ASYMPTOTIC FAITHFULNESS OF THE QUANTUM SU(n) REPRESENTATIONS OF THE MAPPING CLASS GROUPS

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Abstract. We prove that the sequence of projective quantum SU(n) representations of the mapping class group of a closed oriented surface, obtained from the projective flat SU(n)-Verlinde bundles over Teichmüller space, is asymptotically faithful, that is the intersection over all levels of the kernels of these representations is trivial, whenever the genus is at least 3. For the genus 2 case, this intersection is exactly the order two subgroup, generated by the hyper-elliptic involution, in the case of even degree and \( n = 2 \). Otherwise the intersection is also trivial in the genus 2 case.

1. Introduction

In this paper we shall study the finite dimensional quantum SU(n) representations of the mapping class group of a genus \( g \) surface. These form the only rigorously constructed part of the gauge-theoretic approach to Topological Quantum Field Theories in dimension 3, which Witten proposed in his seminal paper [W1]. We discovered the asymptotic faithfulness property for these representations by studying this approach, which we will now briefly describe, leaving further details to section 2 and the references given there.

Let \( \Sigma \) be a closed oriented surface of genus \( g \geq 2 \) and \( p \) a point on \( \Sigma \). Fix \( d \in \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_{SU(n)} \) in the center of \( SU(n) \). Let \( M \) be the moduli space of flat SU(n)-connections on \( \Sigma - p \) with holonomy \( d \) around \( p \).

By applying geometric quantization to the moduli space \( M \) one gets a certain finite rank vector bundle over Teichmüller space \( T \), which we will call the Verlinde bundle \( V_k \) at level \( k \), where \( k \) is any positive integer. The fiber of this bundle over a point \( \sigma \in T \) is \( V_k,\sigma = H^0(M_\sigma, L_\sigma^k) \), where \( M_\sigma \) is \( M \) equipped with a complex structure induced from \( \sigma \) and \( L_\sigma \) is an ample generator of the Picard group of \( M_\sigma \).

The main result pertaining to this bundle \( V_k \) is that its projectivization \( \mathbb{P}(V_k) \) supports a natural flat connection. This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. Now, since there is an action of the mapping class group \( \Gamma \) of \( \Sigma \) on \( V_k \) covering its action on \( T \), which preserves the flat connection in \( \mathbb{P}(V_k) \), we get for each \( k \) a finite dimensional projective representation, say \( \rho_k^{n,d} \), of \( \Gamma \), namely on the covariant constant sections of \( \mathbb{P}(V_k) \) over \( T \). This sequence of projective representations \( \rho_k^{n,d}, k \in \mathbb{N}_+ \) is the quantum SU(n) representations of the mapping class group \( \Gamma \).

For each given \((n, d, k)\), we cannot expect \( \rho_k^{n,d} \) to be faithful. However, V. Turaev conjectured a decade ago (see e.g. [T]) that there should be no non-trivial
element of the mapping class group representing trivially under \( \rho_{k}^{n,d} \) for all \( k \), keeping \((n, d)\) fixed. We call this property *asymptotic faithfulness* of the quantum SU\((n)\) representations \( \rho_{k}^{n,d} \). In this paper we prove Turaev’s conjecture:

**Theorem 1.** Assume that \( n \) and \( d \) are coprime or that \((n, d) = (2, 0)\) when \( g = 2 \). Then we have that

\[
\bigcap_{k=1}^{\infty} \ker(\rho_{k}^{n,d}) = \begin{cases} 
\{1, H\} & g = 2, n = 2 \text{ and } d = 0 \\
\{1\} & \text{otherwise}.
\end{cases}
\]

where \( H \) is the hyperelliptic involution.

This Theorem states that for any element \( \phi \) of the mapping class group \( \Gamma \), which is not the identity element (and not the hyperelliptic involution in genus 2), there is an integer \( k \) such that \( \rho_{k}^{n,d}(\phi) \) is not a multiple of the identity. We will suppress the superscript on the quantum representations and simply write \( \rho_{k} = \rho_{k}^{n,d} \) throughout the rest of the paper.

Our key idea in the proof of Theorem 1 is the use of the endomorphism bundle \( \text{End}(\mathcal{V}_{k}) \) and the construction of sections of this bundle via Toeplitz operators associated to smooth functions on the moduli space \( M \). By showing that these sections are *asymptotically* flat sections of \( \text{End}(\mathcal{V}_{k}) \) (see Theorem 6 for the precise statement), we prove that any element in the above intersection of kernels acts trivially on the smooth functions on \( M \), hence acts by the identity on \( M \) (see the proof of Theorem 7). Theorem 1 now follows directly from knowing which elements of the mapping class group act trivially on the moduli space \( M \).

The assumptions on the pair \((n, d)\) in Theorem 1 are exactly picking out the cases where the moduli space \( M \) is smooth. This means we can apply the work of Bordemann, Meinrenken and Schlichenmaier on Toeplitz operators on smooth Kähler manifolds, in particular their formula for the asymptotics in \( k \) of the operator norm of Toeplitz operators and the asymptotic expansion of the product of two Toeplitz operators. - It is with the use of these results we establish that the Toeplitz operator sections are asymptotically flat with respect to Hitchin’s connection.

In the remaining cases, where \( M \) is singular, we also have a proof of asymptotic faithfulness, where we use the desingularization of the moduli space, but this argument is technically quite a bit more involved. However, together with Michael Christ we have in [AC] extended some of the results of Bordemann, Meinrenken and Schlichenmaier to the case of singular varieties. In [AM] the argument of the present paper will be repeated in the non-coprime case, making use of the results of [AC] to show that Theorem 1 holds in general without the coprime assumption.

The abelian case, i.e. the case where SU\((n)\) is replaced by U\((1)\), was considered in [A2], before we considered the case discussed in this paper. In this case, with the use of Theta-functions, explicit expressions for the Toeplitz operators associated to holonomy functions can be obtained. From these expressions it follows that the Toeplitz operators are not covariant constant even in this much simpler case (although the relevant connection is the one induced from the \( L_{2} \)-projection as shown by Ramadas in [R1]). However, they are asymptotically covariant constant, in fact we find explicit perturbations to all orders in \( k \), which in this case, we argue, can be summed and used to create actual covariant constant sections of the
endomorphism bundle. The result as far as the mapping class group goes, is that
the intersection of the kernels over all \( k \) in that case is the Torelli group.

Returning to the non-abelian case at hand, we know by the work of Laszlo \[La1\],
that \( \mathbb{P}(V_k) \) with its flat connection is isomorphic to the projectivization of the bundle
of conformal blocks for \( sl(n, \mathbb{C}) \) with its flat connection over \( \mathcal{T} \) as constructed
by Tsuchiya, Ueno and Yamada \[TUY\]. This means that the quantum \( SU(n) \)
representations \( \rho_k \) is the same sequence of representations as the one arising from
the space of conformal blocks for \( sl(n, \mathbb{C}) \).

By the work of Reshetikhin-Turaev, Topological Quantum Field Theories have
been constructed in dimension 3 from the quantum group \( U_q sl(n, \mathbb{C}) \) (see \[RT1\],
\[RT2\] and \[T\]) or alternatively from the Kauffman bracket and the Homfly-polynomial
by Blanchet, Habegger, Masbaum and Vogel (see \[BHMV1\], \[BHMV2\] and \[B1\]).

In ongoing work of Ueno joint with this author (see \[AU1\], \[AU2\] and \[AU3\]),
we are in the process of establishing that the TUY construction of the bundle of
conformal blocks over Teichmüller space for \( sl(n, \mathbb{C}) \) gives a modular functor, which
in turn gives a TQFT, which is isomorphic to the \( U_q sl(n, \mathbb{C}) \)-Reshetikhin-Turaev
TQFT. A corollary of this will be that the quantum \( SU(n) \) representations are
isomorphic to the ones that are part of the \( U_q sl(n, \mathbb{C}) \)-Reshetikhin-Turaev TQFT.
Since it is well known that the Reshetikhin-Turaev TQFT is unitary one will get
unitarity of the quantum \( SU(n) \) representations from this. We note that unitarity
is not clear from the geometric construction of the quantum \( SU(n) \) representations.

If the quantum \( SU(n) \) representations \( \rho_k \) are unitary, then we have for all \( \phi \in \Gamma \) that

\[
|\text{Tr}(\rho_k(\phi))| \leq \dim \rho_k.
\]

From the above Theorem \[1\] follows directly, assuming unitarity, that one has the following

**Corollary 1.** Assume that \( n \) and \( d \) are coprime or that \( (n, d) = (2, 0) \) when \( g = 2 \).
Then equality holds in (1) for all \( k \), if and only if

\[
\phi \in \begin{cases} 
\{1, H\} & g = 2, n = 2 \text{ and } d = 0 \\
\{1\} & \text{otherwise}.
\end{cases}
\]

Furthermore, one will get that the norm of the Reshetikhin-Turaev quantum
invariant for all \( k \) and \( n = 2 \) \((n = 3 \text{ in the genus } 2 \text{ case})\) can separate the mapping torus of the identity from the rest of the mapping tori as a purely TQFT consequence of Corollary \[1\].

In this paper we have initiated the program of using of the theory of Toeplitz
operators on the moduli spaces in the study of TQFT’s. The main insight behind
the program is the relation between these Toeplitz operators and Hitchin’s
connection asymptotically in the quantum level \( k \). Here we have presented the
initial application of this program, namely the establishment of the asymptotic
faithfulness property for the quantum representations of the mapping class groups.
However this program can also be used to study other asymptotic properties of
these TQFT’s. In particular we have used them to establish that the quantum
invariants for closed 3-manifolds have asymptotic expansions in \( k \). Topological
consequences of this are that certain classical topological properties are determined
by the quantum invariants, resulting in interesting topological conclusions, including very strong knot theoretical corollaries. Writeup of these further developments of this program is in progress.

We remark that it is also an interesting problem to understand how the Toeplitz operator constructions used in this paper are related to the deformation quantization of the moduli spaces described in [AMR1] and [AMR2]. In the abelian case, the resulting Berezin-Toeplitz deformation quantization was explicitly described in [A2] and it turns out to be equivalent to the one constructed in [AMR2] in this case.

This paper is organized as follows. In section 2 we give the basic setup of applying geometric quantization to the moduli space to construct the Verlinde bundle over Teichmüller space. In section 3 we review the construction of the connection in the Verlinde bundle. We end that section by stating the properties of the moduli space and the Verlinde bundle we will use. We will see that it is only a few elementary properties about the moduli space, Teichmüller space and the general form of the connection in the Verlinde bundle we really need. In section 4 we review the general results about Toeplitz operators on smooth compact Kähler manifolds we use in the following section 5 where we prove that the Toeplitz operators for smooth functions on the moduli space give asymptotically flat sections of the endomorphism bundle of the Verlinde bundle. Finally, in section 6 we prove the asymptotic faithfulness Theorem 1 above.

After the completion of this work Freedman and Walker, together with Wang, have found a proof of the Asymptotic faithfulness property for the SU(2)-BHMV-representations which uses BHMV-technology. Their paper has already appeared [FWW] (see also [M2] for a discussion). As alluded to before, we are working jointly with K. Ueno to establish that these representations are equivalent to our sequence $\rho_{k,0}$. However, as long as this has not been established, our result is logically independent of theirs.

For the SU(2)-BHMV-representations it is already known by the work of Roberts [Ro], that they are irreducible for $k + 2$ prime and that they have infinite image by the work of Masbaum [M1], except for a few low values of $k$.

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2. THE GAUGE THEORY CONSTRUCTION OF THE VERLINDE BUNDLE

Let us now very briefly recall the construction of the Verlinde bundle. Only the details needed in this paper will be given. We refer e.g. to [H] for further details. As in the introduction we let $\Sigma$ be a closed oriented surface of genus $g \geq 2$ and $p \in \Sigma$. Let $P$ be a principal $SU(n)$-bundle over $\Sigma$. Clearly, all such $P$ are trivializable. As above let $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$. Throughout the rest of this paper we will assume that $n$ and $d$ are coprime, although in the case $g = 2$ we also allow $(n, d) = (2, 0)$. Let $M$ be the moduli space of flat $SU(n)$-connections in $P|_{\Sigma - p}$ with holonomy $d$ around $p$. We can identify

$$M = \text{Hom}_d(\tilde{\pi}_1(\Sigma), SU(n))/SU(n).$$
Here $\tilde{\pi}_1(\Sigma)$ is the universal central extension

$$0 \to \mathbb{Z} \to \tilde{\pi}_1(\Sigma) \to \pi_1(\Sigma) \to 1$$

as discussed in [H] and in [AB], and $\text{Hom}_d$ means the space of homomorphisms from $\tilde{\pi}_1(\Sigma)$ to $SU(n)$ which send the image of $1 \in \mathbb{Z}$ in $\tilde{\pi}_1(\Sigma)$ to $d$ (see [H]). When $n$ and $d$ are coprime, $M$ is a compact smooth manifold of dimension $m = (n^2 - 1)(g - 1)$. In general, when $n$ and $d$ are not coprime, $M$ is not smooth, except in the case where $g = 2$, $n = 2$ and $d = 0$. In this case $M$ is in fact diffeomorphic to $\mathbb{CP}^3$. There is a natural homomorphism from the mapping class group to the outer automorphisms of $\tilde{\pi}_1(\Sigma)$, hence $\Gamma$ acts on $M$.

We choose an invariant bilinear form $\{ \cdot , \cdot \}$ on $P = \text{Lie}(SU(n))$, normalized such that $-\frac{1}{6} \{ \theta \wedge [\theta \wedge \theta] \}$ is a generator of the image of the integer cohomology in the real cohomology in degree 3 of $SU(n)$, where $\theta$ is the $g$-valued Maurer-Cartan 1-form on $SU(n)$.

This bilinear form induces a symplectic form on $M$. In fact

$$T_{(A)}M \cong H^1(\Sigma, d_A),$$

where $A$ is any flat connection in $P$ representing a point in $M$ and $d_A$ is the induced covariant derivative in the associated adjoint bundle. Using this identification, the symplectic form on $M$ is:

$$\omega(\varphi_1, \varphi_2) = \int_{\Sigma} \{ \varphi_1 \wedge \varphi_2 \},$$

where $\varphi_i$ are $d_A$-closed 1-forms on $\Sigma$ with values in $\text{ad} P$. See e.g. [H] for further details on this. The natural action of $\Gamma$ on $M$ is symplectic.

Let $\mathcal{L}$ be the Hermitian line bundle over $M$ and $\nabla$ the compatible connection in $\mathcal{L}$ constructed by Freed in [Fr]. This is the content of Corollary 5.22, Proposition 5.24 and equation (5.26) in [Fr] (see also the work of Ramadas, Singer and Weitsman [RSW]). By Proposition 5.27 in [Fr] we have that the curvature of $\nabla$ is $\sqrt{-1} \pi \omega$. We will also use the notation $\nabla$ for the induced connection in $\mathcal{L}^k$, where $k$ is any integer.

By an almost identical construction, we can lift the action of $\Gamma$ on $M$ to act on $\mathcal{L}$ such that the Hermitian connection is preserved (See e.g. [AI]). In fact, since $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(M, \mathbb{Z}) = 0$, it is clear that the action of $\Gamma$ leaves the isomorphism class of $(\mathcal{L}, \nabla)$ invariant, thus alone from this one can conclude that a central extension of $\Gamma$ acts on $(\mathcal{L}, \nabla)$ covering the $\Gamma$ action on $M$. This is actually all we need in this paper, since we are only interested in the projectivized action.

Let now $\sigma \in \mathcal{T}$ be a complex structure on $\Sigma$. Let us review how $\sigma$ induces a complex structure on $M$ which is compatible with the symplectic structure on this moduli space. The complex structure $\sigma$ induces a $*$-operator on 1-forms on $\Sigma$ and via Hodge theory we get that

$$H^1(\Sigma, d_A) \cong \ker(d_A + *d_A *).$$

On this kernel, consisting of the harmonic 1-forms with values in $\text{ad} P$, the $*$-operator acts and its square is $-1$, hence we get an almost complex structure on $M$ by letting $I = I_\sigma = -*$. It is a classical result by Narasimhan and Seshadri (see [NSI]), that this actually makes $M$ a smooth Kähler manifold, which as such, we denote $M_\sigma$. By using the $(0,1)$ part of $\nabla$ in $\mathcal{L}$, we get an induced holomorphic
structure in the bundle $\mathcal{L}$. The resulting holomorphic line bundle will be denoted $\mathcal{L}_\sigma$. See also [H] for further details on this.

From a more algebraic geometric point of view, we consider the moduli space of $S$-equivalence classes of semi-stable bundles of rank $n$ and determinant isomorphic to the line bundle $\mathcal{O}(d[p])$. By using Mumford’s Geometric Invariant Theory, Narasimhan and Seshadri (see [NS2]) showed that this moduli space is a smooth complex algebraic projective variety which is isomorphic as a Kähler manifold to $M_\sigma$. Referring to [DN] we recall that

**Theorem 2** (Drezet & Narasimhan). The Picard group of $M_\sigma$ is generated by the holomorphic line bundle $L_\sigma$ over $M_\sigma$ constructed above:

$$\text{Pic}(M_\sigma) = \langle L_\sigma \rangle.$$ 

**Definition 1.** The Verlinde bundle $\mathcal{V}_k$ over Teichmüller space is by definition the bundle whose fiber over $\sigma \in \mathcal{T}$ is $H^0(M_\sigma, L_k^{\sigma})$, where $k$ is a positive integer.

### 3. The Projectively Flat Connection

In this section we will review Axelrod, Della Pietra and Witten’s and Hitchin’s construction of the projective flat connection over Teichmüller space in the Verlinde bundle. We refer to [H] and [ADW] for further details.

Let $\mathcal{H}_k$ be the trivial $C^\infty(M, \mathcal{L}_k)$-bundle over $\mathcal{T}$ which contains $\mathcal{V}_k$, the Verlinde sub-bundle. If we have a smooth one-parameter family of complex structures $\sigma_t$ on $\Sigma$, then that induces a smooth one-parameter family of complex structures on $M$, say $I_t$. In particular we get $\sigma'_t \in T_{\sigma_t}(\mathcal{T})$, which gives an $I'_t \in H^1(M_{\sigma_t}, T)$ (here $T$ refers to the holomorphic tangent bundle of $M_{\sigma_t}$).

Suppose $s_t$ is a corresponding smooth one-parameter family in $C^\infty(M, \mathcal{L}_k^{\sigma})$ such that $s_t \in H^0(M_{\sigma_t}, L_k^{\sigma_t})$. By differentiating the equation

$$(1 + \sqrt{-1} I'_t) \nabla s_t = 0,$$

we see that

$$\sqrt{-1} I'_t \nabla s + (1 + \sqrt{-1} I_t) \nabla s'_t = 0.$$ 

Hence, if we have an operator

$$u(v) : C^\infty(M, \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k)$$

for all real tangent vectors to Teichmüller space $v \in T(\mathcal{T})$, varying smoothly with respect to $v$, and satisfying

$$\sqrt{-1} I'_t \nabla^{1,0}s_t + \nabla^{0,1}u(\sigma'_t)(s_t) = 0,$$

for all smooth curves $\sigma_t$ in $\mathcal{T}$, then we get a connection induced in $\mathcal{V}_k$ by letting

$$(2) \quad \nabla_v = \hat{\nabla}'_v - u(v),$$

for all $v \in T(\mathcal{T})$, where $\hat{\nabla}'$ is the trivial connection in $\mathcal{H}_k$.

In order to specify the particular $u$ we are interested in, we use the symplectic structure on $\omega \in \Omega^{1,1}(M_\sigma)$ to define the tensor $G = G(v) \in \Omega^0(M_\sigma, S^2(T))$ by

$$v[I_\sigma] = G(v) \omega,$$

where $v[I_\sigma]$ means the derivative in the direction of $v \in T_\sigma(\mathcal{T})$ of the complex structure $I_\sigma$ on $M$. Following Hitchin, we give an explicit formula for $G$ in terms of $v \in T_\sigma(\mathcal{T})$:
The holomorphic tangent space to Teichmüller space at $\sigma \in T$ is given by
\[ T_{\sigma}^{1,0}(T) \cong H^1(\Sigma_\sigma, K^{-1}). \]

Furthermore, the holomorphic co-tangent space to the moduli space of semi-stable bundles at the equivalence class of a stable bundle $E$ is given by
\[ T_{[E]}^* M_\sigma \cong H^0(\Sigma_\sigma, \text{End}_0(E) \otimes K). \]

Thinking of $G(v) \in \Omega^0(M_\sigma, S^2(T))$ as a quadratic function on $T^* = T^* M_\sigma$, we have that
\[ G(v)(\alpha, \alpha) = \int_\Sigma \text{Tr}(\alpha^2)v^{(1,0)} \]
where $v^{(1,0)}$ is the image of $v$ under the projection onto $T^{1,0}(T)$. From this formula it is clear that $G(v) \in H^0(M_\sigma, S^2(T))$ and that $\hat{\nabla}$ agrees with $\hat{\nabla}t$ along the anti-holomorphic directions $T^{0,1}(T)$. From Proposition (4.4) in [H] we have that this map $v \mapsto G(v)$ from $T_\sigma(T)$ to $H^0(M_\sigma, S^2(T))$ is an isomorphism.

The particular $u(v)$ we are interested in is $u_G(v)$, where
\[ u_G(s) = \frac{1}{2(k + n)}(\Delta_G - 2\nabla G \partial F + \sqrt{-1}k f_G)s. \]

The leading order term $\Delta_G$ is the 2nd order operator given by
\[ \Delta_G : C^\infty(M, L^k) \xrightarrow{\nabla^{1,0}} C^\infty(M, T^* \otimes L^k) \xrightarrow{G} C^\infty(M, T \otimes L^k) \xrightarrow{\nabla^{1,0} \otimes 1 + 1 \otimes \nabla^{1,0}} C^\infty(M, T^* \otimes T \otimes L^k) \xrightarrow{\text{Tr}} C^\infty(M, L^k), \]
where we have used the Chern connection in $T$ on the Kähler manifold $(M_\sigma, \omega)$.

The function $F = F_\sigma$ is the Ricci potential uniquely determined as the real function with zero average over $M$, which satisfies the following equation
\[ \text{Ric}_\sigma = 2n\omega + 2\sqrt{-1}\partial \bar{\partial} F_\sigma. \]

We usually drop the subscript $\sigma$ and think of $F$ as a smooth map from $T$ to $C^\infty(M)$.

The complex vector field $G \partial F \in C^\infty(M_\sigma, T)$ is simply just the contraction of $G$ with $\partial F \in C^\infty(M_\sigma, T^*)$.

The function $f_G \in C^\infty(M)$ is defined by
\[ f_G = -\sqrt{-1} v[F], \]
where $v$ is determined by $G = G(v)$ and $v[F]$ means the derivative of $F$ in the direction of $v$. We refer to [ADW] for this formula for $f_G$.

**Theorem 3** (Axelrod, Della Pietra & Witten; Hitchin). The expression (2) above defines a connection in the bundle $\mathcal{V}_k$, which induces a flat connection in $\mathbb{P}(\mathcal{V}_k)$.

Faltings has established this Theorem in the case where one replaces $SU(n)$ with a general semi-simple Lie group (see [Fal]).

We remark about genus 2, that [ADW] covers this case, but [H] excludes this case, however, the work of Van Geemen and De Jong [vGdJ] extends Hitchin’s approach to the genus 2 case.

As discussed in the introduction, we see by Laszlo’s Theorem that this particular connection is the relevant one to study.
It will be essential for us to consider the induced flat connection \( \hat{\nabla}_v \) in the endomorphism bundle \( \text{End}(V_k) \). Suppose \( \Phi \) is a section of \( \text{End}(V_k) \). Then for all sections \( s \) of \( V_k \) and all \( v \in T(T) \) we have that
\[
(\hat{\nabla}_v \Phi)(s) = \hat{\nabla}_v \Phi(s) - \Phi(\hat{\nabla}_v(s)).
\]
Assume now that we have extended \( \Phi \) to a section of \( \text{Hom}(H_k, V_k) \) over \( T \). Then
\[
\hat{\nabla}_v \Phi = \hat{\nabla}_v \Phi - [\Phi, u(v)]
\]
where \( \hat{\nabla}_v \) is the trivial connection in the trivial bundle \( \text{End}(H_k) \) over \( T \).

Let us end this section by summarizing the properties we use about the moduli space in section 5 to prove Theorem 6 which intern implies Theorem 1.

The moduli space \( M \) is a finite dimensional smooth compact manifold with a symplectic structure \( \omega \), a Hermitian line bundle \( L \) and a compatible connection \( \nabla \), whose curvature is \( \frac{1}{2\pi} \omega \). The moduli space \( T \) is a smooth connected finite dimensional manifold, which smoothly parametrizes Kähler structures \( I_\sigma, \sigma \in T \) on \( (M, \omega) \). For any positive integer \( k \), we have inside the trivial bundle \( H_k = T \times C^\infty(M, L^k) \) the finite dimensional subbundle \( V_k \), given by
\[
V_k(\sigma) = H^0(M, L^k_\sigma)
\]
for \( \sigma \in T \). We have a connection in \( V_k \) given by
\[
\hat{\nabla}_v = \hat{\nabla}_v^t - u(v)
\]
where \( \hat{\nabla}_v^t \) is the trivial connection in \( H_k \) and \( u(v) \) is the second order differential operator \( u_{G(v)} \) given in \( \ref{eq:second-order-operator} \). All we will need about the operator \( \Delta_G - 2\nabla_{G\partial F} \), is that there is some finite set of vector fields \( X_r(v), Y_r(v), Z(v) \in C^\infty(M, T), \ r = 1, \ldots, R \) (where \( v \in T(\sigma) \)) all varying smoothly\(^1\) with \( v \in T(T) \), such that
\[
\Delta_{G(v)} - 2\nabla_{G(v)\partial F} = \sum_{r=1}^R \nabla_{X_r(v)} \nabla_{Y_r(v)} + \nabla_{Z(v)}.
\]
This follows immediately from the definition of \( \Delta_{G(v)} \). From this we have the expression
\[
u(v) = \frac{1}{2(k+n)} \left( \sum_{r=1}^R \nabla_{X_r(v)} \nabla_{Y_r(v)} + \nabla_{Z(v)} + nv[F] \right) - \frac{1}{2}v[F].
\]
All we need to use about \( F : T \to C^\infty(M) \) is that it is a smooth function, such that \( F_\sigma \) is real valued on \( M \) for all \( \sigma \in T \).

4. Toeplitz operators on compact Kähler manifolds

In this section \( (N^{2m}, \omega) \) will denote a compact Kähler manifold on which we have a holomorphic line bundle \( L \) admitting a compatible Hermitian connection whose curvature is \( \frac{1}{2\pi} \omega \). On \( C^\infty(N, L^k) \) we have the \( L_2 \)-inner product:
\[
\langle s_1, s_2 \rangle = \frac{1}{m!} \int_N (s_1, s_2) \omega^m
\]
where \( s_1, s_2 \in C^\infty(N, L^k) \) and \( \langle \cdot, \cdot \rangle \) is the fiberwise Hermitian structure in \( L^k \). This \( L_2 \)-inner product gives the orthogonal projection \( \pi : C^\infty(N, L^k) \to H^0(N, L^k) \).

\(^1\)This makes sense when we consider the holomorphic tangent bundle \( T \) of \( M_\sigma \) inside the complexified real tangent bundle \( TM \otimes \mathbb{C} \) of \( M \).
For each \( f \in C^\infty(N) \) consider the associated Toeplitz operator \( T_f^{(k)} \) given as the composition of the multiplication operator \( M_f : H^0(N, L^k) \to C^\infty(N, L^k) \) with the orthogonal projection \( \pi : C^\infty(N, L^k) \to H^0(N, L^k) \), so that
\[
T_f^{(k)}(s) = \pi(f s).
\]

Since the multiplication operator is a zero order differential operator, \( T_f^{(k)} \) is a zero-order Toeplitz operator. Sometimes we will suppress the superscript \((k)\) and just write \( T_f = T_f^{(k)} \).

Let us here give an explicit formula for \( \pi \): Let \( h_{ij} = \langle s_i, s_j \rangle \), where \( s_i \) is a basis of \( H^0(N, L^k) \). Let \( h_{ij}^{-1} \) be the inverse matrix of \( h_{ij} \). Then
\[
\pi(s) = \sum_{i,j} (s, s_i) h_{ij}^{-1} s_j. \tag{8}
\]

This formula will be useful when we have to compute the derivative of \( \pi \) along a one-parameter curve of complex structures on the moduli space.

Suppose we have a smooth section \( X \in C^\infty(N, TN) \) of the holomorphic tangent bundle of \( N \). We then claim that the operator \( \pi \nabla_X \) is a zero-order Toeplitz operator. Suppose \( s_1 \in C^\infty(N, L^k) \) and \( s_2 \in H^0(N, L^k) \), then we have that
\[
X(s_1, s_2) = \langle \nabla_X s_1, s_2 \rangle.
\]

Now, calculating the Lie derivative along \( X \) of \( (s_1, s_2) \omega^m \) and using the above, one obtains after integration that
\[
\langle \nabla_X s_1, s_2 \rangle = -\langle \Lambda d(i_X \omega) s_1, s_2 \rangle,
\]
where \( \Lambda \) denotes contraction with \( \omega \). Thus
\[
\pi \nabla_X = T_{fx}, \tag{9}
\]
as operators from \( C^\infty(N, L^k) \) to \( H^0(N, L^k) \), where \( fx = -\Lambda d(i_X \omega) \). Iterating this, we find for all \( X_1, X_2 \in C^\infty(TN) \) that
\[
\pi \nabla_{X_1} \nabla_{X_2} = T_{fx_2 fx_1 - x_2(fx_1)} \tag{10}
\]
again as operators from \( C^\infty(N, L^k) \) to \( H^0(N, L^k) \).

For \( X \in C^\infty(N, TN) \), the complex conjugate vector field \( \bar{X} \in C^\infty(N, TN) \) is a section of the antiholomorphic tangent bundle, and for \( s_1, s_2 \in C^\infty(N, L^k) \), we have that
\[
\bar{X}(s_1, s_2) = \langle \nabla_{\bar{X}} s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.
\]

Computing the Lie derivative along \( \bar{X} \) of \( (s_1, s_2) \omega^m \) and integrating, we get that
\[
\langle \nabla_{\bar{X}} s_1, s_2 \rangle + \langle (\nabla_X)^* s_1, s_2 \rangle = \langle \Lambda d(i_{\bar{X}} \omega) s_1, s_2 \rangle.
\]

Hence we see that
\[
(\nabla_X)^* = - (\nabla_{\bar{X}} + f_{\bar{X}})
\]
as operators on \( C^\infty(N, L^k) \). In particular, we see that
\[
\pi(\nabla_X)^* \pi = -T_{fx} |_{H^0(N, L^k)} : H^0(N, L^k) \to H^0(N, L^k). \tag{11}
\]
Moreover, it is also proved that the formal generating series for \((\nabla_X)^*\) that
\[
\pi(\nabla_{X_1})^*(\nabla_{X_2})^*h \pi = \pi \tilde{X}_1 \tilde{X}_2(h) \pi + \pi f_{\tilde{X}_1} \tilde{X}_2(h) \pi + \pi f_{\tilde{X}_1} \tilde{X}_2(h) \pi + \pi \tilde{X}_1 (f_{\tilde{X}_2}) h \pi + \pi f_{\tilde{X}_1} f_{\tilde{X}_2} h \pi
\]
as operators on \(H^0(N, L^k)\).

We need the following Theorems on Toeplitz operators. The first is due to Bordemann, Meinrenken and Schlichenmaier (see [BMS]). The \(L^2\)-inner product on \(C^\infty(N, L^k)\) induces an inner product on \(H^0(N, L^k)\), which in turn induces the operator norm \(\| \cdot \|\) on \(\text{End}(H^0(N, L^k))\).

**Theorem 4** (Bordemann, Meinrenken and Schlichenmaier). For any \(f \in C^\infty(N)\) we have that
\[
\lim_{k \to \infty} \| T_f^{(k)} \| = \sup_{x \in N} |f(x)|.
\]

Since the association of the sequence of Toeplitz operators \(T_f^k, k \in \mathbb{Z}_+\) is linear in \(f\), we see from this Theorem, that this association is faithful.

**Theorem 5** (Schlichenmaier). For any pair of smooth functions \(f_1, f_2 \in C^\infty(N)\), we have an asymptotic expansion
\[
T_{f_1}^{(k)} T_{f_2}^{(k)} \sim \sum_{l=0}^{\infty} T_{c_l(f_1, f_2)}^{(k)} k^{-l},
\]
where \(c_l(f_1, f_2) \in C^\infty(N)\) are uniquely determined since \(~\sim\) means the following: For all \(L \in \mathbb{Z}_+\) we have that
\[
\| T_{f_1}^{(k)} T_{f_2}^{(k)} - \sum_{l=0}^{L} T_{c_l(f_1, f_2)}^{(k)} k^{-l} \| = O(k^{-(L+1)}).
\]
Moreover, \(c_0(f_1, f_2) = f_1 f_2\).

This Theorem was proved in [Sch] and is published in [Sch1] and [Sch2], where it is also proved that the formal generating series for the \(c_l(f_1, f_2)\)'s gives a formal deformation quantization of the Poisson structure on \(\hat{N}\) induced from \(\omega\). By examining the proof in [Sch] (or in [Sch1] and [Sch2]) of this Theorem, one observes that for continuous families of functions, the estimates in Theorem 5 are uniform over compact parameter spaces.

5. Toeplitz operators on moduli space and the projective flat connection

Let \(f \in C^\infty(M)\) be a smooth function on the moduli space. We consider \(T_f^{(k)}\) as a section of the endomorphism bundle \(\text{End}(\mathcal{V}_k)\). The flat connection \(\hat{\nabla}\) in the projective bundle \(\mathbb{P}(\mathcal{V}_k)\) induces the flat connection \(\hat{\nabla}^e\) in the endomorphism bundle \(\text{End}(\mathcal{V}_k)\) as described in section 3. We shall now establish that the sections \(T_f^{(k)}\) are in a certain sense asymptotically flat by proving the following Theorem.

**Theorem 6**. Let \(\sigma_0\) and \(\sigma_1\) be two points in Teichmüller space and \(P_{\sigma_0, \sigma_1}\) be the parallel transport in the flat bundle \(\text{End}(\mathcal{V}_k)\) from \(\sigma_0\) to \(\sigma_1\). Then
\[
\| P_{\sigma_0, \sigma_1} T_f^{(k)} - T_f^{(k)} \| = O(k^{-1}),
\]
where \( \| \cdot \| \) is the operator norm on \( H^0(M_{\sigma}, \mathcal{L}^k_{\sigma}) \).

In the proof of this Theorem we will make use of the following Hermitian structure on \( \mathcal{H}_k \)
\[
\langle s_1, s_2 \rangle_F = \frac{1}{m!} \int_M (s_1, s_2)e^{-F} \omega^m,
\]
where we recall that \( F = F_\sigma \) is the Ricci potential, which is a real smooth function on \( M_{\sigma} \) for each \( \sigma \in \mathcal{T} \) determined by equation (4). In Lemma 11 below we will see that \( \langle \cdot, \cdot \rangle_F \) is uniform equivalent to the constant \( L_2 \)-Hermitian structure on \( \mathcal{H}_k \), when both are restricted to \( \mathcal{V}_k \) over any compact subset of \( \mathcal{T} \). The constant \( L_2 \)-Hermitian structure on \( \mathcal{H}_k \) is not asymptotically flat with respect to \( \bar{\nabla} \), but Proposition 12 below shows that the Hermitian structure \( \langle \cdot, \cdot \rangle_F \) restricted to \( \mathcal{V}_k \) is asymptotically flat with respect to \( \bar{\nabla} \). - It therefore induces a Hermitian structure on \( \mathcal{V}_k \otimes \mathcal{V}_k \) (which we also denote \( \langle \cdot, \cdot \rangle_F \)), which is asymptotic flat with respect to \( \bar{\nabla}^c \).

This suggests one considers the smooth function \( t \mapsto |P_{\sigma_0, \sigma} T^{(k)}_{f, \sigma_0} - T^{(k)}_{f, \sigma}|_F \) and establishes an \( O(k^{-1}) \) estimate for its derivative uniformly over \( J \), since by Lemma 13 and the first inequality in (17) below, \( O(k^{-1}) \) control on \( |P_{\sigma_0, \sigma} T^{(k)}_{f, \sigma_0} - T^{(k)}_{f, \sigma}|_F \) implies Theorem 6. An \( O(k^{-1}) \) estimate on \( \bar{\nabla}^c \sigma_1 T^{(k)}_{\sigma_1} \) uniformly over \( J \) would imply this estimate, however we are only able to establish that \( \| \bar{\nabla}^c \sigma_1 T^{(k)}_{\sigma_1} \| \) is \( O(k^{-1}) \) uniformly over the interval, which is considerably weaker because of the \( \frac{m}{2} \)th power of \( k \) in the second inequality in (17) below.

We shall therefore perturb the function \( f \) by adding on sufficiently many terms of the form \( (h_t)_l k^{-l} \) (\( l = 1, \ldots, r > m/2 \)), where \( (h_t)_l \in C^\infty(M), t \in J \), so as to obtain a smooth one-parameter family of functions \( (f_r)_l = f + \sum_{r=1}^r (h_t)_l k^{-l} \). The \( (h_t)_l \)'s are determined inductively in \( l \) such that \( \| \bar{\nabla}^c \sigma_1 T^{(k)}_{\sigma_1} \| \) is \( O(k^{-r-1}) \) uniformly over \( J \). This is the content of Proposition 14 below. That estimate on the covariant derivative of the Toeplitz operator \( T^{(k)}_{(f_r)_l, \sigma_1} \) will allow us to prove that \( \| P_{\sigma_0, \sigma} T^{(k)}_{(f_r)_l, \sigma_0} - T^{(k)}_{(f_r)_l, \sigma_1} \| \) is \( O(k^{-1}) \) by analyzing the derivative of

\[
t \mapsto |P_{\sigma_0, \sigma} T^{(k)}_{(f_r)_l, \sigma_0} - T^{(k)}_{(f_r)_l, \sigma_1}|^2_F.
\]

Since we can arrange that \( (f_r)_0 = f \) and since \( \| T^{(k)}_{(f_r)_l, \sigma_1} - T^{(k)}_{f, \sigma_1} \| \) is \( O(k^{-1}) \), this will allow us to prove Theorem 6.

First however, we need to establish a useful formula for the derivative of the orthogonal projection \( \pi \) along the curve \( \sigma_t \). To this end, consider a basis of covariant constant sections\(^2\) \( s_i = (s_i)_t \), \( i = 1, \ldots, \text{Rank } \mathcal{V}_k \), of \( \mathcal{V}_k \) over the curve \( \sigma_t \):

\[
s'_i = uG(s_i), \quad i = 1, \ldots, \text{Rank } \mathcal{V}_k.
\]

\(^2\)We will from now on mostly suppress the subscript \( t \) on various quantities defined along the curve \( \sigma_t \).
Recall formula (8) for the projection \( \pi : C^\infty(M, \mathcal{L}^k) \to H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t}) \) and compute the derivative along \( \sigma_t \): For any fixed \( s \in C^\infty(M, \mathcal{L}^k) \), we have that

\[
\pi'(s) = \sum_{i,j} (s, s'_i) h_{ij}^{-1} s_j + \sum_{i,j} (s, s_i) (h_{ij}^{-1})' s_j + \sum_{i,j} (s, s_i) h_{ij}^{-1} s_j'.
\]

An easy computation gives that

\[
(h_{ij}^{-1})' = - \sum_{l,r} h_{il}^{-1} (s'_l, s_r) h_{lj}^{-1} (s_i, s_r) h_{lj}^{-1},
\]

so

\[
\pi \pi'(s) = \sum_{i,j} (u_G s_i, s_i) h_{ij}^{-1} s_j + \sum_{i,l,m,j} (s, s_i) h_{il}^{-1} (s_i, s_m') h_{mj}^{-1} s_j + \pi u_G^* (s) - \sum_{m,j} (s, s_m') h_{mj}^{-1} s_j
\]

\[
= \pi u_G^* (s) - \pi u_G^* \pi (s).
\]

Hence we conclude that

\[
\pi \pi' = \pi u_G^* - \pi u_G^* \pi.
\]

Having derived the formula for the derivative of \( \pi \), we now proceed to construct the needed perturbation of \( f \). Let \( r \) be a non-negative integer and let

\[
(f_r)_t = \sum_{i=0}^{r} (h_i)_t k^{-i},
\]

where the \( (h_i)_t \), \( t \in J \), for now are arbitrary smooth one-parameter families of smooth functions on \( M \), however we will fix \( (h_0)_t = f \) for all \( t \in J \). We have that \( T_{f_r}^{(k)} \) is a section of \( \text{End}(\mathcal{V}_k) \) over the curve \( \sigma_t \). According to formula (8), we have that

\[
\nabla_{\sigma_t}^2 (T_{f_r}) = (T_{f_r})' - [u_G, T_{f_r}] = \pi f'_r + \pi f_r - [u_G, \pi f_r].
\]

Since \( \nabla_{\sigma_t}^2 (T_{f_r}) \) is a section of \( \text{End}(\mathcal{V}_k) \), we have of course that

\[
\pi \nabla_{\sigma_t}^2 (T_{f_r}) \pi = \nabla_{\sigma_t}^2 (T_{f_r}) \pi : H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t}) \to H^0(M_{\sigma_t}, \mathcal{L}^k_{\sigma_t}).
\]

**Proposition 1.** Given \( f \) and a non-negative integer \( r \), there exists unique smooth one-parameter families of functions \( (h_i)_t \in C^\infty(M) \), \( i = 1, \ldots, r \) and \( t \in J \) such that

\[
\sup_{t \in J} \| \nabla_{\sigma_t}^2 (T_{f_r}) \| = O(k^{-r-1}),
\]

and \( (h_i)_0 = 0 \), \( i = 1, \ldots, r \).
Proof. For any choice of $h_i$’s, $i = 1, \ldots, r$, we compute using (19) that

$$\pi \hat{\nabla}_{\sigma'}(T_{f_r}) \pi = \sum_{i=1}^{r} \pi h_i' \pi k^{-i}$$

$$+ \sum_{i=0}^{r} (\pi u_G h_i \pi - \pi u_G' h_i \pi ) k^{-i}$$

$$- \sum_{i=0}^{r} (\pi u_G h_i \pi - \pi u_G h_i \pi ) k^{-i}.$$

Using (14) together with (9) and (10) we define the function $H_G \in C^\infty(M)$ independent of $k$, as follows

$$\pi (\Delta G - 2\nabla_G) = \pi H_G.$$  

We also define $H_i \in C^\infty(M)$ which depend on $h_i$ and $G$ but are independent of $k$, by

$$\pi h_i (\Delta G - 2\nabla_G) = \pi H_i.$$  

Using (11) and (12) we further define the function $H^*_G \in C^\infty(M)$ independent of $k$, as follows

$$\pi (\Delta^*_G - 2(\nabla_G)^*)_h \pi = \pi H^*_G \pi.$$  

And we define $H^*_i \in C^\infty(M)$ which depend on $h_i$ and $G$ but are independent of $k$, by

$$\pi (\Delta^*_G - 2(\nabla_G)^*)_h \pi = \pi H^*_i \pi.$$  

Then we have that

$$\pi \hat{\nabla}_{\sigma'}(T_{f_r}) \pi = \sum_{i=1}^{r} \pi h_i' \pi k^{-i}$$

$$- \sum_{i=0}^{r} \frac{1}{2(k+n)} (\pi H^*_G \pi h_i \pi - \pi H^*_i \pi ) k^{-i}$$

$$- \sum_{i=0}^{r} \frac{1}{2(k+n)} (\pi H_G \pi h_i \pi - \pi H_i \pi ) k^{-i}$$

$$+ \sum_{i=0}^{r} \frac{k}{2(k+n)} \sqrt{-1} (\pi f_G \pi h_i \pi - \pi f_G h_i \pi ) k^{-i}$$

$$- \sum_{i=0}^{r} \frac{k}{2(k+n)} \sqrt{-1} (\pi f_G \pi h_i \pi - \pi h_i f_G \pi ) k^{-i}.$$  

Because of (13) and Theorem 4 we see for any $r$, that condition (16) is equivalent to
\[
\sup_{i,j} \| \sum_{i=1}^{r} \pi h_i^* \pi k^{-i} - \frac{1}{2k} \sum_{i=0}^{m} \sum_{j=0}^{r} (-1)^j n^j (\sum_{l=0}^{r} \pi c_l(H_G, h_i) \pi k^{-l} - \pi H_i^* \pi) k^{-i-j}
- \frac{1}{2k} \sum_{i=0}^{r} \sum_{j=0}^{r} (-1)^j n^j (\sum_{l=0}^{r} \pi c_l(H_G, h_i) \pi k^{-l} - \pi H_i^* \pi) k^{-i-j}
+ \frac{\sqrt{-1}}{2k} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{l=1}^{r} (-1)^j n^j \pi c_l(f_G, h_i) \pi k^{-l-i-j+1}
- \frac{\sqrt{-1}}{2k} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{l=1}^{r} (-1)^j n^j \pi c_l(f_G, h_i) \pi k^{-l-i-j+1} \| = O(k^{-r-1}).
\]

Note that the expression inside the norm is a polynomial in $k^{-1}$, say $\sum_{i=1}^{N_r} \pi (C_i) t \pi k^{-i}$, where each coefficient $\pi (C_i) t \pi$ is the Toeplitz operator associated to a smooth one-parameter family $(C_i)_t \in C^\infty(M)$, $t \in J$. We also note that the functions $C_i$, $i = 1, \ldots, r$ do not depend on $r$ and in the same range for $i$ they are all of the following form

\[ C_i = h_i' - D_i \]

where $(D_i)_t \in C^\infty(M)$, $t \in J$, is a linear combination of

\[ H_j^*, H_j, c_l(H_G, h_j), c_l(H_G, h_j), c_l(f_G, h_j), c_l(f_G, h_j) \in C^\infty(J, C^\infty(M)) \]

for $j = 0, \ldots, i - 1$ and $l = 0, \ldots, i$, e.g. for $i = 1$ we get

\[ D_1 = -\frac{1}{2} (H_G h_0 - H_0^* h_0 - H_G h_0 - \sqrt{-1} c_1(f_G, h_0) + \sqrt{-1} c_1(f_G, h_0)). \]

Now observe by Theorem 4 that condition (16) is equivalent to $C_i = 0$ for $i = 1, \ldots, r$. From this it follows that we can inductively uniquely determine the functions $h_i$. First of all, since the coefficient of $k^0$ is zero, we see that (16) holds for $r = 0$. Now assume that we have determined $h_1, \ldots, h_{r-1}$ uniquely such that $(h_i)_{0} = 0$ and

\[ h_i' = D_i, \]

for $i = 1, \ldots, r - 1$. By the above we then observe that condition (16) holds if and only if $C_r = 0$, which means if and only if

\[ h_r' = D_r. \]

But since we require $(h_r)_0 = 0$, we must have that

\[ (h_r)_t = \int_0^t (D_r)_s ds. \]

\[ \square \]

**Lemma 1.** The Hermitian structure on $\mathcal{H}_k$

\[ \langle s_1, s_2 \rangle_F = \frac{1}{m^d} \int_M \langle s_1, s_2 \rangle e^{-F} \omega^m \]
and the constant $L_2$-Hermitian structure on $\mathcal{H}_k$

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^n$$

are equivalent uniformly in $k$ when restricted to $\mathcal{V}_k$ over any compact subset $K$ of $T$.

**Proof.** We clearly have that

$$|s|^2_F \leq \|T^{(k)}_c \| |s|^2,$$

so by Theorem 4 we see that there exists a constant $C$ (depending on $K$) such that

$$|s|_F \leq C |s|,$$

for all $k$. Conversely, we have that

$$|s|^2 = \langle \pi e^{-\frac{1}{2}F} \pi s, s \rangle$$

$$|s|^2 \leq |\langle \pi e^{-\frac{1}{2}F} \pi - \pi e^{-\frac{1}{2}F} \pi \rangle s, s \rangle| + |\langle \pi e^{-\frac{1}{2}F} \pi s, s \rangle|$$

$$\leq \|\pi e^{-\frac{1}{2}F} \pi - \pi e^{-\frac{1}{2}F} \pi \| |s|^2 + \|\pi e^{-\frac{1}{2}F} \pi \| |\pi e^{-\frac{1}{2}F} \pi s| |s|.$$

By Theorem 4 and 5 we see there exists constants $C'$ and $C''$ (again depending on $K$) such that

$$|s| \leq \frac{C'}{k} |s| + C'' |s|_F.$$

But then we have for all sufficiently large $k$ that

$$|s| \leq 2C' |s|_F.$$

Hence we have established the claimed equivalence.

□

Along any smooth one-parameter family of complex structures $\sigma_t$, we have that

$$\frac{d}{dt} \langle s_1, s_2 \rangle_F = \langle \hat{\nabla}^t_{\sigma_t}, s_1, s_2 \rangle_F + \langle s_1, \hat{\nabla}^t_{\sigma_t}, s_2 \rangle_F - \langle \frac{\partial F}{\partial t}, s_1, s_2 \rangle_F.$$

So, if we let

$$E(s) = \frac{d}{dt} |s|^2_F - \langle \hat{\nabla}^t_{\sigma_t}, s \rangle_F - \langle s, \hat{\nabla}^t_{\sigma_t}, s \rangle_F$$

recalling that $\sqrt{-1} f_G = \frac{\partial F}{\partial t}$, $\hat{\nabla} = \hat{\nabla}^t - u(v)$ and formula (7), we have for all sections $s$ of $\mathcal{V}_k$ that

$$E(s) = \frac{1}{2(k+n)} \left( \langle \pi e^{-F} (\Delta_G - 2\nabla_{G\partial F} - \sqrt{-1} f_G) s, s \rangle + \langle s, \pi e^{-F} (\Delta_G - 2\nabla_{G\partial F} - \sqrt{-1} f_G) s \rangle \right).$$

Hence by combining Theorem 4, 6, 9 and 10 we have proved that
Proposition 2. The Hermitian structure \(|\hat{\nabla}|\) is asymptotically flat with respect to the connections \(\hat{\nabla}\), i.e. for any compact subset \(K\) of \(\mathcal{T}\), there exists a constant \(C\) such that for all sections \(s\) of \(\mathcal{V}_k\) over \(K\), we have that
\[
|E(s)| \leq \frac{C}{k + n}|s|^2_F
\]
over \(K\).

We note that this proposition implies the same proposition for sections of \(\text{End}(\mathcal{V}_k)\) with respect to the induced Hermitian structure on \(\mathcal{V}_k = \mathcal{V}_k^* \otimes \mathcal{V}_k\), which we also denote \(\langle \cdot, \cdot \rangle_F\). We denote the analogous quantity of \(E\) for the endomorphism bundle by \(E_e\).

Proof of Theorem 2 Let \(\sigma_i, t \in J\) be a smooth one-parameter family of complex structures such that \(\sigma_i\) is a curve in \(\mathcal{T}\) between the two points in question. By Lemma 1 the Hermitian structures \(\langle \cdot, \cdot \rangle\) and \(\langle \cdot, \cdot \rangle_F\) on \(\mathcal{V}_k\) are equivalent uniformly in \(k\) over compact subsets of \(\mathcal{T}\). It follows there exists constants \(C_1\) and \(C_2\), such that we have the following inequalities over the image of \(\sigma_i\) in \(\mathcal{T}\) for the operator norm \(||\cdot||\) and the norm \(||\cdot||_F\) on \(\text{End}(\mathcal{V}_k)\):
\[
|\cdot| \leq C_1|\cdot|_F \leq C_2\sqrt{P_{g,n}(k)}||\cdot||,
\]
where \(P_{g,n}(k)\) is the rank of \(\mathcal{V}_k\) given by the Verlinde formula. By the Riemann-Roch Theorem this is a polynomial in \(k\) of degree \(m\).

Because of these inequalities, we choose an integer \(r\) bigger than \(m/2\) and let \(f_r\) be as provided by Proposition 1 and we define \(n_k : J \rightarrow [0, \infty)\) by
\[
n_k(t) = |\Theta_k(t)|^2_F
\]
where
\[
\Theta_k(t) : \mathcal{V}_{k,\sigma_i} \rightarrow \mathcal{V}_{k,\sigma_i}
\]
is given by
\[
\Theta_k(t) = P_{\sigma_0,\sigma_i} T_{(f_r),\sigma_0}^{(k)} - T_{(f_r),\sigma_i}^{(k)}.
\]
The functions \(n_k\) are differentiable in \(t\) and we compute that
\[
\frac{dn_k}{dt} = \langle \hat{\nabla}_{\sigma_i}^c T_{(f_r),\sigma_i}^{(k)}, \Theta_k(t) \rangle_F + \langle \Theta_k(t), \hat{\nabla}_{\sigma_i}^c (\Theta_k(t)) \rangle_F + E_e(\Theta_k(t))
\]
\[
= -\langle \hat{\nabla}_{\sigma_i} T_{(f_r),\sigma_i}^{(k)}, \Theta_k(t) \rangle_F - \langle \Theta_k(t), \hat{\nabla}_{\sigma_i}^c T_{(f_r),\sigma_i}^{(k)} \rangle_F + E_e(\Theta_k(t)).
\]
Using the above, we get the following estimate
\[
\left|\frac{dn_k}{dt}\right| \leq 2|\hat{\nabla}_{\sigma_i}^c T_{(f_r),\sigma_i}^{(k)}|_F |\Theta_k(t)|_F + |E_e(\Theta_k(t))|
\]
\[
\leq 2C_2\sqrt{P_{g,n}(k)}||\hat{\nabla}_{\sigma_i}^c T_{(f_r),\sigma_i}^{(k)}||_{n_k}^{1/2} + |E_e(\Theta_k(t))|.
\]
Consequently we can apply Proposition 1 and 2 to obtain that there exists a constant \(C\) such that
\[
\left|\frac{dn_k}{dt}\right| \leq \frac{C}{k} (n_k^{1/2} + n_k).
\]
This estimate implies that
\[
n_k(t) \leq (\exp(\frac{Ct}{2k}) - 1)^2.
\]
But by \[17\] we get that
\[\|P_{\sigma_0,\sigma_0}T_{(f_r)_0,\sigma_0}^{(k)} - T_{(f_r)_1,\sigma_1}^{(k)}\| = \|\Theta_k(1)\| \leq C_1 n_k(1)^{1/2}.\]
The Theorem then follows from these two estimates, since \((f_r)_0 = f\) and
\[\|T_{(f_r)_1,\sigma_1}^{(k)} - T_{f,\sigma_1}^{(k)}\| = O(k^{-1}).\]

\[\square\]

6. Asymptotic Faithfulness

Recall that the flat connection in the bundle \(\mathbb{P}(V_k)\) gives the projective representation of the mapping class group
\[\rho_k : \Gamma \to \text{Aut}(\mathbb{P}(V_k)),\]
where \(\mathbb{P}(V_k) = \text{covariant constant sections of } \mathbb{P}(V_k)\) over Teichmüller space.

**Theorem 7.** For any \(\phi \in \Gamma\), we have that
\[\phi \in \bigcap_{k=1}^{\infty} \ker \rho_k\]
if and only if \(\phi\) induces the identity on \(M\).

**Proof.** Suppose we have a \(\phi \in \Gamma\). Then \(\phi\) induces a symplectomorphism of \(M\) which we also just denote \(\phi\) and we get the following commutative diagram for any \(f \in C^\infty(M)\)
\[
\begin{array}{ccc}
H^0(M_\sigma, L_\sigma^k) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, L_{\phi(\sigma)}^k) \\
T_{f,\sigma}^{(k)} \downarrow & & \downarrow \quad T_{f \circ \phi, \sigma}^{(k)} \\
H^0(M_\sigma, L_\sigma^k) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, L_{\phi(\sigma)}^k)
\end{array}
\]
where \(P_{\phi(\sigma),\sigma} : H^0(M_{\phi(\sigma)}, L_{\phi(\sigma)}^k) \to H^0(M_\sigma, L_\sigma^k)\) on the horizontal arrows refer to parallel transport in the Verlinde bundle itself, whereas \(P_{\phi(\sigma),\sigma}\) refers to the parallel transport in the endomorphism bundle \(\text{End}(V_k)\) in the last vertical arrow. Suppose now \(\phi \in \bigcap_{k=1}^{\infty} \ker \rho_k\), then \(P_{\phi(\sigma),\sigma} \circ \phi^* = \rho_k(\phi) \in \mathbb{C} \text{Id}\) and we get that \(T_{f,\sigma}^{(k)} = P_{\phi(\sigma),\sigma} T_{f \circ \phi, \sigma}^{(k)}\). By Theorem \[14\] we get that
\[
\lim_{k \to \infty} \|T_{f \circ \phi, \sigma}^{(k)} - T_{f,\sigma}^{(k)}\| = \lim_{k \to \infty} \|T_{f,\sigma}^{(k)} - T_{f \circ \phi, \sigma}^{(k)}\| = \lim_{k \to \infty} \|P_{\phi(\sigma),\sigma} T_{f \circ \phi, \sigma}^{(k)} - T_{f,\sigma}^{(k)}\| = 0.
\]

By Bordemann, Meinrenken and Schlichenmaier’s Theorem \[14\] we must have that \(f = f \circ \phi\). Since this holds for any \(f \in C^\infty(M)\), we must have that \(\phi\) acts by the identity on \(M\).

\[\square\]

**Proof of Theorem 7**. Our main Theorem \[14\] now follows directly from Theorem \[14\] since it is known that the only element of \(\Gamma\), which acts by the identity on the moduli space \(M\) is the identity, if \(g > 2\). If \(g = 2\), \(n = 2\) and \(d\) is even, it is contained in the sub-group generated by the hyper-elliptic involution, else it is also the identity.
A way to see this using the moduli space of flat $SL(n, \mathbb{C})$-connections goes as follows: We have that $M$ is a component of the real slice in $\mathcal{M}$, the moduli space of flat $SL(n, \mathbb{C})$-connections on $\Sigma - p$, whose holonomy around $p$ has trace $n \exp(2\pi \sqrt{-1} \frac{d}{n})$. Hence if $\phi$ acts by the identity on $M$, it will also act by the identity on an open neighbourhood of $M$ in $\mathcal{M}$, since it acts holomorphically on $\mathcal{M}$. But since $M$ is connected, $\phi$ must act by the identity on the entire $SL(n, \mathbb{C})$-moduli space $\mathcal{M}$. Now the generalized Teichmüller space $\tilde{T}_p$ of $\Sigma - p$ is also included in $\mathcal{M}$, hence we get that $\phi$ acts by the identity on $\tilde{T}_p$. But then the statement about $\phi$ follows by classical theory of how $\Gamma$ acts on $\tilde{T}_p$.

□

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