Semi-simple extension of the (super) Poincaré algebra

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Abstract

A semi-simple tensor extension of the Poincaré algebra is proposed for the arbitrary dimensions $D$. A supersymmetric also semi-simple generalization of this extension is constructed in the $D = 4$ dimensions. This paper is dedicated to the memory of Anna Yakovlevna Gelyukh.

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1 Introduction

In the papers [1, 2, 3, 4, 5, 6] the Poincaré algebra for the generators of the rotations $M_{ab}$ and translations $P_a$ in $D$ dimensions

$$[M_{ab}, M_{cd}] = (g_{ad}M_{bc} + g_{bc}M_{ad}) - (c \leftrightarrow d),$$

$$[M_{ab}, P_c] = g_{bc}P_a - g_{ac}P_b,$$

$$[P_a, P_b] = 0$$  \hspace{1cm} (1.1)

has been extended by means of the second rank tensor generator $Z_{ab}$ in the following way:

$$[M_{ab}, M_{cd}] = (g_{ad}M_{bc} + g_{bc}M_{ad}) - (c \leftrightarrow d),$$

$$[M_{ab}, P_c] = g_{bc}P_a - g_{ac}P_b,$$

$$[P_a, P_b] = cZ_{ab},$$

$$[M_{ab}, Z_{cd}] = (g_{ad}Z_{bc} + g_{bc}Z_{ad}) - (c \leftrightarrow d),$$

$$[P_a, Z_{bc}] = 0,$$

$$[Z_{ab}, Z_{cd}] = 0,$$  \hspace{1cm} (1.2)

where $c$ is some constant\(^1\).

Such an extension makes common sense, since it is homomorphic to the usual Poincaré algebra (1.1). Moreover, in the limit $c \to 0$ the algebra (1.2) goes to the semi-direct sum of the commutative ideal $Z_{ab}$ and Poincaré algebra (1.1).

It is remarkable enough that the momentum square Casimir operator of the Poincaré algebra under this extension ceases to be the Casimir operator and it is generalized by adding the term linearly dependent on the angular momentum

$$P^a P_a + cZ^{ab} M_{ba} \overset{\text{def}}{=} X_k h^{kl} X_l,$$  \hspace{1cm} (1.3)

where $X_k = \{P_a, Z_{ab}, M_{ab}\}$. Due to this fact, an irreducible representation of the extended algebra (1.2) has to contain the fields of the different masses [4, 7]. This extension with non-commuting momenta has also something in common with the ideas of the papers [8, 9, 10] and with the non-commutative geometry idea [11].

\(^1\)Note that, to avoid the double count under summation over the pair antisymmetric indices, we adopt the rules which illustrated by the following example:

$$[P_a, P_b] = cZ_{ab} = \frac{c}{2}(\delta_a^c \delta_b^d - \delta_a^d \delta_b^c) Z_{cd} = \sum_{c<d} f_{ab}^{cd} Z_{cd} = \frac{1}{2} f_{ab}^{cd} Z_{cd},$$

where $f_{ab}^{cd}$ are structure constants, and so on.
It is interesting to note that in spite of the fact that the algebra (1.2) is not semi-simple and therefore has a degenerate Cartan-Killing metric tensor nevertheless there exists another non-degenerate invariant tensor $h_{kl}$ in adjoint representation which corresponds to the quadratic Casimir operator (1.3), where the matrix $h^{kl}$ is inverse to the matrix $h_{kl}$: $h^{kl}h_{lm} = \delta^k_m$.

There are other quadratic Casimir operators

$$c^2 Z_{ab}Z_{ab}, \quad (1.4)$$

$$c^2 \epsilon^{abcd} Z_{ab}Z_{cd}. \quad (1.5)$$

Note that the Casimir operator (1.5), dependent on the Levi-Civita tensor $\epsilon^{abcd}$, is suitable only for the $D = 4$ dimensions.

It has also been shown that for the dimensions $D = 2, 3, 4$ the extended Poincaré algebra (1.2) allows the following supersymmetric generalization:

$$\{Q_\kappa, Q_\lambda\} = -d (\sigma^{ab} C)_{\kappa\lambda} Z_{ab},$$

$$[M_{ab}, Q_\kappa] = - (\sigma_{ab} Q)_{\kappa},$$

$$[P_a, Q_\kappa] = 0,$$

$$[Z_{ab}, Q_\kappa] = 0 \quad (1.6)$$

with the help of the super-translation generators $Q_\kappa$. In (1.6) $C$ is a charge conjugation matrix, $d$ is some constant and $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, where $\gamma_a$ is the Dirac matrix. Under this supersymmetric generalization the quadratic Casimir operator (1.3) is modified into the following form:

$$P^a P_a + c Z_{ab}M_{ba} - \frac{c}{2d} Q_\kappa (C^{-1})^{\kappa\lambda} Q_\lambda, \quad (1.7)$$

while the form of the rest quadratic Casimir operators (1.4), (1.5) remains unchanged.

In the present paper we propose another possible semi-simple tensor extension of the Poincaré algebra (1.1) and for the case $D = 4$ dimensions we give a supersymmetric generalization of this extension. In the limit this supersymmetrically generalized extension go to the Lie superalgebra (1.2), (1.6).

2 Semi-simple tensor extension

Let us extend the Poincaré algebra (1.1) in the $D$ dimensions by means of the tensor generator $Z_{ab}$ in the following way:

$$[M_{ab}, M_{cd}] = (g_{ad}M_{bc} + g_{bc}M_{ad}) - (c \leftrightarrow d),$$

$$[M_{ab}, P_c] = g_{ac}P_a - g_{ac}P_b,$$
\[ [P_a, P_b] = c Z_{ab}, \]
\[ [M_{ab}, Z_{cd}] = (g_{ad} Z_{bc} + g_{bc} Z_{ad}) - (c \leftrightarrow d), \]
\[ [Z_{ab}, P_c] = \frac{4a^2}{c} (g_{bc} P_a - g_{ac} P_b), \]
\[ [Z_{ab}, Z_{cd}] = \frac{4a^2}{c} [(g_{ad} Z_{bc} + g_{bc} Z_{ad}) - (c \leftrightarrow d)], \]

(2.1)

where \( a \) and \( c \) are some constants. This Lie algebra, when the quantities \( P_a \) and \( Z_{ab} \) are taken as the generators of a homomorphism kernel, is homomorphic to the usual Lorentz algebra. It is remarkable that the Lie algebra (2.1) is \textit{semi-simple} in contrast to the Poincaré algebra (1.1) and extended Poincaré algebra (1.2).

The extended Lie algebra (2.1) has the following quadratic Casimir operators:

\[ C_1 = P^a P_a + c Z^{ab} M_{ba} + 2a^2 M^{ab} M_{ab} \overset{\text{def}}{=} X_k H^k_{kl} X_l, \]

(2.2)

\[ C_2 = c^2 Z^{ab} Z_{ab} + 8a^2 (c Z^{ab} M_{ba} + 2a^2 M^{ab} M_{ab}) \overset{\text{def}}{=} X_k H^k_{2kl} X_l, \]

(2.3)

\[ C_3 = \epsilon^{abcd} [c^2 Z_{ab} Z_{cd} + 8a^2 (c Z_{ba} M_{cd} + 2a^2 M_{ab} M_{cd})]. \]

(2.4)

Note that in the limit \( a \to 0 \) the algebra (2.1) tend to the algebra (1.2) and the quadratic Casimir operators (2.2), (2.3) and (2.4) are turned into (1.3), (1.4) and (1.5), respectively.

The symmetric tensor

\[ H^{kl} = s H^k_{1l} + t H^k_{2l} = H^{lk} \]

(2.5)

with arbitrary constants \( s \) and \( t \) is invariant with respect to the adjoint representation

\[ H^{kl} = H^{mn} U^k_m U^l_n. \]

Conversely, if we demand the invariance with respect to the adjoint representation of the second rank contravariant symmetric tensor, then we come to the structure (2.5) (see also the relation (32) in [6]).

The semi-simple algebra (2.1)

\[ [X_k, X_l] = f^{m}_{kl} X_m \]

has the non-degenerate Cartan-Killing metric tensor

\[ g_{kl} = f^m_{km} f^n_{ln} g_{mn}, \]

which is invariant with respect to the co-adjoint representation

\[ g_{kl} = U^m_k U^n_l g_{mn}. \]
With the help of the inverse metric tensor \( g^{kl} \): \( g^{kl}g_{lm} = \delta^k_m \) we can construct the quadratic Casimir operator which, as it turned out, has the following expression in terms of the quadratic Casimir operators (2.2) and (2.3):

\[
X_k g^{kl} X_l = \frac{1}{8a^2(D-1)} \left[ C_1 + \frac{3 - 2D}{8a^2(D - 2)} C_2 \right],
\]

(2.6)

that corresponds to the particular choice of the constants \( s \) and \( t \) in (2.5).

The extended Poincaré algebra (2.1) can be rewritten in the form:

\[
[N_{ab}, N_{cd}] = (g_{ad}N_{bc} + g_{bc}N_{ad}) - (c \leftrightarrow d),
\]

(2.7)

\[
[L_{AB}, L_{CD}] = (g_{AD}L_{BC} + g_{BC}L_{AD}) - (C \leftrightarrow D),
\]

(2.8)

\[
[N_{ab}, L_{CD}] = 0,
\]

(2.9)

where the metric tensor \( g_{AB} \) has the following nonzero components:

\[
g_{AB} = \{ g_{ab}, g_{D+1D+1} = -1 \}.
\]

(2.10)

The generators

\[
N_{ab} = M_{ab} - \frac{c}{4a^2} Z_{ab}
\]

(2.11)

form the Lorentz algebra \( so(D - 1, 1) \) and the generators

\[
L_{AB} = \{ L_{ab} = \frac{c}{4a^2} Z_{ab}, L_{aD+1} = -L_{D+1a} = \frac{1}{2a} P_a, L_{D+1D+1} = 0 \}
\]

(2.12)

form the algebra \( so(D - 1, 2) \). The algebra (2.7)-(2.9) is a direct sum \( so(D - 1, 1) \oplus so(D - 1, 2) \).

The quadratic Casimir operators \( N_{ab}N^{ab}, L_{AB}L^{AB} \) and \( \epsilon^{abcd}N_{ab}N_{cd} \) of the algebra (2.7)-(2.9) are expressed in terms of the operators \( C_1 \) (2.2), \( C_2 \) (2.3) and \( C_3 \) (2.4) in the following way:

\[
N_{ab}N^{ab} - L_{AB}L^{AB} = \frac{1}{2a^2} C_1,
\]

(2.13)

\[
N_{ab}N^{ab} = \frac{1}{16a^4} C_2,
\]

(2.14)

\[
\epsilon^{abcd}N_{ab}N_{cd} = \frac{1}{16a^4} C_3.
\]

(2.15)

\[\text{Note that in the case } D = 4 \text{ we obtain the anti-de Sitter algebra } so(3, 2).\]
3 Supersymmetric generalization

In the case $D = 4$ dimensions the extended Poincaré algebra (2.1) admits the following supersymmetric generalization:

\[
\{Q_\kappa, Q_\lambda\} = -d \left[\frac{2a}{c} (\gamma^a C)_{\kappa\lambda} P_a + (\sigma^{a\beta} C)_{\kappa\lambda} Z_{a\beta}\right],
\]

\[
[M_{ab}, Q_\kappa] = - (\sigma_{ab} Q)_\kappa,
\]

\[
[P_a, Q_\kappa] = a (\gamma_a Q)_\kappa,
\]

\[
[Z_{ab}, Q_\kappa] = -\frac{4a^2}{c} (\sigma_{ab} Q)_\kappa,
\]

(3.1)

where $Q_\kappa$ are the super-translation generators.

Under such a generalization the Casimir operator (2.2) is modified by adding a term quadratic in the super-translation generators

\[
\tilde{C}_1 = P^a P_a + c Z^{ab} M_{ba} + 2a^2 M^{ab} M_{ab} - \frac{c}{2d} Q_\kappa (C^{-1})^{\kappa\lambda} Q_\lambda
\]

\[
\overset{\text{def}}{=} X_K H_1^{KL} X_L,
\]

(3.2)

whereas the form of the rest quadratic Casimir operators (2.3) and (2.4) is not changed. In (3.2) $X_K = \{P_a, Z_{ab}, M_{ab}, Q_\kappa\}$ is a set of the generators for the also semi-simple extended superalgebra (2.1), (3.1).

The tensor

\[
H^{KL} = v H_1^{KL} + w H_2^{KL} = (-1)^{p_K p_L + p_K + p_L} H^{LK}
\]

(3.3)

is invariant with respect to the adjoint representation

\[
H^{KL} = (-1)^{(p_K + p_M)(p_L + 1)} H^{MN} U^L N U_ M K,
\]

where $p_K = p(K)$ is a Grassmann parity of the quantity $K$. In (3.3) $v$ and $w$ are arbitrary constants and nonzero elements of the matrix $H_2^{KL}$ equal to the elements of the matrix $H_2^{KL}$ followed from (2.3). Again, by demanding the invariance with respect to the adjoint representation of the second rank contravariant tensor $H^{KL} = (-1)^{p_K p_L + p_K + p_L} H^{LK}$, we come to the structure (3.3) (see also the relation (32) in [6]).

The semi-simple Lie superalgebra (2.1) (3.1) has the non-degenerate Cartan-Killing metric tensor $G_{KL}$ (see the relation (A.5) in the Appendix A) which is invariant with respect to the co-adjoint representation

\[
G_{KL} = (-1)^{p_K (p_L + p_N)} U^L N U_ K^M G_{MN}.
\]

With the use of the inverse metric tensor $G^{KL}$

\[
G^{KL} G_{LM} = \delta^K_M
\]
we can construct the quadratic Casimir operator (see the relation (A.8) in the Appendix A) which takes the following expression in terms of the Casimir operators (2.3) and (3.2):

\[ X_K G^{KL} X_L = \frac{1}{20a^2} \left( \tilde{C}_1 - \frac{9}{32a^2} C_2 \right), \]  

(3.4)

that meets the particular choice of the constants \( v \) and \( w \) in (3.3).

In the \( D = 4 \) case the extended superalgebra (2.1), (3.1) can be rewritten in the form of the relations (2.7)-(2.9) and the following ones:

\[ \{ Q_\kappa, Q_\lambda \} = -\frac{4a^2d}{c} (\Sigma^{AB} C)_{\kappa\lambda} L_{AB}, \]  

(3.5)

\[ [L_{AB}, Q_\kappa] = -(\Sigma_{AB} Q)_\kappa, \]  

(3.6)

\[ [N_{ab}, Q_\kappa] = 0, \]  

(3.7)

where

\[ \Sigma_{AB} = \frac{1}{4} [\Gamma_A, \Gamma_B], \quad \Gamma_A = \{ \gamma_a \gamma_5, \gamma_5 \}, \]

\[ \{ \gamma_a, \gamma_b \} = 2g_{ab}, \quad g_{ab} = \text{diag}(-1, 1, 1, 1), \]

\[ \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3. \]

The generators \( N_{ab} \) (2.11) form the Lorentz algebra \( so(3,1) \) and the generators \( L_{AB} \) (2.12), \( Q_\kappa \) form the orthosymplectic algebra \( osp(1,4) \). We see that superalgebra (2.7)-(2.9), (3.5)-(3.7) is a direct sum \( so(3,1) \oplus osp(1,4) \).

In this case the Casimir operator (2.13) is modified by adding a term quadratic in the super-translation generators

\[ N_{ab} N^{ab} - L_{AB} L^{AB} - \frac{c}{4a^2d} Q_\kappa (C^{-1})^{\kappa\lambda} Q_\lambda = \frac{1}{2a^2} \tilde{C}_1, \]

while the form of the quadratic Casimir operators (2.14) and (2.15) is not changed.

### 4 Conclusion

Thus, we proposed the semi-simple second rank tensor extension of the Poincaré algebra in the arbitrary dimensions \( D \) and super Poincaré algebra in the \( D = 4 \) dimensions. It is very important, since under construction of the models it is more convenient to deal with the non-degenerate space-time symmetry. We also constructed the quadratic Casimir operators for the semi-simple extended Poincaré and super Poincaré algebras.

It is interesting to develop the models based on these extended algebras. The work in this direction is in progress.
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A Appendix: Properties of Lie superalgebra

Permutation relations for the generators $X_K$ of Lie superalgebra are

$$[X_K, X_L] \overset{\text{def}}{=} X_K X_L - (-1)^{p_K p_L} X_L X_K = f_{KL}^M X_M. \quad (A.1)$$

Structure constants $f_{KL}^M$ have the Grassmann parity

$$p(f_{KL}^M) = p_K + p_L + p_M = 0 \pmod{2}, \quad (A.2)$$

following symmetry property:

$$f_{KL}^M = -(-1)^{p_K p_L} f_{LK}^M \quad (A.3)$$

and obey the Jacobi identities

$$\sum_{(KLM)} (-1)^{p_K p_M} f_{KN}^P f_{LM}^N = 0, \quad (A.4)$$

where the symbol $(KLM)$ means a cyclic permutation of the quantities $K$, $L$ and $M$. In the relations (A.1)-(A.4) an every index $K$ takes either a Grassmann-even value $k$ ($p_k = 0$) or a Grassmann-odd one $\kappa$ ($p_\kappa = 1$). The relations (A.1) have the following components:

$$[X_k, X_l] = f_{kl}^m X_m,$$

$$\{X_\kappa, X_\lambda\} = f_{\kappa\lambda}^m X_m,$$

$$[X_k, X_\lambda] = f_{k\lambda}^\mu X_\mu.$$

The Lie superalgebra possesses the Cartan-Killing metric tensor

$$G_{KL} = (-1)^{p_N} f_{KM}^N f_{LN}^M = (-1)^{p_K p_L} G_{LK} = (-1)^{p_K} G_{LK} = (-1)^{p_\mu} G_{LK}, \quad (A.5)$$

which components are

$$G_{kl} = f_{km}^n f_{ln}^m - f_{k\mu}^\nu f_{\nu\mu},$$

$$G_{\kappa\lambda} = f_{\kappa\mu}^m f_{\lambda m}^\mu - f_{km}^\mu f_{\lambda\mu}^m,$$

$$G_{k\lambda} = 0.$$
As a consequence of the relations (A.3) and (A.4) the tensor with low indices

\[ f_{KLM} = f_{KL}^N G_{NM} \]  \hspace{1cm} (A.6)

has the following symmetry properties:

\[ f_{KLM} = -(-1)^{pKpL} f_{LKM} = -(-1)^{pLpM} f_{KML}. \]  \hspace{1cm} (A.7)

For a semi-simple Lie superalgebra the Cartan-Killing metric tensor is non-degenerate and therefore there exists an inverse tensor \( G^{KL} \)

\[ G_{KL} G^{LM} = \delta^M_K. \]

In this case, as a result of the symmetry properties (A.7), the quantity

\[ X_K G^{KL} X_L \]  \hspace{1cm} (A.8)

is a Casimir operator

\[ [X_K G^{KL} X_L, X_M] = 0. \]

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