A semiclassical Hamiltonian
for plane waves in loop quantum gravity

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Abstract

This is the first of two papers which study the semiclassical limit of a loop quantum gravity (LQG) canonical quantization of unidirectional plane gravity waves. Initially I formulate an exact LQG in which each triad, $E_x^x$, for example, grasps only one of the $x$ holonomies present at the vertex. Field strengths are non-local, constructed from holonomies which connect two neighboring vertices of the spin network. The equations are simplified using a semiclassical approximation, meaning eigenvalues of the volume operator are assumed to be large enough that the [volume, holonomy] commutators may be replaced by their quantum field theory limits. Additionally, SU(2) holonomies are expanded in sines and cosines, sines are assumed small, and terms quadratic in sines are dropped. In the semiclassical limit many non-local features disappear. However, differences replace derivatives with respect to $z$, the propagation direction; and semiclassical triads grasp both holonomies present at each vertex. Gauge-fixing constraints, as well as the constraint that the wave is traveling only in one direction, are formulated in a language appropriate to LQG. A subsequent paper constructs a sinusoidal solution using a Hilbert space of coherent states tailored to the symmetry of the plane wave case.

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I Introduction

This and a succeeding paper investigate the semiclassical limit of the loop quantum gravity (LQG) theory of gravitational plane waves. The present paper quantizes the theory, using a canonical quantization based on triad and real connection variables, and takes a semiclassical limit. The succeeding paper constructs a Hilbert space based on coherent states, and investigates a sinusoidal solution.

The theory investigated here is non-local. In an LQG theory, operators have support only at vertices and on edges of a three-dimensional simplex. For plane waves propagating in the z direction, this simplex becomes a series of vertices along the z axis, connected by edges along z. "Non-local" means field strengths can be non-local: the holonomic loop which defines each field strength may stretch from one vertex to a nearest-neighbor vertex.

Initial formulations of LQG used local field strengths, i.e. the holonomic loop encircled small areas infinitesimally close to a single vertex. Smolin suggested this formulation is too local to allow for propagation of information from one vertex to the next. In response, Thiemann proposed his "Master Constraint" program, which would allow non-local field strengths, while preserving a constraint algebra free of anomalies. (The two issues, anomalies and non-locality, are closely connected, because the original, local formulation is anomaly free.) The Master Constraint proposal is not examined in this paper. In any case, plane waves may not be the best forum for testing that proposal, because the geometrodynamics constraint algebra for plane waves is known to be anomaly free.

Section describes a candidate non-local theory, and simultaneously lists the approximations appropriate to the classical limit. In that limit most of the non-local features disappear. The theory still has no z derivatives; those are replaced by differences with respect to n, the vertex index.

\[ \partial f / \partial z \rightarrow \frac{[f(n+1) - f(n)]}{\Delta z} \]

Holonomies along z remain non-local, with support along the edge from n to n+1.

Evidently, if LQG is indeed non-local, the experimental study of gravity waves is unlikely to shed much light on the finer details of the
structure of the theory. For this reason, the discussion of the exact non local theory is brief. By the end of section II I have passed to the semiclassical limit, and the remainder of the paper concentrates on quantizing this limit.

There exists a series of increasingly classical formulations of plane wave theory.

\[ \text{LQG} \rightarrow \text{SS} \rightarrow \text{QFT} \rightarrow \text{CGR}, \]

CGR is classical general relativity. QFT is the usual quantized field theory based on triad and connection variables, without holonomies. I stop at the theory labeled SS (for small sine). In QFT the connection field \( A_i \) occurs outside a holonomy. In small sine approximation, the connection field is left inside a sine function, as

\[ \sin\left( \int A \cdot S \right), \]

where \( S \) is an SU(2) rotation generator. This preserves the desirable property that divergences of the connection field are regulated. In the classical limit of small connections, the sine is small; I keep only terms in the constraints which are linear or quadratic in sines.

One may think of the small sine theory as either an approximation to exact LQG; or alternatively, as a model, simpler than exact LQG, but displaying the key feature that makes LQG attractive. SS is a logically coherent scheme for regulating divergences, free of arbitrary cutoffs which must be left finite to avoid blowups; and volumes and areas are quantized.

When considering SS as a model, one would handle \( 1/(\text{volume}) \) divergences as does Thiemann: replace divergent quantities by commutators of volume with holonomies \([11]\). The small sine model is a promising way to test small volume, non classical limits, without going to the full complication of exact LQG.

### A Number of Vertices

This calculation uses a lattice with a fixed number of vertices. Where does the number of vertices enter into the calculation? To obtain the classical limit, I must assume the fields vary slowly from vertex to vertex, so that the discrete structure of the spin network is not obvious. The precise value of the number of vertices is not important, provided the number of vertices is large enough to guarantee
slow variation. I could replace the fixed number of vertices with a
distribution in the number of vertices and nothing would change,
provided the distribution were peaked at a large number.

The restriction to a fixed number of vertices may be more appar-
tent than real, because the classical limit uses coherent states. In a
coherent state, at each vertex \( n \), the values of SU(2) angular mo-
mentum \( L(n) \) are Gaussian distributed. This distribution includes
angular momentum zero. In that sense a coherent state already
includes the possibility of no vertex at \( n \).

Since the distribution is Gaussian, the probability of no vertex
is very small. Presumably spin networks with small numbers of
vertices do not contribute significantly in the classical limit.

B Gauge Fixing

Let the longitudinal direction (direction of propagation) be labeled
\( z \), and label the two transverse directions \( x,y \). Husain and Smolin
choose gauges which simplify the \( \tilde{E} \) and connection fields [7]. They
set off-diagonal elements \( E_{a}^{Z} = E_{A}^{z} = 0 \), with \( a = x,y \) and \( A =
X,Y \). This fixes SU(2) rotations around \( X,Y \) and diffeomorphisms in
transverse directions \( x,y \). They then solve the corresponding con-
straints, which allows them to set to zero the off-diagonal \( A_{z}^{Z} =\)
\( A_{z}^{A} = 0 \). Three constraints survive: the scalar constraint, the vec-
tor constraint for \( z \) diffeomorphisms, and the Gauss constraint for
rotations around \( Z \). (SU(2) \( \rightarrow \) U(1).)

The term “planar” is a slight misnomer: the theory does not have
full planar symmetry in the \( xy \) plane. With suitable choice of co-
ordinates, the Killing vectors become \( \partial/\partial x, \partial/\partial y \), implying that all
functions are independent of \( x \) and \( y \). However, this is translational
invariance, not full planar symmetry, which would require isotropy.
Isotropy is inconsistent with the presence of waves: vibrations of
the usual cloud of test particles are described by an ellipse, which
picks out preferred directions. The translational invariance implies
the ellipse is the same everywhere in the \( xy \) plane.

Since the only \( A_{z}^{I} \) which survives has \( I = Z \), holonomies along the
longitudinal \( z \) direction are quite simple, involving only the rotation
generator \( S_{Z} \) for rotations around \( Z \).

\[
\exp(i \int A_{z}^{Z} \cdot S_{Z}).
\]
Conversely, the transverse holonomies (those along the x and y directions of the spin network) contain no $S_Z$ and involve $S_X, S_Y$ only.

C Topology of the Spin Network

In the z direction the spin network has the topology of the real line. The line is made up of a series of vertices, labeled by integers $n_z$. These are connected by edges, which may be labeled by their endpoints, as $(n_z, n_{z+1})$.

In the transverse directions, there are two possible approaches. The first approach is easiest to relate to the full, three-dimensional case. Give each vertex on the original z axis three integer coordinates: $(n_x = 0, n_y = 0, n_z)$. Construct a three dimensional rectangular lattice by drawing a congruence of lines, all parallel to the original z axis. All lines have the identical arrangement of vertices, but differ in their x and y coordinates, $n_x = \pm 1, \pm 2, \ldots, n_y = \pm 1, \pm 2, \ldots$. Connect neighboring vertices having the same $n_z$ with edges $(n_x, n_{x+1}), (n_y, n_{y+1})$. In this way one fills out a full, three dimensional rectangular lattice.

Each member of the congruence is labeled by a pair of indices $(n_x, n_y)$, and each vertex by a triplet $(n_x, n_y, n_z)$. Because of the translational invariance, physics will be independent of $(n_x, n_y)$. I will refer to this as the "congruence" picture. (This is a slight abuse of notation, since members of a traditional congruence are labeled by continuously varying parameters, rather than discrete integers $(n_x, n_y)$.)

The second method for handling the transverse directions is simpler topologically, but a little harder to relate to the three dimensional case. Construct a small cubic box surrounding each vertex. Equip each face with an outward normal. Call a face positive (negative) if its normal points in the positive (negative) coordinate direction. Consider the holonomy with support on edge $(n_x, n_{x+1})$. It leaves a cube at position $n_x$, passing through the positive x face, then enters the nearest neighbor cube at $n_{x+1}$, through a negative x face. Because of the planar symmetry, the holonomy entering the negative x face of cube $n_{x+1}$ must be identical to the holonomy entering the negative x face of cube $n_x$. Therefore one could give the edge $(n_x, n_{x+1})$ the topology of a circle: imagine that the holonomy leaves cube $n_x$ through the positive x face, travels along $(n_x, n_{x+1})$.
(now a circle rather than a straight line) and reenters \( n_x \) through the negative \( x \) face.

The congruence has now disappeared. There is only a real line \( R \) in the \( z \) direction, and two \( S_1 \) edges leaving each vertex in the \( x \) and \( y \) directions. I will refer to this as the "\( S_1 \) picture". The \( R \times S_1 \times S_1 \) topology is simpler for calculations: but for thinking, I believe it is better to use the congruence: one has more assurance the results will generalize to three dimensions.

In the congruence picture, it is natural to refer to the smallest rectangular area enclosed by \( x \) and \( y \) edges as an "xy plane". I will use this terminology, even though in the \( S_1 \) picture this area has the topology of a torus. Similarly, I will refer to an area bounded by two neighboring congruences in the \( z \) direction and two neighboring \( x \) edges as an "xz plane". In the \( S_1 \) picture this area has the topology of a cylinder.

D Boundary Conditions at Infinity

In Newtonian planar gravity the gravitational potential at infinity does not die off as some power of \( z \), but rather grows linearly. Similarly in planar general relativity, one cannot assume flat space at infinity.

One can assume conformal flatness without loss of generality, however. I assume the \((z,t)\) portion of the metric at infinity assumes the conformal form

\[
-\mathbf{N}^2 + (\mathbf{N}^z)^2 g_{zz} dt^2 + 2\mathbf{N}^z g_{zz} dz dt + g_{zz} dz^2
\]

\[
\rightarrow g_{zz} (-dt^2 + dz^2),
\]

(1)

where \( \mathbf{N} \) and \( \mathbf{N}^z \) are the ADM lapse and shift. This requires the boundary conditions

\[
\mathbf{N}^z \to 0;
\]

\[
\mathbf{N} := \mathbf{N} / \sqrt{g_{zz}} \to 1.
\]

(2)

\( \mathbf{N} \), rather than \( \mathbf{N} \), goes to unity. The underlining is needed because \( \mathbf{N} \) is density weight \(-1\). To simplify the typography, I have not always put tildes over the triads. They are familiar to most readers,
and it is understood the triads are weight 1. $N$, however, is an unfamiliar quantity, and its density weight will play a role in section $X$ when the diffeomorphism gauge is fixed.

E Other Treatments

Bannerjee and Date [8] and Hinterleitner and Major [9] have quantized plane waves within a LQG framework. Both papers use a non-local Hamiltonian. Bannerjee and Date thoroughly discuss regularization issues; Hinterleitner and Major suggest a triad regularization tailored to the special nature of the planar case. (Their regularization is discussed in section $V$.)

Both papers use a Bohr quantization of the transverse degrees of freedom. There are no holonomies along transverse directions $(x,y)$, only holonomies along the longitudinal direction (propagation direction, $z$).

I use holonomies along $x$ and $y$, rather than Bohr quantization. One reason for studying theories in fewer dimensions is to throw light on the behavior of the full theory, which is based on holonomies. In the follow-on paper, because I use holonomies, I am able to study the behavior of transverse angle and angular momentum variables as they vary under the influence of a passing gravitational wave.

Of course there is a price to pay: holonomies involve more variables, therefore are more complex. Part of this complexity comes about because transverse holonomies support a hidden $O(3)$. Also, the semiclassical limit is best described using coherent states, which are superpositions of many holonomies of different spin.

However, although the symmetry may be hidden, it is only an $O(3)$; and the construction of holonomic coherent states closely parallels the construction of the familiar coherent states for the free particle. I believe the price of admission is therefore not that high; and including holonomies can be highly informative.

F Conventions

Throughout, indices from the middle of the alphabet $i, j, \cdots$ range over coordinates $x, y, z$ on the manifold. Indices from the beginning of the alphabet $a, b, \cdots$ range over $x, y$ only, where $z$ is the direction of propagation. Similarly, indices $I, J, K$ range over coordinates
X, Y, Z in the local free-fall frame. Indices A, B · · · range over transverse directions X, Y only.

When expanding 2 x 2 matrices, I use Hermitean sigma matrices, rather than antiHermitean tau matrices, or spin matrices $\sigma_I/2$ in the fundamental representation. A typical Lie group valued operator would be written

$$O_i := O_I^I \sigma_I$$

Generally I will suppress the sigma matrices and suppress bold face for matrices; it should be clear from context which quantities are sigma-valued. I will usually write the operator in eq. (3) simply as $O_i$.

In LQG triads are written as area two-forms,

$$E_I^I(n) dx^i \wedge dx^k \epsilon_{ijk}/2!$$

and connections are written as one-forms, $A^I_j dx^j$. The area and line integrals in the definitions of triad and connection guarantee simpler transformation properties under spatial diffeomorphisms; also, the [ holonomy, triad ] commutator will contain enough integrations to kill the delta function. Usually, I suppress the area and line integrals, and write simply $E_I^I, A^i_j$, e. g.

$$E_I^I dx^j \wedge dx^k \epsilon_{ijk}/2! \to E_I^I.$$

II Approximations

A The Small Sine Approximation

LQG is expected to approach QFT in the limit of weak fields. The usual weak field approximation approximates the holonomies as

$$h^{(i)} := \exp(i \int A_i \cdot S) \approx 1 + i \int A_i \cdot S \quad (\text{QFT}).$$

This expansion replaces a bounded expression by an unbounded one. The following small sine approximation is less drastic, in that the bounded expression is replaced by another bounded expression. I expand the basic spin 1/2 holonomy in sigma matrices:
\( h^{(i)} = \exp(i \sigma \cdot \hat{n} \theta_i / 2) \)
\( = 1 \cos(\theta_i/2) + i \sin(\theta_i/2) \hat{n}^{(i)} \cdot \sigma. \) (6)

\( h^{(i)} \) is a rotation through \( \theta \) around an axis given by \( \hat{n} \). Now expand the expression in powers of sine, keeping out to linear in sine.

\[ h_i \cong 1 + i \sin(\theta_i/2) \hat{n}_i \cdot \sigma + \text{order } \sin^2(\theta_i/2) \text{ (SS)} \] (7)

SS stands for for small sine. This leaves the connection inside a bounded function.

Comparison of eq. (7) to eq. (5) yields the QFT limit of the sine.

\[ i \sin(\theta_i/2) \hat{n}_i \to (i/2) \int A_i^B \text{ (QFT).} \] (8)

The usual weak field expansion of a spin network holonomy (LQG \( \to \) QFT) keeps terms out to order \( A^2 \) in the connection. Similarly in the small sine expansion, I keep terms out to order \( \sin^2 \).

When carrying out an expansion, it is a little simpler to write each holonomy as

\[
\begin{align*}
  h & := \bar{h} + \hat{h}; \\
  \bar{h} & = (h + h^{-1})/2 = 1 \cos(\theta_i/2); \\
  \hat{h} & = (h - h^{-1})/2 = i \sin(\theta_i/2) \hat{n}^{(i)} \cdot \sigma.
\end{align*}
\] (9)

Then

\[ h \cong 1 + \hat{h}, \text{ (SS)} \] (10)

which is a more compact notation not involving explicit factors of \( \sin(\theta/2) \). In this notation, the passage from small sine to QFT is

\[ -2i \hat{h}^i \to \int A_i \cdot \sigma \text{ (QFT).} \] (11)

One might question the validity of this approximation, because the kinematic dot product based on Haar measure integrates over all values of \( \theta_i \), not just small values. To avoid this problem, one can use a coherent state. Suppose the wavefunctional at vertex \( n \) is a function of products of transverse holonomies, or equivalently, \( SU(2) \) rotation matrices having typical spin \( L_a, a = x,y \). Similarly,
suppose the wavefunctional along each $z$ link is a superposition of $U(1)$ $z$ holonomies having typical $Z$ component $M$. Superimpose the $L_a$ and $M$ values so as to form a coherent state peaked at values $< \theta_i >$. If the values of $< \theta_i >$ are small, then matrix elements will be dominated by small values of $\sin(\theta_i/2)$, and the small sine approximation will be valid.

The wavefunctional can be peaked at small $\sin(\theta_i/2)$, only if typical angular momenta $L_a$ (and $z$ components $M$) are moderately large. In a typical SU(2) coherent state, the standard deviations of $\theta_a$ and its conjugate momentum $L_a$ are of order $1/\sqrt{L_a}$ and $\sqrt{L_a}$ respectively. Sharp $\theta_a$ therefore requires moderately large $L_a$, $1/\sqrt{L_a} \ll \pi$. The small sine approximation breaks down if the representations of the rotation group occurring at a given vertex have too small values of total angular momentum.

B The Slow Variation Assumption

In the classical limit one expects slow variation of dynamical quantities from one vertex to the next. Slow variation implies that a plot of the quantity versus vertex index looks like a smooth curve, rather than a union of piecewise smooth segments.

I define a central and forward differences by

\[
\delta_c f(n) = (f(n+1) - f(n-1))/2; \quad \delta_f f(n) = f(n+1) - f(n). \quad (12)
\]

The slow variation assumption is

\[
(\delta f/f) \ll 1, \quad (14)
\]

where $\delta$ may be either difference.

The slow variation assumption also applies to higher differences. Define second differences by

\[
\begin{align*}
\delta_c^{(2)} f(n) &:= [\delta_c f(n+1) - \delta_c f(n-1)] \\
&= [f(n+2) - 2f(n) + f(n-1)]/4;
\end{align*}
\]

\[
\begin{align*}
\delta_f^{(2)} f(n-1) &:= [\delta_f f(n) - \delta_f f(n-1)] \\
&= [f(n+1) - 2f(n) + f(n-1)]. \quad (15)
\end{align*}
\]
If $\delta f/f$ is negligible, $(\delta f/f)(\delta g/g)$ is more so. Let $g = \delta f$.

\[
(\delta f/f)(\delta g/g) = (\delta f/f)(\delta(\delta f)/(\delta f)) = \delta^{(2)}f/f \ll 1.
\]

(16)

The second difference is of second order in small differences.

**C Commutators Can Be Neglected**

In the semiclassical limit, commutators of the basic fields are unimportant. To make this idea more quantitative, I assume commutators obey

\[
[\hat{h}_i, f^{(p)}]/f^{(p)} = \text{order}1/L,
\]

where $f^{(p)}$ is any function of the triads, homogeneous of order p.

This estimate of the commutator is based on the idea that the triads, acting on a rotation matrix at vertex n, bring down a spin matrix, therefore act like angular momentum operators. Let $| L \rangle$ be a state such that typical spins $L_a$ of the transverse holonomies in the state (and typical z components M of the longitudinal holonomies) are large. Then

\[
f^{(p)} | L \rangle = \text{order}(\kappa\gamma L)^p | L \rangle;
\]

\[
[\hat{h}_a, f^{(p)}] | L \rangle = \text{order}(\kappa\gamma)^p L^{p-1} | L \rangle,
\]

(18)

(and similar formulas for $\hat{h}_z$, with L replaced by M). I.e. the commutator with a holonomy removes one power of E, leaving behind a power of $\kappa\gamma$; therefore (commutator of f)/f is order 1/L.

Eq. (18) can be proved by induction, starting from $p = 1$.

\[
[\hat{h}_a, E_A] = [h_a - h_a^{-1}, E_A]/2
\]

\[
= (\kappa/2)[[\sigma_A/2], \hat{h}_a]_+ + [+\sigma_A/2, \hat{h}_a^{-1}]_+]/2
\]

\[
= (\kappa/2)[-\sigma_A/2, \hat{h}_a]_+
\]

\[
= (\kappa/2)[-\sigma_A/2, 1 \cos(\alpha_a/2)]_+
\]

\[
\cong (\kappa/2)(-\sigma_A). (\text{SS})
\]

(19)

The holonomy leaves the vertex through a surface on the positive "a" axis, then returns through a surface on the negative a axis. I am assuming $E_A$ is a sum of two operators, one of which grasps at each surface. The action of $E_A$ then brings down one factor.
of spin generator $\sigma/2$, on each side of the holonomy, resulting in an anticommutator. However, the anticommutator is not the important feature. For any integer, the commutator removes one factor of $\tilde{E}$ and replaces it with a factor of order $\kappa\gamma$.

I use eq. (17) when proving formulas valid in the SS limit. However, there are two other contexts where neglect of commutators is important. A commutator of constraints may not equal a linear combination of constraints because a factor ordering step produces an unwanted commutator (anomaly problem). There are no anomalies in the semiclassical limit.

Also, it may be necessary to neglect commutators when imposing a constraint at the quantum level, because the constraint must be commuted to the right. However, in this case an alternative procedure is available (admittedly, less desirable): impose the constraint at the classical level; then quantize the classical, constrained theory.

### D Integration by Parts is Allowed

The following formula is exactly true for the central difference.

$$
\delta_c (AB)(n) = \delta_c A(n)(1/2) [B(n + 1) + B(n - 1)] \\
+ (1/2) [A(n + 1) + A(n - 1)] \delta_c B(n).
$$

(20)

This formula resembles the distributive law for derivatives, with derivatives replaced by differences. However, the derivative law has $A(n)$ in place of the sums $(1/2)[A(n+1) + A(n-1)]$.

In slow variation limit, I can use eq. (15) to replace the sums $f(n+1) + f(n-1)$ by $2 f(n)$ plus a forward second derivative. Then I can use the slow variation assumption to drop the second derivative.

$$
\delta_c (AB)(n) \approx \delta_c A(n) B(n) + A(n) \delta_c B(n).
$$

(21)

This formula more closely resembles the corresponding relation for derivatives.

A corresponding formula for forward differences is less accurate. I dropped second order quantities $\delta^{(2)}f/f$ to get eq. (21). If central differences are replaced by forward differences, then eq. (21) holds only up to corrections first order in $\delta f/f$.

Eq. (21) may be rewritten in the form

$$
(\delta_c A) B \approx \delta_c (AB) - A(\delta_c) B.
$$

(22)
I will call this the "integration by parts" (IBP) formula, since it is analogous to the differential calculus identity used when integrating by parts.

E Finite Products of Triads

Since I am assuming volume $\neq 0$, I can use the QFT formulas for the cotriads, i.e.

$$e^R_r = \epsilon_{rst}^{RST} E^s_S E^l_T/(2! \mid e \mid).$$  \hspace{1cm} (23)$$

This QFT formula is accurate to order $\sin^2$ in SS approximation.

Proof: I prove the theorem here for transverse indices only. I cannot yet evaluate the theorem for the longitudinal holonomy $\hat{h}_z$, since it is not yet clear whether $\hat{h}_z$ means the holonomy on line $(n,n+1)$, the holonomy on line $(n-1,n)$, or some combination of the two. For transverse indices, there is only one holonomy per vertex and no ambiguity. I evaluate both sides of

$$[\mid e\mid^2, \hat{h}_a] = \mid e\mid [\mid e\mid, \hat{h}_a] + [\mid e\mid, \hat{h}_a] \mid e\mid.$$

1.h.s. $\quad = (\kappa \gamma/2) \text{sgn} \epsilon_{ajk} \epsilon^{AJK} \mid \sigma_A/2, \tilde{h}_a \mid + E^j_j E^K_K/2!$

r.h.s. $\quad = (\kappa \gamma/4) \text{sgn} \{[\mid e\mid, e_a(\text{SS})] + e_a(\text{SS}) \mid e\mid\}$

$$= (\kappa \gamma/4) \text{sgn} \{2 e_a(\text{SS}) \mid e\mid + [[\mid e\mid, e_a]\}.$$  \hspace{1cm} (24)$$

$\text{sgn}$ is the sign of $e$. The commutator on the last line is down by $1/L$ in SS approximation (commutators negligible). Equating the left- and right-hand sides, we find

$$e_a(\text{SS}) = \epsilon_{ajk} \epsilon^{AJK} \sigma_A E^j_j E^K_K/(2! \mid e \mid) + \text{order } \sin^2;$$

$$e_a(\text{SS}) = \text{sgn}(\kappa \gamma/4)^{-1}[\mid e\mid, \hat{h}_a].$$  \hspace{1cm} (25)$$

The leading term in SS approximation is the QFT result, with corrections smaller by order $\sin^2$. More precisely, there can be a linear-in-sine correction to $e_a(\text{SS})$, but it is down by order $1/L$. $\Box$

The "cotriad", defined at eq. (23) and computed at eq. (25), is not the usual cotriad = inverse of the triad. The inverse triad has $e$ rather than $\mid e \mid$ in the denominator. However, the Hamiltonian constraint contains "cotriads", with $\mid e \mid$ rather than $e$. If one is
using \( H \) only as a constraint (i.e. \( H = 0 \)), then \( H \sim 1/|e| \) and \( H \sim 1/e \) are equivalent. They differ only by a constant.

There is some theoretical justification for using \( 1/e \) rather than \( 1/|e| \). In LQG, one considers situations where the volume passes through zero, and \( e \) is less singular at zero.

However, in the present papers I use \( H \) both as constraint and as Hamiltonian (when a surface term is added to the constraint \( H \)).

\[
i dO/dt = [O, H + ST].
\]

When used as part of the Hamiltonian, \( H \) must contain \( 1/|e| \). Therefore if I use the real cotriads \( \sim 1/e \), I must transfer back and forth between two versions of \( H \). Since I do not go near \( e = 0 \), I keep the discussion simple and use only the \( 1/|e| \) form. From now on I drop the quotes on “cotriad”.

Because of gauge fixing, the \( E^i_I \) are block diagonal with a 1x1 longitudinal subblock \( E^z_Z \) and 2x2 transverse subblock \( E^a_A \). The QFT cotriads are also block diagonal. Since QFT and SS cotriads agree to order \( \sin^2 \), the SS cotriads are block diagonal.

The cotriad can be rewritten as a rank two tensor,

\[
\epsilon^{ijk} e^K_k.
\]

If \( e^K_k \) is block diagonal, then at most one of indices \( ijK \) can be longitudinal. Only \( e^{abZ} \), \( e^{azB} \) are non-vanishing.

The following formulas, well known in QFT, are also valid to order \( \sin^2 \) in SS approximation. The first formula states that a triad equals a determinant of cotriads:

\[
e_a^A e_b^B e^{abz} \epsilon^{iK} = e e^{ABZ} \epsilon^{iZ} \epsilon^{jK} sgn. = \epsilon^{ABZ} E^i_Z sgn. \tag{26}
\]

\[
e^J_i E^i_K = |e| \delta(I, J). \tag{27}
\]

This second formula states that basis and cobasis are orthogonal. Other formulas involving products will be developed as needed.

Eqs. (26) and (27) have a special character. Even though they were proved using QFT limit formulas for the leading term in an expansion in powers of sine, neither side contains an explicit factor of \( 1/e \). It is tempting to assume that leading terms for the product of triads and cotriads can be extrapolated to small volume, if the factors of \( 1/e \) cancel; i.e. QFT gives the leading term even in the
small sine model. At small volume, however, this leading term will be correct only to order sine: the commutator in eq. \((25)\) cannot be dropped when volume is small.

F Spin connection and extrinsic curvature are small

I assume the spin connections \(\Gamma_i\) and extrinsic curvatures \(K_i\) are small. Both are expected to be less than order unity in SS approximation.

Although matrix elements of the \(E_i^j\) grow as \(L\) (or \(M\)), matrix elements of the \(\Gamma_i\) remain order unity, because the \(\Gamma_i\) are homogeneous of order zero in the triads. In fact the \(\Gamma_i\) must be less than unity in SS approximation, because they contain a difference. Because of the homogeneity, the difference must occur in a ratio

\[
\delta E/E,
\]

which is small.

Classically, \(\gamma K_i = A_i - \Gamma_i\). I derive the SS \(K_i\) in section \(\text{IV}\) and show

\[
\gamma K_i = -2i \hat{h}_i - \Gamma_i, \text{ (SS)}
\]

not surprising, since the small sine limit of \(-2i \hat{h}_i\) is \(A_i\). The latter is unbounded (QFT), whereas \(-2i \hat{h}_i \sim \sin\) is less than order unity (SS). From the above relation, the \(K_i\) are of order the maximum of \(\Gamma_i\) and \(\hat{h}_i\), both less than order unity. \(\square\)

The unidirectional constraints, to be presented in section \(\text{X}\), imply that \(\delta E/E\) terms and \(K\) terms are of the same order. This is qualitatively reasonable; space derivatives equal time derivatives in a wave theory.

\(\hat{h}\), a combination of \(\Gamma\) and \(K\), is likely to be of the same order as \(\Gamma\). I. e. small sine terms and \(\delta f/f\) terms are likely to be of the same order. The SS Hamiltonian will contain terms of order \(\delta \hat{h}, \Gamma^2,\) and \(K^2\) (times triads). From the above discussion, all three terms will be the same order. None can be neglected.
G Operators As Functional Derivatives

The basic $[\hat{E}, \text{holonomy}]$ commutator suggests that the holonomy can be interpreted as a functional derivative.

$$[E_z^n(n), h_z(n, n + 1)] = (\kappa \gamma / 2)(\sigma_z / 2)(1 / 2)[h_z + h_z^{-1}];$$

$$\Leftrightarrow h_z(n, n + 1) = (\kappa \gamma / 2)(\sigma_z / 2)[h_z + h_z^{-1}] / 2 [-\delta / \delta E_z^n]. \quad (28)$$

The temptation is to extend this to commutators with functions.

$$[E_z^n(r), h_z] = (\kappa \gamma / 2)(r \text{sgn}(z) \sigma_z / 2)(1 / 2)[h_z + h_z^{-1}] \mid E_z^n]\mid^{-1}. \quad (29)$$

The right hand side is the classical Poisson bracket times i. Thiemann and Winkler, however, show that in LQG the replacement (commutator $\rightarrow$ i P. b.) is valid only in a semiclassical limit $^{12}$.

It is instructive to verify their theorem by means of a concrete calculation. This is straightforward for $E_z^n$, since eigenfunctions of $E_z^n$ are easy to construct. To reproduce eq. (28) in the LQG context I must take

$$E_z^n \rightarrow -i \partial / \partial \theta;$$

$$\hat{h}_{z+} \rightarrow [\exp(i \theta / 2) - \exp(-i \theta / 2)](1 / 2).$$

I have specialized to the ++ matrix element, as a representative example.

Now consider a wavefunction which is a sum of eigenfunctions of $E_z^n$.

$$\psi(\lambda) = \exp[i \lambda \theta];$$

$$E_z^n \psi(\lambda) = (\kappa \gamma / 2) \lambda \psi(\lambda). \quad (30)$$

Let $\psi$ be acted on by a commutator involving $[E_z^n]$.  

$$[E_z^n]^{\uparrow} \mid \hat{h}_{z+} \mid \psi(\lambda)$$

$$= [E_z^n]^{\uparrow} \mid (1 / 2)[\psi(\lambda + 1 / 2) - \psi(\lambda - 1 / 2)]$$

$$- \hat{h}_{z+} \mid E_z^n \mid^{\uparrow} \psi(\lambda)$$

$$= \{(1 / 2) \mid \lambda + 1 / 2 \mid \psi(\lambda + 1 / 2) - (1 / 2) \mid \lambda - 1 / 2 \mid \psi(\lambda - 1 / 2)$$

$$- \mid \lambda \mid^{\uparrow} [\psi(\lambda + 1 / 2) - \psi(\lambda - 1 / 2)]\}$$

$$\cong (r / 2) \text{sgn}(\lambda) \mid \lambda \mid^{\uparrow -1} (1 / 2)[\psi(\lambda + 1 / 2) + \psi(\lambda - 1 / 2)]$$

$$= (r / 2) \text{sgn}(\lambda) \mid \lambda \mid^{\uparrow -1} (1 / 2)[h_{z+} + h_{z+}^{-1}] \psi(\lambda). \quad (31)$$
(I suppress factors of $\kappa \gamma / 2$ for typographical clarity.) The last line give the expected functional derivative result, but only because I have expanded $(\lambda \pm 1/2)^r$ and assumed

$$| \lambda | \gg 1/2.$$ 

The functional derivative representation is valid only for large eigenvalues.

III The Euclidean Hamiltonian

This section suggests a form for the exact, LQG, Euclidean Hamiltonian. This Hamiltonian is not the correct Hamiltonian for LQG, but is a necessary first step in constructing that Hamiltonian.

I then apply the approximations described in the previous section. Several features of the exact Hamiltonian persist to the semiclassical version, and I use these features in succeeding sections.

Since my form for the exact Hamiltonian is largely speculative, readers interested only in the semiclassical limit may prefer to skip to the summary at the end of this section on a first read. That summary describes the features which persist to the semiclassical limit and argues that the persistent features are generic to a large class of models.

The quantum field theory expression for the Euclidean Hamiltonian is

$$-NH_e + ST = \int d^3x N(F^I_{jk} E_j^I E^k_K \epsilon_{1JK}/2\kappa | e | ) + ST;$$

$$ST = - (N A^I_a Z^Z E^a_K \epsilon_{1ZK}/\kappa | e | ) |_{z=-\infty}^{+\infty};$$

$$\kappa = 8\pi G;$$

$$|e| \doteq \sqrt{\text{det} \hspace{1mm} E \text{sgn.}}.$$ 

(32)

sgn is the sign of the determinant and $N$ is the lapse. The surface term ST is needed to cancel another surface term which appears when the volume term is varied to produce field equations.

A LQG Field Strengths

I must now replace the QFT field strengths and triads by expressions valid in LQG. The LQG formula for field strength $F$ generalizes the
classical formula

\[ F_{ij} = \lim_{\Delta A \to 0} \left( \prod h \right)/\Delta A(ij), \]  

(33)

where \( \prod h \) is the product of holonomies around the edges of infinitesimal area \( \Delta A(ij) \). A local treatment of the Hamiltonian would construct each \( F \) using small areas at a single vertex \( n_z \), plus triads at that vertex; then sum over vertices. I.e. the basic modular unit would be the vertex.

To construct a non-local Hamiltonian, I use non-local modular units. Since eq. (33) contains an area, I choose the basic modular units to be areas. The areas are bounded by nearest neighbor vertices: i.e. holonomies along each edge of the area run from one vertex to the next, nearest neighbor vertex. For example, the LQG contribution to \( F_{xy} \) from the area bounded by nearest neighbor vertices \((n_x, n_y, n_z), (n_x, n_y + 1, n_z), (n_x + 1, n_y + 1, n_z), (n_x + 1, n_y, n_z)\) is

\[ F_{xy} = i \text{const}. h_y^{-1} h_x^{-1} h_x h_y + \text{H.c.} \quad \text{(LQG)} \]  

(34)

Each \( h_i \) traverses edge \( i \) from \( n_i \) to \( n_i + 1 \); the \( h^{-1} \) traverse in the reverse direction. Reading from right to left, the explicitly written term circulates the xy area in counterclockwise direction. The Hermitean conjugate \( (\text{H.c.}) \) term circulates in the clockwise direction.

The above two terms are not the only possibilities, even though we restrict ourselves to circuits starting from \((n_x, n_y, n_z)\) and continuing in the xy plane to nearest neighbors only. In fact there are eight such terms. The eight correspond to the four vertices in the xy plane which are nearest neighbors to the vertex \((n_x, n_y, n_z)\), times two for clockwise or counterclockwise circuit. The eight may be grouped into four sets of two terms each, after we impose the requirement that the field strength is Hermitean. The full LQG expression for \( F_{xy} \) therefore contains four adjustable constants analogous to the ”const” in eq. (33). I could determine them by carrying out a small sine expansion, and demanding that the expansion have the minimum number of powers of sine beyond those needed to recover the quadratic limit. The discussion would be straightforward but lengthy, and I omit it in the interests of focusing on the semi-classical limit.

If I small-sine expand the contribution eq. (34), using eq. (10), and keep up to order \((\sin)^2\), I get
The $\hat{h}_a$ need only a single argument $n_z$, since all variables are independent of $x$ and $y$. The $(1/2 \, i)$ cancels a corresponding factor coming from the commutator of two sigma matrices. The $(-2i)^2$ factor converts the $\hat{h}_a$ into $A_a$ if I go to the QFT limit; cf. eq. (11).

If the ”const” in eq. (34) is chosen real, then the ”1” terms (unit matrix terms) in the expansion are canceled by the H.c. term. Contributions from the remaining three areas are important in canceling higher powers of sine in the expansion; but each of the four areas yields the same small sine limit, eq. (35).

Eq. (35) has no linear-in-sine terms. These would spoil the QFT limit.

\[
F_{ij} \to (\partial_i A^I_j - \partial_j A^I_i \\
A^I_i A^I_j \epsilon_{IJK} \sigma^I \Delta x^i \Delta x^j) \quad \text{(QFT)}
\]

For $ij = xy$, the linear in $A$ terms are absent, because all fields are independent of $x$ and $y$.

The remaining two field strengths, $F_{za}$ with $a = x,y$, may be constructed in similar fashion. For $F_{zx}$, for example, the relevant plane is $zx$ rather than $xy$; but there are still eight loops, corresponding to four vertices which are nearest neighbors to the vertex $(n_x, n_y, n_z)$, times two, for clockwise or anticlockwise. Four of the holonomy loops travel forward to an edge at $n_z + 1$; the remaining four travel backward to an edge at $n_z - 1$. The four ”forward” loops are

\[
F_{za}(n_z, n_z + 1) = i \text{ const.} [h_z(n_z, n_z + 1)^{-1} h^{-1}_a(n_z + 1) h_z(n_z, n_z + 1) h_a(n_z) \\
- (h_a \leftrightarrow h^{-1}_a) + \text{H.c.}] \quad \text{(LQG)}
\]

The $(h_a \leftrightarrow h^{-1}_a)$ term is the second forward loop; ”H.c.” includes the remaining two loops, which have clockwise ↔ anticlockwise. Since the $h_z$ link two vertices, $F_{za}$ requires two $n_z$ arguments.

In the $xy$ case all four areas had the same small sine limit. In the $za$ case, forward and rearward loops have different SS limits and cannot be combined, because they involve different $x$ holonomies $h_x(n_x \pm 1)$ on different $x$ edges.
Also, the first two forward loops in eq. (37) each contain unwanted \( h_x(n_z) h_x(n_z + 1) \) quadratic terms in the SS limit; the minus sign between these two loops insures that the unwanted terms cancel. Explicitly, the small sine limits are

\[
2 F_{za}(n_z, n_z + 1) \to (-2i) \{-[\hat{h}_z(n_z, n_z + 1), \hat{h}_a(n_z + 1)]
+ [\hat{h}_a(n_z + 1), \hat{h}_a(n_z)]\} - (\hat{h}_a \leftrightarrow -\hat{h}_a);
\]

\[
2 F_{za}(n_z, n_z - 1) \to (-2i) \{-[\hat{h}_z(n_z - 1, n_z), \hat{h}_a(n_z - 1)]
+ [\hat{h}_a(n_z - 1), \hat{h}_a(n_z)]\} - (\hat{h}_a \leftrightarrow -\hat{h}_a). \quad \text{(SS)} \tag{38}
\]

The \( \hat{h} \leftrightarrow -\hat{h} \) terms represent the second loop; the unwanted \( (\hat{h}_a)^2 \) terms cancel out when the two loops are summed.

When eq. (38) is taken one step further to the QFT limit, the \([\hat{h}_z, \hat{h}_a]\) commutator produces the term

\[
A^I_i A^J_j \epsilon_{IJK} \to A^Z_z A^A_a \epsilon_{ZAK}
\]

in agreement with eq. (36); and \(-2i(\hat{h}_a(n_z + 1) - \hat{h}_a(n_z))\) produces the term

\[
\partial_i A^I_i - \partial_A A^I_i
\]

with \( ij = za \).

In the QFT limit the LQG and SS field strengths \( F_{ij} \) become the classical field strengths, times area \( \Delta x^i \Delta x^j \). The QFT limit of \(-2i(\hat{h}_a(n_z + 1) - \hat{h}_a(n_z))\) contains only a factor \( \Delta x^a \) times the difference \( A_a(n_z + 1) - A_a(n_z) \). However, when a \( \Delta x^x \Delta x^a \) is factored out of the field strength, the difference acquires a factor \( 1/\Delta x^x \). This yields the difference approximation for the derivative:

\[
\delta_f(n)g = g(n + 1) - g(n);
\]

\[
\delta_f g/\Delta x^z \approx \partial g/\partial z. \tag{39}
\]

The \( xy \) field strength distorts a spherical cloud of test particles into an ellipsoid. This distortion mimics the behavior of the sphere under free fall in a static gravitational field. The \( za \) field strengths produce the distortions typical of time-varying waves. It is not surprising, therefore, that the \( za \) field strengths possess all the nonlocality.
B LQG Triads and the Lapse

Since the triads are associated with areas and volumes, which are local, we assume the LQG triads are local, with support at the vertices. More precisely: draw a small cube around each vertex. The triads, which are associated with area two forms via $E^a_A \epsilon_{abc} dx^b \wedge dx^c$, live on the six faces of the cube.

When defining the volume operator, typically one assumes each $E^a_A$ grasps at both faces having outward normals in the "a" direction, the positive face (with normal pointing in the positive a direction) and the negative face. This is natural; the volume operator is not directional, and one would expect contributions from all six faces. The volume operator therefore involves the product of three operators, each grasping at two faces.

$$E^i_I := E^i_I(+) + E^i_I(-), \quad (40)$$

where ±, the sign of the normal, indicates where the $\tilde{E}$ grasps.

In the operator expression for Gauss’ Law, however, one needs the difference between the + and - operators, $E^x_Z(+) - E^x_Z(-)$, because Gauss’ Law involves the difference between ingoing and outgoing $Z$ component of spin. It is not enough, therefore, to specify the vertex $n_z$ where the triad has its support; one must also supply an argument ± specifying which face it grasps.

In eq. (32) there are two triads and a volume e associated with each field strength. A given field strength is a holonomy loop passing through four vertices, but at each vertex, the holonomies do not pass through all six faces. One holonomy enters at one face, and a different holonomy leaves at another face. I assume: if a given loop on area ij starts and ends at a vertex $n_z$, then the triads associated with that loop are all evaluated at $n_z$; and the triad $E^i_A$ is given the sign associated with the holonomy passing through face i.

For example, consider the xy holonomy loop beginning at vertex $(n_x, n_y, n_z)$ and passing through the + x face to $(n_x+1, n_y, n_z)$. The loop then continues to $(n_x+1, n_y+1, n_z), \cdots$, finally returning down the y axis through the + y face. Both holonomies pass through + faces; the triads associated with this contribution would be $E^x_A(+) + E^y_A(+)$. The same triads occur in the H.c. loop.

The remaining three pairs of loops involve the remaining three sign pairs: $(\pm, \mp)$ and $(-,-)$. The $F_{xy}$ contribution remains a sum of
four terms, if we group each term and its H.c. together as a single contribution. Each term contains a different sign pair.

A z triad $E^z$ also contributes to each $F_{xy}$ term in $H_e$. The z triad is contained in the factor of volume, $e$. It is not obvious what sign to assign to the z triad, since no xy holonomy passes through a z face. However, a given xy area bounds two volumes, each containing a different z line ($n_z \pm 1, n_z$). One can get either sign, $z = \pm 1$, according as the xy area is interpreted as bounding the volume containing $(n_z, n_z + 1)$, or $(n_z - 1, n_z)$. There is no reason to favor one interpretation over the other, and we therefore make the z triad the sum of the two z signs: $E^z = E^z_1 + E^z_2$.

In the small sine, slow variation limit, the triad functions in $H_e$ lose their dependence on the individual $E^i_I(\pm)$ and depend only on the sums $E^i_I := E^i_I(+) + E^i_I(-)$. Proof: begin with the four loops contributing to $F_{xy}$. All field strengths have the same small sine limit, eq. (35). One can factor this out, leaving a sum over four triad functions.

$$H_e(x y) = F_{xy}(SS) \sum_\eta f[E^x(\eta_x), E^y(\eta_y), e(\eta_x, \eta_y)]$$

where $\eta_a = \pm 1$) indicate the areas grasped by the $\tilde{E}$; and the $E^z$, not indicated explicitly, are already in the desired form.

Now expand each transverse $\tilde{E}$:

$$E^a(\pm) = [(E(+)) - E(-)]^a/2 \pm [(E(+)) - E(-)]^a/2 = E^a/2 \pm \tilde{\delta} E^a/2 = (E^a/2)[1 \pm \tilde{\delta} E^a/E^a].$$

The tilde difference denotes a difference between two triad functions differing only in sign. I insert the expansions of eq. (41) into eq. (41), and power series expand around $\tilde{\delta} E = 0$, assuming the tilde differences are small because of the slow variation assumption.

The expansion contains no terms having an odd number of tilde differences, since the sum in eq. (41) is even under $(+ \leftrightarrow -)$. The expansion of an arbitrary symmetric function of the $E^a_I(\pm)$ begins with the terms

$$f[E(+)] + f[E(-)] = 2 f(E/2) + (1/2!) (\partial^2 f/\partial E^2) (\tilde{\delta} E)^2.$$
The leading term in the expansion contains two more factors of $E$ than the second order term, and is therefore larger than the second order term by a factor

$$\frac{(\delta E)^2}{(E^a)^2} = \text{order}(\delta f/f)^2.$$ 

The second order term can be neglected. Tilde differences have disappeared from the $F_{xy}$ terms.

Now consider the $F_{za}$ terms, for example the forward areas involving $z$ holonomies on edge $(n_z, n_z + 1)$, and $za = zx$. At eq. (38) the sign between two terms was adjusted so as to cancel an unwanted commutator term. This cancelation must be reconsidered; the two terms are now multiplied by different triads $E^x(\pm)$. The unwanted commutators now have a contribution of the form

$$\left[ \hat{h}_z(n_z + 1), \hat{h}_x(n_z) \right] \times (f[E^z(+), E^z(+)] - f[E^z(-), E^z(+)]) 
\cong \left[ \delta_f \hat{h}_x(n_z), \hat{h}(n_z) \right] 2 (\partial f/\partial E^z) \delta E^z. \quad (42)$$

This term is second order in differences and can be dropped.

The remaining terms, those without the unwanted commutators, give the correct QFT limit and are even under $E^a(+) \leftrightarrow E^a(-)$. By the same argument as for the $F_{xy}$ terms, the expansion in powers of $\tilde{E}^a$ may be terminated at the leading term which is independent of $\tilde{E}^x$. Similarly for the rearward loops.

At this point the $F_{za}$ loops have the desired $E^a$ dependence, but forward loops are multiplied by a function of $E^z(+)$, while the rearward loops depend on $E^z(-)$. From the SS limits, eq. (38), both loops contain forward difference terms $\delta_f \hat{h}_a$ and commutator terms $[\hat{h}_z, \hat{h}_a]$.

Consider first the difference terms. When forward and rearward loops are summed, the forward differences become 2 times a central difference.

$$\delta_f h_a(n_z) f[E^z(+)] + \delta_f h_a(n_z - 1) f[E^z(-)] 
\cong \left[ \delta_f h_a(n_z) + \delta_f h_a(n_z - 1) \right] f[E^z] 
+ \left[ \delta_f h_a(n_z) - \delta_f h_a(n_z - 1) \right] (\partial f/\partial E^z) \delta E^z. \quad (43)$$

The middle line is twice a central difference, from definitions eqs. (13) and (15). The last line is down by two factors of $\delta f/f$ and may be dropped.
For the commutator terms one must expand, not only \( f[E_z(\pm)] \), but also \( \hat{h}_z \) and \( \hat{h}_a \).

\[
\hat{h}_z(n_z, n_z \pm 1) = [\hat{h}_z(n_z, n_z + 1) + \hat{h}_z(n_z - 1, n_z)]/2 \pm \delta \hat{h}_z(n_z - 1, n_z)/2;
\]

\[
\hat{h}_a(n_z \pm 1) = \hat{h}_a(n_z) \pm \delta \hat{h}_a(n_z \pm 1). \tag{44}
\]

The commutator terms are then

\[
\sum_{\pm} [\hat{h}_z(n_z, n_z \pm 1), \hat{h}_a(n_z \pm 1)] f[E^i(\pm)]
\]

\[
= [\hat{h}_z(n_z), \hat{h}_a(n_z)] 2 f(E^z) + \text{order} \left( \delta f/f \right)^2. \tag{45}
\]

The two terms linear in \( \delta \hat{h}_z(n_z - 1, n_z) \) and \( \delta E^z \) have opposite signs and cancel. The two terms linear in \( \delta \hat{h}_a \) have the form

\[
[\hat{h}_z(n_z), \delta \hat{h}_a(n_z) - \delta \hat{h}_a(n_z - 1)] f[E^z].
\]

This difference of differences is a second forward difference, which is order \( (\delta f/f)^2 \).

All four \( F_{za} \) loops now depend only on \( E^i \). Dependence on the \( E^i(\pm) \) has disappeared. \( \square \)

After the above replacements, the SS expression for \( H_e \) is

\[
-N H_e + ST = \sum_{nz} N(n_z) \{ F_{Zxy}^Z(n_z) E_J^x E_K^y \epsilon_{ZJK} \\
+ F_{Zza}^A E_Z^a E_B^b \epsilon_{ABZ} \}/\kappa \gamma \mid e \mid + ST;
\]

\[
F_{xy}(n_z) = F_{xy}^Z(n_z) \sigma_Z = (1/2i)[\hat{h}_x(n_z), \hat{h}_y(n_z)](-2i)^2;
\]

\[
F_{za}(n_z) = F_{za}^A(n_z) \sigma_A = (-2i)[- [\hat{h}_z(n_z), \hat{h}_a(n_z)]
+ \delta \hat{h}_a(n_z). \tag{SS} \tag{46}
\]

The \( E \)'s and \( \hat{h}_z \) are sums over two signs, as at eqs. \( \text{(40)} \) and \( \text{(44)} \).

In eq. \( \text{(46)} \) I have assumed the lapse \( N \) is local, associated with a vertex rather than a line or area: \( N = N(n_z) \). The lapse is derivative-free, and is not a connection or field strength. Also, it is a zero-form, not associated with any area.

24
More on $\hat{h}_z$

The replacement of non-local by local holonomies ($\hat{h}_a(n_z\pm1) \rightarrow \hat{h}_a(n_z)$) is a simplification. However, the replacement

$$\hat{h}_z(n_z, n_z \pm 1) \rightarrow [\hat{h}_a(n_z, n_z + 1) + \hat{h}_a(n_z - 1)]/2$$

can be a complication, because some holonomy calculations must be repeated twice, once for transverse holonomies, once for longitudinal. For example, the commutator representation for the longitudinal triad $e_z$ contains different constant factors:

$$e_z(n) = \text{sgn}(\kappa \gamma/8)^{-1} [ e(n), \hat{h}_z(n) ];$$
$$e_z(n \pm 1) = \text{sgn}(\kappa \gamma/16)^{-1} [ e(n \pm 1), \hat{h}_z(n) ];$$
$$e_z(n) = \sigma_Z^{(2)} E(n)/ |e| + \text{order } \sin^2.$$

For the constant factors in the transverse case, see eq. (25). Note the last line, however: in SS approximation the QFT value for all commutators is valid to order $\sin^2$.

Commutators of $\hat{h}_z$ with volume are slightly changed (eq. (47)). However, whenever $E_z^z$ or $\hat{h}_z$ occurs in a sum over vertices, commutators with $E_z^z$ are identical to commutators with transverse $E_A$ because of the slow variation assumption. For example,

$$[ f(m) E_z^z(m), \sum_n \hat{h}_z(n) g(n) ]
= \sum_{n,\pm} (1/2) f [ E_z^z(m), \hat{h}_z(n, n \pm 1) ] g
= \sum_{n,\pm} (1/2) f (\sigma_z^{(2)} \hat{h}_z(n, n \pm 1) [\delta(m, n) + \delta(m, n \pm 1)] g
= (1/4) f(m) \sigma_z [2 g(m) + g(m - 1) + g(m + 1)] + \text{order } \sin^2
= (1/4) f(m) \sigma_z [4 g(m) + \delta_f^{(2)} g(m)].$$

This is identical to the corresponding transverse result,

$$[ f(m) E_A^a(m), \sum_n \hat{h}_a(n) g(n) ] = f(m) \sigma_A g(m) + \text{order } \sin^2.$$

Summary

The model proposed at the start of this section uses an area as its basic module. The holonomic loop associated with a given $(z,a)$ field
strength $F_{za}$ includes only the holonomies associated with that area: $\hat{h}_a(n)$, $\hat{h}_a(n+1)$, and $\hat{h}_z(n,n+1)$, for a loop going from vertex $n$ to vertex $n+1$ and back. Holonomic differences are forward differences, defined within the area.

$$\delta_f \hat{h}_a(n) = \hat{h}_a(n+1) - \hat{h}_a(n).$$

Even the triads in the initial model are area oriented, in the following sense. Surround vertex $n$ with a small cube. The triads are allowed to grasp only at the two faces of the cube where holonomies enter and leave.

After the semiclassical limit is taken, the final model is vertex-oriented, rather than area-oriented: eq. (46) is a sum over vertices. How did this happen? The initial model contains two sets of loops beginning at $n$. One set goes forward, to vertex $n+1$; those loops depend on $\hat{h}_a(n+1)$. The other set goes backward, to vertex $n-1$; those depend on $\hat{h}_a(n-1)$. In the final model, the $\hat{h}_a(nz \pm 1)$ disappear, because the sum over forward and backward loops yields expressions symmetric with respect to $n+1 \leftrightarrow n-1$.

$$\hat{h}_a(nz + 1) + \hat{h}_a(nz - 1) = 2 \hat{h}_a(nz) + \delta_f^{(2)} \hat{h}_a(nz).$$

Slow variation allows the second difference to be dropped.

The $\hat{h}_z(nz,n \pm 1)$ do not quite disappear. However, dependence on the difference

$$\hat{h}_z(nz + 1, nz) - \hat{h}_z(nz, nz - 1)$$

drops out, again because of slow variation and $n + 1 \leftrightarrow n - 1$ symmetry. The semiclassical limit depends on an average at each vertex.

$$\hat{h}_z(nz) := [ \hat{h}_z(nz + 1, nz) + \hat{h}_z(nz, nz - 1) ] / 2$$

Similarly, triads which grasp at only one face are replaced by triads which include both faces.

$$E(\pm)(n) \rightarrow E(n) = E(+)(n) + E(-)(n)$$

Another example of averaging: when forward and backward loops are summed, forward differences disappear, replaced by their average, a central difference.

$$[ \delta_f \hat{h}(n) + \delta_f \hat{h}(n - 1) ] / 2 = \delta(c) \hat{h}(n).$$
These results perhaps are not surprising. They might be expected from any treatment which uses a slow variation assumption and assigns equal weights to forward and backward expressions.

In the introductory section I described congruence and $S_1$ pictures for visualizing the topology of the spin network. The congruence picture was very useful in suggesting the form of the original, exact model.

**IV Extrinsic Curvature and Spin Connection $\Gamma$**

From this point on I drop the subscript $z$ on $n_z$: $n_z \to n$.

Thiemann has proposed a procedure for generating a regulated set of extrinsic curvatures $K_i$. His procedure leads to $K(n)_a$, $a = x,y$, which are weighted averages over holonomies at $n$ and its immediate neighbors. My construction is essentially that of Bannierjee and Date [8], who construct $\tilde{K}(n)_a$ proportional to a single holonomy at $n$. In a slow variation framework the two procedures are indistinguishable because of identities such as

$$K(n + 1) + K(n - 1) = 2 K(n) + \delta^{(2)}_i K \approx 2 K(n).$$

**A Recovering the Lorentzian Hamiltonian**

In the QFT limit, $H_e$ has terms linear and quadratic in the connection $A$. When one expands

$$A = \gamma K + \Gamma,$$

one finds that linear-in-$K$ terms vanish because of an identity obeyed by the triads. $H_e$ is not the Lorentzian Hamiltonian $H$. The terms containing only $\Gamma$ have the correct magnitude but wrong sign; the terms quadratic in $K$ have the wrong magnitude (off by a factor $\gamma^2$). Both deficiencies can be corrected by adding an order $K^2$ term to the Euclidean Hamiltonian. The QFT Lorentzian Hamiltonian $H$ is then

$$H = \sum_n \left[ -(1 + \gamma^2)/2\kappa \right] (K^I_i K^J_j \epsilon_{IJK} e^{ijK} N)(n) - H_e. \quad (QFT)$$

(50)
The transition to LQG requires each $K$ to be replaced by a function of holonomies. The most straightforward replacement is

$$\gamma K_i \rightarrow -2i \hat{h}_i - \Gamma_i \quad (\text{SS}),$$

which is just the SS analog of $\gamma K = A - \Gamma$. For $i = z$, $\hat{h}_i$ is the average defined at eq. (49).

Of course we must now decide on a SS $\Gamma$. In QFT limit the linear-in-$K$ terms cancel out because of an identity obeyed by the cotriads,

$$0 = \partial_i e_{rI} \epsilon^{rmi} + \epsilon_{IJK} \Gamma^J_i e^K_q \epsilon^{qmi}$$

I. e., any function of cotriads has zero covariant divergence; also, the covariant divergence of a weight one rank two tensor, antisymmetric under $m \leftrightarrow i$, contains no Christoffel symbols.

In order for the SS approximation to resemble the classical limit as closely as possible, it is desirable to have the linear-in-$K$ terms vanish also in the SS case, when derivatives are replaced by differences in eq. (52). I use this requirement to determine the SS $\Gamma$.

The first step is to replace the derivative in eq. (52) by a central difference. The Hamiltonian will contain central differences of cotriads because of the $\delta_{(c)} \hat{h}_a e_b \epsilon^{zab}$ terms. These must be integrated by parts to get the $\delta_{(c)}$ off the holonomy and onto the cotriad: see the next section on surface terms. After this IBP, the Hamiltonian will depend on central differences of cotriads (not forward differences).

The Hamiltonian will also contain $\Gamma_j$, and these should be defined so as to be consistent with eq. (52). This is straightforward to arrange: in QFT, eq. (52) can be inverted to obtain a formula for the spin connections in terms of derivatives of cotriads. The inversion uses tensor-algebraic manipulations and employs no property of the derivative; hence the inversion goes through even after the replacement derivative $\rightarrow$ difference.

The inversion is not especially straightforward. I do not go through every step of the proof, but list key steps. Multiply by $e^M_m$ and use

$$e^K_q e^M_m \epsilon^{qmi} = e^i_N \epsilon^{KMN}.$$
The QFT formula, eq. (52), gives

\[ \Gamma^i_M E^i_I = \text{sgn} (\partial_i e_{rI}) e^{rmi} e^M_m + \Gamma \cdot E \delta^M_I; \]
\[ \Gamma \cdot E := \Gamma^j_i E^i_j = (-\text{sgn}/2) (\partial_i e_{rM}) e^{rmi} e^M_m + \Gamma \cdot E \delta^M_I; \] (QFT) (53)

After the replacement, derivative \(\rightarrow\) central difference, we have

\[ \Gamma^i_M E^i_I = \text{sgn} (\delta(e) e_{rI}) e^{rmi} e^M_m + \Gamma \cdot E \delta^M_I; \]
\[ \Gamma \cdot E = \Gamma^j_i E^i_j = (-\text{sgn}/2) (\delta(e) e_{rM}) e^{rmi} e^M_m. \] (SS) (54)

I have included the appropriate \(\Delta x^r\) in all one-forms and connections (not shown explicitly); and the \(E^i\) contain \(\Delta x^j \Delta x^k\). I do not strip off the \(E^j_i\) because the \(\Gamma^j_i\) in the Hamiltonian typically occur contracted with a triad. The identity eq. (52) is now

\[ 0 = \delta(e) e_{rI} e^{rmi} e^M_m + \epsilon_{IKJ} (\Gamma^K_j e^j_q e^{qmi}). \] (SS) (55)

From eq. (54), the small sine \(\Gamma^i_i\) have exactly the same block diagonal structure as the cotriads. A 1 x 1 block contains \(\Gamma^Z_Z\); a 2 x 2 block contains the transverse components \(\Gamma^A_A\).

The foregoing arguments go through whether one uses "cotriads" or cotriads. The original identity eq. (52) is homogeneous in the cotriads; the final formula for \(\Gamma\) contains an even number of cotriads, therefore an even number of factors of \(e\).

B Transverse Trace; \(\Gamma^z\)

A corollary of the block diagonal nature of the cotriads: for \(I = J = Z\) in eq. (54), the term involving \(\delta(e) e_{rJ}\) vanishes. The formula for \(\Gamma \cdot E\) then simplifies to

\[ \Gamma \cdot E = \Gamma^Z_Z E^z_z \]
\[ = (-\text{sgn}/2) (\delta(e) e_{A}) e^{ahz} e^A_b. \] (SS) (56)

I.e. the formula for \(\Gamma^z\) involves a sum over transverse indices only. Furthermore, from the first line, the transverse trace vanishes.

\[ 0 = \Gamma^A_a E^a_A. \] (SS) (57)

The transverse trace vanishes also in QFT.
C The Gauss Identity

In QFT, the Gauss Identity

\[ \partial_i E_i^i + \epsilon_{IJK} A^J_m E^m_K \]  

(58)

can be broken into two parts,

\begin{align*}
0 &= \partial_i E_i^i + \epsilon_{IJK} \Gamma^J_m E^m_K, \\
0 &= \epsilon_{IJK} K^J_m E^m_K.
\end{align*}

(59)

The breakup is possible because the first line vanishes by itself: the covariant divergence of a density one triad vanishes, and involves no Christoffel symbols.

By this point it should be clear that in general a QFT identity involving \( \Gamma \) will hold also in small sine LQG, because the functions cotriad = cotriad \( (E_i^i) \) are given approximately by their QFT values. It is only necessary to replace derivatives by central differences.

In the plane wave case only the U(1) corresponding to rotations around the Z axis survives. Line one above becomes

\[ 0 = \delta_{(c)} E_Z^z + \epsilon_{ZAB} \Gamma^A_m E^m_B. \]  

(60)

This may be proved directly from eq. (54) by contracting with \( \epsilon_{IJZ} \) and using the QFT expressions for the cotriads. Note it is possible to replace

\[ \delta_{(c)} e_b^A e^C_a \rightarrow (1/2) \left( (\delta_{(c)} e_b^A) e^C_a + (\delta_{(c)} e^C_a) e_b^A \right) \]

\[ \cong (1/2) \delta_{(c)} (e_b^A e^C_a) \]  

(61)

because of the antisymmetry of the Levi-Civita tensors.

The second line of eq. (59) must be imposed as a constraint on the Hilbert space. It will turn out to be trivial, because of the assumption of a single polarization. That assumption implies the vanishing of all off diagonal elements, for both K and \( \hat{E} \) tensors.

V The Switch from N to \( \hat{N} \)

For plane waves, the boundary conditions require a shift from lapse \( N \) to a modified lapse

\[ \hat{N}(n) := (N E_z^z / \mid e \mid)(n) \]  

(62)
The lapse $N$ is a scalar under spatial diffeomorphisms. Therefore from eq. (62) $N$ is a rank one contravariant tensor. $N$ has no factors of $\Delta x^i$, but $\bar{N}$ has a factor $1/\Delta z$.

This shift in lapse generates a shift in the Hamiltonian, from $H$ to $\bar{H}$

$$NH = \bar{N} \bar{H};$$
$$\bar{H} = H | e | /E^z_Z.$$

$H$ has $1/|e|$ singularities, whereas $\bar{H}$ has a $1/E^z_Z$ singularity. The various cotriads are changed as follows.

$$N \cdots e^{xyZ} = (N | e | /E^z_Z) \cdots \epsilon^{ZAB} E^x_A E^y_B / |e|$$
$$= N \cdots \epsilon^{ZAB} E^x_A E^y_B / E^z_Z;$$
$$N \cdots \epsilon^{zaA} = N \cdots \epsilon^{zAB} E^a_B. \quad (63)$$

The $F_{xy} e^{xyZ}$ terms in $H_e$ have a $1/E^z_Z$ singularity in the classical limit. The Euclidean Hamiltonian becomes

$$NH_e \rightarrow \bar{N} \bar{H}_e = \sum_n N \{ F_{xy} e^{ZAB} E^x_A E^y_B / E^z_Z$$
$$+ F^{aA} \epsilon^{ZAB} E^a_B \} / \kappa + ST. \quad (64)$$

The field strengths are as at eq. (16).

A Regularization of the Cotriads

The switch to $\bar{N}$ changes the singularities of $H$, therefore requires a change in regularization. There are two possible ways to regulate the $1/E^z_Z$ singularity. The first approach uses the Thiemann prescription for the cotriad $e_z$, eq. (47).

$$e_z(n) := (-8/\kappa \gamma) h_z[-h^{-1}_z(n), |e|(n)] \quad \text{(LQG)}$$
$$\rightarrow (-8/\kappa \gamma) 1[-\hat{h}_z(n), |e|(n)] \quad \text{(SS)}$$
$$\rightarrow \sigma_Z (^{(2)}E / e)(n) \quad \text{(QFT )}$$
$$= \sigma_Z (^{(2)}E / e)(e/e)$$
$$= \sigma_Z e \text{ sgn}/E^z_Z. \quad (65)$$

Note that this regularizes only the $e/E^z_Z$ in

$$N = \bar{N} (|e| / E^z_Z).$$
the cotriads multiplying each field strength are not changed. One must include a second $[\hat{h}, e]$ commutator to regularize that cotriad, exactly as in the usual case.

Hinterleitner and Major [9] suggest an alternative regularization. After the replacement $N = \frac{N}{e/E_z}$, only the $F_{xy}$ terms in $H$ have a singularity in the classical limit, and it is a $1/E_z$ singularity, not a $1/e$ singularity. They replace the $1/E_z$ by two factors of

$$\left(\frac{8}{\kappa \gamma} h_z \right) \left[ h_z(n)^{-1}, \sqrt{sgn(z)E_z(n)} \right] \rightarrow \text{sgn}(z) \sigma / \sqrt{sgn(z)E_z}.$$ (66)

$\text{sgn}(z)$ is the sign of $E_z$.

Support for this regularization comes from the form of the unidirectional constraints, to be introduced in section X. When $H$ is rewritten as an expression quadratic in those constraints, one finds that every term in $H$, including the $F_{xa}$ terms, acquires a $1/E_z$ singularity. (The expression for $H$ uses constraints for both right- and left-moving waves, therefore is valid generally, even if one does not assume unidirectional waves.) One could argue that the volume singularity ($1/e$) is replaced by an area singularity ($1/E_z$) because of the reduction in spatial dimensions.

VI The Surface Term

I must treat the ST, before constructing the physical Hamiltonian $H_e$, because the ST modifies $H_e$ in two ways. Boundary conditions require the shift from the usual lapse $N$ to the lapse $\frac{N}{e}$, which has the simpler boundary conditions at spatial infinity. Also, the formula for $H_e$ is incomplete; I must add a surface term (ST). The $\delta_c (\hat{h}_a) e^{za}$ terms in $H_e$ contain a second derivative, since $\hat{h}_a$, when expressed in terms of tetrads, contains a time derivative. The $\delta_c$ must be integrated by parts onto the cotriad. The IBP brings the Hamiltonian (and Lagrangian) into a standard form with only first derivatives. The IBP generates a total derivative, which becomes a surface term $\hat{h}_a e^{za}$. This term must be canceled, which means a ST must be added to perform the cancelation: If

$$H_e \sim (\delta_c \hat{h}_a) e^{za} = (\delta_c \hat{h}_a e^{za}) - \hat{h}_a (\delta_c e^{za})$$

$$:= -\text{ST} - \hat{h}_a \delta_c e^{za},$$
then
\[ H_e + ST \sim -\hat{h}_a \delta(c) e^{\Delta a}; \text{ and no ST.} \]

The integration by parts shifts the difference onto the triads and changes the sign of a term in the Hamiltonian.

Although the surface term produces only a rather simple change in \( H_e \), the ST must be calculated in detail, because ST is the physical Hamiltonian. \( H_e \) is a constraint, and vanishes when acting on physical states. The ST does not vanish. When I construct a solution in the follow-on paper, I use the ST to compute the total energy of the wave.

From the preceding discussion, the surface term comes entirely from the \( \delta(c) \hat{h} \) terms in \( F_{za} \). I insert eq. (46) for \( F_{za} \) into eq. (64) for \( \tilde{H}_e \).

\[
-\mathcal{N} \tilde{H}_e + ST = \left( \frac{1}{2\kappa} \right) \sum_n \{ \cdots + \left( -2i \right) \left( \delta_c \hat{h}_a^c \right) E_B \epsilon^{ZBC} \mathcal{N}(n) \} + ST
\]

\[
= \left( \frac{1}{2\kappa} \right) \sum_n \{ \cdots - \left( -2i \right) \hat{h}_a^c \delta_c \left[ E_B \epsilon^{ZBC} \mathcal{N}(n) \right] + ST \}
\]

I have used eq. (22) to carry out the manipulation analogous to integration by parts in QFT. The \( \cdots \) denote terms which are derivative-free and do not contribute to the ST.

This is a good point to describe the labeling of the vertices at the surface. The \( \sum_n \) in the Hamiltonian ranges from \( n = \text{min} \) to \( n = \text{mx} \); \( \text{min} \leq n \leq \text{mx} \). However, the spin network itself extends to values \( n < \text{min} \) and \( n > \text{mx} \). Compare classical field theory, where one integrates the Lagrangian or Hamiltonian from \( \text{min} z \) to \( \text{mx} z \), but the space extends beyond these limits.

In principle, the limits (\( \text{min}, \text{mx} \)) can be chosen anywhere. In practice, the limits are chosen to lie in an asymptotic region, so that surface terms generated by integration by parts can be evaluated using boundary conditions. Similarly here, the only restriction on \( \text{min} \) and \( \text{mx} \) is that the system is asymptotic at those values of \( n \); but the spin network does not vanish beyond those limits.

In particular, suppose the construction of the spin connection
predicts that it depends on a central difference
\[
[f(n+1) - f(n-1)]/2.
\]
At \(n = mx\), \(f(n+1)\) is \(f(mx + 1)\). I use \(f(mx + 1)\); I do not assume this quantity is zero.

Since the \(\delta(c)\) connects every other vertex, the total derivative on the last line, eq. (67), gives rise to two surface terms, one from even \(n\) terms and one from odd \(n\). \(\tilde{H}_e\) becomes

\[
-k(N\tilde{H}_e + ST) = \sum_n \{ \cdots - (-2i)\hat{h}_a^C \delta_c \left[ E^a_B \epsilon^{ZBC} N \right](n) \}
\]

\[
+ \left[ N(n)(-2i)\hat{h}_a^C E^a_B \epsilon^{ZBC} \right](n) \left[ \frac{mx}{n_{min}} + \frac{mx+1}{n_{min-1}} \right] (1/2) + ST. \tag{68}
\]

The ST is now chosen so that the last line vanishes.

The 1/2 in the surface term comes from the 1/2 in the central difference. For example, use the definition of central difference, eq. (13), to expand each term in the sum

\[
\sum_{-1}^{+1} \delta(c) f(n) = (1/2) [ f(+2) - f(-2) + f(+1) - f(-1) ].
\]

It is possible to eliminate the holonomy from the surface term. Replace the \((-2i)\hat{h}_a^C\) by \((\gamma K + \Gamma)^C_a\) (eq. (51)). The term involving \(K\),

\[
K^C_a E^a_B \epsilon^{ZBC},
\]

is (one half of) the Gauss constraint, eq. (59), and may be dropped. The term involving \(\Gamma\) may be simplified by using the other half of the constraint.

\[
\Gamma^C_a E^a_B \epsilon^{BC} = \delta(c) E^Z_c.
\]

The surface term is then

\[
ST = -N \delta(c) E^Z_c(n) \left[ \frac{mx}{n_{min}} + \frac{mx+1}{n_{min-1}} \right] (1/2). \tag{69}
\]

VII The Hamiltonian H

In this section I construct \(H\), using eq. (50). At the 3+1 level, the LQG construction parallels a corresponding classical general relativity (CGR) construction:
\[-H_e \sim -|e| R(\Gamma) - \gamma^2 (K^i K^j) \epsilon^{ijk} \epsilon_{ijk} \text{ terms};\]

\[H + H_e \sim (1 + \gamma^2) (K^i K^j) \epsilon^{ijk} \epsilon_{ijk} \text{ terms};\]

\[H = (H + H_e) - H_e \sim -|e| R(\Gamma) + (K^i K^j) \epsilon^{ijk} \epsilon_{ijk} \text{ terms}. \quad \text{(CGR)} \] (70)

\(R\) is the three-dimensional scalar curvature. In \(H_e\), on the first line, I have replaced the connection \(A\) by \(\gamma K + \Gamma\), then expanded out the resultant expression. Evidently in CGR, adding in the \(H + H_e\) term merely changes the coefficient of the \(K^2\) term in \(H_e\).

This will happen also in the SS case, provided I use the same cotriads in both \(H_e\) and \(H_e + H\). If \(H_e\) contains \(\tilde{E}\) which grasp at two faces, then the \(H + H_e\) should do likewise.

\[H + H_e = (1 + \gamma^2) \sum_n N(n) [-K^i K^j \epsilon^{ijk} \epsilon_{ijk}](n)/2\kappa; \quad \text{(71)}\]

\[-H_e + ST = \sum_n N(n) [F^K_{ij} \epsilon^{ijk}]/2\kappa\gamma + ST. \quad \text{(72)}\]

The final Hamiltonian contains three variables: \(K_i, \dot{h}_i\), and \(\epsilon^{ijk}\). They are not independent, and I must decide which variable to eliminate. From eq. (51), I can eliminate either \(K_i\) or \(\dot{h}_i\). Either choice introduces a new, and complicated field, the \(\Gamma_i\).

There is no way of avoiding the \(\Gamma_i\). However, once I specialize to singly polarized waves (which I do in the next section) it will not matter whether I eliminate \(K\) or \(\dot{h}\). The singly polarized \(K\) will be fully diagonal (not just block diagonal), while \(\Gamma\) will be fully off-diagonal. For singly polarized waves the only non-vanishing components are

\[K^X_x, K^Y_y, K^Z_z, \Gamma^X_x, \Gamma^Y_y, \Gamma^Z_z \neq 0. \quad \text{(73)}\]

In this situation, \(K_i\) is just \(-2i\dot{h}_i/\gamma\). Whether one eliminates \(\dot{h}\) or the \(K\)'s is largely a matter of taste or convenience.

I choose to eliminate the \(\dot{h}\). Here I am guided in part by a desire to parallel the classical calculation, which of course has no \(\dot{h}\).

\section{Eliminating Linear in \(K_i\) Terms}

If one replaces \(-2i\dot{h} \rightarrow \gamma K + \Gamma\) in \(H_e\), the result contains \(K^2, K \Gamma,\) and \(\Gamma^2\) terms. The linear in \(K\) terms should cancel because of the
I focus on $-H_e$, since the $H+H_e$ terms are order $K^2$ and contribute no linear in $K$ terms. From eq. (46)

$$-\kappa H_e = \sum_n N(n)(-2i)^2(1/2)\hat{h}^I_m \hat{h}^J_n \epsilon_{IJK} e^{mnK}$$

$$-(-2i)(\hat{h}_a^C) \delta_c (e^{zaD} N) \epsilon_{ZAD}$$

$$\to \sum_n N(1/2)[\gamma K^I_m, \Gamma^J_n + \Gamma^I_m \gamma K^J_n] \epsilon_{IJK} e^{mnK}$$

$$-\gamma K^C_a e \delta_c (e^{zaD} N) \epsilon_{ZAD}. \tag{74}$$

On the third line I drop all but linear in $K$ terms. I have switched from $N$ back to $N$; the cancelation is not special to the plane wave case. The extra factor of $(1/2)$ must be included whenever the $m,n$ sums include, e.g., both $x,y$ and $y,x$. On the third line, I commute the $\Gamma^I_m$ to the far right, neglecting the commutator. (When I shift to single polarization, in a following section, the entire third line vanishes; it is not necessary to assume that a commutator is negligible.) The two terms on this line are equal, canceling the $1/2$. The linear in $K$ terms then cancel if

$$0 = -K^C_a \delta_c (E^a_Z E^a_B \epsilon^{ZBC} N/ | e |)$$

$$+ K^I_m \epsilon_{IJK} \Gamma^J_n \epsilon_{MNK} E^m_M E^n_N / | e |. \tag{75}$$

The $N$ can be taken out of the first square bracket, provided the following expression vanishes.

$$K^C_a (E^a_Z E^a_B \epsilon^{ZBC} / | e |) \delta_c N.$$

In order for it to vanish, I must impose the constraint

$$K^C_a E^a_B \epsilon^{ZBC} = 0.$$

This is just one half of the Gauss constraint, eq. (59). Now replace

$$E^a_Z E^a_B \epsilon^{ZBC} / | e | = -\epsilon_{rC} \epsilon^{raz};$$

$$\epsilon_{MNK} E^m_M E^n_N / | e | \to e^K_q \epsilon^{qmn}. \tag{76}$$

The sums over transverse C and a, first line of eq. (75), may be extended to longitudinal indices ($C,a \to I,m$) because the $\epsilon^{raz}$ causes the $m = z$ contribution to cancel. The result (after slight relabeling of indices) is just the vanishing combination given at eq. (55).
B  H Without Linear-in-K Terms

Since linear in K terms have disappeared, H becomes a sum of terms quadratic in K, plus terms involving only \( \Gamma \). I write the holonomies in \( H_e \), eq. (46), as a sum of K plus \( \Gamma \) terms. As anticipated at eq. (50), adding \( H + H_e \) to \(-H_e\) changes only the coefficient of the \( K^2 \) terms.

\[
\sum_n [1/\kappa] \{-K_x^I K_y^J \epsilon_{IJK} \epsilon^{KMN} E_M^x E_N^y \sum(n)/E_Z^z \\
-K_x^Z K_a^A E_A^a \sum(n) \\
+\Gamma_x^I \Gamma_y^J \epsilon_{IJK} E_M^x E_N^y \epsilon^{MNK} \sum(n)/E_Z^z \\
+\Gamma_z^Z \Gamma_a^A E_A^a \sum(n) \\
-\Gamma_a^A \epsilon_{BA} \delta(c) [\sum \sum_B(n)] \}.
\]  

(77)

VIII  Single Polarization Constraints

I now impose the constraints

\[
E_y^x = E_x^y = 0,
\]

(78)
as well as the consistency conditions

\[
[H + ST, E_x^x] = [H + ST, E_x^y] = 0.
\]

(79)

For brevity I will refer to these two conditions simply as the "single polarization" constraints; but they not only specialize to a single polarization; they also fix the U(1) gauge. If I wanted to specialize to single polarization without fixing the U(1) gauge, I could impose \( E_I^x E_I^y = 0 \).

A  Consistency Conditions I

This section proves that the consistency conditions eq. (79) are satisfied if

\[
K_y^x = K_x^y = 0.
\]

(80)

When proving consistency, one cannot impose the constraints eq. (78) until after all commutations have been carried out. However, if the Hamiltonian contains a term which is a product of two tensors, both of which vanish because of the constraint, then it is permissible to
drop this term before commuting. If A and B both vanish because of the constraint, then

\[
[\tilde{E}, H = AB] = A[\tilde{E}, B] + [\tilde{E}, A] B.
\]

Both terms vanish after the commutation.

It is therefore desirable to inquire what tensors vanish because of the constraints eq. (78). Since the leading term in the SS expansion of the \(e^{ijk}\) is the QFT value \(\propto E^i_M E^j_N \epsilon^{MNK}\), certain components of \(e^{ijk}\) with two indices equal contain at least one off-diagonal triad and must vanish:

\[
e^{xzY} = e^{zxX} = 0;
\]
equivalently,\[
e^Y = e_X = e^Y = 0 \text{ (single pol.)} \quad (81)
\]
(In fact the remaining cotriads \(e^{ijk}\) with two equal indices also vanish, because of the block diagonal nature of the triads.)

The single polarization constraints also affect the spin connection. The \(e^{ab}\) in eq. (55) for \(\Gamma^Z_z E^z_Z\) forces at least one of the triads \(e^a_a, e^A_A\) to be off-diagonal. An \(e^{rmz}\) in eq. (54) forces \(\Gamma^I_j E^j_j\) to vanish when \(I = J\). As a result, \(\Gamma^I_i\) loses its diagonal elements.

\[
\Gamma^Z_z E^z_Z = 0. \text{ (single pol.)} \quad (82)
\]

\[
\Gamma^X_x E^x_X = \Gamma^Y_y E^y_Y = 0. \text{ (single pol.)} \quad (83)
\]

The above results could be left in \(\Gamma^I_i E^i_i\) form because in H \(\Gamma\) always occurs multiplied by an \(\tilde{E}\). However, it is also possible to strip off the \(E^i_i\). The volume determinant in single polarization limit is now a single term,

\[
\sqrt{\text{sgn } E^X_x E^Y_y E^Z_z},
\]

Off diagonal components can be dropped because they occur squared. In SS approximation \(e \neq 0\); therefore diagonal \(E^i_i\) do not vanish and can be stripped off.

The basic \([\tilde{E}, \hat{h}]\) commutator collapses to \([\tilde{E}, \hat{K}]\), because the \(\Gamma\) are constructed from triads which are independent of \(\sin\) to order \(\sin^2\).

\[
[\tilde{E}^i_j, -2i\hat{h}_i] = [\tilde{E}^i_j, \gamma K_i + \Gamma_i] = \gamma [\tilde{E}^i_j, K_i]. \text{ (SS)} \quad (84)
\]
Therefore the only fields in $\tilde{H}$ not commuting with off-diagonal $\tilde{E}$ are the off-diagonal $K^C_C$, $C \neq d$. The terms in $\tilde{H}$, eq. (77), which contain the off-diagonal $K^Y_x$ (for example) are

$$\tilde{H} \sim K^I_i K^J_j E^I_M E^J_N \epsilon_{IJK} \epsilon^{MNK} \sum_n E^z\nu_e \sum_n$$

The commutator eq. (84) with $E^Y_x$ removes the $K^Y_x$ and leaves an expression with either one factor of $E^Y_x$ or one factor of $K^X_y$. □

B Consistency Conditions II

I must now check that the commutators of eq. (80) with $H$ vanish. I can simplify $H$ by dropping all terms which are a product of two vanishing $K$'s or $\tilde{E}$. Eq. (77) for $\tilde{H}$ simplifies to

$$\sum_n [1/\kappa] \{ -K^X_x E^X_x E^y_Y N(n)/E^z\nu_e \sum_n$$

In the sums over $a$, all off-diagonal $E^A_a$ drop out, because multiplied by an off diagonal $K$ or on diagonal $\Gamma$.

On the last line I have retained $N$, temporarily, because it is multiplied by a cotriad $e^zaA$ which will be easier to manipulate in a later step. Note off diagonal $K$’s commute with both $\sum_n$ and $N$, even though the two differ by a factor of $e$, because off-diagonal $\tilde{E}$ occur quadratically in $e$ and have been dropped.

On the last line, I use the distributive property of the difference, eq. (21), to separate off a $\delta(c)N$ term. The term not involving $\delta(c)N$ can be rewritten using eq. (55). The last line becomes

$$-\Gamma^A_a(n) e^{zaA}(\delta(c) N) + 2 N \Gamma^Y_x \Gamma^X_y e^{zyZ}(n). \quad (87)$$

If I insert this result back into eq. (136), I change the sign of the
\[ \Gamma_x^Y \Gamma_y^X \text{ term in that equation.} \]

\[ \mathcal{N} \dot{H} + ST = \sum_n \left( \frac{1}{\kappa} \right) \left\{ -K_A^x K_y^B (2) \dot{\mathcal{E}} \ \mathcal{N}(n) / E_z^x - K_z^n(n) K_a^A E_a^x(n) \mathcal{N}(n) + (\Gamma_x^Y \Gamma_y^X)(n) E_x^x E_y^y \mathcal{N}(n) / E_z^y - \Gamma_a^A(n) \epsilon_B^a E_B^b(n) E_z^z \delta_{(c)} N / |e| \right\}. \] (88)

To prove consistency of the new constraints eq. (80), I must compute their commutators with the \( e^{ijK} \) and \( \Gamma_i \) factors in \( H \). As remarked previously, all off diagonal \( \mathcal{E} \) have dropped out; therefore the \( e^{ijK} \) contain only on diagonal \( \mathcal{E} \) and commute with the \( K \)'s.

Next consider commutators with \( \Gamma_x^X \) or \( \Gamma_y^Y \). From eq. (54) with \( I \neq J \), \( \Gamma_j^I E_j^I \) is a sum of terms with no off diagonal cotriads, plus terms with two off diagonal cotriads. The latter can be dropped. The resulting expression for \( \Gamma_j^I E_j^I \) commutes with the off diagonal \( K \)'s. Note that in the Hamiltonian the off diagonal \( \Gamma \) are always paired with an \( \mathcal{E} \), so that in the foregoing argument I am allowed to use the \( \Gamma_j^I E_j^I \), rather than \( \Gamma_j^I \). \( \Box \)

**IX The Vector Constraint \( H_z \)**

Since the wavefunctional to be constructed is not based on closed loops, the diffeomorphism constraint is not satisfied automatically. It must be treated as an additional constraint and imposed on the state after quantization, in the manner of Dirac.

The classical constraint is

\[ \mathcal{N}^z H_z = \left( \frac{1}{\kappa \gamma} \right) \int d^3 x N^z F_{2a}^A E_a^x. \text{(QFT)} \] (89)

To obtain the SS version, I follow the procedure used previously to construct the Hamiltonian. One can start with \( E^a(\eta_a) \) values which grasp only at surfaces internal to the area around which \( F \) circulates; but, in this case as for the Hamiltonian, the \( E^a(\eta_a) \) are replaced by \( E^a(+) + E^a(-) \) in the slow variation limit.
\[ \kappa \gamma N^z H_z = \sum_n N^z (Tr/2) \{ (-2i) \delta_{(c)} \hat{h}_a(n) E^a(n) \}
\]
\[ -(-2i)[\hat{h}_z(n, n-1), \hat{h}_a(n-1)] E^a(n) \]
\[ -(-2i)[\hat{h}_z(n, n+1), \hat{h}_a(n+1)] E^a(n) \}
\[ = \sum_n N^z (Tr/2) \{ (-2i) \delta_{(c)} \hat{h}_a(n) E^a(n) \}
\]
\[ -(-2i)[\hat{h}_z, \hat{h}_a(n)] E^a(n) \}. \] (90)

On the first three lines I still have both field strengths, the one going forward to \( n+1 \), and the one going backward to \( n-1 \). On the last line I have used the slow variation assumption to replace
\[ \hat{h}_z(n, n \pm 1) \rightarrow \hat{h}_z(n) = [\hat{h}_z(n, n + 1) + \hat{h}_z(n, n - 1)]/2. \]

All these manipulations are exactly as for \( H_z \), section III.

As in the QFT case, one must add in a term proportional to the Gauss constraint to make \( H_z \) into the generator of \( z \) diffeomorphisms. The fourth line of eq. (134) equals
\[ (-2i)^2 \hat{h}_z^Z \epsilon_{AB} \hat{h}_a^A(n) E_B^a(n). \] (91)

We replace \((-2i) \hat{h} \rightarrow \gamma K + \Gamma\). From Gauss, eq. (59), the term involving \( K \) vanishes, and the term involving \( \Gamma \) equals one half of the Gauss constraint, eq. (59). Eq. (134) becomes
\[ \kappa \gamma N^z H_z = \sum_n N^z \{ (-2i) \delta_{(c)} \hat{h}_a^A(n) E^a_A(n) \}
\[ -(-2i) \hat{h}_z(n) \delta_{(c)} E^z_Z \}. \] (92)

For simplicity I will refer to this combination as "\( H_z \)", even though it contains some Gauss.

**X Unidirectional and Diffeomorphism Constraints**

In this section I fix the spatial diffeomorphism gauge, removing the freedom to transform the \( z \) variable; and I specialize to unidirectional waves moving only in the positive \( z \) direction.

The two sets of constraints should be considered at the same time. There are three unidirectional constraints and two diffeomorphism
constraints. However, these five constraints are not independent. A linear combination of unidirectional constraints equals a linear combination of the diffeomorphism constraints. There are only four constraints total, as one sees if all constraints are considered simultaneously.

A Are Dirac Brackets Necessary?

Since unidirectional constraints typically are second class, it is necessary to replace Poisson by Dirac brackets. Dirac brackets often are not pretty. Are there ways of avoiding the introduction of Dirac brackets?

At the classical level, one of the unidirectional constraints is satisfied, if the remaining two unidirectional constraints plus scalar and vector constraints are satisfied [9]. Also, one can write one of the unidirectional constraints as a linear combination of the diffeomorphism constraints. This suggests only one unidirectional constraint survives, so it must be first class (commute with itself).

Eliminating a unidirectional constraint by writing it as a combination of diffeomorphism constraints does simplify the calculations, but does not eliminate completely the need for Dirac brackets. Even if only one unidirectional constraint survives, it does not commute with itself. The generic unidirectional constraint has the form $U = \pi + \delta(c) q = 0$ (time derivative plus $z$ derivative vanishes). The generic commutator is

$$[U(z_1), U(z_2)] = [\pi + \partial_{z_1} q(z_1), \pi + \partial_{z_2} q(z_2)]$$

$$= (-i\hbar \partial_{z_2} \delta(z_1 - z_2) - \partial_{z_1} \delta(z_1 - z_2)]$$

$$= -2i \hbar \partial_{z_2} \delta(z_1 - z_2).$$

I have done the calculation in QFT for the convenience of the reader, but the result in LQG is similar: replace

$$z_1, z_2 \rightarrow n_1, n_2;$$

$$\partial_{z_2} \delta(z_1 - z_2) \rightarrow [\delta(n_1, n_2 + 1) - \delta(n_1, n_2 - 1)]/2$$

$$:= \delta(c)(n_2) \delta(n_1, n_2).$$

The derivative of a Dirac delta has been replaced by the difference of a Kronecker delta. The commutator of a unidirectional constraint with itself does not vanish. The discrete version of the commutator suggests a reason for this behavior: the $\delta(c) q$ term is non-local.
The non-linearity of CGR has nothing to do with the foregoing result. Even in the simplest, linear field theory (real scalar free field) a unidirectional constraint will be second class. It is not possible to avoid Dirac brackets.

Strictly speaking, one should calculate Dirac brackets every time one eliminates variables. However, in the special case that the variables being eliminated are a \((\pi, q)\) pair, the Dirac brackets for the remaining \((\pi, q)\) pairs are trivial, equal to their Poisson brackets. It is not necessary to calculate Dirac brackets when specializing to single polarization, or when fixing the diffeomorphism gauge, below.

When the Dirac brackets are trivial, one can eliminate the \((\pi, q)\) pair in classical general relativity (CGR), before quantization. If a constraint is second-class, with non-trivial Poisson brackets, these must be calculated, and the constraint eliminated, before quantization. Except for occasional comments on regularization, the rest of this section is therefore a temporary return to CGR. Occasionally I may use the phrase "commutes with" when I should say, "has zero (Dirac or Poisson) bracket with".

B The Unidirectional Operator

In conventional wave theory a solution is unidirectional if all fields depend only on \(z - ct\). In general relativity those coordinates are arbitrary, and I must use local free-fall coordinates \(Z - cT\) instead.

In terms of derivatives
\[
\partial_U = \partial_Z - \partial_T; \\
\partial_V = \partial_Z + \partial_T.
\]

The constraint (no \(V\) dependence) is \((\partial_Z + \partial_T) = 0\).

This constraint can be rewritten in terms of \((z,t)\) derivatives.

\[
0 = (\partial_Z + \partial_T) f(Z - cT) \\
= (e^Z_Z \partial_z + e^Z_T \partial_t + e^T_Z \partial_z + e^T_T \partial_t) f \\
= (e^Z_Z \partial_z + 0 + (-N^Z/N) \partial_z + (1/N) \partial_t) f.
\]

I have invoked the usual gauge which fixes the Lorentz boosts and reduces the full Lorentz group to \(SU(2)\): \(e^{i}_{X,Y,Z} = 0\).

I now replace the derivatives \(\partial_z\) and \(\partial_t\) in eq. (95) by Poisson brackets with \(H_z\) and \(\sum H(z) + ST + N^z H_z\) respectively.
\[ 0 = \left[ e^Z_N - N^Z/N(z) \right] \{ f, H_z(z) \} \]
\[ + \left( 1/N \right) \{ f, \left( \bar{N} \bar{H}(z) + ST \right) + N^z H_z(z) \} \]
\[ \propto \left( NE_Z^Z / | e | \right) \{ f, H_z(z) \} + \{ f, \left( \bar{N} \bar{H}(z) + ST \right) \} \]
\[ = \{ f, \bar{N} H_z(z) + (\bar{N} \bar{H}(z) + ST) \}. \] (96)

On the third line I have multiplied through by \( N \). Here I anticipate a later result: given the unidirectional constraints and the diffeomorphism gauge choice, \( N \) cannot vanish.

Strictly speaking \( H_z \) is not the \( z \) derivative operator unless its Lagrange multiplier \( \bar{N} \) is a constant. If \( \bar{N} \) is not a constant, \( \{ f, H_z \} \) does take the \( z \) derivative of \( f \); but it also generates gauge transformations proportional to \( \partial_z \bar{N} \). In the present case this is not a problem; once unidirectional and diffeomorphism constraints are imposed, \( \bar{N} \) will turn out to be a constant.

I write out the constraint eq. (96) explicitly, using eqs. (92) and (88) for \( H_z \) and \( \tilde{H} \).

\[ \bar{N} H_z(z) + \bar{N} \bar{H}(z) + ST = \left( 1/\kappa \right) \sum_n \bar{N} \left\{ (-2i) \delta^{(c)} \hat{h}_{\alpha}^A(n) E_{\alpha}^a(n) \right. \]
\[ \left. -(-2i) \hat{h}_{\alpha}^A(n) \delta^{(c)} E_Z^Z \right\} \]
\[ + \sum_n \left( 1/\kappa \right) \left\{ -K_x^X K_y^Y E_X^y \bar{N} (n)/E_Z^Z \right. \]
\[ \left. -K_X^X(n) K_Y^Y(n) E_{\alpha}^a(n) \bar{N} (n) \right\} + (\Gamma_x^X \Gamma_y^Y)(n) \bar{N} (n)^{2\bar{E}} (n)/E_Z^Z \]
\[ \left. -\Gamma_{\alpha}^A(n) e^{\alpha A(n)} \delta^{(c)} N \right\}. \] (97)

C The Unidirectional Constraints

In a unidirectional theory, eq. (97) commutes with every dynamical variable. Therefore I can construct unidirectional constraints if I commute eq. (97) with any set of independent functions \( f_i \), then set the commutators equal to zero. I choose the \( f_i \) to be three independent functions of the three triads: \( E_Z^Z \), \( ^{2\bar{E}} \), and \( \ln[E_Y^y/E_X^x] \).
Commutation of these three yields the constraints

\[
0 = \{ K^A_a E_a^a + \delta(c) E_Z^a \}/\sqrt{E_Z} := U_1;
\]

\[
0 = \{ K^A_a E_a^a + 2 K^Z Z^a \delta(c) E_Z^a \} /
\]

\[
E_Z \{ (2) E / (2) \tilde{E} + 2 E_Z \delta(c) N \}/\sqrt{E_Z} := U_2;
\]

\[
0 = \{ K^Y_y E_y^y - K^X_x E_X^x \}
\]

\[
E_Z \{ \delta(c) E_Y^y / E_Y^y - \delta(c) E_X^x / E_X^x \} /
\]

\[
\sqrt{E_Z} := U_3. \quad (98)
\]

The appearance of \( \delta(c) N \) in the second constraint may be a bit surprising. This comes from a bracket

\[
\{ N (2) \delta(c) \hat{h} E, (2) \tilde{E} \} = \sum_n \{ (2) \delta(c) \hat{h} A(n) E_A^a(n), (2) \tilde{E} (m) \}
\]

\[
= -\delta(c) (N E_Y^y) E_X^x + (x \leftrightarrow y, X \leftrightarrow Y),
\]

followed by multiplication by \( \sqrt{E_Z}/N \{ (2) \tilde{E} \}. \) In effect, the \( \delta(c) \) has been integrated by parts off the holonomy and onto the \( N \) and \( \tilde{E} \). Similarly, an integration by parts produces the \( \delta(c) \tilde{E} / E \) terms in the third constraint; in that case a \( \delta(c) N \) term cancels out.

The factors of \( 1/\sqrt{E_Z} \) have been added to split up the \( 1/E_Z \) singularity into two parts. (The Hamiltonian is quadratic in the unidirectional constraints.)

These constraints have the right form. The K dependent terms represent time derivatives; the \( \delta(c) \) terms the corresponding space derivatives. The \( (\delta(c) N)/N \) term on the second line may not seem to fit the expected pattern; however, I show in the next section (on diffeomorphism gauge fixing) that this term is required if the second line is to transform as a covariant tensor under spatial diffeomorphisms.

D The Diffeomorphism Constraint

In both QFT and LQG the usual gauge choice which fixes the Lorentz boosts, reducing the full Lorentz group to SU(2), is

\[
\epsilon_{X,Y,Z}^t = 0 = \epsilon_{T}^{x,y,z}. \quad (99)
\]

This gauge still allows transformations

\[
t' = t' (t); \; z' = z' (z, t) \quad (100)
\]
The transverse triads vary with this change in the z coordinate, despite their lack of an explicit z index, because $e$, the volume factor, contains an implicit z subscript. Conversely, the longitudinal triad (has an explicit z index but) does not vary.

$$ E^a_A \propto |e| = \text{sgn} \left( e^Z_z \right) \left( (2) e \right); $$

$$ E^z_Z = |e| e^z_Z = \text{sgn} \left( (2) e \right); $$

$$ N = N E^z_Z / |e| = N e^z_Z. \quad (101) $$

$(2) e$ is the determinant of the 2x2 transverse cotriad matrix, an invariant. Therefore $E^z_Z$ is a scalar, while the $E^a_A$ are rank one covariant tensors. In QFT, therefore, a gauge fixing constraint must involve at least some transverse triads.

Note the last line: the usual lapse $N$ is a scalar, but the new lapse $N$ is a contravariant tensor. Both differences, $\delta_{(c)} \left( \tilde{E} \right) / \left( 2 \tilde{E} \right)$ and $2 \delta_{(c)} N / N$ in eq. (98), have inhomogeneous terms in their diffeomorphism transformation laws. The inhomogeneous terms cancel out in the sum. The quantity

$$ \delta_{(c)} \left( \tilde{E} \right) / \left( 2 \tilde{E} \right) + 2 \delta_{(c)} N / N $$

transforms like a tensor.

I use a gauge fixing function constructed from the two simplest triad functions which are U(1) scalars, $(2) \tilde{E}$ and $E^z_Z$.

$$ 0 = \ln \left[ \frac{(2) \tilde{E}}{(C E^z_Z)^{p+1/2}} \right] := D_1; \quad 0 = 2 K_z E^z + K_a E^a / 2 - p K_a E^a := D_2. \quad (102) $$

$(2) \tilde{E}$ is the determinant of the 2x2 transverse triad matrix. Eq. (102) is a family of gauge choices, depending on a parameter p. In $D_1$ I use $p + 1/2$, rather than p, because at a later point the value $p = 0$ will prove to be special. C is a constant. The second line is the consistency condition, the result of demanding $\{ \tilde{H}, D_1 \} = 0$.

Eq. (102) is by no means the only way of fixing the diffeomorphism gauge. However, it is the simplest. More complex choices found in the literature appear to require advance knowledge of the form of the solution.

On the first line, I chose a gauge choice involving a logarithm, rather than the simpler choice

$$ (2) \tilde{E} - (C E^z_Z)^{p+1/2} = 0. \quad (103) $$
To see the reason for this, recall fixing the diffeomorphism gauge is equivalent to first transforming to new canonical coordinates \((\pi, q)\), then discarding one pair of \((\pi, q)\)'s. Write

\[
E^i_t (dK^I_t / dt) = -dE^i_t / dt K^I_t + \text{total derivative},
\]

then expand:

\[
-K^i_t dE^i_t / dt = -K^i_t E^i d(\ln E^i) / dt \\
\quad -K^2 E^2 d(\ln E^2) / dt - (1/2)K^a E^a d(\ln E^a E^a) / dt \\
\quad - (1/2)(K^y E^y - K^x E^x) d[\ln (E^y / E^x)] / dt \\
\quad + \{(2 K^z E^z + K^a E^a) / 2 - \rho K^a E^a\} d[\ln (2 E^0 / E^0)] / dt.
\]

The fourth and fifth lines are \(D_2\) times the derivative of \(D_1\). One can drop this \((\pi, q)\) pair completely from the theory, without altering the canonical brackets of the other \((\pi, q)\) pairs. The constant \(C\) of eq. (103) arises as a constant of integration. Note the special case \(p = 0\) has a singularity.

The logarithm is not a problem in slow variation approximation, since

\[
\delta_c \ln f(n) = \ln[f(n - 1) + \delta_c f(n)] - \ln f(n - 1)] / 2 \\
\quad = \ln f(n - 1) + \ln[1 + \delta_c f(n) / f(n - 1)] - \ln f(n - 1)] / 2 \\
\quad = \delta_c f(n) / f(n - 1) + \text{order } (\delta f / f)^2 \\
\quad = \delta_c f(n) / f(n) + \text{order } (\delta f / f)^2.
\]

Near \(f = 0\) presumably one would have to define the logarithm by an infinite series,

\[
\ln f(m) = \delta_c f(m_0) / f(m_0) + \delta_c f(m_0 + 2) / f(m_0 + 2) + \cdots + \delta_c f(m) / f(m),
\]

then regulate the terms with \(f(k) \ll 0\).

The popular choice for \(C\) and \(p\), in the classical literature, is \(C = \text{sgn}, p = 1/2\), which implies \(g_{zz} = 1\). (\(\text{sgn}\) is the sign of e.) I leave these constants undetermined until the succeeding paper.

The case \(p = 0\) clearly requires a special discussion. Because the classical literature favors the gauge choice \(p = 1/2\), presumably I will not need the \(p = 0\) case. I do not discuss it.
For \( p \neq 0 \), I can solve eq. (102) or eq. (103) to eliminate one \((p,q)\) pair. The surviving \((\pi,q)\) pair is

\[
\Pi = \frac{2 K_z E^z + K_a E^a / 2 + p K_a E^a}{4p};
\]

\[
Q = \ln (2 \tilde{E}) + (p - 1/2) \ln (C E^z_Z).
\]  \hspace{1cm} (106)

This is the pair indicated schematically by \((p \rightarrow -p)\) in eq. (104). I absorb the factor \(1/4p\) of that equation into the \(\Pi\) (an arbitrary choice); this gives bracket \(\{\Pi, Q\}\) the correct norm. Then

\[
K^A_a E^a_A = \frac{(4p \Pi - D_2)}{2p} \rightarrow 2 \Pi;
\]

\[
2K_z E^z_Z + K^a_a E^a_A / 2 = \frac{(4p, \Pi + D_2)}{2p} \rightarrow 2p \Pi;
\]

\[
2K_z E^z_Z \rightarrow (2p - 1) \Pi.
\]  \hspace{1cm} (107)

Similarly for the triads,

\[
\ln (2 \tilde{E}) = \frac{Q(p + 1/2)}{2p} + D_1(p - 1/2)/2p \rightarrow Q(p + 1/2)/2p;
\]

\[
\ln (C E^z_Z) = \frac{(Q - D_1)}{2p} \rightarrow Q/2p;
\]

\[
\ln (2 \tilde{E} / C E^z_Z) \rightarrow Q(p - 1/2)/2p.
\]  \hspace{1cm} (108)

For example, I can write a typical term in \(\tilde{H}\) as

\[
\cdots K^A_a E^a_A / E^z_Z = \cdots [(4p \Pi - D_2)/2p] C \exp[(-Q + D_1)/2p].
\]

After all \(D_i\) are moved to the right (they are a \((\pi,q)\) pair which commutes with all other \((\pi,q)\) pairs) the constraints \(D_i = 0\) can be imposed on the wavefunction, and the \(D_i\) dropped from the Hamiltonian.

The triads must be written as exponentials of \(Q\). Since this is awkward, I will not eliminate the \(z\) triads, but will use the solution to eq. (108) when computing brackets.

\[
E^z_Z = (1/C) \exp (Q/2p);
\]

\[
\{\Pi, E^z_Z\} = (-1/2p) E^z_Z.
\]  \hspace{1cm} (109)

\textit{Differences} involving triads are relatively easy to eliminate:

\[
\delta_{(c)} (2 \tilde{E} / (2 \tilde{E})) \rightarrow \delta_{(c)} Q(p + 1/2)/2p;
\]

\[
\delta_{(c)} E^z_Z / E^z_Z \rightarrow \delta_{(c)} Q/2p.
\]  \hspace{1cm} (110)

When the unidirectional constraints are reexpressed in terms of these \((\pi,q)\) pairs, the two constraints \(U_1, U_2\) in eq. (98) collapse
to the same constraint, except $U_2$ has an extra term proportional to $(\delta(c)N)/N$. Therefore that expression must vanish. In place of eq. (98) one gets

$$0 = \{\delta(c)N\}/N;$$

$$0 = \{K^A_a E^a_A + \delta(c) E^z_Z\}/\sqrt{E^z_Z} = \{2\Pi + \delta(c) E^z_Z\}/\sqrt{E^z_Z} = U_1;$$

$$0 = \{K^Y_y E^y_Y - K^X_x E^x_X \}
- \delta(c) E^x_Y/E^y_Y \{\delta(c) E^z_X/E^x_X \}/\sqrt{E^z_Z} = U_3. \quad (111)$$

Because of the conformal boundary conditions, $N$ must equal unity.

E Dirac Brackets

The diffeomorphism constraints have been eliminated and we are left with two unidirectional constraints, eq. (111). They are still second class. The Dirac bracket matrix for the two surviving constraints is (rows and columns in order $U_1, U_3)$

$$\{U_i(m), U_j(n)\} = \left\{ \begin{array}{cc}
-8p_A & C \\
-C & +4A
\end{array} \right\} \quad (112)$$

$$\mathcal{A} = \delta(c) (m, n) := (\delta(m, n + 1) - \delta(m, n - 1))/2;$$

$$\mathcal{C} = 2 \left[ \delta(c) E^y_Y/E^y_Y - \delta(c) E^x_X/E^x_X \right] \delta(m, n).$$

The inverse bracket matrix is

$$\{U_j(n), U_k(r)\}^{-1} = \left\{ \begin{array}{cc}
\mathcal{K}^{-1} & -\mathcal{K}^{-1}\mathcal{C}\mathcal{A}^{-1}/4 \\
\mathcal{A}^{-1}\mathcal{K}^{-1}/4, & \mathcal{A}^{-1}/4 - \mathcal{A}^{-1}\mathcal{C}\mathcal{K}^{-1}\mathcal{C}\mathcal{A}^{-1}/16
\end{array} \right\}; \quad (113)$$

$$\mathcal{K} = -8p.A + C\mathcal{A}^{-1}C/4;$$

$$\mathcal{A}^{-1}(n, r) = -\Theta(n - r).$$

There are no $1/\sqrt{E^z_Z}$ factors in the inverse Dirac bracket matrix, because these factors have been absorbed into the constraints.

The theta function is a discrete analog of the usual step function.

$$\Theta(n - r) = 0, \quad n - r \leq 0;$$

$$\Theta(n - r) = 2, \quad n - r > 0, \quad n - r = \text{odd};$$

$$\Theta(n - r) = 0, \quad n - r > 0, \quad n - r = \text{even};$$

$$\delta(c)(n) \Theta(n - r) := [\Theta(n + 1 - r) - \Theta(n - 1 - r)]/2$$
$$= \delta(n, r). \quad (114)$$
The last line is reminiscent of eq. (94), where the derivative of a Dirac delta is shown to have a discrete analog, the difference of a Kronecker delta. Similarly here, the continuous formula

\[ \partial_2 \Theta(z_1 - z_2) = \delta(z_1 - z_2) \]

has a discrete analog, the last line of eq. (114).

I have written the final answer in terms of the more familiar \( E \) rather than the less familiar \( Q \). Since \( E_Z \propto \exp[Q/2p] \), this hides a divergence in \( p \). Again, the case \( p = 0 \) requires separate discussion.

The matrix of constraints contains a field-dependent quantity

\[ C = 2 \left[ \delta(c) \frac{E_y^y}{E_y^y} - \delta(c) \frac{E_x^x}{E_x^x} \right] \delta(1, 2). \]

This is very unusual. In most field theories brackets between unidirectional constraints are field-independent. In weak field limit, \( C \) disappears from the Dirac brackets because \( E_Z \to 1 \), and the off-diagonal commutator

\[ [U_1, U_3] \sim [\Pi, 1/\sqrt{E^z}] \]

vanishes. The presence of \( C \) is therefore a consequence of the nonlinearity of the theory, as represented by the area factors \( E_Z \).

The field dependence prohibits an exact solution for \( K^{-1} \). However, an integral equation for \( K^{-1} \) has a power series solution.

\[ \delta(1, 3) = \sum_2 K(1, 2)K^{-1}(2, 3) \]

\[ = +8p \delta(c) (1)K^{-1}(1, 3) \]

\[ + \sum_2 C(1)(A^{-1/4})(1, 2)C(2)K^{-1}(2, 3). \] (115)

The solution for \( \Theta \), eq. (114), is undetermined up to an additive constant. I have not thought through the boundary conditions on \( \Theta \). I do not need the details of the Dirac brackets; I need to know only that the brackets exist. Then if I encounter an expression containing unidirectional constraints I can commute all constraints to the right to annihilate the wavefunction.

Even when verifying consistency of the unidirectional constraints, one does not need the Dirac brackets. One must use the original Poisson brackets.
On Consistency of Constraints

In one sense second class constraints (such as the unidirectional constraints) are trivially consistent. Once Dirac brackets are introduced, they commute trivially with the other constraints and with each other. However, one must compute Poisson brackets of secondary constraints with the Hamiltonian \((\text{before introducing Dirac brackets})\), to verify there are no additional constraints.

Since the diffeomorphism constraints are a \((\pi, q)\) pair, further commutation of these constraints produces (almost) nothing new. However, the bracket of \(D_2\) with \(H\) gives a Laplace-like equation for \(N\).

\[
0 = 2(E^z_\tilde{Z}) (\delta(c) N) + N \{2 \epsilon_{AB} K^A_x K^B_y (\tilde{E}) \\
+2 K^Z_\tilde{Z}(n) E^z_\tilde{Z} K^A_a(n) E^a_A(n) \\
+(1/2)(\delta(c) E^y_Y / E^x_X - \delta(c) E^y_Y / E^x_X)^2 \\
-(\delta(c) \tilde{E} / (\tilde{E}^2) (\tilde{E}^2)) + (1/2)(\delta(c) E^z_Z)^2 \}/E^z_\tilde{Z}.
\] (116)

The unidirectional constraints, eq. (98), force every term in this equation to cancel, except \(\delta(c) N\) terms. We have rederived \(\delta(c) N = 0\), the first of the unidirectional plus diffeomorphism constraints, eq. (111).

Also, the bracket of \(D_1\) with \(H\) gives

\[
\delta(c) N^x = 0.
\] (117)

The diffeomorphism constraints determine the Lagrange multipliers, exactly as they do in geometrodynamics. Since there are no surprises, and the calculations to verify eqs. (116) and (117) are very similar to those required to verify that the commutators of \(H\) with the unidirectional constraints produce no new constraint, the rest of this section concentrates on the brackets between \(H\) and the unidirectional constraints.

Most of the work involves setting up the calculation. \(H\), eq. (88), must be rewritten. It contains a term involving \(N\) rather than \(\tilde{N}\); and the Hamiltonian contains spin connections \(\Gamma\) which must be expressed in terms of triads in order to compute brackets with extrinsic curvature. One must also compute brackets involving differences. The actual calculation of \(\{H, U_i\}\) is then straightforward and will be sketched briefly.
G  Rewriting $\Gamma$

From eq. (54),

$$\Gamma^X_y E^y_X = \text{sgn} \delta(c) e_y Y e^{yx} e^X_x \quad \text{(single pol.)},$$

plus an additional formula with $x \leftrightarrow y, X \leftrightarrow Y$.

If I difference the equation $e^X_x E^x_X = |e|$, eq. (27), and divide by $e^X_x E^x_X$, I get

$$\delta(c) e^X_x / |e_x| + \delta(c) E^x_X / E^x_X = \delta(c) |e| / |e|,$$

plus a similar equation for $x \rightarrow y$. These equations imply

$$\Gamma^X_x E^x_X + \Gamma^X_y E^y_Y$$

$$= [\delta(c) E^y_Y / E^y_y - \delta(c) E^x_X / E^x_x] e^X_x e^Y_y$$

$$= [\delta(c) E^y_Y / E^y_y - \delta(c) E^x_X / E^x_x] E^Z_Z. \quad (118)$$

The remaining linear combination is one half of Gauss, eq. (60):

$$\Gamma^Y_x E^x_X - \Gamma^X_y E^y_Y = \delta(c) E^Z_Z,$$

Since the $\Gamma_a$ typically occur multiplied by an $E^a_A$, these results will be enough to replace $\Gamma_a$ by functions of triads.

H  Rewriting the Hamiltonian

I expand $H$, eq. (88), in the combinations

$$\Gamma^Y_x E^x_X \pm \Gamma^X_y E^y_Y;$$

$$K^y_y E^y_Y \pm K^x_x E^x_X;$$

$$K^Z_z E^Z_Z.$$

$$\text{NH} + \text{ST} = \sum_n \left\{ \frac{1}{\kappa} \{ -(K^A_A(n))^2 N(n)/(4 E^Z_Z) \right.$$

$$+ (K^Y_y E^y_y - K^X_x E^x_x)^2 N(n)/(4 E^Z_Z) \right.$$

$$- K^Z_Z(n) E^Z_Z K^A_A(n) E^a_A N(n)/E^Z_Z \right.$$

$$+ (\Gamma^Y_y E^y_y + \Gamma^X_x E^x_x)^2(n) N(n)/(4 E^Z_Z) \right.$$

$$- (\Gamma^Y_y E^y_y - \Gamma^X_x E^x_x)^2(n) N(n)/(4 E^Z_Z) \right.$$

$$- \Gamma^A_A(n) e^{zaA}(n) \delta(c) N \left. \right\}. \quad (119)$$
Most of the terms in \( H \) now have a \( 1/E_z^a \) singularity.

The next task is to eliminate the \( N \) from the sixth (last) line of \( \tilde{H} \), eq. (119). I use the QFT value for \( e^{zA} \), then Gauss for a \( \Gamma \tilde{E} \) factor.

\[
\text{line 6} = -\delta(c) E_z^a \bigg[ \delta(c) N + N \left( e_{E_z^a} (e/E_z^a) \right) \bigg]
\]

When eqs. (118) and (120) are inserted into the Hamiltonian, eq. (119) becomes

\[
\sum_n \left\{ \frac{1}{\kappa} \left\{ -(K^A_a E^a_A(n)/4N) \right/ N \right\}
\]

\[\]
I  "Counting" Brackets

To prove consistency I need certain brackets which do not involve differences:

\[ [K(q)_i, E^i_I(m)] = \left[ -2i \hat{h}(q), \tilde{E}(m) \right]/\gamma \]
\[ = 2 \left[ \tilde{h}_i, (\sigma_I/2)_+ (\kappa/2) \right] \delta(q, m) \]
\[ \cong \sigma_I \kappa \delta(q, m). \quad \text{(SS)} \]

This result holds also for \( i = z \), provided \( K \) or \( \tilde{E} \) occurs in a sum over \( q \) or \( m \). In what follows I will usually suppress the \( \sigma_I \kappa \), since this factor does not affect consistency.

The Hamiltonian has been rewritten so as to contain only \( K^I E^i_I \) products. If a \( KE \) product is commuted with a term containing no \( \delta(c) \tilde{E} \), only \( \tilde{E} \) or \( K \), then the \( K^I E^i_I \) product acts as a counting operator (a.k.a. scale operator). For example,

\[ \{ K^z E^z, (K^z)^r (E^z)^s \} = (s - r) (K^z)^r (E^z)^s. \quad \text{(123)} \]

This KE product counts the number of longitudinal \( E \)'s minus number of longitudinal \( K \)'s.

To understand the minus sign for \( (K)^r \), one may think of \( K \) as \( i \kappa \hbar \delta/\delta E \): \( K \) is a negative power of \( E \). For non integer and negative powers of \( s \) or \( r \), the operators \( K^z E^z \) must be interpreted as functional derivatives, as in section [III].

J  Brackets Involving Differences

Since the form \( \delta(c) E/E \) occurs often in the constraints, the following bracket is also needed.

\[ \{ KE(q), \delta(c) E(m)/E(m) \} = \delta(c) E(-1/E) \delta(q, m) \]
\[ + \{ KE(m), [E(m + 1) - E(m - 1)]/2E(m) \} \]
\[ = \delta(c) E(-1/E) \delta(q, m) \]
\[ + [E(q)/E(m)] [\delta(q, m + 1) - \delta(q, m - 1)]/2 \]
\[ := \delta(c) E(-1/E) \delta(q, m) + [E(q)/E(m)] \delta(c)(m) \delta(q, m). \quad \text{(124)} \]

In QFT, a \( [K, \partial_z \tilde{E} \] commutator yields the derivative of a Dirac delta function; in LQG, the commutator yields the difference of a Kronecker delta function as at eq. (94). This difference obeys the identity

\[ \delta(c)(m) \delta(q, m) = -\delta(c)(q) \delta(q, m), \quad \text{(125)} \]
which is the discrete analog of the Dirac delta identity
\[ \partial \delta(x - y) / \partial x = -\partial \delta(x - y) / \partial y. \]

The last line of eq. (124) can be simplified. When the \( \delta(c)(m) \) is integrated by parts off the \( \delta(q, m) \), it will difference the \( 1/E(m) \). This difference will cancel the \( \delta(q, m) \) term. The result is simply
\[ \{\text{KE}(q), \delta(c)(m) E(m) / E(m)\} = \delta(c)(m) \delta(q, m). \] (126)

One can think of the KE product as returning a "count" of zero in this case, since there is one \( \tilde{E} \) in the numerator and one in the denominator. However, the \( \delta(c) \) does not go away.

Eq. (126) has two corollaries.
\[
\{(K_{Y}^{q} / K_{Y}^{q} \pm \delta(c) K_{X}^{x} / K_{X}^{x})(q), (E_{Y}^{q} / E_{Y}^{q} \mp \delta(c) E_{X}^{x} / E_{X}^{x})(m)\} = 0;
\]
\[
\{(K_{Y}^{q} / K_{Y}^{q} \pm \delta(c) K_{X}^{x} / K_{X}^{x})(q), (E_{Y}^{q} / E_{Y}^{q} \mp \delta(c) E_{X}^{x} / E_{X}^{x})(m)\} = 2 \delta(c)(m) \delta(q, m). \] (127)

The commutator without the \( 1/E \) is
\[ \{\text{KE}(q), \delta(c)(m) E(m)\} = \delta(c)(m) \delta(q, m) E(q). \] (128)

One can also eliminate one \((\pi, q)\) pair, using the diffeomorphism gauge, express \( H \) in terms of \((\Pi, Q)\), and check consistency using the reduced variable set. The "counting" and difference brackets in this case become
\[
\{\Pi, (E_{Z}^{z})^{r}\} = \{\Pi, \exp(rQ/2p)/C^{r}\}
\]
\[ = (-r/2p)(E_{Z}^{z})^{r} \delta(q, m); \]
\[
\{\Pi(q), \delta(c)(m) E_{Z}^{z}(m) / E_{Z}^{z}\} = (-1/2p) \delta(c)(m) \delta(q, m). \] (129)

Except for the factor \((1/2p)\), \( \Pi \) is a counting or scale operator for \( E_{Z}^{z} \), rather than \( Q \). If the constraints are expressed in terms of \( \Pi \) and \( E_{Z}^{z} \), then the calculation with a reduced set of variables closely resembles the calculation with the full set of variables.

K Consistency of the \( U_i \)

The machinery is now in place, and the calculations are now straightforward, if lengthy. I spare the reader the details and limit myself
to a few comments. The brackets

\[ \{ H + ST, U_1 \text{ or } U_3 \} \]

involve the brackets computed at eq. (127), as well as counting brackets eq. (123). The overall factor of \( 1/\sqrt{E^z_Z} \), merely gives back a count of \((-1/2)\) times the original \( U_i \), hence is especially easy to handle.

One may check either list of unidirectional constraints: the original list eq. (98), or the list which anticipates the fixing of the diffeomorphism gauge, eq. (111). I choose to check the latter.

How should one interpret the first item on that list, \( \delta(c) \frac{\tilde{N}}{\tilde{N}} = 0 \)? As stressed earlier, \( \tilde{N} \) is not a diffeomorphism scalar, therefore does not automatically commute with the vector constraint. Write

\[ \tilde{N} = N E^z_Z/|e| = N \sqrt{sgn E^z_Z/(2)} \tilde{E}. \]

Then

\[ 2 \delta(c) \frac{\tilde{N}}{\tilde{N}} = 2 \delta(c) \frac{N}{\tilde{N}} + \delta(c) E^z_Z/E^z_Z - \delta(c) \frac{(2) \tilde{E}}{/(2) \tilde{E}} = 0. \] (130)

There is no need to check the first item on the list; it may be interpreted as the equation which determines \( N \).

Can brackets with \( \tilde{N} \) give trouble elsewhere? Strictly speaking, brackets with \( \tilde{N} \) cannot be ignored. Usually, however, those brackets do not give trouble, for two reasons. \( \delta(c) \frac{\tilde{N}}{\tilde{N}} \) does not occur in the Hamiltonian, if we do not eliminate the ST; and \( \tilde{N} \) normally occurs multiplied by constraints \( H \) or \( \tilde{H} \), which vanish. Let \( O \) be an arbitrary operator. When the ST is retained, \( \tilde{N} \) is an overall factor multiplying the Hamiltonian.

\[ \{(H+ST)\tilde{N}, O\} = \tilde{H}\{(E^z_Z/e)\tilde{N}, O\} + \{\tilde{H}, O\}\tilde{N} + \{ST\tilde{N}, O\} \]

\[ = H\{N, O\} + \tilde{H}\{(E^z_Z/e), O\}\tilde{N} + \{\tilde{H}, O\}\tilde{N} + \{ST\tilde{N}, O\} \]

\[ = 0 + 0 + \{\tilde{H}, O\}\tilde{N} + \{ST\tilde{N}, O\}. \] (131)

\( \tilde{N} \) can be extracted from the surface term, because the boundary condition requires \( \tilde{N} \) to be a constant. Despite its contravariant nature, \( \tilde{N} \) is only a "spectator".

One can also calculate consistency by eliminating the pair \((\pi, q)\) set to zero by the diffeomorphism gauge and retaining the surviving
pair \((\Pi, Q)\) (or better, \(\Pi, E_z^x\)). One needs the Hamiltonian in the surviving variables:

\[
\hat{N}\hat{H} + ST = \sum_n \left\{ \frac{1}{\kappa} \left[ \frac{1}{4} \left\{ K_y^Y E_Y^y - K_x^X E_X^x \right\}^2 / E_Z^z \right. \\
+ \frac{1}{4} \left[ E_Z^z (\delta(c) E_Y^y / E_Y^y - \delta(c) E_X^x / E_X^x) \right]^2 / E_Z^z \\
- 2p \left[ \Pi^2 + (\delta(c) E_Z^z / 2E_Z^z)^2 \right] / E_Z^z - \delta(c) E_Z^z \delta(c) \hat{N} \right\}.
\]

(132)

XI Final Form of the Constraint Equations

I now count the number of surviving equations and compare to the number of surviving unknowns. After the single polarization constraints are introduced and the triad matrix becomes diagonal, three diagonal triads remain, plus their associated momenta, plus lapse and shift. The \((\pi, q)\) pair

\( (K_y^Y E_Y^y - K_x^X E_X^x, \ln(E_Y^y / E_x^x)) \)

represents the physical degree of freedom and must be fixed using initial conditions.

One of the remaining \((\pi, q)\) pairs was fixed by the diffeomorphism gauge conditions \(D_i\). Requiring consistency of those constraints fixes lapse and shift.

The two constraints, \(\hat{H} = H_z = 0\), determine the two remaining non-dynamical variables \((\Pi, Q)\). From eqs. (122) and (132), after eliminating the variables fixed by the diffeomorphism gauge:

\[
\hat{H} = \sum_n \left\{ \frac{1}{\kappa} \left[ \frac{1}{4} \left\{ K_y^Y E_Y^y - K_x^X E_X^x \right\}^2 / E_Z^z \right. \\
+ \frac{1}{4} \left[ (\delta(c) E_Y^y / E_Y^y - \delta(c) E_X^x / E_X^x)^2 E_Z^z \\
- 2p \left[ \Pi^2 + (\delta(c) E_Z^z / 2E_Z^z)^2 \right] E_Z^z + \delta(c) (\delta(c) E_Z^z) \right\} \\
= 0.
\]

(133)
\[
\kappa H_z = \sum_n \{ \delta(c) [ K^A_n(n) E_A^a(n) ] - \sum_i K'_i \delta(c) E_i^a \} \\
= \sum_n \{ \delta(c) (2\Pi) - (1/2)(K^Y_E^Y - K^X_E^X) \\
\times (\delta(c) E^y_Y/E^y_Y - \delta(c) E^x_X/E^x_X) - \Pi \delta(c) Q \} \\
= 0; \\
\delta(c) Q = 2p \delta(c) E^z_Z/E^z_Z. 
\]

A fine point: \( \hat{H} + ST \) is not a constraint; it is the true Hamiltonian. To get the constraint, \( \hat{H} \), I must eliminate the ST; I must undo the integration by parts in eq. (132) for \( \hat{H} + ST \).

\[-\delta(c) \sum \delta(c) E^z_Z \rightarrow + \sum \delta(c) (\delta(c) E^z_Z).\]

I am left with \( \hat{H} \), eq. (133), which is an equation for \( E^z_Z \) (i.e. \( Q \)). \( H_z \) becomes the equation for \( \Pi \).

One can carry the simplification of the constraints a step farther, by using the unidirectional constraints to eliminate half the variables. I will use these forms as a starting point in the next paper. Here is a sample of the reasoning: \( \Pi \) can be replaced by \(-\delta(c) E^z_Z \).

\[
\delta(c) E^z_Z = (U_1 - V_1) \sqrt{E^z_Z}/2; \\
2 \Pi = (U_1 + V_1) \sqrt{E^z_Z}/2 \\
= [(-U_1 + V_1)/2 + U_1] \sqrt{E^z_Z} \\
= -\delta(c) E^z_Z. 
\]

The \( V_i \) are the constraints for left-moving waves, obtained from the \( U_i \) by changing the sign of \( \delta(c) \) terms. On the third line, one can always add arbitrary amounts of the \( U_i \), since they can be commuted freely to the right because of the Dirac brackets.

If the unidirectional constraints are used to eliminate all \( K \)'s from \( \hat{H} \), the result is

\[
\hat{H} = \sum_n [1/\kappa] \{ (1/2)(\delta(c) E^y_Y/E^y_Y) \\
- \delta(c) E^z_Z/E^z_Z \} E^z_Z \\
- p (\delta(c) E^z_Z/E^z_Z)^2 E^z_Z + \delta(c) (\delta(c) E^z_Z) \}. 
\]

If the unidirectional constraints are used to eliminate all \( K \)'s from \( H_z \), we get eq. (136) once again. This is not surprising, since unidirectional constraints relate \( t \) and \( z \) derivatives.
In section III I proposed an exact LQG Hamiltonian in which some \( \tilde{E} \) grasp at only one of the two faces surrounding a given vertex. (For example, a cube surrounding the vertex has two faces with normals in the x direction, and some \( E^x \) grasp at only one of the two faces.) There is some motivation for this choice at the level of exact LQG; but the \( \tilde{E} \) which survive to the semiclassical limit all grasp at both faces.

The forward differences of the exact model become central differences in the semiclassical limit. From the semiclassical point of view this is a relatively minor change, since the slow variation assumption implies forward and central differences are approximately equal.

Readers who are familiar with the relation between geometrodynamical variables, Szekeres variables, and (\( K, \tilde{E} \)) variables, will recognize numerous points where I have shifted to combinations of the \( K \) and \( \tilde{E} \) which equal Szekeres or geometrodynamical variables [13]. For example, the ADM \( \pi^{ij} \) are linear combinations of \( K, \tilde{E} \) products. The triad combinations involving logarithms are Szekeres variables. Whatever the superiority of \( K \) and \( \tilde{E} \) at short distances, the traditional combinations hold the edge in the semiclassical limit.

Of course \( K \cdot E \) is not really a geometrodynamical variable, because the "\( K \)" is a holonomy, not a field. This is the fundamental change which leads to quantization of areas and volumes. However, the combination, holonomy times triad, seems to be more appropriate than holonomy alone.

In this paper, the term small sine approximation was used as shorthand to describe a combination of the slow variation and small sine approximations, both applied away from \( e = 0 \). Near \( e = 0 \), one must abandon slow variation and regulate cotriads. If one retains the small sine assumption, however, one has a simplified model which retains the quantization of geometrical quantities and the boundedness characteristic of full LQG.

Also, in a typical cotriad expression,

\[
N E^j_i E^K_k \epsilon^{JK} / \left| e \right| = \sum N E^j_i E^K_k \epsilon^{JK} / E^z_z,
\]

one does not need to replace triads by a cotriad. One can retain the transverse triads, and regulate only \( 1/E^z_z \). Because \( E^z_z \) is already diagonal, its square root is immediate. The transverse triads are then
available to implement a shift to geometrodynamical and Szekeres variables, even near $e = 0$.

The spin connections need to be regulated. From CGR formulas for $\Gamma \cdot E$, eq. (54), with derivatives replaced by differences, plus CGR formulas for the cotriads, we get

$$\Gamma^X_y E^y_x + \Gamma^Y_x E^X_y = sgn \left[ -\delta_{(c)} e^Y_y e^X_x + e^Y_y \delta_{(c)} e^X_x \right]$$

$$= sgn \left( E^z_Z / | e | \right)^2 \left[ \delta_{(c)} (E^X_Y) E^X_Y - E^y_y \delta_{(c)} (E^X_X) \right].$$

(137)

The dangerous overall factor of $\left( E^z_Z / | e | \right)^2$ can be removed by an appropriate choice of gauge. For $p = 1/2$, that factor becomes a constant. The other linear combination follows from Gauss and is free of singularities.

$$\delta_{(c)} E^z_Z = -\Gamma^X_y E^y_y + \Gamma^Y_x E^x_x.$$

I mention one other LQG treatment of the planar case. Mena Marugán and Martín-Benito study inhomogeneities in the early universe [10]. They quantize a homogeneous planar model and add inhomogeneous matter. Their study might not seem to apply to the wave case ("homogeneous" implies no $z$ dependence). However, the models studied in the present paper probably have no singularities at small volume. (Unidirectional plane waves in CGR have coordinate singularities, but no curvature singularities.) To get singularities, presumably one must study colliding gravitational waves, or introduce matter. The latter approach may be simpler, and reference [10] may have shown the way.

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