Role of helicities for the dynamics of turbulent magnetic fields

WOLF-CHRISTIAN MÜLLER† and SHIVA KUMAR MALAPAKA†‡*†‡*

†Max-Planck Institut für Plasmaphysik, Boltzmannstr. 2, D-85748 Garching bei München, Germany
‡Department of Mathematics, University of Leeds, Leeds, LS2 9JT, UK

(Received 7 December 2011; in final form 12 April 2012; first published online 1 June 2012)

Investigations of the inverse cascade of magnetic helicity are conducted with pseudospectral, three-dimensional direct numerical simulations of forced and decaying incompressible MHD turbulence. The high-resolution simulations which allow for the necessary scale separation show that the observed self-similar scaling behavior of magnetic helicity and related quantities can only be understood by taking the full nonlinear interplay of velocity and magnetic fluctuations into account. With the help of the eddy-damped quasi-normal Markovian approximation a probably universal relation between kinetic and magnetic helicities is derived which closely resembles the extended definition of the prominent dynamo pseudoscalar $\alpha$. This unexpected similarity suggests an additional nonlinear quenching mechanism of the current-helicity contribution to $\alpha$.

Keywords: $\alpha$-effect; Magnetic helicity; Kinetic helicity; Large-scale magnetic structures

1. Introduction

Understanding large-scale magnetic structure formation in the Universe is one of the challenging problems in modern astrophysics. In this context, mean-field dynamo theory is a prominent approach (Moffatt 1978, Biskamp 2003, Brandenburg and Subramanian 2005). Based on a homogenization formalism, it describes the generation of large-scale magnetic fields by small-scale turbulent fluctuations of a magnetofluid. As a result, this classical two-scale closure (Krause and Rädler 1980) yields, next to a turbulent diffusivity, a scalar, $\alpha \sim \tau H^K$, that expresses the nonlinear interaction of large-scale field and small-scale turbulence. Here, $\tau$ stands for a correlation time of the turbulent fluctuations and $H^K = (1/2V) \int_V \mathbf{v} \cdot \mathbf{\omega} \, dV$ is the kinetic helicity of the associated velocity field $\mathbf{v}$ with $V$ being the volume under consideration and $\mathbf{\omega} = \nabla \times \mathbf{v}$ defining the vorticity. Statistical closure theory (Pouquet et al. 1976), more specifically the eddy-damped quasi-normal Markovian (EDQNM) approximation, suggests a more complex expression, $\alpha \sim \tau(H^K - H^L)$, that introduces the current helicity $H^L = (1/2V) \int_V \mathbf{b} \cdot \mathbf{j} \, dV$ with $\mathbf{j}$ denoting the electric current density (see also Blackman and Field 2002, Field and

*Corresponding author. Email: kumarshiva@gmail.com; previously at LJLL, UPMC, 4 place Jussieu 75005 Paris, France.

© 2013 Taylor & Francis
Blackman 2002, Subramanian and Brandenburg 2004, Brandenburg and Subramanian 2005). Its name is actually misleading as $H^J$ expresses the helicity of the magnetic field and is in this respect a close relative of the kinetic helicity and, furthermore, also proportional to the total resistive dissipation rate of magnetic helicity (see below).

While $H^K$ is ideally conserved and is spectrally cascading toward smaller scales in the inertial range of three-dimensional Navier–Stokes turbulence, the current helicity has apparently no comparable significance for turbulent dynamics apart from its meaning for the turbulent dynamo. However, with electric current $j = \nabla \times b = -\Delta a$, magnetic field $b = \nabla \times a$ and magnetic vector potential $a$ (both dimensionless), a link to an ideal invariant of three-dimensional incompressible MHD emerges through the magnetic helicity, $H^M = (1/2V) \int (a \cdot b) \, dV$. This quantity characterizing the topology of the magnetic field (Moffatt 1969) is prone to an inverse cascade. The cascade is a robust nonlinear mechanism that creates large-scale order out of the chaotic randomness of small-scale magnetic turbulence presupposing a sufficient separation of large and turbulent small scales in the system in combination with a small-scale supply of magnetic helicity.

This work is motivated by the potential importance of magnetic helicity for the dynamics of large-scale dynamo configurations. This is not to be confused with the related issue of the effect of boundary conditions on the magnetic helicity evolution and the consequences for the dynamo process, a topic that has been subject of a number of investigations (see, e.g., Brandenburg 2009, and the references therein). In this work, an idealized system, homogeneous incompressible MHD turbulence with triply periodic boundary conditions, is investigated by three-dimensional direct numerical simulations in combination with statistical closure theory.

2. Model equations and numerical setup

The dimensionless incompressible MHD equations giving a concise single-fluid description of a plasma are

$$\partial_t \omega = \nabla \times (v \times \omega - b \times j) + \mu_\eta (-1)^{n/2-1} \nabla^n \omega + F_v + \lambda \Delta^{-1} \omega, \quad (1a)$$

$$\partial_t b = \nabla \times (v \times b) + \eta_\mu (-1)^{n/2-1} \nabla^n b + F_b + \lambda \Delta^{-1} b, \quad (1b)$$

$$\nabla \cdot v = \nabla \cdot b = 0. \quad (1c)$$

Relativistic effects are neglected and the mass density is assumed to be unity throughout the system. Other effects such as convection, radiation, and rotation are also neglected. Direct numerical simulations are performed by solving the set of model equations by a standard pseudospectral method (Canuto et al. 1988) in combination with leap-frog integration on a cubic box of linear size $2\pi$ that is discretized with 1024 collocation points in each spatial dimension. Spherical mode truncation is used for alleviating aliasing errors. By solving the equations in Fourier space, the solenoidality of $v$ and $b$ is maintained algebraically.

To observe clear signatures of an inverse cascade of magnetic helicity the system has to contain a source of this quantity at small scales. This is achieved in two different ways resulting in two main configurations: a driven system and a decaying one. In the driven case, the forcing terms $F_v$ and $F_b$ are delta-correlated random processes acting in
a band of wavenumbers $203 \leq k_0 \leq 209$. They create a small-scale background of fluctuations with adjustable amount of magnetic and kinetic helicities. The results reported in this article do not change if kinetic helicity injection is finite. The theoretical results presented in the following do not depend on the setup of the forcing as they presuppose an existing self-similar distribution of energies and helicities. For obtaining such spectra in numerical experiments the magnetic source term $F_b$ is necessary while a finite momentum source $F_v$ speeds up the spectral development significantly. In the decaying case the forcing terms are set to zero and the initial condition represents an ensemble of smooth and random fluctuations of maximum magnetic helicity with respect to the energy content (see below) and a characteristic wavenumber $k_0 = 70$.

To reduce finite-size effects, the simulations are run for 6.7 (forced) and 9.2 (decaying) large-eddy turnover times of the system, respectively. The time unit is defined using the system size and its total energy. Additionally, a large-scale energy sink $\lambda \Delta^{-1}$ with $\lambda = 0.5$ is present for both fields. In the decaying case $\lambda = 0$. The hyperdiffusivities $\mu_n$ and $\eta_n$ are dimensionless dissipation coefficients of order $n$ (always even in these simulations), with $n = 8$ in both runs. They act like higher order realizations of viscosity and magnetic diffusivity, respectively. The magnetic hyperdiffusive Prandtl number $Pr_{mn} = \mu_n/\eta_n$ is set to unity.

The initial conditions to these simulations are smooth fluctuations with random phases having a Gaussian energy distribution peaked around $k_0$ in the decaying and the forced cases. Magnetic and kinetic helicities of the initial state can be controlled in the same way as for the forcing terms (cf. Biskamp and Müller 2000). The initial/force-supplied ratio of kinetic to magnetic energy is unity with an amplitude of 0.05 in the forced case and an amplitude of unity in the decaying case. Hyperviscosity of order $n = 8$ is chosen in the simulations to obtain sufficient scale separation. It is difficult to define an unambiguous Reynolds number owing to the use of hyperviscosity (Malapaka 2009, and the references therein). With the above-mentioned simulation setup, the equations are solved both for decaying and forced cases separately and the results obtained are discussed below.

3. Simulation results

Using the simulation setup described in the previous section, inverse cascading of magnetic helicity with a clear scale separation between large and small scales is established in both forced and decaying cases for wavenumbers $k < k_0$. This is indicated by the spectral flux, $H_k^{HM} = \int_0^k dk' \int d\Omega \{ \hat{b}^* \cdot (\hat{v} \times \hat{b}) \}_{k'=k}$, in both cases depicted in figure 1(b) and taken at $t = 6.7$ and 9.2, respectively, as dissipation of magnetic helicity is negligible (see figure 1(a)). The tilde indicates Fourier transformation and * stands for complex conjugate. The inverse flux in the driven case is constant over a significant spectral interval, indicating equilibrium of source and sink, while the temporal decay of the magnetic helicity reservoir in the decaying case is reflected by the associated non-constant inverse flux. In both cases the characteristic wavenumber of the $H_k^{HM}$-source can be identified as the separation between inverse and direct flux regions. The spectral flux of magnetic helicity has been extensively studied in earlier numerical simulations (see, e.g. Brandenburg 2001, Alexakis et al. 2006). These works, however, are lacking the necessary scale separation to observe self-similar scaling laws. The spectrum of
magnetic helicity exhibits scaling behavior $\sim k^q$ with $q \approx -3.3$ and $q \approx -3.6$ (forced and decaying cases, respectively) which cannot be explained by the straightforward constant-flux reasoning a la Kolmogorov adopted in Pouquet et al. (1976) to interpret their EDQNM results.

In fact, the involved dimensional argument (Alfvénic units), $[H_k^M] = L^4/T^2$ (spectrum), $[\varepsilon_M] = L^3/T^3$ (spectral flux), in combination with the assumption of spectral self-similarity, $H_k^M \sim \varepsilon_M^a k^b$, yields $a = 2/3$, $b = -2$, but does not explicitly include the nonlinear interaction of velocity and magnetic fields (see also Biskamp 2003). Here “$L$” represents length and “$T$” the time. As a first step in the necessary refinement of the theoretical modeling additional consideration of the kinetic helicity $H_k^K$ seems appropriate.

As a consequence of the inverse spectral transfer of magnetic helicity, all magnetic quantities should inherit the observed spectral inverse transfer property. This is indeed the case for the magnetic energy, the electric current density, and the current helicity. These quantities also show self-similar scaling that however, differs to some degree between the two investigated configurations. It is particularly interesting that the residual helicity $H_k^R = |H_k^V - k^2 H_k^M|$ also shows self-similar scaling with $q \approx -1.4$ and $q \approx -1.8$ in the forced and decaying cases, respectively (see Malapaka 2009, for further details). The interaction of the magnetic field with the velocity in a progressing inverse cascade of magnetic helicity appears to be of importance for a better understanding of the observed scaling laws. At high Reynolds numbers, the process of large-scale magnetic structure formation by the inverse cascade is accompanied by a continuous stirring of the velocity field caused by the expanding magnetic field structure. The magnetic stirring of the MHD-fluid leads to a transfer of magnetic to kinetic energy and generates ever larger velocity fluctuations. These also show self-similar scaling, as, for example, reflected by the kinetic helicity spectrum with $q \approx -0.4$ (forced case) and $q \approx 0.4$ (decaying case).

With regard to the finding (e.g. Alexakis et al. 2006) of the pronounced spectral non-locality of the nonlinear interactions underlying $\Pi_k^{H_M}$ a few words about the physical picture of the inverse cascade are in order. The cascading process is realized as a merging of positively aligned and thus mutually attracting current carrying structures (cf. Biskamp and Bremer 1993). It is not necessary that the structures grow in size as they indeed do in the decaying case, as long as the corresponding current densities
increase. This is observed in the simulation with small-scale forcing. As there is no obvious fluid-dynamical constraint on the merging of two current filaments with regard to their size, this picture is consistent with a spectrally non-local inverse cascade of magnetic helicity.

4. Spectral relationship between kinetic and magnetic helicities

A link between kinetic and magnetic helicities can be constructed with the help of dimensional analysis of the magnetic helicity evolution equation in the EDQNM approximation, a statistical closure model discussed, e.g. in Pouquet et al. (1976). Such an approach was successful earlier in describing the turbulent residual energy spectrum, $E_k^R = |E_k^M - E_k^J|$ yielding $E_k^R \sim kE_k^2$ (Müller and Grappin 2005) with $E_k = E_k^M + E_k^K$, which also turns out to be valid in the present simulations, where $E_k^M$ and $E_k^K$ are magnetic and kinetic energies, respectively.

Assuming that the most important nonlinearities involve the turbulent velocity and stationarity of the spectral scaling range of $H_k^M$, a dynamical equilibrium of turbulent advection and the $H_k^M$-increasing effect of helical fluctuations is proposed. This can be formulated straightforwardly using the corresponding dimensionally approximated nonlinear terms from the EDQNM model (for a more detailed derivation see Müller et al. 2012) yielding

$$H_k^K \sim (E_k^K/E_k^M)k^2H_k^M.$$  \hspace{1cm} (2)

This statement about the spectral dynamics of kinetic and magnetic helicities (or, equivalently, kinetic and current helicities since $H_k^J \sim k^2H_k^M$) is also valid for $E_k^K/E_k^M \neq 1$. The agreement of relation (2) with the numerical experiments is however significantly improved by a modification (relation (3)) whose justification is beyond the scope of the presented equilibrium ansatz which basically assumes spectral locality of the inverse cascade

$$H_k^K \sim (E_k^K/E_k^M)^2H_k^M.$$  \hspace{1cm} (3)

Relation (3) is a significant improvement over the earlier relations of similar kind (Pouquet et al. 1976, 2010, Müller and Malapaka 2010). This is shown in figures 2(a) and 3(a), where $\Theta = (E_k^K/E_k^M)\gamma H_k^J/H_k^K$ is shown with $\gamma = 0, 1,$ and $2$ (corresponding to $\Theta_0$, $\Theta_1$, and $\Theta_2$) for the forced and decaying cases, respectively. It is remarkable that relation (3) is only fulfilled in wavenumber intervals where the flux of magnetic helicity is spectrally constant.

This relation brings back the ratio of energies (kinetic to magnetic) into picture, which under the assumption of equipartition of energies was ignored in previous work (Pouquet et al. 1976), while linking the magnetic/current and kinetic helicities. Another interpretation for this expression is the partial Alfvénization of the turbulent flow (Pouquet et al. 2010). Furthermore, it also highlights the influence of kinetic helicity in the inverse cascade of magnetic helicity.

The relation (3) belongs to a class of probably highly universal expressions which are statistically based on the quasi-normal approximation of nonlinear fluxes. It is interesting to note that relation (3) also allows us to determine the spectral scaling
exponent of magnetic helicity from astronomical current-helicity measurements using vector magnetograms (see, e.g. Brandenburg and Subramanian 2005, and the references therein) if kinetic and magnetic energy spectra are also measurable or can be estimated with sufficient accuracy.

The modification of the current-helicity contribution present in relation (3) suggests a corresponding modification to the residual helicity, $H^R = H^K - H^J$, and accordingly to the mean-field dynamo $\alpha$. This, however, has to be taken with care as the present simulations are energetically dominated by the magnetic field although the modifying factor $(E^K/E^M)^2$ should compensate for this. Figures 2(b) and 3(b) allow us to roughly estimate the respective scale-dependent influence of kinetic and magnetic helicity on the modified residual helicity. The spectrum of residual helicity closely follows the spectral kinetic helicity with growing systematic deviations due to the influence of magnetic
helicity at large wavenumbers in both cases. Thus, the modified residual helicity complies with the earlier definitions of $\alpha$ (Pouquet et al. 1976, Krause and Rädler 1980) at large scales.

5. Conclusions

In high-resolution direct numerical simulations of forced and decaying magnetically helical homogeneous MHD turbulence, the nonlinear dynamics of active inverse cascade of magnetic helicity is studied. The simulation results, in particular the observed self-similar spectral scaling of magnetic helicity which contradicts an earlier theoretical explanation (Pouquet et al. 1976), motivate the consideration of velocity field characteristics for the nonlinear evolution of this purely magnetic quantity. This is done with the help of statistical closure theory yielding a possibly universal relation between kinetic and current helicities. The relation is corroborated by the numerical results. Its form, $H_k^K - (E_k^K/E_k^M)^2 H_k^J \sim \text{constant}$, closely resembles the extended definition of the pseudoscalar $\alpha \sim H_k^K - H_k^J$ known from mean-field dynamo theory. The inverse cascade of magnetic helicity is neither a dynamo in itself (as dimensionally $H_k^M \sim k^{-1} E_k^M$) nor even a turbulent cascade in the strict sense, but is a spectral transport process (Müller et al. 2012). It can as a robust and efficient spectral transporter, nevertheless, play a role in the actual realization of turbulent large-scale dynamos like the $\alpha$-dynamo. In this respect, it is interesting that the newly obtained relation includes the squared ratio of kinetic and magnetic energies. This leads to a purely nonlinear quenching of the current-helicity contribution to $\alpha$ that has no direct connection to the dynamo-quenching mechanisms considered so far (of order $(E^M)^{-1}$) in the literature which are seemingly consequences of a combination of boundary conditions and the approximate conservation of magnetic helicity. In this context, it is encouraging that Rheinhardt and Brandenburg (2010) for a homogeneous mean flow with Roberts forcing using a test field method observe $\alpha$-quenching with an $(E^M)^{-2}$ signature. The comparison with this work assumes equivalence of their imposed mean field with the root-mean-square large-scale magnetic fluctuations in the present simulations.

The present relation (3) needs further investigation as it is an additional possible mechanism for dynamo quenching. This new link between kinetic and magnetic helicities in the inverse cascade of magnetic helicity has to be verified in more complex numerical setups such as mean-field dynamos, as well as anisotropic 3D-MHD and isotropic 3D-MHD turbulence with different initial conditions and forcing mechanisms.

Acknowledgments

SKM thanks Prof. B. Despres of JLLL, UPMC, Paris, VI, and CNRS for the financial support they provided to enable attendance at the Rädler fest, Prof. A. Brandenburg and his NORDITA team for hosting him at this fest, as well as Dr. A. Busse currently in Southampton, UK and Prof. D. Hughes at University of Leeds, UK for their help and useful discussions. The authors thank Prof. U. Frisch for useful remarks.
References

Alexakis, A., Mininni, P.D. and Pouquet, A., On the inverse cascade of magnetic helicity. *Astrophys. J.* 2006, **640**, 335–343.

Biskamp, D., *Magnetohydrodynamic Turbulence*. 2003. (Cambridge: Cambridge University Press).

Biskamp, D. and Bremer, U., Dynamics and statistics of inverse cascade processes in 2D magnetohydrodynamic turbulence. *Phys. Rev. Lett.* 1993, **72**, 3819–3822.

Biskamp, D. and Müller, W.-C., Scaling properties of three-dimensional isotropic magnetohydrodynamic turbulence. *Phys. Plasmas* 2000, **7**, 4889–4900.

Blackman, E.G. and Field, G.B., New dynamical mean-field dynamo theory and closure approach. *Phys. Rev. Lett.* 2002, **89**, 265007.

Brandenburg, A., The inverse cascade and nonlinear Alpha-effect in simulations of isotropic helical hydromagnetic turbulence. *Astrophys. J.* 2001, **550**, 824–840.

Brandenburg, A., Advances in theory and simulations of large-scale dynamos. *Space Sci. Rev.* 2009, **144**, 87–104.

Brandenburg, A. and Subramanian, K., Astrophysical magnetic fields and nonlinear dynamo theory. *Phys. Rep.* 2005, **417**, 1–209.

Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang, T.A., *Spectral Methods in Fluid Mechanics*, 1988. (Berlin, New York: Springer-Verlag).

Field, G.B. and Blackman, E.G., Dynamic quenching of the $a^2$ dynamo. *Astrophys. J.* 2002, **572**, 685–692.

Krause, F. and Rädler, K.-H., *Mean Field Magnetohydrodynamics and Dynamo Theory*, 1980. (Berlin: Akademie).

Malapaka, S.K., A study of magnetic helicity in forced and decaying 3D-MHD turbulence, Ph.D. Thesis, University of Bayreuth, Bayreuth, Germany, 2009. Available online at: http://edoc.mpg.de/display.epl?mode=doc&id=464051&col=33&grp=1311 (accessed 4 May 2012).

Moffatt, H.K., The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* 1969, **35**, 117–129.

Moffatt, H.K., *Magnetic Field Generation in Electrically Conducting Fluids*, 1978. (Cambridge: Cambridge University Press).

Müller, W.-C. and Grappin, R., Energy dynamics in magnetohydrodynamic turbulence. *Phys. Rev. Lett.* 2005, **95**, 114502.

Müller, W.-C. and Malapaka, S.K., Understanding nonlinear cascades in magnetohydrodynamic turbulence by statistical closure theory. In *Numerical Modeling of Space Plasma Flows ASTRONUM-2009*, edited by N.V. Pogorelov, E. Audit and G.P. Zank, ASP Conference Series Vol. 429, p. 28, 2010 (Orem, UT: IAU Publications).

Müller, W.-C., Malapaka, S.K. and Busse, A., Inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *Phys. Rev. E* 2012, **85**, 015302.

Pouquet, A., Brachet, M.-E., Lee, E., Mininni, P.D., Rosenberg, D. and Uritsky, V., Lack of universality in MHD turbulence, and the possible emergence of a new paradigm? In *Astrophysical Dynamics: From Stars to Galaxies*, edited by N.H. Brummell, A.S. Brun, M.S. Miesch and Y. Ponty, Proceedings IAU Symposium, Vol. 271, pp. 304–316, 2010. (Cambridge: Cambridge University Press).

Pouquet, A., Frisch, U. and Léorat, J., Strong MHD helical turbulence and the nonlinear dynamo effect. *J. Fluid Mech.* 1976, **77**, 321–354.

Rheinhardt, M. and Brandenburg, A., Test-field method for mean-field coefficients with MHD background. *Astron. Astrophys.* 2010, **520**, A28.

Subramanian, K. and Brandenburg, A., Nonlinear current helicity fluxes in turbulent dynamos and alpha quenching. *Phys. Rev. Lett.* 2004, **93**, 20.