Statistics of pre-localized states in disordered conductors

Vladimir I. Fal’ko and K.B. Efetov

1 Max-Planck-Institut für Festkörperforschung, Heisenbergstr. 1, 70569 Stuttgart, Germany
2 Max-Planck-Institut für Physik Komplexer Systeme, Heisenbergstr. 1, 70569 Stuttgart, Germany
3 Institute of Solid State Physics RAS, Chernogolovka, 142432 Russia
4 L.D. Landau Institute for Theoretical Physics, Moscow, Russia

(November 17, 2021)

I. INTRODUCTION

Localization of a particle by a random potential has been extensively investigated during the past several decades [1–3]. It is well known [5–7] that, at strong disorder, single-particle wave functions are confined and have exponentially decaying tails beyond the scale of the localization length $L_c$. At weak disorder, the localization length can be very large in 1D and 2D conductors, and infinite in 3D. A natural question arises: What is the behavior of the wave functions at distances smaller than the localization length? Despite of its importance, the problem of structure of quantum states of weakly disordered conductors for scales below the length $L_c$ has only recently started to attract interest [3–7].

In particular, one of the issues that has not been explored up to now concerns the way the localized states develop as a consequence of the increase of disorder in an isolated piece of a metal, though a great deal is already known about the extended states in it. Some part of recent results related to extended (metallic-type) states have been obtained by mapping the problem of quantum mechanics in the classically chaotic systems to the Wigner-Dyson random matrix theory [8–17], or using the zero-dimensional supermatrix $\sigma$-model [18–21], which are two equivalent ways of describing disordered and chaotic systems.

Both the advantage and disadvantage of such an approach comes from the statistical equivalence of eigenstates which is usually built into the construction of the random matrix substituting the real dynamics. This reveals the set of universalities of the spectra, the level-level correlations and the transition matrix elements [19] which are similar for a wide variety of objects. For example, the distribution function of local densities of wave functions $|\psi(r_o)|^2$ in a chaotic cavity which one can find in such a way is determined only by the fundamental symmetry of the system and its volume $V \sim L^d$, but is independent of the level of disorder (i.e., of the value of a mean free path $l$) or a physical dimension, $d$.

On the other hand, this approach hides individual features of physically different systems and permits to describe only metallic-type states which equally test the random potential all over the sample. More complex states which can distinguish between the ballistic and diffusive regimes have to be analyzed beyond the conventional random matrix theory. Numerical evidence
for their existence have been obtained by several groups \[10,11,13\]. The goal of the theory to be presented in the present paper is to find manifestations of these precursors of localization among the wave functions of classically diffusive conductors \((p \ll 1, l < L)\). That is, we consider an isolated piece of a disordered metal with dimensions \(l \ll L < L_c\), assuming that the internal 'conductance' \(g\) which one would assign to the 'electric circuit' connecting the observation point (blown up to the mean free path size) with the external surface of the specimen \(c\) is much larger than the conductance quantum, i.e. \(g \gg 1\).

In the following Sections, we perform a statistical analysis of local densities and partly reconstruct spatial structure of those rare states which have locally too high amplitudes (as compared to the average \(V^{-1}\)) to fit into the universal random matrix theory description. As can be suggested on the basis of the theory below, these states are responsible for non-Gaussian tails of distributions of fluctuations of local densities of states and conductances suggested by Altshuler, Kravtsov and Lerner \[3\] and are generic for the long-living current relaxation discussed in Refs. \[9,12\].

Our paper is organized as follows. In Section II, we introduce the notion of the eigenstates statistics (II.A), discuss the universal distributions of metallic-type states (II.B) and, then, sketch the main results of the paper focusing our attention at the localization effects (II.C). Sections III and IV are devoted to the presentation of our theoretical scheme: We derive a reduced supersymmetric \(\sigma\)-model and show that it has a non-trivial saddle-point. The details of the derivation of the saddle-point solutions of the reduced \(\sigma\)-model are given separately for each of the fundamental symmetry classes (unitary, orthogonal and symplectic) in Sections IV.A-C, and the influence of fluctuations around the saddle-point is discussed in Section IV.D and the Appendices. The resulting statistics of wave functions and the structure of the pre-localized states in the conducting regime in 1D, 3D and in the most interesting case of 2D samples are discussed in Sections V, VI and VII, respectively. Section VIII contains a brief summary of our results and their discussion.

II. METALLIC VERSUS PRE-LOCALIZED STATES IN THE EIGENSTATES STATISTICS (PRELIMINARIES AND RESULTS)

In this Section, we give mathematical formulation to the problem of the eigenvalues statistics in disordered systems and consider alternative approaches to its solution. That is what is the subsection A about. The next part B is devoted to the universal statistics of metallic-type of states known in the random matrix theory as the Porter-Thomas distribution. The localization effects which are beyond the random matrix theory approach are discussed in the subsection C, where we give an essence of the obtained results. This subsection is written for the first reading and can be used as a guide through the rest of the text.

A. Definitions of the eigenstates statistics

To define the statistics which we shall be studying in this paper, we first mention that the properly normalized eigenstates \(\{\psi_\alpha\}\) we consider below correspond to a quantum particle in a disordered cavity

\[
\left\{ \frac{P^2}{2M} + U(r) \right\} \psi_\alpha(r) = \epsilon_\alpha \psi_\alpha(r), \psi_\alpha(r) \in S = 0,
\]

where \(U\) is a random potential. The local amplitude \(\psi\) of a wave function at some observation point \(r\) inside the sample, i.e.,

\[ t \equiv |\psi(r)|^2, \] (1)

will be the object of our statistical analysis. In this we employ two related quantities: the distribution function \(f(t)\) of local amplitudes \(t\) averaged over disorder,

\[ f(t) = \Delta \langle \sum_\alpha \delta(t - |\psi_\alpha(r)|^2) \delta(\epsilon - \epsilon_\alpha) \rangle, \] (2)

and the set of generalized inverse participation numbers (IPN) \[\{s\}\] which are the moments of the distribution function \(f\),

\[ t_n = \Delta \langle |\psi_\alpha(r)|^{2n}\delta(\epsilon - \epsilon_\alpha) \rangle \equiv \int_0^\infty t^n f(t) dt. \] (3)

As indicated, \(\langle \rangle\) denotes the averaging over random configurations of a random potential \(U\) in the system. In Eq. (4), the sum is over the full set of states \(\{\psi_\alpha\}\), \(V\) is the volume of the system, and \(\Delta = \langle \nu V \rangle^{-1}\) denotes the mean level spacing with \(\nu = \nu(\epsilon)\) the density of states per unit volume. Since the distribution function \(f\) and wave functions \(\{\psi_\alpha\}\) are normalized, one has the following relations:

\[ t_0 = \int_0^\infty f(t) dt \equiv 1; \quad t_1 = \langle |\psi_\alpha|^2 \rangle \equiv V^{-1}. \]

One can also introduce the distribution \(f_s(\sigma, t)\) of a local spin-density of the wave with \(\sigma = \downarrow, \uparrow\). It is an important quantity for systems with a strong spin-flip scattering. In random matrix theories these are known as a symplectic ensemble. In such a case, the statistics can be formulated in terms of spin-projected eigenstates, e.g. \(t_\downarrow = |\psi_\downarrow(r)|^2\) \[31\]. The distribution of a total local density \(t = t_\downarrow + t_\uparrow\) can be found as the convolution

\[ f_s(t) = \int_0^t f(\downarrow, t - t') f(\downarrow, t') dt'. \] (4)

Historically, the studies of eigenstates in disordered conductors started from Wegner's perturbative calculations of IPN's \[3\]. Due to the equivalence between the descriptions based on the distribution function and the
full set of its moments [24], in most of the later studies [24,26,27], the eigenstates statistics were reconstructed from the set of IPN’s. Alternatively, one can start from calculating directly the entire distribution function [24,26,27], especially regarding the possibility to apply the supersymmetry technique [7]. This alternative approach has already been used for describing the eigenstates statistics over the entire crossover regime from the orthogonal to unitary ensembles (low magnetic fields) [24]. More recently, this construction has been advanced by developing a reduced σ-model which is applicable to closed systems. The reduced σ-model has non-trivial saddle-point solutions which enabled us to consider the localization effects non-perturbatively [22].

The idea to work with the distribution function as a whole has also the additional advantage that it makes possible to select those rare states which do not fit to the universal statistics and study their spatial structure. The latter information is implicit in the cross-correlation function $R(t, r)$,

$$R(t, r) = \Delta \langle \sum_{\alpha} \delta \left( t - |\psi_{\alpha}(r_{o})|^{2} \right) |\psi_{\alpha}(r_{o} + r)|^{2} \delta(\epsilon - \epsilon_{o}) \rangle .$$

(5)

As it will be clear from the calculations below, rare pre-localized states show up as deviations of the function in Eq. (3) from universal distributions at the tails where $t \gg V^{-1}g$, so that the combination $R(t, r)/f(t)$ mimics the envelope $|\psi_{t}(r)|^{2}$ of these states at distances $r = |r| > l$ from the top-amplitude $(t)$ position.

**B. Metallic states and universal statistics**

To find the manifestation of the pre-localized states in the distribution function $f(t)$, we have, for comparison, to give an idea about what would be the form of the distribution function if all states were extended. Qualitatively, the extended states test the realization of a random potential equivalently all over the sample and that’s why their statistics coincides with the Porter-Thomas eigenstates statistics renown in the random matrix theories [24,25]. The recent studies of the properties of the eigenstate of disordered and ballistic chaotic cavities (both using the numerical tools [24,21] or based on the zero-dimensional limit (0D) of the supersymmetric non-linear σ-model [22,22,27]) have confirmed such an expectation.

Depending on the fundamental symmetry class, the Porter-Thomas distributions can be represented as following. For the single-particle Hamiltonian describing a spin-less particle in the system with a broken (e.g., by a magnetic field) time-reversal symmetry (unitary class), the distribution function of local amplitudes and the corresponding IPN’s have the form

$$f_{o}(t) = V \exp \{ - Vt \} , \quad t_{n} = nV^{-n},$$

(6)

whereas in the case of a system with the time-reversal symmetry (orthogonal ensemble),

$$f_{o}(t) = \sqrt{\frac{V}{2\pi t}} \exp \{- \frac{Vt}{2} \} , \quad t_{n} = (2n-1)! \sqrt{\frac{V}{2n}} .$$

(7)

For spin $\frac{1}{2}$ particles which undergo a strong spin-orbit interaction (symplectic ensemble) the Porter-Thomas-type of a distribution can be repeated both for the spin-projected wave functions and for the total density and has the form

$$f(\downarrow, t) = 2Ve^{-2Vt} ; \quad f_{s}(t) = 4V^{2}te^{-2Vt} .$$

(8)

**C. Localization effects and eigenstates statistics beyond the universal limit**

The universal statistics described by Eqs. (6,8) are presented only as a reference point for the subsequent analysis. The rare events which cannot be described on the basis of the statistical equivalence of eigenstates have to be studied using more sophisticated methods, and need the extension of the non-linear σ-model beyond the 0D limit. Details of these calculation are presented in Sections IV-VII, whereas in the forthcoming subsection we sketch only the basic results. In few words, the universal disorder-independent laws work well enough either until this disorder is so weak, that the system behaves as in the nearly ballistic regime, or at small amplitudes $t < V^{-1}\sqrt{g}$. But they are partly broken or, at least, modified after the disorder makes the electron motion diffusive. In one- and two- dimensional conductors, this requires a different statistical treatment of states which have too high splashes of a local amplitude, $t > V^{-1}\sqrt{g}$.

The method of taking into account all finite (i.e., not only small) inhomogeneous variations of the fields used in the supersymmetric field theory is based on the existence of saddle-point solutions of the non-linear σ-model discovered by Muzykantskii and Khmelnitskii [17]. The existence of the saddle-point solution is especially prominent for a reduced version of σ-model formulated and solved in Ref. [32]. An interesting result of Ref. [32] for the quantum diffusion in the dimension $d = 2$ is the multifractality of the states which is in agreement with previous numerical simulations [11,15]. Using the same saddle-point method as for the unitary (u) symmetry class [32], we extend the analysis of 2D systems to the other fundamental symmetry classes (orthogonal, o; symplectic, s; spin-unitary, su) and arrive, again, at the multifractality. The latter is manifested by the following scaling of INP’s,

$$Vt_{n} \propto L^{-(n-1)d^{*}} , \quad d^{*}(n) = 2 - \frac{\beta_{o} - 1}{\beta_{s} + \beta_{o}} .$$

(9)

The fractal (or generalized Rényi [32]) dimensions, $d^{*}(n)$, obey Eq. (9) only for those $n$’s where they are positive.
and are obtained in the leading order in $(2\pi\nu D)^{-1}$, where $D$ is the classical diffusion coefficient. The sensitivity of the derived statistics to boundary conditions, as well as the form of the correlation function $R(t, r)$ which we find in our calculations, enables us to suggest such a behavior of 2D multifractal states which is associated with the power-law envelopes of their tails, $|\psi(x_o + x)|^2 \propto (l/r)^{2\eta}$. Being extended from the position of a rare high amplitude splash $|\psi(r_o)|^2 = t \gg 1$, these tails have exponents $\mu(t)$ individual for each state marked by its own $t$.

The behavior of pre-localized states in a quasi-1D wire within the localization length scale $L_c$ also resembles the power-law localization. Independently of $\eta$, where

\[ \eta = ln \frac{L}{t} \]

and $L \approx \pi \nu D$, this results in integrable, so that no assertion about fractality can be made and the inverse participation numbers $t_n$ tend to take a volume-independent form for $n > g$.

The localization effects in 3D conductors are weak, if disorder is weak enough to keep the system far away from the metal-insulator transition, $p_F l \gg 1$. As a result, the statistics of eigenstates in 3D conductor is most similar to the universal one: Amplitude $t$ scales with $V$, and the mean free path $l$ appears only as an extra parameter both in the distribution function $f = f(V, t, p_F l)$, and inverse participation numbers $t_n \propto V^{-n g(n, p_F l)}$.

Nevertheless, even then, the statistics of rare events shows intriguing deviations from the Porter-Thomas formulae. Both for 2D and 3D diffusive samples we obtain the logarithmically-normal distribution of large local amplitudes of wave functions,

\[ f(t) = \exp \left\{ -\beta \frac{\pi^2 l^2}{\eta_3} \ln^2 T \right\}, \quad T = \frac{V t_n}{2\pi^2 \nu D}, \]

where $\eta_2 = ln \frac{L}{t}$ and $\eta_3 \sim (2l)^{-1}$. Although we study an isolated specimen, this result strikingly resembles the asymptotics of distributions of local density of states or conductance fluctuations found in open systems [3]. This signals about deep physical reasons behind it related, probably, to the properties of random walk paths.

### III. EIGENSTATES PROBLEM IN TERMS OF A NON-LINEAR SUPERSYMMETRIC $\sigma$-MODEL

In the following paragraphs, we formulate the eigenstates statistics problem in terms of the non-linear $\sigma$-model. The details of this technique are described in the review article [7], and below we give only a compressed extraction from it, keeping similar notations.

One can try to use the supersymmetry technique as soon as a physical quantity of interest is expressed in terms of retarded and advanced Green’s functions,

\[ G_{\epsilon}^{R,A}(r, r') = \sum_{\gamma} \frac{|\psi_{\epsilon}(r)|^2}{\epsilon - \epsilon_{\gamma} + i\gamma/2}. \]

In Eq. (11), $\gamma$ is a level broadening. In an isolated sample, one has to take the limit of $\gamma \to 0$. Due to the discreteness of the spectrum of levels $\{\epsilon_{\alpha}\}$, this extracts only $\epsilon_0$, the closest to the current energy slice $\epsilon$. Using the expression in Eq. (11) and taking the limit of $\gamma \to 0$, one can so formalize the statistics of Eq. (12) that

\[ f(t) = \Delta \lim_{\gamma \to 0} \left\{ \int \frac{d r'}{2\pi i} \left( G_{\epsilon}^{A}(r', r') - G_{\epsilon}^{R}(r', r') \right) \delta (t - i\gamma V G_{\epsilon}^{R}(r, r)) \right\}. \]

The reformulation of Eq. (12) in terms of the $\sigma$-model exploits the fact [22] that any product of Green functions,

\[ i^{n+1} \left[ G_{\epsilon}^{R}(r, r) \right]^n G_{\epsilon}^{A}(r', r') = \]

\[ = -\frac{1}{n!} \int |s_1(r')|^2 |s_2(r)|^{2n} e^{-L(\Psi)} D\Psi, \]

can be represented as a functional integral over the 8-component super-vector field $\Psi = (\Psi_1, \Psi_2)$, $\Psi_m = \frac{1}{\sqrt{2}} \left( \chi_m, \chi^*_m, s_m, s^*_m \right)$. The super-vector $\Psi$ is composed of 4 anti-commuting and 4 commuting variables $\chi$ and $s$, respectively. The indices $m = 1, 2$ appear in order to distinguish between advanced and retarded Green functions. Besides $\Psi$, the charge-conjugate field $\bar{\Psi}$ should be defined; one can find this definition in Ref. [7]. The action $L(\Psi)$,

\[ L(\Psi) = i \int \bar{\Psi}(r)[\epsilon - \hat{H}_0 - U(r) - i\frac{2}{\nu} \Lambda] \Psi(r) dr, \]

incorporates both the free-particle Hamiltonian $\hat{H}_0$ and the random impurity potential $U(r)$.

After Gaussian averaging over $U$, we derive a new Lagrangian with an interaction of the super-fields $\Psi$. The interaction term can be decoupled by the functional integration over a super-matrix field $Q$, so that any calculation is finally reduced to the evaluation of a functional integral,

\[ \int DQ \exp\left\{ -F(\{Q\}) \right\} W(\{Q\}), \]

over the slow-varying superfields $\{Q\}(r)$. This manipulation is analogous to the introduction of an effective order parameter in the theory of superconductivity. The free energy which determines weights of different configurations of $Q$ appears after integrating over fast modes and has the form

\[ F[\{Q\}] = \int dr \left[ -\frac{1}{2} \text{Str} \ln \left( -i\hat{H}_0 + \frac{2}{\nu} \Lambda + \frac{Q}{\nu} \right) + \frac{\pi^2}{8\nu} \text{Str} Q^2 \right]. \]

The 'anomalous mean' $Q$ can be found from the self-consistency condition
\[ \pi \nu Q = \int dp \left( -i\hat{H}_0 + \gamma \Lambda /2 + Q /2\tau \right)^{-1} \]  
which minimizes \( F[Q] \).

In the limit of \( \gamma \to 0 \), the solutions of Eq. (17) take the values

\[ Q = V \Delta V \]  

from the degeneracy space of one of the graded symmetry group. This field-theoretical model is strongly non-linear, since the matrix \( V \) satisfies the condition \( \nabla V = 1 \), and operations of the conjugation \( V \to \bar{V} \) and the supertrace (Str) in Eq. (16) are those defined in Ref. [7]. The ‘rotation’ \( V \),

\[ V = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix} \exp \left( \begin{pmatrix} 0 & -iu_2 \theta v_2 \\ -i\nu v_2^2 & 0 \end{pmatrix} \right) \]  

is parametrized with the equal number of commuting and anti-commuting variables. The second matrix in the product in Eq. (14) is composed only of commuting ones. All anti-commuting variables are collected into the matrices \( u_1 \) and \( v_1 \). Smooth spatial variations of the Q-field at the length scale longer than the mean free path \( l \) and the influence of a small but finite value of \( \gamma \) can be taken into account by the effective free energy functional,

\[ F[Q] = \frac{\pi \nu}{4} \int dr \text{Str} \left[ \frac{D}{2} \left( \nabla Q \right)^2 - \gamma \Lambda Q \right]. \]  

The extension of this equation to the case of spin-\( \frac{1}{2} \) particles can be found in Ref. [7], too.

To transform the formula in Eq. (2) into integrals over the \( Q \)-space, we expand the \( \delta \)-function in Eq. (12) into the formal series in \( G^R \), and study the averages for all

\[ \left\langle i^{n+1} G^A_n(r, r') \right\rangle = \int dr G^A_n(r, r') \]  

Using Eq. (13), each of them can be represented as a functional integral over the field \( \Psi \) and then modified into the construction

\[ \lim_{t, \omega \to 0} \int d\zeta_1 d\zeta_2 \frac{(-1)^n \pi!}{(2\pi)^2} \frac{1}{2(n!)} \partial_{\mu} \partial_{\lambda} \left\langle D \Psi e^{-L(\Psi)} \delta L(\Psi) \right\rangle. \]  

In the latter equation, we perform an additional integration over the phases \( \zeta_{1,2} \) (which are hidden into the vectors \( \bar{v}_1 = \sqrt{2}(0,0,e^{i\zeta_1}, e^{-i\zeta_1}, 0,0,0,0,0,0) \), \( \bar{v}_2 = \sqrt{2}(0,0,0,0,0,0,0,0,0,0) \) and add to the Lagrangian from Eq. (14) a weak perturbation \( \delta L \),

\[ \delta L = \int dr' \left[ \mu(\bar{v}_1 \Psi(r'))(\bar{\Psi}(r')v_1) \right. \]  
\[ + \lambda(\bar{v}_2 \Psi(r'))(\bar{\Psi}(r')v_2)\delta(r' - r) \]  

After this, we have to evaluate the functional \( \left\langle \int D\Psi e^{-L(\Psi)} \delta L(\Psi) \right\rangle \). Its exponent differs from that in Eqs. (17-20) only by the perturbation

\[ \delta \hat{H} = i \int dr' \left[ \mu_1 \bar{v}_1 + \lambda v_2 \bar{v}_2 \bar{v}_2 \delta(r' - r) \right] \]  

added to the Hamiltonian \( \hat{H}_0 \) in Eq. (16). The latter results in an additional term \( \delta \hat{F} \) in the free energy functional; we find \( \delta \hat{F} \) by expanding the logarithmical expression in Eq. (16) into the series in \( \mu \) and \( \lambda \). Doing that, we keep only the contributions up to the first order in \( \mu \), whereas ‘cross-terms’ which originate from pairing of \( \Psi \)'s at different coordinates (\( r \) and \( r' \)) can be neglected. As an intermediate step, we obtain

\[ \delta \hat{F} = \frac{\mu}{2} \pi \nu (\bar{v}_1 Q v_1) + \frac{1}{2} \ln \left[ 1 + \lambda \pi \nu (\bar{v}_2 Q v_2) \right], \]  

and after some algebra we arrive at

\[ f(t) = \int \frac{d\zeta}{\pi} \lim_{\tau \to 0} \int DQ \left[ \int \frac{d\zeta}{\pi} \text{Str}(Q(r)) \right] \]  

\[ \delta \left( t - \frac{\pi \gamma}{2} \text{Str}(Q(r_0)) \right) e^{-\hat{F}[Q]} \]  

where

\[ \pi_1 = \begin{pmatrix} \pi_0 \ 0 \\ 0 \ 0 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 0 \ 0 \\ 0 \ \pi_0 \end{pmatrix}, \quad \pi_0 = \begin{pmatrix} 0 \ 0 \\ 0 \ 1 \end{pmatrix} \otimes \tau_0, \]  

and

\[ \hat{Y}(\zeta) = \pi_2 e^{i \zeta \tau_3} (\tau_0 + \tau_1) e^{-i \zeta \tau_3}. \]  

Everywhere below, \( \tau_i \) are the Pauli matrices, and \( \tau_0 \) is \( 2 \times 2 \) unit matrix.

IV. REDUCED \( \sigma \)-MODEL AND ITS SADDLE-POINT SOLUTIONS

Basing on Eq. (21), we can obtain the full statistics of local amplitudes \( |\psi|^2 \) for any regime. As we mentioned before, the universal expressions of Eqs. (18-21) can be derived by assuming the zero-dimensional (0D) limit: the coordinate-dependent field \( Q(r) \) has to be replaced by its value at the observation point, \( Q(r_0) = Q_0 \), which transforms the functional integral in Eq. (21) into a definite integral over \( Q_0 \).

To go beyond the 0D approximation, one should take into account inhomogeneous fluctuations of the field \( Q \). If we integrate over \( Q_0 = V_0 \Delta V_0 \), any of functions in Eqs. (18-21) can be finally expressed in terms of relative rotations of the \( Q \)-field with respect to its value at \( r_0 \). This is the reduced \( \sigma \)-model. For its derivation, it is significant to note that the degeneracy space of the supermatrix \( Q \) is non-compact. Due to this property, the main contribution to the integral in Eq. (21) comes from the region of
large $Q_o$’s where $\text{Str}(\Upsilon Q_o) \propto 1/\gamma$. As a result, finite variations of $Q(r)$ produced by means of local rotations $Q(r) \rightarrow Q(r') = V(r,r')Q(r)V(r',r')$ of the supermatrix field along the non-compact 'direction' can be taken into account consistently, since they cover only relatively small environs of an 'infinitely large' $Q_o$.

Using the decomposition $V(r) = V_o \tilde{V}(r)$, we define supermatrices $\tilde{Q}$ of the reduced $\sigma$-model as

$$\tilde{Q} = \tilde{V} \Lambda \tilde{V}, \quad Q(r) = V_o \tilde{Q}(r) V_o, \quad \tilde{Q}(r_o) = \Lambda. \quad (22)$$

Due to the invariance of the $Q$-space, the transformation of Eq. (22) preserves the form of the gradient term in the free energy $F$ in Eq. (21), whereas the second term in $F$ can be modified as

$$F_2 = -\frac{\pi \nu \gamma}{4} \int dr \text{Str}((\tilde{Q},\tilde{Q})), \quad \tilde{Q}_o = V_o \Lambda V_o. \quad (23)$$

A corresponding substitution can be done in the pre-exponential in Eq. (21), too. The explicit form and exact parametrization of the matrix $Q_o$ varies from one symmetry ensemble to another. Nonetheless, in the limit of $\gamma \rightarrow 0$, those parameters of the $Q$-space which are responsible for its non-compactness appear in the argument of the $\delta$-function in Eq. (21) in the same combination with the factor $\gamma$ that enters to the 'potential' part of the free energy, $F_2$. Therefore, integrating over $Q_o$ in this limit, we eliminate $\gamma$ and convert Eq. (21) into expressions which relate the distribution function $f(t)$ to the generating functionals represented in terms of the fields $\tilde{Q}$.

The parametrization of $Q$-matrices, and, therefore, derivation and form of the reduced $\sigma$-model depend on the fundamental symmetry of the system. In parts A, B and C of this Section we specify this for unitary, orthogonal and symplectic symmetry classes separately, though it turns out that the most essential part of our calculation - the use of the saddle-point solution described in subsection IV.D - is quite similar for all of them.

### A. Unitary ensemble

In the unitary case, the parametrization of the $Q$-field using Eq. (11) includes 'angles’

$$\hat{\theta} = \begin{pmatrix} \theta \tau_0 & 0 \\ 0 & i\theta_1 \tau_0 \end{pmatrix}, \quad 0 < \theta < \pi, \quad 0 < \theta_1 < \infty,$$

where only one of them is imaginary and makes the symmetry group non-compact. Matrices $u_2$ and $v_2$ in Eq. (19) are diagonal and can be trivially eliminated from Eq. (21) as well as the external phases $\zeta$. Other details of the integration over $Q_o$ are the same as those in Ref. 20.

The distribution function $f$

$$f_u(t) = \frac{1}{V} v^2 \frac{d \Phi_u(t)}{dt^2}, \quad (23)$$

and the inverse participation numbers $t_n, n \geq 2$,

$$t_n = \int_0^\infty t^n f(t) dt = \frac{n(n-1)}{2} \int_0^\infty t^{n-2} \Phi_u(t) dt, \quad (24)$$

can be related to the generating functional $\Phi_u(t)$ of the reduced $\sigma$-model,

$$\Phi_u(t) = \int_\tilde{Q}(r_o) = \lambda D\tilde{Q}(r)e^{-F_s[t;\tilde{Q}].} \quad (25)$$

The free energy $F_u$ in Eq. (25) has the form

$$F_u[t, \tilde{Q}] = \int dr \text{Str}\left[ \frac{\pi \nu D}{2} (\nabla \tilde{Q})^2 - \frac{1}{4} \Lambda \Pi \tilde{Q}(r) \right], \quad (26)$$

and we remind that $\tilde{Q}(r_o) = \Lambda$ at the origin. The projection operator $\Pi$ in Eq. (26) is defined as

$$\Pi = \begin{pmatrix} \pi_b & \pi_b & \pi_b \\ \pi_b & \pi_b & 0 \\ \pi_b & 0 & 0 \end{pmatrix}, \quad \pi_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (27)$$

and selects from the $Q$-matrix only its non-compact sector.

The generating functional $\Phi_u(t)$ has several funny features. First, at $t = 0$, it has a completely invariant form, and, therefore, is equal to unity, what corresponds to the normalization of the wave functions,

$$V t_1 = \Phi_u(0) \equiv 1. \quad (28)$$

On the other hand, for any finite $t$, the reduced $\sigma$-model is a model with a broken symmetry, so that the free energy in Eq. (26) can be minimized by an inhomogeneous solution $\tilde{Q}(r)$. Indeed, $\frac{1}{4} \Lambda \Pi$ in the second term in Eq. (26) looks as a field tending to align the matrix $\tilde{Q}$ along the non-compact 'direction' of the $Q$-space (related to the parameter $\theta_1$), whereas the boundary condition at $r_o$, together with the gradient term is a rigidity attempting to prevent that. The competition between these two tendencies results in an optimal configuration of $\tilde{Q}$. To find such an optimal (saddle-point) solution, we use, again, the invariance of the $Q$-space with respect to rotations $\tilde{V}$ and represent $Q$ as

$$\tilde{Q}(r) = V_t(r) \Lambda^{\frac{1}{\nu} \int \Delta t} V_t(r), \quad P = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \quad (29)$$

where a weak perturbation $P$ stands for fluctuations around the saddle-point, and the matrices $B$, $B$ can be decomposed into blocks as follows,

$$B = \begin{pmatrix} s_1 \tau_0 + is_1 \tau_3 & \hat{\delta}_1 \\ \hat{\delta}_1^* & s_2 \tau_0 + is_2 \tau_3 \end{pmatrix}, \quad (30)$$

$$\hat{\delta}_1 = \begin{pmatrix} \sigma_{\alpha} & 0 & 0 \\ 0 & \sigma_{\alpha}^* & 0 \end{pmatrix}. \quad (31)$$

The form of the saddle-point $\tilde{Q} = V_t \Lambda \tilde{V}_t$, follows from the requirement of the absence of linear terms in the expansion $F_u$ into the series
in the perturbation $P$. This selects
\[ V_i = \exp \left( \frac{1}{2} \theta \epsilon e^{-i\chi \tau_3} + \frac{1}{2} \theta e^{i\chi \tau_3} \right), \]  \hspace{1cm} (32)
where the parameter $\theta_i(r)$ satisfies the optimum equation
\[ \Delta \theta_i(r) = -\frac{t}{\pi \tau^D} e^{-\theta}; \quad \chi = \pi \]  \hspace{1cm} (33)
with the boundary conditions $\theta_i(r_o) = 0$ in the origin and $(\mathbf{n} \nabla) \theta_i = 0$ at the surface of a sample. In Eq. (33) $\Delta$ stands for the Laplacian in the real space. This equation is partly similar to the saddle-point equation derived by Muzykantskii and Khmelnitskii \[17\] when studying the problem of long-living current relaxation in open conductors, but it has a different non-linearity and - what is more important - different boundary conditions.

**B. Orthogonal ensemble**

The parametrization of $Q$-matrices in the orthogonal case is more complicated due to a larger number of independent parameters in it. In particular, the non-compact sector of the degeneracy space is parametrized by two 'imaginary angles' - variables $\theta_{1,2}$:
\[ \hat{\theta} = \begin{pmatrix} \theta \tau_0 & 0 \\ 0 & i(\theta \tau_0 + \theta_2 \tau_1) \end{pmatrix}, \quad 0 < \theta < \pi, \quad 0 < \theta_{1,2} < \infty. \]

Unitary matrices $u_2$ and $v_2$ in Eq. (19) have a more complicated form, too,
\[ u_2 = \begin{pmatrix} M & 0 \\ 0 & e^{i\phi \tau_3} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \tau_0 & 0 \\ 0 & e^{i\chi \tau_3} \end{pmatrix}, \quad M = \frac{1 - im\tau^\sigma}{1 + im\tau^\sigma}. \]

where $0 \leq \phi, \chi < 2\pi$, $m_{1,2,3}$ are real, and the number of anti-commuting variables in $u_1$ and $v_1$ is twice as large as in the unitary case.

The integration over $Q_o$ can be performed in a complete analogy with the unitary case, but with several distinguishing features. First of all, in the limit of $\gamma \to 0$ the main contribution comes from the region of the $(\theta_1, \theta_2)$-plane where $\cosh \theta_1 \cosh \theta_2 \sim 1/\gamma$. Since the product $\cosh \theta_1 \cosh \theta_2$ can be large at large $\theta_1$ as well as large $\theta_2$, we end up with the new form of projection operators,
\[ \Pi_o = \begin{pmatrix} \pi_o(\theta) & 0 \\ 0 & \pi_o(\theta) \end{pmatrix}, \quad \pi_o(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_0 + \tau_1 \end{pmatrix}, \]
in the free energy
\[ F_o[t', \tilde{Q}] = F_t + F^{(2)} + F^{(3)} + F^{(4)} + \ldots \]  \hspace{1cm} (31)
Next, in the orthogonal ensemble, one has to keep the external phase factor $e^{i\xi}$ in Eq. (21) until the end of the integration over $Q_o$, which results in the integro-differential relation
\[ f_o(t) = \frac{4}{V\pi \sqrt{t}} \int_{t_1} dz \Phi_o(t + z^2). \]  \hspace{1cm} (35)
between the distribution function $f_o$ and generating functional
\[ \Phi_o(t') = \int_{Q(r_o) = \Lambda} DQ \exp(-F_o[t', \tilde{Q}]). \]  \hspace{1cm} (36)
The generating functional $\Phi_o(t')$ gives directly the inverse participation numbers $t_n$, $n \geq 2$,
\[ t_n = \frac{2}{\sqrt{\pi V}} \int_{t_1}^{t_2} (t')^{(n-2)\Phi_o(t')}dt', \]  \hspace{1cm} (37)
and, again, has the property $\Phi_o(0) = V \tau_1 \equiv 1$.

To study the fluctuations near the saddle-point, we represent $\tilde{Q}$ in the form of Eq. (24), where
\[ B = \frac{B_+ + B_-}{\sqrt{2}}, \]
\[ B_\pm = \begin{pmatrix} \frac{(s_{11}^+ \tau_0 + is_{12}^+ \tau_3)\tau_\pm}{\sigma_{1\pm}^+} & \frac{\hat{\sigma}_{1\pm}^+}{\sigma_{1\pm}^+} \\ \pm \frac{\sigma_{1\pm}^+}{\sigma_{1\pm}^+} & \frac{-(\sigma_{1\pm}^+)^*}{\sigma_{1\pm}^+} \end{pmatrix}, \quad \tau_\pm = \tau_0 \mp \tau_1. \]

In this decomposition, $s_{\alpha\beta}^\pm$ are real numbers, $\sigma_{\alpha\pm}^\pm$ and $(\sigma_{\alpha\pm}^\pm)^*$ - anti-commuting variables. Indices '±' are introduced for the later convenience. Everywhere below, we keep superscript '-' but omit '+'.

Similarly to the unitary case, the free energy $F_o$ which governs the generating functional $\Phi_o$ has the minimum at $\tilde{Q} = V_i \Lambda \tilde{V}_i$,
\[ V_i = \exp \left( \frac{i\phi \tau_3}{8} \sum_{\pm} \theta^\pm \tau_\pm e^{-i\chi \tau_3} \right) \]  \hspace{1cm} (38)
where the variables $\theta_i \equiv \theta_1 + \theta_2$, $\theta_i^- \equiv \theta_1 - \theta_2$ and $\chi, \phi$ satisfy the following equations:
\[ \Delta \theta_i(r) = -\frac{i'}{\pi \tau^D} e^{-\theta_i}; \quad \Delta \theta_i^-(r) = 0, e^{i(\chi \pm \phi)} = -1. \]

The conditions at the origin and boundary are the same, as in the unitary case. The latter gives $\theta_i^- (r) = 0$, and the non-trivial saddle-point is related only to the symmetric variable $\theta_i$. 

7
C. Symplectic ensemble

An analogous investigation of the statistics of spin-polarized electron waves in the case of a strong spin-flip scattering needs an extension of dimensions of $Q$-matrices and the following analysis of the degeneracy space related to their gapless Goldstone modes. The gaps in the spectrum of $Q$’s appear due to a large spin-relaxation rate, $\tau_a^{-1}$, which can be caused both by the spin-orbit coupling built into the material properties or by the spin-flip scattering on a classical randomly oriented static magnetic field. In the former case, the time-reversal symmetry is conserved, while in the latter this invariance is violated by the source of a scattering. Since triplet components of $Q$ correspond to gapfull modes [5,7], only singlet modes have to be taken into account, so that the number of independent variables of $Q$ is the same as in the spinless case. One has to remember only that all the matrix elements of $Q$ are multiplied by the unit $2 \times 2$ spin-matrices.

In this subsection, we work with the distribution $f(\downarrow, t)$ of a local spin density $t \equiv |\psi_i|^2$ of the eigenstates defined by Eq. (6) and above it. An incorporation of spins into Eq. (21) can be done by substituting

$$\pi_{1,2} \to \pi_{1,2} \otimes \tilde{\tau}_0, \quad \gamma \to \gamma \otimes \tilde{\tau}_4,$$

where ‘$\otimes$’ stands for the direct product of matrices, and $\tilde{\tau}_i$ are the Pauli spin operators: $\tilde{\tau}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{\tau}_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The degeneracy space of $Q_o$ is non-compact along a single direction, and the integration over $Q_o$ gives us

$$f(\downarrow, t) = \frac{1}{2\pi} \frac{d^2 \Phi_s(t)}{dt^2}, \quad \Phi_s(t) = \int DQ(t)e^{-F_s(t, Q)},$$

where

$$F_s(t, Q) = \int dr \text{Str} \left[ \frac{\pi \nu D}{2} (\nabla Q)^2 - \frac{1}{2} \Lambda \Pi Q(r) \right],$$

and $\Pi$ is exactly the same as in Eq. (27).

From the point of view of the rest of calculations, the case with a broken time-reversal invariance is equivalent to the spinless unitary symmetry class $\tilde{6}$. That’s why we mark the quantities related to this symmetry with a label ‘$su$’ and generate the distribution function $f_{su}(\downarrow, t)$ form the distribution function $f_s(t)$ in Eqs. (23, 24) as

$$f_{su}(\downarrow, t) = 2f_s(2t) at D \to 2D.$$

As concerns the symplectic ensemble with the time-reversal symmetry, it demands an extra calculation, since in the presence of the spin-orbit interactions it cannot be reduced to the spinless orthogonal one. The parametrization of the $Q$-space in this case is given by Eqs. (49) with

$$\hat{\theta} = \begin{pmatrix} \theta t_0 + \theta' t_1 & 0 \\ 0 & i\theta_1 t_0 \end{pmatrix}; \quad 0 < \theta, \theta' < \pi, \quad 0 < \theta_1 < \infty,$

and

$$u_2 = \begin{pmatrix} e^{i\chi t_0} & 0 \\ 0 & M \end{pmatrix}, \quad u_2 = \begin{pmatrix} e^{i\chi t_0} & 0 \\ 0 & \tau_0 \end{pmatrix}, \quad M = \frac{1 - i\tilde{m}\tilde{\pi}}{1 + i\tilde{m}\tilde{\pi}}$$

where $0 \leq \phi, \chi < 2\pi$, and $m_{1,2,3}$ are real variables. After this, the saddle-point configuration of $Q$ for the symplectic-orthogonal case can be found as $Q = V_t \Lambda V_t$,

$$V_t = \exp \left( \frac{\theta_1}{2} M + \frac{\theta_0}{2} \right),$$

where $\theta_i$ satisfies the optimum equation and the matrix $M$ is chosen in such a way that $M = -\tau_0$,

$$\Delta \theta_i(\mathbf{r}) = -\hat{\tau} \int \pi \nu D e^{-\theta_\tau}, |m| \to \infty, \quad (41)$$

with the boundary conditions $\theta_i(\mathbf{r}) = 0$ and $(\mathbf{n} \nabla) \theta_i = 0$ at the surface.

D. Optimal free energy and fluctuations near the saddle-point

After comparing the saddle-point equations in Eqs. (32, 33, 34), one finds that they are similar in different symmetry classes. The difference between the unitary, orthogonal and symplectic ensembles leads only to different values of a coefficient $\beta$,

$$\beta_o = \frac{1}{2}, \quad \beta_u = 1, \quad \beta_{su} = \beta_s = 2, \quad (42)$$

in the expression for the optimal free energy

$$F_t = \beta \int dr \left\{ \pi \nu D (\nabla \theta)^2 + te^{-\theta} \right\}, \quad (43)$$

and in the higher order terms of the expansion of $F[t, \hat{Q}]$ in the environs of the saddle-point. The generating functional $\Phi(t)$ from Eqs. (27, 36, 39) can be represented in the form

$$\Phi(t) = J(t) \exp(-F_t).$$

In the conducting regime, the value of the optimal free energy determines the leading term in the exponential of the generating functional, whereas the effect of fluctuations around the saddle-point is included into the function $J(t)$,

$$J(t) = \int DP \exp(-F^{(2)} - F^{(3)} - F^{(4)} \ldots).$$

Due to the normalization condition in Eq. (28), the relation $J(0) = 1$ holds exactly, and the contribution from the fluctuations $P$ can be calculated by expanding the exponential in the integrand in Eq. (14) into the series in the higher-order terms $F^{(3,4,\ldots)}$ and performing Gaussian integrations over with the weight $\exp\{-F^{(2)}\}$ determined by the second-order correction to the free energy.

The applicability of such a perturbation theory is justified by the fact that the higher orders are, at least, by the
factor of $(2\pi \nu D)^{-1} \ll 1$ smaller, as compared to what is given by

$$J(t) \approx \int D\mathcal{P} \exp\{-F^{(2)}[t, \mathcal{P}]\}. \quad (45)$$

The latter is nothing but the super-determinant of the Hamiltonian related to the fluctuations around the saddle-point. The value of $J(t)$ differs from unity merely because the symmetry between fermionic and bosonic degrees of freedom is broken by the optimal solution. Since not all the projections of the infinitesimal degrees of freedom is broken by the optimal solution. Since $J(t)$ differs from unity merely because the symmetry between fermionic and bosonic degrees of freedom is broken by the optimal solution. Since not all the projections of the infinitesimal $P$ to the generators of the Lie algebra of the graded symmetry group are equivalently affected by the symmetry breaking, it is convenient to separate in $F^{(2)}$ the terms which feel the existence of the optimal solution from those which do not. Depending on the physical symmetry class, this involves different sets of variables. Nevertheless, after an appropriate diagonalization, quadratic form $F^{(2)}$ can be represented uniquely for all symmetry classes:

$$F^{(2)} = F_t^{(2)} + \zeta F_0^{(2)}, \quad \left\{ \begin{array}{l}
s_{\alpha, su} = 0 \\ s_{\alpha, s} = 1 \end{array} \right. \quad (46)$$

The term $F_0^{(2)}$ in Eq. (46) is composed of fields which are not affected by the symmetry breaking,

$$F_0^{(2)} = 2\pi \nu D \sum_{\alpha=1,2} \int dr \left\{ \delta \sigma_\alpha \delta (\sigma_\alpha^*) + \delta s_\alpha \delta (s_\alpha^*) \right\}.$$

This term does not contribute to the function $J$ in the Gaussian approximation, due to the symmetry between boson and fermion degrees of freedom incorporated in it. On the contrary, the first term in Eq.(46) is the sum over those four pairs of dynamical variables which feel the violation of the boson-fermion symmetry

$$F_t^{(2)} = 2\pi \nu D \int dr \left\{ \sum_{\alpha=1,2} [\partial \sigma_\alpha \partial \sigma_\alpha^* + U_\sigma \sigma_\alpha \sigma_\alpha^*] ight.$$  

$$+ \sum_{\beta, \alpha=1,2} (\partial s_{\alpha, \beta})^2 + U_s s_{\alpha, \beta}^2 \right\}.$$

Due to that, the pre-exponential $J$ can be represented as

$$J = \exp \left\{ \frac{1}{2} \sum_{\alpha} \ln \left( (\chi_\sigma(n))^4 / \prod_{\alpha, \beta} \chi_\alpha^\beta(n) \right) \right\}, \quad (47)$$

where the sum is extended over all the eigenvalues of the spectral problem

$$[-\Delta + U - \chi] \phi = 0, \quad \phi(r_{\sigma}) = 0, \quad n \nabla \phi(S) = 0. \quad (48)$$

As we already mentioned above, in the quadratic approximation, any difference of $J$ from unity is due to the broken symmetry between fermionic and bosonic degrees of freedom in $P$. The broken symmetry in the Hamiltonian $F_t^{(2)}$ is the consequence of the difference between the effective potentials

$$U_\sigma = \frac{1}{4} (\nabla \theta_t)^2 + \frac{l}{2\pi \nu D} e^{-\theta_t},$$

$$U_s^{\alpha \beta} = \frac{k_{\alpha \beta}}{4} (\nabla \theta_t)^2 + \frac{\nu_{a, a}}{2\pi \nu D} e^{-\theta_t}, \quad (49)$$

in the Hamiltonian $F_t^{(2)}$. In Eq. (49), $k_{\alpha,1} = q_{\alpha,1} = 0$ and $k_{1,2} = 4, k_{2,2} = 0, q_{\alpha,2} = 2$. The spectra $\{\chi_\sigma(n)\}$ and $\{\chi_\alpha^\beta(n)\}$ of modes remain gapful, since all $U > 0$. Moreover, due to the sum rule

$$\sum_{\alpha, \beta} U_s^{\alpha \beta} = 4 \sum U_\sigma, \quad (50)$$

their main contribution to $J$ comes from low-lying eigenvalues of Eq. (18). Since the set of $U$’s in Eq. (49) depends on the optimal solution alone, the calculation of the correction to the exponent in this order in $(2\pi \nu D)^{-1}$ can be performed simultaneously for all symmetry classes and is small.

On the other hand, the effect of fluctuations can become important once we want to extend the consideration of samples with the size larger than the localization length. Such a step which is left beyond of the scope of this paper would need some kind of a renormalization of the saddle-point. We would only like to stress that this can be a way to avoid the previously found difficulty [26] [28] to obtain a stable fixed-point in the renormalization group treatment of the localization problem.

The existence of the saddle-point and relatively small contribution from fluctuations in the metallic regime makes it easy to find the form of the cross-correlation function $R(t, r)$ from Eq. (6), too. If we study the envelope of the wave function at large enough distances $r \gg l$ from the position of a high amplitude splash, the reasoning used above can be repeated for $R(t, r)$ with minor modifications, and we arrive at

$$R(t, r)/f(t) \propto e^{-\theta_t(r)}, \quad (51)$$

so that one can say that the envelope of $|\psi(r)|^2$ follows the form of the saddle-point configuration of the reduced $\sigma$-model.

Up to now, we performed our analysis without referring to any specific dimensionality of the system. On the other hand, from Eqs. (13, 23, 24, 18), one can see that the saddle-point solution, and, therefore, the optimal free energy $F_t$ crucially depend on the dimensionality. In the next Sections, we discuss the statistics of local amplitudes of wave functions in 1D, 3D and 2D conductors separately.

V. NEARLY LOCALIZED STATES IN A SHORT DISORDERED WIRE

It is well known that the localization effects are the strongest in one- and quasi-one-dimensional (1D) conductors [33]. Even when disorder is weak, the quantum diffusion of a particle is blocked at the length scale $L_c = \beta 2\pi \nu D$, where $D$ is the classical diffusion coefficient determined by the elastic impurity scattering [3] and $\beta$
is specified by Eq. (14). Since in the quasi-1D wire the effective density of states \( \nu \) is that integrated over cross-sectional width \( w \) or an area, \( w^2 \), the localization length \( L_c \sim l(w/\lambda_F)^d-1 \) can be much longer than the mean free path \( l \). This allows us to consider the short wires \( L < L_c \) with an already developed diffusive regime, and below we analyze the distribution of amplitudes and the shape of unstable states which are the precursors of localization at larger distances. Since there is a lot of known about the 1D case [39, 40], its example can be a good point to compare the results of the saddle-point approach with calculations based on the use of the exact transfer-matrix method [1]. In fact, in the 1D case, the saddle-point calculation is nothing but the 'semiclassical' solution of the effective Schrödinger equation on the Q-space which appears in the transfer-matrix method. Such a 'semiclassics' not only gives the results which are very close to the exact solutions [40], but also enables us to make a statement about the form of typical pre-localized states in the metallic regime.

In the following paragraphs, we apply the scheme of calculus described in the previous section. First of all, we have to solve the saddle-point equation,

\[
\partial_x^2 \theta_i(x) = -\frac{i}{\pi \nu D} \exp(-\theta_i),
\]

and use its solution \( \theta_i(x) \) when calculating the optimal free energy \( F_i \). Due to the condition \( \theta_i(x_o) = 0 \) at the observation point \( x_o \), the latter splits the wire with the length \( L \) into two (not necessarily equal) intervals \( 0 < x < L_i = L-R \). The form of \( \theta_i(x) \) can be found separately in each of them. In dimension one, the differential equation in Eq. (52) can be solved exactly [12], and we represent its general solution in the form

\[
e^{-\theta_i} = \left[ \cos\left(\frac{A_i \sqrt{T_i}}{1-x} \right) \right]^2, \quad x > 0; \quad i = L, R.
\]

Although one can notice that the general form of this solution formally contains a singularity at \( x_\infty = -L_i[\pi/(2A_i \sqrt{T_i}) - 1] \), the latter is illusory since it takes place in the non-physical region \( x < 0 \) and plays no role unless it comes up to the formulation of the limitations to our theory. The requirement \( x_\infty \gg l \) which emerges from the existence of the singular point is related to the conditions on maximal values of gradients permitted by the use of only the lowest-order gradient expansion terms in the free energy functional in Eq. (21). We shall discuss the consequences of this condition at the end of the Section, assuming for a while that it is satisfied. If so, the consistency equations on the parameters \( A_i \) come from the boundary conditions \( \partial_x \theta_i(L_i) = 0 \) at the edges and have the form

\[
A_i = \cos\left( A_i \sqrt{T_i} \right),
\]

where \( T_i \) are defined as

\[
T_i = \frac{\mu^2}{2\pi \nu D}.
\]

The optimal value of the free energy can be found, in its turn, as

\[
F_i = \beta \sum_{i=L, R} \int_{x=0}^{x_L} \left[ 2\sqrt{1-A_i^2 - A_i^2 \sqrt{T_i}} \right].
\]

In general, the exact form of \( F_i \) in Eq. (56) based on the closed set of equations in Eqs. (53-55) can be studied numerically at any values of the parameters included, but a somewhat simpler analytical expression can be written in the asymptotical regions. First of all, we examine the limit of small amplitudes, \( T_i < 1 \), where the exact distribution has to match with the random matrix theory results. At \( T_i < 1 \), the results of Eqs. (54-56) can be expanded into the series in \( T_i \), which gives \( A_i \approx 1 - \frac{1}{2} T_i + \frac{1}{4} T_i^2 + \ldots \) and

\[
F_i \approx V t \left( 1 - \sum_{i=L, R} \frac{T_i L_i}{3 L} + \ldots \right), \quad f^{(1)}(t) \approx V \mu^2, \approx \frac{1}{2} T_i + \frac{1}{4} T_i^2 + \ldots \]

We see that in this limit the Porter-Thomas formulae, Eq. (28), give a good approximation for the body of the distribution function \( f(t) \) which describes those amplitudes \( t \) which are \( t < L^{-1} \sqrt{L_c/L} \). Otherwise, the second term of this expansion, \( V T_i \), becomes larger than unity and strongly affects the probability to detect a too high splash of the wave function.

When \( T_i \gg 1 \), the optimal configuration \( e^{-\theta_i} \) develops at the length scale of \( \lambda = \sqrt{2\pi \nu D/T_i} \), where it can be approximated as

\[
e^{-\theta_i(x)} \sim (\lambda/x)^2,
\]

and gets less sensitive to the presence of boundaries. Indeed, in the limit of \( T_i \gg 1 \), one has \( A_i \approx \frac{\pi}{2} T_i^{-1/2} (1 - T_i^{-1/2} + \ldots) \), and the exact expression for the optimal free energy can be expanded into the series in \( T_i^{-1/2} \),

\[
F_i = 4 \sqrt{\beta L_c t} \left( 1 - \delta_L - \delta_R \right),
\]

where

\[
\delta_i \approx \frac{\pi^2}{8} [T_i^{-1/2} - \frac{1}{2} T_i^{-1} + \ldots], \quad i = L, R.
\]

The leading term in Eq. (58) does not depend on the system length and (in the orthogonal ensemble) coincides with the asymptotical behavior of the distribution function of local amplitudes in infinite wires. The latter has been found by Mirlin and Fyodorov [11] on the basis of the analysis of the transfer-matrix equations derived by Efetov and Larkin [10]. Although we did our calculations for the short-length samples, \( L < L_c = \beta \pi \nu D \), the results for the high amplitude splashes surprisingly agree with those for the infinite geometry, even up to the leading term of pre-exponential factor \( J \). The latter is derived in Appendix A by taking into account fluctuations near the saddle-point. Roughly speaking, the 'semiclassical'
solution of the transfer-matrix equation gives an almost exact result. The full form of the tails of $f(t)$ at $t > g/L$, where $g = L_c/L$, can be represented as

$$f^{(1)}(t) \approx C \sqrt{\frac{\pi}{2L}} \sqrt{\frac{L}{t}} \exp \left[-4\sqrt{\frac{2L}{t}} \left\{1 - \delta_L - \delta_R\right\}\right]$$

and is applicable up to the amplitudes $t \sim L_c/|t| \sim w/(l \lambda_F)$. The latter condition emerges from the requirement of a smoothness of the saddle-point solution, so that its characteristic length scale $\lambda$ should be longer than the mean free path, $\lambda > l$. Otherwise, the singularity of the equation in Eq. (54) comes too close to the physical part of the space ($x > 0$) which would create too large gradients forbidden within the framework of the used approximations.

The distribution function given by Eq. (59) evidences that the states which are responsible for the rare event we discussed in the previous paragraph are (at least, partly) localized. Nevertheless, even for the largest amplitudes $t > L_c/L^2$, the effect of edges is still present, which means that this in not an exponential localization. On the basis of an analysis of the cross-correlation function $R(t, r)$ from Eqs. (3) and (21), we can say, within the range of distances $x < L_c$, the envelope of the density of pre-localized states $\psi_t(x)$ resembles the form of the optimal solution and has the power-law intermediate asymptotics

$$|\psi_t(x)|^2 \propto t e^{-\theta_t} \sim L_c/x^2.$$  

In contrast to the 2D case which we discuss in Section VII, the derived form of a typical wave function has the same exponent for all pre-localized states, independently of the amplitude of their top splashes. Further, the tails of the envelope in Eq. (60) are integrable, so that the inverse participation numbers which one can find on the basis of Eqs. (3) and (24) do not indicate any fractal behavior.

VI. EIGENSTATES STATISTICS IN $D = 3$

The localization effects in weakly disordered 3D conductors are known to be the least pronounced $\Psi$, so that the eigenstates statistics of 3D conductors has to be most similar to the universal one. Nevertheless, even in this case, not all states are described by the Porter-Thomas distribution, and this Section is devoted to the disorder-dependent corrections to formulae in Eqs. (1) in dimension three.

To describe the statistics beyond the universality limit, we have to solve optimum equations (23, 25, 31). For the sake of simplicity, we consider a spherically shaped conducting particle and place the observation point into its center. This enables us to seek for the solutions $\theta_t(r)$ in a symmetric form. Nonetheless, even that does not help us to find the exact form (43) of a general solution of the non-linear equation

$$\Delta \theta_t(r) = \left[r^{-2} \partial_r r^2 \partial_r\right] \theta_t = -\frac{4 \pi \nu}{\lambda_F} e^{-\theta_t},$$

so that we have to develop the following approximate procedure. The non-linear $\sigma$-model, Eq. (23), was derived under the conjecture of smoothly varying $Q$-fields. This implies that the distances shorter than the mean free path $l$ are excluded from our consideration, and the condition $\theta_t(0) = 0$ at the origin has to be substituted by the condition $\theta_t(r_0) = 0$ at the sphere of a radius $r_0 \sim l$. After this, we scale the distances by $l$, so that $\theta_t = \theta_t(r/l)$, and solve the problem iteratively. The iterative procedure appeals to the fact that the Laplace equation which one can get by neglecting the right hand side of Eq. (61) has non-zero solutions and that the parameter

$$\rho = \sqrt{\frac{\pi \nu D}{(l^2 t)}}$$

which appears after rescaling the distances $r$ with the mean free path is large. The latter condition restricts our consideration to the amplitudes $t < (l \lambda_F^2)^{-1}$ which are smaller than the density formed by the forward-and-backward scattered trajectory between two impurities.

As the first step, we expand the desired function $\theta_t(r)$ as

$$\theta_t \approx \theta_t^{(0)} + \theta_t^{(1)}, \quad \theta_t^{(1)} = A(1 - l/r).$$

The first term in it satisfies the Laplace equation but does not satisfy the necessary boundary condition at $r = L$. The term $\theta_t^{(1)}$ is added in order to satisfy the requirement $\partial_r \theta_t = 0$ at the external edge. It must turn to zero at $r = l$ and can be found from the linearized equation

$$\left[u^{-2} \partial_u u^2 \partial_u\right] \theta_t^{(1)} = -\rho^{-2} \exp(-\theta_t^{(0)}), \quad \theta_t^{(1)}(1) = 0,$$

where $u = r/l$. After this, the non-linearity of Eq. (61) transforms into a self-consistent determination of the parameter $A$ from the algebraic equation

$$A = \rho^{-2} \int_1^{L/l} u^2 du \exp(-A(1-u^{-1})) \approx \frac{1}{3 \rho^2} \left(\frac{L}{l}\right)^3 e^{-A},$$

which arises from the requirement $\partial_u \theta_t^{(1)}(L/l) = 0$. One could continue the iterative scheme even further and add corrections which improve the function $\theta_t^{(1)}(x)$ itself, and so on, but this is not necessary for evaluating the leading terms of the optimal free energy $F_t$. So, we stop the iteration after the first step and find that

$$e^{-\theta_t(r)} = \exp(-A(1 - l/r)), \quad r > l.$$  

The combination of the parameters which stands in the right-hand side of Eq. (61) and the self-consistency equation itself can be rewritten in the form

$$Ae^A = T \equiv \frac{V_{te}}{\pi \nu D} \sim Vt/(pl)^2,$$

where $\eta_t \sim (2l)^{-1}$ and the condition $pl \gg 1$ corresponds to the limit of a weak disorder. The optimal free
energy related to the saddle-point configuration can be calculated, too, and has the form

\[ F_t \approx \beta 2 \pi^2 \nu D \frac{A}{\eta_3} \left\{ 1 + \frac{1}{2} \right\} . \]  

(64)

When \( T \ll 1 \), the calculation both of the self-consistent \( A \) and the related value of \( F_t \) can be performed as an expansion into a series in the parameter \( T \), i.e., we approximate \( A \approx T - T^2 + \ldots \) and

\[ F_t = \beta VT \left( 1 - \frac{VT^3}{4 \pi^2 \nu D} + \ldots \right), \quad \eta_3 \approx (2l)^{-1} \times . \]

When \( T \gg 1 \), the leading terms arise from the estimation \( A \approx \ln T \). In each of these two regimes, the generating functional \( \Phi(t) \) has the form

\[ \Phi^{(3)} \approx \begin{cases} \exp \left( -\beta VT + \beta \frac{(VT)^3}{4 \pi^2 \nu D} + \ldots \right), & T < 1, \\ \exp \left( -\beta \nu D \ln^2 T + \ldots \right), & T > 1, \end{cases} \]

(65)

which can be used for evaluating both the distribution function \( f(t) \) and IPN’s using Eqs. (23, 35, 39) and (24, 37). At this point, we have to mention that the coefficient \( \eta_3 \approx (2l)^{-1} \) cannot be determined better than by the order of magnitude. We also remind that different symmetry classes are distinguished by the parameter \( \beta \): \( \beta_u = 1, \beta_o = \frac{1}{2}, \beta_s = 2 \).

Eq. (63) indicates that the noticeable deviations from the universal Porter-Thomas distribution start rising at local densities \( IV \sim p_{FL} \) (the second term in the exponent in Eq. (63) becomes larger than unity) and then develop into the logarithmically-normal asymptotics at \( tV > (p_{FL})^2 \). On one hand the states which generate such an asymptotical tail are not typically metallic. On the other hand, both the form of the envelope,

\[ |\psi_t(r)|^2 \propto \exp\{ -A(1 - 1/r) \}, \]

(66)

which we extract from the shape of the optimal solution in Eq. (62) and the scaling of IPN’s with the integer power of the system volume for any \( n \),

\[ t_n \approx \min \{ \varphi(n), \frac{2 \pi^2 \nu D / \eta_3^n}{\nu} \} \exp \left( \frac{n^2 \eta_3}{\beta 4 \pi^2 \nu D} \right), \]

\[ \varphi_u = n!, \quad \varphi_o = (2n - 1)!!, \quad \varphi_s = n! / 2^n, \]

indicate that these 3D states are not localized in a conventional sense: They always have a finite part of the density ‘equally’ smeared all over the sample. Of course, these extended density-tails decrease when \( t \) approaches the limiting value \( t \sim (\lambda_{FL}^2)^{-1} \), but our methods do not allow us to make a statement about the structure of standing waves at the scale of \( r < l \).

The version of the \( \sigma \)-model we used above also restricts our consideration to the metallic regime \( p_{FL} \approx 1 \). The development of a theory at critical conditions, \( p_{FL} \leq 1 \), requires the use of more sophisticated methods \[ 13 \]. Nevertheless, the common believe which arises from the most of known localization theories \[ 3 \] is that the dimension \( d = 2 \) is critical, so that the analysis of wave function statistics in 2D disordered conductors would manifest the important features of the criticality.

**VII. MULTIFRACTALITY OF EIGENSTATES IN WEAKLY DISORDERED 2D CONDUCTORS**

To find an optimal configuration in the 2D case we limit the length scale of the \( Q \)-field variations from below by the value of a mean free path - similarly to what we discussed in the previous Section. This modifies the boundary conditions into \( \theta_1(r_o) = 0 \) at \( r_o \approx l \). For the sake of simplicity, we consider the sample in the form of a disk (with the radius \( L \)) and place the observation point \( r_o \) right in its center. Then, we seek for an axially symmetric solution \( \theta_t(r) \) of the equation

\[ (r^{-1} \partial_r r \partial_r) \theta_t = -\frac{t}{\pi \nu D} e^{-\theta_t}. \]

### A. Exact solution

This can be done both exactly \[ 42 \] or using an iterative procedure developed in Section VI. The exact solution can be represented in the form

\[ e^{-\theta_t} = \left[ \frac{2(t/r)^{1-A} \left( \frac{(r/r_0)^2 + 1}{1} \right) \left( \frac{(r/r_0)^2 + 1}{1} \right)^2}{\left( \frac{(r/r_0)^2 + 1}{1} \right)^2 - \left( \frac{(r/r_0)^2 + 1}{1} \right)^2} \right]^2 , \]

(67)

where \( \rho = \sqrt{2 \pi D / \nu} \), and \( A \) has to be found from the boundary condition at the sample edge \( r = L \),

\[ \sqrt{A^2 + \rho^2} + A = \frac{(L/\nu)^A}{\rho} \sqrt{1 + A} \]

(68)

After substituting the saddle-point solution \( \theta_t(r) \) from Eqs. (67) to Eq. (43), we also find the optimal free energy,

\[ F_t = \beta 4 \pi^2 \nu D \left\{ \ln \left( \frac{(L/\nu)^{(1+A)}}{\rho^2 (1 - A^2)} \right) + 2(1 - \sqrt{A^2 + \rho^2}) \right\} . \]

Together with Eq. (48), the latter expression can be used for the numerical analysis of the exponential of the distribution function.

The numerical analysis shows that the consistency equation in Eq. (48) has positive roots only if \( \rho > \ln(L/\nu) \gg 1 \), which provides a reasonable limitation to the wave functions amplitudes we can study using this method: We have to restrict the density \( t \) of a splash by the value \( (\lambda_{FL})^{-1} \) related to the density of an electron state bound to the forward-and-backward scattered trajectory between two impurities. At the same time, in the
limit of $\rho \gg 1$, the roots of Eq. (68) can be approximated by $A = 1 - \mu$, where $\mu < 1$. The same conditions give us a possibility to replace the exact solution in Eq. (67) by

$$e^{-\theta_t} \approx (1/r)^{2\mu}, \quad (69)$$

what means that there is an easier way to get a satisfactory approximate solution of the saddle-point equation in $d = 2$ similar to what we did in $d = 3$.

**B. Solution using iterations**

The result of Eq. (69) can also be derived using the iterative scheme discussed in the Section VI. Being approved by the strong inequality $\rho \gg 1$, we solve, first, the linear Laplace equation by choosing its solution in the form which satisfies the boundary conditions at the origin,

$$\theta_t^{(0)} = 2\mu \ln(r/l),$$

where the parameter $\mu$ will be the subject of the next iteration. That’s, we seek for such $\theta_t = \theta_t^{(0)} + \theta_t^{(1)}(r/l)$ which satisfies the boundary condition at the external edge, and where

$$[u^{-1}\partial_u u \partial_u] \theta_t^{(1)} = -2\rho^{-2}u^{-2\mu}, \quad \theta_t^{(1)}(1) = 0, \quad u = r/l.$$

This gives $\theta_t^{(1)}$ in the form

$$\theta_t^{(1)}(u) = \frac{\rho^{-2}}{2(1 - \mu)^2} \left(1 - u^{2(1 - \mu)}\right).$$

The requirement $\partial_u \theta_t(L) = 0$ gives rise to the consistency equation and enables us to formulate the approximate procedure in the closed form,

$$2\mu = \frac{(L/l)^{2-2\mu}}{(1 - \mu)^{\rho^{2}}}.$$

The use of the iterative procedure is formally limited by a requirement $\theta_t^{(1)} \ll 1$.

The parameter $\mu$ in the above equation can be found (with the accuracy controlled by $1/\ln(L/l) \ll 1$) as

$$\mu = \frac{z(T)}{2 \ln(L/l)}, \quad ze^z = T \equiv \frac{t \ln(L/l)}{2\pi \nu T} \quad (70)$$

and varies when the amplitude $t$ changes. For example, the cross-over of the optimal solution to the homogeneous $Q \equiv \Lambda$ occurs in the limit of $T \ll 1$ where one can find that

$$\mu = \frac{1}{2}T(1 - T + ...) / \ln \frac{L}{l}.$$

In the opposite limit of large amplitudes, $T \gg 1$,

$$\mu \approx \frac{1}{2} \ln T / \ln \frac{L}{l} \ll 1.$$

The approximate form of the optimal free energy can be found, in its turn, as

$$F_t \approx \beta 4\pi^2 \nu D \{\mu + \mu^2 \ln \frac{T}{\nu D}\}. \quad (71)$$

When $T \ll 1$, this can be expanded in $T$ as

$$F_t \approx \beta V t \{1 - \frac{T}{2} + ...\}. $$

When $T \gg 1$ (but still $t \ll (lA)^{-1}$), the leading term in the optimal $F_t$ takes the form

$$F_t \approx \beta 2\pi \nu D \ln^2 \frac{T}{\ln(L/l)}.$$

Although the size of the system enters these formulae, the logarithmically weak dependence of $F_t$ on $L$ makes it meaningful to use the derived expressions for an arbitrary position of the observation point inside the sample of an arbitrary convex shape.

The fluctuations around the saddle-point configuration and the resulting pre-exponential factor $J(t)$ for the 2D case are discussed in the Appendix B. All over the condensation regime, their contribution to the value of the generating functional is small, as compared to that of the optimal solution itself.

**C. Distribution function and IPN’s**

All this enables us to calculate the distribution function $f$. For small amplitudes $t < 2\pi \nu D / [L^2 \ln \frac{L}{l}]$, one obtains

$$f^{(2)} \approx V \exp \left(-\beta V t \left[1 - \frac{T}{2} + ...ight]\right) \times \left\{ \begin{array}{ll} \frac{1}{2}, & s, su \\ 1, & 0, \quad u \\ \frac{2}{2}, & 2, \quad s, su \end{array} \right\}, \quad (72)$$

where $T$ is defined in Eq. (70), and

$$\beta_o = \frac{1}{2}; \quad \beta_u = 1; \quad \beta_{s, su} = 2.$$

In the opposite limit, $t > 2\pi \nu D / [L^2 \ln \frac{L}{l}]$, the distribution function takes the form

$$f^{(2)} \sim V \exp \left(- \frac{\nu^2 \nu D}{(L/l)^2} \ln^2 T \right). \quad (73)$$

Eqs. (72) and (73) generalize our earlier result [22] to various symmetry classes. They show that for any of the fundamental symmetries - orthogonal, symplectic and unitary - disorder makes the appearance of high-amplitude splashes of wave functions much more probable than one would expect from the Porter-Thomas formula and, as concerns the most extra-ordinary events, tends the tails to take the logarithmically-normal form. When being written for the orthogonal ensemble, the logarithmically-normal law in Eq. (63) strikingly coincides with the form
of the asymptotics of the distribution of the local density of states and conductance fluctuations in open systems found in Ref. [8], although our theory was made for closed systems and is based on a different scheme of calculations. This agreement reveals the deep relationship of these two results obviously caused by the localization effects. But the localization of wave functions which are responsible for the asymptotic events is not the localization of a particle in a confining potential. The tails of these states do not decay exponentially: Even in the asymptotical regime $T \gg 1$, the size $L$ of the system influences the distribution. The splashes look as if they were formed by focusing the waves by some rare configurations of scatterers. The structure of these states can be anticipated from the way how their distribution feels the boundary or - directly - from the cross-correlation functions $R(t, r)$ in Eqs. (83). Following the form of the optimal solution, the envelope of the density of such a state has a power-law asymptotic tail

$$|\psi_t(r)|^2 \sim e^{-\theta_t(r)} \approx (l/r)^{2\mu}$$

which approaches the limiting $r^{-2}$ dependence for the highest amplitudes $t \sim (l\lambda r)^{-1}$.

Moreover, the form of IPN’s, $t_n$ derived on the basis of Eqs. (24,33) shows such a scaling with the size of a system which implies them a multifractal structure. To find the moments $t_n$ accurately enough, we have to take into account that, although the cross-over to the 0D case looks like the formal limit $T(t) \to 0$, the Porter-Thomas statistics fails unless the condition $TV \ll 2\pi\nu D$ is satisfied (see Eq. (33)). Hence, only first few ratios $t_n$, $2 \leq n < \sqrt{2\pi\nu D}$, can be estimated using a finite polynomial expansion of $f(t)$ into the series in $T$, and their first terms reproduce corrections to the universal statistics found pertubatively in Ref. [2]. Alternatively, we derive the higher order IPN’s from Eqs. (24,38) using the saddle-point method. The moments $t_n$ calculated in both ways are in a good agreement with each other and, in the leading order, can be represented as

$$t_n \approx \min\{\varphi(n),2\pi\nu D/\ln(\psi(\varphi(n)))\} \left(\frac{\nu}{T}\right)^{n+1}d^*/2,$$

where

$$d^*(n) \approx 2 - \frac{n}{4\pi\nu D}.$$ (76)

As one can see from Eqs. (24,73), we end up with such a volume-dependence of the inverse participation numbers $t_n$ that manifests the multifractal behavior of quantum states, Eq. (8). The multifractality seems to be the generic property of 2D disordered systems. The multifractal dimensions in Eq. (75) are calculated in the leading order on the inverse conductivity, so that all over the metallic regime, the dependence of $d^*$ on $n$ and disorder is accurate enough and qualitatively agrees with numerical results [1]. Due to the limitation $t < (l\lambda r)^{-1}$, the above equations work at $n \leq 2\pi\nu D$, so that $n - \delta > 0$, and the fractal dimensions $d^*$ in Eq. (76) are positive.

VIII. DISCUSSIONS

Summarizing the results of the paper, we studied the manifestation of precursors of localization among the eigenstates of isolated disordered conductors with the size smaller than the localization length. In order to detect them, we analyzed the statistics of local amplitudes of wave functions, $t \equiv |\psi|^2$, and, at the tails $t \gg V^{-1}$ found strong deviations from the universal Porter-Thomas distribution (see Eqs. (83)) associated with the typically extended-type behavior. The universal statistics equally describes the quantum states of various classically chaotic systems; it depends on their fundamental symmetry but not on the physical dimensionality or the level of disorder. Such a description can be successfully applied to the most of the states (extended ones) in the metallic regime and gives the body of the distribution function of their local amplitudes. The deviations from the universal laws start rising at the amplitudes $t \sim \sqrt{\pi}/V$ and finally develop into a completely different asymptotics at $t \gg V$. In dimensions $d = 2$ and 3, the form of the asymptotics is described by the logarithmically-normal tails in Eq. (11,33), whereas in $d = 1$ it has a stronger dependence, $f \propto \exp(-4\sqrt{\beta L}c)$. The scheme of calculus (see Sections III and IV) and the similarity between our results for isolated systems and the asymptotics of distributions of the local density of states and fluctuations of other quantities in 1D [10] and 2D [9] conductors indicate that the above-mentioned long tails are strongly influenced by the localization. To answer the question, how the localization develops, we can refer to the fact that the deviations from the Porter-Thomas distribution appear as a so small number of events, $\propto \exp(-\sqrt{\rho})$, that their occurrence near the Fermi level in a specific sample is a typically mesoscopic phenomenon. We interpret this as that the rare top-amplitude splashes are not locally implicit as portions to any state but represent very non-trivial configurations of waves which can be found more and more often if the disorder increases. The analysis of the cross-correlations $R(t, r)$ also indicates that the states which are responsible for locally the highest amplitudes $|\psi(r_n)|^2 > g/V$ have individually specific envelopes of their decaying density far away from the observation point $r_n$. In 1D and 2D systems, the tails of the envelopes obey the power-law dependence $|\psi(r_n + r)|^2 \propto r^{-2\mu}$. In $d = 1$, its exponent is $\mu = 1$, so that the density of these tails is perfectly integrable at long distances, and one could speak about them as about nearly localized ones. In dimension $d = 2$ the exponent $\mu$ is limited by $\mu(t) \leq 1$ (so that it is not the localization in the usual sense) and is individual for the states with different amplitudes of the top-amplitude splash. Such a behavior of pre-localized states in $d = 2$ coexists with the typically multifractal behavior of the inverse participation numbers which has been observed earlier in various numerical simulations at the critical conditions of the localization-delocalization phenomenon.
transition [14,15]. Unfortunately, at the present stage we cannot approach close to the 3D Anderson transition and to check the multifractality globally. Nevertheless, even in $d = 3$ we find the non-trivial logarithmically-asymptotic behavior of the statistics, although the states which seem to be responsible for that are not localized.

The combination of the facts presented above forces us to suggest that the details of the structure and unusual statistics of rare pre-localized states which we discussed in the present paper have something to do with the statistics of extraordinary multiply self-crossing diffusive paths which anomalously often return to the same spatial coordinates. This means that the almost confined states develop because of rare shortened classical trajectories which form a closed loop not only in the real space, but also in the full phase space, since they finally come to the same ‘sell’ $dpdx = \hbar$ where they started. For instance, the most dense configuration could be formed by a forward-and-backward scattered trajectory of a particle bouncing few impurities. An additional argument which supports this scenario relates to the conditions limiting the validity of our theory. Basing on the use of the $\sigma$-model, we have necessarily to cut the linear length scale of the supermatrix $Q$-field variation from below by the value of the mean free path. Nevertheless, the densities which can be described in our approach are limited by the value $1/(l\lambda_F^{-1})$ instead of a naively expected inverse volume $l^{-d}$. This is only possible if the states which we study are locally anysotropic at the fine scale of distances of about $l$ and typically have a snake-like structure with the cross-sectional width $\sim \lambda_F^{d-1}$.

IX. ACKNOWLEDGEMENTS

We are grateful to B. Altshuler, B. Kramer, I. Lerner, B. Simons and M. Schreiber for useful discussions and thank P. Fulde for continuous encouragements during all the time we worked on this problem. We especially acknowledge the discussions with V. Kravtsov and his comments on the manuscript which helped to improve it. One of the authors (VF) acknowledges a partial support from NATO Collaborative Research Grant No. 921333.

X. APPENDIX A: PRE-EXPONENTIAL FACTOR FOR QUASI-1D CASE

In this Appendix, we show some details of calculations of the pre-exponential factor $J$ for quasi-1D case. Due to the condition $\theta_i = 0$ at $x = x_o$, the observation point splits the interval $[0, L]$ into two pieces, and the spectrum $\{\chi(n)\}$ of fluctuations around the saddle-point solution can be found in each interval separately. Therefore, we represent the pre-exponential as a product $J = J_{L}J_{R}$ of contributions from the left- and right-hand-side intervals with lengths $L_{L,R}$, where each of $J_{L,R}$ is determined by the eigenvalues of the Schrödinger equation in Eq. (48) with the symmetry-breaking potentials

$$U_i = \left[ kT_i + \kappa \left( \frac{\sqrt{T_i \pi/2}}{1 + \sqrt{T_i}} \right) \sin \left( \frac{\sqrt{T_i \pi/2} x}{1 + \sqrt{T_i} L_i} \right) \right]^{-2},$$

where $i = L, R$. When $T \ll 1$ (in the paragraph below, we omit indices $L$ and $R$), these potentials can be treated perturbatively. Their first-order corrections cancel due to the sum rule from Eq. (50), so that $J \approx 1 + T^2 \approx 1$. When $T \gg 1$, the same cancellation eliminates contributions from the high-excitation eigenvalues $\chi > (\pi/2L)^2 T$, so that the important contribution comes from the low-energy part of the spectrum, $\chi < (\pi/2L)^2 T$, where one can approximate

$$U \approx (\pi/2L)^2 [k + k'/ \sin^2(\pi x/2L)], \quad k' = k + \kappa.$$

Using this approximation, the spectral problem of 1D Eq. (43) can be solved exactly. To find the exact solution, one has to change variables from $x$ to $y = \cot(\pi x/2L)$ and then seek for solutions in the form $\phi = P_n(y^2)/(1 + y^2)^{\delta(n)}$, where $P_n(y^2)$ are polynomials. This results in the set of eigenvalues $\chi(n)$, $n \geq 0$,

$$\chi(n) = (\pi/2L)^2 \left\{ \left[ 2n + 1 + \sqrt{k' + 1/4} \right]^2 - k \right\}.$$

Being substituted into Eq. (47), this gives the pre-exponential factor $J$ in the main order in $T_{L,R}$:

$$J \approx \exp \left( \sum_{i=L,R} \frac{1}{4} \ln T_i + \text{const} \right) \approx C (T_{L,R})^{1/4}. \quad (77)$$

XI. APPENDIX B: PRE-EXPONENTIAL FACTOR FOR 2D CASE

In the 2D case, the spectrum $\{\chi(n, m)\}$ of fluctuations around the saddle-point should be classified by orbital and radial quantum numbers $n$ and $m$, respectively, and can be found from the eigenvalues of the Hamiltonian

$$\hat{H} = -r^{-1} \partial_r (r \partial_r) + m^2 r^{-2} + U,$$

where $U$ is determined by Eq. (19).

Without any symmetry-breaking, the spectrum of $\chi$’s can be approximated as

$$\chi(0, 0) \approx 2L^{-2}/\ln(L/l) \quad (78)$$

for the lowest mode and as

$$\chi(n, m) \approx \left( \frac{\pi}{T} \right)^2 \left[ n + \frac{1}{4} + \frac{m^2}{16} \right]^2 \quad (79)$$

for higher $n$’s and $m$’s.
The optimal solution breaks the fermion-boson symmetry and induces effective potentials composed of two types of contributions,

\[
\frac{1}{4} k(\partial \theta)^2 \simeq k \mu^2 / r^2 \quad \text{and} \quad \frac{\kappa t}{2\pi \nu D} r^{-\theta} \simeq \kappa \mu L^2 \left( \frac{L}{r} \right)^2 \mu.
\]

In the above equations, the approximate values are given for the most important range of distances \(r \leq L/(z(T)/T)\), and one has to remember that \(\mu < 1\).

For any \(m \neq 0\), the potential \(U\) is relatively small, \(U \ll m^2 / r^2\), and could be treated perturbatively. Due to the sum rule mentioned in Section III.D, Eq. (50), the modes with \(m \neq 0\) contribute only in the second order in \(U\), and what they give to the exponential of \(J\) is of the order of \(\mu^4 \ln(L/l); \mu^2\). With the accuracy we need here regarding the leading terms in \(F_i\), this contribution can be neglected.

The spectrum of low-lying modes \(\{ \chi(n > 0, 0) \}\) is given by the expression

\[
\chi(n, 0) \sim \left( \frac{\mu}{T} \right)^2 \left[ n + \frac{1}{4} + \sqrt{k \frac{\mu}{2}} \right]^2, \quad 0 < n \leq \frac{L}{\pi T}.
\]

The cancellation between different eigenvalues from Eqs. (59) and (60) substituted to the general equation in Eq. (73) produces a multiplier to \(J\) which is of the order of \(e^{-\mu^2} / T\) at \(T \gg 1\). Finally, we get

\[
J = 1 + o(T^2), \quad T \ll 1
\]

\[
J \propto \mu \exp(\mu \ln \frac{L}{T}) \sim L/T, \quad T \gg 1.
\]

This result can be used for all symmetry classes.

[1] P.W. Anderson, Phys. Rev. B 109, 1492 (1958)
[2] D. Thouless, Phys. Rev. Lett. 39, 1792 (1972)
[3] M. Kaveh and N. Mott, J. Phys. C 15, L697 (1982); ibid., L707 (1982)
[4] F. Wegner, Z. Phys. B 35, 207 (1979)
[5] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985), and refs. therein
[6] B. Kramer and A. MacKinnon, Rep. Prog. Phys. 56, 1469 (1993), and refs. therein
[7] K.B. Efetov, Adv. Phys. 32, 53 (1983)
[8] F. Wegner, Z. Phys. B 36, 209 (1980)
[9] B.L. Altshuler, V.E. Kravtsov and I.V. Lerner, in Mesoscopic Phenomena in Solids, ed. B.L. Altshuler et al, Elsevier, Amsterdam 1991, p.449, and refs. therein
[10] H.Aoki, J.Phys. C 16, L205 (1983); Phys. Rev. B 33, 7310 (1985)
[11] M. Schreiber, J. Phys. C 18, 2493, (1985); Phys. Rev. B 31, 6146, (1985); M. Schreiber and H. Grussbach, Phys. Rev. Lett. 67, 607 (1991); H. Grussbach and M. Schreiber, Phys. Rev. B 48, 6650 (1993); ibid. B 51, 663 (1995)
[12] B. Kramer, Y. Ono and T. Ohtsuki, Surf. Sci. 196, 127 (1988)
[13] T. Karawarabayashi and T. Ohtsukki, Phys. Rev. B 51, 10897 (1995)
[14] S. Evangelou and D. Katsanos, J. Phys. A 26, L1243 (1993)
[15] M. Janssen, O. Viehweger, U. Fastenrath and J. Hajdu, Introduction to the Theory of the Integer Quantum Hall Effect, VCH, Weinheim 1994; W. Pook and M. Janssen, Z. Phys. B 82, 295 (1991); U. Fastenrath, M. Janssen and W. Pook, Physica A 191, 401 (1992)
[16] C. Castellani, L. Peliti, J. Phys. A 19, L429 (1986)
[17] B.A. Muzykantskii and D.E. Khmelnitskii, Phys. Rev. B 51, 5480 (1995)
[18] L.P. Gorkov and G.M. Eliashberg, Zh. Eksp. Teor. Fiz. 48, 1407 (1965) (Sov. Phys. JETP 21, 940 (1965))
[19] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey and S.S.M. Wong, Rev. Mod. Phys. 53, 385 (1981); J.J.M. Verbaarschot, H.A. Weidenmuller and M.R. Zirnbauer, Phys. Rep. 129, 367 (1985)
[20] M. Berry, Proc. Roy. Soc. Lond. A 400, 229 (1985)
[21] R.A. Jalabert, A.D. Stone and Y. Alhassid, Phys. Rev. Lett. 68, 3468 (1992)
[22] K.B. Efetov and V.N. Prigodin, Phys. Rev. Lett. 70, 1315 (1993)
[23] A.D. Mirlin and Y.V. Fyodorov, Phys. Rev. Lett. 72, 526 (1994)
[24] B. D. Simons and B. L. Altshuler, Phys. Rev. Lett. 76, 4063 (1993); Phys. Rev. B 48, 5422 (1993)
[25] V. N. Prigodin, K. B. Efetov and S. Iida, Phys. Rev. Lett. 71, 1230 (1993)
[26] V.I. Fal'ko and K.B. Efetov, Phys. Rev. B 50, 11267 (1994)
[27] Y.V. Fyodorov and A.D. Mirlin, JETP Lett. 60, 790 (1994)
[28] 'Intrinsic' conductance \(g\) depends on the dimensionality of the system:

\[
1D : g \sim \frac{L}{T} ; \quad 2D : g \sim \frac{pF}{\ln(L/h)} ; \quad 3D : g \sim (pF)^2.
\]

Here, we speak about the dimensionless conductance, \(g \equiv G/(e^2/h)\).
[29] W. Feller, Introduction to Probability Theory and Its Applications, John Wiley Inc., NY 1966
[30] B.L. Altshuler and V.N. Prigodin, Zh. Eksp. Teor. Fiz. 95, 348 (1989) (Sov. Phys. JETP 68, 198 (1989))
[31] It is worth mentioning that \(f(\langle t, t\rangle)\) is normalized, \(t_0 = 1\), and the normalization of spin-projected wave functions, \(t_1 = 1/(2V)\), produces the same requirement as in Eq. (55).
[32] V.I. Fal'ko and K.B. Efetov, submitted to Phys. Rev. Lett.
[33] B.B. Mandelbrot, J. Fluid Mech. 62, 331 (1974); T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia and B. Shraiman, Phys. Rev. A 33, 1141 (1986); G. Paladin and A. Vulpiani, Phys. Rep. 156, 147 (1987)
[34] By this term, we denote the class of the systems both with the spin-dependent mechanism of scattering and broken time-reversal symmetry. For instance, this can be imagined as a metal with built-in disordered static magnetic field or a random exchange field.
[35] B. DeWitt, *Supermanifolds*, Cambridge University Press, Cambridge 1992

[36] V.E. Kravtsov, I.V. Lerner and V.I. Yudson, Phys. Lett. A 134, 245 (1989)

[37] G. Castilla and S. Chakravarty, Phys. Rev. Lett. 71, 384 (1993)

[38] F. Wegner, Z. Phys. B 78, 33 (1990)

[39] K.B. Efetov and A.I. Larkin, Zh. Eksp. Teor. Fiz. 85, 764 (1983) (Sov. Phys. JETP 58, 444 (1983))

[40] A.D. Mirlin and Y.V. Fyodorov, J. Phys. A 26, L551 (1993)

[41] R. Feynman, Rev. Mod. Phys. 20, 367 (1948)

[42] E. Kamke, *Differential Equations*, Clesea Publ., NY 1971

[43] K.B. Efetov, Zh. Eksp. Teor. Fiz. 92, 638 (1987) (Sov. Phys. JETP 65, 360 (1987))