Transitive partially hyperbolic diffeomorphisms with one-dimensional neutral center

To the Memory of Professor Shantao Liao

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Received May 5, 2019; accepted July 21, 2020; published online August 21, 2020

Abstract In this paper, we study transitive partially hyperbolic diffeomorphisms with one-dimensional topologically neutral center, meaning that the length of the iterate of small center segments remains small. Such systems are dynamically coherent. We show that there exists a continuous metric along the center foliation which is invariant under the dynamics. As an application, we classify the transitive partially hyperbolic diffeomorphisms on 3-manifolds with topologically neutral center.

Keywords partial hyperbolicity, dynamical coherence, conjugacy, transitivity, neutral

MSC(2010) 37D30, 37C15, 37E05, 57M60

Citation: Bonatti C, Zhang J H. Transitive partially hyperbolic diffeomorphisms with one-dimensional neutral center. Sci China Math, 2020, 63: 1647–1670, https://doi.org/10.1007/s11425-019-1751-2

1 Introduction

A $C^1$ diffeomorphism $f$ on a closed manifold $M$ is partially hyperbolic if there exists a $Df$-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

such that $E^s$ is uniformly contracting, $E^u$ is uniformly expanding and $E^c$ has the intermediate behavior; to be precise, there exists an integer $N \in \mathbb{N}$ such that for any $x \in M$, we have the following:

- Contraction and expansion,

$$\|Df^N|_{E^s(x)}\| < \frac{1}{2} \quad \text{and} \quad \|Df^{-N}|_{E^u(x)}\| < \frac{1}{2},$$

- Domination,

$$\|Df^N|_{E^c(x)}\| \cdot \|Df^{-N}|_{E^c(f^N(x))}\| < \frac{1}{2} \quad \text{and} \quad \|Df^N|_{E^c(x)}\| \cdot \|Df^{-N}|_{E^u(f^N(x))}\| < \frac{1}{2}.$$

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Definition 1.1. For a $C^1$ partially hyperbolic diffeomorphism $f$ on $M$, one says that $f$ is neutral along center, if there exists $C > 1$ such that
\[
\frac{1}{C} < \| Df^n |_{E^c(x)} \| < C \quad \text{for any } x \in M \text{ and } n \in \mathbb{Z}.
\]

One says that $f$ is topologically neutral along center if for any $\varepsilon > 0$ there is $\delta > 0$ so that any $C^1$ center path $\sigma$ of length bounded by $\delta$ has all its images $f^n(\sigma), n \in \mathbb{Z}$ bounded in length by $\varepsilon$.

One easily checks that if $f$ is neutral, then $f$ is topologically neutral. However, the reverse is not true: there are partially hyperbolic diffeomorphisms on 3-manifolds, with one-dimensional center bundle, which are topologically neutral but not neutral (see Subsection 2.1).

For partially hyperbolic diffeomorphisms with neutral or topologically neutral center, the center bundle is uniquely integrable due to [26].

A center arc is an equivalence class of locally injective center paths, up to changing the parametrization. A point is a degenerate arc.

Definition 1.2. We will call a function $\ell^c$ defined on the set of arcs the center metric, with the following properties:

- (Positivity) Strictly positive on the non-degenerate arcs, and vanishing on degenerate arcs.
- (Additivity) Consider $\sigma: [a, b] \to M$ a center path and $c \in [a, b]$, and then
  \[
  \ell^c(\sigma_{[a,c]}) + \ell^c(\sigma_{[c,b]}) = \ell^c(\sigma_{[a,b]}).
  \]
- (Continuity) If $\sigma_t$ are center arcs associated with a $C^0$-continuous family of center paths, then $\ell^c(\sigma_t)$ varies continuously with $t$.

1.1 Results in any dimension

Recall that a diffeomorphism on a connected closed manifold $M$ is transitive if it admits a dense orbit. In this paper, we work in the $C^1$ scenario.

Theorem A. Let $f$ be a $C^1$ partially hyperbolic diffeomorphism on a closed manifold $M$. Assume that $f$ has one-dimensional topologically neutral center and $f$ is transitive. Then there exists a center metric which is invariant under $f$ (in other words, the action of $f$ on center leaves is by isometries for this center metric). As a consequence, this center metric is invariant by the strong stable and strong unstable holonomies. Furthermore, the invariant center metric is unique up to multiplying by a (positive) constant.

When the center bundle is orientable and $f$ preserves the orientation of the center, the center metric gives a continuous flow, by following the center leaves at a constant speed. The invariance of the center metric implies that the constant speed flow is invariant under the dynamics. Thus the next result is a straightforward corollary of Theorem A.

Theorem B. Let $f$ be a $C^1$ partially hyperbolic diffeomorphism on a closed manifold $M$. Assume that

- $f$ has one-dimensional topologically neutral center and $f$ is transitive;
- $E^c$ is orientable and $f$ preserves its orientation.

Then there exists a continuous flow $\{\varphi_t\}_{t \in \mathbb{R}}$ on $M$ with the following properties:

- $\{\varphi_t(x)\}_{t \in \mathbb{R}} = F^c(x)$ for any $x \in M$; in particular, $\{\varphi_t\}_{t \in \mathbb{R}}$ has no singularities;
- $f$ commutes with the flow $\varphi_t$, i.e., $f \circ \varphi_t = \varphi_t \circ f$ for any $t \in \mathbb{R}$.

The following result gives the transitivity of a partially hyperbolic diffeomorphism with topologically neutral center provided that the orbit of some point is dense in an open set. Under the setting of partial hyperbolicity and allowing an $\omega$-limit set to contain an open set, the usual way to recover transitivity is to assume accessibility. Here, we strongly use the topologically neutral property.

Proposition 1.3. Let $f$ be a $C^1$ partially hyperbolic diffeomorphism on a closed connected manifold $M$. Assume that

- $f$ has topologically neutral center;
• there is \( y \in M \) whose \( \omega \)-limit set \( \omega(y) \) has non-empty interior.

Then \( f \) is transitive.

As a consequence, one has the following observation which has its own interest and is useful when the center bundle \( E^c \) is not orientable, or \( f \) does not preserve an orientation of it.

**Proposition 1.4.** Let \( f \) be a \( C^1 \) partially hyperbolic diffeomorphism on a closed manifold \( M \). Assume that \( f \) has topologically neutral center and \( f \) is transitive. Let \( \pi : \widetilde{M} \to M \) be a (connected) finite cover of \( M \) and \( \tilde{f} \) be a lift of \( f \) to \( \widetilde{M} \), and \( k > 0 \) be an integer. Then \( \tilde{f}^k \) is transitive.

We remark that in Propositions 1.4 and 1.3, we do not assume the center to be one-dimensional.

Considering non-transitive partially hyperbolic diffeomorphisms with topologically neutral center, we get the following result which may be useful for further studies.

**Proposition 1.5.** Let \( f \) be a \( C^1 \) partially hyperbolic diffeomorphism with one-dimensional topologically neutral center. Then the set of recurrent (resp. positively recurrent) points is saturated by the center leaves.

Let us finish these general results by observing that Theorems A and B are no more true if one removes the transitivity hypothesis: consider the partially hyperbolic diffeomorphism \( f \) built in [8] which is non-transitive and has one-dimensional neutral center; the example is obtained by composing a Dehn twist to the time \( N \)-map of a non-transitive Anosov flow which admits only one transitive attractor, one transitive repeller and two transverse tori \( T_1 \) and \( T_2 \) in the wandering domain; one can assume that the Dehn twist is supported on an orbit segment of \( T_1 \); the dynamics of \( f \) coincides with the time \( N \)-map of the Anosov flow, and hence one has no choice of the center metric on the repeller and the attractor since the dynamics in the orbit of \( T_2 \) coincides with the time \( N \)-map of the Anosov flow; however, one can do a small perturbation in the support of the Dehn twist and one gets a new partially hyperbolic diffeomorphism with neutral center and does not admit the invariant metric. As it is not the main aim of this paper, we will not present all the details.

### 1.2 Classification results in dimension three

Given two diffeomorphisms \( f \) and \( g \) on a closed manifold \( M \), one says that \( f \) is \( C^0 \)-conjugate to \( g \) if there exists a homeomorphism \( h \) on \( M \) such that \( h \circ f = g \circ h \).

Using Theorem A, we obtain the following classification up to conjugacy.

**Theorem C.** Let \( f \) be a \( C^1 \) partially hyperbolic diffeomorphism on a closed 3-manifold \( M \). Assume that \( f \) has one-dimensional topologically neutral center and \( f \) is transitive. Then up to finite lifts and iterates, \( f \) is \( C^0 \)-conjugate to one of the following:

- skew products over a linear Anosov on \( \mathbb{T}^2 \) with the rotations of the circle;
- the time 1-map of a transitive topological Anosov flow.

**Remark 1.6.**

- The example in [8] (see also [10]) shows that the transitivity assumption is necessary: there are partially hyperbolic diffeomorphisms \( f \) on 3-manifolds with neutral center and admitting non-compact center leaves which are not periodic. Thus \( f \) is not conjugated, and not even center-leaf conjugated, to any of the models in Theorem C.
- During the final preparation of this paper, we notice a paper by Carrasco et al. [14] proving a classification result under certain smooth rigid conditions: they considered transitive \( C^\infty \)-partially hyperbolic diffeomorphisms on 3-manifolds satisfying that \( \| Df \|_{E^c(x)} = 1 \) for any \( x \in M \) and the bundles \( E^s, E^c \) and \( E^u \) are smooth, and they proved that such partially hyperbolic diffeomorphisms, up to finite lifts and iterates, are \( C^\infty \)-conjugate to skew products with rotations on the circle fibers or time 1-map of a smooth Anosov flow. Their techniques are also different from the ones in this paper.
- There are many difficulties in classifying such partially hyperbolic diffeomorphisms on higher dimensional manifolds. For example, we use the results by Rosenberg [30] and Gabai [17] and classification results in [9] which only work for 3-manifolds, and in the proof we use the notion of \( n \)-th-intersection which can only be defined in dimension three to prove the center leaves to be periodic in the case where
there are non-compact center leaves; another difficulty in higher dimension is that it is unknown if the center leaves of a compact center foliation are uniformly compact.

As a consequence, one immediately gets the following corollary.

**Corollary 1.7.** Let $f$ be a transitive partially hyperbolic diffeomorphism on a 3-manifold $M$. Assume that $f$ has one-dimensional topologically neutral center. Then $f$ has compact center leaves. Furthermore, if there exist compact center leaves which are non-periodic, then the center foliation is uniformly compact.

Our result is motivated by the following question raised in [25].

**Question.** Does there exist a partially hyperbolic diffeomorphism with isometric action on the center bundle which is robustly transitive?

The evidence in [6, 31] indicates the answer might be negative, but the question remains open.

Let us briefly recall some historical background of this paper. In a talk in 2001, Pujals informally conjectured that the family of transitive partially hyperbolic diffeomorphisms, up to isotopy, falls into three parts: time 1-map of a transitive Anosov flow, linear Anosov on $T^3$ and skew products over linear Anosov maps on $T^2$ with rotations on the circle. Then observed by Bonatti and Wilkinson [9], one has to take finite lifts and iterates into account. Inspired by Pujals’ conjecture, Rodriguez Hertz et al. [26] conjectured that the family of dynamically coherent partially hyperbolic diffeomorphisms, up to finite lifts and iterates, falls into three parts as in the conjecture of Pujals. Some partial results towards these two conjectures have been obtained in [1, 9, 13, 18–20]. Then some counter-examples are constructed in [4, 5, 8]. In [8], Bonatti et al. built a dynamically coherent partially hyperbolic diffeomorphism on a 3-manifold which supports an Anosov flow, and the diffeomorphism neither has a periodic center foliation nor is isotopic to identity (therefore is a counter-example to the Rodriguez Hertz-Rodriguez Hertz-Ures conjecture, and some generalization is obtained in [10]); furthermore, the example in [8] is not transitive. In [4, 5], Bonatti et al. built robustly transitive partially hyperbolic diffeomorphisms on 3-manifolds which do not satisfy Pujals’ conjecture, and the examples in [4] are designed to be non-dynamically coherent, but the dynamical coherence of examples in [5] is still unknown. In the end, we remark that the measure-theoretical properties of partially hyperbolic diffeomorphisms with neutral center have drawn attention of researchers (see, for example, [15, 28]).

2 Preliminaries

In this section, we collect the notions and the known results used in this paper.

2.1 Dynamical coherence

Given a partially hyperbolic diffeomorphism $f$, one says that $f$ is dynamically coherent, if there exist invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$, respectively. When $f$ is dynamically coherent, it naturally induces the center foliation by taking the intersection of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$.

For partially hyperbolic diffeomorphisms, the strong stable and strong unstable bundles are always integrable, and they are integrated into unique $f$-invariant foliations which will be called strong foliations, (see [22]). For the center bundle, the situation is more delicate; even in the one-dimensional center case, there might not exist center foliations (see the examples in [4, 27]).

Recall that $f$ has topologically neutral center if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $C^1$ path $\gamma$ tangent to $E^c$ of length bounded by $\delta$, the length of $f^n(\gamma)$ is bounded by $\varepsilon$ for any $n \in \mathbb{Z}$.

**Theorem 2.1** (See [26, Theorem 7.5]). Let $f$ be a $C^1$ partially hyperbolic diffeomorphism. Assume that $f$ has topologically neutral center. Then $f$ is dynamically coherent. Furthermore, the center bundle is uniquely integrable.

**Remark 2.2.** It is worth noticing that in [26, Theorem 7.5], the plaque expansiveness is also obtained (in this paper, we will not use this fact).
To end this subsection, we show that there exists a transitive partially hyperbolic diffeomorphism whose center is topologically neutral but not neutral.

**Proposition 2.3.** There exists a transitive partially hyperbolic diffeomorphism on $T^3$ with one-dimensional topologically neutral center but not neutral.

**Proof.** Let $R_\alpha$ be an irrational rotation on $S^1 = \mathbb{R}/\mathbb{Z}$. As $R_\alpha$ has no periodic points, one can apply [3, Theorem B] to get a $C^1$ diffeomorphism $h \in \text{Diff}^1(S^1)$ which is $C^1$-close enough to $R_\alpha$ such that

- \( \lim_{n \to \infty} \inf_{x \in S^1} \{ \|Dh^n(x)\|, \|Dh^{-n}(x)\| \} = \infty \);
- $h$ is $C^0$-conjugate to $R_\alpha$.

Let $A$ be a linear Anosov map on $T^2$. Then $F := A \times h$ is a partially hyperbolic diffeomorphism on $T^3$ with one-dimensional center. As $h$ is conjugate to a rotation, $F$ has a topologically neutral but not neutral center bundle. As $h$ is transitive and $A$ is topologically mixing (i.e., for any open sets $U, V \subset T^2$, there exists $N \in \mathbb{N}$ such that $A^n(U) \cap V \neq \emptyset$ for $n \geq N$), $F$ is transitive. \( \square \)

### 2.2 Invariant foliations for partially hyperbolic diffeomorphisms with topologically neutral center

Let $f$ be a partially hyperbolic and dynamically coherent diffeomorphism. Then one has

$$F^{ss}(F^c(x)) := \bigcup_{y \in F^c(x)} F^{ss}(y) \subset F^{cs}(x);$$

one says that the center stable foliation is complete if

$$F^{cs}(x) = F^{ss}(F^c(x)) \quad \text{for any } x \in M.$$

To our knowledge, it is still open if the center stable foliation is complete for all dynamically coherent partially hyperbolic diffeomorphisms. For the case where $f$ is a partially hyperbolic diffeomorphism with one-dimensional neutral center, it has been proved in [33] that its invariant foliations are complete and the topology of the center stable leaves is described.

**Theorem 2.4** (See [33, Theorem A]). Let $f$ be a partially hyperbolic diffeomorphism on a 3-manifold with one-dimensional neutral center. Then one has that

- the center stable and center unstable foliations are complete;
- each leaf of the center stable (resp. center unstable) foliation is a plane, a cylinder or a Möbius band;
- a center stable (resp. center unstable) leaf is a cylinder or a Möbius band if and only if such a center stable (resp. center unstable) leaf contains a compact center leaf.

**Remark 2.5.** Indeed, the second and the third items are the consequences of completeness of center stable (resp. center unstable) foliations.

According to Theorem 2.4, in the case where there is no compact center leaves, the center stable and center unstable leaves are planes, and in this case one can know which manifold supports such a partially hyperbolic diffeomorphism by the following result.

**Theorem 2.6** (See [30, Theorem 3]). Let $M$ be a closed 3-manifold. Assume that there exists a $C^0$-foliation on $M$ whose leaves are all planes. Then $M$ is $T^3$.

**Remark 2.7.** Theorem 2.6 is first proved by Rosenberg [30] assuming that the foliation is $C^2$. Then observed by Gabai, the result holds for $C^0$-foliation due to [23, Theorem 3.1], and the proof can be found in [17, Section 3].

The completeness of center stable and center unstable foliations can also be obtained in the topologically neutral case.

**Proposition 2.8.** Let $f$ be a $C^1$ partially hyperbolic diffeomorphism with topologically neutral center. Then the center stable and center unstable foliations are complete.

The proof follows as that of Theorem A in [33]. Here, we sketch the proof.
Sketch of the proof. By Theorem 2.1, \( f \) is dynamically coherent and the center is uniquely integrable. Furthermore, there exist \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that for any \( x \), if \( y \in \mathcal{F}_{ss}^c(x) \), then \( \mathcal{F}_{\delta_2}^c(y) \) intersects \( \mathcal{F}_{\delta_1}^c(x) \) into a unique point.

If the center stable leaf is not complete, then there exists a point \( x_0 \) such that

\[
\mathcal{F}_{ss}^c(\mathcal{F}_c^e(x_0)) \subseteq \mathcal{F}_{cs}^c(x_0).
\]

In this case, by [9, Proposition 1.3], there exists a strong stable leaf \( \mathcal{F}_{ss}^c(y_0) \subset \mathcal{F}_{cs}^c(x_0) \) such that

- \( \mathcal{F}_{ss}^c(y_0) \) is disjoint from \( \mathcal{F}_{ss}^c(\mathcal{F}_c^e(x_0)) \);
- there exists an arbitrarily short center path \( \sigma \) whose two endpoints are in \( \mathcal{F}_{ss}^c(y_0) \) and \( \mathcal{F}_{ss}^c(\mathcal{F}_c^e(x_0)) \), respectively.

By iterating \( \sigma \) forwardly, for \( n \) large enough \( f^n(\sigma) \) has one endpoint close enough to a point in \( \mathcal{F}_c^e(f^n(x_0)) \) and the other endpoint in \( \mathcal{F}_{ss}^c(f^n(y_0)) \) which is uniformly away from \( \mathcal{F}_c^e(f^n(x_0)) \) in this case, and therefore the length of \( f^n(\sigma) \) can be arbitrarily large contradicting the topologically neutral property.

\[\square\]

2.3 Previous classification results on 3-manifolds

In this subsection, we recall some classification results of partially hyperbolic diffeomorphisms on 3-manifolds which are used in this paper (we refer the readers to a survey [21] and the references therein for more results on classification).

In [9], Bonatti and Wilkinson classified certain transitive partially hyperbolic diffeomorphisms on 3-manifolds. As we are in the setting of dynamical coherence, for simplicity, we will present a weaker version of [9, Theorems 1 and 2].

**Theorem 2.9.** Let \( f \) be a \( C^1 \) partially hyperbolic diffeomorphism on a closed 3-manifold \( M \). Assume that \( f \) is transitive and dynamically coherent.

- If there exists a compact and invariant center leaf \( L \) such that
  \[
  W^s_\delta(L) \cap W^u_\delta(L) \setminus \{L\}
  \]
  contains a compact center leaf for some \( \delta > 0 \), then up to finite lifts, \( f \) is \( C^0 \)-conjugate to a skew product.
- If there exists a compact and periodic center leaf \( L \) such that every center leaf in \( W^s(L) \) is periodic under \( f \), then there exist \( n \in \mathbb{N} \) and \( c > 0 \) such that
  - every center leaf is \( f^n \)-invariant;
  - for any \( x \in M \), the distance of \( x \) and \( f^n(x) \) on the center leaf is bounded by \( c \);
  - the center foliation carries a continuous flow \( C^0 \)-conjugate to an expansive transitive flow.

Given two partially hyperbolic and dynamically coherent diffeomorphisms \( f \) and \( g \) on \( M \), one says that \( f \) is leaf conjugate to \( g \), if there exists a homeomorphism \( h \in \text{Homeo}(M) \) such that for any \( x \in M \),

- \( h(\mathcal{F}_g^c(x)) = \mathcal{F}_f^c(h(x)) \);
- \( h \circ g(\mathcal{F}_g^c(x)) = f \circ h(\mathcal{F}_g^c(x)) \).

Each \( f \in \text{Diff}^1(T^3) \) induces an action on the fundamental group of

\[
\mathbb{T}^3 : f_* : \pi_1(T^3) \mapsto \pi_1(T^3),
\]

which is called the linear part of \( f \).

**Theorem 2.10** (See [19, Theorem 1.3]). Let \( f \) be a dynamically coherent partially hyperbolic diffeomorphism on \( \mathbb{T}^3 \). Then \( f \) is leaf conjugate to its linear part \( f_* \).

As a consequence, one has the following result.

**Proposition 2.11.** Let \( f \) be a partially hyperbolic diffeomorphism on a closed 3-manifold \( M \). Assume that \( f \) has one-dimensional topologically neutral center. Then \( f \) has compact center leaves.
Proof. Theorem 2.1 gives the dynamical coherence of $f$. Assume, on the contrary, that $f$ does not admit any compact center leaves. Then by Theorem 2.4, all the center stable leaves are planes. By Theorem 2.6, one has that $M = \mathbb{T}^3$. By Theorem 2.10, $f$ is leaf conjugate to its linear part
\[ f_* : \pi_1(\mathbb{T}^3) \rightarrow \pi_1(\mathbb{T}^3). \]
Since the center stable leaves are planes, $f$ is isotopic to a linear Anosov map $A = f_*$ on $\mathbb{T}^3$. Therefore, $f$ is semi-conjugate to $A$ (for a proof see, for example, [29]). Moreover, the semi-conjugacy sends the center leaves of $f$ to the center leaves of $A$, and on each leaf the semi-conjugacy maps at most countably many center segments of $f$ into points (see [32]). Let $p$ be a fixed point of $f$. Then
\[ f|_{\mathcal{F}^s(p)} : \mathcal{F}^s(p) \rightarrow \mathcal{F}^c(p) \]
is semi-conjugate to a contracting or expanding affine map on $\mathbb{R}$, which contradicts the neutral property on the center. \hfill \Box

2.4 Hölder theorem

In this subsection, we recall the Hölder theorem for actions on one-dimensional manifolds. The action given by a group $\Gamma$ acting on a manifold $M$ is a free action if each non-trivial element in $\Gamma$ has no fixed points.

Theorem 2.12. Let $\Gamma$ be a group of orientation preserving homeomorphisms acting freely on $\mathbb{R}$ (resp. $S^1$). Then $\Gamma$ is isomorphic to a subgroup of translations on $\mathbb{R}$ (resp. a subgroup of $SO(2)$).

The proof of Theorem 2.12 can be founded in [24, Propositions 2.2.28 and 2.2.29, and Theorem 2.2.23].

3 $\omega$-limit sets with non-empty interior

The aim of this section is to prove Proposition 1.3.

Lemma 3.1. Let $f$ be a $C^1$ partially hyperbolic diffeomorphism. Assume that there is a point $y \in M$ whose $\omega$-limit set $\omega(y)$ has non-empty interior. Then $\omega(y)$ is saturated by strong stable and strong unstable leaves. Furthermore, if $f$ has the topologically neutral center bundle, then $\omega(y)$ is also saturated by center leaves.

Proof. Notice that the interior $\text{Int}(\omega(y))$ of $\omega(y)$ is $f$-invariant. As the positive orbit of $y$ meets the interior of $\omega(y)$, one has $y \in \text{Int}(\omega(y))$. Thus the restriction of $f$ to $\omega(y)$ is a transitive homeomorphism, and therefore there is $x \in \text{Int}(\omega(y))$ so that
\[ \alpha(x) = \omega(x) = \omega(y). \]
Since the orbit of $x$ is dense in $\omega(y)$, it suffices to show that $\mathcal{F}^{ss}(x)$ and $\mathcal{F}^{uu}(x)$ are contained in $\omega(y)$.

Since $x$ is in interior of $\omega(y)$, there exists $\delta_0 > 0$ such that the $\delta_0$-neighborhood of $x$ in $M$ is contained in $\text{Int}(\omega(y))$. For any point $z \in \mathcal{F}^{ss}(x)$, there exists $n_z \in \mathbb{N}$ such that $d(f^n(x), f^n(z)) < \delta_0/2$ for any $n \geq n_z$. As $x$ is recurrent, there exists an integer $N > n_z$ such that $f^N(x)$ is in the $\delta_0/3$-neighborhood of $x$. Therefore $f^N(z) \in \text{Int}(\omega(y))$. By the $f$-invariance of $\text{Int}(\omega(y))$, one has $z \in \text{Int}(\omega(y))$. By the arbitrariness of $z$, one has that $\mathcal{F}^{ss}(x) \subset \omega(y)$. Analogously, one can show that $\mathcal{F}^{uu}(x) \subset \omega(y)$.

Now, we prove the “furthermore” part. Since the $\delta_0$-neighborhood of $x$ is contained in the interior of $\omega(y)$, one has that $\mathcal{F}^{\omega}/2(x) \subset \omega(y)$. As the forward orbit of $x$ is dense in $\omega(y)$, by the topologically neutral property, there exists $\eta_0 > 0$ such that for any point $z \in \omega(y)$, one has that $\mathcal{F}^{\omega}_\eta(z)$ is contained in $\omega(y)$. By the arbitrariness of $z$, one has that $\mathcal{F}^c(x) \subset \omega(y)$ which by the density of the orbit of $x$ implies that $\omega(y)$ is saturated by center leaves. \hfill \Box

Ending the proof of Proposition 1.3. By Lemma 3.1, the set $\omega(y)$ is saturated by strong foliations and center foliation. Any set which is saturated by the three foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$ and $\mathcal{F}^c$ is open. As $\omega(y)$ is compact, one gets $\omega(y) = M$ as $M$ is assumed to be connected. \hfill \Box
4 Existence of invariant center metrics: Proof of Theorem A

Throughout this section, we assume that $f$ is a $C^1$ partially hyperbolic diffeomorphism on a closed connected manifold $M$ with one-dimensional topologically neutral center. By Theorem 2.1, $f$ is dynamically coherent and the center bundle is uniquely integrable.

The aim of this section is to show that if $f$ is transitive, one can define a center metric which is invariant under $f$. In this section, for notational convenience, we use $L$ or $L_i$ to denote a center leaf.

4.1 Limit center maps

**Definition 4.1.** Let $L_1$ and $L_2$ be center leaves of $f$. Consider a map $F: L_1 \to L_2$. We say that $F$ is a limit center map if there is a sequence $n_i \in \mathbb{N}$ with $|n_i| \to \infty$ so that the sequence $f^{|n_i|} L_1$ pointwise converges to $F$.

**Remark 4.2.** By continuity of the center foliation and the topologically neutral property for limit center maps (see Lemma 4.4) and for $f$ along the center, in Definition 4.1, for each compact center path $\sigma \subset L_1$, the convergence of $f^{|n_i|}\sigma$ is uniform.

The next result gives the existence of limit center maps between certain center leaves.

**Lemma 4.3.** Let $f$ be a partially hyperbolic diffeomorphism on $M$ with a one-dimensional, topologically neutral center bundle. For any $x, y \in M$, if $y \in \omega(x)$, then there exists a limit center map from $L_x$ to $L_y$. More precisely, if the sequence $f^{|n_i|}(x)$ converges to $y$ then one can extract a subsequence $\{m_j\}$ of the sequence $\{n_i\}$ so that the restriction $f^{|m_j|}L_x$ converges to a limit center map $F: L_x \to L_y$ with $F(x) = y$.

**Proof.** Let $\{x_i\}_{i \in \mathbb{N}}$ be a dense subset of $L_x$. Assume that for an integer $j \in \mathbb{N}$, one has the subsequences $\{n_k^1\} \subset \{n_k^{-1}\}$ of $\{n_k\}$ such that $f^{|n_k^1|}(x_i)$ converges to a point on $L_{y_j}$ when $k$ tends to infinity for each $l \leq j$. Now, by the topologically neutral property along the one-dimensional center bundle and $f^{|n_k^1|}(x)$ tending to $y$, there exists a subsequence $\{n_k^{j+1}\} \subset \{n_k^2\}$ such that $f^{|n_k^{j+1}|}(x_{j+1})$ converges to a point on $L_{y_j}$.

Then the diagonal argument provides a subsequence $\{m_j\}$ of $\{n_k\}$ such that for each $l$, $f^{|m_j|}(x)$ converges to a point on $L_{y_{j+1}}$ when $j$ tends to infinity. The topologically neutral property and the continuity of the center foliation give that $f^{|m_j|}L_x$ pointwise converges to a limit center map.

Now, we give some basic properties of limit center maps.

**Lemma 4.4.** Let $f$ be a partially hyperbolic diffeomorphism. Assume that the center bundle is one-dimensional and topologically neutral. Then one has the following:

1. The set of limit center maps are uniformly topologically neutral in the following sense: for any $\varepsilon > 0$ small, there exist $\delta > 0$ and $\eta > 0$ such that for any limit center map $F: L_1 \to L_2$, and two points $x, y \in L_1$, one has
   - If $d^c(x, y) < \delta$, then $d^c(F(x), F(y)) < \varepsilon$,
   - If $\varepsilon_0/4 > d^c(x, y) > \varepsilon$, then $d^c(F(x), F(y)) > \eta$,

where $d^c(\cdot, \cdot)$ denotes the distance on center leaves; in particular, $F$ is continuous.

2. Each limit center map $F$ from $L_1$ to $L_2$ is a local homeomorphism and is surjective.

3. If $F: L_1 \to L_2$ and $G: L_2 \to L_3$ are limit center maps, then $G \circ F$ is a limit center map from $L_1$ to $L_3$.

4. If $F: L \to L$ is a limit center map having a fixed point $x \in L$, then
   - $F$ is the identity map of $L$ provided that $F$ is orientation preserving;
   - $F$ is an involution on $L$ (i.e., $F^2 = \text{Id}_L$) provided that $F$ is orientation reversing.

5. If $F: L \to L$ is a limit center map, then $F$ is a homeomorphism.
Proof. By the topologically neutral property, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that any center path \( \sigma \) of length bounded by \( \delta \) has its images \( f^i(\sigma) \) whose length is bounded by \( \varepsilon \) for any \( i \in \mathbb{Z} \); by the continuity of the center foliation, this in particular gives that for any limit center map \( F : L_1 \to L_2 \) and any two points \( x, y \in L_1 \) with \( d^c(x, y) < \delta \), one has

\[
d^c(F(x), F(y)) < \varepsilon.
\]

On the other hand, if there exists \( \varepsilon_1 > 0 \) such that for any \( n \in \mathbb{N} \), there exist a limit center map \( F_n : L^n_1 \to L^n_2 \) and two points \( x_n, y_n \in L^n_1 \) such that

\[
d^c(x_n, y_n) > \varepsilon_1 \quad \text{and} \quad d^c(F_n(x_n), F_n(y_n)) < \frac{1}{n},
\]

i.e., there exist center paths whose length is uniformly bounded from below and some of whose images have length arbitrarily small, which contradicts the topologically neutral property. This proves the first item.

By the definition of limit center maps and the continuity of the center foliation, each limit center map is surjective. Since the center bundle is non-degenerate everywhere, there exists \( \varepsilon_0 > 0 \) such that the length of each compact center leaf is bounded from below by \( \varepsilon_0 \). Then there exists \( \delta_0 \in (0, \varepsilon_0/2) \) such that for any two points \( x \) and \( y \) in the same center leaf with \( d^c(x, y) < \delta_0 \), one has \( d^c(f^i(x), f^i(y)) < \varepsilon_0/4 \) for \( i \in \mathbb{Z} \). Thus, for any limit center map \( F : L_1 \to L_2 \) and any center path \( \sigma \subset L_1 \) of length \( \delta_0 \), the length of \( F(\sigma) \) is bounded by \( \varepsilon_0/4 \); by the topologically neutral property of \( F \), one has that \( F : \sigma \to F(\sigma) \) is injective and therefore is a homeomorphism. This proves the second item.

Given two limit center maps \( F : L_1 \to L_2 \) and \( G : L_2 \to L_3 \), by the second item, the map

\[
G \circ F : L_1 \to L_3
\]

is a local homeomorphism. Let \( \{n_i\} \) and \( \{m_i\} \) be the sequence of integers such that \( f^{n_i}\big|_{L_1} \) and \( f^{m_i}\big|_{L_2} \) converge to \( F \) and \( G \), respectively. Let \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) be a dense subset of \( L_1 \). Let \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) be a sequence of positive numbers such that \( \varepsilon_n \) tends to 0. For \( \varepsilon_1 \) and \( x_1 \), by the choices of \( \{n_i\} \) and \( \{m_i\} \), there exist \( l_1 \in \{n_i\} \) and \( k_1 \in \{m_i\} \) such that \( f^{l_1+k_1}(x_1) \) is \( \varepsilon_1 \)-close to \( G \circ F(x_1) \). Assume that one already has \( |l_1| < |l_2| < \cdots < |l_i| \) which are in \( \{n_j\} \) and \( |k_1| < |k_2| < \cdots < |k_i| \) which are in \( \{m_j\} \) such that \( f^{l_1+k_1}(x_1) \) is \( \varepsilon_i \)-close to \( G \circ F(x_j) \) for any \( j \leq t \leq i \). Once again, by the choice of \( \{n_j\} \) and \( \{m_j\} \), for \( \varepsilon_{i+1} \) and \( x_{i+1} \), there exist \( |l_{i+1}| > |l_i| \) and \( |k_{i+1}| > |k_i| \) such that \( f^{l_{i+1}+k_{i+1}}(x_{i+1}) \) is \( \varepsilon_{i+1} \)-close to \( G \circ F(x_j) \) for \( j \leq i + 1 \). Then one gets a sequence of integers \( n_1 = k_{i+1} + l_1 \) such that for any \( j \), \( f^{n_j}(x_1) \) tends to \( G \circ F(x_j) \) when \( i \) tends to infinity. The topologically neutral property and continuity of the center foliation give that \( f^{n_j}\big|_{L_1} \) pointwise converges to \( G \circ F \). This gives the third item.

Let \( F : L \to L \) be a limit center map with fixed points. If \( F \) preserves the orientation, let \( p \) be a fixed point and \( I \subset L \) be a small center segment such that \( F \big|_I : I \to F(I) \) is a homeomorphism and \( p \) is an endpoint of \( I \). As \( F \) is topologically neutral, all the points on \( I \) are fixed points of \( F \). By the arbitrariness of \( p \) and \( I \), one has that \( F : L \to L \) is \( \text{Id}_L \). If \( F \) reverses the orientation, then by the second item, \( F^2 : L \to L \) is a limit center map with fixed points and preserving the orientation. Therefore \( F^2 \) is \( \text{Id}_L \).

Let \( F : L \to L \) be a limit center map. If \( L \) is homeomorphic to \( \mathbb{R} \), as \( F \) is a local homeomorphism and is surjective, \( F \) is a homeomorphism. If \( L \) is homeomorphic to \( S^1 \), since the limit center map \( F : L \to L \) is a local homeomorphism on \( L \), \( F \) is an endomorphism on \( L \) of degree \( d \in \mathbb{Z} \). If \( |d| \neq 1 \), \( F \) is a covering map and therefore \( F \) has periodic points, and thus there exists \( n \in \mathbb{N} \) such that \( F^n : L \to L \) has fixed points; by the fourth item, \( F^{2n} : L \to L \) is \( \text{Id}_L \) which leads to the contradiction. \( \square \)

Lemma 4.4 motivates the following notions.

Definition 4.5. Consider the center leaves \( L, L_1 \) and \( L_2 \). We denote by \( \mathcal{L}(L) \) (resp. \( \mathcal{L}(L_1, L_2) \)) the set of all limit center maps from \( L \) to \( L \) (resp. from \( L_1 \) to \( L_2 \)).

We denote by \( \mathcal{L}^+(L) \) the subset of orientation preserving limit center maps from \( L \) to \( L \).
We denote by $\mathcal{L}^s(L_1, L_2)$ (resp. $\mathcal{L}^u(L_1, L_2)$) the subset of limit center maps from $L_1$ to $L_2$ obtained as limit of sequences $f^{n_i}|_{L_1}$ with $n_i \to +\infty$ (resp. $n_i \to -\infty$).

We define in the same way $\mathcal{L}^s(L)$ and $\mathcal{L}^u(L)$.

Now, we give the proof of Proposition 1.5.

**Proof of Proposition 1.5.** Let $x$ be a recurrent point. Then by Lemma 4.3, there exists a limit center map $F : L_x \to L_x$ having $x$ as a fixed point. By the third and fourth items in Lemma 4.4, $\text{Id}_{L_x}$ is a limit center map. Therefore, every point on $L_x$ is recurrent. \qed

**Corollary 4.6.** Let $L$ be a center leaf containing $x$ and $y$ with $y \in \omega(x)$. Then every strong stable leaf cuts $L$ in at most one point.

**Proof.** Consider a sequence $m_j \to +\infty$, given by Lemma 4.3, so that $f^{m_j}$ converges to a limit center map $F : L \to L$. Let us argue by contradiction. Assume that $z_1 \neq z_2$ are points in $L$ with $z_2 \in F^s(z_1)$. Thus $F(z_1) = F(z_2)$ and $F$ is not a homeomorphism, contradicting the fifth item of Lemma 4.4. \qed

### 4.2 Limit center maps for transitive diffeomorphisms

Let $f$ be a transitive partially hyperbolic diffeomorphism with a one-dimensional topologically neutral center bundle. We denote

$$\mathcal{N} = \{x \in M : \alpha(x) = \omega(x) = M\}.$$

Then $\mathcal{N}$ is $f$-invariant. As $f$ is transitive, then $\mathcal{N}$ is a residual subset of $M$.

We will build metrics along center leaves in the residual subset $\mathcal{N}$ of $M$ and we will show that the metric we built is $f$-invariant, continuous and invariant under holonomies of strong stable and strong unstable foliations.

In our setting, we show that $\mathcal{N}$ is saturated by center leaves and we give some description of the sets of limit center maps.

**Proposition 4.7.** Let $f$ be a $C^1$ partially hyperbolic diffeomorphism. Assume that $f$ is transitive and has one-dimensional topologically neutral center. Then for any center leaf $L$ containing a point in $\mathcal{N}$, one has the following:

- $L \subset \mathcal{N}$.
- If $L$ is not compact, then there is a homeomorphism $\psi_L : L \to \mathbb{R}$ so that

  $$\mathcal{L}^+(L) = \{\psi_L^{-1} \circ T_t \circ \psi_L, t \in \mathbb{R}\},$$

  where $T_t$ is the translation $T_t : \mathbb{R} \to \mathbb{R}$, $s \mapsto s + t$. In this case, either $\mathcal{L}(L) = \mathcal{L}^+(L)$ or $\mathcal{L}(L)$ is the group of homeomorphisms generated by $\mathcal{L}^+(L)$ and $\psi_L^{-1} \circ (-\text{Id}_\mathbb{R}) \circ \psi_L$. Furthermore, $\psi_L$ is unique up to composition by an affine map of $\mathbb{R}$.

- If $L$ is compact, then there is a homeomorphism

  $$\psi_L : L \to S^1 = \mathbb{R}/\mathbb{Z}$$

  so that

  $$\mathcal{L}^+(L) = \{\psi_L^{-1} \circ R_s \circ \psi_L, t \in S^1\},$$

  where $R_s$ is the rotation $R_s : S^1 \to S^1$, $s \mapsto s + t \mod \mathbb{Z}$. In this case, either $\mathcal{L}(L) = \mathcal{L}^+(L)$ or $\mathcal{L}(L)$ is the group of homeomorphisms generated by $\mathcal{L}^+(L)$ and $\psi_L^{-1} \circ (-\text{Id}_{S^1}) \circ \psi_L$. Furthermore, $\psi_L$ is unique up to composition by a rotation of $S^1$.

**Proof.** Let $\varepsilon_0 > 0$ be the infimum of the lengths of compact center leaves if compact center leaves exist; otherwise one takes $\varepsilon_0 = 1$.

Fix a point $x \in L \cap \mathcal{N}$. Since $L$ is one-dimensional, one gives an orientation to it. For any $\varepsilon \in (0, \varepsilon_0 / 4)$, let $I^+_\varepsilon = [x, x^+_\varepsilon]$ (resp. $I^-_\varepsilon = [x^-_\varepsilon, x^-_\varepsilon]$) be a center segment whose length is $\varepsilon$ and the direction pointing from $x$ to $x^+_\varepsilon$ (resp. $x^-_\varepsilon$) through $I^+_\varepsilon$ (resp. $I^-_\varepsilon$) coincides with the positive (resp. negative) direction of $L$. 

Claim 1. For $\varepsilon \in (0, \varepsilon_0/4)$, there exists a limit center map $F \in \mathcal{L}^+(L)$ (resp. $G \in \mathcal{L}^+(L)$) sending $x$ to a point in $I^+_\varepsilon \setminus \{x\}$ (resp. $I^-_\varepsilon \setminus \{x\}$). Moreover, such limit center maps can be obtained by the forward and backward iterates of $f$, respectively.

Now we give the proof of the claim. We only deal with the case for $I^+_\varepsilon$ and prove the claim only using the fact $\omega(x) = M$ (the other cases follow analogously).

As $\omega(x) = M$, by Lemma 4.3, there exists a limit map $\tilde{F} : L \to L$ sending $x$ to $x^+_\varepsilon$. If $\tilde{F}$ preserves the orientation, one can conclude. If $\tilde{F}$ preserves the orientation, by the fourth item of Lemma 4.4, one has $\tilde{F}^2 = \text{Id}_L$. In this case $\tilde{F}(x^+_\varepsilon) = x$, therefore there exists an $\tilde{F}$-fixed point $z_\varepsilon \in \text{Int} (I^+_\varepsilon)$. Now, consider a limit center map $\hat{H} : L \to L$ sending $x$ to $z_\varepsilon$. If $\hat{H}$ preserves the orientation, one can also conclude. If $\hat{H}$ reverses the orientation, we consider the map $F = \tilde{F} \circ \hat{H}$ which is a limit center map from $L$ to $L$ by the third item of Lemma 4.4 and preserves the orientation of $L$. Since $\hat{H}(x) = z_\varepsilon$ and $\tilde{F}(z_\varepsilon) = z_\varepsilon$, one has $F(x) = z_\varepsilon$. This ends the proof of the claim.

Now, we show that there exist limit center maps preserving the orientation and sending $x$ to any point in $L$. To be precise, we have the following claim.

Claim 2. For any point $y \in L$, there exists a limit center map $F \in \mathcal{L}^+(L)$ which sends $x$ to $y$. Moreover, one can obtain such limit center maps by the forward as well as the backward iterates of $f$.

Now we give the proof of the claim. Consider the set

$$\mathcal{A} := \{ y \in L : \text{there exists } F \in \mathcal{L}^+(L) \text{ such that } F(x) = y \}.$$ 

The claim is reformulated as $\mathcal{A} = L$. It suffices to show that one can obtain limit center maps which preserve the orientation and send $x$ to any point in $L$ by the forward iterates of $f$. The other case would follow analogously.

By Claim 1, the set $\mathcal{A}$ is non-empty. We will show that $\mathcal{A}$ is a closed subset of $L$. Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence of points in $\mathcal{A}$ which tends to $y_0$ according to the distance on $L$. Now, one fixes a small neighborhood of $y_0$. Then one gives an orientation to those center plaques in this neighborhood of $y_0$ according to the orientation of the local center plaque $L_{\text{loc}}(y_0)$ of $y_0$. For any $l \in \mathbb{N}$, take $y_{n_l}$ which is $1/l$-close to $y_0$ on $L$. As $y_{n_l} \in \mathcal{A}$, one can choose $m_l \in \mathbb{N}$ large enough such that $f^{m_l}(x)$ is $1/l$-close to $y_{n_l}$ and

$$f^{m_l} : L_{\text{loc}}(x) \to L_{\text{loc}}(f^{m_l}(x))$$

preserves the orientation of local plaques. Now, one gets a sequence of positive integers $\{m_l\}$ tending to infinity such that $f^{m_l}(x)$ tends to $y_0$ and

$$f^{m_l} : L_{\text{loc}}(x) \to L_{\text{loc}}(f^{m_l}(x))$$

preserves the orientation of local plaques. By Lemma 4.3, there exists a limit center map $F \in \mathcal{L}^+(L)$ with $F(x) = y_0$.

If $\mathcal{A} \neq L$, then there exists an open center path

$$(a, b) \cap \mathcal{A} = \emptyset$$

and one endpoint is in $\mathcal{A}$. Without loss of generality, one can assume $a \in \mathcal{A}$. Let $F_a \in \mathcal{L}^+(L)$ be a map such that $F_a(x) = a$. For $\varepsilon > 0$, consider the center path $I^+_\varepsilon = [x, x^+_\varepsilon]^+$ as before. By Claim 1, there exists $F_\varepsilon \in \mathcal{L}^+(L)$ sending $x$ to a point in the interior of $I^+_\varepsilon$. As the limit center maps are uniformly topological neutral (due to the first item of Lemma 4.4), for $\varepsilon > 0$, the limit center map $F_\varepsilon \circ F_a$ sends $x$ to a point in $(a, b)$ which gives the contradiction. This ends the proof of the claim.

Now, consider the sets

$$\mathcal{B}^+ = \{ y \in L : \omega(y) = M \} \quad \text{and} \quad \mathcal{B}^- = \{ y \in L : \alpha(y) = M \},$$

which are non-empty since $x \in \mathcal{B}^+ \cap \mathcal{B}^-$. The following claim gives that $\mathcal{N}$ is saturated by center leaves.

Claim 3. It holds that $\mathcal{B}^+ = \mathcal{B}^- = L$. 

Now we give the proof of the claim. One only needs to deal with $B^+$ and the case for $B^-$ would follow analogously.

We will first show that $B^+$ is a closed subset of $L$. Let $z_n$ be a sequence of points in $L$ such that $z_n$ tends to a point $z \in L$ according to the distance on $L$. For $\varepsilon > 0$, let $z_n$ be a point close enough to $z$ such that for the shortest center path $\sigma_n$ connecting $z_n$ and $z$, the length of $f^i(\sigma_n)$ is bounded by $\varepsilon/2$ for $i \in \mathbb{Z}$ due to the topologically neutral property on the center bundle. As $z_n \in B$, one can take $m_n \in \mathbb{N}$ large enough such that $d(f^{m_n}(z_n), x) < \varepsilon/2$, which implies that $d(f^{m_n}(z), x) < \varepsilon$. The arbitrariness of $\varepsilon$ gives $x \in \omega(z)$ which implies $z \in B^+$.

Assume, on the contrary, that $B^+$ is not the whole center leaf $L$. Since $B^+$ is closed in $L$, there exists a center path $\sigma = [z, w] \subset L$ such that

- its interior is disjoint from $B^+$;
- one of its endpoints is in $B^+$;
- the orientation pointing from $z$ to $w$ in $\sigma$ coincides with the positive orientation of $L$.

Without loss of generality, one can assume that $w \in B^+$. By the topologically neutral property on the center bundle, there exists $\delta_0 > 0$ such that the length of $f^i(\sigma)$ is bounded from below by $\delta_0$ for any $i \in \mathbb{Z}$. Consider a short center path $[x, p]$ in $L$ such that its length is much smaller than $\delta_0$ and the orientation of $[x, p]$ pointing from $x$ to $p$ coincides with the positive orientation of $L$. As $w \in B^+$, one can apply Claim 2 to $w$ with respect to the forward iterates of $f$, and one gets a limit center map $F : L \to L$ which is orientation-preserving and maps $w$ to $p$. This implies that there exists a point $w_0$ in the interior of $[z, w]$ whose $\omega$-limit set contains $x$. As $x \in N$, one has $w \in B^+$ and one obtains the contradiction. This ends the proof of the claim.

In the following, we will show that $L^+(L)$ is a group; in particular, this implies that the limit center map in $L^+(L)$ sending one specific point to another one is unique. By Claim 2, the third and fourth items of Lemma 4.4, one has

- $\text{Id}_L \in L^+(L)$;
- for any $F, G \in L^+(L)$, $F \circ G \in L^+(L)$.

To prove that $L^+(L)$ is a group, one needs to check that for any $F \in L^+(L)$, there exists $G \in L^+(L)$ such that

$$F \circ G = G \circ F = \text{Id}_L.$$  

Let $F \in L^+(L)$ such that $F(x) = y$ for some $y \in L$. As $L \subset N$, there exists $G \in L^+(L)$ such that $G(y) = x$. Then the limit center map $G \circ F$ has a fixed point. By the fourth item of Lemma 4.4, $G \circ F = \text{Id}_L$. By the fifth item of Lemma 4.4, $F$ and $G$ are homeomorphisms on $L$. Therefore $F \circ G = \text{Id}_L$.

To summarize, the action on $L$ given by the group $L^+(L)$ is free and transitive. By the H"older theorem (i.e., Theorem 2.12), the group $L^+(L)$ is isomorphic to the group of translations (resp. rotations) on $\mathbb{R}$ (resp. $S^1$) if $L$ is homeomorphic to $\mathbb{R}$ (resp. $S^1$). As each orientation reversing limit center map from $L$ to $L$ is an involution, $L(L)$ is a group; moreover, $L(L)$ either coincides with $L^+(L)$ or is the group generated by $L^+(L)$ and $-\text{Id}_L$.

The next remark explains why these properties are key points for the proof of Theorem A.

**Remark 4.8.** The Euclidean metric on $\mathbb{R}$ (resp. on $\mathbb{R}/\mathbb{Z}$) is invariant under the action of the group generated by the translations and $-\text{Id}_\mathbb{R}$ (resp. by the rotations and $-\text{Id}_{S^1}$) and any invariant metric by the set of translations (resp. rotations) is obtained by multiplying the Euclidean metric by a scalar.

Lemma 4.3 gives that for any $x, y \in M$, if $y \in \omega(x)$, then there exists a limit center map from $L_x$ to $L_y$; this allows us to build the connections between the limit center maps on different center leaves.

**Lemma 4.9.** Let $f$ be a $C^1$ partially hyperbolic diffeomorphism. Assume that $f$ is transitive and has one-dimensional topologically neutral center. Then for any two center leaves $L_1$ and $L_2$, each of which contains a point in $N$, one has

- each limit center map from $L_1$ to $L_2$ is a homeomorphism;
- for any limit center maps $F, G \in L(L_1, L_2)$, there are $F_1 \in L(L_1)$ and $F_2 \in L(L_2)$ so that

$$G = F \circ F_1 = F_2 \circ F.$$
Proof. By Proposition 4.7 and the assumption, \( L_1 \cup L_2 \) is contained in \( \mathcal{N} \).

Let \( H \in \mathcal{L}(L_1, L_2) \) be a limit center map. By Lemma 4.3 and the fact that \( L_2 \subset \mathcal{N} \), for a point \( x \in L_1 \), there exists a limit center map \( \Phi : L_2 \to L_1 \) with \( \Phi(H(x)) = x \). By the third item of Lemma 4.4, \( \Phi \circ H \) is a limit center map from \( L_1 \) to \( L_1 \) which is a homeomorphism due to the fourth item of Lemma 4.4. Therefore \( H \) is injective. As \( H \) is surjective, one has that \( H \) is a homeomorphism with \( H^{-1} = \Phi \).

As the center bundle is one-dimensional, one can give an orientation to \( L_1 \) and \( L_2 \) respectively such that \( F \) preserves the orientation. As \( F \) and \( G \) are surjective, there exist \( x_1, x_2 \in L_1 \) such that \( F(x_1) = G(x_2) \). By Proposition 4.7, there exists a limit center map \( F_1 \in \mathcal{L}^+(L_1) \) such that \( F_1(x_2) = x_1 \). Therefore

\[
F \circ F_1(x_2) = G(x_2).
\]

Let \( H : L_2 \to L_1 \) be a limit center map with \( H \circ G(x_2) = x_2 \). If \( G \) also preserves the orientation, then

\[
H \circ (F \circ F_1) = H \circ G
\]

since they have a common fixed point and simultaneously preserve or reverse the orientation, which implies that \( F \circ F_1 = G \). If \( G \) reverses the orientation, then one of the maps \( H \circ G \) and \( H \circ F \circ F_1 \) reverses the orientation, and thus \( \mathcal{L}(L_1) \) contains an involution. By Proposition 4.7, there exists an involution \( R \in \mathcal{L}(L_1) \) having \( x_2 \) as a fixed point, and then the map \( F \circ (F_1 \circ R) \) reverses the orientation. An analogous argument as above gives that \( F \circ (F_1 \circ R) = G \).

Similarly, one can show that there exists \( F_2 \in \mathcal{L}(L_2) \) with \( F_2 \circ F = G \). \( \square \)

The first item of Lemma 4.9 allows us to consider the image \( F_*(\ell) \) of an \( \mathcal{L}(L) \)-invariant metric \( \ell \) on a center leaf \( L \subset \mathcal{N} \) by a limit center map \( F \in \mathcal{L}(L, L_1) \) for \( L_1 \subset \mathcal{N} \), as \( F \) is a homeomorphism. The second item gives that the metric \( F_*(\ell) \) on \( L_1 \) is independent of the choice of \( F \) and is \( \mathcal{L}(L_1) \)-invariant.

**Corollary 4.10.** Consider a center leaf \( L \) containing a point in \( \mathcal{N} \) (equivalently, included in \( \mathcal{N} \)), and fix an \( \mathcal{L}(L) \)-invariant metric \( \ell_L \) on \( L \).

For any center leaf \( L_1 \subset \mathcal{N} \) and any two limit center maps \( F_1, F_2 \in \mathcal{L}(L, L_1) \), the image metrics \( (F_1)_*(\ell_L) \) and \( (F_2)_*(\ell_L) \) are equal, i.e.,

\[
(F_1)_*(\ell_L) = (F_2)_*(\ell_L).
\]

Let us denote by \( \ell_{L_1} \) this metric. Then \( \ell_{L_1} \) is invariant under the action of \( \mathcal{L}(L_1) \).

**Remark 4.11.** Let \( L \subset \mathcal{N} \) be a center leaf and \( \ell_L \) be an \( \mathcal{L}(L) \)-invariant metric on \( L \). Then \( f(L) \subset \mathcal{N} \). Furthermore, for any \( F \in \mathcal{L}(L, f(L)) \), notice that \( f^{-1} \circ F \in \mathcal{L}(L) \), and thus

\[
F_*(\ell_L) = f_*(\ell_L).
\]

To summarize, we get the following proposition.

**Proposition 4.12.** Let \( f \) be a \( C^1 \) partially hyperbolic diffeomorphism. Assume that \( f \) is transitive and has one-dimensional topologically neutral center. Then there is a family \( \{ \ell_L \}_{\text{center leaf in } \mathcal{N}} \) of metrics in the center leaves contained in \( \mathcal{N} \), so that

- for any center leaves \( L_1 \) and \( L_2 \) contained in \( \mathcal{N} \) and any \( F \in \mathcal{L}(L_1, L_2) \), one has
  \[
  F_*(\ell_{L_1}) = \ell_{L_2};
  \]

- for any center leaf \( L \subset \mathcal{N} \), one has
  \[
  f_*(\ell_L) = \ell_{f(L)}.
  \]

Furthermore, if \( \{ \ell_L \}_{\text{center leaf in } \mathcal{N}} \) is another family of metrics satisfying the properties above, then there is \( \lambda > 0 \) so that for any \( L \subset \mathcal{N} \) one has

\[
\tilde{\ell}_L = \lambda \cdot \ell_L.
\]

Thus to prove Theorem A, it remains to show that the family of metrics \( \{ \ell_L \}_{\text{center leaf in } \mathcal{N}} \) extends in a continuous way as a center metric on all \( M \). The main tool for proving that is to check that the family \( \{ \ell_L \} \) is invariant by the holonomies of the strong stable and strong unstable foliations, which is the aim of the next section.
4.3 Holonomy invariance and continuity: Ending the proof of Theorem A

In this subsection, we keep the notations from Subsection 4.2. The following lemma tells us that the strong stable holonomy is well defined restricted to $N$.

**Lemma 4.13.** Let $f$ be a $C^1$ partially hyperbolic diffeomorphism. Assume that $f$ is transitive and has one-dimensional topologically neutral center. Let $L_1$ and $L_2$ be two center leaves contained in $N$ and in the same center-stable leaf $L^{cs}$. Then the holonomy of the strong stable foliation induces a homeomorphism from $L_1$ to $L_2$.

**Proof.** Recall that the center stable foliation has the completeness property (due to Proposition 2.8). Thus our assumption says that $L_2$ is contained in the union of the strong stable leaves through $L_1$ which coincides with $L^{cs}$ and vice versa. According to Corollary 4.6, each strong stable leaf cuts $L_1$ in at most one point, and the same for $L_2$. Then each strong stable leaf in $L^{cs}$ cuts $L_1$ and $L_2$ in exactly one point respectively inducing a 1-to-1 correspondence, and proving the lemma. \square

**Lemma 4.14.** Let $L_1$ and $L_2$ be two center leaves contained in $N$ and in the same center-stable leaf $L^{cs}$. Let $H^{ss}: L_2 \to L_1$ be the holonomy of the strong stable foliation given by Lemma 4.13, and \( \{\ell_L\}_{L \text{ center leaf in } N} \) be a family of metrics in the center leaves, given by Proposition 4.12. Then we have $\ell_{L_1} = (H^{ss})_\ast(\ell_{L_2})$.

**Proof.** Let us fix a sequence $n_i \to +\infty$ so that the restriction $f^{n_i}|_{L_2}$ converges to a limit center map $F \in L^s(L_2, L_1)$.

According to Proposition 4.12 one has $F_\ast(\ell_{L_2}) = \ell_{L_1}$.

On the other hand, as we are iterating positively, points in the same strong stable leaf have the same limit; therefore the restriction $f^{n_i}|_{L_1}$ converges to $F \circ (H^{ss})^{-1} = \tilde{F} \in L^s(L_1, L_1)$.

Thus $$(F \circ (H^{ss})^{-1})_\ast(\ell_{L_1}) = \tilde{F}_\ast(\ell_{L_1}) = \ell_{L_1}.$$ One deduces $$F_\ast((H^{ss})^{-1}_\ast(\ell_{L_1})) = F_\ast(\ell_{L_2}),$$ i.e., $$\ell_{L_2} = (H^{ss})^{-1}_\ast(\ell_{L_1}),$$ which concludes the proof. \square

Remark 4.11 gives the $f$-invariance of the center metric defined on $N$. The next proposition gives a continuous family of metrics on all the center leaves, and therefore ends the proof of Theorem A.

**Proposition 4.15.** Let $f$ be a $C^1$ partially hyperbolic diffeomorphism. Assume that $f$ is transitive and has one-dimensional topologically neutral center. Let $\{\ell_L\}_{L \text{ center leaf in } N}$ be a family of metrics in the center leaves, given by Proposition 4.12. Then this family of metrics in the center leaves in $N$ can be extended in a unique way, by continuity, to all the center leaves, defining a center metric on $M$.

**Proof.** We denote $s = \dim(E^s)$ and $u = \dim(E^u)$. We consider a finite open cover $\{V_i\}$ of $M$ given by compact $C^0$-foliated boxes $\varphi_i: [-2, 2]^s \times [-2, 2]^u \times [-2, 2] \to M$ so that

- $V_i = \varphi_i((-1,1)^s \times (-1,1)^u \times (-1,1))$;
- each square $[-2, 2]^s \times \{y\} \times [-2, 2]$ is contained in a center stable leaf;
- each square $\{x\} \times [-2, 2]^u \times [-2, 2]$ is contained in a center unstable stable leaf;
- for any two points $(x_1, y_1), (x_2, y_2) \in [-1, 1]^s \times [-1, 1]^u$, as both $$W^{ss}_{\text{loc}}((x_1, y_1) \times [-2, 2]) \cap W^{su}_{\text{loc}}((x_2, y_2) \times [-2, 2])$$ and $$W^{su}_{\text{loc}}((x_1, y_1) \times [-2, 2]) \cap W^{ss}_{\text{loc}}((x_2, y_2) \times [-2, 2])$$
consist of a unique center path $L_1$ and $L_2$, respectively, then the local strong stable (resp. strong unstable) holonomy map sends $(x_1, y_1) \times [-1, 1]$ into $L_1$ (resp. $L_2$) and its image is sent by the local strong unstable (resp. strong stable) holonomy map into the interior of $(x_2, y_2) \times [-2, 2]$.

Let us denote

$$U_i = \varphi_i([-2, 2]^s \times [-2, 2]^u \times [-2, 2]).$$

For each $p \in U_i$ (resp. $V_i$), we denote by $L_p \cap U_i$ (resp. $L_p \cap V_i$) the connected component of $L_p \cap U_i$ (resp. $L_p \cap V_i$) containing the point $p$, where $L_p$ denotes the center leaf through $p$.

We define a metric $\ell_i$ on center segments contained in $V_i$ as follows. As $\mathcal{N}$ is a dense subset of $M$, for each point $p \in V_i$, there exists a sequence of points $\{q_n\}_{n \in \mathbb{N}} \subset \mathcal{N}$ with $q_n$ tending to $p$. For $n_1$ and $n_2$ large enough, the intersection $W^s_{\text{loc}}(L_{q_{n_1}} | V_i) \cap W^u_{\text{loc}}(L_{q_{n_2}} | V_i)$ is non-empty and is contained in $\mathcal{N}$. As the center metric is invariant under strong stable and unstable holonomies, by the uniform continuity of the local strong stable and unstable holonomies in $U_i$, one deduces that the center metric on $L_{q_n} | V_i$ uniquely induces a center metric on the $L_p | V_i$, and hence one gets a metric on each center plaque in $V_i$. Moreover, the uniqueness gives the continuity of the center metric in $V_i$. Notice that the center metric on each center path is independent of the choice of $V_i$, which allows us to define the center metric on the whole center leaf. Since the center metric on $\mathcal{N}$ is invariant under the dynamics $f$ and invariant under the strong stable and unstable holonomies, by the continuity of the center metric and the strong stable and unstable holonomies, the center metric is invariant everywhere under the dynamics and the strong stable and unstable holonomies.

\section{Existence of periodic compact center leaves}

In this section, we first work in any dimension and show that for partially hyperbolic diffeomorphisms with one-dimensional topologically neutral center, if there exist compact center leaves, then there exist periodic compact center leaves. Then we give some consequences in dimension three.

The following general result is needed in this part.

\textbf{Lemma 5.1.} Let $f$ be a dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center and $L$ be a compact center leaf. For $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that for any compact center leaf $L'$ in the $\delta$-tubular neighborhood of $L$, the intersection $W^u_\varepsilon(L) \cap W^s_\varepsilon(L')$ consists of finitely many compact center leaves.

\textbf{Proof.} Let $\varepsilon_0 > 0$ be small enough such that one can define an $\varepsilon_0$-tubular neighborhood $V$ of $L$ together with a $C^1$ projection $\pi : V \to L$ such that each fiber $\pi^{-1}(x)$ (for $x \in L$) is transverse to the center foliation.

For $\varepsilon \in (0, \varepsilon_0)$, by the uniform transversality between $E^s \oplus E^c$ and $E^u$, there exists $\delta \in (0, \varepsilon/4)$ such that for any two points $x, y \in M$ with $d(x, y) < \delta$, one has

- the intersection $\mathcal{F}^u_\varepsilon(x) \cap \mathcal{F}^s_\varepsilon(y)$ consists of exactly one point;
- $\mathcal{F}^u_{\varepsilon/2}(x) \cap \mathcal{F}^s_{\varepsilon/2}(y) = \mathcal{F}^u_{\varepsilon/2}(x) \cap \mathcal{F}^s_{\varepsilon/2}(y)$.

Let $L'$ be a compact center leaf in the $\delta$-tubular neighborhood of $L$. Then for any $x \in L$, by the choice of $\delta$, one has that $\mathcal{F}^u_\varepsilon(x) \cap \mathcal{F}^s_\varepsilon(L')$ consists of finitely many points and is $\varepsilon/2$ away from the boundaries of $\mathcal{F}^u_\varepsilon(x)$ and $\mathcal{F}^s_\varepsilon(L')$. This gives that $W^u_\varepsilon(L) \cap W^s_\varepsilon(L')$ consists of finitely many compact center leaves.

In the following, we consider the case where there exists a compact center leaf $\gamma$ for a partially hyperbolic diffeomorphism with topologically neutral center. We will show that one can always find a compact and periodic center leaf. The proof uses the notion of bad sets for a compact lamination introduced in [16] and a Bowen-type shadowing lemma given in the appendix (see also [2, 13]).

\textbf{Proposition 5.2.} Let $f$ be a partially hyperbolic diffeomorphism. Assume that $f$ has one-dimensional topologically neutral center and admits a compact center leaf $\gamma$. Then $f$ has a compact periodic center leaf. Moreover, if $\gamma$ is not periodic, then there exists a compact periodic center leaf whose center stable manifold contains another different compact center leaf.
Proof. If $\gamma$ is periodic, we are done.

Now, we assume that $\gamma$ is non-periodic. Let $x \in \gamma$. Then we consider the $\omega$-limit set $\omega(x)$ of $x$. By the topologically neutral property, there exists a compact $f$-invariant set $\Lambda$ saturated by compact center leaves whose length are uniformly bounded. If $\Lambda$ contains a compact periodic center leaf $L$, then for an arbitrarily small tubular neighborhood of $L$, by the topologically neutral property, there exists $n \in \mathbb{N}$ such that $f^n(\gamma)$ is entirely contained in the tubular neighborhood of $L$, and thus one can apply Lemma 5.1 to conclude.

Now, we only need to deal with the case where $\Lambda$ does not contain periodic center leaves. We define a function $\ell : \Lambda \to \mathbb{R}_+$ by associating $x \in \Lambda$ with the length of the center leaf through $x$. By continuity of the center foliation, the function $\ell$ varies lower semi-continuously. Now we define the bad set for $\ell$. Let us denote $\Lambda_0 = \Lambda$. For $i \in \mathbb{N}$, one defines the $(i+1)$-th bad set by

$$\Lambda_{i+1} = \{ x \in \Lambda_i : \ell |_{\Lambda_i} \text{ is not continuous at } x \}.$$ 

The $f$-invariance of the center foliation implies that $\Lambda_i$ is $f$-invariant. Notice that $\ell |_{\Lambda_i}$ is continuous at $x \in \Lambda_i$ if and only if each holonomy map restricted to $\Lambda_i$ for $F^c(x)$ is identity, and hence the continuous points of $\ell_{\Lambda_i}$ form an open set which implies that $\Lambda_{i+1}$ is compact. Since the length of center leaves in $\Lambda$ are uniformly bounded from above, there exists $i_0 \in \Lambda$ such that $\ell |_{\Lambda_{i_0}}$ is a continuous map. By Proposition A.1, arbitrarily close to $\Lambda_{i_0}$, there exists a compact and periodic center leaf $L$ whose stable manifold contains another compact center leaf.

As an application, we obtain the following consequence on 3-manifolds.

**Proposition 5.3.** Let $f$ be a transitive partially hyperbolic diffeomorphism on a closed 3-manifold $M$. Assume that

- $f$ has one-dimensional topologically neutral center;
- there exist two different compact center leaves which are in the same center stable leaf.

Then up to finite lifts and iterates, $f$ is $C^0$-conjugate to a skew product.

Proof. Let $\gamma$ and $\gamma'$ be two compact center leaves of $f$ which are in the same center stable leaf. By Proposition 5.2, without loss of generality, one can assume that $\gamma$ is a periodic center leaf. Thanks to Proposition 1.4, one can assume that $f(\gamma) = \gamma$ for simplicity.

The compact leaves $\gamma'$ and $\gamma$ bound a region $C$ in $W^s(\gamma)$ which is an annulus or a M"obius band. By the Poincaré-Bendixson theorem, for each point $x \in C$, either $F^c(x)$ is compact or $F^c(x)$ consists of $F^c(x)$ and two compact center leaves in $C$. Since $f$ is transitive, there exist a point $x_0 \in C$ and $n \in \mathbb{N}$ such that $f^{-n}(x_0)$ is in $W^s_{loc}(\gamma)$. Once again, by the Poincaré-Bendixson theorem, there exists a compact center leaf in $F^c(f^{-n}(x_0)) \cap W^s_{loc}(\gamma)$. Since

$$\frac{F^c(f^{-n}(x_0)) \subset f^{-n}(C) \subset W^s(\gamma)},$$

the intersection of stable manifolds and unstable manifolds of $\gamma$ contains an entire compact center leaf. By the first item in Theorem 2.9, modulo finite lifts and iterates, $f$ is $C^0$-conjugate to a skew product.

As a corollary of Propositions 5.2 and 5.3, one has the following consequence.

**Corollary 5.4.** Let $f$ be a partially hyperbolic diffeomorphism on a closed 3-manifold. Assume that

- $f$ is transitive and has one-dimensional topologically neutral center;
- $f$ simultaneously has compact and non-compact center leaves.

Then all the compact center leaves are periodic under $f$.

6 Classification of transitive partially hyperbolic diffeomorphisms with neutral center: Proof of Theorem C

In this section, we first recall the notion of $N$-th intersection of a hyperbolic saddle for surface diffeomorphisms (introduced in [7]) and some properties of $N$-th intersection sets. Then we extend this notion to
the partially hyperbolic setting for a compact periodic center leaf provided that the system is transitive and has topologically neutral center. At last, we give the proof of Theorem C.

6.1 N-th intersection of a hyperbolic saddle

Now, we introduce the notion of N-th intersection for a hyperbolic saddle of a surface diffeomorphism.

Let $f$ be a $C^1$ diffeomorphism on a surface $S$ and $p$ be a hyperbolic saddle. Assume that the stable and unstable manifolds of $p$ are homeomorphic to $\mathbb{R}$. Then for each $x \in W^s(p)$, we denote by $I^s_x$ the compact segment in $W^s(p)$ bounded by $p$ and $x$. Analogously, one defines $I^u_x$ for $x \in W^u(p)$.

**Definition 6.1.** Let $f$ be a $C^1$ diffeomorphism on a surface $S$ and $p$ be a hyperbolic saddle. Assume that there is no homoclinic tangency between the stable and unstable manifolds of $p$. A point $x \in W^s(p) \cap W^u(p) \setminus \{p\}$ is called the N-th intersection of the invariant manifolds of $p$, if

$$\#(I^s_x \cap I^u_x \setminus \{p\}) = N.$$

We define a function

$$x \in W^s(p) \cap W^u(p) \setminus \{p\} \mapsto n(x) \in \mathbb{N}$$

provided that $x$ is the $n(x)$-th intersection of the invariant manifolds of $p$ (see Figure 1).

**Remark 6.2.**

- Notice that the invariant manifolds of $p$ under $f$ coincide with the invariant manifolds of $f^k$ for any $k \in \mathbb{N}^+$, and thus the N-th intersections of invariant manifolds under $f$ coincide with the ones under $f^k$ for any $k \in \mathbb{N}^+$.
- For any $x \in W^s(p) \cap W^u(p) \setminus \{p\}$, one has $f(I^s_x) = I^s_{f(x)}$ and $f(I^u_x) = I^u_{f(x)}$, which implies that $n(x) = n(f(x))$.

In the following, we will show that for each $N \in \mathbb{N}$ there are finitely many homoclinic orbits which are j-th intersection for $j \leq N$.

**Proposition 6.3.** Let $p$ be a hyperbolic saddle of a surface diffeomorphism $f$, and assume that $p$ has no homoclinic tangencies. For any $N \in \mathbb{N}$, one has

$$\#\{\text{Orb}(x) : x \in W^s(p) \cap W^u(p) \setminus \{p\} \text{ and } n(x) \leq N\} < +\infty.$$  

Proof. Denote by $J^{s,+}$ and $J^{s,-}$ the two separatrices of the stable manifold of $p$ and by $J^{u,+}$ and $J^{u,-}$ the two separatrices of the unstable manifold of $p$. Up to replacing $f$ by $f^2$, we may assume that $f$ preserves these separatrices. One only needs to prove the proposition for the intersections between $J^{s,+}$ and $J^{u,+}$, and the rest case would follow analogously.
Let \( x \in J^{s,+}(p) \cap J^{u,+}(p) \) be a homoclinic intersection of \( p \) such that
\[
n(x) = \sup\{n(y) : y \in J^{s,+} \cap J^{u,+} \setminus \{p\} \text{ and } n(y) \leq N \}.
\]
Since for any \( z \in J^{s,+} \cap J^{u,+} \setminus \{p\} \), the number \( n(z) \) is invariant under \( f \). Without loss of generality, one can assume that \( z \in I^s_z \setminus I^s_{f(x)} \). If \( z \not\in I^s_{f(x)} \), then \( f(x) \in I^s_z \cap I^s_u \) which implies that \( n(z) \geq n(x) + 1 \). Therefore, for each homoclinic point \( z \in J^{s,+} \cap J^{u,+} \setminus \{p\} \) with \( n(z) \leq n(x) \), up to finite iterates, one has
\[
z \in (I^s_z \setminus I^s_{f(x)}) \cap I^s_{f(x)}.
\]
Since there are no homoclinic tangencies for \( p \), one has
\[
\#(I^s_z \setminus I^s_{f(x)}) \cap I^s_{f(x)} < \infty,
\]
ending the proof of Proposition 6.3. \( \square \)

### 6.2 \( N \)-th intersection for a periodic compact center leaf

The idea is to “modulo the center foliation”, and we “come to” the surface case and we define the \( N \)-th intersection for a periodic compact center leaf. The difficulty comes from checking that the notion is well defined along the center leaves and is overcome by the center flows given by Theorem B.

Before defining the intersection number for a compact periodic center leaf, we need some preparations.

**Lemma 6.4.** Let \( f \) be a partially hyperbolic diffeomorphism on a closed 3-manifold \( M \). Assume that \( f \) has one-dimensional topologically neutral center and has a periodic compact center leaf \( \gamma \). Then one has
- \( \mathcal{F}^{ss}(x) \cap \gamma = \mathcal{F}^{uu}(x) \cap \gamma = \{x\} \), for each \( x \in \gamma \);
- if \( f \) is transitive, then the intersection of \( W^s(\gamma) \cap W^u(\gamma) \) is dense in \( W^s(\gamma) \) and \( W^u(\gamma) \).

**Proof.** Assume, on the contrary, that there exist \( x, y \in \gamma \) with \( x \in \mathcal{F}^{ss}(y) \). Then by iterating \( x \) and \( y \) forwardly, one gets that for any \( \varepsilon > 0 \), there exists a point \( z_\varepsilon \in \gamma \) such that \( \mathcal{F}^{ss}(z_\varepsilon) \) intersects \( \gamma \) into at least two points, which contradicts the transversality in \( E^{cs} \) between \( E^s \) and \( E^c \).

Let \( k \) be the period of the center leaf \( \gamma \) under \( f \). By Proposition 1.4, \( f^k \) is still transitive and \( \gamma \) is \( f^k \)-invariant. Then one concludes by transitivity (see [9, Corollary 1.2] for a proof in a more general case). \( \square \)

Let us fix some notations before defining the \( N \)-th intersection. Let \( f \) be a partially hyperbolic diffeomorphism on a closed 3-manifold with the following properties:
- \( f \) has one-dimensional topologically neutral center;
- \( f \) admits a periodic compact center leaf \( \gamma \);
- the bundles \( E^s, E^c \) and \( E^u \) are orientable.

For \( x \in W^u(\gamma) \cap W^s(\gamma) \setminus \{\gamma\} \), let \( x^s = \mathcal{F}^{ss}(x) \cap \gamma \) and \( x^u = \mathcal{F}^{uu}(x) \cap \gamma \) (which is unique due to Lemma 6.4). We denote by \( I^{ss}_x \) the compact strong stable segments bounded by \( x \) and \( x^s \). Analogously, one can define \( I^{uu}_x \) associated with \( \gamma \) and \( x \) and \( x^u \). When there is no confusion, we will drop the index \( \gamma \) for simplicity. By the completeness of the invariant foliations, the center leaf \( \mathcal{F}^c(x) \) intersects \( I^{ss}_x \) and \( I^{uu}_x \) into infinitely many points, respectively. Let \( x^*_1 \in \mathcal{F}^c(x) \cap I^{ss}_x \) be a point such that the open strong stable segment \( (x^*_1, x)^{ss} \) is disjoint from \( \mathcal{F}^c(x) \), and let \( x^*_1 \in \mathcal{F}^c(x) \cap I^{uu}_x \) be the point analogously defined for the strong unstable. Then the center segment \( (x^*_1, x)^c \), the strong stable segment \( I^{ss}_x \) and \( \gamma \) bound a compact center stable submanifold and we denote it as \( I^{cs}(x) \); likewise, one gets a compact center unstable submanifold denoted as \( I^{cu}(x) \) (see Figure 2).

To guarantee that the notion is well defined, we need to put more restrictions on the diffeomorphism than the case for a surface diffeomorphism.

**Definition 6.5.** Consider a partially hyperbolic diffeomorphism \( f \) on a closed 3-manifold with the following properties:
- \( f \) is transitive and has one-dimensional topologically neutral center;
- \( f \) admits a periodic compact center leaf \( \gamma \);
• the bundles $E^s, E^c$ and $E^u$ are orientable.

We say that $x \in W^s(\gamma) \cap W^u(\gamma) \setminus \{\gamma\}$ is the $N$-th intersection of $\gamma$ if

$$I^{cu}(x) \cap I^{cs}(x) \setminus \{\gamma\}$$

has exactly $N$ connected components. Then each $x \in W^s(\gamma) \cap W^u(\gamma) \setminus \{\gamma\}$ is associated with a number $n(x) \in \mathbb{N}$ if $x$ is the $n(x)$-th intersection.

**Remark 6.6.** The invariant foliations of $f$ coincide with the corresponding invariant foliations of $f^k$ for any $k \in \mathbb{N}^+$, and hence the $N$-th intersections under $f$ coincide with the ones of $f^k$ for $k \in \mathbb{N}^+$.

**Lemma 6.7.** The intersection number is well defined, i.e., for each $x \in W^s(\gamma) \cap W^u(\gamma) \setminus \{\gamma\}$, one has

1. $n(x) = n(f(x))$;
2. $n(x) = n(y)$ for any $y \in F^c(x)$.

**Proof.** By the invariance of the foliations,

$$I^{cu}(f(x)) = f(I^{cu}(x)) \quad \text{and} \quad I^{cs}(f(x)) = f(I^{cs}(x)),$$

which implies $n(x) = n(f(x))$.

By Remark 6.6 and Proposition 1.4, up to replacing $f$ by $f^2$, one can assume that $f$ preserves the orientation of $E^c$. By Theorem B, there exists a center flow $\{\varphi_t\}_{t \in \mathbb{R}}$ commuting with the strong stable and unstable holonomies. Therefore the center flow preserves the strong stable and unstable foliations. For any point $y \in F^c(x)$, there exists $t_0 \in \mathbb{R}$ such that $\varphi_{t_0}(x) = y$. By definition and the fact that the center flow preserves the strong stable and unstable foliations,

$$\varphi_{t_0}(I^{cs}(x)) = I^{cs}(y) \quad \text{and} \quad \varphi_{t_0}(I^{cu}(x)) = I^{cu}(y),$$

which implies $n(x) = n(y)$ since $\varphi_{t_0}$ is a homeomorphism.

**Proposition 6.8.** Let $f$ be a partially hyperbolic diffeomorphism on a closed 3-manifold with the following properties:

• $f$ is transitive and has one-dimensional topologically neutral center;
• $f$ admits a periodic compact center leaf $\gamma$ and a non-compact center leaf;
• the bundles $E^s, E^c$ and $E^u$ are orientable.
Then for any integer $N \in \mathbb{N}$, there are finitely many center leaves where $n(\cdot)$ is bounded by $N$; in the formula,
\[
\# \{ \mathcal{F}(x) : \mathcal{F}(x) \subset W^s(\gamma) \cap W^u(\gamma) \setminus \{\gamma\} \text{ and } n(x) \leq N \} < \infty.
\]

**Remark 6.9.** Here, we prove the finiteness of center leaves where $n(\cdot)$ is bounded, whereas Proposition 6.3 gives the finiteness of orbits with $n(\cdot)$ bounded.

**Proof of Proposition 6.8.** By Proposition 1.4, up to replacing $f$ by $f^2$, one can assume that $f$ preserves the orientation of the bundles $E^s, E^c$ and $E^u$. As $E^s, E^c$ and $E^u$ are orientable, $\gamma$ separates its stable and unstable manifolds into two connected components, respectively. Thus, we only need to work on one connected component $W^{s,+}(\gamma)$ of $W^s(\gamma) \setminus \gamma$ and one connected component $W^{u,+}(\gamma)$ of $W^u(\gamma) \setminus \gamma$, and the other cases would follow analogously.

Fix $N \in \mathbb{N}$ and $x \in W^{u,+}(\gamma) \cap W^{s,+}(\gamma)$ such that
\[
n(x) = \sup \{ n(y) : y \in W^{u,+}(\gamma) \cap W^{s,+}(\gamma) \text{ and } n(y) \leq N \}.
\]

We keep the notations in the definitions of $I^{cu}(x)$ and $I^{cs}(x)$ (see Figure 2). Let $\{ \phi_t \} \in \mathbb{R}$ be the center flow given by Theorem B. Without loss of generality, one can assume that $x_1^s$ is on the forward orbit of $x$ under the center flow $\{ \phi_t \} \in \mathbb{R}$. Let $x_{s,-}^1$ and $x_{u,+}^1$ be the first and second intersections between the backward orbit of $x$ under the center flow $\{ \phi_t \} \in \mathbb{R}$ and the strong stable manifold of $x$. Then
\[
I^{cs}(x) \subset I^{cs}(x_{s,-}^1) \subset I^{cs}(x_{u,+}^1).
\]

Analogously, one can define $x_{u,-}^1$ in the strong unstable manifold of $x$. Then
\[
I^{cu}(x) \subset I^{cu}(x_{u,-}^1).
\]

As $f$ has non-compact center leaves, by Proposition 5.3, $\gamma$ is the unique compact center leaf in $W^s(\gamma)$ and in $W^u(\gamma)$. By the Poincaré-Bendixson theorem, for each $\mathcal{F}(x) \subset W^{s,+}(\gamma) \cap W^{u,+}(\gamma) \setminus \{\gamma\}$, one has $\gamma \subset \mathcal{F}(x)$. For any $y \in W^{s,+}(\gamma) \cap W^{u,+}(\gamma) \setminus \{\gamma \cup \{x\}\}$, by Lemma 6.7, up to replacing $y$ by a point in $\mathcal{F}(y)$, one can assume that $y$ belongs to the strong unstable segment bounded by $x$ and $x_{s,-}^1$. Then
\[
I^{cu}(x) \subset I^{cu}(y) \subset I^{cu}(x_{u,-}^1).
\]

**Claim 6.10.** If $n(y) \leq N$, then $I^{cs}(y) \subset I^{cs}(x_{u,-}^1)$.

**Proof.** Assume, on the contrary, that $I^{cs}(y) \notin I^{cs}(x_{u,+}^1)$. Then $y \notin I^{cs}(x_{s,-}^1)$ and $I^{cs}(x_{u,+}^1) \subset I^{cs}(y)$, which implies that $\mathcal{F}(y)$ is disjoint from $I^{cs}(x_{s,-}^1) \cap I^{cu}(x_{u,+}^1)$. Since $I^{cs}(x_{s,-}^1) \subset I^{cs}(y)$ and $I^{cs}(x_{u,+}^1) \subset I^{cs}(y)$, the cardinal of the connected components of $(I^{cs}(y) \cap I^{cu}(y))$ is larger than the cardinal of the connected components of $I^{cs}(x_{s,-}^1) \cap I^{cu}(x_{u,+}^1)$, which leads to the contradiction. □

By Claim 6.10, for any $y \in W^{s,+}(\gamma) \cap W^{u,+}(\gamma) \setminus \{\gamma \cup \{x\}\}$ with $n(y) \leq N$, one has
\[
\mathcal{F}(y) \cap (I^{cs}(x_{s,-}^1) \cap I^{cu}(x_{u,-}^1)) \neq \emptyset.
\]

By the compactness of $I^{cs}(x_{s,-}^1)$ and $I^{cu}(x_{u,-}^1)$, and the uniform transversality between $E^c$ and $E^{cu}$, the set $I^{cs}(x_{s,-}^1) \cap I^{cu}(x_{u,-}^1)$ has finitely many connected components, which implies
\[
\# \{ \mathcal{F}(y) : \mathcal{F}(y) \subset W^s(\gamma) \cap W^u(\gamma) \setminus \{\gamma\} \text{ and } n(y) \leq N \} < \infty.
\]

This completes the proof. □

We conclude this subsection by the following result.

**Corollary 6.11.** Under the assumption of Proposition 6.8, all the center leaves in $W^s(\gamma)$ and $W^u(\gamma)$ are periodic under $f$.  

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Proof. We claim that each center leaf in \( F^c(x) \subset W^s(\gamma) \cap W^u(\gamma) \) is periodic under \( f \). By Proposition 6.8, one has
\[
\#\{F^c(y) : F^c(y) \subset W^s(\gamma) \cap W^u(\gamma) \text{ and } n(y) \leq n(x)\} < \infty.
\]
By Lemma 6.7, for each \( k \in \mathbb{Z} \), one has
\[
F^c(f^k(x)) \subset \{F^c(y) : F^c(y) \subset W^s(\gamma) \cap W^u(\gamma) \text{ and } n(y) \leq n(x)\},
\]
which implies that \( F^c(x) \) is periodic under \( f \).

By Lemma 6.4, the intersection \( W^s(\gamma) \cap W^u(\gamma) \) is a dense subset of \( W^s(\gamma) \). As \( W^s(\gamma) \) is a cylinder and \( \gamma \) is periodic under \( f \), in each connected component of \( W^s(\gamma) \setminus \{\gamma\} \), the space of center leaves is identified with \( S^1 \) and \( f \) induces a homeomorphism on it. Therefore, the set of periodic points for the induced maps on \( S^1 \) is dense in \( S^1 \), which implies that the induced maps on \( S^1 \) are periodic.

\[\square\]

6.3 Proof of Theorem C

Now, we are ready to give the proof of Theorem C. The proof is carried out according to the topology of the center stable leaves.

Proof of Theorem C. By Proposition 2.11, \( f \) has compact center leaves.

If there exists a compact center leaf which is non-periodic under \( f \), then by the “moreover” part of Proposition 5.2 there exists a compact periodic center leaf and Proposition 5.3 gives us that \( f \) is, up to finite lifts, \( C^0 \)-conjugate to a skew product. Therefore, up to finite lifts, \( f \) is conjugate to a skew product and also \( f \) preserves a volume on the center fibers \((S^1)\). Thus, \( f \) is conjugate to a skew product of an Anosov diffeomorphism on \( T^2 \) over the rotations on the circle.

It remains to prove the case where all the compact center leaves are periodic under \( f \). By Proposition 1.4, up to finite iterates and lifts, \( f \) satisfies the assumption of Proposition 6.8. By Corollary 6.11 and the second item in Theorem 2.9, up to finite iterates and lifts, each center leaf is \( f \)-invariant. Let \((\varphi_t)_{t \in \mathbb{R}} \) be the center flow given by Theorem B. Let \( x_0 \) be a point whose orbit under \( f \) is dense. Each center leaf is \( f \)-invariant and \( f \) commutes with the center flow. Then there exists \( t_0 \in \mathbb{R} \setminus \{0\} \) such that \( f|_{\varphi(t_0)(x_0)} = \varphi_{t_0} f|_{\varphi(t_0)(x_0)} \). Since the orbit of \( x_0 \) is dense and \( f \) commutes with the center flow, one has \( f = \varphi_{t_0} \). In particular, this implies the center flow is transitive. Moreover, there exists \( \lambda > 0 \) such that for any two points \( x \) and \( y \) on the same strong stable manifold of \( f \), one has
\[
\limsup_{t \to +\infty} \frac{1}{t} \log d(\varphi_t(x), \varphi_t(y)) < -\lambda.
\]

An analogous statement for strong unstable also holds. \[\square\]

Acknowledgements The second author was supported by the European Research Council (Grant No. 692925) and the starting grant from Beihang University. The second author thanks Institut de Mathématiques de Bourgogne and Laboratoire de Mathématiques d’Orsay for hospitality.

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Appendix A  Periodic compact center leaves generated by a uniformly compact lamination

In this section, we prove the existence of periodic compact center leaves near a compact invariant set which is laminated by compact center leaves. The proof adopts a variation of Bowen’s construction [11] of the shadowing lemma for hyperbolic sets which has been used in [13].

Proposition A.1.  Let $f$ be a dynamically coherent partially hyperbolic diffeomorphism on $M$ and $\Lambda$ be a compact invariant set. Assume that
Then for any \( \varepsilon > 0 \), there exists a compact and periodic center leaf \( L_\varepsilon \) in the \( \varepsilon \)-neighborhood of \( \Lambda \). Furthermore, if \( L_\varepsilon \cap \Lambda = \emptyset \), the center stable leaf of \( L_\varepsilon \) contains another compact center leaf different from \( L_\varepsilon \).

**Proof.** As each center leaf in \( \Lambda \) is compact, one associates each center leaf \( L \subset \Lambda \) with a tubular neighborhood \( V_L \) of \( L \) together with the \( C^1 \) projection \( \pi_L : V_L \rightarrow L \) such that for any \( x \in L \), \( \pi_L^{-1}(x) \) is a \( C^1 \) disc of co-dimension \( \dim(L) \) which is transverse to the center foliation (see, for example, [12, Chapter IV, Lemma 2]). As the volumes of the center leaves vary continuously in \( \Lambda \), up to shrinking \( V_L \), one can assume that

- for any \( x \in \Lambda \cap V_L \), the center leaf \( L_x \) is contained in \( V_L \);
- for each \( y \in L \), the intersection \( L_x \cap \pi_L^{-1}(y) \) is unique.

Then by the compactness of \( \Lambda \), there exist the compact center leaves \( L_1, \ldots, L_m \) in \( \Lambda \) such that their tubular neighborhoods \( (V_{L_i}, \pi_{L_i}) \) chosen as above form an open cover of \( \Lambda \) (i.e., \( \Lambda \subset \bigcup_{i=1}^m V_{L_i} \)). For simplicity, we denote \( V_i = V_{L_i} \) and \( \pi_i = \pi_{L_i} \). By a standard argument, one gets \( \delta_0 > 0 \) such that for any center leaf \( L \subset \Lambda \), there exists \( 1 \leq i \leq m \) such that the \( \delta_0 \)-tubular neighborhood of \( L \) is in \( V_i \).

Fix \( \delta \in (0, \delta_0/2) \) and define \( \Lambda(\delta) \) as the set of points \( x \in M \) with the following properties:

- The center leaf \( L_x \) is compact.
- There exists a center leaf \( L \subset \Lambda \) such that \( L_x \) is in the closure of the \( \delta \)-tubular neighborhood of \( L \).
- \( L_x \) intersects each fiber of \( \pi_i \) into a unique point, where \( V_i \) contains the \( \delta_0 \)-tubular neighborhood of \( L \).

By definition, \( \Lambda(\delta) \) is compact. Notice that for any \( \varepsilon \in (0, \delta_0/8) \) small enough one has that for any two points \( x, y \in M \),

- if \( W_{2\varepsilon}^c(x) \cap W_{2\varepsilon}^u(y) \neq \emptyset \) (resp. \( W_{2\varepsilon}^c(x) \cap W_{2\varepsilon}^s(y) \neq \emptyset \)), then such intersection consists of a unique point;
- if \( d(x, y) > \delta \), then \( W_{2\varepsilon}^c(y) \cap W_{2\varepsilon}^u(x) = \emptyset \).

For \( \varepsilon \), there exists \( \delta_1 \in (0, \delta) \) such that for any \( x, y \in M \) with \( d(x, y) < \delta_1 \), one has

- \( F^c_{2\varepsilon}(x) \cap F^c_{2\varepsilon}(y) = F^s_{\varepsilon/2}(x) \cap F^s_{\varepsilon/2}(y) \);
- \( F^u_{2\varepsilon}(x) \cap F^u_{2\varepsilon}(y) \) consists of exactly one point.

**Claim A.2.** Given two compact center leaves \( L_1, L_2 \in \Lambda(\delta) \) satisfying that \( L_1 \) is contained in the \( \delta_1 \)-tubular neighborhood of \( L_2 \), one has

- \( W_{2\varepsilon}^c(L_1) \cap W_{2\varepsilon}^c(L_2) = W_{\varepsilon/2}^c(L_1) \cap W_{\varepsilon/2}^c(L_2) \);
- \( W_{2\varepsilon}^u(L_1) \cap W_{2\varepsilon}^u(L_2) \) consists of exactly one compact center leaf \( L \).

Moreover, for \( x \in L_1 \) (resp. \( x \in L_2 \)), \( L \) intersects \( W_{\varepsilon/2}^c(x) \) (resp. \( W_{\varepsilon/2}^u(x) \)) into a unique point.

**Proof.** By the definition of \( \Lambda(\delta) \), there exists \( 1 \leq i_0 \leq m \) such that \( V_{i_0} \) contains \( L_1 \) and \( L_2 \). Furthermore, for any point \( y \in L_{i_0} \), the transverse section \( \pi_{i_0}^{-1}(y) \) cuts \( L_1 \) and \( L_2 \) into a unique point respectively and we denote them by \( y_1, y_2 \). As \( L_1 \) is contained in the \( \delta_1 \)-tubular neighborhood of \( L_2 \), one has that \( W_{2\varepsilon}^c(y_1) \cap W_{2\varepsilon}^u(y_2) \) (resp. \( W_{2\varepsilon}^u(y_1) \cap W_{2\varepsilon}^u(y_2) \)) consists of a unique point which is \( \varepsilon/2 \)-close to \( y_1 \) and \( y_2 \). By the choice of \( \varepsilon \), the intersection \( W_{2\varepsilon}^c(y_1) \cap W_{\varepsilon/2}^c(L_2) \) (resp. \( W_{2\varepsilon}^u(y_2) \cap W_{\varepsilon/2}^u(L_1) \)) consists of exactly one point which is \( \varepsilon/2 \)-close to \( y_1 \) (resp. \( y_2 \)), which concludes the claim.

Since \( \Lambda \) is compact and \( f \)-invariant, there exists a recurrent point \( x_0 \in \Lambda \). Due to the continuity of the volume of center leaves in \( \Lambda \), and the uniform contraction and expansion along \( E^s \) and \( E^u \) respectively, there exists \( k \in \mathbb{N} \) such that

- \( f^k(L_{x_0}) \) is in the \( \delta_1/4 \)-neighborhood of \( L_{x_0} \);
- \( f^k(W_{\varepsilon/2}^s(y)) \subset W_{\varepsilon/2}^s(f^k(y)) \) and \( f^{-k}(W_{\varepsilon/2}^u(y)) \subset W_{\varepsilon/2}^u(f^{-k}(y)) \) for any \( y \in M \);
- \( \max_{x \in M} \|Df^k|_{E^s(x)}\|, \sup_{x \in M} \|Df^{-k}|_{E^u(x)}\| < 1/4 \).

By Claim A.2, \( W_{\varepsilon/2}^c(f^k(L_{x_0})) \cap W_{\varepsilon/2}^c(L_{x_0}) \) consists of exactly one compact center leaf \( \tilde{L}_1 \subset \Lambda(\delta) \).

Assume that we already get the compact center leaves \( \{\tilde{L}_j\}_{j \leq i-1} \subset \Lambda(\delta) \) such that for \( j < i-1 \), one has

- \( \tilde{L}_j \subset W_{\varepsilon/2}^u(f^j(\tilde{L}_{j-1})) \cap W_{\varepsilon/2}^c(L_{x_0}) \);
\begin{itemize}
  \item \(\hat{L}_j\) intersects \(W_{\varepsilon/2}^u(z)\) into a unique point for each \(z \in f^k(\hat{L}_{j-1})\);
  \item \(\hat{L}_j\) intersects \(W_{\varepsilon/2}^s(z)\) into a unique point for each \(z \in L_{x_0}\).
\end{itemize}

By the choice of \(k\), one has
\[
f^k(\hat{L}_{i-1}) \subset W_{\delta/4}^s(f^k(L_{x_0})),
\]
and then once again by Claim A.2, the intersection
\[
W_{\varepsilon/2}(f^k(\hat{L}_{i-1})) \cap W_{\varepsilon/2}(L_{x_0})
\]
consists of exactly compact center leaf \(\hat{L}_i\) which by definition is contained in \(\Lambda(\delta)\).

Let \(L_i = f^{-ik}(\hat{L}_{2i})\) for each \(i \in \mathbb{N}\). By construction, one has
\begin{itemize}
  \item \(L_i \subset \Lambda(\delta)\);
  \item \(L_i\) is contained in the \(2\varepsilon\)-tubular neighborhood of \(L_{x_0}\);
  \item \(W_{\varepsilon}(L_i)\) (resp. \(W_{\varepsilon}^s(L_i)\)) intersects \(W_{\varepsilon}^s(L_{x_0})\) (resp. \(W_{\varepsilon}^u(f^k(L_{x_0}))\)) into a unique compact center leaf;
  \item \(\{f^j(L_i)\}_{j=-ik}^{ik}\) is in the \(2\varepsilon\)-tubular neighborhood of \(\{f^j(L_{x_0})\}_{j=-ik}^{ik}\) (mod \(k\)).
\end{itemize}

Let \(L\) be an accumulation of \(\{L_i\}_{i \in \mathbb{N}}\). Then \(L\) is a compact center leaf contained in the \(\varepsilon\)-tubular neighborhood of \(L_{x_0}\). Furthermore, \(f^j(L) \subset \Lambda(\delta)\) is contained in the \(2\varepsilon\)-tubular neighborhood of \(f^{j-[j/k]k}(L)\), and thus \(f^k(L)\) has the same property, which implies that the orbit of \(f^k(L)\) follows the orbits of \(L\) in the distance of \(2\varepsilon\). Applying [22, Theorem 6.1(c)] to \(\bigcap_{n \in \mathbb{Z}} f^n(\Lambda(\delta))\), one has
\[
f^k(L) \subset W_{2\varepsilon}^s(L) \cap W_{2\varepsilon}^u(L).
\]

By Claim A.2, one has \(f^k(L) = L\). \(\square\)