A Table of Third and Fourth Order Feynman Diagrams of the Interacting Fermion
Green’s Function

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The Feynman diagrams of the Green’s function expansion of fermions interacting with a non-relativistic 2-body interaction are displayed in first, second and third order of the interaction as 2, 10 and 74 diagrams, respectively. A name convention for the diagrams is proposed and then used to tabulate the 706 diagrams of fourth order. The Hartree-Fock approximation summons up 2, 8, 40 and 224 of them, respectively.

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I. INTRODUCTION

The ground-state expectation value of the $n$-th order term in the expansion of the Green’s function of interacting fermions contains an integral kernel which comprises the ground-state expectation value of the time-ordered product of $4n + 2$ field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$, and $n$ pair interactions $U$, \[ (8.9) \]

\[
\langle | T [ \hat{\psi}^\dagger(n)\hat{\psi}^\dagger(n')U(\tau_0)^{-\sigma\sigma'}\hat{\psi}_\tau(n')\hat{\psi}_\sigma(n) \\
\cdots \hat{\psi}^\dagger(1)\hat{\psi}^\dagger(1')U(11')\lambda_\mu\nu_\lambda \hat{\psi}_\nu(1')(\hat{\psi}_\mu(1)) \\
-n(1)\hat{\psi}^\dagger(0)| \rangle. \tag{1}
\]

The arguments of the field operators are space-time coordinates. The two external vertices at which the test particle is created and destroyed have been labeled 0 and $n + 1$, and the internal vertices in pairs of primed and unprimed integers from 1 to $n$. The Greek indices at the field operators and pair interaction symbolize the additional dependence on spin.

We construct the Feynman diagrams \[2\] by contraction of the time-ordered product with Wick’s theorem \[1, \(\S 8\)\[3\], generating all possible pairs of contractions with a computer program. Disconnected diagrams are silently dropped during the process \[1, \(\S 9\).\]

The singly connected diagrams can often be abandoned in self-consistent field methods, like those related to Dyson’s equations \[2\], but keeping track of them is useful to classify all orders and an ingredient to another hierarchical method of generation as well \[2\].

II. FIRST ORDER AND NOMENCLATURE

For $n = 1$ we obtain two diagrams shown in Fig. \[1\]. The $2n + 1$ Green’s functions of the non-interacting ground state are solid lines, the $n$ pair interactions are dashed lines. In higher orders, the number of diagrams rises quickly, and tabulation of the results as diagrams becomes a laborious and space-consuming process. A tighter text version is described subsequently, and shown as a footer to each diagram up to third order to familiarize the notation. The fourth order diagrams in Section \[V\] are then all listed in the text format.

![FIG. 1: The first order diagrams of the self-energy \[1, \fig{9.7}], the exchange diagram at the left, the direct (or “tadpole”) diagram at the right.](image)

A short name will be constructed that labels each $n$-th order diagram: let the “backbone” of the graph be the sequence of 2 to $2n + 1$ fermion lines which leads from the external coordinate at which the additional particle is created to the external coordinate at which it is annihilated. This path is uniquely defined never to divert into interaction lines; within diagrams of this script it runs straight upwards. Since we will not show disconnected diagrams, this backbone passes at least one vertex and at most $2n$ vertices, from which interaction lines lead to fermion loops or return to the backbone. The vertex names are starting with 1 at the first, and are either (i) increased by one at the next vertex if a new interaction line starts from there, or (ii) use the primed number of the originating vertex, if an interaction ends that started earlier on the backbone. (The interaction lines are not directed: the phrases “start” and “end” refer to a mere book keeping order derived from the virtual tour along all directed fermion lines.) The external start and end coordinates are not part of the name; the first part of the name is the sequence of backbone vertices. Each fermion
loop is separated by a dash from the name constructed so far; diagrams with \( L \) loops have \( L-1 \) dashes in the format backbone-loop1-loop2-...-loopL. Within each loop a vertex is defined at which the loop is entered (see below). The indexing is very similar to the one for the backbone: (i) either a vertex starts a new interaction line which has not been part of any vertex in the backbone or any loop further to the left in the name; then the next higher, new integer is glued to the name, or (ii) the vertex collects an interaction line that was started earlier in the name, in case of which the primed version of this earlier index is attached. Since all loops close in this kind of perturbation theory, the “final” vertex within a loop equals the “entry” vertex: it is not repeated in the name. Within this scheme, the first number after a dash—for the entry vertex—is a primed version of a number more to the left. As a guidance, the arrow for one Green’s function within each loop has been placed near the entry vertex in the graphs. (We tacitly assume that all Green’s functions in a loop share the same direction of rotation; therefore one such arrow—indicative of the order of the arguments of one Green’s function—suffices.)

In addition, a hierarchical scheme of defining which of the loops is collected first in the name (i.e., more to the left) is also needed to maintain a one-to-one correspondence of the name to the diagram. Loops that are reached from the backbone by a single, direct interaction line (first order neighbors) are listed to the left of loops that are started from intermediate loops. Within each of the groups, loops reached by an earlier exit via an interaction line are placed more to the left. If one would create a tree-type structure of the backbone and all loops of one Feynman Diagram, this left-right scheme would mandate that the representation is transversed by first handling the root (i.e., the backbone), then the first nodes descending from the root, then the second nodes descending from the first nodes and so on. The vertex via which each loop is first accessible if this ranking is introduced also defines the entry point.

The intent of these names is (i) to allow fast intuitive recovery of the graph without loss of information in the sense that application of the Feynman rules finds all information to build the corresponding \( 2n \)-fold integral over the internal points, and (ii) one-to-one correspondence between a name and the topology in the sense that each diagram carries only one single name.

Side effects of the naming scheme are: (i) Each integer from 1 to \( n \) shows up once unprimed and once latter primed in the name. (ii) There are between zero and \( n \) dashes in the name.

There are always two possible directions (clockwise, counter-clockwise) to circle a loop as a result of the semi-directed nature of the planar graphs. In low perturbative orders (e.g., in all diagrams of Figs. 1-8), the topology is indifferent to the choice of the direction. In these cases the name does not depend on this choice, and this is consistent with the fact that then there is only one Feynman diagram to be counted and the time-reversed (hermitian) version is the original one. If otherwise the result depends on the direction to start from the loop’s entry vertex, one may (i) either flip this piece of the graph and stay with the (for example clockwise) circulation, as for example with the two graphs in the lower left of Fig. 9 or (ii) simply flip the arrow as done for the two pairs in the lower right portion of Fig. 10.

### III. SECOND ORDER

The \( N(2) = 10 \) second order diagrams, already shown in Fig. 9, contain \( N_i(2) = 4 \) improper (Fig. 2) and \( N_p(2) = 6 \) proper diagrams (Figs. 3-4), where

\[
N(n) = N_i(n) + N_p(n)
\]

is the number of \( n \)-th order diagrams, comprising \( N_i(n) \) improper and \( N_p(n) \) proper diagrams. Although formally defined in the realm of the self energy, we will use the adjectives “proper” and “improper” for the Green’s function itself, as if being stripped of the two factors (particle lines) that (i) create the particle at one external coordinate and annihilate it at the last vertex, and that (ii) create the particle at the last vertex and annihilate it at the other external coordinate.

The diagrams of Fig. 2 are obtained by repeating first order diagrams. In any order \( n \geq 1 \), \( 2^n \) improper diagrams of this type exist because there are \( N(1) = N_p(1) = 2 \) first order diagrams.

![FIG. 2: The \( N_i(2) = 4 \) second order diagrams of the improper self-energy.](image_url)

### IV. THIRD ORDER

The \( N(3) = 74 \) third order diagrams contain \( N_i(3) = 32 \) improper (Figs. 5-7) and \( N_p(3) = 42 \) proper diagrams (Figs. 8-11).
A. improper

The \( n \)th order improper diagrams are generated by concatenation of lower order diagrams, joining pairs at their external vertices. A classification follows by partitioning \( n \) into integer terms \([9]\), here \( n = 3 = 1 + 1 + 1 = 1 + 2 = 2 + 1 \). The diagrams of Fig. 5 are obtained by repeating three first order diagrams according to the decomposition \( 1 + 1 + 1 \), producing \( 2 \times 2 \times 2 = 8 \) diagrams.

The decomposition \( 1 + 2 \) yields the remaining \( 2 \times 6 = 12 \) diagrams of Fig. 6 of leading first order, and the decomposition \( 2 + 1 \) yields \( 6 \times 2 = 12 \) diagrams of Fig. 6. The number \( N_i(n) \) of improper diagrams is given by summation over all ordered partitions weighted by the number

\[
N_i(n) = \sum_{l=1}^{n-1} N_p(l) N(n - l). \tag{3}
\]

There are \( N_p(l) \) ways to choose a “leading” proper diagram that contains what we have labeled vertex 1, and each can be followed in \( N(n - l) \) ways by some diagram of order \( n - l \).

B. proper

The third order diagrams in Fig. 8 are created by blending the second order diagrams into the first order diagram 11′ which becomes the skeleton. The names are derived from the second order “intruder” diagrams by incrementing all numbers in the intruder diagram by one.
FIG. 6: The 12 third order diagrams of the improper self-energy from succession of a first order diagram and a proper second order diagram.

and enclosing the backbone part of the name in a pair of 1 and 1'. The third order diagrams in Fig. 6 are created by blending the second order diagrams into the first order tadpole diagram: In accordance with the rules described above, the names are derived from the second order intruder diagrams by incrementing all numbers in the intruder diagram by one, then placing 1-1' in front.

FIG. 7: The 12 third order diagrams of the improper self-energy from a leading proper second order followed by one of the two first order diagrams.

(Two examples: 1-1’22’33’ from 11’22’, and 1-1’22’3-3’ from 11’2-2’.)

More doubly-connected third order diagrams follow from insertion of any of the two first order diagrams into one of the three internal fermion lines of the proper second order diagrams: Fig. 7 shows those that have not
FIG. 8: The third order diagrams of the proper self-energy from insertion of the 10 second order diagrams into the first order 1-1' "exchange" diagram. Two diagrams, 1231'-2'3' and 1-1'232'3', are shown as F12 and F14 in [10, Fig 4].

FIG. 9: The third order diagrams of the proper self-energy from insertion of the 10 second order diagrams into the first order 1-1' "direct" diagram. Two of these, 1-1'232'3' and 1-1'23-2'3', are shown as F11 and F13 in [10, Fig 4].

yet been part of Fig. 9.

The concluding set of third order proper diagrams in Fig. 11 contains diagrams that cannot be decomposed into a skeleton with insertions.
FIG. 10: Third order diagrams of the proper self-energy from insertion of a first order diagram into a second order fermion line.

FIG. 11: The remaining third order diagrams of the proper self-energy of some “complex” exchange type. All of these are shown as F1–F10 in [10, Fig 4].
V. FOURTH ORDER

The $N(4) = 706$ diagrams of the fourth order perturbation contain $N_i(4) = 2 \times 74 + 6 \times 10 + 2 \times 42 = 292$ improper diagrams of leading order $l = 1, 2$ or $3$ according to (3), and $N_p(4) = 414$ proper diagrams.

A. improper

Guided by the partitioning of the order 4 into $4 = 1 + \ldots = 2 + \ldots = 3 + 1$, the improper diagrams are either (i) a first order diagram followed by a proper or improper third order diagram, or (ii) a proper second order diagram followed by a proper or improper second order diagram, or (iii) a proper third order diagram followed by a first order diagram. This classification leads to the following subsections.

1. Leading First Order

This set contains $N_p(1)N(3) = 2 \times 74$ diagrams since we have 2 first order diagrams and 74 third order diagrams. If the first order diagram is the exchange diagram, the name pattern is $1\ldots 1'$ and is listed in Table I; if the first order diagram is the direct diagram, the name pattern is $1\ldots -1'\ldots$ or $1\ldots 1''$ and is listed in Table II.

| TABLE I: 74 improper fourth order diagrams created by starting with $11'$ and attaching one of the 32 improper third order diagrams (first eight lines) or 42 proper third order diagrams (final eleven lines) |
|---|
| $11'233'2'44'-1'$ | $11'233'3'-1'2'$ |
| $11'233'2'-1'2'$ | $11'233'4'3'$ |
| $11'233'3'-1'2'$ | $11'233'3'2'$ |
| $11'233'3'-1'2'$ | $11'233'4'2'$ |
| $11'233'2'44'$ | $11'233'3'-1'2'$ |
| $11'233'2'44'$ | $11'233'2'44'$ |
| $11'233'2'44'$ | $11'233'2'44'$ |
| $11'233'2'44'$ | $11'233'2'44'$ |

2. Leading Second Order

This set contains $N_p(2)N(2) = 6 \times 10$ diagrams since we have 6 proper second order diagrams and 10 second order diagrams. These six groups form Table III.

3. Leading Third Order

This set contains $N_p(3)N(1) = 42 \times 2$ diagrams since we have 42 proper third order diagrams and 2 first order diagrams. Those that end on the first order exchange diagram have name patterns $\ldots 22\ldots$, $\ldots 33\ldots$, or $\ldots 44\ldots$, and are listed in Table IV. Those that end on the first order tadpole diagram have name patterns $\ldots 2-2'$, $\ldots 3-3'$, or $\ldots 4'$ and are listed in Table V.

B. proper

Some graphs are created by replacing internal Green’s functions in $i$-th order diagrams—which become the skeleton of the $n$-th order diagram ($i < n$)—by one or more diagrams with a total order of $i - n$. 

TABLE III: 60 improper fourth order diagrams created by starting with any of the six proper second order diagrams.

TABLE IV: 42 improper fourth order diagrams created by starting with a proper third order diagram and attaching the 11' diagram.

TABLE V: 42 improper fourth order diagrams created by starting with a proper third order diagram and attaching a 1-1' diagram.

TABLE VI: 74 proper fourth order diagrams from moving a third order diagram into 11'.

TABLE VII: 42 improper fourth order diagrams created by starting with a proper third order diagram and attaching the 1-1' diagram.

1. First Order Skeleton

74 proper fourth order diagrams are created by moving with any of the third order diagrams into the central Green's function of the 11' diagram, creating Table VI. The name pattern is 1...1' or 1...1'-...-

74 proper fourth order diagrams are created by moving with any third order diagram into the tadpole's head of the 1-1' diagram, tabulated in Table VII. The name pattern is 1-1'...-

2. Second Order Skeleton

More proper fourth order diagrams are created by introducing any of the 10 proper or improper second order diagram into one of the three internal Green's functions of the two proper second order diagrams of Fig. 4. Those with skeleton 12-1'2' are in Table VIII, those with skeleton 121'2' in Table IX. Application of this method to the four second order proper diagrams of Fig. 3 generates no new diagrams beyond those already incorporated in the previous section with first order skeletons.

Additional diagrams listed in Table X are created by insertion of two first order diagrams at two different
TABLE VII: 74 proper fourth order diagrams from moving a third order diagram into $1\rightarrow 1'$.

| Diagram 1 | Diagram 2 | Diagram 3 | Diagram 4 |
|-----------|-----------|-----------|-----------|
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |

TABLE VIII: 30 proper fourth order diagrams by insertion of any of the 10 second order diagrams into any of the three internal Green's functions of $12\rightarrow 1'$.

| Diagram 1 | Diagram 2 | Diagram 3 | Diagram 4 |
|-----------|-----------|-----------|-----------|
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |

TABLE IX: 30 proper fourth order diagrams by insertion of any of the 10 second order diagrams into any of the three internal Green's functions of $12\rightarrow 1'$.

| Diagram 1 | Diagram 2 | Diagram 3 | Diagram 4 |
|-----------|-----------|-----------|-----------|
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |

TABLE X: 24 proper fourth order diagrams from insertion of two first-order diagrams at two different places of $12\rightarrow 1'$ (first 12 entries) or of $12\rightarrow 1'$ (second 12 entries).

| Diagram 1 | Diagram 2 | Diagram 3 | Diagram 4 |
|-----------|-----------|-----------|-----------|
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |

TABLE XI: 50 proper fourth order diagrams from insertion of the first order exchange diagram into the "complex" third order diagrams of Fig. 11.

| Diagram 1 | Diagram 2 | Diagram 3 | Diagram 4 |
|-----------|-----------|-----------|-----------|
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |
| $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ | $1\rightarrow 12'34'2'$ |

TABLE XII: 82 proper fourth order diagrams generated by adding exchange interactions to graphs in Fig. 11, and some alternatively by inserting an empty "vacuum polarization" bubble into one of the third order diagrams. We are not trying to build a unique or well defined inheritance scheme toward the third order parent diagrams. Assuming the fourth order diagram is best displayed with a minimum number of crossing lines, we classify them as if bubbles were inserted, where possible, which yields Table XIII.

3. Third Order Skeleton

Each of the 10 diagrams of Fig. 11 has 5 internal Green's functions into which one can can insert one of the first order diagrams. Insertion of $11'$ yields the diagrams of Table XIII insertion of $1\rightarrow 1'$ those of Table XI 100 diagrams in total.

4. complex

The remaining 82 fourth order diagrams could be generated by adding exchange interactions to graphs in Fig. 11 and some alternatively by inserting an empty "vacuum polarization" bubble into one of the third order diagrams. We are not trying to build a unique or well defined inheritance scheme toward the third order parent diagrams. Assuming the fourth order diagram is best displayed with a minimum number of crossing lines, we classify them as if bubbles were inserted, where possible, which yields Table XIII.

The quickest way of visualizing the final set of fourth
TABLE XII: 50 proper fourth order diagrams from insertion of the first order tadpole diagram into the "complex" third order diagrams of Fig. 11.

| Diagram | Diagram | Diagram | Diagram |
|---------|---------|---------|---------|
| 1234-1'3'4'-2' | 123-1'3'4'-2' | 12341'3'2'-4' | 1231'42'2'-3' |
| 12341'3'2'-4' | 1232'1'43'-4' | 12341'3'2'-4' | 1231'42'2'-3' |
| 12341'3'2'-4' | 1231'3'2'-4' | 12341'3'2'-4' | 1231'42'2'-3' |
| 12341'3'2'-4' | 1231'3'2'-4' | 12341'3'2'-4' | 1231'42'2'-3' |
| 12341'3'2'-4' | 1231'3'2'-4' | 12341'3'2'-4' | 1231'42'2'-3' |

TABLE XIII: 25 proper fourth order diagrams from insertion of a fermion loop into an interaction line of one diagram of Fig. 11. The third order reference diagram is added in the column left to the line. Each fourth order diagram is added at most once.

| Diagram | Diagram | Diagram | Diagram |
|---------|---------|---------|---------|
| 121'3-2'3' | 1234-1'3'2'-4' | 121'3-2'3' | 1234-1'3'2'-4' |
| 1232'-1'3'4' | 1234-1'3'2'-4' | 1232'-1'3'4' | 1234-1'3'2'-4' |
| 1232'-1'3'4' | 1234-1'3'2'-4' | 1232'-1'3'4' | 1234-1'3'2'-4' |
| 1232'-1'3'4' | 1234-1'3'2'-4' | 1232'-1'3'4' | 1234-1'3'2'-4' |
| 1232'-1'3'4' | 1234-1'3'2'-4' | 1232'-1'3'4' | 1234-1'3'2'-4' |

VI. STATISTICS OF HARTREE-FOCK TERMS

The Hartree-Fock (HF) approximation to the Green’s function is a particular type of merging the two first order diagrams with the Dyson equation (Fig. 12). It is equivalent to assembling all proper graphs that can be constructed by replacing iteratively any of the internal fermion lines in Fig. 1 by any of the two graphs or by any improper graphs that can be recursively constructed by this method. Any improper diagram that is a chain of diagrams of this type is included as well.

![Fig. 12: Hartree-Fock approximation to the Dyson equation of the Green’s function][1]

In this sense, the improper second order diagrams are all of the HF-class (Fig. 2), some of the proper diagrams are as well (Fig. 3), and two are not (Fig. 4). We introduce the notation

\[ H^{(1)}(n) = H^{(1)}_i(n) + H^{(1)}_p(n), \quad H^{(1)}(n) \leq N(n) \quad (4) \]

for the number of HF diagrams of order \( n \), \( H^{(1)}_i(n) \) of which are improper and \( H^{(1)}_p(n) \) of which are proper, bounded by the total number of diagrams in these subclasses:

\[ H^{(1)}_i(n) \leq N_i(n), \quad H^{(1)}_p(n) \leq N_p(n). \quad (5) \]
All 8 diagrams of Fig. [5] 8 diagrams of Fig. [6] and 8 diagrams of Fig. [7] \( H^{(1)}_i(3) = 24 \) out of \( N_i(3) = 32 \) improper third order diagrams are of the HF-type. 8 diagrams of Fig. [8] 8 diagrams of Fig. [3] and none of the diagrams of Figs. 10, 11 [total: \( H^{(1)}_p(3) = 16 \) out of \( N_p(3) = 42 \) proper third order diagrams] are of HF-type. Accumulating both statistics, \( H^{(1)}(3) = 40 \) out of \( N(3) = 74 \) (54%) of the third order diagrams are of HF-type.

The HF-property is volatile if one looks at improper higher order composite diagrams. Any non-HF diagram within the partitioning destroys it. If all pieces are of HF-type, the property is maintained:

\[
H_i^{(1)}(n) = \sum_{l=1}^{n-l} H_p^{(1)}(l)H_i^{(1)}(n-l).
\]

Applied to the improper fourth order diagrams we get:

\[
H_p^{(1)}(1)H^{(1)}(3) = 2 \times 40 = 80 \text{ out of the 148 diagrams of Section } \text{Table IX} \text{ are of the HF type. } H_p^{(1)}(2)H^{(1)}(2) = 4 \times 8 = 32 \text{ of the 60 diagrams of Section } \text{Table XI} \text{ are of the HF type. With reference to Section } \text{Table X} \text{ } H^{(1)}(3)H^{(1)}(1) = 16 \times 2 = 32 \text{ out of the 84 diagrams of Section } \text{Table XII} \text{ are in the HF class. Adding up, } H^{(1)}(4) = 80 + 32 + 32 = 144 \text{ out of } N(4) = 148 + 60 + 84 = 292 \text{ improper fourth order diagrams are in the HF class.}

The same devaluation occurs if proper higher order diagrams are composites of a skeleton and lower order insertions: all elements (the skeleton and the insertions) must be of the HF class to preserve the property. 2 \times 40 = 80 \text{ out of the 148 diagrams of Section } \text{Table IX} \text{ are of the HF class. None of the 84 + 100 + 82 = 266 diagrams of Sections } \text{Table XII} \text{ is in the HF class. So } H_p^{(1)}(4) = 80 \text{ out of } N_p(4) = 414 \text{ proper fourth order diagrams are in the HF class. In total, } H^{(1)}(4) = 224 \text{ out of } N(4) = 706 \text{ (32\%)} \text{ of the fourth order diagrams are in the HF class.}

Inclusion of those second order diagrams which are not yet part of the HF theory, those of Fig. 11 extends the self-consistent field to the second order self energy. This extends Fig. 12 to Fig. 14 equivalent to Cederbaum’s Eq. (4.26) \[11\] and generalized in \[12\] Fig. 4. This extended theory includes all second order diagrams (by construction), all improper third order diagrams, and all proper third order diagrams built upon a skeleton of lower order. Only the 10 third order diagrams in Fig. 13 remain excluded. The statistics of included third order diagrams becomes 64 out of \( N(3) = 74 \) (86\%). Like in \[6\] we introduce a notation

\[
H^{(2)}(n) = H_i^{(2)}(n) + H_p^{(2)}(n), \quad \text{for the counts in this extended theory. Eq. } \text{11 and the “devaluation rule” remain applicable for the upper indices raised to (2): } H_i^{(2)}(4) = 2 \times 64 + 6 \times 10 + 32 \times 2 = 252.
\]

This evaluation yields Table XVI.

\[
\begin{array}{cccccc}
 n & N(n) & H^{(1)}(n) & H^{(2)}(n) & H_i^{(2)}(n) & H_p^{(2)}(n) \\
 1 & 2 & 2 & 0 & 2 \\
 2 & 10 & 8 & 10 & 4 & 6 \\
 3 & 74 & 40 & 64 & 32 & 32 \\
 4 & 706 & 224 & 464 & 252 & 212 \\
 5 & 8162 & 1344 & 3624 & 2056 & 1568 \\
 6 & 110410 & 8448 & 29744 & 17336 & 12408 \\
\end{array}
\]

![FIG. 13: Extended Hartree-Fock approximation to the Dyson equation of the Green’s function.](image)

\[\]

VII. HIGHER ORDERS

The diagrams of fifth and sixth order are not listed explicitly but available in the files public/Feyn[56]ip.txt on the author’s web page; the first wildcard means \( n \) and the second separates proper from improper diagrams.

The series \( N(n) \) with increasing perturbative order \( n \) is also obtained by insertion of \( l = 0 \) in \[13\] (11), because this choice of a “vertex coupling strength” reduces \[13\] (7) to the theory of bare Coulomb interactions. In sixth and seventh order we have \( N(6) = 110410 \) and \( N(7) = 1708394 \), see entry A000698 in the Encyclopedia of Integer Sequences \[14\], Table 2 in \[15\], Table 1 in \[16\], the list of \( S_n(1) \) in \[17\], and \[17\] (2.3). Recursive application of \[6\] yields Table XVI.

VIII. SUMMARY

The third order of the interacting Green’s function expansion in powers of the 2-fermion interaction contains 42 proper self-energy terms, the fourth order 414 terms.

The rules for recursive construction of the names of the \( N(n) \) topologically distinct diagrams are: (i) Start the name with a bare 1. (ii) Attach all possible combi-
TABLE XVI: Term counts and their decomposition in subsets of improper and proper diagrams. For comparison: the Goldstone expansion needs 84 diagrams at \( n = 3 \) and 3120 at \( n = 4 \), respectively \[11\].

| \( n \) | \( N(n) \) | \( N_i(n) \) | \( N_p(n) \) |
|---|---|---|---|
| 1 | 2 | 0 | 2 |
| 2 | 10 | 4 | 6 |
| 3 | 74 | 32 | 42 |
| 4 | 706 | 292 | 414 |
| 5 | 8162 | 3104 | 5058 |
| 6 | 110410 | 37924 | 72486 |

nations of three types of tokens, (a) bare numbers, (b) primed numbers and (c) dashes, subject to the following constraints: (iii) Each bare number from 1 to \( n \) appears once. This subsequence of bare numbers is sorted in natural order from the left to the right. (iv) Each primed number from \( 1' \) to \( n' \) appears once. A primed number appears somewhere to the right of the associated bare number. (v) Every dash is followed by some primed number. (vi) The set of primed numbers immediately after dashes is sorted: a string \(-j'\) must appear after a string \(-i'\) if \( j > i \). (vii) Within the set of primed numbers after a dash up to the next dash or the end of the name (whichever comes first, ie, up to where the fermion loop ends), the primed number immediately after the dash is the smallest.

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