Spatial mixing and the random-cluster dynamics on lattices

Reza Gheissari1 | Alistair Sinclair2

1Department of Mathematics, Northwestern University, Evanston, Illinois, USA
2Computer Science Division, UC Berkeley, Berkeley, California, USA

Correspondence
Reza Gheissari, Department of Mathematics, Northwestern University, Evanston, IL, USA.
Email: gheissari@northwestern.edu

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Abstract
An important paradigm in the understanding of mixing times of Glauber dynamics for spin systems is the correspondence between spatial mixing properties of the models and bounds on the mixing time of the dynamics. This includes, in particular, the classical notions of weak and strong spatial mixing, which have been used to show the best known mixing time bounds in the high-temperature regime for the Glauber dynamics for the Ising and Potts models. Glauber dynamics for the random-cluster model does not naturally fit into this spin systems framework because its transition rules are not local. In this article, we present various implications between weak spatial mixing, strong spatial mixing, and the newer notion of spatial mixing within a phase, and mixing time bounds for the random-cluster dynamics in finite subsets of $\mathbb{Z}^d$ for general $d \geq 2$. These imply a host of new results, including optimal $O(N \log N)$ mixing for the random cluster dynamics on torii and boxes on $N$ vertices in $\mathbb{Z}^d$ at all high temperatures and at sufficiently low temperatures, and for large values of $q$ quasi-polynomial (or quasi-linear when $d = 2$) mixing time bounds from random phase initializations on torii at the critical point (where by contrast the mixing time from worst-case initializations is exponentially large). In the same parameter regimes, these results translate to fast sampling algorithms for the Potts model on $\mathbb{Z}^d$ for general $d$.

KEYWORDS
Glauber dynamics, metastability, mixing time, phase coexistence, random-cluster model, spatial mixing

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1 | INTRODUCTION

Let $G = (V, E)$ be a finite graph. Configurations of the random-cluster model on $G$ are sets of edges $\omega \subseteq E$, with an associated probability given by the Gibbs distribution:

$$\pi_{G,p,q}(\omega) = \frac{1}{Z_{G,p,q}} p^{\|\omega\|} (1 - p)^{|E| - |\omega|} q^{\text{Comp}(\omega)},$$  

where $\text{Comp}(\omega)$ is the set of connected components in the subgraph $(V, \omega)$ and the normalizing factor $Z_{G,p,q}$ is known as the partition function. Here $p \in [0, 1]$ and $q \in [1, \infty)$ are parameters. Note that the distribution (1.1) generalizes the standard Erdős-Rényi random subgraph model (or edge percolation model) on $G$, with the inclusion of the extra factor $q^{\text{Comp}(\omega)}$. This factor assigns higher weight to configurations with more connected components, and this bias increases with $q$.

The random-cluster model was introduced in the late 1960s by Fortuin and Kasteleyn [20] as a unifying framework for studying percolation, spin systems in statistical physics and random spanning trees; see the book [26] for extensive background. The connection with spin systems is via the $q$-state Potts model, whose configurations are assignments $\sigma \in \{1, \ldots, q\}^V$ of one of $q$ possible spins, or colors, to each vertex of $G$, with associated Gibbs distribution.

$$\pi_{G,p,q}^\text{Potts}(\sigma) = \frac{1}{Z_{G,p,q}^\text{Potts}} \exp(-\beta |\text{Cut}(\sigma)|),$$  

where $\beta \geq 0$ is a parameter, $\text{Cut}(\sigma)$ is the set of edges connecting different spins (i.e., the edges in the multi-way cut induced by $\sigma$), and $Z_{G,p,q}^\text{Potts}$ is the partition function. The case of two spins ($q = 2$) is the Ising model. For integer $q$, under the correspondence $p = 1 - \exp(-\beta)$, the random-cluster and Potts models are intimately related, in the sense that both are marginals of the same joint distribution on edges and spins (the so-called Fortuin–Kasteleyn (FK) distribution [19]). Specifically, if $\sigma$ is chosen from the Potts distribution (1.2), and we include in $\omega$ the edges connecting equal spins in $\sigma$ independently with probability $p = 1 - \exp(-\beta)$, then $\omega$ is distributed according to the random-cluster distribution (1.1); and conversely, if we pick $\omega$ from (1.1) and assign spins $\{1, \ldots, q\}$ independently and u.a.r. to each connected component in $(V, \omega)$, then the resulting spin configuration is distributed according to (1.2). However, the random-cluster model is more general in that, unlike the Potts model, it is defined even when $q$ is not an integer.

In this article, we focus for definiteness on the classical setting where $G = \Lambda_n$ is a box of side-length $n$ in the $d$-dimensional lattice $\mathbb{Z}^d$. In this setting, the random-cluster model undergoes a phase transition at a certain critical value $p = p_c(q, d)$: namely, as $n \to \infty$, if $p > p_c$ then the configuration $\omega$ has w.h.p. a giant component (of size linear in the number of vertices $N = n^d$), while if $p < p_c$ the largest component is w.h.p. of size only $O(\log N)$ [16]. For integer $q \geq 2$, an analogous phase transition occurs in the Potts model: for $\beta > \beta_c(q, d) = -\ln(1 - p_c(q, d))$, the Gibbs distribution (1.2) becomes multi-modal, with each of the $q$ modes corresponding to a phase in which the configuration is dominated by one of the spins, while for $\beta < \beta_c$ correlations between spins decay exponentially with distance and there is no long-range order. The regimes $\beta < \beta_c$ and $\beta > \beta_c$ are referred to as the high-temperature and low-temperature regimes, respectively, because of the interpretation of $\beta$ in statistical physics as an “inverse temperature” parameter. With slight abuse of terminology, we shall use

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1 The random-cluster model is actually defined for all positive real $q$; the case of $q \in (0, 1)$ is interesting in its own right, but is phenomenologically very different from $q \geq 1$ and thus beyond the scope of the current paper.

2 For $q = 1$, this is just the well-known phenomenon of emergence of a giant component in Bernoulli percolation.
the same phrases to refer to the analogous regimes $p < p_c$ and $p > p_c$ in the random-cluster model. For a more precise formulation of this phase transition, see Section 2.

We study the natural local dynamics for the random-cluster model, analogous to the widely studied Glauber dynamics for the Potts model that updates a random spin at each step conditional on the spins of its neighbors. This random-cluster dynamics, which we abbreviate to FK dynamics, at each step picks a random edge $e \in E$ and includes $e$ in the current configuration $\omega$ with the correct conditional probability given $\omega \setminus e$. Thus, if the configuration at time $t$ is $\omega$, then the configuration at time $t + 1$ is obtained as follows:

1. Pick an edge $e \in E$ uniformly at random.
2. Include $e$ in the new configuration with probability

$$
\begin{align*}
\begin{cases}
\frac{p}{q(1-p)+p} & \text{if } e \text{ is a bridge in } \omega \cup \{e\} \text{(i.e., if } |\text{Comp}(\omega \setminus \{e\})| \neq |\text{Comp}(\omega \cup \{e\})|); \\
p & \text{otherwise.}
\end{cases}
\end{align*}
$$

(1.3)

Since this dynamics is irreducible and reversible w.r.t. the Gibbs distribution $\pi$ in (1.1), it converges to this distribution from any initial configuration. As in all such investigations, the main question is the mixing time of the chain, that is, the number of steps until it gets close in total variation distance to $\pi$ from a worst-case initialization. This is relevant both for bounding the running time of a MCMC algorithm that samples configurations from $\pi$, and for understanding the actual behavior of the Markov chain over time.

The analysis of Glauber dynamics has a long and rich history, and the dynamics of the Ising model (and to a lesser extent the Potts model) are by now quite well understood. By comparison, the study of the FK dynamics is less well developed and introduces new technical challenges, mainly due to the fact that the update rule (1.3) necessarily depends on the number of connected components, $|\text{Comp}(\omega)|$, a non-local quantity. Many of the most important results for the Ising/Potts Glauber dynamics on $\mathbb{Z}^d$ relate the mixing time to some form of spatial mixing (i.e., decay of correlations with distance in the Gibbs distribution). Our goal in this article is to overcome the non-locality and develop analogous implications for the FK dynamics. In the following paragraphs we explain each of these implications, along with their consequences for the mixing time of the FK dynamics and sampling from the Potts model in various parameter regimes.

### 1.1 Weak spatial mixing

The most basic notion of spatial mixing is weak spatial mixing (WSM), which expresses the fact that in $\mathbb{Z}^d$, or on a torus $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$, when the status of the edges in some subset $A$ of the configuration are changed, the effect on the distribution of the edges in some other set $B$ decays exponentially with the distance between $A$ and $B$. (For a precise formulation, see Definition 3.1.)

The following theorem says that, whenever the random-cluster model has WSM, the FK dynamics on $\mathbb{T}_n$ has very fast (actually, optimal) mixing time. In general, WSM is believed to be the weakest condition under which one can hope to have optimal mixing [18].

**Theorem 1.1.** Suppose $d \geq 2$, $q \geq 1$, $p$ are such that WSM holds for the random-cluster model. Then the mixing time of the FK dynamics on $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$ is $O(N \log N)$, where $N = n^d$ is the number of vertices.
This implication is the analog of a similar one for Glauber dynamics for the Ising model (the $q = 2$ case of the Potts model), which is a landmark result in the field [34]. Indeed, Harel and Spinka [28] recently observed, towards a different purpose, that the proof technique of [34] in fact extends to more general monotone models than the Ising model, including the random-cluster model.\(^3\) Thus Theorem 1.1 can be deduced fairly easily from the arguments in [28], but we include it here because the important implication for mixing times in finite volumes is not explicitly derived there.

When $d = 2$, WSM is known to hold for the random-cluster model at all non-critical $p$ for all $q$ from [3] together with planar duality. For general $d \geq 3$, WSM was recently shown throughout the high-temperature regime $p < p_c(q, d)$ for all $q$ [16], and also throughout the low-temperature regime $p > p_c(2, d)$ for the special case $q = 2$ [15]. WSM can also easily be seen to hold for all $q$ and $d$ provided the temperature is low enough, that is, $p > p_0(q, d)$ is sufficiently large (see Section 6.1). In fact, WSM is conjectured to hold at all non-critical temperatures $p \neq p_c(q, d)$ for all $q$ and $d$ [15, Section 1.4].

Combining these facts with Theorem 1.1, we immediately obtain the following.

**Corollary 1.1.** For all $d \geq 2$, the FK dynamics on $\mathbb{T}_n$ mixes in $O(N \log N)$ time in the following cases:

(i) Throughout the high-temperature regime $p < p_c(q, d)$ for all $q \geq 1$.

(ii) Throughout the low-temperature regime $p > p_c(2, d)$ for $q = 2$.

(iii) At all sufficiently low temperatures $p > p_0(q, d)$ for all $q \geq 1$.

Moreover, as mentioned above, if as conjectured WSM holds at all $p \neq p_c(q, d)$, then Theorem 1.1 will immediately imply optimal mixing time on $\mathbb{T}_n$ for all non-critical $p$ for all $q$ and $d$.

Corollary 1.1 has interesting implications for sampling algorithms, not only for the random-cluster model but also, in light of the intimate connection with the Potts model, for that as well: see Section 1.2.1.

### 1.2 Strong spatial mixing

Another widely used notion of spatial mixing is known as **strong spatial mixing (SSM)**; this notion is more adapted to finite domains such as $\Lambda_n$, which we recall is a box of side-length $n$ in $\mathbb{Z}^d$. SSM captures the same correlation decay property between two subsets $A$ and $B$ as WSM but in the context of a fixed environment, called a **boundary condition**, which we can understand to be a fixed configuration of edges (or spins) $\xi$ on the exterior of the region $\Lambda_n$; we use the notation $\Lambda_n^\xi$ to denote the presence of boundary condition $\xi$. (See Section 2.1.1 for a precise formulation.) Notice that SSM is in general a stronger notion than WSM since the fixed boundary edges $\xi$ may be very close to the subset $B$ (much closer than the edges $A$ whose status is being varied), which may have the effect of increasing the correlation between $A$ and $B$. We emphasize that this discussion includes the case of a **free** boundary (i.e., no fixed edges/spins) which already causes the results of the previous subsection to break down because the transitivity present in $\mathbb{T}_n$ is lost. Indeed any treatment of dynamics in the finite region $\Lambda_n$ must address the boundary condition.

In the classical literature concerning the Ising and Potts Glauber dynamics, the SSM property *uniformly over all possible boundary conditions* is the standard criterion that implies rapid mixing on $\Lambda_n$, see for example, [12, 34, 35]. Such results generally require uniformity of SSM over boundary conditions because they employ a recursive argument in which one loses control of the specific boundary

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\(^3\)The results of [28, 34] are actually formulated for the infinite lattice $\mathbb{Z}^d$ rather than for the torus $\mathbb{T}_n$. 

conditions present at subsequent levels of the recursion. With the exception of the Ising model [13], and the Potts model in two dimensions [3, 36], where uniform SSM holds throughout the high-temperature regime, this uniform SSM property is believed to break down for $q \geq 3$ in dimensions $d \geq 3$ at some temperature strictly above the critical one.

For the random-cluster model, one cannot expect uniform SSM to hold because boundary conditions can enforce long-range correlations even at very high temperatures (and general boundary conditions can even cause exponentially large mixing times [5]). On the other hand, certain special classes of boundary conditions may exhibit SSM and also be amenable to proving rapid mixing at $p \neq p_c(q, d)$: this approach was taken by [6] in $d = 2$ for the class of side-homogeneous boundary conditions (i.e., all edges present or all absent on each side of $\Lambda_n$), for which SSM followed from planar duality and [2]. Cutoff was then shown in $d \geq 2$ at very high temperatures $p \ll p_c$ [21]. However, even SSM for side-homogeneous boundaries is expected to break down in $d > 2$ and $q \neq 2$ at temperatures $p < p_c$ sufficiently close to the critical point.

In this article, we prove that SSM for the random-cluster model with respect to any fixed boundary condition $\xi$ is enough to imply rapid mixing of the FK dynamics on $\Lambda_n^\xi$. To the best of our knowledge, such boundary-specific implications were not previously known even for the Ising/Potts dynamics. (We note that our argument actually adapts easily to prove the same implication for the Ising Glauber dynamics as well, with potentially interesting implications. See Remark 3.2.)

**Theorem 1.2.** Suppose $d \geq 2$, $q \geq 1$, $p$ are such that SSM holds for the random-cluster model on $\Lambda_n^\xi$. Then the mixing time of the FK dynamics on $\Lambda_n^\xi$ is $O(N \log N)$.

Theorem 1.2 is useful because certain natural boundary conditions can be shown to satisfy SSM all the way up to the critical point, thus implying rapid mixing of the FK dynamics in a largest possible range. The two most canonical boundary conditions to consider are the free condition (denoted $0$) and the wired condition (denoted $1$), corresponding to all edges in $\mathbb{Z}^d \setminus \Lambda_n$ being absent and present, respectively. We are able to prove SSM for these boundary conditions throughout the high-temperature regime, as well as at sufficiently low temperatures, leading to the following corollary of Theorem 1.2.

**Corollary 1.2.** For all $d \geq 2$, the mixing time of the FK dynamics is $O(N \log N)$ in the following settings:

(i) On $\Lambda_n^0$ throughout the high-temperature regime $p < p_c(q, d)$, for all $q \geq 1$.

(ii) On $\Lambda_n^1$ and $\Lambda_n^0$ at all sufficiently low temperatures $p > p_0(q, d)$, for all $q \geq 1$.

**Remark 1.1.** We emphasize that Corollary 1.2 applies equally to any other class of boundary conditions for which one can prove the SSM property. To quote just one further example, we mention “cylindrical” boundary conditions, which are toroidal in a subset of the coordinate directions and all-free or all-wired in the others. See Remark 6.2 for a proof of SSM for these boundaries.

### 1.2.1 Implications for sampling from the Potts model

Corollaries 1.1 and 1.2 have novel implications for the existence of sampling algorithms. Of course, they immediately imply the existence of an efficient sampling algorithm from the random-cluster distribution (1.1) in all the specified regimes. But also, in light of the intimate connection between the Potts and random-cluster models, they lead to near-optimal sampling algorithms for the Potts distribution (1.2) in several new parameter regimes.
Corollary 1.3. For all $d \geq 2$, there is an $O(N \log^3 N)$ time algorithm for sampling from the Gibbs distribution of the $q$-state Potts model in the following scenarios:

(i) On $T_n = (\mathbb{Z}/n\mathbb{Z})^d$ in all the parameter regimes specified in Corollary 1.1.

(ii) On $\Lambda_n^0$ (i.e., no spins are assigned at the boundary) at all high temperatures $\beta < \beta_c(q, d)$, for all $q \geq 1$.

(iii) On $\Lambda_n^k$ (i.e., all boundary spins have color $k$) at low enough temperatures $\beta > \beta_0(q, d)$, for all $q \geq 1$.

Note that, once we sample a random-cluster configuration, we can easily convert it to a Potts configuration using the mechanism described earlier in the introduction. The extra factor of $O(N \log N)$ over the mixing time in Corollary 1.1 comes from the time needed to implement each step of the FK dynamics, which requires dynamic maintenance of the connected components of the configuration [42].

In dimensions $d \geq 3$, Corollary 1.3 goes well beyond what was previously known. In particular, it covers the entire high-temperature regime for all $q$, not only on the torus but also on $\Lambda_n^0$, the region $\Lambda_n$ with free boundary condition. Previously no polynomial time sampling algorithm for the Potts model with $q \geq 3$ was known on $T_n$ or $\Lambda_n^0$, except at sufficiently high temperatures that the stronger property of SSM uniformly over all boundary conditions holds [35], or when $q$ is sufficiently large as a function of $d$ [9]. Furthermore, in the latter case the polynomial is of high degree, in contrast to the near-optimal runtime of Corollary 1.3. We also mention [8] which proves rapid mixing of the Potts Glauber dynamics on general graphs of maximum degree $\Delta$ up to an explicit threshold $\beta \leq \frac{1+o_q(1)}{\Delta - 1} \log q$, that asymptotically in $q$, $\Delta$ is roughly half of the critical $\beta_c$ on the lattice of corresponding degree, that is, $\mathbb{Z}^{\Delta/2}$.

Corollary 1.3 also applies at sufficiently low temperatures for all $q$, both on the torus and on $\Lambda_n$ with free or monochromatic boundary condition. (Moreover, subject to the conjecture mentioned earlier that WSM holds at all off-critical temperatures, the restriction to large $\beta$ is not needed, and our results would then cover all parameter regimes where $O(N \log N)$ mixing time of FK dynamics is to be expected.)

In the low-temperature regime, polynomial time samplers were available for the Ising model ($q = 2$) [27, 31, 40] on arbitrary graphs, but they are of fairly high degree. For the Potts model with $q \geq 3$ on $T_n$ and $\Lambda_n^0$, [9, 29] gave the first polynomial time samplers (again of high degree) in regimes where a suitable cluster expansion converges; this holds for all $q$ at sufficiently low temperatures, and at all temperatures when $q$ is sufficiently large as a function of $d$, but is known to fail at moderate $q$ and $p > p_c(q, d)$.

Remark 1.2. We conclude this discussion of implications for sampling from the Potts model with a mention of the Swendsen–Wang (SW) dynamics, a commonly used global Markov chain that moves on Ising/Potts configurations using the random-cluster representation. In [41], comparison results between the SW and random-cluster dynamics showed that their mixing times are within a factor of $O(N)$ of one another. Like the FK dynamics, the analysis of SW is complicated by its non-locality, and for $d \geq 3$ fast mixing on $T_n$ and $\Lambda_n$ were only known on $q = 2$ [27] and for general $q$ at sufficiently high temperatures that uniform SSM holds for the Potts model [4]. By [41], our results imply that under any of the scenarios (i)–(iii) of Corollary 1.3, the Swendsen–Wang dynamics has $O(N^2 \log N)$ mixing time. While even polynomial mixing time was not previously known throughout these regimes, the true mixing time should be $\Theta(\log N)$. 


1.3 Weak spatial mixing within a phase

As we have mentioned, for the Ising/Potts Glauber dynamics, the notions of WSM and SSM have been the dominant tools for obtaining mixing time bounds at high temperatures $\beta < \beta_c$; as soon as $\beta \geq \beta_c$ both these properties break down, and indeed their mixing times on $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$ become exponentially slow at all $\beta > \beta_c(d)$ [7, 39]. The slowdown is due to the emergence of an exponentially tight bottleneck between the phases (each corresponding to dominance by one of the $q$ colors), which coexist at low temperatures. A generally believed paradigm (which turns out to be quite hard to prove) is that the mixing time, when initialized from an appropriate random mixture of configurations that are representative of each of these phases—-in the Ising case, the all-$+1$ and all-$-1$ configurations with probability 1/2 each—should in fact be fast. A version of this paradigm was recently established in [25] for the Ising model on $\mathbb{T}_n$ in its entire low-temperature coexistence regime using a new notion of spatial mixing known as WSM within a phase. Informally, WSM within a phase requires exponential decay of correlations between sets $A, B$ on $\mathbb{T}_n$ as before, but now under the Gibbs measure restricted to a phase (e.g., to configurations with majority +1 spins), and only when the configuration on $A$ is set to the extremal configuration of that phase (i.e., the all+1 configuration).

Recall that, for the random-cluster model, WSM is expected to hold at all off-critical temperatures $p \neq p_c(q,d)$; the only phase coexistence regime is then exactly at the critical point $p_c(q,d)$, and actually only when $q$ is large enough. The coexistence here is between the high- and low-temperature phases, corresponding to non-existence and existence of a giant component (called the “free” or “disordered” and “wired” or “ordered” phases, respectively). Indeed, at $p_c(q,d)$ when $q$ is large, there is an exponential bottleneck between the free and wired phases on $\mathbb{T}_n$, and the FK dynamics on $\mathbb{T}_n$ slows down exponentially [10, 23]. (Technically in $d \geq 3$, [10] only applies for integer $q$; for completeness, in Appendix B we deduce slow mixing for all $d \geq 2$ and general $q \geq q_0$ at $p = p_c + o(1/n)$.) By analogy with the Ising model, one might conjecture that this slowdown can be overcome by starting the FK dynamics in an appropriate mixture of extremal representatives from its two phases, namely the all-free and all-wired configurations.

In this article, we define a notion of WSM within a phase for the random-cluster model (see Definition 4.1 for the details; this definition is technically more complicated than in the Ising case due to the lack of symmetry between the free and wired phases). We then use this notion to verify the above intuition by first establishing the following upper bound on the mixing time of the FK dynamics within each of its phases when initialized from the corresponding extremal configuration. We capture this by considering the dynamics “restricted to a phase,” by which we mean that transitions to the other phase are rejected.

**Theorem 1.3.** Suppose $d \geq 2$, $q \geq 1$, and $p$ are such that there is an exponential bottleneck between the free and wired phases, and WSM within the wired phase holds. Then the mixing time of the FK dynamics on $\mathbb{T}_n$ restricted to the wired phase, starting from the all-wired configuration, is $O(N \log N \cdot t_{\text{mix}}(A_{\text{ClogN}}^1))$. A symmetrical statement holds for the free phase.

Note that this theorem reduces the mixing time of the FK dynamics restricted to the wired (resp., free) phase initialized from the extremal all-wired (resp., all-free) configuration to that of the

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4We note that the notion of mixing time from a specified (possibly random) initialization is slightly weaker than the standard worst-case mixing time, in that the usual “boosting” property does not apply. See Section 2.3 for details.

5By convention, in the case of the Ising model $q = 2$, the two states are usually taken to be $\{−1, +1\}$ rather than $\{1, 2\}$.

6In $d = 2$, there is coexistence at $p_c(q,d)$ as soon as $q > 4$ [14], and in large dimension as soon as $q > 2$. For smaller values of $q$, the random-cluster model has no coexistence regime, and the mixing time should be polynomial even at $p_c$. [24, 33].
(worst-case initialization) mixing time of the dynamics with wired (resp., free) boundary condition, but at an exponentially smaller scale. The mixing time with wired boundary conditions is expected to be polynomial in the size of the region (by analogy with the conjectured behavior of the low-temperature Ising model with +1 or −1 boundary conditions), in which case the mixing time bound in Theorem 1.3 would be $O(N(\log N)^C)$. However, even without this conjecture we can deduce, via a crude bound on the mixing time at logarithmic scale, that the mixing time implied by Theorem 1.3 is at most quasi-polynomial in all dimensions (and indeed quasi-linear in $d = 2$ using a more refined such bound from [23]).

Finally, we are able to apply Theorem 1.3 to establish a version of the above conjecture about mixing from a random phase initialization in the FK dynamics.

**Theorem 1.4.** Fix $d \geq 2$, $q \geq q_0(d)$ and $p = p_c(q, d)$. The FK dynamics on $\mathbb{T}_n$, initialized from a $\left(\frac{1}{q+1}, \frac{q}{q+1}\right)$-mixture of the all-free and the all-wired configurations, satisfies the following:

1. If $d = 2$, the mixing time is $N^{1+o(1)}$;
2. If $d \geq 3$, the mixing time is $N^{O((\log N)^{d-2})}$.

Given Theorem 1.3, the core of the proof of Theorem 1.4 is to show that, for $q \geq q_0(d)$, the random-cluster model satisfies WSM within both the wired and free phases at $p_c$ (see Lemma 6.3). Theorem 1.3, together with crude mixing time estimates for extremal boundary conditions at logarithmic scale, then implies the stated mixing time bounds within each phase at $p_c$. Simultaneous mixing within each phase at $p_c$ then easily implies mixing for the entire Gibbs distribution (1.1) at $p_c$, provided the all-free and all-wired configurations are correctly weighted in the initialization. We recall that, in contrast, the mixing time on $\mathbb{T}_n$ from a worst-case initialization suffers an exponential slowdown at $p_c$.

A potential shortcoming of Theorem 1.4 is that a priori knowledge of $p_c$ is a delicate matter (there is no exact expression for it in $d \geq 3$). Relatedly, if $p$ is not exactly $p_c(q, d)$ but is microscopically close, the weights accorded to the wired and free phases may both be $\Omega(1)$ but different from the values $(1, q)$ exactly at $p_c$. It turns out that WSM within a phase holds for the wired phase uniformly over $p \geq p_c - o(1/n)$ and for the free phase uniformly over $p \leq p_c - o(1/n)$. These behaviors can therefore be stitched together to ensure that sampling from random phase initializations of the form in Theorem 1.4 satisfies the above bounds at all temperatures, provided the weighting of the free and wired initializations is accordingly adjusted: see Corollary 7.1. In Lemma 7.1, we describe an MCMC algorithm whose run-time is at most $N^{O((\log N)^{d-2})}$ to learn the relative weights of the two phases—when both are non-negligible—to sufficient accuracy by random sampling within each phase, along sequences of $p$ that approach $p_c$ from above and below.

**Remark 1.3.** As stated earlier, the quasi-polynomial mixing times in Theorem 1.4 immediately become polynomial (or indeed $O(N)$) if as conjectured the mixing time of FK dynamics with wired and free boundary is shown to be polynomial at $p_c$. Similarly, the time to approximately learn the relative phase weights for $p \approx p_c$ would become some low-degree polynomial in $N$.

In the same fashion as Corollary 1.3, our Theorem 1.4 has implications for sampling from the Potts model at, and around, its critical point $\beta_c = -\ln(1 - p_c)$. In [9], a polynomial-time algorithm for sampling from the Potts model exactly at $\beta_c(q, d)$ was provided for $q \geq q_0(d)$ using a large-$q$ cluster expansion. (As the authors note, that algorithm suffers from the shortcomings mentioned above concerning the exact identification of $\beta_c$.) As evidenced by the $d = 2$ case, our approach has the potential for yielding essentially optimal-time sampling algorithms at $p_c(q, d)$. Furthermore, we anticipate that
the approach should work at all $q$ for which there is coexistence at $p_c(q, d)$, as this should be sufficient to ensure WSM within the phases.

2 | PRELIMINARIES AND NOTATION

In this section, we introduce our main notation and recall important properties of the random-cluster model and associated Markov chains. We refer the reader to [26, 32] for more on these topics.

2.1 The random-cluster model on $\mathbb{Z}^d$

The random-cluster model on a finite graph $G = (V, E)$ with parameters $p \in [0, 1]$ and $q \geq 1$ is defined in (1.1). In a configuration $\omega \subseteq E$, if edge $e$ belongs to $\omega$ we write $\omega(e) = 1$ and call $e$ wired or open; else we write $\omega(e) = 0$ and call $e$ free or closed. If $x, y$ are in the same connected component of the sub-graph $(V, \omega)$, we write $x \leftrightarrow y$.

Throughout the article, we will focus on the case where $G$ is a rectangular subgraph of the $d$-dimensional integer lattice $\mathbb{Z}^d$ with nearest-neighbor edges $E(\mathbb{Z}^d) = \{ \{u, v\} : u, v \in \mathbb{Z}^d, d_1(u, v) = 1\}$, where $d_1(\cdot, \cdot)$ is $\ell_1$ distance. The subsets of $\mathbb{Z}^d$ we will be interested in are $\Lambda_n = \left[\frac{-n}{2}, \frac{n}{2}\right]^d \cap \mathbb{Z}^d$, with induced edge set denoted $E(\Lambda_n)$. The (inner) boundary vertices of $\Lambda_n$, denoted $\partial \Lambda_n$, are those vertices in $\Lambda_n$ that have neighbors in $\mathbb{Z}^d \setminus \Lambda_n$.

2.1.1 Boundary conditions

A random-cluster boundary condition $\xi$ on $\Lambda_n$ is a partition of $\partial \Lambda_n$ into an arbitrary number of subsets, such that the vertices in each subset are identified with one another. The random-cluster distribution with boundary condition $\xi$, denoted $\pi_{\Lambda_n, p, q}^\xi$, is the same as in (1.1), except that the set $\text{Comp}(\omega)$ is replaced by $\text{Comp}(\omega; \xi)$, the set of connected components with this vertex identification. Thus the boundary condition can be viewed as “ghost wirings” of vertices in the same subset of $\xi$.

The free boundary condition, denoted $\xi = 0$, is the one whose partition of $\partial \Lambda_n$ consists only of singletons. The wired boundary condition, $\xi = 1$, puts all vertices of $\partial \Lambda_n$ in the same subset. There is a natural partial order on boundary conditions, given by $\xi \leq \xi'$ iff $\xi$ is a refinement of $\xi'$. The wired/free boundary conditions are then the maximal/minimal ones under this order.

The periodic boundary condition on $\Lambda_n$ corresponds to the partition that pairs up opposite points on $\partial \Lambda_n$. Under this identification of vertices, the random-cluster model on $\Lambda_n$ is the same as that on the torus $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$. Finally, we will make use of boundary conditions induced by an arbitrary configuration $\eta$ on $\mathbb{Z}^d \setminus \Lambda_n$, corresponding to the partition in which $x, y \in \partial \Lambda_n$ are in the same subset if and only if $x \eta \leftrightarrow y$.

An important property of the random-cluster model that we will use extensively is monoticity in boundary conditions [26, Lemma 4.14]: for any two boundary conditions $\xi \leq \xi'$, we have $\pi_{\Lambda_n, p, q}^\xi \succeq \pi_{\Lambda_n, p, q}^{\xi'}$ where $\succeq$ denotes stochastic domination.

The above definition of boundary conditions and monotonicity in boundary conditions carry over naturally to general domains $A \subset \mathbb{Z}^d$ with associated boundary $\partial A$. 
2.2 Phase transition on $\mathbb{Z}^d$

On $\mathbb{Z}^d$, the random-cluster model is well-known to undergo a percolation phase transition at some $p_c(q, d)$. Classical theory (see e.g., [26, Section 5]) implies that for all $p < p_c$, the probability of a giant component (a component having $\Theta(N)$ many vertices) under $\pi_{\Lambda_n}$ is $o(1)$, whereas when $p > p_c$ the probability of a giant component is $1 - o(1)$. It turns out that the phase transition is sharp and for all $p < p_c$, the component sizes have exponential tails [1, 3, 16]. When $p > p_c$, it is expected that all non-giant components have exponential tails; this is known in $d = 2$ by planar duality, in the $q = 2$ case by [7, 39], and when $q \geq q_0$ for some $q_0(d)$ by a technique known as Pirogov–Sinai theory [11].

At the critical point $p_c$, the behavior is very rich and itself exhibits a transition as one varies $q$. Namely, there exists a particular $q_c(d)$ such that when $q \leq q_c$, the behavior at $p_c$ should be like that of the Ising model, in that component sizes have polynomial decay on their sizes, whereas when $q > q_c$, there is phase coexistence. In this context, that means the model at $p_c$ is in a mixture of two possibilities, one mirroring the $p < p_c$ behavior, and one mirroring the $p > p_c$ behavior. This coexistence behavior is known rigorously in $d = 2$, where $q_c = 4$ [14, 17], and in general dimension when $q \geq q_0$ [11].

2.3 Mixing times of Markov chains

Consider a (discrete-time) Markov chain with transition matrix $P$ on a finite state space $\Omega$, reversible with respect to an invariant distribution $\pi$; denote the chain initialized from $x_0 \in \Omega$ by $(X_t^{x_0})_{t \in \mathbb{N}}$. Its (worst-case) mixing time is given by

$$t_{\text{mix}}(\varepsilon) := \min \left\{ t : \max_{x_0 \in \Omega} \|\mathbb{P}(X_t^{x_0} \in \cdot) - \pi\|_{\text{tv}} \leq \varepsilon \right\},$$

where $\|\mu - \nu\|_{\text{tv}}$ is total variation distance. Typically the mixing time is defined specifically as $t_{\text{mix}}(1/4)$, since for any $\delta$ one has $t_{\text{mix}}(\delta) \leq t_{\text{mix}}(1/4) \log(2\delta^{-1})$. To bound the mixing time, it suffices to bound the coupling time; that is, if we construct a coupling $\mathbb{P}$ of the steps of the chain such that for each $x_0, y_0 \in \Omega$, we have $\mathbb{P}(X_T^{x_0} \neq X_T^{y_0}) \leq 1/4$, then $t_{\text{mix}} \leq T$.

For a probability distribution $\nu$ over $\Omega$, we shall also talk about the mixing time with initialization $\nu$, which is defined as in (2.1) but without the maximization over $x_0$ and with $x_0$ chosen according to $\nu$. In this case the above “boosting” does not apply; however, in this article all of our mixing time bounds from specific initializations suffice to achieve total variation distance $\varepsilon$ equal to any desired inverse polynomial in $N$.

2.4 The FK dynamics

Recall the definition of the random-cluster (FK) dynamics from the introduction. In the presence of boundary conditions $\xi$, the only change is that in the update rule (1.3) the condition for $\epsilon_{t+1}$ to be a bridge is determined by whether its presence changes $\text{Comp}(X_t^{x_0}, \xi)$.

2.4.1 Continuous time

It will be convenient for us to prove our mixing time bounds for the FK dynamics on $\Lambda_n$ in continuous time rather than discrete time. This is defined by assigning every edge in $E(\Lambda_n)$ an i.i.d. rate-1 Poisson clock; if the clock at $e$ rings at time $t_i$, we update $X_t^{x_0}(e)$ according to the update rule (1.3). This definition is the same as taking the continuous-time Markov chain with heat
kernel $H_t = e^{t(L-P)}$, with $P$ being the transition matrix for the discrete-time FK dynamics. Recall (e.g., [32, Theorem 20.3]) that

$$C^{-1} |E(\Lambda_n)| t_{\text{mix}}^{\text{cont}} \leq t_{\text{mix}} \leq C |E(\Lambda_n)| t_{\text{mix}}^{\text{cont}}$$

(2.2)

for some absolute constant $C$, where $t_{\text{mix}}^{\text{cont}}$ is the mixing time of the continuous-time dynamics. (Note that the constant $C$ here depends on the minimum probability in (1.3) of keeping the edge status unchanged, which is a function only of $p, q$.) Thus, it suffices to prove our mixing time bounds in continuous time, reduced by a factor of $E(\Lambda_n)$. (The comparison also holds for the mixing time from a specified initialization.) Abusing notation slightly, from this point onwards we let $(X_t^{\omega_0})_{t \geq 0}$ and all other random-cluster dynamics chains we consider be the continuous-time versions as defined above.

### 2.4.2 Monotonicity and the grand coupling

The FK dynamics can be seen to be monotone in the following sense: if $\omega_0 \geq \omega'_0$, then the law of $X_t^{\omega_0}$ stochastically dominates the law of $X_t^{\omega'_0}$. Moreover, there is a standard choice of grand monotone coupling (using the same instantiation of Poisson clock rings, and the same source of randomness for the edge updates) which simultaneously couples all FK dynamics chains $(X_t^{\omega_0})$ indexed by their initial configuration $\omega_0$, domain $A \subset \mathbb{Z}^d$, and boundary condition $\xi$. It is not difficult to verify that the coupling is monotone in the sense that if $\omega_0 \geq \omega'_0$ and $\xi \geq \xi'$, then $X_t^{\omega_0} \geq X_t^{\omega'_0}$ for all $t \geq 0$: see for example, [26, Section 8.3].

### 2.4.3 The restricted FK dynamics

A crucial tool in our analysis will be the FK dynamics restricted to an increasing event $\hat{\Omega}$ (symmetrically, to a decreasing event $\hat{\Omega}$). This chain, denoted $(\hat{X}_t^{\omega_0})_{t \geq 0}$ (symmetrically, $(\hat{X}_t^{\omega'_0})_{t \geq 0}$) is defined exactly as before, except that if the update in (1.3) would cause $\omega$ not to be in $\hat{\Omega}$ (symmetrically, in $\hat{\Omega}$) we leave $\omega(e)$ unchanged. It is easy to check that the Markov chains $(\hat{X}_t)$ and $(\hat{X}_t)$ are reversible w.r.t. $\hat{\pi} = \pi(\cdot | \hat{\Omega})$ and $\hat{\pi} = \pi(\cdot | \hat{\Omega})$, respectively.

### 2.5 Notational disclaimers

The letter $C > 0$ will be used frequently, indicating the existence of a constant that is independent of $r, n, m$ and so forth but that may depend on $p, d, q$ and may differ from line to line. We use $O, o$, and $\Omega$ notation in the same manner, where the hidden constants may depend on $p, d, q$.

### 3 Spatial and Temporal Mixing for Off-Critical Random-Cluster Models

In this section, we establish the implications of WSM and of SSM with a specific boundary condition for the mixing time of the FK dynamics on, respectively, the torus and a box with that specific boundary condition. Namely, the purpose of this section is to establish Theorems 1.1 and 1.2.

We begin by giving the formal definitions of WSM, and of SSM with boundary condition $\xi$ for the random-cluster model. We formulate these for regions that are boxes in $\mathbb{Z}^d$.

**Definition 3.1.** We say the random-cluster model satisfies WSM if, for every $r$,

$$\|\pi_{\Lambda^r}(\omega(\Lambda_{r/2}) \in \cdot) - \pi_{\Lambda^r}(\omega(\Lambda_{r/2}) \in \cdot)\|_{TV} \leq C e^{-r/C}.$$  

(3.1)

for some constant $C > 0$ (which may depend on $d, q$, and $p$).
To phrase the property of SSM with a specific boundary condition precisely, given a box $\Lambda_n$ with boundary conditions $\xi$, define $B_{m,e}$ to be the graph with vertex set $\{v \in \Lambda_n : d_\infty(v, e) \leq m\}$ and induced edge set $E(B_{m,e})$. When understood from context, $B_{m,e}$ will stand in for its edge set for readability. If $\eta$ is a configuration on $E(\Lambda_n) \setminus E(B_{m,e})$, then by $B'_{m,e}$ we mean the graph $B_{m,e}$ with boundary conditions induced by $\eta$ together with $\xi$.

**Definition 3.2.** We say the random cluster model on $\Lambda_n^\xi$ satisfies SSM with constant $C$ if, for all $e \in E(\Lambda_n)$ and all $m \leq n/2$, we have

$$\|\pi_{B_{m/2,e}}(\omega(B_{m/2,e}) \in \cdot) - \pi_{B_{m/2,e}}(\omega(B_{m/2,e}) \in \cdot)\|_1 \leq Ce^{-m/C}. \quad (3.2)$$

**Remark 3.1.** If Definitions 3.1 and 3.2 hold for some constant $C$, then there exists a constant $C'(C, p, q) > 0$ such that they hold with the right-hand sides of (3.1) and (3.2) replaced by $e^{-r/C'}$ and $e^{-m/C'}$, respectively. This can be readily checked using the fact that for any fixed $r, m$, the TV-distances are bounded strictly away from 1 by forcing all edges to take the same state. In the sequel, we will sometimes use this modified form for convenience.

For convenience, we now restate Theorems 1.1 and 1.2 in the language of continuous time. (The restatement of Theorem 1.2 is also more precise about quantification and dependencies on the SSM constant, because a specific boundary condition may only make sense for a single box-size $n$.)

**Theorem 1.1** (formal). Fix $(p, q, d)$. If the random-cluster model satisfies WSM, then the mixing time of the continuous-time FK dynamics on $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$ is $O(\log N)$.

**Theorem 1.2** (formal). Fix any $C_* > 0$ and $(p, q, d)$. There exists $n_0$ such that for all $n \geq n_0$, if the random cluster model on $\Lambda_n^\xi$ has SSM with constant $C_*$, then the mixing time of the continuous-time FK dynamics on $\Lambda_n^\xi$ is $O(\log N)$.

As mentioned in the introduction, it is fairly straightforward to deduce Theorem 1.1 using the classical arguments of Martinelli and Olivieri [34], which, as noted by Harel and Spinka [28], extend naturally to the random-cluster model. Our more novel contribution is therefore Theorem 1.2, which can be seen as an extension of the implication of Theorem 1.1 to the more delicate setting of a fixed boundary condition. Accordingly, we shall devote the majority of this section to proving Theorem 1.2, and will then for completeness briefly indicate in Section 3.4 how to extract Theorem 1.1 from our proof.

Our approach to proving Theorem 1.2 is a finite-volume analogue of the space-time recursion of [34]. Whereas that argument shows that WSM implies exponential relaxation in the infinite volume $\mathbb{Z}^d$, to the best of our knowledge it has not until now been applied in the presence of specific boundary conditions. Previous proofs for fast mixing in the presence of boundary conditions have gone through recursive schemes that require notions of SSM that apply uniformly over all boundary conditions. Recall that there are parameter regimes in $d \geq 3$ where such uniform SSM is not expected to hold, but the random-cluster model on $\Lambda^\emptyset_n$ or $\Lambda^\emptyset_n$ does satisfy SSM in the form of Definition 3.2.

**Remark 3.2.** Since the proof of Theorem 1.2 primarily used monotonicity, it would equally well work for the Glauber dynamics for the Ising model on $\Lambda_n$ in the presence of boundary conditions $\eta \in \{\pm 1\}^{\mathbb{Z}^d \setminus \Lambda_n}$, assuming only SSM for $\Lambda^\emptyset_n$ and not uniformly over all boundary conditions. It was recently shown in [13] that the Ising model has SSM uniformly over all boundary conditions at all $\beta < \beta_c(d)$ in the absence of an external field. However, there are parameter regimes (e.g., low-temperatures with positive external field...
in $d \geq 3$) where the Ising model is known to have WSM [37] but not SSM uniformly over all boundary conditions (an example bad boundary condition is given in [36]). All the same, even in this parameter regime there are classes of boundary conditions—most obviously the all-$+1$ condition—that do satisfy SSM. Our results can therefore be used to deduce fast mixing for the Ising Glauber dynamics in such settings, may not have followed from known technology.

### 3.1 A recurrence for the disagreement probability at a single edge

Let us begin with some notation. Throughout this section, $n$ and $\xi$ will be fixed such that $\Lambda_n^\xi$ satisfies SSM with constant $C_0$, and $n \geq n_0(C_0)$ to be determined later. For $m \leq n/2$, for $r \leq m$, define the event

$$E_t(m, r, e) := \{X^{1}_{t, B^1_{m,e}}(B_{r,e}) \neq X^{0}_{t, B^0_{m,e}}(B_{r,e})\},$$

and under the grand coupling of $X^{1}_{t, B^1_{m,e}}, X^{0}_{t, B^0_{m,e}}$, let

$$\phi_{m,t}(e) := \mathbb{P}(E_t(m, 0, e)) = \mathbb{P}\left(X^{1}_{t, B^1_{m,e}}(e) \neq X^{0}_{t, B^0_{m,e}}(e)\right).$$

By monotonicity of the grand coupling, we next observe that

$$\phi_{m,s}(e) \geq \phi_{l,t}(e) \text{ for all } m \leq l \text{ and } s \leq t. \quad (3.3)$$

Indeed, $B^1_{t,e}$ stochastically dominates $B^1_{m,e}$ if $m \leq l$ regardless of $\xi$, and vice versa with $0$ boundary. For the monotonicity in time, notice that by the censoring lemma of [38, Theorem 1.1], $X^{1}_{t, B^1_{m,e}}$ is stochastically decreasing in time, and $X^{0}_{t, B^0_{m,e}}$ is stochastically increasing in time, so

$$\phi_{m,s}(e) = \mathbb{P}(X^{1}_{t, B^1_{m,e}}(e) = 1) - \mathbb{P}(X^{0}_{t, B^0_{m,e}}(e) = 1) \geq \mathbb{P}(X^{1}_{t, B^1_{m,e}}(e) = 1) - \mathbb{P}(X^{0}_{t, B^0_{m,e}}(e) = 1) = \phi_{m,t}(e).$$

Define now

$$\phi_{m,t} := \max_{e \in E(\Lambda_n)} \phi_{m,t}(e). \quad (3.4)$$

Observe that (3.3) implies the same monotonicity relation for $\phi_{m,t}$. The main result of this subsection, and indeed the main input into establishing Theorem 1.2 is the following.

**Proposition 3.1.** Suppose that the random-cluster model on $\Lambda_n^\xi$ satisfies SSM with constant $C_*$. Then for every $t \geq 0$, every $m \leq n/4$, and every $r \leq m$,

$$\phi_{2m,2t} \leq e^{-t/C_*} + d(2r)^d \phi_{m,t}. \quad (3.5)$$

**Proof.** Fix $t$ and any edge $e \in E(\Lambda_n)$, and consider the quantity $\phi_{2m,2t}(e)$. We can of course express this as

$$\phi_{2m,2t} = \mathbb{P}(E_t(2m, 0, e), E_t(2m, r, e)^c) + \mathbb{P}(E_t(2m, 0, e), E_t(2m, r, e)). \quad (3.6)$$
Let us consider these two terms separately. For the first term we notice that, under the grand coupling, on the event $\mathcal{E}_i(2m, r, e)^c$ the chains with all possible initializations agree on $B_{r,e}$, so we have

$$X_{1,B_{2m,e}}^1(B_{r,e}) = X_{1,B_{2m,e}}^0(B_{r,e}).$$

By monotonicity in both initialization and boundary conditions, this implies

$$X_{1,B_{2m,e}}^1(B_{r,e}) \leq X_{1,B_{2m,e}}^0(B_{r,e}).$$

The analogous (reverse) inequality holds for $X_{0,B_{2m,e}}^0(B_{r,e})$. In particular, on $\mathcal{E}_i(2m, r, e)^c$, we have

$$X_{1,B_{2m,e}}^1(B_{r,e}) \geq X_{1,B_{2m,e}}^0(B_{r,e}) \geq X_{0,B_{2m,e}}^0(B_{r,e}) \geq X_{0,B_{2m,e}}^0(B_{r,e}).$$

By monotonicity of the FK dynamics and the Markov property, on $\mathcal{E}_i(2m, r, e)^c$, this ordering is maintained for all times beyond $t$, so that in particular

$$X_{1,B_{2m,e}}^1(e) \geq X_{1,B_{2m,e}}^0(e) \geq X_{0,B_{2m,e}}^0(e) \geq X_{0,B_{2m,e}}^0(e).$$

Therefore,

$$\mathbb{P}(\mathcal{E}_2(2m, 0, e), \mathcal{E}_i(2m, r, e)^c) = \mathbb{E}[(X_{1,B_{2m,e}}^1(e) - X_{0,B_{2m,e}}^0(e))\mathbf{1}\{\mathcal{E}_i(2m, r, e)^c\}]$$

$$\leq \mathbb{E}[(X_{1,B_{2m,e}}^1(e) - X_{0,B_{2m,e}}^0(e))\mathbf{1}\{\mathcal{E}_i(2m, r, e)^c\}].$$

Under the grand coupling, $X_{1,B_{2m,e}}^1(e) - X_{0,B_{2m,e}}^0(e) \geq 0$, so we can now drop the indicator to get

$$\mathbb{P}(\mathcal{E}_2(2m, 0, e), \mathcal{E}_i(2m, r, e)^c) = \mathbb{E}[X_{1,B_{2m,e}}^1(e) - X_{0,B_{2m,e}}^0(e)]$$

$$= \pi_{B_{2m,e}}(\omega_e = 1) - \pi_{B_{2m,e}}(\omega_e = 1) \leq \exp(-r/C_\star), \quad (3.7)$$

where the last inequality follows from our assumption that the random-cluster model on $\Lambda_t^n$ satisfies SSM with constant $C_\star$ as $r \leq n/2$ (recall Definition 3.2).

We now turn to the second term in (3.6). Clearly we can write

$$\mathbb{P}(\mathcal{E}_2(2m, 0, e), \mathcal{E}_i(2m, r, e)) = \mathbb{P}(\mathcal{E}_2(2m, 0, e)|\mathcal{E}_i(2m, r, e))\mathbb{P}(\mathcal{E}_i(2m, r, e)). \quad (3.8)$$

A union bound gives

$$\mathbb{P}(\mathcal{E}_i(2m, r, e)) \leq |E(B_{r,e})| \max_{f \in E(B_{r,e})} \mathbb{P}(X_{1,B_{2m,e}}^1(f) \neq X_{0,B_{2m,e}}^0(f)).$$

Since $r \leq m$, for every $f \in B_{r,e}, B_{m,f} \subset B_{2m,e}$, so that by monotonicity

$$\mathbb{P}(\mathcal{E}_i(2m, r, e)) \leq |E(B_{r,e})| \max_{f \in E(B_{r,e})} \mathbb{P}(X_{1,B_{2m,e}}^1(f) \neq X_{0,B_{2m,e}}^0(f)) = |E(B_{r,e})| \max_{f \in E(B_{r,e})} \phi_m(f)$$

$$\leq d(2r)^d \phi_{m,f}. $$
At the same time, by the Markov property and monotonicity of the grand coupling, taking a worst case over the realizations of the coupling in the time interval \([0, t]\), we get

\[
P(E_{2t}(2m, 0, e)|E_t(2m, r, e)) \leq \max_{A_t \in F_t} P(E_{2t}(2m, 0, e)|A_t) \leq P(E_t(2m, 0, e)) = \phi_{2m,t}(e),
\]

where \(F_t\) is the filtration defined by the grand coupling up to time \(t\). The right-hand side is in turn at most \(\phi_{m,t}(e) \leq \phi_{m,t}\) by (3.3). Plugging these bounds into (3.8) we get

\[
P(E_{2t}(2m, 0, e), E_t(2m, r, e)) \leq d(2r)^d \phi_{m,t}^2.
\]

Together with the bound (3.7) on the first term in (3.6), we obtain the desired bound (3.5).

### 3.2 Exponential decay of disagreement probability at a single edge

We next claim that Proposition 3.1 implies exponential decay of \(\phi_{m,t}\) in space and time.

**Corollary 3.1.** For every \(C_r\), there exists \(C(C_r, p, q, d) > 0\) such that if the random-cluster model on \(\Lambda^*_n\) satisfies SSM with constant \(C_r\), then we have

\[
\phi_{t,d} \leq Ce^{-t/C_r} \quad \text{for all} \quad t \leq n/2.
\]

**Proof.** The corollary will follow by setting \(a_k = \phi_{k,k}\) and applying the following lemma.

**Lemma 3.1.** Fix \(d\) and \(C_r\). Suppose \(0 \leq a_k \leq 1\) is a non-increasing (in \(k\)) sequence. There exists \(\varepsilon_0(C_r, d)\) such that if \(a_{k_0} \leq \varepsilon_0\) for some \(k_0\), and \(a_k\) satisfies

\[
a_{2k} \leq d(2r)^d a_k^2 + e^{-r/C_r}, \quad \text{for all} \quad r \leq k \leq n/2,
\]

then there exists \(C = C(k_0, C_r, d)\) such that \(a_k \leq Ce^{-k/C_r}\) for all \(k \leq n/2\).

The proof of the lemma is standard, and we therefore defer it to Appendix A. Let us now reason that we can apply the lemma to the sequence \(a_k = \phi_{k,k}\). Clearly \(0 \leq a_k \leq 1\) as it is a probability, and the fact that the sequence is non-increasing comes from (3.3). Let \(R = R(\varepsilon_0) = R(C_r, d)\) be sufficiently large that \(e^{-R/C_r}\) is less than \(\varepsilon_0/3\) and make sure \(n_0\) is at least \(2R\). Then, let \(t_0(R)\) be given by

\[
\max_{G : |V(G)| \leq 2R^d, |E(G)| \leq 2R^d} t_{\text{mix}}(X_{t_0,G}) \log(3/\varepsilon_0),
\]

where the maximum runs over all possible multi-graphs on at most \((2R)^d\) many vertices with at most \(d(2R)^d\) many edges. (This captures all the possible boundary conditions induced on \(B_r,e\).) Evidently this is a finite set, and thus the maximal mixing time above is some constant \(T(p, q, R, d) = T(C_r, p, q, d)\). Now by sub-multiplicativity in time of the total-variation distance between a worst-pair of initializations, which is at most twice the total-variation distance and which in turn bounds \(\max_{\omega_0} d_{tv}(\omega_0, t)\) in any Markov chain (see e.g., [32]), this implies that

\[
a_{Rt_0(R)} \leq \phi_{R,t_0(R)} \leq \max_{e \in E(\Lambda^*_n)} \left( \max_{\omega_0} \left\| P(X^{\omega_0}_{t_0,B^*_r} \in \cdot) - \pi_{B^*_r} \right\|_{tv} + \max_{\omega_0} \left\| P(X^{\omega_0}_{t_0,B^*_r} \in \cdot) - \pi_{B^*_r} \right\|_{tv} \right),
\]

\[
+ \left\| \pi_{B^*_r} (\omega_e \in \cdot) - \pi_{B^*_r} (\omega_e \in \cdot) \right\|_{tv},
\]

\[\phi_{R,t_0(R)},\]
is at most $\varepsilon_0$. Therefore, taking $k_0 = \max\{R, t_0(R)\}$, which clearly only depends on $C_*, p, q, d$, we have $a_{k_0} \leq \varepsilon_0$. Further, the recurrence relation is satisfied for all $r \leq k \leq n/2$ by Proposition 3.1 whenever the random-cluster model on $\Lambda_n^S$ satisfies SSM with constant $C_*$. Therefore, applying Lemma 3.1, we see that there exists $C(C_*, p, q, d)$ such that $a_k \leq Ce^{-k/C}$ for all $k \leq n/2$. Finally the fact that Corollary 3.1 is for real values of $t$, not just integer, is taken care of by the monotonicity in time (3.3) and a change in the constant $C$ in the bound.

3.3 | Fast mixing with boundary conditions under SSM

We now use Corollary 3.1 to conclude the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Consider the grand coupling of the FK dynamics $(X_t^{0,0}) = (X_t^{0,0})$ on $\Lambda_n^S$. Let $C_0(C_*, p, q, d)$ be the constant obtained from Corollary 3.1, and let $n_0(C_*, p, q, d)$ be such that $2dC_0 \log n \leq n/2$ and $C_0n^{-2d}|E(\Lambda_n)| \leq 1/4$ for all $n \geq n_0$.

By the coupling definition of total variation distance and a union bound, it suffices to show that the following is at most $1/4$ when $t = 2dC_0 \log n$:

$$\mathbb{P}(X_t^1 \neq X_t^0) \leq \sum_{e \in E(\Lambda_n)} \mathbb{P}(X_t^1(e) \neq X_t^0(e)).$$

By monotonicity, this is at most

$$\sum_{e \in E(\Lambda_n)} \mathbb{P}(X_t^1_{t,B_{t,e}^1(e)} \neq X_t^0_{t,B_{t,e}^0(e)}) = \sum_{e \in E(\Lambda_n)} \phi_{e,t} \leq |E(\Lambda_n)| \phi_{e,t}.$$

Applying our bound on $\phi_{e,t}$ from Corollary 3.1, and the choice of $t, n_0$, we see that as long as $n \geq n_0$, under the grand coupling

$$\mathbb{P}(X_t^1 \neq X_t^0) \leq C_0|E(\Lambda_n)|e^{-t/C_0} = C_0n^{-2d}|E(\Lambda_n)| \leq 1/4,$$

concluding the proof.

3.4 | Fast mixing on the torus under WSM

We conclude with brief comments describing the differences (actually, simplifications) of the above argument that recover Theorem 1.1.

**Proof of Theorem 1.1.** The proof can be derived essentially by a simpler version of the proof of Theorem 1.2, so we just explain the necessary modifications. Rather than defining $B_{m,e}^1$ and $B_{m,e}^0$ as boxes that stop at the boundary of the box $\Lambda_n$, consider them as balls on the torus $\mathbb{T}_n$ with wired/free boundaries, respectively. By transitivity of the torus, there is no need to maximize $\phi_{m,t}$ over $e$ ($\phi_{m,t}(e)$ is the same for all $e$), and it therefore suffices to consider an edge $e$ incident to the origin. In the proof, since $B_{r,e}^1, B_{r,e}^0$ do not reach the boundary of $\Lambda_n$ as $r < n/2$, the bounds in the proof all go through as before, and the assumption of WSM is sufficient to establish (3.7).
We next turn to controlling the mixing time from the all-wired and all-free initializations (and mixtures thereof), rather than from a worst-case initialization. The key point will be that, unlike the worst-case mixing time, the mixing time when initialized in the dominant phase will not blow up as $p \to p_c(q, d)$.

To make this precise, we must formalize what we mean by the wired and free phases. We use the notation $\hat{\Omega}$ and $\tilde{\Omega}$ for these respectively, and define them as follows. We emphasize that we have some flexibility with these definitions and they are not as clearly dictated as in the Ising case in [25], where they were determined by the signature of the majority spin. For the random-cluster model, for every $p, q, d$ we define

$$\hat{\Omega} = \{ \omega \subset E(T_n) : |C_1(\omega)| \geq \varepsilon n^d \} ; \quad \tilde{\Omega} = \{ \omega \subset E(T_n) : |C_1(\omega)| \leq \varepsilon n^d \} ,$$

where $C_1(\omega)$ denotes the largest connected component of $\omega$. Here we choose $\varepsilon(p, q, d)$ to be a sufficiently small constant (independent of $n$) such that (4.1) and (4.2) indeed capture the wired and free phases of the measure (in the sense made precise following Definition 4.2 below). In all the parameter regimes we care about, such a choice of $\varepsilon$ exists.

For ease of notation, in what follows we will use $\pi_n$ to denote the random-cluster model on the torus $\pi T_n$. Denote the measure on $T_n$ conditioned on the wired and free phases as follows:

$$\hat{\pi}_n = \pi_n(\cdot | \hat{\Omega}) \quad \text{and} \quad \tilde{\pi}_n = \pi_n(\cdot | \tilde{\Omega}).$$

Throughout this section, we will take $\iota$ to be a placeholder for either of 1 or 0, depending on which of the two phases we are interested in. Our fundamental spatial mixing assumption, as indicated in the introduction, is called WSM within a phase. We formalize this notion as follows.

**Definition 4.1.** We say that WSM holds within the wired phase (with constant $C$) if for all $n$ and all $r \leq n/2$,

$$\| \pi_\Lambda^\iota(\omega(\Lambda_{r/2}) \in \cdot) - \hat{\pi}_n(\omega(\Lambda_{r/2}) \in \cdot) \|_{tv} \leq C e^{-r/C} .$$

Analogously, we say that WSM holds within the free phase (with constant $C$) if for all $n$ and all $r \leq n/2$,

$$\| \pi_\Lambda^\iota(\omega(\Lambda_{r/2}) \in \cdot) - \tilde{\pi}_n(\omega(\Lambda_{r/2}) \in \cdot) \|_{tv} \leq C e^{-r/C} .$$

In words, WSM within the wired (resp., free) phase says that the measure obtained on a box with wired (resp., free) boundary condition is close in total variation in its bulk to the measure on the torus conditioned on having a giant component (resp., not having a giant component).

Besides WSM within a phase, we make a further assumption that the $\iota$-phase is indeed thermodynamically stable, capturing the fact that the $\varepsilon$ we chose in the definitions of (4.1) and (4.2) is a good one.

**Definition 4.2.** Let $\partial \hat{\Omega}$ be the set of configurations in $\hat{\Omega}$ that are one edge-flip away from $\hat{\Omega}^c$. We say the wired phase is exponentially stable with constant $C_1$ if the following holds:

$$\pi_n(\partial \hat{\Omega} | \hat{\Omega}) \leq C_1 e^{-n^{d-1}/C_1} .$$
Similarly, the free phase is exponentially stable with constant $C_1$ if (4.5) holds with $\tilde{\Omega}$ replaced by $\Omega$.

It is widely expected (see e.g., [26, Conjecture 5.103 and Theorem 5.104]) that there is a choice of constant $\epsilon(p, q, d) > 0$ in (4.1) and (4.2) such that when $p < p_c(q, d)$ for all $d, q$ the free phase is exponentially stable, when $p > p_c(q, d)$ for all $d, q$, the wired phase is exponentially stable, and when $q > q_c(d)$ so that the phase transition is discontinuous, both phases are exponentially stable at $p = p_c(q, d)$. It follows from known results that these properties hold in $d = 2$ for all $q > 4$ and when $q \geq q_0(d)$ in general $d$ (see Lemma 5.6 for a short proof). We thus consider this a milder condition than WSM within a phase.

### 4.1 Main reduction

Theorem 1.3 essentially says that we can reduce the mixing time on the torus $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$ of the restricted chain $\hat{X}^1_{t, B_{m, e}}$ to the mixing time at local $O(\log n)$ scales with wired boundary condition (and similarly for the free phase). We state the following mixing time assumption at the local scale that is slightly weaker than a (worst-case) mixing time bound. Suppose there exists a non-decreasing sequence $f(m)$ such that for all $m = O(\log n)$ and all $t > 0$,

$$\max_{e \in E(\mathbb{T}_n)} \| \mathbb{P}(X^1_{t, B_{m, e}}(e) \in \cdot) - \pi_{B_{m, e}}(\omega_e \in \cdot) \|_{tv} \leq e^{-t/f(m)}. \quad (4.6)$$

Of course by transitivity of the torus the left-hand side is independent of $e$, so we may drop the maximum and just consider a box of side-length $m$ centered at some fixed $e \in E(\mathbb{T}_n)$ with $t$ boundary condition.

A (worst-case) mixing time (or inverse spectral gap) bound of $f(m)$ for the FK dynamics on $\Lambda^*_m$ would automatically yield (4.6), but we use the above formulation in case one can obtain better bounds on the exponential rate of relaxation when initialized from the $t$ configuration.

Given such a sequence $f(m)$, fix a constant $K$ sufficiently large, and define the following function:

$$g_n(t) = \max \left\{ m \leq n : mf(m) \leq t \wedge e^{(d-1)/K} \right\}. \quad (4.7)$$

The following theorem is a more formal restatement (in continuous time) of Theorem 1.3. Recall the definition and notation of the FK dynamics restricted to a chain from Section 2.4.3.

**Theorem 1.3** (formal). Suppose that $(p, q, d)$ is such that the wired phase is exponentially stable with constant $C_1$, WSM within the wired phase holds with constant $C_2$, and (4.6) holds with $t = 1$ for a non-decreasing sequence $f(m)$. Then there exist $C_0, K_0$ depending only on $C_1, C_2$ such that if $g_n(t)$ is as in (4.7) for $K = K_0$, then for every $t \geq 0$,

$$\| \mathbb{P}(\hat{X}^1_{t, e} \in \cdot) - \tilde{\pi}_{n} \|_{tv} \leq C_0 n^d \exp(-g_n(t)/C_0).$$

Likewise, if the free phase is exponentially stable with constant $C_1$, WSM within the free phase holds with constant $C_2$, and (4.6) holds with $t = 0$ for a non-decreasing sequence $f(m)$, then for every $t \geq 0$,

$$\| \mathbb{P}(\hat{X}^0_{t, e} \in \cdot) - \tilde{\pi}_{n} \|_{tv} \leq C_0 n^d \exp(-g_n(t)/C_0).$$
Remark 4.1. In order to see that this gives the bound of Theorem 1.3, notice that in order for the right-hand sides to be $o(1)$, we need $g_n(t)$ to be at least $C \log n$ for a large enough $C$.

By the definition of $g_n(t)$ from (4.7), we see that this will happen if $t \geq C \log n \cdot f(C \log n)$.

Taking $f(m)$ to be the worst-case mixing time on $\Lambda^1_m$, then gives the claimed mixing time of $O(\log N \cdot t_{\text{mix}}(\Lambda^1_{c \log n})$ for the restricted chain. (The extra factor of $N$ in Theorem 1.3 comes from the switch between discrete- and continuous-time dynamics.)

With the definitions of the phases of the random-cluster model, and the random-cluster notions of WSM within a phase and exponential stability in hand, our proof of Theorem 1.3 proceeds similarly to the proof of [25, Theorem 3.2]. For the remainder of this section, let $X^0$ denote $X^0_{t,\tau_n}$; similarly, we will drop the $\tau_n$ subscript in other places where the domain is understood to be the torus. Also, we prove everything for the wired phase $t = 1$; the proof for the free phase is completely analogous.

4.2 | Single-edge relaxation within a phase

We begin by proving a rate of relaxation bound on the single-edge marginals of $\hat{X}^1_t$ to $\pi_n$ (and similarly of $X^0_t$ to $\hat{\pi}_n$).

**Proposition 4.1.** Suppose the wired phase is exponentially stable with constant $C_1$, WSM within the wired phase holds with constant $C_2$, and (4.6) holds with $t = 1$ for some non-decreasing sequence $f(m)$. Then there exist $C_0, K_0$ depending only on $C_1, C_2$ such that if $g_n(t)$ is as in (4.7) for $K = K_0$, then for every $e \in E(\mathbb{T}_n)$, for all $t \leq e^{n - 1}/K$,

$$|\mathbb{P}(\hat{X}^1_t(e) = 1) - \pi_n(\omega_e = 1)| \leq C_0 e^{-g_n(t)/C_0}.$$

The first step entails circumventing the absence of monotonicity of the FK dynamics when it is restricted to a phase, using the fact that the hitting time of the boundary $\partial \hat{\Omega}$ is typically exponentially long if the wired phase is exponentially stable. Denote by $\hat{\tau}_u$ the hitting time of $\partial \hat{\Omega}$ of the restricted dynamics $\hat{X}^0_t$. Let $\hat{\tau}$ denote this hitting time when one first draws $x_0 \sim \pi_n$ and then considers $\hat{\tau}_u$. Similarly let $X^\hat{\tau}$ denote the distribution of $X^0_{t,\tau_n}$ when $x_0$ is first drawn randomly from $\hat{\pi}_n$.

**Lemma 4.1.** Suppose $(p, q, d)$ is such that the wired phase is exponentially stable with constant $C_1$. There exists $C(C_1)$ such that for every $t \geq 0$, we have $\mathbb{P}(\hat{\tau}^1 \leq t) \leq \mathbb{P}(\hat{\tau} \leq t) \leq C(t \lor 1)e^{-dt/C}$. 

**Proof.** We begin by observing, by monotonicity of the grand coupling and the definition of the restricted dynamics, that

$$\hat{X}^\hat{\tau}_t \leq X^1_t \leq \hat{X}^1_t \quad \text{for } 0 \leq t \leq \hat{\tau}^\hat{\tau}.$$

In particular, this implies the first inequality in the lemma. To obtain the second inequality, fix a sequence of times $t_1, t_2, \ldots$ at which some Poisson clock rings for the continuous-time FK dynamics in $\mathbb{T}_n$; this process is then a Poisson clock of rate $|E(\mathbb{T}_n)| = O(n^d)$. Conditional on any sequence of clock rings, the chain $\hat{X}^\hat{\tau}_t$ is stationary, and thus distributed as $\hat{\pi}_n$ for all $t_i$. Then by a union bound,

$$\mathbb{P}(\hat{\tau} \leq t) \leq \mathbb{P}(\text{Pois}(|E(\mathbb{T}_n)| \geq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)) + \sum_{i \leq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)} \mathbb{P}(\hat{X}^\hat{\tau}_t \in \partial \hat{\Omega}) \leq Ce^{-n - 1/C} + Cn^{2d-1}(t \lor 1)\hat{\pi}_n(\partial \hat{\Omega}),$$

We begin by observing, by monotonicity of the grand coupling and the definition of the restricted dynamics, that

$$\hat{X}^\hat{\tau}_t \leq X^1_t \leq \hat{X}^1_t \quad \text{for } 0 \leq t \leq \hat{\tau}^\hat{\tau}.$$

In particular, this implies the first inequality in the lemma. To obtain the second inequality, fix a sequence of times $t_1, t_2, \ldots$ at which some Poisson clock rings for the continuous-time FK dynamics in $\mathbb{T}_n$; this process is then a Poisson clock of rate $|E(\mathbb{T}_n)| = O(n^d)$. Conditional on any sequence of clock rings, the chain $\hat{X}^\hat{\tau}_t$ is stationary, and thus distributed as $\hat{\pi}_n$ for all $t_i$. Then by a union bound,

$$\mathbb{P}(\hat{\tau} \leq t) \leq \mathbb{P}(\text{Pois}(|E(\mathbb{T}_n)| \geq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)) + \sum_{i \leq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)} \mathbb{P}(\hat{X}^\hat{\tau}_t \in \partial \hat{\Omega}) \leq Ce^{-n - 1/C} + Cn^{2d-1}(t \lor 1)\hat{\pi}_n(\partial \hat{\Omega}),$$

$$\mathbb{P}(\hat{\tau} \leq t) \leq \mathbb{P}(\text{Pois}(|E(\mathbb{T}_n)| \geq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)) + \sum_{i \leq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)} \mathbb{P}(\hat{X}^\hat{\tau}_t \in \partial \hat{\Omega}) \leq Ce^{-n - 1/C} + Cn^{2d-1}(t \lor 1)\hat{\pi}_n(\partial \hat{\Omega}),$$

$$\mathbb{P}(\hat{\tau} \leq t) \leq \mathbb{P}(\text{Pois}(|E(\mathbb{T}_n)| \geq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)) + \sum_{i \leq n^{d-1}|E(\mathbb{T}_n)|(t \lor 1)} \mathbb{P}(\hat{X}^\hat{\tau}_t \in \partial \hat{\Omega}) \leq Ce^{-n - 1/C} + Cn^{2d-1}(t \lor 1)\hat{\pi}_n(\partial \hat{\Omega}),$$
for a universal constant $C$. We conclude the proof by applying the bound of (4.5) to $\hat{\pi}_n(\partial\Omega)$ and adjusting the constants accordingly.

We are now in position to prove Proposition 4.1.

Proof of Proposition 4.1. Fix $t \geq 0$ and $e \in E(T_n)$. For ease of notation, define

$$\hat{P}_{\partial e}(1) := \mathbb{E}[\hat{X}_t^1(e)] - \hat{\pi}_n[\omega_e] = \mathbb{P}(\hat{X}_t^1(e) = 1) - \hat{\pi}_n(\omega_e = 1),$$

so that the left-hand side of Proposition 4.1 is $|\hat{P}_{\partial e}(1)|$. We begin by lower bounding this quantity, using the fact that under the grand coupling,

$$\hat{P}_{\partial e}(1) = \mathbb{E}[\hat{X}_t^1(e) - \hat{\pi}_t^g(e)] \geq -\mathbb{P}(\hat{\pi}_t^g \leq t).$$

Here we used (4.8) to deduce that the difference of the edge variables is non-negative when $\hat{\pi}_t^g > t$. Thus, by Lemma 4.1, while $t \leq e^{d^2-1}/K$, we have $\hat{P}_{\partial e}(1) \geq -C_3 e^{-d^2-1}/C_1$ for some $C_3(C_1)$.

We now turn to the upper bound. For any $m = m(t) \leq n$, by monotonicity of the FK dynamics,

$$\hat{P}_{\partial e}(1) \leq \mathbb{E}[\hat{X}_t^1(e)] - \mathbb{E}[X_t^1(e)] + \mathbb{E}[X_t^{B_1}(e)] - \hat{\pi}_n[\omega_e].$$

By (4.8), the absolute difference of the first two terms on the right is at most $\mathbb{P}(\hat{\pi}_t^g \leq t)$, which is at most $C_3 e^{-d^2-1}/C_1$ by Lemma 4.1 for $t \leq e^{d^2-1}/K$. Using this and a triangle inequality, we get

$$\hat{P}_{\partial e}(1) \leq C_3 e^{-d^2-1}/C_1 + \left|\pi_{B_1}^1[\omega_e] - \hat{\pi}_n[\omega_e]\right| + \left|\mathbb{E}[X_t^{B_1}(e)] - \pi_{B_1}^1[\omega_e]\right|. \tag{4.9}$$

The second term in (4.9) is seen to be at most $C_2 e^{-m/C_2}$ by the assumption of WSM within the wired phase (4.3). The third term is exactly the quantity controlled by the assumption in (4.6). Combining these, we see that for all $t \leq e^{d^2-1}/K$ we have

$$\hat{P}_{\partial e}(1) \leq C_3 e^{-d^2-1}/C_1 + C_2 e^{-m/C_2} + e^{-1/\mathbb{E}(m)}.$$ 

Next, taking $m = g_n(t)$, we have that $m \leq n$ so that (since $d \geq 2$) the first term is at most $C_3 e^{-m/C_3}$, and $mf(m) \leq t$ so that the third term is at most $e^{-m}$. Combined, this yields the desired upper bound on $\hat{P}_{\partial e}(1)$, concluding the proof.

4.3 | Fast relaxation within a phase

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Consider the total variation distance of interest,

$$|||\mathbb{P}(\hat{X}_t^1 \in \cdot) - \hat{\pi}_n||_v = |||\mathbb{P}(\hat{X}_t^1 \in \cdot) - \mathbb{P}(\hat{\pi}_t^g \in \cdot)||_v.$$


By the definition of total variation distance, we have under the grand coupling,
\[ \|\mathbb{P}(\hat{X}^I_t \in \cdot) - \hat{\pi}\|_{tv} \leq \mathbb{P}(\hat{X}^I_t \neq \hat{X}^\#_t) \leq \mathbb{P}(\hat{X}^I_t \neq \hat{X}^\#_t, \hat{\tau} > t) + \mathbb{P}(\hat{\tau} \leq t). \]

By Lemma 4.1, for all \( t \leq e^n e^{-d-1}/K \), the second term on the right-hand side is at most \( C_3 e^{-d-n/C_3} \). Let us now control the first term. By a union bound,
\[ \mathbb{P}(\hat{X}^I_t \neq \hat{X}^\#_t, \hat{\tau} > t) \leq \sum_{e \in E(T_n)} \mathbb{P}(\hat{X}^I_t(e) \neq \hat{X}^\#_t(e), \hat{\tau} > t) \]
\[ = \sum_{e \in E(T_n)} \mathbb{E}[1(\hat{X}^I_t(e) \neq \hat{X}^\#_t(e))1(\hat{\tau} > t)]. \]

We can rewrite the indicator of the disagreement as the difference \( \hat{X}^I_t(e) - \hat{X}^\#_t(e) \) on the event \( \hat{\tau} > t \), to get
\[ \mathbb{P}(\hat{X}^I_t \neq \hat{X}^\#_t, \hat{\tau} > t) \leq \sum_{e \in E(T_n)} \left( \mathbb{E}[\hat{X}^I_t(e)1(\hat{\tau} > t)] - \mathbb{E}[\hat{X}^\#_t(e)1(\hat{\tau} > t)] \right) \]
\[ \leq \sum_{e \in E(T_n)} \left( \mathbb{E}[\hat{X}^I_t(e)] - \hat{\pi}_n[\omega_e] + \mathbb{P}(\hat{\tau} \leq t) \right). \]

Combining the above and applying Lemma 4.1, we obtain that for all \( t \leq e^n e^{-d-1}/K \),
\[ \|\mathbb{P}(\hat{X}^I_t \in \cdot) - \hat{\pi}\|_{tv} \leq |E(T_n)| \max_{e \in E(T_n)} \left( \mathbb{E}[\hat{X}^I_t(e)] - \hat{\pi}_n[\omega_e] \right) + C_3 n^d e^{-d-1/C_3}. \]

The quantity in the parentheses is bounded by Proposition 4.1 as \( C_0 e^{-g_n(t)/C_0} \) for all \( t \leq e^n e^{-d-1}/K \). The second term can be absorbed into this term since \( g_n(t) \leq n \leq n^d \) for \( d \geq 2 \).

Finally, the constraint on \( t \) can be dropped by the fact that total variation distance of a Markov chain to stationarity is non-increasing, and the fact that, by definition, \( g_n(t) = g_n(e^n e^{-d-1}/K) \) for all \( t \geq e^n e^{-d-1}/K \).

\[ \Box \]

5 | THE LARGE-q CLUSTER EXPANSION OF THE RANDOM-CLUSTER MODEL

In this section, we follow the presentation of [9, 10] to introduce the combinatorial framework used to understand the random-cluster model at large \( q \). We then use this framework to deduce certain estimates on uniform connectivity probabilities, and later, in Section 6, use these estimates to deduce WSM within the wired and free phases on the respective sides of the critical point. Unless specified otherwise, in this section we will always work in the context of the model on the torus \( \mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d \).

A slightly delicate point in this section is that we perform all the calculations in the context of \( n \) finite and \( p \) within \( o(1) \) of \( p_c \), rather than exactly at \( p_c \).

For consistency with [9, 10] we reparametrize \( p \) by the inverse-temperature parameter \( \beta > 0 \):
\[ \beta(p) : = -\log(1 - p). \]

Recall that, in the case of integer \( q \), this is exactly the parameter transformation that translates the random-cluster measure into the Potts measure via the Edwards–Sokal coupling; however, we will use
it for all (not necessarily integer) \( q \geq 1 \). This naturally gives rise to a critical value \( \beta_c(q, d) = \beta_c(p_c(q, d)) \) for all \( d \geq 2 \) and all \( q \geq 1 \). The critical point of the random-cluster model on \( \mathbb{Z}^d \) has the following asymptotic expansion as \( q \) gets large (see, e.g., [26, Theorem 7.34]):

\[
\beta_c(q, d) = \frac{1}{d} \log q - O(q^{-1/d}). \tag{5.1}
\]

As our aim in this section is to do a large-\( q \) cluster expansion, the estimates we are aiming for will hold for all \( \beta \) bigger than some threshold which must diverge with \( q \). In order to include \( \beta_c(q, d) \) in the window we consider for all large \( q \), define

\[
\beta_h(q, d) := \frac{1}{d} \log q - 1, \tag{5.2}
\]

and observe that by (5.1), for \( q \) sufficiently large (depending on \( d \)), we have \( \beta_h(q, d) < \beta_c(q, d) \).

### 5.1 Combinatorial formalism

For a random-cluster configuration \( \omega \) on \( \mathbb{T}_n \), let \( V(\omega) \) be the set of vertices belonging to some edge of \( \omega \), and let \( H_\omega \) denote the subgraph \((V(\omega), \omega)\). Define the (outer edge) boundary of \( \omega \) by

\[
\partial \omega = \{ e \in E \setminus \omega : e \cap V(\omega) \neq \emptyset \}.
\]

This boundary is naturally partitioned into \( \partial_0 \omega \), where \( e \cap V(\omega) \) is a single vertex, and \( \partial_1 \omega \), where \( e \cap V(\omega) \) is a pair of vertices. Using this notation, it is not hard to check that we can write the weight \( W(\omega) \) as

\[
W(\omega) = q^{\text{Comp}(H_\omega)} e^{-c_{\text{dis}}|V \setminus V(\omega)|} e^{-c_{\text{ord}}|V(\omega)|} e^{-\kappa(|\partial_0 \omega| + 2|\partial_1 \omega|)}, \tag{5.3}
\]

where

\[
c_{\text{dis}} = d \beta - \log q ; \quad c_{\text{ord}} = -d \log(1 - e^{-\beta}) ; \quad \kappa = \frac{1}{2} \log(e^\beta - 1). \tag{5.4}
\]

The expansion we perform will be valid for \( \beta > \log 2 \) so that \( \kappa > 0 \). Notice that as long as \( q \) is large, \( \beta > \beta_h(q, d) \) will imply \( \beta > \log 2 \). The exponential decay in \( |\partial_0 \omega| + 2|\partial_1 \omega| \) will then be the most relevant for us. The set \( \partial \omega \) is roughly the boundary between the wired and free portions of the configuration, and therefore will play the role of “contours” between phases near criticality.

To formalize this notion, it will be helpful to view the set \( \omega \) as a subset of the continuum torus \( T_n = (\mathbb{R}/n\mathbb{Z})^d \) of side-length \( n \). Associate to any configuration \( \omega \subset E(\mathbb{T}_n) \) a continuum set \( \omega \) as follows:

1. For \( k = 1, \ldots, d \), let \( h_k(\omega) \) be the set of unit \( k \)-dimensional hypercubes in \( T_n \) with corners in \( \mathbb{T}_n \) all of whose edges are in \( \omega \), that is, \( h_k(\omega) \cap E(\mathbb{T}_n) \subset \omega \).

2. Let

\[
\omega = \left\{ x \in T_n : \min_{k=1,\ldots,d} d_\infty(x, h_k(\omega)) \leq 1/4 \right\}.
\]

Let \( \partial \omega \) denote the boundary of \( \omega \subset T_n \).
5.2 The contour representation of a random-cluster configuration

It will be important to express the random-cluster distribution as a distribution over collections of “contours,” defined as follows.

**Definition 5.1.** A contour is a (maximal) connected component of \( \partial \omega \). The set of all contours for a given configuration \( \omega \) is denoted \( \Gamma(\omega) \).

We can also use \( \partial \omega \) to split the continuum torus \( T_n \) into two parts, one corresponding to the open components and one to their complement, as follows.

**Definition 5.2.** Consider \( T_n \setminus \partial \omega \). Its connected components are labeled ordered if they are a subset of \( \omega \), and disordered if they are a subset of \( T_n \setminus \omega \). The union of the ordered components form \( \omega_{\text{ord}} \) and the union of the disordered components form \( \omega_{\text{dis}} \). Denote by \( \ell_A \) the label of a connected component \( A \) of \( T_n \setminus \partial \omega \).

With this, we obtain the following bijection between configurations and labeled collections of contours.

**Definition 5.3.** A collection of contours \( \Gamma \) and a labeling \( \ell \) are admissible if (i) \( d_{\infty}(\gamma_1, \gamma_2) \geq 1/2 \) for every \( \gamma_1, \gamma_2 \in \Gamma \); and (ii) \( \ell \) assigns to each connected component of \( T_n \setminus \Gamma \) one of \{dis, ord\} in such a way that adjacent connected components get distinct labels.

The map from \( \omega \) to \( (\Gamma, \ell) \) is then a bijection between configurations and admissible collections of contours with a labeling; see [9, Lemma 3.2].

In what follows, for a continuum set \( A \), let \( |A| \) denote the number of vertices in \( A \cap T_n \). (This notation will not cause any confusion since we will never measure the actual volumes of continuum sets.) Also, for a contour \( \gamma \), let \( ||\gamma|| = |\gamma \cap E(T_n)| \) (understood as the cardinality of the discrete set \( \gamma \cap E(T_n) \) with the natural embedding of \( E(T_n) \) in \( T_n \)). Recalling the parameters \( (5.4) \), we can rewrite \( (5.3) \) as

\[
W(\omega) = q^{\text{Comp}(\omega_{\text{ord}})} e^{-c_{\text{dis}}|\omega_{\text{dis}}|} e^{-c_{\text{ord}}|\omega_{\text{ord}}|} \prod_{\gamma \in \Gamma(\omega)} e^{-k||\gamma||}.
\]

Here, for a continuum set \( A \), we understand \( \text{Comp}(A) \) to mean the number of connected components of \( A \).

5.3 Topologically trivial and non-trivial contours

Using the continuum embedding of the random-cluster configuration, we now define two types of contours on the torus: topologically non-trivial and topologically trivial ones. The former are contours that “wrap around” the torus, and are referred to as interfaces in [10].

**Definition 5.4.** A contour is topologically non-trivial if it has an odd number of intersections with the \( i \)th fundamental loop \( \{x \in T_n : x_j \neq 1 \text{ for } j \neq i\} \) for some \( i \). Otherwise, a contour is called topologically trivial. Denote the sets of topologically trivial and non-trivial contours by \( \Gamma_0(\omega) \) and \( \Gamma_1(\omega) \), respectively.

For notational purposes, we use this to partition the random-cluster model’s state space into

\[
\Omega_{\text{tunnel}} = \{ \omega \subset T_n : \Gamma_1(\omega) \neq \emptyset \} \quad \text{and} \quad \Omega_{\text{rest}} = \{ \omega \subset T_n : \Gamma_1(\omega) = \emptyset \}.
\]
The following shows that the existence of topologically non-trivial contours is exponentially unlikely uniformly over all $\beta$, which allows us to essentially disregard topologically non-trivial contours in what follows. We obtain this result by combining [10, Lemma 6.1 (a)] with [16, Theorem 1.2].

**Lemma 5.1.** For each fixed $d \geq 2$, there exist constants $c > 0$ and $q_0$ such that for all $q \geq q_0$,

$$\pi_n(\Omega_{\text{tunnel}}) \leq \exp(-c(\beta \vee 1)n^{d-1}).$$

**Proof.** If $\beta$ is larger than $\beta_h$ from (5.2), then—as noted in [10, lines following (A.1)]—(i) and (ii) in Lemma 6.3 of [10] hold, which is all that is used in establishing item (a) of Lemma 6.1 therein, giving the bound $\exp(-c_1 \beta n^{d-1})$ on the probability of $\Omega_{\text{tunnel}}$ for all $\beta \geq \beta_h$. To stitch this together with a similar bound for $\beta \leq \beta_h$, notice that for every $\beta < \beta_h$, by [16, Theorem 1.2] there exists $c_2(\beta)$ for which one has exponential tails with rate $c_2$ on connected component sizes, and $\Omega_{\text{tunnel}}$ necessitates a connected component of size at least $\Omega(n^{d-1})$. Since $\beta_h < \beta_c$, we can maximize $c_2$ on the closed interval $\beta \in [0, \beta_h]$ and stitch this together with $c_1$ to obtain the claimed bound for all $\beta$.

\[ \blacksquare \]

### 5.4 Geometry of topologically trivial contours

Our aim is now to express the distribution on $\Omega_{\text{rest}}$ as a union of an ordered phase $\Omega_{\text{ord}}$ and a disordered phase $\Omega_{\text{dis}}$, each of which admit convergent polymer expansions in terms of the topologically trivial contours. Abusing notation, from now on, unless otherwise stated, we will use the unqualified term “contour” to mean a topologically trivial contour.

It was observed in [10, Lemma 4.3] that any such contour $\gamma \in \Gamma_0(\omega)$ has the property that $T_n \setminus \gamma$ consists of exactly two connected components, one of which is an exterior $\text{Ext}(\gamma)$ and the other an interior $\text{Int}(\gamma)$, with the internal one being the one that is simply connected following [10, Definition 4.4]. A contour $\gamma$ is called outermost or external if there is no contour $\gamma'$ such that $\text{Int}(\gamma) \subseteq \text{Int}(\gamma')$.

Given an admissible contour collection $\Gamma$, there is a single connected component, denoted $\text{Ext}(\Gamma)$, of $T_n \setminus \Gamma$ that is external in that it is disjoint from $\text{Int}(\gamma)$ for every $\gamma \in \Gamma$. In other words, $\text{Ext}(\Gamma) = \bigcap_{\gamma} \text{Ext}(\gamma)$. Because of its first characterization, $\text{Ext}(\Gamma)$ receives a label among $\{\text{dis, ord}\}$ under any admissible labeling of $\Gamma$. Furthermore notice that, given a collection of admissible contours, knowing the label of $\text{Ext}(\Gamma)$ dictates the labels of all connected components of $T_n \setminus \Gamma$. Note that a labeling $\ell$ of the connected components of $T_n \setminus \Gamma$ can also be interpreted as a labeling of contours, where $\ell_{\gamma} := \ell_{\text{Int}(\gamma)}$. (We warn that this is the opposite labeling convention for contours from that used in [10].)

We next partition $\Omega_{\text{rest}}$ into two sets defined as follows:

$$\Omega_{\text{ord}} = \{ \omega \in \Omega_{\text{rest}} : \ell_{\text{Ext}(\Gamma(\omega))} = \text{ord} \} \quad \text{and} \quad \Omega_{\text{dis}} = \{ \omega \in \Omega_{\text{rest}} : \ell_{\text{Ext}(\Gamma(\omega))} = \text{dis} \}.$$  

We can then express the partition function of the random-cluster model on $T_n$ restricted to $\Omega_{\text{ord}}$ as a sum of products as follows. For any continuum set $A$, we say a contour $\gamma$ is in $A$ if $d_\infty(\gamma, \partial A) \geq 1/2$, and define $G_{\text{ext}}(A)$ to be the set of all mutually external contour collections in $A$ (meaning all contour collections in $A$ in which all the contours are outermost). If we are in $\Omega_{\text{ord}}$, then the label of all outermost
contours must be dis, and vice versa; accordingly, we can divide the set of mutually external contour collections into \( \mathcal{C}_\text{ext}^{\text{ord}}(A) \) and \( \mathcal{C}_\text{ext}^{\text{dis}}(A) \). Then, define

\[
Z_{\text{ord}}(A) := \sum_{\Gamma \in \mathcal{C}_\text{ext}^{\text{ord}}(A)} e^{-c_{\text{ord}}|A\cap \text{Ext}(\Gamma)|} \prod_{\gamma \in \Gamma} e^{-\kappa\|\gamma\|} Z_{\text{dis}}(\text{Int}(\gamma)) ; \tag{5.6}
\]

\[
Z_{\text{dis}}(A) := \sum_{\Gamma \in \mathcal{C}_\text{ext}^{\text{dis}}(A)} e^{-c_{\text{dis}}|A\cap \text{Ext}(\Gamma)|} \prod_{\gamma \in \Gamma} e^{-\kappa\|\gamma\|} q Z_{\text{ord}}(\text{Int}(\gamma)) . \tag{5.7}
\]

In the case \( A = T_n \), these give the partition function restricted to \( \Omega_{\text{ord}} \) and \( \Omega_{\text{dis}} \) respectively:

\[
Z_{\text{dis}}(T_n) = \sum_{\omega \in \Omega_{\text{dis}}} W(\omega) ; \quad q Z_{\text{ord}}(T_n) = \sum_{\omega \in \Omega_{\text{ord}}} W(\omega).
\]

Recursing over Equations (5.6) and (5.7), and letting \( \mathcal{C}_\text{dis}^{\text{ord}}(A) \) and \( \mathcal{C}_\text{ord}^{\text{dis}}(A) \) be the sets of compatible collections of contours in \( A \) all receiving label dis, or ord respectively, we end up with

\[
Z_{\text{ord}}(A) = e^{-c_{\text{ord}}|A|} \sum_{\Gamma \in \mathcal{C}_\text{ord}^{\text{ord}}(A)} \prod_{\gamma \in \Gamma} K_{\text{ord}}(\gamma), \quad \text{where} \quad K_{\text{ord}}(\gamma) = e^{-\kappa\|\gamma\|} Z_{\text{dis}}(\text{Int}(\gamma)) / Z_{\text{ord}}(\text{Int}(\gamma)) ; \tag{5.8}
\]

\[
Z_{\text{dis}}(A) = e^{-c_{\text{dis}}|A|} \sum_{\Gamma \in \mathcal{C}_\text{dis}^{\text{dis}}(A)} \prod_{\gamma \in \Gamma} K_{\text{dis}}(\gamma), \quad \text{where} \quad K_{\text{dis}}(\gamma) = e^{-\kappa\|\gamma\|} q Z_{\text{ord}}(\text{Int}(\gamma)) / Z_{\text{dis}}(\text{Int}(\gamma)). \tag{5.9}
\]

The representation above, and variations on it, will allow us to perform various manipulations of partition functions to establish exponential tails on contour sizes and so forth. The key ingredient to proving exponential tails comes from [10, Lemma 6.3, items (i) and (ii)] which we restate here for the reader’s convenience.

In what follows, for \( i \in \{\text{dis, ord}\} \), define the free energies

\[
f_i^n = -\frac{1}{n^q} \log Z_i(T_n) \quad \text{and} \quad f_i = \lim_{n \to \infty} f_i^n .
\]

**Lemma 5.2** ([10], Lemma 6.3, items (i) and (ii)). Fix \( d \geq 2 \). There exist \( q_0(d), c > 0 \) and functions \( f_{\text{ord}}, f_{\text{dis}} \) such that the following statements hold for \( i \in \{\text{dis, ord}\} \), all \( q \geq q_0 \) and all \( \beta \geq \beta_h \).

(i) Let \( f = \min\{f_{\text{ord}}, f_{\text{dis}}\} \) and let \( a_i = f_i - f \). If \( \gamma \) is such that \( \mathcal{E}_{\text{Ext}}(\gamma) = i \), and \( a_i \text{diam}(\gamma) \leq c\beta \), then

\[
K_i(\gamma) \leq e^{-c\beta\|\gamma\|} .
\]

(ii) If \( A \subset T_n \), then

\[
Z_i(A) \geq e^{-\langle f_i + e^{-c\beta}\rangle |A|} e^{-|\partial A|},
\]

and if \( \zeta \) is the opposite label of \( i \), then

\[
Z_i(A) \leq e^{-\langle f_i - e^{-c\beta}\rangle |A|} e^{2|\partial A|} \max_{\Gamma \in \mathcal{C}_\text{ext}(A)} e^{-\frac{\zeta c}{2} |A\cap \text{Ext}(\Gamma)|} \prod_{\gamma \in \Gamma} e^{-\frac{\zeta c}{2}\|\gamma\|}.
\]
Exponential tails on topologically trivial contours

Given Lemma 5.2, our aim in what follows is to establish that both ordered measures and disordered measures have exponential tails on their contour lengths, that is, $||\gamma||$, even after conditioning on a compatible collection of contours. In what follows, let

$$\pi_{\text{ord}} := \pi_n(\cdot | \Omega_{\text{ord}}) \quad \text{and} \quad \pi_{\text{dis}} := \pi_n(\cdot | \Omega_{\text{dis}}).$$

We formalize the above statement in the following lemma.

**Lemma 5.3.** For all $\beta \geq \beta_c(q,d) - o(n^{-1})$ and $q \geq q_0(d)$, we have for all $v \in T_n$ and all fixed collections $\Gamma_{\text{ext}} \in \mathcal{G}_{\text{ext}}(T_n)$, none of which contain $v$ in their interior,

$$\pi_{\text{ord}}(||\gamma_v|| \geq r | \Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq Ce^{-\beta r/C}.$$

For all $\beta \leq \beta_c(q,d) + o(n^{-1})$, we have for all $\Gamma_{\text{ext}} \in \mathcal{G}_{\text{ext}}(T_n)$, none of which contain $v$ in their interior,

$$\pi_{\text{dis}}(||\gamma_v|| \geq r | \Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq Ce^{-(\beta v_1)r/C}.$$

**Proof.** We first prove the claim for the ordered phase. Fix any collection $\Gamma_{\text{ext}}$, and fix a $\gamma$ compatible with $\Gamma_{\text{ext}}$ such that $v \in \text{Int}(\gamma)$. Following the derivation of (5.8), observe that the weight of configurations in $A$ with ordered external label, having external contour set containing $\Gamma_{\text{ext}}$, is

$$Z_{\text{ord}}(\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) = e^{-c_{\text{ord}}|A|} \sum_{\gamma : \gamma \in \mathcal{G}_{\text{ext}}(A) \land \gamma \in \Gamma_{\text{ext}}} \prod_{\gamma \in \Gamma_{\text{ext}}} K_{\text{ord}}(\gamma). \quad (5.10)$$

With the choice $A = T_n$, we wish to consider the ratio

$$\frac{Z_{\text{ord}}(\Gamma_{\text{ext}} \cup \gamma \subset \Gamma_{\text{ext}})}{Z_{\text{ord}}(\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}})} = \pi_{\text{ord}}(\gamma \in \Gamma_{\text{ext}} | \Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}).$$

This is done by defining a map from contour collections $\Gamma$ with $(\gamma, \Gamma_{\text{ext}}) \in \Gamma_{\text{ext}}$ to contour collections $\Gamma$ having $\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}$, that simply deletes the contour $\gamma$. Evidently, for fixed $\gamma$ this map is injective (the pre-image can be uniquely recovered by adding back $\gamma$). As a result, we obtain

$$\pi_{\text{ord}}(\gamma \in \Gamma_{\text{ext}} | \Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq K_{\text{ord}}(\gamma).$$

Now recall item (i) of Lemma 5.2 which bounds $K_{\text{ord}}(\gamma)$ by $e^{-c||\gamma||}$ when $f_{\text{ord}} \leq f_{\text{dis}}$. In fact, when $\beta$ is such that

$$f_{\text{ord}} \leq f_{\text{dis}} + o(n^{-1}) \quad \text{so that} \quad a_{\text{ord}} \leq o(n^{-1}), \quad (5.11)$$

...
then we have the bound

\[ \pi_{\text{ord}}(\vec{y} \in \Gamma_{\text{ext}}|\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq e^{-c\beta\|\vec{y}\|} \text{ for all } \vec{y} \text{ in } T_n. \]

Here we used the fact that the maximal diameter of \( \vec{y} \) in \( T_n \) is \( n \) (the specific notion of diameter in \( T_n \) used here is given by [10, Section 5.4]). Now, we can sum this bound over all possible \( \vec{y} \) nesting \( v \). Using the fact (see e.g., [10, Item (iii) of Lemma 5.8]) that for some constant \( K = K(d) \), there are at most \( K^r \) many possible \( \vec{y} \)'s nesting a fixed vertex \( v \) and having the property \( \|\vec{y}\| = r \), we have by a union bound

\[ \pi_{\text{ord}}(\|y_v\| \geq r|\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq \sum_{k \geq r} \sum_{\|\vec{y}\| = k} \pi_{\text{ord}}(\vec{y} \in \Gamma_{\text{ext}}|\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \]
\[ \leq \sum_{k \geq r} K^r e^{-c\beta r} \leq Ce^{-\beta r/C}. \]

It remains to show that \( \beta \geq \beta_c - o(n^{-1}) \) implies the assumption of (5.11). When \( \beta \geq \beta_c \), we have \( a_{\text{ord}} = f_{\text{ord}} - \min\{f_{\text{ord}}, f_{\text{dis}}\} = 0 \) (essentially by definition of \( f_{\text{ord}}, f_{\text{dis}} \): see [10, item (iii) of Lemma 6.3]). We claim that \( a_{\text{ord}} \) depends in a uniformly Lipschitz manner on \( \beta \). To see this, differentiate the finite-volume free energies \( f_i^n \) for \( i \in \{\text{dis, ord}\} \) in \( p \) to get

\[ \frac{d}{dp} f_i^n = -\frac{1}{Z_n} \sum_{\omega \in \Omega} \frac{d}{dp} W(\omega) = \pi_i [(1 - p)^{-1} n^{-d} (|E| - p^{-1} |\omega|)], \tag{5.12} \]

which is non-negative and bounded by \( d(1 - p)^{-1} \) uniformly in \( n \). Taking into account the factor \( dp/d\beta = e^{-\beta} \), which is uniformly bounded by 1, we find that the derivative \( \frac{d}{d\beta} f_i^n \) in \( \beta \), evaluated at \( \beta_c \), is at most \( d(1 - p)^{-1} \). The Lipschitz constants of \( f_{\text{ord}} \) and \( f_{\text{dis}} \) are thus bounded by some absolute constant when \( \beta \approx \beta_c \), and as a consequence we find that \( a_i = o(1/n) \) when \( |\beta - \beta_c| = o(1/n) \).

The analogous bound for the disordered phase is completely symmetrical for \( \beta_h \leq \beta \leq \beta_c + o(n^{-1}) \). To see that it extends to the even higher temperature regime \( \beta \in [0, \beta_h] \), we use the results of [16]. Since \( \beta_h < \beta_c \), the role of conditioning on \( \Omega_{\text{dis}} \) is negligible so it suffices to prove the bound under \( \pi_n \) itself. In order for there to be a contour \( \gamma \) such that \( \|y_v\| \geq r \), there must be a connected component in \( \omega \) of diameter at least \( r \). Expose the connected components bounded by the contours of \( \Gamma \), inducing free boundary conditions on \( \text{Ext}(\Gamma) \). We wish to bound the probability of there existing a connected component of size at least \( k \geq r \) at distance at most \( k \) from \( v \) (so that it can nest \( v \)). The fact that we condition on the contours \( \Gamma \) being external is equivalent to there not being any connected component nesting one of those contours, a decreasing event, so by the FKG inequality we can drop that conditioning and consider the probability of there existing a connected component of size at least \( k \geq r \) at distance at most \( k \) from \( v \) under \( \pi_{\text{Ext}(\Gamma)^c} \). This decays exponentially in \( k \) for every \( k \) by a union bound and the exponential tails on connected components from [16].

Using essentially the same argument, we can also obtain a version of Lemma 5.3 that applies when the phase is picked by boundary conditions, rather than by conditioning on \( \Omega_{\text{ord}} \) or \( \Omega_{\text{dis}} \).
**Lemma 5.4.** For all $\beta \geq \beta_c(q, d) - o(n^{-1})$ and $q \geq q_0(d)$ we have, for all fixed collections $\Gamma_{\text{ext}} \in C_{\text{ext}}^{\text{dis}}(\Theta_n)$,

$$\pi_{\Lambda_n}(||\mathcal{Y}_n|| \geq r|\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq Ce^{-\beta r/C}.$$  

Similarly, for all $\beta \leq \beta_c(q, d) + o(n^{-1})$ we have, for all fixed $\Gamma_{\text{ext}} \in C_{\text{ext}}^{\text{ord}}(\Theta_n)$,

$$\pi_{\Lambda_n}(||\mathcal{Y}_n|| \geq r|\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq Ce^{-(\beta v_1)r/C}.$$  

**Proof.** We can do a similar expansion of the partition functions of the random-cluster model on $\Lambda_n^1$ and $\Lambda_n^0$ as done on the torus to arrive at (5.6) and (5.7). One indirect way to do so is to naturally embed $\Lambda_n$ in $\mathbb{T}_{3n}$, and view the boundary conditions as coming either from the all-wired or all-free configurations on $E(\mathbb{T}_{3n} \setminus \Lambda_n)$. For instance, the contour given by $\partial \Theta_n$, with (internal) label ord, would necessarily wire all edges along $\partial \Lambda_n$, which is the same as that model with wired boundary conditions up to a uniform proportionality factor.

This was the argument carried out in [9, Proposition 3.14]. If $Z_{\Lambda_n}^1$ and $Z_{\Lambda_n}^0$ denote the partition functions associated with $\pi_{\Lambda_n}^1$ and $\pi_{\Lambda_n}^0$ respectively, then it was shown there that there exist contours $\mathcal{Y}_n^1, \mathcal{Y}_n^0 \subset \mathbb{T}_{3n}$ such that

$$Z_{\Lambda_n}^1 = qp^{E(\Lambda_n)}d|\text{Int}(\mathcal{Y}_n^1)|Z_{\text{ord}}(\text{Int}(\mathcal{Y}_n^1)) \quad \text{and} \quad Z_{\Lambda_n}^0 = (1 - p)|\mathcal{Y}_n^0|^2Z_{\text{dis}}(\text{Int}(\mathcal{Y}_n^0)). \quad (5.13)$$

At this point, recall that (5.10) applied for general $A$, so we can take $A = \text{Int}(\mathcal{Y}_n^1)$. From this starting point, one establishes that, for every fixed $\mathcal{Y}$ and fixed $\Gamma_{\text{ext}}$,

$$\pi_{\Lambda_n}(\mathcal{Y} \in \Gamma_{\text{ext}}|\Gamma_{\text{ext}} \subset \Gamma_{\text{ext}}) \leq K_{\text{ord}}(\mathcal{Y})$$

exactly as in the previous proof, noting that the extra coefficients in (5.13) cancel out between the numerator and denominator when passing to probabilities. The remainder of the proof is unchanged. \qed

### 5.6 Equivalence of different notions of phases

We now use Lemma 5.3 to establish the following equivalence between the restrictions of the measure $\pi_n$ to $\Omega_{\text{ord}}$ and to $\hat{\Omega}$ when $q \geq q_0$. We emphasize that the reason we presented the results in Section 4 for $\hat{\Omega}$ and $\hat{\Omega}$ is that when $q$ is intermediate, that is, larger than $q_c(d)$ so there is phase coexistence at $p_c$ but not very large, one does not expect Lemma 5.3 to hold, but still the coexistence of $\hat{\Omega}$ and $\hat{\Omega}$ should hold.

In what follows, we use $A \Delta B$ to denote the symmetric difference between sets $A$ and $B$.

**Lemma 5.5.** Fix $d \geq 2$ and $q \geq q_0$. Let $\epsilon < \frac{1}{2}$ be any value used in the definition of $\hat{\Omega}$. There exists $c(\epsilon) > 0$ such that uniformly over all $\beta \geq \beta_c(q, d) - o(n^{-1}),$

$$\pi_n(\Omega_{\text{ord}} \Delta \hat{\Omega}) \leq Ce^{-\beta q d^{d-1}} \quad \text{and} \quad \|\pi_{\text{ord}} - \hat{\pi}_n\|_1 \leq Ce^{-\beta q d^{d-1}}.$$  

An analogous statement holds for $\Omega_{\text{dis}}$ and $\hat{\Omega}$ at all $\beta \leq \beta_c(q, d) + o(n^{-1}).$
The first of these two terms is at most $Ce^{-c\beta \delta^{-1}}$ by Lemma 5.1. For the second term, if \( \beta \geq \beta_c + \frac{c(\delta)}{n} \) is such that \( d_{\text{dis}} > \frac{S}{n} \), for a \( \delta \) to be chosen later, then a combination of the bounds in item (ii) of Lemma 5.2 yields
\[
\pi_n(\Omega_{\text{dis}}) \leq \frac{Z_{\text{dis}}(T_n)}{Z_{\text{ord}}(T_n)} \leq e^{-\alpha_{\text{ord}}|T_n|} e^{-c\beta |T_n|} \leq e^{-\delta \beta \delta^{-1}/2},
\]
when \( n \) is large enough. On the other hand, when \( |\beta - \beta_c| \leq c(\delta)/n \) we use the fact that \( \hat{\Omega} \cap \Omega_{\text{dis}} \) requires that there exists a contour \( \gamma \in \Gamma \) having \( |\text{Int}(\gamma)| \geq \varepsilon n^d \). As long as \( \delta < 1 \) in the choice of \( \beta \) above, the exponential tails on \( |\gamma| \) in Lemma 5.3 apply because \eqref{eq:5.11} is satisfied, and we get by a union bound
\[
\pi_n(\hat{\Omega} \cap \Omega_{\text{dis}}) \leq \pi_{\text{dis}}(\exists v : |\text{Int}(\gamma_v)| \geq \varepsilon n^d) \leq \pi_{\text{dis}}(\exists v : \|\gamma_v\| \geq \varepsilon n^{d-1}) \leq Ce^{-\beta \varepsilon n^{d-1}/C}.
\]

We now turn to bounding \( \pi_n(\Omega_{\text{ord}} \setminus \hat{\Omega}) \). In order for \( \Omega_{\text{ord}} \setminus \hat{\Omega} \) to occur, it must be that \( |\text{Ext}(\Gamma)| \leq \varepsilon n^d \) since \( \text{Ext}(\Gamma) \) is itself a connected component whenever \( \omega \in \Omega_{\text{ord}} \). This event was shown to have probability at most \( e^{-c\beta \delta^{-1}} \) by \cite[Lemma 6.2]{10} uniformly over \( \beta \geq \beta_c \). To extend the bound to \( |\beta - \beta_c| \leq n^{-1} \), we use the fact that the Radon–Nikodym derivative between \( \pi_{n,p} \) and \( \pi_{n,p_c} \) is bounded as
\[
\left\| \frac{\pi_{n,p}}{\pi_{n,p_c}} \right\|_{\infty} \leq (1 + |p - p_c|)^{|E|},
\]
which is clearly \( \exp(o(n^{d-1})) \) if \( |p - p_c| = o(n^{-1}) \), or equivalently if \( |\beta - \beta_c| = o(n^{-1}) \).

The total variation bound on the conditional distributions follows from the fact that, for any event \( A \),
\[
\left| \frac{\pi_n(A \cap \Omega_{\text{ord}})}{\pi_n(\Omega_{\text{ord}})} - \frac{\pi_n(A \cap \hat{\Omega})}{\pi_n(\hat{\Omega})} \right| \leq \frac{1}{\pi_n(\Omega_{\text{ord}})} |\pi_n(A \cap \Omega_{\text{ord}}) - \pi_n(A \cap \hat{\Omega})| + \frac{\pi_n(A \cap \Omega_{\text{ord}})}{\pi_n(\Omega_{\text{ord}})} |\pi_n(\hat{\Omega}) - \pi_n(\Omega_{\text{ord}})| \leq \frac{2}{\pi_n(\Omega_{\text{ord}})} \cdot \pi_n(\Omega_{\text{ord}} \triangle \hat{\Omega}).
\]

Using the fact that \( \pi_n(\Omega_{\text{ord}}) \) is uniformly bounded away from zero for all \( \beta \geq \beta_c \) (see\cite[eq. (6.7)]{10}), and the Radon–Nikodym bound of \eqref{eq:5.14}, the first term above is \( \exp(o(n^{d-1})) \) uniformly over \( \beta \geq \beta_c - o(n^{-1}) \), and the second symmetric difference term is what we already bounded above.

The proof for \( \Omega_{\text{dis}} \) and \( \hat{\Omega} \) is symmetric for \( \beta_h \leq \beta \leq \beta_c + o(1/n) \). To extend it to \( \beta \in [0, \beta_h] \), notice that each of the quantities we would bound in the symmetric proof for the high-temperature regime in the above argument is governed by events that involve the existence of a connected component of size at least \( n^{d-1} \), for which we can use the bound of \cite{16}.

\[\blacksquare\]
5.7 | Exponential stability of the wired and free phases

The aim in this section is to establish exponential stability, as defined in Definition 4.2, for the wired and free phases in the right parameter regimes, when \( q \) is sufficiently large.

**Lemma 5.6.** Fix \( d \geq 2 \) and \( q \geq q_0(d) \). For any \( \varepsilon < \frac{1}{2} \), there exists \( C_1(\varepsilon) \) such that for all \( \beta \geq \beta_c - o(n^{-1}) \)

\[
\hat{\pi}_n(\partial \hat{\Omega}) \leq C_1 e^{-\varepsilon n^d / C_1}.
\]

The symmetric statement holds for \( \check{\pi}_n(\partial \check{\Omega}) \) for all \( \beta \leq \beta_c + o(n^{-1}) \).

**Proof.** By [10, eq. (6.7)], we have that \( \pi_n(\Omega_{ord}) \geq q/(q+1) - o(1) \) when \( p \geq p_c(q,d) \).

More generally using Lemma 5.5 to compare \( \check{\Omega} \) to \( \Omega_{ord} \), and the Radon–Nikodym bound of (5.14), we have

\[
\pi_n(\hat{\Omega}) \geq \exp(-o(n^{d-1})),
\]

uniformly over all \( \beta \geq \beta_c - o(n^{-1}) \). Next consider the mass of \( \partial \hat{\Omega} \) which consists of the set of configurations which are one edge-flip away from \( \hat{\Omega}^{c} \). By Lemma 5.5, and the fact that \( \pi_n(\Omega_{ord}) \geq \exp(-o(n^{d-1})) \) uniformly over \( \beta \geq \beta_c - o(n^{-1}) \), it suffices to provide an \( \exp(-\Omega(n^{d-1})) \) bound for some \( \delta > 0 \) on

\[
\pi_{ord}(\partial \hat{\Omega}) \leq \pi_{ord}(|\text{Ext}(\Gamma)|) \leq \left( \frac{1}{2} + \delta \right)n^d + \pi_{ord}(\exists v : \|r_v\| \geq \delta n^{d-1}).
\]

The inequality here can be seen because in order for a configuration to be one spin-flip away from having no component of size larger than \( \varepsilon n^d \), for any \( \varepsilon < 1/2 \), either its external component is smaller than size \( 1/2 + \delta \), or if not then its external component is its largest component, and it has a cut-edge \( e \in \omega \) such that if the state of \( e \) is changed to closed, then it splits off a component of size at least \( \delta n^d \). In order for this latter situation to occur, there must be a contour having size at least \( \delta n^{d-1} \).

As long as \( q \) is sufficiently large, the first of the two terms of (5.15) has probability at most \( e^{-c\beta n^{d-1}} \) by [10, Lemma 6.2] uniformly over \( \beta \geq \beta_c \), extended to \( \beta \geq \beta_c + o(1/n) \) by the Radon–Nikodym bound we have been using repeatedly above. The second term in (5.15) is bounded by \( Ce^{-c\beta n^{d-1}} \) per Lemma 5.3 and a union bound over \( v \). Altogether, that yields the claimed bound.

The proof for \( \check{\pi}_n(\partial \check{\Omega}) \) is similar for \( \beta_h \leq \beta \leq \beta_c + o(1/n) \). It effectively reduces to bounding the probability of there existing two connected components whose sizes together sum up to \( \varepsilon n^d \), which in particular requires there to be at least one contour under \( \pi_{dis} \) of size \( \varepsilon n^{d-1}/2 \). To extend it to \( \beta \in [0, \beta_h] \), one can work directly with the event on connected components in \( \omega \), and apply the exponential tails on component sizes from [16].

5.8 | The unstable phase near criticality

In order for us to be able to faithfully sample from the random-cluster distribution near the critical point in finite volumes without knowing the exact value of the critical point, it will be important to have an understanding of how close to the critical point the unstable phase becomes negligible. The following lemma shows that the \( o(n^{-1}) \) window of temperatures around \( p_c \) through which the above
estimates apply is sufficient to cover the window of temperatures at which that phase contributes more than exponentially small mass.

**Lemma 5.7.** Fix $d \geq 2$ and $q \geq q_0(d)$ and any $\delta > 0$. For every $p \geq p_c + \Omega(n^{-(1+\delta)})$,

$$
\pi_n(\Omega_{\text{dis}}) \leq \exp(-cn^{d-1-\delta}).
$$

The same bound holds for $\pi_n(\Omega_{\text{ord}})$ as soon as $p \leq p_c - \Omega(n^{-(1+\delta)})$.

**Proof.** Recall from [10] that $\beta_c$ is the unique point at which $f_{\text{ord}} = f_{\text{dis}}$, and when $\beta > \beta_c$ necessarily $f_{\text{dis}} > f_{\text{ord}}$. Our aim is to understand the asymptotic dependence of $a_{\text{dis}} = f_{\text{dis}} - f_{\text{ord}}$ on $(\beta - \beta_c)$ from above. Recall that $f_i = \lim_{L \to \infty} f_i^L$ and by the line above (A.8) in [10], $|f_i - f_i^L| \leq 2e^{-c\beta L}$. Letting $a_{\text{dis}}^L = f_{\text{dis}}^L - f_{\text{ord}}^L$, it suffices therefore to control (uniformly in $L$) the dependence of $a_{\text{dis}}^L$ on $(\beta - \beta_c)$.

For this purpose, differentiating as in (5.12),

$$
\frac{d}{dp} a_{\text{dis}}^L = \frac{d}{dp} (f_{\text{dis}}^L - f_{\text{ord}}^L) = p^{-1}(1-p)^{-1} (\pi_{\text{ord}}[L^{-d}|\omega|] - \pi_{\text{dis}}[L^{-d}|\omega|]),
$$

where $\pi_{\text{ord}}$ and $\pi_{\text{dis}}$ denote those measures on $\mathbb{T}_L$. By [10, Lemma 6.2], when $q$ is sufficiently large, for all $\beta \geq \beta_c$, under $\pi_{\text{ord}}$, $|E(\text{Ext}(\Gamma))| \leq |\omega|$ is itself at least 3|E|/4 with probability $1 - \alpha_L(1)$. On the other hand, while $\beta - \beta_c = o(L^{-1})$, $\pi_{\text{dis}}$ satisfies the exponential tail bound of Lemma 5.3, and as a consequence, when $q$ is sufficiently large, the expected fraction of open edges, which is bounded by $d$ times the expected number of vertices having non-empty $\gamma_v$, is at most $|E|/4$ (say). Combined with the factors $p^{-1}(1-p)^{-1}$ and $\frac{dp}{d\beta}$ we find that, uniformly over $L$, at $\beta_c$, $\frac{d}{d\beta} a_{\text{dis}}^L$ is at least some constant $c_0(p, q, d) > 0$.

Continuity of $a_{\text{dis}}^L$ in $\beta$ (as noted in [10, Appendix A.4]) then implies that, for $\beta > \beta_c$,

$$
a_{\text{dis}}^L(\beta) \geq a_{\text{dis}}^L(\beta_c) + c_0(\beta - \beta_c) + O((\beta - \beta_c)^2).
$$

These being free energies corresponding to absolutely convergent power series, when we pass to the $L \to \infty$ limit, we get the same inequality for $a_{\text{dis}}(\beta)$. Now consider a sequence of $\beta$ such that $\beta = \beta_c + n^{-1+\delta}$ for a fixed $\delta > 0$. At such $\beta$, there is some $c_1$ such that $a_{\text{dis}}(\beta) \geq c_1 n^{-1+\delta}$. Fix this $\beta$ and consider the quantity

$$
\pi_n(\Omega_{\text{dis}}) \leq \frac{Z_{\text{dis}}(T_n)}{Z} \leq \frac{Z_{\text{dis}}(T_n)}{Z_{\text{ord}}(T_n)}.
$$

By combining the two parts of item (ii) in Lemma 5.2, as observed in [9, eq. (30)], this satisfies

$$
\frac{Z_{\text{dis}}(T_n)}{Z_{\text{ord}}(T_n)} \leq \exp(-\frac{1}{4} \min\{a_{\text{dis}} n^d, c \beta n^{d-1}\}).
$$

Applying our bound on $a_{\text{dis}}$ then implies the desired bound at $\beta = \beta_c + n^{-(1+\delta)}$. The fact that the bound holds for all larger $\beta$ follows from monotonicity of the random-cluster model and the fact that $\Omega_{\text{dis}}$ is a decreasing event. The bound on $\pi_n(\Omega_{\text{ord}})$ follows symmetrically.
6 | SPATIAL MIXING PROPERTIES OF THE RANDOM-CLUSTER MODEL

Recall the notions of WSM, SSM, and WSM within a phase from Definitions 3.1, 3.2, and 4.1. In this section, we will establish reductions from these notions of spatial mixing to certain exponential decay properties on contours with label ord or dis. These reductions will make it easier to ascertain the off-critical parameter regimes in which WSM and SSM hold, and will also allow us to establish the WSM within a phase property at large $q$, leveraging the arguments of Section 5. In particular, this section also includes proofs of two of our main results on mixing times, Corollaries 1.1 and 1.2 from the introduction.

Since we are interested in all temperature regimes, we need to consider both ord and dis connectivities simultaneously. Recall the continuum embedding of a random-cluster configuration $\omega$, denoted $\mathcal{G}$ in Section 5. For any two (continuum) points $x, y$, write $x \overset{\text{ord}}{\longleftrightarrow} y := \{ \omega : x, y \text{ are in the same connected component of } \mathcal{G} \}$; $x \overset{\text{dis}}{\longleftrightarrow} y := \{ \omega : x, y \text{ are in the same connected component of } \Lambda \setminus \omega \}$.

Notice that, by construction of $\mathcal{G}$, for vertices $v, w \in \mathbb{Z}^d$, $v \overset{\text{ord}}{\longleftrightarrow} w$ is equivalent to $v$ and $w$ belonging to the same connected component of $\omega$ itself. When $d = 2$, for dual vertices $v, w \in (\mathbb{Z} + \frac{1}{2})^2$, $v \overset{\text{dis}}{\longleftrightarrow} w$ is the same as the usual notion of dual connectivity. In higher dimensions, if one defines $F_*(\omega)$ as the set of unit $(d-1)$-cells that are dual to closed edges, then $v \overset{\text{dis}}{\longleftrightarrow} w$ indicates that $v, w \in (\mathbb{Z} + \frac{1}{2})^d$ are connected through $F_*(\omega)$ (where two dual $(d-1)$-cells are adjacent if they are incident on one another).

6.1 | Weak spatial mixing

The first reduction of this section is from WSM to bulk exponential decay of ord and dis connectivities (i.e., connectivities at distance $\Theta(n)$ from the boundary). The proof is quite standard, and such coupling arguments are common in the literature when working with ord-connectivities, but we include it since the later proofs for SSM and WSM within a phase will build on similar ideas.

**Lemma 6.1.** Suppose $(p, q, d)$ are such that one of the following hold:

$$
\pi_{\Lambda_n}(\partial \Lambda_{m/2} \overset{\text{ord}}{\longleftrightarrow} \partial \Lambda_m) \leq Ce^{-m/C} \quad \text{or} \quad \pi_{\Lambda_n}(\partial \Lambda_{m/2} \overset{\text{dis}}{\longleftrightarrow} \partial \Lambda_m) \leq Ce^{-m/C}.
$$

Then the random-cluster model satisfies WSM.

**Proof.** We start with the implication from the first assumption. We construct a coupling between $\omega^{\theta} \sim \pi_{\Lambda_n}^{\theta}$ and $\omega^1 \sim \pi_{\Lambda_n}^1$ such that they agree on $\Lambda_{n/2}$ on the complement of the event

$$
\left\{ \omega^1 : \partial \Lambda_{n/2} \overset{\text{ord}}{\longleftrightarrow} \partial \Lambda_n \right\}.
$$

The coupling goes as follows. Expose the connected component of $\partial \Lambda_n$ (i.e., the union of the connected components of all vertices in $\partial \Lambda_n$) under $\omega^1$ and call that set $D$. Notice

---

7A $k$-cell is a $k$-dimensional unit hypercube. A $d-1$-cell is dual to an edge $e$ if they share a barycenter and $e$ is normal to the cell.
that this is exactly the set of vertices in Ext(Γ_{\text{ext}}(\omega^1)), and \(\partial D = \Gamma_{\text{ext}}(\omega^1)\). Expose also the values of \(\omega^0(D)\) under the grand monotone coupling with \(\omega^1\). Having revealed \(D\), we have also revealed that all edges of
\[
\partial_{\text{out}} D = \{e = (v, w) : v \in D, w \notin D\}
\]
are closed under \(\omega^1\), which is what gave rise to the contour collection \(\partial D\), all of whose interior labels are dis. Therefore, we can sample the remainder of the configuration of \(\omega^1\) by drawing
\[
\omega^1(E(\Lambda_n \setminus D)) \sim \pi_{\Lambda_n}^1(\cdot | \omega^1(D \cup \partial_{\text{out}} D)),
\]
which by the domain Markov property is exactly the random-cluster distribution on Int(\(\partial D\)) with free boundary conditions. By monotonicity of the coupling of \((\omega^0, \omega^1)\), all edges of \(\omega^0(\partial_{\text{out}} D)\) are also closed, so by the domain Markov property the marginal
\[
\pi_{\Lambda_n}^1(\cdot | \omega^0(D \cup \partial_{\text{out}} D))
\]
is also just a random-cluster distribution on Int(\(\partial D\)) with free boundary conditions. Therefore, we can conclude the coupling simply by setting \(\omega^1(E(\Lambda_n \setminus D)) = \omega^1(E(\Lambda_n \setminus D))\).

Thus \(\omega^1(E(\Lambda_{n/2})) = \omega^0(E(\Lambda_{n/2}))\) as long as \(\Lambda_{n/2} \cap D = \emptyset\). As such,
\[
\|\pi_{\Lambda_n}^1(\omega(\Lambda_{m/2}) \in \cdot) - \pi_{\Lambda_n}^1(\omega(\Lambda_{m/2}) \in \cdot)\|_\text{tv} \leq \pi_{\Lambda_n}^1(\Lambda_{n/2} \cap D \neq \emptyset) = \pi_{\Lambda_n}^1(\partial \Lambda_{m/2} \leftrightarrow \partial \Lambda_m),
\]
which is exactly the first quantity bounded in (6.1).

The proof assuming the bound on the disordered connectivity events in (6.1) proceeds analogously, by revealing Ext(Γ_{\text{ext}}(\omega^0)) using adjacency of the dual \((d - 1)\)-cells we called \(F_\star(\omega)\). Then the set of edges whose states are exposed to reveal the connected component of \(\partial \Lambda_n\) in \(F_\star(\omega)\) induce a wired boundary on the interiors of \(\Gamma_{\text{ext}}(\omega^0)\) as expected, and the domain Markov property can be applied on that set.

**Proof of Corollary 1.1.** By Theorem 1.1 it suffices to establish that WSM holds in all the parameter regimes claimed in Corollary 1.1. In [16] it was shown that the random-cluster model has exponential decay of connectivities of the form of the first bound in (6.1) whenever \(d \geq 2\), \(q \geq 1\) and \(p < p_c(q, d)\). Together with Lemma 6.1 that implies WSM at all \(p < p_c(q, d)\). The fact that the Ising random-cluster model \((q = 2)\) satisfies WSM when \(d \geq 2\) at all \(p > p_c(q, d)\) was directly shown in [15]. Finally, in order to see that the random-cluster model satisfies WSM whenever \(p\) is sufficiently large, one notices that the second item in (6.1) holds at \(p\) sufficiently large by comparison of the set of closed edges to a sub-critical percolation process on the graph induced by the dual notion of adjacency we considered for closed edges.

**Remark 6.1.** It seems plausible that the methods of [10] as outlined in Section 5 are strong enough to show WSM (and in turn SSM on \(\Lambda_n^1\)) for all \(p > p_c(q, d)\) when \(q \geq q_0(d)\) is sufficiently large. The challenge is to establish the second bound of (6.1) at slightly super-critical \(p\). Under \(\pi_{\Lambda_n}^0\) and \(p > p_c(q, d)\) there will necessarily be a long contour confining almost all of the volume of \(\Lambda_n\), so the question reduces to understanding the geometry of this separating surface, which presumably will remain within an \(O(1)\) distance from \(\partial \Lambda_m\) with exponential tails on oscillations away from that.
6.2 Strong spatial mixing

The next reduction of this section is between strong spatial mixing and connectivity decay that holds uniformly over vertices in the box. This reduction will only be applicable for a special class of boundary conditions which are called side-homogeneous.

**Definition 6.1.** Side-homogeneous boundary conditions are those in which a subset of the $2d$ sides of $\partial \Lambda_m$ are selected and all vertices in those sides are in one single element of the boundary partition; all vertices in the other sides of $\partial \Lambda_m$ are singletons. In particular, the all-wired and all-free boundary conditions are side-homogeneous.

**Lemma 6.2.** Fix any side-homogeneous boundary condition $\eta$. Suppose $(p, q, d)$ are such that either:

$$\max_v \pi_{\Lambda_m \cap B_1^v}^{\omega} (\partial B_{r/2,v} \leftrightarrow \partial B_{r,v}) \leq Ce^{-m/C} \quad \text{or} \quad \max_v \pi_{\Lambda_m \cap B_2^v}^{\omega} (\partial B_{r/2,v} \leftrightarrow \partial B_{r,v}) \leq Ce^{-m/C}. \quad (6.3)$$

Then the random-cluster model on satisfies SSM on $\Lambda_n^0$.

**Proof.** Fix any $v \in \Lambda_n$, and for simplicity let $B_1^v$ denote $B_{r,v}$ with boundary conditions that are induced by $\eta$ on $\partial \Lambda_n$ and $I$ on $\partial B_{r,v} \setminus \partial \Lambda_n$. The proof goes by constructing a coupling between $\omega^0 \sim \pi_{\omega}^{\eta}$ and $\omega^1 \sim \pi_{\omega}^{\eta}$, such that they agree on $B_{r/2}$ on the complement of the event

$$\bigg\{ \omega^1 : \partial B_{r/2} \leftrightarrow \partial B_r \setminus \partial \Lambda_n \bigg\}. \quad (6.4)$$

This coupling goes as follows. Expose the connected component of $\partial B_r \setminus \partial \Lambda_n$ under $\omega^1$ and call that set $D$. To relate this to the continuum construction of Section 5, this is the set of vertices in the connected component of $B_r \setminus \Gamma(\omega^1)$ that is incident to $\partial B_r \setminus \partial \Lambda_n$. Since the set $D$ is bounded by a contour in $\Gamma(\omega^1)$, the set of edges $\partial_{out} D$ are revealed to be closed under $\omega^1$, and together with $\partial \Lambda_n$ they enclose the region $B_r \setminus D$. Also, under the grand monotone coupling, expose the configuration $\omega^0(D \cup \partial_{out} D)$, and note that by monotonicity all edges of $\omega^0(\partial_{out} D)$ will also be closed.

We next sample

$$\omega^1(E(B_r \setminus D)) \sim \pi_{\omega}^{\eta}(\cdot | \omega^1(D \cup \partial_{out} D)), \quad \text{which is exactly the random-cluster distribution on } E(B_r \setminus D) \text{ with boundary conditions induced by free on } \partial_{out} D \text{ and } \eta \text{ on the portions of } \partial E(B_r \setminus D) \text{ that intersect } \Lambda_n. \text{ In particular we claim that, because } \eta \text{ is side-homogeneous, the induced boundary conditions on } B_r \setminus D \text{ are independent of the values of } \omega^1(D). \text{ This follows because the vertices of } \partial \Lambda_n \text{ that are incident to } B_r \setminus (D \cup \partial_{out} D) \text{ are either on a wired side of } \eta, \text{ in which case they are wired regardless, or they are on a free side of } \eta, \text{ in which case they cannot be wired up to any other vertex through } \omega^1(D) \text{ because they are not incident to } D. \quad \text{The coupling can now be concluded by setting } \omega^0(E(B_r \setminus D) = \omega^1(E(B_r \setminus D)). \text{ Under the above coupling, } \omega^1(B_{r/2}) = \omega^0(B_{r/2}) \text{ as long as } B_{r/2} \cap D \neq \emptyset. \text{ This means that for any } v,$$

$$\| \pi_{B_1^v}^{\omega} (\omega(B_{r/2} \in \cdot)) - \pi_{B_2^v}^{\omega} (\omega(B_{r/2} \in \cdot)) \|_{tv} \leq \pi_{B_1^v} (\partial B_{r/2} \setminus D \neq \emptyset) = \pi_{B_1^v}^{\omega} (\partial B_{r/2} \leftrightarrow \partial B_r).$$

Maximizing both sides over $v$, this reduces SSM on $\Lambda_n^0$ to the first inequality in (6.3).
The proof assuming the bound on the disordered connectivity events in (6.3) proceeds analogously, by revealing the dis components using the dual notion of connectivity in $F_+(\omega)$. All vertices adjacent to the revealed set $D$ will be wired to one another, and the same claim about the induced boundary conditions on $E(B_r \setminus D)$ being independent of the edge values on $D$ holds.

**Proof of Corollary 1.2.** Let us start with item (i). By Theorem 1.1 it suffices to establish that SSM holds on $\Lambda^0_n$ whenever $p < p_c(q, d)$. We wish to boost exponential decay of connectivities in the bulk, in the form of the first bound in (6.1), to uniform exponential decay of connectivities in the presence of free boundary conditions in the form of the first bound in (6.3) with $\eta = 0$. In order to see this, note that by monotonicity

$$\pi_{\Lambda^0_n \cap B_{r/2,v}}(\partial B_{r/2,v}) \lesssim \pi_{\Lambda^0_n}(\partial \Lambda_{r/2} \leftrightarrow \partial \Lambda_r)$$

for every $v \in \Lambda_m$. For any $d \geq 2$, $q \geq 1$, the latter decays exponentially whenever $p < p_c(q, d)$ by [16], so SSM on $\Lambda^0_n$ holds whenever $p < p_c(q, d)$.

We turn now to item (ii). At $p$ sufficiently large, we can compare the set of closed edges to a sub-critical percolation process on the graph induced by the dual notion of adjacency we considered for closed edges. Since this is a comparison to an independent percolation process, it is independent of the boundary condition $\eta$, and therefore implies that the second bound in (6.3) holds, which by Lemma 6.2 in turn implies SSM for $\Lambda^0_n$ for all side-homogeneous $\eta$.

**Remark 6.2.** For $0 \leq k \leq d$, define the free cylinder $(\mathbb{Z}/n\mathbb{Z})^k \times \Lambda^0_n$ for $\Lambda_n$ in $d - k$ dimensions, and similarly the wired cylinder $(\mathbb{Z}/n\mathbb{Z})^k \times \Lambda^1_n$. As hinted at in Remark 1.1, the same argument used to prove Corollary 1.2 could also be applied to deduce fast mixing on the free cylinder when $p < p_c(q, d)$ and the wired cylinder when $p > p_c(q, d)$. To see this generalization, in the above proof of Corollary 1.2, one simply takes the balls $B_{r,v}$ and $B_{r/2,v}$ to be on the cylinder (i.e., wrapping around in its first $k$ coordinates); with this change, the rest of the argument goes the same way.

### 6.3 Weak spatial mixing within a phase

We conclude the section by using the results of Section 5 to establish WSM within a phase for the random-cluster model at large $q$. This also essentially goes via a reduction to exponential tails on connectivities, but in this case under $\pi_{\text{ord}}$ and $\pi_{\text{dis}}$ as established in Lemma 5.3. The fact that these are conditional measures introduces some complications that are handled with a somewhat more involved coupling argument.

**Lemma 6.3.** Fix $d \geq 2$ and suppose $q \geq q_0(d)$. There is a constant $C(q, d) > 0$ such that the random-cluster model on $\mathbb{T}_n$ satisfies

1. WSM within the free phase with constant $C$ uniformly over $p \leq p_c(q, d) + o(n^{-1})$; and
2. WSM within the wired phase with constant $C$ uniformly over $p \geq p_c(q, d) - o(n^{-1})$.

We will prove the result for the free phase, the proof for the wired phase being analogous. The proof goes by constructing a coupling between $\omega^0 \sim \pi_{\Lambda^0_n}$ and $\tilde{\omega} \sim \tilde{\pi}_n$ such that they agree on $\Lambda_{r/2}$ except with probability $Ce^{-r/C}$. (Here, we are thinking of $\Lambda_r$ as being embedded in $\mathbb{T}_n$ by identifying the vertices of $\mathbb{T}_n$ with those of $\Lambda_n$.) By virtue of Lemma 5.5, it actually suffices to construct the
coupling with \( \omega_{\text{dis}} \sim \pi_{\text{dis}} \) instead. Let \( \mathcal{E}_{\text{dis}} \) be the set of configurations \( \omega \) on \( E(\Lambda_r)^c = E(\mathbb{T}_n) \setminus E(\Lambda_r) \) such that \( \omega \cup E(\Lambda_r) \in \Omega_{\text{dis}} \) (i.e., even if all edges in \( E(\Lambda_r) \) are present, it still forms a configuration in \( \Omega_{\text{dis}} \)). This event is important because it allows us to drop the conditioning in \( \pi_{\text{dis}} \) and apply properties of the unconditional random-cluster measure such as monotonicity and domain Markov.

**Lemma 6.4.** Suppose \( \eta \in \mathcal{E}_{\text{dis}} \). Then

\[
\pi_{\text{dis}}(\omega(E(\Lambda_r)) \in \cdot | \omega(E(\Lambda_r)^c) = \eta) = \pi_n(\omega(E(\Lambda_r)) \in \cdot | \omega(E(\Lambda_r)^c) = \eta) = \pi_{\Lambda_r}(\omega \in \cdot).
\]

**Proof.** The second inequality is simply the domain Markov property. Now consider any event \( A \) on configurations on \( E(\Lambda_r) \). Then,

\[
\pi_{\text{dis}}(A | \omega(E(\Lambda_r)^c) = \eta) = \frac{\pi_n(\omega(E(\Lambda_r)) \in A, \omega(E(\Lambda_r)^c) = \eta, \Omega_{\text{dis}})}{\pi_n(\omega(E(\Lambda_r)^c) = \eta, \Omega_{\text{dis}})}.
\]

But the configuration \( \eta \) (together with empty on \( E(\Lambda_r) \)) is in \( \Omega_{\text{dis}} \), and \( \Omega_{\text{dis}} \) is a decreasing set, so \( \eta \cup \omega(E(\Lambda_r)) \) is always in \( \Omega_{\text{dis}} \). As such, the intersection with \( \Omega_{\text{dis}} \) can be dropped from both the numerator and denominator, giving the desired first equality. \( \blacksquare \)

The main coupling of this section goes as follows. We describe it in the free phase, with the construction for the ordered phase being symmetrical.

**Definition 6.2.** The coupling \( \mathbb{P} \) of \( (\omega^0, \omega_{\text{dis}}) \) is constructed as follows:

1. Sample the configuration \( \omega_{\text{dis}}(E(\Lambda_r)^c) \sim \pi_{\text{dis}} \) where \( E(\Lambda_r)^c = E(\mathbb{T}_n) \setminus E(\Lambda_r) \).
2. If \( \omega_{\text{dis}}(E(\Lambda_r)^c) \notin \mathcal{E}_{\text{dis}} \), sample independently

\[
\omega^0 \sim \pi_{\Lambda_r} \quad \text{and} \quad \omega_{\text{dis}} \sim \pi_{\text{dis}}(\cdot | \omega_{\text{dis}}(E(\Lambda_r)^c)).
\]

3. If \( \omega_{\text{dis}}(E(\Lambda_r)^c) \in \mathcal{E}_{\text{dis}} \), then independently from \( \partial \Lambda_r \) inwards, reveal all connected components of \( \partial \Lambda_r \) under

\[
\omega_{\text{dis}} \sim \pi_n(\omega(E(\Lambda_r)) \in \cdot | \omega(E(\Lambda_r)^c)),
\]

and call that set \( D \). These will evidently be the interiors of all contours labeled ord that intersect \( \partial \Lambda_r \). Thus we will also have revealed their boundary \( \partial_{\text{out}} D \) to be all-closed in \( \omega_{\text{dis}} \), and the boundary conditions induced by the set of revealed edges on \( \Lambda_r \setminus D \) will be all-free. Under the monotone coupling of \( \omega^0 \) to \( \omega_{\text{dis}} \) we can also reveal all edges of \( \omega^0(\partial_r \cup \partial_{\text{out}} D) \) and they will also be all closed on \( \partial_{\text{out}} D \), therefore also inducing free boundary conditions on \( \Lambda_r \setminus D \).

4. Sample \( \omega_{\text{dis}}(E(\Lambda_r \setminus D)) \) according to the random-cluster model on that set with free boundary conditions, and set \( \omega^0(E(\Lambda_r \setminus D)) = \omega_{\text{dis}}(E(\Lambda_r \setminus D)) \).

The validity of this coupling can be seen quite easily with the key point being the use of Lemma 6.4 to ensure that in items (3) and (4), \( \omega_{\text{dis}} \) stochastically dominates \( \omega^0 \) and satisfies the domain Markov property.

**Proof of Lemma 6.3.** Let us begin with the proof for the free phase. As mentioned, by Lemma 5.5, it suffices for us to establish that

\[
\|\pi_{\Lambda_r}^0(\omega(\Lambda_r/2) \in \cdot) - \pi_{\text{dis}}(\omega(\Lambda_r/2) \in \cdot)\|_{\text{tv}} \leq Ce^{-\gamma/2}.
\]
Under the coupling of Definition 6.2, it is evident that

$$\mathbb{P}(\omega^0(\Lambda_r/2) \neq \omega_{\text{dis}}(\Lambda_r/2)) \leq \pi_{\text{dis}}(\omega(E(\Lambda_r)^c) \notin \mathcal{E}_{\text{dis}}) + \pi_{\text{dis}}(\partial \Lambda_r \leftrightarrow \partial \Lambda_r/2).$$

(6.5)

Let us start with the first of these events. In order for a configuration to not be in \(\mathcal{E}_{\text{dis}}\), it must be the case that there is a contour in \(\omega_{\text{dis}}\) that connects opposite sides of \(\partial \Lambda_r\) in \(T_n \setminus \Theta_r\). This is because there must be a connected component \(\mathcal{C}\) of \(\partial \omega_{\text{dis}}(\mathbb{Z}_n \setminus \Lambda_r)\), such that if \(\gamma_{\text{ord}}\) is the contour of \(\omega_{\text{dis}} \cup \Theta_r\) and \(\gamma_{\text{dis}}\) is the contour of \(\omega_{\text{dis}} \setminus \Theta_r\) containing \(\mathcal{C}\), then either:

1. the topological triviality/non-triviality of \(\gamma_{\text{dis}}\) and \(\gamma_{\text{ord}}\) are different; or
2. the topological triviality/non-triviality of the connected component of \(T_n \setminus \gamma_{\text{dis}}\) and \(T_n \setminus \gamma_{\text{ord}}\) containing the open edges of \(\omega_{\text{dis}}\) are different.

The first of these holds if \(\omega_{\text{dis}} \cup E(\Lambda_r) \in \Omega_{\text{tunnel}}\), and the second holds if \(\omega_{\text{dis}} \cup E(\Lambda_r) \in \Omega_{\text{ord}}\). Either of these events necessitates that, in some direction of the torus, there is both an \(\text{ord}\) and a \(\text{dis}\) connection connecting two opposite sides of \(\partial \Lambda_r\), which implies in particular that \(\mathcal{C}\) has length at least \(n - r\). Therefore, by a union bound and Lemma 5.3,

$$\pi_{\text{dis}}(\omega(E(\Lambda_r)^c) \notin \mathcal{E}_{\text{dis}}) \leq \pi_{\text{dis}}(\exists v \in \partial \Lambda_r : ||\gamma_v|| \geq n - r) \leq Ce^{-\beta(n-r)}$$

uniformly over all \(p \leq p_c(q, d) + o(n^{-1})\). Since in the definition of WSM within a phase we only consider \(r \leq n/2\), this is exponentially decaying in \(r\).

The second term of (6.5) is also bounded by a union bound together with Lemma 5.3 via

$$\pi_{\text{dis}}(\partial \Lambda_r \leftrightarrow \partial \Lambda_r/2) \leq \pi_{\text{dis}}(\exists v \in \partial \Lambda_r : ||\gamma_v|| \geq r/2) \leq Ce^{-\beta r}.$$
on mixing times of the FK dynamics restricted to the wired and free phases respectively. We draw attention to the fact that the mixing time bound is uniform over all $p$ even microscopically beyond the critical point in each direction.

**Theorem 7.1.** For every $d \geq 2$, there exists $q_0(d)$ and $C(q, d)$ such that, for all $q \geq q_0$, the following holds uniformly over sequences $p_n \geq p_c - o(1/n)$: if

$$t_n^c = \exp(C(\log n)^{d-1}),$$

then the restricted FK dynamics on $\mathbb{T}_n$ has

$$\|P(\hat{X}_{t_n}^1 \in \cdot) - \hat{\pi}_n\|_{tv} = O(n^{-100}).$$

The analogous claim holds uniformly over $p_n \leq p_c + o(1/n)$ with the free phase instead of the wired phase.

By Lemma 5.7 (together with Lemma 5.1), when $|p_n - p_c| = \Omega(1)$ the restricted FK dynamics in Theorem 7.1 in fact produces a sample that is within total variation distance $n^{-100}$ of $\pi_n$ itself. When $p_n$ is sufficiently close to $p_c$, however, both the wired and free phases have non-negligible mass under $\pi_n$, so that neither of the restricted chains is sufficient to obtain a sample from $\pi_n$. The following corollary shows that, close to the critical point, we can combine the restricted dynamics in each of the two phases to show that the unrestricted FK dynamics initialized from an appropriate mixture of the all-free and all-wired configurations also mixes quickly and produces samples close to $\pi_n$.

**Corollary 7.1.** Fix $d \geq 2$, $q \geq q_0(d)$, and a sequence $|p_n - p_c| = o(1/n)$. Suppose that the sequence $m_n^c$ satisfies $|m_n^c - \pi_n(\hat{\Omega})| \leq \eta_n$ for some sequence $\eta_n = o(1)$. Then the FK dynamics on $\mathbb{T}_n$ initialized from

$$v^{0/1}_n := (1 - m_n^c)\delta_0 + m_n^c\delta_1$$

satisfies the following:

$$\|P(X_{t_n} \in \cdot) - \pi_n\|_{tv} \leq O(n^{-100}) + 4\eta_n.$$

**Proof of Theorem 7.1.** Our aim is to apply Theorem 1.3. First, by Lemma 5.6, $\hat{\Omega}$ is exponentially stable with constant $C_1$ uniformly over all $p \geq p_c - o(1/n)$, and by Lemma 6.3 WSM within the wired phase holds with constant $C_2$ uniformly over $p \geq p_c - o(1/n)$. Thus, by Theorem 1.3, there exist $C_0, K_0$ depending only on $C_1, C_2$ such that if $g_n(t)$ is as in (4.7) for $K = K_0$, then for every $t \geq 0$,

$$\|P(\hat{X}_{t, T_n}^1 \in \cdot) - \hat{\pi}_n\|_{tv} \leq C_0n^d \exp(-g_n(t)/C_0).$$

To plug in a value for $g_n(t)$, we recall that $f(m)$ in its definition is given by (4.6), and as described there, one can plug in for $f(m)$ a worst-case mixing time bound for the dynamics on $\Lambda_{m}^1$.

By a standard canonical paths argument (originating in [30]; see e.g., [22, Proposition 4.2] for a formulation for the random-cluster model with free or wired boundary conditions), the (worst-case) mixing time of the FK dynamics on $\Lambda_{m}^1$ is at most $\exp(C_3m^{d-1})$ for some $C_3(q) > 0$, uniformly over all $p \geq p_c(q)/2$, say, so that certainly $p \geq p_c(q) - o(1/m)$.
In particular, (4.6) is satisfied by the choice $f(m) = C_3 e^{C_4 m^{-1}}$. In turn, $g_n(t)$ of (4.7) satisfies $g_n(t) \geq C_4^{-1}(\log t)^{1/(d-1)} \land n$ for some other constant $C_4$, again uniformly over $p \geq p_c - o(1/n)$. At this point, the desired result follows from a direct application of Theorem 1.3, since $g_n(t_n^*)$ is seen to be smaller than $n^{-100}$ if the constant $C$ in $t_n^*$ exceeds $100C_0C_4 d^2$.

The bound for the free phase is obtained by a symmetrical argument. □

**Proof of Corollary 7.1.** By a triangle inequality, we can control the total variation distance as
\[
\left\| \mathbb{P}\left(X_t^{\nu/1} \in \cdot \right) - \pi_n \right\|_{tv} \leq m_n^* \mathbb{P}(X_t^{\nu/1} \in \cdot) + (1 - m_n^*) \mathbb{P}(X_t^0 \in \cdot) - m_n^* \hat{\pi}_n - (1 - m_n^*) \hat{\pi}_n \right\|_{tv} + 2|m_n^* - \pi_n(\hat{\Omega})|. \tag{7.2}
\]

By the assumption on $m_n^*$, the second term satisfies $2|m_n^* - \pi_n(\hat{\Omega})| \leq 2\eta_n$. By another triangle inequality, the first term in (7.2) is at most
\[
m_n^* \mathbb{P}(X_t^{\nu/1} \in \cdot) - \hat{\pi}_n\|_{tv} + (1 - m_n^*) \mathbb{P}(X_t^0 \in \cdot) - \hat{\pi}_n\|_{tv}.
\]

The two terms on the right are handled analogously, so we only consider one of them. By another triangle inequality, we have
\[
\left\| \mathbb{P}(X_t^{\nu/1} \in \cdot) - \hat{\pi}_n \right\|_{tv} \leq \left\| \mathbb{P}(X_t^0 \in \cdot) - \hat{\pi}_n \right\|_{tv} + \left\| \mathbb{P}(X_t^1 \in \cdot) - \hat{\pi}_n \right\|_{tv}. \tag{7.3}
\]

By (4.8) and the grand coupling, the first term in (7.3) is at most $\mathbb{P}(\mathcal{G}_1 \leq t) \leq C_3 e^{-n^{d-1}/C_3}$ while $t \leq e^{n^{d-1}/K}$. The second term is exactly the term shown in Theorem 7.1 to be $O(n^{-100})$ when $t = t_n^*$. □

**Proof of Theorem 1.4.** Theorem 1.4 is a special case of Corollary 7.1 exactly at $p = p_c$, where we know from [10, Lemma 6.1] and Lemma 5.5 that if $m_n^* = q/(q + 1)$ then the assumption on $m_n^*$ holds with $\eta_n = \exp(-\Omega(n))$.

In order to get the improved bound $t_n^* = n^{o(1)}$ when $d = 2$, we replace the crude canonical paths bound used on the local mixing time quantity $f(m)$ with the $\exp(n^{o(1)})$ bound of [23] on the mixing times on $A^1$ and $A^0$ at $p = p_c$. With that choice, $g_n(t)$ from (4.7) satisfies $g_n(t) \geq C_4^{-1}(\log t)^{o(1)} \land n$, so that $t = \exp((\log n)^{o(1)}) = n^{o(1)}$ suffices to attain a small total variation distance to stationarity. (The extra factor of $N$ in the mixing time is of course due to the change from discrete to continuous time.) □

### 7.2 Efficiently approximating the weights for the random phase initialization

Given a value of $p$, in order to obtain a good sample from $\pi_n$ on $\mathbb{T}_n$ for large $n$ using Theorem 7.1 and Corollary 7.1, it is imperative to have a good approximation $m_n^*$, that is, deduce, for a fixed $p$ and size $n$, the relative weights of $\hat{\Omega}$ and $\hat{\Omega}$. In particular, at finite $n$, it could be that $p$ differs only microscopically from $p_c$ but the correct weighting for the initial distribution in Corollary 7.1 is not $\left( \frac{1}{q+1}, \frac{q}{q+1} \right)$ but rather some other non-negligible pair of weights.

The following lemma describes an MCMC algorithm that is (up to polynomial factors) as efficient as the mixing times in Corollary 7.1 itself, which produces a weight $m_n^*$ that is a good approximation to
\(\pi_n(\hat{\Omega})\) whenever the weight is non-trivial. Our approach is to approximate the partition functions \(\hat{Z}_p, \tilde{Z}_p\) of \(\hat{\Omega}\) and \(\tilde{\Omega}\), respectively, at parameter \(p\) by interpolating from \(p = 1\) down to \(p_c\) with FK dynamics chains restricted to the wired phase, and up from \(p = 0\) to \(p_c\) with FK dynamics chains restricted to the free phase.

We claim that it suffices to approximate \(\pi_n(\hat{\Omega})\) (or, symmetrically, \(\pi_n(\tilde{\Omega})\)) when \(|p - p_c| = o(1/n)\). This is because, by monotonicity, \(\pi_n(\tilde{\Omega})\) decreases as \(p\) grows, and by Lemma 5.7 there will be some \(p\) within \(o(1)\) of \(p_c\) at which \(\tilde{\Omega}\) is exponentially negligible (and symmetrically for \(\pi_n(\hat{\Omega})\)).

**Lemma 7.1.** Let \(d \geq 2, q \geq q_0(d)\), and consider any sequence \(p = p_n\) such that \(|p - p_c| = o(1/n)\). There exists an MCMC-based algorithm that, using \(\text{poly}(n)\) samples from restricted FK dynamics run for time \(t_o\) on each chain, outputs an estimate of \(\pi_n(p_n)\) such that

\[
\mathbb{P}\left(\left| \pi_n(p_n) - \pi_n(\hat{\Omega}) \right| > n^{-100} \right) = O(n^{-100}).
\]

**Proof.** Observe that, by definition,

\[
\pi_n(\hat{\Omega}) = \frac{\hat{Z}_p}{\hat{Z}_p + \tilde{Z}_p}. \tag{7.4}
\]

Our aim is to approximate each of \(\hat{Z}_p\) and \(\tilde{Z}_p\) within \(1 \pm O(n^{-50})\) multiplicative factors; combining these estimates as in (7.4) yields an estimate of \(\pi_n(\hat{\Omega})\) within the same ratio. We will explain how to approximate \(\hat{Z}\) using an increasing sequence of values of \(p\) starting at \(p = 0\); the approximation scheme for \(\tilde{Z}\) works symmetrically using a decreasing sequence starting at \(p = 1\).

For any \(p : |p - p_c| = o(1/n)\), construct an increasing sequence \((p^{(i)})_{i=0,\ldots,K}\) such that \(p^{(0)} = 0, p^{(K)} = p\), and \(p^{(i)} - p^{(i-1)} \leq n^{-d}\) for all \(i\). Then, we can expand \(\hat{Z}_{p^{(i)}}\) as

\[
\hat{Z}_{p^{(K)}} = \hat{Z}_{p^{(0)}} \prod_{1 \leq i \leq K} \frac{\hat{Z}_{p^{(i)}}}{\hat{Z}_{p^{(i-1)}}}. \tag{7.5}
\]

It is now easily seen that each ratio in the product appearing above is actually the expectation of a bounded random variable under the Gibbs measure at parameter \(p^{(i-1)}\). More precisely, letting \(W_{p}(\omega) = p^{\|\omega\|} (1-p)^{|E(T_n)|-\|\omega\|} q^{\text{Comp}(\omega)}\), we have

\[
\frac{\hat{Z}_{p^{(i)}}}{\hat{Z}_{p^{(i-1)}}} = \tilde{X}_{n,p^{(i-1)}}, \frac{W_{p^{(i)}}(\omega)}{W_{p^{(i-1)}}(\omega)} \leq \max_\omega \left( \frac{p^{(i)}(1-p^{(i-1)})}{p^{(i-1)}(1-p^{(i)})} \right)^{|\omega|} \cdot \text{poly}(n). \tag{7.6}
\]

Using the fact that \(p^{(i)} - p^{(i-1)} \leq n^{-d}\) and \(|\omega| \leq |E(T_n)| = O(n^d)\), the above ratio is bounded uniformly by some constant, say \(A\). As a consequence, if for each \(i, \tilde{X}_{i}^{(i-1)}\) is the final configuration after \(t\) steps of the restricted (to \(\hat{\Omega}\)) FK dynamics initialized from \(0\) on the torus \(T_n\) at parameter value \(p^{(i)}\), then we can estimate the values of the ratios in (7.5) via

\[
\mathbb{E} \left[ \frac{W_{p^{(i)}}(\tilde{X}_{i}^{(i-1)})}{W_{p^{(i-1)}}(\tilde{X}_{i}^{(i-1)})} \right] - \tilde{X}_{n,p^{(i-1)}} \left[ \frac{W_{p^{(i)}}(\omega)}{W_{p^{(i-1)}}(\omega)} \right] \leq A \mathbb{E} \left[ (\tilde{X}_{i}^{(i-1)} \in \cdot) - \tilde{X}_{n,p^{(i-1)}} \right]_{TV}. \tag{7.6}
\]

By Theorem 7.1, since \(p^{(i)} \leq p_c + o(1/n)\) for all \(i\), the total variation distance on the right is at most \(n^{-100}\) if \(t \geq t_o\) from (7.1).
Notice further that, algorithmically, we can approximate the expectation on the left-hand side of (7.6) using $\ell^i$ independent runs of the restricted FK dynamics initialized from the $0$ configuration, up to an error of $\epsilon = \ell^{d-1/2}$, except with probability $\exp(-\Omega(\epsilon^{2d}))$. (Here, we are using the uniform boundedness of the ratio of weights to apply standard concentration estimates.) Taking $\ell$ to be a large polynomial in $n$, this error can be absorbed.

Altogether, we can suppose we have a sequence of estimates $a_i$ of the above expectation, so that for all $i$,

$$a_{i-1} - \tilde{\kappa}_{n,\ell^{i-1}} \left[ \frac{W_{\ell^i}(\omega)}{W_{\ell^{i-1}}(\omega)} \right] = O(n^{-100}).$$

By boundedness of the expectation under consideration, we then have

$$a_i = (1 + O(n^{-100}))\tilde{\kappa}_{n,\ell^{i-1}} \left[ \frac{W_{\ell^i}(\omega)}{W_{\ell^{i-1}}(\omega)} \right].$$

In that case, by (7.5), since $K \leq n^d$,

$$Z_{\ell^0} \prod_{1 \leq i \leq K} a_i = Z_{\ell^0} \prod_{1 \leq i \leq K} \left( \frac{\tilde{Z}_{\ell^i}}{\tilde{Z}_{\ell^{i-1}}} \right) (1 + O(n^{-100})) \leq \tilde{Z}_{\ell^K}(1 + O(n^{-100})).$$

Furthermore, $Z_{\ell^0} = Z_0$ is of course easy to compute exactly (it is just the weight of a single, empty configuration). Thus, $\tilde{Z}_{\ell^K}$ can be approximated to multiplicative factor $1 \pm O(n^{-100})$ by a Markov chain approach consisting of polynomially many independent runs at $n^d$ many parameter values $\ell^i$, each of which have run-time bounded by $t^i = \exp(C(\log n)^{d-1})$. Reasoning analogously from the other direction, we can also estimate $\tilde{Z}_{\ell^K}$ with a matching running time, and combine these two estimates to obtain an estimate at $m^*_n$ with the desired accuracy.

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**DATA AVAILABILITY STATEMENT**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**ORCID**

Reza Gheissari https://orcid.org/0000-0003-4236-9407

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APPENDIX A: SOLUTION TO THE RECURRENCE OF LEMMA 3.1

In this section, we establish the bound claimed in Lemma 3.1. The argument is standard—see for example, the more general bound of [28, Lemma 17]—but we include a short proof for self-containedness.

Proof of Lemma 3.1. We prove the bound in two steps, first establishing that the sequence decays at least as a stretched exponential, then boosting this to the desired exponential decay. First of all, take \( r = r(k) := -C_0 \log a_k \) for \( C_0 \) sufficiently large to be specified later. Let

\[
\kappa_\delta := \inf \{ k : a_k \leq 2^{-\delta k} \} \wedge \frac{n}{2}.
\]

If \( \delta \) is sufficiently small as a function of \( C_0 \), then while \( k \leq \kappa_\delta \) we have \( r(k) \leq k \). Plugging this choice of \( r \) into the recurrence relation, for all \( k \leq \kappa_\delta \),

\[
a_{2k} \leq d(2C_0 \log a_k^{-1})^d a_k^2 + a_k^{C_0/C_*}.
\]

There evidently exist \( \epsilon_0 \in (0, 1/2) \) and \( C_0 > 0 \) depending only on \( C_* \) and \( d \), so that for all \( k_0 \leq k \leq \kappa_\delta \),

\[
a_{2k} \leq a_k^{1.99}.
\]

Thus, for all \( l \leq \log_{1.99}(\kappa_\delta/k_0) \), we get

\[
a_{2^l k_0} \leq a_{k_0}^{1.99^l}.
\]
Since \( a_{k_0} \leq 1/2 \), we deduce that for all \( k_0 \leq k \leq \kappa \delta \)
\[
   a_k \leq \left( \frac{1}{2} \right)^{(k/k_0)^{99}} \tag{A1}
\]
using the inequality \( 1.99^J \geq 2^{99J} \).

We now boost this to an exponential decay. Take \( r = r(k) := k \), so that
\[
   a_{2k} \leq d(2k)^d a_k^2 + e^{-k/C_*} \quad \text{for all} \quad k \leq n/2.
\]

Now let
\[
   \psi_k := \sqrt{d(4k)^{d/2}} a_k + e^{-k/2C_*}.
\]
Then, using \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \), we have for all \( k \leq \kappa \delta \),
\[
   \psi_{2k} \leq \sqrt{d(4^{d/2}(2k)^{d/2}} a_k + e^{-k/2C_*} \leq d(4k)^d a_k + \sqrt{d(4^{d/2}(2k)^{d/2}} e^{-k/2C_*}.
\]

In particular, this implies that for every \( k \leq \kappa \delta \),
\[
   \psi_{2k} \leq \psi_k^2 \quad \text{and} \quad \psi_{2\kappa \delta k_0^c} \leq \psi_{k_0^c}^{2\alpha} \quad \text{for every} \quad k_0',
\]
and all \( \alpha \leq \log_2(\kappa \delta / k_0^c) \). Let us find \( k_0' \) such that \( \psi_{k_0'} < 1/2 \), which is equivalent to asking for \( k_0' \) such that
\[
   a_{k_0'} + e^{-k_0'/2C_*} < \frac{1}{2} (4k_0')^{-d}.
\]
From the bound of (A1), there would exist \( k_0'(k_0, C_*, d) \) such that such a bound holds. As a consequence, for all \( k_0' \leq k \leq \kappa \delta \), we would have \( a_k \leq \psi_k \leq 2^{-k/k_0^c} \) which together with the definition of \( \kappa \delta \), for \( \delta \) sufficiently small depending only on \( C_*, d \) implies that for \( k_0' \leq k \leq n/2 \),
\[
   a_k \leq 2^{-k/(\delta^{-1}k_0^c)}.
\]
The requirement that \( k \geq k_0' \) can be absorbed by the pre-factor \( C \), depending only on \( k_0, C_*, d \) in the claimed bound of Lemma 3.1 concluding the proof. \( \blacksquare \)

APPENDIX B: SLOW (WORST-CASE) MIXING OF THE FK DYNAMICS NEAR CRITICALITY

In [10], estimates of the form of Section 5 were used to show that the Swendsen–Wang dynamics has \( \exp(\Omega(n^{d-1})) \) (worst-case) mixing time on \( \mathbb{T}_n \) at \( \beta = \beta_c(q, d) \) and integer \( q \geq q_0(d) \). Together with the comparison results of [41], this implies that at integer \( q \geq q_0(d) \), the mixing time of FK dynamics is \( \exp(\Omega(n^{d-1})) \) at \( p = p_c(q, d) \). For the sake of completeness, we demonstrate that one can similarly deduce \( \exp(\Omega(n^{d-1})) \) mixing time at general (possibly non-integer) \( q \geq q_0(d) \) and \( p = p_c(q, d) \) using

\[
   \text{null}.
\]
the same technology of Section 5. We also use the fact that those estimates apply in microscopic windows about \( p_c \) to show that the exponential slowdown applies in a \( o(1/n) \) window about \( p_c \).

**Proposition B.1.** Fix \( d \geq 2, q \geq q_0(d) \), and suppose \( p = p_n \) is such that \( |p - p_c| = o(1/n) \). Then the mixing time of the FK dynamics on \( \mathbb{T}_n \) is \( \exp(\Omega(n^{d-1})) \).

**Proof.** Recall (see, e.g., [32]) that for a Markov chain with transition matrix \( P \) and stationary distribution \( \pi \),

\[
t_{\text{mix}} \geq \frac{1}{2\Phi_*}, \quad \text{where} \quad \Phi_* = \inf_{A \subset \Omega} \Phi(A) = \inf_{A \subset \Omega} \sum_{\omega \in A, \omega' \notin A} P(\omega, \omega') \pi(\omega) / \pi(A) \pi(A^c).
\]

Evidently, the numerator of \( \Phi(A) \) is at most \( \pi(\partial A) \). The slow mixing will be a consequence of the exponential stability of the wired and free phases \( \hat{\Omega} \) and \( \hat{\Omega} \), which we recall from (4.1) to (4.2) and Definition 4.2. To be precise, in (4.1) and (4.2), fix \( \varepsilon(q, d) \) such that Lemma 5.6 holds.

If \( p \geq p_c \), then take \( \hat{\Omega} \) as the bottleneck set \( A \) and note that

\[
t_{\text{mix}} \geq \frac{\pi_n(\hat{\Omega}) \pi_n(\hat{\Omega})}{2\pi_n(\partial \hat{\Omega})} = \frac{\pi_n(\hat{\Omega})}{2\pi_n(\partial \hat{\Omega})}.
\]

Since \( p \geq p_c \), \( \pi_n(\hat{\Omega}) = \Omega(1) \) while \( \pi_n(\partial \hat{\Omega}) = \exp(-\Omega(n^{d-1})) \) per Lemma 5.6 since \( p \leq p_c + o(1/n) \). The argument when \( p \leq p_c \) proceeds symmetrically, taking \( \hat{\Omega} \) as the set \( A \). \( \blacksquare \)