The optical model of $NN$ interaction without cut-off radius.

O.D. Dalkarov, A. Yu. Voronin
P. N. Lebedev Physical Institute
53 Leninsky pr., 117924 Moscow, Russia

We suggest a regular potential model of $NN$ interaction without any cut-off. The effect of singular terms of OBE potential, modified by annihilation is shown to be repulsive. The experimental data for S- and P-wave scattering lengths are well reproduced.

I. INTRODUCTION

During the last decades numerous nonrelativistic models of $NN$ low energy interaction [1, 2, 3, 4, 5, 6] have been suggested as well as great experimental efforts have been made on the Low Energy Antiproton Ring (LEAR) at CERN. An intriguing problem of possible existing of quasi-nuclear $NN$ states [7] strongly stimulated the mentioned researches. The physical arguments in favor of quasi-nuclear states are the following. The interaction between $N$ and $N$ should be much more attractive than the $NN$ interaction, as it follows from the procedure of $G$-conjugation [1]. Such a strong attraction should produce a spectrum of $NN$ quasi-bound states (so called baryonium). In the same time the range of annihilation, estimated from the position of the nearest to the threshold singularity in Feynman annihilation diagrams, is much smaller than the range of meson exchange forces. This means that baryonium states could be rather narrow to be observed experimentally. It was indeed discovered in LEAR experiments that certain partial waves contain narrow nearthreshold resonances (so called $P$-wave enhancement). Unfortunately, this transparent physical picture has a significant drawback. The $G$-conjugation of $NN$ OBE potential yields in attractive singular terms in $NN$ potential of the type $1/r^3$. (In case of $NN$ these terms are repulsive and play a role of so called short range core). It is well known, that attractive singular potentials produce a collapse [8], i.e. the spectrum of the system is not bounded from below, while the scattering problem has no definite solution. The usual way of dealing with such pathological potentials is to impose, that singular behavior is an artifact of certain approximations (for instance nonrelativistic approximation).

In the absence of the self-consistent theory it is common practice to introduce the phenomenological cut-off radius to regularize the singular behavior of the model at short distance. However the results change dramatically with small variations of the cut-off radius\([9]\), which seriously diminish predictive power of the model. In fact it is not clear if the nearthreshold states are determined by the ”physical part” of the OBEP, or they are artifacts, produced by the ”non-physical”, singular part of the interaction.

The aim of the present study is to analyze the role of singular interaction and suggest a model of $N\bar{N}$ interaction which is free from the non-physical cut-off parameters. The main idea of our approach is that strong enough short range annihilation regularizes the singular attractive potentials, as far as the particle annihilates rather than falls to the center. Mathematically this means that singular potential can be regularized by complex addition to the coupling constant. The main properties of the potentials of the type $-\left(\alpha + i\omega\right)/r^s$, $s \geq 2$ were studied in [10]. It was shown that such potentials with $\omega \neq 0$ become regular and the scattering on such a potential is equivalent to the full absorption of the particles in the scattering center. These results enable to suggest a regular potential model of $N\bar{N}$ interaction, based on OBE potential, but without any cut-off radius. We will show that the overall effect of the ”singular” part of potential, modified by annihilation is repulsive and thus cannot be responsible for new ”false” states. The nearthreshold resonances, which are well reproduced by our model, are determined mainly by long range part of OBEP. We calculate the scattering lengths in S- and P- partial waves for different values of spin, isospin and total momentum and demonstrate that our model reproduces rather well the experimental data.

II. SINGULAR POTENTIALS WITH COMPLEX STRENGTH.

In this section we present the main results concerning the properties of singular potentials with a complex strength. In the following we put $2M = 1$ and consider the interaction strength to be complex $\alpha_s = \text{Re} \alpha_s \pm i\omega$. Near the origin we can neglect all the nonsingular terms of the Shrödinger equation, increasing slower than $1/r^2$. One get the following expression for the wave-function near the origin [11]:
\[ \Phi(r) = \sqrt{r} \left( H^{(1)}_\mu(z) + \exp(2i\delta_0)H^{(2)}_\mu(z) \right) \] (1)

\[ z = \frac{2\sqrt{\omega}}{s - 2} \] (2)

\[ \mu = \frac{2l + 1}{s - 2} \] (3)

Here \( H^{(1)}_\mu(z) \) and \( H^{(2)}_\mu(z) \) are the Hankel functions of order \( \mu \). \( \delta_0 \) is a contribution of the short range part of the singular potential into the scattering phase. It is worth to mention that the variable \( z \) is a semiclassical phase.

Let us replace the singular potential at distance less than \( r_0 \) by the constant \( -\alpha/r^s_0 \), having in mind to tend \( r_0 \to 0 \). Matching the logarithmic derivatives for the "square-well" solution and the solution (1) at small \( r_0 \), one can get for \( \delta_0 \):

\[ \delta_0 = p(r_0)r_0 \] (4)

\[ p(r_0) = \frac{\sqrt{\text{Re} \alpha_s \pm i\omega}}{r_0^{s/2}} \] (5)

Now it is important that the interaction strength \( \alpha_s \) is complex. In the limit \( r_0 \to 0 \) we obtain:

\[ \lim_{r_0 \to 0} \text{Im} \delta_0 = \text{Im} \left( \frac{\sqrt{\text{Re} \alpha_s \pm i\omega}}{r_0^{(s-2)/2}} \right) \to \pm \infty \] (6)

which means, that \( \exp(2i\delta_0) \) is either 0 or \( \infty \) and the linear combination of the Schrödinger equation solutions (1) is uniquely defined in the limit of zero cut-off radius \( r_0 \):

\[ \lim_{r_0 \to 0} \Phi(r) = \begin{cases} \sqrt{r}H^{(1)}_\mu(z) & \text{if } \omega > 0 \\ \sqrt{r}H^{(2)}_\mu(z) & \text{if } \omega < 0 \end{cases} \] (7)

One can see, that \( \omega > 0 \) selects an incoming wave, which corresponds to the full absorption of the particle in the scattering center, while \( \omega < 0 \) selects an outgoing wave, which corresponds to the creation of the particle in the scattering center.

As one can see from (1) and (5) due to the singular character of our potential \( s > 2 \), the above conclusions are valid for any infinitesimal value of \( \omega \). It means, that the sign of an infinitesimal imaginary addition to the interaction constant selects the full absorption or the full creation boundary condition (7). This boundary condition can be formulated in terms of the logarithmic derivative in the origin:

\[ \lim_{r \to 0} \frac{\Phi'(r)}{\Phi(r)} = -i \text{sign}(\omega)p(r) \] (8)

where \( p(r) \) is a classical local momentum (5). (Compare with plane incoming (outgoing) wave boundary condition \( \exp(\mp ipr)/\exp(\mp ipr) = \mp ip \)).

As soon as solution of the Schrödinger equation is uniquely defined, we can calculate the scattering observables. In particular we can now obtain the S-wave scattering length for the potential \(-\alpha_s/r^s \) (for \( s > 3 \)):

\[ a = \exp(\mp ip/(s-2)) \left( \frac{\sqrt{\alpha_s}}{s - 2} \right)^{2/(s-2)} \frac{\Gamma((s-3)/(s-2))}{\Gamma((s-1)/(s-2))} \] (9)

The fact, that in spite \( \text{Im} \alpha_s \to \pm 0 \) the scattering length has nonzero imaginary part is the manifestation of the singular properties of the potential which violates the self-adjointness of the Hamiltonian.

Let us compare the scattering length (9) with that of the repulsive singular potential \( \alpha_s/r^s \). One can get:

\[ a^{rep} = \left( \frac{\sqrt{\alpha_s}}{s - 2} \right)^{2/(s-2)} \frac{\Gamma((s-3)/(s-2))}{\Gamma((s-1)/(s-2))} \] (10)

It is easy to see, that (9) can be obtained from (10) simply by choosing the certain branch (corresponding to an absorption or a creation) of the function \( (\sqrt{\alpha_s})^{2/(s-2)} \) when passing through the branching point \( \alpha_s = 0 \). The
scattering length in a regularized singular potential becomes an analytical function of $\alpha_s$ in the whole complex plane of $\alpha_s$ with a cut along positive real axis. One can see, that the presence of an inelastic component in the singular potential acts in the same way, as a repulsion. It suppresses one of two solutions of the Schrödinger equation and thus eliminates the collapse.

It is easy to see, that the boundary condition of the full absorption (creation) is incompatible with the existence of any bound state. Indeed, one needs both incoming and reflected wave to form a standing wave, corresponding to a bound state. This means, that the regularized singular potential supports no bound states. This is also clear from the mentioned above fact, that the scattering length for a regularized attractive singular potential is an analytical continuation of the scattering length of a repulsive potential.

A. Potential $-\alpha^2/r^2$

Let us now turn to the very important case $-\alpha/r^2$. The wave-function now is:

$$\Phi = \sqrt{r} \left[ J\nu_+ (kr) + \exp(2i\delta_0) J\nu_- (kr) \right]$$

$$\nu_{\pm} = \pm \sqrt{1/4 - \alpha^2}$$

where $k = \sqrt{E}$, and $J\nu_{\pm}$ are the Bessel functions [12]. In the following we will be interested in the values of Re $\alpha^2$ greater than critical Re $\alpha^2 > 1/4$. We use the same cut-off procedure at small $r_0$. Matching the logarithmic derivatives at cut-off point $r_0$ we get for $\exp(2i\delta_0)$:

$$\lim_{r_0 \to 0} \exp(2i\delta_0) = r_0^{\nu_+ - \nu_- \text{ const}} \sim r_0^{2\nu}$$

One can see, that due to the presence of an imaginary addition $\omega$ we get Im $\delta_0 \to \pm \infty$ when $r_0 \to 0$. Again we come to the boundary condition:

$$\lim_{r_0 \to 0} \Phi(r) = \begin{cases} \sqrt{r} J\nu_+ (kr) & \text{if } \omega > 0 \\ \sqrt{r} J\nu_- (kr) & \text{if } \omega < 0 \end{cases}$$

where $\nu_{\pm} = \pm \sqrt{1/4 - \alpha^2}$

For the large argument this function behaves like:

$$\Phi \sim \cos(z - \nu_{\pm} \pi/2 - \pi/4)$$

The corresponding scattering phase is:

$$\delta = \frac{\pm i\pi}{2} \sqrt{\alpha^2 - 1/4 + \pi/4}$$

As one can see, the S-matrix $S = \exp(2i\delta)$ is energy independent. This means that the regularized inverse square potential supports no bound states. The regularized wave-function and the phase-shift are analytical functions of $\alpha^2$ in the whole complex plane with a cut along the axis Re $\alpha^2 > 1/4$.

B. Critical strength of inelastic interaction

Now we would like to determine ”how strong” should be annihilation to regularize the attractive singular potential of order $s$. In other words we would like to find the minimum order of singularity of an infinitesimal imaginary potential required for the regularization of given singular potential. The potential of interest is a sum $-\alpha^s/r^s \mp i\omega/r^t$. Here we keep $\alpha_s$ real. From expressions [4] one immediately comes to the conclusion that the regularization is possible only if $t > s/2 + 1$.

The above statement makes clear the physical sense of suggested regularization. The scattering is insensitive to any details of the short range modification of a singular interaction if the inelastic component of such an interaction behaves more singular than $-1/r^{(s/2+1)}$. 
C. Singular potential and WKB approximation.

The WKB approximation holds if \( |\frac{\partial(1/p)}{\partial r}| \ll 1 \). In case of the zero-energy scattering on a singular potential with \( s > 2 \) this condition is valid for \( r \ll r_{sc} \equiv (2\sqrt{\alpha_s}/s)^{2/(s-2)} \), i.e. near the origin. (For \( s = 2 \) the semiclassical approximation is valid only for \( \alpha \gg 1 \)). The WKB approximation, consistent with the boundary condition \( 7 \) for \( s > 2 \) is:

\[
\Phi = \frac{1}{\sqrt{p(r)}} \exp(\pm i \int \frac{p(r)dr}{r}) \tag{15}
\]

with \( p(r) \) from 15. It follows from the above expression, that in case the WKB approximation is valid everywhere the solution of the Schrödinger equation includes incoming wave only (for distinctness we speak here of absorptive potential). The corresponding S-matrix is equal to zero \( S = 0 \) within such an approximation and insensitive to any modifications of the inner part of potential \( p^2(r) \). The outgoing wave can appear in the solution only in the regions where \( \frac{1}{r} \frac{\partial(1/p)}{\partial r} \gg 1 \)

For the zero energy scattering and \( l = 0 \) this holds for \( r \geq r_{sc} \).

The reflection coefficient \( P \equiv |S|^2 = \exp(-4 \text{Im} \delta) \) which shows the reflected part of the flux has the following form in the low energy limit:

\[ P = 1 - 4k \text{Im} \alpha \]

For the energies \( E \gg E_{sc} \equiv (s/2)^{2s/(s-2)} \alpha_s^{-2/(s-2)} \) the WKB holds everywhere and S-wave reflection becomes exponentially small.

An important conclusion is that any information, which comes from the scattering on an absorptive singular potential is due to a quantum reflection from the tail of such a potential.

III. OPTICAL MODEL OF N\(\bar{N}\) INTERACTION.

From the above results it is clear that the model potential, which behaves at short distance like \( -(\alpha + i\omega)/r^3 \) is regular, i.e. it enables definite unique solution of the scattering problem. Such a potential is absorptive and describes not only elastic, but inelastic scattering as well. We suggest the following model potential:

\[
W = V_{KW} - i \frac{A}{r^3} \exp(-r/\tau) \tag{16}
\]

here \( V_{KW} \) is the real part of Kohno-Weise version of OBE potential[4], but without any cut-off. The parameters of imaginary part of the potential were chosen as follows: \( A=4.7 \text{ GeV} \text{fm}^2 \), \( \tau = 0.4 \text{ fm} \). We have calculated the values of S- and P-scattering lengths in our model potential. The obtained results, (indicated as Reg) together with the results of two Dover-Richard models, (DR1 and DR2), and Kohno-Weise model (KW) are presented in the table.

| State | DR1  | DR2  | KW   | Reg  |
|-------|------|------|------|------|
| \(1\) | S0   | 0.02-i1.12 | 0.1-i1.06 | -0.03-i1.35 | -0.08-i1.16 |
| \(3\) | S1   | 1.17-i0.51 | 1.2-i0.57 | 1.07-i0.62 | 1.05-i0.55 |
| \(1\) | S1   | 1.16-i0.46 | 1.16-i0.47 | 1.24-i0.63 | 1.19-i0.64 |
| \(3\) | S1   | 0.86-i0.63 | 0.87-i0.67 | 0.71-i0.76 | 0.7-i0.65 |
| \(1\) | P1   | -3.33-i0.56 | -3.28-i0.78 | -3.36-i0.62 | -3.19-i0.59 |
| \(3\) | P1   | 0.92-i0.5 | 1.02-i0.43 | 0.71-i0.47 | 0.81-i0.46 |
| \(1\) | P0   | -9.58-i5.2 | -8.53-i3.51 | -8.83-i4.45 | -7.67-i4.74 |
| \(3\) | P0   | 2.69-i0.13 | 2.67-i0.15 | 2.43-i0.11 | 2.46-i0.15 |
| \(1\) | P1   | 5.16-i0.08 | 5.14-i0.09 | 4.73-i0.08 | 4.75-i0.15 |
| \(3\) | P1   | -2.08-i0.86 | -2.02-i0.7 | -2.17-i0.95 | -2.09-i0.79 |
| \(1\) | P2   | 0.04-i0.57 | 0.22-i0.56 | -0.03-i0.88 | -0.12-i0.82 |
| \(3\) | P2   | -0.1-i0.46 | 0.05-i0.55 | -0.25-i0.39 | -0.14-i0.39 |
One can see rather good agreement between the results obtained within the suggested optical model without cut-off and cited above versions of Kohno-Weise and Dover-Richard models.

IV. CONCLUSION

We have found that the scattering observables are insensitive to any details of the short range modification of a singular interaction, if such an interaction includes strong inelastic component. In this case the scattering amplitude can be calculated by solving the Schrödinger equation with the regularized singular potential $-(\alpha_s \pm i0)/r^s$. It was shown that the low energy scattering amplitude on such a potential is determined by the quantum scattering from the region, where WKB approximation fails (the potential tail). The mentioned formalism was used to build an optical model of $NN$ low energy interaction free from uncertainty, related to the cut-off parameter. The good agreement of the results, obtained within our regularization method and within different $NN$ interaction models proves that the long range part (pionic tail) of OBEP plays essential role for the nearthreshold scattering.

V. ACKNOWLEDGEMENT

The research was performed under support of Russian Foundation for Basic Research grant 02-02-16809.

[1] R.J.N. Phillips, Rev. Mod. Phys.39 (1967) 681
[2] O.D. Dalkarov, F. Myhrer, Il Nuovo Cimento 40A (1977), 152
[3] J.Carbonell, O. Dalkarov, K. Protasov, I. Shapiro, Nucl. Phys. A535 (1991), 651
[4] M.Kohno, W. Weise, Nucl. Phys. A454 (1986) 429
[5] Dover, C.B., Richard J.-M.: Phys. Rev. C21, 1466 (1980)
[6] M. Pignone et. al. Phys. Rev. C50 (1994), 2710
[7] I.S. Shapiro, Phys. Rep. 35C (1978) 129
[8] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Pergamon Press, N.Y. 1977
[9] J. Carbonell, Revista Mexicana de Fisica 47 (2001) 70-77, Selected topics on Low Energy Antiproton Physics, e-print arXiv: nucl-th/0103043
[10] A. Yu. Voronin, Phys.Rev.A 67,062706 (2003)
[11] M.F. Mott, H.S.W. Massey, The theory of atomic collisions, Oxford, At The Clarendon Press, 1965
[12] G.N. Watson, A treatise on the theory of Bessel functions, Cambridge University Press, 1922
[13] K.Meetz, Nuov.Cim. 34(1964),690
[14] W.M.Frank, D.J. Lund, R.M. Spector, Rev. Mod. Phys. 43, 36 (1971)
[15] A.M. Perelomov, V.S. Popov, Theor. Mat. Phys.(USSR) 4(1970),664
[16] E. Vogt, G.M. Wannier, Phys.Rev. 95 (1954), 1190
[17] V.D. Skarzhinsky, J. Audretsch, Singular potentials and absorption problem in quantum mechanics, arXiv:quant-ph/0012004 (2000)