Clifford algebras, Spin groups and qubit trees

Alexander Yu. Vlasov

Representations of Spin groups and Clifford algebras derived from the structure of qubit trees are introduced in this work. For ternary trees the construction is more general and reduction to binary trees is formally defined by deletion of superfluous branches. The usual Jordan–Wigner construction also may be formally obtained in this approach by bringing the process up to trivial qubit chain (trunk). The methods can also be used for effective simulation of some quantum circuits corresponding to the binary tree structure. The modeling of more general qubit trees, as well as the relationship with the mapping used in the Bravyi–Kitaev transformation, are also briefly discussed.

I. INTRODUCTION

In earlier work [1] about effective modeling of quantum state transfer in qubit chains a question about a problem to generalize suggested approach to arbitrary graphs was raised. The presented work provides an extension of some methods used for qubit chains in Ref. [1] to qubit trees together with appropriate applications. It is also interesting from point of view of generalizations of Jordan–Wigner transformations [2] to trees and more general graphs discussed in other works [3]–[7].

The approach developed in this work associates representations of Clifford algebras and Spin group with ternary and binary qubit trees. It can be more naturally defined by ternary trees with transition to binary trees using some ‘pruning’. The application of similar ternary trees for fermion-to-qubit mapping was also discussed recently in Ref. [8].

Some preliminaries about Clifford algebras, Spin groups with application to construction of quantum gates are introduced in Section II. Representations of Clifford algebras and Spin groups using ternary qubit trees and deterministic finite automata are defined in Section III together with description of a ‘pruning process,’ i.e., producing new trees by deleting of the branches. The procedure can also be used for construction of binary qubit trees introduced in Section IV. The binary trees can be considered as more natural generalization of some methods touched upon earlier in Ref. [1] due to possibility to use some supplementary tools such as annihilation and creation operators discussed in Section V. The applications of the binary qubit trees to constructions of effectively modeled quantum circuits are outlined in Section VI with some examples appropriate both for theory of quantum computations and communications.

The different scheme of qubit encoding by so-called Fenwick trees was also discussed in Ref. [9] for applications to Bravyi–Kitaev transformation [10]. For trees of arbitrary size the number of children for some qubit nodes in such a case may not be limited. Such models can be encoded by an alternative version of binary trees outlined for completeness in Section VII A together with example of application to Bravyi–Kitaev encoding in Section VII B.

II. PRELIMINARIES

Let us recollect standard properties and definitions for Clifford algebras and Spin groups [11]–[12] necessary in next sections. For the vector space $V = \mathbb{F}^n$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) the Clifford algebra $Cl(V)$ provides linear embedding of vector $v \in V$ with property

$$c : V \rightarrow Cl(V), \quad (c(v))^2 = -|v|^2 \mathbb{1}, \quad (1)$$

where $\mathbb{1}$ is the unit of the algebra and $|v|$ is a norm of the vector. For a vector $v \in V$ with coordinates $v_k$ the embedding is written

$$v = (v_1, \ldots, v_n), \quad c(v) = \sum_{k=1}^{n} v_k \epsilon_k, \quad (2)$$

where $\epsilon_k$ are generators of Clifford algebra. The possibility to work with complex vector spaces $V = \mathbb{C}^n$ is desirable for many models below, but some definitions and examples may be more naturally introduced for real case $V = \mathbb{R}^n$. The Minkowski (pseudo-Euclidean) norm is not considered here and for Euclidean case Eq. (1) can be rewritten using Eq. (2)

$$\{\epsilon_j, \epsilon_k\} = \epsilon_j \epsilon_k + \epsilon_k \epsilon_j = -2\delta_{jk} \mathbb{1}, \quad j, k = 1, \ldots, n. \quad (3)$$

Due to relations Eq. (3) maximal number of different products of generators up to sign is $2^n$ and Clifford algebra with such dimension is called universal and denoted further $Cl(n, \mathbb{F})$. The natural non-universal examples are algebra of Pauli matrices

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

for $V = \mathbb{C}^2$ and the algebra of quaternions $\mathbb{H}$ for 3D real space $V = \mathbb{R}^3$. The dimension of such algebras is not maximal and one generator in such a case could be dropped to satisfy universality condition, but it may be not always justified due to structure of a model.

For complex vector space with even dimension $\mathbb{C}^{2m}$ the universal Clifford algebra $Cl(2m, \mathbb{C})$ may be represented as $2^m \times 2^m$ complex matrix algebra [11]. The generators of $Cl(2m, \mathbb{C})$ can be expressed using so-called Jordan–
The basic property of Spin\((n)\) is expression of orthogonal group as
\[
\hat{s} \varepsilon \psi \hat{s}^{-1} = \varepsilon(\psi'), \quad \psi' = R_3 \psi, \quad R_3 \in \text{SO}(n),
\]
i.e., \(R_3\) is some \(n\)-dimensional rotation. It should be noted, that two elements \(\pm \hat{s} \in \text{Spin}(n)\) in Eq. \(5\) correspond to the same transformation \(R_3 \in \text{SO}(n)\). Thus, Spin\((n)\) group doubly covers SO\((n)\).

The Spin group also can be described as the Lie group. The universal Clifford algebra \(\mathcal{C}f(n) = \mathcal{C}_\mathbb{R}(\mathbb{F}, n)\) is a Lie algebra with respect to the bracket operation
\[
[a, b] = ab - ba, \quad a, b \in \mathcal{C}_n.
\]

For the Lie group Spin\((n)\) the Lie algebra spin\((n)\) is a subalgebra of \(\mathcal{C}_n\), with the basis \(e_j\), \(1 \leq j < k \leq n\). The Lie algebra so\((n)\) of the orthogonal group is isomorphic with spin\((n)\).

The representation of Clifford algebras discussed above has dimension \(2^n\), but the both spin\((n)\) and so\((n)\) have dimension only \(n(n-1)/2\). The Lie algebraic approach is also important due to direct relation with Hamiltonians of quantum gates [13].

There is some subtlety, because exponential map producing an element of the Spin group is \(A_\varepsilon = \exp(\varepsilon a)\), but in the physical applications expressions with the generators are often written with imaginary unit multiplier, e.g., the quantum gates near identity should be written [14]

\[
\delta U = e^{i\epsilon H} \approx 1 + i\epsilon H, \quad \epsilon \to 0.
\]

In such a case the imaginary unit should also appears in anticommutators. For example, the commutator algebra with the bracket operation \([H_a, H_b]\) appears in a proof of two-qubit gates universality [14]. The set of gates represented in such a way is universal if elements \(H\) from Eq. \(10\) generate entire Lie algebra of unitary group by the commutators.

Similar Lie-algebraic approach with Clifford algebras can be used both for construction of universal and non-universal sets of two-qubit gates [13]. The basis of the Lie algebra spin\((2m)\) consists of quadratic elements \(e_j\). The construction of the Lie algebra spin\((2m)\) using Eq. \(5\) represents the Spin\((2m)\) group as some subgroup of the unitary group U\((2^n)\).

Let us consider four consequent generators \(e_2k-1\), \(e_2k\), \(e_2(k+1)\), \(e_2k+2\). The linear combinations of six different quadratic elements produced from the generators for particular representation Eq. \(5\) correspond to Hamiltonians of some one- and two-qubit gates. For different \(k\) it generates the non-universal set of quantum gates on nearest-neighbor qubits often called matchgates [15,16].

The Jordan–Wigner representation of generators for Clifford algebra Eq. \(3\) is not unique. Alternative methods based on tree-like structures are discussed in next sections.

III. TERNARY TREES

Let us consider nine generators
\[
\begin{align*}
\hat{e}_1 &= \hat{i}\hat{\sigma}_x \hat{\sigma}_y \hat{\sigma}_z \mathbb{1} \otimes \mathbb{1}, \\
\hat{e}_2 &= \hat{i}\hat{\sigma}_y \hat{\sigma}_z \mathbb{1} \otimes \mathbb{1} \\
\hat{e}_3 &= \hat{i}\hat{\sigma}_z \mathbb{1} \otimes \mathbb{1} \\
\hat{e}_4 &= \hat{i}\hat{\sigma}_x \hat{\sigma}_y \mathbb{1} \otimes \mathbb{1} \\
\hat{e}_5 &= \hat{i}\hat{\sigma}_y \mathbb{1} \otimes \mathbb{1} \\
\hat{e}_6 &= \hat{i}\hat{\sigma}_x \mathbb{1} \otimes \mathbb{1} \\
\hat{e}_7 &= \hat{i}\hat{\sigma}_x \hat{\sigma}_y \hat{\sigma}_z \\
\hat{e}_8 &= \hat{i}\hat{\sigma}_y \hat{\sigma}_z \\
\hat{e}_9 &= \hat{i}\hat{\sigma}_z \\
\end{align*}
\] (11)

A shorter notation is often used further, e.g.,
\[
\begin{align*}
\hat{e}_1 &= \hat{i}\hat{\sigma}_z, \quad \hat{e}_2 = \hat{i}\hat{\sigma}_y, \quad \hat{e}_3 = \hat{i}\hat{\sigma}_x, \\
\hat{e}_4 &= \hat{i}\hat{\sigma}_x, \quad \hat{e}_5 = \hat{i}\hat{\sigma}_y, \quad \hat{e}_6 = \hat{i}\hat{\sigma}_z, \\
\hat{e}_7 &= \hat{i}\hat{\sigma}_x, \quad \hat{e}_8 = \hat{i}\hat{\sigma}_y, \quad \hat{e}_9 = \hat{i}\hat{\sigma}_z. \\
\end{align*}
\] (11)
where $\hat{\sigma}_\mu$ denotes Pauli matrix $\mu = x, y, z$ acting on qubit with index $j$.

The universal Clifford algebra could be defined using eight generators instead of nine and product of all $\hat{c}_k$ is identity up to possible multiplier with some power of imaginary unit denoted further as

$$\iota \in \{\pm 1, \pm i\}, \quad \iota^4 = 1.$$  \hfill (12)

Nine generators Eq. (11) demonstrate natural threefold symmetries derived from Pauli matrices. The generalization for arbitrary power of three using ternary trees is discussed below. For the initial example Eq. (11) it corresponds to four qubits nodes $j = 1, \ldots, 4$ represented by lower indexes in Eq. (11) with root is $j = 1$ and three child nodes $j = 2, 3, 4$ are associated with three generators each. Such construction can be generalized, e.g., similar example with tree for thirteen qubits is depicted on FIG. 2.

![FIG. 1. Ternary $\mathcal{T}_L$-tree with $L = 3$.](image)

Let us start with three generators $\hat{c}_1^{(3)} = i\hat{\sigma}_x$, $\hat{c}_2^{(3)} = i\hat{\sigma}_y$, $\hat{c}_3^{(3)} = i\hat{\sigma}_z$ for $L = 1$. For any $L > 1$, $3^L$ anticommuting generators for ternary $\mathcal{T}_{L+1}$-tree can be produced by recursion $L \rightarrow L + 1$ using $3^L$ anticommuting generators defined for $\mathcal{T}_L$-tree

$$\hat{c}_{j-2}^{(3+1)} = \hat{c}_j^{(3)} \otimes I \otimes \cdots \otimes I \otimes \hat{\sigma}_x \otimes I \otimes \cdots \otimes I, \quad j = 1, 2, \ldots, 3^L - 2,$$

$$\hat{c}_{j-1}^{(3+1)} = \hat{c}_j^{(3)} \otimes I \otimes \cdots \otimes I \otimes \hat{\sigma}_y \otimes I \otimes \cdots \otimes I, \quad j = 1, 2, \ldots, 3^L - 1,$$

$$\hat{c}_3^{(3+1)} = \hat{c}_j^{(3)} \otimes I \otimes \cdots \otimes I \otimes \hat{\sigma}_z \otimes I \otimes \cdots \otimes I, \quad j = 1, 2, \ldots, 3^L,$$

where $j = 1, \ldots, 3^L$ and the total number of terms in the tensor product is $m_{L+1} = m_L + 3^L$. All generators in Eq. (14) anticommute — in different triples due to terms $\hat{c}_j^{(3)}$ and in the same triple due to terms $\hat{c}_j^{(3)} \mu = x, y, z)$.

Let us prove recursively that any $3^L - 1$ generators between $\hat{c}_j^{(3)}$ generate whole basis for universal Clifford algebra $\mathbf{CL}(2m_L, \mathbb{C})$. Let us start with useful property: the product of all $3^L$ generators is $I$. It is true for $L = 1$, $\hat{c}_k^{(3)}$, $k = 1, 2, 3$ and for any $L + 1$ it is derived directly from Eq. (14). Due to such property any chosen generator up to $\iota$ multiplier is represented as product of all other generators and can be dropped. Thus, any $3^L - 1$ generators between $3^L$ can be used as a basis of $\mathbf{CL}(2m_L, \mathbb{C})$.

The standard basis of $\mathbf{CL}(2m_L, \mathbb{C})$ is naturally expressed as $4^{m_L}$ tensor products using Pauli basis, i.e., three Pauli matrices and $2 \times 2$ unit matrix. Let us show, that the basis can be also represented (not necessary in unique way) by products of $\hat{c}_k^{(3)}$. It is again true for $L = 1$ and $\mathbf{CL}(2, \mathbb{C})$. Let us consider $L + 1$ for some $L \geq 1$ with the basis of $\mathbf{CL}(2m_L, \mathbb{C})$ expressed by products of $\hat{c}_k^{(3)}$. Arbitrary basic element $b$ of $\mathbf{CL}(2m_{L+1}, \mathbb{C})$ can be represented as tensor products with $m_{L+1}$ elements of Pauli basis. The product of three generators for any $j$ in Eq. (14) is

$$\iota \hat{c}_j^{(3)} \otimes I \otimes \cdots \otimes I,$$

so the first $m_L$ terms in $b$ can be rewritten by product of such triples due to previous steps of recursion. Three possible products of two generators with given $j$ in Eq. (14) are

$$\iota \hat{c}_j \otimes \cdots \otimes \hat{c}_j \otimes \hat{\sigma}_x \otimes \cdots \otimes I, \quad \mu = x, y, z,$$

$$m_{L+1} = \sum_{k=0}^{L-1} 3^k \frac{3^L - 1}{2}.$$  \hfill (13)
and remaining last $3^k$ terms of $b$ can be also expressed using products of such pairs. So, any element $b$ of standard basis $\mathcal{C}(2m_{L+1},C)$ with $m_{L+1} = m_L + 3^k$ terms is some product of $\hat{e}^{(3^{k+1})}_k$.

It was also shown, that any element can be expressed up to $i$ as product of other generators. In such a case the construction with one dropped element corresponds to universal Clifford algebra.

Each generator $\hat{e}^{(3^k)}_k$, $k = 1, \ldots, 3^k$ has $m_L = (3^k - 1)/2$ terms in tensor product with only $L$ (non-unit) Pauli matrices, because recursion Eq. (14) appends only one non-unit term for each level. The scheme of such terms may be represented by directed ternary $T_L$-tree with first qubit as root, see FIG. 1. Each triple of generators in Eq. (14) formally corresponds to path from the root of the tree to leaf nodes.

For example, the tree with three levels represented on FIG. 1 may illustrate structure of nine triples with twenty seven generators: $\hat{e}^{(27)}_1 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_2 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_3 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_4 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_5 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_6 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_7 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_8 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $\hat{e}^{(27)}_9 = i\hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$.

The representation with tree provides yet another explanation of anticommutativity of all $\hat{e}^{(m)}_j$. Any two ‘branches’ of tree have some common part corresponding to qubits with the same index and non-unit tensor factors, but only last pair of Pauli matrices in common subsequences (corresponding to ‘fork node’ for pair of branches) may differ. Such approach produces an illustrative argument for the generalization with arbitrary ternary trees.

Let us first extend the model to provide formal definition using some methods from theory of deterministic finite automata (DFA) [19,20]. The model of deterministic finite automaton below uses extension [17] of ternary $T_L$-tree with basic nodes representing qubits and three additional output nodes for each leaf. For more general ternary trees discussed further number of children for any qubit node is added up to three by new output nodes.

Each link is marked by letters $x, y, z$ representing possible transition between nodes, see FIG. 2. The word (sequence of letters $x, y, z$) corresponding to path from the root to output nodes is recognized by deterministic finite automaton. The sequence of nodes generated by such transition represents generator expressed as product of terms with Pauli matrices indexed by number of node and letter. e.g., $xxx \rightarrow \hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $yxy \rightarrow \hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $zyx \rightarrow \hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$, $zzz \rightarrow \hat{\sigma}_1^x\hat{\sigma}_5^z\hat{\sigma}_9^z$ for FIG. 2.

More generally, if some sequence $\mu_1, \mu_2, \ldots, \mu_\ell$ of letters $\mu_k \in \{x, y, z\}$ for $k = 1, \ldots, \ell$ is recognized by deterministic finite automaton and generates sequence of nodes (path)

$$j_1^{\mu_1} \rightarrow j_2^{\mu_2} \rightarrow \ldots \rightarrow j_\ell^{\mu_\ell} \rightarrow \text{output node}$$

with root $j_1 = 1$ and output $j_{\ell + 1}$ is the output node, the generator is

$$\tilde{\hat{e}}_{\text{output}} = i\hat{\sigma}_{j_1}^\mu_1\hat{\sigma}_{j_2}^\mu_2\ldots\hat{\sigma}_{j_\ell}^\mu_\ell = i \prod_{k=1}^\ell \hat{\sigma}_k^{\mu_k}.$$  (16)

The model with deterministic finite automaton and Eq. (16) can be applied for a general ternary tree for a level $\ell$ that is not necessary equal to the maximal $L$ and the number of outbound links for each node may be from zero to three. Let us start with a ternary $T_L$-tree discussed above with maximal number of qubit nodes $m_q = (3^k - 1)/2$ and $n_g = 3^k$ anticommuting generators

$$n_g = 2m_q + 1.$$  (17)

The Eq. (17) is also valid for any subtree.

Other ternary trees can be produced by recursive process of ‘pruning’ discussed below. Let us delete all nodes and generators of subtree $\xi$ originated from node $j_k$ attached to parent node $j_p$ by link with label $\mu_p \in \{x, y, z\}$. Let us also add the new element including only initial common sequence of nodes in products Eq. (16) coinciding for all deleted nodes of the subtree $\xi$

$$\tilde{\hat{e}}_{\xi} = i\hat{\sigma}_{j_1}^{\mu_1}\ldots\hat{\sigma}_{j_p}^{\mu_p}.$$  (18)

The tree and all its subtrees after any deletion also meet Eq. (17), because

$$n'_g = n_g - n_g^\xi + 1 = (2m_q + 1) - (2m_q^\xi + 1) + 1 = 2m_q^\xi + 1,$$

where $n'_g, m_q', n_g^\xi, m_q^\xi$ denote parameters (number of generators, number of qubit nodes) for produced tree and deleted subtree respectively.

The new element Eq. (18) anticommutates with all elements except deleted. Let us also prove that product of $n'_g$ generators for new tree is $i\mathbb{I}$, there $i$ is possible unessential multiplier Eq. (12). For initial ternary $T_L$-tree Eq. (17) is true and the product of all generators
was already calculated earlier. Any subtree of the $\tilde{T}_L$-tree is also ternary $\tilde{T}_{L'}$-tree for some $L' < L$ and product of all generators for such subtree is

$$\prod_{k \in \xi} \hat{\xi}_k = (\hat{\xi}_x^{\xi_k} \pm \hat{\xi}_y^{\xi_k} \pm \hat{\xi}_z^{\xi_k}),$$

because $n_x^\xi$ is odd and $(\hat{\xi}_x^{\xi_k})^2 = -1$. So, after each deletion the products of all generators of deleted trees up to sign are equal with corresponding $\hat{\xi}_k$ and total product of all elements is always $i \cdot \hat{\xi}_k$.

Let us prove, that for any tree with $m_q'$ qubit nodes obtained by such pruning, the products of any subset with $n_g' - 1 = 2m_q'$ generators may be used as a basis of universal Clifford algebra $\mathcal{C}(2m_q', \mathbb{C})$. Let us again for simplicity start with all $n_g' = 2m_q' + 1$ generators, because any generator may be expressed as product of other generators.

Let us note, that each deletion in process of pruning may be treated also as two stage process: (1) to drop multipliers with Pauli matrices for excluded qubit nodes from all products and (2) to remove duplicates from list of generators. The approach is also correct for description of whole pruning as a series of consequent deletions.

Let us consider final tree as subtree of ternary $\tilde{T}_{L'}$-tree. Any element of standard basis of the Clifford algebra for qubits from this subtree can be represented by product of generators of initial tree. If to drop Pauli matrices for qubits from this subtree can be represented by product of all generators of deleted trees

$$\prod_{k \in \xi} \hat{\xi}_k = \hat{\xi}_x^{\xi_k} \hat{\xi}_y^{\xi_k} \hat{\xi}_z^{\xi_k} = \pm \hat{\xi}_x^{\xi_k} \pm \hat{\xi}_y^{\xi_k} \pm \hat{\xi}_z^{\xi_k},$$

any element of standard basis of the Clifford algebra for the final tree is always $\hat{\xi}_x^{\xi_k} \hat{\xi}_y^{\xi_k} \hat{\xi}_z^{\xi_k}$.

Let us consider formal procedure for construction of generators from arbitrary ternary tree produced by the pruning described above:

- Ternary tree should be extended by adding of terminal (output) nodes, i.e., all initial nodes with number of children $n_c < 3$ should be connected with $3 - n_c$ new leafs associated with generators.

- Now all non-terminal (initial) nodes have three output links marked by triple of labels $x, y, z$. Such a tree also may be considered as a deterministic finite automaton.

- Any path from root to terminal node is described by analogue of Eq. (15) with $I$ is level of the node and the generator for each terminal node can be expressed as Eq. (16).

- Formally, a possible sequence of letters $\mu_k \in \{x, y, z\}$ in Eq. (15) corresponds to a word recognized by the deterministic finite automaton and any generator is represented in such a way by product of Pauli matrices Eq. (16). Let us summarize construction of generators using extended ternary tree. Rooted directed ternary tree is defined by set of qubit nodes $j = 1, \ldots, m$ and directed links between pairs of nodes. Any node except root has one parent and up to three children. The links are marked by labels $x, y, z$.

Let us first for any qubit node $j$ define an auxiliary operator (stub) $\hat{\xi}_j$. For root node $j = 1$, $\hat{\xi}_1 = 1$ and for any child node $k$ linked with a parent node $j$ by link with a label $\mu \in \{x, y, z\}$

$$j \xrightarrow{\mu} k : \quad \hat{\xi}_k = \hat{\xi}_j \hat{\sigma}_\mu^j.$$

Now for any node $j$ with less than three children $n_c$ it is necessary to attach $n_g = 3 - n_c$ output generator nodes with appropriate unique indexes $\hat{\xi}$ by new links for missing labels $\mu \in \{x, y, z\}$.

The maximal total number of outbound links for $m$ nodes is $3m$, but $m − 1$ children are qubits nodes (because all of them except root have one parent). Thus, number of generator nodes satisfies Eq. (17)

$$n_g = 3m - (m - 1) = 2m + 1.$$

The generator associated with each such node is defined as

$$\hat{\xi}_j = \hat{\xi}_{j,\mu} = \hat{\xi}_j \hat{\sigma}_\mu^j, \quad j = 1, \ldots, 2m + 1, \quad j = 1, \ldots, m.$$

An alternative notation $\hat{\xi}_{j,\mu}$ is introduced for convenience in Eq. (18). Any generator may be expressed in such a way $\hat{\xi}_j = \hat{\xi}_{j,\mu}$ after choosing of some map to set of consequent indexes $j = j(j, \mu)$, but number of elements $\hat{\xi}_{j,\mu}$ is bigger, $3m > 2m+1$. Redundant $\hat{\xi}_{j,\mu}$ correspond to products of generators denoted earlier as $\hat{\xi}_j$ Eq. (18).

The Eq. (18) together with definition of stub operator Eq. (19) formalizes Eq. (16) used earlier without necessity to introduce an enveloping $\tilde{T}_L$-tree.

For the ternary $\tilde{T}_L$-tree deterministic finite automaton recognizes any sequences with $L$ letters and resulting $3^L$ generators are attached to leafs of qubit tree FIG. 2. Number of nodes for such a tree is $(3^L - 1)/2$ Eq. (13).

For more general ternary tree with $m$ nodes produced with the method discussed above the number of generator leaves (DFA output nodes) on the extended tree is always $2m + 1$. The product of all generators is proportional to identity. It was already discussed that any subset with $2m$ generators may be used for construction of universal Clifford algebra $\mathcal{C}(2m, \mathbb{C})$.

Let us consider yet another formal construction of $\mathcal{C}(2m + 1, \mathbb{C})$ without necessity to get rid of one generator. Let us introduce an auxiliary node with index zero to extend the set of generators to $m + 1$ qubits using straightforward method, cf Eq. (16)

$$\hat{\xi}_j = \hat{\sigma}^z \otimes \hat{\xi}_j, \quad j = 1, \ldots, 2m + 1.$$

The products of $2m + 1$ elements Eq. (21) is $\hat{\sigma}^z$ and, thus, $\mathcal{C}(2m + 1, \mathbb{C})$ can be generated by Eq. (21) using standard representation with block diagonal matrices, see Eq. (1).
The even subalgebra $\mathcal{C}_0$ is generated by products of even number of generators $\hat{e}_j$, Eq. (21). The cancellation of $\hat{\sigma}_0^z$ in products illustrates natural isomorphism

$$\mathcal{C}_0(2m+1, \mathbb{C}) \simeq \mathcal{C}(2m, \mathbb{C})$$

and it also produces representation of Spin($2m+1$) group by all $2m+1$ elements $\hat{e}_j \in \mathcal{C}(2m, \mathbb{C})$.

For $m > 1$ the Spin($2m+2$) can be also represented in a similar way. Let us consider construction of Spin groups as Lie algebras [11] recollected in Section I. In such a case the element may be expressed as exponent of linear combinations of quadratic terms $\hat{\epsilon}_1 \hat{\epsilon}_2$.

Let us again introduce an extra zero node, but for alternative representation of $2m+2$ generators instead of Eq. (21) should be used

$$\hat{\epsilon}_j = \hat{\sigma}^z \otimes \hat{\epsilon}_j, \quad j = 1, \ldots, 2m+1,$$

$$\hat{\epsilon}_0 = \hat{\sigma}^y \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}.$$  \hspace{1cm} (22)

The products of two such elements are either $\mathbb{I} \otimes (\hat{\epsilon}_j \hat{\epsilon}_k)$ or $\hat{\sigma}^z \otimes \hat{\epsilon}_l$, where $j, k, l = 1, \ldots, 2m+1$. The quadratic terms can be expressed as block-diagonal matrices Eq. (1). For $m > 1$ all $\hat{\epsilon}_j \hat{\epsilon}_k$ with $j < k$ and $\hat{\epsilon}_l$ are different and exponents of matrices with linear combination of such elements $\exp(\mathbf{A}) \in \mathcal{C}(2m, \mathbb{C})$ can be used for construction of irreducible representation of Spin($2m+2$). It is not true for $m = 1$ due to $\hat{\epsilon}_1 \hat{\epsilon}_2 = \hat{\epsilon}_3$, e.g., for quaternions or Pauli matrices $\hat{\sigma}_x \hat{\sigma}_y = i \hat{\sigma}_z$.

A standard representation of Clifford algebra may be considered as an extreme case of pruning into a chain of $z$-linked nodes. At least two generators $(x, y)$ are attached to each node with an additional one $(z)$ on the end. Such a degenerate tree corresponds to $2m$ Jordan–Wigner generators Eq. (5)

$$\hat{c}_{2k-1} = i \hat{\sigma}^z_{j} \cdots \hat{\sigma}^z_{k-1} \hat{\sigma}^x_{k},$$

$$\hat{c}_{2k} = i \hat{\sigma}^z_{j} \cdots \hat{\sigma}^z_{k-1} \hat{\sigma}^y_{k}$$

for $k = 1, \ldots, m$ together with Eq. (5)

$$\hat{c}_{2m+1} = i \hat{\sigma}^z_{1} \cdots \hat{\sigma}^z_{2m}.$$  \hspace{1cm} (23)

### IV. BINARY TREES

Binary $\Upsilon_L$-trees can be introduced formally by deleting of all nodes attached to $z$-links of the ternary $\Upsilon_L$-trees, see FIG. 3. The term `binary x-y tree' may be also used sometimes to distinguish that from an alternative construction with deleted $y$-links, but such ‘x-z’ trees are introduced only in Section VII B.

The deterministic finite automaton for such binary tree produces three generators for terminal qubit nodes with maximal level $l = L$, but only one generator for other qubit nodes with $l < L$, see FIG. 4.

The binary $\Upsilon_L$-tree has $2^L - 1$ qubit nodes. With ‘enumeration along levels’ the nodes $j = 1, \ldots, 2^L - 1$ have two children $2j$ and $2j+1$, except leaves $j = 2^L - 1$. Qubits with indexes $j = 1, \ldots, 2^L - 1$ have only one generator node, but three generators are linked to remaining $2^{L-1}$ terminal qubit nodes $k = 2^{L-1}, \ldots, 2^L - 1$ with maximal level $L$. Thus, total number of generators meets Eq. (17)

$$n_g = 2^{L-1} - 1 + 3 \cdot 2^{L-1} = 2^{L+1} - 1 = 2m_q + 1.$$  

Here the ‘redundant’ notation for generators used in
The terms Eq. (28) are trivial for index $j$ corresponding to terminal qubit nodes with three generators
\[ \hat{h}^x_j = \hat{\sigma}^x_j, \quad j = 2^{l-1}, \ldots, 2^1 - 1, \quad \mu = x, y, z. \] (29)

For nodes with single generator first pair of expressions Eq. (28) can be associated with links of binary tree
\[ \hat{h}^z_j = \hat{\sigma}^z_j \hat{\sigma}_j^z, \quad \hat{h}^x_j = \hat{\sigma}^x_j \hat{\sigma}_j^x, \quad j = 1, \ldots, 2^l-1 - 1. \] (30)

It should be noted, that $\hat{h}^z_j$ and $\hat{h}^y_j$ in Eq. (30) correspond to links marked by exchanged labels ($y$ and $x$ respectively, see FIG. 4). Remaining $z$-elements Eq. (28) can be assigned to ‘forks’ with both links
\[ \hat{h}^z_j = \hat{\sigma}^z_j \hat{\sigma}_j^z \hat{\sigma}_j^z, \quad j = 1, \ldots, 2^l-1 - 1. \] (31)

Due to Lie-algebraic approach the linear combinations of quadratic expressions such as Eq. (28) correspond to the Hamiltonians $H$ and the quantum gates can be represented as exponents
\[ \hat{U} = e^{-i\hat{H}\tau} = \exp\left(\tau \sum_{j<k} \hat{h}_{jk} \hat{c}_j \hat{c}_k\right). \] (32)

The Hamiltonians such as Eq. (29) and Eq. (30) generate one- and two-qubit gates and produce non-universal set of quantum gates for representation of Spin group corresponding to Eq. (32). The arbitrary one-qubit gates may be generated by such a way for all terminal qubit nodes due to Eq. (29), but two-qubit gates defined on all links of binary qubit tree are restricted by single-parameter families with Hamiltonians from Eq. (30).

V. ANNIHILATION AND CREATION OPERATORS

Let us split $2m$ generators $\xi_j$ of some Clifford algebra $C_{\theta}(2m, \mathbb{C})$ into two parts with $m$ elements $\xi_j^x, \xi_j^y$ to introduce annihilation and creation (‘ladder’) operators
\[ \hat{a}_j = \xi_j^x + i \xi_j^y, \quad \hat{a}_j^\dagger = \frac{\xi_j^x - i \xi_j^y}{\sqrt{2}}, \quad j = 1, \ldots, m. \] (33)

Due to Eq. (3) the elements satisfy canonical anticommutation relations (CAR)
\[ \{\hat{a}_j, \hat{a}_k\} = \{\hat{a}_j^\dagger, \hat{a}_k^\dagger\} = 0, \quad \{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk} \mathbb{I}, \] (34)

where $j, k = 1, \ldots, m$.

For standard representation of Clifford algebra mentioned earlier Eq. (5) only first $2m$ generators may be used $\xi_j^x = \xi_{2j-1}$, $\xi_j^y = \xi_{2j}$ and thus
\[ \hat{a}_j = \hat{\sigma}^z_j \cdots \hat{\sigma}_{j-1}^z \hat{\sigma}_j, \quad \hat{a}_j^\dagger = \hat{\sigma}^z_1 \cdots \hat{\sigma}^z_{j-1} \hat{\sigma}_j \] (35)

where $j = 1, \ldots, m$ and $\hat{a}, \hat{a}^\dagger$ are $2 \times 2$ matrices
\[ \hat{a} = \frac{\hat{\sigma}^z + i \hat{\sigma}^y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \] (36)
\[ \hat{a}^\dagger = \frac{\hat{\sigma}^z - i \hat{\sigma}^y}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]
with index \( j \) is for position in tensor product, i.e.,
\[
\hat{a}_j \equiv \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \hat{a} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}.
\]

Usual Jordan–Wigner transformation \([2]\) corresponds to standard representation Eq. \((35)\).

Let us also introduce analogue notation \( \hat{n}_k, \hat{\eta}_k \), where
\[
\hat{n} = \hat{a}^\dagger \hat{a} = \frac{\mathbb{1} - \hat{\sigma}_z}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\hat{\eta} = \hat{a}^\dagger \hat{a} = \frac{\mathbb{1} + \hat{\sigma}_z}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Sometimes in physical applications the ladder operators may be considered as primary objects and expressions for generators follow directly from Eq. (33).

\[
e_j' = i(\hat{a}_j + \hat{a}_j^\dagger), \quad e_j'' = \hat{a}_j - \hat{a}_j^\dagger.
\]

The generators \( e_j \) itself due to such representation also often treated as creation operator for particle coinciding with own antiparticle, e.g., Majorana mode \([10] [21]\).

The ladder operators also can be used to express specific subgroup of Spin group corresponding to some of quantum gates generated by restricted set of quadratic Hamiltonians \([11])\). Let us introduce notation
\[
\hat{\Sigma}_{j,k} = \frac{\hat{a}_j^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_j}{2}, \quad \hat{A}_{j,k} = \frac{\hat{a}_j^\dagger \hat{a}_k - \hat{a}_k^\dagger \hat{a}_j}{2i}.
\]

For ‘vacuum’ state
\[
|\varnothing\rangle \equiv |00 \cdots 0\rangle,
\]
\[
\hat{a}_k|\varnothing\rangle = 0 \quad \text{and} \quad \hat{\Sigma}_{j,k}|\varnothing\rangle = \hat{A}_{j,k}|\varnothing\rangle = 0.
\]

Any Hamiltonians \( \hat{H} \) expressed as linear combinations of Eq. (39) also has the same property \( \hat{H}|\varnothing\rangle = 0 \) and quantum gate generated by such Hamiltonian for some parameter \( \tau \)
\[
\hat{U} = \exp(-i\hat{H}\tau)
\]
does not change vacuum state \( \hat{U}|\varnothing\rangle = |\varnothing\rangle \).

Let us for certainty suppose consequent indexes \( 1 \leq j < k \leq m \) in Eq. (39) with special notation for ‘occupation number’ operators \( \hat{n}_k \) and number of ‘particles’ (units in computational basis) operator \( \hat{N} \)
\[
\hat{n}_k = \hat{\Sigma}_{k,k} = \hat{a}_k^\dagger \hat{a}_k, \quad \hat{N} = \sum_{k=1}^m \hat{n}_k.
\]

An important property of the operator Eq. (42) can be derived directly from the definition and Eq. (34)
\[
\hat{N} \hat{a}_j = \hat{a}_j \hat{N} - \hat{a}_j = \hat{a}_j (\hat{N} - \mathbb{1}),
\]
\[
\hat{N} \hat{a}_j^\dagger = \hat{a}_j^\dagger (\hat{N} + \mathbb{1}).
\]

Here again \( \hat{N}|\varnothing\rangle = 0 \) and for states such as
\[
|\Xi_{j_1 \cdots j_N}^{(N)}\rangle = \hat{a}_{j_1}^\dagger \cdots \hat{a}_{j_N}^\dagger |\varnothing\rangle,
\]
\[
1 \leq j_1 < \cdots < j_N \leq m
\]
from consequent application of Eq. (43) for all \( \hat{a}_j^\dagger \) it follows
\[
\hat{N}|\Xi_{j_1 \cdots j_N}^{(N)}\rangle = N|\Xi_{j_1 \cdots j_N}^{(N)}\rangle.
\]

It may be also derived from Eq. (43) or checked directly that quadratic operators Eq. (39) commute with \( \hat{N} \)
\[
\hat{\Sigma}_{j,k} \hat{N} = \hat{N} \hat{\Sigma}_{j,k}, \quad \hat{A}_{j,k} \hat{N} = \hat{N} \hat{A}_{j,k}.
\]

The Hamiltonians \( \hat{H} \) with linear combination of terms Eq. (39) also commute with \( \hat{N} \) and quantum gates \( \hat{U} \) generated by \( \hat{H} \) Eq. \((41)\) respect subspaces composed from states Eq. \((44)\). Such restricted case was introduced initially in Ref. \([16]\) and later discussed as a basic example in Ref. \([1]\).

With standard representation Eq. (35) expression for \( \hat{N} \) Eq. (42) may be rewritten
\[
\hat{N} = \hat{N}^z = \sum_{k=1}^m \hat{n}_k = \sum_{k=1}^m \frac{\mathbb{1} - \hat{\sigma}_k^z}{2} = m \mathbb{1} - \frac{1}{2} \sum_{k=1}^m \hat{\sigma}_k^z
\]
and eigenvalues \( N \) Eq. (43) of the operator correspond to number of units in computational basis, e.g., for \( N = 1 \) there are \( m \) states
\[
|\Xi_k^{(m)}\rangle = \hat{a}_k^\dagger |\varnothing\rangle = |k\rangle,
\]
where, for standard (Jordan–Wigner) representation
\[
|k\rangle = |00 \cdots 0 1 0 \cdots 0\rangle, \quad k = 1, \ldots, m
\]
with only unit in position \( k \) of the computational basis state, but analogue constructions even for binary tree discussed below are more complicated.

Let us now introduce similar constructions for binary tree. The indexation Eq. (26) is used further with first element \( \tilde{e}_1 \) is dropped and the Eq. (33) is applied to partition \( e'_j = \tilde{e}_{2j}, e''_j = \tilde{e}_{2j+1}, j = 1, \ldots, m \). Let us also introduce slightly different notation for binary tree ladder operators
\[
\tilde{a}_j = \frac{\tilde{e}_{2j} + \tilde{e}_{2j+1}}{2i}, \quad \tilde{a}_j^\dagger = \frac{\tilde{e}_{2j} - \tilde{e}_{2j+1}}{2i}
\]
with \( j = 1, \ldots, m \).

Only for terminal nodes \( j = 2^{L-1}, \ldots, 2^L - 1 \) of binary tree with given \( L \) the operators Eq. (50) have more usual form with tensor product of only \( 2 \times 2 \) matrices similarly with Eq. (35). Let us consider simple example with \( L = 2 \) Eq. (27) and first node \( j = 1 \) is not terminal
\[
\tilde{a}_1 = \tilde{\sigma}_x \tilde{\sigma}_z + i \tilde{\sigma}_y \tilde{\sigma}_3, \quad \tilde{a}_1^\dagger = \tilde{\sigma}_x \tilde{\sigma}_z - i \tilde{\sigma}_y \tilde{\sigma}_3.
\]
Other operators for $L = 2$ corresponds to terminal nodes with simpler expressions
\begin{equation}
\begin{aligned}
\hat{a}_2 &= \frac{\sigma_z^t \sigma_z^x + i \sigma_y^z \sigma_y^y}{2} = \sigma_z^t \hat{a}_2, \\
\hat{a}_3 &= \frac{\sigma_y^y \sigma_z^x + i \sigma_y^z \sigma_y^y}{2} = \sigma_y^y \hat{a}_3.
\end{aligned}
\end{equation}

The expressions for operators $\hat{a}_j^\dagger$ are complex conjugations of matrices and often omitted further. Let us rewrite Eq. 51 using projectors Eq. 57
\begin{equation}
\begin{aligned}
\hat{a}_1 &= \sigma_z^t (i\hat{n}_2 - \hat{n}_2)(i\hat{n}_2 + \hat{n}_2) + i \sigma_y^y (i\hat{n}_2 + \hat{n}_2)(i\hat{n}_3 - \hat{n}_3) \\
&= \hat{a}_1 i\hat{n}_2 \hat{n}_3 + \hat{a}_1^\dagger \hat{n}_2 \hat{n}_3 - \hat{a}_1 \hat{n}_2 \hat{n}_3 - \hat{a}_1 \hat{n}_2 \hat{n}_3.
\end{aligned}
\end{equation}

The expression correspond to ‘conditional’ annihilation and creation operators on first qubit controlled by pair of other qubits. More general case discussed below for $L \geq 2$ and $j \geq 1$ is quite similar with appropriate indexes substituted instead of 1, 2, 3 in Eq. 53.

Let us rewrite Eq. 54 with two ranges for internal and terminal nodes using stub operator $\hat{t}_j$ Eq. 23 together with Eq. 26 and Eq. 27
\begin{equation}
\begin{aligned}
\hat{a}_j &= \hat{t}_j \frac{\sigma_z^t \sigma_z^x + i \sigma_y^y \sigma_y^y}{2} = \hat{t}_j \hat{a}_{j,2j}, \\
&= \hat{t}_j \frac{\sigma_z^t \sigma_z^x - i \sigma_y^y \sigma_y^y}{2} = \hat{t}_j \hat{a}_j,
\end{aligned}
\end{equation}

where $\hat{a}_{j,2j}$ is generalization of conditional operator Eq. 53 with index $j$ ‘controlled’ by pair $2j, 2j + 1$

\begin{equation}
\begin{aligned}
\hat{a}_{j,2j} &= \frac{\sigma_z^t \sigma_z^x + i \sigma_y^y \sigma_y^y}{2} \\
&= \hat{t}_j (i\hat{n}_2 \hat{n}_2 + i\hat{n}_2 \hat{n}_{2j + 1}) + \hat{t}_j^\dagger (i\hat{n}_2 \hat{n}_2 - i\hat{n}_2 \hat{n}_{2j + 1}).
\end{aligned}
\end{equation}

An example for $L = 4$ is depicted on FIG. 6. The constructions of $\hat{a}_j$, $\hat{a}_j^\dagger$ include three different nodes for $j = 1, \ldots, 7$ and only one for $j = 8, \ldots, 15$.

Let us now consider analogues of Eq. 39
\begin{equation}
\Sigma_{j,k} = \frac{\hat{a}_j^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_j}{2}, \quad \Lambda_{j,k} = \frac{\hat{a}_j^\dagger \hat{a}_k - \hat{a}_k^\dagger \hat{a}_j}{2i}
\end{equation}

and Eq. 42 for modified number (of ‘particles’) operator
\begin{equation}
\hat{N} = \sum_{k=1}^m \hat{a}_k^\dagger \hat{a}_k = \sum_{k=1}^m \hat{n}_k,
\end{equation}

where $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ are modified ‘occupation number’ operators.

The ‘vacuum state’ Eq. 40 for binary tree also satisfies $\hat{a}_j | \varnothing \rangle = 0$ for any $j$. It is clear for terminal nodes $j \geq 2^{k-1}$, because tensor product for $\hat{a}_j$ includes $\hat{a}_j$

\begin{equation}
\hat{U} = \exp(-i\hat{H}\tau)
\end{equation}
do not change ‘vacuum state’ $\hat{U} | \varnothing \rangle = | \varnothing \rangle$ similarly with $\hat{U}$ in Eq. 41, but must commute with modified operator $\hat{N}$ instead of $\hat{N}$.

Let us consider analogues of states Eq. 44
\begin{equation}
\hat{\Xi}^{(N)}_{j_1, \ldots, j_N} = \hat{a}_{j_N}^\dagger \cdots \hat{a}_{j_1}^\dagger | \varnothing \rangle,
\end{equation}

Quantum gates defined by Eq. 58 due to property $\hat{N}\hat{U} = \hat{U}\hat{N}$ do not change $\hat{N}$, but number of units in elements of computational basis may be not fixed.

Let us consider example of Eq. 59 with single creation operator
\begin{equation}
| \hat{\Xi}^{(1)}_{k} \rangle = \hat{a}_k^\dagger | \varnothing \rangle, \quad \hat{N}| \hat{\Xi}^{(1)}_{k} \rangle = | \hat{\Xi}^{(1)}_{k} \rangle, \quad 1 \leq k \leq m.
\end{equation}

The operators $\hat{a}_k^\dagger$ are obtained from $\hat{a}_k$ Eq. 54 by Hermitian conjugation and $| \hat{\Xi}^{(1)}_{k} \rangle$ is up to phase $\ell$ an element of computational basis with units only in positions corresponding to ‘path’ from root to node $k$. The number of units is equal to level $\ell$ of the node in the tree
\begin{equation}
\hat{N}| \hat{\Xi}^{(1)}_{k} \rangle = \ell_k | \hat{\Xi}^{(1)}_{k} \rangle, \quad \ell_k = [\log_2 k] + 1.
\end{equation}

The eigenvalues of $\hat{N}$ operators Eq. 57 can be expressed directly for computational basis using analogue
of sums Eq. (42) or Eq. (47) with operators $\hat{n}_j$ written for different ranges using Eq. (54)

$$\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j = \frac{\mathbb{1} - \sigma_z^j \sigma_z^{j+1}}{2},$$

$$j = 1, \ldots, 2^{l-1} - 1,$$  

(62a)

$$\hat{n}_j = \frac{\mathbb{1} - \sigma_z^j}{2},$$

$$j = 2^{l-1}, \ldots, 2^l - 1.$$  

(62b)

With quadratic expressions $\hat{h}$ defined earlier Eq. (28) it may be rewritten using Eq. (29) and Eq. (31)

$$\hat{n}_j = \frac{\mathbb{1} - \hat{h}_j^z}{2}, \quad j = 1, \ldots, 2^l - 1.$$  

(63)

Tensor product of $\hat{\sigma}^z$ is diagonal matrices and eigenvalues $\eta_j$ of $\hat{h}_j^z$ Eq. (31) for eigenvectors from computational basis can be expressed as

$$\hat{h}_j^z |n_1, \ldots, n_m\rangle = \eta_j |n_1, \ldots, n_m\rangle,$$

$$\eta_j = (-1)^{n_j+n_2j+n_2j+1} (j < 2^{l-1})$$  

(64)

and due to simple identity

$$\frac{1 - (-1)^k}{2} = k \text{ mod } 2$$

eigenvalues of $\hat{n}_j$ using Eq. (62) and Eq. (64) can be expressed as

$$\hat{n}_j = \begin{cases} n_j \oplus n_{2j} \oplus n_{2j+1}, & \text{for } j = 1, \ldots, 2^{l-1} - 1, \\ n_j, & \text{for } j = 2^{l-1}, \ldots, 2^l - 1, \end{cases}$$  

(65)

where $\oplus$ denotes XOR (exclusive OR) operation for binary values

$$\hat{n}_j = n_j \oplus n_{2j} \oplus n_{2j+1} = (n_j + n_{2j} + n_{2j+1}) \text{ mod } 2.$$  

(66)

The eigenvalue of $\hat{N}$ is

$$\hat{N} = \sum_{j=1}^{m} \hat{n}_j.$$  

(67)

Let us consider an example with single creation operator for node $k$ Eq. (40). The positions of units produce some path from root to $k$. Any triple of nodes in Eq. (65) for $j \neq k$ contains zero or two units and $\hat{n}_j$ is only nonzero element in sum Eq. (67), $\hat{n}_j = \delta_{jk}$, thus, $\hat{N} = 1$.

Let us consider $m$ elements with a single unit in computational basis. Method used above illustrates that $\hat{N} = 1$ only for $j = 1$, but $\hat{N} = 2$ for $j > 1$ due to second unit in sum Eq. (67), because triple for $k = j \text{ div } 2$ in Eq. (65) also contains node $j$. It may be also checked directly, that for given indexing Eq. (26)

$$|\hat{\xi}_j^{(2)}\rangle = \hat{a}_j^\dagger \hat{a}_{j+1} |\varnothing\rangle, \quad j = 2, \ldots, m = 2^l - 1,$$

$$j' = j \text{ div } 2$$  

(68)

is element of computational basis (up to $i$) with single unit in position $j$, see Eq. (49)

$$|\hat{\xi}_j^{(2)}\rangle = \hat{a}_j^\dagger \hat{a}_{j+2} |\varnothing\rangle = \iota_j^j, \quad j = 2, \ldots, m,$$  

(69)

there notation $j \div 2 = j \text{ div } 2$ is used for brevity and both elements in each pair $j \in \{2j', 2j' + 1\}$ are taken into account for $j > 1$. Thus

$$\hat{N} |1\rangle = |1\rangle, \quad \hat{N} |j\rangle = 2 |j\rangle, \quad j > 1.$$  

(70)

However, elements of computational basis with units in both positions $2j'$ and $2j' + 1$ also may be expressed in similar way

$$|\hat{\xi}_j^{(2)}\rangle = \hat{a}_{2j'}^\dagger \hat{a}_{2j'+1} |\varnothing\rangle = |2j', 2j'+1\rangle,$$  

(71)

where notation from Ref. [1] is used

$$|k, k+1\rangle = |00\ldots01\ldots0\rangle_{k-1 \ldots m-k-2}.$$  

(72)

Thus, such a states also belong to subspace corresponding to eigenvalue 2 of operator $\hat{N}$, cf Eq. (70)

$$\hat{N} |2j, 2j+1\rangle = 2 |2j, 2j+1\rangle, \quad 1 \leq j \leq 2^{l-1} - 1.$$  

(73)

Let us recollect that quantum circuits with gates generated by Hamiltonians Eq. (58) can be used for transformation between different states from subspaces with the same eigenvalue of $\hat{N}$.

VI. EFFICIENT SIMULATION

Let us start with analogues of efficient classical simulation considered in Ref. [22] [23] with calculation of expectation values of generators $\hat{e}_j$ for binary trees using exponential representation of gates $\hat{U}$ with ‘quadratic’ Hamiltonians $\hat{H}$ Eq. (22).

Unitary operators $\pm \hat{U}_R \in SU(2^m)$ (elements of Spin group) are corresponding to orthogonal matrix $R$ with property

$$\hat{U}_R \hat{e}_j \hat{U}_R^\dagger = \sum_k R_{kj} \hat{e}_k,$$  

(74)

where summation is applied to actually used set of indexes. For binary tree natural choice may include either $k = 1, \ldots, 2m + 1$ for $\mathcal{C}(2m + 1)$, Spin($2m + 1$) and $R \in SO(2m + 1)$ or $k = 2, \ldots, 2m + 1$ for $\mathcal{C}(2m)$, Spin($2m$) and $R \in SO(2m) \subset SO(2m + 1)$, cf Eq. (27) for $m = 3$.

Here consideration of all generators with $R \in SO(2m+1)$ may be useful, because $\hat{e}_1$ appears in quadratic Hamiltonian in terms for links such as $\hat{h}_1^x$, $\hat{h}_1^y$ in Eq. (50). However, $\hat{e}_1$ is dropped in constructions with creation and annihilation operators Eq. (50).
Evolution of state due to such unitary operators is 
\[ |\phi'\rangle = U_R |\phi\rangle \] 
and expectation value of \( \hat{e}_j \) is
\[
\langle \phi' | \hat{e}_j | \phi' \rangle = \langle \phi | U_R^\dagger \hat{e}_j U_R | \phi \rangle = \sum_k R_{jk} \langle \phi | \hat{e}_k | \phi \rangle, \tag{75}
\]
where order of indexes is changed in comparison with Eq. (74) due to inversion \( U_R^\dagger = U_R^{-1} \). Eq. (75) is the formal algebraic analogue of an equation for matchgates [22] with \( R \in SO(2m) \), but for the different operators \( U_R \), \( \hat{e}_j \) are constructed using binary trees instead of linear chain. The quadratic terms were more suitable in Ref. [22, 23] and analogues of such expressions also can be introduced
\[
\langle \phi' | \hat{e}_{j_1} \hat{e}_{j_2} | \phi' \rangle = \langle \phi | U_R^\dagger \hat{e}_{j_1} \hat{e}_{j_2} U_R | \phi \rangle = \langle \phi | U_R^\dagger (U_R^\dagger \hat{e}_{j_1} U_R) (U_R^\dagger \hat{e}_{j_2} U_R) | \phi \rangle = \sum_{k_1 \neq k_2} R_{k_1j_1} R_{k_2j_2} \langle \phi | \hat{e}_{k_1} \hat{e}_{k_2} | \phi \rangle, \tag{76}
\]
where condition \( k_1 \neq k_2 \) can be used because terms with equal indexes are disappear due to orthogonality of matrix \( R \).

For terminal indexes \( j = 2^{l-1}, \ldots, 2^l - 1 \) quadratic terms \( \hat{e}_{j_1} \hat{e}_{j_2} \) are equal with single Pauli matrix \( \hat{\sigma}_j^Z \) Eq. (29) and expectation value is analogue with Ref. [22, 23]. However, for internal indexes \( j = 1, \ldots, 2^{l-1} - 1 \), \( \hat{e}_j^Z \) are product of three Pauli matrices Eq. (31). It may be written
\[
\langle \phi | \hat{e}_{j_1} \hat{e}_{j_2} | \phi \rangle = \begin{cases} 
\langle \phi | \hat{\sigma}_j^Z \hat{\sigma}_j^Z \hat{\sigma}_{j+1}^Z | \phi \rangle, & j = 1, \ldots, 2^{l-1} - 1 \\
\langle \phi | \hat{\sigma}_j^Z | \phi \rangle, & j = 2^{l-1}, \ldots, 2^l - 1 
\end{cases} \tag{77}
\]
Using definition of \( \hat{n}_j \) Eq. (63) it may be rewritten in agreement with analogue equation for \( \hat{n}_j \) Eq. (65)
\[
\langle \phi | \hat{n}_j | \phi \rangle = \langle \hat{n}_j \rangle = \begin{cases} 
\langle n_j + n_{j+1} | j = 1, \ldots, 2^{l-1} - 1 \\
n_j, & j = 2^{l-1}, \ldots, 2^l - 1 
\end{cases} \tag{78}
\]
where notation \( \langle \cdot \cdot \cdot \rangle \) for expectation value is used, e.g., \( \langle n_j \rangle = p_{1j} \) is probability to measure value 1 for qubit \( j \).

For terminal nodes \( j = 2^{l-1}, \ldots, 2^l - 1 \) the result of qubit measurement in computational basis \( n_j = \hat{n}_j \) can be directly found from Eq. (76). For previous level \( \ell_j = L - j \) with indexes \( j = 2^{L-2}, \ldots, 2^{L-1} - 1 \) it includes an expression with three terms
\[
n_j = \hat{n}_j \oplus \hat{n}_{j+1} = \hat{n}_j \oplus \hat{n}_{j+1} \oplus \hat{n}_{j+2}, \tag{79}
\]
for level \( \ell_j = L - 2 \) expression \( n_j \) via \( \hat{n} \) require seven terms
\[
n_j = \hat{n}_j \oplus \hat{n}_{j+1} \oplus \hat{n}_{j+2} \oplus \hat{n}_{j+4} \oplus \hat{n}_{j+5} \oplus \hat{n}_{j+6} \oplus \hat{n}_{j+7}.
\]
For deeper levels \( \ell_j = L - d \) similar expansions produce \( 2^{d-1} - 1 \) terms
\[
\begin{align*}
n_j &= \left( \hat{n}_j + \sum_{d | j} \hat{n}_k \right) \mod 2, \\
&= \begin{cases} 
\hat{n}_j, & j = 2^{l-1}, \ldots, 2^l - 1 
\end{cases}, \tag{79}
\end{align*}
\]
where \( d(j) \) are all descendants of node \( j \), or, more briefly
\[
\begin{align*}
n_j &= \sum_{k \in s(j)} \hat{n}_k \mod 2.
\end{align*}
\]
where \( s(j) = d(j) \cup \{ j \} \) are all nodes of subtree with root \( j \), including trivial case with single term \( s(j) = \{ j \} \) for terminal qubit nodes.

Thus, an analogue of approach used in Ref. [22, 23] can be applied only either to computation of \( \langle \hat{n}_j \rangle \) or for measurements of separate qubits in terminal nodes. For internal nodes with level \( \ell < L \) even for single qubit measurement outcome should be used more complicated approach similar with applied to multi-qubit outputs in a standard case [21], but with measurement of \( 2^{L-1} \) quantum ‘binary variables’ \( \hat{\sigma}_j^Z \) expressed as XOR operations with qubit values. Thus, despite of some resemblance with matchgate circuits the effective modeling with binary trees devotes special consideration.

Together with possible difficulties for internal nodes it has specific advantages for terminal qubits. Linear combinations of quadratic Hamiltonians Eq. (29) may generate arbitrary rotation and expectation values \( \langle \hat{Z}_j \rangle \) in computational basis Eq. (77) can be extended for efficient simulation of qubit measurement ‘along any axis.’

A pair of terminal qubits with indexes \( j_1, j_2 + 1 \) have common parent \( j = 2^{l-2}, \ldots, 2^{l-1} - 1 \). Let us show, that for parent qubit fixed in state \( |0 \rangle \) any transformation from SU(4) group may be implemented using only quadratic Hamiltonian. The construction with auxiliary qubit uses isomorphism between SU(4) and Spin(6) and similar with a method discussed in Ref. 24.

Let us extend a simpler example \( L = 2, m = 3 \) Eq. (27) to write seven generators associated with the ‘terminal triple’ of qubits with parent node \( 2^{L-2} \leq j < 2^{L-1} \) for arbitrary \( L \geq 2 \)
\[
\begin{align*}
\hat{e}_j &= \hat{\sigma}_j^Z, & \hat{e}_{j+1} &= \hat{\sigma}_j^Z \hat{\sigma}_{j+1}^Z, & \hat{e}_{j+2} &= \hat{\sigma}_j^Z \hat{\sigma}_{j+2}^Z, \tag{80}
\end{align*}
\]

Products of two generators Eq. (80) produces 21 different terms, but only 15 of them do not change parent qubit with state \( |0 \rangle \)
\[
\begin{align*}
\sigma_j^Z \sigma_{j+1}^Z, & \quad \sigma_j^Z \sigma_{j+2}^Z, & \quad \sigma_j^Z \sigma_{j+1}^Z \sigma_{j+2}^Z, & \quad \sigma_j^Z \sigma_{j+1}^Z \sigma_{j+2}^Z, & \quad \sigma_j^Z \sigma_{j+1}^Z \sigma_{j+2}^Z. \tag{81}
\end{align*}
\]
The linear combinations of analogues of terms Eq. (81) without multiplier \( \sigma_j^Z \) would produce arbitrary traceless
Hamiltonian for two qubits, but $\hat{\sigma}^z$ acts as identity on state $|0\rangle$ and so terms Eq. (51) also may generate arbitrary SU(4) transformation of two terminal qubits if common parent qubit is $|0\rangle$.

Let us now consider construction of gates $\hat{U}$ Eq. (58) generated by quadratic combinations Eq. (50) of ladder operators $\hat{a}$ and $\hat{a}^\dagger$ for binary tree. For such a case instead of Eq. (74) an auxiliary matrix $U \in SU(2^m)$ can be introduced for operators $\pm \hat{U}_k \in SU(2^m)$ with formal analogue of well-known relations for ladder operators.

$$\hat{U}_j \hat{a}_k \hat{U}_j^\dagger = \sum_{j=1}^m U_{kj} \hat{a}_j, \quad \hat{U}_j \hat{a}_k^\dagger \hat{U}_j^\dagger = \sum_{j=1}^m U_{jk}^* \hat{a}_j^\dagger, \quad (82)$$

where $\hat{U}_{kj}$ is complex conjugation of coefficients and $U^\dagger = U^{-1}$ for unitary matrix $U$.

A ‘path-state’ $|\vec{k}\rangle$ Eq. (60) satisfies an analogue of equations used in Ref. 11 for $|k\rangle$ defined by Eq. (48) up to trivial change of variables, i.e.,

$$\hat{U}_j |\vec{k}\rangle = \sum_{k=1}^m U_{jk} \hat{a}_k |\vec{k}\rangle = \sum_{k=1}^m U_{jk}^* \hat{a}_k^\dagger |\vec{k}\rangle. \quad (83)$$

Let us consider linear superposition of path states $|\vec{\chi}\rangle = \sum_{k=1}^m \chi_k |\vec{k}\rangle$

$$\hat{U}_j |\vec{\chi}\rangle = \sum_{k=1}^m U_{jk} \chi_k |\vec{k}\rangle = \sum_{k=1}^m U_{jk} \chi_k |\vec{k}\rangle = \sum_{l=1}^m \chi_l |\vec{l}\rangle, \quad (84)$$

$$\chi_l = \sum_{k=1}^m U_{lk}^* \chi_k.$$

The Eq. (84) for ‘single-path’ states ($\vec{N} = 1$) is similar with evolution of ‘single-particle’ case ($N^z = 1$) for qubit chain [11], but for all nodes except of root in binary qubit tree $|\vec{k}\rangle$ belongs to $\vec{N} = 2$ subspace due to Eq. (70). However, the same subspace also includes pairs $|2j, 2j+1\rangle$ Eq. (70) and an analogy with ‘two-particle’ case is also relevant.

For Hamiltonians respecting $\vec{N}$ or $\vec{N}$ the consideration of ‘number-preserving’ subspace is natural for models of state transfer in quantum chains [11, 26] or trees. The two-qubit state can be decomposed into three parts:

$$|\psi\rangle = \sum_{N=0}^{N=2} \frac{N=0}{c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle}, \quad (85)$$

but terms with $N = 1$ and $N = 2$ in Eq. (85) in binary tree for pairs of nodes $2j, 2j+1$ ($0 < j < 2^{L-1}$) are belong to the same subspace $\vec{N} = 2$, and, furthermore, $N = N = 0$ is not affected by $\hat{U}$ Eq. (58) for state transfer.

For two consequent indexes $2j, 2j+1$ three terms with $N \neq 0$ ($N = 2$) in Eq. (85) are generated by applications to $|\psi\rangle$ different pairs of operators between the same triple $\hat{a}_{2j}^\dagger$, $\hat{a}_{2j}$ and $\hat{a}_{2j+1}^\dagger$ due to Eq. (69) and Eq. (71). Thus, result of perfect transfer of such two-qubit pair into new position $2k, 2k+1$ indexes by operator $\hat{U}_k$ should correspond to unitary matrix $U$ with simple constrains on three elements

$$|U_{jk}| = |U_{2j, 2k}| = |U_{2j+1, 2k+1}| = 1. \quad (86)$$

For consideration of perfect transfer with single qubit one condition in Eq. (86) may be superfluous.

The example illustrates possibility of exponential decrease of model dimension from $2^m$ to $m$, but the construction of $U$ with a sequence of steps or appropriate Hamiltonians devotes separate consideration elsewhere. Even reduced problem is more difficult than analogue example for qubit chain because of less trivial structure of graph itself and more complicated properties of modified operators such as $\hat{a}$ and $\hat{a}^\dagger$.

VII. GENERAL TREES

A. Alternative encoding of binary trees

In the binary trees discussed earlier all nodes attached to $z$-links were deleted. Let us consider as an alternative the binary $x-z$ trees with $y$-links collapsed instead. The stub operator $\hat{t}_\alpha$ Eq. (19) for such a tree contains $\hat{\sigma}^x$, $\hat{\sigma}^z$ and generators may contain no more than one $\hat{\sigma}^y$.

Some constructions discussed below become more natural, if new root with index zero is attached by $x$-link. Similar method was briefly mentioned in Section 11 and for $T_\alpha$-tree it produces ‘$T_\alpha^x$-tree’ of height $L$ with $2^L$ nodes.

In such a case appropriate pairs of generators can be chosen to provide necessary coupling of $\hat{\sigma}_x$ and $\hat{\sigma}_y$ for qubits with the same index for specific construction of ladder operators Eq. (53) discussed below, see FIG. 7.

![FIG. 7. Pair of nodes in binary x-z T\alpha^x-tree for L = 3](image-url)
root $\hat{\sigma}_c^z$ is excluded from such a coupling and internal nodes require more complicated expressions for ladder operators

$$
\hat{a}_0 = \frac{\hat{\sigma}_0^x \hat{\sigma}_2^x \hat{\sigma}_3^x + i \hat{\sigma}_0^y}{2}, \quad \hat{a}_1 = \frac{\hat{\sigma}_0^x \hat{\sigma}_2^x \hat{\sigma}_3^x + i \hat{\sigma}_0^y}{2},
$$

(87a)

in comparison with terminal qubit nodes, cf. Eq. (54b)

$$
\begin{align*}
\hat{a}_4 &= \frac{\hat{\sigma}_0^x \hat{\sigma}_2^x \hat{\sigma}_3^x + i \hat{\sigma}_0^y}{2}, \\
\hat{a}_5 &= \frac{\hat{\sigma}_0^x \hat{\sigma}_2^x \hat{\sigma}_3^x + i \hat{\sigma}_0^y}{2}, \\
\hat{a}_6 &= \frac{\hat{\sigma}_0^x \hat{\sigma}_3^x \hat{\sigma}_5^x + i \hat{\sigma}_0^y}{2}, \\
\hat{a}_7 &= \frac{\hat{\sigma}_0^x \hat{\sigma}_3^x \hat{\sigma}_5^x + i \hat{\sigma}_0^y}{2}.
\end{align*}
$$

(87b)

In such construction for each terminal node $j$ there are two generators with terms $\hat{\sigma}_j^x$ and $\hat{\sigma}_j^y$ coupled by natural way Eq. (87a), but generator with $\hat{\sigma}_j^z$ is coupled with some internal node $j'$ linked with $j$ by path $xz\cdots z$ in agreement with Eq. (76a), see FIG. 7.

Let us consider structure of expressions for internal nodes such as Eq. (87a). For some set of nodes (‘chain’) $c = \{c_1, \ldots, c_l\}$ the short notation is used further

$$
\hat{\sigma}_c^z = \hat{\sigma}_{c_1}^z \cdots \hat{\sigma}_{c_l}^z.
$$

(88)

Let us also introduce operators

$$
\begin{align*}
\hat{n}_{\oplus_c} &= \frac{1 - \hat{\sigma}_c^x \cdots \hat{\sigma}_c^x}{2} = \frac{1 - \hat{\sigma}_c^x}{2}, \\
\hat{n}^0_{\oplus_c} &= \frac{1 - \hat{n}_{\oplus_c}}{2} = \frac{1 + \hat{\sigma}_c^x}{2}.
\end{align*}
$$

(89)

Such projectors have eigenvalues expressed as XOR of nodes from set $c$

$$
\hat{n}_{\oplus_c}|n_1, \ldots, n_m\rangle = n_{\oplus_c}|n_1, \ldots, n_m\rangle,
$$

$$
n_{\oplus_c} = n_{c_1} \oplus \cdots \oplus n_{c_l}.
$$

(90)

Specific term from expressions for internal nodes such as Eq. (87a) may be rewritten

$$
\begin{align*}
\hat{a}_j_{\oplus_c} &= \frac{\hat{\sigma}_j^x \hat{n}_{\oplus_c} + i \hat{\sigma}_j^y}{2} + \frac{\hat{\sigma}_j^x + \hat{n}_{\oplus_c} + \hat{\sigma}_j^y - i \hat{\sigma}_j^y}{2} \\
&= \hat{a}_j \frac{\hat{n}_{\oplus_c} + \hat{\sigma}_j^x}{2} + \hat{a}_j \frac{1 - \hat{n}_{\oplus_c}}{2} = \hat{a}_j \hat{n}^0_{\oplus_c} - \hat{a}_j \hat{n}_{\oplus_c}.
\end{align*}
$$

(91)

Such a term is an analogue of conditional ladder operator Eq. (59), because $\hat{a}_j_{\oplus_c}$ is also controlled by few nodes $c_1, \ldots, c_l \in c$.

The analogue of Eq. (54) can be written for binary $x$-$z$ tree $T^z_c$-tree with $2^l$ nodes taking into account new root with index zero, see FIG. 7.

$$
\begin{align*}
\hat{\tilde{a}}_j &= \hat{r}_j \frac{\hat{\sigma}_j^x \hat{n}_{\oplus_c(j)} + i \hat{\sigma}_j^y}{2} = \hat{r}_j \hat{a}_j_{\oplus_c(j)}, \\
&= \hat{r}_j \hat{a}_j_{\oplus_c(j)}, \\
&= \hat{r}_j \hat{a}_j_{\oplus_c(j)},
\end{align*}
$$

(92a)

$$
\begin{align*}
\hat{\tilde{a}}_j &= \hat{r}_j \frac{\hat{\sigma}_j^x \hat{n}_{\oplus_c(j)} + i \hat{\sigma}_j^y}{2} = \hat{r}_j \hat{a}_j_{\oplus_c(j)}, \\
&= \hat{r}_j \hat{a}_j_{\oplus_c(j)},
\end{align*}
$$

(92b)

where $\hat{r}_j$ is stab operator already introduced earlier, cf. Eq. (92) for $L = 3$ with Eq. (87). The index $c(j)$ in Eq. (92) denotes set of nodes $c_1, \ldots, c_l$ attached to given node $j$ via chain of $z$ links.

The generators of Clifford algebra for Eq. (92) in agreement with Eq. (38) can be written

$$
\begin{align*}
\hat{\tilde{c}}_j &= \hat{r}_j \hat{\tilde{c}}_j, \\
\hat{\tilde{c}}'_j &= \hat{r}_j \hat{\tilde{c}}'_j, \\
\hat{\tilde{c}}''_j &= \hat{r}_j \hat{\tilde{c}}''_j, \\
&= \frac{1}{2} \hat{\sigma}_j^x, \\
&= \frac{1}{2} \hat{\sigma}_j^y, \\
&= \frac{1}{2} \hat{\sigma}_j^z.
\end{align*}
$$

(93a)

where $T$ denotes set of terminal nodes, e.g., $j = 2L^1 - 1, \ldots, 2L^1 - 1$ for trees used in examples above.

The analogues of Eq. (62) for quadratic operators are also straightforward

$$
\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j = \frac{1 - \hat{\sigma}_j^x}{2}, \\
\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j = \frac{1 - \hat{\sigma}_j^x}{2}, \\
\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j = \frac{1 - \hat{\sigma}_j}{2}, \\
\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j = \frac{1 - \hat{\sigma}_j}{2},
$$

(93b)

The particular example with $2^L$ nodes is interesting due to direct relation with Bravyi–Kitaev (BK) transformation discussed below in Section 5.8.3 but binary $x$-$z$ tree is also used to represent a general tree ($g$-tree). A node $j$ with $l$ children $c_1, \ldots, c_l$ of such a $g$-tree should be mapped into node $j$ of binary $x$-$z$ tree with $x$-link to only one child node $c_1$ together with chain of nodes $c_1, \ldots, c_l$ connected by $z$-links, see FIG. 8. For construction of ladder operators the last node $c_l$ is coupled with node $j$, cf. Eq. (87a).

![FIG. 8. Multiple children encoding](image)

Such construction has some properties of formalism used earlier due to certain similarity of Eq. (54) and Eq. (62) for binary $x$-$y$ trees with Eq. (92) and Eq. (94) for nodes with arbitrary number of children obtained from binary $x$-$z$ trees using correspondence depicted on FIG. 8.

An analogue of Eq. (65) is

$$
\hat{n}_j = \left\{ \begin{array}{ll} n_j \oplus n_{c_1} \oplus \cdots \oplus n_{c_l}, & j \notin T, \\
&= n_j, & j \in T, \end{array} \right.
$$

(95)

where $c_1, \ldots, c_l \in c(j)$ are indexes used in $\hat{\sigma}_j^z$ from Eq. (94a). It is chain of $z$-linked nodes in node $j$ of initial binary $x$-$z$ tree and the same indexes correspond to $l$ children of node $j$ in the $g$-tree obtained by construction depicted on FIG. 8.
Inverse relation for Eq. (95) is similar with Eq. (79) used earlier for binary $x$-$y$ trees and may be written

$$n_j = \left(\tilde{n}_j + \sum_{k \in D(j)} \tilde{n}_k\right) \mod 2,$$

where $D(j)$ is (possibly empty) set of all descendants of node $j$ for $g$-tree obtained from binary $x$-$z$ tree. The set of nodes $D(j)$ may differs from $d(j)$ for corresponding binary $x$-$z$ tree, because $z$-link to ‘peers’ should not be included in $D(j)$, e.g., on FIG. 9 below $D(3) = \{0, 1, 2\}$, but $d(3) = \{0, 1, 2, 4, 5, 6\}$.

**B. Bravyi–Kitaev transformation**

Let us compare structure of ladder operators Eq. (92) or generators Eq. (93) with analogue constructions used in Bravyi–Kitaev transformation based on Fenwick trees, see Ref. [9] and some earlier works [27, 28]. Analogues of operators Eq. (93) with notation used in Ref. [9] are

$$\hat{c}_j = \hat{Z}_{P(j)} \hat{X}_j \hat{X}_{U(j)},$$
$$\hat{d}_j = \hat{Z}_{C(j)} \hat{Y}_j \hat{X}_{U(j)} = \hat{Z}_{P(j) \setminus F(j)} \hat{Y}_j \hat{X}_{U(j)} (97)$$

where $\hat{X}$, $\hat{Y}$, $\hat{Z}$ denote either Pauli matrices or theirs products similar with Eq. (88), where $U(j)$, $C(j)$, $F(j)$ and $P(j) = C(j) \cup F(j)$ are some set of indexes. It can be rewritten to provide similarity with notations used here

$$\hat{c}_j = i \hat{s}_{P(j)} \hat{\sigma}_j^x \hat{s}_{U(j)},$$
$$\hat{d}_j = i \hat{s}_{C(j)} \hat{\sigma}_j^y \hat{s}_{U(j)},$$

where analogue of Eq. (88) is used for given set of indexes $S(j)$ and Pauli matrix

$$\hat{s}_{S}^\mu = \prod_{x \in S} \hat{\sigma}_x^\mu. (88)$$

Thus, operator Eq. (97) from Ref. [9] corresponds to Eq. (93) if $c(j)$ is denoted as $F(j)$ and stub operator is expressed as

$$\hat{c}_j = \pm \hat{s}_{C(j)} \hat{\sigma}_j^z \hat{s}_{U(j)}. (99)$$

Let us again consider the example with eight qubits. The indexes of nodes in binary $x$-$z$ trees should be changed to conform standard denomination in Bravyi–Kitaev transformation also used in Ref. [9], cf FIG. [7] and FIG. [9]

| BK | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|---|
|    | 7 | 3 | 1 | 5 | 0 | 2 | 4 | 6 |

Ladder operators up to numeration Eq. (100) coincide with Eq. (87) for internal

$$\hat{a}_1 = \hat{\sigma}_x^z \hat{\sigma}_x^0 (\hat{\sigma}_x^1 \hat{\sigma}_0^0 + i \hat{\sigma}_1^y)/2,$$
$$\hat{a}_3 = \hat{\sigma}_x^z \hat{\sigma}_x^0 \hat{\sigma}_z^0 \hat{\sigma}_1^y (\hat{\sigma}_1^0 + i \hat{\sigma}_1^y)/2,$$
$$\hat{a}_5 = \hat{\sigma}_x^z \hat{\sigma}_x^0 \hat{\sigma}_z^0 \hat{\sigma}_1^y (\hat{\sigma}_1^0 + i \hat{\sigma}_1^y)/2,$$
$$\hat{a}_7 = (\hat{\sigma}_x^z \hat{\sigma}_x^0 \hat{\sigma}_z^0 \hat{\sigma}_1^y + i \hat{\sigma}_1^y)/2$$

(101a)

With new indexing Eq. (96) may be rewritten for eight qubits depicted on FIG. [9]

$$\tilde{n}_0 = n_0, \quad \tilde{n}_2 = n_2, \quad \tilde{n}_4 = n_4, \quad \tilde{n}_6 = n_6,$$
$$\tilde{n}_1 = n_1 \oplus n_0, \quad \tilde{n}_5 = n_5 \oplus n_4,$$  
$$\tilde{n}_3 = n_3 \oplus n_1 \oplus n_2, \quad \tilde{n}_7 = n_7 \oplus n_3 \oplus n_5 \oplus n_6.$$  

(102)

The inverse relations Eq. (96) are

$$n_0 = \tilde{n}_0, \quad n_2 = \tilde{n}_2, \quad n_4 = \tilde{n}_4, \quad n_6 = \tilde{n}_6,$$
$$n_1 = \tilde{n}_1 \oplus \tilde{n}_0, \quad n_5 = \tilde{n}_5 \oplus \tilde{n}_4,$$
$$n_3 = \tilde{n}_0 \oplus \tilde{n}_1 \oplus \tilde{n}_2 \oplus \tilde{n}_3,$$
$$n_7 = \tilde{n}_0 \oplus \tilde{n}_1 \oplus \tilde{n}_2 \oplus \tilde{n}_3 \oplus \tilde{n}_4 \oplus \tilde{n}_5 \oplus \tilde{n}_6 \oplus \tilde{n}_7.$$  

(103)

Let us recollect, what $n_j$ corresponds to single qubit with index $j$, but $\tilde{n}_j$ is ‘BK number’ related with set of qubits affected by ‘modified BK creation operator’ $\hat{a}_0$.

In such a way, the set of equations Eq. (103) is in agreement with usual scheme of Bravyi–Kitaev transformation [10] and it corresponds to an example of Fenwick tree with eight nodes considered in Ref. [9] taking into account correspondence between $g$-tree and binary $x$-$z$ tree discussed in Section VII A.
Construction of Clifford algebras associated with some kinds of trees is discussed in presented work. Formally, set of generators can be produced by deterministic finite automaton obtained as the extension of ternary tree by addition some formal output nodes. The binary trees can be formally considered as a reduced case of ternary tree with at least one child for each node is omitted, see FIG. 3. In appropriate cases the trees can be also used for modeling of quantum state transfer along the edges.

Spin group can be expressed using exponents with linear combination of terms quadratic by generators of Clifford algebra. Such terms correspond to Hamiltonians with quadratic expressions by generators of Clifford algebras acting on two or more consequent qubit nodes in the chain.

Both for binary and ternary trees the expressions for generators include sequence of nodes from root to some terminal node. Thus, quadratic expressions represent single node or segment with sequence between two nodes. However, number of formal output nodes of deterministic finite automaton attached to given qubit is $n_o = 3 - n_e$, where $n_e$ is number of children for given qubit in a tree. Thus, for ternary trees internal qubit node may be missing in such sequence and binary trees with $n_o > 0$ are more preferable for some purposes.

The construction with trees naturally produces odd number of generators, but any one of them can be expressed as product of others. Due to such property any generator could be dropped, yet new set with even number of generators may lack of initial symmetry. Anyway, even number of generators decomposed on pairs can be used for definition of creation and annihilation (ladder) operators. Such construction is appropriate for general ternary tree, but it looks more natural for reduced cases such as binary trees or linear chain.

The generators of Clifford algebra $c_j$ in some physical applications can be also treated as creation operators, but particle and antiparticle is equivalent in such a case, because $c_j^2 = 1$. The quadratic expressions with generators are convenient for modeling of state transfer. For a system with $m$ qubits and Hilbert space with dimension $2^m$ quadratic Hamiltonian produces evolution described by matrices of rotations in a space with dimension only $2m$ due to main property of Spin groups Eq. (74).

The Section VII slightly extends initial topic of this paper about effective modeling and state transfer to show relations with so-called fermion-to-qubit mapping for applications in quantum computers. It is shown in Section VII A that a model with general trees often used for such a purposes can be obtained from alternative reduction of ternary tree illustrated on FIG. 8. The particular example with Bravyi–Kitaev transformation is explained in Section VII B.

VIII. CONCLUSION AND DISCUSSION

[1] A. Yu. Vlasov. Effective simulation of state distribution in qubit chains. arXiv:1708.07439 [quant-ph]; Quantum Inf. Process. 17, 269 (2018).
[2] P. Jordan and E. Wigner. Über das Paulische Äquivalenzverbot. Zeitschrift für Physik 47, 631–651 (1928).
[3] S. Backens, A. Shnirman, and Yu. Makhlin. Jordan–Wigner transformations for tree structures. arXiv:1810.02590 [cond-mat]; Scientific Reports 9, 2598 (2019).
[4] A. M. Szczesna. Spins and fermions on arbitrary lattices. Comm. Math. Phys. 98, 513–524 (1985).
[5] T. Ried. Jordan–Wigner transformations and quantum spin systems on graphs. Master’s Thesis, Technische Universität München (2013).
[6] M. Steudtner and S. Wehner. Fermion-to-qubit mappings with varying resource requirements for quantum simulation. New J. Phys. 20, 063010 (2018).
[7] M. Steudtner and S. Wehner. Quantum codes for quantum simulation of fermions on a square lattice of qubits. arXiv:1810.02681 [quant-ph]; Phys. Rev. A 99, 022308 (2019).
[8] Z. Jiang, A. Kalev, W. Mruczkiewicz, and H. Neven. Optimal fermion-to-qubit mapping via ternary trees with applications to reduced quantum states learning. arXiv:1910.10746 [quant-ph] (2019); Quantum 4, 276 (2020).
[9] V. Havlíček, M. Troyer, and J. D. Whitfield. Operator locality in quantum simulation of fermionic models. arXiv:1701.07072 [quant-ph]; Phys. Rev. A 95, 032332 (2017).
[10] S. Bravyi and A. Kitaev. Fermionic quantum computation. arXiv:quant-ph/0003137 Ann. Phys. 298, 210–226 (2002).
[11] J. E. Gilbert and M. A. M. Murray, Clifford Algebras and Dirac Operators in Harmonic Analysis (Cambridge University Press, Cambridge, 1991).
[12] I. R. Porteous, Clifford Algebras and the Classical Groups (Cambridge University Press, Cambridge, 1995).
[13] A. Yu. Vlasov. Clifford algebras and universal sets of quantum gates. arXiv:quant-ph/0010071 Phys. Rev. A 63, 054302 (2001).
[14] D. P. DiVincenzo. Two-bit gates are universal for quantum computation. Phys. Rev. A 51, 1015 – 1022 (1995).
[15] L. G. Valiant. Quantum computers that can be simulated classically in polynomial time. Proc. 33rd Annual ACM STOC, 114–123 (2001).
[16] B. M. Terhal and D. P. DiVincenzo. Classical simulation of noninteracting-fermion quantum circuits. arXiv:quant-ph/0108010 Phys. Rev. A 65, 032325
(2002).

[17] D. E. Knuth, *The art of computer programming*, v.1, 3rd ed., (Addison-Wesley, Reading, MA, 1997).

[18] R. Garnier and J. Taylor, *Discrete Mathematics*, 3rd ed., (CRC Press, New York, NY, 2009).

[19] J. H. Conway, *Regular algebra and finite machines*, (Chapman and Hall, London, 1971).

[20] M. Sipser, *Introduction to the theory of computation*, 3rd ed., (Cengage, Boston, MA, 2012).

[21] F. Wilczek. Majorana returns. *Nature Phys.* **5**, 614–618 (2009).

[22] R. Jozsa and A. Miyake. Matchgates and classical simulation of quantum circuits. arXiv:0804.4050 [quant-ph]; *Proc. R. Soc. London, Ser. A* **464**, 3089–3106 (2008).

[23] R. Jozsa, B. Kraus, A. Miyake, and J. Watrous. Matchgate and space-bounded quantum computations are equivalent. arXiv:0908.1467 [quant-ph]; *Proc. R. Soc. London, Ser. A* **466**, 809–830 (2010).

[24] D. J. Brod. Efficient classical simulation of matchgate circuits with generalized inputs and measurements. arXiv:1602.03539 [quant-ph]; *Phys. Rev. A* **93**, 062332 (2016).

[25] A. Yu. Vlasov. Quantum circuits and Spin(3n) groups. arXiv:1311.1666 [quant-ph]; *Quant. Inf. Comp.* **15**, 235–259 (2015).

[26] S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina. Transfer of arbitrary two-qubit states via a spin chain. arXiv:1502.02458 [quant-ph]; *Phys. Rev. A* **91**, 042321 (2015).

[27] J. T. Seeley, M. J. Richard, and P. J. Love. The Bravyi–Kitaev transformation for quantum computation of electronic structure. arXiv:1208.5986 [quant-ph]; *J. Chem. Phys.* **137**, 224109 (2012).

[28] A. Tranter, S. Sofia, J. Seeley, M. Kaicher, J. McClean, R. Babbush, P. V. Coveney, F. Mintert, F. Wilhelm, and P. J. Love. The Bravyi–Kitaev transformation: Properties and applications. *Intern. J. Quantum Chem.* **115**, 1431–1441 (2015).