Quantum Mechanics with Difference Operators

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Abstract  
A formulation of quantum mechanics with additive and multiplicative (q-) difference operators instead of differential operators is studied from first principles. Borel-quantisation on smooth configuration spaces is used as guiding quantisation method. After a short discussion this method is translated step-by-step to a framework based on difference operators. To restrict the resulting plethora of possible quantisations additional assumptions motivated by simplicity and plausibility are required. Multiplicative difference operators and the corresponding q-Borel kinematics are given on the circle and its N-point discretisation; the connection to q-deformations of the Witt algebra is discussed. For a “natural” choice of the q-kinematics a corresponding q-difference evolution equation is obtained. This study shows general difficulties for a generalisation of a physical theory from a known one to a “new” framework.

Key words: nonlinear Schrödinger equation, Witt algebra, q-deformation, discrete derivative.
1 Introduction

1.1 Motivations

There are different arguments to use difference operators \( D_{[x_1,x_2]} \) instead of differential operators acting on suitable (complex) function spaces. Such operators are built out of difference quotients, e.g.
\[
D_{[x_1,x_2]} f(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_2 \neq x_1, x_1, x_2 \in \mathbb{R};
\]
for \( x_2 \to x_1 \) one gets the usual derivative. They can be generalized e.g. to different types of lattices in \( C \).

We mention some of the reasons:

A. Fundamental Remark:

Heisenberg wrote in 1930 in “The Physical Principles of Quantum Theory” [1]: "...it seems necessary to demand that no concept enters a theory which has not been experimentally verified at least to the same degree of accuracy as the experiments explained by the theory.” Hence one can argue that e.g. difference quotients instead of differentials are a more “physical” mathematical model for momenta, and that Borel sets instead of points are more appropriate to describe localization in physical space-time.

B. Space-Time as a lattice:

There are physical reasons to assume that the configuration space and the time, or both, have a lattice structure which is embedded into a smooth manifold. A physical theory on a given lattice can be formulated with the help of difference operators if e.g. a configuration space is not given but constructed from a representation of a Lie algebra (or a deformed Lie algebra) which contains generators interpreted as position operators. It is reasonable in this framework to identify the spectra of these position operators, which may be continuous and/or discrete, as configuration space [2, 3, 4, 5].

C. Quantum Symmetry:

If one assumes that a theory is based on an algebra which is a deformation in the sense of Drinfeld [6], i.e. a \( q \)-deformation of a Lie algebra, one gets realizations on Hilbert spaces spanned by functions on suitably chosen spaces. The generators of the algebra are given in terms of (discrete) \( q \)-derivatives and one finds, e.g., \( q \)-difference equations for the evolution [7].

D. Numerical Methods:

For approximate solutions of PDE lattice methods difference equations are useful. To check their accuracy, it is reasonable to base already the theory, which yields the PDE, on difference operators.

In all these approaches one expects, that the formulation in terms of difference operators yields in the limiting case of differential operators the “usual” formulation. However, because of the different algebraic properties of difference operators this may not be the case.

1.2 Choices for difference quotients and difference operators.

There are two principally different canonical types of difference quotients acting on appropriately chosen function spaces.

We start with a function space over \( \mathbb{R} \), \( F[\mathbb{R}^1] \), where we have the following two options:
1. The **additive type** (based on the additive unit \( a \in \mathbb{R}^1, a \neq 0 \))

\[
D^a f(x) = \frac{f(x + a) - f(x - a)}{(x + a) - (x - a)} = \frac{f(x + a) - f(x - a)}{2a}.
\]

(2)

2. The **multiplicative type** (based on the multiplicative unit \( q \in \mathbb{R}^1, q \neq 1 \))

\[
D^q f(x) = \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x} = \frac{1}{x} \frac{f(qx) - f(q^{-1}x)}{q - q^{-1}}.
\]

(3)

These quotients can also be viewed as operators acting on function spaces over lattices in \( \mathbb{R}^1 \) of the following types:

1. The **additive type** (uniform \( a \)-lattice)

\[
\mathcal{L}_a := \{ x_0 + ja | j \in \mathbb{Z}, x_0 \in \mathbb{R} \}.
\]

(4)

2. The **multiplicative type** (\( q \)-lattice, \( q \) real)

\[
\mathcal{L}_q := \{ x_0q^j | j \in \mathbb{Z}, x_0 \in \mathbb{R}, x_0 \neq 0 \}.
\]

(5)

The above can be extended to the complex case which we need in order to consider functions on the unit circle \( S^1 \). In particular, for \( F[S^1] \) both lattices appear as uniform N-point discretizations \( S^1_N \) of \( S^1 \) depending on the parametrization of \( S^1 \) through \( \phi \in [0, 2\pi) \) or through \( z = e^{i\phi} \). The additive type occurs for \( \phi_j = \frac{2\pi j}{N} \):

\[
S^1_N(a) := \{ \phi_j | \phi_j = aj, j = 0, \ldots, N - 1 \}, \quad a = \frac{2\pi}{N},
\]

(6)

and \( D^a \) acts on \( F[S^1_N(a)] \). For \( z_j \in \mathbb{C} \) we get the multiplicative type:

\[
S^1_N(q) := \{ z_j | z_j = q^j, j = 0, \ldots, N - 1 \}, \quad q = e^{i\frac{2\pi}{N}},
\]

(7)

and \( D^q \) acts on \( F[S^1_N(q)] \). Note that because of the coordinatisation, \( q \) appears as a phase factor.

From \( q \)-difference quotients we construct \( q \)-difference operators

\[
D^q f(x) = \frac{g_1(q, x) f(qx) - g_2(q, x) f(q^{-1}x)}{x(q - q^{-1})} + g_3(q, x)
\]

(8)

with \( \lim_{q \to 1} g_i(q, x) = 1, i = 1, 2, \lim_{q \to 1} g_3(q, x) = 0 \). There is a plethora of possibilities, indicating that one needs additional physical and mathematical arguments to select a reasonable class of difference operators.

In connection with the use of difference operators we introduce additive and multiplicative shift operators \( K^a \) and \( K^q \). The latter may depend in addition on \( k \in \mathbb{Z} \), acting on suitably chosen (e.g. polynomial) complex functions \( f(x) \) as follows:

\[
K^a f(x) = f(x + a), \quad K^a = \exp(a \frac{d}{dx})
\]

(9)

\[
K^q_k f(x) = f(q^k x), \quad K^q_k = q^{k x \frac{d}{dx}} = \exp \left( (\ln q) k x \frac{d}{dx} \right)
\]

(10)
\( a\)- and \( q\)-numbers (operators) are defined as \((A\) denotes a number or an operator\)

\[
[A]_a = \frac{\exp aA - \exp(-aA)}{2a},
\]

\[
[A]_q = \frac{q^A - q^{-A}}{q - q^{-1}},
\]

with \(\lim_{a\to0}[A]_a = A\) and \(\lim_{q\to1}[A]_q = A\). The difference quotients (2) and (3) – for \(k = 1\) – can be written as

\[
D^a f(x) = \left[\frac{d}{dx}\right]_a f(x) = \frac{K^a - K^{-a}}{2a} f(x),
\]

\[
D^q_k f(x) = \frac{1}{x} \left[ k x \frac{d}{dx}\right]_q f(x) = \frac{1}{x} \frac{K^q_k - K^{-q}}{q - q^{-1}} f(x).
\]

The decomposition rule is

\[
[A + B]_a = [A]_a \exp(\epsilon aB) + [B]_a \exp(-\epsilon aA),
\]

\[
[A + B]_q = [A]_q \epsilon^B q - [B]_q \epsilon^{-A},
\]

with \(\epsilon = \pm 1\).

1.3 A path to \(q\)- and \(a\)- quantum mechanics

A (non relativistic) quantum system is given through a quantisation map \(Q\) which maps a certain class of classical observables under physically and mathematically motivated conditions into the set \(SA(\mathcal{H})\) of self-adjoint operators on a Hilbert space \(\mathcal{H}\). In a first step the kinematics of the classical system, that is the class of generalized position and momentum observables, is quantised. There are different methods to construct such a quantisation map: the canonical quantisation for systems localized on \(\mathbb{R}^n\), the geometric quantisation and the Borel quantisation on smooth manifolds \(M\) and the inhomogeneous current algebra on \(\mathbb{R}^n\). All these quantisations model the momentum observables through a partial differential operator on \(\mathcal{H}\). In a second step, a time dependence is introduced.

If one is interested to formulate quantum mechanics, e.g. on one-dimensional manifolds, with difference operators \(D^a\) or \(D^q\), one has to shape the quantisation map accordingly (\(a\)- and \(q\)-quantisations). The quantised kinematics is expected to yield an evolution difference equation. One gets a Schrödinger equation with difference operators from first principles and not by inserting difference operators into the usual Schrödinger equation.

Because the mentioned quantisation methods are (partly) based on realizations of Lie algebras we expect that \(a\)- or \(q\)-quantizations lead to realizations which contain difference quotients and shift operators. In particular, \(D^q\) should yield results which are equal or similar to those from the representation theory of \(q\)-deformed Lie algebras, including also deformed Lie brackets.

We present in the following paths to construct an \(a\)- and \(q\)-quantisation involving \(D^a\) and \(D^q\). In section 2 we shortly review Borel quantisation \(\mathcal{B}\) on smooth manifolds with the example of \(S^1\). In section 3 we formulate a version for a \(q\)-quantisation (which is equivalent to an implicit \(a\)-quantisation, see 3.2.3) on \(C^\infty(\mathbb{S}^1)\) and \(F[S^1_1(q)]\). The study contains a discussion (section 3.2) of different deformations of the inhomogeneous Witt algebra \(W\). For \(F[S^1_1(q)]\) we calculate a family of evolution difference equations; in the “continuous limit” this family is larger than a family which we got from the quantisation map for systems on \(L^2(S^1, d\Phi)\).

This is a case study of the structure of quantum mechanics based on difference operators modelling momentum observables, a corresponding deformation of the kinematical structure and the resulting dynamics. We present no numerical simulations, which are in addition to the formal roots essential to get further physical insight into the model.
2 Short review of Borel quantisation with application to $S^1$

We consider (non relativistic, point–like) systems $S$ moving and localized on a smooth manifold (with measure $\mu$) and specialize later to $S^1$.

2.1 The kinematics

To model possible localization regions of $S$ on $M$ and possible infinitesimal movements of the regions we choose for the regions Borel sets $B$ from a Borel field $\mathcal{B}(M)$ and for the movements (smooth, complete) vector fields $X \in \text{Vect}_0(M)$. These two geometrical objects are the building blocks of the kinematics $\tilde{K}(M)$ of $S$:

\[
\tilde{K}(M) = (\mathcal{B}(M), \text{Vect}_0(M)).
\]

(17)

Borel sets are displaced through $X$ by its flow $\Phi^X_\tau$ as $B' = \{m' | m' = \Phi^X_\tau(m), m \in B, X \in \text{Vect}_0(M), \tau \in [0,1)\}$. For a quantisation of $\tilde{K}(M)$ one has to construct a map $Q = (Q, P)$ which maps the blocks in $\tilde{K}(M)$ into the set $SA(H)$ of self-adjoint operators on a Hilbert space $H$. It is reasonable to interpret the matrix elements of $Q(B)$, i.e. $(\psi, Q(B)\psi)$, $\psi \in H$, as the probability to find the system localized in $B$ in a state $\psi$. The properties of $\mathcal{B}(M)$ and further physical requirements (e.g. no internal degrees of freedom) show that $Q(B)$ acts on $H$ as the characteristic function $\chi(M)$ of $M$, if $H$ is realized via square integrable functions over $M$, i.e. as $L^2(M, d\mu)$. From the spectral theorem and from $Q(\mathcal{B}(M))$ we infer a quantisation map for $C^\infty(M, \mathbb{R})$

\[
Q : C^\infty(M, \mathbb{R}) \to SA(H), \quad Q(f)\psi = f\psi.
\]

(18)

Hence we can use in the kinematics $C^\infty(M, \mathbb{R})$ instead of $\mathcal{B}(M)$, that is the kinematics can be viewed as an infinite dimensional Lie algebra, more precisely as a semidirect sum of the abelian algebra $C^\infty(M, \mathbb{R})$ and a subalgebra of the Lie algebra of vector fields, and we denote it as $K(M)$ (without tilde) in the following:

\[
K(M) = C^\infty(M, \mathbb{R}) \oplus \text{Vect}_0(M).
\]

(19)

To construct the quantisation map $P$ for $\text{Vect}_0(M)$ we need further assumptions, which we will call $P$-assumptions:

1. The Lie structure of $K(M)$ is conserved.

2. The operator $P(X)$ are – in analogy to the canonical quantisation in $\mathbb{R}^n$ – local differential operators.

With these $P$-assumptions we have the following result [8]:

The $P(X)$ are differential operators of order one with respect to a differential structure on the set $M \times \mathbb{C}$. We characterize this structure (up to isomorphism) through Hermitian line bundles $L$ over $M$ with compatible flat connection $\nabla$. Wave functions are sections $\sigma(m)$ in the bundle and $L^2(M, \mu)$ can be viewed as a space of square integrable sections. Unitary equivalent irreducible maps $Q^{(\alpha, D)}$ – quantisations – are given by a bijective mapping onto the set

\[
(\alpha, D) \in \pi_1^*(M) \times \mathbb{R}.
\]

(20)

$\pi_1^*(M)$ denotes the dual of the first fundamental group of $M$, a topological quantity. $D$ is connected with the algebraic structure of $K(M)$ and characteristic for Borel quantisation. $(\alpha, D)$ are quantum
numbers in the sense of Wigner. $Q$ is labeled by these numbers, i.e. $Q^{(\alpha,D)} = (Q^{(\alpha,D)}, P^{(\alpha,D)})$ and one has $(m \in M)$

$$Q^{(\alpha,D)}(f)\sigma(m) = f(m)\sigma(m) \quad \text{(21)}$$

$$P^{(\alpha,D)}(X)\sigma(m) = \left(-i\nabla_X + \left(-\frac{1}{2}i + D\right) \text{div}_\mu g\right)\sigma(m), \quad \text{(22)}$$

which are self-adjoint operators on a common dense set. Here, $\nabla_X$ denotes the connection in the line bundle $L(M)$ over $M$ and $\text{div}_\mu$ denotes divergence. Note that the quantum number $D$ appears as a real factor in front of $\text{div}_\mu g$ and that the nontrivial topology yields the $\alpha$ dependence of $\nabla_X^\alpha$.

2.2 The dynamics

States of $S$ are modeled via density matrices $W$, i.e. through trace class operators with $\text{Tr}(W) = 1$. We introduce a time dependence for $W$ (in the Schrödinger picture), which is based on $Q^{(\alpha,D)}$, through a quantum analog to the classical relation between time derivatives $\frac{d}{dt}$ of time dependent functions $f(m(t))$ and momenta, i.e. for $M = \mathbb{R}^1$

$$\frac{d}{dt} f(x(t)) \sim p\nabla f . \quad \text{(23)}$$

One can show \cite{12} that (in the Schrödinger picture) one has the following relation for expectation values $(\text{Exp}_W(A) = \text{Tr}(WA))$: 

$$\frac{d}{dt} \text{Tr}(W(t)Q^{(\alpha,D)}(f)) = \text{Tr}(W(t)P^{(\alpha,D)}(X_{\text{grad}f})) , \forall f \in C^\infty(M, \mathbb{R}) \quad \text{(24)}$$

This is a restriction for the evolution of $W(t)$. For pure states it implies, under the condition that pure states evolve into pure states \cite{12}, the following generalized version of the first Ehrenfest relation

$$\frac{d}{dt} \left\langle \sigma(t), Q^{(\alpha,D)}(f)\sigma(t) \right\rangle = \left\langle \sigma(t), P^{(\alpha,D)}(X_{\text{grad}f})\sigma(t) \right\rangle , \quad \forall f \in C^\infty(M, \mathbb{R}) \quad \text{(25)}$$

with a scalar product $(.,.)$ in $L^2(M, d\mu)$.

2.3 The kinematical algebra $K(S^1)$ and a family of evolution equations

We consider now an application of Borel quantization to the case that the configuration space is $S^1$. $S^1$ is topologically nontrivial with $\pi_1(S^1) = \{0, 2\pi\}$, and we denote elements in $\pi_1(S^1)$ as $\alpha$. The flat line bundles over $S^1$ are trivial, the vector fields are $X = X(\phi)\frac{d}{d\phi} \in \text{Vect}_0(S^1)$ and the Hilbert space is $L^2(S^1, d\phi)$. In these coordinates $K(S_1)$ is given by the generators

$$Q^{(\alpha,D)}(f)\psi(\phi) = f(\phi)\psi(\phi) \quad \text{(26)}$$

$$P^{(\alpha,D)}(X)\psi(\phi) = \left(-iX(\phi)\frac{d}{d\phi} + \left(-\frac{1}{2}i + D\right) \frac{dX(\phi)}{d\phi}\right) + \alpha X(\phi)\right)\psi(\phi) . \quad \text{(27)}$$

To analyse the structure of $K(S_1)$ we use a Fourier transform $\mathcal{F}$ with $z = e^{i\phi}$:

$$f(\phi) = \hat{f}(z(\phi)) = \sum_{n=-\infty}^{\infty} f_n z^n \quad \text{(28)}$$

$$X(\phi) = \hat{X}(z(\phi)) = \sum_{n=-\infty}^{\infty} X_n z^n ,$$
\[ f_n = \bar{f}_{-n}, \quad X_n = \bar{X}_{-n}. \] For the \( F \)-transformed quantum kinematics we find
\[
Q^{(\alpha,D)}(f) = \sum_{n=-\infty}^{\infty} f_n z^n \\
P^{(\alpha,D)}(X) = \sum_{n=-\infty}^{\infty} X_n z^n \left( z \frac{d}{dz} + \frac{n}{2} + \alpha + iDn \right).
\]

With the operators
\[
T_n = z^n \\
L_n^\alpha = z^n \left( z \frac{d}{dz} + \frac{n}{2} + \alpha \right)
\]

(29) can be expressed as
\[
Q^{(\alpha,D)}(f) = \sum_{n=-\infty}^{\infty} f_n T_n \\
P^{(\alpha,D)}(X) = \sum_{n=-\infty}^{\infty} X_n (L_n^\alpha + iDnT_n).
\]

The generators \( T_n \) are an Abelian Lie algebra which we denote as \( T \), and for fixed \( \alpha \in [0,2\pi) \) the \( L_n^\alpha \) fulfill the commutation relations
\[
[T_m, T_n] = 0 \\
[L_n^\alpha, T_m] = mL_{m+n}^\alpha \\
[L_n^\alpha, L_m^\alpha] = (n-m)L_{m+n}^\alpha.
\]

(32) and span an inhomogenisation of the Witt algebra \( W \) through \( T \). This gives the algebraic structure of \( K_z(S^1) \) \[9\], where we use the index \( z \) to indicate that it is given in terms of the variable \( z \) as opposed to the angle variable \( \phi \). We have from (32) (we have dropped the upper index \( \alpha, D \) for convenience)
\[
[Q(f),Q(g)] = 0, \quad [P(X),Q(f)] = -iQ(Xf), \quad [P(X),P(Y)] = -iP([X,Y]).
\]

Now we introduce the time dependence for pure states \( \psi(\phi) \in L^2(S^1, d\phi) \) and evaluate the restriction (25) with \( X_{\text{grad}}f = f'(\phi) \frac{d}{d\phi} \left( \equiv \frac{d}{d\phi} \right) \):
\[
\frac{d}{dt} (\psi, f\psi) = \left( \psi, P^{(\alpha,D)} \left( f' \frac{d}{d\phi} \right) \psi \right), \quad \forall f \in C^\infty(S^1, \mathbb{R}).
\]

(33) This implies a generalized continuity equation of Fokker-Planck type for \( \rho = \tilde{\psi}\psi \):
\[
\dot{\rho} = \frac{i}{2} (\tilde{\psi}\psi'' - \tilde{\psi}'\psi') + D\rho'' - \alpha\rho' = -(j_0^\alpha)' + D\rho'',
\]

(34) where
\[
j_0^\alpha = \frac{i}{2} (\tilde{\psi}\psi - \tilde{\psi}'\psi') + \alpha\rho
\]
corresponds for vanishing \( \alpha \) to the usual quantum mechanical current density on \( S^1 \).

This can be derived also by other methods based on \( Q^{(\alpha,D)} \) \[8], \[9\]. We use this information in \[9\] for a general Ansatz for a Schrödinger equation of the type
\[
i\partial_t \psi = H\psi + G(\tilde{\psi}, \psi)\psi
\]
in which $H$ is a linear operator and $G[\bar{\psi}, \psi]$ can be written (formally) as a nonlinear complex function $G[\bar{\psi}, \psi] = G_1[\bar{\psi}, \psi] + iG_2[\bar{\psi}, \psi]$ depending on $\bar{\psi}, \psi$, their derivatives and explicitly on $\phi$ and $t$. Hence $G$ acts as a multiplication operator. This Ansatz leads to a family $F_P$ of Schrödinger equations \[16\], \[17\] on $L^2(S^1, d\phi)$ with $G_2$ enforced by \[34\]

\[
\begin{align*}
    i\partial_t \psi &= -\frac{1}{2} \frac{d^2}{d\phi^2} \psi - i\alpha \frac{d}{d\phi} \psi + i\frac{D}{2p} \left( \frac{d^2}{d\phi^2} \rho \right) \psi + G_1[\bar{\psi}, \psi] \psi. 
\end{align*}
\]

The real part $G_1$ cannot be determined by Borel quantization. Hence a set of (natural) assumptions for $G_1$ motivated by the form of the imaginary part $G_2$, has been introduced \[13\], \[15\]:

1. $G_1$ is proportional to $D$, i. e. vanishing for $D = 0$.
2. $G_1$ is a rational function with derivatives no higher than second order and occurring in the numerator only.
3. $G_1$ is complex homogeneous of order zero, i. e. $G_1[\alpha \psi, \bar{\alpha} \bar{\psi}] = G_1[\psi, \bar{\psi}]$ for all $\alpha \in \mathbb{C}$.

These assumptions restrict $G_1$ in the family $F_P$ to the Doebner-Goldin family (DG-family) $F_{DG}$ \[13\] on $S^1$:

\[
G_1[\bar{\psi}, \psi] := D_1 \left( \frac{j_0^2}{\rho} \right)' + D_2 \rho'' + D_3 \left( \frac{j_0^2}{\rho} \right)^2 + D_4 \left( \frac{j_0^2 \rho'}{\rho^2} \right) + D_5 \left( \frac{\rho'}{\rho^2} \right)^2 
\]

with free real parameters $D_k$, $k = 1, \ldots, 5$.

3 An $a$- and $q$- quantisation of the kinematical algebra $K(S^1)$

3.1 Strategy and remarks

After the review of Borel quantisation and their application to the configuration space $S^1$ we follow our programme to construct an analogous realisation of $K(S^1)$ with difference instead of differential operators. For this we need guiding principles. As in section 2 we use $L^2(S^1, d\Theta)$ or the Hilbert space of functions over the lattice $S_N^1(q)$ with elements $\psi = (\psi(0), \ldots, \psi(N-1))$, $\psi(j) = \psi_j \in \mathbb{C}$, with inner product

\[
(\psi, \phi) = c(\psi, \phi) \sum_{j=0}^{N-1} \bar{\psi}_j \phi_j, \tag{37}
\]

where $c(\psi, \phi)$ is a suitable normalisation. For the operator $Q(f)$ we use the corresponding multiplication operator. If applied to difference or shift operators the P-assumptions in section 2 fail to determine $P(X)$ \[\text{2}\] and we need further principles.

We start from the results for $K(S^1)$ in section 2. Our strategy for $P(X)$ is to replace first order differential operators in $K_2(S^1)$ through “first order” difference operators or shift operators. Another option in the plethora of possibilities is a deformation of the Lie bracket. The following assumptions – called d-assumptions\[\text{3}\] – will be implemented:

1. The difference (shift) operators should be chosen such that the corresponding realisation of $K(S^1)$ is again an algebra with a Lie bracket or a deformed Lie bracket. We cannot expect a realisation isomorphic to $K(S^1)$ but a more general deformation of $K(S^1)$, e. g. a non-commutative and/or non-cocommutative Hopf algebra or a higher order Lie algebra.

\[\text{2}\] To realise $P(X)$ as differential operator we have introduced a differentiable structure on $M \times \mathbb{C}$ via a complex line bundle over $M$; the algebraic properties of $K(M)$ restrict the order of $P(X)$ to one.

\[\text{3}\] d for deforms
2. In the limit \( a \to 0 \) or \( q \to 1 \) the realisation should give the “old” result.

3. If the difference (shift) operators (see e.g. (8)) depend on additional objects (like constants, functions) the number of these objects should be as small as possible and they require a physical interpretation.

4. We model the influence of the topology of \( S^1 \) through the term proportional to \( \alpha \) in \( K_z(S^1) \).

5. The Fokker-Planck type equation for the time dependence of the positional density should have a reasonable interpretation.

We add a more technical remark:
The multiplicative and additive difference operators are constructed from shift operators \( K^a \) and \( K^q \).
The algebra \( K_z(S^1) \) acts on complex polynomial functions \( \hat{f}(z) \) on \( \{ z | z = e^{i\Phi} \} \). To use difference operators in \( K_z(S^1) \) this space must be invariant under the shift operators. For \( K^q \) we have

\[
K^q \hat{f}(z) = \hat{f}(qz), \quad q = e^{i\Phi_0}.
\]  
(38)

Hence it is necessary to use \( q \) which is a phase here. From

\[
\hat{f}(q e^{i\Phi}) = \hat{f}(e^{i\Phi + i\Phi_0})
\]  
(39)

we infer that \( K^q \) acts as an implicit additive shift operator \( I^a \) defined as

\[
I^a \hat{f}(e^{i\Phi}) = \hat{f}(e^{i\Phi + a}), \quad a = i\Phi_0.
\]  
(40)

An explicit additive shift operator \( K^a \) with

\[
K^a \hat{f}(z) = \hat{f}(z + a), \quad a = z_0,
\]  
(41)

does not exist in this space. This is one of the reasons to use \( q \)-quantisations. Furthermore, we denote

\[
z \frac{d}{dz} = N_z .
\]  
(42)

3.2 Deformations of the inhomogeneous Witt algebra

3.2.1 The multiplicative setting: a natural Ansatz

With our \( d \)-assumptions we replace in the generators \( L^\alpha_n \) in \( K_z(S^1) \)

\[
L^\alpha_n = z^n A^\alpha_n, \quad A^\alpha_n = N_z + \frac{n}{2} + \alpha,
\]  
(43)

the differential through \( q \)-difference operators and the generators \( T_n \) of \( T \) (formally) through \( T_n \) with

\[
T_n = T_n .
\]  
(44)

A comparison with (27) shows that the origin of the factor \( z^n \) in (43) is the function \( X(\phi) \) which corresponds to the position observable. The differential \(-i \frac{d}{d\phi}\) in (27) implies \( z \frac{d}{dz} \) and also the term \( \frac{n}{2} \). We assume to handle \( \alpha \) in the same way as \( \frac{n}{2} \). Hence, we try the Ansatz

\[
L^\alpha_n \mapsto L^\alpha_n = z^n [A^\alpha_n]_q, \quad q = \exp i\phi_0 .
\]  
(45)

\footnote{In the discrete case \( a \) and \( q \) thus “feel” the topology of \( S^1 \).}
The $\mathcal{L}_n^\alpha$ should close under Lie brackets, but this is not the case which indicates a more general Ansatz. Indeed, with an additional running parameter $k \in \mathbb{Z}$ and $q = \exp i\phi_0$ fixed, the

$$\mathcal{L}_{n,k}^\alpha = z^n \frac{[kA_n^\alpha]}{[k]_q}, \quad k, n \in \mathbb{Z}, \quad \alpha \in \mathbb{R} \tag{46}$$

form a Lie algebra, a $q$-deformed Witt algebra $W_q$. The commutators are for $j_1 \neq j_2$

$$[\mathcal{L}_{m,j_1}^\alpha, \mathcal{L}_{n,j_2}^\alpha] = \left( \frac{j_1 \frac{\pi}{2} - j_2 \frac{\pi}{2}}{j_1 [j_2]} \right) [j_1 + j_2] \mathcal{L}_{m+n,j_1+j_2}^\alpha + \left( \frac{j_1 \frac{\pi}{2} + j_2 \frac{\pi}{2}}{j_1 [j_2]} \right) \mathcal{L}_{m+n,j_2-j_1}^\alpha \tag{47}$$

and for $j_1 = j_2$

$$[\mathcal{L}_{m,j}^\alpha, \mathcal{L}_{n,j}^\alpha] = \frac{[j(n-m)] [2j]}{[j]^2} \mathcal{L}_{m+n,2j}^\alpha \tag{48}$$

Concerning $d$-assumption 2 we have that

$$\lim_{q \to 1} \mathcal{L}_{n,k}^\alpha = L_n^\alpha. \tag{49}$$

$\mathcal{L}_{n,k}^\alpha$ is a difference operator (cf. (8)) which is symmetric (see section 3.2.3) and bounded (because of $|q| = 1$) in the Fourier space:

$$\mathcal{L}_{m,k}^\alpha \hat{f}(z) = z^n \frac{e^{i\phi_0(\frac{\pi}{2} + \alpha)} \hat{f}(e^{ik\phi_0} z) - e^{-i\phi_0(\frac{\pi}{2} + \alpha)} \hat{f}(e^{-ik\phi_0} z)}{2i \sin(\phi_0 k)} \tag{50}$$

and the additional parameter $k \in \mathbb{Z}$ is a (presumably minimal) method to enforce a closure. The commutator (47), (48) allows to restrict the parameter $k$ to sets

$$C(p, k_0) := \{ k | k = pk_0 \nu, \nu \in \mathbb{N}, k_0 \in \mathbb{Z} \text{ fixed} \}. \tag{51}$$

In the following, we use $k_0 = 1$.

The coupling between $T_n$ and $\mathcal{L}_{n,k}^\alpha$ is not semidirect. From (44), (46) we get

$$\mathcal{L}_{m,k}^\alpha T_n = T_n \mathcal{L}_{m+2n,k}^\alpha \tag{52}$$

i.e. a quadratic coupling. The $\{ \mathcal{L}_{n,k}^\alpha, T_n \}$ with $q = \exp i\phi_0$ span\footnote{If one considers $\alpha \in [0, 2\pi)$ not as a fixed but as a running parameter, the basis $\{ \mathcal{L}_{n,k}^\alpha, T_n | \alpha \in \mathbb{R}, k, n \in \mathbb{Z} \}$ is for $q$ fixed again a quadratic Lie algebra.} the $q$-deformed inhomogeneous Witt algebra $W_q$. For $q \to 1$ we find the (undeformed) inhomogeneous Witt algebra.

### 3.2.2 The Multiplicative setting: other attempts and a Hopf structure

The above “natural” $q$-deformation of the inhomogeneous Witt algebra is one example in the plethora of possible deformations. We mention other attempts obtained from a different background, e.g.,\footnote{If one considers $\alpha \in [0, 2\pi)$ not as a fixed but as a running parameter, the basis $\{ \mathcal{L}_{n,k}^\alpha, T_n | \alpha \in \mathbb{R}, k, n \in \mathbb{Z} \}$ is for $q$ fixed again a quadratic Lie algebra.} and present some “generalisations” of the above “natural” construction. We use in this section the Abelian Lie algebra $K$ spanned by the $q$-shift operators $K_k^q$, $k \in \mathbb{Z}$:

$$[K_k^q, K_k'^q] = 0 \tag{53}$$

with $\lim_{q \to 1} K_k^q = 1$ and with

$$K_k^q T_n = q^{kn} T_n K_k^q \tag{54}$$

and

$$K_k^q \mathcal{L}_{n,k}^\alpha = q^{kn} \mathcal{L}_{n,k}^\alpha K_k^q \tag{55}$$
a) One can enlarge the basis \( \{ L^\alpha_{n,k} \} \) through the basis of the algebra \( K \). The augmented basis \( \{ L^\alpha_{n,k}, K^m \} \) leads to a quadratic Lie algebra which is a noncommutative and noncommutative Hopf algebra – Hopf-q-Witt algebra – \( \mathcal{H}W_q \) with coproduct \( \Delta \), counit \( \epsilon \) and antipode \( \gamma \) given as follows:

\[
\Delta(L^\alpha_{m,j}) = L^\alpha_{m,j} \otimes K_m + K_m \otimes L^\alpha_{m,j} \quad \Delta(K_i) = K_i \otimes K_i
\]

\[
\epsilon(L^\alpha_{m,j}) = 0 \quad \epsilon(K_i) = 1
\]

\[
\gamma(L^\alpha_{m,j}) = -K^{-1}_m L^\alpha_{m,j} K^{-1}_m \quad \gamma(K_i) = K_i^{-1}
\]

This construction is of interest because \( W_q \) is cocommutative: the price for a nontrivial Hopf algebra is the enlargement through an infinite dimensional Lie algebra. An inhomogeneous Hopf-q-Witt algebra with basis \( \{ L^\alpha_{n,k}, K^q_m, T_l \} \) follows from (52), (54). A restriction to \( \{ L^\alpha_{n,k}, T_l \} \) yields the same results as in 3.2.1. and also the same dynamics as in section 4.

b) To change \( \{ L^\alpha_{n,k}, T_l \} \) and the resulting dynamics one option is to deform not only the differential quotients but – generalizing our strategy – also the position operators \( T_n \) (i.e. the \( Q(f) \)). Their deformation forms a non-commutative algebra, which is related to some kind of noncommutative geometry. We try this option and multiply each \( T_n \) with a shift operator \( K^q_\delta \) with fixed \( \delta \in \mathbb{Z} \),

\[
T_n \mapsto \tilde{T}_n = T_n K^q_\delta
\]

Relation (54) implies

\[
\tilde{T}_m \tilde{T}_n = q^{\delta(n-m)} \tilde{T}_n \tilde{T}_m
\]

The \( \tilde{T}_n \) couple to \( \{ L^\alpha_{n,k} \} \) according to

\[
q^{-\delta n} \tilde{T}_n L^\alpha_{m,j} = L^\alpha_{n,j} \tilde{T}_m
\]

which reduces to (52) for \( \delta = 0 \), i.e. \( K^q_0 = 1 \). The deformed algebra \( \{ L^\alpha_{n,k}, \tilde{T}_n \} \) is quadratic. As mentioned in 3.2.1 the factor \( z^n \) is the result of a position observable and one should replace \( z^n = T_n \) now by \( \tilde{T}_n \). This gives \( \tilde{L}^\alpha_{n,k} = L^\alpha_{n,k} K^q_\delta \).

c) Another reasonable Ansatz is to deform the three terms in \( A^\alpha_n \) separately and multiply them with different shift operators:

\[
L^\alpha_n \mapsto A^\alpha_n = z^n \left( q^{\lambda_1} \frac{[\alpha_1 N_1]}{[\alpha_1]} K^q_\beta_1 + q^{\lambda_2} \frac{[\alpha_2 z^n]}{[\alpha_2]} K^q_\beta_2 + q^{\lambda_3} \frac{[\alpha_3 z^n]}{[\alpha_3]} K^q_\beta_3 \right)
\]

with \( I \) denoting the nine real parameters \( \lambda_i, \alpha_i, \beta_i, i = 1, 2, 3 \), depending on \( q, \alpha, n \) and on additional parameters like \( k \). The \( d \)-assumption

\[
\lim_{q \to 1} A^\alpha_n = L^\alpha_n
\]

is fulfilled. The \( A^\alpha_n \) are a special case of \( L^\alpha_{n,k} \) for

\[
\begin{align*}
\lambda_1 &= k(\frac{q}{2} + \alpha), \\
\lambda_2 &= k\alpha, \\
\lambda_3 &= -k\frac{q}{2} \\
\beta_1 &= 0, \\
\beta_2 &= \beta_3 = -k, \\
\alpha_1 &= \alpha_2 = \alpha_3 = k.
\end{align*}
\]

It is not possible to close \( \{ A^\alpha_n \} \) if not at least one running parameter is introduced; the structure of the commutators and the coupling to \( T_n \) or \( \tilde{T}_n \) will be discussed in another context.

d) In the above constructions, the Lie bracket is not deformed. A q-commutator

\[
[A, B]_q := q^{\epsilon(A,B)}AB - q^{\epsilon(A,B)}BA
\]
with real valued functions $r$, $s$ depending on $A$ and $B$ gives another freedom. We consider an example for a $q$-deformation of $W$ with a deformed Lie bracket (63) and deformed generators (60) which contain only two fixed real parameters $a$ and $b$ and no running parameter, like $k$ in (66). For simplicity, we treat the case $\alpha = 0$. If we choose in $A_{0} I_{n}$ the set $I$ as

$$I: \lambda_{1} = n(a + 1)b, \quad \lambda_{2} = nab$$

$$\alpha_{1} = -\beta_{1} = b, \quad \alpha_{2} = -\beta_{2} = 2b$$

and a $q$-commutator for $[A_{0}^{q}, A_{m}^{q}]_{q}$ with

$$r(n, m) = -2mb, \quad s(n, m) = -2nb$$

then it closes for $A_{0}^{q}$:

$$[A_{0}^{q}, A_{m}^{q}]_{q} = \frac{(q^{-2bn} - q^{-2bm})}{b} A_{n+m}^{q}.$$  \hspace{1cm} (66)

For $q \to 1$ the Witt algebra $W$ is obtained. The price which one has to pay in order to have no running parameter $k \in \mathbb{Z}$ but only two fixed parameters is a $q$-commutator.

### 3.2.3 An implicit additive setting

Concerning deformations of $K_{z}(S^{1})$ through additive shift operators we refer to the fact (see (38),(40)) that a multiplicative shift $K^{q} f(z) = f(qz), \ |q| = 1$, can be viewed as an implicit additive shift $I^{a} f(z(\phi)) = f(z(\phi + a)), a = i\phi_{0}$. Such a shift reproduces the results in 3.2.1. Accordingly, we use as a deformation of $L_{n}^{q}$

$$L_{n}^{q} \rightarrow \mathcal{M}_{n,k}^{q}$$

with

$$\mathcal{M}_{n,k}^{q} = z^{n} \frac{[kA]_{a}^{q}}{[k]_{a}}, \quad a = i\phi_{0}$$

which gives

$$\mathcal{M}_{n,k}^{q} = L_{n,k}^{q}.$$  \hspace{1cm} (69)

For the deformation types in 3.2.2 similar results hold. If one wants non-implicit additive shifts $K^{q} f(\phi) = f(\phi + \phi_{0})$ one has to work with $K_{\phi}(S^{1})$, that is the kinematical algebra parametrized by $\phi$ instead of $z$.

### 3.2.4 On Witt algebra deformations

Five deformation types of the Witt algebra $W$ and its inhomogenisation have been analysed. A deformation $W_{q}$ is given in section 3.2.1 via a Lie algebra with a running integer parameter $k$. The inhomogenisation of $W_{q}$ is quadratic. A nontrivial Hopf algebra $H W_{q}$ and a corresponding inhomogenisation are obtained if $W_{q}$ is enlarged through an Abelian algebra of $q$-shifts (see 3.2.2 - a). For the inhomogeneous Hopf-$q$-Witt algebra we get the same dynamics as for the inhomogeneous $W_{q}$ as we demonstrate later. If one deforms not only $W$ but also the commutative algebra with basis $\{T_{n}\}$ one finds non-commutative position operators and hence some link to a noncommutative geometry (3.2.2-b). Generalisations of $W_{q}$ were given in 3.2.2 - c. A deformation of the Lie bracket $[.,] \rightarrow [.,]_{q}$ in 3.2.2 - leads for a special family of generalised $W_{q}$ to a Lie algebra without a running parameter but with two fixed parameters. Implicit additive deformations are equivalent to $q$-deformations (3.2.3). For non-implicit additive deformations one has to use $K_{\phi}(S^{1})$. In the limit $q \to 1$ one gets for all types $W$, or respectively, its inhomogenisation. There are further types obtained from a different background. We mention [15 - 19] among other attempts.
If one analyses the different types of $q$-deformations with respect to how the $d$-assumptions are “fulfilled”, it seems that (44) in 3.2.1 is a preferred choice (“natural” Ansatz) $W = \{L^0_n\}$ is a Lie algebra. The running parameter $k$ in the deformation $\{L^0_{n,k}\}$ behaves such that it can be interpreted as “internal degree of freedom” which couples to $W$. It becomes apparent in the evolution equation (see 4.), e.g. for the $N$-point discretisation of $S^1$ with $q = \exp i\frac{2\pi}{N}$, and may be interpreted as follows: A representation of a (non Abelian) Lie algebra in terms of differentials is local in the sense that differentials are local objects because they depend only on a small neighbourhood of the point at which they are evaluated. A difference operator, on the contrary, when evaluated at $x$ depends on $x$ and $q^k x$ or $x + ka$ for some $k \in \mathbb{Z}$. Hence a representation of a “deformed” Lie algebra via difference operators depends on $k$, which we view as internal degree of freedom, and thus each point $x$ depends, via the discrete derivative, on this one-dimensional “internal” space $S_{int}$, which is an indication for how coarse-grained the difference operator is with respect to the underlying discrete space.

There are further peculiarities: A differential quotient is related via its inverse to a corresponding integral which is used to construct $L^2(S^1, d\phi)$. For a $q$-difference quotient the Jackson integral is the appropriate notion. One can apply such an integral for a function space over $S^1$ to model a $q$-inner product; however, this is not convenient here.

The choice of an inner product is relevant if one wants the position and momentum operators to be symmetric (or even self-adjoint). In the Fourier space the $L^0_{n,k}$ and $T_n$ are symmetric, also the $Q(f)$ and $P(X)$, and for the shift operator

$$(K^q_k)^* = K^{-q}_k$$

holds.

### 3.3 Deformations of the kinematical algebra $K_z(S^1)$

The deformation of the inhomogeneous Witt algebra (47), (52) leads almost directly to a deformation $K_z(S^1)_q$ of $K_z(S^1)$; for the other deformation types in 3.2.2 (except 3.2.2.- d), the results are the same or similar.

Insert $L^0_{n,k}$ and $T_n$ for $L^0_n$ and $T^n$, respectively, in (71). The origin of the term $iDn$ is $\frac{d}{\partial \phi}X(\phi) = iz \frac{d}{dz} \hat{X}(z)$ in (72). According to our strategy, we replace

$$\frac{d}{d\phi} \rightarrow i[N_z]_q$$

in (72), thus

$$D\frac{d}{d\phi}X(\phi) \rightarrow iD \sum_n [N_z]_q X_n z^n = \sum_n iD[n]_q X_n z^n. \quad (71)$$

With (44), (10), (72) the deformation $K_z(S^1)_q$ is (the upper index ($\alpha, D$) is dropped)

$$Q_q(f) = \sum_n f_n T_n$$

$$P_q(X) = \sum_n X_n (L^0_{n,k} + i[n]_q DT_n), k \in S_{int} \quad (72)$$

For any $k$, i.e. for any point $k$ in the internal space $S_{int}$, we get a momentum operator $P^k_q$. The $Q_q$ are independent of $k$. $P^k_q$ applied to $f(e^{i\phi})$ gives a linear combination of $\hat{f}(e^{i(\phi+k\phi)})$ and $\hat{f}(e^{i\phi})$ which depends on $n$ and is multiplied by $z^n$.

For $S^1_N$ and the corresponding Hilbert space (72) one can see directly the (“non local”) action: For $\phi_j = \frac{2\pi j}{N}, j = 0, \ldots, N-1, q = \exp i\frac{2\pi}{N}$, the momentum $P^k_q$ at the point $j$ depends on $j$ and $j \pm k$. This implies $k \in C(1,1)$.

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*Because the $d$-assumptions are not “sharp”, also other types can be viewed as “natural”.

*For a deformed Lie bracket the situation may change.
Dynamics from a $q$-deformed kinematical algebra $K_z(S^1)_q$

4.1 Strategy

In section 2 a time dependence for pure states in the Schrödinger representation was introduced through a generalized Ehrenfest relation (25) with a quantisation $Q(\alpha,D)$ of the kinematical algebra $K(M)$ as an input. We use the same construction here with $K_z(S^1)_q$ as input. Hence the time dependence of $\psi_t$ is restricted through

$$\frac{d}{dt}(\psi_t, Q_q(f)\psi_t) = (\psi_t, P^k_q(\text{grad} f)\psi_t), \quad \forall f \in C^\infty(S^1).$$  \hspace{1cm} (73)

We have not used the option to substitute the time derivative $\frac{d}{dt}$ through a reasonable time difference operator because we want to analyze the right hand side of (73), that is the space depending part. For a discretisation using a $t$-difference operator see [17]. For $M = \mathbb{R}^n$, $n = 1, 2$, a $q$-deformation of the Schrödinger equation through a group theoretical approach has been considered in [7, 24].

Relation (73) depends on $k \in S_{\text{int}}$ but the dynamics do not couple different points $k$ in the internal space. The derivation of (25) is based on the operators $Q_q$ and $P^k_q$; the algebraic relations of this kinematics, which couples different points of $S_{\text{int}}$, are not used in the derivation.\footnote{It would be interesting to introduce coupling in $S_{\text{int}}$ such that one gets for $q \to 1$ the non-deformed result.}

As we have mentioned before we focus our consideration on $S^1_N$ and a Hilbert space with elements $\psi_t = (\psi_t(0), \ldots, \psi_t(N-1))$, $\psi_t(l \pm N) = \psi_t(l)$, with inner product (57). As shown in 1.2 we have to choose $q = \exp i\frac{2\pi}{N}$. The Fourier transformation is

$$\psi_t(l) = \sum_{k=1}^{N-1} \psi_{kl} z^k_t, \quad z_t = \exp i\frac{2\pi}{N}l.$$  \hspace{1cm} (74)

The time dependence in the discrete case is restricted by (73) with the corresponding inner product and valid for all $f = (f_0, \ldots, f_{N-1})$, $f_j \in \mathbb{R}$. In Fourier space, the situation for $S^1$ and $S^1_N$ differ through the Fourier sums and $q$. However, we treat formally both cases together and specify later.

4.2 The deformed generalized $q$-Ehrenfest relation for $S^1_N$

On the left hand side of (73) we insert the expression for $Q_q(f)$ in (72) and on the right hand side, we use $\text{grad} f(\phi) = \frac{d}{d\phi} f(\phi) = iNq \hat{f}(z)$. Following (71) this implies with the Fourier transform (28) and (74)

$$\text{grad} = i \sum_n [n]_q f_n z^n$$  \hspace{1cm} (75)

and

$$P^k_q(\text{grad} f) = \sum_n i[n]_q f_n (C^k_{n,k} + iD[n]_q T_n).$$  \hspace{1cm} (76)

Because (73) holds for all $f$ we have for the $t$-dependence of the probability density $\rho_t = \bar{\psi}_t\psi_t$

$$\dot{\rho}_t = \sum_{l,j} \bar{\psi}_t(l)z^{-l}\psi_t(j)z^j\alpha_{l,j}$$  \hspace{1cm} (77)

$$\alpha_{l,j} = i(l-j)q \left( \frac{[k(j+\frac{1}{2}(l-j)+\alpha)]_q}{[k]_q} + iD[l-j]_q \right)$$  \hspace{1cm} (78)

7It would be interesting to introduce coupling in $S_{\text{int}}$ such that one gets for $q \to 1$ the non-deformed result.
This is a $q$-version of the Fokker Planck type equation (34):

$$\dot{\rho}_t = -\frac{\partial}{\partial \phi}(I_0^\alpha - D\frac{\partial}{\partial \phi}\rho_t)$$  \hspace{1cm} (79)

$$I_0^\alpha = \frac{i}{2}(\frac{\partial}{\partial \phi}\bar{\psi}\psi - \bar{\psi}\frac{\partial}{\partial \phi}\psi) + \alpha \rho$$  \hspace{1cm} (80)

To show this, replace in (79) and (80) $\frac{\partial}{\partial \phi}$ by $i[N_{z}]_{q}$ and construct a deformed current $I_0^{\alpha,k}$ such that

$$\dot{\rho}_t = -i[N_{z}]_{q}(I_0^{\alpha,k} - iD[N_{z}]_{q}\rho_t)$$  \hspace{1cm} (81)

yields (77). Such a current exists and is given by ($q = \exp{i\Phi_0}$)

$$I_0^{\alpha,k} = \frac{1}{|\omega_{k}|} \left[ \left( \frac{i}{2}(N_{z} + \alpha) \right)_{q}\psi \left( K_q^{2} \exp(i\frac{k}{2}\alpha \Phi_0)\bar{\psi} \right) \right.$$

$$\left. - \left( K_q^{2} \exp(-i\frac{k}{2}\alpha \Phi_0)\psi \left( \frac{i}{2}(N_{z} - \alpha) \right)_{q}\bar{\psi} \right) \right]$$  \hspace{1cm} (82)

For $q \to 1$, $I_0^{\alpha,k}$ gives $\bar{j}_0^\alpha$ in (34) and (formally) $I_0^{\alpha,k} = -I_0^{\alpha,k}$ holds.

The shift operators $K_q^{2} \pm \frac{k}{2}$ act on $\bar{\psi}(l)$ and on $\psi(l)$ in both terms in $I_0^{\alpha,k}$ and yield $\psi(l \pm \frac{k}{2})$. In the case of $S^1_N$ we assure that $l \pm \frac{k}{2}$ corresponds to a lattice point in $S^1_N$. This implies (see (51))

$$k \in C(p, 1), p \text{ even}$$  \hspace{1cm} (83)

thus $k = p\nu$, $\nu \in \mathbb{N}$. Hence, the interaction between next neighbours via the discrete internal space $S_{int}$ depends on the choice of $p$ and $\nu$.

### 4.3 The Evolution Equation

The $q$-Fokker-Planck type equation (34) restricts the evolution of $\psi_t$. In the undeformed case the Ansatz (note the multiplication by $\psi$)

$$i\dot{\psi}_t\bar{\psi}_t = (H\psi_t + G(\bar{\psi}_t, \psi_t)\bar{\psi}_t)$$  \hspace{1cm} (84)

implies in the usual Fokker-Planck type equation (34) a certain form for $H$ and $G_2$. An analogous method fails in (82) because the $K^q$ act on $\bar{\psi}$ and on $\psi$. One possibility to overcome this is an Ansatz for $i\dot{\psi}_t\bar{\psi}_t$ which contains $N_L$ suitably chosen shifts (we skip the index $t$ in $\psi$), and a splitting of the linear part into $N_L$ units, each associated with a different shift $S^l$ of $\bar{\psi}$, as follows:

$$i\dot{\psi}\bar{\psi} = \sum_{l=1}^{N_i}(H^l_q(\bar{\psi}, \psi)\psi)(S^l\bar{\psi}) + (G_q(\bar{\psi}, \psi)\psi)(R\bar{\psi})$$  \hspace{1cm} (85)

$S^l$ and $R$ are shifts which may be expressed as $(a_1K^q_{k_1} + a_2K^q_{k_2})$ with $a_1, a_2 \in \mathbb{C}$, $k_1, k_2 \in C(p, 1)$: $H^l_q$ for each $l$ is a complex linear difference operator, $G_q$ is a (formal) nonlinear multiplication operator (see the analogy to 2.3). A corresponding expression follows for $i\dot{\bar{\psi}}\bar{\psi}$. The Ansatz was used in (86).

We indicate the calculation:

Split $S^l$, $R$, $\psi$, $\bar{\psi}$, $H^l_q$, $G_q$ in real (index 1) and imaginary (index 2) parts. Insert from (84) $\dot{\psi}\bar{\psi}$ and $\dot{\bar{\psi}}\psi$ in the left hand side of (34) and get a relation between $H^l_q, G_q, S^l$ and $R$, $i = 1, 2$. We collect the linear operators on $\psi$ and on $\bar{\psi}$ in $F_L$ to determine $H^l_q$ and $S^l$ and the nonlinear terms in $F_{NL}$ which give some information on $G_q(\bar{\psi}, \psi)$ and $R$:

$$i\dot{\psi}\bar{\psi} = F_L + F_{NL}$$  \hspace{1cm} (86)
Then a straightforward calculation shows that
\[
F_L = \left( [N] q_{ \frac{1}{2} } \right) \left( \psi \right) (S_{q,1} \bar{\psi}) \\
- \left( \frac{1}{2} \frac{\partial}{\partial q} \right) \left( \left( \left| [N] q_{ \frac{1}{2} } \bar{\psi} \right) \left( \left| [N] q_{ \frac{1}{2} } \psi \right) \right) \right) \\
+ \left( \left( \left| [N] q_{ \frac{1}{2} } \psi \right) \left( \left| [N] q_{ \frac{1}{2} } \bar{\psi} \right) \right) \right)
\]
with
\[
S_{q,1} = \frac{1}{2} \left( K_{q1} + K_{q2} \right), \\
S_{q,2} = 0, \\
a_1 = 2.
\]
As \( q \to 1 \) (\( \Phi_0 \to 0 \)) one has by construction
\[
\lim_{q \to 1} F_L = \left[ \frac{1}{2} N^2 + \alpha N \right] \bar{\psi}.
\]
For a suitable Ansatz for \( F_{NL} \) remember that in the non-deformed case \( G = G_1 + iG_2 \) where the term \( G_2 \) was given through \([54]\); \( G_2 \) could be zero (the corresponding DG-equation for \( M = \mathbb{R}^3 \) was given in \([12, 13]\)). Hence we assume here the same, i.e.
\[
G^I_{q} = iG^I_{q,2} \quad \text{with} \quad G^I_{q,1} = 0.
\]
Because \( G^I_{q,2} \) is a real multiplication operator, this implies after a straightforward calculation using \([54]\), its complex conjugate, and \([57]\):
\[
G^I_{q,2} \left( \psi, \psi \right) = C^I_q - D|N^2| \rho
\]
with
\[
C^I_q = \psi R(\psi) - D|N^2| \rho
\]
\[\quad \text{with} \]
\[
\lim_{q \to 1} C^I_q \to 0
\]
holds and gives the nonlinear imaginary term in \([54]\). With \( F_{NL} = i(G^I_{q,2}(\bar{\psi}, \psi))(R\psi) \) we arrive at a nonlinear difference equation for \( \psi \)
\[
i \dot{\psi} = \psi^{-1} (F_L + iG^I_{q,2}(\bar{\psi}, \psi) \psi R\bar{\psi})
\]
In general, admitting also \( G^I_{q,1} \neq 0 \), one finds again for \( q \to 1 \) the imaginary term \([54]\). In this setting, there are \( G_q \) and \( R \) such that the term \( \frac{D\rho^2}{\rho} \) in the DG-family \([54]\) is reproduced in the limit (see Theorem 7.15 in \([17]\)). However, the choice \( R_1 = 0 \) or \( R_2 = 0 \) (see \([16, 17]\)) gives equations not in the DG family, e.g. a nonlinear term proportional to
\[
\frac{\psi^\prime \psi^\prime - \bar{\psi}^\prime \bar{\psi}^\prime}{\psi^\prime \bar{\psi} - \psi \bar{\psi}^\prime}.
\]
For \( S^I_N \) we find \( N \) coupled nonlinear difference equations. To illustrate the time dependence of \( \bar{\psi}(l) \), \( l = 0, \ldots, N - 1 \), \( \psi(l + N) = \psi(l) \), we consider the case \( G^I, \alpha = 0 \). Hence the second part in \([54]\) vanishes. Evaluating \([54]\) with \([57]\) and \([51]\) we see that the detailed form of the family of evolution
equations depends on \( \nu \) (because \( k = 2\nu, \nu \in \mathbb{N} \), (83), \( D \in \mathbb{R} \) and on the shift \( R \):

\[
i \dot{\psi}(l) = \frac{1}{\psi(l)} \frac{1}{\sin \frac{2\pi}{N} \sin \frac{2\pi k}{N}} \left\{ -1/8(2\psi(l + k_+) - \psi(l + k_-) - \psi(l - k_-)) \right. \\
\left. (\bar{\psi}(l + k_+) + \bar{\psi}(l + k_-)) + \frac{1}{2i} \frac{1}{2 \text{Re}(\psi(l)R\bar{\psi}(l))} ((\psi(l + k_+) - \psi(l + k_-) + \psi(l - k_-)) \bar{\psi}(l + k_-) - \bar{\psi}(l + k_-) + \psi(l - k_-) \bar{\psi}(l + k_-)) + \text{cc.} \right. \\
- \frac{1}{\text{Re}(\psi(l)R\bar{\psi}(l))} \frac{D}{2} \left( \psi(l + 2)\bar{\psi}(l + 2) - 2\psi(l)\bar{\psi}(l) + \psi(l - 2)\bar{\psi}(l - 2) \right) .
\]

(96)

A convenient choice for the shift operator is \( R = 1 \). There exists an evolution equation for any even \( k \in \mathbb{N} \). (96) shows which points interact with a given point \( l \). We give no numerical study on this evolution equation in this report.

5 Summary and Outlook

This study, which also reviews some earlier work [16], [25], shows some possibilities to develop from first principles a framework for quantum mechanics on a configuration space \( M \), such that momentum observables are represented through difference operators whereas position observables are quantised through multiplication operators.

We have applied Borel quantisation. This method is based on the kinematical algebra \( K(M) \) spanned by usual position observables; momentum observables are quantized with \( P \)-assumptions through certain differential operators. The dynamics is introduced via a generalized Ehrenfest relation and yields for pure states a family \( \mathcal{F}_P \) of nonlinear differential equations which contain as a “natural” subfamily the DG family \( \mathcal{F}_{DG} \).

For the development of a framework involving difference operators we start with \( K(M) \) – not from a kinematics \( \widetilde{K}(M) \) with Borel sets – and replace the quantized vector fields through difference operators. This is a highly non unique procedure. A minimal condition we require is that we get the results from Borel quantisation in a suitable limit, but one needs further assumptions, which we have called \( d \)-assumptions (in analogy to the \( P \)-assumptions). The guiding principles were simplicity, plausibility and some “physical feeling”. We gave the results for difference operators of the multiplicative type – so called \( q \)-deformations of \( K(M) \) in the case of \( M = S^1 \) and its discretisation \( S^1_N \).
Kinematical algebra (with usual position operators):

The quantisation of $K(S^1)$ is connected with a representation of the Witt algebra $W$. Hence our construction yields certain $q$-deformations of $W$ as an interesting by-product of our study. The family $\mathcal{F}_{DG} \subset \mathcal{F}_P$ can be obtained from the large family $\mathcal{F}_P$ in the limit $q \to 1$. There are “natural” members in $\mathcal{F}_P$ which are not connected with $\mathcal{F}_{DG}$ in this limit. We have not tried to realise the dotted line in Fig. 1.

The generalization of an existing physical theory requires the input of extra information to pave a path into a new structure - the right choice of such information is difficult; here, it was formulated in terms of the $d$-assumptions. The formulation of quantum gauge field theory in the setting of noncommutative geometry faces similar problems.

The step-by-step procedure to implement difference operators may be compared with walking in a marsh jumping from one stable looking blade of grass to the next. For a motivation of quantum mechanics with difference operators a deeper understanding and possibly a new view on the structure of our space–time could give some necessary information.

Finally we remark that nonlinear DG-equations have recently found attention in string theory [26] and we hope that this study of discretizations of DG-equations over the configuration space $S^1$ from first principles will also be relevant in this context.

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