FUNCTIONAL SPACES AND OPERATORS CONNECTED WITH SOME LÉVY NOISES

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Abstract

We review some recent developments in white noise analysis and quantum probability. We pay a special attention to spaces of test and generalized functionals of some Lévy white noises, as well as to the structure of quantum white noise on these spaces.

1 Gaussian white noise and Fock space

Since the work of Hida [10] of 1975, Gaussian white noise analysis has become an established theory of test and generalized functions of infinitely many variables, see e.g. [11, 6] and the references therein.

Let us shortly recall some basic results of Gaussian analysis. In the space $L^2(\mathbb{R}) := L^2(\mathbb{R}, dx)$, consider the harmonic oscillator

$$(Hf)(t) := -f''(t) + (t^2 + 1)f(t), \quad f \in C_0^\infty(\mathbb{R}).$$

This operator is self-adjoint and we preserve the notation $H$ for its closure. For each $p \in \mathbb{R}$, define a scalar product

$$(f, g)_p := (H^p f, g)_{L^2(\mathbb{R})}, \quad f, g \in C_0^\infty(\mathbb{R}).$$

Let $S_p$ denote the Hilbert space obtained as the closure of $C_0^\infty(\mathbb{R})$ in the norm $\| \cdot \|_p$ generated by the scalar product $(\cdot, \cdot)_p$. Then, for any $p > q$, the space $S_p$ is densely and continuously embedded into $S_q$, and if $p - q > 1/2$, then this embedding is of Hilbert–Schmidt type. Furthermore, for each $p > 0$, $S_{-p}$ is the dual space of $S_p$ with respect to zero space $L^2(\mathbb{R})$, i.e., the dual paring $(f, \varphi)$ between any $f \in S_{-p}$ and any $\varphi \in S_p$ is obtained as the extension of the scalar product in $L^2(\mathbb{R})$. The
above conclusions are, in fact, corollaries of the fact that the sequence of Hermite functions on $\mathbb{R}$,

$$e_j = e_j(t) = (\sqrt{\pi}2^j j!)^{-1/2}(-1)^j(e^{t^2/2}(d/dt)^j e^{-t^2}, \quad j \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\},$$

forms an orthonormal basis of $L^2(\mathbb{R})$ such that each $e_j$ is an eigenvector of $H$ with eigenvalue $(2j + 2)^2$.

Then

$$S := \text{proj lim}_{p \to \infty} S_p$$

is the Schwartz space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}$, and its dual

$$S' = \text{ind lim}_{p \to \infty} S_{-p}$$

is the Schwartz space of tempered distributions.

We denote by $\mathcal{C}(S')$ the $\sigma$-algebra on $S'$ which is generated by cylinder sets in $S'$, i.e., by the sets of the form

$$\{\omega \in S' : (\langle \omega, \varphi_1 \rangle, \ldots, \langle \omega, \varphi_N \rangle) \in A\},$$

where $\varphi_1, \ldots, \varphi_N \in S$, $N \in \mathbb{N}$, and $A \in B(\mathbb{R}^N)$.

By the Minlos theorem, there exists a unique probability measure $\mu_G$ on $(S', \mathcal{C}(S'))$ whose Fourier transform is given by

$$\int_{S'} e^{i\langle \omega, \varphi \rangle} d\mu_G(\omega) = \exp \left[ - (1/2)\|\varphi\|^2_{L^2(\mathbb{R})} \right], \quad \varphi \in S. \quad (1)$$

The measure $\mu_G$ is called the (Gaussian) white noise measure. Indeed, using formula (1), it is easy to see that, for each $\varphi \in S$,

$$\int_{S'} \langle \omega, \varphi \rangle^2 d\mu_G(\omega) = \|\varphi\|^2_{L^2(\mathbb{R})}. $$

Hence, extending the mapping

$$L^2(\mathbb{R}) \supset S \ni \varphi \mapsto \langle \cdot, \varphi \rangle \in L^2(S', \mu_G)$$

by continuity, we obtain a random variable $\langle \cdot, f \rangle \in L^2(S', \mu_G)$ for each $f \in L^2(\mathbb{R})$. Then, for each $t \in \mathbb{R}$, we define

$$X_t := \begin{cases} \langle \cdot, 1_{[0,t]} \rangle, & t \geq 0, \\ -\langle \cdot, 1_{[t,0]} \rangle, & t < 0. \end{cases} \quad (2)$$

It is easily seen that $(X_t)_{t \in \mathbb{R}}$ is a version of Brownian motion, i.e., finite-dimensional distributions of the stochastic process $(X_t)_{t \in \mathbb{R}}$ coincide with those of Brownian
motion. We now informally have, for all \( t \in \mathbb{R} \), \( X_t(\omega) = \int_0^t \omega(t) \, dt \), so that \( X'_t(\omega) = \omega(t) \). Thus, elements \( \omega \in \mathcal{S}' \) can be thought of as paths of the derivative of Brownian motion, i.e., Gaussian white noise.

Let us recall that the symmetric Fock space over a real separable Hilbert space \( \mathcal{H} \) is defined as

\[
\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathcal{H})n!.
\]

Here \( \mathcal{F}^{(n)}(\mathcal{H}) := \mathcal{H}_C^n \), where \( \odot \) stands for symmetric tensor product and the lower index \( C \) denotes complexification of a real space. Thus, for each \( (f^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\mathcal{H}) \),

\[
\|(f^{(n)})_{n=0}^{\infty}\|_\mathcal{F}(\mathcal{H})^2 = \sum_{n=0}^{\infty} \|f^{(n)}\|_{\mathcal{F}^{(n)}(\mathcal{H})}^2 n!.
\]

The central technical point of the construction of spaces of test and generalized functionals of Gaussian white noise is the Wiener–Itô–Segal isomorphism \( I_G \) between the Fock space \( \mathcal{F}(L^2(\mathbb{R})) \) and the complex space \( L^2(\mathcal{S}' \to \mathbb{C}, \mu) \), which, for simplicity of notations, we will denote by \( L^2(\mathcal{S}', \mu) \).

There are different ways of construction of the isomorphism \( I_G \), e.g., using multiple stochastic integrals with respect to Gaussian random measure. For us, it will be convenient to follow the approach which uses the procedure of orthogonalization of polynomials, see e.g. [6] for details.

A function \( F(\omega) = \sum_{i=0}^{n} \langle \omega^{\odot i}, f^{(i)} \rangle \), where \( \omega \in \mathcal{S}' \), \( n \in \mathbb{Z}_+ \), and each \( f^{(i)} \in \mathcal{S}_C^{\odot i} \), is called a continuous polynomial on \( \mathcal{S}' \), and \( n \) is called the order of the polynomial \( F \). The set \( \mathcal{P} \) of all continuous polynomials on \( \mathcal{S}' \) is dense in \( L^2(\mathcal{S}', \mu_G) \). For \( n \in \mathbb{Z}_+ \), let \( \mathcal{P}^{(n)} \) denote the set of all continuous polynomials on \( \mathcal{S}' \) of order \( \leq n \), and let \( \mathcal{P}^{(n)}_G \) be the closure of \( \mathcal{P}^{(n)} \) in \( L^2(\mathcal{S}', \mu_G) \). Let \( \mathcal{P}^{(n)}_G \) stand for the orthogonal difference \( \mathcal{P}^{(n)}_G \oplus \mathcal{P}^{(n-1)}_G \) in \( L^2(\mathcal{S}', \mu_G) \). Then we easily get the orthogonal decomposition

\[
L^2(\mathcal{S}', \mu_G) = \bigoplus_{n=0}^{\infty} \mathcal{P}^{(n)}_G.
\]

Next, for any \( f^{(n)} \in \mathcal{S}_C^{\odot n} \), we define \( \langle \omega^{\odot n}, f^{(n)} \rangle :_G \) as the orthogonal projection of \( \langle \omega^{\odot n}, f^{(n)} \rangle \) onto \( \mathcal{P}^{(n)}_G \). The set of such projections is dense in \( \mathcal{P}^{(n)}_G \). Furthermore, for any \( f^{(n)}, g^{(n)} \in \mathcal{S}_C^{\odot n} \), we have:

\[
\int_{\mathcal{S}'} \langle \omega^{\odot n}, f^{(n)} \rangle :_G \times \langle \omega^{\odot n}, g^{(n)} \rangle :_G \, d\mu_G(\omega) = \langle f^{(n)}, g^{(n)} \rangle_{\mathcal{F}^{(n)}(L^2(\mathbb{R}))} n!.
\]  

Let \( \mathcal{F}_{\text{fin}}(\mathcal{S}) \) denote the set of all sequences \( (f^{(n)})_{n=0}^{\infty} \) such that each \( f^{(i)} \in \mathcal{S}_C^{\odot i} \) and for some \( N \in \mathbb{N} \) \( f^{(n)} = 0 \) for all \( n \geq N \). The \( \mathcal{F}_{\text{fin}}(\mathcal{S}) \) is a dense subset of \( \mathcal{F}(L^2(\mathbb{R})) \).
For any $f = (f^{(n)})_{n=0}^\infty \in F_{\text{fin}}(S)$, we set
\[
(I_G f)(\omega) = \sum_{n=0}^\infty \langle \omega \otimes n, f^{(n)} \rangle_G \in L^2(S', \mu_G).
\]
(4)

By (3), we can extend $I_G$ by continuity to get a unitary operator
\[
I_G : F(L^2(\mathbb{R})) \to L^2(S', \mu_G).
\]
For any function $f^{(n)} \in F_{\text{fin}}(L^2(\mathbb{R}))$, we will use the evident notation:
\[
\langle \omega \otimes n, f^{(n)} \rangle_G.
\]
Then, for each $f = (f^{(n)})_{n=0}^\infty \in F(L^2(\mathbb{R}))$, $I_G f$ is given by formula (4).

For each $\kappa \in [-1, 1]$ and $p \in \mathbb{R}$, we denote
\[
F_\kappa(S_p) := \bigoplus_{n=0}^\infty F^{(n)}(S_p)(n!)^{1+\kappa},
\]
and for each $\kappa \in [0, 1]$ we set
\[
F_\kappa(S) := \text{proj lim}_{p \to \infty} F_\kappa(S_p).
\]
It is easy to see that each $F_\kappa(S)$ is a nuclear space. Furthermore, the dual space of $F_\kappa(S)$ with respect to zero space $F(L^2(\mathbb{R}))$ is
\[
F_{-\kappa}(S') := \text{ind lim}_{p \to \infty} F_{-\kappa}(S_{-p}).
\]
Thus, we get a standard triple
\[
F_\kappa(S) \subset F(L^2(\mathbb{R})) \subset F_{-\kappa}(S').
\]
The test space $\mathcal{F}_1(S)$ is evidently the smallest one between the above spaces, whereas its dual space $\mathcal{F}_{-1}(S')$ is the biggest one.

For each $F = (F^{(n)}) \in \mathcal{F}_{-1}(S')$, the $S$-transform of $F$ is defined by

$$
(SF)(\xi) := \sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n} \rangle, \quad \xi \in \mathcal{S}_C,
$$

provided the series on the right hand side of (6) converges absolutely.

The $S$-transform uniquely characterizes an element of $\mathcal{F}_{-1}(S')$. More exactly, let $\text{Hol}_0(\mathcal{S}_C)$ denote the set of all (germs) of functions which are holomorphic in a neighborhood of zero in $\mathcal{S}_C$. The following theorem was proved in [16].

**Theorem 1.1** The $S$-transform is a one-to-one map between $\mathcal{F}_{-1}(S')$ and $\text{Hol}_0(\mathcal{S}_C)$.

Note that the choice of $\kappa > 1$ would imply that the $S$-transform is not well-defined on $\mathcal{F}_{-\kappa}(S')$. On the other hand, all the spaces $\mathcal{F}_{-\kappa}(S')$ and $\mathcal{F}_{\kappa}(S)$ with $\kappa \in [0, 1]$ admit a complete characterization in terms of their $S$-transform, see e.g. [6, 19, 16].

Taking into account that the product of two elements of $\text{Hol}_0(\mathcal{S}_C)$ remains in this set, one defines a Wick product $F_1 \diamond F_2$ of $F_1, F_2 \in \mathcal{F}_{-1}(S')$ through the formula

$$
S(F_1 \diamond F_2) = S(F_1)S(F_2).
$$

Furthermore, if $F \in \mathcal{F}_{-1}(S')$ and $f$ is a holomorphic function in a neighborhood of $(SF)(0)$ in $\mathbb{C}$, then one defines $f^\circ(F) \in \mathcal{F}_{-1}(S')$ through

$$
S(f^\circ(F)) = f(SF).
$$

Using the unitary operator $I_G$, all the above definitions and results can be reformulated in terms of test and generalized functions on $S'$ whose dual paring is generated by the scalar product in $L^2(S', \mu_G)$. In particular, one defines spaces of test functions $(\mathcal{S})^\kappa_G := I_G\mathcal{F}_{\kappa}(S)$ and their dual spaces $(\mathcal{S})^\kappa_G^*$, $\kappa \in [0, 1]$. For $\kappa = 0$, these are the Hida test space and the space of Hida distributions, respectively (e.g. [11, 8]). For $\kappa \in (0, 1)$, these spaces were introduced and studied by Kondratiev and Streit [19], and for $\kappa = 1$, by Kondratiev, Leukert and Streit [16]. Note also that, in the case of a Gaussian product measure, such spaces for all $\kappa \in [0, 1]$ were studied by Kondratiev [15].

The Wick calculus of generalized Gaussian functionals based on the definitions (7), (8) and the unitary operator $I_G$ has found numerous applications, in particular, in fluid mechanics and financial mathematics, see e.g. [12, 9].

Additionally to the description of the above test spaces $(\mathcal{S})^\kappa_G$ in terms of their $S$-transform, one can also give their inner description, e.g. [19, 16]. Let $\mathcal{E}(\mathcal{S}_C)$ denote
the space of all entire functions on $S'_C$. For each $\beta \in [1, 2]$, we denote by $E^\beta_{\min}(S'_C)$ the subset of $E(S'_C)$ consisting of all entire functions of the $\beta$-th order of growth and minimal type. That is, for any $\Phi \in E^\beta_{\min}(S'_C)$, $p \geq 0$, and $\varepsilon > 0$, there exists $C > 0$ such that
\[ |\Phi(z)| \leq C \exp(\varepsilon |z|^{\beta-p}), \quad z \in S_{-p,C}. \]
Next, we denote by $E^\beta_{\min}(S')$ the set of all functions on $S'$, which are obtained by restricting functions from $E^\beta_{\min}(S'_C)$ to $S'$. The following theorem unifies the results of [19, 16], see also [6, 11],

**Theorem 1.2** For each $\kappa \in [0, 1]$, we have
\[ (S)_{\kappa}^{G} = E^{2/(1+\kappa)}_{\min}(S'). \tag{9} \]

The equality (9) is understood in the sense that, for each $f = (f^{(n)}) \in F_{\kappa}(S)$, the following realization was chosen for $\Phi := I_G f$:
\[ \Phi(\omega) = \sum_{n=0}^{\infty} \langle \omega^{\otimes n} : G, f^{(n)} \rangle, \quad \omega \in S', \]
where $\omega^{\otimes n} : G \in S'^{\otimes n}$ is defined by the recurrence relation
\[ \omega^{\otimes 0} : G = 1, \quad \omega^{\otimes 1} : G = \omega, \]
\[ \omega^{\otimes (n+1)} : G(t_1, \ldots, t_{n+1}) = \left( \omega^{\otimes n} : G(t_1, \ldots, t_n) \omega(t_{n+1}) \right)^\sim - n \left( \omega^{\otimes (n-1)} : G(t_1, \ldots, t_{n-1}) \delta(t_{n+1} - t_n) \right)^\sim, \quad n \in \mathbb{N}. \]
Here $\delta(\cdot)$ denotes the delta function at zero and $(\cdot)^\sim$ denotes symmetrization.

For each $t \in \mathbb{R}$, we define an annihilation operator at $t$, denoted by $\partial_t$, and a creation operator at $t$, denoted by $\partial_t^\dagger$, by
\[ (\partial_t f^{(n)})(t_1, \ldots, t_{n-1}) := nf^{(n)}(t_1, \ldots, t_{n-1}, t), \quad f^{(n)} \in S_C^{\otimes n}, \]
\[ \partial_t^\dagger F^{(n)} := \delta_t \otimes F^{(n)}, \quad F^{(n)} \in S'_C^{\otimes n}, \]
where $\delta_t$ denotes the delta function at $t$. The operators $\partial_t$ and $\partial_t^\dagger$ can then be extended to linear continuous operators on $F_{\kappa}(S)$ and $F_{-\kappa}(S')$, respectively, and $\partial_t^\dagger$ becomes the dual operator of $\partial_t$. Then,
\[ A^- (\varphi) = \int_{\mathbb{R}} \varphi(t) \partial_t dt, \]
\[ A^+ (\varphi) = \int_{\mathbb{R}} \varphi(t) \partial_t^\dagger dt, \]
and so by (5),

$$A_G(\varphi) = \int_{\mathbb{R}} \varphi(t)(\partial_t + \partial_t^\dagger) dt,$$

the above integrals being understood in the Bochner sense. The operators

$$W_G(t) := \partial_t + \partial_t^\dagger, \quad t \in \mathbb{R},$$

are called quantum Gaussian white noise. Realized on $$(S)_G^1$$, the operator $\partial_t$ becomes the operator of Gâteaux differentiation in direction $\delta_t$:

$$(\partial_t F)(\omega) = \lim_{\varepsilon \to 0} (F(\omega + \varepsilon \delta_t) - F(\omega))/\varepsilon, \quad F \in (S)_G^1, \ t \in \mathbb{R}, \ \omega \in S',$$

see e.g. [11].

2 Poisson white noise

The measure $\mu_P$ of (centered) Poisson white noise is defined on $$(S', C(S'))$$ by

$$\int_{S'} e^{i\langle \omega, \varphi \rangle} d\mu_P(\omega) = \exp \left[ \int_{\mathbb{R}} (e^{i\varphi(t)} - 1 - i\varphi(t)) dt \right], \quad \varphi \in S.$$

Under $\mu_P$, the stochastic process $$(X_t)_{t \in \mathbb{R}}$$, defined by (2), is a centered Poisson process, which is why $\omega \in S'$ can now be thought of as a path of Poisson white noise (in fact, $\mu_P$ is concentrated on infinite sums of delta functions shifted by $-1$).

The procedure of orthogonalization of continuous polynomials in $L^2(S', \mu_P)$ leads to a unitary isomorphism $I_P$ between the Fock space $F(L^2(\mathbb{R}))$ and $L^2(S', \mu_P)$. The counterpart of formula (5) now looks as follows:

$$A_P(\varphi) = A^+(\varphi) + A^0(\varphi) + A^-(\varphi),$$

where $A^0(\varphi)$ is the neutral operator:

$$(A^0(\varphi) f^{(n)})(t_1, \ldots, t_n) := \left( \sum_{i=1}^n \varphi(t_i) \right) f^{(n)}(t_1, \ldots, t_n), \quad f^{(n)} \in S_{C^n}. $$

In terms of the operators $\partial_t$ and $\partial_t^\dagger$, the neutral operator has the following representation:

$$A^0(\varphi) = \int_{\mathbb{R}} \varphi(t) \partial_t^\dagger \partial_t dt.$$

Thus, the quantum (centered) Poisson white noise is given by

$$W_P(t) = \partial_t + \partial_t^\dagger \partial_t + \partial_t^\dagger$$

(11)
Next, using the unitary operator $I_P$, we obtain a scale of spaces of test functions $(S)_{P}^{\kappa}$ and generalized functions $(S')_{P}^{-\kappa}$. For any $f^{(n)} \in S_{C}^{\otimes n}$, the orthogonal projection $\langle \omega^{\otimes n} : f^{(n)} \rangle_{P}$ has a $\mu_P$-version $\langle \omega^{\otimes n} : P, f^{(n)} \rangle$, where $\omega^{\otimes n} : P \in S'_{C}^{\otimes n}$ are given by the following recurrence relation (see [22]):

$$\omega^{\otimes 0} : P = 1, \quad \omega^{\otimes 1} : P = \omega,$$

$$\omega^{\otimes (n+1)} : P(t_1, \ldots, t_{n+1}) = (\omega^{\otimes n} : P(t_1, \ldots, t_n) \omega(t_{n+1}))^{\sim}$$

$$- n(\omega^{\otimes (n-1)} : P(t_1, \ldots, t_{n-1}) 1(t_n) \delta(t_{n+1} - t_n))^{\sim}$$

$$- n(\omega^{\otimes n} : P(t_1, \ldots, t_n) \delta(t_{n+1} - t_n))^{\sim}, \quad n \in \mathbb{N}.$$  

However, in the Poisson case, the following statement holds [22].

**Theorem 2.1** For each $\omega \in S'$, denote $D(\omega) := (\omega^{\otimes n} : P)_{n=0}^{\infty}$. Then $D(\omega) \in F_{-1}(S')$, and if $\omega \neq 0$, then $D(\omega) \notin F_{-\kappa}(S')$ for all $\kappa \in [0, 1)$.

It is straightforward to see that the $D(\omega)$ in the Poisson realization is just the delta function at $\omega$, denoted by $\delta_\omega$. Thus, Theorem 2.1 implies:

**Corollary 2.1** For each $\omega \in S'$, $\delta_\omega \in (S')_{P}^{-1}$, and if $\omega \neq 0$, then $\delta_\omega \notin (S')_{P}^{-\kappa}$ for all $\kappa \in [0, 1)$.

The above corollary shows that the test spaces $(S)_{P}^{\kappa}$ with $\kappa \in [0, 1)$ do not possess nice inner properties, and therefore they are not appropriate for applications. On the other hand, we have [20]:

**Theorem 2.2** We have:

$$(S)_{P}^{1} = E_{\min}^{1}(S').$$

Thus, $E_{\min}^{1}(S')$ appears to be a universal space for both Gaussian and Poisson white noise analysis.

The annihilation operator $\partial_t$ realized on $(S)_{P}^{1}$ becomes a difference operator [22]:

$$(\partial_t F)(\omega) = F(\omega + \delta_t) - F(\omega), \quad F \in (S)_{P}^{1}, \quad t \in \mathbb{R}, \quad \omega \in S'.$$

We note that one can also study a more general white noise measure of Poisson type for which the corresponding quantum white noise is given by

$$\partial_t + \lambda \partial_t^{\dagger} \partial_t + \partial_t^{\dagger}, \quad (12)$$

where $\lambda \in \mathbb{R}_+, \lambda \neq 0$, see e.g. [26].
3 Lévy white noise and extended Fock space

We will now discuss the case of a Lévy white noise without Gaussian part. Let $\nu$ be a probability measure on $(\mathbb{R}, B(\mathbb{R}))$ such that $\nu(\{0\}) = 0$. We will also assume that there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}} \exp(\varepsilon |s|) \nu(ds) < \infty$. The latter condition implies that $\nu$ has all moments finite, and moreover, the set of all polynomials on $\mathbb{R}$ is dense in $L^2(\mathbb{R}, \nu)$.

We define a centered Lévy white noise as a probability measure $\mu_\nu$ on $(S', C(S'))$ with Fourier transform

$$\int_{S'} e^{i \langle \omega, \varphi \rangle} \mu_\nu(d\omega) = \exp \left[ \kappa \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{is\varphi(t)} - 1 - is\varphi(t)) \frac{1}{s^2} \nu(ds) dt \right], \quad \varphi \in S,$$

where $\kappa > 0$. For notational simplicity, we will assume that $\kappa = 1$, which is a very weak restriction.

Under the measure $\mu_\nu$, the stochastic process $(X_t)_{t \in \mathbb{R}}$, defined by (2), is a centered Lévy process with Lévy measure $\frac{1}{s^2} \nu(ds)$:

$$\int_{S'} e^{iuX_t(\omega)} \mu_\nu(d\omega) = \exp \left[ |t| \int_{\mathbb{R}} (e^{isu\text{sign}(t)} - 1 - isu\text{sign}(t)) \frac{1}{s^2} \nu(ds) dt \right], \quad u \in \mathbb{R}.$$

Hence, $\omega \in S'$ can be thought of as a path of Lévy white noise.

By the above assumptions, the set $\mathcal{P}$ of all continuous polynomials on $S'$ is dense in $L^2(S', \mu_\nu)$. Therefore, through the procedure of orthogonalization of polynomials, one gets an orthogonal decomposition

$$L^2(S', \mu_\nu) = \bigoplus_{n=0}^{\infty} \mathfrak{P}_\nu^{(n)}$$

and the set of all orthogonal projections $\langle \omega^{\otimes n}, f^{(n)} \rangle_{\mu_\nu}$ of $\langle \omega^{\otimes n}, f^{(n)} \rangle$ onto $\mathfrak{P}_\nu^{(n)}$ is dense in $\mathfrak{P}_\nu^{(n)}$.

However, in contrast to the Gaussian and Poisson cases, the scalar product of any $\langle \omega^{\otimes n}, f^{(n)} \rangle_{\nu}$ and $\langle \omega^{\otimes n}, g^{(n)} \rangle_{\nu}$ in $L^2(S', \mu_\nu)$ is not given by the scalar product of $f^{(n)}$ and $g^{(n)}$ in the Fock space, but by a much more complex expression, see [24] for an explicit formula in the general case, and formulas [14], [15] below in a special case. Still it is possible to construct a unitary isomorphism $I_\nu$ between the so-called extended Fock space $F_\nu(L^2(\mathbb{R})) = \bigoplus_{n=0}^{\infty} F_\nu^{(n)}(L^2(\mathbb{R}))n!$ and $L^2(S', \mu_\nu)$. In the above construction, $F_\nu^{(n)}(L^2(\mathbb{R}))$ is a Hilbert space that is obtained as the closure of $\mathcal{S}_c^{\otimes n}$ in the norm generated by the scalar product

$$\langle f^{(n)}, g^{(n)} \rangle_{F_\nu^{(n)}(L^2(\mathbb{R}))} := \frac{1}{n!} \int_{S'} \langle \omega^{\otimes n}, f^{(n)} \rangle_{\nu} \times \langle \omega^{\otimes n}, g^{(n)} \rangle_{\nu} \mu_\nu(d\omega).$$
We next set
\[ A_\nu(\varphi) := I_\nu^{-1}(\cdot, \varphi) \cdot I_\nu, \quad \varphi \in S. \]

Our next aim is to derive an explicit form of the action of these operators. This can be done [24, 7], however, the property that \( F_{\text{fin}}(S) \) is invariant under the action of \( A_\nu(\varphi) \), generally speaking, does not hold. The following theorem [7, 24, 23] identifies all the Lévy noises for which this property is preserved.

**Theorem 3.1** Under the above assumptions, the property
\[ A_\nu(\varphi)F_{\text{fin}}(S) \subset F_{\text{fin}}(S), \quad \varphi \in S, \]
holds if and only if \( \nu \) is the measure of orthogonality of a system of polynomials \((p_n(t))_{n=0}^\infty \) on \( \mathbb{R} \) which satisfy the following recurrence relation:
\[
 tp_n(t) = \sqrt{(n+1)(n+2)}p_{n+1}(t) + \lambda(n+1)p_n(t) + \sqrt{n(n+1)}p_{n-1}(t),
\]
\[ n \in \mathbb{Z}_+, \quad p_0(t) = 1, \quad p_{-1}(t) = 0, \quad (13) \]
for some \( \lambda \in \mathbb{R} \).

Let us consider the situation described in Theorem 3.1 in more detail. We will denote by \( \nu_\lambda \) the measure \( \nu \) which corresponds to the parameter \( \lambda \in \mathbb{R} \) through (13). We first mention that the condition of orthogonality of the polynomials satisfying (13) uniquely determines the measure \( \nu_\lambda \). It is easy to show that, for \( \lambda > 0 \), the measure \( \nu_{-\lambda} \) is the image of the measure \( \nu_\lambda \) under the mapping \( \mathbb{R} \ni t \mapsto -t \in \mathbb{R} \), which is why we will only consider the case \( \lambda \geq 0 \). In fact, we have (see e.g. Ref. [8]): for \( \lambda \in [0, 2) \),
\[
 \nu_\lambda(ds) = \frac{\sqrt{4-\lambda^2}}{2\pi} \left| \Gamma(1 + i(4 - \lambda^2)^{-1/2}s) \right|^2 \times \exp \left[ -s2(4 - \lambda^2)^{-1/2} \arctan \left( \lambda(4 - \lambda^2)^{-1/2} \right) \right] ds
\]
(\( \nu_\lambda \) is a Meixner distribution), for \( \lambda = 2 \)
\[
 \nu_2(ds) = \chi_{(0,\infty)}(s)e^{-s}s ds
\]
(\( \nu_2 \) is a gamma distribution), and for \( \lambda > 2 \)
\[
 \nu_\lambda(ds) = (\lambda^2 - 4) \sum_{k=1}^{\infty} \left( \frac{\lambda - \sqrt{\lambda^2 - 4}}{\lambda + \sqrt{\lambda^2 - 4}} \right)^k k \delta_{\sqrt{\lambda^2 - 4}k}
\]
(\( \nu_\lambda \) is now a Pascal distribution).

In what follows, we will use the lower index \( \lambda \) instead of \( \nu_\lambda \).
The stochastic process \((X_t)_{t \in \mathbb{R}}\) under \(\mu_\lambda\) is a Meixner process for \(|\lambda| < 2\), a gamma process for \(|\lambda| = 2\) and a Pascal process for \(|\lambda| > 2\). In other words, for each \(t \in \mathbb{R}, t \neq 0\), the distribution of the random variable \(X_t\) under \(\mu_\lambda\) is of the same class of distributions as the measure \(\nu_\lambda\).

Next, for each \(\lambda\), the scalar product in the space \(F^{(n)}(L^2(\mathbb{R})) = F^{(n)}_\lambda(L^2(\mathbb{R}))\) is given as follows \cite{23, 24}. For each \(\alpha \in \mathbb{Z}_+, 1\alpha_1 + 2\alpha_2 + \cdots = n, n \in \mathbb{N}\), and for any function \(f^{(n)} : \mathbb{R}^n \to \mathbb{R}\) we define a function \(D_\alpha f^{(n)} : \mathbb{R}^{|\alpha|} \to \mathbb{R}\) by setting

\[
(D_\alpha f^{(n)})(t_1, \ldots, t_{|\alpha|}) := f^{(n)}(t_1, \ldots, t_{\alpha_1}, t_{\alpha_1+1}, t_{\alpha_1+1}, t_{\alpha_1+2}, \ldots, t_{\alpha_1+2}, t_{\alpha_2}, t_{\alpha_2+1}, t_{\alpha_2+1}, \ldots).
\] (14)

Here \(|\alpha| := \alpha_1 + \alpha_2 + \cdots\). Then, for any \(f^{(n)}, g^{(n)} \in \mathcal{S}^\omega_{\mathbb{C}}\),

\[
(f^{(n)}, g^{(n)})_{F^{(n)}(L^2(\mathbb{R}))} = n! \sum_{\alpha \in \mathbb{Z}_+: 1\alpha_1 + 2\alpha_2 + \cdots = n} \frac{n!}{\alpha_1! 1^{\alpha_1} 2^{\alpha_2} \cdots} \times \int_{\mathbb{R}^{|\alpha|}} (D_\alpha f^{(n)})(t_1, \ldots, t_{|\alpha|}) \times (D_\alpha g^{(n)})(t_1, \ldots, t_{|\alpha|}) dt_1 \cdots dt_{|\alpha|}.
\] (15)

The following theorem \cite{23, 24} describes the action of \(A_\lambda(\varphi)\) on \(\mathcal{F}_{\text{fin}}(\mathcal{S})\).

**Theorem 3.2** For each \(\lambda \in \mathbb{R}\) and \(\varphi \in \mathcal{S}\), we have on \(\mathcal{F}_{\text{fin}}(\mathcal{S})\):

\[
A_\lambda(\varphi) = A^+(\varphi) + \lambda A^0(\varphi) + \mathfrak{A}^-(\varphi).
\]

Here \(\mathfrak{A}^-(\varphi)\) is the restriction to \(\mathcal{F}_{\text{fin}}(\mathcal{S})\) of the adjoint operator of \(A^+(\varphi)\) in \(F(L^2(\mathbb{R})) = \bigoplus_{n=0}^\infty F^{(n)}(L^2(\mathbb{R})) n!\), and

\[
\mathfrak{A}^-(\varphi) = A^-(\varphi) + A_1^-(\varphi),
\]

where

\[
(A_1^-(\varphi)f^{(n)})(t_1, \ldots, t_{n-1}) = n(n-1)(\varphi(t_1)f^{(n)}(t_1, t_2, t_3, \ldots, t_n))\sim.
\] (16)

Furthermore, each \(A_\lambda(\varphi)\) is essentially self-adjoint on \(\mathcal{F}_{\text{fin}}(\mathcal{S})\).

We see that \(A_\lambda(\varphi)\) has creation, neutral, and annihilation parts. Therefore, the family of self-adjoint commuting operators \((A_\lambda(\varphi))_{\varphi \in \mathcal{S}}\) is a Jacobi field in \(F(L^2(\mathbb{R}))\) (compare with \cite{3} and the references therein).
From Theorem 3.2, one concludes that, for any \( f^{(n)} \in \mathcal{S}_C^{\otimes n} \), the orthogonal projection \( \langle \omega^{(n)}, f^{(n)} \rangle_{\lambda} \) has a \( \mu_{\lambda} \)-version \( \langle \omega^{\otimes n};_{\lambda}, f^{(n)} \rangle \), where \( \omega^{\otimes n};_{\lambda} \in \mathcal{S}'^{\otimes n} \) are given by the following recurrence relation:

\[
\omega^{\otimes 0};_{\lambda} = 1, \quad \omega^{\otimes 1};_{\lambda} = \omega,
\]

\[
\omega^{(n+1)};_{\lambda}(t_1, \ldots, t_{n+1}) = \left( \omega^{\otimes n};_{\lambda}(t_1, \ldots, t_n) \omega(t_{n+1}) \right) - n \left( \omega^{(n-1)};_{\lambda}(t_1, \ldots, t_{n-1}) 1(t_n) \delta(t_{n+1} - t_n) \right) - \lambda n \left( \omega^{\otimes n-1};_{\lambda}(t_1, \ldots, t_n) \delta(t_{n+1} - t_n) \right), \quad n \in \mathbb{N}.
\]

We will now construct a space of test functions. It is possible to show \[17\] that the Hilbert space \( \mathcal{F}_1(\mathcal{S}_1) \) is densely and continuously embedded into \( F(L^2(\mathbb{R})) \). This embedding is understood in the sense that \( \mathcal{F}_1(\mathcal{S}_1) \) is considered as the closure of \( \mathcal{F}_{\min}(\mathcal{S}) \) in the respective norm. Therefore, the nuclear space \( \mathcal{F}_1(\mathcal{S}) \) is densely and continuously embedded into \( F(L^2(\mathbb{R})) \). We will denote by \( (\mathcal{S})^1_{\lambda} \) the image of \( \mathcal{F}_1(\mathcal{S}) \) under \( I_{\lambda} \).

Using a result from the theory of test and generalized functions connected with a generalized Appell system of polynomials \[18, 14\], one proves the following theorem.

**Theorem 3.3** For each \( \lambda \in \mathbb{R} \),

\[
(\mathcal{S})^1_{\lambda} = \mathcal{E}^1_{\min}(\mathcal{S}').
\]

Thus, \( \mathcal{E}^1_{\min}(\mathcal{S}') \) appears to be a universal space for our purposes.

Taking to notice that \( (\mathcal{S})^1_{\lambda} \) is the image of \( \mathcal{F}_1(\mathcal{S}) \) under \( I_{\lambda} \), we will identify the dual space \( (\mathcal{S})^1_{\lambda} \) with \( \mathcal{E}^{-1}_{\lambda}(\mathcal{S}) \). Notice, however, that now the dual pairing between elements of \( (\mathcal{S})^1_{\lambda} \) and \( (\mathcal{S})^1_{\lambda} \) is obtained not through the scalar product in \( L^2(\mathcal{S}', \mu_{\lambda}) \), or equivalently in \( F(L^2(\mathbb{R})) \), but through the scalar product in the usual Fock space \( \mathcal{F}(L^2(\mathbb{R})) \). In particular, such a realization of the dual space \( (\mathcal{S})^1_{\lambda} \) is convenient for developing Wick calculus on it.

By \[16\], the operator \( A_1^-(\varphi) \) has the following representation through the operators \( \partial_t \) and \( \partial_t^1 \):

\[
A_1^-(\varphi) = \int_{\mathbb{R}} \varphi(t) \partial_t^1 \partial_t \partial_t \, dt.
\]

Therefore, by Theorem 3.2, we get:

\[
A_\lambda(\varphi) = \int_{\mathbb{R}} \varphi(t) \left( \partial_t^1 + \lambda \partial_t \partial_t + \partial_t \partial_t \partial_t \right) dt.
\]

Hence, the corresponding quantum white noise, denoted by \( W_\lambda(t) \), is given by

\[
W_\lambda(t) = \partial_t^1 + \lambda \partial_t \partial_t + \partial_t \partial_t \partial_t.
\]
Realized on the space \((S)^1_\lambda\), the operator \(\partial_t\) acts as follows \cite{23}:

\[
(\partial_t F)(\omega) = \int_{\mathbb{R}} \frac{F(\omega + s\delta_t) - F(\omega)}{s} \nu_\lambda(ds), \quad F \in (S)^1_\lambda, \quad t \in \mathbb{R}, \quad \omega \in S'.
\]

4 The square of white noise algebra

As we have seen in Sections 1 and 2, the Gaussian white noise is just the sum of the annihilation operator \(\partial_t\) and the creation operator \(\partial_t^\dagger\), whereas the Poisson white noise is obtained by adding to the Gaussian white noise a constant times the product of \(\partial_t^\dagger\) and \(\partial_t\). Let us also recall that the operators \(\partial_t, \partial_t^\dagger, t \in \mathbb{R}, \omega \in S'\). satisfy the canonical commutation relations:

\[
[\partial_t, \partial_s] = [\partial_t^\dagger, \partial_s^\dagger] = 0,
\]

\[
[\partial_t, \partial_s^\dagger] = \delta(t - s),
\]

(18)

where \([A, B] := AB - BA\).

It was proposed in \cite{2} to develop a stochastic calculus for higher powers of white noise, in other words, for higher powers of the operators \(\partial_t, \partial_t^\dagger\). This problem was, in fact, influenced by the old dream of T. Hida that the operators \(\partial_t, \partial_t^\dagger\) should play a fundamental role in infinite-dimensional analysis.

We will now deal with the squares of \(\partial_t, \partial_t^\dagger\). The idea is to introduce operators \(B_t\) and \(B_t^\dagger\) which will be interpreted as \(\partial_t^2\) and \((\partial_t^\dagger)^2\), to derive from (18) the commutation relations satisfied by \(B_t, B_t^\dagger, N_t := B_t^\dagger B_t\) and then to consider the quantum white noise \(B_t + B_t^\dagger + \lambda N_t\), where \(\lambda \in \mathbb{R}\) (compare with (10), (11), and (12)). However, when doing this, one arrives at the expression \(\delta(\cdot)^2\) — the square of the delta function. It was proposed in \cite{3, 4} to carry out a renormalization procedure by employing the following equality, which may be justified in the framework of distribution theory:

\[
\delta(\cdot)^2 = c\delta(\cdot).
\]

Here \(c \in \mathbb{C}\) is arbitrary. This way one gets the following commutation relations:

\[
[B_t, B_s^\dagger] = 2c\delta(t - s) + 4\delta(t - s)N_s, \\
[N_t, B_s^\dagger] = 2\delta(t - s)B_s^\dagger, \\
[N_t, B_s] = -2\delta(t - s)B_s, \\
[N_t, N_s] = [B_t, B_s] = [B_t^\dagger, B_s^\dagger] = 0. 
\]

(19)

As usual in mathematical physics, the rigorous meaning of the commutation relations (19) is that they should be understood in the smeared form. Thus, we introduce the smeared operators

\[
B(\varphi) := \int_{\mathbb{R}} \varphi(t)B_t dt, \quad B^\dagger(\varphi) := \int_{\mathbb{R}^d} \varphi(t)B_t^\dagger dt, \quad N(\varphi) := \int_{\mathbb{R}} \varphi(t)N_t dt,
\]
where $\varphi \in \mathcal{S}$, and then the commutation relations between these operators take the form

\[
\begin{align*}
[B(\varphi), B^\dagger(\psi)] &= 2c \langle \varphi, \psi \rangle + 4N(\varphi\psi), \\
[N(\varphi), B^\dagger(\psi)] &= 2B^\dagger(\varphi\psi), \\
[N(\varphi), B(\psi)] &= -2B(\varphi\psi), \\
[N(\varphi), N(\psi)] &= [B(\varphi), B(\psi)] = [B^\dagger(\varphi), B^\dagger(\psi)] = 0, \quad \phi, \psi \in \mathcal{S}.
\end{align*}
\]

(20)

The operator algebra with generators $B(\varphi), B^\dagger(\varphi), N(\varphi), \varphi \in \mathcal{S}$, and a central element 1 with relations (20) is called the square of white noise (SWN) algebra.

In [3], it was shown that a Fock representation of the SWN algebra exists if and only if the constant $c$ is real and strictly positive. In what follows, it will be convenient for us to choose the constant $c$ to be 2, though this choice is not essential.

Using the notations of Section 3, we define

\[
\begin{align*}
B_t &= 2(\partial_t + \partial^\dagger_t \partial_t), \quad B^\dagger_t = 2\partial_t^\dagger, \quad N_t = 2\partial_t^\dagger \partial_t.
\end{align*}
\]

Then it is straightforward to show that the corresponding smeared operators

\[
\begin{align*}
B(\varphi) &= 2\mathcal{A}(\varphi), \quad B^\dagger = 2A^+(\varphi), \quad N(\varphi) = 2A^0(\varphi)
\end{align*}
\]

(21)

form a representation of a SWN algebra in $F(L^2(\mathbb{R}))$. We also refer to [1] for a unitarily equivalent representation, see also [25].

Thus, the quantum white noises $W_\lambda(t), \lambda \in \mathbb{R}$, (see (17)) can be thought of as a class of (commuting) quantum processes obtained from the SWN algebra (21).

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