The Cauchy problem for a higher order shallow water type equation on the circle

Wei Yan
Department of Mathematics and information science, Henan Normal University,
Xinxiang, Henan 453007, P.R.China
Email: yanwei19821115@sina.cn

Yongsheng Li
Department of Mathematics, South China University of Technology
Email: Guangzhou, Guangdong 510640, P. R. China

and

Jianhua Huang
College of Science, National University of Defense and Technology,
Changsha, P. R. China 410073

Abstract. In this paper, we investigate the Cauchy problem for a higher order shallow water type equation

\[ u_t - u_{txx} + \partial_x^{2j+1} u - \partial_x^{2j+3} u + 3uu_x - 2ux_{xx} - uu_{xxx} = 0, \]

where \( x \in \mathbb{T} = \mathbb{R}/2\pi \) and \( j \in \mathbb{N}^+ \). Firstly, we prove that the Cauchy problem for the shallow water type equation is locally well-posed in \( H^s(\mathbb{T}) \) with \( s \geq -\frac{j-2}{2} \) for arbitrary initial data. By using the I-method, we prove that the Cauchy problem for the shallow water type equation is globally well-posed in \( H^s(\mathbb{T}) \) with \( \frac{2j+1-j^2}{2j+1} < s \leq 1 \). Our results improve the result of A. A. Himonas, G. Misiolek (Communications in partial Differential Equations, 23(1998), 123-139; Journal of Differential Equations, 161(2000), 479-495.)

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Short Title: Cauchy problem for a shallow water type equation

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1. Introduction

In this paper, we consider the Cauchy problem for a higher order shallow water type equation

\[ u_t - u_{txx} + \partial_x^{2j+1} u - \partial_x^{2j+3} u + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \quad (1.1) \]

\[ u(x,0) = u_0(x), \quad x \in T = \mathbb{R}/2\pi, \quad (1.2) \]

which is considered as the higher modification of the Camassa-Holm equation. Rewrite (1.1) as follows:

\[ u_t + \partial_x^{2j+1} u + \frac{1}{2} \partial_x(u^2) + \partial_x(1 - \partial_x^2)^{-1} \left[ u^2 + \frac{1}{2} u_x^2 \right] = 0, \quad (1.3) \]

which was derived by Camassa and Holm as a nonlinear model for water wave motion in shallow channels with the aid of an asymptotic expansion directly in the Hamiltonian for Euler equations [6, 8]. Omitting the last term yields

\[ u_t + \partial_x^{2j+1} u + \frac{1}{2} \partial_x(u^2) = 0. \quad (1.4) \]

When \( j = 1 \), equation (1.1) reduces to the Korteweg-de Vries (KdV) equation

\[ u_t + u_{xxx} + \frac{1}{2} \partial_x(u^2) = 0. \quad (1.5) \]

Kenig et al. [21, 22] proved that \( s = -3/4 \) is the critical Sobolev index for the KdV equation in real line and proved that the Cauchy problem for the periodic KdV equation is locally well-posed in \( H^s(0, 2\pi \lambda) \) with \( s \geq -\frac{1}{2} \) and \( \lambda \geq 1 \). Bourgain [4] proved that the Cauchy problem for the periodic KdV equation is ill-posed in \( H^s(0, 2\pi \lambda) \) with \( s < -\frac{1}{2} \) and \( \lambda \geq 1 \). Colliander et al. [7] proved that the Cauchy problem for the periodic KdV equation is globally well-posed in \( H^s(0, 2\pi \lambda) \) with \( s \geq -\frac{1}{2} \) and \( \lambda \geq 1 \). Kappeler and Topalov [17, 18] proved the global well-posedness of the KdV and the defocusing mKdV equations in \( H^s(0, 2\pi \lambda) \) for respectively \( s \geq -1 \) and \( s \geq 0 \) and \( \lambda \geq 1 \) with a solution-map which is continuous from \( H^{-1}(0, 2\pi \lambda)( L^2(0, 2\pi \lambda)) \) into \( C(R; H^{-1}(0, 2\pi \lambda)) \) \((C(R; L^2(0, 2\pi \lambda))\)) with \( \lambda \geq 1 \). Molinet [25, 27] proved that the Cauchy problem for the periodic KdV equation is ill-posed in \( H^s(0, 2\pi \lambda) \) with \( s < -1 \) and \( \lambda \geq 1 \) in the sense that the solution-map associated with the KdV equation is discontinuous for the \( H^s(T) \) topology for \( s < -1 \).
Lots of people have investigated the Cauchy problem for (1.3), for instance, see [5, 6, 8, 11–13, 19, 20, 24, 26, 30–32]. Himonas and Misiolek [11] proved that the Cauchy problem for (1.1) is locally well-posed for small initial data in $H^s(T)$ with $s \geq \frac{2-j}{2}$ and globally well-posed in $H^1(T)$. Himonas and Misiolek [12] proved that the Cauchy problem for (1.1) with $j = 1$ is locally well-posed for arbitrary initial data in $H^s(T)$ with $s \geq \frac{2}{3} - j_2$ and globally well-posed in $H^{1/2}(T)$. Gorsky [10] proved that the Cauchy problem for (1.1) with $j = 1$ is locally well-posed in $H^1(T)$ for small initial data. Li and Yang [26] prove that the Cauchy problem for (1.1) with $j = 1$ is locally well-posed in $H^s(T)$ for $1/2 < s < 1$ and globally well-posed in $H^s(T)$ for $2/3 < s < 1$ with the aid of $I$-method. Olson [20] proved that the Cauchy problem for (1.1) is locally well-posed in $H^s(\mathbb{R})$ with $s > s'$, where $1/4 \leq s' < 1/2$. Yan et.al [24] prove that the Cauchy problem for (1.1) is locally well-posed in $H^s(\mathbb{R})$ with $s > -j + \frac{5}{4}$ and is globally well-posed in $H^1(\mathbb{R})$. Yan et al [31] prove that the Cauchy problem for (1.1) is locally well-posed in $H^s(\mathbb{R})$ with $s = -j + \frac{5}{4}$, $j \geq 2$, $j \in \mathbb{N}^+$ and ill-posed in $\dot{H}^s(\mathbb{R})$ with $s < -j + \frac{5}{4}$.

In this paper, by establishing some bilinear estimates and the fixed point Theorem, we prove that the Cauchy problem for (1.1) is locally well-posed in $H^s(T)$ with $s \geq \frac{2-j}{2}$; by using the $I$-method, we prove that the problem is globally well-posed in $H^s(T)$ with $\frac{2j+1-j^2}{2j+1} < s \leq 1$.

We give some notations before stating the main results. $0 < \epsilon < \frac{1}{1000(2j+1)}$ and $\epsilon' = \frac{1}{100(2j+1)}$. $C$ is a positive constant which may vary from line to line. $A \sim B$ means that $|B| \leq |A| \leq 4|B|$. $A \gg B$ means that $|A| \geq 4|B|$. $a \lor b = \max\{a, b\}$. $a \land b = \min\{a, b\}$. Let $\eta(t)$ the smooth function supported in $[-1, 2]$ and equals to 1 in $[0, 1]$. Let $\Psi \in C^\infty_0(\mathbb{R})$ be an even function such that $\Psi \geq 0$, supp $\Psi \subset [-\frac{3}{2}, \frac{3}{2}]$, $\Psi = 1$ on $[-\frac{5}{4}, \frac{5}{4}]$ and $v_k = \Psi(2^{-k}\xi) - \Psi(2^{-k+1}\xi)$.

For $k = k_1 + k_2$, we define

$$|k_{\min}| = \{|k|, |k_1|, |k_2|\}, \quad |k_{\max}| = \{|k|, |k_1|, |k_2|\}.$$ 

Throughout this paper, $\hat{Z} := Z - \{0\}$ and $\hat{Z}^+ := Z^+ - \{0\}$. Denote $dk$ by the normalized counting measure on $\hat{Z}$:

$$\int a(k)dk = \sum_{k \in \hat{Z}} a(k).$$
Denote $\mathcal{F}_x f$ by the Fourier transformation of a function $f$ defined on $[0, 2\pi]$ with respect to the space variable

$$\mathcal{F}_x f(k) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ikx} f(x) dx.$$ 

and we have the Fourier inverse transformation formula

$$f(x) = \int e^{ikx} \mathcal{F}_x f(k) dk = \sum_{k \in \hat{Z}} e^{ikx} \mathcal{F}_x f(k).$$

Denote $\mathcal{F}_t f$ by the Fourier transformation of a function $f$ with respect to the time variable

$$\mathcal{F}_t f(\tau) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\tau} f(t) dt.$$ 

and we have the Fourier inverse transformation formula

$$f(t) = \int e^{it\tau} \mathcal{F}_t f(\tau) d\tau.$$ 

We define

$$S(t)\phi(x) = \int e^{ikx} e^{ikt^{2}+1} \mathcal{F}_x \phi(k) dk.$$ 

We define the space-time Fourier transform $\mathcal{F} f(k, \tau)$ for $k \in \hat{Z}$ and $\tau \in \mathbb{R}$ by

$$\mathcal{F} f(k, \tau) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} e^{-ikx} e^{-it\tau} f(x, t) dx dt$$

and this transformation is inverted by

$$v(x, t) = \int \int e^{ikx} e^{it\tau} \mathcal{F} f(k, \tau) dk d\tau.$$ 

We define

$$\mathcal{F}_x [J^*_s \phi] (k) = \langle k \rangle^s \mathcal{F}_x \phi(k), \mathcal{F}_t [J^*_t \phi] (\tau) = \langle \tau \rangle^s \mathcal{F}_x \phi(\tau).$$

Thus, by using the above definitions, we have that

$$\|f\|_{L^2([0, 2\pi])} = \|\mathcal{F}_x f\|_{L^2(\hat{Z})},$$

$$\int_{0}^{2\pi} f(x)g(x) dx = \int \mathcal{F}_x f(k) \overline{\mathcal{F}_x f(k)} dk,$$

$$\mathcal{F}_x (fg) = \mathcal{F}_x f * \mathcal{F}_x g = \int \mathcal{F}_x f(k - k_1) \mathcal{F}_x g(k_1) dk_1.$$
Let 

\[ P(k) = k^{2j+1}, \sigma = \tau + P(k), \quad \sigma_l = \tau_l + P(k_l), \quad l = 1, 2. \]

For \( s < 1 \), we define

\[ \mathcal{F}_x I u(k) = m(k) \mathcal{F}_x u(k), \]

where \( m(k) = \left( \frac{|k|}{N} \right)^{1-s} \) if \( |k| > 2N \), \( m(k) = 1 \) if \( |k| \leq N \). We define the Sobolev space \( H^s(0, 2\pi) \) with the norm

\[ \| f \|_{H^s(T)} = \| \mathcal{F}_x f(k) \langle k \rangle^s \|_{L^2(k)} \]

and define the \( X_{s, b} \) spaces for \( 2\pi \)-periodic KdV via the norm

\[ \| u \|_{X_{s, b}(T \times \mathbb{R})} = \| \langle k \rangle^s \langle \tau + P(k) \rangle^b \mathcal{F} u(k, \tau) \|_{L^2(k \tau)}. \]

and define the \( Y_s \) space defined via the norm

\[ \| u \|_{Y_s} = \| u \|_{X_{s, \frac{1}{2}}} + \| \langle k \rangle^s \mathcal{F} u(k, \tau) \|_{L^2(k \tau) L^1(\tau)}. \]

and define the \( Z_s \) space defined via the norm

\[ \| u \|_{Z_s} = \| u \|_{X_{s, -\frac{1}{2}}} + \| \langle k \rangle^s \mathcal{F} u(k, \tau) \|_{L^2(k \tau) L^{\frac{1}{2}}(\tau)}. \]

We define

\[ \| u \|_{X_{s, b}^\delta} = \inf \left\{ \| v \|_{X_{s, b}}, \quad v|_{[0, \delta]} = u \right\}, \]

\[ \| u \|_{Y_s^\delta} = \inf \left\{ \| v \|_{Y_s}, \quad v|_{[0, \delta]} = u \right\}. \]

The main result of this paper are as follows.

**Theorem 1.1.** Let \( s \geq -\frac{j^2}{2} \) and \( u_0 \) be \( 2\pi \)-periodic function and zero \( x \)-mean. Then the Cauchy problems (1.1)(1.2) are locally well-posed in \( H^s(T) \).

**Theorem 1.2.** Let \( \frac{2j+1-j^2}{2j+1} < s \leq 1 \) and \( u_0 \) be \( 2\pi \)-periodic function and zero \( x \)-mean. Then the Cauchy problem (1.1)(1.2) is globally well-posed in \( H^s(T) \). More precisely, for any \( T > 0 \), let \( u_0 \) be \( 2\pi \)-periodic function and zero \( x \)-mean, then the Cauchy problems (1.1)(1.2) are globally well-posed on \([0, T]\) in \( H^s(T) \) with \( \frac{2j+1-j^2}{2j+1} < s \leq 1 \). Moreover,

\[ \sup_{t \in [0, T]} \| u(\cdot, t) \|_{H^s} \leq C T^{\frac{1-s}{j(1-s)}} \| u_0 \|_{H^s(T)}^{\frac{j}{j(1-s)}} \| u_0 \|_{H^s(T)}^{\frac{-j}{j(1-s)}}, \quad (1.6) \]
where
\[ f(j) = \frac{(2j + 1)}{j - 3(2j + 1)\epsilon}. \]

The rest of the paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we establish the bilinear estimate. In Section 4, we give the proof of Theorem 1.1. In Section 5, we give the proof of Theorem 1.2.

2. Preliminaries

In this section, we make some preliminaries which are crucial in establishing the Theorem 1.1.

Lemma 2.1. Let \( u_l \) with \( l = 1, 2 \) be \( L^2(\mathbb{Z} \times \mathbb{R}) \)-real valued functions. Then for any \((l_1, l_2) \in \mathbb{N}^2\)
\[
\|(\Psi_{l_1} u_1) \ast (\Psi_{l_2} u_2)\|_{L^2_{xt}} \leq C (2^{l_1} \wedge 2^{l_2})^{1/2} (2^{l_1} \vee 2^{l_2})^{\frac{1}{2(j+1)}} \|\Psi_{l_1} u_1\|_{L^2} \|\Psi_{l_2} u_2\|_{L^2}. \tag{2.1}
\]

Proof. As the proof of [4, 28], we can assume that \( \text{supp } u_l \subset \left\lbrace (\tau, k) \in \mathbb{R} \times \mathbb{Z}^+ \right\rbrace \). By using the Cauchy-Schwarz in \((\tau_1, k_1)\), we have that
\[
\|(\Psi_{l_1} u_1) \ast (\Psi_{l_2} u_2)\|^2_{L^2} = \int_{\mathbb{R}} \sum_{\tau \in \mathbb{R}} \int_{\mathbb{Z}} \sum_{k_1 \in \mathbb{Z}} (\Psi_{l_1} u_1)(\tau_1, k_1)(\Psi_{l_2} u_2)(\tau - \tau_1, k - k_1) d\tau_1 \left| d\tau \right|^2 d\tau
\]
\[
\leq C \alpha(\tau, k) \int_{\mathbb{R}} \sum_{\tau_1 \in \mathbb{Z}} \sum_{k_1 \in \mathbb{Z}} \left| (\Psi_{l_1} u_1)(\tau_1, k_1)(\Psi_{l_2} u_2)(\tau_1, k_1) \right|^2 d\tau_1 d\tau
\]
\[
\leq C \sup_{\tau \in \mathbb{R}, k \in \mathbb{Z}} \alpha(\tau, k) \|\Psi_{l_1} u_1\|^2_{L^2} \|\Psi_{l_2} u_2\|^2_{L^2}, \tag{2.2}
\]
where
\[
\alpha(\tau, k) \leq C \# \Lambda_1(\tau, k),
\]
here
\[
\Lambda_1(\tau, k) = \left\lbrace (\tau_1, k_1) \in \mathbb{R} \times \mathbb{Z}^+ / k_1 \in \mathbb{Z}^+, \langle \sigma_1 \rangle \sim 2^{l_1}, \langle \sigma_2 \rangle \sim 2^{l_2} \right\}.
\]
For fixed $\tau, \xi \neq 0$, We define $M' = \tau + (-1)^j \frac{k^{2j+1}}{4^j}$ and let $E_1$ and $E_2$ be the projections of $\Lambda_1$ onto the $k_1$-axis and $\tau_1$-axis, respectively. It is easily checked that

\[
\left(\tau + (-1)^n \frac{k^{2j+1}}{4^j}\right) - \left(\tau_1 + (-1)^j k_1^{2j+1}\right) - \left(\tau_2 + (-1)^j k_2^{2j+1}\right)
\]

\[
= (-1)^j \left[ k_1^{2j+1} + k_2^{2j+1} - \frac{k^{2j+1}}{4^j} \right]
\]

\[
= (-1)^j k \left( k_1 - \frac{k}{2} \right)^2 F(k, k_1),
\]

where

\[
F(k, k_1) = C_{2j+1}^2 \left( \frac{1}{2} \right)^{2j-2} k^{2j-2} + C_{2j+1}^4 \left( \frac{1}{2} \right)^{2j-2} k^{2j-4} \left( k_1 - \frac{k}{2} \right)^2 + \cdots + C_{2j+1}^{2j} \left( k_1 - \frac{k}{2} \right)^{2j-2}.
\]

From (2.3), we have that there exist two constant $C_1, C_2 > 0$ such that

\[
\frac{|C_1 (2^{l_1} + 2^{l_2}) - M'|}{|kF(k, k_1)|} \leq \frac{3}{4} (k_1 - k_2)^2 \leq \frac{|C_2 (2^{l_1} + 2^{l_2}) + M'|}{|kF(k, k_1)|}.
\]

(2.4)

When $k^{2j+1} > 2^{l_1} \lor 2^{l_2}$, from (2.4), we have that

\[
\# E_2 \leq \mes E_2 + 1 \leq 2 \left[ \frac{|C_1 (2^{l_1} + 2^{l_2}) + M'|}{|kF(k, k_1)|} - \frac{|C_2 (2^{l_1} + 2^{l_2}) - M'|}{|kF(k, k_1)|} \right]^{1/2} + 1
\]

\[
\leq C \left( \frac{2^{l_1} \lor 2^{l_2}}{|k^{2j+1}|} \right)^{1/2} + 1 \leq C (2^{l_1} \lor 2^{l_2})^{\frac{1}{2j+1}}.
\]

(2.5)

When $0 \leq k^{2j+1} \leq 2^{l_1} \lor 2^{l_2}$, since $0 \leq k_1 \leq k$, we have that

\[
\# E_2 \leq \# \{k_1, \ 0 \leq k_1^{2j+1} \leq 2^{l_1} \lor 2^{l_2}\} \leq C (2^{l_1} \lor 2^{l_2})^{\frac{1}{2j+1}}.
\]

(2.6)

From (2.2), it is easily checked that

\[
\# E_1 \leq \mes E_1 + 1 \leq C (2^{l_1} \land 2^{l_2}).
\]

(2.7)

Combining (2.2) with (2.5)-(2.7), we have that

\[
\| (\Psi_{l_1} u_1) \ast (\Psi_{l_2} u_2) \|_{L^2} \leq C (2^{l_1} \land 2^{l_2})^{1/2} (2^{l_1} \lor 2^{l_2})^{\frac{1}{2j+1}} \| \Psi_{l_1} u_1 \|_{L^2} \| \Psi_{l_2} u_2 \|_{L^2}.
\]

(2.8)

We have completed the proof of Lemma 2.1.

**Lemma 2.2.** Let $v(x, t)$ be a $2\pi$-periodic function. Then

\[
\| v \|_{L^4_{x,t}} \leq C \| v \|_{X^0_{\frac{(l+1)}{2(2j+1)}, (T \times \mathbb{R})}}.
\]

(2.9)
**Proof.** By using the triangle inequality, let \( l_1 = l + l_2 \) with \( l \in N \), by using \((2.1)\), we have that

\[
\|v\|_{L_{t,x}^2}^2 = \|v\|_{L_x^2}^2 = \|F v \ast F v\|_{L_t^2} \leq \sum_{l_1 \geq 0} \sum_{l_2 \geq 0} \|\Psi_{l_1} F v \ast \Psi_{l_2} F v\|_{L_t^2}
\]

\[
\leq C \sum_{l_1 \geq 0} \sum_{l_2 \geq 0} \|\Psi_{l_1} F v \ast \Psi_{l_2} F v\|_{L_t^2}
\]

\[
\leq C \sum_{l_1 \geq 0} \sum_{l_2 \geq 0} 2^{l_1/2} 2^{(l_2+l)/2(2j+1)} \|\Psi_{l_1+t} F v\|_{L_t^2} \|\Psi_{l_2} F v\|_{L_t^2}
\]

\[
\leq C \sum_{l_1 \geq 0} \sum_{l_2 \geq 0} 2^{l_1/2} 2^{l_2/2(2j+1)} \sum_{l_2 \geq 0} 2^{l_1/2} \left( \sum_{l_2 \geq 0} 2^{l_1/2} \|\Psi_{l_2} F v\|_{L_t^2}^2 \right)^{1/2} \left( \sum_{l_2 \geq 0} 2^{l_1/2} \|\Psi_{l_2} F v\|_{L_t^2}^2 \right)^{1/2}
\]

\[
\leq C \|v\|_{X_{0, (2j+1)} \times (0, 2\pi \times R)}^2 \lesssim \left( \int_0^{2\pi} v^{4/3} (x, t) dx \right)^{3/4}
\]

From \((2.10)\), we have \((2.9)\).

We have completed the proof of Lemma 2.2.

**Remark:** In line -3 of page 493 in [12], Himonas and Misiolek presented the conclusion of Lemma 2.2, however, the proof process is not given.

**Lemma 2.3.** Let \( v(x, t) \) be a \( 2\pi \)-periodic function. Then

\[
\|v\|_{X_{0, (2j+1)} \times (0, 2\pi \times R)} \leq C \|v\|_{L_{t,x}^{4/3}} = \left( \int_0^{2\pi} v^{4/3} (x, t) dx \right)^{3/4}
\]

**Proof.** Combining the Lemma 2.2 with the duality, we have Lemma 2.3.

**Lemma 2.4.** Let

\[
k = k_1 + k_2, \tau = \tau_1 + \tau_2,\]

\[
\sigma = \tau + (-1)^j k_2^{2j+1}, \sigma_l = \tau_l + (-1)^j k_l^{2j+1}, l = 1, 2.
\]

Then

\[
3\max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \leq |\sigma - \sigma_1 - \sigma_2| = |k_2^{2j+1} - k_1^{2j+1} - k_2^{2j+1}| \sim |k_{\min}| |k_{\max}|^{2j}.
\]

For the proof of Lemma 2.4, we refer the readers to Lemma 2.5 in [31].
Lemma 2.5. For $k \in \hat{Z}$, $k_j \in \hat{Z}(j = 1, 2)$ and dyadic $M \geq 1$ and $\epsilon' = \frac{1}{100(2^j + 1)}$, we have that

$$\text{mes}\left\{ \mu \in \mathbb{R} : \mid \mu \mid \sim M, \mu = k^{2j+1} - k_1^{2j+1} - k_2^{2j+1} + O((\mid k_{\min} \mid k_{\max}^{2j})^{\epsilon'}) \right\} \leq CM\frac{100(2^j+1)}{2^{2j}}.$$  

(2.12)

\textbf{Proof.} Without loss of generality, we can assume that $|k_1| \geq |k_2|$. When $|k| \geq |k_1|$ which yields that $|k_1| \leq |k| \leq 2|k_1|$, from

$$\mu = k^{2j+1} - k_1^{2j+1} - k_2^{2j+1} + O((|k_{\min}|k_{\max}^{2j})^{\epsilon'}),$$

(2.13)

we have that $C_1|k|^{2j} \leq |\mu| \leq C_2|k|^{2j+1}$ since $k_1, k_2 \in \hat{Z}$. Thus, we have that $|\mu| \sim |k|^{p}, p \in [2j, 2j + 1]$. Thus, $|k_1^{2j-1}k_2| \sim M^{1-\frac{1}{p}}, p \in [2j, 2j + 1]$. Consequently, we have that

$$\text{mes}\left\{ \mu \in \mathbb{R} : \mid \mu \mid \sim M, \mu = k^{2j+1} - k_1^{2j+1} - k_2^{2j+1} + O((\mid k_{\min} \mid k_{\max}^{2j})^{\epsilon'}) \right\} \leq CM\frac{100(2^j+1)}{2^{2j}p}\leq CM\frac{100(2^j+1)}{2^{2j}p}.$$  

(2.14)

When $|k_1| \geq |k|$, from (2.13), we have that $C_1|k_1|^{2j} \leq |\mu| \leq C_2|k_1|^{2j+1}$ since $k_1, k_2 \in \hat{Z}$. Thus, we have that $|\mu| \sim |k_1|^p, p \in [2j, 2j + 1]$. Thus, $|k_1^{2j-1}k_2| \sim M^{1-\frac{1}{p}}, p \in [2j, 2j + 1]$. Consequently, we have that

$$\text{mes}\left\{ \mu \in \mathbb{R} : \mid \mu \mid \sim M, \mu = k^{2j+1} - k_1^{2j+1} - k_2^{2j+1} + O((\mid k_{\min} \mid k_{\max}^{2j})^{\epsilon'}) \right\} \leq CM\frac{100(2^j+1)}{2^{2j}p}\leq CM\frac{100(2^j+1)}{2^{2j}p}.$$  

(2.15)

We have completed the proof of Lemma 2.5.

Lemma 2.6. Let $\phi$ be $2\pi$-periodic function. Then

$$\|\eta(t)S(t)\phi\|_{Y^{s}_{\delta}} \leq C\|\phi\|_{H^{s}}.$$  

(2.16)

\textbf{Proof.} To obtain (2.16), it suffices to prove that

$$\left\|\eta(t)\eta\left(\frac{t}{\delta}\right)S(t)\phi\right\|_{Y^{s}_{\delta}} \leq C\|\phi\|_{H^{s}}.$$  

(2.17)

From Lemma 7.1 of [7], we have that

$$\left\|\eta(t)\eta\left(\frac{t}{\delta}\right)S(t)\phi\right\|_{Y^{s}_{\delta}} \leq C\|\eta\left(\frac{t}{\delta}\right)\phi\|_{H^{s}} \leq C\|\phi\|_{H^{s}}.$$  

(2.18)

We have completed the Lemma 2.6.
Lemma 2.7. Let $F$ be $2\pi$-periodic function. Then
\[
\left\| \eta(t) \int_0^t S(t-\tau)F(\tau)d\tau \right\|_{Y^\delta_s} \leq C \|\eta\left(\frac{t}{\delta}\right) F\|_{Z_s}.
\] (2.19)

Proof. To obtain (2.19), it suffices to prove that
\[
\left\| \eta(t) \int_0^t S(t-\tau)F(\tau)d\tau \right\|_{Y^\delta_s} \leq C \left\| \eta\left(\frac{t}{\delta}\right) F \right\|_{Z_s}
\] (2.20)
which follows from Lemma 7.2 of [7].

We have completed the proof of Lemma 2.7.

Lemma 2.8. Let
\[
\Omega(k) = \{ \mu \in \mathbb{R} : |\mu| \sim M, \mu = k_1^{2j+1} - k_2^{2j+1} - k_2^{2j+1} + O\left(|k_{\min}|k_{\max}^{2j+i}\right) \}
\]
Then
\[
\int (\mu)^{-1} \chi_{\Omega(k)}(\mu) d\mu \leq C.
\] (2.21)

Proof. Combining Lemma 2.6 with the proof of page 737 in [7], we have Lemma 2.10.

Lemma 2.9. Let $s \in \mathbb{R}$ and $\delta \in (0,1)$, then for $-\frac{1}{2} < b < b' \leq 0$ or $0 \leq b < b' < \frac{1}{2}$, we have that
\[
\left\| \eta\left(\frac{t}{\delta}\right) u \right\|_{X^{\delta,s}_{0,b}} \leq C\delta^{b-b'} \|u\|_{X^{\delta,s}_{0,b'}}.
\] (2.22)

For the proof of Lemma 2.9, we refer the readers to Lemma 1.10 of [10].

Lemma 2.10. For $u \in X^{\delta,s}_{\sigma,b}$ there exists $\tilde{u}$ with $u|_{[0,\delta]} = \tilde{u}$, such that for $s \leq \sigma$, we have that
\[
\|u\|_{X^{\delta,s}_{\sigma,b}} = \|\tilde{u}\|_{X^{\delta,s}_{\sigma,b}}.
\]

For the proof of Lemma 2.10, we refer the readers to Lemma 1.6 of [10].

Lemma 2.11. Let $s \in \mathbb{R}$ and $0 < \epsilon < \frac{1}{10000(2j+1)}$ and
\[
F(k,\tau) = \langle k \rangle^s \langle \sigma \rangle^{1/2} \mathcal{F} \left( \eta\left(\frac{t}{\delta}\right) \tilde{u} \right) (k,\tau),
\] (2.23)
where $F \in L^2$. Then
\[
\left\| \mathcal{F}^{-1} \left( \frac{F}{\langle \sigma \rangle^{1/2}} \right) \right\|_{L^4} \leq C\delta^{\frac{1}{2(2j+1)} - \epsilon} \|F\|_{L^2}.
\] (2.24)
Proof. From (2.23) and Lemmas 2.2, 2.9, we have that
\[
\left\| \mathcal{F}^{-1} \left( \frac{F}{\langle \sigma \rangle^{1/2}} \right) \right\|_{L^4} = \left\| \eta \left( \frac{t}{\delta} \right) J_x^s \tilde{u} \right\|_{L^4}
\]
\[
\leq C \left\| \eta \left( \frac{t}{\delta} \right) J_x^s \tilde{u} \right\|_{X_0, \frac{j+1}{2j+1}}
\]
\[
\leq C \delta^{\frac{j}{2j+1}} \epsilon \left\| \eta \left( \frac{t}{\delta} \right) J_x^s \tilde{u} \right\|_{X_0, \frac{j}{2j+1}}
\]
\[
\leq C \delta^{\frac{j}{2j+1}} \epsilon \left\| \eta \left( \frac{t}{\delta} \right) \tilde{u} \right\|_{L^4}
\]
\[
= C \delta^{\frac{j}{2j+1}} \epsilon \left\| F \right\|_{L^2}.
\]
(2.25)

We have completed the proof of Lemma 2.11.

Remark: Lemma 2.11 improves the result of Lemma 3.2 in [12] with \( \mu = 2j + 1 \).

Lemma 2.12. Let
\[
\sigma = \tau + (-1)^j k^{2j+1}, \sigma_l = \tau_l + (-1)^j k_l^{2j+1}, l = 1, 2.
\]
and \( s \in \mathbb{R} \) and \( 0 < \epsilon < \frac{1}{10000(2j+1)} \) and
\[
G_l(k_l, \tau_l) = \langle k_l \rangle^s \langle \sigma_l \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u}_l \right)(k_l, \tau_l), l = 1, 2,
\]
(2.26)
where \( G_l \in L^2, l = 1, 2. \) Then
\[
\left\| \langle \sigma \rangle^{-\frac{1}{2} + \epsilon} \int_{k = k_1 + k_2, \tau = \tau_1 + \tau_2} \prod_{l=1}^2 G_l(k_l, \tau_l) dk_1 d\tau_1 \right\|_{L^2} \leq C \delta^{\frac{j}{2j+1}} \epsilon \prod_{l=1}^2 \left\| G_l \right\|_{L^2}.
\]
(2.27)
Proof. By using Lemmas 2.3, 2.4.2.11, we have that

\[
\left\| \left( \sigma^{-\frac{3}{4}+\epsilon} \int_{k=k_1+k_2}^{k_1+k_2} \prod_{l=1}^{2} G_l(k_l, \tau_l) \frac{d k_l d \tau_l}{(\sigma_2)^{1/2}} \right) \right\|_{L^2} \\
= \left\| \left( \eta \left( \frac{t}{\delta} \right) J_x^s \tilde{u}_2 \right) \mathcal{F}^{-1}(G_1) \right\|_{X_{0, -\frac{3}{4}+\epsilon}} \\
\leq C \delta^{\frac{3}{2(j+1)}-\epsilon} \left\| \left( \eta \left( \frac{t}{\delta} \right) J_x^s \tilde{u}_2 \right) \mathcal{F}^{-1}(G_1) \right\|_{X_{0, -\frac{3}{2(j+1)}}} \\
\leq C \delta^{\frac{3}{2(j+1)}} \left\| \mathcal{F}^{-1}(G_1) \right\|_{L^2} \left\| J_x^s \eta \left( \frac{t}{\delta} \right) \tilde{u}_2 \right\|_{L^4_{x,t}} \\
\leq C \delta^{\frac{3}{2(j+1)}} \left\| \mathcal{F}^{-1}(G_1) \right\|_{L^2} \left\| J_x^s \eta \left( \frac{t}{\delta} \right) \tilde{u}_2 \right\|_{L^4_{x,t}} \\
\leq C \delta^{\frac{3}{2(j+1)}-2\epsilon} \left\| \mathcal{F}^{-1}(G_1) \right\|_{L^2} \left\| J_x^s \eta \left( \frac{t}{\delta} \right) \tilde{u}_2 \right\|_{X_{0, \frac{1}{2}-\epsilon}} \\
\leq C \delta^{\frac{3}{2(j+1)}-2\epsilon} \left\| \mathcal{F}^{-1}(G_1) \right\|_{L^2} \left\| J_x^s \eta \left( \frac{t}{\delta} \right) \tilde{u}_2 \right\|_{X_{0, \frac{1}{2}}} \\
\leq C \delta^{\frac{3}{2(j+1)}-2\epsilon} \prod_{l=1}^{2} \left\| G_l \right\|_{L^2}.
\]

We have completed the proof of Lemma 2.12.

3. Bilinear estimates

In this section, we establish some important bilinear estimates which are the core of this paper.

Lemma 3.1. Let \( u_l(x, t) \) with \( l = 1, 2 \) which are zero \( x \)-mean for all \( t \) be \( 2\pi \)-periodic functions of \( x \) and \( s \geq \frac{2-j}{2} \). For \( \epsilon < \frac{1}{10000(2j+1)} \), then we have that

\[
\left\| \partial_x (1 - \partial_x^2)^{-1} \left[ \prod_{l=1}^{2} \partial_x \eta \left( \frac{t}{\delta} \right) u_l \right] \right\|_{X_{s,-\frac{1}{2}}} \leq C \delta^{\frac{3}{2(j+1)}-2\epsilon} \prod_{l=1}^{2} \left\| u_l \right\|_{X_{s, \frac{1}{2}}}.
\]

Proof. Let \( \tilde{u} \) and \( \tilde{u}_1, \tilde{u}_2 \) be the extension of \( u, u_1, u_2 \), respectively, according to Lemma 2.10, we have that

\[
\left\| u \right\|_{X_{s, \frac{1}{2}}} = \left\| \tilde{u} \right\|_{X_{s, \frac{1}{2}}}, \left\| u_l \right\|_{X_{s, \frac{1}{2}}} = \left\| \tilde{u}_l \right\|_{X_{s, \frac{1}{2}}}, \quad l = 1, 2.
\]

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By duality and the Plancherel identity, for \( u \in X_{-s, \frac{1}{2}}^\delta \), to obtain (3.1), it suffices to prove that
\[
\int_{k = k_1 + k_2} \left| \frac{kk_1 k_2}{1 + k^2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u} \right) (k, \tau) \prod_{l=1}^2 \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u}_l \right) (k_1, \tau_1) \right| dk_1 d\tau_1 dk d\tau \\
\leq C \delta^{\frac{1}{2} - 2s} \| u \|_{X_{-s, \frac{1}{2}}^\delta} \prod_{l=1}^2 \| u_l \|_{X_{-s, \frac{1}{2}}^\delta} = C \delta^{\frac{1}{2} - 2s} \| u \|_{X_{-s, \frac{1}{2}}^\delta} \prod_{l=1}^2 \| \tilde{u}_l \|_{X_{s, \frac{1}{2}}^\delta}.
\]
(3.2)
Without loss of generality, we can assume that \( \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u} \right) (k, \tau) \geq 0(l = 1, 2) \) and \( \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u} \right) (k, \tau) \geq 0 \). Let
\[
F(k, \tau) = \langle k \rangle^{-s} \langle \sigma \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u} \right) (k, \tau),
\]
\[
F_l(k_l, \tau_l) = \langle k_l \rangle^s \langle \sigma_l \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \tilde{u}_l \right) (k_l, \tau_l), \quad l = 1, 2,
\]
\[
K_1(k_1, \tau_1, k, \tau) = \frac{|kk_1 k_2| \langle k \rangle^s}{(1 + k^2) \langle \sigma \rangle^{1/2} \prod_{l=1}^2 \langle k_l \rangle^s \langle \sigma_l \rangle^{1/2}}.
\]
To obtain (3.2), it suffices to prove that
\[
\int_{k = k_1 + k_2} K_1(k_1, \tau_1, k, \tau) F(k, \tau) \prod_{l=1}^2 F_l(k_l, \tau_l) dk_1 d\tau_1 dk d\tau \\
\leq C \delta^{\frac{1}{2} - 2s} \| F \|_{L^2} \prod_{l=1}^2 \| F_l \|_{L^2}.
\]
(3.3)
From the mean zero condition, we can assume that \( k \neq 0, k_l \neq 0(l = 1, 2) \).

Since \( \min \{|k|, |k_1|, |k_2|\} \geq 1 \), from Lemma 2.4, we have that one of the following three cases must occur:

(a) \(|\sigma| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j}\),

(b) \(|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j}\),

(c) \(|\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j}\).

When (a) \(|\sigma| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j}\), we have that
\[
K_1(k_1, \tau_1, k, \tau) = \frac{|kk_1 k_2| \langle k \rangle^s}{(1 + k^2) \langle \sigma \rangle^{1/2} \prod_{l=1}^2 \langle k_l \rangle^s \langle \sigma_l \rangle^{1/2}} \leq C \frac{|k|^{s - \frac{3}{2}} \prod_{l=1}^2 k_l^{2 - s}}{\prod_{l=1}^2 \langle \sigma_l \rangle^{1/2}}; \quad (3.4)
\]
if \( \frac{2 - s}{2} \leq s \leq \frac{3}{2} \), from (3.4), we have that
\[
K_1(k_1, \tau_1, k, \tau) \leq C \frac{1}{\prod_{l=1}^2 \langle \sigma_l \rangle^{1/2}}; \quad (3.5)
\]
if \( s \geq \frac{3}{2} \), since \( s \geq \frac{2-j}{2} \), we have that

\[
K_1(k_1, \tau_1, k, \tau) \leq C \frac{|k|^{s-\frac{3}{2}} [\max \{|k_1|, |k_2|\}]^{\frac{2-j}{2}-s} [\min \{|k_1|, |k_2|\}]^{\frac{2-j}{2}-s}}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}
\]

\[
\leq C \frac{[\max \{|k_1|, |k_2|\}]^{-\frac{1+j}{2}} [\min \{|k_1|, |k_2|\}]^{\frac{2-j}{2}-s}}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}
\]

\[
\leq \frac{C}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}};
\]

(3.6)

from (3.5)-(3.6), by using the Plancherel identity and the Hölder inequality as well as Lemma 2.11, we have that

\[
\int_{\mathbb{R}^{2+k}} \int_{k = k_1 + k_2} \frac{K_1(k_1, \tau_1, k, \tau) F(k, \tau) \prod_{l=1}^{2} F_l(k_l, \tau_l) dk_1 d\tau_1 dk d\tau}{\langle \sigma_l \rangle^{1/2}}
\]

\[
\leq C \int_{k = k_1 + k_2} \frac{F(k, \tau) \prod_{l=1}^{2} F_l(k_l, \tau_l) dk_1 d\tau_1 dk d\tau}{\langle \sigma_l \rangle^{1/2}}
\]

\[
\leq C \|F^{-1}(F)\|_{L^2} \prod_{l=1}^{2} \|F_l^{-1}(F_l)\|_{L^2}^{1/2}
\]

\[
\leq C \delta^{\frac{j}{1+j} - \epsilon} \|F\|_{L^2} \prod_{l=1}^{2} \|F_l\|_{L^2}^{1/2}.
\]

(3.7)

When \((b)\) : \(|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j} \), by using the proof similar to (3.5)-(3.6), we have that

\[
K_1(k_1, \tau_1, k, \tau) \leq \frac{C}{\langle \sigma_l \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}};
\]

(3.8)

by using the Cauchy-Schwarz inequality and Lemma 2.12, we have that

\[
\int_{\mathbb{R}^{2+k}} F(k, \tau) \left( \langle \sigma \rangle^{-1/2} \int_{k = k_1 + k_2} \frac{\prod_{l=1}^{2} F_l(k_l, \tau_l)}{\langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 \right) dk d\tau
\]

\[
\int_{\mathbb{R}^{2+k}} F(k, \tau) \left( \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} \int_{k = k_1 + k_2} \frac{\prod_{l=1}^{2} F_l(k_l, \tau_l)}{\langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 \right) dk d\tau
\]

\[
\leq C \|F(k, \tau)\|_{L^2} \left\| \langle \sigma \rangle^{-1/2+\epsilon} \int_{k = k_1 + k_2} \frac{\prod_{l=1}^{2} F_l(k_l, \tau_l)}{\langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 \right\|_{L^2_{\langle k \rangle}}
\]

\[
\leq C \delta^{\frac{j}{1+j} - \epsilon} \|F\|_{L^2} \prod_{l=1}^{2} \|F_l\|_{L^2}.
\]

When \((c)\) : \(|\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j} \), this case can be proved similarly to case \((b)\) : \(|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j} \).
We have completed the proof of Lemma 3.1.

**Lemma 3.2.** Let \( u_l(x, t) \) with \( l = 1, 2 \) which are zero \( x \)-mean for all \( t \) be \( 2\pi \)-periodic functions of \( x \) and \( s \geq -\frac{l}{2} \). For \( \epsilon < \frac{1}{10000(2j+1)} \), then we have that

\[
\left\| \partial_x \left[ \prod_{l=1}^{2} \eta \left( \frac{t}{\delta} \right) u_l \right] \right\|_{X^s_{l, \frac{1}{2}}} \leq C\delta^{\frac{j}{j+1}} 2^{\epsilon} \prod_{l=1}^{2} \| u_l \|_{X^s_{l, \frac{1}{2}}}.
\]

**Proof.** Let \( \tilde{u} \) and \( \tilde{u}_1, \tilde{u}_2 \) be the extension of \( u, u_1, u_2 \), respectively, according to Lemma 2.10, we have that

\[
\| u \|_{X^s_{l, \frac{1}{2}}} = \| \tilde{u} \|_{X^s_{l, \frac{1}{2}}}, \quad \| u_l \|_{X^s_{l, \frac{1}{2}}} = \| \tilde{u}_l \|_{X^s_{l, \frac{1}{2}}}, \quad l = 1, 2.
\]

By duality and the Plancherel identity, for \( u \in X^s_{-l, \frac{1}{2}} \), it suffices to prove that

\[
\int_{k = k_1 + k_2}^{k_1 + k_2} \| u \|_{X^s_{-l, \frac{1}{2}}} \left( \prod_{l=1}^{2} \| u_l \|_{X^s_{l, \frac{1}{2}}} \right) dk_1 d\tau_1 dk d\tau 
\]

\[\leq C\delta^{\frac{j}{j+1}} 2^{\epsilon} \prod_{l=1}^{2} \| u_l \|_{X^s_{l, \frac{1}{2}}} = C\delta^{\frac{j}{j+1}} 2^{\epsilon} \| u \|_{X^s_{l, \frac{1}{2}}}.
\]

Without loss of generality, we can assume that \( \mathcal{F} \left( \eta \left( \frac{4}{\delta} \right) \tilde{u}_l \right) (k_l, \tau_l) \geq 0(l = 1, 2) \) and \( \mathcal{F} \left( \eta \left( \frac{4}{\delta} \right) \tilde{u} \right) (k, \tau) \geq 0 \). Let

\[
F(k, \tau) = \langle k \rangle^{-s} \langle \sigma \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{4}{\delta} \right) \tilde{u} \right) (k, \tau),
\]

\[
F_l(k_l, \tau_l) = \langle k_l \rangle^{s} \langle \sigma_l \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{4}{\delta} \right) \tilde{u}_l \right) (k_l, \tau_l), \quad l = 1, 2,
\]

\[
K_2(k_1, \tau_1, k, \tau) = \frac{|k|^s}{\langle \sigma \rangle^{1/2} \prod_{l=1}^{2} \langle k_l \rangle^{s} \langle \sigma_l \rangle^{1/2}}.
\]

To obtain (3.10), it suffices to prove that

\[
\int_{k = k_1 + k_2}^{k_1 + k_2} K_2(k_1, \tau_1, k, \tau) F(k, \tau) \prod_{l=1}^{2} F_l(k_l, \tau_l) dk_1 d\tau_1 dk d\tau 
\]

\[\leq C\delta^{\frac{j}{j+1}} 2^{\epsilon} \| F \|_{L^2} \prod_{l=1}^{2} \| F_l \|_{L^2}.
\]

From the mean zero condition, we can assume that \( k \neq 0, k_l \neq 0(l = 1, 2) \). Since
min \{ |k|, |k_1|, |k_2| \} \geq 1$, from Lemma 2.4, we have that one of the following three cases

(a) : \( |\sigma| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C |k_{\min}| |k_{\max}|^{2j} \),

(b) : \( |\sigma_1| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C |k_{\min}| |k_{\max}|^{2j} \),

(c) : \( |\sigma_2| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C |k_{\min}| |k_{\max}|^{2j} \).

When (a) : \( |\sigma| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C |k_{\min}| |k_{\max}|^{2j} \), we have that

\[
K_2(k_1, \tau_1, k, \tau) = \frac{|k|^4}{\langle \sigma \rangle^{1/2} \prod_{l=1}^{2} (k_l)^s \langle \sigma_l \rangle^{1/2}} \leq C \frac{|k|^{s+\frac{1}{2}} \prod_{l=1}^{2} k_l^{-\frac{s}{2}}}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}; \tag{3.12}
\]

if \(-\frac{1}{2} \leq s \leq -\frac{1}{2}\), from (3.12), we have that

\[
K_2(k_1, \tau_1, k, \tau) \leq \frac{C}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}; \tag{3.13}
\]

if \(s \geq -\frac{1}{2}\), since \(s \geq -\frac{1}{2}\), we have that

\[
K_2(k_1, \tau_1, k, \tau) \leq C \frac{|k|^{s+\frac{1}{2}} \max \{ |k_1|, |k_2| \}^{-\frac{s}{2}} \min \{ |k_1|, |k_2| \}^{-\frac{s}{2}}}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} \leq C \frac{\max \{ |k_1|, |k_2| \}^{-\frac{s}{2}} \min \{ |k_1|, |k_2| \}^{-\frac{s}{2}}}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} \leq \frac{C}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}; \tag{3.14}
\]

from (3.13)-(3.14), by using the Plancherel identity and the Hölder inequality and Lemma 2.11, we have that

\[
\int_{\mathbb{R}^2} \int_{k_1 + k_2 \tau_1 + \tau_2} K_2(k_1, \tau_1, k, \tau) F(k, \tau) \prod_{l=1}^{2} F_l(k_l, \tau_l) dk_1 d\tau_1 dk d\tau \leq C \int_{k_1 + k_2 \tau_1 + \tau_2} \frac{F(k, \tau) \prod_{l=1}^{2} F_l(k_l, \tau_l)}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau \leq C \| \mathcal{F}^{-1}(F) \|_{L^2} \prod_{l=1}^{2} \left\| \mathcal{F}^{-1} \left( \frac{F_l}{\langle \sigma_l \rangle^{1/2}} \right) \right\|_{L^2} \leq C \delta^{-\frac{1}{2j+1}} \| F \|_{L^2} \prod_{l=1}^{2} \| F_l \|_{L^2}. \tag{3.15}
\]

When (b) : \( |\sigma_1| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C |k_{\min}| |k_{\max}|^{2j} \), by using the proof similar to (3.13)-(3.14), we have that

\[
K_2(k_1, \tau_1, k, \tau) \leq \frac{C}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}}; \tag{3.16}
\]
by using the Cauchy-Schwarz inequality and Lemma 2.12, we have that

$$\int_{\mathbb{R}^2} \int_{k = k_1 + k_2} K_2(k_1, \tau_1, k, \tau) F(k, \tau) \prod_{l=1}^2 F_l(k_l, \tau_l) dk_1 d\tau_1 dk d\tau$$

$$\leq C \int_{\mathbb{R}^2} F(k, \tau) \left( \mathcal{O}^{-1/2} \int_{k = k_1 + k_2} \prod_{l=1}^2 \frac{F_l(k_l, \tau_l)}{(\sigma^2)^{1/2}} dk_1 d\tau_1 \right) dk d\tau$$

$$\leq C \int_{\mathbb{R}^2} F(k, \tau) \left( \mathcal{O}^{-1/2} + \epsilon \int_{k = k_1 + k_2} \prod_{l=1}^2 \frac{F_l(k_l, \tau_l)}{(\sigma^2)^{1/2}} dk_1 d\tau_1 \right) dk d\tau$$

$$\leq C \|F\|_{L^2} \left( \mathcal{O}^{-1/2} + \epsilon \right) \int_{k = k_1 + k_2} \prod_{l=1}^2 \frac{F_l(k_l, \tau_l)}{(\sigma^2)^{1/2}} dk_1 d\tau_1 \right) dk d\tau$$

$$\leq C \delta_{2^{j+1}}^{-2\epsilon} \|F\|_{L^2} \prod_{l=1}^2 \|F_l\|_{L^2}.$$  

When \( (c) : |\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}||k_{\max}|^{2j} \), this case can be proved similarly to case \( (b) : |\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}||k_{\max}|^{2j}. \)

We have completed the proof of Lemma 3.2.

**Lemma 3.3.** Let \( u_l(x, t) \) with \( l = 1, 2 \) which are zero \( x \)-mean for all \( t \) be \( 2\pi \)-periodic functions of \( x \) and \( s \geq -\frac{i+2}{2} \). For \( \epsilon < \frac{1}{10000(2j+1)} \), then we have that

$$\left\| \partial_x (1 - \partial_x^2)^{-1} \left[ \prod_{l=1}^2 \left[ \eta \left( \frac{t}{\delta} \right) u_l \right] \right] \right\|_{X^{\delta, s}} \leq C \delta_{2^{j+1}}^{-2\epsilon} \prod_{l=1}^2 \|u_l\|_{X^{\delta, s}}. \quad (3.17)$$

Lemma 3.3 can be proved similarly to Lemma 3.2.

**Lemma 3.4.** Let \( v_l(x, t) \) with \( l = 1, 2 \) which are zero \( x \)-mean for all \( t \) be \( 2\pi \)-periodic functions of \( x \). For \( s \geq \frac{2j-2}{2} \), we have that

$$\left\| \int_{k = k_1 + k_2} \frac{|k_k| k_2 \langle k \rangle^s}{\langle \sigma \rangle (1 + k^2)} \prod_{l=1}^2 \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \bar{v}_l \right) (k_l, \tau_l) dk_1 d\tau_1 \right\|_{(L^2(\mathbb{R}^2) L^1(\tau))}$$

$$\leq C \delta_{2^{j+1}}^{-3\epsilon} \prod_{l=1}^2 \|v_l\|_{X^{\delta, s}}. \quad (3.18)$$

**Proof.** Let \( \bar{v}_1, \bar{v}_2 \) be the extension of \( v_1, v_2 \), respectively, according to Lemma 2.10, we have that

$$\|v_l\|_{X^{\delta, s}} = \|\bar{v}_l\|_{X^{\delta, s}}, \quad l = 1, 2.$$
Without loss of generality, we can assume that \( \mathcal{F} \left( \eta t^\delta \tilde{v}_l \right)(k_l, \tau_l) \geq 0 \) for \( l = 1, 2 \) and \( \mathcal{F} \left( \eta t^\delta \tilde{v} \right)(k, \tau) \geq 0 \). Let

\[
G_l(k_l, \tau_l) = \langle k_l \rangle^s \langle \sigma_l \rangle^{1/2} \mathcal{F} \left( \eta t^\delta \tilde{v}_l \right)(k_l, \tau_l), \quad l = 1, 2,
\]

\[
K_3(k_1, \tau_1, k, \tau) = \frac{|k_k k_2| \langle k \rangle^s}{(1 + k^2) \langle \sigma \rangle \prod_{l=1}^2 \langle k_l \rangle^s \langle \sigma_l \rangle^{1/2}}.
\]

To obtain (3.18), it suffices to prove that

\[
\left\| \int_{k = k_1 + k_2} \int_{\tau = \tau_1 + \tau_2} K_3(k_1, \tau_1, k, \tau) \prod_{l=1}^2 G_l(k_l, \tau_l) dk_l d\tau_l \right\|_{(L^2(k) L^1(\tau))} \leq C \delta^{2j+1} - 3\epsilon \prod_{l=1}^2 \left\| G_l \right\|_{L^2}.
\]

(3.19)

Since \( \min \{|k|, |k_1|, |k_2|\} \geq 1 \), from Lemma 2.4, we know that one of the following three cases must occur:

- (a) : \(|\sigma| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j},
- (b) : \(|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j},
- (c) : \(|\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j}.

When (a) : \(|\sigma| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}| |k_{\max}|^{2j}, \) If \( \langle \sigma_1 \rangle \geq C|k_{\min}|^{\epsilon'} |k_{\max}|^{2j \epsilon'} \), in this case, by using the proof similar to (3.5)-(3.6), we have that

\[
K_3(k_1, \tau_1, k, \tau) \leq \frac{|k|^{s-\frac{3}{2}} \prod_{l=1}^2 \langle k_l \rangle^{2-s} \langle \sigma \rangle^{\frac{1}{2}+\epsilon} \langle \sigma_1 \rangle^{\frac{1}{2}-\epsilon} \langle \sigma_2 \rangle^{\frac{1}{2}}} \langle \sigma \rangle^{\frac{1}{2}+\epsilon} \langle \sigma_1 \rangle^{\frac{1}{2}-\epsilon} \langle \sigma_2 \rangle^{\frac{1}{2}} \leq \frac{C}{\langle \sigma \rangle^{\frac{1}{2}+\epsilon} \langle \sigma_1 \rangle^{\frac{1}{2}-\epsilon} \langle \sigma_2 \rangle^{\frac{1}{2}}},
\]

(3.20)

by using (3.20), the Cauchy-Schwarz inequality and the Plancherel identity as well as
Lemmas 2.3, 2.13, then we have that
\[
\left\| \frac{1}{2} \int_{k = k_1 + k_2}^{k_1} \frac{G_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} dk_l d\tau_l \right\|_{L^2(k) L^1(d\tau)} \leq C \left\| \int_{k = k_1 + k_2}^{k_1 + k_2} \frac{G_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} dk_l d\tau_l \right\|_{L^2_k L^2_\tau} \leq C \left\| \mathcal{F}^{-1} \left( \frac{G_1}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \right) \right\|_{L^4_{x,t}} \left\| \mathcal{F}^{-1} \left( \frac{G_2}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} \right) \right\|_{L^4_{x,t}} \leq C \left\| \eta \left( \frac{t}{\delta} \right) \tilde{v}_1 \right\|_{X^{s, \frac{j+1}{2}+1}_{t, t} \left( \frac{t}{\delta} \right) \left( \frac{t}{\delta} \right)} \left\| \eta \left( \frac{t}{\delta} \right) \tilde{v}_2 \right\|_{X^{s, \frac{j+1}{2}+1}_{t, t} \left( \frac{t}{\delta} \right) \left( \frac{t}{\delta} \right)} \leq C \delta^{j+1} - 3 \prod_{l=1}^{2} \left\| \tilde{v}_l \right\|_{X^{s, \frac{j+1}{2}+1}_{t, t}} \leq C \delta^{j+1} - 3 \prod_{l=1}^{2} \left\| \tilde{v}_l \right\|_{X^{s, \frac{j+1}{2}+1}_{t, t}} \leq C \delta^{j+1} - 3 \prod_{l=1}^{2} \left\| G_l \right\|_{L^2}.
\]

If \( \langle \sigma_2 \rangle \geq C |k_{\min}|^{\epsilon'} |k_{\max}|^{2j \epsilon'} \), this case can be proved similarly to case \( \langle \sigma_1 \rangle \geq C |k_{\min}|^{\epsilon'} |k_{\max}|^{2j \epsilon'} \).

If \( \langle \sigma_l \rangle \leq C |k_{\min}|^{\epsilon'} |k_{\max}|^{2j \epsilon'} \), \( l = 1, 2 \), in this case we have that
\[
\mu = k^{2j+1} - k_1^{2j+1} - k_2^{2j+1} + O(\langle |k_{\min}| |k_{\max}|^{2j} \rangle^{\epsilon'})
\]
and
\[
K_3(k_1, \tau_1, k, \tau) \leq C \frac{|k|^{s-3/2} \prod_{l=1}^{2} |k_l|^{2j+1-s}}{\langle \sigma \rangle^{1/2} \prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}.
\]
by using the proof similar to (3.5)-(3.6), we have that
\[
K_3(k_1, \tau_1, k, \tau) \leq \frac{C}{\langle \sigma \rangle^{1/2} \prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}}.
\]

 Consequently, by using (3.14) and the Cauchy-Schwartz inequality with respect to \( \tau \) and
Lemmas 2.8, 2.11, we have that

\[
\left\| \langle \sigma \rangle^{-\frac{1}{2}} \chi_{\Omega(k)} \int_{k=k_1+k_2 \atop \tau = \tau_1 + \tau_2} \prod_{l=1}^2 G_l(k_l, \tau_l) \, dk \right\|_{L^2(k \Omega)} \leq C \left\| \left( \int \langle \sigma \rangle^{-1} \chi_{\Omega(k)} \, d\tau \right)^{1/2} \int_{k=k_1+k_2 \atop \tau = \tau_1 + \tau_2} \prod_{l=1}^2 G_l(k_l, \tau_l) \, dk \right\|_{L^2(\tau)} \\
\leq C \left( \int \langle \sigma \rangle^{-1} \chi_{\Omega(k)} \, d\tau \right)^{1/2} \left\| \int_{k=k_1+k_2 \atop \tau = \tau_1 + \tau_2} \prod_{l=1}^2 G_l(k_l, \tau_l) \, dk \right\|_{L^2(\tau)} \\
\leq C \delta^{\frac{1}{2}j+1} - 2e^2 \prod_{l=1}^2 \| G_l \|_{L^2}, \tag{3.23}
\]

where

\[
\Omega(k) = \left\{ \mu \in \mathbb{R} : |\mu| \sim M, \mu = C|k_{\min}|k_{\max}|^{2j} + O(\langle |k_{\min}|k_{\max}|^{2j} \rangle^{'}) \right\}.
\]

When (b) : \(|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}|k_{\max}|^{2j}\). by using the proof similar to (3.5)-(3.6), we have that

\[
K_3(k_1, \tau_1, k, \tau) \leq C \frac{|k|^{-\frac{3}{2}} \prod_{l=1}^2 \langle k_l \rangle^{\frac{2j-4-s}{2}} \langle \sigma \rangle \langle \sigma_2 \rangle^{\frac{1}{2}}}{\langle \sigma_1 \rangle^{\frac{1}{2}}}, \tag{3.24}
\]

by using the Cauchy-Schwarz inequality with respect to \(\tau\) and Lemma 2.12, we have that

\[
\left\| \int_{k=k_1+k_2 \atop \tau = \tau_1 + \tau_2} \prod_{l=1}^2 G_l(k_l, \tau_l) \, dk \right\|_{L^2(k \Omega)} \leq C \left( \int \langle \sigma \rangle^{-1} \chi_{\Omega(k)} \, d\tau \right)^{1/2} \left\| \int_{k=k_1+k_2 \atop \tau = \tau_1 + \tau_2} \prod_{l=1}^2 G_l(k_l, \tau_l) \, dk \right\|_{L^2(\tau)} \\
\leq C \delta^{\frac{1}{2}j+1} - 2e^2 \prod_{l=1}^2 \| G_l \|_{L^2}. 
\]

When (c) : \(|\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}|k_{\max}|^{2j}\). This case can be proved similarly to (b) : \(|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq C|k_{\min}|k_{\max}|^{2j}\).

We have completed the proof Lemma 3.4.

**Lemma 3.5.** Let \(v_l(x, t)\) with \(l = 1, 2\) which are zero x-mean for all \(t\) be 2π-periodic
functions of $x$. For $s \geq -\frac{3}{2}$, we have that

$$
\left\| \frac{\langle k \rangle^s}{\langle \sigma \rangle^s} \prod_{l=1}^{2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) v_l \right) (k_l, \tau_l) dk_1 d\tau_1 \right\|_{L^2(k)L^1(\tau)} \\
\leq C\delta^{j+1 - 2\epsilon} \prod_{l=1}^{2} \| v_l \|_{X^{\frac{\delta}{2}, \frac{s}{2}}}. \tag{3.25}
$$

By using the proof similar to Lemma 3.4, we can obtain Lemma 3.5.

**Lemma 3.6.** Let $v_l(x,t)$ with $l = 1, 2$ which are zero $x$-mean for all $t$ be $2\pi$-periodic functions of $x$. For $s \geq -\frac{3}{2}$, we have that

$$
\left\| \frac{\langle k \rangle^s}{\langle \sigma \rangle^s} \prod_{l=1}^{2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) v_l \right) (k_l, \tau_l) dk_1 d\tau_1 \right\|_{L^2(k)L^1(\tau)} \\
\leq C\delta^{j+1 - 2\epsilon} \prod_{l=1}^{2} \| v_l \|_{X^{\frac{\delta}{2}, \frac{s}{2}}}. \tag{3.26}
$$

By using the proof similar to Lemma 3.4, we can obtain Lemma 3.6.

**Lemma 3.7.** Let $u_l(x,t)$ with $l = 1, 2$ which are zero $x$-mean for all $t$ be $2\pi$-periodic functions of $x$. Then

$$
\left\| \partial_x (1 - \partial^2_x)^{-1} \prod_{l=1}^{2} \left[ \partial_x \eta \left( \frac{t}{\delta} \right) u_l \right] \right\|_{Z_s} \leq C\delta^{j+1 - 2\epsilon} \prod_{l=1}^{2} \| u_l \|_{Y^s}. \tag{3.27}
$$

Combining Lemma 3.1 with Lemma 3.4, we have Lemma 3.7.

**Lemma 3.8.** Let $u(x,t)$ with $l = 1, 2$ which are zero $x$-mean for all $t$ be $2\pi$-periodic functions of $x$. Then

$$
\left\| \partial_x \left[ \prod_{l=1}^{2} \eta \left( \frac{t}{\delta} \right) u_l \right] \right\|_{Z_s} \leq C\delta^{j+1 - 2\epsilon} \prod_{l=1}^{2} \| u_l \|_{Y^s}. \tag{3.28}
$$

Combining Lemma 3.2 with Lemma 3.5, we have Lemma 3.8.

**Lemma 3.9.** Let $u(x,t)$ with $l = 1, 2$ which are zero $x$-mean for all $t$ be $2\pi$-periodic functions of $x$. Then

$$
\left\| \partial_x (1 - \partial^2_x)^{-1} \prod_{l=1}^{2} \left[ \eta \left( \frac{t}{\delta} \right) u_l \right] \right\|_{Z_s} \leq C\delta^{j+1 - 2\epsilon} \prod_{l=1}^{2} \| u_l \|_{Y^s}. \tag{3.29}
$$
Combining Lemma 3.3 with Lemma 3.6, we have Lemma 3.9.

4. Proof of Theorem 1.1

Now we are in a position to prove Theorem 1.1. We define

\[
\Phi(u) = \eta(t)S(t)\phi - \eta(t) \int_0^t S(t - t')\eta(t')A(u)dt',
\]

\[
B = \{ u \in Y_\delta : \| u \|_{Y_\delta} \leq 2C\| \phi \|_{H^s(T)} \},
\]

(4.1)

where

\[
A(u) = \frac{1}{2} \partial_x \left[ \eta \left( \frac{t}{\delta} \right) u \right]^2 + (1 - \partial_x^2)^{-1} \left[ \eta \left( \frac{t}{\delta} \right) u \right]^2 + \frac{1}{2}(\eta \left( \frac{t}{\delta} \right) u_x)^2.
\]

By using Lemmas 2.8-2.9, 3.7-3.9, for sufficiently small \( \delta > 0 \), we have that

\[
\delta^\frac{j-1}{2} \| \phi \|_{H^s(T)} \leq \frac{1}{4},
\]

which yields that

\[
\| \Phi(u) \|_{Y_\delta} \leq \| \eta(t)S(t)\phi \|_{Y_\delta} + \left| -\frac{1}{2} \eta(t) \int_0^t S(t - t')\eta(t')A(u)dt' \right|_{Y_\delta}
\]

\[
\leq C_1\| \phi \|_{H^s(T)} + C \left\| \eta \left( \frac{t}{\delta} \right) A(u) \right\|_{Z_\delta}
\]

\[
\leq C\| \phi \|_{H^s(T)} + C\delta^{\frac{j-1}{2} - 3\epsilon} \| u \|_{Y_\delta}^2
\]

\[
\leq C\| \phi \|_{H^s(T)} + C\delta^{\frac{j-1}{2} - 3\epsilon} \| \phi \|_{H^s(T)}^2 \leq 2C\| \phi \|_{H^s(T)}
\]

(4.2)

For \( u, v \in B \), for sufficiently small \( \delta > 0 \), we have that

\[
\| \Phi(u) - \Phi(v) \|_{Y_\delta}
\]

\[
\leq C\delta^{\frac{j-1}{2} - 3\epsilon} (\| u \|_{Y_\delta} + \| v \|_{Y_\delta}) \| u - v \|_{Y_\delta}
\]

\[
\leq 2C\delta^{\frac{j-1}{2} - 3\epsilon} \| \phi \|_{H^s(T)} \| u - v \|_{Y_\delta}
\]

\[
\leq \frac{1}{2}\| u - v \|_{Y_\delta}.
\]

(4.3)

From (4.3), by using the fixed point Theorem, we have that there exists a \( u \) such that \( \Phi(u) = u \). The proof of the remainder of Theorem 1.1 is standard.

We have completed the proof of Theorem 1.1.

5. Modified energy
In this section, we give the almost conserved law which can be used to extend the local solution to the Cauchy problem for (1.1) to the global solution to the Cauchy problem for (1.1).

**Lemma 5.1.** Let \( \frac{2-j}{2} \leq s < 1 \) and \( u \) be the solution to the Cauchy problem for (1.1) on \([0, \delta]\). Then

\[
|\int_0^\delta \int_T \partial_x^2(I\eta \left( \frac{t}{\delta} \right) u) \left[ I(\eta \left( \frac{t}{\delta} \right) u)^2 - (\eta \left( \frac{t}{\delta} \right) Iu)^2 \right] \, dx \, dt| 
\leq C\delta^{\frac{j-1}{j+1}} N^{-j} \|u\|_{X_1^s}^3 .
\]

**Proof.** To obtain (5.1), it suffices to prove that

\[
\int_{k=k_1+k_2}^{k_1+k_2} \frac{|k|^3 |m(k) - m(k_1)m(k_2)|}{\prod_{l=1}^2 m(k_l)} \times \left| \mathcal{F}(\eta \left( \frac{t}{\delta} \right) \bar{u})(\tau, k) \prod_{l=1}^2 \mathcal{F}(\eta \left( \frac{t}{\delta} \right) \bar{u}_l)(\tau_l, k_l) \right| \, dk \, d\tau \, dk \, d\tau
\leq C\delta^{\frac{j-1}{j+1}} N^{-j} \|\bar{u}\|_{X_1^s} \prod_{l=1}^2 \|\bar{u}_l\|_{X_1^s}.
\]

(5.2)

where

\[
\|\bar{u}\|_{X_1^s} = \|u\|_{X_1^s}, \quad \|\bar{u}_l\|_{X_1^s} = \|u_l\|_{X_1^s}, \quad l = 1, 2.
\]

Let

\[
H_l(k_1, \tau_1) = \langle k_1 \rangle \langle \sigma_1 \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \bar{u} \right)(k_1, \tau_1), \quad l = 1, 2,
\]

\[
H(k, \tau) = \langle k \rangle \langle \sigma \rangle^{1/2} \mathcal{F} \left( \eta \left( \frac{t}{\delta} \right) \bar{u} \right)(k, \tau).
\]

To prove (5.2), it suffices to prove

\[
\int_{k=k_1+k_2}^{k_1+k_2} \frac{|m(k) - m(k_1)m(k_2)| |k|^3 H(k, \tau) \prod_{l=1}^2 H_l(k_1, \tau_l) m(k_1)m(k_2) \langle \sigma \rangle^{1/2} \langle k \rangle \prod_{l=1}^2 \langle \sigma_l \rangle^{1/2} \langle k_l \rangle} {m(k_1)m(k_2) \langle \sigma \rangle^{1/2} \langle k \rangle \prod_{l=1}^2 \langle \sigma_l \rangle^{1/2} \langle k_l \rangle} \, dk \, d\tau \, dk \, d\tau
\leq C\delta^{\frac{j-1}{j+1}} N^{-j} \|H\|_{L^2} \prod_{l=1}^2 \|H_l\|_{L^2}.
\]

(5.3)

We define \( A = A_1 \cup A_2 \cup A_3 \), where

\[
A = \left\{ (k_1, \tau_1, k, \tau) \in \left( \mathbb{Z} \times \mathbb{R} \right)^2 : k = k_1 + k_2, \tau = \tau_1 + \tau_2, \, |k_1| \leq |k_2|, \, |k_2| \geq \frac{N}{2} \right\}
\]

\[
A_1 = \left\{ (k_1, \tau_1, k, \tau) \in A : |k_1| \ll |k_2|, \, |k_1| \leq N \right\}
\]

\[
A_2 = \left\{ (k_1, \tau_1, k, \tau) \in A : |k_1| \ll |k_2|, \, |k_1| \geq N \right\}
\]

\[
A_3 = \left\{ (k_1, \tau_1, k, \tau) \in A : |k_1| \sim |k_2| \right\}.
\]
The integrals corresponding to \( A_j \) for \( j = 1, 2, 3 \) will be denoted by \( I_1, I_2, I_3 \). We consider cases

(a) : \(|\sigma| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C|k_{\min}|k_{\max}^{2j}\),

(b) : \(|\sigma_1| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C|k_{\min}|k_{\max}^{2j}\),

(c) : \(|\sigma_2| = \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C|k_{\min}|k_{\max}^{2j}\).

1. Estimate of \( I_1 \). By using the mean value Theorem, we have that

\[
m(k_1 + k_2) - m(k_1)m(k_2) = m'(\theta k_1 + k_2)k_1,
\]

thus in region \( A_1 \), we have that \(|\theta k_1 + k_2| \sim |k_2|\) which yields that

\[
\left| \frac{m(k_1 + k_2) - m(k_1)m(k_2)}{m(k_1)m(k_2)} \right| = \frac{|m(k_1 + k_2) - m(k_2)|}{m(k_2)} \leq \frac{m'(\theta k_1 + k_2)|k_1|}{|k_2|} \leq C|k_1|, \quad (5.4)
\]

When (a) is valid, by using (5.4), the Plancherel identity and Hölder inequality as well as Lemma 2.11, we have that in this case the left hand side of (5.3) can be bounded by

\[
\int \frac{|k_1||k|^3H(k, \tau)\prod_{l=1}^2 H_l(k_l, \tau)}{|k_2|<\sigma^{1/2}<k_2\prod_{l=1}^2<\sigma_l^{1/2}<k_l>\prod_{l=1}^2<k_l>dk_1d\tau_1dkd\tau} \leq C \int \frac{|k|^{-j}|k_1|^{-1/2}H(k, \tau)\prod_{l=1}^2 H_l(k_l, \tau)}{\prod_{l=1}^2<\sigma_l^{1/2}<k_l>\prod_{l=1}^2<k_l>\prod_{l=1}^2<k_l>dk_1d\tau_1dkd\tau}
\]

\[
\leq CN^{-j}\|H\|_L^2 \prod_{l=1}^2 \left| \mathcal{F}_{k_1}^{-1} \left( \frac{H_l}{<\sigma_l^{1/2}>} \right) \right|_{L^2} \leq CN^{-j}\delta_{2j+1}^{-2\epsilon}\|H\|_{L^2}^2 \prod_{l=1}^2 \|H_l\|_{L^2}^2.
\]

When (b) is valid, by using (5.4), the Plancherel identity and Hölder inequality as well as Lemma 2.12, we have that in this case the left hand side of (5.3) can be bounded by

\[
\int \frac{|k_1||k|^3H(k, \tau)\prod_{l=1}^2 H_l(k_l, \tau)}{|k_2|<\sigma^{1/2}<k_2\prod_{l=1}^2<\sigma_l^{1/2}<k_l>\prod_{l=1}^2<k_l>dk_1d\tau_1dkd\tau} \leq C \int \frac{|k|^{-j}|k_1|^{-1/2}H(k, \tau)\prod_{l=1}^2 H_l(k_l, \tau)}{\prod_{l=1}^2<\sigma_l^{1/2}<k_l>\prod_{l=1}^2<k_l>\prod_{l=1}^2<k_l>dk_1d\tau_1dkd\tau}
\]

\[
\leq CN^{-j} \int \frac{\langle \sigma \rangle^{-1/2+t}H(k, \tau)\prod_{l=1}^2 H_l(k_l, \tau)}{\langle \sigma_2 \rangle^{1/2}<\sigma^{1/2}>\prod_{l=1}^2<k_l>\prod_{l=1}^2<k_l>dk_1d\tau_1dkd\tau}
\]

\[
\leq CN^{-j}\left\| \int \frac{\langle \sigma \rangle^{-1/2+t}H(k, \tau)\prod_{l=1}^2 H_l(k_l, \tau)}{\langle \sigma_2 \rangle^{1/2}<\sigma^{1/2}>\prod_{l=1}^2<k_l>\prod_{l=1}^2<k_l>dk_1d\tau_1} \right\|_{L^2} \|H\|_{L^2}
\]

\[
\leq CN^{-j}\delta_{2j+1}^{-2\epsilon}\|H\|_{L^2}^2 \prod_{l=1}^2 \|H_l\|_{L^2}^2.
\]
When (c) is valid, this case can be proved similarly to case (b).

2. Estimate of $I_2$. In this case, we have that
\[
\frac{|m(k_1 + k_2) - m(k_1)m(k_2)|}{m(k_1)m(k_2)} \leq \max \{m(k_1 + k_2), m(k_2)\} \leq \frac{C}{m(k_1)} \leq C \left( \frac{|k_1|}{N} \right)^{-s}.
\]

When (a) is valid, we have that in this case the left hand side of (5.3) can be bounded by
\[
\int_{k = k_1 + k_2 \atop \tau = \tau_1 + \tau_2} |k_1|^{-s}|k_1|^3 N^s N\frac{H(k, \tau) \prod_{l=1}^{2} H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau \leq C \int_{k = k_1 + k_2 \atop \tau = \tau_1 + \tau_2} |k_1|^{-s-\frac{3}{2}} N^s |k_1|^{1-j} N^1 \frac{H(k, \tau) \prod_{l=1}^{2} H_l(k_l, \tau_l)}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau, \quad (5.5)
\]
if $-s - \frac{3}{2} \leq 0$, by using the Plancherel identity and the Hölder inequality as well as Lemma 2.11, we have that (5.5) can be bounded by
\[
C \int_{k = k_1 + k_2 \atop \tau = \tau_1 + \tau_2} \frac{N^{-s-\frac{3}{2}} N^s N^{1-j} H(k, \tau) \prod_{l=1}^{2} H_l(k_l, \tau_l)}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau \leq CN^{-j-\frac{1}{2}} \int_{k = k_1 + k_2 \atop \tau = \tau_1 + \tau_2} \frac{H(k, \tau) \prod_{l=1}^{2} H_l(k_l, \tau_l)}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau \leq CN^{-j-\frac{1}{2}} \|H\|_{L^2} \prod_{l=1}^{2} \|H_l\|_{L^2},
\]
if $-s - \frac{3}{2} \geq 0$, since $s \geq \frac{2-j}{2}$, by using the Plancherel identity and the Hölder inequality as well as Lemma 2.11, we have that (5.5) can be bounded by
\[
C \int_{k = k_1 + k_2 \atop \tau = \tau_1 + \tau_2} \frac{|k|^{-s-\frac{3}{2}} |k|^{1-j} H(k, \tau) \prod_{l=1}^{2} H_l(k_l, \tau_l)}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau \leq CN^{-j-\frac{1}{2}} \int_{k = k_1 + k_2 \atop \tau = \tau_1 + \tau_2} \frac{H(k, \tau) \prod_{l=1}^{2} H_l(k_l, \tau_l)}{\prod_{l=1}^{2} \langle \sigma_l \rangle^{1/2}} dk_1 d\tau_1 dk d\tau \leq CN^{-j-\frac{1}{2}} \|H\|_{L^2} \prod_{l=1}^{2} \|H_l\|_{L^2}.
\]
When \((b)\) is valid, by using \((5.4)\) and the Plancherel identity and the Hölder inequality as well as Lemma 2.11, we have that in this case the left hand side of \((5.3)\) can be bounded by
\[
\int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{|k_1|^{-s} |k|^3 N^s H(k, \tau) \prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2}} dk_1 d\tau_1 dk d\tau
\leq C \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{|k_1|^{-s-\frac{3}{2}} N^s |k|^{1-j} H(k, \tau) \prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 dk d\tau,
\]
if \(-s - \frac{3}{2} \leq 0\), by using the Plancherel identity and the Hölder inequality as well as Lemma 2.12, we have that \((5.6)\) can be bounded by
\[
C \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{N^{-s-\frac{3}{2}} N^s N^{1-j} H(k, \tau) \prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 dk d\tau
\leq CN^{-j-\frac{1}{2}} \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{H(k, \tau) \prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 dk d\tau
\leq CN^{-j-\frac{1}{2}} \left\| \left\langle \sigma \right\rangle^{-\frac{1}{2}+\epsilon} \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{\prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 \right\| \|H\|_{L^2}
\leq CN^{-j-\frac{1}{2}} \delta_{\frac{3}{2}\tau_1 \tau}^{-2\epsilon} \|H\|_{L^2} \prod_{l=1}^2 \|H_l\|_{L^2}.
\]
if \(-s - \frac{3}{2} \geq 0\), since \(s \geq \frac{2-j}{2}\), \((5.6)\) can be bounded by
\[
C \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{|k|^{-s-\frac{3}{2}} N^s |k|^{1-j} H(k, \tau) \prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_1 \rangle^{1/2}} dk_1 d\tau_1 dk d\tau
\leq C \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{|k|^{-s-\frac{3}{2}} N^s H(k, \tau) \prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma \rangle^{1/2} \langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 dk d\tau
\leq CN^{-j-\frac{1}{2}} \left\| \left\langle \sigma \right\rangle^{-\frac{1}{2}+\epsilon} \int_{\substack{k = k_1 + k_2 \\
\tau = \tau_1 + \tau_2}} \frac{\prod_{l=1}^2 H_l(k_l, \tau_l)}{\langle \sigma_2 \rangle^{1/2}} dk_1 d\tau_1 \right\| \|H\|_{L^2}
\leq CN^{-j-\frac{1}{2}} \delta_{\frac{3}{2}\tau_1 \tau}^{-2\epsilon} \|H\|_{L^2} \prod_{l=1}^2 \|H_l\|_{L^2}.
\]
When \((c)\) is valid, this case can be proved similarly to case \((b)\).

3. Estimate of \(I_3\). In this case, we have that
\[
\frac{|m(k_1 + k_2) - m(k_1)m(k_2)|}{m(k_1)m(k_2)} \leq C \prod_{l=1}^2 \left( \frac{|k_l|}{N} \right)^{-s}.
\]
When \((a)\) is valid, by using \((5.7)\) and the Plancherel identity and the Hölder inequality as well as Lemma 2.11, since \(\frac{2-j}{2} \leq s \leq 1\), we have that in this case the left hand side of \((5.3)\) can be bounded by

\[
\int \frac{|k^3|k_1|^{-2s} N^{2s} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\left<\sigma\right>^{1/2}\left<k\right> \prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau \leq C \int \frac{|k_1|^{-2s-2-j} N^{2s}|k|^{5/2} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\left<k\right> \prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau
\]

\[
\leq C \int \frac{|k_1|^{-2s-\frac{1}{2}-j} N^{2s} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau
\]

\[
\leq CN^{-j-\frac{1}{2}} \int \frac{|k|^{-2s-\frac{1}{2}-j} N^{2s} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau
\]

\[
\leq CN^{-j-\frac{1}{2}} \|H\|_{L^2} \prod_{i=1}^2 \left\|\mathcal{F}^{-1} \left( \frac{H_i}{\left<\sigma_i\right>^{1/2}} \right) \right\|_{L^2_{x_t}}
\]

\[
\leq CN^{-j-\frac{1}{2}} \delta \frac{j}{j+1}-2^\epsilon \|H\|_{L^2} \prod_{i=1}^2 \|H_i\|_{L^2}.
\]

When \((b)\) is valid, by using \((5.7)\) and the Plancherel identity and the Hölder inequality as well as Lemma 2.12, since \(\frac{2-j}{2} \leq s \leq 1\), we have that in this case the left hand side of \((5.3)\) can be bounded by

\[
\int \frac{|k^3|k_1|^{-2s} N^{2s} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\left<\sigma\right>^{1/2}\left<k\right> \prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau \leq C \int \frac{|k_1|^{-2s-2-j} N^{2s}|k|^{5/2} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\left<k\right> \prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau
\]

\[
\leq C \int \frac{|k_1|^{-2s-\frac{1}{2}-j} N^{2s} H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau
\]

\[
\leq C N^{-j-\frac{1}{2}} \int \frac{H(k, \tau) \prod_{i=1}^2 H_i(k_i, \tau_i)}{\prod_{i=1}^2 \left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1dkd\tau
\]

\[
\leq C N^{-j-\frac{1}{2}} \left\|\langle\sigma\rangle^{-\frac{1}{2}+\epsilon} \int \frac{\prod_{i=1}^2 H_i(k_i, \tau_i)}{\left<\sigma_i\right>^{1/2}} \, dk_1d\tau_1 \right\|_{L^2_{x_t}} \|H\|_{L^2}
\]

\[
\leq C N^{-j-\frac{1}{2}} \delta \frac{j}{j+1}-2^\epsilon \|H\|_{L^2} \prod_{i=1}^2 \|H_i\|_{L^2}.
\]

When \((c)\) is valid, this case can be proved similarly to case \((b)\).

We have completed the proof of Lemma 5.1.
Lemma 5.2. Let $\frac{j-2}{2} \leq s < 1$ and $u$ be the solution to the Cauchy problem for (1.1) on $[0, \delta]$. Then

$$\left| \int_0^\delta \int_\Omega \left( \partial_x(Iu) \left[ I(u_x^2) - (\partial_x Iu)^2 \right] \right) \, dx \, dt \right| \leq C \delta \frac{j}{2j+1} 2e N^{-j} \|Iu\|_{X^{s, 1/2}_1}^3.$$  \hspace{1cm} (5.8)

Lemma 5.2 can be proved similarly to Lemma 5.1.

Lemma 5.3. Let $-\frac{j}{2} \leq s < 1$ and $u$ be the solution to the Cauchy problem for (1.1) on $[0, \delta]$. Then

$$\left| \int_0^\delta \int_\Omega \left( \partial_x(Iu) \left[ Iu^2 - (Iu)^2 \right] \right) \, dx \, dt \right| \leq C \delta \frac{j}{2j+1} 2e N^{-j} \|Iu\|_{X^{s, 1/2}}^3.$$  \hspace{1cm} (5.9)

Lemma 5.3 can be proved similarly to Lemma 5.1.

Lemma 5.4. Let $\frac{2-j}{2} \leq s < 1$ and $u$ be the solution to the Cauchy problem for (1.1) on $[0, \delta]$. Then

$$\left| \|Iu(\delta)\|_{H^1}^2 - \|Iu(0)\|_{H^1}^2 \right| \leq C \delta \frac{j}{2j+1} 2e N^{-j} \|Iu\|_{X^{s, 1/2}}^3.$$  \hspace{1cm} (5.10)

Proof. By using a proof similar to (4.3) of [26], we have that

$$\|Iu(\delta)\|_{H^1}^2 - \|Iu(0)\|_{H^1}^2 = \int_0^\delta \int_\Omega \left( 1 - \partial_x^2 \right) \partial_x(Iu) \left[ Iu^2 - (Iu)^2 \right] \, dx \, dt$$
$$+ 2 \int_0^\delta \int_\Omega \partial_x(Iu) \left[ Iu^2 - (Iu)^2 \right] \, dx \, dt$$
$$+ \int_0^\delta \int_\Omega \left( \partial_x(Iu) \left[ I(u_x^2) - (\partial_x Iu)^2 \right] \right) \, dx \, dt.$$ \hspace{1cm} (5.11)

Proof. To prove (5.11), it suffices to prove that

$$\left| \|Iu(\delta)\|_{H^1}^2 - \|Iu(0)\|_{H^1}^2 \right| \leq \int_0^\delta \int_\Omega \left( 1 - \partial_x^2 \right) \partial_x(Iu) \left[ Iu^2 - (Iu)^2 \right] \, dx \, dt$$
$$+ 2 \int_0^\delta \int_\Omega \partial_x(Iu) \left[ Iu^2 - (Iu)^2 \right] \, dx \, dt$$
$$+ \int_\Omega \partial_x(Iu) \left[ I(u_x^2) - (\partial_x Iu)^2 \right] \, dx \, dt \leq C \delta \frac{j}{2j+1} 2e N^{-j} \|Iu\|_{X^{s, 1/2}}^3.$$ \hspace{1cm} (5.12)

(5.12) can be obtained from Lemmas 5.1-5.3.

We have completed the proof of Lemma 5.4.

6. Proof of Theorem 1.2
We give Theorem 5.1 which is a variant of Theorem 1.1 before giving the proof of Theorem 1.2.

We consider the Cauchy problem for

\[(Iu)_t + \partial_x^{2j+1}(Iu) + \frac{1}{2}\partial_x I(u^2) + \partial_x (1 - \partial_x^2)^{-1} I \left[ u^2 + \frac{1}{2}u_x^2 \right] = 0, \quad (6.1) \]

\[Iu(x, 0) = Iu_0(x). \quad (6.2)\]

**Theorem 6.1.** Let \(s \geq -\frac{j - 2}{2}\) and \(u_0\) be \(2\pi\)-periodic function and zero \(x\)-mean and \(Iu_0 \in H^1(T)\). Then the Cauchy problems (6.1)(6.2) are locally well-posed.

**Proof.** Let \(Iu = v\), we define

\[G(v) = \eta(t)S(t)v(0) + \eta(t) \int_0^t \left[ \frac{1}{2}\partial_x I(\eta \left( \frac{t}{\delta} \right) u)^2 + \partial_x (1 - \partial_x^2)^{-1} I \left[ \eta \left( \frac{t}{\delta} \right) u^2 + \frac{1}{2}(\eta \left( \frac{t}{\delta} \right) u_x)^2 \right] \right] dt'.\]

and

\[B = \left\{ u \in Y^\delta : \|Iu\|_{Y^\delta} \leq 2C\|Iu_0\|_{H^1(T)} \right\}, \quad (6.3)\]

and

\[E = \frac{1}{2}\partial_x I(\eta \left( \frac{t}{\delta} \right) u)^2 + \partial_x (1 - \partial_x^2)^{-1} I \left[ \eta \left( \frac{t}{\delta} \right) u^2 + \frac{1}{2}(\eta \left( \frac{t}{\delta} \right) u_x)^2 \right] \]

\[- \frac{1}{2}\partial_x (\eta \left( \frac{t}{\delta} \right) Iv)^2 - \partial_x (1 - \partial_x^2)^{-1} \left[ \eta \left( \frac{t}{\delta} \right) Iv^2 + \frac{1}{2}(\eta \left( \frac{t}{\delta} \right) Iv_x)^2 \right].\]

Thus, we have that

\[G(v) = \eta(t)S(t)Iu_0 + \eta(t) \int_0^t \left[ E + \partial_x (1 - \partial_x^2)^{-1} \left[ \eta \left( \frac{t}{\delta} \right) Iv^2 + \frac{1}{2}(\eta \left( \frac{t}{\delta} \right) Iv_x)^2 \right] \right] dt'.\]

By using Lemmas 3.7-3.9, 5.1-5.3, we have that

\[\|G(v)\|_{Y^\delta} \leq \|\eta(t)S(t)Iu_0\|_{Y^\delta} + \|\eta(t) \int_0^t \left[ \partial_x (1 - \partial_x^2)^{-1} \left[ \eta \left( \frac{t}{\delta} \right) Iv^2 + \frac{1}{2}(\eta \left( \frac{t}{\delta} \right) Iv_x)^2 \right] \right] dt'\|_{Y^\delta} \]

\[+ \left\| \eta(t) \int_0^t E dt' \right\|_{Y^\delta} \leq C\|Iu_0\|_{H^1} + C\delta^{-j+1} \|v\|_{Y^\delta}^2 \leq 2C\|Iu_0\|_{H^1}.\]
Thus, $G$ maps $B$ into $B$. By using Lemmas 3.7-3.9, 5.1-5.3, we have that
\[ \|G(u) - G(v)\|_{Y^\delta} \leq \frac{1}{2} \|u - v\|_{Y^\delta}. \]

$G$ is a contraction mapping.

We have completed the proof of Theorem 5.1.

Now we are in a position to prove Theorem 1.2. For $u_0 \in H^s(T)$, from Theorem 5.1, we have that $u$ exists on $[0, \delta]$ and
\[ \delta \sim \|Iu_0\|_{H^{\frac{2j+1}{2}}} - \frac{2}{3(2j+1)\epsilon}. \] (6.4)

From Theorem 5.1, we have that
\[ \|Iu\|_{Y^\delta} \leq 2C\|Iu_0\|_{H^1}. \] (6.5)

Combining (6.5) with Lemma 5.4, we have that
\[ \|Iu(\delta)\|_{H^1} \leq \|Iu_0\|_{H^1} + CN^{-j}\delta^{\frac{2}{2j+1} - 3}\|Iu_0\|^3_{H^1}. \] (6.6)

If
\[ CN^{-j}\delta^{\frac{2}{2j+1} - 3}\|Iu_0\|^3_{H^1} \leq 3\|Iu_0\|^2_{H^1}, \] (6.7)
then, we have that
\[ \|Iu(\delta)\|_{H^1} \leq 2\|Iu_0\|_{H^1}, \] (6.8)

thus, we can consider $u(\delta)$ as the initial data, repeat the above process and extend the local solution on $[0, \delta]$ to the local solution on $[\delta, 2\delta]$. To extend the local solution to the global on time interval $[0, T]$, we need to extend $[T\delta^{-1}]$ times, from (6.7), it suffices to prove that
\[ CN^{-j}\delta^{\frac{2}{2j+1} - 3}\|Iu_0\|^3_{H^1} T\delta^{-1} \leq 3\|Iu_0\|^2_{H^1}, \] (6.9)

It is easily checked that
\[ \|u\|_{H^s} \leq \|Iu_0\|_{H^1} \leq CN^{1-s}\|u\|_{H^s}. \] (6.10)

Combining (6.4), (6.10) with (6.9), we have that
\[ CTN^{\left[\frac{(2j+1)(1-s)}{3-\frac{2}{2j+1}}\right] - j}\|u_0\|_{H^s}^{2j+1} \leq 1. \] (6.11)
Let \( f(j) = \frac{(2j+1)}{j-(2j+1)} \). To obtain (6.11), it suffices to choose \( s > \frac{2j+1-j^2}{2j+1} \) and

\[
N = \left( CT\|u_0\|_{H^s}^{f(j)}\right)^{\frac{1}{j-f(j)(1-s)}}.
\] (6.12)

From the above iteration process, we have that

\[
\sup_{t \in [0,T]} \|u(\cdot, t)\|_{H^s} \leq 2\|Iu_0\|_{H^s},
\]

\[
\leq CN^{1-s}\|u_0\|_{H^s} \leq C \left( CT\|u_0\|_{H^s}^{f(j)}\right)^{\frac{1}{j-f(j)(1-s)}} \|u_0\|_{H^s},
\]

\[
\leq CT^{\frac{1}{j-f(j)(1-s)}}\|u_0\|_{H^s}^{\frac{1}{j-f(j)(1-s)}}.
\] (6.13)

We have completed the proof of Theorem 1.2.

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