The Tetrahexahedric Calogero Model

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Abstract—We consider the spherical reduction of the rational Calogero model (of type $A_{m-1}$, without the center of mass) as a maximally superintegrable quantum system. It describes a particle on the $(n-2)$-sphere in a very special potential. A detailed analysis is provided of the simplest non-separable case, $n = 4$, whose potential blows up at the edges of a spherical tetrahexahedron, tesselating the two-sphere into 24 identical right isosceles spherical triangles in which the particle is trapped. We construct a complete set of independent conserved charges and of Hamiltonian intertwiners and elucidate their algebra. The key structure is the ring of polynomials in Dunkl-deformed angular momenta, in particular the subspaces invariant and antiinvariant under all Weyl reflections, respectively.

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1. SOME HISTORY

The Calogero model has a 45-year history, starting in 1971 with the original Calogero paper [1]. Ten years later Olshanetsky and Perelomov generalized the $A_{m-1}$ model to arbitrary finite-dimensional Lie algebras and demonstrated their classical [2] and quantum [3] integrability. In 1983, the superintegrability of the Calogero-Moser system was established by Wojciechowski [4]. Starting with their seminal 1990 paper [5] on commutative rings of partial differential operators and Lie algebras, Veselov and Chalykh initiated a series of works on intertwiners (shift operators) and the exact energy spectrum for integer couplings (multiplicities). In parallel, employing the differential-difference operators associated to reflection groups and introduced by Dunkl [6], Heckman gave an elementary construction for commuting charges and intertwiners [7]. The first investigation of the spherical reduction of the rational Calogero model (here called ‘angular Calogero model’) goes back to M. Feigin in 2003 [8]. The $A_2$ and $A_3$ cases were analyzed classically in 2008 by Hakobyan, Nersessian and Yeghikyan [9], and five years later the quantum energy spectra and eigenstates were derived for all angular Calogero models by M. Feigin, Lechtenfeld and Polychronakos [10]. More recently, M. Feigin and Hakobyan presented a deeper analysis of the algebra of Dunkl angular momentum operators, and just now the $A_2$ and $A_3$ angular models have been reconsidered on the quantum level by the authors [12]. This talk reviews their results.

2. THE ANGULAR (RELATIVE) CALOGERO MODEL

In the first half of the talk, let us introduce the spherical reduction of rational $A_{m-1}$ Calogero model and present some of its salient features. In an $n$-particle quantum phase space with particle coordinates $x^\mu$ and momenta $p_\mu$, where $\mu = 1, 2, \ldots, n$, subject to $[x^\mu, p_\nu] = i\delta^\mu_\nu$, the rational Calogero Hamiltonian (after separating the center of mass) reads

$$H = \sum_{\mu < \nu} \left[ \frac{1}{2n} (p_\mu - p_\nu)^2 + \frac{g(g-1)}{(x^\mu - x^\nu)^2} \right]$$

(1)

The strength of the inverse-square two-body potential is parametrized by a real coupling constant $g$ (which could be taken $\geq \frac{1}{4}$). In the ‘relative’ $(n-1)$-dimensional phase space, a radial coordinate and momentum are defined via

$$\frac{1}{n} \sum_{\mu < \nu} (x^\mu - x^\nu)^2 = r^2$$

and

$$\frac{1}{n} \sum_{\mu < \nu} (p_\mu - p_\nu)^2 = p_r^2 + \frac{1}{r^2} L^2 + \frac{(n-2)(n-4)}{4r^2}$$

(2)

It is convenient to switch to $n-1$ ‘relative’ coordinates $y^i$ and momenta $p_i$, with $i = 1, 2, \ldots, n-1$,

$$r^2 = \sum_{i=1}^{n-1} (y^i)^2, \quad p_i = p_{y^i}$$

$$L_{y^i} = -i(y^i p_{y^i} - y^i p_i), \quad L^2 = -\sum_{i<j} L_{y^i}^2$$

(3)
In terms of polar coordinates \((r, \theta)\) on \(\mathbb{R}^{n-1}\), the Hamiltonian takes the form

\[
H = \frac{1}{2} p_r^2 + \frac{(n - 2)(n - 4)}{8r^2} + \frac{1}{r^2} H_\Omega
\]

with \(H_\Omega = \frac{1}{2} L^2 + U(\theta)\),

where the angular potential is

\[
U(\theta) = r^2 \sum_{\mu < \nu} g(g - 1) \frac{(x^\mu - x^\nu)^2}{(x^\mu - x^\nu)^2} = r^2 \sum_{\alpha \in \mathcal{R}_+} \frac{g(g - 1)}{2} \sum_{\alpha \in \mathcal{R}_+} \cos^{-2} \theta_\alpha.
\]

(5)

Here, we introduced the \(A_{n-1}\) positive root system \(\mathcal{R}_+\) and the angle \(\theta_\alpha\) between the point \(\theta \in \mathbb{S}^{n-2}\) and the root \(\alpha\). \(H_\Omega\) is the angular \((\text{relative})\) Calogero Hamiltonian, our object of interest.

In the position representation, we pass to differential operators,

\[
p_i \mapsto -i\partial_i \Rightarrow p_r \mapsto -i \left( \partial_r + \frac{n - 2}{2r} \right),
\]

so our Hamiltonian operators become

\[
H \mapsto -\frac{1}{2} \left( \partial_r^2 + \frac{n - 2}{r} \partial_r \right) + \frac{1}{r^2} H_\Omega = S^{-1} \left[ -\frac{1}{2} \left( \partial_r^2 + \frac{n - 2}{r} \partial_r \right) - \frac{(n - 2)(n - 4)}{4r^2} \right] + \frac{1}{r^2} H_\Omega \]

\[
H_\Omega \mapsto -\frac{1}{2} \sum_{i < j} (y^i \partial_j - y^j \partial_i)^2 + \frac{1}{r^2} \sum_{\alpha \in \mathcal{R}_+} \frac{g(g - 1)}{(\alpha \cdot y)^2}
\]

with \(S = r^{n-2} \).

The spectrum and the eigenfunctions of \(H\) are known,

\[
H \Psi_{E,q} = E \Psi_{E,q} \quad \text{with} \quad E \in \mathbb{R}_{\geq 0}
\]

and \(\Psi_{E,q}(r, \theta) = r^{-\frac{n-3}{2}} J_{q+(n-3)/2} (\sqrt{2Er}) v_q(\theta)\),

where we took advantage of the conformal invariance to separate in polar coordinates. The angular wave function \(v_q(\theta)\) is an eigenfunction of the angular Hamiltonian, whose spectrum is also in the literature,

\[
H_\Omega v_q = \varepsilon_q v_q \quad \text{with} \quad \varepsilon_q = \frac{1}{2} q(q + n - 3)
\]

and

\[
q = \frac{1}{2} m(n - 1) + \ell
\]

where \(\ell = 3\ell_3 + 4\ell_4 + \ldots + n\ell_n \in \mathbb{N}_0\).

The degeneracy of energy level \(\varepsilon_q\) is given by

\[
\text{deg}(\varepsilon_q) = p_n(\ell) - p_n(\ell - 1) - p_n(\ell - 2) + p_n(\ell - 3)
\]

(10)

with the restricted partitions \(p_n(\ell)\) given by the simple generating function

\[
p_n(\ell) := \sum_{\ell'=0}^{\infty} p_n(\ell') r^\ell' = \prod_{m=1}^n (1 - t^m)^{-1}.
\]

(11)

Relevant for this talk are the cases of \(n = 3\) and 4,

\[
\text{deg}_3(\ell) = \begin{cases} 0 & \text{for } \ell = 1, 2 \mod 3 \\ 1 & \text{for } \ell = 0 \mod 3, \end{cases}
\]

(12)

\[
\text{deg}_4(\ell) = \begin{cases} \ell & \text{for } \ell = 1, 2, 5 \mod 12 \\ 1 & \text{for } \ell = \text{else mod 12} \end{cases}
\]

All the interesting nontrivial structure is hidden in the angular eigenfunctions:

\[
v_q(\theta) \equiv v_q^{(e)}(\theta) \sim r^{n-3+q} \prod_{\mu = 3}^n \sigma_\mu((\mathcal{D}_\mu),)\ell^{\mu} \Delta^{r^{3-n(n-1)}},
\]

(13)

which employs the Vandermonde \(\Delta\) and the (mutually commuting) Dunkl operators \(\mathcal{D}_\mu\) as arguments in the \(\mu\)th Newton sum \(\sigma_\mu(y) = \sum_i (y_i)^\mu\),

\[
\Delta = \prod_{\alpha \in \mathcal{R}_+} \alpha \cdot y
\]

(14)

and \(\mathcal{D}_\mu \mapsto \partial_i - g \sum_{\alpha \in \mathcal{R}_+} \alpha_i \cdot y s_\alpha\),

where \(s_\alpha\) denotes the reflection on the hyperplane orthogonal to the root \(\alpha\). These wave functions contain a factor of \(\Delta^{r^\ell}\) and are directly related to Dunkl-deformed Weyl-symmetric harmonic polynomials,

\[
v_q^{(e)}(\theta) = r^{-q} \Delta^{r^\ell} h_q^{(e)} \text{ with } H(\Delta^{r^\ell} h_q^{(e)}) = 0.
\]

(15)

The \(\mathcal{D}_\mu, y^i\) and \(s_\alpha\) form a rational Cherednik algebra. The restriction \textquoteleft res\textquoteright{} of its elements to Weyl-invariant functions yields important differential operators, in particular our Hamiltonians. To make this explicit, we \textquoteleft Dunkl-deform\textquoteright{} not only the linear momenta, \(\partial_i \mapsto \mathcal{D}_i\) but also the angular momenta,

\[
L_i \mapsto -\left( y^i \partial_j - y^j \partial_i \right) \Rightarrow \mathcal{L}_{ij} = -(y^i \mathcal{D}_j - y^j \mathcal{D}_i),
\]

(16)

and define the \textquoteleft pre-Hamiltonians\textquoteright

\[
\mathcal{H} = -\frac{1}{2} \sum_i \mathcal{D}_i^2
\]

and

\[
\mathcal{H}_\Omega = -\frac{1}{2} \sum_{i < j} \mathcal{D}_{ij}^2 + \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} g \Sigma_\alpha s_\alpha (g \Sigma_\alpha s_\alpha + n - 3),
\]

(17)

whose Weyl-symmetric restriction produce

\[
H = \text{res}(\mathcal{H})
\]

and

\[
H_\Omega = \text{res}(\mathcal{H}_\Omega) = \frac{1}{2} \text{res}(\mathcal{L}_j^2) + \varepsilon_q(\ell = 0).
\]

(18)
The Cherednik subalgebra generated by the $L_{ij}$ and
the Weyl reflections is given by the relations

$$\begin{align*}
[L_{ij}, L_{kl}] &= L_{ik} L_{jl} - L_{jk} L_{il} + \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl}, \\
[L_{ij}, L_{kl}] &= L_{ij} L_{kl} - L_{kl} L_{ij} + \delta_{ij} L_{kl} - \delta_{kl} L_{ij}.
\end{align*}$$

(19) (20)

$$[F_{ij}, F_{kl}] = 0,$$

(21)

$$[F_{ij}, L_{kl}] = \delta_{ik} L_{jl} - \delta_{ik} L_{jl},$$

with $F_{ij} = \{1 + g \sum_{k \neq i} s_k\}$ for $i \neq j$.

(22)

It is a ‘Dunkl deformation’ of $so(n-1)$, with $H_{\Omega}$ being the Casimir invariant.

A hallmark of Calogero models is their isospectrality, which is characterized by the existence of intertwinning (or shift) operators relating the energy spectra at couplings $g$ and $g + 1$. This concept is well established for the full rational model, but is also works in the angular submodel. There, angular intertwiners are differential operators $M_s$ in $\theta$ of some order $s$, constructed with the following recipe,

$$M_s = \text{res}(M_s)$$

with $M_s$ is Weyl antiinvariant

(23)

in $L_{ij}$ of degree $s$.

Since $[L_{ij}, H_{\Omega}] = 0$ and $M_s$ has no $r$ dependence, it follows that

$$[M_s, H_{\Omega}] = 0 \Rightarrow M_s^{(g)} H_{\Omega}^{(g)} = H_{\Omega}^{(g+1)} M_s^{(g)}$$

and $M_s^{(g)} \nu^{(g)} - \nu^{(g+1)} \sim_{-m(n-1)/2}^{m(n-1)/2}$.

(24)

The adjoint $M_s^{(g)} = M_s^{(-g)}$ intertwines in the opposite direction, i.e. $M_s^{(-g)} \nu^{(g)} - \nu^{(g+1)} \sim_{m(n-1)/2}^{-m(n-1)/2}$. It follows that for integer $g$ we can obtain the angular eigenfunctions more directly by successively applying intertwiners to the free eigenfunctions, say at $g = 1$,

$$\nu^{(g)} \sim M_s^{(-1)} M_s^{(-2)} \cdots M_s^{(0)} \nu^{(1)} \sim_{-m(n-1)/2}^{m(n-1)/2}.$$ (25)

An important issue is the existence of conserved charges beyond the Hamiltonian $H_{\Omega}$. Obviously,

$$[M_s, H_{\Omega}] = 0 = [L_{ij}, H_{\Omega}],$$

but this need not provide new quantities. However, any Weyl-invariant polynomial $\mathcal{C}_t(L_{ij})$ of some degree $t$ gives rise to a conserved charge,

$$\mathcal{C}_t = \text{res}(\mathcal{C}_t)$$

commutes with $H_{\Omega}$.

(26)

We already know of $C_0 = 1$ and $C_2 = -\text{res}(L^2)$ but expect $2n - 5$ algebraically independent constants of motion (beyond $C_0$) in a superintegrable theory. Other than the Liouville charges in the full Calogero model, they will generically mix under the intertwining action,

$$M_s^{(g)} C_t^{(g)} = \sum_{s'\neq s} \Gamma_{s'}^{(g)} (g) C_t^{(g+1)} M_s^{(g)}$$ (27)

with some coefficient functions $\Gamma_{s'}^{(g)} (g)$.}

3. WARMUP: THE HEXAGONAL OR PÖSCHL-TELLER MODEL

Let us illustrate the structures just mentioned on the first nontrivial example, which at $n = 3$ is the $A_2$ model. Its spherical reduction (to the unit circle) is known as the Pöschl–Teller model, but we call it ‘hexagonal’ because the potential is singular at angles $\phi = (2k + 1)\pi/6$. The relation between the 3 particle coordinates $x^i$ and the 2 Jacobi relative coordinates $y^i$ orthogonal to the center of mass $X$ is

$$x^1 = X + \frac{1}{\sqrt{2}} y^1 + \frac{1}{\sqrt{6}} y^2,$$

$$\frac{\partial}{\partial x^i} = \frac{1}{3} \frac{\partial}{\partial y^i} + \frac{1}{\sqrt{6}} \frac{\partial}{\partial y^2},$$

(28)

$$x^2 = X - \frac{1}{\sqrt{2}} y^1 + \frac{1}{\sqrt{6}} y^2,$$

$$\frac{\partial}{\partial x^i} = \frac{1}{3} \frac{\partial}{\partial y^i} - \frac{1}{\sqrt{6}} \frac{\partial}{\partial y^2} + \frac{1}{\sqrt{6}} \frac{\partial}{\partial y^2},$$

$$x^3 = X - \frac{2}{\sqrt{6}} y^2,$$

$$\frac{\partial}{\partial x^i} = \frac{1}{3} X - \frac{2}{\sqrt{6}} \frac{\partial}{\partial y^2}.$$ (29)

Performing the polar decomposition and introducing a complex coordinate,

$$y^1 = r \cos \phi \quad \text{and} \quad y^2 = r \sin \phi$$

$$\Rightarrow w := y^1 + iy^2 = re^{i\phi},$$

the angular Hamiltonian takes the form

$$H_{\Omega} = \frac{1}{2} \left( w \partial_w - \bar{w} \partial_{\bar{w}} \right)^2$$

+ $g(g - 1) \frac{18(w \bar{w})}{(w^3 + \bar{w}^3)^2}$

(30)

since

$$U(\phi) = \frac{g(g - 1)}{2} \sum_{k=0,1,2} \cos^{-2} \left( \phi + k \frac{2\pi}{3} \right)$$

$$= \frac{9}{2} g(g - 1) \cos^{-2} (3\phi) = g(g - 1) \frac{18(w \bar{w})}{(w^3 + \bar{w}^3)^2}.$$ (31)

Its spectrum depends on a single quantum number $\ell = 3l_3$, with $l_3 \in \mathbb{N}_0$, $\varepsilon_{q} = \frac{1}{2} q^2$.

(32)

with $q = 3g + \ell = 3(g + l_3)$ and $deg(\varepsilon_q) = 1$.

Since the third Newton sum is $\sigma_3(w, \bar{w}) = w^3 - \bar{w}^3$, the angular wave functions are constructed as

$$\nu_q(\phi) \equiv \nu^{(g)}_q(\phi)$$

$$\sim r^q \left( \frac{\partial^3}{\partial \phi^3} - \frac{\partial^3}{\partial \phi^3} \right) \Delta \Delta r^{-6\varepsilon} = r^{-q} \Delta^2 \nu^{(g)}_q \left( w^3, \bar{w}^3 \right),$$

(33)
where the ingredients are

$$\Delta \sim w^3 + \bar{w}^3 - r^3 \cos(3\Phi)$$ \hspace{1cm} (34)

$$= \partial_w - g \left\{ \frac{1}{w + \bar{w}} s_0 + \frac{\rho}{\rho w + \rho \bar{w}} s_+ + \frac{\bar{\rho}}{\bar{\rho} w + \bar{\rho} \bar{w}} s_- \right\}$$ \hspace{1cm} (35)

with \( \rho = e^{2i\nu/3} \).

The application of the Dunkl operators can be evaluated analytically, arriving at

$$\hat{\mathcal{h}}_{\ell}^{(g>0)} = \sum_{k=0}^{\ell_1} (-1)^k \frac{\Gamma(g + k) \Gamma(g + \ell_3 - k)}{\Gamma(g) \Gamma(1 + k) \Gamma(1 + \ell_3 - k)} w^{\ell_3 - 3k} \bar{w}^{3k}.$$ \hspace{1cm} (36)

The table below lists some low-lying hexagonal wave functions, abbreviating \( (m\bar{m}) := w^{3m} \bar{w}^{3\bar{m}} \).

| \(\ell\) | \(\hat{\mathcal{h}}_{\ell}^{(0)}\) | \(\hat{\mathcal{h}}_{\ell}^{(1)}\) | \(\hat{\mathcal{h}}_{\ell}^{(2)}\) |
|---|---|---|---|
| 0 | (00) | (00) | (00) |
| 3 | (10) - (01) | (10) - (01) | (10) - (01) |
| 6 | (20) + (02) | (20) - (11) + (02) | 3(20) - 4(11) + 3(02) |
| 9 | (30) - (03) | (30) - (21) + (12) - (03) | 4(30) - 6(21) + 6(12) - 4(03) |
| 12 | (40) + (04) | (40) - (31) + (22) - (13) + (04) | 5(40) - 8(31) + 9(22) - 8(13) + 5(04) |

The simplest Weyl antiinvariant build from \( \mathcal{L}_{12} \) is the Dunklized angular momentum itself,

$$M_1 = i(w \partial_w - \bar{w} \partial_{\bar{w}}) \hspace{1cm} (37)$$

$$- i(w \partial_{\bar{w}} + \bar{w} \partial_w)$$

whose Weyl-symmetric restriction gives a most simple angular intertwiner,

$$M_1 = i(w \partial_w - \bar{w} \partial_{\bar{w}}) - 3ig \frac{w^3 - \bar{w}^3}{w + \bar{w}} \hspace{1cm} (38)$$

$$= \Delta^{(g>0)} (w \partial_w - \bar{w} \partial_{\bar{w}}) \Delta^{-g} = \partial_\theta + 3g \tan 3\Phi,$$

which allows for an even simpler recursion relation for the hexagonal wave functions,

$$\hat{\mathcal{h}}_{\ell}^{(g+1)} = \Delta^{-1} (w \partial_w - \bar{w} \partial_{\bar{w}}) \hat{\mathcal{h}}_{\ell}^{(g)}.$$ \hspace{1cm} (39)

Iterating this recursion is an easier way to construct these wave functions from the ground state.

Because

$$(M_1^+ M_1)^{(g)} = -2H^{(g)} + 9g^2 = -\mathrm{res}(\mathcal{L}^2) = -C_2^{(g)},$$ \hspace{1cm} (40)

there is no further conserved charge besides the angular Hamiltonian in the hexagonal model.

### 4. TETRAHEXAHEDRIC MODEL: THE SPECTRUM

Now we pass to the next and more interesting case, \( n = 4 \). This angular model is quite new and describes a particle on the two-sphere with a non-separable potential. We call it tetrahexahedric because the singular loci of the potential are six great circles which form the edges of a spherical polyeder called tetrahexahedron. Therefore, the particle is trapped in one of 24 identical fundamental domains (the faces), which have the shape of a (spherical) right isosceles triangle. It is convenient to pass to Walsh–Hadamard relative coordinates (due to \( A_4 = D_3 \)):

$$x^1 = X + \frac{1}{2} (x + y + z), \hspace{1cm} \partial_x^1 = \frac{1}{4} \partial_X + \frac{1}{2} (\partial_x + \partial_y + \partial_z),$$

$$x^2 = X + \frac{1}{2} (x - y - z), \hspace{1cm} \partial_x^2 = \frac{1}{4} \partial_X + \frac{1}{2} (\partial_x - \partial_y - \partial_z),$$

$$x^3 = X + \frac{1}{2} (-x + y - z),$$

$$x^4 = X + \frac{1}{2} (-x - y + z), \hspace{1cm} \partial_x^4 = \frac{1}{4} \partial_X + \frac{1}{2} (-\partial_x + \partial_y + \partial_z),$$

and introduce spherical coordinates

$$x = r \sin \theta \cos \phi, \hspace{0.5cm} y = r \sin \theta \sin \phi, \hspace{0.5cm} z = r \cos \theta.$$ \hspace{1cm} (42)
The angular momenta and the spherical Laplacian take the familiar form

\[ L_x = -(y \partial_y - z \partial_z), \quad L_y = -(z \partial_y - x \partial_z), \quad L_z = -(x \partial_y - y \partial_z), \quad L^2 = -(L_x^2 + L_y^2 + L_z^2) \]

and the angular Hamiltonian reads

\[ H_{\Omega} = \frac{1}{2} L^2 + U(\theta, \phi) \]

with

\[ U(\theta, \phi) = 2g(g - 1)(x^2 + y^2 + z^2) \left( \frac{x^2 + y^2}{(x^2 - y^2)^2} + \frac{y^2 + z^2}{(y^2 - z^2)^2} + \frac{z^2 + x^2}{(z^2 - x^2)^2} \right) \]

\[ = 2g(g - 1) \left[ \frac{1}{\sin^2 \theta \cos^2 \phi} \cos^2 \theta + \sin^2 \theta \cos^2 \phi \right] \]

\[ + \cos^2 \theta + \sin^2 \theta \sin^2 \phi \]

\[ \left( \frac{1}{\cos^2 \theta - \sin^2 \theta \sin^2 \phi} \right)^2 \]

The tetrahexahedric energy spectrum is given by

\[ \epsilon_q = \frac{1}{2} q(q + 1) \text{ with } q = 6g + \ell \]

\[ = 6g + 3\ell_3 + 4\ell_4 \text{ and } \ell_3, \ell_4 \in \mathbb{N}_0. \]

The corresponding wave functions can be computed from

\[ v^{(g)}_{\ell}(\theta, \phi) - r^{q+1}(\mathcal{D}_x \mathcal{D}_y \mathcal{D}_z)^{f_1} \times (\mathcal{D}_x^4 + \mathcal{D}_y^4 + \mathcal{D}_z^4)^{f_2} \Delta^q r^{12q} = r^{q} \Delta^q \tilde{h}_{\ell}^{(g)}(x, y, z), \]

with \( \Delta = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2) \)

and the linear Dunkl operators

\[ \mathcal{D}_x = \partial_x - \frac{g}{x + y} s_{x+y} - \frac{g}{x - y} s_{x-y} \]

\[ - \frac{g}{z + x} s_{x+z} - \frac{g}{z - x} s_{x-z} \]

\[ \mathcal{D}_y = \partial_y - \frac{g}{y + x} s_{y+x} - \frac{g}{y - x} s_{y-x} \]

\[ - \frac{g}{z + y} s_{y+z} - \frac{g}{z - y} s_{y-z} \]

including the elementary reflections constituting the \( S_4 \) Weyl group action,

\[ s_{x+y} : (x, y, z) \mapsto (-y, -x, +z), \]

\[ s_{x-y} : (x, y, z) \mapsto (+y, +x, +z), \]

\[ s_{y+z} : (x, y, z) \mapsto (+x, -z, -y), \]

\[ s_{y-z} : (x, y, z) \mapsto (+x, +z, +y), \]

\[ s_{z+x} : (x, y, z) \mapsto (-z, +y, -x), \]

\[ s_{z-x} : (x, y, z) \mapsto (+z, +y, +x). \]

The following table lists the low-lying tetrahexahedric wave functions for \( g = 0 \) and \( g = 1 \), using the notation

\[ \{rst\} := x^r y^s z^t + x^r y^t z^s + x^s y^r z^t + x^s y^t z^r + x^t y^r z^s + x^t y^s z^r. \]

| \( \ell_3 \) | \( \ell_4 \) | \( \tilde{h}_{\ell_3, \ell_4}^{(0)} \) |
|---|---|---|
| 0 | 0 | \{000\} |
| 1 | 0 | \{111\} |
| 0 | 1 | \{400\} - 3\{220\} |
| 2 | 0 | \{600\} - 15\{420\} + 30\{222\} |
| 1 | 1 | 3\{511\} - 5\{331\} |
| 0 | 2 | \{800\} - 28\{620\} + 35\{440\} |
| 3 | 0 | 9\{711\} - 63\{531\} + 70\{333\} |
| 2 | 1 | \{1000\} - 45\{820\} + 42\{640\} + 504\{462\} - 630\{442\} |
| 1 | 2 | 5\{911\} - 60\{731\} + 63\{551\} |
| 4 | 0 | 36\{1200\} - 2376\{1020\} + 2445\{840\} - 4612\{822\} + 4893\{660\} - 2152\{562\} + 1793\{744\} |
| 0 | 3 | 101\{1200\} - 6666\{1020\} + 47100\{840\} + 8685\{822\} - 42609\{660\} - 40530\{642\} + 3377\{444\} |
We note that these are eigenfunctions of the free model, \( H_0 = \frac{1}{2} L^2 \), since the potential is absent at \( g = 0 \) or 1, but they are \( S_4 \) invariant. The interacting eigenfunctions are of the same form, only the coefficients depend on \( g \).

5. TETRAHEXAHEDRIC MODEL: INTERTWINER AND INTEGRABILITY

In order to construct the intertwiners of the tetrahexahedric model, one starts with the angular Dunkl operators,

\[
\mathcal{L}_x = L_x + g\left( \frac{z}{x-y} s_{s-x} - \frac{z}{x+y} s_{s+x} - \frac{y}{x-z} s_{z-x} + \frac{y}{z-x} s_{x-z} \right),
\]

\[
\mathcal{L}_y = L_y + g\left( \frac{x}{y-z} s_{s-y} - \frac{x}{y+z} s_{s+y} - \frac{y}{y-x} s_{y-x} + \frac{y}{y+z} s_{x+y} \right),
\]

\[
\mathcal{L}_z = L_z + g\left( \frac{y}{z-x} s_{s-z} - \frac{y}{z+y} s_{s+y} - \frac{x}{z-y} s_{z-y} + \frac{x}{z+y} s_{y+z} \right).
\]

It turns out that the simplest Weyl antiinvariant is cubic,

\[
\mathcal{M}_3 = -\frac{1}{6} (\mathcal{L}_x \mathcal{L}_y \mathcal{L}_z + \mathcal{L}_x \mathcal{L}_z \mathcal{L}_y + \mathcal{L}_y \mathcal{L}_z \mathcal{L}_x),
\]

and taking the Weyl-symmetric reduction we obtain a first angular intertwiner,

\[
\begin{align*}
M_3 &= y^2 z \partial_{xx} - yz^2 \partial_{xy} + \frac{1}{2} (y^2 - z^2) \partial_{xx} \\
&\quad + 4g \frac{yz}{y^2 - z^2} (yz \partial_{xx} + x^2 \partial_{xy} - zx \partial_{xy}) \\
&\quad + g \left[ 2gy^2 z^2 \left( \frac{8g}{(x^2 - y^2)(z^2 - x^2)} \right) \\
&\quad - \frac{16g}{(z^2 - x^2)(y^2 - z^2)} - \frac{2g-1}{(x^2 - y^2)} + \frac{2g-1}{(z^2 - x^2)} \right] \times x^2 \\
&\quad - \frac{2z^2}{z^2 - x^2} \frac{y^2 + z^2}{y^2 - z^2} x \partial_x + 2g(g-1)(g+2) \quad (53)
\end{align*}
\]

\[
\begin{align*}
M_3 &= y^2 z \partial_{xx} - yz^2 \partial_{xy} + \frac{1}{2} (y^2 - z^2) \partial_{xx} \\
&\quad + 4g \frac{yz}{y^2 - z^2} (yz \partial_{xx} + x^2 \partial_{xy} - zx \partial_{xy}) \\
&\quad + g \left[ 2gy^2 z^2 \left( \frac{8g}{(x^2 - y^2)(z^2 - x^2)} \right) \\
&\quad - \frac{16g}{(z^2 - x^2)(y^2 - z^2)} - \frac{2g-1}{(x^2 - y^2)} + \frac{2g-1}{(z^2 - x^2)} \right] \times x^2 \\
&\quad - \frac{2z^2}{z^2 - x^2} \frac{y^2 + z^2}{y^2 - z^2} x \partial_x + 2g(g-1)(g+2) \quad (53)
\end{align*}
\]

\[
\begin{align*}
M_3 &= y^2 z \partial_{xx} - yz^2 \partial_{xy} + \frac{1}{2} (y^2 - z^2) \partial_{xx} \\
&\quad + 4g \frac{yz}{y^2 - z^2} (yz \partial_{xx} + x^2 \partial_{xy} - zx \partial_{xy}) \\
&\quad + g \left[ 2gy^2 z^2 \left( \frac{8g}{(x^2 - y^2)(z^2 - x^2)} \right) \\
&\quad - \frac{16g}{(z^2 - x^2)(y^2 - z^2)} - \frac{2g-1}{(x^2 - y^2)} + \frac{2g-1}{(z^2 - x^2)} \right] \times x^2 \\
&\quad - \frac{2z^2}{z^2 - x^2} \frac{y^2 + z^2}{y^2 - z^2} x \partial_x + 2g(g-1)(g+2) \quad (53)
\end{align*}
\]

In the 'potential-free frame', attained by a similarity transformation, it simplifies to
\[
\Delta^{-\varepsilon} M_2 \Delta^{-\varepsilon} - y^2 z \partial_{xzx} - y z^2 \partial_{xzy} \\
+ \frac{1}{2} (y^2 - z^2) \partial_{xx} + 2g \frac{y^2 z^2 (y^2 - z^2)}{(x^2 - y^2)(x^2 - z^2)} \partial_{xx} \\
+ 4g \frac{y^2 z^2}{x^2 - z^2} \partial_{xz} \\
+ 2gx \left[ \frac{y^2 (x^2 + 3z^2)}{(x^2 - z^2)^2} - z^2 \frac{(x^2 + 3y^2)}{(x^2 - y^2)^2} \right] \partial_x \\
+ \text{cyclic permutations.}
\] 

(54)

The next independent antiinvariant is sextic,

\[
M_6 = (\mathcal{L}_x^4, \mathcal{L}_y^4) - (\mathcal{L}_y^4, \mathcal{L}_z^4) + (\mathcal{L}_x^4, \mathcal{L}_z^4),
\]

(55)

and gives rise to a rather lengthy expression (not displayed) for a second intertwiner \( M_6 \). We expect that \( \Delta^{-\varepsilon} M_6 \Delta^{-\varepsilon} \) is more compact. All higher angular intertwiners can be reduced to \( M_3 \) and \( M_6 \).

Let us finally take a look at the conserved charges in this model. It is not hard to see that they are generated by

\[
J_k := \text{res}(\mathcal{L}_x^k + \mathcal{L}_y^k + \mathcal{L}_z^k) \quad \text{for} \quad k = (0,2,4,6),
\]

(56)

with \( J_0 = C_0 = 1 \)

(57)

Higher conserved charges are algebraically dependent, e.g.

\[
6J_2 = 8J_2 + 3J_4J_4 - 6J_4J_2 + J_2 J_2 J_2 \\
- 12(8 + 5g + 12g^2)J_6 + 4(34 + 23g + 30g^2)J_4J_2 \\
- 8(5 + 3g + 3g^2)J_4J_2 \\
+ 24(13 + 15g - 102g^2 - 72g^3)J_4 \\
- 4(43 + 70g - 252g^2 - 144g^3)J_2J_2 \\
- 48(1 + 3g)(1 + 4g)(1 + 12g)J_2.
\]

(58)

Any word in \( \{J_2, J_4, J_6\} \) is conserved, but there are some relations in their algebra. Namely, \( J_6 \) and \( J_2 \) span the center, and

\[
\{J_2, J_4\} = \{J_2, J_6\} = 0 \quad \text{but} \quad \{J_4, J_6\} \neq 0,
\]

(59)

so \( J_2, J_4, J_6 \) are two independent new words. The basic intertwining relations read

\[
M_3 J_2^{(\varepsilon)} = (J_2^{(\varepsilon+1)} - 6(7 + 12g))M_3^{(\varepsilon)}, \\
M_3 J_4^{(\varepsilon)} = (J_4^{(\varepsilon+1)} - 4(11 + 12g)J_2^{(\varepsilon+1)}) \\
+ 48(26 + 73g + 48g^2)M_3^{(\varepsilon)} + 2M_6^{(\varepsilon)}, \\
M_3 J_6^{(\varepsilon)} = (J_6^{(\varepsilon+1)} - (35 + 36g)J_4^{(\varepsilon+1)}) \\
- 3(7 + 4g)J_4^{(\varepsilon+1)}J_2^{(\varepsilon+1)} \\
+ 2(1111 + 2668g + 1392g^2)J_3^{(\varepsilon+1)} \\
+ 96(457 + 1933g + 2717g^2 + 1368g^3 + 144g^4)M_3^{(\varepsilon)} \\
+ (3J_2^{(\varepsilon+1)} - (115 + 200g + 48g^2))M_6^{(\varepsilon)}.
\]

(60)

Particular conserved quantities are obtained by intertwining ‘back and forth’, e.g.

\[
M_3 J_2 = 12J_6 - 18J_4 + 6J_2 J_2 \\
- 6(1 + 16g - 48g^2)J_4 + 3(13 + 24g - 48g^2)J_2J_2 \\
+ 12(1 + 3g)(1 + 4g)(1 - 12g)J_6, \\
M_3 J_6 = -12J_6 + 12J_6 J_4J_2 \\
- \frac{16}{3} J_2 J_4 J_2 J_2 + 2J_4 J_4 - 14J_2 J_2 J_2 \\
+ 6J_4 J_2 J_4 J_2 J_2 - \frac{2}{3} J_4 J_4 J_2 J_2 J_2 + \text{lower-order terms},
\]

and similarly for \( M_4 M_6 \) and \( M_6 M_4 \). An additional set of ‘odd’ conserved charges appears due to the equality

\[
H_\Omega^{(\varepsilon)} = H_\Omega^{(\varepsilon+\varepsilon)} \quad \text{(here } \varepsilon = 3 \text{ or } 6),
\]

(62)

\[
\Rightarrow Q^{(\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon)} := M_3^{(\varepsilon)} M_3^{(\varepsilon)} \cdots M_3^{(\varepsilon)} = H_\Omega^{(\varepsilon)} Q^{(\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon)}.
\]

Combining all charges one ends up with a \( \mathbb{Z}_2 \) graded nonlinear algebra generated by \( \{Q, J_2, J_4, J_6\} \).

6. SUMMARY AND OUTLOOK

Let us summarize. We have presented a geometrical picture of a superintegrable but not separable potential on \( S^{n-2} \). The full set of conserved charges is characterized by the Weyl invariants built from the Dunkl-deformed angular momenta. Their form and action on the conserved charges was elucidated in the \( n \) (Pöschl–Teller or hexagonal) \( n \) (tetrahexahedric) cases. For integer coupling states into the picture.

Their form and action on the conserved charges was elucidated in the \( n \) (Pöschl–Teller or hexagonal) and \( n \) (tetrahexahedric) cases. For integer coupling states into the picture.

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REFERENCES

1. F. Calogero, “Solution of a three-body problem in one dimension,” J. Math. Phys. 10, 2191–2196 (1969).

2. M. A. Olshanetsky and A. M. Perelomov, “Classical integrable finite-dimensional systems related to lie algebras,” Phys. Rep. 71, 313–400 (1981).

3. M. A. Olshanetsky and A. M. Perelomov, “Quantum integrable systems related to lie algebras,” Phys. Rep. 94, 313–404 (1983).

4. S. Wojciechowski, “Superintegrability of the Calogero-Moser system,” Phys. Lett. A 95, 279–281 (1983).

5. O. A. Chalykh and A. P. Veselov, “Commutative rings of partial differential operators and lie algebras,” Comm. Math. Phys. 126, 597–611 (1990).

6. C. F. Dunkl, “Differential-difference operators associated to reflection groups,” Trans. Am. Math. Soc. 311, 167–183 (1989).

7. G. J. Heckman, “A remark on the Dunkl differential-difference operators – harmonic analysis on reductive groups,” Progr. Math. 101, 181–191 (1991).

8. M. V. Feigin, “Intertwining relations for the spherical parts of generalized Calogero operators,” Theor. Math. Phys. 135, 497–509 (2003).

9. T. Hakobyan, A. Nersessian, and V. Yeghikyan, “The cuboctahedric Higgs oscillator from the rational Calogero model,” J. Phys. A: Math. Theor. 42, 205206 (2009); arXiv:0808.0430 [hep-th].

10. M. Feigin, O. Lechtenfeld, and A. Polychronakos, “The quantum angular Calogero-Moser model,” J. High Energy Phys. 1307, 162 (2013); arXiv:1305.5841 [math-ph].

11. M. Feigin and T. Hakobyan, “On Dunkl angular momentum algebra,” J. High Energy Phys. 1511, 107 (2015); arXiv:1409.2480 [math-ph].

12. F. Correa and O. Lechtenfeld, “The tetrahexahedric angular Calogero model,” J. High Energy Phys. 1510, 191 (2015); arXiv:1508.04925 [hep-th].