MULTIDIMENSIONAL EXPONENTIAL DIVISOR FUNCTION
OVER GAUSSIAN INTEGERS

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Abstract. Let \( \tau_{k}^{(e)} : \mathbb{Z} \to \mathbb{Z} \) be a multiplicative function such that \( \tau_{k}^{(e)}(p^a) = \sum_{d_1 \cdots d_k = a} 1 \). In the present paper we introduce generalizations of \( \tau_{k}^{(e)} \) over the ring of Gaussian integers \( \mathbb{Z}[i] \). We determine their maximal orders by proving a general result and establish asymptotic formulas for their average orders.

1. Introduction

Exponential divisor function \( \tau^{(e)} : \mathbb{Z} \to \mathbb{Z} \) introduced by Subbarao in [3] is a multiplicative function such that

\[
\tau^{(e)}(p^a) = \tau(a),
\]

where \( \tau : \mathbb{Z} \to \mathbb{Z} \) stands for the usual divisor function, \( \tau(n) = \sum_{d|n} 1 \). Erdős estimated its maximal order and Subbarao proved an asymptotic formula for \( \sum_{n \leq x} \tau^{(e)}(n) \). Later Wu [12] gave more precise estimate:

\[
\sum_{n \leq x} \tau^{(e)}(n) = Ax + Bx^{1/2} + O(x^{\theta_{1,2} + \varepsilon}),
\]

where \( A \) and \( B \) are computable constants, \( \theta_{1,2} \) is an exponent in the error term of the estimate \( \sum_{ab \leq x} 1 = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{\theta_{1,2} + \varepsilon}) \). The best modern result [2] is that \( \theta_{1,2} \leq 1057/4785 \).

One can consider multidimensional exponential divisor function \( \tau_{k}^{(e)} : \mathbb{Z} \to \mathbb{Z} \) such that

\[
\tau_{k}^{(e)}(p^a) = \tau_k(a),
\]

where \( \tau_k(n) \) is a number of ordered \( k \)-tuples of positive integers \((d_1, \ldots, d_k)\) such that \( d_1 \cdots d_k = n \). So \( \tau^{(e)} \equiv \tau_{2}^{(e)} \). Toth [11] investigated asymptotic properties of \( \tau_{k}^{(e)} \) and proved that for arbitrarily \( \varepsilon > 0 \)

\[
\sum_{n \leq x} \tau_{k}^{(e)}(n) = C_k x + x^{1/2} S_{k-2}(\log x) + O(x^{w_k + \varepsilon}),
\]

where \( S_{k-2} \) is a polynomial of degree \( k - 2 \) and \( w_k = (2k - 1)/(4k + 1) \).

In the present paper we generalize multidimensional exponential divisor function over the ring of Gaussian integers \( \mathbb{Z}[i] \). Namely we introduce multiplicative functions \( \tau_{k}^{(e)} : \mathbb{Z} \to \mathbb{Z}, t_k^{(e)}, t_k^{(e)} : \mathbb{Z}[i] \to \mathbb{Z} \) such that

\[
(1) \quad \tau_{k}^{(e)}(p^a) = \tau_k(a), \quad t_k^{(e)}(p^a) = \tau_k(a), \quad t_k^{(e)}(p^a) = t_k(a),
\]

where \( p \) is prime over \( \mathbb{Z} \), \( p \) is prime over \( \mathbb{Z}[i] \), \( \tau_k(a) \) is a number of ordered \( k \)-tuples of non-associated in pairs Gaussian integers \((d_1, \ldots, d_k)\) such that \( d_1 \cdots d_k = a \).

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The aims of this paper are to determine maximal orders of \( \tau^{(e)}_k \), \( \tau^{(e)}_{k^*} \), \( t^{(e)}_k \), \( t^{(e)}_{k^*} \) and to provide asymptotic formulas for \( \sum_{n \leq x} \tau^{(e)}_k(n) \), \( \sum'_{N(\alpha) \leq x} \tau^{(e)}_{k^*}(\alpha) \), \( \sum'_{N(\alpha) \leq x} t^{(e)}_k(\alpha) \), \( \sum'_{N(\alpha) \leq x} t^{(e)}_{k^*}(\alpha) \). A theorem on the maximal order of multiplicative functions over \( \mathbb{Z}[i] \), generalizing [9], is also proved.

2. Notation

Let us denote the ring of Gaussian integers by \( \mathbb{Z}[i] \), \( N(a+bi) = a^2 + b^2 \).

In asymptotic relations we use \( \sim \), \( \asymp \), Landau symbols \( O \) and \( o \), Vinogradov symbols \( \ll \) and \( \gg \) in their usual meanings. All asymptotic relations are written for the argument tending to the infinity.

Letters \( p \) and \( q \) with or without indexes denote Gaussian primes; \( p \) and \( q \) denote rational primes.

As usual \( \zeta(s) \) is Riemann zeta-function and \( L(s, \chi) \) is Dirichlet \( L \)-function. Let \( \chi_4 \) be the single nonprincipal character modulo 4, then \( Z(s) = \zeta(s)L(s, \chi_4) \) is Hecke zeta-function for the ring of Gaussian integers.

Real and imaginary components of the complex \( s \) are denoted as \( \sigma := \Re s \) and \( t := \Im s \), so \( s = \sigma + it \).

We use abbreviations \( \log x := \log \log x \), \( l\log x := \log \log \log x \).

Notation \( \sum' \) means a summation over non-associated elements of \( \mathbb{Z}[i] \), and \( \prod' \) means the similar relative to multiplication. Notation \( a \sim b \) means that \( a \) and \( b \) are associated, that is \( a/b \in \{\pm 1, \pm i\} \). But in asymptotic relations \( \sim \) preserves its usual meaning.

Letter \( \gamma \) denotes Euler–Mascheroni constant. Everywhere \( \varepsilon > 0 \) is an arbitrarily small number (not always the same).

We write \( f \ast g \) for the notation of the Dirichlet convolution

\[
(f \ast g)(n) = \sum_{d|n} f(d)g(n/d).
\]

3. Preliminary lemmas

We need following auxiliary results.

**Lemma 1** (Gauss criterion). Gaussian integer \( p \) is prime if and only if one of the following cases complies:

- \( p \sim 1 + i \),
- \( p \sim p \), where \( p \equiv 3 \pmod{4} \),
- \( N(p) = p \), where \( p \equiv 1 \pmod{4} \).

In the last case there are exactly two non-associated \( p_1 \) and \( p_2 \) such that \( N(p_1) = N(p_2) = p \).

**Proof.** See [1, §34].

**Lemma 2.**

\[
\sum'_{N(p) \leq x} 1 \sim \frac{x}{\log x},
\]

\[
\sum'_{N(p) \leq x} \log N(p) \sim x,
\]
Proof. Taking into account Gauss criterion and the asymptotic law of the distribution of primes in the arithmetic progression we have

\[
\sum_{N(p)\leq x} 1 \sim \# \{ p \mid p \equiv 3 \pmod{4}, p \leq \sqrt{x} \} + 2 \# \{ p \mid p \equiv 1 \pmod{4}, p \leq x \} \sim
\]

\[
\sim \frac{\sqrt{x}}{\phi(4) \log x/2} + 2 \frac{x}{\phi(4) \log x} = \frac{x}{\log x}.
\]

A partial summation with the use of (2) gives us the second statement of the lemma.

Lemma 3. For \( k \geq 2 \)

(4) \[
\max_{n \geq 1} \frac{\log \tau_k(n)}{n} = \frac{\log k}{2}.
\]

Proof. Taking into account

\[
\tau_k(p^a) = \binom{k + a - 1}{a} \leq k^a,
\]

for \( \Omega(n) := \sum_{p \mid |n} a \) we have \( \tau_k(n) \leq k^{\Omega(n)} \leq k^{\log_2 n} \). This implies

\[
\frac{\log \tau_k(n)}{n} \leq \frac{\log_2 n}{n} \log k \leq \frac{\log k}{2},
\]

because \( n^{-1} \log_2 n \) is strictly decreasing for \( n \geq 2 \). But

\[
\frac{\log \tau_k(2)}{2} = \log k.
\]

Lemma 4. For \( k \geq 2 \)

(5) \[
\max_{n \geq 1} \frac{\log t_k(n)}{n} = \frac{1}{2} \log \left( \frac{k + 1}{2} \right).
\]

Proof. Let \( k_2 := \binom{k+1}{2} \). Lemma 4 implies that

\[
t_k(2^a) = \binom{k + 2a - 1}{2a} \leq k_2^a,
\]

\[
t_k(p^a) = \binom{k + a - 1}{a} \leq k^a \leq k_2^a \quad \text{if} \quad p \equiv 3 \pmod{4},
\]

\[
t_k(p^a) = \binom{k + a - 1}{a}^2 \leq k_2^{2a} \quad \text{if} \quad p \equiv 1 \pmod{4}.
\]

Let us define

\[
\Omega_1(n) := \sum_{p \equiv 1 \pmod{4}} a, \quad \Omega_2(n) := \sum_{p \equiv 1 \pmod{4}} a.
\]

Then \( t_k(n) \leq k^{2\Omega_1(n)}k_2^{\Omega_2(n)} \). Consider

\[
f(x, y) = \frac{x \log k^2 + y \log k_2}{5x^2y},
\]

then \( n^{-1} \log t_k(n) \leq f(\Omega_1(n), \Omega_2(n)) \). One can verify that if \( x \geq 1 \) or \( y \geq 1 \) then

\[
f(x + 1, y) \leq f(x, y), \quad f(x, y + 1) \leq f(x, y),
\]

because \( \log k_2 + \log k^2 < 5 \log k_2 \). So

\[
\max_{x, y \geq 0} f(x, y) = \max \{ f(1, 0), f(0, 1) \} = \frac{\log k_2}{2}.
\]
But
\[ \frac{\log t_2(2)}{2} = \frac{\log k_2}{2}. \]

**Lemma 5.** Let \( F: \mathbb{Z} \to \mathbb{C} \) be a multiplicative function such that \( F_k(p^a) = f(a) \), where \( f(n) \ll n^\beta \) for some \( \beta > 0 \). Then
\[ \limsup_{n \to \infty} \frac{\log F_k(n) \log n}{\log n} = \sup_{n \geq 1} \frac{\log f(n)}{n}. \]

**Proof.** See [9].

**Lemma 6.** Let \( f(t) \geq 0 \). If
\[ \int_1^T f(t) \, dt \ll g(T), \]
where \( g(T) = T^\alpha \log^\beta T, \alpha \geq 1 \), then
\[ I(T) := \int_1^T \frac{f(t)}{t} \, dt \ll \begin{cases} T^{\beta+1} & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases} \]

**Proof.** Let us divide the interval of integration into parts:
\[ I(T) \leq \sum_{k=0}^{\log_2 T} \int_{T/2^k}^{T/2^{k+1}} \frac{f(t)}{t} \, dt < \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_{T/2^k}^{T/2^{k+1}} f(t) \, dt \ll \sum_{k=0}^{\log_2 T} g(T/2^k). \]
Now the lemma’s statement follows from elementary estimates.

**Lemma 7.** Let \( T > 10 \) and \( |d - 1/2| \ll 1/\log T \). Then we have the following estimates
\[ \int_{-T}^{d+iT} |\zeta(s)|^4 \frac{ds}{s} \ll \log^5 T, \]
\[ \int_{-T}^{d+iT} |L(s, \chi_4)|^4 \frac{ds}{s} \ll \log^5 T, \]
for growing \( T \).

**Proof.** The statement is the result of the application of Lemma [9] to the estimates [6] Th. 10.1, p. 75.

**Lemma 8.** Define \( \theta > 0 \) such that \( \zeta(1/2 + it) \ll t^\theta \) as \( t \to \infty \), and let \( \eta > 0 \) be arbitrarily small. Then
\[ \zeta(s) \ll \begin{cases} |t|^{1/2 - (1-2\theta)/2}, & \sigma \in [0, 1/2], \\ |t|^{2\theta(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\theta(1-\sigma)} \log^{2/3} |t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3} |t|, & \sigma \geq 1. \end{cases} \]
The same estimates are valid for \( L(s, \chi_4) \) also.

**Proof.** The statement follows from Phragmén–Lindelöf principle, exact and approximate functional equations for \( \zeta(s) \) and \( L(s, \chi_4) \). See [4] and [10] for details.

The best modern result [3] is that \( \theta \leq 32/205 + \varepsilon. \)
4. Main results

First we give maximal orders of $\tau_k(\alpha)$, $\tau_k(\beta)$, $t_k(\alpha)$ and $t_k(\beta)$.

The following theorem generalizes Lemma 5 in [9] to Gaussian integers; the proof’s outline follows the proof of Lemma 5 in [9].

**Theorem 1.** Let $F : \mathbb{Z}[i] \rightarrow \mathbb{C}$ be a multiplicative function such that $F(p^\alpha) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then

$$\limsup_{\alpha \to \infty} \frac{\log F(\alpha) \log N(\alpha)}{\log N(\alpha)} = \sup_{n \geq 1} \frac{\log f(n)}{n} := K_f.$$

**Proof.** Let us fix arbitrarily small $\varepsilon > 0$. Firstly, let us show that there are infinitely many $\alpha$ such that

$$\frac{\log F(\alpha) \log N(\alpha)}{\log N(\alpha)} > K_f - \varepsilon.$$

By definition of $K_f$ we can choose $l$ such that

$$(\log f(l))/l > K_f - \varepsilon/2.$$ 

It follows from (3) that for $x \geq 2$ inequality

$$\sum_{N(p) \leq x} \log N(p) > Ax$$

holds, where $0 < A < 1$.

Let $q$ be an arbitrarily large Gaussian prime, $N(q) \geq 2$. Consider

$$r = \sum_{N(p) \leq N(q)} 1, \quad \alpha = \prod_{N(p) \leq N(q)} p^\alpha.$$ 

Then $F(\alpha) = (f(l))^r$ and we have

$$\sum_{N(p) \leq N(q)} \log N(p) > AN(q),$$

$$\log F(\alpha) = r \log f(l) \geq \frac{\log N(\alpha) \log f(l)}{\log N(\alpha)} l.$$ 

But (3) implies

$$\log A + \log N(q) < \log \frac{\log N(\alpha)}{l} \leq \log N(\alpha),$$

so $\log N(q) < \log N(\alpha) - \log A$. Then it follows from (7) that

$$\frac{\log F(\alpha) \log N(\alpha)}{\log N(\alpha)} > \frac{\log N(\alpha) \log f(l)}{l \log N(\alpha) - \log A} (K_f - \varepsilon/2) > K_f - \varepsilon.$$

Second, let us show the existence of $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ we have

$$\frac{\log F(n) \log N(\alpha)}{\log N(\alpha)} < (1 + \varepsilon)K_f.$$

Let us choose $\delta \in (0, \varepsilon)$ and $\eta \in (0, \delta/(1 + \delta))$. Suppose $N(\alpha) \geq 3$, then we define

$$\omega := \omega(\alpha) = \frac{(1 + \delta)K_f}{\log N(\alpha)}, \quad \Omega := \Omega(\alpha) = \log^{1-\eta} N(\alpha).$$
By choice of $\delta$ and $\eta$ we have
\[ \Omega^\omega = \exp(\omega \log \Omega) = \exp((1 - \eta)(1 + \delta)K_f) > e^{K_f}. \]
Suppose that the canonical expansion of $\alpha$ is
\[ \alpha \sim p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{b_1} \cdots q_s^{b_s}, \]
where $N(p_k) \leq \Omega$ and $N(q_k) > \Omega$. Then
\[ \frac{F(\alpha)}{N^{\omega b_k}(p_k)} = \prod_{k=1}^{r} \frac{f(a_k)}{N^{\omega a_k}(p_k)} \prod_{k=1}^{s} \frac{f(b_k)}{N^{\omega b_k}(q_k)} := \Pi_1 \cdot \Pi_2. \]
But since $\Omega^\omega > e^{K_f}$ and $K_f \geq (\log f(b_k))/b_k$ then
\[ \frac{f(b_k)}{N^{\omega b_k}(q_k)} < \frac{f(b_k)}{\Omega^{b_k}} < \frac{f(b_k)}{e^{K_f} b_k} \leq 1 \]
and it follows that $\Pi_2 \leq 1$. Consider $\Pi_1$. From the statement of the theorem we have $f(n) \ll n^\beta$, so
\[ \frac{f(a_k)}{N^{\omega a_k}(p_k)} \ll \frac{a_k^\beta}{(\omega a_k)^\beta} \ll \omega^{-\beta}. \]
Then
\[ \log \Pi_1 \ll \Omega \log \omega^{-\beta} \ll \log^{1-n} N(\alpha) \ll \log N(\alpha) = o\left(\frac{\log N(\alpha)}{\log \log N(\alpha)}\right) \]
And finally by (5) we get
\[ \log F(n) = \omega \log n + \log \Pi_1 + \log \Pi_2 = \frac{(1 + \delta)K_f \log n}{\log n} + \frac{(\varepsilon - \delta)K_f \log n}{\log n}. \]

**Theorem 2.**

\begin{align*}
\limsup_{n \to \infty} \frac{\log \tau_k(n) \log n}{\log n} &= \frac{\log k}{2}, \\
\limsup_{n \to \infty} \frac{\log \tau_k^{(c)}(n) \log n}{\log n} &= \frac{1}{2} \log \binom{k + 1}{2}, \\
\limsup_{\alpha \to \infty} \frac{\log \tau_k^{(c)}(\alpha) \log N(\alpha)}{\log N(\alpha)} &= \frac{\log k}{2}, \\
\limsup_{\alpha \to \infty} \frac{\log \tau_k^{(c)}(\alpha) \log N(\alpha)}{\log N(\alpha)} &= \frac{1}{2} \log \binom{k + 1}{2}.
\end{align*}

**Proof.** Statements follow from (4), (5), Lemma 5 and Theorem 1. 

A simple corollary of the Theorem 2 is that
\[ \tau_k^{(c)}(n) \ll n^\varepsilon, \quad t_k^{(c)}(\alpha) \ll N^\varepsilon(\alpha), \quad t_k^{(c)}(\alpha) \ll N^\varepsilon(\alpha). \]

We are ready to provide asymptotic formulas for sums of $\tau_k^{(c)}(n)$, $t_k^{(c)}(\alpha)$, $t_k^{(c)}(\alpha)$. Let us denote
\[ G_k(s) := \sum_n \tau_k^{(c)}(n)n^{-s}, \quad T_k(s) := \sum_n \tau_k^{(c)}(n), \]
\[ F_k(s) := \sum_\alpha t_k^{(c)}(\alpha)N^{-s}(\alpha), \quad M_k(s) := \sum_\alpha t_k^{(c)}(\alpha), \]
\[ F_k(s) := \sum_\alpha t_k^{(c)}(\alpha)N^{-s}(\alpha), \quad M_k(s) := \sum_\alpha t_k^{(c)}(\alpha). \]
Lemma 9.  
\[(10)\] \[G_k(s) = \zeta(s)(2s)\zeta(-k^2+k-2)/2(2s)\zeta(-k^2+k)/2(3s)\zeta(-k^2+7k^2-6k)/12(4s) \times \zeta(5k^4-6k^3-5k^2+6k)/24(5s)g_k(s),\]
\[(11)\] \[F_k(s) = Z(s)Z^{-1}(2s)Z^{-1}(k-k^2)/2(5s)Z^{-1}(k^2+6k-5k^2)/6(6s) \times \]
\[Z(k^2-4k^2+3k)/2(7s)Z(3k^4-26k^3+57k^2-34k)/24(8s)f_k(s),\]
\[(12)\] \[F_k(s) = Z(s)Z^{-1}(k^2+k-2)/2(2s)Z^{-1}(k^2+k)/2(3s)Z^{-1}(k^2+7k^2-6k)/12(4s) \times \]
\[Z(5k^4-6k^3-5k^2+6k)/24(5s)f_k(s),\]
where Dirichlet series $f_k(s)$ are absolutely convergent for $\Re s > 1/9$ and Dirichlet series for $g_k(s)$ are absolutely convergent for $\Re s > 1/6$.

Proof. The statements can be directly verified with the help of the Bell series for corresponding functions. For example, for $t_k(e)$ we have following representation:
\[
\left(\sum_{a=0}^{\infty} t_k^{(e)}(p^a)x^a\right)(1-x)(1-x^2)k-1(1-x^5)(k-k^2)/2(1-x^6)(k^2+6k^3-5k^2)/6 \times
\]
\[(1-x)^7(k^2-4k^2+3k)/2(1-x)^8(3k^4-26k^3+57k^2-34k)/24 = 1 + O(x^9).\]
Then (11) follows from the representation of $F_k$ and $Z$ in the form of infinite products by $p$:
\[F_k(s) = \prod_p \left(\sum_{a=0}^{\infty} t_k^{(e)}(p^a)x^a\right), \quad Z(s) = \prod_p (1-p^{-s})^{-1}.\]
Identities (10) and (12) can be proved the same way. □

Let us define $\alpha := (1,2,\ldots,2)$,
\[\tau(\alpha; n) := \sum_{d_0d_1^2d_2^2 = n} 1, \quad T(\alpha; x) := \sum_{n \leq x} \tau(\alpha; n) = \sum_{d_0d_1^2d_2^2 \leq x} 1,\]
Due to [5, Th. 6.10] we have
\[(13)\] \[T(\alpha; x) = C_1 x + x^{1/2}Q(\log x) + O(x^{w_1+\varepsilon}),\]
where $Q$ is a polynomial with computable coefficients, $\deg Q = l - 1$, and $w_l \leq (2l + 1)/(4l + 5)$. For some special values of $l$ better estimates of the error term can be obtained. For example, $w_1 \leq 1057/4785$ (see [2]) and $w_2 \leq 8/25$ due to [5] (6.16).

Theorem 3.  
\[T_k^*(x) = A_k x + x^{1/2}P_k(\log x)O(x^{v_k+\varepsilon}),\]
where $P_k$ is a polynomial with computable coefficients, $\deg P_k = (k^2 + k - 4)/2$, and $v_k = \max(w(k^2+k-2)/2, 1/3)$.

Proof. Let $l = (k^2 + k - 2)/2$. Identity (11) implies
\[(14)\] \[\tau_k^{(e)} = \tau(\alpha; \cdot) * f, \quad T_k^*(x) = \sum_{n \leq x} T(\alpha; x/n)f(n),\]
where series $\sum_{n=1}^{\infty} f(n)n^{-\sigma}$ are absolutely convergent for $\sigma > 1/3$. 
One can plainly estimate:

\[(15) \quad \sum_{n>x} \frac{f(n)}{n} \ll x^{-2/3+\varepsilon} \sum_{n>x} \frac{f(n)}{n^{1/3+\varepsilon}} \ll x^{-2/3+\varepsilon}, \]

\[(16) \quad \sum_{n>x} \frac{f(n) \log^a n}{n^{1/2}} \ll x^{-1/6+\varepsilon} \sum_{n>x} \frac{f(n) \log^a n}{n^{1/3+\varepsilon}} \ll x^{-1/6+\varepsilon}. \]

Substituting estimates \((13), (15)\) and \((16)\) into \((14)\) we get

\[T_{k^*}(x) = C_1 x \sum_{n \leq x} \frac{f(n)}{n} + x^{1/2} \sum_{n \leq x} \frac{f(n)Q(x/n)}{n^{1/2}} + O(x^{w_1+\varepsilon}) + O(x^{1/3+\varepsilon}) =
\]

\[= A_k x + x^{1/2} F_k (\log x) + O(x^{w_k+\varepsilon}). \]

\[\blacksquare\]

**Lemma 10.**

\[(17) \quad \res_{s=1} F_k(s) x^s/s = C_k x, \quad \res_{s=1} F_{k^*}(s) x^s/s = C_{k^*} x, \]

where

\[(18) \quad C_k = \frac{\pi}{4} \prod_p \left(1 + \sum_{a=2}^\infty \frac{\tau_k(a) - \tau_k(a-1)}{N^a(p)} \right), \]

\[(19) \quad C_{k^*} = \frac{\pi}{4} \prod_p \left(1 + \sum_{a=2}^\infty \frac{\ell_k(a) - \ell_k(a-1)}{N^a(p)} \right). \]

**Proof.** As a consequence of the representation \((11)\) we have

\[\frac{F_k(s)}{Z(s)} = \prod_p \left(1 + \sum_{a=1}^\infty \frac{\tau_k(a)}{N^{a+1}(p)} \right) \left(1 - p^{-1}\right) = \prod_p \left(1 + \sum_{a=2}^\infty \frac{\tau_k(a) - \tau_k(a-1)}{N^{a+1}(p)} \right), \]

and so function \(F_k(s)/Z(s)\) is regular in the neighbourhood of \(s = 1\). At the same time we have

\[\res_{s=1} Z(s) = L(1, \chi_k) \res_{s=1} \zeta(s) = \frac{\pi}{4}, \]

which implies \((13)\). The proof of \((19)\) is similar. \[\blacksquare\]

Numerical values of \(C_k\) and \(C_{k^*}\) can be calculated in PARI/GP \([7]\) with the use of the transformation

\[\prod_p f(N(p)) = f(2) \prod_{p=4k+1} f(p^2) \prod_{p=4k+3} f(p^2) \]

due to Lemma \((11)\). For example,

\[C_2 \approx 1,156,101, \quad C_{2^*} \approx 1,524,172. \]

**Theorem 4.**

\[(20) \quad M_k(x) = C_k x + O(x^{1/2} \log^{3+4(k-1)/3} x), \]

\[(21) \quad M_{k^*}(x) = C_{k^*} x + O(x^{1/2} \log^{3+2(k^2+k-2)/3} x), \]

where \(C_k\) and \(C_{k^*}\) were defined in \((13)\) and \((19)\).

**Proof.** By Perron formula and by \((1)\) for \(c = 1 + 1/\log x, \log T \ll \log x\) we have

\[M_k(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F_k(s) \frac{x^s}{s} ds + O \left(\frac{x^{1+\varepsilon}}{T}\right). \]
Suppose $d = 1/2 - 1/\log x$. Let us shift the interval of integration to $[d - iT, d + iT]$. To do this consider an integral about a closed rectangle path with vertexes in $d - iT$, $d + iT$, $c + iT$ and $c - iT$.

There are two poles in $s = 1$ and $s = 1/2$ inside the contour. The residue at $s = 1$ was calculated in [17]. The residue at $s = 1/2$ is equal to $D x^{1/2}$, $D$ is const and will be absorbed by error term (see below).

Identity (11) implies

$$F_k(s) = Z(s)Z^{k-1}(2s)f_k(s),$$

where $f_k(s)$ is regular for $\Re s > 1/3$, so for each $\varepsilon > 0$ it is uniformly bounded for $\Re s > 1/3 + \varepsilon$.

Let us estimate the error term using Lemma 7 and Lemma 8. The error term absorbs values of integrals about three sides of the integration’s rectangle. We take into account $Z(s) = \zeta(s)L(s, \chi_4)$. On the horizontal segments we have

$$\int_{d+iT}^{c+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s} ds \ll \max_{\sigma \in [d,c]} Z(\sigma + iT)Z^{k-1}(2\sigma + 2iT)x^{\sigma}T^{-1} \ll x^{1/2}T^{2\sigma-1}\log^{4(k-1)/3}T + xT^{-1}\log^{4/3}T,$$

It is well-known that $\zeta(s) \sim (s-1)^{-1}$ in the neighborhood of $s = 1$. So on $[d, d+i]$ we get

$$\int_{d}^{d+i} Z(s)Z^{k-1}(2s)\frac{x^s}{s} ds \ll x^{1/2} \int_{0}^{1} \zeta^{k-1}(2d + 2it)dt \ll x^{1/2} \int_{0}^{1} \frac{1}{|it - 1/\log x|^{k-1}} \ll x^{1/2} \log^{k-1} x,$$

and for the rest of the vertical segment we have

$$\int_{d+i}^{D+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s} ds \ll \left( \int_{1}^{T} |\chi(1/2+it)|^{4} \left| T \right|^{1/4} \left| \left( \int_{1}^{T} \left| Z(1+2it) \right| \frac{1}{t} \right|^{1/4} \right| \left( \int_{1}^{T} \left| Z(1+2it) \right|^{2(k-1)} \frac{1}{t} \right|^{1/2} \right) \ll x^{1/2} \log^{5}T \cdot \log^{k(k-1)/3+1}T^{1/2} \ll x^{1/2} \log^{3+4(k-1)/3}T,$$

The choice $T = x^{1/2+\varepsilon}$ finishes the proof of (20).

The proof of (21) is similar, but due to (12) one have replace $k - 1$ by $(k^2 + k - 2)/2$.

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