ON THE CONVERGENCE PROPERTIES OF A SMOOTHING APPROACH FOR MATHEMATICAL PROGRAMS WITH SYMMETRIC CONE COMPLEMENTARITY CONSTRAINTS

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ABSTRACT. This paper focuses on a class of mathematical programs with symmetric cone complementarity constraints (SCMPCC). The explicit expression of C-stationary condition and SCMPCC-linear independence constraint qualification (denoted by SCMPCC-LICQ) for SCMPCC are first presented. We analyze a parametric smoothing approach for solving this program in which SCMPCC is replaced by a smoothing problem \( P_\varepsilon \) depending on a (small) parameter \( \varepsilon \). We are interested in the convergence behavior of the feasible set, stationary points, solution mapping and optimal value function of problem \( P_\varepsilon \) when \( \varepsilon \to 0 \) under SCMPCC-LICQ. In particular, it is shown that the convergence rate of Hausdorff distance between feasible sets \( F_\varepsilon \) and \( F_0 \) is of order \( O(|\varepsilon|) \) and the solution mapping and optimal value of \( P_\varepsilon \) are outer semicontinuous and locally Lipschitz continuous at \( \varepsilon = 0 \) respectively. Moreover, any accumulation point of stationary points of \( P_\varepsilon \) is a C-stationary point of SCMPCC under SCMPCC-LICQ.

1. Introduction. In this paper, we consider a class of mathematical programs with symmetric cone complementarity constraints (SCMPCCs) of the form:

\[
\begin{align*}
\min_{z \in \mathbb{R}^n} & \quad f(z) \\
\text{s. t.} & \quad g(z) \leq 0, \quad h(z) = 0, \\
& \quad G(z) \in K, \quad H(z) \in K, \\
& \quad \langle G(z), H(z) \rangle = 0,
\end{align*}
\]

where \( f : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}^q, \ h : \mathbb{R}^n \to \mathbb{R}^p, \ G, \ H : \mathbb{R}^n \to \mathcal{V} \) are all twice continuously differentiable mappings, \( \mathcal{V} \) is a \( m \)-dimensional real Euclidean space, \( A := (\mathcal{V}, \langle \cdot, \cdot \rangle, \circ) \) is a Euclidean Jordan algebra, and \( K \) is a symmetric cone in

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A (see Section 2). This model provides a general framework for various existing optimization problems with complementarity constraints, such as mathematical program with semidefinite cone complementarity constraints (SDCMPCC), mathematical program with second-order cone complementarity constraints (SOCMPCC) and mathematical program with (nonlinear) complementarity constraints (MPCC).

SCMPCC is a special case of mathematical program with equilibrium constraints (MPEC) in which the equilibrium constraints are defined by a symmetric cone complementarity system. It has applications not only in engineering design and economics (see [13, 14]) but also in the optimization problems themselves, such as symmetric cone bilevel programs, inverse problems over symmetric cones, robust optimizations and so on. In particular, a bilevel program yields a SCMPCC when the lower level symmetric cone program is replaced by the corresponding KKT conditions. Robust optimization is a recent approach to deal with the optimization problems involve uncertainty. For a robust bilevel programming problem, in which the only uncertain data set in the lower level problem is given by a system of symmetric cone inequalities, it follows from [1] that if the lower level problem is equivalently replaced by its KKT conditions, the robust problem can be represented as a SCMPCC.

SCMPCC is a class of problems difficult to solve since Robinson’s constraint qualification fails to hold at each feasible point. Hence the classical KKT condition may not hold at a local minimizer. In consequence, with the help of Euclidean Jordan algebras, we aim to derive explicit expressions of C-stationary point and propose the SCMPCC-linear independence constraint qualification for SCMPCC by extending the results on MPCC [13, 14, 16].

Moreover, we intend to construct a smoothing approximation problem of SCMPCC and analyze the convergence properties of the feasible set, the solution mapping, the optimal value function and the stationary point. For MPCC, there are enormous publications on the optimal theories and regularized or smoothing methods, see [2, 3, 6, 9, 12, 17] for reference. This list is far from being completed. Recently, smoothing approaches for SOCMPCC and SCMPCC have received much attention, see [20, 22, 24] for instance. Zhang, Zhang and Wu [24] discuss the convergence properties of a smoothing approach for solving SOCMPCC and demonstrate that any accumulation point of the sequence of stationary points to the sequence of smoothing problems, when smoothing parameters decrease to zero, is a C-stationary point, M-stationary point and S-stationary point under different conditions. Yan and Fukushima [22] propose a smoothing algorithm for SCMPCC based on a class of Chen-Mangasarian smoothing functions and showed the convergence to C-stationary points under certain conditions.

Among all papers about the smoothing methods for MPCC, Bouza and Still [2] analyze a parametric smoothing approach for solving MPCC in which MPCC is replaced by a perturbed problem $P_\tau$ and show that under generic assumptions the solutions of $P_\tau$ converge to a solution of MPCC with a rate $O(\sqrt{\tau})$ and the Hausdorff distance between the feasible sets of $P_\tau$ and MPCC is of order $O(\sqrt{\tau})$. A natural question is raised, whether is this result true for a smoothing approximation of SCMPCC problem? We will answer this question in this paper.

This paper is organized as follows. In Section 2, we briefly review some basic concepts and conclusions in nonsmooth analysis and variational analysis, and introduce the Jordan algebra and some of its properties used in subsequent analysis. Section 3 is devoted to deriving the C-stationary point and SCMPCC-LICQ of SCMPCC.
In Section 4, we construct a smoothing approximation of SCMPCC and discuss the convergence behavior of the feasible set, stationary points, solution mapping and optimal value of the smoothing approximation problem.

The following notations are used throughout the paper. Let $X$ and $Y$ be two finite dimensional real Euclidean spaces. For a given set $S \subseteq X$, $\text{conv}S$ denotes the convex hull of $S$. Let $d(x, S) = \min\{\|x - y\| : y \in S\}$ for $x \in X$, where $\|\cdot\|$ is the Euclidean norm. Denote $B_\delta(x) := \{x : \|x - x\| \leq \delta\}$. Let $I$ be the identity operator, i.e., $Ix = x$ for all $x \in X$. For an operator $A$, $A^*$ denotes the adjoint operator of $A$.

We say that the operator $A$ is onto if equation $Ax = 0$ implies $x = 0$ and say $A$ is nonsingular if the equation $Ax = 0$ has a unique solution $x = 0$. For a differentiable mapping $H : X \to Y$ and $z \in X$, we denote by $JH(z)$ the Jacobian of $H$ at $z$ and $\nabla H(z) := JH(z)^*$.

2. Preliminaries.

2.1. Background in nonsmooth analysis and variational analysis. In this subsection, we briefly review some concepts and results in nonsmooth analysis and variational analysis. First we present the definitions of Bouligand-subdifferential and generalized Jacobian of a Lipschitz continuous function.

**Definition 2.1.** Let $X$ and $Y$ be two finite dimensional Hilbert spaces. Let $O$ be an open set in $X$ and $F : O \subseteq X \to Y$ be a locally Lipschitz continuous function on $O$. We denote by $D_F$ the set of Fréchet-differentiable points of $F$ in $O$. Then, the Bouligand-subdifferential of $F$ at $x \in O$, denoted as $\partial_B F(x)$, is

$$
\partial_B F(x) := \left\{ \lim_{k \to \infty} JF(x^k) \mid x^k \in D_F, x^k \to x \right\}.
$$

The generalized Jacobian in the sense of Clarke [4] is the convex hull of $\partial_B F(x)$, i.e.,

$$
\partial F(x) = \text{conv}\{\partial_B F(x)\}.
$$

Next, we introduce concepts of semicontinuity and continuity of set-valued mappings in [15, Definition 5.4], epi-continuity of function-valued mappings in [15, Definition 7.1] and Pompeiu-Hausdorff distance between two closed nonempty sets in [15, Example 4.13].

**Definition 2.2.** A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous at $\bar{x}$ if $\lim \sup_{x \to \bar{x}} S(x) \subseteq S(\bar{x})$, but inner semicontinuous at $\bar{x}$ if $\lim \inf_{x \to \bar{x}} S(x) \supseteq S(\bar{x})$, where

$$
\lim \sup_{x \to \bar{x}} S(x) : = \{u \mid \exists x^k \to \bar{x}, \exists u^k \to u \text{ with } u^k \subseteq S(x^k)\},
$$

$$
\lim \inf_{x \to \bar{x}} S(x) : = \{u \mid \forall x^k \to \bar{x}, \exists u^k \to u \text{ with } u^k \subseteq S(x^k)\}.
$$

$S$ is continuous at $\bar{x}$ if both conditions are satisfied, namely $\lim \lim_{x \to \bar{x}} S(x) = S(\bar{x})$.

**Definition 2.3.** For a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, the function-valued mapping $\varepsilon \to f(\cdot, \varepsilon)$ is said to be epi-continuous at $\bar{\varepsilon}$, denoted as $\varepsilon \lim_{\varepsilon \to \bar{\varepsilon}} f(\cdot, \varepsilon) = f(\cdot, \bar{\varepsilon})$, if

$$
\lim_{\varepsilon \to \bar{\varepsilon}} \text{epi} f(\cdot, \varepsilon) = \text{epi} f(\cdot, \bar{\varepsilon}),
$$

where $\text{epi} f(\cdot, \varepsilon) := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x, \varepsilon) \leq \alpha\}$.
Definition 2.4. For two sets $C, D \subset \mathbb{R}^n$ closed and nonempty, the Pompeiu-Hausdorff distance between $C$ and $D$ is the quantity
\[
\mathcal{H}(C, D) := \sup_{z \in \mathbb{R}^n} |d(z, C) - d(z, D)| = \max_{z \in C} \max_{z \in D} d(z, C).
\]

A sequence $\{C^k\}$ is said to converge with respect to Pompeiu-Hausdorff distance to $C$ when
\[
\lim_{k \to \infty} \mathcal{H}(C^k, C) = 0.
\]

The following implicit function theorem for locally Lipschitz continuous function in [19, Lemma 1] is crucial for the convergence analysis in Section 4.

Lemma 2.5. Suppose $H : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is a locally Lipschitz continuous function in an open neighborhood of $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with $H(x, y) = 0$. If every element in $\Pi_x H(x, y)$, the projection of $H(x, y)$ onto the space $\mathcal{X}$, is nonsingular, then there exist a neighborhood $U(y)$ of $y$ and a unique locally Lipschitz continuous function $x(\cdot) : U(y) \to \mathcal{X}$ satisfying $x(y) = x$ such that for every $y \in U(y)$,
\[
H(x(y), y) = 0.
\]

2.2. Euclidean Jordan algebras and the Jacobian of Löwner operators.

We give a brief introduction to Euclidean Jordan algebras. Details on Euclidean Jordan algebras can be found in Koecher’s lecture notes [11] and the monograph by Faraut and Korányi [7].

A Euclidean Jordan algebra is a triple $(\mathcal{V}, \langle \cdot, \cdot \rangle, \circ) := A$, where $(\mathcal{V}, \langle \cdot, \cdot \rangle, \circ)$ is a real $n$-dimensional inner product space and $(x, y) \mapsto x \circ y : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a bilinear mapping which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{V}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{V}$ where $x^2 := x \circ x$,
(iii) $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in \mathcal{V}$.

We call $x \circ y$ the Jordan product of $x$ and $y$. In general, the Jordan product is not associative; i.e., $(x \circ y) \circ z \neq x \circ (y \circ z)$ for all $x, y, z \in \mathcal{V}$. In addition, we assume that there exists an element $e$ (called the identity element) such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{V}$. Given a Euclidean Jordan algebra $A$, define the set of squares as $\mathcal{K} := \{x^2 \mid x \in \mathcal{V}\}$. From Theorem III.2.1 in [7], $\mathcal{K}$ is a symmetric cone in $A$. In other words, $\mathcal{K}$ is self-dual closed convex cone, and for any two elements $x, y \in \text{int} \mathcal{K}$, there exists an invertible linear transformation $\Gamma : \mathcal{V} \to \mathcal{V}$ such that $\Gamma(\mathcal{K}) = \mathcal{K}$ and $\Gamma(x) = y$. For $x \in \mathcal{V}$, let $l := l(x)$ be the smallest positive integer such that the set $\{e, x, x^2, \ldots, x^l\}$ is linearly dependent. Then $l$ is said to be the degree of $x$, which is denoted by $\deg(x)$. The rank of $A$ denoted by $\text{rk}(A)$ is defined as $\text{rk}(\mathcal{A}) := \max(\deg(x) \mid x \in \mathcal{V})$. Let $\dim(\mathcal{V})$ denote the dimension of $\mathcal{V}$. Obviously, $\text{rk}(\mathcal{A}) \leq \dim(\mathcal{V})$. An element $c \in \mathcal{V}$ is an idempotent if $c^2 = c \neq 0$. An idempotent element is primitive if it cannot be written as a sum of two idempotents.

A complete system of orthogonal idempotents in $A$ is a finite set $\{c_1, c_2, \ldots, c_k\}$ of idempotents where $c_i \circ c_j = 0$ for all $i \neq j$, and $c_1 + c_2 + \cdots + c_k = e$. A Jordan frame in $A$ is a complete system of orthogonal primitive idempotents. The number of elements of any Jordan frame equals the positive integer $\text{rk}(A)$.

We now review the following spectral decomposition theorem of an element in a Euclidean Jordan algebra.

Theorem 2.6. (Spectral Decomposition Type II (Theorem III.1.2, [7])) Let $A$ be a Euclidean Jordan algebra with rank $r$. Then for $x \in \mathcal{V}$ there exist a Jordan frame
Theorem 2.7. Let \( \{c_1, c_2, \cdots, c_r\} \) and real numbers \( \lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x) \), arranged in decreasing order \( \lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x) \), such that
\[
x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r. \tag{1}
\]
The numbers \( \lambda_i(x) \) \((i = 1, 2, \cdots, r)\), which are uniquely determined by \( x \), are the eigenvalues of \( x \). We call (1) the spectral decomposition of \( x \).

Note that the Jordan frame \( \{c_1, c_2, \cdots, c_r\} \) in (1) depends on \( x \). We do not write this dependence explicitly sometimes for simplicity of notation. Let \( C(x) \) be the set consisting of all such Jordan frames at \( x \), then by [18, Proposition 3.2], \( C(x) \) is outer semicontinuous at \( x \).

Next, we recall the Peirce decomposition theorem on the space \( \mathbb{V} \), where a Jordan frame \( \{c_1, c_2, \cdots, c_r\} \) is fixed beforehand. In this case, define the following subspaces
\[
\mathbb{V}_{ij} := \{ x \in \mathbb{V} | x \circ c_i = x \} \quad \text{and} \quad \mathbb{V}_{ij} := \{ x \in \mathbb{V} | x \circ c_i = \frac{1}{2} x = x \circ c_j \}, \quad i \neq j,
\]
for \( i, j \in \{1, 2, \cdots, r\} \).

**Theorem 2.7.** (Theorem IV.2.1, [7]) Let \( \{c_1, c_2, \cdots, c_r\} \) be a given Jordan frame in a Euclidean Jordan algebra \( \mathbb{A} \) of rank \( r \). Then \( \mathbb{V} \) is the orthogonal direct sum of spaces \( \mathbb{V}_{ij} \) \((i \leq j)\). Furthermore,
\begin{enumerate}[(i)]  
\item \( \mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subseteq \mathbb{V}_{ii} + \mathbb{V}_{jj} \);
\item \( \mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subseteq \mathbb{V}_{ik}, \) if \( i \neq k \);
\item \( \mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\}, \) if \( \{i, j\} \cap \{k, l\} = \emptyset \).
\end{enumerate}

For each \( x \in \mathbb{V} \), we define the Lyapunov transformation \( L(x) : \mathbb{V} \to \mathbb{V} \) by \( L(x)y = x \circ y \) for all \( y \in \mathbb{V} \), which is a symmetric operator in the sense that \( \langle L(x)y, z \rangle = \langle y, L(x)z \rangle \) for all \( y, z \in \mathbb{V} \). Meanwhile, the operator \( Q(x) := 2L^2(x) - L(x^2) \) is called the quadratic representation of \( x \). We say two elements \( x, y \in \mathbb{V} \) operator commute if \( L(x)L(y) = L(y)L(x) \). By Lemma X.2.2 in [7], for a given Jordan frame \( \{c_1, c_2, \cdots, c_r\} \), it is easy to see that \( c_i, c_j \) operator commute and \( L(c_i)L(c_j) = L(c_j)L(c_i) \) for any \( i, j \in \{1, 2, \cdots, r\} \).

We introduce Löwner operators on a Euclidean space in Sun and Sun [18] below.

**Definition 2.8.** Let \( x = \sum_{i=1}^{r} \lambda_i(x)c_i \) and \( p : \mathbb{R} \to \mathbb{R} \) be a real-valued function. We define the corresponding Löwner operator (function) \( P: \mathbb{V} \to \mathbb{V} \) as
\[
P(x) := \sum_{i=1}^{r} p(\lambda_i(x))c_i = p(\lambda_1(x))c_1 + p(\lambda_2(x))c_2 + \cdots + p(\lambda_r(x))c_r.
\]

When \( p(t) = t_+ = \max\{0, t\}, t \in \mathbb{R} \), this becomes the metric projection operator
\[
\Pi_K(x) := (\lambda_1(x))c_1 + (\lambda_2(x))c_2 + \cdots + (\lambda_r(x))c_r
\]
on to a symmetric cone \( \mathcal{K} \). Note that \( x \in \mathcal{K} \) (respectively, \( x \in \text{int} \mathcal{K} \)) if and only if \( \lambda_i(x) \geq 0 \) (respectively, \( \lambda_i(x) > 0 \), \( i = 1, 2, \cdots, r \)). For any \( x \in \mathcal{K} \), define \( \sqrt{x} := \sum_{i=1}^{r} \sqrt{\lambda_i(x)}c_i \).

We consider the differentiability of the Löwner operator \( P(\cdot) \). Suppose that \( p \) is differentiable at \( \tau_i, i = 1, 2, \cdots, r \). Define the first divided difference \( p^{[1]}(\tau) \) of \( p \) at \( \tau := (\tau_1, \tau_2, \cdots, \tau_r)^T \in \mathbb{R}^r \) as the \( r \times r \) symmetric matrix with the \( ij \)-th entry given by
\[
[p^{[1]}(\tau)]_{ij} = \begin{cases} 
\frac{p(\tau_i) - p(\tau_j)}{\tau_i - \tau_j}, & \text{if } \tau_i \neq \tau_j, \\
p'(\tau_i), & \text{if } \tau_i = \tau_j,
\end{cases}
\]
for \( i, j = 1, 2, \cdots, r \).
A direct implication of Theorem 3.2 in [18] is the following property of the Jacobian of Löwner operator $P(\cdot)$.

**Theorem 2.9.** Let $x = \sum_{i=1}^{r} \lambda_i(x)c_i$. Then, $P(\cdot)$ is (continuously) differentiable at $x$ if and only if for each $i \in \{1, 2, \ldots, r\}$, $p$ is (continuously) differentiable at $\lambda_i(x)$. In this case, the Jacobian $J P(x)$ is given by

$$JP(x) = 2 \sum_{i\neq j, i,j=1}^{r} \lambda_i(x), \lambda_j(x)_{\mu} L(c_i)L(c_j) + \sum_{i=1}^{r} p'(\lambda_i(x))Q(c_i).$$

Furthermore, $JP(x)$ is a linear and symmetric operator from $V$ into itself.

Define the three index sets of positive, zero, and negative eigenvalues of $\bar{x} \in V$, respectively, by

$$\alpha := \{i \mid \lambda_i(\bar{x}) > 0\}, \quad \beta := \{i \mid \lambda_i(\bar{x}) = 0\}, \quad \text{and} \quad \gamma := \{i \mid \lambda_i(\bar{x}) < 0\}.$$  

Denote

$$\mathbb{R}_{\geq}[\beta] := \{z \in \mathbb{R}_{\geq}^{[\beta]} \mid z_1 \geq \cdots \geq z_{|\beta|}, z_i \neq 0, i = 1, 2, \cdots, |\beta|\},$$  

$$\mathcal{U}_{[\beta]} := \{\Omega \in S^{[\beta]} \mid \Omega = \lim_{k \to \infty} p_k^1(z^k), z^k \to 0, z^k \in \mathbb{R}_{\geq}^{[\beta]}\}.$$  

Combining Proposition 2.5 with Proposition 2.6 in [21], it yields the following result on the Bouligand subdifferential of $\Pi_K(\cdot)$ at $\bar{x} \in V$.

**Theorem 2.10.** Let $\bar{x} = \sum_{i=1}^{r} \lambda_i(\bar{x})\hat{c}_i$. Then, $V \in \partial_B \Pi_K(\bar{x})$ if and only if there exist $\Omega_{\beta \gamma} \in \mathcal{U}_{[\beta]}$ and a system of orthogonal idempotents $\{\hat{c}_i\}_{i \in \beta}$ such that $\sum_{i \in \beta} \hat{c}_i = \sum_{i \notin \beta} \hat{c}_i$, $\{\hat{c}_i\}_{i \in \alpha \cup \gamma} \cup \{\hat{c}_i\}_{i \in \beta}$ form a Jordan frame of $\Lambda$ at $\bar{x}$ and

$$V = 2 \sum_{i \neq j, i,j=1}^{r} a_{ij} L(\hat{c}_i)L(\hat{c}_j) + \sum_{i=1}^{r} b_i Q(\hat{c}_i) + 2 \sum_{i \neq j, i,j \in \beta} \Omega_{ij} L(\hat{c}_i)L(\hat{c}_j) + \sum_{i \in \beta} \Omega_{ii} Q(\hat{c}_i),$$

with

$$a_{ij} = \begin{cases} \max \{0, \lambda_i(\bar{x})\} + \max \{0, \lambda_j(\bar{x})\}, & (i, j) \notin \beta \times \beta, \\ \frac{\lambda_i(\bar{x})}{\lambda_j(\bar{x})}, & (i, j) \in \beta \times \beta, \\ 0, & (i, j) \notin \beta \times \beta, \\ 0, & (i, j) \in \beta \times \beta, \end{cases} \quad b_i = \begin{cases} 1, & i \in \alpha \\ 0, & i \notin \beta \cup \gamma. \end{cases}$$

3. C-stationary condition and SCMPCC-LICQ. In this section, we consider the C(larke)-stationary condition by reformulating SCMPCC as a nonsmooth problem:

$$\min_{z \in \mathbb{R}^n} f(z) \quad \text{s. t.} \quad g(z) \leq 0, \quad h(z) = 0, \quad G(z) - \Pi_K(G(z) - H(z)) = 0.$$  

From [10, Proposition 6], we know that the reformulation NS-SCMPCC is equivalent to SCMPCC. The same with MPCC case, the C-stationary condition introduced below is the nonsmooth KKT condition of NS-SCMPCC by using the Clarke subdifferential. Let $\mathcal{F}$ denote the feasible set of SCMPCC.

**Definition 3.1.** Let $\bar{z} \in \mathcal{F}$ and $\bar{x} := G(\bar{z}) - H(\bar{z})$ have the spectral decomposition $\bar{x} = \sum_{i=1}^{r} \lambda_i(\bar{x})\hat{c}_i$. We say that $\bar{z}$ is a C-stationary point of SCMPCC if there exist multipliers $(\lambda, \mu, \sigma^G, \sigma^H) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{V} \times \mathbb{V}$ and $V \in \partial \Pi_K(\bar{x})$ such that

$$\nabla f(\bar{z}) + J g(\bar{z})^T \lambda + J h(\bar{z})^T \mu + J G(\bar{z})^* \sigma^G + J H(\bar{z})^* \sigma^H = 0,$$  

$$\nabla G(z) - \Pi_K(G(z) - H(z)) = 0.$$  

\[ \lambda \geq 0, \quad \langle \lambda, g(\tilde{z}) \rangle = 0, \quad \text{(4)} \]

\[ \sigma^H = V(\sigma^G + \sigma^H). \quad \text{(5)} \]

The C-stationary point defined in Definition 3.1 is proposed in the form of Clarke subdifferential of \( \Pi_C(\cdot) \). To develop explicit expressions of C-stationary condition, we should derive explicit expressions of equation (5) in Definition 3.1. First, we need some notations. Let \( \bar{c} := \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_r\} \) be a given Jordan frame in a Euclidean Jordan algebra \( A \) of rank \( r \). Then for any \( t \in \mathbb{R} \), \( 1 \leq i \leq r \), \( t\bar{c}_i \circ \bar{c}_i = t\bar{c}_i^2 = t\bar{c}_i \), namely, for any \( t \in \mathbb{R} \), \( t\bar{c}_i \in \mathbb{V}_i \). Therefore it follows from Theorem 2.7, any element \( \sigma \in \mathbb{V} \) can be expressed by \( \sigma = \sum_{i=1}^r (\sigma_i) \bar{c}_i + \sum_{1 \leq k < l \leq r} (\sigma_{kl}) \bar{c}_k \bar{c}_l \), where \( (\sigma_i) \in \mathbb{R} \), \( i = 1, 2, \ldots, r \), and \( (\sigma_{kl}) \in \mathbb{V}_{kl} \), \( 1 \leq k < l \leq r \). Denote

\[ (\sigma_{\alpha \alpha}) := \sum_{\alpha \in \alpha} (\sigma_\alpha) \bar{c}_\alpha + \sum_{k < l, k \in \alpha, l \in \alpha} (\sigma_{kl}) \bar{c}_k \bar{c}_l, \quad (\sigma_{\alpha \beta}) := \sum_{k \in \alpha} \bar{c}_k \sum_{l \in \beta} (\sigma_{kl}), \quad (\sigma_{\beta \gamma}) := \sum_{k \in \beta} \bar{c}_k \sum_{l \in \gamma} (\sigma_{kl}), \quad (\sigma_{\alpha \gamma}) := \sum_{\alpha \in \alpha} (\sigma_{\alpha \gamma}). \]

Theorem 2.7 also implies

\[ \bar{c}_j \circ \sum_{i=1}^r (\sigma_i) \bar{c}_i = \sum_{i=1}^r (\sigma_i) \bar{c}_j \circ \bar{c}_i = (\sigma_j) \bar{c}_j \]

for any \( j, 1 \leq j \leq r \), and

\[ \bar{c}_j \circ \sum_{1 \leq k, l \leq r} (\sigma_{kl}) \bar{c}_k \bar{c}_l = \sum_{j \neq k, l} (\sigma_{kl}) \bar{c}_j \bar{c}_k + \sum_{1 \leq k < j} (\sigma_{jkl}) + \sum_{1 \leq j < l} (\sigma_{jkl}) \]

\[ = \frac{1}{2} \sum_{k < j} (\sigma_{jak}) + \frac{1}{2} \sum_{l > j} (\sigma_{jkl}), \]

then for \( i < j \), we have

\[ L(\bar{c}_i)L(\bar{c}_j) = \bar{c}_i \circ (\bar{c}_j \circ \sigma) = \bar{c}_i \circ \left[ (\sigma_{\alpha}) \bar{c}_j + \frac{1}{2} \sum_{k=1}^{j-1} (\sigma_{k}) \bar{c}_k + \frac{1}{2} \sum_{l=1}^{j-1} (\sigma_{l}) \bar{c}_l \right] = \frac{1}{4} (\sigma_{\alpha}) \bar{c}_j, \quad \text{(6)} \]

\[ \langle \sigma, L(\bar{c}_i)L(\bar{c}_j) \rangle = \langle \bar{c}_i \circ \sigma, \bar{c}_j \circ \sigma \rangle = \left( \frac{1}{2} (\sigma_{\alpha}) \bar{c}_j, \frac{1}{2} (\sigma_{\alpha}) \bar{c}_j \right) = \frac{1}{4} \| (\sigma_{\alpha}) \bar{c}_j \|^2, \quad \text{(7)} \]

and

\[ Q(\bar{c}_i) \sigma \]

\[ = 2\bar{c}_i \circ (\bar{c}_i \circ \sigma) - \bar{c}_i \circ \sigma \]

\[ = 2\bar{c}_i \circ \left[ (\sigma_{\alpha}) \bar{c}_i + \frac{1}{2} \sum_{k=1}^{i-1} (\sigma_{k}) \bar{c}_k + \frac{1}{2} \sum_{l=i+1}^{r} (\sigma_{l}) \bar{c}_l \right] \]

\[ - \left[ (\sigma_{\alpha}) \bar{c}_i + \frac{1}{2} \sum_{k=1}^{i-1} (\sigma_{k}) \bar{c}_k + \frac{1}{2} \sum_{l=i+1}^{r} (\sigma_{l}) \bar{c}_l \right] \]

\[ = 2 \left[ (\sigma_{\alpha}) \bar{c}_i + \frac{1}{4} \sum_{k=1}^{i-1} (\sigma_{k}) \bar{c}_k + \frac{1}{4} \sum_{l=i+1}^{r} (\sigma_{l}) \bar{c}_l \right] \]
therefore, it is easily verified that

\[ \langle \sigma, Q(\tilde{e}_i) \sigma \rangle = ||(\sigma \tilde{e}_i)\tilde{e}_i||^2. \] (9)

Thus it is easily verified that

\[
\begin{align*}
(\sigma_\epsilon)_{\alpha\alpha} &= 2 \sum_{i \neq j, i, j \in \alpha} L(\bar{e}_i)L(\bar{e}_j)\sigma + \sum_{i \in \alpha} Q(\bar{e}_i)\sigma, \\
(\sigma_\epsilon)_{\beta\beta} &= 2 \sum_{i \neq j, i, j \in \beta} L(\bar{e}_i)L(\bar{e}_j)\sigma + \sum_{i \in \beta} Q(\bar{e}_i)\sigma, \\
(\sigma_\epsilon)_{\gamma\gamma} &= 2 \sum_{i \neq j, i, j \in \gamma} L(\bar{e}_i)L(\bar{e}_j)\sigma + \sum_{i \in \gamma} Q(\bar{e}_i)\sigma,
\end{align*}
\] (10)

In virtue of Definition 2.1, for any \( V \in \partial \Pi_K(\bar{x}) \), there exist \( V^k \in \partial B \Pi_K(\bar{x}) \), \( k = 1, 2, \cdots, N \) such that

\[ V = \sum_{k=1}^{N} a_k V^k, \]

where \( a_k \geq 0, \sum_{k=1}^{N} a_k = 1 \). Let \( \Omega_{ij}^k \in U_{|\beta|} \) and \( \{ \tilde{e}_i^k \}_{i \in \beta} \) be a system of orthogonal idempotents associated with \( V^k \) in Theorem 2.10. Denote

\[ W^k = 4 \sum_{i < j, i, j \in \beta} \Omega_{ij}^k L(\tilde{e}_i)L(\tilde{e}_j) + \sum_{i \in \beta} \Omega_{ii}^k Q(\tilde{e}_i^k). \]

Then it follows from Theorem 2.10,

\[ V^k = 2 \sum_{i \neq j, i, j = 1}^{r} a_{ij} L(\bar{e}_i)L(\bar{e}_j) + \sum_{i=1}^{r} b_i Q(\bar{e}_i) + W^k, \]

hence

\[ V = \sum_{k=1}^{N} a_k V^k = 2 \sum_{i \neq j, i, j = 1}^{r} a_{ij} L(\bar{e}_i)L(\bar{e}_j) + \sum_{i=1}^{r} b_i Q(\bar{e}_i) + \sum_{k=1}^{N} a_k W^k \]

\[ = 2 \sum_{i \neq j, i, j \in \alpha} L(\bar{e}_i)L(\bar{e}_j) + \sum_{i \in \alpha} Q(\bar{e}_i) + 4 \sum_{i \in \alpha, j \in \beta} L(\bar{e}_i)L(\bar{e}_j) \]

\[ + 4 \sum_{i \in \alpha} \sum_{j \in \gamma} \frac{\lambda_i(\bar{x})}{\lambda_i(\bar{x}) - \lambda_j(\bar{x})} L(\bar{e}_i)L(\bar{e}_j) + \sum_{k=1}^{N} a_k W^k, \]

therefore,

\[ V(\sigma^G + \sigma^H) = (\sigma_{\epsilon^G})_{\alpha\alpha} + (\sigma_{\epsilon^H})_{\alpha\alpha} + (\sigma_{\epsilon^G})_{\alpha\beta} + (\sigma_{\epsilon^H})_{\alpha\beta} + \sum_{k=1}^{N} a_k W^k(\sigma^G + \sigma^H) \]

\[ + 4 \sum_{i \in \alpha} \sum_{j \in \gamma} \frac{\lambda_i(\bar{x})}{\lambda_i(\bar{x}) - \lambda_j(\bar{x})} [(\sigma_{\epsilon^G})_{ij} + (\sigma_{\epsilon^H})_{ij}]. \]

On the other hand,

\[ \sigma^H = (\sigma_{\epsilon^H})_{\alpha\alpha} + (\sigma_{\epsilon^H})_{\alpha\beta} + (\sigma_{\epsilon^H})_{\beta\beta} + (\sigma_{\epsilon^H})_{\alpha\gamma} + (\sigma_{\epsilon^H})_{\beta\gamma} + (\sigma_{\epsilon^H})_{\gamma\gamma}. \]
Then (5) implies that
\[
\langle \sigma^G, \zeta \rangle = 0, \quad \langle \sigma^H, \zeta \rangle = 0, \quad \langle \sigma^H, \varphi \rangle = 0, \quad \langle \sigma^H, \psi \rangle = 0,
\]
\[
\begin{aligned}
\sum_{i \in \alpha} \sum_{j \in \gamma} \lambda_i(x) [\langle \sigma^G, \zeta \rangle_{ij} + \langle \sigma^H, \varphi \rangle_{ij}] &= \sum_{i \in \alpha} \sum_{j \in \gamma} (\sigma^H)_{ij}, \\
\sum_{k=1}^N a_k W^k (\sigma^G + \sigma^H) &= (\sigma^H)_{\beta \beta}.
\end{aligned}
\]

Next, we show that \((\langle \sigma^H, \beta \beta \rangle, (\sigma^G, \beta \beta \rangle) \geq 0\). It is easily verified that
\[
(W^k)^2 = 16 \sum_{i<j, i,j \in \beta} (\Omega^k_{ij})^2 L^2(c_i^k) L^2(c_j^k) L^2(c_i^k) + \sum_{i \in \beta} (\Omega^k_{ii})^2 Q(c_i^k).
\]
Combining this with (6)-(9) and \(0 \leq \Omega^k_{ij} \leq 1\), one has for any \(\sigma \in \mathbb{V}\),
\[
\begin{aligned}
\langle \sigma, (W^k - (W^k)^2) \sigma \rangle &= \langle \sigma, W^k \sigma \rangle - \langle \sigma, (W^k)^2 \sigma \rangle \\
&= \sum_{i \in \beta} (\Omega^k_{ii} - (\Omega^k_{ii})^2) \langle \sigma, Q(c_i^k) \sigma \rangle \\
&\quad + \sum_{i<j, i,j \in \beta} [4\Omega^k_{ij} \langle L(c_i^k) L(c_j^k) \rangle - 16(\Omega^k_{ij})^2 \langle L(c_i^k) L(c_j^k) \rangle] \\
&= \sum_{i \in \beta} (\Omega^k_{ii} - (\Omega^k_{ii})^2) \| (\sigma_c^k) c_i^k \|^2 + \sum_{i<j, i,j \in \beta} (\Omega^k_{ij} - (\Omega^k_{ij})^2) \| (\sigma_c^k)_{ij} \|^2 \geq 0.
\end{aligned}
\]
Noting that \(W^k (\sigma^G + \sigma^H) = W^k ((\sigma^G, \beta \beta) + (\sigma^H, \beta \beta))\), then
\[
\begin{aligned}
\langle (\sigma^H, \beta \beta), (\sigma^G, \beta \beta) \rangle &= \sum_{k=1}^N a_k W^k (\sigma^G, \beta \beta) + (\sigma^H, \beta \beta) - \sum_{k=1}^N a_k W^k (\sigma^G, \beta \beta) + (\sigma^H, \beta \beta) \\
&\geq \sum_{k=1}^N a_k (W^k ((\sigma^G, \beta \beta) + (\sigma^H, \beta \beta)) - W^k (\sigma^G, \beta \beta) + (\sigma^H, \beta \beta)) \\
&= \sum_{k=1}^N a_k ((\sigma^G, \beta \beta) + (\sigma^H, \beta \beta), (W^k - (W^k)^2) ((\sigma^G, \beta \beta) + (\sigma^H, \beta \beta)) \geq 0.
\end{aligned}
\]
Employing above discussions, we get the explicit expression of C-stationary point of SCMPCC below.

**Definition 3.2.** Let \(\tilde{x} \in \mathcal{F}\) and \(x := G(\tilde{x}) - H(\tilde{x})\) have the spectral decomposition \(\tilde{x} = \sum_{i=1}^r \lambda_i(\tilde{x}) \tilde{c}_i\). We say that \(\tilde{x}\) is a C-stationary point of SCMPCC if there exists multiplier \((\lambda, \mu, \sigma^G, \sigma^H) \in \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{V} \times \mathbb{V}\) such that (3)-(4) hold and
\[
\begin{aligned}
\langle \sigma^G, \zeta \rangle = 0, \quad \langle \sigma^H, \zeta \rangle = 0, \quad \langle \sigma^H, \varphi \rangle = 0, \quad \langle \sigma^H, \psi \rangle = 0, \\
\sum_{j \in \alpha} \sum_{l \in \gamma} \lambda_j(x) \lambda_l(x) \langle (\sigma^G)_{jl} + (\sigma^H)_{jl} \rangle &= \sum_{j \in \alpha} \sum_{l \in \gamma} (\sigma^H)_{jl}, \\
\langle (\sigma^G, \beta \beta), (\sigma^H, \beta \beta) \rangle &\geq 0.
\end{aligned}
\]
Now we show that the C-stationary condition for SCMPCC could be reduced to the C-stationary condition for MPCC, SOCMPCC or SDCMPCC when $\mathcal{K}$ is the nonnegative orthant, a second-order cone, or a positive semidefinite cone.

**Example 1.** To see that the C-stationary condition for SCMPCC coincides with the C-stationary condition in the MPCC case, see [2, 6, 9, 12, 13, 14, 16, 17], we consider the case $\mathcal{V} = \mathbb{R}$ with the inner product and Jordan product defined by $\langle x, y \rangle = x \circ y = xy$ for $x, y \in \mathbb{R}$. Then $(\mathbb{R}, \langle \cdot, \cdot \rangle, \circ)$ forms an Euclidean Jordan algebra, and $\mathbb{R}_+$ is its cone of squares. The set $\bar{c} := \{1\}$ is the unique Jordan frame. Then the SCMPCC is reduced to the MPCC case

\[
\min_{z \in \mathbb{R}^n} f(z)
\text{ s.t. } g(z) \leq 0, \quad h(z) = 0,
\quad G(z) \geq 0, \quad H(z) \geq 0,
\quad \langle G(z), H(z) \rangle = 0,
\]

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^q$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $G$, $H : \mathbb{R}^n \to \mathbb{R}^m$ are all continuously differentiable mappings.

Let $\bar{z}$ be a feasible point of MPCC, then $\bar{x} := G(\bar{z}) - H(\bar{z})$ has the spectral decomposition $\bar{x} = (G(\bar{z}) - H(\bar{z})) \cdot 1$. If $G(\bar{z}) > 0$ and $H(\bar{z}) = 0$, we have $\beta = \gamma = \emptyset$, then $\sigma_G = (\sigma_G^\beta)^{\alpha} = 0$. If $G(\bar{z}) = 0$ and $H(\bar{z}) > 0$, we have $\alpha = \beta = \emptyset$, then $\sigma_H = (\sigma_H^\beta)^{\gamma} = 0$. If $G(\bar{z}) = 0$ and $H(\bar{z}) = 0$, we have $\alpha = \gamma = \emptyset$, then $\langle \sigma_G^\beta, \sigma_H^\beta \rangle = 0$. Therefore, $\bar{z}$ is a C-stationary point for the MPCC problem.

**Example 2.** Now we show that the C-stationary condition for SCMPCC coincides with the C-stationary condition in the SOCMPCC case when we consider the Euclidean Jordan algebra $\mathbb{A} = (\mathbb{R}^m, \langle \cdot, \cdot \rangle, \circ)$, $m \geq 2$. $x \in \mathbb{R}^m$ is written as $x = (x_1, x_2^T)^T$ with $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{m-1}$. The inner product is $(x, y) = x^T y$ and the Jordan product is defined by

$$x \circ y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1^T y \\ x_2^T y + x_1 y_2 \end{pmatrix}.$$ 

The cone of squares of $\mathbb{A}$ is the second order cone defined by $\mathcal{K}_m = \{(x_1, x_2^T)^T \in \mathbb{R}^m | x_1 \geq \|x_2\|\}$. The identity element in $\mathbb{A}$ is $e = (1, 0^T)^T$. The set $c = \{c_1, c_2\}$ is a Jordan frame given by $c_i = \frac{1}{2}(1, (-1)^i w^T)^T$ for $i = 1, 2$ with any $w \in \mathbb{R}^{m-1}$ satisfying $\|w\| = 1$. In virtue of Theorem 2.7, $V_{11} = \mathbb{R} c_1$, $V_{22} = \mathbb{R} c_2$ and $V_{12} = \{(0, x_2^T)^T \in \mathbb{R}^m | x_2^T w = 0\}$. Then SCMPCC is reduced to the SOCMPCC case

\[
\min_{z \in \mathbb{R}^n} f(z)
\text{ s.t. } g(z) \leq 0, \quad h(z) = 0,
\quad G(z) \in \mathcal{K}_m, \quad H(z) \in \mathcal{K}_m,
\quad \langle G(z), H(z) \rangle = 0,
\]

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^q$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $G$, $H : \mathbb{R}^n \to \mathbb{R}^m$ are all continuously differentiable mappings. Let $\bar{z}$ be a feasible point of SOCMPCC, and $\bar{x} := G(\bar{z}) - H(\bar{z})$ has the spectral decomposition $\bar{x} = \lambda_1(\bar{c}) c_1 + \lambda_2(\bar{c}) c_2$ with

$$\lambda_1(\bar{x}) = \bar{x}_1 - \|\bar{x}_2\|, \quad \lambda_2(\bar{x}) = \bar{x}_1 + \|\bar{x}_2\|, \quad \bar{c}_1 = \frac{1}{2}(1, -\frac{x_2^T}{\|x_2\|})^T \quad \text{and} \quad \bar{c}_2 = \frac{1}{2}(1, \frac{x_2^T}{\|x_2\|})^T.$$ 

Let $\bar{c} = \{\bar{c}_1, \bar{c}_2\}$. It follows from Theorem 2.7, any $\sigma \in \mathbb{R}^m$ can be expressed by

$$\sigma = (\sigma_{\bar{c}})_1 \bar{c}_1 + (\sigma_{\bar{c}})_2 \bar{c}_2 + (\sigma_{\bar{c}})_{12}$$
with \((\sigma_\epsilon)_1, (\sigma_\epsilon)_2 \in \mathbb{R}, (\sigma_\epsilon)_12 \in \mathbb{V}_{12}\).

If \(\bar{x}_1 > \|\bar{x}_2\|\), we have \(\lambda_1(\bar{x}) > 0, \lambda_2(\bar{x}) > 0,\) then \(\alpha = \{1, 2\}, \beta = \gamma = \emptyset\). Thus
\[
\sigma^G = (\sigma^G_\epsilon)_{\alpha\alpha} = 0.
\]

If \(\bar{x}_1 < \|\bar{x}_2\|\), we have \(\lambda_1(\bar{x}) < 0, \lambda_2(\bar{x}) < 0,\) then \(\alpha = \emptyset, \beta = \emptyset, \gamma = \{1, 2\}\). Thus
\[
\sigma^H = (\sigma^H_\epsilon)_{\gamma\gamma} = 0.
\]

If \(-\|\bar{x}_2\| < \bar{x}_1 < \|\bar{x}_2\|\), it holds that \(\lambda_1(\bar{x}) < 0, \lambda_2(\bar{x}) > 0,\) then \(\alpha = \{2\}, \beta = \emptyset, \gamma = \{1\}\). It follows from (11), (12) and (13) that \((\sigma^G_\epsilon)_{\alpha\alpha} = (\sigma^G_\epsilon)_2\bar{c}_2 = 0,\)
\[(\sigma^H_\epsilon)_{\gamma\gamma} = (\sigma^H_\epsilon)_1\bar{c}_1 = 0\]
and
\[
\frac{\lambda_2(\bar{x})}{\lambda_2(\bar{x}) - \lambda_1(\bar{x})} [((\sigma^G_\epsilon)_12 + (\sigma^H_\epsilon)_12] = (\sigma^H_\epsilon)_12.
\]

Then \((\sigma^G_\epsilon)_12 = -\frac{\lambda_1(\bar{x})}{\lambda_2(\bar{x}) - \lambda_1(\bar{x})} (\sigma^G_\epsilon)_12\) is \[\frac{\|\bar{x}_2\| - \bar{x}_1}{\|\bar{x}_2\| + \bar{x}_1} (\sigma^H_\epsilon)_12,\]
therefore
\[
\sigma^G + \sigma^H = (\sigma^G_\epsilon)_1\bar{c}_1 + (\sigma^G_\epsilon)_2\bar{c}_2 + (\sigma^G_\epsilon)_12 + (\sigma^H_\epsilon)_1\bar{c}_1 + (\sigma^H_\epsilon)_2\bar{c}_2 + (\sigma^H_\epsilon)_12
\]
\[= (\sigma^H_\epsilon)_2\bar{c}_2 + (\sigma^G_\epsilon)_1\bar{c}_1 + \frac{2\|\bar{x}_2\|}{\|\bar{x}_2\| + \bar{x}_1} (\sigma^H_\epsilon)_12.
\]
Thus
\[
\frac{1}{2} \left( \frac{1}{\|\bar{x}_2\|} I + \frac{\bar{x}_1 T}{\|\bar{x}_2\|} \left( I - \frac{\bar{x}_2\bar{x}_2 T}{\|\bar{x}_2\|^2} \right) \right) (\sigma^G + \sigma^H)
\]
\[= \frac{1}{2} \left( \frac{1}{\|\bar{x}_2\|} I + \frac{\bar{x}_1 T}{\|\bar{x}_2\|} \left( I - \frac{\bar{x}_2\bar{x}_2 T}{\|\bar{x}_2\|^2} \right) \right) \left( (\sigma^H_\epsilon)_2\bar{c}_2 + (\sigma^G_\epsilon)_1\bar{c}_1 + \frac{2\|\bar{x}_2\|}{\|\bar{x}_2\| + \bar{x}_1} (\sigma^H_\epsilon)_12 \right)
\]
\[= (\sigma^H_\epsilon)_2\bar{c}_2 + (\sigma^H_\epsilon)_12 = \sigma^H_\epsilon,
\]
where \(I\) is the identity matrix.

If \(\bar{x}_1 = \|\bar{x}_2\| > 0,\) we have \(\lambda_1(\bar{x}) = 0, \lambda_2(\bar{x}) > 0,\) then \(\alpha = \{2\}, \beta = \{1\}\) and \(\gamma = \emptyset\). By (11), \((\sigma^G_\epsilon)_1\bar{c}_1 = (\sigma^G_\epsilon)_2\bar{c}_2 = 0, (\sigma^G_\epsilon)_2\bar{c}_2 = 0,\) then \(\sigma^G = (\sigma^G_\epsilon)_1\bar{c}_1 = (\sigma^G_\epsilon)_{\beta\beta}\). On the other hand, \(\sigma^H = (\sigma^H_\epsilon)_1\bar{c}_1 + (\sigma^H_\epsilon)_2\bar{c}_2 + (\sigma^H_\epsilon)_12,\) then \(\langle \sigma^H, \bar{c}_1 \rangle = (\sigma^H_\epsilon)_1\). Therefore, it follows from (14) that
\[
\langle \sigma^G, \bar{c}_1 \rangle \langle \sigma^H, \bar{c}_1 \rangle = (\sigma^G_\epsilon)_{\alpha\alpha} (\sigma^H_\epsilon)_{11}
\]
\[= (\sigma^G_\epsilon)_{1\bar{c}_1} (\sigma^H_\epsilon)_{11} = (\sigma^G_\epsilon)_{1\bar{c}_1} (\sigma^H_\epsilon)_{1\bar{c}_1}
\]
\[= (\sigma^G_\epsilon)_{\bar{c}_1\bar{c}_1} (\sigma^H_\epsilon)_{\bar{c}_1\bar{c}_1} \geq 0.
\]

If \(\bar{x}_1 = -\|\bar{x}_2\| < 0,\) we have \(\lambda_1(\bar{x}) < 0, \lambda_2(\bar{x}) = 0,\) then \(\alpha = \emptyset, \beta = \{2\}\) and \(\gamma = \{1\}\). By (12), \((\sigma^H_\epsilon)_{\gamma\gamma} = (\sigma^H_\epsilon)_1\bar{c}_1 = 0, (\sigma^H_\epsilon)_{\gamma\gamma} = (\sigma^H_\epsilon)_{12} = 0,\) then \(\sigma^H = (\sigma^H_\epsilon)_2\bar{c}_2 = (\sigma^H_\epsilon)_{\beta\beta}\). On the other hand, \(\sigma^G = (\sigma^G_\epsilon)_1\bar{c}_1 + (\sigma^G_\epsilon)_2\bar{c}_2 + (\sigma^G_\epsilon)_12,\) then \(\langle \sigma^G, \bar{c}_2 \rangle = (\sigma^G_\epsilon)_2\). Therefore, by (14),
\[
\langle \sigma^G, \bar{c}_2 \rangle \langle \sigma^H, \bar{c}_2 \rangle = (\sigma^G_\epsilon)_2 (\sigma^H_\epsilon)_{2\bar{c}_2}
\]
\[= (\sigma^G_\epsilon)_2 (\langle \sigma^H_\epsilon)_2\bar{c}_2, \bar{c}_2 \rangle = (\sigma^G_\epsilon)_2 (\sigma^H_\epsilon)_2\bar{c}_2
\]
\[= (\sigma^G_\epsilon)_2 (\sigma^H_\epsilon)_{\beta\beta} \geq 0.
\]
If \( \bar{x}_1 = \| \bar{x}_2 \| = 0 \), we have \( \lambda_1(\bar{x}) = 0 \), \( \lambda_2(\bar{x}) = 0 \), then \( \alpha = \gamma = 0 \) and \( \beta = \{1, 2\} \), hence \( \sigma^G = (\sigma^G_\beta)_\beta \) and \( \sigma^H = (\sigma^H_\beta)_\beta \). It follows from (14) that \( \langle \sigma^G, \sigma^H \rangle = \langle (\sigma^G_\beta)_\beta, (\sigma^H_\beta)_\beta \rangle \geq 0 \).

Combining the six cases above with Definition 4.4 in [23], \( \bar{x} \) is a C-stationary point of SCMPCC.

**Example 3.** We show that the C-stationary condition for SCMPCC coincides with the C-stationary condition in SDCMPCC case when we consider the Euclidean Jordan matrices with the inner product and Jordan product defined, respectively, by \( \langle X, Y \rangle = \text{Trace}(XY) \) and \( X \circ Y := \frac{XY + YX}{2} \). Its cone of squares \( S^m_+ \) is the set of all positive semidefinite symmetric matrices. The identity element is the identity matrix \( I \). Then SCMPCC is reduced to the SDCMPCC case

\[
\text{(SDCMPCC)} \quad \min_{z \in \mathbb{R}^n} \quad f(z) \\
\text{s. t.} \quad g(z) \leq 0, \quad h(z) = 0, \\
G(z) \in S^m_+, \quad H(z) \in S^m_+, \quad \langle G(z), H(z) \rangle = 0,
\]

where \( f : \mathbb{R}^n \to \mathbb{R}, \quad g : \mathbb{R}^n \to \mathbb{R}^q, \quad h : \mathbb{R}^n \to \mathbb{R}, \quad G, \quad H : \mathbb{R}^n \to \mathbb{S}^m \) are all continuously differentiable mappings. Let \( \bar{x} \) be a feasible point of SCMPCC and \( A := G(\bar{x}) - H(\bar{x}) \) have the spectral decomposition \( A = \bar{P}\bar{A}^T \), where \( \lambda_1(A), \lambda_2(A), \ldots, \lambda_m(A) \in \mathbb{R}^m \) and \( \Lambda = \text{diag}(\lambda(A)) \). For any \( x \in \mathbb{R}^m \), \( \text{diag}(x) \) denotes the diagonal matrix. The set \( \bar{c} = \{P_1, P_2, \ldots, P_m \} \) is a Jordan frame, where \( P_i \) is the \( i \)-th column vector of \( P \). Let \( \alpha := \{i|\lambda_i(A) > 0\} \), \( \beta := \{i|\lambda_i(A) = 0\} \) and \( \gamma := \{i|\lambda_i(A) < 0\} \). Then for any \( \Gamma \in \mathbb{S}^m_+ \), it follows from Theorem 2.7 and (10) that

\[
\Gamma = (\Gamma_\bar{c})_\alpha + (\Gamma_\bar{c})_\alpha \beta + (\Gamma_\bar{c})_\beta \gamma + (\Gamma_\bar{c})_\alpha \gamma + (\Gamma_\bar{c})_\gamma ,
\]

where

\[
(\Gamma_\bar{c})_\alpha \alpha = \sum_{i,j \in \alpha, i \neq j} P_i P_i^T \Gamma P_j P_j^T + \sum_{i \in \alpha} P_i P_i^T \Gamma P_i P_i^T ,
\]

\[
(\Gamma_\bar{c})_\alpha \beta = \sum_{i,j \in \beta, i \neq j} P_i P_i^T \Gamma P_j P_j^T + \sum_{i \in \beta} P_i P_i^T \Gamma P_i P_i^T ,
\]

\[
(\Gamma_\bar{c})_\beta \gamma = \sum_{i,j \in \gamma, i \neq j} P_i P_i^T \Gamma P_j P_j^T + \sum_{i \in \gamma} P_i P_i^T \Gamma P_i P_i^T ,
\]

\[
(\Gamma_\bar{c})_\gamma \gamma = \sum_{i,j \in \gamma, i \neq j} P_i P_i^T \Gamma P_j P_j^T + \sum_{i \in \gamma} P_i P_i^T \Gamma P_i P_i^T .
\]

Let \( \tilde{\Gamma} := \bar{P}^T \Gamma \bar{P} \). Let \( \tilde{J}_{1,2} \) denote the \( |J_1| \times |J_2| \) sub-matrix of \( \tilde{\Gamma} \) obtained by removing all the rows of \( \tilde{J} \) not in \( J_1 \) and all the columns of \( \tilde{J} \) not in \( J_2 \). By (11), \( (\Gamma_\bar{c})_\alpha \alpha = 0 \) implies \( (\Gamma_\bar{c})^{\alpha \alpha} = P^{\alpha \alpha}_\bar{c} \Gamma P_\alpha = P^{\alpha \alpha}_\alpha \Gamma P_\alpha = 0 \), and \( (\Gamma_\bar{c})_\alpha \beta = 0 \) implies \( (\Gamma_\bar{c})^{\alpha \beta} = P^{\alpha \beta}_\bar{c} \Gamma P_\beta = P^{\alpha \beta}_\alpha \Gamma P_\beta = 0 \) and \( (\Gamma_\bar{c})^{\alpha \gamma} = P^{\alpha \gamma}_\bar{c} \Gamma P_\gamma = P^{\alpha \gamma}_\alpha \Gamma P_\gamma = 0 \)
Finally, it follows from (14) that

\[ P^T_\beta (\Gamma^G)_{\alpha \beta} P_\alpha = 0. \]

In the same way, by (12) and (13), we obtain \((\Gamma^U)^{\gamma \gamma} = 0, (\Gamma^U)^{\beta \gamma} = 0, (\Gamma^U)^{\gamma \beta} = 0\) and for any \(i \in \alpha\) and \(j \in \gamma\),

\[
\frac{\lambda_i(A)}{\lambda_i(A) - \lambda_j(A)} \left[ (\Gamma^G)^{ij} + (\Gamma^U)^{ij} \right] = \frac{\lambda_i(A)}{\lambda_i(A) - \lambda_j(A)} P^T_i (\Gamma^G + \Gamma^U) P_j
\]

\[
= \frac{\lambda_i(A)}{\lambda_i(A) - \lambda_j(A)} P^T_i (\Gamma^G)^{ij} P_j
\]

\[
= P^T_i (\Gamma^H)^{ij} P_j = P^T_i (\Gamma^U)^{ij} P_j = (\Gamma^H)^{ij}.
\]

Finally, it follows from (14) that

\[
((\Gamma^G)^{\beta \beta}, (\Gamma^H)^{\beta \beta}) = (P^T_\beta (\Gamma^G)^{\beta \beta} P_\beta, P^T_\beta (\Gamma^H)^{\beta \beta} P_\beta) = (P^T_\beta (\Gamma^G)^{\beta \beta} P_\beta, P^T_\beta (\Gamma^H)^{\beta \beta} P_\beta)
\]

\[
= ((\Gamma^G)^{\beta \beta}, (\Gamma^H)^{\beta \beta}) \geq 0,
\]

which implies that \(\tilde{z}\) is a C-stationary of SDCMPCC in [5].

Next we introduce SCMPCC-linear independence constraint qualification, which will play an important role in Section 4. Let \(I(\tilde{z}) := \{i \mid g_i(\tilde{z}) = 0, i = 1, 2, \ldots, q\}\).

**Definition 3.3.** Let \(\tilde{z} \in F\) and \(\bar{x} := G(\tilde{z}) - H(\tilde{z})\) have the spectral decomposition \(\bar{x} = \sum_{i=1}^r \lambda_i(\bar{x}) \bar{c}_i\). We say that SCMPCC linear independence constraint qualification (denoted by SCMPCC-LICQ) holds at \(\tilde{z}\) if there is no nonzero multiplier \((\lambda_I(\tilde{z}), \mu, \sigma^G, \sigma^H) \in \mathbb{R}^{f(\tilde{z})} \times \mathbb{R}^p \times \mathcal{V} \times \mathcal{V}\) such that

\[
J^G h(\tilde{z})^T \lambda_I(\tilde{z}) + J^H h(\tilde{z})^T \mu + J G(\tilde{z})^* \sigma^G + J H(\tilde{z})^* \sigma^H = 0,
\]

\[
(\sigma^G)_{\gamma \gamma} = 0, \quad (\sigma^G)_{\beta \beta} = 0,
\]

\[
(\sigma^H)_{\gamma \gamma} = 0, \quad (\sigma^H)_{\beta \beta} = 0,
\]

\[
\sum_{j \in \alpha} \sum_{i \in \gamma} \lambda_j(\bar{x}) [(\sigma^G)^{ji} + (\sigma^H)^{ji}] = \sum_{j \in \alpha} \sum_{i \in \gamma} (\sigma^H)^{ji}.
\]

SCMPCC-LICQ is the analogue of the well-known MPCC-LICQ in [16] and could be reduced to the corresponding constraint qualification in the cases of SDCMPC, SOCMPCC or MPCC by [5, 16, 23]. We now give the following result on the stability of SCMPCC-LICQ.

**Lemma 3.4.** If SCMPCC-LICQ holds at \(\tilde{z} \in F\), then there exists a neighborhood \(U(\tilde{z})\) of \(\tilde{z}\) such that SCMPCC-LICQ holds at any \(z \in F \cap U(\tilde{z})\).

**Proof.** Suppose, to the contrary, that there exist a neighborhood \(U_1(\tilde{z})\) of \(\tilde{z}\) and a sequence \(\{\tilde{z}^k\} \subset F \cap U_1(\tilde{z})\) such that \(\lim_{k \to \infty} \tilde{z}^k = \tilde{z}\) and SCMPCC-LICQ does not hold at \(z^k\), for each \(k\). Let \(\bar{x} := G(\tilde{z}) - H(\tilde{z})\) and \(x^k := G(z^k) - H(z^k)\) have the spectral decomposition \(\bar{x} = \sum_{i=1}^r \lambda_i(\bar{x}) \bar{c}_i\) and \(x^k = \sum_{i=1}^r \lambda_i(x^k) c_i(x^k)\) respectively. Denote \(\bar{c} := \{\bar{c}_1, \ldots, \bar{c}_r\}\) and \(c^k := \{c_1(x^k), \ldots, c_r(x^k)\}\). Define the three index sets of eigenvalues at \(x^k\) by

\[
\alpha^k := \{i \mid \lambda_i(x^k) > 0\}, \quad \beta^k := \{i \mid \lambda_i(x^k) = 0\}, \quad \gamma^k := \{i \mid \lambda_i(x^k) < 0\}.
\]
Since SCMPCC-LICQ does not hold at each \( z^k \), there exists at least a nonzero multiplier \( (\lambda^k_{I(z^k)}, \mu^k, \sigma^{Gk}, \sigma^{HK}) \in \mathbb{R}^{|I(z^k)|} \times \mathbb{R}^p \times \mathbb{R} \times \mathbb{R} \) such that

\[
\mathcal{J} g_{I(z^k)}(z^k)^{T} \lambda^k_{I(z^k)} + \mathcal{J} h(z^k)^{T} \mu^k + \mathcal{J} G(z^k)^{T} \sigma^{Gk} + \mathcal{J} H(z^k)^{T} \sigma^{HK} = 0,
\]

(15)

\[
(\sigma^{Gk})_{\alpha^*\alpha} = 0, \quad (\sigma^{Gk})_{\alpha^*\beta} = 0,
\]

(16)

\[
(\sigma^{HK})_{\lambda^*\gamma} = 0, \quad (\sigma^{HK})_{\beta^*\gamma} = 0,
\]

(17)

\[
\sum_{j \in \alpha^*} \sum_{l \in \gamma^*} \lambda_j(x^k) - \lambda_i(x^k) = \sum_{j \in \alpha^*} \sum_{l \in \gamma^*} (\sigma^{HK})_{jl}.
\]

(18)

As \( I(z^k) \subseteq I(\tilde{z}) \), (15) can be reformulated as

\[
\mathcal{J} g_{I(\tilde{z})}(z^k)^{T} \lambda^k_{I(\tilde{z})} + \mathcal{J} h(z^k)^{T} \mu^k + \mathcal{J} G(z^k)^{T} \sigma^{Gk} + \mathcal{J} H(z^k)^{T} \sigma^{HK} = 0,
\]

(19)

with \( \lambda^k_{I(\tilde{z})} \).

As \( \|\lambda^k_{I(\tilde{z})}, \mu^k, \sigma^{Gk}, \sigma^{HK}\| \neq 0 \), we divide (16)-(19) by \( \|\lambda^k_{I(\tilde{z})}, \mu^k, \sigma^{Gk}, \sigma^{HK}\| \) and let \( (\bar{\lambda}^k_{I(\tilde{z})}, \bar{\mu}^k, \bar{\sigma}^{Gk}, \bar{\sigma}^{HK}) \) be the normalized vector of multipliers. Then, by taking a subsequence if necessary, we may assume that the latter sequence converges to \( (\bar{\lambda}^k_{I(\tilde{z})}, \bar{\mu}, \bar{\sigma}^{G}, \bar{\sigma}^{H}) \) with \( \|\lambda^k_{I(\tilde{z})}, \bar{\mu}, \bar{\sigma}^{G}, \bar{\sigma}^{H}\| = 1 \). Also, as \( \mathcal{E}(\cdot) \) is outer semicontinuous at \( \tilde{z} \), without loss of generality, we assume that \( \lim_{k \to \infty} \hat{c}_i = \hat{c}_i \) with \( \hat{c}_i = \bar{c}_i \) when \( i \in \alpha \cup \gamma \) and \( \sum_{i \in \beta} \hat{c}_i = \sum_{i \in \beta} \bar{c}_i \), then \( \hat{c} := \{\hat{c}_1, \ldots, \hat{c}_r\} \) forms a Jordan frame of \( A \). Choosing \( k_0 \) sufficiently large, fix \( \hat{\alpha} := \alpha^{k_0}, \hat{\beta} := \beta^{k_0} \) and \( \hat{\gamma} := \gamma^{k_0} \), then we have \( \hat{\alpha} \supseteq \alpha, \hat{\beta} \supseteq \beta \) and \( \hat{\gamma} \supseteq \gamma \). Let \( k \to \infty \) in (15)-(19), it holds that

\[
\mathcal{J} g_{I(\tilde{z})}(\tilde{z})^{T} \bar{\lambda}^k_{I(\tilde{z})} + \mathcal{J} h(\tilde{z})^{T} \bar{\mu} + \mathcal{J} G(\tilde{z})^{T} \bar{\sigma}^{G} + \mathcal{J} H(\tilde{z})^{T} \bar{\sigma}^{H} = 0,
\]

(20)

\[
(\bar{\sigma}^G)_{\alpha^*\alpha} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\beta} = 0,
\]

(21)

\[
(\bar{\sigma}^H)_{\gamma^*\gamma} = 0, \quad (\bar{\sigma}^H)_{\beta^*\gamma} = 0,
\]

hence, by Theorem 2.7, we get

\[
(\bar{\sigma}^G)_{\alpha^*\alpha} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\beta} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\gamma} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\gamma} = 0,
\]

\[
(\bar{\sigma}^H)_{\gamma^*\gamma} = 0, \quad (\bar{\sigma}^H)_{\beta^*\gamma} = 0, \quad (\bar{\sigma}^H)_{\beta^*\gamma} = 0,
\]

where “\( \set{\alpha} \)” is an index set including subscripts in \( \hat{\alpha} \) but not in \( \alpha \). Then

\[
(\bar{\sigma}^G)_{\alpha^*\alpha} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\beta} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\gamma} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\gamma} = 0,
\]

\[
(\bar{\sigma}^H)_{\beta^*\gamma} = 0, \quad (\bar{\sigma}^H)_{\beta^*\gamma} = 0, \quad (\bar{\sigma}^H)_{\beta^*\gamma} = 0
\]

Noticing that \( \hat{c}_i = \bar{c}_i \) when \( i \in \alpha \cup \gamma \), and \( \sum_{i \in \beta} \hat{c}_i = \sum_{i \in \beta} \bar{c}_i \), it holds that

\[
(\bar{\sigma}^G)_{\alpha^*\alpha} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\beta} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\gamma} = 0, \quad (\bar{\sigma}^G)_{\alpha^*\gamma} = 0,
\]

\[
(\bar{\sigma}^H)_{\beta^*\gamma} = 0,
\]

(22)

(23)
and
\[
(s^G_{\epsilon})_{\alpha,\beta} = 4 \sum_{i \in \alpha, j \in \beta} L(\hat{c}_i)L(\hat{c}_j)\sigma^G = 4 \sum_{i \in \alpha, j \in \beta} L(\hat{c}_i)L(\hat{c}_j)\sigma^G
\]
\[
= 4 \sum_{i \in \alpha, j \in \beta} L(\hat{c}_i)L(\hat{c}_j)\sigma^G = (s^G_{\epsilon})_{\alpha,\beta} = 0,
\]
\[
(s^H_{\epsilon})_{\beta,\gamma} = 4 \sum_{i \in \beta, j \in \gamma} L(\hat{c}_i)L(\hat{c}_j)\sigma^H = 4 \sum_{i \in \beta, j \in \gamma} L(\hat{c}_i)L(\hat{c}_j)\sigma^H
\]
\[
= 4 \sum_{i \in \beta, j \in \gamma} L(\hat{c}_i)L(\hat{c}_j)\sigma^H = (s^H_{\epsilon})_{\beta,\gamma} = 0.
\]

As SCMPCC-LICQ holds at \( \bar{z} \in F \), together with (20)-(25), we get \( \bar{\lambda}_{I(\bar{z})} = 0 \), \( \bar{\mu} = 0 \), \( \bar{\sigma}^G = 0 \) and \( \bar{\sigma}^H = 0 \), which contract with \( \| (\bar{\lambda}_{I(\bar{z})}, \bar{\mu}, \bar{\sigma}^G, \bar{\sigma}^H) \| = 1 \). Then the proof is completed.

4. Smoothing approximation approach. For any \( x, y \in K \), we have from [8, Theorem 5.13] that \( \langle x, y \rangle \geq 0 \). Similar to Proposition 4.1 in [5], Robinson’s constraint qualification fails to hold at each feasible point of SCMPCC, hence SCMPCC is a difficult class of optimization problems. To avoid this difficulty, we intend to employ a smoothing approximation approach to solve SCMPCC. As described in the beginning of Section 3, SCMPCC is equivalent to the nonsmooth problem NS-SCMPCC. We construct a smoothing approximation of NS-SCMPCC and focus on discussing the convergence properties of the smoothing approximation problem.

For any \( \varepsilon \in \mathbb{R} \), define \( \psi_{\varepsilon} : \mathbb{R} \to \mathbb{R} \) by
\[
\psi_{\varepsilon}(t) := \sqrt{t^2 + \varepsilon^2}, \quad t \in \mathbb{R}.
\]

Let \( x \in V \) have the spectral decomposition \( x = \sum_{i=1}^{r} \lambda_i(x)c_i \). Then the corresponding Löwner’s operator \( \Psi_{\varepsilon} : V \to V \) takes the following form
\[
\Psi_{\varepsilon}(x) = \sum_{i=1}^{r} \sqrt{\lambda_i^2(x) + \varepsilon^2}c_i = \sqrt{x^2 + \varepsilon^2}e,
\]
which can be treated as a smoothing approximation to the “absolute value” function \( |x| := \sqrt{x^2} \). Thus \( \frac{1}{2} \left( x + \sqrt{x^2 + \varepsilon^2}e \right) \) is the smoothing approximation to the projection function \( \Pi_K(x) \). A well known NCP function is the minimum function
\[
\phi_{\min}(a, b) = \min\{a, b\} = a + b - \sqrt{(a - b)^2}, \quad a, b \in \mathbb{R},
\]
the parameterized NCP function \( \phi_{\varepsilon} \) is defined by
\[
\phi_{\varepsilon}(a, b) = a + b - \sqrt{(a - b)^2 + \varepsilon^2}.
\]

In this section, we consider the following parametric smoothing problem with the parameter \( \varepsilon \neq 0 \) as an approximation of SCMPCC,
\[
(P_{\varepsilon}) \quad \min_{z \in \mathbb{R}^n} f(z) \quad \text{s. t.} \quad g(z) \leq 0, \quad h(z) = 0, \quad \Phi_{\varepsilon}(z) = 0,
\]
where \( \Phi_{\varepsilon}(z) = G(z) + H(z) - \sqrt{(G(z) - H(z))^2 + \varepsilon^2}e \) is the corresponding Löwner’s operator of the parameterized NCP function \( \phi_{\varepsilon} \). When \( \varepsilon = 0 \), problem \( P_0 \) is equivalent to the nonsmooth problem SCMPCC. Let \( F_{\varepsilon} \) denote the feasible set.
of $P_\varepsilon$. Obviously, $F_0 = F$, and we will not distinguish the feasible set $F_0$ from $F$ in the rest of this section.

For the smoothing approximation problem $P_\varepsilon$, we stress that most of the times we view $\varepsilon$ as a parameter, and this explains the notation adopted, where $\varepsilon$ is a subscript. However, in some cases, especially in the proofs, we shall view $\varepsilon$ as a dependent variable. Hence, from this point of view, the function $\Phi$ depends on the two variables $(\varepsilon, z)$. Actually, for any $z \in \mathbb{R}^n$, $\Phi_\varepsilon(z)$ is Lipschitz continuous with respect to $\varepsilon$. Let $x := G(z) - H(z)$ have the spectral decomposition $x = \sum_{i=1}^r \lambda_i(x) e_i$. Then

$$\|\Phi_0(z) - \Phi_\varepsilon(z)\| = \| \sum_{i=1}^r \left( \sqrt{\lambda_i(x)^2 + \varepsilon^2} - \sqrt{\lambda_i(x)^2} \right) c_i \| \leq \| \sum_{i=1}^r \varepsilon c_i \| = \| \varepsilon \| \leq a |\varepsilon|$$

(26)

holds for any $z \in \mathbb{R}^n$ with $a := \|\varepsilon\|$.

Before discussing the convergence behavior of the smoothing method, we give the following assumption on $P_\varepsilon$ first.

**Assumption 4.1.** For all $\varepsilon \in \mathbb{R}$, $F_\varepsilon$ is contained in a compact set $X \subset \mathbb{R}^n$.

Under this assumption, global solutions of SCMPCC and $P_\varepsilon$ exist. Actually, this assumption does not mean a restriction since sometimes it is treatable to add some box constraints, such as $|z_i| \leq M$, $i = 1, 2, \cdots, n$ for some large $M > 0$, to the constraints $g(z) \leq 0$.

4.1. **Convergence rate of $F_\varepsilon$.** In this subsection, we consider the convergence behavior of the feasible set $F_\varepsilon$ from local and global viewpoints when $\varepsilon \to 0$, and quantify the convergence rate for $F_\varepsilon$. The following lemma will view $\varepsilon$ as a dependent variable and illustrate the nonempty of $F_\varepsilon$ under SCMPCC-LICQ.

**Lemma 4.1.** Suppose that SCMPCC-LICQ holds at $\bar{z} \in F$. Then there exist a constant $\varepsilon_0 > 0$ and a unique continuous function $z(\cdot) : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ satisfying $z(0) = \bar{z}$ such that for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $z(\varepsilon) \in F_\varepsilon$.

**Proof.** Let $\bar{x} := G(\bar{z}) - H(\bar{z})$ have the spectral decomposition $\bar{x} = \sum_{i=1}^r \lambda_i(\bar{z}) e_i$. Suppose that $\lambda \in \mathbb{R}^{[I(\bar{z})]}$, $\mu \in \mathbb{R}^p$ and $\sigma \in V$ satisfy

$$J g_{I_\sigma}(\bar{z})^T \lambda + J h(\bar{z})^T \mu + \Pi_\varepsilon^* \sigma = 0,$$

(27)

with $U \in \partial \Phi_0(\bar{z})$. Since

$$\Phi_0(z) = G(z) + H(z) - \sqrt{(G(z) - H(z))^2} = 2 \left[ G(z) - \Pi_\varepsilon (G(z) - H(z)) \right],$$

there exists $V \in \partial \Pi_\varepsilon(\bar{x})$ such that

$$U^* \sigma = 2 [J G(\bar{z}) - V (J G(\bar{z}) + J H(\bar{z}))]^* \sigma = 2 J G(\bar{z})^* \left( \sigma - V \sigma \right) + 2 J H(\bar{z})^* V \sigma.$$

(28)

By Theorem 2.10, there exist $\xi^k \in [0, 1]$, $\Omega_{k\beta}^k \in U_{[\beta]}$ and systems of orthogonal idempotents $\{\tilde{e}_i\}_{i \in \beta}$, satisfying $\sum_{k=1}^N \xi^k = 1$ and $\sum_{i \in \beta} \xi^k \tilde{e}_i = \sum_{i \in \beta} e_i$, $k = 1, 2, \cdots, N$, such that

$$V = 2 \sum_{i \neq j, \; i, j = 1}^r a_{ij} L(\tilde{e}_i) L(\tilde{e}_j) + \sum_{i=1}^r b_i Q(\tilde{e}_i) + 2 \sum_{k=1}^N \sum_{i \neq j, \; i, j \in \beta} \xi^k \Omega_{ij}^k L(\tilde{e}_i^k) L(\tilde{e}_j^k)$$

$$+ \sum_{k=1}^N \sum_{i \in \beta} \xi^k \Omega_{ij}^k Q(\tilde{e}_i^k),$$

(29)

where $a_{ij}$ and $b_i$ are defined by (2), $i, j = 1, 2, \cdots, r$. Since

$$I = 2 \sum_{i \neq j, i, j = 1}^r L(c_i)L(c_j) + \sum_{i = 1}^r Q(c_i),$$

we can verify that

$$[(\sigma - V\sigma)_\varepsilon]_{\alpha} = 0, \quad [(\sigma - V\sigma)_\varepsilon]_{\beta} = 0, \quad [(V\sigma)_\varepsilon]_{\gamma} = 0, \quad [(V\sigma)_\varepsilon]_{\beta\gamma} = 0,$$

and

$$\sum_{i \in \alpha} \sum_{j \in \gamma} \lambda_i(\tilde{x}) \lambda_j(\tilde{x}) \{[(\sigma - V\sigma)_\varepsilon]_{ij} + [(V\sigma)_\varepsilon]_{ij}\} = \sum_{i \in \alpha} \sum_{j \in \gamma} [(V\sigma)_\varepsilon]_{ij}.$$ (30)

Since SCMPCC-LICQ holds at $\tilde{z}$, together with (27)-(30), we have $\lambda = 0$, $\mu = 0$ and $\sigma = 0$, which implies

$$\mathcal{A} := \begin{bmatrix} Jg(I_\varepsilon(\tilde{z})} & Jh(\tilde{z}) & U \end{bmatrix},$$

from $\mathbb{R}^n$ to $\mathbb{R}^{[I(\varepsilon)]} \times \mathbb{R}^p \times \mathbb{V}$, is onto. Since $\mathbb{V}$ is a $m$-dimensional real Euclidean space, it is isometric isomorphism to $\mathbb{R}^m$, then there exists a matrix $B \in \mathbb{R}^{n \times J}$ with $J = n - m - p - |I(\varepsilon)|$ such that

$$\mathcal{A} := \begin{bmatrix} \mathcal{A} \\ B \end{bmatrix},$$

from $\mathbb{R}^n$ to $\mathbb{R}^{[I(\varepsilon)]} \times \mathbb{R}^p \times \mathbb{V} \times \mathbb{R}^J$, is nonsingular.

Now we define a function $F_\varepsilon : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{[I_\varepsilon(\tilde{z})]} \times \mathbb{R}^p \times \mathbb{V} \times \mathbb{R}^J$ by

$$F_\varepsilon(\varepsilon, z) = \begin{bmatrix} g(I_\varepsilon(\tilde{z})} \\ h(\varepsilon) \\ \Phi_\varepsilon(\varepsilon) \\ B(\varepsilon - \varepsilon) \end{bmatrix}. \quad (31)$$

Obviously, $F_\varepsilon(0, \tilde{z}) = 0$ and any element in $\Pi_\varepsilon \partial F_\varepsilon(0, \tilde{z})$ is nonsingular. By Lemma 2.5, there exist a constant $\varepsilon_0 > 0$ and a unique continuous function $z(\cdot) : [-\varepsilon_0, \varepsilon_0] \rightarrow \mathbb{R}^n$ satisfying $z(0) = \tilde{z}$ such that for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $F_\varepsilon(\varepsilon, z(\varepsilon)) = 0$, hence $g(I_\varepsilon(\tilde{z}(\varepsilon))) = 0, h(z(\varepsilon)) = 0, \Phi_\varepsilon(\varepsilon) = 0$. As $g_i(\tilde{z}) < 0$ for $i \notin I(\varepsilon)$, we have $g_i(z(\varepsilon)) \leq 0$ for $i \notin I(\varepsilon)$ by decreasing the constant $\varepsilon_0 > 0$ if necessary. Thus $z(\varepsilon) \in \mathcal{F}_\varepsilon$.

A conclusion about the convergence of $\mathcal{F}_\varepsilon$ at $\varepsilon = 0$ will be given in the following theorem.

**Theorem 4.2.** If SCMPCC-LICQ holds at any feasible point $z \in \mathcal{F}$, then

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon = \mathcal{F}. \quad (32)$$

**Proof.** From Definition 2.2 of outer semicontinuous, for any $\tilde{z} \in \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon$, there exists a sequence $\{(z^k, \varepsilon^k)\} \subseteq \mathbb{R}^n \times \mathbb{R}$ such that $\lim_{k \rightarrow \infty} z^k = \tilde{z}, \lim_{k \rightarrow \infty} \varepsilon^k = 0$ and $z^k \in \mathcal{F}_{\varepsilon^k}$, i.e., $g(z^k) \leq 0, h(z^k) = 0, \Phi_{z^k}(z^k) = 0$. By the continuity of functions $g$, $h$ and $\Phi$, let $k \rightarrow \infty$, we have $g(\tilde{z}) \leq 0, h(\tilde{z}) = 0$ and $\Phi_0(\tilde{z}) = 0$, namely $\tilde{z} \in \mathcal{F}$. The inclusion $\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon \subseteq \mathcal{F}$ holds.
Next, we prove the inclusion $\mathcal{F} \subseteq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon$. Since SCMPCC-LICQ holds at any feasible point of SCMPCC, by Lemma 4.1, for any $\bar{z} \in \mathcal{F}$, there exist a constant $\varepsilon_0 > 0$ and a continuous function $z(\cdot): [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ satisfying $z(0) = \bar{z}$ such that for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $z(\varepsilon) \in \mathcal{F}_\varepsilon$. Therefore, for any $\tilde{z} \in \mathcal{F}$, we can find $z(\varepsilon) \in \mathcal{F}_\varepsilon$ such that $z(\varepsilon) \to \tilde{z}$ for any $\varepsilon \to 0$, which means $\mathcal{F} \subseteq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon$. Then we obtain the equation (32).

By making use of the implicit function theorem, we are now able to show the local convergence rate for $\mathcal{F}_\varepsilon$.

**Lemma 4.3.** Let SCMPCC-LICQ hold at $\tilde{z} \in \mathcal{F}$ and $a := \|e\|$, then

(a) there exist $\varepsilon_1 > 0$, $\delta_1 > 0$, $L_1 > 0$ such that for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$, for any $\tilde{z} \in \mathcal{F} \cap \mathbb{B}_{\delta_1}(\tilde{z})$, there is a point $z_\varepsilon \in \mathcal{F}_\varepsilon$ with $\|z_\varepsilon - \tilde{z}\| \leq aL_1|\varepsilon|$

(b) there exist $\varepsilon_2 > 0$, $\delta_2 > 0$, $L_2 > 0$ such that for all $\varepsilon \in [-\varepsilon_2, \varepsilon_2]$, for any $\tilde{z} \in \mathcal{F}_\varepsilon$, there is a point $z_\varepsilon \in \mathcal{F}$ with $\|z_\varepsilon - \tilde{z}\| \leq aL_2|\varepsilon|$

**Proof.** Let $\mathcal{X} := \mathbb{R}^{I(\tilde{z})} \times \mathbb{R}^p \times \mathbb{V} \times \mathbb{R}^J$ with $J = n - m - p - |I(\tilde{z})|$. First, we prove (a). Let $F_{\tilde{z}}(\varepsilon, z)$ be defined in (31), $y = F_{\tilde{z}}(\varepsilon, z)$, then $F_{\tilde{z}}(\cdot, \cdot)$ is locally Lipschitz continuous at $(0, \tilde{z})$. Since SCMPCC-LICQ holds at $\tilde{z} \in \mathcal{F}$, similar to the proof in Lemma 4.1, every element in $\Pi_{\tilde{z}} \partial F_{\tilde{z}}(0, \tilde{z})$ is nonsingular. Then applying Lemma 2.5, there exist a neighborhood $U(\tilde{y})$ of $\tilde{y} := 0$, a neighborhood $U(\tilde{y})$ of $\tilde{y} := F_{\tilde{z}}(0, \tilde{z}) = 0$ and a unique locally Lipschitz continuous function $z(\cdot, \cdot): U(\tilde{y}) \times U(\tilde{y}) \to \mathbb{R}^n$ with a Lipschitz modulus $L_1$, such that $z(\varepsilon, \tilde{y}) = \tilde{z}$ and for every $(\varepsilon, y) \in U(\tilde{y}) \times U(\tilde{y})$, $y = F_{\tilde{z}}(\varepsilon, z(\varepsilon, y))$. Then we can find $\varepsilon_1 > 0$, $\delta_1 > 0$ satisfying $[-\varepsilon_1, \varepsilon_1] \subseteq U(\tilde{y})$, $\mathbb{B}_{\delta_1}(\tilde{y}) \subseteq z(U(\tilde{y}), U(\tilde{y}))$ such that for any $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$, $\zeta \in \mathcal{F} \cap \mathbb{B}_{\delta_1}(\tilde{y})$, we have $y := F_{\tilde{z}}(\varepsilon, z(\varepsilon, y))$. By the uniqueness in Lemma 2.5, one has $\zeta = z(\varepsilon, y)$. Taking $y_\varepsilon := F_{\tilde{z}}(0, \tilde{z})$, we have $y_\varepsilon \in U(\tilde{y})$, hence $F_{\tilde{z}}(\varepsilon, z(\varepsilon, y_\varepsilon)) = y_\varepsilon = F_{\tilde{z}}(0, \tilde{z})$. Since $\tilde{z} \in \mathcal{F} \cap \mathbb{B}_{\delta_1}(\tilde{y})$, we have $z(\varepsilon, y_\varepsilon) \in \mathcal{F}_\varepsilon$ by decreasing $\varepsilon_1 > 0$ if necessary. Let $z_\varepsilon := z(\varepsilon, y_\varepsilon)$, it holds that

$$\|z_\varepsilon - \tilde{z}\| = \|z(\varepsilon, y_\varepsilon) - z(\varepsilon, \tilde{y})\| \leq L_1\|y_\varepsilon - \tilde{y}\| = L_1\|\Phi_\varepsilon(\tilde{z}) - \Phi_0(\tilde{z})\| \leq aL_1|\varepsilon|,$$

the last inequality comes from (26).

Now, we prove (b). We define a function $\hat{F}_{\tilde{z}}: \mathbb{R}^n \to \mathcal{X}$ as follows:

$$y := \hat{F}_{\tilde{z}}(z) = \begin{pmatrix} g_{\tilde{z}}(z) \\ h(z) \\ \Phi_0(z) \\ B(z - \tilde{z}) \end{pmatrix}.$$

Then we have $\hat{F}_{\tilde{z}}(0) = 0$ and $\hat{F}_{\tilde{z}}(\cdot)$ is locally Lipschitz continuous at $\tilde{z}$. Since SCMPCC-LICQ holds at $\tilde{z} \in \mathcal{F}$, similar to the proof in Lemma 4.1, every element in $\partial \hat{F}_{\tilde{z}}(\tilde{y})$ is nonsingular. Then applying Lemma 2.5, there exist a neighborhood $U(\tilde{y})$ of $\tilde{y} := 0$ and a unique locally Lipschitz continuous function $\hat{z}(\cdot, \cdot): U(\tilde{y}) \to \mathbb{R}^n$ with a Lipschitz modulus $L_2$, such that $\hat{z}(\tilde{y}) = \tilde{z}$ and for every $y \in U(\tilde{y})$, $y = \hat{F}_{\tilde{z}}(\tilde{y}(y))$. Then there exists $\varepsilon_2 > 0$, $\delta_2 := \delta_2(\tilde{z}) > 0$ such that $\mathbb{B}_{\delta_2}(\tilde{z}) \subseteq \hat{z}(U(\tilde{y}))$ and for any $\varepsilon \in [-\varepsilon_2, \varepsilon_2]$, $z_\varepsilon \in \mathcal{F}_\varepsilon \cap \mathbb{B}_{\delta_2}(\tilde{z})$, we have $\hat{y}_\varepsilon := \hat{F}_{\tilde{z}}(z_\varepsilon) \in U(\tilde{y})$, then $\hat{y}_\varepsilon = \hat{F}_{\tilde{z}}(\tilde{y}(\hat{y}_\varepsilon))$. By the uniqueness, $\tilde{z}_\varepsilon = \hat{z}(\tilde{y}_\varepsilon)$. Taking

$$\hat{y}_\varepsilon := \begin{pmatrix} g_{\tilde{z}}(\tilde{z}_\varepsilon) \\ h(\tilde{z}_\varepsilon) \\ 0 \\ B(\tilde{z}_\varepsilon - \tilde{z}) \end{pmatrix},$$
Theorem 4.5. Suppose that Assumption 4.1 holds and SCMPCC-LICQ holds at 
Haudorff distance.

Lemma 4.3 yields the local convergence rate of \( F_\varepsilon \) near \( \tilde{z} \in F \). Now we are 
interested in the global convergence rate under the SCMPCC-LICQ assumption.

**Lemma 4.4.** Suppose that Assumption 4.1 holds and SCMPCC-LICQ holds at each 
point \( z \in F \) and \( a := \|e\| \). Then there are \( \varepsilon_0 > 0 \), \( L_1 > 0 \), \( L_2 > 0 \) such that for all 
\( \varepsilon \in [\varepsilon_0, \varepsilon_0] \), the following holds: for each \( \tilde{z} \in F \), there exists \( \tilde{z}_\varepsilon \in F_\varepsilon \) with

\[
\|\tilde{z}_\varepsilon - \tilde{z}\| \leq a L_1 |\varepsilon|,
\]

and for any \( \tilde{z}_\varepsilon \in F_\varepsilon \), there exists a point \( \hat{z}_\varepsilon \in F \) satisfying

\[
\|\hat{z}_\varepsilon - \tilde{z}_\varepsilon\| \leq a L_2 |\varepsilon|.
\]

**Proof.** We first prove (34). To extend the analysis from a local to a global statement, 
we employ a compactness argument. The union \( \bigcup_{\tilde{z} \in F} B_{\delta(\tilde{z})}(\tilde{z}) \) forms an open cover 
of the compact feasible set \( F \subset \mathbb{R}^n \). So, by definition of compactness, we can choose a 
finite cover: i.e., points \( z_k \in F \), \( k = 1, 2, \ldots, N \), such that with \( \delta_k = \delta(z_k) \) the 
set \( \bigcup_{k=1,2,\ldots,N} B_{\delta_k}(\tilde{z}_k) \) provides an open cover of \( F \). We can choose some \( \delta_0 > 0 \) 
(small) such that

\[
F + \delta_0 B \subset \bigcup_{k=1,2,\ldots,N} B_{\delta_k}(\tilde{z}_k).
\]

In view of [15, Proposition 5.12], Assumption 4.1 and Theorem 4.2 imply that for the 
given \( \delta_0 > 0 \), there exists \( \varepsilon_0 > 0 \) sufficiently small such that for all \( -\varepsilon_0 \leq \varepsilon \leq \varepsilon_0 \):

\[
F_\varepsilon \subset F + \delta_0 B,
\]

thus

\[
F_\varepsilon \subset F + \delta_0 B \subset \bigcup_{k=1,2,\ldots,N} B_{\delta_k}(\tilde{z}_k).
\]

By construction, for any \( \tilde{z}_\varepsilon \in F_\varepsilon \), it must be contained in at least one of the balls 
\( B_{\delta_k}(\tilde{z}_k) \), \( k \in \{1, 2, \ldots, N\} \). Then it follows from Lemma 4.3(b) that, for \( \tilde{z}_\varepsilon \in F_\varepsilon \), 
there exists a point \( \hat{z}_\varepsilon \in F \) satisfying

\[
\|\hat{z}_\varepsilon - \tilde{z}_\varepsilon\| \leq a L_2 |\varepsilon|,
\]

where \( L_2^k \) is correspond to \( \tilde{z}_k \). Let \( L_2 = \max\{L_2^k | k = 1, 2, \ldots, N\} \), we have (34) 
holds.

The convergence result (33) could be obtained in a similar way. \( \square \)

The convergence rate between the feasible sets of MPCC and its perturbed prob-
lem \( P_\varepsilon \) in [2] is of order \( O(\sqrt{\varepsilon}) \). Although we use a different smoothing function 
for the SCMPCC problem in this paper, we have a similar conclusion below on the 
convergence rate between the feasible sets \( F_\varepsilon \) and \( F \) with respect to the Pompeiu-
Haudorff distance.

**Theorem 4.5.** Suppose that Assumption 4.1 holds and SCMPCC-LICQ holds at each 
point \( z \in F \). Then, there exist \( \varepsilon_0 > 0 \) and \( L > 0 \) such that, for any \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \),

\[
\mathbb{H}(F_\varepsilon, F) \leq L |\varepsilon|.
\]

**Proof.** Let \( L := \max\{L_1, L_2\} \) in Lemma 4.4, then the conclusion holds by the 
definition of Pompeiu-Haudorff distance. \( \square \)
4.2. Convergence behavior of the optimal value function and solution mapping for $P_\varepsilon$. By the compactness assumption, a global minimizer of $P_\varepsilon$ always exists, assuming $F_\varepsilon \neq \emptyset$. Now we consider how close the optimal solution set and optimal value of problem $P_\varepsilon$ are from that of SCMPCC when $\varepsilon \to 0$. Let

$$\bar{f}(z, \varepsilon) := \begin{cases} f(z), & z \in F_\varepsilon, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\kappa(\varepsilon) := \inf \{ f(z) | z \in F_\varepsilon \},$$

$$S(\varepsilon) := \arg\min \{ f(z) | z \in F_\varepsilon \}.$$ 

The relationship between the convergence of a set-valued mapping and the epi-convergence of the indicator function of the set-valued mapping is discussed.

**Lemma 4.6.** For a set-valued mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}^n$, the following equivalence holds

$$\lim_{\varepsilon \to 0} \mathcal{T}(\varepsilon) = \mathcal{T}(0) \iff e^\varepsilon \lim_{\varepsilon \to 0} \delta_{\mathcal{T}(\varepsilon)}(\cdot) = \delta_{\mathcal{T}(0)}(\cdot).$$

**Proof.** Noting that

$$\text{epi}[\delta_{\mathcal{T}(\varepsilon)}(\cdot)] = \{(z, \alpha) | z \in \mathcal{T}(\varepsilon), \alpha \geq 0\} = \mathcal{T}(\varepsilon) \times \mathbb{R}$$

and

$$\lim_{\varepsilon \to 0} \mathcal{T}(\varepsilon) \times \mathbb{R} = \mathcal{T}(0) \times \mathbb{R} = \text{epi}[\delta_{\mathcal{T}(0)}(\cdot)],$$

we obtain the equivalence. \qed

The following lemma discusses the epi-convergence of the sum of a function-valued mapping and a continuous function.

**Lemma 4.7.** Let $\psi(\cdot)$ satisfy

$$e^\varepsilon \lim_{\varepsilon \to 0} \psi(\varepsilon)(\cdot) = \psi(0)(\cdot)$$

and $\psi_1$ be a continuous function, then

$$e^\varepsilon \lim_{\varepsilon \to 0} [\psi(\varepsilon)(\cdot) + \psi_1(\cdot)] = \psi(0)(\cdot) + \psi_1(\cdot).$$

**Proof.** The conclusion is obvious by the definition of epi-convergence. \qed

**Theorem 4.8.** Let SCMPCC-LICQ hold at each $z \in \mathcal{F}$. The function-valued mapping $\varepsilon \to \bar{f}(\cdot, \varepsilon)$ is epi-continuous at $\varepsilon = 0$.

**Proof.** In view of Theorem 4.2, $\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon = \mathcal{F}$. Together with the continuity of $\bar{f}(\cdot)$, the result comes from Lemma 4.6 and Lemma 4.7. \qed

The main convergence results on optimal value function and solution mapping of $P_\varepsilon$ are obtained below.

**Theorem 4.9.** Let Assumption 4.1 hold and SCMPCC-LICQ hold at each point $z \in \mathcal{F}$. Then the optimal value function $\kappa(\varepsilon)$ is locally Lipschitz continuous and the solution mapping $S(\varepsilon)$ is outer semicontinuous at $\varepsilon = 0$.

**Proof.** Under Assumption 4.1, there exist $\varepsilon_0 > 0$ and $M > 0$ such that for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $\mathcal{F}_\varepsilon \subset [-M \mathbf{1}_n, M \mathbf{1}_n]$, where $\mathbf{1}_n$ is a vector in $\mathbb{R}^n$ with all entries being ones. Choose a constant $\bar{\alpha}$ such that $\text{lev}_{\leq \alpha} f \neq \emptyset$, for any $\alpha \leq \bar{\alpha}$, $\text{lev}_{\leq \alpha} \bar{f}(\cdot, \varepsilon) = \text{lev}_{\leq \alpha} f \cap \mathcal{F}_\varepsilon \subset [-M \mathbf{1}_n, M \mathbf{1}_n]$, the level set is uniformly bounded for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, i.e., $\bar{f}$ is level-bounded in $z$ uniformly for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Hence we have from [15, Theorem 7.41] that $\kappa(\varepsilon)$ is continuous and $S(\varepsilon)$ is outer semicontinuous at $\varepsilon = 0$.\qed
Now we prove the locally Lipschitz continuity of $\kappa(\cdot)$ at $\varepsilon = 0$. Suppose $\tilde{z} \in F$ is a global minimizer of SCMPCC, then $\kappa(0) = f(\tilde{z})$. Since SCMPCC-LICQ holds at $\tilde{z} \in F$, from Lemma 4.4, there exist $\bar{\varepsilon} > 0$, $L_1 > 0$ such that for all $\varepsilon \in [0, \bar{\varepsilon}]$, there is a point $z_\varepsilon \in F_\varepsilon$ with $\|z_\varepsilon - \tilde{z}\| \leq a L_1 |\varepsilon|$, where $a = \|\varepsilon\|$. Then, there exists a constant $L' > 0$ such that
\[
\kappa(\varepsilon) - \kappa(0) \leq f(z_\varepsilon) - f(\tilde{z}) \leq L' \|z_\varepsilon - \tilde{z}\| \leq a L' L_1 |\varepsilon|,
\] (35) in which the second inequation comes from the twice differentiability of function $f(z)$. On the other hand, let $\bar{z}_\varepsilon \in F_\varepsilon$ be a global minimizer of $P_\varepsilon$, then $\kappa(\varepsilon) = f(\bar{z}_\varepsilon)$. As SCMPCC-LICQ holds at each point $z \in F$, by Lemma 4.4, for the given $\bar{\varepsilon} > 0$, there exists a constant $L_2 > 0$ and a point $\hat{z}_\varepsilon \in F$ satisfying $\|\hat{z}_\varepsilon - \bar{z}_\varepsilon\| \leq a L_2 |\varepsilon|$. Then we have
\[
\kappa(\varepsilon) - \kappa(0) \geq f(\hat{z}_\varepsilon) - f(\bar{z}_\varepsilon) \geq -L' \|\hat{z}_\varepsilon - \bar{z}_\varepsilon\| \geq -a L' L_2 |\varepsilon|,
\] (36) in which the second inequation comes from the twice differentiability of function $f(z)$. Let $L := \max\{a L' L_1, a L' L_2\}$, in view of (35) and (36), one has
\[
\|\kappa(\varepsilon) - \kappa(0)\| \leq L |\varepsilon|.
\] The proof is completed. 

Theorem 4.9 tells us that the optimal solution mapping $S(\varepsilon)$ is outer semicontinuous at $\varepsilon = 0$, and the convergence rate for the optimal value function $\kappa(\varepsilon)$ is of order $O(|\varepsilon|)$.

4.3. Local convergence to C-stationary point. Based on the explicit expressions of C-stationary point and SCMPCC-LICQ proposed in Section 3, we consider the limiting behavior of the stationary points of problem $P_\varepsilon$ when $\varepsilon \to 0$.

The following lemma states that SCMPCC-LICQ carries to the standard LICQ for the smoothing approximation problem $P_\varepsilon$ if $|\varepsilon| > 0$ is sufficiently small. The idea of this lemma is the same with which in [24, Lemma 4.1], we omit the proof here.

Lemma 4.10. If SCMPCC-LICQ holds at $\tilde{z} \in F$, then there exist a neighborhood $U(\tilde{z})$ of $\tilde{z}$ and a scalar $\varepsilon_0 > 0$ such that for every $\varepsilon \in [\varepsilon_0, 0) \cup (0, \varepsilon_0]$, LICQ holds at every $z \in U(\tilde{z}) \cap F_\varepsilon$ of $P_\varepsilon$.

We now extend the result on the local convergence to C-stationary point of SOCMPCC in [24, Theorem 4.1] to the SCMPCC case under SCMPCC-LICQ. The same proof techniques as in [24] are adopted, we sketch the proof here since SCMPCC is a general case for SOCMPCC. A similar result is also obtained by [22, Theorem 4.3] under different assumptions.

Theorem 4.11. Let $\{z^k\} \subseteq \mathbb{R}$ be convergent to 0, and $z^k \in F_{z^k}$ be a stationary point of problem $P_{z^k}$ for each $k$. Suppose that $\tilde{z}$ is an accumulation point of sequence $\{z^k\}$, then if SCMPCC-LICQ holds at $\tilde{z}$, $\tilde{z}$ is a C-stationary point of SCMPCC.

Proof. We assume that
\[
\lim_{k \to \infty} z^k = \tilde{z}.
\] (37) Since all the functions involved in SCMPCC are continuous, $F$ is closed and hence $\tilde{z} \in F$. Let $\bar{x} := G(\tilde{z}) - H(\tilde{z})$ and $x^k := G(z^k) - H(z^k)$ have the spectral decomposition $\bar{x} = \sum_{i=1}^r \lambda_i(\bar{x})e_i$ and $x^k = \sum_{i=1}^r \lambda_i(x^k)e_i(x^k)$, respectively. It follows from the SCMPCC-LICQ assumption and Lemma 4.10, for any sufficiently large $k,$
bounded for \( i,j \in \mathbb{I} \), and hence, by the stationarity of \( z^k \), there exist unique Lagrange multipliers \( \lambda^k \in \mathbb{R}^q, \mu^k \in \mathbb{R}^p \) and \( \sigma^k \in \mathbb{V} \) such that

\[
\nabla f(z^k) + Jg(z^k)^T \lambda^k + Jh(z^k)^T \mu^k + J\Phi_{\varepsilon^k}(z^k)^* \sigma^k = 0, \tag{38}
\]

\[
g(z^k) \leq 0, \quad \lambda^k \geq 0, \quad h(z^k) = 0, \quad \Phi_{\varepsilon^k}(z^k) = 0, \tag{39}
\]

\[
g(z^k)^T \lambda^k = 0. \tag{40}
\]

It follows from (40) that \( \lambda^k_0 = 0, l \notin I(z^k) \). Now suppose that, for all sufficiently large \( k \), (38)-(40) hold and in addition, \( I(z^k) \subseteq I(\bar{z}) \) and the operator

\[
B(\varepsilon^k, z^k) = \begin{bmatrix}
Jg_{\varepsilon^k}(z^k) \\
Jh(z^k) \\
J\Phi_{\varepsilon^k}(z^k)
\end{bmatrix}
\]

is onto since LICQ holds at \( z^k \). It follows from (38) that

\[
-\nabla f(z^k) = B(\varepsilon^k, z^k)^* \begin{bmatrix}
\lambda^k_{I(z^k)} \\
\mu^k \\
\sigma^k
\end{bmatrix}, \tag{41}
\]

In consequence, from (37) and (41), all the multiplier sequences \( \{\lambda^k_0 | l \in I(\bar{z})\} \), \( \{\mu^k\} \) and \( \{\sigma^k\} \) are convergent when \( k \to \infty \). Let \( \lambda \in \mathbb{R}^q, \mu \in \mathbb{R}^p \) and \( \sigma \in \mathbb{V} \) be denoted as follows

\[
\lambda_l = \begin{cases} 
\lim_{k \to \infty} \lambda^k_l, & l \in I(\bar{z}), \\
0, & l \notin I(\bar{z}) \end{cases}, \quad \mu = \lim_{k \to \infty} \mu^k, \quad \sigma = \lim_{k \to \infty} \sigma^k. \tag{42}
\]

Since

\[
J\Phi_{\varepsilon^k}(z^k)^* \sigma^k = JG(z^k)^* \left[ \sigma^k - JW_{\varepsilon^k}(x^k)\sigma^k \right] + JH(z^k)^* \left[ \sigma^k + JW_{\varepsilon^k}(x^k)\sigma^k \right],
\]

(38) can be written as

\[
\nabla f(z^k) + Jg(z^k)^T \lambda^k + Jh(z^k)^T \mu^k + JG(z^k)^* \left[ \sigma^k - JW_{\varepsilon^k}(x^k)\sigma^k \right] + JH(z^k)^* \left[ \sigma^k + JW_{\varepsilon^k}(x^k)\sigma^k \right] = 0, \tag{43}
\]

where

\[
JW_{\varepsilon^k}(x^k) = 2 \sum_{i \neq j, k=1}^r a_{ij}(\varepsilon^k, x^k)L(c_i(x^k))L(c_j(x^k)) + \sum_{i=1}^r a_{ii}(\varepsilon^k, x^k)Q(c_i(x^k))
\]

and \( a_{ij} (\varepsilon, x) \) is defined by

\[
a_{ij}(\varepsilon, x) = \begin{cases} 
\frac{\lambda_i(x) + \lambda_j(x)}{\sqrt{\lambda_i(x)^2 + \varepsilon^2} + \sqrt{\lambda_j(x)^2 + \varepsilon^2}} & \text{if } \lambda_i(x) \neq \lambda_j(x), \\
\frac{\lambda_i(x)}{\lambda_i(x)}, & \text{if } \lambda_i(x) = \lambda_j(x).
\end{cases}
\]

Let \( k \to \infty \). Since \( \mathcal{C}(\cdot) \) is outer semicontinuous at \( \bar{x} \) and \( a_{ij}(\varepsilon^k, x^k) \) are uniformly bounded for \( i, j = 1, 2, \ldots, r \), by taking a subsequence if necessary, we may assume the system of orthogonal idempotents \( \{c_1(x^k), c_2(x^k), \ldots, c_r(x^k)\} \) converges to a
system of orthogonal idempotents \( \hat{c} := \{ \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_r \} \) with \( \hat{c}_i = \bar{c}_i \) when \( i \in \alpha \cup \gamma \) and \( \sum_{i \in \beta} \hat{c}_i = \sum_{i \in \beta} \bar{c}_i \), and

\[
A_{ij}(\bar{x}, \bar{x}) \rightarrow \bar{a}_{ij} := \begin{cases} 
1, & \text{if } (i, j) \in (\alpha \times \alpha) \cup (\alpha \times \beta) \cup (\beta \times \alpha), \\
\frac{\lambda_i(\bar{x}) + \lambda_j(\bar{x})}{\lambda_i(\bar{x}) - \lambda_j(\bar{x})}, & \text{if } (i, j) \in \alpha \times \gamma, \\
\frac{\bar{a}_{ij}}{\lambda_i(\bar{x}) + \lambda_j(\bar{x})}, & \text{if } (i, j) \in \beta \times \beta, \\
\frac{\lambda_i(\bar{x}) - \lambda_j(\bar{x})}{\lambda_i(\bar{x}) + \lambda_j(\bar{x})}, & \text{if } (i, j) \in \gamma \times \alpha, \\
-1, & \text{if } (i, j) \in (\gamma \times \gamma) \cup (\gamma \times \beta) \cup (\beta \times \gamma).
\]

Then we have \( \lim_{k \to \infty} JW_{\epsilon k}(x^k) = \mathcal{L} \) with \( \mathcal{L} = 2 \sum_{i \neq j, i, j = 1}^r \bar{a}_{ij} L(\hat{c}_i)L(\hat{c}_j) + \sum_{i=1}^r \bar{a}_{ii} Q(\hat{c}_i) \).

Denote

\[
\sigma^G := \lim_{k \to \infty} \left[ \sigma^k - JW_{\epsilon k}(x^k)\sigma^k \right], \quad \sigma^H := \lim_{k \to \infty} \left[ \sigma^k + JW_{\epsilon k}(x^k)\sigma^k \right],
\]

we have

\[
\sigma^G = [\mathcal{I} - \mathcal{L}]\sigma = 2 \sum_{i \neq j, i, j = 1}^r (1 - \bar{a}_{ij})L(\hat{c}_i)L(\hat{c}_j)\sigma + \sum_{i=1}^r (1 - \bar{a}_{ii})Q(\hat{c}_i)\sigma,
\]

\[
\sigma^H = [\mathcal{I} + \mathcal{L}]\sigma = 2 \sum_{i \neq j, i, j = 1}^r (1 + \bar{a}_{ij})L(\hat{c}_i)L(\hat{c}_j)\sigma + \sum_{i=1}^r (1 + \bar{a}_{ii})Q(\hat{c}_i)\sigma.
\]

Then we can verify that

\[
(\sigma^G_c)_{i\alpha} = 0, \quad (\sigma^G_c)_{i\beta} = 0, \quad (\sigma^G_c)_{ij} = \frac{4\lambda_j(\bar{x})}{\lambda_i(\bar{x}) - \lambda_j(\bar{x})}L(\hat{c}_i)L(\hat{c}_j)\sigma, \quad i \in \alpha, j \in \gamma, \quad (45)
\]

\[
(\sigma^H_c)_{i\gamma} = 0, \quad (\sigma^H_c)_{i\beta} = 0, \quad (\sigma^H_c)_{ij} = \frac{4\lambda_j(\bar{x})}{\lambda_i(\bar{x}) - \lambda_j(\bar{x})}L(\hat{c}_i)L(\hat{c}_j)\sigma, \quad i \in \alpha, j \in \gamma, \quad (46)
\]

and

\[
(\sigma^G_{\bar{c}})_{i\beta} = 2 \sum_{i \neq j, i, j \in \beta}^r (1 - \bar{a}_{ij})L(\hat{c}_i)L(\hat{c}_j)\sigma + \sum_{i \in \beta}^r (1 - \bar{a}_{ii})Q(\hat{c}_i)\sigma, \quad (47)
\]

\[
(\sigma^H_{\bar{c}})_{i\beta} = 2 \sum_{i \neq j, i, j \in \beta}^r (1 + \bar{a}_{ij})L(\hat{c}_i)L(\hat{c}_j)\sigma + \sum_{i \in \beta}^r (1 + \bar{a}_{ii})Q(\hat{c}_i)\sigma, \quad (48)
\]

Since \( \hat{c}_i = \bar{c}_i \) for \( i \in \alpha \cup \gamma \) and \( \sum_{i \in \beta} \hat{c}_i = \sum_{i \in \beta} \bar{c}_i \), it holds that

\[
(\sigma^G_{\bar{c}})_{i\alpha} = (\sigma^G_{\bar{c}})_{i\alpha}, \quad (49)
\]

\[
(\sigma^G_{\bar{c}})_{i\beta} = 4 \sum_{i \in \alpha} L(\hat{c}_i) \sum_{j \in \beta} L(\hat{c}_j)\sigma^G = 4 \sum_{i \in \alpha} L(\hat{c}_i) \sum_{j \in \beta} L(\hat{c}_j)\bar{\sigma}^G = (\sigma^G_{\bar{c}})_{i\beta}, \quad (50)
\]

\[
(\sigma^G_{\bar{c}})_{ij} = 4L(\hat{c}_i)L(\hat{c}_j)\sigma^G = 4L(\hat{c}_i)L(\hat{c}_j)\bar{\sigma}^G = (\sigma^G_{\bar{c}})_{ij}, \quad i \in \alpha, j \in \gamma, \quad (51)
\]

similarly,

\[
(\sigma^H_{\bar{c}})_{i\gamma} = (\sigma^H_{\bar{c}})_{i\gamma}, \quad (\sigma^H_{\bar{c}})_{i\beta} = (\sigma^H_{\bar{c}})_{i\beta}, \quad (\sigma^H_{\bar{c}})_{ij} = (\sigma^H_{\bar{c}})_{ij}, \quad i \in \alpha, j \in \gamma. \quad (52)
\]
And
\[(σ_c^G)_{ββ} = 2 \sum_{i,j,λ,j,i,j} λ(\bar{c}_i)L(\bar{c}_j)σ_{β}^G + \sum_{i,j,λ,j,i,j} Q(\bar{c}_i)σ_{β}^G = 2 \sum_{i,j,λ,j,i,j} λ(\bar{c}_i)L(\bar{c}_j)σ_{β}^G - \sum_{i,j,λ,j,i,j} L(\bar{c}_i)σ_{β}^G
\]
\[= 2 \sum_{i,j,λ,j,i,j} λ(\bar{c}_i)L(\bar{c}_j)σ_{β}^G - \sum_{i,j,λ,j,i,j} L(\bar{c}_i)σ_{β}^G = \sum_{i,j,λ,j,i,j} L(\bar{c}_i)σ_{β}^G = (σ_c^G)_{ββ}, \]
similarly, \((σ_c^H)_{ββ} = (σ_c^H)_{ββ}\).

Since the functions \(G, H, g\) and \(h\) are continuous, let \(k \to ∞\) in (38), we have from (42),(43) and (44) that
\[\nabla f(\bar{z}) + \mathcal{J}g(\bar{z})^T λ + \mathcal{J}h(\bar{z})^T μ + \mathcal{J}G(\bar{z})^*σ^G + \mathcal{J}H(\bar{z})^*σ^H = 0,
\]
namely, (3) holds for the multipliers \(λ, μ, σ^G\) and \(σ^H\). Also, we have (4), (11)-(13) hold immediately from (39)-(40), (45)-(52). Then the rest of the proof is to show (14).

From (47)-(48) and (53), we have
\[\langle (σ_c^G)_{ββ}, (σ_c^H)_{ββ} \rangle = \sum_{i,j,λ,j,i,j} (1 - \bar{a}_{ij}^2)\| (σ_c)_{ij} \|^2 + \sum_{i,λ,j,i,j} (1 - \bar{a}_{ii}^2)\| (σ_c)_{i,λ} \|^2 ≥ 0.
\]
Then the proof is completed. \(\square\)

5. Conclusion. The explicit expressions of C-stationary points and SCMPCC-LICQ of SCMPCC are introduced and a parametric smoothing scheme for SCMPCC is studied. We discuss the convergence behavior of feasible set \(F_ε\), stationary points, solution mapping and optimal value function of \(P_ε\) respectively.

This paper takes ideas from the convergence analysis of MPCCs in [2, 9, 17], but it is not a simple extension of these papers. The convergence of the feasible set \(F_ε\) is discussed by the tool of implicit function theorem for Lipschitz function and variational analysis in [15] and the convergence behavior of solution mapping and optimal value of \(P_ε\) is based on the epi-continuity of function-valued mappings in [15].

The proposed smoothing approach is only conceptual since it assumes the symmetric cone programs \(P_ε\) to be solved in each iteration. In our future work, we plan to exploit some efficient algorithms for SCMPCC based on the convergence results of this paper. Also, only the explicit expressions of C-stationary condition are introduced, what about the M- and S-stationary conditions?

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