FAST AND OBLIVIOUS ALGORITHMS FOR DISSIPATIVE AND 2D WAVE EQUATIONS

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Abstract. The use of time-domain boundary integral equations has proved very effective and efficient for three dimensional acoustic and electromagnetic wave equations. In even dimensions and when some dissipation is present, time-domain boundary equations contain an infinite memory tail. Due to this, computation for longer times becomes exceedingly expensive. In this paper we show how oblivious quadrature, initially designed for parabolic problems, can be used to significantly reduce both the cost and the memory requirements of computing this tail. We analyse Runge-Kutta based quadrature and conclude the paper with numerical experiments.

Key words. fast and oblivious algorithms, convolution quadrature, wave equations, boundary integral equations, retarded potentials, contour integral methods.

AMS subject classifications. 65R20, 65L06, 65M15, 65M38

1. Introduction. Certain wave problems exhibit the property that behind the wave front travelling at a finite speed there exists a smooth tail. The simplest examples of this phenomenon are the scalar wave equation in even dimensions or damped wave equation in any dimension. Recently the interest in the numerical solution of scalar and vector linear wave equations by means of time-domain boundary integral equations has risen sharply [2, 6, 13, 21, 22, 26, 27, 28, 29, 31, 32]. Two main approaches to the discretization are time-space Galerkin methods [3] and Convolution Quadrature (CQ) [23].

The difficulty that arises in applying time-domain boundary integral methods to dissipative or two-dimensional wave equation can best be appreciated by comparing the free space Green’s functions for the two and three dimensional acoustic wave equation

\[ G_{2D}(t, x) = \frac{H(t - |x|)}{2\pi \sqrt{t^2 - |x|^2}} \quad G_{3D}(t, x) = \frac{\delta(t - |x|)}{4\pi |x|}, \]

where \( H(\cdot) \) is the Heaviside function and \( \delta(\cdot) \) is the Dirac delta. Here we see that whereas in three dimensions the support of the Green’s function is on the time-cone \( t = |x| \), in two dimensions the Green’s function is non-zero for all \( t > |x| \) and the decay in \( t \) is slow. This infinite tail forces an infinite memory on time-domain boundary integral equation based methods that results in expensive long-time computations. This will affect any numerical method based on time-domain integral equations – in particular both time-space Galerkin and Convolution Quadrature. The smoothness of this tail has been used in [10] and [5] to speed up computations and more importantly for the current work, in [5] it was noticed that this tail is due to an operator of parabolic type; the precise meaning of this is explained in the next section. This fact was used in [5] for purely theoretical purposes whereas in this paper it will be used

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to develop a fast algorithm with reduced memory requirements. Our new algorithm applied to the tail of the linear hyperbolic problems has essentially the same properties as the algorithms developed in [25, 30] for parabolic problems. To be more precise, let \( n_0 > \text{diam}(\Omega)/h \) where \( h \) is the time-step and \( \Omega \) is the scatterer; in other words \( n_0 > |x|/h \) for all \( |x| \) in the above. Fast CQ algorithms for hyperbolic problems, see [4], can compute the first \( n_0 \) steps in \( O(n_0 \log^2 n_0) \) or \( O(n_0 \log n_0) \) time and \( O(n_0) \) memory and history. Denoting by \( N = n - n_0 \) the number of time-step after \( n_0 \), in the following we discuss the additional cost required for a target accuracy \( \varepsilon \).

- The computational complexity is \( O \left( \log \left( \frac{1}{\varepsilon} \right) N \log(N) \right) \).
- The number of evaluations of the transfer operator \( K \) is reduced from \( O(N) \) to \( O \left( \log \left( \frac{1}{\varepsilon} \right) \log(N) \right) \).
- The memory requirements are reduced from \( O(N) \) to \( O \left( \log \left( \frac{1}{\varepsilon} \right) \log(N) \right) \).

The structure of the paper is as follows. In the next section we consider an abstract setting that covers the motivating application described above. Next, Runge-Kutta based Convolution Quadrature is introduced. In the main part of the paper, the description and analysis of an efficient scheme to compute the convolution weights is described. The paper concludes with a more detailed description of the time-domain boundary integral method and with the results of numerical experiments.

2. The abstract setting. Let \( K(\lambda, d) : U_\delta \times \mathbb{R}_{>0} \rightarrow \mathbb{C} \) be analytic as a function of \( \lambda \) in the sector

\[
U_\delta = \{ \lambda \in \mathbb{C} : |\text{Arg} \lambda| < \pi - \delta \}, \quad \delta \in (0, \pi/2).
\]

and for some \( \mu \in \mathbb{R} \) bounded as

\[
|e^{\lambda d}K(\lambda, d)| \leq C(d)|\lambda|^\mu, \quad \lambda \in U_\delta.
\]  \hspace{1cm} (2.1)

Note that this in turn implies the standard bound for hyperbolic operators

\[
|K(\lambda, d)| \leq C(d)|\lambda|^\mu, \quad \text{for } \text{Re} \lambda > 0.
\]  \hspace{1cm} (2.2)

We are interested in computing convolutions

\[
u(t) = \int_0^t k(t - s, d)g(s)ds,
\]  \hspace{1cm} (2.3)

where \( k(t, d) \) is the inverse Laplace transform of \( K(\lambda, d) \). If \( \mu < -1 \), \( k(t, d) \) as a function of \( t \) is continuous and the above convolution is a well-defined continuous function for integrable \( g \). For \( \mu \geq -1 \), the convolution is defined directly via the inverse Laplace transform

\[
u(t) = \frac{1}{2\pi i} \int_{\gamma+i\mathbb{R}} e^{\lambda t}K(\lambda, d)Lg(\lambda)d\lambda,
\]

where \( \sigma > 0 \) and \( Lg(\lambda) = \int_0^\infty e^{-\lambda t}g(t)dt \) denotes the Laplace transform. For data \( g \) such that its Laplace transform decays sufficiently fast, the inverse Laplace transform above defines a continuous function, see [23].

Our aim is to describe an efficient algorithm for the computation of (2.3). An important aspect of the algorithm is that it should be effective for a range of \( 0 < d \leq R \) where \( R > 0 \) is given. So far in the literature, fast algorithms have been considered in
the non-sectorial, hyperbolic, case, i.e., kernels satisfying \(2.2\), \([9, 4]\). In the sectorial, parabolic, case, i.e., kernels bounded as

\[
|K(\lambda, d)| \leq C(d)|\lambda|^p, \quad \lambda \in U_{\delta},
\]

fast algorithms which further allow huge memory savings are available \([25, 17, 30, 18]\). Our kernel is strictly speaking non-sectorial, but after multiplication with \(e^{\lambda d}\) becomes sectorial; this is what was meant by tail being due to a parabolic operator. Naturally, this special class of operators requires its own fast algorithms.

3. **Runge-Kutta convolution quadrature.** Time discretization methods used in this paper, are based on \(A\)-stable Runge-Kutta methods \([14]\). We employ standard notation for an \(s\)-stage Runge-Kutta discretization based on the Butcher tableau described by the matrix \(A = (a_{ij})_{i,j=1}^s \in \mathbb{R}^{s \times s}\) and the vectors \(b = (b_1, \ldots, b_s)^T \in \mathbb{R}^s\) and \(c = (c_1, \ldots, c_s)^T \in [0, 1]^s\). The corresponding stability function is given by

\[
r(z) = 1 + zb^T(I - zA)^{-1}1,
\]

where

\[
1 = (1, 1, \ldots, 1)^T.
\]

Note that \(A\)-stability is equivalent to the condition \(|r(z)| \leq 1\) for \(\text{Re} \ z \leq 0\). In the following we collect all the assumptions on the Runge-Kutta method. These are satisfied by, for example, Radau IIA and Lobatto IIIC families of Runge-Kutta methods.

**Assumption 1.**

(a) The Runge-Kutta method is \(A\)-stable with (classical) order \(p \geq 1\) and stage order \(q \leq p\).

(b) The stability function satisfies \(|r(iy)| < 1\) for all real \(y \neq 0\).

(c) \(r(\infty) = 0\).

(d) The Runge-Kutta coefficient matrix \(A\) is invertible.

Since \(r(z)\) is a rational function, the above assumptions imply that

\[
r(z) = O(z^{-1}), \quad |z| \to \infty.
\]

We define the weight matrices \(W_n\) corresponding to the operator \(K\) by

\[
\sum_{n=0}^{\infty} W_n(d)\zeta^n = K\left(\frac{\Delta(\zeta)}{h}, d\right),
\]

where

\[
\Delta(\zeta) = \left(1 + \frac{\zeta}{1 - \zeta}1b^T\right)^{-1}.
\]

Denoting by \(\omega_n = (\omega_1^n, \ldots, \omega_s^n)\) the last row of \(W_n\), the approximation to the convolution integral \((2.3)\) at time \(t_{n+1} = (n+1)h\) is given by

\[
u_{n+1} = \sum_{j=0}^{n} \sum_{i=1}^{s} \omega_{n-j}^i(d)g(t_j + c_i h) = \sum_{j=0}^{n} \omega_{n-j}(d)g_j
\]

with the column vector \(g_j = g(t_j + ch) = (g(t_j + c_i h))_{i=1}^s\).
The convergence order of this approximation has been investigated in [24] for parabolic problems, i.e., for sectorial \( K(\lambda) \), and in [7] and [8] for hyperbolic problems, i.e., for non-sectorial operators.

With the row vector \( e_n(z) = (e_1^n(z), \ldots, e_s^n(z)) \) defined as the last row of the \( s \times s \) matrix \( E_n(z) \) given by

\[
(\Delta(\zeta) - zI)^{-1} = \sum_{n=0}^{\infty} E_n(z) \zeta^n,
\]

(3.6)

we obtain an integral formula for the weights

\[
\omega_n(d) = \frac{h}{2\pi i} \int_\Gamma K(\lambda, d)e_n(h\lambda) d\lambda.
\]

(3.7)

This representation follows from Cauchy’s formula and the definition of the weights in (3.3), with the integration contour \( \Gamma \) chosen so that it and surrounds the poles of \( e_n(h\lambda) \). An explicit expression for \( e_n \) is given by

\[
e_n(z) = r(z)^n q(z),
\]

(3.8)

with the row vector \( q(z) = b^T(I - zA)^{-1} \); cf. [24, Lemma 2.4]. The A-stability assumption implies that the poles of \( r(z) \) are all in the right-half plane. Further, due to the decay of the rational function \( r(z) \), see (3.2), for \( n > \mu + 1 \), the contour \( \Gamma \) can be deformed into the imaginary axis.

For the weight matrices it holds

\[
W_n(d) = \frac{h}{2\pi i} \int_\Gamma K(\lambda, d)E_n(h\lambda) d\lambda.
\]

(3.9)

By [24, Lemma 2.4], for \( n \geq 1 \), \( E_n(z) \) is the rank-1 matrix given by

\[
E_n(z) = r(z)^{n-1}(I - zA)^{-1} \frac{1}{2} b^T(I - zA)^{-1}.
\]

(3.10)

The Runge-Kutta approximation of the inhomogeneous linear problem

\[
y'(t) = \lambda y(t) + g(t), \quad y(0) = 0
\]

(3.11)

at time \( t_{n+1} \) is given by

\[
y_{n+1}(\lambda) = h \sum_{j=0}^{n} e_{n-j}(h\lambda)g_j
\]

(3.12)

and thus the approximation of the convolution integral in (3.5) can be rewritten as [24, Proposition 2.4]

\[
u_{n+1} = \frac{1}{2\pi i} \int_\Gamma K(\lambda, d)y_{n+1}(\lambda)d\lambda.
\]

(3.13)

In the next section we discuss how to approximate the integral in (3.7) by an efficient quadrature rule. It turns out that there exists \( s > 0 \) such that for \( n_0 + s < n < n_0 + sB \) for an offset \( n_0 \) proportional to \( d/h \) the quadrature error decays exponentially in the number of quadrature nodes. The convergence rate depends on \( B > 1 \), but is independent of \( s \). This is the key observation for the fast algorithm explained in section 4.2, where we split the sum in (3.12) and use contour quadrature to approximate (3.13).
4. Efficient quadrature for the computation of weights. If $K(\lambda)$ is a sectorial operator, in [17] it is shown that the contour in the integral formula for the weights $\omega_n(d)$ in (3.7) can be chosen as the left branch of a hyperbola with center at the origin and foci on the real axis. In [30] both hyperbolas and Talbot contours are tested and shown to work in practice. In the case of kernels considered here, the contour must not have a real part extending to $-\infty$. Proposition 4 shows that cutting the hyperbola at a finite real part, commits a small error. For the proof we will require the following technical lemma.

**Lemma 2.** Let

$$\gamma(\xi) = \inf_{-\xi \leq \text{Re} z \leq 0} \frac{\log|r(z)|}{\text{Re} z}.$$  

Then $\gamma(\xi) \in (0, 1]$ for $\xi > 0$, it monotonically increases as $\xi \to 0$ and

$$|r(z)| \leq |e^{\gamma(\xi)z}|,$$

for all $z$ in the strip $-\xi \leq \text{Re} z \leq 0$.

Further, there exists $\rho > 0$ such that

$$|r(z)| \leq |e^{2z}|,$$

for all $z$ in the strip $0 \leq \text{Re} z \leq \rho$.

**Proof.** By assumption $|r(z)| < 1$ for all $\text{Re} z < 0$ and $r(\infty) = 0$, hence $\gamma(\xi) > 0$. Further, $\gamma(\xi)$ cannot be greater than 1, since this would contradict the approximation property

$$r(z) = e^z + O(z^{p+1}).$$

The bound on $r(z)$ is clear by the definition of $\gamma(\xi)$.

The remaining statement can be proved similarly, making sure that $\rho$ is less than the real part of any singularity of $r(z)$; for a similar statement see Lemma 1 in [17].

**Remark 3.** Some numerically obtained values of $\gamma$ are given in Table 4.1. For backward Euler $\gamma(\xi)$ is given explicitly by

$$\gamma(\xi) = \frac{\log(1 + \xi)}{\xi}.$$  

In order to reduce the number of constants in the estimates, we have chosen not to be as precise about the bound for the case $\text{Re} z > 0$. The optimality of the estimates has nevertheless not been significantly affected.

With this we have that the integrand in (3.7) can be bounded as

$$|K(\lambda, d)r(\lambda h)^n| \leq C(d)|\lambda|^\mu e^{-\lambda d/n} r(\lambda h)^n \leq C(d)|\lambda|^\mu e^{\text{Re} \lambda/n} (\gamma(\xi) - d/n)$$  

(4.1)
Fig. 1. Contour $\Gamma_0$ is depicted by the solid line and $\Gamma_{-1}$ and $\Gamma_1$ by the dashed lines.

and thus it decreases exponentially with $-\Re \lambda t_n$ as long as $0 < -\Re \lambda h < \xi$ and $d/t_n < \gamma(\xi)$. This suggests replacing contour $\Gamma$ in (3.1) with a finite section of a hyperbola:

$$\Gamma_0 = \nu \varphi([−a, a]), \quad \varphi(x) = 1 - \sin(\alpha - ix), \quad \nu > 0.$$  \hfill (4.2)

We want the endpoint of the finite hyperbola to be in the left-half complex plane, i.e.,

$$\Re \varphi(a) = (1 - \sin \alpha \cosh a) < 0 \iff \cosh a > 1/\sin \alpha.$$  \hfill (4.3)

The right-most point on the hyperbola is given by

$$\nu \sup_{x \in [−a, a]} \Re \varphi(x) = \nu \varphi(0) = \nu(1 - \sin \alpha) < \nu.$$  \hfill (4.4)

The error committed in replacing $\Gamma$ with $\Gamma_0$ is investigated next.

**Proposition 4.** Let $d, \delta > 0$ and $\mu$ be given such that $K(\lambda, d)$ satisfies (2.1) and let $\alpha$ and $\alpha \in (0, \pi/2 - \delta)$ be given such that $\cosh a > 1/\sin \alpha$. Then for $h, \nu_0 > 0$, $m > \mu$, $0 < \nu \leq \nu_0$ and $t_n - m > d/\gamma(\xi)$ with $\xi = h \nu_0 |\Re \varphi(a)|$,

$$\bigg| \frac{h}{2\pi i} \int_{\Gamma_0} K(\lambda, d) e_n(\lambda h) d\lambda \bigg| \leq C |\nu \varphi(a)|^{\mu-m} h^{-m} e^{\nu |\Re \varphi(a)| (\gamma(\xi) t_n - m - d)},$$

where $C = \text{const} \cdot C(d)$, with $C(d)$ in (2.1).

**Proof.** Let us choose the contour in (3.1) as $\Gamma = \Gamma_{-1} + \Gamma_0 + \Gamma_1$, where $\Gamma_0$ is defined by $\{ \bar{w} - i \rho \mid \rho \in [0, \infty) \}$ and $\Gamma_1 = \{ w + i \rho \mid \rho \in [0, \infty) \}$, where $w = \nu \varphi(a)$; see Figure 1.

To prove the required result we need to bound

$$\frac{h}{2\pi i} \int_{\Gamma_{\pm 1}} K(\lambda, d) e_n(\lambda h) d\lambda.$$  \hfill (4.5)

From the definition of $\Gamma_{\pm 1}$, (4.1), property (2.1) of $K(\lambda)$ and the fact that $r(z) = O(z^{-1})$ it follows that for $\lambda \in \Gamma_{-1} \cup \Gamma_1$

$$h |K(\lambda, d)r(\lambda h^n)| \leq C h^{1-m} |\lambda|^{\mu-m} e^{\Re \lambda (\gamma t_n - m - d)} = C h^{1-m} |\lambda|^{\mu-m} e^{\nu |\Re \varphi(a)| (\gamma t_n - m - d)},$$

where $C = \text{const} \cdot C(d)$, with $C(d)$ in (2.1).
Hence
\[ h \int_{\Gamma_{\pm 1}} \| K(\lambda, d) r(\lambda h)^n b^T (I - \lambda h A)'^{-1} \| \]
\[ \leq C h^{1-m} e^{\nu \Re \varphi(a)(\gamma t_{n-m}-d)} \int_{\Gamma_{\pm 1}} |\lambda|^{\mu-m} \| (I - \lambda h A)'^{-1} \| d\lambda \]
\[ \leq C h^{-m} e^{\nu \Re \varphi(a)(\gamma t_{n-m}-d)} \int_{\Gamma_{\pm 1}} |\lambda|^{\mu-m-1} d\lambda \]
\[ \leq C |\nu \varphi(a)|^{\mu-m} h^{-m} e^{\nu \Re \varphi(a)(\gamma t_{n-m}-d)}. \]
finishes the proof. \( \square \)

The next step is to define a quadrature on the interval \([-a, a]\) and hence on the contour \(\Gamma_0\). Since the integrand is small at the edges of the interval, trapezoidal quadrature is a good choice and the following result gives the quadrature error.

**Proposition 5.** Let \(d, \delta > 0\) and \(\mu\) be given such that \(K(\lambda, d)\) satisfies (2.1). Let \(\alpha \in (0, \pi/2 - \delta)\) and \(a > 0\) be such that \(\cosh a > 1/\sin \alpha\) and \(0 < b < \min(\alpha, \pi/2 - (\delta + \alpha))\). Assume that \(h \neq 0\) are small enough so that \(h \nu_0(1 - \sin(\alpha - b)) < \rho\) with \(\rho\) as in Lemma 4. For \(L \in \mathbb{N}\) and \(\tau = a/L\) let
\[ \xi = h\nu_0 \sup_{y \in [-b, b]} |\Re \varphi(a + \tau/2 + iy)| = -h\nu_0(1 - \sin(\alpha + b) \cosh(a + \tau/2)). \] (4.5)

Then for
\[ f(x) = \frac{\nu h}{2\pi i} K(\nu \varphi(x)) e_{\alpha}(\nu \varphi(x)) h \varphi'(x) \]
and
\[ I = \int_{-a}^{a} f(x) dx, \quad I_L = \frac{a}{L} \sum_{k=-L}^{L} f(x_k), \]
where \(x_k = k\tau\), and any \(0 < \nu < \nu_0\) it holds that
\[ \| I - I_L \| \leq C \left( \frac{h - e^{2\nu a/\pi} h^{1/\mu} \max(1, (\cosh(a + \tau/2)^{(1+\mu)})}}{2\pi} \right), \]
for \(m\) the smallest integer with \(m > \mu\), and
\[ C = C(d, \sin(\alpha + b)) \nu^{1+\mu} \max(1, (\cosh(a + \tau/2)^{(1+\mu)})). \]

**Proof.** Let us first suppose that \(f\) is analytic and bounded as \(\| f(z) \| \leq M\) for \(z \in R_{\tau} = \{ w \in \mathbb{C} : -a - \tau \leq \Re w \leq a + \tau, -b < \Im w < b \}\). Then it follows from Lemma 12 in the Appendix that
\[ \| I - I_L \| \leq \frac{2M}{e^{2\tau b/\pi} - 1} + \frac{\log 2}{\pi} \tau \sup_{y \in [-b, b]} \| f(a + \tau/2 + iy) - f(-a - \tau/2 + iy) \|. \] (4.6)
To finish the proof we need to first bound \(f\) in the rectangles \(R_{\tau}\) and \(R_{\tau/2}\), in a similar fashion as in [19].

By the assumptions on \(K\), for \(b\) such that \(0 < b < \min(\alpha, \pi/2 - (\delta + \alpha))\), the integrand is analytic in the rectangle \(R_{\tau}\). For any \(x, y \in \mathbb{R}\)
\[ |\varphi(x + iy)| = \cosh x - \sin(\alpha + y) \quad \text{and} \quad |\varphi'(x + iy)| = \sqrt{\cosh^2 x - \sin^2(\alpha + y)}, \]
\[ \cosh x - \sin(\alpha + y) \leq \frac{2M}{e^{2\tau b/\pi} - 1} + \frac{\log 2}{\pi} \tau \sup_{y \in [-b, b]} \| f(a + \tau/2 + iy) - f(-a - \tau/2 + iy) \|. \] (4.6)

leading to the following estimates for \( z = x + iy \in \mathbb{R} \tau \)

\[
1 - \sin(\alpha + b) \leq |\varphi(x + iy)| \leq \cosh(\alpha + \tau) - \sin(\alpha - b),
\]

\[
|\varphi'(x + iy)| \leq \sqrt{\cosh^2(\alpha + \tau) - \sin^2(\alpha - b)},
\]

\[
\left| \frac{\varphi'(x + iy)}{\varphi(x + iy)} \right| \leq \sqrt{\frac{1 + \sin(\alpha + b)}{1 - \sin(\alpha + b)}}.
\]

Thus, if \( \mu \leq -1 \) in (2.2), by using the second part of Lemma 2 we can bound the integrand for \( z \in \mathbb{R} \tau \) (and thus the constant \( M \) in (4.6)) as

\[
\|f(z)\| \leq \frac{h}{2\pi} C(d) \left(1 + \sin(\alpha + b)\right)^{1/2} \left(1 - \sin(\alpha + b)\right)^{1/2+\mu} \nu^{1+\mu} e^{2\nu t_n}.
\]

Using now that \( \text{Re} \varphi(\pm a \pm \tau/2 + iy) < 0 \) we can estimate as in Proposition 4, with \( m = 0 \),

\[
\|f(\pm a \pm \tau/2 + iy)\| \leq \frac{h}{2\pi} C(d) \left(1 + \sin(\alpha + b)\right)^{1/2} \left(1 - \sin(\alpha + b)\right)^{1/2+\mu} \nu^{1+\mu} e^{\nu t_n} \text{Re} \varphi(a+\tau/2-ib)(\gamma t_n/t_n).
\]

For \( \mu > -1 \), we obtain instead

\[
\|f(z)\| \leq \frac{h}{2\pi} C(d) \sqrt{\frac{1 + \sin(\alpha + b)}{1 - \sin(\alpha + b)}} \nu^{1+\mu} \left(\cosh(\alpha + \tau/2)\right)^{1+\mu} e^{2\nu t_n}
\]

and

\[
\|f(\pm a \pm \tau/2 + iy)\| \leq \frac{h^{-m}}{2\pi} C(d) \sqrt{\frac{1 + \sin(\alpha + b)}{1 - \sin(\alpha + b)}} \nu^{1+\mu} \left(\cosh(\alpha + \tau/2)\right)^{1+\mu} e^{\nu t_n} \text{Re} \varphi(a+\tau/2-ib)(\gamma t_n/t_n - m).
\]

This finishes the proof. \( \square \) Combining the two propositions gives the following theorem.

**Theorem 6.** Under the conditions of Proposition 5 and with the same definition of \( I_L \)

\[
\|\omega_n(d) - I_L\| \leq C \left(\frac{e^{2t_n \nu}}{e^{2\pi b/\tau} - 1} + (1 + \tau) (h \nu \cosh a)^{-m} e^{\nu(1-\sin(\alpha - b) \cosh a)(\gamma t_n/t_n - m)} \right),
\]

with \( m = \lceil \mu \rceil \) and

\[
C = C(d, \sin(\alpha + b)) \nu^{1+\mu} \max\{1, (\cosh(\alpha + \tau/2))^{1+\mu}\}.
\]

### 4.1. A non-optimal choice of parameters

We will first show that a good choice of the parameters exists resulting in an efficient algorithm. The optimal choice of parameters is discussed later.

To set the stage let us deal with the first term in the error estimate of Theorem 4.5 in a way that is standard for oblivious quadrature algorithms. Here time is split into
ever increasing intervals. A novelty is that we require the first interval to start at some $n_0h > d$.

Therefore, let $t_n = nh \in [hn_0 + hB^\ell, hn_0 + h2B^{\ell+1}]$, $\ell \geq 0$, and denote by $T_\ell$ the right-end point of this interval, i.e., $T_\ell = hn_0 + h2B^{\ell+1}$. Choose $\nu_\ell = \frac{c_0}{T_\ell}$ and $a_\ell = c_1 \log T_\ell$ for some constants $B > 1$, $c_0 > 0$, $c_1 > 0$. For $\tau$ small enough, i.e., $L = a/\tau$ big enough, the first term can be made arbitrarily small. In fact for the first term to be smaller than $\epsilon_1$ we need

$$L \propto \log T_\ell \log \frac{1}{\epsilon_1}.$$

To simplify the details of the analysis of the second term, let us assume $m = 0$ in Theorem 4.5 and let $\xi$ be given by the formula

$$\xi(\nu_\ell, a_\ell) = h\nu_\ell|\Re \varphi(a_\ell)| = h\nu_\ell(\cosh a_\ell \sin \alpha - 1) \sim h,$$

where in the last step we used $\nu_\ell \sim \frac{1}{T_\ell}$ and $a_\ell \sim \log T_\ell$. In fact $\xi$ is given by a somewhat more complicated formula, but the inclusion of all the details would not change the asymptotic results we give here; for an optimal choice of parameters these details are of importance. We can thus write the second term in the estimate of Theorem 4.5 as

$$\epsilon_2 = e^{-\xi(\nu_\ell, a_\ell)/(\gamma(\xi(\nu_\ell, a_\ell))t_n - d)} \sim e^{-\gamma(h)(t_n - d)}.$$

Therefore as $L$ is increased we expect the error to decrease exponentially fast until it reaches $\epsilon_2$; see Figure 4. Note that if we make $n_0$ large enough, this error can also be controlled. For later intervals, i.e., for larger $\ell$, the second term quickly becomes insignificantly small.

**4.2. The fast and oblivious algorithm.** The algorithm here is similar to the fast and oblivious algorithm described in [30], but with a shift by $n_0 = [d/(h\gamma(\xi(n_0, a_0)))]$ as explained above.

For $N_\ell$ the smallest integer such that $n < n_0 + 1 + B + \sum_{\ell=2}^{N_\ell} B^\ell$ the convolution (3.5) is split into $N_\ell$ sums

$$u_{n+1}^{(0)} = u_{n+1}^{(0)} + \sum_{\ell=2}^{N_\ell} u_{n+1}^{(\ell)}, \quad (4.9)$$

where for suitable $b_\ell$ given below

$$u_{n+1}^{(0)} = \sum_{j=b_1}^{n} \omega_{n-j} g_j \quad \text{and} \quad u_{n+1}^{(\ell)} = \sum_{j=b_\ell - 1}^{b_\ell - 1} \omega_{n-j} g_j.$$

In view of the discussion in Section 4.1 the $b_\ell$ are chosen such that for $j = b_\ell, b_\ell + 1, \ldots, b_{\ell-1}$ we have $t_{n-j} = (n-j)h \in [hn_0 + hB^\ell, hn_0 + h2B^{\ell+1}]$, where $n_0$ is a fixed integer offset with $n_0 \geq d/(h\gamma(\xi)).$ Inserting (3.7) and using (3.8) we obtain

$$u_{n+1}^{(\ell)} = \sum_{j=b_\ell}^{b_{\ell-1}} \frac{h}{2\pi i} \int_{\Gamma} K(\lambda, d)e_{n-j}(h\lambda) d\lambda g_j \quad (4.10)$$

$$= \frac{1}{2\pi i} \int_{\Gamma} r(h\lambda)^{n-(b_{\ell-1} - 1)} K(\lambda, d)y^{(\ell)}(h\lambda) d\lambda \quad (4.11)$$
with \( y^{(\ell)}(\lambda; b_{\ell-1}, b_{\ell}) \), similar to (3.12), the Runge-Kutta approximation to (3.11) at time \( t = hb_{\ell-1} \) with initial value \( y^{(\ell)}(hb_{\ell}) = 0 \). The contour integral \( \int_{\Gamma} \) is approximated by the \( \ell \)-dependent approximation given in Proposition 5. Instead of keeping all the history, i.e. \( g_j \) for \( j = 0, \ldots, n \) in memory, for evaluating the convolution the algorithm requires only three copies of the Runge-Kutta solution \( y^{(\ell)}(\lambda; b_{\ell}, b_{\ell-1}) \) for \( \ell = 2 \ldots N_\ell \) and each \( \lambda_k^{(\ell)} = \nu_k \varphi(\tau_k) \), which are calculated step by step. Details can be found in [30]. As \( N_\ell \) is proportional to \( \log(n - n_0) \), the memory requirement thus grows like \( \mathcal{O}(\log(n - n_0)) \) and the operation count as \( \mathcal{O}((n - n_0) \log(n - n_0)) \).

4.3. Optimal choice of parameters. We begin with a corollary of Theorem 6.

**Corollary 7.** Assume that \( t_n \in [t_0, \Lambda t_0] \), for given \( t_0 > 0 \) and \( \Lambda \geq 1 \), and that there exists \( 0 < D < 1 \) such that \( d \leq D \gamma(\xi) t_0 \). Then for every \( \theta \in (0, 1) \), the following choice of parameters

\[
\tau = \frac{a(\theta)}{L}, \quad \nu = \frac{\pi b L \theta}{N_0 a(\theta)},
\]

with

\[
a(\theta) = \text{arccosh} \left( \frac{\gamma(\xi)(1 - D) \theta + 2 \Lambda(1 - \theta)}{\gamma(\xi)(1 - D) \theta \sin(\alpha - b)} \right)
\]

yields the uniform error estimate

\[
|\omega_n(d) - \mathbf{1}_L| \leq C \exp \left( -\frac{2\pi b L (1 - \theta)}{a(\theta)} \right),
\]

where \( C \) includes all non exponentially growing terms in the bound (4.8). The above choice of parameters is independent of \( h \).

**Proof.** The stated choice for \( \tau \) and \( \nu \) guarantees that exponents in the bound (4.8) with \( D \) in place of \( d \) are equal and negative, with

\[
2\Lambda \nu t_0 = \theta \frac{2\pi b}{\tau}, \quad \theta \in (0, 1)
\]

and

\[
(\theta - 1) \frac{2\pi b}{\tau} = \gamma t_0 (1 - \sin(\alpha - b) \cosh a).
\]

\[ \square \]

**Remark 8.** Notice that \( \gamma(\xi) \) and \( a(\theta) \) are nonlinearly related via (4.5). In our experiments we have opted to fix the value of \( \gamma(\xi) \in (0, 1) \) and then use the error estimate in Corollary 7 to compute \( \theta, a(\theta) \) and \( \tau(\theta) \). This strategy may lead to an underestimation of the stability function according to Lemma 6. Still, our numerical results show that reasonable values of \( \gamma \) and good choices for the rest of parameters are easy to find for prescribed accuracies. These parameters depend on \( \delta, \mu \) and \( d \) in (2.1) but not on the particular application of our method. Results for different values of \( \gamma(\xi) \) are displayed in Figures 5 and 6, for a particular example.

The effect of round-off errors can be included in the analysis in the same way as in [20], leading to a minimization problem for the choice of \( \theta \). In the simplest case of
Fig. 2. Absolute (left) and relative (right) errors in the computation of weights $\omega_n(d)$ for the 3 stage Radau IIA method of order 5 for $d = 0.1$, number of quadrature points $L = 10$ and basis $B = 10$ (top row) and $L = 15$ and $B = 5$ (bottom row).

analysis in [20], the propagation of the errors in the evaluation of $K$, that we denote by $\varepsilon$, is governed by the exponentially growing term

$$\varepsilon e^{\lambda \sigma} = \varepsilon (\theta)^{-\theta/2}, \quad \text{with} \quad (\theta) = e^{-2\pi b L/\alpha (\theta)}.$$

Then from Corollary 7 we deduce that the total error estimate is of the form

$$\varepsilon (\theta)^{-\theta/2} + \varepsilon (\theta)^{1-\theta}. \quad \text{(4.12)}$$

The choice $\theta = 1/L$ above guarantees the boundedness of the term in $\varepsilon$ and the control of the error propagation, giving a convergence rate like $O(\varepsilon^{-c L/\ln L})$. A better choice of $\theta$ can be obtained by minimizing (4.12) for given $L$, $\alpha$, $b$, and $\Lambda$.

4.4. A numerical experiment. We illustrate Theorem 6 and Corollary 7 by considering in (4.10) the generating function $K(\lambda, d) = K_0(\lambda d)$

where $K_0(\cdot)$ is a modified Bessel function [1]. This function satisfies the bounds (2.2) and (2.4) with $\mu = -1/2$ as proved in [1]. In Figures 2 and 3 we show the error in the approximation of the convolution weights in (4.10) along approximation intervals of the form $[hn_0 + hB^\ell, hn_0 + 2hB^{\ell+1}]$, with $B = 5$ and $B = 10$ and for two different values of the distance parameter $d$, namely $d = 0.1$ (Fig. 2) and $d = 0.01$ (Fig. 3).

The lower row of error curves corresponds to the case $B = 5$, where we take $L = 15$ quadrature nodes on the hyperbola and consider seven approximation intervals, i.e. $\ell = 2, \ldots, 8$. The upper row of error curves, with the larger error corresponds to $B = 10$, $L = 10$ and $\ell = 2, \ldots, 6$. The other parameters that determine the hyperbola are as follows $\alpha = 0.9$, $\gamma(\xi) = 0.8$, $b = 0.6$ and $n_0 = [d/\gamma(\xi)/h]$.

5. An application. As explained in the introduction, the initial motivation for investigating operators satisfying (2.1) comes from the application of time-domain boundary integral operators for wave propagation in cases where the strong Huygens’ principle does not hold. Most common examples of the latter are propagation of acoustic waves in two dimensions or in a dissipative medium and propagation of viscoelastic waves. Here, we will give a brief introduction to time-domain boundary integral equations – for more background information see [11].
Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain with boundary $\Gamma = \partial \Omega$ and let $\Omega^c = \mathbb{R}^d \setminus \Omega$ be its complement. Let $u$ be a causal solution of the dissipative wave equation in $\mathbb{R}^d \setminus \Gamma$

$$\partial_t^2 u(t, x) + \alpha \partial_t u(t, x) - \Delta u(t, x) = 0, \quad t \in [0, T], x \in \mathbb{R}^d \setminus \Gamma$$

$$u(t, x) = g(t, x), \quad t \in [0, T], x \in \Gamma$$

(5.1)

where $\alpha \geq 0$ is a constant and $g(t, x)$ is a given boundary data.

The solution of (5.1) can be represented as a single layer potential

$$u(t, x) = \int_0^t \int_{\Gamma} k(t - \tau, |x - y|) \phi(\tau, y) d\Gamma_y d\tau,$$  

(5.2)

where the boundary density $\phi$ is the unique solution of the boundary integral equation, see [23]: Find $\phi$ such that

$$g(t, x) = \int_0^t \int_{\Gamma} k(t - \tau, |x - y|) \phi(\tau, y) d\Gamma_y d\tau, \quad \text{for all } (t, x) \in [0, T] \times \Gamma.$$  

(5.3)

Explicit formulas for the kernel function are complicated, see [11], and even unavailable in the literature for two dimensions and $\alpha > 0$. Nevertheless, the Laplace transform of the kernel function

$$K(\lambda, d) = (\mathcal{L}k)(\lambda, d) = \int_0^\infty e^{-\lambda t} k(t, d) dt,$$

the so-called transfer function, is easily written down explicitly:

$$K(\lambda, d) = \begin{cases} \frac{1}{2\pi} K_0(\sqrt{\lambda^2 + \alpha \lambda d}) & \text{in two dimensions} \\ \frac{e^{-\sqrt{\lambda^2 + \alpha \lambda d}}}{\pi \lambda d} & \text{in three dimensions} \end{cases}$$

(5.4)

where $K_0(\cdot)$ is a modified Bessel function. The transfer function satisfies the bounds

$$\text{[2.2]} \quad \text{and} \quad \text{[2.1]},$$

as first noticed in [5].

**Lemma 9.** For a fixed $d > 0$, the function $K(s, d)$ given in (5.4) satisfies (2.1)
with
\[\mu = \begin{cases} -\frac{1}{2} & \text{in two dimensions} \\ 0 & \text{in three dimensions} \end{cases}\]

Proof. The bound \((2.1)\) is obvious in the 3D case and for the two dimensional case follows from the asymptotic expansions for large arguments of \(K_0(z)\), see [1]. In [5, Lemma 3.2] the proof of the result is given for the two dimensional case and \(\alpha = 0\) and for three dimensional case and general \(\alpha \geq 0\). The remaining case is a consequence of these results as shown next

\[\begin{align*}
&\left| e^{\lambda d} K_0(\sqrt{\lambda^2 + \alpha \lambda d}) \right| = \left| e^{\lambda d - \sqrt{\lambda^2 + \alpha \lambda d}} \right| e^{\sqrt{\lambda^2 + \alpha \lambda d} K_0(\sqrt{\lambda^2 + \alpha \lambda d})} \\
&\leq C(\sigma, d) \sqrt{\lambda^2 + \alpha \lambda}^{-1/2} \\
&\leq C(\sigma, d) |\lambda|^{-1/2}.
\end{align*}\]

This now allows us to discretize the time convolution in (5.3) using convolution quadrature:

\[g_n(x) = \sum_{j=0}^{n} \int_{\Gamma} \omega_{n-j}(|x-y|)\phi_j(y)d\Gamma_y, \quad n = 0, 1, \ldots, T/h = N,\]

(5.5)

where the weights \(\omega_{n-j}(|x|)\) are determined from the kernel function \(K(s, |x|)\) and a choice of linear multistep or Runge-Kutta method in the same way as in the previous sections. To simplify the description of the method, we will confine ourselves to single stage RK methods, i.e., the backward Euler method. Numerical results will be for multistage RK based discretization, details of implementing these can be found in [4]. Note that we have used the notation \(g_n(x) = g(t_n, x)\) (and \(\phi_n(x) \approx \phi(t_n, x)\)) above; in the case of multistage method these would be vectors of size \(s\) as before. The convergence of such an approximation of the integral operators has been first analyzed in [23] for linear multistep methods and then in [7, 8] for Runge-Kutta methods.

To obtain a fully discrete system we need to discretize (5.5) in space as well. Here we will make use of a standard Galerkin boundary element method. In order to do this, let \(\{\Gamma_1, \Gamma_2, \ldots, \Gamma_M\}\) with \(\bigcup \Gamma_j = \Gamma\) be a boundary element mesh and let \(S_h = \text{Span}\{b_1, b_2, \ldots, b_M\}\) be a subspace of \(H^{-1/2}(\Gamma)\), in particular let it be the space of piece-wise constant functions with the basis defined by

\[b_i(x) = \begin{cases} 1 & \text{if } x \in \Gamma_i, \\ 0 & \text{otherwise}. \end{cases}\]

Writing (5.5) in a variational form and discretizing by the Galerkin method we obtain the fully discrete system: Find \(\phi_{kj}\) such that

\[\int_{\Gamma} g_n(x)b_k(x)dx = \sum_{j=0}^{n} \int_{\Gamma} \omega_{n-j}(|x-y|)\phi_{kj}b_k(y)b_l(x)d\Gamma_y d\Gamma_x,\]

13
\[ n = 0, 1, \ldots, T/h = N. \] It will be convenient for the later discussion to rewrite this system in a matrix notation
\[ \mathbf{g}_n = \sum_{j=0}^{n} \mathbf{W}_{n-j} \phi_j. \tag{5.6} \]

The simplest way of applying the techniques developed in this paper would be to apply oblivious quadrature for times \( t_n > \text{diam}(\Omega)/\gamma(\xi) \). A more efficient approach would be to split the matrices \( \mathbf{W}_j \) into distance classes. For example given a positive constant \( d_1 < \text{diam}(\Omega) \) we could split each matrix into two as follows
\[
\begin{pmatrix}
W^{(1)}_k \\
W^{(2)}_k
\end{pmatrix}_{ij} = \begin{cases}
(W_k)_{ij} & \text{if dist}(\Gamma_i, \Gamma_j) \leq d_1, \\
0 & \text{otherwise}
\end{cases}, \quad W^{(2)}_k = W_k - W^{(1)}_k \tag{5.7}
\]

Then the above sum could be split as
\[ \mathbf{g}_n = \sum_{j=0}^{n} \mathbf{W}^{(1)}_{n-j} \phi_j + \sum_{j=0}^{n} \mathbf{W}^{(2)}_{n-j} \phi_j \]
and the new method applied to the first sum for \( t_n > d_1/\gamma(\xi) \) and to the second for \( t_n > \text{diam}(\Omega)/\gamma(\xi) \), i.e., this way we can obtain savings earlier for the computation of the first sum.

**Remark 10.** In this example it is important that the quadrature used to compute the weights \( \mathbf{W}^{(1)}_j \) or \( \mathbf{W}^{(2)}_j \) is valid for a range of distances \( d \). To compute the relevant parameters we proceed as explained in Section 4 taking \( d = d_1 \) or \( d = \text{diam}(\Omega) \). As shown in Corollary 7 these parameters are then valid also for any \( \tilde{d} < d \).

With this the stage is set for applying the oblivious quadrature techniques of the previous section to the setting of time-domain boundary integral equations described above. The algorithm is adapted to solve convolution integrals such as the one in (5.3) in the same way as explained in [30, Section 4.2]. From (5.7) we see that the scheme is implicit in the vector of stages \( \phi_n \). The fast and oblivious algorithm is then applied to deal with the evaluation of the history term in the right hand side of the linear system
\[ \mathbf{W}_0 \phi_n = \mathbf{g}_n - \sum_{j=0}^{n-1} \mathbf{W}_{n-j} \phi_j. \]

Results of numerical experiments are given in the next section.

6. Numerical experiments.

6.1. Wave equation in two dimensions. Let \( \Omega \subset \mathbb{R}^2 \) be a disk with radius one. We solve (5.1) with \( \alpha = 0 \) and \( g(x, t) = t^4e^{-2t} \). We integrate from \( t = 0 \) to \( T = 40 \) with step-size \( h = T/400 \), i.e. \( N = 400 \), and discretize in space with equally large patches of size \( \approx 2\pi/100 \), i.e. \( M = 100 \). Because of the symmetries of the circle, see [28], the solution is space independent. It is shown in Figure 4 on the time interval \( t \in [0, 40] \). The splitting of the weights into two distance classes is done by setting the parameter \( d_1 = \sqrt{2} \) in Equation (5.7), such that the entries of \( \mathbf{W} \) are divided into two equally large distance classes. In all experiments we chose the basis \( B = 5 \), which defines the splitting in (5.3) together with the offset \( n_0 = \lfloor d/h/\gamma(\xi) \rfloor \), where \( d \) depends on the distance class and is either 2 or \( \sqrt{2} \).
The evolution of the error for contour parameters $\alpha = 0.98$ and $b = 0.33$ for different $L$ and $\gamma(\xi)$ is shown in Figure 5. Increasing $L$ and decreasing $\gamma(\xi)$ yields more accurate results. The error is measured against a reference solution calculated with a standard CQ algorithm described in [9]. Convergence in the number of quadrature nodes $L$ for different $\gamma(\xi)$ is shown in Figure 6. The error here is measured in the sup norm. Only if $\gamma(\xi)$ is small enough increasing $L$ improves the result. In this case we observe exponential convergence in $L$ in agreement with Theorem 6. The choice of $\alpha$ and $b$ is done experimentally.

Remark 11. A better choice of parameters might be feasible including the angle $\alpha$ in the optimization process and eliminating $b$, as it is done in [33] for the numerical inversion of the Laplace transform. In our example we have tested the parameters from [33] on a purely experimental basis and the convergence rates are actually better. However we point out that the theory in [33] does not apply to our situation. Another issue with the parameters from [33] is that the propagation of the errors in the evaluations of the Laplace transform $K$ is not under control, as can be observed in Figure 7 when the error curves reach the accuracy of the reference solution, about $10^{-10}$. A further study of the optimal parameters in the context of 2D and damped wave equations might be the topic of future research.

7. Appendix. We modify the proof given in [15] to show the following result.

Lemma 12. Let $f$ be analytic and bounded as $|f(z)| \leq M$ for $z \in R\tau_0 = \{w \in \mathbb{C} : -a - \tau_0 \leq \Re w \leq a + \tau_0, -b < \Im w < b\}$ and some $\tau_0 > 0$. Further, let

$$I = \int_{-a}^{a} f(x)dx, \quad I_L = \frac{a}{L} \sum_{k=-L}^{L} f(x_k),$$
Fig. 5. Evolution of the error: For different $L$ with fixed $\gamma(\xi) = 0.6$ (top) and for different $\gamma(\xi)$ and fixed $L = 26$ (bottom).

Fig. 6. Convergence in the number of contour quadrature nodes $L$.

where $x_k = k\tau$, $\tau = a/L$ and $0 < \tau \leq \tau_0$. Then

$$|I - I_L| \leq \frac{2M}{e^{2\pi b/\tau} - 1} + \frac{\log 2}{\pi} \sup_{y \in [-b,b]} |g_{\tau/2}(y)|,$$

where $g_{\tau/2}(y) = f(a + \tau/2 + iy) - f(-a - \tau/2 + iy)$. 
Proof. Let $\Gamma$ be the boundary of the rectangle $R_{\tau/2} \subset R_{\tau_0}$ with $\Gamma_1$ the top and bottom sides and $\Gamma_2$ the left and right sides of the rectangle. Using residue calculus

$$I_L = \int_{\Gamma} f(z)(2i)^{-1} \cot(\pi z/\tau) dz$$

whereas using the analyticity of $f$

$$I = \int_{\Gamma} f(z)u(z) dz$$

where

$$u(z) = \begin{cases} -\frac{1}{2} & \text{Im } z > 0, \\ \frac{1}{2} & \text{Im } z < 0. \end{cases}$$

Combining these two formulas we have

$$I_L - I = \int_{\Gamma} f(z)S(z) dz$$

where $S(z) = (2i)^{-1} \cot(\pi z/\tau) - u(z)$ or when simplified

$$S(z) = \begin{cases} \frac{1}{1-e^{-i\pi z/\tau}} & \text{Im } z > 0, \\ \frac{1}{e^{i\pi z/\tau} - 1} & \text{Im } z < 0. \end{cases}$$

We now split the error into two terms

$$I_L - I = I_1 + I_2$$
corresponding to \( \Gamma_1 \) and \( \Gamma_2 \). The first term is easily bounded to give the first term in the above error estimate. Noticing that 
\[
S(\pm a \pm \tau/2 + iy) = S(\tau/2 + iy)
\]
we see that
\[
|I_2| = \left| \int_{-b}^{b} g_{\tau/2}(y)S(\tau/2 + iy)dy \right| \leq \sup_{y \in [-b,b]} |g_{\tau/2}(y)| \int_{-b}^{b} |S(\tau/2 + iy)dy
\]
\[
\leq \sup_{y \in [-b,b]} |g_{\tau/2}(y)| \int_{-\infty}^{\infty} |S(\tau/2 + iy)dy = \frac{\log 2}{\pi} \tau \sup_{y \in [-b,b]} |g_{\tau/2}(y)|,
\]
where we have used
\[
\int_{0}^{\infty} |S(\tau/2 + iy)dy = \int_{0}^{\infty} \frac{1}{1 + e^{2\pi y/\tau}} dy = \tau \int_{0}^{\infty} \frac{1}{1 + e^{2\pi u}} du = \frac{\log 2}{2\pi} \tau
\]
and similarly
\[
\int_{-\infty}^{0} |S(\tau/2 + iy)dy = \frac{\log 2}{2\pi} \tau. \tag{\ref{eq:log2tau}}
\]

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