REPRESENTATIONS OF HERMITIAN COMMUTATIVE ∗-ALGEBRAS
BY UNBOUNDED OPERATORS

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Dedicated to Christian Berg on the occasion of his sixty-fifth birthday

ABSTRACT. We give a spectral theorem for unital representations of
Hermitian commutative unital ∗-algebras by possibly unbounded op-
erators on pre-Hilbert spaces. A better result is known for the case in
which the ∗-algebra is countably generated.

1. STATEMENT OF THE MAIN RESULT

Our main result is the following theorem:

THEOREM 1. Let π be a unital representation of a Hermitian commutative
unital ∗-algebra A on a pre-Hilbert space H ≠ {0}.

The operators π(a) (a ∈ A) are essentially normal in the completion.

There is a spectral measure P on a subset of Δ*(A), acting on the com-
pletion of H, such that the closure of the operator π(a) in the completion
of H is given by

\[ \pi(a) = \int \hat{a} \, dP \quad (a \in A). \]

We then have that

(i) the spectral resolution of the normal operator \( \pi(a) \) with \( a \in A \) is the
image \( \hat{a} P = P \circ \hat{a}^{-1} \) of \( P \) under the function \( \hat{a} \),

(ii) so the normal operators \( \pi(a) \) (\( a \in A \)) commute spectrally (strongly),

(iii) a bounded operator \( b \) on the completion of \( H \) commutes with the
normal operators \( \pi(a) \) (\( a \in A \)) if and only if \( b \) commutes with \( P \).

Notation and terminology are explained in the following section. A
proof is given in section 5 below. For countably generated ∗-algebras,
a better result is available, cf. Savchuk and Schmüdgen [13] Theorem 7 p.
60 f.]. Cf. Schmüdgen [16] Theorem 7.23 p. 148 f.] & [14 p. 237]; also see
[2, 3, 12] for related results.

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thanks go to Torben Maack Bisgaard as well as other.
2. TERMINOLOGY AND NOTATION

2.1. Representations. Let \((H, \langle \cdot, \cdot \rangle)\) be a pre-Hilbert space. Denote by \(\text{End}(H)\) the algebra of linear operators mapping \(H\) to \(H\). Let \(A\) be some \(*\)-algebra. By a representation of \(A\) on \(H\) we shall understand an algebra homomorphism \(\pi\) from \(A\) to \(\text{End}(H)\) such that

\[
\langle \pi(a)x, y \rangle = \langle x, \pi(a^*)y \rangle \quad \text{for all} \quad a \in A, \quad \text{and all} \quad x, y \in H.
\]

The range \(\text{ran}(\pi) := \pi(A)\) then is a \(*\)-algebra as is easily seen. (Please note that \(\text{End}(H)\) need not be a \(*\)-algebra for this.)

2.2. Unitisation and Spectrum. We require a unit in an algebra to be non-zero. The notation \(e\) is reserved for units.

We denote the unitisation of an algebra \(A\) by \(\tilde{A}\). (If the algebra \(A\) is unital, then the unitisation \(\tilde{A}\) is defined to be \(A\) itself.)

The spectrum \(\text{spec}(a) = \text{spec}_A(a)\) of an element \(a\) of an algebra \(A\) is defined as the set of complex numbers \(\lambda\) such that \(\lambda e - a\) is not invertible in \(\tilde{A}\). The spectrum in an algebra is the same as in the unitisation.

2.3. Homomorphisms and Spectra. Some algebra homomorphism between unital algebras is called unital if it takes unit to unit.

If \(\pi : A \to B\) is an algebra homomorphism, then

\[
\text{spec}_B(\pi(a)) \setminus \{0\} \subset \text{spec}_A(a) \setminus \{0\} \quad \text{for all} \quad a \in A, \quad (1)
\]

if \(A, B, \pi\) are unital, then \(\text{spec}_B(\pi(a)) \subset \text{spec}_A(a)\) for all \(a \in A\). \((2)\)

For a proof, see e.g. [17, Theorem (7.10) p. 36].

If \(A\) is an algebra, we denote by \(\Delta(A)\) the set of multiplicative linear functionals on \(A\). The Gel'fand transform of \(a \in A\) is denoted by \(\hat{a}\).

We equip the spectrum \(\Delta(A)\) of an algebra \(A\) with the relative topology induced by the weak* topology. For an algebra \(A\), we have that

\[
\hat{a} \left( \Delta(A) \right) \subset \text{spec}(a) \quad \text{for all} \quad a \in A, \quad (3)
\]

cf. e.g. [17] Proposition (20.6) p. 78], or use \((1)\) & \((2)\) above and the fact that \(0 \in \text{spec}(a)\) if \(A\) is not unital, as then \(A\) is a proper two-sided ideal in \(\tilde{A}\). That is, “the range of the Gel'fand transform of \(a\) is contained in the spectrum of \(a\)”.

If \(A\) is a \(*\)-algebra, we denote by \(\Delta^*(A)\) the set of Hermitian multiplicative linear functionals on \(A\).

2.4. Hermitian \(*\)-Algebras. A \(*\)-algebra is called Hermitian if the spectrum of each of its Hermitian elements is real.

The above formula \((1)\) also shows that if \(\pi\) is a \(*\)-algebra homomorphism from a Hermitian \(*\)-algebra \(A\) to another \(*\)-algebra, then the range \(*\)-algebra \(\pi(A)\) is Hermitian. In particular, if \(\pi\) is a representation of a Hermitian \(*\)-algebra \(A\) on a pre-Hilbert space, then the range \(\text{ran}(\pi) = \pi(A)\) is a Hermitian \(*\)-algebra.

The above formula \((3)\) implies that a multiplicative linear functional on a Hermitian \(*\)-algebra \(A\) is Hermitian, \(\Delta^*(A) = \Delta(A)\). (Real on Hermitian.)
2.5. Spectral Measures. The Borel $\sigma$-algebra is the $\sigma$-algebra generated by the open sets of a Hausdorff space. A resolution of identity is a spectral measure $P$ defined on the Borel $\sigma$-algebra of a Hausdorff space such that the Borel probability measures $\omega \mapsto \langle P(\omega)x, x \rangle$ are inner regular (and so outer regular) for all unit vectors $x$ in the Hilbert space on which $P$ acts. (See [17, Observation (98.5) p. 403].) The support of $P$ then is well-defined, see [17, Definition (48.38) p. 223]. For spectral integrals, cf. [17, § 48 p. 213 ff., § 60 p. 269 ff.]. Also see [15, 4.3.1 p. 74 f., 4.3.2 p. 75 ff.]. We also write $P$ for the range of $P$.

For images of spectral measures, see [17, Definition (48.28) p. 220].

2.6. Commutativity for Unbounded Operators. See [15, sections 5.5 - 5.6 p. 101 ff.]. A possibly unbounded operator $N$ in a Hilbert space is called normal if it is densely defined, closed and $N^*N = NN^*$, cf. e.g. [11] 13.29 p. 368]. It then is maximally normal, cf. e.g. [11] 13.32 p. 370].

Let $N$ be a (possibly unbounded) normal operator in a Hilbert space $\neq \{0\}$. A bounded operator $T$ is said to commute with $N$ if and only if the domain of $N$ is invariant under $T$, and if $TNx = NTx$ for all $x$ in the domain of $N$. This is equivalent to $TN \subset NT$. The Spectral Theorem says that this is the case if and only if $T$ commutes with the spectral resolution of $N$. See for example [11, Theorem 13.33 p. 371].

A bounded operator $T$ commutes with a (possibly unbounded) self-adjoint operator $A$ in a Hilbert space $\neq \{0\}$ if and only if $T$ commutes with the Cayley transform of $A$, cf. [17, Lemma (61.15) p. 282].

One says that two (possibly unbounded) normal operators in a Hilbert space $\neq \{0\}$ commute spectrally (or strongly), if their spectral resolutions commute, cf. [15, Proposition 5.27 p. 107]. For two self-adjoint operators this is the case if and only if their Cayley transforms commute.

A (possibly unbounded) operator $N$ in a Hilbert space $\neq \{0\}$ is normal if and only if $N = R + iS$ where $R$ and $S$ are self-adjoint and commute spectrally. (See [15, Proposition 5.30 p. 108 f.].)

3. Reminders and Preliminaries

3.1. The Cayley Transformation in Algebras. We next give a few remarks on Cayley transforms in algebras. (See also [17, § 15 p. 62 f.].) (The theory of Cayley transforms of densely defined symmetric operators in a Hilbert space is supposed to be known. We refer the reader to [11] 7.4A pp. 520 - 528], [18, Chapter 8 pp. 229 - 247], [15, sections VI.13.1-2 pp. 283 - 290].)

First, note that an element $a$ of an algebra $A$ commutes with an invertible element $b$ of $A$ if and only if $a$ commutes with $b^{-1}$.

Next, the Rational Spectral Mapping Theorem says that if $r$ is a non-constant rational functional without pole on the spectrum of an element $a$ of an algebra, then

$$\text{spec}(r(a)) = r(\text{spec}(a)).$$

See for example [17, Theorem (7.9) p. 35].
Let $a$ be an element of an algebra $A$ such that $\text{spec}(a)$ does not contain $-i$. Then
\[ g = (a - ie) (a + ie)^{-1} \in \tilde{A} \]
is called the Cayley transform of $a$. The Rational Spectral Mapping Theorem implies that 1 is never in the spectrum of a Cayley transform. (Because the corresponding Moebius transformation does not assume the value 1.) In particular $e - g$ is invertible in $\tilde{A}$. A simple computation then shows that $a$ can be regained from $g$ as the inverse Cayley transform
\[ a = i (e - g)^{-1} (e + g). \]

If $A$ is a $*$-algebra, and if $a$ is Hermitian, then $g$ is unitary, i.e. $g^* g = gg^* = e$. This follows from the above commutativity property.

If $A$ is a Hermitian $*$-algebra, then the Cayley transformation of a Hermitian element of $A$ is well-defined.

3.2. The Archetypal Spectral Theorem.

**REMARK 2** (spectral theorem). Let $C$ be a commutative $C^*$-algebra of bounded linear operators on a Hilbert space $H \neq \{0\}$ containing the unit operator. There then exists a unique resolution of identity $R$ on the spectrum $\Delta(C) = \Delta^*(C)$ with
\[ c = \int \hat{c} dR \quad \text{for all} \quad c \in C. \]
One says that $R$ is the spectral resolution of $C$. A bounded operator on $H$ commutes with the operators $c \in C$ if and only if it commutes with $R$. The support of $R$ is all of $\Delta^*(C)$. (For a proof, see e.g. [11, Theorem 12.22 p. 321 f.] or [17, Theorem (49.1) p. 224].)

4. A SUFFICIENT CRITERION FOR ESSENTIAL NORMALITY

If $a$ is a Hermitian element of a $*$-algebra $A$ and if either $i$ or $-i$ does not belong to $\text{spec}(a)$ then neither of them belongs to $\text{spec}(a)$. (Because for an arbitrary element $b$ of $A$, one has $\text{spec}(b^*) = \overline{\text{spec}(b)}$ as is easily seen.)

**PROPOSITION 3.** Let $\pi$ be a representation of a $*$-algebra $A$ on some pre-Hilbert space $H$. Assume that $a$ is a Hermitian element of $A$ whose spectrum does not contain either $i$ or $-i$. Then $\pi(a)$ is essentially self-adjoint.

**Proof.** The operator $\pi(a)$ is symmetric in $H$. It follows from the preceding remark and formula (1) of subsection 2.3 that the spectrum of $\pi(a)$ in $\text{End}(H)$ does not contain $i$ nor $-i$. Hence $\pi(a) \pm iI$ are invertible in $\text{End}(H)$, and thus surjective onto $H$. This implies that $\pi(a)$ is essentially self-adjoint. See [11 Proposition 7.4.13 p. 522].

**PROPOSITION 4.** Let $\pi$ be a representation of $*$-algebra $A$ on a pre-Hilbert space $\neq \{0\}$. Let $a, b$ be commuting Hermitian elements of $A$ each of whose spectrum does not contain either $i$ or $-i$. The self-adjoint closures of $\pi(a)$ and $\pi(b)$ then commute spectrally.
Proof. This is so because in this the case the Cayley transforms of $\pi(a)$ and $\pi(b)$ are commuting bijective isometries of the pre-Hilbert space. \hfill \Box

**Lemma 5.** Let $A, B$ be two essentially self-adjoint operators in a Hilbert space $\neq \{0\}$, whose closures $\overline{A}, \overline{B}$ commute spectrally. Then $A + iB$ is essentially normal and its closure is $\overline{A + iB}$.

**Proof.** As noted in subsection 2.4, an operator $N$ is normal if and only if $N = R + iS$ where $R$ and $S$ are self-adjoint and commute spectrally. This implies that $\overline{A + iB}$ is normal. In particular $\overline{A + iB}$ is closed. It follows that $A + iB$ is closable and its closure is easily seen to extend the closed operator $\overline{A + iB}$. Therefore $\overline{A + iB} = \overline{A + iB}$. \hfill \Box

**Theorem 6.** Let $\pi$ be a representation of a $*$-algebra $A$ on a pre-Hilbert space $\neq \{0\}$. Let $a, b$ be commuting Hermitian elements of $A$ each of whose spectrum does not contain either $i$ or $-i$, and put $c := a + ib$. The operators $\pi(a)$ and $\pi(b)$ are essentially self-adjoint and their closures commute spectrally. The operator $\pi(c)$ is essentially normal, and one has

$$\pi(c) = \pi(a) + i\pi(b).$$

**Proof.** This follows from the preceding three items. \hfill \Box

**Corollary 7.** Let $\pi$ be a representation of some commutative Hermitian $*$-algebra $A$ on a pre-Hilbert space $\neq \{0\}$. Then for each element $c$ of $A$ the operator $\pi(c)$ is essentially normal. If $c = a + ib$ with $a, b$ Hermitian elements of $A$, then the operators $\pi(a)$ and $\pi(b)$ are essentially self-adjoint, their closures commute spectrally, and one has

$$\pi(c) = \pi(a) + i\pi(b).$$

Things like these have been noted before, cf. [14, Corollary 8.1.20 p. 210 & Corollary 9.1.4 p. 237] or [16, Corollary 4.12 p. 67 & Theorem 7.11 p. 143]. I came to them through a good question.

**5. Proof of the Main Result**

Let $\pi(A)$ be a Hermitian commutative unital $*$-algebra of linear operators on a pre-Hilbert space $H \neq \{0\}$, such that $\langle \pi(a)^* x, y \rangle = \langle x, \pi(a) y \rangle$ holds for all $\pi(a) \in \pi(A)$ and $x, y \in H$. Then $\Delta(\pi(A)) = \Delta^* (\pi(A))$. Cf. 2.4.

Now let $\pi(a) \in \pi(A)$. The linear operator $\pi(a)$ is essentially normal. To see this, apply corollary 7 to the identical representation of $\pi(A)$. We shall identify $\pi(a)$ with its normal closure $\overline{\pi(a)}$. The operators $\overline{\pi(a)}$ then form a $*$-algebra. Let $\pi(A)_{\text{Hermitian}}$ denote the set of Hermitian elements of $\pi(A)$.

Let $\kappa : a \to (a - ie)(a + ie)^{-1}$ denote the Cayley transformation in the Hermitian $*$-algebra $\pi(A)$, or in the complex plane, cf. subsection 3.1.

The unitary Cayley transforms $\kappa(\overline{\pi(a)}) \in \pi(A)$ of the self-adjoint operators $\overline{\pi(a)} \in \pi(A)_{\text{Hermitian}}$ generate a unital commutative $*$-algebra of bounded operators $B$ that is $\subset \pi(A)$, as $\pi(A)$ is Hermitian. Let $R$ be the spectral resolution of the completion $C$ of $B$, cf. reminder 2 in subsection 3.2.
We shall map $\Delta^*(C)$ to $\Delta^*(\pi(A))$. Let $\tau \in \Delta^*(C)$. Then $\tau\big|_B \in \Delta^*(B)$ as $B$ is unital in $C$. We next use that $B \subset \pi(A)$, as noted above.

**Statement 8.** The function $\tau\big|_B \in \Delta^*(B)$ extends uniquely to $\sigma \in \Delta^*(\pi(A))$.

**Proof.** The set 
\[ S := \{ c \in B : c \text{ is invertible in } \pi(A) \} \]
is a unital $\ast$-subsemigroup of $B$. Hence the set of fractions 
\[ F := \{ c^{-1} b \in \pi(A) : c \in S, \ b \in B \} \]
forms a unital $\ast$-subalgebra of $\pi(A)$.

We shall prove that $F$ is all of $\pi(A)$. Since $F$ is a complex vector space, it suffices to prove that $F$ contains every element of $\pi(A)_{\text{sa}}$. Let $\pi(a) \in \pi(A)_{\text{sa}}$, and let $g \in B$ be its Cayley transform. Cf. subsection 3.1. Then $\pi(a)$ is the inverse Cayley transform of $g$:
\[ \pi(a) = i(e - g)^{-1}(e + g). \]
This says that $\pi(a)$ is of the form $c^{-1} b$ where $b, c \in B$ with $c$ invertible in $\pi(A)$. That is, $\pi(a) \in F$. Thus $F$ contains every element of $\pi(A)_{\text{sa}}$, as was to be shown.

For $\pi(a) = c^{-1} b \in \pi(A)$ with $c \in S, \ b \in B$, put $\sigma(\pi(a)) := \tau(c)^{-1} \tau(b)$. Please note here that $\tau(c) \in \text{spec}_{\pi(A)}(c)$ is never zero for $c \in S \subset B \subset \pi(A)$, cf. formula (3) of subsection 3.1. Please note that this definition is independent of the representatives $c \in S, \ b \in B$ of $\pi(a) \in \pi(A)$. It is easy to verify that this definition extends $\tau\big|_B$ uniquely to a Hermitian multiplicative linear functional $\sigma$ on all of $\pi(A)$. \qed

**Statement 9.** For $\pi(a) \in \pi(A)_{\text{sa}}$ and $\tau \in \Delta^*(C)$ we have 
\[ \kappa(\pi(a)) = \kappa(\pi(a)\sigma). \]

**Proof.** Indeed, by the homomorphic nature of $\sigma \in \Delta^*(\pi(A))$, we then find 
\[ \tau(\kappa(\pi(a))) = \sigma(\kappa(\pi(a))) = \kappa(\sigma(\pi(a))). \quad \square \]

A spectral measure $Q$ on a subset of $\Delta^*(\pi(A))$ is defined as the image measure $f(R)$ of $R$ under the above injective map $f : \Delta^*(C) \to \Delta^*(\pi(A))$, see section 2.5.

**Statement 10.** The spectral measure $Q$ satisfies 
\[ \pi(a) = \int \pi(a) \, dQ \quad \text{for all } \pi(a) \in \pi(A). \]
Proof. For $\pi(a) \in \pi(A)_\text{sa}$, we compute
\[
\kappa(\pi(a)) = \int \kappa(\pi(a)) \, dR
= \int \kappa(\pi(a)) \, dQ \quad \text{see statement [9]}
= \int \kappa \left( \int \text{id} \, d\pi(a)(Q) \right) \quad \text{see [17] (60.10) p. 272}
= \kappa \left( \int \pi(a) \, dQ \right) \quad \text{see again [17] (60.10) p. 272}
\]
or, by injectivity of the Cayley transformation
\[
\pi(a) = \int \pi(a) \, dQ.
\]

Let now $\pi(c) \in \pi(A)$ be arbitrary, and let $\pi(a), \pi(b)$ be Hermitian elements of $\pi(A)$ with $\pi(c) = \pi(a) + i\pi(b)$. Corollary [7] (applied to the identical representation of $\pi(A)$) says that $\overline{\pi(c)}$ is normal and that
\[
\overline{\pi(c)} = \overline{\pi(b)} + i\overline{\pi(a)} = \int \overline{\pi(a)} \, dQ + i \int \overline{\pi(b)} \, dQ
\]
\[
\subset \int \overline{\pi(c)} \, dQ,
\]
\text{cf. [17] Theorem (60.14) p. 273}. Since the member on the right hand side is normal, cf. [17], Theorem (60.23) p. 275, we get by maximal normality of $\overline{\pi(c)}$ (subsection 2.6) that
\[
\overline{\pi(c)} = \int \overline{\pi(c)} \, dQ.
\]

Let now $\pi$ be a unital representation of a unital $\ast$-algebra $A$ on a pre-Hilbert space $H \neq \{0\}$. Assume that its image $\pi(A)$ is commutative and Hermitian, e.g. $A$ commutative and Hermitian. Please note that then $\Delta^\ast(\pi(A)) = \Delta(\pi(A))$. Continue with the notation as above.

Consider the map $\pi^\ast$ adjoint to $\pi$, given by
\[
\pi^\ast : \Delta^\ast(\pi(A)) \to \Delta^\ast(A)
\]
\[
\sigma \mapsto \pi^\ast(\sigma) = \sigma \circ \pi.
\]
This is well-defined as the $\ast$-algebras are unital. So a spectral measure $P$ on a subset of $\Delta^\ast(A)$ is defined as the image $\pi^\ast(Q)$ of $Q$ under the injective map $\pi^\ast$.

**Statement 11.** The spectral measure $P$ satisfies
\[
\overline{\pi(a)} = \int \overline{a} \, dP \quad \text{for all} \quad a \in A.
\]
Proof. We compute

\[ \pi(a) = \int \pi(a) \, dQ \]
\[ = \int \sigma(\pi(a)) \, dQ(\sigma) \]
\[ = \int (\sigma \circ \pi)(a) \, dQ(\sigma) \]
\[ = \int \hat{a}(\sigma \circ \pi) \, dQ(\sigma) \]
\[ = \int (\hat{a} \circ \pi^*)(\sigma) \, dQ(\sigma) \]
\[ = \int (\hat{a} \circ \pi^*) \, dQ \]
\[ = \int \hat{a} \, d\pi^*(Q) = \int \hat{a} \, dP \] \hfill \Box

Statements (ii) and (iii) of the main result, theorem 1, follow as in [17, Proposition (49.11) p. 229]. In order to prove statement (iv), we note that if a bounded linear operator \( b \) on the completion of \( H \) commutes with \( P \), then it commutes with all operators \( \pi(a) \ (a \in A) \), by statement (i). To prove the converse, we note that if a bounded linear operator on the completion of \( H \) commutes with all operators \( \pi(a) \ (a \in A) \), then it commutes with all operators in the commutative \( \mathbb{C}^* \)-algebra \( \mathbb{C} \), by subsection 2.6 and so with its spectral resolution \( R \), and then with its image \( Q \), and thus with the image \( P \) thereof. This finishes the proof of the main result, theorem 1.

6. APPENDIX: DERIVATION OF SOME KNOWN CONSEQUENCES

**Reminder 12** (the GNS construction). Let \( A \) be a unital \(*\)-algebra, and let \( \varphi \) be a positive linear functional on \( A \). (That is, \( \varphi(a^* a) \geq 0 \) for all \( a \in A \).) Assume that \( \varphi \) is non-zero. One defines a positive semidefinite sesquilinear form on \( A \) by putting

\[ \langle a, b \rangle = \varphi(b^* a) \quad (a, b \in A). \]

This sesquilinear form is a so-called Hilbert form, in that it satisfies

\[ \langle ab, c \rangle = \langle b, a^* c \rangle \quad \text{for all} \quad a, b, c \in A. \]

It follows that the isotropic subspace

\[ I := \{ a \in A : \langle a, b \rangle = 0 \text{ for all } b \in B \} \]

is a left ideal in \( A \). The quotient space \( H := A/I \) is a pre-Hilbert space when equipped with the inner product

\[ \langle a + I, b + I \rangle = \langle a, b \rangle \quad (a, b \in A). \]
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(This follows from the Cauchy-Schwarz inequality.) The assignment

$$\pi(a)(b + I) := ab + I \quad (a, b \in A)$$

defines a unital representation $\pi$ of $A$ on the pre-Hilbert space $H \neq \{0\}$.

(This uses the fact that the sesquilinear form is a Hilbert form.) The vector $e + I$ is (algebraically) cyclic for the representation $\pi$, and the positive linear functional $\varphi$ can be regained from the cyclic representation $\pi$ via

$$\varphi(a) = \langle \pi(a)(e + I), (e + I) \rangle \quad \text{for all } a \in A.$$

REMARK 13 (semiperfect). A commutative unital $*$-algebra $A$ is said to be semiperfect, if every positive linear functional $\varphi$ on $A$ with $\varphi(e) = 1$ has a representing measure $\mu$ in the sense that

$$\varphi(a) = \int \hat{a} \, d\mu \quad \text{for all } a \in A$$

for some probability measure $\mu$ on a subset of $\Delta^*(A)$.

THEOREM 14. A Hermitian commutative unital $*$-algebra is semiperfect.

Proof. Let $\varphi$ be a positive linear functional on a Hermitian commutative unital $*$-algebra $A$ such that $\varphi(e) = 1$. Consider the cyclic unital representation $\pi$ associated to $\varphi$ by the GNS construction. Let $P$ be the spectral measure provided by Theorem 1. Now $\varphi$ can be regained from $\pi$ via

$$\varphi(a) = \langle \pi(a)(e + I), (e + I) \rangle \quad \text{for all } a \in A.$$

We may then take the probability measure given by

$$\mu : \Delta \rightarrow \langle P(\Delta)(e + I), (e + I) \rangle$$

as a representing measure.

□

LEMMA 15. A $*$-algebra, which is a field different from $\mathbb{C}$, carries no positive linear functionals different from zero. (This applies for example to the field $\mathbb{C}(x)$ of rational functions endowed with the involution making the variable $x$ Hermitian, cf. [8].)

Proof. Let $A$ be a $*$-algebra, which is a field different from $\mathbb{C}$. We first note that the elements of $A \setminus Ce$ have empty spectrum because $A$ is a field. It follows that $A$ is Hermitian. It also follows that there is no multiplicative linear functional on $A$. Indeed, for $a \in A \setminus Ce$, we would run into a contradiction with formula (3) of subsection 2.3, by the fact that $\text{spec}(a)$ is empty as noted above. The statement follows now from the semiperfectness of $A$ guaranteed by the preceding theorem [14] □

THEOREM 16. A $*$-algebra, which is a field different from $\mathbb{C}$, does not have any representation on pre-Hilbert spaces other than the zero representations. (This applies for example to the field $\mathbb{C}(x)$ of rational functions endowed with the involution making the variable $x$ Hermitian.)
**Corollary 17** (See [14, Theorem 2.1.12 p. 38] for a stronger result). A *-algebra of operators on a pre-Hilbert space which is a field consists of scalar multiples of the identity operator.

**Reminder 18** (the radical). Let $A$ be an algebra. The radical $\text{rad}(A)$ of $A$ is the set of those elements $a$ of $A$ such that $e + ba$ is invertible in the unitisation of $A$ for all $b \in A$ cf. [5, Proposition III.24.16 (ii) p. 125]. Please note that this definition of the radical can be given in terms of the spectrum in the algebra.

An element $a$ of $\text{rad}(A)$ satisfies $\text{spec}_A(a) \subseteq \{0\}$, [5, Proposition III.24.16 (i) p. 125].

An algebra is called radical if it coincides with its own radical. For example, the radical of an algebra is a radical algebra. (This follows from [5, Corollary III.24.20 p. 126].)

We get a conceptually easy proof of an essential theorem of Torben Maack Bisgaard [22] below.

**Observation 19** (see e.g. [7, Proposition I.3.62 p. 95]). A multiplicative linear functional on an algebra vanishes on the radical.

**Proof.** Indeed, we have

\[ \tau(a) \in \text{spec}(a) \subseteq \{0\} \]

for any multiplicative linear functional $\tau$ and any $a$ in the radical, as can be seen from the second to last statement of the preceding reminder [18] and from formula (3) from subsection 2.3. □

**Observation 20.** The radical of a *-algebra is a Hermitian *-algebra.

**Proof.** Let $A$ be a *-algebra. It is easily seen from the definition that the radical $\text{rad}(A)$ is a *-subalgebra of $A$, using the involution on the definition as in reminder [18] as well as the elementary facts that $\text{spec}(a^*) = \text{spec}(a)$ for any $a \in A$ and that $\text{spec}(ab) \setminus \{0\} = \text{spec}(ba) \setminus \{0\}$, cf. [17, Proposition (7.12) p. 37]. Using the fact that $\text{rad}(A)$ is a radical algebra, we find that an element $a$ of $\text{rad}(A)$ satisfies $\text{spec}_{\text{rad}(A)}(a) \subseteq \{0\}$, as is seen from the preceding reminder [18]. This implies in particular that the radical $\text{rad}(A)$ is Hermitian. □

**Corollary 21.** A positive linear functional on some commutative unital *-algebra vanishes on the radical.

**Proof.** Consider a commutative unital *-algebra $A$. Then $\mathbb{C}e + \text{rad}(A)$ is a unital Hermitian *-algebra by the preceding observation [20] using that the spectrum in an algebra is the same as in the unitisation, cf. section 2.2. Thus $\mathbb{C}e + \text{rad}(A)$ is semiperfect by the above theorem [14] The statement follows now from observation [19]. □

The assumption of commutativity can be dropped, by considering a suitable commutative *-subalgebra, as we shall do in the following proof.
Theorem 22 (Torben Maack Bisgaard [4]). A positive linear functional on a unital \(*\)-algebra vanishes on the radical.

Proof. Let \(A\) be a unital \(*\)-algebra, let \(\varphi\) be a positive linear functional on \(A\), and let \(c \in \text{rad}(A)\). To prove that \(\varphi(c) = 0\), it suffices to show that \(\varphi(c^*c) = 0\) by the Cauchy-Schwarz inequality \(|\varphi(c)| \leq \varphi(e)\varphi(c^*c)\). Please note that \(c^*c \in \text{rad}(A)\) as well. The second commutant \(B\) of \(c^*c\) in \(A\) is a commutative unital \(*\)-subalgebra of \(A\), cf. [17, Observation (19.6) p. 74], such that \(\text{spec}_B(b) = \text{spec}_A(b)\) for all \(b \in B\), cf. [17, Lemma (19.8) p. 74]. It follows that \(\text{rad}(A) \cap B \subseteq \text{rad}(B)\), by the possibility to define the radical through the spectrum. In particular \(c^*c \in \text{rad}(B)\). The statement follows now from the preceding theorem 21 applied to the commutative unital \(*\)-algebra \(B\).

\(\Box\)

Reminder 23 (extensibility). Let \(A\) be a \(*\)-algebra. A positive linear functional on \(A\) is called extensible, if it can be extended to a positive linear functional on the unitisation of \(A\). Cf. [17, Proposition (35.3) p. 160 f.].

For example, if \(\pi\) is a representation of \(A\) on a pre-Hilbert space \(H\), then the positive linear functionals given by

\[\varphi(a) := \langle \pi(a)x, x \rangle \quad (a \in A)\]

are extensible for all vectors \(x \in H\), by considering a representation of the unitisation of \(A\) extending \(\pi\).

Corollary 24. An extensible positive linear functional on a \(*\)-algebra vanishes on the radical.

Proof. Consider the unitisation \(\tilde{A}\) of a \(*\)-algebra \(A\), and use that the radical of \(A\) is contained in (and thus coincides with) the radical of \(\tilde{A}\), by applying the definition of the radical as in reminder 18 and the fact that \(\text{rad}(\tilde{A})\) is a subspace of \(\tilde{A}\) and does not contain the unit, as the spectrum of an element in a radical consists of zero at most, cf. reminder 18.

\(\Box\)

Theorem 25. A representation of a \(*\)-algebra on a pre-Hilbert space vanishes on the radical.

Corollary 26. A radical \(*\)-algebra does not have any representation on pre-Hilbert spaces other than the zero representations.

This applies for instance to convolution \(*\)-algebras on an interval of the positive half-line. Cf. [10] vol. I, § 2.IV.15 p. 179 ff.; vol. II, § 6.V.1 p. 156 f.] or [9] § 6.V.1 p. 369 f.]

Reminder 27 (semisimple). An algebra is called semisimple if its radical consists of 0 alone.

Reminder 28 (faithful). A representation of a \(*\)-algebra on a pre-Hilbert space is called faithful if it is injective.

Corollary 29. A \(*\)-algebra that admits a faithful representation on a pre-Hilbert space is semisimple, by theorem 25.
**Corollary 30** (see [16, Corollary 7.24 p. 150]). If a Hermitian commutative unital $\ast$-algebra $A$ admits a faithful unital representation $\pi$ on a pre-Hilbert space, then the set of Hermitian characters $\Delta^\ast(A)$ separates the points of $A$.

**Proof.** Let $a \in A$. Then $\pi(a) \neq 0$ as $\pi$ is faithful. So there exists $\tau \in \Delta^\ast(A)$ with $\tau(a) \neq 0$, by the main result[1].

The $\ast$-radical is defined as the intersection of all representations of a $\ast$-algebra on pre-Hilbert spaces. To make that rigorous, cf. Schmüdgen [16, p. 81 f.]. Theorem[25] says that the radical of any $\ast$-algebra is contained in the $\ast$-radical. This is of concern to factoring over the radical.

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