The Generalized Odd Log-Logistic Fréchet Distribution for Modeling Extreme Values

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Abstract

We introduce a new extension of the Fréchet distribution for modeling the extreme values. The new model generalizes eleven distributions at least, five of them are quite new. Some important mathematical properties of the new model are derived. We assess the performance of the maximum likelihood estimators (MLEs) via a simulation study. The new model is better than some other important competitive models in modeling the breaking stress data, the glass fibers data and the relief time data.

Key Words: Fréchet distribution; Extreme Values; Moments; Estimation; Odd Log-Logistic Family.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

A statistical field known as extreme value theory or extreme value analysis (EVA) studies extreme departures from the median of probability distributions. It aims to determine the likelihood of events that are more extreme than any previously recorded events from a given ordered sample of a given random variable. Extreme value analysis is frequently employed in a variety of fields, including geological engineering, finance, earth sciences, and structural and structural engineering. The EVA, for instance, might be used in the hydrology profession to calculate the likelihood of an exceptionally significant flooding occurrence, like the 100-year flood. Similar to this, a coastal engineer would aim to determine the 50-year wave and construct the structure appropriately while designing a breakwater (for more details, see Kotz and Nadarajah (2000), Afify et al. (2016a,b) and Salah et al. (2020)). The Fisher-Tippett-Gnedenko theorem only states that if the distribution of a normalised maximum converges, then the limit has to be one of a specific class of distributions. The role of the extremal types theorem for maxima is similar to that of the central limit theorem for averages, with the exception that the central limit theorem applies to the average of a sample from any distribution with finite variance. It doesn’t say that the normalised maximum distribution converges.

The Fisher-Tippet-Gnedenko theorem in statistics is a broad conclusion of the extreme value theory concerning the asymptotic distribution of extreme order statistics. It is also known as the Fisher-Tippett theorem or the extreme value theorem. Only one of three alternative distributions, the Gumbel distribution, the Fréchet distribution, or the Weibull distribution, can be reached by the maximum of a sample of iid random variables after sufficient renormalization. The Fréchet (Fr) model is one of the most important distributions in modeling extreme values. The Fr model was originally proposed by Fréchet (1927). It has many applications in ranging, accelerated life testing, earthquakes, the floods, the wind speeds, the horse racing, the rainfall, queues in supermarkets and sea waves. Some new Fréchet versions can be cited by Jahanshahi et al. (2019) (for the Burr X Fréchet extension for modeling the extreme values with some mathematical properties, classical and Bayesian analysis), Al-Babtain et al. (2020a,b), Elsayed and Yousof
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A RV $X$ is said to have the Fr distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$g_{\alpha,\theta}(x) = ba^b x^{-(b+1)} \exp \left[ - \left( \frac{a}{x} \right)^b \right] |_{x \geq 0} \quad (1)$$

and

$$G_{\alpha,\theta}(x) = \exp \left[ - \left( \frac{a}{x} \right)^b \right] |_{x \geq 0}, \quad (2)$$

where $a > 0$ is a scale parameter and $b > 0$ is a shape parameter. For $b = 2$, we get the Inverse Rayleigh (IR) model. For $a = 1$ we get the Inverse Exponential (IEx) model.

Recently, Cordeiro et al. (2016) proposed a new class of distributions called the generalized odd log-logistic-G (GOLL-G) family with two extra shape parameters. For an arbitrary baseline CDF $G(\xi)(x)$, the CDF of the GOLL-G family is given by

$$F_{\alpha,\theta,\xi}(x) = \frac{G_{\xi}(x)^{\alpha \theta}}{G_{\xi}(x)^{\alpha \theta} + \left[ 1 - G_{\xi}(x) \right]^\theta}. \quad (3)$$

The PDF corresponding to (3) is given by

$$f_{\alpha,\theta,\xi}(x) = \frac{\alpha \theta G_{\xi}(x) G_{\xi}(x)^{\alpha \theta - 1} \left[ 1 - G_{\xi}(x) \right]^{\alpha - 1}}{\left\{ G_{\xi}(x)^{\alpha \theta} + \left[ 1 - G_{\xi}(x) \right]^\theta \right\}^2}. \quad (4)$$

For $\theta = 1$ we get the OLL-G family (Gleaton and Lynch (2006)). For $\alpha = 1$ we get the Proportional reversed hazard rate G family (Gupta and Gupta (2007)). Here, we define a new Fr model based on Cordeiro et al. (2016) called generalized odd log-logistic Fr (GOLL-Fr) family and provide some plots of its PDF and hazard rate function (HRF) $[h_{\alpha,\theta,a,b}(x)]$. The GOLLFr CDF is given by

$$F_{\alpha,\theta,a,b}(x) = \frac{\exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right]}{\exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] + \left\{ 1 - \exp \left[ -\theta \left( \frac{a}{x} \right)^b \right] \right\}^\alpha}. \quad (5)$$

The new CDF in (5) can be used for presenting a new discrete model for modeling the count data (see Aboraya et al. (2020), Chesneaux et al. (2022), Ibrahim et al. (2021a) and Yousof et al. (2021) for more details). The PDF corresponding to (5) is given by

$$f_{\alpha,\theta,a,b}(x) = \frac{\alpha \theta ba^b x^{-(b+1)} \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] \left\{ 1 - \exp \left[ -\theta \left( \frac{a}{x} \right)^b \right] \right\}^{\alpha - 1}}{\left\{ \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] + \left\{ 1 - \exp \left[ -\theta \left( \frac{a}{x} \right)^b \right] \right\}^\alpha \right\}^2}. \quad (6)$$

The new model in (6) can be used in regression modeling and in assessing various estimations methods as recently presented by Altun et al. (2022), Yousof et al. (2022), Korkmaz et al. (2022) and Aboraya et al. (2022). The HRF for the new model can be get from $f_{\alpha,\theta,a,b}(x) / \left[ 1 - F_{\alpha,\theta,a,b}(x) \right]$. Let $\tau = \inf \left\{ x \mid G(x;\xi) > 0 \right\}$, the asymptotics of the CDF, PDF and HRF as $x \to \tau$ are given by

$$F_{\alpha,\theta,a,b}(x) \sim \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] \mid_{x \to \tau}.$$
\[
f_{\alpha,\theta,a,b}(x) \sim \alpha \theta ba^bx^{-(b+1)} \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] |_{x \to \tau}
\]

and

\[
h_{\alpha,\theta,a,b}(x) \sim \alpha \theta ba^bx^{-(b+1)} \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] |_{x \to \tau}.
\]

The asymptotics of CDF, PDF and HRF as \( x \to \infty \) are given by

\[
1 - F_{\alpha,\theta,a,b}(x) \sim \left( \theta \left\{ 1 - \exp \left[ -\left( \frac{a}{x} \right)^b \right] \right\} \right)^{\alpha} |_{x \to \infty},
\]

\[
f_{\alpha,\theta,a,b}(x) \sim \alpha \theta ba^bx^{-(b+1)} \exp \left[ -\left( \frac{a}{x} \right)^b \right] \left\{ 1 - \exp \left[ -\left( \frac{a}{x} \right)^b \right] \right\}^{\alpha-1} |_{x \to \infty}
\]

and

\[
h_{\alpha,\theta,a,b}(x) \sim \frac{\alpha ba^bx^{-(b+1)} \exp \left[ -\left( \frac{a}{x} \right)^b \right]}{1 - \exp \left[ -(\frac{a}{x})^b \right]} |_{x \to \infty}.
\]

Table 1 provides the some sub-models of the GOLLFr model. As illustrated in Table 1, the new model generalizes eleven sub-model, five of them are quite new.

Table 1: Sub-models of the GOLLFr model.

| N | \( \alpha \) | \( \theta \) | \( a \) | \( b \) | Reduced model | Reduced CDF | Author |
|---|---|---|---|---|---|---|---|
| 1 | 2 | GOLLIR | \( \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] \) | New |
| 2 | 1 | GOLLIEx | \( \exp \left[ -\alpha \theta \left( \frac{a}{x} \right)^b \right] \) | New |
| 3 | 1 | PRHFr | \( \exp \left[ -\theta \left( \frac{a}{x} \right)^b \right] \) | Gusmao et al. (2011) |
| 4 | 1 | 2 | PRHIR | \( \exp \left[ -\theta \left( \frac{a}{x} \right)^2 \right] \) | Gusmao et al. (2011) |
| 5 | 1 | 1 | PRHIEx | \( \exp \left[ -\theta \left( \frac{a}{x} \right)^b \right] \) | Gusmao et al. (2011) |
| 6 | 1 | OLLFr | \( \exp \left[ -\alpha \left( \frac{a}{x} \right)^b \right] \) | Yousof et al. (2018) |
| 7 | 1 | 2 | OLLIR | \( \exp \left[ -\alpha \left( \frac{a}{x} \right)^b \right] \) | Yousof et al. (2018a) |
| 8 | 1 | 1 | OLLIEx | \( \exp \left[ -\alpha \left( \frac{a}{x} \right)^b \right] \) | Yousof et al. (2018a) |
| 9 | 1 | 1 | Fr | \( \exp \left[ -(\frac{a}{x})^b \right] \) | Fréchet (1927) |
| 10 | 1 | 2 | IR | \( \exp \left[ -(\frac{a}{x})^2 \right] \) | Trayer (1964) |
| 11 | | IEx | \( \exp \left[ -(\frac{a}{x})^b \right] \) | Keller and Kamath (1982) |
Some plots of the GOLLFr PDF and HRF are given in Figure 1 to illustrate some of its characteristics.

![Figure 1: Plots of the GOLLFr PDF and HRF.](image)

For simulation of this new model, we obtain the quantile function (QF) of $X$ (by inverting (5)), say $x_u = Q(u) = F^{-1}(u)$, as

$$x_u = a \left( - \ln \left\{ \left[ \frac{u}{1-u} \right]^{\frac{1}{\alpha}} \right\}^{\frac{1}{\beta}} \right)^{-\frac{1}{\theta}}.$$  

Equation (7) is used for simulating the new model (see Section 5).

2. Mathematical properties

2.1. Useful representations

Based on generalized binomial expansions and after some algebra, the PDF in (6) can be expressed as

$$f(x) = \sum_{k=0}^{\infty} b_k \pi_{(1+k)}(x; a, b),$$  

where

$$b_k = \frac{\alpha \theta}{1+k} \sum_{i,j=0}^{l} \sum_{k=0}^{l-k} (-1)^{i+k+l} \left( \begin{array}{c} -2 \\ i \\ j \end{array} \right) \left( \begin{array}{c} -\alpha \left( i+1 \right) \\ j \end{array} \right) \left( \begin{array}{c} \alpha \theta \left( i+1 \right) + \theta j - 1 \\ l \end{array} \right) \left( \begin{array}{c} 1 \\ k \end{array} \right),$$

and $\pi_{(1+k)}(x; a, b)$ is the PDF of the Fr model with scale parameter $a \left[ (1 + k) \right]^\frac{1}{\alpha}$ and shape parameter $b$. So, the new density (6) can be expressed as a double linear mixture of the Fr density. Then, several structural properties of the new model can be obtained from Equation (8) and those properties of the Fr model. By integrating Equation (8), the CDF of $X$ becomes

$$F(x) = \sum_{k=0}^{\infty} b_k \Pi_{(1+k)}(x; a, b),$$

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where $\Pi(1+k)(x; a, b)$ is the CDF of the Fr distribution with scale parameter $a$ ($ck)^{\frac{1}{b}}$ and shape parameter $b$. The $r^{th}$ ordinary moment of $X$ is given by

$$
\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) \, dx,
$$

then we obtain

$$
\mu'_r = \sum_{k=0}^{\infty} b_k a^r (1 + k)^{\frac{1}{b}} \Gamma \left(1 - \frac{r}{b} \right), \forall \ b > r, \tag{10}
$$

where

$$
\Gamma(1 + \omega) \mid_{\omega \in \mathbb{R}^+} = \omega! = \prod_{h=0}^{\omega-1} (\omega - h).
$$

Setting $r = 1, 2, 3$ and $4$ in (10), we have

$$
E(X) = \mu'_1 = \sum_{k=0}^{\infty} b_k a (1 + k)^{\frac{1}{b}} \Gamma \left(1 - \frac{1}{b} \right), \forall \ b > 1,
$$

$$
E(X^2) = \mu'_2 = \sum_{k=0}^{\infty} b_k a^2 (1 + k)^{\frac{2}{b}} \Gamma \left(1 - \frac{2}{b} \right), \forall \ b > 2,
$$

$$
E(X^3) = \mu'_3 = \sum_{k=0}^{\infty} b_k a^3 (1 + k)^{\frac{3}{b}} \Gamma \left(1 - \frac{3}{b} \right), \forall \ b > 3,
$$

and

$$
E(X^4) = \mu'_4 = \sum_{k=0}^{\infty} b_k a^4 (1 + k)^{\frac{4}{b}} \Gamma \left(1 - \frac{4}{b} \right), \forall \ b > 4,
$$

where $E(X) = \mu'_1$ is the mean of $X$. The skewness (Skew$(X)$) and kurtosis (Kur$(X)$) measures can be calculated via (10) using well-known relationships. $E(X)$, variance (Var$(X)$), Skew$(X)$ and Kur$(X)$ of the GOLLFr distribution are computed numerically for some selected values of parameter $\alpha, \theta, a$ and $b$ using the R software. We conclude that, the Skew$(X)$ can range in the interval (1, 19706) and always positive, whereas the Kur$(X)$ varies in the interval (1, 388338). The parameters $\theta$ and $a$ don’t control neither the Skew$(X)$ nor the Kur$(X)$ as illustrated below in Table 2, on the other hand parameters $\alpha$ and $b$ control Skew$(X)$) and Kur$(X)$. $E(X)$ decreases as $\alpha$ and $b$ increases. $E(X)$ increases as $a$ and $\theta$ increases.
Table 2: Mean, variance, skewness and kurtosis of the GOLLFr distribution.

| α  | θ  | a  | b  | \(E(X)\) | Var(X) | Skew(X) | Kur(X) |
|----|----|----|----|---------|--------|---------|--------|
| 10 | 10 | 1  | 1  | 14.59026 | 3.815803 | 0.8830787 | 6.295525 |
| 2  | 3.811433 | 0.0632438 | 0.3176427 | 4.419477 |
| 5  | 1.706911 | 0.002005427 | 0.231448 | 4.309573 |
| 10 | 1.306376 | 0.0002927583 | 0.2534783 | 4.321513 |
| 15 | 1.195007 | 0.0001087731 | 0.2321448 | 4.309573 |
| 20 | 1.142944 | 5.594415 \times 10^{-05} | 0.2217203 | 4.281838 |
| 25 | 1.112803 | 3.393171 \times 10^{-05} | 0.215345 | 4.281838 |
| 30 | 1.142944 | 2.577334 \times 10^{-06} | 0.2217203 | 4.281838 |
| 4  | 2.5 | 1  | 1.5 | 2.405262 | 0.1952397 | 1.419549 | 9.958146 |
| 5  | 4.810524 | 0.7809586 | 1.419549 | 9.958146 |
| 10 | 5.905664 | 4.880991 | 1.419549 | 9.958146 |
| 20 | 7.792571 | 0.6454427 | 1.419549 | 9.958146 |
| 50 | 11.24234 | 2.339019 | 1.419549 | 9.958146 |
| 100 | 14.83436 | 4.072468 | 1.419549 | 9.958146 |
| 200 | 19.57405 | 1.155017 | 1.419549 | 9.958146 |
| 500 | 28.23948 | 1.155017 | 1.419549 | 9.958146 |

The \(r^{th}\) incomplete moment, say \(\varphi_r(t)\), of \(X\) can be expressed, from (9), as

\[
\varphi_r(t) = \int_{-\infty}^{t} x^r f(x) dx = \sum_{k=0}^{\infty} b_k \int_{-\infty}^{0} x^r \pi_{(1+k)}(x; a, b) dx
\]

\[
= \sum_{k=0}^{\infty} b_k a^r [(1+k)^{\frac{r}{b}} \gamma(1 - \frac{r}{b}, [(1+k)(\frac{a}{t})^b])], \forall b > r, \tag{11}
\]

where \(\gamma(\omega, q)\) is the incomplete gamma function.

\[
\gamma(\omega, q)|_{(\omega\neq0, -1, -2,...)} = \int_{0}^{q} t^{\omega-1} \exp(-t) dt
\]

\[
= \frac{q^{\omega}}{\omega} {\text{1F1}}[\omega; \omega + 1; -q]
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (\omega + k)} q^{\omega+k},
\]

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and $\text{$_1F_1$}[\cdot, \cdot, \cdot]$ is a confluent hypergeometric function. The first incomplete moment given by (11) with $r = 1$ as

$$\varphi_1(t) = \sum_{k=0}^{\infty} w_k a [(1 + k)]^{1 \over b} \Gamma \left( 1 - \frac{1}{b}, [(1 + k)] \left( \frac{a}{t} \right)^b \right), \forall b > 1.$$

### 2.2. Moment generating function (MGF)

The MGF $M_X(t) = E(e^{tX})$ of $X$ can be derived from equation (8) as

$$M_X(t) = \sum_{k=0}^{\infty} b_k M_{(1+k)}(t; a, b),$$

where $M_{(1+k)}(t; a, b)$ is the MGF of the Fr model with scale parameter $a [(1 + k)]^{1 \over b}$ and shape parameter $b$, then

$$M_X(t) = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (t^r b_k / r!) a^r [(1 + k)]^{1 \over b} \Gamma \left( 1 - \frac{r}{b}, \left( \frac{a}{t} \right)^b \right), \forall b > r.$$

We aslo can determine the generating function of $g_{a,b}(x)$ by setting $y = x^{-1}$, the MGF can be written as

$$M(t; a, b) = ba^b \int_0^{\infty} \exp \{ t/y \} y^{b-1} \exp \left\{ -(ay)^b \right\}.$$

By expanding the first exponential and calculating the integral, we have

$$M(t; a, b) = ba^b \int_0^{\infty} \sum_{m=0}^{\infty} \frac{(t^m/m!)}{m!} \exp \{ t/y \} y^{b-m-1} \exp \left\{ -(ay)^b \right\}.$$

Consider the Wright generalized hypergeometric function (Wright (1935)) defined by

$$p\Psi q \left[ \begin{array}{c} a_1, A_1, \ldots, a_p, A_p \\ b_1, B_1, \ldots, b_q, B_q \\ x \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j + A_j n)}{\prod_{j=1}^{q} (b_j + B_j n)} \frac{x^n}{n!}.$$

Then, $M(t; a, \beta)$ can be written as

$$M(t; a, b) = \Psi \left[ \begin{array}{c} (1, -{1 \over b}) \\ a t \end{array} \right].$$

Combining expressions (10) and (12), we obtain the MGF of $X$, say $M(t)$, as

$$M(t) = \sum_{k=0}^{\infty} b_k \left\{ \Psi \left[ \begin{array}{c} (1, -{1 \over b}) \\ a [c(k + 1)]^{1 \over b} t \end{array} \right] \right\}.$$

### 2.3. Residual life and reversed residual life functions

The $n^{th}$ moment of the residual life

$$m_n(t) = E[(X - t)^n | X > t, n = 1, 2, \ldots]$$

the $n^{th}$ moment of the residual life of $X$ is given by

$$m_n(t) = \int_0^{\infty} (x - t)^n dF(x) \frac{1}{1 - F(t)}.$$
Therefore,
\[
m_n(t) = \frac{a^n}{1 - F(t)} \sum_{k=0}^{\infty} \zeta_k \left[ (1 + k) \right]^{\frac{a}{b}} \Gamma \left( 1 - \frac{n}{b}, \left[ (1 + k) \right] \left( \frac{a}{t} \right)^b \right), \forall b > n,
\]
where
\[
\zeta_k = b_k \sum_{r=0}^{n} \binom{n}{r} (-t)^r,
\]
and
\[
\Gamma \left( \omega, q \right) \mid x \geq 0 = \int_{q}^{\infty} t^{\omega-1} \exp(-t) \, dt,
\]
and
\[
\Gamma \left( \omega, q \right) + \gamma \left( \omega, q \right) = \Gamma \left( \omega \right).
\]
The \(n^{th}\) moment of the reversed residual life, say
\[
M_n(t) = E \left[ (t - X)^n \mid X \leq t, t > 0 \right. \text{ and } n = 1, 2, \ldots
\]
uniquely determines \(F(x)\). We obtain
\[
M_n(t) = \frac{\int_{0}^{t} (t - x)^n dF(x)}{F(t)}.
\]
Then, the \(n^{th}\) moment of the reversed residual life of \(X\) becomes
\[
M_n(t) = \frac{a^n}{F(t)} \sum_{k=0}^{\infty} \eta_k \left[ (1 + k) \right]^{\frac{a}{b}} \gamma \left( 1 - \frac{n}{b}, \left[ (1 + k) \right] \left( \frac{a}{t} \right)^b \right), \forall b > n,
\]
where
\[
\eta_k = b_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.
\]
3. Maximum likelihood estimation (MLE)
Let \(x_1, \ldots, x_n\) be a random sample from the GOLLFr distribution with parameters \(\alpha, \theta, a\) and \(b\). Let \(\Theta = (\alpha, \theta, a, b)^\top\) be the \(4 \times 1\) parameter vector. For determining the MLE of \(\Theta\), we have the log-likelihood function
\[
\ell = \ell(\Theta) = n \log \left( \alpha \theta c \left( \frac{a}{x_i} \right)^b \right) - (b + 1) \sum_{i=1}^{n} \log \left( x_i \right) - \alpha \theta \sum_{i=1}^{n} \left( \frac{a}{x_i} \right)^b
\]
\[
+ 2 \sum_{i=1}^{n} \log \left( \exp \left[ -\alpha \theta \left( \frac{a}{x_i} \right)^b \right] + \left\{ 1 - \exp \left[ -\theta \left( \frac{a}{x_i} \right)^b \right] \right\} \right)
\]
\[
+ (\alpha - 1) \sum_{i=1}^{n} \log \left\{ 1 - \exp \left[ -\theta \left( \frac{a}{x_i} \right)^b \right] \right\}.
\]
The components of the score vector, \(\mathbf{L}(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left( \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b} \right)^\top\), are available if needed. Setting \(\mathbf{L}_\alpha = \mathbf{L}_\theta = \mathbf{L}_a = 0\) and solving them simultaneously yields the MLE \(\hat{\Theta} = (\hat{\alpha}, \hat{\theta}, \hat{a}, \hat{b})^\top\). To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize \(\ell\). For interval estimation of the parameters, we obtain the \(5 \times 5\) observed information matrix
\[
\mathbf{J}(\Theta) = \left\{ \frac{\partial^2 \ell}{\partial r \partial s} \right\} \text{ (\forall r, s = \alpha, \theta, a, b)},
\]
whose elements can be computed numerically.

4. Simulation studies
We simulate the GOLLFr model by taking \(n=50, 100, 200, 500\) and \(1000\) using (7). For each sample size, we evaluate the sample means and standard deviations (SDs) using the optim function of the R software. Then, we repeat this
process 1000 times. Values in Table 3 indicate that the empirical means approach to the true parameter values when the sample size $n$ increases. The SDs decrease when the sample size $n$ increases as expected. These results are in agreement with first-order asymptotic theory.

| Parameters | $n = 50$ | $n = 100$ |
|------------|----------|-----------|
| $5.5, 1.0, 0.1, 1.1$ | 5.77822, 0.04799, 1.08389 | 5.69167, 0.04799, 1.08389 |
| $1.0, 1.0, 0.1, 1.0$ | 1.17971, 0.04028, 1.09449 | 1.05001, 0.03571, 0.50965 |
| $2.0, 5.0, 1.0, 1.0$ | 1.75894, 0.94989, 0.97457 | 1.17971, 1.01439, 0.50965 |
| $5.0, 3.0, 1.0, 1.5$ | 4.87732, 1.02135, 1.38780 | 5.82595, 5.21645, 3.78133 |
| $1.5, 2.5, 1.5, 1.5$ | 1.34786, 1.76762, 1.55591 | 3.01830, 2.18634, 1.84569 |
| $5.0, 3.0, 5.0, 4.0$ | 4.82595, 5.21645, 3.78133 | 3.01830, 2.18634, 1.84569 |
| $3.0, 2.0, 2.0, 4.0$ | 2.61830, 1.84569, 4.14765 | 3.01830, 2.18634, 1.84569 |
5. Real data modeling

We consider the Cramér-Von Mises and the Anderson-Darling \([W^*, A^*]\) and the Kolmogorov-Smirnov (KS) statistic. The \(W^*\) and \(A^*\) statistics are given by

\[
W^* = (1 + 1/2n) \left[1/ (12n) + \sum_{j=1}^{n} W_j\right],
\]

where \(n\) is the sample size.

\[
A^* = \sqrt{n} \left|\frac{\sum_{j=1}^{n} F_{obs}(X_j) - \frac{n+1}{2}}{\frac{n}{2}}\right|
\]

Here, \(F_{obs}(x)\) is the observed cumulative distribution function.

| Parameters | \(n = 200\) | \(n = 500\) | \(n = 1000\) |
|------------|-------------|-------------|-------------|
| 5.5, 1.0, 0.1, 1.1 | 5.55859 | 5.50283 | 5.50001 |
| (0.01201) | (0.00339) | (0.1515) |
| 1.0, 1.0, 0.1, 1.0 | 1.0123 | 1.00201 | 1.00011 |
| (0.0548) | (0.00231) | (0.2071) |
| 2.0, 5.0, 1.0, 1.0 | 2.0006 | 2.00399 | 1.99996 |
| (0.023955) | (0.00960) | (0.0667) |
| 5.0, 2.0, 1.0, 1.5 | 4.9942 | 4.99121 | 4.99011 |
| (0.19074) | (0.03298) | (0.00851) |
| 1.5, 2.5, 1.5, 1.5 | 1.49944 | 1.49944 | 1.49901 |
| (0.3276) | (0.00091) | (0.00851) |
| 5.0, 3.0, 5.0, 4.0 | 4.99121 | 4.9929 | 4.99088 |
| (0.03990) | (0.2015) | (0.00088) |
| 3.0, 2.0, 2.0, 4.0 | 2.87466 | 2.90367 | 3.00441 |
| (0.05271) | (0.3165) | (0.03274) |

The Generalized Odd Log-Logistic Fréchet Distribution for Modeling Extreme Values
and
\[ A^* = nA_{(n)} + A_{(n)}n^{-1} \sum_{j=1}^{n} a_j, \]
where
\[ W_j = \left[ z_i - \frac{(2j - 1)}{(2n)} \right]^2, \]
\[ A_{(n)} = 1 + \frac{9}{4}n^{-2} + \frac{3}{4}n^{-1}, \]
and
\[ A_j = (2j - 1) \log \left[ z_i (1 - z_{n-j+1}) \right], \]
where \( z_i = F(y_j) \) and the \( y_j \)'s values are the ordered observations. We compare the fits of the GOLLFr distribution with other models such as Fréchet (Fr), Kumaraswamy Fréchet (KFr), exponentiated Fréchet (EFr), beta Fréchet (BFr), transmuted Fréchet (TFr), Marshal-Olkin Fréchet (MOFr) and McDonald Fréchet (McFr) distributions given by:

**EFr**:
\[ f_{EFr}(x) = \alpha b^a x^{-(b+1)} \exp \left\{ -(a/x)^b \right\} \left\{ 1 - \exp \left\{ -(a/x)^b \right\} \right\} \alpha^{-1}; \]

**BFr**:
\[ f_{BFr}(x) = b B^{-1}(\alpha, c)x^{-(b+1)} \exp \left\{ -\alpha(a/x)^b \right\} \left\{ 1 - \exp \left\{ -\alpha(a/x)^b \right\} \right\} \alpha^{-1}; \]

**KFr**:
\[ f_{KFr}(x) = \alpha c b^a x^{-(b+1)} \exp \left\{ -\alpha(a/x)^b \right\} \left\{ 1 - \exp \left\{ -\alpha(a/x)^b \right\} \right\} \alpha^{-1}; \]

**TFr**:
\[ f_{TFr}(x) = b^a x^{-(b+1)} \exp \left\{ -(a/x)^b \right\} \left\{ 1 + \alpha - 2\alpha \exp \left\{ -(a/x)^b \right\} \right\}; \]

**MOFr**:
\[ f_{MOFr}(x) = \alpha b^a x^{-(b+1)} \exp \left\{ -(a/x)^b \right\} \left\{ \alpha + (1 - \alpha) \exp \left\{ -(a/x)^b \right\} \right\} \alpha^{-2}; \]

**McFr**:
\[ f_{McFr}(x) = \lambda b^a x^{-(b+1)} B^{-1}(\alpha, c) \exp \left\{ -(a/x)^b \right\} \left\{ \exp \left\{ -(a/x)^b \right\} \right\} \alpha^{-1} \times \left\{ 1 - \left( \exp \left\{ -(a/x)^b \right\} \right)^\alpha \right\} \alpha^{-1}; \]

**OLLEFr**:
\[ f_{OLLLEFr}(x) = \beta b^a x^{-(b+1)} \exp \left\{ -\beta \left( \frac{a}{x} \right)^b \right\} \times \left\{ \exp \left\{ -\beta \left( \frac{a}{x} \right)^b \right\} \left\{ 1 - \exp \left\{ -\beta \left( \frac{a}{x} \right)^b \right\} \right\} \right\} \alpha^{-1} \times \left\{ 1 - \left( \exp \left\{ -\beta \left( \frac{a}{x} \right)^b \right\} \right)^\alpha \right\} \alpha^{-1}. \]

The parameters of the above densities are all positive real numbers except for the TFr distribution for which \( |\alpha| \leq 1. \) For more different symmetric and asymmetric real-life data sets see Brito et al. (2017), Merovci et al. (2017, 2020), Nascimento et al. (2019), Hamedani et al. (2017, 2018, 2019, 2021), Shehata and Yousif (2021a,b), Shehata et al. (2021), Chesneau and Yousif (2021), Chesneau et al. (2022), Elgohari and Yousif (2020a,b, 2021), Elgohari et al. (2021), Karamikabir et al. (2020), Korkmaz et al. (2018a,b, 2020) and Hamedani et al. (2022).

### 5.1. Breaking stress data

The 1st data set is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006) and these data are used by Mahmoud and Mandouh (2013) to fit the transmuted Fréchet distribution. The data are: 0.920, 0.9280, 0.997, 0.99710, 1.061, 1.1170, 1.162, 1.1830, 1.187, 1.1920, 1.196,
1.2130, 1.215, 1.21990, 1.22, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.2590, 1.261, 1.2630, 1.276, 1.310, 1.321, 1.3290, 1.331, 1.3370, 1.351, 1.3590, 1.388, 1.4080, 1.449, 1.44970, 1.45, 1.4590, 1.471, 1.4750, 1.477, 1.480, 1.489, 1.5010, 1.507, 1.5150, 1.53, 1.53040, 1.533, 1.5440, 1.5443, 1.5520, 1.556, 1.562, 1.5660, 1.585, 1.5860, 1.5990, 1.602, 1.6140, 1.6160, 1.6170, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.759, 1.7770, 1.794, 1.7990, 1.806, 1.8140, 1.8160, 1.828, 1.830, 1.884, 1.8920, 1.944, 1.9720, 1.9840, 1.9870, 2.02, 2.03040, 2.03500, 2.037, 2.0430, 2.0460, 2.059, 2.1110, 2.165, 2.6860, 2.778, 2.9720, 3.504, 3.8630 and 5.3060. Figure 2 gives the total time test (TTT) plot for data set $I$. It indicates that the empirical HRFs of data sets $I$ is increasing.

The statistics ($W^\star$, $A^\star$, K-S and p-value) of all fitted models are presented in Table 4. The MLEs and corresponding standard errors are given in Table 5. The GOLLFr distribution in Table 4 gives the lowest values the $W^\star$, $A^\star$, K-S and the biggest value of the p-value statistics as compared to other extensions of the Fr models, and therefore the new one can be chosen as the best model. Figure 3 gives the estimated density, estimated CDF, P-P plot and estimated HRF for data set $I$. 

Figure 2: TTT plot for data set $I$
Table 4: $W^\star$, $A^\star$, K-S and p-value for data set I.

| Model | Goodness of fit criteria |          |          |        |
|-------|--------------------------|----------|----------|--------|
|       |                          | $W^\star$| $A^\star$| K-S    | p-value |
| GOLLFr |                         | 0.0624   | 0.4837   | 0.0638 | 0.8101  |
| OLLEFr |                         | 0.1203   | 0.9639   | 0.5561 | $2.2 \times 10^{-16}$ |
| OLLEIR |                         | 0.1553   | 1.21197  | 0.65497| $2.2 \times 10^{-16}$ |
| OLLIR  |                         | 0.15532  | 1.21201  | 0.6550 | $2.2 \times 10^{-16}$ |
| Fr     |                         | 0.1090   | 0.7657   | 0.0874 | 0.4282  |
| KFr    |                         | 0.0812   | 0.6217   | 0.0759 | 0.6118  |
| EFr    |                         | 0.1091   | 0.7658   | 0.0874 | 0.4287  |
| BFr    |                         | 0.0809   | 0.6207   | 0.0757 | 0.6147  |
| TFr    |                         | 0.0871   | 0.6209   | 0.0782 | 0.5734  |
| MOFr   |                         | 0.0886   | 0.6142   | 0.0763 | 0.5168  |
| McFr   |                         | 0.1333   | 1.0608   | 0.0807 | 0.5332  |
### Table 5: MLEs and their standard errors (in parentheses) for data set I.

| Model | Estimates |         |         |         |         |
|-------|-----------|---------|---------|---------|---------|
|       | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{c}$ | $\hat{a}$ | $\hat{b}$ |
| GOLLFr| 1.6989    | 2.5203  | 0.9779  | 2.8584  |         |
|       | (0.655)   | (27.6)  | (3.75)  | (1.0037)|         |
| OLLEFr| 0.1351    | 3.7216  | 0.9296  | 21.319  |         |
|       | (0.011)   | (0.0034)| (0.0033)| (0.0034)|         |
| OLLEIR| 0.4946    | 0.067   | 1.74262 | 2       |         |
|       | (0.04135) | (0.7195)| (9.3007)|         |         |
| OLLIR | 0.49459   | 0.45242 | 2       |         |         |
|       | 0.04135   | 0.03869 |         |         |         |
| Fr    |           |         | 1.3968  | 4.3724  |         |
|       |           |         | (0.0336)| (0.3278)|         |
| KFr   | 0.8489    | 1.6239  | 1.6341  | 3.4208  |         |
|       | (16.083)  | (0.6979)| (9.049) | (0.7635)|         |
| EFr   | 0.9395    | 1.4169  | 0.9395  |         |         |
|       | (3.543)   | (2.568) | (0.3278)|         |         |
| BFr   | 0.7346    | 1.5830  | 1.6684  | 3.5112  |         |
|       | (1.5290)  | (0.7132)| (0.7662)| (0.9683)|         |
| TFr   | −0.7166   | 1.2656  | 4.7121  |         |         |
|       | (0.2616)  | (0.0579)| (0.3657)|         |         |
| MOFr  | 0.0033    | 6.2296  | 1.2419  |         |         |
|       | (0.0009)  | (1.0134)| (0.1181)|         |         |
| McFr  | 0.8503    | 44.423  | 19.859  | 0.0203  | 46.974  |
|       | (0.1353)  | (25.100)| (6.706) | (0.0060)| (21.871)|
5.2. Glass fibers data

The 2nd data set is generated data to simulate the strengths of glass fibers which was given by Smith and Naylor (1987). The data set is: 1.0140, 1.0810, 1.082, 1.1850, 1.2230, 1.2480, 1.2670, 1.2710, 1.2720, 1.2750, 1.2760, 1.278, 1.2860, 1.288, 1.2920, 1.304, 1.3060, 1.355, 1.361, 1.3640, 1.379, 1.4090, 1.426, 1.4590, 1.460, 1.4760, 1.481, 1.4840, 1.501, 1.5060, 1.5240, 1.5260, 1.5350, 1.541, 1.5680, 1.579, 1.5810, 1.591, 1.5930, 1.602, 1.6660, 1.67, 1.684, 1.6910, 1.704, 1.7310, 1.735, 1.7470, 1.748, 1.7570, 1.8000, 1.806, 1.8670, 1.876, 1.878, 1.910, 1.9160, 1.9720, 2.0120, 2.456, 2.5920, 3.1970 and 4.1210. Figure 4 gives the TTT plot for data set II. It indicates that the empirical HRFs of data sets II is increasing.
The statistics ($W^\star$, $A^\star$, K-S and p-value) of the fitted models are provided in Table 6. The MLEs and corresponding standard errors are given in Table 7. From Table 6, the GOLLFr distribution gives the lowest values the $W^\star$, $A^\star$, K-S and the biggest value of the p-value statistics as compared to further Fr models, and therefore the new one can be chosen as the best model. Figure 5 gives the estimated density, estimated CDF, P-P plot and estimated HRF for data set II.
Table 6: \( W^*, A^*, K-S \) and p-value for data set II.

| Model     | Goodness of fit criteria |
|-----------|--------------------------|
|           | W^*         | A^*     | K-S       | p-value      |
| GOLLFr    | 0.05084     | 0.4117  | 0.06901   | 0.9040       |
| GOLLIR    | 0.05250     | 0.4530  | 0.070334  | 0.8926       |
| OLLEFr    | 0.10487     | 0.8325  | 0.55196   | \( 6.661 \times 10^{-16} \) |
| OLLEIR    | 0.1502      | 1.14697 | 0.67949   | \( 6.661 \times 10^{-16} \) |
| OLLIR     | 0.15021     | 1.14697 | 0.67951   | \( 6.661 \times 10^{-16} \) |
| Fr        | 0.0707      | 0.5332  | 0.0772    | 0.8185       |
| KFr       | 0.0634      | 0.4981  | 0.0715    | 0.8810       |
| EFr       | 0.0707      | 0.5332  | 0.0772    | 0.8187       |
| BFr       | 0.0640      | 0.5008  | 0.0716    | 0.8804       |
| TFr       | 0.0655      | 0.4939  | 0.0735    | 0.8470       |
| MOFr      | 0.0629      | 0.4902  | 0.0813    | 0.7685       |
| McFr      | 0.1161      | 0.9193  | 0.0831    | 0.7455       |
Table 7: MLEs and their standard errors (in parentheses) for data set II.

| Model   | Estimates            |          |          |          |          |
|---------|----------------------|----------|----------|----------|----------|
|         | \( \hat{\alpha} \)  | \( \hat{\theta} \) | \( \hat{\gamma} \) | \( \hat{\alpha} \) | \( b \)   |
| GOLLFr  | 1.472169             | 3.474396 | 1.015180 | 4.000147 | (0.63)   |
|         | (0.0129)             | (0.00879)| (0.129)  | (0.000147)|          |
| OLLEFr  | 0.1449               | 0.00879  | 1.2997   | 24.878   | (0.0129) |
|         | (0.000)              | (0.000)  | (0.000)  | (0.000)  |          |
| GOLLIR  | 3.032                | 2.1568   | 0.86599  | 2        | (0.325)  |
|         | (0.325)              | (9.145)  | (1.836)  |          |          |
| OLLEIR  | 0.5025               | 0.0716   | 1.7048   | 2        | (0.0529) |
|         | (0.0529)             | (1.13062)| (13.47)  |          |          |
| OLLIR   | 0.50251              | 0.45599  | 2        |          | (0.052946)|
|         | 0.048652             |          |          |          |          |
| Fr      | 1.4108               | 5.4377   | (0.0344) | (0.5192) |
| KFr     | 0.2855               | 1.2824   | 1.9142   | 4.7731   | (9.1338) |
|         | (0.6388)             | (12.836) | (13.134) |          |          |
| EFr     | 0.9059               | 1.4367   | 5.4379   |          | (2.764)  |
|         | (4.324)              | (0.5193) |          |          |          |
| BFr     | 1.2996               | 1.3945   | 4.7927   |          | (4.4378) |
|         | (0.6640)             | (0.9304) | (1.4641) |          |          |
| TFr     | 0.7778               | 1.5491   | 4.3139   |          | (0.2477) |
|         | (0.0655)             | (0.5849) |          |          |          |
| MOFr    | 0.0023               | 5.2383   | 1.4537   |          | (0.0004) |
|         | (0.8209)             | (0.1650) |          |          |          |
| McFr    | 56.227               | 14.953   | 0.0073   | 29.104   | (30.539) |
|         | (4.733)              | (0.0013) | (11.304) |          |          |

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5.3. Relief time data

The 3rd data set (wingo data) represents a complete sample from a clinical trial describe a relief time (in hours) for 50 arthritic patients. The data set is: 0.700, 0.84, 0.58, 0.500, 0.55, 0.82, 0.590, 0.71, 0.720, 0.610, 0.62, 0.49, 0.54, 0.36, 0.360, 0.71, 0.35, 0.64, 0.84, 0.550, 0.59, 0.29, 0.75, 0.460, 0.46, 0.60, 0.60, 0.360, 0.52, 0.68, 0.800, 0.55, 0.84, 0.340, 0.34, 0.700, 0.490, 0.56, 0.710, 0.61, 0.570, 0.73, 0.75, 0.440, 0.44, 0.81, 0.80, 0.870, 0.29 and 0.500. Figure 6 gives the TTT plot for data set III. It indicates that the empirical HRFs of data sets III is increasing.
Figure 6: TTT plot for data set **III**

Table 8: $W^\star$, $A^\star$, K-S and p-value for data set **III**.

| Model  | $W^\star$ | $A^\star$ | K-S     | p-value       |
|--------|-----------|-----------|---------|--------------|
| GOLLFr | 0.15615   | 1.1276    | 0.10476 | 0.6427       |
| GOLLIR | 0.19551   | 1.3498    | 0.11008 | 0.5797       |
| OLLEFr | 0.1577    | 1.09876   | 0.53498 | $7.436 \times e^{-13}$ |
| Fr     | 0.3233    | 2.0301    | 0.1506  | 0.2066       |
| EFr    | 0.3233    | 2.0301    | 0.1506  | 0.2064       |
| TFr    | 0.2823    | 1.8152    | 0.1370  | 0.3045       |
### Table 9: MLEs and their standard errors (in parentheses) for data set III.

| Model   | Estimates          |            |            |            |            |
|---------|--------------------|------------|------------|------------|------------|
|         | \( \hat{\alpha} \) | \( \hat{\theta} \) | \( \hat{c} \) | \( \hat{a} \) | \( \hat{b} \) |
| GOLLFr  | 2.899              | 0.09875    | 2.4072     | 1.34989    |             |
|         | (0.8263)           | (0.05666)  | (0.8460)   | (0.34177)  |             |
| GOLLIR  | 1.961              | 0.111      | 1.4123     | 2          |             |
|         | (0.234)            | (0.000)    | (0.000)    |             |             |
| OLLEFr  | 0.0669             | 0.00459    | 0.3558     | 32.561     |             |
|         | (0.0076)           | (0.0028)   | (0.0047)   | (0.006)    |             |
| Fr      |                    |            |            | 0.4859     | 3.2078     |
|         |                    |            |            | (0.0227)   | (0.3263)   |
| EFr     | 0.9047             | 0.5013     | 3.2077     |             |             |
|         | (18.784)           | (3.2444)   | (0.3263)   |             |             |
| TFr     | −0.5816            | 0.4400     | 3.4974     |             |             |
|         | (0.2787)           | (0.0290)   | (0.3527)   |             |             |
The statistics \((W^*, A^*, K-S\) and p-value\) of all fitted models are presented in Table 8. The MLEs and corresponding standard errors are given in Table 9. From Table 8, the GOLLFr distribution gives the lowest values the \(W^*, A^*,\) K-S and the biggest value of the p-value statistics as compared to further Fr models, and therefore the new one can be chosen as the best model. Figure 5 gives the estimated density, estimated CDF, P-P plot and estimated HRF for data set III.

6. Concluding remarks

We introduce a new distribution called GOLLFr distribution for modeling the extreme values. The proposed model provides generalization for eleven distributions at least, five of them are quite new. The sample mean and the standard deviations are evaluated using a maximum likelihood method via a simulation study. The MLEs for GOLLFr distribution provides satisfying results. Some important mathematical properties of the new model are derived. The skewness of the new model always positive, whereas the kurtosis can be more than (or less than) 3. The new model is better than some other important competitive versions of the Fréchet model in modeling the breaking stress data, the glass fibers...
data and the relief time data. In our upcoming work, we can apply many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test and Bagdonavičius-Nikulin goodness-of-fit test as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a-f), Yadav et al. (2020), Goual and Yousof (2020), Aidi et al. (2021) Yadav et al. (2020, 2022) and Ibrahim et al. (2021a, 2022), among others.

References

1. Aboraya, M., Ali, M. M., Yousof, H. M. and Ibrahim, M. (2022). A New Flexible Probability Model: Theory, Estimation and Modeling Bimodal Left Skewed Data. Pakistan Journal of Statistics and Operation Research, 18(2), 437-463.
2. Aboraya, M., M. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). A new family of discrete distributions with mathematical properties, characterizations, Bayesian and non-Bayesian estimation methods. Mathematics, 8, 1648.
3. Ahmed, B. and Yousof, H. M. (2022). A New Group Acceptance Sampling Plans based on Percentiles for the Weibull Fréchet Model. Statistics, Optimization & Information Computing, forthcoming.
4. Al-babtain, A. A., Elbatal, I., & Yousof, H. M. (2020a). A New Flexible Three-Parameter Model: Properties, Clayton Copula, and Modeling Real Data. Symmetry, 12(3), 440.
5. Al-Babtain, A. A., Elbatal, I., & Yousof, H. M. (2020b). A new three parameter Fréchet model with mathematical properties and applications. Journal of Taibah University for Science, 14(1), 265-278.
6. Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Nofal, Z. M. and Ahmad, M. (2016a). The Kumaraswamy Marshall-Olkin Fréchet distribution with applications. Journal of ISOSS, 2(2), 151-168.
7. Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Ortega, E. M. and Nofal, Z. M. (2016b). The Weibull Fréchet distribution and its applications. Journal of Applied Statistics, 43(14), 2608-2626.
8. Altun, E., Alizadeh, M., Yousof, H. M. and Hamedani, G. G. (2022). The Gudermannian generated family of distributions with characterizations, regression models and applications, Studia Scientiarum Mathematicarum Hungarica, 59 (2), 93-115.
9. Barreto-Souza, W. M., Cordeiro, G. M. and Simas, A. B. (2011). Some results for beta Fréchet distribution. Commun. Statist. Theory-Meth., 40, 798-811.
10. Brito, E., Cordeiro, G. M., Yousof, H. M., Alizadeh, M. and Silva, G. O. (2017). Topp-Leone Odd Log-Logistic Family of Distributions, Journal of Statistical Computation and Simulation, 87(15), 3040–3058.
11. Chesneau, C. and Yousof, H. M. (2021). On a special generalized mixture class of probabilistic models. Journal of Nonlinear Modeling and Analysis, 3(1), 71-92.
12. Chesneau, C., Yousof, H. M., Hamedani, G. and Ibrahim, M. (2022). A New One-parameter Discrete Distribution: The Discrete Inverse Burr Distribution: Characterizations, Properties, Applications, Bayesian and Non-Bayesian Estimations. Statistics, Optimization & Information Computing, forthcoming.
13. Cordeiro, G. M., Alizadeh, M.M., Ozel, G., Hosseini, B., Ortega, E. M. M. and Altun, E. (2016). The generalized odd log-logistic family of distributions: properties, regression models and applications, Journal of Statistical Computation and Simulation, 87(5), 908-932.
14. Elgohari, H. and Yousof, H. M. (2020a). A Generalization of Lomax Distribution with Properties, Copula and Real Data Applications. Pakistan Journal of Statistics and Operation Research, 16(4), 697-711.
15. Elgohari, H. and Yousof, H. M. (2021). A New Extreme Value Model with Different Copula, Statistical Properties and Applications. Pakistan Journal of Statistics and Operation Research, 17(4), 1015-1035.
16. Elgohari, H. and Yousof, H. M. (2020b). New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling. Statistics, Optimization & Information Computing, 8(4), 972-993.
17. Elgohari, H., Ibrahim, M. and Yousof, H. M. (2021). A New Probability Distribution for Modeling Failure and Service Times: Properties, Copulas and Various Estimation Methods. Statistics, Optimization & Information Computing, 8(3), 555-586.
18. Elsayed, H. A. H. and Yousof, H. M. (2020). The generalized odd generalized exponential Fréchet model: univariate, bivariate and multivariate extensions with properties and applications to the univariate version. Pakistan Journal of Statistics and Operation Research, 529-544.
19. Fréchet, M. (1927). Sur la loi de probabilité de l’écart maximum. Ann. de la Soc. polonaises Math, 6, 93–116.
20. Gusmao, F. R. S., Ortega, E. M. M. and Cordeiro, G. M. (2011). The generalized inverse Weibull distribution. Statistical Papers 52, 591-619.
21. Gleaton, J. U. and Lynch, J.D. (2006). Properties of generalized loglogistic families of lifetime distributions. Journal of Probability and Statistical Science, 4, 51-64.
22. Goual, H., Yousof, H. M. and Ali, M. M. (2019). Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. Pakistan Journal of Statistics and Operation Research, 15(3), 745-771.
23. Goual, H. and Yousof, H. M. (2020). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. Journal of Applied Statistics, 47(3), 393-423.
24. Goual, H., Yousof, H. M. and Ali, M. M. (2020). Lomax inverse Weibull model: properties, applications, and a modified Chi-squared goodness-of-fit test for validation. Journal of Nonlinear Sciences & Applications, 13(6), 330-353.
25. Gupta, R. C. and Gupta, R. D. (2007). Proportional reversed hazard rate model and its applications. J Statist Plan Inference.137, 3525–3536.
26. Haq, M. A. ul, Yousof, H. M., & Hashmi, S. (2017). A New Five-Parameter Fréchet Model for Extreme Values. Pakistan Journal of Statistics and Operation Research, 13(3), 617-632.
27. Hamedani, G. G., Altun, E, Korkmaz, M. C., Yousof, H. M. and Butt, N. S. (2018). A new extended G family of continuous distributions with mathematical properties, characterizations and regression modeling. Pak. J. Stat. Oper. Res., 14(3), 737-758.
28. Hamedani, G. G., Korkmaz, M. C., Butt, N. S. and Yousof, H. M. (2021). The Type I Quasi Lambert Family: Properties, Characterizations and Different Estimation Methods. Pakistan Journal of Statistics and Operation Research, 17(3), 545-558.
29. Hamedani, G. G., Korkmaz, M. Ç., Butt, N. S. and Yousof H. M. (2022). The Type II Quasi Lambert G Family of Probability Distributions. Pakistan Journal of Statistics and Operation Research, forthcoming.
30. Hamedani, G. G. Rasekhi, M., Najib, S. M., Yousof, H. M. and Alizadeh, M., (2019). Type II general exponential class of distributions. Pak. J. Stat. Oper. Res., XV (2), 503-523.
31. Hamedani, G. G. Yousof, H. M., Rasekhi, M., Alizadeh, M., Najibi, S. M. (2017). Type I general exponential class of distributions. Pak. J. Stat. Oper. Res., XIV (1), 39-55.
32. Ibrahim, M., Ali, M. M. and Yousof, H. M. (2021a). The discrete analogue of the Weibull G family: properties, different applications, Bayesian and non-Bayesian estimation methods. Annals of Data Science, forthcoming.
33. Ibrahim, M., Hamedani, G. G., Butt, N. S. and Yousof, H. M. (2022). Expanding the Nadarajah Haghighi Model: Characterizations, Properties and Application. Pakistan Journal of Statistics and Operation Research, forthcoming.
34. Ibrahim, M., Handique, L., Chakraborty, S., Butt, N. S. and M. Yousof, H. (2021b). A new three-parameter xgamma Fréchet distribution with different methods of estimation and applications. Pakistan Journal of Statistics and Operation Research, 17(1), 291-308.
35. Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H. and Hamedani, G. G. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. Communications for Statistical Applications and Methods, 26(5), 473-495.
36. Jahanshahi, S.M.A., Yousof, H. M. and Sharma, V.K. (2019). The Burr X Fréchet Model for Extreme Values: Mathematical Properties, Classical Inference and Bayesian Analysis. Pak. J. Stat. Oper. Res., 15(3), 797-818.
37. Keller, A. Z. and Kamath, A. R. (1982). Reliability analysis of CNC machine tools. Reliab Eng 3, 449–473.
38. Karamikabir, H., Afshari, M., Yousof, H. M., Alizadeh, M. and Hamedani, G. (2020). The Weibull Topp-Leone Generated Family of Distributions: Statistical Properties and Applications. Journal of The Iranian Statistical Society, 19(1), 121-161.
39. Korkmaz, M. Ç., Altun, E., Chesneau, C. and Yousof, H. M. (2022). On the unit-Chen distribution with associated quantile regression and applications. Mathematica Slovaca, 72 (2022), No. 3, 765-786.
40. Korkmaz, M. Ç., Altun, E., Yousof, H. M. and Hamedani, G. G. (2020). The Hjorth’s IDB Generator of Distributions: Properties, Characterizations, Regression Modeling and Applications. Journal of Statistical Theory and Applications, 19(1), 59-74.
41. Korkmaz, M. Ç., Yousof, H. M. and Ali, M. M. (2017). Some theoretical and computational aspects of the odd Lindley Fréchet distribution. İstatistikçiler Dergisi: İstatistik ve Aktü Erya, 10(2), 129-140.
42. Korkmaz, M. Ç. Yousof, H. M. and Hamedani G. G. (2018a). The exponential Lindley odd log-logistic G family: properties, characterizations and applications. Journal of Statistical Theory and Applications,17(3),
43. Korkmaz, M. C., Yousof, H. M., Hamedani G. G. and Ali, M. M. (2018b). The Marshall–Olkin generalized G Poisson family of distributions, Pakistan Journal of Statistics, 34(3), 251-267.

44. Kotz, S. and Nadarajah, S. (2000). Extreme value distributions: theory and applications. Imperial College Press, London.

45. Krishna, E., Jose, K. K., Alice, T. and Risti, M. M. (2013). The Marshall–Olkin Fréchet distribution. Communications in Statistics-Theory and Methods, 42, 4091-4107.

46. Mahmoud, M. R. and Mandouh, R. M. (2013). On the transmuted Fréchet distribution. Journal of Applied Sciences Research, 9, 5553-5561.

47. Mansour, M. M., Ibrahim, M., Aidi, K., Shafique Butt, N., Ali, M. M., Yousof, H. M. and Hamed, M. S. (2020a). A New Log-Logistic Lifetime Model with Mathematical Properties, Copula, Modified Goodness-of-Fit Test for Validation and Real Data Modeling. Mathematics, 8(9), 1508.

48. Mansour, M. M., Butt, N. S., Ansari, S. I., Yousof, H. M., Ali, M. M. and Ibrahim, M. (2020b). A new exponentiated Weibull distribution’s extension: copula, mathematical properties and applications. Contributions to Mathematics, 1 (2020) 57–66. DOI: 10.47443/cm.2020.0018

49. Mansour, M., Korkmaz, M. C., Ali, M. M., Yousof, H. M., Ansari, S. I., & Ibrahim, M. (2020c). A generalization of the exponentiated Weibull model with properties, Copula and application. Eurasian Bulletin of Mathematics, 3(2), 84-102.

50. Mansour, M., Rasekhi, M., Ibrahim, M., Aidi, K., Yousof, H. M. and Elrazik, E. A. (2020d). A New Parametric Life Distribution with Modified Bagdonavičius–Nikulin Goodness-of-Fit Test for Censored Validation, Properties, Applications, and Different Estimation Methods. Entropy, 22(5), 592

51. Mansour, M., Yousof, H. M., Shehata, W. A. and Ibrahim, M. (2020e). A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. Journal of Nonlinear Science and Applications, 13(5), 223-238.

52. Mansour, M. M., Butt, N. S., Yousof, H. M., Ansari, S. I. and Ibrahim, M. (2020f). A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling Skewed and Symmetric Real Data Sets. Pakistan Journal of Statistics and Operation Research, 16(2), 373-386.

53. Merovci, F., Yousof, H. M. and Hamedani, G. G. (2020). The Poisson Topp Leone Generator of Distributions for Lifetime Data: Theory, Characterizations and Applications. Pakistan Journal of Statistics and Operation Research, 16(2), 343-355.

54. Merovci, F., Alizadeh, M., Yousof, H. M. and Hamedani G. G. (2017). The exponentiated transmuted-G family of distributions: theory and applications, Communications in Statistics-Theory and Methods, 46(21), 10800-10822.

55. Nascimento, A. D. C., Silva, K. F., Cordeiro, G. M., Alizadeh, M. and Yousof, H. M. (2019). The odd Nadarajah-Haghighi family of distributions: properties and applications. Studia Scientiarum Mathematicarum Hungarica, 56(2), 1-26.

56. Nadarajah, S. and Kotz, S. (2003). The exponentiated exponential distribution, Statistica, Available online at http://interstat.statjournals.net/YEAR/2003/abstracts/03 12001.php.

57. Nadarajah, S. and Kotz, S. (2008). Sociological models based on Fréchet random variables. Quality and Quantity, 42, 89-95.

58. Nichols, M. D, Padgett, W. J. (2006). A Bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International, 22, 141-151.

59. Salah, M. M., El-Morshedy, M., Eliwa, M. S. and Yousof, H. M. (2020). Expanded Fréchet Model: Mathematical Properties, Copula, Different Estimation Methods, Applications and Validation Testing. Mathematics, 8(11), 1949.

60. Shehata, W. A. M. and Yousof, H. M. (2021a). The four-parameter exponentiated Weibull model with Copula, properties and real data modeling. Pakistan Journal of Statistics and Operation Research, 17(3), 649-667.

61. Shehata, W. A. M. and Yousof, H. M. (2021b). A novel two-parameter Nadarajah-Haghighi extension: properties, copulas, modeling real data and different estimation methods. Statistics, Optimization & Information Computing, forthcoming.

62. Shehata, W. A. M., Yousof, H. M. and Aboraya, M. (2021). A Novel Generator of Continuous Probability Distributions for the Asymmetric Left-skewed Bimodal Real-life Data with Properties and Copulas. Pakistan Journal of Statistics and Operation Research, 17(4), 943-961.

63. Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and bayesian estimators for the
three-parameter Weibull distribution. Applied Statistics, 36, 358-369.

64. Wright, E. M. (1935). The asymptotic expansion of the generalized hypergeometric function. Journal of the London Mathematical Society, 10, 286-293.

65. Yadav, A. S., Goual, H., Alotaibi, R. M., Ali, M. M. and Yousof, H. M. (2020). Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. Symmetry, 12(1), 57.

66. Yadav, A. S., Shukl, S., Goual, H., Saha, M. and Yousof, H. M. (2022). Validation of Xgamma Exponential Model via Nikulin-Rao-Robson Goodness-of-Fit-Test under Complete and Censored Sample with Different Methods of Estimation. Statistics, Optimization & Information Computing, 10(2), 457-483.

67. Yousof, H. M., Afify, A. Z., Abd El Hadi, N. E., Hamedani, G. G. and Butt, N. S. (2016). On six-parameter Fréchet distribution: properties and applications. Pakistan Journal of Statistics and Operation Research, 281-299.

68. Yousof, H. M., Ali, M. M., Hamedani, G. G., Aidi, K. & Ibrahim, M. (2022). A new lifetime distribution with properties, characterizations, validation testing, different estimation methods. Statistics, Optimization & Information Computing, 10(2), 519-547.

69. Yousof, H. M., Altun, E., & Hamedani, G. G. (2018a). A new extension of Fréchet distribution with regression models, residual analysis and characterizations. Journal of Data Science, 16(4), 743-770.

70. Yousof, H. M., Butt, N. S., Alotaibi, R. M., Rezk, H., Alomani, G. A., & Ibrahim, M. (2019). A new compound Fréchet distribution for modeling breaking stress and strengths data. Pakistan Journal of Statistics and Operation Research, 15(4), 1017-1035.

71. Yousof, H. M., Chesneau, C., Hamedani, G. and Ibrahim, M. (2021). A New Discrete Distribution: Properties, Characterizations, Modeling Real Count Data, Bayesian and Non-Bayesian Estimations. Statistica, 81(2), 135-162.

72. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). The Two-parameter Xgamma Fréchet Distribution: Characterizations, Copulas, Mathematical Properties and Different Classical Estimation Methods. Contributions to Mathematics, 2 (2020), 32-41.

73. Yousof, H. M., Rasekhi, M., Altun, E. and Alizadeh, M. (2018b). The extended odd Fréchet family of distributions: properties, applications and regression modeling. International Journal of Applied Mathematics and Statistics, 30(1), 1-30.

74. Zaharim, A., Najid, S.K., Razali, A.M. and Sopian, K. (2009). Analysing Malaysian wind speed data using statistical distribution. In Proceedings of the 4th IASME/WSEAS International conference on energy and environment, Cambridge, UK.