Scaled Rate Optimization for Beta-Binomial Models

Inon Sharony

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Inon.Sharony@gmail.com

Abstract

Rates of binomial processes are modeled using beta-binomial distributions (for example, from Beta Regression). We treat the offline optimization scenario and then the online one, where we optimize the exploration-exploitation problem. The rates given by two processes are compared through their distributions, but we would like to optimize the net payout (given a constant value per successful event, unique for each of the processes). The result is an analytically-closed, probabilistic, hypergeometric expression for comparing the payout distributions of two processes. To conclude, we contrast this Bayesian result with an alternative frequentist approach and find 4.5 orders of magnitude improvement in performance, for a numerical accuracy level of 0.01%.

Keywords: Bayesian, beta, binomial, hypergeometric, rate

1 Introduction

Modeling some proportion quantity is essentially different from independently modeling a numerator and denominator, and rate prediction is a specific example of proportion. In some contexts, we would like to compare two rate processes, which are competing in the context of some portfolio optimization[1]. Furthermore, we will assume that our optimization is performed on time-scales comparable with those of the underlying rate process, and is therefore treated as an online learning problem. To treat the offline problem, exact statistical tests can be used. For binomial processes, the joint probability for the data is given by the multivariate hypergeometric distribution. See section §A.

Within the Multi-Armed Bandit picture, a competitor has some intrinsic payout distribution, and we are tasked with finding an optimal solution to the exploration-exploitation problem. In this paper, I will derive an analytically closed expression...
closed expression which optimizes the payout, given that each competitor has a binomial probability distribution for success, and some unique payout value for a successful trial.  

The event of interest is modeled as a binomial process with parameter \( \phi \), where \( m \) and \( n \) will denote the number of trials and successes, respectively: \((m, n) \sim \text{Bin}(\phi)\). Therefore, the probability density function (PDF) given the rate parameter \( \phi \), of a potential observation of \( n \) wins out of \( m \) trials, is

\[
\Pr(m, n|\phi) = \binom{m}{n} \phi^n (1 - \phi)^{m-n} \tag{1.1}
\]

Competing rate processes would manifest in different values for the rate parameter, \( \phi \), which is represented by some unknown underlying distribution, which we would like to model using empirical data.

### 1.1 Beta-binomial model

To compare rate models, we’d like to compare the probabilities of the models given some observation data, \( \Pr(\phi|m, n) \). The conjugate prior of a variable \( (\phi) \) drawn from a binomial distribution is a beta-binomial distribution, \( \phi \sim \text{Beta}(\alpha, \beta) \):

\[
\Pr(\phi|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \phi^{\alpha-1} (1 - \phi)^{\beta-1} \tag{1.2}
\]

where \( B(\cdot) \) is the beta function.

The Beta Regression model of Ferrari & Cribari-Neto\cite{2} is used to learn the probabilistic distribution of the rate parameter \( \phi \) of a given binomial process, given its observations. Specifically, the Beta Regression model is a regression towards the underlying distribution of \( \phi \), from the observed data: \( \alpha \) is one plus the observed number of wins, and \( \beta \) is one plus the observed number of losses. To choose the optimal of two competing rate processes, we compare the distributions of the rates of these processes.

### 1.2 Comparison of rates of beta-binomial processes\cite{3, 4}

Given some observational data \( (\alpha_A, \beta_A, \alpha_B, \beta_B) \) for two beta-binomial processes \( A \) and \( B \), the probability that the underlying rate of process \( B \), \( \phi_B \), is higher than that of process \( A \), \( \phi_A \), is (section §B):

\footnote{This value will be assumed to be slowly-changing, relative to all other time-scales. For example, the rate may fluctuate at an hourly resolution, or higher, but the payout will change only on the order of days or weeks.}
\[
\Pr(\phi_B > \phi_A|\alpha_A, \beta_A, \alpha_B, \beta_B) = \int_0^1 d\phi_A \Pr(\phi_A|\alpha_A, \beta_A) \int_0^1 d\phi_B \Pr(\phi_B|\alpha_B, \beta_B)
\]

\[
= \frac{1}{B(\alpha_A, \beta_A)} \sum_{i=1}^{\alpha_B} B(\alpha_A - 1 + i, \beta_B + \beta_A) \frac{\beta_B - 1 + i}{B(i, \beta_B)}
\]

Being combinatoric functions, it is sometimes more convenient to calculate via the logarithms of the beta functions:

\[
\Pr(\phi_B > \phi_A|\alpha_A, \beta_A, \alpha_B, \beta_B) \equiv \sum_{i=1}^{\alpha_B} \exp(S(\alpha_A, \beta_A, \beta_B, i))
\]

\[
S(\alpha_A, \beta_A, \beta_B, i) \equiv \ln B(\alpha_A - 1 + i, \beta_B + \beta_A) - \ln B(i, \beta_B) - \ln(\beta_B - 1 + i) - \ln B(\alpha_A, \beta_A)
\]

All terms must enter the exponential (even the prefactor of the sum) to avoid numerical overflow.

### 2 Comparison of payouts of beta-binomial processes

Since we are interested in optimizing not the success ratio itself, but the payout (given that one process may have a lower rate, but higher payout value), we will now derive the expression for \( \Pr(\phi_B > \gamma \phi_A|\alpha_A, \beta_A, \alpha_B, \beta_B) \) given some ratio of the payouts, \( \gamma > 1 \) (without loss of generality).

\[
\Pr(\phi_B > \gamma \phi_A|\alpha_A, \beta_A, \alpha_B, \beta_B) \equiv \int_{\gamma \phi_A}^{\gamma - 1} d\phi_A \Pr(\phi_A|\alpha_A, \beta_A) \int_0^1 d\phi_B \Pr(\phi_B|\alpha_B, \beta_B)
\]

\[
= \int_0^{\gamma - 1} d\phi_A \int_0^1 d\phi_B \frac{\phi_A^{\alpha_A - 1}(1 - \phi_A)^{\beta_A - 1}}{B(\alpha_A, \beta_A)} \frac{\phi_B^{\alpha_B - 1}(1 - \phi_B)^{\beta_B - 1}}{B(\alpha_B, \beta_B)}
\]

\[
= \int_0^{\gamma - 1} \frac{\phi_A^{\alpha_A - 1}(1 - \phi_A)^{\beta_A - 1}}{B(\alpha_A, \beta_A)} \int_{\gamma \phi_A}^1 \frac{\phi_B^{\alpha_B - 1}(1 - \phi_B)^{\beta_B - 1}}{B(\alpha_B, \beta_B)} d\phi_B d\phi_A
\]

\[
= \int_0^{\gamma - 1} \frac{\phi_A^{\alpha_A - 1}(1 - \phi_A)^{\beta_A - 1}}{B(\alpha_A, \beta_A)} \left[ 1 - I_{\phi_A}(\alpha_B, \beta_B) \right] d\phi_A
\]

\[
= \int_0^{\gamma - 1} \frac{\phi_A^{\alpha_A - 1}(1 - \phi_A)^{\beta_A - 1}}{B(\alpha_A, \beta_A)} \left[ 1 - \sum_{i=0}^{\alpha_B - 1} \frac{\phi_A(1 - \phi_A)^{\beta_B}}{(\beta_B + i) B(1 + i, \beta_B)} \right] d\phi_A
\]

\[
= \sum_{i=0}^{\alpha_B - 1} \int_0^{\gamma - 1} \frac{\phi_A^{\alpha_A - 1}(1 - \phi_A)^{\beta_A - 1}}{B(\alpha_A, \beta_A)} \frac{\phi_A(1 - \phi_A)^{\beta_B}}{(\beta_B + i) B(1 + i, \beta_B)} d\phi_A
\]
We now perform a transformation to remove the explicit factor $\gamma$ from the integral boundary: $\gamma \phi_A \mapsto \phi_A' \Rightarrow d\phi_A \mapsto \gamma^{-1} d\phi_A'$.

$$\Pr (\phi_B > \gamma \phi_A) = \int_0^1 \gamma^{-1} d\phi_A' \int_{\phi_A'}^1 \frac{\gamma^{-\alpha_A} \phi_A'^{-1} (1 - \phi_A'/\gamma)^{\beta_A - 1} \phi_B^{-1} (1 - \phi_B)^{\beta_B - 1}}{B(\alpha_A, \beta_A) B(\alpha_B, \beta_B)}$$

(2.7)

In appendix section §D we show how one of the integrals, identified as Euler’s hypergeometric integral, is solved.

$$\Pr (\phi_B > \gamma \phi_A) = \frac{\gamma^{-\alpha_A}}{B(\alpha_A, \beta_A)} \sum_{i=0}^{\alpha_B - 1} \frac{B(\alpha_A + i, \beta_B + 1)}{(\beta_B + i) B(1 + i, \beta_B)} \, _2F_1 (1 - \beta_A, \alpha_A + i; \alpha_A + i + \beta_B + 1; \gamma^{-1})$$

(2.8)

Again, for computational efficiency, we give the logarithmic expression

$$\Pr (\phi_B > \gamma \phi_A) = \sum_{i=0}^{\alpha_B - 1} \exp \{ C(\alpha_A, \beta_A, \gamma) + S(\alpha_A + i, \beta_B, i) + F(\alpha_A + i, \beta_A, \beta_B, \gamma) \}$$

(2.9)

$$C(\alpha_A, \beta_A, \gamma) \equiv -\alpha_A \ln \gamma - \ln B(\alpha_A, \beta_A)$$

(2.10)

$$S(a, \beta_B, i) \equiv \ln B(a, \beta_B + 1) - \ln B(1 + i, \beta_B) - \ln (\beta_B + i)$$

(2.11)

$$F(a, \beta_A, \beta_B, \gamma) = \ln \, _2F_1 (1 - \beta_A, a; a + \beta_B + 1; \gamma^{-1})$$

(2.12)

Some computational libraries have direct support of general hypergeometric functions\(^3\), and other lack it. Luckily for those cases\(^4\), our formula is eligible to be implemented using Jacobi polynomials (section §F)\(^5\)

$$F(a, \beta_A, \beta_B, \gamma) = \ln \, _2F_1 (a + \beta_B, \beta_B - \beta_A + 1; 1 - 2\gamma^{-1}) + \ln B(\beta_B - \beta_A + 1, \beta_A - 1)$$

(2.13)

It shouldn’t be difficult to understand, therefore, how a simple benchmark of this formula would outperform the equivalent frequentist method by orders of magnitude.

### 3 Results

A frequentist approach to parameter estimation requires some number of samples in order to predict the rate with a given level of confidence (see \(^6\)). Since

\(^3\)For example, in Python SciPy and JVM MIPAV.

\(^4\)e.g. Apache Commons Math.

\(^5\)In SciPy, the Jacobi polynomials are actually defined in terms of the hypergeometric function.

\(^6\)section §G
the arrival time of events is Poisson distributed, the rate at which we can gather data samples to estimate the rate parameter of each of the competing processes decays exponentially.

“The advantage of Bayesian formulas over the traditional frequentist formulas is that you don’t have to collect a preordained sample size in order to get a valid result.”[3]

Bayesian calculation (see H) shows improvement of 4.5 orders of magnitude in speed over a Frequentist implementation, where 10 million samples are required to achieve numerical accuracy to within 0.01%, on the random samples generated.

4 Discussion

We noted the hypergeometric distribution involved in exactly solving the offline problem. For the online problem, we derived an analytically-closed, probabilistic, hypergeometric expression for comparing the payout distributions of two beta-binomial rate processes. The cost of the frequentist approach turns out to be prohibitively high for very sparse data, such as highly-hierarchic or otherwise “wide” models.

5 Acknowledgments

I would like to thank Bill Tilly for the preliminary exposition, Evan Miller for posting his analytical formula for comparison of rates of beta-binomial processes, Chris Stucchio for analyzing its asymptotics, and to all three for choosing to share their research openly and freely. I would like to thank Professor Raydonal Ospina Martínez for his encouragement in writing this report.

A Exact binomial data

Given the data in the following contingency table,

|       | Asset 1 | . . . | Asset N | Marginal Totals |
|-------|---------|------|---------|-----------------|
| wins  | n₁      | . . . | nₙ      | n_{tots} = \sum nᵢ |
| losses| o₁      | . . . | oₙ      | o_{tots} = \sum oᵢ |
| trials| m₁      | . . . | mₙ      | m_{tots} = \sum mᵢ |

Table 1: Contingency table of binomial data

where \( mᵢ = nᵢ + oᵢ \).
A.1 Fisher’s exact test

The joint probability for the data is given by the multivariate hypergeometric distribution, \( n_i \sim HG(m_i, m_{tot}, n_{tot}) \), and the exact statistical test is Fisher’s exact test.

Denoting the contingency table elements \( a_{ij} \) (column-major form),

\[
\Pr \left\{ \{a_{ij}\} \right\} = \frac{\left( \prod_i \left( \sum_j a_{ij} \right)! \right) \left( \prod_j \left( \sum_i a_{ij} \right)! \right)}{\left( \sum_{i,j} a_{ij} \right)! \left( \prod_{i,j} a_{ij}! \right)} \tag{A.1}
\]

which in our case reduces to

\[
\Pr \left( n_i; m_i, m_{tot}, n_{tot} \right) = \frac{\left( \begin{array}{c} n_{tot} \\ n_i \end{array} \right) \left( \begin{array}{c} m_{tot} - n_{tot} \\ m_i - n_i \end{array} \right)}{\left( \begin{array}{c} m_{tot} \\ m_i \end{array} \right)} \tag{A.2}
\]

\[
= \frac{\left( \begin{array}{c} n_{tot} \\ n_i \end{array} \right) \left( \begin{array}{c} o_{tot} \\ o_i \end{array} \right)}{\left( \begin{array}{c} m_{tot} \\ m_i \end{array} \right)} \tag{A.3}
\]

\[
= \frac{\left( n_{tot}! o_{tot}! \right) \left( m_1! m_2! \right)}{\left( m_{tot}! n_1! n_2! o_1! o_2! \right)} \tag{A.4}
\]

A.2 Significance and confidence

We would like to compare the data generated by two such models, and we begin our analysis with the null hypothesis that the two models have identically distributed underlying rates. The null hypothesis is rejected if this is supported by observational evidence. That is, if the probability that the observed evidence combined from both models (assuming i.i.d. rates) is lower than some significance level (e.g. p-value lower than 5%), we can reject the null hypothesis.

A.2.1 Single-tailed test\(^7\)

If \( \sum_M \Pr \left( n_i^M; m_i^M, m_{tot}^M, n_{tot}^M \right) < p \), where \( M \) signifies the model, then the null hypothesis can be rejected on the grounds the evidence provided by the two models differ in a more extreme way than they would had the models been equivalent.

\(^7\)If the evidence provided by the two models is very lopsided, we should prefer a two-tailed test. For example, if only one of the models is the incumbent, and therefore we have vastly more observations for it.
A.2.2 Power analysis

Likelihood-ratio test Following the Neyman-Pearson lemma, which states that the most powerful significance \( \alpha \) level test (p-value) is the likelihood ratio test, we denote the likelihood (and log-likelihood)

\[
L (M; n_i^M, m_i^M, m^M_{\text{tot}}, n^M_{\text{tot}}) = \Pr (n_i^M; m_i^M, m^M_{\text{tot}}, n^M_{\text{tot}}) \tag{A.5}
\]

\[
\ell (M; n_i^M, m_i^M, m^M_{\text{tot}}, n^M_{\text{tot}}) = \ln \Pr (n_i^M; m_i^M, m^M_{\text{tot}}, n^M_{\text{tot}}) \tag{A.6}
\]

\[
= \ln \left( \frac{\Gamma (n^M_{\text{tot}}) \Gamma (o^M_{\text{tot}})}{\Gamma (m^M_{\text{tot}})} \prod_i \Gamma (m_i) \prod_i \Gamma (o_i) \right) \tag{A.7}
\]

\[
= \ln \left( \frac{\text{Beta} (n^M_{\text{tot}}, o^M_{\text{tot}})}{\prod_i \text{Beta} (n_i, o_i)} \right) \tag{A.8}
\]

\[
= \ln \text{Beta} (n^M_{\text{tot}}, o^M_{\text{tot}}) - \sum_i \ln \text{Beta} (n_i, o_i) \tag{A.9}
\]

Comparing models using Wilk's theorem Define the alternative hypothesis as the model with more degrees of freedom, \( \nu_D \equiv \nu_1 - \nu_0 \geq 0 \), and the test statistic \( D \)

\[
D = -2 \ln \Lambda = 2 [\ell (M = H_1) - \ell (M = H_0)] \tag{A.10}
\]

The probability distribution of \( D \) tends to a \( \chi^2_{\nu_D} \) distribution as the sample size tends to infinity.

The use of this theorem is in approximating the limit of the p-value for large sample sets, via the tabulated probability density distribution of \( \lim_{n^M_{\text{tot}} \to \infty} \Pr (D) = \chi^2_{\nu_D} (D) \).

Note that the test statistic here is chi-squared, which makes some assumptions on the distribution of the samples.

A non-parametric test: Kolmogorov-Smirnov For a compared quantity \( x \), we denote the empirical CDF (ECDF) of \( x \) (for example from a histogram of \( x \)) by \( F_K (x) \), where \( K \in \{ A, B \} \) and \( m_K \) is the number of impressions given to \( K \). The Kolmogorov-Smirnov statistic is

\[
D_{m_A, m_B} \equiv \sup_x |F_B (x) - F_A (x)| \tag{A.11}
\]

The null hypothesis (B is not different from A) is rejected at level \( \alpha \) if

\[
D_{m_A, m_B} > c (\alpha) \sqrt{\frac{\sum K m_K}{\prod K m_K}} \tag{A.12}
\]

\[
c (\alpha) \equiv \sqrt{-\frac{1}{2} \ln \left( \frac{\alpha}{2} \right)} \tag{A.13}
\]
B  Probabilistic comparison of the rates of two processes

The probability, given some observational data \((\alpha_A, \beta_A, \alpha_B, \beta_B)\) for two beta-binomial processes \(A\) and \(B\), that the underlying rate of process \(B\), \(\phi_B\), is higher than that of process \(A\), \(\phi_A\), is:

\[
\Pr(\phi_B > \phi_A | \alpha_A, \beta_A, \alpha_B, \beta_B) = \int_0^1 d\phi_A \Pr(\phi_A|\alpha_A, \beta_A) \int_0^1 d\phi_B \Pr(\phi_B|\alpha_B, \beta_B)
\]

\[
= \int_0^1 d\phi_A \int_{\phi_A}^1 d\phi_B \frac{\phi_A^{\alpha_A-1} (1 - \phi_A)^{\beta_A-1} \phi_B^{\alpha_B-1} (1 - \phi_B)^{\beta_B-1}}{B(\alpha_A, \beta_A) B(\alpha_B, \beta_B)}
\]

\[
= \int_0^1 d\phi_A \left[ \frac{\phi_A^{\alpha_A-1} (1 - \phi_A)^{\beta_A-1}}{B(\alpha_A, \beta_A)} \right] \int_{\phi_A}^1 d\phi_B \frac{\phi_B^{\alpha_B-1} (1 - \phi_B)^{\beta_B-1}}{B(\alpha_B, \beta_B)}
\]

\[
= 1 - \int_0^1 \frac{\phi_A^{\alpha_A-1} (1 - \phi_A)^{\beta_A-1}}{B(\alpha_A, \beta_A)} I_{\phi_A}(\alpha_B, \beta_B) d\phi_A
\]

where \(I_{\phi}(\alpha, \beta)\) is just shorthand for the regularized incomplete beta function. We next use a lemma (section §C) to simplify this expression to

\[
\Pr(\phi_B > \phi_A) = 1 - \int_0^1 \frac{\phi_A^{\alpha_A-1} (1 - \phi_A)^{\beta_A-1}}{B(\alpha_A, \beta_A)} \left( 1 - \sum_{i=0}^{\alpha_B-1} \frac{\phi_A^{i} (1 - \phi_A)^{\beta_B}}{(\beta_B + i) B(i + 1, \beta_B)} \right) d\phi_A
\]

\[
= 1 - 1 + \int_0^1 \frac{\phi_A^{\alpha_A-1} (1 - \phi_A)^{\beta_A-1}}{B(\alpha_A, \beta_A)} \sum_{i=0}^{\alpha_B-1} \frac{\phi_A^{i} (1 - \phi_A)^{\beta_B}}{(\beta_B + i) B(i + 1, \beta_B)} d\phi_A
\]

\[
= \sum_{i=0}^{\alpha_B-1} \int_0^1 \frac{\phi_A^{\alpha_A+i-1} (1 - \phi_A)^{\beta_A+\beta_B-1}}{(\beta_B + i) B(\alpha_A, \beta_A) B(i + 1, \beta_B)} d\phi_A
\]

\[
= \sum_{i=0}^{\alpha_B-1} \frac{1}{(\beta_B + i) B(\alpha_A, \beta_A) B(i + 1, \beta_B)} \int_0^1 \phi_A^{\alpha_A+i-1} (1 - \phi_A)^{\beta_A+\beta_B-1} d\phi_A
\]

\[
= \sum_{i=0}^{\alpha_B-1} \frac{B(\alpha_A + i, \beta_A + \beta_B)}{(\beta_B + i) B(\alpha_A, \beta_A) B(i + 1, \beta_B)} \int_0^1 \phi_A^{\alpha_A+i-1} (1 - \phi_A)^{\beta_A+\beta_B-1} d\phi_A
\]

where in the last line we multiplied and divided by \(B(\alpha_A + i, \beta_A + \beta_B)\), and then the integral term is just the integral of the distribution Beta \((\alpha_A + i, \beta_A + \beta_B)\), which is unity.
Re-indexing,

\[
\Pr(\phi_B > \phi_A|\alpha_A, \beta_A, \alpha_B, \beta_B) = \sum_{i=0}^{\alpha_B-1} \frac{B(\alpha_A + i, \beta_B + \beta_A)}{(\beta_B + i) B(1 + i, \beta_B) B(\alpha_A, \beta_A)} \quad (B.10)
\]

\[
= \frac{1}{B(\alpha_A, \beta_A)} \sum_{i=1}^{\alpha_B} \frac{B(\alpha_A - 1 + i, \beta_B + \beta_A)}{(\beta_B - 1 + i) B(i, \beta_B)} \quad (B.11)
\]

Chris Stucchio has published an asymptotic analysis of this formula[5].

C Lemma for the regularized incomplete beta function

Recursively iterating

\[
I_x(\alpha, \beta) = I_x(\alpha - 1, \beta) - \frac{x^{\alpha-1} (1 - x)^{\beta}}{(\alpha - 1) B(\alpha - 1, \beta)} \quad (C.1)
\]

Until the base case

\[
I_x(1, \beta) = 1 - (1 - x)^{\beta} \quad (C.2)
\]

We get

\[
I_x(\alpha, \beta) = 1 - (1 - x)^{\beta} - \sum_{i=1}^{\alpha-1} \frac{x^{\alpha-1} (1 - x)^{\beta}}{(\alpha - i) B(\alpha - i, \beta)} \quad (C.3)
\]

Subsuming the zeroth term into the sum:

\[
I_x(\alpha, \beta) = 1 - \sum_{i=0}^{\alpha-1} \frac{x^i (1 - x)^{\beta}}{B(\beta + i) B(1 + i, \beta)} \quad (C.4)
\]

D Solution of Euler’s hypergeometric integral

Starting from

\[
\Pr(\phi_B > \gamma \phi_A) = \gamma^{-\alpha_A} \int_0^1 d\phi_A' \int_{\phi_A'}^1 d\phi_B \frac{\phi_A'^{\alpha_A-1} (1 - \phi_A'/\gamma)^{\beta_{A'}-1}}{B(\alpha_A, \beta_A)} \frac{\phi_B^{\alpha_B-1} (1 - \phi_B)^{\beta_B-1}}{B(\alpha_B, \beta_B)} \quad (D.1)
\]

we now employ the (exact) binomial expansion:
\[(1 - \phi_{A'})^{\beta_A - 1} = \sum_{k=0}^{\beta_A - 1} \left( -\frac{\beta_A - 1}{k} \right) \sum_{k=0}^{\beta_A - 1} \left( -\frac{\beta_A - 1}{k} \right) (-1)^k \phi_{A'}^{\gamma-k} \]  
(D.2)

so:

\[
\Pr (\phi_B > \gamma\phi_A) = \gamma^{-\alpha_A} \int_0^1 d\phi_{A'} \int_0^1 d\phi_B \frac{\phi_{A'}^{\alpha_A - 1}}{B(\alpha_A, \beta_A)} \sum_{k=0}^{\beta_A - 1} (-1)^k \left( -\frac{\beta_A - 1}{k} \right) \phi_{A'}^{\gamma-k} \phi_{B}^{\beta_B - 1} \left( 1 - \phi_B \right)^{\beta_B - 1} \frac{B(\alpha_B, \beta_B)}{B(\alpha_B, \beta_B)} \]  
(D.3)

\[
\Pr (\phi_B > \gamma\phi_A) = \gamma^{-\alpha_A} \int_0^1 d\phi_{A'} \int_0^1 d\phi_B \sum_{k=0}^{\beta_A - 1} (-1)^k \left( -\frac{\beta_A - 1}{k} \right) \gamma^{-k - \alpha_A} \frac{\phi_{A'}^{\alpha_A + k - 1}}{B(\alpha_A, \beta_A)} \int_0^1 d\phi_{A'} \phi_{A'}^{\alpha_A + k - 1} [1 - I_{\phi_{A'}}(\alpha_B, \beta_B)] \]  
(D.4)

using the same lemma from section §C,

\[
I_{\phi_{A'}}(\alpha_B, \beta_B) = 1 - \sum_{i=0}^{\beta_B - 1} \frac{\phi_{A'}^{\beta_B} (1 - \phi_{A'})^{\beta_B}}{(\beta_B + i) B(1 + i, \beta_B)} \]  
(D.5)

we get

\[
\Pr (\phi_B > \gamma\phi_A) = \sum_{k=0}^{\beta_A - 1} (-1)^k \left( -\frac{\beta_A - 1}{k} \right) \gamma^{-k - \alpha_A} \frac{\phi_{A'}^{\alpha_A + k - 1}}{B(\alpha_A, \beta_A)} \int_0^1 d\phi_{A'} \phi_{A'}^{\alpha_A + k - 1} \left[ 1 - 1 + \sum_{i=0}^{\beta_B - 1} \frac{\phi_{A'}^{\beta_B} (1 - \phi_{A'})^{\beta_B}}{(\beta_B + i) B(1 + i, \beta_B)} \right] \]  
(D.6)

\[
\Pr (\phi_B > \gamma\phi_A) = \sum_{k=0}^{\beta_A - 1} \sum_{i=0}^{\beta_B - 1} (-1)^k \left( -\frac{\beta_A - 1}{k} \right) \gamma^{-k - \alpha_A} \frac{\phi_{A'}^{\alpha_A + k + i - 1}}{B(\beta_B + i) B(1 + i, \beta_B)} \]  
(D.7)

we define \( \alpha_{A'} \equiv \alpha_A + k + i \) and \( \beta_{B'} \equiv \beta_B + 1 \)
\[
\Pr(\phi_B > \gamma \phi_A) = (D.11)
\]
\[
= \sum_{k=0}^{\beta_A - 1} \sum_{i=0}^{\alpha_B - 1} (-1)^k \binom{\beta_A - 1}{k} \frac{\gamma^{-k - \alpha_A}}{(\beta_B + i) B(1 + i, \beta_B) B(\alpha_A, \beta_A)} \int_0^1 d\phi_A' \phi_A'^{\alpha_A' - 1} (1 - \phi_A')^{\beta_A' - 1}
\]
\[
= \sum_{k=0}^{\beta_A - 1} \sum_{i=0}^{\alpha_B - 1} (-1)^k \binom{\beta_A - 1}{k} \frac{\gamma^{-k - \alpha_A} B(\alpha_A', \beta_B')}{(\beta_B + i) B(1 + i, \beta_B) B(\alpha_A, \beta_A)}
\]
\[
\text{where in the last step we multiplied and divided by } B(\alpha_A', \beta_B') \text{ and integrated over the entire Beta}(\alpha_A', \beta_B') \text{ distribution to get unity.}
\]
\[
\Pr(\phi_B > \gamma \phi_A) = (D.14)
\]
\[
= \frac{\gamma^{-\alpha_A}}{B(\alpha_A, \beta_A)} \sum_{i=0}^{\alpha_B - 1} \frac{1}{(\beta_B + i) B(1 + i, \beta_B)} \sum_{k=0}^{\beta_A - 1} (-\gamma)^{-k} \binom{\beta_A - 1}{k} B(\alpha_A + i + k, \beta_B + 1)
\]
\[
\text{(D.15)}
\]

Using a definition of the Gauss hypergeometric series, detailed in the appendix section §E, we identify the second sum as proportional to the Gauss hypergeometric function

\[
\Pr(\phi_B > \gamma \phi_A) = (D.16)
\]
\[
= \frac{\gamma^{-\alpha_A}}{B(\alpha_A, \beta_A)} \sum_{i=0}^{\alpha_B - 1} \frac{B(\alpha_A + i, \beta_B + 1)}{(\beta_B + i) B(1 + i, \beta_B)} \, _2F_1\left(1 - \beta_A, \alpha_A + i; \alpha_A + i + \beta_B + 1; \gamma^{-1}\right)
\]
\[
\text{(E.1)}
\]

**E Hypergeometric series**

The hypergeometric function is defined for $|z| < 1$ by the power series

\[
_2F_1\left(x_1, x_2; y; z\right) = \sum_{n=0}^{\infty} \frac{(x_1)_n^+ (x_2)_n^+ z^n}{(y)_n^+ n!}
\]
\[
\text{(E.1)}
\]

where $(q)_n^+$ is the rising factorial or Pochhammer symbol (written to avoid confusion with $(q)_n$ which also refers to the falling factorial)

\[
(q)_n = \frac{\Gamma(q + n)}{\Gamma(q)}
\]
\[
\text{(E.2)}
\]

Using the following series expansion for a hypergeometric function with a non-positive integer parameter:

\[
\sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(x_2)_n^+}{(y)_n^+} z^n = _2F_1\left(-m, x_2; y; z\right)
\]
\[
\text{(E.3)}
\]
in our case, \((m = \beta_A - 1, x_2 = \alpha_A + i; y = \alpha_A + i + \beta_B + 1; z = \gamma^{-1})\), and we will introduce the following variables for convenience: \(a \equiv \alpha_A + i\) and \(b \equiv \beta_B + 1\).

\[
_2F_1\left(- (\beta_A - 1), a; a + b; \gamma^{-1}\right) = \sum_{k=0}^{\beta_A-1} (-1)^k \binom{\beta_A - 1}{k} \frac{(a)^+_k}{(a + b)_k^+} \gamma^{-k} \quad (E.4)
\]

\[
= \sum_{k=0}^{\beta_A-1} (-\gamma)^{-k} \binom{\beta_A - 1}{k} \frac{\Gamma(a + k)}{\Gamma(a)} \frac{\Gamma(a + b + k)}{\Gamma(a + b + k)} \quad (E.5)
\]

\[
= \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{k=0}^{\beta_A-1} (-\gamma)^{-k} \binom{\beta_A - 1}{k} \frac{\Gamma(a + k)}{\Gamma(a + b + k)} \quad (E.6)
\]

\[
= \frac{\Gamma(b)}{\Gamma(a + b)} \sum_{k=0}^{\beta_A-1} (-\gamma)^{-k} \binom{\beta_A - 1}{k} \frac{\Gamma(a + k)}{\Gamma(b)} \quad (E.7)
\]

\[
= \frac{1}{\Gamma(a + b)} \sum_{k=0}^{\beta_A-1} (-\gamma)^{-k} \binom{\beta_A - 1}{k} \frac{\Gamma(a + k)}{\Gamma(b)} \quad (E.8)
\]

F Evaluation of the hypergeometric function using a Jacobi polynomial

We start from the following identity

\[
_2F_1\left(-m, m + x + 1 + y; x + 1; z\right) = \frac{m!}{(x + 1)_m^+} P_m^{(x,y)}(1 - 2z) \quad (F.1)
\]

Next, we use the following variable transformations to recover the hypergeometric function in the form we used above

\[
a = -m \Rightarrow m = -a
\]

\[
c = y + 1 \Rightarrow y = c - 1
\]

\[
b = m + x + 1 + y \Rightarrow x = b - m - y - 1 = b + a - c + 1 - 1 = b + a - c
\]

and now we have

\[
_2F_1\left(a, b; c; z\right) = \frac{(-a)!}{(c)^+_{-a}} P_{-a}^{(c-1,b+a-c)}(1 - 2z) \quad (F.2)
\]

\[
= \frac{\Gamma(-a)}{\Gamma(c - a) / \Gamma(c)} P_{-a}^{(c-1,b+a-c)}(1 - 2z) \quad (F.3)
\]

\[
= B(c, a) P_{-a}^{(c-1,b+a-c)}(1 - 2z) \quad (F.4)
\]
for $2F_1\left(1 - \beta_A, a; a + \beta_B + 1; \gamma^{-1}\right)$ we find the following variable identities:

\begin{align*}
z &= \gamma^{-1} \\
m &= \beta_A - 1 \\
x + 1 &= a + \beta_B + 1 \Rightarrow x = a + \beta_B \\
m + x + 1 + y &= a \Rightarrow y = a - m - x - 1 = a - \beta_A + 1 - a - \beta_B + 1 = \beta_B - \beta_A + 2
\end{align*}

And finally we have

\begin{align*}
2F_1\left(1 - \beta_A, a; a + \beta_B + 1; \gamma^{-1}\right) &= \frac{(\beta_A - 1)!}{(\beta_B - \beta_A + 1)^{\beta_A} - 1} P_{\beta_A}^{(a + \beta_B, \beta_B - \beta_A + 2)} (1 - 2\gamma^{-1}) \\
&= \Gamma(\beta_A - 1) \frac{\Gamma(\beta_B - \beta_A + 1)}{\Gamma(\beta_B - \beta_A + 1 + \beta_A - 1)} P_{\beta_A}^{(a + \beta_B, \beta_B - \beta_A + 2)} (1 - 2\gamma^{-1}) \quad \text{(F.6)} \\
&= B(\beta_B - \beta_A + 1, \beta_A - 1) P_{\beta_A}^{(a + \beta_B, \beta_B - \beta_A + 2)} (1 - 2\gamma^{-1}) \quad \text{(F.7)}
\end{align*}

**G Sequential frequentist approach**[6]

The key insight in Ben Tilly’s article[7] is that if users are randomly assigned to two groups, and the two groups have the same conversion rate, then the sequence of successes from the two groups is mathematically equivalent to a series of random coin flips.

The following procedure is derived from the analysis of the gambler’s ruin problem for this one-dimensional random walk,

**Algorithm 1** Simple sequential A/B testing (Evan Miller)

1. At the beginning of the experiment, choose a sample size $N$.
2. Assign subjects randomly to the treatment and control, with 50% probability each.
3. Track the number of incoming successes from the treatment group. Call this number $T \equiv n^M_{tot}$.
4. Track the number of incoming successes from the control group. Call this number $C \equiv n_{tot} - n^M_{tot}$.
5. If $d \equiv T - C = n^M_{tot} - (n_{tot} - n^M_{tot}) = 2n^M_{tot} - n_{tot}$ reaches $2\sqrt{N}$, stop the test. Declare the treatment to be the winner.
6. If $n_{tot} = T + C$ reaches $N$, stop the test. Declare no winner.
A reference to the proof for step 5 is given in the original post.

Samples should be i.i.d. between the models (C and T). Ideally, online testing of T vs C should be done on mutually exclusive sets to avoid effects of interactions between C and T.

### G.1 Power and significance

Given a model with $n^M_{tot}$ total wins such that $n_{tot} = n^M_{tot} + (n^M_{tot} - d^*_M)$ and that the sum of the wins by both models is $n_{tot}$.

\[
\alpha > \sum_{j=1}^{n_{tot}} \frac{n^M_{tot}}{j} \left( \frac{j}{(d^*_M + j)/2} \right)^2 \quad \text{(G.1)}
\]

\[
\beta > 1 - \sum_{j=1}^{n_{tot}} \frac{n^M_{tot}}{j} \left( \frac{j}{(d^*_M + j)/2} \right) \left( \frac{1}{2 + \delta_M} \right)^{(j-d^*_M)/2} \left( \frac{1 + \delta_M}{2 + \delta_M} \right)^{(j+d^*_M)/2} \quad \text{(G.2)}
\]

\[
= 1 - \sum_{j=1}^{n_{tot}} \frac{n^M_{tot}}{j} \left( \frac{j}{(d^*_M + j)/2} \right) (2 + \delta_M)^{-j} (1 + \delta_M)^{(d^*_M+j)/2} \quad \text{(G.3)}
\]

where $\delta_M = n^M_{tot} / (n_{tot} - n^M_{tot})$ is the lift.

For example, for $\alpha = 5\%$, $\beta = 20\%$ and $\delta = 50\%$, we get $n_{tot} = 170$ and $d^*_M = 26$.

So, to see if we can get 50% lift with a p-value of 5% and 80% power, we should look for a 26 win margin in favor of the treatment, or give up if we reach 170 overall wins.

### H Numerical comparison of Bayesian vs. frequentist calculations

```python
from time import perf_counter
from numpy import exp, log, mean, nan, reciprocal
from numpy.random import beta, random
from scipy.special import betaln, binom, hyp2f1

def frequentist(alpha_a, beta_a, alpha_b, beta_b, gamma, n):
    return mean(beta(alpha_b, beta_b, size=n) > gamma * beta(alpha_a, beta_a, size=n))

def pr_b_gt_pr_ga(alpha_a, beta_a, alpha_b, beta_b, gamma):
    assert gamma > 1
```

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result = 0
m = beta_a - 1
b = beta_b + 1
z = 1 / gamma
c = -alpha_a * log(gamma) - betaln(alpha_a, beta_a)

for i in range(alpha_b):
    a = alpha_a + i
    s = betaln(a, b) - betaln(i + 1, beta_b) - log(beta_b + i)
    f = log(hyp2f1(-m, a, a + b, z))
    result += exp(c + s + f)
return result

def main():
    i, j = 0, 0
    hg_times, freq_times = 0, 0

    while j < 10:
        alpha_a, beta_a, alpha_b, beta_b, gamma = map(lambda x: int(x + 1), reciprocal(random(5)))
        i += 1
        if alpha_a > beta_a or alpha_b > beta_b:
            continue
        start = perf_counter()
        hg = pr_b_gt_pr_ga(alpha_a, beta_a, alpha_b, beta_b, gamma)
        stop = perf_counter()
        hg_times += stop - start
        if hg is None or hg == nan:
            continue
        print("hg: ", hg, stop - start)
        start = perf_counter()
        freq = frequentist(alpha_a, beta_a, alpha_b, beta_b, gamma, pow(10, 7))
        stop = perf_counter()
        freq_times += stop - start
        print("freq ", freq, stop - start)
        j += 1
    print(hg_times / j, freq_times / j)

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