Information and Sufficiency on the Stock Market

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Abstract—It is well-known that there are a number of relations between theoretical finance theory and information theory. Some of these relations are exact and some are approximate. In this paper we will explore some of these relations and determine under which conditions the relations are exact. It turns out that portfolio theory always leads to Bregman divergences. The Bregman divergence is only proportional to information divergence in situations that are essentially equal to the type of gambling studied by Kelly. This can be related an abstract sufficiency condition.

I. INTRODUCTION

The relation between gambling and information theory has been known since Kelly [1]. Later Kelly’s theory has been extended to trading of assets, but the link to information theory is weaker than in the case of gambling [2]. In both gambling theory and more general portfolio theory logarithmic terms appear because we are interested in the exponential growth rate. In this paper we shall demonstrate that portfolio theory consist of two parts. The general part is related to Bregman divergences and this part is shared with a number of other convex optimization problems. If a sufficiency condition is imposed on the general theory we arrive at a theory where the Bregman divergence reduces to information divergence. The sufficiency is essentially equal to Kelly’s theory of gambling.

The general theory of convex optimization and Bregman divergences has a number of important applications. In each of the applications we get a strong link to information theory if a sufficiency condition is imposed. Therefore sufficiency conditions will lead to strong relations between the different applications.

In information theory an important goal is to compress. As long as we restrict to uniquely decodable codes we get a Bregman divergence. The sufficiency condition corresponds to allowing codewords real valued length which is relevant when we allow block codes with no upper limit on the block length. This leads to the wide spread use of information divergence in information theory. The link between information divergence and the notion of sufficiency was emphasized already by Kullback and Leibler in 1951 in the paper entitled “Information and Sufficiency” [3].

In statistics the idea of scoring rules has its roots in the 1920’s in the Dutch book theorem by Ramsay and de Finetti. McCathy [4] studied scoring rules in a more systematic way and Dawid, Lauritzen and Parry [5] have recently extended the notion of proper local scoring rules. Proper scoring rules leads to Bregman divergences and sufficiency lead to local proper scoring rules. The basic result is that any strictly local proper scoring rule is proportional to logarithmic score. The link between information theory and statistics is now very well established [6].

Convex optimization also appear in thermodynamics and statistical mechanics where the goal is to extract as much energy as possible from some physical system. The notion of entropy obviously play an important role in both theories, but the best interpretation has been debated ever since Shannon decided to call his quantity entropy. Since all these theories are related we also get a link between finance theory and physics so there is a whole topic called econophysics where ideas from physics are applied to economic systems. We hope that the present paper will help to understand to what extend quantities in finance are really proportional to quantities in information theory, statistics, or physics.

The general idea of using Bregman divergences for convex optimization was presented in [7]. In the present paper we will develop the theory further. Therefore there will be some overlap between then the presentation in [7] and the present paper. The second goal of this paper is apply the general theory to portfolio theory.

II. OPTIMIZATION

Assume that our knowledge of a system can be represented by an element in a convex set $S$ that we will call the state space. The simplest case of a state space is the simplex of probability measures on a set. In quantum information theory the state space is the set of density matrices on a Hilbert space. For states $s_0$ and $s_1$ and $t \in [0, 1]$ the convex combination $(1-t) \cdot s_0 + t \cdot s_1$ is identified with the mixed state where $s_0$ is taken with probability $1-t$ and the state $s_1$ is taken with probability $t$. The pure states are the extreme points of the state space. For simplicity we will assume that the state space is a finite dimensional convex compact set.

Let $A$ denote a subset of the feasible measurements such that $a \in A$ maps $S$ into a distribution on the real numbers i.e. a random variable. The elements of $A$ may represent actions (decisions) that lead to a payoff like the score of a statistical decision, the energy extracted by a certain interaction with the system, (minus) the length of a codeword of the next encoded input letter using a specific code book, or the revenue of using a certain portfolio. If the action $a$ is applied to the state $s$ then we get a random variable $a(s)$ that we will allow to take...
values in $\mathbb{R} \cup \{-\infty\}$. For each $s \in S$ we define $F(s) = \sup_{a \in A} E[a(s)]$. Without loss of generality we may assume that the set of actions $A$ is closed so that we may assume that there exists $a \in A$ such that $F(s) = E[a(s)]$ and in this case we say that $a$ is optimal for $s$. We note that $F$ is convex but $F$ need not be strictly convex.

**Definition 1.** If $F(s)$ is finite the regret of the action $a$ is defined by

$$D_F(s, a) = F(s) - E[a(s)]$$

(1)

**Proposition 2.** The regret $D_F$ has the following properties:

- $D_F(s, a) \geq 0$ with equality if $a$ is optimal for $s$.
- If $\hat{a}$ is optimal for the state $\hat{s} = \sum t_i \cdot s_i$ where $(t_1, t_2, \ldots, t_F)$ is a probability vector then

$$\sum t_i \cdot D_F(s_i, a) = \sum t_i \cdot D_F(s_i, \hat{a}) + D_F(\hat{s}, a).$$

- $\sum t_i \cdot D_F(s_i, a)$ is minimal if $a$ is optimal for $\sum t_i \cdot s_i$.

If the state is $s_1$ but one acts as if the state were $s_2$ one suffers a regret that equals the difference between what one achieves and what could have been achieved.

**Definition 3.** If $F(s_1)$ is finite the regret is defined by

$$D_F(s_1, s_2) = \inf_a D_F(s, a)$$

(2)

where the infimum is taken over actions $a$ that are optimal for $s_2$.

If there exists a unique action $a$ such that $F(s) = E[a(s)]$ then $F$ is differentiable which implies that the regret can be written as a Bregman divergence in the following form

$$D_F(s_1, s_2) = F(s_1) - (F(s_2) + \langle s_1 - s_2, \nabla F(s_2) \rangle).$$

(3)

In the context of forecasting and statistical scoring rules the use of Bregman divergences dates back to [8].

Bregman divergences satisfy the Bregman identity

$$\sum t_i \cdot D_F(s_i, \hat{s}) = \sum t_i \cdot D_F(s_i, \hat{s}) + D_F(\hat{s}, \hat{s})$$

but if $F$ is not differentiable this identity can be violated. If the state $s_2$ has the unique optimal action $a_2$ then

$$F(s_1) = D_F(s_1, s_2) + E[a_2(s_1)].$$

(4)

so the function $F$ can be reconstructed from $D_F$ except for an affine function of $s_1$. Similarly the divergence $D_F$ is uniquely determined by the function $F$.

Consider the case where the state is not known exactly but we know that $s \in S$ for some set of states. The minimax regret of the set $S$ is defined as

$$C_F = \inf_a \sup_s D_F(s, a).$$

Using general minimax results we get

$$C_F = \sup_t \inf_s \sum_i t_i \cdot D_F(s_i, a)$$

where the supremum is taken over all probability vectors $t'$ supported on $S$. This result can be improved.

**Theorem 4.** If $(t_1, t_2, \ldots, t_n)$ is a probability vector on the states $s_1, s_2, \ldots, s_n$ with $\tilde{s} = \sum t_i \cdot s_i$ and $a_{opt}$ is the optimal action for $\tilde{s}$ then

$$C_F \geq \inf_a \sum t_i \cdot D_F(s_i, a) + D_F(\tilde{s}, a_{opt}).$$

If $a$ is an action and $s_{opt}$ is optimal then

$$\sup_i D_F(s_i, a) \geq C_F + D_F(s_{opt}, a).$$

**III. SUFFICIENCY**

Let $(s_0)_p$ denote a family of states and let $F$ denote an affine transformation $\mathcal{S} \to \mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ denote state spaces. Then $F$ is said to be sufficient for $(s_0)_p$ if there exists an affine transformation $\Psi : \mathcal{S} \to \mathcal{S}$ such that $\Psi(F(s_0)) = s_0$.

We define a transformation $F$ to be an isomixture if $F$ has the form $F = \sum_{i=1}^k p_i \cdot F_i$ where $(p_1, p_2, \ldots, p_k)$ is a probability vector and $F_i$ is a isometry, i.e. a bijective transformation of the state space itself. We say that the regret $D_F$ on the state space $S$ satisfies the iso-sufficiency property if

$$D_F(F(s_1), F(s_2)) = D_F(s_1, s_2)$$

(5)

for any isomixture $S \to S$ that is sufficient for $(s_1, s_2)$. The notion of sufficiency as a property of divergences was introduced in [9]. The crucial idea of restricting the attention to transformations of the state space into itself was introduced in [10].

The center of a convex set $S$ is the point of set in $S$ that are invariant under isometries of $S$. Note that the center is convex and non-empty [11]. If the center of the state space is not a point there are many Bregman divergences that satisfy the sufficiency condition.

**Proposition 5.** Let $G$ denote the set of isometries of a state space $S$ and let $\mu$ denote the Haar probability measure on $G$. Let $\Phi$ denote the projection $s \to \int g(s) \, d\mu$. Let $F$ denote a concave function on the center of $S$. Then $D_F(F(s_1), F(s_2))$ defines a Bregman divergence on $S$ that satisfies the iso-sufficiency condition.

**Proposition 6.** Assume that $S$ is a state space. If the divergence $D_F$ satisfies the iso-sufficiency property then there exists a $\hat{D}$ such that

$$\hat{D}_F(s_1, s_2) = D_F(s_1, s_2)$$

and $\hat{D}(\Phi(s)) = \hat{D}(s)$.

If the state space is a one dimensional simplex then the only sufficient transformation is the reflection and the above condition on $F$ is sufficient to conclude that Equation (5) holds.

**Proposition 7.** If the state space has the shape of a ball then any function $F$ on the ball that is concave and invariant under rotations satisfies the iso-sufficiency condition.

**Proof:** Assume that the isomixture $F$ is sufficient for $(s_0, s_1)$. Then $F$ is also sufficient for any affine combination of $s_0$ and $s_1$. In particular we may replace $s_0$ and $s_1$ by affine combinations for the form $s_{t_i} = (1 - t_i) \cdot s_0 + t_i \cdot s_1$ that are
extreme points in $S$. Since $\Phi$ is assumed to be sufficient it maps $s_i$, into an extreme points. Hence $\Phi$ acts as a rotation on the intersection of the state space and the affine span of $s_1, s_2$ and $U$. Since $F$ is invariant under rotations the divergence $D_F$ is also invariant under rotations implying that $D_F(\Phi(s_1), \Phi(s_2)) = D_F(s_1, s_2)$.

The simplest case of a ball is an interval, which corresponds to the probability measures on a binary alphabet. This special case was discussed in [10]. The balls in dimensions 2, 3, and 5 correspond to density matrices of a 2 dimensional Hilbert space over the real numbers, over the complex numbers, and over the quaternions.

We say that the states $s_0$ and $s_1$ are orthogonal and write $s_1 \perp s_2$ if there exists an affine function $\phi : S \to [0, 1]$ such $\phi(s_0) = 0$ and $\phi(s_1) = 1$. The following theorem can be proved by the same technique as [11] Thm. 4 except that we will make sufficient projections by taking the mean actions of a groups equipped with the Haar probability measure.

**Theorem 8.** Assume that the state space $S$ satisfies the following properties:

1. For and two pure states $s_1$ and $s_2$ there exists an isometry of $S$ such that $\Phi(s_1) = s_2$.
2. For any three pure states $s_1, s_2, s_3$ such that $s_1 \perp s_3$ and $s_2 \perp s_3$ there exists an isometry of $S$ such that $\Phi(s_1) = s_2$ and $\Phi(s_3) = s_3$.
3. The state space has at least three orthogonal pure states.
4. Any state can be written as a mixture of orthogonal pure states.

If the regret $D_F$ satisfies the iso-sufficiency property given by Equation 5 then $D_F$ is uniquely determined except for a multiplicative factor.

**Remark 9.** Condition 4 seems to be redundant, but we have not been able to prove this.

When the state space is a simplex the uniquely determined divergence is information divergence and when the state space is density matrices on a complex Hilbert space we get quantum relative entropy.

**Lemma 10.** Assume that the state space satisfies the conditions in Theorem 8. If $s_0 \perp s_1$ then any optimal action $a$ for $s_1$ satisfies $E[a(s_0)] = -\infty$.

**Proof:** Since $s_0$ and $s_1$ are orthogonal and the conditions in the previous theorem are fulfilled the we have that the regret restricted to the line segment $\{ t \in [0, 1] \mid (1 - t) s_0 + t s_1 \}$ is proportional to information divergence, but information divergence equals $\infty$ for orthogonal distributions so $D_F(s_0, s_1) = \infty$. Hence $\inf_a (F(s_0) - E[a(s_0)]) = \infty$ where the infimum is taken over actions that are optimal for $s_1$. Therefore $E[a(s_0)] = -\infty$ for any action $a$ that is optimal for $s_1$. 

**IV. PORTFOLIO THEORY**

Let $X_1, X_2, \ldots, X_k$ denote price relatives for a list of $k$ assets. For instance $X_5 = 1.04$ means that asset no. 5 increases its value by 4%.

**Example 11.** A special asset is the safe asset where the price relative is 1 for any feasible price relative vector. Investing in this asset corresponds to place the money at a safe place with interest rate equal to 0%.

A portfolio is an asset given by a probability vector $\vec{b} = (b_1, b_2, \ldots, b_k)$ where for instance $b_5 = 0.3$ means that 30% of the money is invested in asset no. 5. The total price relative is $X_1 \cdot b_1 + X_2 \cdot b_2 + \cdots + X_k \cdot b_k = \langle \vec{X}, \vec{b} \rangle$. If an asset has the property that the price relative is only positive for one of the feasible price relative vectors, then we may call it a gambling asset. For any set of possible assets we may extend the set of assets by a number of ideal gambling assets so that any of the possible assets can be written as a portfolio of the ideal gambling assets. This can be done without changing the set of feasible price relative vectors. Therefore the set of possible portfolios may be considered as a convex subset of a set of portfolios of some ideal gambling assets.

We now consider a situation where the assets are traded once every day. For a sequence of price relative vectors $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ and a constant re-balancing portfolio $\vec{b}$ the wealth after $n$ days is

$$S_n = \prod_{i=1}^{n} \langle \vec{X}_i, \vec{b} \rangle$$

This is proportional to the doubling rate where

$$E[\log \langle \vec{X}, \vec{b} \rangle] = \exp \left( n \cdot E \left[ \log \langle \vec{X}, \vec{b} \rangle \right] \right)$$

Note that in [2] and [7] it was tacitly assumed that a unique optimal portfolio exists but this is not always the case. Here we will not assume uniqueness.

**Definition 12.** Let $\vec{b}_1$ and $\vec{b}_2$ denote two portfolios. We say that $\vec{b}_1$ dominates $\vec{b}_2$ if $\langle X_j, \vec{b}_1 \rangle \geq \langle X_j, \vec{b}_2 \rangle$ for any $j = 1, 2, \ldots, n$. We say that $\vec{b}_1$ strictly dominates $\vec{b}_2$ if $\langle X_j, \vec{b}_1 \rangle > \langle X_j, \vec{b}_2 \rangle$ for any $j = 1, 2, \ldots, n$.

For a vector $\vec{v} = (v_1, v_2, \ldots, v_k) \in \mathbb{R}^k$ the support supp $\vec{v}$ is the set of indices $i$ such that $v_i > 0$. We note that if $\vec{b}_1$ strictly dominates $\vec{b}_2$ if and only if there exists an $i \in \supp(\vec{b}_2)$ such that $\vec{b}_1$ strictly dominates $\vec{e}_i$ where $\vec{e}_i$ denotes the $i$th basis vector. The consequence is that we may remove assets number $i$ if $\vec{e}_i$ is strictly dominated because one will never put any money on that particular asset. Similarly, $\vec{b}_1$ dominates $\vec{b}_2$ if and only if there exists an $i \in \supp(\vec{b}_2)$ such that $\vec{b}_1$ dominates $\vec{e}_i$. We do not decrease the maximal doubling rate by removing assets that are dominated, but sometimes
assets that are dominated but not strictly dominated may lead to non-uniqueness of the optimal portfolio.

**Definition 13.** A set \( A \) of assets is said to dominate the set of assets \( B \) if any asset in \( B \) is dominated by a by a portfolio of assets in \( A \).

**Proposition 14.** If \( \vec{b}_0 \) is optimal for the distribution \( \vec{v} \), then the support of \( \vec{b} \) is a subset of the support of \( \vec{v} \).

**Proof:** If \( P = \vec{v} \) then \( E \left[ \log \left( X, \vec{b} \right) \right] = \log \left( \vec{v}, \vec{b} \right) \). The portfolio \( \vec{b} \) is a probability distribution over stocks so if we let \( \vec{b}^* \) denote the conditional distribution of \( \vec{b} \) on the support of \( \vec{v} \). Then

\[
\log \left( \vec{v}, \vec{b}_0 \right) \leq \log \left( \vec{v}, \vec{b}^* \right)
\]

with equality if and only if the support of \( \vec{b} \) is a subset of the support of \( \vec{v} \). Therefore \( \vec{b} = \vec{b}_0 \) implies that the support of \( \vec{b} \) is a subset of the support of \( \vec{v} \).

Let \( \vec{b}_P \) denote a portfolio that is optimal for \( P \). The regret of choosing a portfolio according to \( Q \) when the distribution is \( P \) is given by the Bregman divergence

\[
W \left( \vec{b}_P, P \right) - W \left( \vec{b}_Q, P \right).
\]

If \( \vec{b}_Q \) is not uniquely determined we take a minimum over all \( \vec{b} \) that are optimal for \( Q \).

**Example 15.** If the assets are orthogonal gambling assets we get the type of gambling described by Kelly. There will be a one-to-one correspondence between price relative vectors and assets. For a probability distribution \( P \) over price relative vectors the optimal portfolio \( \vec{b}_P \) is a vector with the same coordinates as the probability vector \( P \). We have

\[
W \left( \vec{b}_P, P \right) - W \left( \vec{b}_Q, P \right) = D \left( P \parallel Q \right)
\]

so the sufficiency condition is fulfilled in gambling.

If a set of possible assets it embedded as a subset \( C \) in a set of ideal gambling assets then \( C \) may be identified with a convex set of probability distributions. Now maximizing \( W \left( \vec{b}, P \right) \) over possible portfolios \( \vec{b} \) is the same as minimizing the regret given by (9) over \( Q \in C \) in the set of portfolios over ideal gambling assets. Therefore \( \vec{b}_Q \) may be identified with a reversed information projection of \( Q \) on \( C \).

As proved in (2) the regret satisfies

\[
W \left( \vec{b}_P, P \right) - W \left( \vec{b}_Q, P \right) \leq D \left( P \parallel Q \right).
\]

In the set of portfolios over ideal assets there is a one-to-one correspondence between mixed states and portfolios. Therefore maximizing \( W \left( \vec{b}, P \right) \) over \( \vec{b} \) in the original set of portfolios corresponds to minimizing the regret \( W \left( \vec{b}_P, P \right) - W \left( \vec{b}_Q, P \right) \) over \( Q \) which again corresponds to minimizing \( D \left( P \parallel Q \right) \) under the condition that \( \vec{b}_Q \in C \) in a set of portfolios on orthogonal gambling assets. The inequality (10) therefore states that information divergence decreases when probability measures are projected (reverse information projection) into a convex set. Here we should note that information divergence is convex but not strictly convex in the second argument. Therefore the reversed information may be non-unique.

**V. Sufficient Portfolios**

**Lemma 16.** Assume that there are only two price relative vectors and that the set of assets is minimal dominating. If the Bregman divergence

\[
W \left( \vec{b}_P, P \right) - W \left( \vec{b}_Q, P \right)
\]

is proportional to information divergence \( D \left( P \parallel Q \right) \) then there are only two gambling assets.

**Proof:** Let

\[
\vec{X} = (X_1, X_2, \ldots, X_k)
\]

\[
\vec{Y} = (Y_1, Y_2, \ldots, Y_k)
\]

denote the two price relative vectors. If \( P = (s, t) \) then the vector \( \vec{b} = (b_1, b_2, \ldots, b_n) \) is log-optimal if and only if

\[
sb_1X_1 + \cdots + sb_kX_k + tY_i \leq 1
\]

for all \( i \in \{1, 2, \ldots, k\} \) with equality if \( b_i > 0 \). Since we have assumed that none of the assets are dominated by other portfolios only two of these inequalities can hold with equality. Therefore we may assume that only \( b_1 \) and \( b_2 \) are positive. Hence we may assume that there are only two assets.

Let \( \delta_1 \) denote the measure concentrated on \( \vec{X} \) and let \( \delta_2 \) denote the measure concentrated on \( \vec{Y} \). Since the measures \( \delta_1 \) and \( \delta_2 \) are orthogonal Lemma [10] we have that \( W \left( \vec{b}_i, \delta_i \right) = -\infty \). Now

\[
W \left( \vec{b}_i, \delta_i \right) = E_{\delta_i} \left[ \log \left( \vec{X}, \vec{b}_i \right) \right] = \log \left( \vec{X}, \vec{b}_i \right)
\]

so that \( \left< \vec{X}, \vec{b}_i \right> = 0 \). Since the support of \( \vec{b}_i \) is a subset of the support of \( \vec{X} \) we have that \( \vec{b}_i \perp \vec{b}_j \). Therefore \( \vec{b}_i \) and \( \vec{b}_j \) must be proportional to the basis vectors. Since \( \vec{b}_i \) and \( \vec{b}_j \) are vectors in a 2-dimensional space and their coordinates are non-negative we have that \( \vec{b}_i \) must proportional to a basis vector. Since \( \left< \vec{X}, \vec{b}_j \right> = 0 \) for \( i \neq j \) we have that \( \vec{X} \) is parallel with \( \vec{b}_i \).

**Theorem 17.** Assume that none of the assets are dominated by a portfolio of the other assets. If the Bregman divergence

\[
W \left( \vec{b}_P, P \right) - W \left( \vec{b}_Q, P \right)
\]

is proportional to information divergence \( D \left( P \parallel Q \right) \) the measures \( P \) and \( Q \) are supported by \( k \) distinct price relative vectors of the form \( (0, o_1, 0, \ldots, 0), (0, o_2, 0, \ldots, 0) \), until \( (0, 0, \ldots, o_k) \).
Proof: Assume that there exists a constant $c > 0$ such that
\[ W\left(\tilde{b}_P, P\right) - W\left(\tilde{b}_Q, P\right) = c \cdot D(P\| Q). \tag{13} \]
If $\tilde{b}_P = \tilde{b}_Q$ then
\[ W\left(\tilde{b}_P, P\right) - W\left(\tilde{b}_Q, P\right) = 0 \]
and $D(P\| Q) = 0$ and $P = Q$. Therefore the mapping $P \rightarrow \tilde{b}_P$ is injective. The vectors $\tilde{b}_P$ form a simplex with $k$ extreme points. Therefore the simplex of probability measures $P$ has at most $k$ extreme points, so $P$ is supported on at most $k$ distinct vectors that we will denote $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k$.

Assume that $\tilde{X}$ and $\tilde{Y}$ are two vectors of price relatives. Then Equation (13) holds for probability vectors restricted to the set $\{\tilde{X}, \tilde{Y}\}$. From Lemma 16 it follows that $\tilde{X}$ and $\tilde{Y}$ are orthogonal. Therefore all the price relative vectors are orthogonal, and have disjoint supports. Since the price relative vectors have disjoint support, an asset can only have a positive price relative for one of the price relative vectors. Therefore each price relative vector has one asset that dominates any other asset in the support of the price relative vector. Since we have assumed none of the assets are dominated each price relative vector is supported on a single asset.

If the price relative vectors are as in Theorem 17 we are in the situation of gambling introduced by Kelly [1].

Corollary 18. Assume that the Bregman divergence
\[ W\left(\tilde{b}_P, P\right) - W\left(\tilde{b}_Q, P\right) \tag{14} \]
satisfies the sufficiency condition for probability measures $P$ and $Q$ supported on $k \geq 3$ price relative vectors. Then the set of possible assets contain $k$ gambling assets and any other asset is dominated by a portfolio on the gambling assets.

Example 19. If the Bregman divergence satisfies the sufficiency condition and one of the assets is the safe asset then there exists a portfolio $\tilde{b}$ such that $b_i \cdot \alpha_i \geq 1$ for all $i$. Equivalently $b_i \geq \alpha_i^{-1}$, which is possible if and only if $\sum \alpha_i^{-1} \leq 1$. One say that the gamble is fair if $\sum \alpha_i^{-1} = 1$. If the gamble is superfair, i.e. $\sum \alpha_i^{-1} < 1$, then the portfolio $b_i = \alpha_i^{-1}/\sum \alpha_i^{-1}$ gives a price relative equal to $(\sum \alpha_i^{-1})^{-1} > 1$ independently of what happens, which is a Dutch book.

Corollary 20. Assume that there are at least three distinct price relative vectors. The Bregman divergence (14) satisfies the sufficiency condition if and only if $W\left(\tilde{b}_P, P\right) - W\left(\tilde{b}_Q, P\right) = 0$ implies $P = Q$.

Proof: If Equation (9) does not hold then we do not have sufficiency so the set of possible portfolios can be identified with a convex and proper subset of the set of all portfolios on a set of gambling assets. Then we just have to find to distributions $P$ and $Q$ that have the same reversed information projection into the set of possible portfolios.

VI. CONCLUSION

The link between portfolio theory and information theory works on two levels. Parts of the theory can be stated and proved on the level of convex optimization, where Bregman divergences and related concepts play a central role. If we further impose a sufficiency condition we have, essentially, to restrict our attention to gambling as described by Kelly. Adding certain assets that are dominated does not make any significant changes to the theory. In the case of gambling the correspondence between portfolio theory and information theory becomes perfect. Therefore the link between general portfolio theory and information theory is conveyed by gambling theory.

Information divergence was introduced by Kullback and Leibler in the paper entitled “On Information and Sufficiency”. In the present paper we have made the notion of sufficiency more explicit for portfolio theory. The introduction of ideal gambling assets parallels the use of microscopic states as opposed to macroscopic states in physics. For microscopic states we have reversibility and conservation of energy. Similarly, gambling corresponds to two-person zero sum games where money is the conserved quantity. As we have seen these correspondences are consequences of the sufficiency condition.

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