ARITHMETIC LEVEL RAISING ON CERTAIN QUATERNIONIC UNITARY SHIMURA VARIETY

HAINING WANG

ABSTRACT. In this article we prove an arithmetic level raising theorem for the symplectic group of degree four in the ramified case. This result is a key step towards the Beilinson–Bloch–Kato conjecture for certain Rankin-Selberg motives associated to orthogonal groups within the framework of the Gan–Gross–Prasad conjecture. The theorem itself can be also viewed as an analogue of the Ihara’s lemma or the Tate conjectures for special fibers of Shimura varieties at ramified characteristics. The proof relies heavily on the description of the supersingular locus of certain quaternionic unitary Shimura variety which is closely related to the classical Siegel threefold.

Contents

1. Introduction 2
1.1. Main results 5
1.2. Applications and prospects 8
1.3. Notations and conventions 9
Acknowledgements 10
2. Quaternionic unitary Shimura variety 10
2.1. Quaternionic unitary groups 10
2.2. The local quaternionic unitary Shimura variety 12
2.3. The quaternionic unitary Shimura variety 15
3. Level raising conditions 16
4. Jacquet-Langlands correspondence for GSp_4 22
4.1. Matching of orbital integrals 22
4.2. Stabilization of trace formula 23
5. Monodromy on the cohomology of the quaternionic unitary Shimura variety 26
5.1. Picard-Lefschetz formula 27
5.2. Descriptions of α and β 28

Key words and phrases. Shimura varieties, Arithmetic level raising, Vanishing cycles.
5.3. Nearby cycles on certain Shimura varieties
6. The level raising matrix
   6.1. A Hecke operator identity
   6.2. Computing the level raising matrix
7. Tate cycles on the quaternionic unitary Shimura variety
   7.1. Computing the supersingular matrix
   7.2. Tate classes in the quaternionic unitary Shimura variety
8. Galois deformation rings and cohomology of Shimura varieties
   8.1. Galois representation for $\text{GSp}_4$
   8.2. Galois deformation rings
9. Arithmetic level raising and applications
   9.1. Proof of Theorem 9.5
   9.2. Level raising for $\text{GSp}_4$
References

1. INTRODUCTION

Let $f$ be an elliptic modular newform of weight 2, which is normalized of level $\Gamma_0(N)$ for some positive integer $N$. Let $l \geq 3$ be a prime and we fix an isomorphism $\iota_l : \mathbb{C} \cong \mathbb{Q}_l$. Let $E = \mathbb{Q}(f)$ be the Hecke field of $f$ and suppose $\lambda$ is the place of $E$ over $l$ determined by $\iota_l$. We write $E_\lambda$ for the localization of $E$ at $\lambda$ with valuation ring $\mathcal{O}_\lambda$ and whose residual field is $k$. Then we can attach a Galois representation

$$\rho_{f,\lambda} : G_\mathbb{Q} \to \text{GL}_2(E_\lambda)$$

to $f$ such that $\rho_{f,\lambda}$ is unramified outside $Nl$ and is characterized by the property that the trace of Frobenius at a prime $p$ not dividing $Nl$ equals to the $p$-th Fourier coefficient of $f$. Let $\overline{\rho}_{f,\lambda} : G_\mathbb{Q} \to \text{GL}_2(k)$ be the residual representation which we assume is absolutely irreducible. Then Ribet [Rib84] proves the following level raising theorem.

**Theorem 1.1** (Ribet). Suppose that $\overline{\rho}_{f,\lambda}$ is absolutely irreducible and $p$ is a prime away from $Nl$. If we have

$$a_p^2 \equiv (p + 1)^2 \mod \lambda,$$

then there exists a weight 2 normalized newform $g$ of level $\Gamma_0(Np)$ such that $\overline{\rho}_{f,\lambda} \cong \overline{\rho}_{g,\lambda}$.

We will refer to the condition $a_p^2 \equiv (p + 1)^2 \mod \lambda$ as the level raising condition. Let $\alpha_p, \beta_p$ be the eigenvalues of $\overline{\rho}_{f,\lambda}(\text{Frob}_p)$. Then the level raising condition is equivalent to $\{\alpha_p, \beta_p\} = \{1, p\}$ or $\{\alpha_p, \beta_p\} = \{-1, -p\}$ which is further equivalent to the condition that
\( \alpha_p/\beta_p = p^{\pm 1} \). Let \( \pi \) be the automorphic representation of \( \text{GL}_2(\mathbb{A}_\mathbb{Q}) \) associated to \( f \). Since the eigenvalues of \( \rho_{f, \lambda}(\text{Frob}_p) \) correspond to the Satake parameters of \( \pi_p \) modulo \( \lambda \), the level raising condition can be interpreted as the condition for \( \pi_p \) to degenerate to the twisted Steinberg representation under the reduction by \( \lambda \). This representation theoretic interpretation sheds some lights on the question of generalizing the above theorem to automorphic forms on higher rank reductive groups. One of the themes of this article is to establish a level raising result for cuspidal automorphic representations of \( \text{GSp}_4(\mathbb{A}_\mathbb{Q}) \) of general type in the sense of Arthur [Art04].

Due to the importance of level raising theorems, there are quite a few investigations in this direction using various different methods. We give a survey of some these methods. The original method of Ribet exploits the natural integral structure of the cohomology of the modular curves \( \text{X}_0(N) \) and \( \text{X}_0(Np) \). In particular, it relies on the so-called Ihara’s lemma, which roughly speaking, says that the fundamental group of the special fiber of \( \text{X}_0(N) \) modulo \( p \) is generated by the Frobenius elements supported on the supersingular points of the special fiber of \( \text{X}_0(N) \). It is quite difficult to generalize the Ihara’s lemma for other reductive groups but fortunately there is a method of proving the level raising theorem using modularity lifting theorems. In particular, it relies on the Ihara’s avoidance technique of Taylor. For an implementation of this type of method, see for example [Gee11]. The third method comes from part of Bertolini–Darmon’s first reciprocity law [BD03, Theorem 5.15, Corollary 5.18]. Nowadays, the third method is known as the arithmetic level raising in the terminology of [LTXZZ]. We will go into some details on this third method later as it directly inspires the present article. Finally, there is a more representation theoretic method due to Clozel [Clo] which relies heavily on the theory of universal modules in mod \( l \) representation theory of reductive groups. Note in particular, following this method Sorensen [Sor09] proves a level raising theorem for \( \text{GSp}_4 \). We will make some comments on the relation of his results and the result in this article in the last subsection of this introduction. For the moment, we only remark that these representation theoretic methods seem to work well even when the residual Galois representation has small image, for example, in the endoscopic or the CAP cases.

The first step of proving the arithmetic level raising theorem is to transfer \( f \) to an automorphic form on a definite quaternion algebra \( \overline{B} \) unramified at \( p \) using the Jacquet-Langlands correspondence. Then \( f \) is viewed as an element of \( \mathcal{O}_\lambda(\text{Sh}(\overline{B}, \mathcal{K}_H)) \) where \( \text{Sh}(\overline{B}, \mathcal{K}_H) \) is the Shimura set of \( \overline{B} \) and \( \mathcal{K}_H \) is a suitable level structure which is unramified at \( p \). As we know that if \( f \) satisfies the level raising condition, then its associated automorphic representation \( \pi \) has its local component \( \pi_p \) degenerates to a twisted Steinberg representation which transfers to a representation of the quaternion division algebra over \( \mathbb{Q}_p \), it makes sense to consider the nearby quaternion algebra \( B \) which is obtained by switching the local invariant at \( \infty \) and \( p \) of \( \overline{B} \). From \( B \), we can construct a Shimura curve \( \text{Sh}(B, \mathcal{K}_{1w}) \) where \( \mathcal{K}_{1w} \) is a suitable level structure such that its local model is same as the classical modular curve with Iwahori level at \( p \). Let \( \mathbb{T}^\Sigma \) be the Hecke algebra unramified away from a finite set \( \Sigma \) of non-archimedean places of \( \mathbb{Q} \) containing all the places such that \( \pi \) is ramified at and all the places where \( \overline{B} \) is ramified at. Then the Hecke eigenvalues of \( f \) furnishes a homomorphism \( \varphi_\lambda : \mathbb{T}^\Sigma \to \mathcal{O}_\lambda \). Let \( m = \ker(\mathbb{T}^\Sigma \to \mathcal{O}_\lambda \to \mathcal{O}_\lambda/\lambda) \), we say \( m \) is the maximal
ideal of $\mathbb{T}^2$ corresponding to $\overline{\rho}_{f,\lambda}$. Let $m$ be the $\lambda$-adic valuation of $a_p^2 - (p + 1)^2$. The arithmetic level raising theorem in this setting says the following.

**Theorem 1.2** (Bertolini-Darmon). Let $f$ be an elliptic modular newform of level $\Gamma_0(N)$ and weight 2 as above and suppose it can be transferred to an automorphic form on $\overline{B}$. Let $p$ be a level raising prime for $f$. Assume that the residual representation $\overline{\rho}_{f,\lambda}$

1. is absolutely irreducible and has image containing $\text{GL}_2(\mathbb{F}_p)$;
2. is $l$-isolated in the sense of [BD05, Definition 1.2].

Then we have a canonical isomorphism

$$\mathcal{O}_\lambda / \lambda^m [\text{Sh}(B, K_H)]_m \sim H^1_{\text{sing}}(\mathbb{Q}_p^2, H^1(\text{Sh}(B, K_{I_w}), \mathcal{O}_\lambda / \lambda^m)_m).$$

The righthand side of the above isomorphism is the singular quotient of the Galois cohomology group $H^1(\mathbb{Q}_p^2, H^1(\text{Sh}(B, K_{I_w}), \mathcal{O}_\lambda / \lambda^m)_m)$. The theorem implies that the maximal ideal $m$ appears in the support of the cohomology $H^1(\text{Sh}(B, K_{I_w}), \mathcal{O}_\lambda)$ and since $B$ is ramified at $p$, the automorphic representations carried by $H^1(\text{Sh}(B, K_{I_w}), \mathcal{O}_\lambda)$ are necessarily ramified and hence gives rise to a level raising form $g$ by the Jacquet–Langlands correspondence. Thus the above arithmetic level raising theorem implies the level raising theorem.

The isomorphism

$$\mathcal{O}_\lambda / \lambda^m [\text{Sh}(B, K_H)]_m \sim H^1_{\text{sing}}(\mathbb{Q}_p^2, H^1(\text{Sh}(B, K_{I_w}), \mathcal{O}_\lambda / \lambda^m)_m).$$

is really an analogue of the Ihara’s lemma for the Shimura curve ramified at $p$. Here the set

$$\mathcal{O}_\lambda / \lambda^m [\text{Sh}(B, K_H)]_m$$

comes from the parametrizing set of the irreducible components of the supersingular locus of the special fiber of $\text{Sh}(B, K_{I_w})$ which turns out to be the whole special fiber. This isomorphism is also an analogue of the Tate conjecture proved by in Xiao and Zhu in [XZ] which, roughly speaking, says that the middle degree cohomology of the special fiber of a Shimura variety of good reduction should be generated by Tate classes coming from their supersingular locus when we localize the cohomology at a sufficiently generic maximal ideal.

It is natural to ask if one can generalize the above arithmetic level raising result to higher dimensional Shimura varieties and deduce level raising theorems for automorphic forms on other reductive groups. For this direction, we only mention the following results.

- In [Liu], Liu proves the arithmetic level raising theorem for certain quaternionic Hilbert modular threefold at a ramified prime. Parallel to this, the author proves the arithmetic level raising theorem for triple product of Shimura curves at a ramified prime in [Wang22a]. These results are tailored to have applications to construct Euler system in the frame work of the arithmetic Gan–Gross–Prasad conjecture for $\text{SO}_3 \times \text{SO}_4$.
- In a recent breakthrough, the authors of [LTXZZ] establish the arithmetic level raising theorem for unitary Shimura varieties at a ramified prime. This is closely...
related to the arithmetic Gan–Gross–Prasad conjecture for $U_n \times U_{n+1}$ and its associated Beilinson–Bloch–Kato conjecture. Indeed, the main result of [LTXZZ] establishes one direction in the rank 0 and rank 1 case of the Beilinson–Bloch–Kato conjecture attached to the Rankin–Selberg motive of $U_n \times U_{n+1}$.

In this article, we will establish the arithmetic level raising theorem for $\text{GSp}_4$ over $\mathbb{Q}$ and discuss its applications to level raising results for cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$. Our main results will be described in more details in the next subsection.

1.1. Main results. Let $\pi$ be a cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ of general type in the sense of Arthur [Art04]. This means that $\pi$ is non-endoscopic and non-CAP. Another way to define this is by defining $\pi$ to be of general type if $\pi$ transfers to a cuspidal automorphic representation $\Pi$ of $\text{GL}_4(\mathbb{A}_\mathbb{Q})$. We assume that $\pi$ is of weight $(3, 3)$ with trivial central character. Suppose $E$ is a strong coefficient field for $\pi$ in the sense of Definition 8.3. We fix as before an isomorphism $\iota_l : \mathbb{C} \cong \overline{\mathbb{Q}_l}$ which induces the place $\lambda$ of $E$ over $l$. We can attach to such $\pi$ a continuous homomorphism $\rho_{\pi, \lambda} : \text{G}_{\mathbb{Q}} \to \text{GSp}_4(E_\lambda)$ characterized by the property that for a prime $p$ such that $\pi_p$ is unramified, we have
\[
\det(X - \rho_{\pi, \lambda}(\text{Frob}_p)) = Q_p(X)
\]
where $Q_p(X)$ is the Hecke polynomial given by
\[
X^4 - a_{p,2}X^3 + (pa_{p,1} + (p^2 + p)a_{p,0})X^2 - p^3a_{p,0}a_{p,2}X + p^6a_{p,0}^2
\]
whose coefficients $a_{p,0}, a_{p,1}, a_{p,2}$ are the eigenvalues of the spherical Hecke operators
\[
T_{p,0} = \text{char}(\text{GSp}_4(\mathbb{Z}_p)) \begin{pmatrix} p & \cdot & \cdot \\ \cdot & p & \cdot \\ \cdot & \cdot & p \end{pmatrix} \text{GSp}_4(\mathbb{Z}_p)),
\]
\[
T_{p,2} = \text{char}(\text{GSp}_4(\mathbb{Z}_p)) \begin{pmatrix} p & \cdot & \cdot \\ \cdot & p & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \text{GSp}_4(\mathbb{Z}_p)),
\]
\[
T_{p,1} = \text{char}(\text{GSp}_4(\mathbb{Z}_p)) \begin{pmatrix} p^2 & \cdot & \cdot \\ \cdot & p & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \text{GSp}_4(\mathbb{Z}_p)),
\]
which generate the spherical Hecke algebra for $\text{GSp}_4(\mathbb{Q}_p)$ acting on the space $\pi_p^{\text{GSp}_4(\mathbb{Z}_p)}$ of spherical vectors of $\pi_p$. The four roots $[\alpha_p, \beta_p, \gamma_p, \delta_p]$ of the Hecke polynomial $Q_p(X)$ are called the Hecke parameters of $\pi$ at $p$ which we can arrange to be in the form $[\alpha_p, \beta_p, p^3\beta_p^{-1}, p^3\alpha_p^{-1}]$ under our assumptions imposed on $\pi$. We will further assume that $\pi_p$ is an unramified type I representation in the classification of Sally–Tadic [ST93] and Schmidt [Sch05]. See the beginning of section 3 for a review of this notion.
Definition 1.3. Suppose $p$ is a prime such that $\pi_p$ is an unramified type I representation and $m$ is a positive integer. We say $p$ is level raising special for $\pi$ of depth $m$ if

1. $l \nmid p^2 - 1$;
2. the Hecke parameters $[\alpha_p, \beta_p, p^3\beta_p^{-1}, p^3\alpha_p^{-1}]$ of $\pi_p$ satisfy simultaneously the following three conditions,

\begin{align*}
\beta_p + p^3\beta_p^{-1} &\equiv u(p + p^2) \pmod{\lambda} \text{ for some } u \in \{\pm 1\} \\
\alpha_p + p^3\alpha_p^{-1} &\not\equiv \pm(p + p^2) \pmod{\lambda} \\
\alpha_p + \beta_p + p^3\beta_p^{-1} + p^3\alpha_p^{-1} &\not\equiv \pm2(p + p^2) \pmod{\lambda}
\end{align*}

and such that the $\lambda$-adic valuation of $\beta_p + p^3\beta_p^{-1} - u(p + p^2)$ is exactly $m$.

In terms of the classification of non-supercuspidal irreducible admissible representations of $\text{GSp}_4(\mathbb{Q}_p)$ of Sally–Tadic [ST93] and Schmidt [Sch05], this condition means $\pi_p$, which is an unramified type I representation, degenerates into a ramified representation of type IIa under the reduction modulo $\lambda$. In the theory of local newforms for $\text{GSp}_4$ of [BS], representations of type IIa can be viewed as the local newforms of paramodular level. Moreover, representations of type IIa can be transferred to a representation of the inner form $\text{GU}_2(D)$ of $\text{GSp}_4(\mathbb{Q}_p)$ for the quaternion division algebra $D$ over $\mathbb{Q}_p$ under the Jacquet–Langlands correspondence. This puts us in a similar setting as in the $\text{GL}_2$ case discussed in the beginning of the introduction. We therefore will proceed analogously to prove the arithmetic level raising theorem.

Let $\pi$ be as above and $p$ be a level raising special prime for $\pi$ of depth $m$. Let $\overline{B}$ be the definite quaternion algebra ramified at $q\infty$ where $q$ is a prime which will play a completely auxiliary role in this article. This restriction on the discriminant of the quaternion algebra $\overline{B}$ is completely unnecessary and is only assumed for simplicity. The quaternionic unitary group $G(\overline{B}) = \text{GU}_2(\overline{B})$ of degree 2 over $\mathbb{Q}$ is an inner form of $G = \text{GSp}_4$ over $\mathbb{Q}$. It defines a Shimura set $\text{Sh}(\overline{B}, K_H)$ where $K_H$ is a suitable level structure which in particular is hyperspecial at $p$. Similar to the $\text{GL}_2$-case, we will also switch the local invariant of $\overline{B}$ at $p$ and $\infty$ to obtain the indefinite quaternion algebra $B$ ramified at $pq$ and in this case we have the quaternionic unitary group $G(B) = \text{GU}_2(B)$ of degree 2 over $\mathbb{Q}$ which is another inner form of $G = \text{GSp}_4$. There is an associated PEL-type Shimura variety $\text{Sh}(B, K_{Pa})$ which classifies abelian fourfolds with quaternionic multiplication and principal polarizations. This Shimura variety is the central geometric object in this article, its level structure $K_{Pa}$ is chosen so that its integral model over $\mathbb{Z}_p$ has the same local model as the classical Siegel modular threefold with paramodular level at $p$. In particular, its integral model $X_{Pa}(\overline{B})$ has isolated quadratic singularities at its special fiber. This allows us to compute the monodromy action on the cohomology of this Shimura variety using the Picard-Lefschetz formula. We will write the special fiber of $X_{Pa}(B)$ as $\overline{X}_{Pa}(B)$ and we will use the same notation $X_{Pa}(B)$ to denote its base change to the algebraic closure $\overline{F}$ of $\mathbb{F}_p$.

Now we can state our main result on arithmetic level raising for $\text{GSp}_4$ in the ramified case. Let $\Sigma$ be a finite set of places of $\mathbb{Q}$ containing all the places at which $\pi$ is ramified. Then the Hecke parameters of $\pi$ defines a morphism $\phi_\Sigma : \mathbb{T}^\Sigma \to E$ and its $\lambda$-adic avatar
φπ,λ: TΣ → Oλ. Using the same terminology as in the GL2 case, we say
\[ m_π = \ker(T^T T \rightarrow O_λ \rightarrow O_λ/λ) \]
is the maximal ideal corresponding to \( \overline{\rho}_{π,λ} \) and we define
\[ m = T^{Σ(p)} \cap m_π. \]

**Theorem 1.4** (Arithmetic level raising for GSp4). Suppose that \( π \) is a cuspidal automorphic representation of GSp4(AQ) of general type with weight (3,3) and trivial central character. Let \( p \) be a level raising special prime for \( π \) of depth \( m \). We assume further that

1. The prime \( l \geq 3 \);
2. \( \overline{\rho}_{π,λ} \) is rigid for \((Σ_{min}, Σ_{lr})\) as in Definition 8.6;
3. \( \overline{\rho}_{π,λ} \) is absolutely irreducible and the image of \( \overline{\rho}_{π,λ}(G_Q) \) contains GSp4(Fl);
4. The cohomology groups \( H^d(Sh(B, K_Pa), O_λ) \) of the Shimura variety \( Sh(B, K_Pa) \) is concentrated in degree 3;
5. \( m \) is in the support of \( O_λ[Sh(B, K_H)] \).

Then we have an isomorphism
\[ H^1_{sing}(Q_p^2, H^2_c(Sh(B, K_Pa), O_λ/λ^m(1))) \sim H^1_λ[Sh(B, K_H)] \]
In particular \( m \) occurs in the support of \( H^4(Sh(B, K_Pa), O_λ) \).

Here the cohomology \( H^d_c(Sh(B, K_Pa), O_λ)m \) in the above theorem are understood as the cohomology of the Shimura variety \( Sh(B, K_Pa) \) base changed to \( \overline{Q} \). To prove the theorem, we identify \( H^2_c(Sh(B, K_Pa), O_λ) \) with the nearby cycle cohomology \( H^2_c(\overline{X}_{Pa}(B), R\Psi(O_λ)) \) and apply the Picard–Lefschetz formula to calculate the monodromy action on the cohomology \( H^2_c(\overline{X}_{Pa}(B), R\Psi(O_λ)) \) in terms of the space of vanishing cycles on \( \overline{X}_{Pa}(B) \). Then the most difficult step in the proof of the main theorem is to prove that the Tate cycles coming from the supersingular locus of the Shimura variety \( \overline{X}_{Pa}(B) \) generate the degree 2 and degree 4 cohomology of the special fiber \( \overline{X}_{Pa}(B) \) localized at a generic maximal ideal. More precisely, we prove the following theorem.

**Theorem 1.5.** Let \( m \) be a generic maximal ideal, we have the following isomorphisms.

1. There is an isomorphism
   \[ O_λ[Sh(\overline{B}, K_H)]^⊕2_m \sim H^2_c(\overline{X}_{Pa}(B), O_λ(1))_m. \]
2. There is an isomorphism
   \[ H^4(\overline{X}_{Pa}(B), O_λ)_m \sim O_λ[Sh(\overline{B}, K_H)]^⊕2_m. \]

We will not explain in detail the meaning of generic maximal ideal in the introduction but refer the reader to Definition 7.8. The appearance of the two copies of \( O_λ[Sh(\overline{B}, K_H)]_m \) is related to the fact that the irreducible components of the supersingular locus of \( \overline{X}_{Pa}(B) \) are naturally parametrized by two copies of \( Sh(\overline{B}, K_H) \). This fact along with other finer properties of the supersingular locus are obtained in author’s work [Wang19a, Wang19b] and Oki’s work [Oki22].
Coming back to the above theorem, the following interesting picture emerges: it seems that the low degree cohomology of the special fiber $\overline{X}_{P_0}(B)$ will only be related to oldforms via the Jacquet-Langlands correspondence while the middle degree cohomology will contribute to the Galois representations of the newforms. This should be compared to a conjecture of Arthur predicting that the low degree cohomology of the Shimura variety (not the special fiber) should be generated by automorphic forms coming from theta correspondences. Note that a similar picture is confirmed for the special fiber of certain unitary Shimura variety at a ramified characteristic, see \cite[Proposition 6.3.1]{LTXZZ}. Another related results to Theorem 1.5 is that the cohomology of the special fiber $\overline{X}_{P_0}(B)$ with coefficient in $O_\lambda$ are all torsion free after localization at a generic maximal ideal $m$. This seems to be a quite difficult question in general, the author can not prove it directly using only geometric techniques. Fortunately, this can be proved using techniques from the theory of Galois deformations developed in \cite{LTXZZa} for unitary Shimura varieties. In the companion article \cite{Wang22a}, we adapt their methods to our setting and prove a freeness result of the cohomology of the quaternionic unitary Shimura variety and the quaternionic unitary Shimura set over suitable universal deformation rings.

1.2. Applications and prospects. As mentioned in the discussion for the GL$_2$ case, the arithmetic level raising theorem is not only an interesting theorem in its own right but also has applications in many other related problems. We finish this introduction by discussing some of these applications.

1.2.1. Level raising for GSp$_4$. The most direct application of our arithmetic level raising theorem is to deduce a level raising theorem for cuspidal automorphic representations of GSp$_4(\mathbb{A}_\mathbb{Q})$ of general type. More precisely, we can deduce the following theorem from the arithmetic level raising theorem.

**Theorem 1.6.** Let $\pi$ be an automorphic representation of GSp$_4(\mathbb{A}_\mathbb{Q})$ of general type with trivial central character and weight $(3, 3)$. Suppose that $\pi$ satisfy all the assumptions in Theorem 1.4. Suppose $p$ is a level raising special prime for $\pi$.

Then there exists an automorphic representation $\pi'$ of GSp$_4(\mathbb{A}_\mathbb{Q})$ of general type with weight $(3, 3)$ and trivial central character such that

1. the component $\pi'_p$ at $p$ is of type IIa;
2. we have an isomorphism of Galois representation as in the Construction $\overline{\rho}_{\pi, \lambda} \cong \overline{\rho}_{\pi', \lambda}$.

In this case we will say $\pi'$ is a level raising of $\pi$.

Sorensen proves a related level raising result for GSp$_4$ in \cite{Sor09}. We comment that the level raising condition in his theorem is different from our's, in fact he assumes that $\pi_p$ is congruent to the trivial representation modulo $\lambda$. Also the focus of his article is different, while we assume $\pi$ is of general type throughout this article, $\pi$ is mainly of Saito-Kurokawa type in \cite{Sor09} which makes it possible to have applications in the Bloch-Kato conjecture in the GL$_2$ setting.
1.2.2. Beilinson–Bloch–Kato conjecture. The arithmetic level raising theorem is the first step towards the Beilinson–Bloch–Kato conjecture of certain Rankin-Selberg product motive for $SO_4 \times SO_5$ within the framework of the Gan–Gross–Prasad conjecture. In a sequel of this work, we will show that if certain period integral of on the definite Shimura set $\text{Sh}(\mathcal{B}, K_H)$ is non-zero modulo $\lambda$, then certain Bloch–Kato Selmer group of certain Rankin–Selberg product motive for $SO_4 \times SO_5$ is zero. This period integral should have direct relations to the Rankin-Selberg $L$-function for automorphic forms on $SO_4 \times SO_5$ once the the Global Gan–Gross–Prasad conjecture and the Ichino–Ikeda conjecture [II10] is confirmed. However, as far as the author’s knowledge, this is only known in certain endoscopic or CAP cases.

1.2.3. Level lowering theorem for $GSp_4$. In the companion article [Wang22b], we will prove certain level lowering result for cuspidal automorphic representations $\pi$ of $GSp_4(\mathbb{A}_F)$ whose component at the level lowering prime $p$ admits paramodular fixed vectors. The approach in that article follows the famous approach of Ribet in his proof of Serre’s epsilon conjecture for modular forms. In particular, we need to apply the so called $p,q$-switch trick in order to realize the representation $\pi$ in the cohomology of the quaternionic unitary Shimura variety $\text{Sh}(B, K_{Pa})$ studied in this article. To achieve this, one has to first take an auxiliary prime $q$ and raise the level of $\pi$ at $q$ and our Theorem 1.6 makes this possible.

It would be interesting to generalize the results of this article to cuspidal automorphic representations $\pi$ of $GSp_4(\mathbb{A}_F)$ where $F$ is a totally real field. The main reason we make this restriction in this article is that the results in [Wang19a, Wang19b] and [Oki22] are obtained only for the quaternionic unitary Shimura variety over $\mathbb{Q}$. As far as the author’s knowledge, the current Rapoport–Zink uniformization theorem does not apply to those Shimura varieties studied in this article if one considers totally real field. However, we still believe the description of the supersingular locus should be similar if one is able to extend our results to the totally real field case. It is also reasonable to ask if our result works for those $\pi$ of general weights. This is only a matter of formalism if we restrict the Hodge-Tate weights of $\rho_{\pi, \lambda}$ in the regular Fontaine-Laffaille range. Finally, our results and computations should shed some lights on the general problem of realizing arithmetic level raising on general orthogonal type Shimura varieties at ramified characteristic.

1.3. Notations and conventions. We will use common notations and conventions in algebraic number theory and algebraic geometry. The cohomology of schemes appear in this article will be understood as computed over the étale sites.

For a field $K$, we denote by $\overline{K}$ a separable closure of $K$ and put $G_K := \text{Gal}(\overline{K}/K)$ the Galois group of $K$. Suppose $K$ is a number field and $v$ is a place of $K$, then $K_v$ is the completion of $K$ at $v$ with valuation ring $\mathcal{O}_v$ whose maximal ideal is also written as $v$ if no danger of confusion could arise. Let $\text{Art} : K_v^\times \to \mathbb{W}^{ab}_K$ be the local Artin map sending the uniformizers to the geometric Frobenius element $\text{Frob}_p$. We let $\phi_v$ be the arithmetic Frobenius element at $v$. We let $\mathbb{A}_K$ be the ring of adèles over $K$ and $\mathbb{A}_K^{(\infty)}$ be the subring of finite adèles. Let $l \geq 3$ be a prime and we fix an isomorphism $\iota_l : \mathbb{C} \cong \mathbb{Q}_l$. We will
denote by $\epsilon_l$ the $l$-adic cyclotomic character. The prime $l$ will be the residue characteristic of the coefficient

Let $K$ be a local field with ring of integers $O_K$ and residue field $k$. We let $I_K$ be the inertia subgroup of $G_K$. Suppose $M$ is a $G_K$-module. Then the finite part $H^1_{\text{fin}}(K, M)$ of $H^1(K, M)$ is defined to be $H^1(k, M^I_K)$ and the singular part $H^1_{\text{sing}}(K, M)$ of $H^1(K, M)$ is defined to be the quotient of $H^1(K, M)$ by the image of $H^1_{\text{fin}}(K, M)$.

Let $p$ be a prime and let $F$ be an algebraically closed field containing $\mathbb{F}_p$. We denote by $\sigma$ be the Frobenius element on $F$. Let $W_0 = W(F)$ be the ring of Witt vectors of $F$ and $K_0 = W(F)_\mathbb{Q}$ its fraction field, then $\sigma$ extends to an action on $W_0(F)$. Let $F_{p^d}$ be the finite field of $p^d$ elements. Then $\mathbb{Z}_{p^d}$ is defined to be $W(F_{p^d})$. Let $M_1 \subset M_2$ be two $W_0$-modules we write $M_1 \subset^d M_2$ if the $W_0$ colength of the inclusion is $d$.

If $R$ is ring and $L$ is an $R$-module and $R'$ is an $R$-algebra, we define $L_{R'} = L \otimes_R R'$. Let $X$ be a scheme or a formal scheme over $R$, we write $X_{R'}$ its base change to $R'$. Let $M_1 \subset M_2$ be two $R$-modules, then we write $M_1 \subset_{GM} M_2$ if $M_2/M_1 = \text{Gr} M$.

For our convention, $GSp_4$ be the reductive group over $\mathbb{Z}$ defined by

$$GSp_4 = \{ g \in GL_4 : gJg^t = c(g)J \}$$

where $J$ is the antisymmetric matrix given by \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} for $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and where $c$ is the similitude character of $GSp_4$. By definition, we have an exact sequence

$$1 \rightarrow \text{Sp}_4 \rightarrow GSp_4 \rightarrow \mathbb{G}_m \rightarrow 1.$$ 

Acknowledgements. The author would like to thank Liang Xiao for all his help and encouragement throughout the course of writing this article. The author is also grateful to Yifeng Liu for very helpful conversations in Shanghai. Finally, the author would like to thank his colleagues at SCMS Kei-Yuen Chan, Chen Jiang and Zhiyuan Li for some useful discussions.

2. Quaternionic unitary Shimura variety

2.1. Quaternionic unitary groups. We will recall the definition of the quaternionic unitary groups of degree 2 and some of their parahoric subgroups. Then we define integral models of certain quaternionic unitary Shimura variety with paramodular level structure and some related quaternionic unitary Shimura set. This quaternionic unitary Shimura variety is closely related the classical Siegel threefold with para modular level structure. Their relation is similar to that of a classical modular curve with a Shimura curve for an indefinite quaternion algebra.

Let $B = B_{pq}$ be an indefinite quaternion algebra which is ramified exactly at two distinct primes $p$ and $q$ which are distinct from the prime $l$. Let $*$ be a nebentype involution on $B$ and $O_B$ be a maximal order fixed by $*$. We have the quaternionic unitary group
\( \mathbf{G}(\mathcal{B}) = \text{GU}_2(\mathcal{B}) \) defined by an alternating form \((\cdot, \cdot)\) on \( \mathcal{V} = \mathcal{B} \oplus \mathcal{B} \) with the property that \((bx, y) = (x, b^*y)\) for \( x, y \in V \) and \( b \in \mathcal{B} \). Thus we define

\[
\mathbf{G}(\mathcal{B})(\mathbb{Q}) = \{ g \in \text{GL}_B(V) : (gx, gy) = c(g)(x, y), \ c(g) \in \mathbb{Q}^\times \}.
\]

For a definite choice of such alternating form, see \([KR00]\) or \([Oki22]\). In the local setting, let \( D \) be the quaternion division algebra over \( \mathbb{Q}_p \). Then we can define the group \( \text{GU}_2(D) \) over \( \mathbb{Q}_p \) in the same way as above and we have \( \mathbf{G}(\mathcal{B})(\mathbb{Q}_p) = \text{GU}_2(D)(\mathbb{Q}_p) \). Note that the group \( \text{GU}_2(D) \) is non-quasi-split but \( \text{GU}_2(D)/\mathbb{Q}_{p^2} \cong \text{GSp}_4/\mathbb{Q}_{p^2} \).

Let \( \mathcal{B} = \mathcal{B}_\infty \) be the definite quaternion algebra ramified at the archimedean place and the prime \( p \). Let \( * \) be a main type involution on \( \mathcal{B} \) and \( \mathcal{O}_\mathcal{B} \) be a maximal order fixed by \( * \).

We have the quaternionic unitary group \( \mathbf{G}(\mathcal{B}) = \text{GU}_2(\mathcal{B}) \) defined by an alternating form \((\cdot, \cdot)\) on \( \mathcal{V} = \mathcal{B} \oplus \mathcal{B} \) with the property that \((bx, y) = (x, b^*y)\) for \( x, y \in V \) and \( b \in \mathcal{B} \). Thus

\[
\mathbf{G}(\mathcal{B})(\mathbb{Q}) = \{ g \in \text{GL}_B(V) : (gx, gy) = c(g)(x, y), \ c(g) \in \mathbb{Q}^\times \}
\]

and \( \mathbf{G}(\mathcal{B})(\mathbb{Q}) \) is anisotropic modulo the center. Note we have \( \mathbf{G}(\mathcal{B})(\mathbb{Q}_p) = \text{GSp}_4(\mathbb{Q}_p) \) as \( p \) is unramified in \( \mathcal{B} \).

For the split reductive group \( \mathbf{G} = \text{GSp}_4 \) over \( \mathbb{Z} \), we will fix a datum \( \mathbf{T} \subset \mathbf{B} = \mathbf{TU} \)

where \( \mathbf{T} \) is the split torus of the diagonal matrices and \( \mathbf{B} \) is the Borel subgroup of the upper triangular matrices. In the local situation, we will abuse the notations and write \( \mathbf{G}, \mathbf{B} = \mathbf{TU} \) for the above group schemes but considered over \( \mathbb{Z}_p \). The affine Dynkin diagram of type \( \tilde{C}_2 \) is given by

\[
\begin{align*}
0 & \quad - & \quad 1 \quad - & \quad 2 \\
\end{align*}
\]

where the nodes 0 and 2 are special and 1 is not special in the sense that the parahoric subgroup corresponding to 0 and 2 are special parahoric subgroups of \( \mathbf{G}(\mathbb{Q}_p) \). These nodes correspond to the so-called vertex lattices in a symplectic space \( \mathcal{V} \) over \( \mathbb{Q}_p \).

Let \( \mathcal{L}_{\{i\}} \) be the set of vertex lattices of type \( i \in \{0, 1, 2\} \), they are by definition given by

\[
\begin{align*}
\mathcal{L}_{\{0\}} &= \{ \text{vertex lattice } \Lambda_0 \text{ of type 0 in } V: \ p\Lambda_0^\vee \subset^4 \Lambda_0 \subset^0 \Lambda_0^\vee \}; \\
\mathcal{L}_{\{2\}} &= \{ \text{vertex lattice } \Lambda_2 \text{ of type 2 in } V: \ p\Lambda_2^\vee \subset^0 \Lambda_2 \subset^4 \Lambda_2^\vee \}; \\
\mathcal{L}_{\{1\}} &= \{ \text{vertex lattice } \Lambda_1 \text{ of type 1 in } V: \ p\Lambda_1^\vee \subset^2 \Lambda_1 \subset^2 \Lambda_1^\vee \}. 
\end{align*}
\]

We will usually write a vertex lattice \( \Lambda_1 \) of type 1 as \( \Lambda_{Pa} \) as it is related to the paramodular subgroup. For a vertex lattice \( \Lambda_i \) with \( i \in \{0, 1, 2\} \), we denote its stabilizer in \( \text{GSp}(V) \cong \text{GSp}_4(\mathbb{Q}_p) \) by \( K_{\{0\}}, K_{\{2\}} \) and \( K_{\{1\}} \).

- The groups \( K_{\{0\}} \) and \( K_{\{2\}} \) are hyperspecial subgroups of \( \text{GSp}_4(\mathbb{Q}_p) \) and are conjugate to \( \text{GSp}_4(\mathbb{Z}_p) \).
- The group \( Pa = K_{\{1\}} \) is the so-called paramodular subgroup.
- The group \( Kl = K_{\{01\}} \) is the so-called Klingen parahoric which is the stabilizer of a pair of vertex lattices \( (\Lambda_0, \Lambda_1) \) of type 0 and 1 with the relation \( p\Lambda_0^\vee \subset p\Lambda_1^\vee \subset \Lambda_1 \subset \Lambda_0 \).
• The group $\text{Sie} = K_{(02)}$ is the so-called Siegel parahoric which is the stabilizer of a pair of vertex lattices $(\Lambda_0, \Lambda_2)$ of type 0 and 2 with the relation $p\Lambda_0^\vee \subset p\Lambda_2^\vee = \Lambda_2 \subset \Lambda_0$.

• The group $\text{Iw} = K_{(012)}$ is the Iwahoric subgroup of $\text{GSp}_4(\mathbb{Q}_p)$ which is the preimage of the group $B(\mathbb{F}_p)$ in $G(\mathbb{Z}_p)$. We define the pro-p Iwahori subgroup $\text{Iw}_1$ to be the preimage of $U(\mathbb{F}_p)$ in $G(\mathbb{Z}_p)$.

The quaternionic unitary group $GU_2(D)$ is the unique inner form of $\text{GSp}_4$ over $\mathbb{Q}_p$, it has the same affine Dynkin diagram of type $\tilde{C}_2$

but with the Frobenius acting on the diagram by permuting the node 0 and the node 2. Therefore the node 0 and the node 2 should be identified when we consider vertex lattices for $GU_2(D)$. We define a vertex lattice of type $\{0,2\}$ as a pair of lattices $(\Lambda_0, \Lambda_2)$ where $\Lambda_0$ is a vertex lattice of type 0 and $\Lambda_2$ is a vertex lattice of type 2 with the relation $p\Lambda_0^\vee \subset p\Lambda_2^\vee = \Lambda_2 \subset \Lambda_0$. The stabilizer of a vertex lattice $(\Lambda_0, \Lambda_2)$ of type $\{0,2\}$ in $GU_2(D)$ by $K_{\{0,2\}}^D$ and a vertex lattice $\Lambda_1$ of type 1 in $GU_2(D)$ by $K_{\{1\}}^D$. The parahoric subgroup $K_{\{0,2\}}^D$ splits to the Siegel parahoric while the parahoric subgroup $K_{\{1\}}^D$ splits to the paramodular subgroup over $\mathbb{Z}_p^2$. Therefore we will sometimes refer to them as the Siegel parahoric subgroup and the paramodular subgroup in $GU_2(D)$. Later we will use the notation $P_0^D$ for the group $K_{\{1\}}^D$.

2.2. The local quaternionic unitary Shimura variety. Let $N$ be an isocrystal of height 8 over $\mathbb{F}$ which is isotypic of slope $\frac{1}{2}$. Let $\iota : D \to \text{End}(N)$ be an action of $D$ on $N$. We can write $D$ as $\mathbb{Q}_p^{x,y} + \mathbb{Q}_p\Pi$ with $\Pi^2 = p$. Suppose $N$ is equipped with an alternating form $(\cdot, \cdot) : N \times N \to K_0$ with the property that $(Fx, y) = (x, Vy)^\sigma$ and $(bx, y) = (x, b^* y)$ where $b^*$ is the image of $b$ under the oben type involution given by

$$x^* = \sigma(x), \ x \in \mathbb{Q}_p^{x,y}$$

$$\Pi^* = \Pi.$$

The action of $\mathcal{O}_D$ decomposes $N$ into $N = N_0 \oplus N_1$. Restricting the alternating form $(x, y)_0 := (x, \Pi y)$ to $N_0$ gives an identification between the group $GU_0(N)$ and $GSp(N_0)$ over $W_0$ cf. [RZ96, 1.42]. We will use covariant Dieudonné theory throughout this article.

A Dieudonné lattice $M$ is a lattice in $N$ with the property that $pM \subset FM \subset M$. A Dieudonné lattice is superspecial if $F^2M = pM$ and in this case $F = V$. We are concerned with a Dieudonné lattice $M$ with an additional endomorphism $\iota : \mathcal{O}_D \to \text{End}(M)$. We will represent $\mathcal{O}_D$ by $\mathbb{Z}_p^{x,y} + \mathbb{Z}_p\Pi$ and hence we can decompose $M$ as $M_0 \oplus M_1$ with an additional operator $\Pi$ that swaps the two components. The alternating form $(\cdot, \cdot)$, restricted to $M$, induces a pairing $(\cdot, \cdot) : M_0 \times M_1 \to W_0$. We will fix a $p$-divisible group $\mathfrak{X}$ over $\mathbb{F}$ of dimension 4 and height 8 with quaternionic multiplication $\iota_\mathfrak{X} : \mathcal{O}_D \to \text{End}(\mathfrak{X})$ and with polarization $\lambda : \mathfrak{X} \to \mathfrak{X}^\vee$ whose isocrystal agrees with $N$ with its additional structures discussed above. Let $\text{Nilp}$ be the category of $W_0$-schemes over which $p$ is locally nilpotent.
obtained by Oki in [Oki22] using a different method.

The main result of [Wang19a] concerns the structure of the group $J_{\sigma}$ over $Spf W$.

We require that $\iota_{X}$ satisfies the Kottwitz condition
\[
\det(T - \iota_{X}(c)|\text{Lie}(X)) = (T^{2} - \text{Trd}^{0}(c)T + \text{Nrd}^{0}(c))^{2}
\]
for $c \in O_{D}$, where $\text{Trd}^{0}(c)$ is the reduced trace of $c$ and $\text{Nrd}^{0}(c)$ is the reduced norm of $c$. For $\rho_{X} : X \times S \mathcal{S} \to X \times F \mathcal{S}$, we require that $\rho_{X}^{-1} \circ \lambda_{X} \circ \rho_{X} = c(\rho_{X})\lambda_{X}$ for a $Q_{p}$-multiple $c(\rho_{X})$.

This moduli problem is representable by a formal scheme $\mathcal{N}_{Pa}$, locally formally of finite type over $Spf W_{0}$. The formal scheme $\mathcal{N}_{Pa}$ can be decomposed into an open and closed formal subschemes: $\mathcal{N}_{Pa} = \bigsqcup_{i \in L} \mathcal{N}_{Pa}(i)$ where each $\mathcal{N}_{Pa}(i)$ is isomorphic to $\mathcal{N}_{Pa}(0)$. Let $b$ be the $\sigma$-conjugacy class corresponding to the isocrystal $N$, the space $\mathcal{N}_{Pa}$ admits an action of the group $J_{b}(\mathbb{Q}_{p}) \cong \text{GSp}_{2}(\mathbb{Q}_{p})$. Denote by $\mathcal{M}_{Pa}$ the underlying reduced subscheme of $\mathcal{N}_{Pa}(0)$. The main result of [Wang19a] concerns the structure of $\mathcal{M}_{Pa}$. The same result is also obtained by Oki in [Oki22] using a different method.

**Theorem 2.1.** The scheme $\mathcal{M}_{Pa}$ admits the Bruhat-Tits stratification
\[\mathcal{M}_{Pa} = \mathcal{M}_{Pa}^{0}(0) \sqcup \mathcal{M}_{Pa}^{2}(0) \sqcup \mathcal{M}_{Pa}^{0}(02) \sqcup \mathcal{M}_{Pa}(1).\]

1. The stratum $\mathcal{M}_{Pa}^{0}(0)$ has its irreducible components indexed by the set $L_{\{0\}}$ of vertex lattices of type $\{0\}$. For each vertex lattice $\Lambda_{0} \in L_{\{0\}}$, the corresponding irreducible component $\mathcal{M}_{Pa}^{0}(\Lambda_{0})$ is isomorphic to a Deligne–Lusztig variety whose closure is given by the projective surface
\[Z_{0}^{1}Z_{1} - Z_{0}^{2}Z_{2} + Z_{1}^{0}Z_{2} = 0\]
in a suitable coordinate system $[Z_{0} : Z_{1} : Z_{2} : Z_{3}]$ of $\mathbb{P}^{3}$. We write $\mathcal{M}_{Pa}(0)$ for the closure of $\mathcal{M}_{Pa}^{0}(0)$ and $\mathcal{M}_{Pa}(\Lambda_{0})$ for the closure of $\mathcal{M}_{Pa}^{0}(\Lambda_{0})$.

2. The stratum $\mathcal{M}_{Pa}^{2}(0)$ has its irreducible components indexed by the set $L_{\{2\}}$ of vertex lattices of type $\{2\}$. For each vertex lattice $\Lambda_{2} \in L_{\{2\}}$, the corresponding irreducible component $\mathcal{M}_{Pa}^{2}(\Lambda_{2})$ is isomorphic to a Deligne–Lusztig variety whose closure is given by the projective surface
\[Z_{0}^{1}Z_{0} - Z_{0}^{2}Z_{3} + Z_{2}^{0}Z_{1} - Z_{1}^{0}Z_{2} = 0\]
in a suitable coordinate system $[Z_{0} : Z_{1} : Z_{2} : Z_{3}]$ of $\mathbb{P}^{3}$. We write $\mathcal{M}_{Pa}(2)$ for the closure of $\mathcal{M}_{Pa}^{2}(2)$ and $\mathcal{M}_{Pa}(\Lambda_{2})$ for the closure of $\mathcal{M}_{Pa}^{2}(\Lambda_{2})$.

3. The stratum $\mathcal{M}_{Pa}^{0}(02)$ has its irreducible components indexed by a pair of vertex lattices $(\Lambda_{0}, \Lambda_{2})$ with the relation $p\Lambda_{0}^{\vee} \subset p\Lambda_{2}^{\vee} = \Lambda_{2} \subset \Lambda_{0}$. For such a pair $(\Lambda_{0}, \Lambda_{2})$, the corresponding irreducible component $\mathcal{M}_{Pa}^{0}(\Lambda_{0}, \Lambda_{2})$ is the intersection
of $\mathcal{M}_P^0(\Lambda_0)$ and $\mathcal{M}_P^0(\Lambda_2)$, and its closure $\mathcal{M}_P(\Lambda_0, \Lambda_2)$ is isomorphic to the projective line $\mathbb{P}^1$.

(4) The stratum $\mathcal{M}_P\{1\}$ is a discrete set of points indexed by the set $\mathcal{L}_{\{1\}}$ of vertex lattices of type 1. For a lattice $\Lambda_1$ of type 1, the point $\mathcal{M}_P(\Lambda_1)$ corresponds to a superspecial $p$-divisible group.

(5) For a pair $(\Lambda_0, \Lambda'_0)$ of vertex lattices of type 0, suppose $\Lambda_1 = \Lambda_0 \cap \Lambda'_0$ is a vertex lattice of type 1. Then the irreducible components $\mathcal{M}_P(\Lambda_0)$ and $\mathcal{M}_P(\Lambda'_0)$ intersect at the point $\mathcal{M}_P(\Lambda_1)$. Similarly, for a pair $(\Lambda_2, \Lambda'_2)$ of vertex lattices of type 2, suppose $\Lambda_1 = \Lambda_2 + \Lambda'_2$ is a vertex lattice of type 1. Then the irreducible components $\mathcal{M}_P(\Lambda_2)$ and $\mathcal{M}_P(\Lambda'_2)$ intersect at the point $\mathcal{M}_P(\Lambda_1)$.

We will need to recall some ingredients going into the proof of the above theorem. We will also calculate the tangent space at a smooth point. Given a point $x = (X, \lambda_X, \iota_X, \rho_X) \in \mathcal{M}_P(\mathbb{F})$, we denote by $M = M_0 \oplus M_1$ the Dieudonné lattice attached to $x$. Then the set $\mathcal{M}_P(\mathbb{F})$ can be identified with the set

$$\{ M \in N : M^\perp = M, pM \subset M, pM \subset \Pi M \subset M \}$$

where $M^\perp$ is the integral dual of $M$ in $N$ with respect to $(\cdot, \cdot)$. Then projecting $M$ to $D = M_0$ identifies this set with

$$\{ D \subset N_0 : pD^\vee \subset D \subset pD^\vee, p\tau D^\vee \subset p\tau D \subset p\tau D \}$$

where $D^\vee$ is the integral dual of $D$ with respect to the pairing $(\cdot, \cdot)_0$ on $N_0$ defined by $(x, y)_0 = (x, \Pi y)$ for $x \in M_0$ and $y \in M_0$ and $\tau$ is given by $\Pi^{-1}$.

2.2.1. The scheme $\mathcal{M}_P^0(\Lambda_0)$. Consider $\mathcal{M}_P^0(\Lambda_0) = \mathcal{M}_P(\Lambda_0) - \mathcal{M}_P\{1\}$ which corresponds to the set of points where $D + \tau(D)$ is $\tau$-stable and gives rise to a vertex lattice $\Lambda_0$ of type 0. Notice that $D$ and $\tau D$ inserts in the chains

$$p\Lambda_0 \subset pD^\vee \subset D \cap \tau D \subset D \subset \Lambda_0$$
$$p\Lambda_0 \subset p\tau D^\vee \subset D \cap \tau D \subset \tau D \subset \Lambda_0.$$

Taking the quotient by $p\Lambda_0$, we obtain two complete isotropic flags

$$pD^\vee/p\Lambda_0 \subset D \cap \tau D/p\Lambda_0 \subset D/\Lambda_0 \subset \Lambda_0/p\Lambda_0$$
$$p\tau D^\vee/p\Lambda_0 \subset D \cap \tau D/p\Lambda_0 \subset \tau D/\Lambda_0 \subset \Lambda_0/p\Lambda_0$$

in the symplectic space $\Lambda_0/p\Lambda_0$. Now it is clear that the latter two isotropic flags agree with the Deligne–Lusztig variety defined in [DL76 2.4] and its equation is worked out in [Wang19, Theorem 3.6]. In particular, let $x$ be a smooth point in $\mathcal{M}_P(\Lambda_0)$, then we have the following lemma about its tangent space at $x$.

**Lemma 2.2.** Let $x$ be a point in $\mathcal{M}_P^0(\Lambda_0)$, then the tangent space $T_{\mathcal{M}_P(\Lambda_0), x}$ is given by

$$0 \rightarrow \text{Hom}(pD^\vee/p\Lambda_0, D \cap \tau D/pD^\vee) \rightarrow T_{\mathcal{M}_P(\Lambda_0), x} \rightarrow \text{Hom}(p\tau D^\vee/p\Lambda_0, D \cap \tau D/p\tau D^\vee) \rightarrow 0.$$ 

**Proof.** This is a standard computation using Grothendieck–Messing theory and follows from the above discussions. We only point out that the extension comes from lifting
paramodular subgroup given by $G_1$ acts on $V$ sending $z \mapsto p \tau z$. This represents certain PEL-type moduli problem. Let $V = B$ be a point in $M_{Sp}(\Lambda_2)$ and we assume $K = K_p \cap \Lambda_2$. Taking the quotient by $\Lambda_2$, we obtain two complete isotropic flags

$$D/\Lambda_2 \subset D + \tau D/\Lambda_2 \subset D^\vee/\Lambda_2 \subset \Lambda_2^\vee_2/\Lambda_2;$$
$$\tau D/\Lambda_2 \subset D + \tau D/\Lambda_2 \subset \tau D/\Lambda_2 \subset \Lambda_2^\vee_2/\Lambda_2.$$

in the symplectic space $\Lambda_2^\vee_2/\Lambda_2$. Then the tangent space is calculated in the same way as in the previous lemma.

**Lemma 2.3.** Let $x$ be a point in $M_{Sp}(\Lambda_2)$, then the tangent space $T_{M_{Sp}(\Lambda_2),x}$ is given by

$$0 \to \text{Hom}(D/\Lambda_2, D + \tau D) \to T_{M_{Sp}(\Lambda_2),x} \to \text{Hom}(\tau D/\Lambda_2, D + \tau D/\tau D) \to 0.$$

**Proof.** This is a standard computation using Grothendieck–Messing theory and follows from the above discussions. We only point out that the extension comes from lifting $D/\Lambda_2$ first and then lifting $\tau D/\Lambda_2$ to $\mathbb{F}[\varepsilon]/\varepsilon^2$. Overall the computation is similar to that of [LTXZZ, A.2.2], we skip the details. □

### 2.2.2. The scheme $M_{pa}(\Lambda_2)$

The discussion for $M_{pa}(\Lambda_2)$ runs completely parallel to that of $M_{pa}(\Lambda_0)$. In this case $M_{pa}(\Lambda_2) = M_{pa}(\Lambda_2) - M_{pa}\{1\}$ corresponds to the set of points where $D \cap \tau D$ is $\tau$-stable and gives rise to the vertex lattice $\Lambda_2$ of type 2. Moreover the lattices $D$ and $\tau D$ inserts into the chain

$$\Lambda_2 \subset D \subset D^\vee \subset \Lambda_2^\vee$$

respectively. Taking the quotient by $\Lambda_2$, we obtain two complete isotropic flags

$$D/\Lambda_2 \subset D + \tau D/\Lambda_2 \subset D^\vee/\Lambda_2 \subset \Lambda_2^\vee/\Lambda_2;$$
$$\tau D/\Lambda_2 \subset D + \tau D/\Lambda_2 \subset \tau D/\Lambda_2 \subset \Lambda_2^\vee/\Lambda_2.$$

in the symplectic space $\Lambda_2^\vee/\Lambda_2$. Then the tangent space is calculated in the same way as in the previous lemma.

**Lemma 2.3.** Let $x$ be a point in $M_{pa}(\Lambda_2)$, then the tangent space $T_{M_{pa}(\Lambda_2),x}$ is given by

$$0 \to \text{Hom}(D/\Lambda_2, D + \tau D) \to T_{M_{pa}(\Lambda_2),x} \to \text{Hom}(\tau D/\Lambda_2, D + \tau D/\tau D) \to 0.$$

**Proof.** This is a standard computation using Grothendieck–Messing theory and follows from the above discussions. We only point out that the extension comes from lifting $D/\Lambda_2$ first and then lifting $\tau D/\Lambda_2$ to $\mathbb{F}[\varepsilon]/\varepsilon^2$. Overall the computation is similar to that of [LTXZZ, A.2.2], we skip the details. □

### 2.3. The quaternionic unitary Shimura variety

We define the integral model of the quaternionic unitary Shimura variety with paramodular level $X_{pa}(B)$ over $\mathbb{Z}(p)$ which represents certain PEL-type moduli problem. Let $V = B \oplus B$ considered as a vector space of dimension 8 over $\mathbb{Q}$. Suppose that $V$ is equipped with an alternating form $(\cdot, \cdot)$ such that $(x, by) = (b^*x, y)$. Then we recall that

$$G(B)(\mathbb{Q}) = \{g \in GL_B(V) : (gx, gy) = c(g)(x, y), \ c(g) \in \mathbb{Q}^\times\}.$$ 

Since $B$ is split at $\infty$, $G(B)(\mathbb{R}) = GSp(4)(\mathbb{R})$. Let $h : \mathbb{G}_m \to G(B)(\mathbb{C})$ be the cocharacter sending $z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. Then $h$ defines a decomposition $V_\mathbb{C} = V_1 \oplus V_2$ where $h(z)$ acts on $V_1$ by $z$ and on $V_2$ by $\bar{z}$. We fix an open compact subgroup $K = K_p K_1$ of $G(B)(\mathbb{A}_\mathbb{Q}^{(\infty)})$ and we assume $K_1$ is sufficiently small and $K_p = K_{1_p} = G(B)(\mathbb{Z}_p)$ is the paramodular subgroup given by

$$G(B)(\mathbb{Z}_p) = \{g \in GL_{O_B}(V_{O_B}) : (gx, gy) = c(g)(x, y), \ c(g) \in \mathbb{Z}_p^\times\}.$$
for $V_{O_B} = O_B \oplus O_B$ and the restriction of $(\cdot, \cdot)$ to $V_{O_B}$. Let $\Lambda = O_B \oplus O_B$, then $G(B)(\mathbb{Z}_p)$ is the stabilizer of $\Lambda \otimes \mathbb{Z}_p = O_D \oplus O_D$. This datum defines a Shimura variety $Sh(B, K_{Pa})$ over $\mathbb{Q}$ such that its $\mathbb{C}$ points are given by

$$Sh(B, K_{Pa})(\mathbb{C}) \cong G(B)(\mathbb{Q}) \backslash \mathcal{H}^\pm \times G(B)(A^\infty_\mathbb{Q})/K^pK_p$$

where $\mathcal{H}^\pm$ is the lower and upper Siegel half spaces of dimension 3.

To the datum $(B, *, V, (\cdot, \cdot), h, A)$, we associate the following moduli problem $X_{Pa}(B)$ over $\mathbb{Z}(p)$.

- $A$ is an abelian scheme of relative dimension 4 over $S$;
- $\lambda : A \to A^\vee$ is a prime to $p$ polarization which is $O_B$-linear;
- $\iota : O_B \to \text{End}_S(A)$ is a morphism such that $\lambda \circ \iota(a^*) = \iota(a)^\vee \circ \lambda$ and satisfies the Kottwitz condition
  $$\det(T - \iota(a)|\text{Lie}(A)) = (T^2 - \text{Trd}^0(a)T + \text{Nrd}^0(a))^2$$
  for all $a \in O_B$ with $\text{Trd}^0$ the reduced trace on $B$ and $\text{Nrd}^0$ the reduced norm on $B$;
- $\eta^p : V \otimes_{\mathbb{Q}} A^{(\infty p)}_\mathbb{Q} \to V^{(p)}(A)$ is a $K^p$-orbit of $B \otimes_{\mathbb{Q}} A^{(\infty p)}_\mathbb{Q}$-linear isomorphisms where
  $$V^{(p)}(A) = \prod_{v \neq p} T_{Av}(A) \otimes A^{(\infty p)}_\mathbb{Q}.$$

We require that $\eta^p$ is compatible with the Weil-pairing on the righthand side and the form $(\cdot, \cdot)$ on the lefthand side.

This moduli problem is representable by a quasi-projective scheme $X_{Pa}(B)$ over $\mathbb{Z}(p)$ whose generic fiber is given by $Sh(B, K_{Pa})$. We will only use the integral model $X_{Pa}(B)$ base changed to $W_0$ in this article. Therefore we will abuse the notation and write $X_{Pa}(B)$ for this base change from here on. The special fiber will be denoted by $\overline{X}_{Pa}(B)$. The singularity of the integral model of this Shimura variety is well-understood and the local model is written down in [Wang19a, 2.3]. In particular, we have the following result.

**Lemma 2.4.** The scheme $X_{Pa}(B)$ over $W_0$ is regular whose singular locus is concentrated on its special fiber $\overline{X}_{Pa}(B)$ with isolated quadratic singularities.

Suppose that the singular locus is written as $\overline{X}_{Pa}^{\text{sing}}(B)$. The complete local ring of $X_{Pa}(B)$ at a singular point in $\overline{X}_{Pa}^{\text{sing}}(B)$ can be written as

$$W_0[Z_1, Z_2, Z_3, Z_4]/(Z_1Z_2 - Z_3Z_4 - p).$$

Moreover points in $\overline{X}_{Pa}^{\text{sing}}(B)$ are precisely those points whose underlying $p$-divisible groups are superspecial.

**Remark 2.5.** There is a natural semi-stable model $X_{Pa}^{\text{Bl}}(B)$ of $X_{Pa}(B)$ by blowing up $X_{Pa}(B)$ at $\overline{X}_{Pa}^{\text{sing}}(B)$ which we will only implicitly use in the next section, the description of this semi-stable model is given in [5.2] of the next section.
There is an arithmetic minimal compatification of the scheme $X_{Pa}(B)$ whose boundary components are the zero-dimensional cusps. There is also a good theory of toroidal compatification of $X_{Pa}(B)$. Since we will not use them explicitly, we will not make these theories precise in this article.

We are interested in the supersingular locus $\overline{X}_{Pa}(B)^{ss}$ of $X_{Pa}(B)$ considered as a reduced closed subscheme. The uniformization theorem of Rapoport–Zink translates this problem to the problem of describing the corresponding Rapoport–Zink space. More precisely, we have the following isomorphism.

**Theorem 2.6.** There is an isomorphism of $\mathbb{F}$-schemes

$$\overline{X}_{Pa}(B)^{ss} \cong I(\mathbb{Q}) \backslash N_{Pa,\text{red}} \times G(\overline{B})(\mathbb{A}_{\mathbb{Q}}^{(\infty)})/K^p.$$ 

Here $I$ is an inner form of $\text{GU}_2(B)$ with $I(\mathbb{Q}_p) = J_b(\mathbb{Q}_p)$ in our case. More precisely, it is defined in the following way. Let $(A, \iota, \lambda, \eta)$ be a supersingular point in $\overline{X}_{Pa}$ whose associated $p$-divisible group is given by $\mathbb{X}$ which we use as a base point to define the Rapoport–Zink space. Then $I$ is defined to be the algebraic group over $\mathbb{Q}$ whose $\mathbb{Q}$-points $I(\mathbb{Q})$ are given by the group of quasi-isogenies in $\text{End}(A)_{\mathbb{Q}}$ which preserve the polarization $\lambda$. In fact, it is not difficult to see that this group $I$ is exactly the group $G(\overline{B}) = \text{GU}_2(\overline{B})$.

**Definition 2.7.** Let $K$ be an open compact subgroup of $G(\overline{B})(\mathbb{A}^{(\infty)}_{\mathbb{Q}})$. We define the Shimura set for the group $G(\overline{B})$ with level $K$ by

$$\text{Sh}(\overline{B}, K) = G(\overline{B})(\mathbb{Q}) \backslash G(\overline{B})(\mathbb{A}^{(\infty)}_{\mathbb{Q}})/K.$$ 

When $K_p = G(\overline{B})(\mathbb{Z}_p) \cong G(\mathbb{Z}_p)$, we will write the Shimura set as $\text{Sh}(\overline{B}, K_H)$ to emphasize the level structure at $p$ is hyperspecial.

**Construction 2.8.** When we study the supersingular locus of the Shimura variety $\overline{X}_{Pa}(B)$, these Shimura sets appear naturally, and in this setting we will prefer to use another set of notations.

1. For $i = 0, 1, 2$, we will write the Shimura set $\text{Sh}(\overline{B}, K^pK_{\{i\}})$ as $Z_{\{i\}}(\overline{B})$ and will refer to the Shimura set $Z_{\{i\}}(\overline{B})$ as the Shimura set for $G(\overline{B})$ with $K_{\{i\}}$-level structure.
2. Since $K_{\{0\}} \cong K_{\{2\}} \cong G(\mathbb{Z}_p)$, sometimes, we identify the Shimura sets $Z_{\{0\}}(\overline{B})$ and $Z_{\{2\}}(\overline{B})$ with $\text{Sh}(\overline{B}, K_H)$, in which case we will write them commonly as $Z_H(\overline{B})$.
3. Similarly, we sometimes prefer the notation $Z_{Pa}(\overline{B})$ for the Shimura set $Z_{\{1\}}(\overline{B})$ sometimes.

The Rapoport–Zink uniformization theorem immediately transfers the results in Theorem 2.1 to a description of the supersingular locus $\overline{X}_{Pa}(B)^{ss}$.

**Theorem 2.9.** We have the following descriptions for the regular scheme $X_{Pa}(B)$ over $W_0$.

1. The special fiber $\overline{X}_{Pa}(B)$ has isolated quadratic singularities whose singular locus $\overline{X}_{Pa}^{\text{sing}}(B)$ is parametrized by the Shimura set $Z_{\{1\}}(\overline{B})$. 

| | |
(2) The supersingular locus $\mathfrak{X}_{\text{Pa}}(\mathfrak{B})$ of $\mathfrak{X}_{\text{Pa}}(\mathfrak{B})$ is pure of dimension 2. The irreducible components of $\mathfrak{X}_{\text{Pa}}(\mathfrak{B})$ are parametrized by the Shimura set $Z_{\{0\}}(\mathfrak{B})$ and $Z_{\{2\}}(\mathfrak{B})$.
(3) The irreducible components are all isomorphic to a projective surface of the form

$$Z_{\{0\}}^pZ_0 - Z_{\{2\}}^pZ_2 + Z_2^pZ_1 - Z_1^pZ_2 = 0$$

in a suitable coordinate system $[Z_0 : Z_1 : Z_2 : Z_3]$ of $\mathbb{P}^3$.

Proof. Notice that we can write the Shimura set $Z_{\{i\}}(\mathfrak{B})$ in the form

$$Z_{\{i\}}(\mathfrak{B}) = G(\mathfrak{B})(\mathbb{Q})/J_b(\mathbb{Q}_p)/K_{\{i\}} \times G(\mathfrak{B})(\mathbb{A}_\mathbb{Q}^{(\infty p)})/K_p.$$

This follows from the fact that $J_b(\mathbb{Q}_p) \cong GSp_4(\mathbb{Q}_p)$.

Then statements (1) and (2) follow from the immediately from our previous discussions and the fact that $J_b(\mathbb{Q}_p)$ acts transitively on the set of vertex lattices of type $\{i\}$ for $i \in \{0, 1, 2\}$. The last part follows from the Rapoport–Zink uniformization theorem and our local result Theorem 2.1.

The final task in this section is to compute the normal bundle for an irreducible component of the supersingular locus in the special fiber. Suppose $DL(\Lambda_i)$ is such an irreducible component corresponding to a vertex lattice $\Lambda_i$ of type $i = 0, 2$. We denote by $DL^{\square}(\Lambda_i)$, the complement in $DL(\Lambda_i)$ of the set of singular points $\mathfrak{X}_{\text{Pa}}^{\text{sing}}(\mathfrak{B})$. Then we have an exact sequence

$$0 \to T_{DL^{\square}(\Lambda_i)} \to T_{\mathfrak{X}_{\text{Pa}}^{\square}(\mathfrak{B})|DL^{\square}(\Lambda_i)} \to N_{DL^{\square}(\Lambda_i)}(\mathfrak{X}_{\text{Pa}}^{\square}(\mathfrak{B})) \to 0$$

where $T_\ast$ is the tangent bundle of $\ast$, $\mathfrak{X}_{\text{Pa}}^{\square}(\mathfrak{B}) = \mathfrak{X}_{\text{Pa}}(\mathfrak{B}) - \mathfrak{X}_{\text{Pa}}^{\text{sing}}(\mathfrak{B})$ is the smooth locus and where $N_{DL^{\square}(\Lambda_i)}(\mathfrak{X}_{\text{Pa}}^{\square}(\mathfrak{B}))$ is the normal bundle of $DL^{\square}(\Lambda_i)$ in $\mathfrak{X}_{\text{Pa}}^{\square}(\mathfrak{B})$.

Lemma 2.10. The normal bundle $N_{DL^{\square}(\Lambda_i)}(\mathfrak{X}_{\text{Pa}}^{\square}(\mathfrak{B}))$ is isomorphic to $O_{DL(\Lambda_i)}(-2p)|DL^{\square}(\Lambda_i)$ and the normal bundle $N_{DL(\Lambda_i)}(\mathfrak{X}_{\text{Pa}}(\mathfrak{B}))$ is isomorphic to $O_{DL(\Lambda_i)}(-2p)$.

Proof. For simplicity, we only treat the case when $i = 0$ and the proof for the case when $i = 2$ is almost the same with trivial modifications.

By the deformation theory of Serre–Tate and Grothendieck–Messing, to calculate tangent spaces at points on $\mathfrak{X}_{\text{Pa}}(\mathfrak{B})$, it suffices to consider the underlying $p$-divisible group of the point. Let $x \in DL^{\square}(\Lambda_0)$, we have already calculated the $T_{DL^{\square}(\Lambda_0), x}$, it is given by

$$0 \to \text{Hom}(pD^\vee/p\Lambda_0, D \cap \tau D/pD^\vee) \to T_{DL^{\square}(\Lambda_0), x} \to \text{Hom}(p\tau D^\vee/p\Lambda_0, D \cap \tau D/p\tau D^\vee).$$

where $D = M_0$ where $M = M_0 \oplus M_1$ is the Dieudonné module of the $p$-divisible group underlying the point $x$. Given a Dieudonné lattice $M$ with an additional endomorphism $\iota : O_D \to \text{End}(M)$ and it comes from to a point $x \in \mathfrak{X}_{\text{Pa}}(\mathfrak{B})$, then the Dieudonné module $M$ needs to satisfy the following conditions

$$pM_0 \subset VM_1 \subset M_0;$$
$$pM_0 \subset PM_1 \subset M_0;$$
$$pM_1 \subset VM_0 \subset M_1.$$
By applying $\tau = \Pi V^{-1}$ to the three relations, we translate these relations to

$$p\tau D \subset pD^\vee \subset \tau D;$$
$$p\tau D \subset p\tau D^\vee \subset \tau D;$$
$$pD \subset p\tau D^\vee \subset D.$$

It then follows that $\mathcal{T}_{\mathcal{X}_{\mathbb{P}_a}(B), x}$ is the extension

$$0 \to \text{Hom}(pD^\vee/p\tau D, \tau D \cap D/pD^\vee) \to \mathcal{T}_{\mathcal{X}_{\mathbb{P}_a}(B), x} \to \text{Hom}(p\tau D^\vee/pA_0, \tau D \cap D/p\tau D^\vee) \to 0.$$

Thus the quotient of $\mathcal{T}_{\mathcal{D}L(\Lambda_0), x}$ by $\mathcal{T}_{\mathcal{X}_{\mathbb{P}_a}(B), x}$ is given by $\text{Hom}(p\Lambda_0/p\tau D, \tau D \cap D/p\Lambda_0)$. Varying $x$, we see $p\Lambda_0/p\tau D \cong \Lambda_0/\tau D$ corresponds to $O_{\mathcal{D}L(\Lambda_0)}(\nu)$ where $\nu$ is the normalized valuation such that $\nu(p) = |p| = p^{-1}$. In this case, we say $\tau$ is a type I representation. If the characters $\chi_1, \chi_2, \sigma$ are unramified, then we say $\tau$ is an unramified type I representation. This terminology comes from the work of [ST93] and [Sch05].

The second part follows from the first part and the fact that $\mathcal{D}L(\Lambda_0)$ is normal and that the singular locus we discard is of codimension 2 in $\mathcal{X}_{\mathbb{P}_a}(B)$. $\square$

### 3. Level raising conditions

Let $\chi_1, \chi_2, \sigma$ be characters of $\mathbb{Q}_p^\times$, we consider the principal series for $G(\mathbb{Q}_p) = \text{GSp}_4(\mathbb{Q}_p)$ given by

$$\chi_1 \times \chi_2 \times \sigma := \text{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \otimes \sigma.$$

This is defined by the normalized induction from the Borel subgroup for the character given by

$$\chi_1 \otimes \chi_2 \otimes \sigma : \begin{pmatrix} a & * & * & * \\ b & * & * & * \\ cb^{-1} & * & * & * \\ ca^{-1} \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c).$$

This is irreducible if and only if none of the characters $\chi_1, \chi_2, \chi_1\chi_2^{\pm 1}$ is equal to $\nu^{\pm 1}$ where $\nu$ is the normalized valuation such that $\nu(p) = |p| = p^{-1}$. In this case, we say $\tau$ is a type I representation. If the characters $\chi_1, \chi_2, \sigma$ are unramified, then we say $\tau$ is an unramified type I representation. This terminology comes from the work of [ST93] and [Sch05].

The local Langlands correspondence associates $\tau = \chi_1 \times \chi_2 \times \sigma$ with the $L$-parameter given by

$$\text{rec}_{GT}(\tau) = \begin{pmatrix} \chi_1\chi_2\sigma & \chi_1\sigma \\ \chi_2\sigma & \sigma \end{pmatrix}.$$
if $\tau$ is irreducible. We will be interested in the case when $\tau$ is unramified. Then the representation $\text{rec}_{GT}(\tau \otimes |c|^{-3/2})$ has the characteristic polynomial at the geometric Frobenius $\text{Frob}_p$ given by

$$Q_p(X) = X^4 - a_{p,2}X^3 + (pa_{p,1} + (p^3 + p)a_{p,0})X^2 - p^3a_{p,2}a_{p,0}X + p^6a_{p,0}^2$$

where $a_{p,0}, a_{p,1}, a_{p,2}$ are the eigenvalues of the spherical operators $T_{p,0}, T_{p,1}, T_{p,2}$ on the space $\tau^{G(\mathbb{Z}_p)}$ of spherical vectors in $\tau$, where

$$T_{p,0} = \text{char}(G(\mathbb{Z}_p)) \left( \begin{array}{ccc} p & & \\ & p & \\ & & p \end{array} \right) G(\mathbb{Z}_p);$$

$$T_{p,2} = \text{char}(G(\mathbb{Z}_p)) \left( \begin{array}{ccc} p & & \\ & 1 & \\ & & 1 \end{array} \right) G(\mathbb{Z}_p);$$

$$T_{p,1} = \text{char}(G(\mathbb{Z}_p)) \left( \begin{array}{ccc} p^2 & & \\ & p & \\ & & 1 \end{array} \right) G(\mathbb{Z}_p).$$

We call the roots of this polynomial the Hecke parameters of $\tau$. Let

$$[\alpha_p, \beta_p, \gamma_p, \delta_p] = [\alpha_p, \beta_p, \xi_p, \beta_p^{-1}, \xi_p, \alpha_p^{-1}]$$

be the Hecke parameters of $\tau$ with $\xi_p$ the similitude character of $\text{rec}_{GT}(\tau \otimes |c|^{-3/2})$ evaluated at the $\text{Frob}_p$. We are interested in the case $\xi_p = p^3$. Without loss of generality, we can assume that

$$p^3\alpha_p^{-1} = p^{3/2}\chi_1\chi_2\sigma \circ \text{Art}^{-1}(\text{Frob}_p);$$
$$p^3\beta_p^{-1} = p^{3/2}\chi_1\sigma \circ \text{Art}^{-1}(\text{Frob}_p);$$
$$\beta_p = p^{3/2}\chi_2\sigma \circ \text{Art}^{-1}(\text{Frob}_p);$$
$$\alpha_p = p^{3/2}\sigma \circ \text{Art}^{-1}(\text{Frob}_p).$$

The level raising representation can be thought of as the representation where the $\chi_1\sigma$, $\chi_2\sigma$ part of the principal series degenerates into a twisted Steinberg representation by a quadratic character. More formally, we would like to have a congruence of the form

$$\beta = p^{3/2}\chi_2\sigma \circ \text{Art}^{-1}(\text{Frob}_p) \equiv p^{3/2}\chi\nu^{1/2} \circ \text{Art}^{-1}(\text{Frob}_p) = up \mod \lambda$$
$$p^3\beta^{-1} = p^{3/2}\chi_1\sigma \circ \text{Art}^{-1}(\text{Frob}_p) \equiv p^{3/2}\chi\nu^{-1/2} \circ \text{Art}^{-1}(\text{Frob}_p) = up^2 \mod \lambda$$

for some quadratic character $\chi$ and $u = \chi(p) \in \{\pm 1\}$.

**Definition 3.1.** Let $p$ be prime and let $\tau$ be the unramified type I representation as above. Then we say $p$ is level raising special for $\tau$ if the following two conditions are satisfied.

1. $l \mid p^2 - 1$;
(2) the Hecke parameters $[\alpha_p, \beta_p, p^3\beta^{-1}, p^3\alpha_p^{-1}]$ of $\tau$ satisfy simultaneously the following three conditions:

$$
\begin{align*}
\beta_p + p^3\beta_p^{-1} &\equiv u(p + p^2) \mod \lambda \text{ for some } u \in \{\pm 1\} \\
\alpha_p + p^3\alpha_p^{-1} &\not\equiv \pm(p + p^2) \mod \lambda \\
\alpha_p + p^3\alpha_p^{-1} + \beta_p + p^3\beta_p^{-1} &\not\equiv \pm2(p + p^2) \mod \lambda.
\end{align*}
$$

We say $p$ is level raising special for $\pi$ of depth $m$ if it is level raising special for $\tau$ and the $\lambda$-adic valuation of

$$\beta_p + p^3\beta_p^{-1} - u(p + p^2)$$

is exactly $m$.

Examine the classification result of [ST93] and [Sch05]. This can be understood as the congruence between the unramified type I representation and the ramified type II representation. Recall a representation $\tau$ is of type II if $\tau$ is an irreducible constituent of the representation $\nu^{-1/2}\chi \times \nu^{1/2}\chi \times \sigma$ for the normalized valuation $\nu(p) = p^{-1}$. More precisely, it is called of type IIa if it is of the form $\chi\text{St}_{\text{GL}_2} \times \sigma$ and it is called of type IIb if it is of the form $\chi\text{GL}_2 \times \sigma$. Both of these two representations have their common semi-simple part of the $L$-parameter given by

$$\text{rec}_{GT}(\tau) = \begin{pmatrix}
\chi^2\sigma \\
\nu^{1/2}\chi\sigma \\
\nu^{-1/2}\chi\sigma \\
\sigma
\end{pmatrix}.$$

While the $L$-parameter of $\chi\text{GL}_2 \times \sigma$ has trivial monodromy $N = 0$, the $L$-parameter of the type IIa representation $\chi\text{St}_{\text{GL}_2} \times \sigma$ has monodromy $N$ conjugate to

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
$$

Our level raising condition can be understood as a condition for congruence between type I representation and a special type of the type IIa representation. In the work of Sorensen [Sor09], he further assumes that $[\alpha_p, \beta_p, p^3\beta^{-1}, p^3\alpha_p^{-1}]$ is congruent to $[1, p, p^2, p^3]$. In fact, his level raising condition corresponds to the case when $\tau$ is congruent to the trivial representation.

Let $\pi$ be a cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ which is of general type. We assume that $\pi$ has trivial central character and weight $(k_1, k_2) = (3, 3)$.

**Definition 3.2.** Suppose $\pi$ is a cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ as above and is an unramified principal series at $p$. Then we say $p$ is level raising special for $\pi$ of depth $m$ if $p$ is level raising special of depth $m$ for $\pi_p$ in the sense of Definition 3.1.

Let $\Sigma_\pi$ be the minimal set of non-archimedean places of $\mathbb{Q}$ such that $\pi$ is unramified away from the places in $\Sigma_\pi$. 

**Construction 3.3.** Let \( \pi \) be as above and \( \Sigma \) be a finite set of non-archimedean places of \( \mathbb{Q} \) containing the set \( \Sigma_\pi \). Let 
\[
\mathcal{T}^\Sigma = \bigotimes_{v \notin \Sigma} \mathbb{Z}[\mathcal{G}(\mathbb{Z}_v) \backslash \mathcal{G}(\mathbb{Q}_v)/\mathcal{G}(\mathbb{Z}_v)]
\]
be the abstract Hecke algebra unramified away from \( \Sigma \).

1. To such \( \pi \), we can attach a homomorphism 
\[
\phi_\pi : \mathcal{T}^\Sigma \to \mathbb{C}
\]
determined by the Hecke parameters of \( \pi \) at each place \( v \) outside of \( \Sigma \). More precisely, \( \phi_\pi \) will send 
\[
T_{0,v} \mapsto a_{v,0}, \\
T_{1,v} \mapsto a_{v,1}, \\
T_{2,v} \mapsto a_{v,2},
\]
where \( a_{v,0} = 1 \) as \( \pi \) has trivial central character and \( a_{v,1} \) and \( a_{v,2} \) are the eigenvalues of \( T_{v,1} \) and \( T_{v,2} \) on the space of spherical vectors of \( \tau \) for each \( v \) outside of \( \Sigma \).

2. In fact, the image of \( \phi_\pi \) is contained in a number field \( E \) called the coefficient field of \( \pi \). We further assume that the image of \( \phi_\pi \) is contained in the ring of integers \( \mathcal{O}_E \), that is we have a homomorphism 
\[
\phi_\pi : \mathcal{T}^\Sigma \to \mathcal{O}_E.
\]
Let \( \lambda \) be a prime of \( E \) induced by \( \iota \) over \( l \) and \( \mathcal{O}_\lambda \) be the valuation ring of \( E_\lambda \). Then \( \phi_\pi \) induces a morphism 
\[
\phi_{\pi,\lambda} : \mathcal{T}^\Sigma \to \mathcal{O}_\lambda.
\]
which we will call the \( \lambda \)-adic avatar of \( \phi_\pi \).

3. We introduce the following two ideals given by 
\[
m = \mathcal{T}^{\Sigma \cup \{p\}} \cap \ker(T^\Sigma \phi_{\pi,\lambda} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda/\lambda),
\]
\[
n = \mathcal{T}^{\Sigma \cup \{p\}} \cap \ker(T^\Sigma \phi_{\pi,\lambda} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda/\lambda^m).
\]
of the Hecke algebra \( \mathcal{T}^{\Sigma \cup \{p\}} \) unramified away from \( \Sigma \cup \{p\} \).

### 4. Jacquet-Langlands Correspondence for GSp\(_4\)

**4.1. Matching of orbital integrals.** In this subsection we will work locally at a prime \( p \). Let \( G \) be the quaternionic unitary group \( \text{GU}_2(D) \) and \( \mathbf{G} \) be the group \( \text{GSp}_4 \) over \( \mathbb{Q}_p \) throughout this subsection. We fix an inner twisting \( \psi : G \to \mathbf{G} \), that is \( \psi \) is an isomorphism when base changed to \( \overline{\mathbb{Q}}_p \) such that \( \sigma(\psi)\psi^{-1} \) is an inner automorphism of \( \mathbf{G} \). This \( \psi \) is defined over \( \mathbb{Q}_p^2 \) and defines an injection from the semisimple stable conjugacy classes in \( \mathbf{G}(\mathbb{Q}_p) \). Let \( \gamma \in \mathbf{G}(\mathbb{Q}_p) \) be a semisimple element and \( f \in C_c^\infty(\mathbf{G}^{\text{ad}}(\mathbb{Q}_p)) \). We recall the orbital integral is by defined by 
\[
O^G_\gamma(f) = \int_{\mathbf{G}(\mathbb{Q}_p) \backslash G(\mathbb{Q}_p)} f(g^{-1}\gamma g) dg.
\]
where $G_\gamma$ is the centralizer for $\gamma$ in $G(\mathbb{Q}_p)$. Let $\tilde{\gamma}$ be the set of representatives of the stable conjugacy class of $\gamma$ modulo the center. The stable orbital integral is

$$SO_\gamma^G(f) = \sum_{\tilde{\gamma}} e(G_{\tilde{\gamma}}) O_{\tilde{\gamma}}^G(f)$$

where $e(\cdot)$ is the Kottwitz sign given in [Ko83]. Let $f_p^G \in C_c^\infty(G^\text{ad}(\mathbb{Q}_p))$ and $f_p^G \in C_c^\infty(G^\text{ad}(\mathbb{Q}_p))$. Then we say $f_p^G$ and $f_p^G$ have matching orbital integrals if

$$SO_\gamma^G(f_p^G) = SO_\gamma^G(f_p^G)$$

if $\gamma'$ is in the semisimple conjugacy class given by $\psi(\gamma)$ and $SO_\gamma^G(f_p^G) = 0$ for $\gamma$ not coming from $G$. It follows from [Wa97] one can always find such function with matching orbital integrals. For the global case, one defines a global stable orbital integral as a product of local ones.

Let $e_K$ be the characteristic function of $K$ and $\eta$ be the Atkkin-Lehner element given by

$$\eta = \begin{pmatrix} 1 & 1 \\ p & p \end{pmatrix}$$

Let $P_a$ be the paramodular subgroup of $G(\mathbb{Q}_p)$ and $P_a^D$ be the paramodular subgroup of $\text{GU}_2(D)$. Then the following matching theorem is due to Sorensen [Sor09, Theorem B].

**Theorem 4.1 (Sorensen).** The functions $e_{\eta P_a}$ and $e_{P_a^D}$ have matching orbital integrals.

4.2. **Stabilization of trace formula.** In this subsection, we will consider the global situation. Therefore we will let $G$ be the quaternionic unitary group $G(B)$ or $G$ over $\mathbb{Q}$ and $G$ be the group $\text{GSp}_4$ over $\mathbb{Q}$. Up to equivalence, $G$ and $G$ admits a unique non-trivial elliptic endoscopic triple $(H, s, \xi)$ consisting of the following data.

- The endoscopic group $H$ is
  $$H = GL_2 \times GL_2 / G_m$$
  where $G_m$ acts by identifying $x$ with $(x, x^{-1})$. The dual group of $H$ is given by
  $$\hat{H} = \{(g, g') \in GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) : \det(g) = \det(g')\}.$$

- The element $s \in \hat{G}$ is given by
  $$s = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

- The homomorphism $\xi : \hat{H} \to \hat{G}$ whose image is given by $Z_{\hat{G}}(s)^\circ$ is defined by
  $$\xi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \to \begin{pmatrix} a' & b' \\ a & b \\ c & d \end{pmatrix} \begin{pmatrix} c' & d' \end{pmatrix}.$$
Recall that a semisimple element \( \delta \in H(\mathbb{Q}_p) \) is said to be a \((G, H)\)-regular if \( \alpha(\delta) \neq 1 \) for every root \( \alpha \) of \( G \) that does not come from \( H \).

**Theorem 4.2.** For every test function \( f^G \in C_c^\infty(G(\mathbb{Q}_p)) \), there exists a matching function \( f^H \in C_c^\infty(H(\mathbb{Q}_p)) \) which means

\[
SO^H_\delta(f^H) = \sum_{\gamma} \Delta_{G,H}(\delta, \gamma) e(G_\gamma)O^G_\gamma(f^G)
\]

for all \((G, H)\)-regular semisimple \( \delta \in H(\mathbb{Q}_p) \). Here the sum on \( \gamma \) runs through a set of representatives for conjugacy classes in \( G(\mathbb{Q}_p) \) in the stable conjugacy class associated to \( \delta \) and \( \Delta_{G,H}(\delta, \gamma) \) is the Langlands–Shelstad transfer factors.

**Proof.** This follows from the general theorem of Waldapurger [Wal97] and the standard fundamental lemma due to Hales in [Hal97].

We consider the invariant trace formula for \( G \) and \( G \) but only for the discrete terms. We have spectral expansions for the invariant distributions \( I^G_{\text{disc}} \) and \( I^G_{\text{disc}} \): for smooth functions \( f^G \) and \( f^G \) for \( G(\mathbb{A}_Q) \) and \( G(\mathbb{A}_Q) \)

\[
I^G_{\text{disc}}(f^G) = \sum_{\pi} m^G_{\text{disc}}(\pi) \text{tr} \pi(f^G)
\]

\[
I^G_{\text{disc}}(f^G) = \sum_{\pi} m^G_{\text{disc}}(\pi) \text{tr} \pi(f^G)
\]

where the sums run over discrete automorphic representations of \( G(\mathbb{A}_Q) \) and \( G(\mathbb{A}_Q) \). We can and will assume that all these representations have trivial central characters. For the endoscopic group \( H \), we can define the invariant distribution \( I^H_{\text{disc}} \) and similarly we have a spectral expansion

\[
I^H_{\text{disc}}(f^H) = \sum_{\rho} m^H_{\text{disc}}(\rho) \text{tr} \rho(f^H)
\]

where \( \rho \) runs through discrete automorphic representations of \( H(\mathbb{A}_Q) \) and \( f^H \) is a smooth function for \( H(\mathbb{A}_Q) \). When \( \pi \) (resp. \( \bar{\pi} \)) is cuspidal but not a CAP representation, then \( m^G_{\text{disc}}(\bar{\pi}) \) (resp. \( m^G_{\text{disc}}(\pi) \)) is the automorphic multiplicity of \( \bar{\pi} \) (resp. \( \pi \)). The distribution \( I^H_{\text{disc}} \) is not stable but can be stabilized by subtracting suitable endoscopic terms. In fact, we have the following theorem of Arthur developed in [Art01], [Art02] and proved in [Art03].

**Theorem 4.3 (Arthur).** Let \( f^G \) and \( f^G \) be smooth functions as above.

1. The distribution \( ST^G_{\text{disc}} \) defined by

\[
ST^G_{\text{disc}}(f^G) = I^G_{\text{disc}}(f^G) - \frac{1}{4} I^H_{\text{disc}}(f^H)
\]

is a stable distribution. Here \( f^H \) is a function that has matching orbital integrals with \( f^G \).
(2) The distribution $I^G_{\text{disc}}$ can be expressed by

$$I^G_{\text{disc}}(f^G) = ST^G_{\text{disc}}(f^G_G) + \frac{1}{4}I^H_{\text{disc}}(f^G_H).$$

Here $f^G_H$ and $f^G_G$ are functions that have matching orbital integrals with $f^G$.

There is a global character identity

$$\text{tr} \rho(f^H) = \sum \Delta_{G,H}(\rho, \pi) \text{tr} \pi(f^G).$$

Here $f^H$ and $f^G$ have matching orbital integrals and $\pi$ runs through all irreducible admissible representations of $G(\mathbb{A}_Q)$. The term $\Delta_{G,H}(\rho, \pi)$ is the global transfer factor which is a product of local transfer factors. We have arrived the following lemma.

Lemma 4.4. Suppose that $f^G$ and $f^G$ are functions as above with matching orbital integrals. Then following trace identity holds:

$$\sum_{\pi} (m_{\text{disc}}^G(\pi) - \frac{1}{4} \sum_{\rho} m^H\text{disc}(\rho) \Delta_{G,H}(\rho, \pi)) \text{tr} \pi(f^G) = \sum_{\rho} (m^G_{\text{disc}}(\pi) - \frac{1}{4} \sum_{\rho} m^H\text{disc}(\rho) \Delta_{G,H}(\rho, \pi)) \text{tr} \pi(f^G).$$

The following theorem is the main result of this subsection which can be seen as a special case of the refined Jacquet-Langlands transfer for $\text{GSp}_4$ over $\mathbb{Q}$ and its inner form $G$. Note that the global Jacquet-Langlands transfer for automorphic representations of general type for $G(\mathbb{A}_Q)$ and its inner forms are constructed in [RW21, Theorem 11.4, Theorem 11.5]. Moreover the local character identity in our setting is proved in the work of Chan–Gan [CG15], we therefore expect that more general version of the theorem below may be within reach.

Theorem 4.5. Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A}_Q)$ of general type with weight $(3,3)$ and trivial central character.

1. Suppose that $\overline{\pi}(\text{Sh}(B,K)[t^q \pi^\infty])$ is nontrivial for some open compact $K$ of $G(B)$ which is paramodular at $q$. Then we can complete $\pi^\infty$ to a cuspidal automorphic representation $\Pi$ of $G(\mathbb{A}_Q)$. Moreover for all such $\Pi$, its local component $\Pi_q$ at $q$ is of type IIa. Moreover the automorphic multiplicity of those $\pi$ completing $\pi^\infty$ in $\overline{\pi}(\text{Sh}(B,K))$ is the same as the automorphic multiplicity of $\Pi$.

2. Suppose that $\Pi_\infty^p(\text{Sh}(B,K_{Pa}), \mathcal{O}_l)[t^p \pi^\infty]$ is non-trivial for some open compact $K_{Pa}$ which is paramodular at $p$. Then we can complete $\pi^\infty$ to a cuspidal automorphic representation $\Pi$ of $G(\mathbb{A}_Q)$. Moreover for all such $\Pi$, both $\Pi_p$ and $\Pi_q$ are of type IIa. Moreover the automorphic multiplicity of those $\pi$ completing $\pi^\infty$ in $\Pi_\infty^p(\text{Sh}(B,K_{Pa}), \mathcal{O}_l)$ is the same as the automorphic multiplicity of $\Pi$.

Proof. For the first statement, note that $\pi^\infty$ can be completed to an automorphic representation $\overline{\pi}$ of $G(B)$ whose component $\overline{\pi}_q$ is paraspherical. This means that $\overline{\pi}_q$ admits
fixed vectors by the paramodular subgroup \( \text{Pa}^D \) for the quaternion division algebra \( D' \) over \( \mathbb{Q}_q \). We apply the trace identity in Lemma 4.4 and fix \( \pi^\infty \). Then we have

\[
\sum_{\bar{\pi}_{q,\infty}} m^G_{\text{disc}}(\bar{\pi}_\infty \otimes \bar{\pi}_q \otimes \pi^\infty) \text{tr} \bar{\pi}_{q,\infty}(f_{q,\infty}^G(\bar{\pi})) = \sum_{\Pi_{q,\infty}} m^G_{\text{disc}}(\Pi_\infty \otimes \Pi_q \otimes \pi^\infty) \text{tr} \Pi_{q,\infty}(f_{q,\infty}^G).
\]

For any pair of functions \( f_{q,\infty}^G(\bar{\pi}) \) and \( f_{q,\infty}^G \) with matching orbital integrals for \( G(\mathbb{B}) \) and \( G \) at \( \infty q \). To see this, since \( \pi \) is of general type, the transfer factors in the trace identity in Lemma 4.4 must vanish. Using Shelstad’s character identity [She79] and the same argument as in [Sor09, 4.3.2], we can further simplify the above identity to

\[
\sum_{\bar{\pi}_q} m^G_{\text{disc}}(\bar{\pi}_\infty \otimes \bar{\pi}_q \otimes \pi^q) \text{tr} \bar{\pi}_q(f_{q}^G(\bar{\pi})) = \sum_{\Pi_q} m^G_{\text{disc}}(\Pi_\infty \otimes \Pi_q \otimes \pi^q) \text{tr} \Pi_q(f_{q}^G).
\]

Since \( e_{q,\text{Pa}} \) and \( e_{p,\text{Pa}'} \) have matching orbital integrals and \( \bar{\pi}_q \) is paraspherical, the Atkin–Lehner element \( \eta \) has positive trace on \( \Pi_{q,\text{Pa}} \) by the local character identity in [CG15, Proposition 11.1]. Then \( \Pi_q \) must be of type IIa since an unramified type I representation is traceless for \( \eta \). It follows from the independence of characters on \( G(\mathbb{B})(\mathbb{Q}_q) \) and \( G_{\mathbb{Q}_q} \) that \( m^G_{\text{disc}}(\bar{\pi}_\infty \otimes \bar{\pi}_q \otimes \pi^q) \) is the same as \( m_{\text{disc}}^G(\Pi_\infty \otimes \Pi_q \otimes \pi^q) \). Then the first part is clear.

For the second part, the argument is the same except that we don’t have to use Shelstad’s character identity over \( \mathbb{R} \) and that we have to use the same argument twice for \( p \) and \( q \), one at a time. \( \square \)

## 5. Monodromy on the cohomology of the quaternionic unitary Shimura variety

Let \( S = \text{Spec}(\mathbb{R}) \) be the spectrum of a henselian DVR. We choose a uniformizer \( \pi \) of \( R \). We denote by \( s \) the closed point of \( S \) and by \( \eta \) the generic point of \( S \). We assume the residue field \( k(s) \) at \( s \) is of characteristic \( p \). Let \( \pi \) be a geometric point of \( S \) over \( s \). Let \( \overline{\eta} \) be a separable closure of \( \eta \).

For a morphism \( f : X \to S \), we obtain by base change the following natural inclusion maps

\[
X_{\overline{\eta}} \hookrightarrow X_{\eta} \hookleftarrow X_{\overline{\eta}}.
\]

Let \( K \in \text{D}^+(X_{\overline{\eta}}, \Lambda) \) with coefficient in \( \Lambda = \mathbb{Z}/l^m \) or \( \mathbb{Z}_l \), then we define the nearby cycle complex \( R\Psi(K) \in \text{D}^+(X_{\overline{\eta}}, \Lambda) \) by \( R\Psi(K) = \overline{\eta} \otimes_{\mathbb{Z}} \mathbb{Z}/lK|_{X_{\overline{\eta}}} \). For \( K \in \text{D}^+(X_{\overline{\eta}}, \Lambda) \), the adjunction map defines the specialization map \( sp \) and the following distinguished triangle

\[
K|_{X_{\overline{\eta}}}[1] \xrightarrow{sp} R\Psi(K|_{X_{\overline{\eta}}}) \to R\Phi(K) \to K|_{X_{\overline{\eta}}}[1]
\]
The complex $R\Phi(K)$ is known as the vanishing cycle complex and is the mapping cone of the specialization map $sp$. If $f$ is proper, then we have

$$R\Gamma(X, K|_{X_s}) \cong R\Gamma(X, R\Psi(K|_{X_s}))$$

and in general we always have the following long exact sequence

$$\cdots \to H^i(X, K|_{X_s}) \xrightarrow{sp} H^i(X, R\Psi(K|_{X_s})) \to H^i(X, R\Phi(K)) \to \cdots$$

The first map $sp$ is called the specialization map and the cohomology of vanishing cycles $H^i(X_s, R\Phi(K))$ measures the defect of $sp$ from being an isomorphism. If we assume that $f$ is smooth, then $H^i(X_s, R\Phi(K))$ vanishes and the specialization map is an isomorphism.

5.1. **Picard-Lefschetz formula.** We recall the formalism of the Picard–Lefschetz formula. Suppose that $f : X \to S$ is a regular, flat, finite type morphism of relative dimension $n$ which is smooth outside a finite collection $\Sigma$ of closed points in $X_s$. In this subsection, we only consider the trivial coefficient $\Lambda$ for simplicity and all the results recalled here can be extended to more general constructible coefficients. We are particularly interested in the cohomology $H^n(X_s, R\Psi(\Lambda))$ and the action of the monodromy on it.

Under the above assumptions, we have $R\Phi(\Lambda)|_{X_s} - \Sigma = 0$ and moreover

$$R\Phi(\Lambda) = \bigoplus_{\sigma \in \Sigma} R^n\Phi_\sigma(\Lambda)$$

is concentrated at degree $n$. Therefore the distinguished triangle (5.1) gives the following long exact sequence

$$\text{(5.2)}$$

$$0 \to H^n(X, \Lambda) \xrightarrow{sp} H^n(X, R\Psi(\Lambda)) \xrightarrow{\partial} \bigoplus_{\sigma \in \Sigma} R^n\Phi_\sigma(\Lambda) \to H^{n+1}(X_s, \Lambda) \to H^{n+1}(X, R\Psi(\Lambda)) \to 0.$$ 

Assume furthermore that every $\sigma \in \Sigma$ is an ordinary quadratic singularity and that $H^n(X, R\Psi(\Lambda)) = H^n(X_s, \Lambda)$. The first condition means $X$ is étale locally near $x$ isomorphic to

- $V(Q) \subset A^{2r}_S$ if $n = 2r - 1$ where $Q = \sum_{1 \leq i \leq r} X_i X_{i+r} + \pi$;
- $V(Q) \subset A^{2r+1}_S$ if $n = 2r$ where $Q = \sum_{1 \leq i \leq r} X_i X_{i+r} + X^2_{2r+1} + \pi$.

Then in this case, we have

$$R^n\Phi_\sigma(\Lambda) \cong \Lambda$$

is free of rank 1 over $\Lambda$. We will call $\bigoplus R^n\Phi_\sigma(\Lambda)(r)$ the space of vanishing cycles. The localized cohomology $H^n_{[\sigma]}(X_s, R\Psi(\Lambda(r - 1)))$ is the dual of $R^n\Phi_\sigma(\Lambda(r))$ under Poincare duality. We will call $\bigoplus_{\sigma \in \Sigma} H^n_{[\sigma]}(X_s, R\Psi(\Lambda(r - 1)))$ the dual space of vanishing cycles.

Let $I_{\eta} \subset \text{Gal}(\overline{\eta}/\eta)$ be the inertia group and let $\xi \in I_{\eta}$. Then we have the local variation map

$$\text{Var}_\sigma(\xi) : R^n\Phi_\sigma(\Lambda(r - 1)) \to H^n_{[\sigma]}(X_s, R\Psi(\Lambda(r - 1)))$$
as in [Il02, 1.4.2] and the action of $\xi - 1$ on $H^n(X_{\eta}, \Lambda(r - 1))$ can be factored as

\[
\begin{array}{c}
H^n(X_{\eta}, \Lambda(r - 1)) \rightarrow \bigoplus_{\sigma \in \Sigma} R^n\Phi_\sigma(\Lambda(r - 1)) \xrightarrow{\alpha} H^{n+1}(X_{\eta}, \Lambda(r - 1)) \\
\downarrow \xi - 1 \hspace{1cm} \downarrow \text{Var}_\Sigma(\xi) \\
H^c_n(X_{\eta}, \Lambda(r - 1)) \leftarrow \bigoplus_{\sigma \in \Sigma} H^n_{(\sigma)}(X_{\eta}, R\Psi(\Lambda(r - 1))) \xleftarrow{\beta} H^{n-1}_c(X_{\eta}, \Lambda(r - 1))
\end{array}
\]

where $\text{Var}_\Sigma(\xi) = \bigoplus_{\sigma \in \Sigma} \text{Var}_\sigma(\xi)$.

One has also a Frobenius equivariant version

\[
\begin{array}{c}
H^n(X_{\eta}, \Lambda(r)) \rightarrow \bigoplus_{\sigma \in \Sigma} R^n\Phi_\sigma(\Lambda(r)) \xrightarrow{\alpha} H^{n+1}(X_{\eta}, \Lambda(r)) \\
\downarrow N \hspace{1cm} \downarrow N_\Sigma \\
H^c_n(X_{\eta}, \Lambda(r - 1)) \leftarrow \bigoplus_{\sigma \in \Sigma} H^n_{(\sigma)}(X_{\eta}, R\Psi(\Lambda(r - 1))) \xleftarrow{\beta} H^{n-1}_c(X_{\eta}, \Lambda(r - 1))
\end{array}
\]

where $N$ is the monodromy operator and $N_\Sigma = \bigoplus_{\sigma \in \Sigma} N_\sigma$ are local monodromy operators.

We will loosely refer to this diagram as the Picard-Lefschetz formula for $X_{\eta}$. The upper half of this diagram is usually referred to as the specialization exact sequence whereas the lower half is usually referred to as the co-specialization exact sequence.

Note that the monodromy filtration of $H^c_n(X_{\eta}, \Lambda)$ is then determined by the Picard-Lefschetz formula

\[
0 \subset \text{Gr}_{-1}(r - 1) \subset \text{Gr}_0(r - 1) \subset \text{Gr}_1(r - 1)
\]

The successive quotient of the filtration is given by

\[
\begin{align*}
\text{Gr}_{-1}(r - 1) &= \text{Gr}_{-1} H^c_n(X_{\eta}, \Lambda(r - 1)) = \text{Coker}(\beta) \\
\text{Gr}_0(r - 1) &= \text{Gr}_0 H^c_n(X_{\eta}, \Lambda(r - 1)) = H^c_n(X_{\eta}, \Lambda(r - 1)) \\
\text{Gr}_1(r - 1) &= \text{Gr}_1 H^c_n(X_{\eta}, \Lambda(r - 1)) = \text{Ker}(\alpha).
\end{align*}
\]

The filtration induces an filtration $F_1 H^1(I_{Q_{p^2}}, H^c_n(X_{\eta}, \Lambda(r - 1)))$ on the quotient module

\[
H^1(I_{Q_{p^2}}, H^c_n(X_{\eta}, \Lambda(r - 1))).
\]

The monodromy map $N$ factor through

\[
H^p(X_{\eta}, \Lambda(r)) \rightarrow \text{Gr}_1(r - 1) \xrightarrow{N} \text{Gr}_{-1}(r - 1) \rightarrow H^c_n(X_{\eta}, \Lambda(r - 1)).
\]

We summarize the discussion in the following proposition.

**Proposition 5.4.** We have an isomorphism

\[
F_{-1} H^1(I_{Q_{p^2}}, H^c_n(X_{\eta}, \Lambda(r - 1))) \cong \frac{\text{Coker}(\beta)}{N_\Sigma \text{Ker}(\alpha)}.
\]
The map $\alpha$ induces an isomorphism
$$F_1^{-1}(I_{Q_p^2}, H^r_c(X_{\eta}, \Lambda(r - 1))) \cong \text{Coker}(\alpha \circ N^{-1} \circ \beta).$$

Proof. The first isomorphism follows directly from the above discussion. The second isomorphism follows from applying $\alpha$ to the first isomorphism and the fact that $N_\Sigma$ is an isomorphism in our case. \hfill \Box

5.2. Descriptions of $\alpha$ and $\beta$. It is important to realize the maps $\alpha$ and $\beta$ as certain Gysin and restriction maps. For this purpose, we need to introduce a semi-stable model of $X$ over $S = \text{Spec } R$.

- Let $X_{\text{Bl}}$ be the blow-up of $X$ at the singular points $\sigma \in \Sigma$. The scheme $X_{\text{Bl}}$ is semi-stable and its special fiber has the form $X^s_{\text{Bl}} = \tilde{X}^s_{\text{Bl}} + \sum_{\sigma \in \Sigma} D_{\sigma}$ as a divisor in $X^s_{\text{Bl}}$.
- The scheme $\tilde{X}^s_{\text{Bl}}$ is the strict transform of $X^s$ under this blow-up and $D = \sum_{\sigma \in \Sigma} D_{\sigma}$ is the exceptional divisor. Note the smooth scheme $\tilde{X}^s_{\text{Bl}}$ is also the blow-up of $X^s$ at the singular locus $\Sigma$.
- Each $D_{\sigma}$ is the hypersurface of $P^{n+1} = \text{Proj } k[s][X_1, \ldots, X_n, T]$ defined by $\overline{Q}_\sigma - \overline{\pi}_\sigma T^2 = 0$ where $Q_\sigma$ is a smooth quadric and $\overline{\pi}_\sigma$. Let $C_\sigma$ be the hyperplane defined by $T = 0$ which is also a smooth quadric. We have $\tilde{X}^s_{\text{Bl}} \cap D_{\sigma} = C_\sigma$.
- The exceptional divisor of the blow-up $\tilde{X}^s_{\text{Bl}}$ is precisely $\sum_{\sigma \in \Sigma} C_\sigma$.

Lemma 5.5. The cohomology of the quadric $C_\sigma$ is given by the formulas below.

1. If $n - 1 = 2r - 1$ is odd, then we have
$$H^i(C_\sigma, \Lambda(i/2)) = \begin{cases} \Lambda \eta^{i/2} & \text{if } i \text{ even, } 0 \leq i \leq 2(n - 1); \\ 0 & \text{otherwise.} \end{cases}$$

2. If $n - 1 = 2r - 2$ is even, then we have
$$H^i(C_\sigma, \Lambda(i/2)) = \begin{cases} \Lambda \eta^{i/2} & \text{if } i \text{ even, } 0 \leq i \leq 2(n - 1), i \neq n - 1; \\ \text{rank two module over } \Lambda & \text{if } i = n - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Here $\eta \in H^2(C_\sigma, \Lambda(1))$ is the class of the hyperplane section.

Remark 5.6. Note that $(\eta^{(n-1)/2})$ is always contained in $H^{n-1}(C_\sigma, \Lambda((n-1)/2))$.

Proof. The result is well-known. See [SGA7 II, Theorem 3.3] for the proof. The last claim is given in [SGA7 II, Theorem 3.3 (iii)(c)]. \hfill \Box

We are particularly interested in the middle degree cohomology $H^{n-1}(C_\sigma, \Lambda(r - 1))$ in the case $n - 1$ is even. We call the module
$$H^{n-1}(C_\sigma, \Lambda(r - 1))^\bullet := (\eta^{(n-1)/2})^\perp$$
the primitive part of $H^{n-1}(C_\sigma, \Lambda(r-1))$ and

$$H^{n-1}(C_\sigma, \Lambda(r-1)) := H^{n-1}(C_\sigma, \Lambda(r-1))/\eta^{(n-1)/2}$$

the primitive quotient of $H^{n-1}(C_\sigma, \Lambda(r-1))$.

**Lemma 5.7.** We have the following descriptions of the space of vanishing cycles and the dual space of vanishing cycles.

1. The space of vanishing cycles $R^n \Phi_\sigma(\Lambda)(r)$ at $\sigma$ can be identified with the primitive part $H^{n-1}(C_\sigma, \Lambda(r-1))^\bullet$ of $H^{n-1}(C_\sigma, \Lambda(r-1))$.
2. The dual space of vanishing cycles $H^n_{\{\sigma\}}(X_\sigma, R\Psi(\Lambda)(r-1))$ at $\sigma$ can be identified with the primitive quotient $H^{n-1}(C_\sigma, \Lambda(r-1))^\bullet$ of $H^{n-1}(C_\sigma, \Lambda(r-1))$.
3. Let $\eta_\sigma^\bullet$ be a generator of the space $H^{n-1}(C_\sigma, \Lambda(r-1))^\bullet$. Then the self-intersection number of $\eta_\sigma^\bullet$ is $-2$.

**Proof.** This is also well-known. See [SGA7 II, 2.2.5] and [Ill02, Remarks 2.7]. □

The above discussions allows us to describe the maps $\alpha$ and $\beta$ as certain Gysin or restriction maps as in the proposition below.

**Proposition 5.8.** We have the following descriptions of the maps $\alpha$ and $\beta$.

1. The space of vanishing cycles $\bigoplus_{\sigma \in \Sigma} R^n \Phi_\sigma(\Lambda)(r)$ and the map $\alpha$ inserts in the following commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{\sigma \in \Sigma} H^{n+1}(C_\sigma, \Lambda(r-1)) & \xrightarrow{\sim} & \bigoplus_{\sigma \in \Sigma} H^{n+1}(C_\sigma, \Lambda(r)) \\
& \uparrow \omega & \\
\bigoplus_{\sigma \in \Sigma} H^{n-1}(C_\sigma, \Lambda(r-1)) & \xrightarrow{\omega} & H^{n+1}(X_\sigma, \Lambda(r)) \\
& \uparrow & \\
\bigoplus_{\sigma \in \Sigma} R^n \Phi_\sigma(\Lambda)(r) & \xrightarrow{\alpha} & H^n(X_\sigma, \Lambda(r))
\end{array}
$$

The left and right columns are exact.
(2) The dual space of vanishing cycles $\bigoplus_{\sigma \in \Sigma} H^0_{(\sigma)}(X_{\Sigma}, R\Psi(\Lambda)(r - 1))$ and the map $\beta$ inserts in the following commutative diagram

$$\begin{array}{c}
\bigoplus_{\sigma \in \Sigma} H^0_{(\sigma)}(X_{\Sigma}, R\Psi(\Lambda)(r - 1)) \\ \cap \beta \leftarrow H^0_{c}(X_{\Sigma}, \Lambda(1 - r)) \\
\bigoplus_{\sigma \in \Sigma} H^0_{c}(C_{\sigma}, \Lambda(1 - r)) \uparrow \beta \\
\bigoplus_{\sigma \in \Sigma} H^0_{c}(C_{\sigma}, \Lambda(2 - r)) \sim \bigoplus_{\sigma \in \Sigma} H^0_{c}(C_{\sigma}, \Lambda(2 - r))
\end{array}$$

The left and right columns are exact.

Proof. The second part is essentially the dual statement of the first part. Therefore we will only prove the first part.

For the first part, we have $R^n\Phi(\Lambda(r)) = H^0_{c}(X_{\Sigma}, \Lambda(r - 1))$ and

$$H^0_{c}(C_{\sigma}, \Lambda(1 - r)) \sim Ker(H^0_{c}(C_{\sigma}, \Lambda(2 - r)) \cup \eta) \rightarrow H^0_{c}(C_{\sigma}, \Lambda(2 - r))$$

by Lemma 2.10. The left column is therefore exact. The right column is the blow-up exact sequence. The commutativity of the diagram follows by definition.

5.3. Nearby cycles on certain Shimura varieties. Suppose $f : X \rightarrow S$ is a proper morphism. Then the proper base change theorem implies that we have an isomorphism

$$H^i(X_{\eta}, \Lambda) \cong H^i(X_{\Sigma}, R\Psi(\Lambda)).$$

Note that if $f : X \rightarrow S$ is not proper, then we don’t always have the above isomorphism. However, in the setting of Shimura varieties with good compactification in sense of [LS18a] and [LS18b], we do have such isomorphism.

In this article we are only concerned with the Shimura variety $X_{Pa}(B)$ and this Shimura variety belongs to the case of

* (Nm): A flat integral model defined by taking normalization of a characteristic 0 PEL type moduli problem over a product of good reduction integral models of smooth PEL type moduli problem

in the classification of [LS18a] and [LS18b]. The following theorem summarizes the results we need.

**Theorem 5.9** ([LS18b Corollary 4.6]). Suppose $X$ is a Shimura variety that is in the case of (Nm) and let $V$ be an automorphic étale sheaf defined as in [LS18b §3]. Then the canonical adjunction morphisms

$$H^i(X_{\eta}, V) \rightarrow H^i(X_{\Sigma}, R\Psi(V)).$$
and
\[ H^i_c(X_{\overline{\mathbb{Q}}}, R\Psi(V)) \to H^i_c(X_{\overline{\mathbb{Q}}}, V) \]
are isomorphisms for all \( i \).

Using this theorem, we will freely identify the cohomology of the generic fiber of the quaternionic unitary Shimura variety \( X_{Pa}(B) \) and the nearby cycle cohomology of the special fiber \( \overline{X_{Pa}(B)} \) for the rest of this article.

6. The level raising matrix

Let \( \pi \) be a cuspidal automorphic representation of general type with weight \((3, 3)\) and trivial central character. Let \( \Sigma \) be the finite set of non-archimedean places of \( \mathbb{Q} \) containing the prime \( q \) but not \( p \) where \( pq \) is the discriminant of the quaternion algebra \( B \). We let
\[ T^\Sigma = \bigotimes_{v \not\in \Sigma} \mathbb{Z}[G(\mathbb{Z}_v) \backslash G(\mathbb{Q}_v)/G(\mathbb{Z}_v)] \]
be the abstract Hecke algebra unramified away from \( \Sigma \). Suppose that the datum \((\pi, \Sigma)\) satisfy the following additional properties.

- The representation \( \pi \) is unramified away from \( \Sigma \). The Hecke parameter of \( \pi \) at \( p \) is given by \([\alpha_p, \beta_p, p^3\beta_p^{-1}, p^3\alpha_p^{-1}]\).
- We have the homomorphism \( \phi_{\pi, \lambda} : T^\Sigma \to \mathcal{O}_\lambda \) as in Construction 3.3 (2) which gives the maximal ideal
\[ m = T^\Sigma \lambda(p) \cap \ker(T^\Sigma \overset{\phi_{\pi, \lambda}}{\to} \mathcal{O}_\lambda \to \mathcal{O}_\lambda/\lambda) \]
as in Construction 3.3 (3).

We fix the datum \((\pi, \Sigma, m)\) in this section. We are eventually interested in understanding the singular quotient
\[ H^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, H^3_c(X_{Pa}(B), R\Psi(\mathcal{O}(1)))_m) \]
A first step, we have the following proposition.

**Proposition 6.1.** Suppose that \( p^2 - 1 \) is invertible in \( \mathcal{O}_{\lambda}/\lambda \). We define
\[ H_m = H^3_c(X_{Pa}(B), R\Psi(\mathcal{O}(1)))_m . \]
There is then an exact sequence
\[ 0 \to F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_m) \to H^1_{\mathrm{sing}}(\mathbb{Q}_{p^2}, H_m) \to H^3_c(X_{Pa}(B), \mathcal{O}(1))_m G_{\mathbb{Q}_{p^2}}^H \to 0 . \]

**Proof.** We consider the two tautological exact sequences
\[ 0 \to F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_m) \to H^1(I_{\mathbb{Q}_{p^2}}, H_m) \to \frac{H^1(I_{\mathbb{Q}_{p^2}}, H_m)}{F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_m)} \to 0 \]
(6.2)
\[ 0 \to F_0H_m / F_{-1}H_m \to \frac{H^1(I_{\mathbb{Q}_{p^2}}, H_m)}{F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_m)} \overset{}{\to} \text{Gr}_1H_m \to 0 . \]
The latter exact sequence rewrites to

\[ (6.3) \quad 0 \to H^3_c(X_{Pa}(B), O_{\lambda}(1)) \to \frac{H^1(I_{Q_p^2}, H_m)}{F_{-1}H^1(I_{Q_p^2}, H_m)} \to \text{Ker}(\alpha)_m \to 0 \]

by (5.3) and where the last term is the kernel of

\[ \bigoplus_{\sigma \in \Sigma} R^3\Phi(\sigma(O_{\lambda}(2))) \to H^3_c(X_{Pa}, O_{\lambda}(2))_m. \]

Taking invariant under \( G_{F_{p^2}} \) for the short exact sequence (6.3), we have an isomorphism

\[ H^3_c(X_{Pa}(B), O_{\lambda}(1))_{G_{F_{p^2}}} \cong \left( \frac{H^1(I_{Q_p^2}, H_m)}{F_{-1}H^1(I_{Q_p^2}, H_m)} \right)_{G_{F_{p^2}}}. \]

Taking invariant under \( G_{F_{p^2}} \) for the first short exact sequence of (6.2), we immediately obtain the desired exact sequence

\[ 0 \to F_{-1}H^1(I_{Q_p^2}, H_m) \to H^1_{\text{sing}}(Q_p^2, H_m) \to H^3_c(X_{Pa}(B), O_{\lambda}(1))_{G_{F_{p^2}}}. \]

\[ \square \]

**Construction 6.4.** Recall that the irreducible components of the supersingular locus of \( X_{Pa}(B) \) are parametrized by the Shimura set \( Z_{(0)}(B) \) and \( Z_{(2)}(B) \). Therefore we have two natural Gysin morphisms

\[ (6.5) \]

\[ \text{inc}^{(0)}_{\text{inc}} : O_{\lambda}[Z_{(0)}(B)] \cong H^0(X_{Pa_{(0)}}, O_{\lambda}) \to H^2_c(X_{Pa}(B), O_{\lambda}(1)) \]

\[ \text{inc}^{(2)}_{\text{inc}} : O_{\lambda}[Z_{(2)}(B)] \cong H^0(X_{Pa_{(2)}}, O_{\lambda}) \to H^2_c(X_{Pa}(B), O_{\lambda}(1)) \]

and the two natural restriction morphisms

\[ (6.6) \]

\[ \text{inc}^{(0)}_{\text{res}} : H^4(X_{Pa}(B), O_{\lambda}(2)) \to H^4(X_{Pa_{(0)}}, O_{\lambda}(2)) \cong O_{\lambda}[Z_{(0)}(B)] \]

\[ \text{inc}^{(2)}_{\text{res}} : H^4(X_{Pa}(B), O_{\lambda}(2)) \to H^4(X_{Pa_{(2)}}, O_{\lambda}(2)) \cong O_{\lambda}[Z_{(2)}(B)]. \]

The above construction fits into the following diagram which sets up the level raising matrix.
Construction 6.8. We use the above diagram to define the following four maps

\[ T_{lr}(00) = \text{inc}^*_{(0)} \circ \alpha \circ N^{-1}_\Sigma \circ \beta \circ \text{inc}^{(0)} \]
\[ T_{lr}(02) = \text{inc}^*_{(2)} \circ \alpha \circ N^{-1}_\Sigma \circ \beta \circ \text{inc}^{(0)} \]
\[ T_{lr}(20) = \text{inc}^*_{(0)} \circ \alpha \circ N^{-1}_\Sigma \circ \beta \circ \text{inc}^{(2)} \]
\[ T_{lr}(22) = \text{inc}^*_{(2)} \circ \alpha \circ N^{-1}_\Sigma \circ \beta \circ \text{inc}^{(2)} \]

We call the resulting matrix

\[ T_{lr} = \begin{pmatrix} T_{lr}(00) & T_{lr}(02) \\ T_{lr}(20) & T_{lr}(22) \end{pmatrix} \]

We can localize the above diagram at the maximal ideal \( m \) and write the resulting matrix as \( T_{lr,m} \), however all the matrix entries will still be denoted by same symbols without referring to \( m \). If we need to consider this matrix modulo \( m \), then we will denote it by \( T_{lr}/m \). Again the entries of this matrix however will be denoted the same notation without referring to \( m \) again.

### 6.1. A Hecke operator identity

We will identify the entries of the level raising matrix with certain Hecke operators, we first prove some auxiliary results about the spherical Hecke algebra of \( G = \text{GSp}_4 \) over \( \mathbb{Q}_p \). The computation in this subsection is completely done in the local set up and hence in this subsection \( G \) will be understood as a group over \( \mathbb{Q}_p \). Let \( G(\mathbb{Z}_p) = \text{GSp}_4(\mathbb{Z}_p) \) be the hyperspecial subgroup of \( G(\mathbb{Q}_p) \). Consider
the spherical Hecke algebra $T_p = \mathbb{Z}[G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)]$. We have the classical Satake transform

$$S : T_p \to \mathbb{Z}[p^{-1}][X^*(\hat{T})]^{W_G}$$

where $W_G$ is the absolute Weyl group of $G$. We fix a datum $T \subset B \subset G$ of the diagonal torus contained in the upper triangular Borel subgroup as in the global case. This determines a positive Weyl chamber $P^+$ in $X_+(T) \cong X^*(\hat{T})$. An element $\nu \in X^*(\hat{T})$ determines an irreducible representation $V_\nu$ of the dual group $\hat{G} = \text{GSpin}(5)$ whose character is denoted by $\chi_\nu \in R(\hat{G}) = \mathbb{Z}[X^*(\hat{T})]^{W_G}$.

It is well-known that $T_p$ is isomorphic to $\mathbb{Z}[T^\pm_{p,0}, T_{p,1}, T_{p,2}]$ where

$$T_{p,0} = c_{\nu_0} = \text{char}(G(\mathbb{Z}_p) \left( \begin{array}{ccc} p & p & 1 \\ p & p & 1 \\ p & p & 1 \end{array} \right) G(\mathbb{Z}_p))$$
for $\nu_0 = (1, 1, 1, 1)$

$$T_{p,2} = c_{\nu_2} = \text{char}(G(\mathbb{Z}_p) \left( \begin{array}{ccc} p^2 & p^2 & 1 \\ p^2 & p^2 & 1 \\ p^2 & p^2 & 1 \end{array} \right) G(\mathbb{Z}_p))$$
for $\nu_2 = (1, 1, 0, 0)$

$$T_{p,1} = c_{\nu_1} = \text{char}(G(\mathbb{Z}_p) \left( \begin{array}{ccc} p & p & 1 \\ p & p & 1 \\ p & p & 1 \end{array} \right) G(\mathbb{Z}_p))$$
for $\nu_1 = (2, 1, 1, 0)$

and where $\text{char}(\cdot)$ is the characteristic function.

The following spherical Hecke operator will appear naturally in later computations

$$T_{p^2,2} = c_{2\nu_2} = \text{char}(G(\mathbb{Z}_p) \left( \begin{array}{ccc} p^2 & p^2 & 1 \\ p^2 & p^2 & 1 \\ p^2 & p^2 & 1 \end{array} \right) G(\mathbb{Z}_p))$$
for $2\nu_2 = (2, 2, 0, 0)$.

We will also need the following two Hecke operators in $\mathbb{Z}[K_{\{2\}} \backslash G(\mathbb{Q}_p) / K_{\{0\}}]$

$$T_{02} = \text{char}(K_{\{2\}} \left( \begin{array}{ccc} p & p & 1 \\ p & p & 1 \\ p & p & 1 \end{array} \right) K_{\{0\}})$$

$$T_{20} = \text{char}(K_{\{0\}} \left( \begin{array}{ccc} p^{-1} & p^{-1} & 1 \\ p^{-1} & p^{-1} & 1 \\ p^{-1} & p^{-1} & 1 \end{array} \right) K_{\{2\}}).$$
In the computations below, we will follow mostly the notations and conventions in [Gross] and use results from there freely. We have the following useful formula [Gross, 3.12]

\[ p^{(\nu, \rho)} \chi_\nu = S(c_\nu) + \sum_{\xi < \nu} d_\nu(\xi) S(c_\xi). \]

The integers \(d_\nu(\xi)\) are given by \(P_{\xi, \nu}(p)\) for the Kazhdan-Lusztig polynomial \(P_{\xi, \nu}\) evaluated at \(p\) and \(2\rho\) is the sum of the positive roots. Since the weight \(\nu_2\) is minuscule, we immediately have

\[ p^3 \chi_{\nu_2} = S(c_{\nu_2}) = S(T_{p, 2}). \]

On the other hand for the non-minuscule weight \(\nu_1\), we have

\[ p^2 \chi_{\nu_1} = S(c_{\nu_1}) + S(c_0) = S(T_{p, 1}) + 1. \]

The following identity is the main result of this subsection. It can perhaps be proved by a massive double coset computation whereas our proof uses crucially the Satake isomorphism to simplify the computation. This identity seems not to exist already in the literature and therefore we give full details which we think are of independent interest.

**Proposition 6.9.** The following identity holds in \(T_p\)

\[ T_{p^2, 2} = T_{p, 2}^2 - (p + 1)T_{p, 1} - (p + 1)(p^2 + 1). \]

**Proof.** The representation \(V_{\nu_1}\) is the 5-dimensional orthogonal representation and \(V_{\nu_2}\) is the standard representation. Therefore \(\wedge^2 V_{\nu_2} = V_{\nu_1} \ominus \mathbb{C}\). We also have \(\wedge^2 V_{\nu_1} = V_{2\nu_2}\) is the adjoint representation of dimension 10. Thus we have

\[
\begin{align*}
V_{\nu_2} \otimes^2 V_{\nu_2} &= V_{2\nu_2} + \wedge^2 V_{\nu_2} \\
&= V_{2\nu_2} + V_{\nu_1} + \mathbb{C}
\end{align*}
\]

Taking characters, we have \(\chi_{\nu_2}^2 = \chi_{2\nu_2} + \chi_{\nu_1} + 1\) and hence \(p^3 \chi_{\nu_2}^2 = p^3 \chi_{2\nu_2} + p^3 \chi_{\nu_1} + p^3\). Therefore

\[
\begin{align*}
p^3 \chi_{2\nu_2} &= p^3 \chi_{\nu_2}^2 - p^3 \chi_{\nu_1} - p^3 \\
&= S(T_{p, 2}^2) - pS(T_{p, 1}) - (p + p^3).
\end{align*}
\]

On the other hand, we also have

\[
\begin{align*}
p^3 \chi_{2\nu_2} &= S(c_{2\nu_2}) + d_{2\nu_2}(\nu_1)S(c_{\nu_1}) + d_{2\nu_2}(0) \\
&= S(T_{p^2, 2}) + d_{2\nu_2}(\nu_1)S(T_{p, 1}) + d_{2\nu_2}(0).
\end{align*}
\]

- \(d_{2\nu_2}(\nu_1) = 1\); This follows from the fact that \(P_{\nu_1, 2\nu_2}\) has degree strictly less than \(\langle 2\nu_2 - \nu_1, \rho \rangle = 1\) and the constant coefficient is 1.
- \(d_{2\nu_2}(0) = 1 + p^2\); This follows from the fact that \(2\nu_2\) is the highest weight of the adjoint representation, which implies that \(P_{0, 2\nu_2}(p) = \sum_{i=1} m_i \) where \(m_i\) are the exponents of the Weyl group of type \(C_2\) which are 1 and 3, see [Gross 4.6].

Equating the two expressions of \(p^3 \chi_{2\nu_2}\) gives the desired result. \(\Box\)
6.2. Computing the level raising matrix. We now calculate the entries of the level raising matrix \( T_{lr} \) term by term. The main result is the following proposition.

**Proposition 6.10.** The level raising matrix \( T_{lr} \) is given by

\[
T_{lr} = -2 \left( \begin{array}{cc}
T_{p,0}^{-1}T_{p,1} + (p+1)(p^2+1) & (p+1)T_{02} \\
(p+1)T_{20} & T_{p,0}^{-1}T_{p,1} + (p+1)(p^2+1)
\end{array} \right).
\]

**Proof.** To begin the proof, note that we can identify both

\[
\bigoplus_{\sigma \in \mathbb{Z} \{0\}} H^3(\mathfrak{B} \mathcal{O}_\lambda(1))
\]

and

\[
\bigoplus_{\sigma \in \mathbb{Z} \{1\}} R^3 \Phi_\sigma(\mathcal{O}_\lambda(2))
\]

with \( \mathcal{O}_\lambda[Z_{Pa}(\mathcal{B})] \) ignoring the Galois actions. The level raising diagram (6.11) reduces to

\[
\begin{array}{ccc}
\mathcal{O}_\lambda[Z_{\{0\}}(\mathcal{B})] & \xrightarrow{\text{inc}^{\{0\}}} & \mathcal{O}_\lambda[Z_{\{2\}}(\mathcal{B})] \\
\downarrow & & \downarrow \\
H^2(X_{Pa}(\mathcal{B}), \mathcal{O}_\lambda(1)) & \xleftarrow{\beta} & \mathcal{O}_\lambda[Z_{Pa}(\mathcal{B})] \\
\downarrow & & \downarrow \\
H^4(X_{Pa}(\mathcal{B}), \mathcal{O}_\lambda(2)) & \xleftarrow{\alpha} & \mathcal{O}_\lambda[Z_{\{2\}}(\mathcal{B})]
\end{array}
\]

(6.11)

By Proposition 5.8, the map \( \alpha \) can be realized as a Gysin map and \( \beta \) can be realized as a restriction map. The situation is similar to that of [LTXZZ Proposition 5.8.8]. We proceed to analyze each entry.

6.2.1. \( T_{lr,\{00\}} \). Since \( Z_{\{i\}}(\mathcal{B}) \) corresponds to the set of vertex lattices of type \( i \) for \( i \in \{0, 1, 2\} \). Therefore to understand the map \( T_{lr,\{00\}} = \text{inc}^{\{0\}} \circ \alpha \circ \Sigma^{-1} \circ \beta \circ \text{inc}^{\{0\}} \), we need to understand the possible relative position of vertex lattices \( \Lambda_0, \Lambda_{Pa} \) and \( \Lambda'_0 \) of type 0, 1 and 0. It not hard to see that the only possible situation is that \( \Lambda_{Pa} \subset^1 \Lambda_0 \) and \( \Lambda_{Pa} \subset^1 \Lambda'_0 \). There are two possible cases.

Case (1): \( \Lambda_0 = \Lambda'_0 \). Then we have \( p\Lambda_0 \subset^1 \Lambda_{Pa} \subset^2 \Lambda_{Pa} \subset^1 \Lambda_0 \). Giving such a \( \Lambda_{Pa} \) is equivalent to giving a complete flag in \( \Lambda_0/p\Lambda_0 \). The number of such flags is given by \( [G(\mathbb{Z}_p) : \mathbb{K}] = (p+1)(p^2+1) \) where \( \mathbb{K} \) is the Klingen parahoric.
Case (2): \( \Lambda_0 \neq \Lambda_0' \). Then we have \( \Lambda_0 \cap \Lambda_0' = \Lambda_{Pa} \) and the relative position between \( \Lambda_0 \) and \( \Lambda_0' \) is given by

\[
K_{\{0\}} \begin{pmatrix} p & 1 \\ 1 & p^{-1} \end{pmatrix} K_{\{0\}}.
\]

This operator is the same as \( T_{p,0}^{-1} T_{p,1} \) by identifying \( K_{\{0\}} \) with the hyperspecial group \( G(\mathbb{Z}_p) \). Therefore the entry \( T_{lr,\{0\}} \) is given by

\[
T_{lr,\{0\}} = -2(T_{p,0}^{-1} T_{p,1} + (p + 1)(p^2 + 1)).
\]

Here the coefficient \(-2\) comes from the self-intersection number of the class \( \eta_\sigma \) as in Lemma 5.7.

6.2.2. \( T_{lr,\{22\}} \). To understand the map \( T_{lr,\{22\}} = \text{inc}^*_{\{2\}} \circ \alpha \circ N_{\Sigma}^{-1} \circ \beta \circ \text{inc}^{(2)}_\Sigma \), we need to understand the possible relative position of vertex lattices \( \Lambda_2, \Lambda_{Pa} \) and \( \Lambda_2' \) of type 2, 1 and 2. It not hard to see that the only possible way is that \( \Lambda_2 \subset \Lambda_{Pa} \) and \( \Lambda_2' \subset \Lambda_{Pa} \).

The rest of the analysis is completely the same as in the case of \( T_{lr,\{0\}} \). Once we identify \( K_{\{2\}} \) with the hyperspecial group \( G(\mathbb{Z}_p) \), the entry \( T_{lr,\{22\}} \) is given by

\[
T_{lr,\{22\}} = -2(T_{p,0}^{-1} T_{p,1} + (p + 1)(p^2 + 1)).
\]

Here the coefficient \(-2\) comes from the self-intersection number of the class \( \eta_\sigma \) as in Lemma 5.7.

6.2.3. \( T_{lr,\{02\}} \). To understand the map \( T_{lr,\{02\}} = \text{inc}^*_{\{2\}} \circ \alpha \circ N_{\Sigma}^{-1} \circ \beta \circ \text{inc}^{(0)}_\Sigma \), we need to understand the possible relative position of vertex lattices \( \Lambda_0, \Lambda_{Pa} \) and \( \Lambda_2' \) of type 0, 1 and 2. We find that the only possible situation is that

\[
\Lambda_2 \subset \Lambda_{Pa} \subset \Lambda_0 = \Lambda_0^\vee \subset \Lambda_{Pa}^\vee \subset \Lambda_2^\vee.
\]

It follows each such \( \Lambda_{Pa} \) corresponds to a line in the two dimensional space \( \Lambda_0/\Lambda_2 \).

Therefore the entry \( T_{lr,\{02\}} \) is given by

\[
T_{lr,\{02\}} = -2(p + 1)\text{char}(K_{\{0\}} \begin{pmatrix} p & 1 \\ 1 & \end{pmatrix} K_{\{2\}}) = -2(p + 1)T_{02}.
\]

6.2.4. \( T_{lr,\{20\}} \). To understand the map \( T_{lr,\{20\}} = \text{inc}^*_{\{0\}} \circ \alpha \circ N_{\Sigma}^{-1} \circ \beta \circ \text{inc}^{(2)}_\Sigma \), we need to understand the possible relative positions of the vertex lattices \( \Lambda_0, \Lambda_{Pa} \) and \( \Lambda_2' \) of type 0, 1 and 2. We find that the only possible way is that

\[
\Lambda_2 \subset \Lambda_{Pa} \subset \Lambda_0 = \Lambda_0^\vee \subset \Lambda_{Pa}^\vee \subset \Lambda_2^\vee.
\]

Each \( \Lambda_{Pa} \) corresponds to a line in the two dimensional space \( \Lambda_0/\Lambda_2 \).
Therefore the entry $\mathcal{T}_{lr, \{20\}}$ is given by

$$\mathcal{T}_{lr, \{20\}} = -2(p + 1)\text{char}(K_{\{0\}}) \begin{pmatrix} p^{-1} & p^{-1} \\ 1 & 1 \end{pmatrix} K_{\{2\}} = -2(p + 1)T_{02}.$$ 

This finishes the proof of Proposition 6.10. □

We will compute the determinant of the level raising matrix and the supersingular matrix that we will define in the next section. For this reason, we will next compute the composite $T_{20} \circ T_{02}$ and $T_{02} \circ T_{20}$ in terms of elements in the spherical Hecke algebra.

**Lemma 6.12.** The composite $T_{20} \circ T_{02}$ and $T_{02} \circ T_{20}$ are both given by

$$T_{p,0}^{-1}T_{p^2,2} + (p + 1)T_{p,0}^{-1}T_{p,1} + (p^2 + 1)(p + 1).$$

As a result, if $T_{p,0}$ acts trivially, then $T_{20} \circ T_{02} = T_{02} \circ T_{20} = T_{p,2}^2$.

**Proof.** The proof of this lemma is similar to that of the above proposition.

6.2.5. $T_{20} \circ T_{02}$. Let $\Lambda_0$ be a vertex lattice of type 0. Then $T_{20} \circ T_{02}(\Lambda_0)$ classifies a pair of vertex lattices $(\Lambda_2, \Lambda'_0)$ of type 2 and 0. We need to understand all the possible relative position of the vertex lattices $\Lambda_0, \Lambda_2$ and $\Lambda'_0$. The only possible situation is given below

$$p\Lambda_0^\vee \subset 2 \Lambda_2 \subset 2 \Lambda_0 = \Lambda_0^\vee \subset 2 \Lambda'_2 \quad \text{and} \quad p\Lambda'_0^\vee \subset 2 \Lambda'_2 \subset 2 \Lambda_0 = \Lambda'_0^\vee \subset 2 \Lambda'_2.$$ 

Note that $\Lambda_2$ determines an isotropic subspace of the symplectic space $\Lambda_0 \cap \Lambda'_0 / p\Lambda_0^\vee + p\Lambda'_0^\vee$ of possible dimensions in the set $\{0, 2, 4\}$.

When $\dim_{\mathbb{F}} \Lambda_0 \cap \Lambda'_0 / p\Lambda_0^\vee + p\Lambda'_0^\vee = 0$, then

$$\Lambda_0 \cap \Lambda'_0 \subset 2 \Lambda_0 \quad \text{and} \quad \Lambda_0 \cap \Lambda'_0 \subset 2 \Lambda'_0.$$ 

Hence this case contributes to the $T_{20} \circ T_{02}$ by the double coset operator

$$K_{\{0\}} \begin{pmatrix} p & p^{-1} \\ p & 1 \end{pmatrix} K_{\{0\}}.$$ 

Once we identify $K_{\{0\}}$ with $G(\mathbb{Z}_p)$, we see this is the same as $T_{0,p}^{-1}T_{p^2,2}$. 

When $\dim_{\mathbb{F}} \Lambda_0 \cap \Lambda'_0 / p\Lambda_0^\vee + p\Lambda'_0^\vee = 2$, there are $p + 1$ choices of $\Lambda_2$. And we have

$$\Lambda_0 \cap \Lambda'_0 \subset 1 \Lambda_0 \quad \text{and} \quad \Lambda_0 \cap \Lambda'_0 \subset 1 \Lambda'_0.$$
Hence this case contributes to the $T_{20} \circ T_{02}$ by the double coset operator

$$\begin{pmatrix} p \\ 1 \\ p^{-1} \end{pmatrix} K_{\{0\}}.$$  

Once we identify $K_{\{0\}}$ with $G(\mathbb{Z}_p)$, we see this is the same as $(p + 1)T_{0,p}^{-1}T_{p,1}.$

When $\dim F \Lambda_0 \cap \Lambda_0' / p\Lambda_0'' + p\Lambda_0'' = 4$, there are $[G(\mathbb{Z}_p) : \text{Sie}]) = (p + 1)(p^2 + 1)$ choices of $\Lambda_2$. And we have

$$\begin{align*}
\Lambda_0 \cap \Lambda_0' &\subset^0 \Lambda_0 \\
\Lambda_0 \cap \Lambda_0' &\subset^0 \Lambda_0'.
\end{align*}$$

Hence this case contributes to the $T_{20} \circ T_{02}$ by the constant $(p^2 + 1)(p + 1)$. All in all, we obtain the following formula

$$6.13 \quad T_{20} \circ T_{02} = T_{-1,0}T_{p} - T_{p,1} + (p + 1)(p^2 + 1)(p + 1).$$

6.2.6. $T_{02} \circ T_{20}$. The computation is almost the same to the above. Let $\Lambda_2$ be a vertex lattice of type 2. Then $T_{02} \circ T_{20}(\Lambda_2)$ classifies pair of vertex lattices $(\Lambda_0, \Lambda_2)$ of type 0 and 2. We need to understand the relative position of vertex lattices $\Lambda_0$, $\Lambda_2$ and $\Lambda_2'$. The only possible situation is given below

$$\begin{align*}
p\Lambda_0' &\subset^2 \Lambda_2 \subset^2 \Lambda_0 = \Lambda_0' \subset^2 \Lambda_2' \\
p\Lambda_0' &\subset^2 \Lambda_2' \subset^2 \Lambda_0 = \Lambda_0' \subset^2 \Lambda_2'.
\end{align*}$$

Note that $\Lambda_0$ is determined by an isotropic subspace of the symplectic space $\Lambda_2 \cap \Lambda_2'/\Lambda_2 + \Lambda_2'$ of possible dimensions $\{0, 2, 4\}$.

When $\dim F \Lambda_2 \cap \Lambda_2'/\Lambda_2 + \Lambda_2' = 0$, then in this case we have

$$\begin{align*}
\Lambda_2 &\subset^2 \Lambda_2 + \Lambda_2' \\
\Lambda_2' &\subset^2 \Lambda_2 + \Lambda_2'.
\end{align*}$$

Hence this case contributes to the $T_{02} \circ T_{20}$ by the double coset operator

$$\begin{pmatrix} p^{-1} \\
p^{-1} \\ p \end{pmatrix} K_{\{2\}}.$$  

Once we identify $K_{\{2\}}$ with $G(\mathbb{Z}_p)$, we see this is the same as $T_{0,p}^{-1}T_{p,2}.$

When $\dim F \Lambda_2' \cap \Lambda_2'/\Lambda_2 + \Lambda_2' = 2$, there are $p + 1$ choices of $\Lambda_0$. And in this case we have

$$\begin{align*}
\Lambda_2 &\subset^1 \Lambda_2 + \Lambda_2' \\
\Lambda_2' &\subset^1 \Lambda_2 + \Lambda_2'.
\end{align*}$$
Hence this case contributes to the $T_{20} \circ T_{02}$ by the Hecke operator

$$\begin{pmatrix} p^{-1} & 1 \\ 1 & p \end{pmatrix} K_{(2)}.$$ 

Once we identify $K_{(2)}$ with the hyperspecial subgroup $G(\mathbb{Z}_p)$, we see this is the same as $(p+1)T_{0,p}^{-1}T_{p,1}$.

When $\dim F \Lambda_2^0 \cap \Lambda_2^0/\Lambda_2 + \Lambda_2' = 4$, there are $|G(\mathbb{Z}_p) : \text{Sie}| = (p+1)(p^2+1)$ choices of $\Lambda_0$.

And in this case we have

$$\Lambda_2 \subset \Lambda_2 + \Lambda_2'$$

$$\Lambda_2' \subset \Lambda_2 + \Lambda_2'.$$

Hence this case contributes to the $T_{20} \circ T_{02}$ by the constant $(p^2+1)(p+1)$. All in all, we obtain the following formula

$$(6.14) \quad T_{20} \circ T_{02} = T_{0,p}^{-1}T_{p,2} + (p+1)T_{0,p}^{-1}T_{p,1} + (p^2+1)(p+1).$$

The last claim in the lemma follows immediately from Proposition 6.15. This finishes the proof of the lemma.

The final result of this section is the computation of the determinant of the level raising matrix modulo $m$.

**Proposition 6.15.** Let $(\pi, \Sigma, m)$ be the datum considered as in the beginning of the section. Suppose the maximal ideal $m$ appears in the support of $\mathcal{O}_H(\overline{\mathcal{B}})$. Let $[\alpha_p, \beta_p, p^3 \beta^{-1}, p^3 \alpha^{-1}]$ be the Hecke parameter of $\pi$ at $p$. Then the determinant of the level raising matrix modulo $m$ is given by

$$\det T_{lr}/m = 4p^{-2} \prod_{u=\pm 1} (\alpha_p + p^3 \alpha^{-1} - up(p+1))(\beta_p + p^3 \beta^{-1} - up(p+1)).$$

**Proof.** Since $\pi$ has trivial central character, $T_{p,0}$ acts trivially on $\mathcal{O}_H(\overline{\mathcal{B}})/m$. We therefore have $T_{20} \circ T_{02} = T_{02} \circ T_{20} = T_{p,2} + (p+1)T_{p,1} + (p^2+1)(p+1) = T_{p,2}^2$ by Lemma 6.13. Then

$$\det T_{lr}/m = 4((T_{p,1} + p(p+1))^2 - (p+1)^2 T_{20} T_{02})$$

$$= 4(T_{p,1} + p(p+1) - (p+1)T_{p,2})(T_{p,1} + p(p+1) + (p+1)T_{p,2})$$

$$= 4p^{-2}[(\alpha_p + p^3 \alpha^{-1})(\beta_p + p^3 \beta^{-1}) - p(p+1)(\alpha_p + \beta_p + p^3 \alpha^{-1} + p^3 \beta^{-1} + p^2(p+1)^2)]$$

$$[\alpha_p + p^3 \alpha^{-1})(\beta_p + p^3 \beta^{-1}) + p(p+1)(\alpha_p + \beta_p + p^3 \alpha^{-1} + p^3 \beta^{-1} + p^2(p+1)^2)]$$

$$= 4p^{-2} \prod_{u=\pm 1} (\alpha_p + p^3 \alpha^{-1} - up(p+1))(\beta_p + p^3 \beta^{-1} - up(p+1)).$$

This finishes the proof. 

\[\square\]
7. Tate cycles on the quaternionic unitary Shimura variety

In this section, our goal is to establish the following principle: the localized cohomology group $H^2_c(X_{P\alpha}(B), O_{\lambda}(1))_m$ is generated by cycles coming from the supersingular locus. On the dual side, similar result also holds for $H^4(X_{P\alpha}(B), O_{\lambda}(2))_m$, however we need to show this cohomology group contains no $O_{\lambda}$-torsion first. We will establish the full result simultaneously with our main theorem on arithmetic level raising in Corollary 9.8. Let $(\pi, \Sigma, m)$ be the datum considered as in the beginning of the last section.

**Construction 7.1.** Consider the supersingular locus $\mathcal{X}_{P\alpha}^{ss}(B)$, we have the following restriction morphism

$$H^2_c(X_{P\alpha}(B), O_{\lambda}(1)) \to H^2_c(X_{P\alpha}(B), O_{\lambda}(1))$$

and we denote the composite of this restriction morphism with the natural map

$$H^2_c(X_{P\alpha}(B), O_{\lambda}(1)) \to H^2(X_{P\alpha}^{ss}(B), O_{\lambda}(1))$$

by

$$inc^*_{ss} : H^2_c(X_{P\alpha}(B), O_{\lambda}(1)) \to H^2(X_{P\alpha}^{ss}(B), O_{\lambda}(1)).$$

We also have the natural Gysin map

$$inc_{ss}^! : H^2(X_{P\alpha}^{ss}(B), O_{\lambda}(1)) \to H^4(X_{P\alpha}(B), O_{\lambda}(2)).$$

We can fit the two maps $(inc^*_{\{ss\}}, inc_{ss}^!)$ in the above construction in the diagram below.

\[
\begin{array}{ccc}
O_\lambda[\mathbb{Z}_{\{0\}}(\overline{B})] & \xrightarrow{inc_{\{0\}}} & O_\lambda[\mathbb{Z}_{\{2\}}(\overline{B})] \\
\downarrow{inc^*_{\{ss\}}} & & \downarrow{inc_{\{ss\}}^!} \\
H^2_c(X_{P\alpha}(B), O_{\lambda}(1)) & \xrightarrow{inc_{\{ss\}}^*} & H^2(X_{P\alpha}^{ss}(B), O_{\lambda}(1)) \\
\downarrow{inc_{\{ss\}}} & & \downarrow{inc_{\{ss\}}^!} \\
H^4(X_{P\alpha}(B), O_{\lambda}(2)) & \xrightarrow{inc_{\{0\}}} & O_\lambda[\mathbb{Z}_{\{0\}}(\overline{B})] \\
& & \xrightarrow{inc_{\{2\}}} \quad O_\lambda[\mathbb{Z}_{\{2\}}(\overline{B})]
\end{array}
\]

The above diagram sets up the intersection matrix for the supersingular locus which we will elaborate below.
Construction 7.3. We obtain naturally the following four maps from the above diagram
\[ \mathcal{T}_{\text{ss},\{00\}} = \text{inc}^*_0 \circ \text{inc}^{\{\text{ss}\}}_1 \circ \text{inc}^{\{\text{ss}\}}_2 \circ \text{inc}^*_1 \]
\[ \mathcal{T}_{\text{ss},\{02\}} = \text{inc}^*_0 \circ \text{inc}^{\{\text{ss}\}}_1 \circ \text{inc}^{\{\text{ss}\}}_2 \circ \text{inc}^*_1 \]
\[ \mathcal{T}_{\text{ss},\{20\}} = \text{inc}^*_0 \circ \text{inc}^{\{\text{ss}\}}_1 \circ \text{inc}^{\{\text{ss}\}}_2 \circ \text{inc}^*_1 \]
\[ \mathcal{T}_{\text{ss},\{22\}} = \text{inc}^*_0 \circ \text{inc}^{\{\text{ss}\}}_1 \circ \text{inc}^{\{\text{ss}\}}_2 \circ \text{inc}^*_1 \]

The resulting matrix
\[ \mathcal{T}_{\text{ss}} = \begin{pmatrix} \mathcal{T}_{\text{ss},\{00\}} & \mathcal{T}_{\text{ss},\{02\}} \\ \mathcal{T}_{\text{ss},\{20\}} & \mathcal{T}_{\text{ss},\{22\}} \end{pmatrix} \]
will be referred to as the supersingular matrix. We can localize the above diagram at the maximal ideal \( m \) and write the resulting matrix as \( \mathcal{T}_{\text{ss},m} \), however all the matrix entries will still be denoted by symbols without referring to \( m \). If we need to consider this matrix modulo \( m \), then we will denote it by \( \mathcal{T}_{\text{ss}/m} \). The entries of this matrix will be denoted without referring to \( m \).

7.1. Computing the supersingular matrix. We compute the entries of the supersingular matrix using the same procedure as we did for computing the level raising matrix.

Proposition 7.4. The supersingular matrix \( \mathcal{T}_{\text{ss}} \) is given by
\[ \mathcal{T}_{\text{ss}} = \begin{pmatrix} 4p^2(p+1)^2 + T_{20} \circ T_{02} & -4p(p+1)T_{02} \\ -4p(p+1)T_{20} & 4p^2(p+1)^2 + T_{02} \circ T_{20} \end{pmatrix} \]

Proof. We will prove the proposition in an entry by entry manner.

7.1.1. \( \mathcal{T}_{\text{ss},\{00\}} \). Recall \( \mathcal{T}_{\text{ss},\{00\}} = \text{inc}^*_0 \circ \text{inc}^*_{\{\text{ss}\}} \circ \text{inc}^*_{\{\text{ss}\}} \circ \text{inc}^*_1 \). If the maps
\[ H^2(\overline{X}_{Pa}(B),\mathcal{O}_\Lambda(1)) \xrightarrow{\text{inc}^*_0} H^2(\overline{X}_{Pa}(B),\mathcal{O}_\Lambda(1)) \xrightarrow{\text{inc}^{\{\text{ss}\}}_1} H^4(\overline{X}_{Pa}(B),\mathcal{O}_\Lambda(2)) \]
factor through \( H^2(\overline{X}_{Pa,\{0\}}(B),\mathcal{O}_\Lambda(1)) \). Then the contribution is \( 4p^2(p+1)^2 \). This follows from the fact that for each \( \Lambda_0 \in \mathcal{L}_{\{0\}} \) the degree of the variety \( DL(\Lambda_0) \) has degree \( (p+1) \) and the normal bundle has degree \( -2p \) by Lemma [2.10] If the maps
\[ H^2(\overline{X}_{Pa}(B),\mathcal{O}_\Lambda(1)) \xrightarrow{\text{inc}^{\{\text{ss}\}}_1} H^2(\overline{X}_{Pa}(B),\mathcal{O}_\Lambda(1)) \xrightarrow{\text{inc}^*_0} H^4(\overline{X}_{Pa}(B),\mathcal{O}_\Lambda(2)) \]
factor through \( H^2(\overline{X}_{Pa,\{2\}}(B),\mathcal{O}_\Lambda(1)) \), then as we have seen before we need to understand the relative position of the vertex lattices \( \Lambda_0, \Lambda_2 \) and \( \Lambda'_0 \) of type 0, 2 and 0. The only possible situation is given by
\[ p\Lambda_0^\vee \subset^2 \Lambda_2 \subset^2 \Lambda_0 = \Lambda_0^\vee \subset^2 \Lambda_2^\vee \\
p\Lambda'_0^\vee \subset^2 \Lambda_2 \subset^2 \Lambda'_0 = \Lambda'_0^\vee \subset^2 \Lambda_2^\vee. \]
As we have seen, this is parametrized by \( T_{20} \circ T_{02} \). Therefore this entry is given by
\[ \mathcal{T}_{\text{ss},\{00\}} = (p+1)^2 + T_{20} \circ T_{02}. \]
7.1.2. $\mathcal{T}_{ss, (02)}$. If the maps
\[ H^2(X_{Pa}(B), O_\lambda(1)) \xrightarrow{\text{inc}^*_{ss}} H^2(X_{Pa}(B), O_\lambda(1)) \xrightarrow{\text{inc}^*_{ss}} H^4(X_{Pa}(B), O_\lambda(2)) \]
factor through $H^2(X_{Pa,(0)}(B), O_\lambda(1))$. This case contributes by $-2p(p+1)T_{02}$. If the maps factor through $H^2(X_{Pa,(2)}(B), O_\lambda(1))$. This case contributes by $-2p(p+1)T_{02}$. Therefore the total contribution gives the entry
\[ T_{ss, (02)} = -4p(p+1)T_{02}. \]

7.1.3. $\mathcal{T}_{ss, (20)}$. The computation is the same as in the case for $T_{ss, (02)}$. Thus this entry is given by
\[ T_{ss, (20)} = -4p(p+1)T_{20}. \]

7.1.4. $\mathcal{T}_{ss, (22)}$. The computation is exactly the same as in the case for $T_{ss, (00)}$. Thus this entry is given by
\[ T_{ss, (22)} = 4p^2(p+1)^2 + T_{02} \circ T_{20}. \]
This finishes the proof of Proposition 7.5. □

**Proposition 7.5.** Let $(\pi, \Sigma, m)$ be the datum considered as in the beginning of the section. Suppose the maximal ideal $m$ appears in the support of $O_\lambda[Z_H(B)]$. Suppose that $[\alpha_p, \beta_p, \alpha_p^{-1}, \beta_p^{-1}]$ is the Hecke parameter of $\pi$ at $p$. Then determinant of the supersingular matrix modulo $m$ is given by
\[ \det T_{ss/m} = \prod_{u=\pm 1} \left( \alpha_p + \beta_p + p^3 \alpha_p^{-1} + p^3 \beta_p^{-1} - 2up(p+1) \right)^2. \]

**Proof.** Since $\pi$ has trivial central character, then $T_{p,0}$ acts trivially on $O_\lambda[Z_H(B)]/m$. We therefore have
\[ T_{20} \circ T_{02} = T_{02} \circ T_{20} = T_{p^2,2} + (p+1)T_{p,1} + (p^2+1)(p+1) = T_{p^2,2} \]
by Lemma 6.13. Then we have
\[ \det T_{ss/m} = \left( T_{p^2,2} + 4p^2(p+1) \right)^2 - 16p^2(p+1)^2T_{p^2,2} \]
\[ = \left( T_{p^2,2} + 4p^2(p+1) - 4p(p+1)T_{p,2} \right) \]
\[ = \left( T_{p^2,2} + 4p^2(p+1) + 4p(p+1)T_{p,2} \right) \]
\[ = (\alpha_p + \beta_p + p^3 \beta_p^{-1} + p^3 \alpha_p^{-1} - 2p(p+1))^2 \]
\[ (\alpha_p + \beta_p + p^3 \beta_p^{-1} + p^3 \alpha_p^{-1} + 2p(p+1))^2 \]
\[ = \prod_{u=\pm 1} \left( \alpha_p + \beta_p + p^3 \beta_p^{-1} + p^3 \alpha_p^{-1} - 2up(p+1) \right)^2. \]
□
7.2. Tate classes in the quaternionic unitary Shimura variety. We fix the datum \((\pi, \Sigma, m)\) as in the beginning of this section. Let \(p\) be a level raising special prime for \(\pi\) of length \(m\). We make the following assumption regarding to the vanishing of the nearby cycle cohomology of the quaternionic unitary Shimura variety.

**Assumption 7.6.** Let \(H^d_{(c)}(\overline{X}_{\mathbb{P}^1(B)}, R\Psi(\mathcal{O}_\lambda))_m\) be the usual or compact support nearby cycle cohomology of the special fiber \(\overline{X}_{\mathbb{P}^1(B)}\). We assume the maximal ideal \(m\) satisfies the following properties

1. The natural morphism \(H^d_{(c)}(\overline{X}_{\mathbb{P}^1(B)}, R\Psi(\mathcal{O}_\lambda))_m \rightarrow H^d(\overline{X}_{\mathbb{P}^1(B)}, R\Psi(\mathcal{O}_\lambda))_m\) is an isomorphism for all \(d\).
2. The cohomology \(H^d_{(c)}(\overline{X}_{\mathbb{P}^1(B)}, R\Psi(\mathcal{O}_\lambda))_m = 0\) for \(d \neq 3\), and that the middle degree cohomology \(H^3_{(c)}(\overline{X}_{\mathbb{P}^1(B)}, R\Psi(\mathcal{O}_\lambda))_m\) is a finite free \(\mathcal{O}_\lambda\) module.

**Remark 7.7.** The first assumption is usually known in literature as that of \(m\) being a non-Eisenstein ideal. It is satisfied if \(m\) is associated to a cuspidal automorphic representation of general type. The second assumption is more serious. This is reasonable assumption in light of the recent work of Caraiani-Scholze [CS17, CS19] and recent advances by Koshikawa [Kos19, Kos21]. Although they deal with certain unitary Shimura varieties, the methods are general enough to treat at least our cases: if one assumes that \(\pi\) has an unramified type I component which is generic modulo \(l\), then combining the proof of Koshikawa [Kos21] with a semi-perversity result of the pushforward along the Hodge-Tate period map [San22], one can prove that \(m\) satisfies (2).

If we make the assumption that \(X_{\mathbb{P}^1(B)}\) has good reduction at \(l\) and that \(\pi\) is ordinary at \(l\), then the method of [MT02] could be used to show that the Assumption 7.6 is fulfilled.

We also conjecture that the product of Shimura curves for \(B\) embedded in \(Sh(B, K_{\mathbb{P}^1})\) has an affine complement. Then the above assumption can be proved if one assumes that the residual Galois representation has large image. This is the analogue of the fact that the Igusa divisor on the classical Siegel threefold has an affine complement.

**Definition 7.8.** If \(m\) satisfies either of the two conditions below, we call \(m\) a generic maximal ideal.

1. We say \(m\) is generic level raising at \(p\) if it is level raising special at \(p\) and Assumption 7.6 is satisfied.
2. We say \(m\) is generic non-level raising at \(p\) if the Hecke parameters for \(\pi\) satisfies
   \[
   \alpha_p + p^3\alpha_p^{-1} \not\equiv \pm(p + p^2) \mod \lambda \\
   \beta_p + p^3\beta_p^{-1} \not\equiv \pm(p + p^2) \mod \lambda
   \]
   and Assumption 7.6 is satisfied.

**Theorem 7.9.** Let \(m\) be a generic maximal ideal as in the above definition, we have the following statements.

1. The map from Construction 6.4
   \[(\text{inc}^{(0)} + \text{inc}^{(2)}): \mathcal{O}_\lambda[Z_{(0)}(B)]_m \oplus \mathcal{O}_\lambda[Z_{(2)}(B)]_m \rightarrow H^2_{(c)}(X_{\mathbb{P}^1(B)}, \mathcal{O}_\lambda(1))_m\]
is an isomorphism.

(2) The map from Construction 6.4

\( \text{inc}_{\{0\},m}^* \circ \text{inc}_{\{2\},m}^* : H^4(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_\lambda(2))_m \to \mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \)

is surjective and whose kernel is the torsion part of \( H^4(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_\lambda(2))_m \).

**Proof.** Suppose for the moment that \( \mathfrak{m} \) is generic level raising at \( p \). Then the determinant of the supersingular matrix \( \det \mathcal{T}_{ss/\mathfrak{m}} \) is given by

\[
\prod_{u=\pm 1} (\alpha_u + \beta_u + p^3 \beta^{-1}_u + p^3 \alpha^{-1}_u - 2up(p + 1))^2.
\]

This is non-zero by our definition of the level raising condition. This immediately implies that we have an injection

\[
\mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \hookrightarrow H_c^2(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_\lambda(1))_m
\]

by the Nakayama's lemma. Note that \( H_c^2(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_\lambda(1))_m \) is a free \( \mathcal{O}_\lambda \)-module as it injects into

\[
H_c^3(\overline{X}_\mathcal{P}(\mathcal{B}), R\Psi(\mathcal{O}_\lambda)(1))_m
\]

which is torsion free by the genericness of \( \mathfrak{m} \). Therefore to finish the proof in this case, we need to show that

\[
2\text{rank}_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[Z_H(\overline{B})]_m \geq \text{rank}_{\mathcal{O}_\lambda} H_c^2(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_\lambda(1))_m.
\]

For this purpose, let \( \overline{\pi} \) be an automorphic representation of \( G(\mathbb{A}_\mathbb{Q}) \) of general type. It suffices to show that

\[
2\text{dim}_{\mathbb{Q}_l} [\mathcal{H}_l[Z_H(\overline{B})]]_{[\ell \overline{\pi}^\infty \pi]} \geq \text{dim}_{\mathbb{Q}_l} H_c^2(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_l(1))_{[\ell \overline{\pi}^\infty \pi]}.
\]

Consider the co-specialization exact sequence

\[
0 \to H_c^2(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_l(1))_{[\ell \overline{\pi}^\infty \pi]} \to \bigoplus_{\sigma \in \mathcal{Z}_l(\mathcal{B})} H_{\sigma}^3(\overline{X}_\mathcal{P}(\mathcal{B}), R\Psi(\overline{\mathcal{O}_l})(1))_{[\ell \overline{\pi}^\infty \pi]} \to H_c^3(\overline{X}_\mathcal{P}(\mathcal{B}), \mathcal{O}_l(1))_{[\ell \overline{\pi}^\infty \pi]} \to 0
\]

We can canonically identify the space

\[
\bigoplus_{\sigma \in \mathcal{Z}_l(\mathcal{B})} H_{\sigma}^3(\overline{X}_\mathcal{P}(\mathcal{B}), R\Psi(\overline{\mathcal{O}_l})(1))_{[\ell \overline{\pi}^\infty \pi]}
\]

with the space \( \mathcal{U}_l[Z_{\mathcal{P}_l}(\mathcal{B})]_{[\ell \overline{\pi}^\infty \pi]} \). Note that to complete \( \overline{\pi}^\infty \pi \) to a representation \( \overline{\pi} \) of \( G(\mathbb{A}_\mathbb{Q}) \) occurring in \( H_c^3(\overline{X}_\mathcal{P}(\mathcal{B}), R\Psi(\overline{\mathcal{O}_l})(1)) \), \( \overline{\pi}_p \) must be of type IIa as a representation of \( G(\mathbb{Q}_p) \) by the proof of Theorem 4.5. On the other hand, those \( \overline{\pi}_p \) completing \( \overline{\pi}^\infty \pi \) in \( \mathcal{U}_l[Z_{\mathcal{P}_l}(\mathcal{B})]_{[\ell \overline{\pi}^\infty \pi]} \) can be either the unramified principal series or of type IIa. In the proof of Theorem 4.5 we have seen that the Jacquet–Langlands correspondence preserves the automorphic multiplicity. Thus by the uniqueness of local newforms [BS Theorem 5.6.1]
or [Sch05, Theorem 2.3.1], those \( \pi_p \) completing \( \pi_\infty \) in \( H^2_c(\mathcal{X}_{Pa}(B), \overline{\Omega}_l(1))[tl\pi_\infty] \) can only be unramified unless the map

\[
\bigoplus_{\sigma \in \mathbb{Z}/(1)} H^3_{\{\sigma\}}(\mathcal{X}_{Pa}(B), R\Psi(\overline{\Omega}_l)(1))[tl\pi_\infty] \rightarrow H^3_c(\mathcal{X}_{Pa}(B), R\Psi(\overline{\Omega}_l)(1))[tl\pi_\infty]
\]

is zero. However this is impossible as otherwise, it would imply the isomorphism

\[
H^3_c(\mathcal{X}_{Pa}(B), R\Psi(\overline{\Omega}_l)(1))[tl\pi_\infty] \rightarrow H^3_c(\mathcal{X}_{Pa}(B), \overline{\Omega}_l(1))[tl\pi_\infty]
\]

which is absurd by considering the inertia action. By the old forms principle [BS, Theorem 5.6.1] or [Sch05, Theorem 2.3.1], we have the desired

\[
2 \dim_{\mathbb{Q}_l}[\mathbb{Z}_H(B)][t_l\pi_\infty] \geq \dim_{\mathbb{Q}_l}H^2_c(\mathcal{X}_{Pa}(B), \overline{\Omega}_l(1))[t_l\pi_\infty].
\]

Next we show the same result under the assumption that \( m \) is generic non-level raising. Suppose that \( m \) is not level raising special. Then the determinant of the level raising matrix \( \det T_{lr}/m \) is given by

\[
\det T_{lr}/m = 4p^{-2} \prod_{u=\pm 1} (\alpha_p + p^3\alpha_p^{-1} - up(p+1))(\beta_p + p^3\beta_p^{-1} - up(p+1)).
\]

This is non-zero by our assumption. This immediately implies that we have an injection

\[
\mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \hookrightarrow H^2_c(\mathcal{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m
\]

by the Nakayama’s lemma. The rest of the argument is the same as in the previous case. Finally, the second part of the proposition is dual to the first part. In particular, the map

\[
(\text{inc}^*_{\{0\}, m}, \text{inc}^*_{\{2\}, m}) : H^4(\mathcal{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m \rightarrow \mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m
\]

is surjective. Using the same argument as in the first part, we see that \( (\text{inc}^*_{\{0\}, m}, \text{inc}^*_{\{2\}, m}) \) is an isomorphism up to the torsion of \( H^4(\mathcal{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m \). The second part now follows.

We finish this section with several remarks about this theorem. First of all, this result can be understood as the statement that the cohomology of the special fiber \( H^2_c(\mathcal{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m \) is generated by Tate cycles coming from the supersingular locus. This result along with [LTXZZ, Proposition 6.3.1] can be seen as an analogue of the main result of [XZ] in the bad reduction case for low degree cohomology groups localized at generic maximal ideals. It seems that these cohomology groups always correspond to oldforms. It would be of great interest to formulate some predictions in a similar manner of [XZ]. Secondly, we have established the following analogue of the Ihara’s lemma in the setting of definite quaternionic unitary groups.

**Corollary 7.10.** There is an injection

\[
\text{Ih}_m : \mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \hookrightarrow \mathcal{O}_\lambda[Z_{\{1\}}(\overline{B})]_m.
\]
Proof. We have shown in the above proof that the composite of the two maps
\[
\mathcal{O}_\lambda[Z_{\{0\}}(\mathcal{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\mathcal{B})]_m \xrightarrow{(\text{inc}_{m,1}^{(0)} + \text{inc}_{m,1}^{(2)})} H^2_\epsilon(X_{\text{Pa}}(B), \mathcal{O}_\lambda(1))_m
\]

\[
H^3_{\epsilon}(X_{\text{Pa}}(B), \mathcal{O}_\lambda(1))_m \rightarrow \bigoplus_{\sigma \in Z_{\{1\}}(\mathcal{B})} H^3_{\epsilon}(X_{\text{Pa}}(B), R\Psi(\mathcal{O}_\lambda)(1)) = \mathcal{O}_\lambda[Z_{\{1\}}(\mathcal{B})]_m
\]
gives rise to an injection which we call
\[
\mathfrak{l}_m : \mathcal{O}_\lambda[Z_{\{0\}}(\mathcal{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\mathcal{B})]_m \hookrightarrow \mathcal{O}_\lambda[Z_{\{1\}}(\mathcal{B})]_m.
\]

Note that there is no natural degeneracy map from neither \(\mathcal{O}_\lambda[Z_{\{0\}}(\mathcal{B})]_m\) nor \(\mathcal{O}_\lambda[Z_{\{2\}}(\mathcal{B})]_m\) to \(\mathcal{O}_\lambda[Z_{\{1\}}(\mathcal{B})]_m\). Instead, these maps should be the analogues of the level raising operators in the sense of [BS]. Finally suppose that \(m\) is generic level raising at \(p\), we will show that \(H^4(X_{\text{Pa}}(B), \mathcal{O}_\lambda(2))_m\) is torsion free following the method of [LTXZZ] using Galois deformation argument.

8. Galois deformation rings and cohomology of Shimura varieties

In this section, we recall the main results in [Wang22a] where we use the method of [LTXZZa] to show that the cohomology of the quaternionic unitary Shimura varieties are free over certain universal deformation rings. These results are necessary as we lack of torsion vanishing results on the cohomology of the special fibers of these quaternionic unitary Shimura varieties.

8.1. Galois representation for \(\text{GSp}_4\). First we recall some results on associating Galois representations to cuspidal automorphic representations of \(\text{GSp}_4(\mathbb{A}_\mathbb{Q})\) and the local-global compatibility of Langlands correspondence. Let \(\pi\) be a cuspidal automorphic representation of \(\text{GSp}_4(\mathbb{A}_\mathbb{Q})\) with trivial central character. We assume that \(\pi\) has weight \((k_1, k_2)\) and trivial central character. In this article, we will only consider discrete automorphic representation of \(\text{GSp}_4(\mathbb{A}_\mathbb{Q})\) that is of general type. This means there is a cuspidal automorphic representation \(\Pi\) of \(\text{GL}_4(\mathbb{A}_\mathbb{Q})\) of symplectic type such that for each place \(v\) of \(\mathbb{Q}\), the \(L\)-parameter obtained from \(\text{rec}_{\text{GT}}(\pi_v)\) by composing with the embedding \(\text{GSp}_4 \hookrightarrow \text{GL}_4\) is precisely \(\text{rec}_{\text{GL}_4}(\Pi_v)\) which is the Langlands parameter attached to \(\Pi_v\). Here \(\Pi\) is of symplectic type if the partial \(L\)-function \(L^S(s, \Pi, \lambda^2)\) has a pole at \(s = 1\) for any finite set \(S\) of places of \(\mathbb{Q}\). In this case, we say \(\Pi\) is the transfer of \(\pi\).

We now recall some general results of Mok and Sorensen on the existence of Galois representations attached to \(\pi\). This theorem is very general and we temporarily drop the assumption that \(\pi\) is of general type.

**Theorem 8.1.** Suppose \(\pi\) is a cuspidal automorphic representation of \(\text{GSp}_4(\mathbb{A}_\mathbb{Q})\) of weight \((k_1, k_2)\) where \(k_1 \geq k_2 \geq 3\) and \(k_1 \equiv k_2 \mod 2\). Suppose that \(\pi\) has trivial central character. Let \(l \geq 5\) be a fixed prime. Then there is a continuous semisimple representation
\[
\rho_{\pi, l} : G_\mathbb{Q} \rightarrow \text{GSp}_4(\overline{\mathbb{Q}_l})
\]
satisfying the following properties.

1. \( c \circ \rho_{\pi,\ell} = \epsilon^{-3}_l; \)
2. For each finite place \( v \), we have
   \[
   \text{WD}(\rho_{\pi,\ell}|_{G_{\mathbb{Q}v}})^{F-ss} \cong \text{rec}_{\text{GT}}(\pi_v \otimes |c|^{-3/2})^{ss};
   \]
3. The local representation \( \rho_{\pi,\ell}|_{G_{\mathbb{Q}l}} \) is de Rham with Hodge–Tate weights
   \[
   \left(2 - \frac{k_1 + k_2}{2}, -\frac{k_1 - k_2}{2}, 1 + \frac{k_1 - k_2}{2}, -1 + \frac{k_1 + k_2}{2}\right).
   \]
4. If \( \pi \) is unramified at \( l \), then \( \rho_{\pi,\ell}|_{G_{\mathbb{Q}v}} \) is moreover crystalline at \( l \).
5. If \( \rho_{\pi,\ell} \) is irreducible, then for all finite place \( v \), \( \rho_{\pi,\ell}|_{G_{\mathbb{Q}v}} \) is pure.

**Remark 8.2.** Several remarks about this theorem are in order. The construction of the Galois representation in this case is given in \cite{Lau97, Lau05, Wei05, Tay93} using the Langlands–Kottwitz method. The statements about local-global compatibilities are mostly proved in \cite{Sor10} and completed in \cite{Mok05}. In fact the authors use the strong transfer of cuspidal automorphic representation from \( \text{GSp}_4 \) to \( \text{GL}_4 \) to construct the desired Galois representation in the totally real field case. The Harish-Chandra parameter \((\mu_1, \mu_2)\) of \( \pi \) used more often has the following relation with the weight of \( \pi \) used in this theorem:

\[
\begin{align*}
k_1 &= \mu_1 + 1, \\
k_2 &= \mu_2 + 2.
\end{align*}
\]

In this article, we will be interested in the case \( \pi \) of general type, and has trivial central character, and weights \((3, 3)\). In our normalization, this is the case when these representation appear in the cohomology of quaternionic unitary Shimura varieties with trivial coefficient.

**Definition 8.3.** Let \( \pi \) be a cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}_{\mathbb{Q}}) \) as in the previous remark. We say a number field \( E \subset \mathbb{C} \) is a strong coefficient field of \( \pi \) if for every prime \( \lambda \) of \( E \) there exits a continuous homomorphism

\[
\rho_{\pi,\lambda} : G_{\mathbb{Q}} \to \text{GSp}_4(E_{\lambda})
\]

up to conjugation, such that for every isomorphism \( \iota_\ell : \mathbb{C} \xrightarrow{\sim} \mathbb{Q}_\ell \) inducing the prime \( \lambda \), the representations \( \rho_{\pi,\lambda} \otimes_{E_{\lambda}} \mathbb{Q}_\ell \) and \( \rho_{\pi,\iota_\ell} \) are conjugate.

Let \( p \) be a place at which \( \pi \) is an unramified type I representation. Then the Frobenius eigenvalues of \( \rho_{\pi,\lambda}(\text{Frob}_p) \) agree with the Hecke parameter \([\alpha_p, \beta_p, \gamma_p, \delta_p]\) of \( \pi_p \) by Theorem 8.1 (2). We will always make the following assumption on \( \rho_{\pi,\lambda} \).

**Assumption 8.4.** The Galois representation \( \rho_{\pi,\lambda} : G_{\mathbb{Q}} \to \text{GSp}_4(E_{\lambda}) \) is residually absolutely irreducible.

The above assumption allows us to define the residual Galois representation

\[
\varphi_{\pi,\lambda} : G_{\mathbb{Q}} \to \text{GSp}_4(k)
\]

which is unique up to conjugation. Note it also follows that \( \pi \) is necessarily of general type if assumption 8.4 is effective.
8.2. Galois deformation rings. We summarize the main results in [Wang22a]. Suppose we are given a Galois representation \( \overline{\rho} : G_{\mathbb{Q}} \to \text{GSp}(k) \). We will write the restriction \( \overline{\rho}|_{G_{\mathbb{Q}}^v} : G_{\mathbb{Q}}^v \to \text{GSp}(k) \) by \( \overline{\rho}_v \).

Let \( \Sigma_{\text{min}} \) and \( \Sigma_{\text{lr}} \) be two sets of non-archimedean places of \( \mathbb{Q} \) away from \( l \) and such that

- \( q \) is contained in \( \Sigma_{\text{lr}} \);
- \( \Sigma_{\text{min}}, \Sigma_{\text{lr}} \) and \( \{ p \} \) are mutually disjoint;
- for every \( v \in \Sigma_{\text{lr}} \cup \{ p \} \), the prime \( v \) satisfies \( l \nmid (v^2 - 1) \).

**Definition 8.6.** For a Galois representation \( \overline{\rho} : G_{\mathbb{Q}} \to \text{GSp}(k) \) and \( \psi = \epsilon_l^{-3} \), we say the pair \( \overline{\rho} \) is rigid for \( (\Sigma_{\text{min}}, \Sigma_{\text{lr}}) \) if the followings are satisfied.

- For every \( v \in \Sigma_{\text{min}} \), every lifting of \( \overline{\rho}_v \) is minimally ramified in the sense of [Wang22a, Definition 3.4];
- For every \( v \in \Sigma_{\text{lr}} \), the generalized eigenvalues of \( \overline{\rho}_v(\phi_v) \) contain the pair \( \{ v^{-1}, v^{-2} \} \) exactly once;
- At \( v = l \), \( \overline{\rho}_v \) is regular Fontaine–Laffaille crystalline as in [Wang22a, Definition 3.10];
- For \( v \not\in \Sigma_{\text{min}} \cup \Sigma_{\text{lr}} \cup \{ l \} \), the representation \( \overline{\rho}_v \) is unramified.

Suppose that \( \overline{\rho} \) is rigid for \( (\Sigma_{\text{min}}, \Sigma_{\text{lr}}) \). We consider a global deformation problem in the sense of [Wang22a, Definition 2.2] of the form

\[
S^* = (\overline{\rho}, \psi, \Sigma_{\text{min}} \cup \Sigma_{\text{lr}} \cup \{ p \} \cup \{ l \}, \{ D_v \}_{v \in \Sigma_{\text{min}} \cup \Sigma_{\text{lr}} \cup \{ p \} \cup \{ l \}})
\]

where \( * = \{ \text{ram}, \text{unr}, \text{mix} \} \).

- For \( v \in \Sigma_{\text{min}} \), \( D_v \) is the local deformation problem \( D_v^{\text{min}} \) classifying all the minimal ramified liftings defined in [Wang22a, Definition 3.4];
- For \( v \in \Sigma_{\text{lr}} \cup \{ q \} \), \( D_v \) is the local deformation problem \( D_v^{\text{ram}} \) defined as in [Wang22a, Definition 3.6 (2)] classifying certain ramified liftings;
- For \( v = l \), \( D_v \) is the local deformation problem \( D_v^{\text{FL}} \) defined as in [Wang22a, Definition 3.1.1] classifying regular Fontaine–Laffaille crystalline liftings;
- For \( v = p \), depending on \( * \in \{ \text{ram}, \text{unr}, \text{mix} \} \), we have
  - \( D_v \) is the local deformation problem \( D_v^{\text{unr}} \) which is given by [Wang22a, Definition 3.6 (2)] when \( * = \text{unr} \);
  - \( D_v \) is the local deformation problem \( D_v^{\text{ram}} \) which is given by [Wang22a, Definition 3.6 (3)] when \( * = \text{ram} \);
  - \( D_v \) is the local deformation problem \( D_v^{\text{mix}} \) defined in [Wang22a, Definition 3.6 (1)] when \( * = \text{mix} \);
- For each global deformation problem \( S^* \) with \( * = \{ \text{ram}, \text{unr}, \text{mix} \} \), we have its corresponding universal deformation ring denoted by \( R^* = R^\text{univ}_S \) for \( * = \{ \text{ram}, \text{unr}, \text{mix} \} \).

**Construction 8.7.** When \( B \) is indefinite, we choose the level \( K = K_{Pa} \) for the Shimura variety \( \text{Sh}(B, K_{Pa}) \) in the following way.

- For \( v \not\in \Sigma_{\text{lr}} \cup \Sigma_{\text{min}} \) or \( v = l \), then \( K_v \) is hyperspecial;
- For \( v \in \Sigma_{\text{lr}} \setminus \{ p, q \} \), then \( K_v \) is the paramodular subgroup of \( \text{GSp}(\mathbb{Z}_v) \).
• For $v \in \Sigma_{\text{min}}$, then $K_v$ is contained in the pro-$v$ Iwahori subgroup $Iw_1(v)$;
• For $v = p, q$, then $K_v$ is the paramodular subgroup $G(B)(\mathbb{Z}_v)$ of $GU_2(D)$ and $GU_2(D')$ respectively;

Let $\Sigma_{Pa} = \Sigma_{\text{min}} \cup \Sigma_{lr} \cup \{p, q\}$. We denote by $T^{\text{ram}}$ the image of $T^{\Sigma_{Pa}}$ in $\text{End}_{\mathcal{O}_\lambda}(H^3(X_{Pa}(B), \mathcal{R}\Psi(\mathcal{O}_\lambda)))$.

**Construction 8.8.** We choose the level $\mathcal{K} = K_H$ for the Shimura variety $Sh(\mathcal{B}, K_H)$ when $\mathcal{B}$ is definite in such the following way.

• For $v \not\in \Sigma_{lr} \cup \Sigma_{\text{min}}$ or $v = l$ or $v = p$, then $K_v$ is hyperspecial;
• For $v \in \Sigma_{lr}$, then $K_v$ is the paramodular subgroup of $GSp(\mathbb{Z}_v)$;
• For $v \in \Sigma_{\text{min}}$, then $K_v$ is contained in the pro-$v$ Iwahori subgroup $Iw_1(v)$;
• For $v = q$, then $K_v$ is the paramodular subgroup $G(B)(\mathbb{Z}_v)$ of $GU_2(D')$.

Let $\Sigma_H = \Sigma_{\text{min}} \cup \Sigma_{lr} \cup \{q\}$. We denote by $T^{\text{unr}}$ the image of $T^{\Sigma_H}$ in $\text{End}_{\mathcal{O}_\lambda}(\mathcal{O}_\lambda[Z_H(\mathcal{B})])$.

Let $\overline{\rho}$ be a Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \text{GSp}_4(\mathbb{k})$ and let $m$ be the maximal ideal corresponding to $\overline{\rho}$ in $T^\Sigma$ for $\Sigma \in \{\Sigma_{Pa}, \Sigma_H\}$, in the sense that the characteristic polynomial $\det(X - \overline{\rho}(\text{Frob}_v))$ is congruent to the Hecke polynomial $Q_v(X)$ given by

$$X^4 - T_{v,2}X^3 + (vT_{v,1} + (v^3 + v)T_{v,0})X^2 - v^3T_{v,0}T_{v,2}X + v^6T_{v,0}^2$$

modulo $m$ for each $v$ not in $\Sigma$. We will also assume that the image of $\overline{\rho}$ contains $\text{GSp}_4(\mathbb{F}_l)$ and that

$$H^d_c(X_{Pa}(B), \mathcal{R}\Psi(\mathcal{O}_\lambda))_m = H^d_c(X_{Pa}(B), \mathcal{R}\Psi(\mathcal{O}_\lambda))_{m'} = 0$$

for $d \neq 3$ where $m' = m \cap T^\Sigma$.

The following theorem is the main result of [Wang22a] proved using the Taylor-Wiles method.

**Theorem 8.9.** Let $\overline{\rho}$ and $m$ be given as above. Suppose the following assumptions hold.

(D1) the image of $\overline{\rho}(G_{\mathbb{Q}})$ contains $\text{GSp}_4(\mathbb{k})$;
(D2) $\overline{\rho}$ is rigid for $(\Sigma_{\text{min}}, \Sigma_{lr})$;
(D3) For every finite set $\Sigma'$ of nonarchimedean places containing $\Sigma$ and every open compact subgroup $K'$ of $K$ satisfying $K'_v = K_v$ for all $v \not\in \Sigma'$, we have

$$H^d_c(X_{Pa}(B), \mathcal{R}\Psi(\mathbb{k}))_{m'} = 0$$

for $d \neq 3$ where $m' = m \cap T^\Sigma$.

Then the following holds true.

(1) If $T^{\text{unr}}_m$ is non-zero, then we have an isomorphism of complete intersection rings:

$$R^{\text{unr}} = T^{\text{unr}}_m$$

and $\mathcal{O}_\lambda[Z_H(\mathcal{B})]_m$ is a finite free $T^{\text{unr}}_m$-module.
If $T^\text{ram}_m$ is non-zero, then we have an isomorphism of complete intersection rings:

$$R^\text{ram} = T^\text{ram}_m$$

and $H^3(X_{Pa}(B), R\Psi(O_\lambda))_m$ is a finite free $T^\text{ram}_m$-module.

**Remark 8.10.** Let $\pi$ be a cuspidal automorphic representation of general type. Assume that $\overline{\rho}_{\pi, \lambda}(G_\mathbb{Q})$ contains $\text{GSp}_4(\mathbb{F}_l)$ for sufficiently large $l$, then we have shown that $\overline{\rho}_{\pi, \lambda}$ is indeed rigid, see [Wang22a, Theorem 5.5].

### 9. Arithmetic level raising and applications

Now we are ready to prove our main result on arithmetic level raising on the quaternionic Shimura varieties studied in the previous sections. First we recall the setting we are in. Let $\pi$ be a cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ with weight $(3,3)$ and trivial central character. Suppose that $\pi$ is of general type and $E$ be the coefficient field of $\pi$. Let $\Sigma_\pi$ be the minimal set of finite places outside of which $\pi$ is unramified. We fix an isomorphism $\iota_l : \mathbb{C} \cong \mathbb{Q}_l$ which induces a place $\lambda$ in $E$ over $l$. We fix the following datum.

- A finite set $\Sigma_{\text{min}}$ of places of $\mathbb{Q}$ contained in $\Sigma_\pi$;
- A finite set $\Sigma_{\text{lr}}$ of places of $\mathbb{Q}$ which is disjoint from $\Sigma_{\text{min}}$ and contained in $\Sigma_{\text{lr}}$;
- A finite set of places $\Sigma$ containing $\Sigma_{\text{min}} \cup \Sigma_{\text{lr}}$ which is away from $l$.
- We have, by Construction 3.3 (2), a homomorphism
  $$\phi_\pi : T^\Sigma \rightarrow O_E.$$  
  and its $\lambda$-adic avatar $\phi_{\pi, \lambda} : T^\Sigma \rightarrow O_\lambda$ for the valuation ring $O_\lambda \subset E_\lambda$.
- We can associate to it a Galois representation
  $$\rho_{\pi, \lambda} : G_\mathbb{Q} \rightarrow \text{GSp}_4(E_\lambda)$$
  which we assume is residually absolutely irreducible as in Assumption 8.4.
- Let $p$ be a prime which is level raising special of length $m$ for $\pi$ in the sense of Definition 3.2.
- Let $q$ be a prime such that $l \mid q^2 - 1$. Suppose that $\pi$ is of type IIa and $q$ is contained in $\Sigma_{\text{lr}}$. In this case $\pi_q$ admits Jacquet-Langlands transfer to a representation of the group $\text{GU}_2(D')$ where $D'$ which is the quaternion division algebra over $\mathbb{Q}_q$.
- We choose the level $K_{Pa}$ and $\Sigma_{Pa}$ as in Construction 8.7 for the Shimura variety $\text{Sh}(\mathbb{B}, K_{Pa})$ whose integral model is $X_{Pa}(B)$ and whose special fiber is $X_{Pa}(\mathbb{B})$; and choose the level $K_H$ and $\Sigma_H$ as in Construction 8.8 for the Shimura set $\text{Sh}(\mathbb{B}, K_H)$ which is also denoted by $Z_H(\mathbb{B})$.
- We introduced two ideals
  $$m = T^{\Sigma_{\text{lr}}}(p) \cap \ker(T^\Sigma \phi_{\pi, \lambda} \rightarrow O_\lambda \rightarrow O_\lambda / \lambda^m)$$
  $$n = T^{\Sigma_{\text{lr}}}(p) \cap \ker(T^\Sigma \phi_{\pi, \lambda} \rightarrow O_\lambda \rightarrow O_\lambda / \lambda^m)$$
  as in Construction 3.3 (3) and we assume that $m$ is generic level raising at $p$ in the sense of 7.8.
Construction 9.2. We define the following map
\[ \nabla : \mathcal{O}_\lambda[\mathbb{Z}_H(B)] \oplus \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})] \to \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})] \]
by sending
\[ (x, y) \to (T_{p,0}^{-1}T_{p,1} + (p + 1)(p^2 + 1))x - (p + 1)T_{20}y \]
for \((x, y) \in \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})] \oplus \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]\).

We also write
\[ \nabla : \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \oplus \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \to \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \]
for the localization of \(\nabla\) at \(m\).

Note that the composition of the level raising matrix
\[ T_{\text{lr},m} : \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \oplus \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \to \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \oplus \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m \]
with the map \(\nabla_m\) is given by sending \((x, y) \to \det T_{\text{lr},m} \cdot x\) for \((x, y) \in \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})] \oplus \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]\) where we have
\[ \det T_{\text{lr},m} = (T_{p,0}^{-1}T_{p,1} + (p + 1)(p^2 + 1))^2 - (p + 1)^2T_{20} \circ T_{02} \]
by Proposition 6.10.

Proposition 9.3. Let \(p\) be a level raising special prime for \(\pi\) and \(m\) be the maximal ideal as above. We have a surjective morphism
\[ F_{-1}H^1(I_{Q_p^2}, H^3_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m) \to \mathcal{O}_\lambda[\mathbb{Z}_H(\overline{B})]_m / \det T_{\text{lr},m} \]
whose kernel can be identified with the torsion part of \(H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m\).

Proof. Note that, by Proposition 5.4, we have an isomorphism
\[ F_{-1}H^1(I_{Q_p^2}, H^3_c(\mathbb{X}_{Pa}(B), R\Psi(\mathcal{O}_\lambda)(1))_m) \sim \text{Coker}(\alpha \circ N^{-1}_\Sigma \circ \beta)_m. \]

On the other hand, by Proposition 7.9 we have an isomorphism
\[ (\text{inc}_{\{0\},m} + \text{inc}_{\{2\},m}) : \mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \to H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m \]
and a surjection
\[ (\text{inc}_{\{0\},m}, \text{inc}_{\{2\},m}) : H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m \to \mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \]
whose kernel is given by the torsion part of \(H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m\).

It follows that we have a surjection
\[ F_{-1}H^1(I_{Q_p^2}, H^3_c(\mathbb{X}_{Pa}(B), R\Psi(\mathcal{O}_\lambda)(1))_m) \to \frac{\mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m}{\mathcal{O}_\lambda[Z_{\{0\}}(\overline{B})]_m \oplus \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m \cup \mathcal{O}_\lambda[Z_{\{2\}}(\overline{B})]_m}, \]
The kernel of this map is given by
\[ \frac{H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m^{\text{tor}}}{\text{Im}(\alpha \circ N^{-1}_\Sigma \circ \beta)_m \cap H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m^{\text{tor}}}. \]
Composing this with the map $\nabla_m : O_\lambda[Z_H(B)]_m \oplus O_\lambda[Z_H(B)]_m \to O_\lambda[Z_H(B)]_m$, we therefore obtain a map

$$F_1 H^1(I_{Q,p}, H^2_c(X_{Pa}(B), R\Psi(O_\lambda)(1))_m) \to O_\lambda[Z_H(B)]_m/\det T_{r,m},$$

which is surjective by Nakayama’s lemma.

It remains to show that $\text{Im}(\alpha \circ N^{-1}_\Sigma \circ \beta)_m \cap H^4(X_{Pa}(B), \mathcal{O}_\lambda(2))^{\text{tor}}_m$ is zero. By [Nek07 Proposition 4.2.2(1)], the group $F_1 H^1(I_{Q,p}, H^2_c(X_{Pa}(B), R\Psi(E_\lambda)(1))_m)$ vanishes. It follows that the $\mathcal{O}_\lambda$-rank of $\text{Im}(\alpha \circ N^{-1}_\Sigma \circ \beta)_m$ is the same as the $\mathcal{O}_\lambda$-rank of $H^4(X_{Pa}(B), \mathcal{O}_\lambda(2))_m$.

Next, using the excision exact sequences, we have isomorphisms

$$(9.4) \quad H^4(X_{Pa}(B), E_\lambda(2)) \xrightarrow{\sim} H^4(X_{Pa}^{\text{sing}}(B), E_\lambda(2))$$

$$H^2_c(X_{Pa}^{\text{sing}}(B), E_\lambda(1)) \xrightarrow{\sim} H^2_c(X_{Pa}(B), E_\lambda(1))$$

where $X_{Pa}^{\text{sing}}(B) = X_{Pa}(B) - X_{Pa}^{\text{sing}}(B)$, using the fact that the singular locus $X_{Pa}^{\text{sing}}(B)$ is zero dimensional. Therefore Poincare duality for $X_{Pa}^{\text{sing}}(B)$ implies that we have

$$\dim_{E_\lambda} H^4(X_{Pa}(B), E_\lambda(2))_m = \dim_{E_\lambda} H^2_c(X_{Pa}(B), E_\lambda(1))_m.$$ 

Since the source $H^2_c(X_{Pa}(B), \mathcal{O}_\lambda(1))_m$ of $\text{Im}(\alpha \circ N^{-1}_\Sigma \circ \beta)_m$ is free over $\mathcal{O}_\lambda$, the intersection

$$\text{Im}(\alpha \circ N^{-1}_\Sigma \circ \beta)_m \cap H^4(X_{Pa}(B), \mathcal{O}_\lambda(2))^{\text{tor}}_m$$

is trivial. The proposition is proved. \qed

Now we are ready to state and prove our main result on arithmetic level raising for the quaternionic Shimura variety studied in this article. We first recall a list of running assumptions below.

- We assume that $H^i_c(X_{Pa}(B), R\Psi(O_\lambda))_m = 0$ for $i \neq 3$, and that $H^3_c(X_{Pa}(B), R\Psi(O_\lambda))_m \cong H^3(X_{Pa}(B), R\Psi(O_\lambda))_m$ is a finite free $O_\lambda$ module. This is Assumption $\mathcal{Z}$6.
- We assume that for $\pi$, the associated Galois representation $\rho_{\pi,\lambda} : G_\mathbb{Q} \to \text{GSp}_4(E_\lambda)$ is residually absolutely irreducible. This is Assumption $\mathcal{S}$4.

**Theorem 9.5** (Arithmetic level raising for GSp$_4$). Suppose that $\pi$ is a cuspidal automorphic representation of GSp$_4(\mathbb{A}_\mathbb{Q})$ of general type. We assume that the assumptions in $\mathcal{Z}$6, $\mathcal{S}$4 recalled above are effective. Let $p$ be a level raising special prime for $\pi$ of depth $m$. We assume further that

1. $l \geq 3$ and $l \nmid p^2 - 1$;
2. $\overline{\rho}_{\pi,\lambda}$ is rigid for $\langle \Sigma_{\text{min}}, \Sigma_{\text{lr}} \rangle$ as in Definition $\mathcal{S}$6;
3. The image of $\overline{\rho}_{\pi,\lambda}(G_\mathbb{Q})$ contains GSp$_4(F_l)$;
4. $m$ is in the support of $O_\lambda[Z_H(B)]$ and is generic level raising.

Then following holds.
(1) There is an isomorphism
\[ H^1_{\text{sing}}(\mathbb{Q}_p^2, H^3_\xi(X_{P\alpha}(B), R\Psi(\mathcal{O}_\lambda(1)))_m) \xrightarrow{\sim} \mathcal{O}_\lambda[\text{det} T_\text{Ir}]. \]

(2) The above isomorphism induces an isomorphism
\[ H^1_{\text{sing}}(\mathbb{Q}_p^2, H^3_\xi(X_{P\alpha}(B), R\Psi(\mathcal{O}_\lambda/\lambda^m(1)))_m) \xrightarrow{\sim} \mathcal{O}_\lambda/\lambda^m[\text{det} T_\text{Ir}]. \]

9.1. Proof of Theorem 9.5. Consider the universal deformation problem
\[ S^* = (\mathcal{F}_{\pi, \lambda}, \psi, \Sigma_{\text{ss}}, \Sigma_{\text{lr}} \cup \{p\} \cup \{l\}, \{D_v\}_{v \in \Sigma_{\text{ss}} \cup \Sigma_{\text{lr}} \cup \{p\} \cup \{l\}} \]
where \( * = \{\text{ram}, \text{unr}, \text{mix}\} \). Recall that Spf \( R^\text{ram} \) and Spf \( R^\text{unr} \) are subspaces of Spf \( R^\text{mix} \).

Thus we have surjections
\[ R^\text{mix} \twoheadrightarrow R^\text{ram}, \]
\[ R^\text{mix} \twoheadrightarrow R^\text{unr}. \]

We define \( R^\text{cong} = R^\text{unr} \otimes R^\text{mix} \). We have a universal lifting
\[ \rho^\text{mix} : G_\mathbb{Q} \rightarrow \text{GSp}_4(R^\text{mix}). \]

It also induces two representations
\[ \rho^\text{ram} : G_\mathbb{Q} \rightarrow \text{GSp}_4(R^\text{ram}) \]
\[ \rho^\text{unr} : G_\mathbb{Q} \rightarrow \text{GSp}_4(R^\text{unr}) \]

by the above surjections.

Denote by \( P_{\mathbb{Q}_p} \) be the maximal closed subgroup of the inertia subgroup \( I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p} \) of pro-order coprime to \( l \). Then \( G_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \cong \mathbb{Z}/l \mathbb{Z} \rtimes \hat{\phi}_p^\mathbb{Z} \) is a \( p \)-tame group. By definition, \( \rho^\text{mix} \) is trivial on \( P_{\mathbb{Q}_p} \). Let \( v_0 \) and \( v_1 \) be eigenvectors in \( k^4 \) with eigenvalues \( p^{-2} \) and \( p^{-4} \), respectively. By Hensel’s lemma, they lift to \( v_0 \) and \( v_1 \) in \( (R^\text{mix})^{\otimes 4} \) for \( \rho^\text{mix}(\hat{\phi}_p^2) \) with eigenvalues \( s_0 \) and \( s_1 \) lifting \( p^{-2} \) and \( p^{-4} \). Let \( x \in R^\text{mix} \) be the unique element such that \( \rho^\text{mix}(t)v_1 = xv_0 + v_1 \). It follows from the relation
\[ \rho^\text{mix}(\hat{\phi}_p^2)\rho^\text{mix}(t)\rho^\text{mix}(\hat{\phi}_p^{-2}) = \rho^\text{mix}(t)p^2 \]
that \( x(s - p^{-2}) = 0 \). By the definitions of \( R^* \) for \( * = \{\text{ram}, \text{unr}, \text{mix}\} \), we have
\[ R^\text{unr} = R^\text{mix}/(x) \]
\[ R^\text{ram} = R^\text{mix}/(s_0 - p^{-2}) \]
\[ R^\text{cong} = R^\text{mix}/(x, s_0 - p^{-2}). \]

By Theorem 8.9, \( \mathcal{O}_\lambda[\text{det} T_\text{Ir}]/\mathbb{Z} \) is a free \( T_\text{unr} \)-module of rank \( d_\text{unr} \). Note that the characteristic polynomial of \( \rho^\text{mix}(\hat{\phi}_p^2) \) can be written as \( (X - s_0)(X - p^{-6}s_0 - 1)Q(X) \) where \( Q(X) \) is a polynomial in \( R_{\text{unr}}[X] \) whose reduction in \( \mathcal{O}_\lambda/\lambda[X] = k[X] \) does not contain \( p^{-2} \) and \( p^{-4} \) as its roots.

By Proposition 6.13 and the relations
\[ (\alpha_p + p^3\alpha_p^{-1} - p(p + 1))(\alpha_p + p^3\alpha_p^{-1} + p(p + 1)) = \alpha_p^2 + p^6\alpha_p^{-2} - p^2(p^2 + 1), \]
\[ (\beta_p + p^3\beta_p^{-1} - p(p + 1))(\beta_p + p^3\beta_p^{-1} + p(p + 1)) = \beta_p^2 + p^6\beta_p^{-2} - p^2(p^2 + 1), \]
we have
\[(\det T_{l,m}) \cdot \mathcal{O}_L[Z_H(B)]_m = (s_0 - p^{-2}) \cdot \mathcal{O}_L[Z_H(B)]_m.\]
Thus we have
\[\mathcal{O}_L[Z_H(B)]_m / \det T_{l,m} \cong \mathcal{O}_L[Z_H(B)]_m \otimes_{\mathbb{R}^\text{cong}} \mathbb{R}^\text{cong}\]
and it follows that \(\mathcal{O}_L[Z_H(B)]_m / \det T_{l,m}\) is a free \(\mathbb{R}^\text{cong}\)-module of rank \(d_{\text{unr}}\).

Since we have a surjection
\[F_{-1}H^1(I_{Q_{p^2}}, H^3_c(X_{Pa}(B), R\Psi(O_L(1)))_m) \twoheadrightarrow \mathcal{O}_L[Z_H(B)]_m / \det T_{l,m}\]
by Proposition 9.3 we have \(T^\text{ram}_m\) is non-zero. Moreover \(H^3_c(X_{Pa}(B), R\Psi(O_L(1)))_m\) is a free \(\mathbb{R}^\text{ram}\)-module by Theorem 8.9 (2), say of rank \(d_{\text{ram}}\). Consider the \(\mathbb{R}^\text{ram}\)-module
\[H := \text{Hom}_{G_Q}((\mathbb{R}^\text{ram})^\oplus 4, H^3_c(X_{Pa}(B), R\Psi(O_L(1)))_m)\]
where \(G_Q\) acts on \((\mathbb{R}^\text{ram})^\oplus 4\) via \(\rho^\text{ram}\) which is free of rank \(d_{\text{ram}}\). It follows that we have an isomorphism
\[H^3_c(X_{Pa}(B), R\Psi(O_L))_m \cong H \otimes_{\mathbb{R}^\text{ram}} (\mathbb{R}^\text{ram})^\oplus 4.\]

It follows from that
\[H^1_{\text{sing}}(Q_{p^2}, H^3_c(X_{Pa}(B), R\Psi(O_L(1)))_m) \cong H \otimes_{\mathbb{R}^\text{ram}} H^1_{\text{sing}}(Q_{p^2}, (\mathbb{R}^\text{ram})^\oplus 4(1))\]
We still denote by \(v_0\) and \(v_1\) the projections of \(v_0\) and \(v_1\) in \((\mathbb{R}^\text{ram})^\oplus 4\) of \(G_Q\)-modules. Then a straightforward computation shows that
\[H^1_{\text{sing}}(Q_{p^2}, (\mathbb{R}^\text{ram})^\oplus 4(1)) \cong R^\text{ram}v_0 / xv_0 \cong R^\text{cong}.\]
Hence we have
\[H^1_{\text{sing}}(Q_{p^2}, H^3_c(X_{Pa}(B), R\Psi(O_L(1)))_m)\]
is a free \(R^\text{cong}\)-module of rank \(d_{\text{ram}}\).

**Lemma 9.6.** We have an equality
\[d_{\text{unr}} = d_{\text{ram}}.\]

**Proof.** Let \(\eta^\text{unr} \in \text{Spec } R^\text{unr}[1/l](\mathbb{Q}_{l})\) be in the support of \(\mathcal{O}_L[Z_H(B)]_m\) which gives rise to an automorphic representation \(\pi^\text{unr}\) of \(\text{GSp}_4(\mathbb{A}_{\mathbb{Q}})\) such that \(\rho_{\pi, l} \otimes_k \mathbf{F}_l\) and \(\rho_{\eta^\text{unr}, l}\) are residually isomorphic. Then we have
\[d_{\text{unr}} = \dim_{\mathbf{F}_l} \mathbb{Q}_{l}[Z_H(B)]_l[t_{\eta^\text{unr}}].\]
Similarly, let \(\eta^\text{ram} \in \text{Spec } R^\text{ram}[1/l](\mathbb{Q}_{l})\) be in the support of \(H^3_c(X_{Pa}(B), R\Psi(O_L(1)))_m\) which gives rise to an automorphic representation \(\pi^\text{ram}\) of \(\text{GSp}_4(\mathbb{A}_{\mathbb{Q}})\) such that \(\rho_{\pi^\text{ram}, l}\) is residually isomorphic to \(\rho_{\pi, l} \otimes_k \mathbf{F}_l\). Then we have
\[4d_{\text{ram}} = \dim_{\mathbf{F}_l} H^3_c(X_{Pa}(B), R\Psi(O_L))_l[t_{\eta^\text{ram}}].\]
Note that \(\pi^\text{unr}\) and \(\pi^\text{ram}\) are necessarily of general type. By considering the transfer of \(\pi^\text{unr}\) and \(\pi^\text{ram}\) to \(\text{GL}_4(\mathbb{A}_{\mathbb{Q}})\), we have \(d_{\text{ram}} = d_{\text{unr}}\) by [LTZZZ, Lemma 6.4.2].
Proof of Theorem 9.3. Now we can prove our main theorem on arithmetic level raising. We have proved that we have a surjection
\[ F^{-1}H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \to O_L[Z_H(B)]_m \]
where the target is free of rank \( d_{\text{unr}} \) over \( R^{\text{cong}} \). On the other hand, we have an injection
\[ F^{-1}H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \to H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \]
and the target is free of rank \( d_{\text{ram}} \) over \( R^{\text{cong}} \). Since \( d_{\text{unr}} = d_{\text{ram}} \), we have the desired isomorphism
\[
H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \xrightarrow{\sim} O_L[Z_H(B)]_m / \det T_{\text{fr}}.
\]
This finishes the proof of the first part of the theorem.

For the second part, by the freeness of \( H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m \) as a \( T_{\text{ram}}^m \)-module. It follows that the natural map
\[ H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) / n \xrightarrow{\sim} H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m / n) \]
is an isomorphism. We also have the short exact sequence as in Proposition 6.1
\[
0 \to F^{-1}H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \to H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \\
\to H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) / F^{-1}H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \to 0
\]
which is split by the \( G_{\mathbb{Q}_{p^2}} \)-action. And it follows that
\[ F^{-1}H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) / n \xrightarrow{\sim} F^{-1}H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m / n) \]
is an isomorphism. Since we have shown above that there is an isomorphism
\[ F^{-1}H^1(I_{Q,p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) \xrightarrow{\sim} H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m), \]
it follows that
\[ H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) / n \xrightarrow{\sim} H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m / n). \]
On the other hand, we have \( T_{\text{ram}}^m / n \cong O_L / \lambda^m \) and thus
\[
H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m) / n \cong H^1_{\text{sing}}(\mathbb{Q}_{p^2}, H_c^3(X_{Pa}(B), R\Psi(O_L/\lambda^m(1)))_m)
\]
by the freeness of \( H_c^3(X_{Pa}(B), R\Psi(O_L(1)))_m \) as \( T_{\text{ram}}^m \)-module. Finally the second claim follows from the observation that
\[
O_L[Z_H(B)]_m / \det T_{\text{fr}} \cong O_L[Z_H(B)]_m \otimes_{R^{\text{unr}}} R^{\text{cong}} \\
\cong O_L[Z_H(B)]_m \otimes_{R^{\text{unr}}} T_{\text{ram}}^m / n \\
\cong O_L[Z_H(B)]_m \otimes O_L / \lambda^m
\]
and the isomorphism 9.7. □

Corollary 9.8. We maintain the assumptions in Theorem 9.3.
(1) $H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda)_m$ is a torsion free $\mathcal{O}_\lambda$-module. In particular, the map from Construction 6.4
\[ (\text{inc}_{[0],m}, \text{inc}_{[2],m}) : H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m \to \mathcal{O}_\lambda[Z_{[0]}(\mathbb{B})]_m \oplus \mathcal{O}_\lambda[Z_{[2]}(\mathbb{B})]_m \]
is an isomorphism.
(2) $H^3(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda)_m$ and $H^3_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda)_m$ are both torsion free $\mathcal{O}_\lambda$-modules.

**Proof.** Recall we have proved that the kernel of the natural map
\[ F_{-1}^1(I_{\mathbb{Q}_p^d}, H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m) \to \mathcal{O}_\lambda[Z_H(B)]_m / \det T_{r,m} \]
can be identified with the torsion part of $H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m$. We have shown in the above proof that this surjection is an isomorphism and the result follows.

For the second part, we have an injection
\[ H^3(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m \hookrightarrow H^3(\mathbb{X}_{Pa}(B), R\Psi(\mathcal{O}_\lambda(2)))_m \]
and $H^3(\mathbb{X}_{Pa}(B), R\Psi(\mathcal{O}_\lambda(2)))_m$ is a torsion free $\mathcal{O}_\lambda$-module by our assumption. Thus $H^3(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m$ is torsion free.

Next, we show the cohomology group $H^3_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m$ is torsion free. We consider the long exact sequence
\[ \cdots \to H^3_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m \to H^3(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m \to H^3_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(1))_m \to \cdots \]
induced by the short exact sequence
\[ 0 \to \mathcal{O}_\lambda \to \mathcal{O}_\lambda / \lambda \to 0. \]

We need to show the injection
\[ H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m / \lambda \hookrightarrow H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(1))_m \]
is in fact surjective. For this, note that we have established that $H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m$ is torsion free and thus $\dim_k H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m / \lambda = \dim_{E_\lambda} H^2_c(\mathbb{X}_{Pa}(B), E_\lambda(1))_m$. But we have
\[ \dim_k H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(1))_m = \dim_k H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(2))_m; \]
\[ \dim_{E_\lambda} H^2_c(\mathbb{X}_{Pa}(B), E_\lambda(1))_m = \dim_{E_\lambda} H^4(\mathbb{X}_{Pa}(B), E_\lambda(2))_m. \]
by the Poincare duality and (9.14). By the considering the same long exact sequence and the reasoning as above but applied to $H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(2))_m$, we have
\[ \dim_k H^4(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(2))_m = \dim_{E_\lambda} H^4(\mathbb{X}_{Pa}(B), E_\lambda(2))_m \]
and hence
\[ \dim_k H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(1))_m = \dim_{E_\lambda} H^2_c(\mathbb{X}_{Pa}(B), E_\lambda(1))_m. \]
It follows that $H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda(1))_m / \lambda \hookrightarrow H^2_c(\mathbb{X}_{Pa}(B), \mathcal{O}_\lambda/\lambda(1))_m$ is in fact surjective and we are done.
9.2. Level raising for GSp\(_4\). Let \(\pi\) be a cuspidal automorphic representation of GSp\(_4(\mathbb{A}_\mathbb{Q})\) of general type with weight \((3,3)\) and trivial central character. Let \(p\) be a level raising special prime for \(\pi\) of length \(m\). From the arithmetic level raising theorem proved in the last subsection, we can deduce a level raising theorem for \(\pi\) which can be applied to deduce level raising theorems in [Wang22b]. This is the content of this subsection.

Suppose the component \(\pi_q\) of \(\pi\) at the prime \(q\) is of type IIa. Then \(\pi\) admits a global Jacquet–Langlands transfer to an automorphic representation of \(G(\mathbb{B})(\mathbb{A}_\mathbb{Q})\). In particular the maximal ideal \(m\) associated to \(\pi\) in the Construction 3.3 (3) lies in the support of \(O_\lambda[Z_H(\mathbb{B})]\) as a \(T_{\Sigma,}^{\cup\{p\}}\)-module.

**Theorem 9.9.** Let \(\pi\) be an automorphic representation of GSp\(_4(\mathbb{A}_\mathbb{Q})\) as above. Suppose that \(\pi\) satisfy all the assumptions in Theorem 9.5. Let \(p\) be a level raising special prime for \(\pi\) of length \(m\).

Then there exists an automorphic representation \(\pi'\) of GSp\(_4(\mathbb{A}_\mathbb{Q})\) of general type with weight \((3,3)\) and trivial central character such that

1. the component \(\pi'_p\) at \(p\) is of type IIa;
2. we have an isomorphism of the residual Galois representation

\[ \overline{\rho}_{\pi,\lambda} \cong \overline{\rho}_{\pi',\lambda}. \]

In this case we will say \(\pi'\) is a level raising of \(\pi\).

**Proof.** As remark in the above, the localization \(O_\lambda[Z_H(\mathbb{B})]_m\) is non-zero. By Theorem 9.5, we have an isomorphism

\[ H^1_{\text{sing}}(Q_p^2, H^3_c(\mathbb{X}_{P_\lambda}(B), R\Psi(O_\lambda(1)))_m) \cong O_\lambda[Z_H(\mathbb{B})]_m/\det T_{\frak{H}}. \]

In particular \(H^3_c(\mathbb{X}_{P_\lambda}(B), R\Psi(O_\lambda(1)))_m\) is non-zero. Since \(H^3_c(\mathbb{X}_{P_\lambda}(B), R\Psi(O_\lambda(1)))_m\) is a free \(R_{\text{ram}}\)-module, we can find a point \(\eta_{\text{ram}} \in \text{Spec } R_{\text{ram}}[1/l]\) in the support of

\[ H^3_c(\mathbb{X}_{P_\lambda}(B), R\Psi(O_\lambda(1)))_m. \]

Then we can find an automorphic representation \(\pi'\) of GSp\(_4(\mathbb{A}_\mathbb{Q})\) which satisfy the isomorphism

\[ \overline{\rho}_{\pi,\lambda} \cong \overline{\rho}_{\pi',\lambda}. \]

In particular \(\pi'\) is of general type as \(\overline{\rho}_{\pi',\lambda}\) is absolutely irreducible. By the Picard-Lefschetz formula applied to the Shimura variety \(X_{P_\lambda}(B)\), we see that the monodromy of the associated Weil-Deligne representation of \(\rho_{\pi',\lambda}\) is necessarily of the form

\[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

up to conjugation. Since \(\pi'\) is of general type and hence tempered at \(p\), it follows that \(\pi'\) is of type IIa by the local Langlands correspondence for non-supercuspidal representations, see for example [Sch05, Table 2]. The theorem is proved. \qed
References

[Art01] J. Arthur, A stable trace formula. I. General expansions, J. Inst. Math. Jussieu 1 (2002), no. 2, 175–277.

[Art02] J. Arthur, A stable trace formula. II. Global descent, Invent. Math. 143 (2001), no. 1, 157–220.

[Art03] J. Arthur, A stable trace formula. III. Proof of the main theorem, Ann. of Math. (2) 158 (2003), 769–837.

[Art04] J. Arthur, Automorphic representations of GSp(4) in Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, (2004), 65–81.

[BD05] M. Bertolini and H. Darmon, Iwasawa’s main conjecture for elliptic curves over anticyclotomic $Z_p$-extensions, Ann. of Math. (2) 162, (2005), 1–64.

[BS] R. Brooks and R. Schmidt, Local newforms for GSp(4), Lecture Notes in Mathematics, 1918 (2007), Springer, Berlin, viii+307.

[Clo] L. Clozel, Ribet’s theorem for U(3), Amer. J. Math., 122 (2000), 1265–1287.

[CHT] L. Clozel, M. Harris and R. Taylor Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations, Publ. Math. Inst. Hautes Études Sci. 63 (1991), 281–332.

[CG15] P-S. Chan and W-T. Gan, The local Langlands conjecture for GSp(4) III: Stability and twisted endoscopy, J. Number Theory 146 (2015), 69–133.

[CS17] A. Caraiani and P. Scholze, On the generic part of the cohomology of non-compact unitary Shimura varieties, Ann. of Math. (2), 186 (2017), no. 3, 649–766.

[CS19] A. Caraiani and P. Scholze, On the generic part of the cohomology of compact unitary Shimura varieties, arXiv:1909.01898v1, preprint.

[DL76] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. (2) 103 (1976), no. 1, 103–161.

[Gee11] T. Gee, Automorphic lifts of prescribed types, Math. Ann. 350 (1), (2011), 107–144.

[Gross] B. Gross, On the Satake isomorphism, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser. Vol. 254, Cambridge Univ. Press, Cambridge, 1998.

[Hal97] T. Hales, The fundamental lemma for Sp(4), Proc. American. Math. Soc. 125 (1) (1997), 301–308.

[Ill10] A. Ichino and T. Ikeda, On the Periods of Automorphic Forms on Special Orthogonal Groups and the Gross–Prasad Conjecture, Geom. Funct. Anal. 19 (2010), 1378–1425.

[Ill02] L. Illusie, Sur la Formule de Picard-Lefschetz, Algebraic Geometry 2000, Azumino, Adv. Stud. Pure Math. 36, Math. Soc. Japan, Tokyo (2002), 249–268.

[Kos19] T. Koshikawa, Vanishing theorems for the mod p cohomology of some simple Shimura varieties, arXiv:1910.03147, preprint, (2019).

[Kos21] T. Koshikawa, On the generic part of the cohomology of local and global Shimura varieties, arXiv:2106.10602, preprint, (2021).

[Kot83] R. Kottwitz, Sign changes in harmonic analysis on reductive groups, Trans. Amer. Math. Soc. 278 (1983), 289–297.

[KR00] S. Kudla and M. Rapoport, Cycles on Siegel threefolds and derivatives of Eisenstein series, Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 5, 695–756.

[Lau97] G. Laumon, Sur la cohomologie à supports compacts des variétés de Shimura pour GSp(4)Q, Compositio Math. 105 (1997), no. 3, 267–359.

[Lau05] G. Laumon, Fonctions zêtas des variétés de Siegel de dimension trois, Formes automorphes. II. Le cas du groupe GSp(4)Q. 302 (2005), 1–66.

[Liu] Y. Liu, Bounding cubic-triple product Selmer groups of elliptic curves, J. Eur. Math. Soc. (JEMS). 21 (2017) no.5 1411–1508.

[LS13] K-W. Lan and J. Suh, Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties, Adv. Math. 242 (2013), 228–286.

[LS18b] K-W. Lan and B. Stroh, Nearby cycles of automorphic étale sheaves, Compos. Math. 154 (2018), no. 1, 80–119.
K-W. Lan and B. Stroh, Nearby cycles of automorphic étale sheaves II, Cohomology of arithmetic groups, Springer Proc. Math. Stat. 245 (2018), Springer, Cham, 83–106.

Y. Liu, Y. Tian, L. Xiao, W. Zhang and X. Zhu, On the Beilinson-Bloch-Kato conjecture for Rankin-Selberg motives, arXiv:1912.11942, online first article in Invent. Math. (2022).

Y. Liu, Y. Tian, L. Xiao, W. Zhang and X. Zhu, Deformation of rigid conjugate self-dual Galois representations, arXiv:2108.06998, preprint, (2021).

A. Mokrane and J. Tilouine, Cohomology of Siegel varieties with p-adic integral coefficients and applications, Cohomology of Siegel varieties, Asterisque no. 280 (2002), 1–95.

C. Mok, Galois representations attached to automorphic forms on GL2 over CM fields, Compos. Math. 150 (2014), no. 4, 523–567.

Y. Oki, On supersingular loci of Shimura varieties for quaternionic unitary groups of degree 2, Manuscripta Math. 167 (2022), 263–343.

K. Ribet, Congruence relations between modular forms, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 503–514, PWN, Warsaw, 1984.

D. Shelstad, Characters and inner forms of a quasi-split group over \( \mathbb{R} \), Compositio. Math. 39 (1979), 11–45.

M. Rosner and R. Weissauer, Global liftings between inner forms GSp4, arXiv:2103.14715, preprint, 2021.

M. Rapoport and T. Zink, Period spaces for \( p \)-divisible groups, Annals of Mathematics Studies 141, Princeton University Press, Princeton, NJ (1996), xxii+324.

M. Santos, Upcoming thesis at LSNTG.

R. Schmidt, Iwahori-spherical representations of GSp(4) and Siegel modular forms of degree 2 with square-free level, J. Math. Soc. Japan, 57 (2005), no.1, 259–293.

R. Schmidt, Packet structure and paramodular forms, Trans. Amer. Math. Soc., 370 (2018) no.5, 3085–3112.

D. Shelstad, Characters and inner forms of a quasi-split group over \( \mathbb{R} \), Compositio. Math. 39 (1979), 11–45.

P. Sally and M. Tadić, Induced representations and classifications for GSp(2, \( F \)) and Sp(2, \( F \)), Mém. Soc. Math. France (N.S.) 52 (1993), 75–133.

C. Sorensen, Level-raising for Saito–Kurokawa forms, Compositio Math. 145 (2010), no. 4, 915–953.

C. Sorensen, Galois representations attached to Hilbert-Siegel modular forms, Doc. Math. 15 (2010), 623–670.

R. Taylor, Galois representations associated to Siegel modular forms of low weight, Duke Math. J. 63 (1991), 281–332.

R. Taylor, On the \( l \)-adic cohomology of Siegel threefolds, Invent. Math. 114 (1993), 289–310.

H. Wang, On the Bruhat–Tits stratification of a quaternionic unitary Shimura variety, Math. Ann. 376 (2020), 1107–1144.

J. Waldspurger, Le lemme fondamental implique le transfert, Compositio. Math. 105 (1997), 153–236.

H. Wang, On quaternionic unitary Rapoport–Zink spaces with parahoric level structures, arXiv:1909.12263, online first article in IMRN (2020).

H. Wang, Arithmetic level raising on triple product of Shimura curves and Gross–Schoen Diagonal cycles I: Ramified case, arXiv:2004.00555, to appear in Algebra Number Theory.

H. Wang, Deformation of rigid Galois representations and cohomology of certain quaternionic unitary Shimura varieties, preprint.

H. Wang, Level lowering on Siegel modular threefold of paramodular level, preprint.
[Wei05] R. Weissauer, Four dimensional Galois representations, Formes automorphes. II. Le cas du groupe GSp(4), Astérisque, 302, 67–150.

[XZ] L. Xiao and X. Zhu, Cycles on Shimura varieties via geometric Satake, arXiv:1707.05700, preprint.

Haining Wang
Shanghai Center for Mathematical Sciences,
Fudan University,
No.2005 Songhu Road,
Shanghai, 200438, China.

Email address: wanghaining1121@outlook.com