Real space statistical properties of standard cosmological models

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Abstract. After reviewing some basic relevant properties of stationary stochastic processes (SSP), we discuss the properties of the so-called Harrison-Zeldovich like spectra of mass density perturbations. These correlations are a fundamental feature of all current standard cosmological models. Examining them in real space we note they imply a sub-poissonian normalized variance in spheres $\sigma^2_M(R) \sim R^{-4} \ln R$. In particular this latter behaviour is at the limit of the most rapid decay ($\sim R^{-4}$) of this quantity possible for any stochastic distribution (continuous or discrete). In a simple classification of all SSP into three categories, we highlight with the name “super-homogeneous” the properties of the class to which models like this, with $P(0) = 0$, belong. In statistical physics language they are well described as lattice or glass-like. We illustrate their properties through two simple examples: (i) the “shuffled” lattice and the One Component Plasma at thermal equilibrium.

INTRODUCTION

In standard theories of structure formation in cosmology the density field in the early Universe is described as a perfectly uniform and isotropic matter distribution, with superimposed tiny fluctuations characterized by some particular correlation properties (e.g. [1]). These fluctuations are believed to be the initial seeds from which, through a complex dynamical evolution, galaxies and galaxy structures have emerged. In particular, in all the standard models, the initial fluctuations are taken to be Gaussian and with a power spectrum (PS) $P(k)$ satisfying the so-called Harrison-Zeldovich (HZ) condition of being proportional to $k$ at small $k$ [2].

The present paper has two main purposes [3]. Firstly, to clarify the real space statistical properties of the mass density fluctuations common to all the standard cosmological models (which have been almost completely overlooked in the literature on the subject). And secondly, through this discussion, to relate and compare these models of the primordial Universe to correlated systems encountered in statistical physics.

In particular we find that all these standard cosmological models are characterized by a “superhomogeneous” (or superuniform) matter distribution. This means that mass fluctuations over sufficiently large spatial scales are “sub-poissonian”, i.e. they increase with the spatial scale more slowly than in a random poissonian matter distribution. Moreover, it is shown that the scaling of mass fluctuations satisfying the HZ condition approaches the slowest possible for any stochastic matter distribution. We will see that in the context of usual statistical physics this kind of behavior is recovered in the case of...
lattice-like or glass-like particle distributions or in the so called One Component Plasma (OCP) \[4\] at thermal equilibrium. In this context we present some simple recipe to build particle distributions satisfying the HZ condition at small $k$. This last point can be useful first of all in the context of $N$-body simulations for the study of structure formation from primordial fluctuations through a gravitational dynamics \[5\].

**BASIC CONCEPTS OF CORRELATION ANALYSIS**

Inhomogeneities of the mass density field in cosmology are described using the general framework of stationary stochastic processes (hereafter SSP). Let us consider in general the description of a continuous or a discrete homogeneous mass distribution $\rho(\vec{r})$ in terms of such a process. A stochastic process is completely characterized by its “probability density functional” $\mathcal{P}[\rho(\vec{r})]$ which gives the probability that the result of the stochastic process is the density field $\rho(\vec{r})$ (e.g. see Gaussian functional distributions \[6\]). For a discrete mass distribution, the space (e.g. the infinite three dimensional space) is divided into sufficiently small cells and the stochastic process consists in occupying or not any cell with a point-particle following certain correlated probabilities, and $\rho(\vec{r})$ can be written in general as:

$$\rho(\vec{r}) = \sum_{i=1}^{\infty} \delta(\vec{r} - \vec{r}_i),$$

where $\vec{r}_i$ is the position vector of the particle $i$ of the distribution.

The word “stationarity” refers in the present context to the spatial stationarity of the process, and means that the functional $\mathcal{P}[\rho(\vec{r})]$ is invariant under spatial translation. We suppose also that the distribution is statistically isotropic (invariance of $\mathcal{P}[\rho(\vec{r})]$ under spatial rotation), and has a well defined positive average value:

$$\langle \rho(\vec{r}) \rangle = \rho_0 > 0,$$

where $\langle ... \rangle$ is the ensemble average over all the possible realizations of the stochastic process, i.e. the average over the functional $\mathcal{P}[\rho(\vec{r})]$. Moreover it is usually assumed that $\mathcal{P}[\rho(\vec{r})]$ is ergodic. This means that spatial averages in single infinite realization coincide with the ensemble averages.

The quantity $\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \ldots \rho(\vec{r}_l) \rangle$ is called the complete $l$-point correlation function. In the discrete case $\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \ldots \rho(\vec{r}_l) \rangle dV_1, dV_2, \ldots, dV_l$ gives the a priori probability of finding $l$ particles, in a single realization, placed in the infinitesimal volumes $dV_1, dV_2, \ldots, dV_l$ respectively around $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_l$. In both cases the statistical stationarity and isotropy (hereafter SSI) imply that the $l$-point correlation functions, for any $l$, depend only on the scalar relative distances among the $l$ points \[7\].

Let us analyze in further detail the auto-correlation properties of these systems. As aforementioned, in our hypothesis, $\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle$ depends only on $r_{12} = |\vec{r}_1 - \vec{r}_2|$.

The reduced two-point correlation function $\tilde{\xi}(r)$ is defined by:

$$\langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle \equiv \rho_0^2 \left[ 1 + \tilde{\xi}(r_{12}) \right].$$
The correlation function $\tilde{\xi}(r)$ is one way to measure the "persitence of memory" of spatial variations in the mass density [8]. For a discrete distribution of particles this means that $\rho_0 \tilde{\xi}(r) d^3r$ measures the excess of probability of finding a particle in a small volume $d^3r$ at a distance $r$ from another fixed particle, with respect to the a priori probability $\rho_0 d^3r$. Obviously, since $\rho_0$ must be the average density in any single realization in the infinite volume limit, $\tilde{\xi}(r)$ satisfies the condition of vanishing in the limit $r \to \infty$.

**The Poisson particle distribution**

Let us consider the paradigm of a stochastic homogeneous point-mass distribution: the Poisson case (see [7]). It can be defined as follows: let us partition the space in cubic cells of volume $\Delta V$, and then occupy each cell independently of the others, with a point-particle with a probability $n_0 \Delta V$, where $n_0 > 0$ and $\Delta V \ll n_0^{-1}$. It is simple to show that in the limit $\Delta V \to 0$, one can write:

$$\rho_0 = n_0 \quad (1)$$
$$\tilde{\xi}(r) = \frac{\delta(\vec{r})}{\rho_0} \quad (2)$$

where $\delta(\vec{r})$ is the usual Dirac delta function. Equation (2) is a direct consequence of the fact that there is no correlation between different spatial points in the definition of the stochastic process. That is, the reduced correlation function $\tilde{\xi}$ has only the so called diagonal part. It is simply shown that this diagonal part is present in the reduced two-point correlation functions of any statistically homogeneous discrete distribution of particles with correlations. Therefore in general [9] for a SSI distribution of particles the reduced correlation function can be written as

$$\tilde{\xi}(r) = \frac{\delta(\vec{r})}{\rho_0} + \xi(r)$$

where $\xi$ is the non-diagonal parts which is meaningful only for $r > 0$.

**THE MASS VARIANCE IN A SPHERE**

In this section we consider the amplitude of the mass fluctuations in a generic sphere of radius $R$ with respect to the average mass. First let $M(R) = \int_{C(R)} \rho(\vec{r}) d^3r$ be the mass (for a discrete distribution the number of particles) inside the sphere $C(R)$ of radius $R$ and then volume $\|C(R)\| = \frac{4\pi R^3}{3}$. The normalised mass variance is defined as

$$\sigma^2_M(R) = \frac{\langle M(R)^2 \rangle - \langle M(R) \rangle^2}{\langle M(R) \rangle^2}.$$
It is simple to show that $\sigma^2_M(R)$ in a stationary stochastic mass density can be rewritten as:

$$\sigma^2_M(R) = \frac{1}{\|C(R)\|^2} \int_{C(R)} d^3r_1 \int_{C(R)} d^3r_2 \tilde{\xi}(|\vec{r}_1 - \vec{r}_2|).$$ (3)

This formula will be useful in the following for a complete classification of the mass density fields with respect to the large scale behavior of mass fluctuations.

Note that since $\tilde{\xi}(r) \to 0$ for $r \to \infty$, then

$$\lim_{R \to \infty} \sigma^2_M(R) = 0.$$

This is nothing but the condition of existence of a well defined average density $\rho_0$.

A classification of mass fluctuations

If we apply Eq. (3) to the case of a Poisson point process as above defined, we find

$$\sigma^2_M(R) = \frac{1}{\rho_0\|C(R)\|} \equiv \frac{1}{\langle M(R) \rangle},$$

that is $\langle \Delta M^2(R) \rangle \simeq \|C(R)\|$. The same widespread behavior is found in systems characterized by mainly positive short range correlations (e.g. a perfect gas at high temperature).

We can characterize an arbitrary mass distribution with respect to the behavior of mass fluctuations as follows. Let us suppose $\sigma^2_M(R) \sim R^{-\alpha}$ at large $R$, then:

- If $0 < \alpha < 3$ the system has critical fluctuations, typical of the order parameter of a thermodynamical system at the critical point of a second order phase transition. As we show below, this happens in the case of mainly positive long range correlations such that $\int d^3r \tilde{\xi}(r) = +\infty$;
- If $\alpha = 3$ we have a substantially poissonian system, characterized by mainly positive short range correlations, such that $\int d^3r \tilde{\xi}(r) = c$ where $0 < c < +\infty$;
- If $\alpha > 3$ the system has sub-poisssonian fluctuations typical of lattice or glass-like systems with an almost ordered arrangement of mass perturbations (of particles in the discrete case), characterized by a balance between positive and negative correlations, such that $\int d^3r \tilde{\xi}(r) = 0$. For this reason we call these systems superhomogeneous. We show also below that $\alpha < 4$ ($\alpha < d + 1$ in $d$ dimensions) for any stochastic mass distribution. As shown below all the standard cosmological models for primordial mass fluctuations belong to this class with $\alpha = 4$ (for a difference between a Poisson and a superhomogeneous distribution see Fig. [1]).
THE POWER SPECTRUM $P(K)$

Much more used in cosmology than $\tilde{\xi}(r)$ is the equivalent $k$-space quantity, the power spectrum $P(\vec{k})$ which is defined as

$$P(\vec{k}) = \lim_{V \to \infty} \frac{\left< |\delta_p(\vec{k})|^2 \right>}{V}$$

where $\delta_p(\vec{k}) = \int_V d^3r e^{-i\vec{k} \cdot \vec{r}} (\rho(\vec{r}) - \rho_0)/\rho_0$ is the Fourier integral in the volume $V$ of the normalized fluctuation field $(\rho(\vec{r}) - \rho_0)/\rho_0$. In a SSI matter distribution this depends only on $k = |\vec{k}|$ and can be written as

$$P(\vec{k}) \equiv P(k) = \int d^3r e^{-i\vec{k} \cdot \vec{r}} \tilde{\xi}(r) = 4\pi \int_0^\infty dr r^2 \sin(kr) \tilde{\xi}(r) . \quad (4)$$

It follows from its definition that $P(k) \geq 0$ (this property and Eq. (4) constitute the so-called Khinchin theorem of SSP’s). In particular for a Poisson particle distribution one has $P(k) = \rho_0^{-1}$. At last note the important relation $P(0) = \int d^3r \tilde{\xi}(r)$ for any SSP.

The cosmological power spectrum, and the HZ condition

In current cosmological models it is generally assumed that the power spectrum $P(k) \sim k$ at small $k$. This is the famous Harrison-Zeldovich (HZ) condition. It is believed to describe the “primordial” fluctuations at very early times, the putative remnants of a period of “inflation” prior to the ordinary Big Bang phase [1,10]. The HZ behaviour is appropriately cut-off at a short distance scale (i.e. for $k$ larger than some cut-off $k_c$). The type of the cut-off depends on the single cosmological model (e.g. Cold Dark Matter, Hot Dark Matter, etc.). The reason for the HZ condition is tied to considerations about the consistency of the Friedmann-Robertson-Walker (FRW) metrics under perturbations (see [1,10]): any other spectrum will give mass fluctuations which dominate over the homogeneous background either at some time in the future or past.
One can see [3] that in all the standard cosmological models the corresponding ξ(r) is positive at small r and has a negative tail ∼ −r^{−4} at large r with only one zero.

The power spectrum vs. the mass variance

Quite generally, by studying Eq. (3), it is not difficult to show [3] that for a spectrum \( P(k) \sim k^n \) for \( k \to 0 \) (note that the Poisson-like system are given by \( n = 0 \)) and appropriately cut-off at large k, one has at large R

\[
\sigma^2_M(R) \sim \begin{cases} 
1/R^{3+n} & \text{if } n < 1 \\
\log(R)/R^4 & \text{if } n = 1 \\
1/R^4 & \text{if } n > 1
\end{cases}
\]

In terms of the non-normalized quantity \( \langle \Delta M^2(R) \rangle \sim \sigma^2_M(R)R^6 \), Eq. (3) says that for \( n > 0 \) (i.e. \( \int d^3r \tilde{\xi}(r) = 0 \)) we have a scaling behavior of \( \langle \Delta M^2 \rangle \) as a function of R slower than for Poisson fluctuations, which correspond to \( n = 0 \) (i.e. \( \int d^3r \tilde{\xi}(r) = c > 0 \)), with the limiting behavior for \( n \geq 1 \) corresponding to quadratic mass fluctuations which are proportional to the surface area of the sphere. These systems are thus characterised by surface fluctuations, ordered (or homogeneous enough, one could say) to give this very particular behaviour. The case \( n < 0 \) (i.e. \( \int d^3r \tilde{\xi}(r) = +\infty \)) is typical of critical phenomena in which long range mainly positive correlations determine a “super-poissonian” behavior of quadratic fluctuations.

Equation (5) shows clearly that cosmological models, characterized by the HZ condition \( (n = 1) \), are not only superhomogeneous, but are at the limit of the slowest possible behavior of quadratic mass fluctuations with the spatial scale.

TWO “SUPERHOMOGENEOUS” EXAMPLES

In this section we present two examples of superhomogeneous point-particle distribution: the shuffled lattice [5] and the One Component Plasma (OCP) [4].

1) In the case of particles placed on the sites of a regular cubic lattice (in any dimension d), one has [3] \( \rho_0 = a^{-d} \) where a is the lattice spacing and:

\[
P_l(\vec{k}) = (2\pi)^d \sum_{\vec{H} \neq 0} \delta(\vec{k} - \vec{H}),
\]

where the sum is extended to all the vector of the reciprocal lattice but the origin \( \vec{H} = 0 \). Therefore we can say that around \( k = 0 \) \( P_l(k) \sim k^{+\infty} \), and then it is the most superhomogeneous particle distribution with \( \sigma^2_M(R) \sim R^{-(d+1)} \).

At this point let us introduce a stochastic displacement field, in which each particle is displaced from its lattice site \( \vec{R} \) of a random and statistically isotropic vector \( \vec{u}_R \), each particle independently of the others. One can show [11] that, if the probability density of the displacements \( p(u) \) has a finite variance, then the resulting particle distribution has
a power spectrum $P_{sl}(k) \sim k^2$ at small $k$. If instead, the variance is infinite, and $p(u)$ has a tail going as $u^{-a}$ (with $d < a < d + 2$ for normalizability, but with infinite variance), then $P_{sl}(k) \sim k^{a-d}$ at small $k$. Therefore, choosing appropriately $a$ we can obtain any kind of superhomogeneous particle distribution with the exponent of $P_{sl}(k)$ at small $k$ ranging from 0 to 2. In particular for $a = d + 1$ ($a = 4$ in three dimension) we obtain the HZ condition.

2) Let us now present briefly another model taken from equilibrium statistical physics displaying superhomogeneous mass fluctuations: the OCP. It is a system of interacting point particles carrying a unit positive charge and mass repulsing each other via a Coulomb potential $V(r) = 1/r$, in a continuous uniform background negatively charged giving overall charge neutrality [4]. One can show quite generally that, at thermodynamic equilibrium the system has only one phase, and at small $k$ the power spectrum of the particle distribution satisfies:

$$P(k) \sim \frac{k^2}{4\pi\rho_0^2\beta},$$

where $\rho_0$ is the average density of particles and $\beta$ is the usual inverse temperature. It is very interesting to note that [12] if, instead of having the Coulomb potential $V(r) \sim r^{-1}$, one has a potential $V(r) \sim r^{-2}$ one would obtain

$$P(k) \sim \frac{k}{2\pi^2n^2\beta},$$

that is the HZ condition. These results can be very important if applied to $N$-body simulations of gravitational dynamics to study structure formation starting from the primordial mass fluctuations [5]. In fact, in this context a great importance is given to the accurate preparation of the initial conditions for the simulation. For instance, this can be set through a modified OCP with $V(r) \sim r^{-2}$ at large scales and an appropriate modified behavior at smaller scales [12].

Both 1) and 2) provide particle distributions similar to the right side picture of Fig. 1.

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