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Bhaskar, Siddharth

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THICKET DENSITY

SIDDHARTH BHASKAR

Abstract. We define a new type of “shatter function” for set systems that satisfies a Sauer-Shelah type dichotomy, but whose polynomial-growth case is governed by Shelah’s 2-rank instead of VC dimension. We identify the rate of growth of this shatter function, the quantity analogous to VC density, with Shelah’s ω-rank.

§1 Introduction

The shatter function is a function of type \( N \to N \) that measures of the complexity of a set system. The shatter function of any set system satisfies the Sauer-Shelah dichotomy: it is either the binary exponential function \( n \mapsto 2^n \), or is polynomially bounded. Whether or not the shatter function is polynomially bounded or exponential depends on whether a certain integer parameter, the VC dimension, is finite or infinite. In the finite case, the growth rate of the shatter function is a real number called the VC density.

VC density was discovered by Vapnik and Chervonenkis [11] and found important applications in probability theory, combinatorial geometry, and computational learning theory [1]. The relevance of VC density to theories without the independence property was pointed out by Laskowski [7], and subsequently developed by Aschenbrenner et al. [1].

In the present paper, we associate a new function of type \( N \to N \) with any set system, which we call the thicket shatter function. It also satisfies the Sauer-Shelah dichotomy, but the quantity that distinguishes between polynomial and exponential growth is an instance of Shelah’s local 2-rank, and its rate of growth is an instance of Shelah’s local ω-rank. In this context, we call these two quantities thicket dimension and thicket density to emphasize the analogy.

Seen from another angle, our work can be read as a new way to calculate Shelah’s local ω-rank using the asymptotic growth of certain finite combinatorial objects. Notably, this can be performed in any model of a theory, not just a saturated one.

Our work was foreshadowed by Tiuryn, whose Lemma 3.6 in [10] contains a special case of our Theorem 4.1 below, the Sauer-Shelah dichotomy for thicket shatter functions. It is remarkable that he was concerned with problems in dynamic

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1 For an excellent exposition, see Assouad [2].
logic—at best a distant relative of model theory, and even further from geometry and computational learning theory.

**Organization of this paper** In Section 2, we introduce thicket dimension, shatter function, and density, and prove the thicket version of the Sauer-Shelah dichotomy. In Section 5, we identify thicket density with a local model-theoretic rank, and in Section 6, we formulate a notion of degree, or multiplicity, for thicket density.

### §2 Trees

Our fundamental objects of study are set systems and binary trees. The latter comes in two varieties: ordered, and unordered, and this refers to whether we distinguish left from right children of non-leaves. Trees are normatively ordered; we shall say “unordered trees,” when we mean it.

**Definition 2.1.** A tree is either a single leaf or an ordered pair of subtrees, which we call left and right. An unordered tree is either a single leaf or an unordered pair of subtrees.

Notice that this definition allows for both finite trees and infinite trees of depth $\omega$. In a set-theoretic account, a tree $T$ would be defined as a prefix-closed subset of $2^{<\omega}$ such that for every $u \in 2^{<\omega}$, $u0 \in T \iff u1 \in T$. We prefer the “coinductive data type” definition presented here, which has the advantage of giving a more succinct definition of unordered trees. However, we will freely imagine a tree as a set of vertices, some of which are leaves, equipped with a partial order for the ancestral relation. We trust that this will cause no difficulty.

**Definition 2.2.** For a tree $T$, and vertices $u, v, w \in T$, we say

- $u \prec v$ in case $v \neq u$, but $v$ is contained in the subtree with root $u$,
- $u \prec_L v$ if $v$ is contained in the subtree whose root is the left child of $u$, and
- $u \prec_R v$ if $v$ is contained in the subtree whose root is the right child of $u$.

For fixed $v \in T$, the set of vertices $P(v) = \{ u : u \prec v \}$ is linearly ordered by $\prec$ and is partitioned by $P_L(v) = \{ u : u \prec_L v \}$ and $P_R(v) = \{ u : u \prec_R v \}$. For example in Figure 2.1, $P_L(\ell_3) = \{ u \}$ and $P_R(\ell_3) = \{ v, w \}$.

For vertices $u, v$ in an unordered binary tree $T$, we cannot of course say that $u \prec_L v$ or $u \prec_R v$. What we can say is that $v$ and $w$ are contained in the same subtree of $u$, or different subtrees of $u$. 

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**Figure 2.1.** A binary tree with leaves $\ell_1, \ell_2, \ell_3, \ell_4$ and non-leaves $u, v, w$. Depending on the context, this could be ordered or unordered.
Definition 2.3. For an unordered tree \( T \), and vertices \( u, v, w \in T \), we say
- \( u \prec v \) in case \( v \neq u \), but \( v \) is contained in the subtree with root \( u \),
- \( v \sim u \) \( w \) in case \( u \sim v \), \( u \sim w \), and \( v \) and \( w \) lie in the same subtree of \( u \), and
- \( v \perp u \) \( w \) in case \( u \perp v \), \( u \perp w \), and \( v \) and \( w \) lie in different subtrees of \( u \).

Notice that, for any \( v \) and \( w \), there is a unique \( u \) such that \( v \perp u \) \( w \). For example, if we interpret the tree in Figure 2.1 to be unordered, then \( \ell_2 \sim \ell_3 \), \( \ell_2 \sim v \) \( \ell_3 \), but \( \ell_2 \perp w \) \( \ell_3 \).

An important example of trees are the finite, balanced trees.

Definition 2.4. The tree \( B_0 \) is a single leaf. The tree \( B_{n+1} \) is the ordered pair of trees \( (B_n, B_n) \). Similarly, we can define the unordered tree \( B_n^0 \).

Definition 2.5. An embedding of the tree \( T_1 \) into the tree \( T_2 \) is an injection of the vertices of \( T_1 \) into the vertices of \( T_2 \) that preserves the \( \prec_L \) and \( \prec_R \) relations. An embedding of the unordered tree \( T_1 \) into the unordered tree \( T_2 \) is an injection of the vertices of \( T_1 \) into the vertices of \( T_2 \) that preserves the \( \prec \), \( \sim \), and \( \perp \) relations. The dimension \( d \) of a tree (respectively, unordered tree) \( T \) is the largest \( n \) such that \( B_n \) (respectively, \( B_n^0 \)) can be embedded into \( T \), or \( \infty \) if there are arbitrarily large such \( n \).

Remark 2.6. For finite (ordered or unordered) trees \( T \), dimension satisfies the following useful recursive identity. If \( T \) is a leaf, then \( d = 0 \). Otherwise, if \( d_1 \) and \( d_2 \) are the dimensions of its two subtrees, then

\[
d = \begin{cases} 
\max\{d_1, d_2\} & \text{if } d_1 \neq d_2 \\
 d_1 + 1 & \text{if } d_1 = d_2.
\end{cases}
\]

§3 Labeled trees and their solutions

A set system \( (X, \mathcal{F}, \in) \) is a two-sorted structure, with sorts \( X \) and \( \mathcal{F} \), and equipped with a single binary relation \( \in \subseteq X \times \mathcal{F} \). Usually, we shall just write \( (X, \mathcal{F}) \), suppressing \( \in \). A typical example of a set system is obtained by taking \( X \) to be some set, \( \mathcal{F} \) to be a family of subsets of \( X \), and \( \in \) to be the containment relation. (But, in general, elements in \( \mathcal{F} \) need not be extensional.) Given any set system \( (X, \mathcal{F}, \in) \), its dual is the set system \( (\mathcal{F}, X, \in^*) \), where \( F \in^* x \iff x \in F \). Clearly, the dual of the dual of any set system is identical to the original.

Definition 3.1. Let \( (X, \mathcal{F}) \) be a set system and \( T \) be a tree. An \( X \)-labeling of \( T \) is an assignment of elements of \( X \) to non-leaves of \( T \). If \( T \) is an \( X \)-labeled tree with labeling \( u \mapsto x_u \), and if \( v \in T \) is a leaf, then we say \( F \in \mathcal{F} \) solves \( v \) in case \( (\forall u \prec v)(x_u \in F \iff u \prec_L v) \), or equivalently, \( F \cap P(v) = P_L(v) \). In the special case that \( T \) has depth 0, i.e., is a single leaf \( v \), the quantifier \( \forall u \prec v \) is vacuous, so \( v \) has a solution in \( \mathcal{F} \) iff \( \mathcal{F} \) is nonempty.

If \( T \) is an unordered tree, an \( X \)-labeling is an assignment of vertices of \( T \) to elements of \( X \). The solution of a single leaf is meaningless, but we can still speak of a solution to the whole tree.

Definition 3.2. If \( T \) is an \( X \)-labeled unordered tree and \( L \subseteq T \) is the set of its leaves, a solution to \( T \) (in \( \mathcal{F} \)) is an assignment \( L \rightarrow \mathcal{F} \), denoted \( v \mapsto F_v \), satisfying

\[
\begin{align*}
  v_1 \sim u v_2 & \implies x_u \in F_{v_1} \iff x_u \in F_{v_2}, \\
  v_1 \perp u v_2 & \implies x_u \in F_{v_1} \not\iff x_u \in F_{v_2}.
\end{align*}
\]
In other words, any pair of sets labeling two leaves must disagree on whether they include the element of $X$ labeling their most recent common ancestor, and agree on all other common ancestors.

**Definition 3.3.** Say that a set system $(X,F)$ admits a tree $T$ in case there is an $X$-labeling of $T$ such that each leaf of $T$ has a solution in $F$. Similarly, $(X,F)$ admits an unordered tree $T$ in case there exists an $X$-labeling of $T$ which has a solution in $F$. If a set system does not admit a tree or an unordered tree, the n it forbids that tree.

**Remark 3.4.** For a typical tree $T$, it is a much stronger statement to say that $(X,F)$ admits $T$, rather than $(X,F)$ admits the underlying unordered tree of $T$. Similarly, it is much stronger to say that $(X,F)$ forbids the underlying unordered tree of $T$, rather than $(X,F)$ forbids $T$. However, when $T = B_n$, there is no difference: admitting or forbidding $B_n$ is equivalent to admitting or forbidding $B_n^o$.

**Remark 3.5.** Suppose $T$ is an unordered tree with subtrees $T_1$ and $T_2$. Then if $(X,F)$ admits $T$, there is an $X$-labeling of $T$ with a solution in $F$. Any solution can be partitioned into the leaves labeling $T_1$ and the leaves labeling $T_2$. If $x$ is the label of the root of $T$, then the two parts of this solution disagree on $x$. Let $F_x$ and $F_x^o$ be the elements of $F$ containing and excluding $x$ respectively. Then, either $(X,F_x)$ admits $T_1$ and $(X,F_x^o)$ admits $T_2$, or $(X,F_x)$ admits $T_2$ and $(X,F_x^o)$ admits $T_1$.

Conversely, suppose that for some $x \in X$, $(X,F_x)$ admits $T_1$ and $(X,F_x^o)$ admits $T_2$. Then $(X,F)$ admits $T$, by combining the $X$-labelings of $T_1$ and $T_2$ into an $X$-labeling of $T$, labeling the root by $x$. Similarly, if $(X,F_x)$ admits $T_2$ and $(X,F_x^o)$ admits $T_1$, then $(X,F)$ admits $T$.

**Definition 3.6.** The thicket dimension $\dim(X,F)$ is the largest $n$ such that $B_n$ is admissible, or $\infty$ if there are arbitrarily large such $n$. If there are no such $n$, equivalently if $F$ is empty, the dimension is $-1$.

**Remark.** This is a well-known quantity that occurs in many different contexts. The thicket dimension of $(X,F)$ is equal to Shelah’s rank $R(x = x, \{\varphi\}, 2)$, where $\varphi$ is the formula $x \in F$, relative to the theory $\text{Th}(X,F)$ [9,6]. It is also called Littlestone dimension in the context of computational learning theory [4]. Hodges calls it the branching index [6].
Definition 3.7. For a given set system $(X, \mathcal{F})$, let $\rho(n)$ be the maximum, as $T$ varies over $X$-labelings of $B_n$, of the number of leaves of $T$ with solutions in $\mathcal{F}$. The resulting function $\rho : \mathbb{N} \to \mathbb{N}$ is the thicket shatter function associated with $(X, \mathcal{F})$.

Remark 3.8. The thicket shatter function of any set system is bounded above by the binary exponential function $n \mapsto 2^n$.

Dual Quantities Given a set system $(X, \mathcal{F})$ and a tree $T$, an $\mathcal{F}$-labeling of $T$ is an assignment of vertices of $T$ to elements of $\mathcal{F}$. If $T$ is an $\mathcal{F}$-labeled tree with labeling $u \mapsto F_u$, and if $v \in T$ is a leaf, then we say $x \in X$ is a solution to $v$ in case $(\forall u \prec v)(F_u \in x \iff u \prec_L v)$. This allows us to define corresponding “dual” versions of thicket dimension and the thicket shatter function, which are identical, respectively, to the thicket dimension and thicket shatter function of the dual set system $(X, \mathcal{F})^*$.

VC dimension and dual VC dimension are bound within a single exponent; that is, each is bound by 2 raised to the power of the other. On the other hand, thicket dimension and dual thicket dimension are bound within a double exponent. We do not know whether this is known to be tight.

§4 The Sauer-Shelah dichotomy

We now come to our first central result, that the Sauer-Shelah Lemma, relating thicket shatter function to VC dimension, holds verbatim in the thicket context.

Theorem 4.1. Let $\chi(n, -1) = 0$ and, for $k < \omega$, $\chi(n, k) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$. For any set system $(X, \mathcal{F})$ and $k \in \{-1\} \cup \omega$,

\[
\dim(X, \mathcal{F}) = \infty \implies \forall n \, \rho(n) = 2^n \\
\dim(X, \mathcal{F}) \leq k \implies \forall n \, \rho(n) \leq \chi(n, k).
\]

The first implication is immediate: if $\dim(X, \mathcal{F}) = \infty$, then $(X, \mathcal{F})$ admits $B_n$ for each $n$, so $\rho(n) = 2^n$. If however $\dim(X, \mathcal{F}) \leq k$, then $(X, \mathcal{F})$ forbids $B_{k+1}$. The conclusion follows by the second sentence of the next theorem.

Theorem 4.2. For every finite unordered tree $T$ of dimension $d$, there’s a function $f(n) \in O(n^{d-1})$ such that if any set system $(X, \mathcal{F})$ forbids $T$, then $\rho(n) \leq f(n)$. If, in particular, $T$ is $B_d$, then $\rho(n) \leq \chi(n, d - 1)$.

Proof. Suppose that $(X, \mathcal{F})$ forbids $T$. We proceed by induction on the construction of $T$. If $T$ has dimension 0, it is the single leaf $B_0$. If $(X, \mathcal{F})$ forbids $T$, then $\mathcal{F}$ must be empty, so $\rho(n) = 0$, which is $O(n^{-1})$.

Otherwise, suppose that $T$ has subtrees $T_1$ and $T_2$. Suppose their dimensions are $d_1$ and $d_2$, and suppose that $f_1$ and $f_2$ are given by induction. For $x \in X$, let $\mathcal{F}_x$ be the collection of those sets in $\mathcal{F}$ that include $x$, and let $\mathcal{F}_x$ be the collection of those which exclude $x$. Let $\rho_x(n)$ and $\rho_x(n)$ be the thicket shatter functions of $(X, \mathcal{F}_x)$ and $(X, \mathcal{F}_x)$ respectively. It is easy to see that for any $x \in X$, $\rho(n) \leq \rho_x(n) + \rho_x(n)$; simply take the $X$-labeled tree witnessing $\rho(n)$ and observe that the solutions to its leaves can be partitioned into those containing $x$ and those not containing $x$.

Let $P_1(x)$ express that $(X, \mathcal{F}_x)$ admits $T_1$, and $Q_1(x)$ express that $(X, \mathcal{F}_x)$ admits $T_1$. Then by Remark 3.8

\[(X, \mathcal{F}) \text{ admits } T \iff \exists x \in X \left( (P_1(x) \land Q_2(x)) \lor (P_2(x) \land Q_1(x)) \right).\]
Reasoning propositionally, \((X, \mathcal{F})\) forbids \(T\) if and only if for all \(x \in X\),
\[
(\neg P_1(x) \land \neg P_2(x)) \lor (\neg P_1(x) \land \neg Q_1(x)) \lor (\neg Q_2(x) \land \neg P_2(x)) \lor (\neg Q_2(x) \land \neg Q_1(x)).
\]
By induction, this implies, for all \(x \in X\),
\[
(\rho_x \leq f_1 \land \rho_x \leq f_2) \lor (\rho_x \leq f_1 \land \rho_x \leq f_1) \lor (\rho_x \leq f_2 \land \rho_x \leq f_2) \lor (\rho_x \leq f_2 \land \rho_x \leq f_1),
\]
where, e.g., \(\rho_x \leq f_1\) abbreviates \(\forall n \in \mathbb{N} \rho_x(n) \leq f_1(n)\). Label the four disjuncts (i)-(iv).

Suppose that for some \(x \in X\), case (ii) holds. Then \(\rho(n) \leq \rho_x(n) + \rho_x(n) \leq 2f_1(n)\). The right-hand side is \(O(n^{d_1})\), which is \(O(n^d)\), and we are done. Similar reasoning applies if for some \(x \in X\), case (iii) holds. If neither (ii) nor (iii) holds for any \(x\), then
\[
\forall x \in X \left((\rho_x \leq f_1 \land \rho_x \leq f_2) \lor (\rho_x \leq f_2 \land \rho_x \leq f_1)\right),
\]
in which case,
\[
\forall x \in X \left((\rho_x \leq f_1 \lor \rho_x \leq f_1) \lor (\rho_x \leq f_2 \lor \rho_x \leq f_2)\right).
\]

I claim that for any function \(g\), \(\langle x \rangle(\rho_x \leq g \lor \rho_x \leq g) \implies \rho < \int g\), where \(\int g\) is bounded by \(\langle f g \rangle(n) = 1 + \sum_{k < n} g(k)\). So \(\rho\) would be bounded above by \(\int f_1\) and \(\int f_2\). If \(d_1 = d_2\), then \(d = d_1 + 1\), and \(\rho\) would be bounded by a function in \(O(n^{d_1+1})\), which is \(O(n^d)\). If \(d_1 \neq d_2\), then suppose without loss of generality that \(d_1 < d_2\). Then \(\rho\) would be bounded above by \(\int f_1\), a function in \(O(n^{d_1+1})\), which is again \(O(n^d)\), which completes the proof.

It remains to justify the claim. We show that \(\rho(n) \leq \langle f g \rangle(n)\) by induction on \(n\). If \(n = 0\), \(\rho(0) \leq 1 \leq \langle f g \rangle(n)\). Otherwise, consider the labeled tree \(W\) witnessing \(\rho(n)\), and let \(r\) be the label of its root. By hypothesis either \(\rho_r\) or \(\rho_{\bar{r}}\) is bounded above by \(g\). Therefore, either the number of solutions to the left of the root or the number of solutions to the right must be bounded by \(g(n-1)\). The number of solutions in the remaining half is bounded by \(\rho(n-1)\), which by induction is at most \(\langle f g \rangle(n-1)\). Therefore the total number of solutions is at most \(g(n-1) + \langle f g \rangle(n-1) = \langle f g \rangle(n)\).

This concludes the proof of the first sentence. Finally, we must show that if \(T\) is \(B_d\), \(\rho(n)\) is bounded by \(\chi(n, d-1)\). The base cases \(d = 0\) and \(d = 1\) are the same as before. Otherwise, if \((X, \mathcal{F})\) forbids \(B_d\), then for any \(x \in X\), either \((X, \mathcal{F}_x)\) forbids \(B_{d-1}\) or \((X, \mathcal{F}_x)\) forbids \(B_{d-1}\). Therefore, for any \(x \in X\), \(\rho_x \leq \varphi(\cdot, d-1)\), or \(\rho_x \leq \varphi(\cdot, d-1)\). Hence, by the above claim, \(\rho \leq \int \varphi(\cdot, d-1)\), which is \(\varphi(\cdot, d)\). \(\square\)

The VC density of a set system is defined to be the “growth rate” of the shatter function. Analogously,

**Definition 4.3.** Thicket density \(\text{dens}(X, \mathcal{F})\) is defined by
\[
\inf\{c \in \mathbb{R} : \rho(n) \in O(n^c)\},
\]
where \(\rho(n)\) is the thicket shatter function of \((X, \mathcal{F})\). In case \(\rho(n) \in O(n^c)\) for all \(c \in \mathbb{R}\), \(\text{dens}(X, \mathcal{F}) = -\infty\), and in case \(\rho(n) \in O(n^c)\) for no \(c \in \mathbb{R}\), \(\text{dens}(X, \mathcal{F}) = \infty\). The dual thicket density is defined with the dual thicket shatter function \(\rho^*\) instead of \(\rho\).

**Remark 4.4.** By Theorem 4.4 if \(\dim(X, \mathcal{F}) = k < \infty\), then \(\rho(n) \leq \chi(n, k)\), which is \(O(n^k)\). Therefore, \(\text{dens}(X, \mathcal{F}) \leq \dim(X, \mathcal{F})\) for any set system \((X, \mathcal{F})\), as \(\mathbb{R} \cup \{-\infty, \infty\}\)-valued quantities.
Figure 4.1. The unordered tree of Lemma 4.5.

**Lemma 4.5.** The thicket density of \((X, F)\) is \(-\infty\), 0, or \(\geq 1\), depending on whether the cardinality of \(F\) modulo extensionality is zero, positive and finite, or infinite.

**Proof.** If \(F\) is empty, then \(\rho(n) = 0\), and \(\text{dens}(X, F) = -\infty\). If \(F\) has nonempty but has finitely many elements modulo extensionality, say at most \(B\), then for all \(n\) \(\rho(n) \leq B\). Therefore, \(\text{dens}(X, F) = 0\). On the other hand, if \(F\) has infinitely many elements modulo extensionality, then we can extract a sequence \((x_i, F_i)_{i<\omega}\) such that for any \(i < j\), \(F_i\) and \(F_j\) agree on \(x_h\) for \(h < i\), but disagree on \(x_i\). (Given \(F\), pick some \(x \in X\) such that both \(F_x\) and \(F_{\bar{x}}\) are nonempty. One of them, call it \(F^*\) must be infinite; pick some \(F\) come from the other, \(F^\dagger\). Replace \(F\) by \(F^*\) and repeat.) Then the unordered tree pictured in Figure 4.1 has a solution, and, by truncating the tree at depth \(n\) for each \(n\), we observe that \(\rho(n) \geq n + 1\), hence \(\text{dens}(X, F) \geq 1\). \(\Box\)

An alternate formulation of density Theorem 4.1 says roughly that, for set systems of finite thicket dimension, trees which are balanced cannot have very many realized leaves. Then it stands to reason that admissible trees, i.e. those with every leaf realized, must be far from balanced. This idea yields another way to define thicket density.

**Definition 4.6.** Let \(\sigma(n)\) be the minimum depth of any finite \(X\)-labeled tree with at least \(n\) realized leaves.

If \((X, F)\) has infinite thicket density, then \(\sigma(n) = \lfloor \log_2(n) \rfloor\), as witnessed by a balanced binary tree. If \((X, F)\) has zero density, then \(\sigma(n)\) is undefined for arbitrarily large \(n\). The interesting case is when \((X, F)\) has finite and positive density, in which case \(\sigma(n)\) is bounded below by \(n^\epsilon\) for some \(\epsilon > 0\). In fact, the rate of growth of \(\sigma\) is the inverse of the density:

**Lemma 4.7.** For any set system \((X, F)\) of finite positive density \(\delta\), if \(\epsilon = \sup\{c \in \mathbb{R} : \sigma(n) \in \Omega(n^c)\}\), then \(\epsilon = \frac{1}{\delta}\).

**Proof.** Let \(T\) be a finite \(X\)-labeled binary tree with \(n\) realized leaves, and let \(\Delta\) be the depth of \(T\). Create a new \(X\)-labeled tree \(T'\) by taking the balanced binary tree of depth \(\Delta\), superimposing the labeling of \(T\), and labeling all remaining non-leaves arbitrarily. Then \(T'\) also has at least \(n\) realized leaves, which implies that \(n \leq \rho(\Delta)\). Since \(T\) is arbitrary, by taking the \(T\) with the least possible \(\Delta\), we obtain \(n \leq \rho(\sigma(n))\), for all \(n \in \mathbb{N}\). This implies that \(\epsilon \geq \frac{1}{\delta}\).

On the other hand, let \(T\) be a finite, balanced \(X\)-labeled binary tree of depth \(\Delta\) and \(n\) realized leaves. Minimally prune \(T\) to obtain a tree \(T'\) such that every leaf
The depth of such trees grows like $\Omega(\sqrt{n})$, where $n$ is the size.

Hence $\delta \leq 1/\epsilon$. Hence $\delta \epsilon = 1$, and we are done. \hfill \Box

§5 Rank

In this section, we establish an equivalence between thicket density and “Shelah’s local $\omega$-rank” $R(p, \{\varphi\}, \omega)$, the local analogue of Morley rank\footnote{We do not know of a commonly accepted name for this rank. However, it is equivalent to $R(p, \{\varphi\}, \infty)$, and the Cantor-Bendixson rank of the space of $\varphi$-types.}. From a model-theoretic perspective, the significance of this result is probably reversed, i.e., we identify $R(p, \{\varphi\}, \omega)$ with the thicket density of a particular set system.

Concretely, given any partitioned first-order formula $\varphi(x, y)$, and for any model $M$ of a complete theory $T$, define a set system $(M^x, M^y, \in)$ by $a \in b \iff M \models \varphi(a, b)$. Then, roughly speaking, we can calculate the rank $R(x = x, \{\varphi\}, \omega)$ of a finite $\varphi$-type modulo $T$ using the thicket shatter function of the set system $(M^x, M^y, \in)$. Since the thicket shatter function depends only on $T$ and not the particular choice of $M$, we can carry out this calculation in any model $M \models T$, not only a sufficiently saturated one.

In this section, fix a set system $(X, \mathcal{F})$, and a sufficiently saturated model $M$ of $\text{Th}(X, \mathcal{F}, \in)$ (so that we can calculate $\omega$-rank). Let $M_X$ and $M_\mathcal{F}$ be the two sorts of $M$. Let $\varphi$ be the formula $x \in \mathcal{F}$. (We use typewriter script to distinguish syntactic variables, e.g., $x$, from values, e.g., $x \in X$.) By a $\varphi$-formula, we mean a formula of the form $\varphi(a, \mathcal{F})$ or $\neg \varphi(a, \mathcal{F})$ for some $a \in M_X$. Whenever we assert that some sentence holds, we always mean relative to the model $M$.

By a finite $\varphi$-type, we mean a conjunction of $\varphi$-formulas, including the empty conjunction $\top$. Two finite $\varphi$-types $p$ and $q$ are contradictory in case there exists some $a \in M_X$ such that one of $\{\varphi(a, \mathcal{F}), \neg \varphi(a, \mathcal{F})\}$ occurs as a conjunct in $p$, and the other occurs as a conjunct in $q$. In this case, we say that $p$ and $q$ disagree on $\varphi(a, \mathcal{F})$. By $p(\mathcal{F})$ we mean the subfamily of $\mathcal{F}$ satisfying the type $p$.  

Figure 4.2. A $N$-labeled tree with all leaves realized in $\binom{N}{2}$. The depth of such trees grows like $\Omega(\sqrt{n})$, where $n$ is the size.
Definition 5.1. For any unordered tree $T$, let $L$ be its set of leaves and $N$ be its set of non-leaves. Let $\mathcal{L}_T$ be the signature \{ $\in$ \} expanded by a fresh set of constant symbols $\{a_u : u \in N\}$ of type $X$, and $\{b_v : v \in L\}$ of type $\mathcal{F}$. Given in addition a finite $\varphi$-type $p$, define a first-order $\mathcal{L}_T$-theory $Adm^T_p$ by:

1. $p(b_v)$ for any $v \in L$,
2. $\varphi(a_u, b_v, \ldots) \not\rightarrow \varphi(a_u, b_w, \ldots)$ if $v, w \in L^2$, $u \in N$, and $v \sim_u w$, and
3. $\varphi(a_u, b_v, \ldots) \leftrightarrow \varphi(a_u, b_w, \ldots)$ if $v, w \in L^2$, $u \in N$, and $v \sim_u w$.

Lemma 5.3. Suppose that $M$ is sufficiently saturated, the consistency of $\text{Th}(X, \mathcal{F})$ is equivalent the admissibility of $T$ in $(M_X, p(M_F))$. (Concretely: the existence of injections $u \mapsto a_u : N \to M_X$ and $v \mapsto b_v : L \to M_F$ such that properties (1)-(3) of Definition 5.1 hold in $M$.)

Remark 5.2. The following hold of $Adm^T_p$:

- If $(X, p(\mathcal{F}))$ admits $T$, then $\text{Th}(X, \mathcal{F})$ is consistent with $Adm^T_p$. If $T$ is finite, then the converse holds as well.
- Since $M$ is sufficiently saturated, $\text{Th}(X, \mathcal{F})$ is equivalent the admissibility of $T$ in $(M_X, p(M_F))$.

Lemma 5.3. Suppose that $T$ is a finite-dimensional unordered tree and that there is an embedding of the unordered tree $S$ into $T$. Then if $\text{Th}(X, \mathcal{F})$ is consistent with $Adm^T_p$, it is consistent with $Adm^S_p$.

Proof. First observe that for every vertex $v \in T$, the subtree rooted at $v$ contains some leaf. For otherwise (since each non-leaf has two children) $T$ would have a complete infinite binary subtree, and not be finite-dimensional. Fix an embedding $\iota : S \to T$, and let $j$ map leaves of $S$ to leaves of $T$, such that for each leaf $\ell \in S$, $j(\ell)$ is contained in the subtree rooted at $\iota(\ell)$. Then notice that for any non-leaf $s$ and leaf $\ell$ in $S$, $s \prec_\ell \in S$ if $\iota(s) \prec j(\ell)$ in $T$. Moreover, if $\ell'$ is another leaf of $S$, then $\ell \sim_\ell \ell'$ in $S$ iff $j(\ell) \sim_{j(\ell)} j(\ell')$ in $T$. Similarly, $\ell \sim_\ell \ell'$ in $S$ iff $j(\ell) \sim_{j(\ell)} j(\ell')$ in $T$.

Suppose that $T$ is admissible in $(M_X, p(M_F))$, witnessed by the labeling $u \mapsto a_u$, $v \mapsto b_v$. For any non-leaf $s \in S$, label it by $a_{\iota(s)}$, and for any leaf $t \in S$, label it by $b_{j(t)}$. Since the relations $\sim$ and $\prec$ are preserved by the map $\iota$ on non-leaves and $j$ on leaves, properties (2) and (3) of Definition 5.1 carry over from $T$ to $S$. Since $j$ maps leaves to leaves, property (1) of Definition 5.1 carries over from $T$ to $S$. Hence, $S$ is admissible in in $(M_X, p(M_F))$. $\square$

Remark 5.5. If $R^\varphi(p) \geq \omega$, then $R^\varphi(p) = \infty$. (Pillay Exercise 6.53) Hence $R^\varphi$ takes values in $\omega \cup \{-\infty, \infty\}$.}

Footnote: cf. Definition 5.4
In fact, we can rearrange the quantifiers in the second clause to be a little bit stronger, using some simple combinatorics:

**Lemma 5.6.** Suppose that \( \{ p_i : i < \omega \} \) is a sequence of pairwise contradictory finite \( \varphi \)-types. Then there is an infinite set \( S \subseteq \omega \) such that for any \( r \in S \), there exists \( a \in M_X \), such that for any \( s > r \) in \( S \), \( p_r \) and \( p_s \) disagree on \( \varphi(a,F) \).

**Proof.** For any infinite set \( S \subseteq \omega \), let \( m \) be its least element. Since \( p_m \) is finite, there must be some \( a \in M_X \) and an infinite subset \( S' \subseteq S \setminus \{ m \} \) such that \( p_m \) disagrees with \( p_n \) on \( \varphi(a,F) \) for any \( n \in S' \). Let \( S_0 = \omega \), and for each \( i < \omega \), obtain \( S_{i+1} \) from \( S_i \) in this manner. Obtain \( m_i \) and \( a_i \) from \( S_i \) as above. Then \( S_0 \supset S_1 \supset S_2 \supset \ldots \), \( m_0 < m_1 < m_2 < \ldots \), and \( m_i \in S_j \iff i \geq j \). Moreover, for every \( i < j \) \( p_{m_i} \) and \( p_{m_j} \) disagree on \( \varphi(a_i,F) \). Therefore, we may take \( S = \{ m_0, m_1, m_2, \ldots \} \). \( \square \)

**Definition 5.7** (\( k \)-branching trees). The 0-branching tree \( T_0 \) is a single leaf. For \( k \geq 1 \), the \( k \)-branching tree \( T_k \) is the unordered binary tree with subtrees \( T_k \) and \( T_{k-1} \).

**Remark 5.8.** \( T_k \) has dimension \( k \), and the number of vertices in \( T_k \) at depth \( n \) is \( \chi(n,k) \).

**Theorem 5.9.** \( R^\varphi(p) \geq k \) if and only if \( \text{Th}(X,F) \cup \text{Adm}_{T_k}^p \) is consistent.

**Proof.** We work in \( M \), by Remark 5.2 and induct on \( k \). If \( k = 0 \), then \( R^\varphi(p) \geq k \) iff there exists some \( b \in M_\varphi \) such that \( p(b) \), iff the tree \( T_0 \) is admissible in \( (M_X, p(M_\varphi)) \), iff \( \text{Th}(X,F) \cup \text{Adm}_{T_0}^p \) is consistent.

Otherwise, suppose that \( R^\varphi(p) \geq k + 1 \), and let \( \{ p_i \}_{i<\omega} \) witness this. By Lemma 5.6, there is an infinite set \( S \subseteq \omega \) such that for any \( r \in S \), there is some \( a \in M_X \),
such that for any \( s > r \) in \( S \), \( p_r \) and \( p_s \) disagree on \( \varphi(a, F) \). Let \( s(i) \) be the \( i \)-th element of \( S \), and \( a^{(i)} \) witness the \( \varphi \)-formula distinguishing \( p_{s(i)} \) from \( p_{s(j)} \), for \( j > i \).

Let \( N \) and \( L \) be the sets of non-leaves and leaves, respectively, of \( T_{k+1} \). The tree \( T_{k+1} \) can be partitioned into a single infinite spine plus countably many copies of \( T_k \). Let \( N = \bigcup_{i < \omega} N_i \cup N_s \), where \( N_i \) is the set of non-leaves of the \( i \)-th copy of \( T_k \), and \( N_s \) are the vertices down the spine. Let \( L = \bigcup_{i < \omega} L_i \), where \( L_i \) is the set of leaves of the \( i \)-th copy of \( T_k \).

For each \( i < \omega \), we have \( R^\varphi(p \land p_{s(i)}) \geq k \). Therefore, by induction, there are injections \( u \mapsto a_u : N_i \to M_X \) and \( v \mapsto b_v : L_i \to M_F \) satisfying properties (1), (2), and (3). By taking the union over all \( i \), we have a map \( v \mapsto b_v : L \to M_F \). Similarly, we can get a map \( u \mapsto a_u : N \to M_X \) if we specify \( a_u \) for \( u \in N_s \). But for \( u \in N_s \) of distance \( i \) from the root, simply let \( a_u = a^{(i)} \).

For any \( v \in L \), \( b_v \) satisfies \( p \land p_{s(i)} \) for some \( i \); in particular it satisfies \( p \). Thus we have verified (1), and it remains to verify (2) and (3). Fix two leaves \( v, w \in L^2 \) and a common ancestor \( u \in N \).

- If for some \( i, v, w \in L_i \) and \( u \in N_i \), then the relations \( v \perp_u w, v \sim_u w \) are identical in the ambient tree \( T_{k+1} \) and the \( i \)-th copy of \( T_k \), hence (2) and (3) are inherited from the given maps \( N_i \to M_X \) and \( L_i \to M_F \).
- If for some \( i \), \( v, w \in L_i \) and \( u \notin N_i \), then \( u \) must be the vertex on the spine of distance \( i \) from the root, and \( v \sim_u w \). Therefore, \( a_u = a^{(i)} \), and since \( v, w \) satisfy \( p_{s(i)}(b_v) \) and \( b_u \) must agree on \( \varphi(a_u, F) \).
- If for some \( i < j, v \in L_i \) and \( w \in L_j \), then \( u \) must be some vertex on the spine of distance at most \( i \) from the root; moreover, \( p_{s(i)}(b_v) \) and \( p_{s(j)}(b_w) \).

Therefore, \( b_v \) and \( b_w \) disagree on \( \varphi(a^{(i)}, F) \), and agree on \( \varphi(a^{(i)}, F) \) for all \( 0 \leq i' < i \). But, \( a_u = a^{(i)} \) just in case \( v \perp_u w, v \sim_u w \). This concludes the forward direction.

In the other direction, suppose that we have injections \( u \mapsto a_u : N \to M_X \) and \( v \mapsto b_v : L \to M_F \) satisfying (1), (2), and (3). Let \( N_s \), \( N_i \), and \( L_i \) be as above, and let \( a^{(i)} \) be the sequence of vertices along the spine. For \( i < \omega \), define \( p_i = \{ \varphi^*(a^{(i)}, F) \} \cup \{ \neg \varphi^*(a^{(i)}, F) : i' < i \} \), where \( \varphi^*(a^{(i)}, F) \) is either \( \varphi(a^{(i)}, F) \) or \( \neg \varphi(a^{(i)}, F) \), depending on which one the leaves in \( L_i \) satisfy.

For each \( i \), by restricting the maps \( u \mapsto a_u \) and \( v \mapsto b_v \) to \( N_i \) and \( L_i \) respectively, we get injections which inherit (2) and (3) from the original function. Towards (1), notice that for each \( v \in L_i \), \( b_v \) satisfies \( p \land p_i \), by definition of \( p_i \). Hence, by induction, \( R^\varphi(p \land p_i) \geq k \). Since for each \( i < j \), \( p_i \) and \( p_j \) disagree on \( \varphi(a^{(i)}, F) \), we have that \( R^\varphi(p) \geq k + 1 \), which concludes the proof.

\[ \square \]

**Theorem 5.10.** The thicket density of \( (X, p(F)) \) is identical to \( R^\varphi(p) \), as \( \mathbb{R}_{\geq 0} \cup \{-\infty, \infty\} \)-valued quantities.

**Proof.** We show that both quantities are bounded above by each other.

For some \( 0 \leq k < \omega \), suppose that \( R^\varphi(p) \geq k \). By Theorem 5.9, \( \text{Th}(X, F) \) is consistent with \( \text{Adm}^T_{\varphi, b} \). By Remark 5.2, for any finite subtree \( S \) of \( T_k \), \( \text{Adm}^S_{\varphi, b} \) is consistent, so \( (X, p(F)) \) admits \( S \). But if we obtain \( S \) by truncating \( T_k \) to depth \( n \), then \( S \) has \( O(n^k) \) leaves, thus ensuring \( \text{dens}(X, F) \geq k \). Hence \( R^\varphi(p) \leq \text{dens}(X, p(F)) \).
Figure 5.2. By Theorem 5.9, \( R^e(p) \geq 2 \) is equivalent to the admissibility of the unordered tree \( T_2 \) in \( M \), such that every \( b_{ij} \) satisfies \( p \).

Conversely, suppose that for some \( 0 \leq k < \omega \), \( R^e(p) \not\geq k \). By Theorem 5.9, \( \text{Adm}^T_k \) is inconsistent with \( \text{Th}(X, \mathcal{F}) \). By compactness, some finite fragment of \( \text{Adm}^T_k \) is inconsistent with \( \text{Th}(X, \mathcal{F}) \), and hence there exists a finite subtree \( S \) of \( T_k \) such that \( \text{Adm}^S_p \) is inconsistent with \( \text{Th}(X, \mathcal{F}) \). By Remark 5.2, \((X, p(\mathcal{F}))\) forbids \( S \). Since \( T_k \) has dimension \( k \), \( S \) has dimension at most \( k \). By Theorem 4.2, \( \text{dens}(X, p(\mathcal{F})) \leq k - 1 \). Hence \( \text{dens}(X, p(\mathcal{F})) \leq R^e(p) \). \( \square \)

As an immediate corollary, we deduce that thicket density is integer-valued, in contrast to VC density. This fact was first proven by James Freitag and Dhruv Mubayi, using elementary combinatorics, and a very short elementary proof has been found by Ross Berkowitz. (Both of these results are unpublished and were communicated to us in person.)

§6 Degree

Each model-theoretic rank has an associated notion of degree (sometimes called multiplicity); roughly, given a formula \( p \), this is the maximum number of pairwise contradictory extensions of \( p \) which each have rank no less than \( p \). For any fixed rank, this degree must be absolutely bounded by some cardinal \( \kappa \), where the rank of a formula is at least \( \alpha + 1 \) if there are at least \( \kappa \) many pairwise contradictory extensions of rank \( \alpha \). For example, we can define the degree \( D \) associated with the rank \( R^e \) as follows.
Def. 6.1. For any finite ϕ-type p, let D(p) be the greatest integer n < ω such that there exist pairwise contradictory finite ϕ-types p₁, . . . , pₙ, such that for each 1 ≤ i ≤ n, Rϕ(p ∧ pᵢ) = Rϕ(p).⁴

In this section, we develop a notion of degree that seems natural and appropriate for thicket density. Like in Section 5 we fix a set system (X, F) and a sufficiently saturated model M of Th(X, F). For a finite ϕ-type p to have parameters from X means every conjunct of p is of the form ϕ(x, F) or ¬ϕ(x, F) for some x ∈ X. For such a type p, let dens(p) abbreviate dens(X, p(F)), where p(F) = {F ∈ F : p(F)}.

Def. 6.2. Given a vertex v in an X-labeled tree T, define the finite ϕ-type pv(F) to be the conjunction of literals ϕ(x, F) for u ≺L v, and ¬ϕ(xu, F) for u ≺R v. Then pv is a finite ϕ-type with parameters from X, and F solves v if and only if pv(F).

Def. 6.3. Let p be a finite ϕ-type with parameters from X. An X-labeled tree T factors p in case dens(p ∧ pᵢ) = dens(p), for each leaf v of T. Furthermore, T irreducibly factors p if, in addition, no proper extension T′ of T factors p. We say that p is irreducible if it is irreducibly factored by a single leaf. We say that (X, F) is irreducible in case the empty type is.

Lemma 6.4. For every finite ϕ-type p with parameters from X, if 0 ≤ dens(p) < ∞, then p is irreducibly factored by some finite tree.

Proof. Fix the domain X. For each finite type p ∈ Sϕ(F), we build an X-labeled tree Π(p) irreducibly factoring p via a nondeterministic construction.

- If p is irreducible, Π(p) is a single leaf.
- Otherwise, let x ∈ X satisfy dens(p ∧ ϕ(x, F)) = dens(p ∧ ¬ϕ(x, F)) = dens(p). Let Π(p) have root x and left and right subtrees Π(p ∧ ϕ(x, F)) and Π(p ∧ ¬ϕ(x, F)) respectively.

We claim that this construction terminates after finitely many steps. For otherwise, there would be infinite branch x₀, x₁, x₂, . . . and an infinite sequence p₀ ⊆ p₁ ⊆ p₂ ⊆ . . . such that for each i < ω, either pᵢ₊₁ = pᵢ ∧ ϕ(xᵢ, F) or pᵢ₊₁ = pᵢ ∧ ¬ϕ(xᵢ, F), and dens(pᵢ) = dens(pᵢ ∧ ϕ(xᵢ, F)) = dens(pᵢ ∧ ¬ϕ(xᵢ, F)). Let pᵢ be the extension of pᵢ that is not pᵢ₊₁. Then, for each i < ω dens(pᵢ) = dens(p), and for each i < j < ω, pᵢ and pⱼ agree on ϕ(xᵢ, F) for h < i, and disagree on ϕ(xᵢ, F).

Let k = dens(p). By Theorems 5.9 and 5.10, for each i < ω, we can find a consistent labeling of Tk by elements of M, such that each leaf satisfies pᵢ. By stringing these trees along a spine labeled by the sequence x₀, x₁, x₂, . . ., we get a consistent labeling of Tk₊₁ by elements of M, such that each leaf satisfies p. But this implies that dens(p) ≥ k + 1, a contradiction.

We now show that even though there may not be a unique tree that irreducibly factors a set system, the number of vertices in any such tree is the same. Even stronger, the partition induced by such a tree is unique up to rearrangement by pieces of strictly smaller density. This is analogous to the situation for, e.g., Morley rank.

⁴If there were arbitrarily large such n, the there must exist an infinite family pᵢ of pairwise contradictory ϕ-types, such that for each i, R(p ∧ pᵢ) = R(p), contradicting Def. 5.1.
Lemma 6.5. Let \( p \) be a finite type with parameters from \( X \). Suppose \( \delta = \text{dens}(p) \) is finite and nonnegative, and suppose \( T_1 \) and \( T_2 \) are \( X \)-labeled trees that each irreducibly factor \( p \). Then \( T_1 \) and \( T_2 \) have the same number of leaves. A fortiori, there is a bijection between the leaves of \( T_1 \) and the leaves of \( T_2 \), such that \( \text{dens}(p \wedge (p_v \Delta p_w)) < \delta \) for any bijective pair \((v, w)\).

Proof. For \( i \in \{1, 2\} \), let \( L_i \) be the set of leaves of \( T_i \). Since \( T_1 \) and \( T_2 \) each factor \( p \), \( \text{dens}(p \wedge p_v) = \text{dens}(p \wedge p_w) = \delta \) for each \( v \in L_1 \) and \( w \in L_2 \).

For any fixed \( v \in L_1 \), \( \text{dens}(p \wedge p_v) = \max\{\text{dens}(p \wedge p_v \wedge p_w) : w \in L_2\} \)\(^5\) Hence, for some \( w \in L_2 \), \( \text{dens}(p \wedge p_v \wedge p_w) = \delta \). However, such a \( w \) must be unique; if \( \text{dens}(p \wedge p_v \wedge p_w) = \text{dens}(p \wedge p_v \wedge p_w') = \delta \), then \( T_1 \) does not irreducibly factor \( p \): the leaf \( v \) can be replaced by a non-leaf labeled by the least common ancestor of \( w \) and \( w' \) in \( T_2 \).

Reasoning symmetrically, for any \( w \in L_2 \), there is a unique \( v \in L_1 \) such that \( \text{dens}(p \wedge p_v \wedge p_w) = \delta \). Hence, the relation \( R(v, w) \iff \text{dens}(p \wedge p_v \wedge p_w) = \delta \) is the graph of a bijection, so \( |L_1| = |L_2| \). Furthermore, for any bijective pair \((v, w)\), \( \text{dens}(p \wedge (p_v \Delta p_w)) = \max\{\text{dens}(p \wedge p_v \wedge p_w), \text{dens}(p \wedge p_v \wedge p_w') : v' \neq v, w' \neq w\} \), and this quantity must be strictly less than \( \delta \), since we maximize over finitely many densities all strictly less than \( \delta \).

Hence, we can now define

Definition 6.6. For any finite \( \varphi \)-type \( p \) with parameters from \( X \), the thicket degree \( \text{deg}(p) \) is the number of leaves of any tree irreducibly factoring \( p \). The degree \( \text{deg}(X, \mathcal{F}) \) is defined to be \( \text{deg}(\top) \), the degree of the empty conjunction.

Remark 6.7. Given any tree \( T \) irreducibly factoring \( p \), the types \( p_v \) are pairwise contradictory as \( v \) varies over the leaves of \( T \), and moreover \( R^\varphi(p \wedge p_v) = R^\varphi(p) \) for any leaf \( v \). Hence \( \text{deg}(p) \leq D(p) \).

Remark 6.8. By a finite \( \varphi^* \)-type \( q \) with parameters from \( \mathcal{F} \), we mean a finite conjunction of formulas of the form \( \varphi(x, \mathcal{F}) \) and \( \neg \varphi(x, \mathcal{F}) \), where \( \mathcal{F} \) ranges over \( \mathcal{F} \). We can define the dual degree, \( \text{deg}^*(q) \), by switching the roles of \( X \) and \( \mathcal{F} \).

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\(^5\)In general, \( \text{dens}(X, \mathcal{F} \cup \mathcal{G}) = \max\{\text{dens}(X, \mathcal{F}), \text{dens}(X, \mathcal{G})\} \). The thicket shatter function on the left is bounded below by both thicket shatter functions on the right, and bounded above by their sum.
THICKET DENSITY

throughout. Concretely, the dual degree of $q$ is the number of leaves of any $\mathcal{F}$-labeled binary tree $T$, such that $\text{dens}^\ast(q \land q_v) = \text{dens}^\ast(q)$ for any leaf $v$ of $T$, and $T$ is maximal with respect to this property. (The dual degree of $q$ is simply the degree of $q$ in the dual set system.)

Remark 6.9. Even though thicket density is equivalent to the rank $R$, the thicket degree can differ from the degree $D$ associated with the rank $R$. It suffices, and is easier, to exhibit a difference between the corresponding dual quantities. Let $X$ be any infinite set, and let $\mathcal{F}$ be a partition of $X$ with arbitrarily large finite sets, but no infinite set. Then $\text{dens}^\ast(X, \mathcal{F}) \geq 1$, since $X$ is infinite, but for any $F \in \mathcal{F}$, $\text{dens}^\ast(F, F \upharpoonright F) = 0$, as $F$ is finite (cf. Lemma 4.5). Hence, $(X, \mathcal{F})$ is (dually) irreducible: any $\mathcal{F}$-labeled tree that irreducibly factors $(X, \mathcal{F})$ must be a single leaf, and hence $\text{deg}(X, \mathcal{F}) = 1$. On the other hand, using compactness, we can find an element $b \in M_\mathcal{F}$ that has infinitely many members and infinitely many non-members among elements of $M_X$. Therefore, $R^\ast(\varphi(x, b))$ and $R^\ast(\neg \varphi(x, b))$ are both at least 1, so $D^\ast(\top) \geq 2$.

§7 Conclusion and future work

We have established a relationship between a measure of asymptotic growth of certain finite objects with the ordinal rank of an infinite object, a common phenomenon in the tradition relating infinitary and finitary combinatorics. There are several lines of inquiry that our work raises. For example, from a technical standpoint, we do not know what other information about a set system is contained in its thicket shatter function. We suspect that other invariants (like the leading coefficient) might encode something interesting. Furthermore, there are many identities that the rank $R^\varphi$ is known to satisfy, and it would be interesting to see if they might admit a purely combinatorial proof.

The similarity between the Sauer-Shelah Lemma and the present “thicket” version naturally raises the question of whether they both share a general setting. Chase and Freitag [3] answer this question positively by formulating a shatter function for op-rank of Guingona and Hill [5], which interpolates Shelah’s 2-rank with VC dimension, and establishing a dichotomy which interpolates both versions of the Sauer-Shelah Lemma. Most questions about the corresponding notion of density remain open, for example, what values it takes “between” the VC and thicket cases.

Finally, as we mentioned in the Introduction, this work was originally inspired by reading Tiuryn’s paper [10] separating deterministic from nondeterministic dynamic logic. Several branches of computer science, such as dynamic logic, descriptive complexity theory, and program schematology, are concerned with programs that operate over first-order structures. A basic construction in programming language theory is the unwind of a program; roughly, transforming a finitary program with recursive calls into an infinitary program without. A fundamental invariant of the unwind of a program its its underlying decision tree. Many algorithmic lower bounds obtained from lower bounds of decision trees, for example, the $\Omega(n \log n)$ lower bound on deterministic comparison sorting.

We can leverage the thicket shatter function, if it is bounded by a polynomial, to prove stronger lower bounds that what we would otherwise be able to show using binary decision trees. (In fact, Tiuryn does just this.) We suspect that there might be more applications of our results to computer science, in particular, in
establishing lower bounds for programs which branch using a model-theoretically tame family of conditional tests.

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DIKU, University of Copenhagen.