Existence of proper weak solutions to the Navier-Stokes-Fourier system

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Abstract

The existence of proper weak solutions of the Dirichlet-Cauchy problem constituted by the Navier-Stokes-Fourier system which characterizes the incompressible homogeneous Newtonian fluids under thermal effects is studied. We call proper weak solutions such weak solutions that verify some local energy inequalities in analogy with the suitable weak solutions for the Navier-Stokes equations. Finally, we deal with some regularity for the temperature.

Keywords: Navier-Stokes-Fourier system, Joule effect, suitable weak solutions

MSC2000: 76D03, 35Q30, 80A20

1 Introduction

We study the existence and the regularity of weak solutions for the Dirichlet-Cauchy problem constituted by the Navier-Stokes-Fourier system which is one of the prominent mathematical problems for full thermodynamical systems describing flows of incompressible fluids (see for example [8, 9, 23] and the references therein). The presence of the temperature dependent viscosity into the momentum equation and the Joule effect into the energy equation are the main contributions on the nonlinear behavior of the coupled system

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of partial differential equations. Other problem which arises from fluid thermomechanics is the Boussinesq approximation describing the dynamics in the planet mantle (see [22] and the references therein). In fact, it is a simpler coupled system which does not include the Joule effect. For instance, in [12, 5, 17] even the viscosity is considered constant.

Although recently the $C(0, T; C^0, c(\Omega))$-regularity for any weak solution is found to the non-stationary Stokes system in 3D [24], the non-uniqueness and smoothness of weak solutions are related with the non-uniqueness of Leray-Hopf’s solutions to the first initial boundary value problem for the three-dimensional Navier-Stokes equations. For any class of weak solutions of the non-stationary N-S equations in three-dimensional spaces, it is known that a weak solution cannot be used as a test function in the weak variational formulation. Indeed, in the N-S-F system, the viscosity is a temperature dependent function and the temperature is a solution of a parabolic equation with $L^1$-data due to the existence of the Joule effect. By these reasons, in the work [6, 25] (and some references therein) the N-S-F system is constituted by momentum and total energy equations in order to the $L^1$ dissipative term in the internal energy equation is obtained by approximative methods using the weakly sequential lower semicontinuity of the norm. On the other hand, in the N-S system the study of the class of suitable weak solutions is being of interest since [7, 20, 26, 27], if homogeneous Dirichlet boundary conditions are enforced (see for instance [15, 16]).

In this work, we prove the existence of weak solutions to the problem under study and that among the weak solutions at least the existence of one proper weak solution is guaranteed, i.e. such weak solutions that verify some local energy inequalities. Moreover, we show a higher integrability of the gradient of the velocity, using the reverse Hölder inequality, which is used in the proof of existence of weak solutions in order to deal with the $L^1$-dissipative term in the energy equation. Some regularity for the temperature appears as a direct consequence. We refer to [11] for the proof of the higher summability of the gradient of the velocity in the stationary case. The regularity of strong two-dimensional solutions was solved by the author in the paper [10] if some smallness on the data is considered. The study of the partial regularity for the velocity of the fluid will be an ongoing future work.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain sufficiently regular and $T > 0$. 

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Let us consider the boundary-value problem of the N-S-F system:
\begin{align}
\partial_t \mathbf{u} - \text{div}(\mu(\theta) \nabla \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \mathbf{f} - \nabla p \quad \text{in } Q_T := \Omega \times [0, T]; \\
\text{div } \mathbf{u} &= 0 \quad \text{in } Q_T; \\
\partial_t \theta - \text{div}(k(\theta) \nabla \theta) + \mathbf{u} \cdot \nabla \theta &= \mu(\theta) |D\mathbf{u}|^2 \quad \text{in } Q_T; \\
\mathbf{u}|_{t=0} &= \mathbf{u}_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \Omega; \\
\mathbf{u} &= 0, \quad \theta = 0 \quad \text{on } \partial \Omega \times [0, T],
\end{align}

where \( p \) denotes the pressure, \( \mu \) the viscosity, \( \theta \) the temperature, \( \mathbf{u} \) the velocity of the fluid and \( D\mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \), \( \mathbf{f} \) denotes the given external body forces, \( k \) denotes the conductivity and it is assumed constant, \( k(\theta) \equiv k \). Notice that the assumption of a constant conductivity is not necessary for the proof of existence of weak solutions. It is a sufficient condition for the proof of existence of the local energy inequalities to the temperature solution. For simplicity, the constant density is assumed equal to one, and we do not consider the existence of the external heat source, since the heating dissipative term is the main mathematical difficulty. The product of two tensors is given by \( D : \tau = D_{ij} \tau_{ij} \) and the norm by \( |D|^2 = D : D \).

The initial conditions are given in (3). For the sake of clarity we found convenient that the boundary conditions which are given in (4) are assumed homogeneous Dirichlet conditions.

The outline of the paper is as follows: in next section we establish the appropriate functional framework and we present the main results. The Section 3 is devoted to the proof of the existence result (Theorem 2.1). In Section 4 we prove some regularity result (Proposition 2.1).

## 2 Assumptions and main results

Here we assume that \( \Omega \subset \mathbb{R}^n \) is a bounded open set such that its boundary \( \partial \Omega \in C^2 \). In the framework of Lebesgue and Sobolev spaces, for \( 1 \leq q \leq \infty \), we introduce \([18]\)
\[ J_{0}^{1,q}(\Omega) = \{ \mathbf{u} \in W_{0}^{1,q}(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \} \]
with norm
\[ \| \cdot \|_{1,q,\Omega} = \| \nabla \cdot \|_{q,\Omega}, \]
and we set
\[ W_{q}^{2,1}(Q_T) := \{ v \in L^q(0, T; W_{2,q}^2(\Omega)) : \partial_t v \in L^q(Q_T) \}. \]
For any set $A$, we write $(u,v)_A := \int_A uv$ whenever $u \in L^q(A)$ and $v \in L^{q'}(A)$, where $q' = q/(q-1)$ is the conjugate exponent to $q$, or simply $(\cdot, \cdot)$ whenever there exists no confusing at all, and we use the symbol $(\cdot, \cdot)$ to denote a generic duality pairing, not distinguished between scalar and vector fields. We denote by bold the vector spaces of vector-valued or tensor-valued functions.

The following assertions on data are assumed as well as the following assumptions on the physical parameters appearing in the equations are established.

(A1) $f : Q_T \to \mathbb{R}^n$ is given such that $f \in L^{2(1+\epsilon_0)}(Q_T)$ with $\epsilon_0 > 0$;

(A2) $\mu : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$0 < \mu_\# \leq \mu(s) \leq \mu_\#, \quad \forall s \in \mathbb{R};$$  \hspace{1cm} (5)

(A3) $u_0 \in L^2(\Omega)$, $\theta_0 \in L^1(\Omega)$ such that

$$\text{div } u_0 = 0 \quad \text{in } \Omega; \quad \text{ess inf}_{x \in \Omega} \theta_0(x) \geq 0.$$  \hspace{1cm} (6)

**Definition 2.1** We say that the triple $(u, p, \theta)$ is a weak solution to the Navier-Stokes-Fourier (N-S-F) problem (1)-(4) in $Q_T$ if

$$u \in U := L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; J^{1/2}_0(\Omega)), \quad p \in L^{(n+2)/n}(Q_T),$$

$$\theta \in E := L^\infty(0,T; L^1(\Omega)) \cap L^q(0,T; W^{1,q}_0(\Omega)), \quad q < \frac{n+2}{n+1},$$

$$\partial_t u \in X := L^2(0,T; W^{-1,2}(\Omega)) \cap L^{(n+2)/n}(0,T; W^{-1,(n+2)/n}(\Omega)),$$

$$\partial_t \theta \in L^1(0,T; W^{-1,\ell}(\Omega)), \quad \frac{1}{\ell} = \frac{n}{q(n+1)} + \frac{n}{2(n+2)}$$

and satisfies the variational formulation

$$\langle \partial_t u, v \rangle + \int_{Q_T} (\mu(\theta) Du : Dv + (u \cdot \nabla) u \cdot v) = \int_{Q_T} (f \cdot v + p \text{ div } v) \quad \forall v \in L^\infty(0,T; W^{1,\infty}_0(\Omega)), \quad u_{|t=0} = u_0 \text{ in } \Omega;$$  \hspace{1cm} (7)

$$\langle \partial_t \theta, \phi \rangle + \int_{Q_T} (k \nabla \theta - \theta u) \cdot \nabla \phi dx dt = \int_{Q_T} \mu(\theta) |Du|^2 \phi dx dt$$

$$\forall \phi \in L^\infty(0,T; W^{1,\infty}_0(\Omega)), \quad \theta_{|t=0} = \theta_0 \text{ in } \Omega.$$  \hspace{1cm} (8)
The embedding $X \hookrightarrow L^{(n+2)/n}(0,T; W^{-1,(n+2)/n}(\Omega))$ occurs for $n = 2, 3$. For every $u \in U \hookrightarrow L^{2(n+2)/n}(Q_T)$, the convective term verifies $(u \cdot \nabla)u \in L^{(n+2)/(n+1)}(Q_T)$ and consequently $(u \cdot \nabla)u \cdot u \notin L^1(Q_T)$. Moreover, the advection term $u \cdot \nabla \theta \notin L^1(Q_T)$ if $\theta \in \mathcal{E}$ and $q < (n + 2)/(n + 1)$. Remark that $\theta \in \mathcal{E} \hookrightarrow L^{q(n+1)/n}(Q_T)$ and $\theta u \in L^\ell(Q_T)$ for $\ell \geq 1$, i.e. $q > 1$ if $n = 2$ and $q \geq 15/14$ if $n = 3$ ($q \geq 2(n + 2)/((n + 4)(n + 1))$).

**Definition 2.2** We say that a weak solution $(u, p, \theta)$ of the N-S-F problem is proper in the following sense. The local energy inequality holds

$$
\int_\Omega \frac{|u(x, t) - a|^2}{2} \varphi^2(x, t) dx + \int_{Q_t} \mu(\theta)|Du|^2 \varphi^2 d\tau \leq \int_{Q_t} \mu(\theta) |Du| (u \cdot \nabla \varphi) + \int_{Q_t} |u - a|^2 \varphi (\partial_t \varphi + u \cdot \nabla \varphi) + 2 \int_{Q_t} \rho u \cdot \nabla \varphi d\tau + \int_{Q_t} f (u - a) \varphi^2 d\tau,
$$

for all $\varphi \in C^\infty_0(Q_T)$, a.e. $t \in ]0, T[\]$ and for any $a \in \mathbb{R}^n$, and two more local energy inequalities hold

$$
\int_{Q_t} \sqrt{\zeta + \theta^2} \psi (x, t) dx + \zeta k \int_{Q_t} \frac{|\nabla \theta|^2}{(\zeta + \theta^2)^{3/2}} \psi d\tau \leq \int_{Q_t} \sqrt{\zeta + \theta^2} (\partial_t \psi + k \Delta \psi + u \cdot \nabla \psi) + \int_{Q_t} \mu(\theta)|Du|^2 \frac{\theta \psi}{(\zeta + \theta^2)^{1/2}},
$$

$$
\xi k \int_{Q_t} \frac{|\nabla \theta|^2}{(1 + \theta)^{\xi + 1}} \psi d\tau + \int_{Q_t} \mu(\theta)|Du|^2 \frac{\psi}{(1 + \theta)^{\xi}} d\tau \leq \frac{1}{1 - \xi} \int_\Omega ((1 + \theta)^{-\xi} \psi) (x, t) dx - \frac{1}{1 - \xi} \int_{Q_t} (1 + \theta)^{-\xi} (\partial_t \psi + k \Delta \psi + u \cdot \nabla \psi) d\tau,
$$

for any $\zeta > 0$ and $0 < \xi < 1$, and for all $\psi \in C^\infty_0(Q_T)$ such that $\psi \geq 0$, a.e. $t \in ]0, T[\]$.

**Remark 2.1** Two local energy inequalities are stated for the temperature because from (10) if we take $\zeta \to 0^+$ we obtain $\theta \in L^\infty(0, T; L^1_{loc}(\Omega))$ and from (11) after some computations we get $\nabla \theta \in L^q_{loc}(Q_T)$. 

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Theorem 2.1 \((n = 2, 3)\) Under the assumptions \((A1)-(A3)\), the N-S-F problem defined by \((7)-(8)\) has proper weak solutions. Moreover, \(\nabla u \in L^2_{\text{loc}}(Q_T)\) for \(0 < \epsilon < \min\{(4 - n)/(3n), 1/(n + 2), \epsilon_0\}\) and \(\theta \geq 0\) in \(Q_T\).

Some interior regularity is proved.

Proposition 2.1 (Interior regularity) Let \((u, p, \theta)\) be a weak solution in accordance to Theorem 2.1. Then \(\theta \in W^{2,1}_{1+\epsilon,\text{loc}}(Q_T)\). In particular, \(\theta \in L^{(n+2)(1+\epsilon)/(n-2\epsilon)}_{\text{loc}}(Q_T)\).

Henceforth \(C\) will denote different positive constants depending on the data, but not on the unknown functions \(u, p\) or \(\theta\).

3 Proof of Theorem 2.1

The proof of Theorem 2.1 is split in each subsection. In sections 3.1, 3.2 and 3.3 for reader’s convenience we delineate the main arguments concerning the existence of approximate solutions (for details, see [6]). The sections 3.4, 3.5 and 3.6 are new and they are the main contributions for the desired existence result.

3.1 The Faedo-Galerkin method

For \(\nu, \epsilon > 0\) fixed, there exists \(\{u^{N,M}, p^{N,M}, \theta^{N,M}\}_{N,M \in \mathbb{N}}\) being of the form [30] Carathéodory Theorem]

\[
\begin{align*}
\mathbf{u}^{N,M} \in \langle \mathbf{w}^1, \cdots, \mathbf{w}^N \rangle & \iff \mathbf{u}^{N,M}(x, t) = \sum_{j=1}^{N} c_j^{N,M}(t) \mathbf{w}^j(x), \\
p^{N,M}(x, t) & = \mathcal{F}_\epsilon(u^{N,M}(x, t)), \\
\theta^{N,M} \in \langle \mathbf{w}^1, \cdots, \mathbf{w}^M \rangle & \iff \theta^{N,M}(x, t) = \sum_{j=1}^{M} d_j^{N,M}(t) \mathbf{w}^j(x)
\end{align*}
\]

where \(\{(\mathbf{w}^j, \mathbf{w}^j)\}_{j \in \mathbb{N}}\) is a basis of \(W^{1,\beta}_0(\Omega) \times W^{1,\beta}_0(\Omega)\) with \(\beta > n\), \(\mathcal{F}_\epsilon\) is the continuous functional such that maps \(u \in W^{1,2}_0(\Omega)\) into \(p \in W^{2,2}(\Omega)\) which is
the solution of the homogeneous Neumann problem for the Laplace equation (see, for instance, [13])

\[ \varepsilon \Delta p(t) = \text{div} \ u(t) \quad \text{in} \ \Omega \]
\[ \nabla p(t) \cdot n = 0 \quad \text{on} \ \partial \Omega \]
\[ \int_{\Omega} p(t) dx = 0, \]

which satisfies, a.e. \( t \in ]0, T[ , \)

\[ \varepsilon \| p(t) \|_{2, 2, \Omega} \leq C(\Omega) \| u(t) \|_{1, 2, \Omega}, \quad \forall u(t) \in W_{0}^{1, 2}(\Omega) \]
\[ \varepsilon \| p(t) \|_{1, r, \Omega} \leq C(\Omega, r) \| u(t) \|_{r, \Omega}, \quad \forall u(t) \in W_{0}^{1, 2}(\Omega) \cap L^{r}(\Omega), \quad r > 1. \]

The functions \( c_{N}^{N, M} = (c_{1}^{N, M}, \cdots, c_{N}^{N, M}) \) and \( d_{N}^{N, M} = (d_{1}^{N, M}, \cdots, d_{M}^{N, M}) \) solve the following system of ordinary differential equations, for every \( M, N \in \mathbb{N} , \)

\[ \frac{d}{dt}(u^{N, M}, w^{j}) - (M_{\nu}(u^{N, M}) \otimes u^{N, M}, \nabla w^{j}) + (\mu(\theta^{N, M})Du^{N, M}, Dw^{j}) - \\
\quad - (F_{\varepsilon}(u^{N, M}), \nabla \cdot w^{j}) = (f, w^{j}), \quad j = 1, \cdots, N; \quad (12) \]
\[ \frac{d}{dt}(\theta^{N, M}, w^{j}) - (\theta^{N, M}M_{\nu}(u^{N, M}), \nabla w^{j}) + k(\nabla \theta^{N, M}, \nabla w^{j}) = \\
\quad = (\mu(\theta^{N, M})|Du^{N, M}|^{2}, w^{j}), \quad j = 1, \cdots, M, \quad (13) \]

under the initial conditions \( u_{0}^{N}, \theta_{0}^{N, M} \) given by the projections of \( u_{0} \) and the mollification \( \theta_{0}^{N} \) of \( \theta_{0} \) (after extending \( \theta_{0} \) by zero outside \( \Omega \)), respectively, onto linear hulls of the base’s vectors. Finally, \( M_{\nu} \) is the divergenceless part of the Helmholtz-mollification decomposition

\[ M_{\nu}(u) := (\chi u) * \omega - \nabla h \]

with \( \omega \) denoting a mollifier with support in a ball of radii \( \nu , \)

\[ \chi(x) = \begin{cases} 
0 & \text{if dist}(x, \partial \Omega) \leq 2\nu \\
1 & \text{elsewhere,} \end{cases} \]

and \( h \) is the Helmholtz-mollification decomposition, that is,

\[ \Delta h = \text{div}[(\chi u) * \omega] \quad \text{in} \ \Omega \]
\[ \nabla h \cdot n = 0 \quad \text{on} \ \partial \Omega \]
\[ \int_{\Omega} h dx = 0. \]
Moreover, the standard estimates hold, independently of $M$, [19, 21]

$$\sup_{t \in [0,T]} \|u^{N,M}(t)\|_{2,\Omega}^2 + \mu_\#\|Du^{N,M}\|_{2,Q_T}^2 +$$

$$\varepsilon\|\nabla p^{N,M}\|_{2,Q_T}^2 \leq \|u_0\|_{2,\Omega}^2 + C\|f\|_{2,Q_T}^2; \quad (14)$$

$$\sup_{t \in [0,T]} \|\theta^{N,M}(t)\|_{2,\Omega}^2 + k\|\nabla \theta^{N,M}\|_{2,Q_T}^2 \leq \|\theta_0\|_{2,\Omega}^2 + C(N); \quad (15)$$

$$\left\|\frac{d}{dt}u^{N,M}\right\|_{L^2(0,T)} \leq C(N); \quad (16)$$

$$\|\partial_t \theta^{N,M}\|_{2,W^{-1,2}(\Omega)} \leq C(N, \nu). \quad (17)$$

Hence, the initial value problem (12)-(13) has a global-in-time solution and passes to the limit as $M$ tends to infinity ($N$ fixed) by standard compactness arguments, resulting

$$\frac{d}{dt}(u^N, w^j) - (\mathfrak{M}_\nu(u^N) \otimes u^N, \nabla w^j) + (\mu(\theta^N)Du^N, Dw^j) -$$

$$- (F_\varepsilon(u^N), \nabla \cdot w^j) = (f, w^j), \quad j = 1, \cdots, N, \quad \text{a.e. } t \in [0,T]; \quad (18)$$

$$\langle \partial_t \theta^N, \phi \rangle - (\theta^N \mathfrak{M}_\nu(u^N), \nabla \phi) + k(\nabla \theta^N, \nabla \phi) =$$

$$= (\mu(\theta^N)|Du^N|^2, \phi), \quad \forall \phi \in L^2(0,T; H^1_0(\Omega)). \quad (19)$$

Moreover, the minimum principle holds, i.e. $\theta^N \geq 0$ a.e. in $Q_T$.

In order to pass to the limit on $N$, when $N$ tends to infinity, the estimates (15)-(17) are no more valid. Thus, we recall the additional estimates (for details, see [5])

$$\|\theta^N\|_{\infty,L^1(\Omega)} \leq \mu_\#\|Du^N\|_{1,Q_T} + T\|\theta_0\|_{1,\Omega} + |Q_T|/2; \quad (20)$$

$$\|\nabla \theta^N\|_{Q_{Q_T}}^q \leq C \left( \mu_\#\|Du^N\|_{1,Q_T} + \|\theta_0\|_{1,\Omega} \right) \times \|\theta^N\|_{\infty,L^1(\Omega)}^{(2-q)/(2n)}; \quad (21)$$

$$\|\partial_t u^N\|_{2,W^{-1,2}(\Omega)} \leq C \left( \nu + 1 + \frac{1}{\varepsilon} + 1 \right) \|u^N\|_{2,W^{-1,2}(\Omega)} \|f\|_{2,Q_T}^2; \quad (22)$$

$$\|\partial_t \theta^N\|_{1,W^{-1,2}(\Omega)} \leq \|\nabla \theta^N\|_{Q_{Q_T}} + C(\nu)\|\theta^N\|_{Q_{Q_T}} + \mu_\#\|Du^N\|_{1,Q_T}; \quad (23)$$

for every exponent $1 < q < 2 - n/(n + 1)$ (cf. [3, 4]). Moreover we have the strong convergences

$$\nabla p^N \rightharpoonup \nabla p_\varepsilon \text{ in } L^2(Q_T); \quad (24)$$

$$\mu(\theta^N)|Du^N|^2 \rightharpoonup \mu(\theta_\varepsilon)|Du_\varepsilon|^2 \text{ in } L^1(Q_T). \quad (25)$$

Then, the initial value problem (18)-(19) passes to the limit as $N$ tends to infinity ($\varepsilon$ fixed), concluding the below quasi-compressible approximative problem.
3.2 The quasi-compressible approximative problem

From Section 3.1, for each $\varepsilon > 0$, there exists $(u_\varepsilon, p_\varepsilon, \theta_\varepsilon)$ in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega)) \times L^2(0, T; W^{1,2}(\Omega)) \times E$ such that $\partial_t u_\varepsilon \in L^2(0, T; W^{-1,2}(\Omega))$ and $\partial_t \theta_\varepsilon \in L^1(0, T; W^{-1,2}(\Omega))$, and it satisfies

$$
\langle \partial_t u_\varepsilon, v \rangle + \int_{\Omega} \nabla u_\varepsilon : v \otimes M_\nu(u_\varepsilon) \, dx + \int_{\Omega} \mu(\theta_\varepsilon) D u_\varepsilon : Dv \, dx = \int_{\Omega} (f - \nabla p_\varepsilon) \cdot v \, dx, \quad \forall v \in W^{1,2}_0(\Omega), \quad \text{a.e. } t \in ]0, T[; \quad (26)
$$

$$
\langle \partial_t \theta_\varepsilon, \phi \rangle + \int_{Q_T} M_\nu(u_\varepsilon) \cdot \nabla \theta_\varepsilon \phi dz + k \int_{Q_T} \nabla \theta_\varepsilon \cdot \nabla \phi dz = \int_{Q_T} \mu(\theta_\varepsilon) |D u_\varepsilon|^2 \phi dz, \quad \forall \phi \in L^\infty(0, T; W^{-1,q'}_0(\Omega)); \quad (27)
$$

$$
\varepsilon \int_{\Omega} \nabla p_\varepsilon \cdot \nabla \phi \, dx + \int_{\Omega} \text{div } u_\varepsilon \phi \, dx = 0, \quad \forall \phi \in W^{1,2}(\Omega); \quad (28)
$$

$$
u_\varepsilon(\cdot, 0) = u_0, \quad \theta_\varepsilon(\cdot, 0) = \theta_0.
$$

In order to pass to the limit, when $\varepsilon$ tends to zero ($\nu$ fixed), the estimate (22) is no more valid. To estimate $p_\varepsilon$ independently of $\varepsilon$ we choose $v = \nabla \eta_\varepsilon$ as a test function in (26), where $\eta_\varepsilon$ is the solution of the following homogeneous Neumann problem for the Laplace equation (see, for instance, [13])

$$
\Delta \eta_\varepsilon(t) = p_\varepsilon(t) - \frac{1}{|\Omega|} \int_{\Omega} p_\varepsilon(t) \, dx \text{ in } \Omega
$$

$$
\nabla \eta_\varepsilon(t) \cdot n = 0 \text{ on } \partial \Omega
$$

$$
\int_{\Omega} \eta_\varepsilon(t) \, dx = 0,
$$

which satisfies

$$
\| \eta_\varepsilon(t) \|_{2, Q_T} \leq C(\Omega) \| p_\varepsilon(t) \|_{2, Q_T} \quad \text{a.e. } t \in ]0, T[. \quad (29)
$$

After some technical computations, it results (for details, see [6])

$$
\| p_\varepsilon \|_{2, Q_T}^2 \leq C(\nu),
$$

and consequently

$$
\| \partial_t u_\varepsilon \|_{2, W^{-1,2}(\Omega)}^2 \leq C(\nu). \quad (30)
$$
Note that $\mu(\theta_\nu)||Du_\nu||^2 \rightarrow \mu(\theta_\nu)||Du_\nu||^2$ holds in $L^1(Q_T)$ (compare to (25)) and it is not required the strong convergence of the pressure $\nabla p_\varepsilon \rightarrow \nabla p_\nu$ in $L^2(Q_T)$ (compare to (24)) since (28) holds for $\phi = p_\varepsilon$. Then, the initial value problem (26)-(27) passes to the limit as $\varepsilon$ tends to zero ($\nu$ fixed), concluding the below regularized problem.

3.3 The regularized problem

For each $\nu \in \mathbb{N}$, there exists $(u_\nu, p_\nu, \theta_\nu)$ in $U \times L^2(Q_T) \times E$ such that $\partial_t u_\nu \in L^2(0,T; W^{1,2}(\Omega))$ and $\partial_\nu \theta_\nu \in L^1(0,T; W^{-1,q}(\Omega))$, and it satisfies

$$
\langle \partial_t u_\nu, v \rangle + \int_\Omega \nabla u_\nu \cdot v \otimes \mathcal{M}_\nu(u_\nu) dx + \int_\Omega \mu(\theta_\nu)|Du_\nu|^2 dx = \int_\Omega f \cdot v dx + \int_\Omega p_\nu \text{div} v dx, \quad \forall v \in W^{1,2}_0(\Omega), \quad \text{a.e. } t \in ]0, T[; (31)
$$

$$
\langle \partial_\nu \theta_\nu, \phi \rangle + \int_{Q_T} \mathcal{M}_\nu(u_\nu) \cdot \nabla \theta_\nu \phi dx dt + k \int_{Q_T} \nabla \theta_\nu \cdot \nabla \phi dx dt = \int_{Q_T} \mu(\theta_\nu)|Du_\nu|^2 \phi dx dt, \quad \forall \phi \in L^\infty(0,T; W^{1,q'}_0(\Omega)); (32)
$$

$$
u_\nu(\cdot,0) = u_0, \quad \theta_\nu(\cdot,0) = \theta_0.
$$

Now in order to pass to the limit in (31)-(32) as $\nu$ tends to infinity, neither (23) nor (30) are valid. Following the argument of [6], we decompose the pressure $p_\nu$ into $p_\nu := p_{\nu,1} + p_{\nu,2}$ such that the two particular pressures, $p_{\nu,1}$ and $p_{\nu,2}$, belong to bounded sets of $L^{(n+2)/n}(Q_T)$ and $L^2(Q_T)$, respectively, independently of $\nu$. For each $t \in ]0, T[,$ $p_{\nu,1}$ is the unique solution to the problem

$$
\int_\Omega p_{\nu,1}(t) dx = 0, \quad -\langle p_{\nu,1}(t), \Delta \phi \rangle = \langle u_\nu \otimes \mathcal{M}_\nu(u_\nu)(t), D\nabla \phi \rangle,
$$

for all $\phi \in W^{2,2}(\Omega)$ such that $\nabla \phi \cdot n = 0$ on $\partial \Omega$, and define $p_{\nu,2} := p_\nu - p_{\nu,1}$. Then $p_{\nu,2}$ solves at each time level

$$
\langle p_{\nu,2}, \Delta \phi \rangle = \int_\Omega \mu(\theta_\nu)|Du_\nu|^2 \phi dx dt - \int_\Omega f \cdot \nabla \phi dx,
$$

for all $\phi \in W^{2,2}(\Omega)$ such that $\nabla \phi \in W^{1,2}_0(\Omega)$, and the following estimate holds

$$
\|p_{\nu,1}\|_{(n+2)/n,Q_T} \leq C; \quad \|p_{\nu,2}\|_{2,Q_T} \leq C.
$$
Thus we conclude the following uniform estimates

\[
\|\partial_t u_\nu\|_X \leq C \left( 1 + \|u_\nu\|_{2, W_0^{1,2}(\Omega)}^2 + \|u_\nu\|_{2(n+2)/n, QT}^2 \right); \quad (33)
\]

\[
\|\theta_\nu u_\nu\|_{1, W^{-1,1}(\Omega)} \leq C; \quad \|\partial_t \theta_\nu\|_{1, W^{-1,1}(\Omega)} \leq C, \quad (34)
\]

for \(1/\ell = n/[q(n+1)] + n/[2(n+2)] < 1\) since \(q < (n+2)/(n+1)\) and \(n < 4\). Then under compactness arguments \cite{29}, \((u, p, \theta)\) satisfies the limit variational formulation \cite{27}. However, with the above estimates we only obtain

\[
\langle \partial_t \theta, \phi \rangle - \int_{QT} \theta u \cdot \nabla \phi dx dt + k \int_{QT} \nabla \theta \cdot \nabla \phi dx dt \geq (\mu(\theta)|Du|^2, \phi),
\]

for all \(\phi \in C^1(Q_T)\) such that \(\phi \geq 0\) \cite{6}. So we will need to prove an additional estimate.

### 3.4 Higher integrability of \(\nabla u_\nu\)

First let us remark that \((31)-(32)\) may be rewritten as

\[
\langle \partial_t u_\nu, v \rangle + \int_{\Omega} \left( \mu(\theta_\nu) Du_\nu : Dv + v \otimes M_\nu(u_\nu) : \nabla u_\nu \right) dx = \int_{\Omega} (f \cdot v + p_\nu \text{div } v) dx, \quad \text{a.e. } t \in [0, T] \quad \forall v \in W_0^{1,2}(\Omega),
\]

\[
\left. u_\nu \right|_{t=0} = u_0 \text{ in } \Omega;
\]

\[
\langle \partial_t \theta_\nu, \phi \rangle + \int_{\Omega} \left( k \nabla \theta_\nu - \theta_\nu M_\nu(u_\nu) \right) \cdot \nabla \phi dx = \int_{\Omega} \mu(\theta_\nu)|Du_\nu|^2 \phi dx, \quad \text{a.e. } t \in [0, T] \quad \forall \phi \in W_0^{1,\infty}(\Omega), \quad \theta_\nu \bigg|_{t=0} = \theta_0 \text{ in } \Omega.
\]

Let us prove the required estimate.

**Theorem 3.1** Let \((u_\nu, p_\nu, \theta_\nu) \in \mathcal{U} \times L^{(n+2)/n}(QT) \times \mathcal{E}\) verify the system \((31)-(32)\) then \(\nabla u_\nu \in L^{2(1+\epsilon)}_{\text{loc}}(QT)\) for \(0 < \epsilon < \min\{(4-n)/(3n), 1/(n+2), \epsilon_0\}\) and the following estimate holds:

\[
\|\nabla u_\nu\|_{2(1+\epsilon), Q(z_0, R)} \leq C \left( \|\nabla u_\nu\|_{2, Q(z_0, 2R)} + \|u_\nu\|_{2(n+2)/n, Q(z_0, 2R)} + \|f\|_{2(1+\epsilon_0), Q(z_0, 2R)} + \|p_\nu\|_{(n+2)/n, Q(z_0, 2R)}^{1/2} \right),
\]

for any cylinder \(Q(z_0, 2R) := B(x_0, 2R) \times [t_0 - (2R)^2, t_0] \subset QT\).
Proof. We denote the points of the space-time cylinder by \( z = (x, t) \) and employ a shorthand notation \( dz = dxdt \). We write \( \int_{Q(z,R)} v := \frac{1}{\omega_n R^{n+2}} \int_{Q(z,R)} v \) whenever \( v \in L^1(Q_T) \), where \( \omega_n \) denotes the measure of the \( n \)-dimensional unit ball in \( \mathbb{R}^n \). For every \( z_0 = (x_0, t_0) \in Q_T \) and \( R > 0 \) small enough such that \( Q(z_0, 2R) \subset Q_T \), in order to prove the higher integrability of \( \nabla u_\nu \) (cf. \[1, 4, 28 \] Gehring Lemma), it is sufficient to show the following reverse estimate

\[
\int_{Q(z_0,R)} |\nabla u_\nu|^2 dz \leq \delta \int_{Q(z_0,2R)} |\nabla u_\nu|^2 dz + \frac{B_1}{R^{n+1}} \int_{Q(z_0,2R)} |u_\nu|^2 dz + \frac{B_2}{R} \int_{Q(z_0,2R)} |u_\nu|^3 dz + B_3 \int_{Q(z_0,2R)} |f|^2 dz + R \int_{Q(z_0,2R)} |p_\nu|^{(n+2)/n} dz
\]  

for some \( \delta \in [0,1] \) and positive constants \( B_1, B_2, B_3 \), independent of \( u_\nu, p_\nu \) and \( \theta_\nu \), considering that, for \( n < 4 \),

\[
\begin{align*}
  u_\nu & \in L^{2(n+2)/n}(Q_T), \quad \frac{2(n+2)}{n} > 2; \\
  |u_\nu|^{3/2} & \in L^{4(n+2)/(3n)}(Q_T), \quad \frac{4(n+2)}{3n} > 2; \\
  f & \in L^{2(1+\epsilon_0)}(Q_T), \quad \epsilon_0 > 0; \\
  |p_\nu|^{(n+2)/n} & \in L^1(Q_T).
\end{align*}
\]

Thus, we take \( 0 < \epsilon < \min \{2/n, (4 - n)/(3n), 1/(n + 2), \epsilon_0 \} = \min \{(4 - n)/(3n), 1/(n + 2), \epsilon_0 \} \).

Adapting the argument used in \[2\], let \( \varphi \in C_0^\infty(Q(z_0,2R)) \) be a cut-off function such that \( \varphi \equiv 1 \) in \( Q(z_0, R) \) and \( |\nabla \varphi| \leq C/R, |\partial_t \varphi| \leq C/R^2 \) in \( Q(z_0, 2R) \). Choose \( v = \varphi^2 u_\nu \) as a test function in (35), then

\[
\begin{align*}
  &\int_{Q_t} \frac{1}{2} \frac{d}{dt} (\varphi^2 |u_\nu|^2) - \int_{Q_t} \varphi |u_\nu|^2 \partial_t \varphi + \int_{Q_t} \mu(\theta_\nu) \varphi^2 |Du_\nu|^2 + \\
  &+ 2 \int_{Q_t} \mu(\theta_\nu) \varphi Du_\nu : (u_\nu \otimes \nabla \varphi) = I + \int_{Q_t} \varphi^2 f \cdot u_\nu + 2 \int_{Q_t} p_\nu \varphi u_\nu \cdot \nabla \varphi
\end{align*}
\]

where \( I \) corresponds to the convective term

\[
I = \int_{Q_t} M_\nu(u_\nu) \otimes u_\nu : (\varphi^2 \nabla u_\nu + 2 \varphi \nabla \varphi \otimes u_\nu) dz
\]
\[ = \int_{Q_t} (M_\nu(u_\nu) \cdot \nabla \varphi) |u_\nu|^2 \varphi \, dz. \]

Applying the assumption (5) and Hölder and Young inequalities, it follows
\[
\int_{\Omega} \frac{1}{2}(\varphi^2 |u_\nu|^2)(t) \, dx + \mu\int_{Q_t} \varphi^2 |Du_\nu|^2 \, dx \, d\tau \leq \int_{Q_t} \varphi |u_\nu|^2 |\partial_t \varphi| + \\
+ \delta \int_{Q_t} \varphi |\nabla u_\nu|^2 + C(\delta, \mu^\#) \int_{Q_t} \varphi |u_\nu|^2 |\nabla \varphi|^2 + |I| + \\
+ \frac{1}{2} \int_{Q_t} \varphi^2 |f|^2 + \frac{1}{2} \int_{Q_t} \varphi^2 |u_\nu|^2 + \frac{nR}{n+2} \int_{Q_t} |p_\nu|^{(n+2)/n} \varphi^{(n+2)/(2n)} + \\
+ \frac{2}{n+2} \int_{Q_t} \varphi^{(n+2)/4} |u_\nu|^{(n+2)/2} |\nabla \varphi|^{(n+2)/2}. 
\]

By Korn inequality the following estimate holds
\[
\int_{Q_t} \varphi^2 |\nabla u_\nu|^2 \leq 2 \int_{Q_t} \varphi^2 |Du_\nu|^2 + 4 \int_{Q_t} |\nabla \varphi|^2 |u_\nu|^2. 
\]

According to the properties of \( \varphi \) it arises
\[
\mu\int_{Q(z_0,R)} |\nabla u_\nu|^2 \leq \delta \int_{Q(z_0,2R)} |\nabla u_\nu|^2 + \frac{C(\delta, \mu^\#, n)}{R^{n+1}} \int_{Q(z_0,2R)} |u_\nu|^2 \, dz + \\
+ |I| + C \int_{Q(z_0,2R)} |f|^2 \, dz + \frac{nR}{n+2} \int_{Q(z_0,2R)} |p_\nu|^{(n+2)/n} \, dz, 
\]

observing that for \( R < 1 \) we have \( R^2 > R^{n+1} = R^{n/2+(n+2)/2} \). Since
\[
|I| \leq \frac{C}{R} \int_{Q(z_0,2R)} |M_\nu(u_\nu)||u_\nu|^2 \, dz \leq \\
\leq \frac{C}{R} \|M_\nu(u_\nu)||_{3,Q(z_0,2R)} \|u_\nu||^2_{3,Q(z_0,2R)} \leq \frac{C}{R} \|u_\nu||^3_{3,Q(z_0,2R)} 
\]

then we conclude (38).

### 3.5 Existence of weak solutions

From estimates (14), (20)-(21), (33)-(34) and (37), independent on \( \nu \), we can extract a subsequence, still denoted by \((u_\nu, p_\nu, \theta_\nu)\), verifying (31)-(32) and
\[
\text{weakly* in } L^\infty(0,T; L^2(\Omega)); 
\]

then we conclude (38).
\[ \nabla u_\nu \rightharpoonup \nabla u \quad \text{weakly in} \quad L^2(Q_T); \quad (39) \]
\[ \partial_t u_\nu \rightharpoonup \partial_t u \quad \text{weakly in} \quad \mathcal{X}; \]
\[ u_\nu \rightharpoonup u \quad \text{strongly in} \quad L^m(Q_T), \quad \text{for} \quad 1 \leq m < 2(n+2)/n; \quad (40) \]
\[ \theta_\nu \rightharpoonup \theta \quad \text{weakly in} \quad L^q(0,T;W^{1,q}_0(\Omega)), \quad \text{for} \quad 1 < q < 2 - n/(n+1); \quad (41) \]
\[ p_\nu \rightharpoonup p \quad \text{weakly in} \quad L^{(n+2)/n}(Q_T); \quad (42) \]
\[ p_{\nu,1} \rightharpoonup p_1 \quad \text{strongly in} \quad L^m(Q_T), \quad \text{for} \quad 1 \leq m < (n+2)/n; \]
\[ p_{\nu,2} \rightharpoonup p_2 \quad \text{weakly in} \quad L^2(Q_T). \]

Since \( \text{div} u_\nu = 0 \) and \( u_\nu|_{\partial \Omega \times [0,T]} = 0 \), it follows from the definition and properties of Helmholtz decomposition that
\[ M_\nu(u_\nu) \in L^\infty(0,T;L^m(\Omega)) \quad \text{for all} \quad m \in [1,\infty), \]
\[ M_\nu(u_\nu) \rightarrow u \quad \text{strongly in} \quad L^m(Q_T), \quad \text{for} \quad 1 \leq m < 2(n+2)/n. \]

Then, the initial value problem \([31]-[32]\) passes to the limit as \( \nu \) tends to infinity, concluding the required problem \([7]-[8]\).

### 3.6 Local energy inequalities

#### 3.6.1 Proof of the local energy inequality \([9]\)

Let \( \varphi \in C^\infty_0(Q_T) \) and choose \( v = \varphi^2(u_\nu - a) \) as a test function in \([35]\) for an arbitrary \( a \in \mathbb{R}^n \), arguing as in Section 3.4 then
\[
\frac{1}{2}\|\varphi(u_\nu - a)\|_{L^2(Omega)}^2(t) + \int_{Q_t} \mu(\theta_\nu)\varphi^2|Du_\nu|^2dxdt = \]
\[
= \int_{Q_t} \varphi|u_\nu - a|^2\partial_t \varphi dxdt - \int_{Q_t} \mu(\theta_\nu)\varphi Du_\nu : (u_\nu - a) \otimes \nabla \varphi dxdt + \]
\[
+ \int_{Q_t} (M_\nu(u_\nu) \cdot \nabla \varphi) u_\nu \cdot (u_\nu - a) \varphi dxdt + \]
\[
+ \int_{Q_t} \varphi^2 f \cdot (u_\nu - a) + 2 \int_{Q_t} p_\nu \varphi(u_\nu - a) \cdot \nabla \varphi dxdt. \]

Next, arguing as in Section 3.5 we can pass to the limit as \( \nu \) tends to infinity concluding \([9]\).
3.6.2 Proof of the local energy inequality (10)

Going back to the solution \((u^N, p^N, \theta^N) \in (L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega))) \times L^2(0, T; W^{2,2}(\Omega)) \times E\) of the initial value problem (18)-(19) we can choose \(\phi = \theta^N(\zeta + (\theta^N)^2)^{-1/2}\psi \in L^2(0, T; W^{1,2}_0(\Omega)))\), for \(\zeta > 0\) and a non-negative function \(\psi \in C_0^\infty(Q_T)\), as a test function in (19). First, we calculate separately the following terms

\[
\langle \partial_t \theta^N, \phi \rangle = \int_\Omega \left(\sqrt{\zeta + (\theta^N)^2}\psi\right)(x,t)\,dx - \int_{Q_T} \sqrt{\zeta + (\theta^N)^2} \partial_t \psi(x,\tau)\,dx\,d\tau
\]

\[
(M_\nu(u^N), \nabla \theta^N \phi) = \int_{Q_T} M_\nu(u^N) \cdot \nabla \left(\sqrt{\zeta + (\theta^N)^2}\right) \psi\,dx\,d\tau
\]

\[
= - \int_{Q_T} \sqrt{\zeta + (\theta^N)^2} \left(M_\nu(u^N) \cdot \nabla \psi\right)\,dx\,d\tau
\]

\[
(\nabla \theta^N, \nabla \phi) = \zeta \int_{Q_T} \frac{|
abla \theta^N|^2}{(\zeta + (\theta^N)^2)^{3/2}} \psi\,dx\,d\tau + \int_{Q_T} \nabla \left(\sqrt{\zeta + (\theta^N)^2}\right) \cdot \nabla \psi\,dx\,d\tau
\]

\[
= \zeta \int_{Q_T} \frac{|
abla \theta^N|^2}{(\zeta + (\theta^N)^2)^{3/2}} \psi\,dx\,d\tau - \int_{Q_T} \sqrt{\zeta + (\theta^N)^2} \Delta \psi\,dx\,d\tau.
\]

Thus, we conclude

\[
\int_{Q_T} \left(\sqrt{\zeta + (\theta^N)^2}\psi\right)(x,t)\,dx + \zeta k \int_{Q_T} \frac{|
abla \theta^N|^2}{(\zeta + (\theta^N)^2)^{3/2}} \psi\,dx\,d\tau =
\]

\[
= \int_{Q_T} \sqrt{\zeta + (\theta^N)^2} \left(\partial_t \psi + k \Delta \psi + M_\nu(u^N) \cdot \nabla \psi\right)\,dx\,d\tau +
\]

\[
+ \int_{Q_T} \mu(\theta^N)|Du^N|^2 \frac{\theta^N \psi}{\sqrt{\zeta + (\theta^N)^2}}\,dx\,d\tau. \quad (43)
\]

Since \(\psi \in C_0^\infty(Q_T)\) such that \(\psi \geq 0\) we can pass successively to the limit as \(N\) tends to infinity and \(\varepsilon\) and \(\nu\) tend to zero, obtaining (10). Indeed, still denoting by \((u^N, p^N, \theta^N)\) the subsequence extracted from the solutions \((u^N, p^N, \theta^N)\) of (18)-(19) it verifies

\[
u^N \rightharpoonup u \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega));
\]

\[
\nabla u^N \rightharpoonup \nabla u \quad \text{weakly in } L^2(Q_T);
\]

\[
\partial_t u^N \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0, T; W^{-1,2}(\Omega));
\]

\[
u^N \rightarrow u \quad \text{strongly in } L^m(Q_T), \text{ for } 1 \leq m < 2(n+2)/n;
\]
\[ \theta^N \rightharpoonup \theta \quad \text{weakly in} \quad L^q(0,T;W_0^{1,q}(\Omega)), \quad \text{for} \quad 1 < q < 2 - n/(n+1); \]
\[ \theta^N \rightarrow \theta \quad \text{strongly in} \quad L^m(Q_T), \quad \text{for} \quad 1 \leq m < q(n+1)/n; \]
\[ p^N \rightharpoonup p \quad \text{weakly in} \quad L^2(0,T;W^{2,2}(\Omega)); \]

where \((u_\varepsilon, p_\varepsilon, \theta_\varepsilon)\) is a solution of (26)-(27). Moreover, considering that \(\psi \geq 0\),

\[
\frac{\nabla \theta^N}{(\zeta + (\theta^N)^2)^{3/4}} \rightharpoonup \frac{\nabla \theta_\varepsilon}{(\zeta + (\theta_\varepsilon)^2)^{3/4}} \quad \text{weakly in} \quad L^2(Q_T),
\]

the lower semicontinuity property of the \(L^2\)-norm and

\[
\mu(\theta^N) \frac{\theta^N}{\sqrt{\zeta + (\theta^N)^2}} \rightharpoonup \mu(\theta_\varepsilon) \frac{\theta_\varepsilon}{\sqrt{\zeta + (\theta_\varepsilon)^2}} \quad * \text{-weakly in} \quad L^\infty(Q_T),
\]

and (25) hold, then the equality (43) delivers to the local energy inequality (10) satisfied by \((u_\varepsilon, p_\varepsilon, \theta_\varepsilon)\).

Next, still denoting by \((u_\varepsilon, p_\varepsilon, \theta_\varepsilon)\) the subsequence extracted from the solutions \((u_\varepsilon, p_\varepsilon, \theta_\varepsilon)\) of (26)-(27) it verifies

\[
u_\varepsilon \rightharpoonup u \quad \text{weakly* in} \quad L^\infty(0,T;L^2(\Omega));
\]
\[ u_\varepsilon \rightarrow u \quad \text{weakly in} \quad L^2(0,T;W_0^{1,2}(\Omega));
\]
\[ l_t u_\varepsilon \rightharpoonup \theta \quad \text{weakly in} \quad L^2(0,T;W^{-1,2}(\Omega));
\]
\[ u_\varepsilon \rightarrow u \quad \text{strongly in} \quad L^m(Q_T), \quad \text{for} \quad 1 \leq m < 2(n+2)/n;
\]
\[ \theta_\varepsilon \rightarrow \theta_\nu \quad \text{weakly in} \quad L^q(0,T;W_0^{1,q}(\Omega)), \quad \text{for} \quad 1 < q < 2 - n/(n+1);
\]
\[ \theta_\varepsilon \rightarrow \theta_\nu \quad \text{strongly in} \quad L^m(Q_T), \quad \text{for} \quad 1 \leq m < q(n+1)/n;
\]
\[ p_\varepsilon \rightarrow p_\nu \quad \text{weakly in} \quad L^2(Q_T).
\]

where \((u_\nu, p_\nu, \theta_\nu)\) is a solution of (31)-(32). Analogously to the above argument the local energy inequality (10) arises for \((u_\nu, p_\nu, \theta_\nu)\).

In conclusion, the convergences in Section 3.5 imply that the limit \(\theta\) verifies \(\theta \geq 0\) in \(Q_T\) and the local energy inequality (10) holds for the weak solution \((u, p, \theta)\) such that verifies (9).

### 3.6.3 Proof of the local energy inequality (11)

Proceeding as in Section 3.6.2, we go back to the existence of the solution \((u^N, p^N, \theta^N) \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))) \times L^2(0,T;W^{2,2}(\Omega)) \times \mathcal{E})\)
of the initial value problem $(18)-(19)$. Next we can choose $\phi = (1 + \theta^N)^{-\xi} \psi \in L^2(0, T; W^{1,2}_0(\Omega))$, for $0 < \xi < 1$ and a non-negative function $\psi \in C^\infty_0(Q_T)$, as a test function in $(19)$. Now, calculating separately the following terms

\[
\begin{align*}
(\partial_t \theta^N, \phi) &= \frac{1}{1 - \xi} \int_{\Omega} (1 + \theta^N)^{1-\xi} \psi(x, t) dx - \\
&\quad - \frac{1}{1 - \xi} \int_{Q_t} (1 + \theta^N)^{1-\xi} \partial_t \psi(x, \tau) dx d\tau \\
(M_\nu(u^N), \nabla \theta^N \phi) &= - \frac{1}{1 - \xi} \int_{Q_t} (1 + \theta^N)^{1-\xi} M_\nu(u^N) \cdot \nabla \psi dx d\tau \\
(\nabla \theta^N, \nabla \phi) &= - \xi \int_{Q_t} \frac{\nabla \theta^N}{(1 + \theta^N)^{\xi+1}} \psi - \frac{1}{1 - \xi} \int_{Q_t} (1 + \theta^N)^{1-\xi} \Delta \psi.
\end{align*}
\]

Thus, we conclude

\[
\begin{align*}
\xi k \int_{Q_t} \frac{\nabla \theta^N}{(1 + \theta^N)^{\xi+1}} \psi dx d\tau + \int_{Q_t} \mu(\theta^N) |Du^N|^2 \psi (1 + \theta^N)^{\xi} dx d\tau &= \\
= \frac{1}{1 - \xi} \int_{\Omega} (1 + \theta^N)^{1-\xi} \psi(x, t) dx - \\
&- \frac{1}{1 - \xi} \int_{Q_t} (1 + \theta^N)^{1-\xi} \left( \partial_t \psi + k \Delta \psi + M_\nu(u^N) \cdot \nabla \psi \right) dx d\tau.
\end{align*}
\]

Since $\psi \in C^\infty_0(Q_T)$ such that $\psi \geq 0$ we can pass successively to the limit as $N$ tends to infinity and $\varepsilon$ and $\nu$ tend to zero, obtaining $(11)$.

## 4 Regularity of $\theta$ (Proposition 2.1)

Let $(u, p, \theta)$ be a weak solution in accordance to Theorem 2.1 that is, it satisfies in the sense of distributions

\[
\partial_t \theta - k\Delta \theta = -\nabla \theta + \mu(\theta) |Du|^2 \quad \text{in } Q_T. \tag{44}
\]

Thanks to Theorem 2.1 we have

\[
u \in L^\infty(0, T; L^2(\Omega)) \cap L^{2(1+\varepsilon)}(0, T; W^{1,2}_{loc}(\Omega)) \hookrightarrow L^{4(n+2)(1+\varepsilon)/n}_{loc}(Q_T), \quad |\nabla u|^2 \in L^{1+\varepsilon}_{loc}(Q_T).
\]
Thus we get
\[ u \cdot \nabla \theta \in L^{\kappa}_{\text{loc}}(Q_T), \quad \frac{1}{\kappa} = \frac{n}{4(n+2)(1+\epsilon)} + \frac{1}{q}. \]

Since \( q < \frac{n+2}{n+1} \) it follows
\[ 1 < \kappa < \frac{4(n+2)(1+\epsilon)}{5n + 4 + 4\epsilon(n+1)}, \quad \text{for every } \epsilon > 0. \]

In particular, \( \kappa > 1 + \epsilon \) if \( \epsilon < \frac{4-n}{4n+4} \).

Let us split the proof into two cases.

**n=2** Since we have \( u \cdot \nabla \theta \in L^{1+\epsilon}_{\text{loc}}(Q_T) \) supposing \( \epsilon < 1/6 \), the classical theory for the heat equation \( (44) \) leads \( \theta \in W^{2,1}_{1+\epsilon,\text{loc}}(Q_T) \).

**n=3** The classical theory for the heat equation \( (44) \) leads \( \theta \in W^{2,1}_{\kappa,\text{loc}}(Q_T) \hookrightarrow L^{5\kappa/(5-\kappa)}(0,T;W^{1,5\kappa/(5-\kappa)}(\Omega)) \). Thus we get \( u \cdot \nabla \theta \in L^{1+\epsilon}_{\text{loc}}(Q_T) \) supposing \( \epsilon < 1/6 \), and the bootstrap argument guarantees the required result.

Finally, the last assertion is valid from the embedding
\[ W^{2,1}_{1+\epsilon}(Q_T) \hookrightarrow L^\varsigma(Q_T), \quad \frac{2}{n+2} = \frac{1}{1+\epsilon} - \frac{1}{\varsigma}. \]

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