Abstract. This study considers \(q\)-Gaussian distributions and stochastic differential equations with both multiplicative and additive noises. In the \(M\)-dimensional case a \(q\)-Gaussian distribution can be theoretically derived as a stationary probability distribution of the multiplicative stochastic differential equation with both mutually independent multiplicative and additive noises. By using the proposed stochastic differential equation a method to evaluate a default probability under a given risk buffer is proposed.

1. Introduction

Values of risky assets such as stocks, bonds, and loans sometimes unpredictably fluctuate and are not mutually independent of one another. Since financial markets are complex systems consisting of many interacting elements, it is almost undesirable to completely model price fluctuations from a microscopic point of view. One of the most prominent approaches seems to manage situations based on characteristics of fluctuations in complex systems, such as power-law distributions [1, 2].

Recently regarding the fat-tailness of distributions of price changes (extremal events) the risk management for risky assets based on financial time series is becoming a crucial issue for financial institutions [3]. The risk management is divided into two parts: an estimation of risks of assets and an assignment of their amount (portfolio) within a limited monetary buffer.

Under the assumption of Gaussianity one sometimes has a tendency to underestimate risks which one is potentially facing. Therefore it is important to establish methodology to evaluate risks of own portfolio taking into account their power-law tailness [4].

Recently the framework of Tsallis statistics has been proposed in order to analyze the fluctuations which come from the complex systems. It is well-known that maximization of Tsallis entropy under constraints on normalization and variance for its escort distribution leads to a \(q\)-Gaussian distribution, and that the \(q\)-Gaussian distribution has power-law tails when \(q > 1\). On the other hand various types of dynamical models of which probability has power law tails. If one can combine statistics and dynamics for analyzing the financial time series with actual data, then one expects to improve its estimation in risk assessment.

In this article the \(q\)-Gaussian distribution and a stochastic differential equation with both multiplicative and additive noises are considered in order to analyze multiple time series of a financial market.
This article is organized as follows. In Sec. 2 a brief explanation of Tsallis entropy is shown. In Sec. 3 a stochastic differential equation with both multiplicative and additive noises generating \( q \)-Gaussian fluctuations is introduced. In Sec. 4 a portfolio optimization technique under \( q \)-Gaussian fluctuations is proposed. In Sec. 5 results of both numerical simulations and empirical calculations are shown. Sec. 6 is devoted to concluding remarks.

2. \( q \)-Gaussian on Tsallis statistics

\( q \)-Gaussian distributions are successively studied on the context of complex systems or Tsallis statistics. Tsallis entropy \([5]\) is a generalization of entropy and its maximization

\[
S_q[p] = \frac{1 - \int_{-\infty}^{\infty} p(x)^q dx}{q - 1}, \quad q > 1
\]

under conditions,

\[
\int_{-\infty}^{\infty} p(x) dx = 1,
\]

\[
\int_{-\infty}^{\infty} p(x) x^2 dx = \sigma_q^2,
\]

where \( p(q)(x) = p^q(x)/\int p^q(x) dx \) is called an escort probability, yields a \( q \)-Gaussian distribution

\[
p(x) = \frac{1}{Z_q} \left( 1 + \frac{q - 1}{(3 - q) \sigma^2} x^2 \right)^{\frac{1}{3-q}},
\]

\[
Z_q^{-1} = \left( \frac{q - 1}{(3 - q) \sigma^2} \right)^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)},
\]

where \( \Gamma(x) \) is defined as

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (x > 0).
\]

3. Random multiplicative Processes

It is well known that several types of stochastic differential equations can generate \( q \)-Gaussian fluctuations \([8, 9, 10, 11]\). Nonlinear stochastic differential equations \([8]\), Markov switching model \([9, 12]\), superstatistics of Gaussian processes with a stochastic temperature sampled from a \( \chi^2 \)-distribution \([10]\), and a linear stochastic differential equation with both multiplicative and additive noises \([7, 11]\) have been extensively studied in the context of power law fluctuations.

A stochastic differential equation with both multiplicative and additive noises \([11]\)

\[
\frac{dx}{dt} = [\nu + \xi(t)]x(t) + \eta(t),
\]

where \( \xi(t) \) and \( \eta(t) \) represent Gaussian white noises having

\[
\langle \xi(t) \rangle = 0, \quad \langle \xi(t_1) \xi(t_2) \rangle = 2D_\xi \delta(t_1 - t_2),
\]

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1) \eta(t_2) \rangle = 2D_\eta \delta(t_1 - t_2),
\]

\[
\langle \xi(t) \eta(t) \rangle = 0,
\]

yields the Ito/Stratonovich Fokker-Planck equation

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} [\nu x P] + \frac{\partial^2}{\partial x^2} (D_\xi x^2 + D_\eta) P,
\]

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} [(\nu + D_\xi) x P] + \frac{\partial^2}{\partial x^2} (D_\xi x^2 + D_\eta) P.
\]
From Eqs. (11) and (12) under the assumption that $\nu < 0$, $D_\xi > 0$, and $D_\eta > 0$, one obtains a $q$-Gaussian stationary distribution

$$P_{eq}(x) = \frac{1}{Z} \left(1 + \frac{D_\xi}{D_\eta} x^2\right)^{-(\alpha+1)/2},$$  \hspace{1cm} (13)$$

$$Z^{-1} = \left(\frac{D_\eta}{D_\xi}\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\alpha}{(\alpha+1)}\right),$$  \hspace{1cm} (14)$$

where $Z$ represents a partition function, and $\alpha$ denotes a characteristic exponent given by $\alpha = -2\nu/D_\xi + 1$ (Ito calculus) and $\alpha = -2\nu/D_\xi$ (Stratonovich calculus).

Furthermore the $M$-dimensional version of Eq. (7) may be describe as

$$\frac{dx_i}{dt} = [\nu + \xi(t)] \sum_{j=1}^{M} A_{ij} x_j(t) + \eta_i(t),$$  \hspace{1cm} (15)$$

where $A_{ij}$ is a semi-defined matrix, $\eta_i(t)$ represents mutually independent Gaussian white noises,

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t_1) \xi(t_2) \rangle = 2D_\xi \delta(t_1 - t_2),$$

$$\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t_1) \eta_j(t_2) \rangle = 2D_\eta \delta_{ij} \delta(t_1 - t_2),$$

$$\langle \eta_i(t) \xi(t) \rangle = 0.$$  \hspace{1cm} (16)$$

(17)$$

(18)$$

One can find a unitary matrix $Q_{ij}$ which orthogonalizes $A_{ij}$ by means of linear transformation $y_i = \sum_{j=1}^{M} Q_{ij} x_j$ and obtain its eigenvalues $\lambda_i$ ($i = 1, \ldots, M$). Applying the linear transformation to Eq. (15) yields

$$\frac{dy_i}{dt} = [\nu + \xi(t)] \lambda_i y_i(t) + \eta'_i(t),$$  \hspace{1cm} (19)$$

where and $\eta'_i(t) = \sum_{j=1}^{M} Q_{ij} \eta_j(t)$. According to properties of white Gaussian noises with homogeneous noise strength and a unitary matrix $\eta'_i(t)$ are also white Gaussian noises. Therefore one obtains a stationary solution from the Ito/Stratonovich Fokker-Planck equation

$$\frac{\partial P_i}{\partial t} = -\frac{\partial}{\partial y_i} \left[\nu \lambda_i y_i P_i\right] + \frac{\partial^2}{\partial y_i^2} \left(\lambda_i^2 D_\xi y_i^2 + D_\eta P_i\right),$$

$$\frac{\partial P_i}{\partial t} = -\frac{\partial}{\partial y_i} \left[\nu \lambda_i + D_\xi \lambda_i^2\right] y_i P_i + \frac{\partial^2}{\partial y_i^2} \left(\lambda_i^2 D_\xi y_i^2 + D_\eta P_i\right),$$

$$\lambda_i^2 D_\xi y_i^2 + D_\eta P_i,$$  \hspace{1cm} (20)$$

(21)$$

corresponding to Eq. (19)

$$P_j(y_j) = \frac{1}{Z_j} \left(1 + \lambda_j^2 D_\xi y_j^2\right)^{-(\alpha_j+1)/2},$$

$$Z_j^{-1} = \lambda_j^{-1} \left(D_\eta/2\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\alpha_j}{(\alpha_j+1)}\right),$$

$$\lambda_j^2 D_\xi y_j^2 + D_\eta P_i,$$  \hspace{1cm} (22)$$

(23)$$

where $\alpha_j = -2\nu/(\lambda_j D_\xi) + 1$ (Ito calculus) and $\alpha_j = -2\nu/(\lambda_j D_\xi)$ (Stratonovich calculus). Furthermore the joint probability density function in terms of $x_1, \ldots, x_M$ can be expressed as

$$P(x_1, \ldots, x_M) = \prod_{j=1}^{M} \frac{1}{Z_j} \left(1 + \lambda_j^2 D_\xi \sum_{i=1}^{M} \sum_{k=1}^{M} Q_{ij} Q_{kj} x_i x_k\right)^{-(\alpha_j+1)/2}.$$  \hspace{1cm} (24)$$
Therefore the $M$-dimensional $q$-Gaussian distribution may be expressed as

$$p(x_1, \ldots, x_M) = \prod_{j=1}^{M} \frac{1}{Z_{q_j}(\beta_j)} (1 + \beta_j (q_j - 1) \sum_{i=1}^{M} \sum_{k=1}^{M} Q_{ij}^k Q_{kj} x_i x_k)^{\frac{1}{1-q_j}}, \quad (25)$$

by using the unitary matrix $Q_{ij}$ which diagonalizes $A_{ij}$. Furthermore from Eq. (24) the covariance matrix defined as

$$Cov_{ij} = \int dx_1 \ldots \int dx_M x_i x_j P(x_1, \ldots, x_M), \quad (26)$$

is given by

$$Cov_{ij} = \left( \frac{D_{\xi}}{D_{\xi}} \right)^{\frac{1}{2}} \sum_{k=1}^{M} \lambda_k^2 \frac{\Gamma\left(\frac{\alpha_k}{2}\right) \Gamma\left(\frac{\alpha_k+1}{2}\right)}{\Gamma\left(\frac{\alpha_k+1}{2}\right)^2} \int_{-\infty}^{\infty} dy_k Q_{ik} Q_{kj} (27)$$

$$= \frac{\pi}{2} \left( \frac{D_{\eta}}{D_{\xi}} \right)^{2} \sum_{k=1}^{M} \lambda_k^2 \frac{\Gamma\left(\frac{\alpha_k}{2}\right) \Gamma\left(\frac{\alpha_k+1}{2}\right)}{\Gamma\left(\frac{\alpha_k}{2}\right)^2} Q_{ik} Q_{kj}. \quad (28)$$

According to the theory on stable distributions [14], the finite variance of stable processes is satisfactory when a characteristic exponent is greater than 2. Hence the expression Eq. (28) is convergent for $\alpha_k > 2$ ($k = 1, \ldots, M$). Namely, $\lambda_k$ should be $\lambda_k < 2\nu/D_\xi$ (Ito calculus) and $\lambda_k < \nu/D_\xi$ (Stratonovich calculus). Therefore eigenvalues of covariance matrix $Cov_{ij}$ are given by

$$\mu_k = g(\lambda_k) = \frac{\pi}{2} \left( \frac{D_{\eta}}{D_{\xi}} \right)^{2} \lambda_k^2 \frac{\Gamma\left(\frac{\alpha_k}{2}\right) \Gamma\left(\frac{\alpha_k+1}{2}\right)}{\Gamma\left(\frac{\alpha_k+1}{2}\right)^2} \left( 0 < \lambda_k < -\alpha/\nu/D_\xi \right), \quad (29)$$

where $\alpha = 2$ (Ito calculus) and $\alpha = 1$ (Stratonovich calculus). Eigenvectors corresponding to $\mu_k$ are further obtained from the unitary matrix $Q_{ij}$. Since $\mu_k$ is a monotonically increasing function in terms of $\lambda_k$, $\mu_k \to 0$ ($\lambda_k \to 0$), and $\mu_k \to \infty$ ($\lambda_k \to -\alpha/\nu/D_\xi$) (see. Fig. 1), an inverse function of $g(\lambda_k)$ exists. Here we describe the inverse function of $g(x)$ as $g^{-1}(x)$.

![Figure 1. The relationship between $\mu_k$ and $\lambda_k$.](image)

Practically it is important to estimate $A_{ij}$ from an empirical covariance matrix $\tilde{C}ov_{ij}$ calculated from observed multiple time series with a finite length instead of $Cov_{ij}$. Let
\( r_i(t) \quad (i = 1, \ldots, M; t = 1, \ldots, T) \) denote multiple financial time series of log returns. Then the empirical covariance matrix is defined as

\[
\tilde{\text{Cov}}_{ij} = T^{-1} \sum_{t=1}^{T} r_i(t)r_j(t) - \left( T^{-1} \sum_{t=1}^{T} r_i(t) \right) \left( T^{-1} \sum_{t=1}^{T} r_j(t) \right).
\]

(30)

One can diagonalize the empirical covariance matrix \( \tilde{\text{Cov}}_{ij} \) into diagonal matrix \( \tilde{\Lambda}_{ij} = \tilde{\mu}_i \delta_{i,j} \) with unitary matrix \( \tilde{Q}_{ij} \). Then one has a relation

\[
\tilde{\lambda}_k = g^{-1}(\tilde{\mu}_k)
\]

(31)

and one obtains a relationship between \( A_{ij} \) and \( \tilde{\text{Cov}}_{ij} \)

\[
A_{ij} = \sum_{k=1}^{M} g^{-1}(\tilde{\mu}_k) \tilde{Q}_{ik} \tilde{Q}_{kj}.
\]

(32)

4. Portfolio and risks

The portfolio is one of well-known methods to reduce total risks of risky assets by possessing various types of assets. This is mathematically described as follows. The correlation matrix of \( M \) risky assets \( r_i \) is described as \( \text{Cov}_{ij} \), and their means \( m_i \). Then the variance of total assets \( R_P(t) = \sum_{i=1}^{M} w_i r_i(t) \) is given by

\[
\sigma_P^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} \text{Cov}_{ij} w_i w_j,
\]

(33)

where \( w_i \) represents a possessing amount of risky assets \( i \).

Therefore if we know \( w_i \), then we can evaluate the total risk of own portfolio. However this representation has a tendency to evaluate the total risk less than the realized risk due to the Gaussian assumption. Fig. 2 shows \( q \)-Gaussian distribution which is calculated from time series generated by Eq. (15) at \( M = 1 \). The solid red curve represents a normal distribution with the same variance of that \( q \)-Gaussian distributions, which is drawn as the blue curve in Fig. 2. It is found that the probability of the \( q \)-Gaussian distribution for values greater than \( 3-\sigma \) is higher than one of the normal distribution for values greater than \( 3-\sigma \). Namely, the variance and the Gaussian assumption can under-evaluate extremal events of risky assets.

Let us consider the probability density function of \( R_P(t) \). According to Eq. (32) we can obtain \( A_{ij} \) from the empirical time series covariance matrix \( \text{Cov}_{ij} \). Then we may obtain the distribution of the total assets in portfolio \( w_i \). Since the total assets can be modeled as

\[
R_p(t) = \sum_{j=1}^{M} \sum_{i=1}^{M} Q_{ji} w_i y_j(t),
\]

(34)

they are expressed as a superposition of independent random variables \( y_j(t) \) sampled from Eq. (22), i.e. random variables generated from Eq. (19). Hence we can evaluate the default probability of a buffer \( h \) with the portfolio \( w_i \) from \( \text{Prob}\{R_p < -h\} \) by using Eq. (22).

5. Numerical simulations

If one computes empirical covariance matrix \( \text{Cov}_{ij} \) from multiple financial time series, then one can obtain eigenvalues \( \tilde{\mu}_k \) and unitary matrix \( Q_{ij} \) by orthogonalizing it. Moreover by using Eq.
Figure 2. A dashed green curve is the $q$-Gaussian distribution which is computed from time series obtained from Eq. (15) at $D_\xi = 0.1$, $D_\eta = 2.0$, and $\nu = 2.0$. A dotted blue curve is obtained from Eq. (13) at $D_\xi = 0.1$, $D_\eta = 2.0$, and $\nu = 2.0$. A $q$-Gaussian distribution and normal distribution. A solid red curve represents zero-mean normal distribution with the same variance as these $q$-Gaussian distributions.

(32) we can estimate $A_{ij}$ of the multiplicative Langevin equation shown in Eq. (15). Furthermore from Eq. (29) $D_\eta/D_\xi$ is estimated as

$$\frac{D_\xi}{D_\eta} = \frac{\pi}{2} \frac{1}{\lambda_1^{\frac{a+1}{2}} - \tilde{\mu}_1} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)^2},$$

(35)

where $\tilde{\lambda}_1$ represents the first eigenvalue of the empirical covariance matrix, $\tilde{\mu}_1$ the first eigenvalue of $A_{ij}$, and $\tilde{\alpha}_1$ the characteristic exponent estimated from the cumulative distribution for the superposition of log returns by the first eigenvector $x_1(t) = \sum_{j=1}^{M} r_j(t)Q_{1j}$. $\nu/D_\xi$ is further given by $\nu/D_\xi = -(\alpha_1 - a + 1)/(2\nu)$ at $a = 2$ (Ito calculus) and $a = 1$ (Stratonovich).

Under a given $A_{ij}$ we can generate multiple time series with the same covariance matrix as the empirical estimation and the same power law exponents as of the empirical marginal distribution from Eq. (15). As an example, Fig. 3 shows a sample path which is computed with $A_{ij} = \sum_{i=1}^{M} \sum_{m=1}^{M} \Lambda_{ij} Q_{il} Q_{mj}$ given by a normally distributed random unitary matrix $Q_{ij}$ and $\Lambda_{ij} = \lambda_i \delta_{ij}$ as uniformly distributed random eigenvalues $\lambda_i$ ranging from -1 to -0.5 in the case of $M = 2$.

Here we compare a default probability of multiple time series obtained from Eq. (15) under a weight with it obtained from Eq. (34) with the same distributions as their superpositions.

Fig. 4 shows the default probability of the portfolio $w_i = (0.25, 0.5, 0.25)$ in terms of a buffer $h$ from Eq. (15) for $M = 3$. We set $A_{ij}$ as a matrix of which eigenvalues are sampled from a uniform distribution $U(-1, -0.5)$, $\nu = 2.0$, $D_\eta = 0.1$, and $D_\xi = 2.0$. Obviously the default probability is a monotonically decreasing function in terms of the buffer $h$.

6. Conclusion
We discussed a method to evaluate the total risk of portfolio by means of multi-dimensional $q$-Gaussian distributions and stochastic differential equations with both multiplicative and additive noises. The multi-dimensional $q$-Gaussian distributions were derived as a stationary distribution for the stochastic differential equation.
Figure 3. A 2-dimensional scatter plot of dynamical variable obtained from Eq. (15) at $D_\xi = 0.1$, $D_\eta = 2.0$, and $\nu = 2.0$.

Figure 4. A plot of the default probability in terms of a risk buffer $h$.

By using the stochastic differential equation with multiplicative and additive noises a method to evaluate the default probability for a given risk buffer under a given portfolio was proposed. From numerical simulations dependence of the default probability on risk buffer size was computed.

As future work empirical computation of default probability should be conducted in order to cope with heavy tailed fluctuations of multiple financial time series by means of the proposed method.

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