From Bit-Parallelism to Quantum: Breaking the Quadratic Barrier

Massimo Equi
Arianne Meijer - van de Griend
Veli Mäkinen

Department of Computer Science
University of Helsinki, Finland

December 2021

Abstract

Many problems that can be solved in quadratic time have bit-parallel speed-ups with factor $w$, where $w$ is the computer word size. For example, edit distance of two strings of length $n$ can be solved in $O(n^2/w)$ time. In a reasonable classical model of computation, one can assume $w = \Theta(\log n)$. There are conditional lower bounds for such problems stating that speed-ups with factor $n^\epsilon$ for any $\epsilon > 0$ would lead to breakthroughs in complexity theory. However, these conditional lower bounds do not cover quantum models of computing. Moreover, it is open if problems like edit distance can be solved in truly sub-quadratic time using quantum computing. To partially address this question, we study another bit-parallel algorithm for a problem that admits a quadratic conditional lower bound, and show how to convert its bit-parallelism into a realistic quantum algorithm that attains speed-up with factor $n$. The technique we use is simple and general enough to apply to many similar bit-parallel algorithms, where dependencies are local. However, it does not immediately yield a faster algorithm for more complex problems like edit distance, whose bit-parallel dynamic programming solutions require breaking more global dependencies. We hope that this initial study sheds some light on how, in general, bit-parallelism could be converted to quantum parallelism.

1 Introduction

Exact string matching problem is to decide if a pattern string $P$ appears as a substring of a text string $T$. In the classical models of computation, this problem can be solved in $O(|P| + |T|)$ time [7]. Different quantum algorithms for this basic problem have been developed [10][11][13], resulting into different solutions, the best of which finds a match in $O(\sqrt{|T|}(\log^2 |T| + \log |P|))$ time [10] with
high probability. These assume the pattern and text are stored in quantum registers, requiring thus $O(|P| + |T|)$ qubits to function. Moreover, these approaches implicitly assume that strings of the size of the pattern could fit into a (quantum) memory word; an assumption that, if dropped, would introduce a linear dependence with $P$ in the time complexity.

In the classical models of computation, an analogy for the first storage assumption is to assume the text has been preprocessed for subsequent queries. For example, one can build a Burrows-Wheeler transform -based index structure for the text in time $O(|T|)$ [4], assuming $T \in \{1, 2, \ldots, \sigma\}^*$, where $\sigma \leq |T|$. Then, one can query the pattern from the index in $O(|P| \log \log \sigma)$ time [4, Theorem 6.2]. In this light, quantum models can offer only limited benefit over the classical models for exact string matching.

Motivated by this difficulty in improving linear-time solvable problems using quantum approaches, let us consider problems known to be solved in quadratic time. For example, approximate string matching problem is such a problem: decide if a pattern string $P$ is within edit distance $k$ from a substring of a text string $T$, where edit distance is the number of single symbol insertions, deletions, and substitution needed to convert a string to another. This problem can be solved using bit-parallelism in $O(|P|/w|T|)$ time [8], under the Random Access Memory (RAM) model with computer word size $w$. A reasonable assumption is that $w = \Theta(\log |T|)$, so that this model reflects the capacity of classical computers. Thus, when $|P| = |T| = n$, this bit-parallel algorithm for approximate string matching takes time $\Omega(n^2)$ for all $\epsilon > 0$, as $\log n = o(n^\epsilon)$ for all $\epsilon > 0$. It is believed that this quadratic bound cannot be significantly improved, as there is a matching conditional lower bound saying that if approximate pattern matching could be solved in $O(|P||T|^{1-\epsilon})$ time with some $\epsilon > 0$, then the Orthogonal Vector Hypothesis (OVH) and therefore the Strong Exponential Time Hypothesis (SETH) would not hold [2]. As these hypotheses are about classical models of computation, it is natural to ask if the quadratic barrier could be broken using quantum computation.

In this quest for breaking the quadratic barrier, we study another problem with a bit-parallel solution and a conditional lower bound. Consider exact pattern matching on a graph, that is, consider deciding if a pattern string $P \in \Sigma^*$ equals a labeled path in a graph $G = (V,E)$, where $V$ is the set of nodes and $E$ is the set of edges. Here we assume the nodes $v$ of the graph are labeled by $\ell(v) \in \Sigma$ and a path $v_1 \to v_2 \to \cdots v_t$, $(v_i,v_{i+1}) \in E$ for $1 \leq i < t$, spells string $\ell(v_1)\ell(v_2)\cdots\ell(v_t)$. There is an OVH lower bound conditionally refuting an $O(|P||E|^{1-\epsilon})$ or $O(|P|^{1-\epsilon}|E|)$ time solution [6]. This conditional lower bound holds even if graph $G$ is a level DAG [6]: for every two nodes $v$ and $w$, holds the property that every path from $v$ to $w$ has the same length. On DAGs, this string matching on labeled graphs (SMLG) problem can be solved in $O(|P|/w|E|)$ time [12], so the status of this problem is identical to that of approximate pattern matching on strings. However, the simplicity of the bit-parallel solution for SMLG on level DAGs enables a connection to quantum computation. We consider a specific model of quantum computation, the Quantum Random Access Memory (QuRAM) model, in which we have access to “quantum arrays”, and
we assume that integer values like $|P|$, $|V|$ or $|E|$ fit into a (quantum) memory word. Under this model, we turn the bit-parallel solution into a quantum algorithm that solves SMLG on level DAGs with high probability in $O(|E| + |V|)$ time, breaking through the classical quadratic conditional lower bound. As far as we know, this is the first time the quadratic barrier has been broken using quantum computation for a problem that admits a classical conditional lower bound based on SETH and OVH. Moreover, the bit-parallel strategy allow us to use a limited number of qubits, that is, our qubits space complexity on level DAGs is $O(|P| + \max L)$, where $\max L$ is the number of nodes of the level with the highest number nodes. We remark that $\max L$ is always less or equal to the width of the graph, that is the minimum number of non-disjoint paths needed to cover all the nodes.

As mentioned above, in some previous works \cite{10, 11, 13} (and references in \cite{13}) algorithms have been proposed to solve string matching in plain text in the QuRAM model, under the assumption that the entire text, or at least text sub-strings of the same length of the pattern, fit into a memory word. We find this assumption to be too restrictive, as even the classical RAM model does not adopt it, since in such a model of computation many operations would become trivial. Instead, our algorithm works without the need for such an assumption, and has the same time complexity as the previous solutions, if they are all compared under our QuRAM model of computation. This is because, using our QuRAM model, a $O(|P|)$ term would appear in the time complexities of these previous algorithms.

The paper is structured as follows. We revisit exact pattern matching and derive a simple quantum algorithm for it, in order to introduce the quantum machinery. Then we give a brute-force quantum algorithm for SMLG, which we later improve on level DAGs. This improvement is based on extending the Shift-And algorithm \cite{3}, whose quantum version we first study for exact pattern matching, and finally extend for level DAGs. We conclude with discussion on some connections to other related problems.

In what follows, we assume the reader is familiar with the basic notions in quantum computing as covered in textbooks \cite{9}.

2 String Matching in Plain Text

A quantum computer, with access to QuRAM, can solve the problem of finding an exact match for a string pattern $P$ into a text string $T$ in time $O(|P| + \sqrt{|T|})$, with high probability. We explain a simple solution to this problem. We assume to have $T$ and $P$ stored in quantum registers $|T\rangle\langle P|$, register $|S\rangle$ initialized to $|1\rangle$, and register $|I\rangle$ initialized to $|0\rangle$. We prepare quantum register $|I\rangle$ in an equally balanced superposition spanning all the text positions, that is $|I\rangle \leftarrow 1/\sqrt{2^{|T|}} \sum_{i=0}^{|T|-1} |i\rangle$ (assuming $|T|$ to be a power of 2 for simplicity). Each individual state $|i\rangle$ in the superposition represents a computation starting at position $i$ in the text. In each of these computations, we scan $T[i..i + |P| - 1]$ and try to match each character with $P[0..|P| - 1]$, storing the intermediate
results of such comparisons in register $|S\rangle$. More precisely, at iteration $j$, $0 \leq j \leq |P| - 1$, we compute a logic and between $T[i + j] = P[j]$ and the value in $|S\rangle$. This procedure is correct because at the first step $|S\rangle = |1\rangle$, thus we simply compute and store $T[i + j] = P[j]$ in $|S\rangle$. At iteration $j$, we assume by induction that register $|S\rangle$ stores a 1 if $T[i \ldots i + j] = P[0 \ldots j]$, thus computing $T[i + j] = P[j] \wedge S$ tells us if we are extending a match or not. At the end, we can run Grover’s search algorithm to retrieve the superposition states where $|S\rangle = |1\rangle$, and then measure register $|I\rangle$ to locate the ending position of a match. We illustrate this procedure in Algorithm 1.

**Algorithm 1**: An algorithm for solving exact string matching in plain text, which achieves $O(|P| + \sqrt{|T|})$ time complexity thanks to the use of QuRAM.

1. $|\psi\rangle \leftarrow |T\rangle |P\rangle |I\rangle |S\rangle$;
2. $|S\rangle \leftarrow |1\rangle$;
3. $H^\otimes|T||I|$;
4. for $j \in [0,|P| - 1]$ do
   5. $|S\rangle \leftarrow |T[I] = P[j] \wedge S\rangle$;
   6. $|I\rangle \leftarrow |I \oplus 1\rangle$;
7. Run Grover’s search algorithm with oracle $f(T, PI, S) = S$;
8. $idx \leftarrow$ measurement of register $|I\rangle$;
9. $idx \leftarrow idx - 1$;
10. return $idx$;

In the following example we have $T = \text{AABABABB}$ and $P = \text{ABA}$.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $T$ = | A | A | B | A | B | A | B | B |

|   | 0 | 1 | 2 |
|---|---|---|
| $P$ = | A | B | A |

We demonstrate how Algorithm 1 works by simulating it on text $T$ and pattern string $P$. In the following example, we depict one term of the superposition per row, omitting the amplitudes and the plus sign, as the amplitudes are all the
same and the plus sign does not add important information.

\[
|T\rangle |P\rangle |0\rangle |1\rangle \rightarrow \\

|T\rangle |P\rangle |0\rangle |1\rangle \rightarrow |T\rangle |P\rangle |0\rangle |1\rangle \rightarrow |T\rangle |P\rangle |0\rangle |1\rangle \rightarrow |T\rangle |P\rangle |0\rangle |1\rangle \rightarrow |T\rangle |P\rangle |0\rangle |1\rangle \rightarrow |T\rangle |P\rangle |0\rangle |1\rangle \\

\]

3 String Matching in Labeled Graphs

3.1 Quantum Brute-force Algorithm for SMLG

In SMLG we are given pattern string \( P \) with characters in alphabet \( \Sigma \) and a node-labeled graph \( G = (V, E) \), with labelling function \( \ell : V \rightarrow \Sigma \). We are asked to find a path (or, actually, a walk) \( \pi = v_1, v_2, \ldots, v_{|P|} \) in \( G \) such that \( \ell(v_1) \circ \ell(v_2) \circ \ldots \circ \ell(v_{|P|}) = P \), where \( \circ \) denotes string concatenation.

Generalizing the idea we presented for plain text, we provide a brute-force quantum algorithm solving SMLG running in time \( O(|P| \log |V|) \), or just \( O(|P|) \) if \( |V| \) can be stored in a single quantum register. The idea is to list all possible paths of length \( |P| \) in the graph and then mark those ones that are actual matches for \( P \). Consider \( |P| \) qubits registers \( |V_k\rangle, 1 \leq k \leq |P| \), each consisting of \( \log |V| \) qubits, plus one additional qubit \( |e\rangle \). We set \( |e\rangle \) to \( |1\rangle \). We set every \( |V_k\rangle \) in an equally balanced superposition, so that every state in such superposition represent a different node in \( V \). Thus, we have all the nodes in \( V \) listed \( |P| \) times. For \( k \in [1, |P| - 1] \), we access the nodes represented by \( |V_k\rangle \) and \( |V_{k+1}\rangle \), and we check that an edge exists between these two nodes and that \( \ell(V_k) = P[k] \). This is done by updating the value of \( |e\rangle \) as \(|e\rangle \leftarrow |e\rangle \wedge (V_k, V_{k+1}) \in E \wedge L[V_k] = P[k] \rangle \), where \( L \) is and array in our QuRAM that implements the \( \ell \) function. Then, for \( k = |P| \), we handle the last comparison updating \( |e\rangle \) as \(|e\rangle \leftarrow |e\rangle \wedge L[V_k] = P[k] \rangle \).

Notice that every time we perform the check \((V_k, V_{k+1}) \in E \) what we are doing is accessing the QuRAM using \( |V_k\rangle \) and \( |V_{k+1}\rangle \) as indexes. Since these two register are in a superposition, we are actually checking every pair of nodes
at the same time. Thus, what the whole algorithm is doing is accessing every possible \( |P| \)-tuple of nodes \((v^{(1)}, v^{(2)}, \ldots, v^{(|P|)})\), checking that it is an actual walk in the graph, and checking that the labels match the pattern.

We now show an example where \( |P| = 3 \) and \( |V| = 8 \).

\[
\begin{array}{ccc}
V_1 & V_2 & V_3 \\
|000\rangle & |000\rangle & |000\rangle \\
|001\rangle & |001\rangle & |001\rangle \\
|010\rangle & |010\rangle & |010\rangle & \text{e} \\
|011\rangle & |011\rangle & |011\rangle \\
|100\rangle & |100\rangle & |100\rangle & \otimes |1\rangle \\
|101\rangle & |101\rangle & |101\rangle \\
|110\rangle & |110\rangle & |110\rangle \\
|111\rangle & |111\rangle & |111\rangle \\
\end{array}
\]

The superposition is indicated with the tensor product notation, otherwise it would take too much space, but it is indeed representing every possible tuple \((v^{(1)}, v^{(2)}, v^{(3)}) \in V \times V \times V\). Assume that we can check in constant time whether an edge between two nodes exists. For instance, we could assume to have adjacency matrix \( A_G \) of \( G \) stored in our QuRAM, that we can access to check the existence of an edge. For nodes \((v^{(1)}, v^{(2)}, v^{(3)})\), our algorithm computes \( A_G[v^{(1)}][v^{(2)}] \wedge A_G[v^{(2)}][v^{(3)}] \wedge e \) and stores the result in \(|e\rangle\), which is an operation that we can perform in \( O(|P|) \) time by scanning the superposition from left to right. For instance, for nodes \((011, 101, 011)\), we have \(|e\rangle \leftarrow |A_G[011][101] \wedge A_G[101][011] \wedge 1\rangle\). In this way, the states of the superposition representing an actual path in the graph are marked with \(|e\rangle = |1\rangle\).

Then, we check whether the pattern matches these paths, that is we compute \( \ell(v^{(1)}) = P[0] \wedge \ell(v^{(2)}) = P[1] \wedge \ell(v^{(3)}) = P[2] \wedge e \), and we store the result in \(|e\rangle\). This operation also takes \( O(|P|) \) for the same reason as above. At this point, the states of the superposition marked with \(|e\rangle = |1\rangle\) represent a path in \( G \) with labels matching \( P \). Thus, we can retrieve these states with Grover’s algorithm (or, to better put it, we can amplify the amplitudes of the states with \(|e\rangle = |1\rangle\), and then perform a measurement). the overall time complexity is then \( O(|P| + \sqrt{|V|}) \). However, the space complexity is \( O(|V|^2) \) qubits, because we are assuming to have access to the entire adjacency matrix of the graph.

### 3.2 The Classical Shift-And Algorithm

We first introduce the classic shift-and algorithm for matching a pattern against a text and generalize it to work on graphs. Then, we show how the bit-vector data structure of that algorithm can be represented as a superposition of a logarithmic number of qubits. This approach allows to achieve better performances than the brute force algorithm for special types of graphs.

In the shift-and algorithm, we use bit vector \( D \) of the same length of pattern \( P \) to represent which of its prefixes are matching the text during the computation. Assuming integer-alphabet \( \Sigma \), we also initialize bidimensional array \( B \)
of size $|P| \times |\Sigma|$ so that $B[j][c] = 1$ if and only if $P[j] = c$, and $B[j][c] = 0$ otherwise. The algorithm starts by initializing vector $D$ to zero and array $B$ as specified above. Then, we scan whole text $T$ performing the next three operations for each $T[i]$, $i \in [1, n]$:

1. $D \leftarrow D + 1$;
2. $D \leftarrow D \land B[T[i]]$;
3. if $D[|P|] = 1$, return yes;
4. $D \leftarrow D << 1$.

Operation 1 sets the least significant bit of $D$ to 1, which is needed to test $P[1]$ against $T[i]$. Operation 2 computes a bit-wise and between $D$ and the column of $B$ corresponding to character $T[i]$. Remember that $B[j][T[i]] = 1$ means $P[j] = T[i]$, thus this operation leaves each bit $D[j]$ set to 1 if and only if it was already set to 1 before this step and the $j$-th character of the pattern matches the current character of the text. At this point, if bit $D[|P|]$ is set to 1 we have found a match for $P$, and Operation 3 will return yes. For the other positions, if bit $D[j]$ is set to 1, then we know that prefix $P[1...j]$ matches $T[i-j+1...i]$, and Operation 4 shift the bits in $D$ by one position, so that in the next iteration we will check whether $P[j+1]$ matches $T[i+1]$. We illustrate a simulation of the algorithm in Figure 1. In a labeled DAG $G = (V,E)$, each node $v$ has a single-character label $\ell(v)$. We generalize the shift-and algorithm to labeled DAGs by computing a different bit-vector $D_v$ for each node $v \in V$, initializing them to zero. Consider a DFS visit of DAG $G$. When visiting node $v$, each bit-vector $D_w$ of its in-neighbor $w \in \text{in}(v)$ represents a set of prefixes of $P$ matching a path in the graph ending in $w$. Thus, we merge all of this information together by taking the bit-wise or of all of the in-neighbors of $v$, that is we replace Operation 1 with $D_v \leftarrow 1 + \bigvee_{w \in \text{in}(v)} D_w$. Operations 2 and 3 are performed as before. An example of the state of the data structures at a generic iteration $i$ is shown in Figure 2 and the body of the iteration now is:

1. $D_v \leftarrow 1 + \bigvee_{w \in \text{in}(v)} D_w$;
2. $D \leftarrow D \land B[T[i]]$;
3. if $D[|P|] = 1$, return yes;
4. $D \leftarrow D << 1$.

### 3.3 Quantum Bit-Parallel Algorithm for Level DAGs

We make the classic techniques work in a quantum setting for a special class of DAGs, which we call level DAGs. A level DAG is a DAGs in which, for every two nodes $v$ and $w$, every path from $v$ to $w$ has the same length. Given any node
$T = \text{AAABB}$

$P = \text{AABB}$

$B | [A] | [B]$

|   | 0 | 1 | 0 |
|---|---|---|---|
| 0 | 1 | 0 |
| 1 | 1 | 0 |
| 2 | 0 | 1 |
| 3 | 0 | 1 |

$D \text{ init}$

$T[0] = \text{A}$

$T[1] = \text{A}$

$T[2] = \text{A}$

$T[3] = \text{B}$

$T[4] = \text{B}$

Figure 1: The classical ShiftAnd algorithm for matching pattern $P$ in text $T$. Matrix $B$ represents the positions at which each character is found in $P$. In this figure, we combined the shift operation of an iteration with the $+1$ operation of the next iteration.
Figure 2: The adaptation of the classical algorithm for matching pattern $P$ in DAG $G$. Each bit-vector $D_v$ represent the result after the merging of the bit-vectors of the in-neighbours of $v$ and before the shifting.
Figure 3: The adaptation of the classical algorithm for matching pattern $P$ in level DAG $G$. Each bit-vector $D_v$ represent the result after the merging of the bit-vectors of the in-neighbours of $v$ and before the shifting.

$s$ with no in-neighbours, we call level (or depth) the length $d(v)$ of any path from $s$ to $v$; the level $d(G)$ (or depth) of DAG $G$ is then $d(G) = \max\{d(v) : v \in V\}$. The DAG from Figure 2 is not a level DAG due to the horizontal edges, but without those it would be a level DAG, as in Figure 3.

Our approach aims to represent each bit vector $D_v$ with a single qubit $q_v$ set up in a proper superposition, and realize the bit-wise operations as parallel operations across such superposition. To this end, we use quantum register $I$ of size $O(\log |P|)$ to generate the superposition, one quantum register $D_v$ for each node $v \in V$ consisting of one qubit, ancilla qubits $A$, $R$ and $R'$ and additional register $C$ of size $O(\log |P|)$. Moreover, we assume to have access to a quantum random access memory (QuRAM).

Assume all the quantum registers to be initialized to $|0\rangle$. The algorithms starts by setting quantum register $I$ in a balanced superposition, by applying the Hadamard gate on each one of its qubits. In the following, we use the notation $Q|j\rangle$ to refer to the sub-state of register $Q$ that appears in the $j + 1$-th term of the superposition summation. For example, if $I = a|00\rangle + b|01\rangle +$
\[
\begin{array}{cccccccc}
I & D_1 & D_2 & D_n & A & R & R' \\
|00\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle \\
|01\rangle & |0\rangle & |0\rangle & \ldots & |0\rangle & |0\rangle & |0\rangle \\
|10\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle \\
|11\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle & |0\rangle \\
\end{array}
\]

Figure 4: The superposition resulting after putting register \( I \) in a equally balanced superposition. Each row represents a term of the sum in the superposition, where the amplitudes and the plus sign have been omitted.

c \( |10\rangle + d \|11\rangle \), then \( |I_2\rangle = |10\rangle \). Figure 4 shows a schematic representation of the resulting superposition. The rest of the algorithm maintains almost the same overall structure, with the exception of one necessary adaptation that we will discuss later. The main change consists in the fact that the four operations listed above have to be converted into operations on the quantum registers, to make them work across the superposition. This translation to bit-parallelism to superposition parallelism is the core of our technique, and we now describe how to apply it to each operation.

Operation 1 can be broken down into two simpler operations: computing the bit-wise \( or \) and adding 1. If we denote \( D_v \) as the qubit of current node \( v \), adding 1 means to set \( D_v[0] \) to \( 1 \), because \( D_v[0] \) is always guaranteed to be \( |0\rangle \) for any node \( v \) that we have not visited yet. This operation can be implemented by applying a \( CX \) gate on \( D_v \), using \( I \) as control register and \( 0 \) as the \( CX \) guard, an operation which we write as \( CX(0, I, D_v) \). This has the effect of flipping \( D_v[0] \), thus setting it to \( |1\rangle \), while leaving \( D_v[1], \ldots, D_v[|P| - 1] \) unchanged. The bit-wise \( or \) can be implemented as a series of updates \( D_v \leftarrow D_v \lor D_w \), one for each \( w \in in(v) \), because in the superposition this has the effect of a term-wise \( or \).

Operation 2 can similarly be implemented as \( D_v \leftarrow D_v \land B[I][\ell(v)] \), a term-wise \( and \) that uses the QuRAM to simultaneously access all the entries \( B[0][\ell(v)], \ldots, B[|P| - 1][\ell(v)] \) and compute \( D_v[j] \leftarrow D_v[j] \land B[j][\ell(v)] \), for every \( j \in [0, |P| - 1] \).

Operation 3 is now converted into updating the last sub-state \( R[|P| - 1] \) of ancilla qubit \( R \) as \( R[|P| - 1] \leftarrow R[|P| - 1] \lor D_v[|P| - 1] \), leaving \( R[0], \ldots, R[|P| - 2] \) unchanged. We implement this operation by performing \( CX(|P| - 1, I, A) \) and then computing \( R' \leftarrow (R' \lor D_v) \land A \) and finally \( R \leftarrow R \lor R' \). We then reset ancilla qubits \( A \) and \( R' \) to \( |0\rangle \). The need for this procedure will be clear later, and the fact that an occurrence was found or not will be reported by Grover’s Search phase at the end of the algorithm.

Operation 4 is the crucial one. Our goal is to cyclically shift the values of the bits across the superposition, meaning that this shift operation should perform the update \( D_v[j+1] \leftarrow D_v[j] \) on the values of \( D_v \), where \( j \) and \( j + 1 \) are intended modulo \(|P|\). We observe that, if we consider register \( I \) and any other register \( Q \), the shift \( Q[j+1] \leftarrow Q[j] \) is equivalent to \( I[j] \leftarrow I[j+1] \). This, in turn, can be
implemented as $I_j \leftarrow I_j + 1$, since the state of $I$ is a balanced superposition of every value between 0 and $|P| - 1$, thus $I_j$ simply stores $j$.

As mentioned above, we have to change the overall structure of the algorithm to handle the fact that each time that we perform the new shift operation, we are actually shifting all the vectors, not just the current $D_v$. This poses a problem when we process a node at a certain level and another node of the same level has already been processed. To overcome this situation, we visit the graph one level at a time, and we update all the $D_v$ vectors of that level before performing the shift. Algorithm 2 shows the entire procedure.

Algorithm 2: The algorithm for testing whether pattern string $P$ has a match in level DAG $G$, running in $O(|E| + \sqrt{|P|})$ time.

**Input:** Level DAG $G$, pattern string $P$.

**Output:** yes, if $P$ has a match in $G$, no otherwise.

1 $I \leftarrow |0\rangle$;
2 for $v \in V$ do
3 $D_v \leftarrow |0\rangle$;
4 $A \leftarrow |0\rangle$;
5 $R \leftarrow |0\rangle$;
6 $R' \leftarrow |0\rangle$;
7 $I^\otimes \log |P| I$;
8 for $s \in V$ such that $d(s) = 1$ do
9 $CX(0, I, D_s)$;
10 $D_s \leftarrow D_s \land B[I[\ell(v)]]$;
11 for $l \leftarrow 1$ to $d(G)$ do
12 for $v \in V$ such that $d(v) = l$ do
13 $CX(0, I, D_v)$;
14 $D_v \leftarrow \bigvee_{w \in \text{in}(v)} D_w$;
15 $D_v \leftarrow D_v \land B[I[\ell(v)]]$;
16 $CX(|P| - 1, I, A)$;
17 $R' \leftarrow D_v \land A$;
18 $R \leftarrow R \lor R'$;
19 $CX(|P| - 1, I, A)$;
20 $R' \leftarrow A$;
21 $I \leftarrow I + 1$;
22 Run Grover’s search with oracle function $f(I, D_1, \ldots, D_{|V|}, A, R, R') = R$;
23 Measure $R$ and return yes if $R = |1\rangle$, no otherwise.

As last step of the algorithm, we run Grover’s search that uses as oracle function an identity function returning the value of register $R$. Thus, if there is at least one $R|j| = |1\rangle$, the measurement at the end of the search will return $R = |1\rangle$ with high probability. Otherwise, if for every $j$ holds that $R|j| = |0\rangle$,
than the measurement of Grover’s search will yield $R_{jj} = |0\rangle$.

To prove the correctness of Algorithm 2, we formalise the key properties in the following lemmas. We start by ensuring that the shift operation provides the desired result.

**Lemma 1.** Operation $I \leftarrow I + 1$ transforms state

$$|\psi\rangle = \sum_{j=0}^{n-1} \alpha_j |Q_1_j\rangle \ldots |Q_n_j\rangle$$

into state

$$|\psi'\rangle = \sum_{j=0}^{n-1} \alpha_j |J_{j+1 \mod |P|} |Q_1_j\rangle \ldots |Q_n_j\rangle$$

for any number $n$ of qubits.

**Proof.** We observe that the state of $I$ is a balanced superposition of every value between 0 and $|P| - 1$, that is $I_j$ stores $j$, thus

$$|\psi\rangle = \sum_{j=0}^{n-1} \alpha_j |I_j\rangle |Q_1_j\rangle \ldots |Q_n_j\rangle.$$  

When we perform $I \leftarrow I + 1$, the effect is that of performing $I_j \leftarrow I_j + 1 \mod |P|$ for every $0 \leq j \leq |P| - 1$. After such operation, $I_j = j + 1 \mod |P|$, for every $0 \leq j \leq |P| - 1$, thus $|\psi\rangle$ has been transformed into $|\psi'\rangle$.  

Then, we need to guarantee two invariant conditions that have to hold after executing the inner for-loop (line 12).

**Lemma 2.** After running the inner for-loop in Algorithm 2 $i$ times (and before performing $I \leftarrow I + 1$), for every $D_v$ such that $d(v) = i$ holds that $D_v|j = |1\rangle$ if and only if there exists a path in $G$ ending at $v$ and matching $P[0, j]$.

**Proof.** We proceed by induction on the number $i$ of times that we run the inner for-loop.

**Base case**, $i = 0$. In this case, vectors $D_v$ such that $d(v) = 0$ are those with in-degree zero, initialized by the first for-loop, while the inner for-loop has never run. For each such $v$, the initialization for-loop initially sets $D_v|0$ to $|1\rangle$ ($CX$ operation), and then resets it to $|0\rangle$ if and only if it does not match $B[0]|\ell(v)|$ (and operation). Thus, the only $D_v|0$ set to $|1\rangle$ are the ones matching $P[0, 0]$, while $D_v|1, \ldots, D_v|P-1$ are correctly left to $|0\rangle$.

**Inductive case**, $1 \leq i \leq d(G)$. After running the inner for-loop $i$ times, we have to perform $I \leftarrow I + 1$ before running the for-loop for the $i+1$-th time. Assuming the inductive hypothesis and using Lemma 1, operation $I \leftarrow I + 1$ makes every $D_w$ with $d(w) = i$ such that $D_w|j = |1\rangle$ if and only if there is a match for $P[1, j-1]$ in $G$ ending at $w$. Then we run the for-loop for the $i+1$-th time. Notice that in any previous iteration of the for-loop it could never be
that case that $d(v) = i + 1$, and we update $D_v$ if and only if $d(v) = i + 1$, thus every $D_v|j$ is currently set to $|0\rangle$. At this point, for each $v$ such that $d(v) = i + 1$, we set $D_v|0 \rightarrow |1\rangle$ with operation $CX(0, I, D_v)$. Then, we perform the bitwise 
or and the bitwise and, after which $D_v|j = |1\rangle$ if and only if there is a match 
for $P[1\times j - 1]$ in $G$ ending at $v$ and $\ell(v)$ matches $P[j]$, where $(w, v) \in E$. This 
means that $D_v|j = |1\rangle$ if and only if a match for $P[1\times j - 1]$ ending at $w$ can be 
extended to a match for $P[1\times j]$ ending at $v$ using edge $(w, v)$.

**Lemma 3.** After running the inner for-loop in Algorithm 3 $i$ times (and before 
performing $I \leftarrow I + 1$), there exists at least one $j$ such that $R|j = |1\rangle$ if and only 
if there exists at least one $v$ such that $P$ has a match in $G$ ending at $v$, where 
$d(v) \leq i$.

**Proof.** Base case, $i = 0$. In this case, nodes $v$ such that $d(v) = 0$ are those with 
in-degree zero, while the inner for-loop has never run. Since we are visiting only 
single-node paths and we are assuming that pattern $P$ has length at least two, 
there can be no match for $P$ ending at these nodes. Correctly, every $R|j = |0\rangle$.

Inductive case, $1 \leq i \leq d(G)$. Right before running the inner for-loop 
for the $i + 1$-th time, every $D_v$ with $d(v) = i$ is such that $D_v|j = |1\rangle$ if and 
only if there is a match for $P[1\times j - 1]$ in $G$ ending at $v$. This follows from the 
same reasoning as in the proof of Lemma 2 and from the inductive hypothesis.

While running the inner for-loop for the $i + 1$-th time, we observe that during 
an iteration for a single node $v$ the statement of Lemma 2 restricted to only $v$ 
holds after line 15. This means that $D_v ||P|-1 \rightarrow |1\rangle$ if and only if $P[0\times |P| - 1]$ 
has a match ending at $v$. At this point we set up register $A$ with operation 
$CX(0, I, D_v)$, and then we run $R' \leftarrow D_v \land A$. This way, $R'[0\times |P| - 2]$ are set 
to $|0\rangle$ since $A[0] = |0\rangle \ldots A[|P| - 2] = |0\rangle$, while $R'|P|-1 = D_v|P|-1$ since $A|P|-1 = |1\rangle$. Operation $R \leftarrow R'$ leaves $R[0\times |P| - 2]$ unchanged, while $R|P|-1$ is set 
to $|1\rangle$ either if it was already $|1\rangle$ or if $D_v|P|-1 = |1\rangle$. Then, $A$ and $R'$ are reset 
to $|0\rangle$. If $R|P|-1$ was already $|1\rangle$, then the inductive hypothesis guarantees 
that at some earlier iteration a path matching $P$ was already found. If we 
turned $R|P|-1$ to $|1\rangle$ in this iteration, we know that this happened if and only 
if $D_v|P|-1 = |1\rangle$, which means that there is a path ending at $v$ that matches 
$P[0\times |P| - 1]$ (that is, the full pattern $P$). Either way, we are guaranteed that 
$R|P|-1 = |1\rangle$ if and only if there is a path matching $P$ that ends at node $v$, 
where $d(v) \leq i$.

The correctness of the algorithm follows from the previous lemmas combined 
with few additional observations.

**Theorem 1.** Given a pattern string $P$ of length at least 2 and level DAG $G$, if 
$P$ does not have a match in $G$ Algorithm 2 returns no; if $P$ has a match in $G$ 
Algorithm 2 returns yes with high probability.

**Proof.** After running the inner for-loop of Algorithm 2 $d(G)$ times we exit also 
the outer loop, having visited all the nodes. Thanks to Lemma 3 we know that 
there is at least one sub-state $R|j$ of register $R$ set to $|1\rangle$ if and only if there is 
a node $v$ such that $P$ has a match in $G$ ending at $v$, where $d(v) \leq d(G)$. 

14
moreover, condition $d(v) \leq d(G)$ means any node in the graph. Thus, if $P$ has no match in $G$, every sub-state $R_j$ is set to $|0\rangle$, the measurement will output a $|0\rangle$, thus the algorithm will return no. If $P$ has at least one match in $G$, there is at least one $R_j = |1\rangle$, which will be amplified by Grover’s search. In turn, the algorithm will measure a $|1\rangle$ and return yes with high probability. \hfill \square

Finally, the time complexity of our algorithm is linear in the size of the graph.

**Theorem 2.** The time complexity of Algorithm 2 is $O(|E| + |V|)$ in the QuRAM model, and the space complexity is $O(|P| + |V|)$.

**Proof.** The $or$ operation that we perform to update $D_v$ in the inner $for$-loop processes every edge once, and for each edge we perform a qubit $or$. The nested $for$-loops scan every node once and, for each node, we perform a constant number of quantum operations. We assume that $\log |P|$ fits in a quantum memory word (as it would be the case in a classical setting). Since we are scanning each node and edge once and each time we do a constant number of operations, the time complexity is $O(|E| + |V|)$. Matrix $B$ occupy $O(|P| |\Sigma|)$ qubits, where $\Sigma$ is the alphabet, and we use $|V|$ qubits $D_v$. Since we assume a constant alphabet, the space complexity is $O(|P| + |V|)$. \hfill \square

We can improve Algorithm 2 to run in the same complexity but using only $O(maxL)$ qubits, where $maxL$ is the number of nodes of the level with the highest number nodes. Notice that $maxL \leq W$ always, where $W$ is the minimum number of non-disjoint paths needed to cover all the nodes, that is the width of the DAG.

**Theorem 3.** The problem of SMLG on level DAG $G = (V, E)$ and pattern $P$ can be solved in $O(|E| + \sqrt{|P|})$ time using $O(|P| + maxL)$ qubits.

**Proof.** We can modify Algorithm 2 to satisfy the statement of the theorem. Instead of using a qubit $D_v$ for each node $v \in V$, we use $2maxL$ qubits, $X_1, \ldots, X_{maxL}$ and $Y_1, \ldots, Y_{maxL}$. This is because, by definition of width, for each level $l$ there are at most $maxL$ nodes $v$ such that $d(v) = l$. We also use classical arrays $A_X$ and $A_Y$, both f size $|V|$, which map every node $v$ to the qubit representing its bit-vector (we use two arrays for symmetry with the qubits but, since they will store the same values, one would be enough). We were already implicitly using a similar data structure in the unmodified Algorithm 2 but in that case we could assume to initialize it once at the beginning and never change it later. Here, we need to handle these arrays explicitly.

At each iteration $i$ of the first $for$-loop (line 8), we visit each node $s \in V$ such that $d(s) = 1$ and we set $A_X[s] \leftarrow i$, which means that qubit $X_i$ represents the bit-vector for node $s$. We then perform the same operations of this $for$-loop as before, replacing $D_v$ with $X_i$.

At each iteration $j$ of the inner $for$-loop (line 12), we set $A_X[s] \leftarrow j$ and $A_Y[s] \leftarrow j$, then we perform the same operations as before replacing $D_v$ with $Y_{A_Y[v]}$ and $D_w$ with $X_{A_X[w]}$. After the execution of this entire $for$-loop, right
before or right after $I \leftarrow I + 1$, we switch the role of $X$ and $Y$ by setting $X_k \leftarrow Y_k$, $Y_k \leftarrow |0\rangle$, for each $1 \leq k \leq maxL$.

Notice that the positions of $A_X$ and $A_Y$ that we access at iteration $j$ are those of the current node, initialized at the beginning of the same iteration, and of the in-neighbors of said node, which have been initialized at some previous iteration. Thus, the positions of $A_X$ and $A_Y$ are always correctly initialized when we access them. Moreover, the update $X_k \leftarrow Y_k$, $Y_k \leftarrow |0\rangle$, for each $1 \leq k \leq maxL$, preserves the following invariant: after performing this update $i$ times, thus having run the inner for-loop $i$ times, qubits $X_1, \ldots, X_{maxL}$ store the bit-vectors of the nodes of level $j$. This holds true after the initialization for-loop, and can be seen to hold true by an induction argument on $i$ similar to the ones in the previous lemmas. We conclude that, thanks to this invariant, the algorithm is correct.

The new update operation $X_k \leftarrow Y_k$, $Y_k \leftarrow |0\rangle$, for each $1 \leq k \leq maxL$, takes at most $O(maxL)$ time for each level $l$, thus $O(|V|)$ time in total, not affecting the overall time complexity. Instead, the number of qubits that we are using is $2maxL$ plus a constant, that is the ancilla qubits and register $I$; thus, we use $O(|P| + maxL)$ qubits in total. This is because data structure $O(B)$ requires $O(|P|\Sigma)$ space to be stored, where $\Sigma$ is the alphabet, that we assume to have constant size.

4 Discussion

Although we studied level DAGs here just to give an example of breaking the quadratic barrier, such DAGs have connections to earlier literature. Namely, degenerate strings [1] are a special case of level DAGs, so the algorithms developed here apply directly on them. We leave it as open question if similar solutions can be derived for arbitrary DAGs, or e.g. for elastic degenerate strings [5]. Or more generally, can bit-parallel algorithms with more complex dependencies, like in the case of approximate pattern matching, be turned into quantum algorithms?

Acknowledgements

We would like to thank Sabrina Maniscalco for giving useful feedback on technical details of the quantum computing framework.

References

[1] Mai Alzamel, Lorraine A. K. Ayad, Giulia Bernardini, Roberto Grossi, Costas S. Iliopoulos, Nadia Pisanti, Solon P. Pissis, and Giovanna Rosone. Comparing degenerate strings. Fundam. Informaticae, 175(1-4):41–58, 2020.
[2] Arturs Backurs and Piotr Indyk. Edit distance cannot be computed in strongly subquadratic time (unless SETH is false). *SIAM J. Comput.*, 47(3):1087–1097, 2018.

[3] Ricardo A. Baeza-Yates and Gaston H. Gonnet. A new approach to text searching. *Commun. ACM*, 35(10):74–82, 1992.

[4] Djamal Belazzougui, Fabio Cunial, Juha Kärkkäinen, and Veli Mäkinen. Linear-time string indexing and analysis in small space. *ACM Trans. Algorithms*, 16(2), March 2020.

[5] Giulia Bernardini, Nadia Pisanti, Solon P. Pissis, and Giovanna Rosone. Approximate pattern matching on elastic-degenerate text. *Theor. Comput. Sci.*, 812:109–122, 2020.

[6] Massimo Equi, Roberto Grossi, Veli Mäkinen, and Alexandru I. Tomescu. On the complexity of string matching for graphs. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece*, volume 132 of *LIPIcs*, pages 55:1–55:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[7] Donald E. Knuth, James H. Morris Jr., and Vaughan R. Pratt. Fast pattern matching in strings. *SIAM J. Comput.*, 6(2):323–350, 1977.

[8] Gene Myers. A fast bit-vector algorithm for approximate string matching based on dynamic programming. *J. ACM*, 46(3):395–415, 1999.

[9] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010.

[10] Pradeep Niroula and Yunseong Nam. A quantum algorithm for string matching. *npj Quantum Information*, 7(1):1–5, 2021.

[11] H Ramesh and V Vinay. String matching in o(n+m) quantum time. *Journal of Discrete Algorithms*, 1(1):103–110, 2003. Combinatorial Algorithms.

[12] Mikko Rautiainen, Veli Mäkinen, and Tobias Marschall. Bit-parallel sequence-to-graph alignment. *Bioinform.*, 35(19):3599–3607, 2019.

[13] Kapil Kumar Soni and Ashwini Kumar Malviya. Design and analysis of pattern matching algorithms based on quram processing. *Arabian Journal for Science and Engineering*, 46(4):3829–3851, 2021.