nu} \leq 0 \) on \( \partial \Omega \) (see [20, Lemme 2.1.1]) and hence also \( \frac{\partial \varepsilon}{\partial \nu} \leq 0 \) on \( \partial \Omega \). We define

\[
y_{\varepsilon} := \|U_\varepsilon(\cdot, 0)\|_{L^\infty(\Omega)} + \frac{\varepsilon}{2} \|\nabla V_\varepsilon(\cdot, 0)\|_{L^\infty(\Omega)}^2 > 0,
\]

and denote by \( y_\varepsilon : [0, \infty) \rightarrow \mathbb{R} \) the function solving

\[
\begin{cases}
y'_\varepsilon(t) + y_\varepsilon(t) = \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon}, & t > 0, \\
y_\varepsilon(0) = y_{\varepsilon 0}.
\end{cases}
\]

By the comparison theorem we obtain that

\[
0 \leq u_\varepsilon(\cdot, t) = U_\varepsilon(\cdot, t) + 1 \leq y(t) + 1 \leq c_1 e^{-t} + 1 + \frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon},
\]

where \( c_1 := \|U_\varepsilon(\cdot, 0)\|_{L^\infty(\Omega)} + \frac{\varepsilon}{2} \|\nabla V_\varepsilon(\cdot, 0)\|_{L^\infty(\Omega)}^2 \). In view of (2.1) we complete the proof, seeing that actually \( T_{\text{max}, \varepsilon} = \infty \).

In the next lemma we aim at deriving a bound for \( \nabla v_\varepsilon \). We restrict the admissible values for \( \varepsilon \) to a smaller range than in Lemma 2.2 in order to establish the estimate independently of \( \varepsilon \), in contrast to the right-hand side of (2.4).

**Lemma 2.3.** There exist \( c_2 > 0 \) and \( \lambda_1 > 0 \) such that

\[
\|\nabla v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_2 (1 + e^{-t} + e^{-(1+\lambda_1)t})
\]

for all \( t \geq 0 \) and all \( \varepsilon \in [0, \frac{2\mu}{n}) \).
Proof. We note that
\[
\frac{(\mu - 1)^2 + n\varepsilon}{4\mu - n\varepsilon} \leq \frac{(\mu - 1)^2 + 2\mu}{4\mu - 2\mu} = \frac{\mu^2 + 1}{2\mu}
\]
for all \( \varepsilon \in \left[0, \frac{2\mu}{n}\right) \), so that according to Lemma 2.2 the estimate
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 e^{-t} + 1 + \frac{\mu^2 + 1}{2\mu} \leq c_1 + 1 + \frac{\mu^2 + 1}{2\mu} =: c_3
\]
holds for all \( t > 0 \) and all \( \varepsilon \in [0, \frac{2\mu}{n}) \). By means of a variation-of-constants representation for \( v \), we have
\[
\nabla v_\varepsilon(\cdot, t) = \nabla e^{t(\Delta - 1)} v_{\text{init}} + \int_0^t \nabla e^{(t-s)(\Delta - 1)} u_\varepsilon(\cdot, s) \, ds \quad \text{for all } t > 0.
\]
Known smoothing estimates for the Neumann heat semigroup in \( \Omega \) (more precisely: the limit case \( p \to \infty \) in [31, Lemma 1.3 (iii)]) provide us with constants \( c_4 > 0 \) and \( \lambda_1 > 0 \) such that
\[
\|\nabla e^\sigma \varphi\|_{L^\infty(\Omega)} \leq c_4 e^{-\lambda_1 \tau} \|\nabla \varphi\|_{L^\infty(\Omega)} \quad \text{for all } \tau > 0 \text{ and all } \varphi \in W^{1,\infty}(\Omega).
\]
Accordingly,
\[
\|\nabla e^{t(\Delta - 1)} v_{\text{init}}\|_{L^\infty(\Omega)} \leq c_4 e^{-(1+\lambda_1)t} \|\nabla v_{\text{init}}\|_{L^\infty(\Omega)}.
\]
Similarly employing [31, Lemma 1.3 (ii)], we gain \( c_5 > 0 \) such that
\[
\int_0^t \|\nabla e^{(t-s)(\Delta - 1)} u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)} \, ds \leq c_5 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(1+\lambda_1)(t-s)} \|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)} \, ds
\]
\[
\leq c_3 c_5 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(1+\lambda_1)(t-s)} \, ds
\]
\[
\leq c_3 c_5 \int_0^\infty \sigma^{-\frac{1}{2}} e^{-(1+\lambda_1)\sigma} \, d\sigma
\]
for all \( t > 0 \). Therefore we have (2.5) for all \( t > 0 \) and all \( \varepsilon \in [0, \frac{2\mu}{n}) \), with an obvious definition of \( c_2 > 0 \). \( \square \)

3 Local-in-time convergence to the Fisher–KPP equation

In this section we shall prove the convergence of solutions of (1.3) to those of the Fisher-KPP equation (1.5) on some interval \([0, T]\). We will begin with the key ingredient of both the proof on finite and on eventual time intervals: a differential inequality that will first lead to an estimate of \( L^{2k}\)-norms of the difference \( \omega_\varepsilon \):

**Lemma 3.1.** Let \( k \geq 1 \) be an integer. Then there is \( c_6(k) > 0 \) such that for all \( \varepsilon \in [0, \frac{2\mu}{n}) \)
\[
\omega_\varepsilon := u_\varepsilon - u \quad \text{in } \Omega \times (0, \infty)
\]
satisfies
\[
\frac{d}{dt} \int_\Omega \omega_\varepsilon^{2k} \leq \varepsilon^{2k} c_6(k) + \mu k \int_\Omega \omega_\varepsilon^{2k} + 2k \mu \int_\Omega \omega_\varepsilon^{2k}(1 - u_\varepsilon - u)
\]
on \( (0, \infty) \).
Proof. We immediately see that \( \omega_\varepsilon \) satisfies
\[
(\omega_\varepsilon)_t = \Delta \omega_\varepsilon - \varepsilon \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \mu \omega_\varepsilon - \mu (u_\varepsilon + u) \omega_\varepsilon \quad \text{in } \Omega \times (0, \infty).
\]
Multiplying the above equation by \( \omega_\varepsilon^{2k-1} \) and integrating over \( \Omega \), we can calculate
\[
\frac{d}{dt} \int_\Omega \omega_\varepsilon^{2k} = -2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} |\nabla \omega_\varepsilon|^2 + \varepsilon 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} u_\varepsilon \nabla \omega_\varepsilon \cdot \nabla v_\varepsilon \\
+ 2k \mu \int_\Omega (1 - u_\varepsilon - u) \omega_\varepsilon^{2k} \tag{3.1}
\]
on \( (0, \infty) \). Two successive applications of Young’s inequality reveal that with some \( c_7 = c_7(k) > 0 \) we have
\[
\varepsilon 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} u_\varepsilon \nabla \omega_\varepsilon \cdot \nabla v_\varepsilon \leq 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} |\nabla \omega_\varepsilon|^2 + \frac{\varepsilon^2 k(2k-1)}{2} \int_\Omega \omega_\varepsilon^{2k-2} u_\varepsilon^2 |\nabla v_\varepsilon|^2 \\
\leq 2k(2k-1) \int_\Omega \omega_\varepsilon^{2k-2} |\nabla \omega_\varepsilon|^2 + k \mu \int_\Omega \omega_\varepsilon^{2k} \\
+ c_7(k) \varepsilon^{2k} \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2k} \tag{3.2}
\]
for all \( \varepsilon \in [0, \frac{2\mu}{\mu}] \) and on \( (0, \infty) \), where thanks to Lemma \ref{Lemma2.2} and Lemma \ref{Lemma2.3} we may further estimate
\[
c_7(k) \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2k} \leq c_6(k) \quad \text{on } (0, \infty) \tag{3.3}
\]
for some \( c_6(k) > 0 \) independent of \( \varepsilon \in [0, \frac{2\mu}{\mu}] \), so that the combination of \ref{3.2}, \ref{3.3} and \ref{3.4} finally yields \ref{3.1}.

**Corollary 3.2.** Let \( k \geq 1 \) be an integer and let \( c_6(k) \) be as in Lemma \ref{Lemma2.4}. Then, for any \( \varepsilon \in [0, \frac{2\mu}{\mu}] \), the function \( u_\varepsilon \) satisfies
\[
\|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^{2k}(\Omega)} \leq \sqrt{c_6(k)} \varepsilon e^{\frac{\mu}{2} t} \quad \text{for any } t > 0.
\]

**Proof.** Nonnegativity of \( u_\varepsilon \) and \( u \) together with Lemma \ref{Lemma2.4} show that
\[
\frac{d}{dt} \int_\Omega \omega_\varepsilon^{2k} \leq 3k \mu \int_\Omega \omega_\varepsilon^{2k} + c_6(k) \varepsilon^{2k} \quad \text{on } (0, \infty),
\]
which upon an ODE comparison argument and radication readily results in the Corollary, due to the fact that \( \omega_\varepsilon(\cdot, 0) \equiv 0 \).

We employ semigroup techniques to upgrade these estimates to uniform bounds.

**Lemma 3.3.** Let \( q > \frac{n}{2} \) and \( p > n \). There is \( c_8 > 0 \) such that for any \( T > 0 \) and any \( z_0 \in C^0(\overline{\Omega}) \), \( f \in C^0((0, T); C^1(\overline{\Omega}; \mathbb{R}^N)) \), \( g \in C^0(\overline{\Omega} \times (0, T)) \), the solution \( z \) of
\[
z(\cdot, 0) = z_0 \quad \text{in } \Omega, \quad \partial_\nu z|_{\partial \Omega} = 0, \quad z_t = \Delta z + \nabla \cdot f + g \text{ in } \Omega \times (0, T)
\]
for all \( t \in (0, T) \) satisfies
\[
\|z(\cdot, t)\|_{L^\infty(\Omega)} \leq c_8 \left( \|f\|_{L^\infty((0, T); L^p(\Omega))} + \|g\|_{L^\infty((0, T); L^q(\Omega))} + \|z_0\|_{L^\infty(\Omega)} + \|z\|_{L^\infty((0, T); L^q(\Omega))} \right).
\]
Proof. Aided by $L^p$-$L^q$ estimates for the heat semigroup similar to those in [31] Lemma 1.3 i) and iv)], we let $c_9 > 0$ and $c_{10} > 0$ be such that

$$
\|e^{L\Delta w}\|_{L^\infty(\Omega)} \leq c_9 (1 + t^{-\frac{1}{p}}) \|w\|_{L^q(\Omega)} \quad \text{for all } w \in L^q(\Omega)
$$

and

$$
\|e^{L\Delta \cdot \varphi}\|_{L^\infty(\Omega)} \leq c_{10} (1 + t^{-\frac{1}{2} - \frac{1}{p}}) e^{-\lambda_1 t} \|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in L^p(\Omega; \mathbb{R}^n).
$$

Then for $t \in (0, 2] \cap (0, T)$ we obtain

$$
\|z(\cdot, t)\|_{L^\infty(\Omega)} \leq \|z_0\|_{L^\infty(\Omega)} + c_{10} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1 (t-s)} \|f(\cdot, s)\|_{L^p(\Omega)} ds + c_9 \int_0^t (1 + (t-s)^{-\frac{1}{p}}) \|g(\cdot, s)\|_{L^q(\Omega)} ds
$$

$$
\leq \left(1 + c_{10} \int_0^t (1 + \tau^{-\frac{1}{2}}) e^{-\lambda_1 \tau} d\tau + c_9 \int_0^t (1 + \tau^{-\frac{1}{p}}) d\tau\right) \left(\|z_0\|_{L^\infty(\Omega)} + \|f\|_{L^\infty((0,T);L^p(\Omega))} + \|g\|_{L^\infty((0,T);L^q(\Omega))}\right).
$$

For $t \in (2, \infty) \cap (0, T)$, on the other hand, we have

$$
z(\cdot, t) = e^{L\Delta} z(\cdot, t-1) + \int_0^1 e^{(1-s)\Delta} \nabla \cdot f(\cdot, t-1+s) ds + \int_0^1 e^{(1-s)\Delta} g(\cdot, t-1+s) ds
$$

and hence may estimate

$$
\|z(\cdot, t)\|_{L^\infty(\Omega)} \leq 2c_9 \|z(\cdot, t-1)\|_{L^\infty(\Omega)} + \int_0^1 c_9 (1 + (1-s)^{-\frac{1}{p}}) e^{-\lambda_1 (1-s)} \|f(\cdot, t-1+s)\|_{L^p(\Omega)} ds
$$

$$
+ \int_0^1 c_9 (1 + (1-s)^{-\frac{1}{p}}) \|g(\cdot, t-1+s)\|_{L^q(\Omega)} ds
$$

$$
\leq \left(2c_9 + \int_0^1 c_{10} (1 + \tau^{-\frac{1}{2}}) e^{-\lambda_1 \tau} d\tau + \int_0^1 c_9 (1 + \tau^{-\frac{1}{p}}) d\tau\right) \left(\|z\|_{L^\infty((0,T);L^\infty(\Omega))} + \|f\|_{L^\infty((0,T);L^p(\Omega))} + \|g\|_{L^\infty((0,T);L^q(\Omega))}\right).
$$

A consequence for the model under consideration is the following.

**Corollary 3.4.** Let $k > \frac{n}{2}$ be an integer. There is $c_{11} > 0$ such that for any $\varepsilon \in [0, \frac{2k}{n}]$ and any $t > 0$

$$
\|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{11} (\varepsilon + \|u_\varepsilon - u\|_{L^\infty((0,T);L^{2k}(\Omega))}).
$$

**Proof.** The function $\omega_\varepsilon := u_\varepsilon - u$ solves $(\omega_\varepsilon)_t = \Delta \omega_\varepsilon + \varepsilon \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \mu \omega_\varepsilon - \mu u_\varepsilon (u_\varepsilon + u)$ and hence from Lemma [3.3] we can take $c_{12} > 0$ such that

$$
\|\omega_\varepsilon(\cdot, \tau)\|_{L^\infty(\Omega)} \leq c_{12} (\varepsilon \|u_\varepsilon \nabla v_\varepsilon\|_{L^\infty(\Omega)} + \|\mu \omega_\varepsilon - \mu u_\varepsilon (u_\varepsilon + u)\|_{L^\infty((0,T);L^{2k}(\Omega))} + \|\omega_\varepsilon\|_{L^\infty((0,T);L^{2k}(\Omega))})
$$

$$
\leq c_{12} (\varepsilon \|u_\varepsilon \nabla v_\varepsilon\|_{L^\infty(\Omega)} + \|\mu \omega_\varepsilon - \mu u_\varepsilon (u_\varepsilon + u)\|_{L^\infty((0,T);L^{2k}(\Omega))})
$$

$$
+ c_{12} (\mu (1 + \|u\|_{L^\infty(\Omega)}) + 1) \|\omega_\varepsilon\|_{L^\infty((0,T);L^{2k}(\Omega))}
$$

holds for any $\varepsilon \in [0, \frac{2k}{n}]$ and any $\tau \in (0, t)$. Using the uniform bounds on $u$, $u_\varepsilon$, $\nabla v_\varepsilon$ that have been provided by Lemma [2.2] and Lemma [2.3] we obtain the conclusion.

**Corollary 3.5.** For any $T > 0$

$$
u_\varepsilon \rightarrow u \quad \text{uniformly in } \overline{\Omega} \times [0, T] \text{ as } \varepsilon \searrow 0.
$$

**Proof.** This results from straightforward combination of Corollary 3.2 and Corollary 3.4.
4 Large time behaviour in both systems

Corollary 4.3 takes care of convergence of $u_\varepsilon$ to $u$ on finite time intervals. Seeing that $u_\varepsilon$ and $u$ both converge to 1 as $t \to \infty$, we still have hope that they will be close to each other on intervals of the form $(T, \infty)$. We will, nevertheless, need such information in a much more quantitative form – and this is what we prepare in the present section. After recalling a well-known estimate for the Laplacian in $\Omega$ supplemented with homogeneous Neumann boundary conditions, we obtain bounds for $U_\varepsilon$ in the domain of some fractional power of this operator and of $\Delta u_\varepsilon$ in $L^\infty(\Omega)$ that, together with the uniform lower bound of $u_\varepsilon(\cdot, t)$ for some positive time $t$ (Lemma 4.4), can consequently be turned into precisely those quantitative lower bounds for $u_\varepsilon$ and $u$ we will need in the proof of Theorem 1.1 in Section 5.

We fix any number $\hat{\mu} \in (0, \mu) \cap (0, 1)$ and given $p > 1$ we let $A = A_p$ denote the realization of the operator $-\Delta + \hat{\mu}$ under homogeneous Neumann boundary condition in $L^p(\Omega)$.

**Lemma 4.1.** $A$ is sectorial and thus possesses closed fractional powers $A^\eta$ for arbitrary $\eta > 0$, and the corresponding domains $D(A^\eta)$ are known to have the embedding property

$$D(A^\eta) \hookrightarrow W^{2,\infty}(\Omega) \quad \text{if} \quad 2\eta - \frac{n}{p} > 2. \quad (4.1)$$

Moreover, if $(e^{-tA})_{t \geq 0}$ denotes the corresponding analytic semigroup, then for each $\eta > 0$ there exists $c_{13}(p, \eta) > 0$ such that

$$\|A^\eta e^{-tA} \varphi\|_{L^p(\Omega)} \leq c_{13}(p, \eta)t^{-\eta}\|\varphi\|_{L^p(\Omega)} \quad (4.2)$$

for all $t > 0$ and each $\varphi \in L^p(\Omega)$.

**Proof.** See [11] Theorem 1.4.3 and Theorem 1.6.1. \qed

**Lemma 4.2.** For all $p > 1$ and any $\eta \in (0, \frac{1}{2})$ there exists $c_{14} > 0$ such that

$$\|A^\eta U_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq c_{14} \quad (4.3)$$

holds for all $t \geq 2$ and all $\varepsilon \in [0, \frac{2\mu}{\hat{\mu}})$, where $U_\varepsilon \equiv u_\varepsilon - 1$.

**Proof.** According to standard estimates for the Neumann heat semigroup (see e.g. [8] Lemma 3.3]), we can find $c_{15} > 0$ such that

$$\|e^{t\hat{\mu}\nabla \cdot \varphi}\|_{L^p(\Omega)} \leq c_{15}(1 + t^{-\frac{1}{2}})\|\varphi\|_{L^p(\Omega)} \quad (4.4)$$

all $\tau > 0$ and any $\varphi \in C^1((\Omega; \mathbb{R}^n)$ such that $\varphi \cdot \nu = 0$ on $\partial \Omega$. We represent $U_\varepsilon$ according to

$$U_\varepsilon(\cdot, t) = e^{(t-1)\hat{\mu}\nabla \cdot \varphi}(\cdot, 1) - \varepsilon \int_1^t e^{(t-s)\hat{\mu}\nabla \cdot \varphi}(u_\varepsilon \nabla u_\varepsilon(\cdot, s))ds$$

$$-\mu \int_1^t e^{(t-s)\hat{\mu}\nabla \cdot \varphi} U_\varepsilon^2(\cdot, s)ds$$

$$= e^{-(\mu-\hat{\mu})(t-1)}e^{-(t-1)A}U_\varepsilon(\cdot, 1) - \varepsilon \int_1^t e^{-(\mu-\hat{\mu})(t-s)}e^{-\frac{t-s}{t-s}}A e^{-\frac{\mu}{t-s}}\hat{\mu}\nabla \cdot (u_\varepsilon \nabla u_\varepsilon(\cdot, s))ds$$

$$-\mu \int_1^t e^{-(\mu-\hat{\mu})(t-s)}e^{-(t-s)A}U_\varepsilon^2(\cdot, s)ds \quad \text{for all } t > 1.$$
Hence we can calculate that
\[
\|A^n U_\varepsilon(t)\|_{L^p(\Omega)} \leq e^{-(\mu - \bar{\nu})(t-1)}\|A^n e^{-(t-1)A} U_\varepsilon(\cdot, 1)\|_{L^p(\Omega)}
\]
\[+\varepsilon \int_1^t e^{-(\mu - \bar{\nu})(t-s)} \|A^n e^{-\frac{s}{2} A} e^{\frac{s}{2} \Delta} \cdot (u_\varepsilon \nabla v_\varepsilon(\cdot, s))\|_{L^p(\Omega)} ds
\]
\[+\mu \int_1^t e^{-(\mu - \bar{\nu})(t-s)} \|A^n e^{-(t-s)A} U_\varepsilon^2(\cdot, s)\|_{L^p(\Omega)} ds \quad \text{for all } t > 1.
\]

Herein, (4.2) and (4.4) allow us to estimate
\[
\|A^n e^{-(t-1)A} U_\varepsilon(\cdot, 1)\|_{L^p(\Omega)} \leq c_{13}(p, \eta)(t-1)^{-\eta}\|U_\varepsilon(\cdot, 1)\|_{L^p(\Omega)},
\]
\[
\|A^n e^{-\frac{s}{2} A} e^{\frac{s}{2} \Delta} \cdot (u_\varepsilon \nabla v_\varepsilon(\cdot, s))\|_{L^p(\Omega)} \leq c_{13}(p, \eta)c_{15}\left(\frac{t-s}{2}\right)^{-\eta}\left(1 + \left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\right)\|u_\varepsilon \nabla v_\varepsilon(\cdot, s)\|_{L^p(\Omega)},
\]
\[
\|A^n e^{-(t-s)A} U_\varepsilon^2(\cdot, s)\|_{L^p(\Omega)} \leq c_{13}(p, \eta)(t-s)^{-\eta}\|U_\varepsilon^2(\cdot, s)\|_{L^p(\Omega)} ds
\]
for \(1 < s < t\). Together with the finiteness of \(\int_0^\infty e^{-(\mu - \bar{\nu})s} \sigma^{-\eta}(1 + \sigma^{-\frac{1}{2}}) d\sigma\) and \(\int_0^\infty e^{-(\mu - \bar{\nu})\sigma} \sigma^{-\eta} d\sigma\), Lemma 2.2 and Lemma 2.3 establish the existence of \(c_{14} > 0\) such that (4.5) holds for all \(t \geq 2\). □

An important consequence of this estimate is that it provides some control over \(\Delta v_\varepsilon\):

**Lemma 4.3.** There exists \(c_{16} > 0\) such that for all \(\varepsilon \in [0, \frac{2\mu}{\eta}]\)
\[
\|\Delta v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{16}
\]
for all \(t \geq 3\).

**Proof.** We fix an arbitrary \(\gamma \in (1, \frac{3}{2})\) and then can choose positive numbers \(\eta\) and \(p\) such that
\[
\gamma - 1 < \eta < \frac{1}{2}, \quad p > \frac{n}{2(\gamma - 1)}.
\]
Then \(2\gamma - \frac{n}{p} > 2\gamma - 2(\gamma - 1) = 2\). According to a variation-of-constants formula associated with the second equation in (2.3), we can write
\[
V_\varepsilon(t) = e^{(t-2)(\Delta - 1)} V_\varepsilon(\cdot, 2) + \int_2^t e^{(t-s)(\Delta - 1)} U_\varepsilon(\cdot, s) ds
\]
\[
= e^{-(1-\bar{\nu})(t-2)} e^{-(t-s)A} V_\varepsilon(\cdot, 2) + \int_2^t e^{-(1-\bar{\nu})(t-s)} e^{-(t-s)A} U_\varepsilon(\cdot, s) ds,
\]
for all \(t \geq 2\), and hence (4.1) implies that
\[
\|V_\varepsilon(\cdot, t)\|_{W^{2, \infty}(\Omega)} \leq c_{17}\|A^7 V_\varepsilon(\cdot, t)\|_{L^p(\Omega)}
\]
\[
\leq c_{17} e^{-(1-\bar{\nu})(t-2)} \|A^7 e^{-(t-2)A} V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)}
\]
\[+c_{17} \int_2^t e^{-(1-\bar{\nu})(t-s)} \|A^7 e^{-(t-s)A} U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds
\]
\[= c_{17} e^{-(1-\bar{\nu})(t-2)} \|A^7 e^{-(t-2)A} V_\varepsilon(\cdot, 2)\|_{L^p(\Omega)}
\]
\[+c_{17} \int_2^t e^{-(1-\bar{\nu})(t-s)} \|A^7 e^{-(t-s)A} U_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds
\]
for all \(t \geq 2\), and hence (4.1) implies that...
with some $c_{17} > 0$. Using (4.3) and (4.2) to estimate

$$\| A^{\gamma} \eta e^{-(t-s)A^\alpha U_\varepsilon(\cdot, s)} \|_{L^p(\Omega)} \leq c_{13}(p, \gamma - \eta)(t-s)^{-(\gamma-\eta)} c_{14} \quad \text{for any } 2 < s < t,$$

and taking into account the boundedness of $c_{17} e^{-(1-\hat{\mu})(t-2)} \| A^{\gamma} e^{-(2)A^\alpha U_\varepsilon(\cdot, 2)} \|_{L^p(\Omega)}$ on $(3, \infty)$ due to

$$\| A^{\gamma} e^{-(t-2)A^\alpha U_\varepsilon(\cdot, 2)} \|_{L^p(\Omega)} \leq c_{13}(p, \gamma) (t-2)^{-\gamma} \| V_\varepsilon(\cdot, 2) \|_{L^p(\Omega)}$$

$$\leq c_{13}(p, \gamma) \left( \| e^{2(\Delta-1)(v_{\text{init}} - s))} \|_{L^p(\Omega)} + \int_0^2 \| e^{(2-s)(\Delta-1)} U_\varepsilon(\cdot, s) \|_{L^p(\Omega)} ds \right), \ t \in (3, \infty),$$

and Lemma 2.2 we obtain $c_{16} > 0$ such that (4.5) holds.

We will now establish lower bounds for $u_\varepsilon(x, 3)$, which are independent of $\varepsilon$. In contrast to the final assertion of Theorem 1.1, this lemma relies on our assumption $u_{\text{init}} \neq 0$.

**Lemma 4.4.** There exist $c_{18} > 0$ and $\varepsilon_1 \in (0, \frac{2\mu}{n})$ such that

$$u_\varepsilon(x, 3) \geq c_{18}$$

for all $x \in \Omega$ and for all $\varepsilon \in [0, \varepsilon_1)$.

**Proof.** We will use a contradiction argument. If there exist $(\varepsilon_j)_{j \in \mathbb{N}}$ with $\lim_{j \to \infty} \varepsilon_j = 0$ and $(x_j)_{j \in \mathbb{N}} \subset \Omega$ such that

$$\lim_{j \to \infty} u_\varepsilon(x_j, 3) = 0,$$  \hspace{1cm} (4.6)

then we can take a subsequence $(x_{jk})_{k} \subset (x_j)_{j}$ satisfying that there exists $x_0 \in \overline{\Omega}$ such that

$$\lim_{k \to \infty} x_{jk} = x_0.$$

Thanks to Corollary 3.5 and (4.6) we deduce

$$u(x_0, 3) = \lim_{k \to \infty} u_{\varepsilon_{jk}}(x_{jk}, 3) = 0.$$

However, we obtain by the strong maximum principle that

$$u(x_0, 3) > 0,$$

which is contradiction. \hfill \square

We are now able to estimate $u_\varepsilon$ and $u$ from below. The following lemma can be viewed as a one-sided quantitative statement on the long-term behaviour of $u_\varepsilon$ and $u$.

**Lemma 4.5.** Let $\varepsilon_2 := \min\{\varepsilon_1, \frac{1}{4c_{16}}\} \leq \frac{2\mu}{n}$ with $\varepsilon_1$ taken from Lemma 4.4 and $c_{16}$ as defined in Lemma 4.3. Then there is $c_{19} > 0$ such that

$$\inf_{x \in \Omega} \inf_{\varepsilon \in [0, \varepsilon_2]} u_\varepsilon(x, t) \geq \frac{1 - c_{19}e^{-\frac{\varepsilon_1}{t}}}{1 + c_{19}e^{-\frac{\varepsilon_1}{t}}} \quad \text{for all } t \geq 3. \quad (4.7)$$
Proof. From the first equation in (1.3) and (5.5) we see that
\[
(u_ε)_t = \Delta u_ε - ε \nabla u_ε \cdot \nabla v_ε - ε u_ε \Delta v_ε + μ u_ε - μ u_ε^2 \geq \Delta u_ε - ε \nabla u_ε \cdot \nabla v_ε + (μ - c_{16} ε) u_ε - μ u_ε^2 \quad \text{in } Ω
\]
for \( t \geq 3 \). We choose \( y_0 \in (0, \frac{1}{2}) \) to be a positive number such that \( y_0 < \frac{μ - c_{16} ε}{μ} \) and \( y_0 \leq \inf_{x ∈ (0, ε)} \inf_{x ∈ Ω} u_ε(x, 3) \) (cf. Lemma 4.3) and put
\[
y_ε(t) := \frac{μ - c_{16} ε}{μ + (\frac{μ - c_{16} ε}{y_0} - μ) e^{-(μ - c_{16} ε)(t - 3)}}, \quad t \geq 3.
\]
Then \( y_ε : [3, ∞) → ℝ \) is the solution to the ODE initial value problem
\[
\begin{align*}
y_ε'(t) &= (μ - c_{16} ε)y_ε(t) - μ y_ε(t)^2, \quad t > 3, \\
y_ε(3) &= y_0.
\end{align*}
\]
Apparently, \( 0 < \frac{μ - c_{16} ε}{y_0} - μ \leq \frac{μ}{y_0} \) and due to \( ε < \frac{μ}{2c_{16}} \), we may employ the estimate \( e^{-(μ - c_{16} ε)(t - 3)} \leq e^{-\frac{μ}{2} t + \frac{μ}{2 c_{16}}} \) to see that
\[
y_ε(t) ≥ \frac{μ - c_{16} ε}{μ + \frac{μ - c_{16} ε}{y_0} - μ e^{-\frac{μ}{2} t + \frac{μ}{2 c_{16}}}} = \frac{1 - \frac{c_{16} ε}{μ}}{1 + c_{19} e^{-\frac{μ}{2} t}} \quad \text{for all } t ≥ 3
\]
if we let \( c_{19} := \frac{1}{y_0 e^{\frac{μ}{2 c_{16}}}} \). In light of a comparison lemma we can deduce (1.7). \( □ \)

5 Global-in-time convergence: Proof of Theorem 1.1

With these explicit and quantitative uniform lower bounds for \( u_ε \) and \( u \), everything has been prepared to revisit the differential inequality of Lemma 3.1 and turn our attention to the proof of Theorem 1.1. In fact, we only have to show the following:

Lemma 5.1. There are \( c_{20} > 0 \) and \( ε_0 > 0 \) such that
\[
\|u_ε(\cdot, t) - u(\cdot, t)\|_{L^∞(Ω)} ≤ c_{20} ε \quad \text{for any } t ∈ (0, ∞) \text{ and any } ε ∈ [0, ε_0).
\]

Proof. We let \( k > \frac{2}{3} \) be an integer and with \( ε_2 \) as in Lemma 4.6 and \( c_{16} \) taken from Lemma 4.3 we set \( ε_0 := \min\{ε_2, \frac{μ}{2c_{21}}\} \). In accordance with Lemma 4.6 we then choose \( T > 0 \) such that on \( Ω × (T, ∞) \)
\[
u > \frac{7}{8} \text{ and } u_ε > \frac{7}{8} \quad \text{for any } ε ∈ [0, ε_0).
\]
By Corollary 3.2 and Corollary 3.3 there is \( c_{21} > 0 \) such that
\[
\|u_ε - u\|_{L^∞(Ω × (0, T))} ≤ c_{21} ε \quad \text{for all } ε ∈ [0, ε_0).
\]
Moreover, Lemma 3.1 ensures that (with \( c_6(k) \) as defined there)
\[
\frac{d}{dt} \int Ω \omega_ε^{2 k} ≤ \varepsilon^{2k} c_6(k) + μ k \int Ω \omega_ε^{2 k} + 2 μ k \int Ω (1 - u - u_ε) ω_ε^{2 k}
\]
\[
≤ \varepsilon^{2k} c_6(k) - \frac{μ k}{2} \int Ω \omega_ε^{2 k} \quad \text{for all } ε ∈ [0, ε_0), t ∈ (T, ∞),
\]
where we have used that $1 - u - u_\varepsilon < 1 - \frac{7}{8} - \frac{7}{8} = -\frac{1}{4}$ on $\Omega \times (T, \infty)$. Therefore,

$$\int_\Omega \omega_k^2(\cdot, t) \leq \int_\Omega \omega_k^2(\cdot, T)e^{-\frac{\mu k}{2}(t-T)} + \frac{2c_6(k)}{\mu k} \varepsilon^{2k} \leq \left(\frac{c_2k}{2} + \frac{2c_6(k)}{\mu k}\right)\varepsilon^{2k}$$

for all $t > T$ and all $\varepsilon \in (0, \varepsilon_0)$, and hence we have found $c_{22} > 0$ such that

$$\|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^{2k}(\Omega)} \leq c_{22}\varepsilon \quad \text{for all } t > T, \varepsilon \in (0, \varepsilon_0).$$

A further application of Corollary 3.4 shows that hence for some $c_{23} > 0$

$$\|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^{\infty}(\Omega)} \leq c_{23}\varepsilon \quad \text{for all } t > T, \varepsilon \in (0, \varepsilon_0).$$

Together with (5.1), this proves the lemma and hence also Theorem 1.1.

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