BASE TREE PROPERTY

BOHUSLAV BALCAR, MICHAL DOUCHA, AND MICHAEL HRUŠÁK

Abstract. Building on previous work from [1] we investigate $\sigma$-closed partial orders of size continuum. We provide both an internal and external characterization of such partial orders by showing that (1) every $\sigma$-closed partial order of size continuum has a base tree and that (2) $\sigma$-closed forcing notions of density $\mathfrak{c}$ correspond exactly to regular suborders of the collapsing algebra $\text{Coll}(\omega_1, 2^\omega)$.

We further study some naturally occurring examples of such partial orders.

Introduction

A partially ordered set $(P, \leq)$ is $\sigma$-closed if every countable decreasing sequence of elements of $P$ has a lower bound. In this note we study $\sigma$-closed partial orders of size continuum. Orders of this type naturally arise in combinatorial and descriptive set-theory, topology and analysis.

An essential example is the collapsing algebra $\text{Coll}(\omega_1, 2^\omega)$, i.e. the completion, in the sense of Boolean algebra, of the complete binary tree of height $\omega_1$. This forcing notion has several presentations:

- $(\text{Fn}(\omega_1, \{0, 1\}, \omega_1), \supseteq)$ - ordering for adding a new subset of $\omega_1$,
- $(\text{Fn}(\omega_1, \mathbb{R}, \omega_1), \supseteq)$ - ordering for the consistency of the continuum hypothesis,
- $(\text{Fn}(2^\omega, \{0, 1\}, \omega_1), \supseteq)$ - ordering for adding $\mathfrak{c}$-many subsets of $\omega_1$,
- the natural ordering for adding a $\omega$-sequence,
- Jech’s forcing for adding a Suslin tree by countable conditions.

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All these orderings are forcing equivalent, in fact, they have isomorphic base trees (see Theorem 2.1 for the term base tree). We refer to [13] for definitions of these orderings.

Consider now the set $[\omega]^\omega$ of all infinite sets of natural numbers ordered by inclusion. This order is not $\sigma$-closed, but it is also not separative\(^1\). The separative quotient of $([\omega]^\omega, \subseteq)$ are the positive elements in the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$. In [11] the surprising fact that also $\mathcal{P}(\omega)/\text{fin}$ has a base tree was established. It was then studied in [8], [10], [19].

Since then many other naturally occurring examples were studied ([2], [3]) and in each case the methods of [11] were used to prove the corresponding Base Tree Theorem.

In this note we prove this general fact for all partial orders with a dense $\sigma$-closed subset of size continuum. We also identify the $\sigma$-closed forcings of size continuum as the regular subalgebras of the collapsing algebra $\text{Coll}(\omega_1, 2^\omega)$.

We then present some of the standard examples and review the relevant published results.

We note that similar but somewhat more general notions were studied in [12].

1. Main results

The height of a partial order $(P, \leq)$, $h(P)$ shortly, is the minimal cardinality of a system of open dense subsets of $P$ such that the intersection of the system is not dense. An equivalent definition involves maximal antichains: $h(P)$ is equal to the minimal cardinality of a system of maximal antichains from $P$ that do not have a common refinement. For a Boolean algebra $B$ we define $h(B)$ as the height of the ordering $(B \setminus \{0\}, \leq)$, where $\leq$ is the canonical ordering on $B$. If $B$ is complete, it coincides with its distributivity number. Recall that the distributivity number of $B$ is the least cardinal $\kappa$ such that there exists a matrix $\langle u(\alpha, \beta) : \beta \in I_\alpha, \alpha < \kappa \rangle$ of elements from $B$ such that $\bigwedge_{\alpha < \kappa} \bigvee_{\beta \in I_\alpha} u(\alpha, \beta) \neq \bigvee_{f \in \prod_{\alpha < \kappa} I_\alpha} \bigwedge_{\alpha < \kappa} u(\alpha, f(\alpha))$. We will deal mostly with non-atomic orderings but for completeness we allow atomic orderings in the definition too. Thus, if $(P, \leq)$ is atomic, i.e. there is a set of minimal elements such that every other element is above one of them, then we set $h(P) = \infty$. Note that for non-atomic orderings height is always a regular cardinal.

\(^1\)Recall that a partial order $P$ is separative if whenever $p, q$ are elements of $P$ such that $p \not\leq q$, there is an $r \in P$ such that $r \leq p$ and $r \perp q$. 

The height is a forcing invariant, that means every dense subset of an ordering has the same height. In particular, \( h(P) = h(\text{RO}(P)) \).

**Fact 1.1.** For an ordering \( P \), \( h(P) \) is the minimal cardinal \( \kappa \) such that forcing with \( P \) adds a new function from \( \kappa \) to ordinals. In particular, forcing with \( P \) preserves all cardinals less or equal to \( \kappa \).

An ordering \( P \) is homogeneous in height if for every \( p \in P \) \( h(\downarrow p) = h(P) \). The following proposition shows that every partial order can be decomposed into factors homogeneous in height.

**Proposition 1.2.** Let \( B \) be a complete Boolean algebra. Then \( B \cong \prod_{b \in I} B \upharpoonright b \), where \( I \) is a partition of unity and \( B \upharpoonright b \) is homogeneous in height for every \( b \in I \).

Moreover, \( h(B \upharpoonright a) \neq h(B \upharpoonright b) \) if \( a \neq b \) for \( a, b \in I \).

**Proof.** Let \( A \) be the set of all atoms, then \( B \upharpoonright \bigvee A \) is the first factor homogeneous in the height \( \infty \).

Next, we work with an atomless complete algebra \( B_0 = B \upharpoonright (\neg \bigvee A) \) \((B \cong B_0 \times B \upharpoonright \bigvee A)\). Let \( (D_\alpha)_{\alpha < h(B_0)} \) be the system of open dense subsets of \( B_0 \) such that \( \bigcap_{\alpha < h(B_0)} D_\alpha \) is not dense. Let \( A_1 \) be the subset of elements of \( B_0 \) witnessing the non-density, i.e. \( \downarrow a \cap \bigcap_{\alpha < h(B_0)} D_\alpha = \emptyset \) for every \( a \in A_1 \). We claim that for every \( a \in A_1 \) \( B \upharpoonright a \) is homogeneous in the height (with height \( h(B_0) \)). Assume not, then there is some \( a \in A_1 \) and \( b < a \) such that \( h(B \upharpoonright b) < h(B \upharpoonright a) \). Thus, there is a system \( (S_\alpha)_{\alpha < h(B_0)} \) of open dense subsets of \( B \upharpoonright b \) with a non-dense intersection below \( b \). However, if we set \( D_\alpha = S_\alpha \cup B_0 \downarrow b \) then we get a system of open dense subsets in \( B_0 \) without a dense intersection less than \( h(B_0) \), that is a contradiction.

We take the join \( \bigvee A_1 \) of all elements from \( A_1 \) and the factor \( B \upharpoonright \bigvee A_1 \) is homogeneous in height. We continue with the remainder \( B_1 = B_0 \upharpoonright (\neg \bigvee A_1) \) and in the same way get a set \( A_2 \) of elements witnessing the non-density of the intersection of a system of open dense subsets of size \( h(B_1) \). It is possible that \( h(B_1) = h(B_0) \). In this case, we join the elements of \( A_2 \) with the elements of \( A_1 \). In the opposite case, \( B_1 \upharpoonright \bigvee A_2 \) is a new factor homogeneous in height.

We continue similarly until we treat all elements of \( B \). We end up with the desired decomposition. \( \square \)

**Definition 2** (Base tree property). An ordering \((P, \leq)\) has the Base Tree Property (we shall shortly say it has the BT-property) if it contains a dense subset \( D \subseteq P \) with the following three properties:

- it is atomless; i.e. for every \( d \in D \) there are elements \( d_1, d_2 \in D \) below \( d \) such that \( d_1 \perp d_2 \)
- it is \( \sigma \)-closed
- \(|D| \leq c\)

It can be easily seen that assuming the Continuum Hypothesis, all partial orders with the BT-property are forcing equivalent with \( Coll(\omega_1, 2^\omega) \) and, consequently have a base tree. In fact, the following is true in ZFC.

**Theorem 2.1 (The base tree theorem).** Let \( (P, \leq) \) be an ordering homogeneous in height with the BT-property. Then there are \( h(P) \) maximal antichains \( (T_\alpha)_{\alpha < h(P)} \subseteq P \) such that:

(i) \( (T = \bigcup_{\alpha < h(P)} T_\alpha, \geq) \) is a tree of height \( h(P) \), where \( T_\alpha \) is the \( \alpha \)-th level of the tree,

(ii) each \( t \in T \) has \( c \) immediate successors,

(iii) \( T \) is dense in \( P \).

\( T \) is called a base tree of \( P \).

**Proof.** We need to work with some dense subset guaranteed by the definition of the BT-property rather than with \( P \) itself. To avoid introducing new symbols and sets, we assume \( P \) itself has the properties.

We use the definition of height. So we can find a system \( (A_\alpha)_{\alpha < h(P)} \) of dense open subsets with a non-dense intersection. However, we need to ensure the intersection to be empty. For this, we will work in the completion \( \text{RO}(P) \). Suppose \( \bigcap_{\alpha < h(P)} A_\alpha \) is not empty. Consider the element \( a = \bigvee (\bigcap_{\alpha < h(P)} A_\alpha) \in \text{RO}(P) \). Since \( P \) is homogeneous in height, also \( \text{RO}(P) \) is homogeneous in height, and there is a system \( (\bar{A}_\alpha)_{\alpha < h(P)} \) of dense open sets of \( P \) below \( a \) of the same size such that their intersection is not dense below \( a \). We replace \( \downarrow a \cap A_\alpha \) by \( \bar{A}_\alpha \) (i.e. \( (A_\alpha \setminus \downarrow a) \cup \bar{A}_\alpha \)). We get a new system \( (A'_\alpha)_{\alpha < h(P)} \) of dense open subsets of \( P \) with a non-dense intersection. If this intersection is again non-empty, we again consider the element \( a \geq b = \bigvee (\bigcap_{\alpha < h(P)} A'_\alpha) \in \text{RO}(P) \) and continue similarly. We can repeat this procedure until we get the desired system \( (B_\alpha)_{\alpha < h(P)} \) of dense open subsets with an empty intersection.

Next, we extract from each dense open set \( B_\alpha \) a maximal antichain \( C_\alpha \). We claim that for every \( p \in P \) there is at least one maximal antichain \( C_\alpha \) and elements \( a, b \in C_\alpha \) such that \( p \) is compatible with both of them: suppose that for some \( p \in P \) and for every \( \alpha < h(P) \) there is only one element \( c_\alpha \) from \( C_\alpha \) that is compatible with \( p \). However, \( p \) is then, in fact, below \( c_\alpha \) (since if \( p \not\leq c_\alpha \) then there is a \( p_0 \leq p \) that is disjoint with \( c_\alpha \) but necessarily compatible with another element of
Corollary 2.2. For an ordering \((P, \leq)\), the following statements are equivalent:

(i) \(P\) has the BT-property,

(ii) \(P\) has a dense subset with the BT-property,

(iii) Every dense subset of \(P\) has the BT-property,
(iv) RO($P$) has the BT-property.

Proof. Note that (i)⇒(ii) and (iii)⇒(iv) follow from the definition. It suffices to prove (ii)⇒(iii), (iv)⇒(i) is then a consequence.

We need to find a dense subset of a given dense subset that is, moreover, σ-closed and of size $\mathfrak{c}$, atomlessness is clear.

Assuming (ii), we have a base tree $T$, we are given a dense subset $D$ and we show that there is a σ-closed dense subset $S \subseteq D$.

We make $S$ from maximal antichains. For every $t \in T_0$ we find a maximal antichain $A_t \subseteq D$ below the element $t$. $\bigcup_{t \in T_0} A_t$ is the first maximal antichain $S_0$.

Then for every $s \in S_0$ we find a maximal antichain $M_s \subseteq T$ below $s$. Let $P_1 \subseteq T$ be a maximal antichain from $T$ refining $\bigcup_{s \in S_0} A_s$ and $T_1$. Again, for every $p \in P_1$ we find a maximal antichain $A_p \subseteq D$ from $D$, the union $\bigcup_{p \in P_1} A_p$ is $S_1$.

Isolated steps are treated similarly. We need not omit $P_\alpha$ to be refining the tree level $T_\alpha$. Then we refine it to $S_\alpha \subseteq D$.

For a limit $\alpha$ we take a refinement $P_\alpha$ of all $P_\beta$’s for $\beta < \alpha$ (which is also a refinement of $S_\beta$’s) together with $T_\alpha$. Then we again refine it to $S_\alpha \subseteq D$.

The resulting set $S = \bigcup_{\alpha < b(P)} S_\alpha$ is dense and σ-closed. We ensured density by refining all levels of $T$. For σ-closedness observe that for every countable descending chain $s_0 \geq s_1 \geq \ldots$ from $S$, where $s_n \in S_{\alpha_n}$, there is an interwined descending chain $p_0 \geq p_1 \geq \ldots$ such that $p_0 \geq s_0 \geq p_1 \geq s_1 \geq \ldots$, where $p_n \in P_\alpha$. This interwined chain has a lower bound $p$ in $P_\alpha$, where $\alpha = \sup \{ \alpha_n : n \in \omega \}$, and $p$ has some successor $s \in S$.

In other words, having a σ-closed dense set is preserved by forcing equivalence among separative partial orders of size continuum. On the other hand, Zapletal in [20] has constructed a model in which the Continuum Hypothesis holds and there are two forcing equivalent separative partial orders of size $\aleph_2$ one σ-closed and the other without a σ-closed dense set. One has to wonder whether such a pair exists in ZFC.

Question 2.3. Are there, in ZFC, two separative partial orders which are forcing equivalent such that one is σ-closed and the other does not have a σ-closed dense set? Can such partial orders be found as dense subsets of the completion of $\text{Fn}(\omega_1, \mathfrak{c}^+)$?

Finally, using this internal characterization of the partial orders with the BT-property one can easily deduce the following external characterization.
Theorem 2.4.

(1) Let \((P, \leq)\) be an ordering with the BT-property. Then \(\text{RO}(P)\) is a regular subalgebra of \(\text{Coll}(\omega_1, c)\).

(2) Let \(B\) be a complete atomless regular subalgebra of \(\text{Coll}(\omega_1, c)\). Then \(B\) has the BT-property.

Proof. We prove (1): Let \(D \subseteq P\) be its dense subset witnessing the BT-property. Then \(D \times \text{Fn}(\omega_1, \{0, 1\}, \omega_1)\) with induced product ordering clearly has the BT-property, the height is \(\omega_1\), thus it is forcing equivalent with the complete Boolean algebra \(\text{Coll}(\omega_1, c)\). Note that there is a regular embedding \(e : D \rightarrow D \times \text{Fn}(\omega_1, \{0, 1\}, \omega_1)\) defined as \(e(d) = (d, 1)\) where 1 is the biggest element in \(\text{Fn}(\omega_1, \{0, 1\}, \omega_1)\), i.e. the empty set. \(e\) is extended onto \(\bar{e} : \text{RO}(P) \rightarrow \text{Coll}(\omega_1, c)\) mapping \(\text{RO}(P)\) on a regular subalgebra of \(\text{Coll}(\omega_1, c)\).

Now we prove (2): We use the result of [17] that every atomless game-closed complete Boolean algebra of density at most \(c\) has a \(\sigma\)-closed dense subset (of size at most \(c\) in fact). We note that the converse is trivial; see also [17]. Since \(\text{Fn}(\omega_1, \{0, 1\}, \omega_1)\) contains a \(\sigma\)-closed dense subset it is game-closed. Now let \(B\) be any complete atomless regular subalgebra, it is still game-closed and of density at most \(c\). Thus it has a \(\sigma\)-closed dense subset which is atomless and of size at most \(c\) thus it witnesses that \(B\) has the BT-property. \(\square\)

3. Classical examples

The Boolean algebra \(\mathcal{P}(\omega)/\text{fin}\) is a prototype of an ordering with the BT-property. Recall the definitions of the cardinal invariants \(p, t\) ([6]). It was proved recently by M. Malliaris and S. Shelah ([15]) that \(p = t\). We shall discuss these cardinal invariants on other orderings too.

The second fundamental example is \((\text{Dense}(\mathbb{Q}), \subseteq)\), where \(\text{Dense}(\mathbb{Q})\) is a set of all dense subsets in rationals. The situation here is similar with the previous example, it is not separative and the ordering \((\text{Dense}(\mathbb{Q}), \subseteq)\) itself does not satisfy the BT-property. The separative modification is \((\text{Dense}(\mathbb{Q}), \subseteq_{\text{nwd}})\), where \(A \subseteq_{\text{nwd}} B\) if \(A \setminus B\) is nowhere dense in \(\mathbb{Q}\), has the BT-property. This ordering is studied in [2].

Let \(p_\mathbb{Q}, t_\mathbb{Q}, h_\mathbb{Q}\) be the cardinal invariants of \((\text{Dense}(\mathbb{Q}), \subseteq_{\text{nwd}})\) defined in the same way as their counterparts in \([\omega]^\omega, \subseteq^+\). It was proved in [2] that \(p_\mathbb{Q} = p\) and \(t_\mathbb{Q} = t\) (thus \(p_\mathbb{Q} = t_\mathbb{Q} = t\)) whereas \(h_\mathbb{Q}\) and \(h\) are incomparable in ZFC, \(h_\mathbb{Q} < h\) and \(h_\mathbb{Q} > h\) are both consistent (see [2] and [7]); and \(h_\mathbb{Q} = h\) too, of course.
For the third example, let $A$ be the Cantor algebra, i.e. the algebra of all clopen subsets of $2^\omega$, and consider the countable product $A^\omega$ modulo the ideal $\text{Fin} \subseteq A^\omega$, where $\text{Fin} = \{ f \in A^\omega : |\{ n : f(n) \neq 0 \}| < \omega \}$. It satisfies the BT-property, moreover, $A^\omega/\text{Fin}$ is homogeneous.

$s_{A^\omega/\text{Fin}} = t$ and $h(A^\omega/\text{Fin}) \leq \min\{ h, \text{add}(A) \}$ (3) and it is consistent that $h(A^\omega/\text{Fin}) < h$ (3,9).

For any Boolean algebra $B$ let us consider an infinite product $B^\omega$. Let $J$ be an ideal on $\omega$. By $I_J \subseteq B^\omega$ we denote the ideal $\{ f \in B^\omega : \{ n \in \omega : f(n) \neq 0 \} \in J \}$. The quotient algebra $B^\omega/I_J$ consists of equivalence classes where $f, g \in B^\omega$ are equivalent if $\{ n : f(n) \neq g(n) \} \in J$ ($f \triangle g \in I_J$ equivalently). We state and prove a simple criterion for when such a product has the BT-property.

**Theorem 3.1.** Let $B$ be a Boolean algebra and $J$ an ideal on $\omega$. Then the reduced product $B^\omega/I_J$ has the BT-property if and only if $B$ contains a dense subset of size $\mathfrak{c}$ and (either $\mathcal{P}(\omega)/J$ is $\sigma$-closed or $J$ is a maximal ideal), and (either $\mathcal{P}(\omega)/J$ or $B$ is atomless).

**Proof.** Since $B^\omega/I_J$ contains a dense subset of size $\mathfrak{c}$ if and only if $B$ contains a dense subset of size less or equal to $\mathfrak{c}$ the requirement on the cardinality is satisfied.

Suppose that $\mathcal{P}(\omega)/J$ is not $\sigma$-closed. Let $(X_n)_{n \in \omega}$ be a descending chain of infinite subsets of $\omega$ such that the chain $([X_n])_{n \in \omega}$ does not have a lower bound in $\mathcal{P}(\omega)/J$, where $[X_n]$ is the equivalence class containing $X_n$. We define a descending chain $([f_n])_{n \in \omega} \subseteq B^\omega/I_J$ as follows: $f_n(i) = 1$ if $i \in X_n$ and $f_n(i) = 0$ otherwise (it is the image of the chain $([X_n])_{n \in \omega}$ via the regular embedding of $\mathcal{P}(\omega)/J$ into $B^\omega/I_J$). Suppose that it has a lower bound $[f]$. Then the support of $f$, i.e. the set $\{ i : f(i) \neq 0 \}$, would determine a lower bound for $([X_n])_{n \in \omega}$.

Next we use the fact mentioned in 3 that $B^\omega/I_J$ can be written as an iteration $\mathcal{P}(\omega)/J \ast B^\omega/\hat{U}$, where $\hat{U}$ is a name for an ultrafilter added by $\mathcal{P}(\omega)/J$. For $[f] \in B^\omega/I_J$ we define $\Phi([f]) = (\{ i : f(i) \neq 0 \}, [f])$, where $[f]$ is a name for an equivalence class containing $f$ in $B^\omega/\hat{U}$. $\Phi$ is easily verified to be a dense embedding which proves the fact.

Now observe that an ultrapower of any Boolean algebra is $\sigma$-closed. For a countable descending chain we can choose representatives of equivalence classes $(f_n)_{n \in \omega}$ so that support $f_0 = \omega$, support $f_1 \supseteq$ support $f_2 \supseteq$ support $f_3 \supseteq \ldots$ and $\bigcap_{n \in \omega}$ support $f_n = \emptyset$ since the ultrafilter is non-principal. Then we set $f(i) = f_n(i)$ if $n$ is the smallest number such that $i \in \text{support } f_n \setminus \text{support } f_{n+1}$. $f$ clearly determines the lower bound for the chain. Hence, we conclude that $B^\omega/I_J$ is $\sigma$-closed since an iteration of two $\sigma$-closed forcings is.
To check atomlessness, if $\mathcal{P}(\omega)/J$ is atomless then for any $f \in B^\omega$, where the support of $f$ is not in $J$, we can always split the support of $f$ into two disjoint infinite sets both outside of $J$, restrict $f$ to these sets and make two disjoint elements of $B^\omega/I_J$ below $[f]$. If $B$ is atomless then we can always find two disjoint successors coordinatewise. Finally, suppose that $\mathcal{P}(\omega)/J$ has an atom $[A]$ and $B$ has an atom $b$. Then $f \in B^\omega$ defined so that $f(n) = b$ for $n \in A$ and $f(n) = 0$ for $n \notin A$ determines an atom in $B^\omega/I_J$. □

4. On classes of ideals ordered by reverse inclusion

We shall deal with orderings that consist of ideals on $\omega$ of some type ordered by reverse inclusion. We assume that all such ideals extend the ideal of finite subsets of $\omega$. Since every ideal on $\omega$ can be considered as a subset of the Cantor space we can speak about the topological, resp. measure-theoretical characterizations of such ideals.

4.1. Non-tall ideals. An ideal $I$ on $\omega$ is tall if for every $X \in [\omega]^\omega$ there is infinite $Y \subseteq X$ that belongs to $I$. Consider the set $\mathcal{I}$ of all non-tall ideals on $\omega$ ordered by reverse inclusion.

First of all, this ordering is not separative. However, for every $A \in [\omega]^\omega$ consider the ideal $I_A$ of all subsets of $\omega$ that have a finite intersection with $A$. $I_A$ is a non-tall ideal and $B \subseteq^* A$ implies $I_B \supseteq I_A$. Moreover, for every non-tall ideal $I$ and some infinite set $A$ almost disjoint with every element of $I$, $I_A \supseteq I$. Thus we see that $([\omega]^\omega, \subseteq)$ is isomorphic with a dense subset of $(\mathcal{I}, \supseteq)$ and of its separative modification showing that the separative modification of $(\mathcal{I}, \supseteq)$ has the BT-property, however it is forcing equivalent to $([\omega]^\omega, \subseteq^*)$.

4.2. $F_\sigma$ ideals. Consider the ordering of all $F_\sigma$ ideals on $\omega$ denoted as $\mathfrak{F}$ ordered by reverse inclusion. The study of this ordering was initiated by C. Laflamme in [14] and also studied in [11].

It is immediate from the definition that $\mathfrak{F}$ is $\sigma$-closed. To show that it is atomless, consider any $F_\sigma$ ideal $I$. Since it is not maximal there is a subset $A \subseteq \omega$ such that neither $A$ nor $\omega \setminus A$ belong to $I$. We extend $I$ by adding $A$ to obtain an ideal $I_A$; similarly, we obtain an ideal $I_{\omega \setminus A}$. They are disjoint, we must prove they are $F_\sigma$. We do it for $I_A$. Write $I$ as $\bigcup_n F_n$ where each $F_n$ is closed, thus compact. The mapping $\pi$ that sends $X$ to $X \cup A$ is continuous, thus $\pi[I] = \bigcup_n \pi[F_n]$ is still $F_\sigma$ and the downward closure of $\pi[I]$ is still $F_\sigma$ and equal to $I_A$. Since there are only $c$-many $F_\sigma$ ideals we just proved that $\mathfrak{F}$ has the BT-property. However, we do not know what the height of this ordering is.
4.3. Summable ideals. Consider the ordering \((c_0^+ \setminus \ell^1, \leq^*)\) where \(c_0^+\)
the set of all sequences of positive reals that tend to zero and \(\ell^1\) the set of all sequences of reals whose sum converges. The order relation \(\leq^*\) is almost domination, i.e. \(f \leq^* g\) if \(\{n : g_n > f_n\}\) is finite. The investigation of this ordering was initiated by P. Vojt\v{a}š in [13]. \((c_0^+ \setminus \ell^1, \leq^*)\) is not separative but we will show that the separative quotient is isomorphic to the set \(\mathcal{I}_\Sigma\) of all summable ideals ordered by inverse inclusion.

An ideal \(\mathcal{I}\) is summable if there exists \(\bar{f} \in c_0^+ \setminus \ell^1\) such that \(\mathcal{I} = \{A : \sum_{n \in A} f(n) < \infty\}\). Note that any summable ideal is an \(F_\sigma\) \(P\)-ideal, thus \(\mathcal{I}_{\Sigma}\) is a subordering of \(\mathcal{F}\).

We check that \(\mathcal{I}_{\Sigma}\) has the BT-property. Let us verify atomlessness. Let \(I\) be a summable ideal determined by a sequence \((a_n)\), and \(A \in I\). Then \(\sum_{i \in \omega \setminus A} a_i\) diverges; we divide \(\omega \setminus A\) into two infinite subsets \(B_1\) and \(B_2\) such that the appropriate sums both diverge. We make new sequences \((b_n)\) and \((c_n)\) so that \(b_i = a_i\) for \(i \in A \cup B_1\) and \(b_i = c_i\) for \(i \in B_2\), where \((z_n)\) is an arbitrary converging sequence of positive reals. \((c_n)\) is defined similarly, just \(B_1\) and \(B_2\) change their roles. Both \((b_n)\) and \((c_n)\) diverge. We denote the appropriate summable ideals \(I_b\) and \(I_c\). It is clear that \(I_b, I_c \supseteq I\) and that they are disjoint.

Let \((I_j)\) be an increasing (inclusion) sequence of summable ideals. Let \(a_i^\\infty\) be the sequence of positive reals that determines the ideal \(I_j\). We may assume that \((a_n^0)\) is finite. The

To verify separativeness, consider ideals \(I_a\) and \(I_b\) corresponding to sequences \((a_n)\) and \((b_n)\), such that \(I_a \not\supseteq I_b\), i.e. there is a set \(B \in I_b\) which does not belong to \(I_a\). That means \(\sum_{k \in B} b_k < \infty\) but \(\sum_{k \in B} a_k = \infty\). If \(\omega \setminus B\) belongs to \(I_a\) then \(I_a\) and \(I_b\) are already disjoint, if this is not the case then we make a new sequence \((c_n)\) such that \(c_n = a_n\) for \(n \in B\) and \(\sum_{k \in \omega \setminus B} c_k < \infty\). The corresponding ideal \(I_c\) is below \(I_a\) and disjoint with \(I_b\).

It is easy to check that if \((a_n)\) \(\approx_{\text{sep}} (b_n)\), i.e. \(\forall (c_n) \in (c_0^+ \setminus \ell^1, \leq^*) ((c_n) \perp (a_n) \iff (c_n) \perp (b_n))\), then \((a_n)\) and \((b_n)\) determine the same summable ideal and the mapping \(\Phi : (c_0^+ \setminus \ell^1, \leq^*) \to (\mathcal{I}_\Sigma, \supseteq)\), defined as \(\Phi((c_n)) = \{A \subseteq \omega : \sum_{n \in A} c_n < \infty\}\), is an onto homomorphism of orderings preserving the disjointness.
relation. And the preimage of each summable ideal is precisely an equivalence class of sequences in \( \approx_{\text{sep}} \).

**Proposition 4.1.** \( t((c_0^+ \setminus \ell^1, \leq^*) = t.\)

**Proof.** To simplify the notation we will write \( \bar{a} \) instead of \((a_n)_{n=0}^\infty\). Let \((\bar{a}_\alpha)_{\alpha < \kappa}\) be a descending chain of sequences from \((c_0^+ \setminus \ell^1, \leq^*)\) of length \( \kappa < t \). We use the methods from \([3]\) to show it has a lower bound.

For each \( \alpha < \kappa \) let \( h_\alpha : \omega \to \omega \) be a function such that \( \forall n \in \omega(\frac{1}{n_{\alpha(n)}} \leq \bar{a}_{\alpha,n}) \). Since \( \kappa < t \leq b \), there is a function \( h \in \omega^\omega \) that almost dominates all \( h_\alpha \)'s, i.e. \( h \geq^* h_\alpha \) for all \( \alpha < \kappa \).

Similarly, for each \( \alpha < \kappa \) let \( f_\alpha : \omega \to \omega \) be a function such that \( \forall n \in \omega(\sum_{f_{\alpha}(n) \leq i < f_{\alpha}(n+1)} a_\alpha(i) > 1) \). Since \( \kappa < t \leq b \), there is a function \( f \in \omega^\omega \) that almost dominates all \( f_\alpha \)'s, i.e. \( f \geq^* f_\alpha \) for all \( \alpha < \kappa \). Define \( g \in \omega^\omega \) recursively so that \( g(0) = f(0) \) and \( g(n+1) = f(g(n)+1) \). Note that for every \( \alpha < \kappa \) and all but finitely many \( n \)’s \( \sum_{g(n) \leq i < g(n+1)} a_\alpha(i) > 1 \) since \( g(n) < f_\alpha(g(n)) < f_\alpha(g(n) + 1) \leq g(n) + 1 \). We denote \( I_n \) the interval \([g(n), g(n+1)]\).

For every \( n \), we denote \( F_n \) the following set of functions \( \{ F : \text{dom}(F) \subseteq I_n \land \text{rng}(F) \subseteq \{ \tfrac{1}{2|I_n|}, \tfrac{2}{2|I_n|}, \ldots, 1 \} \land \sum_{i \in \text{dom}(F)} F(i) > \tfrac{1}{2} \} \). Let \( \mathcal{F} = \bigcup_{n \in \omega} F_n \). In the following we shall treat \( \mathcal{F} \) as \( \omega \).

For every \( \bar{a}_\alpha \), let \( X_\alpha = \{ F : \exists n(F \in F_n \land \forall i \in \text{dom}(F)(F(i) \leq \bar{a}_\alpha(i))) \} \)

An easy pigeon-hole type argument shows that it is infinite for every \( \alpha < \kappa \). It is also clear that \( X_\beta \setminus X_\alpha \) is finite for \( \alpha < \beta \). Since \( \kappa < t \), there is a lower bound \( X \subseteq \mathcal{F} \). By shrinking it if necessary, we can assume that \( |X \cap F_n| \leq 1 \) for every \( n \). Finally, we define a sequence \( \bar{a} \) as follows:

For every \( m \in \omega \), if there exists \( F \in X \) such that \( m \in \text{dom}(F) \) then we set \( \bar{a}(m) = F(m) \). Otherwise, we set \( \bar{a}(m) = \frac{1}{h(m)} \). It is now easy to check that \( \bar{a} \) is the desired lower bound.

To prove the converse, let us at first prove that \( t((c_0^+ \setminus \ell^1, \leq^*) \leq b \.

Suppose the contrary. Let \((b_\alpha)_{\alpha < b}\) be a system of almost increasing functions from \( \omega^\omega \) without an upper bound, \( \pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) a bijection and \((l_n)_{n=0}^\infty\) a strictly decreasing sequence from \( \ell^1 \) such that \( l_n < \frac{1}{n} \) for every \( n \). We define a descending chain of sequences from \((c_0^+ \setminus \ell^1, \leq^*)\) \((\bar{a}_\alpha)_{\alpha < b}\) as follows: \( \bar{a}_\alpha(\pi(1, k)) = l_k \) for \( k \leq b_\alpha(0) \), for \( l > b_\alpha(0) \) we set \( \bar{a}_\alpha(\pi(1, l)) = \frac{1}{l} \); generally, \( \bar{a}_\alpha(\pi(n, k)) = l_k \) for \( k \leq b_\alpha(n-1) \), for \( l > b_\alpha(n-1) \) we set \( \bar{a}_\alpha(\pi(n, l)) = \frac{1}{l} \).

Let \( \bar{a} \) be a lower bound for this chain. Define a function \( f \) by \( f(n) = \min\{ k : \bar{a}(\pi(n, k)) > l_k \} \). It is easy to check that \( f \) almost dominates \((b_\alpha)_{\alpha < b}\), a contradiction.
Now assume that \( t < t((c_0^+ \setminus \ell^1, \leq^*)) \). Let \((X_\alpha)_{\alpha < t} \subseteq [\omega]^{\omega} \) be a descending chain without a lower bound. We define \( f_\alpha \in \omega^\omega \) for every \( \alpha < t \) so that \( f_\alpha(n) = k \) such that \(|X_\alpha \cap [f_\alpha(n) - 1, f_\alpha(n)])| \geq n + 1. \) Since \( t < t((c_0^+ \setminus \ell^1, \leq^*)) \leq b \), we can again find \( g \in \omega^\omega \) such that for every \( \alpha < t \) and for almost all \( n \)’s \(|X_\alpha \cap [g(n) - 1, g(n))]| \geq n + 1. \)

Define a chain \((\bar{a}_\alpha)_{\alpha < t}\) of sequences as follows: \( \bar{a}_{\alpha,n} = \frac{1}{k} \) if \( n \in X_\alpha \cap [g(k - 1), g(k)) \); if no such \( k \) exists then let \( \bar{a}_{\alpha,n} = l_n \).

Finally, let \( \bar{a} \) be a lower bound for this descending chain and define a lower bound \( X = \{ n : \bar{a}_n > l_n \} \) for the chain \((X_\alpha)_{\alpha < t} \).

4.4. Meager and null ideals. Next we consider the class of all meager ideals \( \mathcal{M} \) and the set of all ideals \( \mathcal{N} \) of measure zero; i.e. those ideals that are meager sets and null sets respectively in the Cantor space topology. Simultaneously, we study the set of all hereditary meager \( \mathcal{M}_H \) and hereditary null \( \mathcal{N}_H \) ideals, where an ideal \( I \) is hereditary meager (null) if for every \( X \in I^+ \) the restriction \( I \upharpoonright X = \{ A \in I : A \subseteq X \} \) is meager (null) in the Cantor space \( 2^X \).

It is obvious they are both \( \sigma \)-closed. We show they are atomless, what the separative quotient for meager ideals is and that there is no dense subset in both of these orderings that has cardinality \( c \). In fact, there are \( 2^c \) mutually disjoint elements in both orderings. Let us note that by \( \approx_{\text{sep}} \) we denote the "separative equivalence" (in the ordering of meager ideals); i.e. \( I \approx_{\text{sep}} J \) if and only if for any meager ideal \( K \), \( K \) is disjoint with \( I \) iff \( K \) is disjoint with \( J \).

We will use the following characterizations of meager and null ideals.

Proposition 4.2 (Talagrand; see for example Theorem 4.1.2 [H]). An ideal \( I \) on \( \omega \) is meager if and only if there is a partition \((P_i)_{i \in \omega} \) of \( \omega \) into finite sets such that \( \bigcup_{i \in A} P_i \in I \) iff \( A \) is finite.

Proposition 4.3 (Bartoszyński; Theorem 4.1.3 [H]). An ideal \( I \) on \( \omega \) is null if and only if there exists an infinite system \( \{ A_n : n \in \omega \} \) such that

1. each \( A_n \) is a finite set consisting of finite subsets of \( \omega \)
2. \( \forall n \neq \omega (\bigcup A_n \cap \bigcup A_m = \emptyset) \)
3. \( \sum_{n \in \omega} \mu \{ X \subseteq \omega : \exists a \in A_n (a \subseteq X) \} < \infty \)
4. for every \( X \in I \exists \infty n \exists a \in A_n (X \cap a = \emptyset) \)

where \( \mu \) is the Lebesgue measure on the Cantor space.

Proposition 4.4. There is a mapping \( \Phi : (\mathcal{M}, \supseteq) \to (\mathcal{M}_H, \supseteq) \) such that \( \forall I \in \mathcal{M} (\Phi(I) \supseteq I \land \Phi(I) \approx_{\text{sep}} I) \).

Note that it follows that for any meager ideal \( I \), \( \Phi(I) \) is the least element in the equivalence class of \( \approx_{\text{sep}} \) containing \( I \).
Proof. For a meager ideal $I$ consider the set

$$\tilde{I} = \{ A \subseteq \omega : I \upharpoonright A \text{ is not meager} \}.$$ 

Let $(P_n)_{n \in \omega}$ be the partition of $\omega$ witnessing $I$ is meager (from Proposition 4.2). $\tilde{I}$ is a hereditary meager ideal containing $I$. To see that it is meager check that $(P_n)_{n \in \omega}$ still works. Let $A \in \tilde{I}$ be arbitrary. Since $A \not\in \tilde{I}$ we have $I \upharpoonright A$ is meager, so there is a partition $(Q_n)_{n \in \omega}$ of $A$ into finite sets such that $\bigcup_{i \in C} Q_i \in I$ iff $C$ is finite. If $\tilde{I} \upharpoonright A$ were not meager then there would be an infinite set $C \subseteq \omega$ such that $B = \bigcup_{i \in C} Q_i \in \tilde{I} \upharpoonright A$. $I \upharpoonright B$ would have to be nonmeager but then there would be an infinite set $D \subseteq C$ such that $\bigcup_{i \in D} Q_i \in I \upharpoonright A$, a contradiction.

It remains to prove that $\Phi(X) \approx_{\text{sep}} X$ for any $X \in \mathcal{M}$. But obviously if a meager ideal $I$ is compatible with a meager ideal $J$, then $\Phi(I)$ is compatible with $\Phi(J)$ and so also with $J$. Each equivalence class of meager ideals has its minimal element, the corresponding hereditary meager ideal. \hfill \Box

Question 4.5. We do not know whether the previous proposition also holds true for the class of null ideals. Let $I$ be a null ideal, is $\tilde{I} = \{ A \subseteq \omega : I \upharpoonright A \text{ is not null} \}$ a (hereditary) null ideal?

It is easy to prove that $\tilde{I} = \{ A \subseteq \omega : I \upharpoonright A \text{ is not null} \}$ is an ideal though. It is immediate that $\tilde{I}$ extends $I$. Let us check that it is downward closed. Let $A \in \tilde{I}$ and $B \subseteq A$. Suppose that $B \not\in \tilde{I}$. Then $I \upharpoonright B$ is null. We use Proposition 4.3 to obtain an infinite system $\{ A_n : n \in \omega \}$ witnessing it. It follows from that proposition that the same system would witness that $I \upharpoonright A$ is null as well, which is a contradiction.

Let $A, B \in \tilde{I}$, we may assume they are disjoint. Then realize that $X \to (X \cap A, X \cap B)$ is a measure preserving homeomorphism from $A \cup B$ to $A \times B$ and it follows from the Fubini theorem that $I \upharpoonright A \times I \upharpoonright B$, so also $I \upharpoonright A \cup B$, is not null. Thus $\tilde{I}$ is closed under taking finite unions.

Corollary 4.6. $(\mathcal{M}, \supseteq)$ and $(\mathcal{N}, \supseteq)$ are atomless, not separative, the separative quotient of $(\mathcal{M}, \supseteq)$ is isomorphic to the ordering $(\mathcal{M}_H, \supseteq)$ of all hereditary meager ideals via the mapping $\Phi$.

Proof. To prove they are atomless, let $I$ be an arbitrary meager ideal, let $A$ and $B$ be two infinite subsets of $\omega$ such that $A \cup B = \omega$ and neither $A$ nor $B$ is in $\Phi(I)$ ($\Phi(I)$ is meager, thus not maximal). Extend $I$ by $A$ and by $B$ to obtain two disjoint ideals $X_A$ and $X_B$ that are easily verified to be meager. The proof for null ideals is similar.

We claim they are not separative. Consider some maximal ideal $M$ on odd natural numbers and the ideal $F$ of finite sets on even
numbers. Then let \( I = \{ A \cup B : A \in M \land B \in F \} \) and \( J = \{ A \cup B : A \) is an arbitrary subset of odd natural numbers \( \land B \in F \} \) be two ideals, both easily verified to be meager and null. However, \( I \) and \( J \) are equivalent in the separative modification for both classes (meager and null) of ideals.

On the other hand, if \( I \) and \( J \) are two hereditary meager ideals such that \( I \nsubseteq J \), then there is an infinite set \( A \in J \) that is not in \( I \). Let \( K \) be an ideal generated by \( I \cup \{ \omega \setminus A \} \), it is clearly meager and \( \Phi(K) \supseteq I \) is disjoint with \( J \).

It remains to show that \( \Phi \) defines an isomorphism between the separative quotient of \( (\mathcal{M}, \supseteq) \) and an ordering \( (\mathcal{N}, \supseteq) \). But \( \Phi \) obviously preserves the inclusion relation and the disjointness between ideals and from the previous proposition, \( \Phi(I) \approx_{\text{sep}} I \) for any \( I \in \mathcal{M} \), so we are done. \( \square \)

Finally, we show there is no dense subset of these orderings with size \( c \), thus preventing them to have the BT-property.

**Theorem 4.7.** There are \( 2^c \) ideals that are both meager and null and that are mutually disjoint (in both \( (\mathcal{M}, \supseteq) \) and \( (\mathcal{N}, \supseteq) \)).

In particular, neither \( (\mathcal{M}, \supseteq) \) nor \( (\mathcal{N}, \supseteq) \) has the BT-property.

**Proof.** Let \( (I_n)_{n \in \omega} \) be a partition of \( \omega \) into intervals such that \( |I_n| = n + 1 \). For any \( X \subseteq \omega \) let \( X^I \) be the set

\[
\bigcup_{m \in X} \{ k \in \omega : \exists n (k \text{ is the } m\text{-th element of } I_n) \}
\]

It is clear that if \( J \) is an ideal on \( \omega \) then \( \{ X^I : X \in J \} \) is a base of an ideal; we shall denote this ideal as \( I_J \).

Now let \( \mathcal{M} \) be the set of all maximal ideals on \( \omega \), its size is \( 2^c \). For any \( J \in \mathcal{M} \), we make an ideal \( I_J \) and obtain a system \( \mathcal{J} \) of \( 2^c \) ideals. The disjointness of two ideals \( J_1, J_2 \in \mathcal{M} \) is easily seen to be preserved for \( I_{J_1} \) and \( I_{J_2} \).

**Claim 1** \( \mathcal{J} \) is a system of meager ideals.

The interval partition \( (I_n)_{n \in \omega} \) works for all ideals from \( \mathcal{J} \). Assume that some set \( X \in I_J \), where \( I_J \in \mathcal{J} \), contains a union of infinitely many intervals. It is easy to check that once it contains the whole interval \( I_n \) then it contains all previous intervals. Thus, we conclude that \( X = \omega \) which is a contradiction.

**Claim 2** \( \mathcal{J} \) is a system of null ideals.
We use characterization of null ideals from 4.3. For \( m \leq n \), let \( i^m_n \) be the \( m \)-th element of \( I_n \). Let \( A_n = \{a_n = \{i^m_n : n \leq m \leq 2n - 1\}\} \). These sets satisfy the first three conditions from 4.3. Let \( X \in I_J \), where \( I_J \in J \), be a given set. \( X \subseteq Y^I \) for some \( Y \in J \). Then it is easy to check that \( A_n \cap X = \emptyset \) for \( n \in \omega \setminus Y \) and \( |\omega \setminus Y| = \omega \); thus we also verified the last fourth condition and proved that every \( I_J \) is null. □

5. Products

If \( P \) and \( Q \) are two orderings with the BT-property then their product \( P \times Q \) has again the BT-property and the height is less or equal to the minimum of heights of the original orderings. To see this, just realize that if \( B \) is a regular subalgebra of \( C \) then \( h(B) \geq h(C) \), and that \( \text{RO}(P) \) is a regular subalgebra of \( \text{RO}(P \times Q) \). The same holds for countable products and iterations. Let us mention the case of products of \( ([\omega]^{< \alpha}, \subseteq^*) \). By \( h_n \), for \( 2 \leq \alpha \leq \omega \) we denote \( h(([\omega]^{< \alpha}, \subseteq^*)^\alpha) \). It immediately follows that \( t(([\omega]^{< \alpha}, \subseteq^*)^\alpha) = t \) for any \( 2 \leq \alpha \leq \omega \). Shelah and Spinas in [16] proved that consistently for any \( n \in \omega \) \( h_{n+1} < h_n \). However, the following question remains open.

Question 5.1. Does it hold in ZFC that \( h_\omega = \min \{h_n : n \in \omega\} \)?

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The Center for Theoretical Study, Charles University in Prague, Jilská 1, Prague, Czech Republic
E-mail address: balcar@cts.cuni.cz

Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, Prague, Czech Republic
E-mail address: m.doucha@post.cz

Instituto de Matemáticas, UNAM, Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México.
E-mail address: michael@matmor.unam.mx