A NONSTANDARD GROWTH STEKLOV OPTIMIZATION PROBLEM WITH VOLUME CONSTRAINT

ARIEL SALORT, BELEM SCHVAGER AND ANALÍA SILVA

Abstract. In this article we study an optimal design problem for a nonstandard growth Steklov eigenvalues ruled by the $g$–Laplacian operator. More precisely, given $\Omega \subset \mathbb{R}^n$ and $\alpha, c > 0$ we analyze existence and symmetry properties of solution of the optimization problem $\inf \{ \lambda(\alpha, E) : E \subset \Omega, |E| = c \}$, where, for a suitable function $u(\alpha, E)$, $\lambda(\alpha, E)$ solves

$$\begin{cases}
- \operatorname{div}(g(|\nabla u|) \frac{\nabla u}{|\nabla u|}) + (1 + \alpha \chi_E) g(|\nabla u|) \frac{\nabla u}{|\nabla u|} = 0 & \text{in } \Omega, \\
g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \eta = \lambda g(|u|) \frac{u}{|u|} & \text{on } \partial \Omega
\end{cases}$$

being $g$ the derivative of a Young function, and $\eta$ the unit outward normal derivative.

We analyze the behavior of the optimization problem as $\alpha$ approaches infinity and its connection of the trace embedding for Orlicz-Sobolev functions.

1. Introduction

The literature on optimization problems is very wide, from the classical cases of isoperimetrical problems to the most recent applications including elasticity and spectral optimization. Only to mention some references and motivations, we refer the reader to the books of Allaire [1], Bucur and Buttazzo [2], Henrot [16], Pironneau [25] and Sokolowski and Zolésio [27], where a huge amount of shape optimization problems is introduced. Optimization problems in the more general form can be stated as follows: given a cost functional $\mathcal{F}$ and a class of admissible domains $\mathcal{D}$, solve the minimization problem

$$\min \{ \mathcal{F}(D) : D \in \mathcal{D} \}.$$ 

In recent years there has been an increasing amount of interest in optimization problems for power-like functionals, see [1, 2, 3, 4, 8, 9, 11, 12, 13, 16, 21, 25, 27] for instance. Moreover, optimization problems describing non-local phenomena have been approached recently [5, 11, 14, 24]. However, optimization problems of the form (1.1), where the state equation to be solved on $D$ involves behaviors more general than powers are less common in the literature. Additional drawbacks can arise in these class of problems due to the possible lack of homogeneity of the functional. We cite for instance the articles [6, 22, 23, 30].

Inspired in [3, 11], in this manuscript we study the existence of an optimal configuration for a minimization problem ruled by the nonlinear degenerate and possibly

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not homogeneous operator $g$-Laplacian defined as $\Delta_g u := \text{div}(g(|\nabla u|) \nabla u / |\nabla u|)$, where $G(t) = \int_0^t g(s) \, ds$ is a Young function fulfilling the following growth condition

$$(L) \quad 1 < p^- \leq \frac{t g(t)}{G(t)} \leq p^+ < \infty \quad \text{for all } t \in \mathbb{R}^+,$$

for fixed constants $p^+$ and $p^-$. These kind of problems appears naturally when studying general optimal design problems, and they are usually formulated as problems of minimization of the energy, stored in the design under a prescribed loading. Solutions of these problems are unstable to perturbations of the loading, and the optimal design problem is formulated as minimization of the stored energy under the most unfavorable loading. This most dangerous loading is one that maximizes the stored energy over the class of admissible functions. Finally, the problem is reduced to minimization involving the behavior on the boundary. See [3, 26] for more details.

We describe now our problem and main aims. Given a bounded open set $\Omega \subset \mathbb{R}^n$, a fixed subset $E$ of $\Omega$ and a fixed number $\alpha \in \mathbb{R}$, we consider the following quantity

$$(1.2) \quad \lambda(\alpha, E) := \min_{v \in A} \left\{ \int_{\Omega} G(|\nabla v|) + G(|v|) \, dx + \alpha \int_E G(|v|) \, dx \right\},$$

where the class $A$ of admissible functions is given by

$$A := \left\{ v \in W^{1,G}(\Omega) : \text{ such that } \int_{\partial \Omega} G(|v|) \, d\mathcal{H}^{n-1} = 1 \right\}.$$

See Section 2 for precise definitions. Existence of such a number is guaranteed by the compactness of the embedding of the Orlicz-Sobolev space $W^{1,G}(\Omega)$ into $L^G(\partial \Omega)$ under suitable hypothesis on $G$. In fact, (1.2) turns out to be an eigenvalue of the following problem with Steklov boundary condition:

$$(1.3) \quad \begin{cases} -\Delta_g u + (1 + \alpha \chi_E) g(|u|) \frac{u}{|u|} = 0 & \text{in } \Omega \\ g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \eta = \mu g(|u|) \frac{u}{|u|} & \text{on } \partial \Omega, \end{cases}$$

where $\eta$ denotes the outer normal to $\partial \Omega$ and $\mu$ is a real parameter.

Our main goal is to optimize the eigenvalue $\lambda(\alpha, E)$ with respect to the class of sets $E \subset \Omega$ of fixed volume, i.e., given fixed values of $\alpha$ and $c \in [0, |\Omega|]$, to analyze the existence of solution of the optimization problem

$$(1.4) \quad \lambda(\alpha, c) := \inf \{ \lambda(\alpha, E) : E \subset \Omega, |E| = c \}.$$ Such a solution, when exists, will be referred as a classical solution of (1.4).

In order to prove the existence of an optimal configuration of (1.4), we adopt the strategy introduced in [11] and we consider a relaxed version of the problem. However, now we have an extra difficulty due to the loss of homogeneity of the equation. Indeed, instead of minimizing eigenvalues of (1.3), we consider eigenvalues of the relaxed eigenvalue problem

$$(1.5) \quad \begin{cases} -\Delta_g u + (1 + \alpha \phi) g(|u|) \frac{u}{|u|} = 0 & \text{in } \Omega \\ g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \eta = \Lambda g(|u|) \frac{u}{|u|} & \text{on } \partial \Omega. \end{cases}$$

where the function $\phi$ belongs to the class
\[ B := \left\{ \phi \in L^\infty(\Omega) \text{ such that } 0 \leq \phi \leq 1 \text{ and } \int_\Omega \phi \, dx = c \right\}. \]
That is, instead of (1.2) we consider the eigenvalue $\Lambda(\alpha, \phi)$ given by
\begin{equation}
\Lambda(\alpha, \phi) := \min_{v \in A} \left\{ \int_\Omega G(|\nabla v|) + G(|v|) \, dx + \alpha \int_\Omega \phi G(|v|) \, dx \right\},
\end{equation}
and the relaxed version of (1.4) becomes to optimize (1.6) over the class $B$, i.e.,
\begin{equation}
\Lambda(\alpha, c) := \inf \{ \Lambda(\alpha, \phi) : \phi \in B \}.
\end{equation}
Existence of a solution $u = u(\alpha, \phi) \in A$ of (1.5) is derived from the direct method of the calculus of variations, and the Harnack’s inequality in this setting ensure the strict positive of $u$ in $\Omega$. Any minimizer of (1.7) will be called an optimal configuration of the data $(\alpha, c)$. If $\phi$ is an optimal configuration and $u$ satisfies (1.6), then $(u, \phi)$ is called an optimal pair (or solution).

Our first result established that an optimal pair of the relaxed problem allows to recover a solution of the original problem (1.4). More precisely, if $(u, \phi)$ is an optimal pair, then $\phi = \chi_D$, for some measurable set $D \subset \Omega$. Moreover, the set $D$ is shown to be a sublevel set of $u$.

**Theorem 1.1.** For any $\alpha > 0$ and $c \in [0, ||\Omega||]$ there exists an optimal pair. Moreover, any optimal pair $(u, \phi)$ has the following properties:

(i) $u \in C^{1, \gamma}(\Omega)$ for same $\gamma \in (0, 1)$.

(ii) There exists an optimal configuration $\phi = \chi_D$, where $D$ is a sublevel set of $u$, that is, there is a number $t \geq 0$ such that $D = \{u \leq t\}$.

(iii) Every level set $\{u = s\}$, has Lebesgue measure zero.

Theorem 1.1 clearly boils down to a minimization problem ruled by the $p$−Laplacian operator when $G(t) = t^p$, $p > 1$, and recovers the main result of [11].

When in particular $\Omega$ is the unit ball $B_1$, we obtain existence of a spherically symmetric optimal configuration:

**Theorem 1.2.** Fix $\alpha > 0$ and $c \in (0, |B_1|)$, there exists an optimal pair $(u, \chi_D)$ of (1.6) such that $u$ and $D$ are spherically symmetric.

Once the set $E$ is fixed in (1.2), it is easy to see that when $\alpha \to \infty$ the quantity $\lambda(\alpha, E)$ converge to the minimizing function vanishes on $E$, i.e.,
\begin{equation}
\lambda(\alpha, E) := \lim_{\alpha \to \infty} \lambda(\alpha, E) = \inf_{v \in A, \phi \equiv 0} \left\{ \int_\Omega G(|\nabla v|) + G(|v|) \, dx \right\}.
\end{equation}
The natural limit optimization problem in this case is
\begin{equation}
\lambda(\infty, c) := \inf \{ \lambda(\infty, E) : E \subset \Omega, |E| = c \}.
\end{equation}
A natural question is whether the optimal configuration of (1.4) converges to these of (1.9) when $\alpha \to \infty$. The following result answers positively to that issue.
Theorem 1.3. For any sequence $\alpha_k \to \infty$ and optimal pairs $(D_k, u_k)$ of (1.4) there exists a subsequence, still denoted $\alpha_k$, and an optimal pair $(D, u)$ of (1.9) such that
\[
\lim_{k \to \infty} \chi_{D_k} = \chi_D \text{ weakly* in } L^\infty(\Omega),
\]
\[
\lim_{k \to \infty} u_k = u \text{ strongly in } W^{1,G}(\Omega).
\]
Moreover, $u > 0$ in $\Omega \setminus D$.

The paper is organized as follows. In Section 2 we introduce the notation and basic facts on Orlicz-Sobolev spaces used along the manuscript. Section 3 is devoted to study the link between the minimization problems (1.2) and (1.6) and the eigenvalue problems (1.3) and (1.5). In Sections 4 and 5 we prove our main results stated in Theorems 1.1, 1.2 and 1.3. Finally in Section 6 we state some possible generalizations.

2. Preliminaries

In this section we introduce some notation and basic results on Orlicz-Sobolev spaces that we will use in this paper.

2.1. Young functions. An application $G: \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a Young function if it admits the integral formulation $G(t) = \int_0^t g(\tau) \, d\tau$, where the right continuous function $g$ defined on $[0, \infty)$ has the following properties:
\begin{align*}
(g_1) & \quad g(0) = 0, \quad g(t) > 0 \text{ for } t > 0, \\
(g_2) & \quad g \text{ is non-decreasing on } (0, \infty), \\
(g_3) & \quad \lim_{t \to \infty} g(t) = \infty.
\end{align*}

From these properties it is easy to see that a Young function $G$ is continuous, non-negative, strictly increasing and convex on $[0, \infty)$.

We will assume the following growth condition on Young functions: there exist fixed constants $p^\pm$ such that
\[
1 \leq p^- \leq \frac{t g(t)}{G(t)} \leq p^+ < \infty, \quad \text{for all } t > 0.
\]

The following properties on Young functions are well-known. See for instance [17] for the proof of these results.

Lemma 2.1. Let $G$ be a Young function satisfying (2.1) and $a, b \geq 0$. Then
\begin{align*}
(L_1) \quad & \min\{a^{p^-}, a^{p^+}\} G(b) \leq G(ab) \leq \max\{a^{p^-}, a^{p^+}\} G(b), \\
(L_2) \quad & G(a + b) \leq C(G(a) + G(b)) \quad \text{with } C := 2p^+, \\
(L_3) \quad & G \text{ is Lipschitz continuous}.
\end{align*}

Condition (2.1) is known as the $\Delta_2$ condition or doubling condition and, as it is showed in [17, Theorem 3.4.4], it is equivalent to the right hand side inequality in (2.1).
The complementary Young function of a Young function $G$ is defined as
\[ \tilde{G}(t) := \sup_{s \geq 0} \{ st - G(s) \}. \]

It is easy to see that the left hand side inequality in (2.1) is equivalent to assume that $\tilde{G}$ satisfies the $\Delta_2$ condition.

2.2. **Orlicz-Sobolev spaces.** Given a Young function $G$ and a bounded set $\Omega$ we consider the spaces
\[ L^G(\Omega) := \{ u : \mathbb{R} \to \mathbb{R} \text{ measurable such that } \Phi_{G,\Omega}(u) < \infty \}, \]
\[ L^G(\partial \Omega) := \{ u : \mathbb{R} \to \mathbb{R} \text{ measurable such that } \Phi_{G,\partial \Omega}(u) < \infty \}, \]
\[ W^{1,G}(\Omega) := \{ u \in L^G(\Omega) \text{ such that } \Phi_{G,\Omega}(\|\nabla u\|) < \infty \}, \]
where modulars are defined as
\[ \Phi_{G,\Omega}(u) := \int_\Omega G(|u|) \, dx, \]
\[ \Phi_{G,\partial \Omega}(u) := \int_{\partial \Omega} G(|u|) \, dH^{n-1}. \]

These spaces are endowed with the so called **Luxemburg norm** defined as follows
\[ \|u\|_{L^G(\Omega)} = \inf \left\{ \lambda > 0 : \Phi_{G,\Omega}(\frac{u}{\lambda}) \leq 1 \right\}, \]
\[ \|u\|_{L^G(\partial \Omega)} = \inf \left\{ \lambda > 0 : \Phi_{G,\partial \Omega}(\frac{u}{\lambda}) \leq 1 \right\}, \]
\[ \|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}, \]
and are reflexive and separable Banach spaces if and only if $G$ and $\tilde{G}$ satisfies the $\Delta_2$ condition.

For the sake of simplicity, given $u \in W^{1,G}(\Omega)$ and $\phi \in L^\infty(\Omega)$ we denote
\[ \Phi_{G,\phi,\Omega}(u) := \int_\Omega \phi G(|u|) \, dx, \]
\[ \Phi_{1,G,\Omega}(u) := \int_\Omega G(|u|) + G(|\nabla u|) \, dx. \]

We conclude this subsection by recalling that under our assumptions, boundedness of the modular implies boundedness of the norm (see for instance [7, Lemma 2.1.12]).

**Lemma 2.2.** Let $G$ be a Young function satisfying (2.1). Then, if $\Phi_{G,\partial \Omega}(u)$ (resp. $\Phi_{G,\Omega}(u)$) is bounded, then $\|u\|_{L^G(\partial \Omega)}$ (resp. $\|u\|_{L^G(\Omega)}$) is bounded.

2.3. **Some embedding results.** In order to guarantee compact embeddings the following conditions will be assumed
\[ (2.2) \quad \int_0^1 \frac{G^{-1}(s)}{s^{1+\frac{1}{n}}} \, ds < \infty \quad \text{and} \quad \int_1^\infty \frac{G^{-1}(s)}{s^{1+\frac{1}{n}}} \, ds = \infty. \]

For any Young function satisfying (2.2), the **Sobolev critical function** is defined as
\[ G^{-1}_*(t) = \int_0^t \frac{G^{-1}(s)}{s^{\frac{1}{n}}} \, ds. \]
Combining [17, Theorems 7.4.6 and 7.4.6] with [6, Example 6.3] the following embedding holds.

**Proposition 2.3.** Let $G$ be a Young function satisfying (2.1) and (2.2). Let $\Omega \subset \mathbb{R}^n$ be a $C^{0,1}$ bounded open subset. Then the embeddings

$$W^{1,G}(\Omega) \hookrightarrow L^G(\Omega), \quad W^{1,G}(\Omega) \hookrightarrow L^G(\partial\Omega)$$

are compact.

3. The minimization problems and their related eigenvalue problems

From now on, we will always assume that $G$ is a Young function satisfying (2.1) and (2.2).

We start this section by studying some basic properties of the relaxed minimization problem stated in (1.6).

Given $\alpha > 0$ and a function $\phi \in L^\infty(\Omega)$, in light of the definition of the set $A$ observe that the quantity defined in (1.6) can be reformulated as

$$(3.1) \quad \Lambda(\alpha, \phi) := \inf_{v \in A} \mathcal{I}(v)$$

where the functionals $\mathcal{I}, \mathcal{J} : W^{1,G}(\Omega) \to \mathbb{R}$ are defined as

$$\mathcal{I}(v) := \Phi_{1,G,\Omega}(v) + \alpha \Phi_{G,\phi,\Omega}(v), \quad \mathcal{J}(v) := \Phi_{G,\partial\Omega}(v).$$

Since $G$ and its conjugated function $\tilde{G}$ satisfy the $\Delta_2$ condition, it is straightforward to see that $\mathcal{I}$ and $\mathcal{J}$ are class $C^1(\Omega)$ and their Fréchet derivatives $\mathcal{I}'$ and $\mathcal{J}'$ are defined from $W^{1,G}(\Omega)$ into its dual space and are given by

$$\langle \mathcal{I}'(u), v \rangle = \int_\Omega g(|u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla v \, dx + \int_\Omega (1 + \alpha \phi)g(|u|)\frac{u}{|u|}v \, dx, \quad \forall u, v \in W^{1,G}(\Omega)$$

$$\langle \mathcal{J}'(u), v \rangle = \int_{\partial\Omega} g(|u|)\frac{u}{|u|}v \, d\mathcal{H}^{n-1}.$$

We start by proving that the infimum in (3.1) is attained.

**Proposition 3.1.** Given $\alpha > 0$ and $\phi \in L^\infty(\Omega)$ there exists $u \in A$ solving (3.1).

**Proof.** This is a consequence of the direct method in the calculus of variations. Indeed, let $\{u_k\}_{k \in \mathbb{N}}$ be a minimizing sequence for $\Lambda(\alpha, \phi)$, that is

$$(3.2) \quad u_k \in A \quad \text{and} \quad \lim_{k \to \infty} \Phi_{1,G,\Omega}(u_k) + \alpha \Phi_{G,\phi,\Omega}(u_k) = \Lambda(\alpha, \phi).$$

Observe that $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $L^G(\partial\Omega)$ and for $k$ big enough, $\Phi_{1,G,\Omega}(u_k) \leq 1 + \Lambda(\alpha, \phi)$. Therefore, since $W^{1,G}(\Omega)$ is a reflexive space, due Lemma 2.2 together with the compact embedding stated in Proposition 2.3 there exists a function $u \in$
\(W^{1, G}(\Omega)\) and a sub-sequence \(\{u_{k_j}\}_{j \in \mathbb{N}} \subset \{u_k\}_{k \in \mathbb{N}}\) such that

\[
\begin{align*}
  u_{k_j} &\to u \text{ weakly in } W^{1, G}(\Omega), \\
  u_{k_j} &\to u \text{ strongly in } L^G(\Omega), \\
  u_{k_j} &\to u \text{ strongly in } L^G(\partial \Omega), \\
  u_{k_j} &\to u \text{ a.e. in } L^G(\Omega).
\end{align*}
\]

The strong convergence in \(L^G(\partial \Omega)\) implies that \(\Phi_{G, \partial \Omega}(u) = 1\), and therefore \(u \in \mathcal{A}\). Hence, by definition

\[
\Lambda(\alpha, \phi) \leq \Phi_{1, G, \Omega}(u) + \alpha \Phi_{G, \phi, \Omega}(u).
\]

On the other hand, since modulars are lower semi-continuous, by Fatou’s Lemma we get

\[
\Phi_{1, G, \Omega}(u) + \alpha \Phi_{G, \phi, \Omega}(u) \leq \liminf_{j \to \infty} \Phi_{1, G, \Omega}(u_{k_j}) + \alpha \Phi_{G, \phi, \Omega}(u_{k_j}) = \Lambda(\alpha, \phi).
\]

From the last two inequalities the lemma follows. \(\square\)

We define the Euler-Lagrange equation corresponding to the constrained minimization problem (3.1): given \(\alpha > 0\) and \(\phi \in L^\infty(\Omega)\) we consider the following eigenvalue problem with Steklov boundary condition

\[
\begin{aligned}
  -\Delta u + g(|u|)\frac{\partial u}{|\partial \Omega|} + \alpha \phi g(|u|)\frac{\partial u}{|\partial \Omega|} &= 0 \quad \text{in } \Omega, \\
  g(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \eta = \mu g(|u|)\frac{u}{|u|} \quad &\text{on } \partial \Omega,
\end{aligned}
\]

where \(\eta\) denotes the outer unit normal derivative on \(\partial \Omega\). We say that \(\mu \in \mathbb{R}\) is an eigenvalue of (3.3) with associated eigenfunction \(u \in \mathcal{A}\) if for any \(v \in W^{1, G}(\Omega)\) it holds that

\[
\langle \mathcal{I}'(u), v \rangle = \mu \langle \mathcal{J}'(u), v \rangle.
\]

We prove now that (3.1) is actually an eigenvalue of the previous Steklov problem.

**Proposition 3.2.** The number \(\Lambda(\alpha, \phi)\) defined in (3.1) is an eigenvalue of (3.3).

**Proof.** Proposition 3.1 guarantees that there is \(u \in \mathcal{A}\) such that \(\Lambda(\alpha, \phi) = \frac{\mathcal{I}(u)}{\mathcal{J}(u)}\), i.e., \(u\) solves the minimization problem 3.1. Hence, for any \(v \in W^{1, G}(\Omega)\) we get

\[
0 = \langle \frac{\mathcal{I}(u)}{\mathcal{J}(u)}', v \rangle = \langle \frac{\mathcal{I}'(u)\mathcal{J}(u) - \mathcal{I}(u)\mathcal{J}'(u)}{\mathcal{J}(u)^2}, v \rangle
\]

\[
= \langle \frac{\mathcal{I}'(u)}{\mathcal{J}(u)}, v \rangle - \langle \frac{\mathcal{I}(u)\mathcal{J}'(u)}{\mathcal{J}(u)^2}, v \rangle = \frac{1}{\mathcal{J}(u)} \langle \mathcal{I}'(u), v \rangle - \frac{1}{\mathcal{J}(u)} \langle \mathcal{I}(u), \mathcal{J}'(u), v \rangle
\]

which gives that

\[
\langle \mathcal{I}'(u), v \rangle = \frac{\mathcal{I}(u)}{\mathcal{J}(u)} \langle \mathcal{J}'(u), v \rangle = \Lambda(\alpha, \phi) \langle \mathcal{J}'(u), v \rangle
\]

for all \(v \in W^{1, G}(\Omega)\), concluding the proof. \(\square\)

The next result states some useful properties of minimizers of the relaxed problem.

\[3.3\]
Proposition 3.3. Let $u \in A$ be a solution of (3.1). Then $u \in C^{1,\gamma}(\overline{\Omega})$ and $u > 0$ in $\Omega$.

Proof. Let $u \in A$ be a solution of (3.1). By the regularity theory of [28, Corollary 3.1] we have that $u \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. Moreover, since $|u|$ also solves (3.1), we can assume that $u \geq 0$ in $\Omega$, and by the strong maximum principle stated in [19] we have that either $u > 0$ in $\Omega$ or $u$ is constant. If there would exist $x \in \Omega$ such that $u(x) = 0$, then we would have $u = 0$ in $\Omega$. Therefore $u > 0$ in $\Omega$. $\Box$

We conclude this section by remarking that a completely analogous analysis can be done to link the quantity $\lambda(\alpha, E)$ defined in (1.2) and the eigenvalue problem (1.3):

Proposition 3.4. The number $\lambda(\alpha, E)$ defined in (1.2) is an eigenvalue of (1.3).

4. AN EXISTENCE RESULT

This section is devoted to prove the existence of an optimal configuration of (1.4) and to analyze some properties which it fulfills, namely, Theorem 1.1.

Proof of Theorem 1.1. We first prove existence for a fixed $\alpha$ and $c$. For the sake of simplicity we set $\Lambda = \Lambda(\alpha, c)$ and $\Lambda(\phi) = \Lambda(\alpha, \phi)$.

Let $\{\phi_k\}_{k \in \mathbb{N}}$ be a minimizing sequence, i.e.,

$$0 \leq \phi_k \leq 1, \quad \int_{\Omega} \phi_k \, dx = c \quad \text{and} \quad \lim_{k \to \infty} \Lambda(\phi_k) = \Lambda.$$ (4.1)

In light of Proposition 3.1, for each $k \in \mathbb{N}$ let $u_k \in A$ be such that

$$\Lambda(\phi_k) = \inf \{v \in A : \Phi_{1,G,\Omega}(v) + \alpha \Phi_{G,\phi_k,\Omega}(v)\}$$

$$= \Phi_{1,G,\Omega}(u_k) + \alpha \Phi_{G,\phi_k,\Omega}(u_k).$$

Since $\phi_k \leq 1$ for all $k \in \mathbb{N}$ we have that $\Lambda(\phi_k)$ is uniformly bounded in $k$, and then $\{u_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $W^{1,G}(\Omega)$. Therefore, from Proposition 2.3 up to a subsequence, there exist $u \in W^{1,G}(\Omega)$ and $\phi \in L^\infty(\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } W^{1,G}(\Omega),$$

$$u_k \rightarrow u \quad \text{strongly in } L^G(\Omega),$$

$$u_k \rightarrow u \quad \text{strongly in } L^G(\partial\Omega),$$

$$\phi_k \rightharpoonup^* \phi \quad \text{weakly* in } L^\infty(\Omega).$$

The strong convergence of $u_k$ in $L^G(\partial\Omega)$ gives that $\Phi_{G,\partial\Omega}(u) = 1$ and the weak* convergence in $L^\infty(\Omega)$ of $\phi_k$ gives that $0 \leq \phi \leq 1$ and $\int_{\Omega} \phi \, dx = c$.

Hence, taking limit in (4.1) and using the lower semicontinuity of the modular we get

$$\Lambda = \lim_{k \to \infty} \Lambda(\phi_k) = \liminf_{k \to \infty} \Phi_{1,G,\Omega}(u_k) + \alpha \Phi_{G,\phi_k,\Omega}(u_k)$$

$$\geq \Phi_{1,G,\Omega}(u) + \alpha \Phi_{G,\phi,\Omega}(u).$$ (4.2)
On the other hand, since by definition of $\Lambda$, we have

$$
\Lambda \leq \Lambda(\phi) \leq \Phi_{1,G,\Omega}(u) + \alpha \Phi_{G,\phi,\Omega}(u).
$$

By (4.2) and (4.3) we obtain that $(u, \phi)$ is an optimal pair.

The proof of item (i) follows as a consequence of the regularity theory for quasi-linear elliptic equations in Orlicz Sobolev spaces. More precisely, following the arguments of [15] and [19], in [28, Corollary 3.1] is stated the boundary regularity for weak solutions of the $g$-Laplacian, which gives that $u \in C^{1,\gamma}(\Omega)$ for some $\gamma \in (0, 1)$.

Now, by the Bathtub Principle [20, Theorem 1.15], the minimization problem

$$
\inf_{\phi \in B} \Phi_{G,\phi,\Omega}(u)
$$

is solved by $\phi(x) = \chi_D(x)$, where $D$ is given by $D = \{x: u < t\} \cup K \cdot \{x: u = t\}$ and the number $K$ is chosen such that $K|\{x: u = t\}| = c - |\{x: u < t\}|$, which gives $|D| = c$. This principle also ensures that

$$
\{u < t\} \subset D \subset \{u \leq t\}
$$

where $t = \sup \{s: |\{u < s\}| \leq c\}$.

Therefore, (4.2) and (4.4) yield

$$
\Lambda \geq \inf_{\phi \in B} \{\Phi_{1,G,\Omega}(u) + \alpha \Phi_{G,\phi,\Omega}(u)\} = \Phi_{1,G,\Omega}(u) + \alpha \Phi_{G,\chi_D,\Omega}(u).
$$

Moreover, since $\phi$ and $u$ are admissible in the characterization of $\Lambda$, we get

$$
\Lambda = \inf_{\phi \in B} \inf_{v \in A} \{\Phi_{1,G,\Omega}(v) + \alpha \Phi_{G,\phi,\Omega}(v)\} \leq \Phi_{1,G,\Omega}(u) + \alpha \Phi_{G,\chi_D,\Omega}(u),
$$

and then $(u, \chi_D)$ is an optimal pair.

Finally, if $(u, \chi_D)$ is any solution and $\mathcal{R}_s = \{u = s\}$ for any $s > 0$. Since, $\nabla u = 0$ a.e. on $\mathcal{R}_s$, we get $\Delta_g u = 0$ on $\mathcal{R}_s$. Then

$$
(1 + \chi_D)g(|u|)\frac{u}{|u|} = 0 \text{ a.e. on } \mathcal{R}_s.
$$

Since $u = s > 0$ on $\mathcal{R}_s$, we have

$$
(1 + \chi_D)g(|u|)\frac{u}{|u|} > 0 \text{ on } \mathcal{R}_s,
$$

from where $|\mathcal{R}_s| = 0$. This proves (iii). In particular, when $s = t$ we get the last assertion in (ii).

In the following proposition we obtain the derivative of $\Lambda(\alpha, \phi)$ in the direction of $f$ in the class $\mathcal{F}$ given by

$$
\mathcal{F} := \left\{ f: f \leq 0 \text{ in } \{\phi = 1\}, f \geq 0 \text{ in } \{\phi = 0\}, \int_{\Omega} f \, dx = 0 \right\}.
$$

**Proposition 4.1.** Let $f \in \mathcal{F}$, then the right derivative of $\Lambda(\alpha, \phi)$ in the direction of $f \in \mathcal{F}$ is given by

$$
\Lambda'(\alpha, \phi)(f) = \lim_{t \to 0} \frac{\Lambda(\alpha, \phi + tf) - \Lambda(\alpha, \phi)}{t} = \alpha \Phi_{G,f,\Omega}(u)
$$

where $u$ is of solution of $\Lambda(\alpha, \phi)$. 
Proof. Let us consider the function \( \phi_t = \phi + tf, \) \( t \geq 0. \) Since \( f \in \mathcal{F} \) and \( \phi \) is an admissible function, it follows that \( \phi_t \) is admissible for every \( t \geq 0 \) small enough.

Consider an eigenfunction \( u_t \) of \( \Lambda(\alpha, \phi_t) \) and an eigenfunction \( u \) of \( \Lambda(\alpha, \phi) \). First, using \( u_t \) in the variational formulation of \( \Lambda(\alpha, \phi) \) we obtain

\[
\frac{\Lambda(\alpha, \phi_t) - \Lambda(\alpha, \phi)}{t} \geq \alpha \Phi_{G,f,\Omega}(u_t).
\]

Moreover, taking \( u \) in the variational formulation of \( \Lambda(\alpha, \phi_t) \) we obtain

\[
\frac{\Lambda(\alpha, \phi_t) - \Lambda(\alpha, \phi)}{t} \leq \alpha \Phi_{G,f,\Omega}(u).
\]

Now, using \( v = G^{-1}(|\partial \Omega|^{-1}) \) as a test function in the characterization of \( \Lambda(\alpha, \phi_t) \) we obtain that the family \( u_t \) is bounded in \( W^{1,G}(\Omega) \) for \( 0 < t \leq t_0 \). Then, from Proposition 2.3, \( u_t \to u \) strongly in \( L^G(\Omega) \) when \( t \to 0 \). Finally, taking limit as \( t \to 0 \) in (4.6) and (4.7), we obtain (4.5) as desired.

As a direct consequence of Proposition 4.5 we can provide for an alternative proof to the fact that the optimal set is a sublevel set of \( u \), claim which was already proved in the second part of (ii) from Theorem 1.1.

**Corollary 4.2.** Let \( u \) be a solution to (1.4). Then the optimal set \( D \) satisfies \( D = \{ u \leq t \} \) for some \( t \geq 0 \).

Proof. Since in the first part of (ii) from Theorem 1.1 we have already proved that \( \chi_D \) realizes the minimum of \( \Lambda(\alpha, \phi) \), we have that for all \( f \in \mathcal{F}, \)

\[
\Lambda'(\alpha, \phi)(f) = \alpha \Phi_{G,f,\Omega}(u) \geq 0.
\]

Given a point \( x_0 \in D \) of positive density i.e., for every \( \varepsilon \geq 0, |B(x_0, \varepsilon) \cap D| > 0, \) and \( x_1 \in (\Omega \setminus D) \) also with positive density, we can take a function \( f \in \mathcal{F} \) of the form \( f := M_{\chi T_1} - M_{\chi T_0} \in \mathcal{F} \) with \( T_0 \subset B(x_0, \varepsilon) \cap D \) and \( T_1 \subset B(x_1, \varepsilon) \cap (\Omega \setminus D) \) and \( M^{-1} := |T_0| = |T_1|. \) Hence,

\[
0 \leq \alpha \Phi_{G,f,\Omega}(u) = \alpha \int_{\Omega} (M_{\chi T_1} - M_{\chi T_0}) G(|u|) dx
= \alpha \frac{1}{|T_1|} \int_{T_1} G(|u|) dx - \alpha \frac{1}{|T_0|} \int_{T_0} G(|u|) dx,
\]

and taking limit when \( \varepsilon \to 0 \) and using the continuity of \( u \) (from Theorem 1.1 \( u \in C^{1,\gamma}(\Omega) \)) we have that \( G(u(x_0)) \leq G(u(x_1)) \). Therefore by the monotonicity of \( G \), we obtain \( u(x_0) \leq u(x_1) \). We conclude that \( D = \{ u \leq t \} \) as we wanted to prove.

We conclude this section by proving Theorem 1.2. For that end, we recall some basic results on spherical symmetrization of functions on Orlicz-Sobolev spaces.

Given a measurable set \( \Omega \subset \mathbb{R}^n \), the spherical symmetrization \( \Omega^* \) of \( \Omega \) with respect to an axis given by a unit vector \( e_k \) reads as follows: for each positive number \( r \), take the intersection \( \Omega \cap \partial B(0, r) \) and replace it by the spherical cap of the same \( H^{n-1} \)–measure and center \( re_k \). Hence, \( \Omega^* \) is the union of these caps.
Now, the spherical symmetrization $u^*$ of a measurable function $u: \Omega \to \mathbb{R}_+$ is constructed by symmetrizing the super-level sets so that, for all $t$

$$\{u^* \geq t\} = \{u \geq t\}^*.$$  

We refer to [18] for more details.

Denoting by $B_1$ the ball of unit radius centered at the origin, the following properties on symmetrization can be found in [6, Theorem 4.1] (see also [18]).

**Proposition 4.3.** Let $u \in W^{1,G}(B_1)$ and $u^*$ be its spherical symmetrization. Then, $u^* \in W^{1,G}(B_1)$. Moreover,

(i) $\Phi_{G,B_1}(u^*) = \Phi_{G,B_1}(u)$,

(ii) $\Phi_{G,\partial B_1}(u^*) = \Phi_{G,\partial B_1}(u)$,

(iii) $\Phi_{G,B_1}(|\nabla u^*|) \leq \Phi_{G,B_1}(|\nabla u|)$,

(iv) $\Phi_{G,(\alpha \chi_D),B_1}(u^*) \leq \Phi_{G,\alpha \chi_D,B_1}(u)$,

where $D \subset B_1$ and $(\alpha \chi_D)^* = -(-\alpha \chi_D)^*$.

We are now in position to prove our symmetrization result for solutions.

**Proof of Theorem 1.2.** Given $\alpha > 0$ and $c \in (0, |B_1|)$, let $(u, \chi_D)$ be an optimal pair to (1.6) and let $u^*$ be the spherical symmetrization of $u$. Consider the set $D^*$ defined by $\chi_{D^*} = (\chi_D)^*$.

Observe that from item (ii) in Proposition 4.3, $u^* \in \mathcal{A}$ since $u \in \mathcal{A}$. Then, using Proposition 4.3 again we get

$$\lambda(\alpha, D^*) = \inf_{v \in \mathcal{A}} \{\Phi_{1,G,B_1}(v) + \Phi_{G,(\alpha \chi_D),B_1}(v)\}$$

$$\leq \Phi_{1,G,B_1}(u^*) + \Phi_{G,(\alpha \chi_D),B_1}(u^*)$$

$$\leq \Phi_{1,G,B_1}(u) + \alpha \Phi_{G,\chi_D,B_1}(u) = \lambda(\alpha, D).$$

Finally, since $|D^*| = c = |D|$, $(u^*, \chi_{D^*})$ is also an optimal pair to (1.6). \[ \square \]

5. LIMIT AS $\alpha \to \infty$

Fixed a subset $E \subset \Omega \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, in our first result we have focused on the study of the optimization problem

$$\lambda(\alpha, c) := \inf \{\lambda(\alpha, E) : E \subset \Omega, |E| = c\}$$

where the quantity $\lambda(\alpha, E)$ defined in (1.2) is an eigenvalue of (1.3). In this section, fixed the value of $c$ we analyze the the behavior of $\lambda(\alpha, c)$ as $\alpha \to \infty$. We recall that the corresponding limit problem is defined as

$$\lambda(\infty, c) := \inf \left\{\lim_{\alpha \to \infty} \lambda(\alpha, E) : E \subset \Omega, |E| = c\right\} = \inf \{\lambda(\infty, E) : E \subset \Omega, |E| = c\}.$$  

Our concern now is to analyze the convergence as $\alpha \to \infty$ of the optimal configurations $\lambda(\alpha, c)$ obtained in Theorem 1.1 to those of $\lambda(\infty, c)$. A key result for that purpose is to study a monotonicity property of $\lambda(\infty, c)$ with respect to the parameter $c$.

**Lemma 5.1.** $\lambda(\infty, c)$ is strictly monotonically increasing in $c$. 

Proof. We split the proof in three steps.

**Step 1.** First we show that
\begin{equation}
\inf_{|E|=c} \lambda(\infty, E) = \inf_{|E|\geq c} \lambda(\infty, E).
\end{equation}

It is clear by definition of infimum that
\[
\inf_{|E|=c} \lambda(\infty, E) \geq \inf_{|E|\geq c} \lambda(\infty, E).
\]

On the other hand, if \(v\) is a test function for a set of measure larger or equal to \(c\) it is also a test function for a set of measure \(c\). Thus, the two infima coincide.

**Step 2.** We prove that, if \(u\) is a extremal for \(\lambda(\infty, c)\) then \(|\{x : u(x) = 0\}| = c\).

Suppose by contradiction that \(|\{x : u(x) = 0\}| < c\). We may assume that \(E\) is closed. Let us take a small ball \(B\) so that \(|E\setminus B| \geq c\) with \(B\) centered at a point in \(\partial E \cap \partial \Omega_1\), where \(\Omega_1\) is the connected component of \(\Omega\setminus E\) such that \(\partial \Omega \subset \partial \Omega_1\). We can pick the ball \(B\) in such a way that \(|E \cap B| \geq 0\). In particular \(|\{x : u(x) = 0\} \cap B| \geq 0\). Since that \(u\) is an extremal for \(\lambda(\infty, c)\) and \(|E \setminus B| > c\), it is an extremal for \(\lambda(\infty, |E \setminus B|)\). Thus, it holds that
\[
-\Delta_g u + g(|u|) \frac{u}{|u|} = 0 \quad \text{in} \quad \Omega \setminus (E \setminus B) = (\Omega \setminus E) \cup B.
\]

Now, as \(u \geq 0\), by the strong maximum principle (see [19, Lemma 6.4]) there holds that either \(u \equiv 0\) or \(u > 0\) in each connected component of \((\Omega \setminus E) \cup B\). Since \(u \not\equiv 0\) on \(\partial \Omega\) we get, in particular, that \(u > 0\) in \(B\). This is a contradiction to the choice of the ball \(B\). Therefore
\[
|\{x : u(x) = 0\}| = c.
\]

**Step 3.** Finally, we show that \(\lambda(\infty, c)\) is strictly monotonically increasing in \(c\).

From step 1, we get that \(\lambda(\infty, c)\) is non decreasing respect to \(c\). On the other hand, let \(0 < c_1 < c_2 < |\Omega|\) such that \(\lambda(\infty, c_1) = \lambda(\infty, c_2)\) and let \(u\) be an extremal for \(\lambda(\infty, c_2)\) then, by step 2 \(|\{u = 0\}| = c_2\). But since \(u\) is an admissible function for \(\lambda(\infty, c_1)\), it is an extremal for \(\lambda(\infty, c_1)\) with \(|\{x : u(x) = 0\}| > c_1\). This contradicts step 2, and therefore \(\lambda(\infty, c)\) is strictly monotonically increasing in \(c\).

Now we are in position to prove our second main result.

**Proof of Theorem 1.3.** Observe that in light of Theorem 1.1, if \((u_\alpha, \chi_{D_\alpha})\) is an optimal pair to \(\Lambda(\alpha, c)\), then \((u_\alpha, D_\alpha)\) is an optimal pair to \(\Lambda(\alpha, c)\). Then, if \(u_\alpha \to u_\infty\) and \(D_\alpha \to D_\infty\) in some suitable sense as \(\alpha \to \infty\), we would get that \((u_\infty, D_\infty)\) is an optimal pair to \(\Lambda(\infty, c)\). Let us identify such a limit optimal pair.

We split the proof in two steps.

**Step 1.** We characterized the limit of \(\Lambda(\alpha, c)\) as \(\alpha \to \infty\).

Given \(c > 0\) and \(\alpha > 0\), let \((u_\alpha, \chi_{D_\alpha})\) be a solution of
\begin{equation}
\Lambda(\alpha, c) = \inf_{u \in A} \frac{\varphi_{G, \frac{\alpha}{\varphi_{G, \alpha, \varphi}}}(u)}{\Phi_{G, \partial \Omega}(u)}.
\end{equation}

Let \(u_0 \in W^{1,G}(\Omega)\) and \(D_0 \subset \Omega\) be such that \(|D_0| = c\) and \(G(|u_0|)\chi_{D_0} = 0\). Then, we have that
\[
\Lambda(\alpha, c) \leq \frac{\varphi_{G, \frac{\alpha}{\varphi_{G, \alpha, \varphi}}}(u_0)}{\Phi_{G, \partial \Omega}(u_0)} = \frac{\varphi_{G, \frac{\alpha}{\varphi_{G, \alpha, \varphi}}}(u_0)}{\Phi_{G, \partial \Omega}(u_0)} := K.
\]
with $K$ independent of $\alpha$. Thus $\Lambda(\alpha, c)$ is a bounded sequence in $\mathbb{R}$ and it is clearly increasing. Consequently, $\{u_\alpha\}_{\alpha > 0}$ is bounded in $W^{1, G}(\Omega)$. Moreover $\{\chi_{D_{\alpha}}\}_{\alpha > 0}$ is bounded in $L^\infty(\Omega)$. Therefore for Proposition 2.3, we may choose a sequence $\alpha_k$, $u_{\infty} \in W^{1, G}(\Omega)$ and $\phi_{\infty} \in L^\infty(\Omega)$ such that

\begin{equation}
(5.3) \quad u_{\alpha_k} \rightharpoonup u_{\infty} \text{ weakly in } W^{1, G}(\Omega),
\end{equation}

\begin{equation}
(5.4) \quad u_{\alpha_k} \to u_{\infty} \text{ strongly in } L^G(\Omega),
\end{equation}

\begin{equation}
(5.5) \quad u_{\alpha_k} \to u_{\infty} \text{ strongly in } L^G(\partial\Omega),
\end{equation}

\begin{equation}
(5.6) \quad \chi_{D_{\alpha_k}} \rightharpoonup \phi_{\infty} \text{ weakly* in } L^\infty(\Omega).
\end{equation}

By (5.5) we get that $\|u_{\infty}\|_{L^G(\partial\Omega)} = 1$ and by (5.6) we can assure that $0 \leq \phi_{\infty} \leq 1$ and $\int_{\Omega} \phi_{\infty} \, dx = c$, i.e., $\phi_{\infty} \in \mathcal{B}$ and $u_{\infty} \in \mathcal{A}$. Finally, by (5.4) and (5.6), it also holds $\Phi_{G, \chi_{D_{\alpha_k}}} \leq \Phi_{\infty, \chi_{D_{\alpha_k}}} = \Phi_{G, \phi_{\infty}, \Omega}(u_{\infty})$ as $\alpha \to \infty$.

Observe that since $u_{\alpha_k}$ is minimizer,

$$
\alpha_k \Phi_{G, \chi_{D_{\alpha_k}}} \leq \Lambda(\alpha_k, c) \leq K \quad \text{for all } k \in \mathbb{N}.
$$

Then, $0 \leq \Phi_{G, \chi_{D_{\alpha_k}}} \leq \frac{K}{\alpha_k}$, from where, when $k \to \infty$ we get $\int_{\Omega} \phi_{\infty} G(|u_{\infty}|) \, dx = 0$, which gives that

\begin{equation}
(5.7) \quad \phi_{\infty} G(|u_{\infty}|) = 0 \text{ a.e. in } \Omega.
\end{equation}

Moreover, since $\Lambda(\alpha, c)$ is bounded and increasing, there exists $\Lambda_{\infty}$ such that

$$
\Lambda_{\infty} = \lim_{\alpha \to \infty} \Lambda(\alpha, c) = \lim_{k \to \infty} \Phi_{1, G, \Omega}(u_{\alpha_k}) + \alpha_k \Phi_{G, \chi_{D_{\alpha_k}}}(u_{\alpha_k})
\geq \liminf_{k \to \infty} \Phi_{1, G, \Omega}(u_{\alpha_k})
\geq \Phi_{1, G, \Omega}(u_{\infty})
\geq \inf_{\phi \in \mathcal{B}} \inf_{u \in \mathcal{A}} \Phi_{1, G, \Omega}(u)
\geq \Lambda(\alpha_k, c), \quad \text{for all } k \in \mathbb{N},
$$

where we have used the lower semicontinuity of the modular with respect to the weak convergence and the definition of $\Lambda$.

From the previous computations we deduce that

$$
\Lambda_{\infty} = \inf_{\phi \in \mathcal{B}} \inf_{u \in \mathcal{A}} \Phi_{1, G, \Omega}(u) = \Phi_{1, G, \Omega}(u_{\infty}).
$$

**Step 2.** We identify the limit optimal pair.

If we consider the set $D_{\infty} = \{\phi_{\infty} > 0\}$ we get that $b := |D_{\infty}| \geq a$. Suppose that $b > a$, then by Lemma 5.1 we get $\lambda(\infty, a) < \lambda(\infty, b)$, but, on the other hand

$$
\lambda(\infty, b) = \inf_{E \subset \Omega} \lambda(\infty, E) \leq \lambda(\infty, D_{\infty}) = \inf_{v \in W^{1, G}} \Phi_{1, G, \Omega}(v)
\leq \Phi_{1, G, \Omega}(u_{\infty}) = \Lambda_{\infty} \leq \lambda(\infty, a),
$$

where we have used (5.7) and the characterization of $\Lambda_{\infty}$ from the previous step. This contradiction implies that $|D_{\infty}| = c$ and then $\phi_{\infty} = \chi_{D_{\infty}}$. 

Finally, let us prove that \(D_\infty = \{u_\infty = 0\}\). Indeed, if we take \(x \in D_\infty\), from (5.7) we get \(G(|u_\infty|) = 0\) and then \(u_\infty(x) = 0\), so \(D_\infty \subset \{u_\infty = 0\}\). Let us see that the claim \(D_\infty \subset \{u_\infty = 0\}\) leads to a contradiction. Observe that in the second step of Lemma 5.1 we proved that if \(u\) is a extremal for \(\lambda(\infty, c)\) then \(\|\{x : u(x) = 0\}\| = c\). This gives that an optimal pair \((u_\alpha, \chi_{D_\alpha})\) to \(\Lambda(\alpha, c)\) satisfies \(\|\{x : u_\alpha(x) = 0\}\| = c\) and because of the convergences in step i, \(\|\{x : u_\infty(x) = 0\}\| = c\). Then, if we suppose that \(D_\infty \subset \{u_\infty = 0\}\), we obtain that \(c = |D_\infty| < \|\{x : u_\infty(x) = 0\}\| = c\), which is absurd.

The proof is now completed.

\[\square\]

6. Generalizations and final remarks

The techniques used in our results are flexible enough to prove existence of an optimal solution to problems involving eigenvalues with different homogeneity. More precisely, fixed \(\alpha > 0\) and \(c \in [0, |\Omega|]\), given a Young function \(G\) satisfying (2.1) and (2.2), and a Young function \(H\) increasing more slowly than \(G\) in the sense that

\[
\lim_{t \to \infty} \frac{H(t)}{G(\lambda t^p)} = 0
\]

for any \(\lambda > 0\), the results of Theorems 1.1, 1.2 and 1.3 can be extended to optimization problems having the form

\[
\lambda(\alpha, c) := \inf \{\lambda(\alpha, E) : E \subset \Omega, |E| = c\}
\]

where

\[
\lambda(\alpha, E) := \min_{v \in \mathcal{A}} \left\{ \int_{\Omega} G(|\nabla v|) + G(|v|) \, dx + \alpha \int_{E} G(|v|) \, dx \right\},
\]

being \(\mathcal{A}\) the class of admissible functions given by

\[
\mathcal{A} := \left\{ v \in W_{1,G}^{1}(\Omega) : \text{ such that } \int_{\partial \Omega} H(|v|) \, dH^{n-1} = 1 \right\}.
\]

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(A. Salort) **DEPARTAMENTO DE MATEMÁTICA FCEyN - UNIVERSIDAD DE BUENOS AIRES AND IMAS - CONICET. CIUDAD UNIVERSITARIA, PABELLÓN I (C1428EGA) AV. CANTILO 2160. BUENOS AIRES, ARGENTINA.**

*Email address*: asalort@dm.uba.ar

*URL*: http://mate.dm.uba.ar/~asalort/

(A. Silva) **INSTITUTO DE MATEMÁTICA APLICADA SAN LUIS (IMASL), UNIVERSIDAD NACIONAL DE SAN LUIS, CONICET. EJERCITO DE LOS ANDES 950, D5700HHW, SAN LUIS, ARGENTINA.**

*Email address*: acsilva@unsl.edu.ar

*URL*: https://analiasilva.weebly.com

(B.B. Schwager) **INSTITUTO DE MATEMÁTICA APLICADA SAN LUIS (IMASL), UNIVERSIDAD NACIONAL DE SAN LUIS, CONICET. EJERCITO DE LOS ANDES 950, D5700HHW, SAN LUIS, ARGENTINA.**

*Email address*: bbschwager@unsl.edu.ar