Log-unimodality for free positive multiplicative Brownian motion

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Abstract

We prove that the marginal law \( \sigma_t \boxplus \nu \) of free positive multiplicative Brownian motion is log-unimodal for all \( t > 0 \) if \( \nu \) is a multiplicatively symmetric log-unimodal distribution, and that \( \sigma_t \boxplus \nu \) is log-unimodal for sufficiently large \( t \) if \( \nu \) is supported on a suitably chosen finite interval. Counterexamples are given when \( \nu \) is not assumed to be symmetric or having a bounded support.

1 Introduction

This paper is a continuation of the first two authors' works \[7, 8\] on the unimodality of free Brownian motions.

Since its first appearance in \[2\], free multiplicative Brownian motion has been an object of interest in free probability. For examples, Biane showed that free unitary multiplicative Brownian motion can be approximated by \( U_N \)-valued Brownian motion as \( N \) tends to infinity, and he calculated the moments and the density of the marginal laws of free unitary multiplicative Brownian motion in \[2\] and \[3\]. Notably, Kemp \[12\] and Cébron \[5\] introduced another type of free multiplicative Brownian motion approximated by \( GL_N \)-valued Brownian motion as the size \( N \) of matrices tends to infinity.

Biane defined in \[4, \text{Definition 4.2}\] free positive multiplicative increment processes, which contain as a special case free positive multiplicative Brownian motion. Given a probability measure \( \nu \) on \( \mathbb{R}^+ := (0, \infty) \), the free positive multiplicative Brownian motion with initial distribution \( \nu \) has the marginal laws \( \{ \sigma_t \boxplus \nu : t \geq 0 \} \), where \( \sigma_t \) is the \( \boxplus \)-infinitely divisible measure whose \( \Sigma \)-transform is given by

\[
\Sigma_{\sigma_t}(z) = \exp \left( \frac{t}{2} \cdot \frac{z+1}{z-1} \right).
\]

Zhong proved in \[15\] that \( \sigma_t \boxplus \nu \) is absolutely continuous with a continuous density relative to the Lebesgue measure on \( \mathbb{R}^+ \); Section 2.2 of the present paper summarizes the results.

In this paper, we address the unimodality of \( \sigma_t \boxplus \nu \). Recall that a positive Borel measure \( \mu \) on \( \mathbb{R} \) is said to be unimodal with mode \( a \) if \( \mu = c\delta_a + f(x) \, dx \) where \( c \in [0, \infty) \) and \( f : \mathbb{R} \to [0, \infty) \) is non-decreasing on \( (-\infty, a) \) and non-increasing on \( (a, \infty) \). In view of the analytic apparatus of free multiplicative convolution, we find that it is more natural and appropriate to study the unimodality of the push-forward measure \( x \, d(\sigma_t \boxplus \nu)(x) \) than that of \( \sigma_t \boxplus \nu \) itself. It turns out that the unimodality of \( x \, d(\sigma_t \boxplus \nu)(x) \) is equivalent to the unimodality of the push-forward measure \( \log^*(\sigma_t \boxplus \nu) \) by the logarithmic function \( \log : \mathbb{R}^+ \to \mathbb{R} \). We refer to Section 3 for a detailed discussion of this log-unimodality.

Our main results are as follows. We prove in Theorem 4.2 that \( \sigma_t \boxplus \nu \) is log-unimodal for every \( t > 0 \) if the initial distribution \( \nu \) is log-unimodal and symmetric about 1 with respect to the multiplication on \( \mathbb{R}^+ \). In particular, \( \sigma_t \) itself is log-unimodal for all \( t > 0 \). In Theorem 4.4, we show that if \( \nu \) is supported on a closed interval \([\alpha, \beta]\) where \( \beta^4 < 2\alpha^3 \beta + 3\alpha^4 \), then the process \( \sigma_t \boxplus \nu \) becomes log-unimodal for sufficiently large \( t \). The log-unimodality may fail
when \( \nu \) is not symmetric or if \( \nu \) has an unbounded support, see Theorem 4.5 and Example 4.8.

The paper is organized as follows. After recalling basic results of free convolution in Section 2, we introduce and investigate the class of log-unimodal measures in Section 3. The main results are proved in Section 4.

2 Preliminaries

2.1 Free multiplicative convolution

The free multiplicative convolution \( \mu \boxtimes \nu \) of probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}_+ \) is defined to be the distribution of \( X^\frac{1}{2} Y X^\frac{1}{2} \), where \( X, Y \) are free independent, non-negative self-adjoint operators affiliated with a finite von Neumann algebra and having the distributions \( \mu \) and \( \nu \), respectively.

For a probability measure \( \mu \) on \( \mathbb{R}_+ \), we define
\[
\psi_\mu(z) = \int_0^\infty \frac{xz}{1-xz} d\mu(x), \quad z \in \mathbb{C} \setminus [0, +\infty),
\]
and
\[
\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}.
\]

It is shown in [1] that the function \( \eta_\mu \) has an analytic compositional inverse \( \eta_\mu^{-1} \) defined in a neighborhood of \((-\infty, 0)\). Accordingly, we define the \( \Sigma \)-transform of \( \mu \) by
\[
\Sigma_\mu(z) = \frac{\eta_\mu^{-1}(z)}{z}.
\]
The measure \( \mu \) is uniquely determined by its \( \Sigma \)-transform.

For probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}_+ \), their free multiplicative convolution \( \mu \boxtimes \nu \) is determined by
\[
\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z) \Sigma_\nu(z), \quad z \in (-\infty, 0).
\]

2.2 Density function of \( \sigma_t \boxtimes \nu \)

We review Zhong’s density formula [15] as follows. Fix a probability measure \( \nu \) on \( \mathbb{R}_+ \). Define a function \( u_t : (0, \infty) \to [0, \pi) \) and a set \( V_{t,\nu} \) by
\[
\begin{align*}
u_t(r) &= \inf \left\{ \theta \in (0, \pi) : \frac{\sin \theta}{\theta} \int_0^\infty r^\left( \frac{\xi}{\sqrt{1 + r^2 \xi^2 - 2r \xi \cos \theta}} \right) d\nu(\xi) \leq \frac{1}{t} \right\}, \quad r > 0, \\
V_{t,\nu} &= \left\{ r > 0 : \int_0^\infty \frac{r^\xi}{(1-r\xi)^2} d\nu(\xi) > \frac{1}{t} \right\}.
\end{align*}
\]

It is shown in [15] that the function \( u_t \) is continuous on \((0, +\infty)\) and that \( r \in V_{t,\nu} \) if and only if \( \nu_t(r) > 0 \); in which case, \( \nu_t(r) \) is the unique solution \( \theta \in (0, \pi) \) of the equation
\[
\frac{\sin \theta}{\theta} \int_0^\infty \frac{r^\xi}{1 + r^2 \xi^2 - 2r \xi \cos \theta} d\nu(\xi) = \frac{1}{t}.
\]

Moreover, the map
\[
\Lambda_{t,\nu}(r) = r \exp \left( \frac{t}{2} \int_0^\infty \frac{r^2 \xi^2 - 1}{(1-r \xi \cos(\nu_t(r)))^2} d\nu(\xi) \right), \quad r > 0,
\]
is a homeomorphism from \((0, \infty)\) to \((0, \infty)\).

The measure \(\sigma_t \otimes \nu\) is Lebesgue absolutely continuous with a continuous density \(q_t\) given by

\[
xq_t(x) = \frac{u_t \left( \Lambda_{t, \nu}^{-1}(1/x) \right)}{\pi t}, \quad x \in (0, \infty).
\]

(2.1)

Thus, the support of \(\sigma_t \otimes \nu\) is the closure \(\{x > 0 : 1/x \in \Lambda_{t, \nu}(V_{t, \nu})\}\), and the function \(q_t\) is analytic on the set \(\{x > 0 : 1/x \in \Lambda_{t, \nu}(V_{t, \nu})\}\).

### 3 Log-unimodal distributions

#### 3.1 Characterizations of log-unimodality

**Definition 3.1.** A positive Borel measure \(\mu\) on \(\mathbb{R}^+\) is said to be log-unimodal with mode \(a \in \mathbb{R}^+\) if its push-forward \(\log_* \mu\) by the logarithm function \(\log : \mathbb{R}^+ \to \mathbb{R}\) is a unimodal measure with mode \(\log a\) on \(\mathbb{R}\).

**Lemma 3.2.** A positive Borel measure \(\mu\) on \(\mathbb{R}^+\) is log-unimodal with mode \(a \in \mathbb{R}^+\) if and only if the measure \(x \, d\mu(x)\) is unimodal with mode \(a\).

**Proof.** Assume that \(\mu\) is log-unimodal with mode \(a\). Then there are \(c \geq 0\) and a function \(f : \mathbb{R} \to [0, \infty)\) which is non-decreasing on \((-\infty, \log a)\) and non-increasing on \((\log a, \infty)\) such that

\[
(\log_* \mu)(dx) = c\delta_{\log a} + f(x) \, dx, \quad x \in \mathbb{R}.
\]

It follows that

\[
\mu(dx) = c\delta_a + \frac{f(\log x)}{x} \, dx, \quad x > 0,
\]

and therefore,

\[
x \mu(dx) = ca \delta_a + f(\log x) dx, \quad x > 0.
\]

Since the logarithm function \(\log : \mathbb{R}^+ \to \mathbb{R}\) is strictly increasing, the function \(x \mapsto f(\log x)\) is non-decreasing on \((0, a)\) and non-increasing on \((a, \infty)\). It follows that \(x \mu(dx)\) is unimodal with mode \(a > 0\). The converse implication is proved in the same way.

The next result shows that if the distribution of a positive random variable \(X\) is log-unimodal, then so is the distribution of its multiplicative inverse \(X^{-1}\). The proof is similar to that of Lemma 3.2, and the details are left to the interested reader.

**Proposition 3.3.** Let \(a > 0\). Let \(\mu\) be a positive Borel measure on \(\mathbb{R}^+\) and denote \(d\mu^{-1}(x) = d\mu(1/x)\). The following conditions are equivalent.

1. \(\mu\) is log-unimodal with mode \(a\).
2. \(\mu^{-1}\) is log-unimodal with mode \(1/a\).

We now prove that the class of log-unimodal probability measures is weakly closed. Note that the family of unimodal probability measures on \(\mathbb{R}\) is weakly closed, see [13].

**Lemma 3.4.** The set of log-unimodal probability measures is closed with respect to weak convergence of probability measures on \(\mathbb{R}^+\).

**Proof.** Let \(\{\mu_n\}_{n \in \mathbb{N}}\) be a sequence of log-unimodal probability measures on \(\mathbb{R}^+\). Assume that there is a probability measure \(\mu\) on \(\mathbb{R}^+\) such that \(\mu_n \Rightarrow \mu\) as \(n \to \infty\). By the continuous mapping theorem, the push-forward measures \(\log_* \mu_n\) converge weakly to \(\log_* \mu\), and therefore the limit distribution \(\log_* \mu\) is unimodal on \(\mathbb{R}\). By definition, this means that \(\mu\) is log-unimodal.  

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We characterize log-unimodal distributions by their moment generating functions. The next result is a generalization of Isii’s characterization [11, Theorem 3.2] for unimodal probability measures on $\mathbb{R}$.

**Theorem 3.5.** Let $\tau$ be a positive Borel measure on $\mathbb{R}$ such that

$$
\int_{\mathbb{R}} \frac{\tau(dx)}{1 + x^2} < \infty.
$$

We define the associated Pick function

$$
P_{\tau}(z) = \int_{\mathbb{R}} \frac{1 + xz}{(x - z)(1 + x^2)} \tau(dx), \quad z \in \mathbb{C}^+.
$$

Then $\tau$ is unimodal with mode $c \in \mathbb{R}$ if and only if

$$
\Im[(z - c) P'_{\tau}(z)] \leq 0, \quad z \in \mathbb{C}^+.
$$

**Proof.** The “only if” part is easier to prove. Thus, assume that $\tau$ is unimodal and observe that $\tau_n(dx) = 1_{[-n,n]}(x)\tau(dx)$ is a unimodal measure on $\mathbb{R}$ with mode $c$ for each positive integer $n > |c|$, and that

$$
P_{\tau_n}(z) = \int_{\mathbb{R}} \left( \frac{1}{x - z} - \frac{x}{1 + x^2} \right) \tau_n(dx) = - \int_{\mathbb{R}} \frac{x}{1 + x^2} \tau_n(dx) + \int_{\mathbb{R}} \frac{1}{x - z} \tau_n(dx).
$$

Isii’s result [11, Theorem 3.2] implies that

$$
\Im[(z - c) P'_{\tau_n}(z)] \leq 0, \quad z \in \mathbb{C}^+.
$$

Since $P_{\tau_n}$ converges to $P_{\tau}$ pointwisely in $\mathbb{C}^+$, the desired inequality follows.

The “if” part needs a more detailed analysis. Thus, we assume $\Im[(z - c) P'_{\tau}(z)] \leq 0$ for all $z \in \mathbb{C}^+$.

Since $P_{\tau_n}$ converges to $P_{\tau}$ pointwisely in $\mathbb{C}^+$, the desired inequality follows.

The “if” part needs a more detailed analysis. Thus, we assume $\Im[(z - c) P'_{\tau}(z)] \leq 0$ for all $z \in \mathbb{C}^+$. We also assume $c = 0$; the general case follows by a translation of $\tau$ and $P_{\tau}$ by the amount of $c$. In what follows, the functions log and arg are defined continuously on $\mathbb{C} \setminus i(-\infty, 0]$ such that $\log 1 = 0$ and $\arg 1 = 0$.

Since $Q(z) = -z P'_{\tau}(z)$ is a Pick function defined for $z \in \mathbb{C}^+$, there exist $a \geq 0$, $b \in \mathbb{R}$, and a finite Borel measure $\rho$ such that

$$
Q(z) = az + b + \int_{\mathbb{R}} \frac{1 + xz}{x - z} d\rho(x), \quad z \in \mathbb{C}^+.
$$

(*3.1*)

**Step 1.** Fix $0 < \delta < \kappa < \infty$ such that $\rho$ is continuous at $\delta$ and $\kappa$, and define

$$
F(t) = \int_{[\delta, t]} \frac{1 + x^2}{x} d\rho(x), \quad t \in [\delta, \kappa].
$$

We will prove that there exists a continuous function $E$ on $\mathbb{C}^+ \cup (\delta, \kappa)$ such that $\Im[E(z)]$ is a constant for all $z \in (\delta, \kappa)$ and

$$
P_{\tau}(z) = \int_{\delta}^{\kappa} \frac{F(x)}{z - x} dx + E(z), \quad z \in \mathbb{C}^+.
$$

(*3.2*)

To this end, we first notice that

$$
P'_{\tau}(z) = -a - \frac{b}{z} + \int_{\mathbb{R}} \frac{1 + xz}{z(z - x)} d\rho(x).
$$

Integrating the both sides from $i$ to $z$ with Fubini’s theorem yields

$$
P_{\tau}(z) = \gamma - az - b \log z + \int_{\mathbb{R}} k(x, z) d\rho(x), \quad z \in \mathbb{C}^+,
$$

(*4*)
where \( \gamma = P_r(i) + ai + b \log i \) and
\[
k(x, z) = \int_i^z \frac{1 + xw}{w(w - x)} \, dw, \quad x \in \mathbb{R}, \quad z \in (\mathbb{C}^+ \cup \mathbb{R}) \setminus \{0\}, \ x \neq z,
\]
\[
= \int_i^z \left( -\frac{1}{xw} + \frac{1 + x^2}{x(w - x)} \right) \, dw
\]
\[
= \frac{1 + x^2}{x} \left[ \log(z - x) - \log(i - x) - \frac{1}{1 + x^2}(\log z - \log i) \right]
\]
denoted by \( \ell(x, z) \) if \( x \neq 0 \).

Now \( P_r \) is of the form
\[
P_r(z) = \int_\delta^\kappa k(x, z) \, d\rho(x) + E_0(z), \quad z \in \mathbb{C}^+,
\]
where \( E_0 \) is the continuous function on \( \mathbb{C}^+ \cup (\delta, \kappa) \) defined by
\[
E_0(z) = \gamma - az - b \log z + \int_{(-\infty, \delta)} k(x, z) \, d\rho(x) + \int_{(\kappa, \infty)} k(x, z) \, d\rho(x).
\]
By integration by parts we obtain
\[
\int_\delta^\kappa k(x, z) \, d\rho(x) = \int_\delta^\kappa \ell(x, z) \, dF(x) = [\ell(x, z) F(x)]^z_{x=\delta} - \int_\delta^\kappa \partial_x \ell(x, z) \, F(x) \, dx
\]
\[
= \ell(\kappa, z) F(\kappa) + \int_\delta^\kappa \left[ \frac{1}{i - x} - \frac{1}{1 + x^2} \cdot \frac{x(2 \log z - \pi i)}{(1 + x^2)^2} \right] F(x) \, dx.
\]

In summary, the function \( P_r \) is of the desired form \( \ell(x, z) \) where \( E \) is defined by
\[
E(z) = E_0(z) + \ell(\kappa, z) F(\kappa) - \int_\delta^\kappa \left[ \frac{1}{i - x} + \frac{x(2 \log z - \pi i)}{(1 + x^2)^2} \right] F(x) \, dx.
\]
Moreover, \( \Re[E(z)] \) and \( \Re[E_0(z)] \) are constant functions on \( (\delta, \kappa) \), because
\[
\Re[\ell(x, z)] = \arg(z - x) - \arg(i - x) + \frac{\pi}{2(1 + x^2)}
\]
\[
= \begin{cases} 
\pi - \arg(i - x) + \frac{\pi}{2(1 + x^2)}, & x \geq \kappa, \\
- \arg(i - x) + \frac{\pi}{2(1 + x^2)}, & x \leq \delta
\end{cases}
\]
is a constant function for \( z \in (\delta, \kappa) \).

**Step 2.** We apply the Stieltjes inversion formula to \( \ell(x, z) \). For convenience, we denote by \( T \) the Stieltjes transform of the measure \( F(x) 1_{[a, b]}(x) \, dx \) and write \( P_r(z) = T(z) + E(z) \). For two continuity points \( s, t \in (\delta, \kappa) \) of \( \tau \) with \( s < t \), we have
\[
\tau([s, t]) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_s^t \Re[P_r(x + i\varepsilon)] \, dx
\]
\[
= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_s^t \Re[T(x + i\varepsilon)] \, dx + \frac{1}{\pi} \int_s^t \Re[E(x)] \, dx
\]
\[
= \int_s^t (\gamma - F(x)) \, dx,
\]
where \( \gamma = \pi^{-1} \Re[E(x)] \) is a constant, possibly depending on \( \delta \) and \( \kappa \). Therefore, \( \tau \) is Lebesgue absolutely continuous on \( (\delta, \kappa) \) with density \( p \) given by
\[
p(x) = \gamma - F(x) = \gamma - \int_\delta^x \frac{1 + y^2}{y} \, d\rho(y), \quad x \in (\delta, \kappa).
\]
This shows that \( p \) is non-increasing on \( (\delta, \kappa) \). Letting \( \delta \to 0 \) and \( \kappa \to \infty \), we conclude that \( p \) is non-increasing on \( (0, \infty) \). Similarly, we can prove that \( \tau \) has a non-decreasing density \( p \) on \( (-\infty, 0) \) as well, and hence \( \tau \) is unimodal with mode 0. \( \square \)
Remark 3.6. The formula (3.8) below shows a finer relationship among \( \tau, \rho \) and \( p \). To prove it, we first notice that the constant \( \gamma \) is actually independent of \( \kappa \) by virtue of (3.3).

Since \( p \) is non-increasing and nonnegative on \((0, \infty)\), the limit \( \alpha = \lim_{x \to \infty} p(x) \) exists in \([0, \infty)\), implying further that \( \int_\delta^\infty \frac{1+y^2}{y} d\rho(y) < \infty \). If we take \( p \) to be right-continuous, we obtain

\[
p(x) = \alpha + \int_{(x, \infty)} \frac{1+y^2}{y} d\rho(y), \quad x > 0. \tag{3.4}
\]

Similarly, we have

\[
p(x) = \beta + \int_{(-\infty,x]} \frac{1+y^2}{|y|} d\rho(y), \quad x < 0 \tag{3.5}
\]

for some constant \( \beta \geq 0 \) and

\[
\int_\mathbb{R} |y| d\rho(y) < \infty. \tag{3.6}
\]

The relation \( -zP'_\tau(z) = Q(z) \) and the dominated convergence theorem yield

\[
\tau(\{0\}) = \lim_{\epsilon \to 0^+, z = i\epsilon} z^2 P'_\tau(z) = -\lim_{\epsilon \to 0^+, z = i\epsilon} zQ(z) = \rho(\{0\}). \tag{3.7}
\]

So we have

\[
\tau(dx) = p(x) \, dx + \rho(\{0\}) \delta_0. \tag{3.8}
\]

Conversely, given a finite Borel measure \( \rho \) on \( \mathbb{R} \) with \( \int_\mathbb{R} |y| d\rho(y) < \infty \) and given two constants \( \alpha, \beta \geq 0 \), we define a measure \( \tau \) by (3.8) where \( p \) is defined through (3.4) – (3.5), then \( \tau \) is unimodal with mode 0 such that \( \int_\mathbb{R} \frac{1}{x^2} d\tau(x) < \infty \).

Applying integration by parts to \( -zP'_\tau(z) \) together with formulas (3.4) – (3.8) leads to

\[
Q(z) = -zP'_\tau(z) = \alpha - \beta + \int_\mathbb{R} \frac{1+x^2}{x-z} d\rho(x),
\]

which means \( a = 0 \) and \( b = \alpha - \beta + \int_\mathbb{R} x d\rho(x) \) in (3.1).

**Proposition 3.7.** A probability measure \( \mu \) on \( \mathbb{R}_+ \) is log-unimodal with mode \( c \in \mathbb{R}_+ \) if and only if the following inequality holds:

\[
\exists \left[ z(1-cz)\psi'_\mu(z) \right] \geq 0, \quad z \in \mathbb{C}^+. \tag{3.9}
\]

**Proof.** By Lemma 3.2, \( \mu \) is log-unimodal with mode \( c \) if and only if the measure \( \tau(dx) = x \, d\mu(x) \) on \( \mathbb{R}_+ \) is unimodal with mode \( c \). The result follows from Theorem 3.5 and the fact

\[
P'_\tau(z) = \int_{\mathbb{R}_+} \frac{x}{(z-x)^2} d\mu(x) = \frac{1}{z^2} \psi'_\mu \left( \frac{1}{z} \right), \quad z \in \mathbb{C} \setminus [0, +\infty).
\]

**Remark 3.8.** This is to be compared with a similar characterization in the unitary case [2] Theorem 7.16.

**Example 3.9.** The following distributions are log-unimodal.

1. (Half normal distribution) If \( t > 0 \) and \( X \sim \mathcal{N}(0,t) \), then the distribution \( \rho_t \) of \( |X| \) is called a half normal distribution and its density function is given by

\[
\frac{2}{\sqrt{2\pi} t} \exp \left( -\frac{x^2}{2t^2} \right) 1_{\mathbb{R}_+}(x).
\]

The measure \( x \rho_t(dx) \) is unimodal with mode \( \sqrt{t} \). Therefore the half normal distribution \( \rho_t \) is log-unimodal with mode \( \sqrt{t} \).
(2) (Gamma distribution) The probability measure
\[ \gamma_{\theta,p}(dx) = \frac{\theta^{-p}}{\Gamma(p)} x^{p-1} e^{-x/\theta} 1_{\mathbb{R}_+}(x) \, dx, \quad p, \theta > 0, \]
is log-unimodal with mode \( p\theta \). Furthermore the inverse gamma distribution is also log-unimodal by Proposition 3.3.

(3) (Beta distribution) The probability measure
\[ \beta_{p,q}(dx) = \frac{1}{B(p,q)} x^{p-1} (1-x)^{q-1} 1_{(0,1)}(x) \, dx, \quad p, q > 0, \]
is log-unimodal with mode \( s \), where
\[ s = \begin{cases} \frac{p}{p+q-1}, & p > 0, \quad q > 1, \\ 1, & p > 0, \quad 0 < q \leq 1. \end{cases} \]

(4) (Marchenko-Pastur distribution) The probability measure
\[ \pi(dx) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} 1_{(0,4]}(x) \, dx, \]
is log-unimodal with mode 2. It is known that the measure \( \pi^{-1} \) is the positive free stable laws with index 1/2. By Proposition 3.3, the measure \( \pi^{-1} \) is log-unimodal with mode 1/2.

(5) (Positive Boolean stable laws with index \( \alpha \in (0,1) \)) The probability measure
\[ b_\alpha(dx) = \frac{\sin \alpha \pi}{\pi} \frac{x^{\alpha-1}}{x^{2\alpha} + 2x^\alpha \cos \alpha + 1} 1_{(0,\infty)}(x) \, dx, \]
is log-unimodal with mode 1.

3.2 Log-unimodality under classical multiplicative convolution

For two probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}_+ \), their classical multiplicative convolution \( \mu \ast \nu \) is the distribution of \( XY \), where \( X \) and \( Y \) are independent positive random variables distributed according to \( \mu \) and \( \nu \). In this section we study the log-unimodality of measures under the convolution \( \ast \).

We first look at symmetric measures. Recall that a Borel measure \( \mu \) on \( \mathbb{R}_+ \) is said to be multiplicatively symmetric if \( \mu = \mu^{-1} \). The free Brownian motion \( \sigma_t \) is multiplicatively symmetric for all \( t > 0 \).

A measure \( \nu \) on \( \mathbb{R} \) is said to be additively symmetric if the push-forward measure \( \exp_*(\nu) \) by the exponential map \( e^z : \mathbb{R} \to \mathbb{R}_+ \) is multiplicatively symmetric. In other words, \( \nu \) being additively symmetric means that the mass distribution of \( \nu \) is symmetric with respect to the origin.

Since the exponential map turns the classical additive convolution \( \ast \) to the multiplicative convolution \( \otimes \) and the additively symmetric unimodal probability measures on \( \mathbb{R} \) are preserved by the convolution \( \ast \) (see [13, Exercise 29.22] or the original article of Wintner [14, Theorem XIII]), we obtain the following result.

Proposition 3.10. If \( \mu \) and \( \nu \) are multiplicatively symmetric log-unimodal probability measures on \( \mathbb{R}_+ \), then \( \mu \ast \nu \) is multiplicatively symmetric and log-unimodal.

Next, we consider the so-called strong unimodality. A probability measure \( \mu \) on \( \mathbb{R}_+ \) is said to be \( \otimes \)-strongly log-unimodal if for all log-unimodal distributions \( \nu \), the convolution \( \mu \oplus \nu \) is log-unimodal. The notion of strong unimodality relative to other convolutions is defined analogously. By virtue of the exponential map again, the next result follows immediately.
**Proposition 3.11.** A probability measure \( \mu \) on \( \mathbb{R}^+ \) is \( \bigcirc \)-strongly log-unimodal if and only if \( \log_* \mu \) is \( \ast \)-strongly unimodal on \( \mathbb{R} \).

**Example 3.12.** The log-normal distribution is \( \bigcirc \)-strongly log-unimodal since the normal distribution is \( \ast \)-strongly unimodal.

For \( b \in (0, \pi) \), we define the following probability measure:

\[
\lambda_b(dx) = \frac{c_b}{1 - 2x \cos b + x^2} dx, \quad x \in \mathbb{R}^+,
\]

where \( c_b = \sin b / (\pi - b) \) is a normalizing constant such that \( \lambda_b(\mathbb{R}^+) = 1 \). By examining its density directly, it is easy to see that \( \lambda_b \) is multiplicatively symmetric and log-unimodal with mode 1 for all \( b \in (0, \pi) \).

**Lemma 3.13.** The measure \( \lambda_b \) is \( \bigcirc \)-strongly log-unimodal on \( \mathbb{R}^+ \) if and only if \( \cos b \leq 0 \).

**Proof.** The density function \( f(x) = \frac{d \log_* \lambda_b}{dx}(x) \) is easily seen to be

\[
f(x) = \frac{c_b e^x}{1 - 2e^x \cos b + e^{2x}}, \quad x \in \mathbb{R}.
\]

Define the function \( g(x) = \log f(x) \). Then we have

\[
g''(x) = \frac{2e^x (\cos b - 2e^x + e^{2x} \cos b)}{(1 - 2e^x \cos b + e^{2x})^2}.
\]

One can see that \( g'' \leq 0 \) on \( \mathbb{R} \) if and only if \( \cos b \leq 0 \). Thus, the function \( g \) is concave downward on \( \mathbb{R} \) if and only if \( \cos b \leq 0 \). A result of Ibragimov \cite{10} shows that the concavity of \( g \) is equivalent to the \( \ast \)-strong unimodality of \( \log_* \lambda_b \), whence the desired result follows from Proposition 3.11. \( \square \)

### 4 Main results

We begin with a criterion for the log-unimodality of \( \sigma_t \boxtimes \nu \). Recall that \( q_t \) denotes the density function of \( \sigma_t \boxtimes \nu \) in Zhong’s formula.

**Lemma 4.1.** Let \( t > 0 \). The following conditions are equivalent.

1. The measure \( \sigma_t \boxtimes \nu \) is log-unimodal.
2. For each \( a \in (0, 1/t) \), the equation

\[
\frac{r}{c_{ant}} \cdot \frac{d(\lambda_{ant} \boxtimes \nu^{-1})}{dx}(r) = \frac{a \pi}{\sin(a \pi t)} \sin(a \pi t)
\]

about \( r \) has at most two solutions in \( \mathbb{R}^+ \). Recall that the measure \( \lambda_{ant} \) and the normalization constant \( c_{ant} \) are defined in Section 3.2 and \( d\nu^{-1}(x) = d\nu(1/x) \).

3. For each \( R \in (0, \pi) \), the equation

\[
\frac{\sin R}{R} \int_0^\infty \frac{r \xi}{1 + r^2 \xi^2 - 2r \xi \cos R} d\nu(\xi) = \frac{1}{t}
\]

about \( r \) has at most two solutions in \( \mathbb{R}^+ \).

**Proof.** This proof is similar to \cite{7 Lemma 3.1], and we present it here for the sake of completeness.

The equivalence between (1) and (2) is obvious. We show that the measure \( \sigma_t \boxtimes \nu \) is log-unimodal if and only if for each \( a \in (0, 1/t) \), the equation

\[
\int_0^\infty \frac{r \xi}{1 + r^2 \xi^2 - 2r \xi \cos(a \pi t)} d\nu(\xi) = \frac{a \pi}{\sin(a \pi t)}
\]

(4.1)
in $r$ has at most two solutions in $\mathbb{R}_+$. The latter condition is easily seen to be equivalent to (3) from the substitution $R = a\pi t$.

Assume that $\sigma_t \boxtimes \nu$ is log-unimodal. By Zhong’s formula \eqref{eq:zhong}, we have

$$xq_t(x) = \frac{u_t(\Lambda_{t,\nu}^{-1}(1/x))}{\pi t}.$$

By Lemma 3.2, the measure $xq_t(x)dx$ is unimodal. Since the continuous density $xq_t(x)$ is analytic whenever it is positive, the graph of $xq_t(x)$ has no plateau above the real line. It follows that for each $a \in (0, 1/t)$, the equation $u_t(\Lambda_{t,\nu}^{-1}(1/x)) = a\pi t$ in $x$ has at most two solutions in $\mathbb{R}_+$, that is, the equation $u_t(r) = a\pi t$ in $r$ has at most two solutions in $\mathbb{R}_+$. Finally, since

$$\frac{\sin(u_t(r))}{u_t(r)} \int_0^\infty \frac{r\xi}{1 + r^2\xi^2 - 2r\xi \cos(u_t(r))}d\nu(\xi) = \frac{1}{t},$$

we conclude that the equation \eqref{eq:nut} has at most two solutions in $\mathbb{R}_+$. Notice that there is no need to investigate the solvability of $u_t(\Lambda_{t,\nu}^{-1}(1/x)) = a\pi t$ when $a \geq 1/t$, because the angle $u_t < \pi$.

Conversely, suppose now \eqref{eq:nut} has at most two solutions. In order to derive a contradiction, we assume that $\sigma_t \boxtimes \nu$ is not log-unimodal. This means that there exists $a \in (0, 1/t)$ such that the equation $xq_t(x) = a$ has at least three distinct solutions $x_1$, $x_2$, and $x_3$. We put $r_i = \Lambda_{t,\nu}^{-1}(1/x_i)$ and deduce that the equation $u_t(r_i) = a\pi t$ holds for $i = 1, 2, 3$. It follows that the equation \eqref{eq:nut} has solution $r = r_1, r_2, r_3$, a contradiction. Therefore the measure $\sigma_t \boxtimes \nu$ must be log-unimodal.

We next address the symmetry and log-unimodality of $\sigma_t \boxtimes \nu$. The identity $(X^{\frac{1}{2}}YX^{\frac{1}{2}})^{-1} = X^{-\frac{1}{2}}Y^{-1}X^{-\frac{1}{2}}$ for invertible non-negative self-adjoint operators $X, Y$ affiliated with a finite von Neumann algebra readily shows that $(\mu \boxtimes \nu)^{-1} = \mu^{-1} \boxtimes \nu^{-1}$, so that the free convolution of two multiplicatively symmetric measures is multiplicatively symmetric. In particular, $\sigma_t \boxtimes \nu$ is multiplicatively symmetric whenever $\nu$ is.

Our first result is a free analogue of Proposition 3.9 in the context of free Brownian motion. Yet, interestingly enough, the proof replies on Proposition 3.9.

**Theorem 4.2.** If $\nu$ is multiplicatively symmetric and log-unimodal, then so are $\sigma_t \boxtimes \nu$ for all $t > 0$. In particular, the measure $\sigma_t$ itself is log-unimodal.

**Proof.** Fix $t > 0$. Since $\lambda_{a\pi t}$ and $\nu^{-1}$ are log-unimodal and multiplicatively symmetric, and since the density of $\sigma_t \boxtimes \nu$ is continuous without having plateaux, Proposition 3.9 implies that the measure $\lambda_{a\pi t} \circledast \nu^{-1}$ is log-unimodal and hence

$$\# \left\{ r > 0 : r \cdot \frac{d(\lambda_{a\pi t} \circledast \nu^{-1})}{dx}(r) = \frac{a\pi}{\sin(a\pi t)} \right\} \leq 2, \quad a \in (0, 1/t).$$

By Lemma 4.1, the measure $\sigma_t \boxtimes \nu$ is log-unimodal.

If we take $\nu = \delta_1$, then $\sigma_t = \sigma_t \boxtimes \delta_1$ is log-unimodal. \hfill \Box

**Problem 4.3.** If $\mu$ and $\nu$ are multiplicatively symmetric and log-unimodal on $\mathbb{R}_+$, is the free convolution $\mu \boxtimes \nu$ log-unimodal?

We next show the eventual log-unimodality of $\sigma_t \boxtimes \nu$ when $\nu$ is supported on a suitable finite interval.

**Theorem 4.4.** Let $\nu$ be a probability measure supported on $[\alpha, \beta]$, where $0 < \alpha < \beta$ such that $\beta^4 - 3\alpha^4 < 2\alpha^3 \beta$. If

$$t \geq D_{\alpha, \beta} = \frac{2\beta^2(\alpha + \beta)^2\pi}{\sqrt{4\alpha^6\beta^2 - (3\alpha^4 - \beta^4)^2}},$$

then $\sigma_t \boxtimes \nu$ is log-unimodal.
Proof. We set
\[ \Theta_R(r) = \frac{\sin R}{R} \int_{0}^{\infty} \frac{r\xi}{1 + r^2\xi^2 - 2r\xi \cos R} d\nu(\xi) \]
for our purpose. We aim to prove that for each \( R \in (0, \pi) \) and \( t \geq D_{\alpha,\beta} \), the equation \( \Theta_R(r) = \frac{1}{t} \) has at most two solutions. This suffices thanks to Lemma [11]. Note that
\[ \Theta'_R(r) = \frac{\sin R}{R} \int_{\alpha}^{\beta} \frac{\xi(1 - r^2\xi^2)}{(1 - 2r\xi \cos R + r^2\xi^2)^2} d\nu(\xi), \]
\[ \Theta''_R(r) = \frac{\sin R}{R} \int_{\alpha}^{\beta} \frac{2\xi^2(3\xi^3 - 3r\xi + 2\cos R)}{(1 - 2r\xi \cos R + r^2\xi^2)^3} d\nu(\xi). \]
For all \( 0 < r < 1/\beta \) and \( \xi \in [\alpha, \beta] \), we have \( 1 - r^2\xi^2 > 0 \) and hence \( \Theta'_R(r) > 0 \). Similarly, we have \( \Theta'_R(r) < 0 \) if \( r > 1/\alpha \). So, the function \( \Theta_R \) is strictly monotonic on \( \mathbb{R}_+ \setminus [1/\beta, 1/\alpha] \), and therefore the equation \( \Theta_R(r) = \frac{1}{t} \) can only have at most two solutions on this complement.

We next consider \( r \in [1/\beta, 1/\alpha] \) and distinguish two cases according to whether \( \cos R \leq \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \) or not.

Case I: \( \cos R \leq \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \). For each \( \xi \in [\alpha, \beta] \) and \( 1/\beta < r < 1/\alpha \), we have
\[ r^3\xi^3 - 3r\xi + 2\cos R < \frac{\beta^3}{\alpha^3} - 3\frac{\alpha}{\beta} + 2\cos R = 2 \left( \frac{\beta^4 - 3\alpha^4}{2\alpha^3\beta} + \cos R \right) \leq 0, \]
showing that \( \Theta''_R(r) < 0 \). This means that the function \( \Theta_R \) is concave down and the derivative \( \Theta'_R \) is strictly decreasing on the interval \([1/\beta, 1/\alpha]\). Then the intermediate value theorem shows that the derivative \( \Theta'_R \) has a unique zero \( p \) in the closed interval \([1/\beta, 1/\alpha]\), and hence the critical point \( p \) is the unique local maximizer for the function \( \Theta_R \) in \([1/\beta, 1/\alpha]\). This analysis on the graph of \( \Theta_R \) shows that there are at most two solutions for \( \Theta_R(r) = \frac{1}{t} \) in this case.

Case II: \( \cos R > \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \). In this case we have
\[ \sin \frac{R}{R} > \sin \left( \frac{\cos^{-1} \left( \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \right)}{\pi} \right) = \frac{1}{\pi} \sqrt{1 - \left( \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \right)^2}, \quad \frac{\pi}{2} < R < \pi, \]
and
\[ \sin \frac{R}{R} > \frac{2}{\pi} > \frac{1}{\pi} \sqrt{1 - \left( \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \right)^2}, \quad 0 < R < \frac{\pi}{2}. \]
On the other hand, observe that
\[ 1 - 2r\xi \cos R + r^2\xi^2 \leq 1 + 2r\xi + r^2\xi^2 = (1 + r\xi)^2 \leq \left( 1 + \frac{\beta}{\alpha} \right)^2 = \frac{(\alpha + \beta)^2}{\alpha^2}, \quad \xi \in [\alpha, \beta]. \]
It follows that
\[ \Theta_R(r) > \frac{1}{\pi} \sqrt{1 - \left( \frac{3\alpha^4 - \beta^4}{2\alpha^3\beta} \right)^2} \times \frac{\alpha^2}{(\alpha + \beta)^2} \times \frac{\alpha}{\beta} = \frac{1}{D_{\alpha,\beta}} \geq \frac{1}{t} \]
for all \( r \in [1/\beta, 1/\alpha] \) and \( R \in (0, \pi) \). This shows that the equation \( \Theta_R(r) = \frac{1}{t} \) has no solutions in \([1/\beta, 1/\alpha]\).

In all cases, we have shown that for all \( R \in (0, \pi) \), the equation \( \Theta_R(r) = \frac{1}{t} \) has at most two solutions if \( t \geq D_{\alpha,\beta} \).

We follow the ideas in [9, 7] to construct probability measures \( \nu \) such that (i) the masses of \( \nu \) escape to either 0 or \( +\infty \), and (ii) \( \sigma_t \otimes \nu \) is not log-unimodal for any \( t > 0 \).

Theorem 4.5. Let \( \{w_n\}_{n \in \mathbb{N}} \) and \( \{a_n\}_{n \in \mathbb{N}} \) be two sequences in \( \mathbb{R}_+ \) such that
• $a_n > a_{n+1}$ for all $n \in \mathbb{N}$,
• $\lim_{k \to \infty} a_k = 0$ and $\lim_{k \to \infty} \frac{a_k a_{k+1}(a_k + a_{k+1})}{(a_k - a_{k+1})^2} = 0$,
• $\sum_{n=1}^{\infty} w_n = 1$ and $\sum_{n=1}^{\infty} w_n a_n^{-1} < \infty$.

Let $\nu = \sum_{n=1}^{\infty} w_n \delta_{a_n}$. Then both $\sigma_t \otimes \nu$ and $\sigma_t \otimes \nu^{-1}$ are not log-unimodal for every $t > 0$.

Proof. Define the function

$$f(r) = \int_0^\infty \frac{rx}{(1-rx)^2}d\nu(x) = \sum_{n=1}^{\infty} \frac{w_n a_n^r}{(1-a_n r)^2}, \quad r > 0.$$ 

We set

$$b_k = \frac{a_{k+1}^{-1} + a_k^{-1}}{2}, \quad k \in \mathbb{N},$$

and observe that

$$|1-a_n b_k| \geq \frac{a_n}{2} (a_{k+1}^{-1} + a_k^{-1}), \quad n, k \in \mathbb{N}.$$ 

As $k \to \infty$, we have

$$f(b_k) = \sum_{n=1}^{\infty} \frac{w_n a_n b_k}{(1-a_n b_k)^2} \leq \frac{4b_k}{(a_{k+1}^{-1} + a_k^{-1})^2} \sum_{n=1}^{\infty} w_n a_n^{-1} = \frac{2a_k a_{k+1}(a_k + a_{k+1})}{(a_k - a_{k+1})^2} \sum_{n=1}^{\infty} w_n a_n^{-1} \to 0.$$ 

Recall that $V_{t,\nu} = \{r \in \mathbb{R}_+ : f(r) > 1/t\} = \{r \in \mathbb{R}_+ : u_t(r) > 0\}$. The above limit implies that for each $t > 0$, there exists an integer $K(t) > 0$ such that $f(b_k) < 1/t$ for all $k \geq K(t)$. So, for $k \geq K(t)$, the closure of $V_{t,\nu}$ does not contain $b_k$. Represent the open set $V_{t,\nu}$ as a disjoint union of open intervals, we conclude that the closure $\overline{V_{t,\nu}}$ contains at least two disjoint closed intervals. (None of these two intervals is a singleton set, because the function $u_t$ is continuous.) Therefore, under the homeomorphism $1/\Lambda_{t,\nu}$, the support $\text{supp}(\sigma_t \otimes \nu) = (1/\Lambda_{t,\nu})(\overline{V_{t,\nu}})$ contains two disjoint closed intervals. It follows that the support $\text{supp}(\log_+ (\sigma_t \otimes \nu)) = \log(\text{supp}(\sigma_t \otimes \nu))$ of the push-forward measure $\log_+ (\sigma_t \otimes \nu)$ also contains two disjoint closed intervals, so that the continuous density $d\log_+ (\sigma_t \otimes \nu)/dx$ vanishes in between these two intervals. We conclude that $\log_+ (\sigma_t \otimes \nu)$ is not unimodal, that is, $\sigma_t \otimes \nu$ is not log-unimodal.

Since $\sigma_t \otimes \nu^{-1} = (\sigma_t \otimes \nu)^{-1}$, Proposition 3.3 implies that $\sigma_t \otimes \nu^{-1}$ is not log-unimodal for any $t > 0$. \qed

Remark 4.6. The proof of the preceding result actually shows that $\sigma_t \otimes \nu$ is not unimodal.

Problem 4.7. Can we delete the assumption $\beta^4 - 3\alpha^4 < 2\alpha^3 \beta$ in Theorem 4.4? More precisely, if $\nu$ is a probability measure supported on $[\alpha, \beta]$ such that $0 < \alpha < \beta$, does there exist a number $D \geq 0$ (depending on $\nu$) such that $\sigma_t \otimes \nu$ is log-unimodal for all $t \geq D$?

Now, it is fairly easy to construct counterexamples of Theorem 4.4 when $\nu$ has an unbounded support.

Example 4.8. If $w_n = \frac{945}{\pi^6 n^6}$ and $a_n = n^{-4}$ for all $n \in \mathbb{N}$, then $a_n \leq 1$ and

$$\lim_{k \to \infty} \frac{a_k a_{k+1}(a_k + a_{k+1})}{(a_k - a_{k+1})^2} = 0.$$
Moreover, we have
\[ \sum_{n \geq 1} \frac{w_n}{a_n} = \frac{945}{\pi^6} \sum_{n \geq 1} \frac{1}{n^2} = \frac{945}{\pi^6} \times \frac{\pi^2}{6} = \frac{315}{2} < \infty. \]

Theorem 4.5 shows that both
\[ \sigma_t \boxtimes \left( \frac{945}{\pi^6} \sum_{n=1}^{\infty} \frac{1}{n^6} \delta_{n+4} \right) \]
and \[ \sigma_t \boxtimes \left( \frac{945}{\pi^6} \sum_{n=1}^{\infty} \frac{1}{n^6} \delta_{n+4} \right) \]
are not log-unimodal for any \( t > 0 \).

We conclude this paper with counterexamples of Theorem 4.2 when \( \nu \) is not multiplicatively symmetric. Recall that a probability measure \( \mu \) on \( \mathbb{R}_+ \) is said to be \( \boxtimes \)-strongly log-unimodal if for all log-unimodal distributions \( \nu \), the free convolution \( \mu \boxtimes \nu \) is log-unimodal.

**Theorem 4.9.** There exists \( t_0 > 0 \) such that \( \sigma_t \) is not \( \boxtimes \)-strongly log-unimodal for any \( t \in (0, t_0] \).

**Proof.**

**Step 1.** We show that \( \sigma_{t_1} \) is not \( \boxtimes \)-strongly log-unimodal for some \( t_1 > 0 \), that is, there are \( t_1 > 0 \) and a log-unimodal measure \( \nu \) such that \( \sigma_{t_1} \boxtimes \nu \) is not log-unimodal.

By Lemma 3.13, there is a log-unimodal distribution \( \nu \) such that \( \lambda_1 \boxtimes \nu^{-1} \) is not log-unimodal, where
\[ \lambda_1(dx) = \frac{c_1}{1 - 2x \cos 1 + x^2} dx, \quad x \in \mathbb{R}_+. \]

It follows that there exists \( d > 0 \) such that
\[ \# \left\{ r > 0 : r \cdot \frac{d(\lambda_1 \boxtimes \nu^{-1})}{dx}(r) = d \right\} \geq 3. \]

Now take \( a > 0 \) and \( t_1 > 0 \) such that
\[ a \pi t_1 = 1, \quad \frac{a \pi}{\sin(a \pi t_1)} = \frac{d}{c_1}. \]

Indeed, one can choose
\[ a = \frac{d \sin 1}{c_1 \pi}, \quad t_1 = \frac{c_1}{d \sin 1}. \]

In view of Lemma 4.1, the measure \( \sigma_{t_1} \boxtimes \nu \) is not log-unimodal.

**Step 2.** Let
\[ I = \{ t \in (0, \infty) : \sigma_t \text{ is not } \boxtimes \text{-strongly log-unimodal} \}. \]

We claim that if \( t \in I \) and \( s \in (0, t) \), then one has either \( s \in I \) or \( t - s \in I \). To see this, we take a log-unimodal distribution \( \mu \) such that \( \sigma_t \boxtimes \mu \) is not log-unimodal, and we observe the identity
\[ \sigma_s \boxtimes (\sigma_{t-s} \boxtimes \mu) = \sigma_t \boxtimes \mu. \] (4.2)

If \( \sigma_{t-s} \boxtimes \mu \) is not log-unimodal then \( t - s \in I \). If \( \sigma_{t-s} \boxtimes \mu \) is log-unimodal then \( s \in I \) because of (4.2).

**Step 3.** Let
\[ t_2 = \max \{ t \in [0, t_1] : \sigma_t \text{ is } \boxtimes \text{-strongly log-unimodal} \}. \]

Note that the above subset of \([0, t_1]\) is closed by Lemma 3.4. Also, it is non-empty since \( \sigma_0 = \delta_1 \) is \( \boxtimes \)-strongly log-unimodal. So the maximum exists. Moreover, since \( \sigma_{t_1} \) is not \( \boxtimes \)-strongly log-unimodal, we have \( t_2 \not\in [0, t_1] \).

If \( t_2 = 0 \) then \( (0, t_1) \subset I \) and we may take \( t_0 = t_1 \).

If \( t_2 > 0 \) then \( t_2 + \epsilon \in I \) for every \( \epsilon \in (0, t_1 - t_2) \]. Applying Step 2 to \( t = t_2 + \epsilon \) and \( s = \epsilon \), we conclude that \( \epsilon \in I \), since we know \( t_2 = t - s \not\in I \). This argument shows that \( [0, t_1 - t_2] \subset I \) and we may take \( t_0 = t_1 - t_2 \).
Remark 4.10. This result is contrasted with the fact that the log-normal distributions are all \( \ast \)-strongly log-unimodal.

Problem 4.11. Is it true that \( \sigma_t \) is not \( \tilde{\ast} \)-strongly log-unimodal for any \( t > 0 \)?

Acknowledgment
The first-named author is granted by JSPS kakenhi (B) 19K14546. Support of the third author came from the NSERC Canada Discovery Grant RGPIN-2016-03796. This research is an outcome of Joint Seminar supported by JSPS and CNRS under the Japan-France Research Cooperative Program.

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