Gravitationally coupled magnetic monopole and conformal symmetry breaking

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Abstract

We consider a Georgi-Glashow model conformally coupled to gravity. The conformally invariant action includes a triplet of scalar fields and SO(3) non-Abelian gauge fields. However, the usual mass term $\mu^2 \phi^2$ is forbidden by the symmetry and this role is now played by the conformal coupling of the Ricci scalar to the scalar fields. Spontaneous symmetry breaking occurs via gravitation. The spherically symmetric solutions correspond to localized solitons (magnetic monopoles) in asymptotically anti-de-Sitter (AdS) space-time and the metric outside the core of the monopole is found to be Schwarzschild-AdS. Though conformal symmetry excludes the Einstein-Hilbert term in the original action, it emerges in the effective action after spontaneous symmetry breaking and dominates the low-energy/long-distance regime outside the core of the monopole.

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1 Introduction

This paper is based on an invited talk given at Theory Canada IV in June, 2008 at the Centre de Recherche Mathématique, Université de Montréal. The talk discussed a previously published paper [4]. In these proceedings we summarize the paper and also highlight various points in the conclusion section that were not discussed in much detail in [4].

A conformal transformation is a rescaling of the metric at every coordinate $x$. Such a transformation alters the curvature of spacetime but in such a way that the light cones, and hence the causal structure of spacetime, is preserved. This renders conformal symmetry of potential interest. However, before one can incorporate such a symmetry there are two things to consider. The introduction of a scale, such as a mass term in the action, breaks the symmetry. Nature is not conformally invariant and one therefore needs a mechanism that breaks the original symmetry. Secondly, though some well-known massless field equations are conformally invariant, such as Maxwell’s equations in 3+1 dimensions, others such as the Klein-Gordon equation for a massless scalar field are not. One must add an extra term to the usual scalar field kinetic term to obtain an invariant equation. This extra term is the conformal coupling of the Ricci scalar to the scalar field. Interestingly, it plays a dual role. Besides rendering the action for the scalar field conformally invariant it also acts as part of a symmetry breaking potential.

Though conformal symmetry forbids the inclusion of the Einstein-Hilbert term in the original action, we find that the effective action after symmetry breaking is dominated by Einstein gravity (General Relativity) in the long distance or low energy regime. The particular model we consider is a Georgi-Glashow model [1] conformally coupled to gravity. As in the usual Georgi-Glashow model we include a triplet of scalar fields and SO(3) non-abelian gauge fields. However, the usual mass term $\mu^2 |\phi|^2$ is forbidden by the conformal symmetry and is replaced by $R |\phi|^2/6$ i.e. the Ricci scalar conformally coupled to the scalar fields. Our solutions have a non-singular core representing localized soliton solutions of the magnetic monopole type in an asymptotically anti-de-Sitter (AdS) space-time. The metric for the spherically symmetric case is Schwarzschild-AdS outside the core of the monopole and we obtain numerical solutions for the scalar fields, the gauge fields and the metric both inside and outside the core of the monopole. Conformally
coupled matter/gravity and also spontaneous breaking of conformal invariance has been considered by Demir, by Odintsov and by Mannheim and Kazanas [2]. The latter work is the most relevant to ours. They consider the conformally coupled scalar/gravitational field equations and find for a negative scalar field self-coupling that a mass scale is generated in the theory and that the metric corresponds to de-Sitter space-time. Other work of interest for the case of Einstein gravity or string inspired gravity with Yang-Mills and Higgs fields can be found in [3].

2 Metric, scalar and gauge fields in a conformally invariant action

We construct an action which is both conformally invariant and generally covariant (our metric signature is \((+,-,-,-)\)). A conformal transformation is a scaling of the metric at every spacetime point:

\[ g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \]

where \(\Omega = \Omega(x)\) is a smooth positive function. The gravitational part of the action is given by the Weyl tensor squared instead of the Einstein-Hilbert term:

\[ S = \int \sqrt{|g|} C_{\beta\mu\nu\rho} C^{\beta\mu\nu\rho} d^4x \equiv \int \sqrt{|g|} C^2 d^4x. \]

This is the unique gravitational action constructed using only the metric, which is invariant under (local) conformal and coordinate transformations. We consider a gauge potential as \(n\) 4-vectors \(A^a_\mu\). The field strength is then given by

\[ F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \]  

(2.1)

where \(C^a_{bc}\) are the structure constants of the Lie algebra of the gauge group. The conformal transformation for the gauge field is defined as \(A'_{\mu}^a = A_{\mu}^a\) and this yields \(F'_{\mu\nu}^a = F_{\mu\nu}^a\). The Higgs boson is defined as \(n\) scalars \(\phi^a\). The gauge covariant derivative of Higgs bosons is given by

\[ D_\mu \phi^a = \nabla_\mu \phi^a + C^a_{bc} A^b_\mu \phi^c. \]  

(2.2)
The action for the Higgs bosons in interaction with the gauge field is given by the standard form (with implicit summation over \( k \) assumed)

\[
S = \int \sqrt{|g|} \left( D_\mu \phi^k D^\mu \phi_k - \frac{1}{4e^2} F^k_{\mu\nu} F^{\mu\nu}_k \right) d^4 x.
\]

(2.3)

A conformal transformation for the Higgs is defined as \( \phi'^a = \Omega^{-1} \phi^a \) and the term \( D_\mu \phi^k D^\mu \phi_k \) in (2.3) is not conformally invariant. However, by adding the extra term \( \frac{1}{6} R \phi^k \phi_k \) i.e. the conformal coupling of the Ricci scalar to the scalar fields, the action is made conformally invariant. We are also free to add a \( \phi^4 \) self-coupling term \( \lambda^2 (\phi^k \phi_k)^2 \), which is conformally invariant by itself. This will be part of a symmetry breaking potential. The complete conformally invariant action is then given by [4]

\[
S = \int \sqrt{|g|} \left( C^2 - \frac{1}{4e^2} F^2 + (D\phi)^2 + \frac{1}{6} R \phi^2 - \lambda^2 \phi^4 \right) d^4 x
\]

(2.4)

where \( C^2 = C_{\beta\mu\nu\rho} C^{\beta\mu\nu\rho} \), \( F^2 = F_{\mu\nu}^k F^{\mu\nu}_k \) and \( \phi^2 = \phi^k \phi_k \) with implicit summation over \( k \). The above is a conformally invariant action that includes the interaction between the metric, the gauge fields and the Higgs bosons. The quantity \( e \) is the gauge coupling constant.

### 2.1 Symmetry breaking via gravitation

In (2.4) the last two terms can be seen as a non-minimal potential for the scalar field

\[
V(\phi) = -\frac{1}{6} R \phi^2 + \lambda^2 \phi^4.
\]

(2.5)

Note that the usual mass term \( \mu^2 \phi^2 \) is absent from the potential. The term with \( \phi^2 \) is proportional to the Ricci scalar \( R \), which is not the mass of Higgs field, nor a constant. The above potential can provide for spontaneous breaking of the conformal and gauge symmetry. We consider the case of positive quartic scalar self-coupling, \( \lambda^2 > 0 \) which is required for stability in flat backgrounds. Spontaneous symmetry breaking can occur, for example, when the scalar curvature is a constant positive value, namely, in AdS space-times. It is too great of a restriction to impose that \( R \) is a positive constant everywhere and we make the Ansatz that the AdS geometry is only asymptotic, that is the Ricci scalar goes to a positive constant only in the limit for which the
independent variables are at the boundary of the particular system of coordinates we will consider. In asymptotically AdS space-times, then, we will set the numerical value of the Ricci scalar at the boundary of that specific system of coordinates to be equal to $12\lambda^2 v^2$, with $v$ an arbitrary constant. In this case, $\phi^2 = v^2$ is the solution which asymptotically gives the spontaneous breakdown of the gauge symmetry, giving mass to some of the gauge fields via the Higgs mechanism. The conformal symmetry is therefore broken in the process.

3 Spherically symmetric construction: metric, gauge and scalar fields

In the Georgi-Glashow model the basic assumption is that we have a triplet of Higgs bosons and a triplet of gauge fields for which the symmetry group is $SO(3)$. The geometrical configuration of the spacetime we want to study is chosen to be stationary and spherically symmetric, i.e. the distribution of energy does not depend on time and it is isotropically distributed around the origin, hence, the symmetry group of the spatial isometries is also $SO(3)$.

This structure for spatial isometries allows us to consider 3 linearly independent 4-dimensional Killing vectors in spherical coordinates $(t, r, \theta, \phi)$, given as follows: $\xi^{(1)} = (0, 0, \cos \varphi, -\sin \varphi \cot \theta)$, $\xi^{(2)} = (0, 0, -\sin \varphi, -\cos \varphi \cot \theta)$, $\xi^{(3)} = (0, 0, 0, 1)$, for which $[\xi^{(i)}, \xi^{(j)}] = \varepsilon_{ijk} \xi^{(k)}$, the correct representation of the symmetry group for $SO(3)$.

The line element in spherical coordinates has to be isotropic, that is there exists a system of coordinates in which it has the form

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(3.6)

It has been noted by Mannheim and Kazanas ([5]) that, via a sequence of coordinate and conformal transformations, one can always bring the general static, spherically symmetric metric to a form in which $A = 1/B$. Thus it is justified to set $A = 1/B = 1 + h(r)$ for a smooth function $h(r)$, so that

$$ds^2 = (1 + h(r))dt^2 - \left(\frac{1}{1 + h(r)}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(3.7)
Given this line element, we can compute the curvature tensors (see [4]). Asymptotically the spacetime is AdS where the Ricci scalar is given by

$$\lim_{r \to \infty} R(r) = 12\lambda^2 v^2$$  \hspace{1cm} (3.8)

We note that $SO(3)$ is at the same time the spatial and internal (gauge) group of symmetries. The three gauge fields can then be expressed in terms of the three above Killing vectors (using spherical coordinates): $A_i^\mu = q(r^2) \xi_i^{(\mu)}$, where $q(r^2)$ is a function of $r^2$. For the Higgs field we obtain $\phi^a = f(r^2) \frac{r^a}{r}$ where $f(r^2)$ is another function of $r^2$. Instead of working with $q(r^2)$ it is more convenient to work with the function $a(r^2)$ defined by

$$1 + r^2 q(r^2) = a(r^2).$$  \hspace{1cm} (3.9)

The field strength and the gauge covariant derivatives can now be written down explicitly for this theory. The equations of motion for each field are obtained by varying the effective action with respect to the radial functions $h(r), f(r), a(r)$. This yields the equations [4]

$$(rh)^{\prime\prime\prime\prime} = \frac{r}{2}(2f'^2 - ff'') + \frac{3a'^2}{2re^2}$$  \hspace{1cm} (3.10)

$$((1 + h)r^2 f')' = f(2a^2 + r^2(2\lambda^2 f^2 - \frac{(r^2 h)''}{6r^2}))$$  \hspace{1cm} (3.11)

$$((1 + h)a')' = a(2f^2 e^2 + \frac{a^2 - 1}{r^4})$$  \hspace{1cm} (3.12)

where the prime is the derivative with respect to $r$.

### 3.1 Results: analytical and numerical solutions

Outside a core region the vanishing Nöther’s density allows a non-trivial solution $a = 0$. This is non-trivial because there is a long range monopole like magnetic field. It is possible to solve the field equations exactly for this external solution. A solution with spontaneous symmetry breaking of the conformal symmetry requires that $f$ reaches a nonzero arbitrary constant $v$ asymptotically. We consider the analytical solution with $f = v$ identically. The only possible solution for the metric is given as $h = \frac{1}{r} + \lambda^2 v^2 r^2$. One
solution for the exterior region is then \( a(r) = 0, f(r) = v \) and \( h(r) = -\frac{\beta}{r} + \lambda^2 v^2 r^2 \) for any value of the free parameter \( \beta \) and for any positive value of the parameter \( v \). The quadratic term, which is dominant at large distances, gives rise to a constant, positive Ricci scalar for the positive \( \lambda^2 v^2 \). Hence, the space-time is asymptotically anti-de-Sitter. Other analytical solutions are discussed in [4]. Outside the core region we therefore obtain Schwarzschild-AdS spacetime. The behaviour at shorter distances inside the core requires numerical solutions (see Figure 1). In the core it is clearly seen that Einstein gravity no longer dominates.

The solution \( a = 0 \) means, from (3.9), that \( q(r^2) \) behaves as \(-1/r^2\). This gives rise to a non-zero purely radial magnetic field strength \( B^{(k)r} = \frac{q(k)}{|\phi|} \frac{1}{r^2} \) that is proportional to the Higgs field, in the internal space [4]. Locally we can choose (apart from the origin) \( \vec{\phi} = (0, 0, v) \) and then the third component of the gauge field is \( B^{(3)r} = \frac{1}{r^2} \). This third component corresponds to the \( SO(2) \) unbroken symmetry; hence, it can be identified with the Abelian magnetic field.

We can look for numerical solutions to the field equations 3.10, 3.11 and 3.12. Such a numerical solution includes the behaviour of the core region (i.e.

![Figure 1: Plot of \( a(r) \), \( f(r) \) and \( h(r)/r^2 \) for \( e = 1 \).](image)
the magnetic monopole) as well as the external region. The field equations need eight initial conditions to be given at the origin which specify a unique solution. A simple program finds the solutions of the type that we are looking for i.e. where asymptotically \( f = v \) and \( h = \lambda^2 v^2 \). A set of initial conditions can be chosen to give the solution presented in Figure 1. The most important point about the numerical solution is that the core is non-singular. This is where the magnetic monopole resides (the region in the neighborhood of \( r = 0 \) where \( a(r) \) is significant). The spacetime is AdS asymptotically with Schwarzschild in the intermediate region. However, it is easy to see that the Schwarzschild solution does not extend to \( r = 0 \) in the core region as there is no Schwarzschild singularity there.

4 Conclusion: discussion and future work

We have focused thus far on the broken sector where the exterior solution is Schwarzschild AdS. There is also a non-broken sector where \(|\phi| = 0\) asymptotically. This yields the more general vacuum solution originally found by Mannheim and Kazanas [5]. The hallmark of that solution is the presence of a linear potential \( \gamma r \) (\( \gamma \) is an integration constant) besides the usual \( 1/r \) Newtonian potential. This arises because in the non-broken sector one actually has a fourth order gravity theory due to the presence of the conformally invariant \( C^2 \) term, the Weyl tensor squared. Unlike General Relativity, this yields a fourth order poisson equation which has a linear potential as part of its solution. What is interesting is that spontaneous symmetry breaking eliminates altogether this long range linear potential even though the \( C^2 \) term is still present in the Lagrangian. The spontaneous symmetry breaking acts as a “switch” from fourth order gravity to second order gravity at large distances. Outside the core, one finds the Schwarzschild metric only (in an AdS background). Let us investigate this further.

Einstein gravity (General Relativity) is not present in the original action. In fact, the original Lagrangian looks like fourth order Weyl gravity with matter coupled to the Ricci scalar. However Einstein gravity is induced via spontaneous symmetry breaking. This can be seen by expanding the action \(^{2.3}\) about \(|\phi| = v\). One of the terms in the expansion is \((1/6) v^2 R\). This is nothing other than the Einstein-Hilbert term with \( v^2/6 \) identified with \( 1/(16 \pi G) \) where \( G \) is Newton’s gravitational constant. Another term in
the expansion is $\lambda^2 v^4$ arising from $(1/6)R\phi^2 - \lambda^2 \phi^4$ where $R = 12 \lambda^2 v^2$ in AdS spacetime. This acts as a cosmological constant. In the broken sector, where $|\phi| = v$ asymptotically, Einstein gravity (General Relativity) with a cosmological constant dominates over the higher-derivative $C^2$ term in the low energy or long distance regime. This is why there is no linear term in the potential. Besides inducing Einstein gravity, an important and interesting feature of the solution is that there is no singularity at $r = 0$. The numerical results show a regular magnetic monopole solution in the interior. The core region is clearly not dictated by the Schwarzschild metric. This makes sense because in the core region it is fourth order gravity that now dominates over the Einstein-Hilbert term. The roles switch in the interior.

Inducing Einstein gravity starting with a different theory may be important in general. At present, theoretical physics is faced with the cosmological constant problem. Since gravity couples to energy it should couple to vacuum energy. This should lead to a very large cosmological constant. However, this is not observed. This is not a quantum gravity problem though the two problems may be ultimately related. There is no reason to doubt the existence of quantum vacuum fluctuations since they are fundamental, stemming from the uncertainty principle. Yet, when we include matter vacuum fluctuations in the calculation of the expectation value of the energy-momentum tensor we obtain an absurdly large cosmological constant. GR is a very successful theory and has been tested experimentally in both the weak and strong field regimes. One possibility then is that GR is an induced theory and emerges after matter vacuum fluctuations are included in a fundamentally different theory. What we have shown in this paper is that the idea of induced gravity, at least at the classical level, is a realizable project. Moreover, it is not realized in an arbitrary fashion. The original theory is motivated by symmetry, in our case conformal symmetry. Contact with nature is then achieved via spontaneous symmetry breaking. Again, we see the powerful and fruitful role that both symmetries and symmetry-breaking play.

For future work, it would be interesting and important to add quantum corrections to the action. These contribute higher-derivative geometrical terms such as the Ricci tensor squared $R_{\alpha\beta}R^{\alpha\beta}$ and Ricci scalar squared $R^2$ to the gravitational part of the action [6]. The action is then no longer conformally invariant and introduces a fundamental length scale into the theory. This would ultimately fix the value $v$ for the expectation value of the scalar field. As already discussed, the value of $v$ is related to Newton’s
constant $G$, a cosmological constant and massive scalars. These will then have a non-trivial relation to each other that includes Planck’s constant. This should be interesting.

How will adding quantum corrections affect the equations of motion? The corrections are higher-derivative terms and should not affect the long distance regime in the broken sector (after spontaneous symmetry breaking). Einstein gravity with cosmological constant should still dominate in this regime but now with parameters fixed by the introduction of a length scale. What will definitely change is the numerical solution in the interior, the core region, since the new higher derivative terms affect this high-energy regime. A question of interest is whether the magnetic monopole solution will remain regular at $r = 0$.

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