The dynamical role of initial correlation in the exactly solvable dephasing model

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We investigate the effects of the initial correlation on the dynamics of open system in the exactly solvable pure dephasing model. We show that the role of the initial correlation come into play through a phase function and a weight factor, which would perform oscillations during time evolution, and find that the decoherence of a qubit coupled to a boson bath is more enhanced with respect to a spin bath in the short time. We also demonstrate that the trace distance between two states of a qubit can increase above its initial value, and that the initial correlation can provide another resource for the damply oscillation and revival of the entanglement of two qubits. We finally investigate the dependence of the crossover of decoherence from the dynamical enhancement to suppression under the bang-bang pulse control on the initial correlation and the statistics of the bath constituents.

PACS numbers:

I. INTRODUCTION

The conventional method of studying open quantum system is the quantum master equation in the Lindblad form, which relies on two assumptions [1]. The first is that the coupling of the system-bath interaction is weak, i.e. Born-approximation, and the other is that the relaxation time of bath is much shorter than the response time of system, i.e. Markov-approximation. Then the second order perturbation is applied and the memory effect of the bath can be neglected for simplification. If we ignore these assumptions, as for the rigorous treatment of a quantum Brownian particle interacting with a heat bath [2], a generalized exact master equation or Hu-Paz-Zhang equation is derived, which does not have Lindblad form. For general open systems, it is usually impossible to solve such exact master equations analytically.

Besides these two assumptions, there is another underlying assumption in deriving most of master equations in literature [1, 2], namely the open system is completely isolated from bath initially. This uncorrelated initial condition is much useful when the system-bath coupling is sufficiently weak and when we are only interested in the long time behavior of system. In such cases, the evolution of the system can be simply described by the completely positive map [3] acting only on the system state, which leads to many profound results, e.g. the trace distance between two states of the open system can not increase above its initial value [4, 5]. Although the uncorrelated initial condition is widely used, it is mainly based on mathematical idealization or simplicity rather than physical considerations, and thus not always well justified when the system-bath coupling is strong and when we study the transient behavior of system at short times [3, 4].

Thus the effects of initial correlations on the dynamics of open systems have been intensively discussed recently [4, 6]. It is argued in that the initial correlation can be witnessed by the increase of the trace distance of two system states over its initial value. Some other witnesses for initial correlations have been proposed in [8], such as purity, quantum discord, entanglement, etc. Moreover, it has been discussed that the initial condition can significantly influence the time development of system based on some solvable models [9]. In the pure dephasing model with the boson bath and the correlated initial condition, the decoherence function can have some sharp peaks in comparison with the uncorrelated initial condition [7, 10].

In this paper we address the influence of the initial correlation on the dynamics of the open system in the exactly solvable pure dephasing model. We show that the role of the initial correlation comes into play through a phase function and a weight factor, which take into account the memory effect of bath and might give rise to coherence oscillation during the time evolution. In particular, for the short time, the decoherence of a qubit coupled to a boson bath is more enhanced with respect to a spin bath. We also demonstrate that the trace distance between two states of a qubit can increase above its initial value, and that the entanglement of two qubits, each locally interacting with an independent bath, can damply oscillate and revive to a large amount being comparable to the uncorrelated initial condition [11]. Moreover, it is known that the decoherence can be effectively inhibited with the bang-bang control pulses [12], and depends on the statistics of the bath constituents [13] without considering initial correlations. So we finally extend them and investigate how the initial correlations affect the decoherence and its crossover from the dynamical enhancement to suppression under the pulse control.

The organization of this paper is as follows. In Sec. II we review the main features of two pure dephasing models with boson and spin baths respectively. The dynamics without initial correlations are analyzed in Sec. III, including the bang-bang control pulses. We then study the influence of the initial correlation on the dynamics of the qubit in Sec. IV. The numerical results are given in Sec. V. Our conclusion is given in Sec. VI.
II. MODELS FOR PURE DEPHASING

The Hamiltonian describing a two-state system (S), i.e. qubit coupled to a bath (B) for pure qubit-dephasing in the boson-bath model can be written as \[9, 12, 14\]

\[H = H_S + H_B + H_{int}\]
\[= \omega_0 S_z + \sum_k \omega_k a_k^\dagger a_k + S_z \sum_k g_k (a_k^\dagger + a_k), (1)\]

where \(\omega_0\) is the excited energy of the qubit, and \(S_z\) is the spin matrix of \(z\) component with basis \(2S_z|\pm\rangle = \pm|\pm\rangle\). The annihilation and creation operators \(a_k\) and \(a_k^\dagger\) are the bath mode with frequency \(\omega_k\). The coupling strength between the qubit and the \(k\)th bath mode is denoted by \(g_k\). Suppose the initial state of the system at \(t = 0\) is given by a density matrix \(\rho(0)\), then the state at time \(t\) is

\[\rho(t) = e^{-iHt} \rho(0) e^{iHt}. \quad (2)\]

The reduced density matrix of the qubit is the partial trace taken over the bath modes of the total density matrix, \(\rho_S = \text{Tr}_B \rho\).

It would be convenient to work in the interaction picture, where the new state and time evolution are given by

\[\begin{align*}
\varrho(t) &= e^{i(H_S + H_B)t} \rho(t) e^{-i(H_S + H_B)t}, \\
U(t, t_0) &= \mathcal{T} \exp[-i \int_{t_0}^t ds H(s)], \\
\mathcal{H}(t) &= S_z \sum_k g_k (e^{i\omega_k t} a_k^\dagger + e^{-i\omega_k t} a_k). \quad (3)
\end{align*}\]

Here the notation \(\mathcal{T}\) represents the time-ordered product. The evolution of the new density matrix can be written as

\[\varrho(t) = U(t, t_0) \varrho(t_0) U^\dagger(t, t_0). \quad (4)\]

In order to further simplify \(U(t, t_0)\), we use the generalized Baker-Hausdorff formula \[11,\]

\[\mathcal{T} \exp \left[ i \int_{t_0}^t ds A(s) \right] = \exp \left[ i \int_{t_0}^t ds A(s) \right] \times \exp \left\{ -\frac{1}{2} \int_{t_0}^t ds_1 \int_{t_0}^{s_1} ds_2 [A(s_1), A(s_2)] \right\}, \quad (5)\]

which is satisfied if the commutator \([A(s_1), A(s_2)]\) is a \(c\)-number. For the interaction Hamiltonian \(\mathcal{H}(s)\) we have

\[\mathcal{H}(t) = -\frac{i}{2} \sum_k g_k^2 \sin \omega_k (t - t'). \quad (6)\]

With Eqs. (3) and (4), the evolution operator \(U(t, t_0)\) can be simplified as

\[U(t, t_0) = U_+ (t, t_0)|+\rangle\langle+| + U_- (t, t_0)|-\rangle\langle-|, \quad (7)\]

where the unitary operators \(U_{\pm}\) are

\[U_{\pm} = \exp \left\{ -i f(t - t_0) \pm \frac{1}{2} \sum_k \left\{ \xi_k (t - t_0) e^{i\omega_k t_0} a_k^\dagger - \xi_k^* (t - t_0) e^{-i\omega_k t_0} a_k \right\} \right\}, \quad (8)\]

The functions \(f(t)\) and \(\xi_k(t)\) are determined by the coupling to the bath,

\[\begin{align*}
f(t) &= \sum_k \frac{g_k^2}{4\omega_k^2} (\omega_k t - \sin \omega_k t), \\
\xi_k(t) &= \frac{g_k}{\omega_k} (1 - e^{i\omega t}). \quad (9)
\end{align*}\]

Similarly, the Hamiltonian for a qubit coupled to a spin-bath is given by \[13\]

\[H = H_S + H_B + H_{int}\]
\[= \omega_0 S_z + \sum_k \omega_k \sigma_k^z + S_z \sum_k g_k (\sigma_k^+ + \sigma_k^-), \quad (10)\]

where the coupling strength \(g_k\) is identical to that of Eq. (1), \(\omega_k\) denotes the excited energy of the \(k\)th bath spin \(\sigma_k\). It is worth to point out that we have introduced an extra factor of \(1/2\) in \(H_B\) that is different from the corresponding Hamiltonian in \[13\] to make the excited energy same as the photon bath, which leads to the same evolutions for qubit in the two baths with zero temperature at weak coupling regime and \(t \to 0\) as shown below. This spin-bath model can be solved by expressing the Hamiltonian \(H\) as

\[H = H^+|+\rangle\langle+| + H^-|-\rangle\langle-|, \quad (11)\]

The unitary evolution operator \(U(t) = e^{-iHt}\) is given by

\[\begin{align*}
U(t) &= U_+ |+\rangle\langle+| + U_- |-\rangle\langle-|, \\
U_{\pm} &= e^{\mp i\omega t/2} \prod_k U_{k}^\pm, \quad (12)
\end{align*}\]

where the unitary \(U_k^\pm\) is

\[\begin{align*}
U_k^\pm &= \cos \Omega_k t / 2 - i \left( \frac{\omega_k}{\Omega_k} \sigma_k^z \pm \frac{g_k}{\Omega_k} \sigma_k^x \right) \sin \frac{\Omega_k t}{2}, \\
\Omega_k &= \sqrt{\omega_k^2 + g_k^2}. \quad (13)
\end{align*}\]

It can be seen from Eqs. (7) and (12) that the populations of the \(|\pm\rangle\) states, \(\rho_{\pm}^\pm(t)\) and \(\rho_{\pm}^-\) do not change with time, namely the pure dephasing model can not de-
scribe the relaxation process of the qubit to equilibrium. However, the exact solutions of these simple models with initial correlations still capture the essential features of decoherence in the more complicated dissipative models.

III. DEPHASING WITHOUT INITIAL CORRELATIONS

The uncorrelated initial condition means that the initial state of the total system is a direct product of the qubit state and the bath state,

\[ \rho(0) = \rho_S(0) \otimes \rho_B \]
\[ \rho_S(t) = \text{Tr}_B[U \rho_S(0) \otimes \rho_B U^+] \] (14)

Note the map from \( \rho_S(0) \) to \( \rho_S(t) \) is completely positive under the uncorrelated initial condition. Usually, the initial bath is assumed to be in the thermal equilibrium at temperature \( T = 1/\beta, \rho_B = e^{-\beta H_B}/Z_B \) with the normalization \( Z_B = \text{Tr}_B[e^{-\beta H_B}] \). Because the evolution is governed by the unitary operator of the form \( U_+ |+\rangle \langle +| + U_- |\rangle \langle -| \), the coherence of the qubit \( \rho_S^\pm \) evolves in time as

\[ \rho_S^\pm(t) = e^{-i\omega t} e^{-\Gamma_\pm(t)} \rho_S^\pm(0), \]
\[ \Gamma_\pm(t) = -\ln \left( \text{Tr}_B[U \rho_S U^+] \right) \]

where \( \Gamma_\pm(t) \) denotes the decoherence function for the boson-bath model. Using Eq. (3) and the identity

\[ \text{Tr}_B[\rho_B \exp \left( \sum_k \eta_k a_k^\dagger - \eta_k^* a_k \right)] = \exp \left( -\sum_k |\eta_k|^2 \coth \frac{\beta \omega_k}{2} \right), \] (16)

we obtain

\[ U_+ U_+ = \exp \left( \sum_k \left[ \xi_k(t) a_k^\dagger - \xi_k^*(t) a_k \right] \right), \]
\[ \Gamma_\pm(t) = \sum_k g_k^2 \coth \frac{\beta \omega_k}{2} \frac{1 - \cos \omega_k t}{\omega_k^2}. \] (17)

For the spin-bath model under the uncorrelated initial condition, we use Eqs. (12), (13), and (14) to get

\[ \rho_S^\pm(t) = e^{-i\omega t} e^{-\Gamma_\pm(t)} \rho_S^\pm(0), \]
\[ \Gamma_\pm(t) = -\ln \left( \text{Tr}_B[\rho_B \prod_k (U_k^\dagger U_k^+)] \right) \]

\[ = -\sum_k \ln \left( 1 - 2 g_k^2 \frac{\omega_k t}{\Omega_k} \sin \frac{\Omega_k t}{2} \right). \] (18)

Eqs. (17) and (18) are the exact results for the decoherence functions under the uncorrelated initial conditions in the pure dephasing models. Unlike the boson-bath model, the decoherence function for the spin-bath model is independent of temperature and can lead to complete dephasing when \( 1 - 2(g_k^2/\Omega_k^2) \sin^2(\Omega_k t/2) = 0 \) at certain time if one of the coupling strength \( g_k \geq \omega_k \). This is quite different from Eq. (17) in the boson-bath model, which does not have complete dephasing even if all the couplings satisfy \( g_k \geq \omega_k \). If the size of bath is small, the coherence can revive to a large amount at certain time. Moreover, due to the renormalized spin-bath mode frequency \( \Omega_k > \omega_k \), it induces faster dynamics than boson-bath at zero temperature despite the same bath coupling spectrum.

In the short time case \( t \to 0 \), we see that both Eqs. (17) and (18) perform the quadratic behavior, namely \( \Gamma(t) \propto t^2 \). On the other hand, in the weak coupling limit \( g_k \ll \omega_k \) and \( \Omega_k \approx \omega_k \), we can see Eq. (18) approaches to (17) at zero temperature since \( -\ln(1 - x) \approx x \) for \( x \ll 1 \). It is known that for the boson-bath model the non-Markovian master equation up to the second order perturbation can give the exact result for the decoherence function even in the strong coupling limit because the expansion of the evolution unitary operator can be truncated at the second term, which is not legitimate for the spin-bath model. The quantum master equation in the second-order approximation is

\[ \frac{d\rho_S(t)}{dt} = -\int_0^t ds \text{Tr}_B[\mathcal{H}(t), \mathcal{H}(s), \rho_S(s) \otimes \rho_B]]. \] (19)

For the pure dephasing model, it leads to

\[ \frac{d\rho_S^\pm(t)}{dt} = -\int_0^t ds \mathcal{K}(t, s) g_S^\pm(t), \]

where the kernel for the boson bath is given by

\[ \mathcal{K}(t, s) = \sum_k g_k^2 \cos[\omega_k(t - s)] \coth \frac{\beta \omega_k}{2}. \] (21)

The solution of Eq. (20) gives the decoherence function \( \Gamma_{\sigma}^{ME}(t) \) of the form (17), which is valid as long as the second order analysis is legitimate, i.e. \( g_k/\omega_k \leq 1 \). On the other hand, in the spin-bath model the kernel is

\[ \mathcal{K}(t, s) = \sum_k g_k^2 \cos[\omega_k(t - s)]. \] (22)

The solution of Eq. (20) is then

\[ \Gamma_{\sigma}^{ME}(t) = \sum_k g_k^2 \frac{\omega_k}{\omega_k}(1 - \cos \omega_k t), \] (23)

which is also independent of temperature and approaches to (18) only when \( g_k/\omega_k \ll 1 \). Therefore, the condition of the validity of the second order master equation for
the spin-bath is more stringent than that of the boson-bath. We also note that coherence dynamics described by Eqs. (17) and (13) in both models are non-divisible and non-Markovian according to definition in (16).

Next, we analyze the effect of the dynamical decoupling pulses along \( x \) direction at intervals \( \tau \) under the uncorrelated initial condition [12 13]. The simplest bang-bang pulses that can significantly reduce the dephasing rate is realized by frequent \( \pi \) pulses along \( x \) direction applied on the qubit. That is after such a pulse, the qubit states change as \( |\pm \rangle \to |\mp \rangle \) in the Schrödinger picture. In the presence of the decoupling pulses, at time \( t = 2N\tau \), the evolution unitary operator becomes

\[
U(t, 0) = U_\mp |+\rangle \langle +| + U_- |\rangle \langle -|,
\]

where

\[
U_k = e^{i\omega_k t/2}U_\mp (t, t - \tau) \cdots U_\mp (2\pi, \tau)U_\pm (\tau, 0),
\]

and the phase factors \( e^{\pm i\omega_k t/2} \) appear due to the state transformation from the Schödinger picture to the interaction picture. Substituting Eqs. (8) into (24) and neglecting state-independent global phase factors that are irrelevant to the density matrix, we can treat the factors in Eq. (24) as commuting operators and get

\[
U_\mp = e^{\pm i\omega_k t/2} \exp \left\{ \pm \frac{1}{2} \sum_k \left[ \eta_k (\tau) a_k^\dagger - \eta_k^\dagger (\tau) a_k \right] \right\},
\]

where

\[
\eta_k (\tau) = \xi_k (\tau) (1 - e^{i\omega_k \tau}) \sum_{n=1}^N e^{2i(n-1)\omega_k \tau}.
\]

Repeat the steps to derive Eq. (17), we have

\[
\rho_S^\pm (t) = e^{-\Gamma_S^\pm (t)} \rho_S^\mp (0), \quad \Gamma_S^\pm (t) = -\sum_k \ln (1 - 8F_k^2),
\]

\[
F_k = \frac{\sin N\phi_k}{\sin \phi_k} \frac{g_k \omega_k}{\Omega_k} \sin^2 \frac{\Omega_k \tau}{2},
\]

where \( \phi_k = \cos^{-1} x_k \). Under the periodic pulse control, the expression for \( \Gamma_S^\pm (t) \) is still temperature-independent and approaches to \( \Gamma_S^\pm (t) \) in (28) at zero temperature as in the weak coupling limit \( g_k/\omega_k \ll 1 \) and \( \phi_k \approx \omega_k \tau \), which can also be obtained from the second order analysis.

IV. DEPHASING WITH INITIAL CORRELATIONS

One natural way to implement the initial correlation between the qubit and bath is to use the positive operator-value measurement (POVM) \( \mathbb{M} E_m \) acting only on the qubit, where the whole system is in the thermal equilibrium state \( \rho = e^{-\beta H}/\text{Tr}[e^{-\beta H}] \), then after the action of POVM, the whole system becomes to be

\[
\rho(0) = \frac{1}{Z} \sum_m E_m e^{-\beta H} E_m^\dagger,
\]

where the factor \( Z \) is the normalization of \( \rho(0) \),

\[
Z = \sum_{\pm} \text{Tr}_B [e^{-\beta Hz}],
\]

\[
H^z = \frac{\omega_0}{2} + \sum_k \omega_k a_k^\dagger a_k \pm \frac{1}{2} \sum_k g_k (a_k^\dagger + a_k),
\]
where the notation $u_{\pm} \equiv \sum_m (\pm |E_m|^2 E_m| \pm)$, and the fact that the total Hamiltonian is diagonal in the basis of $S_z$ has been used.

In such way we prepare an initial state by measurement, instead of the supposed uncorrelated state Eq. (13). In contrast to the form of (13), the initial density matrix (38) is expressed in terms of the total Hamiltonian $H$ and takes into account the initial qubit-bath correlation through the interaction term $H_{\text{int}}$ of $H$. Consequently, the bath is no longer in thermal equilibrium initially, and its initial state becomes

$$
\rho_B(0) = \text{Tr}_S[\rho(0)] = \frac{1}{Z} \sum_m \text{Tr}_S[E_m e^{-\beta H} E_m^\dagger]
= \frac{1}{Z} \sum_{\pm} \rho_{\pm} e^{-\beta H_{\pm}}.
$$

(38)

Eq. (38) is quite different from the thermal state $\rho_B = e^{-\beta H_B}/Z_B$ which does not contain the interaction terms. Even the resulting state sometimes takes the initial product form $\rho_S \otimes \rho_B(\rho_S)$ where $\rho_B$ depends on $\rho_S$, the dynamical map would be very different from the situation of $\rho_B$ and $\rho_S$ being purely independent. For the purely independent case, the map from $\rho_S(0)$ to $\rho_S(t)$ is linear, whereas the map for the former case is non-linear. Therefore, in the following we will not distinguish the usual classification of quantum and classical correlations. For example in (17) it was shown that even the classical correlation to entangle oscillation. So the product states $\rho_S \otimes \rho_B(\rho_S)$ are also correlated in our general sense.

Now we calculate the evolution of coherence $\rho_S^+(t)$ with the initial correlation introduced by Eq. (38). Using Eq. (4), we first write

$$
g_S^+(t) = \text{Tr}[\sigma \cdot \dot{\sigma}(t)]
= \frac{1}{Z} \sum_m \text{Tr}_B \text{Tr}_S[\sigma \cdot U(t) E_m e^{-\beta H} E_m^\dagger U(t)]
= \frac{1}{Z} \sum_{m, \pm} \langle \pm | E_m^\dagger \sigma_0 E_m | \pm \rangle \text{Tr}_B[U_1 U_0 e^{-\beta H_{\pm}}]
= \frac{1}{Z} \sum_{\pm} w_{\pm} \text{Tr}_B[U_1 U_0 e^{-\beta H_{\pm}}],
$$

(39)

where the factor $w_{\pm} = \sum_m \langle \pm | E_m^\dagger \sigma_0 E_m | \pm \rangle$. As $\text{Tr}_B[U_1 U_0 e^{-\beta H_B}]$ appeared in Eq. (15), we need to find the expression for $\text{Tr}_B[U_1 U_0 e^{-\beta H_{\pm}}]$. Note the identity

$$
\frac{1}{Z_B} \text{Tr}_B[e^{-\beta H_{\pm}} e^{\eta_k \sigma_4 - \bar{\eta}_k \sigma_3}] = \text{Tr}_B[\rho_B e^{\eta_k \sigma_4 - \bar{\eta}_k \sigma_3}]
\times \exp \left[ \beta \sum_k \frac{g_k^2}{4\omega_k} \pm \frac{\beta \omega_k}{2} \pm i\Theta \right],
$$

(40)

where a unitary transformation that diagnoses $H_{\pm}$ was performed inside the trace function, and the phase is

$$
\Theta = \sum_k \frac{g_k}{2\omega_k} (\eta_k^* - \eta_k).
$$

(41)

Using Eqs. (16) and (17), we get the final result for the coherence

$$
\rho_S^+(t) = \rho_S^+(0) e^{-\Gamma_s(t)} \sum_{\pm} w_{\pm} e^{\frac{\beta \omega_k}{2} \pm \pm i\Theta_k(t)}
= \rho_S^+(0) e^{-\Gamma_s(t)} [\cos \Theta_k(t) + i W \sin \Theta_k(t)],
$$

(42)

where the decoherence function $\Gamma_s(t)$ has the same form of (17). The phase function is given by

$$
\Theta_k(t) = \sum_k g_k^2 \sin \omega_k t \frac{\omega_k}{w_k^2},
$$

(43)

and the weight factor is

$$
W = \frac{w_{+} e^{-\beta \omega_0/2} - w_{-} e^{\beta \omega_0/2}}{w_{+} e^{-\beta \omega_0/2} + w_{-} e^{\beta \omega_0/2}},
$$

(44)

which take into account the memory effect of bath and the role of initial correlation. Here the initial coherence is given by

$$
\rho_S^+(0) = \frac{w_{+} e^{-\beta \omega_0/2} + w_{-} e^{\beta \omega_0/2}}{w_{+} e^{-\beta \omega_0/2} + w_{-} e^{\beta \omega_0/2}}.
$$

(45)

Similar results have been obtained in (10).

For the spin-bath model, we get

$$
\rho_S^+(t) = \rho_S^+(0) e^{-\Gamma_s(t)} [\cos \Theta_s(t) + i W \sin \Theta_s(t)]
\times \prod_k \sqrt{1 + \eta_k(t)^2},
$$

(46)

where

$$
\Theta_s(t) = \sum_k \tan^{-1} \eta_k(t),
$$

$$
\eta_k(t) = \frac{g_k^2}{2\beta \omega_k} \tanh \frac{\omega_k \sin \Theta_k t}{1 - 2g_k^2 \sin^2 \Theta_k t}. \quad (47)
$$

The memory effect of bath is taken into account by the temperature dependent function $\eta_k(t)$, which also makes the decoherence function temperature independent. For convenience, we express Eqs. (12) and (16) in a compact form,

$$
\rho_S^+(t) = \rho_S^+(0) e^{-\Gamma_s(t)} [\cos \Theta(t) + i W \sin \Theta(t)]
\times \prod_k \left[1 + \eta_k(t)^2\right]^{1/2},
$$

(48)

where the exponent $s = 0$ for the boson bath and $s = 1$ for the spin bath. At $T = 0$ and $g_k \to 0$,
where the weight factor is $W_c = -\tanh \beta \omega_0 < 1$. The result for the uncorrelated initial condition is obtained by putting $\Theta = \eta_k = 0$ in the above equations. If we replace the initial state by $|\Phi\rangle = (|++\rangle + |--\rangle)/\sqrt{2}$, the effect of initial correlation would cancel out in the phase function, namely $\Theta = 0$ in the entanglement measure.

The dynamical decoupling effect of the periodic pulses can be derived as in the previous section. The final results have the same forms of (42) and (46) with the replacements $\Gamma \rightarrow \Gamma^\pi$, $\Theta_a \rightarrow \Theta_a^\pi$, and $\eta_k \rightarrow \eta_k^\pi$, where

$$\Theta_a^\pi(t) = \sum_k g_k^2 \tan \frac{\omega_k \tau}{2} \frac{1 + \cos \omega_k t}{\omega_k^2},$$

$$\eta_k^\pi(t) = \frac{2 g_k^2 \tanh \frac{\omega_k \tau}{2} \tan \frac{\Omega_k}{2} \sin^2 N\phi_k}{\left(1 + \frac{g_k^2}{\Gamma_k} \tan^2 \Theta_a^\pi \cos 2N\phi_k\right)}.$$

Obviously, in the limit of the pulse interval $\tau \rightarrow 0$, we have $\Gamma^\pi \rightarrow 0$, $\Theta_a^\pi \rightarrow 0$ and $\eta_k^\pi \rightarrow 0$, which means that the decoherence are inhibited significantly. For the finite pulse interval, the behaviors of the two models are quite different as discussed below.

V. NUMERICAL RESULT

Because the decoherence function for the boson bath linearly depends on the coupling strength $g_k^2$, we can treat the bath within the classical noise description [13, 19]. By introducing the spectral density of the boson-bath, $J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k)$, we can express the following results:

$$\delta(\omega - \omega_k) = g^2 \int_0^\infty d\omega N(\omega),$$

where the second line holds for the all equal couplings with the boson bath, and the number distribution of bath mode is denoted by $N(\omega)$. Then the summation over the bath modes in the decoherence functions can be transformed into integral. For example, Eq. (17) becomes

$$\Gamma_a(t) = \int d\omega J(\omega) \coth \frac{\beta \omega_1}{2} \frac{1 - \cos \omega t}{\omega},$$

$$\rightarrow g^2 \int d\omega N(\omega) \coth \frac{\beta \omega_1}{2} \frac{1 - \cos \omega t}{\omega}.$$
The solution of coherence function in order to avoid the possible increases \[20\]. Hence we only concern the beginning evolution, which can give almost complete decoherence as the size of spin bath increases by randomly choosing coupling constants \(g_k\). This difficulty can be avoided by keeping the coupling constants \(g_k\) fixed, which can give almost complete decoherence as the size of spin bath increases \[20\]. Hence we only concern the beginning evolution of coherence function in order to avoid the possible divergence problems for spin bath in the following.

For the numerical analysis, we choose the operator \(E_m\) to be

\[
E_m = I + \frac{1}{2}(\sigma_x + \sigma_z),
\]

where the renormalized frequency is \(\Omega = \sqrt{\omega^2 + g^2}\). It needs to point out that the above recipe only works well at the beginning of the evolution for the spin bath model due to the possible zeroes of the argument such as in Eq. \[57\]. It can give large coherence revival at large time even with a huge size bath which is just an artifact of our particular assumption. This difficulty can be avoided by randomly choosing coupling constants \(g_k\), which can give almost complete decoherence as the size of spin bath increases \[20\]. Hence we only concern the beginning evolution of coherence function in order to avoid the possible divergence problems for spin bath in the following.

For the numerical analysis, we choose the operator \(E_m\) to be

\[
E_m = I + \frac{1}{2}(\sigma_x + \sigma_z),
\]

and the constants \(u_\pm\) and \(w_\pm\) are thus given by

\[
u_\pm = \frac{3}{2} \pm 1, \quad w_\pm = \frac{1}{2} \pm \frac{1}{4},
\]

Suppose the coupling constant \(g = 0.02\), the temperature \(\beta = 2\), and the energy \(\omega_0 = 0.1\). We also set the number distribution of bath mode be the Ohmic spectrum with a high frequency cut-off, \(N(\omega) = \lambda N_0 e^{-\omega/\Lambda}\) with the cut-off \(\Lambda = 5\) and \(N_0 = 2.5 \times 10^4\), where the weak & strong bath corresponds to \(\lambda = 1\) & 10.

In Fig. \ref{fig:1} we show the free evolution of the coherence in the bath with and without initial correlation. It can be seen that the initial correlation can lead to damped coherence oscillation for the strong bath, and reduce the coherence to some extent for the weak bath. We also need to point out that the above recipe only works well for the weak (strong) boson & spin bath. (b) The evolution of the trace distance of two states in the Ohmic bath spectral density. The coherence in the weak (thin lines) and correlated (thick lines) initial conditions with \(g_k\). For example, Eq. \(18\) can thus be written as

\[
\sigma(\tau) = \int d\omega N(\omega) \ln \left(1 - 2 g^2 \frac{\Omega t}{\Omega^2} \sin^2 \frac{\Omega t}{2}\right), \quad (57)
\]

where the renormalized frequency is \(\Omega = \sqrt{\omega^2 + g^2}\). It needs to point out that the above recipe only works well at the beginning of the evolution for the spin bath model due to the possible zeroes of the argument such as in Eq. \[57\]. It can give large coherence revival at large time even with a huge size bath which is just an artifact of our particular assumption. This difficulty can be avoided by randomly choosing coupling constants \(g_k\), which can give almost complete decoherence as the size of spin bath increases \[20\]. Hence we only concern the beginning evolution of coherence function in order to avoid the possible divergence problems for spin bath in the following.

For the numerical analysis, we choose the operator \(E_m\) to be

\[
E_m = I + \frac{1}{2}(\sigma_x + \sigma_z),
\]

and the constants \(u_\pm\) and \(w_\pm\) are thus given by

\[
u_\pm = \frac{3}{2} \pm 1, \quad w_\pm = \frac{1}{2} \pm \frac{1}{4},
\]

Suppose the coupling constant \(g = 0.02\), the temperature \(\beta = 2\), and the energy \(\omega_0 = 0.1\). We also set the number distribution of bath mode be the Ohmic spectrum with a high frequency cut-off, \(N(\omega) = \lambda N_0 e^{-\omega/\Lambda}\) with the cut-off \(\Lambda = 5\) and \(N_0 = 2.5 \times 10^4\), where the weak & strong bath corresponds to \(\lambda = 1\) & 10.
see that for the short time, the boson bath dephases the qubit to more degree than the spin bath with the same initial condition.

The evolution of the trace distance $D(\rho_S(t), \tilde{\rho}_S(t))$ is plotted in Fig. 2 (a), which can take nonzero value at $t > 0$ witnessing the initial quantum correlation. Fig. 2 (b) shows that the entanglement measure, i.e. concurrence $C$, can revive to a large amount in contrast to the uncorrelated initial condition for the strong bath, which only appears as the two qubits are coupled to a common strong bath without initial correlation [22]. It can be seen that both quantities can perform damped oscillations for the strong baths. The reason for the entanglement oscillation with initial correlation is that the preparation scheme on the two qubits induces some entanglement between the two independent baths at the beginning, which can return into qubits at later time via the non-Markovianity of the evolution [11]. Fig. 2 (b) also shows that the entanglement without initial correlation always larger than the envelop of the entanglement with initial correlation.

Fig. 3 (a) shows the dependence of the coherence at time $t = 2$ in the strong baths on the pulses applied at a rate $\tau = 1/N$, from which we see that the initial correlation always reduces the efficiency of the control pulses relative to the uncorrelated case. Fig. 3 (b) shows the evolution of the relative change of the coherence $\delta = |\rho_{S+}^g - \rho_{S-}^g| / \rho_{S+}^g$ under pulse control applied at a rate $\tau = 0.15$ with respect to its free evolution in the weak baths. It indicates that the $\pi$ pulse control suppresses the decoherence for the boson bath, while the same control can only suppress the decoherence for the spin bath in the short time duration, and then crossovers to the enhancement of the decoherence as time becomes longer. Moreover, the decoherence is more strongly suppressed in the boson bath than in the spin bath.

In Fig. 4, we plot the crossover of the qubit coherence from the suppression to the enhancement under the bang-bang control pulses as a function of the pulse interval ($\tau$) versus the pulse number ($N$). It is shown that the initial correlation affects the crossover of decoherence to some extent only at the beginning of evolution, and induces relatively more changes for the boson (weak) bath than for the spin (strong) bath. The underlying reason is that the bath’s memory of the initial correlation would be lost as time being longer. On the other hand, the higher temperature for the boson bath, the more efficient for the control pulses to make the crossover occur. As shown in Fig. 4, the crossovers for the correlated and uncorrelated initial conditions in the limit of $t \rightarrow \infty$ are essentially coincident with each other. Asymptotically for the boson bath, note that

$$\Gamma_s(t) = \frac{2\omega^2}{\omega^2 t} \frac{\sin^2 \frac{\omega t}{2}}{\omega^2 t} \left( \frac{\omega}{\omega^2 t} \right) \left( \frac{\omega}{\omega^2 t} \right)$$

and

$$\frac{\Gamma_s(t)}{t} = 2g^2 \int_0^\infty d\omega N(\omega) \coth \frac{\beta \omega}{2} \tan^2 \frac{\omega t}{2\omega^2 t} \frac{\sin^2 \frac{\omega t}{2}}{\omega^2 t}$$

$$\rightarrow \frac{\pi}{2} \kappa_\beta(0) \text{ when } t \rightarrow \infty,$$

(60)

$$\lim_{t \rightarrow \infty} \left( \frac{\sin^2 \frac{\omega t}{2}}{\omega^2 t} \right) = 2\pi \delta(\omega),$$

$$\lim_{t \rightarrow \infty} \left( \frac{\sin^2 \frac{\omega t}{2}}{\omega^2 t} \tan^2 \frac{\omega t}{2} \right) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \times \left[ \delta \left( \omega + \frac{\pi}{2} (2n+1) \right) + \delta \left( \omega - \frac{\pi}{2} (2n+1) \right) \right].$$

Here the pulse interval $\tau$ is fixed and $t = 2N\tau$ while the pulse number $N \rightarrow \infty$. The crossover at distant time between the two regimes takes place at $\tau = \tau^*$ where $\tau^*$ is determined by the equation,

$$\frac{4}{\pi} \kappa_\beta \left( \frac{\pi}{2} \right) = \frac{2}{\pi} \kappa_\beta(0),$$

(62)

the solution of which is given by $\tau^* = 0$ at zero temperature and $\tau^* \approx 0.289$ at $\beta = 2$ same as the numerical results shown in Fig. 4. For the spin bath, the crossover under control shows the temperature insensitivity instead.

VI. CONCLUSION

We addressed the effects of initial correlation on the dynamics of open system in the pure dephasing models with the boson and spin baths. We found that the initial correlation can reveal the memory of bath during the time evolution. The decoherence of a qubit coupled to a boson bath is more enhanced with respect to a spin bath in the short time. We also demonstrated that the trace distance between two states of qubit can increase above its initial value witnessing the initial non-classical correlation between the qubit and bath, and that the entanglement of two qubits, locally interacting with an independent bath, can damply oscillate and revive to a large amount compared to the uncorrelated initial condition. We finally showed that the initial correlation affects the crossover of decoherence from the dynamical enhancement to suppression under the bang-bang control pulses to some extent only at the beginning of evolution, and induces more changes for the boson (weak) bath than for the spin (strong) bath. On the other hand, the higher temperature for the boson bath, the more efficient for the control pulses to make the crossover occur. For the spin bath, the crossover under control shows the temperature insensitivity instead.
Acknowledgments

The authors would like to acknowledge support from NSFC grand No. 11147137.