BERGMAN KERNELS AND ALGEBRAIC STRUCTURE OF LIMIT SPACE FOR A SEQUENCE OF ALMOST KÄHLER-RICCI SOLITONS

WENSHUAI JIANG, FENG WANG, AND XIAOHUA ZHU*

Abstract. In this paper, we give a lower bound of Bergman kernels for a sequence of almost Kähler-Einstein Fano manifolds, or more general, a sequence of almost Kähler-Ricci solitons. This generalizes a result by Donaldson-Sun and Tian for Kähler-Einstein manifolds sequence with positive scalar curvature.

1. Introduction

Let \((M^n, g)\) be an \(n\)-dimensional Fano manifold with its Kähler form \(\omega_g\) in \(2\pi c_1(M)\). Then \(g\) induces a Hermitian metric \(h\) of the anti-canonical line bundle \(K^{-1}_M\) such that \(\text{Ric}(K^{-1}_M, h) = \omega_g\). Also \(h\) induces a Hermitian metric (for simplicity, we still use the notation \(h\)) of \(l\)-multiple line bundle \(K^{-1}_M\).

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As usual, the $L^2$-inner product on $H^0(M,K_M^{-l})$ is given by
\begin{equation}
(s_1, s_2) = \int_M \langle s_1, s_2 \rangle_h dv_g, \quad \forall \ s_1, s_2 \in H^0(M,K_M^{-l}).
\end{equation}
Choosing an unit orthogonal basis $\{s_i\}$ of $H^0(M,K_M^{-l})$ with respect to the inner product $\langle \cdot, \cdot \rangle$ in (1.1), we define the Bergman kernel of $(M,K_M^{-l}, h)$ by
\begin{equation}
\rho_l(x) = \sum_i |s_i|^2_h(x).
\end{equation}
Clearly, $\rho_l(x)$ is independent of the choice of basis $\{s_i\}$. In [T4], Tian proposed a conjecture for the existence of uniform lower bound of $\rho_l(x)$:

**Conjecture 1.1.** Let $\{(M_i,g^i)\}$ be a sequence of $n$-dimensional Kähler-Einstein manifolds with constant scalar curvature $n$. Then there exists an integer number $l_0$ such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ with property:
\begin{equation}
\rho_{l_0}(M_i,g^i) \geq c_l,
\end{equation}
where $c_l$ depends only on $l,n$.

The above conjecture was recently proved by Donalson-Sun [DS] and Tian [T6], independently. The main idea in their proofs is to use the Hörmander $L^2$-estimate to construct peak holomorphic sections by solving $\bar{\partial}$-equation. This idea can go back to Tian’s orginal work in [T1] (see also a survey paper by him [T4]). In fact he used the idea to prove the conjecture for the Kähler-Einstein surfaces more than twenty years ago [T2].

The estimate (1.2) is usually called the partial $C^0$-estimate. Very recently, (1.2) was generalized to a sequence of conical Kähler-Einstein manifolds by Tian [T5]. As an application of (1.2) he gives a proof of the famous Yau-Tian-Donaldson’s conjecture for the existence problem of Kähler-Einstein metrics with positive scalar curvature. Chen-Donaldson-Sun also give a proof of the conjecture in [CDS] independently.

**Theorem 1.2** (Tian, Chen-Donaldson-Sun). A Fano manifold admits a Kähler-Einstein metric if and only if it is $K$-stable.

The $K$-stability was first introduced by Tian [T3] and it was generalized by Donaldson in terms of test-configurations [Do].

In this paper, we want to generalize the estimate (1.2) to a sequence of almost Kähler-Einstein Fano manifolds defined in [TW], or more general, a sequence of almost Kähler-Ricci solitons (see Definition 7.3 in Section 7). Namely, we prove

**Theorem 1.3.** Let $\{(M_i,g^i)\}$ be a sequence of almost Kähler-Einstein Fano manifolds or a sequence of almost Kähler manifolds which admit almost
Bergman Kernels and algebraic structure

Kähler-Ricci solitons) with dimension $n$. Then there exists an integer number $l_0$ such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ with property:

$$\rho_{l_0}(M_i, g^i) \geq c_l,$$

where the constant $c_l$ depends only on $l, n$, and some geometric uniform constants (cf. Section 9).

Theorem 1.3 implies

**Corollary 1.4.** Let $\{(M_i, g^i)\}$ be a sequence of almost Kähler-Einstein Fano manifolds (or a sequence of Kähler manifolds which admit almost Kähler-Ricci solitons). Then there exists a subsequence $\{i_k\} (i_k \to \infty)$ of $\{i\}$ such that $\{(M_{i_k}, g^{i_k})\}$ converges to a normal Fano variety in Gromov-Hausdorff topology, which admits a Kähler-Einstein metric (or a Kähler-Ricci soliton) on the smooth part of variety.

In case of Kähler-Einstein manifolds with positive scalar curvature, Corollary 1.4 has been proved by Donaldson-Sun [DS]. They used an argument described by Tian in [T4]. Li in his Ph.d thesis also gave a proof following the Tian’s argument [L1]. For the readers’ convenience, we will give a sketch of proof to Corollary 1.4 parallel to Li’s proof (cf. Section 8). We note that the existence of Kähler-Einstein metrics on the smooth part of variety follows from Cheeger-Colding-Tian compactness theorem for a sequence of Kähler-Einstein manifolds [CCT]. By using Ricci flow to smooth Kähler metrics, Tian and B. Wang recently extended Cheeger-Colding-Tian compactness theorem to a sequence of almost Kähler-Einstein manifolds [TW], and F. Wang and Zhu to a sequence of almost Kähler-Ricci solitons [WZ2].

There are important examples of almost Kähler-Einstein metrics and almost Kähler-Ricci solitons:

1) Tian and B. Wang constructed a family of almost Kähler-Einstein metrics $g_t (t \to 1)$ arising from solutions of certain complex Monge-Ampère equations on a Fano manifold with the Mabuchi’s $K$-energy bounded from below [TW].

2) Tian constructed a family of almost Kähler-Einstein metrics $g_t (t \to 1)$ modified from conical Kähler-Einstein metrics on a Fano manifold whose corresponding conical angles go to $2\pi$ [T4].

3) F. Wang and Zhu constructed a family of almost Kähler-Ricci solitons $g_t (t \to 1)$ arising from solutions of certain complex Monge-Ampère equations on a Fano manifold with the modified $K$-energy bounded from below [WZ1], [WZ2]. Thus by Theorem 1.3 we have

**Corollary 1.5.** Let $g_t (t \to 1)$ be a family of almost Kähler-Einstein metrics (or almost Kähler-Ricci solitons) on a Fano manifold constructed above
Bergman Kernels and algebraic structure

Then there exists an integer number $l_0$ such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ independent of $t$ with property:

$$\rho_{l_0}(g_t) \geq c_l > 0.$$  

(1.4)

It was proved recently by Li that the lower boundedness of $K$-energy is equivalent to the $K$-semistability [L2]. Li’s proof depends on the construction of test-configurations in Theorem 1.2 by studying conical Kähler-Einstein metrics. There should be an analogy of Li’s result to describe the modified $K$-energy in sense of “$K$”-semistability.

**Question 1.6.** Is there a direct proof (without using conical Kähler-Einstein metrics as in the proof of Theorem 1.2) for that the $K$-stability implies the $K$-energy bounded from below?

A same question was proposed by Paul in his recent paper [Pa]. He proved there that the $K$-stability is equivalent to the properness of $K$-energy in the space of Kähler metrics induced by the Bergman Kernels. Thus as pointed by Tian in [T6], [T7] (also see [T3]), (1.4) will give a new proof of Theorem 1.2 if we know that the $K$-stability implies the $K$-energy bounded from below.

As in the proof of Theorem 1.2, we need to construct peak holomorphic sections by solving $\bar{\partial}$-equation to prove Theorem 1.3. Because there is a lack of local strong convergence of $\{(M_i, g^i)\}$, we shall smooth the sequence to approximate the original one by Ricci flow as in [TW], [WZ2]. This approximation will depend on points in the Gromov-Hausdorff limit space of $\{(M_i, g^i)\}$, so it depends on the time $t$ in the Ricci flow. Thus we need to give estimates for the scalar curvatures and Kähler potentials along the flow for small time $t$ (cf. Section 2, 3, 7). Another technical part is in the construction of peak holomorphic sections by using the rescaling method as in [T6], [DS], which will depend on the choice of Kähler metrics evolved in the Ricci flow in our case (cf. Section 5, 6, 7).

The organization of paper is as follows. In Section 2, we give some estimates for scalar curvatures and Kähler potentials along the Ricci flow, then, in Section 3, we use them to give the $C^0$-estimate and the gradient estimate for holomorphic sections on multiple line bundles of $K^{-1}_M$. Section 4 is devoted to construct almost peak holomorphic sections by using the trivial bundle on the tangent cone. The peak holomorphic sections, which depend on time $t$, will be constructed in Section 5. Theorem 1.3 will be proved in Section 6, 7, according to almost Kähler-Einstein metrics and almost Kähler-Ricci solitons, respectively, while its proof is completed in Section 9. In Section 8, we prove Corollary 1.4.
Bergman Kernels and algebraic structure

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2. Estimates from Kähler Ricci flow

In this section, we give some necessary estimates for the scalar curvatures and Kähler potentials along the Kähler-Ricci flow. Let \((M, g)\) be an \(n\)-dimensional Fano manifold with its Kähler form \(\omega \in 2\pi c_1(M)\). Let \(g_t = g(\cdot, t)\) be a solution of normalized Kähler Ricci flow,

\[
\begin{aligned}
\frac{\partial}{\partial t} g &= -\text{Ric}(g) + g, \\
g_0 &= g(\cdot, 0) = g.
\end{aligned}
\]

(2.1)

Recall an estimate for Sobolev constants of \(g_t\) by Zhang [Zh].

Lemma 2.1. Let \(g_t\) be the solution of (2.1). Suppose that there exists a \(L^2\)-Sobolev constant \(C_s\) of \(g\) such that the following inequality holds,

\[
(\int_M f^{\frac{2n}{n-1}} dv_g)^{\frac{n-1}{n}} \leq C_s(\int_M f^2 dv_g + \int_M |\nabla f|^2 dv_g), \quad \forall \ f \in C^1(M).
\]

(2.2)

Then there exist two uniform constants \(A = A(C_s, -\inf_M R(g), V)\) and \(C_0 = C_0(C_s, -\inf_M R(g), V)\) such that for any \(f \in C^1(M)\) it holds

\[
(\int_M f^{\frac{2n}{n-1}} dv_{g_t})^{\frac{n-1}{n}} \leq A(\int_M (|\nabla f|^2 + (R_t + C_0) f^2) dv_{g_t}),
\]

(2.3)

where \(R_t\) are scalar curvatures of \(g_t\).

By using the Moser iteration method, we have

Lemma 2.2. Let \(\Delta = \Delta_t\) be Laplace operators associated to \(g_t\). Suppose that \(f \geq 0\) satisfies

\[
(\frac{\partial}{\partial t} - \Delta)f \leq af, \quad \forall \ t \in (0, 1),
\]

(2.4)

where \(a \geq 0\) is a constant. Then for any \(t \in (0, 1)\), it holds

\[
\sup_{x \in M} f(x, t) \leq \frac{C}{t^{\frac{a+1}{p}}} \left( \int_0^t \int_M |f(x, \tau)|^p dv_{g_\tau} d\tau \right)^{\frac{1}{p}},
\]

(2.5)

where \(C = C(a, p, C_s, -\inf R(g), V)\) and \(C_s\) is the \(L^2\)-Sobolev constant of \(g\) in (2.2).
Bergman Kernels and algebraic structure

Proof. Multiplying both sides of (2.4) by $f^p$, we have

$$\int_M f^p f'_\tau \, dv_{g_r} - \int_M f^p \Delta f \, dv_{g_r} \leq a \int_M f^{p+1}.$$

Taking integration by parts, we get

$$\frac{1}{p+1} \int_M (f^{p+1})'_\tau \, dv_{g_r} + \frac{4p}{(p+1)^2} \int_M |\nabla f^{p+1}|^2 \, dv_{g_r} \leq a \int_M f^{p+1} \, dv_{g_r}.$$

Since

$$\frac{d}{d\tau} \int_M f^{p+1} \, dv_{g_r} = \int_M (f^{p+1})'_\tau \, dv_{g_r} + \int_M f^{p+1} (n - R) \, dv_{g_r},$$

we deduce

$$\frac{1}{p+1} \frac{d}{d\tau} \int_M f^{p+1} \, dv_{g_r} + \frac{1}{p+1} \int_M Rf^{p+1} \, dv_{g_r} + \frac{4p}{(p+1)^2} \int_M |\nabla f^{p+1}|^2 \, dv_{g_r} \leq (a + \frac{n}{p+1}) \int_M f^{p+1} \, dv_{g_r}.$$

It turns

$$\frac{d}{d\tau} \int_M f^{p+1} \, dv_{g_r} + \int_M (R + C_0)f^{p+1} \, dv_{g_r} + 2 \int_M |\nabla f^{p+1}|^2 \, dv_{g_r} \leq ((p+1)a + n + C_0) \int_M f^{p+1} \, dv_{g_r}.$$

(2.6)

For any $0 \leq \sigma' \leq \sigma \leq 1$, we define

$$\psi(\tau) = \begin{cases} 
0, \tau \leq \sigma' t \\
\frac{\tau - \sigma' t}{(\sigma - \sigma') t}, \sigma' t \leq \tau \leq \sigma t \\
1, \sigma t \leq \tau \leq t.
\end{cases}$$

Then by (2.6), we have

$$\frac{d}{d\tau} \left( \psi \int_M f^{p+1} \, dv_{g_r} \right) + \psi \int_M [(R + C_0)f^{p+1} + 2|\nabla f^{p+1}|^2] \, dv_{g_r} \leq \left[ \psi((p+1)a + n + C_0) + \psi' \right] \int_M f^{p+1} \, dv_{g_r}.$$

It follows

$$\sup_{\sigma \leq \tau \leq t} \int_M f^{p+1} \, dv_{g_r} + \int_{\sigma t}^t \int_M [(R + C_0)f^{p+1} + 2|\nabla f^{p+1}|^2] \, dv_{g_r} \leq ((p+1)a + n + C_0 + \frac{1}{(\sigma - \sigma') t}) \int_{\sigma t}^t \int_M f^{p+1} \, dv_{g_r}. $$

6
Thus by Lemma 2.1, we get
\[
\int_{\sigma t}^{t} \int_{M} f^{(p+1)(1+\frac{1}{n})} dv_{g_{r}}
\leq \left( \int_{\sigma t}^{t} \int_{M} f^{p+1} dv_{g_{r}} \right)^{\frac{1}{n}} \left( \int_{M} f^{(p+1)n-1} \right)^{\frac{n-1}{n}}
\leq \left( \sup_{\sigma t \leq r \leq t} \int_{M} f^{p+1} dv_{g_{r}} \right)^{\frac{1}{n}} \int_{\sigma t}^{t} \int_{M} \left( (R + C_{0})f^{p+1} + 2|\nabla f^{p+1}|^{2} \right) dv_{g_{r}}
\]
(2.7)
\[
\leq A(p+1)a + n + C_{0} + \frac{1}{(\sigma - \sigma')^{\frac{n+1}{n}}} \left( \int_{\sigma \tau}^{t} \int_{M} f^{p+1} dv_{g_{r}} \right)^{\frac{n+1}{n}}.
\]
By choosing \( \sigma' = \frac{1}{2} + \frac{1}{4}\sigma_{k}, \sigma = \frac{1}{2} + \frac{1}{4}\sigma_{k+1} \), where \( \sigma_{k} = \sum_{l=0}^{k}(\frac{1}{2})^{l-1} \), and replacing \( p \) by \( p_{k+1} = (p_{k})^{\frac{n+1}{n}} - 1 \) with \( p_{0} = p \) in (2.7), then iterating \( k \) we will get the desired estimate (2.5).

By Lemma 2.2, we prove

**Proposition 2.3.** Let \( u = u_{t} \) and \( R = R_{t} \) be Ricci potentials and scalar curvatures of solutions \( g_{t} \) in (2.1), respectively. Suppose that \( (M, g) \) satisfies

\[
\text{Ric}(g) \geq -\Lambda^{2} g \ 	ext{and} \ \text{diam}(M, g) \leq D.
\]
(2.8)

Then there exists a constant \( C(n, , \Lambda, D) \) such that

\[
|\nabla u|^{2}(x, t)
\leq \frac{C}{t(n+1)(n+\frac{3}{2})} \int_{\frac{t}{2}}^{t} \int_{M} |R - n| dv_{g_{r}}, \ \forall \ 0 < t \leq 1
\]
(2.9)

and

\[
|R - n|(x, t)
\leq \frac{C}{t(n+1)(n+\frac{3}{2})+n} \int_{\frac{t}{2}}^{t} \int_{M} |R - n| dv_{g_{r}}, \ \forall \ 0 < t \leq 1.
\]
(2.10)

**Proof.** By a direct computation, we have the the following evolution formulas for \( |\nabla u| \) and \( R \), respectively,

\[
\frac{\partial}{\partial t} - \Delta |\nabla u|^{2} = \Delta |\nabla u|^{2} - |\nabla \nabla u|^{2} - |\nabla u|^{2} \leq |\nabla u|^{2}
\]
(2.11)

and

\[
\frac{\partial}{\partial t} - \Delta R = \Delta R + R - n + |\text{Ric}(g) - g|^{2}.
\]
(2.12)

It follows

\[
\frac{\partial}{\partial t} - \Delta (R + n\Lambda + |\nabla u|^{2})
= R - n - |\nabla \nabla u|^{2} + |\nabla u|^{2} \leq R + n\Lambda + |\nabla u|^{2}.
\]
(2.13)
Bergman Kernels and algebraic structure

Note that \( R(g_t) + n\Lambda \geq 0 \) by the maximum principle. It was proved in [1] that there exists a uniform constant \( C = C(\Lambda, D) \) such that

\[
\int_0^1 \int_M (R + n\Lambda + |\nabla u|^2) dv_g dt \leq C.
\]

Then by Lemma 2.2 we obtain

(2.14) \( (R + n\Lambda + |\nabla u|^2)(x, t) \leq \frac{C}{t^{n+1}}. \)

In particular,

(2.15) \( |\nabla u|^2(x, t) \leq \frac{C}{t^{n+1}}, \quad \text{and} \quad R \leq \frac{C}{t^{n+1}}. \)

Next we estimate the \( C^0 \)-norm of \( u_t \). By Lemma 2.1 we have the \( L^2 \)-Sobolev inequality,

\[
\left( \int_M f^{2n-1} dv_{g_t} \right)^{\frac{2n-1}{n}} \leq A \left( \int_M (|\nabla f|^2 + (R(x, t) + C_0 f^2) dv_{g_t}) \right) \\
\leq A \left( \int_M (|\nabla f|^2 + \frac{C}{t^{n+1}} f^2) dv_{g_t} \right).
\]

The inequality implies (cf. [He], [Ye]),

\[
\text{vol}(B(x, 1)) \geq C t^{(n+1)}, \quad \forall \ x \in M.
\]

Since \( \text{vol}(M) = V \), it is easy to obtain

\[
\text{diam}(M, g_t) \leq \frac{V}{C t^{n(n+1)}}.
\]

Thus by (2.15), we get

(2.16) \( \text{osc}_{M} u(x, t) \leq \frac{C}{t^{(n+1)(n+1)}}. \)

By (2.16), we can improve (2.15) to (2.9). In fact, by applying Lemma 2.2 to (2.11), we have

\[
|\nabla u|^2(x, t) \leq \frac{C}{t^{n+1}} \int_T^t \int_M |\nabla u|^2 dv_{g_t} d\tau \\
= \frac{C}{t^{n+1}} \int_T^t \int_M -u \Delta u dv_{g_t} d\tau \\
\leq \frac{C}{t^{n+1}} \text{osc}_{(x, \tau) \in M \times [\frac{T}{2}, T]} |u|(x, \tau) \int_T^t \int_M |R - n| dv_{g_t} d\tau \\
\leq \frac{C'}{t^{(n+1)(n+1)+\frac{3}{2}}} \int_T^t \int_M |R - n| dv_{g_t} d\tau,
\]

where the constant \( C' \) depends only on \( n, \Lambda, D \). This proves (2.9).
Bergman Kernels and algebraic structure

To get (2.10), we use the evolution equation as same as (2.13),
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (|\nabla u|^2 + R - n) = R - n - |\nabla \nabla u|^2 + |\nabla u|^2 \leq |\nabla u|^2 + R - n.
\]

Then applying Lemma 2.2, we see
\[
(|\nabla u|^2 + R - n)_+ \leq \frac{C}{t^{n+1}} \int_M \int_M |\nabla u|^2 + R - n |dv_g|d\tau \leq \frac{C}{t^{(n+1)(n+\frac{3}{2})}} \int_M |R - n|dv_g|d\tau.
\]

Thus by (2.9), it follows
\[
(R - n)_+ \leq \frac{C}{t^{(n+1)(n+\frac{3}{2})}} \int_M |R - n|dv_g|d\tau \leq A(t).
\]

On the other hand, by the evolution equation (2.12) of $R$,
\[
\left( \frac{\partial}{\partial t} - \Delta \right) R = R - n + |\nabla \bar{\nabla} u|^2,
\]
we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (A(T) + n - R) \leq A(T) + n - R.
\]

Hence applying Lemma 2.2 again, we get
\[
(A(t) + n - R)(x, t) \leq \frac{C''}{t^{n+1}} \int_M |n - R|dv_g|d\tau + \frac{A(t)VC}{t^n}.
\]

Therefore, inserting (2.18) into the above estimate, we obtain (2.10). \hfill \square

3. Estimates for holomorphic sections

In this section, we use the estimates in Section 2 to give the $C^0$-estimate and the gradient estimate for holomorphic sections with respect to $g_t$. Let $(M^n, g)$ be a Fano manifold and $L = K_M^{-1}$ its anti-canonical line bundle with induced Hermitian metric $h$ by $g$. We begin with the following lemma.

Lemma 3.1. Suppose that the Ricci potential $u$ of $g$ satisfies
\[
\|\nabla u\|_g \leq 1.
\]
Then for $s \in H^0(M, L^l)$ we have

$$
\|s\|_h + l^{-\frac{1}{2}} \|\nabla s\|_h \leq C(C_s, n) l^{\frac{1}{2}} \left( \int_M |s|^2 dv_g \right)^{\frac{1}{2}},
$$

where $C_s$ is the Sobolev constant of $(M, g)$.

**Proof.** Note that

$$
\Delta |s|_h^2 = |\nabla s|_h^2 - nl |s|_h^2.
$$

It follows

$$
-\Delta |s|_h^2 \leq nl |s|_h^2.
$$

Thus applying the standard Moser iteration method to (3.3), we get

$$
\|s\|_h \leq C(C_s, n) l^{\frac{1}{2}} \left( \int_M |s|^2 dv_g \right)^{\frac{1}{2}}.
$$

On the other hand, we have the following Bochner formula,

$$
\Delta | \nabla s|_h^2 = | \nabla \nabla s|_h^2 + | \nabla s|_h^2 - (n + 2)l |s|_h^2 + \langle \text{Ric}(\nabla s, \cdot), \nabla s \rangle.
$$

Then we can also apply the Moser iteration to obtain a $L^\infty$-estimate for $|\nabla s|_h^2$ as done for $|s|_h^2$. In fact, it suffices to deal with the extra integral terms like $\langle \text{Ric}(\nabla s, \cdot), \nabla s \rangle$. But those terms can be controlled by the integral of $(|\nabla \nabla s|_h^2 + |\nabla s|_h^2)|\nabla s|_h^{2p}$ by taking integral by parts with the help of the condition (3.1) (cf. [WZ2], [TZZZ]). As a consequence, we obtain

$$
\|\nabla s\|_h \leq C(C_s, n) l^{\frac{1}{2}} \left( \int_M |s|^2 dv_g \right)^{\frac{1}{2}} \leq C(C_s, n) l^{\frac{1}{2} + \frac{1}{2p}} \left( \int_M |s|^2 dv_g \right)^{\frac{1}{2p}}.
$$

Therefore, combining (3.4) and (3.5), we derive (3.2).

**Lemma 3.2.** Let $(M, g)$ be a Fano manifold which satisfies (3.1) as in Lemma 3.1. Let $\bar{\partial}$-operator be defined for smooth sections on $(M, L^l)$ $(l \geq 4n)$ with the induced metric $h$. Then for any $\sigma \in C^\infty(\Gamma(M, L^l))$, there exists a solution $v \in C^\infty(\Gamma(M, L^l))$ such that $\bar{\partial}v = \bar{\partial}\sigma$ with property:

$$
\int_M |v|^2 \leq 4l^{-1} \int_M |\bar{\partial}\sigma|^2.
$$

**Proof.** The existence part comes from the Hörmander $L^2$-theory. We suffice to verify (3.6), which is equal to prove that the first eigenvalue $\lambda_1(\bar{\partial}, L^l)$ of $\Delta_{\bar{\partial}}$ is greater than $\frac{1}{4}$, where $\Delta_{\bar{\partial}}$ denotes the Laplace operator defined on $L^2(T^*M \otimes L^l)$.

Note that the following two identities hold for any $\theta \in \Omega^{0,1}(L^l)$,

$$
\Delta_{\bar{\partial}} \theta = \bar{\nabla}^* \bar{\nabla} \theta + \text{Ric}(\theta, \cdot) + l\theta
$$

and

$$
\Delta_{\bar{\partial}} \theta = \bar{\nabla}^* \bar{\nabla} \theta - (n - 1)l\theta.
$$
Bergman Kernels and algebraic structure

It follows

$$\Delta_\theta \bar{\theta} = (1 - \frac{1}{2n}) \bar{\nabla}^* \nabla + (1 - \frac{1}{2n}) \text{Ric}(\theta, .) + \frac{1}{2n} \bar{\nabla}^* \nabla \bar{\theta} + \frac{l}{2} \theta. \quad (3.7)$$

Then with the help of condition (3.1), a direct computation shows

$$\int_M \langle \Delta_\theta \bar{\theta}, \theta \rangle \nabla \nabla \theta = (1 - \frac{1}{2n}) \int_M |\bar{\nabla} \theta|^2 + \frac{1}{2n} \int_M |\nabla \theta|^2 + \frac{1}{2} \int_M |\theta|^2$$

$$+ (1 - \frac{1}{2n}) \int_M (|\theta|^2 + \langle \nabla \nabla u(\theta, .), \theta \rangle)$$

$$\geq (1 - \frac{1}{2n}) \int_M |\bar{\nabla} \theta|^2 + \frac{1}{2n} \int_M |\nabla \theta|^2 + \frac{1}{2} \int_M |\theta|^2$$

$$+ (1 - \frac{1}{2n}) \int_M |\theta|^2 - (1 - \frac{1}{2n}) \int_M \langle \nabla u, ((\nabla \theta, \theta) + \langle \theta, \bar{\nabla} \theta \rangle) \rangle$$

$$\geq (1 - \frac{1}{2n}) \int_M |\bar{\nabla} \theta|^2 + \frac{1}{2n} \int_M |\nabla \theta|^2 + \frac{1}{2} \int_M |\theta|^2$$

$$+ (1 - \frac{1}{2n}) \int_M |\theta|^2 - (1 - \frac{1}{2n}) \int_M \frac{1}{2n}(|\nabla \theta|^2 + |\nabla \theta|^2 + n|\theta|^2$$

$$\geq \frac{l}{2} - n \int_M |\theta|^2. \quad (3.8)$$

Now we can choose $l \geq 4n$ to get that $\lambda_1(\bar{\theta}, L) \geq \frac{l}{4}$ as required. \qed

**Remark 3.3.** If the upper bound of $|\nabla u|$ is replaced by a constant $C$, the coefficient at the last inequality in (3.8) will be $\frac{l}{2} - nC^2$. Then by choosing $l \geq 4nC^2$, one can also get (3.6). This was proved in [TZha]. It can be also proved that the estimate (3.2) in Lemma 3.1 still holds under the condition $\|\nabla u\| \leq C$.

Recall that a sequence of almost Kähler-Einstein Fano manifolds $(M_i, J_i, g^i)$ satisfy (TW):

\begin{align*}
i) \quad & \text{Ric}(g^i) \geq -\Lambda^2 g^i \text{ and diam}(M_i, g^i) \leq D; \\
ii) \quad & \int_{M_i} |\text{Ric}(g^i) - g^i| dv_{g^i} \to 0; \\
iii) \quad & \int_0^1 \int_{M_i} |R - n| dv_{g^i} \to 0, \text{ as } i \to \infty. \quad (3.9)
\end{align*}

Here $g^i$ are normalized so that $\omega_{g^i} \in 2\pi c_1(M_i)$ and $g^i_t$ are the solutions of (2.1) with the initial metrics $g^i$. We note that $\text{vol}(M_i, g^i) = (2\pi)^n c_1(M_i)^n \geq V$ for some uniform constant $V$ by the normalization.
Applying Lemma 3.1 and Lemma 3.2 to almost Kähler-Einstein manifolds with the help of gradient estimate (2.9) in Proposition 2.3, we have the following proposition.

**Proposition 3.4.** Let \( \{(M_i, g^i)\} \) be a sequence of almost Kähler-Einstein metrics which satisfy (3.9). Then for any \( t \in (0, 1) \) there exist integers \( N = N(t), l_0 = l_0(t) \) and a uniform constant \( C = C(t) \) such that for any \( i \geq N \) and \( l \geq l_0 \) the following property holds:

\[
\|s\|_{h^i_t} + l^{-\frac{2}{n}}\|\nabla s\|_{h^i_t} \leq Cl^2\left(\int_M |s|^2dv_{g^i_t}\right)^{\frac{1}{2}}
\]

and

\[
\int_{M_i} |v|^2_{h^i_t} \leq 4l^{-1}\int_{M_i} |\bar{\partial}\sigma|^2.
\]

Here \( s \in H^0(M_i, K_{-l}M_i) \) and the norm of \( |\cdot|_{h^i_t} \) are induced by \( g^i_t \).

**Proof.** A well-known result shows that the \( L^2 \)-Sobolev constants \( C_s \) of \( (M_i, g^i) \) depend only on the constants \( \Lambda, D \) and \( V \). Then by (2.9) in Proposition 2.3, for any \( t \in (0, 1) \), there exists \( N = N(t) \) such that

\[
\|\nabla u^i\|_{h^i_t} \leq 1, \quad \forall \ i \geq N,
\]

where \( u^i \) are Ricci potentials of \( g^i_t \). Thus we can apply Lemma 3.1 to get (3.10). Similarly, we can get (3.11) by Lemma 3.2. \( \square \)

4. **Construction of locally approximate holomorphic sections**

Let \( \{(M_i, g^i)\} \) be a sequence of almost Kähler-Einstein manifolds as in Section 3 and \( (M_\infty, g_\infty) \) its Gromov-Hausdorff limit. It was proved by Tian and B. Wang that the regular part \( \mathcal{R} \) of \( M_\infty \) is an open Kähler manifold and the codimension of singularities of \( M_\infty \) is at least 4 [TW]. Moreover, according to Proposition 5.1 in that paper, we have

**Lemma 4.1.** Let \( x \in M_\infty \). Then there exist constants \( \epsilon = \epsilon(n) \) and \( r_0 = r_0(n, C) \) such that if \( \text{vol}(B_x(r)) \geq (1 - \epsilon)\omega_{2n}r^{2n} \) for some \( r \leq r_0 \), then \( B_x\left(\frac{r}{2}\right) \subseteq \mathcal{R} \), and

\[
\text{Ric}(g_\infty) = g_\infty, \quad \|\nabla^l\text{Rm}\|_{C^0(B_x(\frac{r}{2}))} \leq \frac{C}{r^{l+2}},
\]

where the constant \( C \) depends only on \( l \), and the constants \( \Lambda \) and \( D \) in (3.9).

Recall that a tangent cone \( C_x \) at \( x \in M_\infty \) is a Gromov-Hausdorff limit defined by

\[
(C_x, g_x, x) = \lim_{j \to \infty} (M_\infty, g^\infty_{\frac{r_j}{2}}, x),
\]

(4.1)
where \( \{ r_j \} \) is some sequence which goes to 0. Without the loss of generality, we may assume that \( l_j = \frac{1}{r_j} \) are integers. Since \( (C_x, g_x, x) \) is a metric cone, 
\[
g_x = \text{hess} \frac{\rho_x^2}{2},
\]
where \( \rho_x = \text{dist}(x, \cdot) \) is a distance function staring from \( x \) in \( C_x \).

Denote the regular part of \( (C_x, g_x, x) \) by \( CR \), which consists of points in \( C_x \) with flat cones. By Lemma 4.1, we prove

**Lemma 4.2.** \( CR \) is an open Kähler-Ricci flat manifold. Moreover, for any compact set \( K \subset CR \), there exist a sequence of \( (K_j \subset \mathcal{R}, \frac{1}{r_j}g_\infty) \) which converges to \( K \) in \( C^\infty \)-topology.

**Proof.** Let \( \epsilon \) be a small number chosen as in Lemma 4.1. Then for any \( y \in CR \), there exists some small \( r \) such that \( \hat{B}_y(r) \subset C_x \) and
\[
\text{vol} (\hat{B}_y(r)) \geq (1 - \frac{\epsilon}{2})\omega_{2n}r^{2n}.
\]
Thus there exists a sequence of \( y_\alpha \subset C_x \) such that
\[
\text{vol} (B_{y_\alpha}(rr_\alpha)) \geq (1 - \epsilon)\omega_{2n}(rr_\alpha)^{2n},
\]
where the sequence \( \{r_\alpha\} \) is chosen as in (4.1). By Lemma 4.1, it follows
\[
\|\text{Rm}(\tilde{g}_\infty)\|_{C^1(\hat{B}_{y_\alpha}(\frac{1}{r})))} \leq \frac{C_l}{r^{l+2}},
\]
where \( \tilde{g}_\infty = \frac{g_\infty}{r_\alpha^2} \) and \( \hat{B}_{y_\alpha}(\frac{1}{r}) \subset M_\infty \) is a \( \frac{1}{r} \)-geodesic ball with respect to the metric \( \tilde{g}_\infty \). Hence, by Cheeger-Gromov compactness theorem [GW], \( (\hat{B}_{y_\alpha}(\frac{1}{r}), \tilde{g}_\infty) \) converge to \( (\hat{B}_y(\frac{1}{r}), g_x) \) in \( C^\infty \)-topology. In particular, \( B_{y_\alpha}(\frac{rr_\alpha}{r}) \subset \mathcal{R} \) and \( \hat{B}_y(\frac{1}{r}) \subset CR \). This implies that \( CR \) is an open manifolds.

Moreover, \( CR \) is a Kähler-Ricci flat manifold since each \( (B_{y_\alpha}(\frac{rr_\alpha}{r}), g_\infty) \) is an open Kähler-Einstein manifold. If \( K \) is a compact set of \( CR \), then by taking finite small geodesic covering balls, one can find a sequence \( \{ (K_j \subset \mathcal{R}, \frac{1}{r_j}g_\infty) \} \) which converges to \( (K, g_x) \) in \( C^\infty \)-topology. \( \square \)

Define an open set \( V(x; \delta) \) of \( CR \) by
\[
(4.2) \quad V(x; \delta) = \{ y \in C_x | \text{dist}(y, S_x) \geq \delta, d(y, x) \leq \frac{1}{\delta} \},
\]
where \( S_x = C_x \setminus CR \). The following lemma shows that there exists a “nice” cut-off function on \( C_x \) which supported on \( V(x; \delta) \).

**Lemma 4.3.** For any \( \eta, \delta > 0 \), there exist some \( \delta_1 < \delta \) and a cut-off function \( \beta \) on \( C_x \) which supported in \( V(x; \delta_1) \) with property: \( \beta = 1, \) in \( V(x; \delta) \);
\[
\int_{C_x} |\nabla \beta|^2 e^{-\frac{\rho_x^2}{2}} dv_{g_x} \leq \eta.
\]

**Lemma 4.3** is in fact a corollary of following fundamental lemma.
Lemma 4.4. Let \((X^m, d, \mu)\) be a measured metric space such that

\[
C_0 r^m \geq \mu(B_y(r)) \geq \frac{1}{C_0} r^m, \quad \forall \ r \leq 1, \ y \in X.
\]

Let \(Z\) be a closed subset of \(X\) with \(\mathcal{H}^{m-2}(Z) = 0\). Suppose that there exists a nonnegative function \(f \leq 1\) on \(X\) such that

\[
\int_X f d\mu \leq 1.
\]

Then for any \(x \in X, \ \eta > 0\) and \(\delta > 0\), there exist a positive \(\delta_1 \leq \delta\) and a cut-off function \(\beta \geq 0\), which supported in \(B_x(\frac{1}{\delta_1}) \setminus Z_{\delta_1}\) with property:

\[
\beta = 1 \quad \text{in} \quad B_x(\frac{1}{\delta_1}) \setminus Z_{\delta};
\]

\[
\int_X f|\text{Lip}(\beta)|^2 d\mu \leq \eta.
\]

Here \(Z_{\delta_1} = \{x' \in X| \text{dist}(x', Z) \leq \delta_1\}\) and \(\text{Lip}(\beta) = \sup_{w \rightarrow z} |\frac{f(w) - f(z)}{d(w,z)}|\).

Proof. Let \(R \geq \sqrt{\frac{\eta}{\theta} + \frac{\delta}{\delta_1}}\). Since \(\mathcal{H}^{m-2}(Z) = 0\), then for any \(\kappa > 0\), we can take finite distance balls \(B_x(r_i)\) \((r_i \leq \delta)\) to cover \(B_x(R) \cap Z\) such that

\[
\sum r_i^{m-2} \leq \kappa.
\]

Moreover, by (4.3), we may assume that for any \(y \in X\) there are at most \(N = N(C_0, m)\) balls containing \(y\).

Let \(\zeta : \mathbb{R} \rightarrow \mathbb{R}\) be a cut-off function which satisfies:

\[
\zeta(t) = 1, \quad \text{for} \ t \leq 1; \ \zeta(t) = 0, \quad \text{for} \ t \geq 1; \ |\zeta'(t)| \leq 2.
\]

Set

\[
\beta(y) = \zeta\left(\frac{\epsilon}{d(y, x)}\right)\zeta\left(\frac{d(y, x)}{R}\right)\Pi(1 - \zeta\left(\frac{d(y, x_i)}{r_i}\right)),
\]

where \(\epsilon \leq \frac{\delta}{2}\). Then it is easy to see that \(\beta\) is supported in \(B_x(R) \setminus \cup B_x(\frac{r_i}{2})\) with \(\beta \equiv 1\) in \(B_x(\frac{1}{\delta_1}) \setminus Z_{\delta}\). Moreover,

\[
\int_X f|\text{Lip}(\beta)|^2 d\mu \leq C_0 N \sum r_i^{m-2} + 4C_0 \epsilon^{m-2} + \frac{4}{R^2} \leq C_0 N \kappa + 4C_0 \epsilon^{m-2} + \frac{\eta}{2}.
\]

Thus, if we choose \(\epsilon\) and \(\kappa\) such that \(C_0 N \kappa + 4C_0 \epsilon^{m-2} \leq \frac{\eta}{2}\), then we get (4.4). By choosing \(\delta_1 \leq \min\{\frac{\kappa}{2}, \frac{1}{\delta_1}\}\) so that

\[
Z_{\delta_1} \cap B_x(R) \subseteq \cup B_x(\frac{r_i}{2}),
\]

we also get \(\text{supp}(\beta) \subset B_x(\frac{1}{\delta_1}) \setminus Z_{\delta_1}\). Hence \(\beta\) satisfies all conditions required in the lemma.

\[\square\]
Bergman Kernels and algebraic structure

**Proof of Lemma 4.3**. Applying Lemma 4.3 to $X = \mathbb{C}$, $Z = S_x$, $f = e^{-\frac{\rho}{2}}$, we get the lemma. □

By Lemma 4.2 we see that for any $\delta > 0$ there exists a sequence of $K_j \subset (M_\infty, r_j^{-2} g_\infty)$ which converge to $V(x; \delta)$. Let $L_0 = (C_x, \mathbb{C})$ be the trivial holomorphic bundle over $C_x$ with a hermitian metric $h_0 = e^{-\frac{\rho}{2}}$. Then $h_0$ induces the Chern connection $\nabla_0$ with its curvature

$$\text{Ric}(L_0, \nabla_0) = g_x.$$ 

In the following we show that a sufficiently large multiple line bundles of $K_R^{-1}|K_j$ will approximate to $L_0$ over $V(x; \delta)$. This is in fact an application of the following fundamental lemma.

**Lemma 4.5.** Let $(V, g)$ be a $C^2$ open Riemannian manifold and $U, U' \subset\subset V$ are two pre-compact open subsets of $V$ with $\bar{U} \subset\subset U'$. Then for any positive number $\epsilon$, there exist a small number $\delta = \delta(U', g, \epsilon)$ and a positive integer $N = N(U, g, \epsilon)$, which depends on the fundamental group of $U$, the metric $g$ on $U$, and the small $\epsilon$ such that the following is true: if a hermitian complex line bundle $(L, h)$ over $V$ with associated connection $\nabla$ satisfies

$$|\text{Ric}^\nabla|_g \leq \delta, \text{ in } U',$$

then there exist a positive integer $l \leq N$ and a section $\psi$ of $L^\otimes l$ over $U$ with $|\psi|_h \equiv 1$ which satisfies

$$|D^\otimes l \psi|_{h, g} \leq \epsilon, \text{ in } U.$$

**Proof.** The proof seems standard. Here we give a proof from [15]. First we show that $(L, U)$ is a flat bundle with respect to some connection. Notice that $L$ is topologically trivial as long as $\delta$ is small, since the first Chern class lies in the secondary integral cohomology group. Let $B_{x_i}(r_i)$ ($r_i \leq 1$) be finite convex geodesic balls in $V$ such that $\bar{U} \subset \cup B_{x_i}(r_i) \subset U'$. Then for $y \in B_{x_i}(r_i)$ there exists a minimal geodesic curve $\gamma_y$ in $B_{x_i}(r_i)$, which connects $x_i$ and $y$. Picking any vector $s_i \in L_{x_i}$, with $|s_i| = 1$ and using the parallel transportation, we define a parallel vector field for any $y \in B_{x_i}(r_i)$,

$$e_i(y) = \text{Para}_{\gamma_y}(s_i).$$

In particular, $De_i|_{x_i} = 0$. Let $T$ be a vector field, which is tangent to $\gamma_y$, and $X$ another vector field with $[T, X] = 0$. Then

$$D_T[D_X e_i] = D_X[D_T e_i] + \text{Ric}^\nabla(T, X)e_i = \text{Ric}^\nabla(T, X)e_i.$$ 

By the condition (4.5), it follows

$$|De_i|_{C^0(B_{x_i}(r_i))} \leq C(U', g)||\text{Ric}^\nabla||_{(U', g)} \leq C(U', g)\delta.$$ 

15
Bergman Kernels and algebraic structure

This implies that the transformation function $g_{ij}$ of $L$ is nearly constant in $B_{x_i} \cap B_{x_j}$. Hence, there exists some complex function $f_i$ over $B_{x_i}(r_i)$ such that

$$(4.9) \quad |Df_i| \leq C(U', g)\delta << 1,$$

and the transition function for $\tilde{e}_i = f_i e_i$ is constant. As a consequence, we can define another metric $h'$ with associated connection $\nabla'$ on $L$ such that

$$|\tilde{e}_i|_{h'} = 1 \text{ and } D^{\nabla'} \tilde{e}_i = 0.$$ 

In fact, if we set $\nabla' = \nabla + \alpha$, then

$$D^{\nabla'} \tilde{e}_i = D\nabla(f_i e_i) + \alpha(\tilde{e}_i) = f_i D\nabla e_i + df_i \otimes e_i + \alpha(\tilde{e}_i).$$

Thus $\alpha$ is uniquely determined by requiring $D^{\nabla'}(\tilde{e}_i) = 0$. Therefore, $(L, \nabla')$ is a flat bundle over $U$ with respect to $\nabla'$. Moreover, by (4.8) and (4.9), we have

$$||\nabla - \nabla'||_{(U, g)} \leq C(U', g)||Ric\nabla||_{(U', g)} \leq C(U', g)\delta.$$

Next we note that the holonomy group of a flat bundle over $U$ is an element of $Hom(\pi_1(U), S^1) \cong G \times \mathbb{T}^k$ for some finite group $G$ with order $m_1$, where $k$ is the Betti number of $\pi_1(U)$. By the pigeon-hole principle, we see that for any $\gamma$-neighborhood $W \subseteq \mathbb{T}^k$ of the identity there exists a positive integer $m_2 = m_2(\gamma)$ such that for any element $\rho \in \mathbb{T}^k$, $\rho^a \in W$ for some number $a (1 \leq a \leq m_2)$. As a consequence, for any element $t \in G \times \mathbb{T}^k$, there exists $l (1 \leq l \leq N = m_1 m_2)$ such that $t^l \in W$. Hence, there exists $l$ such that

$$||\nabla^{\otimes l} - \nabla''||_{(U, g)} \leq C(U', g)\gamma(\delta),$$

where $\nabla''$ is a connection on $L$ with the trivial holonomy group. Let $\psi$ be a normalized section induced by a global parallel section of $(L, \nabla'')$ so that $|\psi|_{h} \equiv 1$. Then it is easy to see

$$|D^{\nabla^{\otimes l}} \psi| \leq C(U', g)N\delta + C(U, g)\gamma \leq \epsilon,$$

as long as $\delta$ is small enough. The lemma is proved. \qed

**Proposition 4.6.** Let $x \in M_{\infty}$ and $\delta_1 > 0$. Then for any $\epsilon > 0$, there exist positive two integer $N = N(V(x; \delta_1), \epsilon)$ and $l \leq N$ such that there exist a large integer $j_0$, a sequence of $K_j \subseteq M_{\infty}$ and a sequence of pairs of isomorphisms $(\phi_j, \psi_j)$ for $j \geq j_0$ with property:

$$L_0 \xrightarrow{\psi_j} K_R^{H_j} |_{K_j}$$

$$\downarrow \quad \downarrow$$

$$V(x; \delta_1) \xrightarrow{\phi_j} K_j,$$

(4.10)
Bergman Kernels and algebraic structure

which satisfy

$$\phi_j^*(l_j g_\infty) \to g_x, \text{ as } j \to \infty,$$

and

$$|\psi_j| \equiv 1 \text{ and } |D\psi_j|_{g_x} \leq \epsilon, \text{ in } V(x; \delta_1).$$

**Proof.** Define an open set $U$ of $CR$ by

$$U = U(x; \epsilon_1, \epsilon_2, R) = \{ y \in C_x \mid \text{dist}(\bar{y}, S_x) \geq \epsilon_1, \epsilon_2 \leq d(y, x) \leq R \},$$

where $\bar{y}$ is the projection to the section $Y$ of $C_x = C(Y)$. Then there exist some $\epsilon_1, \epsilon_2$ and $R$ such that

$$V(x; \delta_1) \subseteq U(x; \epsilon_1, \epsilon_2, R).$$

Moreover, we can choose a sequence of integers $l_j = 1/\sqrt{N}$ such that

$$(M_\infty, l_j g_\infty, x) \to (C_x, g_x, x), \text{ as } j \to \infty.$$ 

Hence by Lemma 4.2, there exist a sequence of $\tilde{K}_j \subseteq M_\infty$ and a sequence of diffeomorphisms $\tilde{\phi}_j$ from $U(x; \epsilon_1, \epsilon_2, \sqrt{l_j})$ to $\tilde{K}_j$ such that $\tilde{\phi}_j^*(l_j g_\infty) \to g_x$, where $N = N(U, g_x, \epsilon)$ is a large integer as determined in Lemma 4.5.

Let $h_\infty$ be the induced hermitian metric on $K^{-1}_R$ induced by $g_\infty$ on the regular part $R$ of $M_\infty$. We consider product complex line bundles

$$L_j = \tilde{\phi}_j^*(K^{-l_j}_R, h_\infty^{\otimes l_j}) \otimes (L_0, h_0)^*,$$

with induced connections $\nabla_j$ by $h_\infty$ and $h_0$. Clearly,

$$\|\text{Ric}^{\nabla_j}\|_{(U', g_x)} \leq \delta \ll 1,$$

as long as $j$ is large enough, where $U' \subset \subset CR$ is an open set such that $U \subset \subset U'$. Applying Lemma 4.5 to $L_j$ over $U'$, we see that there exist some positive integer $l \leq N$ and a section $\psi'$ on $L_j^{\otimes l}$ such that

$$|D^{\nabla_j^{\otimes l}} \psi'|_{U(x; \epsilon_1, \sqrt{\epsilon_2}, \epsilon_2, \sqrt{l_j}, R)} \leq \epsilon.$$ 

Let $Y_{\epsilon_1} = U(x; \epsilon_1, \epsilon_2, R) \cap Y$ and $\tilde{\psi}$ an extension section over $U(x; \epsilon_1, \epsilon_2, \sqrt{\epsilon_1}, R)$ of the restriction of $\psi'$ on $Y_{\epsilon_1}$ by the parallel transportation along rays from $x$. Then by the formula (4.7), it is easy to see

$$|D^{\nabla_j^{\otimes l}} \tilde{\psi}|_{U(x; \epsilon_1, \epsilon_2, \sqrt{\epsilon_1}, R, g_x)} \leq \frac{\sqrt{7}}{\epsilon_2} (\epsilon + \delta) + R\delta.$$ 

(4.11)
Thus we have pairs of isomorphisms $(\tilde{\phi}_j, \tilde{\psi}_j)$ with property:

\[
L_0 \xrightarrow{\tilde{\psi}_j} K_{M_\infty}^{-l_j} K_j \xrightarrow{\phi_j} (U(x; \epsilon_1, \epsilon_2, R), g_x) \xrightarrow{\psi_j} (K_j, l_j g_\infty),
\]

(4.12)

which satisfy

\[
|\mathcal{D}\tilde{\psi}_j|_{g_x} \leq 2 \sqrt{l} \frac{\epsilon}{\epsilon_2},
\]

(4.13)
as long as $j$ is large enough.

Rescaling $U(x; \epsilon_1, \epsilon_2, R)$ into $U(x; \epsilon_1, \epsilon_2, R)$ by

\[
\mu_l : y \mapsto \frac{y}{\sqrt{l}}, y \in U(x; \epsilon_1, \epsilon_2, R).
\]

We have isometrics

\[
\mu_l^* L_0 \cong L_0, \mu_l^* g_x = \frac{g_x}{l}.
\]

By (4.12), it follows

\[
L_0 \xrightarrow{\tilde{\psi}_j \circ (\mu_l^*)^{-1}} K_{M_\infty}^{-l_j} K_j \xrightarrow{\phi_j \circ \mu_l} (U(x; \epsilon_1, \epsilon_2, R), g_x) \xrightarrow{\psi_j \circ (\mu_l^*)^{-1}} (K_j, l_j g_\infty).
\]

(4.14)

Let

\[
\phi_j = \tilde{\phi}_j \circ \mu_l, \text{ and } \psi_j = \tilde{\psi}_j \circ (\mu_l^*)^{-1}.
\]

Note that $V(x; \delta_1) \subseteq U(x; \epsilon_1, \epsilon_2, R)$. Then $K_j = \phi_j(V(x; \delta_1))$ is well-defined. Hence, rescaling the metric $\frac{g_x}{l}$ back to $g_x$, we get from (4.13),

\[
|\mathcal{D}\psi_j|_{g_x} \leq \frac{2 \epsilon}{\epsilon_2}, \text{ in } V(x; \delta_1).
\]

(4.15)

On the other hand, since $|\psi_j| \equiv 1$ on $Y_{\epsilon_1}$, by (4.15), it is easy to see

\[
||\psi_j| - 1| \leq CR^2 \frac{\epsilon}{\epsilon_2} << 1, \text{ in } V(x; \delta_1),
\]
as long as $\epsilon$ is small enough, where $C$ is a uniform constant. Thus by normalizing $\psi_j$ so that $|\psi_j| \equiv 1$ in $V(x; \delta_1)$, we get $|D\psi_j|_{g_x} \leq C' \frac{\epsilon}{\epsilon_2}$ for some uniform constant $C'$. Replacing $C' \frac{\epsilon}{\epsilon_2}$ by $\epsilon$, we prove the proposition. \(\square\)

Proposition 4.6 will be used to construct peak sections of holomorphic line bundles over a sequence of Kähler manifolds in next section as done in [T5].
5. \(\bar{\partial}\)-EQUATION AND CONSTRUCTION OF HOLOMORPHIC SECTIONS

In this section, we give a construction of peak holomorphic sections by solving \(\bar{\partial}\)-equation on a smoothing sequence of almost Kähler-Einstein manifolds in [TW]. We will use the rescaling method as done for the Kähler-Einstein manifolds sequence in [DS], [T6].

**Proposition 5.1.** Let \(\{ (M_i, g^i) \} \) be a sequence of almost Kähler-Einstein Fano manifolds as in Section 3 and \((M_\infty, g_\infty)\) be its Gromov-Hausdorff limit. Then for any sequence of \(p_i \in M_i\) which converges to \(x \in M_\infty\), there exist two large number \(l_x\) and \(i_0\), and a small time \(t_x\) such that for any \(i \geq i_0\) there exists a holomorphic section \(s_i \in \Gamma(K_{M_i}^{-l_x}, h_{t_x}^i)\) which satisfies

\[
\int_{M_i} |s_i|^2_{h_{t_x}^i} \, dv_{g_{t_x}^i} \leq 1 \quad \text{and} \quad |s_i|_{h_{t_x}^i} (p_i) \geq \frac{1}{8},
\]

where \(g_i^j\) is a solution of (2.1) with the initial metric \(g_i^j\) and \(h_{t_x}^i\) is the hermitian metric of \(K_{M_i}^{-l_x}\) induced by \(g_{t_x}^i\).

**Proof.** As in Section 4, let

\[
(C_x, \omega_x, x) = \lim_{j \to \infty} (M_\infty, g_\infty, \frac{g_{t_x}^i}{t_j^2}, x).
\]

Choose a \(\delta\) so that \(\delta \leq (2\pi)^{-\frac{n}{2}} C_1\), where \(C_1\) is a constant chosen as in (3.10). We consider the \(\bar{\partial}\)-equation for sections on the trivial line bundle \(L_0 = (V(x; \delta), \mathbb{C})\),

\[
\bar{\partial}\sigma = f, \quad \forall \ f \in \Gamma^\infty(L_0 \otimes (T^*V)^{(0,1)}).
\]

Then the standard \(C^0\)-estimate for the elliptic equation shows

\[
|\sigma|_{C^0(V(x; 2\delta))} \leq C_2 (|f|_{C^0(V(x; \delta))} + \delta^{-n} \int_{V(x; \delta)} |\sigma|^2 dv_g) \frac{1}{2},
\]

where the constant \(C_2\) depends on the metric \(g_x\).

Let \(0 < \eta \leq \frac{\delta^{2n}}{1000C_2}\) and \(\beta\) a cut-off function supported in \(V(x; \delta_1)\) constructed in Lemma 4.3. Let \(K_j\) be the sequence of open sets in \(M_\infty\) which converge to \(V(x; \delta_1)\) and \(\psi_j\) be the sequence of isomorphisms from \(L_0\) to \(K_{M_i}^{-l_j}|_{K_j}\) constructed in Proposition 4.6. Set \(\tau_j = \psi_j(\beta e)\), where \(e\) is a unit base of \(L_0\). Then \(\{\tau_j\}\) is a sequence of smooth sections of \(K_{M_i}^{-l_j}\) supported in \(\psi_j(V(x; \delta_1))\). Moreover, \(\tau_j\) satisfies the following property as long as \(j\) is
Bergman Kernels and algebraic structure

large enough:

\[ i) \|\tau_j\|^2_{C^0(V(x;\delta)\cap B_x(3\delta))} \geq \frac{3}{4} e^{-3\delta^2} \geq \frac{1}{2}; \]

\[ ii) \int_{M_\infty} |\tau_j|^2 dv_{g_\infty} \leq \frac{3}{2} \frac{r_{j+2}}{l} (2\pi)^n; \]

\[ iii) \bar{\partial}_{J_\infty} \tau_j \leq \frac{\eta}{8}, \text{ in } V(x;\delta); \]

\[ iv) \int_{M_\infty} |\bar{\partial}_{J_\infty} \tau_j|^2 dv_{g_\infty} \leq \frac{3}{2} \frac{r_{j+2}}{l} \frac{\eta}{l^{n-1}}. \]

(5.3)

On the other hand, from the proof of Lemma 4.2, we see that there exists \( t_0 \), which depends on \( V(x;\delta_1) \) such that for any sufficiently large \( j \) it holds

\[ \text{vol}(B_y(\sqrt{t_0} \frac{r_j}{l})) \geq (1-\epsilon) \text{vol}(B_y(\sqrt{t_0} \frac{r_j}{l})), \forall y \in K_j, \]

where \( \epsilon \) is a small constant chosen as in Lemma 4.1. Then by the pseudo-locality theorem in [TW], there exist \( t_0' \leq t_0 \), and a sequence of sets \( B_i \subset M_i \) and a sequence of diffeomorphisms \( \varphi_i : K_j \rightarrow B_i \) such that

\[ \varphi_i^* g(t_0' \frac{r_j^2}{l}) \rightarrow g_\infty, \]

\[ \varphi_i^* J_i \rightarrow J_\infty, \]

\[ \varphi_i^* K_{M_i}^{-1} \rightarrow K_{CR}^{-1}, \]

in \( C^\infty \)-topology, where \( g^i(t) = g^i \). Thus, if we let \( v_i = (\varphi_i)_* \tau_{j_0} \in \Gamma(M_i, K_{M_i}^{-l_{j_0}}) \) for some large integer \( l_{j_0} = \frac{1}{r_{j_0}} \), then there exists a large integer \( N \) such that for any \( i \geq N \) it holds:

\[ i') \ |v_i|_{h_{l_{j_0}}}^i \geq \frac{3}{8}, \text{ in } (\varphi_i \circ \psi_{j_0})(V(x;2\delta)\cap B_x(3\delta)); \]

\[ ii') \int_{M_i} |v_i|^2_{h_{l_{j_0}}} dv_{g_{l_{j_0}}} \leq \frac{5}{4} \frac{r_{j_0}^{2n-2}}{l} \frac{\eta}{l^{n-1}}; \]

\[ iii') \ |\bar{\partial}_{J_i} v_i|_{h_{l_{j_0}}} \leq \frac{1}{4} \eta, \text{ in } (\varphi_i \circ \psi_{j_0})(V(x;\delta)); \]

\[ iv') \int_{M_i} |\bar{\partial}_{J_i} v_i|^2_{h_{l_{j_0}}} dv_{g_{l_{j_0}}} \leq \frac{5}{4} \frac{r_{j_0}^{2n-2}}{l} \frac{\eta}{l^{n-1}}. \]

(5.4)

Here \( t_x = t_0' r_{j_0}^2 / l \) and \( h_{l_{j_0}}^i \) are hermitian metrics of \( K_{M_i}^{-l_{j_0}} \) induced by \( g_{l_{j_0}}^i \).

By Solving \( \bar{\partial} \)-equations for \( K_{M_i}^{-l_{j_0}} \)-valued (0,1)-form \( \sigma_i \),

\[ \bar{\partial} \sigma_i = \bar{\partial} v_i, \text{ in } M_i, \]

we get the \( L^2 \)-estimates from (3.6) and \( iv' \) in (5.4),

\[ \|\sigma_i\|^2_{L^2(M_i, g_{l_{j_0}})} \leq \frac{4}{l l_{j_0}} \int_{M_i} |\bar{\partial}_{J_i} v_i|^2 dv_{g_{l_{j_0}}} \leq \frac{5\eta}{l^n l_{j_0}^{n-1}}. \]

(5.5)
Bergman Kernels and algebraic structure

Hence, by (5.2) and iii’) in (5.4), we derive

\[
|\sigma_i|_{h^i_{tx}}(q) \\
\leq 2C_2\left( \sup_{(\varphi_i \circ \psi_j)(V(x; \delta))} |\bar{\partial}u_i|_{h^i_{tx}} + \delta^{-n}[(ll_{j0})^n \int_{(\varphi_i \circ \psi_j)(V(x; \delta))} |\sigma_i|^2_{h^i_{tx}} dv_{g_{tx}}]^{\frac{1}{2}} \right) \\
\leq 2C_2\left( \frac{1}{4}\eta + \delta^{-n}[(ll_{j0})^n \int_{M_i} |\sigma_i|^2 dv_{g_{tx}}]^{\frac{1}{2}} \right) \\
\leq 2C_2\left( \frac{1}{4}\eta + \delta^{-n}[(ll_{j0})^n \frac{5n}{l_{j0}^2 ll_{j0}^2n-2}]^{\frac{1}{2}} \right) \\
\leq 5C_2\left( \frac{1}{4}\eta + \delta^{-n} \sqrt{\eta} \right) \leq \frac{1}{8}, \ \forall \ q \in (\varphi_i \circ \psi_j)(V(x; 2\delta)).
\]

(5.6)

Let \( s_i = v_i - \sigma_i \). Then \( s_i \) is a holomorphic section of \( K^{-ll_{j0}}_{M_i} \). By i’) in (5.4) and (5.6), we have

\[
|s_i|_{h^i_{tx}}(q_1) \geq \frac{3}{8} - \frac{1}{8} = \frac{1}{4}, \ \forall \ q_1 \in (\varphi_i \circ \psi_j)(V(x; 2\delta) \cap B_x(3\delta)).
\]

Moreover, by ii’ in (5.4), it is easy to see that

\[
\int_{M_i} |s_i|^2_{h^i_{tx}} dv_{g_{tx}} \leq 2\int_{M_i} |v_i|^2_{h^i_{tx}} dv_{g_{tx}} + \int_{M_i} |\sigma_i|^2_{h^i_{tx}} dv_{g_{tx}} \\
\leq 4(2\pi)^n \frac{r_{j0}^{2n}}{l^n}.
\]

(5.7)

Thus by the estimate (3.10), we get

\[
\|\nabla s_i\|_{h^i_{tx}} \leq \sqrt{4(2\pi)^n C_1} \sqrt{l_{j0}^{-1}}.
\]

Since \( d(p_i, q_1) \leq 4\frac{r_{j0}}{\sqrt{l}} \delta \), we deduce

\[
|s_i(p_i)|_{h^i_{tx}} \geq |s_j(q_1)| - 4\frac{r_{j0}}{\sqrt{l}} \delta \|\nabla s_i\|_{h^i_{tx}} \\
\geq |s_i|_{h^i_{tx}}(q_1) - 8\sqrt{(2\pi)^n C_1} \delta \geq \frac{1}{8}.
\]

This proves the theorem while \( l_x \) is chosen by \( ll_{j0} \).

\( \square \)

6. PROOF OF THEOREM 1.3–I

In this section, we use the estimate in Section 5 to give a lower bound of \( \rho_{ll_{j0}}(x) \) for a sequence of almost Kähler-Einstein manifolds.
Bergman Kernels and algebraic structure

**Theorem 6.1.** Let \((M_i, g^i)\) be a sequence of almost Kähler-Einstein manifolds as in Section 3 and \((M_\infty, g_\infty)\) be its Gromov-Hausdorff limit. Then there exists an integer \(l_0 > 0\) such that for any integer \(l > 0\) there exists a uniform constant \(c_l > 0\) with property:

\[
\rho_{l_0}(M_i, g^i) \geq c_l, \quad \forall i,
\]

where \(l_0\) depends only on the limit \((M_\infty, g_\infty)\) and \(c_l\) depends only on \((M_\infty, g_\infty)\) and \(l\).

The proof of Theorem 6.1 depends on the following lemma.

**Lemma 6.2.** Let \((M, g)\) be a Fano manifold with \(\omega_g \in 2\pi c_1(M)\) which satisfies

\[
\text{Ric}(g) \geq -\Lambda^2 g \quad \text{and} \quad \text{diam}(M, g) \leq D.
\]

Let \(g_t\) be a solution of (2.1) with the initial metric \(g\). Then there exists a small \(t_0 = t_0(l, \Lambda, D)\) such that the following is true: if \(s \in \Gamma(M, K^{-l}_M)\) is a holomorphic section with \(\int_M |s|_{h_t}^2 dv_{g_t} = 1\) for some \(t \leq t_0\) which satisfies

\[
|s|_{h_t}^2(p) \geq c > 0,
\]

then

\[
|s|_{h_t}^2(p) \geq c' > 0 \quad \text{and} \quad \int_M |s|_{h_t}^2 dv_g \leq c'',
\]

where \(h_t\) and \(h\) are hermitian metrics of \(K^{-l}_M\) induced by \(g_t\) and \(g\), respectively, and \(c', c'' > 0\) are uniform constants depending only on \(c, l, \Lambda\) and \(D\).

**Proof.** Let \(\omega_{g_t} = \omega_g + \sqrt{-1} \partial \bar{\partial} \phi\). Namely, \(\phi\) are potentials of \(g_t\). Then \(\phi = \phi(x, t)\) satisfies

\[
\frac{\partial}{\partial t} \phi = \log \left( \frac{\omega_{g} + \sqrt{-1} \partial \bar{\partial} \phi}{\omega_g^n} \right) + \phi - f_g,
\]

where \(f_g\) is the Ricci potential of \(g\) normalized by

\[
\int_M f_g dv^n_g = 0.
\]

Since

\[
\Delta f_g = R(g) - n \geq -(n - 1) \Lambda^2 - n,
\]

by using the Green formula, we see

\[
f_g(x) \leq -\int_M G(x, \cdot) \Delta f_g \leq C(\Lambda, D).
\]

Thus applying the maximum principle to (6.4), it follows

\[
\phi \geq -C(\Lambda, D).
\]
On the other hand, integrating both sides of (6.4), we have
\[
\frac{d}{dt} \int_M \phi dv g = \int_M \log \left( \frac{\omega g + \sqrt{-1} \partial \bar{\partial} \phi^n}{\omega_g^n} \right) dv g + \int_M \phi dv g - \int_M f dv g \\
\leq \int_M \phi dv g + C,
\]
It follows
\[
\int_M \phi dv g \leq C e^t \leq eC.
\]
Hence by using the Green formula to \( \phi \), we can also get
\[
\phi \leq C' \lambda, \Lambda, D.
\]
As a consequence, we derive
\[
(6.5) \quad e^{-C't} | \cdot |_h \leq | \cdot |_{h_t} = e^{-\lambda t} | \cdot |_h \leq e^{\lambda t} | \cdot |_h.
\]
Therefore to prove Proposition 6.2, we suffice to prove

**Claim 6.3.** Let \( s \in \Gamma(M, K^{-l}_M) \) be a holomorphic section. Suppose that
\[
\int_M |s|^2_{h} dv g = 1.
\]
Then
\[
(6.6) \quad \int_M |s|^2_{h_t} dv g_t \geq c(l, \Lambda, D) > 0.
\]
Since
\[
\frac{\partial}{\partial t} \left( \frac{\omega g + \sqrt{-1} \partial \bar{\partial} \phi^n}{\omega_g^n} \right) = \Delta' \phi \frac{\partial \phi}{\partial t} = -R(g_t) + n \leq \lambda = \lambda(\Lambda),
\]
(6.7)
\[
\text{vol } g_t(\Omega) \leq e^{\lambda t} \text{vol } g(\Omega), \forall \Omega \subset M.
\]
It follows
\[
\text{vol } g_t(\Omega) = V - \text{vol } g_t(M \setminus \Omega) \geq V - e^{\lambda t} \text{vol } g(M \setminus \Omega) \\
\geq \text{vol } g(\Omega) - 2V \lambda t.
\]
(6.8)
By the estimate (3.4), we see
\[
|s(x)|_{h}^2 \leq H = H(\Lambda, D).
\]
Then
\[
\int_0^H \text{vol } g \{ x \in M \mid |s(x)|_{h}^2 \geq s \} ds = \int_M |s|^2_{h} dv g.
\]
Bergman Kernels and algebraic structure

Hence, by using (6.5), and (6.7) and (6.8), we get
\[
\int_M |s|^2_{h_t} \, dv_{g_t} \geq \int_0^H \text{vol}_{g_t}\{x \in M \mid |s(x)|^2_h \geq s\}ds \\
\geq \int_0^H \text{vol}_{g_t}\{x \in M \mid |s(x)|^2_h \geq e^{C't} s\}ds \\
\geq e^{-C't} \int_0^e^{C't} \text{vol}_{g_t}\{x \in M \mid |s(x)|^2_h \geq s\} - 2V\lambda t)ds \\
\geq e^{-C't}(1 - 2VHe^{C't}).
\]

Therefore, by choosing \(t_0 \leq (4VHe^{C't})^{-1}\), we derive (6.6). The claim is proved.

\[\square\]

Proof of the Theorem 6.1: By Proposition 5.1, we see that for any \(x \in M_\infty\) and a sequence \(\{p_i \subset M_i\}\) which converges to \(x\), there exist two large number \(l_x\) and \(i_0\), a small time \(t_x\) such that there exists a holomorphic section \(s_i \in \Gamma(K_{M_i}^{-l_x} h_i^{l_x})\) for any \(i \geq i_0\) with \(\int_{M_i} |s_i|^2_{h_i^{l_x}} \, dv_{g_i} \leq 1\) which satisfies
\[
|s_i|_{h_i^{l_x}}(p_i) \geq \frac{1}{8},
\]
where \(h_i^{l_x}\) is the hermitian metric of \(K_{M_i}^{-l_x}\) induced by \(g_i^{l_x}\). By Lemma 6.2, it follows that there exists a constant \(c(\lambda, \Lambda, D)\) and a holomorphic section \(\hat{s}_i \in \Gamma(K_{M_i}^{-l_x}, h_i)\) for any \(i \geq i_0\) with \(\int_{M_i} |\hat{s}_i|^2_{h_i} \, dv_{g_i} = 1\) which satisfies
\[
|\hat{s}_i|_{h_i}(p_i) \geq c_x = c(\lambda, \Lambda, D),
\]
where \(h_i\) is the hermitian metric of \(K_{M_i}^{-l_x}\) induced by \(g_i\).

Let \(C = C(C_S, n)\) be the constant as in (3.2), which depending only on \(\Lambda\) and \(D\). For each \(x\), we choose \(r_x = \frac{C}{2} l_x^{-\frac{n}{2}} \sqrt{\lambda} C\). Then by the estimate (3.2), we get
\[
|\hat{s}_i|_{h_i}(q) \geq \frac{c_x}{2}, \quad \forall \ q \in B_{p_i}(r_x).
\]

Take \(N\) balls \(B_{x_a}(\frac{r_a}{2})\) to cover \(M_\infty\). Then it is easy to see that there exists \(i_1 \geq i_0\) such that \(\bigcup_{\alpha} B_{p_a^{i_0}}(r_{x_a}) = M_i\) for any \(i \geq i_1\), where \(\{p_a^{i_0}\}\) is a set of \(N\) points in \(M_i\). This shows that for any \(q \in M_i\) \((i \geq i_1)\) there exist a ball \(B_{p_a^{i_0}}(r_{x_a})\) and a holomorphic section \(s^i_{x_a} \in \Gamma(K_{M_i}^{-l_x}, h_i)\) such that \(q \in B_{p_a^{i_0}}(r_{x_a})\), and \(\int_{M_i} |s^i_{x_a}|^2_{h_i} \, dv_{g_i} = 1\) and
\[
|s^i_{x_a}|_{h_i}(q) \geq c = \min_{\alpha} \{c_{x_{a_\alpha}}\} > 0.
\]

Set \(l_0 = \prod_{\alpha} l_{x_{a_\alpha}}\). Then by using a standard method (cf. [DS], [T5]), for any \(q \in M_i\) \((i \geq i_1)\), one can construct another holomorphic section \(s \in\)
Bergman Kernels and algebraic structure

\[ \Gamma(K^{-l_0}_M, h_i) \] based on holomorphic sections \( s^i_\alpha \) such that \( \int_{M_i} |s|^{2}_{h_i} \, dv_{g_i} = 1 \) and

\[ |s|_{h_i}(q) \geq c' > 0, \]

where \( c' = c'(l, c) \). This proves the theorem for \( l = 1 \). One can also prove the theorem for general \( l \geq 1 \) as above while \( l_0 \) replaced by \( ll_0 \).

\[ \Box \]

7. Proof of Theorem 1.3—II

In this section, we prove Theorem 1.3 in case of almost Kähler-Ricci solitons. We assume that a Fano manifold \((M, g)\) admits a non-trivial holomorphic vector field \( X \), where \( X \) lies in an reductive Lie subalgebra \( \eta_r \) of space of holomorphic vector fields, and \( g \) is \( K_X \)-invariant with \( \omega_g \in 2\pi c_1(M) \) [TZ]. We also suppose that \( g \) satisfies the following geometric conditions:

1) \( \text{Ric}(g) + L_X g \geq -\Lambda^2 g, \ |X|_g \leq A \) and \( \text{diam} \ (M, g) \leq D \);

2) \( R \geq -C_0 \).

(7.1)

In particular, under the condition i), \( g \) has a uniform \( L^2 \)-Sobolev constant \( C_s = C_s(\Lambda, A, D) \) (cf. [WZ2]). We note that the volume of \((M, g)\) is uniformly bounded below by the normalized condition \( \omega_g \in 2\pi c_1(M) \) and it is uniformly bounded above by the volume comparison theorem [WW].

Now we consider the following modified Kähler-Ricci flow with the above initial Kähler metric \( g \),

\[ \begin{aligned}
\frac{\partial}{\partial t} g &= -\text{Ric}(g) + g + L_X g, \\
g_0 &= g(\cdot, 0) = g.
\end{aligned} \]

(7.2)

Clearly, all solutions \( g_t \ (t \in (0, \infty)) \) of (7.2) are \( K_X \)-invariant.

Since the Sobolev constant \( g \) is uniformly bounded below, by Zhang’s result [Zh], we have an analogy to Lemma 2.1 as follows.

**Lemma 7.1.** All solutions \( g_t \) of (7.2) have Sobolev constants \( C_s = C_s(\Lambda, A, D) \) uniformly bounded below. Namely, the following inequalities hold,

\[ (\int_M f^{2+n} \, dv_{g_t})^{\frac{n-1}{n}} \leq C_s(\int_M f^{2}(R + \hat{C}_0) \, dv_{g_t} + \int_M |\nabla f|^2 \, dv_{g_t}), \]

where \( f \in C^1(M) \) and \( \hat{C}_0 \) is a uniform constant depending only on the lower bound \( C_0 \) of scalar curvature \( R \) of \( g \).

**Lemma 7.2.** Let \( \Delta = \Delta_t \) be the Laplace operator associated to \( g_t \). Suppose that \( f \equiv 0 \) satisfies

\[ (\frac{\partial}{\partial t} - (\Delta + X))f \leq af, \]

(7.3)
where a is a constant. Then for any $t \in (0,1)$, we have

$$
\sup_{x \in M} f(x,t) \leq \frac{C_1(A, A, D, C)}{t^{n+1}} \left( \int_{\frac{t}{2}}^{t} |f(x, \tau)|^p d\nu_{g, \tau} d\tau \right)^{\frac{1}{p}}.
$$

(7.4)

**Proof.** As in the proof of Lemma 2.2, multiplying both sides of (7.3) by $f^p$, we have

$$
\int_M f^p f'_r d\nu_{g, r} + p \int_M |\partial f|^2 f^{p-1} d\nu_{g, r} - \int_M (\partial \theta, \partial f) f^p d\nu_{g, r} \leq a \int_M f^{p+1} d\nu_{g, r}.
$$

On the other hand, by (7.2), it is easy to see

$$
\int_M f^p f'_r d\nu_{g, r} = \frac{1}{p+1} \frac{d}{d\tau} \left( \int_M f^{p+1} d\nu_{g, r} \right) + \frac{1}{p+1} \int_M (R - n - \Delta \theta) f^{p+1} d\nu_{g, r}.
$$

Thus we get

$$
\frac{1}{p+1} \frac{d}{d\tau} \left( \int_M f^{p+1} d\nu_{g, r} \right) + \frac{1}{p+1} \int_M (R - n) f^{p+1} d\nu_{g, r} + p \int_M |\partial f|^2 f^{p-1} d\nu_{g, r} \leq a \int_M f^{p+1} d\nu_{g, r}.
$$

It follows

$$
\frac{d}{d\tau} \int_M f^{p+1} d\nu_{g, r} + \int_M (R + \hat{C}_0) f^{p+1} d\nu_{g, r} + 2 \int_M |\nabla f^{p+1}|^2 \leq ((p+1)a + n + C_0) \int_M f^{p+1} d\nu_{g, r}.
$$

(7.5)

Note that (7.5) is similar to (2.6). Therefore, we can follow the argument in the proof of Lemma 2.2 to obtain (7.4).

Recall that according to [WZ2] a sequence of weak almost Kähler-Ricci solitons $(M_i, J_i, g^i, X_i)$ $(i \to \infty)$ satisfy the condition i) in (7.1) and

$$
iii) \int_{M_i} |\text{Ric}(g^i) - g^i - L_{X_i} g^i| d\nu_{g^i} \to 0, \text{ as } i \to \infty.
$$

(7.6)

As in [WZ2], we shall further assume that the solutions $g^i_t$ of (7.1) with the initial metrics $g^i$ satisfy

$$
iii) |X^i|_{g^i_t} \leq \frac{B}{\sqrt{t}};
$$

(7.7)

$$
vii) \int_0^1 dt \int_{M_i} |R(g^i_t) - \Delta g^i_t - n| d\nu_{g^i_t} \to 0, \text{ as } i \to \infty,
$$
Bergman Kernels and algebraic structure

where $B$ is a uniform constant. It was proved that under the conditions $i)$ of (7.1), and (7.6) and (7.7) there exists a subsequence of $\{(M_i, J_i, g^i, X_i)\}$ which converges to a Kähler-Ricci soliton away from singularities of Gromov-Hausdorff limit with codimension 4.

**Definition 7.3.** Fano manifolds $(M_i, J_i, g^i, X_i)$ are called a sequence of almost Kähler-Ricci solitons if (7.1), (7.6) and (7.7) are satisfied.

**Lemma 7.4.** Let $(M_i, J_i, g^i, X_i)$ be a sequence of almost Kähler-Ricci solitons. Then there exists a uniform constant $C = C(\Lambda, D, B, C_0)$ such that for any $t \in (0, 1)$ there exists $N = N(t)$ such that for any $i \geq N$ it holds

$$|\nabla h_i|^2 \leq C \quad \text{and} \quad |R^i| \leq C.$$ 

*Proof.* By

$$(\frac{\partial}{\partial t} - (\Delta + X))|\nabla (h - \theta)|^2$$

$$=-|\nabla \nabla (h - \theta)|^2 - |\nabla \nabla (h - \theta)|^2 + |\nabla (h - \theta)|^2$$

$$\leq |\nabla (h - \theta)|^2,$$ 

we apply Lemma 7.2 to get

$$|\nabla (h - \theta)|^2 \leq \frac{C}{t^{n+1}} \int_0^t \int_M |\nabla (h - \theta)|^2 dv_{g_\tau} d\tau$$

$$= \frac{C}{t^{n+1}} \int_0^t \int_M (\theta - h)\Delta (h - \theta) dv_{g_\tau} d\tau$$

$$\leq \frac{C}{t^{n+1}} \int_0^t \int_M osc_M (h - \theta) |R - n - \Delta \theta| dv_{g_\tau} d\tau.$$ 

By (2.16), it follows

$$(7.8) \quad |\nabla (h - \theta)|^2 \leq \frac{C}{t^{n+1}(n+\frac{3}{2})} \int_0^t \int_M |R - n - \Delta \theta| dv_{g_\tau} d\tau.$$ 

On the other hand, by the evolution equation of $(\Delta + X)(h - \theta)$ [CTZ],

$$(\frac{\partial}{\partial t} - (\Delta + X))[(\Delta + X)(h - \theta)]$$

$$= (\Delta + X)(h - \theta) + |\nabla \nabla (h - \theta)|^2,$$ 

we have

$$(\frac{\partial}{\partial t} - (\Delta + X))[(\Delta + X)(h - \theta) + |\nabla (h - \theta)|^2]$$

$$\leq (\Delta + X)(h - \theta) + |\nabla (h - \theta)|^2.$$
Then applying Lemma 7.2, we get
\[
(\Delta + X)(h - \theta) + |\nabla (h - \theta)|^2 \\
\leq \frac{C}{t^{n+1}} \int_{\frac{t}{2}}^{t} \int_{M} |(\Delta + X)(h - \theta) + |\nabla (h - \theta)|^2| \text{dv}_{g_{\tau}} d\tau.
\]

(7.10)

Note that by iii) in (7.7) we have
\[
\int_{\frac{t}{2}}^{t} \int_{M} |X(h - \theta)| \text{dv}_{g_{\tau}} d\tau \\
\leq B \text{vol}(M) \left[ \int_{\frac{t}{2}}^{t} \int_{M} |\nabla (h - \theta)|^2 \text{dv}_{g_{\tau}} d\tau \right]^{\frac{1}{2}}.
\]

It follows from (2.16),
\[
\int_{\frac{t}{2}}^{t} \int_{M} |(\Delta + X)(h - \theta) + |\nabla (h - \theta)|^2| \text{dv}_{g_{\tau}} d\tau \\
\leq \int_{\frac{t}{2}}^{t} \int_{M} |R - n - \Delta \theta| \text{dv}_{g_{\tau}} d\tau \\
+ CB \text{vol}(M) \frac{1}{t^{\frac{n+1}{2}}(n+\frac{1}{2})} \left[ \int_{\frac{t}{2}}^{t} \int_{M} |R - n - \Delta \theta| \text{dv}_{g_{\tau}} d\tau \right]^{\frac{1}{2}} \]

+ C \frac{1}{t^{(n+1)(n+\frac{1}{2})}} \int_{\frac{t}{2}}^{t} \int_{M} |R - n - \Delta \theta| \text{dv}_{g_{\tau}} d\tau.
\]

Thus inserting the above inequality into (7.10), we derive
\[
(\Delta + X)(h - \theta) + |\nabla (h - \theta)|^2 \\
\leq \frac{C}{t^{(n+1)(n+\frac{1}{2})}} \left( \int_{\frac{t}{2}}^{t} \int_{M} |R - n - \Delta \theta| \text{dv}_{g_{\tau}} d\tau \right.
\]

\[
+ \left[ \int_{\frac{t}{2}}^{t} \int_{M} |R - n - \Delta \theta| \text{dv}_{g_{\tau}} d\tau \right]^{\frac{1}{2}}.
\]

(7.11)

Combining (7.9) and (7.11), we see that for any $t \in (0, 1)$ there exists $N = N(t)$ such that
\[
\frac{1}{\sqrt{t}} \nabla (h - \theta) \leq 1 \text{ and } R - n - \Delta \theta \leq 1, \forall i \geq N(t).
\]

(7.12)

It follows
\[
\Delta \theta = -|\nabla \theta|^2 - X(h - \theta) - \theta \leq C.
\]

As a consequence, we get $R \leq C$, and so $|R| \leq C$. By (7.12), we have
\[
\Delta \theta \geq R - n - 1 \geq -C.
\]
Bergman Kernels and algebraic structure

Thus

$$|\nabla \theta|^2 = -X(h - \theta) - \theta - \Delta \theta \leq C.$$  

Again by (7.12), we prove that $|\nabla h| \leq C$. □

By Lemma 7.1 and the scalar curvature estimate in Lemma 7.4, we see that for any $t \in (0, 1)$ there exists an integer $N = N(t)$ such that the Sobolev constant $C_s^i$ of $g_i^t$ is uniformly bounded for any $i \geq N$. Then by the gradient estimate of Kähler potentials in Lemma 7.4, we can follow the arguments in Lemma 3.1 and Lemma 3.2 (also see Remark 3.3) to get an analogy of Proposition 3.4.

**Proposition 7.5.** Let $(M_i, g_i^t)$ be a sequence of almost Kähler-Ricci solitons which satisfy (7.1), (7.6) and (7.7). Then for any $t \in (0, 1)$ there exist integers $N = N(t)$ and $l_0 = l_0(t)$, and a uniform constant $C = C(t)$ such that for any $i \geq N$ and $l \geq l_0$ the following property holds:

$$\|s\|_{h_i^t} + l^{-\frac{1}{2}}\|
abla s\|_{h_i^t} \leq Cl^\frac{1}{2} \left( \int_{M_i} |s|^2 dv_{g_i^t} \right)^\frac{1}{2}$$  

and

$$\int_{M_i} |v|_{h_i^t}^2 \leq 4l^{-1} \int_{M_i} |\bar{\partial}\sigma|_{h_i^t}^2,$$

where $s \in H^0(M_i, K_{M_i}^{-1})$ and the norm of $|\cdot|_{h_i^t}$ is induced by $g_i^t$.

By Proposition 7.5, we can follow the arguments in Proposition 5.1 and Theorem 6.1 to prove

**Theorem 7.6.** Let $(M_i, g_i^t)$ be a sequence of almost Kähler-Ricci solitons and $(M_\infty, g_\infty)$ be their Gromov-Hasudorff limit. Then there exists an integer $l_0 > 0$ which depending only on $(M_\infty, g_\infty)$ such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ with property:

$$\rho_{l_0}(M_i, g_i^t) \geq c_l, \ \forall \ i,$$

where $l_0$ depends only on the limit $(M_\infty, g_\infty)$ and $c_l$ depends only on $(M_\infty, g_\infty)$ and $l$.

**Proof.** We give a sketch of proof of Theorem 7.6.

Step 1. By the rescaling method as in proof of Proposition 5.1 with the helps of Proposition 7.5 and the pseudo-locality theorem in [WZ2], we have an analogy of Proposition 5.1. For any sequence of $p_i \in M_i$ which converge to $x \in M_\infty$, there exist two large number $l_x$ and $i_0$, and a small time $t_x$.
Bergman Kernels and algebraic structure

such that for any \( i \geq i_0 \), there exists a holomorphic section \( s_i \in \Gamma(K^{-l}_{M_i}, h^{i}_{tx}) \) which satisfies

\[
(7.16) \quad \int_{M_i} |s_i|^2_{h^{i}_{tx}} \, dv_{g^{i}_{tx}} \leq 1 \quad \text{and} \quad |s_i|_{h^{i}_{tx} (p_i)} \geq \frac{1}{8},
\]

where \( g^{i}_{tx} \) is a solution of (7.2) with the initial metric \( g^i \) and \( h^{i}_{tx} \) is the hermitian metric of \( K^{-l}_{M_i} \) induced by \( g^{i}_{tx} \).

Step 2. We can compare the \( C^0 \)-norm of holomorphic sections with respect to the varying metrics \( g_t \) evolved in the flow (7.2). In fact, we have

**Lemma 7.7.** Let \( (M, g) \) be a Fano manifold with \( \omega_g \in 2\pi c_1(M) \) which satisfies (7.1), and \( g_t \) a solution of (7.2) with the initial metric \( g \). Then there exists a small \( t_0 = t_0(l, \Lambda, D) \) such that the following is true: if \( s \in \Gamma(M, K^{-l}_{M}) \) is a holomorphic section with

\[
(7.17) \quad \int_{M} |s|^2_{h_{t}} \, dv_{g_{t}} = 1
\]

for some \( t \leq t_0 \) which satisfies

\[
(7.18) \quad |s|_{h_{t}} (p) \geq c > 0,
\]

then there is a holomorphic section \( s' \) of \( K^{-l}_{M} \) which satisfies

\[
|s'|(h_{t}) (p) \geq c' > 0 \quad \text{and} \quad \int_{M} |s'|^2_{h} \, dv_{g} \leq c'',
\]

where \( h_t \) and \( h \) are the hermitian metrics of \( K^{-l}_{M} \) induced by \( g_t \) and \( g \), respectively, and the constants \( c' \) and \( c'' \) depend only on \( c, l, \Lambda, A, C_0 \) and \( D \).

**Proof of Lemma 7.7.** Let \( \Phi_t \) be a one-parameter subgroup generated by \( -X \). Then \( \Phi_t^* g_t \) is a solution of (2.1). It is clear that (7.17) also holds for \( \Phi_t^* s, \Phi_t^* g_t, \Phi_t^* h_t \) and the condition (7.18) is equivalent to \( |\Phi_t^* s|_{\Phi_t^* h_t} (\Phi_t^{-l} (p)) \geq c \). Since the Green functions associated to the metric \( g \) is bounded below under the condition i) of (7.1) (cf. [Ma], [CTZ]), we can follow the argument in Lemma 6.2 for the metrics \( \Phi_t^* g_t \) to obtain

\[
|\Phi_t^* s|_{h_t} (\Phi_t^{-l} (p)) \geq \tilde{c} \quad \text{and} \quad \int_{M} |\Phi_t^* s|^2_{h_t} \, dv_{g_t} \leq c'',
\]

where the constant \( \tilde{c} \) depends only on \( c, l, \Lambda, A \) and \( D \). Let \( s' = \Phi_t^* s \). Then by the gradient estimate of \( |\nabla s'| \leq C(l, \Lambda, D, C_0, A) \), we have

\[
|s'|_h (p) \geq |s'|_{h} (\Phi_t^{-l} (p)) - C(\Lambda, D, C_0, A) A t \geq c'.
\]

This proves Lemma 7.7. \( \square \)
Bergman Kernels and algebraic structure

Step 3. By using the covering argument as in Theorem 5.1, together with the results in Step 1 and Step 2, we can finish the proof of Theorem 7.6.

\[ \square \]

8. Proof of Corollary 1.4

In this section, for simplicity, we just give a proof of Corollary 1.4 in case of almost Kähler-Einstein manifolds with dimension \( n \). We assume that a sequence of almost Kähler-Einstein manifolds \( (M_i, g^i) \) with a limit \( (M_\infty, g_\infty) \) in Goromov-Hausdorff topology satisfies the partial \( C^0 \)-estimate,

\[ (8.1) \quad \rho_i(M_i, g^i) \geq c_l > 0, \]

for some integer \( l \). Then, as an application of (8.1), we have

\[ (8.2) \quad H^0(M_i, K_{M_i}^{-m}) \subseteq H^0(M_i, K_{M_i}^{-(m-l)}) \otimes H^0(M, K_{M_i}^{-l}), \]

where \( m \geq l(n + 2 + [\Lambda^2]) \) is any integer and the constant \( \Lambda \) is a uniform lower bound of Ricci curvature of \( (M_i, g^i) \) (cf. Proposition 7, [L1]).

We need a strong version of (8.1) as follows.

**Lemma 8.1.** For two different points \( x, y \in M_\infty \), there exist \( \ell = \ell(n, \Lambda, D, x, y) \), which is a multiple of \( l \), and two sections \( s_x, s_y \in H^0(M_i, K_{M_i}^{\ell}) \) such that

\[ (8.3) \quad |s_x(p_i)|_{h_i} = |s_y(q_i)|_{h_i} = 1 \quad \text{and} \quad s_x(q_i) = s_y(p_i) = 0, \]

where \( p_i \to x, q_i \to y \).

**Proof.** As in the proof of Proposition 5.1, we can choose two compact sets \( V(x; \delta^x_i), V(y; \delta^y_j) \) in \( C_x \) and \( C_y \), respectively, such that \( \phi_i \circ \psi_j(V(x; \delta^x_i)) \) and \( \phi_i \circ \psi_j(V(y; \delta^y_j)) \) are disjoint as long as \( j \) and \( i \) are large enough. Let \( v^x_i, \sigma^x_i, s^x_i \in \Gamma(M_i, K_{M_i}^{-l}) \) and \( v^y_j, \sigma^y_j, s^y_j \in \Gamma(M_i, K_{M_i}^{-l}) \) be sections associated \( x \) and \( y \), respectively. We may assume that \( l_x = l_y = \ell \) for a multiple of \( l \). Moreover, by the \( C^0 \)-estimate of \( \sigma^x_i \) in \( V(x; \delta^x) \) in (5.6), we see that \( |s^x_i(q_i)| \) is small. Similarly, \( |s^y_j(p_i)| \) is also small. Now we define holomorphic sections

\[ (8.4) \quad \tilde{s}^i_x = s^x_i - \frac{s^y_j(q_i)}{s^y_j(q_i)} s^y_i \quad \text{and} \quad \tilde{s}^i_y = s^y_i - \frac{s^x_i(p_i)}{s^x_i(p_i)} s^x_i. \]

Clearly, \( \tilde{s}_x(q_i) = \tilde{s}_y(p_i) = 0 \). Then \( s_x = \frac{\tilde{s}^i_x}{|\tilde{s}^i_x(p_i)|_{h_i}} \) and \( s_y = \frac{\tilde{s}^i_y}{|\tilde{s}^i_y(q_i)|_{h_i}} \) will satisfy (8.3).

\[ \square \]

By Lemma 8.1, we prove

\[ \square \]

\[ \text{1} \text{There is a generalization of (8.2) under the Bakry-Emery Ricci condition in Appendix.} \]
**Proposition 8.2.** Let \( \{(M_i, g^i)\} \) be a sequence of Fano manifolds with Ricci bounded from below and diameter bounded from above, and \((M_\infty, g_\infty)\) its limit in Gromov-Hausdorff topology. Suppose that \( \text{(8.1)} \) and \( \text{(8.3)} \) in Lemma 8.1 hold. Then \( M_\infty \) is homeomorphic to an algebraic variety.

**Proof.** By \( \text{(8.1)} \), for any \( k \), we can define
\[
T_{kl,i} : M_i \to \mathbb{CP}^N,
\]
where \( N+1 = \dim H^0(M_i, K_{M_i}^{-kl}) \) is constant if \( i \) is large enough. Since \( T_{kl,i} \) is uniformly Lipschitz by \( \text{(8.1)} \), we get a limit map
\[
T_{kl,\infty} : M_\infty \to \mathbb{CP}^N.
\]
On the other hand, the images \( W_i^{kl} \) of \( T_{kl,i} \) have a chow limit \( W^{kl} \), which coincides with the image of the map \( T_{kl,\infty} \). Thus \( T_{kl,\infty} \) maps \( M_\infty \) onto \( W^{kl} = T_{kl,\infty}(M_\infty) \). It remains to show that \( T_{(n+2+|\Lambda|^2)l,\infty} \) is injective.

By Lemma 8.3, for any \( x, y \in M_\infty \), there are \( p_i \to x \) and \( q_i \to y \), and \( s_x, s_y \in H^0(M_\infty, K_{M_\infty}^{-k_1l}) \) for some \( k_1 \) such that
\[
|s_x|_{h_i(p_i)} = |s_y|_{h_i(q_i)} = 1 \quad \text{and} \quad s_x(q_i) = s_y(p_i) = 0.
\]
This means \( T_{kl,\infty}(x) \neq T_{kl,\infty}(y) \). We claim
\[
T_{(n+2+|\Lambda|^2)l,\infty}(x) \neq T_{(n+2+|\Lambda|^2)l,\infty}(y).
\]
In fact, if \( \text{(8.6)} \) is not true, it is easy to see \( T_{kl,\infty}(x) = T_{kl,\infty}(y) \) for any \( i \leq n+2+|\Lambda|^2 \). Then by \( \text{(8.2)} \), it follows
\[
T_{kl,\infty}(x) = T_{kl,\infty}(y), \quad \forall \, k,
\]
which is contradict to \( \text{(8.5)} \). Thus we prove that \( T_{(n+2+|\Lambda|^2)l,\infty} \) is injective.

**Proof of Corollary 1.4.** By the Gromov-Hausdorff compactness theorem, there exists a subsequence \( \{(M_{i_k}, g^{i_k})\} \) of \( \{(M_i, g^i)\} \), which converges to \( (M_\infty, g_\infty) \). By Proposition 8.2, we know that \( M_\infty \) is an algebraic variety. We can further prove that it is a normal variety.

Let \( H^0(M_\infty, K_{M_\infty}^{-k_0l}) \) be a space of bounded holomorphic sections of \( K_{M_\infty}^{-k_0l} \) with respect to the induced metric by \( g_\infty \), where \( k = n + 2 + |\Lambda|^2 \). Then for any compact set \( K \subseteq \mathcal{R} \subseteq M_\infty \), we know that there are \( t_K > 0 \) and \( K_i \subseteq M_i \) such that \( (K_i, g_i(t_K)) \) converge to \( (K, g_\infty) \) smoothly. Thus by the argument in Proposition 5.1 and Lemma 6.2, we can identify \( H^0(M_\infty, K_{M_\infty}^{-k_0l}) \) with the limit of \( H^0(M_i, K_{M_i}^{-k_0l}) \). But, by Proposition 8.2 the latter is the same as \( H^0(W^{k_0l}, \mathcal{O}(1)) \). This will imply that \( M_\infty \) is homeomorphic to the normalization of \( W^{k_0l} \). For the details, see [T5] and [DS].

□
9. Conclusion

In the proofs of Theorem 6.1 and Theorem 7.6, the constants $c_i$ in the estimates (6.1) and (7.6) may depend on the limit $(M_\infty, g_\infty)$. In this section, we show that $c_1$ just depends on $n, l_0$ and $l$, and the geometric uniform constants $\Lambda$ and $D$ in (3.9), or the constants $\Lambda, D, C_0$ and $B$ in (7.1) and (7.7). Thus we complete the proof of Theorem 1.3. For simplicity, we just consider the case of almost Kähler-Einstein Fano manifolds below.

Set a class of Fano manifolds by

$$K_{\Lambda, D} = \{ (M^n, g) | \omega_g \in 2\pi c_1(M), \text{Ric}(g) \geq -(n-1)\Lambda^2, \text{diam}(M, g) \leq D \}.$$ 

It is known that $K_{\Lambda, D}$ is precompact in Gromov-Hausdorff topology. Moreover, by Cheeger-Colding theory in [CC], any Gromov-Hausdorff limit $M_\infty$ in $K_{\Lambda, D}$ contains singularities with codimension at least 2 and each tangent cone at $x \in M_\infty$ is a metric cone $C_x$, which also contains singularities with codimension at least 2.

Let $K^0_{\Lambda, D}$ be a subset of $K_{\Lambda, D}$ such that $H^{2n-2}(\text{Sing}(C_x)) = 0$ for any $x \in M_\infty$, where $M_\infty$ is any Gromov-Hausdorff limit in $K^0_{\Lambda, D}$. Then according to the proofs in Proposition 5.1 and Theorem 6.1, we have

**Proposition 9.1.** Let $(M, g) \in K^0_{\Lambda, D}$ and $g_t$ a solution of (2.1) with the initial metric $g$. Then there exist a small number $\delta = \delta(\Lambda, D, n)$ and a large integer $l_0 = l_0(n, \Lambda, D)$ such that the following is true: if $g$ satisfies

$$\int_0^1 \int_M |R - n| dv_{g_t} dt \leq \delta,$$

then for any integer $l$ there exists a uniform constant $c = c(n, l, \Lambda, D) > 0$ such that

$$\rho_{l_0}(M, g) \geq c.$$

**Proof.** By Theorem 6.1, we see that for any $Y \in \bar{K}^0_{\Lambda, D}$, there exist a small number $\delta_Y > 0$, a large integer $l_Y$ and a uniform constant $c_Y > 0$ such that if $M \in K_{\Lambda, D}$ satisfies

$$d_{GH}((M, g), (Y, g_Y)) \leq \delta_Y, \int_0^1 \int_M |R - n| dv_{g_t} dt \leq \delta_Y,$$

then

$$\rho_{l_Y}(M, g) \geq c_Y.$$ 

Since $\bar{K}_{\Lambda, D}$ is compact, we can cover it by finite balls $B_{Y_i}(\delta_{Y_i})(1 \leq i \leq N)$ in Gromov-Hausdorff topology. Putting $l_0 = \Pi l_{Y_i}, \delta = \min\{\delta_{Y_i}\}$ and $c = \min\{c_{Y_i}\}$. Then we get (9.2) for $l = 1$, if $(M, g)$ satisfies (9.1). (9.2) is also true for general $l$ as in the proof of Theorem 6.1.

(1.3) in Theorem 1.3 follows from (9.2).
Bergman Kernels and algebraic structure

10. Appendix

In this appendix, we use the following Siu’s lemma to generalize the finite generation formula \((8.2)\) under the Bakry-Emery Ricci condition \([Si]\).

**Lemma 10.1.** Let \((M^n, g)\) be a compact complex manifold, \(G\) a holomorphic line bundle, \(E\) a holomorphic line bundle with a hermitian metric \(e^{-\psi}\) whose Ricci curvature is positive. Let \(\{s_i\}_{1 \leq i \leq p}\) be a basis of \(H^0(M, G)\) and \(|s|^2 = \Sigma_{i=1}^p |s_i|^2\). Then for any \(f \in H^0(M, (n+k+1)G + E + K_M)\) which satisfies

\[
\int_M \frac{|f|^2 e^{-\psi}}{|s|^{2(n+k+1)}} dv_g < +\infty,
\]

there are some \(h_i \in H^0(M, (n+k)G + E + K_M)\) \((k \geq 1)\) such that \(f = \Sigma_{i=1}^p h_i \otimes s_i\) and each \(h_i\) satisfies

\[
\int_M \frac{|h_i|^2 e^{-\psi}}{|s|^{2(n+k)}} dv_g \leq \frac{n+k}{k} \int_M \frac{|f|^2 e^{-\psi}}{|s|^{2(n+k+1)}} dv_g.
\]

**Proposition 10.2.** Let \((M, g)\) be a Kähler manifold with

\[
\text{Ric}(g) + \text{Hess} u \geq -Cg,
\]

where \(X = \nabla u\) is a holomorphic vector field and \(|u| \leq A\). Assume that

\[
(10.1) \quad c' \geq \rho_l(M, g) \geq c > 0
\]

for some \(l \in \mathbb{N}\). Then for any \(s \in H^0(M, K_{M}^{-m})\) with \(m \geq (n+2)l + C + 1\), there are \(u_i \in H^0(M, K_{M}^{-m-l})\) such that \(s = \Sigma_{i=0}^N u_i \otimes s_i\), where \(\{s_i\}\) is an orthonormal basis of \(H^0(M, K_{M}^{-l})\). Moreover, each \(u_i\) satisfies

\[
(10.2) \quad \int_M |u_i|^2 h_{\otimes m-l} dv_g \leq (n+1) e^{2A} \left( \frac{c'}{c} \right)^\frac{m}{n} \int_M |s|^2 h_{\otimes m} dv_g.
\]

**Proof.** Putting \(L = K_{M}^{-1}\) and \(m - C - 1 = (n+k+1)l + r\) \((0 \leq r < l)\), we decompose \(mL\) as

\[
mL = (n+k+1)(lL) + ((m - (n+k+1)l)L - K_M) + K_M.
\]

Let \(h\) and \(\omega^n_g\) be two hermitian metrics on \(L\) such that

\[
\text{Ric}(L, h) = g, \quad \text{Ric}(L, \omega^n_g) = \text{Ric}(g).
\]

Denote the line bundle \((m - (n+k+1)l)L - K_M\) by \(E\). Then \(h_1 = h_{\otimes m-(n+k+1)l} \otimes e^{-u} \otimes \omega^n_g\) is a hermitian metric on \(E\). It is easy to see

\[
\text{Ric}(E, h_1) = (m - (n+k+1)l)\omega_g + \text{Ric}(g) + \sqrt{-1}\partial\bar{\partial} u \geq \omega_g.
\]

Now applying the above lemma to \(G = ll, s_i, E\) and \(f = s\), we see that there are \(u_i \in H^0(M, (n+k)G + E + K_M)\) such that

\[
\int_M \frac{|u_i|^2 h_{\otimes (n+k)} \otimes h_1}{(\Sigma_{i=0}^N |s_i|^2 h_{\otimes l})^{n+k}} dv_g \leq \frac{n+k}{k} \int_M \frac{|s|^2 h_{\otimes (n+k+1)} \otimes h_1}{(\Sigma_{i=0}^N |s_i|^2 h_{\otimes l})^{n+k+1}} dv_g.
\]
Bergman Kernels and algebraic structure

The above is equivalent to

\[
\int_M \frac{|u_i|_{h^{m-i}}^2}{(\Sigma_{i=0}^N |s_i|^2_{h^m})^{n+k}} e^{-u} dv_g \leq \frac{n+k}{k} \int_M \frac{|s_i|_{h^m}^2}{(\Sigma_{i=0}^N |s_i|^2_{h^m})^{n+k+1}} e^{-u} dv_g.
\]

By (10.1), it follows

\[
\frac{1}{e^{2A}} \int_M |u_i|_{h^{m-i}}^2 dv_g \leq \frac{n+k}{ke^{C_{n+k+1}}} \int_M |s_i|_{h^m}^2 dv_g,
\]

which implies (10.2) immediately. \[\Box\]

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WENSUAI JIANG, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING, 100871, CHINA

FENG WANG, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING, 100871, CHINA

XIAOHUA ZHU, SCHOOL OF MATHEMATICAL SCIENCES AND BICMR, PEKING UNIVERSITY, BEIJING, 100871, CHINA, xhzhu@math.pku.edu.cn