Abstract—Population protocols are a formal model of computation by identical, anonymous mobile agents interacting in pairs. It has been shown that their computational power is rather limited: They can only compute the predicates expressible in Presburger arithmetic. Population protocols are oblivious, in the sense that their behavior only depends on the number of agents in each state of the current configuration, and nothing else. Obliviousness has advantages for applications where agents want to reveal as little as possible about their trajectories in a computation. We investigate the computational power of oblivious protocols. We first show that, under a weak assumption, oblivious protocols can only compute number predicates $\varphi : \mathbb{N}^n \to \{0, 1\}$ in $\text{NSPACE}(n)$ (with the input written, as usual, in binary), while all predicates computed by population protocols are in $\text{DSPACE}(\log n)$, thus proving an exponential gap. Then we introduce broadcast consensus protocols, in which agents can also broadcast signals to all other agents. We prove that they compute all predicates in $\text{NSPACE}(n)$, reaching the theoretical limit for oblivious protocols. Finally, we conduct the first systematic comparison of different models introduced in the literature (population protocols, broadcast protocols, community protocols, and mediated protocols) with respect to their computational power and their privacy guarantees.

I. INTRODUCTION

Population protocols are a theoretical model for the study of ad hoc networks of tiny computing devices without any infrastructure [1], intensely investigated in recent years (see e.g. [2], [3], [4], [5]). The model postulates a “soup” of indistinguishable agents that behave identically, and only have a fixed number of bits of memory, i.e., a finite number of local states. Agents repeatedly interact in pairs, changing their states according to a joint transition function. A global fairness condition ensures that every finite sequence of interactions that becomes enabled infinitely often is also executed infinitely often. The purpose of a population protocol is to allow the agents to collectively compute some information about their global initial state. For example, assume that initially each agent picks a boolean value. The many majority protocols described in the literature allow the agents to eventually reach a stable consensus on the value chosen by a majority of the agents. More formally, let $x_0$ and $x_1$ denote the initial numbers of agents that pick 0 and 1; majority protocols compute the predicate $\varphi(x_0, x_1) : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ given by $\varphi(x_0, x_1) = (x_1 \geq x_0)$.

In a seminal paper, Angluin et al. proved that population protocols compute exactly the predicates expressible in Presburger arithmetic [1], [6]. So, for example, the agents can decide if they are at least a certain number, if at least $2/3$ of the agents picked 1, or, more generally, if the vector $(x_1, x_2, \ldots, x_n)$ representing the number of agents that picked option 1, 2, \ldots, $n$ in an election with $n$ choices is a solution of a system of linear inequalities. On the other hand, the agents cannot decide if they are a square or a prime number, or if the product of the number of votes for options 1 and 2 exceeds the number of votes for option 3.

For many purposes, the class of Presburger predicates is fairly small. For this reason, much work has been devoted to designing more powerful formalisms and analyzing their expressive power. In the community model of Guerraoui and Ruppert, agents have unique identities, represented by integers [7]. When two agents meet, they can communicate their identity to the other agent. Further, each agent can store a constant number of identities and compare identities (i.e., determine which one is the largest), although they cannot perform arithmetic on them. In the community model, a configuration of $n$ agents is represented by a word $q_1q_2\cdots a_n$, where $q_i$ is the current state of the $i$-th agent, i.e., the agent with the $i$-th largest identity. The main result of [7] is that community protocols can solve precisely the decisions problem in $\text{NSPACE}(n \log n)$. In other words, for every nondeterministic $(n \log n)$-space bounded Turing machine over alphabet $\Sigma$, there is a protocol with set of states $\Gamma \supseteq \Sigma$ such that for every input $x \in \Sigma^n$, if the protocol is started with a community of $n$ agents in the configuration $x$, all agents eventually reach the correct consensus on whether the machine accepts or rejects.

Motivated by Guerraoui and Ruppert’s model, Michail, Chatzigiannakis and Spirakis have investigated whether the computational model of population protocols can be increased without assigning identities to the processes [8]. They introduce mediated population protocols, a very ingenious model in which agents communicate through unidirectional channels with a finite number of bits of memory. More precisely, the agents at the source and target of a channel are called the initiator and responder, respectively, and the rules are of the form “if the initiator is in state $q_1$, the responder in state $q_2$, and the channel in state $q_3$, then they respectively move to states $q_1'$, $q_2'$, and $q_3'$, represented as $(q_1, q_2, q_3) \rightarrow (q_1', q_2', q_3')$. They prove that mediated population protocols can solve precisely the decision problems in symmetric $\text{NSPACE}(n^2)$, i.e., the set of languages $L \in \text{NSPACE}(n^2)$ such that $x \in L$ implies $y \in L$ for every permutation $y$ of $x$. 
While in mediated population protocols agents do not have built-in identities, the model allows an agent $A$ to dynamically “tag” another agent $B$, being sure that the tag will remain until their next interaction. Indeed, the execution of rule $(q_1, q_2, q_3) \rightarrow (q_1', q_2', q_3')$ by initiator $a$, responder $b$, and channel $ab$ can also be interpreted as $a$ “tagging” $b$ with $q_3'$; Since the value of $ab$ can only be changed by transitions involving $a$, there is no way for $b$ to get rid of this tag without $a$'s consent. So, while the agents of mediated population protocols have no identities and behave identically (two properties called anonymity and uniformity in [8]), agents can lose their anonymity with respect to other processes in the course of the computation. It is not difficult to imagine scenarios in which agents, for this reason, might not be willing to engage in a mediated protocol. For example, consider an ad-hoc network of agents performing financial transactions. After $a$ paying to $b$ a sum, $b$ might not be interested in carrying a tag that will identify him or her to $a$ the next time they interact. Such “tagging” is not possible in the population protocol model. Indeed, in population protocols agents currently occupying the same state have identical possible futures, a fact following from a fundamental property of population protocols: the global state, usually called configuration, is completely determined by the number of agents in each state, represented by a multiset of states. In other words, the configuration already provides full information about the future behavior of the protocol. We call such protocols oblivious, since they “forget” any information about the agents apart from how many of them are in each state.

In this paper we investigate if and how much can the expressive power of population protocols be extended while maintaining obliviousness. We first show that, under very mild assumptions, models enjoying obliviousness cannot compute predicates beyond NSPACE($n$), meaning that there is an $O(n)$-space-bounded nondeterministic Turing machine which accepts exactly the tuples $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$, encoded in binary, such that $\varphi(n_1, n_2, \ldots, n_k)$ holds. We then prove that population protocols can only compute (a strict subset of) predicates in DSPACE($\log n$), and so are very far away of the theoretical limit. (This is a slight improving on the result of [1], showing inclusion in NL.)

NSPACE($n$) is a large class containing all standard predicates used in the literature to argue that population protocols are not sufficiently expressive, like multiplication and exponentiation, and many more. This leads us to the second part of the paper, containing our main result: broadcast consensus protocols reach the NSPACE($n$) upper bound. Broadcast protocols were introduced by Emerson and Namjoshi in [9] to describe bus-based hardware protocols, and then used and further studied in many other contributions, e.g. [10], [11], [12], [13], [14]. In broadcast protocols, agents can perform binary interactions, as in the population protocol model, but an agent can also broadcast a signal to all other agents, which are guaranteed to react to it. Broadcast protocols are a natural computational model, easy to implement with current technology on mobile agents moving in a limited area. We show that they reach the theoretical NSPACE($n$) upper bound, even if only one agent is allowed to broadcast, and the agent can only broadcast one signal.

In the third and last part of the paper, we present general conclusions about the impact of anonymity and obliviousness on the computational power of distributed systems computing by consensus. In particular, a careful comparison with [7], [8] shows that oblivious protocols are exponentially more expressive than population protocols, and non-oblivious protocols are exponentially more expressive than oblivious ones.

II. Preliminaries

A. Multisets

A multiset over a finite set $E$ is a mapping $M : E \rightarrow \mathbb{N}$. The set of all multisets over $E$ is denoted $\mathbb{N}^E$. For every $e \in E$, $M(e)$ denotes the number of occurrences of $e$ in $M$. We sometimes denote multisets using a set-like notation, e.g. $\{f, g, g\}$ is the multiset $M$ such that $M(f) = 1$, $M(g) = 2$ and $M(e) = 0$ for every $e \in E \setminus \{f, g\}$. Addition and comparison are extended to multisets componentwise, i.e. $(M + M')(e) \defeq M(e) + M'(e)$ for every $e \in E$, and $M \leq M' \iff M(e) \leq M'(e)$ for every $e \in E$. We define multiset difference as $(M \ominus M')(e) \defeq \max(M(e) - M'(e), 0)$ for every $e \in E$. The empty multiset is denoted $\emptyset$ and, for every $e \in E$, we write $e \lessdot e$. Finally, in the paper we use the following notations for $M \in \mathbb{N}^E$:

- the support of $M$ is $|M| \defeq \{ e \in E : M(e) > 0 \}$
- the size of $M$ is $|M| \defeq \sum_{e \in E} M(e)$

B. Oblivious consensus protocols

We introduce a generic computational model following the computation-by-consensus paradigm introduced in [1], [6] for population protocols. The model postulates an arbitrary number of agents over a common set of states. Its main feature is the requirement that the future behavior of a protocol depends only on the current number of agents in each state, and not on their identities. In particular, the protocol is oblivious of past states and of the identities of the agents.

A population $P$ over a finite set $E$ is a multiset $P \in \mathbb{N}^E$ such that $|P| \geq 2$. The set of all populations over $E$ is denoted by Pop($E$). An oblivious consensus protocol with leaders is a tuple $\mathcal{P} = (Q, \text{Step}, \Sigma, L, I, O)$ where

- $Q$ is a non-empty finite set of states,
- $\text{Step} \subseteq \text{Pop}(Q) \times \text{Pop}(Q)$ is the step relation between populations,
- $\Sigma$ is a non-empty finite input alphabet,
- $I : \Sigma \rightarrow Q$ is the input function mapping input symbols to states,
- $L \in \mathbb{N}^Q$ is the multiset of leaders, and
- $O : Q \rightarrow \{0, 1\}$ is the output function mapping states to boolean values.

Following the standard convention, we call the populations of Pop($Q$) configurations. Intuitively, a configuration $C$ describes a collection of identical finite-state agents with $Q$ as
set of states, containing \( C(q) \) agents in state \( q \) for every \( q \in Q \), and at least two agents in total.

We write \( C \rightarrow C' \) if \((C,C') \in \text{Step} \), and \( C \rightarrow^* C' \) if \((C,C') \in \text{Step}^* \), the reflexive and transitive closure of \text{Step}. If \( C \rightarrow^* C' \), then we say that \( C' \) is \textit{reachable from} \( C \).

An execution is an infinite sequence of configurations \( C_0 C_1 \cdots \) such that \( C_i \rightarrow C_{i+1} \) for every \( i \in \mathbb{N} \). An execution \( C_0 C_1 \cdots \) is \textit{fair} if for every step \( C \rightarrow C' \) the following holds: if \( C_i = C \) for infinitely many indices \( i \in \mathbb{N} \), then \( C_j = C' \) for infinitely many indices \( j \in \mathbb{N} \).

If \( O(p) = O(q) \) for every \( p,q \in [C] \), then \( C \) is a \textit{consensus configuration}, and \( O(C) \) denotes the unique output of the states in \([C]\). An execution \( C_0 C_1 \cdots \) \textit{stabilizes} to \( b \in \{0,1\} \) if there exists \( n \in \mathbb{N} \) such that \( C_i \) is a consensus configuration and \( O(C_i) = b \) for every \( i \geq n \).

We now explain the roles of the input function \( I \) and the multiset \( L \) of leaders. The elements of \( \text{Pop}(\Sigma) \) are called \textit{inputs}. For every input \( X \in \text{Pop}(\Sigma) \), let \( I(X) \in \text{Pop}(Q) \) denote the configuration defined by

\[
I(X)(q) = \sum_{\sigma \in \Sigma, I(\sigma) = q} X(\sigma) \quad \text{for every} \quad q \in Q.
\]

A configuration \( C \) is \textit{initial} if \( C = I(X) + L \) for some input \( X \). Intuitively, the agents of \( I(X) \) encode the input, while those of \( L \) are a fixed number of agents, traditionally called leaders, that perform the computation together with the agents of \( I(X) \). We now recall the standard definition of predicate computed by an oblivious protocol, as introduced in [1], [6].

**Definition 1:** An oblivious protocol \( P \) over an alphabet \( \Sigma \) computes a predicate \( \varphi: \text{Pop}(\Sigma) \rightarrow \{0,1\} \) for every input \( X \in \text{Pop}(\Sigma) \), every fair execution of \( P \) starting at the initial configuration \( I(X) + L \) stabilizes to \( \varphi(X) \).

Throughout the paper, we assume \( \Sigma = \{A_1, \ldots, A_k\} \) for some \( k > 0 \). Abusing language, we identify population \( M \in \text{Pop}(\Sigma) \) to vector \( \alpha = (M(A_1), \ldots, M(A_k)) \), and say that \( P \) computes a \textit{number predicate} \( \varphi: \mathbb{N}^k \rightarrow \{0,1\} \) of arity \( k \).

**Remark 2:** The classical definition of a computable predicate imposes no restrictions on the resources needed to compute it: a Turing machine computing the predicate may use arbitrary time and space. However, the definition of [1], [6] does. Indeed, consider for example a number predicate \( \varphi \) of arity 1. The definition postulates that \( \varphi(n) \) must be computed by a configuration with exactly \( n \) agents, plus a constant number of leaders. Since each agent has a constant amount of memory, i.e. \( \log |Q| \) bits, this implicitly limits the memory that the protocol can use. The definition of [1], [6] corresponds to the intuition that the purpose of a protocol computing \( \varphi(x) \) is to allow the agents to learn information about their number, i.e., to know whether their number \( n \) satisfies \( \varphi(n) \) or not.

Following the example of [7], [8], we characterize the expressive power of different classes of oblivious protocols. We do so in terms of standard complexity classes, defined as usual in terms of the resources needed by a Turing machine to recognize the number predicate. Since Turing machines naturally recognize languages, and not number predicates, we define precisely what we mean. The language associated to a number predicate \( \varphi: \mathbb{N}^k \rightarrow \{0,1\} \) is:

\[
L_\varphi \equiv \{ \binom{n_1}{} \# \cdots \# \binom{n_k}{k} : \varphi(n_1, \ldots, n_k) = 1 \},
\]

where \( \binom{n}{k} \) denotes the binary encoding of \( n \). We say that a Turing machine \( M \) \textit{computes} \( \varphi \) if \( L(M) = L_\varphi \). For each complexity class \( C \) of languages, we say that a predicate \( \varphi: \mathbb{N}^k \rightarrow \{0,1\} \) belongs to \( C \) if \( L_\varphi \in C \).

**Remark 3:** We follow the convention of writing Turing machines numerical inputs in binary. We divert from [7], [8], which, as we shall see in Section VII, write inputs in unary. We do so for consistency with statements like “PRIMES \( \in \text{P} \)” or “linear programming is in \( \text{P} \)”, which assume binary encoding.

### III. An upper bound on the expressive power of oblivious protocols

We show that under a very weak assumption on the step relation, oblivious protocols can only compute number predicates in \( \text{NSPACE}(n) \). The assumption is that the step relation itself is computable in \( \text{NSPACE}(n) \). Observe that the step relation of a consensus protocol with \( k \) states can be seen as a number predicate of arity \( 2k \).

**Theorem 4:** Let \( \mathcal{P} = (Q, \text{Step}, \Sigma, I, L, O) \) be an oblivious protocol computing a number predicate \( \varphi \). If \( \text{Step} \in \text{NSPACE}(n) \), then \( \varphi \in \text{NSPACE}(n) \).

**Proof:** Let \( \ell \) be the arity of \( \varphi \). We show that there is a nondeterministic Turing machine, running in \( O(n) \) space, that decides, on input \( x \in \mathbb{N}^k \), whether \( \varphi(x) = 1 \) holds.

Let \( d \equiv |I(x)| \). Let \( G = (V,E) \) be the graph where \( V \) is the set of all configurations of \( \mathcal{P} \) of size \( d \), and \( (C,C') \in E \) iff \( C \rightarrow C' \). Every node of \( V \) can be stored using at most \( \ell \cdot \log d \) space and so, since \( \ell \) is fixed, in space \( O(\log d) \). Observe that, since \( C \rightarrow C' \) implies \( |C| = |C'| \), \( V \) contains all configurations reachable from \( C_0 \).

We claim that \( \varphi(x) = 1 \) iff \( G \) contains a configuration \( C \) satisfying the two following properties:

1. \( C_0 \rightarrow C \);
2. every configuration reachable from \( C \), including \( C \) itself, is a 1-consensus.

If \( \varphi(x) = 1 \), then such a configuration \( C \) exists by definition. For the other direction, assume some configuration \( C \) satisfies both properties. By assumption, there exists a fair execution starting at \( C_0 \) that converges to 1. Thus, since \( \mathcal{P} \) is well-specified, every fair execution starting at \( C_0 \) converges to 1, and so \( \varphi(x) = 1 \).

By the claim, it suffices to exhibit a nondeterministic Turing machine \( TM_{12}(C) \) that runs in \( O(\log |C|) \) space and accepts a configuration \( C \) iff \( C \) satisfies properties 1 and 2.

We first observe that there is a nondeterministic Turing machine \( TM(C) \) that runs in \( O(\log |C|) \) space and accepts the configurations \( C \) satisfying the following property:

there exists a configuration \( C' \) such that \( C \rightarrow C' \) and \( C' \) is not a 1-consensus.

The machine \( TM(C) \) starts at \( C \), guesses a path of configurations step by step, and checks that the final configuration \( C'' \) is
not a 1-consensus. While guessing the path, the machine only stores two configurations at any given time, and so, since every configuration reachable from \( C \) has the same size as \( C \), and since Step \( \in \text{NSPACE}(n) \), the machine only uses \( O(\log |C|) \) space. Checking whether \( C' \) is not a 1-consensus can be done in constant space.

Now we use the fact that space complexity classes are closed under complement [15]. Since NSPACE\((n) = \text{coNSPACE}(n)\), there exists a nondeterministic Turing machine \( TM(C) \) that, given as input a configuration \( C \), decides in \( O(\log |C|) \) space whether every configuration reachable from \( C \) is a consensus.

The machine \( TM_{12}(C) \) first guesses a configuration \( C' \) reachable from \( C \), proceeding as in the description of \( TM(C) \), and then simulates \( TM(C') \). Clearly, the machine runs in \( O(\log |C|) \) space.

The assumption that \( \text{Step} \in \text{NSPACE}(n) \) is very weak. In particular, NSPACE\((n) \) includes all relations computable in Presburger arithmetic. This is a consequence of the following:

**Proposition 5:** For every Presburger formula \( \varphi \), it is the case that \( \varphi \in \text{DSPACE}(\log n) \).

**Proof:** Let us fix some Presburger formula \( \varphi \) with \( d \) free variables. We must show that there exists a deterministic logspace Turing machine that accepts \( L_\varphi \). We first describe a language closely related to \( L_\varphi \). Let \( \Sigma = \{0, 1\}^d \), let \( x_\epsilon = 0 \) and, for every word \( w \in \Sigma^* \), let \( x_w \in \mathbb{N}^d \) be such that

\[
x_w(i) = \text{val}(w_1(i) \cdot w_2(i) \cdots w_{|w|}(i))
\]

where \( \text{val}(u) \) is the numerical value of \( u \) interpreted as a binary sequence with its most significant bit first. For example, the vector \( x_w \) associated to \( w = (1, 0)(1, 1)(0, 0)(1, 1) \) is \((13, 5)\) since \( 1101_2 = 13 \) and \( 0101_2 = 5 \).

Since \( \varphi \) is a Presburger formula, there exists a deterministic finite automaton \( A \) that accepts the language

\[
X_\varphi = \{ w \in \Sigma^* : \varphi(x_w) = 1 \}.
\]

Note that the usual constructions to translate a Presburger formula into an automaton yield a nondeterministic finite automaton \( B \) with least-significant-bit-first encoding (see e.g. [16]). This is not an issue since the language of \( B \) can be reversed and the resulting automaton can be determined, by standard results on regular languages.

Observe that a Turing machine can decide whether \( \varphi(x) = 1 \) by simulating \( A \) on the binary encoding of \( x \). However, we have to be careful since the encoding of \( x \) in \( X_\varphi \) and \( L_\varphi \) differ: \( X_\varphi \) interleaves the bits of each component of \( x \), while \( L_\varphi \) does not. With this in mind, let us give a deterministic Turing machine \( M_\varphi \):

1. \( M \) receives an input \( \text{bin}(n_1)\#\text{bin}(n_2)\# \cdots \#\text{bin}(n_d) \);
2. \( M \) has a counter \( j \), initially set to 1, that indicates the index of the next bit that must be read in each \( n_i \);
3. \( M \) stores the initial state \( q_0 \) of \( A \);
4. \( M \) iteratively fetches bits \( n_1(j), n_2(j), \ldots, n_d(j) \);
5. \( M \) simulates one step of \( A \) on \( (n_1(j), n_2(j), \ldots, n_d(j)) \), and updates the state of \( A \);
6. \( M \) increments \( j \) and repeats step 4 until all bits have been fetched;
7. \( M \) accepts its input iff the state reached in \( A \) is final.

It is readily seen that \( M \) accepts \( L_\varphi \). The only detail of concern is that the \( \text{bin}(n_i) \)'s may have different lengths. This can be addressed by implicitly producing “\#” as a padding bit whenever \( j \) exceeds the length of \( \text{bin}(n_i) \), i.e. whenever the reading head hits the next “\#” before seeing \( j \) cells.

It remains to argue that \( M \) works in logarithmic space. Let \( n \) be the length of the input. Note that \( \varphi \) and \( d \) are both fixed, and hence independent from \( n \). Thus, tuples \( (n_1(i), n_2(i), \ldots, n_d(i)) \) and states of \( A \) take constant space and can be stored in the control states of \( M \). Moreover, counter \( j \) does not exceed \( n \), so its binary encoding can be stored on a work tape of length \( O(\log n) \).

By Proposition 5, we immediately get:

**Theorem 6:** Let \( \mathcal{P} \) be an oblivious consensus protocol whose step relation is expressible in Presburger arithmetic.
If \( \mathcal{P} \) computes a number predicate \( \varphi \), then \( \varphi \in \text{NSPACE}(n) \).

### A. Population protocols

Population protocols are the class of oblivious protocols in which steps are caused by binary interactions between agents. More precisely, at every step two agents in states \( p, q \) interact according to a joint transition function and change their states simultaneously to some states \( p', q' \). Since states \( p, q \) need not be different, we get the following formal definition:

**Definition 7:** Let \( Q \) be a set of states and let \( R \subseteq (Q \times Q) \times (Q \times Q) \). We say that \( R \) is a set of rendez-vous transitions. We write \( (p, q) \xrightarrow{\text{Step}} (p', q') \) to denote that \( (p, q, p', q') \in R \).

Relation \( \text{Step}_{R} : \text{Pop}(Q) \rightarrow \text{Pop}(Q) \) is defined by: \( (C, C') \in \text{Step}_{R} \) iff there exists \( (p, q, p', q') \in R \) such that \( C \geq (p, q) \) and \( C' = C' \leq (p', q') \). An oblivious protocol \( \mathcal{P} = (Q, \text{Step}, \Sigma, I, L, O) \) is a population protocol if there exists a finite set \( R \) of rendez-vous transitions such that \( \text{Step} = \text{Step}_{R} \).

It follows immediately from the definition that the step relation of population protocols is expressible in Presburger arithmetic, and so, by Theorem 6, they can only compute number predicates in \( \text{NSPACE}(n) \). However, their expressive power is far below \( \text{NSPACE}(n) \). Indeed, it was shown in [1] that they can only compute predicates in \( \text{NL} \), and we can now even go a bit further. Since population protocols compute exactly the predicates expressible in Presburger arithmetic [6], by Proposition 5 we have:

**Corollary 8:** Population protocols can only compute number predicates in \( \text{DSPACE}(\log n) \).

This raises the main question of this paper: Is there a natural class of oblivious protocols that reaches the \( \text{NSPACE}(n) \) bound? The next section gives an affirmative answer.

### IV. BROADCAST CONSENSUS PROTOCOLS

Broadcast protocols are a model introduced by Emerson and Namjoshi in [9] to describe bus-based hardware protocols, such as those for cache coherency. The model has also been applied to the verification of multithreaded programs [12], and to idealized modeling of control problems for living organisms [17], [14]. Its theory has been further studied in [10], [11], [13].
Agents of broadcast protocols can communicate in pairs, as in population protocols, and, additionally, they can also communicate by means of a reliable broadcast: An agent can broadcast a signal to all other agents, which after receiving the signal move to a new state. Broadcasts are routinely used in wireless ad-hoc and sensor networks (see e.g. [18], [19]), and so they are easy to implement on the same kind of systems targeted by population protocols. They can also model computation-by-consensus paradigm.

For this, we present a broadcast consensus protocol for the single computation step. (Much unlike population protocols,

\[ L = 0, O(q) = 1 \iff q = 1, \] and the sets, \( R,B \) of transitions are defined as follows:

- \( R \) contains two rendez-vous transitions \( s, \bar{s} \):
  \[ s: (x, x) \mapsto (\bar{s}, 0) \text{ and } \bar{s}: (\bar{s}, \bar{x}) \mapsto (\bar{x}, \bar{x}). \]

- \( B \) contains four broadcast transitions \( t_0, t_1, \bar{t}, \bar{t}, r \):
  \[ t_0: x \mapsto 0; \bar{t} \mapsto 0; t_1: \bar{t} \mapsto 1; \bar{t} \mapsto 0; \bar{r} \mapsto x \mapsto x; \bar{r} \mapsto \perp \mapsto x; \bar{r} \mapsto \perp \mapsto x. \]

Intuitively, \( \mathcal{P} \) repeatedly halves the number of agents in state \( x \), and it accepts iff it never obtains an odd remainder. More precisely, \( \mathcal{P} \) is intended to work as follows:

1. \( s \) and \( \bar{s} \) occur repeatedly, until it is not further possible, which results in moving half of the agents in state \( x \) to \( \bar{x} \), and moving the other half to state \( 0 \);
2. if a single agent remains in \( x \), then an odd remainder was obtained and hence transition \( t_0 \) moves all agents to consensus \( 0 \);
3. otherwise, if a single agent remains in state \( \bar{s} \) and no agent remains in \( x \), then all the halvings were successful, and hence transition \( t_1 \) moves all agents to consensus \( 1 \);
4. otherwise, transition \( \bar{r} \) moves all agents in states \( \bar{x} \) to \( x \), and Step 1 is repeated.

It is easy to show that the protocol produces a consensus, and the right consensus, if it executes transitions as intended. However, an arbitrary execution may not follow the above steps. For this reason, transitions \( \bar{t}, t_0 \) and \( t_1 \) can detect errors by moving agents to state \( \perp \). This way, if an error has been detected, then, by fairness, \( r \) will eventually reset the agents back to the initial configuration and, by fairness, this will go on until the transitions are finally fired as intended.

**Proposition 11:** The broadcast consensus protocol \( \mathcal{P} \) described above computes the number predicate \( \varphi \), defined as \( \varphi(x) = 1 \) iff \( x > 1 \) and \( x \) is a power of two.

**B. Leaderless broadcast protocols**

A broadcast protocol \( \mathcal{P} = (Q,R,B,\Sigma,I,L,O) \) is leaderless if \( L = \emptyset \). Since leaders can significantly simplify the design of a protocol, or improve its runtime, the literature has studied if leaderless protocols have the same expressive power as protocols with leaders. For broadcast consensus protocols, it is easy to see that leaderless protocols compute the same predicates as the general class. We only sketch the argument, omitting the details. First, it is easy to show that a broadcast protocol with leader multiset \( L \) can be simulated by a protocol with one single leader: Indeed, the protocol can be designed so that the first task of the leader is to “recruit” the other leaders of \( L \) from among the agents. Second, a broadcast protocol with one leader can be simulated by a leaderless protocol because, loosely speaking, a broadcast protocol can elect a leader in one single computation step. (Much unlike population protocols,
where efficient leader election is a non-trivial, much studied problem, see [5] for a recent survey.) Indeed, if initially all agents are in a state, say \( q \), then a broadcast \( q \rightarrow \ell; f \), where \( f(q) = q' \), sends exactly one agent to a leader state \( \ell \), and all other agents to state \( q' \). It is easy to construct \( \mathcal{P}' \) using this feature, and the details are omitted.

In the rest of the paper we use protocols with leaders, but all results (except Proposition 23) remain valid for leaderless protocols.

V. Broadcast consensus protocols compute all number predicates in NSPACE\( (n) \)

We prove the main theorem of the paper: every number predicate in NSPACE\( (n) \) is computable by broadcast consensus protocols. The proof is involved, and we start by describing its structure. In Section V-A, we show that it suffices to prove that every number predicate in NSPACE\( (n) \) is silently semi-computable. In the rest of the section, we proceed to prove this in three steps. Loosely speaking, we show that:

- predicates computable by nondeterministic Turing machines in \( \mathcal{O}(n) \) space can also be computed by counter machines whose counters are polynomially bounded in \( n \) (Section V-B);
- predicates computed by polynomially bounded counter machines can also be computed by \( n \)-bounded counter machines, in which the sum of the values of all counters never exceeds their initial sum \( n \) (Section V-C);
- predicates computed by \( n \)-bounded counter machines can be silently semi-computed by broadcast protocols (Section V-D).

Finally, Section V-E puts all parts of the proof together.

A. Silent semi-computation

Recall that, loosely speaking, a protocol computes \( \varphi \) if it converges to 1 for inputs that satisfy \( \varphi \), and it converges to 0 for inputs that do not satisfy \( \varphi \). Additionally, a protocol \( \text{silently computes } \varphi \) if convergence to \( b \in \{0, 1\} \) happens by reaching a terminal \( b \)-consensus, i.e., a configuration \( C \) that is a \( b \)-consensus and from which one can only reach \( C \) itself. (Intuitively, the protocol eventually becomes “silent” because no agent changes state anymore.) Finally, a protocol \( \text{silently semi-computes } \varphi \) if it reaches a terminal 1-consensus for inputs that satisfy \( \varphi \), and no terminal configuration for other inputs:

Definition 12: A broadcast consensus protocol \( \mathcal{P} \) \( \text{silently semi-computes an } m \)-ary predicate \( \varphi \) if for every \( \alpha \in \mathbb{N}^m \) the following properties hold:

1) if \( \varphi(\alpha) = 1 \), then every fair execution of \( \mathcal{P} \) starting at \( I(\alpha) \) eventually reaches a terminal configuration satisfying \( O(C) = 1 \).

2) if \( \varphi(\alpha) = 0 \), then no fair execution of \( \mathcal{P} \) starting at \( I(\alpha) \) eventually reaches a terminal configuration.\(^1\)

We show that if a predicate and its complement are both silently semi-computable by broadcast protocols, say \( \mathcal{P}_1 \) and \( \mathcal{P}_0 \), then the predicate is also computable by a broadcast protocol \( \mathcal{P} \) which, intuitively, under input \( \alpha \) behaves as follows. At every moment in time, \( \mathcal{P} \) is simulating either \( \mathcal{P}_1 \) or \( \mathcal{P}_0 \). Initially, \( \mathcal{P} \) simulates \( \mathcal{P}_0 \). Assume \( \mathcal{P} \) is simulating \( \mathcal{P}_1 \) and the current configuration is \( C \). If \( C \) is a terminal configuration of \( \mathcal{P}_1 \), then \( \mathcal{P} \) terminates too. Otherwise, \( \mathcal{P} \) nondeterministically chooses to continue the simulation, or to “reset” the computation to \( I_{1-\varphi}(\alpha) \), i.e., start simulating \( \mathcal{P}_{1-\varphi} \). Conditions 1 and 2 ensure that exactly one of \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) can reach a terminal configuration, namely \( \mathcal{P}_{\varphi(\alpha)} \). Fairness ensures that \( \mathcal{P} \) will eventually reach a terminal configuration of \( \mathcal{P}_{\varphi(\alpha)} \), and so, by condition 1, that it will always reach the right consensus. So \( \mathcal{P} \) silently computes \( \varphi \).

The “reset” is implemented by means of a broadcast that sends every agent to its initial state in the configuration \( I_{1-\varphi}(\alpha) \); for this, the states of \( \mathcal{P} \) are partitioned into classes, one for each input symbol \( x \in X \). Every agent moves only within the states of one of the classes, and so every agent “remembers” its initial state in both \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \).

Lemma 13: Let \( \varphi \) be an \( m \)-ary predicate, and let \( \overline{\varphi} \) be the predicate defined by \( \overline{\varphi(\alpha)} \equiv 1 - \varphi(\alpha) \) for every \( \alpha \in \mathbb{N}^m \). Further let \( \mathcal{P}_1 \) and \( \mathcal{P}_0 \) be broadcast protocols that silently semi-compute \( \varphi \) and \( \overline{\varphi} \), respectively. The following holds: there exists a protocol \( \mathcal{P} \) that silently computes \( \varphi \).

Proof: Let \( \mathcal{P}_1 = (Q_1, R_1, B_1, \Sigma, I_1, L_1, O_1) \) and \( \mathcal{P}_0 = (Q_0, R_0, B_0, \Sigma, I_0, L_0, O_0) \) be protocols that silently semi-compute \( \varphi \) and \( \overline{\varphi} \), respectively. Assume w.l.o.g. that \( Q_1 \) and \( Q_0 \) are disjoint. We construct a protocol \( \mathcal{P} = (Q, R, B, X, I, L, O) \) with one extra leader that computes \( \varphi \).

For the sake of clarity we refrain from giving a fully formal description, but we provide enough details to show that the design idea above can indeed be implemented. In particular, in our description, the protocol has 3-way rendezvous transitions, i.e. in which three different agents interact. While this is not allowed by the definition, we have shown in Lemma 3 of [20] that k-way transitions can be simulated by ordinary 2-way transitions.

States and mappings. The set of states of \( \mathcal{P} \) is defined as:

\[
Q \overset{\text{def}}{=} \{0, 1, 0, \overline{1}\} \cup X \times (Q_1 \cup Q_0).
\]

The protocol has a single leader that starts in state 0: \( L \overset{\text{def}}{=} \{0\} \). When the leader is at state \( i \), the protocol simulates \( \mathcal{P}_i \). When the leader is at state \( \overline{1} \), it has decided to stop the simulation of \( \mathcal{P}_i \) and to switch to the simulation of \( \mathcal{P}_{1-i} \). If a (nonleader) agent is at state \( (x, q) \), we say that \( x \) is its origin and that \( q \) is its position. The initial position of a nonleader is its initial state in \( \mathcal{P}_0 \), i.e. \( (x) \overset{\text{def}}{=} (x, I_0(x)) \). Transitions will be designed so that agents may update their position, but not their origin.

Simulation transitions. We define transitions that proceed with the simulation of \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) as follows. For every \( i \in \{0, 1\} \), every \( x, y \in X \), and every rendez-vous transition \( (q, r) \rightarrow (q', r') \) of \( R_i \), we add the following 3-way rendez-vous transition to \( R \):

\[
(i, (x, q), (y, r)) \rightarrow (i, (x, q'), (y, r')).
\]
For every broadcast transition \( q \to q' \); \( f \) of \( B \); and every \( x \in X \), we add the following broadcast transition to \( B \):
\[
(x, q) \mapsto (x, q'); f'
\]
where \( f' \) only acts on \( Q_i \) by \( f'(y, r) \equiv (y, f(r)) \) for every \( (y, r) \in X \times Q_i \).

**Mode-changing transitions.** We define transitions that stop the simulation of \( P_i \), and move the leader to state \( \top \) to indicate that the population must be reset to the initial configuration of \( P_{i-1} \). For every \( i \in \{1, 0\} \), every \( x, y \in X \), and every non-silent rendez-vous transition \( (q, r) \to (q', r') \in R_i \), we add the following rendez-vous transition to \( R \):
\[
(i, (x, q), (y, r)) \mapsto (i, (x, q), (y, r)).
\]
Moreover, for every \( i \in \{1, 0\} \), every \( x \in X \), and every non-silent broadcast transition \( q \to q'; f \) of \( B_i \), we add the following rendez-vous transition to \( R \):
\[
(i, (x, q)) \mapsto (i, (x, q)).
\]

**Reset transitions.** We define transitions that trigger a new simulation of \( P_{i-1} \). For every \( i \in \{1, 0\} \), let \( f_i : Q \to Q \) be the function defined as \( f_i(x, q) \equiv (x, L_i(x)) \) for every \( x \in X \) and \( q \in Q_i \), and as the identity for all other states. For every \( i \in \{1, 0\} \), we add the following broadcast transition to \( B \):
\[
(1 - i) \mapsto (1 - i); f_{1-1}.
\]

Using Lemma 13 we now prove:

**Proposition 14:** If every number predicate in \( \text{NSPACE}(n) \) is silently semi-computable by broadcast consensus protocols, then every number predicate in \( \text{NSPACE}(n) \) is silently computable (and so computable) by broadcast consensus protocols.

**Proof:** Assume every number predicate in \( \text{NSPACE}(n) \) is silently semi-computable by broadcast consensus protocols, and let \( \varphi \) be a number predicate in \( \text{NSPACE}(n) \). We resort to the powerful result stating that number predicates in \( \text{coNSPACE}(n) \) and \( \text{NSPACE}(n) \) coincide. This is an immediate corollary of the \( \text{NSPACE}(n) = \text{NSPACE}(n) \) theorem for languages [15], [21], [22], and the fact that one can check in constant space whether a given word encodes a vector of natural numbers. Thus, both \( \varphi \) and \( \overline{\varphi} \) are number predicates in \( \text{NSPACE}(n) \), and so, by assumption, silently semi-computable by broadcast consensus protocols. By Lemma 13, they are silently computable by broadcast consensus protocols.

**B. Simulation of Turing machines by counter machines**

We show that nondeterministic Turing machines working in \( O(n) \) space can be simulated by counter machines whose counters are polynomially bounded in \( n \), and so that both models compute the same number predicates.

A \( k \)-counter machine \( M \) is a tuple \((Q, X, \Delta, m, q_0, q_a, q_r)\) where
\[ \quad \cdot m \leq k \text{ is the number of counter inputs;} \]
\[ \quad \cdot q_0 \in Q \text{ is the initial state;} \]
\[ \quad \cdot q_a \in Q \text{ is the accepting state;} \]
\[ \quad \cdot q_r \in Q \setminus \{q_a\} \text{ is the rejecting state.} \]

A configuration of \( M \) is a pair \((q, v) \in Q \times \mathbb{N}^k \) consisting of a control state \( q \) and counter values \( v \). For every \( i \in [k] \), we denote the value of counter \( x_i \) in \( C \) by \( C(x_i) \equiv v_i \).

The size of \( C \) is \( |C| \equiv \sum_{i=1}^k C(x_i) \).

Let \( e_i \) denote the \( i \)-th row of the \( k \times k \) identity matrix.

For every \( i \in [k] \), we define the following labelled transition relation \( \rightarrow \rightarrow \) between configurations:
\[
(q, v) \xrightarrow{\text{inc}(x_i)} (r, v + e_i) \iff (q, \text{inc}(x_i), r) \in \Delta; \quad (q, v) \xrightarrow{\text{dec}(x_i)} (r, v - e_i) \iff (q, \text{dec}(x_i), r) \in \Delta \text{ and } v_i > 0; \quad (q, v) \xrightarrow{\text{zro}(x_i)} (r, v) \iff (q, \text{zro}(x_i), r) \in \Delta \text{ and } v_i = 0; \quad (q, v) \xrightarrow{\text{nzr}(x_i)} (r, v) \iff (q, \text{nzr}(x_i), r) \in \Delta \text{ and } v_i > 0; \quad (q, v) \xrightarrow{\text{nop}(r)} (r, v) \iff (q, \text{nop}(r), r) \in \Delta.
\]

For every \( \alpha \in \mathbb{N}^m \), the initial configuration of \( M \) with input \( \alpha \) is defined as:
\[
C_\alpha \equiv (q_0, (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0))_.
\]

We say \( M \) accepts \( \alpha \) if there exist counter values \( v \in \mathbb{N}^k \) satisfying \( C_\alpha \rightarrow \rightarrow (q_a, v) \). We say \( M \) rejects \( \alpha \) if \( M \) does not accept \( \alpha \) and for all configurations \( C' \) with \( C_\alpha \rightarrow \rightarrow C' \), there exist counter values \( v \in \mathbb{N}^k \) satisfying \( C' \rightarrow \rightarrow (q_r, v) \). We say \( M \) computes a number predicate \( \varphi : \mathbb{N}^m \to \{0, 1\} \) if \( M \) accepts all inputs \( \alpha \) such that \( \varphi(\alpha) = 1 \), and rejects all \( \alpha \) such that \( \varphi(\alpha) = 0 \).

A \( k \)-counter machine \( M \) is \( f(n) \)-bounded if \( |C| \leq f(|\alpha|) \) holds for every initial configuration \( C_\alpha \) and for every configuration \( C \) reachable from \( C_\alpha \). It is well-known that counter machines can simulate Turing machines:

**Theorem 15 ([23, Theorem 3.1]):** A number predicate is computable by an \( s(n) \)-space-bounded Turing machine iff it is computable by a \( 2^{s(n)} \)-bounded counter machine.

In [23], a weaker version of Theorem 15 is proven that applies to deterministic Turing/counter machines only. However, the proof can be easily adapted to the nondeterministic setting we consider here.

**Corollary 16:** A number predicate is in \( \text{NSPACE}(n) \) iff it is computable by a polynomially bounded counter machine.

**Proof:** The Turing machine computes \( \varphi(\alpha) \) on input \( \alpha \) in \( O(\sum_{i=1}^k |\text{bin}(\alpha_i)|) \)-space, and hence in \( O(|\log|\alpha||) \)-space since \( |\text{bin}(\alpha_i)| \in O(|\log \alpha_i|) \). Apply now Theorem 15.

**C. Simulation of polynomially bounded counter machines by \( n \)-bounded counter machines**

We give a proof sketch of the following lemma:

**Lemma 17:** For every polynomially bounded counter machine that computes some predicate \( \varphi \), there exists an \( n \)-bounded counter machine that computes \( \varphi \).
Proof: We sketch the main idea of the proof. Details can be found in the Appendix. Let $c \in \mathbb{N}_{>0}$ and let $M$ be an $n^c$-bounded counter machine. To simulate $M$ by an $n$-bounded counter machine $M'$, we need some way to represent any value from $[0, n^c]$ by means of counters with values in $[0, n]$. If $n$ would be fixed, then any number in $[0, n^c]$ could be encoded in base $n+1$ over $c$ counters. However, $n$ refers to the size of the input and hence it is an arbitrary number which is not fixed. Thus, we introduce a more technical representation that relies on this idea, but does not assume $n$ to be fixed. We encode a number $\ell \in [0, n^c]$ as a multiset $M_\ell = \{ v_1, \ldots, v_n \}$ of binary vectors of dimension $c$, containing exactly $n$ elements. $M_\ell$ satisfies that $\sum_{i=1}^{n} v_i$ is the encoding of $\ell$ in base $n+1$. For every counter of $M$, there is one counter in $M'$ for each of the $2^c$ binary vectors of dimension $c$. Thus, the number of counters remains linear as $2^c$ is a constant. The counter for vector $v \in \{0, 1\}^c$ stores the number of copies of $v$ in $M_\ell$, i.e., the number $|M_\ell(v)|$. Since $|M_\ell| = n$, the sum of all counter values is bounded by $n$.

The rest of the proof shows how to simulate a (non)zero-test, an increment, and a decrement of $M$ using the new representation. This is not difficult, but technical, and the reader is referred to the Appendix.

D. Simulation of $n$-bounded counter machines by broadcast consensus protocols

Let $M = (Q, X, \Delta, m, q_0, q_a, q_r)$ be an $n$-bounded counter machine that computes some predicate $\varphi: \mathbb{N}^m \rightarrow \{0, 1\}$, i.e., $\varphi(\alpha) = 1$ iff $M$ accepts $\alpha$. We construct a broadcast protocol $\mathcal{P} = (Q', R, B, \Sigma, I, L, O)$ that silently semi-computes $\varphi$.

States and mappings. Let $X' \equiv X \cup \{idle, err\}$. The states of $\mathcal{P}$ are defined as

$$Q' \equiv Q \times \{0, 1\} \cup X' \times X \times \{0, 1\}.$$  

The protocol will be designed in such a way that there is always exactly one agent, called the leader, in states $Q \times \{0, 1\}$. Whenever the leader is in state $(q, b)$, we say that its position is $q$, and its opinion is $b$. Every other agent will remain in a state from $X' \times X \times \{0, 1\}$. Whenever a nonleader agent is in state $(x, y, b)$, we say that its position is $x$, its origin is $y$, and its opinion is $b$. Intuitively, the leader is in charge of storing the control state of $M$, and the nonleaders are in charge of storing the counter values of $M$.

The protocol has a single leader whose initial position is the initial control state of $M$, i.e., $L \equiv \{(q_0, 0)\}$. Moreover, every nonleader agent initially has its origin set to its initial position, which will remain unchanged by definition of the forthcoming transition relation: $I(x) \equiv (x, x, 0)$ for every $x \in X$. The output of each agent is its opinion:

$$O(q, b) \equiv b \quad \text{for every } q \in Q, x \in X', y \in X, b \in \{0, 1\}.$$  

We now describe how $\mathcal{P}$ simulates the instructions of $M$.

Dec/incrementation. For every transition $q \xrightarrow{\text{dec}(x)} r \in \Delta$, every $y \in X$ and every $b, b' \in \{0, 1\}$, we add to $R$ the rendezvous transition:

$$(q, b), (x, y, b') \mapsto (r, b), (idle, y, b').$$  

These transitions change the position of one agent from $x$ to $idle$, and thus decrement the number of agents in position $x$.

Similarly, for every transition $q \xrightarrow{\text{inc}(x)} r$, every $y \in X$ and every $b, b' \in \{0, 1\}$, we add to $R$ the rendez-vous transition:

$$(q, b), (idle, y, b') \mapsto (r, b), (x, y, b').$$  

These transitions change the position of an idle agent to $x$, and thus increment the number of agents in position $x$. If no agent is in position $err$, then at least one idle agent is available when a counter needs to be incremented, since $M$ is $n$-bounded.

Nonzero-tests. For every $q \xrightarrow{\text{nz}(x)} r \in \Delta$, every $y \in X$ and every $b, b' \in \{0, 1\}$, we add to $R$ the rendez-vous transition:

$$(q, b), (x, y, b') \mapsto (r, b), (x, y, b').$$  

These transitions can only be executed if there is at least one agent in position $x$, and thus only if the value of $x$ is nonzero.

Zero-tests. Let $f_{err}: Q' \rightarrow Q'$ be the function that maps every nonleader to the error position, i.e.

$$f_{err}(q, b) \equiv (q, b) \quad \text{for every } q \in Q, b \in \{0, 1\},$$  

$$f_{err}(x, y, b) \equiv (err, y, b) \quad \text{for every } x \in X', y \in X, b \in \{0, 1\}.$$  

For every transition $q \xrightarrow{\text{err}(x)} r \in \Delta$ and every $b \in \{0, 1\}$, we add to $B$ the broadcast transition:

$$(q, b) \mapsto (r, b); f_{err}.$$  

If such a transition occurs, then nonleaders in position $x$ move to $err$. Thus, an error is detected iff the value of $x$ is nonzero.

To recover from errors, $\mathcal{P}$ can be reset to its initial configuration as follows. Let $f_{rst}: Q' \rightarrow Q'$ be the function that sends every state back to its origin, i.e.

$$f_{rst}(q, b) \equiv (q_0, 0) \quad \text{for every } q \in Q, b \in \{0, 1\},$$  

$$f_{rst}(x, y, b) \equiv (y, y, 0) \quad \text{for every } x \in X', y \in X, b \in \{0, 1\}.$$  

For every $y \in X$ and every $b \in \{0, 1\}$, we add the following broadcast transition to $B$ to reset $\mathcal{P}$ to its initial configuration:

$$(err, y, b) \mapsto (y, y, 0); f_{rst}.$$  

Acceptance. For every $q \in Q \setminus \{q_a\}$ and $b \in \{0, 1\}$, we add the following broadcast transition to $B$:

$$(q, b) \mapsto (q_0, 0); f_{rst}.$$  

Intuitively, as long as the leader’s position differs from the accepting control state $q_a$, it can reset $\mathcal{P}$ to its initial configuration. This ensures that $\mathcal{P}$ can try all computations.

Let $f_{one}: Q' \rightarrow Q'$ be the function that changes the opinion of each state to 1, i.e.

$$f_{one}(q, b) \equiv (q, 1) \quad \text{for every } q \in Q, b \in \{0, 1\},$$  

$$f_{one}(x, y, b) \equiv (x, y, 1) \quad \text{for every } x \in X', y \in X, b \in \{0, 1\}.$$  

For every $b \in \{0, 1\}$, we add the following transition to $B$:

$$t_{\text{one},b}: (q_a, b) \to (q_a, 1); f_{\text{one}}.$$

Intuitively, these transitions change the opinion of every agent to 1. If such a transition occurs in a configuration with no agent in $err$, then no agent can change its state anymore, and the stable consensus 1 has been reached.

**Correctness.** Let us fix some some input $\alpha \in \mathbb{N}^m$. Let $C_0$ and $D_0$ be respectively the initial configurations of $\mathcal{M}$ and $\mathcal{P}$ on input $\alpha$. Abusing notation, for every $D \in \text{Pop}(Q')$, let

$$D(x) \equiv \sum_{(x,y,b) \in Q'} D(x,y,b).$$

The two following propositions state that every execution of $\mathcal{M}$ has a corresponding execution in $\mathcal{P}$ and vice versa. The proofs are routine.

**Proposition 18:** Let $C$ be a configuration of $\mathcal{M}$ such that $C$ is in control state $q$ and $C_0 \xrightarrow{*} C$. There exists a configuration $D \in \text{Pop}(Q')$ such that $D_0 \xrightarrow{*} D$ and

- $D(x) = C(x)$ for every $x \in X$,
- $D(\text{err}) = 0$,
- $D(q,b) = 1$ for some $b \in \{0,1\}$.

**Proposition 19:** Let $D \in \text{Pop}(Q')$ be such that $D_0 \xrightarrow{*} D$. If $D(\text{err}) = 0$, then there is a configuration $C$ of $\mathcal{M}$ such that $C_0 \xrightarrow{*} C$ and

- $C(x) = D(x)$ for every $x \in X$,
- if $D(q,b) = 1$ for some $(q,b) \in Q'$, then $C$ is in control state $q$.

We may now prove that $\mathcal{P}$ silently semi-computes $\varphi$:

**Proposition 20:** For every $n$-bounded counter machine $\mathcal{M}$ that computes some predicate $\varphi$, there exists a broadcast consensus protocol that silently semi-computes $\varphi$.

**Proof:** We show that $\mathcal{P}$ silently semi-computes $\varphi$ by proving the two properties of Definition 12. Let $\alpha$ be an input.

1) Assume $\varphi(\alpha) = 1$. Then $\mathcal{M}$ accepts $\alpha$, and so there is a configuration $C$ such that $C_0 \xrightarrow{*} C$ and $\mathcal{C}$ is in control state $q_a$. By Proposition 18, there exists some configuration $D \in \text{Pop}(Q')$ satisfying $D_0 \xrightarrow{*} D$, $D(\text{err}) = 0$ and $D(q_a,b) = 1$. Since $\mathcal{M}$ halts when reaching $q_a$, the only transition enabled at $D$ is $t_{\text{one},b}$, and its application yields a terminal configuration $D'$ of consensus 1. Further, every configuration reachable from $D_0$, where the leader is not in position $q_a$ or where some nonleader is in position $err$, can be set back to $D_0$ via some reset transition. Therefore, every fair execution of $\mathcal{P}$ starting at $I(\alpha) = C_0$ will eventually reach $D'$.

2) Assume $\varphi(\alpha) = 0$. We prove by contradiction that no configuration $D$ reachable from $D_0$ is terminal. Assume the contrary. We must have $D(q_a,1) = 1$, $D(\text{err}) = 0$ and $O(D) = 1$, for otherwise some broadcast transition with $f_{\text{rst}}$ or $f_{\text{one}}$ would be enabled. From this and by Proposition 19, there exists some configuration $C$ of $\mathcal{M}$ in control state $q_a$ and satisfying $C_0 \xrightarrow{*} C$. Thus, $\mathcal{M}$ accepts $\alpha$, contradicting $\varphi(\alpha) = 0$. \[\square\]

**E. Main Theorem**

We prove our main result by means of the chain of simulations described in this section:

**Theorem 21:** Broadcast consensus protocols compute exactly the number predicates in $\text{NSPACE}(n)$.

**Proof:** By Corollary 10, broadcast consensus protocols only compute number predicates in $\text{NSPACE}(n)$. For the other direction, let $\varphi$ be a number predicate in $\text{NSPACE}(n)$. Since $\text{NSPACE}(n) = \text{coNSPACE}(n)$, the complement predicate $\overline{\varphi}$ is also in $\text{NSPACE}(n)$. Thus, $\varphi$ and $\overline{\varphi}$ are computable by $O(n)$-space-bounded nondeterministic Turing machines. By Theorem 15, $\varphi$ and $\overline{\varphi}$ are computable by polynomially bounded counter machines. By Proposition 17, $\varphi$ and $\overline{\varphi}$ are silently semi-computable by broadcast consensus protocols. By Proposition 14, $\varphi$ is silently computable by a broadcast consensus protocol. \[\square\]

Actually, the proof shows this slightly stronger result:

**Corollary 22:** The number predicates computed and silently computed by broadcast consensus protocols coincide, i.e., broadcast consensus protocols silently compute exactly the number predicates in $\text{NSPACE}(n)$.

**VI. Subclasses of Broadcast Consensus Protocols**

While broadcasting is a natural, well understood, and much used communication mechanism, it also consumes far more energy than rendez-vous communication. In particular, agents able to broadcast are more expensive to implement. In this section, we briefly analyze which restrictions can be imposed on the broadcast model without reducing its computational power. We show that $\text{NSPACE}(n)$ can be computed by protocols satisfying two properties:

1) only one agent broadcasts; all other agents only use rendez-vous communication.

2) the broadcasting agent only needs to send one signal, meaning that the receivers’ response is independent of the broadcast signal.

Finally, we show that a third restriction does decrease the computational power. In simulations of the previous section, broadcasts are often used to “reset” the system. Since computational models with resets have been devoted quite some attention [24], [25], [26], we investigate the computational power of protocols with resets.

**A. Protocols with only one broadcasting agent**

Loosely speaking, a broadcast protocol with one broadcasting agent is a broadcast protocol $\mathcal{P} = (Q, R, B, \Sigma, I, L, O)$ with a set $Q_L$ of leader states such that $L = \{q\}$ for some $q \in Q_L$ (i.e., there is exactly one leader), and whose transitions ensure that the leader always remains within $Q_L$, that no other agent enters $Q_L$, and that only agents in $Q_L$ can trigger broadcast transitions. Protocols with multiple broadcasting agents are easy to simulate by protocols with one broadcasting agent, say $b$. Instead of directly broadcasting, an agent communicates with $b$ by rendez-vous, and delegates to $b$ the task of executing the broadcast. More precisely, a
broadcast transition \( q \mapsto q' \); \( f \) is simulated by a rendezvous transition \((q, q') \mapsto (q_{aux}, q, f)\), followed by a broadcast transition \( q_{aux} \mapsto q'\).

B. Single-signal broadcast protocols

In single-signal protocols the receivers’ response is independent of the broadcast signal. Formally, a broadcast protocol \((Q, R, B, \Sigma, I, O)\) is a single-signal protocol if there exists a function \( f : Q \to Q \) such that \( B \subseteq Q^2 \times \{ f \} \).

Proposition 23: Predicates computable by broadcast consensus protocols are also computable by single-signal broadcast protocols.

Proof: We give a proof sketch, details can be found in the appendix. We simulate a broadcast protocol \( \mathcal{P} \) by a single-signal protocol \( \mathcal{P}' \). The main point is to simulate a broadcast step \( C_1 \overset{q_{aux}}{\rightarrow} q_{aux} \rightarrow C_2 \) of \( \mathcal{P} \) by a sequence of steps of \( \mathcal{P}' \).

In \( \mathcal{P} \), an agent at state \( q_1 \), say \( a \), moves to \( q_2 \), and broadcasts the signal with meaning “react according to \( g \)”. Intuitively, in \( \mathcal{P}' \), agent \( a \) broadcasts the unique signal of \( \mathcal{P}' \), which has the meaning “freeze”. An agent that receives the signal, say \( b \), becomes “frozen”. Frozen agents can only be “awaken” by a rendez-vous with \( a \). When the rendez-vous happens, \( a \) tells \( b \) which state it has to move to according to \( g \).

The problem with this procedure is that \( a \) has no way to know if it has already performed a rendez-vous with all frozen agents. To solve this problem, frozen agents can spontaneously move to a state err signalling “I am tired of waiting”. If an agent is in this state, then eventually all agents go back to their initial states, reinitializing the computation. This is achieved by letting agents in state err move to their initial states while broadcasting the “freeze” signal.

C. Protocols with reset

In protocols with reset, all broadcasts transitions reset the protocol to its initial configuration. Formally, a population protocol with reset is a broadcast protocol \( \mathcal{P} = (Q, R, B, \Sigma, I, O) \) such that for every initial configuration \( C_0 \) and every finite execution \( C_0 \overset{q_1}{\rightarrow} C_2 \cdots C_k \), the following holds: \( C_k \overset{b}{\rightarrow} C' \) implies \( C' = C_0 \) for every \( b \in B \) and every \( C' \in \text{Pop}(Q) \).

Proposition 24: Every predicate computable by a population protocol with reset is Presburger-definable, and thus computable by a standard population protocol.

Proof: Let \( \mathcal{P} = (Q, R, B, \Sigma, I, O) \) be a well-specified population protocol with reset. Let \( \rightarrow \) be the reflexive and transitive closure of the relation \( \rightarrow \) of \( \mathcal{P}' \overset{\text{def}}{=} (Q, R, B, \Sigma, I, O) \). For every \( X \subseteq \text{Pop}(Q) \), let

\[ \text{Pre}^*(X) \overset{\text{def}}{=} \{ q \in Q : q \overset{r}{\rightarrow} \text{ for some } r \in X \}. \]

Let \( C', C'' \in \text{Pop}(Q) \) denote the mutual-reachability relation of \( \mathcal{P}' \). A bottom strongly connected component (BSCC) is a non-empty set of pairwise mutually reachable configurations closed under reachability. We call a configuration \( C \) bottom if \( C \in X \) for some BSCC \( X \). Further let \( B \) denote the set of all bottom configurations of \( \mathcal{P}' \). Recall from [27] that an execution \( C_0 \overset{q_1}{\rightarrow} C_2 \cdots \overset{q_i}{\rightarrow} C \) is fair if and only if \( C_i \in B \) for all but finitely many indices \( i \).

Let \( \mathcal{N} \) be the set of configurations of \( \mathcal{P}' \) from which no reset can occur, i.e. let

\[ \mathcal{N} \overset{\text{def}}{=} \{ C \in \text{Pop}(Q) : \forall C' \in \text{Pop}(Q), \forall t \in B, \]

\[ \text{if } C \overset{t}{\leftrightarrow} C', \text{ then } t \text{ is disabled in } C' \}. \]

For every \( b \in \{0, 1\} \), let \( F_b \overset{\text{def}}{=} S_b \cap B \cap \mathcal{N} \), where \( S_b \) is the set of \( b \)-stable configurations:

\[ S_b \overset{\text{def}}{=} \{ C \in \text{Pop}(Q) : \forall C', \text{ if } C \overset{b}{\leftrightarrow} C', \text{ then } O(C') = b \}. \]

We claim that the set of accepting initial configurations of \( \mathcal{P} \), denoted \( I_1 \), equals \( \text{Pop}(I) \cap (S_1 \cup \text{Pre}^*(F_1)) \). Let us prove the claim. Let \( C_0 \) be from the latter set. Either \( C_0 \in S_1 \) or \( C_0 \in \text{Pre}^*(F_1) \). If \( C_0 \in S_1 \), then \( C_0 \) is 1-stable in \( \mathcal{P}' \). Notice that every 1-stable initial configuration of \( \mathcal{P}' \) is also 1-stable in \( \mathcal{P} \), and consequently \( C_0 \) is accepting. If \( C_0 \in \text{Pre}^*(F_1) \), then by definition of \( F_1 \), there is a BSCC \( X \subseteq B \) reachable from \( C_0 \) such that \( O(C) = 1 \) and such that resets are disabled for every \( C \in X \). Since no reset is enabled in \( X \), set \( X \) is a BSCC not just in \( \mathcal{P}' \), but also in \( \mathcal{P} \). Hence, at least one fair execution of \( \mathcal{P} \), starting in \( C_0 \), stabilizes to 1 and thus, by well-specification of \( \mathcal{P} \), all fair executions of \( \mathcal{P} \) starting in \( C_0 \) stabilize to 1. The converse direction is proven analogously.

It remains to show that \( I_1 \) is Presburger-definable. For this, we make use of the following results from [27]:

- \( S_0, S_1 \rightarrow \) and \( B \) are Presburger-definable;
- for every Presburger-definable sets \( X, F_0, F_1 \subseteq \text{Pop}(Q) \), if \( X \cap \text{Pre}^*(F_0) \) and \( X \cap \text{Pre}^*(F_1) \) form a partition of \( X \), then both sets are Presburger-definable.

By the above, \( I' \overset{\text{def}}{=} \text{Pop}(I) \setminus (S_0 \cup S_1) \) is a boolean combination of Presburger-definable sets, and hence Presburger-definable too. Similarly, \( F_b \) is Presburger-definable for every \( b \in \{0, 1\} \) through the Presburger formula \( \psi_b(C) \):

\[ \psi_b(C) \overset{\text{def}}{=} \left( C \in B \right) \land \left( \forall C' : \left[ \left( C \overset{b}{\leftrightarrow} C' \right) \implies \left( C'(q) = 0 \right) \right] \right). \]

Since \( F_0, F_1 \) and \( I' \) are Presburger-definable, and since sets \( I_0', I_1' \overset{\text{def}}{=} I' \cap \text{Pre}^*(F_0) \) and \( I_0', I_1' \overset{\text{def}}{=} I' \cap \text{Pre}^*(F_1) \) form a partition of \( I' \), it follows by the above observations that both \( I_0' \) and \( I_1' \) are Presburger-definable. Moreover, it is relatively straightforward to see that the following equalities hold:

\[ I_1 = \text{Pop}(I) \cap (S_1 \cup \text{Pre}^*(F_1)) = \text{Pop}(I) \cap (S_1 \cup (I' \cap \text{Pre}^*(F_1))) = \text{Pop}(I) \cap (S_1 \cup I_1'). \]

Thus, \( I_1 \) is Presburger-definable since it is a boolean combination of Presburger-definable sets.

VII. A COMPARATIVE STUDY OF THE EXPRESSIVE POWER OF CONSENSUS PROTOCOLS

We compare the four models mentioned in this paper: population protocols, community protocols, mediated protocols, and broadcast protocols. While the comparison is technically simple, it requires to define some notions with precision.
All four models postulate a finite set of states $Q$, input alphabet $\Sigma$, and input and output functions $I, O$. However, not all assume that agents have no identities, or that only agents can store information.

**Community protocols** assume an infinite set $Id$ of agent identifiers, e.g. $N$, and a total order $\prec \subseteq Id \times Id$, e.g., the natural order $\prec$. Each agent is assigned a unique identifier, and so for $a \in Id$, we speak of “the agent $a$”. Each agent can store up to a constant number of identifiers. At each step, two agents, say $a_1, a_2$, interact and move to new states. Assume $a_1$ and $a_2$ are in states $q_1, q_2$ with store values $A_1, A_2$. The new states $q'_1, q'_2$ and store values $A'_1, A'_2$ are a function of $q_1, q_2, a_1, a_2, A_1, A_2$. However, the function only depends on $\prec \cap (A \times A)$, where $A = \{a_1, a_2\} \cup A_1 \cup A_2$, and not on the set $A$ itself (technically, the function must return the same value for any two sets $A, A'$ such that there is a $\prec$-preserving bijection between $A$ and $A'$).

A possible transition of a community protocol is:

If agents $a_1$ and $a_2$ are in state $p$, and $a_1 \prec a_2$, then agents $a_1$ and $a_2$ can move resp. to states $q_1$ and $q_2$.

**Mediated protocols** postulate a set of agents communicating by means of point-to-point channels, one for each pair of agents. Channels have a constant amount of memory. This is formalized by introducing a set $Q$ of agent states and a disjoint set $S$ of channel or edge states. Transitions are of the form $(q_1, q_2, s) \mapsto (q'_1, q'_2, s')$, with the following meaning:

If two agents are in state $q_1$ and $q_2$, and the channel between them is in state $s$, then the agents move to states $q'_1$ and $q'_2$, and the channel moves to state $s'$.

Observe that the transitions, and hence the step relation induced by them, do not depend on the identifiers of the agents.

A first axis for the comparison are the privacy properties preserved by the different classes. In particular, one of the motivations for mediated protocols was to design a model with expressive power similar to the power of community protocols, but satisfying anonymity, which was however not formally defined in [8]. While for the purposes of [8] this is no problem, we need a definition here to compare anonymity with a second natural property, indistinguishability, which holds for oblivious protocols, but not for mediated or community protocols.

The second axis of the comparison is the expressive power of the models. Together with the first axis, it describes the price of privacy. We determine for each model the class of number predicates computable by the model. For the comparison, we need a uniform definition of “a protocol computes a number predicate”. Fortunately, Definition 1 applies to all models. In particular, recall that a predicate $\varphi$ of arity 1 is computable by a class of protocols if there is a protocol in the class such that for every number $n$, the value $\varphi(n)$ can be computed by the protocol with $n$ agents in the unique initial state.

**A. Defining anonymity and indistinguishability**

Let $Id$ be a set of identifiers or names, and let $Q$ be a set of states. A trajectory is a finite sequence of states. A trace of length $k$ is a mapping $\tau$ that assigns to every $a \in Id$ a trajectory $\tau(a)$ of length $k$. A trace language is a set of traces (possibly of different lengths). In the rest of the section, we identify a protocol with the set of traces it can execute, defined in the expected way.

Intuitively, anonymity guarantees that by observing the trajectories of the agents in one execution, one cannot gain any information about their identities.

Given a trace $\tau$ and a permutation $\pi : Id \to Id$, we define $\tau \circ \pi$ as the trace given by $(\tau \circ \pi)(a) \defeq \tau(\pi(a))$.

**Definition 25**: A trace language $L$ preserves anonymity if $\tau \in L \iff (\tau \circ \pi) \in L$ for every trace $\tau$ and permutation $\pi$.

Now we introduce a second property, called indistinguishability. To motivate it, consider a “House of Cards” protocol with the following states:

| State | Intended meaning |
|-------|------------------|
| $M$  | “agent knows that the US president is a Murderer” |
| $N$  | “agent knows Nothing” |
| $C$  | “agent goes to the Cinema” |
| $J$  | “agent talks to a Journalist” |
| $D$  | “agent Dies of a heart attack” |

Assume a detective informs the president that two agents have executed the trajectories $\{MCJ, NCD\}$. Even if the president ignores the identities of the agents, s/he has learned crucial information. Imagine now the agents are identical twins. The detective observes that both enter the cinema and later, at the end of the film, leave it. In this case, the detective and president only learn that the trajectories are either $\{MCJ, NCD\}$ or $\{MCD, NCJ\}$, a very different situation! In the twin case, agents that occupy the same state at the same time are indistinguishable: We may know the set of their past and future trajectories, but not how these two sets are connected.

We call this property indistinguishability. Let us formalize it. Given two traces $\tau_1$ and $\tau_2$, define their concatenation $\tau_1 \tau_2$ by $(\tau_1 \tau_2)(a) \defeq \tau_1(a) \tau_2(a)$ for every $a \in Id$. A permutation is compatible with $\tau$ if for every $a \in Id$, the trajectories $\tau(a)$ and $\tau(\pi(a))$ start in the same state.

**Definition 26**: A trace language $L$ preserves indistinguishability if for every two traces $\tau, \tau'$ and for every permutation $\pi$ compatible with $\tau'$, we have: $\tau \tau' \in L \iff \tau(\pi \circ \tau')(\pi(a)) \in L$.

It follows immediately from the definition that oblivious consensus protocols preserve indistinguishability. The converse does not hold: Indeed, one can design protocols in which the step relation depends not only on the current configuration, but also on previous ones. Those protocols preserve indistinguishability, but are not oblivious. In fact, the following “equation” is a direct consequence of the definitions:

Obliviousness = Indistinguishability + Memorylessness, where memorylessness means that steps from a configuration $C$ depend only on $C$, and not on the past of the computation.

**B. Comparison**

We go through the four models, describing the privacy properties they satisfy and their computational power.
1) **Community protocols:** Community protocols do not preserve anonymity.

**Example.** Consider a community protocol with \{p, q_1, q_2\} as set of states and the following unique transition:

If agents \(a_1\) and \(a_2\) are in state \(p\), and \(a_1 \prec a_2\), then agents \(a_1\) and \(a_2\) can move resp. to states \(q_1\) and \(q_2\).

Mapping \(\tau\) with \(\tau(a_1) = p \cdot q_1\) and \(\tau(a_2) = p \cdot q_2\) is a trace of the protocol, but \(\tau \circ \pi\) with \(\pi(a_1) = a_2\), \(\pi(a_2) = a_1\) is not.

In [7], Guerraoui and Ruppert show the following result. Let \(M\) be a nondeterministic Turing machine, over alphabet \(\Sigma\), running in \(O(n \log n)\) space. There is a community protocol such that, for every input word \(w \in \Sigma^*\) of length \(n\), the following holds: \(M\) accepts \(w\) (resp. rejects \(w\)) iff the instance of the protocol in which \(n\) agents with identities \(1, 2, \ldots, n\) are initially put in states \(I(w_1), I(w_2), \ldots, I(w_n)\) converges to 1 (resp. 0). Moreover, the \(O(n \log n)\) space bound is tight, i.e., the property does not hold for any function \(f \in \omega(n \log n)\).

Guerraoui and Ruppert state their result as: “community protocols compute exactly the predicates in \(O(n \log n)\)”. However, in our setting the result is actually far more impressive. Indeed, consider any number predicate \(\varphi \in \text{NSPACE}(2n\log n)\) of arity \(k\). For such a predicate, there is a Turing machine \(M\) running in \(O(n \log n)\)-space such that for every \(x \in \mathbb{N}^k\), the machine accepts the unary encoding of \(x\), i.e. \(w = 1^{x_1} \# 1^{x_2} \# \cdots \# 1^{x_k}\), iff \(\varphi(x) = 1\). By the result of [7], the execution of \(M\) on \(w\) can be simulated by the instance of a community protocol with a total of \(x_1 + x_2 + \ldots + x_k\) agents; more precisely, for every \(1 \leq i \leq k\) we place \(x_i\) agents in the state corresponding to the \(i\)-th input. Therefore, we obtain:

**Proposition 27 ([7]):** Community protocols compute exactly the number predicates contained in \(\text{NSPACE}(2n\log n)\).

2) **Mediated protocols:** Since, by definition, transitions of mediated protocols do not depend on agent identities, mediated protocols satisfy anonymity. However, they do not satisfy indistinguishability. It is easy to construct an instance of a mediated protocol with three agents, two twins \(a, b\) and a detective \(d\), modeling the “House of Cards” scenario. In addition to the states \(M, N, C, J, D\) of the twins, the detective can be in states \(F\) (follow) and \(K\) (Kill). The detective is initially in state \(F\) (follow), the twins in states \(M\) and \(N\). All transitions involve a twin and the detective. When the twins go to the cinema, the detective updates the state of their communication channel by means of transitions \((M, F, 0) \rightarrow (C, F, 1)\) and \((N, F, 0) \rightarrow (C, F, 0)\), respectively. This amounts to “tagging” the twin in state \(M\) with a 1 before she enters the cinema, and tagging her sister with a 0. Transition \((C, F, 1) \rightarrow (J, K, 1)\) allows the detective to know that the dangerous twin is the one that after exiting the cinema is meeting the journalist, and hence to move to state \(K\). The appendix describes the mediated protocol in detail, and describes a trace \(\tau \tau'\) of the protocol such that \(\tau(\tau' \circ \pi)\) is not a trace, where \(\pi\) is the mapping that permutes the twins. Intuitively, \(\tau\) and \(\tau'\) describe the behavior before and after entering the cinema.

In [8], Michail, Chatzigiannakis and Spirakis show that a mediated protocol can simulate a Turing machine running in \(\text{NSPACE}(n^2)\). Applying the same observations as for community protocols, we obtain:

**Proposition 28 ([8]):** Community protocols compute exactly the number predicates contained in \(\text{NSPACE}(2n^2)\).

This shows that anonymity can be achieved without losing expressive power. (The fact that the expressive power looks even higher is due to the fact that channels are not considered as agents, even though they have states.)

3) **Broadcast protocols and population protocols:** Oblivious protocols preserve anonymity and indistinguishability. This follows immediately from the definition. As we have seen, (reasonable) oblivious protocols can only compute predicates in \(\text{NSPACE}(n)\). Broadcast protocols reach the theoretical bound, while population protocols stay “exponentially far” from it, and can only compute predicates in \(\text{DSpace}(\log n)\).

It was shown in [20] that extending the population protocol model with interactions between \(k\) agents, for any fixed \(k\), does not increase the expressive power. So the main difference between population and broadcast protocols is the fact that the latter exhibit global interactions involving all agents.

**VIII. Conclusion**

We have studied the expressive power of oblivious consensus protocols, in which the step relation depends only on the number of agents in each state, and not on their identities. We have shown that there is a natural \(\text{NSPACE}(n)\) upper bound for their expressive power, but that population protocols are very far from achieving it. We have then introduced broadcast consensus protocols, and proved that they reach this bound. Finally, we have conducted the first comparative and systematic study of the expressive power of consensus protocols, including community protocols and mediated protocols, two models introduced in [7] and [8]. We have studied the trade-off between expressive power and privacy properties. The results of [7], [8], and this paper show the following:

- Population protocols satisfy anonymity and indistinguishability, but they only compute a strict subset of the number predicates in DSPACE(\(\log n)\).
- Broadcast protocols satisfy anonymity and indistinguishability, and compute exactly the number predicates in \(\text{NSPACE}(n)\). No other (reasonable) class of oblivious protocols is strictly more expressive.
- Mediated protocols compute exactly the predicates in \(\text{NSPACE}(2n^2)\), but they do not satisfy indistinguishability.
- Community protocols compute exactly the predicates in \(\text{NSPACE}(2n \log n)\), but they satisfy neither anonymity nor indistinguishability.

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APPENDIX
BEYOND PRESBURGER ARITHMETIC: PROOF OF PROPOSITION 11

Proposition 11: The broadcast consensus protocol $\mathcal{P}$ described above computes the number predicate $\varphi$, defined as $\varphi(x) = 1$ if $x > 1$ and $x$ is a power of two.

Proof: As mentioned in the main text, if the protocol executes transitions as intended, then it produces a consensus, and the right consensus. If the protocol deviates from the intended sequence, we say that it commits an error. The protocol detects an error by executing transitions $\overline{s}$, $t_0$ and $t_1$. If an error is detected, then transition $r$ eventually resets the agents back to the initial configuration. So it only remains to prove that every error is eventually detected.

Let $C_0$, $C$, $C'$ be configurations and let $u$ be a transition such that $C_0$, $\xrightarrow{u} C$, $\xrightarrow{u} C'$, $u \neq r$ and $C'(|\bot) = 0$.

We claim that:
- If $s$ or $\overline{s}$ is enabled in $C$, then $u \in \{s, \overline{s}\}$;
- else if $C(x) = 1$, then $u = t_0$;
- else if $C(\overline{s}) = 1$ and $C(x) = 0$, then $u = t_1$;
- else $u = \overline{s}$.

The claim states that if no error is detected in some step, then the step takes the intended transition, or equivalently: if some step takes the wrong transition, then an error is detected.

We now prove the claim:
- Assume $s$ or $\overline{s}$ is enabled in $C$ and $u \notin \{s, \overline{s}\}$.
  By assumption, we have $C(x) \geq 2$ or $C(\overline{s}) \geq 2$, and $u \in \{\overline{s}, t_0, t_1\}$. By definition of $u$, we have $C'(\bot) > 0$ which is a contradiction, hence $u \in \{s, \overline{s}\}$.
- Assume $s$ and $\overline{s}$ are disabled in $C$, $C(x) = 1$ and $u \neq t_0$.
  We have $u \in \{\overline{s}, t_1\}$ and, by definition of $u$, we have $C'(\bot) > 0$ which is a contradiction, hence $u = t_0$.
- Assume $s$ and $\overline{s}$ are disabled in $C$, $C(\overline{s}) = 1$, $C(x) = 0$ and $s \neq t_1$. Since $C(x) = 0$, transition $t_0$ is disabled in $C$. Thus, $u = \overline{s}$. Since $C(\overline{s}) = 1$, we have $C'(\bot) > 0$ which is a contradiction, hence $s = t_1$.

SIMULATION OF POLYNOMIALLY BOUNDED COUNTER MACHINES BY $n$-BOUNDED COUNTER MACHINES: PROOF OF LEMMA 17

Lemma 17: For every polynomially bounded counter machine that computes some predicate $\varphi$, there exists an $n$-bounded counter machine that computes $\varphi$.

Proof: Let $c \in \mathbb{N}_{>0}$ and let $\mathcal{M} = (Q, X, \Delta, m, q_0, q_a, q_r)$ be an $n^c$-bounded counter machine.

Counter values representation. To simulate $\mathcal{M}$ by an $n$-bounded counter machine $\overline{\mathcal{M}}$, we need some way to represent any counter value from $[0, n^c]$ with counters with values from $[0, n]$. If $n$ would be fixed, then any number $\ell \in [0, n^c]$ could be encoded in base $n + 1$ over $c$ counters. For example, if $c = 3$ and $n = 4$, then

$59 = 2 \cdot 5^2 + 1 \cdot 5^1 + 4 \cdot 5^0$,

and hence 59 can be represented by $(2, 1, 4)$. More generally, any number from $[0, n^c]$ can be represented this way by a vector of $[0, n]^c$. However, $n$ refers to the size of the input and hence it is an arbitrary number which is not fixed. Thus, we cannot directly use the described encoding. Instead, we introduce a more technical representation that relies on this idea, but does not assume $n$ to be fixed.

Let us reconsider the case where $c = 3$ and $n = 4$. The representation of 59 was $(2, 1, 4)$, which is equal to $(0, 0, 1) + (0, 1, 0) + (1, 0, 1) + (1, 1, 1)$. We can summarize this by the multiset $\{2 \cdot (0, 0, 1), (1, 0, 1), (1, 1, 1)\}$ or graphically as a table:

| 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 2   | 0   | 0   | 1   | 0   | 0   | 1   |

This multiset is our representation of 59. Observe that the size of the multiset is $4 = n$. This is no coincidence: any number from $[0, n^c]$ can be represented this way by a multiset of size $n$ made of binary vectors. We call such a multiset a linear-multiset since it yields an $n$-bounded representation.

Note that the domain of a linear-multiset has $2^n$ elements, which is a fixed constant independent from $n$. Moreover, note that the above linear-multiset is unique for 59, but this is generally not the case. For example, the two following linear-multisets both represent $57 = 2 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$:

| 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 0   | 0   | 0   | 1   | 1   | 0   |

One could make the representation canonical by fixing some total order on the domain, but this will not be necessary for our construction.

Counters and states. More formally, a linear-multiset is a multiset $M$ over $\{0, 1\}^c$. For every counter $x$, we represent each entry of a linear-multiset associated to $x$ by $2^c$ counters:

$\overline{x} = \{x_b : b \in \{0, 1\}^c\}$.

The set of counters of $\overline{\mathcal{M}}$ is $\overline{\mathcal{X}} = \bigcup_{\ell \in X} \overline{x}$.

Machine $\overline{\mathcal{M}}$ has the same arity as $\mathcal{M}$: counter $x_0...0_1$ is made an input counter for every $x_\ell \in X$. The control states of $\overline{\mathcal{M}}$ form a superset of $Q$, and the initial, accepting and rejecting states are the same.

We now describe the transition relation of $\overline{\mathcal{M}}$ by explaining how to simulate instructions $\text{zro}(x)$, $\text{nzr}(x)$, $\text{inc}(x)$ and $\text{dec}(x)$ over $\overline{x}$, for every $x \in X$. Instruction $\text{nop}$ is trivially simulated by nop itself. Let us fix $x \in X$, some $n$ and a linear-multiset $M$ of size $n$ corresponding to $\overline{x}$. All our constructions will preserve the invariant $\sum_{b \in \{0, 1\}^c} x_b = n$, which yields $n$-boundedness.

(Non)zero-tests. We have $x = 0$ iff $[M] = \{0 \ldots 0\}$, i.e. iff $\sum_{y \in \overline{x} \setminus x_0...0_0} y = 0$. Thus, instruction $\text{zro}(x)$ is simulated by $2^c - 1$ sequential zero-tests over $\overline{x}$. Similarly, $\text{nzr}(x)$ is simulated by parallel nonzero-tests. Both gadgets are depicted in Figure 1. Note that nop instructions are further added within the gadget for $\text{zro}(x)$ to ensure that $\overline{\mathcal{M}}$ does not block where $\mathcal{M}$ would not block.
Fig. 1. Gadgets for the simulation of instructions \((q, \text{zro}(x), r) \in \Delta\) (top gadget) and \((q, \text{nzr}(x), r) \in \Delta\) (bottom gadget).

Incrementation. Let \(v \in [0, n]^c\) be the base-\((n+1)\) vector representation of \(x\). For example, recall that for \(c = 3, n = 4\) and \(x = 59\), we have \(v = (2, 1, 4)\). Incrementing \(x\) amounts to incrementing \(v\) in base-\((n+1)\) arithmetic. While \(M\) does not have direct access to \(v\), it is possible to check whether \(v(i) = n\) by checking whether every element of \(M\) has its \(i\)-th bit set to 1, or equivalently:

\[
\bigwedge_{b \in \{0, 1\}^c \atop b(i) = 0} (x_b = 0).
\]

The latter test is independent from \(n\). Moreover, it is sufficient for implementing incrementation. If \(v(c) \neq n\), then \(M\) decrements \(x_{b_0}\) and increments \(x_{b_1}\) for some \(b \in \{0, 1\}^{c-1}\). Otherwise, an “overflow” occurs and hence \(M\) flips the last bit of each element of \(M\) to 0, and repeats the process on the next bit. The flipping can be done by a gadget assigning:

\[
\begin{align*}
x_{b_0} & := x_{b_0} + x_{b_1}, \\
x_{b_1} & := 0,
\end{align*}
\]

for every \(\{0, 1\}^{c-1}\). Note that overflows can occur at most \(c-1\) times since \(M\) is \(n^c\)-bounded. The overall construction is depicted in Figure 2 for the case \(c = 3\).

Decrementation. This case is symmetric to incrementation:

1) Decrementing \(x\) amounts to decrementing \(v\) in base-\((n+1)\) arithmetic;
2) \(M\) cannot access \(v\) but it can check whether \(v(i) = 0\) by testing \(\bigwedge_{b, b(i) = 1} (x_b = 0)\);
3) \(M\) starts from the last index \(i := c\);
4) If \(v(i) \neq 0\), then \(v(i)\) is decremented by flipping the \(i\)-th bit of some element of \(M\) from 1 to 0;
5) Otherwise \(v(i)\) is set to \(n\) by flipping the \(i\)-th bit of every element of \(M\) to 1;
6) Step 4 is repeated with \(i := i - 1\).

Single-signal broadcast protocols: proof of Proposition 23

Proposition 23: Predicates computable by broadcast consensus protocols are also computable by single-signal broadcast protocols.

Proof: By Corollary 22, it suffices to show that for every broadcast protocol \(\mathcal{P} = (Q, R, B, \Sigma, I, L, O)\) that silently computes a predicate there is a single-signal protocol \(\mathcal{P}' = (Q', R', B', \Sigma, I', O')\) that computes the same predicate.

States and mappings. Let

\[
Q' \triangleq Q^2 \cup (Q^2 \times B) \cup (Q^2 \times \{\text{frozen, err, reset}\}).
\]

Every state of \(Q'\) has two or three components. As in the construction of Lemma 13, the first two components describe the position of the agent, and its origin. Agents never change their origin, and so they know which state to return to if they are told to reset.

The third component can be either a broadcast transition, or one of \(\{\text{frozen, err, reset}\}\). The intended meaning of agent \(a\) being in a state with third component \(c\) is as follows:

- \(c \in B\): agent \(a\) is in charge of simulating \(c\) by first freezing all other agents and then performing a rendez-vous with each of them;
- \(c = \text{frozen}\): agent \(a\) is currently waiting for a rendez-vous with the broadcasting agent that told it to freeze;
- \(c = \text{err}\): agent \(a\) has decided not to wait any longer for the rendez-vous with the broadcasting agent, or a new broadcast signal has been sent before the simulation of the previous broadcast is completed;
- \(c = \text{reset}\): agent \(a\) is in charge of telling other agents to reset.

The input and output mappings are defined respectively by

\[
\begin{align*}
I'(x) & \triangleq (I(x), I(x)) \text{ for every } x, \\
O' & \triangleq O \circ pos,
\end{align*}
\]

where \(pos\) is the function that maps every state from \(Q'\) to its position, i.e. \(pos(q) \triangleq q_1\) for every \(q = (q_1, q_2) \in Q^2\) and every \(q = (q_1, q_2, x) \in Q' \setminus Q^2\).

We now describe how \(\mathcal{P}'\) simulates \(\mathcal{P}\).

Initiation of a broadcast simulation. The “freeze” signal used by the protocol is described by the following function \(f: Q' \rightarrow Q'\):

\[
f(q) \triangleq \begin{cases} 
(q, \text{frozen}) & \text{if } q \in Q^2, \\
(r, \text{err}) & \text{if } q = (r, x) \in Q' \setminus Q^2.
\end{cases}
\]

Intuitively, when an agent broadcasts this signal, it tells all agents in “normal” states to freeze. In an error-free simulation, all agents are in such normal states, and so the broadcasts sends all other agents to a “frozen state” \((r, \text{frozen})\).

When an already “frozen” agent receives the freeze signal, this means that the simulation of a broadcast or reset is not completed before another broadcast or reset is initiated, and the frozen agent assumes an “error state”. An error state
indicates that the population must be reset at some point in the future.

For every broadcast transition \( t: q \rightarrow q', g \) from \( B \) and every \( r \in Q \), we add to \( B' \) the broadcast transition:

\[
(q, r) \rightarrow (q', r, t); f. 
\]

The agent in state \((q', r, t)\) is in charge of simulating the effect of the broadcast transition \( t \) via rendez-vous with the other agents.

**Simulation of rendez-vous.** For every rendez-vous transition \((q_1, q_2) \rightarrow (q'_1, q'_2) \in R \) and every \((r_1, r_2) \in Q^2\), we add to \( R' \) the rendez-vous transition:

\[
((q_1, r_1), (q_2, r_2)) \rightarrow ((q'_1, r_1), (q'_2, r_2))
\]

**Receiver’s response.** For every broadcast transition \( t = (q_1, q_2, g) \in B, (q, r) \in Q^2 \) and every origin \( q_0 \in Q \), we add to \( R' \) the transition:

\[
((q_2, q_0, t), (q, r, \text{frozen})) \rightarrow ((q_2, q_0, t), (g(q), r)).
\]

The transitions defined thus far would suffice if broadcasts were always simulated correctly. But we cannot rule out the initiation of a broadcast simulation before a previous simulation is completed, which yields an agent in an error state. Additional transitions are thus needed for error handling. Whenever an agent is in an error state, the population must be eventually reset to start a new, clean simulation attempt. In the implementation of the reset we must ensure that “illegitimate” agents, that have not yet been reset, cannot interact with “legitimate” agents, that have already been reset.

**Initiation of a reset.** Agents in an error state may transition to a reset state in order to initiate a reset. The agent in a reset state is in charge of implementing the reset via rendez-vous with the other agents. When a reset is initiated, there should be precisely one agent in a reset state, while all other agents are temporarily *disabled*, for otherwise some agent’s state could be modified by some other “illegitimate” agent before the reset is completed, and a reset agent would have no means to distinguish between “legitimate” agents and “illegitimate” agents.

We implement the initiation of a reset by a broadcast. For every \((q, r) \in Q^2\), we add to \( B' \) the broadcast transition:

\[
(q, r, err) \rightarrow (r, r, reset); f.
\]

**Reset to origin.** For every \( q \in Q^2 \) and every \((q, r, x) \in Q' \setminus Q^2\), we add to \( R' \) the transition:

\[
((q, \text{reset}), (q, r, x)) \rightarrow ((q, \text{reset}), (r, r)).
\]

**Completion of a reset.** A reset is (perhaps prematurely) completed when an agent in a reset state resets itself. For every \( q \in Q^2, x \in Q' \), we add to \( R' \) the transition:

\[
((q, \text{reset}), x) \rightarrow (q, x).
\]

**From frozen to error.** It may be the case that a reset agent resets itself to origin before all frozen agents have been reached. To avoid that frozen agents wait forever to be “unfrozen”, frozen agents can non-deterministically decide to assume an error state, thereby initiating a new reset.

For every \( q \in Q^2 \) and every \( x \in Q' \), we add to \( R' \) the transition:

\[
((q, \text{frozen}), x) \rightarrow ((q, err), x).
\]

\( P' \) computes the same predicate as \( P \): Since \( P \) is silent, every fair execution of \( P \) reaches a terminal configuration, and by construction of \( P' \), every correct simulation of \( P \) in \( P' \) eventually reaches a terminal configuration of the same consensus.

An error in the simulation may occur in one of two cases: Either another broadcast signal is initiated before the simulation of the last broadcast or a reset is completed, or the agent in charge of a reset reverts to its initial state before all other agents have been reset. In the former case, at least one

![Diagram](image-url)
agent is sent to an error state, which will eventually lead to a reset. In the latter case, at least one agent in an error state or a frozen agent remains. Frozen agents can non-deterministically choose to turn to error states, and thus eventually initiate a reset. In either case, the population is eventually reset to its initial configuration. Fairness guarantees that the simulation is eventually executed correctly.

Note that silentness of protocol $P$ is crucial for the correctness of the construction: Since $P$ is silent, broadcast signals are bound to cease to occur in every correct simulation, and thus all agents eventually remain unfrozen forever, hence we may safely demand that frozen agents non-deterministically turn to error states, which allows us to handle incomplete resets.

**Mediated Protocols Violate Indistinguishability**

We describe the mediated protocol formalizing the scenario described in Section VII.

**Example.** Consider a mediated protocol with states $Q \equiv \{M, N, C, J, D, F, K\}$ and transitions:

- $t_1: (M, F, 0) \rightarrow (C, F, 1)$
- $t_2: (N, F, 0) \rightarrow (C, F, 0)$
- $t_3: (C, F, 1) \rightarrow (J, K, 1)$
- $t_4: (C, F, 0) \rightarrow (J, F, 0)$
- $t_5: (C, F, 0) \rightarrow (D, F, 1)$
- $t_6: (C, F, 0) \rightarrow (D, F, 0)$

Let $\tau$ and $\tau'$ be the mappings with trajectories:

- $\tau(a_1) = MCC$  
- $\tau'(a_1) = CJ$
- $\tau(d) = FFF$  
- $\tau'(d) = FK$

- $\tau(a_2) = NNC$  
- $\tau'(a_2) = D D$.

The mapping $\sigma = \tau \tau'$ with trajectories:

- $\sigma(a_1) = MCCJ$
- $\sigma(d) = FFFK$
- $\sigma(a_2) = N N C D D$.

is a trace of the protocol, corresponding to executing the sequence diagrammatically described by:

- $a_1 \leftarrow \begin{array}{c|ccc} \ & M & C & C & C & J \\ \hline d & F & F & C & C & J \\ a_2 & N & N & C & D & D \\
\end{array}$

Intuitively, $t_1$ “tags” $a_1$, what allows us to execute $t_3$ later on. Consider now the mapping $\rho = \tau(\tau' \circ \pi)$, where $\pi$ permutes the twins. The mapping gives the trajectories:

- $\rho(a_1) = MCCDD$
- $\rho(d) = FFFK$
- $\rho(a_2) = N N C J$.

This mapping is not a trace of the protocol. Intuitively, after executing $t_1 t_2 t_6$, it is not possible to execute $t_3$, because $a_2$ is not the twin tagged by $t_1$:

- $a_1 \leftarrow \begin{array}{c|ccc} \ & M & C & C & D & D \\ \hline d & F & F & C & D & D \\ a_2 & N & N & C & C & J \\
\end{array}$