THE DOUBLE EULERIAN POLYNOMIAL AND INVERSION TABLES

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Abstract. We show that the pair \((\text{des}, \text{ides})\) of statistics on the set of permutations has the same distribution as the pair \((\text{asc}, \text{row})\) of statistics on the set of inversion tables, proving a conjecture of Visontai. The common generating function of these pairs is the double Eulerian polynomial.

The double Eulerian polynomial. The double Eulerian polynomial \(A_n(u, v)\) enumerates the number of descents of a permutation and its inverse,

\[ A_n(t, s) = \sum_{\pi \in S_n} u^{\text{des}(\pi)} v^{\text{des}(\pi^{-1})}. \]

It is a natural generalization of the classical Eulerian polynomial \(A_n(u, 1)\). The latter polynomial is well-known to be positive in the basis \((u^i(1 + u)^{n-i})_{i=0}^n\). This has been proved in several ways; notably by geometric means [1], and by an elegant bijective argument by Foata and Strehl (see [3] for an excellent exposition).

There is no analogous result for the double Eulerian polynomial, though there is a conjectured one by Gessel [2]: \(A_n(u, v)\) should be integral and positive in the basis \((uv)^i(u + v)^j(1 + uv)^{n-2i-j})_{i,j}\). Visontai [5] gave explicit formulas for the coordinates of \(A_n(u, v)\) in this basis, but was unable to prove that they are positive, nor that they are integers. He also conjectured a new way of defining \(A_n(u, v)\), as follows (this is our main theorem).

Theorem 1. For all \(n\),

\[ A_n(u, v) = \sum_{e \in I_n} u^{\text{asc}(e)} v^{\text{row}(e)}. \]

Here, \(I_n\) is the set of inversion tables of length \(n\), and \((\text{asc}, \text{row})\) are two statistics, all defined below. We will spend the remainder of this note proving Theorem 1 after giving the necessary definitions.

We identify permutations \(w\) of length \(n\) with words \(w_1 \ldots w_n\), and with permutation diagrams \[\{(i, w_i) : 1 \leq i \leq n\}\], which we read as Cartesian coordinates: a point \((x, y)\) refers to a point \(x\) steps to the right and \(y\) steps up from \((0,0)\). See Figures 1 and 2 for examples. An inversion table \(e = e_1 \ldots e_n\) of length \(n\) is any sequence of positive integers satisfying \(1 \leq e_i \leq i\) for all \(i\) (this differs slightly from the notation in [1], which we otherwise follow). We will identify these with marked staircases, examples of which are given in Figures 3 and 4. For a permutation \(w\), let \(\text{DES}(w) = \{i : w_i > w_{i+1}\}\) be the set of descent positions, and \(\text{IDES}(w) = \text{DES}(\pi^{-1})\) be the descent set of the inverse permutation. For an inversion table \(e\), we define \(\text{ASC}(e) = \{i : e_i < e_{i+1}\}\) and \(\text{ROW}(e) = \{e_i : 1 \leq i \leq n\} - \{1\}\). Note that the strict

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For children whose row statistic equals descents satisfying \( r \in \Phi \) between the two trees, which takes a permutation isomorphic as labeled rooted trees. We will do this by producing an isomorphism on indeterminates \( u_1, u_2, \ldots \) and \( v \).

**Examples.** We give the values of the statistics of the permutation \( \pi \) in Figure 2 and of the inversion table \( e \) in Figure 1. We have \( \text{DES}(\pi) = \{1, 4, 5, 7, 8\} \), \( \text{IDES}(\pi) = \{1, 3, 4, 6, 8\} \), \( \text{ASC}(e) = \{1, 2, 4, 5, 7, 8\} \) and \( \text{ROW}(e) = \{2, 3, 4, 5, 7\} \). Thus \( \text{DES}(\pi) = 5 \), \( \text{ides}(\pi) = 5 \), \( \text{asc}(e) = 6 \) and \( \text{row}(e) = 5 \).

To prove Theorem 1, we will prove the stronger statement

\[
\sum_{\pi \in S_n} u^{\text{DES}(\pi)} v^{\text{ides}(\pi)} = \sum_{e \in S_n} u^{\text{ASC}(e)} v^{\text{row}(e)}.
\]

By Möbius inversion, equation 1 holds if and only if we have

\[
\sum_{\pi \in S_n \atop S \subseteq \text{DES}(\pi)} s^{\text{ides}(\pi)} = \sum_{e \in S_n \atop S \subseteq \text{ASC}(e)} s^{\text{row}(e)},
\]

for each \( S \subseteq [n-1] \). The idea now is to fix a subset \( S \) of positive integers and induc on \( n \) (in a sense to be specified) to prove equation 2.

Thus fix a subset \( S \subseteq [1, 2, \ldots] \). We define two rooted labeled trees \( T_3 \) and \( T_1 \), as follows. The vertices of \( T_3 \) are all permutations \( w_1 \ldots w_n \) whose length \( n \) satisfies \( n \notin S \) (in particular, the empty permutation, of length 0, is a node). A permutation \( \pi \) of length \( r + s \) is a child of another permutation \( \pi' \) of length \( r \), where \( s \) is smallest such that \( s \geq 1 \) and \( r + s \notin S \), if the first \( r \) letters of \( \pi \) induce the same permutation as \( \pi' \). It follows that the empty permutation is the root node. Each node in \( T_3 \) is labeled by the pair \((n, k)\), where \( n \) is the length of the permutation and \( k \) is its number of inverse descents.

Similarly, the vertices of \( T_1 \) are all inversion tables \( e_1 \ldots e_n \) whose length \( n \) satisfies \( n \notin S \), and an inversion table \( e \) of length \( r + s \) is a child of \( e' = e'_1 \ldots e'_r \) if \( e = e'_1 \ldots e'_r e_{r+1} \ldots e_{r+s} \) (with the same condition on \( s \) as for \( T_3 \)). Each node is labeled \((n, k)\), where \( n \) is the length of the inversion table and \( k \) is its value of the row statistic. The empty inversion table is the root node.

To prove equation 2 for our fixed set \( S \), it suffices to prove that \( T_3 \) and \( T_1 \) are isomorphic as labeled rooted trees. We will do this by producing an isomorphism \( \Phi \) between the two trees, which takes a permutation \( \pi \) of length \( n \) with \( k \) inverse descents satisfying \( S \cap [n] \subseteq \text{DES}(\pi) \) to some inversion table \( e = \Phi(\pi) \) of length \( n \), satisfying \( \text{row}(e) = k \) and \( S \cap [n] \subseteq \text{ASC}(e) \).

We will construct \( \Phi \) inductively. Let \( \Phi \) map the root of \( T_3 \) to the root of \( T_1 \). Suppose that we have already defined \( \Phi(\pi') = e' \). We will show, for each \( k \), that the number of children of \( \pi' \) with \( k \) inverse descents equals the number of children of \( e' \) whose row statistic equals \( k \). This allows us to extend \( \Phi \) to all the children of \( \pi' \). Thus fix a permutation \( \pi' \) of length \( r \) and an inversion table \( e' \) of length \( r \) such that \( \text{ides}(\pi') = \text{row}(e') = p \), say. Suppose \( s \) is smallest such that \( s \geq 1 \) and \( r + s \notin S \). For children \( \pi \) of \( \pi' \), we call the first \( r \) letters the early part, and the last \( s \) letters the late part. The children \( \pi \) are determined in a bijective way by \((r + 1)\)-tuples \((x_0, \ldots, x_r)\) of nonnegative integers \( x_i \) with sum \( s \). The bijection is given by letting

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1. that is, if we renumber the first \( r \) letters of \( \pi \) by \( 1, \ldots, r \), we get the permutation \( \pi' \).
2. To prove this is a bijection, note that the last \( r \) letters of \( \pi \) form a decreasing word since \( \pi \) is a child (which implies that \( r, r + 1, \ldots, r + s - 1 \) are descents).
Figure 1. A permutation \( \pi' = 325641 \) together with an \( 6 + 1 \)-tuple \( (x_1, \ldots, x_7) = (0, 0, 2, 0, 0, 1, 0) \) determines the child \( \pi \) of \( \pi' \) given in Figure 2. For all the examples in Figures 1-4, the fixed set \( S \) can be taken to be \{1, 4, 7, 8\}.

\( x_i \) be the number of letters in the late part of \( \pi \) which are between (in value) the \( i \)th and \((i + 1)\)st largest letters of the early part of \( \pi \). Moreover, the number of inverse descents of \( \pi \) is

\[
\text{ides}(\pi') + \sum_{i \in T} (x_i - 1) + \sum_{i \notin T} x_i,
\]

where \( T = \text{IDES}(\pi') \). This is hopefully made clear by Figures 1 and 2.

On the other hand we consider children \( e \) of \( e' \), of length \( r + s \). Such \( e \) are in bijection with subsets \( T \subseteq \{1, 2, \ldots, r + s\} \) of size \( s \) by letting 

\[
T = \{e_i: r < i \leq r + s\}
\]

(note that this is a set since \( \{s + 1, \ldots, r - 1\} \subseteq \text{ASC}(e) \) is a child). Moreover, 

\[
\text{row}(e) = \text{row}(e') + \#(T \cap \text{ROW}(e')).
\]

By the preceding two paragraphs, all that remains is to prove that the number of \( (x_0, \ldots, x_r) \) with

\[
\sum_{i=0}^{r-1}(x_i - 1) + \sum_{i=p}^{r} x_i = t \text{ and } \sum_{i=0}^{r} x_i = s \quad \text{(here, we have reordered the } x_i\text{'s, which clearly does not affect the count)}
\]

equals the number of \( T \subseteq [r + s] \) such that \( |T| = s \) and \( |T \cap \{p + 2, p + 3, \ldots\}| = t \), for all nonnegative integers \( r, s, p, t \).

These two counts are easily seen to be

\[
\sum_{a=0}^{t} \binom{p+1}{s-t} \binom{s-a-1}{a} \binom{a+r-p-1}{a},
\]

and

\[
\binom{p+1}{s-t} \binom{r+s-p-1}{t},
\]

respectively. That they are equal is a classical fact. This finishes the proof of Theorem 1.

**Final remarks.** It is interesting to note that while the statement of Theorem 1 is symmetric in \( t \) and \( s \), the proof is not. We have failed to generalize the Theorem to one with two set-valued statistics. If we define \( \text{maj}(\pi) \) to be the sum of the elements in \( \text{DES}(\pi) \), and \( \text{amaj}(e) \) to be the sum of the elements in \( \text{ASC}(e) \), then the proof shows that the pairs \( (\text{maj}, \text{ides}) \) and \( (\text{amaj}, \text{row}) \) are equidistributed. There does not seem to be an obvious generalisation involving \( \text{maj}, \text{amaj} \) and sums of \( \text{IDES} \) and \( \text{ROW} \). Finally, it follows from Theorem 1 that \( \text{asc} \) and \( \text{row} \) are equidistributed (since \( \text{des} \) and \( \text{ides} \) are). I do not know of a direct proof of this fact.

**References**

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\( ^3 \)We use the notation \( x_+ = x \) for \( x \geq 0 \) and \( x_+ = 0 \) for \( x < 0 \).
Figure 2. The child $\pi = 527961843$ referred to in Figure 1. We have $\text{ides}(\pi) = \text{ides}(\pi') + (2 - 1)_+ + 1 = 5$.

Figure 3. An inversion table $\pi' = 123135$ and a set $T = \{1, 4, 7\}$ determines the child $\pi$ of $\pi'$ given in Figure 4.

Figure 4. The child $\pi = 123135147$ to $\pi'$ referred to in Figure 3.

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