THE AVERAGE SIGNATURE OF GRAPH LINKS

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ABSTRACT. We compute the average Tristram—Levine signature of any graph link with positive weights in a three sphere, generalizing the results of Kirby and Melvin. The main tools are the Neumann’s algorithm for computing the equivariant signatures of graph links and the Reciprocity Law for Dedekind sums.

1. Graph links and their signatures

1.1. Introduction. Let \( L \subset S^3 \) be a link. If \( S \) is a Seifert matrix for \( L \), the Tristram–Levine signature of \( L \) is a piecewise constant function from the unit circle in \( \mathbb{C} \) to \( \mathbb{Z} \) given by

\[
\sigma_L(t) = \text{signature of the hermitian form } (1 - t)S + (1 - \overline{t})S^T.
\]

The Tristram–Levine signature does not depend on the choice of the Seifert matrix. We consider also the average signature of \( L \) defined as

\[
\overline{\sigma}(L) = \int_0^1 \sigma_L(e^{2\pi ix}) \, dx.
\]

In this paper we provide an algorithm of computing \( \overline{\sigma}(L) \) if \( L \) is an arbitrary graph link in \( S^3 \) (see [12] for definition of the graph link) in terms of the underlying graph.

The motivation to study \( \overline{\sigma}(L) \) is the following.

• If \( K \) is a knot, then \( \overline{\sigma}(K) \) is equal to the \( \rho \)-invariant associated with the representation \( \pi_1(S^3 \setminus K) \to \mathbb{Z} \) given by the abelianization, compare [7,8]. There are many applications of \( \rho \)-invariants in obstructing sliceness or studying the structure of the topological concordance group, see [9,6,10] and many, many others.

• The average signature is in general hard to compute. For a knot \( K \), the Tristram–Levine signature has discontinuities at the roots of the Alexander polynomial. Computing the average signature usually involves finding roots of the Alexander polynomial on the unit circle, a task which can be highly non-trivial by Galois theory. If \( K \) is a torus knot, the computations can be done and the result is known, see [13,17,11,19].

• In [6,10,11] the average signature of links with pairwise linking number 0 was studied to detect sliceness of some knots.

• The main motivation comes from singularity theory. In [11] the average signature of an algebraic knot was related to an invariant of the singular point, called the M-number.
Using this relation, a bound for $M$-numbers under a deformation of cuspidal singular points was obtained [2]. We expect a similar relation to hold for general, that is not necessary cuspidal, algebraic links. This would allow to extend results from [2] to more general classes of deformations. Computing the average signature is the first step towards establishing such a relation. As algebraic links in $S^3$ are all graph links by [12], this first step is done in the present paper.

The structure of the article is the following. After an overview of the necessary background on graph links and Dedekind sums, we provide in Theorem 2.3.4 an algorithm for computing the average signature of any splice component, see (2.2.7). Then we use splice additivity of signatures, Lemma 2.2.8, to obtain a general formula for the average signature. Unfortunately, the result involves many Dedekind sums.

And then we start using the Reciprocity Law. An inductive argument, given in Section 3, allows us to simplify Dedekind sums and after somewhat lengthy, but rather straightforward computations, we obtain a desired result.

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1.2. Review of the theory of graph links. To fix the notation we recall the terminology of splice links. We refer to [12] or [18] for a more detailed exposition.

Throughout the paper a graph $\Gamma$ will denote a collection $(\mathcal{V}, \mathcal{A}, \mathcal{E})$ of the ordinary vertices $\mathcal{V}$, the arrowhead vertices $\mathcal{A}$ and the edges $\mathcal{E}$. For an ordinary vertex $v \in \mathcal{V}$, we denote by $\nu(v)$ its valency, that is the number of incident edges. Vertices with valency 3 or more are called nodes, those with valency 1 are called leaves. We assume that there are no vertices of valency 2 and that there is at least one node. The valency of an arrowhead vertex is always 1. We will assume that $\Gamma$ is a tree, that is, it is connected and has no loops.

The graph $\Gamma$ is also assumed to have the following labelling by non-negative integers: each arrowhead vertex $a \in \mathcal{A}$ is labelled by an integer $m_a$ called the multiplicity. On each edge $e \in \mathcal{E}$ connecting two vertices $v, w \in \mathcal{A} \cup \mathcal{V}$ there are two positive integer weights $d_{ve}$ and $d_{we}$, the first one near $v$ (it is called the weight of $e$ adjacent to $v$), the other one near $w$. For any two edges $e$ and $e'$ incident to the same node $v$, the weights $d_{ve}$ and $d_{we'}$ are assumed to be coprime. A weight adjacent to an arrowhead or a leaf is always equal to 1 (usually it is omitted when one draws a graph). An example of a graph is presented on Figure 1. Another one is on Figure 4.

Definition 1.2.1. A tree decorated as above is called a splice graph or an Eisenbud–Neumann graph.

For a node $v$ we denote by $d_v$ the product of all weights $d_{ve}$ over all edges $e$ incident to $v$. If $v$ is an arrowhead or a leaf, we define $d_v$ in a different way. Let $e$ be the unique vertex incident to $v$ and let $w$ be its other end. It is necessarily a node. Then $w$ is called the nearest node to $v$, we denote it by $v^\#$, and $d_{we}$ is the nearest weight to $v$. we shall denote it by $d_v^\#$. The weight of $v$ is defined as $d_v := d_w/d_{we}^2$. Note, that $d_v$ is not necessarily an integer.
For any two vertices $v, w \in A \cup V$ we define the *linking number* $\text{lk}(v, w)$ as the product of all the weights adjacent to, but not lying on, the shortest path connecting $v$ with $w$ in $\Gamma$, see [12, page 84]. If $v$ is a node, we define its *multiplicities* $m_v$ as the sum

$$m_v = \sum_{a \in A} \text{lk}(a, v).$$

For example, for the graph in Figure 1 and the vertex $v_6$ the multiplicity is $4 \cdot 3 + 4 \cdot 3 + 4 \cdot 2 = 32$.

In [12] it is shown that each graph such that the multiplicity of each arrowhead is 1, gives rise to a link in a graph 3-manifold; the arrowheads correspond to components of the link. If the multiplicity of some arrowheads are not all equal to 1, we speak about a *multilink*, see also [12].

There is a notion of a Seifert surface and a Seifert matrix for a multilink, the Tristram–Levine signature for multilinks is defined by (1.1.1) and the average signature is as in (1.1.2).

We consider only links in $S^3$, therefore we add a following assumption on the graph.

**Assumption 1.2.2.** *For any node at most 2 adjacent weights are different than 1.*

We point out that it is technically possible to give a formula for the average signature in the general case, but one encounters additional problems in sections 3.3 and 3.4.

### 1.3. Splicing and splice components.

There is one important procedure, namely the splicing of two graphs. It is easier to describe the inverse operation, which consists on cutting an edge into two halves and changing them into arrowheads as on Figure 2.

**Figure 2.** The graph $\Gamma$ on the left is a result of splicing of $\Gamma_1$ and $\Gamma_2$. 
Here $m_1$ and $m_2$ are the multiplicities of the newly appeared arrowheads. They are uniquely determined by the condition that if we cut the graph, the multiplicities of all nodes and leaves inside $\Gamma_1$ and $\Gamma_2$ are preserved. Splicing is the reverse procedure, it consists on taking two arrowheads of the two graphs and joining them to form an edge connecting two graphs. In general, it is impossible to splice two graphs without some conditions on the multiplicities of the arrowhead vertices $m_1$ and $m_2$. The whole procedure is described in details in [12] pages 20–33 or in [18] Section 9.

Given a graph $\Gamma$, we can decompose it as a union of so-called splice components, where each splice component contains exactly one node. A splice component is presented on Figure 3.

![Figure 3. A splice component.]

2. Signatures of a graph link

2.1. Dedekind sums. We begin with recalling the definition of the sawtooth function:

\[
\langle x \rangle = \begin{cases} 
\{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\
0 & x \in \mathbb{Z},
\end{cases}
\]

where $\{x\}$ denotes the fractional part. Given the above notation we introduce the Dedekind sums.

**Definition 2.1.1.** For two numbers $p, q$ such that $q > 0$ we define the Dedekind sum as

\[
s(p, q) = \sum_{j=0}^{q-1} \langle \frac{j}{q} \rangle \langle \frac{pj}{q} \rangle.
\]

We have several well known facts.

\[
s(ap, aq) = s(p, q) \quad \text{for any } a \in \mathbb{Z}_{>0}
\]

\[
s(p', q) = s(p, q) \quad \text{if } pp' = 1 \mod q
\]

\[
s(-p, q) = -s(p, q)
\]

\[
s(p + aq, q) = s(p, q) \quad \text{for any } a \in \mathbb{Z}.
\]

The most important relation we shall use is the Reciprocity Law. We refer to [23] for an excellent survey.
**Proposition 2.1.3** (Reciprocity Law). If $p, q$ are coprime, then

\[ s(p, q) + s(q, p) = \frac{1}{12} \left( \frac{p}{q} + \frac{q}{p} + \frac{1}{pq} - 3 \right). \]

Motivated by the above result we define

\[ R(p, q) = \frac{p}{q} + \frac{q}{p} + \frac{\gcd(p, q)^2}{pq} - 3. \]

Then for all $p, q > 0$, (2.1.4) translates into

\[ s(p, q) + s(q, p) = \frac{R(p, q)}{12}. \]

We will also use the following generalization of (2.1.4).

**Proposition 2.1.7** ([22, Theorem 7]). If $p, q, u, v$ are positive integers such that $\gcd(p, q) = \gcd(u, v) = 1$ and $p', q'$ are such that $pp' + qq' = 1$, then

\[ s(p, q) + s(u, v) = s(p'u - q'v, t) - \frac{1}{4} + \frac{1}{12} \left( \frac{q}{qt} + \frac{v}{qt} + \frac{t}{qt} \right), \]

where $t = pv + qu$.

### 2.2. Formulae for signatures involving Dedekind sums.

Let $\Gamma$ be a splice graph and $L = L_\Gamma$ the corresponding graph (multi)link. In this subsection we shall show how to compute the average signature of $L$ from the graph $\Gamma$. We write $\sigma(\Gamma)$ for $\sigma(L_\Gamma)$.

Let for $\lambda \in S^1$, the quantity $\sigma^-_\lambda$ denote the equivariant signature of $L$, as defined for instance in [19].

**Lemma 2.2.1** (see [20, Theorem 5.3]). The signatures $\sigma^-_\lambda$ are additive under splicing. For a splice component as in Figure 3 let us define $m_j = 0$ for $j = k + 1, \ldots, n$. Let $\beta_j$ be chosen so that

\[ \beta_j \alpha_1 \ldots \widehat{\alpha_j} \ldots \alpha_n \equiv 1 \pmod{\alpha_j}. \]

Such $\beta_j$ exist because $\alpha_1, \ldots, \alpha_n$ are pairwise coprime. The multiplicity of the central vertex is equal to

\[ m = \sum_{j=1}^{n} \alpha_1 \ldots \widehat{\alpha_j} \ldots \alpha_n m_j. \]

Finally, let $s_j = (m_j - \beta_j m)/\alpha_j$ (it is easy to see that $s_j \in \mathbb{Z}$). Then if $\lambda = e^{2\pi ip/q}$ with $p, q$ coprime, then

\[ \sigma^-_\lambda = \begin{cases} 0 & \text{if } q \text{ does not divide } m \\ 2 \sum_{j=1}^{n} s_j \frac{p}{q} & \text{if } q \text{ divides } m. \end{cases} \]

We have a well-known lemma, for convenience of the reader we present a sketch of proof.
Lemma 2.2.3. Let \( L \) be a graph link and \( \Gamma \) the underlying graph. If \( \zeta = e^{2\pi i x} \) is not a root of the Alexander polynomial of \( L \), then the Tristram–Levine signature of \( L \) is related to the equivariant signatures by the following equation.

\[
\sigma(\zeta) = -\sum_{y>x} \sigma_{e^{2\pi iy}} - \sum_{y<x} \sigma_{e^{2\pi iy}} + 1 - \#\Gamma,
\]

where \( \#\Gamma \) is the number of number of arrowheads of the graph (that is the number of components of \( L \)).

**Proof.** The jumps of the Tristram–Levine signatures are given by the equivariant signatures, compare [14]. This implies that

\[
\sigma(\zeta) = -\sum_{y>x} \sigma_{e^{2\pi iy}} - \sum_{y<x} \sigma_{e^{2\pi iy}} + \lim_{\zeta \rightarrow 1} \sigma(\zeta).
\]

It remains to show that \( \lim_{\zeta \rightarrow 1} \sigma(\zeta) = 1 - \#\Gamma \).

This appears to be a folklore result. Since we did not find a good reference in the literature, we present a sketch of a proof, referring to [12, 19, 15, 3] for some auxiliary results.

The restriction that the weights are positive implies by [12, Theorem 11.2] that \( L \) is fibered. Let \( \Sigma \) be a fiber and let \( S \) be the corresponding Seifert matrix. It is non-degenerate. A Jordan block decomposition of \( S^{-1}ST \) gives a decomposition \( H_1(\Sigma; \mathbb{R}) = U_{\neq 1} \oplus U_{=1} \) such that \( S \) has a block structure with respect to this decomposition, \( S = S_{\neq 1} \oplus S_{=1} \) and \( S_{\neq 1}^{-1}S_{\neq 1} \) has eigenvalues different than 1 and \( S_{=1}^{-1}S_{=1} \) has eigenvalues equal to 1. By [12, Corollary 11.5] \( S_{=1}^{-1}S_{=1}^{T} \) is actually the identity matrix.

The proof splits now into two parts. The first part is that for \( \zeta \) sufficiently close to 1 the signature \( \sigma(\zeta) \) is equal to the signature of the hermitian form \( (1 - \zeta)S_{=1} + (1 - \bar{\zeta})S_{=1}^{T} \).

This follows for example from [3, Proposition 4.14] combined with the symmetry property of H-numbers [3, Lemma 3.4(a)].

The second part is to show that the signature of the above hermitian form is \( 1 - \#\Gamma \).

Since \( S - ST = S(1 - S^{-1}ST) \), the intersection form on \( U_{\neq 1} \) is non-degenerate and on \( U_{=1} \) it is zero. But this means that \( U_{=1} \) is the image of \( H_1(\partial \Sigma) \) in \( H_1(\Sigma) \), in other words \( U_{=1} \) is spanned by components \( L_1, \ldots, L_n \) of \( L \), subject to the relation \( L_1 + \ldots + L_n = 0 \). It follows that \( S_{=1} \) can be viewed as the linking matrix of \( L_1, \ldots, L_{n-1} \), where we define \( \text{lk}(L_j, L_i) \) as \( \text{lk}(L_j, L_1 + \ldots + L_{j-1} + L_{j+1} + L_n) \), compare [19, page 321]. In particular \( S_{=1} \) is symmetric and again by [19, page 321] \( (1 - \zeta)S_{=1} + (1 - \bar{\zeta})S_{=1}^{T} \) is equal to \( -(\#\Gamma - 1) \) as desired. \( \square \)

*Remark 2.2.5.* If \( L \) is a graph multilink such that each arrowhead has non-negative multiplicities, then [22, 23] still holds with the exception that \( \#\Gamma \) should be understood as the number of arrowhead vertices with non-zero multiplicities. The argument is slightly more complicated since the Seifert matrix in general has to be decomposed into \( S_0 \oplus S_{\neq 1} \oplus S_{=1} \), where \( S_0 \) is a zero matrix and \( S_{\neq 1} \), \( S_{=1} \) are as above. An argument very similar to the one in [4, Section 3] shows that if the multilink has components \( L_1, \ldots, L_n \) with multiplicities \( m_1, \ldots, m_n \) and for some \( k, m_j \geq 1 \) if \( j \leq k \) and \( m_j = 0 \) if \( j > k \), then \( S_0 \) has size \( (m_1 - 1) + \ldots + (m_k - 1) \) and \( S_{=1} \) has size \( k - 1 \) and is symmetric negative definite.
It follows from Lemma 2.2.3 and Lemma 2.2.1 that for a splice component as above, the Tristram–Levine signature is given by the following formula:

\begin{equation}
\sigma(e^{2\pi i x}) = 1 - \#\Gamma + \sum_{j=1}^{n} \sum_{i=1}^{m} \langle i s \rangle \delta_{i},
\end{equation}

where we use notation from Lemma 2.2.1 and \( \delta_{i} = +1 \) if \( x > i/m \) and \( -1 \) if \( x < i/m \). The formula (2.2.6) holds as long as \( mx \) is not an integer.

Integrating (2.2.6) over the interval \([0,1]\) yields the following result

\begin{equation}
\sigma(\Gamma) = 1 - \#\Gamma - 4 \sum_{j=1}^{n} s(s_{j}, m),
\end{equation}

which is due, essentially to Némethi [16, Corollary 4.2].

The equivariant signatures are splice invariant by [20, Theorem 5.3]. If we splice two graphs \( \Gamma_{1} \) and \( \Gamma_{2} \) to a graph \( \Gamma \), and either \( m_{1} \) or \( m_{2} \) is zero (notation as in Figure 2), then \( \#\Gamma = \#\Gamma_{1} + \#\Gamma_{2} \), compare Remark 2.2.5. If \( m_{1}m_{2} \neq 0 \), then \( \#\Gamma = \#\Gamma_{1} + \#\Gamma_{2} - 1 \). This gives the following fact, which we state now for future reference.

**Lemma 2.2.8** (Splice additivity of signatures). Let \( \Gamma \) be a graph, let us cut it into two components \( \Gamma_{1} \) and \( \Gamma_{2} \). Then \( \sigma(\Gamma) = \sigma(\Gamma_{1}) + \sigma(\Gamma_{2}) + \eta \), where \( \eta = 1 \) if both newly appeared arrowheads have non-zero multiplicities, otherwise \( \eta = 0 \).

### 2.3. The function \( S(\Gamma) \)

We shall define the function \( S(\Gamma) \), which we shall use to compute the average signature of the underlying link.

**Definition 2.3.1.** For a graph \( \Gamma \) we define

\[
S(\Gamma) = S_{\text{link}}(\Gamma) + S_{\text{node}}(\Gamma) + S_{\text{leaf}}(\Gamma) + S_{\text{edge}}(\Gamma) + S_{\text{arr}}(\Gamma),
\]

where

- **(linking) \( S_{\text{link}} \)** is twice the total linking number, that is
  \[
  S_{\text{link}} = \sum_{a,a' \in A, a \neq a'} \text{lk}(a,a').
  \]

- **(nodes) \( S_{\text{node}} \)** is the contribution of nodes
  \[
  S_{\text{node}} = \sum_{v \in V: \nu(v) > 2} d_{v}(\nu(v) - 2).
  \]

- **(leaves) \( S_{\text{leaf}} \)** comes from leaves of \( \Gamma \)
  \[
  S_{\text{leaf}} = \sum_{v \in V: \nu(v) = 1} -d_{v}.
  \]

- **(edges) \( S_{\text{edge}} \)** is a sum of contributions of those edges that connect nodes. Let \( e \) connects nodes \( v \) and \( w \) with multiplicities \( m_{v} \) and \( m_{w} \). Suppose that upon cutting
Γ along $e$, the edge $e$ becomes two arrowheads with multiplicities $\mu_v$ and $\mu_w$. Set $c = \gcd(\mu_v, \mu_w)$. The contribution of the edge $e$ to $S_{\text{edge}}$ is equal to

$$
\begin{cases}
    c^2 \left( \frac{d_v}{\mu_v m_v} + \frac{d_w}{\mu_w m_w} - \frac{1}{\mu_v \mu_w} \right) & \text{if } \mu_v \mu_w \neq 0 \\
    \frac{1}{d_v} - \frac{d_v}{d_v^2} & \text{if } \mu_w = 0 \\
    \frac{1}{d_w} - \frac{d_w}{d_w^2} & \text{if } \mu_v = 0.
\end{cases}
$$

- **(arrowheads)** $S_{\text{arr}}$ is a contribution of arrowheads.

$$
S_{\text{arr}} = \sum_{a \in A} \frac{1}{m^a_#},
$$

where we recall that $a^#$ is the nearest node to the arrowhead $a$.

**Remark 2.3.2.**
- The formulae for $S_{\text{node}}$ and $S_{\text{leaf}}$ look similarly and one could combine these two contributions in one term. However, the values of $d_v$ for $v$ a node and $v$ a leaf are different and computed in a different way.
- There is a similarity between the function $S(\Gamma)$ and the function $W(\Gamma)$ defined in [5, Section 4]. The difference is equal to $S_{\text{link}} + S_{\text{edge}} + S_{\text{arr}}$ and $S_{\text{link}}$ has a clear topological meaning. We expect that the quantity $S_{\text{edge}} + S_{\text{arr}}$ is small if $\Gamma$ is a graph of a link of a singularity. We know it is between 0 and 2/9, if $\Gamma$ is a link of a unibranched singular point, see [1].

**Example 2.3.3.** Let us consider the link on Figure 4 (it is taken from [12, page 147], only we changed the multiplicity of one arrowhead vertex from 2 to 1). Both nodes of the graph are non-free. The quantity $S(\Gamma)$ is computed as follows.

- $S_{\text{link}} = 2 \cdot (2 \cdot 2 \cdot 3) = 24$. The 2 in front comes from the fact that we compute the linking for each pair of arrowheads twice.

- $S_{\text{node}} = 26 \cdot (3 - 2) + 6 \cdot (4 - 2) = 38$.

- $S_{\text{leaf}} = -\frac{6}{32} - \frac{6}{22} - \frac{132}{22} = -\frac{2}{3} - \frac{3}{13} - \frac{13}{2}$.

- To compute $S_{\text{edge}}$ we observe that there is one edge connecting nodes. The multiplicities of the nodes are $M_v = 38$ and $M_w = 18$, upon cutting the edge, the multiplicities of the two arrowheads are $m_v = 6$ and $m_w = 2$, hence $c = 2$. We get

$$
S_{\text{edge}} = 4 \left( \frac{13}{38 \cdot 6} + \frac{1}{18 \cdot 2} - \frac{1}{6 \cdot 2} \right) = \frac{1}{171}.
$$

- $S_{\text{arr}}$ is readily computed to be $\frac{1}{38} + \frac{1}{18} = \frac{14}{171}$.

We see that $S(\Gamma) = \frac{1015}{19}$.

Now we can state the main result of this article.

**Theorem 2.3.4.** If $L$ is a graph link in $S^3$ other than the unknot or the Hopf link and $\Gamma$ is the graph representing it, then

$$
\overline{\sigma}(L) = -\frac{1}{3} S(\Gamma).
$$

The proof of Theorem 2.3.4 is given in Section 3 now let us provide some examples.
Example 2.3.5. Assume that $\Gamma$ has one arrowhead vertex (so it represents a knot), see Figure 5. With the notation from Figure 5 we have

$S_{\text{link}} = 0$

$S_{\text{node}} = \sum_{j=1}^{n} p_j q_j$

$S_{\text{leaf}} = -\frac{p_1}{q_1} - \sum_{j=1}^{n} \frac{q_j}{p_j}$

$S_{\text{edge}} = \sum_{j=1}^{n-1} \frac{1}{p_j q_j} - \frac{p_2}{q_2} - \ldots - \frac{p_n}{q_n}$

$S_{\text{arr}} = \frac{1}{p_n q_n}$

Adding this up we obtain

$S(\Gamma) = \sum (p_j - 1/p_j)(q_j - 1/q_j)$.

It is known, see for example [1], that $\sigma = -\frac{1}{3} S(\Gamma)$ in this case.
Example 2.3.6. For the link from Example 2.3.3 we can compute the signature directly using (2.2.7). The vertex on the left contributes to

\[-1 - 4(s(-19, 38) + s(-20, 38) + s(1, 38)) = -1 - 4 \cdot \frac{45}{19}.
\]

The vertex of the right contributes to

\[-1 - 4(s(1, 18) + s(-9, 18) + s(2, 18) + s(-12, 18)) = -1 - 4 \cdot \frac{11}{6}.
\]

Hence the average signature is equal to \(-\frac{1015}{57}\).

We point out that Theorem 2.3.4 does not hold for general multilinks. As formula (3.1.3) suggests, for general multilinks one cannot avoid Dedekind sums.

3. Proof of Theorem 2.3.4

3.1. Some terminology used in the proof. We begin with the following definition, which is given to make precise notions, which are intuitively obvious.

Definition 3.1.1. A path in a splice graph \(\Gamma\) is a collection of nodes \(v_1, \ldots, v_k \in V\) and edges \(e_1, \ldots, e_{k-1} \in E\) such that \(e_j\) connects \(v_j\) to \(v_{j+1}\) and all the nodes \(v_1, \ldots, v_k\) are distinct. The length of a path is the number of nodes occurring on the path. The diameter of a graph \(\Gamma\), denoted \(l(\Gamma)\), is the maximum of length over all paths on \(\Gamma\).

A graph with diameter 1 has exactly one node. A graph with diameter 2 has exactly two nodes. However, a graph with diameter 3 can have arbitrary many nodes. For instance, a graph in Figure 7 has length 3.

Definition 3.1.2. For any positive integers \(a, b\), the elementary graph \(\Gamma(a, b)\) is a graph as on Figure 6.

Applying (2.2.7) to \(\Gamma(a, b)\) we obtain a formula, which we will use several times in the future.

\[(3.1.3) \quad \sigma(\Gamma(a, b)) = -1 - \frac{1}{3} (R(1, a(b + 1)) + R(b, (b + 1)a) - R(1, a) - 12s(a, b)).
\]

We shall use elementary graphs to transform a graph of a multilink into a graph of a link in a controlled way.
Definition 3.1.4. Let \( \Gamma \) be a graph with one arrowhead of multiplicity \( m \neq 1 \) and the multiplicity of any other arrowhead is equal to 1. The completion of \( \Gamma \) is a graph \( \bar{\Gamma} \) obtained by

- If \( m > 1 \): splicing a graph \( \Gamma(m, b) \) to \( \Gamma \) along the arrowhead with multiplicity \( m \), where \( b \) is a unique positive integer for which this is possible.
- If \( m = 0 \): replacing the arrowhead with multiplicity 1 by a leaf.

We shall need the last notion.

Definition 3.1.5. A graph \( \Gamma \) is simple if all but possibly one of its splice components are elementary.

3.2. Proof of Theorem 2.3.4 up to a few lemmas. We begin with the following lemma.

Lemma 3.2.1. Theorem 2.3.4 holds for all simple graphs of diameter 3, for which all the outer nodes (that is those that are adjacent to one node) are elementary.

Proof. Let \( \Gamma \) be a simple graph of diameter 3 such that all the outer nodes are elementary. Then the only non-elementary node can be the central one. Since the graph represents a link in \( S^3 \), at most two adjacent weights to the central node might be different than 1, we denote them \( p \) and \( q \) (this also covers the case that \( p = 1 \) or \( q = 1 \)). According to whether the edge of the central node is adjacent to a leaf or not, we have three cases depicted on Figures 7, 8 and 9. We prove Theorem 2.3.4 by a direct computation: in Section 3.3 we compute it for graphs from Figure 7. In Section 3.4 we show, how do the computations change in the case of graphs from Figure 8 and Figure 9.

The main argument for passing from the special cases of low diameter to the general case, and in fact the gist of the proof of Theorem 2.3.4 is given by the following proposition.

Proposition 3.2.2. Let \( \Gamma \) be a graph of a link and \( e \) an edge connecting two nodes. Let \( \Gamma_1 \) and \( \Gamma_2 \) be the graphs resulting with cutting \( \Gamma \) along \( e \) as in Figure 2. Let \( \bar{\Gamma}_1 \) and \( \bar{\Gamma}_2 \) be the completions along the arrowheads that appear as a result of cutting \( e \).

If Theorem 2.3.4 holds for \( \bar{\Gamma}_1 \) and \( \bar{\Gamma}_2 \), then it holds for \( \Gamma \).

Proposition 3.2.2 is proved in Sections 3.5 and 3.6.

Corollary 3.2.3. Theorem 2.3.4 holds for all graphs of length 1 and for all simple graphs of length 2.

Proof. We could achieve the result by direct computations, we shall present an argument using Lemma 3.2.1 and Proposition 3.2.2.

Let \( \Gamma \) be a graph with a single node. If it has exactly one arrowhead, it is a graph of a knot and the statement follows by Example 2.3.5. Assume that \( \Gamma \) has more than one arrowheads. We splice \( \Gamma \) with two elementary diagrams and obtain a simple graph of length 3, so Lemma 3.2.1 applies.

If \( \Gamma \) has length 2, it has two nodes \( v_1 \) and \( v_2 \). Suppose that the splice component corresponding to \( v_2 \) is elementary. If there are no arrowhead vertices adjacent to \( v_1 \), it follows that \( \Gamma \) is a splice diagram of a knot, and the statement follows. If there is an arrowhead vertex, we splice to it an elementary graph. We obtain a simple graph of length 3 such the outer nodes are elementary, thus we can use Lemma 3.2.1 to conclude the proof in this case.
Conclusion of a proof of Theorem 2.3.4

Consider a graph \( \Gamma \) of diameter \( l \) and suppose there are \( k \) different paths of length \( l \) on \( \Gamma \). Suppose \( l > 3 \) and let us choose a path consisting on edges \( e_1, \ldots, e_{l-1} \). We cut \( \Gamma \) along \( e_2 \) and complete the resulting graphs to \( \bar{\Gamma}_1 \) and \( \bar{\Gamma}_2 \).

It is clear from the construction that for \( j = 1, 2 \), \( \bar{\Gamma}_j \) either has smaller diameter than \( \Gamma \); or it has the same diameter, but lower number of different paths of length \( l \). An inductive step lets us reduce the proof of Theorem 2.3.4 to the case of graphs of diameter 3 and less.

Let us consider a graph of diameter 2. If it has two non-elementary nodes, we cut \( \Gamma \) along the edge connecting the two nodes. The resulting completed graphs \( \bar{\Gamma}_1 \) and \( \bar{\Gamma}_2 \) both have diameter 2 and are simple, therefore Corollary 3.2.3 applies.

Let us now consider a graph \( \Gamma \) of diameter 3. If an outer edge \( v \) of \( \Gamma \) is not elementary, we cut \( \Gamma \) along the unique edge \( e \) connecting \( v \) to another node in \( \Gamma \). We complete the two graphs to \( \bar{\Gamma}_1 \) that contains \( v \), and \( \bar{\Gamma}_2 \). Then \( \bar{\Gamma}_1 \) has two nodes and is simple. If \( \bar{\Gamma}_2 \) has diameter 2, we conclude the proof. In general \( \bar{\Gamma}_2 \) still might have diameter 3, yet it has one less non-elementary outer node. We repeat the procedure for \( \bar{\Gamma}_2 \) until we arrive to the case, when \( \bar{\Gamma}_2 \) has no non-elementary outer nodes and we conclude by Lemma 3.2.1.

3.3. Theorem 2.3.4 for a graph on Figure 7

Let us begin with setting up a notational convention for Sections 3.3 and 3.4.

In Section 3.3 \( \Gamma \) denotes a splice graph as presented in Figure 4, in Section 3.4 \( \Gamma \) is the graph in Figure 8 or in Figure 9. The splice component of \( \Gamma \) will be denoted by subscripts. The central splice components will be \( \Gamma_{cen} \) and the components with multiplicities \( m_1, \ldots, m_k \) shall be denoted by \( \Gamma_1, \ldots, \Gamma_k \). The splice component adjacent to the edge with weight \( p \) shall be denoted \( \Gamma_p \) (it is absent in Figure 7), and the component adjacent to the edge with weight \( q \) shall be denoted \( \Gamma_q \); the latter one is present only in Figure 9. To distinguish the last two components from \( \Gamma_j \) for \( j = p \) or \( j = q \), we use roman fonts as subscripts.

The components \( \Gamma_j, j = 1, \ldots, k, \Gamma_p \) and \( \Gamma_q \) are all elementary of type \( \Gamma(m_j, M_j), \Gamma(m_p, M_p) \) and \( \Gamma(m_q, M_q) \), where \( M_j, M_p, M_q \) will be computed.
Now we pass to the case of graph on Figure 7. We have
\[ M_j = M - pqm_j, \]
where
\[ M = pq \sum_{j=1}^{k} m_j \]
is the multiplicity of \( \Gamma_{cen} \). Summing (3.1.3) for \( \Gamma(m_1, M_1) \) up to \( \Gamma(m_k, M_k) \) we obtain the following quantity:

\[
(3.3.1) \quad -k - \frac{1}{3} \sum_{j=1}^{k} \left( R(1, m_j (M_j + 1)) + R(M_j, (M_j + 1)m_j) - R(1, m_j) - 12s(m_j, M_j) \right).
\]

To compute \( \sigma(\Gamma_{cen}) \) we use the algorithm from Section 2.2. We have \( \alpha_1 = \cdots = \alpha_k = 1 \), we introduce \( \alpha_p = p \) and \( \alpha_q = q \). Then \( \beta_1 = \cdots = \beta_k = 0 \), \( \beta_p = q' \) and \( \beta_q = p' \), where \( p' \) and \( q' \) are such that \( pp' + qq' = 1 \). Then \( s_j = m_j \) for \( j = 1, \ldots, k \) and \( s_p = -q'M/p, s_q = -p'M/q \). Therefore \( \sigma(\Gamma_{cen}) \) is equal to

\[
(1 - k) - 4 \sum_{j=1}^{k} s(m_j, M) - 4s(-q'M/p, M) - 4s(-p'M/q, M).
\]

The latter formula can be transformed using (2.1.2) to

\[
(3.3.2) \quad (1 - k) - 4 \sum_{j=1}^{k} \left( \frac{1}{12} R(m_j, M) - s(M_j, m_j) \right) + 4s(q', p) + 4s(p', q).
\]

Combining (3.3.1) and (3.3.2) with splice additivity of \( \sigma \) (Lemma 2.2.8), we obtain the following formula, which does not involve Dedekind sums anymore.

\[
\sigma(\Gamma) = 1 - k - \frac{1}{3} \sum_{j=1}^{k} \left( R(1, m_j (M_j + 1)) + R(M_j, (M_j + 1)m_j) + R(m_j, M) - R(m_j, M_j) - R(1, m_j) \right) + \frac{1}{3} R(p, q).
\]

We will now substitute (2.1.5) for \( R \). As the formulae become more involved, to save the space we present a formula for \(-3\sigma(\Gamma)\). Here \( c_j \) denotes \( \gcd(m_j, M_j) = \gcd(m_j, M) \).

\[
\sum_{j=1}^{k} \left( m_j(M_j + 1) + \frac{2}{m_j(M_j + 1)} + m_j + \frac{m_j}{M_j} + \frac{M_j}{m_j(M_j + 1)} + \frac{c_j^2}{M_j(M_j + 1)m_j} + \frac{M_j}{m_j} - \frac{M_j}{m_j} - \frac{c_j^2}{m_jM_j} - \frac{2}{m_j} - m_j \right) - \frac{p}{q} - \frac{q}{p} - \frac{1}{pq}
\]

\[
(3.3.4)
\]
There are some cancellations in the above formula. We have \( \sum \frac{m_j}{M_j} - \frac{1}{pq} = 0 \), \( \sum \frac{M_j}{m_j} - \frac{M}{m_j} = kpq \) and \( \frac{1}{m_j(M_j+1)} + \frac{M_j}{m_j(M_j+1)} - \frac{1}{m_j} = 0 \). We observe that

- \( S_{\text{link}} = \sum m_j M_j \);
- \( S_{\text{node}} = kpq + \sum m_j \);
- \( S_{\text{leaf}} = -\sum \frac{1}{m_j} - \frac{k}{q} - \frac{2}{p} \);
- \( S_{\text{edge}} = \sum c_j^2 \left( \frac{1}{m_j M_j} + \frac{1}{m_j M_j (M_j+1)} - \frac{1}{m_j M_j} \right) \);
- \( S_{\text{arr}} = \sum \frac{1}{m_j (M_j+1)} \).

Hence we obtain

\[
-3\bar{\sigma} = S_{\text{link}} + S_{\text{node}} + S_{\text{leaf}} + S_{\text{edge}} + S_{\text{arr}} = S(\Gamma)
\]

as expected. Theorem 2.3.4 holds in that case.

3.4. Graphs from Figures 8 and 9. We shall now consider the graph from Figure 8. To compute \( \bar{\sigma} \) we will again sum over contributions of all splice components. We shall denote \( M = q m_p + pq \sum M_j \), and \( M_j = M - pq m_j \). Furthermore let \( M_p = q \sum m_j \). Then \( M \) is the multiplicity of the central node and the elementary graphs \( \Gamma_1, \ldots, \Gamma_k, \Gamma_p \) are of type \( \Gamma(m_1, M_1), \ldots, \Gamma(m_k, M_k), \Gamma(m_p, M_p) \) respectively. The value of \( \bar{\sigma} \) of these graphs is given by (3.1.3).

To compute \( \bar{\sigma}(\Gamma_{\text{cen}}) \) we choose \( p' \) and \( q' \) so that \( pp' + qq' = 1 \). Then, by computations in Section 2.2 we obtain

\[
(3.4.1) \quad \bar{\sigma}(\Gamma_{\text{cen}}) = -k - 4 \left( \sum s(m_j, M) + s(-q'M/q, M) + s((m_p - q'M)/p, M) \right).
\]
We recall that \( M = pM_p + qM_p \), so \( (m_p - q'M)/p = p'm_p - q'M_p \). Applying Proposition 2.1.7 we obtain

\[
(3.4.2) \quad s((m_p - q'M)/p, M) = s(p, q) + s(m_p, M_p) + \frac{1}{4} - \frac{1}{12} \left( \frac{q^2 c_p^2}{M_p M} + \frac{M_p}{qM} + \frac{M}{qM_p} \right),
\]

where \( c_p = \gcd(m_p, M_p) \). Adding (3.4.1) to the sum of expressions from (3.1.3) we obtain.

\[
\overline{\sigma} = \frac{1}{3} \sum (R(1, m_j(M_j + 1)) + R(M_j, M_j(M_j + 1)) + R(m_j, M) - R(m_j, M_j) - R(1, m_j)) +
\frac{1}{3} (R(1, m_p(M_p + 1)) + R(M_p, m_p(M_p + 1)) - R(1, m_p)) +
\frac{1}{3} - \frac{1}{3} \left( \frac{q^2 c_p^2}{M_p M} + \frac{M_p}{qM} + \frac{M}{qM_p} \right)
\]

Substituting again (2.1.5) for \( R \) we obtain a formula for \(-3\overline{\sigma}(\Gamma)\), which differs from (3.3.4) only in the last two lines.

\[
(3.4.3) \quad \sum_{j=1}^k \left( m_j(M_j + 1) + \frac{2}{m_j(M_j + 1)} + m_j + \frac{m_j}{M_j} + \frac{M_j}{M_j(M_j + 1)} + \frac{c_j^2}{M_j(M_j + 1)m_j} +
\frac{m_j}{M_j} + \frac{c_j^2}{m_j M_j} - \frac{m_j}{M_j} - \frac{c_j^2}{m_j M_j} - \frac{2}{m_j} - m_j \right) -
\sum m_p(M_p + 1) + \frac{2}{m_p(M_p + 1)} + \frac{m_p}{M_p(M_p + 1)} + \frac{c_p^2}{M_p(M_p + 1)m_p} + \frac{m_p}{M_p} - \frac{2}{m_p} - \frac{q^2 c_p^2}{M_p M} - \frac{M_p}{qM} - \frac{M}{qM_p}
\]

The terms with \( c_j \) and \( c_p \) are equal to

\[
(3.4.4) \quad \sum \left( \frac{c_j^2}{M_j(M_j + 1)m_j} + \frac{c_j^2}{m_j M_j} - \frac{c_j^2}{m_j M_j} \right) + \frac{c_p^2}{M_p(M_p + 1)m_p} - \frac{q^2 c_p^2}{M_p M}
\]

Observe that \( \frac{q^2}{m_p M} - \frac{q^2}{m_p M} = -\frac{q^2}{M_p M} \), so (3.4.4) is actually equal to \( S_{\text{edge}}(\Gamma) \). The terms \( \sum (m_j/M) - (M_p/qM) \) cancel, indeed \( M_p = q \sum m_j \). Then we can also simplify \( \frac{m_p^2}{M_p M} - \frac{qM}{qM_p} = \frac{-q^2}{q} \). Repeating the procedure from Section 3.3 we conclude that \( \overline{\sigma}(\Gamma) = -\frac{1}{3} S(\Gamma) \). We omit straightforward but tedious details.

Essentially the same argument works for the graph on Figure 9. We need to use the trick from (3.4.2) twice, otherwise the proof is essentially the same.

3.5. **Proof of Proposition 3.2.2. First case.** Let \( a \) and \( b \) be the multiplicites of arrowheads corresponding to the edge \( e \) of the \( \Gamma \), which we cut to obtain graphs \( \Gamma_1 \) and \( \Gamma_2 \). We suppose that \( a \) is the multiplicity of the arrowhead belonging to graph \( \Gamma_1 \), and \( b \) the multiplicity of an arrowhead belonging to \( \Gamma_2 \). Let \( v \) and \( w \) be vertices of \( \Gamma \) adjacent to the edge \( e \), such that
$v \in \Gamma_1$ and $w \in \Gamma_2$. We consider two cases. The first one is when neither $a$ nor $b$ are equal to 0. We will deal with the other in Section 3.6.

If $ab \neq 0$, $\tilde{\Gamma}_1$ is obtained from $\Gamma_1$ by splicing it with the graph $\Gamma(a, b)$ and $\tilde{\Gamma}_2$ is obtained from $\Gamma_2$ by splicing with the graph $\Gamma(b, a)$. By splice additivity Lemma 2.2.8 we have:

$$\sigma(\Gamma) + \sigma(\Gamma(a, b)) + \sigma(\Gamma(b, a)) + 1 = \sigma(\tilde{\Gamma}_1) + \sigma(\tilde{\Gamma}_2).$$
We combine (3.1.3) and the analogous expression for $\Gamma(b,a)$ to obtain.

$$-3 - \sigma(\Gamma(a,b)) - 3\sigma(\Gamma(b,a)) =$$

$$= R(1, a(b+1)) + R(1, b(a+1)) + R(a, (a+1)b) + R(b, (b+1)a) -$$

$$- R(a, b) - R(1, a) - R(1, b) + 3 =$$

$$= 2ab + a + b - \frac{1}{a} - \frac{1}{b} +$$

$$\frac{c^2}{ab(a+1)} + \frac{c^2}{ab(b+1)} - \frac{c^2}{ab} + \frac{1}{a(b+1)} + \frac{1}{b(a+1)},$$

where $c = \gcd(a, b)$.

If Theorem 2.3.4 holds for $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, then it holds for $\tilde{\Gamma}$ if and only if the last complicated expression is equal to $-S(\Gamma) + S(\tilde{\Gamma}_1) + S(\tilde{\Gamma}_2)$. Let us compare contributions to $S$. We shall need the following result.

**Lemma 3.5.2.** We have $S_{\text{link}}(\Gamma) - S_{\text{link}}(\tilde{\Gamma}_1) - S_{\text{link}}(\tilde{\Gamma}_2) = -2ab$.

**Proof.** Let $a_{\text{new}}^1$ and $a_{\text{new}}^2$ be the arrowheads of $\Gamma_1$, respectively of $\Gamma_2$, which appear as a result of cutting $\Gamma$ along the edge $e$. Let $a_1^j, \ldots, a_k^j$ be the other arrowheads of $\Gamma_j$, $j = 1, 2$. These arrowheads might be regarded also as arrowheads lying on $\Gamma$.

By the arguments in [12, page 28] we have

$$a = \sum_{t=1}^{k_2} \text{l}_{\Gamma_2}(a_{\text{new}}^2, a_t^2), \quad b = \sum_{t=1}^{k_1} \text{l}_{\Gamma_1}(a_{\text{new}}^1, a_t^1),$$

where the subscripts denote on which graph is the linking number computed. Furthermore for any $r = 1, \ldots, k_1$ and any $t = 1, \ldots, k_2$ we have by [12, Proposition 1.2]

$$\text{l}_{\Gamma}(a_r^1, a_t^2) = \text{l}_{\Gamma_1}(a_r^1, a_{\text{new}}^1) \cdot \text{l}_{\Gamma_2}(a_{\text{new}}^2, a_t^2).$$

Substituting the last two equations into the definition of $S_{\text{link}}$, after straightforward computations we obtain the desired result. $\square$

Given Lemma 3.5.2 we can show the difference $S(\Gamma) - S(\tilde{\Gamma}_1) - S(\tilde{\Gamma}_2)$.

- **$S_{\text{link}}(\Gamma) - S_{\text{link}}(\tilde{\Gamma}_1) - S_{\text{link}}(\tilde{\Gamma}_2) = -2ab$** by Lemma 3.5.2.
- **$S_{\text{node}}(\Gamma) - S_{\text{node}}(\tilde{\Gamma}_1) - S_{\text{node}}(\tilde{\Gamma}_2) = -a - b$** (the nodes from $\Gamma(a,b)$ and $\Gamma(b,a)$ contribute);
- **$S_{\text{leaf}}(\Gamma) - S_{\text{leaf}}(\tilde{\Gamma}_1) - S_{\text{leaf}}(\tilde{\Gamma}_2) = \frac{1}{a} + \frac{1}{b}$** (there is a contribution of a single leaf in $\Gamma(a,b)$ and $\Gamma(b,a)$);
- **$S_{\text{edge}}(\Gamma) - S_{\text{edge}}(\tilde{\Gamma}_1) - S_{\text{edge}}(\tilde{\Gamma}_2) = -c^2(\frac{1}{ab(a+1)} + \frac{1}{ab(b+1)} - \frac{1}{ab})$, where $c = \gcd(a,b)$.** To explain this, observe that we cut an edge $e$ of $\Gamma$ and obtain two new edges: one in $\tilde{\Gamma}_1$ and the other one in $\tilde{\Gamma}_2$. The contribution of the edge on $\Gamma$ is $c^2(\frac{d_{\text{new}}}{m_{\text{new}}} + \frac{d_{\text{new}}}{m_{\text{new}}b} - \frac{1}{ab})$. The new edge on $\tilde{\Gamma}_1$ contributes by $c^2(\frac{d_{\text{new}}}{m_{\text{new}}} + \frac{1}{ab(b+1)} - \frac{1}{ab})$, the contribution of the new edge on $\tilde{\Gamma}_2$ is $c^2(\frac{d_{\text{new}}}{m_{\text{new}}} + \frac{1}{ab(a+1)} - \frac{1}{ab})$. The formula follows.
- **$S_{\text{arr}}(\Gamma) - S_{\text{arr}}(\tilde{\Gamma}_1) - S_{\text{arr}}(\tilde{\Gamma}_2) = -\frac{1}{a(b+1)} - \frac{1}{a(b+1)}$; in fact, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have exactly two new arrowheads as compared to $\Gamma$.**
In particular we see that

\[ S(\Gamma) - S(\tilde{\Gamma}_1) - S(\tilde{\Gamma}_2) = -3(\sigma(\Gamma(a,b)) + \sigma(\Gamma(b,a)) + 1). \]

We use now this equation together with (3.5.1). Since 

\[ S(\tilde{\Gamma}_j) = -3\sigma(\tilde{\Gamma}_j) \quad (j = 1, 2) \] by the assumption of the proposition, the proof is finished.

\[ \Gamma \]
\[ \Gamma_1 \quad \Gamma_2 \]
\[ \text{Splice} \]
\[ \Gamma_1 \quad \Gamma_2 \]

Figure 11. Cutting graph along a free edge.

3.6. Proof of Proposition 3.2.2. Second case. We shall suppose that \( b = 0 \). Then it cannot happen that \( a = 0 \), for otherwise the graph has no arrowheads at all.

Observe that in that case \( \sigma(\Gamma_2) = \sigma(\tilde{\Gamma}_2) \). Since \( b = 0 \), we have \( \sigma(\Gamma(a,b)) = 0 \) (the graph represents an unknot), hence \( \sigma(\Gamma_1) = \sigma(\tilde{\Gamma}_1) \) by Lemma 2.2.8 (notice that \( \eta = 0 \) in this case).

In particular, we have

\[ \sigma(\Gamma) = \sigma(\Gamma_1) + \sigma(\Gamma_2) = \sigma(\tilde{\Gamma}_1) + \sigma(\tilde{\Gamma}_2). \]

Now we look at the difference \( S(\Gamma) - S(\tilde{\Gamma}_1) - S(\tilde{\Gamma}_2) \). Recall that the edge that is cut is denoted by \( e \) and it connects vertices \( v \) and \( w \).

- \( S_{\text{link}}(\Gamma) - S_{\text{link}}(\tilde{\Gamma}_1) - S_{\text{link}}(\tilde{\Gamma}_2) = 0 \). The argument is as in Lemma 3.5.2. This time \( b = 0 \);
- \( S_{\text{node}}(\Gamma) - S_{\text{node}}(\tilde{\Gamma}_1) - S_{\text{node}}(\tilde{\Gamma}_2) = -a \) (contribution of the vertex of \( \Gamma(a,b) \));
- \( S_{\text{leaf}}(\Gamma) - S_{\text{leaf}}(\tilde{\Gamma}_1) - S_{\text{leaf}}(\tilde{\Gamma}_2) = \frac{1}{a} + \frac{d_w}{d_w - d_v} \), the contribution is from the leaf of \( \Gamma(a,b) \) and from the new leaf of \( \tilde{\Gamma}_2 \);
- \( S_{\text{edge}}(\Gamma) - S_{\text{edge}}(\tilde{\Gamma}_1) - S_{\text{edge}}(\tilde{\Gamma}_2) = \left( \frac{1}{d_v} - \frac{d_w}{d_v} \right) - \left( \frac{1}{d_w} - a \right) \). The first expression is the contribution from the edge \( e \) in \( \Gamma \), the other comes from the new edge in \( \tilde{\Gamma}_1 \) connecting \( \Gamma_1 \) to \( \Gamma(a,b) \);
- \( S_{\text{arr}}(\Gamma) - S_{\text{arr}}(\tilde{\Gamma}_1) - S_{\text{arr}}(\tilde{\Gamma}_2) = -\frac{1}{a} \).

In particular we see that \( S(\Gamma) = S(\tilde{\Gamma}_1) + S(\tilde{\Gamma}_2) \). By the induction assumption we conclude the proof of Proposition 3.2.2 as in the case \( b \neq 0 \) in Section 3.5.
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