COMPUTATION OF MILNOR NUMBERS AND CRITICAL VALUES AT INFINITY

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Abstract. We describe how to compute topological objects associated to a complex polynomial map of \( n \geq 2 \) variables with isolated singularities. These objects are: the affine critical values, the affine Milnor numbers for all irregular fibers, the critical values at infinity, and the Milnor numbers at infinity for all irregular fibers. Then for a family of polynomials we detect parameters where the topology of the polynomials can change. Implementation and examples are given with the computer algebra system SINGULAR.

1. Introduction

1.1. Review on the local case. Let \( g : \mathbb{C}^n, 0 \to \mathbb{C}, 0 \) be a germ of polynomial map with isolated singularities. One of the most important topological object attached to \( g \) is its local Milnor number [Mi]:

\[
\mu_0 = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/\text{Jac}(g)
\]

where \( \text{Jac}(g) = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}) \) is the Jacobian ideal of \( g \). It is possible to compute \( \mu_0 \) with the help of a Gröbner base. For example such a computation motivates the computer algebra system SINGULAR, [GPS].

Now we consider a family \( (g_s)_{s \in [0,1]} \), with \( g_s : \mathbb{C}^n, 0 \to \mathbb{C}, 0 \) germs of isolated singularities, such that \( g_s \) is a smooth function of \( s \). To each \( s \in [0,1] \) we associate the local Milnor number \( \mu_0(g_s) \). The main topological result for families is Lê-Ramanujam-Timourian \( \mu \)-constant theorem.

Theorem 1 ([LR, Ti]). If \( n \neq 3 \) and \( \mu_0(g_s) \) is constant \( (s \in [0,1]) \) then the family \( (g_s)_{s \in [0,1]} \) is a topologically trivial family.

1.2. Motivation and aims for the global case. Now we consider a polynomial function \( f : \mathbb{C}^n \to \mathbb{C} \). The study of the topology of \( f \) is not just the glueing of local studies because of the behaviour of \( f \) at infinity, see [Br]. To the polynomial \( f \) we attach “Milnor numbers” \( \mu, \lambda \) and finite sets of critical values \( \mathcal{B}_{\text{aff}}, \mathcal{B}_{\infty}, \mathcal{B} = \mathcal{B}_{\text{aff}} \cup \mathcal{B}_{\infty} \) (see the definitions below). The first aim of this work is to compute these objects and to give the topology of the fibers \( f^{-1}(c) \) for all \( c \in \mathbb{C} \).

There is a global version of the local \( \mu \)-constant theorem (see Theorem 2) where the Milnor number \( \mu_0 \) is replaced by a Milnor multi-integer \( m = \dots \)
i.e. that the set

\[ \{ \nu \mid \mu \geq \nu \} \]

be the degree of\( f \) at the point \( (\bar{x}, \bar{z}) \). Let \( \mu_{\bar{x}} \bar{z} \) be the local Milnor number of \( \bar{x}, \bar{z} \). Let \( \mu_{\bar{x}} \bar{z} = \sum_{c \in \mathbb{C}} \mu_{c} \)

be the affine critical values and the affine Milnor number.

1.3. Critical values. Let \( f : \mathbb{C}^{n} \rightarrow \mathbb{C} \) be a polynomial map, \( n \geq 2 \). By a result of Thom [Th] there is a minimal set of critical values \( \mathcal{B} \) of point of \( \mathbb{C} \) such that \( f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \rightarrow \mathbb{C} \setminus \mathcal{B} \) is a fibration.

1.4. Affine singularities. We suppose that affine singularities are isolated i.e. that the set \( \{ x \in \mathbb{C}^{n} \mid \text{grad}_x f = 0 \} \) is a finite set. Let \( \mu_{c} \) be the sum of the local Milnor numbers at the points of \( f^{-1}(c) \). Let

\[ \mathcal{B}_{\text{aff}} = \{ c \mid \mu_{c} > 0 \} \quad \text{and} \quad \mu = \sum_{c \in \mathbb{C}} \mu_{c} \]

be the affine critical values and the affine Milnor number.

1.5. Singularities at infinity. See [Br]. Let \( d \) be the degree of \( f : \mathbb{C}^{n} \rightarrow \mathbb{C} \), let \( f = f_{d} + f_{d-1} + \cdots + f_{0} \) where \( f_{j} \) is homogeneous of degree \( j \). Let \( \bar{f}(x, z) \) (with \( x = (x_{1}, \ldots, x_{n}) \)) be the homogenisation of \( f \) with the new variable \( z \): \( \bar{f}(x, z) = f_{d}(x) + f_{d-1}(x)z + \cdots + f_{0}(x)z^{d} \). Let

\[ X = \{(x : z, t) \in \mathbb{P}^{n} \times \mathbb{C} \mid \bar{f}(x, z) - cz^{d} = 0 \} \]

Let \( \mathcal{H}_{\infty} \) be the hyperplane at infinity of \( \mathbb{P}^{n} \) defined by \( (z = 0) \). The singular locus of \( X \) has the form \( \Sigma \times \mathbb{C} \) where

\[ \Sigma = \{(x : 0) \mid \frac{\partial f_{d}}{\partial x_{1}} = \cdots = \frac{\partial f_{d}}{\partial x_{n}} = f_{d-1} = 0 \} \subset \mathcal{H}_{\infty}. \]

We suppose that \( f \) has isolated singularities at infinity that is to say that \( \Sigma \) is finite. This is always true for \( n = 2 \). We say that \( f \) has strong isolated singularities at infinity if

\[ \Sigma' = \{(x : 0) \mid \frac{\partial f_{d}}{\partial x_{1}} = \cdots = \frac{\partial f_{d}}{\partial x_{n}} = 0 \} \]

is finite.

For a point \( (x : 0) \in \mathcal{H}_{\infty} \), assume, for example, that \( x = (x_{1}, \ldots, x_{n-1}, 1) \) and set \( \bar{x} = (x_{1}, \ldots, x_{n-1}) \) and

\[ F_{\bar{x}}(\bar{x}, z) = \bar{f}(x_{1}, \ldots, x_{n-1}, 1) - cz^{d}. \]

Let \( \mu_{\bar{x}}(F_{\bar{x}}) \) be the local Milnor number of \( F_{\bar{x}} \) at the point \( (\bar{x}, 0) \). If \( (x : 0) \in \Sigma \) then \( \mu_{\bar{x}}(F_{\bar{x}}) > 0 \). For a generic \( s \), \( \mu_{\bar{x}}(F_{s}) = \nu_{\bar{x}} \), and for finitely many \( c \), \( \mu_{\bar{x}}(F_{c}) > \nu_{\bar{x}} \). We set \( \lambda_{c, \bar{x}} = \mu_{\bar{x}}(F_{c}) - \nu_{\bar{x}} \), \( \lambda_{c} = \sum_{(x:0) \in \Sigma} \lambda_{c, \bar{x}} \). Let

\[ \mathcal{B}_{\infty} = \{ c \in \mathbb{C} \mid \lambda_{c} > 0 \} \quad \text{and} \quad \lambda = \sum_{c \in \mathbb{C}} \lambda_{c} \]

be the critical values at infinity and the Milnor number at infinity.
We can now describe the set of critical values $B$ as follows (see [HL] and [Pa]):

$$B = B_{aff} \cup B_{\infty}.$$  

Moreover by [HL] and [ST] for all $c \in \mathbb{C}$, $f^{-1}(c)$ has the homotopy type of a wedge of $\mu + \lambda - \mu_c - \lambda_c$ spheres of real dimension $n - 1$.

1.6. Families of polynomials. To a polynomial we associate its Milnor multi-integer $m = (\mu, \#B_{aff}, \lambda, \#B_{\infty}, \#B)$. Two polynomials maps $f, g : \mathbb{C}^n \rightarrow \mathbb{C}$ are topologically equivalent if there exist homeomorphisms $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and $\Psi : \mathbb{C} \rightarrow \mathbb{C}$ such that $f \circ \Phi = \Psi \circ g$. The Milnor multi-integer is a topological invariant, that is to say if $f$ and $g$ are topologically equivalent then $m(f) = m(g)$. We recall a result of [Bo, BT] that is kind of converse of this property.

Let $(f_s)_{s \in [0,1]}$ be a family of polynomials, such that $f_s$ has strong isolated singularities at infinity and isolated affine singularities for all $s \in [0,1]$. For each $s \in [0,1]$ we consider the Milnor multi-integer of $f_s$, $m(f_s) = (\mu(s), \#B_{aff}(s), \lambda(s), \#B_{\infty}(s), \#B(s))$. We suppose that the coefficients of the family are polynomials in $s$ and that the degree $\deg f_s$ is constant.

Theorem 2 ([Bo, BT]). Let $n \neq 3$. If $m(f_s)$ is constant ($s \in [0,1]$), then $f_0$ is topologically equivalent to $f_1$.

How to verify the hypotheses from a computable point of view? It is not possible to compute $m(f_s)$ for infinitely many $s \in [0,1]$. But in fact $m(f_s)$ is constant except for finitely many $s$, we denote by $S$ the set of these critical parameters.

In paragraph 4 we give a computation of a finite set $S'$ such that

$$S \subset S'.$$

Now to check if a value $s \in S'$ is in $S$ we compute $m(f_s)$ and we compare it with $m(f_{s'})$ where $s'$ is any value of $[0,1] \setminus S'$; now $s \in S$ if and only if $m(f_s) \neq m(f_{s'})$.

1.7. Implementation. The results of this paper have been implemented in two libraries critic and defpol. The first one enables to calculate all the objects defined above: $B_{aff}, \mu, \mu_c$ for $c \in B_{aff}$; $B_{\infty}, \lambda, \lambda_c$ for $c \in B_{\infty}$. These programs are written for SINGULAR, [GPS]. It is based on polar curves and on the article of D. Siersma and M. Tibăr, [ST]. For polynomials in two variables ($n = 2$) a program in MAPLE has been written by G. Bailly-Maitre, [BM], based on a discriminant formula of HÀ H.V., [Ha]. For families of polynomials the second library computes a finite set $S'$ that contains the critical parameters.

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2. Milnor numbers and critical values in affine space

2.1. Milnor number. The computation of the affine Milnor number \( \mu \) is easy and well-known (see [GPS] for example). Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \). Let \( J \) be the Jacobian ideal of the partial derivative \((\partial f/\partial x_i)_i\). Then \( \mu \) is the vector space dimension (over \( \mathbb{C} \)) of a Gröbner basis of the quotient \( \mathbb{C}[x_1, \ldots, x_n]/J \).

2.2. Critical values. We add a new variable \( t \). We consider the variety \( C = \{(x, t) \in \mathbb{C}^n \times \mathbb{C} \mid f(x) - t = 0 \text{ and } \text{grad}_xf = 0\} \).

The critical values are the projection of \( C \) on the \( t \)-coordinate: \( B_{\text{aff}} = \text{pr}_t(C) \).

2.3. Milnor number of a fiber. Set \( c \in \mathbb{C} \). We would like to compute \( \mu_c \), the sum of the Milnor numbers of the points of \( f^{-1}(c) \). Let \( J \) be the Jacobian ideal of \( f \) and set \( x \) a critical point. We denote by \( J_x \) the localization of \( J \) at \( x \). Let \( I_x = (t - c, J_x) \), the dimension of \( I_x \) is equal to the Milnor number of \( f \) at \( x \). For \( k \geq 1 \) we consider \( K_x^k = ((f - t)^k, I_x) \). Then \( f(x) = c \) if and only if \( K_x \) has non-zero dimension (as a vector space). Moreover if \( f(x) = c \) then, by the Nullstellensatz, \( (f - t)^k \) is in \( I_x \) for a sufficiently large \( k \). For such a \( k \), the dimension of \( K_x \) is the Milnor number at \( x \) if \( f(x) = c \), and it is 0 otherwise. Such a \( k \) is less or equal to the Milnor number at \( x \), but \( k \) can often be chosen much less. The minimal \( k \) is the first integer such that the vector space dimension of \( K_x^k \) is equal to the one of \( K_x^{k+1} \).

3. Milnor numbers and critical values at infinity

We give the computation of the objects at infinity and its implementation in SINGULAR. We will suppose that \( f \) has isolated singularities at infinity, in fact computations are valid for a larger class of polynomials but it cannot be computed if \( f \) belongs to this class. The algorithm is based on the article of D. Siersma and M. Tibăr, [ST], that gives critical values at infinity and Milnor numbers at infinity with the help of polar curves.

3.1. Working space. We will work in \( \mathbb{P}^n \times \mathbb{C} \), with the homogeneous coordinates of \( \mathbb{P}^n \): \((x_1 : \ldots : x_n : z)\); we still need \( t \) which is a parameter or a variable depending on the context.

We recall that

\[
X = \left\{(x : z, t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x, z) - tz^d = 0\right\}.
\]

The part at infinity of \( X \) is \( X_\infty = X \cap (\mathcal{H}_\infty \times \mathbb{C}) \):

\[
X_\infty = \left\{(x : 0, t) \in \mathbb{P}^n \times \mathbb{C} \mid f^d(x) = 0\right\},
\]

Where \( f = f^d + f^{d-1} + \cdots \) is the decomposition in homogeneous polynomials. In SINGULAR, we write:
ring r = 0, (x(1..n),z,t), dp;
poly f = ...;
poly fH = homog(f,z)-t*z^deg(f);
ideal X = fH;
ideal Xinf = z, fH;

3.2. Polar curve. Let $k$ be in $\{1, \ldots, n\}$. The polar curve $\mathcal{P}$ is the critical locus of the map $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^2$ defined for $x = (x_1, \ldots, x_n)$ by $\phi(x) = (f(x), x_k)$:

$$\mathcal{P} = \left\{ x \in \mathbb{C}^n \mid \frac{\partial f}{\partial x_i}(x) = 0, \forall i \neq k \right\}.$$ 

We have that $\mathcal{P}$ is a curve or is void. We call $\mathcal{P}_H$ the projective closure of $\mathcal{P}$. This curve intersects the hyperplane at infinity $H_\infty$ in finitely many points.

$$\text{ideal } P = \text{diff}(f,x(1)), \ldots, \text{diff}(f,x(k-1)), \text{diff}(f,x(k+1)), \ldots;\text{ideal } PH = \text{homog}(P,z);$$

The former objects can be viewed in $X$, we will also denote by $\mathcal{P}_H$ the set $(\mathcal{P}_H \times \mathbb{C}) \cap X$. In the chart $x_k = 1$, we denote the curve $\mathcal{P}_H$ by $\bar{\mathcal{C}}$. The “real” polar curve $\bar{\mathcal{C}}$ in this chart is the closure of $\bar{\mathcal{C}} \setminus X_\infty$:

$$\text{ideal } Cbar = x(k)-1, PH, X;$$
$$\text{ideal } C = \text{sat}(Cbar,Xinf)[1];$$

3.3. Critical values at infinity. We need the following result of [ST]. A value $c$ is a critical values at infinity if and only there is coordinate $x_k$ and a point $(x:0,t)$ in $X_\infty$ (with $x_k \neq 0$) such that $(x:0,t) \in C$. That is to say $\mathcal{B}_\infty$ is the projection of $\mathcal{C}_\infty = X_\infty \cap C$ on the space of parameters $t \in \mathbb{C}$. Then the critical values are computed with:

$$\text{ideal } Cinf = z, C;$$
$$\text{poly } Binf = \text{eliminate}(Cinf,x(1)x(2) \ldots x(n)z)[1];$$

The set of critical values at infinity are the roots of the polynomial $Binf$, which belongs to $\mathbb{C}[t]$.

3.4. Milnor numbers at infinity. Actually the results in [ST] are more precise. For a fixed $t$, let $X_t = \{(x,z,t) \in X\}$, this is a projective model for the fiber $f^{-1}(t)$.

**Theorem 3** ([ST]). The Milnor number at infinity at a point $(x:0,t) \in C_\infty$ is given by the intersection number (in $X$) of $C$ with $X_t$ at $(x:0,t)$.

So, for $c \in \mathcal{B}_\infty$, the Milnor number at infinity $\lambda_c$ (for the chart $x_k \neq 0$), is equal to the sum of all intersection numbers of $X_c$ and $C$ in $X_\infty$.

We compute an ideal $I$ which correspond to $X_c \cap C$, then we only deals with points at infinity by intersecting it this set with $z^q = 0$, for a sufficiently large $q$.

$$\text{number } c = \ldots;$$
$$\text{ideal } Xc = t-c, X;$$
$$\text{ideal } I = Xc, C;$$
ideal $K = z^q, I$; // $q >> 1$
lambda$dac = vdim(std(K));

Once we have computed $\lambda_c$ for all $c \in B_\infty$, we have $\lambda = \sum_{c \in B_\infty} \lambda_c$.

4. FAMILIES OF POLYNOMIALS

Let $(f_s)_{s \in [0,1]}$ be a family of complex polynomials in $n$ variables. We suppose that the coefficients are polynomial functions of $s$ and that for all $s \in [0,1]$, $f_s$ has affine isolated singularities and strong isolated singularities at infinity. The implementation is similar to the one of paragraph 3 and will be omitted.

4.1. Change in affine space. It is not possible to compute infinitely many $\mu(s)$, so we have to detect a change of $\mu(s)$. The Milnor numbers $\mu(s)$ changes if and only if some critical points escape at infinity. Then we can detect critical parameters for $\mu$ as follows: Let $J = \{(x_1, \ldots, x_n, s) \in \mathbb{C}^n \times \mathbb{C} | \frac{\partial f_1}{\partial x_1} = \ldots, \frac{\partial f_n}{\partial x_n} = 0\}$ be the set of critical points (that corresponds to the Jacobian ideal in $\mathbb{C}[x_1, \ldots, x_n]$). Let $J$ be the homogeneization of $J$ with the new variable $z$, while $s$ is considered as a parameter. The part at infinity of $J$ corresponds to the ideal $J_\infty = J \cap (z = 0)$, and the affine part of $J$ is $J_{aff} = J \setminus J_\infty$. Now the critical parameters for $\mu$ is $pr_s(J_{aff}) \subset \mathbb{C}$, where $pr_s$ is the projection to the $s$-coordinate.

It is possible to compute $B_{aff}(s)$ for all $s \in [0,1]$ by a direct extension of the work of paragraph 2. Then we can compute the parameters where the cardinal of this set changes.

4.2. Change at infinity. Again it is not possible to compute infinitely many $\lambda(s)$. We extend the definition of paragraph 3 by adding a parameter $s$. We set $d = \deg f_s$ and

$$X = \{(x : z, t, s) \in \mathbb{P}^n \times \mathbb{C} \times \mathbb{C} | \bar{f}_s(x, z) - tz^d = 0\}.$$ 

The part at infinity of $X$ is $X_\infty = X \cap (H_\infty \times \mathbb{C} \times \mathbb{C})$:

$$X_\infty = \{(x : 0, t, s) \in \mathbb{P}^n \times \mathbb{C} | f_s^d(x) = 0\}.$$ 

The polar “curve” is

$$\mathcal{P} = \{(x, s) \in \mathbb{C}^n \times \mathbb{C} | \frac{\partial f_s}{\partial x_i}(x) = 0, \forall i \neq k\}.$$ 

In the chart $x_k = 1$ we denote the homogeneization of $\mathcal{P}$ (with $s$ a parameter) by $\mathcal{C}$, and the “real” polar curve $\mathcal{C}$ in this chart is the closure of $\mathcal{C} \setminus X_\infty$.

The part at infinity of $\mathcal{C}$ is $\mathcal{C}_\infty = \mathcal{C} \cap X_\infty$.

Let $B_\infty(s) = pr_1\{(x : 0, t, s) \in \mathcal{C}_\infty\}$. For a generic $s'$, $B_\infty(s') = B_\infty(s')$. Then the critical parameters for $B_\infty(s)$ is included in the set of parameters where $\#B_\infty(s)$ fails to be equal to $\#B_\infty(s')$ (in fact $B_\infty(s)$ may be infinite).

We set $X_s = \{(x : z, c, s) \in X | (x : 0, c, s) \in \mathcal{C}_\infty\}$, for non-critical parameters it corresponds to union of the irregular fibers at infinity. Now a change of $\lambda$ corresponds a change in the value of the intersection multiplicity of
the polar curve $C$ with $X_\lambda$. The critical parameters for $\lambda$ are given as the projection to the $s$-coordinate of

$$\frac{\partial C \cap X_\lambda}{\partial \mathcal{C}_\lambda \cap (z = 0)}.$$ 

At last we compute parameters where the cardinal of $\mathcal{B}(s) = \mathcal{B}_{aff}(s) \cup \mathcal{B}_\infty(s)$ changes.

5. Examples

5.1. Briançon polynomial. The following example shows how to use the program once you have started SINGULAR. We have to load the library critic.lib, then we set the ring, with $n+1$ variables, the last variable will able to have the critical values (as the zeroes of a polynomial) in return. The following code gives critical values at infinity of Briançon polynomial.

```
LIB "critic.lib";
ring r = 0, (x,y,t), dp;
poly s = xy+1;
poly p = x*s+1;
poly f = 3*y*p^3+3*p^2*s-5*p*s-s;
crit(f);
```

The result is:

```
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of 3t^2+16t
> Milnor number at infinity : 4
> Details of critical values at infinity :
> t 1
> 3t+16 3
```

This shows, that there is no affine critical value (as the root of the polynomial 1) and that $B_\infty = \{0, -16/3\}$ (as the root of the polynomial $t$ and $3t + 16$) are the critical values at infinity, with Milnor number at infinity respectively equal to 1 and 3.

5.2. More variables. Let $f(a,b,c,d) = a + a^4b + b^2c^3 + d^5$ be the example of Choudary-Dimca, [CD] and [ACD]. This polynomial has isolated singularities at infinity. The only singularity is a singularity at infinity for the critical value 0. Let’s check it.

```
ring r = 0, (a,b,c,d,t), dp;
poly f = a+a^4*b+b^2*c^3+d^5;
crit(f);
```

The result is:

```
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of t
> Milnor number at infinity : 8
```
5.3. A family. We give example of deformation, we first need to load the library defpol.lib, then we set a ring in \( n + 1 \) variables, where the last variable is the parameter of the deformation. For instance we consider the deformation \( f_s(x,y) = y(1-sx)(y-(s-1)x) \).

```plaintext
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = y*(1-sx)*(y-(s-1)*x);
parCrit(f);
> Critical parameters are included in the roots of s^2-s
```

Then the critical parameters are \( s = 0 \) and \( s = 1 \).

5.4. A trivial family. Another deformation is \( f_s(x,y) = x(x^3y + sx^2 + s^2x+1) \).

```plaintext
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = x*(x^3*y+s*x^2+s^2*x+1);
parCrit(f);
> Critical parameters are included in the roots of 1
```

Then \( m(f_s) \) and the degree are constant; by Theorem 2 it implies that for all \( s, s' \in \mathbb{C} \), \( f_s \) and \( f_{s'} \) are topologically equivalent.

5.5. Combination. We consider the family \( f_s(x,y) = (x-s^2-1)(x^2y+1) \).

```plaintext
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = (x-s^2-1)*(x^2*y+1);
parCrit(f);
> Critical parameters are included in the roots of s^2+1
```

For a generic value we have

```plaintext
LIB "critic.lib";
ring r = (0,s), (x,y,t), dp;
poly f = (x-s^2-1)*(x^2*y+1);
crit(f);
> Affine critical values are the roots of t
> Affine Milnor number : 1
> Critical values at infinity are the roots of t+(s^2+1)
> Milnor number at infinity : 1
```

And for a critical parameter (\( s = i \) or \( s = -i \)):

```plaintext
ring r = (0,s), (x,y,t), dp;
minpoly = s^2+1;
poly f = (x-s^2-1)*(x^2*y+1);
crit(f);
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of t
```
Milnor number at infinity : 1

References

[ACD] E. Artal-Bartolo, P. Cassou-Noguès and A. Dimca, Sur la topologie des polynômes complexes, Singularities (Oberwolfach, 1996), Progr. Math., 162, Birkhäuser, Basel, 317-343, 1998.

[BM] G. Bailly-Maître, Monodromies des polynômes de deux variables complexes, Thèse de l’université de Bordeaux, 2000.

[Bo] A. Bodin, Invariance of Milnor numbers and topology of complex polynomials, Comment. Math. Helv., 78, 134-152, 2003.

[BT] A. Bodin and M. Tibăr, Topological equivalence in families of complex polynomials, preprint.

[Br] S.A. Broughton, Milnor numbers and the topology of polynomials hypersurfaces, Inv. Math., 92, 217-241, 1988.

[CD] A. Choudary and A. Dimca, Complex hypersurfaces diffeomorphic to affine spaces, Kodai Math. J., 17, 171-178, 1994.

[GPS] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 2.0: a computer algebra system for polynomial computations. Centre for computer algebra, university of Kaiserslautern, 2001. http://www.singular.uni-kl.de.

[Ha] HÀ H.V., Sur la fibration globale des polynômes de deux variables complexes, C. R. Acad. Sci. Paris, 309, 231-234, 1989.

[HL] HÀ H.V. and LÊ D.T., Sur la topologie des polynômes complexes, Acta Mathematica Vietnamica, 9, 21-32, 1984.

[LR] LÊ D.T. and C.P. Ramanujam, The invariance of Milnor’s number implies the invariance of the topological type, Amer. J. Math., 98, 67-78, 1976.

[Mi] J. Milnor, Singular points of complex hypersurfaces, Ann. of Math., Studies 61, Princeton University Press, 1968.

[Pa] A. Parusiński, On the bifurcation set of complex polynomial with isolated singularities at infinity, Compositio Math., 97, 369-384, 1995.

[ST] D. Siersma and M. Tibăr, Singularities at infinity and their vanishing cycles, Duke Math. J., 80, 771-783, 1995.

[Th] R. Thom, Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc., 75, 249-312, 1969.

[Ti] J.G. Timourian, The invariance of Milnor’s number implies topological triviality, Amer. J. Math., 99, 437-446, 1977.