GROWTH OF NONSYMMETRIC OPERADS

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Abstract. The paper concerns the Gelfand-Kirillov dimension and the generating series of nonsymmetric operads. An analogue of Bergman’s gap theorem is proved, namely, no finitely generated locally finite nonsymmetric operad has GK-dimension strictly between 1 and 2. For every \( r \in \{0\} \cup \{1\} \cup [2, \infty) \) or \( r = \infty \), we construct a single-element generated nonsymmetric operad \( P \) such that \( \text{GKdim}(P) = r \). We also provide counterexamples to two expectations of Khoroshkin and Piontkovski about the generating series of operads.

0. Introduction

Let \( \mathbb{F} \) be a base field. An algebra stands for a unital associative algebra over \( \mathbb{F} \) unless otherwise stated. The Gelfand-Kirillov dimension (GK-dimension for short) of an algebra \( A \) is defined to be

\[
\text{GKdim}(A) := \sup_V \left\{ \limsup_{n \to \infty} \log_n \left[ \dim(\sum_{i=0}^{n} V^i) \right] \right\}
\]

where the supremum is taken over all finite dimensional subspaces \( V \) of \( A \). Similarly, one can define the GK-dimension of nonassociative algebras. GK-dimension is a standard and powerful invariant for investigating associative and nonassociative algebras. We refer to [KL00] for more background and properties of the GK-dimension of algebras and modules.

The range of possible values for the GK-dimension of an algebra is

\[
\mathbb{R}_{\text{GKdim}} := \{0\} \cup \{1\} \cup [2, \infty) \cup \{\infty\}.
\]

The gap between 0 and 1 follows easily from the definition of GK-dimension. The existence problem of algebras \( A \) with \( 1 < \text{GKdim}(A) < 2 \) was open for some years until Bergman [KL00, Theorem 2.5] proved that no such algebras exist. Bergman’s gap theorem is also valid for some other classes of algebras, for example, Jordan algebras [MZ96] and dialgebras [ZCY20]. However, there exist Lie algebras [Pet97] and Jordan superalgebras [PS19] with GK-dimension strictly between 1 and 2.

The notion of an operad was first introduced by Boardman-Vogt [BV73] and May [May72] in late 1960s and early 1970s in the study of iterated loop spaces. Since 1990s, due to Ginzburg-Kapranov’s Koszul duality theory of operads [GK94], Kontsevich’s [Kont93] and Tamarkin’s [Tam08] operadic approach to the formality theorem, as well as Getzler’s study on topological field theories [Get94, Get95], operad theory has become an important tool in homological algebra, category theory, algebraic geometry and mathematical physics.

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In this paper, we investigate the GK-dimension and the generating series of nonsymmetric operads. The GK-dimension of locally finite operads was defined in [KP15, p. 400] and [BYZ20, Definition 4.1]. Let $\mathcal{P}$ be a locally finite operad, i.e., an operad $\mathcal{P}$ with each $\mathcal{P}(i)$ finite dimensional over the base field $F$. The GK-dimension of $\mathcal{P}$ is defined to be

$$\text{GKdim}(\mathcal{P}) := \limsup_{n \to \infty} \log_n \left( \sum_{i=0}^{n} \dim F \mathcal{P}(i) \right).$$

Our main result is an analogue of Bergman’s gap theorem:

**Theorem 0.1.** No finitely generated locally finite nonsymmetric operad has GK-dimension strictly between 1 and 2.

For the definition of a finitely generated operad, see Definition 1.16. Bergman’s gap theorem for associative algebras was proved by counting specific words that satisfy a set of conditions and form a monomial basis of the algebra under consideration, equivalently, by counting a set of single-branched tree monomials (defined in Section 4) where the algebra is interpreted as an operad. This method cannot be extended directly to prove the gap theorem for nonsymmetric operads since the tree monomials we want to count are not necessarily single-branched. To overcome the above difficulty, we divide the underlying tree into three subtrees such that the “big” subtree is single-branched and thus we can count its tree monomials similarly as for the case of associative algebras. The other two subtrees are both “small” such that their tree monomials are well controlled.

Note that if we drop the condition “finitely generated” in Theorem 0.1, then the statement does not hold. In fact, as Example 3.4 shows, any positive real number is the GK-dimension of some nonsymmetric operad.

Theorem 0.1 serves as an essential ingredient of the next result. Recall that $R_{\text{GKdim}}$ is defined in (E0.0.1).

**Theorem 0.2.**

1. If $\mathcal{P}$ is a finitely generated locally finite nonsymmetric operad, then $\text{GKdim}(\mathcal{P}) \in R_{\text{GKdim}}$.

2. If $r \in R_{\text{GKdim}}$, then there is a single-element generated single-branched locally finite nonsymmetric operad $\mathcal{P}$ such that $\text{GKdim}(\mathcal{P}) = r$.

The Gelfand-Kirillov dimension of an operad is closely related to the generating series that is defined as follows. Let $\mathcal{P}$ be a locally finite operad. The generating series of $\mathcal{P}$ is defined to be the formal power series [KP15 (0.1.2)]

$$G_{\mathcal{P}}(z) := \sum_{n=0}^{\infty} \dim F \mathcal{P}(n) z^n.$$

We recall the following definition.

**Definition 0.3.** Let $F(z) := \sum_{n \geq 0} f(n)z^n$ be a formal power series or a $C^{\infty}$-function where $f(n) \in \mathbb{R}$ for all $n$.

1. $F(z)$ is called holonomic (also called D-finite or differentiably finite) if it satisfies a nontrivial linear differential equation with polynomial coefficients.

2. $F(z)$ is called differential algebraic if it satisfies a nontrivial algebraic differential equation with polynomial coefficients.
It is well-known that
\[
\text{rational } \implies \text{algebraic (over } \mathbb{R}(z)) \implies \text{holonomic } \implies \text{differential algebraic}
\]
where the second implication is \cite[Theorem 2.1]{Sta80}. Several researchers have recently been studying holonomic and differential algebraic property of \( \mathcal{G}_{\mathcal{P}}(z) \) \cite{Ber14, KP15}. In particular, Khoroshkin-Piontkovski showed that, under moderate assumptions, operads with a finite Gröbner basis have rational, or algebraic, or differential algebraic generating series \cite[Theorems 0.1.3, 0.1.4, and 0.1.5]{KP15}. In \cite[Section 4]{KP15} Khoroshkin-Piontkovski listed several expectations and conjectures, one of which is

**Expectation 0.4.** \cite[Expectation 2]{KP15} The generating series of a generic finitely presented nonsymmetric operads is algebraic over \( \mathbb{Z}[z] \).

We construct a finitely presented nonsymmetric operad such that the generating series is not holonomic (hence, not algebraic), which provides a (non-generic) counterexample to the above Expectation \cite[Example 6.4]{KP15}. We also prove

**Proposition 0.5.** Let \( r \in R_{\mathcal{GKdim}} \setminus \{0\} \). Then there is a single-element generated locally finite nonsymmetric operad \( \mathcal{P} \) with \( \mathcal{GKdim}(\mathcal{P}) = r \) and \( \mathcal{G}_{\mathcal{P}}(z) \) not being holonomic. As a consequence, \( \mathcal{G}_{\mathcal{P}}(z) \) is neither rational nor algebraic in this case. Therefore such an operad is a counterexample to \cite[Expectation 3]{KP15}.

**Remark 0.6.** Constructions \cite[1.3, 5.1, and 7.1]{KP15} provide useful constructions of nonsymmetric operads (or symmetric ones in Construction 7.1) from graded algebras (or monomial algebras in Construction 5.1). A lot of algebraic properties of graded (or monomial) algebras can be transformed to the corresponding properties of the related operads. Using this idea, in addition to Proposition 0.5 and Example 6.4, we obtain a potential counterexample to \cite[Expectation 1]{KP15} in Example 7.5.

This paper mainly concerns nonsymmetric operads. In the final section we touch upon the symmetric ones. The following is easy to prove.

**Theorem 0.7.**

1. Let \( \mathcal{P} \) be a finitely generated locally finite symmetric operad. Then \( \mathcal{GKdim}(\mathcal{P}) \in R_{\mathcal{GKdim}} \cup (1, 2) \).
2. For every \( r \in R_{\mathcal{GKdim}} \setminus (2, 3) \), there exists a finitely generated locally finite symmetric operad \( \mathcal{P} \) such that \( \mathcal{GKdim}(\mathcal{P}) = r \).
3. For every \( r \in R_{\mathcal{GKdim}} \setminus (\{0\} \cup \{1\} \cup (2, 3)) \), there exists a finitely generated locally finite symmetric operad \( \mathcal{P} \) such that \( \mathcal{GKdim}(\mathcal{P}) = r \) and that \( \mathcal{G}_{\mathcal{P}}(z) \) is not holonomic.

In light of \cite[Ber14], it would be very interesting to determine which classes of operads have rational (resp. algebraic, holonomic, differential algebraic) generating series. Theorem 0.7 suggests that, generically, \( \mathcal{G}_{\mathcal{P}}(z) \) is not holonomic.

**Question 0.8.** Let \( r \in (1, 2) \cup (2, 3) \). Is there a finitely generated locally finite symmetric operad \( \mathcal{P} \) such that \( \mathcal{GKdim}(\mathcal{P}) = r \)?

In view of Theorem 0.7(2,3) the following result of \cite{BYZ20} is quite surprising, see \cite{BYZ20} for details.

**Theorem 0.9.** Let \( \mathcal{P} \) be a 2-unitary locally finite symmetric operad. Then \( \mathcal{GKdim}(\mathcal{P}) \) is either an integer or the infinity. If \( \mathcal{GKdim}(\mathcal{P}) \) is finite, then \( \mathcal{P} \) is finitely generated and \( \mathcal{G}_{\mathcal{P}}(z) \) is rational.
This paper is organized as follows. Section 1 recalls basic definitions and properties of nonsymmetric operads. Section 2 introduces Gröbner-Shirshov bases of nonsymmetric operads. Section 3 contains definitions, examples and properties of the GK-dimension of nonsymmetric operads. In Section 4, we prove the main result, Theorem 0.1, and in Section 5, we prove Theorem 0.2. In Section 6, we study the generating series of operads and prove Proposition 0.3. Finally, Section 7 provides some comments and examples about symmetric operads.

1. Preliminaries

Let $\mathbb{F}$ be the base field and $\mathbb{F}^* := \mathbb{F} \setminus \{0\}$. Let $\mathbb{N}$ denote the natural numbers and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. To save space, some non-essential details are omitted here and there. But we try to provide as much detail as possible for the proof of the main result – Theorem 0.1.

1.1. Ns operads. The following definition is copied from [BD16, Definition 3.2.2.3] (see also [LV12, Chapter 5]).

**Definition 1.1.** A nonsymmetric operad or simply ns operad is a collection of vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ equipped with an element $\text{id} \in \mathcal{P}(1)$ (called the identity element) and maps (called partial compositions), for $1 \leq i \leq n$,

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1), \quad \alpha \otimes \beta \mapsto \alpha \circ_i \beta,$$

which satisfy the following properties for all $\alpha \in \mathcal{P}(n)$, $\beta \in \mathcal{P}(m)$ and $\gamma \in \mathcal{P}(r)$:

(i) (sequential axiom)

$$(E1.1.1) \quad (\alpha \circ_i \beta) \circ_j \gamma = \alpha \circ_i (\beta \circ_{j-1} \gamma) \quad \text{for } i \leq j \leq i + m - 1;$$

(ii) (parallel axiom)

$$(E1.1.2) \quad (\alpha \circ_i \beta) \circ_j \gamma = \begin{cases} (\alpha \circ_{j-m+1} \gamma) \circ_i \beta, & i + m \leq j \leq n + m - 1, \\ (\alpha \circ_j \gamma) \circ_{i+r-1} \beta, & 1 \leq j \leq i - 1; \end{cases}$$

(iii) (unit axiom)

$$(E1.1.3) \quad \text{id} \circ_1 \alpha = \alpha, \quad \alpha \circ_i \text{id} = \alpha \quad \text{for } 1 \leq i \leq n.$$
Example 1.2. [LV12 p.137] A unital associative algebra $A$ can be interpreted as an operad $P$ with $P(1) = A$ and $P(n) = 0$ for all $n \neq 1$, and the compositions in $P$ are given by the multiplication of $A$.

The following construction is due to Dotsenko [Dot19].

Construction 1.3. [Dot19 Definition 3.2(2)]. Let $A := \oplus_{i \geq 0} A_i$ be an $N$-graded algebra with unit $1_A$. Suppose that $A$ has a graded augmentation $\epsilon : A \to \mathbb{F}$ such that $m := \ker \epsilon$ is a maximal graded ideal of $A$. Let $P_A(0) = 0$ and $P_A(n) = A_{n-1}$ for all $n \geq 1$. Define compositions as follows

$$\circ_i : P_A(m) \otimes P_A(n) \to P_A(n + m - 1),$$

$$a_{m-1} \otimes a_{n-1} \mapsto\begin{cases} c a_{m-1} & a_{n-1} = c 1_A, c \in \mathbb{F}, \\ a_{m-1} a_{n-1} & a_{n-1} \in m, i = 1, \\ 0 & a_{n-1} \not\in m, i \neq 1. \end{cases}$$

It is easy to check by definition that $P_A := \{P_A(n)\}_{n \geq 0}$ is an operad with id := 1$_A$. This operad is called the min-envelope operad of $A$. Piontkovski [Pio17 Theorem 3.1] used this construction to produce a counterexample to [BD16, Conjecture 10.4.1.1]. See [Dot19 Definition 3.2(1)] for another related construction.

If $A$ is generated by $A_1$, then $P_A$ is generated by $P_A(2)$. By Lemma [6.3] $A$ is finitely generated as an algebra if and only if $P_A$ is finitely generated as an operad.

For an $N$-graded locally finite algebra $A$, its Hilbert series is defined to be

$$(E1.3.1) \quad H_A(z) = \sum_{n=0}^{\infty} \dim A_n z^n$$

in the same way of defining the generating series of an operad.

By the construction given in Construction 1.3 one sees that

(E1.3.2) \quad $G_{P_A}(z) = z H_A(z).$

A morphism $\phi$ between two operads $P$ and $P'$ is a collection of linear maps $\phi_n : P(n) \to P'(n)$, $n \geq 0$, such that $\phi_1(id_P) = id_{P'}$ and

$$\phi_m(u) \circ_i \phi_n(v) = \phi_{m+n-1}(u \circ_i v)$$

for all $u \in P(m)$, $v \in P(n)$, $1 \leq i \leq m$, $n \geq 0$. If each $P(n)$ is a subspace of $P'(n)$, we call $P$ a subcollection of $P'$, and write $P \subseteq P'$.

An operation alphabet (or generating operations) is a collection $X = \{X(n)\}_{n \geq 0}$ of sets $X(n)$. The number $n$ is called the arity of an element $x \in X(n)$ and denoted by $\text{Ar}(x) = n$.

Let $X$ be an operation alphabet. The free operad over $X$ is an operad $F(X)$ equipped with an inclusion $\eta : X \to F(X)$ (i.e., a collection of inclusions $\eta_n : X(n) \to F(X)(n)$ for all $n \geq 0$) which satisfies the following universal property: any map $f : X \to P$, where $P$ is an operad, extends uniquely to an operad morphism $\hat{f} : F(X) \to P$. [BD16, Section 3.3].

1.2. Planar rooted tree. Similar to the case of associative algebras, an operad can be presented by generators and relations, i.e., as a quotient of a free operad. In this subsection, we introduce a language for working with planar rooted trees, which will be used later to construct free operads. We mainly follow the ideas in [BD16, Section 3.3] and use the language introduced in [BD16] with minor modification for the rest of the paper, see Remark 1.6.
Definition 1.4. A rooted tree $\tau$ consists of

- a finite set $\text{Vert}(\tau)$ of vertices, which is a disjoint union
  \[ \text{Vert}(\tau) = \text{Int}(\tau) \sqcup \text{Leaves}(\tau) \sqcup \{r\}, \]
  where elements of the (possible empty) set $\text{Int}(\tau)$ are called internal vertices of $\tau$, elements of the nonempty set $\text{Leaves}(\tau)$ are called leaves of $\tau$, and the element $r$ is called the root of $\tau$ and denoted by $\text{Root}(\tau)$; and

- a parent function $\text{Parent} = \text{Parent}_\tau: \text{Vert}(\tau) \setminus \{r\} \to \text{Vert}(\tau) \setminus \text{Leaves}(\tau)$ such that $|\text{Parent}^{-1}(r)| = 1$, $|\text{Parent}^{-1}(v)| \geq 1$ for all $v \in \text{Int}(\tau)$, and the connectivity condition is satisfied: for each vertex $v \neq r$, there are an $h \in \mathbb{N}^*$ and vertices $v_0 = r, v_1, \ldots, v_{h-1}, v_h = v$, such that $v_i = \text{Parent}(v_{i+1})$ for $0 \leq i \leq h - 1$.

When we draw a rooted tree on the plane, the root is always drawn at the bottom of the tree; a hollow circle presents an internal vertex while a black solid circle presents a leaf or the root; a solid segment between two vertices indicates that the lower vertex is the parent of the higher one. See Figure 1 for examples.

![Figure 1. Rooted trees](attachment:image.png)

- Let $\tau$ be a rooted tree. The following lemma is straightforward.

Lemma 1.5. Suppose $v \in \text{Vert}(\tau) \setminus \text{Root}(\tau)$, and $h$ and $v_i$ for $0 \leq i \leq h$ are the same as in Definition 1.4. Then the number $h$ and the sequence $v_0, v_1, \ldots, v_h$ are uniquely determined by $v$, called respectively the height of $v$ (denoted by $h(v)$) and the path from root to $v$.

The height of $\tau$ is the maximal height of its internal vertices, i.e.,

\[ h(\tau) := \max\{h(v) : v \in \text{Int}(\tau)\}. \]

For $v \in \text{Vert}(\tau) \setminus \text{Leaves}(\tau)$, elements in $\text{Child}(v) := \text{Parent}^{-1}(v)$ are called children of $v$. More generally we define the descendants of $v$ to be

\[ \text{Child}^\infty(v) := \{v' \in \text{Vert}(\tau) : \text{Parent}^i(v') = v \text{ for some } i \in \mathbb{N}^*\}. \]

The number of children of $v$ (respectively, the number of leaves of $\tau$) is called the arity of $v$ (respectively, of $\tau$), denoted by $\text{Ar}(v)$ (respectively, $\text{Ar}(\tau)$). Define the weight of $\tau$ to be $\text{wt}(\tau) := |\text{Int}(\tau)|$. The only tree with weight 0 is called the trivial (rooted) tree, denoted by $\tau_0$, the only two vertices of which are the root $\text{Root}(\tau_0)$ and the leaf in $\text{Child}(\text{Root}(\tau_0))$, see Figure 1. For a nontrivial rooted tree, the only child of the root must be an internal vertex.

When working with more than one rooted trees, we usually use subscripts (e.g., the height $h_r(v)$) to indicate the tree under consideration. We write a map $\varphi$: 

\[ \varphi : \text{Vert}(\tau) \to \text{Vert}(\tau'), \]

such that $\varphi(v) = \varphi_r(v)$ for each leaf $v$.
Vert(τ) → Vert(τ′) between the vertex sets of rooted trees τ and τ′ as φ: τ → τ′ for short. An isomorphism from τ to τ′ is a bijective map φ: τ → τ′ that respects the parent functions, i.e.,

φ(Parent(τ(v))) = Parent(τ′(φ(v))), ∀v ∈ Vert(τ) \ Root(τ).

We say τ is isomorphic to τ′ if there is an isomorphism from τ to τ′. It is easy to see that an isomorphism also respects the type of each vertex, more precisely, if φ: τ → τ′ is an isomorphism, then v ∈ Vert(τ) is a leaf (respectively, an internal vertex, the root) in τ if and only if so is φ(v) in τ′.

**Remark 1.6.** The only difference between our rooted trees and those in [BD16, Section 3.3] is that an internal vertex of our rooted tree always has positive arity while an internal vertex in the sense of [BD16] may have zero arity. Note that the existence of internal vertices of arity zero makes [BD16, Proposition 3.4.1.6] false. (see Example 2.3 for a counterexample).

**Definition 1.7.** A planar rooted tree (PRT, for short) is a rooted tree together with a planar structure, i.e., a rooted tree with a total order on Child(v) for each v ∈ Vert(τ) \ Leaves(τ).

The planar structure of a PRT τ induces a total order on Vert(τ): Suppose u and u′ are two different vertices and consider the paths from root to u and u′:

\[ v_0 = r, v_1, \ldots, v_{h(u)}; \quad v_0' = r, v_1', \ldots, v_{h(u')}. \]

We say u < u′ if one of the following holds:

(i) h(u) < h(u′) and v_0 = v_0′, v_1 = v_1′, \ldots, v_{h(u)} = v_{h(u)}′;
(ii) there is \(1 \leq k \leq \min\{h(u), h(u')\} - 1\) such that v_0 = v_0′, v_1 = v_1′, \ldots, v_k = v_k′ and v_{k+1} < v_{k+1}′.

If there is no confusion, we usually use positive integers to denote the leaves, i.e., Leaves(τ) = \{1, 2, \ldots, Ar(τ)\}, to indicate the order on Leaves(τ) in the obvious way. We draw a PRT on the plane in a way that the planar order on Child(v) is determined by ordering the corresponding vertices left-to-right. A PRT τ is isomorphic to PRT τ′ if there exists a rooted tree isomorphism φ: τ → τ′ such that φ(u) < φ(v) whenever u < v for u, v ∈ Vert(τ).

**Example 1.8.** Let τ and τ′ be PRTs as drawn in Figure 2. It is easy to see that τ is isomorphic to τ′ as rooted trees. However, they are not isomorphic as PRTs.

![Figure 2](image-url) Non-isomorphic PRTs that are isomorphic as rooted trees

The following lemma is clear.
Lemma 1.9. [BD16] Definition 3.4.2.1] Let $\tau$ be a PRT. Suppose that $\emptyset \neq V' \subseteq \text{Int}(\tau)$ satisfies the following conditions

(i) there is a unique $v' \in V'$ such that $\text{Parent}_\tau(v') \notin V'$; and
(ii) for each $v'' \in V'$ there exist $h \in \mathbb{N}^*$ and vertices $v_1 = v', v_2, \ldots, v_h = v''$ such that $v_i = \text{Parent}_\tau(v_{i+1})$ for all $1 \leq i \leq h - 1$.

Then $V'$ defines a PRT $\tau'$ as follows

$$\text{Root}(\tau') = \text{Parent}_\tau(v'), \quad \text{Int}(\tau') = V', \quad \text{Leaves}(\tau') = \left( \bigcup_{v \in V'} \text{Child}(v) \right) \setminus V',$$

and the parent function and the planar structure of $\tau'$ are the restrictions of the parent function and the planar structure of $\tau$. We call $\tau'$ a subtree of $\tau$, denote $\tau' \subseteq \tau$.

Note that, in Lemma 1.9, condition (i) implies that each $v_i$ in condition (ii) belongs to $V'$.

Let $\tau$ be a PRT, $v \in \text{Int}(\tau)$, and $V' = (\text{Int}(\tau) \cap \text{Child}_\infty(v)) \cup \{v\}$. Then $V'$ satisfies the conditions in Lemma 1.9 and thus defines a subtree $\tau'$ of $\tau$. We call this subtree the maximal subtree of $\tau$ rooted at $\text{Parent}(v)$ and containing internal vertex $v$. Given $r' \in \text{Vert}(\tau)$ and $v' \in \text{Int}(\tau) \cap \text{Child}(r')$, it is easy to see that there is a unique maximal subtree of $\tau$ that is rooted at $r'$ and contains $v'$. We denote it by MaxSub$_\tau(r', v')$.

1.3. Free ns operads. Let $\mathcal{X}$ be an operation alphabet. A labelling of a nontrivial PRT $\tau$ is a map $x : \text{Int}(\tau) \to \mathcal{X}$ such that $\text{Ar}(\tau(v)) = \text{Ar}(v) = |\text{Child}(v)|$ for all $v \in \text{Int}(\tau)$. A nontrivial tree monomial in $\mathcal{X}$ is a pair $T = (\tau, x)$ of a nontrivial PRT $\tau$ and a labelling $x$ of $\tau$. Define the tree monomial for the trivial tree to be the trivial tree monomial, denoted by $1 := (\tau_0, \emptyset)$. The PRT $\tau$ is called the underlying tree of tree monomial $T$, denoted by $\text{Tr}(T) := \tau$. More generally, if $W$ is a set of tree monomials, denote $\text{Tr}(W) := \{\tau : \tau = \text{Tr}(T), T \in W\}$. The arity and weight of a tree monomial $T = (\tau, x)$ are defined as $\text{Ar}(T) := \text{Ar}(\tau)$ and $\text{wt}(T) := \text{wt}(\tau)$ respectively. Let $\text{TM}(\mathcal{X})$ denote the set of tree monomials in $\mathcal{X}$ together with the trivial tree monomial.

Example 1.10. Suppose that $\mathcal{X}(1) = \{a\}$ and $\mathcal{X}(2) = \{b, c\}$. In Figure 3 are examples of tree monomials in $\mathcal{X}$.

![Figure 3. Tree monomials](image)

Definition 1.11. Suppose $T = (\tau, x) \in \text{TM}(\mathcal{X})$ and $\tau'$ is a subtree of $\tau$. Then the tree monomial $T' := (\tau', x|_{\text{Int}(\tau')})$ is called a submonomial of $T$. The tree monomial
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T is said divisible by \( T_1 = (\tau_1, x_1) \in \text{TM}(\mathcal{X}) \) if \( \tau \) contains a subtree \( \tau'_1 \) isomorphic to \( \tau_1 \) via \( \phi : \tau'_1 \to \tau_1 \) and \( x(v) = x_1(\phi(v)) \) for all \( v \in \text{Int}(\tau'_1) \).

A tree polynomial in \( \mathcal{X} \) with coefficients in \( \mathbb{F} \) is an \( \mathbb{F} \)-linear combination of tree monomials of the same arity. The support of a tree polynomial \( f \), denoted by \( \text{Supp}(f) \), is the set of all tree monomials that appear in \( f \) with nonzero coefficients. The arity of \( f \) is defined to be the arity of a tree monomial in \( \text{Supp}(f) \). Denote the vector space of all tree polynomials of arity \( n \geq 1 \) by \( \mathcal{T}(\mathcal{X})(n) \) and \( \mathcal{T}(\mathcal{X})(0) = 0 \). (Note that we only consider \( \mathcal{X} \) with \( \mathcal{X}(0) = \emptyset \) in this paper.) Let \( \mathcal{T}(\mathcal{X}) := \{ \mathcal{T}(\mathcal{X})(n) \}_{n \geq 0} \). In order to make \( \mathcal{T}(\mathcal{X}) \) an operad, we need to define compositions of tree polynomials. We first define the graftings of PRTs.

**Definition 1.12.** \[\text{BD16, Definition 3.3.3.2}\] Suppose that \( \tau_1 \) and \( \tau_2 \) are PRTs. Let \( l \in \text{Leaves}(\tau_1) = \{1, 2, \ldots, \text{Ar}(\tau_1)\} \). We define a PRT \( \tau_1 \circ_l \tau_2 \), called the result of partial grafting of \( \tau_2 \) to \( \tau_1 \) at \( l \), as follows:

\[
\text{Root}(\tau_1 \circ_l \tau_2) := \text{Root}(\tau_1),
\text{Int}(\tau_1 \circ_l \tau_2) := \text{Int}(\tau_1) \cup \text{Int}(\tau_2),
\text{Leaves}(\tau_1 \circ_l \tau_2) := \text{Leaves}(\tau_1) \cup \text{Leaves}(\tau_2) \setminus \{l\};
\]

the parent function and the planar structure on \( \tau_1 \circ_l \tau_2 \) are induced respectively by the respective parent functions and planar structures of \( \tau_1 \) and \( \tau_2 \) with two exceptions: for the only vertex \( v \in \text{Child}_{\tau_2}(\text{Root}(\tau_2)) \), define \( \text{Parent}_{\tau_1 \circ_l \tau_2}(v) := \text{Parent}_{\tau_1}(l) \); the total order needed by the planar structure puts \( v \) in the place of \( l \).

Partial graftings of PRTs induce partial compositions of tree monomials.

**Definition 1.13.** Given two tree monomials \( T_1 = (\tau_1, x_1) \) and \( T_2 = (\tau_2, x_2) \), we define the partial composition \( T_1 \circ_l T_2 \) for \( 1 \leq l \leq \text{Ar}(T_1) \) to be the tree monomial \( T = (\tau, x) \), where \( \tau = \tau_1 \circ_l \tau_2 \) and

\[
x(v) := \begin{cases} x_1(v) & v \in \text{Int}(\tau_1), \\
x_2(v) & v \in \text{Int}(\tau_2). \end{cases}
\]

**Example 1.14.** Let \( T_1 \) and \( T_2 \) be the same as in Example 1.10; see Figure 3. All possible compositions \( T_1 \circ_l T_2 \) are demonstrated in Figure 4.

![Figure 4. Partial compositions of tree monomials](image-url)
Extending compositions of tree monomials by multilinearity to the collection $T(\mathcal{X})$ gives partial compositions of tree polynomials:

\[ \circ_i : T(\mathcal{X})(n) \otimes T(\mathcal{X})(m) \to T(\mathcal{X})(n + m - 1), \alpha \otimes \beta \mapsto \alpha \circ_i \beta, \quad 1 \leq i \leq n. \]

The following lemma is easy to prove.

**Lemma 1.15.** Equipped with the partial compositions defined above, $T(\mathcal{X})$ is the free reduced ns operad generated by $\mathcal{X}$.

An ideal $\mathcal{I}$ of an operad $\mathcal{P}$ is a subcollection of $\mathcal{P}$ such that each composition $f \circ_i g$ belongs to $\mathcal{I}$ whenever $f$ or $g$ belongs to $\mathcal{I}$. Suppose $\mathcal{S}$ is a subcollection of $\mathcal{P}$. The ideal of $\mathcal{P}$ generated by $\mathcal{S}$, denoted by $\langle \mathcal{S} \rangle$, is the smallest (by inclusion) ideal of $\mathcal{P}$ containing $\mathcal{S}$.

We are ready now to define a presentation of an operad by generators and relations.

**Definition 1.16.** Suppose that an operad $\mathcal{P}$ is the quotient of the free operad $T(\mathcal{X})$ by some ideal $\mathcal{I}$, and that $\mathcal{I}$ is generated by a subcollection $\mathcal{R} \subset \mathcal{I}$. We say that the operad $\mathcal{P}$ is presented by generators $\mathcal{X}$ and relations $\mathcal{R}$. We call $\mathcal{P}$ finitely generated (resp. finitely presented) if $\mathcal{P}$ can be presented by a finite set $\mathcal{X}$ of generators and a finite dimensional subcollection $\mathcal{R}$ of relations.

### 2. Gröbner-Shirshov bases of ns operads

In this section, we follow the ideas in [BD16, Chapter 3] to introduce Gröbner-Shirshov basis (also known as Gröbner basis) theory for ns operads. Similarly to the case of associative algebras, the Gröbner-Shirshov basis method is useful for the computation of GK-dimension of an operad.

Let $\mathcal{X}$ be an operation alphabet and $\mathcal{X}^*$ be the free monoid generated by $\mathcal{X}$. Recall that a total order $\succ$ on $\mathcal{X}^*$ is called a monomial order on $\mathcal{X}^*$ if $\succ$ is a well-order and $u_1 \succ u_2$ implies $u_1 u_3 \succ u_2 u_3$ and $u_3 u_1 \succ u_3 u_2$ for all $u_1, u_2, u_3 \in \mathcal{X}^*$.

A collection of total orders $\succ_n$ of $TM(\mathcal{X})(n)$, $n \geq 0$, is called a monomial order on $TM(\mathcal{X})$ if the following conditions are satisfied:

(i) each $\succ_n$ is a well-order;

(ii) each partial composition is a strictly increasing function in each of its arguments, i.e., if $T_0, T_0' \in TM(\mathcal{X})(m)$, $T_1, T_1' \in TM(\mathcal{X})(n)$, $1 \leq i \leq m$, then

\[
T_0 \circ_i T_1 \succ_{m+n-1} T_0' \circ_i T_1 \quad \text{if} \quad T_0 \succ_m T_0', \quad T_0 \circ_i T_1 \succ_{m+n-1} T_0 \circ_i T_1' \quad \text{if} \quad T_1 \succ_n T_1'.
\]

Now we introduce a monomial order on $TM(\mathcal{X})$. Let $T = (\tau, x)$ be a tree monomial. For each leaf $l$ of $\tau$, we record the labels of internal vertices of the path from the root to $l$, forming a word in alphabet $\mathcal{X}$. The sequence of these words, ordered by the planar structure on $\text{Leaves}(\tau)$, is called the path sequence of the tree monomial $T$, denoted by $\text{Path}(T)$.

**Example 2.1.** The path sequences of the tree monomials in Figure 3 are

\[
\text{Path}(T_1) = (ab, ab), \quad \text{Path}(T_2) = (b, bc, bc), \\
\text{Path}(T_3) = (bc, bc, b), \quad \text{Path}(T_4) = (bc, bc, bb, bb).
\]
It is not difficult to prove that a tree monomial is uniquely determined by its path sequence, see [BD16, Lemma 3.4.1.4]. Given a monomial order > on \(X^*\), we define an order (still denoted by >), called the path extension of the monomial order on \(X^*\) on \(\text{TM}(X)\) by using path sequences of tree monomials. Suppose \(T, T' \in \text{TM}(X)\), \(\text{Path}(T) = (u_1, u_2, \ldots, u_m)\) and \(\text{Path}(T') = (u_1', u_2', \ldots, u_n')\). We say \(T > T'\) if either \(m > n\), or \(m = n\) and there exists \(1 \leq i \leq m\) such that \(u_1 = u_1', u_2 = u_2', \ldots, u_{i-1} = u_{i-1}', u_i > u_i'\).

The following lemma is copied from [BD16, Proposition 3.4.1.6].

**Lemma 2.2.** Suppose \(X = \biguplus_{n \geq 1} X(n)\), namely, \(X(0) = \emptyset\). The path extension of a monomial order on \(X^*\) is a monomial order on \(\text{TM}(X)\).

Note that the above statement does not hold if we allow internal vertices of arity zero in a PRT. See the following example.

**Example 2.3.** Let \(X = \{a, b\}\) with \(a \in X(2)\) and \(b \in X(0)\). Consider the tree monomials \(T_1\) and \(T_2\) in Figure 5. The path sequences of the tree monomials are \(\text{Path}(T_1) = (ab, a)\), \(\text{Path}(T_2) = (a, ab)\).

Under the path extension of a degree-lexicographic order on \(X^*\), we have \(T_1 > T_2\). However, if we denote \(T_3\) the tree monomial with only one internal vertex labeled by \(b\), then the compositions \(T_1 \circ_1 T_3\) and \(T_2 \circ_1 T_3\) are equal, which shows that the path extension is not a monomial order.

Fix a monomial order on \(\text{TM}(X)\). Suppose \(g = a_1 U_1 + a_2 U_2 + \cdots + a_n U_n \in \mathcal{T}(X)\) where \(n \geq 1\), each \(a_i \in F^*\), \(U_i \in \text{TM}(X)\) and \(U_1 > U_2 > \cdots > U_n\). Then \(U_1\) (respectively, \(a_1, a_1 U_1\)) is called the leading monomial (respectively, leading coefficient, leading term) of \(g\), denoted by \(\overline{g}\) (respectively, \(\text{lc}(g), \text{lt}(g)\)). We say \(g\) is monic if \(\text{lc}(g) = 1\).

Suppose \(\mathcal{G} \subseteq \mathcal{T}(X)\). Let \(\mathcal{F} := \{\overline{g} : g \in \mathcal{G}\}\) and \(\text{Irr}(\mathcal{G}) := \{u \in \text{TM}(X) : u \text{ is not divisible by } \overline{g}, \forall g \in \mathcal{G}\}\).

A tree polynomial \(f\) is reduced with respect to \(\mathcal{G}\) if \(\text{Supp}(f) \subseteq \text{Irr}(\mathcal{G})\). We say \(\mathcal{G}\) is self-reduced if, for all \(g \in \mathcal{G}\), \(g\) is monic and reduced with respect to \(\mathcal{G} \setminus \{g\}\).

Note that, for an ideal \(\mathcal{I}\) of \(\mathcal{T}(X)\), the vector space \(F\mathcal{I}\) spanned by the leading monomials \(\overline{\mathcal{I}}\) is also an ideal (see [BD16, Proposition 3.4.3.1]), which is exactly the ideal generated by \(\overline{\mathcal{I}}\), i.e., \((\overline{\mathcal{I}}) = F\mathcal{I}\).
Definition 2.4. Let \( I \) be an ideal of \( \mathcal{T}(\mathcal{X}) \). A subcollection \( \mathcal{G} \) of \( I \) is called a Gröbner-Shirshov basis for \( I \) (or for the quotient operad \( \mathcal{T}(\mathcal{X})/I \)) if the ideal generated by \( \mathcal{G} \) coincides with that generated by \( I \), i.e., \((\mathcal{G}) = (I)\).

It is clear that an ideal \( I \subseteq \mathcal{T}(\mathcal{X}) \) is a Gröbner-Shirshov basis for \( I \).

Proposition 2.5. \([\text{BD16}, \text{Proposition 3.4.3.4}]\) Let \( I \) be an ideal of \( \mathcal{T}(\mathcal{X}) \) and \( G \subseteq I \).

Then \( G \) is a Gröbner-Shirshov basis for \( I \) if and only if \( \text{Irr}(G) \) forms an \( \mathbb{F} \)-basis for the quotient \( \mathcal{T}(\mathcal{X})/I \).

3. Gelfand-Kirillov dimension

We refer to \([\text{KL00}]\) for basics about the Gelfand-Kirillov dimension of associative algebras. The Gelfand-Kirillov dimension of a locally finite operad is defined in \([\text{KP15}, \text{p.400}]\) and \([\text{BYZ20}, \text{Definition 4.1}]\).

Definition 3.1. \([\text{KP15}, \text{p.400}]\) \([\text{BYZ20}, \text{Definition 4.1}]\) Let \( \mathcal{P} \) be a locally finite operad. The Gelfand-Kirillov dimension, or GK-dimension for short, of \( \mathcal{P} \) is defined to be

\[
\text{GKdim}(\mathcal{P}) := \limsup_{n \to \infty} \frac{\log n}{\sum_{i=0}^{n} \dim \mathcal{P}(i)}
\]

where \( \dim \) stands for \( \dim_{\mathbb{F}} \).

When we talk about the GK-dimension of an operad \( \mathcal{P} \), we always implicitly assume that \( \mathcal{P} \) is locally finite. By Lemma 3.2 below, we might only consider the GK-dimension of reduced (or reduced connected) operads from now on.

Given two subcollections \( \mathcal{V} \) and \( \mathcal{W} \) of \( \mathcal{P} \), let \( \mathcal{V} \circ \mathcal{W} \) be the subcollection of \( \mathcal{P} \) spanned by all elements of the form \( v \circ_i w \) for \( v \in \mathcal{V}, w \in \mathcal{W} \) and \( 1 \leq i \leq \text{Ar}(v) \). Given a subcollection \( \mathcal{V} \) of \( \mathcal{P} \), let \( \mathcal{V}^0 = (0, F, 0, 0, \ldots) \), and inductively, let \( \mathcal{V}^m = \mathcal{V}^{m-1} \circ \mathcal{V} \) for \( m \geq 1 \). It is clear that \( \mathcal{V}^m = \{\mathcal{V}^m(n)\}_{n \geq 0} \) where \( \mathcal{V}^m(n) \) is the subspace of \( \mathcal{P}(n) \) spanned by all elements of arity \( n \) that have the following form

\[
((\cdots ((a_1 \circ_j a_2) \circ_j a_3) \circ_j \cdots) \circ_j a_{m-1}) a_m), \quad \text{each } a_i \in \mathcal{V}.
\]

We call \( \mathcal{V} \) a generating subcollection of \( \mathcal{P} \) if

\[
\mathcal{P} = \sum_{m \geq 0} \mathcal{V}^m := \left\{ \sum_{m \geq 0} \mathcal{V}^m(n) \right\}_{n \geq 0}.
\]

We say \( \mathcal{P} \) is a finitely generated operad with a finite generating alphabet \( \mathcal{X} \), if \( \mathcal{P} \) has a finite dimensional generating subcollection \( \mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0} \) where \( \mathcal{V}(n) \) is the space spanned by \( \mathcal{X}(n) \).

The following lemma is easy to prove and its proof is omitted.

Lemma 3.2. Let \( \mathcal{P} \) be a finitely generated locally finite operad.

(1) We define a reduced operad associated with \( \mathcal{P} \):

\[
\mathcal{P}_r := (0, \mathcal{P}(1), \mathcal{P}(2), \mathcal{P}(3), \ldots).
\]

Then \( \mathcal{P}_r \) is finitely generated and locally finite.

(2) We define a reduced connected operad

\[
\mathcal{P}_{rc} := (0, F, \mathcal{P}(2), \mathcal{P}(3), \ldots).
\]

Then \( \mathcal{P}_{rc} \) is finitely generated and locally finite.
The following lemma gives a characterization of the GK-dimension of a finitely generated operad.

**Lemma 3.3.** Suppose \( \mathcal{P} \) is a locally finite operad generated by a finite dimensional subcollection \( \mathcal{V} \). Then

\[
\text{GKdim}(\mathcal{P}) = \limsup_{n \to \infty} \log_n \left( \dim \left( \sum_{i=0}^{n} \mathcal{V}^i \right) \right).
\]

**Proof.** Let \( t \) be the maximal arity of nonzero elements in \( \mathcal{V} \) and \( s \) be an integer such that \( (\mathcal{P}(0), \mathcal{P}(1), \ldots, \mathcal{P}(t), 0, 0, \ldots) \subseteq \sum_{j=0}^{s} \mathcal{V}^j \).

By the definition of \( t \), we have \( \mathcal{V} \subseteq (\mathcal{P}(0), \mathcal{P}(1), \ldots, \mathcal{P}(t), 0, 0, \ldots) \). Then, for every \( n > 0 \),

\[
\sum_{i=0}^{n} \mathcal{V}^i \subseteq (\mathcal{P}(0), \mathcal{P}(1), \ldots, \mathcal{P}(nt - (n - 1)), 0, 0, \ldots) \subseteq \sum_{i=0}^{ns} \mathcal{V}^i.
\]

(E3.3.1) Consequently,

\[
\dim \left( \sum_{i=0}^{n} \mathcal{V}^i \right) \leq \dim \left( \sum_{i=0}^{nt-(n-1)} \mathcal{P}(i) \right) \leq \dim \left( \sum_{i=0}^{ns} \mathcal{V}^i \right), \quad n > 0.
\]

Thus we have that

\[
\text{GKdim}(\mathcal{P}) = \limsup_{n \to \infty} \log_n \left( \dim \left( \sum_{i=0}^{n} \mathcal{P}(i) \right) \right) = \limsup_{n \to \infty} \log_n \left( \dim \left( \sum_{i=0}^{n} \mathcal{V}^i \right) \right).
\]

\( \square \)

The following example shows that, in general, the GK-dimension of an operad is not the supremum of the GK-dimensions of its finitely generated suboperads.

**Example 3.4.** Let \( \alpha \) be any positive real number. Let \( \mathcal{P} \) be the operad generated by infinitely many elements in \( \mathcal{X} \) such that

\[
\mathcal{X} = \{ \mathcal{X}(n) \}_{n \geq 0}, \quad \mathcal{X}(0) = \mathcal{X}(1) = \emptyset, \quad |\mathcal{X}(n)| = |n^\alpha| - |(n-1)^\alpha|, \quad \forall n \geq 2,
\]

and subject to relations

\[
x_m \circ_i x_n = 0, \quad \forall x_m \in \mathcal{X}(m), x_n \in \mathcal{X}(n), \quad m, n \geq 2, \quad 1 \leq i \leq m.
\]

Then it is easy to check that \( \text{GKdim}(\mathcal{P}) = \alpha > 0 \) but \( \text{GKdim}(\mathcal{P}') = 0 \) for all finitely generated suboperad \( \mathcal{P}' \) of \( \mathcal{P} \).

We will use the following nice construction which is related to Warfield’s example [War84].

**Example 3.5.** We fix a real number \( r \) strictly between 2 and 3 and let \( q \) be \( \frac{r-1}{2} \) which is strictly between \( \frac{1}{2} \) and 1. Let \( A \) be the quotient algebra \( \mathbb{F}(x_1, x_2)/J \) generated by two elements \( x_1, x_2 \) of degree 1 and modulo the monomial ideal \( J \) generated by monomials having degree \( \geq 3 \) in \( x_2 \) together with all monomials of the form

\[
x_1^i x_2 x_1^l x_2^j
\]
satisfying \( j < n - \lfloor n^q \rfloor \) where \( n = i + j + l + 2 \).

By an easy counting, \( \dim A_0 = 1 \), \( \dim A_1 = 2 \), and for each \( n \geq 2 \),

\[
\dim A_n = 1 + n + \sum_{j=n-\lfloor n^q \rfloor}^{n-2} (n-1-j) = 1 + n + \sum_{p=1}^{\lfloor n^q \rfloor - 1} p.
\]

As a consequence, we have

1. \( \dim A_n \) is strictly increasing.
2. \( \dim A_n = 1 + n + \frac{1}{2}(\lfloor n^q \rfloor - 1)(\lfloor n^q \rfloor) \sim \frac{1}{2}n^{2q} = \frac{1}{2}n^r - 1 \).
3. \( \text{GKdim}(A) = r \).

Using the example above, we obtain the range of possible values for the GK-dimension of an associative algebra.

**Lemma 3.6.**

1. For every integer \( d \in \mathbb{N}^* \), there is a graded algebra \( A \) such that \( \text{GKdim}(A) = d \) and \( \{\dim A_i\}_{i=0}^{\infty} \) is weakly increasing.
2. There is a graded algebra \( A \) such that \( \text{GKdim}(A) = \infty \).
3. For \( d \in R_{\text{GKdim}} \setminus \{0\} \), there is a graded algebra \( A \) such that \( \text{GKdim}(A) = d \) and \( \{\dim A_i\}_{i=0}^{\infty} \) is strictly increasing.
4. All algebras in parts (1,2,3) can be taken to be monomial algebras (hence connected graded) finitely generated in degree 1.

**Proof.**
(1) We can take \( A \) to be the commutative polynomial ring \( \mathbb{F}[x_1, \ldots, x_d] \).
(2) We can take \( A \) to be the free algebra \( \mathbb{F}\langle x_1, x_2 \rangle \).
(3) If \( d \) is an integer, it follows from part (1). If \( d = \infty \), it follows from part (2). Now we let \( d \) be a finite non-integral real number \( > 2 \). Let \( n \) be \( \lfloor d \rfloor - 2 \) and \( r = d - n \). Then \( n \geq 0 \) and \( 2 < r < 3 \). Let \( A \) be the algebra given in Example 3.5 and let \( A' = A[x_1, \ldots, x_n] \). Then the assertion follows.
(4) It is well-known (see, e.g., [Bel15, Remark 4.1]) that given a finitely generated associative algebra \( A \), there is a finitely generated monomial algebra \( B = \mathbb{F}\langle X \rangle / I \) such that \( \text{GKdim}(A) = \text{GKdim}(B) \), where \( \mathbb{F}\langle X \rangle \) is the free associative algebra generated by a finite set \( X \) and \( I \) is an ideal consisting of words in \( X \). Since \( A \) is connected graded, we even have that \( A \) and \( B \) have the same Hilbert series. \( \square \)

Below is a weak version of Theorem 0.2(2).

**Proposition 3.7.** For any \( r \in R_{\text{GKdim}} \), there exists a finitely generated operad \( P \) such that \( \text{GKdim}(P) = r \).

**Proof.** By Lemma 3.6(3), there is a finitely generated connected graded monomial algebra \( A \) such that \( \text{GKdim}(A) = r \). By Construction 1.3, there is a finitely generated operad \( P \) such that \( \dim P(n) = \dim A_{n-1} \) for all \( n \geq 1 \). Therefore, by definition, \( \text{GKdim}(P) = \text{GKdim}(A) = r \). The assertion follows. \( \square \)

**Proposition 3.8.** Suppose \( P \) is a finitely generated and locally finite ns operad.

1. \( \text{GKdim}(P) = 0 \) if and only if \( P \) is finite dimensional.
2. \( \text{GKdim}(P) \) cannot be strictly between 0 and 1.

**Proof.** (1) The “if” part is clear. For the “only if” part, suppose \( \dim(P) = \infty \). Since \( P \) is finitely generated, there is a finite-dimensional subcollection \( (1_P \in P) \) of \( P \) that generates \( P \). We claim that \( Y^{m+1} \neq Y^m \) for every \( m \). Suppose to the contrary
that \( V^{m+1} = V^m \) for some \( m \). Then by induction, one sees that \( V^n = V^m \) for every \( n > m \). So \( P = \cup_{n>m} V^n = V^m \), which is finite dimensional. This yields a contradiction. Therefore we proved the claim, and consequently, \( \dim V^m \geq m + 1 \) for every \( m \). By Lemma 3.3,

\[
\text{GKdim}(P) = \limsup_{n \to \infty} \log_n \left( \sum_{i=0}^{m} \dim V^i \right) \geq \lim_{n \to \infty} \log_n (n + 1) = 1,
\]
a contradiction.

(2) See the proof of part (1). \( \square \)

The part (2) of the above proposition, as in the case of associative algebras, there is a gap between 0 and 1 for the GK-dimensions of finitely generated operads. This is false if \( P \) is infinitely generated, see Example 3.4 and [BYZ20, Corollary 6.12].

A monomial operad means a quotient of free operad by an ideal generated by tree monomials. The following lemma implies that given an operad, there exists a monomial operad with the same GK-dimension.

**Lemma 3.9.** Suppose \( I \supseteq T(X) \). Then \( \text{GKdim}(T(X)/I) = \text{GKdim}(T(X)/(I)) \).

**Proof.** Note that both \( I \) and \( (I) \) are ideals and thus Gröbner-Shirshov bases. Now the statement follows from Proposition 2.5 and \( \text{Irr}(I) = \text{Irr}((I)) \). \( \square \)

4. **Bergman’s gap theorem**

4.1. **Single-branched ns operads.** In this subsection, we will introduce so-called single-branched tree monomials and study their properties, which will be used in the next subsection to prove Bergman’s gap theorem for operads.

Let \( \tau \) be a PRT. An internal vertex \( v \) is called a top internal vertex if its children are all leaves. A branch of \( \tau \) is a path from the root to a top internal vertex \( v \), denoted by \( \text{bran}(v) \). A PRT is called a single-branched tree if it has exactly one branch. The planar structure of \( \tau \) induces an order on branches of \( \tau \):

\[
\text{bran}(v) > \text{bran}(v') \quad \text{if} \quad h(v) > h(v'), \quad \text{or} \quad h(v) = h(v') \quad \text{and} \quad v > v'.
\]

The maximal branch of \( \tau \) is called the pivot branch of \( \tau \). Denote by \( \text{Pivot}(\tau) \) the set of all vertices of \( \tau \) that belong to the pivot branch of \( \tau \). The top internal vertex of \( \tau \) belonging to \( \text{Pivot}(\tau) \) is called the pivot top internal vertex, denoted by \( \text{TIV}(\tau) \).

A tree monomial is called single-branched if its underlying tree is single-branched. An operad is called single-branched if it has an \( F \)-basis that consists of single-branched tree monomials.

A tree monomial \( T \) is called right normal if it can be written in terms of partial compositions of generating operations with parentheses from right to left, i.e.,

\[
(T4.0.1) \quad T = (x_1 \circ_i (\cdots (x_{n-2} \circ_{i_{n-2}} (x_{n-1} \circ_{i_{n-1}} x_n)) \cdots)).
\]

Left normal tree monomials are defined similarly, see (E3.1.1). Note that every tree monomial is left normal, but not necessarily right normal. For a right normal tree monomial, we usually write without parentheses (and/or composition symbols \( \circ \) sometimes) for short when no confusion arises, e.g.,

\[
(T4.0.2) \quad T = x_1 \circ_i x_2 \circ_i \cdots \circ_i x_{n-1} \circ_i x_n = x_1 x_2 \cdots x_n.
\]

The following lemma is straightforward.

**Lemma 4.1.** Let \( T \) be a tree monomial. Then the following statements are equivalent:
Example 4.2. In Figure 3, $T_1, T_2$ and $T_3$ are single-branched and $T_4$ is not single-branched.

A single-branched tree monomial $w = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n$ is called \emph{periodic} if there exists a positive integer $p < n$ such that $x_j = x_{j+p}$ for all $1 \leq j \leq n - p$ and $i_j = i_{j'} + p$ for all $1 \leq j' \leq n - p - 1$. The integer $p$ is called a local period of $w$ and the smallest local period of $w$ is called the minimal period of $w$. Given a periodic tree monomial $w = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n$ with minimal period $p$, by using its minimal period, we can extend $w$ to the left and/or to the right to get a new single-branched tree monomial which contains $w$ as a submonomial. More precisely, $w$ can be extended to 

$$w_{m,l} := x_{-m} \circ_{i_{-m}} \cdots \circ_{i_{-1}} x_0 \circ_{i_0} x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n \circ_{i_n} \cdots \circ_{i_{l-1}} x_l$$

where $m \geq -1$, $l \geq n$, for each integer $q$ between $-m$ and $l$ (say, $-m \leq q = ps + r \leq l$, $s, r \in \mathbb{Z}$, $1 \leq r \leq p$), $x_q = x_r$ and $\circ_{i_q} = \circ_{i_r}$ (except $i_0$). It is clear that $p$ is still the minimal period of $w_{m,l}$. If a positive integer $p'$ is a local period of all extensions $w_{m,l}$ of $w$, we call $p'$ a period of $w$. For example, given 

$$w = a \circ_1 a \circ_1 b \circ_1 a \circ_1 a \circ_1 b \circ_1 a \circ_1 a,$$

then 3 is the minimal period of $w$; 6 is a period of $w$; 7 is a local period (but not a period) of $w$.

The following lemma on periods is easy to prove.

Lemma 4.3. Suppose $w$ is a single-branched tree monomial of minimal period $p$. If $l$ is a period of $w$, then $p$ divides $l$.

\textbf{Proof.} Suppose to the contrary that $l = pq + r$ with $q, r \in \mathbb{N}$ and $0 < r < p$. Assume $w = x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n$. Consider the extension 

$$w_{l,n+p} = x_{-l} \circ_{i_{-l}} \cdots \circ_{i_{-1}} x_0 \circ_{i_0} x_1 \circ_{i_1} x_2 \circ_{i_2} \cdots \circ_{i_{n-1}} x_n \circ_{i_n} \cdots \circ_{i_{n+p-1}} x_{n+p}.$$

For all $1 \leq i \leq n - r$, since $l$ and $pq$ are local periods of $w_{l,n+p}$, we have that $x_{i+r} = x_{i+l-pq} = x_{i-pq} = x_i$. Thus $r$ is a local period of $w$, contradicting the minimality of $p$. \hfill \Box

The following is an analogue of [KL00] Lemma 2.3.

Lemma 4.4. Let $w$ be a single-branched tree monomial of height $n > 0$. Suppose that $w$ is periodic with minimal period $p < n$ and has two equal submonomials

(E4.4.1) \quad $x_{i+1} \circ_{a_{i+1}} \cdots \circ_{a_{i+r-1}} x_{i+r} = x_{j+1} \circ_{a_{j+1}} \cdots \circ_{a_{j+r-1}} x_{j+r}$

of height $r \geq p$, $0 \leq i < j$. Then $p$ divides $j - i$.

\textbf{Proof.} We modify the proof of [KL00] Lemma 2.3 to take care of the composition indices. For the manipulations described below, if the tree monomial $w$ is too short, consider it in its extensions. Since $r \geq p$, $i + r \geq i + p$. Similarly, $j + r \geq j + p$. Since $p$ is the minimal period of $w$,

$$x_i = x_{i+p} = x_{j+p} = x_j.$$
By periodicity, we have the following equality of two sets
\[
\{a_{\alpha_1}, a_{\alpha_1+1}, \ldots, a_{\alpha_1+p-1}\} = \{a_{\alpha_j}, a_{\alpha_j+1}, \ldots, a_{\alpha_j+p-1}\}.
\]
By (E4.4.1), we also have
\[
\leq\beta_0
\]
Lemma 4.5. Suppose \(\alpha, \beta, \gamma\) have height \(r\) which forces that \(\alpha = \beta\). Thus \(w\) has two equal submonomials
\[
x_1 \circ a_{\alpha_1} \circ a_{\alpha_1+1} \cdots \circ a_{\alpha_1+r-1} x_{l+r} = x_j \circ a_{\alpha_j} \circ a_{\alpha_j+1} \cdots \circ a_{\alpha_j+r-1} x_{j+r},
\]
which have height \(r + 1\). Similarly we can extend the above two submonomials to any height \(\geq r\).

Next we show that \(j - i\) is a period of \(w\). In any extension \(w_{m,m'}\) of \(w\) (for some \(m \geq 0\) and \(m' \geq n\)), let \(-m \leq l \leq m' - (j - i)\) and \(t \in \mathbb{Z}\) such that \(l + tp = i + s\), \(0 \leq s \leq p - 1\). Then
\[
x_{l+(j-i)} = x_{l+(j-i)+tp} = x_{l+s+(j-i)} = x_{j+s} = x_{l+tp} = l
\]
for \(-m \leq l \leq m' - (j - i)\) and
\[
\alpha_{l+(j-i)} = \alpha_{l+(j-i)+tp} = \alpha_{l+s+(j-i)} = \alpha_{j+s} = \alpha_{l+tp} = \alpha_l
\]
for \(-m \leq l \leq m' - (j - i) - 1\). Hence \(w\) has a period \(j - i\). Now it follows from Lemma 4.3 that \(p\) divides \(j - i\). \(\Box\)

We modify [KL00, Lemma 2.4] and obtain the following

**Lemma 4.5.** Suppose \(X\) is a finite operation alphabet. Let \(W\) be a set of single-branched tree monomials on \(X\) such that all submonomials of elements in \(W\) still belong to \(W\). Suppose that, for some positive integer \(d \geq 3\), \(W\) contains at most \(d - 1\) tree monomials of height \(d\). Then \(W\) contains at most \((d - 1)^3\) tree monomials of height \(h\) for all \(h \geq d\).

**Proof.** Let \(w\) be a single-branched tree monomial of height \(h\) of the form (E4.0.2) and let \(W_h\) be the subset of \(W\) consisting of \(w \in W\) of height \(h\).

If \(h \leq 2d - 1\), then \(w\) is completely determined by its top and bottom submonomials of height \(d\). There are at most \((d - 1)^2\) possibilities for this case. So
\[
|W_h| \leq (d - 1)^2 < (d - 1)^3.
\]

If \(2d \leq h \leq 3d - 2\), then \(w\) is completely determined by its top and bottom submonomials of height \(d\) and the submonomial \(x_d \cdots x_{2d-1}\). There are at most \((d - 1)^3\) possibilities for this case. So
\[
|W_h| \leq (d - 1)^3.
\]

For other cases, we need to use the following claim.

**Claim:** Suppose \(w\) has height \(\geq 2d - 1\). Then \(w = w_1 w_2 w_3 = w_1 \circ \beta_1 w_2 \circ \beta_2 w_3\) where \(w_2\) is periodic of minimal period \(p \leq d - 1\), \(h(w_2) \geq p + d\), \(h(w_1) \leq d - p\) and \(h(w_3) \leq d - p\). The periodic submonomial \(w_2\) is \(x_{j+1} \cdots x_{r}\) for some integers \(j \leq d - p\) and \(r \geq h - (p - d)\).

Note that our lemma for \(h \geq 3d - 1(> 2d - 1)\) follows from the claim. In fact, by the claim, \(w = x_1 \cdots x_h\) for \(h \geq 3d - 1\) is uniquely determined by its bottom and top submonomials of height \(d\) (each has at most \(d - 1\) possibilities by assumption), and the periodic submonomial \(w'_2 := x_{d-p+1} \cdots x_{h-(d-p)}\). Note that \(w'_2\) is a submonomial of \(w_2\) with minimal period \(p \leq d - 1\). Hence \(w'_2\) has at most
$d - 1$ possibilities, determined by it top submonomial of height $d$. As a result, there are at most $(d - 1)^3$ possibilities for $w$ as desired.

**Proof of the Claim:** We prove this claim by induction on $h$.

Initial step: Suppose $h = 2d - 1$. Then $w = x_1 x_2 \cdots x_{2d - 1}$ has $d$ submonomials of height $d$, all belonging to $W$ by assumption. Hence two of these tree monomials must be equal, say

\[ x_{j+1} \cdots x_{j+d} = x_{j+p+1} \cdots x_{j+p+d} \]

where $j \geq 0$, $p \geq 1$, $j + p \leq d - 1$, $p$ chosen as small as possible. Then $x_{j+1} \cdots x_{j+d}$ and $x_{j+p+1} \cdots x_{j+p+d}$ have a nonempty overlap and thus

\[ w_2 := x_{j+1} \cdots x_{j+p+d} \]

is a periodic submonomial (of $w$) of minimal period $p \leq d - 1$ and of height $h(w_2) = p + d$. The bottom submonomial $w_3 := x_1 \cdots x_j$ and the top submonomial $w_3 := x_{j+p+d+1} \cdots x_{2d - 1}$ are of height $\leq d - p - 1 < d - p$. Thus the claim holds for $h = 2d - 1$ and we finish the initial step.

Inductive step: Suppose the claim is true for an $h \geq 2d - 1$. Consider $w = x_1 x_2 \cdots x_{h+1} \in W$ of height $h + 1$. By hypothesis,

\[ w = x_1 (x_2 \cdots x_j) (x_{j+1} \cdots x_{j+r}) (x_{j+r+1} \cdots x_{h+1}) \]

where $h(x_2 \cdots x_j) = j - 1 \leq d - p$, $h(x_{j+r+1} \cdots x_{h+1}) = h - j - r + 1 \leq d - p$, and $x_{j+1} \cdots x_{j+r}$ is periodic of minimal period $p \leq d - 1$ and height $r \geq p + d$. If $j - 1 < d - p$, i.e., $j \leq d - p$, then the claim follows by taking

\[ w_1 = x_1 \cdots x_j, \quad w_2 = x_{j+1} \cdots x_{j+r}, \quad w_3 = x_{j+r+1} \cdots x_{h+1}. \]

Now suppose $j - 1 = d - p$. We will show that the periodicity of

\[ x_{j+1} \cdots x_{j+r} = x_{j+1} \circ_{\alpha_{j+1}} \cdots \circ_{\alpha_{j+r-1}} x_{j+r} \]

can be extended down to include the term $x_j$, which completes our inductive step. Note that

\[ j + r = d - p + 1 + r \geq d - p + 1 + d + p > 2d. \]

Thus $x_2 \cdots x_{2d}$ is a proper submonomial of $x_1 \cdots x_{j+r}$, and contains two equal submonomials of height $d$, say,

\[ x_{i+1} \cdots x_{i+d} = x_{i+n+1} \cdots x_{i+n+d}, \]

where $i \geq 1$, $n \geq 1$ and $i + n \leq d$. Since $j = d - p + 1 \leq d$, both of these submonomials have an overlap with the periodic tree monomial $x_{j+1} \cdots x_{j+r}$ and the overlap contains at least their top (i.e., right) $d - (j - 1) = p$ terms. By Lemma 4.4, we have that $p$ divides $n$ (say, $n = cp \geq p$ for some integer $c > 0$) and hence

\[ i + d > d \geq j = d - p + 1 \geq d - n + 1 \geq i + 1. \]

Thus $x_j = x_{j+n}$ and $\alpha_j = \alpha_{j+n}$. Note that $j + 1 \leq j + p < j + r$ and $j + 1 \leq j + n < j + d < j + r$. By the periodicity, we have that

\[ x_{j+p} = x_{j+n-(c-1)p} = x_{j+n} = x_j \quad \text{and} \quad x_{j+n} = \alpha_{j+n} = \alpha_j, \]

i.e., the periodicity of $x_{j+1} \cdots x_{j+r}$ extends down to include $x_j$ as desired.

This finishes the inductive step, then the claim, and finally the assertion in the lemma.

The following example shows that the original [KL00, Lemma 2.4] is not true for the setting of operads, i.e., if we replace $d - 1$ by $d$ in Lemma 4.5 then the statement does not hold any further.
Example 4.6. Let $W$ be the set of single-branched tree monomials of the form
\[ x \circ_{i_1} x \circ_{i_2} \cdots \circ_{i_{n-1}} x \]
where $\text{Ar}(x) = 2$, $n \geq 1$, at most one composition index $i_j$ equals 2 and the other indices are all 1. It is easy to see that all submonomials of elements in $W$ are still in $W$ and that, for $n \geq 1$, $W$ contains exactly $n$ distinct monomials of height $n$.

Lemma 4.7. Let $\mathcal{P}$ be a locally finite operad with $\text{GKdim}(\mathcal{P}) < 2$. Let $a$ and $b$ be two positive real numbers.

1. There are infinitely many integer $n$ such that $\dim(\mathcal{P}(n)) < an - b$.
2. Suppose $\mathcal{P}$ is finitely generated by $\mathcal{V}$. Then there are infinitely many integer $n$ such that $\dim(\sum_{i=0}^{n} \mathcal{V}^i)/(\sum_{i=0}^{n-1} \mathcal{V}^i) < an - b$.

Proof. (1) Suppose to the contrary that $\dim(\mathcal{P}(n)) \geq an - b$ for all $n \gg 0$. Then, there is an $m$, such that
\[ \text{GKdim}(\mathcal{P}) = \limsup_{n \to \infty} \log_n(\sum_{i=0}^{n} \dim(\mathcal{P}(i))) \geq \limsup_{n \to \infty} \log_n(\sum_{i=m}^{n} (ai - b)) = 2, \]
yielding a contradiction. The proof of part (2) is similar to the above. \hfill $\square$

4.2. Gap theorem for ns operads. In this subsection, we first prove three lemmas and then use them to prove an analogue of Bergman’s gap theorem for finitely generated ns operads.

Roughly speaking, the following lemma says that, if the GK-dimension of a finitely generated operad is less than 2, then there exists a uniform upper bound for the heights of maximal subtrees that are rooted at the pivot branch and do not contain the pivot top internal vertex.

Lemma 4.8. Let $\mathcal{X}$ be a finite operation alphabet and $\mathcal{P} = \mathcal{T}(\mathcal{X})/\mathcal{I}$. Write $\mathcal{V} = F\mathcal{X}$. Suppose $\dim(\sum_{i=0}^{d} \mathcal{V}^i)/(\sum_{i=0}^{d-1} \mathcal{V}^i) \leq d - 2$ for some $d$ (or $\text{GKdim}(\mathcal{P}) < 2$). Then there exists a positive integer $M_1$ such that, for all $\tau \in \text{Tr(\text{Irr}(\mathcal{I})))}$,
\[ h(\text{MaxSub}_{\tau}(v, v')) < M_1 \]
where $v \in \text{Pivot}(\tau)$, $v' \in \text{Int}(\tau) \setminus \text{Pivot}(\tau)$.

Proof. Suppose to the contrary that for each $n \in \mathbb{N}$ there is a tree monomial $T_n \in \text{Irr}(\mathcal{I})$ with underlying PRT $\tau_n$ and a maximal subtree $\tau_n' := \text{MaxSub}_{\tau_n}(v_n, v_n')$ such that $h_n := h(\tau_n') \geq n$, where $v_n \in \text{Pivot}(\tau_n)$ and $v_n' \in \text{Int}(\tau_n) \setminus \text{Pivot}(\tau_n)$. Assume that the vertices in $\text{Pivot}(\tau_n')$ are labelled from $v_n$ to $\text{TIV}(\tau_n')$ by generating operations $x, y_1, y_2, \ldots, y_n$, and that the vertices in $\text{Int}(\tau_n) \cap \text{Pivot}(\tau_n)$ with height $\geq h(v_n)$ are labelled from $v_n$ to $\text{TIV}(\tau_n)$ by $x, x_1, x_2, \ldots, x_m$ (see Figure 6, where only a subset of $\text{Vert}(\tau_n)$ are drawn and the pivot branch is drawn vertically).

It follows from the definition of pivot branch that $m \geq h_n \geq n$. Consider the submonomial $T_n''$ of $T_n$ with underlying PRT $\tau_n'' \subseteq \tau_n$ for which $\text{Root}(\tau_n'') = \text{Parent}_{\tau_n}(v_n)$ and
\[ \text{Int}(\tau_n'') = \text{Pivot}(\tau_n') \cup \{v \in \text{Pivot}(\tau_n) \cap \text{Int}(\tau_n) : h(v) \geq h(v_n)\}. \]
Without loss of generality, we assume $T_n''$ is of the form in Figure 6 and denote it by
\[ T_n'' = (x \circ_{j} (x_1 x_2 \cdots x_m)) \circ_{i} (y_1 y_2 \cdots y_{h_n}) \]

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where $i$ and $j$ are proper composition indices and $i < j$. The following are distinct submonomials of $T''_n$ of weight $n$:

$$(x \circ_j (x_1 \cdots x_s)) \circ_i (y_1 \cdots y_{n-s-1}), \ 1 \leq s \leq n - 1.$$ 

These submonomials are pairwise distinct since their underlying PRTs are pairwise distinct. Thus $\text{Irr}(I)$ contains at least $n - 1$ elements of weight $n$, namely, $\dim(P_{n \geq 0} V_i) / (P_{n-1 \geq 0} V_i) \geq n - 1$ for all $n$, yielding a contradiction. □

The following lemma says that, under suitable conditions, if an internal vertex $v$ in the pivot branch has a child that is an internal vertex and not in the pivot branch, then $v$ must be either “close” to the root or “close” to the top of the PRT.

**Lemma 4.9.** Let $\mathcal{X}$ be a finite operation alphabet and $\mathcal{P} = T(\mathcal{X})/I$. Write $\mathcal{V} = F\mathcal{X}$. Assume $\dim(\sum_{i=0}^{d} V_i) / (\sum_{i=0}^{d-1} V_i) \leq d - 3$ for some $d \geq 3$ (or $\text{GKdim}(\mathcal{P}) < 2$). Suppose that $W := \{ T \in \text{Irr}(I) : T \text{ not single-branched } \} \neq \emptyset$.

Then there exists a positive integer $M_2$ such that, for all $\tau \in \text{Tr}(W)$ and all $v \in \text{Pivot}(\tau)$ such that there is a maximal subtree $\text{MaxSub}(v, v')$ for some $v' \in \text{Int}(\tau) \setminus \text{Pivot}(\tau)$, the following inequality holds

$$\min\{h(v), h(\tau) - h(v)\} < M_2.$$

**Proof.** Suppose to the contrary that for each $n \in \mathbb{N}^*$ there exist $T_n \in W$ with $\text{Tr}(T_n) = \tau_n$, $v_n \in \text{Pivot}(\tau_n)$, and $v'_n \in \text{Int}(\tau_n) \setminus \text{Pivot}(\tau_n)$ such that $\tau_n$ has a maximal subtree $\text{MaxSub}(v_n, v'_n)$ and $\min\{h(v_n), h(\tau_n) - h(v_n)\} \geq n$. Without loss of generality, we suppose that $\text{Parent}(v'_n) = v_n$. Consider the subtree $\tau'_n \subseteq \tau_n$ that has internal vertices

$$\text{Int}(\tau'_n) = (\text{Int}(\tau_n) \cap \text{Pivot}(\tau_n)) \cup \{v'_n\}.$$ 

Assume that the vertex $v'_n$ is labelled by generating operation $x'$ and that the internal vertices in $\text{Pivot}(\tau_n)$ are labelled from bottom to top by generating operations

$$x_m, x_{m-1}, \ldots, x_1, x, y_1, y_2, \ldots, y_q$$

for some integers $m \geq n - 1$, $q \geq n$ (see Figure 7, where only a subset of $\text{Vert}(\tau'_n)$ are drawn).
Consider following submonomial of $T'_{n}$ of weight $n + 1$

$$(x_s x_{s-1} \cdots x_1 x y_1 y_2 \cdots y_{n-s-1}) \circ_i x', \; 1 \leq s \leq n - 1$$

with suitable composition indices $i_s$. These submonomials are pairwise distinct since so are their underlying PRTs. Thus $\text{Irr}(\mathcal{I})$ contains at least $n - 1$ elements of weight $n + 1$, namely, $\dim(\sum_{i=0}^{n+1} V^i)/(\sum_{i=0}^{n} V^i) \geq n - 1$, yielding a contradiction. 

The following lemma is a generalization of Lemma 4.5.

**Lemma 4.10.** Retain the hypotheses of Lemma 4.9. Then there exist positive integers $d_1$ and $d_2$ such that $\text{Irr}(\mathcal{I})$ contains at most $d_1$ tree monomials of weight $h$ for all $h \geq d_2$.

**Proof.** Let $M_1$ and $M_2$ be the same as in Lemmas 4.8 and 4.9. Suppose $T \in \text{Irr}(\mathcal{I})$ and denote $\tau := \text{Tr}(T)$ and $n := \text{wt}(\tau)$. Assume that $h(\mathcal{I})$ is large enough for the following decomposition (say, $h(\tau) > 2(M_1 + 2M_2)$). Let $T_1$, $T_2$ and $T_3$ be the submonomials of $T$ whose underlying PRTs $\tau_1$, $\tau_2$ and $\tau_3$ have internal vertices

\begin{align*}
\text{Int}(\tau_1) &= \{ v \in \text{Int}(\tau) : h(v) \leq M_1 + M_2 \}, \\
\text{Int}(\tau_2) &= \{ v \in \text{Int}(\tau) : M_1 + M_2 \leq h(v) \leq h(\tau) - M_2 \}, \\
\text{Int}(\tau_3) &= \{ v \in \text{Int}(\tau) : h(v) \geq h(\tau) - M_2 \}.
\end{align*}

These subtrees are well-defined. In fact, by Lemmas 4.8 and 4.9, within each $\text{Int}(\tau_i)$ vertices are connected by the parent function of $\tau$ and $\tau$ has a unique internal vertex $v_1$ of height $M_1 + M_2$ (respectively, $v_2$ of height $h(\tau) - M_2$), whose parent is the root of $\tau_2$ (respectively, $\tau_3$). Note that, by Lemmas 4.8 and 4.9, all $v \in \text{Int}(\tau) \setminus \text{Pivot}(\tau)$ are contained in $\text{Int}(\tau_1) \cup \text{Int}(\tau_3)$ and thus $\tau_2$ is single-branched. See Figure 8 for an example of such decomposition, where only a subset of the internal vertices of $\tau$ are drawn and all leaves are omitted.

Set $a := \max\{ \text{Ar}(x) : x \in \mathcal{X} \}$ and $c := |\mathcal{X}|$. Then

(E4.10.1) $M_1 + M_2 = h(\tau_1) \leq \text{wt}(\tau_1) < M_3 := 1 + a + a^2 + \cdots + a^{M_1+M_2}$
Figure 8. Decomposition of $\tau$, where $h(v_1) = M_1 + M_2$, $h(v_2) = h(\tau) - M_2$, $v_1 \in \text{Int}(\tau_1) \cap \text{Int}(\tau_2)$ and $v_2 \in \text{Int}(\tau_2) \cap \text{Int}(\tau_3)$. 

and

(E4.10.2) \[ M_2 + 1 = h(\tau_3) \leq \text{wt}(\tau_3) < M_3. \]

There are only finitely many, say $M_4$, PRTs of weight $< M_3$ and thus there are at most $c^{M_3} \cdot M_4$ tree monomials of weight $< M_3$. It follows from $\text{wt}(\tau_1) + \text{wt}(\tau_2) + \text{wt}(\tau_3) = n + 2$ and inequalities (E4.10.1) and (E4.10.2) that $h(\tau_2) = \text{wt}(\tau_2) > n - 2M_3 + 2$.

Thus, for all $n \geq d + 2M_3 - 2$, we have $h(\tau_2) > n - 2M_3 + 2 \geq d$. By Lemma 4.5 there are at most $(d - 1)^3$ possibilities for $T_2$ for all $n \geq d + 2M_3 - 2$. Since $T$ is uniquely determined by its submonomials $T_1$, $T_2$ and $T_3$, there are at most $(c^{M_3} \cdot M_4)^2(d - 1)^3$ possibilities for $T$. Therefore, there exist positive integers $d_1 := (c^{M_3} \cdot M_4)^2(d - 1)^3$ and $d_2 := d + 2M_3 - 2$

such that $\text{Irr}(I)$ contains at most $d_1$ tree monomials of weight $h$ for all $h \geq d_2$. \[ \square \]

Now we are ready to prove our main result.

**Theorem 4.11.** Let $\mathcal{X}$ be a finite operation alphabet and $\mathcal{P} = \mathcal{T}(\mathcal{X})/I$. Write $\mathcal{V} = \mathbb{F}\mathcal{X}$. Assume $\dim(\sum_{i=0}^{d} \mathcal{V}^i)/(\sum_{i=0}^{d-1} \mathcal{V}^i) \leq d - 3$ for some $d \geq 3$ (or $\text{GKdim}(\mathcal{P}) < 2$). Then there exist positive real numbers $a, b$ such that $\dim \sum_{i=0}^{n} \mathcal{P}(i) \leq an + b$ for all $n \geq 0$. As a consequence $\text{GKdim}(\mathcal{P}) \leq 1$.

**Proof.** Define

$\dim \sum_{i=0}^{n} \mathcal{V}^i$
for all $n \in \mathbb{N}$. By Lemma 4.10 there exist positive integers $d_1$ and $d_2$ such that \( \text{Irr}(I) \) contains at most $d_1$ tree monomials of weight $h$ for all $h \geq d_2$. Thus for all $n \geq d_2$, setting $n := d_2 + q$, we have that

\[
d_V(n) \leq d_V(d_2) + qd_1 = d_V(d_2) + (n - d_2)d_1,
\]

and the function on the right hand side is linear in $n$. Now the assertion follows from (E3.3.1). The consequence is clear. \( \square \)

Theorem 0.1 follows from Theorem 4.11 easily.

5. Single-generated ns operads

We say an operad $P$ is single-generated (or single-element generated) if it is generated by an operation alphabet $\mathcal{X}$ that consists of a single element. In this section, we will describe an approach that can be used to construct a single-generated single-branched operad from a finitely generated monomial algebra. This procedure is called the operadization of a finitely generated monomial algebra, which is different from that given in Construction 1.3.

Fix an integer $d \geq 2$. Denote $X_0 = \{x_1, x_2, \ldots, x_d\}$ and $X_0^*$ the free monoid generated by $X_0$. (Note that $X_0$ is not the generating alphabet $X$ of an operad). Let $\mathbb{F}(X_0) := \mathbb{F}(x_1, x_2, \ldots, x_d) = \bigoplus_{l \geq 0} \mathbb{F}(X_0)_l$ be the free graded algebra generated by $X_0$ with $\deg(x_i) = 1$ ($1 \leq i \leq d$).

We always assume that the tree monomials $T_d$ and $R_{i,j}^\vee$ ($1 \leq i < j \leq d$) are defined as follows, see Figure 9:

![Figure 9](image)

Suppose that $\tilde{Q}$ is the operad presented by generator $\mathcal{X} = \{a\}$ and relations \{(a $\circ_j a) \circ_{i} a$, $1 \leq i < j \leq d$\}, then $\tilde{Q}$ is a single-branched operad. We also say $\tilde{Q}$ is the operad generated by $T_d$ and subject to the relations $R_{i,j}^\vee$ ($1 \leq i < j \leq d$).

Identifying the tree monomial $T_d$ and its labelling $a$, we define the following $\mathbb{F}$-linear operadization map, which is an isomorphism of vector spaces,

\[
O : \mathbb{F}(X_0) \longrightarrow \tilde{Q}_{\geq 2}
\]

by setting

\[
O(w) = O(x_{i_1}x_{i_2} \cdots x_{i_k}) = a \circ_{i_1} a \circ_{i_2} \cdots \circ_{i_k} a := (a \circ_{i_1} (a \circ_{i_2} (\cdots \circ_{i_k} a)))
\]

for any $w = x_{i_1}x_{i_2} \cdots x_{i_k} \in X_0^*$. See (E4.0.1) and (E4.0.2).
Now we are ready to give the following construction, in the proof of which we describe an operadization of a finitely generated monomial algebra.

**Construction 5.1.** For any finitely generated monomial algebra $A = \mathbb{F}\langle X_0 \rangle / I$ with $I \subseteq \bigoplus_{i \geq 2} \mathbb{F}\langle X_0 \rangle_i$ being a monomial ideal of the free algebra, there exists a single-generated single-branched ns operad $Q_A$ such that

$$GK\dim(Q_A) = GK\dim(A).$$

**Proof.** Since $I$ is a monomial ideal of $\mathbb{F}\langle X \rangle$, it is routine to check that $O(I)$ is an ideal of $eQ$.

Define the operad $Q_A$ to be the quotient operad $eQ / O(I)$. Noting that

$$\dim Q_A(n) = \begin{cases} 
1, & \text{when } n = 1 \text{ or } d, \\
\dim A_l, & \text{when } n = (l + 1)d - l \text{ and } l \geq 1, \\
0, & \text{otherwise},
\end{cases}$$

we obtain that

$$GK\dim(Q_A) = GK\dim(A).$$

□

**Definition 5.2.** Suppose $d := |X_0| \geq 2$. The single-generated single-branched ns operad $Q_A = eQ / O(I)$ defined in Construction 5.1 is called the **operadization** of the finitely generated monomial algebra $A = \mathbb{F}\langle X_0 \rangle / I$, where $I \subseteq \bigoplus_{i \geq 2} \mathbb{F}\langle X_0 \rangle_i$ is a monomial ideal of the free algebra.

Note that Construction 5.1 is related to some ideas presented in [DMR20, DT20]. For example, a special case of this construction is given in [DT20, Section 7.1.4].

Next we will present some examples of single-generated single-branched operads by using the operadization procedure described above.

**Example 5.3.** Let $d = 2$. Suppose $T_2, R_{1,2}^\vee, R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}$ are defined as in Figure 10.

(1) If $Q$ is the operad generated by $T_2$ and subject to the relation $R_{1,2}^\vee$, then we can check that $\dim Q(n) = 2^{n-2}$ for $n \geq 2$. Hence in this case $GQ(z) = z + z^2$ and $GK\dim(Q) = \infty$.

(2) Define $Q$ to be the operad generated by $T_2$ and subject to the relations $R_{1,2}^\vee, R_{1,1}$. Then we can check that $Q$ is an operadization of the graded algebra $A := \mathbb{F}\langle x_1, x_2 \rangle / (x_1^2)$. and

$$\dim Q(n) = \dim Q(n-1) + \dim Q(n-2) \text{ for } n \geq 2.$$ 

Hence in this case $GQ(z) = \frac{z}{1 - z^2 + z^2}$, $GK\dim(Q) = \infty$, and we call $Q$ the **Fibonacci operad**.

(3) Define $Q$ to be the operad generated by $T_2$ and subject to the relations $R_{1,2}^\vee, R_{2,1}, R_{2,2}$. Then $Q$ is an operadization of the graded algebra $\mathbb{F}\langle x_1, x_2 \rangle / (x_2x_1, x_2^3)$. and

$$\dim Q(n) = \begin{cases} 
1, & \text{when } n = 1 \text{ or } 2, \\
2, & \text{when } n \geq 3, \\
0, & \text{when } n = 0.
\end{cases}$$

Hence $GQ(z) = z + z^2 + \frac{z^3}{1 - z^2}$ and $GK\dim(Q) = 1$. 
If we consider a graded algebra $A = F\langle x_1, x_2 \rangle / \langle x_1x_2 - x_2x_1 \rangle$ which is not a monomial algebra. It is easy to check that $O(I)$ is not an ideal of $\bar{Q}$. So it requires more work to understand the quotient operad $\bar{Q}/(O(I))$ in this case.

Now we are ready to prove Theorem 0.2.

Proof of Theorem 0.2. (1) Let $P$ be a finitely generated locally finite ns operad. It is clear that $\text{GKdim}(P) \geq 0$. By Proposition 3.8, $\text{GKdim}(P)$ is not strictly between 0 and 1. By Theorem 0.1, $\text{GKdim}(P)$ is not between 1 and 2. Therefore $\text{GKdim}(P) \in R_{\text{GKdim}}$, see \eqref{E0.0.1} for the definition of $R_{\text{GKdim}}$.

(2) Let $r \in R_{\text{GKdim}}$. Let $A$ be a finitely generated algebra of GK-dimension $r$. By [Bel15, Remark 4.1], we can assume $A$ is a monomial algebra finitely generated in degree 1. By Construction 5.1, $Q_A$ is a single-generated single-branched locally finite operad with $\text{GKdim}(Q_A) = r$. The assertion follows.

6. Generating series and exponential generating series

The generating series of an operad is defined in \eqref{E0.2.1}. By \cite{KKP15} (0.1.1), the exponential generating series of an operad $P$ is defined to be

\begin{equation}
E_P(z) := \sum_{n \geq 0} \frac{\dim P(n)}{n!} z^n.
\end{equation}

Clearly, $G_P(z)$ (resp. $E_P(z)$) contains more information than $\text{GKdim}(P)$. A list of (exponential) generating series of operads are given in [Zin12, KKP15]. Several authors have recently been studying the holonomic and differential algebraic properties of $G_P(z)$ (resp. $E_P(z)$), see Definition 0.3.
By [Sta80] Theorem 1.5, $F(z) := \sum_{n=0}^{\infty} f(n)z^n$ is holonomic if and only if the sequence $\{f(n)\}_{n \geq 0}$ satisfies a recurrence relation of the form
\begin{equation}
\tag{E6.0.2}
 f(n) = q_1(n)f(n-1) + \cdots + q_{m}(n)f(n-m), \quad n > 0
\end{equation}
for some fixed $m$ and rational functions $q_1(x), \ldots, q_m(x)$. For all examples given in Example 5.3, $G_P(z)$ are rational, and consequently, holonomic. Using [E6.0.2], it is easy to see that
\begin{equation}
\tag{E6.0.3}
 G_P(z) \text{ is holonomic if and only if so is } E_P(z).
\end{equation}

For two integers $a$ and $b$, let $[a, b]_N := [a, b] \cap \mathbb{N}$. Suppose the sequence $\{f(n)\}_{n \geq 0}$ has infinitely many nonzeros. Let
\[ \Phi_F = \{ n \mid f(n) = 0 \} \]
and write $\Phi$ as
\[ \Phi = [i_1, j_1]_N \cup [i_2, j_2]_N \cup \cdots \]
where $i_s \leq j_s \leq i_{s+1} - 2$ for all $s \geq 1$. Then the following is true.

**Lemma 6.1.** Let $F(z) = \sum_{n=0}^{\infty} f(n)z^n$ with infinitely many nonzero coefficients. Write $\Phi_F = \bigcup_{i \geq 1} [i_s, j_s]_N$. Suppose that $\limsup_{n \to \infty} (j_s - i_s) = \infty$. Then $F(z)$ is not holonomic.

**Proof.** Suppose to the contrary that $F(z)$ is holonomic. Then [E6.0.2] holds for some $m$. Pick any $s$ such that $j_s - i_s \geq m + 1$. By the definition of $\Phi_F$, $f(n) = 0$ for all $i_s \leq n \leq j_s$. By [E6.0.2] and induction, we have that $f(n) = 0$ for all $n \geq i_s + m$. Therefore $F(z)$ has only finitely many nonzero coefficients, a contradiction. \hfill $\square$

The next example shows that we can easily construct a monomial algebra such that its Hilbert series is not holonomic.

**Example 6.2.** Let $U$ be the monomial algebra $\mathbb{F}(x_1, x_2)/I$ where $\deg x_1 = \deg x_2 = 1$ and $I$ is a span of monomials of the form
\begin{itemize}
    \item [(a)] all monomials having degree $\geq 3$ in $x_2$,
    \item [(b)] $x_i^j x_2^k$ with $i > 0$ or $j > 0$,
    \item [(c)] $x_2 x_i^j x_2$ where $(2m + 1)^{2m+1} - 1 \leq i \leq (2m + 2)^{2m+2} - 1$ for some integer $m \geq 0$.
\end{itemize}
Let $\Lambda = \bigcup_{m=0}^{\infty} [(2m + 1)^{2m+1} + 1, (2m + 2)^{2m+2} + 1]_\mathbb{N}$ which is a subset of $\mathbb{N}$. Let $\Lambda^c = \mathbb{N} \setminus \Lambda$. Define
\[ \delta_{\Lambda}(n) = \begin{cases} 0 & n \in \Lambda, \\ 1 & n \in \Lambda^c. \end{cases} \]
Then $U$ has an $\mathbb{F}$-basis \( \{x_i^j\}_{i \geq 0} \cup \{x_i^j x_2\}_{i \geq 0} \cup \{x_2 x_i^j\}_{i \geq 0} \cup \{x_2 x_i^j x_2\}_{i + 2 \in \Lambda^c} \) and its Hilbert series is
\[ H_U(z) = 1 + 2z + \sum_{n=2}^{\infty} (3 + \delta_{\Lambda}(n)) z^n = 1 + 2z + \frac{3z^2}{(1 - z)} + V(z) \]
where $V(z) = \sum_{n=2}^{\infty} \delta_{\Lambda}(n) z^n$. Since $V(z)$ satisfies the hypothesis of Lemma 6.1 it is not holonomic. Let $P_U$ be the construction given in Construction 1.3. It is easy to see that $\text{GKdim}(P_U) = \text{GKdim}(U) = 1$. 

and that
\[ G_{P_U}(z) = zH_U(z) = z(1 + 2z + \frac{3z^2}{1 - z}) + zV(z) \]
where \( z(1 + 2z + \frac{3z^2}{1 - z}) \) is holonomic, but \( zV(z) \) is not (by Lemma 6.1 again). Therefore \( G_{P_U}(z) \) is not holonomic by [Bel14 Holonomic Theorem 2].

Let \( Q_U \) be the construction given in Construction 5.1 Then \( \text{GKdim}(Q_U) = 1 \) and
\[ G_{Q_U}(z) = \sum_{d \geq 1} H_U(z^d) = z + z^dH_U(z^{d-1}) = z + z^2H_U(z) \]
\[ = \sum_{d = 1} \left( 1 + 2z + \frac{3z^2}{1 - z} \right) + z^2V(z) \]
as \( d = \text{dim} U_1 = 2 \). By an argument similar to the previous paragraph, \( G_{Q_U}(z) \) is not holonomic.

Now we are ready to prove Proposition 0.5.

**Proof of Proposition 0.5.** Let \( r \in R_{\text{GKdim}} \setminus \{0\} \). We claim that there is a monomial algebra \( B \) finitely generated in degree 1 such that \( \text{GKdim}(B) = r \) and that \( H_B(z) \) is not holonomic. By [Bel15 Remark 4.1], there is a monomial algebra \( A \) finitely generated in degree 1 such that \( \text{GKdim}(A) = r \). If \( H_A(z) \) is not holonomic, then we are done by setting \( B = A \). If \( H_A(z) \) is holonomic, let \( B = \mathbb{F} \oplus (A_{\geq 1} \oplus U_{\leq 1}) \). Then \( B \) is a monomial algebra finitely generated in degree 1 such that \( \text{GKdim}(B) = r \) and that \( H_B(z) = H_A(z) + H_U(z) = 1 \) Since \( H_U(z) \) is not holonomic, by [Bel14 Holonomic Theorem 2], \( H_B(z) \) is not holonomic. So we proved the claim.

Let \( \mathcal{P} = Q_B \) as in Construction 5.1 Then \( \mathcal{P} \) is single-generated, \( \text{GKdim}(\mathcal{P}) = \text{GKdim}(B) = r \) and \( G_{\mathcal{P}} = z + z^dH_B(z^{d-1}) \). Since \( H_B(z) \) is not not holonomic, it is routine to show that \( G_{\mathcal{P}}(z) \) is not holonomic. \( \square \)

The following lemma is easy and the proof is analogous to the one given in the quadratic case, see the proof of [Dot19 Corollary 4.2(i)].

**Lemma 6.3.** Let \( A \) be a connected graded locally finite algebra and let \( \mathcal{P}_A \) be the operad constructed in Construction 1.3

1. \( A \) is finitely generated if and only if so is \( \mathcal{P}_A \).
2. \( A \) is finitely presented if and only if so is \( \mathcal{P}_A \).

For the rest of this section, we construct a finitely presented locally finite ns operad such that its generating series is not holonomic. Therefore we give a “non-generic” counterexample to [KP15 Expectation 2] (see Expectation 0.4).

Let us recall some history in the setting of noncommutative graded algebra. In 1972 Govorov conjectured that the Hilbert series of a finitely presented graded algebra is rational [Gov72]. This was shown to be false, for example, by Shearer in [She80] by constructing an irrational (but algebraic) Hilbert series of a finitely generated graded algebra. Shearer mentioned a similar construction giving also an example with a transcendental (but still holonomic) Hilbert series. His third example, involving the generating function of the number of partitions, indeed has its Hilbert series not holonomic. A similar example was given in [Smi76] which contains the following example as a special case.

**Example 6.4.** Suppose \( \text{char } \mathbb{F} = 0 \). Let \( L \) be the graded Lie algebra with basis \( \{e_1, e_2, \cdots, e_n, \cdots\} \) with \( \deg e_i = i \) for all \( i \) and Lie bracket determined by
\[ [e_i, e_j] = (i - j)e_{i+j}, \quad \forall i \neq j. \]
This is a subalgebra of the Witt algebra. Let $A$ be the universal enveloping algebra of $L$. Then $A$ has intermediate growth and is generated by $e_1$ and $e_2$ (deg $e_i = i$) and subject to the relations $e_3e_2 - e_2e_3 = e_5$ and $e_5e_2 - e_2e_5 = 3e_7$. So $A$ is finitely presented, but not generated in degree 1.

It is easy to see that

$$H_A(z) = \prod_{i=1}^{\infty} \frac{1}{(1 - z^i)}.$$ 

Note that $H_A(z)$ is equal to $P(z) := \sum_{n=0}^{\infty} p(n)z^n$ where $p(n)$ is the number of partitions of $n$. 

Now we list some facts about $P(z)$:

- (P1) $P(z)$ is not holonomic.
- (P2) $zP(z)$ is not holonomic.
- (P3) $Q(z) := z(zP(z))' = \sum_{n=0}^{\infty} p(n)(n + 1)z^{n+1}$ is not holonomic.

Note that Parts (P2,P3) follow from part (P1) immediately.

Now let $\mathcal{P}_A$ be the operad given by Construction 1.3. By Lemma 6.3, $\mathcal{P}_A$ is finitely generated. By (E1.3.2)

$$G_{\mathcal{P}_A}(z) = zH_A(z) = zP(z)$$

which is not holonomic by (P2). Therefore $\mathcal{P}_A$ is a “non-generic” counterexample to [KP15, Expectation 2].

Note that there is a quadratic algebra with similar properties on its Hilbert series, but with 14 generators and 96 quadratic relations [Koc15, Theorem 1(iv)]. The algebra $A$ could be replaced by the Kocak’s example.

**Remark 6.5.** Using the same proof as above, one sees that a version of Proposition 0.5 holds for exponential generating series. See (E6.0.1) for the definition of the exponential generating series.

### 7. Comments on symmetric operads

In this final section we make some comments on symmetric operads and prove the main result of this section, namely, Theorem 0.7.

First of all we refer to [LV12, Chapter 5] or [BYZ20, Definition 1.2] for the definition of a symmetric operad.

Given a graded algebra with augmentation, there is an easy construction of a symmetric operad similar to the one given in Construction 1.3.

**Construction 7.1.** Let $A := \bigoplus_{i \geq 0} A_i$ be a locally finite $\mathbb{N}$-graded algebra with unit $1_A$. Suppose that $A$ has a graded augmentation $\epsilon : A \to \mathbb{F}$ such that $\mathfrak{m} := \ker \epsilon$ is a maximal graded ideal of $A$. We let $C_n$ be the cyclic group of order $n$ and elements in $C_n$ are denoted by $\{1, \cdots, n\}$.

Let $S_A(0) = 0$ and $S_A(n) = A_{n-1} \otimes \mathbb{F}C_n$ for all $n \geq 1$. Elements in $S_A(n)$ are $\mathbb{F}$-linear combinations of $(a, i)$ where $a \in A_{n-1}$ and $1 \leq i \leq n$. The $S$-action on $S_A(n)$ is determined by

$$(a, i) \ast \sigma = (a, \sigma^{-1}(i))$$

for all $\sigma \in S_n$. 

Define partial compositions as follows, for \(1 \leq s \leq m\),
\[
\circ_s : \mathcal{S}_A(m) \otimes \mathcal{S}_A(n) \to \mathcal{S}_A(n + m - 1),
\]
\[
(a_{m-1}, i) \otimes (a_{n-1}, j) \mapsto \begin{cases}
(c_{a_{m-1}, i}) & a_{n-1} = c_{1A}, c \in \mathbb{F}, \\
(a_{m-1}a_{n-1}, i + j - 1) & a_{n-1} \in \mathcal{M}, s = i, \\
0 & a_{n-1} \in \mathcal{M}, s \neq i.
\end{cases}
\]

We claim that that \(\mathcal{S}_A := \{\mathcal{S}_A(n)\}_{n \geq 0}\) is a symmetric operad with identity \(id = (1_A, 1)\). We give a sketch proof below.

We refer to [LV12] Section 5.3.4 for the partial definition of a symmetric operad. In fact, a symmetric operad \(\mathcal{P}\) is an \(\mathcal{S}\)-module satisfying the axioms of a nonsymmetric operad [Definition 1.1] and the following two additional equations
\[
\mu \circ_s (\nu * \sigma) = (\mu \circ_s \nu) * \sigma'
\]
\[
(\mu * \phi) \circ_s \nu = (\mu \circ_{\phi(s)} \nu) * \phi''
\]
where \(\mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n)\), \(1 \leq s \leq m, \sigma \in \mathcal{S}_n, \phi \in \mathcal{S}_m\) and where \(\sigma' = 1_m \circ_s \sigma\) and \(\phi'' = \phi \circ_s 1_n\). See [LV12] Section 5.3.4 and [BYZ20] (E1.2.1) and (E8.1.3) for the explanation of \(\sigma'\) and \(\phi''\). We first verify (E7.1.1) and (E7.1.2) and then the rest of axioms given in Definition 1.1 for \(\mathcal{P} := \mathcal{S}_A\).

**Verification of (E7.1.1):** Write \(\mu = (a_{m-1}, i)\) and \(\nu = (a_{n-1}, j)\). If \(n = 1\) and \(a_{n-1} = c_{1A}\) (or if \(n = 1\) and \(a_{n-1} \in \mathcal{M}\), then \(j = 1\) and \(\sigma = 1_1 \in \mathcal{S}_1\) and \(\sigma' = 1_{n+1} \in \mathcal{S}_{n+1}\). Clearly (E7.1.1) holds. If \(n \geq 2\) and \(s \neq i\), both sides of (E7.1.1) are zero. It remains to consider the case when \(n \geq 2\) and \(s = i\). Write
\[
\sigma = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ 1 & 2 & \cdots & n \end{pmatrix}
\]
where by convention \(k_i = \sigma^{-1}(i)\) for all \(i\). Then, by definition,
\[
\sigma' = \begin{pmatrix} 1 & \cdots & i - 1 & k_1 + i - 1 & k_2 + i - 1 & \cdots & k_n + i - 1 & i + n & \cdots & n + m - 1 \\ 1 & \cdots & i - 1 & i & i + 1 & \cdots & n + i - 1 & i + n & \cdots & n + m - 1 \end{pmatrix}.
\]
In this case, we have
\[
\text{LHS of (E7.1.1)} = (a_{m-1}, i) \circ_i [(a_{n-1}, j) * \sigma] = (a_{m-1}, i) \circ_i (a_{n-1}, \sigma^{-1}(j))
\]
\[
= (a_{m-1}a_{n-1}, i + \sigma^{-1}(j) - 1) = (a_{m-1}a_{n-1}, i + k_j - 1)
\]
\[
\text{RHS of (E7.1.1)} = [(a_{m-1}, i) \circ_i (a_{n-1}, j)] * \sigma' = (a_{m-1}a_{n-1}, i + j - 1) * \sigma'
\]
\[
= (a_{m-1}a_{n-1}, (\sigma')^{-1}(i + j - 1)) = (a_{m-1}a_{n-1}, k_j + i - 1).
\]
Therefore (E7.1.1) holds.

**Verification of (E7.1.2):** Recycle the letter \(k_i\) and write
\[
\phi = \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}
\]
using the convention of (E7.1.3). If \(a_{n-1} = c_{1A}\), then \(n = j = 1\) and \(\phi'' = \phi \circ_s 1_1 = \phi\). Consequently, \(\phi^{-1}(i) = k_i = (\phi'')^{-1}(i)\). In this case,
\[
\text{LHS of (E7.1.2)} = [(a_{m-1}, i) * \phi] \circ_s (c_{1A}, 1) = (a_{m-1}, \phi^{-1}(i)) \circ_s (c_{1A}, 1)
\]
\[
= (ca_{m-1}, \phi^{-1}(i)) = (ca_{m-1}, k_i)
\]
\[
\text{RHS of (E7.1.2)} = [(a_{m-1}, i) \circ_{\phi(s)} (c_{1A}, 1)] * \phi'' = (ca_{m-1}, i) * \phi''
\]
\[
= (ca_{m-1}, i) * \phi = (ca_{m-1}, k_i).
\]
Hence (E7.1.2) holds. Next we assume that \( a_{n-1} \in \mathfrak{m} \). If \( s \neq \phi^{-1}(i) \), then both sides of (E7.1.2) are zero. It remains to consider the case when \( s = \phi^{-1}(i) \) or \( i = \phi(s) \). By definition,

\[
\phi'' = \begin{pmatrix}
  k_1' & \cdots & k_{i-1}' & k_i' & k_i' + 1 & \cdots & k_i' + n - 1 & k_{i+1}' & \cdots & k_m' \\
  1 & \cdots & i-1 & i & i+1 & \cdots & i+n-1 & i+n & \cdots & m+n-1
\end{pmatrix}
\]

where \( k_i' = \left\{ \begin{array}{ll}
  k_i' + n - 1 & k_t > s \\
  k_t & k_t \leq s
\end{array} \right. \). In particular,

\[
(\phi'')^{-1}(i + j - 1) = k_i' + j - 1 = s + j - 1 = k_i + j - 1 = \phi^{-1}(i) + j - 1.
\]

Now we compute

LHS of (E7.1.2) = \([ (a_{m-1}, i) \ast \phi ] \circ_s (a_{n-1}, j) = ((a_{m-1}, i) \ast \phi) \circ_{\phi^{-1}(i)} (a_{n-1}, j) = (a_{m-1}a_{n-1}, \phi^{-1}(i) + j - 1)\]

RHS of (E7.1.2) = \([ (a_{m-1}, i) \ast \phi(a_s(a_{n-1}, j)) \ast \phi'' \circ_s (1_A, 1) = (a_{m-1}a_{n-1}, \phi''(a_{n-1}, j)) = (a_{m-1}a_{n-1}, \phi''(i + j - 1)) = (a_{m-1}a_{n-1}, \phi^{-1}(i) + j - 1)\]

which implies that (E7.1.2) holds in this case. So we verified (E7.1.2) for all cases.

**Verification of (E1.1.3):** It follows easily from the definition that \((1_A, 1) \circ_1 (a_{n-1}, j) = a_{n-1}, j) = (a_{n-1}, j) \circ_s (1_A, 1)\) for all \( 1 \leq s \leq n \), which is (E1.1.3).

**Verification of (E1.1.2):** Recall that (E1.1.2) is equivalent to

(E7.1.4) \( (\lambda \circ_s \mu) \circ_{s+1+m} \nu = (\lambda \circ_t \nu) \circ_s \mu \)

for \( \lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n) \) and \( 1 \leq s < t \leq l \). Write \( \lambda = (a_{l-1}, i), \mu = (a_{m-1}, j) \) and \( \nu = (a_{n-1}, k) \). If the LHS of (E7.1.4) were nonzero, we must have \( s = i \) and \( s + j - 1 = t - 1 + m \). But \( s < t \) and \( j \leq m \) which contradict the equation \( s + j - 1 = t - 1 + m \). So the LHS of (E7.1.4) is zero. For a similar reason, the RHS of (E7.1.4) is zero. Hence (E7.1.4) holds.

**Verification of (E1.1.1):** Recall that (E1.1.1) is equivalent to

(E7.1.5) \( (\lambda \circ_s \mu) \circ_{s+1+t} \nu = (\lambda \circ_t \nu) \circ_s \mu \)

for \( \lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n), 1 \leq s, t \leq l \) and \( 1 \leq t \leq m \). Write \( \lambda = (a_{l-1}, i), \mu = (a_{m-1}, j) \) and \( \nu = (a_{n-1}, k) \) as before. If the LHS of (E7.1.5) is nonzero, then we must have both \( s = i \) and \( t = j \). Similarly, if the RHS of (E7.1.5) is nonzero, then \( s = i \) and \( t = j \). So it suffices to consider the case when \( s = i \) and \( t = j \). In this case, both sides of (E7.1.5) are equal to \((a_{l-1}a_{m-1}a_{n-1}, i + j + k - 2)\). Hence (E7.1.5) holds.

Now we have verified all axioms of a symmetric operad for \( \mathcal{P} := \mathcal{S}_A \) and therefore \( \mathcal{S}_A \) is a symmetric operad.

It is obvious that

(E7.1.6) \( G_{\mathcal{S}_A}(z) = z(zH_A(z))' \).

**Lemma 7.2.** Let \( A \) be a connected graded algebra and let \( \mathcal{S}_A \) be the symmetric operad provided in Construction 7.1.
(1) If \(A\) is finitely generated, then \(S_A\) is finitely generated as a symmetric operad (resp. as a nonsymmetric operad).
(2) If \(A\) is finitely presented, then \(S_A\) is finitely presented as a nonsymmetric operad.
(3) If \(A\) is finitely presented, then \(S_A\) is finitely presented as a symmetric operad.

\textbf{Proof.} We continue to use the notation introduced in Construction 7.1

(1) Let \(m = \bigoplus_{i \geq 1} A_i\). Suppose \(V \subseteq m\) is a finite dimensional graded subspace that generates \(A\). Let \(\{v_1, \cdots, v_e\}\) be a basis of \(V\). We claim that \(S_A\) is generated by \(E\) such that \(E\) denotes the relation ideal generated by \(V\). Note that \(\text{Ar}(E(v_i, j)) = \text{deg}(v_i) + 1\). Since every element in \(S_A\) is of the form \((a_{n-1}, d)\) where \(1 \leq d \leq n\) and \(a_{n-1}\) is generated by \(V\) in \(A\), it can be generated by \((v_i, j)\) by partial compositions. Therefore \(S_A\) is finitely generated as a nonsymmetric operad (resp. as a symmetric operad).

(2) Suppose \(A\) is generated by a finite dimensional graded subspace \(V\) and subject to a finite dimensional relation subspace \(W := \sum_{j=1}^{r} f_j\). Then \(A\) is the factor algebra \(\mathbb{F}(V)/(W)\) where \(\mathbb{F}(V)\) is the free algebra generated by \(V\) and where \((W)\) denotes the relation ideal generated by \(W\). Note that \(\mathbb{F}(V)\) is connected graded. Let \(E\) be defined as in the proof of part (1) and let \(F^\ns(E)\) be the free nonsymmetric operad generated by \(E\) (see [1012, 5.9.6] for the definition).

Our first step is to define the relation subspace \(R\) of the \(F^\ns(E)\) such that \(S_A \cong F^\ns(E)/(R)\). For each homogeneous element \(g\) in the free algebra \(\mathbb{F}(V)\), fix an expression of \(g\) as a linear combination of possibly repeated monomials

\[ g = \sum c_i v_{i_1} \cdots v_{i_u} \]

where the sum is over \(i := (i_1, \cdots, i_u)\) and where \(c_i\) are scalars in \(\mathbb{F}\). Note that monomials \(\{v_{i_1} \cdots v_{i_u}\}\) in expression (E7.2.1) are not assumed to be distinct. For each term \(v_{i_1} \cdots v_{i_u}\) appearing in (E7.2.1), let \(\{k_{i_s}\}_{s=1}^u\) be a sequence of integers such that \(1 \leq k_{i_s} \leq \text{deg}(v_{i_s}) + 1\) for all \(1 \leq s \leq u\). We will use \(k_i\) to denote \(\{k_{i_s}\}_{s=1}^u\). Define

\[ |k_i| := \sum_{s=1}^u k_{i_s} - u + 1. \]

Fix any integer \(d\) between 1 and \(\text{deg}(g) + 1\). For each \(i := (i_1, \cdots, i_u)\) appearing in (E7.2.1), pick any sequence \(\{k_{i_s}\}_{s=1}^u\) such that \(|k_i| = d\). Such a sequence \(\{k_{i_s}\}_{s=1}^u\) is called a \(d\)-sequence. Let \(k_d\) be the collection of \(d\)-sequences associated to expression (E7.2.1).

Now we define the following element in the free operad \(F^\ns(E)\)

\[ r_{k_d}(g) := \sum_{k_i \in k_d} c_i (v_{i_1}, k_{i_1}) \circ (v_{i_2}, k_{i_2}) \circ \cdots \circ (v_{i_u}, k_{i_u}) \]

where each \((v_{i_1}, k_{i_1}) \circ (v_{i_2}, k_{i_2}) \circ \cdots \circ (v_{i_u}, k_{i_u})\) is a right normal tree monomial defined in (E4.0.1)-(E4.0.2).

For example, let

\[ h = v_1 v_4 - 2v_2 v_3 + 3v_1 v_4 \]

be in \(\mathbb{F}(V)\) where \(\text{deg} v_i = i\) for \(1 \leq i \leq 4\). Note that the first monomial in the above expression of \(h\) equals the third one. Let \(d = 4\) and we pick three \(d\)-sequences.
\{1, 4\}, \{3, 2\} and \{2, 3\} corresponding to three monomials in the expression of \( h \). Then
\[
q_k(h) = (v_1, 1) \circ_1 (v_4, 4) - 2(v_2, 3) \circ_3 (v_3, 2) + 3(v_1, 2) \circ_2 (v_4, 3).
\]

Let \( g' \) be another homogeneous element of degree equal to \( \deg(g) \) and fix an expression of \( g' \) similar to (E7.2.1). Let \( c \) and \( c' \) be scalars in \( \mathbb{F} \) and let \( f = cg + c'g' \) with expression of \( f \) induced by the expression of \( g \) and \( g' \). Let \( k_d \) (resp. \( k'_d \)) be a collection of \( d \)-sequences of \( g \) (resp. \( g' \)) corresponding to the monomials appeared in (E7.2.1). Then the disjoint union \( k_d \cup k'_d \) is a collection of \( d \)-sequences corresponding to the expression of \( f \). It follows from the definition that
\[
q_{k_d}(-k'_d)(f) = crk_d(g) + c'rk'_d(g').
\]

For any given \( d \), there are only finitely many possibilities for \( d \)-sequences \( \{k_i\}_{s=1}^u \), and consequently, for \( q_{k_d}(g) \).

Now let \( f_i \) be an element in the relation space \( W \) and fix an expression of \( f_i \) as in (E7.2.1). It is easy to check that all of
\[
q_{k_d}(f_i) = 0
\]
are relations of \( S_A \) for \( 1 \leq d \leq \deg f_i \) and different choices of \( d \)-sequences. By the definition of \( S_A \), the following are also relations of \( S_A \):
\[
\phi : \mathcal{F}^{ns}(E) \to S_A
\]
sending \( (v_i, j) \in E \subseteq \mathcal{F}^{ns}(E) \) to \( (v_i, j) \in S_A \). It is clear that \( \phi \) maps relations defined in (E7.2.3) to zero. Therefore \( \phi \) induces naturally a surjective morphism of nonsymmetric operads
\[
\Phi : \mathcal{F}^{ns}(E) \to S_A
\]
sending \( (v_i, j) \in E \subseteq \mathcal{F}^{ns}(E) \) to \( (v_i, j) \in S_A \). Our next step is to show that the natural map \( \phi \) is injective (consequently, bijective).

We need the following notion of leading form. For each tree monomial \( t \in \mathcal{F}^{ns}(E)/(R) \) of the form
\[
t = (v_{i_1}, j_{i_1}) \circ_{j_1} (v_{i_2}, j_{i_2}) \circ_{j_2} \cdots \circ_{j_{m-1}} (v_{i_m}, j_m),
\]
there exist finitely many elements of the form
\[
t' = (v_{i_1}, j'_{i_1}) \circ_{j'_{i_1}} (v_{i_2}, j'_{i_2}) \circ_{j'_{i_2}} \cdots \circ_{j'_{m-1}} (v_{i_m}, j'_{m}),
\]
such that \( j'_1 + \cdots + j'_{m} = j_1 + \cdots + j_m \) and \( 1 \leq j'_l \leq \deg(v_{i_l})+1 \) for \( 1 \leq l \leq m \). Call the largest tree monomial (under the path-lexicographic order) among these elements the leading form of \( t \), denote it by \( L(t) \). Let \( R_{7.2.6} \) be the ideal of \( \mathcal{F}^{ns}(E) \) generated
by relations given in \((E7.2.6)\). It follows from \((E7.2.6)\) that \(L(t) - t \in (R_{7.2.6})\) and it is clear that \(L(t) - L(t) = 0\) for any choice of \((j_1', \ldots, j_m')\). As a consequence, modulo the ideal \((R_{7.2.6})\), \(r_{k_d}(g)\) is independent of the expressions of \(g\) given in \((E7.2.1)\) and the choices of \(k_d\) – the collection of \(d\)-sequences. By pre-composing with \(\pi\), the argument in this paragraph together with \((E7.2.3)\) implies that \(\pi r_{k_d}(g)\) is independent of the expressions of \(g\) and the choices of \(d\)-sequences and that \(\pi r_{k_d}\) is a well defined \(\mathbb{F}\)-linear map from \(\mathcal{F}(V) \rightarrow \mathcal{F}^{ns}(E)/(R)\).

Fix any \(n\), let \(\{a_\alpha\}_{\alpha \in I_n}\) be a monomial basis of \(A_{n-1}\) for some index set \(I_n\). Then \(\{(a_\alpha, i)\}_{\alpha \in I_n, 1 \leq i \leq n}\) is an \(\mathbb{F}\)-linear basis of \(S_A(n)\). For any \(a_\alpha = v_{\alpha,1} v_{\alpha,2} \cdots v_{\alpha,m_\alpha}\), let \(b'_{\alpha,i} \in \mathcal{F}^{ns}(E)/(R)\) be a monomial of the following form

\[
b'_{\alpha,i} = (v_{\alpha,1}, j_1') \circ_{k_1} (v_{\alpha,2}, j_2') \circ_{k_2} \cdots \circ_{k_{m_\alpha - 1}} (v_{\alpha,m_\alpha}, j_{m_\alpha}') \in \mathcal{F}^{ns}(E)/(R)
\]

where \(j_1' + \cdots + j_{m_\alpha}' = i + m_\alpha - 1\) and \(1 \leq j_l' \leq \deg(v_{\alpha,i}) + 1\) for \(1 \leq l \leq m_\alpha\). Then \(b'_{\alpha,i} \in \phi^{-1}(a_\alpha, i)\). Take \(b_{\alpha,i}\) to be the largest one among the monomials of this form, i.e., \(b_{\alpha,i} := L(b'_{\alpha,i})\), which is independent of the choice of \(b'_{\alpha,i}\). By definition, both \(b_{\alpha,i}\) and \(b'_{\alpha,i}\), considered as elements in \(\mathcal{F}^{ns}(E)\), are of the form \(r_{k_l}(a_\alpha)\) for some choices of \(i\)-sequences \(k\). By \((E7.2.6)\), \(b_{\alpha,i} = b'_{\alpha,i} \in \mathcal{F}^{ns}(E)/(R)\).

The conclusion of this paragraph is that \(b_{\alpha,i} = \pi r_{k_i}(a_\alpha)\) for \(1 \leq i \leq n\) and any collection of \(i\)-sequences.

By definition, any monomial \(s \in \mathcal{F}^{ns}(E)/(R)(n)\) (or in \(\mathcal{F}^{ns}(E)(n)\)) is left normal, namely,

\[
s = (\cdots ((v_{i_1}, j_1) \circ_{k_1} (v_{i_2}, j_2)) \circ_{k_2} \cdots \circ_{k_{m_\alpha - 1}} (v_{i_{m_\alpha}}, j_{m_\alpha})).
\]

Using the relations of the form \((E7.2.5)\), \(s \in \mathcal{F}^{ns}(E)/(R)(n)\) is either zero or equal to

\[
s = (v_{i_1}, j_1) \circ_{k_1} (v_{i_2}, j_2) \circ_{k_2} \cdots \circ_{k_{m_\alpha - 1}} (v_{i_{m_\alpha}}, j_{m_\alpha}).
\]

namely, it is right normal, see \((E4.0.1)\)–\((E4.0.2)\). For the rest of the proof of part (2), we will only use right normal tree monomials in \(\mathcal{F}^{ns}(E)\) and these will simply be called monomials. For each monomial \(s\), we will freely replace \(s\) by \(L(s)\) and vice versa as \(L(s) = s\) in \(\mathcal{F}^{ns}(E)/(R)\) by \((E7.2.6)\).

To prove \(\phi\) is injective, it suffices to show that

\[
dim_{\mathbb{F}}(S_A(n)) \geq \dim_{\mathbb{F}} \mathcal{F}^{ns}(E)/(R)(n)
\]

for all \(n\).

Then it is enough to show that every \(s\) in \((E7.2.8)\) can be presented as a linear combination of \(\{b_{\alpha,i}\}_{\alpha \in I_n, 1 \leq i \leq n}\) in \(\mathcal{F}^{ns}(E)/(R)(n)\).

Since \(Ar((v_{i_1}, j_1)) + \cdots + Ar((v_{i_m}, j_m)) = n + m - 1\), the element \(v_{i_1} v_{i_2} \cdots v_{i_m} \in A_{n-1}\) and it can be presented in the free algebra generated by \(V\) as follows:

\[
v_{i_1} v_{i_2} \cdots v_{i_m} = \sum_{\alpha} c_{\alpha} a_\alpha + \sum_{\gamma} c_{\gamma} v_{\gamma,1} \cdots v_{\gamma,p} g_{\gamma} v_{\gamma,p+1} \cdots v_{\gamma,q} \in \mathcal{F}^{ns}(E)/(R)(n)
\]

where \(c_{\alpha}, c_{\gamma} \in \mathbb{F}\) and \(g_{\gamma} \in \{f_j \mid 1 \leq j \leq w\}\). Rewrite the above equation as

\[
v_{i_1} v_{i_2} \cdots v_{i_m} - \sum_{\alpha} c_{\alpha} a_\alpha = \sum_{\gamma} c_{\gamma} v_{\gamma,1} \cdots v_{\gamma,p} g_{\gamma} v_{\gamma,p+1} \cdots v_{\gamma,q}.
\]

We claim that \(r_{k_d}(g) \in (R)\), or equivalently, \(\pi r_{k_d}(g) = 0\), where \(g\) is the right-hand side of \((E7.2.9)\). Note that each monomial in right-hand side of \((E7.2.9)\) has degree \(n - 1\). If the claim holds, then, for \(d := \sum_{w=1}^{m} j_w - m + 1\),

\[
s = \pi r_{k_d}(v_{i_1} v_{i_2} \cdots v_{i_m}) = \pi r_{k_d}(\sum_{\alpha} c_{\alpha} a_\alpha) = \sum_{\alpha} c_{\alpha} \pi r_{k_d}(a_\alpha) = \sum_{\alpha} c_{\alpha} b_{\alpha,d}
\]
in $\mathcal{F}^{ns}(E)/(R)$. Now we prove the claim. Since $\pi r_{k_{\text{f}}}(-)$ is additive, we may assume that $\pi$ only has one term, namely,

\[(E7.2.10) \quad g = v_1 \cdots v_pf_{p+1} \cdots v_q\]

where $f$ is one of $f_i$ in $W$. Write $f$ as a linear combination of monomials, say $\{v_{f,1} \cdots v_{f,m_f}\}$. Then $(E7.2.10)$ can be considered as an expression of $g$ which is a linear combination of monomials of the form

\[(E7.2.11) \quad s_h := v_1 \cdots v_p v_{f,1} \cdots v_{f,m_f} v_{p+1} \cdots v_q.\]

For any $d$-sequence $k_\bullet$ corresponding to $(E7.2.11)$, it can be decomposed into $k(b)_\bullet := \{k_l\}_{l=1}^p$ and $k(m)_\bullet := \{k_{f,l}\}_{l=1}^{m_f}$, and $k(e)_\bullet := \{k_l\}_{l=p+1}$ such that $1 \leq k_l \leq \deg(v_l) + 1$ for $1 \leq l \leq q$, $1 \leq k_{f,l} \leq \deg(v_{f,l}) + 1$ for $1 \leq l \leq m_f$, and

$$\sum_{l=1}^q k_l + \sum_{l=1}^{m_f} k_{f,l} - q - m_f + 1 = d.$$ 

Note that $k(b)_\bullet$ (resp. $k(f)_\bullet$, $k(e)_\bullet$) is the beginning part (resp. the middle part, the ending part) of $k_\bullet$. For different $s_h$, since $\deg v_{f,1} \cdots v_{f,m_f} = \deg f$ which is independent of the individual monomial $v_{f,1} \cdots v_{f,m_f}$, one can easily choose $k_\bullet$ such that $k(b)_\bullet$ and $k(e)_\bullet$ are independent of the middle part $v_{f,1} \cdots v_{f,m_f}$. Therefore

$$r_{k_{\text{d}}}(s_h) = r_{k(b)_{d_{b}}} (v_1 \cdots v_p) \circ_{d_{b}} r_{k(f)_{d_{f}}} (v_{f,1} \cdots v_{f,m_f}) \circ_{d_{f}} r_{k(e)_{d_{e}}} (v_{p+1} \cdots v_q)$$

for some fixed $k(b)_{d_{b}}$, $k(e)_{d_{e}}$ and $d_f$. This fact implies that

$$r_{k_{\text{d}}}(g) = r_{k(b)_{d_{b}}} (v_1 \cdots v_p) \circ_{d_{b}} r_{k(f)_{d_{f}}} (f) \circ_{d_f} r_{k(e)_{d_{e}}} (v_{p+1} \cdots v_q) \in (R).$$

So we proved that $r_{k_{\text{d}}}(g) \in (R)$ as desired.

(3) By part (1), there is a surjective morphism of symmetric operads

$$\Psi : \mathcal{F}^{sg}(E) \to \mathcal{S}_A$$

where $\mathcal{F}^{sg}(E)$ is the free symmetric operad generated by $E$. Let $R$ be the relation subspace defined in the proof of part (2). It is clear that $\Psi$ maps $R$ to 0. Hence $\Psi$ induces naturally a surjective morphism of symmetric operads

$$\psi : \mathcal{F}^{sg}(E)/(R * S) \to \mathcal{S}_A$$

where $R * S$ is the space generated by $f * \sigma$ for all $f \in R$ and $\sigma \in S_{\text{Aut}(f)}$. By the universal property, there is a morphism of nonsymmetric operads $\mathcal{F}^{ns}(E) \to \mathcal{F}^{sg}(E)/(R * S)$ which induces a morphism of nonsymmetric operads $\mathcal{F}^{ns}(E)/(R) \to \mathcal{F}^{sg}(E)/(R * S)$. So we have the following sequence of morphisms

$$\mathcal{F}^{ns}(E)/(R) \xrightarrow{\theta} \mathcal{F}^{sg}(E)/(R * S) \xrightarrow{\psi} \mathcal{S}_A.$$

By part (2), the composition is an isomorphism and consequently $\theta$ is injective. For simplicity, we consider $\theta$ as an inclusion and identity $f \in \mathcal{F}^{ns}(E)/(R)$ with $\theta(f) \in \mathcal{F}^{sg}(E)/(R * S)$.

It follows from the Equivariance Axiom [BYZ20 Definition 1.2(OP3')], every element in $\mathcal{F}^{sg}(E)/(R * S)$ is a linear combination of elements of the form $s * \sigma$ where $s \in \mathcal{F}^{ns}(E)/(R)$ and $\sigma \in S$. It remains to show the claim that every element of the form $s * \sigma$ is in fact in $\mathcal{F}^{ns}(E)/(R)$. By the proof of part (2), we may further assume that

$$s = (v_{i_1}, j_1) \circ_{j_1} (v_{i_2}, j_2) \circ_{j_2} \cdots \circ_{j_{m-1}} (v_{i_m}, j_m)$$
where \( v_i \in E \) and \( 1 \leq j_s \leq \text{deg}(v_i) + 1 \). Let \( d \) be \( \sum_{s=1}^{m} j_s - m + 1 \). Note that \( \{j_s\}_{s=1}^{m} \) can be replaced by any \( d \)-sequence in the above formula. Let \( \sigma \) be any permutation in \( S_{\text{Ar}(s)} \). Using induction on \( m \), relations of form (E7.2.6), and the fact
\[
(v_i, j) \ast \tau = (v_i, \tau^{-1}(j))
\]
for all \( \tau \in S_{\text{Ar}(v_i, j)} \) we obtain that
\[
s \ast \sigma = (v_i, j'_1) \circ j'_1 (v_i, j'_2) \circ j'_2 \cdots \circ j'_{m-1} (v_i, j'_m)
\]
where \( d' := \sum_{s=1}^{m} j'_s - m + 1 \) is equal to \( \sigma^{-1}(d) \). (The above equation can also be seen from the tree presentation of \( s \) and \( s \ast \sigma \).) Again \( \{j'_s\}_{s=1}^{m} \) can be any \( d' \)-sequence by relations (E7.2.6). Hence \( s \ast \sigma \in F_{ns}(E)/(R) \) as desired. \( \square \)

Note that Construction 7.1 can be viewed as a symmetric version of Construction 1.3. We are wondering if there is a symmetric version of Construction 5.1. If \( A \) is a commutative graded algebra, there is another construction of a symmetric operad associated to \( A \), see [DK10, Section 4.2].

Suppose a symmetric operad \( P \) is finitely generated by a finite alphabet \( X \). Let \( V = \mathbb{F}X \), and for every \( m \geq 0 \), \( V^m \) is defined as in (E3.1.1). For any subcollection \( W \subseteq P \), let \( W_S = \{W(n) \otimes S_n\}_{n \geq 0} \). Now let \( V_S^m \) denote \( (V^m)_S \), so \( V_S^m(n) \) is a right \( S_n \)-module for all \( n \). We have a symmetric version of Lemmas 3.2 and 3.3.

Lemma 7.3. Suppose \( P \) is a locally finite symmetric operad generated by a finite dimensional subcollection \( V \).

(1) Then
\[
\text{GKdim}(P) = \limsup_{n \to \infty} \log_n \left( \dim \left( \sum_{i=0}^{n} V^i \right) \right).
\]

(2) The reduced symmetric operad associated with \( P \), defined as in Lemma 3.2(1), is finitely generated and locally finite.

(3) The reduced connected symmetric operad associated with \( P \), defined as in Lemma 3.2(2) is finitely generated and locally finite.

We have a version of Proposition 3.8 for symmetric operads.

Lemma 7.4. Suppose \( P \) is a finitely generated locally finite symmetric operad.

(1) \( \text{GKdim}(P) = 0 \) if and only if \( P \) is finite dimensional.

(2) \( \text{GKdim}(P) \) cannot be strictly between 0 and 1.

Now we are ready to show Theorem 0.7.

Proof of Theorem 0.7. (1) Let \( P \) be a finitely generated and locally finite symmetric operad. It is clear that \( \text{GKdim}(P) \geq 0 \). The assertion follows from Lemma 7.4(2).

(2) It is well-known that, if \( r \in \mathbb{N} \), then there are finitely generated and locally finite symmetric operads \( P \) such that \( \text{GKdim}(P) = r \). Now suppose that \( r \) is not an integer and \( r \in R_{\text{GKdim}} \setminus (2,3) \). Then \( r > 3 \). By Lemma 3.6(3), there is a connected graded algebra \( A \) such that \( \text{GKdim}(A) = r - 1 \) and \( f(n) := \dim A_n \) is increasing. By Construction 7.1 and Lemma 7.2 there is a locally finite and finitely generated symmetric operads \( S_A \) such that \( \dim S_A(n) = nf(n-1) \) for all \( n \in \mathbb{N} \).
Then, using the fact that \( f(n) \) are increasing,

\[
\text{GKdim}(S_A) = \limsup_{n \to \infty} \log_n \left( \sum_{i=0}^{n} \dim S_A(i) \right) = \limsup_{n \to \infty} \log_n \left( \sum_{i=0}^{n} i f(i - 1) \right)
\]

\[
\leq \limsup_{n \to \infty} \log_n (n \sum_{i=0}^{n} f(i - 1)) = 1 + \text{GKdim}(A),
\]

\[
\text{GKdim}(S_A) = \limsup_{n \to \infty} \log_n \left( \sum_{i=0}^{n} i f(i - 1) \right) \geq \limsup_{n \to \infty} \log_n \left( \sum_{i=\lfloor n/2 \rfloor + 1}^{n} f(i - 1) \right)
\]

\[
\geq \limsup_{n \to \infty} \log_n \left( \sum_{i=\lfloor n/2 \rfloor + 1}^{n} f(i - 1) \right) = 1 + \text{GKdim}(A).
\]

Therefore \( \text{GKdim}(S_A) = \text{GKdim}(A) + 1 = r \).

(3) Let \( U \) be the finitely generated graded algebra given in Example 6.2. So \( \text{GKdim}(U) = 1 \). Let \( S_U \) be the finitely generated locally finite symmetric operad constructed in Construction 7.1 and Lemma 7.2. By the proof of part (2), \( \text{GKdim}(S_U) = 2 \). In fact, its generating series is

\[
G_{S_U}(z) = z(zH_U(z))' = \frac{z(1 + 2z + 2z^2 - 2z^3)}{(1 - z)^2} + \bar{V}(z)
\]

where

\[
\bar{V}(z) = \sum_{n=2}^{\infty} (n + 1)\delta_A(n)z^{n+1}.
\]

By Lemma 6.1, \( \bar{V}(z) \) and hence \( G_{S_U}(z) \) are not holonomic.

Let \( r \geq 2 \) be any real number such that there is a finitely generated locally finite symmetric operad \( P \) with \( \text{GKdim}(P) = r \). If \( G_P \) is not holonomic, then we are done. Otherwise, \( G_P \) is holonomic. Now consider a new operad \( Q := P \oplus S_U \) with \( G_Q = G_P + G_{S_U} \). By [Ber14, Holonomic Theorem 2], \( G_Q \) is not holonomic. Since \( \text{GKdim}(S_U) = 2 \), we have \( \text{GKdim}(Q) = \text{GKdim}(P) = r \). The assertion follows. \( \square \)

We finish the paper with a potential counterexample to [KP15, Expectation 1].

**Example 7.5.** Let \( A \) be the connected graded algebra in Example 6.4 and let \( S_A \) be the operad given in Construction 7.1. Then \( S_A \) is a finitely presented symmetric operad by Lemma 7.2. By (E7.1.6), its generating series is

\[
G_{S_A}(z) = z(zH_A(z))' = z(zP(z))'
\]

which is not holonomic by Property (P3) in Example 6.4. Therefore \( S_A \) is a “non-generic” counterexample to the symmetric version of [KP15, Expectation 2].

Since \( G_{S_A}(z) \) is not holonomic, by (E6.0.3), \( E_{S_A}(z) \) is not holonomic. We conjecture that \( E_{S_A}(z) \) is not differential algebraic. If this is the case, then \( S_A \) is a “non-generic” counterexample to [KP15, Expectation 1].

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