NOTIONS FOR RSA INTEGERS

Full version

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Abstract. The key-generation algorithm for the RSA cryptosystem is specified in several standards, such as PKCS#1, IEEE 1363-2000, FIPS 186-3, ANSI X9.44, or ISO/IEC 18033-2. All of them substantially differ in their requirements. This indicates that for computing a “secure” RSA modulus it does not matter how exactly one generates RSA integers. In this work we show that this is indeed the case to a large extend: First, we give a theoretical framework that will enable us to easily compute the entropy of the output distribution of the considered standards and show that it is comparatively high. To do so, we compute for each standard the number of integers they define (up to an error of very small order) and discuss different methods of generating integers of a specific form. Second, we show that factoring such integers is hard, provided factoring a product of two primes of similar size is hard.

Keywords. RSA integer, output entropy, reduction. ANSI X9.44, FIPS 186-3, IEEE 1363-2000, ISO/IEC 18033-2, NESSIE, PKCS#1, GnuPG, OpenSSL, OpenSwan, SSH.

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1. Introduction

An RSA integer is an integer that is suitable as a modulus for the RSA cryptosystem as proposed by Rivest, Shamir & Adleman (1977, 1978):

“You first compute $n$ as the product of two primes $p$ and $q$:

\[ n = p \cdot q. \]

These primes are very large, ‘random’ primes. Although you will make $n$ public, the factors $p$ and $q$ will be effectively hidden from everyone else due to the enormous difficulty of factoring $n$.”

Also in earlier literature such as Ellis (1970) or Cocks (1973) one does not find any further restrictions. In subsequent literature people define RSA integers similarly to Rivest, Shamir & Adleman: Crandall & Pomerance (2001) note that it is “fashionable to select approximately equal primes but sometimes one runs some further safety tests”. In more applied works such as Schneier (1996) or Menezes et al. (1997) one can read that for maximum
security one chooses two (distinct) primes of equal length. Also von zur Gathen & Gerhard (2003) follow a similar approach. On suggestion of B. de Weger, Decker & Moree (2008) define an RSA integer to be a product of two primes $p$ and $q$ such that $p < q < rp$ for some parameter $r \in \mathbb{R}_{>1}$. Real world implementations, however, require concrete algorithms that specify in detail how to generate RSA integers. This has led to a variety of standards, notably the standards PKCS#1 (Jonsson & Kaliski 2003), ISO 18033-2 (International Organization for Standards 2006), IEEE 1363-2000 (IEEE working group 2000), ANSI X9.44 (Accredited Standards Committee X9 2007), FIPS 186-3 (NIST 2009), the standard of the RSA foundation (RSA Laboratories 2000), the standard set by the German Bundesnetzagentur (Wohlmacher 2009), and the standard resulting from the European NESSIE project (Preneel et al. 2003). All of those standards define more or less precisely how to generate RSA integers and all of them have substantially different requirements. This reflects the intuition that it does not really matter how one selects the prime factors in detail, the resulting RSA modulus will do its job. But what is needed to show that this is really the case?

Following Brandt & Damgård (1993) a quality measure of a generator is the entropy of its output distribution. In abuse of language we will most of the time talk about the output entropy of an algorithm. To compute it, we need estimates of the probability that a certain outcome is produced. This in turn needs a thorough analysis of how one generates RSA integers of a specific form. If we can show that the outcome of the algorithm is roughly uniformly distributed, the output entropy is closely related to the count of RSA integers it can produce. It will turn out that in all reasonable setups this count is essentially determined by the desired length of the output, see Section 5. For primality tests there are several results in this direction (see for example Joye & Paillier 2006) but we are not aware of any related work analyzing the output entropy of algorithms for generating RSA integers.

Another requirement for the algorithm is that the output should be ‘hard to factor’. Since this statement does not even make sense for a single integer, this means that one has to show that the restrictions on the shape of the integers the algorithm produces do not introduce any further possibilities for an attacker. To prove this, a reduction has to be given that reduces the problem of factoring the output to the problem of factoring a product of two primes of similar size, see Section 8. Also there it is necessary to have results on the count of RSA integers of a specific form to make the reduction work. As for the entropy estimations, we do not know any related work on this. A conference version of this article, focusing on the analysis of standardized RSA key-generators only, was published in Loebenberger & Nüsken (2011).

In the following section we will develop a formal framework that can handle all possible definitions for RSA integers. After discussing the necessary number theoretic tools in Section 3, we give explicit formulae for the count of such integers which will be used later for entropy estimations of the various standards for RSA integers. In Section 4 we show how our general framework can be instantiated, yielding natural definitions for several types of RSA integers (as used later in the standards). The section afterwards compares in more detail the relations of the different notions. Section 6 gives a short overview on generic constructions for fast algorithms that generate such integers almost uniformly. At this point
we will have described all necessary techniques to compute the output entropy, which we
discuss in Section 7. The following section resolves the second question described above
by giving a reduction from factoring special types of RSA integers to factoring a product of
two primes of similar size. We finish by applying our results to various standards for RSA
integers in Section 9.

2. RSA integers in general

If one generates an RSA integer it is necessary to select for each choice of the security
parameter the prime factors from a certain region. This security parameter is typically an
integer \( k \) that specifies (roughly) the size of the output. We use a more general definition
by asking for integers from the interval \([x/r, x]\), given a real \( x \) and a parameter \( r \)
(possibly depending on \( x \)). Clearly, this can also be used to model the former selection
process by setting \( x = 2^k - 1 \) and \( r = 2 \). Let us in general introduce a notion of RSA
integers with tolerance \( r \) as a family

\[ A := (A_x)_{x \in \mathbb{R}_{>1}} \]

of subsets of the positive quadrant \( \mathbb{R}^2_{>1} \), where for every \( x \in \mathbb{R}_{>1} \)

\[ A_x \subseteq \left\{ (y, z) \in \mathbb{R}^2_{>1} \middle| \frac{x}{r} < yz \leq x \right\}. \]

The tolerance \( r \) shall always be larger than 1. We allow here that \( r \) varies with \( x \), which of
course includes the case when \( r \) is a constant. Typical values used for RSA are \( r = 2 \)
or \( r = 4 \) which fix the bit-length of the modulus more or less. Now an \( A \)-integer \( n \) of size \( x \)
—for use as a modulus in RSA — is a product \( n = pq \) of a prime pair \((p, q) \in A_x \cap (\mathbb{P} \times \mathbb{P})\),
where \( \mathbb{P} \) denotes the set of primes. They are counted by the associated prime pair counting
function \( \#A \) for the notion \( A \):

\[ \#A : \mathbb{R}_{>1} \rightarrow \mathbb{N}, \quad x \mapsto \# \{ (p, q) \in \mathbb{P} \times \mathbb{P} \mid (p, q) \in A_x \}. \]

Thus every \( A \)-integer \( n = pq \) is counted once or twice in \( \#A \) \( (x) \) depending on whether only
\((p, q) \in A_x \) or also \((q, p) \in A_x \), respectively. We call a notion symmetric if for all choices
of the parameters the corresponding area in the \((y, z)\)-plane is symmetric with respect to the
main diagonal, i.e. that \((y, z) \in A_x \) implies also \((z, y) \in A_x \). If to the contrary \((y, z) \in A_x \)
implies \((z, y) \notin A_x \) we call the notion antisymmetric. When we are only interested in
the associated RSA integers we can always require symmetry or antisymmetry, yet many
algorithms proceed in an asymmetric way.

Note that varying \( r \) do not occur in standards and implementations for RSA integers,
analyzed in Section 9. However, there are still quite natural notions in which a varying \( r \)
occurs: Consider for example the notion where the primes \( p, q \) are selected from the interval
\([x^{1/4}, x^{1/2}]\). Then we obtain the product \( pq \in [x^{1/2}, x] \). This corresponds to the notion
Figure 2.1: A generic notion of RSA integers with tolerance \( r \). The gray area shows the parts of the \((\ln y, \ln z)\)-plane which is counted. It lies between the tolerance bounds \( \ln x \) and \( \ln \frac{r}{r} \). The dashed lines show boundaries as imposed by \([c_1, c_2]\)-balanced. The dotted diagonal marks the criterion for symmetry.

discussed in Section 4.2 with \( r = \sqrt{x} \). Indeed, all the counting theorems in Section 4 can handle such large \( r \). However, the error term is correspondingly large.

Certainly, we will also need restrictions on the shape of the area we are analyzing: If one considers any notion of RSA integers and throws out exactly the prime pairs one would be left with a prime-pair-free region and any approximation for the count of such a notion based on the area would necessarily have a tremendously large error term. However, for practical applications it turns out that it is enough to consider regions of a very specific form. Actually, we will most of the time have regions whose boundary can be described by graphs of certain smooth functions, see Definition 3.3(ii).

For RSA, people usually prefer two prime factors of roughly the same size, where size is understood as bit length. Accordingly, we call a notion of RSA integers \([c_1, c_2]\)-balanced iff additionally for every \( x \in \mathbb{R}_{>1} \)

\[
A_x \subseteq \left\{ (y, z) \in \mathbb{R}_{>1}^2 \mid y, z \in [x^{c_1}, x^{c_2}] \right\},
\]

where \( 0 < c_1 \leq c_2 \) can be thought of as constants or — more generally — as smooth functions in \( x \) defining the amount of allowed divergence subject to the side condition that \( x^{c_1} \) tends to infinity when \( x \) grows. If \( c_1 > \frac{1}{2} \) then \( A_x \) is empty, so we will usually assume \( c_1 \leq \frac{1}{2} \). In order to prevent trial division from being a successful attacker it would be sufficient to require \( y, z \in \Omega(\ln^k x) \) for every \( k \in \mathbb{N} \). Our stronger requirement still seems reasonable and indeed equals the condition Maurer (1995) required for secure RSA moduli, as the supposedly most difficult factoring challenges stay within the range of our attention.
As a side-effect this greatly simplifies our approximations later. The German Bundesnetzagentur uses a very similar restriction in their algorithm catalog (Wohlmacher 2009). We can — for a fixed choice of parameters — easily visualize any notion of RSA integers by the corresponding region $A_x$ in the $(y, z)$-plane. It is favorable to look at these regions in logarithmic scale: writing $y = e^\nu$ and $z = e^\zeta$, we depict the region $(\ln A)_x$ in the $(\nu, \zeta)$-plane corresponding to the region $A_x$ in the $(y, z)$-plane, i.e. $(\nu, \zeta) \in (\ln A)_x \iff (y, z) \in A_x$. We obtain a picture like in Figure 2.1.

Often the considered integers $n = pq$ are also subject to further side conditions, like $\gcd((p - 1)(q - 1), e) = 1$ for some fixed public RSA exponent $e$. Most of the number theoretic work below can easily be adapted, but for simplicity of exposition we will often present our results without those further restrictions and just point out when necessary how to incorporate such additional properties.

In Wohlmacher (2009) it is additionally required that the primes $p$ and $q$ are not too close to each other. We ignore this issue here, since the probability that two primes are very close to each other would be tiny if the notion from which $(p, q)$ was selected is sufficiently large. If necessary, we are able to modify our notions such that also this requirement is met.

In order to count the number of $A$-integers we have to evaluate

$$\#A(x) = \sum_{(p, q) \in A_x} 1.$$

If we follow the intuitive view that a randomly generated number $n$ is prime with probability
$rac{1}{\ln n}$, we expect that we have to evaluate integrals like
\[
\int_{A_x} \frac{1}{\ln y \ln z} \, dz \, dy,
\]
while carefully considering the error between those integrals and the above sums. In logarithmic scale we obtain expressions of the form $\int_{(\ln A)\mathbb{Z}} \frac{e^{\rho + \zeta}}{\rho \zeta} \, d\rho \, d\zeta$. To get an understanding of these functions, in Figure 2.2 some contour lines of the inner function are depicted. From the figure we observe that pairs $(\nu, \zeta)$ where $\nu + \zeta$ is large have a higher weight in the overall count.

As we usually deal with balanced notions the considered regions are somewhat centered around the main diagonal. We will show in Section 8 that if factoring products of two primes is hard then it is also hard to factor integers generated from such notions.

3. Toolbox

We will now develop the necessary number theoretic concepts to obtain formulæ for the count of RSA integers that will later help us to estimate the output entropy of the various standards for RSA integers. In related articles, like Decker & Moree (2008) one finds counts for one particular definition of RSA integers. We believe that in the work presented here for the first time a sufficiently general theorem is established that allows to compute the number of RSA integers for all reasonable definitions.

We assume the Riemann hypothesis throughout the entire paper. The main terms are the same without this assumption, but the error bounds one obtains are then much weaker. We use the following version of the prime number theorem:

**Prime Number Theorem 3.1 (Von Koch 1901, Schoenfeld 1976).** If (and only if) the Riemann hypothesis holds, then for $x \geq 2657$
\[
|\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \ln x,
\]
where $\text{li}(x) := \int_0^x \frac{dt}{\ln t}$.

We first state a quite technical lemma that enables us to do our approximations:

**Lemma 3.2 (Prime sum approximation).** Let $f$, $f$, $\widehat{f}$ be functions $[B, C] \to \mathbb{R}_{>1}$, where $B, C \in \mathbb{R}_{>1}$ such that $\widehat{f}$ and $\widetilde{f}$ are piecewise continuous, $\widehat{f} + \widetilde{f}$ is either weakly decreasing, weakly increasing, or constant, and for $p \in [B, C]$ we have the estimate
\[
|f(p) - \widehat{f}(p)| \leq \widetilde{f}(p).
\]

Further, let $\widehat{E}(p)$ be a positive valued, continuously differentiable function of $p$ bounding $|\pi(p) - \text{li}(p)|$ on $[B, C]$. (For example, under the Riemann hypothesis we can take $\widehat{E}(p) = \cdots$)
Then
\[ \sum_{p \in \mathbb{P} \cap [B,C]} f(p) - \tilde{g} \leq \hat{g} \]
with
\[ \tilde{g} = \int_B^C \frac{\tilde{f}(p)}{\ln p} \, dp, \]
\[ \hat{g} = \int_B^C \frac{\tilde{f}(p)}{\ln p} \, dp + 2(\tilde{f} + \hat{f})(B)\hat{E}(B) + 2(\tilde{f} + \hat{f})(C)\hat{E}(C) + \int_B^C (\tilde{f} + \hat{f})(p)\hat{E}'(p) \, dp. \]

In the special case when \( \tilde{f} + \hat{f} \) is constant we have the better bound
\[ \hat{g} = \int_B^C \frac{\hat{f}(p)}{\ln p} \, dp + (\tilde{f} + \hat{f})(B)(\hat{E}(B) + \hat{E}(C)). \]

**Proof.** The proof can be done analogously to the proof of Lemma 2.1 in Loebenberger & Nüsken (2010): First, rewrite \( \sum_{p \in \mathbb{P} \cap [B,C]} f(p) \) as a Stieltjes integral \( \int_B^C f(p) \, dp \). Then integrate by parts, estimate \( \pi \), and finally integrate by parts 'backwards'.

Next we formulate a lemma specialized to handle RSA notions. We cannot expect to obtain an approximation of the number of prime pairs by the area of the region unless we make certain restrictions.

The following definition describes the restrictions that we use. As you will notice, it essentially enforces a certain monotonicity that allows the error estimation.

**Definition 3.3.** Let \( A \) be a notion of RSA integers with tolerance \( r \).

(i) The notion \( A \) is graph-bounded iff there are (at least) integrable boundary functions \( B_1, C_1 : \mathbb{R}_{>1} \to \mathbb{R}_{>1} \) and \( B_2, C_2 : \mathbb{R}_{>1}^2 \to \mathbb{R}_{>1} \) such that we can write
\[ A_x = \left\{ (y, z) \in \mathbb{R}_{>1}^2 \mid B_1(x) < y \leq C_1(x), \quad B_2(y, x) < z \leq C_2(y, x) \right\}, \]
where for all \( x \in \mathbb{R}_{>1} \) and all \( y \in ]B_1(x), C_1(x)[ \) we have \( 1 < B_1(x) \leq C_1(x) \leq x \) and \( 1 < B_2(y, x) \leq C_2(y, x) \leq x \).

(ii) The notion \( A \) is monotone at \( x \) (relative to the error bound \( \hat{E} \)) for some \( x \in \mathbb{R}_{>1} \) iff it is graph-bounded and the function
\[ \int_{B_2(p,x)}^{C_2(p,x)} \frac{1}{\ln q} \, dq + \hat{E}(B_2(p,x)) + \hat{E}(C_2(p,x)) \]
is either weakly increasing, weakly decreasing, or constant as a function in \( p \) restricted to the interval \([B_1(x), C_1(x)]\). If not mentioned otherwise we refer to the error bound given by \( \hat{E}(p) = \frac{1}{8\pi} \sqrt{p} \ln p \).

We call the notion \( A \) monotone iff it is monotone at each \( x \in \mathbb{R}_{>1} \) where \( A_x \neq \emptyset \).

(iii) The notion \( A \) is piecewise monotone iff there is a parameter \( m \in \mathbb{N} \) such that

\[
A_x := \bigcup_{j=1}^{m} A_{j,x},
\]

where \( A_{j,x} \) are all monotone notions of RSA integers of tolerance \( r \). Note that we may also allow \( m \) to depend on \( x \).

For (i) note that \( B_1(x) = C_1(x) \) allows to describe an empty set \( A_x \), and otherwise the inequality \( B_2(y, x) \neq C_2(y, x) \) makes sure that all four bounding functions are determined by \( A_x \) as long as \( y \in [B_1(x), C_1(x)] \). This condition enforces that \( A_x \) is (path) connected. We do not need that but also it does no harm. For (iii) observe that in the light of a multi-application of Lemma 3.6 we would be on the safe side if we require \( m \in \ln^{O(1)} x \). At the extreme \( m \in o \left( c_1 x^{\frac{1}{c_4}} \ln x \right) \) with \( c = \max(2c_2 - 1, 1 - 2c_1) \) is necessary for any meaningful result generalizing Lemma 3.6. As in particular (ii) is rather weird to verify we provide an easily checkable, sufficient condition for monotonicity of a notion.

**Lemma 3.4.** Assume \( A \) is a graph-bounded notion of RSA integers with tolerance \( r \) given by continuously differentiable functions \( B_1, C_1: \mathbb{R}_{>1} \to \mathbb{R}_{>1} \) and \( B_2, C_2: \mathbb{R}_{>1}^2 \to \mathbb{R}_{>1} \). Finally, let \( x \in \mathbb{R}_{>1} \) be such that

- the function \( B_2(p, x) \) is weakly decreasing in \( p \) and
- the function \( C_2(p, x) \) is weakly increasing in \( p \)

for \( p \in [B_1(x), C_1(x)] \), or vice versa. As usual let \( \hat{E}(p) \) be the function given by \( \hat{E}(p) = \frac{1}{8\pi} \sqrt{p} \ln p \). Then the notion \( A \) is monotone at \( x \) (relative to \( \hat{E} \)).

**Proof.** The goal is to show that the function

\[
h(p) := \int_{B_2(p,x)}^{C_2(p,x)} \frac{1}{\ln q} \, dq + \hat{E}(B_2(p,x)) + \hat{E}(C_2(p,x))
\]

is weakly increasing or weakly decreasing in \( p \). We write \( B'_2(p, x) \) and \( C'_2(p, x) \), respec-
tively, for the derivative with respect to \( p \). Note that

\[
h'(p) := \left( \frac{1}{\ln C_2(p, x)} + \frac{2 + \ln C_2(p, x)}{16\pi \sqrt{C_2(p, x)}} \right) C'_2(p, x)
\]

\[
> 0
\]

\[
- \left( \frac{1}{\ln B_2(p, x)} - \frac{2 + \ln B_2(p, x)}{16\pi \sqrt{B_2(p, x)}} \right) B'_2(p, x).
\]

Some simple calculus shows that the second underbraced term is always positive since
\( B_2(p, x) > 1 \). Thus if \( B_2(p, x) \) is weakly decreasing and \( C_2(p, x) \) is weakly increasing, we have that \( h(p) \) is weakly increasing. If on the other hand \( B_2(p, x) \) is weakly increasing and \( C_2(p, x) \) is weakly decreasing it follows that \( h(p) \) is weakly decreasing. \( \square \)

Clearly, the conditions of the lemma are not necessary. We can easily extended it, for example, as follows:

**Lemma 3.5.** Assume \( \mathcal{A} \) is a graph-bounded notion of RSA integers with tolerance \( r \) given by continuously differentiable functions \( B_1, C_1 : \mathbb{R}_{>1} \to \mathbb{R}_{>1} \) and \( B_2, C_2 : \mathbb{R}_{>1}^2 \to \mathbb{R}_{>1} \). Further, individually for each \( x \in \mathbb{R}_{>1} \), the functions \( B_2(p, x) \) and \( C_2(p, x) \) are both weakly increasing in \( p \) for \( p \in \{ B_1(x), C_1(x) \} \). Then there are two monotone notions \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \) with tolerance \( r \), both having \( \mathcal{A}^i_x \subseteq \mathbb{R}_{\geq B_1(x)} \times \mathbb{R}_{\geq B_2(B_1(x), x)} \) for all \( x \), such that \( \mathcal{A} = \mathcal{A}^1 \setminus \mathcal{A}^2 \).

**Proof.** Let \( A(x) := B_2(B_1(x), x) \). We define two \([c_1, c_2]\)-balanced graph-bounded notions \( \mathcal{A}^1, \mathcal{A}^2 \) of RSA integers by the following: the first notion \( \mathcal{A}^1 \) is defined by the functions \( B_1^1 := B_1 \), \( C_1^1 := C_1 \), \( B_2^1(p, x) := A(x) \) and \( C_2^1 := C_2 \). The second notion \( \mathcal{A}^2 \) is defined by the functions \( B_1^2 := B_1 \), \( C_1^2 := C_1 \), \( B_2^2(p, x) := A(x) \) and \( C_2^2 := B_2 \). Since \( x/r < B_1(x)B_2(B_1(x), x) = B_1(x)A(x) \) both new notions have tolerance \( r \) as well. Then \( \mathcal{A}^1, \mathcal{A}^2 \) are by Lemma 3.4 both monotone and \( \mathcal{A} = \mathcal{A}^1 \setminus \mathcal{A}^2 \). \( \square \)

A similar result with \( B_2 \) and \( C_2 \) both weakly decreasing is more difficult to obtain while simultaneously retaining the tolerance. A particularly difficult example is the maximal notion \( \mathcal{M}^{r,c_1} \) given by \( \mathcal{M}^{r,c_1} = \{ (y, z) \in \mathbb{R}_{>1}^2 \mid \frac{x}{r} < y \leq x \wedge y, z \geq x^{c_1} \} \). The following lemma covers all the estimation work. Notice that we could in principle obtain explicit values for the \( O() \) constant based on Lemma 3.2 but the expressions are rather ugly.

**Lemma 3.6 (Two-dimensional prime sum approximation for monotone notions).** Assume that we have a monotone \([c_1, c_2]\)-balanced notion \( A \) of RSA integers with tolerance \( r \), where \( 0 < c_1 \leq c_2 \). (The values \( r, c_1, c_2 \) are allowed to vary with \( x \).) Then under the Riemann hypothesis there is a value \( \bar{a}(x) \in \left[ \frac{1}{4c_2^2}, \frac{1}{4c_1^2} \right] \) such that

\[
\#A(x) \in \bar{a}(x) \cdot \frac{4 \text{area}(A_x)}{\ln^2 x} + O \left( c_1^{-1} x^{\frac{3+c_1}{4}} \right),
\]
where \( c = \max (2c_2 - 1, 1 - 2c_1) \).

Note that the following proof gives a precise expression for \( \tilde{a}(x) \), namely

\[
\tilde{a}(x) = \frac{1}{4} \iint_{A_x} \frac{1}{\ln p \ln q} \, dp \, dq.
\]

It turns out that we can only evaluate \( \tilde{a}(x) \) numerically in our case and so we tend to estimate also this term. Then we often obtain \( \tilde{a}(x) \in 1 + o(1) \). Admittedly, this mostly eats up the advantage obtained by using the Riemann hypothesis. However, we accept this because it still leaves the option of going through that difficult evaluation and obtain a much more precise answer. If we do not use the Riemann hypothesis we need to replace \( O \left( c_{1}^{-1} x^{3 + \varepsilon} \right) \) with \( O \left( x \ln \frac{k}{x} \right) \) for any \( k > 2 \) of your choice.

**Proof.** Fix any \( x \in \mathbb{R}_{>1} \). In case \( \text{area}(A_x) = 0 \) the claim holds with any desired \( \tilde{a}(x) \) and zero big-Oh term. We can thus assume that the area is positive. As the statement is asymptotic and \( x^{c_1} \) tends to \( \infty \) with \( x \) we can further assume that \( x^{c_1} \geq 2657 \). Abbreviating \( \tilde{h}(x) = 4 \text{area}(A_x) \ln x \), we prove that there exists a value \( \tilde{a}(x) \in \left[ \frac{1}{4c_2}, \frac{1}{4c_1} \right] \) such that

\[
\left| \#A(x) - \tilde{a}(x) \cdot \tilde{h}(x) \right| \leq \tilde{h}(x)
\]

with \( \tilde{h}(x) = \frac{1}{4\pi c_1} \left( 7 - 6c_2 + \frac{12}{\ln x} \right) x^{\frac{1+c_2}{2}} + \frac{1}{8\pi^2} \cdot x^{\frac{1+c_2}{2} + \frac{2}{\ln x}} + \frac{1}{4\pi c_1} \left( 1 + \frac{4}{\ln x} \right) x^{1-\frac{c_2}{2}}. \)

This is slightly more precise and implies the claim.

Since the given notion is \([c_1, c_2]\)-balanced with tolerance \( r \) for any \( (y, z) \in A_x \) we have \( \frac{x}{r} \leq yz \leq x \) and \( y, z \in [x^{c_1}, x^{c_2}] \) which implies \( \ln y, \ln z \in [c_1, c_2] \ln x \). Equivalently, we have

(3.7) \quad x^{c_1} \leq B_1(x) \leq C_1(x) \leq x^{c_2}

and for \( y \in ]B_1(x), C_1(x)[ \) we have

(3.8) \quad \frac{x}{ry} \leq B_2(y, x) < C_2(y, x) \leq \frac{x}{y}

and

(3.9) \quad x^{c_1} \leq B_2(y, x) < C_2(y, x) \leq x^{c_2}.

From (3.8) we infer that for all \( y \in ]B_1(x), C_1(x)[ \) we have

(3.10) \quad \frac{x}{r} \leq yB_2(y, x) \leq x \quad \text{and} \quad \frac{x}{r} \leq yC_2(y, x) \leq x.\]
In order to estimate
\[
\#A(x) = \sum_{p \in \mathbb{P} \cap [B_1(x), C_1(x)]} \sum_{q \in \mathbb{P} \cap [B_2(p, x), C_2(p, x)]} 1,
\]
we apply Lemma 3.2 twice. Since \(x^{c_1} \geq 2657\) and so \(B_2(p, x) \geq 2657\) for the considered \(p\) we obtain for the inner sum
\[
\left| \sum_{q \in \mathbb{P} \cap [B_2(p, x), C_2(p, x)]} 1 - \tilde{g}_1(p, x) \right| \leq \tilde{g}_1(p, x),
\]
where
\[
\tilde{g}_1(p, x) = \int_{B_2(p, x)} \frac{1}{\ln q} \, dq,
\]
\[
\tilde{g}_1(p, x) = \tilde{E}(B_2(p, x)) + \tilde{E}(C_2(p, x)),
\]
since we can use the special case of constant functions in Lemma 3.2. Because we are working under the restriction that the notion is monotone, i.e. \(\tilde{g}_1(p, x) + \tilde{g}_1(p, x)\) is monotone, we are able to apply the lemma a second time. Since \(x^{c_1} \geq 2657\) and so \(B_1(x) \geq 2657\) we obtain
\[
\left| \sum_{p \in \mathbb{P} \cap [B_1(x), C_1(x)]} \sum_{q \in \mathbb{P} \cap [B_2(p, x), C_2(p, x)]} 1 - \tilde{g}_2(x) \right| \leq \tilde{g}_2(x),
\]
where
\[
\tilde{g}_2(x) = \int_{B_1(x)} \int_{B_2(p, x)} \frac{1}{\ln p \ln q} \, dq \, dp,
\]
\[
\tilde{g}_2(x) = \frac{1}{8\pi} \int_{B_1(x)} \int_{B_2(p, x)} \left( \sqrt{B_2(p, x)} \ln B_2(p, x) + \sqrt{C_2(p, x)} \ln C_2(p, x) \right) \cdot \left( \frac{1}{\ln p} + \frac{\ln p + 2}{2\sqrt{\pi}} \right) \, dp
\]
\[
+ \frac{1}{4\pi} \sqrt{B_1(x)} \ln B_1(x) \int_{B_2(p, x)} \frac{1}{\ln q} \, dq
\]
\[
+ \frac{1}{4\pi} \sqrt{C_1(x)} \ln C_1(x) \int_{B_2(p, x)} \frac{1}{\ln q} \, dq
\]
\[
+ \frac{1}{32\pi} \sqrt{B_1(x)} \ln B_1(x) \left( \sqrt{B_2(B_1(x), x)} \ln (B_2(B_1(x), x)) + \sqrt{C_2(B_1(x), x)} \ln (C_2(B_1(x), x)) \right)
\]
\[
+ \frac{1}{32\pi} \sqrt{C_1(x)} \ln C_1(x) \left( \sqrt{B_2(C_1(x), x)} \ln (B_2(C_1(x), x)) + \sqrt{C_2(C_1(x), x)} \ln (C_2(C_1(x), x)) \right)
\]
\[
+ \frac{1}{8\pi} \int_{B_1(x)} \int_{B_2(p, x)} \frac{\ln p + 2}{2\sqrt{\pi} \ln q} \, dq \, dp.
\]
It remains to estimate \(\tilde{g}_2(x)\) and \(\tilde{g}_2(x)\) suitably sharply.

For \((p, q) \in A_x\) we frequently use the estimate \(\ln p, \ln q \in [c_1, c_2] \ln x\). For the main term we obtain
\[
\tilde{g}_2(x) \in \left[ \frac{1}{4c_2^2}, \frac{1}{4c_1^2} \right] \cdot \frac{4 \text{area}(A_x)}{\ln^2 x}.
\]
We also read off the exact expression \( \bar{a}(x) = \frac{\ln^2 x}{4 \text{area}(A_1)} \tilde{g}_2(x) \).

We treat the error term \( \tilde{g}_2(x) \) part by part. For the first term we obtain

\[
\frac{1}{8\pi} \int_{B_1(x)}^{C_1(x)} \left( \sqrt{B_2(p, x)} \ln B_2(p, x) + \sqrt{C_2(p, x)} \ln C_2(p, x) \right) \cdot \left( \frac{1}{\ln p} + \frac{\ln p + 2}{2\sqrt{p}} \right) \ dp
\]

\[
\leq \frac{1}{4\pi} \int_{x^2}^{x^2} \sqrt{\frac{x}{p}} \ln \left( \frac{x}{p} \right) \cdot \frac{3}{\ln p} \ dp
\]

\[
\leq \frac{3}{4\pi} \frac{1}{c_1 \ln x} \int_{x^2}^{x^2} \sqrt{\frac{x}{p}} \ln \left( \frac{x}{p} \right) \ dp
\]

\[
\leq \frac{3}{2\pi} \frac{1}{c_1} \left( 1 - c_2 + \frac{2}{\ln x} \right) x^{1+c_2} \in O \left( c_1^{-1} x^{1+c_2} \right),
\]

where we used in the second line that \( \frac{\ln p + 2}{2\sqrt{p}} \leq \frac{2}{\ln p} \) for all \( p \geq 2 \). Basic calculus shows that \( \frac{\ln(p \ln p + 2)}{2\sqrt{p}} \) is maximal at \( p = \exp(\sqrt{5} + 1) \), where it is less than 1.68. For the fourth line note that

\[
\int \sqrt{\frac{x}{p}} \ln \left( \frac{x}{p} \right) \ dp = 2p \sqrt{\frac{x}{p}} \left( \ln \left( \frac{x}{p} \right) + 2 \right).
\]

The definite integral is not greater than this function evaluated at \( p = x^{c_2} \) since \( c_1 \leq \frac{1}{2} \).

Using \( c_2 \geq 0 \) gives the claim.

The second term yields

\[
\frac{1}{8\pi} \sqrt{B_1(x)} \ln B_1(x) \int_{B_2(B_1(x), x)}^{C_2(B_1(x), x)} \frac{1}{\ln q} \ dq
\]

\[
\leq \frac{1}{8\pi c_1 \ln x} \sqrt{B_1(x)C_2(B_1(x), x)} \ln B_1(x)
\]

\[
\leq \frac{1}{8\pi c_1} x^{1+c_2} \in O \left( c_1^{-1} x^{1+c_2} \right),
\]

since we have \( \sqrt{B_1(x)}C_2(B_1(x), x) \sqrt{C_2(B_1(x), x)} \leq x^{1+c_2} \) and \( \ln B_1(x) \leq \ln x \).

Similarly we obtain for the third term

\[
\frac{1}{8\pi} \sqrt{C_1(x)} \ln C_1(x) \int_{B_2(C_1(x), x)}^{C_2(C_1(x), x)} \frac{1}{\ln q} \ dq
\]

\[
\leq \frac{1}{8\pi c_1} x^{1+c_2} \in O \left( c_1^{-1} x^{1+c_2} \right),
\]

using \( \sqrt{C_1(x)}C_2(C_1(x), x) \sqrt{C_2(C_1(x), x)} \leq x^{1+c_2} \) and \( \ln C_1(x) \leq \ln x \).

The fourth term yields

\[
\frac{1}{32\pi^2} \sqrt{B_1(x)} \ln B_1(x) \left( \sqrt{B_2(B_1(x), x)} \ln B_2(B_1(x), x) + \sqrt{C_2(B_1(x), x)} \ln C_2(B_1(x), x) \right)
\]

\[
\leq \frac{1}{16\pi^2} \sqrt{x} \ln^2 x \in O \left( x^{1+c_2} \right),
\]
where we used (3.10) and the (very weak) bound \( \ln B_1(x), \ln C_2(p, x) \leq \ln x \). The fifth term can be treated similarly. We finish by observing for the sixth term

\[
\frac{1}{8\pi} \int_{B_1(x)}^{C_1(x)} \int_{B_2(p, x)}^{C_2(p, x)} \frac{\ln p + 2}{2\sqrt{p} \ln q} \, dq \, dp \\
\leq \frac{1}{8\pi} \frac{1}{c_1 \ln x} \int_{B_1(x)}^{C_1(x)} \int_{B_2(p, x)}^{C_2(p, x)} \frac{\ln p}{\sqrt{p}} \, dq \, dp \\
\leq \frac{1}{8\pi} \frac{1}{c_1 \ln x} \int_{x^{c_1}}^{x^{c_2}} \frac{\ln p}{\sqrt{p}} \, dp \\
\leq \frac{1}{8\pi} \frac{1}{c_1 \ln x} \cdot x \cdot \int_{x^{c_1}}^{x^{c_2}} \frac{\ln p}{p^{3/2}} \, dp \\
\leq \frac{1}{4\pi} \frac{1}{c_1} \left( 1 + \frac{4}{\ln x} \right) x^{1 - c_1} \\
\in O \left( c_1^{-1} x^{1 - \frac{3}{2}} \right)
\]

using \( B_1(x) \geq x^{c_1}, c_1 \leq \frac{1}{2} \), and

\[
\int \frac{\ln p}{p^{3/2}} \, dp = -\frac{2(\ln p + 2)}{\sqrt{p}}.
\]

This completes the proof. \( \square \)

In specific situations one may obtain better estimates. In particular, when we substitute \( C_2(p, x) \) by \( x/p \) in the estimation of the sixth summand of the error we may loose much.

Of course we can generalize this lemma to notions composed of few monotone ones. We leave the details to the reader. As mentioned before, in many standards the selection of the primes \( p \) and \( q \) is additionally subject to the side condition that \( \gcd((p - 1)(q - 1), e) = 1 \) for some fixed public exponent \( e \) of the RSA cryptosystem. To handle these restrictions, we prove

**Theorem 3.11.** Let \( e \in \mathbb{N}_{>2} \) be a public RSA exponent and \( x \in \mathbb{R} \). Then under the Extended Riemann Hypothesis we have for the number \( \pi_e(x) \) of primes \( p \leq x \) with \( \gcd(p - 1, e) = 1 \) that

\[
\pi_e(x) \in \phi_1(e) \phi(e) \cdot \text{li}(x) + O \left( \sqrt{x} \ln x \right),
\]

where \( \text{li}(x) = \int_{0}^{x} \frac{1}{\ln t} \, dt \) is the integral logarithm, \( \phi(e) \) is Euler’s totient function and

(3.12)
\[
\frac{\phi_1(e)}{\phi(e)} = \prod_{\ell | \phi(e)} \left( 1 - \frac{1}{\ell - 1} \right).
\]
We first show that the number of elements in $\mathbb{Z}_e^\times \cap (1 + \mathbb{Z}_e^\times)$ is exactly $\varphi_1(e)$. Write $e = \prod_{\ell | e} \ell^{f(\ell)}$. Observe that by the Chinese Remainder Theorem we have

$$\mathbb{Z}_e^\times \cap (1 + \mathbb{Z}_e^\times) = \bigoplus_{\ell | e} \left( \mathbb{Z}_{\ell^{f(\ell)}}^\times \cap (1 + \mathbb{Z}_{\ell^{f(\ell)}}^\times) \right)$$

and each factor in this expression has size $(\ell - 2)\ell^{f(\ell) - 1}$. Multiplying up all factors gives

$$\#(\mathbb{Z}_e^\times \cap (1 + \mathbb{Z}_e^\times)) = \prod_{\ell | e} \left( 1 - \frac{1}{\ell} \right) \left( 1 - \frac{1}{\ell} \right) \ell^{f(\ell)} = \varphi_1(e).$$

To show the result for $\pi_e(x)$ note that Oesterlé (1979) implies the following quantitative version of Dirichlet’s theorem on the number $\pi_{e,a}(x)$ of primes $p \leq x$ with $p \equiv a \in \mathbb{Z}_e$ when $\gcd(a,e) = 1$ under the Extended Riemann Hypothesis

$$\left| \pi_{e,a}(x) - \frac{1}{\varphi(e)} \cdot \text{li}(x) \right| \leq \sqrt{x} (\ln x + 2 \ln e).$$

This is also documented in Bach & Shallit (1996, Theorem 8.8.18). We now have to sum over $\varphi_1(e)$ residue classes and so obtain

$$\pi_e(x) \in \frac{\varphi_1(e)}{\varphi(e)} \cdot \text{li}(x) + \mathcal{O} \left( \varphi_1(e) \sqrt{x} \ln x \right),$$

which proves the claim. $\square$

This theorem shows that the prime pair approximation in Lemma 3.6 can be easily adapted to RSA integers whose prime factors satisfy the conditions of Theorem 3.11 (when assuming the Extended Riemann Hypothesis), since the density of such primes differs for every fixed $e$ essentially just by a multiplicative constant compared to the density of arbitrary primes.

### 4. Some common definitions for RSA integers

We will now give formal definitions of three specific notions of RSA integers. In particular, we consider the following example definitions within our framework:

- The number theoretically inspired notion following Decker & Moree. Note that this occurs in no standard and no implementation.

- The simple construction given by just choosing two primes in given intervals. This construction occurs in several standards, like the standard of the RSA foundation (RSA Laboratories 2000), the standard resulting from the European NESSIE project (Preneel et al. 2003) and the FIPS 186-3 standard (NIST 2009). Also open source implementations of OpenSSL (Cox et al. 2009), GnuPG (Skala et al. 2009) and the GNU crypto library GNU Crypto (Free Software Foundation 2009) use some variant of this construction.
○ An algorithmically inspired construction which allows one prime being chosen arbitrarily and the second is chosen such that the product is in the desired interval. This was for example specified as the IEEE standard 1363 (IEEE working group 2000), Annex A.16.11. However, we could not find any implementation following this standard.

Figure 4.1: Three notions of RSA integers.

4.1. A number theoretically inspired notion. In Decker & Moree (2008) on the suggestion of B. de Weger, the number $C_r(x)$ of RSA integers up to $x$ was defined as the count of numbers whose two prime factors differ by at most a factor $r$, namely

$$C_r(x) := \# \left\{ n \in \mathbb{N} \mid \exists p, q \in \mathbb{P} : n = pq \land p < q < rp \land n \leq x \right\}.$$ 

Written as a notion of RSA integers in the sense above, we analyze

$$A^{DM(r)}(x) := \left\{ (y, z) \in \mathbb{R}^2 \left| \frac{y}{r} < z < ry \land \frac{x}{r} < yz \leq x \right\} \right\}_{x \in \mathbb{R} > 1}.$$ 

Note that the prime pair counting function of this notion is closely related to the function $C_r(x)$: Namely we have

$$\#A^{DM(r)}(x) = 2 \left( C_r(x) - C_r \left( \frac{x}{r} \right) \right) + \pi \left( \sqrt{x} \right) - \pi \left( \sqrt{\frac{x}{r}} \right),$$
where the last part is comparatively small. We now analyze the behavior of the function  \( \# A_{DM(r)}(x) \) under the Riemann hypothesis. Similar to Decker & Moree (2008), we rewrite

\[
(4.2) \quad \frac{1}{2} \cdot \# A_{DM(r)}(x) = \sum_{p \in \mathcal{P} \cap \frac{r}{\sqrt{r}}} \sum_{q \in \mathcal{P} \cap \frac{r}{\sqrt{r}}} 1 + \sum_{p \in \mathcal{P} \cap \frac{r}{\sqrt{r}}} \sum_{q \in \mathcal{P} \cap \frac{r}{\sqrt{r}}} 1 + \frac{\pi(\sqrt{x}) - \pi(\sqrt{\frac{r}{x}})}{2}.
\]

With these bounds we obtain using Lemma 3.6:

**Theorem 4.3.** Under the Riemann hypothesis we have

\[
\# A_{DM(r)}(x) \in \tilde{a}(x) \frac{4x}{\ln^2 x} \left( \ln r - \frac{\ln r}{r} \right) + O \left( x^{\frac{3}{4} - \frac{1}{2}} \right)
\]

with \( \tilde{a}(x) \in \left[ 1 - \frac{\ln x}{\ln x + \ln r}, \left( 1 + \frac{2 \ln r}{\ln x - 2 \ln r} \right)^2 \right] \). This makes sense as long as \( r \in O(x^{\frac{1}{2} - \varepsilon}) \) for some \( \varepsilon > 0 \). If additionally \( \ln r \in o(\ln x) \) then \( \tilde{a}(x) \in 1 + o(1) \).

You may want to sum this up as \( \# A_{DM(r)}(x) \in (1 + o(1)) \frac{4x}{\ln^2 x} \left( \ln r - \frac{\ln r}{r} \right) \). However, you then forego the option of actually calculating \( \tilde{a}(x) \).

**Proof.** Consider \( x \) large enough such that all sum boundaries are beyond 2657, i.e. \( \sqrt{x} \geq 2657 \). By definition \( A_{DM(r)} \) is a notion of tolerance \( r \). Further it is \([c_1, c_2]\)-balanced with \( c_1 = \log_x \left( \frac{\sqrt{x}}{r} \right) = \frac{1}{2} - \frac{\ln x}{2 \ln r} \) and \( c_2 = \log_x \left( \sqrt{\frac{r}{x}} \right) = \frac{1}{2} + \frac{\ln r}{2 \ln x} \). As depicted next to (4.1), we treat the upper half of the notion as the union of those two notions matching the two double sums in (4.2), which both inherit being \([c_1, c_2]\)-balanced of tolerance \( r \). Considering the inner bounds \( \frac{r}{p} \) to \( rp \) and \( p \) to \( \frac{r}{p} \), respectively, as a function of the outer variable \( p \), we observe that the lower and upper bound in each case have opposite monotonic behavior and thus by Lemma 3.4 each part is a monotone notion. We can thus apply Lemma 3.6. Under the restriction \( \ln r \in o(\ln x) \) we have \( c_1, c_2 \in \frac{1}{2} + o(1) \), which implies that \( \frac{1}{c_i} \in 4 \left( 1 + o(1) \right) \) for both \( i \in \{1, 2\} \). Computing the area of the two parts yields

\[
\int_{\sqrt{x}}^{\sqrt{x}/r} \int_{\frac{r}{p}}^{rp} 1 \, dp \, dq = \frac{1}{2} \cdot x \left( 1 - \frac{\ln r}{r} - \frac{1}{r} \right)
\]

and

\[
\int_{\sqrt{x}}^{\sqrt{x}/r} \int_{p}^{\frac{r}{p}} 1 \, dq \, dp = \frac{1}{2} \cdot x \left( \ln r - 1 + \frac{1}{r} \right).
\]

For the error term we obtain \( O(x^{\frac{3}{4} - \frac{1}{2}}) \) noting that the number \( \pi(\sqrt{x}) \) of prime squares up to \( x \) is at most \( \sqrt{x} \). \( \square \)
Actually, we can even prove that the error term is in $O\left(\frac{x^3}{r^{1/4}}\right)$. We lost this in the last steps of the proof of Lemma 3.6 when we replaced $C_2(p, x) = rp$ by $x/p$.

4.2. A fixed bound notion. A second possible definition for RSA integers can be stated as follows: We consider the number of integers smaller than a real positive bound $x$ that have exactly two prime factors $p$ and $q$, both lying in a fixed interval $[B, C]$, in formula:

$$\pi_{B,C}^2(x) := \# \left\{ n \in \mathbb{N} \left| \exists p, q \in P \cap [B, C]: n = pq \land n \leq x \right. \right\}.$$

To avoid problems with rare prime squares, which are also not interesting when talking about RSA integers, we instead count

$$\kappa_{B,C}^2(x) := \# \left\{ (p, q) \in (P \cap [B, C])^2 \left| pq \leq x \right. \right\}.$$

Such functions are treated in Loebenberger & Nüsken (2010).

In the context of RSA integers we consider the notion

$$\mathcal{A}_{FB(r, \sigma)}^x := \left\{ (y, z) \in \mathbb{R}_{> 1}^2 \left| \frac{x}{r} < y, z \leq \sqrt{r^\sigma x} \land yz \leq x \right. \right\} \quad x \in \mathbb{R}_{> 1}^2$$

with $\sigma \in [0, 1]$. The parameter $\sigma$ describes the (relative) distance of the restriction $yz \leq x$ to the center of the rectangle in which $y$ and $z$ are allowed. We split the corresponding counting function into two double sums:

$$\# \mathcal{A}_{FB(r, \sigma)}^x (x) = \sum_{p \in P \cap \left[\sqrt{r}, \sqrt{r^{\sigma} x}\right]} \sum_{q \in P \cap \left[\frac{1}{r}, \sqrt{r^{\sigma} x}\right]} \frac{1}{q},$$

$$\quad + \sum_{p \in P \cap \left[\sqrt{r^{\sigma} x}, \sqrt{r^{1/2} x}\right]} \sum_{q \in P \cap \left[\frac{1}{r}, \sqrt{r^{\sigma} x}\right]} \frac{1}{q}.$$

The next theorem follows directly from Loebenberger & Nüsken (2010) but we can also derive it from Lemma 3.6 similar to Theorem 4.3.

**Theorem 4.6.** We have under the Riemann hypothesis

$$\# \mathcal{A}_{FB(r, \sigma)}^x (x) \in \widetilde{a}(x) \frac{4x}{\ln^2 x} \left( \sigma \ln r + 1 - \frac{2}{r \ln x} + \frac{1}{r} \right) + O\left(\frac{x^{3/4}}{r^{1/4}}\right)$$

with $\widetilde{a}(x) \in \left[ \left(1 - \frac{\sigma \ln r}{\ln x + \sigma \ln r}\right)^2, \left(1 + \frac{\ln r}{\ln x - \ln r}\right)^2 \right]$. If additionally $\ln r \in o(\ln x)$ then $\widetilde{a} \in 1 + o(1)$. 
PROOF. Let $x$ be such that all sum boundaries are beyond $2657$. By definition $A^{FB(r,\sigma)}$ is a notion of tolerance $r$. Further for all $\sigma \in [0,1]$ it is clearly $[c_1,c_2]$-balanced with $c_1 = \log_x \sqrt{\frac{r}{x}} = \frac{1}{2} - \frac{\ln \sqrt{r}}{\ln x}$ and $c_2 = \log_x \sqrt{r^\sigma x} = \frac{1}{2} + \frac{\sigma \ln r}{2\ln x}$. As depicted next to (4.4), we treat the notion as the union of two notions corresponding to the two double sums in (4.5), which are both $[c_1,c_2]$-balanced of tolerance $r$.

Consider the inner bounds $\sqrt{\frac{r}{x}}$ to $\sqrt{\frac{r^\sigma x}{x}}$ and $\sqrt{\frac{r}{x}}$ to $x$ respectively, as a function of the outer variable $p$ (while $\sigma$ is fixed): We observe that the lower and upper bound in the first case are constant and in the second case consist of a constant lower bound and an weakly decreasing upper bound. Thus by Lemma 3.4 each part is a monotone notion and we can apply Lemma 3.6.

As in the proof of Theorem 4.3, we have under the additional restriction $\ln r \in o(\ln x)$ that $\frac{1}{c_i} \in (4 + o(1))$ for both $i \in \{1,2\}$. Computing the area of the two parts yields

$$\int_{\sqrt{\frac{r^\sigma x}{x}}}^{\sqrt{\frac{r}{x}}} \int_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{x}{y}}} 1 \, dq \, dp = x \left( \sigma \ln r + \frac{1}{r(1+\sigma)/2} - \frac{1}{r(1-\sigma)/2} \right)$$

and

$$\int_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{r^\sigma x}{x}}} \int_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{x}{y}}} 1 \, dq \, dp = x \left( 1 - \frac{1}{r(1-\sigma)/2} - \frac{1}{r(1+\sigma)/2} + \frac{1}{r} \right)$$

For the error term we obtain $O \left( x^{3/4} r^{1/4} \right)$.

4.3. An algorithmically inspired notion. A third option to define RSA integers is the following notion: Assume you wish to generate an RSA integer between $\frac{x}{r}$ and $x$, which has two prime factors of roughly equal size. Then algorithmically we might first generate the prime $p$ and afterward select the prime $q$ such that the product is in the correct interval. As we will see later, this procedure does — however — not produce every number with the same probability, see Section 6. Formally, we consider the notion

$$(4.7) \quad A^{ALG_1(r)} := \left\{ (y,z) \in \mathbb{R}_+^2 \left| \begin{array}{l} \frac{\sqrt{x}}{r} < y \leq \sqrt{x}, \\
\frac{x}{y} < z \leq \frac{x}{y}, \\
\frac{1}{r} < yz \leq x \end{array} \right. \right\}_{x \in \mathbb{R}_+}.$$

We proceed with this notion similar to the previous one. By observing

$$(4.8) \quad \#A^{ALG_1(r)}(x) = \sum_{p \in \mathbb{P} \cap \left[ \frac{\sqrt{x}}{r}, \sqrt{x} \right]} \sum_{q \in \mathbb{P} \cap \left[ \sqrt{x}, x \right]} 1 + \sum_{p \in \mathbb{P} \cap \left[ \frac{\sqrt{x}}{r}, \sqrt{x} \right]} \sum_{q \in \mathbb{P} \cap \left[ \frac{x}{y}, \sqrt{x} \right]} 1,$$

and again applying Lemma 3.6 and Lemma 3.4 we obtain
THEOREM 4.9. We have under the Riemann hypothesis
\[
\# \mathcal{A}^{\text{ALG}_1}(r) (x) \in \bar{a}(x) \frac{4x}{\ln^2 x} \left( \ln r - \frac{\ln r}{r} \right) + O \left( x^{\frac{3}{4}} r^{\frac{1}{2}} \right)
\]
with \( \bar{a}(x) \in \left[ \left( 1 - \frac{2 \ln r}{\ln x + 2 \ln r} \right)^2, \left( 1 + \frac{2 \ln r}{\ln x - 2 \ln r} \right)^2 \right] \). If additionally \( \ln r \in o(\ln x) \) then \( \bar{a} \in 1 + o(1) \).

PROOF. Again let \( x \) be such that all sum boundaries are beyond 2657. By definition \( \mathcal{A}^{\text{ALG}_1}(r) \) is a notion of tolerance \( r \). Further it is clearly \([c_1, c_2]\)-balanced with \( c_1 = \log_x \sqrt{x} = \frac{1}{2} - \frac{\ln r}{\ln x} \) and \( c_2 = \log_x r \sqrt{x} = \frac{1}{2} + \frac{\ln r}{\ln x} \). As depicted next to (4.7), we treat the notion as the union of two notions corresponding to the two double sums in (4.8), which are both \([c_1, c_2]\)-balanced of tolerance \( r \).

If we consider the inner bounds \( \sqrt{x} \) to \( \frac{x}{p} \) and \( \frac{x}{rp} \) to \( \sqrt{x} \), respectively, as a function of the outer variable \( p \), we observe that in both cases one of them is constant and the other decreasing. Furthermore by Lemma 3.4 each part is a monotone notion. We can thus apply Lemma 3.6.

As for the previous notions we have under the additional restriction \( \ln r \in o(\ln(x)) \) that \( \frac{1}{c_i^2} \in 4 \left( 1 + o(1) \right) \) for both \( i \in \{1, 2\} \). Computing the area of the two parts yields
\[
\int_{\sqrt{x}/r}^{\sqrt{x}} \int_{\frac{x}{rp}}^{\frac{x}{p}} 1 \, dq \, dp = x \left( 1 - \frac{\ln r}{r} - \frac{1}{r} \right)
\]
and
\[
\int_{\sqrt{x}/r}^{\sqrt{x}} \int_{\frac{x}{rp}}^{\sqrt{x}} 1 \, dq \, dp = x \left( \ln r - 1 + \frac{1}{r} \right).
\]
For the error term we obtain \( O \left( x^{\frac{3}{4}} r^{\frac{1}{2}} \right) \). \( \square \)

Note that we also could have employed Lemma 3.5, but in this particular case we decided to use another split of the notion.

The IEEE standard P1363 suggest a slight variant, both generalize to
\[
\mathcal{A}^{\text{ALG}(r, \sigma)}(x) := \left\{ (y, z) \in \mathbb{R}_{>1}^2 \left| \begin{array}{l}
\sigma^{-1} \sqrt{x} < y \leq \sigma \sqrt{x}, \\
\frac{x}{yp} < z \leq \frac{x}{y}, \\
\frac{x}{z} < yz \leq x
\end{array} \right. \right\}
\]
with \( \sigma \in [0, 1] \). Now, our notion above is \( \mathcal{A}^{\text{ALG}(r, 0)} \), and the IEEE variant is \( \mathcal{A}^{\text{ALG}(r, \frac{1}{2})} \). By similar reasoning as above we obtain

(4.10)
THEOREM 4.11. We have under the Riemann hypothesis
\[
\#A^{\text{ALG}}(r,\sigma)(x) \in \tilde{a}(x) \left( \frac{4x}{\ln^2 x} \left( \ln r - \frac{\ln r}{r} \right) + O \left( x^{\frac{3}{4}} r^{\frac{1}{2}} \right) \right),
\]
with \(\tilde{a}(x) \in \left[ \left( 1 - \frac{2\sigma' \ln x}{\ln x + 2\sigma' \ln r} \right)^2, \left( 1 + \frac{2(1+\sigma) \ln x}{\ln x - 2(1+\sigma) \ln r} \right)^2 \right],\) where \(\sigma' = \max(\sigma, 1 - \sigma).\) If additionally \(\ln r \in o(\ln x)\) then \(\tilde{a}(x) \in 1 + o(1)\) \(\square\)

4.4. Summary. As we see, all notions, summarized in Figure 4.1, open a slightly different view. However the outcome is not that different, at least the numbers of described RSA integers are quite close to each other, see Section 5.

Current standards and implementations of various crypto packages mostly use the notions \(A^{\text{FB}}(4,0), A^{\text{FB}}(4,1), A^{\text{FB}}(2,0)\) or \(A^{\text{ALG}}(2,1/2)\). For details see Section 9.

5. Arbitrary notions

The preceding examinations show that the order of the analyzed functions differ by a factor that only depends on the notion parameters, i.e. on \(r\) and \(\sigma\), summarizing:

THEOREM. Assuming \(\ln r \in o(\ln x)\) and \(r > 1\) and \(\sigma \in [0, 1]\) we have
\[
\begin{align*}
(i) \quad & \#A^{\text{DM}}(r)(x) \in (1 + o(1)) \left( \frac{4x}{\ln^2 x} \left( \ln r - \frac{\ln r}{r} \right) \right), \\
(ii) \quad & \#A^{\text{FB}}(r,\sigma)(x) \in (1 + o(1)) \left( \frac{4x}{\ln^2 x} \left( \sigma \ln r + 1 - \frac{2}{r} \right) + \frac{1}{r} \right), \\
(iii) \quad & \#A^{\text{ALG}}(1)(r)(x) \in (1 + o(1)) \left( \frac{4x}{\ln^2 x} \left( \ln r - \frac{\ln r}{r} \right) \right). \quad \square
\end{align*}
\]
It is obvious that the three considered notions with many parameter choices cover about the same number of integers.

To obtain a much more general result, we consider the following maximal notion

\[
\mathcal{M}^{r,c_1} := \left\{ (y, z) \in \mathbb{R}_+^2 \left| \begin{array}{c}
x^{c_1} < y \leq x^{1-c_1}, \\
x^{c_1} < z \leq x^{1-c_1}, \\
\frac{x}{r} < yz \leq x \\
x \in \mathbb{R}_+ \end{array} \right. \right\}
\]

All of the notions discussed in Section 4 are subsets of this notion. Using the same techniques as above, we obtain:

THEOREM 5.2. For \(\ln r \in o(\ln x)\) we have under the Riemann hypothesis
\[
\begin{align*}
(i) \quad & \text{For } c_1 \leq \frac{1}{2} - \frac{\ln r}{2 \ln x} \text{ and for some } \ell \text{ and large } x \text{ additionally } c_1 > \frac{1}{2} - \ln^\ell x \ln r, \text{ we have that} \\
& \#A^{\mathcal{M}^{r,c_1}}(x) \in \tilde{a}(x) \left( \frac{4x}{\ln^2 x} \left( 1 - 2c_1 \right) \left( 1 - \frac{1}{r} \right) \ln x - 1 + \frac{\ln r + 1}{r} \right) + O \left( e^{-1} x^{1-\frac{3}{4}} \ln^{\ell+1} \right),
\end{align*}
\]
(ii) when \( c_1 > \frac{1}{2} - \frac{\ln r}{2 \ln x} \), we obtain the fixed bound notion

\[
\# \mathcal{M}^{r,c_1}(x) \in \tilde{a}(x) \frac{4x}{\ln^2 x} \left( (1 - 2c_1) \ln x + \frac{1}{x(1 - 2c_1)} - 1 \right) + O \left( c_1^{-1} \cdot x^{1 - \frac{c_1}{2}} \right).
\]

This is independent of \( r \).

In both cases \( \tilde{a}(x) \in \left[ \frac{1}{4(1 - c_1)^2} \cdot \frac{1}{4c_1^2}, 1 \right] \). In particular for \( c_1 \in \frac{1}{2} + o(1) \) we have \( \tilde{a}(x) \in 1 + o(1) \).

Case (i) considers the case where the notion \( \mathcal{M}^{r,c_1} \) looks like a thin band. The other alternative (ii) treats the case where the notion is actually a triangle, namely the notion \( A_{\text{FB}}(x^{1 - 2c_1}, 1) \). In the former case we have to make sure that the band is not too long so that we may apply Lemma 3.6 for not too many pieces. As noted after Definition 3.3, the first case could still be somewhat extended.

**Proof.** As usual let \( x \) be such that all sum boundaries are beyond 2657. By definition \( \mathcal{M}^{r,c_1} \) is a notion of tolerance \( r \). Further it is clearly \( \left[ c_1, 1 - c_1 \right] \)-balanced. For \( c_1 > \frac{1}{2} - \frac{\ln r}{2 \ln x} \) the result follows directly from Theorem 4.6, since \( \mathcal{M}^{r,c_1} \) is simply the fixed bound notion \( A_{\text{FB}}(x^{1 - 2c_1}, 1) \).

For \( c_1 \leq \frac{1}{2} - \frac{\ln r}{2 \ln x} \) we treat the notion as the sum of several monotone, \( [c_1, 1 - c_1] \)-balanced notions of tolerance \( r \) by triangulating the maximal notion as indicated in the picture next to (5.1). The number \( m \) of necessary cuts is \( (1 - 2c_1) \frac{\ln x}{\ln r} \) which is in \( O \left( \ln^{\ell + 1} x \right) \) by assumption. This gives by Lemma 3.6 the claim. \( \square \)

We obtain

**Theorem 5.3.** Let \( c_1, c_2 \in \frac{1}{2} + o(1), r > 1 \) with \( \ln r \in \Omega \left( \frac{1 - 2c_1}{\ln^2 x} \right) \cap o(\ln x) \) be possibly \( x \)-dependent values, and \( a \in [0, 1] \) constant. Consider a piecewise monotone notion \( A \) of RSA integers with tolerance \( r \) such that for large \( x \in \mathbb{R}_{>1} \) we have area \( A_x \geq ax \). Then

\[
\# A(x) = \frac{4x}{\ln^2 x} \cdot \tilde{a}(x)
\]

where \( \tilde{a}(x) \in o(\ln x) \) and \( a(x) \geq a - \varepsilon(x) \) for some \( \varepsilon(x) \in o(1) \).

In particular, the prime pair counts of two such notions can differ by at most a factor of order \( o(\ln x) \).

**Proof.** Let \( A \) be as specified. Assume \( x \) to be large enough to grant that area \( A_x \geq ax \) and \( x^{c_1} > 2657 \). Without loss of generality we assume \( c_1 + c_2 \leq 1 \). Otherwise we replace \( c_2 = 1 - c_1 \). Denote \( c := \max(2c_2 - 1, 1 - 2c_1) \), this now is always in \( [0, 1] \). By Lemma 3.6 we obtain

\[
\# A(x) \geq a - \frac{4x}{\ln^2 x} - \tilde{a}(x), \quad \tilde{a}(x) \in O \left( x^{\frac{3+\varepsilon}{4}} \right).
\]

To provide an upper bound, we consider the \( [c_1, 1 - c_1] \)-balanced maximal notion (5.1). As mentioned above we have for all \( x \in \mathbb{R}_{>1} \) that \( A_x \subseteq A^{r,c_1}_x \), and so \( \# A(x) \leq \# M^{r,c_1}(x) \).
Figure 5.1: Enclosing notions of RSA integers using others.

Note that $c_1 \leq \frac{1}{2}$, as otherwise $A_x$ would be empty rather than having area at least $a_x$. By assumption we have $c_1 \in \frac{1}{2} + o(1)$ and thus $0 \leq 1 - 2c_1 \in o(1)$. Now the claim follows from Theorem 5.2 and the assumption $\ln r \in o(\ln x)$.

In the following we will analyze the relation between the proposed notions in more detail. Namely, we carefully check how each of the notions can be enclosed in terms of the others. Clearly the fixed bound notions $A_{FB(r, \sigma)}$ enclose each other:

**Lemma 5.4.** For $r \in \mathbb{R}_{>1}$, $x \in \mathbb{R}_{>1}$ and $\sigma, \sigma' \in [0, 1]$ with $\sigma \leq \sigma'$ we have

$$\# A_{FB(\sqrt{r}, 1)} (x/\sqrt{r}) \leq \# A_{FB(r, 0)} (x) \leq \# A_{FB(r, \sigma)} (x) \leq \# A_{FB(r, \sigma')} (x) \leq \# A_{FB(r, 1)} (x)$$

**Proof.** For the first inequality simply observe that $x/\sqrt{r} \leq x$. The remaining inequalities follow from the fact that $\sqrt{r^\sigma x} \leq \sqrt{r^\sigma' x}$ whenever $\sigma \leq \sigma'$. □

We can also enclose different notions by each other:

**Lemma 5.5.** For $r \in \mathbb{R}_{>1}$ and $x \in \mathbb{R}_{>1}$ we have

$$\frac{1}{2} \# A_{FB(r, 1)} (x) \leq \frac{1}{2} \# A_{DM(r)} (x) \leq \# A_{ALG(r)} (x) \leq \# A_{FB(r^2, 1)} (x)$$

**Proof.** We prove every inequality separately. For an easier understanding of the proof a look at Figure 5.1 is advised:

\[ \frac{1}{2} \# A_{FB(r, 1)} (x) = \frac{1}{2} \sum_{p \in \mathbb{P} \cap \sqrt{r} \mathbb{Z}} \sum_{q \in \mathbb{P} \cap \sqrt{r} \mathbb{Z}} 1 = \sum_{p \in \mathbb{P} \cap \sqrt{r} \mathbb{Z}} \sum_{q \in \mathbb{P} \cap \sqrt{r} \mathbb{Z}} \frac{1}{p} \]

due to the restriction $p < q$. This is exactly the second summand in (4.2).
$\frac{1}{2} \#A_{DM}(r)(x) \leq \#A_{ALG}(r)(x)$: Consider again the double sum (4.2). We expand the summation area for $q$ (thus increasing the number of prime pairs we count) in order to obtain the sum (4.8) for the algorithmic notion: For the first summand we obtain from $p \leq \sqrt{\frac{x}{r}}$ that $rp \leq \frac{x}{p}$ and for the second summand from the same argument that $\frac{x}{rp} \leq p$. The third summand disappears while doing this, since the squares (which are counted by the third summand) are now counted by the second summand. Thus we can bound the whole sum from above by changing the summation area for $q$ in this way.

$\#A_{ALG}(r)(x) \leq \#A_{FB}(r^2, 1)(x)$: We proceed as in the previous step, by replacing in the sum (4.8) the summation area for $q$: Since $p \leq \sqrt{x}$, we obtain $\frac{x}{rp} \geq \frac{\sqrt{x}}{r}$. Now since $\sqrt{x} \leq r \sqrt{x}$ the claim follows.

We actually can enclose the Decker & Moree notion even tighter by the fixed bound notion:

**Lemma 5.6.** For $r \in \mathbb{R}_{>1}$ and $x \in \mathbb{R}_{>1}$ we have

$$\#A_{FB}(r, 1)(x) \leq \#A_{DM}(r)(x) \leq \#A_{FB}(r^2, \frac{1}{2})(x).$$

**Proof.** Assume $\frac{r}{x} < p < q \leq \sqrt{x}$ and $pq \leq x$. Then $\frac{p}{x} < pq \leq x$ and $q \leq rp$. If on the other hand $\frac{x}{p} < pq \leq x$ and $p < q < rp$, then $\frac{x}{pq} < \frac{1}{x}pq < p^2 < q^2 < rpq \leq rx$ and the claim follows. □

All the inclusion described above are compatible to the result from Theorem 5.3. However, many of the explicit inclusions are much tighter.

### 6. Generating RSA integers

In this section we analyze how to generate RSA integers properly. It completes the picture and we found several implementations overlooking this kind of arguments.

We wish that all the algorithms generate integers with the following properties:

- If we fix $x$ we should with at least overwhelming probability generate integers that are a product of a prime pair in $A_x$.
- These integers (not the pairs) should be selected roughly uniformly at random.
- The algorithm should be efficient. In particular, it should need only few primality tests.

For the first point note that we usually use probabilistic primality tests with a very low error probability, for example Miller (1976), Rabin (1980), Solovay & Strassen (1977), or Artjuhov (1966/67). Deterministic primality tests are also available but at present for these purposes by far too slow.
6.1. Rejection sampling. Assume that \( \mathcal{A} \) is a \([c_1, c_2]\)-balanced notion of RSA integers with tolerance \( r \). The easiest approach for generating a pair from \( \mathcal{A} \) is based on von Neumann’s rejection sampling method. For this the following definition comes in handy:

**Definition 6.1 (Banner).** A banner is a graph-bounded notion of RSA integers such that for all \( x \in \mathbb{R}_{>1} \) and for every prime \( p \in [B_1(x), C_1(x)] \) the number \( f_x(p) \) of primes in the interval \([B_2(p, x), C_2(p, x)]\) is almost independent of \( p \) in the following sense: \[ \frac{\max\{f_x(p) \mid p \in [B_1(x), C_1(x)] \}}{\min\{f_x(p) \mid p \in [B_1(x), C_1(x)] \}} \in 1 + o(1). \]

For example, a rectangular notion, where \( B_2(p, x) \) and \( C_2(p, x) \) do not depend on \( p \), is a banner. Now given any notion \( \mathcal{A} \) of RSA integers we select a banner \( \mathcal{B} \) of (almost) minimal area enclosing \( \mathcal{A} \). Note that there may be many choices for \( \mathcal{B} \). We can easily generate elements in \( \mathcal{B}^x \cap \mathbb{N}^2 \). Select first an appropriate \( y \in [B_1(x), C_1(x)] \cap \mathbb{N} \), second an appropriate \( z \in [B_2(p, x), C_2(p, x)] \cap \mathbb{N} \). By the banner property this chooses \((y, z)\) almost uniformly. We obtain the following straightforward Las Vegas algorithm:

**Algorithm 6.2. Generating an RSA integer (Las Vegas version).**

**Input:** A notion \( \mathcal{A} \), a bound \( x \in \mathbb{R}_{>1} \).

**Output:** An integer \( n = pq \) with \((p, q) \in \mathcal{A}^x \).

1. \( \text{Repeat} \ 2–4 \)
2. \( \text{Repeat} \)
3. \( \text{Select} \ (y, z) \) at random from \( B_x \cap \mathbb{N}^2 \) as just described.
4. \( \text{Until} \ (y, z) \in \mathcal{A}^x \).
5. \( \text{Until} \ y \text{ prime and } z \text{ prime.} \)
6. \( p \leftarrow y, q \leftarrow z. \)
7. \( \text{Return} \ pq. \)

The expected repetition count of the inner loop is \( \frac{\#B^x}{\#A^x} \) which is roughly \( \frac{\text{area}(\mathcal{B}^x)}{\text{area}(\mathcal{A}^x)} \). The expected number of primality tests is about \( \frac{\text{area}(\mathcal{B}^x)}{\#A^x} \). By Theorem 5.3 this is for many notions in \( \mathcal{O}(\ln^2 x) \). We have seen implementations (for example the one of GnuPG) where the inner and outer loop have been exchanged. This increases the number of primality tests by the repetition count of the inner loop. For \( A^{FB(r, 1)} \) this is a factor of about

\[
\frac{\#A^{FB(r, 2, 0)}(r, x)}{\#A^{FB(r, 1)}(x)} \sim \frac{\frac{1}{r} - 2 + r}{\ln r + \frac{1}{r} - 1} = \frac{(r - 1)^2}{r(\ln r - 1) + 1},
\]

which for \( r = 2 \) is equal to \( 2.58 \pi \) and even worse for larger \( r \). Also easily checkable additional conditions, like \( \gcd((p - 1)(q - 1), e) = 1 \), should be checked before the primality tests to improve the efficiency.

*Side remark: to indicate how a real number was rounded we append a special symbol. Examples: \( \pi = 3.1414 \), \( \pi = 3.142 \), \( \pi = 3.1416 \), \( \pi = 3.14159 \). The height of the platform shows the size of the left-out part and the direction of the antenna indicates whether actual value is larger or smaller than displayed. We write, say, \( e = 2.72 \), \( e = 2.71 \) as if the shorthand were exact.
6.2. Inverse transform sampling. Actually we would like to avoid generating out-of-bound pairs completely. Then a straightforward attempt to construct such an algorithm looks the following way:

**Algorithm 6.3. Generating an RSA integer (non-uniform version).**

Input: A notion $A$, a bound $x \in \mathbb{R}_{>1}$.
Output: An integer $n = pq$ with $(p, q) \in A_x$.

1. Repeat
2. Select $y$ uniformly at random from $\{y \in \mathbb{R} \mid \exists z \in \mathbb{N}: (y, z) \in A_x\} \cap \mathbb{N}$.
3. Until $y$ prime.
4. $p \leftarrow y$.
5. Repeat
6. Select $z$ uniformly at random from $\{z \in \mathbb{R} \mid (p, z) \in A_x\} \cap \mathbb{N}$.
7. Until $z$ prime.
8. $q \leftarrow z$.
9. Return $pq$.

The main problem with Algorithm 6.3 is that the output it produces typically is not uniform since the sets $\{z \in \mathbb{R} \mid (p, z) \in A_x\} \cap \mathbb{N}$ do not necessarily have the same cardinality when changing $p$. To retain uniform selection, we need to select the primes $p$ non-uniformly with the following distribution:

**Definition 6.4.** Let $A$ be a notion of RSA integers with tolerance $r$. For every $x \in \mathbb{R}_{>1}$ the associated cumulative distribution function of $A_x$ is defined as

$$F_{A_x} : \mathbb{R} \rightarrow [0, 1], \quad y \mapsto \frac{\text{area}(A_x \cap ([1, y] \times \mathbb{R}))}{\text{area}(A_x)}.$$

In fact we should use the function $G_{A_x} : \mathbb{R} \rightarrow [0, 1], \quad y \mapsto \frac{\#(A_x \cap ([1, y] \times \mathbb{P}))}{\#A_x}$, in order to compute the density but computing $G_{A_x}$ (or its inverse) is tremendously expensive. Fortunately, by virtue of Lemma 3.6 we know that $F_{A_x}$ approximates $G_{A_x}$ quite well for monotone, $[c_1, c_2]$-balanced notions $A$. So we use the function $F_{A_x}$ to capture the distribution properties of a given notion of RSA integers. As can be seen by inspection, in practically relevant examples this function is sufficiently easy to handle, see Table 6.1. Using this we modify Algorithm 6.3 such that each element from $A_x$ is selected almost uniformly at random:

**Algorithm 6.5. Generating an RSA integer.**

Input: A notion $A$, a bound $x \in \mathbb{R}_{>1}$.
Output: An integer $n = pq$ with $(p, q) \in A_x$.

1. Repeat
2. Select \( y \) with distribution \( F_{A_x} \) from \( \{ y \in \mathbb{R} \mid \exists z : (y, z) \in A_x \} \cap \mathbb{N} \).
3. Until \( y \) prime.
4. \( p \leftarrow y \).
5. Repeat
6. Select \( z \) uniformly at random from \( \{ z \in \mathbb{R} \mid (p, z) \in A_x \} \cap \mathbb{N} \).
7. Until \( z \) prime.
8. \( q \leftarrow z \).
9. Return \( pq \).

As desired, this algorithm generates any pair \((p, q) \in A_x \cap (\mathbb{P} \times \mathbb{P})\) with almost the same probability. In order to generate \( y \) with distribution \( F_{A_x} \) one can use inverse transform sampling, see for example Knuth (1998):

**Theorem 6.6 (Inverse transform sampling).** Let \( F \) be a continuous cumulative distribution function with inverse \( F^{-1} \) for \( u \in [0, 1] \) defined by

\[
F^{-1}(u) := \inf \{ x \in \mathbb{R} \mid F(x) = u \}.
\]

If \( U \) is uniformly distributed on \([0, 1]\), then \( F^{-1}(U) \) follows the distribution \( F' \).

**Proof.** We have \( \text{prob}(F^{-1}(U) \leq x) = \text{prob}(U \leq F(x)) = F(x) \). \( \square \)

The expected number of primality tests now is \( \mathcal{O} (\ln x) \): If \( A \) is \([c_1, 1]\)-balanced then \( F_{A_x}(y) = 0 \) as long as \( y \leq x^{c_1} \). The exit probability of the first loop is \( \text{prob}(y \text{ prime}) \) where \( y \) is chosen according to the distribution \( F'_{A_x} \). Thus

\[
\text{prob}(y \text{ prime}) \sim \int_1^x \frac{F'_{A_x}(y)}{\ln y} \, dy \in \left[ \frac{1}{\ln x}, \frac{1}{c_1 \ln x} \right]
\]

and we expect \( \mathcal{O} (\ln x) \cap \Omega (c_1 \ln x) \) repetitions of the upper loop until \( y \) is prime. Of course we have to take into account that for each trial \( u \) an inverse \( F_{A_x}^{-1}(u) \) has to be computed — at least approximately —, yet this cost is usually negligible compared to a primality test, see Table 6.1.

**6.3. Other constructions.** There are variants around, where the primes are selected differently: Take an integer randomly from a suitable interval and increase the result until the first prime is found. This has the advantage that the amount of randomness needed is considerably lower and by optimizing the resulting algorithm can also be made much faster. The price one has to pay is that the produced primes will not be selected uniformly at random: Primes \( p \) for which \( p - 2 \) is also prime will be selected with a much lower probability than randomly selected primes of a given length. As shown in Brandt & Damgård (1993) the output entropy of such algorithms is still almost maximal and also generators based on these kind of prime-generators might be used in practice.
Table 6.1: Non-cumulative density functions with respect to \( y \).

| Notion \( \mathcal{A} \) | \( F'_{\mathcal{A}_x} \) | Plot |
|-------------------------|---------------------------|------|
| \( \mathcal{A}_{DM}(r) \) | \[
\begin{cases} 
\frac{2(r^2-x^2)}{y(r-1)\ln r - 2xy^2} & \text{if } \frac{\sqrt{r}}{r} < y \leq \sqrt{x}, \\
\frac{y(r-1)}{x} & \text{if } \sqrt{x} < y \leq \sqrt{r}, \\
0 & \text{otherwise}.
\end{cases}
\] | ![Plot](image1.png) |
| \( \mathcal{A}_{FB}(r,\sigma) \) | \[
\begin{cases} 
\sqrt{r} \left( \frac{1+\sigma^2}{r} - 1 \right) & \text{if } \frac{\sqrt{r}}{r} < y \leq \sqrt{\sigma r}, \\
\sqrt{\sigma r} \ln r + \frac{\sigma r}{r} & \text{if } \sqrt{\sigma r} < y \leq \sqrt{r}, \\
0 & \text{otherwise}.
\end{cases}
\] | ![Plot](image2.png) |
| \( \mathcal{A}_{ALG_1}(r) \) | \[
\begin{cases} 
\frac{1}{y\ln r} & \text{if } \frac{\sqrt{r}}{r} < y \leq \sqrt{x}, \\
0 & \text{otherwise}.
\end{cases}
\] | ![Plot](image3.png) |

### 6.4. Comparison.
We have seen that Algorithm 6.2 and 6.5 are practical uniform generators for any symmetric or antisymmetric notion.

Note that Algorithm 6.2 and 6.5 may, however, still produce numbers in a non-uniform fashion: In the last step of both algorithms a product is computed that corresponds to either one pair or two pairs in \( \mathcal{A}_x \). To solve this problem we have two choices: Either we replace \( \mathcal{A} \) by its symmetric version \( S \) defined by \( S_x := \{ (y,z) \in \mathbb{R}_+^2 \mid (y,z) \in \mathcal{A}_x ) \lor (z,y) \in \mathcal{A}_x \} \), or by its, say, top half \( T \) given by \( T_x := \{ (y,z) \in S_x \mid z \geq y \} \) before anything else.

It is now relatively simple to instantiate the above algorithms using the notions proposed in Section 4: Namely for an algorithm following the Las Vegas approach, one simply needs to find suitable banner that encloses the desired notion. In order to instantiate Algorithm 6.5 we need to determine the inverse of the corresponding cumulative distribution function for the respective notion (see Table 6.1). Still Algorithm 6.2 and 6.5 are practically uniform generators for any symmetric or antisymmetric notion. Considering run-times we observe that Algorithm 6.5 is much faster, but we have to use inverse transform sampling to generate the first prime. Despite the simplicity of the approaches some common implementations use corrupted versions of Algorithm 6.2 or 6.5 as explained below. Essentially, they buy some extra simplicity by relaxing the uniformity requirement.

### 7. Output entropy

The entropy of the output distribution is an important quality measure of a generator. For primality tests several analyses were performed, see for example Brandt & Damgård (1993) or Joye & Paillier (2006). For generators of RSA integers we are not aware of any work in this direction.

Let \( \mathcal{A}_x \) be any monotone notion. Consider a generator \( G_d \) that produces a pair of primes \((p,q) \in \mathcal{A}_x \) with distribution \( d \). Seen as random variables, \( G_d \) induces two random variables \( P \) and \( Q \) by its first and the second coordinate, respectively. The entropy of the generator

---

**Table 6.1:** Non-cumulative density functions with respect to \( y \).
\( G_\rho \) is given by

\[
H(G_\rho) = H(P \times Q) = H(P) + H(Q \mid P),
\]

where \( H \) denotes the entropy and the conditional entropy is given by

\[
H(Q \mid P) = - \sum_{p \in P} \operatorname{prob}(P = p) \sum_{q \in Q} \operatorname{prob}(Q = q \mid P = p) \log_2(\operatorname{prob}(Q = q \mid P = p)).
\]

If \( \rho \) is the uniform distribution \( U \) we obtain the maximal entropy, which we can approximate by Lemma 3.6, namely

\[
H(G_U) = \log_2(\#A(x)) \approx \log_2(\text{area}(A_x)) - \log_2(\ln x) + 1
\]

with an error of very small order. The algorithms from Section 6, however, return the product \( P \cdot Q \). The entropy of this random variable is at most \( H(P \times Q) \) and can be at most one bit smaller than this:

\[
H(P \cdot Q) = - \sum_{n \in \mathbb{N}, (p,q) \in A_x} \operatorname{prob}(P \cdot Q = n) \log_2(\operatorname{prob}(P \cdot Q = n)) \\
\geq - \sum_{(p,q) \in A_x} \operatorname{prob}(P \times Q = (p,q)) \log_2(2 \operatorname{prob}(P \times Q = (p,q))) \\
= H(P \times Q) - 1.
\]

One should note here that in real-world implementations the generation of the primes might be biased, for example when one uses the above mentioned quite natural generator \textsc{Primeinc}, analyzed in Brandt & Damgård 1993. \textsc{Primeinc} chooses an integer and then outputs the first prime equal to or larger than this number. Note that Algorithm 6.2 and Algorithm 6.5 do not depend on any prime generator but sample integers until they are prime. However, this is not the case in many standards and implementations discussed in Section 9.

To estimate the entropy of an RSA generator \( G = P \times Q \) when employing prime generators \( P \) and \( Q \) with with entropy-loss at most \( \varepsilon_P \) and \( \varepsilon_Q \) then the resulting generator will by (7.1) have an entropy-loss of at most \( \varepsilon_P + \varepsilon_Q \) when compared to the same generator employing generators that produce primes uniformly at random.

Interestingly, some of the standards and implementations in Section 9 (like the standard IEEE 1363-2000 or the implementation of GNU Crypt0) still do not generate every possible outcome with the same probability, even if uniform prime generators are employed: Namely, if one selects the prime \( p \) uniformly at random and afterwards the prime \( q \) uniformly at random from an appropriate interval then this might be a non-uniform selection process if for some choices of \( p \) there are less choices for \( q \).

If in general the probability distribution \( \rho \) is close to the uniform distribution, say \( \rho(p,q) \in [2^{-\varepsilon}, 2^\varepsilon] \frac{1}{\#A(x)} \) for some fixed \( \varepsilon \in \mathbb{R}_{>0} \), then the entropy of the resulting generator \( G_\rho \) can
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be estimated as follows:

\[ H(G_\varphi) = - \sum_{(p,q) \in A_x} \varphi(p,q) \log_2(\varphi(p,q)) \]

\[ \leq \sum_{(p,q) \in A_x} \varphi(p,q) \left[ \log_2(#A(x)) - \varepsilon, \log_2(#A(x)) + \varepsilon \right] \]

\[ = [H(G_U) - \varepsilon, H(G_U) + \varepsilon] \]

and since the entropy of the uniform distribution is maximal, this implies that

\[ H(G_U) - \varepsilon \leq H(G_\varphi) \leq H(G_U). \]

8. Complexity theoretic considerations

We are about to reduce factoring products of two comparatively equally sized primes to the problem of factoring integers generated from a sufficiently large notion. As far as we know there are no similar reductions in the literature.

We consider finite sets \( M \subset \mathbb{N} \times \mathbb{N} \), in our situation we actually have only prime pairs. The multiplication map \( \mu_M \) is defined on \( M \) and merely multiplies, that is, \( \mu_M: M \rightarrow \mathbb{N}, (y,z) \mapsto y \cdot z \). The random variable \( U_M \) outputs uniformly distributed values from \( M \).

An attacking algorithm \( F \) gets a natural number \( \mu_M(U_M) \) and attempts to find factors inside \( M \). Its success probability \( \text{suc}_{F}(M) = \text{prob}(F(\mu_M(U_M)) \in \mu_M^{-1}(\mu_M(U_M))) \)

measures its quality in any fixed-size scenario. We call a countably infinite family \( C \) of finite sets of pairs of natural numbers hard to factor iff for any probabilistic polynomial time algorithm \( F \) and any exponent \( s \) for all but finitely many \( M \in C \) the success probability \( \text{suc}_{F}(M) \leq \ln^{-s} x \) where \( x = \max \mu_M(M) \). In other words: the success probability of any probabilistic polynomial time factoring algorithm on a number chosen uniformly from \( M \in C \) is negligible. That is equivalent to saying that the function family \( (\mu_M)_{M \in C} \) is one-way.

Integers generated from a notion \( A \) are hard to factor iff for any sequence \( (x_i)_{i \in \mathbb{N}} \) tending to infinity the family \( (A_{x_i} \cap (P \times P))_{i \in \mathbb{N}} \) is hard to factor. This definition is equivalent to the requirement that for all probabilistic polynomial time machines \( F \), all \( s \in \mathbb{N} \), there exists a value \( x_0 \in \mathbb{R}_{>1} \) such that for any \( x > x_0 \) we have \( \text{suc}_{F}(A_x) \leq \ln^{-s} x \). Since \( \mathbb{R} \) is first-countable, both definitions are actually equal. This can be easily shown by considering the functions \( g_{s,F}: \mathbb{R}_{>1} \rightarrow \mathbb{R}, x \mapsto \text{suc}_{F}(A_x) \cdot \ln^s x \). The first definition says that each function \( g_{s,F} \) is sequentially continuous (after swapping the initial universal quantifiers). The second definition says that each function \( g_{s,F} \) is continuous. In first-countable spaces sequentially continuous is equivalent to continuous.

For any polynomial \( f \) we define the set \( R_f = \{(m,n) \in \mathbb{N} | m \leq f(n) \land n \leq f(m)\} \) of \( f \)-related positive integer pairs. Denote by \( \mathbb{P}^{(m)} \) the set of \( m \)-bit primes. We can now formulate the basic assumption:
Assumption 8.2 (Intractability of factoring). For any unbounded positive polynomial \( f \) integers from the \( f \)-related prime pair family \( (\mathbb{P}^{(m)} \times \mathbb{P}^{(n)})_{(m,n) \in R_f} \) are hard to factor.

This is exactly the definition given by Goldreich (2001). Note that this assumption implies that factoring in general is hard, and it covers the supposedly hardest factoring instances. Now we are ready to state that integers from all relevant notions are hard to factor.

Theorem 8.3. Let \( \ln r \in \Omega \left( \frac{1-2c_1}{\ln^2 x} \right) \) for some \( \ell \) and \( A \) be a piecewise monotone, \( [c_1, c_2] \)-balanced notion for RSA integers of tolerance \( r \), where \( c_1 \) is bounded away from zero with growing \( x \), and \( A \) has large area, namely, for some \( k \) and large \( x \) we have \( \text{area} \ A_x \geq \frac{x}{\ln^2 x} \). Assume that factoring is difficult in the sense of Assumption 8.2. Then integers from the notion \( A \) are hard to factor.

Actually, under the given conditions Assumption 8.2 can be weakened: we only need that integers from the family of linearly related prime pairs are hard to factor. There is a tradeoff between the strength of the needed assumption on factoring and the assumption on \( c_1 \). If we relax the restriction on \( c_1 \) in the statement of the theorem to the requirement that \( c_1^2 \ln x \) tends to infinity with growing \( x \), we need that integers from the family of quadratically related prime pairs are hard to factor.

Proof. Assume that we have an algorithm \( F \) that factors integers generated uniformly from the notion \( A \). Our goal is to prove that this algorithm also factors certain polynomially related prime pairs successfully. In other words: its existence contradicts the assumption that factoring in the form of Assumption 8.2 is difficult.

By assumption, there is an exponent \( s \) so that for any \( x_0 \) there is \( x > x_0 \) such that the assumed algorithm \( F \) has success probability \( \text{succ}_F(A_x) \geq \ln^{-s} x \) on inputs from \( A_x \). We are going to prove that for each such \( x \) there exists a pair \( (m_0, n_0) \), the entries both from the interval \( [c_1 \ln x - \ln 2, c_2 \ln x + \ln 2] \), such that \( F \) executed with an input from image \( \mu_{\mathbb{P}^{m_0}, \mathbb{P}^{n_0}} \) still has success probability at least \( \ln^{-(s+k)} x \). By the interval restriction, \( m_0 \) and \( n_0 \) are polynomially (even linearly) related, namely \( m_0 < \frac{2c_2}{c_1} n_0 \) and \( n_0 < \frac{2c_2}{c_1} m_0 \) for large \( x \). By the assumption on \( c_1 \) the fraction \( \frac{2c_2}{c_1} \) is bounded independent of \( x \). So that contradicts Assumption 8.2.

First, we cover the set \( A_x \) with small rectangles. Let \( S_{m,n} := \mathbb{P}^{(m)} \times \mathbb{P}^{(n)} \) and \( I_x := \{(m,n) \in \mathbb{N}^2 \mid S_{m,n} \cap A_x \neq \emptyset\} \) then

\[
A_x \cap \mathbb{P}^2 \subseteq \bigcup_{(m,n) \in I_x} S_{m,n} =: S_x.
\]  

Next we give an upper bound on the number \( \#S_x \) of prime pairs in the set \( S_x \) in terms of the number \( \#A(x) \) of prime pairs in the original notion: First, since each rectangle \( S_{m,n} \) extends by a factor 2 along each axis we overshoot by at most that factor in each direction,
that is, we have for \( c'_1 = c_1 - (1 + 2c_1) \frac{ln^2 x}{ln x} \) and all \( x \in \mathbb{R}_{>1} \)
\[
S_x \subset \mathcal{M}_{4x}^{16r,c'_1} = \left\{ (y, z) \in \mathbb{R}^2 \mid y, z \geq \frac{1}{2} x^{c_1} \land \frac{x}{4r} < yz \leq 4x \right\}.
\]
Provided \( x \) is large enough we can guarantee by Theorem 5.2 that
\[
\#S_x \leq \#\mathcal{M}_{4x}^{16r,c'_1} (4x) \leq \frac{8x}{e^{c'_1} ln x}.
\]
On the other hand side we apply Lemma 3.6 for the notion \( A_x \) and use that \( A_x \) is large by assumption. Let \( c = \max(2c_2 - 1, -2c_1) \). Then we obtain for large \( x \) with some \( e_A(x) = O(c_1^{-1} x^{\frac{3x}{2}}) \).

\[
\#A(x) \geq \frac{area(A_x)}{c_2^2 ln^2 x} - e_A(x) \geq \frac{x}{2c_2^2 ln^{k+2} x}.
\]
Together we obtain
\[
(8.5) \quad \frac{\#A(x)}{\#S_x} \geq \frac{c_1^2}{16c_2^2 ln^{k+1} x} \geq \ln^{-(k+2)} x.
\]

By assumption we have \( succ_F(A_x) \geq \ln^{-s} x \) for infinitely many values \( x \). Thus \( F \) on an input from \( S_x \) still has large success even if we ignore that \( F \) might be successful for elements on \( S_x \setminus A_x \).

\[
succ_F(S_x) \geq succ_F(A_x) \frac{\#A(x)}{\#S_x} \geq \ln^{-(k+s+2)} x.
\]

Finally choose \((m_0, n_0) \in I_x\) for which the success of \( F \) on \( S_{m_0,n_0} \) is maximal. Then \( succ_F(S_{m_0,n_0}) \geq succ_F(S_x) \). Combining with the previous we obtain that for infinitely many \( x \) there is a pair \((m_0, n_0)\) where the success \( succ_F(S_{m_0,n_0}) \) of \( F \) on inputs from \( S_{m_0,n_0} \) is still larger than inverse polynomial: \( succ_F(S_{m_0,n_0}) \geq \ln^{-(k+s+2)} x \).

For these infinitely many pairs \((m_0, n_0)\) the success probability of the algorithm \( F \) on \( S_{m_0,n_0} \) is at least \( \ln^{-(k+s+2)} x \) contradicting the hypothesis. \( \square \)

All the specific notions that we have found in the literature fulfill the criterion of Theorem 8.3. Thus if factoring is difficult in the stated sense then each of them is invulnerable to factoring attacks. Note that the above reduction still works if the primes \( p, q \) are due to the side condition \( \gcd((p - 1)(q - 1), e) = 1 \) for a fixed integer \( e \) (see Theorem 3.11). We suspect that this is also the case when one employs safe primes.

Mihăilescu (2001) shows a theorem which is in some respect more general than our considerations and seems to imply our Theorem 8.3. To that end one has to show that one of \( A_x \) and \( P(m) \times P(n) \), with suitably chosen \((m, n)\) depending on \( x \), is 'polynomially dense' in the other. The result would be more general since also the used distribution on \( A_x \), rather than being uniform, is allowed to be 'polynomially bounded'. Our proof is of a different nature and thus may well be of independent interest. Also it may be adapted to polynomially bounded distributions on \( A \).
9. Impact on standards and implementations

In order to get an understanding of the common implementations, it is necessary to consult the main standard on RSA integers, namely the standard PKCS#1 (Jonsson & Kaliski 2003). However, one cannot find any requirements on the shape of RSA integers there. Interestingly, they even allow more than two factors for an RSA modulus. Also the standard ISO 18033-2 (International Organization for Standards 2006) does not give any details besides the fact that it requires the RSA integer to be a product of two different primes of similar length. A more precise standard is set by the German Bundesnetzagentur (Wohlmacher 2009). They do not give a specific algorithm, but at least require that the prime factors are not too small and not too close to each other. We will now analyze several standards which give a concrete algorithm for generating an RSA integer. In particular, we consider the standard of the RSA foundation (RSA Laboratories 2000), the IEEE standard 1363 (IEEE working group 2000), the NIST standard FIPS 186-3 (NIST 2009), the standard ANSI X9.44 (Accredited Standards Committee X9 2007) and the standard resulting from the European NESSIE project (Preneel et al. 2003).

9.1. RSA-OAEP. The RSA Laboratories (2000) describe the following variant:

ALGORITHM 9.1. Generating an RSA number for RSA-OAEP and variants.

Input: A number of bits $k$, the public exponent $e$.
Output: A number $n = pq$.

1. Pick $p$ from $[\left\lfloor 2^{(k-1)/2} \right\rfloor + 1, \left\lceil 2^{k/2} \right\rceil - 1] \cap \mathbb{P}$ such that $\gcd(e, p - 1) = 1$.
2. Pick $q$ from $[\left\lfloor 2^{(k-1)/2} \right\rfloor + 1, \left\lceil 2^{k/2} \right\rceil - 1] \cap \mathbb{P}$ such that $\gcd(e, q - 1) = 1$.
3. Return $pq$.

This will produce a number from the interval $[2^{k-1} + 1, 2^k - 1]$ and no cutting off. The output entropy is maximal. So this corresponds to the notion $\mathcal{A}_{FB(2,0)}$ generated by Algorithm 6.5. The standard requires an expected number of $k \ln 2$ primality tests if the gcd condition is checked first. Otherwise the expected number of primality tests increases to $\frac{\varphi(e)}{\varphi_1(e)} \cdot k \ln 2$, see (3.12). We will in the following always mean by the above notation that the second condition is checked first and afterwards the number is tested for primality. For the security Theorem 8.3 applies.

9.2. IEEE. IEEE standard 1363-2000, Annex A.16.11 (IEEE working group 2000) introduces our algorithmic proposal:
Algorithm 9.2. Generating an RSA number, IEEE 1363-2000.

Input: A number of bits \( k \), the odd public exponent \( e \).
Output: A number \( n = pq \).

1. Pick \( p \) from \( \left[ 2^{\left\lfloor \frac{k-1}{2} \right\rfloor}, 2^{\left\lceil \frac{k+1}{2} \right\rceil} - 1 \right] \cap \mathbb{P} \) such that \( \gcd(e, p - 1) = 1 \).
2. Pick \( q \) from \( \left[ \frac{2^{k-1} + 1}{p}, \left\lceil \frac{2^k}{p} \right\rceil \right] \cap \mathbb{P} \) such that \( \gcd(e, q - 1) = 1 \).
3. Return \( pq \).

Since the resulting integers are in the interval \( [2^{k-1}, 2^k - 1] \) this standard follows \( A^{\text{ALG}}(2,1/2) \) generated by a corrupted variant of Algorithm 6.5 using an expected number of \( k \ln 2 \) primality tests like the RSA-OAEP standard. The notion it implements is neither symmetric nor antisymmetric. The selection of the integers is not done in a uniform way, since the number of possible \( q \) for the largest possible \( p \) is roughly half of the corresponding number for the smallest possible \( p \). Since the distribution of the outputs is close to uniform, we can use the techniques from Section 7 to estimate the output entropy to find that the entropy-loss is less than 0.69 bit. The (numerically approximated) values in Table 9.1 gave an actual entropy-loss of approximately 0.03 bit.

9.3. NIST. We will now analyze the standard FIPS 186-3 (NIST 2009). In Appendix B.3.1 of the standard one finds the following algorithm:

Algorithm 9.3. Generating an RSA number, FIPS186-3.

Input: A number of bits \( k \), a number of bits \( \ell < k \), the odd public exponent \( 2^{16} < e < 2^{256} \).
Output: A number \( n = pq \).

1. Pick \( p \) from \( \left[ \sqrt{2}^{2^{k/2-1}}, 2^{k/2} - 1 \right] \cap \mathbb{P} \) such that \( \gcd(e, p - 1) = 1 \) and \( p \pm 1 \) has a prime factor with at least \( \ell \) bits.
2. Pick \( q \) from \( \left[ \sqrt{2}^{2^{k/2-1}}, 2^{k/2} - 1 \right] \cap \mathbb{P} \) such that \( \gcd(e, p - 1) = 1 \) and \( q \pm 1 \) has a prime factor with at least \( \ell \) bits and \( |p - q| > 2^{k/2 - 100} \).
3. Return \( pq \).

In the standard it is required that the primes \( p \) and \( q \) shall be either provably prime or at least probable primes. The mentioned large (at least \( \ell \)-bit) prime factors of \( p \pm 1 \) and \( q \pm 1 \) have to be provable primes. We observe that also in this standard a variant of the notion \( A^{\text{FB}}(2,0) \) generated by Algorithm 6.5 is used. The output entropy is thus maximal. However, we do not have any restriction on the parity of \( k \), such that the value \( k/2 \) is not necessarily
an integer. Another interesting point is the restriction on the prime factors of $p \pm 1$, $q \pm 1$. Our notions cannot directly handle such requirements, but this may possibly be achieved by appropriately modifying the prime number densities in the proof of Lemma 3.6.

The standard requires an expected number of slightly more than $k \ln 2$ primality tests. It is thus slightly less efficient than the RSA-OAEP standard. For the security the remarks from the end of Section 8 apply.

9.4. ANSI. The ANSI X9.44 standard (Accredited Standards Committee X9 2007), formerly part of ANSI X9.31, requires strong primes for an RSA modulus. Unfortunately, we could not access ANSI X9.44 directly and are therefore referring to ANSI X9.31-1998. Section 4.1.2 of the standard requires that

- $p - 1$, $p + 1$, $q - 1$, $q + 1$ each should have prime factors $p_1$, $p_2$, $q_1$, $q_2$ that are randomly selected primes in the range $2^{100}$ to $2^{120}$,
- $p$ and $q$ shall be the first primes that meet the above, found in an appropriate interval, starting from a random point,
- $p$ and $q$ shall be different in at least one of their first 100 bits.

The additional restrictions are similar to the ones required by NIST. This procedure will have an output entropy that is close to maximal (see Section 7).

9.5. NESSIE. The European NESSIE project gives in its security report (Preneel et al. 2003) a very similar algorithm:

**Algorithm 9.4. Generating an RSA number, NESSIE standard.**

Input: A number of bits $\ell$, the odd public exponent $e$.
Output: A number $n = pq$.

1. Pick $p$ from $[2^{\ell-1}, 2^\ell - 1] \cap \mathbb{P}$ such that $\gcd(e, p - 1) = 1$.
2. Pick $q$ from $[2^{\ell-1}, 2^\ell - 1] \cap \mathbb{P}$ such that $\gcd(e, q - 1) = 1$.
3. Return $pq$.

The resulting integer $n$ is from the interval $[2^{2\ell - 2}, 2^{2\ell} - 1]$ and thus corresponds to the fixed-bound notion $A^{\text{FB}(4,0)}$ generated by Algorithm 6.5. The output entropy is thus maximal. Note the difference to the standard of the RSA foundation: Besides the fact, that in the standard of the RSA laboratories some sort of rounding is done, the security parameter $\ell$ is treated differently: While for the RSA foundation the security parameter describes the (rough) length of the output, in the NESSIE proposal it denotes the size of the two prime factors. For comparison let $k = 2\ell$. The standards requires an expected number of $2k \ln 2$ primality tests. It is thus as efficient as the RSA-OAEP standard. For the security Theorem 8.3 applies.
9.6. OpenSSL. We now turn to implementations: For OpenSSL (Cox et al. 2009), we refer to the file `rsa_gen.c`. Note that in the configuration the routine used for RSA integer generation can be changed, while the algorithm given below is the standard one. OpenSSH (de Raadt et al. 2009) uses the same library. Refer to the file `rsa.c`. We have the following algorithm:

**Algorithm 9.5. Generating an RSA number in OpenSSL.**

**Input:** A number of bits $k$.

**Output:** A number $n = pq$.

1. Pick $p$ from $2\left\lfloor \frac{k-1}{2} \right\rfloor , 2\left\lfloor \frac{k+1}{2} \right\rfloor - 1 \cap \mathbb{P}$.

2. Pick $q$ from $2\left\lfloor \frac{k-3}{2} \right\rfloor , 2\left\lfloor \frac{k-1}{2} \right\rfloor - 1 \cap \mathbb{P}$.

3. Return $pq$.

This is nothing but a rejection-sampling method with no rejections of a notion similar to the fixed-bound notion $\mathcal{A}_{FB(4,0)}$ generated by Algorithm 6.2. The output entropy is thus maximal. The result the algorithm produces is always in $[2^{k-2}, 2^{k-1}]$. It is clear that this notion is antisymmetric and the factors are on average a factor 2 apart of each other. The implementation runs in an expected number of $k \ln 2$ primality tests. The public exponent $e$ is afterwards selected such that $\gcd((p-1)(q-1), e) = 1$. It is thus slightly more efficient than the RSA-OAEP standard. For the security Theorem 8.3 applies.

9.7. Openswan. In the open source implementation Openswan of the IPsec protocol (Richardson et al. 2009) one finds a rejection-sampling method that is actually implementing the notion $\mathcal{A}_{FB(4,0)}$ generated by a variant of Algorithm 6.2. We refer to the function `rsasigkey` in the file `rsasigkey.c`:

**Algorithm 9.6. Generating an RSA number in Openswan.**

**Input:** A number of bits $k$.

**Output:** A number $n = pq$.

1. Pick $p$ from $2\left\lfloor \frac{k-2}{2} \right\rfloor , 2\left\lfloor \frac{k}{2} \right\rfloor - 1 \cap \mathbb{P}$.

2. Pick $q$ from $2\left\lfloor \frac{k-2}{2} \right\rfloor , 2\left\lfloor \frac{k}{2} \right\rfloor - 1 \cap \mathbb{P}$.

3. Return $pq$.

Note that here the notion is actually symmetric. However still the uniformly at random selected number $pq$ will not always have the same length. The implementation runs in an expected number of $k \ln 2$ primality tests and output entropy is maximal. Again the public exponent $e$ is afterwards selected such that $\gcd((p-1)(q-1), e) = 1$. It is thus as efficient as the RSA-OAEP standard. For the security Theorem 8.3 applies.
9.8. GnuPG. Also GnuPG (Skala et al. 2009) uses rejection-sampling of the fixed-bound notion \( A^{FB(2,1)} \) generated by a variant of Algorithm 6.2, implying that the entropy of its output distribution is maximal.

**Algorithm 9.7.** Generating an RSA number in GnuPG.

**Input:** A number of bits \( k \).

**Output:** A number \( n = pq \).

1. Repeat 2–3
2. Pick \( p \) from \( \left[ 2\left\lfloor \frac{k-1}{2}\right\rfloor, 2\left\lfloor \frac{k+1}{2}\right\rfloor - 1 \right] \cap \mathbb{P} \).
3. Pick \( q \) from \( \left[ 2\left\lfloor \frac{k+1}{2}\right\rfloor, 2\left\lfloor \frac{k-1}{2}\right\rfloor - 1 \right] \cap \mathbb{P} \).
4. Until \( \text{len}(pq) = 2\left\lceil \frac{k}{2} \right\rceil \)
5. Return \( pq \).

The hatched region in the picture above shows the possible outcomes that are discarded. We refer here to the file rsa.c. The algorithm is given in the function `generate_std` and produces always numbers with either \( k \) or \( k + 1 \) bits depending on the parity of \( k \). Note that the generation procedure indeed first selects primes before checking the validity of the range. This is of course a waste of resources, see Section 6.

The implementation runs in an expected number of roughly \( 2.589 \cdot (k + 1) \ln 2 \) primality tests. It is thus less efficient than the RSA OAEP standards. Like in the other so far considered implementations, the public exponent \( e \) is afterwards selected such that \( \gcd((p-1)(q-1), e) = 1 \). For the security Theorem 8.3 applies.

9.9. GNU Crypto. The GNU Crypto library (Free Software Foundation 2009) generates RSA integers the following way. Refer here to the function `generate` in the file RSAKeyPairGenerator.java.

**Algorithm 9.8.** Generating an RSA number in GNU Crypto.

**Input:** A number of bits \( k \).

**Output:** A number \( n = pq \).

1. Pick \( p \) from \( \left[ 2\left\lfloor \frac{k-1}{2}\right\rfloor, 2\left\lfloor \frac{k+1}{2}\right\rfloor - 1 \right] \cap \mathbb{P} \).
2. Repeat
3. Pick \( q \) from \( \left[ 2\left\lfloor \frac{k+1}{2}\right\rfloor, 2\left\lfloor \frac{k-1}{2}\right\rfloor - 1 \right] \).
4. Until \( \text{len}(pq) = k \) and \( q \in \mathbb{P} \).
5. Return \( pq \).

The arrow in the picture points to the results that will occur with higher probability. Also here the notion \( A^{FB(2,1)} \) is used, but the generated numbers will not be uniformly distributed, since for a larger \( p \) we have much less choices for \( q \). Since the distribution of the outputs is
Table 9.1: Overview of various standards and implementations. The numbers in parentheses give the entropy loss for each algorithm in per mille. As explained in the text, the entropy of the standards is slightly smaller than the values given due to the fixed public exponent $e$. FIPS 186-3 has a small entropy loss because of the requirement of strong primes. Generators based on nonuniform prime generation suffer extra entropy loss, see page 28.

not close to uniform, we could only compute the entropy for real-world parameter choices numerically (see Table 9.1). For all choices the loss was less than 0.63 bit. The implementation is as efficient as the RSA-OAEP standard.

The Free Software Foundation provides GNU Classpath, which generates RSA integers exactly like the GNU Crypto library, i.e. following $A_{FB(2,1)}$. We refer to the source file named RSAKeyPairGenerator.java. As in the other so far considered implementations the public exponent $e$ is randomly selected afterwards such that $\gcd((p - 1)(q - 1), e) = 1$. Like in the IEEE and the ANSI standard this does not impose practical security risks, but it does not meet the requirement of uniform selection of the generated integers.

9.10. Summary. It is striking to observe that not a single analyzed implementation follows one of the standards described above. The only standards all implementations are compliant to are the standards PKCS#1 and ISO 18033-2, which themselves do not specify
anything related to the integer generation routine. We found that also the requirements from the algorithm catalog of the German Bundesnetzagentur (Wohlmacher 2009) are not met in a single considered implementation, since it is never checked whether the selected primes are too close to each other. The implementation that almost meets the requirements is the implementation of OpenSSL. Interestingly there are standards and implementations around that generate integers non-uniformly. Prominent examples are the IEEE and the ANSI standards and the implementation of the GNU Crypto library. This does not impose practical security issues, but it violates the condition of uniform selection.

10. Conclusion

We have seen that there are various definitions for RSA integers, which result in substantially differing standards. We have shown that the concrete specification does not essentially affect the (cryptographic) properties of the generated integers: The entropy of the output distribution is always almost maximal, generating those integers can be done efficiently, and the outputs are hard to factor if factoring in general is hard in a suitable sense. It remains open to incorporate strong primes into our model. Also a tight bound for the entropy of non-uniform selection is missing if the distribution is not close to uniform.

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