Constraints on Vacuum Energy
from Structure Formation and
Nucleosynthesis

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Abstract. This paper derives an upper limit on the density ρΛ of dark energy based on
the requirement that cosmological structure forms before being frozen out by the eventual
acceleration of the universe. By allowing for variations in both the cosmological parameters
and the strength of gravity, the resulting constraint is a generalization of previous limits.
The specific parameters under consideration include the amplitude Q of the primordial den-
sity fluctuations, the Planck mass Mpl, the baryon-to-photon ratio η, and the density ratio
ΩM/Ωb. In addition to structure formation, we use considerations from stellar structure and
Big Bang Nucleosynthesis (BBN) to constrain these quantities. The resulting upper limit on
the dimensionless density of dark energy becomes ρΛ/Mpl4 < 10−90, which is ∼30 orders of
magnitude larger than the value in our universe ρΛ/Mpl4 ∼ 10−120. This new limit is much
less restrictive than previous constraints because additional parameters are allowed to vary.
With these generalizations, a much wider range of universes can develop cosmic structure
and support observers. To constrain the constituent parameters, new BBN calculations are
carried out in the regime where η and G = Mpl−2 are much larger than in our universe. If the
BBN epoch were to process all of the protons into heavier elements, no hydrogen would be
left behind to make water, and the universe would not be viable. However, our results show
that some hydrogen is always left over, even under conditions of extremely large η and G, so
that a wide range of alternate universes are potentially habitable.
1 Introduction

Our current understanding of the universe falls in a curious regime. On one hand, we can explain the basic properties of the universe with relatively simple equations and relatively few basic cosmological parameters. Observations of the Cosmic Microwave Background Radiation [26] and the expansion rate [19, 22] provide us with a description of the universe with a precision approaching a few percent [23]. On the other hand, one of the required parameters in this paradigm is the energy density of the vacuum, denoted here as $\rho_{\Lambda}$ and often called the dark energy. Existing observations indicate that the density of the dark energy is nearly constant with time, so that it acts much like a cosmological constant. In addition, the value of the vacuum energy density is comparable to the closure density of the universe at the present cosmological epoch, so that the vacuum energy density is much smaller than the benchmark value given by the Planck mass, i.e.,

$$\frac{\rho_{\Lambda}}{M_{\text{Pl}}^4} \sim 10^{-120} \ll 1.$$  \hspace{1cm} (1.1)

This expression, and the rest of the paper, is written in units where $\hbar = c = k_B = 1$. One would like an explanation for this extreme ordering of energy scales, but no general consensus currently exists [31].

In the absence of a definitive prediction for the energy density of the vacuum, many researchers have argued that the value of $\rho_{\Lambda}$ is constrained by anthropic considerations [7–9]. In this context, the value of $\rho_{\Lambda}$ and other fundamental constants must have values that allow for the formation of structure in the universe and the possibility of the existence of observers. An upper bound on the energy density of the vacuum can be derived from the requirement that galaxy formation occurs before the the universe becomes dominated by the cosmological constant. The first treatment of this problem [30] found that the upper bound on the vacuum energy density must be at least as large as $500\rho_0 > \rho_{\Lambda}$, where $\rho_0$ is the current density of the universe. This value was obtained under that assumption that quasars must form by redshift...
z = 4.5. Weaker bounds — corresponding to larger values of the upper bound — can be derived by relaxing the assumption that structures must form by the current epoch or that the perturbations must begin with the small amplitudes realized in our universe [4, 17, 18]. These generalizations allow the upper limit to be much larger, i.e., \( \rho_\Lambda < 10^9 \rho_0 \). Given that the vacuum energy density could be a billion times larger than its observed value, anthropic arguments are not overly constraining.

The goal of this paper is to reexamine the upper limit on the density of dark energy for a more general class of universes by allowing additional parameters to vary. Here we consider variations in the amplitude \( Q \) of the primordial density fluctuations, the baryon-to-photon ratio \( \eta \), the total matter density \( \Omega_M \), the baryon density \( \Omega_b \), and the strength of gravity (given by the Planck mass \( M_{\text{pl}} \)). Because different values for the gravitational constant (equivalently, \( M_{\text{pl}} \)) affect stellar structure, we require that stars are operational, which enforces an upper limit on the strength of gravity (a lower limit on \( M_{\text{pl}} \)). The values of \( \eta \) and \( M_{\text{pl}} \) affect the yields from Big Bang Nucleosynthesis (BBN). Here we carry out new BBN calculations in the regime where the values of both \( \eta \) and \( M_{\text{pl}} \) are much larger than in our universe, and use the results to enforce the requirement that enough hydrogen remains to provide water. These calculations explore a new regime of parameter space and illustrate the difficulty of rendering the universe lifeless due to the BBN epoch.

A secondary goal of this paper is to consider the overarching issue of the possible fine-tuning of the universe [6, 10, 27, 28]. It is often claimed that even small variations in the fundamental constants of physics and/or the parameters of cosmology would make it impossible for the universe to develop complex structures and hence observers [7, 8, 14, 20]. Here we explicitly consider the possible range of values for \( \rho_\Lambda \). In addition, in order to evaluate our constraint on the energy density of the vacuum, we must consider the range of possible variations for the fluctuation amplitude \( Q \), the baryon to photon ratio \( \eta \), the Planck mass \( M_{\text{pl}} \), and the ratio \( \Omega_M/\Omega_b \). We find that all of these quantities, as well as the vacuum energy density \( \rho_\Lambda \) itself, can vary over wide ranges without rendering the universe inhospitable.

This paper is organized as follows. We first provide a brief review of the basic elements of structure formation and then derive the constraint on \( \rho_\Lambda \) resulting from the requirement that structure form before the universe becomes vacuum dominated (Section 2). The resulting constraint depends on the values of the parameters \((Q, \eta, M_{\text{pl}}, \Omega_M/\Omega_b)\); the allowed range of these parameters are discussed and constrained in Section 3. The paper concludes in Section 4 with a summary of the results and a discussion of their implications.

## 2 Constraint on the Energy Density of the Vacuum

### 2.1 Definitions

The equation of motion for the scale factor \( a(t) \) of the universe has the form

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left[ \rho_\Lambda + \rho_M + \rho_R \right],
\]

where we have assumed that the universe is spatially flat, which would be the case if inflation occurs under standard conditions. The densities on the right-hand-side of the equation correspond to the vacuum energy (\( \rho_\Lambda \)), matter (\( \rho_M \)), and radiation (\( \rho_R \)). For the sake of definiteness, we assume that the universe experiences both radiation dominated and matter dominated epochs. Suppose that the universe has total energy density \( \rho_{\text{eq}} \) at the epoch of
equality. If we use this value as a reference density scale, then we can define a corresponding reference time scale

\[ t_{eq} \equiv \left( \frac{8\pi G\rho_{eq}}{3} \right)^{-1/2}, \] (2.2)

and the equation of motion reduces to the form

\[ \left( \frac{\dot{a}}{a} \right)^2 = \Omega_V + \Omega_M a^{-3} + \Omega_R a^{-4}, \] (2.3)

where \( \Omega_V = \rho_V/\rho_{eq}, \Omega_M = \rho_M/\rho_{eq}, \) and \( \Omega_R = \rho_R/\rho_{eq}. \) The time variable in the reduced equation of motion is related to the physical time \( t_{phys} \) (coordinate time) according to \( t = t_{phys}/t_{eq}. \) Flatness of the universe also implies the constraint

\[ \Omega_V + \Omega_M + \Omega_R = 1. \] (2.4)

If we also let the scale factor \( a = 1 \) at the epoch of equality, then \( \Omega_M = \Omega_R \equiv \Omega, \) and we redefine \( \lambda = \Omega_V = 1 - 2\Omega. \) The equation of motion now becomes

\[ \left( \frac{\dot{a}}{a} \right)^2 = \lambda + \Omega \left[ a^{-3} + a^{-4} \right], \] (2.5)

where \( \Omega \approx 1/2 \) and \( \lambda \ll 1. \) For future reference, the temperature of the universe at the epoch of equality can be written in the form \([27, 28]\)

\[ T_{eq} = \eta m_p \frac{\Omega_M}{\Omega_b}, \] (2.6)

where the parameter \( \eta \) is the baryon to photon ratio and \( m_p \) is the proton mass. Notice also that we are ignoring the neutrino contribution to the energy density of radiation.

### 2.2 Review of Perturbation Growth

The usual equation of motion \([16]\) for the growth of density fluctuations \( \delta_k = \rho_k/\rho_0 \) in an expanding universe has the form

\[ \ddot{\delta}_k + 2\frac{\dot{a}}{a} \dot{\delta}_k + \left[ \frac{v^2 k^2}{a^2} - 4\pi G \rho_M \right] \delta_k = 0. \] (2.7)

This equation is written in physical units, where \( \rho_k \) is the density perturbation, \( k \) is the wavenumber of the perturbation, and \( \rho_M \) is the density of the matter. Here we consider only long wavelength modes \( k \ll k_J, \) where \( k_J \) is the Jeans wavenumber, so that we can drop the subscript and ignore the pressure term. Converting to the dimensionless units of the previous section, using the reference time scale \( t_{eq} \) and density scale \( \rho_{eq}, \) the equation of motion for \( \delta \) becomes

\[ \ddot{\delta} + 2\frac{\dot{a}}{a} \dot{\delta} - \frac{3}{2} \Omega a^{-3} \delta = 0. \] (2.8)

The remaining equation of motion for the Hubble expansion has the form

\[ \left( \frac{\dot{a}}{a} \right)^2 = \lambda + \Omega \left[ a^{-3} + a^{-4} \right], \] (2.9)
where \( \lambda \equiv 1 - 2\Omega \). By definition, growth starts at \( t = 1 \), when the initial conditions are

\[
a = 1, \quad \delta = 1, \quad \text{and} \quad \dot{\delta} = 0.
\]  
(2.10)

Note that since the equation of motion for the density perturbation is linear, we can assume any value for the starting state and re-scale it later.

With the above formulation, we have a one parameter family of models for structure formation. Moreover, as long as \( \lambda \neq 0 \), the perturbation \( \delta \to \delta_f = \text{constant} \) in the limit \( t \to \infty \). For the initial condition \( \delta(t = 1) = 1 \), the final value \( \delta_f \) thus represents the total amount of growth available for a given value of the vacuum contribution. This growth factor is shown in Figure 1, plotted here as a function of \( \lambda \). To leading order, we expect the amplitude of a density perturbation to grow linearly with the scale factor \( a(t) \) during the matter dominated era and then to saturate (freeze out) when the universe becomes vacuum dominated. The total growth factor should be approximately \( \delta_{\text{est}} \sim (2\lambda)^{-1/3} \). This factor is shown as the dashed curve in Figure 1. Note that the approximate expression (dashed curve) closely follows the more exact value (solid curve) calculated by numerically evaluating equations (2.9) and (2.8) subject to the initial conditions of (2.10).

Notice also that the linear treatment is approximate. We can correct for this deficiency by writing the growth factor in the form

\[
\delta_{\text{est}} = A\lambda^{-1/3},
\]  
(2.11)

where \( A \) is a dimensionless parameter of order unity. In the original derivation of this constraint [30], this parameter is given by \( A = (500/729)^{1/3} \approx 0.882 \). Although we include the factor \( A \) for completeness here, the exact value will not matter; as shown below, the bound changes by 30 orders of magnitude.

### 2.3 Constraint on Vacuum Energy

We can now write down the constraint due to the requirement that cosmological structures can form. The initial size of perturbations is set by the parameter \( Q \), and the growth factor is given by equation (2.11). In order for the perturbations to become nonlinear before being frozen out due to the acceleration of the expansion, we require

\[
\lambda < A^3Q^3.
\]  
(2.12)

In our universe the parameter \( Q \sim 10^{-5} \) [25, 26]. In any universe, we expect \( Q \ll 1 \) in order for habitable galactic structures to form [3, 20, 27, 28], so that the above constraint also implies \( \lambda \ll 1 \). As a result, for the time scales and temperature scales evaluated at the epoch of equality, any corrections due to the nonzero value of \( \lambda \) must be small. Notice also that this constraint is essentially the same as that given by equation (29) of [27], or by equation (4) of [17].

Recall that \( \lambda \) is the fractional contribution of the vacuum to the energy density at the epoch of equality. It is useful to write the constraint (2.12) in terms of the (constant) energy density \( \rho_\Lambda \) of the vacuum, i.e.,

\[
\frac{\rho_\Lambda}{M_{\text{pl}}^4} \leq A^3Q^3\frac{\rho_{\text{eq}}}{M_{\text{pl}}^4} = (2A^3\alpha_R)Q^3\eta^4\left(\frac{m_p}{M_{\text{pl}}^4}\right)^4\left(\frac{\Omega_M}{\Omega_b}\right)^4,
\]  
(2.13)
Figure 1. Growth factor as a function of the vacuum energy density, expressed here as the fraction of the total energy density at the time of equality (between radiation and matter). The solid blue curve shows the result from numerical integration, whereas the dashed red curve shows the approximate result $\delta_f = (2\lambda)^{-1/3}$ (see text).

where we have scaled $\rho_\Lambda$ by its “natural” value implied by the Planck mass. To obtain the final equality, we have used the expression $\rho_{eq} = 2a_RT_{eq}^4$, where $a_R = \pi^2/15$ is the radiation density constant, along with the expression (2.6) for the crossover temperature $T_{eq}$.

In our universe, $m_p/M_{pl} \sim 10^{-19}$, $\eta \sim 10^{-9}$, $\Omega_M \sim 6\Omega_b$, and $Q \sim 10^{-5}$, so the dimensionless vacuum energy $\rho_\Lambda/M_{pl}^4 \sim 10^{-124}$. In order to place an upper bound on this quantity, we need to separately constrain the possible ranges for the parameters $Q$, $\eta$, $M_{pl}$, and $\Omega_M/\Omega_b$.

3 Limits on the Input Parameters

As outlined above, if we use the values of $Q$, $\eta$, and $M_{pl}$ found in our universe, then the bound on the vacuum energy density from equation (2.13) leads to a value of $\rho_\Lambda$ roughly comparable to that inferred by observational data. Instead of using the observed values for $(Q, \eta, M_{pl})$, however, we instead need to find upper bounds on these quantities. These bounds are discussed below.

3.1 Density Fluctuation Amplitude

Constraints on the fluctuation amplitude $Q$ have been considered by several previous authors [3, 27]. If the fluctuation amplitude is larger, then galaxies form earlier in cosmological history,
when the background density is larger. This ordering of time scales results in galaxies that are denser. If galaxies are too dense, then planets in habitable orbits can be disrupted through scattering interaction with passing stars [27]. Since galaxies have a wide range of densities, however, this effect does not render the entire galaxy uninhabitable. Instead, it limits the fraction of stars that could in principle harbor habitable planets [3]. If we take an optimistic view, the fluctuation amplitude $Q$ could be as large as $Q \sim 10^{-2}$, which implies that about half of the stars in a galaxy the size of our Milky Way would remain habitable. This value of $Q$, in turn, allows the bound in equation (2.13) to be larger (than for the parameters in our universe) by a factor of $10^9$.

3.2 Baryon to Photon Ratio

Bang Nucleosynthesis (BBN) provides a constraint on the value of the baryon to photon ratio $\eta$. In our universe, this ratio must be $\eta \sim 10^{-9}$ in order for the early universe to produce (roughly) the observed abundances of the light elements (deuterium, lithium, and helium). The abundances of these nuclear species could be different in other universe, varying by large factors, with no detrimental effects. A universe could end up sterile, however, if the BBN epoch processes all of the hydrogen into helium or other heavier nuclei. Such a universe, with little or no hydrogen, would not have the basic raw materials to make water.

Calculations of the BBN epoch show that the baryon-to-photon ratio $\eta$ can be increased by many orders of magnitude and still allow substantial hydrogen to remain unprocessed. Here we present results computed using the BURST code [12] for BBN, which is updated from the standard version of the BBN code [24, 29]. To start, we fix all of the parameters at their standard values but allow variations in the value of the baryon-to-photon ratio $\eta$. Note that the gravitational constant $G$ is varied in the following section, and then both $\eta$ and $G$ are allowed to vary at the same time.

Figure 2 shows the resulting BBN yields for a range of $\eta$ values. For small values of $\eta$, much smaller than the values for our universe, the mass fraction of helium-4 is small — even smaller than that of deuterium. As $\eta$ increases, the nuclear reaction rates increase, and the mass fraction of helium-4 increases. The abundances of deuterium and helium-3 decrease with increasing $\eta$, as they are burned into helium-4.

In the limit of large $\eta$, the mass fraction of helium-4 reaches a limiting value of $Y_4 \sim 0.3$. This limit corresponds to the regime where essentially all of the neutrons are burned into helium-4. Notice also that the abundances of the other nuclear species decrease with increasing $\eta$. With all of the neutrons incorporated into helium-4, none are left for constructing the remaining light elements. For example, with $\eta = 10^{-6}$, after helium-4 the next most abundant nuclear species is helium-3, with a mass fraction of only $3 \times 10^{-7}$. As a result, a large fraction of protons always remains to potentially be incorporated into water. We can thus increase the value of $\eta$ by a factor of at least $10^3$ and allow the universe to remain viable. This increase, in turn, allows the limit of equation (2.13) to increase by a factor of $10^{12}$.

3.3 Gravitational Constant (Planck Mass)

Stellar considerations show that the the gravitational constant can be larger than the value in our universe, but “only” by a factor of $\sim 2 \times 10^5$ [1, 2]. This limit arises from the requirement that working stars exist, more specifically that stable nuclear burning states exist, the stellar mass is larger than the minimum value enforced by degeneracy pressure, and that the lower mass limit for stars does not exceed the upper mass limit. The constraint of equation (2.13)
Figure 2. BBN yields as a function of the baryon to photon ratio $\eta$. The curves show the resulting mass fraction at the end of the BBN epoch for helium-4 (red), as well as the corresponding mass fractions $X_i$ for deuterium (green), helium-3 (cyan), and lithium-7 (purple). The blue curve shows the mass fraction of free protons. The vertical black line marks the estimated value of $\eta = 6 \times 10^{-10}$ for our universe.

is proportional to $G^2 \sim M_{\text{pl}}^{-4}$, so this limit can be increased by a factor of $\sim 4 \times 10^{10}$ due to the allowed range of the gravitational constant.

Possible variations in the gravitational constant can also change the predictions of BBN due to corresponding changes in the expansion rate of the universe, where $H = \dot{a}/a \propto G^{1/2}$. Figure 3 shows the yields from BBN, where the gravitational constant is varied over six orders of magnitude, from 100 times smaller than the value in our universe to $10^4$ times larger. For most of the range shown, the abundances of all of the light elements increase with $G$. For sufficiently large values of $G$, however, the expansion rate is so fast that not all of the neutrons can be made into helium-4. As a result, the mass fraction of helium-4 has a maximum value of $Y_4 \sim 0.53$, which occurs at $G/G_0 \sim 100$. Even with this maximum mass fraction of helium-4, the universe retains about half of its protons to make water. As result, BBN does not greatly constrain the habitability of universes in this context.

We can understand the BBN yields shown in Figure 3 as follows. For small values of $G$, the expansion rate is slow, and the freezing of the weak interactions occurs later in cosmic history. As a result, protons and neutrons remain in Nuclear Statistical Equilibrium (NSE) longer and the $n/p$ ratio is smaller. As $G$ increases, the expansion rate increases, freeze-out of weak interactions occurs earlier, and the $n/p$ ratio is larger. Figure 4 shows the ratio $n/p$ as a function of temperature for different values of the gravitational constant. The curves for different values of $G$ show the basic trend outlined above, where weaker gravity allows the
Figure 3. BBN yields as a function of the gravitational constant, $G/G_0$, scaled to the value in our universe. The curves show the resulting mass fraction at the end of the BBN epoch for helium-4 (red), as well as the corresponding mass fractions $X_i$ for deuterium (green), helium-3 (cyan), and lithium-7 (purple). The blue curve shows the mass fraction of free protons. The vertical black line marks the value of $G$ found in our universe.

The $n/p$ ratio to track its NSE value (shown as the black dashed curve) to lower temperatures, thereby resulting in lower neutron abundances during the BBN epoch. At sufficiently late times, free neutrons decay, and $n/p \to 0$. Note that the temperature decreases with increasing time, but the relation $T_{cm}(t)$ depends on the value of the gravitational constant. This trend is illustrated in Figure 4, where the circles mark the location on the curves where time $t = 1$ sec, and the squares delimit $t = 1000$ sec. Since most of the neutrons are processed into helium-4, its abundance generally grows with increasing $G$. The abundances of deuterium and helium-3 also increase. As noted above, for sufficiently large values of $G$, the abundance of helium-4 decreases again. A partial explanation is provided by Figure 4, which shows that the $n/p$ ratio is not monotonic with increasing strength of gravity. The value of $n/p$ at the end of the BBN epoch (right side of the figure) increases as $G/G_0$ increases from unity to 100, but then decreases with further increase in $G/G_0$ from 100 to $10^6$. This decrease in the number of neutrons leads to less helium production for large values of $G/G_0$.

3.4 Generalized Constraints from BBN

Figure 5 shows the abundance of hydrogen in the dual parameter space of $G/G_0$ versus $\eta$. We plot contours of constant mass fraction for all hydrogen isotopes, $X_H$, i.e., the summation of the single-proton hydrogen and deuterium mass fractions

$$X_H = X_{1H} + X_{2H}.$$  

(3.1)
Figure 4. Neutron to proton ratio during the BBN epoch. The ratio $n/p$ is plotted versus the temperature parameter $T_{cm}$ for different choices of the gravitational constant, as labeled (see Ref. [13] for a precise definition of $T_{cm}$). The black dashed curve shows the $n/p$ ratio expected in Nuclear Statistical Equilibrium. The yellow circles (magenta squares) mark the locations on the curves where time $t = 1$ second (1000 seconds).

Figures 2 and 3 both show that when we maximize either $\eta$ or $G/G_0$ while holding the other parameter fixed, hydrogen results as the most abundant element. We would expect that BBN would produce a majority of hydrogen when we maximize both parameters. Figure 5 shows the opposite behavior: $X_H$ falls to below 10% in the upper right-hand corner. The error in the logic resides in the phasing of the freeze-out of two sets of reactions: the weak interactions responsible for setting the ratio of neutrons to protons (denoted $n/p$); and the strong and electromagnetic nuclear reactions responsible for the synthesis of elements with mass number $A > 1$. There are six weak interactions which dictate $n/p$, schematically shown as the forward and reverse reactions

$$\nu_e + n \leftrightarrow p + e^-,$$

$$e^+ + n \leftrightarrow p + \nu_e,$$

$$n \leftrightarrow p + e^- + \nu_e,$$  

where $e^\pm$ denotes positrons and electrons, $\nu_e$ denotes the electron neutrino, and $\nu_e$ denotes the electron antineutrino. As long as the rates associated with the reactions in equations (3.2) – (3.4) are rapid, the ratio $n/p$ will stay in equilibrium and decrease as the universe expands and the temperature decreases. The lepton-capture rates corresponding to the forward and reverse reactions in equations (3.2) and (3.3) both scale as the fifth power of temperature [11]. The Hubble expansion rate scales as the second power of temperature, so that the lepton-capture
rates will always be larger than the expansion rate at early times, and become subsidiary at later times. If we increase the strength of the gravitational constant, we precipitate an earlier epoch when the expansion rate surpasses the lepton capture rates. The result is that $n/p$ goes out of equilibrium earlier, dictating a larger free neutron abundance at this time.

Figure 3 shows that the end state of most free neutrons is their provisional incorporation into $^4$He nuclei. For $1 \lesssim G/G_0 \lesssim 100$, the rate of decrease in $X_{^1H}$ is close to the rate of increase in $X_{^4He}$. However, Figure 3 also shows a local minimum for $X_{^1H}$ when $G/G_0 \simeq 100$. At this point, the Hubble expansion rate is fast enough that the reactions for synthesis of $^4$He, primarily the strong reaction $^3$He($^3$He, 2p)$^4$He, begin to freeze-out at an earlier epoch. The overabundance of neutrons either resides in other nuclei, specifically deuterium (D) and $^3$He, or continues to exist as free neutrons. The forward rate in equation (3.4) is free neutron decay which is invariant with temperature at late times. Any remaining free neutrons at the conclusion of BBN will transmute to free protons (see Figure 4). Figure 3 shows that $^1$H mass fraction increases at a larger absolute rate than those of D and $^3$He, implying that $n/p$ decreases. The mass fractions of D and $^3$He are continuing to increase at the end of the parameter range in Figure 3, but they will reach a local maximum eventually, much like the extant maxima for $^4$He and $^7$Li. To preserve the overabundance of neutrons which exist after the freeze-out of the lepton capture rates, the strong and electromagnetic rates must be more rapid. Those rates are proportional to varying powers of $\eta$, depending on the number of

Figure 5. Contours of the remaining Hydrogen mass fraction (including all isotopes) after BBN for universes with varying $\eta$ and $G/G_0$. The thick contour marked 0.75 corresponds to hydrogen abundances comparable to that of our universe. The red star marks the location of our universe in the plane. The hydrogen abundance falls below 10 percent for the upper right part of the diagram, where $G/G_0 \sim 10^6$ and $\eta \sim 10^{-6}$. 

input particles in any particular reaction. Conversely, the rates for the reactions in equations (3.2) – (3.3) are insensitive to $\eta$, implying that $n/p$ will be roughly the same for all $\eta$ at the epoch when the lepton-capture rates freeze out. The nuclear reactions are more rapid with increasing $\eta$, implying a later epoch of nuclear freeze-out, implying a longer span of time for the assembly of the larger nuclei. Figure 5 shows for any value of $G/G_0$, increasing $\eta$ increases $n/p$, or conversely decreases $X_H$.

In any case, the results of Figure 5 show that the mass fraction of hydrogen remaining after the epoch of BBN drops to below 0.10 in the upper corner of the diagram, where $\eta \sim 10^{-6}$ and $G/G_0 \sim 10^6$. Although some protons must remain after BBN in order for the universe to have the raw material for water, the minimum mass fraction is not known. Here we use 10 percent as a benchmark value. The resulting upper limit for the gravitational constant, $G/G_0 < 10^6$ is comparable to that obtained by requiring stars to function [1, 2]. The value of $\eta$ can be larger than that in our universe by a factor of more than 1000.

Although we use the 10 percent hydrogen abundance as a benchmark for habitable universes, it is of interest to consider the location of the contour for $X_H = 0$ in an extended version of the parameter space shown in Figure 5. To answer this question, we consider the neutron to proton ratio $n/p$ in chemical equilibrium,

$$n/p = \exp \left(-\frac{\delta m_{np}}{T} + \frac{\mu_e}{T} - \xi_{\nu_e} \right), \tag{3.5}$$

where $\delta m_{np} \sim 1.3$ MeV is the mass difference between a neutron and a proton, $\mu_e$ is the chemical potential of the electrons, and $\xi_{\nu_e} = \mu_{\nu_e}/T_{cm}$ is the electron neutrino degeneracy parameter [13]. The parameter $\xi_{\nu_e}$ is a co-moving invariant if we scale the electron neutrino chemical potential with a temperature-like quantity different from the plasma temperature. In this setting, $T_{cm}$ is the co-moving temperature parameter and allows for the neutrinos to be out of thermal equilibrium with the primeval plasma [13]. Both $\mu_e/T$ and $\xi_{\nu_e}$ are small compared to $\delta m_{np}/T$, so $n/p$ is strictly less than unity. Figures 2 and 3 show that for $n/p < 1$, the proton excess is mainly preserved in $^1$H. Therefore, under the assumptions that $\mu_e/T$ and $\xi_{\nu_e}$ are small, the $X_H = 0$ contour is never reached. The contours in Figure 5 continue to be spaced at larger intervals when both $\eta$ and $G/G_0$ are increased. The contour for $X_H = 0$ would therefore be an asymptote assuming $\mu_e/T$ and $\xi_{\nu_e}$ are small.

The assumptions outlined above may not hold for every possible universe within the multiverse. The electron chemical potential is proportional to the baryon asymmetry, characterized by $\eta$. If $\eta$ sufficiently large, of order $\delta m_{np}/T \sim 1$, then the ratio $n/p$ could be larger than unity. However, for $\eta$ close to unity, BBN would occur under matter-dominated conditions (see the following subsection) where inhomogeneities are important [3, 15]. This scenario is much different than the parameter space considered in Figure 5, where the background medium is homogeneous and radiation-dominated. Alternatively, if the parameter $\xi_{\nu_e}$ is large and negative, i.e., if there exists an overabundance of antineutrinos to neutrinos, then BBN continues to occur in radiation-dominated conditions with $n/p > 1$. In this case we would expect very little $^1$H and possibly a significant fraction of nuclei heavier than $^4$He — depending on the specific value of $n/p$. These alternate scenarios should be explored in future work, but are beyond the scope of this present paper.

### 3.5 Ordering of Time Scales and the Ratio of Densities

In order for the universe to produce structure within the paradigm of the standard cosmological model [16], the epoch of equality of matter and radiation must occur after the epoch of
big bang nucleosynthesis. Since we are not considering variations in the nuclear properties, the energy of nuclear reactions must be of order a few MeV. We can parameterize this level by writing

\[ T_{\text{bbn}} = bm_e , \]  

(3.6)

where \( m_e \) is the electron mass (0.511 MeV) and \( b \) is a constant of order unity. The ordering constraint implies that \( T_{\text{eq}} < T_{\text{bbn}} \) or \( \eta \Omega_M/\Omega_b < b m_e/ m_p \sim 10^{-3} \).

As discussed above, if we require the universe to retain at least 10 percent of its hydrogen after BBN, then the baryon to photon ratio \( \eta < 10^{-6} \) (see Figure 5). With the value for \( \eta \) and the ratio \( \Omega_M/\Omega_b \approx 6 \) found in our universe, the epoch of BBN falls well before that of equality.

In the bound on the energy density of the vacuum, given by equation (2.13), the product \( \eta (\Omega_M/\Omega_b) \) appears on the right-hand-side. If we set \( \eta = 10^{-6} \) in order to evaluate the bound on \( \rho_\Lambda \), then equation (3.7) implies that the largest value of the density ratio must be \( \Omega_M/\Omega_b \approx 100 \), which is larger than the value in our universe by a factor of \( \sim 15 \).

The bound of equation (2.12) also indicates that the time of vacuum domination \( t_\Lambda \) must occur after the time of matter domination \( t_{\text{eq}} \). This result, in conjunction with that of equation (3.7), indicates that the time scales must obey the ordering

\[ t_{\text{BBN}} < t_{\text{eq}} < t_\Lambda . \]  

(3.8)

4 Conclusion

The main result of this paper is an upper bound on the density \( \rho_\Lambda \) of the vacuum energy, given by equation (2.13), which represents a generalization of previous treatments. If we evaluate the right-hand-side of the equation using the values for \( (Q, \eta, M_{\text{pl}}) \) found in our universe, the result is roughly comparable to the ratio \( \rho_\Lambda/M_{\text{pl}}^4 \) that is observed. This finding is essentially a restatement of the coincidence problem – in our universe structure has formed recently and the energy density is now becoming dominated by the vacuum. This paper shows (Section 3) that a viable alternate universe could have larger fluctuation amplitude \( Q \) (by a factor of \( \sim 10^3 \)), larger baryon to photon ratio \( \eta \) (by a factor of \( \sim 10^3 \)), and stronger gravity (so that \( M_{\text{pl}} \) is smaller by a factor of \( \sim 10^3 \)). With the maximum value of \( \eta \), the ratio of densities \( \Omega_M/\Omega_b \) can be larger by a factor of \( \sim 15 \). With these generalizations, the density of the vacuum energy can be much larger than that of our universe: The constraint takes the form \( \rho_\Lambda/M_{\text{pl}}^4 < 10^{-90} \) so that \( \rho_\Lambda \) could be larger than the observed value by \( \sim 30 \) orders of magnitude.

This generalization of the bound on \( \rho_\Lambda \) is significant. As emphasized in the original paper of Weinberg [30], if the maximum allowed value for the energy density \( \rho_\Lambda \) is much larger than the empirically allowed value, “then we would have to conclude that the anthropic principle does not explain why the cosmological constant is as small as it is”. Given that \( \rho_\Lambda \) could be \( \sim 30 \) orders of magnitude larger, relative to the Planck scale, and still allow structure to form, we argue that the anthropic principle has limited predictive power in this context.

A related outcome of this work is the finding that universes have difficulty processing all of their protons into heavier elements, even under extreme conditions. More specifically, we have considered variations in both \( \eta \) and \( G \), and determined the fraction of nucleons that
are synthesized into non-hydrogen isotopes (Figure 5). The hydrogen mass fraction remains greater than 10 percent even when the parameters $\eta$ and $G$ are larger than $10^3$ and $10^6$ times the values realized in our universe. Moreover, the results from Section 3.4 indicate that some hydrogen almost always remains after the epoch of BBN. In order to process all of the protons into heavier elements, the neutron to proton ratio must exceed unity ($n/p > 1$), which in turn requires extreme conditions (e.g., matter dominated BBN, large $\nu_e$ degeneracy parameter, and/or large $\eta \sim 1$). These results show that the BBN epoch does not represent an example of fine-tuning. The input parameters can be varied by many orders of magnitude and still allow the universe to remain viable.

The results of this paper have implications for the broader issue of the possible fine-tuning of the universe. The main result of this paper is that the density of the vacuum energy $\rho_\Lambda$ could be larger by more than 30 orders of magnitude and still allow for structure formation (with appropriate adjustments to other parameters). Of course, the density $\rho_\Lambda$ could also be 30 orders of magnitude smaller and the universe would continue to develop cosmic structure. We thus argue that the value of $\rho_\Lambda$ need not be overly fine-tuned for the universe to produce observers.

The results of this paper also show that the epoch of Big Bang Nucleosynthesis generally does not leave the universe with no hydrogen. Specifically, the mass fraction of hydrogen remains larger than 10 percent even when the baryon to photon ratio $\eta$ is larger by a factor of $10^3$ and the gravitational constant $G$ is larger by a factor of $10^6$. The density ratio $\Omega_M/\Omega_b$ can be larger by a factor of $\sim 15$ if $\eta$ takes on its maximum value, but can be much larger for smaller values of $\eta$. As a result, the universe can remain chemically viable without fine-tuning the parameters of the BBN epoch.

Finally, we have used previous results to constrain the other relevant parameters of the problem. The amplitude $Q$ of the primordial density fluctuations spans a range of a factor $\sim 10^4$, from 10 times smaller [27, 28] to $10^3$ times larger [3]. Additional considerations of stellar structure show that working stars can exist for values of $G$ and the fine-structure constant $\alpha$ that vary over many orders of magnitude [1]. This claim holds up in the face of additional constraints, including that the stars have sufficiently hot surface temperatures and long lifetimes [2], and that stars can form within their parental galaxies [21, 32]. Taken together, all of these results indicate that the universe is not overly fine-tuned, in that both the constants of physics ($\alpha, G$) and the cosmological parameters ($Q, \eta, \rho_\Lambda, \Omega_M/\Omega_b$) can vary over a wide range and still allow the universe to develop astrophysical structures and perhaps even life.

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