Abstract
The main objective of this work is to generalize the concept of fuzzy $\sigma$-algebra by introducing the notion of fuzzy $\gamma$-algebra. Characterization and examples of the proposed generalization are presented, as well as several different properties of fuzzy $\gamma$-algebra are proven. Furthermore, the relationship between fuzzy $\gamma$-algebra and fuzzy algebra is studied, where it is shown that the fuzzy $\gamma$-algebra is a generalization of fuzzy algebra too. In addition, the notion of restriction, as an important property in the study of measure theory, is studied as well. Many properties of restriction of a nonempty family of fuzzy subsets of fuzzy power set are investigated and it is shown that the restriction of fuzzy $\gamma$-algebra is fuzzy $\gamma$-algebra too.

Keywords: $\sigma$-algebra, Algebra, Measure, Fuzzy set, Outer measure.
1. Introduction and Basic concepts

Wang [1] in 2009 studied some types of collections of sets which are generalizations of σ-algebra such as ring, σ-ring and proved some important results of these concepts, where a nonempty class \( \mathcal{H} \) is called a ring iff \( E,F \in \mathcal{H}, \) then \( E \cap F \in \mathcal{H} \). In 2019 Ahmed and Ebrahim [2] introduced some generalizations of σ-algebra and σ-ring. Many other authors are interested in studying σ-algebra and σ-ring, for example, see [3], [4] and [5]. Zadeh [6] in 1965 first introduced the concept of the fuzzy set where \( \mathcal{X} \) is a nonempty set, then a fuzzy set \( F \) in \( \mathcal{X} \) is defined as a set of ordered pairs \( \{(\omega, \mu_F(\omega)) : \omega \in \mathcal{X}\} \) where \( \mu_F : \mathcal{X} \to [0,1] \) is a function such that for every \( \omega \in \mathcal{X} \), \( \mu_F(\omega) \) represents the degree of membership of \( \omega \) in \( F \). Brown [7] studied some types of fuzzy sets such as fuzzy power set, empty fuzzy set, universal fuzzy set, the complement of a fuzzy set, the union of two fuzzy sets and intersection of two fuzzy. Ahmed et al. [8-11] first introduced the concept of fuzzy σ-algebra and fuzzy algebra where \( \mathcal{X} \) is a nonempty set and \( \mathcal{P}^*(\mathcal{X}) \) be a fuzzy power set of \( \mathcal{X} \). A nonempty collection \( \mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X}) \) is said to be a fuzzy σ-algebra of sets over a fuzzy set \( \mathcal{X}^* = \{(\omega, 1) : \forall \omega \in \mathcal{X}\} \), if the following conditions are satisfied:

1. \( \emptyset^* \in \mathcal{H}^* \), where \( \emptyset^* = \{(\omega, 0) : \forall \omega \in \mathcal{X}\} \).
2. If \( E \in \mathcal{H}^* \), then \( E^c \in \mathcal{H}^* \).
3. If \( E_1, E_2, \ldots \in \mathcal{H}^* \), then \( \bigcap_{k=1}^{\infty} E_k \in \mathcal{H}^* \).

If condition 3 is satisfied only for finite sets, then \( \mathcal{H}^* \) is said to be a fuzzy algebra over a fuzzy set \( \mathcal{X}^* \).

Another generalization of the fuzzy σ-algebra introduced in this paper, which is a fuzzy \( \gamma \)-algebra. The main aim of this chapter is to study this generalization and introduce some of its basic properties, examples and some characterizations of them.

**Definition (1.1)** [4]

Let \( \mathcal{X} \neq \emptyset \). A collection \( \mathcal{H} \) is called σ-ring if and only if the following conditions hold:

1. If \( F, E \in \mathcal{H} \), then \( F \setminus E \in \mathcal{H} \).
2. If \( E_1, E_2, \ldots \in \mathcal{H} \), then \( \bigcup_{k=1}^{\infty} E_k \in \mathcal{H} \).

**Definition (1.2)** [8]

Let \( \mathcal{X} \neq \emptyset \). A collection \( \mathcal{H} \) is called σ-field if and only if the following conditions hold:

1. \( \mathcal{X} \in \mathcal{H} \).
2. If \( F \in \mathcal{H} \), then \( F^c \in \mathcal{H} \).
3. If \( E_1, E_2, \ldots \in \mathcal{H} \), then \( \bigcup_{k=1}^{\infty} E_k \in \mathcal{H} \).

**Proposition (1.3)** [1]

Every σ-field is a σ-ring.

**Definition (1.4)** [9]

Let \( \mathcal{X} \) be a nonempty set. Then the union of the two fuzzy sets \( F \) and \( E \) in \( \mathcal{X} \) with respective membership functions \( \mu_F(\omega) \) and \( \mu_E(\omega) \) is a fuzzy set \( \mathcal{G} \) in \( \mathcal{X} \) whose membership function is related to those of \( F \) and \( E \) by \( \mu_G(\omega) = \max_{\omega \in \mathcal{X}} \{\mu_F(\omega), \mu_E(\omega)\} \). In symbols:

\[
\mathcal{G} = \mathcal{F} \cup \mathcal{E} \iff \mathcal{G} = \{(\omega, \max_{\omega \in \mathcal{X}} \{\mu_F(\omega), \mu_E(\omega)\}) : \omega \in \mathcal{X}\}.
\]
Definition (1.5) [10]
Let \( \mathcal{X} \) be a nonempty set. Then the intersection of two fuzzy sets \( F \) and \( E \) in \( \mathcal{X} \) with respective membership functions \( \nu_F(\omega) \) and \( \nu_E(\omega) \) is a fuzzy set \( G \) in \( \mathcal{X} \) whose membership function is related to those of \( F \) and \( E \) by \( \nu_G(\omega) = \min_{\omega \in \mathcal{X}} \{ \nu_F(\omega), \nu_E(\omega) \} \). In symbols:
\[
G = F \cap E \iff G = \{ (\omega, \min_{\omega \in \mathcal{X}} \{ \nu_F(\omega), \nu_E(\omega) \}) : \omega \in \mathcal{X} \}.
\]

Definition (1.6) [6]
Let \( \mathcal{X} \) be a nonempty set and \( F \) is a fuzzy sets in \( \mathcal{X} \). Then the complement of a fuzzy set \( F \) is denoted by \( F^c \) and defined as:
\[
F^c = \{ (\omega, 1 - \nu_F(\omega)) : \omega \in \mathcal{X} \}.
\]

Proposition (1.7) [11]
Every fuzzy \( \sigma \)-algebra is a fuzzy algebra.

2. The main results:
In this section, we introduce the concept of fuzzy \( \gamma \)-algebra which is a generalization for the concept of the fuzzy \( \sigma \)-algebra and fuzzy algebra. Also, we present many properties of fuzzy \( \gamma \)-algebra.

Definition (2.1):
Let \( \mathcal{X} \) be a nonempty set. A nonempty collection \( \mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X}) \) is said to be a fuzzy \( \gamma \)-algebra of sets (\( \gamma \)-field) over a fuzzy set \( \mathcal{X}^* \), if the following conditions are satisfied:

1. \( \emptyset^*, \mathcal{X}^* \in \mathcal{H}^* \), where \( \emptyset^* = \{ (\omega, 0) : \forall \omega \in \mathcal{X} \} \) and \( \mathcal{X}^* = \{ (\omega, 1) : \forall \omega \in \mathcal{X} \} \).
2. If \( E_1, E_2, ..., E_n \in \mathcal{H}^* \), then \( \bigcup_{k=1}^n E_k \in \mathcal{H}^* \).

Definition (2.2):
Let \( \mathcal{X} \) be a nonempty set and \( \mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X}) \) be a fuzzy \( \gamma \)-algebra (\( \gamma \)-field) over a fuzzy set \( \mathcal{X}^* \). Then the pair \( (\mathcal{X}^*, \mathcal{H}^*) \) is said to be fuzzy measurable space relatively to fuzzy \( \gamma \)-algebra.

Example (2.3):
Let \( \mathcal{X} = \{a, b\} \) and \( \mathcal{H}^* = \{\emptyset^*, \{(a, 0.1), (b, 0.6)\}, \{(a, 0.3), (b, 0.5)\}\} \). Then the pair \( (\mathcal{X}^*, \mathcal{H}^*) \) is said to be fuzzy measurable space relatively to fuzzy \( \gamma \)-algebra.

Example (2.4):
Let \( \mathcal{X} = \{a, b\} \) and \( \mathcal{H}^* = \{\emptyset^*, \{(a, 0.4), (b, 0.2)\}\} \). Then \( \mathcal{H}^* \) is not a fuzzy \( \gamma \)-algebra over a fuzzy set \( \mathcal{X}^* \), because \( \{(a, 0.4), (b, 0.2)\} \in \mathcal{H}^* \) and \( \{(a, 0.3), (b, 0.5)\} \in \mathcal{H}^* \), but \( \{(a, 0.4), (b, 0.2)\} \cup \{(a, 0.3), (b, 0.5)\} = \{(a, \text{Max}\{0.4, 0.3\}), (b, \text{Max}\{0.2, 0.5\})\} \) \( \notin \mathcal{H}^* \).

Proposition (2.5):
Let \( \mathcal{X} \) be an infinite set and \( \mathcal{H}^* = \{\emptyset^*, \mathcal{X}^*, \text{all } E \subseteq \mathcal{X}^* \text{ s.t. } E^c \text{ is finite}\} \). Then \( (\mathcal{X}^*, \mathcal{H}^*) \) is a fuzzy measurable space relatively to fuzzy \( \gamma \)-algebra.
Proof: 
From the definition of $\mathcal{H}^*$, we get $\emptyset^*, X^* \in \mathcal{H}^*$. Let $E_1, E_2, \ldots, E_n \in \mathcal{H}^*$. Then $E_k^c$ is finite for all $k = 1, 2, \ldots, n$. Hence, $\bigcap_{k=1}^n E_k^c$ is finite. Now, since 
\[
\bigcap_{k=1}^n E_k^c = \{(\omega, \min_{k=1,2,\ldots,n} \{1 - \varphi_{E_k}(\omega)\}) : \omega \in \mathcal{X} \}
\]
\[
= 1 - \{(\omega, \max_{k=1,2,\ldots,n} \{\varphi_{E_k}(\omega)\}) : \omega \in \mathcal{X} \}
\]
\[
= 1 - (U_{k=1}^n E_k)^c
\]
Then $U_{k=1}^n E_k \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ and $(X^*, \mathcal{H}^*)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra.

Proposition(2.6):
Let $X$ be a nonempty set and $F$ be a fuzzy set such that $\emptyset^* \neq F \subseteq X^*$ and let $E \subseteq X^*$ denote to $\varphi_E \leq \varphi_{X^*}$. If $\mathcal{H}^* = \emptyset^* \cup \{E \subseteq X^* : F \subseteq E\}$. Then $(X^*, \mathcal{H}^*)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra.

Proof:
From the definition of $\mathcal{H}^*$, we get $\emptyset^* \in \mathcal{H}^*$. Since $X^* \subseteq X^*$ and $F \subseteq X^*$, then $X^* \in \mathcal{H}^*$. Let $E_1, E_2, \ldots, E_n \in \mathcal{H}^*$. Then $F \subseteq E_k$ for all $k = 1, 2, \ldots, n$ and hence $F \subseteq U_{k=1}^n E_k$. Thus $U_{k=1}^n E_k \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*$. Consequently, $(X^*, \mathcal{H}^*)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra.

Proposition(2.7):
Let $(X^*, \mathcal{H}^*)$ be a fuzzy measurable space relatively to fuzzy $\gamma$–algebra and let $E \subseteq X^*$. Define $\mathcal{H}^*_1 = \{F \subseteq X^* : F \cup E \in \mathcal{H}^*\}$, then $(X^*, \mathcal{H}^*_1)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra.

Proof:
Since $\mathcal{H}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*$, then $X^* \in \mathcal{H}^*$. By hypothesis $E \subseteq X^*$ implies that $X^* = X^* \cup E$ and hence $X^* \in \mathcal{H}^*_1$. Consider $E = \emptyset^*$, then $\emptyset^* \cup X^*$. Let $E_1, E_2, \ldots, E_n \in \mathcal{H}^*_1$. Then by definition of $\mathcal{H}^*_1$ we have, $U_{k=1}^n E_k \subseteq E \in \mathcal{H}^*$ for all $k = 1, 2, \ldots, n$. Hence, $U_{k=1}^n (E_k \cup E) \in \mathcal{H}^*$ because $\mathcal{H}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ this implies that $(U_{k=1}^n (E_k \cup E)) \subseteq \mathcal{H}^*$. Thus $U_{k=1}^n E_k \in \mathcal{H}^*_1$ by definition of $\mathcal{H}^*_1$. Therefore, $\mathcal{H}^*_1$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ and $(X^*, \mathcal{H}^*_1)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra.

Proposition(2.8):
Let $(X^*, \mathcal{H}^*)$ be a fuzzy measurable space relatively to fuzzy $\gamma$–algebra and $F \subseteq X^*$. If $\mathcal{H}^*_1 = \{G \subseteq X^* : F \cap E \subseteq G \text{ for some } E \in \mathcal{H}^*\}$ Then $(X^*, \mathcal{H}^*_1)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra.

Proof:
Since $(X^*, \mathcal{H}^*)$ is a fuzzy measurable space relatively to fuzzy $\gamma$–algebra, then $\emptyset^*, X^* \in \mathcal{H}^*$. By hypothesis $F \subseteq X^*$ implies that $F \cap X^* \subseteq X^*$, hence by definition of $\mathcal{H}^*_1$ we have, $X^* \in \mathcal{H}^*_1$. Now, $F \cap \emptyset^* = \emptyset^* \subseteq \emptyset^*$, hence by definition of $\mathcal{H}^*_1$ we have, $\emptyset^* \in \mathcal{H}^*_1$. Let
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\[ G_1, G_2, \ldots, G_n \in \mathcal{H}_1. \] Then there is \( E_k \in \mathcal{H} \) such that \( F \cap E_k \subseteq G_k \) where \( k=1,2,\ldots, n \). So, we have \( U_{k=1}^n G_k \supseteq U_{n=1}^n (F \cap E_k) = (F \cap E_1) \cup (F \cap E_2) \cup \ldots \cup (F \cap E_n) = F \cap (E_1 \cup E_2 \cup \ldots \cup E_n) = F \cap (U_{k=1}^n E_k). \)

But \( E_k \in \mathcal{H}^* \) and \( \mathcal{H}^* \) is a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \), then \( U_{k=1}^n E_k \in \mathcal{H}^* \), hence by definition of \( \mathcal{H}_1 \) we have, \( U_{k=1}^n G_k \in \mathcal{H}_1 \). Therefore, \( \mathcal{H}_1 \) is a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \) and \( (X^*, \mathcal{H}_1) \) is a fuzzy measurable space relatively to fuzzy \( \gamma \)-algebra.

**Proposition (2.9):**

Let \( X \) be an infinite set and let \( X^* = \{(\omega, 1) : \forall \omega \in X \} \). If \( \mathcal{H}^* = \{E \subseteq X^* : E \) is infinite fuzzy set\}. Then \( (X^*, \mathcal{H}^*) \) is a fuzzy measurable space relatively to fuzzy \( \gamma \)-algebra.

**Proof:**

Since \( X \) be an infinite set, then each of \( \emptyset^*, X^* \) be an infinite fuzzy set, but \( \emptyset^* \subseteq X^* \) and \( \emptyset^* \subseteq X^* \), then by definition of \( \mathcal{H}^* \) we have, \( \emptyset^*, X^* \in \mathcal{H}^* \). Let \( E_1, E_2, \ldots, E_n \in \mathcal{H}^* \). Then \( E_k \) is an infinite fuzzy set for every \( k=1,2,\ldots, n \) and hence \( U_{k=1}^n E_k \) is an infinite fuzzy set. Therefore, \( U_{k=1}^n E_k \in \mathcal{H}^* \) and \( \mathcal{H}^* \) is a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \).

**Proposition (2.10):**

Let \( \{\mathcal{H}_i\}_{i \in I} \) be a collection of fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \). Then \( \bigcap_{i \in I} \mathcal{H}_i \) is a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \).

**Proof:**

Since \( \mathcal{H}_i \) is a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \), \( \forall i \in I \), then \( \emptyset^*, X^* \in \mathcal{H}_i \), \( \forall i \in I \). Hence \( \emptyset^* \cap \bigcap_{i \in I} \mathcal{H}_i \) and \( \emptyset^*, X^* \in \bigcap_{i \in I} \mathcal{H}_i \). Let \( E_1, E_2, \ldots, E_n \in \bigcap_{i \in I} \mathcal{H}_i \). Then \( E_1, E_2, \ldots, E_n \in \mathcal{H}_i \), \( \forall i \in I \). Since \( \mathcal{H}_i \) is a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \), \( \forall i \in I \), then \( \bigcap_{k=1}^n E_k \in \mathcal{H}_i \), \( \forall i \in I \). Hence, \( \bigcap_{k=1}^n E_k \in \bigcap_{i \in I} \mathcal{H}_i \), therefore \( \bigcap_{i \in I} \mathcal{H}_i \) is a fuzzy \( \gamma \)-algebra over \( X^* \).

**Definition (2.11):**

Let \( X \) be a nonempty set and \( \mathfrak{X} \subseteq \mathcal{P}(X) \). Then the intersection of all fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \), which includes \( \mathfrak{X} \), is said to be the fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \) that is generated by \( \mathfrak{X} \) and denoted by \( \gamma(\mathfrak{X}) \), that is,

\[ \gamma(\mathfrak{X}) = \bigcap \left\{ \mathcal{H}_i : \mathcal{H}_i \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \right\} \]

where \( \mathcal{H}_i \supseteq \mathfrak{X}, \forall i \in I \).

**Proposition (2.12):**

Let \( X \) be a nonempty set and \( \mathfrak{X} \subseteq \mathcal{P}(X) \). Then \( \gamma(\mathfrak{X}) \) is the smallest fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \) that includes \( \mathfrak{X} \).

**Proof:**

From the definition of \( \gamma(\mathfrak{X}) \), we have:

\[ \gamma(\mathfrak{X}) = \bigcap \left\{ \mathcal{H}_i : \mathcal{H}_i \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \right\} \]

Hence \( \gamma(\mathfrak{X}) \) is fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \). To prove \( \gamma(\mathfrak{X}) \supseteq \mathfrak{X} \). For each \( i \in I \) let \( \mathcal{H}_i \) be a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \) that includes \( \mathfrak{X} \). Then \( \mathfrak{X} \subseteq \bigcap_{i \in I} \mathcal{H}_i \), thus \( \mathfrak{X} \subseteq \gamma(\mathfrak{X}) \).

Now, let \( \mathcal{H}^* \) be a fuzzy \( \gamma \)-algebra over a fuzzy set \( X^* \) that includes \( \mathfrak{X} \). Then
\[ \bigcap \{ \mathcal{H}_i^* : \mathcal{H}_i^* \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \text{ and } \mathcal{H}_i^* \supseteq X^*, \forall i \in I \} \subseteq \mathcal{H}^* , \text{ hence } \gamma(\mathcal{X}^*) \subseteq \mathcal{H}^* . \text{ Therefore, } \\
\gamma(\mathcal{X}^*) \text{ is the smallest fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \text{ that includes } \mathcal{X}^* .

\textbf{Example (2.13):}\\
\text{Let } X = \{a, b, c\} \text{ and } \\
\mathcal{X}^* = \{(a,0.4), (b,0.2), (c,0.3)\}, \{(a,0.2),(b,0.3),(c,0.4)\} . \text{ Then } \\
\gamma(\mathcal{X}^*) = \{\emptyset^*, \{(a,0.4),(b,0.2),(c,0.3)\}, \{(a,0.2),(b,0.3),(c,0.4)\}, \{(a,0.4),(b,0.3),(c,0.4)\} , \\
X^* = \{(a,1),(b,1),(c,1)\} . \text{ Therefore, } \gamma(\mathcal{X}^*) \text{ is the smallest fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \text{ that includes } \mathcal{X}^* .

\textbf{Proposition (2.14):}\\
\text{Let } X \text{ be a nonempty set and } \mathcal{X}^* \subseteq \mathcal{P}^*(X) . \text{ Then } \gamma(\mathcal{X}^*) = \mathcal{X}^* \text{ if and only if } \mathcal{X}^* \text{ is fuzzy } \gamma \text{-algebra over a fuzzy set } X^* .

\textbf{Proof:}\\
\text{Let } \mathcal{X}^* \subseteq \mathcal{P}^*(X) \text{ and let } \gamma(\mathcal{X}^*) = \mathcal{X}^* . \text{ Since } \gamma(\mathcal{X}^*) \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* , \text{ then } \mathcal{X}^* \text{ is fuzzy } \gamma \text{-algebra over a fuzzy set } X^* . \\
\text{Conversely, suppose that } \mathcal{X}^* \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* . \text{ Since } \gamma(\mathcal{X}^*) \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \text{ which includes } \mathcal{X}^* , \text{ then } \gamma(\mathcal{X}^*) \supseteq \mathcal{X}^* . \text{ But } \mathcal{X}^* \text{ is fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \text{ such that } \mathcal{X}^* \supseteq \mathcal{X}^* \text{ and } \gamma(\mathcal{X}^*) \text{ is the smallest fuzzy } \gamma \text{-algebra over a fuzzy set } X^* \text{ that includes } \mathcal{X}^* , \text{ then } \gamma(\mathcal{X}^*) \subseteq \mathcal{X}^* \text{ and hence } \gamma(\mathcal{X}^*) = \mathcal{X}^* .

\textbf{Proposition (2.15):}\\
\text{Every fuzzy } \sigma \text{-algebra over a fuzzy set } X^* \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* .

\textbf{Proof:}\\
\text{Let } \mathcal{H}^* \text{ be a fuzzy } \sigma \text{-algebra over a fuzzy set } X^* . \text{ Then by definition of fuzzy } \sigma \text{-algebra, we have } \emptyset^* \in \mathcal{H}^* \text{ and hence } \emptyset^* \subseteq \mathcal{H}^* . \text{ Since } \emptyset^* \subseteq X^* , \text{ then } X^* \in \mathcal{H}^* . \text{ Let } E_1, E_2, ..., E_n \in \mathcal{H}^* . \text{ Consider, } E_m = \emptyset^* \text{ for all } m > n, \text{ then we get } \\
E_1, E_2, ..., E_n, E_{n+1}, E_{n+2}, ... \in \mathcal{H}^* . \text{ Hence, from the definition of fuzzy } \sigma \text{-algebra we have, } \\
E_1^c, E_2^c, ..., E_n^c, E_{n+1}^c, E_{n+2}^c, ... \in \mathcal{H}^* \text{ and } \bigcap_{k=1}^{\infty} E_k^c \in \mathcal{H}^* , \text{ thus } \bigcap_{k=1}^{\infty} E_k^c \subseteq \mathcal{H}^* . \text{ But } \\
\bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{n} E_k \bigcup E_{n+1} \bigcup E_{n+2} ... = \bigcup_{k=1}^{n} E_k \bigcup \emptyset^* \bigcup \emptyset^* \bigcup ... = \bigcup_{k=1}^{n} E_k \\
\text{Thus } \bigcup_{k=1}^{\infty} E_k \in \mathcal{H}^* . \text{ Therefore, } \mathcal{H}^* \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^* . \\
\text{In general, the converse of Proposition (2.15) is not true as shown in the following example:}

\textbf{Example (2.16):}\\
\text{Let } X = \{a, b, c\} \text{ and } \\
\mathcal{H}^* = \{\emptyset^*, \{(a,0.4),(b,0.2),(c,0.3)\}, \{(a,0.2),(b,0.3),(c,0.4)\} \subseteq \mathcal{H}^* , \text{ but not fuzzy } \sigma \text{-algebra , because } \\
\{(a,0.4),(b,0.2),(c,0.3)\} \subseteq \mathcal{H}^* , \text{ but } \\
\{(a,0.4),(b,0.2),(c,0.3)\} \subseteq \mathcal{H}^* \} . \text{ Therefore, } \mathcal{H}^* \text{ is a fuzzy } \gamma \text{-algebra over a fuzzy set } X^*.
Proposition (2.17):
Let $X$ be a nonempty set and $X^{*} \subseteq \mathcal{P}^{*}(X)$. Then $\gamma(X^{*}) \subseteq \sigma(X^{*})$, where $\sigma(X^{*})$ is the smallest fuzzy $\sigma$-algebra over a fuzzy set $X^{*}$ that includes $X^{*}$.

Proof:
Let $X$ be a nonempty set and $X^{*} \subseteq \mathcal{P}^{*}(X)$. Then $\sigma(X^{*})$ is a fuzzy $\sigma$-algebra over a fuzzy set $X^{*}$ that includes $X^{*}$. Since every fuzzy $\sigma$-algebra over a fuzzy set $X^{*}$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^{*}$, then $\sigma(X^{*})$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^{*}$ that includes $X^{*}$. But $\gamma(X^{*})$ is the smallest fuzzy $\gamma$-algebra over a fuzzy set $X^{*}$ that includes $X^{*}$ this implies that $\gamma(X^{*}) \subseteq \sigma(X^{*})$.

Remark (2.18):
Every fuzzy algebra over a fuzzy set $X^{*}$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^{*}$.

Proof:
The result follows from the definition of fuzzy algebra.
In general, the converse of the previous Remark is not true as shown in the following example:

Example(2.19):
Let $X = \{a, b, c\}$ and $\mathcal{H}^{*} = \emptyset^{*}, \{(a, 0.6), (b, 0.3), (c, 0.1)\}$, $\{(a, 0.4), (b, 0.2), (c, 0.4)\}$, $\{(a, 0.6), (b, 0.3), (c, 0.4)\}, X^{*}$. Then $\mathcal{H}^{*}$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^{*}$ but it is not fuzzy algebra, because $\{(a, 0.6), (b, 0.3), (c, 0.1)\} \in \mathcal{H}^{*}$, but $\{(a, 0.6), (b, 0.3), (c, 0.1)\} \subseteq \{(a, 0.4), (b, 0.7), (c, 0.9)\} \notin \mathcal{H}^{*}$.

Proposition (2.20):
Let $X$ be a nonempty set and $X^{*} \subseteq \mathcal{P}^{*}(X)$. Then $\gamma(X^{*}) \subseteq \mathcal{AL}(X^{*})$, where $\mathcal{AL}(X^{*})$ is the smallest fuzzy algebra over a fuzzy set $X^{*}$ that includes $X^{*}$.

Proof:
It is clear, so that it is omitted.

Definition(2.21):
Let $\mathcal{H}^{*}$ be a nonempty collection of fuzzy subsets of fuzzy power set $\mathcal{P}^{*}(X)$ of a nonempty set $X$ that is $\mathcal{H}^{*} \subseteq \mathcal{P}^{*}(X)$ and let $T^{*}$ be a nonempty fuzzy subset of a fuzzy set $X^{*}$ that is $T^{*} \subseteq X^{*}$. Then the restriction of $\mathcal{H}^{*}$ on $T^{*}$ is denoted by $\mathcal{H}^{*}_{T^{*}}$, which is defined as follows:
$\mathcal{H}^{*}_{T^{*}} = \{ F : F = E \cap T^{*}, \text{ for some } E \in \mathcal{H}^{*} \}$.

Example(2.22):
Let $X = \{\omega_{1}, \omega_{2}\}$ and $\mathcal{H}^{*} = \left\{ \{(\omega_{1}, 0.6), (\omega_{2}, 0.2)\}, \{(\omega_{1}, 0.3), (\omega_{2}, 0.5)\} \right\}$. Consider $T^{*} = \{(\omega_{1}, 0.5), (\omega_{2}, 0.5)\}$ and $\mathcal{H}^{*}_{T^{*}} = \left\{ \{(\omega_{1}, 0.5), (\omega_{2}, 0.2)\}, \{(\omega_{1}, 0.3), (\omega_{2}, 0.5)\} \right\}$. Put, $E_{1} = \{(\omega_{1}, 0.6), (\omega_{2}, 0.2)\}$, $E_{2} = \{(\omega_{1}, 0.3), (\omega_{2}, 0.5)\}$, $F_{1} = \{(\omega_{1}, 0.5), (\omega_{2}, 0.2)\}$, $F_{2} = \{(\omega_{1}, 0.3), (\omega_{2}, 0.5)\}$. Then $E_{1}, E_{2} \in \mathcal{H}^{*}$ and $F_{1}, F_{2} \in \mathcal{H}^{*}_{T^{*}}$. 

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Now, $E_1 \cap T^* = \{(\omega_1, \text{Min}\{0.6,0.5\}), (\omega_2, \text{Min}\{0.2,0.5\})\}$
$\quad = \{(\omega_1, 0.5), (\omega_2, 0.2)\} = F_1.$

$E_2 \cap T^* = \{(\omega_1, \text{Min}\{0.3,0.5\}), (\omega_2, \text{Min}\{0.5,0.5\})\}$
$\quad = \{(\omega_1, 0.3), (\omega_2, 0.5)\} = F_2.$ This implies that for any $F \in \mathcal{H}_{I_r}^*$ there is $E \in \mathcal{H}^*$ such that $F = E \cap T^*.$

Therefore, $\mathcal{H}_{I_r}^*$ is the restriction of $\mathcal{H}^*$ on $T^*.$

**Proposition (2.23):**

Let $\mathcal{H}^*$ be a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ and let $T^*$ be a nonempty fuzzy subset of a fuzzy set $X^*.$ Then $\mathcal{H}_{I_r}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $T^*.$

**Proof:**

Since $\mathcal{H}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*$, then $\emptyset^*, X^* \in \mathcal{H}^*.$ Since $T^* \subseteq X^*$, then $\varphi_{T^*}(\omega) \leq \varphi_{X^*}(\omega)$ for all $\omega \in X$ and hence

$X^* \cap T^* = \{\omega, \text{inf}\{\varphi_{X^*}(\omega), \varphi_{T^*}(\omega)\} : \forall \omega \in X\} = \{\omega, \varphi_{T^*}(\omega) : \forall \omega \in X\} = T^*$

Therefore, $T^* \in \mathcal{H}_{I_r}^*.$ Now, $\emptyset \cap T^* = \{\omega, \text{inf}\{\varphi_{\emptyset}(\omega), \varphi_{T^*}(\omega)\} : \forall \omega \in X\}$

$\quad = \{\omega, \varphi_{\emptyset}(\omega) : \forall \omega \in X\} = \emptyset^*.$

Then $\emptyset^* \in \mathcal{H}_{I_r}^*.$ Let $F_1, F_2, \ldots, F_n \in \mathcal{H}_{I_r}^*.$ Then there are $E_1, E_2, \ldots, E_n \in \mathcal{H}^*$ such that $F_k = E_k \cap T^*$ for all $k=1,2,\ldots, n$ which implies that

$\bigcup_{k=1}^n F_k = \bigcup_{k=1}^n (E_k \cap T^*) = (\bigcup_{k=1}^n E_k) \cap T^*.$

Since $\mathcal{H}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $X^*,$ then $\bigcup_{k=1}^n E_k \in \mathcal{H}^*$ and hence by definition of $\mathcal{H}_{I_r}^*,$ we get $\bigcup_{k=1}^n F_k \in \mathcal{H}_{I_r}^*.$ Therefore, $\mathcal{H}_{I_r}^*$ is a fuzzy $\gamma$–algebra over a fuzzy set $T^*.$

**Proposition (2.24):**

Let $\mathcal{H}^*$ be a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ and $E \subseteq T^* \subseteq X^*.$ If $E \in \mathcal{H}^*,$ then $E \in \mathcal{H}_{I_r}^*.$

**Proof:**

Let $\mathcal{H}^*$ be a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ and let $E \subseteq T^* \subseteq X^*.$ Suppose that $E \in \mathcal{H}^*,$ since $E \subseteq T^*,$ then $\varphi_E(\omega) \leq \varphi_{T^*}(\omega)$ $\forall \omega \in X.$ So, we have

$E \cap T^* = \{\omega, \text{inf}\{\varphi_E(\omega), \varphi_{T^*}(\omega)\} : \forall \omega \in X\}$

$\quad = \{\omega, \varphi_E(\omega) : \forall \omega \in X\} = E$

Therefore, $E \in \mathcal{H}_{I_r}^*.$

**Proposition (2.25):**

Let $\mathcal{H}^*$ be a fuzzy $\gamma$–algebra over a fuzzy set $X^*$ and let $T^*$ be a nonempty fuzzy subset of a fuzzy set $X^*$ such that $T^* \in \mathcal{H}^*.$ Then $\{E \subseteq T^* : E \in \mathcal{H}^*\} \subseteq \mathcal{H}_{I_r}^*.$

**Proof:**

Let $F \in \{E \subseteq T^* : E \in \mathcal{H}^*\}.$ Then $F \subseteq T^*$ and $F \in \mathcal{H}^*,$ hence $\varphi_F(\omega) \leq \varphi_{T^*}(\omega)$ $\forall \omega \in X.$ So, we have

$F \cap T^* = \{\omega, \text{inf}\{\varphi_F(\omega), \varphi_{T^*}(\omega)\} : \forall \omega \in X\}$

$\quad = \{\omega, \varphi_F(\omega) : \forall \omega \in X\} = F.$

Which implies that $F \in \mathcal{H}_{I_r}^*.$ Therefore, $\{E \subseteq T^* : E \in \mathcal{H}^*\} \subseteq \mathcal{H}_{I_r}^*.$
Proposition (2.26):

Let $\mathcal{H}^*$ be a fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{X}^*$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$ such that $\mathcal{T}^* \in \mathcal{H}^*$ and $G \cap \mathcal{T}^* \in \mathcal{H}^*$ whenever $G \in \mathcal{H}^*$. Then $\mathcal{H}_{\mathcal{T}^*} = \{ E \subseteq \mathcal{T}^* : E \in \mathcal{H}^* \}$.

Proof:

Let $F \in \{ E \subseteq \mathcal{T}^* : E \in \mathcal{H}^* \}$. Then $F \subseteq \mathcal{T}^*$ and $F \in \mathcal{H}^*$. Hence, $F \cap \mathcal{T}^* = F$, which implies that $F \in \mathcal{H}_{\mathcal{T}^*}$. Therefore, $\{ E \subseteq \mathcal{T}^* : E \in \mathcal{H}^* \} \subseteq \mathcal{H}_{\mathcal{T}^*}$.

Let $F \in \mathcal{H}_{\mathcal{T}^*}$. Then there exists $E \in \mathcal{H}^*$ such that $F = E \cap \mathcal{T}^*$. Since $E, \mathcal{T}^* \in \mathcal{H}^*$, then $E \cap \mathcal{T}^* \in \mathcal{H}^*$, thus $F \in \mathcal{H}^*$. On the other hand, $F = E \cap \mathcal{T}^*$ which implies that $\sigma_F(\omega) = \sigma_{E \cap \mathcal{T}^*}(\omega), \forall \omega \in \mathcal{X}$

Thus $F \subseteq \mathcal{T}^*$. Hence, $F \in \{ E \subseteq \mathcal{T}^* : E \in \mathcal{H}^* \}$ implies that $\mathcal{H}_{\mathcal{T}^*} \subseteq \{ E \subseteq \mathcal{T}^* : E \in \mathcal{H}^* \}$. Therefore, $\mathcal{H}_{\mathcal{T}^*} = \{ E \subseteq \mathcal{T}^* : E \in \mathcal{H}^* \}$.

Proposition (2.27):

Let $\mathcal{X}$ be a nonempty set and $\mathcal{X}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$. Then $\gamma(\mathcal{X}^*_{\mathcal{T}^*})$ is a fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$.

Proof:

The result follows from Proposition(2.12) and Proposition(2.23).

Proposition (2.28):

Let $\mathcal{X}$ be a nonempty set and $\mathcal{X}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$. Then $\gamma(\mathcal{X}^*_{\mathcal{T}^*})$ is the smallest fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathcal{H}^*_{\mathcal{T}^*}$, where

$\gamma(\mathcal{X}^*_{\mathcal{T}^*}) = \bigcap \{ \mathcal{H}^*_{\mathcal{T}^*} : \mathcal{H}^*_{\mathcal{T}^*} \text{ is a fuzzy } \gamma \text{- algebra over a fuzzy set } \mathcal{T}^* \}$

Proof:

From Proposition (2.10), we get $\gamma(\mathcal{X}^*_{\mathcal{T}^*})$ is a fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$. To prove that $\gamma(\mathcal{X}^*_{\mathcal{T}^*}) \supseteq \mathcal{H}^*_{\mathcal{T}^*}$, suppose that $\mathcal{H}^*_{\mathcal{T}^*}$ is a fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathcal{X}^*_{\mathcal{T}^*}$, then $\mathcal{T}^* \subseteq \bigcap \{ \mathcal{H}^*_{\mathcal{T}^*} : \mathcal{H}^*_{\mathcal{T}^*} \text{ is a fuzzy } \gamma \text{- algebra over a fuzzy set } \mathcal{T}^* \}$.

Hence, $\gamma(\mathcal{X}^*_{\mathcal{T}^*}) \supseteq \mathcal{H}^*_{\mathcal{T}^*}$. Now, let $\mathcal{H}^*_{\mathcal{T}^*}$ be a fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathcal{X}^*_{\mathcal{T}^*}$. Then $\mathcal{H}^*_{\mathcal{T}^*} \supseteq \gamma(\mathcal{X}^*_{\mathcal{T}^*})$. Therefore $\gamma(\mathcal{X}^*_{\mathcal{T}^*})$ is the smallest fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$ which contains $\mathcal{X}^*_{\mathcal{T}^*}$.

Lemma (2.29):

Let $\mathcal{X}$ be a nonempty set and $\mathcal{X}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$, define a collection $\mathcal{H}^*$ as:

$\mathcal{H}^* = \{ E \subseteq \mathcal{X}^* : E \cap \mathcal{T}^* \in \gamma(\mathcal{X}^*_{\mathcal{T}^*}) \}$. Then $\mathcal{H}^*$ is a fuzzy $\gamma$- algebra over a fuzzy set $\mathcal{X}^*$. 

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Proof:

Since $\gamma(\mathcal{I}^*_{\mathcal{T^*}})$ is fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$, then $\mathcal{T}^* \subseteq X^*$, $\mathcal{T}^* = X^* \cap \mathcal{T}^*$ and $X^* \in \mathcal{H}^*$. Also, $\emptyset = \emptyset^* \cap \mathcal{T}^*$, then $\emptyset^* \in \mathcal{H}^*$. Let $E_1, E_2, \ldots, E_n \in \mathcal{H}^*$. Then $(E_i \cap \mathcal{T}^*) \in \gamma(\mathcal{I}^*_{\mathcal{T}^*})$, for all $i=1,2,\ldots,n$. Hence, $(\bigcap_{i=1}^n E_i) \cap \mathcal{T}^* \in \gamma(\mathcal{I}^*_{\mathcal{T}^*})$, thus $\bigcup_{i=1}^n E_i \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^*$.

Theorem (2.30):

Let $X$ be a nonempty set and $\Xi \subseteq \mathcal{P}^*(X)$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $X^*$. Then $\gamma(\mathcal{I}^*_{\mathcal{T}^*}) = \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$.

Proof:

Let $F \in \mathcal{I}^*_{\mathcal{T}^*}$, then by Definition(2.21) $F = E \cap \mathcal{T}^*$, for some $E \in \mathcal{I}^*$. But $\mathcal{I}^* \subseteq \gamma(\mathcal{I}^*)$, then $E \in \gamma(\mathcal{I}^*)$, thus $F \in \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$, hence $\mathcal{I}^*_{\mathcal{T}^*} \subseteq \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$. By Proposition(2.28), we have $\gamma(\mathcal{I}^*_{\mathcal{T}^*})$ is the smallest fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathcal{I}^*_{\mathcal{T}^*}$. From Proposition(2.27), $\gamma(\mathcal{I}^*)_{\mathcal{T}^*}$ is fuzzy $\gamma$-algebra over a fuzzy set $\mathcal{T}^*$, then $\gamma(\mathcal{I}^*_{\mathcal{T}^*}) \subseteq \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$. Now, define a collection $\mathcal{H}^*$ as: $\mathcal{H}^* = \{E \subseteq X^* : E \cap \mathcal{T}^* \in \gamma(\mathcal{I}^*_{\mathcal{T}^*})\}$. Then $\mathcal{H}^*$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^*$ by Lemma (2.29). Let $G \in \mathcal{I}^*$, then $G \cap \mathcal{T}^* \subseteq \mathcal{I}^*_{\mathcal{T}^*}$, but $\mathcal{I}^*_{\mathcal{T}^*} \subseteq \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$ implies that $G \cap \mathcal{T}^* \in \gamma(\mathcal{I}^*_{\mathcal{T}^*})$, hence by definition of $\mathcal{H}^*$ we get $G \in \mathcal{H}^*$ and $\mathcal{I}^* \subseteq \mathcal{H}^*$. Let $F \in \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$. Then $F = E \cap \mathcal{T}^*$, for some $E \in \gamma(\mathcal{I}^*)$. $\gamma(\mathcal{I}^*)$ is the smallest fuzzy $\gamma$-algebra over a fuzzy set $X^*$ which includes $\mathcal{I}^*$ by Proposition (2.12) and $\mathcal{H}^*$ is a fuzzy $\gamma$-algebra over a fuzzy set $X^*$ which contains $\mathcal{I}^*$, then $\gamma(\mathcal{I}^*) \subseteq \mathcal{H}^*$ and $E \in \mathcal{H}^*$, hence by definition of $\mathcal{H}^*$ we get $E \cap \mathcal{T}^* \in \gamma(\mathcal{I}^*_{\mathcal{T}^*})$, thus $F \in \gamma(\mathcal{I}^*_{\mathcal{T}^*})$. Consequently, $\gamma(\mathcal{I}^*)_{\mathcal{T}^*} \subseteq \gamma(\mathcal{I}^*_{\mathcal{T}^*})$. Therefore, $\gamma(\mathcal{I}^*_{\mathcal{T}^*}) = \gamma(\mathcal{I}^*)_{\mathcal{T}^*}$.

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