Accurate masses for dispersion-supported galaxies

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ABSTRACT
We derive an accurate mass estimator for dispersion-supported stellar systems and demonstrate its validity by analyzing resolved line-of-sight velocity data for globular clusters, dwarf galaxies, and elliptical galaxies. Specifically, by manipulating the spherical Jeans equation we show that the mass enclosed within the 3D deprojected half-light radius \( r_{1/2} \) can be determined with only mild assumptions about the spatial variation of the stellar velocity dispersion anisotropy as long as the projected velocity dispersion profile is fairly flat near the half-light radius, as is typically observed. We find \( M_{1/2} = 3G^{-1}\left<\sigma_{\text{los}}^2\right>_1 r_{1/2} \approx 4G^{-1}\left<\sigma_{\text{los}}^2\right> R_e \), where \( \left<\sigma_{\text{los}}^2\right> \) is the luminosity-weighted square of the line-of-sight velocity dispersion and \( R_e \) is the 2D projected half-light radius. While deceptively familiar in form, this formula is not the virial theorem, which cannot be used to determine accurate masses unless the radial profile of the total mass is known \textit{a priori}. We utilize this finding to show that all of the Milky Way dwarf spheroidal galaxies (MW dSphs) are consistent with having formed within a halo of mass approximately \( 3 \times 10^9 \, M_\odot \), assuming a ΛCDM cosmology. The faintest MW dSphs seem to have formed in dark matter halos that are at least as massive as those of the brightest MW dSphs, despite the almost five orders of magnitude spread in luminosity between them. We expand our analysis to the full range of observed dispersion-supported stellar systems and examine their dynamical I-band mass-to-light ratios \( T_{1/2}^I \). The \( T_{1/2}^I \) vs. \( M_{1/2} \) relation for dispersion-supported galaxies follows a U-shape, with a broad minimum near \( T_{1/2}^I \approx 3 \) that spans dwarf elliptical galaxies to normal ellipticals, a steep rise to \( T_{1/2}^I \approx 3,200 \) for ultra-faint dSphs, and a more shallow rise to \( T_{1/2}^I \approx 800 \) for galaxy cluster spheroids.

Key words: Galactic dynamics, dwarf galaxies, elliptical galaxies, galaxy formation, dark matter

1 INTRODUCTION
Mass determinations for dispersion-supported galaxies based on only line-of-sight velocity measurements suffer from a notorious uncertainty associated with not knowing the intrinsic 3D velocity dispersion. The difference between radial and tangential velocity dispersions is usually quantified by the stellar velocity dispersion anisotropy, \( \beta \). Many questions in galaxy formation are affected by our ignorance of \( \beta \), including our ability to quantify the amount of dark matter in the outer parts of elliptical galaxies (Romanowsky et al. 2003; Dekel et al. 2003), to measure the mass profile of the Milky Way from stellar halo kinematics (Battaglia et al. 2002; Dehnen et al. 2006), and to infer accurate mass distributions in dwarf spheroidal galaxies (dSphs) (Gilmore et al. 2007; Strigari et al. 2007).

Here we use the spherical Jeans equation to show that for each dispersion-supported galaxy, there exists one radius within which the integrated mass as inferred from the line-of-sight velocity dispersion is largely insensitive to \( \beta \), and that this radius is approximately equal to \( r_3 \), the location where the log-slope of the 3D tracer density profile is \( -3 \).

\[^1\] In this paper we will often refer to the stellar number density profile, but this work is applicable to any tracer system, including planetary nebulae and globular clusters that trace galaxy potentials, and galaxies that trace galaxy cluster potentials.
indicate a luminosity-weighted average. For a wide range of \( \sigma \) on line-of-sight kinematic data. We will use these generalized equation to provide general mass likelihood solutions based on line-of-sight kinematic data. Will refer to a deprojected, three-dimensional (3D) radius. (i.e., \( \frac{d \ln n_s}{d \ln r} \approx -3 \)). Moreover, the mass within \( r_5 \) is well characterized by a simple formula that depends only on quantities that may be inferred from observations:

\[
M(r_5) = 3G^{-1} \langle \sigma_{\text{los}}^2 \rangle r_5, \tag{1}
\]

where \( M(r) \) is the mass enclosed within a sphere of radius \( r \), \( \sigma_{\text{los}} \) is the line-of-sight velocity dispersion, and the brackets indicate a luminosity-weighted average. For a wide range of stellar light distributions that describe dispersion-supported galaxies, \( r_5 \) is close to the 3D deprojected half-light radius \( r_{1/2} \) and therefore we may also write:

\[
M_{1/2} \equiv M(r_{1/2}) \approx 3G^{-1} \langle \sigma_{\text{los}}^2 \rangle r_{1/2}, \tag{2}
\]

\[
\simeq 4G^{-1} \langle \sigma_{\text{los}}^2 \rangle R_\odot, \tag{3}
\]

\[
\simeq 930 \left( \langle \sigma_{\text{los}}^2 \rangle \text{ km}^2\text{s}^{-2} \right) \left( \frac{R_\odot}{\text{pc}} \right) M_\odot.
\]

In the second line we have used \( R_\odot \simeq (3/4) r_{1/2} \) for the 2D projected half-light radius. This approximation is accurate to better than 2% for exponential, Gaussian, King, Plummer, and Sérsic profiles (see Appendix B for useful fitting formulae).

As we show below, Equation 2 is understood under the assumption that the observed stellar velocity dispersion profile is relatively flat near \( R_\odot \). Clearly, one can write down self-consistent models that violate this assumption. In these cases, the mass uncertainty is minimized at a radius other than \( r_{1/2} \), and Equation 2 will no longer be as accurate. However, the velocity dispersions of real galaxies in the Universe (including those we consider below) do appear to be rather flat near the half-light radius, thus validating the use of Equation 2.

In the next section we discuss the spherical Jeans equation and our method for determining generalized, maximum-likelihood mass profile solutions based on line-of-sight velocity measurements. As a point of comparison we also discuss the virial theorem as a mass estimator for spherical systems. In §3 we derive Equation 2 and show that it works using real galaxy data, and explain why the \( \beta \) uncertainty is minimized at \( r \approx r_5 \approx r_{1/2} \) for line-of-sight kinematics. In §4 we present two examples of how \( M_{1/2} \) determinations can be used to inform models of galaxy formation: first, we show that the \( M_{1/2} \) vs. \( r_{1/2} \) relationship for Milky Way dSph galaxies provides an important constraint on the type of dark matter halos they were born within; and second, we examine the dynamical half-light mass-to-light ratios for the full range of dispersion-supported stellar systems in the Universe and argue that this relationship can be used to inform models of feedback. We conclude in §5.

In this paper the symbol \( R \) will always refer to a projected, two-dimensional (2D) radius and the symbol \( r \) will refer to a deprojected, three-dimensional (3D) radius.

### 2 REVIEW AND METHODOLOGY

In what follows we review the virial theorem as a mass estimator for spherical systems, introduce the Jeans equation, and present our numerical methodology for using the Jeans equation to provide general mass likelihood solutions based on line-of-sight kinematic data. We will use these generalized mass solutions to evaluate our \( M_{1/2} \) estimator in §3.

#### 2.1 The Scalar Virial Theorem

The scalar virial theorem (SVT) is perhaps the most popular equation used to provide rough mass constraints for spherical galaxies (e.g., Poveda 1954; Tully & Fisher 1977; Busarello et al. 1997). Consider a spherically symmetric dispersion-supported galaxy with a total gravitating mass profile \( M(r) \), which includes a 3D stellar mass density \( \rho_* \equiv m_* n_* (r) \) that truncates at a radius \( r_{1\text{m}} \). \( M_1(r) \) quantifies the distribution of stellar mass per normalized number while the stellar number density \( n_* (r) \) is normalized to integrate to unity over the stellar volume. If \( m_* (r) \) is assumed to be constant, then the SVT can be expressed as:

\[
4\pi G \int_0^{r_{1\text{m}}} n_* (r) M(r) r \, dr = \int V \langle n_* (r) \sigma_{\text{los}}^2 (r) \rangle \, d^3 r \tag{3}
\]

\[
= \langle \sigma_{\text{los}}^2 \rangle = 3 \langle \sigma_{\text{los}}^2 \rangle.
\]

Note that the luminosity-weighted average of the square of the total velocity dispersion \( \sigma_{\text{los}} \) is independent of \( \beta \), and thus if one knows the number density (either by recording the position of every single star, or by making an assumption about how the observed surface brightness relates to the number density), the SVT provides an observationally-applicable constraint on the integrated mass profile within the stellar extent of the system.

Unfortunately, the constraint associated with the SVT is not particularly powerful as it allows a family of acceptable solutions for \( M(r) \). This point was emphasized by Merritt (1985, Appendix A), who considered two extreme possibilities for \( M(r) \) (a point mass and a constant density distribution) to show that the SVT constrains the total mass \( M_t \) within the stellar extent \( r_{1\text{m}} \) to obey

\[
\frac{\langle \sigma_{\text{los}}^2 \rangle}{\langle r_{1\text{m}}^{-1} \rangle} \leq G M_t \leq \frac{r_{1\text{m}}^3 \langle \sigma_{\text{los}}^2 \rangle}{(\langle r_{1\text{m}}^2 \rangle)}, \tag{4}
\]

where \( \langle r_{1\text{m}}^{-1} \rangle \) and \( \langle r_{1\text{m}}^2 \rangle \) are moments of the stellar distribution. The associated constraint is quite weak. For example, if we assume \( n_* (r) \) follows a King (1962) profile with \( r_{1\text{m}}/R_\odot = 5 \) (typical for Local Group dwarf spheroidal galaxies) Equation 4 allows a large uncertainty in the mass within the stellar extent: \( 0.7 \langle \sigma_{\text{los}}^2 \rangle \leq G M_t / r_{1\text{m}} \leq 20 \langle \sigma_{\text{los}}^2 \rangle \).

Another common way to express the SVT is to first define a gravitational radius \( r_g \equiv GM_t^2 / |W| \) (Binney & Tremaine 2008), where \( W \) is the potential energy, which depends on the unknown mass profile. By absorbing our ignorance of the mass profile into \( r_g \), we can write the total mass as

\[
M_t = G^{-1} \langle \sigma_{\text{los}}^2 \rangle r_g = 3G^{-1} \langle \sigma_{\text{los}}^2 \rangle r_g. \tag{5}
\]

In the literature it is common to rewrite Equation 5 as

\[
M_t = k G^{-1} \langle \sigma_{\text{los}}^2 \rangle R_\odot, \tag{6}
\]

where \( k = 3 r_g / R_\odot \) is referred to as the ‘virial coefficient’. If one wishes to re-express this version of the SVT in a form analogous to what we have in Equation 2 we need to relate \( M_t \) to the mass enclosed within \( r_{1/2} \), which again requires

\[2\] The total mass density need not truncate at this radius.
knowledge of the mass profile \( M(r) = f(r) M_1 \):

\[
M(r_{1/2}) = f(r_{1/2}) M_1 = f(r_{1/2}) k G^{-1} \langle \sigma_{los}^2 \rangle r_{1/2}
\]

(7)

\[
c(r_{1/2}) G^{-1} \langle \sigma_{los}^2 \rangle R_c.
\]

Note that the value of \( c(r_{1/2}) \) depends on the (unknown) mass profile through both \( f(r_{1/2}) \) and \( k \). Below, using an alternative analysis, we show that \( c(r_{1/2}) = 4 \) under circumstances that are fairly general for observed galaxies.

2.2 The Spherical Jeans Equation

Given the relative weakness of the SVT as a mass estimator, the spherical Jeans equation provides an attractive alternative. It relates the total gravitating potential \( \Phi(r) \) of a spherically symmetric, dispersion-supported, collisionless stationary system to its tracer velocity dispersion and tracer number density, under the assumption of dynamical equilibrium with no streaming motions:

\[
-\frac{n_\star d\Phi}{dr} = \frac{d(n_\star \sigma^2)}{dr} + 2\beta n_\star \sigma^2/r.
\]

(8)

Here \( \sigma_\star(r) \) is the radial velocity dispersion of the stars/tracers and \( \beta(r) \equiv 1 - \sigma^2 / \sigma^2_\star \) is a measure of the velocity anisotropy, where the tangential velocity dispersion \( \sigma_\theta = \sigma_\phi = \sigma_\phi \). It is informative to rewrite the implied total mass profile as

\[
M(r) = \frac{r \sigma^2_\star}{G} (\gamma_\star + \gamma_\sigma - 2\beta),
\]

(9)

where \( \gamma_\star \equiv -d \ln n_\star / d \ln r \) and \( \gamma_\sigma \equiv -d \ln \sigma^2 / d \ln r \). Without the benefit of tracer proper motions (or some assumption about the form of the distribution function), the only term on the right-hand side of Equation 8 that can be determined by observations is \( \gamma_\star \), which follows from the projected surface brightness profile under some assumption about how it is related to the projected stellar number density \( \Sigma_\star(R) \).

Via an Abel inversion (Appendix A) we map \( n_\star \) in a one-to-one manner with the spherically deprojected observed surface brightness profile (i.e., we assume that the number density traces the light density). As we discuss below, \( \sigma_\star(r) \) can be inferred from \( \sigma_{los}(R) \) measurements, but this mapping depends on \( \beta(r) \), which is free to vary.

2.3 Mass Likelihoods from Line-of-Sight Velocity Dispersion Data

Line-of-sight kinematic data provides the projected velocity dispersion profile \( \sigma_{los}(R) \). In order to use the Jeans equation one must relate \( \sigma_{los} \) to \( \sigma_\star \) (as first shown by Binney & Mamon [1982]):

\[
\Sigma_\star \sigma^2_{los}(R) = \int_{R^2} \frac{n, \sigma^2_\star}{r^2} \left[ 1 - \frac{R^2}{r^2} \beta(r) \right] \frac{dr^2}{\sqrt{r^2 - R^2}}.
\]

(10)

It is clear then that there exists a degeneracy associated with using the observed \( \Sigma_\star(R) \) and \( \sigma_{los}(R) \) profiles to determine an underlying mass profile \( M(r) \) at any radius, as uncertainties in \( \beta \) will affect both the mapping between \( \sigma_\star \) and \( \sigma_{los} \) in Equation 10 and the relationship between \( M(r) \) and \( \sigma_\star \) in Equation 9.

One technique for handling the \( \beta \) degeneracy and providing a fair representation of the allowed mass profile given a set of observables is to consider general parameterizations for \( \beta(r) \) and \( M(r) \) and then to undertake a maximum likelihood analysis to constrain all possible parameter combinations. In what follows, we use such a strategy to derive meaningful mass likelihoods for a number of dispersion-supported galaxies with line-of-sight velocity data sets. We will use these general results to test our proposed mass estimator. Our general technique is described in the supplementary section of Strigari et al. (2008) and in Martinez et al. (2009). We refer the reader to these references for a more complete discussion.

Briefly, for our fiducial procedure we model the stellar velocity dispersion anisotropy as a three-parameter function

\[
\beta(r) = (\beta_1 - \beta_0) \frac{r^2}{r^2 + r^2_\beta} + \beta_0,
\]

(11)

and model the total mass density distribution using the six-parameter function

\[
\rho_{tot}(r) = \frac{\rho_s e^{-r/r_{cut}}}{(r/r_\star)^\alpha [1 + (r/r_\star)^\delta]^{(\beta - \gamma)/\alpha}}.
\]

(12)

For our marginalization, we adopt uniform priors over the following ranges: \( \log_{10}(0.2 r_{1/2}) < \log_{10}(r_{1/2}) < \log_{10}(r_{lim}) \); \(-10 < \beta_1 < 0.1; -10 < \beta_0 < 0.1; \log_{10}(0.2 r_{1/2}) < \log_{10}(r_{1/2}) < \log_{10}(r_{high}) \); \( 0 < \gamma < 2; 3 < \delta < 5; \) and \( 0.5 < \alpha < 3 \), where we remind the reader that \( r_{lim} \) is the truncation radius for the stellar density. The variable \( r_{cut} \) allows the dark matter halo profile to truncate at some radius beyond the stellar extent and we adopt the uniform prior \( \log_{10}(r_{lim}) < \log_{10}(r_{cut}) < \log_{10}(r_{high}) \) in our marginalization. For distant galaxies we use \( r_{high} = 10 r_{lim} \) and for satellite galaxies of the Milky Way we set \( r_{high} \) equal to the Roche limit for a 10\(^3\) M\(_\odot\) point mass. In practice, this allowance for \( r_{cut} \) is not important for our purposes because we focus on integrated masses within the stellar radius.

We also investigate the effects of a more radical model for the stellar velocity dispersion anisotropy that allows \( \beta(r) \) to have an extremum within the limiting radius. The specific form we use in this second model is

\[
\beta(r) = \beta_0 + (\beta_1 - \beta_0) \left( \frac{r}{r_\beta} \right)^2 \exp \left[ 2 - \frac{r}{r_\beta} \right],
\]

(13)

which allows for mild and large variations within the stellar extent depending on the value of \( r_\beta \). We use the same priors for this functional form as those for our fiducial model (Equation 11). A caveat that bears mentioning is that neither of our \( \beta(r) \) profiles allow for multiple extrema, but they do allow for large variations in \( \beta(r) \) with radius. Our motivation for investigating these large variations is not based on physical arguments for their existence, but rather to see if the validity of our mass estimator breaks down.

Below we apply our marginalization procedure to resolved kinematic data for MW dSphs, MW globular clusters, and elliptical galaxies. Since MW dSphs and globular masses for spheroidal galaxies

\[\sigma_\star \text{ and } \sigma_{los} \text{ in Equation 10} \text{ and the relationship between } M(r) \text{ and } \sigma_\star \text{ in Equation 9.}\]

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clusters are close enough for individual stars to be resolved, we consider the joint probability of obtaining each observed stellar velocity given its observational error and the predicted line-of-sight velocity dispersion from Equations (8) and (10). In modeling the line-of-sight velocity distribution for any system, we must take into account that the observed distribution is a convolution of the intrinsic velocity distribution, arising from the distribution function, and the measurement uncertainty from each individual star. If we assume that the line-of-sight velocity distribution can be well-described by a Gaussian, which is observationally consistent with the best-studied samples (see, e.g., Walker et al. 2007), then the probability of obtaining a set of line-of-sight velocities \( \mathcal{V} \) given a set of model parameters \( \mathcal{M} \) is described by the likelihood

\[
P(\mathcal{V} | \mathcal{M}) = \frac{1}{\sqrt{2\pi \epsilon_i^2}} \exp \left[ -\frac{1}{2} \frac{(\bar{v} - \bar{v}^*)^2}{\sigma_{th,i}^2 + \epsilon_i^2} \right].
\]

The product is over the set of \( N \) stars, where \( \bar{v} \) is the average velocity of the galaxy. As expected, the total error at a projected position is a sum in quadrature of the theoretical intrinsic dispersion, \( \sigma_{th,i}(\mathcal{M}) \), and the measurement error \( \epsilon_i \). We generate the posterior probability distribution for the mass at any radius by multiplying the likelihood by the prior distribution for each of the nine \( \beta(r) \) and \( \rho_{tot}(r) \) parameters as well as the observationally derived parameters and associated errors. If we assume that the line-of-sight velocity distribution can be well-described by a Gaussian, which is observationally consistent with the best-studied samples (see, e.g., Walker et al. 2007), then the probability of obtaining a set of line-of-sight velocities \( \mathcal{V} \) given a set of model parameters \( \mathcal{M} \) is described by the likelihood

\[
P(\mathcal{V} | \mathcal{M}) = \frac{1}{\sqrt{2\pi \sigma_{th,i}^2 + \epsilon_i^2}} \exp \left[ -\frac{1}{2} \frac{(\bar{v} - \bar{v}^*)^2}{\sigma_{th,i}^2 + \epsilon_i^2} \right].
\]

Before moving on, we note that the Gaussian assumption made here is not entirely general, and thus is a limiting aspect of our mass modeling. While most dSph velocity distributions are consistent with Gaussian to within membership errors and errors associated with the possibility of binary star populations (Minor et al. 2010), a small amount of excess kurtosis is measured in the outer parts of some systems (Lokas 2009).

For elliptical galaxies that are located too far for individual stellar spectra to be obtained, we analyze the resolved dispersion profiles with the likelihood

\[
P(D | \mathcal{M}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi \epsilon_i^2}} \exp \left[ -\frac{1}{2} \frac{(D_i - \sigma_{th,i})^2}{\epsilon_i^2} \right],
\]

where the product is over the set of \( N \) dispersion measurements \( D_i \) and \( \epsilon_i \) is the reported error of each measurement.

5 The volume of parameter space changes depending on the number of free parameters used to fit the photometry of each system, along with the availability of photometric uncertainties. For each MW dSph we have taken care to ensure that we used what we consider to be the most reliable photometry that include observational errors.
| Galaxy         | Distance [kpc] | Luminosity \[\log_{10} L_\odot \, V\] | \(r_{\text{lm}}\) [arcmin] | 2D \(R_\circ\) [pc] | 3D \(r_{1/2}\) [pc] | \(\sqrt{\sigma_{\text{los}}^2}\) [km s\(^{-1}\)] | \(M_{1/2}\) [\(10^9\) M\(_\odot\)] | \(\Upsilon_V\) [M\(_\odot\)/L\(_\odot\), V] |
|---------------|---------------|-----------------------------------|---------------------------|-------------------|----------------|------------------------|-----------------|----------------------|
| Carina (723)  | 105.5 ± 2\(^a\) | \(4.3 \pm 0.1 \times 10^3\) \(^b\) | 8.8 ± 1.2 \(^c\) | 28.8 ± 3.6 \(^d\) | 254 ± 28 | 334 ± 37 | 6.4 ± 0.2 | 9.56 ± 0.95 \times 10^6 | 44 ± 13 |
| Draco (206)   | 76 ± 5 \(^d\) | \(2.2 \pm 0.1 \times 10^3\) \(^b\) | 7.63 ± 0.04 \(^e\) | 45.1 ± 0.6 \(^f\) | 220 ± 11 | 291 ± 14 | 10.1 ± 0.5 | 2.11 ± 0.33 \times 10^7 | 200 ± 50 |
| Fornax (2409) | 147 ± 3 \(^d\) | \(1.7 \pm 0.1 \times 10^3\) \(^b\) | 13.7 ± 1.2 \(^g\) | 71.1 ± 4.0 \(^h\) | 714 ± 40 | 944 ± 53 | 10.7 ± 0.2 | 7.39 ± 0.41 \times 10^7 | 8.7 ± 2.8 |
| Leo I (305)   | 254 ± 18 \(^i\) | \(5.0 \pm 1.0 \times 10^6\) \(^j\) | 6.21 ± 0.95 \(^k\) | 11.70 ± 0.87 \(^l\) | 295 ± 49 | 388 ± 64 | 9.0 ± 0.4 | 2.21 ± 0.24 \times 10^7 | 8.8 ± 3.4 |
| Leo II (168)  | 233 ± 15 \(^m\) | \(7.8 \pm 1.0 \times 10^5\) \(^n\) | 2.64 ± 0.19 \(^o\) | 9.33 ± 0.47 \(^p\) | 177 ± 13 | 233 ± 17 | 6.6 ± 0.5 | 7.25 ± 1.19 \times 10^6 | 19 ± 2 |
| Sculptor (1355) | 86 ± 5 \(^q\) | \(2.5 \pm 0.1 \times 10^6\) \(^r\) | 5.8 ± 1.6 \(^s\) | 76.5 ± 5.0 \(^t\) | 282 ± 41 | 375 ± 54 | 9.0 ± 0.2 | 2.25 ± 0.16 \times 10^6 | 18 ± 6 |
| Sextans (423) | 96 ± 3 \(^u\) | \(5.9 \pm 1.0 \times 10^5\) \(^v\) | 16.6 ± 1.2 \(^w\) | 160.0 ± 50.0 \(^x\) | 768 ± 47 | 1019 ± 62 | 7.1 ± 0.3 | 3.49 ± 0.56 \times 10^5 | 120 ± 48 |
| Ursa Minor (212) | 77 ± 4 \(^y\) | \(3.9 \pm 1.0 \times 10^5\) \(^z\) | 17.9 ± 2.1 \(^{m'}\) | 77.9 ± 8.9 \(^{m''}\) | 445 ± 44 | 588 ± 58 | 11.5 ± 0.6 | 5.56 ± 0.79 \times 10^5 | 290 ± 140 |

Note: Galaxies are grouped from top to bottom as pre-SDSS/classical MW dSphs, post-SDSS MW dSphs, dwarf elliptical galaxies (dEs), and elliptical galaxies (Es).

Within the parentheses next to each MW dSph is the number of stars analyzed. The dSphs with errors on \(r_{\text{lm}}\) fit with King profiles (where \(R_\circ = r_{\text{lm}}\)). Those without sources for \(r_{\text{lm}}\) are estimated from Figure 1 of Martin et al. (2008a) (we found that our \(M_{1/2}\) determinations were largely insensitive to the choice of reasonable \(r_{\text{lm}}\) values). Except for Leo T, all of the post-SDSS dwarfs are fit with truncated exponential light distributions (where \(R_\circ\) is the exponential scale length derived from the half-light radius). The dEs and Es are fit with truncated Sérsic profiles, where each limiting radius is not usually quoted in the literature. Also note that errors on the masses are approximately normal in log10(\(M_\odot\)). Lastly, note that the quoted errors in the luminosities and in the dynamical mass-to-light ratios were derived in this paper and are also approximately log-normal. For the classical dSphs we took into account the errors in the apparent magnitudes and the errors in the distance estimates. For the post-SDSS dSphs we considered the quoted errors in absolute magnitudes.

References: Values in column 5 (2D \(R_\circ\)) for the classical MW dSphs and Leo T, and the values in columns 6-9 for all the MW dSphs derived in this paper from the quoted elliptical fits to the surface brightness profiles from the cited sources (this convention differs from the means that are sometimes quoted from the equivalent elliptical fits (see, e.g., Section 3 of Irwin & Hatzidimitriou 1993). Except for Hercules and Leo T, values in columns 2-5 of the post-SDSS MW dSphs are from Martin et al. (2008b). Lastly, the values in columns 5-9 for the dEs and Es are derived in this paper. The individual references are as follows: a) Pietrzynski et al. (2009) b) Rederived from apparent magnitudes listed in Mateo (1998), c) Irwin & Hatzidimitriou (1993), d) Bonanos et al. (2004), e) Ségal et al. (2007), f) Bellazzini et al. (2004), g) Snolli et al. (2007), h) Bellazzini et al. (2003), i) Coleman et al. (2007), j) Pietrzynski et al. (2008), k) Lee et al. (2003), l) Carrera et al. (2002), m) RGB tracers from Palma et al. (2003), n) Dall’Ora et al. (2006), o) Martin et al. (2008b), p) Greco et al. (2008), q) Belokurov et al. (2007), r) Sand et al. (2009), s) de Jong et al. (2008), t) Okamoto et al. (2008), u) Zucker et al. (2006a), v) Willman et al. (2005a), w) Derived from Prunil & Herné (1998), x) McConnachie et al. (2005), y) Simien & Prunil (2002), z) Quoted from NASA/IPAC Extragalactic Database, aa) Simien & Prunil (2001), bb) Simien & Prunil (1997c), cc) Simien & Prunil (1997b), dd) Kormendy et al. (2009), who present similar parameters to those the originally derived in Ferrarese et al. (2006).

\(^a\) \(^b\) \(^c\) \(^d\) \(^e\) \(^f\) \(^g\) \(^h\) \(^i\) \(^j\) \(^k\) \(^l\) \(^m\) \(^n\) \(^o\) \(^p\) \(^q\) \(^r\) \(^s\) \(^t\) \(^u\) \(^v\) \(^w\) \(^x\) \(^y\) \(^z\) \(^{m'}\) \(^{m''}\)
3 MINIMIZING THE ANISOTROPY DEGENERACY

3.1 Expectations

Qualitatively, one might expect that the degeneracy between the integrated mass and the assumed anisotropy parameter will be minimized at some intermediate radius within the stellar distribution. Such an expectation follows from considering the relationship between $\sigma_{\text{los}}$ and $\sigma_r$.

At the projected center of a spherical, dispersion-supported galaxy ($R = 0$), line-of-sight observations project onto the radial component with $\sigma_{\text{los}} \sim \sigma_r$, while at the edge of the galaxy ($R = r_{\text{lim}}$), line-of-sight velocities project onto the tangential component with $\sigma_{\text{los}} \approx \sigma_\theta$. For example, consider a galaxy that is intrinsically isotropic ($\beta = 0$). If this system is analyzed using line-of-sight velocities under the false assumption that $\sigma_r > \sigma_t$ ($\beta > 0$) at all radii, then the total velocity dispersion at $r \approx 0$ would be underestimated while the total velocity dispersion at $r \approx r_{\text{lim}}$ would be overestimated. Conversely, if one were to analyze the same galaxy under the assumption that $\sigma_r < \sigma_t$ ($\beta < 0$) at all radii, then the total velocity dispersion would be overestimated near the center and underestimated near the galaxy edge. It is plausible then that there exists some intermediate radius where attempting to infer the enclosed mass from only line-of-sight kinematics is minimally affected by the unknown value of $\beta$.

These qualitative expectations are borne out explicitly in Figure 1 where we present inferred mass profiles for the Carina dSph galaxy for several choices of constant $\beta$. The right-hand panel shows the same data analyzed using our full likelihood analysis, where we marginalized over the fairly general $\beta(r)$ profile presented in Equation 1. We use 723 stellar velocities from Walker et al. (2009a) with the constraint that their membership probabilities (which are based on a combination of stellar velocity and metallicity) are greater than 0.9, and in projection they lie within 650 pc of the center (which is below the lower limit of $r_{\text{lim}}$ given in Table 1). The average velocity error of this set is approximately 3 km s$^{-1}$. Each line in the left panel of Figure 1 shows the median likelihood of the cumulative mass value at each radius for the value of $\beta$ indicated. The 3D half-light radius and the limiting stellar radius are marked for reference. As expected, forcing $\beta > 0$ produces a systematically lower (higher) mass at a small (large) radius compared to $\beta < 0$. This of course demands that every pair of $M(r)$ profiles analyzed with different assumptions about $\beta$ cross at some intermediate radius. Somewhat remarkable is the fact that every pair intersects at approximately the same radius. We see that this radius is very close to the deprojected 3D half-light radius $r_{1/2}$. The right-hand panel in Figure 1 shows the full mass likelihood as a function of radius (which allows for a radially varying anisotropy), where the shaded bands illustrate the 68% and 95% likelihood contours, respectively. The likelihood contour also pinches near $r_{1/2}$, as this mass value is the most constrained by the data.

By examining each of the well-sampled dSph kinematic data sets (Muñoz et al. 2005; Koch et al. 2007; Mateo et al. 2008; Walker et al. 2009a), in more detail, we find that the error on mass near $r_{1/2}$ is always dominated by measurement errors (including the finite number of stars) rather than the $\beta$ uncertainty, while the mass errors at both smaller and larger radii are dominated by the $\beta$ uncertainty (and thus are less affected by measurement error). We now explain this result by examining the Jeans equation in the context of observables.

3.2 Why is the mass within half-light radius insensitive to velocity dispersion anisotropy?

Here we present the derivation of Equations 1 and 2. We start by analytically showing that there exists a radius $r_{\text{eq}}$ within which the dynamical mass will be minimally affected by the velocity dispersion anisotropy, $\beta(r)$. We then consider two cases of interest for observed dispersion-supported systems. First, we consider the case when the velocity dispersion anisotropy is spatially constant and show that $r_{\text{eq}} \approx r_3$ where $r_3$ is an observable defined such that $\gamma_s \equiv -d \ln n_s / d \ln r = 3$ at $r = r_3$. Second, we extend our analysis to allow for non-constant $\beta(r)$ and show that under mild assumptions about the variation of $\beta(r)$, the mass within radius $r_3$ is insensitive to the velocity dispersion anisotropy.

While the steps outlined above provide a deeper insight into Equation 1, the essence of our arguments can be laid out in a few lines. We begin by rewriting the Jeans equation such that $\beta(r)$ dependence is absorbed into the definition of $\sigma_{\text{los}}^2 = \sigma_r^2 + \sigma_\theta^2 + \sigma_\phi^2 = (3 - 2\beta)\sigma_r^2$:

$$GM(r)r^{-1} = \sigma_{\text{los}}^2(r) + \sigma_t^2(r) (\gamma_s + \gamma_\sigma - 3).$$

We then note that if $\gamma_s(r_3) < 3$ (as our numerical computations show it must be for flat observed $\sigma_{\text{los}}(R)$ profiles), then as $r = r_3$ the mass depends only on $\sigma_{\text{los}}$ and we may write

$$M(r_3) \approx \frac{G^{-1} \sigma_{\text{los}}^2(r_3) r_3}{3} \approx G^{-1} \langle \sigma_{\text{los}}^2 \rangle r_3,$$

where the last line is Equation 1. We remind the reader that the brackets indicate a luminosity-weighted average over the entire system. In the above chain of arguments we have used the relation $\langle \sigma_{\text{los}}^2 \rangle \approx \sigma_{\text{los}}^2(r_3)$. We will show why this is a good approximation in Section 3.2.2.

Finally, we show in Appendix B that the log-slope of $n_s$ is approximately 3 at the deprojected half-light radius $r_3 \approx r_{1/2}$ for most common light profiles, and therefore the last line of Equation 1 provides our mass estimator (Equation 2). For example, $r_3 \approx 0.94 r_{1/2}$ for a Plummer profile and $r_3 \approx 1.15 r_{1/2}$ for King (1962) profiles and for the family of Sérsic (1968) profiles with $n = 0.5$ to 10. The relationships between $r_{1/2}$ and the observable scale radii for various commonly-used surface density profiles are provided in Appendix B.

6 van der Marel et al. (2000) demonstrated a comparable result with more restrictive conditions.

7 A similar effect was discussed but not fully explored in Strigari et al. (2007a).
3.2.1 Existence of a radius $r_{\text{eq}}$ where the mass profile is minimally affected by anisotropy

Consider a velocity dispersion-supported stellar system that is well studied, such that $\Sigma_*(R)$ and $\sigma_{\text{los}}(R)$ are determined accurately by observations. If we model this system’s mass profile using the Jeans equation, any viable solution will keep the quantity $\Sigma_*(R)\sigma_{\text{los}}^2(R)$ fixed to within allowable errors. With this in mind, we rewrite Equation (10) in a form that is invertible, isolating the integral’s $R$-dependence into a kernel:

$$\Sigma^2 r_{\text{los}}(R) = \int_{R_2}^{\infty} \left[ \frac{n_\nu \sigma_*^2}{(1-\beta)^{1/2}} + \int_{r_2}^{\infty} \frac{\beta n_\nu \sigma_*^2}{2r^2 \sqrt{\gamma^2 - r^2}} \frac{dr^2}{\sqrt{\gamma^2 - r^2}} \right].$$

We explain this derivation in Appendix A, where we also perform an Abel inversion to solve for $\sigma_*(r)$ and $M(r)$ in terms of directly observable quantities (while we were writing this paper we learned that Mamon & Boué 2010 had independently performed a similar analysis.)

Because Equation (18) is invertible, the fact that the left-hand side is an observed quantity and independent of $\beta$ implies that the term in brackets must be well determined regardless of a chosen $\beta$. This allows us to equate the isotropic integrand with an arbitrary anisotropically defined integrand:

$$n_\nu \sigma_*^2 |_{\beta=0} = \frac{\beta}{G} (\gamma_* + \gamma_* + \gamma_* - 3).$$

We now take a derivative with respect to $\ln r$ and subtract Equation (8) to obtain the following result

$$M(r; \beta) - M(r; 0) \equiv \frac{\beta(r) \sigma_*^2(r)}{G} (\gamma_* + \gamma_* + \gamma_* - 3).$$

We remind the reader that $\gamma_* \equiv -d\ln n_\nu/d\ln r$ and $\gamma_* \equiv -d\ln \beta/d\ln r$. Following the same nomenclature, $\gamma_* \equiv -d\ln \beta/d\ln r = -\beta^t/\beta$, where $\beta^t$ denotes a derivative with respect to $\ln r$.

Equation (20) reveals the possibility of a radius $r_{\text{eq}}$ where the term in parentheses goes to zero, such that the enclosed mass $M(r_{\text{eq}})$ is minimally affected by our ignorance of $\beta(r)$.\footnote{For $\beta$ profiles that are close to isotropic, solving for $r_{\text{eq}}$ is not necessary, as the right-hand side of Equation (20) is close to 0 everywhere.}

$$\gamma_* (r_{\text{eq}}) = 3 - \gamma_* (r_{\text{eq}}) - \gamma_* (r_{\text{eq}}).$$

While in principle one needs to know $\gamma_\beta$ in order to determine $r_{\text{eq}}$, we argue below that this term must be small for realistic cases that correspond to observed galaxies.\footnote{Note that for anisotropic parameterizations that become close to isotropic, $\gamma_\beta$ may be large. However, the combination $\beta \gamma_\beta = \beta^t$ is still well-behaved.} Given this, a solution for $r_{\text{eq}}$ must exist. One can see this immediately, as analyzing the luminosity-weighted average of Equation (18) in conjunction with the scalar virial theorem (Equation 12) requires that $\langle (\gamma_* + \gamma_* - 3) \sigma_*^2 \rangle = 0$. Since $\sigma_*^2(r)$ is positive definite, it must be true that there exists at least one radius where $\gamma_* = 3 - \gamma_*$. More specifically, for typically observed stellar profiles, $\gamma_\beta(r)$ changes from being close to zero (cored) in the center to larger than 3 in the outer parts (to keep the stellar mass finite). (For example, $\gamma_\beta$ for a Plummer profile transitions from 0 to 5.)

8 For $\beta$ profiles that are close to isotropic, solving for $r_{\text{eq}}$ is not necessary, as the right-hand side of Equation (20) is close to 0 everywhere.
9 Note that for anisotropic parameterizations that become close to isotropic, $\gamma_\beta$ may be large. However, the combination $\beta \gamma_\beta = \beta^t$ is still well-behaved.
10 The integral is actually number-weighted, but we map number density to luminosity density in a one-to-one manner.

The changes in $\gamma_\sigma(r)$ are more benign (see Equation 14). Putting these facts together, we see that unless $\gamma_\sigma$ is very large in magnitude, Equation (21) will have a solution.

In order to determine the value of $M(r_{\text{eq}})$ we manipulate Equation (20) in order to isolate the relationship between $\gamma_\sigma^2(r)$ and $\langle \sigma_{\text{los}}^2 \rangle$:

$$\gamma_\sigma^2(r) \langle \sigma_{\text{los}}^2 \rangle = [\gamma_* + \gamma_* (1 - \beta) + \beta^t] \sigma_*^2.$$  \hspace{1cm}  (22)

Here, the quantity $\gamma_\sigma^2(r)$ is dimensionless and depends only on observable functions:

$$\gamma_\sigma^2(r) = \frac{1}{n_\nu r(r) \langle \sigma_{\text{los}}^2 \rangle} \int_{r_2}^{\infty} \frac{d(\Sigma^2 r_{\text{los}})}{dR^2} \frac{dR^2}{\sqrt{R^2 - r^2}}.$$  \hspace{1cm}  (23)

Note that in the limit where $\sigma_{\text{los}}$ is constant we have $\gamma_\sigma^2(r) = \gamma_\sigma^2(r)$, which arises by utilizing an Abel inversion (Equation 12). Now we may use Equations (10) and (22) to show

$$M(r_{\text{eq}}) = \gamma_\sigma^2 (r_{\text{eq}}) G^{-1} \langle \sigma_{\text{los}}^2 \rangle r_{\text{eq}}.$$  \hspace{1cm}  (24)

As mentioned above, for generic cases the value of $r_{\text{eq}}$ will depend on $\beta(r)$ and thus our ignorance of $\beta(r)$ is now translated to $r_{\text{eq}}$. However, as we discuss in the next section, if the observed $\sigma_{\text{los}}(R)$ does not vary much compared to $\Sigma_*(R)$ (as is true for most spheroidal systems), then $r_{\text{eq}} \approx r_{V}$ and $\gamma_\sigma (r_{\text{eq}}) \approx 3$. More generally, each galaxy will have a different $r_{\text{eq}}$ which can be searched for numerically using Equation (20) in conjunction with the family of $M(r)$ and $\beta(r)$ profiles that solve the Jeans equation. When we actually perform this analysis on real galaxies using our maximum likelihood approach, we find that the likelihoods for $r_{\text{eq}}$ peak near $r_{V} \approx r_{V}/2$.

3.2.2 Spatially constant velocity dispersion anisotropy

In this section, we assume that $\beta(r)$ is constant and show that $r_{\text{eq}}$ is close to $r_{V}$. We start with Equation (22) and set $\beta = 0$ to yield:

$$\gamma_\sigma (r_{V}) \langle \sigma_{\text{los}}^2 (r_{V}) \rangle \frac{1 - \beta}{3 - 2\beta} \approx 3 \langle \sigma_{\text{los}}^2 (r_{V}) \rangle - \sigma_{\text{los}}^2 (r_{V}).$$  \hspace{1cm}  (25)

We have assumed that $\sigma_{\text{los}}$ varies slowly with radius such that $\gamma_\sigma \approx 3$. Of course, physically, $\sigma_{\text{los}}$ has to decrease as $R$ approaches the stellar limiting radius, but we find numerically that the relation above is still a good approximation as long as the variations in the observed $\sigma_{\text{los}}$ are mild at $R \approx R_{\text{e}}$. Equation (25) tells us that if $\gamma_\sigma(r_{V})$ is small and $\beta$ is constant, then $\sigma_{\text{los}}^2 (r_{V}) \approx 3 \langle \sigma_{\text{los}}^2 (r_{V}) \rangle$. This provides one justification for the second step in Equation (17).

We now turn to a more detailed computation of $\langle \sigma_{\text{los}}^2 (r_{V}) \rangle$ to elucidate the role of $\gamma_\sigma$, without explicitly assuming that $\sigma_{\text{los}}(R)$ is constant. Consider the average total velocity dispersion written explicitly as an integral over $\sigma_{\text{los}}^2$:

$$\langle \sigma_{\text{los}}^2 \rangle = 4\pi \int_{-\infty}^{\infty} r^3 n_\nu \sigma_*^2 (3 - 2\beta) \frac{d^3 r}{r^2}.$$ \hspace{1cm}  (26)

In realistic cases, $n_\nu$ will vary significantly with radius from a flat inner profile with $\gamma_* = 0$ at small $r$ to a steep profile with $\gamma_* > 3$ at large $r$. Thus the integral is expected to be single peaked unless $\sigma_*$ varies in an unexpectedly strong way to compensate for the behavior of $n_\nu$. However, since observed $\sigma_{\text{los}}$ profiles do not vary much with position in the sky, $\sigma_*(r)$ must also vary smoothly with radius (at least for constant $\beta$; see Equation (13)). Thus the integrand will peak
at \( r = r_\sigma \) such that \( \gamma_\sigma (r_\sigma) + \gamma_\sigma(r_\sigma) = 3 \). We may then use a saddle point approximation after a Taylor expansion of the natural logarithm of the integrand about \( r_\sigma \), approximating the integral as a Gaussian
\[
\langle \sigma_{\text{tot}}^2 \rangle \simeq 4\pi A(r_\sigma) \int_{-\infty}^{\infty} \exp \left[ -\frac{K(r_\sigma)}{2} \left( \ln \left( \frac{r}{r_\sigma} \right) \right)^2 \right] \, \mathrm{d} \ln r
\]
\[
\simeq 4\pi \sqrt{\frac{2\pi}{K(r_\sigma)}} A(r_\sigma).
\]
(27)

where
\[
A(r) = r^3 n_s(r) \sigma_{\text{tot}}^2(r), \quad \text{and} \quad K(r) = \gamma'_\sigma(r) + \gamma'_\sigma(r).
\]
(28)

Similarly, since \( r^3 n_s \) peaks at \( r_\sigma \), one can repeat the analysis of the previous paragraph to write
\[
1 = 4\pi \int_{-\infty}^{\infty} r^3 n_s \, \mathrm{d} \ln r \simeq 4\pi \sqrt{\frac{2\pi}{\gamma'_\sigma(r_\sigma)}} r^3 n_s(r_\sigma).
\]
(29)

The term \( A(r_\sigma) \) computed at \( \gamma_\sigma + \gamma_\sigma = 3 \) is different from \( A(r_\sigma) \) at second order in \( \gamma_\sigma(r_\sigma) \). Thus, even for moderate values of \( \gamma_\sigma(r_\sigma) \) we may replace \( A(r_\sigma) \) in Equation (27) with \( A(r_\sigma) \) to find (with the aid of Equations (28) and (29))
\[
3\langle \sigma_{\text{tot}}^2 \rangle = \langle \sigma_{\text{tot}}^2 \rangle \simeq \sqrt{\frac{\gamma'_\sigma(r_\sigma)}{\gamma'_\sigma(r_\sigma) + \gamma'_\sigma(r_\sigma)}} r^3 n_s(r_\sigma) \approx \sigma_{\text{tot}}^2(r_\sigma).
\]
(30)

The last approximation arises by neglecting the first order correction in \( \gamma_\sigma \), enabling us to evaluate the terms inside of the square root at \( r = r_\sigma \). Our numerical mass estimates show that the observational error is larger than that due to the neglect of the \( \gamma_\sigma \) term.

Next we take the derivative of Equation (22) at \( r = r_\sigma \):
\[
\gamma'_\sigma(r_\sigma) + \gamma'_\sigma(r_\sigma) \cong \gamma'_\sigma(r_\sigma) \frac{3 - 2\beta}{3 - 3\beta}, \quad \text{(31)}
\]

where we have neglected \( \gamma_\sigma(r_\sigma) \). From this expression, we see that it is only for values of \( \beta \) close to unity that the last step in Equation (23) is not a good approximation. Such large values of constant \( \beta \), however, are disfavored by the Jeans equation when considering realistic dispersion profiles. This may be seen by taking a derivative of the Jeans equation (Equation (19)) at \( r = r_\sigma \) to write
\[
\gamma'_\sigma(r_\sigma) + \gamma'_\sigma(r_\sigma) \simeq (3 - 2\beta)(2 - \gamma_\sigma), \quad \text{(32)}
\]

where we neglected the \( \gamma_\sigma(r_\sigma) \) term and where we set \( M(r) = M(r_\sigma)(r/r_\sigma)^{3-\gamma_\sigma} \). Combining this with Equation (31) we require that
\[
1 - \beta \simeq \frac{\gamma'_\sigma(r_\sigma)}{6 - 3\gamma_\sigma}, \quad \text{(33)}
\]

which shows that \( \beta \) values close to 1 are disfavored because observations reveal that \( \gamma'_\sigma(r_\sigma) \) is of order unity for systems in equilibrium.\(^{11}\) With regard to large negative \( \beta \) values, these extremes are preferred when \( \gamma_\sigma \lesssim 2 \). We remind the reader that in the above arguments we have neglected \( \gamma_\sigma(r_\sigma) \) in keeping with our focus on systems with flat observed velocity dispersion profiles (see Equation (19)).

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\(^{11}\) Note that if \( \gamma_\sigma > 2 \), Equation (33) yields the unphysical result of \( \beta > 1 \), implying that \( \gamma_\sigma(r_\sigma) \) should not be neglected.

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As an aside, we note that even if we knew \( \beta(r) \), uncertainties in the inner stellar profile will limit how well we recover the slope of the total density profile \( \gamma_\sigma \) at \( r_\sigma \).

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Given this, Equation (30) can be considered a good approximation. That is, \( 3 \langle \sigma_{\text{tot}}^2 \rangle \simeq \sigma_{\text{tot}}^2(r_\sigma) \) if \( \beta \) is constant and as long as the observed \( \sigma_{\text{tot}} \) does not vary much with position on the sky. Our full numerical analysis of observed spectroscopic data show that this is indeed the preferred solution of the Jeans equation. This realization, together with Equation (19) allows us to derive our mass estimator presented in Equation (2) with \( r_{1/2} \approx r_\sigma \).

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3.2.3 General velocity dispersion anisotropy

Here we provide a qualitative understanding of why our mass estimator works well in the general \( \beta(r) \) case. We begin by reconsidering the derivation of \( \langle \sigma_{\text{tot}}^2 \rangle \), now allowing \( \beta \) to vary with radius. It is clear that the peak in the integrand in Equation (26) will shift to a position where \( \gamma_\sigma + \gamma_\sigma + 3\beta/(3 - 2\beta) = 3 \). This even if \( \gamma_\sigma \) is moderately small, the peak may be shifted due to the third term. For small values of \( \beta \), the typical \( \beta'/(3 - 2\beta) \) values are also small in our parameterizations (Equations (11) and (13)) and hence the peak is close to \( r_\sigma \) as in the constant \( \beta \) case. For large negative values of \( \beta \), the peak of the \( \langle \sigma_{\text{tot}}^2 \rangle \) integrand is essentially at \( r_{eq0} \), but this does not imply that \( r_{eq0} \) is close to \( r_\sigma \). However, if \( \beta(r_\sigma) \) is not small, then \( \beta'(r_\sigma) \) is constrained by Equation (22) This can be realized because the term that determines the shift in the peak of Equation (22) for large negative \( \beta(r_\sigma) \) values is
\[
\gamma_\sigma(r_\sigma) + \beta'(r_\sigma)/(3 - 2\beta(r_\sigma)) \propto 3 \langle \sigma_{\text{tot}}^2 \rangle(r_\sigma) - \sigma_{\text{tot}}^2(r_\sigma).
\]
(34)

The simplest solution to this equation and Equation (22) which is consistent with the Jeans equation is \( 3 \langle \sigma_{\text{tot}}^2 \rangle(r_\sigma) \approx \gamma_\sigma(r_\sigma) \) and \( r_{eq0} \approx r_\sigma \). Our full mass likelihoods derived from analyzing observed data confirm this expectation.

Since we have argued that the mass enclosed within \( r_\sigma \) should be approximately independent of \( \beta(r) \), we may now derive this mass by simply using Equation (2) with \( \beta = 0 \) at \( r = r_\sigma \):
\[
M(r_\sigma) = \frac{3 \pi}{G} \frac{\sigma^2(r_\sigma)}{\gamma_\sigma(r_\sigma)} \approx 3 \frac{\sigma^2(r_\sigma)}{G \beta = 0}. \quad \text{(35)}
\]

This is again Equation (2) with \( r_{1/2} \approx r_\sigma \). In the second line we are using the fact that \( 3 \sigma^2 = \sigma_{\text{tot}}^2 \) for \( \beta = 0 \) and our result from the previous section that \( \sigma_{\text{tot}}^2(r_\sigma) \approx \sigma_{\text{tot}}^2 \).

It is worth emphasizing that the ideal radius for mass determination is \( r_\sigma \) and not \( r_{1/2} \). As one moves away from \( r_\sigma \), the uncertainty in \( \beta(r) \) will start dominating over kinematic (or photometric) errors. However, typically the observational errors on both \( r_\sigma \) and \( \langle \sigma_{\text{tot}}^2 \rangle \) are large enough that the slight (\( \sim 15\% \)) difference between \( r_{1/2} \) and \( r_\sigma \) will not matter. For this reason we have opted to present our results using the more familiar deprojected half-light radius in what follows. We find that for constant \( \beta \) or for our monotonically varying \( \beta(r) \) form, both \( M(r_{1/2}) \) and \( M(r_\sigma) \) are equally well constrained by the data sets we consider when analyzing the population as a whole.
Figure 2. *Left:* The half-light masses for Milky Way dSphs (green squares, blue diamonds, red circles), galactic globular clusters (yellow stars), dwarf ellipticals (cyan triangles), and ellipticals (pink inverted triangles). The vertical axis shows masses obtained using our full likelihood analysis. The horizontal axis shows mass estimates based on our mass estimator, Equation 2. The inset focuses on the pre-SDSS (classical) dSphs, where the dotted lines indicate a 10% scatter in our mass estimator. *Right:* Errors on half-light masses for Milky Way dSphs. The vertical axis shows the 68% error width derived from our full likelihood analysis and the horizontal axis shows the error width calculated by straightforward error propagation using Equation 2. The agreement between the two demonstrates that errors on the mass determinations within the 3D deprojected half-light radius $r_{1/2}$ are dominated by observational uncertainties rather than theoretical uncertainties associated with $\beta(r)$. In both plots and in the inset the solid line indicates the one-to-one relation. The stellar velocities used to derive the globular cluster (GC) masses (in conjunction with photometry from Harris (1996)) were obtained from (lowest to highest mass): NGC 5053 (Yan & Cohen 1996), NGC 6171 (Piatek et al. 1994), NGC 288 (Pryor et al. 1991), NGC 104 (Mayer et al. 1983), NGC 362 (Fischer et al. 1993), NGC 5272 (Pryor et al. 1988), and NGC 2419 (Bonnard et al. 2000). The kinematic data for the classical dSphs were taken from Muñoz et al. (2003), Koch et al. (2004), Mateo et al. (2008), and Walker et al. (2009), and data for the post-SDSS dSphs were taken from Muñoz et al. (2006), Simon & Geha (2007). Geha et al. (2011), and Willman et al. (in preparation). The kinematic data for the ellipticals are as follows (from lowest to highest mass): NGC 185 (De Rijcke et al. 2006), NGC 855 (Simien & Prugniel 2000), NGC 4478 (Simien & Prugniel 1997a), NGC 731 (Simien & Prugniel 2000), NGC 3853 (Simien & Prugniel 1997b), and NGC 499 (Simien & Prugniel 1997c). The photometric data for the MW dSphs, dEs, and ellipticals are referenced in Table 1.

These specific dwarf ellipticals and ellipticals were chosen because they had extended kinematic data (to $R_e$) and showed little rotation. To demonstrate the accuracy of the normalization in our formula we add an inset into Figure 2 which zooms in to the region populated by the so-called “classical” (pre-SDSS) MW dSphs, since they have the most well-measured and spatially extended stellar velocity distributions and well-studied photometry. The dashed lines indicate $\pm10\%$ variation about the predicted relation. In the right-hand panel of Figure 2 we demonstrate that Equation 2 also provides a good measure of uncertainties on $M_{1/2}$ for the MW dSphs (compare to Figure C1). The errors on the vertical axis are $68\%$ likelihoods derived from our analysis, while the errors along the horizontal axis are calculated by simply propagating the observational errors on $r_{1/2}$ and $\sigma_{los}$ through Equation 2. This rough agreement is consistent with the $M_{1/2}$...
uncertainty being dominated by observational errors as opposed to the uncertainty in $\beta$, as expected.

It is worth emphasizing that Equation 2 is not able to capture the full uncertainty on the half-light mass in cases where the kinematic data does not constrain $\sigma_{los}$ beyond $R_\odot$. While our full likelihood procedure naturally takes into account any limitations in the data and factors them into the resultant mass uncertainty, Equation 2 was derived under the assumption that $\sigma_{los}$ remains constant out beyond $R \approx R_\odot$. The lack of extended kinematic data is manifest in the more massive galaxies presented in Figure 2. A careful examination of the dEs and regular Es (those with $M_{1/2} > 10^9 M_\odot$) reveals that the errors on the ordinate axis are on average 0.05 dex larger than the errors on the abscissa. Therefore, in cases where extended kinematics are not available, if one is willing to assume that an unmeasured velocity dispersion profile does not fall too sharply within $\sim 1.5 R_\odot$ (as is seen in most galaxies with measured dispersion profiles that extend this far), then our proposed estimator should provide an accurate description of the half-light mass and the associated uncertainty (via simple error propagation). If one does not wish to accept the assumption of a flat $\sigma_{los}$ profile, then adding an error of 0.05 dex to the propagated mass error provides a reasonable means to allow for a range of $\beta$ profiles.

We note that all of the mass modeling presented so far has been done by allowing $\beta(r)$ to vary according to the profile in Equation 11. This allows for $\beta(r)$ to vary monotonically with three free parameters. All of the results quoted in Table 1 allow for this sort of spatial variation in $\beta(r)$. Though this profile is fairly general and has the added virtue that it is reminiscent of the anisotropy of cold dark matter particles found in numerical simulations (e.g., [1997]), we have also performed our analysis using the $\beta(r)$ form in Equation 13 which allows for an extremum within the stellar light distribution. We find that even with this unusual family of $\beta(r)$ profiles, no bias in the mass estimates exists (within either $r_{1/2}$ or $r_{1/2}^e$) between the two $\beta(r)$ forms. However, the errors on $M_{1/2}$ increased by roughly 0.05 dex when the (rather extreme) second $\beta(r)$ form was used. The errors on $M(r_{1/2})$ were slightly less affected. Hence Equation 11 becomes preferable to Equation 2 for the most general $\beta(r)$ profiles, as long as the required photometric measurements (for $r_{1/2}$) and kinematic data sets (for $\sigma_{los}^2$) are good enough to warrant the need for 10% accuracy.

Before moving on, we mention that in Appendix C we perform a similar test using our full mass modeling procedure against a popular mass estimator for dSphs known as the Illingworth (1976) approximation. We show that the Illingworth formula fails both because it systematically under-predicts masses and because it under-predicts mass uncertainties. The main reason for the failure is that it was derived for mass-follows-light globular clusters using $\beta = 0$. It was never intended to be generally applicable to dark-matter dominated systems like dSphs.

Lastly, in Appendices C2 and C3 we compare Equation...
4 DISCUSSION

We have shown that the integrated mass within the half-light radius of spherically symmetric, dispersion-supported systems is very well constrained by line-of-sight kinematic observations with only mild assumptions about the spatial variation of the stellar velocity dispersion anisotropy: $M_{1/2} = 3G^{-1}(\sigma_{\text{los}}^2)_{1/2}$. Mass determinations at larger and smaller radii are much more uncertain because of the uncertainty in $\beta(r)$. In the following two subsections we use $M_{1/2}$ determinations to examine the dark matter halos of MW dSphs and to explore the mass-luminosity relation in dispersion-supported galaxies as a function of mass scale.

4.1 Dwarf spheroidal satellite galaxies of the Milky Way

As an example of the utility of $M_{1/2}$ determinations, both panels of Figure 3 present $M_{1/2}$ vs. $r_{1/2}$ for MW dSph galaxies. We have used our full mass likelihood approach in deriving these masses and associated error bars, though had we simply used Equation 2 the result would have been very similar. In interpreting this figure, it is important to emphasize that the galaxies represented here span almost five orders of magnitude in luminosity. Relevant parameters for each of the galaxies are provided in Table 1. The symbol types labeled on the plot correspond to three wide luminosity bins (following the same scheme represented in Figure 2). Note that among galaxies with the same half-light radii, there is no clear trend between luminosity and density. We return to this noteworthy point below.

It is interesting now to compare the data points in Figure 3 to the integrated mass profile $M(r)$ predicted for $\Lambda$CDM halos of a given $M_{\text{halo}}$ mass. We define $M_{\text{halo}}$ as the halo mass corresponding to an overdensity of 200 compared to the critical density. In the limit that dark matter halo mass profiles $M(r)$ map in a one-to-one way with their $M_{\text{halo}}$ mass (Navarro et al. 1997), then the points on this figure may be used to estimate an associated halo mass for each galaxy. The association is not perfect for three reasons: 1) some scatter exists in halo concentration at fixed mass and redshift (e.g., Jing 2000; Bullock et al. 2001); 2) the mapping between $M(r)$ and $M_{\text{halo}}$ evolves slightly with redshift (e.g., Bullock et al. 2001); and 3) the MW satellites all reside within subhalos, which tend to lose mass after accretion from the field (see Kazantzidis et al. 2004). Nevertheless, we may still examine the median $M(r)$ dark matter halo profile for a given $M_{\text{halo}}$ in order to provide a reasonable estimate their progenitor halo masses prior to accretion onto the Milky Way.

The solid line in the left panel of Figure 3 shows the mass profile for a NFW (Navarro et al. 1997) dark matter halo at $z = 0$ with a halo mass $M_{\text{halo}} = 3 \times 10^9 M_\odot$. We have used the median concentration ($c = 11$) predicted by the Bullock et al. (2001) mass-concentration model updated by Macciò et al. (2008) for WMAP5 $\Lambda$CDM parameters. The dashed lines indicate the expected 68% scatter about the median concentration at this mass. The orange dot-dashed line shows the expected $M(r)$ profile for the same mass halo at $z = 3$ (corresponding to a concentration of $c = 4$), which provides an estimate of the scatter that would result from the scatter in infall times. We see that each MW dSph is consistent with inhabiting a dark matter halo of mass $\sim 3 \times 10^9 M_\odot$ (Strigari et al. 2008; Walker et al. 2009b) recently submitted an article that presented a similar result for Milky Way dSphs by examining the mass within a radius $r = R_s$ rather than $r = r_{1/2}$ as we have done. Note that since $R_s \approx 0.75 r_{1/2}$, the mass within $r = R_s$ is still somewhat constrained without prior knowledge of $\beta$.

The right panel in Figure 3 shows the same data plotted along with the median mass profiles for several different halo masses. Clearly, the data are also consistent with MW dSphs populating dark matter halos of a wide range in $M_{\text{halo}}$. As described in Strigari et al. (2008), there is a weak power-law relation between a halo’s inner mass and its total mass (e.g., $M(300\text{pc}) \propto M_{\text{halo}}$ at $M_{\text{halo}} \approx 10^8 M_\odot$), and this makes a precise mapping between the two difficult. Nevertheless, several interesting trends are manifest in the comparison.

First, all of the MW dSphs are associated with halos more massive than $M_{\text{halo}} \approx 10^8 M_\odot$. This provides a very stringent limit on the fraction of the baryons converted to stars in these systems. More importantly, there is no systematic relationship between dSph luminosity and the $M_{\text{halo}}$ mass profile that they most closely intersect. The ultra-faint dSph population (red circles) with $L_V < 10,000 L_\odot$ is equally likely to be associated with the more massive dark matter halos as are classical dSphs that are more than 1,000 times brighter (green squares). Indeed, a naive interpretation of the right-hand panel of Figure 3 shows that the two least luminous satellites (which also have the smallest $M_{1/2}$ and $r_{1/2}$ values) are associated with halos that are either more massive than any of the classical MW dSphs (green squares), or have abnormally large concentrations (reflecting earlier collapse times) for their halo mass.

This general behavior is difficult to reproduce in models constructed to confront the Milky Way satellite population (e.g., Koposov et al. 2009; Li et al. 2009; Macciò et al. 2009; Muñoz et al. 2009; Salvadori & Ferrara 2009; Busha et al. 2010; Kravtsov 2010), which typically predict a noticeable trend between halo infall mass and dSph luminosity. It is possible that we are seeing evidence for a new scale in galaxy formation (Strigari et al. 2008) or that there is a systematic bias that makes less luminous galaxies that sit within low-mass halos more difficult to detect than their more massive counterparts (Bovill & Ricotti 2009; Bullock et al. 2009).

4.2 The global population of dispersion-supported stellar systems

A second example of how accurate $M_{1/2}$ determinations may be used to constrain galaxy formation scenarios is presented in Figure 4, where we examine the relationship between the half-light mass $M_{1/2}$ and the half-light I-band luminosity $L_{1/2} = 0.5 L_i$ for the full range of dispersion-supported stellar systems in the Universe: globular clusters, dSphs, dwarf ellipticals, ellipticals, brightest cluster galaxies, and extended cluster spheroids. Each symbol type is matched to a galaxy type as detailed in the caption. We provide three representations of the same information in order to high-
Figure 4. Left: The half I-band luminosity $L_{1/2}$ vs. half-light mass $M_{1/2}$ for a broad population of spheroidal galaxies. Middle: The dynamical I-band half-light mass-to-light ratio $\Upsilon_{I}^{1/2}$ vs. $M_{1/2}$ relation. Right: The equivalent $\Upsilon_{I}^{1/2}$ vs. total I-band luminosity $L_{I} = 2L_{1/2}$ relation. The solid line in the left panel guides the eye with $M_{1/2} = L_{1/2}$ in solar units. The solid, colored points are all derived using our full mass likelihood analysis and their specific symbols/colors are linked to galaxy types as described in Figure 2. The I-band luminosities for the MW dSph and GC population were determined by adopting M92’s $V − I = 0.88$. All open, black points are taken from the literature as follows. Those with $M_{1/2} > 10^{8}M_{\odot}$ are modeled using Equation 3 with $\sigma_{los}$ and $r_{1/2}$ culled from the compilation of Zaritsky et al. (2006): triangles for dwarf ellipticals (Geha et al. 2003), inverse triangles for ellipticals (Jørgensen et al. 2000), plus signs for brightest cluster galaxies (Oegerle & Hoessel 1991), and asterisks for cluster spheroids, which, following Zaritsky et al. (2006), include the combination of the central brightest cluster galaxy and the extended intracluster light. Stars indicate globular clusters, with the subset of open, black stars taken from Pryor & Meylan (1993).

light different aspects of the relationships: $M_{1/2}$ vs. $L_{1/2}$ (left panel); the dynamical I-band mass-to-light ratio within the half-light radius $\Upsilon_{I}^{1/2}$ vs. $M_{1/2}$ (middle panel); and $\Upsilon_{I}^{1/2}$ vs. total I-band luminosity $L_{I}$ (right panel).

Masses for the colored points are derived using our full mass likelihood approach and follow the same color and symbol convention as in Figure 2. All of the black points that represent galaxies were modeled using Equation 3 with published $\sigma_{los}$ and $r_{1/2}$ values from the literature. The middle and right panels are inspired by (and qualitatively consistent with) Figures 9 and 10 from Zaritsky et al. (2006), who presented estimated dynamical mass-to-light ratios as a function of $\sigma_{los}$ for spheroidal galaxies that spanned two orders of magnitude in $\sigma_{los}$.

We note that the asterisks in Figure 4 are cluster spheroids (Zaritsky et al. 2006), which are defined for any galaxy cluster to be the sum of the extended low-surface brightness intracluster light component and the brightest cluster galaxy’s light. These two components are difficult to disentangle, but the total light tends to be dominated by the intracluster piece. One might argue that the total cluster spheroid is more relevant than the brightest cluster galaxy because it allows one to compare the dominant stellar spheroids associated with individual dark matter halos over a very wide mass range self consistently. Had we included analogous diffuse light components around less massive galaxies (e.g., stellar halos around field ellipticals) the figure would change very little, because halo light is of minimal importance for the total luminosity in less massive systems (see Purcell et al. 2007). One concern is that the central cluster spheroid mass estimates here suffer from a potential systematic bias because they rely on the measured velocity dispersion of cluster galaxies for $\sigma_{los}$ rather than the velocity dispersion of the cluster spheroid itself, which is very hard to measure (Zaritsky et al. 2006). For completeness, we have included brightest cluster galaxies on this diagram (plus signs) and they tend to smoothly fill in the region between large elliptical galaxies (inverse triangles) and the cluster spheroids (asterisks).

There are several noteworthy aspects to Figure 4 which are each highlighted in a slightly different fashion in the three panels. First, as seen most clearly in the middle and right panels, the dynamical half-light mass-to-light ratios of spheroidal galaxies in the Universe demonstrate a minimum at $\Upsilon_{I}^{1/2} \simeq 2 − 4$ that spans a remarkably broad range of masses $M_{1/2} \simeq 10^{8−11}M_{\odot}$ and luminosities $L_{I} \simeq 10^{5.5−10.5}L_{\odot}$. It is interesting to note the offset in the average dynamical mass-to-light ratios between globular clusters and $L_{*}$ ellipticals, which may suggest that even within $r_{1/2}$, dark matter may constitute the majority of the mass content of $L_{*}$ elliptical galaxies. Nevertheless, it seems that dark matter plays a clearly dominant dynamical role ($\Upsilon_{I}^{1/2} \gtrsim 5$) within $r_{1/2}$ in only the most extreme systems (see similar results by Dabringhausen et al. 2008; Forbes et al. 2008, who study slightly more limited ranges of spheroidal galaxy luminosities). The dramatic increase in dynamical half-light mass-to-light ratios at both smaller and larger mass and luminosity scales is indicative of a decrease in the efficiency of galaxy formation in the smallest and largest dark matter halos. It is worth mentioning that a qualitatively similar trend in the relationship between $M_{halo}$ and $L$ must exist if $\Lambda$CDM is to explain the luminosity function of galaxies (e.g.,

13 The masses for the open, black stars (globular clusters) were taken directly from Pryor & Meylan (1993).

14 In addition, concerns exist with the assumption of dynamical equilibrium. However, Willman et al. (2004) demonstrated with a simulation that using the intracluster stars as tracers of cluster mass is accurate to $\sim 10\%$. 

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The two main assumptions we have made in this work are that the systems that we are analyzing are spherically symmetric and are in dynamical equilibrium. Testing the accuracy of Equation (2) as a function of ellipticity will be an important future step.

As an example of the usefulness of the $M_{1/2}$ estimator, we applied our result to the dSph satellite population of the Milky Way and specifically used the observed $M_{1/2}$ vs $r_{1/2}$ relation to associate a dark matter halo $M_{\text{halo}}$ mass to each galaxy. By allowing for the expected scatter in halo concentrations at fixed mass, we showed that all of the MW dSphs are consistent with inhabiting dark matter halos of mass $M_{\text{halo}} \simeq 3 \times 10^8 M_\odot$. We also showed that a range of $M_{\text{halo}}$ values from $\sim 10^8 M_\odot$ to $3 \times 10^{11} M_\odot$ is allowable as well, but that no trend exists between the associated $M_{\text{halo}}$ and galaxy luminosity, despite the fact that these galaxies span over four orders of magnitude in luminosity. Specifically, the lowest luminosity dSphs ($L_V \simeq 500 L_\odot$) are at least as dense as, if not more dense than, the brightest MW dSphs ($L_V \simeq 10^7 L_\odot$) when normalized against the inner power-law mass profiles expected in ΛCDM halos. This last point is difficult to reproduce in models that assume a monotonic mapping between $M_{\text{halo}}$ and galaxy luminosity. It is worth emphasizing that none of the MW dSphs are associated with dark matter halos smaller than $M_{\text{halo}} \gtrsim 10^8 M_\odot$, and this alone provides a very tight constraint on the fraction of baryons converted to stars in these systems. Of course, these results assume that no systematic biases in the kinematic data for dSph galaxies are present. One particular worry is the effect of binary stars. Minor et al. (2010) estimate that medium-to-high binary fractions can inflate velocity dispersions by up to 20% in the smallest dSphs. This will have to be taken into account in future work, at least for the classical dwarfs that only have $\sim 10\%$ errors on their $M_{1/2}$ estimates.

We went on to explore the relationship between $M_{1/2}$ and $L_4$ in dispersion-supported galaxies, spanning the full range in I-band luminosity and mass from globular clusters ($L_4 \simeq 10^5 L_\odot$) to intra-cluster light spheroids ($L_4 \simeq 10^{12} L_\odot$). Globular clusters excluded, the $\Upsilon_{1/2}$ vs. $M_{1/2}$ relation for dispersion-supported galaxies follows a U-shape, with a broad minimum near $\Upsilon_{1/2} \simeq 3$ that spans dwarf elliptical galaxies to normal elliptical galaxies, a steep rise to $\Upsilon_{1/2} \simeq 3,200$ for ultra-faint dSphs, and a more shallow rise to $\Upsilon_{1/2} \simeq 20$ for brightest cluster ellipticals. If we include intra-cluster light spheroids in the analysis, the rise continues to $\Upsilon_{1/2} \simeq 800$ for the largest galaxy clusters.

Lastly, we note that Equation (2) can be rewritten succinctly in terms of the circular velocity at $r_{1/2}$ as

\[ V_{\text{circ}}(r_{1/2}) = \sqrt{3} \left( \sigma_{\text{los}}^2 \right). \]  

(36)

It is clear then that the maximum circular velocity of the dark matter halo hosting such a dispersion-supported galaxy must obey $V_{\text{max}} \geq \sqrt{3} \left( \sigma_{\text{los}}^2 \right)$.

In summary, we have shown that the dynamical mass within the deprojected half-light radius of dispersion-supported galaxies can be measured accurately with only line-of-sight stellar velocity measurements. We have provided a simple formula that allows this mass to be computed given the measured luminosity-weighted square of the line-of-sight velocity dispersion and the half-light radius. This result opens up new opportunities to explore...
the relationships between stellar properties and the masses of galaxies spanning approximately ten orders of magnitude in luminosity.

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Here we derive a single expression for the mass profile of spheroidal galaxies $M(r; \beta)$ as a function of the observable combination $\Sigma_* \sigma_*^2 (R)$. 

We begin by manipulating the standard equation for $\sigma_\text{los}$ in order to isolate the $R$ dependence into an integral kernel:

\[
\Sigma_* \sigma_*^2 (R) = \int_{R_2}^{\infty} n_* \sigma_*^2 (r) \left[ 1 - \frac{R^2}{r^2} \beta(r) \right] \frac{dr^2}{\sqrt{r^2 - R^2}} \tag{A1}
\]

\[
= \int_{R_2}^{\infty} n_* \sigma_*^2 \left( 1 - \beta(r) r^2 + \beta(r^2 - R^2) \right) \frac{dr^2}{\sqrt{r^2 - R^2}} \]

\[
= \int_{R_2}^{\infty} \frac{n_* \sigma_*^2 (1 - \beta)}{\sqrt{r^2 - R^2}} dr^2 - \left( \int_{R_2}^{\infty} \frac{\beta n_* \sigma_*^2}{r^2} d\int dr^2 \right) \bigg|_{R_2}^{\infty} + \int_{R_2}^{\infty} \left( \frac{\beta n_* \sigma_*^2}{r^2} d\int dr^2 \right) \frac{dr^2}{2 \sqrt{r^2 - R^2}} \]

\[
= \int_{R_2}^{\infty} \frac{n_* \sigma_*^2}{(1 - \beta)} + \int_{R_2}^{\infty} \frac{\beta n_* \sigma_*^2}{2 r^2} d\int dr^2 \frac{dr^2}{\sqrt{r^2 - R^2}},
\]

where we employed an integration by parts to achieve the third equality. Note that we have set the middle term on the third line to zero by making the physically-motivated assumption that the combination $\beta n_* \sigma_*^2$ falls faster than $r^{-1}$ at large $r$.

With this crucial manipulation in place, we may now utilize the following Abel inversion

\[
f(x) = \int_x^{\infty} \frac{g(t) dt}{\sqrt{t - x}} \Rightarrow g(t) = -\frac{1}{\pi} \int_t^{\infty} \frac{df}{dx} \frac{dx}{\sqrt{x - t}} \tag{A2}
\]

to solve for

\[
g(r^2) = n_* \sigma_*^2 (1 - \beta) + \int_{\text{los}}^{\infty} \beta n_* \sigma_*^2 d\ln \tilde{r} \tag{A3}
\]

in terms of the observable combination $f(R^2) = \Sigma_* \sigma_*^2 (R^2)$:

\[
g(r^2) = -\frac{1}{\pi} \int_{r^2}^{\infty} \frac{d(\Sigma_* \sigma_*^2)}{dR^2} \frac{dR^2}{\sqrt{R^2 - r^2}} \tag{A4}
\]

In order to isolate $n_* \sigma_*^2$ we equate Equations [A3] and [A4] and then differentiate the resulting expression with respect to $\ln r$ (denoted by $\gamma$)

\[
\frac{(n_* \sigma_*^2)'}{(1 - \beta)^{-1}} - (n_* \sigma_*^2) (\beta + \beta') = -\frac{2 r^2}{\pi} \int_{r^2}^{\infty} \frac{d^2(\Sigma_* \sigma_*^2)}{(dR^2)^2} \frac{dR^2}{\sqrt{R^2 - r^2}} \tag{A5}
\]

and employ the integrating factor

\[
h(r) = \exp \left[ -\int_{\text{los}}^{\ln r} \frac{\beta + \beta'}{1 - \beta} d\ln \tilde{r} \right] \tag{A6}
\]

with the constant $a$ chosen such that the value of the integrand goes to zero at the lower limit:

\[
n_* \sigma_*^2 (r; \beta) = \frac{h^{-1}}{\pi} \int_{r^2}^{\infty} \frac{d^2(\Sigma_* \sigma_*^2)}{(dR^2)^2} \frac{dR^2}{\sqrt{R^2 - r^2}} h \frac{d\tilde{r}^2}{\beta - 1} = \frac{h^{-1}}{\pi} \int_{r^2}^{\infty} \left[ \int_{r^2}^{\infty} h \frac{1}{\beta - 1} \frac{d\tilde{r}^2}{\sqrt{R^2 - r^2}} \right] \frac{d^2(\Sigma_* \sigma_*^2)}{(dR^2)^2} dR^2. \tag{A7}
\]

If one wishes to adopt a parametric form for $\beta(r)$, $n_* \sigma_*^2$ can be determined using Equation [A7] and then inserted into the Jeans equation to find the cumulative mass profile $M$. Note that nothing guarantees a physical mass profile (i.e., the mass never decreases); given a very large number of stellar velocities with very low measurement errors, one can restrict the anisotropy such that a physical mass is derived.

If $\beta(r)$ is assumed to be constant, then the inner integral of the right-hand side of Equation [A7] can be written in terms of the incomplete Beta function:

\[
B_x (p, q) \equiv \int_0^x y^{p-1} (1 - y)^{q-1} dy. \tag{A8}
\]

By utilizing the substitution $u = 1 - r^2 / R^2$, we find

\[
n_* \sigma_*^2 (r; \beta) = \frac{h^{-1}}{\pi(\beta - 1)} \int_{r^2}^{\infty} R^{1 - \beta} B_{1 - \frac{2\beta}{2(1 - \beta)}} \left( \frac{1}{2}, \frac{2 - 3 \beta}{2(1 - \beta)} \right) \frac{d^2(\Sigma_* \sigma_*^2)}{(dR^2)^2} dR^2. \tag{A9}
\]
By solving the Jeans equation we can derive the mass by first taking a derivative of Equation (A7) and then inserting the form derived in Equation (A9)

\[ M(r; \beta) = \frac{1}{G(\beta - 1)n}(\sqrt{2\pi})^2 \int_{0}^{\infty} R^2 d^2 \left(\frac{\Sigma_{los}^2}{(dR)^2}\right) K(r; R; \beta) dR^2 \]

where

\[ K(r; R; \beta) = \frac{2r^2/R^3}{\sqrt{1 - r^2/R^2}} + \beta \frac{3 - 2\beta}{(R^2)} B^2 \left( \frac{1}{2}; \frac{2 - 3\beta}{2(1 - \beta)} \right). \]

With this relation we have replaced the dependence of deriving the mass of a dispersion-supported system from the unknown radial dispersion \( \sigma_r(r) \) with the second derivative of the observable combination \( \Sigma_{los}^2(R) \). Note that determining the slope of the mass profile will require an additional derivative, and thus we will require extremely accurate observational constraints on both the light profile and the line-of-sight velocity dispersion. We conjecture that the data will need to be so precise that the assumption of spherical symmetry will no longer do the data proper justice, and thus new derivations must be explored.

### APPENDIX B: USEFUL CONVERSIONS FROM 2D TO 3D HALF-LIGHT RADII

In this Appendix we present scaling relations to derive the 3D deprojected half-light radius \( r_{1/2} \) from the observed 2D projected half-light radius \( R_e \) for several commonly used stellar distributions. For the Hernquist (1990) profile, \( R_e = r_{core} \) and \( c_k \equiv \log_{10}(r_{1/2}/r_{core}) \). A Sérsic profile is defined as \( I(R) = I(0)e^{-b_n(R/R_0)^{1/n}} \), where \( b_n \) is chosen such that \( R_0 = R_e \). Note that although the exponential and Gaussian profiles are special cases of the \( n=1 \) and \( n=0.5 \) Sérsic profiles, the \( R_e/R_0 \) relations are different due to the definitions of their scale radii: an exponential profile is defined as \( R_e = r_{core} \equiv 1.678 \) and a Gaussian profile is defined as \( R_e = r_{core} \equiv 1.15 \).

| Profile | \( R_e/R_0 \) | \( r_{1/2}/R_e \) | \( r_{3}/r_{1/2} \) |
|---------|---------------|-----------------|-----------------|
| Exponential | 1.678 | 1.329 | 1.15 |
| Gaussian | 1.178 | 1.307 | 1.13 |
| King \( (c_k=0.70) \) | 1.185 | 1.322 | 1.13 |
| Plummer | 1.000 | 1.305 | 0.94 |
| Sérsic \( (n=2) \) | 1.000 | 1.342 | 1.16 |
| Sérsic \( (n=4) \) | 1.000 | 1.349 | 1.17 |
| Sérsic \( (n=8) \) | 1.000 | 1.352 | 1.18 |

We do not include the NFW profile due to the fact that the mass is divergent (thus, \( r_{1/2} \rightarrow \infty \)). We also do not include the Einasto (1965) profile in this table because it does not well represent the baryonic tracer number density of most galaxies. The Hernquist (1990) profile is sometimes used in place of a Sérsic profile due to the ease of analytic manipulation. But we caution, as was pointed out in the original paper, that the projected central surface brightness diverges logarithmically. This can cause \( \gamma_o \) to be quite large in magnitude, thus affecting the solution to Equation 21 more profoundly than if the more well-behaved Sérsic profile is used to model a tracer population.

Returning to the relations presented in the above table, for a King profile

\[ R_e/R_0 = 0.5439 + 0.10444c_k + 1.5618c_k^2 - 0.7559c_k^3 + 0.2572c_k^4 \]  \hspace{1cm} (B1)

to better than 2% accuracy for \( 0.30 \leq c_k \leq 3.00 \), and to better than 1% accuracy for \( 0.40 \leq c_k \leq 3.00 \). Also,

\[ r_{1/2}/R_e = 1.3088 + 0.0159c_k + 0.0066c_k^2 - 0.0035c_k^3 + 0.0004c_k^4 \]  \hspace{1cm} (B2)

to better than 0.04% accuracy for \( 0.30 \leq c_k \leq 3.00 \). Thus, the dominant error is in the relation between \( R_e \) and \( R_0 \).

In regard to the family of Sérsic profiles, as stated above, \( R_0 \equiv R_e \). To relate \( r_{1/2} \) to \( R_e \), we utilize the following fit, which Lima Neto et al. (1999) state is valid to 0.25% accuracy after testing against the numerical integration of the family of Sérsic profiles corresponding to \( 0.10 \leq n^{-1} \leq 2.0 \):

\[ r_{1/2}/R_e = 1.3560 - 0.0293n^{-1} + 0.0023n^{-2}. \]  \hspace{1cm} (B3)

Thus, \( r_{1/2}/R_e \approx 4/3 \) is accurate to better than 2% for most surface brightness profiles used to describe observed stellar systems. We also note that this result has been shown before for a wide range of Sérsic profiles in Ciotti (1991) and for the Plummer (1911) profile in Spitzer (1987).
Figure C1. Left: The masses within the stellar extent for Milky Way dSphs. The vertical axis shows masses derived using individual stellar kinematics with our full likelihood procedure (see text) and the horizontal axis shows the “Illingworth approximation”, which is routinely used in the literature as a mass estimate for dSphs. Right: Errors on these masses for Milky Way dSphs. The vertical axis shows the 68% error width derived from our full likelihood analysis and the horizontal axis shows the error width calculated by straightforward error propagation using Equation C1. Note that this approximation tends to underestimate masses by up to an order of magnitude (left) and also under-estimates the relative error on the mass significantly (right). In both plots the solid line indicates the one-to-one relation.

APPENDIX C: COMPARISON WITH OTHER MASS APPROXIMATION FORMULAE

C1 Illingworth Formula

Due to the large amount of attention that dSphs have received since new discoveries [Willman et al. 2005a,b; Zucker et al. 2006a,b; Belokurov et al. 2006, 2007; Sakamoto & Hasegawa 2006; Irwin et al. 2007; Walsh et al. 2007] were found in the public data releases of the SDSS (York et al. 2000), we will discuss an estimator that is often used to determine their masses. Because many dSphs look like larger versions of low-concentration globular clusters, the Illingworth formula (derived by Illingworth (1976) for application only to globular clusters) is often used to estimate the masses of dSphs (e.g., Seitzer & Frogel 1985; Suntzeff et al. 1993; Hargreaves et al. 1994; Mateo 1998; Simon & Geha 2007). Two explicit assumptions made by this formula are that the stellar velocity dispersion is isotropic and that the mass distribution follows a King (1966) light distribution. Under these assumptions, the total mass within the stellar extent \( r_{\text{lim}} \) is stated as

\[
M_{\text{Ilw}} = 167 \mu r_{\text{core}} \sigma_0^2 G^{-1},
\]

where \( \sigma_0 \) is the central line-of-sight velocity dispersion of the system, \( r_{\text{core}} \) is the King core radius and \( \mu \) is a parameter that depends on the King concentration, \( c_k \equiv \log_{10}(r_{\text{lim}}/r_{\text{core}}) \). It is common in the literature to set \( \mu = 8 \) (incorrectly) for all dSphs based on a rough estimate provided in Table 4 of Mateo (1998). By adopting a value of \( \mu \) without any error, many published mass uncertainties for dSphs do not properly include light profile uncertainties, which are typically only factored in from the error on \( r_{\text{core}} \). More important, however, is the implicit assumption that mass follows light in this formulation. While this is a reasonable assumption for globular cluster systems, the majority of the mass in dwarf galaxies does not necessarily follow the shape of their baryonic tracers (e.g., Sofue & Rubin 2001; Walker et al. 2005, 2007; Penarrubia et al. 2008), as they are likely to be deeply embedded inside of dark matter halos (e.g., White & Rees 1978).

The left panel of Figure C1 compares the masses \( M(r_{\text{lim}}) \) of Milky Way dSphs derived using our general approach to Equation C1. Symbol types correspond to luminosity bins, as indicated. For the general mass likelihoods, we analyze the kinematics of individual stars (Muñoz et al. 2005, 2006; Koch et al. 2007; Martin et al. 2007; Simon & Geha 2007; Mateo et al. 2008; Walker et al. 2009a; Geha et al. 2009). Willman et al., in preparation, in conjunction with the distances and stellar surface density profile parameters listed in Table 1. For the Illingworth approximation, we use the same observational datasets to calculate \( \sigma_0 \) (which is very close to the luminosity-weighted dispersion since the dispersion profiles for the MW dSphs are nearly constant with radius) and we follow the common practice of setting \( \mu = 8 \). Clearly, \( M_{\text{Ilw}} \) systematically underestimates

\[16\]

We only accept stars whose projected distances lie within the lower limit of \( r_{\text{lim}} \) (see Table 1). For kinematics with assigned membership probabilities, we only accept those with \( p \geq 0.9 \).
Figure C2. Left: Ratio of the luminosity-weighted square of the dispersion within projected radius R divided by the luminosity-weighted square of the dispersion integrated to infinity for two different dispersion models. The dashed lines (model 1) are derived by considering the median dispersion profile model presented in Equation 1 of Cappellari et al. (2006): \( \sigma_{\text{los}}(R) \propto R^{-0.066} \). The solid lines (model 2) use the same relation to within only one effective radius. After \( R = R_e \), the dispersion profile is assumed to be flat as a function of projected radius. The three projected Sérsic surface brightness profiles modeled are, from top to bottom, \( n=6 \) (blue), \( n=4 \) (red), and \( n=2 \) (black).

Right: Comparison of the derived V-band mass-to-light ratios derived using our general two-component spherical Jeans models (x-axis) compared to those obtained under the assumption that mass follows light and \( \beta = 0 \) (y-axis). The solid line represents the one-to-one relation. The four galaxies modeled, from top to bottom, are NGC 4478, NGC 731, NGC 185, and NGC 855.

the mass with this value of \( \mu \). This systematic difference follows from the fact that \( M_{\text{Ilw}} \) forces the mass profile to truncate at \( r_{\text{lim}} \) while the data prefer models where the mass distribution continues beyond the stellar extent.

However, the most dramatic difference between the full mass likelihoods and the Illingworth approximation is in the implied uncertainty. Errors on the vertical axis represent the 68% width from the median of our derived mass likelihoods, while the symbol placement is indicative of the median of the likelihood. The errors on the horizontal axis propagate the observational errors on \( r_{\text{core}} \) and \( \sigma_0 \) using Equation C1. It is clear that using this equation underestimates the relative error on the mass. As we have discussed, the uncertainty in the mass within \( r_{\text{lim}} \) is dominated by the velocity anisotropy, which is not accounted for in the \( M_{\text{Ilw}} \) equation, as it was derived under the assumption of isotropy. The right panel of Figure C1 shows a comparison between the log\( _{10} \) mass error in both cases.

In conclusion, the Illingworth approximation, which was derived to only be applied on globular clusters, is a very poor estimate of the mass and mass uncertainty for dSph galaxies.

C2 Spitzer Formula

In this subsection we slightly modify the mass estimator presented in Spitzer (1969), by halving their total mass, to better compare to our mass estimator:

\[
M_{\text{Spitzer}}^{1/2} = 3.75 G^{-1} \langle \sigma_{\text{los}}^2 \rangle_{1/2}. \tag{C2}
\]

Despite the fact that this equation was derived by analyzing polytropes with indices between \( n=3 \) and \( n=5 \), which is a very restrictive class of mass densities that describe multi-component galaxies, our coefficient in Equation C2 is only 20% under the Spitzer coefficient of 3.75. Lokas & Mamon (2001) find coefficients in much better agreement with ours when analyzing a wide variety of NFW halos, which better represent the mass density of real galaxies.

C3 Cappellari et al. Dynamical Mass-to-Light Ratio

Using axisymmetric Jeans modeling, Cappellari et al. (2006) (hereafter C06) empirically find the following relation assuming a single-component mass-follows-light (MFL) density distribution:

\[
\frac{M}{L} = \frac{5 \langle \sigma_{\text{los}}^2 \rangle_{n} R_e}{G L} \Rightarrow \gamma_{1/2}^{C06} = \frac{2.5 \langle \sigma_{\text{los}}^2 \rangle_{n} R_e}{G L_{1/2}}, \tag{C3}
\]
where $\langle \sigma^2_{\text{los}} \rangle_{R_e}$ is the luminosity-weighted square of the line-of-sight dispersion within $R_e$. In practice, C06 determined $\langle \sigma^2_{\text{los}} \rangle_{R_e}$ by extrapolating the measured luminosity-weighted square of the dispersion within the observational aperture $R_{\text{ap}}$. The physical aperture size varies depending on the system, but it typically corresponds to $R_{\text{ap}} \approx 0.7 R_e$ for the data that C06 analyzed. We continue the convention presented within this paper, where $\Upsilon_{1/2}$ is the dynamical mass-to-light ratio within the 3D deprojected half-light radius $r_{1/2}$. Let us rewrite our Equation (2) in order to facilitate comparison with the C06 relation:

$$\Upsilon_{1/2}^{\text{W10}} = \frac{4 \langle \sigma^2_{\text{los}} \rangle_{R_e}}{GL_{1/2}},$$

where we remind the reader that $\langle \sigma^2_{\text{los}} \rangle$ is the luminosity-weighted square of the line-of-sight dispersion over the entire galaxy. In the limit that the observed velocity dispersion profile is perfectly flat at all radii we would expect $\langle \sigma^2_{\text{los}} \rangle_{R_e} = \langle \sigma^2_{\text{los}} \rangle$. In this case the C06 estimator (with a coefficient of 2.5) is smaller by $\sim 40\%$ compared to ours (with a coefficient of 4).

We explore three possible reasons for this difference in the coefficients. First, our analysis is explicitly spherical while the C06 models are axisymmetric. C06 addresses this concern by comparing their axisymmetric model results to spherical Jeans model results under the assumption that the velocity dispersion is isotropic ($\beta = 0$). In this comparison, they find little difference in their dynamical mass-to-light ratios. While this is reassuring, it is not entirely general given the assumption of $\beta = 0$ in their comparison. It remains to be seen if the geometric freedom becomes important in comparison to more general spherical models with variable $\beta$. In principle, projection effects can add an additional $\sim 20 - 30\%$ uncertainty to spherical mass estimates as discussed, for example, by Gavazzi (2003). However, it would be surprising if these effects were systematic in biasing mass estimates.

A second possible reason for the difference in our coefficients is that $\langle \sigma^2_{\text{los}} \rangle_{R_e} \neq \langle \sigma^2_{\text{los}} \rangle$. Indeed, the median dispersion profile studied by C06 falls with projected radius as $\sigma_{\text{los}}(R) \propto R^{-0.066}$ over the range probed by their data such that we would expect $\langle \sigma^2_{\text{los}} \rangle_{R_e} > \langle \sigma^2_{\text{los}} \rangle$. In the left panel of Figure C2 we plot the ratio of the luminosity weighted square of the velocity dispersion as measured within an aperture radius $R$ ($\langle \sigma^2_{\text{los}} \rangle_R$) to the total luminosity-weighted square of the velocity dispersion ($\langle \sigma^2_{\text{los}} \rangle$) as a function of $R$ for two models of $\sigma_{\text{los}}(R)$ velocity dispersion profiles and several light profiles (Sérsic profiles with $n = 2$, 4, and 6). Model 1 curves (dashed) assume the median C06 power-law for $\sigma_{\text{los}}(R)$ extends to all $R$. Model 2 curves (solid) assume the median C06 power-law for $\sigma_{\text{los}}(R)$ until $R = R_e$, and then a flat dispersion profile for larger radii. This modification of the C06 model is motivated by the behavior of dispersion profiles of galaxies seen in high quality kinematics that extend out to several effective radii (e.g., Proctor et al. 2009; Weijmans et al. 2009; Geha et al. 2010). As can be seen in Figure C2 we expect $\langle \sigma^2_{\text{los}} \rangle_{R_e}/\langle \sigma^2_{\text{los}} \rangle \approx 1.1 - 1.2$ for the typical aperture size ($R_{\text{ap}} \approx 0.7 R_e$) in the data that C06 analyzed. This result allows us to approximate the C06 formula as $\Upsilon_{1/2}^{\text{C06}} \approx 2.8 \langle \sigma^2_{\text{los}} \rangle_{R_e}/(GL_{1/2})$, bringing the ratio of the C06 coefficient and our coefficient to within $\sim 30\%$.

A third difference between our method and that of C06 is that we have allowed the dark matter mass profile to be distinct from the light profile, while C06 assume that mass follows light. In principle, the MfL assumption can impose a bias because we expect the dynamical mass-to-light ratio to increase with radius. In the right panel of Figure C2 we explore this issue by comparing the dynamical V-band mass-to-light ratios of four galaxies derived using our general methodology to those derived under the assumption of MfL and $\beta = 0$. This MfL model mirrors that shown by C06 to reproduce their axisymmetric results. The four galaxies modeled (from top to bottom: NGC 4478, NGC 731, NGC 185, and NGC 855) were chosen as they had the lowest $\Upsilon_{1/2}$ values in Table 1. We see that two of the galaxies have median MfL mass-to-light ratios that are lower by $\sim 35\%$ than those derived for the general spherical case ($\Upsilon_{1/2}$). The other two galaxies do not show large differences ($\Upsilon_{1/2}$). Thus, it is possible that the MfL assumption can give rise to biases as large as $30\%$, even in systems that are not dark matter dominated.

Future investigations that allow for non-spherical, multicomponent mass models will be important for investigating the advantages and limitations of the current set of assumptions that are often used in Jeans analyses.

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17 The reason for the abscissa errors being larger than the ordinate errors is related to the additional freedom that we allow in our modeling, particularly with regard to $\beta$, as we discuss in Section 3.

18 We note that the $\Upsilon_{1/2}$ values derived from an anisotropic MfL model agree well with those in Table 1, as expected from Equation (2).