A Bicharacteristic Scheme for the Numerical Computation of Two-Dimensional Converging Shock Waves

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ABSTRACT

A 2d unsteady bicharacteristic scheme with shock fitting is presented and its characteristic step, shock point step and boundary step are described. The bicharacteristic scheme is compared with an UNO scheme and the Moretti scheme. Its capabilities are illustrated by computing a converging, deformed shock wave.

INTRODUCTION

Commonly used difference schemes are capable of computing complex flow patterns and shock configurations, but they smear out shocks and other discontinuities. To avoid the effects of shock smearing, shock fitting has to be used. This is most naturally done by the method of bicharacteristics.

Characteristic Normal Form

Introducing the substantial derivate $D_0 = D_t = \partial_t + (\vec{v}\nabla)$, a directional derivative operator $D_B := (\vec{B}\nabla) = \partial_t + ((\vec{v} - a\vec{g})\nabla)$ along a bicharacteristic, the abbreviations $v_g := \vec{v}\vec{g}$, $\partial_g := \vec{g}\nabla$ and the logarithmic pressure $P$ and density $\varrho$, the Euler equations can be written as

\[
\begin{align*}
D_t \varrho &= \frac{1}{\gamma}D_t P & \text{energy equation} \\
D_t P &= -\gamma \nabla \vec{v} & \text{continuity equation} \\
D_t \vec{v} &= -\frac{a^2}{\gamma} \nabla P & \text{momentum equation}
\end{align*}
\]

or, in characteristic normal form:

| characteristic differential equation | direction |
|-------------------------------------|-----------|
| $D_0 \varrho = \frac{1}{\gamma}D_0 P$ | $\frac{1}{\vec{v}}$ |
| $D_B v_g - \frac{a}{\gamma} D_B P = a(\nabla \vec{v} - \partial_g v_g)$ | $\frac{1}{\vec{v} - a\vec{g}}$ |

where the characteristic differential equations (CDEs) are valid only on their specific characteristic surfaces, i.e. the particle path $\vec{C}_0 = (1, \vec{v})$ and the Monge cone, generated by the bicharacteristics $\vec{B} = (1, \vec{v} - a\vec{g})$. The unit vector $\vec{g}$ singles out a specific bicharacteristic on the Monge cone (fig. 1).

Characteristic Step

The bicharacteristic scheme was constructed on a Cartesian grid, in analogy to Hartree’s method [1]. Using an educated guess of $\vec{v}$ and $a$ in a grid point at the new time level the bicharacteristics are drawn backwards in the directions of the coordinate lines. At the intersection points of these bicharacteristics with the old time level, called 'footpoints’, the quantities $P, \varrho, \vec{v}$ are interpolated in second order from the grid points with a 2d nine point least-squares method (fig. 1). Using the discretized CDEs, the quantities $P, \varrho, \vec{v}$ then can be determined in first order at the new time level. A corrector step, which uses the first order values of $\vec{v}$ and $a$ to reconstruct the bicharacteristics, yields the quantities $P, \varrho, \vec{v}$ in second order, eliminating the lateral derivatives on the new time level according to Butler [2].

Treatment Of Shocks

To begin with, a shock point is moved using its previous velocity. At its new location, the (oblique) Rankine-Hugoniot conditions are used to determine the downstream values. With these values, a bicharacteristic is drawn backwards antiparallel to the shock front normal (fig. 2). The correct position of the new shock point is found by iteration, such that both the Rankine-Hugoniot conditions and the differential equation along the bicharacteristic are fulfilled.

Shock points are moved along their trajectories. The density of shock points on a shock front can be kept constant by creating or deleting shock points. Two pointers are attached to each shock point, pointing to

Figure 1: Characteristic Step
its predecessor and successor. Though all shock points are stored in an arbitrary sequence in a single array, shock contours and their normal vectors can nevertheless be reconstructed with the help of the pointers. New shock points are detected with an algorithm according to Moretti [3] at each new time level.

**Shock Point Step**

When a new time level is computed, it is first done regardless of all discontinuities. Then, the shock points are moved. Invalid grid points, which were computed by drawing bicharacteristics crossing a shock surface, are updated afterwards with the help of a boundary step:

When a Monge cone is intersected by a boundary, e.g. a shock wave, the bicharacteristics are no longer drawn backwards in coordinate directions but in directions normal and tangential to the boundary (fig. 3). This improves the accuracy of the step considerably. The intersection points of the bicharacteristics with the boundary are determined, the footpoint values are interpolated at the boundary surface, and the grid point is updated. The boundary step is only of first order because the lateral derivatives cannot be eliminated with the Butler procedure in case the bicharacteristics of a monge cone are cut off at different time levels.

**Interpolation**

In the vicinity of discontinuity surfaces the interpolation algorithm used for ordinary grid points becomes ill-conditioned, producing instabilities. The most preferable way of interpolation in this case proved to be a two-dimensional, second order least-squares pattern, sized $7 \times 7$ grid points, in which all upstream grid points are ignored, but in which all shock points of this area are taken into account. According to our experience an interpolation in smaller regions will render the scheme unstable.

**COMPARISON WITH OTHER NUMERICAL SCHEMES**

The bicharacteristic scheme was tested with an analytical solution, and computations of converging shock waves were compared with the corresponding results of a UNO scheme and the Moretti scheme (a $\lambda$-scheme working with shock fitting). As expected, the bicharacteristic scheme turned out to be superior to the UNO scheme regarding the prediction of position and velocity of the shock front. The shock fitting algorithm provides exact information on position, direction of propagation, and local Mach number of the shock wave.

The Moretti scheme [4] proved to be comparable in accuracy to the bicharacteristic scheme (fig. 4). The aforementioned is faster (by a factor 1.5, assuming Courant Number 2.0). This is due to the time consuming interpolations in the bicharacteristic scheme. We found that the convergence of the shock front position towards a known solution by refinement of the grid is slightly faster in the bicharacteristic scheme, due to its boundary step.

Recently, Nasuti and Onofri [5] extended Moretti's original shock fitting algorithm to handle triple points. We implemented their extensions in our version of the $\lambda$-scheme and added some further improvements (to...
We found that the shock fitting algorithm proposed by these authors still has some disadvantages. For example, the fragmentation of the shock front, described in the next paragraph occurs too late and is not enough pronounced in case the shock contour has an unfavourable orientation to the computational grid; whereas our shock fitting algorithm is not influenced by the relative position of shock front and computational grid.

**COMPUTATION OF CONVERGING SHOCK WAVES**

The computation shown here (fig. 5 and 6) started with a slightly deformed shock wave of Mach No. 2.5, at radius 1.0. Isopycnics of a time step immediately before and some time after fragmentation occurs are shown in Figure 5 and 6, respectively. Presently, the computation of converging shock waves extends to the instant of fully developed fragmentation, just before reflection of the leading shock. The extension of the method of bicharacteristics to proceed beyond this point is under work and does not pose any fundamental problems.

Calculating converging shock waves with a bicharacteristic scheme, one has to make sure that the Mach stem will develop correctly. As mentioned above, information on the downstream values of the physical quantities reaches a shock point by transportation along a retrograde bicharacteristic. The footpoint of this bicharacteristic generally is located only fractions of a grid width apart from the shock front. However, the physical mechanism which generates the Mach stem is a density hump which gradually steepens as the shock wave converges. Finally, this density hump takes the form of a bow shock \[6, 7\]. Emerging gradually from a compression wave, its shock profile is smeared out similar to shocks in common difference schemes over a number of grid points. For this reason the retrograde bicharacteristic sees only a slight increasing of e.g. density values where a marked density hump should be located. Therefore, the local shock velocity could be calculated too small and, hence, the formation of the Mach stem could be delayed.

To overcome this problem, the developing bow shock has to be detected with a pattern recognition algorithm, and consecutively be treated with the shock fitting algorithm.

As mentioned above, the density hump gives rise to new developing shocks in the vicinity of the triple points (fig. 7). These lateral waves are not fitted yet because of the considerable effort, this might take. However, the computed solution is acceptable as long as the lateral waves remain sufficiently weak. Depending on our initialisation the Mach Number of the lateral waves was only slightly above 1.0 and therefore the results can be considered as valid.

**OUTLOOK**

The following improvements seem feasible:
- The treatment of intersecting shock point trajectories should be reconsidered.
- Newly developing shock fronts and discontinuity lines should be detected and fitted as proposed by Moretti.

The bicharacteristic scheme with shock fitting presented here provides great accuracy and physical insight and allows a variety of applications: It could be used to compute, e.g., channel flows, flows around wings, MHD problems, star pulsation, and non-equilibrium flows. The scheme can also profitably be used as a standard to estimate the capabilities of other schemes.
Due to the shock fitting algorithm the leading shock wave, which is represented here by density values exactly fulfilling the Rankine-Hugoniot relations, is not smeared out. The eminent area between lateral shock wave and slip line consists of those fluid particles which passed both the leading shock wave and the lateral wave.

**REFERENCES**

[1] Hartree, D. R., "Some practical methods of using characteristics in the calculation of non-steady compressible flows". US Atomic Energy Comm. Report AECU-2713 (1953)

[2] Butler, D. S., The numerical solution of hyperbolic systems of partial differential equations in three independent variables, Proceedings of the Royal Society Vol. A255, (1960), p. 232-252

[3] Moretti, G., Detection and fitting of two-dimensional shocks, Notes Num. Fluid Mech. Vol 20, (1987), p. 239-246

[4] Moretti, G., Efficient Euler Solver with many applications, AIAA Journal, Vol. 26 No. 6, (1988), p. 655-660

[5] Nasuti, F., Onofri, M., Analysis of Unsteady Supersonic Viscous Flows by a Shock-Fitting Technique, AIAA Journal, Vol. 34, No.7, (1996), p. 1428-1434

[6] Watanabe, M., Takayama, K., Stability of converging cylindrical shock waves, Shock waves, Springer Verlag, (1991), p. 149-160

[7] Ben Dor, G. Shock Wave Reflection Phenomena, Springer Verlag, (1992)

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