THE DOUBLE DIFFERENCE PROPERTY FOR THE CLASS OF
LOCALLY HÖLDHER CONTINUOUS FUNCTIONS

RASHID A. ALIEV, AYSEL A. ASGAROVA, AND VUGAR E. ISMAILOV

Abstract. In this paper, we show that the pair of classes of locally Hölder
continuous functions (considered on $\mathbb{R}$ and $\mathbb{R}^2$, respectively) has the double
difference property.

1. Introduction

The notions difference property and double difference property are due to de
Bruijn [1] and Laczkovich [5], respectively. These properties for various classes
of real functions were investigated by many authors. We refer the reader to
Laczkovich’s survey paper [6] for a detailed source of information on this topic.

For a fixed function $g : \mathbb{R} \to \mathbb{R}$ and any $h \in \mathbb{R}$ we define the difference function
$
\Delta_h g : \mathbb{R} \to \mathbb{R}
$
by

$$
\Delta_h g(x) = g(x + h) - g(x)
$$

and the double difference function $Dg : \mathbb{R}^2 \to \mathbb{R}$ by

$$
Dg(x, y) = g(x + y) - g(x) - g(y).
$$

Let $\mathcal{F}$ be a class of functions defined on $\mathbb{R}$ and $\mathcal{F}_2$ be a class of functions defined on
$\mathbb{R}^2$. The class $\mathcal{F}$ is said to have the difference property if every function $g : \mathbb{R} \to \mathbb{R}$,
for which $\Delta_h g \in \mathcal{F}$ for each $h \in \mathbb{R}$, is of the form $g = f + A$, where $f \in \mathcal{F}$ and
$A$ is an additive function (see [1]). A function $A$ is called additive if it satisfies
the Cauchy functional equation $A(x + y) = A(x) + A(y)$. The pair $(\mathcal{F}, \mathcal{F}_2)$ is said
to have the double difference property if whenever $Dg \in \mathcal{F}_2$ holds for a function
$g : \mathbb{R} \to \mathbb{R}$, then $g$ is of the form $g = f + A$, where $f \in \mathcal{F}$ and $A$ is additive (see
[5]).

de Bruijn [1] was the first who showed that the class of continuous functions
has the difference property and thus resolved Erdős’s famous conjecture. He also
proved that the difference property holds for a large number of essential function
classes (see [1] [2]). Some of these classes are

1) $C^k(\mathbb{R})$, functions with continuous derivatives up to order $k$;
2) $C^\infty(\mathbb{R})$, infinitely differentiable functions;
3) analytic functions;
4) functions which are absolutely continuous on any finite interval;
5) functions having bounded variation over any finite interval;
6) algebraic polynomials;
7) trigonometric polynomials;
8) Riemann integrable functions.

2000 Mathematics Subject Classification. 26B05, 39A70, 39B22, 47B39.

Key words and phrases. Cauchy functional equation; additive function; difference property;
double difference property; modulus of continuity; Hölder continuity.
However, the class $\mathcal{L}$ of Lebesgue measurable functions fails to have this property if we assume the continuum hypothesis (see [1, 5]). It was conjectured by Erdős that every function $g : \mathbb{R} \to \mathbb{R}$ for which $\Delta_h g(x)$ is measurable for each $h$, is of the form $g = f + A + S$, where $f$ is measurable, $A$ is additive and $S$ has the property that $\Delta_h S(x) = 0$ for almost all $x$. Laczkovich [3] solved this conjecture affirmatively and moreover proved that the pair $(\mathcal{L}, \mathcal{L}_2)$ has the double difference property, where $\mathcal{L}_2$ denotes the class of Lebesgue measurable functions defined on $\mathbb{R}^2$. It was also proved in [5] that the double difference property holds for Baire functions. Later Tabor and Tabor [9] proved that the class $C^n(X,Y)$ of $n$-times continuously differentiable functions defined on a real normed space $X$ and taking values in a real Banach space $Y$ has the double difference property. Kotlicka [4] showed that several pairs of classes of functions have the double difference property. Among them there are approximately continuous functions, pointwise continuous functions, essentially continuous functions (considered on $\mathbb{T}$ and $\mathbb{T}^2$, respectively, where $\mathbb{T}$ is the torus).

In [7], Tabor proved that the pair of classes of Lipschitz functions defined on a metric semigroup $G$ and $G \times G$, respectively, with values in a reflexive Banach space $E$ has the double difference property. For finite dimensional Banach spaces $X$ and $Y$, Tabor and Tabor [8] showed that the double difference property holds for the pair of classes of $Y$-valued Lipschitz functions defined on a convex set $K \subset X$ such that $0 \in K$ and on the set

$$C(K) = \{(x, y) \in X \times X : x \in K, y \in K, x + y \in K\},$$

respectively. Consequently, the double difference property holds for the pair of classes of real Lipschitz functions defined on an interval $I$ containing zero and on the set $C(I)$, respectively.

In this paper, we prove that for any $\alpha \in (0, 1]$ the pair of classes of locally Hölder real continuous functions (considered on $\mathbb{R}$ and $\mathbb{R}^2$, respectively) with exponent $\alpha$ has the double difference property.

2. Main result

We start this section with the definition of modulus of continuity of a multivariate function and some notation. Let $f(x) = f(x_1, ..., x_s)$, $s \geq 1$, be any $s$-variable function defined on a set $\Omega \subset \mathbb{R}^s$. The function

$$\omega(f; \delta; \Omega) = \sup \{|f(x) - f(y)| : x, y \in \Omega, \ |x - y| \leq \delta\}, \ 0 \leq \delta \leq diam\Omega,$$

is called the modulus of continuity of $f$ on $\Omega$. We will also use the notation $\omega_0(f; \delta; \Omega)$, which stands for the function $\omega(f; \delta; \Omega \cap \mathbb{Q}^s)$. Here $\mathbb{Q}^s$ denotes the space of $s$-dimensional vectors with rational coordinates. Clearly, $\omega_0(f; \delta; \Omega)$ makes sense if the set $\Omega \cap \mathbb{Q}^s$ is not empty. Note that we always have the inequality $\omega_0(f; \delta; \Omega) \leq \omega(f; \delta; \Omega)$ and the strong equality $\omega_0(f; \delta; \Omega) = \omega(f; \delta; \Omega)$ holds for continuous $f$ and certain sets $\Omega$. For example, this holds if for any $x, y \in \Omega$ with $|x - y| \leq \delta$ there exist sequences $\{x_n\}, \{y_n\} \subset \Omega \cap \mathbb{Q}^s$ such that $x_n \to x, y_n \to y$ and $|x_n - y_n| \leq \delta$, for all $n$. There are many sets $\Omega$, which satisfy this property.

The class $H^{(\alpha)}_0(\mathbb{R}^s)$ of locally Hölder continuous functions with exponent $\alpha$ is defined as the class of functions $f$ for which $\omega(f; \delta; \Omega) \leq K\delta^\alpha$ for any compact set $\Omega \subset \mathbb{R}^s$. Here $K$ depends on $\Omega$. 

Our main result is the following theorem.

**Theorem 2.1.** Assume a function \( g : \mathbb{R} \rightarrow \mathbb{R} \) is such that the bivariate function \( g(x + y) - g(x) - g(y) \) is locally Hölder continuous with exponent \( \alpha \). Then there exist a function \( f \in H^\alpha_{\text{loc}}(\mathbb{R}) \) and an additive function \( A \) such that \( g = f + A \).

To prove this theorem we need the following auxiliary lemma.

**Lemma 2.1.** Assume a function \( F \in C(\mathbb{R}^2) \) has the form

\[
F(x, y) = g(x + y) - g(x) - g(y),
\]

where \( g \) is an arbitrarily behaved function. Then the following inequality holds

\[
\omega_{\mathbb{Q}}(g; \delta; [-M, M]) \leq 2\delta |g(1) - g(0)| + 3\omega(F; \delta; [-M, M]^2),
\]

where \( \delta \in (0, \frac{1}{2}) \cap \mathbb{Q} \) and \( M \geq 1 \).

**Proof.** Consider the function \( h(t) = g(t) - g(0) \) and write (2.1) in the form

\[
G(x, y) = h(x + y) - h(x) - h(y),
\]

where

\[
G(x, y) = F(x, y) + g(0).
\]

Note that the functions \( g \) and \( h \), as well as the functions \( F \) and \( G \), have the common modulus of continuity. Thus we prove the lemma if we prove it for the pair \( \langle G, h \rangle \).

Since \( h(0) = 0 \), it follows from (2.3) that

\[
G(x, 0) = G(0, y) = 0.
\]

Obviously, for any real number \( x \),

\[
G(x, x) = h(2x) - 2h(x); \\
G(x, 2x) = h(3x) - h(x) - h(2x); \\
\cdots \\
G(x, (k-1)x) = h(kx) - h(x) - h((k-1)x).
\]

We obtain from the above equalities that

\[
h(2x) = 2h(x) + G(x, x), \\
h(3x) = 3h(x) + G(x, x) + G(x, 2x), \\
\cdots \\
h(kx) = kh(x) + G(x, x) + G(x, 2x) + \cdots + G(x, (k-1)x).
\]

Thus for any nonnegative integer \( k \),

\[
h(x) = \frac{1}{k}h(kx) - \frac{1}{k} [G(x, x) + G(x, 2x) + \cdots + G(x, (k-1)x)].
\]
Note that in (2.9), the whole number part of \( r \) the pair \((m, k)\) each pair. For example, the last formula in the (2.9) gives us the following equality

\[
h\left(\frac{p}{n}\right) = \frac{1}{m_0} h \left(1 - \frac{p_1}{n}\right)
\]

(2.6) \(-\frac{1}{m_0} \left[ G\left(\frac{p}{n}, \frac{p}{n}\right) + G\left(\frac{p}{n}, \frac{2p}{n}\right) + \cdots + G\left(\frac{p}{n}, (m_0 - 1)\frac{p}{n}\right) \right].
\)

On the other hand, since it follows from (2.6) that

\[
h\left(\frac{p_1}{n}, 1 - \frac{p_1}{n}\right) = h(1) - h\left(\frac{p_1}{n}\right) - h\left(1 - \frac{p_1}{n}\right),
\]

(2.7)

\[-\frac{1}{m_0} h\left(\frac{p_1}{n}\right).\]

Put \( m_1 = \left[\frac{m}{p}\right], p_2 = n - m_1p_1. \) Clearly, \( 0 \leq p_2 < p_1. \) Similar to (2.7), we can write that

\[
h\left(\frac{p_1}{n}\right) = \frac{h(1)}{m_1} - \frac{1}{m_1} \left[ G\left(\frac{p_1}{n}, \frac{p_1}{n}\right) + \cdots + G\left(\frac{p_1}{n}, (m_1 - 1)\frac{p_1}{n}\right) + G\left(\frac{p_2}{n}, 1 - \frac{p_1}{n}\right) \right]
\]

(2.8)

\[-\frac{1}{m_1} h\left(\frac{p_2}{n}\right).\]

Let us make a convention that (2.7) is the 1-st and (2.8) is the 2-nd formula. One can continue this process by defining the chain of pairs \((m_2, p_3), \( (m_3, p_4) \) until the pair \((m_{k-1}, p_k)\) with \( p_k = 0 \) and writing out the corresponding formulas for each pair. For example, the last \( k \)-th formula will be of the form

\[
h\left(\frac{p_k-1}{n}\right) = \frac{h(1)}{m_{k-1}} - \frac{1}{m_{k-1}} \left[ G\left(\frac{p_{k-1}}{n}, \frac{p_{k-1}}{n}\right) + \cdots + G\left(\frac{p_{k-1}}{n}, (m_{k-1} - 1)\frac{p_{k-1}}{n}\right) + G\left(\frac{p_k}{n}, 1 - \frac{p_k}{n}\right) \right]
\]

(2.9)

Note that in (2.9), \( h\left(\frac{p_k}{n}\right) = 0 \) and \( G\left(\frac{p_k}{n}, 1 - \frac{p_k}{n}\right) = 0. \) Considering now the \( k \)-th formula in the \((k - 1)\)-th formula, then the obtained formula in the \((k - 2)\)-th formula, and so forth, we will finally arrive at the equality

\[
h\left(\frac{p}{n}\right) = h(1) \left[ \frac{1}{m_0} - \frac{1}{m_0m_1} + \cdots + \frac{(-1)^{k-1}}{m_0m_1 \cdots m_{k-1}} \right]
\]

\[-\frac{1}{m_0} \left[ G\left(\frac{p}{n}, \frac{p}{n}\right) + \cdots + G\left(\frac{p}{n}, (m_0 - 1)\frac{p}{n}\right) + G\left(\frac{p_1}{n}, 1 - \frac{p_1}{n}\right) \right].\]
\[ + \frac{1}{m_0m_1} \left[ G \left( \frac{p_1}{n}, \frac{p_1}{n} \right) + \cdots + G \left( \frac{p_1}{n}, (m_1 - 1) \frac{p_1}{n} \right) + G \left( \frac{p_2}{n}, 1 - \frac{p_2}{n} \right) \right] + \cdots + \]

\[ \frac{(-1)^k}{m_0m_1 \cdots m_{k-1}} \left[ G \left( \frac{p_{k-1}}{n}, \frac{p_{k-1}}{n} \right) + \cdots + G \left( \frac{p_{k-1}}{n}, (m_{k-1} - 1) \frac{p_{k-1}}{n} \right) \right]. \]

Taking into account (2.4), we obtain from (2.10) that

\[ |h\left(\frac{p}{n}\right)| \leq \left[ \frac{1}{m_0} - \frac{1}{m_0m_1} + \cdots + \frac{(-1)^{k-1}}{m_0m_1 \cdots m_{k-1}} \right] |h(1)| \]

\[ + \left[ 1 + \frac{1}{m_0} + \cdots + \frac{1}{m_0 \cdots m_{k-2}} \right] \omega \left( \frac{p}{n}; [0,1]^2 \right). \]

Since \( m_0 \leq m_1 \leq \cdots \leq m_{k-1} \), it is not difficult to see that in (2.11)

\[ \frac{1}{m_0} - \frac{1}{m_0m_1} + \cdots + \frac{(-1)^{k-1}}{m_0m_1 \cdots m_{k-1}} \leq \frac{1}{m_0} \]

and

\[ 1 + \frac{1}{m_0} + \cdots + \frac{1}{m_0 \cdots m_{k-2}} \leq \frac{m_0}{m_0 - 1}. \]

Considering the above two inequalities in (2.11) we obtain that

\[ |h\left(\frac{p}{n}\right)| \leq \left[ \frac{1}{m_0} - \frac{1}{m_0m_1} + \cdots + \frac{(-1)^{k-1}}{m_0m_1 \cdots m_{k-1}} \right] \frac{m_0}{m_0 - 1} \omega \left( \frac{p}{n}; [0,1]^2 \right). \]

Since \( m_0 = \left\lceil \frac{n}{p} \right\rceil \geq 2 \), it follows from (2.12) that

\[ |h\left(\frac{p}{n}\right)| \leq 2p \frac{|h(1)|}{n} + 2\omega \left( \frac{p}{n}; [0,1]^2 \right). \]

Let now \( \delta \in (0, \frac{1}{2}) \cap \mathbb{Q} \) be a rational increment, \( M \geq 1 \) and \( x, x + \delta \) be two points in \([-M, M] \cap \mathbb{Q}\). By (2.3), (2.4) and (2.13) we can write that

\[ |h(x + \delta) - h(x)| \leq |h(\delta)| + |G(x, \delta)| \leq 2\delta |h(1)| + 3 \omega \left( G; \delta; [-M, M]^2 \right). \]

Now (2.2) follows from (2.14) and the definitions of \( h \) and \( G \).

\[ \square \]

**Remark 1.** Under the assumptions of Lemma 2.1, the restriction of \( g \) to the set of rational numbers is uniformly continuous on any interval \([-M, M] \cap \mathbb{Q}\) and hence continuous on \( \mathbb{Q} \).

Now we are ready to prove Theorem 2.1.
Proof. Let us put
\begin{equation}
F(x, y) = g(x + y) - g(x) - g(y)
\end{equation}
and consider the function
\[ u(t) = g(t) - [g(1) - g(0)]t. \]
Obviously, \( u(1) = u(0) \) and
\begin{equation}
F(x, y) = u(x + y) - u(x) - u(y).
\end{equation}
By Lemma 2.1, the restriction of \(- \) to every interval \([M, M] \cap \mathbb{Q}\). Denote this restriction by \( v \).

Let \( y \) be any real number and \( \{y_n\}_{n=1}^{\infty} \) be any sequence of rational numbers converging to \( y \). We can choose \( M > 0 \) so that \( y_n \in [M, M] \) for any \( n \in \mathbb{N} \). It follows from the uniform continuity of \( v \) on \([M, M] \cap \mathbb{Q}\) that the sequence \( \{v(y_n)\}_{n=1}^{\infty} \) is Cauchy. Thus there exists a finite limit \( \lim_{n \to \infty} v(y_n) \). It is not difficult to see that this limit does not depend on the choice of \( \{y_n\}_{n=1}^{\infty} \).

Let \( f \) denote the following extension of \( v \) to the set of real numbers.
\[ f(y) = \begin{cases} v(y), & \text{if } y \in \mathbb{Q}; \\ \lim_{n \to \infty} v(y_n), & \text{if } y \in \mathbb{R} \setminus \mathbb{Q} \text{ and } \{y_n\} \text{ is a sequence in } \mathbb{Q} \text{ tending to } y. \end{cases} \]

In view of the above arguments, \( f \) is well defined on the whole real line. Let us prove that \( f \) is the function we seek.

Consider an arbitrary point \((x, y) \in \mathbb{R}^2\) and a sequence of points \( \{(x_n, y_n)\}_{n=1}^{\infty} \) with rationale coordinates tending to \((x, y)\). Taking into account (2.16), we can write that
\begin{equation}
F(x_n, y_n) = v(x_n + y_n) - v(x_n) - v(y_n), \quad \text{for all } n = 1, 2, \ldots,
\end{equation}

since \( v \) is the restriction of \( u \) to \( \mathbb{Q} \). Tending \( n \to \infty \) in both sides of (2.17) we obtain that
\begin{equation}
F(x, y) = f(x + y) - f(x) - f(y).
\end{equation}

Set \( A = g - f \). It follows from (2.15) and (2.18) that \( A \) is additive. Let us now prove that \( f \in H_{\alpha}^{(\infty)}(\mathbb{R}) \). Since \( v(1) = v(0) \) we obtain from (2.16) and (2.2) that for \( \delta \in (0, \frac{1}{2}) \cap \mathbb{Q}, \ M \geq 1 \) and any numbers \( a, b \in [-M, M] \cap \mathbb{Q}, \ |a - b| \leq \delta \), the following inequality holds
\begin{equation}
|v(a) - v(b)| \leq 3\omega(F; \delta; [-M, M]^2).
\end{equation}

Consider now any real numbers \( r_1 \) and \( r_2 \) satisfying \( r_1, r_2 \in [-M, M], \ |r_1 - r_2| \leq \delta \) and take sequences \( \{a_n\}_{n=1}^{\infty} \subset [-M, M] \cap \mathbb{Q}, \ \{b_n\}_{n=1}^{\infty} \subset [-M, M] \cap \mathbb{Q} \) with the property \( |a_n - b_n| \leq \delta, \ n = 1, 2, \ldots, \) and tending to \( r_1 \) and \( r_2 \), respectively. By (2.19),
\[ |v(a_n) - v(b_n)| \leq 3\omega(F; \delta; [-M, M]^2). \]
If we take limits on both sides of the above inequality, we obtain that
\begin{equation}
|f(r_1) - f(r_2)| \leq 3\omega \left( F; \delta; [-M, M]^2 \right),
\end{equation}
which means that $f$ is uniformly continuous on $[-M, M]$ and hence it is continuous on the whole real line.

It follows from (2.20) that

\begin{equation}
\omega \left( f; \delta; [-M, M] \right) \leq 3\omega \left( F; \delta; [-M, M]^2 \right).
\end{equation}

Note that in (2.21) $\delta$ is a rational number from the interval $(0, \frac{1}{2})$. Since the modulus of continuity of a continuous function is continuous from the right (see [3]), it follows that, in fact, (2.21) is valid for all $\delta \in \left[0, \frac{1}{2}\right)$. Since $F \in H_{\alpha}^{(\text{loc})}(\mathbb{R}^2)$, we obtain from (2.21) that

\begin{equation}
\omega \left( f; \delta; [-M, M] \right) \leq K\delta^{\alpha}, \text{ where } 0 \leq \delta < \frac{1}{2}.
\end{equation}

Let now $\frac{1}{2} \leq \delta \leq 2M$. We can write that

\begin{equation}
\omega \left( f; \delta; [-M, M] \right) \leq 2\|f\|_{C([-M, M])} \leq 2^{1+\alpha} ||f||_{C([-M, M])} \delta^{\alpha}, \text{ where } \frac{1}{2} \leq \delta \leq 2M.
\end{equation}

The inequalities (2.22) and (2.23) show that $f$ is Hölder continuous on $[-M, M]$ with exponent $\alpha$. Since $M$ is an arbitrary number not less than 1 and any compact $\Omega$ is contained in a closed interval of the form $[-M, M]$, we obtain that $f \in H_{\alpha}^{(\text{loc})}(\mathbb{R})$.

\textbf{Remark 2.} The above proof shows that for any compact set $\Omega \subset \mathbb{R}$ the pair of Hölder continuous function classes $H_{\alpha}(\Omega)$ and $H_{\alpha}(\Omega \times \Omega)$ has the double difference property. This holds, in particular, for the pair of classes of Lipschitz functions defined on $\Omega$ and $\Omega \times \Omega$, respectively. The last assertion complements the corresponding result of J. Tabor and J. Tabor [8] in the real space setting (see Introduction).

\textbf{Remark 3.} Theorem 2.1 is not only an existence result. It’s proof gives a recipe for constructing the function $f$. It also allows us to estimate the modulus of continuity of $f$ in terms of the modulus of continuity of $g(x + y) - g(x) - g(y)$ (see (2.21)).

\textbf{References}

[1] N.G. de Bruijn, Functions whose differences belong to a given class, Nieuw Arch. Wiskd. 23 (1951), 194–218.
[2] N.G. de Bruijn, A difference property for Riemann integrable functions and for some similar classes of functions, Nederl. Akad. Wetensch. Proc. 55 (1952), 145–151.
[3] I. M. Kolodii, F. Hil’debrand, Certain properties of the modulus of continuity, (Russian) Mat. Zametki 9 (1971), 495–500.
[4] E. Kotlicka, The double difference property for some classes of functions. Real Anal. Exchange 25 (1999/00), no. 1, 463–467.
[5] M. Laczkovich, Functions with measurable differences, Acta Math. Hungar. 35 (1980), 217–235.
[6] M. Laczkovich, The difference property, Paul Erdős and His Mathematics I, 363–410, Bolyai Soc. Math. Stud. 11, 2002.
[7] J. Tabor, Lipschitz stability of the Cauchy and Jensen equations, Results Math. 32 (1997), no. 1-2, 133–144.
[8] J. Tabor and J. Tabor, Local stability of the Cauchy and Jensen equations in function spaces, Aequationes Math. 58 (1999), no. 3, 296–310.
[9] J. Tabor and J. Tabor, Stability of the Cauchy type equations in the class of differentiable functions, J. Approx. Theory 98 (1999), no. 1, 167–182.

INSTITUTE OF MATHEMATICS AND MECHANICS, NAS OF AZERBAIJAN, BAKU, AZERBAIJAN
Baku State University, Baku, Azerbaijan

E-mail address: aliyevrashid@mail.ru

AZERBAIJAN UNIVERSITY OF LANGUAGES, BAKU, AZERBAIJAN

E-mail address: asgarova2016@mail.ru

INSTITUTE OF MATHEMATICS AND MECHANICS, NAS OF AZERBAIJAN, BAKU, AZERBAIJAN

E-mail address: vugaris@mail.ru