Deformations of WZW models

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Abstract
Current–current deformations for WZW models of semi-simple compact groups are discussed in a sigma model approach. We start with the Abelian rank one group $U(1)$. Afterwards, we keep the rank one but allow for non-Abelian structures by considering $SU(2)$. Finally, we present the general case of rank larger than one.

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1. Introduction

Often, a consistent string background belongs to a ‘family’ of consistent backgrounds which can be deformed into each other by continuously changing parameters. A simple example is the bosonic string compactified on an $n$-dimensional torus $T^n = U(1)^n$. In this case the parameters which can be continuously changed are the constant background metric and $B$ field, which can be put into an $n \times n$ matrix $G + B$. Out of these, backgrounds are equivalent if they are related by a T-duality transformation. The dualities are given by the automorphisms $O(n, n, \mathbb{Z})$ of the even self-dual charge lattice.

The aim of this paper is to consider $U(1)^n \subset G$ where $G$ is a semi-simple compact group. The deformation of the Cartan torus is a non-trivial modification of the discussion above since the geometry is not a product geometry. As a warm up we will consider the trivial case of a circle compactified string, i.e. $G = U(1)$. Next, we will move to the non-Abelian rank one group $SU(2)$ and finally the general case of a semi-simple compact otherwise arbitrary group will be presented. This paper is based on [1].

2. Rank one, dimension one: $U(1)_R$

The worldsheet action for a circle compactified direction $X$ is given by

$$ S = \frac{1}{2\pi \alpha'} \int d^2 z R^2 \partial \bar{\partial} X \partial \bar{\partial} X, $$

(1)
where $X$ is compactified on a unit circle and the radius of the actual circle is encoded in the target space metric $G_{XX} = R^2$. The corresponding equation of motion can be given the interpretation of a chiral or an anti-chiral conversation law

$$\partial_i J = \partial_i \bar{J} = 0, \quad \text{with} \quad J = \partial_i X, \quad \text{and} \quad \bar{J} = \partial_i X. \quad (2)$$

These currents are conformal primaries of dimension $(1, 0)$ and $(0, 1)$, respectively. Hence the product $J \bar{J}$ is a marginal operator. It is easy to see that an infinitesimal change in the \textquote{deformation parameter} $R$ corresponds to a marginal perturbation,

$$\delta S = \frac{R \delta R}{\pi \alpha'} \int d^2 z J \bar{J}, \quad (3)$$

and hence all circle compactifications can be obtained from the string compactified on a unit circle and exact marginal deformations. As we will see, the situation in less trivial cases is slightly more complicated as the form of the chiral and anti-chiral currents can depend on the value of the deformation parameter.

In the remainder of this section we will discuss some constructions which are trivial in this simple case but turn out to be useful in more general cases to be discussed later. First, we formulate the circle compactified string as a coset theory $(U(1) \times U(1)) / U(1)$. (In later applications the first $U(1)$ factor will be a subgroup of $G$ whose size cannot be chosen arbitrarily in a straightforward way.) The sigma model action for the product group is

$$S = \frac{1}{2 \pi \alpha'} \int d^2 z (\partial_i X \partial_i X + R^2 \partial_i Y \partial_i Y). \quad (4)$$

The coset action is obtained by first gauging the isometry $X \rightarrow X + c$ and $Y \rightarrow Y + c$. This means that we promote the global symmetry to a local one by replacing partial derivatives with covariant ones,

$$\partial_i X \rightarrow \partial_i X + A_i, \quad \partial_i Y \rightarrow \partial_i Y + A_i, \quad (5)$$

where the gauge field $A_i$ transforms as $A_i \rightarrow A_i - \partial_i c$. Next, we gauge fix, e.g. $Y = 0$ and eliminate the gauge field by solving its algebraic equation of motion. This results in

$$S = \frac{1}{2 \pi \alpha'} \int d^2 z R^2 \partial_i X \partial_i X, \quad (6)$$

with $R^2 = R^2 / (1 + R^2)$. Hence, we obtain all circle compactified strings with $R \in (0, 1)$. Radii larger than one can be generated by T-duality. The unit circle, however, can be reached only as a limit in this construction. Therefore, we will focus on a different construction in the rest of the paper, namely the orbifold construction. Let $k$ be a positive integer. Then the circle compactified models can be written as

$$\left( \frac{U(1)_k}{U(1)} \right)^k / \mathbb{Z}_k = U(1)_k / \mathbb{Z}_k = U(1)_R. \quad (7)$$

where the $\mathbb{Z}_k$ action is chosen such that it reduces the size of the circle by a factor $1/k$.

3. Rank one, dimension three: $SU(2)$

Next, we keep the rank of the group to be one but increase its dimensionality, i.e. we consider strings on an $SU(2)$ group manifold. The worldsheet action for strings on group manifolds is the WZW model action

$$S^{WZW} = S^{kin} + S^{WZ} = \frac{k}{4 \pi} \int L^{kin} + \int_R \omega^{WZ}. \quad (8)$$
where the level \( k \) is a positive integer, \( B \) is an auxiliary three-dimensional manifold whose boundary is the worldsheet \( \Sigma \) and

\[
\omega^{WZ} = \frac{1}{2} \text{Tr}(g^{-1} \partial g)^3, \quad L^{\text{kin}} = \text{Tr}(\partial_x g g^{-1}) = -(g^{-1} \partial_x g, g^{-1} \partial_x g). \tag{9}
\]

In order to be specific, we parametrize the \( SU(2) \) group element by Euler angles,

\[
g = \cos x \cos \theta - i \sin x \sin \theta \sigma^1 + i \sin x \cos \theta \sigma^2 + i \cos x \sin \theta \sigma^3, \tag{10}
\]

such that the action (8) now reads

\[
S^{\text{WZW}} = \frac{k}{2\pi} \int d^2 z \left[ \partial_x \partial_\theta \partial_\delta + \cos^2 x \partial_x \partial_\delta \partial_\theta + \cos^2 x \partial_x \partial_\delta \partial_\theta + \cos^2 x (\partial_x \partial_\delta \partial_\theta - \partial_x \partial_\delta \partial_\theta) \right]. \tag{11}
\]

Since the metric and \( B \) field do not depend on two of the coordinates there is an \( O(2, \mathbb{R}) \) group generating new conformal backgrounds. An \( O(1, 1, \mathbb{R}) \) subgroup corresponds to exact marginal deformations [2, 3]. (For general backgrounds not depending on a certain number of coordinates the discussion can be found in [4], for an algebraic discussion, see [5, 6].)

Exact marginal deformations of the \( SU(2) \) model (or its non-compact version) have also been constructed using the coset method in [7]. In either of these papers one can find the action for the deformed model to be

\[
S^R = \frac{k}{2\pi} \int d^2 z \left\{ \sin^2 x \partial_\delta \partial_\theta + \cos^2 x \partial_\delta \partial_\theta + \frac{R^2 \cos^2 x}{\cos^2 x + R^2 \sin^2 x} \partial_\delta \partial_\theta \right\}, \tag{12}
\]

The deformation parameter \( R \) runs from zero to infinity and the undeformed model (11) is obtained for \( R = 1 \). In addition, there is a non-trivial dilaton which will be discussed later. To confirm that changing \( R \) corresponds to an exact marginal deformation we first note that due to the equations of motion the following chiral and anti-chiral currents are conserved:

\[
J = k \frac{\sin^2 x \partial_\delta \partial_\theta - \cos^2 x \partial_\delta \partial_\theta}{\cos^2 x + R^2 \sin^2 x}, \quad \bar{J} = k \frac{\sin^2 x \partial_\delta \partial_\theta + \cos^2 x \partial_\delta \partial_\theta}{\cos^2 x + R^2 \sin^2 x}. \tag{13}
\]

Now, it is easy to see that an infinitesimal change in the deformation parameter corresponds to a marginal current–current perturbation,

\[
S^{R+\delta R} = S^R - \frac{k}{2\pi} \delta R^2 \int d^2 z \bar{J} J. \tag{14}
\]

From an algebraic perspective it has been argued that this class of deformed \( SU(2) \) models can be described as an orbifold [3, 8, 9],

\[
\text{Deformed model} = \left( SU(2)_k \times U(1) \right) / \mathbb{Z}_k. \tag{15}
\]

However, this is also easy to see by T-dualizing the sigma model for the orbifold [1]. The model within the bracket of (15) has the action

\[
S = \frac{k}{2\pi} \int d^2 z \left[ \partial_x \partial_x + \tan^2 x \partial_\theta \partial_\theta + \frac{1}{R^2} \partial_y \partial_y \right], \tag{16}
\]

where \( x \) and \( \theta \) are coordinates on the coset \( SU(2)/U(1) \) whereas \( y \) is compactified on a unit circle. (Further, we have employed that \( R \) and \( 1/R \) are related by duality.) Now, we T-dualize the \( \theta + y \) direction, and in doing so we will incorporate the \( \mathbb{Z}_k \) orbifold below. The first step in performing the T-duality is to gauge constant shifts in \( \theta + y \), i.e. to introduce a gauge field \( A \) and replace partial derivatives with covariant ones,

\[
\partial_\theta \rightarrow \partial_\theta + A_\theta / 2, \quad \partial_y \rightarrow \partial_y + A_y / 2. \tag{17}
\]
Next, we constrain the gauge field to be locally pure gauge $A_i = \partial_i \phi$ by adding a Lagrange multiplier term
\[ \frac{1}{2\pi} \int d^2 z \lambda F_{z\bar{z}} \] (18)
to the action, where $F_{z\bar{z}}$ is the field strength of the gauge field. In order to absorb this pure gauge into a field redefinition of $\theta$ and $y$ we have to consider the global properties of $\theta + y$ and $\phi$. Following [10], we add a topological term
\[ S_{\text{top}} = -\frac{1}{2\pi} \int d\lambda A \] (19)
and specify that $\lambda$ is compactified on a circle with radius $k$, $\lambda \equiv \lambda + 2\pi k$. For a torus worldsheet the topological term takes the form
\[ S_{\text{top}} = kn_a \oint A + kn_b \oint A, \]
where $n_{a,b}$ are the winding numbers of $\lambda$ around the two cycles of the torus labelled by $a$ and $b$. Summing over these winding modes results in the constraint that $\phi$ is compactified on a circle of radius $1/k$. In order to absorb $\phi$ in a field redefinition we have to shrink the size of the $\theta + y$ circle in the product model. This is exactly what the orbifold group $\mathbb{Z}_k$ in (15) does. Finally, integrating out the gauge field instead of $\lambda$ provides the T-dual model which is found to be (12) after replacing $\lambda = k\tilde{\theta}$.

4. Semi-simple compact groups

In this section we sketch the orbifold construction for the general case of a semi-simple compact group $G$. More details and explicit formulae can be found in [1]. Again, from algebraic considerations it can be argued that the deformed models are given by an orbifold [11],
\[ \text{Deformed model} = \frac{(G/H \times U(1)^r)}{\Gamma_k}, \] (20)
where $H$ denotes the Cartan subgroup, $r$ the rank of the group $G$ and $E = G + B$ is the constant background on the $r$-dimensional torus $U(1)^r$. The orbifold group is
\[ \Gamma_k = (\text{Weight lattice})/k(\text{Lattice of long roots}). \] (21)
The action on a representative $g$ of the coset $G/H$, for example, is
\[ \Gamma_k : g \rightarrow \exp(i\tilde{\varphi} \tilde{H}/k)g, \] (22)
where $\tilde{\varphi}$ takes values in the weight lattice. The set of Cartan generators can be diagonalized with the eigenvalues being the weights $\tilde{\mu}_i$ ($i = 1, \ldots, d - r$) ($d$ is the dimension of $G$). If $\tilde{\varphi}$ is $k$ times a long root the action is trivial since the long root lattice is dual to the weight lattice. The notation (21) can be thought of as the analogue of defining $\mathbb{Z}_k$ as $\mathbb{Z}$ mod $k$, whereas (22) is the analogue of generating $\mathbb{Z}_k$ by the $k$th root of unity. If we choose the generators of $U(1)^r$ to be the same as the Cartan generators of $G$ the orbifold acts on $U(1)^r$ as in (22) with $g$ replaced by a $U(1)^r$ element.

In order to confirm these statements on a sigma model level, we first take for $G/H$ the vectorially gauged WZW model. That is, for $h$ being an element of the Cartan subgroup, we promote the global symmetry $g \rightarrow hgh^{-1}$ of (8) to a local one by introducing gauge fields $A$ transforming as $A_i \rightarrow A_i + h\partial_i h^{-1}$. Next, we add the WZW model action for $U(1)^r$ where, as described above, it is convenient to generate a $U(1)^r$ element $y$ by the Cartan generators of $G$. For Abelian groups the WZW model does not contain the WZ term. On the $U(1)^r$ model we allow, however, for a general background such that its action is
\[ S^{U(1)^r} = \frac{k}{4\pi} \int d^2 z (y^{-1} \partial_z y, E_{y}^{-1} \partial_{\bar{z}} y), \] (23)
with \( E \) being a non-degenerate \( r \times r \) constant matrix. The next step is to perform a T-duality in this product and implement during this process also the orbifold. The global symmetry with respect to which we T-dualize is

\[
g \rightarrow fgf, \quad y \rightarrow f^2y,
\]

where \( f \) is an element of the Cartan subgroup of \( G \). This global symmetry is gauged in terms of a gauge field \( B \) transforming as \( B_i \rightarrow B_i + f^{-1} \partial_i f \). It turns out that gauging the symmetry (24) destroys the local gauge invariance under \( g \rightarrow hgh^{-1} \). (In asymmetrically gauged WZW models, usually a constraint relating the two gauge fields is imposed [12].) Here, however, we want to perform a T-duality and have to add a Lagrange multiplier term and a topological term (cf (18) and (19)). Assigning suitable transformation properties to the Lagrange multipliers repairs the vector gauge invariance. The global properties of the Lagrange multipliers are chosen such that they are compactified with respect to the long root lattice. Summing over the corresponding winding modes specifies the global properties of the gauge group with elements \( f \) such that a pure gauge can be absorbed in a field redefinition if the starting model is the orbifold (20). Finally, integrating out the gauge fields \( B \) (gauge fix, e.g., \( y = 1 \)) and the gauge field \( A \) (gauge fix, e.g., \( \lambda = 0 \)) yield the T-dual sigma model

\[
S = S^{WZW} + \frac{k}{2\pi} \int d^2z \langle P\ Ad_g - R^{-1}P\ g - 1\ \partial_g\ g - 1, P\ g - 1\ \partial\ g \rangle,
\]

where \( P \) denotes the projector on the Cartan subalgebra, \( Ad_g \) the adjoint action with \( g \) and \( R = (ET - P)/(ET + P) \). We observe that the bi-invariant metric and \( B \) field are deformed. Hence, the original \( G \times G \) chiral/anti-chiral symmetry of the WZW model is broken to \( U(1)^{r} \times U(1)^{r} \). Indeed, from (25) one finds the following conserved currents:

\[
J = kR^{-1}(1 - RR^T)(P\ Ad_g - R^{-1})^{-1}P\ g - 1\ \partial_g\ g - 1,
\]

\[
\bar{J} = -kR^{-T}(1 - R^TR)(P\ Ad_g^{-1} - R^{-T})^{-1}P\ g - 1\ \partial_g\ g.
\]

An infinitesimal change in the deformation parameter changes the action by

\[
\delta S = \frac{1}{2\pi k} \int d^2z \langle R(R^TR - 1)^{-1}(\delta R^{-1})(1 - RR^T)^{-1}RJ, \bar{J} \rangle,
\]

which corresponds to a marginal current–current operator.

We should also remark that these models have been obtained in a coset description for symmetric \( E \) in [13] and for general \( E \) in [1], again with the restriction that the undeformed model is contained only as a limiting case. To conclude this section we discuss the non-trivial dilaton present in all deformed models. There are several arguments for the source of a non-trivial dilaton. Integrating out gauge fields means solving Gaussian integrals which provide a determinant. On the other hand, when integrating up a marginal perturbation to an exact marginal deformation one should change path integral measures such that they are covariant with respect to the deformed background. In any case, the beta function equations are solved only with a non-trivial dilaton and, indeed, this is one way to compute its form. A more elegant prescription is given in [14]. The idea is to compare the physical state condition that the Hamiltonian minus some normal ordering constant should annihilate ground states with the wave equation of the corresponding effective target space field. The wave operator is a second-order differential operator on \( G \) depending on the target space metric and the dilaton. The worldsheet Hamiltonian on the other hand can be expressed in terms of affine currents [15], which act on ground state wavefunctions as left- and right-invariant vector fields [16]. Comparing the two operators acting on the ground state wavefunctions allows us to determine the dilaton. The result is (for details, see [1]) that \( e^{-2\Phi} \sqrt{G} \) does not change under the deformation. (Here, \( \Phi \) denotes the dilaton and \( G \) the determinant of the target space metric.)
5. Conclusions

We have derived the metric, $B$ field and dilaton of current–current deformed WZW models for a general semi-simple compact group. As in the torus case the deformation can be viewed as deforming an even self-dual charge lattice [1]. Hence, the moduli space is duality $\frac{O(r, r)}{O(r) \times O(r)}$. Since there are additional structures in the model the duality group is the intersection of automorphisms for even self-dual lattices $O(r, r, \mathbb{Z})$ with the self-duality group of the undeformed WZW model which is $\tilde{W} \times \tilde{W} \rtimes$ outer automorphisms, where $\tilde{W}$ denotes the affine Weyl group [17].

For the future, it is planned to add D-branes which have been studied for the $SU(2)$ case in [18]. Another interesting issue might be to add orientifolds. The action is invariant under worldsheet parity reversal if this is combined with $g \rightarrow cg^{-1}$, where $c$ is in the centre of $G$, and $E \rightarrow E^T$. Therefore, in orientifolds $E$ has to be symmetric modulo a duality transformation, i.e. the $B$ field is quantized. As in the flat case it should be interesting to study this phenomenon on group manifolds. Further possible extensions are to non-semisimple and non-compact groups.

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