On the genesis of Post constraint in modern electromagnetism

Akhlesh Lakhtakia\(^1\)

Computational & Theoretical Materials Sciences Group (CATMAS)  
Department of Engineering Science & Mechanics  
Pennsylvania State University, University Park, PA 16802–6812, USA

Abstract: The genesis of the Post constraint is premised on two attributes of modern electromagnetism: (i) its microscopic nature, and (ii) the status of $\mathbf{e}(\mathbf{x}, t)$ and $\mathbf{b}(\mathbf{x}, t)$ as the primitive electromagnetic fields. This constraint can therefore not arise in EH–electromagnetism, wherein the primitive electromagnetic fields are the macroscopic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$. Available experimental evidence against the Post constraint is incomplete and inconclusive.

Keywords: Electromagnetic theories; Free space; Macroscopic physics; Magnetoelectric materials; Microphysics; Post constraint; Tellegen medium

1 Introduction

Ever since its enunciation in 1962 [1], the Post constraint has been an enigma. It was ignored for over three decades by the electromagnetics community for reasons that will probably be extracted only by future historians of science. It arose from obscurity like a phoenix in 1994 in the context of linear, nonreciprocal, biisotropic mediums [2], and since then has been the subject of discussion in the complex–mediums electromagnetics research community.

A remarkable feature of the Post constraint is that it permits a sharp distinction between two widely prevalent conceptions of electromagnetic phenomena. The genesis of the Post constraint lies in the microphysical basis of modern electromagnetism, whereby the (necessarily macroscopic) constitutive functions must be conceived as piecewise homogeneous entities and can therefore not vary continuously in spacetime. In contrast, EH–electromagnetism is essentially macroscopic, and its principles seem to be inimical to the validity of the Post constraint. Available experimental evidence does not negate the Post constraint, but cannot be held to be conclusive either.

These issues are discussed in this essay. Section 2 is an exposition of modern electromagnetism encompassing both the microscopic and the macroscopic levels. Section 3 presents the rationale for and the genesis of the Post constraint. The characteristics of EH–electromagnetism relevant to the Post constraint are given in Section 4, while experimental evidence is reviewed in Section 5. Finally, in Section 6 the constitutive equations of free space are deduced in relation to the Post constraint.

\(^{1}\)Tel: +1 814 863 4319; Fax: +1 814 865 9974; E–mail: akhlesh@psu.edu
2 Modern Electromagnetism

Electromagnetism today is a microscopic science, even though it is mostly used in its macroscopic form. It was certainly a macroscopic science when Maxwell unified the equations of Coulomb, Gauss, Faraday, and Ampère, added a displacement current to Ampère’s equation, and produced the four equations to which his name is attached. Although Maxwell had abandoned a mechanical basis for electromagnetism during the early 1860s, and even used terms like molecular vortices, a close reading [3] of his papers will convince the reader that Maxwell’s conception of electromagnetism — like that of most of his contemporaries — was macroscopic.

By the end of the 19th century, that conception had been drastically altered [4]. Hall’s successful explanation of the eponymous effect, the postulation of the electron by Stoney and its subsequent discovery by Thomson, and Larmor’s theory of the electron precipitated that alteration. It was soon codified by Lorentz and Heaviside, so that the 20th century dawned with the acquisition of a microphysical basis by electromagnetism. Maxwell’s equations remained unaltered in form at macroscopic length scales, but their roots now lie in the fields engendered by microscopic charge quantums. The subsequent emergence of quantum mechanics did not change the form of the macroscopic equations either, although the notion of a field lost its determinism and an inherent uncertainty was recognized in the measurements of key variables [5].

2.1 Microscopic Maxwell Postulates

The microscopic fields are just two: the electric field $\tilde{E}(x, t)$ and the magnetic field $\tilde{B}(x, t)$. These two are accorded the status of primitive fields in modern electromagnetism. Both fields vary extremely rapidly as functions of position $x$ and time $t$. Their sources are the microscopic charge density $\tilde{c}(x, t)$ and the microscopic current density $\tilde{j}(x, t)$, where

$$\tilde{c}(x, t) = \sum_{\ell} q_\ell \delta [x - x_\ell(t)], \quad (1)$$

$$\tilde{j}(x, t) = \sum_{\ell} q_\ell v_\ell(t) \delta [x - x_\ell(t)]. \quad (2)$$

$\delta(\bullet)$ is the Dirac delta function; while $x_\ell(t)$ and $v_\ell(t)$ are the position and the velocity of the point charge $q_\ell$. Uncertainties in the measurements of the positions and the velocities of the discrete point charges open the door to quantum mechanics, but we need not traverse that path here.

All of the foregoing fields and sources appear in the microscopic Maxwell postulates:

$$\nabla \cdot \tilde{E}(x, t) = \varepsilon_0^{-1} \tilde{c}(x, t), \quad (3)$$

$$\nabla \times \tilde{B}(x, t) - \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \tilde{E}(x, t) = \mu_0 \tilde{j}(x, t), \quad (4)$$

$$\nabla \cdot \tilde{B}(x, t) = 0, \quad (5)$$

$$\nabla \times \tilde{E}(x, t) + \frac{\partial}{\partial t} \tilde{B}(x, t) = 0. \quad (6)$$

2The lower-case letter signifies that the quantity is microscopic, while the tilde”indicates dependence on time.
In these equations and hereafter, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ H/m are the permittivity and the permeability of free space (i.e., vacuum), respectively. The first two postulates are inhomogeneous differential equations as they contain source terms on their right sides, while the last two are homogeneous differential equations.

2.2 Macroscopic Maxwell Postulates

Macroscopic measuring devices average over (relatively) large spatial and temporal intervals. Therefore, spatiotemporal averaging of the microscopic quantities appears necessary in order to deduce the macroscopic Maxwell postulates from (3)–(6). Actually, only spatial averaging is necessary [6], because it implies temporal averaging due to the finite magnitude of the universal maximum speed $(\epsilon_0\mu_0)^{-1/2}$. Denoting the macroscopic charge and current densities, respectively, by $\tilde{\rho}(\mathbf{x}, t)$ and $\tilde{\mathbf{J}}(\mathbf{x}, t)$, we obtain the macroscopic Maxwell postulates

\[
\nabla \cdot \tilde{\mathbf{E}}(\mathbf{x}, t) = \epsilon_0^{-1} \tilde{\rho}(\mathbf{x}, t), \tag{7}
\]

\[
\nabla \times \tilde{\mathbf{E}}(\mathbf{x}, t) - \epsilon_0\mu_0 \frac{\partial}{\partial t} \tilde{\mathbf{E}}(\mathbf{x}, t) = \mu_0 \tilde{\mathbf{J}}(\mathbf{x}, t), \tag{8}
\]

\[
\nabla \cdot \tilde{\mathbf{B}}(\mathbf{x}, t) = 0, \tag{9}
\]

\[
\nabla \times \tilde{\mathbf{E}}(\mathbf{x}, t) + \frac{\partial}{\partial t} \tilde{\mathbf{B}}(\mathbf{x}, t) = 0, \tag{10}
\]

which involve the macroscopic primitive fields $\tilde{\mathbf{E}}(\mathbf{x}, t)$ and $\tilde{\mathbf{B}}(\mathbf{x}, t)$ as the spatial averages of $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, respectively. From (7) and (8), a macroscopic continuity equation for the source densities can be derived as

\[
\nabla \cdot \tilde{\mathbf{J}}(\mathbf{x}, t) + \frac{\partial}{\partial t} \tilde{\rho}(\mathbf{x}, t) = 0. \tag{11}
\]

2.3 Familiar Form of Macroscopic Maxwell Postulates

Equations (7)–(10) are not the familiar form of the macroscopic Maxwell postulates, even though they hold in free space as well as in matter. The familiar form emerges after the recognition that matter contains, in general, both free charges and bound charges. Free and bound source densities can be decomposed as

\[
\tilde{\rho}(\mathbf{x}, t) = \tilde{\rho}_{so}(\mathbf{x}, t) - \nabla \cdot \tilde{\mathbf{P}}(\mathbf{x}, t), \tag{12}
\]

and

\[
\tilde{\mathbf{J}}(\mathbf{x}, t) = \tilde{\mathbf{J}}_{so}(\mathbf{x}, t) + \frac{\partial}{\partial t} \tilde{\mathbf{P}}(\mathbf{x}, t) + \nabla \times \tilde{\mathbf{M}}(\mathbf{x}, t). \tag{13}
\]

This decomposition is consistent with (11), provided the free source densities obey the reduced continuity equation

\[
\nabla \cdot \tilde{\mathbf{J}}_{so}(\mathbf{x}, t) + \frac{\partial}{\partial t} \tilde{\rho}_{so}(\mathbf{x}, t) = 0. \tag{14}
\]

The free source densities represent “true” sources which can be externally impressed. Whereas $\tilde{\mathbf{J}}_{so}(\mathbf{x}, t)$ is the source current density, $\tilde{\rho}_{so}(\mathbf{x}, t)$ is the source charge density.
Bound source densities represent matter in its macroscopic form and are, in turn, quantified by the polarization $\tilde{P}(x, t)$ and the magnetization $\tilde{M}(x, t)$. Both $\tilde{P}(x, t)$ and $\tilde{M}(x, t)$ are nonunique to the extent that they can be replaced by $\tilde{P}(x, t) - \nabla \times \tilde{A}(x, t)$ and $\tilde{M}(x, t) + (\partial/\partial t) \tilde{A}(x, t)$, respectively, in (12) and (13) without affecting the left sides of either equation.

Polarization and magnetization are subsumed in the definitions of the electric induction $\tilde{D}(x, t)$ and the magnetic induction $\tilde{H}(x, t)$ as follows:

\begin{align*}
\tilde{D}(x, t) &= \varepsilon_0 \tilde{E}(x, t) + \tilde{P}(x, t), \\
\tilde{H}(x, t) &= \mu_0^{-1} \tilde{B}(x, t) - \tilde{M}(x, t).
\end{align*}

Then, (7)–(10) metamorphose into the familiar form of the macroscopic Maxwell postulates:

\begin{align*}
\nabla \cdot \tilde{D}(x, t) &= \tilde{\rho}_{so}(x, t), \\
\nabla \times \tilde{H}(x, t) - \frac{\partial}{\partial t} \tilde{D}(x, t) &= \tilde{J}_{so}(x, t), \\
\nabla \cdot \tilde{B}(x, t) &= 0, \\
\nabla \times \tilde{E}(x, t) + \frac{\partial}{\partial t} \tilde{B}(x, t) &= 0.
\end{align*}

Let us note, in passing, that the fields $\tilde{d}(x, t)$ and $\tilde{h}(x, t)$ do not exist in microphysics, matter being an ensemble of point charges in free space.

### 2.4 Linear Constitutive Relations

The induction fields at some point in spacetime $(x, t)$ can depend locally on the primitive fields at the same $(x, t)$. This dependence can be spatially nonhomogeneous (i.e., dependent on space $x$) and/or can vary with time $t$ (i.e., age). In addition, the induction fields at $(x, t)$ can depend nonlocally on the primitive fields at some $(x - x_h, t - t_h)$, where the spacetime interval $(x_h, t_h)$, $t_h \geq 0$, must be timelike in order to be causally influential [7, pp. 85–89]. Thus, the most general linear constitutive relations [8]

\begin{align*}
\tilde{D}(x, t) &= \int \int \tilde{\xi}(x; x_h, t_h) \cdot \tilde{E}(x - x_h, t - t_h) \, dx_h \, dt_h \\
&\quad + \int \int \tilde{\zeta}(x; x_h, t_h) \cdot \tilde{B}(x - x_h, t - t_h) \, dx_h \, dt_h \quad (21)
\end{align*}

and

\begin{align*}
\tilde{H}(x, t) &= \int \int \tilde{\xi}(x; x_h, t_h) \cdot \tilde{E}(x - x_h, t - t_h) \, dx_h \, dt_h \\
&\quad + \int \int \tilde{\zeta}(x; x_h, t_h) \cdot \tilde{B}(x - x_h, t - t_h) \, dx_h \, dt_h \quad (22)
\end{align*}

can describe any linear medium — indeed, the entire universe after linearization. The integrals extend only over the causal values of $(x_h, t_h)$, but that does not restrict the analysis presented here.
3 The Post Constraint

Four second–rank tensors appear in the foregoing constitutive relations: \( \tilde{\varepsilon} \) is the permittivity tensor, \( \tilde{\nu} \) is the impermeability tensor, while \( \tilde{\xi} \) and \( \tilde{\zeta} \) are the magnetoelectric tensors. Together, these four tensors contain 36 scalar functions; but the Post constraint indicates that only 35, at most, are independent. This was clarified elsewhere [9] using 4–tensor notation, but we revisit the issue here for completeness. Let us therefore express the magnetoelectric tensors as

\[
\tilde{\xi}(x, t; x_h, t_h) = \tilde{\alpha}(x, t; x_h, t_h) + \frac{1}{6} I \tilde{\Psi}(x, t; x_h, t_h),
\]

and

\[
\tilde{\zeta}(x, t; x_h, t_h) = \tilde{\beta}(x, t; x_h, t_h) - \frac{1}{6} I \tilde{\Psi}(x, t; x_h, t_h),
\]

where \( I \) is the identity tensor and the scalar function

\[
\tilde{\Psi}(x, t; x_h, t_h) = \text{Trace}\left( \tilde{\xi}(x, t; x_h, t_h) - \tilde{\zeta}(x, t; x_h, t_h) \right).
\]

Therefore,

\[
\text{Trace}\left( \tilde{\alpha}(x, t; x_h, t_h) - \tilde{\beta}(x, t; x_h, t_h) \right) \equiv 0.
\]

3.1 Rationale for the Post Constraint

Let us recall that (19) and (20) do not contain the induction fields \( \tilde{D}(x, t) \) and \( \tilde{H}(x, t) \). Hence, (21) and (22) must be substituted only in (17) and (18); thus,

\[
\int \int \nabla \cdot \left( \tilde{\varepsilon}(x, t; x_h, t_h) \cdot \tilde{E}(x - x_h, t - t_h) \right) d\mathbf{x}_h dt_h
\]

\[
+ \frac{1}{6} \int \int \tilde{\Psi}(x, t; x_h, t_h) \left( \nabla \cdot \tilde{B}(x - x_h, t - t_h) \right) d\mathbf{x}_h dt_h
\]

\[
+ \frac{1}{6} \int \int \left( \nabla \tilde{\Psi}(x, t; x_h, t_h) \right) \cdot \tilde{B}(x - x_h, t - t_h) d\mathbf{x}_h dt_h
\]

\[
= \tilde{\rho}_{so}(x, t)
\]
and

\[\begin{align*}
\int \int \nabla \times \left( \tilde{\beta}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) + \tilde{\alpha}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
- \int \int \frac{\partial}{\partial t} \left( \tilde{\beta}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) + \tilde{\alpha}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
- \frac{1}{6} \int \int \tilde{\Psi}(\mathbf{x}, t; \mathbf{x}_h, t_h) \left( \nabla \times \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) + \frac{\partial}{\partial t} \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
- \frac{1}{6} \int \int \left( \nabla \tilde{\Psi}(\mathbf{x}, t; \mathbf{x}_h, t_h) \times \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) d\mathbf{x}_h dt_h \\
- \frac{1}{6} \int \int \left( \frac{\partial}{\partial t} \tilde{\Psi}(\mathbf{x}, t; \mathbf{x}_h, t_h) \right) \cdot \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
= \tilde{J}_{so}(\mathbf{x}, t) .
\end{align*}\]

(28)

The second integral on the left side of (27) is null-valued by virtue of (19); likewise, the third integral on the left side of (28) is null-valued by virtue of (20). Therefore, the four macroscopic Maxwell postulates now read as follows:

\[\begin{align*}
\int \int \nabla \cdot \left( \tilde{\beta}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) + \tilde{\alpha}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
+ \frac{1}{6} \int \int \left( \nabla \tilde{\Psi}(\mathbf{x}, t; \mathbf{x}_h, t_h) \right) \cdot \tilde{B}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
= \tilde{\rho}_{so}(\mathbf{x}, t) ,
\end{align*}\]

(29)

\[\begin{align*}
\int \int \nabla \times \left( \tilde{\beta}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) + \tilde{\alpha}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
- \int \int \frac{\partial}{\partial t} \left( \tilde{\beta}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) + \tilde{\alpha}(\mathbf{x}, t; \mathbf{x}_h, t_h) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
- \frac{1}{6} \int \int \left( \nabla \tilde{\Psi}(\mathbf{x}, t; \mathbf{x}_h, t_h) \times \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) \right) d\mathbf{x}_h dt_h \\
- \frac{1}{6} \int \int \left( \frac{\partial}{\partial t} \tilde{\Psi}(\mathbf{x}, t; \mathbf{x}_h, t_h) \right) \cdot \tilde{E}(\mathbf{x} - \mathbf{x}_h, t - t_h) d\mathbf{x}_h dt_h \\
= \tilde{J}_{so}(\mathbf{x}, t) ,
\end{align*}\]

(30)

\[\nabla \cdot \tilde{B}(\mathbf{x}, t) = 0 ,
\]

(31)

\[\nabla \times \tilde{E}(\mathbf{x}, t) + \frac{\partial}{\partial t} \tilde{B}(\mathbf{x}, t) = 0 .
\]

(32)
Differentiation of the product of two functions is distributive. Hence, the thirty-five independent constitutive scalars in \( \tilde{\epsilon}, \tilde{\alpha}, \tilde{\beta}, \) and \( \tilde{\nu} \) occur in (29)–(32) in two ways: (i) by themselves, and (ii) through their space– and time–derivatives. In contrast, the thirty-sixth constitutive scalar \( \tilde{\Psi} \) does not occur in (29)–(32) by itself. Thus, \( \tilde{\Psi} \) vanished from the macroscopic Maxwell postulates like the Cheshire cat, but left behind its derivatives like the cat’s grin.

This is an anomalous situation, and its elimination leads to the Post constraint.

### 3.2 Post’s Conclusions

In a seminal contribution on the covariant structure of modern electromagnetism [1], Post made a distinction between functional and structural fields. Functional fields specify the state of a medium, and are exemplified by \( \tilde{E} \) and \( \tilde{B} \). Structural fields, exemplified by the constitutive tensors, specify the properties of the medium. Formulating the Lagrangian and examining its Eulerian derivative [1, Eq. 5.31], Post arrived at the conclusion that

\[
\tilde{\Psi}(x, t; x_h, t_h) \equiv 0 \quad (33)
\]

even for nonhomogeneous mediums [1, p. 130]. Furthermore, he held that the space– and time–derivatives of \( \tilde{\Psi}(x, t; x_h, t_h) \) are also identically zero, so that [1, p. 129]

\[
\begin{aligned}
\nabla \tilde{\Psi}(x, t; x_h, t_h) & \equiv 0 \\
\frac{\partial}{\partial t} \tilde{\Psi}(x, t; x_h, t_h) & \equiv 0
\end{aligned}
\quad (34)
\]

Equations (33) and (34) may appear to be independent but are not, because the derivatives of a constant function are zero. Equation (33) alone is called the Post constraint.

### 3.3 Recognizable Existence of \( \tilde{\Psi} \)

Whether \( \tilde{\Psi} \) is identically null–valued or not is a moot point. The real issue is whether it has a recognizable existence or not. This stance was adopted by Lakhtakia and Weiglhofer [10].

Let us recall that all matter is microscopic. Despite the convenience proffered by continuum theories, those theories are merely approximations. Constitutive functions are macroscopic entities arising from the homogenization of assemblies of microscopic charge carriers, with free space serving as the reference medium [11]. In any small enough portion of spacetime that is homogenizable, the constitutive functions are uniform. When such a portion will be interrogated for characterization, it will have to be embedded in free space. Accordingly, the second integral on the left side of (29) as well as the third as well as the fourth integrals on the left side of (30) would vanish during the interrogation for fields inside and outside that piece. Therefore, the principle of parsimony (attributed to a 14th century monk [12]) enjoins the acceptance of (33).

### 3.4 Nature of the Post Constraint

When linear mediums of increasing complexity are investigated, the nature of the Post constraint can appear to vary. For instance, were investigation confined to isotropic mediums [13], the
condition $\tilde{\Psi} \equiv 0$ can resemble a *reciprocity constraint*. But it is not, because it does not impose any transpose–symmetry requirements on $\tilde{\varepsilon}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{\nu}$ [14, Eqs. 23].

Another possibility is to think that the Post constraint negates the generalized duality transformation [15], but actually it does not when it is globally applied at the microscopic level [16, pp. 203–204]. Finally, the Post constraint is not a gauge transformation — i.e., a $\Psi$–independent field $\tilde{A}$ cannot be found to replace $\tilde{P}$ and $\tilde{M}$ by $\tilde{P} - \nabla \times \tilde{A}$ and $\tilde{M} + (\partial/\partial t) \tilde{A}$, respectively, in order to eliminate $\tilde{\Psi}$.

The Post constraint is actually a *structural constraint*. Post may have been inspired towards it in order to eliminate a pathological constitutive relation [1, Eq. 3.20], [17], and then established a covariance argument for it. Physically, this constraint arises from the following two considerations:

• The Ampère–Maxwell equation (containing the induction fields) should be independent of the Faraday equation (containing the primitive fields) at the macroscopic level, just as the two equations are mutually independent at the microscopic level.

• The constitutive functions must be characterized as piecewise uniform, being born of the spatial homogenization of microscopic entities. Therefore, if a homogeneous piece of a medium with a certain set of electromagnetic response properties cannot be recognized, the assumption of continuously nonhomogeneous analogs of that set is untenable.

### 4 EH–Electromagnetism

Time–domain electromagnetic research is a distant second to frequency–domain electromagnetic research, as measured by the numbers of publications as well as the numbers of researchers. Much of frequency–domain research at the macroscopic level also commences with the familiar form (17)–(20) of the Maxwell postulates, but the roles of $\tilde{H}$ and $\tilde{E}$ are interchanged [11].

Thus, constitutive relations are written to express $\tilde{D}$ and $\tilde{B}$ in terms of $\tilde{E}$ and $\tilde{H}$. Specifically, the linear constitutive relations (21) and (22) are replaced by

\[
\tilde{D}(x, t) = \int \int \tilde{A}(x, t; x_h, t_h) \cdot \tilde{E}(x - x_h, t - t_h) \, dx_h \, dt_h
+ \int \int \tilde{B}(x, t; x_h, t_h) \cdot \tilde{H}(x - x_h, t - t_h) \, dx_h \, dt_h
\]

and

\[
\tilde{B}(x, t) = \int \int \tilde{C}(x, t; x_h, t_h) \cdot \tilde{E}(x - x_h, t - t_h) \, dx_h \, dt_h
+ \int \int \tilde{D}(x, t; x_h, t_h) \cdot \tilde{H}(x - x_h, t - t_h) \, dx_h \, dt_h,
\]

with $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ as the constitutive tensors. This version of electromagnetism is called the EH–electromagnetism in this essay.
At first glance, the difference between the modern and the EH versions may not appear to be significant, particularly for linear mediums at the macroscopic level. The frequency–domain versions of the constitutive tensors $\tilde{A}$, etc., can also be microscopically motivated in much the same way as the frequency–domain versions of $\tilde{\varepsilon}$, etc., are. Yet, there is a huge difference: The Faraday equation contains only the primitive fields while the Ampère–Maxwell equation contains only the induction fields, in modern electromagnetism, and can therefore be independent of each other just as at the microscopic level. But each of the two equations contains a primitive field and an induction field in EH–electromagnetism — hence, it is impossible for the two equations to be independent of each other at the macroscopic level. This central difference between the two versions of electromagnetism is often a source of great confusion.

### 4.1 Post Constraint

As both the Faraday and the Ampère–Maxwell equations (at the macroscopic level) contain a primitive field and an induction field in EH–electromagnetism, it appears impossible to derive the Post constraint in the EH version. Not surprisingly, current opposition to the validity of the Post constraint invariably employs the EH version [15, 18], and older constructs that presumably invalidate the Post constraint are also based on EH–electromagnetism [19, 20, 21]. The major exception to the previous statement is the work of O’Dell [22, pp. 38–44], but it is fatally marred by the assumption of purely instantaneous — and, therefore, noncausal — constitutive relations. Simply put, the Post constraint is valid in modern electromagnetism but probably invalid in EH–electromagnetism.

But we hold modern electromagnetism to be truer than its EH counterpart [6, 23, 24, 25]. Accordingly, the Post constraint can translated from the former to the latter, in certain circumstances. For example, let us consider a spatially homogeneous, temporally invariant and spatially local medium: $\tilde{\varepsilon}(x, t; x_h, t_h) \equiv \tilde{\varepsilon}(t_h) \delta(x_h)$, etc. Employing the temporal Fourier transform

$$\tilde{Z}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(x, \omega) \exp(-i\omega t) \, d\omega,$$

(37)

where $\omega$ is the angular frequency and $i = \sqrt{-1}$, we see that (21) and (22) transform to

$$D(x, \omega) = \varepsilon(\omega) \cdot E(x, \omega) + \xi(\omega) \cdot B(x, \omega) \right\},$$

$$H(x, \omega) = \zeta(\omega) \cdot E(x, \omega) + \nu(\omega) \cdot B(x, \omega) \right\},$$

(38)

while (35) and (36) yield

$$\frac{D(x, \omega)}{B(x, \omega)} = \frac{A(\omega)}{C(\omega)} \cdot \frac{E(x, \omega)}{H(x, \omega)} + \frac{B(\omega)}{D(\omega)} \cdot \frac{E(x, \omega)}{H(x, \omega)} \right\}.$$

(39)

With the assumption that $D(\omega)$ is invertible, the Post constraint

$$\Psi(\omega) \equiv 0$$

(40)

translates into the condition [26]

$$\text{Trace} \left( \frac{B(\omega)}{D(\omega)} \cdot D^{-1}(\omega) + D^{-1}(\omega) \cdot C(\omega) \right) \equiv 0$$

(41)

Whereas all quantities decorated with a tilde $\tilde{\cdot}$ are real–valued, their undecorated counterparts are complex–valued in general.
for EH–electromagnetism; equivalently,
\[
\text{Trace} \left[ \mathcal{D}^{-1}(\omega) \cdot \left( \mathcal{B}(\omega) + \mathcal{C}(\omega) \right) \right] \equiv 0.
\]  \hspace{1cm} (42)

We must remember, however, that (42) is probably undervivable within the framework of EH–electromagnetism, but is simply a translation of (40).

5 Experimental Evidence

Fundamental questions are answered by a convergence of theoretical constructs and diverse experimental evidence. On this basis, modern electromagnetism is well–established, which provides confidence in the validity of the Post constraint. Furthermore, incontrovertible experimental results against the Post constraint are unknown. Nevertheless, the constraint is experimentally falsifiable, and available experimental evidence presented against it must not be dismissed lightly. Let us examine that evidence now.

5.1 Magnetoelectric Materials

Anisotropic materials with magnetoelectric tensors are commonplace. Typically, such properties are exhibited at low frequencies and low temperatures. Although their emergence in research literature can be traced back to Pierre Curie [27], a paper published originally in 1959 [20] focused attention on them. O’Dell wrote a famous book on these materials [22] in 1970.

A significant result of O’Dell [22, Eq. 2.64], although derived for spatiotemporally uniform and spatiotemporally local mediums (i.e., \( \tilde{\varepsilon}(x, t; x_h, t_h) = \delta(x_h) \delta(t_h) \), etc.), is often used in frequency–domain literature on spatiotemporally uniform and spatially local mediums as follows:
\[
\text{Transpose} \left( \tilde{\xi}(\omega) \right) = -\zeta(\omega).
\]  \hspace{1cm} (43)

This equation is often held to allow materials for which \( \Psi(\omega) \neq 0 \). More importantly, this equation is widely used in the magnetoelectric research community to reduce experimental tedium in characterizing magnetoelectric materials. Yet this equation is based on a false premise: that materials (as distinct from free space) respond purely instantaneously [22, p. 43]. Hence, experimental data obtained after exploiting (43) cannot be trusted [28].

The false premise can be traced back to Dzyaloshinskii’s 1959 paper [20], wherein EH–electromagnetism was used. Astrov [29] examined the variation of \( \mathcal{C}(\omega) \) of \( \text{Cr}_2\text{O}_3 \) with temperature at 10 kHz frequency. Folen et al. [30] measured \( \mathcal{C}(\omega) \) of \( \text{Cr}_2\text{O}_3 \) at 1 kHz frequency and presumably equated it to \( \mathcal{B}(\omega) \) by virtue of the 1959 antecedent [31], but did not actually measure \( \mathcal{B}(\omega) \).\(^4\) Rado and Folen [32, 33] verified the existences of both \( \mathcal{B}(\omega) \) and \( \mathcal{C}(\omega) \) for the same substance, and they also established that both quantities are temperature–dependent, but they too did not measure \( \mathcal{B}(\omega) \). Similar deficiencies in other published reports have been detailed elsewhere [28]. Recently, Raab [34] has rightly called for comprehensive and complete characterization of magnetoelectric materials, with (43) not assumed in advance but actually subjected to a test.

\(^4\)This and a large fraction of other published reports do not seem to recognize that \( \mathcal{C}(\omega) \), etc., are complex–valued quantities, but treat them as real–valued quantities.
5.2 Tellegen Medium

Take a fluid medium in which permanent, orientable, electric dipoles exist in abundance. Stir in small ferromagnetic particles with permanent magnetic dipole moments, ensuring that each electric dipole moment cleaves together with a parallel magnetic dipole moment, to form a Tellegen particle [18]. Shake well for a homogeneous, isotropic suspension of Tellegen particles. This is the recipe that Tellegen [19] gave for the so-called Tellegen medium, after he had conceptualized the gyrator.

The frequency–domain constitutive relations of this medium may be set down as

\[
\begin{align*}
\mathbf{D}(x, \omega) &= A(\omega) \mathbf{E}(x, \omega) + B(\omega) \mathbf{H}(x, \omega) \\
\mathbf{B}(x, \omega) &= B(\omega) \mathbf{E}(x, \omega) + D(\omega) \mathbf{H}(x, \omega)
\end{align*}
\]

with the assumption of temporal invariance, spatial homogeneity, spatial locality, and isotropy. Furthermore, (44) apply only at sufficiently low frequencies [35].

Gyrators have been approximately realized using other circuit elements, but the Tellegen medium has never been successfully synthesized. Tellegen’s own experiments failed [19, p. 96] Neither has the Tellegen medium been observed in nature. Hence, non–zero values of \(B(\omega)\) of actual materials are not known. A fairly elementary exercise shows that the recognizable existence of this medium is tied to that of irreducible magnetic sources [36, 37]. As the prospects of observing a magnetic monopole are rather remote [38, 39], for now it is appropriate to regard the Tellegen medium as chimerical.

5.3 Tellegen Particle

Each particle in Tellegen’s recipe is actually a uniaxial particle [40]. Because the recipe calls for the suspension to be homogeneous, the particles cannot be similarly oriented. However, if all particles were similarly oriented in free space, and the number density \(N_p\) of the particles is very small, the frequency–domain constitutive relations of the suspension at sufficiently low frequencies will be

\[
\begin{align*}
\mathbf{D}(x, \omega) \simeq \varepsilon_0 \mathbf{E}(x, \omega) + N_p \left( \pi^{(ee)}(\omega) \cdot \mathbf{E}(x, \omega) + \pi^{(eh)}(\omega) \cdot \mathbf{H}(x, \omega) \right) \\
\mathbf{B}(x, \omega) \simeq \mu_0 \mathbf{H}(x, \omega) + N_p \left( \pi^{(he)}(\omega) \cdot \mathbf{E}(x, \omega) + \pi^{(hh)}(\omega) \cdot \mathbf{H}(x, \omega) \right)
\end{align*}
\]

wherein \(\pi^{(ee)}, \text{ etc.}\), are the polarizability tensors of a Tellegen particle in free space.

A recent report [18] provides experimental evidence on the existence of \(\pi^{(eh)}\) for a Tellegen particle made by sticking a short copper wire to a ferrite sphere biased by a quasistatic magnetic field parallel to the wire. However, this work can not lead to any significant finding against the validity of the Post constraint for the following two reasons:

- Although a quantity proportional to the magnitude of \(\text{Trace}(\pi^{(eh)})\) was measured, a similar measurement of \(\text{Trace}(\pi^{(he)})\) was not undertaken; instead, the identity

\[
\text{Trace}(\pi^{(he)}(\omega)) = \text{Trace}(\pi^{(eh)}(\omega))
\]

11
was assumed without testing. This deficiency in experimentation is similar to that for magnetolectric materials mentioned in Section 5.1.

- The Post constraint is supposed to hold rigorously for linear electromagnetic response with respect to the total electromagnetic field, which is constituted jointly by the bias magnetic field as well as the time–harmonic electromagnetic field. As discussed by Chen [41], the ferrite is therefore a nonlinear material.

Incidentally, the biased–ferrite–metal–wire modality for Tellegen particles is likely to be very difficult to implement to realize the Tellegen medium of Section 5.2.

5.4 Summation of Experimental Evidence

On reviewing Sections 5.1–5.3, it becomes clear that experimental evidence against the validity of the Post constraint is incomplete and inconclusive, in addition to being based either on the false premise of purely instantaneous response and/or derived from EH–electromagnetism.

6 Post Constraint and Free Space

Although the Post constraint holds for modern electromagnetism, which has a microscopic basis in that matter is viewed as an assembly of charge–carriers in free space, before concluding this essay it is instructive to derive the constitutive equations of free space back from the macroscopic constitutive equations (21) and (22).

Let us begin with free space being spatiotemporally invariant and spatiotemporally local; then, \( \tilde{\varepsilon}(x, t; x_h, t_h) = \tilde{\varepsilon}_{fs} \delta(x_h) \delta(t_h) \), etc., and (21) and (22) simplify to

\[
\begin{align*}
\tilde{D}(x, t) &= \tilde{\varepsilon}_{fs} \cdot \tilde{E}(x, t) + \tilde{\xi}_{fs} \cdot \tilde{B}(x, t) \\
\tilde{H}(x, t) &= \tilde{\zeta}_{fs} \cdot \tilde{E}(x, t) + \tilde{\nu}_{fs} \cdot \tilde{B}(x, t)
\end{align*}
\]

The free energy being a perfect differential, and because the constitutive relations (47) do not involve convolution integrals, it follows that [1, Eq. 6.14]

\[
\text{Transpose}(\tilde{\xi}_{fs}) = -\tilde{\zeta}_{fs}.
\]

With the additional requirement of isotropy, we get

\[
\begin{align*}
\tilde{D}(x, t) &= \tilde{\varepsilon}_{fs} \cdot \tilde{E}(x, t) + \tilde{\xi}_{fs} \cdot \tilde{B}(x, t) \\
\tilde{H}(x, t) &= -\tilde{\xi}_{fs} \cdot \tilde{E}(x, t) + \tilde{\nu}_{fs} \cdot \tilde{B}(x, t)
\end{align*}
\]

The subsequent imposition of the Post constraint means that \( \xi_{fs} = 0 \), and the constitutive relations

\[
\begin{align*}
\tilde{D}(x, t) &= \tilde{\varepsilon}_{fs} \cdot \tilde{E}(x, t) \\
\tilde{H}(x, t) &= \tilde{\nu}_{fs} \cdot \tilde{B}(x, t)
\end{align*}
\]

finally emerge. The values \( \tilde{\varepsilon}_{fs} = \varepsilon_0 \) and \( \tilde{\nu}_{fs} = 1/\mu_0 \) are used in SI [25]. Although Lorentz–reciprocity was not explicitly enforced for free space, it emerges naturally in this exercise [42]. Alternatively, it could have been enforced from the very beginning, and it would have led to \( \tilde{\xi}_{fs} = 0 \) [43].
7 Concluding Remarks

Despite the fact that the mathematical forms of the macroscopic Maxwell postulates are identical in modern electromagnetism as well as in EH–electromagnetism, the two are very physically very different. Modern electromagnetism is held to be basic; hence, the answers to all fundamental questions must be decided within its framework. Thereafter, if necessary, its equations can be transformed into the frequency domain and then into those of EH–electromagnetism — and the resulting equations may be used to solve any problems that a researcher may be interested in. The reverse transition from EH–electromagnetism to modern electromagnetism can lead to false propositions.

Acknowledgment

Occasional discussions with Dr. E.J. Post are gratefully acknowledged.

References

[1] Post EJ: Formal Structure of Electromagnetics. North–Holland, Amsterdam, The Netherlands 1962; Dover Press, New York, NY, USA 1997.

[2] Lakhtakia A, Weiglhofer WS: Are linear, nonreciprocal, bi-isotropic media forbidden? IEEE Trans. Microw. Theory Techn. 42 (1994) 1715–1716.

[3] Simpson TK: Maxwell on the Electromagnetic Field: A Guided Study. Rutgers University Press, New Brunswick, NJ, USA 1997.

[4] Buchwald JZ: From Maxwell to Microphysics. University of Chicago Press, Chicago, IL, USA 1985.

[5] Schwinger J (ed): Selected Papers on Quantum Electrodynamics. Dover Press, New York, NY, USA 1958.

[6] Jackson JD: Classical Electrodynamics, 3rd ed. Wiley, New York, NY, USA 1999; Sec. 6.6.

[7] Lucas JR, Hodgson PE: Spacetime and Electromagnetism. Clarendon Press, Oxford, United Kingdom 1990.

[8] Lakhtakia A, Weiglhofer WS: Are field derivatives needed in linear constitutive relations? Int. J. Infrared Millim. Waves 19 (1998) 1073–1082.

[9] Lakhtakia A, Weiglhofer WS: Constraint on linear, spatiotemporally nonlocal, spatiotemporally nonhomogeneous constitutive relations. Int. J. Infrared Millim. Waves 17 (1996) 1867–1878.

[10] Weiglhofer WS, Lakhtakia A: The Post constraint revisited. Arch. Elektr. Über. 52 (1998) 276–279.
[11] Weiglhofer WS: Constitutive characterization of simple and complex mediums. In: Weiglhofer WS, Lakhtakia A: Introduction to Complex Mediums for Optics and Electromagnetics. SPIE Press, Bellingham, WA, USA 2003.

[12] http://wotug.kent.ac.uk/parallel/www/occam/occam-bio.html (Consulted on Dec 28, 2003).

[13] Lakhtakia A: Tellegen media: a fecund but incorrect speculation. Speculat. Sci. Technol. 18 (1995) 1–8.

[14] Kong JA: Theorems of bianisotropic media. Proc. IEEE 60 (1972) 1036–1046.

[15] Serdyukov AN, Sihvola AH, Tretyakov SA, Semchenko IV: Duality in electromagnetics: application to Tellegen media. Electromagnetics 16 (1996) 41–51.

[16] Lakhtakia A, Weiglhofer WS: On the application of duality to Tellegen media. Electromagnetics 17 (1997) 199–204.

[17] Lakhtakia A, Weiglhofer WS: On a constraint on the electromagnetic constitutive relations of nonhomogeneous linear media. IMA J. Appl. Math. 54 (1995) 301–306.

[18] Tretyakov SA, Maslokov SI, Nefedov IS, Viitanen AJ, Belov PA, Sanmartín A: Artificial Tellegen particle. Electromagnetics 23 (2003) 665–680.

[19] Tellegen BDH: The gyrator, a new electric network element. Phillips Res. Repts. 3 (1948) 81–101.

[20] Dzyaloshinskii IE: On the magneto–electrical effect in antiferromagnets. Sov. Phys. JETP 10 (1960) 628–629.

[21] Folen VJ, Rado GT, Stalder EW: Anisotropy of the magnetoelectric effect in Cr2O3. Phys. Rev. Lett. 6 (1961) 607–608.

[22] O’Dell TH: The Electrodynamics of Magneto–electric Media. North–Holland, Amsterdam, The Netherlands 1970.

[23] Scharf G: From Electrostatics to Optics. Springer, Berlin, Germany 1994; Chap. 4

[24] López Dávalos A, Zanette D: Fundamentals of Electromagnetism. Springer, Berlin, Germany 1999; Chap. 10.

[25] Post EJ: Separating field and constitutive equations in electromagnetic theory. In: Weiglhofer WS, Lakhtakia A: Introduction to Complex Mediums for Optics and Electromagnetics. SPIE Press, Bellingham, WA, USA 2003.

[26] Weiglhofer WS, Lakhtakia A: Uniformity constraint on recently conceptualised linear uniaxial bianisotropic media. Electron. Lett. 30 (1994) 1656–1657.

[27] Schmid H: Magnetoelectric effects in insulating magnetic materials. In: Weiglhofer WS, Lakhtakia A: Introduction to Complex Mediums for Optics and Electromagnetics. SPIE Press, Bellingham, WA, USA 2003.
[28] Lakhtakia A: An investigative report on the constitutive relations of linear magnetoelectric media. Int. J. Infrared Millim. Waves 15 (1994) 1363–1372.

[29] Astrov DN: The magnetoelectric effect in antiferromagnetics. Sov. Phys. JETP 11 (1960) 708–709.

[30] Folen VJ, Rado GT, Stalder EW: Anisotropy of the magnetoelectric effect in Cr₂O₃. Phys. Rev. Lett. 6 (1961) 607–608.

[31] Rado GT: Mechanism of the magnetoelectric effect in an antiferromagnet. Phys. Rev. Lett. 6 (1961) 609–610.

[32] Rado GT, Folen VJ: Observation of the magnetically induced magnetoelectric effect and evidence for antiferromagnetic domains. Phys. Rev. Lett. 7 (1961) 310–311.

[33] Rado GT, Folen VJ: Magnetoelectric effects in antiferromagnetics. J. Appl. Phys. 33 (1962) 1126–1132.

[34] Raab RE: Some unmeasured crystal properties. Cryst. Res. Technol. 38 (2003) 202–214.

[35] Lakhtakia A: Selected Papers on Linear Optical Composite Materials. SPIE Press, Bellingham, WA, USA 1996.

[36] Lakhtakia A: The Tellegen medium is “a Boojum, you see”. Int. J. Infrared Millim. Waves 15 (1994) 1625–1630.

[37] Dmitriev V: Reply to “Further comments on ‘Returning to the Post constraints’ ”. Microw. Opt. Technol. Lett. 31 (2001) 402–403.

[38] Jeon H, Longo MJ: Search for magnetic monopoles trapped in matter. Phys. Rev. Lett. 75 (1995) 1443–1446.

[39] Hagiwara K et al. (Particle Data Group): Review of particle physics. Phys. Rev. D 66 (2002) 010001.

[40] Weiglhofer WS, Lakhtakia A, Monzon JC: Maxwell–Garnett model for composites of electrically small uniaxial objects. Microw. Opt. Technol. Lett. 6 (1993) 681684.

[41] Chen HC: Theory of Electromagnetic Waves. McGraw–Hill, New York, NY, USA 1983; p. 267.

[42] Weiglhofer WS, Lakhtakia A: Causality and natural optical activity (chirality). J. Opt. Soc. Am. A 13 (1996) 385–386.

[43] Bokut BV, Penyaz VA, Serdyukov AN: Dispersion sum rules in the optics of naturally gyrotropic media. Opt. Spectrosc. (USSR) 50 (1981) 511–513.