LINES ON PROJECTIVE VARIETIES

J.M. LANDSBERG

Definition. A variety (or manifold) \( X \subset \mathbb{P}^N \) (or \( \mathbb{A}^N \)) is covered by lines if through a general point \( x \in X \) there passes a finite number of lines contained in \( X \).

It was known classically that a surface covered by lines can contain at most two lines through a general point. In [MP] it was shown that a 3-fold covered by lines contains at most 6 lines through a general point.

I prove:

Theorem 1. Let \( X^n \subset \mathbb{P}^{n+a} \) be covered by lines. Then there are at most \( n! \) lines passing through a general point of \( X \).

The theorem holds for any ground field of characteristic zero. The same conclusion is valid for analytic varieties in affine or projective space if one asks that the lines be contained in the closure of \( X \). The same conclusion even holds in the \( C^\infty \) category if one replaces “general point” by “every point” in the hypotheses.

Theorem 1 will be a consequence of theorem 2:

Theorem 2. Let \( X^n \subset \mathbb{P}^{n+1} \) be a hypersurface and let \( x \in X \) be a general point. Let \( \Sigma^X \subset \mathbb{P}T_x X \) denote the tangent directions to lines having contact to order \( \lambda \) with \( X \) at \( x \).

(The notation is such that \( \Sigma^1 = \mathbb{P}T_x X \).) If there is an irreducible component \( \Sigma_0^k \subset \Sigma^k \) such that \( \dim \Sigma_0^k > n - k \) then \( \Sigma_0^k \subset \Sigma^\infty \), i.e., all lines corresponding to points of \( \Sigma_0^k \) are contained in \( X \).

Note that the expected dimension of \( \Sigma^k \) is \( n - k \).

Corollary. Let \( X^n \subset \mathbb{P}^{n+a} \) be a variety such that through a general point \( x \in X \) there is a \( p \)-plane having contact to order \( n - p + 2 \). Then \( X \) is uniruled by \( \mathbb{P}^p \)'s.

The corollary is not expected to be optimal for most values of \( k \), see [L2, theorem 4].

Proof of theorem 2. We use notation as in [L1] and [L2]. We choose a basis \( e_1, \ldots, e_n \) of \( T_x X \) such \( e_1 \) is a general point of \( \Sigma_0^k \), and \( T_{[e_1]} \Sigma^k = \mathbb{P}\{e_1, e_2, \ldots, e_p\} \). By hypothesis \( p - 1 = \dim \Sigma_0^k > n - k \).

Let \( 1 \leq \alpha, \beta \leq n \). We let \( r_{\alpha, \beta} \) denote the coefficients of the second fundamental form \( F_2 \) of \( X \) at \( x \) and \( r_{\alpha_1, \ldots, \alpha_i} \) denote the coefficients of \( F_i \). Sometimes we write \( r_{\alpha_1, \ldots, \alpha_i}^i = r_{\alpha_1, \ldots, \alpha_i} \) to keep track of \( i \).

We let \( 2 \leq s, t \leq p, p + 1 \leq j, l \leq n \).

1991 Mathematics Subject Classification. primary 53, secondary 14.

Key words and phrases. uniruled varieties, ruled manifolds moving frames, osculating spaces, projective differential geometry, second fundamental forms.
By our normalizations, we have \( r^1_{1,1,1} = 0 \) and \( r^h_{1,1,s} = 0 \) for \( 2 \leq s \leq p \) and \( 2 \leq \lambda \leq k \). We will show that \( r^h_{1,1,1} = 0 \) and \( r^h_{1,1,1,s} = 0 \) for \( 2 \leq s \leq p \) and for all \( h \), showing that there is a \( p \)-dimensional space of lines passing through \( X \) at \( x \).

The technique of proof is the same as in [L2], namely we will use our lower order equations to solve for the connection forms \( \omega^j \) in terms of the semi-basic forms \( \omega^j \) and then plug into the higher \( F_h \) to obtain the vanishing of \( F_h(e_1, \ldots, e_1) \) and \( F_h(e_1, \ldots, e_1, e_s) \). Using the formalism for the \( F_h \) developed in from [L1], we obtain that for \( \lambda \leq k \):

\[
r^\lambda_{1,1,1,i} \omega^i = -r^{\lambda-1}_{1,1,1,j} \omega^j
\]

These are \( k - 2 \) equations for the \( n - p \) one-forms \( \omega^j \) in terms of the \( n - p \) semi-basic forms \( \omega^j \). Recall that \( n - p \leq k - 2 \). The system is solvable as were the \( (k-2) \times (n-p) \) matrix \( (r^\lambda_{1,1,1,j}) \) not of maximal rank, there would be additional directions in the tangent space to \( \Sigma^k \). Thus we have

\[
\omega^j \equiv 0 \text{ mod } \{ \omega^{p+1}, \ldots, \omega^n \}
\]

so the equation

\[
r^{k+1}_{1,1,1,\beta} \omega^\beta = -r^k_{1,1,1,j} \omega^j
\]

implies \( r^{k+1}_{1,1,1} = r^{k+1}_{1,1,1,2} = \ldots = r^{k+1}_{1,1,1,p} = 0 \) and thus the line through \([e_1]\) has contact to order \( k + 1 \) and moreover \( T_{[e_1]} \Sigma^{k+1} = T_{[e_1]} \Sigma^k \). Now one can use these equations iteratively to show the same holds to order \( k + 2 \) and all orders, i.e., the component of \( \Sigma^k \) containing \([e_1]\) equals the component of \( \Sigma^\infty \) containing \([e_1]\). \( \square \)

Proof of theorem 1. Without loss of generality in theorem 1 we may assume \( X \) is a hypersurface as one can reduce to this case by linear projection. First note that \( n! \) is the expected bound in the sense that in \( \mathbb{P}T_x X \) one has the ideal \( I \) generated by \( F_2, F_3 \ldots F_n \) defining the variety \( \Sigma^n \subset \mathbb{P}T_x X \) of all lines having contact with \( X \) at \( x \) to order \( n \). (Note that the polynomials of degree greater than two are not well defined individually but the ideal \( I \) is.) Since \( \mathbb{P}T_x X \) is a \( \mathbb{P}^{n-1} \), if \( \Sigma^n \) is zero-dimensional, it is at most \( n! \) points and we are done. If \( \dim \Sigma^n > 0 \) then theorem 2 applies. \( \square \)

Concluding remarks. Theorem 1 generalizes theorem 1 of [L2], which says that if \( \Sigma^{n+1} \neq \emptyset \), then \( \Sigma^{n+1} = \Sigma^\infty \). It is sharp for hypersurfaces as a general hypersurface of degree \( n \) will be covered by \( n! \) lines. It is unlikely theorem 1 is sharp in higher codimension, however the following example due to F. Zak shows that one cannot hope to do too much better: Let \( Y^{n-1} \) be a hypersurface with \((n-1)!\) lines passing through a general point and let \( X = Seg(Y \times C) \) where \( C \subset \mathbb{P}M \) is a curve. One can arrange for the codimension of \( X \) to be arbitrarily large in this way and \( X \) has \((n-1)!\) lines passing through a general point.

One could hope to now do a classification in the spirit of [MP]. For example, \( n! \) should only be possible for hypersurfaces of degree \( n \). An interesting class of examples with a small number of lines is obtained by taking linear sections of uniruled homogeneous varieties.

Acknowledgements. It is a pleasure to thank E. Mezzetti and F. Zak for useful conversations.
References

[L1] J.M. Landsberg, Differential-geometric characterizations of complete intersections, Journal of Differential Geometry 44 (1996), 32-73.

[L2] J.M. Landsberg, Is a linear space contained in a submanifold? - On the number of derivatives needed to tell, J. reine angew. Math. 508 (1999), 53-60.

[MP] E. Mezzetti and D. Portelli, On threefolds covered by lines, Abh. Math. Sem. Univ. Hamburg 70 (2000), 211-238.

School of Mathematics, Georgia Institute of Technology, 686 Cherry St., Skiles Bldg.
Atlanta, GA 30332-0160 USA
E-mail address: jml@math.gatech.edu