Generalized rank-constrained matrix approximations

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Abstract

In this paper we give an explicit solution to the rank constrained matrix approximation in Frobenius norm, which is a generalization of the classical approximation of an \( m \times n \) matrix \( A \) by a matrix of rank \( k \) at most.

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1 Introduction

Let \( \mathbb{C}^{m \times n} \) be set of \( m \times n \) complex valued matrices, and denote by \( \mathcal{R}(m, n, k) \subseteq \mathbb{C}^{m \times n} \) the variety of all \( m \times n \) matrices of rank \( k \) at most. Fix \( A = [a_{ij}]_{i,j=1}^{n,m} \in \mathbb{C}^{m \times n} \). Then \( A^* \in \mathbb{C}^{n \times m} \) is the conjugate transpose of \( A \), and \( ||A||_F := \sqrt{\sum_{i,j=1}^{n,m} |a_{ij}|^2} \) is the Frobenius norm of \( A \). Recall that the singular value decomposition of \( A \), abbreviated here as SVD, is given by \( A = U_A \Sigma_A V_A^* \), where \( U_A \in \mathbb{C}^{m \times m}, V_A \in \mathbb{C}^{n \times n} \) are unitary matrices, \( \Sigma_A := \text{diag}(\sigma_1(A), \ldots, \sigma_{\min(m,n)}(A)) \in \mathbb{C}^{m \times n} \) is a generalized diagonal matrix, with the singular values \( \sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq 0 \) on the main diagonal. The number of positive singular values of \( A \) is \( r \), which is equal to the rank of \( A \), denoted by \( \text{rank} \; A \). Let \( U_A = [u_1 \; u_2 \; \ldots \; u_m], V_A = [v_1 \; v_2 \; \ldots \; v_n] \) be the representations of \( U, V \) in terms of their \( m, n \) columns respectively. Then \( u_i \) and \( v_i \) are called the left and the right singular vectors of \( A \), respectively, that correspond to the singular value \( \sigma_i(A) \). Let

\[
P_{A,L} := \sum_{i=1}^{\text{rank} \; A} u_i u_i^* \in \mathbb{C}^{m \times m}, \quad P_{A,R} := \sum_{i=1}^{\text{rank} \; A} v_i v_i^* \in \mathbb{C}^{n \times n},
\]

be the orthogonal projections on the range of \( A \) and \( A^* \) respectively. Denote by

\[
A_k := \sum_{i=1}^{k} \sigma_i(A) u_i v_i^* \in \mathbb{C}^{m \times n}
\]

for \( k = 1, \ldots, \text{rank} \; A \). For \( k > \text{rank} \; A \) we define \( A_k := A \ (= A_{\text{rank} \; A}) \). For \( 1 \leq k < \text{rank} \; A \), the matrix \( A_k \) is uniquely defined if and only if \( \sigma_k(A) > \sigma_{k+1}(A) \).
The enormous application of SVD decomposition of $A$ in pure and applied mathematics, is derived from the following approximation property:

$$\min_{X \in \mathcal{R}(m,n,k)} ||A - X||_F = ||A - A_k||_F, \quad k = 1, \ldots$$  \hfill (1.2)

The latter is known as the Eckart-Young theorem [2]. We note that the work [2] implied a number of extensions. We cite [4, 5, 7, 8] as some recent references. Another application of SVD is a formula for the Moore-Penrose inverse $A^\dagger := V_A \Sigma_A^\dagger U_A^*$ of $A$, where $\Sigma_A^\dagger := \text{diag}(\frac{1}{\sigma_1(A)}, \ldots, \frac{1}{\sigma_{\text{rank}_A(A)}}, 0, \ldots, 0) \in \mathbb{C}^{n \times m}$. See for example [1].

## 2 Main Result

Below, we provide generalizations of the classical minimal problem given in (1.2).

**Theorem 2.1** Let matrices $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{q \times n}$ be given. Then

$$X = B^\dagger(P_{B,L}AP_{C,R})_kC^\dagger$$  \hfill (2.1)

is a solution to the minimal problem

$$\min_{X \in \mathcal{R}(p,q,k)} ||A - BXC||_F,$$  \hfill (2.2)

having the minimal $||X||_F$. This solution is unique if and only if either

$$k \geq \text{rank} \ P_{B,L}AP_{C,R} \quad \text{or} \quad 1 \leq k < \text{rank} \ P_{B,L}AP_{C,R}$$

and

$$\sigma_k(P_{B,L}AP_{C,R}) > \sigma_{k+1}(P_{B,L}AP_{C,R}).$$

**Proof of Theorem 2.1** Recall that the Frobenius norm is invariant under the multiplication from the left and the right by the corresponding unitary matrices. Hence $||A - BXC||_F = ||\tilde{A} - \Sigma_B \tilde{X} \Sigma_C||_F$, where $\tilde{A} := U_B^*AV_C$ and $\tilde{X} := V_B^*XU_C$. Clearly, $X$ and $\tilde{X}$ have the same rank and the same Frobenius norm. Thus, it is enough to consider the minimal problem $\min_{\tilde{X} \in \mathcal{R}(p,q,k)} ||\tilde{A} - \Sigma_B \tilde{X} \Sigma_C||_F$.

Let $s = \text{rank} \ B$ and $t = \text{rank} \ C$. Clearly if $B$ or $C$ is a zero matrix, then $X = 0$ is the solution to the minimal problem (2.2). In this case either $P_{B,L}$ or $P_{C,R}$ are zero matrices, and the theorem holds trivially in this case.

Let us consider the case $1 \leq s, 1 \leq t$. Define $B_1 := \text{diag}(\sigma_1(B), \ldots, \sigma_s(B)) \in \mathbb{C}^{s \times s}, C_1 := \text{diag}(\sigma_1(C), \ldots, \sigma_t(C)) \in \mathbb{C}^{t \times t}$. Partition $\tilde{A}$ and $\tilde{X}$ into four block matrices $A_{ij}$ and $X_{ij}$ with $i, j = 1, 2$ so that $\tilde{A} = [A_{ij}]_{i,j=1}^2$ and $\tilde{X} = [X_{ij}]_{i,j=1}^2$, where $A_{11}, X_{11} \in \mathbb{C}^{s \times t}$. (For certain values of $s$ and $t$, we may have to partition $\tilde{A}$ or $\tilde{X}$ to less than four block matrices.) Next, observe that $Z := \Sigma_B \tilde{X} \Sigma_C = [Z_{ij}]_{i,j=1}^2$, where $Z_{11} = B_1 X_{11} C_1$ and all other blocks $Z_{ij}$ are zero matrices. Since $B_1$ and $C_1$ are invertible we deduce

$$\text{rank} \ Z = \text{rank} \ Z_{11} = \text{rank} \ X_{11} \leq \text{rank} \ \tilde{X} \leq k.$$
The approximation property of \((A_{11})_k\) yields the inequality 
\[ ||A_{11} - Z_{11}||_F \geq ||A_{11} - (A_{11})_k||_F \]
for any \(Z_{11}\) of rank \(k\) at most. Hence for any \(Z\) of the above form,
\[ ||\hat{A} - Z||_F^2 = ||A_{11} - Z_{11}||_F^2 + \sum_{2<i+j\leq 4} ||A_{ij}||_F^2 \geq ||A_{11} - (A_{11})_k||_F^2 + \sum_{2<i+j\leq 4} ||A_{ij}||_F^2. \]

Thus \(\hat{X} = [X_{ij}]_{i,j=1}^2\), where \(X_{11} = B_{11}^{-1}(A_{11})_kC_{11}^{-1}\) and \(X_{ij} = 0\) for all \((i, j) \neq (1, 1)\) is a solution to the problem \(\operatorname{min}_{\hat{X} \in \mathcal{R}(p,q,k)} ||\hat{A} - \Sigma_B\hat{X}\Sigma_C||_F\) with the minimal Frobenius form. This solution is unique if and only if the solution \(Z_{11} = (A_{11})_k\) is the unique solution to the problem \(\operatorname{min} ||A_{11} - Z_{11}||_F\). This happens if either \(k \geq \text{rank } A_{11}\) or \(1 \leq k < \text{rank } A_{11}\) and \(\sigma_k(A_{11}) > \sigma_{k+1}(A_{11})\). A straightforward calculation shows that \(\hat{X} = \Sigma_B^T(P_{C,R}^{-1}AP_{C,R})k\Sigma_C^T\). Thus, a solution of (2.2) with the minimal Frobenius norm is given by
\[
X = B^T U_B P_{B,R}^T V_C C^T (P_{C,R}^{-1}AP_{C,R})_k C^T.
\]

This solution is unique if and only if either \(k \geq \text{rank } P_{B,R}AP_{C,R}\) or \(1 \leq k < \text{rank } P_{B,R}AP_{C,R}\) and \(\sigma_k(P_{B,R}AP_{C,R}) > \sigma_{k+1}(P_{B,R}AP_{C,R})\).

A special case of the minimal problem (2.2), where \(X\) is a rank one matrix and \(C\) the identity matrix, was considered by Michael Elad [3] in the context of image processing.

### 3 Examples

First observe that the classical approximation problem given by (1.2) is equivalent to the case \(m = p, n = q, B = I_m, C = I_n\). (Here, \(I_m\) is the \(m \times m\) identity matrix.) Clearly \(P_{m,L} = I_m, P_{n,R} = I_n, I_m^T = I_m, I_n^T = I_n\). In this case we obtain the classical solution \(B^T(P_{B,R}AP_{C,R})_k C^T = A_k\).

Second, if \(p = m, q = n\) and \(B, C\) are non-singular, then \(\text{rank } (BXC) = \text{rank } X\). In this case, \(P_{B,L} = I_m\) and \(P_{C,R} = I_n\), and the solution to (2.2) is given by
\[
X = B^{-1}A_k C^{-1}.
\]

Next, a particular case of the problem (2.2) occurs in study of a random vector estimation (see, for example, [9, 6]) as follows. Let \((\Omega, \Sigma, \mu)\) be a probability space, where \(\Omega\) is the set of outcomes, \(\Sigma\) a \(\sigma\)-field of measurable subsets \(\Delta \subset \Omega\) and \(\mu : \Sigma \mapsto [0, 1]\) an associated probability measure on \(\Sigma\) with \(\mu(\Omega) = 1\). Suppose that \(\mathbf{x} \in L^2(\Omega, \mathbb{R}^m)\) and \(\mathbf{y} \in L^2(\Omega, \mathbb{R}^n)\) are random vectors such that \(\mathbf{x} = (x_1, \ldots, x_m)^T\) and \(\mathbf{y} = (y_1, \ldots, y_n)^T\) with \(x_i, y_j \in L^2(\Omega, \mathbb{R})\) for \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\), respectively. Let \(E_{xy} = [e_{ij,xy}]_{1 \times n} \in \mathbb{R}^{m \times n}, E_{yy} = [e_{jk,yy}]_{n \times n} \in \mathbb{R}^{n \times n}\) be correlation matrices with entries
\[
e_{ij,xy} = \int_{\Omega} x_i(\omega)y_j(\omega)d\mu(\omega), \quad e_{jk,yy} = \int_{\Omega} y_j(\omega)y_k(\omega)d\mu(\omega),
\]
for \(i = 1, \ldots, m, \quad j, k = 1, \ldots, n, \quad \omega \in \Omega.\)
The problems considered in [9, 6] are reduced to finding a solution to the problem
(2.2) with \( A = E_{xy}E_{yy}^{1/2} \), \( B = I_n \) and \( C = E_{yy}^{1/2} \) where we write \( E_{yy}^{1/2} = (E_{yy}^{1/2})^\dagger \).
Let the SVD of \( E_{yy}^{1/2} \) be given by \( E_{yy}^{1/2} = V_n \Sigma V_n^* \) and let rank \( E_{yy}^{1/2} = r \). Here, \( V_n = [v_1, \ldots, v_n] \) with \( v_i \) the \( i \)-th column of \( V_n \). By Theorem 2.1, the solution
to this particular case of the problem (2.2) having the minimal Frobenius norm is
given by \( X = (E_{xy}E_{yy}^{1/2}V_r V_r^*)_k E_{yy}^{1/2} \), where \( E_{yy}^{1/2} V_r V_r^* = E_{yy}^{1/2} \). Therefore, \( X = (E_{xy} E_{yy}^{1/2})_k E_{yy}^{1/2} \). The conditions for the uniqueness follow directly from Theorem
2.1.

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