Cylindrically and axially symmetric wormholes. Throats in vacuum?

K. A. Bronnikov\textsuperscript{a,b,c,} and M. V. Skvortsova\textsuperscript{a,3}

\textsuperscript{a} Center for Gravitation and Fundam. Metrology, VNII MS, Ozyornaya ul. 46, Moscow 119361, Russia;  
\textsuperscript{b} Institute of Gravitation and Cosmology, PFUR, ul. Mikhalko-Maklaya 6, Moscow 117198, Russia;  
\textsuperscript{c} I. Kant Baltic Federal University, ul. A. Nevskogo 14, Kaliningrad 236041, Russia

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This brief review discusses the existence conditions of wormhole throats and wormholes as global configurations in general relativity under the assumptions of cylindrical and axial symmetries. It is pointed out, in particular, that wormhole throats can exist in static, cylindrically symmetric space-times under slightly different conditions as compared with spherical symmetry. In cylindrically symmetric space-time with rotation, throats can exist in the presence of ordinary matter or even in vacuum; however, there are substantial difficulties in obtaining asymptotically flat wormhole configurations without exotic matter: such examples are yet to be found. Some features of interest are discussed in static, axially symmetric configurations, including wormholes with singular rings and wrongly seeming regular wormhole throats in the Zipoy-Voorhees vacuum space-time.

1 Introduction

Wormholes, the hypothetic narrow “bridges” or “tunnels” connecting different large or infinite regions of space-time or even different Universes, have become a subject of active discussion in the recent decades. Their possible existence can lead to physical effects of great interest, such as realizable time machines or shortcuts between distant parts of the Universe, in particular, across black hole horizons [1–3]. Unusual observable effects can be predicted if wormholes exist on astrophysical scales of lengths and times [4–7].

As is well known, the existence of a static wormhole geometry in the framework of general relativity requires the presence of “exotic”, or phantom matter, that is, matter violating the null energy condition (NEC), at least in a neighborhood of the throat [1–3, 8]. This conclusion, however, rests on the assumption that the throat is a compact 2D surface, having a finite (minimum) area [8]. In other words, a wormhole entrance looks from outside as a local object like a star or a black hole.

Since macroscopic exotic matter has not been observed in laboratory or in the Universe (except for the possible phantom dark energy), it is natural to try to obtain phantom-free\textsuperscript{4} wormholes (or at least throats) by abandoning some of the assumptions of the Hochberg-Visser (HV) theorem [8]. One of the ways is to reject compactness and to consider, as the simplest assumption, cylindrical symmetry. Then, instead of starlike structures, we deal with objects infinitely extended along a certain direction, like cosmic strings. One can also consider nonstatic, rotating configurations, which can be done again in the framework of cylindrical symmetry. Besides, some well-known static, axially symmetric space-times possess wormhole properties [9, 10], but all of them contain singular rings, violating the regularity requirement of the HV theorem. We here briefly describe all such space-times.

In addition, we discuss a somewhat unexpected phenomenon in another family of vacuum static, axially symmetric space-times, namely, a branch of the Zipoy-Voorhees class of solutions [9, 11]: a seeming existence of regular 2-surfaces of minimum area (i.e., throats) contrary to the HV theorem. One could even suspect that there is a loophole in the conditions of the HV theorem, by analogy with the “topological censorship” theorem [12]. It turns out, however, that the “suspicious” surfaces are not minimal, and the HV theorem works quite well in this case.

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\textsuperscript{2}e-mail: kb20@yandex.ru

\textsuperscript{3}e-mail: milenas577@mail.ru

\textsuperscript{4}That is, respecting the NEC, and with nonnegative matter density
2 Static cylindrical wormholes

In static, spherically symmetric space-times with the general metric
\[ ds^2 = A(u)dt^2 - A(u)du^2 - r^2(u)(d\theta^2 + \sin^2\theta d\varphi^2) \] (1)
(\( u \) being an arbitrary admissible radial coordinate), we say that there is a wormhole geometry if at some \( u = u_0 \) the function \( r(u) \) has a regular minimum \( r(u_0) > 0 \) (which is then called a throat) and, on both sides of this minimum, \( r(u) \) grows to much larger values than \( r(u_0) \). It is supposed that, at least in some range of \( u \) containing \( u_0 \), the function \( A(u) \) is smooth, finite and positive, which guarantees regularity and absence of horizons. The stress-energy tensor (SET) of matter compatible with this symmetry has the form
\[ T_{\mu\nu} = \text{diag}(\rho, -p_r, -p_\perp, -p_\varphi), \]
where \( \rho \) is the matter density and \( p_r \) and \( p_\perp \) are pressures in the radial and angular directions, respectively. Then, one of the Einstein equations has the form (in units with \( c = G = 1 \))
\[ 2Ar''/r = -8\pi(\rho + p_r), \] (2)
where the prime stands for \( d/du \). A minimum of \( r(u) \) requires \( r'' > 0 \), hence \( \rho + p_r < 0 \) at the throat,\(^5\) which means NEC violation. It is the simplest manifestation of the HV theorem.

In the static, cylindrically symmetric metric
\[ ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\rho(u)}d\varphi^2 \] (3)
(\( u \) is any admissible radial coordinate, \( z \in \mathbb{R} \) and \( \varphi \in [0, 2\pi] \) are the longitudinal and angular ones), there are two reasonable analogues of the squared spherical radius \( r^2 \) in (1): the squared circular radius \( r^2(u) = e^{\beta(u)} \) and the area function \( a(u) = e^{\mu+\beta} \). Accordingly [13], cylindrical wormhole throats can be defined as regular minima of \( r(u) \) (to be called \( r \)-throats) or \( a(u) \) (to be called \( a \)-throats).

Then [13], taking the most general SET compatible with (3), \( T_{\mu\nu} = \text{diag}(\rho, -p_r, -p_\perp, -p_\varphi) \), and using proper combinations of the Einstein equations, we arrive at the following conditions that must hold at and near the throat:
\[ \rho - p_r - p_\perp + p_\varphi < 0 \quad (r\text{-throat}), \] (4)
\[ \rho < p_r < 0 \quad \rho < p_r \leq 0 \quad (a\text{-throat}). \] (5)
Thus an \( r \)-throat does not necessarily require violation of any of the standard energy conditions, whereas at and near an \( a \)-throat there is always a region with negative energy density \( \rho \).

All that concerned only local conditions at the throat. However, to obtain a wormhole observable as a stringlike source of gravity from an otherwise very weakly curved or even flat environment, we should require the existence of two spatial infinities on both sides of the throat, i.e., such values \( u = u_\pm \) that \( r = e^\beta \rightarrow \infty \) and the metric is either flat or corresponds to the gravitational field of a cosmic string. In other words, at both ends of the \( u \) range we must have \( \beta \rightarrow \infty \) and finite limits of \( \gamma \) and \( \mu \).

Hence such a configuration should contain both \( r \)- and \( a \)-throats (not necessarily coinciding), and the latter inevitably requires \( \rho < 0 \) by (5). We thus obtain a no-go theorem [13]:

*A static, cylindrically symmetric wormhole with two flat or stringlike asymptotic regions cannot exist if the energy density \( \rho \) is everywhere nonnegative.*

Accordingly, all numerous examples of static phantom-free cylindrical wormholes ([13] and references therein) are not asymptotically flat and contain only \( r \)-throats.

3 Rotating cylindrical wormholes

A stationary cylindrically symmetric metric with rotation can be written as
\[ ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}d\varphi]^2 - e^{2\alpha(u)}du^2 + e^{2\rho(u)}dz^2 + e^{2\beta(u)}d\varphi^2, \] (6)
where the second line gives the three-dimensional line element. The definitions of \( r \)- and \( a \)-throats are the same as before in terms of \( e^\beta \) and \( e^\rho \). A new feature as compared to the static case is the emergence of a vortex gravitational field described as a 4-curl of the tetrad \( e^\mu_\nu \): its kinematic characteristic is the angular velocity of tetrad rotation [14]
\[ \omega^\mu = \frac{1}{2}e^{\mu\nu\rho\sigma}e^\nu_{\mu}e^\rho_{\nu}\sigma, \] (7)
where the Latin letters $m, n, \ldots$ stand for Lorentz indices. It turns out that in the gauge $\alpha = \mu$ the vortex $\omega = \sqrt{\omega_0^2 - \omega^2}$ is [15–17]

$$\omega = \frac{1}{2} (E e^{-2\gamma})' e^{\gamma - \beta - \mu}. \quad (8)$$

Furthermore, the off-diagonal component of the Ricci tensor $R^3_0$ is, in the same gauge, given by

$$\sqrt{-g} R^3_0 = (\omega e^{2\gamma + \mu})' g := \det (g_{\mu \nu}). \quad (9)$$

Assuming that our rotating reference frame is comoving to the matter source of gravity, that is, the azimuthal flow $T^3_0 = 0$, we find from $R^3_0 = 0$ that

$$\omega = \omega_0 e^{-\mu - 2\gamma}, \quad \omega_0 = \text{const.} \quad (10)$$

As a result [17], the diagonal components of the Ricci tensor $R^\mu_\mu$ can be written as the corresponding components $s R^\mu_\mu$ for the static metric (3) plus the $\omega$-dependent addition

$$\omega R^\mu_\mu = \omega^2 \text{diag}(-2, 2, 0, 2), \quad (11)$$

The Einstein tensor $G^\mu_\nu = R^\mu_\nu - \frac{1}{2} g^\mu_\nu R$ splits in a similar manner, $G^\mu_\nu = s G^\mu_\nu + \omega G^\mu_\nu$, where

$$\omega G^\mu_\nu = \omega^2 \text{diag}(-3, 1, -1, 1). \quad (12)$$

One can check that the tensors $s G^\mu_\nu$ and $\omega G^\mu_\nu$ (each separately) satisfy the “conservation law” $\nabla_\alpha G^\alpha_\mu = 0$ with respect to the static metric (3).

Then, according to the Einstein equations $G^\mu_\nu = -8\pi T^\mu_\nu$, the tensor $\omega G^\mu_\nu/(8\pi)$ behaves as an additional SET with very exotic properties (thus, the effective energy density is $-3\omega^2/(8\pi) < 0$), acting in the auxiliary metric (3). In its presence, the existence conditions for wormhole throats read

$$p - p_r - p_\phi + p_\varphi < \omega^2/(4\pi) \quad (r-\text{throat}),$$

$$p - p_r < \omega^2/(4\pi), \quad p_r \leq \omega^2/(8\pi) \quad (a-\text{throat}). \quad (14)$$

It is much easier to fulfill them than in the static case, as confirmed by a number of examples [15–17].

For example, if matter is represented by a massless, minimally coupled scalar field with the Lagrangian $L_s = g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$, the Einstein-scalar equations lead to the following solution with the metric (6):

$$e^{2\beta} = \frac{e^{2hu}}{2 \omega_0 s(k, u)}, \quad e^{2\gamma} = \frac{e^{2hu}}{e^{2\gamma}}, \quad e^{2\gamma} = \frac{e^{2hu}}{e^{2\gamma}}, \quad e^{2\alpha} = \frac{e^{4h - 2m}}{u}, \quad \omega = \frac{e^{m - 2hu}}{2s(k, u)}, \quad \phi = C u,$$

$$E = e^{2hu} [E_0 s(k, u) - s'(k, u)], \quad (15)$$

where the function $s(k, u)$ is defined as

$$s(k, u) = \begin{cases} k^{-1} \sinh ku, & k > 0, \ u \in \mathbb{R}_+; \\ u, & k = 0, \ u \in \mathbb{R}_+; \\ k^{-1} \sin ku, & k < 0, \ 0 < u < \pi/|k|; \end{cases} \omega_0, E_0, h, k, m, C \text{ are integration constants obeying the relation}$$

$$k^2 \text{sign } k = 4(h^2 - 2hm) - 32\pi C^2. \quad (16)$$

This solution is written using the harmonic coordinate $u$ corresponding to the gauge condition $\alpha = \beta + \gamma + \mu$. If the scalar charge $C$ is zero, it is a vacuum solution.

In all branches of the solution, $r \to \infty$ and $e^{\gamma} \to 0$ as $u \to 0$. In the same limit, the vortex $\omega \to \infty$, indicating a singularity. At the other end of the $u$ range, the situation is more diverse:

1. $k > 0$. At large $u$, $e^{2\beta} \sim e^{(2h-k)u}$ and $e^{2\gamma} \sim e^{(2h+k)u}$, hence a wormhole with an $r$-throat exists if $0 < k < 2h$; we also have $e^\gamma \to \infty$ at large $u$. A wormhole with an $a$-throat exists if $0 < k < 2(h - m)$. 

2. $k = 0$. At large $u$ we have $e^{2\beta} \sim u^{-1} e^{2hu}$ and $e^{2\gamma} \sim u e^{2hu}$, hence we have a wormhole geometry with an $r$-throat if $h > 0$ and with an $a$-throat if $h - m > 0$. In addition, $e^\gamma \to \infty$ at large $u$. 

3. $k < 0$. A wormhole geometry (with both kinds of throats) is described by all solutions with $k < 0$. At both ends, $e^\beta \to \infty$ and $e^{\gamma} \to 0$, while $e^{3+\gamma}$ and $e^{\mu}$ remain finite, and $\omega \sim e^{-2\gamma} \to \infty$.

We conclude that rotating cylindrical vacuum and scalar-vacuum space-times are quite generically of wormhole nature, and, in particular, the question asked in the title is answered “yes”.

Though, none of these rotating wormhole solutions are asymptotically flat. An attempt to remove this shortcoming was made in [17] by using a cut-and-paste procedure: on both sides of the throat, a wormhole solution is matched to a properly chosen region of flat space (in a rotating reference frame) at some surfaces $\Sigma_-$ and $\Sigma_+$. It was shown, however, that if the throat region is described by the above vacuum or scalar-vacuum solutions, one or both thin shells appearing on $\Sigma_-$ and $\Sigma_+$ inevitably violate the NEC. Thus, although phantom-free rotating wormhole solutions are easily found, exotic matter is still necessary for obtaining asymptotic flatness.
4 Static, axially symmetric wormholes and throats

Consider the Zipoy-Voorhees vacuum static, axially symmetric space-times [9,11] with the metric

$$ds^2 = e^{2\gamma}dt^2 - L^2 e^{2\gamma+2\eta}(x^2 + \varepsilon y^2)$$

$$\times \left( \frac{dx^2}{x^2 + \varepsilon} + \frac{dy^2}{1 - y^2} \right) - L^2 e^{-2\gamma}(x^2 + \varepsilon)(1 - y^2)d\phi^2,$$

where $L$ is a certain (arbitrary) length scale, $\varepsilon = 0, \pm 1$ designates three branches of the solution; $y \in (-1,+1)$ is a coordinate resembling latitude in spherical symmetry, $\phi \in [0,2\pi)$ is the azimuthal angle, while $x$, whose range depends on $\varepsilon$, resembles the spherical radius.

For $\varepsilon = -1$, the range of $x$ is $x > 1$ and

$$e^{2\gamma} = \left( \frac{x - 1}{x + 1} \right)^m, \quad e^{2\eta} = \left( \frac{x^2 - 1}{x^2 - y^2} \right)^{m^2};$$

for $\varepsilon = 0$, $x > 0$, and

$$e^{2\gamma} = e^{-2m/x}, \quad e^{2\eta} = e^{-m^2(1-y^2)/x^2};$$

for $\varepsilon = +1$, $x \in \mathbb{R}$ and

$$e^{2\gamma} = e^{-2m\cot^{-1}x}, \quad e^{2\eta} = \left( \frac{x^2 + y^2}{x^2 + 1} \right)^{m^2}. \quad (20)$$

All three families have a Schwarzschild asymptotic at large $x$ with the mass $M = Lm$. The whole Schwarzschild solution is, however, only restored from the family $\varepsilon = -1$, in the case $m = 1$ (so that $M = L$), and the standard coordinates $r$ and $\theta$ are given by $r = Lx$ and $y = \cos \theta$.

In the family $\varepsilon = +1$, the second spatial infinity $x \to -\infty$ is also flat, but the Schwarzschild mass is there equal to $-M$. (This feature is similar to that of the so-called anti-Fisher wormholes, described by solutions to the Einstein-scalar equations with a massless phantom scalar field [3, 18–20].) The space-time is everywhere regular except on the ring $x = 0$, $y = 0$ whose radius is equal to $e^{-m}L$. The whole configuration is of wormhole type and has been named [10] a ring wormhole. The disk $x = 0$ plays the role of a throat, and the singularity at its edge ($y = 0$) is actually a price paid for wormhole existence in vacuum instead of exotic matter. Such ring wormholes have been discussed in a much more general context of D-dimensional Einstein and dilaton gravity in [10].

It should be stressed that $x > 0$ is not the upper half-space, as could be wrongly imagined, but the whole 3-space sheet, connected with another sheet, $x < 0$, through the disk $x = 0$. Crossing this disk (i.e., threading the ring), a hypothetic observer leaves one world and enters the other. To return back, he or she should thread the ring once again, not necessarily from the same side out of which he/she has appeared in this second world.

Let us now discuss the family $\varepsilon = -1$. In this case, there is no second spatial infinity, instead, the limit $x \to 0$ is an attracting singularity due to $g_{tt} \to 0$ (except for the case $m = 1$ where it is the Schwarzschild horizon). However, the radii of circles $x = \text{const}$, $y = \text{const}$ infinitely grow as $x \to 0$ if $m > 1$. Moreover, the area of 2D sections $x = \text{const}$ diverges as $x \to 0$ in the case $m > 2$. Recalling the definition of a throat as a 2D surface of minimum area, one can suspect that there must be a throat at some finite $x$, contrary to the HV theorem. The situation is illustrated by Fig.1, showing that the surfaces $x = \text{const}$ do have a minimum area if $m > 2$ and do not if $m \leq 2$.

A further study shows, however, that a surface $x = \text{const}$ cannot be a throat. Indeed, the minimality condition in terms of the extrinsic curvature $K_{ab}$ of a 2-surface in 3D space is that the trace of $K_{ab}$ should be zero [8], which, for a surface $x = \text{const}$, leads to $(\partial/\partial x)(\ln |g_{yy}| + \ln |g_{zz}|) = 0$. This condition gives a certain function $x(y)$ with a distinct $y$ dependence, contrary to the assumption $x = \text{const}$. That is, the assumption that some surface $x = \text{const}$ is a throat leads to a contradiction.

For a more general 2-surface $x = x(y)$, the area is given by a functional of $x(y)$,

$$A = 2\pi \int_{-1}^1 dy \left[ g_{\phi\phi} \left( \frac{dx}{dy} \right)^2 g_{xx} + g_{yy} \right]^{1/2}, \quad (21)$$

![Figure 1](image-url)
Figure 2: Areas $A$ of surfaces $x(y) = a + by^2$ in spaces (17) $\varepsilon = -1$ and with $m = 3$ vs. $a$ and $b$. A minimal surface is absent. The section $b = 0$, at which $x = \text{const}$ (the right edge), has a minimum where $A \approx 478$ in the units used, while at the lowest point on the lower edge of the plot (this surface touches the singularity $x = 1$) $A \approx 382$.

whose minimum would describe a throat. The corresponding variational equation is too complex to try to solve it. Instead, we can use the simplest version of Ritz’s direct approximation method, namely, to consider the family of surfaces

$$x(y) = a + by^2, \quad a = \text{const} \geq 1, \quad b = \text{const}$$

and seek a minimum of $A$ among different pairs of numbers $(a,b)$. A numerical calculation leads to the conclusion that no such minimum exists for any $m$. The situation is illustrated by Fig. 2 showing the areas $A$ of the surfaces (22) for $m = 3$. Consequently, there are no throats in the $\varepsilon = -1$ vacuum space-times with the metric (17), in full agreement with the HV theorem.

In conclusion, we would like to mention two more important classes of axially symmetric wormholes with phantom sources that are not described here, namely, rotating axially symmetric wormholes (see [21,22] and references therein) and static multi-wormhole space-times [23].

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