SPIKE SOLUTIONS FOR A MASS CONSERVATION REACTION-DIFFUSION SYSTEM

SHIN-ICHIRO EI
Department of Mathematics
Faculty of Science, Hokkaido University
Sapporo 060-0810, Japan

SHYUH-YAUR TZENG
Department of Mathematics
National Changhua University of Education
Changhua, Taiwan

-dedicated to 70th birthday of Prof. Wei-Ming Ni-

ABSTRACT. This article deals with a mass conservation reaction-diffusion system. As a model for studying cell polarity, we are interested in the existence of spike solutions and some properties related to its dynamics. Variational arguments will be employed to investigate the existence questions. The profile of a spike solution looks like a standing pulse. In addition, the motion of such spikes in heterogeneous media will be derived.

1. Introduction. Consider the following reaction-diffusion system:

\[ \frac{\partial u_1}{\partial t} = d \Delta u_1 + f(u_1, u_2), \quad (1.1) \]
\[ \frac{\partial u_2}{\partial t} = \Delta u_2 - f(u_1, u_2), \quad (1.2) \]

\[ u_1(x + e_i, t) = u_1(x, t), \quad u_2(x + e_i, t) = u_2(x, t), \quad (1.3) \]
\[ u_1(x, 0) = \psi_1(x), \quad u_2(x, 0) = \psi_2(x). \quad (1.4) \]

Here \( \{e_i \mid i = 1, 2, ..., N\} \) is the standard basis of \( \mathbb{R}^N \). System (1.1)-(1.2) has been proposed as a mathematical model that account for gradient sensing and signal amplification in cell polarity. Since the case \( d = 1 \) is not under consideration, without loss of generality it is assumed that \( 0 < d < 1 \).

Observe that adding (1.1) to (1.2) yields

\[ \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} = d \Delta u_1 + \Delta u_2, \quad (1.5) \]

thus for any solution \((u_1(x, t), u_2(x, t))\) of (1.1)-(1.4), the total mass

\[ \int_{\Omega} u_1(x, t) + u_2(x, t) \, dx \quad (1.6) \]

2010 Mathematics Subject Classification. Primary: 35B25, 35J50, 35K57; Secondary: 37L6.
Key words and phrases. Reaction-diffusion system, mass conservation, spike solution, cell polarity, localized pattern, standing pulse, invariant manifold.

* Corresponding author: Shin-Ichiro EI.
is conserved in all the time. As pointed out in literature [16, 18, 23], a mass conserved reaction-diffusion system is one of the fundamental principles of cell polarity.

A stationary solution \((\hat{u}_1(x), \hat{u}_2(x))\) of (1.1)-(1.3) satisfies
\[
\hat{u}_1(x + e_i) = \hat{u}_1(x), \quad \hat{u}_2(x + e_i) = \hat{u}_2(x).
\tag{1.7}
\]

From a simple observation, \((\hat{u}_1 + \hat{u}_2)\) is a bounded harmonic function on \(\mathbb{R}^N\). Then the Liouville’s Theorem shows that \(\hat{u}_1 + \hat{u}_2 = c\) for some \(c \in \mathbb{R}\). Thus we are led to studying
\[
-d\Delta u = f(u, c - du),
\]
\[
u(x + e_i) = u(x), \quad i = 1, 2, ..., N.
\tag{1.9}
\]

If \(u\) is a non-constant solution of (1.8)-(1.9), setting \((\hat{u}_1, \hat{u}_2) = (u, c - du)\) yields a non-constant stationary solution of (1.1)-(1.3).

Motivated by [21, 25, 22] and cell biology, we are interested in localized patterns in mass conserved reaction-diffusion system. Variational methods will be employed to obtain non-constant stationary solutions of (1.1)-(1.3), as stated in the following theorem.

**Theorem 1.1.** There is a \(d^* > 0\) such that if \(d < d^*\) then (1.1)-(1.3) possesses infinite number of non-constant stationary solutions.

As to capture an interesting phenomena in the cellular process, the generation of spike solutions has attracted a great deal of attention. For the FitzHugh-Nagumo equations, a spike solution has been obtained [2, 5, 6, 11, 25] by showing the existence of standing pulse solutions. The related stability questions were investigated in [10], using the Maslov index [8]. Such localized patterns [9, 13, 27] and waves [3, 4, 29] often result from a balance between dispersion and nonlinearity. For the existence of spike solutions of (1.8)-(1.9), the following result will be established.

**Theorem 1.2.** If \(d\) is sufficiently small, there exist infinite number of spike solutions.

In Section 2 we start with the variational framework to employ the Mountain Pass Theorem. It will be shown Section 3 that if \(d\) is small, the critical points obtained by the Mountain Pass Theorem are non-constant stationary solutions. A further analysis in Section 4 shows that the profile of such a stationary solution possesses one spike. Moreover, it is known [17] that for certain types of \(f\), the existence of a Lyapunov functional for the flow \((u_1(x, t), u_2(x, t))\) generated by (1.1)-(1.3) and we brief discuss the Lyapunov stability in Section 5. Heterogeneity is the most important and ubiquitous type of external perturbation observed in natural environments. When a reaction-diffusion system is perturbed by some small heterogeneity, some new interesting phenomena [7, 28] have been observed. With zero being an eigenvalue due to the translation free mode by periodic boundary conditions [12], we study the associated eigenfunctions for the linearization and the adjoint operator in Section 6. Such eigenfunctions play important roles in studying the motion of spike in heterogeneous media, as to be demonstrated in Section 7.

2. **Variational framework for stationary solutions.** In this section variational methods will be used to show the existence of stationary solutions \((u_1(x), u_2(x))\) of (1.1)-(1.3). As to study a model related to cell polarity, we focus on some examples for \(N = 2, 3\), while most of arguments can be applied to the case in other dimensions.
Setting $f_c(u) = f(u, c - du)$, we seek the solutions of the elliptic boundary value problem

\begin{align}
-d\Delta u &= f_c(u), \\
 u(x + e_i) &= u(x), \quad i = 1, 2, ..., N.
\end{align}

(2.1) (2.2)

It is assumed that

(f1) There exist $\xi_0 \leq 0 < \xi_+$ such that $f_c(\xi_0) = f_c(\xi_+) = 0$.
(f2) $f'_c(\xi_0) < 0$ and $f'_c(\xi_+) > 0$.
(f3) $f''_c(\xi) > 0$ if $\xi > \xi_+$ and $\int_{\xi_0}^{\xi} f_c(s)ds > 0$ for some $\xi > \xi_+$.

In what follows we take $\Omega = (-\frac{1}{2}, \frac{1}{2})^N$. For fixed $c$, we define $F_c(\xi) = \int_{\xi_0}^{\xi} f_c(s)ds$ if $\xi > \xi_0$, and it is identically to zero otherwise. Consider

$$J_c(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - F_c(u^+)dx.$$  (2.3)

This functional is defined on the Hilbert space

$$E = \{u \in H^1_{loc}(\mathbb{R}^N)||u(x + e_i) = u(x)|| \quad i = 1, 2, ..., N\}$$

equipped with the norm

$$\|u\| = [\int_{\Omega} (|\nabla u|^2 + u^2)dx]^{\frac{1}{2}}.$$

We will employ the critical point theory [24] to investigate the critical points of $J$ to obtain the non-constant solutions of (2.1)-(2.2). There are many functions satisfy the conditions (f1)-(f4). For instance, $f(u_1, u_2) = -u_1 + u_1^3 + u_2^3$ and $f(u_1, u_2) = (u_1 + u_2)^3 - u_1$. It is not difficult to check the following properties hold.

(i) If $u \in E$ and $u^+$ is not zero almost everywhere, then $\lim_{t \to \infty} J_0(tu) = -\infty$.
(ii) For any given small $|c|$, there is a $\theta \in (0, \frac{1}{2})$ such that $\theta f_c(u)u - F_c(u)$ is bounded below on $\mathbb{R}$.
(iii) There is a $\delta > 0$ and a $r > 0$ such that $\inf_{u \in E, \|u\| = r} J_0(u) \geq \delta$.
(iv) $J_c(u)$ converges uniformly to $J_0(u)$ on any given bounded subset of $E$ as $c \to 0$.

A well-known critical point theory is the Mountain Pass Theorem [1]. In the next section we apply such a theorem to find non-constant stationary solutions of (1.1)-(1.3) when (i)-(iv) hold.

3. Existence of stationary solutions. Let $d^*$ be a small positive number and $d < d^*$. We now investigate the existence of solutions of

\begin{align}
-d\Delta w &= -w + w^3 + (c - dw)^3, \\
w(x + e_i) &= w(x), \quad i = 1, 2, ..., N.
\end{align}

(3.1) (3.2)

Such solutions are the critical points of $J_c$ defined by (2.3).

To simplify the presentation, we may consider $f(u_1, u_2) = -u_1 + u_1^3 + u_2^3$ to illustrate the essence of detailed arguments. With

$$f_c(u) = f(u, c - du) = -u + u^3 + (c - du)^3,$$

it is clear that $f_0$ has three roots, namely $-\frac{1}{\sqrt{1-\frac{c}{d}}}, 0$ and $\frac{1}{\sqrt{1-\frac{c}{d}}}$. We first consider the case $c = 0$.

**Lemma 3.1.** If $c = 0$ and $d^* < \frac{1}{200\sqrt{\pi}}$, then (3.1)-(3.2) has a non-constant solution $u_0$. 
Proof. Note that when $c = 0$, 

$$J_0(u) = \int_{\Omega} \left( \frac{d}{2} |\nabla u|^2 + \frac{1}{2} u^2 \right) - \frac{1}{4} d^3 u^4 \, dx$$

and $J_0(0) = 0$. The other constant solution is $\frac{1}{\sqrt{1 - d^3}}$.

By Sobolev imbedding theorem, there exists $k_0 > 0$ such that

$$\int_{\Omega} u^4 \, dx \leq k_0^4 \|u\|^4$$

(3.4)

if $u \in E$. Hence

$$J_0(u) = \int_{\Omega} \left( \frac{d}{2} |\nabla u|^2 + \frac{1}{2} u^2 \right) \, dxdy - \int_{\Omega} \frac{1}{4} (1 - d^3) u^4 \, dxdy,$$

(3.5)

$$\geq \frac{d}{2} \|u\|^2 - \frac{1}{4} k_0^4 \|u\|^4,$$

$$= \|u\|^2 \left( \frac{d}{2} - \frac{1}{4} k_0^4 \|u\|^2 \right),$$

from which we obtain

$$\inf_{\|u\|=\sqrt{\frac{d}{k_0^4}}} J_0(u) \geq \frac{d^2}{4k_0^4}.$$ (3.6)

Next we claim: there is an $\bar{e} \in E$ with $\|\bar{e}\| > \sqrt{\frac{d}{k_0^4}}$ and $J_0(\bar{e}) < 0$. Set

$$e(x_1, ..., x_N) = \left\{ \begin{array}{ll}
  x_1...x_N(x_1 - 1)...(x_N - 1), & x \in \Omega, \\
  0, & x \notin \Omega,
\end{array} \right.$$ (3.7)

and $e_d(x) = e\left(\frac{1}{\sqrt{d}}x\right)$. Straightforward calculation yields

$$J_0(se_d) < 0$$

if $s$ is large and

$$\max_{s \in [0, \infty)} J_0(se_d) = \frac{d^2 (10N + 1)^2}{4\sqrt{1 - d^3}} \left( \frac{7}{10} \right)^N.$$ (3.8)

Define

$$\beta_0 = \beta_0(d) = \inf_{\theta \in [0, 1]} \max_{\gamma \in C([0, 1]; E)} J_0(\gamma(\theta))|\gamma \in C([0, 1]; E), \gamma(0) = 0, \gamma(1) = se_d \}$$

with $s$ being sufficiently large. Applying the Mountain Pass Theorem yields a critical point $u_0$ of $J_0$ and

$$\beta_0 = J_0(u_0) \geq \frac{d^2}{4k_0^4} > 0.$$ (3.9)

Moreover (3.8) implies that

$$\beta_0 \leq \frac{d^2 (10N + 1)^2}{4\sqrt{1 - d^3}} \left( \frac{7}{10} \right)^N < J_0(\frac{1}{\sqrt{1 - d^3}}).$$ (3.10)

Hence $u_0$ is a non-constant solution.

\[ \square \]

Remark 1. (a) As an consequence of the Maximum Principle, $u_0 > 0$ on $\Omega$.

(b) It follows from (3.9) and (3.10) that $\beta_0 = O(d^{\frac{2}{5}})$ if $d$ is small.

Lemma 3.2. If $S = \{ u \in E | u \neq 0, J_0'(u)u = 0 \}$ then $\beta_0 = \inf_{u \in S} J_0(u)$. 
Proof. Set 

\[ \sigma = \inf_{u \in S} J_0(u). \]

It follows from Lemma 3.1 that 

\[ \sigma \leq \beta_0. \]

Next we show \( \sigma \geq \beta_0. \) Recall that 

\[ J_0(u) = \int_{\Omega} \left( \frac{d}{2} |\nabla u|^2 + \frac{1}{2} u^2 \right) dx - \int_{\Omega} \frac{1}{4} (1 - d^3) u^4 dx. \]

For any \( u \in S, \) it is easily check that 

\[ J_0(hu) \text{ is decreasing if } h > 1, \]

\[ \lim_{h \to \infty} J_0(hu) = -\infty \]

and 

\[ \max_{h \geq 0} J_0(hu) = J_0(u). \]

Moreover \( J_0(se_d) < 0 \) and \( J_0(hse_d) \) is decreasing for \( h \geq 1. \) Clearly 

\[ \lim_{h \to \infty} J_0(h[(1 - \theta)u + \theta se_d]) = -\infty \text{ uniformly on } \theta \in [0, 1]. \]

Hence there is a continuous path \( \gamma, \) starting with 0 and passing through \( u \) and ending at \( se_d, \) on which 

\[ \max_{0 \leq \theta \leq 1} J_0(\gamma(\theta)) = J_0(u) \]

holds. Now the proof is complete. \( \square \)

Next we turn to the case of \( c < 0. \)

Lemma 3.3. For \( d < d^* \), there is a \( c_\ast \in (-d, 0) \) such that, for \( c \in (c_\ast, 0], \) (3.1)-(3.2) has a non-constant solution \( u_c. \)

Proof. Observe that 

\[ |J_c(u) - J_0(u)| \leq |c|d^2k_0^3\|u\|^3 + \frac{c^4}{d} \]

(3.11) and 

\[ J_c(0) = \frac{c^4}{4d}. \]

Pick a \( c_\ast \in (-d, 0) \) with \( |c_\ast| \) being sufficiently small. It is easy to verify, for \( c \in (c_\ast, 0], \) that 

\[ J_c(0) < \frac{d^2}{16k_0^3} \]

\[ \inf_{\|u\|=\sqrt{\frac{d}{16k_0^3}}} J_c(u) \geq \frac{3d^2}{16k_0^3}. \]

(3.12) and \( \lim_{s \to \infty} J_c(se_d) = -\infty. \) Define 

\[ \beta_c = \inf \{ \max_{\gamma \in [0, 1]} J_c(\gamma(\theta)) | \gamma \in C([0, 1]; E), \gamma(0) = 0, \gamma(1) = se_d \} \]

with \( s \) being sufficiently large.

With \( J_c \) being a \( C^2 \) functional on \( E, \) applying the standard Deformation Lemma [24] yields a Palais-Smale sequence \( \{w_n\} \) in \( E \) such that as \( n \to \infty \)

\[ J_c(w_n) \to \beta_c, \]

(3.13)

\[ J'_c(w_n) \to 0 \text{ strongly in } E^{-1}, \]

(3.14)

where \( E^{-1} \) is the dual of \( E. \) Moreover by (3.12) 

\[ \frac{3d^2}{16k_0^3} \leq \beta_c \]

(3.15)
We claim: \{w_n\} is uniformly bounded in \(E\). Indeed for large \(n\), by (3.13) and (3.14)
\[
J_c(w_n) = \int_{\Omega} \left( \frac{d}{2} \nabla |w_n|^2 + \frac{1}{2} w_n^2 \right) dx + \int_{\Omega} \left[ -\frac{1}{4} |w_n|^4 + \frac{1}{4d} (c - dw_n)^4 \right] dx
= \beta_c + o(1)
\]
and
\[
J'(w_n)(w_n) = \int_{\Omega} \left( d \nabla |w_n|^2 + w_n^2 \right) dx + \int_{\Omega} \left[ \frac{1}{12} |w_n|^4 + \frac{1}{3} (c - dw_n)^3 w_n + \frac{1}{4d} (c - dw_n)^4 \right] dx
= \beta_c + o(1) - \frac{o(1)}{3} \|w_n\|.
\]
Note that
\[
\frac{1}{12} u^4 + \frac{1}{3} (c - du)^3 u + \frac{1}{4d} (c - du)^4 = \left( \frac{1}{12} - \frac{d^3}{12} \right) u^4 + \frac{e}{2} du^2 - \frac{2e^3}{3} u + \frac{e^4}{4d}.
\]
For any \(u(x) \in \mathbb{R}\),
\[
\frac{1}{12} u^4 + \frac{1}{3} (c - du)^3 u + \frac{1}{4d} (c - du)^4 + \frac{e^4}{4d} > 0. \quad (3.19)
\]
Thus combining (3.18) and (3.19) yields
\[
\frac{d}{6} \|w_n\|^2 + \frac{o(1)}{3} \|w_n\| \leq \beta_c + o(1) + \frac{e^4}{4d}. \quad (3.20)
\]
which implies that \(\{w_n\}\) is uniformly bounded in \(E\). Therefore along a subsequence \(w_n \rightharpoonup u_c\) weakly for some \(u_c \in E\) and \(w_n \rightarrow u_c\) almost everywhere on \(\Omega\) as \(n \rightarrow \infty\). Then (3.14) together with the Sobolev imbedding theorem gives
\[
J'(u_c)(\varphi) = 0.
\]
for every \(\varphi \in E\). Also, (3.16) and (3.17) imply
\[
J_c(w_n) - \frac{1}{2} J'(w_n)(w_n) = \int_{\Omega} \left( \frac{1}{4} u_n^4 + \frac{1}{4d} (c - dw_n)^4 + \frac{1}{2} (c - dw_n)^3 w_n \right) dx
= \beta_c + o(1) - \frac{o(1)}{2} \|w_n\|.
\]
This together with the Sobolev imbedding theorem gives
\[
J_c(u_c) = J_c(u_c) - \frac{1}{2} J'(u_c)(u_c)
= \int_{\Omega} \left( \frac{1}{4} u_c^4 + \frac{1}{4d} (c - du_c)^4 + \frac{1}{2} (c - du_c)^3 u_c \right) dx
= \beta_c.
\]
Next we claim that
\[
\beta_c \leq \beta_0 + M_0 |c| \quad (3.22)
\]
with $M_0$ being a constant depending on $d$ only. Note that there exists a $\gamma \in C([0,1]; E)$ such that $\gamma(0) = 0$, $u_0 \in \gamma([0,1])$ $J(\gamma(1)) < 0$ and $\max_{\theta \in [0,1]} J_0(\gamma(\theta)) = \beta_0$. This together with (3.11) completes the proof of (3.22). Since $J(u_c) = \beta_c = O(d)$, $u_c$ is a non-constant solution.

**Lemma 3.4.** For fixed $d \in (0, d^*)$,
\[
\lim_{c \to 0} \beta_c = \beta_0.
\]

**Proof.** By (3.22)
\[
\limsup_{c \to 0} \beta_c \leq \beta_0.
\]

Since
\[
J_c(u_c) - \frac{1}{3} J_c'(u_c)u_c = \frac{1}{6} \int_\Omega (d |\nabla u_c|^2 + u_c^4)dx + \int_\Omega \left[ \frac{1}{12} u_c^4 + \frac{1}{3}(c - du_c)^3 u_c + \frac{1}{4d} (c - du_c)^4 \right] dx = \beta_c
\]
and
\[
\frac{1}{12} u_c^4 + \frac{1}{3}(c - du)^3 u_c + \frac{1}{4d} (c - du)^4 + \frac{c^4}{36d} > 0,
\]
it follows from (3.15) and (3.22) that
\[
\frac{d}{6} \| u_c \|^2 \leq \beta_c + \frac{c^4}{36d},
\]
This together with (3.22) yields
\[
\frac{d}{6} \| u_c \|^2 \leq \beta_0 + M_0|c| + \frac{c^4}{36d}.
\]

Take a sequence $c_n \to 0$ as $n \to \infty$. By (3.26), the sequence $\{u_{c_n}\}$ is bounded in $E$. Hence along a subsequence $u_{c_n} \rightharpoonup w_0$ weakly for some $w_0 \in E$ and $u_{c_n} \to w_0$ almost everywhere on $\Omega$. Since
\[
J_{c_n}(u_{c_n})(\varphi) = \int_\Omega (\nabla u_{c_n} \cdot \nabla \varphi + u_{c_n} \varphi)dx + \int_\Omega [-u_{c_n}^3 \varphi - (c_n - du_{c_n})^3 \varphi]dx = 0
\]
for $\varphi \in E$, invoking the Sobolev imbedding theorem gives
\[
J_0'(w_0)(\varphi) = 0.
\]
Since
\[
J_c(u_c) - \frac{1}{2} J_c'(u_c)u_c = \int_\Omega \left[ \frac{1}{2} u_c^2 + \frac{1}{2}(c - du_c)^3 u_c + \frac{1}{4d} (c - du_c)^4 \right] dx = \beta_c \geq \frac{M_0}{16k_3^3}.
\]
Applying the Sobolev imbedding theorem gives
\[
\beta_{c_n} = J_{c_n}(u_{c_n}) \to J_0(w_0),
\]
\[
J_0(w_0) = J_0(w_0) - \frac{1}{2} J_0'(w_0)(w_0) = \frac{1}{4} (1 - \delta_3) \int_\Omega w_0^4 dx \geq \frac{3d^2}{16k_3^3}.
\]
Since $w_0 \in S$, it follows from Lemma 3.2 that
\[
\lim_{n \to \infty} \beta_{c_n} = J_0(w_0) \geq \beta_0.
\]
In fact, the above proof shows that (3.30) holds for any sequence $c_n \to 0$. Thus the proof is complete.

Let
\[
A_c = \{ w \in E | J_c(w) = \beta_c, J_c'(w) = 0 \text{ in } E^{-1} \}.
\]
Lemma 3.5. \( \lim_{c \to 0} \sup_{w \in A_c} \left( \inf_{v \in A_0} \|w - v\| \right) = 0. \)

Proof. Suppose \( u \in A_c \) then

\[
\beta_c = \frac{1}{2} J_c(u) \geq \frac{1}{2} J_c'(u)(u)
\]

\[
= \frac{1}{6} \int_{\Omega} (d|\nabla u|^2 + u^2)dx + \int_{\Omega} \left[ \frac{1}{12} u^4 + \frac{1}{3} (c - du)^3u + \frac{1}{4d}(c - du)^4 \right] dx.
\]

The same argument as in (3.20) shows that the set \( \cup_{0 < |c| < c^*} A_c \) is uniformly bounded in \( E \). For each \( c > 0 \), pick a \( w_c \in A_c \). Now consider a sequence \( \{w_{cn}\} \) with \( c_n \to 0 \) as \( n \to \infty \). Then along a subsequence, \( w_{cn} \rightharpoonup w_0 \) weakly for some \( w_0 \in E \) and \( w_{cn} \to w_0 \) almost everywhere on \( \Omega \). This together with Sobolev imbedding theorem implies

\[
J_0'(w_0)(\varphi) = 0
\]

for \( \varphi \in E \). Moreover,

\[
\frac{3d^2}{16k_0^2} \leq J_0(w_0) - \frac{1}{2} J_c'(w_0)(w_0) = \frac{1}{4} (1 - d^3) \int_{\Omega} w_0^3 dx.
\]

It follows from Lemma 3.2 and the Sobolev imbedding theorem that

\[
J_0(w_0) \geq \beta_0 \tag{3.31}
\]

and

\[
J_{cn}(w_{cn}) \to J_0(w_0) \tag{3.32}
\]

as \( n \to \infty \). This together with Lemma 3.4 yields

\[
J_0(w_0) = \beta_0.
\]

Observe that

\[
\beta_{cn} = J_{cn}(w_{cn}) - \frac{1}{4} J_{cn}'(w_{cn})w_{cn}
\]

\[
= \frac{1}{4} \int_{\Omega} (d|\nabla w_{cn}|^2 + w_{cn}^2)dx + \frac{c}{4d} \int_{\Omega} (c - d w_{cn})^3 dx.
\]

Likewise

\[
\beta_0 = \frac{1}{4} \int_{\Omega} (d|\nabla w_0|^2 + w_0^2)dx.
\]

Then Lemma 3.4 implies

\[
\frac{1}{4} \|w_{cn}\|^2 \to \frac{1}{4} \|w_0\|^2 \tag{3.33}
\]

as \( n \to \infty \). Together with \( w_{cn} \rightharpoonup w_0 \) weakly in \( E \), we conclude that

\[
\|w_{cn} - w_0\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.34}
\]

Since the above proof shows that (3.34) holds for any sequence \( c_n \to 0 \). The proof is complete.

Remark 2. The case \( c > 0 \) can be treated similarly, we do not give the detail of the proof.

From Lemma 3.1, Lemma 3.3 and Remark 2, the proof of Theorem 1.1 is complete.
4. Spike solutions. We now investigate the profile of $u_0$ obtained by Lemma 3.1. The function $u_0$ depends on the value of $d$. Set $v(x) = u_0(d \frac{x}{N})$ and $\Omega_d = \{x | d \frac{x}{N} \in \Omega\}$. Straight forward calculation gives

$$\int_{\Omega_d} v(x)^4 dx = \frac{1}{d} \int_{\Omega} u_0^4 dx,$$

$$\int_{\Omega_d} v(x)^2 dx = \frac{1}{d} \int_{\Omega} u_0^2 dx$$

and

$$\int_{\Omega_d} |\nabla v|^2 dx = \int_{\Omega} |\nabla u_0|^2 dx.$$

Then

$$\frac{1}{d} J_0(u_0) = \int_{\Omega_d} \frac{1}{2} (|\nabla v|^2 + v^2) - \frac{(1 - d^4)}{4} v^4 dx$$

and

$$\frac{1 - d^3}{d} \beta_0(d) = \int_{\Omega_d} \frac{1}{2} (|\nabla \hat{v}|^2 + \hat{v}^2) - \frac{1}{4} \hat{v}^4 dx,$$  \hspace{1cm} (4.1)

if

$$\hat{v}(x) = \sqrt{1 - d^3} v(x) = \sqrt{1 - d^3 u_0(d \frac{x}{N})}.$$  \hspace{1cm} (4.2)

In (4.2), $\hat{v}$, $v$ and $u_0$ are functions depending on $d$. In what follows, we take $d_n = \frac{1}{n}$, $n \in \mathbb{N}$, and denote $\Omega_{\frac{1}{n}}$ by $\Omega(n)$. We replace $\hat{v}$ by $v_n$ to clarify its dependence on $d$. Moreover by taking translation if necessary, we always assume that

$$\int_{\Omega} v_n^2 dx = \max_{z \in \mathbb{Z}^N} \int_{z + \Omega} v_n^2 dx,$$  \hspace{1cm} (4.3)

where $z \in \mathbb{Z}^N$, a lattice point of $\mathbb{R}^N$. Set

$$\alpha(v_n) = \int_{\Omega} v_n^2 dx.$$  \hspace{1cm} (4.4)

It has been shown that the positive solution of

$$\begin{cases} 
- \Delta g + g = g^3, \\
g \in H^1(\mathbb{R}^N). 
\end{cases}$$  \hspace{1cm} (4.5)

is a radially symmetric and it is unique up to translation. We may assume that $g$ is radially symmetric at the origin. Then

$$g(0) = \max_{x \in \mathbb{R}^N} g(x).$$  \hspace{1cm} (4.6)

Let

$$\sigma_g = \int_{\mathbb{R}^N} \frac{1}{2} (|\nabla g|^2 + g^2) - \frac{1}{4} g^4 dx.$$  \hspace{1cm} (4.7)

It is known that $g(|x|)$ is a decreasing function with exponentially decaying as $|x| \to \infty$.

Let $E_n = \{u \in H^1_{loc}(\mathbb{R}^N) | u(x + ne_i) = u(x) \text{ for } i = 1, 2, ..., N\}$.

**Lemma 4.1.** Given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\frac{1}{n} < \delta$ and $u_0 \in A_0$ then

$$\|v_n - g\|_{E_n} < \varepsilon.$$  \hspace{1cm} (4.8)
Proof. Let
\[ Q(v_n) = \frac{1}{2} \int_{\Omega(n)} (|\nabla v_n|^2 + v_n^2)dx - \frac{1}{4} \int_{\Omega(n)} v_n^4dx. \]

By (4.1)
\[ Q(v_n) = \frac{1 - d_n^3}{d_n} \beta_0(d_n) \tag{4.9} \]
and
\[ Q'(v_n)v_n = \int_{\Omega(n)} (|\nabla v_n|^2 + v_n^2)dx - \int_{\Omega(n)} v_n^4dx = 0. \tag{4.10} \]

Combining (4.9) and (4.10) gives
\[ Q(v_n) - \frac{1}{4} Q'(v_n)(v_n) = \frac{1}{4} \int_{\Omega(n)} (|\nabla v_n|^2 + v_n^2)dx = \frac{1 - d_n^3}{d_n} \beta_0(d_n), \tag{4.11} \]
\[ Q(v_n) - \frac{1}{2} Q'(v_n)(v_n) = \frac{1}{4} \int_{\Omega(n)} v_n^4dx = \frac{1 - d_n^3}{d_n} \beta_0(d_n). \tag{4.12} \]

Note that a slightly modified version of the proof of Lemma 3.2 shows that
\[ \limsup_{n \to \infty} \frac{1 - d_n^3}{d_n} \beta_0(d_n) \leq \sigma_g. \tag{4.13} \]

We claim there is a \( C_0 > 0 \) such that
\[ \alpha(v_n) \geq \frac{C_0}{\sigma_g} \tag{4.14} \]
if \( n \) is large enough. Indeed applying the Holder Inequality and Solovol imbedding theorem, there is a \( C_1 > 0 \) such that
\[ \int_{z + \Omega} v_n^4dx \leq \left( \int_{z + \Omega} v_n^2dx \right)^{\frac{1}{2}} \left( \int_{z + \Omega} v_n^6dx \right)^{\frac{1}{2}}. \tag{4.15} \]

It follows from (4.11), (4.12) and (4.15) that
\[ \frac{1 - d_n^3}{d_n} \beta_0(d_n) = \frac{1}{4} \int_{\Omega(n)} v_n^4dx \leq \frac{1}{4} \sum_{z \in \mathbb{Z}^N \cap \Omega(n)} \int_{z + \Omega} v_n^4dx \leq C_1 (\alpha(v_n))^\frac{3}{2} \|v_n\|_{E_n}^3; \tag{4.16} \]
that is,
\[ \frac{1 - d_n^3}{d_n} \beta_0(d_n) \leq C_1 (\alpha(v_n))^\frac{3}{2} \left( \frac{1 - d_n^3}{d_n} \beta_0(d_n) \right)^\frac{3}{2}. \]

This together with (4.13) gives
\[ (\sigma_g)^{-1} \leq \left( \frac{1 - d_n^3}{d_n} \beta_0(d_n) \right)^{-1} \leq C_1^2 \alpha(v_n), \tag{4.17} \]
which completes the proof of (4.14).

Next we show that
\[ \liminf_{n \to \infty} \frac{1 - d_n^3}{d_n} \beta_0(d_n) \geq \sigma_g. \tag{4.18} \]

By (4.11) and (4.13), the sequence \( \{\|v_n\|_{E_n}|n \in \mathbb{N}\} \) is bounded. Hence along a subsequence \( v_n \) converges weakly to \( g \) and \( v_n \to g \) pointwise on \( \mathbb{R}^N \). Moreover for \( \varphi \in C_0^1(\mathbb{R}^N) \),
\[ \int_{\mathbb{R}^N} (\nabla v_n \cdot \nabla \varphi + v_n \varphi)dx \to \int_{\mathbb{R}^N} (\nabla g \cdot \nabla \varphi + g \varphi)dx \tag{4.19} \]
as \( n \to \infty \). Together with (4.12) and Fatou’s Lemma yields
\[
\liminf_{n \to \infty} \frac{1 - d_n^3}{d_n} \beta_0(d_n) \geq \frac{1}{4} \int_{\mathbb{R}^N} g^4 dx = \sigma_g
\] (4.20)
and consequently
\[
\lim_{n \to \infty} \frac{1 - d_n^3}{d_n} \beta_0(d_n) = \sigma_g.
\] (4.21)
This together with (4.21) implies
\[
\lim_{n \to \infty} \|v_n\|_{E_n}^2 = \|g\|_{H^1(\mathbb{R}^2)}^2.
\] (4.22)
Then for large \( j \),
\[
\|v_n - g\|_{E_j}^2 = \|v_n\|_{E_j}^2 - 2 \int_{\Omega(j)} (\nabla v_n \cdot \nabla g + v_n g) dx + \|g\|_{E_j}^2 \to 0
\] (4.23)
as \( n \to \infty \). Now the proof is complete.

In view of Lemma 3.5 and Lemma 4.1, the proof of Theorem 1.2 is complete.

5. Lyapunov stability. Consider the flow generated by (1.1)-(1.4). A class of functions \( f \) investigated in [17] is \( f(u_1, u_2) = g(u_1 + u_2) + bu_2 \) with \( b > 0 \). In particular if \( g(\xi) = -\theta_1 \xi + \xi^3 \) and \( \theta_1 > 0 \) then
\[
f_c(u) = f(u, c - du) = -[\theta_1 + (b - \theta_1)d]u + [(1 - d)u + c]^3 + bc.
\] (5.1)
In particular, the existence results for the stationary solutions and spike solutions obtained in Section 3 and Section 4 are also applicable to \( f(u_1, u_2) = (u_1 + u_2)^3 - u_1 \), since
\[
f_c(u) = f(u, c - du) = -u + [(1 - d)u + c]^3 + c.
\] (5.2)
Recall that by the mass conservation
\[
\int_{\Omega} u_1(x, t) + u_2(x, t) dx = \int_{\Omega} \psi_1(x) + \psi_2(x) dx
\] (5.3)
for all \( t > 0 \). If \( \lim_{t \to \infty}(u_1(x, t), u_2(x, t)) \to (w_c(x), c - dw_c(x)) \) then
\[
\int_{\Omega} c + (1 - d)w_c(x) dx = \int_{\Omega} \psi_1(x) + \psi_2(x) dx.
\]
Define
\[
\Phi(u_1, u_2) = \int_{\Omega} \frac{d}{2} |\nabla (u_1 + u_2)|^2 + \frac{1}{2} |\nabla (du_1 + u_2)|^2 + H(u_1 + u_2) dx,
\] (5.4)
where \( H(\xi) = \frac{1}{2} \xi^2 - \frac{(1-d)}{4} \xi^4 \). The Lyapunov stability for the stationary solutions of (1.1)-(1.3) follows from a result given by [17]:

**Theorem 5.1.** If \( (u_1(x, t), u_2(x, t)) \) is a solution of (1.1)-(1.4) then
\[
\frac{d}{dt} \Phi(u_1, u_2) = -\int_{\Omega} |\nabla (du_1 + u_2)|^2 + (1 + d)[\frac{d}{dt}(u_1 + u_2)]^2 dx.
\] (5.5)
Lemma 6.1. Let $S(x) = (\hat{u}_1(x), \hat{u}_2(x))$ ($x \in \Omega \subset \mathbb{R}^N$) be a stationary solution of (1.1)-(1.3). We look at the linearization of (1.1)-(1.3) at $(\hat{u}_1(x), \hat{u}_2(x))$. With zero being an eigenvalue due to the translation free mode by periodic boundary conditions, we study the associated eigenfunctions and adjoint eigenfunctions.

First, we rewrite (1.1)-(1.2) in the vector form as

$$U_t = D\Delta U + f(U)a,$$

(6.1)

where $U = t(u_1, u_2)$, $D = \text{diag}\{d, 1\}$ and $a = t(1, -1)$. Then the linearized operator $L$ with respect to $S(x)$ is given by $LU = D\Delta U + (\nabla f(S) \cdot U)a$, where $\nabla f(S) = t(f_{u_1}(S), f_{u_2}(S))$, and the adjoint operator $L^*$ of $L$ is $L^*U = D\Delta U + (a \cdot U)\nabla f(S)$. Now, we derive the eigenfunctions and the adjoint eigenfunctions associated to 0 eigenvalue, which will be important in forthcoming papers to consider the various motions of spikes as in [15] and [19].

**Lemma 6.1.** The stationary solution $S(x)$ of (6.1) is given in the form $S(x) = s(x)D^{-1}a + a'$ for a scaler function $s(x)$ and a constant vector $a' \in \mathbb{R}^2$.

**Proof.** $S(x)$ satisfies

$$D\Delta S + f(S)a = 0$$

(6.2)

and hence $\Delta S = -f(S)D^{-1}a$ holds, which shows the result by virtue of the periodic boundary conditions. \qed

Here we note that $s(x)$ satisfies $\Delta s = -f(S)$, which means that $s(x)$ is given by

$$s(x) = d\hat{u}_1(x) + c_1 = -\hat{u}_2(x) + c_2$$

(6.3)

for constants $c_1$, $c_2$. Differentiating (6.2) with respect to $x_j$, we have $LS_{x_j} = 0$ ($j = 1, \cdots, N$). Thus the eigenspace associated to 0 is a $N$-dimensional space.

Here we construct $N$ independent adjoint eigenfunctions $\{\Phi^*_j\}_{j=1, \cdots, N}$ satisfying $L^*\Phi^*_j = 0$. Let $L^*\Phi^* = 0$, that is, $D\Delta \Phi^* + (a \cdot \Phi^*)\nabla f(S) = 0$. Then

$$\Delta \Phi^* + (a \cdot \Phi^*)D^{-1}\nabla f(S) = 0$$

(6.4)

holds and the inner product with $a$ leads

$$\Delta m + m(D^{-1}\nabla f(S) \cdot a) = 0,$$

(6.5)

where $m = m(x) = (\Phi^*(x) \cdot a)$.

On the other hand, substituting $S(x) = s(x)D^{-1}a + a'$ into (6.2), we have $(\Delta s)a + f(S)a = 0$ and $\Delta s + f(S) = 0$ holds. Taking $x_j$ derivative of this equation, it is obtained that

$$\Delta s_{x_j} + s_{x_j}(\nabla f(S) \cdot D^{-1}a) = 0$$

(6.6)

by $S_{x_j} = s_{x_j}D^{-1}a$. (6.5) and (6.6) are the same equations and we may take $m(x) = s_{x_j}(x)$. Thus we can define $\Phi^*_j(x)$ from (6.4) by the solution of

$$\Delta \Phi^*_j + s_{x_j}D^{-1}\nabla f(S) = 0.$$  

(6.7)

The solvability and the linear independence are clear. In [19], the motion of spike was treated in the case of $N = 1$ by solving (6.7) explicitly.
7. Motion of a spike on heterogeneous media. In this section, we consider the equation (1.1)-(1.3) with inhomogeneity as follows:

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= d \Delta u_1 + \{ f(u_1, u_2) + \epsilon g(x, u_1, u_2) \}, \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - \{ f(u_1, u_2) + \epsilon g(x, u_1, u_2) \}
\end{aligned}
\]  

(7.1)

for \(0 < \epsilon \ll 1\) and a function \(g\), which is also written in a vector form

\[
U_t = D \Delta U + \{ f(U) + \epsilon g(x, U) \} a
\]

(7.2)
as in (6.1). (7.1) has a conservative quantity \( \int_\Omega \{ u_1 + u_2 \} dx \). It is expressed as

\[
\int_\Omega \{ u_1 + u_2 \} dx = \sqrt{2} \int_\Omega (U \cdot a^*) dx = \sqrt{2} \langle U, a^* \rangle_2,
\]

(7.3)

where \( \langle \cdot, \cdot \rangle_2 \) denotes the inner product in \( L^2(\Omega) \) and \( a^* = \frac{1}{\sqrt{2}} t(1,1) \) is taken such that \( \|a^*\|_2 = 1 \) and \( \langle a^* \rangle_2 = 0 \).

We assume (7.1) with \( \epsilon = 0 \), that is, (6.1) has a stable stationary solution \( S(x) = (\hat{u}_1(x), \hat{u}_2(x)) \). Then the function \( S(x - \hat{x}) \) is also a stationary solution of (6.1) for any \( \hat{x} = (\hat{x}_1, \cdots, \hat{x}_N) \in \Omega \). For \( \epsilon > 0 \), representing the solution as \( U(t, x) = S(x - \hat{x}(t)) + V(t, x - \hat{x}(t)) \) and substituting it into (7.2), we have by \( z = x - \hat{x}(t) \)

\[
V_t - (\frac{d\hat{x}}{dt} \cdot \nabla_z V) = D \Delta_z V + (\nabla f(S) \cdot a) + \epsilon g(z + \hat{x}(t), S) a + (\frac{d\hat{x}}{dt} \cdot \nabla_z S) + G(z, V) a
\]

for \( |G(z, V)| \leq O(|V|^2 + \epsilon|V|) \). That is,

\[
V_t = LV + \epsilon g(z + \hat{x}(t), S(z)) a + (\frac{d\hat{x}}{dt} \cdot \nabla_z (S + V)) + G(z, V) a.
\]

(7.4)

First, we intuitively derive the motion of \( \hat{x}(t) \) by assuming that \( |V|, \frac{d\hat{x}}{dt} \) are sufficiently small and omitting higher products of them formally. Then the lowest order equation of \( V \) is

\[
V_t = LV + \epsilon g(z + \hat{x}(t), S(z)) a + (\frac{d\hat{x}}{dt} \cdot \nabla_z S).
\]

(7.5)

Taking \( L^2 \) inner product of (7.5) with the adjoint eigenfunction \( \Phi_i^* \), we get

\[
M \frac{d\hat{x}}{dt} = \epsilon b,
\]

(6.6)

where \( M = \{m_{ij}\}_{1 \leq i, j \leq N} \) with \( m_{ij} = \langle S_{x_i}, \Phi_i^* \rangle_2 \), \( \frac{d\hat{x}}{dt} = \{ \frac{d\hat{x}_1}{dt}, \cdots, \frac{d\hat{x}_N}{dt} \} \) and \( b = \{b_1, \cdots, b_N\} \) with \( b_i = - \langle g(z + \hat{x}(t), S(z)) a, \Phi_i^* \rangle_2 \).

(6.6) is a general form of the equation describing the motion of \( \hat{x}(t) \). In order to know more explicit dynamics, we assume a sufficiently large \( \Omega \), say \( \Omega_K = [-K, K]^N \) for \( K >> 1 \). It implies that the stationary solution \( S(x) \) is close to a radially symmetric solution, say \( \hat{S}(r) \) for \( r = |x| \). Precisely speaking, \( |S(x) - \hat{S}(r)| \) is sufficiently small on a bounded domain including origin and \( |S(x)| \leq O(\epsilon^{-\gamma r}) \) holds. Hence by assuming that \( \Omega_K \) is sufficiently large, we may formally expect that the \( L^2 \) inner product on \( \Omega_K \) is replaced by the one on the whole space \( \mathbb{R}^N \) and \( S = \hat{S}(r) = (\hat{u}_1(r), \hat{u}_2(r)) \). Under the above reduction, we can derive an explicit form of (7.6) as follows: Denoting the polar coordinate by \( x = r e(\Theta) \) for \( r \geq 0 \) and \( e(\Theta) = (e_1(\Theta), \cdots, e_N(\Theta)) \) with \( |e(\Theta)| = 1 \) and \( \Theta = (\theta_1, \cdots, \theta_{N-1}) \),
Lemma 7.1. We find $\Phi_j^*(x) = \Psi_j^*$ for $x \in \mathbb{R}^n$. This hypothesis has not been proved, but it is strongly suggested by Remark 3.

First, we calculate the matrix $M$. The component $m_{ij}$ is given by

$$m_{ij} = \langle \Phi^*_i, S_{x_j} \rangle = \langle e_i \phi^*, e_j S_r \rangle_2$$

$$= \int_0^\infty r^{N-1}(\phi^*(r) \cdot S_r(r)) dr \int_{S^{N-1}} e_i(\Theta) \cdot e_j(\Theta) d\Theta$$

$$= m_0 \delta_{ij},$$

where $m_0 = \int_0^\infty r^{N-1}(\phi^*(r) \cdot S_r(r)) dr$ and $\delta_{ij}$ denotes the Kronecker’s $\delta$. Thus we know $M = m_0 I_N$ with identity matrix $I_N$ with order $N$.

In order to compute $b$, we assume $g(x, U) = g_1(U)g_2(x)$, which is naturally derived as the first order term of Taylor expansion when a small heterogeneity is given to $f$ as $f(U, 1 + \varepsilon g_2(x))$. In this case, $b$ is given by

$$b_i = -\langle g(z + \hat{x}(t), S(z) a, \Phi^*_i \rangle_2 = -\int_{\mathbb{R}^N} g_1(S(|z|))(a \cdot \Phi^*_i(z) g_2(z + \hat{x}(t)) dz.$$  

As the simplest case, we put $g_2(x) = \frac{1}{\pi r^2}(\text{Dirac function})$. Then

$$b_i = -g_1(S(|-\hat{x}(t)|))(a \cdot \Phi^*_i(-\hat{x}(t))) = -g_1(S(|-\hat{x}(t)|))(a \cdot \Psi^*(-\hat{x}(t))),$$

holds. Thus we formally get the gradient flow like equation

$$m_0 \frac{d\hat{x}}{dt} = -g_1(S(|-\hat{x}(t)|)) \nabla(a \cdot \Psi^*(-\hat{x}(t)))$$

$$= -g_1(S(|-\hat{x}(t)|))(a \cdot \Psi^*(-\hat{x}(t))) |\hat{x}(t)|^{-1} -\hat{x}(t)|.$$  

In the following, we give the rigorous result on the motion (7.6) of a spike. Let $S(x)$ and $L$ be a stationary solution of (7.1) with $\varepsilon = 0$ and the linearized operator with respect to $S(x)$. $L$ is given by $LU = D\Delta U + (\nabla f(S) \cdot U) a$ as mentioned previously. Since the quantity $\langle U, a^* \rangle_2$ is conserved in time $t$, we put $X = \{ U \in \{L^2(\Omega)\}^2 : \langle U, a^* \rangle_2 = 0 \}$. Note that $S_{x_j} \in X$. We denote the restriction of $L$ in $X$ by $L_X = L|_X$ and the adjoint operator of $L_X$ in $X$ by $L^*_X = (L_X)^*$. We assume the linear stability of $S(x)$ by the following hypothesis:

Hypothesis H1) $L_X$ satisfies $\Sigma(L_X) = \Sigma_0 \cup \Sigma_1$, where $\Sigma(L_X)$ is a spectral set of $L_X$ and $\Sigma_0 = \{0\}$, $\Sigma_1 \subset \{ z \in \mathbb{C} : \text{Re}(z) < -\gamma_0 \}$ for a positive constant $\gamma_0$. 0 is a semi-simple eigenvalue.

Remark 3. This hypothesis has not been proved, but it is strongly suggested by Theorem 5.1.

By H1), the adjoint operator $L^*_X$ also satisfies H1).

Lemma 7.1. $L^*_X$ is given by $L^*_X U = L^* U - \langle L^* U, a^* \rangle_2 a^*$ for $U \in X$, where $L^* U = D\Delta U + (a \cdot U) \nabla f(S)$ as given in the previous section. The adjoint eigenfunctions $\Phi^*_{X_j}$ ($j = 1, \cdots, N$) associated with $0$ eigenvalue of $L^*_X$ are given by $\Phi^*_{X_j} = \Phi^*_j - \langle \Phi^*_j, a^* \rangle_2 a^*$, where $\Phi^*_j$ are the ones constructed in Section 5.
Lemma 7.1 is easily proved while we refer [19].

Remark 4. By Lemma 7.1, $\langle U, \Phi_{x_j}^* \rangle_2 = \langle U, \Phi_j^* \rangle_2$ holds for $U \in X$.

Theorem 7.2. Assume H1) and $1 \leq N \leq 3$. If the initial data $U(0, x)$ of (7.2) is sufficiently close to $S(x - \hat{x}_0)$ for $\hat{x}_0 \in \Omega$, then for the solution $U(t, x)$ of (7.2),
\[
\|U(t) - S(x - \hat{x}(t))\|_\infty \leq O(\varepsilon)
\]
uniformly for $t > 0$ and $x \in \Omega$ holds and $\hat{x}(t)$ satisfies $M\frac{d\hat{x}}{dt} = \varepsilon b + O(\varepsilon^2)$.

Proof. We define the shift operator $\tau(\hat{x})$ by $\langle \tau(\hat{x})U(x) = U(x - \hat{x})$. Let $E_0 = \text{span}\{S_1, \ldots, S_N\}$, $E_0^+ = \{U \in X \mid \langle U, \Phi_j^* \rangle_2 = 0 (j = 1, \ldots, N)\}$. We set the subsets in $\text{span}\{S_1, \ldots, S_N\}$.

We note that almost all inner products in the proof should be taken with $\Phi^*_j$, uniformly for $t > 0$ and $x \in \Omega$. Here we note that almost all inner products in the proof should be taken with $\Phi^*_j$, but Remark 4 says that it is enough to take the products with $\Phi^*_j$ in the space $X$.

We set the subsets in $X$ by $M_0 = \{S(x - \hat{x}) \mid \hat{x} \in \Omega\}$ and $W_\delta = \{\tau(\hat{x})S + V \mid \hat{x} \in \Omega, \|V\|_2 < \delta, V \in X\}$. Then the following lemmas hold.

Lemma 7.3. ([14, 15]) There exists homeomorphism $\Pi(\hat{x}) : E_0^+ \rightarrow E_0^+ (\hat{x})$ satisfying $\|\Pi(\hat{x})\|_2 \leq C_1, \|\Pi^{-1}(\hat{x})\|_2 \leq C_1, \|\Pi_2(\hat{x})\|_2 \leq C_1$ for a constant $C_1 > 0$ and $(\Pi(V))_{\hat{x}_0} \in E_0(\hat{x})$.

Lemma 7.4. ([14, 15]) Suppose to take $\delta > 0$ sufficiently small. Then for $U \in W_\delta$, there uniquely exist $\hat{x} \in \Omega$ and $V \in E_{\hat{x}_0}$ such that $U = \tau(\hat{x})S + \Pi(\hat{x})V$ with $\|V\|_2 < C_2\delta$ for a constant $C_2 > 0$.

Substituting $U(t, x) = \tau(\hat{x}(t))S + \Pi(\hat{x}(t))V(t)$ for $V(t) \in E_0^+$ into (7.2), we have
\[
\Pi(\hat{x})V_t + (\Pi_2(\hat{x})V) \frac{d\hat{x}}{dt} = D\Delta_\omega \Pi(\hat{x})V + (\nabla f(\tau(\hat{x})S) \cdot \Pi(\hat{x})V)a + \varepsilon g(x, \tau(\hat{x})S)a + (\frac{d\hat{x}}{dt}) \cdot \nabla_x \tau(\hat{x})S + G_1(x, \Pi(\hat{x})V)a
\]
for $|G_1(z, U)| \leq O(|U|^2 + \varepsilon|U|)$, which leads
\[
\Pi(\hat{x})V_t = L(\hat{x})\Pi(\hat{x})V + \varepsilon g(x, \tau(\hat{x})S)a + (\frac{d\hat{x}}{dt}) \cdot \nabla_x \tau(\hat{x})S + \Pi_2(\hat{x})V) + G_1(x, \Pi(\hat{x})V)a,
\]
where $L(\hat{x})U = D\Delta U + (\nabla f(\tau(\hat{x})S) \cdot U)a$. Taking the $L^2$ inner product of (7.8) with $\tau(\hat{x})\Phi_i$, it follows
\[
0 = \varepsilon \langle g(\cdot + \hat{x}, S)a, \Phi_i^* \rangle_2 + \langle m_i + P_i(\hat{x}, V) \cdot \frac{d\hat{x}}{dt} \rangle + \langle G_1(x, \Pi(\hat{x})V)a, \tau(\hat{x})\Phi_i \rangle_2
\]
for $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)$.

Let $X_\omega$ be the fractional powered space by $L$ embedded into $L^\infty(\Omega)$, which is possible by taking $N/4 < \omega < 1$. Let $W(D_1) = \{V \in X_\omega \cap E_0^+ \mid \|V\|_\omega \leq D_1\varepsilon\}$. Since $|P_i(\hat{x}, V)| \leq C_0\|V\|_\omega$ and the matrix $M = \{m_{ij}\}$ is invertible from H1), $(M + P(\hat{x}, V))^{-1}$ exists for $V \in W(D_1)$ and (7.9) leads
\[
\frac{d\hat{x}}{dt} = H_1(\hat{x}, V),
\]
where
\[
\mathbf{P}(\hat{x}, V) = (\mathbf{P}_1(\hat{x}, V), \cdots, \mathbf{P}_N(\hat{x}, V)),
\]
\[
-b(\hat{x}, V) = \varepsilon (g(\hat{x} + \hat{x}, S) a, \Phi_1^*)_2 + \langle G_1(x, \Pi(\hat{x})V) a, \tau(\hat{x}) \Phi_1 \rangle_2,
\]
\[
\cdots, \varepsilon (g(\hat{x} + \hat{x}, S) a, \Phi_N^*)_2 + \langle G_1(x, \Pi(\hat{x})V) a, \tau(\hat{x}) \Phi_N \rangle_2,
\]
\[
\mathbf{H}_1(\hat{x}, V) = (M + \mathbf{P}(\hat{x}, V))^{-1} b(\hat{x}, V).
\]

Thus we see from (7.8) by taking the projection \( R(\hat{x}) \)
\[
\begin{cases}
\frac{d\hat{x}}{dt} = \mathbf{H}_1(\hat{x}, V), \\
V_t = A(\hat{x})V + \mathbf{H}_2(\hat{x}, V),
\end{cases}
\]
\[(7.10)\]

where \( A(\hat{x}) = \Pi^{-1}(\hat{x})L(\hat{x})\Pi(\hat{x}) \) and
\[
\mathbf{H}_2(\hat{x}, V) = \Pi^{-1}(\hat{x})R(\hat{x}) \{ \varepsilon g(x, \tau(\hat{x})S) a + (\mathbf{H}_1(\hat{x}, V) \cdot \nabla_x (\tau(\hat{x})S)) + G_1(x, \Pi(\hat{x})V) a \}.
\]

Note that \( \mathbf{H}_2(\hat{x}, V) \in E_0^+ \) by the definition of \( \mathbf{H}_1(\hat{x}, V) \) and \( R(\hat{x})(\Pi(\hat{x})V)_{\hat{x}_j} = 0. \)
Since \( \mathbf{H}_j \) satisfy
\[
|\mathbf{H}_1(\hat{x}, V)| \leq C_4 \{ \varepsilon + \|V\|_\omega \},
\]
\[
\|\mathbf{H}_2(\hat{x}, V)\|_2 \leq C_4 \{ \varepsilon + \varepsilon \|V\|_\omega + \|V\|_\omega^2 \},
\]
\[
|\mathbf{H}_1(\hat{x}, V) - \mathbf{H}_1(\hat{x}', V')| \leq C_4 \varepsilon \{ |\hat{x} - \hat{x}'| + D_1 \|V - V'\|_\omega \},
\]
\[
\|\mathbf{H}_2(\hat{x}, V) - \mathbf{H}_2(\hat{x}', V')\|_2 \leq C_4 \varepsilon \{ |\hat{x} - \hat{x}'| + D_1 \|V - V'\|_\omega \},
\]
we can show the existence of a stable invariant manifold
\[
M_\varepsilon = \{ \tau(\hat{x})S + \Pi(\hat{x})\sigma_\varepsilon(\hat{x}) \mid \hat{x} \in \Omega \} \subset W_\delta
\]
satisfying
\[
\sigma_\varepsilon(\hat{x}) \in E_0^+, \|\sigma_\varepsilon(\hat{x})\|_\omega \leq D_\varepsilon, \|\sigma_\varepsilon(\hat{x}) - \sigma_\varepsilon(\hat{x}')\|_\omega \leq D_\varepsilon |\hat{x} - \hat{x}'| \quad (7.11)
\]
by taking appropriate \( D_1 > 0 \) and \( D_2 > 0 \) in quite a similar manner to [14].

Along the invariant manifold \( M_\varepsilon \), \( \hat{x}(t) \) satisfies \( \frac{d\hat{x}}{dt} = \mathbf{H}_1(\hat{x}, \sigma_\varepsilon(\hat{x})) \). Since \( b(\hat{x}) = \varepsilon b + O(\varepsilon^2), \mathbf{P}(\hat{x}, \sigma_\varepsilon(\hat{x})) = O(\varepsilon) \) hold, we see \( \mathbf{H}_1(\hat{x}, \sigma_\varepsilon(\hat{x})) = \varepsilon M^{-1}b + O(\varepsilon^2) \), which directly leads \( \frac{d\hat{x}}{dt} = \varepsilon b + O(\varepsilon^2). \)

**Acknowledgments.** Part of the work was done when S.-I. Ei was visiting National Center for Theoretical Sciences, Taiwan, and Tseng was visiting Hokkaido University. Ei was supported by the Grant-in-aid for Scientific Research 26310212, JSPS and JST CREST Grant Number JPMJCR14D3 to S.E., Japan. Tseng was supported in part by the Ministry of Science and Technology of Taiwan, MOST 105-2115-M-007-009-MY3, MOST 106-2115-M-007-011, MOST 107-2115-M-018-01 and MOST 108-2115-M-018-002. The authors would like to thank Chao-Nien Chen for stimulating and valuable discussion.
REFERENCES

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, 14 (1973), 394–381.

[2] C.-N. Chen and Y. S. Choi, Standing pulse solutions to FitzHugh-Nagumo equations, *Arch. Rational Mech. Anal.*, 206 (2012), 741–777.

[3] C.-N. Chen and Y. S. Choi, Traveling pulse solutions to FitzHugh-Nagumo equations, *Calculus of Variations and Partial Differential Equations*, 54 (2015), 1–45.

[4] C.-N. Chen, Y. S. Choi and N. Fusco, The Γ-limit of traveling waves in the FitzHugh-Nagumo system, *J. Differential Equations*, 267 (2019), 1805–1835.

[5] C.-N. Chen, Y.-S. Choi and X. F. Ren, Bubbles and droplets in a singular limit of the FitzHugh-Nagumo system, *Interfaces and Free Boundaries*, 20 (2018), 165–210.

[6] C.-N. Chen, Y.-S. Choi, Y. Y. Hu and X. F. Ren, Higher dimensional bubble profiles in a singular limit of the FitzHugh-Nagumo system, *SIAM J. Math. Anal.*, 50 (2018), 5072–5095.

[7] C.-N. Chen, S.-I. Ei and S.-Y. Tseng, Heterogeneity induced effects for pulse dynamics in FitzHugh-Nagumo type systems, *Physica D: Nonlinear Phenomena*, 382/383 (2018), 22–32.

[8] C.-N. Chen and X. J. Hu, Maslov index for homoclinic orbits of Hamiltonian systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24 (2007), 589–603.

[9] C.-N. Chen and X. J. Hu, Stability criteria for reaction-diffusion systems with skew-gradient structure, *Communications in Partial Differential Equations*, 33 (2008), 189–208.

[10] C.-N. Chen and X. J. Hu, Stability analysis for standing pulse solutions to FitzHugh-Nagumo equations, *Calculus of Variations and Partial Differential Equations*, 49 (2014), 827–845.

[11] C.-N. Chen and K. Tanaka, A variational approach for standing waves of FitzHugh-Nagumo type systems, *J. Differential Equations*, 257 (2014), 109–144.

[12] C.-N. Chen and S.-Y. Tseng, Periodic solutions and their connecting orbits of Hamiltonian systems, *J. Differential Equations*, 177 (2001), 121–145.

[13] A. Doelman, P. van Heijster and T. J. Kaper, Pulse dynamics in a three-component system: Existence analysis, *J. Dynam. Differential Equations*, 21 (2009), 73–115.

[14] S.-I. Ei, The motion of weakly interacting pulses in reaction-diffusion systems, *J. Dynam. Differential Equations*, 14 (2002), 85–137.

[15] S.-I. Ei and J. C. Wei, Dynamics of metastable localized patterns and its application to the interaction of spike solutions for the Gierer-Meinhardt systems in two spatial dimension, *Japan J. Ind. Appl. Math.*, 19 (2002), 181–226.

[16] S. Ishihara, M. Otsuji and A. Mochizuki, Transient and steady state of mass-conserved reaction-diffusion systems, *Phys. Rev. E*, 75 (2007), 015203(R).

[17] S. jimbo and Y. Morita, Lyapunov function and spectrum comparison for a reaction-diffusion system with mass conservation, *J. Differential Equations*, 255 (2013), 1657–1683.

[18] J. Keener and J. Sneyd, *Mathematical Physiology. Vol. I: Cellular Physiology*, Second edition, Interdisciplinary Applied Mathematics, 8/I. Springer, New York, 2009.

[19] M. Kuwamura, S. Seirin-Lee and S.-I. Ei, Dynamics of localized unimodal patterns in reaction-diffusion systems for cell polarization by extracellular signaling, *SIAM J. APPL. MATH.*, 78 (2018), 3238–3257.

[20] E. Latos and T. Suzuki, Global dynamics of a reaction-diffusion system with mass conservation, *J. Math. Anal. Appl.*, 411 (2014), 107–118.

[21] Y. Morita and T. Ogawa, Stability and bifurcation of nonconstant solutions to a reaction-diffusion system with conservation of mass, *Nonlinearity*, 23 (2010), 1387–1411.

[22] W.-M. Ni and J. C. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.*, 48 (1995), 731–778.

[23] M. Otsuji, S. Ishihara, C. Co, K. Kaibuchi, A. Mochizuki and S. Kuroda, A mass conserved reaction-diffusion system captures properties of cell polarity, *PLoS Comput. Biol.*, 3 (2007), 1040–1054.

[24] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 1986.

[25] C. Reinecke and G. Sweers, A positive solution on $\mathbb{R}^n$ to a equations of FitzHugh-Nagumo type, *J. Differential Equations*, 153 (1999), 292–312.

[26] T. Suzuki and S. Tasaki, Stationary Fix-Caginalp equation with non-local term, *Nonlinear Anal.*, 71 (2009), 1329–1349.
[27] P. van Heijster, C.-N. Chen, Y. Nishiura and T. Teramoto, Localized patterns in a three-component FitzHugh-Nagumo model revisited via an action functional, *J. Dynam. Differential Equations*, 30 (2018), 521–555.

[28] P. van Heijster, C.-N. Chen, Y. Nishiura and T. Teramoto, Pinned solutions in a heterogeneous three-component FitzHugh-Nagumo model, *J. Dyn. Differ. Equ.*, 31 (2019), 153–203.

[29] E. Yanagida, Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations, *J. Math. Biol.*, 22 (1985), 81–104.

Received March 2019; revised July 2019.

E-mail address: Eichiro@math.sci.hokudai.ac.jp
E-mail address: sytzeng@cc.ncue.edu.tw