A NECESSARY AND SUFFICIENT CONDITION FOR RADIAL PROPERTY
OF POSITIVE ENTIRE SOLUTIONS OF $\Delta^2 u + u^{-q} = 0$ IN $\mathbb{R}^3$

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ABSTRACT. In this article, we are concerned with the following geometric equation

$$\Delta^2 u = -u^{-q} \quad \text{in } \mathbb{R}^3$$

for $q > 0$. Recently in [GWZ18], Guo, Wei and Zhou have established the relationship between the radial symmetry and the exact growth rate at infinity of a positive entire solution of that equation as $1 < q < 3$. The aim of this paper is to obtain the similar result in the case $q > 3$ thanks to the method of moving plane.

1. INTRODUCTION

In this article, we are interested in studying a necessary and sufficient condition for positive entire solutions to be radially symmetric of the following geometric equation

$$\Delta^2 u = -u^{-q} \quad \text{in } \mathbb{R}^3,$$  

provided that $q > 0$. Eq. (1.1) has attracted many mathematicians over years since its root from the prescribed $Q$-curvature problem in conformal geometry. We refer interested readers to [CX09] for further information.

The existence of a positive entire solution of (1.1) was first proved in [CX09] that there holds $q > 1$, necessarily. As $q > 1$, McKenna and Reichel [KR03] looked for the radial solutions of Eq. (1.1) via shooting method. To be precise, Eq. (1.1) was transformed into the following initial value problem

$$\begin{cases}
\Delta^2 u = -u^{-q}, & r \in (0, R_{\text{max}}(\beta)), \\
u(0) = 1, & u'(0) = 0, \\
\Delta u(0) = \beta > 0, & (\Delta u)'(0) = 0.
\end{cases}$$

(1.2)

Here $R_{\text{max}}(\beta)$ is the largest radius of the interval of existence of the solution. The result in [KR03] asserted that there exists a unique threshold parameter $\beta^*$ such that $R_{\text{max}}(\beta) = \infty$ if $\beta \geq \beta^*$. The authors also claimed that $u_\beta > u_{\beta^*}$ in $(0, \infty)$ for $\beta > \beta^*$ thanks to a comparison principle stated in [KR03, Lemma 3.2]. Then, we say that $u_{\beta^*}$ is the (unique) minimal entire radial solution of (1.1) and $(u_\beta)_{\beta \geq \beta^*}$ are a family of non–minimal radial solutions of (1.1). Moreover, the asymptotic behavior of all radially symmetric solutions of (1.1) was classified in [DFG10, Gue12, DN17]. A complete picture of radial entire solutions of Eq. (1.1) was demonstrated in [DN17, Table 1.1]. To be precise,

$$u_{\beta^*}(r) = \begin{cases}
O(r^{\frac{q}{q-1}}) & \text{if } 1 < q < 3, \\
O(r \log r) & \text{if } q = 3, \\
O(r) & \text{if } q > 3,
\end{cases}$$

(1.3)
and
\[ u_2(r) = O(r^2) \quad \text{if } q > 1 \text{ and } \beta > \beta^*. \tag{1.4} \]

Having (1.3) and (1.4) at hand, Guo, Wei and Zhou study further the radial property of singular positive entire solutions of (1.1) in [GWZ18]. The authors show the necessary and sufficient condition to claim that a positive entire solution \( u(x) \) of (1.1) as \( 1 < q < 3 \) which grows like a minimal radial entire solution of (1.1) at infinity is actually a minimal radial entire solution of (1.1). And that goal was also achieved for a positive regular entire solution \( u \) of (1.1) which has the asymptotic behavior at infinity as that of a non–minimal entire radial solution as \( q > 1 \).

In view of Guo-Wei-Zhou’s result, we answer the same question for a positive singular entire solution \( u \) of (1.1) which admits the asymptotic behavior at infinity as that of a minimal radial entire solution as \( q > 3 \). The main theorem is as follows.

**Theorem 1.1.** Let \( q > 3 \) and \( u \in C^4(\mathbb{R}^3) \) be a positive entire solution of (1.1). Then, \( u \) is a minimal radial entire solution of (1.1) if and only if there exists \( 0 < \vartheta < 1 \) and \( L > 0 \) such that
\[ |x|^{-1}u(x) - L = o(|x|^{-\vartheta}) \tag{1.5} \]
as \( |x| \to \infty \).

Inspired by the previous articles [Zou95, Guo02, GW07, GHZ15, GW18], where the authors studied that problem on equations of the form (1.1) involving Laplacian and bi–Laplacian also, the method of moving plane is still our key ingredient to show the necessary part of the main theorem. We first collect in Section 2 some basic properties of the eigenvalues and eigenfunctions of Laplacian and bi–Laplacian on \( S^2 \) and introduce the Kelvin transform of solution \( u \), i.e.
\[ y = \frac{x}{r^2}, \quad r = |x| > 0, \quad v(y) = |x|^{-1}u(x) - L. \tag{1.6} \]

Section 3 is devoted to establish an upper bound for
\[ W(s) := \left( \int_{S^2} w^2(s, \theta) d\theta \right)^{1/2} \tag{1.7} \]
where \( y = (s, \theta), s = |y| = r^{-1} \) and \( w(s, \theta) := v(s, \theta) - \overline{v}(s) \), \( \overline{v} \) is the spherical average of \( v \) on \( S^2 \), i.e.
\[ \overline{v}(s) = \frac{1}{|S^2|} \int_{S^2} v(s, \theta) d\theta. \tag{1.8} \]

Hence, by exploiting further estimates for \( v \) and \( \overline{v} \) near \( s = 0 \) in the next section, we deduce the asymptotic expansion at infinity of \( u \) and \( \Delta u \) in Theorem 4.2, which is crucial in our argument. In the final step, we transform (1.1) into a system of two partial differential equations as follows
\[
\begin{cases}
-\Delta u = w & \text{in } \mathbb{R}^3, \\
-\Delta w = -u^{-q} & \text{in } \mathbb{R}^3
\end{cases} \tag{1.9}
\]
and apply the method of moving plane to the system (1.9) to ensure the radial property. Eventually, we conclude the main theorem by noticing that the sufficiency follows from [Gue12, Theorem 1.3].

### 2. Preliminaries

In this section, we first state here the basic properties of the Laplace operator on \( S^2 \). It is well–known from [CH62] that the eigenvalues of the operator \(-\Delta_{S^2}\) are given by
\[ \lambda_k = k(k+1) \quad (k \in \mathbb{N}), \]
with the multiplicity $m_k = 2k + 1$ and we will denote the corresponding eigenfunctions by $Q_1^k, Q_2^k, \ldots, Q_{m_k}^k$. Without restricting the generality, we assume that
\[
\{Q_1^k(\theta), Q_2^k(\theta), \ldots, Q_{m_k}^1(\theta), Q_2^2(\theta), \ldots, Q_{m_k}^2(\theta), Q_3^2(\theta), \ldots\}
\]
is a standard normalized basis of $H^2(S^2)$. As indicated in [GHZ15, Lemma 2.1], the eigenvalues of $\Delta^2_{S^2}$ are of the form $\lambda_k^2(k \in \mathbb{N})$ with the same multiplicity. Hence, we obtain
\[
\int_{S^2} |\nabla_v w|^2 d\theta \geq 2 \int_{S^2} w^2 d\theta
\]
and
\[
\int_{S^2} |\Delta_v w|^2 d\theta \geq 4 \int_{S^2} w^2 d\theta
\]
for any function $w$ orthogonal to $Q_1^k$. Thanks to the bootstrap argument, we deduce that
\[
\max_{\theta \in \mathbb{S}^2} |Q_j^k(\theta)| \leq D_k, \quad \max_{\theta \in \mathbb{S}^2} |\nabla_v Q_j^k(\theta)| \leq E_k
\]
for $1 \leq j \leq m_k$, where
\[
D_k := C(1 + \lambda_k + \lambda_k^2 + \cdots + \lambda_k^{m_k}), \quad E_k := C(1 + \lambda_k + \lambda_k^2 + \cdots + \lambda_k^{m_k})
\]
with a positive constant $C$ independent of $k$ and $\tau_1, \tau_2 \in \mathbb{N}$ greater than 2.

Recall here the Kelvin transform (1.6), we obtain that the function $v$ satisfies
\[
\partial_v^4 v + 4s^{-1}\partial_v^3 v + 2s^{-4}\Delta_v v + 2s^{-2}\Delta_v(\partial_v^2 v) + s^{-4}\Delta^2_v v + s^{q-7}(v + \kappa)^{-q} = 0,
\]
which is a consequence of the following computation
\[
\Delta^2_{S^2} u = \left(\partial_v^4 + 4r^{-1}\partial_v^3 + 2r^{-4}\Delta_v + 2r^{-2}\Delta_v\partial_v^2 + r^{-4}\Delta^2_v\right)u.
\]
Furthermore, there exists two positive constants $M$ and $s^*$ depending only on $u$ such that
\[
\lim_{|y| \to 0} v(y) = 0, \quad |\nabla^l v(y)| \leq \frac{M}{s^*} \quad \text{for } s = |y| \leq s^*,
\]
due to the standard elliptic theory. Next, a direct calculation shows that $v$ and $w$ respectively fulfill
\[
\partial_v^4 v + 4s^{-1}\partial_v^3 v + s^{q-7}(v + L)^{-q} = 0,
\]
and
\[
\partial_v^4 w + 4s^{-1}\partial_v^3 w + 2s^{-4}\Delta_v w + 2s^{-2}\Delta_v(\partial_v^2 w) + s^{-4}\Delta^2_v w - s^{-4}g(w) = 0,
\]
where
\[
g(w) = s^{q-3}(v + L)^{-q} - s^{q-3}(v + L)^{-q}
\]
\[
= -qs^{q-3}\left[ (\xi(s, \theta) + L)^{-q-1}w(s, \theta) - (\xi(s, \theta) + L)^{-q-1}w(s, \theta) \right]
\]
and $\xi(s, \theta)$ is between $v(s, \theta)$ and $\overline{v}(s)$. Let denote
\[
\zeta(s) = \max_{\theta \in \mathbb{S}^2} |g(w)|
\]
\[
= \max_{\theta \in \mathbb{S}^2} \left| -qs^{q-3}\left[ (\xi(s, \theta) + L)^{-q-1}w(s, \theta) - (\xi(s, \theta) + L)^{-q-1}w(s, \theta) \right] \right|
\]
We see that $\zeta(s) = O(s^{q-3})$ and $\xi(s, \theta) \to 0$ as $s \to 0$. 
3. AN UPPER BOUND OF \( W(s) \) FOR \( s \) SMALL

This section is devoted to give a priori estimate of \( W(s) \), introduced in (1.7). We prove the following proposition.

**Proposition 3.1.** There exists a sufficiently small \( s_0 \) and \( C > 0 \) independent of \( s_0 \) such that for \( s \in (0, s_0) \)

\[ W(s) \leq C s. \]  

**Proof.** It is worth noting that \( w \in H^2(S^2) \subseteq L^2(S^2) \) and \( \overline{w}(s) = 0 \). Then, we have the expansion

\[ w(s, \theta) = \sum_{k=1}^{m_k} \sum_{j=1}^{m_k} w_j^k(s) Q_j^k(\theta). \]  

(3.2)

Substituting (3.2) into (2.5), we deduce that \( w_j^k(s) \) with \( 1 \leq j \leq m_k \) is such that

\[ \partial_j^k w_j^k + 4s^{-1} \partial_j^k w_j^k - 2\lambda_k s^{-2} \partial_j^k w_j^k - (2\lambda_k - \lambda_k^2) s^{-4} w_j^k = s^{-4} g_j^k(s), \]  

(3.3)

where

\[ g_j^k(s) = \int_{\mathbb{R}^2} g(w) Q_j^k(\theta) d\theta = \int_{\mathbb{R}^2} f'(\xi(s, \theta)) w(s, \theta) Q_j^k(\theta) d\theta. \]

which is bounded by

\[ |g_j^k(s)| \leq C |\zeta(s) W(s) = O(s^{-3}) W(s) \]  

for \( s \) near 0. Furthermore, to prove (3.1), we only need to consider the case

\[ |g_j^k(s)| = o_s(1) |w_j^k(s)|. \]

Indeed, note that \( g_j^k(s) \) and \( w_j^k(s) \) are Fourier coefficients of \( f'(\xi) w(s, \theta) \) and \( w(s, \theta) \), respectively. One obtains

\[ \|f'(\xi) w(s, \theta)\|_{L^2(S^2)} \leq \zeta(s) \|w(s, \theta)\|_{L^2(S^2)} = o_s(1) \|w\|_{L^2(S^2)}, \]

which yields

\[ \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |g_j^k(s)|^2 = o_s(1) \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |w_j^k(s)|^2. \]

Set

\[ G_s = \{(j, k) : k \geq 1, 1 \leq j \leq m_k \text{ such that } |g_j^k(s)| = o_s(1) |w_j^k(s)|\}, \]

\[ B_s = \{(j, k) : k \geq 1, 1 \leq j \leq m_k \text{ such that } g_j^k(s) \neq o_s(1) w_j^k(s)\}. \]

Now, we show that there exists \( 0 < \tilde{s} < s^* \) (\( s^* \) is given in (2.3)) and \( C > 0 \) independent of \( j, k \) and \( s \) such that for any \( 0 < s < \tilde{s} \) we have

\[ |g_j^k(s)| \geq C |w_j^k(s)| \]

for \((j, k) \in B_s\). Indeed, we contradict that there exists \( c_n \to 0 \) and \( s_n \to 0 \) as \( n \to \infty \) such that

\[ |g_j^{k_n}(s_n)| \leq c_n |w_j^{k_n}(s_n)| \]

for large \( n \) and \((j_n, k_n) \in B_{s_n}\). Then,

\[ |g_j^{k_n}(s_n)| \leq o_s(1) |w_j^{k_n}(s_n)| \]

for large \( n \), which contradicts \((j_n, k_n) \in B_{s_n}\). Hence, for any \( s \in (0, \tilde{s}) \),

\[ \sum_{(j, k) \in B_s} |w_j^k(s)|^2 \leq C^{-2} \sum_{(j, k) \in B_s} |g_j^k(s)|^2 = o_s(1) \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |w_j^k(s)|^2. \]

Then, we assume that in the rest of the proof of Proposition 3.1, there holds

\[ |g_j^k(s)| = o_s(1) |w_j^k(s)| \]

(3.4)
Hence, there exists a constant $C$. In view of (3.3), we will show that

$$\text{Case 1:}$$

by considering two following cases.

Let $t = -\ln s$ and $z_j(t) = w_j^k(s)$. Then, (3.3) becomes

$$\partial_t^4 z_j^k + 2\partial_t^2 z_j^k - (1 + 2\lambda_k)\partial_t z_j^k - 2(1 + \lambda_k)\partial_s z_j^k + \lambda_k(\lambda_k - 2)z_j^k = f_j^k(t),$$

where $f_j^k(t) = g_j^k(e^{-t})$. The corresponding characteristic polynomial of (3.5) is

$$\mu^4 - 2\mu^3 - (1 + 2\lambda_k)\mu^2 - 2(1 + \lambda_k)\mu + \lambda_k(\lambda_k - 2) = 0,$$

which has the following roots

$$\mu_1^{(k)} = -k - 2, \quad \mu_2^{(k)} = -k, \quad \mu_3^{(k)} = k - 1, \quad \mu_4^{(k)} = k + 1.$$

Due to the variation of parameters formula, we will show that

$$|z_j^k(t)| = O(e^{-k t})$$

by considering two following cases.

**Case 1:** $k \geq 2$. Due to the fact that $z_j^k(t)$ tends to 0 as $t$ tends to $\infty$, one obtains

$$z_j^k(t) = A_{j,1}e^{-(k+2)t} + A_{j,2}e^{-kt}$$

$$+ B_{j,3} \int_t^\infty e^{(k+1)(t-\tau)} f_j^k(\tau)d\tau + B_{j,4} \int_t^\infty e^{(k-1)(t-\tau)} f_j^k(\tau)d\tau$$

$$+ B_{j,5} \int_t^\infty e^{-(k+2)(t-\tau)} f_j^k(\tau)d\tau + B_{j,6} \int_t^\infty e^{-(k-2)(t-\tau)} f_j^k(\tau)d\tau.$$ 

Hence, there exists a constant $C$ depending only on $B_{j,i}^k$ ($i = 1, 2, 3, 4$) such that

$$|z_j^k(t)| \leq O(e^{-kt}) + C \int_t^\infty e^{(i-1)(t-\tau)} |f_j^k(\tau)|d\tau + C \int_T^\infty e^{-k(t-\tau)} |f_j^k(\tau)|d\tau.$$  

(3.7)

In view of (3.4), we obtain that $|f_j^k(t)| = o_t(1)|z_j^k(t)|$ for $t \in (T, \infty)$. Then, we substitute it into (3.7) to get that

$$|z_j^k(t)| \leq O(e^{-kt}) + C \int_t^\infty e^{(k-1)(t-\tau)} o_t(1)|z_j^k(\tau)|d\tau$$

$$+ C \int_T^\infty e^{-k(t-\tau)} o_t(1)|z_j^k(\tau)|d\tau.$$  

(3.8)

Note that for any $\epsilon$ small enough, there exists $t$ large such that $o_t(1) < \epsilon$. Let

$$K_1(t) = \int_t^\infty e^{(k-1)(t-\tau)} |z_j^k(\tau)|d\tau,$$

and

$$K_2(t) = \int_T^\infty e^{-k(t-\tau)} |z_j^k(\tau)|d\tau.$$ 

Clearly $\lim_{t \to \infty} K_1(t) = \lim_{t \to \infty} K_2(t) = 0$ by $\lim_{t \to \infty} z_j^k(t) = 0$. Hence,

$$(K_2 - K_1)'(t) = 2|z_j^k(t)| - (k-1)K_1(t) - kK_2(t)$$

$$\leq 2C\epsilon(K_1(t) + K_2(t)) - (k-1)K_1(t) - kK_2(t) + O(e^{-kt})$$

$$\leq O(e^{-kt}),$$

which yields

$$K_1(t) - K_2(t) \leq O(e^{-kt}),$$

or

$$K_1(t) \leq K_2(t) + O(e^{-kt}).$$

This implies that

$$K_2'(t) = |z_j^k(t)| - kK_2(t) \leq O(e^{-kt}) + (2C\epsilon - k)K_2(t).$$
Thus, \( K_2(t) = O(e^{2C_\varepsilon e^{-kt}}) \). Plugging it into (3.8), one gets that \( |z^k(t)| = O(e^{-kt}) \).

**Case 2:** \( k = 1 \). One has

\[ \mu_1^{(1)} = -3 < \mu_2^{(1)} = -1 < \mu_3^{(1)} = 0 < \mu_4^{(1)} < 2. \]

We also obtain that

\[
z_j^1(t) = A_{j,1}^1 e^{-t} + A_{j,3}^1 e^{-3t} - B_4^1 \int_t^\infty e^{2(t-\tau)} f_j^1(\tau) d\tau
- B_3^1 \int_t^\infty f_j^1(\tau) d\tau + B_2^1 \int_T^t e^{-(t-\tau)} f_j^1(\tau) d\tau + B_1^1 \int_T^t e^{-3(t-\tau)} f_j^1(\tau) d\tau.
\]

It then follows that

\[
|z_j^1(t)| = O(e^{-(t-T)}) + C \int_T^\infty |o_1(1)z_j^1(\tau)| d\tau,
\]

for \( 1 \leq j \leq m_1 \) and \( t > T \). Under the assumption (1.5), one gets \( |w(s, \theta)|^2 \leq Cs^{2\theta} \) for \( s \) near 0. Thus,

\[
\int_0^s \xi^{-1} |w_j^1(\xi)| d\xi \leq \int_0^s \xi^{-1} \left( \int_{\mathbb{S}^2} w^2(s, \theta) d\theta \right)^{1/2} d\xi \leq C \int_0^s \xi^{-(1-\theta)} d\xi < \infty.
\]

Equivalently, we have just shown that \( \int_t^\infty |z_j^1(\tau)| d\tau < \infty \). Let us define

\[ K(t) = \int_t^\infty |z_j^1(\tau)| d\tau, \]

that gives

\[ -K'(t) = |z_j^1(t)| \leq O(e^{-t}) + C\varepsilon K(t), \]

which yields \( K(t) = O(e^{-t}) \). From this, we turn back to (3.9) to get that \( |z_j^1(t)| = O(e^{-t}) \).

Thanks to (3.6), one has

\[
\sum_{k=1}^\infty \sum_{j=1}^{m_k} |z_j^k(t)| \leq O(e^{-t}) + O\left( \sum_{k=2}^\infty k m_k e^{-kt} \right).
\]

For \( t > T_1 > T \), one has

\[
\sum_{k=2}^\infty k m_k e^{-k(t-T)} = O(e^{-2(t-T)}),
\]

by observing

\[
\lim_{k \to \infty} \frac{(k+1)m_{k+1} e^{-(k+1)(t-T)}}{km_k e^{-k(t-T)}} = e^{-(t-T)} \lim_{k \to \infty} \frac{(k+1)m_{k+1}}{km_k} = e^{-(t-T)} < \frac{1}{2}.
\]

Then,

\[
\sum_{k=1}^\infty \sum_{j=1}^{m_k} |z_j^k(t)| \leq O(e^{-t}),
\]

which implies (3.1) for \( 0 < s < s_0 = e^{-T_1} \). Proof of Proposition 3.1 is complete. \( \square \)
4. EXPANSION OF $u$ AT INFINITY

In this section, the main result is given in Theorem 4.2 below. To carry out our analysis, the following propositions are needed.

**Proposition 4.1.** Let $v$ be solution of Eq. (2.2). Then, there exist positive constants $M = M(v)$ and $\varpi \in (0, 1/10)$ such that

$$\begin{cases}
|\nabla(s)| \leq Ms^q, & |\nabla^2(s)| \leq Ms^{q-1}, \\
|\nabla^3(s)| \leq Ms^{q-2} & \text{if } 3 < q < 4,
\end{cases}$$

$$\begin{cases}
|\nabla(s)| \leq Ms^{1-\varpi}, & |\nabla^2(s)| \leq Ms^{-\varpi}, \\
|\nabla^3(s)| \leq Ms^{-1-\varpi} & \text{if } q \geq 4
\end{cases}$$

(4.1)

for $s$ small enough. Furthermore, there also holds

$$\int_{\mathbb{R}^n} v^2(s, \theta)d\theta \leq \begin{cases}
Ms^{2(q-3)} & \text{if } 3 < q < 4, \\
Ms^{2(1-\varpi)} & \text{if } q \geq 4.
\end{cases}$$

(4.2)

**Proof.** Similar to Proposition 3.1, we also transform (2.4) into

$$\Delta_t \nabla + 2\Delta_t \nabla^2 - 2\partial_t \nabla = h(\nabla) + O(e^{-t}),$$

(4.3)

where $t = -\ln s, \nabla(t) = \nabla(s)$ and

$$h(\nabla) = s^{q-3}(\nabla + L)^{-q} = O(e^{-(q-3)t}).$$

(4.4)

The corresponding characteristic polynomial of (4.3) is

$$\mu^4 + 2\mu^3 - \mu^2 - 2\mu = 0,$$

which has four roots $-1, 0, 1$ and $-2$. Notice that $\nabla(t)$ tends to 0 as $t$ tends to infinity, hence

$$\nabla(t) = A_1e^{-t} + A_2e^{-2t} + B_1\int_t^\infty e^{(t-\tau)\nabla(\tau)}d\tau + B_2\int_t^\infty \nabla(\tau)d\tau + B_3\int_{-\infty}^t e^{-(t-\tau)\nabla(\tau)}d\tau + B_4\int_T^t e^{-2(t-\tau)\nabla(\tau)}d\tau,$$

(4.5)

where $\nabla(t) = h(\nabla(t)) + O(e^{-t})$. In addition, (4.4) leads us to

$$\nabla(t) = O(e^{-\min(q-3,1)t}).$$

(4.6)

Plugging (4.6) into (4.5), one has

$$|\nabla(t)| \leq \begin{cases}
Me^{-(1-\varpi)t} & \text{if } q \geq 4, \\
Me^{-(q-3)t} & \text{if } 3 < q < 4
\end{cases}$$

(4.7)

for large $t$ and sufficiently small $\varpi$, which is equivalent to

$$|\nabla(s)| \leq \begin{cases}
Ms^{1-\varpi} & \text{if } q \geq 4, \\
Ms^{q-3} & \text{if } 3 < q < 4
\end{cases}$$

(4.8)

for small $s$. In order to deduce the rest of (4.1), we calculate the first and second derivative $\nabla(t)$ and notice that $\nabla^2(s) = -\nabla(t)e^t$ and that $\nabla^3(s) = (\nabla^2(t) + \nabla(t))e^t$. (4.2) is obtained by Proposition 3.1 and (4.8). \hfill \Box

**Proposition 4.2.** Let $\tau \geq 0$ be an integer, $v$ be a solution of (2.2) and $\varpi$ be defined as in Proposition 4.1. Then, there exists $M = M(v, \tau) > 0$ such that for $s$ small enough,

$$\max_{|y|=s} |D^\tau v(y)| \leq \begin{cases}
Ms^{1-\varpi-\tau} & \text{if } 3 < q < 4, \\
Ms^{1-\varpi} & \text{if } q \geq 4.
\end{cases}$$

(4.9)
Proof. For the case \( \tau = 0 \), we obtain
\[
\max_{\theta \in \mathbb{S}^2} |z(t, \theta)| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |z_j^k(t)| \max_{\theta \in \mathbb{S}^2} |Q_j^k(\theta)| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} D_k |z_j^k(t)|,
\]
where \( z(t, \theta) = w(s, \theta) \) and \( D_k \) is given in (2.1). Note that
\[
\lim_{k \to \infty} \frac{(k + 1)m_{k+1}D_{k+1}}{km_kD_k} = 1,
\]
then a same argument of that in the proof of Proposition 3.1 implies that there exists positive \( C \) independent of \( t \) and large \( T \) such that for \( t > T \),
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} D_k |z_j^k(t)| = O\left( \sum_{k=2}^{\infty} km_kD_k e^{-k(t-T)} \right) + O(e^{-(t-T)}) \leq Ce^{-t},
\]
which yields
\[
\max_{\theta \in \mathbb{S}^2} |z(t, \theta)| \leq Ce^{-t}.
\]
Equivalently,
\[
\max_{\theta \in \mathbb{S}^2} |w(s, \theta)| \leq Cs
\]
for \( 0 < s < e^{-\tau} \). Combining with (4.1), this guarantees (4.9) when \( \tau = 0 \). For the case \( \tau = 1 \), we see that
\[
|\nabla w|^2 = w_s^2 + \frac{1}{s^2} |\nabla \theta w|^2.
\]
One has
\[
w_s(s, \theta) = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (u_j^k)'(s)Q_j^k(\theta).
\]
Hence,
\[
\max_{\theta \in \mathbb{S}^2} |w_s(s, \theta)| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} D_k |(u_j^k)'(s)|.
\]
Notice that \((u_j^k)'(s) = -(z_j^k)'(t)e^t\), hence
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} D_k |(u_j^k)'(s)| \leq O(1) + O\left( \sum_{k=2}^{\infty} km_kD_k s^{k-1} \right).
\]
This implies that there exists \( M_1 = M_1(v) \) such that for \( s \) sufficiently small,
\[
\max_{\theta \in \mathbb{S}^2} |w_s(s, \theta)| \leq M_1. \tag{4.10}
\]
We also obtain that
\[
|\nabla \theta w(s, \theta)| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |u_j^k(s)||\nabla Q_j^k| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} E_k |u_j^k(s)|,
\]
and notice that
\[
\lim_{k \to \infty} \frac{(k + 1)m_{k+1}E_{k+1}}{km_kE_k} = 1.
\]
Consequently, one gets that there also exists \( M_2 = M_2(v) \) such that for \( s \) sufficiently small,
\[
\max_{\theta \in \mathbb{S}^2} |\nabla \theta w(s, \theta)| \leq M_2 s. \tag{4.11}
\]
Thanks to (4.10) and (4.11), we deduce that
\[
|\nabla w| \leq M_1^2 + M_2^2,
\]
i.e. (4.9) holds for \( \tau = 1 \). By differentiating \( w(s, \theta) \), we conclude the other cases of \( \tau \). Proof of Proposition 4.2 is complete. \( \square \)
Lemma 4.1. Let \( \tilde{w}(s, \theta) = w(s, \theta)/s \), there holds
\[
\lim_{s \to 0^+} \tilde{w}(s, \theta) = V(\theta),
\]
where \( V(\theta) = \theta \cdot x^* \) for some \( x^* \in \mathbb{R}^3 \) fixed and \( \theta \in \mathbb{S}^2 \).

Proof. By computing directly, it follows from (2.5) that \( \tilde{w} \) is the solution of the following equation
\[
\partial_s^4 \tilde{w} + 8s^{-1} \partial_s^3 \tilde{w} + 12s^{-2} \partial_s^2 \tilde{w} + 2s^{-4} \Delta_\theta \tilde{w} + 4s^{-2} \Delta_\theta (\partial_s \tilde{w}) + 2s^{-3} \Delta_\theta (\partial_s^2 \tilde{w}) + s^{-4} \Delta_\theta^2 \tilde{w} = s^{-4} g(\tilde{w}),
\]
where
\[
g(\tilde{w}) = -qs^{q-3} \left[ (\xi(s, \theta) + L)^{-q-1} \tilde{w}(s, \theta) - \frac{q}{q-1} \xi(s, \theta) \right]
\]
and \( \xi(s, \theta) \) is between \( v(s, \theta) \) and \( \overline{v}(s) \). Notice that \( \overline{w} = \overline{w}/s = 0 \), hence
\[
\tilde{w}(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \tilde{w}_j^k(s)Q_j^k(\theta),
\]
where \( \tilde{w}_j^k(s) = w_j^k(s)/s \). A direct computation shows that
\[
\partial_s^4 \tilde{w}_j^k + 8s^{-1} \partial_s^3 \tilde{w}_j^k + (12 - 2\lambda_k)s^{-2} \partial_s^2 \tilde{w}_j^k + 4\lambda_k s^{-3} \partial_s \tilde{w}_j^k + (\lambda_k^2 - 2\lambda_k) s^{-4} \tilde{w}_j^k = s^{-4} \tilde{g}_j^k(s),
\]
where \( \tilde{g}_j^k(s) = \int_{\mathbb{S}^2} g(\tilde{w})Q_j^k(\theta)d\theta \). Notice that
\[
|\tilde{g}_j^k(s)| \leq O(s^{q-3})\tilde{W}(s),
\]
where \( \tilde{W}(s) = \left( \int_{\mathbb{S}^2} |\tilde{v}(s, \theta)|^2 d\theta \right)^{1/2} \).

Next, we repeat the process used in Proposition 3.1. Making change of variable \( t = -\ln s \) and letting \( \tilde{z}_j^k(t) = \tilde{w}_j^k(s) \). Hence
\[
\partial_s^4 \tilde{z}_j^k - 2\mu_1 \partial_s^3 \tilde{z}_j^k - (1 + 2\lambda_k)\partial_s^2 \tilde{z}_j^k + 2(1 + \lambda_k)\partial_s \tilde{z}_j^k + \lambda_k \tilde{z}_j^k = \tilde{g}_j^k(t),
\]
where \( \tilde{g}_j^k(t) = \tilde{g}_j^k(s) \). The corresponding characteristic polynomial of (4.16) is
\[
\tilde{\mu}^4 - 2\tilde{\mu}^3 - (1 + 2\lambda_k)\tilde{\mu}^2 + 2(1 + \lambda_k)\tilde{\mu} + \lambda_k (\lambda_k - 2) = 0,
\]
which has 4 real roots as follows
\[
\tilde{\mu}_{1,2}^{(k)} = -k - 1, \quad \tilde{\mu}_2^{(k)} = -k, \quad \tilde{\mu}_3^{(k)} = k, \quad \tilde{\mu}_4^{(k)} = k + 2.
\]
It follows from (3.6) that \( \lim_{s \to 0^+} \tilde{w}_j^k(s) = 0 \) for \( k \geq 2 \) and that \( \tilde{w}_j^1(s) \) is bounded for \( s \) near 0. Hence, \( \tilde{z}_j^1(t) \) is bounded for \( t \) near infinity. We also note that \( |\tilde{W}(s)| = O(1) \) for \( s \) near 0 from (3.1). Thus, (4.15) tells us that \( |\tilde{g}_j^k(s)| = O(s^{q-3}) \), which means \( |\tilde{g}_j^k(t)| = O(e^{-(q-3)t}) \) for \( t \) near infinity. Combining those above facts, one gets for \( t > T \) large enough
\[
\tilde{z}_j^1(t) = C + Ae^{-2t} + B_1 \int_t^\infty e^{3\tau}O(e^{-(q-3)\tau})d\tau + B_2 \int_t^\infty e^\tau O(e^{-(q-3)\tau})d\tau + B_3 \int_T^t O(e^{-(q-3)\tau})d\tau + B_4 \int_T^t e^{-2\tau}O(e^{-(q-3)\tau})d\tau.
\]
This implies that \( \tilde{z}_j^1(t) \) converges to a constant as \( t \to \infty \), i.e. \( \tilde{w}_j^1(s) \) converges to a constant as \( s \to 0 \) for all \( 1 \leq j \leq m_1 \). Note that \( Q_1^1(\theta), Q_2^1(\theta), \ldots \) and \( Q_{m_1}^1(\theta) \) are the eigenfunctions corresponding to the eigenvalue \( \lambda_1 = 2 \). Thus, we deduce that
\[
\lim_{s \to 0^+} \tilde{w}(s, \theta) = V(\theta),
\]
where \( V(\theta) \equiv 0 \) or one of the first eigenfunctions of \(-\Delta_{\mathbb{S}^2}\). From the well–known result [Zou95, Lemma 8.1], we conclude that \( V(\theta) = \theta \cdot x^* \) for some \( x^* \in \mathbb{R}^3 \) fixed. \( \square \)

We sum up the previous results.

**Theorem 4.1.** Let \( v \) be a solution of (2.2) and \( \tilde{w} \) defined in Lemma 4.1. Then, \( v(y) = \overline{\nu}(s) + s\tilde{w}(s) \), with \( \overline{\nu} \) and \( \tilde{w} \) satisfy the following properties.

1. For \( s \) sufficiently small,
   \[
   |\overline{\nu}(s)| = O(s^{1-\varpi}), \quad |\overline{\nu}'(s)| = O(s^{-\varpi}), \quad |\overline{\nu}''(s)| = O(s^{-1-\varpi}) \quad q \geq 4,
   \]
   or
   \[
   |\overline{\nu}(s)| = O(s^{q-3}), \quad |\overline{\nu}'(s)| = O(s^{q-4}), \quad |\overline{\nu}''(s)| = O(s^{q-5}) \quad 3 < q < 4.
   \]

2. For any nonnegative integers \( \tau_1 \) and \( \tau_2 \),
   \[
   |D_\theta^\tau D_s^{\tau_2} \tilde{w}(s, \theta)| = O(s^{-\tau_2}).
   \]

Furthermore, \( \tilde{w}(s, \theta) \) converges uniformly in \( C^\tau_{\nu}(\mathbb{S}^2) \) to \( V(\theta) \) which is 0 or one of the first eigenfunctions of \( \Delta_{\theta} \) on \( \mathbb{S}^2 \) as \( s \to 0 \).

With Theorem 4.1 in hand, we are able to obtain the asymptotic expansion of the solution \( u \) of (1.1).

**Theorem 4.2.** Let \( u \) be a solution of (1.1) under the assumption (1.5). Then, \( u \) and \( -\Delta u \) admit the expansion

\[
\begin{cases}
  u(x) = L + \xi(r) + \frac{\eta(r, \theta)}{r}, \\
  -\Delta u(x) = -\frac{1}{r} \left( 2L + \xi_1(r) + \frac{\eta_1(r, \theta)}{r} \right).
\end{cases}
\]  

(4.17)

at infinity, where

\[
\begin{aligned}
  \xi_1(r) &= - \left( r^2 \xi'' + 6r \xi' + 6\xi \right), \\
  \eta_1(r, \theta) &= - \left( r^2 \partial_\theta^2 \eta + 4r \partial_\eta + 2\eta + \Delta_\theta \eta \right).
\end{aligned}
\]

The functions \( \xi \) and \( \eta \) satisfy the following properties.

1. \( \xi(r) = r^{-1} \overline{\nu}(r) - L \) and for \( r \) large enough, we have
   \[
   |\xi(r)| = O(r^{-1+\varpi}), \quad |\xi'(r)| = O(r^{-2+\varpi}), \quad |\xi''(r)| = O(r^{-3+\varpi}) \quad \text{if} \quad q \geq 4,
   \]
   or
   \[
   |\xi(r)| = O(r^{-(q-3)}), \quad |\xi'(r)| = O(r^{-(q-2)}), \quad |\xi''(r)| = O(r^{-(q-1)}) \quad \text{if} \quad 3 < q < 4.
   \]

2. Let \( \tau_1 \) and \( \tau_2 \) be two non–negative integers. There exists a positive constant \( M = M(u, \tau_1, \tau_2) \) such that
   \[
   |r^{\tau_2} D_\theta^\tau D_s^{\tau_2} \eta(r, \theta)| \leq M \quad \text{and} \quad |\eta_1(r, \theta)| \leq M \quad \text{for} \quad r \to \infty.
   \]

3. Let \( \tau \) be a non–negative integer. Then \( \eta(r, \theta) \) tends to \( V(\theta) \) uniformly in \( C^\tau(\mathbb{S}^2) \) as \( r \to \infty \), where \( V(\theta) \) is given in Lemma 4.1.
5. Proof of Theorem 1.1

In order to give the proof of Theorem 1.1, we use the method of moving plane. For \( \gamma \in \mathbb{R} \), let us define the hyperplane \( \Gamma_\gamma = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_1 = \gamma \} \). For any \( x \in \mathbb{R}^3 \), we denote the reflection point of \( x \in \mathbb{R}^3 \) about \( \Gamma_\gamma \) by \( x^\prime \) and the first component of \( x \in \mathbb{R}^3 \) by \( (x)_1 \). Our next lemma is a consequence of Theorem 4.2 and the proof follows from [Zou95, Lemma 8.2] with a slight modification.

**Lemma 5.1.** Let \( u \) be solution of \((1.1)\) under the assumption \((1.5)\).

1. Let \( \{ \gamma_j \} \) be a real sequence that converges to \( \gamma \in \mathbb{R} \cup \{ \infty \} \) and \( \{ x^j \} \) be an unbounded sequence in \( \mathbb{R}^3 \) with \( (x^j)_1 < \gamma_j \) for all \( j \geq 1 \). There holds,

\[
\lim_{j \to \infty} \frac{|x^j|}{\gamma - (x^j)_1} \left( u(x^j) - u((x^j)\gamma) \right) = -2L\gamma - 2(x_0)_1, \tag{5.1}
\]

where \( x_0 \) is given in Lemma 4.1.

2. We have

\[
\frac{\partial u}{\partial x_1} \geq 0 \quad \text{if} \quad x_1 \geq \gamma_0 + 1 \quad \text{and} \quad |x| \geq M,
\]

for some constants \( M = M(u) \), where

\[
\gamma_0 = \frac{- (x_0)_1}{2L}.
\]

**Proof.** Part (1). Without restricting the generality, we assume that

\[
\lim_{j \to \infty} \frac{x^j}{|x^j|} = \overline{\gamma} \in \mathbb{S}^2,
\]

and that \( \gamma_j = \gamma \) for all \( j \) since the next arguments work equally well for the sequence \( \{ \gamma_j \} \).

It follows from the expansion of \( u \) in \((4.17)\) that

\[
\frac{|x^j|}{\gamma - (x^j)_1} \left( u(x^j) - u((x^j)\gamma) \right) = \frac{L|x^j|}{\gamma - (x^j)_1} \left( |x^j| - |(x^j)\gamma| \right) \quad := I
\]

\[
+ \frac{|x^j|}{\gamma - (x^j)_1} \left( |x^j|\xi(|x^j|) - |(x^j)\gamma|\xi(|(x^j)\gamma|) \right) \quad := II
\]

\[
+ \frac{|x^j|}{\gamma - (x^j)_1} \left( \eta(|x^j|, \theta^j) - \eta(|(x^j)\gamma|,(\theta^j)\gamma) \right) \quad := III.
\]

Computing directly,

\[
|x^j|(|x^j| - |(x^j)\gamma|) = \frac{4|x^j|\gamma((x^j)_1 - \gamma)}{|x^j| + |(x^j)\gamma|},
\]

which implies

\[
\lim_{j \to \infty} I = -4L\gamma. \tag{5.3}
\]

Here we have used the fact that \( |x^j|/(x^j)\gamma| \to 1 \) as \( j \to \infty \). About \( II \), there is \( \beta_j \) between \( |x^j| \) and \( |(x^j)\gamma| \) such that

\[
II = \frac{|x^j|}{\gamma - (x^j)_1} \left( |x^j|\xi(|x^j|) - |(x^j)\gamma|\xi(|(x^j)\gamma|) \right) = \frac{-4\gamma|x^j|}{|x^j| + |(x^j)\gamma|} \left( \xi(\beta_j) + \beta_j\xi'(\beta_j) \right).
\]
For large $j$, it can be seen from Theorem 4.2 that
\[
|\beta_j| |\xi'(\beta_j)| + |\xi(\beta_j)| \leq \begin{cases} 2M|\beta_j|^{-1-\varpi} & \text{if } q \geq 4, \\ 2M|\beta_j|^{-q-3} & \text{if } 3 < q < 4. \end{cases}
\]
Hence,
\[
II \leq \begin{cases} O(|x^j|^{-1-\varpi}) \to 0 & \text{for } q \geq 4, \\ O(|x^j|^{-q-3}) \to 0 & \text{for } 3 < q < 4. \end{cases} \tag{5.4}
\]
We now deal with the last term $III$ in (5.2) by splitting $III$ into two terms $III_1$ and $III_2$ as follows,
\[
III_1 := \frac{|x^j|}{\gamma - (x^j)_1} \left( \eta(|x^j|, \theta^j) - \eta(|x^j|, (\theta^j)\gamma) \right)
\]
and
\[
III_2 := \frac{|x^j|}{\gamma - (x^j)_1} \left( \eta(|x^j|, (\theta^j)\gamma) - \eta(|x^j|, (\theta^j)\gamma) \right).
\]
Using the mean value theorem, there exists $\beta_j$ between $|x^j|$ and $|(x^j)\gamma|$ such that
\[
III_2 = \frac{|x^j|(|x^j| - |(x^j)\gamma|) D_x \eta(\beta_j, (\theta^j)\gamma)}{\gamma - (x^j)_1},
\]
which implies
\[
|III_2| = \left| \frac{-4\gamma|x^j|D_x \eta(\beta_j, (\theta^j)\gamma)}{|x^j| + |(x^j)\gamma|} \right| \leq O(|x^j|^{-1}) \to 0 \text{ as } j \to \infty. \tag{5.5}
\]
To estimate $III_1$, we suppose that
\[
\lim_{j \to \infty} \frac{(x^j)^\gamma}{|(x^j)\gamma|} = \hat{\theta}
\]
and consider two cases $\overline{\theta} \neq \hat{\theta}$ and $\overline{\theta} = \hat{\theta}$. If $\overline{\theta} \neq \hat{\theta}$, we observe that
\[
\frac{(\gamma - (x^j)_1, 0, 0)}{|x^j|} \to \frac{1}{2}(\hat{\theta} - \overline{\theta}) \text{ as } j \to \infty.
\]
Hence
\[
\frac{\gamma - (x^j)_1}{|x^j|} \to \frac{1}{2}(\hat{\theta} - \overline{\theta})_1 \text{ as } j \to \infty,
\]
which yields
\[
III_1 \to \frac{2(V(\overline{\theta}) - V(\hat{\theta}))}{(\hat{\theta} - \overline{\theta})_1} = \frac{2(\hat{\theta} - \overline{\theta}) \cdot x_0}{(\hat{\theta} - \overline{\theta})_1} = -2(x_0)_1. \tag{5.6}
\]
If $\overline{\theta} = \hat{\theta}$, we obtain that there exists a point $\beta^j$ between $\theta^j$ and $(\theta^j)\gamma$ on a geodesic on $S^2$ such that
\[
\eta(|x^j|, \theta^j) - \eta(|x^j|, (\theta^j)\gamma) = \nabla_\theta \eta(|x^j|, \beta^j) (\theta^j - (\theta^j)\gamma) = \nabla_\theta \eta(|x^j|, \beta^j) \cdot \left( \frac{x^j}{|x^j|} - \frac{(x^j)\gamma}{|x^j|} \right) + \nabla_\theta \eta(|x^j|, \beta^j) \cdot (x^j)\gamma \left( \frac{1}{|x^j|} - \frac{1}{|(x^j)\gamma|} \right)
\]
\[
= \frac{-2(\gamma - (x^j)_1)}{|x^j|} \left( \partial_{\theta_1} \eta(|x^j|, \beta^j) + O\left( \frac{1}{|x^j|} \right) \right).
\]
Consequently,
\[
III_1 = -2\left( \partial_{\theta_1} \eta(|x^j|, \beta^j) + O\left( \frac{1}{|x^j|} \right) \right) \to -2\partial_{\theta_1} V(\hat{\theta}) = -2\partial_{\theta_1} V(\overline{\theta}) = -2(x_0)_1 \tag{5.7}
\]
as $j \to \infty$. Plugging (5.3), (5.4), (5.5), (5.6) and (5.7) into (5.2), one gets our desired limit (5.1).
Part (2). Suppose that there exists an unbounded sequence \( x^j \) such that
\[
\frac{\partial u}{\partial x_1}(x^j) < 0, \quad (x^j)_1 \geq \gamma_0 + 1 \quad \forall j \in \mathbb{N}.
\]
Thus, there exists a bounded sequence of positive numbers \( \{a_j\} \) such that
\[
u(x^j) > \nu(x_{a_j}), \quad x_{a_j} = x^j + (2a_j, 0, \ldots, 0) \quad \forall j \in \mathbb{N}.
\]
Let define \( \gamma_j = (x^j)_1 + a_j > (x^j)_1 \) to get that
\[
\frac{|x^j|}{\gamma_j - (x^j)_1} \left( u(x^j) - \nu((x^j)^{\gamma}) \right) > 0. \tag{5.8}
\]
We split our next arguments in two cases.

Case 1. \( \lim \inf_j \gamma_j < \infty \). Due to a passage to a subsequence, we can assume that \( \gamma_j \to \gamma \geq \gamma_0 + 1 \) as \( j \to \infty \). Thus, (5.1) implies that
\[
\lim_{j \to \infty} \frac{|x^j|}{\gamma_j - (x^j)_1} \left( u(x^j) - \nu((x^j)^{\gamma}) \right) = -2L\gamma - 2(x_0)_1 \leq -2L < 0,
\]
a contradiction to (5.8).

Case 2. \( \gamma = \infty \). We first observe that
\[
\lim_{j \to \infty} \frac{|x^j|}{\gamma_j - (x^j)_1} = 1
\]
and that \( \gamma_j \leq |(x^j)^{\gamma_j}| \) since the definition of \( \gamma_j \). Repeating the arguments above, we obtain that
\[
\frac{L|x^j|}{\gamma_j - (x^j)_1} \left( |x^j| - |(x^j)^{\gamma_j}| \right) = -4L\gamma_j(1 + o(1)), \tag{5.9}
\]
that
\[
\frac{|x^j|}{\gamma_j - (x^j)_1} \left( \eta(|x^j|, (\theta^j)^{\gamma_j}) - \eta(|(x^j)^{\gamma_j}|, (\theta^j)^{\gamma_j}) \right) = O \left( \frac{\gamma_j}{|x^j|} \right) = O(1), \tag{5.11}
\]
and that
\[
\frac{|x^j|}{\gamma_j - (x^j)_1} \left( \eta(|x^j|, (\theta^j)^{\gamma_j}) - \eta(|(x^j)^{\gamma_j}|, (\theta^j)^{\gamma_j}) \right) = O \left( \frac{\gamma_j}{|x^j|} \right) = O(1). \tag{5.12}
\]
Combining (5.9), (5.10), (5.11) and (5.12) gives us
\[
\frac{|x^j|}{\gamma_j - (x^j)_1} \left( u(x^j) - u((x^j)^{\gamma}) \right) = -4L\gamma_j(1 + o(1)) + O(\gamma_j^{\infty}) + O(1) + O(1) \to -\infty
\]
if \( q \geq 4 \) or
\[
\frac{|x^j|}{\gamma_j - (x^j)_1} \left( u(x^j) - u((x^j)^{\gamma}) \right) = -4L\gamma_j(1 + o(1)) + O(\gamma_j^{1-q}) + O(1) + O(1) \to -\infty
\]
if \( 3 < q < 4 \). Those contradict (5.8) again. Thus, we conclude the proof of the lemma. \( \Box \)

Proof of Theorem 1.1. We transform (1.1) into the system of two second order elliptic equations
\[
\begin{align*}
-\Delta u &= w \quad \text{in } \mathbb{R}^3, \\
-\Delta w &= -u^{-q} \quad \text{in } \mathbb{R}^3.
\end{align*}
\tag{5.13}
\]
Then, we establish the following lemma whose proof mimics that of [Troy81, Lemma 4.2] and [GWZ18, Lemma 5.3].

**Lemma 5.2.** Let \( \gamma \in \mathbb{R} \) and \((u, w)\) be a positive entire solution of (5.13). Suppose that
\[
\begin{align*}
    u(x) &\leq u(x^\gamma), & w(x) &\neq u(x^\gamma) \text{ if } x_1 < \gamma. 
\end{align*}
\]
Then, we claim that
\[
\begin{align*}
    u(x) &< u(x^\gamma), & w(x) &< w(x^\gamma) \text{ if } x_1 < \gamma, 
\end{align*}
\]
and that
\[
\begin{align*}
    \frac{\partial u}{\partial x_1} > 0, & \quad \frac{\partial w}{\partial x_1} > 0 \quad \text{on } \Gamma_\gamma. 
\end{align*}
\]

Now we are in position to prove Theorem 1.1. To demonstrate the sufficiency of Theorem 1.1, we show that there exists \( \gamma' > 0 \) such that
\[
\begin{align*}
    u(x) &< u(x^\gamma), & w(x) &< w(x^\gamma) \text{ for } \gamma \geq \gamma' \text{ and } (x)_1 < \gamma. 
\end{align*}
\]
We assume that (5.16) is false. By Lemma 5.2, there exists two sequences \( \{\gamma^j\}_{j \geq 1} \subset \mathbb{R} \) and \( \{x^j\}_{j \geq 1} \subset \mathbb{R}^3 \) such that \( \lim_{j \to \infty} \gamma^j = \infty, (x^j)_1 < \gamma^j \) and
\[
\begin{align*}
    u(x^j) &\geq u(y^j), & y^j = (x^j)^{\gamma^j} \quad \text{for all } j. 
\end{align*}
\]
Notice that \( \lim_{j \to \infty} |y^j| = \infty \), which implies \( \lim_{j \to \infty} u(y^j) = \infty \). Hence, \( |x^j| \) tends to infinity. From Lemma 5.1, we obtain
\[
(x^j)_1 \leq \gamma_0 + 1 = \frac{(x_0)_1}{L} + 1 
\]
for large \( j \). Hence, for any \( \beta > \gamma_0 + 1 \), there holds
\[
\begin{align*}
    u(x^j) &\geq u(y^j) \geq u((x^j)^{\beta}) \quad \text{for large } j 
\end{align*}
\]
since \( ((x^j)^{\gamma^j})_1 > ((x^j)^{\beta})_1 \) for large \( j \) and \( u(x) \) tends to 0 as \( |x| \) tends to infinity. We thus use Lemma 5.2 again to get that
\[
0 \leq \frac{|x^j|}{\beta - (x^j)_1} \left( u(x^j) - u((x^j)^{\beta}) \right) = -2L\beta - 2(x_0)_1 < 0. 
\]
This contradiction shows us (5.16). The rest of the sufficient part is followed by that of [Zou95, Theorem 1.1] and [GHZ15, Theorem 1.1], then we omit the details here.

Now, we consider the necessary part of the proof. As pointed out in [Gue12, Theorem 1.6], any minimal radial entire solution \( u(r) \) of (1.1) as \( q > 3 \) satisfies
\[
r^{-1}u(r) - L = \begin{cases} 
O(r^{-1}) & \text{if } q > 4, \\
O(r^{-1} \log r) & \text{if } q = 4, \\
O(r^{3-q}) & \text{if } 3 < q < 4, 
\end{cases} 
\]
for some positive \( L \). This implies
\[
r^{-1}u(r) - L = o(r^{-\vartheta}) 
\]
for \( r \) large and \( \vartheta \in (0, 1) \). Thus, we conclude the necessity and the proof of Theorem 1.1. \( \square \)

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