Geometrical symmetries in atomic nuclei:  
From theory predictions to experimental verifications

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Abstract. In the lectures delivered at the 2012 Predeal School an overview has been presented of the contemporary theory of the nuclear geometrical (shape) symmetries. The formalism combines two most powerful theory tools applicable in the context: The group- and group-representation theory together with the modern realistic mean-field theory. We suggest that all point-groups of symmetry of the mean-field Hamiltonian, sufficiently rich in symmetry elements (as discussed in the text) may lead to the magic numbers that characterise such a group in analogy with the spherical magic gaps characterising nuclear sphericity. We discuss in simple terms the mathematical and physical arguments for the presence of such symmetries in nuclei. In our opinion: It is not so much the question of Whether? - but rather: Where in the Nuclear Chart several of the point group-symmetries will be seen? We focus our presentation on the tetrahedral symmetry with the magic numbers calculated to be 32, 40, 56, 64, 70, 90 and 136, and discuss qualitatively the problem of the formulation of the experimental criteria which would allow for the final discovery of the tetrahedral symmetry in subatomic physics.

1. Introduction: Nuclear interactions and nuclear shapes
The notion of geometrical symmetries in physics is usually associated with the notion of shapes. In the present context these will be the nuclear shapes. However, already at this point an attentive reader may become alert. Indeed, the nuclei are quantum systems, whereas shapes are usually identified with 3-dimensional surfaces, say Σ, with a generic equation of the form: \( f(x, y, z) = 0 \), the latter usually referred to as ‘classical’. This is because, strictly speaking, there is no difference in terms of the mathematical structure between \( \Sigma \) referring to, e.g., the surface of the rugby-ball and \( \Sigma \) - to a nuclear form. And yet: Some physicists talk about shapes of nuclei as if the latter were metal toys... But even if it should make sense to talk about surfaces describing ‘shapes of quantum systems’: To what extent should such imaginary surfaces be useful to inform us about nuclear interactions?

To approach the solution of our puzzle recall, that according to the results of numerous experiments: a. The volume of a nucleus is approximately equal to the sum of the volumes of the constituent nucleons, what implies that nuclei can be considered as ‘tightly packed’ with nuclear-matter balls (read: nucleons) of \( \approx 2.5 \) Fm diameter; b. Independently, the strong nuclear interactions in the vacuum are of a short range, comparable with the nucleonic sizes. Because
of the short-range of the nuclear interactions, nucleonic density in a nucleus, \( \rho(x, y, z) \), falls off to zero within a very narrow layer, sometimes referred to as nuclear skin. The existence of such a thin layer in which nucleonic density decreases from, say 90% to, say, 10% of the maximum value suggests that the associated notion of an auxiliary surface \( \Sigma \), defined by the equation:

\[
\rho(x, y, z) = \frac{1}{2} \rho_{\text{max}},
\]

can be used to depict conveniently the area of space in which bulk of the nucleons can be found i.e. inside of such a surface.

Let us consider Hamiltonian of an \( A \)-body system, \( \hat{H} \), represented with the help of the two-body nucleon-nucleon interactions:

\[
\hat{H} = \hat{t} + \frac{1}{2} \sum_{i \neq j} \hat{v}_{ij}.
\]

Such a hypothesis may seem simplistic these days since there exists an increasing evidence for the presence of the nuclear three-body forces - but it will be amply sufficient in the present context. This interaction Hamiltonian is by construction Hermitian, symmetric under the exchange of particles \( i \) and \( j \), translation and rotation invariant (thus scalar) etc. This last condition, however, is challenging our imagination: How come, that the Hamiltonian whose orientation with respect to the reference frame cannot be defined (\( \hat{H} \) is scalar) can produce non-spherical surfaces allowing to talk about nuclear shapes, their orientation and symmetries in our 3D space? Understanding of this part of the puzzle leads us to the concept of the spontaneous symmetry breaking according to which the symmetry of the lowest energy solution may turn out to be different from that of the underlying Hamiltonian. Indeed, according to numerous experiments, atomic nuclei are in a great majority non-spherical (deformed) and we are lead to conclude that, in reality, the short-range nuclear forces must be present in the non-spherical ‘cavities of space’ - inside deformed surfaces delimiting the nuclear density distributions introduced earlier.

We arrive in this way at the basic concept of one of the most successful nuclear theories, the nuclear mean-field theory, according to which the nucleonic motion in a nucleus is governed, to an approximation, by a generally non-spherical one-body potential whose action is limited essentially to the area of the nucleonic presence: the area of the nucleonic spatial density - thus defined by the surface \( \Sigma \). To be more specific, let us denote the observables needed to describe the quantum motion of a nucleon by \( \hat{x} \equiv \{ \hat{r}, \hat{\rho}, \hat{s}, \hat{t} \} \), where we have introduced the operators of position, momentum, spin and isospin, respectively. According to the mean-field approximation, our many-body Hamiltonian takes the form (\( \hat{t}_i \) - kinetic energy of the \( i^{th} \) nucleon):

\[
\hat{H}(\hat{x}_1, \hat{x}_2, \ldots \hat{x}_A) \approx \sum_{i=1}^{A} \hat{h}(\hat{x}_i) \quad \text{where} \quad \hat{h}(\hat{x}_i) = \hat{t}_i + \hat{v}(\hat{x}_i),
\]

and where the nuclear mean-field operator, \( \hat{v} \), remains common for all the nucleons. One of the most successful phenomenological realisation of such an interaction is based on the so-called Woods-Saxon deformed potential\(^1\) composed of the nuclear central- and the spin-orbit terms \( \hat{v} \equiv V_{\text{cent}}(\hat{r}; V_o, r_o, a_o) + V_{\text{so}}(\hat{r}, \hat{\rho}, \hat{s}; V_{\text{so}}, r_{\text{so}}, a_{\text{so}}) \) together with the Coulomb term for protons.

### 2. Nuclear mean-field Hamiltonian and groups of symmetry

It becomes clear from the expressions in footnote\(^1\), that if the potential-generating surface-\( \Sigma \) is invariant under a certain symmetry operation, say \( \hat{g} \), so is the generated mean-field potential \( \hat{v} \).

It then follows that if a group \( G \) is a group of symmetry of \( \Sigma \) [read: \( \forall \hat{g} \in G, \text{ acting with } \hat{g} \text{ on } \Sigma \leaves \text{ the latter unchanged} \) then \( G \) is also a group of symmetry of the mean-field Hamiltonian.

To examine the symmetry properties of the nuclear mean-field problem we actually need to be able to solve an inverse problem: Given group \( G = \{ \hat{O}_1, \hat{O}_2, \ldots \hat{O}_f \} \) whose impact on the nucleonic motion in the nucleus we wish to examine. How to construct a realistic Hamiltonian

\(^1\) The expression for the central Woods-Saxon potential is \( V_{\text{cent}} \equiv V_c/(1 + \exp[\text{dist}_{\Sigma}(\hat{r})/a_c]) \), whereas the one for the spin-orbit term has the form \( V_{\text{so}} \equiv \{ \hat{\nabla} \{ V_{\text{so}}/(1 + \exp[\text{dist}_{\Sigma}(\hat{r})/a_{\text{so}}]) \} \times \hat{p} \} \cdot \hat{s} \). In both expressions \( \text{dist}_{\Sigma}(\hat{r}) \) denotes the distance from the actual nucleonic point-position \( \hat{r} \) to the nuclear surface \( \Sigma \).
invariant under all transformations in $G$? To solve this problem let us introduce the basis of spherical harmonics, \{Y_{\lambda\mu}(\vartheta, \varphi)\}, and the corresponding expansion expressing $\Sigma$ under the condition that the volume contained within $\Sigma$ does not depend on the actual deformation. The latter condition is taken care of by introducing the constant-volume condition-function $c(\{\alpha\})$

\[ \Sigma : \ R(\vartheta, \varphi) = R_o c(\{\alpha\}) [1 + \sum_{\lambda=2}^{\lambda_{\text{max}}} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}^* Y_{\lambda\mu}(\vartheta, \varphi)] \quad \text{with} \quad R_o = r_o A^{1/3}. \quad (2) \]

The condition of invariance reads now: $\forall \hat{O} : \Sigma \hat{O} \Sigma' \equiv \Sigma$. The latter can be written down as

\[ \sum_{\lambda=2}^{\lambda_{\text{max}}} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}^* \{ \hat{O} Y_{\lambda\mu}(\vartheta, \varphi) \} = \sum_{\lambda=2}^{\lambda_{\text{max}}} \sum_{\mu'=-\lambda}^{\lambda} \alpha_{\lambda\mu'}^* Y_{\lambda\mu'}(\vartheta, \varphi). \quad (3) \]

In what follows we will need a representation of the operators $\hat{O} \in G$ adapted to the action on the spherical harmonics. The operators of proper rotations can be written down as

\[ \hat{O} \rightarrow \hat{R}(\Omega) \equiv \exp(i\alpha j_z) \exp(i\beta j_y) \exp(i\gamma j_z) \quad \text{where} \quad \Omega \equiv \{\alpha, \beta, \gamma\} - \text{Euler angles}, \quad (4) \]

and where the matrix elements of operators $\hat{R}(\Omega)$ within the basis of states $\{|\lambda\mu\rangle\}$ are called Wigner functions. We have: $\langle \lambda'\mu'| \hat{R}(\Omega) |\lambda\mu\rangle \equiv D_{\mu'\mu}^{\lambda\lambda}(\Omega)$, according to the usual notation. Within this notation the invariance condition for the considered surface takes the form (cf. [1])

\[ \sum_{\lambda=2}^{\lambda_{\text{max}}} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}^* \{ \hat{O} Y_{\lambda\mu}(\vartheta, \varphi) \} = \sum_{\lambda=2}^{\lambda_{\text{max}}} \sum_{\mu'=-\lambda}^{\lambda} \alpha_{\lambda\mu'}^* D_{\mu'\mu}^{\lambda\lambda}(\Omega) Y_{\lambda\mu'}(\vartheta, \varphi). \quad (5) \]

The latter can be written down in the form of an identity as

\[ \sum_{\mu'=-\lambda}^{\lambda} \sum_{\lambda=2}^{\lambda_{\text{max}}} \{ \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}^* D_{\mu'\mu}^{\lambda\lambda}(\Omega) - \alpha_{\lambda\mu'}^* \} Y_{\lambda\mu'}(\vartheta, \varphi) = 0, \quad \forall \vartheta, \varphi. \quad (6) \]

Spherical harmonics are linearly independent from where it follows the system of equations

\[ \sum_{\mu=-\lambda}^{\lambda} \{ D_{\mu'\mu}^{\lambda\lambda}(\Omega_k) - \delta_{\mu'\mu} \} \alpha_{\lambda\mu}^* = 0; \quad k = 1, 2, \ldots, f. \quad (7) \]

Above, $\Omega_k$ are fixed sets of Euler angles corresponding to $O_k$. For instance for a four-fold $O_z$-axis we find $\Omega = \{\pi/2, 0, 0\}$ etc. Solutions can be taken as eigen-vectors of the $(2\lambda + 1) \times (2\lambda + 1)$ matrix $D_{\mu'\mu}^{\lambda\lambda}(\Omega_k)$ with the eigen-value equal +1.

The following observations can be made. Firstly, for a fixed representation of a given group $G$ with the help of the triplets of Euler angles $\{\Omega_k; \ k = 1, 2, \ldots, f\}$, relation (7) represents $f$ systems of algebraic equations for the ensembles of coefficients $\{\alpha_{\lambda\mu}\}$ for each $\lambda = 2, 3, \ldots, \infty$. Such systems are often strongly over-defined i.e. they have more equations than unknowns and generally may only have some non-trivial solutions precisely because of the imposed symmetry (see below for particular examples). Secondly, let us notice that the above system of equations is uniform. This allows to select, e.g. $\alpha_{\lambda\mu=0}$ as an independent parameter, which uniquely fixes all the other non-zero components. Therefore, when solutions (i.e. families of surfaces represented at each given order $\lambda$ by the set of parameters $\{\alpha_{\lambda\mu}\}$ exist, they can be parametrized at each order $\lambda$ with the help of $(2\lambda + 1)$ real parameters\(^2\) out of which only one is independent!

In this way we may consider having prepared the ‘workshop’ allowing to construct the mean-field Hamiltonian at any predefined point-group symmetry.

\(^2\) Observe that for nuclear surfaces $\Sigma$ to be real, one needs the condition $\alpha_{\lambda\mu}^* = (-1)^\mu \alpha_{\lambda-\mu}$ what limits the freedom of originally $(4\lambda + 1)$ real parameters to $(2\lambda + 1)$ real parameters only.
3. Groups of symmetry and symmetry-implied conditions for nuclear stability
So far we have addressed important fundamental issues in the context of our lectures such as:
How come that the rotation-insensitive many-body Hamiltonians may simulate the behaviour of
quantum systems whose orientation in space can be defined since the system is non-spherical?
Or: How to construct the mean-field Hamiltonian with an arbitrarily predefined symmetry so
that we may analyse symmetry’s impact on the system of interest through realistic calculations?
In what follows we will discuss the link between the properties of the nucleonic single-particle
spectra directly influenced by the shapes of the nuclear surface and in particular the link between
the level spacing and the total nuclear energy - thus the overall nuclear stability.

3.1. Single-particle energy-gaps and total energy minima: The mean-field view-point
In this section we wish to discuss yet another fundamental aspect: In order that symmetries
of quantum systems become of ‘practical’ interest it is necessary that stationary states of the
system manifest such symmetries so that the associated phenomena can be studied in laboratory.
Stationary states arise when the total energy of the system possesses a local minimum at the
configurations manifesting a geometrical symmetry and it will be now our goal to explain how
such a link can be obtained within the mean-field nuclear theories.

Contemporary realistic calculations of nuclear energies can be performed within the nuclear
mean-field theory framework, either using the self-consistent Hartree-Fock and/or constrained
Hartree-Fock type approaches or phenomenological Hamiltonians like the ones based on the
Woods-Saxon potential described earlier. Our discussion leading to Eq. (7) suggests that the
phenomenological models in which the symmetry of the generating surface can be predefined
in advance are the best suited for the studies of the nuclear symmetry, the argument valid in
addition to the argument of the predictive power, superior in the case of the phenomenological
as compared to the present-day self-consistent models.

Figure 1. Schematic illustration:
The nucleonic single particle energy
levels in function of an ensemble
of the deformation parameters, the
latter represented with the help of
a single axis, as calculated within
the mean-field theories. Gaps in the
spectra represent configurations of
increased stability.

Figure 2. Schematic illustration:
Total energy calculations, either
Hartree-Fock or phenomenological,
give the local minima, when the
numbers of nucleons correspond to
occupation of all the levels below
the gap, figure 1. For neighbouring
particle numbers the minima are
usually less pronounced.

Whereas self-consistent models give directly an approximation of the total nuclear energies,
the phenomenological models use the so-called Strutinsky method allowing to combine the
macroscopic liquid drop model energy, \( E_{ld} \), with the so-called shell-energies, \( E_{shell} \), to approximate the final result: \( E_{total} = E_{ld} + E_{shell} \). Whichever of the two techniques used, the final outcome is similar: The gaps in the single-particle spectra, figure 1, imply the presence of the total energy minima. In figure 2 they correspond to a larger (left) and a smaller (right) particle numbers, particles occupying the lowest single particle levels below the gaps. It then follows that, semi-qualitatively, the search for the stable nuclear configurations represented by the local minima on the total energy surfaces is strongly related to the search of the big gaps in the single-nucleon energy spectra.

3.2. Search for maximum energy-gaps with group-representation theory as a guideline

Given a mean-field theory Hamiltonian \( \hat{H} \) and a group: \( \mathcal{G} = \{ O_1, O_2, \ldots, O_f \} \). Assume, that \( \mathcal{G} \) is a symmetry group of \( \hat{H} \), i.e. \( [\hat{H}, O_k] = 0 \) with \( k = 1, 2, \ldots, f \). Let irreducible representations of \( \mathcal{G} \) be \( \{ R_1, R_2, \ldots, R_r \} \) and denote their dimensions \( \{ d_1, d_2, \ldots, d_r \} \), respectively. One can show that the eigenvalues \( \{ \varepsilon_\nu \} \) of the problem can be split into families of non-interacting levels

\[
\hat{H}\psi_\nu = \varepsilon_\nu \psi_\nu : \quad \{\varepsilon_\nu\} \rightarrow \{\varepsilon^{(1)}_\nu\} \oplus \{\varepsilon^{(2)}_\nu\} \oplus \ldots \oplus \{\varepsilon^{(r)}_\nu\} \tag{8}
\]

where \( \{\varepsilon_\nu\} \) denotes the ensemble of all the levels belonging to the irreducible representation \( R_\kappa \) and thus characterised by the same degeneracy \( d_\kappa \). When deformation changes while respecting the symmetry (see below) - the levels belonging to the same irreducible representation vary ‘repelling each other’, the mechanism known in quantum theories as Landau-Zener effect. Thus they fill the whole available energy interval defined by the depth of the potential well, cf. figure 3. Following this repulsion mechanism, the average level spacing increases, very roughly, by the factor of “\( r \)”.

Consequently, a relatively big level-spacing is created in all the \( r \) families with the increased chance that by superposing all the levels in one common energy window, some big gaps will ‘survive’ (right-hand side of figure 3). Let us note in passing that levels belonging to different irreducible representations cross each other in function of deformation without what we call ‘level-interaction’ - where from the term ‘non-interacting’ in the text above equation (8).

From what has been said, it becomes clear that the key-message from the group theory considerations is this:

Respecting certain geometrical symmetries by a nucleus will increase its chances for the biggest gaps in the single-particle spectra and thus for a possible existence of the energy minima at stable group-symmetric configurations if the underlying symmetry group has:

A. Possibly largest number of irreducible representations, and

B. The irreducible representations in question are of the highest possible dimensions.

Indeed, since the dimension of a given irreducible representation is equal to the degeneracy of the corresponding levels, it follows that the second of the above conditions may play a helpful role...
in increasing even more the average level-spacing. It then becomes clear that the next logical step to follow will be to determine which known symmetry groups fulfil the above conditions in the most satisfactory manner?

The groups of symmetry of finite bodies in the 3D space are very well known in particular from the molecular physics considerations. One can show that all the symmetry elements of all these groups must leave at least one point of the body unchanged - and so the symmetry groups in question are referred to as (molecular) point-groups. There exist a well known hierarchy referred to as “32 point-group symmetry chain” illustrated, in figure 4.

Figure 4. The highest in hierarchy of the 32 symmetry point-groups is $O_h$, the octahedral group, which is composed of 48 symmetry elements. Several groups in this diagram are of potential interest for the theory of nuclear structure; of particular interest for us (see text) is the tetrahedral group, $T_d$, next in hierarchy, composed of 24 symmetry elements. The latter does not contain inversion and can serve for modelling certain important properties of the negative parity rotational bands in nuclei. The triaxial quadrupole-deformed nuclei are described by $D_{2h}$, lower in the hierarchy. [Based on Koster et al., ref. [2]]

The scheme shown in figure 4 contains the so-called classical (or single) as opposed to double point groups. Without entering into mathematical details let us mention that to describe the properties of motion of Fermions whose wave-function have a spinor character we have to assure that, in accordance with the general and well known rules of quantum mechanics, a rotation of a Fermion wave-function through the angle of $2\pi$ must change the sign of such a wave function. To ensure such a property one adds, after H. Bethe, a special new element to each of the single groups what, again in agreement with the well know group properties, actually doubles the number of the groups elements, where from the name double point-groups. The double groups have very different mathematical properties, as compared to the prototype single groups, and differ in particular in numbers and dimensions of the corresponding irreducible representations.

One finds that among 32 double point groups all but octahedral and tetrahedral ones have two-dimensional irreducible representations. The exception from this rule are four-dimensional representations in the groups $O^D_h$ and $T^D_d$, which will therefore receive a particular attention in what follows. Moreover, the octahedral double point group has 6 irreducible representations whereas the octahedral one has 3 (cf. also overlays 71 and 72 of ref. [3]).

Let us mention in passing the particular and fundamental role in quantum physics of the two-dimensional irreducible representations leading to the two-fold (the so-called Kramers, or spin-down - spin-up) degeneracies, which take place for all systems of Fermions whose Hamiltonians are not explicitly time-dependent - thus time-reversal invariant - in particular in all non-spherical nuclei whose the so-called Nilsson orbitals carry always the Kramers degeneracy.

Inspection of figure 4 strongly suggests that there may be several point groups totally unexplored so far in the nuclear structure considerations and our preliminary calculations indicate that some of them are probably present in certain super-deformed nuclei, groups such as, e.g., $C_{4h}$, $D_{3d}$ or $D_{3h}$. In what follows we focus on tetrahedral (also octahedral) groups[4].
4. Focus on nuclear tetrahedral and octahedral symmetries

To be able to examine, through realistic calculations, the symmetry implications of the two symmetry-groups for nuclei, we have applied a simple formalism leading to equation (7) in order to arrive at the computer programmable expressions for the tetrahedral-symmetric surfaces. As already mentioned, only *special combinations* of spherical harmonics, $Y_{\lambda\mu}$, may form a basis for surfaces with tetrahedral symmetry and *only* for $\lambda$-odd. The three lowest-order solutions are:[1]\

\[
\begin{align*}
\lambda = 3 &: \quad \alpha_{3,\pm2} \equiv t_3 \\
\lambda = 5 &: \quad \text{no solution possible} \\
\lambda = 7 &: \quad \alpha_{7,\pm2} \equiv t_7; \quad \alpha_{7,\pm6} \equiv -\sqrt{\frac{11}{13}} \cdot t_7 \\
\lambda = 9 &: \quad \alpha_{9,\pm2} \equiv t_9; \quad \alpha_{9,\pm6} \equiv +\sqrt{\frac{28}{135}} \cdot t_9
\end{align*}
\] (9)

Above, symbols $t_3$, $t_7$ and $t_9$ represent the independent tetrahedral symmetry deformations of order 3, 7 and 9. The equation for the tetrahedral-symmetric nuclear surface in the third order will therefore be proportional to \( \{1 + t_3[Y_{3,\pm2}(\theta, \phi) + Y_{3,-2}(\theta, \phi)]\} \), whereas the one in the seventh order, to \( \{1 + t_7[Y_{7,\pm2}(\theta, \phi) + Y_{7,-2}(\theta, \phi) - \sqrt{\frac{11}{13}}Y_{7,\pm6}(\theta, \phi) + \sqrt{\frac{11}{13}}Y_{7,-6}(\theta, \phi)]\} \), etc. Analogous expressions for the octahedral symmetry are illustrated in the overlays 129-138 of Ref. [3]. Let us note that the classical prototypes of the Platonic figures with the symmetries discussed here are: A ‘pyramid’ with a triangular basis (tetrahedron) and a ‘diamond shape’ (octahedron). The mathematical properties of the solutions illustrated guarantee that the surfaces remain symmetric with respect to all the symmetry elements of their respective groups for all values of parameters $t_3$, $t_7$, $o_4$ etc.; although in this context it does not make sense to say that one surface is ‘more tetrahedral than the other’, nevertheless it makes sense to say that a given tetrahedral symmetric surface is more deformed than the other.

5. Towards experimental verification

Theoretical analysis of the problem of symmetries must aim, among others, at establishing their possible measurable signs. We will arbitrarily introduce two lines of reasoning more related to pedagogy of the discussion rather than the mathematical rigour: Certain criteria can be viewed upon as necessary vs. sufficient conditions whereas some others as static vs. dynamic conditions. Often the necessary conditions are simpler to be presented. They have the form of statements:
Suppose a nuclear quantum state is generated by the tetrahedral-symmetric Hamiltonian - What are the implied characteristic features of such states? Sufficient conditions follow an opposite logic: By establishing a certain property we will be sure that the studied system possesses a given symmetry. The latter conditions (if exist) are usually more difficult to establish experimentally.

We will begin with the historically first and pedagogically simplest (although somewhat naive) a selection of necessary static criteria. Later we will argue that although the related physical properties are worthwhile discussing, they will need to be super-seeded with the dynamical (as opposed to static) ones for reasons that will also be discussed.

5.1. Possible experimental signals of tetrahedral symmetry in nuclei: Static, necessary criteria

Given the fact that the tetrahedral symmetric surfaces are expressed with the help of the harmonics of the order \( \lambda = 3 \), viz. \( Y_{3\pm2} \), the signals of the corresponding shapes can be sought in the negative parity rotational bands in nuclei. This is in analogy to one-phonon negative-parity octupole bands (vibrational or static, see below) associated with the axially symmetric pear-shape nuclei, phonon solutions constructed with the help of the \( Y_{30} \) spherical harmonic.

Consider an academic case of four families of octupole deformed nuclear surfaces, defined using \( \lambda = 3 \) deformations: \( \alpha_{30}, \alpha_{31}, \alpha_{32} \) and \( \alpha_{33} \). Introduce next the uniform charge-density distribution, \( \rho_{\Sigma}(\vec{r}) \), and express the multipole moments as usual by

\[
\rho_{\Sigma}(\vec{r}) = \begin{cases} 
\rho_0 : & \vec{r} \in \Sigma \\
0 : & \vec{r} \notin \Sigma
\end{cases} \rightarrow Q_\lambda \mu = \int \rho_{\Sigma}(\vec{r}) r^\lambda Y_{\lambda\mu} d^3\vec{r},
\]

(10)

to finally calculate one by one the associated quadrupole moments as functions of \( \alpha_{3\mu} \), using the Taylor expansion valid approximately for small deformations. Somewhat schematically:

\[
Q_\lambda \mu(\alpha) \approx Q_\lambda \mu|_{\alpha=0} + Q''_\lambda \mu|_{\alpha=0} \Delta \alpha + \frac{1}{2} Q'''_\lambda \mu|_{\alpha=0} \Delta \alpha \Delta \alpha,
\]

(11)

and after setting in equation (10) \( \{ \lambda = 2 \) and \( \mu = 0 \) we obtain from (11):

\[
\begin{align*}
\alpha_{30} : & \quad Q_{20} = \frac{15}{2\sqrt{5\pi}} \cdot \alpha_{30} \cdot \rho_0 R_0^5, \\
\alpha_{31} : & \quad Q_{20} = \frac{15}{4\sqrt{5\pi}} \cdot \alpha_{31} \alpha_{3-1} \cdot \rho_0 R_0^5, \\
\alpha_{32} : & \quad Q_{20} = 0, \\
\alpha_{33} : & \quad Q_{20} = \frac{125}{2\sqrt{5\pi}} \cdot \alpha_{3+3} \alpha_{3-3} \cdot \rho_0 R_0^5.
\end{align*}
\]

(12) \hspace{1cm} (13) \hspace{1cm} (14) \hspace{1cm} (15)

Among \( \lambda = 3 \) deformations, only the tetrahedral one, \( \alpha_{32} \equiv t_3 \), leads to \( Q_2 = 0 \) - all other octupole deformations induce some quadrupole moments! This observation brings us to the historically first a suggestion that there must exist rotational bands with the usual energy sequences, \( E_I \sim I(I+1) \), since the nuclei in question are non-spherical, however, with none or weak E2-transitions. Although there is nothing wrong with this suggestion, a deeper founded analysis discussed below indicates the existence of alternative manifestations of the tetrahedral symmetry (cf. class-1 solutions in figure 11 below).

Let us summarise: In the discussed purely static scenario a nucleus with the pure tetrahedral deformation (all the others assumed vanishing) is expected to have the E2 transitions vanishing or being rather weak, whereas the presence of the Coriolis interactions is expected to increase the quadrupole polarisation with increasing spin. These mechanisms are schematically illustrated in figure 7. An example of the corresponding experimental data is presented in figure 8. Why are these observations not a direct proof of the presence of the tetrahedral symmetry in the compared nuclei? In experiment, the missing signals from the E2 transitions may be a result of either very weak E2 or, alternatively, very strong, competing E1 towards the ground-state band.
The negative parity bands are traditionally interpreted in terms of either statical octupole minima - cf. one of the best known situations of this type illustrated in figure 9 - or in terms of the octupole vibrations. In this latter case (not shown) it is assumed that the octupole α_{30} deformation is zero but the total energy landscape is flat and the nucleus undergoes a relatively large-amplitude octupole (pear-shape) vibrational motion.

The nuclear rotational structure in this case is represented by two bands of alternating parity with the common moment of inertia what causes an energetic inter-twingling of the alternating parity states. Since the pear-shape deformation implies unequal shifts of the proton and the neutron centres of mass a certain dipole moment is generated. It follows that the competition between the stretched quadrupole and dipole transitions causes a characteristic zig-zag decay pattern as seen in the figure. The measured ratios of the reduced transition probabilities happen to be constant and nearly independent of spin as shown in the accompanying table, whereas in the case of a static tetrahedral deformation an alternative scenario can be envisaged. First of all the negative parity bands, if associated with the tetrahedral minima which are placed higher
in energy, are expected to be shifted upwards in the energy scale and as a consequence the
zig-zag pattern of the E1 transitions becomes energetically impossible. Taking into account
the mechanism of the zero-point (vibrational) motion in the quadrupole direction the most probable
quadrupole dynamical deformation defined as $\langle |\alpha| \rangle \sim \int |\alpha| \phi_2(n(\alpha)) d\alpha \neq 0$ is calculated to be
different from zero implying certain, possibly weak quadrupole moments, thus E2 transitions.
With a possibly weak E1 moments and increasing Coriolis-induced E2 moments a possible
scenario will be that of an increasing $B(E2)/B(E1)$ ratio as indeed observed, cf. table 1. Observe
that the results for the static octupole case and all the other ones differ on the average by an
order to two orders of magnitude. We conclude that the underlying quantum mechanisms must

### Table 1. The experimental values of the reduced transition ratios $B(E2)_{in}/B(E1)_{out}$ in $10^6 Fm$^2
for the lowest negative parity bands in the Rare Earth nuclei close to the tetrahedral magic
numbers $Z = 64, 70$ and $N = 90$ - compared to the same quantity in the static-octupole case of
$^{222}$Th illustrated in figure 9. [Experimental data from ref. [5].]

| Spin  | $^{152}$Gd | $^{156}$Gd | $^{154}$Gd | $^{160}$Er | $^{164}$Er | $^{162}$Yb | $^{164}$Yb | $^{222}$Th |
|-------|------------|------------|------------|------------|------------|------------|------------|-----------|
| 19$^-$ | -          | 50         | -          | -          | -          | -          | -          | +0.3      |
| 17$^-$ | -          | 16         | -          | -          | -          | -          | -          | +0.4      |
| 15$^-$ | -          | 6          | 60         | 24         | -          | -          | -          | +0.4      |
| 13$^-$ | 14         | 7          | 15         | 18         | 23         | -          | 17         | +0.3      |
| 11$^-$ | 4          | 15         | 5          | 10         | 0          | 11         | 11         | +0.4      |
| 9$^-$  | 4          | 0          | 0          | 0          | 0          | 0          | 0          | +0.4      |
| 7$^-$  | 0          | 0          | 0          | 0          | 0          | 0          | 0          | +0.4      |

Figure 9. Microscopic Strutinsky-type calculations with the universal Woods-Saxon potential
calculations for $^{222}$Th in function of the quadrupole and pear-shape (standard octupole)
deformations [left]. In the middle - The experimental results of the octupole band associated with
the static octupole minima. Observe the characteristic ‘zig-zag’ E1-transition pattern associated
with the static octupole minima. The measured ratios of the $B(E2)_{in}/B(E1)_{out}$ in $10^6 e \cdot Fm$
show spin-independence of this quantity. [Experimental data from Ref. [5].]
be different - without being able to affirm having seen the tetrahedral symmetry signal; for that the absolute reduced transition probabilities might be necessary - see below.

5.2. Towards convincing signals of tetrahedral symmetry in nuclei: Sufficient dynamic criteria

Convincing criteria for identification of the tetrahedral symmetry in subatomic systems must be constructed in formal analogy to the corresponding criteria developed in molecular physics. For that purposes the Schrödinger equation for the collective nuclear motion (rotational and vibrational) must be solved and the corresponding transition intensity ratios which happen to depend sensitively on the symmetry of the system should be calculated. This work has not been accomplished yet but several steps have already been undertaken. Without entering into details let us refer to [6] whose main results are qualitatively illustrated in figures 10-12 showing that in addition to purely quadrupole-symmetric solution (left) and purely tetrahedral solution (right) there exist a new, so-called class-1 solutions which combine the simultaneous presence of two very different-symmetry solutions in one single quantum state (middle). Using the wave functions like the ones whose absolute-square values are presented in the figure the electromagnetic transition probabilities can be calculated. [Interested reader is referred to a recent review [7] of the newest progress for details.]

6. Summary and Conclusions

In these lecture series we have discussed and illustrated the consequences of our strategic suggestion about symmetry-groups, nuclei and their mean fields. Although we have focussed on the tetrahedral symmetry the underlying idea is more general and can be formulated as follows:

Each symmetry group sufficiently rich in symmetry elements may generate magic numbers that characterise the underlying symmetry in full analogy to the spherical magic numbers 8, 20, 28, 50, 82 and 126 that characterise the nuclear sphericity. In particular, the tetrahedral stability can be associated with the following

\[ \text{Tetrahedral Magic Numbers: 32, 40, 56, 64, 70, 90 and 136} \] (16)

and, more precisely, with the following tetrahedral doubly magic nuclei:

\[ \text{3 This phrasing, which cannot be made more precise at this point, alludes to the fact that} \]
\[ \text{several point-group symmetries listed in the diagram in figure 4, especially in the parts of the} \]
\[ \text{chain close to the ‘mother group’ } O_h, \]
\[ \text{when applied to the realistic mean-field Hamiltonians generate big single-particle gaps - ‘big’ and} \]
\[ \text{‘sufficiently strong’ referring to the fact that on the total energy surfaces stable minima are} \]
\[ \text{generated.} \]
Characteristically, whereas the spherical symmetry corresponds to a fixed geometry (no sphere is more spherical than the other) the point-group symmetry-driven magic-numbers may correspond to various areas on the (e.g. tetrahedral) deformation axis. In other words: Point-groups allow for ‘variable deformation’ of nuclear objects, whereas perfectly preserving the symmetry properties.

Tetrahedral and octahedral symmetries, if identified in nature, will very likely cause switching certain accents of importance in nuclear structure research, particularly in the experimental objectives of various measurements. Today, as it seems, the nuclear physics community has adopted thinking about shapes, symmetry and geometry, through the perspective of the so-called prolate-oblate shape coexistence (quadrupole axial shapes), the so-called tri-axial shapes (mainly quadrupole) and occasionally the so-called octupole (axial pear-shape) vibrations and rotations. Because of the rather frequent appearance of those keywords in the literature, some colleagues, mainly from the outside of the field, refer to the quadrupole/octetopole shapes as ‘trivial’. Limiting our thinking to the quadrupole/octetopole shapes and trying to force all interpretations of the rotational motion with the help of those is (probably) historically justified but counterproductive today. Since literally thousands of projects have been focussed on the rotational band properties and the associated effective moments of inertia - a very poor tool for testing the geometry and shapes - the electromagnetic transition probes, technically more difficult, have been used much more seldom and the corresponding literature is scarce.

To approach the issue of the experimental verification of the presence of the hypothetical ‘non-trivial’ symmetries such as those listed in the upper half of the diagram in figure 4 requires precise measurements of the transition intensities, branching rations and life-times. Those and possibly other measurements would allow for establishing as precisely as possible, first of all, the E1, E2 and M1 reduced transition probabilities. Such an information seems indispensable in view of the fact that various symmetries with various and usually distinct character of the irreducible representations generate various types of forbidden transitions and of selection rules in the form of very characteristic branching ratios which can serve as the ‘smoking gun signal’ for the possible presence of those exotic symmetries.

On the theory side we will need to implement an already rather well known formalism for constructing the physical solutions (wave-functions) which transform according to the irreducible representations of the group studied. These techniques, although formally rather well known, are not simple in the applications to e.g. determinantal microscopic wave functions which will be needed to evaluate the reduced transition probabilities and the theoretical values of the characteristic branching ratios using various projection techniques. The discovery of the possible presence of what we called class-1 solutions (figure 11) mixing the dominating tetrahedral symmetry with the totally different quadrupole one is an extremely intriguing new structure in low-energy subatomic physics - fascinating, but possibly complicating the interpretations.

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