A LIMITED-RANGE CALDERÓN-ZYGMUND THEOREM
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Abstract. We work with singular integral operators whose kernels satisfy a condition weaker than the typical Hörmander smoothness estimate. We give two proofs of a weak-type \((q, q)\) inequality for these operators and, via interpolation, obtain \(L^p(\mathbb{R}^n)\) estimates for the operators for a certain range of \(p\). One proof of the weak-type estimate uses the Calderón-Zygmund decomposition while the other proof uses ideas first given by Nazarov, Treil, and Volberg.

1. Introduction

The classical theory of singular integral operators was introduced by Calderón and Zygmund in [2] and says that for certain kernels defined on \(\mathbb{R}^n \setminus \{0\}\), the weak-type \((1, 1)\) bound holds for the associated singular integral operator, assuming that an \(L^s(\mathbb{R}^n)\) bound is known for some \(1 < s \leq \infty\). Hörmander extended this theory in [6] to more general kernels \(K\) satisfying the smoothness condition

\[ [K]_{H} := \sup_{y \in \mathbb{R}^n} \int_{|x| \geq 2|y|} |K(x - y) - K(x)| \, dx < \infty. \]

The Hörmander condition is an \(L^1(\mathbb{R}^n)\)-type smoothness condition and has some variants. For example, Watson introduced the following \(L^r(\mathbb{R}^n)\) versions in [13]: for \(1 \leq r \leq \infty\), we say a kernel \(K\) is in the class \(H^r\) if

\[ [K]_{H^r} := \sup_{R > 0} \sup_{y \in \mathbb{R}^n} \sum_{m=1}^{\infty} (2^m R)^{-\frac{a}{r'}} \left[ \int_{|x| \geq 2^m R} |K(x - y) - K(x)|^r \, dx \right]^{\frac{1}{r}} < \infty, \]

where \(r'\) is the Hölder conjugate of \(r\). Observe that Watson’s condition coincides with Hörmander’s condition when \(r = 1\), and

\[ H^\infty \subseteq \cdots \subseteq H^3 \subseteq H^2 \subseteq H^1 = H. \]

In this paper we focus on another set of conditions defined as follows.

Definition 1. Let \(1 \leq r \leq \infty\). A kernel \(K\) defined on \(\mathbb{R}^n \setminus \{0\}\) is in the class \(H_r\) if

\[ [K]_{H_r} := \sup_{R > 0} \left[ \frac{1}{v_n R^n} \int_{|y| \leq R} \left( \int_{|x| \geq 2R} |K(x - y) - K(x)|^r \, dx \right)^{\frac{1}{r}} \right] < \infty, \]

where \(v_n\) is the volume of the unit ball \(B(0, 1)\) in \(\mathbb{R}^n\).

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Remark 1. If Corollary 1. Under the hypotheses of Theorem 1, the operator results regarding multilinear and weighted Calderón-Zygmund theory.

in the nonhomogeneous setting, given in [8]. See [10–12] for applications of the Nazarov, Treil, and Volberg’s proof for the weak-type (1, 1) inequality and is an adaptation of the classical proof given in [2]. The second proof is motivated by Nazarov, Treil, and Volberg’s proof for the weak-type (1, 1) inequality in the nonhomogeneous setting, given in [8]. See [10–12] for applications of the Nazarov, Treil, and Volberg technique to multilinear and weighted settings. Refer to [5,7,9] for related results regarding multilinear and weighted Calderón-Zygmund theory.

By interpolation we obtain the following corollary.

Corollary 1. Under the hypotheses of Theorem 1, the operator $T$ is bounded on $L^p(\mathbb{R}^n)$ for $p$ in the interval $(\min(s', q), \max(q', s))$.

Remark 1. If $q > 1$ and $s < \infty$, then the interval $(\min(s', q), \max(q', s))$ is properly contained in $(1, \infty)$, hence in this case we have a limited-range Calderón-Zygmund theorem.

Remark 2. The constant $A$ does not appear in the conclusion of Theorem 1. The estimate $|K(x)| \leq \frac{A}{|x|^n}$ is only needed to ensure that the operator $T$ is well-defined for a dense class of functions.

Remark 3. The conclusions in Theorem 1 and Corollary 1 also follow under the weaker hypothesis that $T$ is bounded from $L^{s,1}(\mathbb{R}^n)$ to $L^{s,\infty}(\mathbb{R}^n)$. Here $L^{s,r}(\mathbb{R}^n)$ is the usual Lorentz space.

Remark 4. Theorem 1 and Corollary 1 are also valid if the original kernel is not of convolution type.
Remark 5. As in the case $q = 1$, there are natural vector-valued extensions of Theorem 1 and Corollary 1, in the spirit of \[1\].

2. Calderón-Zygmund Decomposition Method

The first proof of Theorem 1 relies on the $L^q(\mathbb{R}^n)$ version of the Calderón-Zygmund decomposition. See \[3,4\] for details on the decomposition.

Proof. Fix $f \in L^q(\mathbb{R}^n)$ and $\alpha > 0$. We will show that

$$\{|\{ |Tf| > \alpha \}| \leq C_{n,s,q}(B + [K]_{H^s'})^q \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.$$  

Apply the $L^q$-form of the Calderón-Zygmund decomposition to $f$ at height $\gamma \alpha$ (the constant $\gamma > 0$ will be chosen later), to write $f = g + b = g + \sum_{j=1}^{\infty} b_j$, where

1. $\|g\|_{L^\infty(\mathbb{R}^n)} \leq 2^{\frac{\gamma}{2}} \gamma \alpha$ and $\|g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)}$,
2. the $b_j$ are supported on pairwise disjoint cubes $Q_j$ satisfying $\sum_{j=1}^{\infty} |Q_j| \leq (\gamma \alpha)^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q$,
3. $\|b_j\|_{L^q(\mathbb{R}^n)} \leq 2^{n+q}(\gamma \alpha)^q |Q_j|$, 
4. $\int_{Q_j} b_j(x) \, dx = 0$, and 
5. $\|b\|_{L^q(\mathbb{R}^n)} \leq 2^{\frac{n+q}{2}} \|f\|_{L^q(\mathbb{R}^n)}$ and $\|b\|_{L^1(\mathbb{R}^n)} \leq 2(\gamma \alpha)^{1-q} \|f\|_{L^q(\mathbb{R}^n)}^q$.

Now,

$$\{|\{ |Tf| > \alpha \}| \leq \left| \{ |Tg| > \frac{\alpha}{2} \} \right| + \left| \{ |Tb| > \frac{\alpha}{2} \} \right|.$$  

Assume first that $s < \infty$. Choose $\gamma = (B + [K]_{H^s'})^{-1}$. Using Chebyshev’s inequality, the bound of $T$ on $L^s(\mathbb{R}^n)$, property (1), and trivial estimates, we have that

$$\left| \{ |Tg| > \frac{\alpha}{2} \} \right| \leq 2^s \alpha^{-s} \|Tg\|_{L^s(\mathbb{R}^n)}^s \leq (2B)^s \alpha^{-s} \|g\|_{L^s(\mathbb{R}^n)}^s \leq 2^{s+n+\frac{m}{2}} B \alpha^{-s}(\gamma \alpha)^s \|g\|_{L^q(\mathbb{R}^n)}^q \leq 2^{s+n+\frac{m}{2}} (B + [K]_{H^s'})^q \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.$$  

We next control the second term. Let $c_j$ denote the center of $Q_j$, let $Q_j^* := Q(c_j, 2\sqrt{n}l(Q_j))$ be the cube centered at $c_j$ and having side length $2\sqrt{n}$ times the side length of $Q_j$, and set $\Omega^* := \bigcup_{j=1}^{\infty} Q_j^*$. Then

$$\left| \{ |Tb| > \frac{\alpha}{2} \} \right| \leq |\Omega^*| + \left| \left\{ x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \frac{\alpha}{2} \right\} \right|.$$  

Notice that since $|Q_j^*| = (2\sqrt{n})^n |Q_j|$ and by property (2), we have

$$|\Omega^*| \leq \sum_{j=1}^{\infty} |Q_j^*| = (2\sqrt{n})^n \sum_{j=1}^{\infty} |Q_j| \leq (2\sqrt{n})^n (B + [K]_{H^s'})^q \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.$$
It remains to control the last term. Use Chebyshev’s inequality, property (4), Fubini’s theorem, Hölder’s inequality, property (3), and property (2) to estimate

\[
\left| \left\{ \mathbb{R}^n \setminus \Omega^* : |Tb| > \frac{\alpha}{2} \right\} \right| \leq 2\alpha^{-1} \int_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)| dx
\]

\[
\leq 2\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega^*} |Tb_j(x)| dx
\]

\[
\leq 2\alpha^{-1} \sum_{j=1}^{\infty} \int_{Q_j} \left[ \int_{\mathbb{R}^n \setminus \Omega^*} |K(x - y) - K(x - c_j)| dx \right] |b_j(y)| dy
\]

\[
\leq 2\alpha^{-1} \sum_{j=1}^{\infty} \left\| \int_{\mathbb{R}^n \setminus \Omega^*} |K(x - \cdot) - K(x - c_j)| dx \right\|_{L^q(Q_j)} \|b_j\|_{L^s}
\]

\[
\leq 2\alpha^{-1} \sup_{j \in \mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega^*} |K(x - \cdot) - K(x - c_j)| dx \right\|_{L^q(Q_j, \frac{dy}{|Q_j|^q})} \sum_{j=1}^{\infty} |Q_j|
\]

\[
\leq 2\gamma^{n+2} \sup_{j \in \mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega^*} |K(x - \cdot) - K(x - c_j)| dx \right\|_{L^q(Q_j, \frac{dy}{|Q_j|^q})} \sum_{j=1}^{\infty} |Q_j|
\]

\[
\leq 2\gamma^{n+2} \alpha^{-q} \| f \|_{L^q(\mathbb{R}^n)} \sup_{j \in \mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega^*} |K(x - \cdot) - K(x - c_j)| dx \right\|_{L^q(Q_j, \frac{dy}{|Q_j|^q})}.
\]

For each \( j \), setting \( R_j = \frac{\sqrt{n}}{2} l(Q_j) \), we have

\[
Q_j \subseteq B(c_j, R_j) \subseteq B(c_j, 2R_j) \subseteq Q_j^*.
\]

where \( B(x, r) \) denotes the ball centered at \( x \) and with radius \( r \). Then the factor involving the supremum is less than or equal to

\[
\sup_{j \in \mathbb{N}} \left[ \int_{B(c_j, R_j)} \left( \int_{\mathbb{R}^n \setminus B(c_j, 2R_j)} |K(x - y) - K(x - c_j)| dx \right)^{\frac{q}{n}} dy \right]^{\frac{1}{q}} \frac{1}{|Q_j|}.
\]

which is bounded by \( \left( \frac{\sqrt{n}}{2} \right)^n v_n [K]_{H_{q'}} \) by changing variables \( x' = x - c_j \), \( y' = y - c_j \) and by replacing the supremum over \( R_j \) by the supremum over all \( R > 0 \).

Putting all of the estimates together, we get

\[
\left| \left\{ |Tf| > \alpha \right\} \right| \leq \left( 2^{s-n+\frac{2n}{q}} + (2\sqrt{n})^n + \frac{2^{n+2-n} \alpha}{4} \right) (B + [K]_{H_{q'}})^{\frac{q}{n}} \alpha^{-q} \| f \|_{L^q(\mathbb{R}^n)}.
\]

When \( s = \infty \), set \( \gamma = 2^{n+2} (4([K]_{H_{q'}} + B))^{-1} \). Then

\[
\| Tg \|_{L^\infty(\mathbb{R}^n)} \leq B \| g \|_{L^\infty(\mathbb{R}^n)} \leq 2\gamma B \alpha \leq \frac{\alpha}{4},
\]

so

\[
\left| \left\{ |Tg| > \frac{\alpha}{2} \right\} \right| = 0.
\]

The part of the argument involving \( \left\{ |Tb| > \frac{\alpha}{2} \right\} \) is the same as in the case \( s < \infty \). \qed
3. Nazarov, Treil, Volberg Method

We provide a second proof of Theorem 1. This proof is motivated by the argument given by Nazarov, Treil, and Volberg in [8]. See also [10–12] for other applications of this technique.

Proof. Fix \( f \in L^q(\mathbb{R}^n) \) and \( \alpha > 0 \). We will show that
\[
|\{|Tf| > \alpha\}| \leq C_{n,s,q}(B + |K|H_{s'})^q\alpha^{-q}\|f\|_{L^q(\mathbb{R}^n)}^q.
\]
By density, we may assume \( f \) is a nonnegative continuous function with compact support. Set
\[
\Omega := \{M(f^q) > (\gamma\alpha)^q\}
\]
where \( \gamma > 0 \) is to be chosen later and where \( M \) denotes the Hardy-Littlewood maximal operator. Apply a Whitney decomposition to write
\[
\Omega = \bigcup_{j=1}^{\infty} Q_j,
\]
a disjoint union of dyadic cubes where
\[
2\text{diam}(Q_j) \leq d(Q_j, \mathbb{R}^n \setminus \Omega) \leq 8\text{diam}(Q_j).
\]
Put
\[
g := f \mathbb{1}_{\mathbb{R}^n \setminus \Omega}, \quad b := f \mathbb{1}_\Omega, \quad \text{and} \quad b_j := f \mathbb{1}_{Q_j}.
\]
Then
\[
f = g + b = g + \sum_{j=1}^{\infty} b_j,
\]
where we claim that
(1) \( \|g\|_{L^\infty(\mathbb{R}^n)} \leq \gamma\alpha \) and \( \|g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} \),
(2) the \( b_j \) are supported on pairwise disjoint cubes \( Q_j \) satisfying
\[
\sum_{j=1}^{\infty} |Q_j| \leq 3^n(\gamma\alpha)^{-q}\|f\|_{L^q(\mathbb{R}^n)}^q,
\]
(3) \( \|b_j\|_{L^q(\mathbb{R}^n)} \leq (17\sqrt{n})^n(\gamma\alpha)^q|Q_j| \), and
(4) \( \|b\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} \) and \( \|b\|_{L^1(\mathbb{R}^n)} \leq (17\sqrt{n})^n3^n(\gamma\alpha)^{1-q}\|f\|_{L^q(\mathbb{R}^n)}^q\).

Indeed, since for any \( x \notin \Omega \), we have
\[
|g(x)|^q = |f(x)|^q \leq M(f^q)(x) \leq (\gamma\alpha)^q,
\]
it follows that \( \|g\|_{L^\infty(\mathbb{R}^n)} \leq \gamma\alpha \). Since \( g \) is a restriction of \( f \), we have \( \|g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} \), and so (1) holds. Using the weak-type \((1,1)\) bound for \( M \) with \( \|M\|_{L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)} \leq 3^n \), we obtain property (2) as follows
\[
\sum_{j=1}^{\infty} |Q_j| = |\Omega| \leq 3^n(\gamma\alpha)^{-q}\|f\|_{L^q(\mathbb{R}^n)}^q.
\]
Addressing (3) and (4), let \( Q_j^* := Q(c_j, 17\sqrt{n}|Q_j|) \) be the cube with the same center as \( Q_j \) but side length \( 17\sqrt{n} \) times as large. Then \( Q_j^* \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset \), so there is a point \( x \in Q_j^* \) such
that \( M(f^q)(x) \leq (\gamma \alpha)^q \). In particular, \( \int_{Q_j} |f(y)|^q dy \leq (\gamma \alpha)^q |Q_j| \). Since \( |Q_j^*| = (17\sqrt{n})^n |Q_j| \), we have

\[
\|b_j\|_{L^q(\mathbb{R}^n)}^q = \int_{Q_j} |f(y)|^q dy \leq \int_{Q_j} |f(y)|^q dy \leq (\gamma \alpha)^q |Q_j| = (17\sqrt{n})^n (\gamma \alpha)^q |Q_j|.
\]

This proves (3). We use Hölder’s inequality, property (3), and property (2) to justify property (4)

\[
\|b\|_{L^1(\mathbb{R}^n)} = \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} \|b_j\|_{L^q(\mathbb{R}^n)} |Q_j| \frac{1}{\gamma} \leq (17\sqrt{n})^\frac{n}{q} (\gamma \alpha)^{1-q} \|f\|_{L^q(\mathbb{R}^n)}^q.
\]

Now,

\[
|\{|Tf| > \alpha\}| \leq \left| \left\{ |Tg| > \frac{\alpha}{2} \right\} \right| + \left| \left\{ |Tb| > \frac{\alpha}{2} \right\} \right|.
\]

Assume first that \( s < \infty \). Choose \( \gamma = (B + [K]_{H_q})^{-1} \). Use Chebyshev’s inequality, the bound of \( T \) on \( L^s(\mathbb{R}^n) \), and property (1) to see

\[
\left| \left\{ |Tg| > \frac{\alpha}{2} \right\} \right| \leq 2s \alpha^{-s} \|g\|_{L^s(\mathbb{R}^n)}^s \leq (2B)^s \alpha^{-s} \|g\|_{L^s(\mathbb{R}^n)}^s \leq (2B)^s (\gamma \alpha)^{s-q} \alpha^{-q} \|g\|_{L^q(\mathbb{R}^n)}^q \leq 2s (B + [K]_{H_q})^q \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.
\]

We will now control the second term. Let \( E_j \) be a concentric dilate of \( Q_j \); precisely,

\[
E_j := Q(c_j, r_j),
\]

where \( c_j \) is the center of \( Q_j \) and \( r_j > 0 \) is chosen so that

\[
|E_j| = \frac{1}{(17\sqrt{n})^{\frac{n}{q}} \gamma \alpha} \int_{Q_j} b_j(x) \, dx.
\]

Note that such \( E_j \) exist since the function \( r \mapsto |Q(x, r)| \) is continuous for each \( x \in \mathbb{R}^n \). Applying Hölder’s inequality and property (3), we have

\[
|E_j| = \frac{1}{(17\sqrt{n})^{\frac{n}{q}} \gamma \alpha} \int_{Q_j} b_j(x) \, dx \leq \frac{1}{(17\sqrt{n})^{\frac{n}{q}} \gamma \alpha} |Q_j| \frac{1}{\gamma} \|b_j\|_{L^q(\mathbb{R}^n)} \leq |Q_j|.
\]

Since \( E_j \) is a cube with the same center as \( Q_j \), and since \( |E_j| \leq |Q_j| \), the containment \( E_j \subseteq Q_j \) holds. In particular, the \( E_j \) are pairwise disjoint. Set

\[
E := \bigcup_{j=1}^{\infty} E_j.
\]

Then

\[
\left| \left\{ |Tb| > \frac{\alpha}{2} \right\} \right| \leq I + II + III,
\]

where

\[
I = |\Omega|,
\]

\[
II = \left| \left\{ x \in \mathbb{R}^n \setminus Q_j : \left( b - \frac{(17\sqrt{n})^{\frac{n}{q}} \gamma \alpha \mathbb{1}_E \right)(x) \frac{\alpha}{4} \right\} \right|,
\]

and

\[
III = \left| \left\{ (17\sqrt{n})^{\frac{n}{q}} \gamma \alpha |T(\mathbb{1}_E)| \frac{\alpha}{4} \right\} \right|.
\]
The control of I follows from property (2),

$$|\Omega| = \sum_{j=1}^{\infty} \leq 3^n (B + [K]_{H_q}) \|f\|_{L^q(\mathbb{R}^n)}^q.$$  

For II, use Chebyshev’s inequality, the fact that \( \int_{Q_j} b_j(y) - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j}(y) \, dy = 0 \), Fubini’s theorem, and Hölder’s inequality to estimate

\[
II \leq 4\alpha^{-1} \int_{\mathbb{R}^n \setminus \Omega} \left| T \left( b - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j} \right) (x) \right| \, dx \\
\leq 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega} \left| T \left( b_j - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j} \right) (x) \right| \, dx \\
\leq 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega} \int_{Q_j} |K(x - y) - K(x - c_j)| \left| b_j(y) - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j}(y) \right| \, dy \, dx \\
= 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{Q_j} \left( \int_{\mathbb{R}^n \setminus \Omega} |K(x - y) - K(x - c_j)| \, dx \right) \left| b_j(y) - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j}(y) \right| \, dy \\
\leq 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega} |K(x - y) - K(x - c_j)| \, dx \left\| b_j - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)} \\
\leq 4\alpha^{-1} \sup_{j \in \mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x - y) - K(x - c_j)| \, dx \right\|_{L^q'(Q_j, \frac{dy}{|Q_j|})} \\
\times \sum_{j=1}^{\infty} |Q_j| \left( \frac{1}{q} \right) \left\| b_j - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)}.
\]

Using the triangle inequality, property (3), and the fact that \(|E_j| \leq |Q_j|\), we have

\[
\left\| b_j - (17\sqrt{n})^\frac{q}{n} \gamma \alpha \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)} \leq \|b_j\|_{L^q(\mathbb{R}^n)} + (17\sqrt{n})^\frac{q}{n} \gamma \alpha |E_j|^\frac{1}{q} \leq 2(17\sqrt{n})^\frac{q}{n} \gamma \alpha |Q_j|^\frac{1}{q}.
\]

Using the above estimate and property (2), we control

\[
II \leq 8(17\sqrt{n})^\frac{q}{n} \gamma \sup_{j \in \mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x - y) - K(x - c_j)| \, dx \right\|_{L^q'(Q_j, \frac{dy}{|Q_j|})} \sum_{j=1}^{\infty} |Q_j| \\
\leq 8(17\sqrt{n})^\frac{q}{n} 3^n \lambda^{1-q} \alpha^{-q} \|f\|^q_{L^q(\mathbb{R}^n)} \sup_{j \in \mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x - y) - K(x - c_j)| \, dx \right\|_{L^q'(Q_j, \frac{dy}{|Q_j|})}.
\]

For each \( j \), setting \( R_j = \frac{\sqrt{\pi}}{2} l(Q_j) \), we have

\( Q_j \subseteq B(c_j, R_j) \subseteq B(c_j, 2R_j) \subseteq \Omega \).

Then the supremum is bounded by

\[
\sup_{j \in \mathbb{N}} \left[ \int_{B(c_j, R_j)} \left( \int_{\mathbb{R}^n \setminus B(c_j, 2R_j)} |K(x - y) - K(x - c_j)| \, dx \right)^{\frac{q'}{q}} \frac{dy}{|Q_j|^\frac{1}{q'}} \right]^{\frac{1}{q'}}.
\]
which is bounded by \( \left( \frac{\sqrt{n}}{s} \right)^n v_n[K]_{H_q} \) by changing variables \( x' = x - c_j, y' = y - c_j \) and by replacing the supremum over \( R_j \) by the supremum over all \( R > 0 \). Therefore

\[
II \leq 8(17\sqrt{n})^{\frac{n}{2}} \left( \frac{3\sqrt{n}}{2} \right)^n v_n(B + [K]_{H_q})^q \|f\|_L^q(\mathbb{R}^n).
\]

To control III, use Chebyshev’s inequality, the bound of \( T \) on \( L^s(\mathbb{R}^n) \), the fact that \( |E| \leq |\Omega| \), and property (2) to estimate

\[
III \leq 4^s(17\sqrt{n})^{\frac{n}{2}} \gamma^s \int_{\mathbb{R}^n} |T(1_E)(x)|^s \, dx
\leq 4^s(17\sqrt{n})^{\frac{n}{2}} \gamma^s B^s |E|
\leq 4^s(17\sqrt{n})^{\frac{n}{2}} |\Omega|
\leq 4^s(17\sqrt{n})^{\frac{n}{2}} \gamma^s 3^n (B + [K]_{H_q})^q \|f\|_L^q(\mathbb{R}^n).
\]

Putting the estimates together, we get

\[
|\{ |Tf| > \alpha \}| \leq \left( 2^s + 3^s + 8(17\sqrt{n})^{\frac{n}{2}} \left( \frac{3\sqrt{n}}{2} \right)^n v_n + 4^s(17\sqrt{n})^{\frac{n}{2}} 3^n \right) \left( B + [K]_{H_q} \right)^q \alpha^{-q} \|f\|_L^q(\mathbb{R}^n).
\]

Since we assumed that \( f \) was nonnegative, we must double the constant above to prove the statement for general \( f \in L^q(\mathbb{R}^n) \).

When \( s = \infty \), set \( \gamma = (4(B + [K]_{H_q}))^{-1} \). Then

\[
\|Tg\|_{L^\infty(\mathbb{R}^n)} \leq B\|g\|_{L^\infty(\mathbb{R}^n)} \leq B\gamma \alpha \leq \frac{\alpha}{4},
\]

so \( |\{ |Tg| > \frac{\alpha}{2} \}| = 0 \). The part of the argument involving the set \( \{ |Tb| > \frac{\alpha}{2} \} \) is the same as in the case \( s < \infty \). \qed

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