Drivers, hitting times, and weldings in Loewner’s equation

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Funding information
National Science Foundation, Grant/Award Number: DMS-1928930

Abstract
In addition to conformal weldings $\varphi$, simple curves $\gamma$ growing in the upper half plane generate driving functions $\xi$ and hitting times $\tau$ through Loewner’s differential equation. While the Loewner transform $\gamma \mapsto \xi$ and its inverse $\xi \mapsto \gamma$ have been carefully examined, less attention has been paid to the maps $\xi \mapsto \tau \mapsto \varphi$. We study the continuity properties of these latter transformations and show that uniform driver convergence implies uniform hitting time convergence and uniform welding convergence, even when the corresponding curves do not converge. Welding convergence implies neither hitting time nor driver convergence, while hitting time convergence implies driver convergence in (at least) the case of constant drivers. As an application, we show that a curve $\gamma$ of finite Loewner energy can be well approximated by an energy minimizer that matches $\gamma$’s welding on a sufficiently-fine mesh.

MSC 2020
30C55, 34A12 (primary), 60J67 (secondary)

1 | INTRODUCTION AND MAIN RESULTS

1.1 | Loewner’s equation and associated transforms

A hundred years ago, Charles Loewner [25] showed the evolution of conformal maps $g_t$ from the slit disk $\mathbb{D}\setminus \gamma([0,t])$ back to $\mathbb{D}$, where $\gamma$ is a curve growing into $\mathbb{D}$ from its boundary, satisfy a
differential equation that in effect transforms \( \gamma \) into a continuous driving function \( \lambda(t) = g_t(\gamma(t)) \) taking values on \( \partial \mathbb{D} \). Loewner’s approach played an important role in de Branges’ proof \([4]\) of the Bieberbach conjecture in 1985, and received renewed interest following the ground-breaking 2000 work of Schramm \([33]\), who showed that the random curves generated by using a Brownian driving function give the only-possible conformally invariant scaling limits of a number of discrete models from statistical physics. These processes, typically normalized to live in the upper half plane \( \mathbb{H} \) now instead of Loewner’s \( \mathbb{D} \), and known as \textit{Schramm–Loewner–Evolutions} \( \text{SLE}_k \), have been intensely studied since and continue to be a topic of active research.\(^\dagger\)

In the setting of the upper half plane, Loewner’s method takes a simple curve \( \gamma : [0, T] \to \mathbb{H} \cup \{x\} \) and produces a real-valued driving function. In the process, however, it also produces a hitting time function \( \tau \) and a conformal welding \( \varphi \). To see this, consider the reversed Loewner flow in \( \mathbb{H} \), described by normalized conformal maps \( h_t : \mathbb{H} \to \mathbb{H} \setminus \gamma_t \) that satisfy the ordinary differential equation

\[
\frac{\partial}{\partial t} h_t(z) = \frac{-2}{h_t(z) - \xi(t)}, \quad h_0(z) = z
\]  

(see §2 for precise definitions and any unexplained terminology). Here, the \( h_t \) map \( \mathbb{H} \) to the complement of the curve \( \gamma_t \) (or more generally “compact \( \mathbb{H} \)-hull”) generated by the continuous function \( \xi : [0, T] \to \mathbb{R} \) on \( [0, t] \). The dynamics in (1) extends to points \( x \in \mathbb{R} \) away from \( \xi(0) \), which by the ODE flow toward the driving function \( \xi \) until the hitting time

\[
\tau(x) := \inf\{ t \geq 0 : \liminf_{s \uparrow t} |h_s(x) - \xi(s)| = 0 \}.
\]  

(2)

When \( \xi \) is sufficiently regular, \( \gamma_t \) neither intersects itself nor the real line, other than at its base \( \xi(t) \), and in this case, the function \( x \mapsto \tau(x) \) is continuous, and exactly two points hit \( \xi(t) \) to be “welded together” at each time \( t \) (see Lemma 3.1 below). By (1), this pair started on opposite sides of \( \xi(0) \), and we obtain another map, the conformal welding \( \varphi \), by sending \( x < \xi(0) \) to the unique \( y = \varphi(x) > \xi(0) \), which satisfies \( \tau(x) = \tau(y) \).

We thus have a natural sequence of mappings

\[
\gamma \mapsto \xi \mapsto \tau \mapsto \varphi.
\]  

(3)

The first arrow is the Loewner transform, and both it and its inverse have been carefully studied.\(^\ddagger\) The latter maps have received less attention, however, and we motivate our study of them by summarizing what is currently known.

\(^\dagger\) The literature is vast, and a nonexhaustive sampling of references is \([1, 2, 7–11, 13, 14, 19, 21, 30, 34, 35, 51]\).

\(^\ddagger\) Results are typically formulated in terms of the forward driver \( \lambda \) and include: \( \gamma \mapsto \lambda \) is continuous as a map from \( C([0, T]) \) to \( C([0, T]) \) \([17, \text{Thm. 6.2}], [50, \text{Thm. 1.8}], \) but not uniformly so \([23, \text{Figure 6}] \). The inverse map \( \lambda \mapsto \gamma \) is not continuous with respect to capacity parametrization on the curves \( \gamma \) \([18, \text{Ex. 4.49}] \), but it is when the \( \gamma \) are equipped with a Carathéodory-type topology on their normalized Riemann maps \([18, \text{Prop. 4.47}] \). In addition, restricting to more regular \( \lambda \) yields continuity for \( \lambda \mapsto \gamma \) with respect to the supremum norm (or finer topologies) on the \( \gamma \). See \([23, \text{Thm. 4.1}] \) and \([48, \text{Thm. 3.3}] \) for \( \lambda \) with locally small Hölder-1/2 norm, \([10, \text{Thm. 2(v), (vi)}] \) for \( \lambda \) of finite Loewner energy, and \([37, \text{Thm. 1.2}] \) for \( \lambda \) of “locally regular” bounded variation. Additionally, estimates in \([41, \text{§3}] \) and \([42, \text{§2}] \) can be used to give continuity for “weakly Hölder-1/2” \( \lambda \); see \([38] \) for an application to \( \text{SLE}_2 \) approximation. See also \([36, \text{Thm. 1.2}] \) for continuity in the case of “bidirectionally and generically close” \( \lambda \).

We also note that some authors call \( \xi \mapsto \gamma \) the Loewner transform. We reserve the term for \( \gamma \mapsto \xi \), however, since this was the direction in Loewner’s original work.
1.2 Motivation from known results

Lind [22] used connections between \( \xi, \tau \) and \( \varphi \) as part of her argument that a driver \( \xi \) with Hölder-\( 1/2 \) seminorm \( |\xi|_{1/2} < 4 \) generates a simple curve. She proved that, given \( |\xi|_{1/2} < 4 \), \( \xi \) welds exactly two points at each time, \( \tau(x) \approx (x - \xi(0))^2 \), and \( \varphi \) satisfies \( (\varphi(x - h) - \varphi(x))/\varphi(x) - \varphi(x + h) \approx 1 \) [22, Lemma 3, Corollary 1, Lemma 4]. In our Lemma 3.1, we generalize the first result to hold whenever \( \xi \) generates a simple curve (which is known to be a broader class than the \( \xi \) satisfying \( |\xi|_{1/2} < 4 \)).

A connection between \( \xi \) to \( \tau \) appeared in Tran and Yuan's work on the topological support of SLE\( _{\kappa} \) [39], where they proved that if two drivers \( \xi \) and \( \tilde{\xi} \) are \( \delta \)-close in sup norm with \( \xi(0) = \tilde{\xi}(0) < y \), then \( \tau(y + \delta) \leq T \) implies \( \tilde{\tau}(y) \leq T \) [39, Lemma 5.1], where \( \tau \) and \( \tilde{\tau} \) are the hitting times under the flows generated by \( \xi \) and \( \tilde{\xi} \), respectively. We give a new, simplified proof of a slightly more general result in Lemma 3.2, and expand on the idea to show continuity properties of \( \xi \mapsto \tau \) and \( \tau \mapsto \varphi \).

Another recent work [3] studied the connection between \( \xi \) and \( \varphi \) in the case that \( \xi\kappa(t) = \sqrt{\kappa B(t)} \) is a Brownian motion run at speed \( \kappa \). In it, the authors showed that the conformal welding \( \xi \mapsto \varphi_{\kappa}(\omega, x) \) associated to the resulting SLE\( _{\kappa} \) has a modification that is a.s. jointly continuous in \( \kappa \) and \( x \) for \( (\kappa, x) \in [0,4] \times (-\infty,0] \). Note that this is a type of continuity statement about the driver-to-welding map \( \xi \mapsto \varphi \), since when \( \kappa' \) is sufficiently close to \( \kappa \), the Brownian drivers \( \xi_{\kappa'}(\omega, t) = \sqrt{\kappa' B(\omega, t)} \) and \( \xi_{\kappa}(\omega, t) = \sqrt{\kappa B(\omega, t)} \) are close on a finite interval \([0, T]\) for fixed \( \omega \). We generalize this joint continuity in \( x \) and \( \xi \) to hold for any drivers producing simple curves in Lemma 4.4.

One component of the proof in [3] was to show that, a.s., for all \( 0 \leq \kappa \leq 4 \) simultaneously, the hitting times \( x \mapsto \tau_{\kappa}(\omega, x) \) are continuous and strictly increasing on either side of \( 0 = \xi_{\kappa}(0) \) [3, Prop. 4.1(a)]. This actually served as inspiration for the present study, as we wondered what could be said always using deterministic Loewner theory, and not just with probability 1. Our Lemma 3.1 shows that \( \tau \) always has these properties when \( \xi \) generates a simple curve. In addition, the authors showed [3, Prop. 4.1(b)] that \( \kappa \mapsto \tau_{\kappa}(\omega, \cdot) \) is continuous as a map from the reals to the space of continuous functions, which is a type of continuity statement about \( \xi \mapsto \tau \). We extend this to all \( \xi \) generating simple curves in Theorem 3.4.

This latter type of continuity question, for maps of functions to functions, is where our main interest lies, and this is what we study for \( \xi \mapsto \varphi \), \( \xi \mapsto \tau \) and \( \tau \mapsto \varphi \). Our other results mentioned above serve as tools for our investigation or are corollaries of our main theorems.

1.3 Main results

We equip spaces of continuous functions with the uniform norm, and restrict to the class of \( \xi, \tau \), and \( \varphi \) corresponding to simple curves \( \gamma \). This is natural to obtain continuous hitting times, as well as conformal weldings in the classical sense of the term. We obtain the following main results. Note that we state some of these informally; in each case, see the referenced result for the precise statement, as well as Remark 1.1 below.

(i) \( \xi \mapsto \tau \) is continuous (Theorem 3.4) but not uniformly so (Lemma 3.6). That is, if \( \xi_n \) and \( \xi \) generate simple curves and \( \xi_n \xrightarrow{u} \xi \), then \( \tau_n \xrightarrow{u} \tau \). However, we can find \( \xi_n, \tilde{\xi}_n \) with \( \|\xi_n - \tilde{\xi}_n\|_\infty \to 0 \) but where \( \|\tau_n - \tilde{\tau}_n\|_\infty > \epsilon \). In addition, \( (x, \xi) \mapsto \tau(x; \xi) \) is pointwise jointly continuous (Lemma 3.7).
(iii) $\xi \mapsto \tau^{-1}$ is Lipschitz continuous, with optimal Lipschitz constant 1 (Theorem 3.8).

(iii) $\tau \mapsto \varphi$ is continuous (Theorem 4.1), but not uniformly so (Lemma 4.3). In addition, $(x; \xi) \mapsto \varphi(x; \xi)$ is pointwise jointly continuous (Lemma 4.4).

(v) $\tau \mapsto \xi$ is well defined whenever the corresponding curve is conformally removable, and $\tau \mapsto \xi$ is continuous at $\tau_C$, the hitting time function of the constant driver $C(t) \equiv C \in \mathbb{R}$ (Theorem 3.13). That is, if $\tau(\cdot; \xi) = \tau(\cdot; \tilde{\xi})$ for curves in this class, then $\xi = \tilde{\xi}$. Furthermore, if $\xi_n$ are drivers corresponding to simple curves with hitting times $\tau_n$ satisfying $\tau_n \rightharpoonup \tau_C$, then $\xi_n \rightharpoonup C$.

(vi) Both $\varphi \mapsto \tau$ and $\varphi \mapsto \xi$ are well defined whenever the corresponding curve is conformally removable (Lemma 4.8). However, neither $\varphi \mapsto \tau$ nor $\varphi \mapsto \xi$ is continuous (Lemma 4.9).

Put succinctly, driver convergence implies hitting time and welding convergence, and is stronger than welding convergence. Additionally, in at least the case of constant drivers, hitting time convergence implies driver convergence.

1.3.1 Discussion

One of the contributions of these results is that no regularity is assumed on the drivers $\xi_n, \xi$ other than that they belong to $S([0, T])$, the class of drivers generating simple curves on $[0, T]$. It is well known, however, that the inverse Loewner transform $\xi \mapsto \gamma$ is not continuous: there exist $\xi_n, \xi \in S([0, T])$ such that $\|\xi_n - \xi\|_{[0, T]} \to 0$ but where the corresponding curves $\gamma_n$ do not even have subsequential limits in their half-plane capacity parametrizations, let alone uniformly converge [18, Ex. 4.49]. Our results in (i) and (iv) above say that the uniform topologies on $\tau$ and $\varphi$ are oblivious to this pathological behavior of the $\gamma_n$. That is, the $\tau_n$ and $\varphi_n$ generated by $\xi_n$ still converge to the $\tau$ and $\varphi$ generated by $\xi$, even though the $\gamma_n$ do not converge to $\gamma$. Furthermore, by result (ii) above, one even has quantitative convergence of the points $x_n < 0 < y_n$ welded at a given time $t$ by $\xi_n$ to the points $x < 0 < y$ welded at the same time by $\xi$.

We also comment that our continuity result for $\tau \mapsto \xi$ in (v) above is rather preliminary, as it only covers continuity at hitting times corresponding to constant drivers $C$. This result is not immediately obvious, however, as we show in Lemma 4.9 that the map $\varphi \mapsto \xi$ is not continuous at weldings corresponding to constant drivers $C$. So, our results show that the hitting times $\tau$ are “better behaved” than the conformal weldings $\varphi$. We also find it interesting that the proof of the continuity of $\tau \mapsto \xi$ is not entirely trivial, despite the simplicity of the statement; see §3.4.

We attempted to build a driver with large oscillations that we suspected could be a counterexample to continuity of $\tau \mapsto \xi$ for more general $\tau$, but numerical simulations showed that our construction still converged for relatively smooth data. We describe this construction and the simulations in §3.4.1 as positive evidence for a broader result.

To show that maps are not continuous or not uniformly continuous, we often construct explicit examples, and we point out one construction that may be surprising. To prove $\tau \mapsto \varphi$ is not uniformly continuous for result (iii) above, we actually show that $\tau \mapsto \varphi$ is, in some sense, as nonuniformly continuous as possible. Specifically, our example yields that for any $\epsilon > 0$ and
$N > 0$, there exists $\tau, \tilde{\tau}$ and corresponding weldings $\varphi, \tilde{\varphi}$ where $\|\tau - \tilde{\tau}\|_{\infty} < \varepsilon$ but $\|\varphi - \tilde{\varphi}\|_{\infty} > N$. See Lemma 4.6. Recall that this is despite the fact that $\tau \mapsto \varphi$ is (point-wise) continuous.

We also highlight two other contributions. In §4.4, we show that our results for the $\xi \mapsto \varphi$ map in (iv) imply that, for any finite collections of pairs $\{(x_j, y_j)\}_{j=1}^N$ with

$$x_N < x_{N-1} < \cdots < x_1 < y_1 < \cdots < y_N,$$

there exists a curve of minimal Loewner energy which welds each $x_j$ to $y_j$. We use this in Theorem 4.11 to show that we can closely approximate any given finite-energy curve $\gamma([0, T])$ by an energy minimizer $\gamma_n$ whose welding agrees with $\gamma$’s on a sufficiently-fine discretization. Here, “closely approximate” means that the capacity parametrizations of $\gamma$ and $\gamma_n$ are uniformly close (see Theorem 4.11 for the precise statement). This approximation scheme is related, although not identical, to the welding zipper algorithm of Donald Marshall; see the discussion at the start of §4.4.

Finally, in the Appendix we list two integral formulas relating all our main actors $\xi, \tau, \varphi$ that appear to have thus far escaped notice in the literature. These could be useful for creating bounds and estimates.

Remark 1.1. Some of the statements in the list above, as alluded to, are imprecise as there is technicality regarding domains. For instance, for the $\xi \mapsto \tau$ result in (i), we assume the uniform convergence $\|\xi_n - \xi\|_{\infty[0, T]} \to 0$ on a fixed time interval $[0, T]$. However, that does not imply that all the hitting time functions $\tau_n, \tau$ share a common domain (see Example 3.5), and so we prove that if the domains of $\tau_n$ and $\tau$ are $[a_n, b_n]$ and $[a, b]$, respectively, then $a_n \to a$, $b_n \to b$ and $\tau_n \to \tau$ on any compact subinterval $[c, d] \subset (a, b)$. (In addition, our result in (ii) gives sharp quantitative control on $|a_n - a|$ and $|b_n - b|$.)

In this sense, we mean $\xi \mapsto \tau$ is continuous, and many of the other results above are similar. We are thus often using “continuity” in a somewhat informal sense; in particular, we do not attempt to equip the space $\tilde{\mathcal{C}}$ of continuous functions with different domains with a topology $\tilde{\mathcal{C}}$ that would make, for instance, $\xi \mapsto \tau$ continuous from $(C([0, T]), \|\cdot\|_{\infty[0, T]})$ to $(\tilde{\mathcal{C}}, \tilde{\mathcal{C}})$, although this may be possible.

### 1.4 Methods

For results concerning the $\xi \mapsto \tau$, $\tau \mapsto \varphi$ and $\xi \mapsto \varphi$ maps, we primarily rely on Lemma 3.1 combined with the surprising power of Lemma 3.2 and formula (25). Proofs that maps are not continuous or not uniformly continuous are based on new, explicitly-constructed examples. Some fresh machinery was needed to say anything about the $\tau \mapsto \xi$ direction, and our main tool here is Lemma 3.11, which says that the farther a driver welds two points $x_0 < y_0$ from their initial average $(x_0 + y_0)/2$, the lower the hitting time.

### 1.5 Organization

In §2 we establish notation, review the basics of deterministic Loewner chains, and prove some elementary lemmas. Section 3 deals with drivers $\xi$ and hitting times $\tau$; we prove our main lemmas in §3.1, the results on $\xi \mapsto \tau$ and $\xi \mapsto \tau^{-1}$ in §3.2 and §3.3, respectively, and cover the $\tau \mapsto \xi$ map.
in §3.4. Section 4 moves on to consider weldings \( \varphi \), proving continuity of \( \xi \mapsto \varphi \) and \( \tau \mapsto \varphi \) in §4.1 and §4.2, respectively. We show by example in §4.3 that neither \( \varphi \mapsto \tau \) nor \( \varphi \mapsto \xi \) is continuous. We give our application of Theorem 4.1 to minimal-energy curves in §4.4, and we conclude with open problems in §5, and then the Appendix.

## 2 | NOTATION AND PRELIMINARIES

### 2.1 | Basics of Loewner theory

In this section, we establish notation and recall basic results concerning the Loewner equation; see [17] or [18] for more thorough treatments. We frame the theory largely in terms of the reverse/upward maps \( h_t \), as they induce the hitting times \( \tau \).

Indeed, \( h_t : \mathbb{H} \to \mathbb{H} \setminus \gamma_t([0, t]) \) in (1) is the unique conformal map that satisfies

\[
h_t(z) = z + O(1/z), \quad z \to \infty.
\]

In other words, \( h_t \) fixes \( \infty \), and scaling and translation only occur locally around \( \gamma_t \), not at \( \infty \). We assume that \( \gamma_t \) is parametrized by half-plane capacity, in which case the above expansion is actually

\[
h_t(z) = z - \frac{2t}{z} + O(1/z^2), \quad z \to \infty,
\]

and we say that the half-plane capacity \( h_{\text{cap}}(\gamma_t) \) of \( \gamma_t \) is \( 2t \) (so note time \( t \) corresponds to \( h_{\text{cap}} \) \( 2t \)). The extension of \( h_t(z) \) to the real line maps \( \xi(0) \) to the tip \( \gamma_t(t) \) of the curve \( \gamma_t \) generated by \( \xi \) on \([0, t] \), whereas the base of \( \gamma_t \) is at \( \xi(t) \). Note that we use the subscript on \( \gamma_t \) to emphasize the fact that the entire curve changes as \( t \) progresses: it both grows from its base and is conformally deformed.

We can actually define \( h_{\text{cap}}(K) \) whenever \( K \subset \mathbb{H} \) is a compact \( \mathbb{H} \)-hull, which is to say, \( K \) is bounded, relatively closed in \( \mathbb{H} \), and \( \mathbb{H} \setminus K \) is simply connected. In this case, there is again a unique conformal map \( h_K : \mathbb{H} \to \mathbb{H} \setminus K \) satisfying (4) at \( \infty \), and with expansion

\[
h_K(z) = z - \frac{a_1}{z} + O(1/z^2), \quad z \to \infty.
\]

We define \( h_{\text{cap}}(K) := a_1 \), which is positive when \( K \neq \emptyset \). It follows from the definition that \( h_{\text{cap}} \) satisfies \( h_{\text{cap}}(K + x) = h_{\text{cap}}(K) \)

\[
h_{\text{cap}}(rK) = r^2 h_{\text{cap}}(K)
\]

for \( r \geq 0 \). Also, \( K_1 \subset K_2 \) implies \( h_{\text{cap}}(K_1) \leq h_{\text{cap}}(K_2) \). Considering \( K = \overline{B_1(0)} \cap \mathbb{H} \) as a concrete example, we find that \( h_K^{-1}(z) = z + 1/z \), and thus

\[
h_{\text{cap}} \left( \overline{B_1(0)} \cap \mathbb{H} \right) = 1.
\]

We will interchangeably call the dynamics given by (1) the “upward Loewner flow” and the “reverse Loewner flow.” The former is not entirely standard, but is natural from the point of view.
that the curves $\gamma_t$ grow upward into $\mathbb{H}$ from $\xi$’s position in $\mathbb{R}$. The downward or forward map $g_t$ is the unique map from $\mathbb{H}\setminus\gamma([0,t])$ to $\mathbb{H}$ that is $z + O(1/z)$ near infinity, and here, the expansion corresponding to (5) is

$$g_t(z) = z + \frac{2t}{z} + O(1/z^2), \quad z \to \infty,$$

while the corresponding Loewner equation is

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z. \quad (8)$$

The $h_t$ and $g_t$, $\gamma_t$ and $\gamma$, and $\xi$ and $\lambda$ are all related, as we proceed to explain. Let $\gamma : [0,T] \to \mathbb{H} \cup \{x\}$ be a fixed, simple curve with $\gamma(0) = x$ (which is what our notation for the range means).

For us, the downward/forward driving function $\lambda$ is just the reversal of $\xi$, $\lambda(t) = \xi(T-t)$. (If we wish to normalize to have $\xi(0) = 0$, we may take $\lambda(t) = \xi(T-t) - \xi(T)$, or equivalently, $\xi(t) = \lambda(T-t) - \lambda(T)$.)

We typically write $\lambda$ for the downward driving function and $\xi$ for the upward.

Under this reversal, the curve $\gamma_t$ generated by $\xi$ on $[0,t]$ is the conformal image $g_{T-t}(\gamma([T-t,T]))$ of the last $t$ units $\gamma([T-t,T])$ of $\gamma$ under $g_{T-t}$. That is, we have the following.

**Lemma 2.1.** Let $T < \infty$ and $\gamma : [0,T] \to \mathbb{H} \cup \{x\}$ be a simple curve with driving function $\lambda$, and let $\xi$ be the reversal $\xi(t) = \lambda(T-t)$. The $h_t$ and $g_t$ maps satisfying (1) and (8) for $\xi$ and $\lambda$, respectively, are related by

$$h_t = g_{T-t} \circ g_t^{-1}, \quad 0 \leq t \leq T. \quad (9)$$

**Proof.** Note that the equation holds for $t = 0$ and that by (8), $g_{T-t} \circ g_t^{-1}$ also satisfies (1). We conclude that (9) holds for all $t$ by uniqueness of solution to ODE’s. \qed

So, for the $g_t$, the underlying $\gamma$ is a fixed curve that is growing at its tip, whereas in the case of the $h_t$, $\gamma_t$ grows at its base. We work almost exclusively with the $h_t$, but explain this background to help the intuition of readers more accustomed to the downward/forward flow given by the $g_t$.

Recall that driving functions $\xi, \lambda$ are always continuous and thus members of $C([0,T])$. We write $C_0$ for the set of continuous functions starting at zero, and $S, S_0$ for those $\xi \in C, C_0$ that generate a simple curve $\gamma^\xi$ in upward Loewner flow, and we append the time domain $[0,T]$ to the notation for function spaces as needed. By “$\xi$ generating a simple curve,” we mean that the final curve $\gamma^\xi$ generated on time $[0,T]$ by $\xi$ does not self-intersect and also does not touch $\mathbb{R}$ other than at its base: $\gamma^\xi \cap \mathbb{R} = \{\xi(T)\}$. It is not hard to see that this is equivalent to saying that the curve generated by $\xi$ on any time interval $[t_1, t_2] \subset [0,T]$ has these same two properties.

Recall that well-known elements of $S$ include linear drivers [15], drivers with one-sided Hölder-1/2 norm less than four [22, 23, 52] and, a.s., scaled Brownian motion $\sqrt{\kappa}B(t)$ when $0 \leq \kappa \leq 4$ [30].

Points $x_0 < \xi(0) < y_0$ under (1) flow along the real line toward $\xi$, and we interchangeably write

$$x(t) = x(t; \xi) = h_t(x_0; \xi) = h_t(x_0)$$

for the image of $x_0$ after $t$ units of time under driver $\xi$, and similarly for $y(t)$.
2.2 Hitting times

For \( \xi \in C([0, T]) \) and \( x_0 \in \mathbb{R} \setminus \xi(0) \), the dynamics in (1) becomes

\[
\dot{x}(t) = -2 \frac{x(t) - \xi(t)}{x(t) - \xi(t)}, \quad x(0) = x_0. \tag{10}
\]

As noted in the introduction, the hitting time \( \tau(x_0) \) of \( x_0 \) is then

\[
\tau(x) = \inf \{ t \geq 0 : \liminf_{s \to t} |x(s) - \xi(s)| = 0 \}, \tag{11}
\]

and so, the flow \( t \mapsto x(t; \xi) \) is well defined on \([0, \tau(x_0))\). Note that if there are no such times \( t \) for some \( x \), then \( \tau(x) = +\infty \), and we say that \( x \) is not welded by \( \xi \). We also remark that (11) defines \( \tau \) regardless of whether or not \( \xi \) generates a simple curve.

Suppose \( x_0 \neq \xi(0) \) and \( \tau(x_0) < \infty \). By (10), \( x_0 \) flows monotonically toward \( \xi \), and so, \( |x(t)| \) is bounded by the maximum of \( |x_0| \) and \( \max_{[0, \tau(x_0))] |\xi(t)| \). In particular, \( \lim_{t \to \tau(x_0)} x(t) \) exists, and by (11) must be \( \xi(\tau(x_0)) \). Thus, we can extend \( t \mapsto x(t) \) from \([0, \tau(x_0))\) to \([0, \tau(x_0)]\) by setting \( x(\tau(x_0)) = \xi(\tau(x_0)) \), and we have

\[
\tau(x) = \inf \{ t \geq 0 : \lim_{s \to t} |x(s) - \xi(s)| = 0 \} = \inf \{ t \geq 0 : x(t) = \xi(t) \}. \tag{12}
\]

We recall that if \( t \mapsto \xi(t) \) generates the curve \( \gamma_t \) on \([0, t]\), then the driver for the scaled curve \( r \gamma_t, r > 0 \), is

\[
\tau(\xi(t) = \tau(t/r^2). \tag{13}
\]

The hitting times for \( r \gamma_t \) have a similar transformation rule.

**Lemma 2.2.** Let \( \tau \) be the hitting times generated by \( \xi \), and let \( \tau_r \) be those corresponding to the scaled driver \( \xi_r(t) = r \xi(t/r^2) \), where \( r > 0 \). Then

\[
\tau_r(x) = r^2 \tau(x/r). \tag{14}
\]

Note that the formula is intuitively obvious: we take the time \( \tau(x/r) \) for the corresponding point \( x/r \) of the unscaled curve and then apply the capacity scaling rule \( hcap(rK) = r^2 hcap(K) \). The reader is invited to make this into a proof.

We also note that for \( \xi \in C_0 \),

\[
\tau(x; -\xi) = \tau(-x; \xi). \tag{15}
\]

We conclude our introduction to hitting times with two examples — one trivial (but still instructive), and one less elementary.

**Example 2.3.** In the case of the zero driver \( 0(t) \equiv 0 \), it is not hard to see that the map satisfying (1) is \( h_t(z) = \sqrt{z^2 - 4t} \), and so, the points mapping to the base of the curve \( 0 = 0(t) \) under the
extension of \( h_t \) to \( \mathbb{R} \) are \( \pm 2\sqrt{t} \), yielding the hitting times
\[
\tau(x; 0) = \frac{x^2}{4}, \quad x \in \mathbb{R}.
\] (16)

The \( Cx^2 \) formula appears frequently, and so, may be informally considered as the “prototypical hitting time.” Other instances include:

- Lind’s generalization of (16), which showed that if \( \xi \)'s Hölder-1/2 seminorm satisfies \( |\xi|_{1/2} < 4 \), then \( \tau(x; 0) = x^2 \) [22, Corollary 1].
- Hitting times for \( \text{SLE}_\kappa \). For the reverse \( \text{SLE}_\kappa \) flow with \( 0 < \kappa < 4 \) (i.e., the solution to (1) with the driver \( \xi(t) = \sqrt{\kappa}B(t) \), \( B(t) \) standard Brownian motion), the hitting time of \( x_0 \neq 0 \) is a random variable
\[
T_{\kappa}(x_0) = \frac{x_0^2}{\kappa} X_{\kappa},
\] (17)
where \( X_{\kappa} \) has Inverse-Gamma\( \left( \frac{1}{2} + \frac{2}{\kappa}, \frac{1}{2} \right) \) distribution. In particular,
\[
\mathbb{E}(T_{\kappa}(x_0)) = \frac{x_0^2}{4 - \kappa}.
\]

A sketch of how to see (17) is as follows: note that for \( \xi(t) = \sqrt{\kappa}B(t) \), \( H_t(x) = \frac{1}{\sqrt{\kappa}} h_t(\sqrt{\kappa}x) - B(t) \) is distributed as a Bessel process of dimension \( d = 1 - \frac{4}{\kappa} \), and so, the hitting time \( \tilde{T}_{\kappa}(x_0) \) for the origin of \( x_0 > 0 \) has distribution \( x_0^2 X_{\kappa} \) by [20, Prop. 2.9]. The capacity scaling rule (6) then gives (17).

We formulate the second example as a lemma.

**Lemma 2.4.** For \( c \in \mathbb{R} \setminus \{0\} \), the hitting times \( \tau_c \) of the linear driver \( \xi_c(t) = ct \) are
\[
\tau_c(x) = \frac{x}{c} + \frac{2}{c^2} \log \left( \frac{2}{2 + cx} \right),
\] (18)
where \(-2/c < x \) when \( c \) is positive, and \( x < -2/c \) when \( c \) is negative. The hitting time is infinite for other \( x \).

The formula (18) is not unrelated to the \( Cx^2 \) formula; see (28) below, for instance.

**Proof.** We suppose first that \( c = 1 \) and note that the Loewner equation for the centered process \( H_t(z) := h_t(z) - t \) is
\[
\dot{H}_t = -\frac{2 + H_t}{H_t} = -\frac{dH_t}{dF}
\]
for the function \( F(H_t) = H_t - 2\log(2 + H_t) \) (we are adapting the clever idea in [15, §3] to the upward-flow setting). Thus, \( \partial_t(F(H_t)) = -1 \), yielding for \( x(t) = h_t(x_0) \) that
\[
x(t) - t - 2 \log(2 + x(t) - t) - (x_0 - 2 \log(2 + x_0)) = -t.
\]
If $\tau(x_0; \xi) < \infty$, we send $t \to \tau$ in the above and find

$$-2 \log(2) - x_0 + 2 \log(2 + x_0) = -\tau,$$

(19)

and so, $x_0 > -2$ and we have (18) in the case $c = 1$.

For general $c > 0$, by (13), we see that scaling the curves $\gamma_t$ generated by $\xi(t) = t$ by $r = 1/c$ yields the driver $\xi_c(t) = ct$, and so, (18) for general $c > 0$ follows from (19) and (14). For $\bar{c} = -c < 0$, we then apply (15). □

2.3 | Conformal welding

We show in Lemma 3.1 that, for $\xi \in S$, $x \mapsto \tau(x)$ is strictly increasing as one moves away from $\xi(0)$. We can thus think of $\tau$ as consisting of two invertible functions, the left and the right hitting times, which we denote by

$$\tau_- := \tau|_{x \leq \xi(0)} \quad \text{and} \quad \tau_+ := \tau|_{y \geq \xi(0)}.$$

(20)

As mentioned in the introduction, the conformal welding associated to $\xi \in S$ is the homeomorphism of intervals on either side of $\xi(0)$ that satisfies $\tau(x) = \tau(\varphi(x))$ for all $x$ with $\tau(x) < \infty$. We take the convention that $\varphi$ maps from the left of $\xi(0)$ to the right, and so more precisely, for $x \leq \xi(0)$,

$$\tau_-(x) = \tau_+(\varphi(x)) \quad \text{or} \quad \tau_+^{-1} \circ \tau_-(x) = \varphi(x).$$

(21)

The welding can also be defined in terms of the maps $h_t$ via $h_t(\varphi(x)) = h_t(x)$ for $t \geq \tau(x)$.

2.4 | Other notation

For a curve welding $\varphi : [-a, 0] \to [0, b]$, when one (or both) of $-a, b$ is infinite, we mean that $\varphi$ is defined on $(-a, 0]$ and $\lim_{x \to a^+} \varphi(x) = b$.

For a continuous function $f$ on an interval $[a, b]$, we write $\|f\|_{\infty[a,b]} := \max_{x \in [a,b]} |f(x)|$. Of course, $f_n \xrightarrow{u} f$ on $[a, b]$ means $\|f - f_n\|_{\infty[a,b]} \to 0$.

We write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$, and $A(x) \asymp B(x)$ means that there exists constant $C \geq 1$ such that

$$\frac{1}{C}B(x) \leq A(x) \leq CB(x)$$

for all values of $x$.

2.5 | Elementary lemmas

We will use the following two easy lemmas. The first is similar to Dini’s theorem and specifies a situation where we can upgrade from pointwise to uniform convergence.
Lemma 2.5. Let \([a, b] \subset \mathbb{R}\) be a closed interval and \(f_n : [a, b] \to \mathbb{R}\) a sequence of functions such that each \(f_n\) is nonincreasing or nondecreasing. Suppose \(f : [a, b] \to \mathbb{R}\) is continuous and \(f_n \to f\) pointwise on \([a, b]\). Then we have \(\|f_n - f\|_{\infty[a,b]} \to 0\).

Note that we do not assume that the \(f_n\) are continuous or that the monotonicity across the sequence is the same, that is, some \(f_n\) may be increasing and some decreasing.

Proof. Choose \(\delta > 0\) such that \(|f(x) - f(y)| < \varepsilon/2\) whenever \(a \leq x, y \leq b\) satisfy \(|x - y| < \delta\), and let \(a = x_0 < x_1 < \cdots < x_m = b\) be a partition of \([a, b]\) of mesh size less than \(\delta\). Let \(N\) be large enough so that \(|f_n(x_j) - f(x_j)| < \varepsilon/2\) for all \(j \in \{1, \ldots, m\}\) whenever \(n \geq N\). Choosing such an \(n\), suppose that \(f_n\) is nondecreasing. Then, for any \(x_j < x < x_{j+1}\),

\[
f_n(x) - f(x) \leq f_n(x_{j+1}) - f(x_{j+1}) + f(x_{j+1}) - f(x) < \varepsilon,
\]

and similarly,

\[
f(x) - f_n(x) \leq f(x) - f(x_j) + f(x_j) - f_n(x_j) < \varepsilon.
\]

The argument is similar when \(f_n\) is nonincreasing. \(\square\)

Lemma 2.6. Let \(f_n, f : \mathbb{R} \to \mathbb{R}\) be continuous and strictly increasing with \(f_n \to f\) point-wise. Then \(f_n^{-1} \to f^{-1}\) point-wise.

Proof. Let \(y \in \mathbb{R}\), fix \(\varepsilon > 0\) and set \(x := f^{-1}(y)\). By monotonicity, \(f(x - \varepsilon) < f(x) < f(x + \varepsilon)\). Let \(\delta := |f(x) - f(x - \varepsilon)| \wedge |f(x) - f(x + \varepsilon)|\), and choose \(N\) such that \(|f_n(x) \pm \varepsilon) - f_n(x) \pm \varepsilon) < \delta/2\) whenever \(n \geq N\). Then \(f_n(x - \varepsilon) = y < f_n(x + \varepsilon)\), and consequently, \(x - \varepsilon < f_n^{-1}(y) < x + \varepsilon\) by monotonicity. \(\square\)

3 | CONTINUITY PROPERTIES OF DRIVERS TO HITTING TIMES AND DRIVERS TO INVERSE HITTING TIMES

3.1 | Lemmas

We establish some tools before proving our continuity results. Our first lemma gives basic properties of the hitting times \(x \mapsto \tau(x; \xi)\) when \(\xi \in S\). This is a generalization of [3, Prop. 4.1(a)] and [22, Lemma 3]; the former, because this always holds, rather than only almost surely, and the latter, because we only assume \(\xi \in S([0, T])\), not that \(\xi \in \text{Höll}(1/2)\) with \(|\xi|_{1/2} < 4\).

Lemma 3.1. If \(\xi \in S([0, T]), 0 < T \leq \infty, x \mapsto \tau(x; \xi) = \tau(x; \xi)\) is continuous. It is strictly increasing for \(x \geq \xi(0)\) and strictly decreasing for \(x \leq \xi(0)\). In particular, for each \(0 < t \leq T\), there are exactly two points \(x < \xi(0) < y\) such that \(\tau(x; \xi) = t = \tau(y; \xi)\).

Proof. Without loss of generality \(\xi(0) = 0\), and by symmetry, it suffices to prove continuity and monotonicity for \(y > 0\). If \(0 < y_1 < y_2\) and \(t < \tau(y_1)\), then (1) yields

\[
\frac{d}{dt}(y_2(t) - y_1(t)) = \frac{2(y_2(t) - y_1(t))}{(y_2(t) - \xi(t))(y_1(t) - \xi(t))} > 0.
\]
Thus, the points are getting further apart for \( t < \tau(y_1) \), showing \( y_1(\tau(y_1)) = \xi(\tau(y_1)) < y_2(\tau(y_1)) \). Since \( \xi \) is continuous, \( \tau(y_1) < \tau(y_2) \), and we see \( y \mapsto \tau(y) \) is strictly increasing.

In particular, \( \tau \) can only have jump discontinuities. To see that this actually does not happen, pick \( 0 < y_1 < y_3 \) arbitrarily and let \( t_2 \in (\tau(y_1), \tau(y_2)) \). We show that there exists \( y_2 \) with \( \tau(y_2) = t_2 \). Indeed, map up with \( h_{t_2} \). Since the curve \( \gamma_{t_2} \) generated by \( \xi([0, t_2]) \) is simple, the prime end \( \xi(t_2) \) of \( \mathbb{H} \setminus \gamma_{t_2} \) corresponds to exactly two preimages \( x_2 < 0 < y_2 \) under the extension of \( h_{t_2} \).

Since
\[
h_{t_2}(y_2) = \xi(t_2),
\]
\( \tau(y_2) \leq t_2 \) by definition of the hitting time.

Suppose that \( y_2 \)'s hitting time is actually earlier, that is,
\[
\tau(y_2) = : t'_2 < t_2,
\]
which means we also have that \( h_{t'_2}(y_2) = \xi(t'_2) \). Let \( \tilde{h}_t \) be the map which, for \( t \geq t'_2 \), solves
\[
\partial_t \tilde{h}_t(z) = -2 \tilde{h}_t(z) - \xi(t), \quad \tilde{h}_{t'_2}(z) = z,
\]
that is, which maps to the complement of the curve segment generated by \( \xi \) for times \( t \geq t'_2 \). Then, for \( t > t'_2 \), \( \tilde{h}_t \) sends \( \xi(t'_2) \) to the tip of the nontrivial curve segment generated on \([t'_2, t]\) by \( \xi \), which thus has positive imaginary part by the assumption \( \xi \in S \) and the fact that its capacity is \( 2(t - t'_2) \) (see [17, Lemma 4.2] for the latter). Since \( h_{t_2} = \tilde{h}_{t_2-t'_2} \circ h_{t'_2} \), we thus see \( \text{Im} h_{t_2}(y_2) > 0 \), which contradicts (23). We conclude that \( \tau(y_2) = t_2 \) and that there are no jump discontinuities in \( \tau \). \( \square \)

We will need the following two observations stemming from the Loewner equation (1). First, for fixed \( \delta \in \mathbb{R} \), we have the shifting formula
\[
h_t(x + \delta; \xi + \delta) = h_t(x; \xi) + \delta.
\]
This holds for any \( x \in \mathbb{H} \) for all \( t \) (since solutions starting in \( \mathbb{H} \) last forever) and for \( x \in \mathbb{R} \) when \( 0 \leq t \leq \tau(x) \).

Second, given two drivers \( \xi_1 \leq \xi_2 \) and a point \( y_0 \) with \( \xi_2(0) < y_0 \) and a time \( t \leq \tau(y_0; \xi_2) \), driver monotonicity combined with the Loewner equation yield
\[
h_t(y_0; \xi_2) \leq h_t(y_0; \xi_1).
\]

The following lemma gives the key inequality for our \( \xi \mapsto \tau \) arguments. While we borrow the statement from [39], we offer a new, succinct proof, and also drop the requirement that the drivers start at the same location. As in [39], we do not assume that the drivers generate simple curves.

**Lemma 3.2** [39, Lemma 5.1]. Suppose \( \xi, \tilde{\xi} \in C([0, T]) \) with \( \|\xi - \tilde{\xi}\|_{C[0, T]} \leq \delta \). Fix \( y \) with \( \tilde{\xi}(0) < y \). If \( t < \tau(y; \tilde{\xi}) \), then \( t < \tau(y + \delta; \xi) \). Similarly, for \( x < \tilde{\xi}(0) \), if \( t < \tau(x; \tilde{\xi}) \), then \( t < \tau(x - \delta; \xi) \).
**Proof.** By symmetry, it suffices to prove the statement for \( y > \tilde{\xi}(0) \). Observing \( t < \tau(y; \tilde{\xi}) \) is equivalent to \( \tilde{\xi}(t) < h_t(y; \tilde{\xi}) \), we use (25) and (26) to conclude
\[
\xi(t) \leq \tilde{\xi}(t) + \delta < h_t(y; \tilde{\xi}) + \delta = h_t(y + \delta; \tilde{\xi} + \delta) \leq h_t(y + \delta; \xi),
\]
and hence \( t < \tau(y + \delta; \xi) \). \( \square \)

**Corollary 3.3.** Suppose \( \xi, \tilde{\xi} \in C([0, T]), 0 < T \leq \infty, \) with \( \| \xi - \tilde{\xi} \|_{\infty[0, T]} \leq \delta \). Then, for any \( y > \xi(0) + \delta \),
\[
\tau(y - \delta; \xi) \leq \tau(y; \tilde{\xi}) \leq \tau(y + \delta; \xi).
\]
Similarly, for \( x < \xi(0) - \delta \), \( \tau(x - \delta; \xi) \geq \tau(x; \tilde{\xi}) \geq \tau(x + \delta; \xi) \).

As no assumption is made on the finiteness of the hitting times, part of the statement of the corollary is that (27) holds whether or not each of the \( \tau \)'s is finite or infinite. Note also that we again do not assume that \( \xi, \tilde{\xi} \in S \).

**Proof.** By symmetry, it suffices to consider \( y > \xi(0) + \delta \). If \( \tau(y - \delta; \xi) = \infty \), then by using \( t = n \) for large \( n \) in Lemma 3.2, we find \( n < \tau(y; \tilde{\xi}) \), and so, \( \tau(y; \tilde{\xi}) = \infty \) as well. Thus, (27) holds when at least two of the members are infinite.

Suppose that \( \tau(y; \tilde{\xi}) < \infty \). Since \( y > \tilde{\xi}(0) \) by assumption, by using times \( \ell_n = \tau(y; \tilde{\xi}) - 1/n \) in Lemma 3.2, we obtain the right-most inequality in (27) in the limit. The assumption \( \tau(y; \tilde{\xi}) < \infty \) and Lemma 3.2 also yield \( \tau(y - \delta; \xi) < \infty \), and then exchanging the roles of \( \tilde{\xi} \) and \( \xi \) and replacing \( y \) with \( y - \delta \) in the above argument yields the left inequality as well. \( \square \)

### 3.2 Continuity of drivers to hitting times

Plotting the formula (18) for hitting times \( \tau_c \) of the linear drivers \( \xi(t) = ct \) for various \( c \), as in Figure 1, shows that the graphs approach a parabola as \( c \to 0 \). And indeed,
\[
\tau_c(x) = \frac{x^2}{4} - \frac{cx^3}{12} + O(c^2), \quad c \to 0.
\]
Comparing (16), we see that $\tau(\cdot; ct)$ is converging to $\tau(\cdot; 0)$. Our first continuity result says that this is no accident, but always happens for uniformly converging $\xi_n, \xi \in S$. 

**Theorem 3.4.** Let $0 < T \leq \infty$ and $\xi, \xi_n \in S([0, T])$ be drivers with hitting times $\tau(\cdot; \xi) : [a, b] \to [0, T]$ and $\tau(\cdot; \xi_n) : [a_n, b_n] \to [0, T]$, respectively, where $-\infty \leq a < b \leq \infty$, $-\infty \leq a_n < b_n \leq \infty$. If $\xi_n \xrightarrow{w} \xi$ on $[0, T]$, then $a_n \to a$, $b_n \to b$, and $\tau(x; \xi_n) \xrightarrow{w} \tau(x; \xi)$ on any $[c, d] \subset (-a, b)$.

We preface the proof with several comments.

- In contrast to some of the results in §3.1, we restrict here to $S$ to avail ourselves of the continuity of $\tau$ from Lemma 3.1.
- With regard to the technicality that we cannot include the endpoints $-a$ and $b$ in the convergence, see Example 3.5 below.
- We give the sharp quantitative control

$$\max\{ |a_n - a|, |b_n - b| \} \leq \| \xi_n - \xi \|_{\infty}[0, T]$$

for the domain endpoints in Theorem 3.8 below.
- This generalizes [3, Prop. 4.1(b)] to hold for all drivers generating simple curves, and not just the a.s. Brownian motion case.
- The notation is slightly imprecise when one of the interval endpoints is $\infty$. For instance, if $a = -\infty$ and $b \in \mathbb{R}$, we mean that $\tau(\cdot; \xi)$ is finite on $(-\infty, b)$ and $a_n$ is either identically $-\infty$ or is eventually less than any $-N$ for all large $n$. If $T = \infty$, we mean that $\xi, \xi_n$ are defined on $[0, \infty)$.

**Proof.** Fix an interval $[c, d] \subset (a, b)$ and $\varepsilon > 0$, and choose $\delta$ small enough such that the following three conditions hold: $[c - \delta, d + \delta] \subset (a, b)$,

$$\max\{ \tau(\xi(0) - 2\delta; \xi), \tau(\xi(0) + 2\delta; \xi) \} < \varepsilon, \quad (29)$$

and the modulus of continuity $\omega$ of $\tau(\cdot; \xi)$ on $[c - \delta, d + \delta]$ satisfies

$$\omega(\delta; [c, d]) := \sup\{ |\tau(u; \xi) - \tau(v; \xi)| : c - \delta \leq u, v \leq d + \delta, |u - v| \leq \delta \} \leq \varepsilon. \quad (30)$$

By Corollary 3.3, the first condition yields that $\tau(x; \xi_n) < \infty$ for all $x \in [c, d]$ when $n$ is large. For $x \in [c, \xi(0) - \delta] \cup (\xi(0) + \delta, d]$, by (27), again, we have

$$|\tau(x; \xi_n) - \tau(x; \xi)| \leq \omega(\delta; [c, d]) < \varepsilon.$$

If $\xi(0) - \delta \leq x \leq \xi(0) + \delta$, then

$$|\tau(x; \xi_n) - \tau(x; \xi)| \leq \tau(x; \xi_n) + \tau(x; \xi) \leq \tau(x; \xi_n) + \varepsilon.$$
by (29). Furthermore, using monotonicity and the fact that \( \xi_n(0) \in [\xi(0) - \delta, \xi(0) + \delta] \) for all large \( n \), we see

\[
\tau(x; \xi_n) \leq \max\{\tau(\xi(0) - \delta; \xi_n), \tau(\xi(0) + \delta; \xi_n)\} \\
\leq \max\{\tau(\xi(0) - 2\delta; \xi), \tau(\xi(0) + 2\delta; \xi)\} < \epsilon
\]

by (27) and (29). We conclude that \( \tau(\cdot; \xi_n) \) is uniformly close to \( \tau(\cdot; \xi) \) on \([c, d]\) for large \( n \).

For the welding interval endpoints, by symmetry, it suffices to show that \( b_n \to b \). Suppose first that \( b < \infty \) and let \( \epsilon > 0 \). Since by the above \( \tau(b - \epsilon; \xi_n) \to \tau(b - \epsilon; \xi) < \infty \), monotonicity of the hitting times yields \( b - \epsilon \leq b_n \) for large \( n \), and thus, \( b \leq \liminf_{n \to \infty} b_n \). On the other hand, however,

\[
+\infty = \tau(b + \epsilon/2; \xi) \leq \tau(b + \epsilon/2 + \|\xi_n - \xi\|_{\infty}; \xi_n) \leq \tau(b + \epsilon; \xi_n)
\]

for all large \( n \), where the first inequality is by (27). Thus, \( b_n \leq b + \epsilon \) for all large \( n \), showing \( \limsup_{n \to \infty} b_n \leq b \).

If \( b = +\infty \), then by the first paragraph, we have \( \tau(x; \xi_n) \to \tau(x; \xi) \) for any fixed, finite \( x > \xi(0) \). This implies \( b_n \geq x \) for all large \( n \), and thus, \( b_n \to \infty \).

\[\square\]

**Example 3.5.** We cannot conclude that the convergence of hitting times in Theorem 3.4 extends all the way to the endpoints of the welding interval \([a, b]\), since, for instance, we may have \( b_n < b \) for all \( n \). For example, consider \( \xi \equiv 0 \) on \( 0 \leq t \leq 1/4 \), which generates the vertical line segment \( \gamma = [0, i] \). For the curves generating \( \xi_n \), set \( \alpha_n := 1 - \frac{1}{n} \) and consider straight line segments \( \gamma_n \) from 0 to

\[
\gamma_n(\tau_n) = \alpha_n^{-\frac{1}{n}}(1 - \alpha_n)^{\frac{1}{n}} e^{i\alpha_n \pi} = \left(1 + \frac{4}{n^2} + O(n^{-4})\right) e^{i\alpha_n \pi}.
\]

The conformal map \( F_n : \mathbb{H} \to \mathbb{H} \setminus \gamma_n([0, \tau_n]) \) taking 0 to the tip \( \gamma_n(\tau_n) \) that satisfies \( F_n(z) = z + O(1) \) as \( z \to \infty \) is explicitly

\[
F_n(z) = \left(z - \sqrt{\frac{\alpha_n}{1 - \alpha_n}}\right)\left(z + \sqrt{\frac{1 - \alpha_n}{\alpha_n}}\right)^{1 - \alpha_n}
\]

\[
= z + \frac{4}{\sqrt{n^2 - 4}} - \frac{1}{2z} + O(z^{-2}), \quad z \to \infty
\]

(see [27, The Slit Algorithm], e.g.). From this, see that the centered welding \( \varphi_n : [-a_n, 0] \to [0, b_n] \) for \( \gamma_n \) has endpoints

\[
-a_n = -\sqrt{\frac{1 - \alpha_n}{\alpha_n}}, \quad b_n = \sqrt{\alpha_n} \frac{1}{1 - \alpha_n} = \sqrt{\frac{1/2 - 1/n}{1/2 + 1/n}} < 1,
\]

that the total time for \( \gamma_n \) is \( \tau_n \equiv 1/4 \) and that the driver \( \lambda_n \) for \( \gamma_n \) has terminal value \( \lambda_n(1/4) = 4/\sqrt{n^2 - 4} \). As it is well known that the driver is \( \lambda_n(t) = C_n \sqrt{t} \) (see [18, Example 4.12], e.g.), \( \lambda_n \) monotonically increases in \( t \), and so, the reversed drivers \( \xi_n(t) := \lambda_n(1/4 - t) - \lambda_n(1/4) \in \)
\(C_0([0,1/4])\) converge uniformly to zero on \([0,1/4]\) as \(n \to \infty\). However, as they only weld \([-a_n, b_n]\), we see

\[
+\infty \equiv \tau(1; \xi_n) \not\to \tau(1; \xi) = 1/4.
\]

While the mapping \(\xi \mapsto \tau\) is continuous in the sense of Theorem 3.4, the next lemma says that there is no global modulus of continuity.

**Lemma 3.6.** There exist two sequences of drivers \(\xi_n, \bar{\xi}_n \in S_0([0, T])\) such that \(\|\xi_n - \bar{\xi}_n\|_{\infty[0, T]} \to 0\) but where for some points \(y_n\) and fixed \(\varepsilon > 0\),

\[
\tau(y_n; \xi_n) < \infty, \quad \tau(y_n; \bar{\xi}_n) < \infty, \quad \text{and} \quad |\tau(y_n; \xi_n) - \tau(y_n; \bar{\xi}_n)| \geq \varepsilon
\]

for all \(n\).

**Proof.** We first construct \(\xi_n \in S_0\) and \(\bar{\xi}_n \in S\), and then modify the construction so that all drivers are in \(S_0\). To that end, we begin by defining

\[
\xi(t) = \xi(t, \delta) := \begin{cases} 
-\frac{t}{\delta} & 0 \leq t \leq \delta \\
-1 & \delta < t,
\end{cases}
\]

and \(\bar{\xi}(t) = \bar{\xi}(t, \delta) := \xi(t) - \delta\). Note that \(\xi, \bar{\xi} \in S\) since \(\xi\) is piece-wise linear. Considering \(y_0 = y_{0,\delta} := 2\delta\), we have

\[
y_0(t) := y_0(t; \xi) = 2\delta - \frac{t}{\delta}
\]

explicitly solves the Loewner equation (1) on \(0 \leq t \leq \delta\) for driver \(\xi\), and thus, \(y_0(\delta) = 2\delta - 1\). We conclude by (16) that

\[
\tau(y_0; \xi) = \delta + \left(\frac{2\delta}{4}\right)^2 = \delta + \delta^2.
\]

(32)

The explicit movement \(\bar{y}_0(t) := y_0(t; \bar{\xi})\) of \(y_0\) under \(\bar{\xi}\) is more complicated, on the other hand, but a very coarse estimate will suffice for our purposes. Noting that

\[
\frac{d}{dt} \bar{y}_0(0) = -\frac{2}{3\delta} > -\frac{1}{\delta} = \frac{d}{dt} \xi(0),
\]

we see that \(\bar{y}_0(t) - \bar{\xi}(t)\) is increasing on \([0, \delta]\), and so,

\[
\bar{y}_0(\delta) - \bar{y}_0(0) \geq -\frac{2}{3\delta} \delta = -\frac{2}{3},
\]

(33)

yielding

\[
\tau(y_0; \bar{\xi}) \geq \delta + \left(\frac{1}{3} + 3\delta\right) > \frac{1}{36}.
\]

(34)
Thus, setting \( \xi_n(t) := \xi(t, 1/n) \), \( \tilde{\xi}_n(t) := \tilde{\xi}(t, 1/n) \), and \( y_n := y_{0,1/n} \), we have \( |\tau(y_n; \xi_n) - \tau(y_n; \tilde{\xi}_n)| \) bounded below while the drivers become arbitrarily close as \( n \) grows.

We can adjust this construction so that \( \tilde{\xi}(\cdot, \delta) \in S_0 \) by starting \( \tilde{\xi} \) at zero and having it move linearly in time \( \delta^2 \) to \( -\delta \) (i.e., extremely fast), whereas \( \xi(\cdot, \delta) \) remains at zero for \( 0 \leq t \leq \delta^2 \). We then proceed as in the above construction, setting \( y_0 = y_{0,\delta} \) to be the point that has image \( y(\delta^2; \xi) = 2\delta \) under \( \xi \), which is explicitly \( y_0 = 2\sqrt{2}\delta \). Then, \( 2\delta = y(\delta^2; \xi) < y(\delta^2; \tilde{\xi}) \), and so, we still obtain the lower bound in (34) for \( \tau(y_0; \xi) \), while from (32), \( \tau(y_0; \xi) = \delta + 2\delta^2 \).

We also easily have the following pointwise joint continuity of \( (x, \xi) \mapsto \tau(x; \xi) \).

**Lemma 3.7.** Let \( 0 < T \leq \infty \) and \( \xi \in S([0, T]) \), and suppose that \( \tau(\cdot; \xi) \) is finite on the interval \( [-a, b] \). If \( x \in (a, b) \) and \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon, x, \xi) \) such that whenever \( |\tilde{x} - x| < \delta \) and \( \tilde{\xi} \in S([0, T]) \) satisfies \( \|\tilde{\xi} - \xi\|_{[0, T]} < \delta \),

\[
|\tau(\tilde{x}; \tilde{\xi}) - \tau(x; \xi)| < \epsilon.
\]

**Proof.** Let \( \omega(\cdot; \tau) \) be the modulus of continuity of the uniformly continuous function \( \tau(\cdot; \xi) \) on \( [-a, b] \), and suppose first that \( x = \xi(0) \). Choose \( \delta \) such that \( [x - 2\delta, x + 2\delta] \subset [-a, b] \) and \( \omega(2\delta; \tau) < \epsilon \). By hypothesis \( \tilde{\xi}(0) < x + \delta \), and so, Lemma 3.2 yields

\[
\tau(x + \delta; \tilde{\xi}) \leq \tau(x + 2\delta; \xi) < \epsilon,
\]

showing by monotonicity that \( \tau(\tilde{x}; \tilde{\xi}) < \epsilon \) for all \( \tilde{x} \in [\tilde{\xi}(0), x + \delta] \). A parallel argument holds for \( \tilde{x} \in [x - \delta, \tilde{\xi}(0)] \), and so, we conclude that

\[
\tau(\tilde{x}; \tilde{\xi}) = |\tau(\tilde{x}; \tilde{\xi}) - \tau(x; \xi)| < \epsilon
\]

whenever \( |\tilde{x} - x| < \delta \) and \( \|\tilde{\xi} - \xi\| < \delta \).

In the case that \( x \neq \xi(0) \), by symmetry, we may assume \( \xi(0) < x \), and we choose \( \delta \) such that \( [x - 2\delta, x + 2\delta] \subset (\xi(0), b] \) and \( \omega(\delta; \tau) < \epsilon/2 \). Then, if \( |\tilde{x} - x| < \delta \) and \( \|\tilde{\xi} - \xi\| < \delta \), we see from (27) that

\[
|\tau(\tilde{x}; \tilde{\xi}) - \tau(\tilde{x}; \xi)| < \frac{\epsilon}{2},
\]

and thus,

\[
|\tau(\tilde{x}; \tilde{\xi}) - \tau(x; \xi)| \leq |\tau(\tilde{x}; \tilde{\xi}) - \tau(\tilde{x}; \xi)| + |\tau(\tilde{x}; \xi) - \tau(x; \xi)| < \epsilon.
\]

**3.3 Continuity of drivers to inverse hitting times**

The hitting times \( \tau(\cdot; \xi) \) associated to \( \xi \in S([0, T]) \) are strictly monotonic on either side of \( \xi(0) \) by Lemma 3.1, and so we may consider them as two invertible functions \( \tau_{\pm} \), as in (20). The following lemma says that the maps \( \xi \mapsto \tau_{+}^{-1} \) and \( \xi \mapsto \tau_{-}^{-1} \) are Lipschitz continuous with Lipschitz constant 1.
Theorem 3.8. For \( \xi, \hat{\xi} \in S([0, T]) \), \( 0 < T \leq \infty \), we have

\[
\| \tau_+^{-1}(\cdot; \xi) - \tau_+^{-1}(\cdot; \hat{\xi}) \|_{\infty[0, T]} \leq \| \xi - \hat{\xi} \|_{\infty[0, T]}.
\] (35)

Furthermore, \( C = 1 \) is the best-possible Lipschitz constant.

Proof. We show (35) for \( \tau_+^{-1} \); the argument for \( \tau_-^{-1} \) is similar. We start with two observations. Setting \( \| \xi - \hat{\xi} \|_{\infty[0, T]} =: \delta \), we first note by (25) and (26) that

\[
h_t(y_0; \xi) + \delta = h_t(y_0 + \delta; \xi + \delta) \leq h_t(y_0 + \delta; \hat{\xi})
\] (36)

for all \( t \leq T \) and \( \xi(0) \leq y_0 \) such that \( t \leq \tau_+(y_0; \xi) \) (which implies \( t \leq \tau_+(y_0 + \delta; \hat{\xi}) \) by Lemma 3.2). We second observe that if for some driver \( \eta \), we have \( \eta(0) < y_1 < y_1 + \epsilon < y_2 \), then

\[
h_t(y_1; \eta) + \epsilon < h_t(y_2; \eta)
\] (37)

whenever \( t \leq \tau_+(y_1; \eta) \), as follows from the growth (22) of intervals in the upward flow.

Now, fix \( t_0 \in [0, T] \) and consider \( y_0 := \tau_+(t_0; \xi) \) and \( \tilde{y}_0 := \tau_+(t_0; \hat{\xi}) \), and suppose, without loss of generality, that \( y_0 \leq \tilde{y}_0 \). We proceed by contradiction: if \( y_0 + \delta + \epsilon < \tilde{y}_0 \) for some \( \epsilon > 0 \), then by (36),

\[
h_{t_0}(y_0; \xi) + \delta + \epsilon \leq h_{t_0}(y_0 + \delta; \hat{\xi}) + \epsilon,
\]

while by (37), we have

\[
h_{t_0}(y_0 + \delta; \hat{\xi}) + \epsilon < h_{t_0}(\tilde{y}_0, \tilde{\xi}),
\]

and thus, combining these yields

\[
\| \xi - \hat{\xi} \|_{\infty[0, T]} + \epsilon \leq h_{t_0}(\tilde{y}_0; \tilde{\xi}) - h_{t_0}(y_0; \xi) = \tilde{\xi}(t_0) - \xi(t_0),
\]

a contradiction. Hence, \( \tilde{y}_0 - y_0 \leq \delta \), as claimed.

Optimality of the constant is immediately evident by example: take the constant drivers \( \xi(t) \equiv 0 \) and \( \hat{\xi}(t) \equiv \epsilon \), for which

\[
\tau_+^{-1}(0; \hat{\xi}) - \tau_+^{-1}(0; \xi) = \epsilon = \| \xi - \hat{\xi} \|_{\infty}.
\]

Note that \( C = 1 \) is still sharp under the more restrictive condition that \( \xi, \hat{\xi} \in S_0([0, T]) \). Indeed, consider \( \xi_2(t) \equiv 0 \) and \( \hat{\xi}_2(t) = t/\delta \) on \([0, \delta]\) for some small \( \delta > 0 \), with \( \hat{\xi}_2(t) \equiv 1 \) for \( t \geq \delta \). We see from (16) that \( \tau_+^{-1}(\delta; \xi_2) = 2\sqrt{\delta} \), while \( \tau_+^{-1}(\delta; \hat{\xi}_2) \geq \hat{\xi}_2(\delta) = 1 \), and thus,

\[
|\tau_+^{-1}(\delta; \xi) - \tau_+^{-1}(\delta; \hat{\xi})| \geq 1 - 2\sqrt{\delta} = (1 - 2\sqrt{\delta})\| \xi - \hat{\xi} \|_{\infty}.
\]

□
3.4 Preliminary results on continuity properties of hitting times to drivers

In this section, we begin to explore the question of whether the map $\tau \mapsto \xi$ can be defined, and if so, whether or not it is continuous.

We first comment that moving from hitting times to drivers is more subtle than moving from drivers to hitting times. One indication of this is that the proof that $\tau \mapsto \xi$ is continuous at the hitting times $\tau(\cdot; C)$ of constant drivers $C(t) \equiv C$ is, to our surprise, not entirely trivial, and seems to require new machinery.

Another indication is the breakdown of monotonicity properties that one has in the $\xi \mapsto \tau$ direction. Indeed, given two drivers $\xi, \tilde{\xi} \in S_0([0, T])$, recall that if $\xi \leq \tilde{\xi}$, then, similar to (26), we have that the corresponding hitting times satisfy

$$\tau_- \leq \tilde{\tau}_- \quad \text{and} \quad \tilde{\tau}_+ \leq \tau_+$$

(38)

on their common domains. However, the converse implication does not hold: assuming the hitting times satisfy (38) does not imply $\xi \leq \tilde{\xi}$, as illustrated by the following example.

**Example 3.9.** Consider $\xi(t) \equiv 0$ and the driver $\tilde{\xi}$ that begins at 0 but then moves linearly with slope $1/\epsilon$ to reach value 1 at time $\epsilon$. Then, at time 1, say, $\tilde{\xi}$ moves to $-\epsilon$ linearly with slope $-(1 + \epsilon)/\epsilon$, then immediately back to value 1 with slope $(1 + \epsilon)/\epsilon$. See Figure 2. We have $\xi, \tilde{\xi} \in S_0([0, 1 + 2\epsilon])$ and the drivers are not monotone, but we claim that their hitting times still satisfy (38) when $\epsilon$ is sufficiently small.

To see why, first consider the left inequality

$$\tau_- \leq \tilde{\tau}_-.$$  

(39)

Since $\xi < \tilde{\xi}$ on $(0, 1]$, $\tau_-^{-1}(1) < \tilde{\tau}_-^{-1}(1) < 0$, and we automatically have (39) for all

$$x \in [\tau_-^{-1}(1) \lor \tilde{\tau}_-^{-1}(1), 0] = [\tau_-^{-1}(1), 0].$$  

(40)

Note $-2 = \tau_-^{-1}(1) =: x_0$ by (16). Since $\tilde{\xi} - \xi = 1$ on $[\epsilon, 1]$, the flow $\tilde{x}$ of the same initial point $x_0$ under $\tilde{\xi}$ satisfies $\tilde{x}(1) < -\delta(\epsilon) < x(1) = 0$, where $\delta$ is some function that decreases in $\epsilon$. As $\epsilon \to 0^+$, $\tilde{x}(1)$ approaches the position of the flow of $x_0$ under one unit of time with the constant driver $\eta \equiv$...
1, which is \(-\sqrt{(x_0 - 1)^2 - 4} = -\sqrt{5}\) (recall the \(h_t\) map from Example 2.3). Thus, for small \(\varepsilon\), we have \(\tilde{x}(1 + 2\varepsilon) < \tilde{\xi}(1 + 2\varepsilon)\), which shows \(x_0 < \tilde{\tau}-1(1 + 2\varepsilon)\). Therefore, \(x \in [\tilde{\tau}-1(1 + 2\varepsilon), \tilde{\tau}-1(1)]\) implies

\[
\tilde{\tau}_-(x) > 1 = \tau_-(x_0) > \tau_-(x)
\]
since \(0 > x > x_0\). Combined with the above monotonicity on the interval in (40), this shows that (39) holds on the common domain \([\tilde{\tau}-1(1 + 2\varepsilon), 0]\) of \(\tau_-\) and \(\tilde{\tau}_-\).

That \(\tau_+ \geq \tilde{\tau}_+\) on \([0, \tilde{\tau}^{-1}(1 + 2\varepsilon)]\) is similarly clear: by (16), we have

\[
\tau^{-1}_+(1 + 2\varepsilon) = 2\sqrt{1 + 2\varepsilon} = 2 + O(\varepsilon),
\]
while \(\tilde{\xi}\) has already eaten points past this on \(t \in [0, 1]\) when \(\varepsilon\) is small, as \(\tilde{\tau}_+^{-1}(1) \approx 3\).

In short, to say much about the \(\tau \mapsto \xi\) direction, one has to find a mechanism for handling rapid oscillations in \(\xi\). The machinery we construct below works in the case of a constant driver \(\tau(\cdot; C)\), which, without loss of generality, we may assume to be the zero driver 0. Our principal tool is Lemma 3.11, which says that for \(\tilde{\xi}\) welding \(x_0 < \tilde{\xi}(0) < y_0\) at time \(\tau\), the further \(\tilde{\xi}(\tau)\) is from \((x_0 + y_0)/2\), the smaller \(\tau\) necessarily is. We also find it helpful to write \(\xi\) as a convex combination of the points it will weld, which leads to the following integral representation for the interval length \(y(t) - x(t)\).

**Lemma 3.10.** Suppose \(\xi \in C([0, \tau])\) welds initial points \(x_0 < \xi(0) < y_0\) at time \(\tau\). Let \(I(t) := y(t) - x(t)\) be the interval length at time \(t\) and let \(\alpha : [0, \tau) \to (0, 1)\) be the convex combination coefficient yielding

\[
\xi(t) = (1 - \alpha(t))x(t) + \alpha(t)y(t).
\]

Then,

\[
I(t) = \sqrt{I(0)^2 - 4 \int_0^t ds \frac{\alpha(s)(1 - \alpha(s))}{\alpha(s)(1 - \alpha(s))}}
\]

for all \(0 \leq t \leq \tau\).

Recall that, as usual, \(x(t) = h_t(x_0; \xi)\) and \(y(t) = h_t(y_0; \xi)\), with the \(h_t\) maps satisfying (1). Note that for the zero driver, the image of a point \(y_0 > 0\) is \(t \mapsto \sqrt{y_0^2 - 4t}\), yielding an interval length of

\[
I(t) = 2\sqrt{y_0^2 - 4t} = \sqrt{I_0^2 - 16t}.
\]

The formula (42) generalizes this expression.

**Proof.** For \(0 \leq t < \tau\), the Loewner equation (1) yields

\[
\frac{d}{dt} (I(t)^2) = \frac{-4I(t)^2}{(y(t) - \xi(t))(\xi(t) - x(t))} = \frac{-4}{\alpha(t)(1 - \alpha(t))},
\]
yielding (42). As \( t \to \tau \), \( I(t) \to I(\tau) = 0 \), which shows that the integral in the radical is convergent and that the formula also holds when \( t = \tau \).

Observe that when \( t = \tau \), (42) says
\[
I(0)^2 = \int_0^\tau \frac{4ds}{\alpha(s)(1 - \alpha(s))} \geq 16\tau
\]
by calculus on the function \( x \mapsto \frac{4}{x(1-x)} \), thus showing
\[
\tau \leq \frac{(y_0 - x_0)^2}{16}.
\]

This maximum time is achieved by the driver that is constantly the average of \( x_0 \) and \( y_0 \) by (43) (or (16)).

**Lemma 3.11.** Suppose \( \xi \in C([0,\tau]) \) welds initial points \( x_0 < \xi(0) < y_0 \) satisfying \( x_0 + y_0 = 0 \) at time \( \tau \) with \( |\xi(\tau)| = \delta \). Then, \( \tau \leq f(\delta) \) for some function \( f \) that is strictly decreasing in \( \delta \) and satisfies \( f(0) = \frac{(y_0 - x_0)^2}{16} \).

Note that we do not require \( \xi \in S([0,\tau]) \), only that \( \xi \) welds \( x_0 \) and \( y_0 \) at time \( \tau \). Our proof also gives an explicit function \( f \) (see (49)–(51)), although we make no claim as to its sharpness.

**Proof.** The \( \delta = 0 \) case holds by (45) (with equality attained for the zero driver). Suppose \( \delta > 0 \). By the symmetry of \( x_0 \) and \( y_0 \), it suffices to suppose \( \xi(\tau) = \delta \). With \( \alpha(t) \) as in (41), define \( \alpha_0 := \frac{1}{2} + \frac{\delta}{4y_0} \) so that
\[
(1 - \alpha_0)x_0 + \alpha_0y_0 = \frac{\delta}{2},
\]
and consider the set
\[
T_0 := \{ 0 \leq t \leq \tau : 1 - \alpha_0 < \alpha(t) < \alpha_0 \},
\]
which is relatively open in \([0,\tau]\) and thus the intersection of \([0,\tau]\) with a finite or countable collection of open intervals. For a constant \( \varepsilon_0 = \varepsilon_0(\delta, y_0) > 0 \) to be determined below, we either have (i) \( |T_0^c| < \varepsilon_0 \) or (ii) \( |T_0^c| \geq \varepsilon_0 \), with \( |T_0^c| \) the Lebesgue measure of \( T_0^c \cap [0,\tau] \). We proceed to explicitly construct a suitable bounding function for \( \tau \) in each of these two cases.

As the argument’s intuition may be easily obscured, we first explicitly state it: we think of \( T_0 \) as the collection of “reasonable” times where \( \xi(t) \) is close to the average \((x(t) + y(t))/2\). If \( |T_0| \) is large, as in case (i), then \( \tau \) cannot be too large, or else \( y \) would move past \( \delta > 0 \) (as for the zero driver and \( \tau \) approaching \((y_0 - x_0)^2/16\)). If \( |T_0| \) is small, as in (ii), then the interval \( I(t) \) is collapsing quickly since \( \xi(t) \) is close to one of \( x(t), y(t) \) often, which also makes the welding time small.

We proceed with case (i), \(|T_0| > \tau - \varepsilon_0\). The point \( y(t) \) is constantly moving toward \( \xi(t) \) under the upward Loewner flow, and we bound \( \tau \) by estimating how far it can travel over times \( t \in T_0 \).
As noted above in (44),

\[
\int_0^t \frac{4ds}{\alpha(s)(1 - \alpha(s))} \geq 16t,
\]  

(46)

and so, \(I(t) \leq \sqrt{I(0)^2 - 16t}\) by (42). Thus, using the Loewner equation (1), we find for \(t \in T_0\) that

\[
-\dot{y}(t) = \frac{2}{(1 - \alpha(t))I(t)} \geq \frac{2}{\alpha_0 \sqrt{I(0)^2 - 16t}},
\]

and that therefore the cumulative change \(\Delta y|_{T_0}\) of \(y\) over \(t \in T_0\) satisfies

\[
y_0 - \delta \geq -\Delta y|_{T_0} \geq \int_{T_0} \frac{2dt}{\alpha_0 \sqrt{I(0)^2 - 16t}} \geq \int_0^{\lvert T_0 \rvert} \frac{2dt}{\alpha_0 \sqrt{I(0)^2 - 16t}} = \frac{I(0)}{4\alpha_0} - \frac{\sqrt{I(0)^2 - 16\lvert T_0 \rvert}}{4\alpha_0}.
\]

Here, the first inequality is by the assumption that \(y(\tau) = \delta\), whereas the second line is because the integrand is increasing in \(t\). We thus have

\[
\sqrt{I(0)^2 - 16\lvert T_0 \rvert} \geq I(0) - 4\alpha_0(y_0 - \delta) = \frac{\delta(y_0 + \delta)}{y_0} > 0,
\]

and therefore,

\[
\frac{I(0)^2}{16} - \frac{\delta^2(y_0 + \delta)^2}{16y_0^2} \geq \lvert T_0 \rvert > \tau - \epsilon_0. \tag{47}
\]

Choosing

\[
\epsilon_0 := \frac{\delta^2(y_0 + \delta)^2}{32y_0^2} > 0, \tag{48}
\]

Equation (47) yields

\[
\tau \leq \frac{(y_0 - x_0)^2}{16} - \frac{\delta^2(y_0 + \delta)^2}{32y_0^2} =: f_1(\delta), \tag{49}
\]

which is strictly decreasing in \(\delta\) and satisfies \(f_1(0) = (y_0 - x_0)^2 / 16\).
We turn, then, to the second case \( |T^c_0| \geq \epsilon_0 \), with \( \epsilon_0 = \epsilon_0(\delta) \) still given by (48). Using (44), the symmetry of \( x \mapsto x(1-x) \) around \( 1/2 \), and (46), we find that

\[
\frac{(y_0 - x_0)^2}{4} = \int_{T_0^c} \frac{ds}{\alpha(s)(1-\alpha(s))} + \int_{T_0} \frac{ds}{\alpha(s)(1-\alpha(s))} \\
\geq \frac{|T_0^c|}{\alpha_0(1-\alpha_0)} + 4|T_0| \\
= \frac{(1-4\alpha_0(1-\alpha_0))|T_0^c|}{\alpha_0(1-\alpha_0)} + 4\tau \\
\geq \frac{(1-4\alpha_0(1-\alpha_0))\epsilon_0}{\alpha_0(1-\alpha_0)} + 4\tau
\]

since \( 1-4\alpha_0(1-\alpha_0) = \delta^2/(4y_0^2) > 0 \). We thus have

\[
\tau \leq \frac{(y_0 - x_0)^2}{16} - \left( \frac{1}{4\alpha_0(1-\alpha_0)} - 1 \right) \epsilon_0(\delta) =: f_2(\delta),
\]

which again is strictly decreasing in \( \delta \) (since \( \alpha_0 \neq 1/2 \)) with initial value \( f_2(0) = (y_0 - x_0)^2/16 \). Combining the cases, we thus see

\[
\tau \leq f_1(\delta) \lor f_2(\delta) =: f(\delta),
\]

which is a function with the claimed properties.

We will use a notion of conformal removability for our main result of this section. We formulate the definition as in the statement of [16, Theorem 1.1].

**Definition 3.12.** A set \( E \subset \mathbb{H} \) is **conformally removable in** \( \mathbb{H} \) if whenever \( F : \mathbb{H} \to \mathbb{H} \) is a homeomorphism that is conformal off of \( E \), then \( F \) is actually conformal on all of \( \mathbb{H} \).

It is standard to define a set \( E \subset \mathbb{C} \) as **conformally removable** (now there is no reference domain) if whenever \( F : \mathbb{C} \to \mathbb{C} \) is a homeomorphism conformal off of \( E \), then \( F \) extends to be conformal on all of \( \mathbb{C} \) (and is thus M"obius). The sets \( E \) of relevance to us are curves \( \gamma \) generated by \( \xi \in S \), and for these conformal removability in \( \mathbb{H} \) implies conformal removability in this latter, standard sense. Indeed, if \( \gamma \) is conformally removable in \( \mathbb{H} \) and \( F : \mathbb{C} \to \mathbb{C} \) is a homeomorphism conformal off of \( \gamma \), then the composition \( G \circ F \) is M"obius, where \( G : F(\mathbb{H}) \to \mathbb{H} \) is a Riemann map. Thus, \( F \) is conformal on \( \mathbb{H} \), and the remaining (potential) singularity \( \gamma \cap \mathbb{R} = \{x\} \) is inconsequential by Riemann’s removability theorem.

The following result says that \( \tau \mapsto \xi \) is well defined whenever \( \tau \) comes from a curve \( \gamma \) that is conformally removable in \( \mathbb{H} \), and that \( \tau \mapsto \xi \) is continuous at the hitting time function of the zero driver.

**Theorem 3.13.**

(i) Suppose that \( \xi, \xi \in S_0([0,T]), T < \infty \), generate the same hitting time function \( \tau : [-a, b] \to \mathbb{R}_{\geq 0} \), and that furthermore, the curve \( \gamma \) generated by \( \xi \) is conformally removable in \( \mathbb{H} \). Then \( \xi = \xi \).
(ii) If \( \tau_n : [a_n, b_n] \to \mathbb{R} \) are hitting times for drivers \( \xi_n \in S_0([0, T_n]) \) with \( a_n \to a < \infty, b_n \to -a \), and where \( \tau_n \) converges uniformly to the hitting time function \( \tau_0 \) of the zero driver on compact subsets of \((a, -a)\), then \( \xi_n \) converges uniformly to 0 on any \([0, T'] \subset [0, T)\), where \( T = \tau_0(a) = a^2/4 \).

We preface the proof with several comments.

• The proof of part (i) is standard, using the fact that conformal removability implies that the conformal welding uniquely determines the curve. We recall this argument in the proof.

The class of such \( \gamma \) includes, for example, those generating quasi-slit half-planes, which are \( \gamma \) for which \( \mathbb{H} \setminus \gamma = f(\mathbb{H} \setminus [0, i]) \) for some \( K \)-quasi-conformal map \( f : \mathbb{H} \to \mathbb{H} \) that fixes \( \infty \) \([22, 26]\).

Indeed, since such \( \gamma \) do not meet \( \mathbb{R} \) tangentially, the union of \( \gamma \) and its reflection \( \gamma^* \) across the real line are the image of an interval under a quasi-conformal map of \( \mathbb{C} \), which is conformally removable (see \([49, \text{Cor. 5.4}]\), e.g.). Hence, given a homeomorphism \( F : \mathbb{H} \to \mathbb{H} \) conformal off of \( \gamma \), extending \( F \) to the lower half plane by Schwarz reflection yields a homeomorphism of \( \mathbb{C} \) conformal off of \( \gamma \cup \gamma^* \), and so, removability of \( \gamma \cup \gamma^* \) shows that \( F \) is, in particular, conformal on \( \mathbb{H} \).

Conformally-removable curves may also be far rougher than images of intervals under quasi-conformal maps, however. A recent work \([16]\), for instance, shows that the trace of \( \text{SLE}_4 \) is almost surely conformally removable in \( \mathbb{H} \). \( \text{SLE}_4 \) is almost surely neither a quasi-arc nor even the boundary of a Hölder domain, however. Theorem 3.13(i) says that the hitting time function generated by a sampling of \( \text{SLE}_4 \) almost surely uniquely determines its Brownian driver.

• In contrast to part (ii) of the theorem, we will see in Lemma 4.9 below that \( \varphi \mapsto \xi \) is not continuous at \( \varphi_0 \), the welding for the zero driver.

• We also note that for part (i), the special case of the constant driver immediately follows from Lemma 3.11: if \( \tau(x; \xi) = x^2/4 \) on some interval \([-y_0, y_0]\), then the lemma yields \( \xi(y^2/4) = 0 \) for each \( 0 \leq y \leq y_0 \).

**Proof.**

(i) Since the hitting time function \( \tau \) generated by both drivers is identical, by \((21)\), both curves have the same conformal welding \( \varphi : [-a, 0] \to [0, b] \). It is known that conformal removability implies that the welding uniquely determines the curve (see the next paragraph for details), and so, \( \gamma = \tilde{\gamma} \), where \( \gamma \) and \( \tilde{\gamma} \) are the curves generated by \( \xi \) and \( \tilde{\xi} \), respectively. Taking the Loewner transform yields that the drivers are identical on \([0, T]\), as claimed.

For reader’s convenience, we proceed to recall the argument that conformal removability in \( \mathbb{H} \) implies that the welding uniquely determines the curve. Let \( h_T : \mathbb{H} \to \mathbb{H} \setminus \gamma \) be the upward Loewner map for \( \gamma \), and write \( f = h_T|_{\text{Re}(z) \leq 0} \) and \( g = h_T|_{\text{Re}(z) \geq 0} \) for the restrictions of \( h_T \) to the two sides of the imaginary axis. Use the analogous notation for \( \tilde{\gamma} \), \( \tilde{h}_T \), and \( \tilde{f} \) and \( \tilde{g} \). The extensions of \( f \) and \( g \) and \( \tilde{f} \) and \( \tilde{g} \) to \([-a, 0]\) and \([0, b]\) determine the conformal weldings, and by assumption \( \varphi = g^{-1} \circ f = \tilde{g}^{-1} \circ \tilde{f}, \) or

\[
\tilde{g} \circ g^{-1}(z) = \tilde{f} \circ f^{-1}(z), \quad z \in \gamma_T.
\]

This equality shows that

\[
F(w) := \begin{cases} 
\tilde{f} \circ f^{-1}(w) & w \in h_T(\{\text{Re}(z) \leq 0\}), \\
\tilde{g} \circ g^{-1}(w) & w \in h_T(\{\text{Re}(z) \geq 0\})
\end{cases}
\]
is a well-defined homeomorphism of $\mathbb{H}$ that is conformal off of $\gamma_T$. By removability, $F$ is thus a real Möbius transformation, which by construction fixes $\infty$, and so is of the form $F(w) = aw + b$ for some $a, b \in \mathbb{R}$. This shows $\tilde{h}_T(z) = a\tilde{h}_T(z) + b$, and sending $z \to \infty$ and recalling (5) yields $a = 1$ and $b = 0$. We conclude $h_T = \tilde{h}_T$, showing $\gamma = \tilde{\gamma}$.

(ii) We use Arzela–Ascoli to show that $\{\xi_n\}$ is precompact on any $[0, T') \subset [0, T)$, and then show that all subsequential limits are $0$.

Since $a_n < \xi_n(t) < b_n$ for all $0 \leq t \leq T$ and every $n$, the sequence $\{\xi_n\}$ is uniformly bounded. If it is not equicontinuous on $[0, T') \subset [0, T)$, then there exist a subsequence $\{\xi_{n_k}\}, \varepsilon_1 > 0$, and times $t_{n_k}, t' \in [0, T']$ satisfying $t_{n_k} \to t'$, but where

$$|\xi_{n_k}(t_{n_k}) - \xi_{n_k}(t')| \geq \varepsilon_1$$

for all $k$. Indeed, lack of equicontinuity yields $\varepsilon > 0$ and a subsequence $\{\xi_{n_k}\}$ with times $u_{n_k}, v_{n_k} \in [0, T')$ such that $|u_{n_k} - v_{n_k}| \to 0$ but where $|\xi_{n_k}(u_{n_k}) - \xi_{n_k}(v_{n_k})| \geq \varepsilon$. By moving to a further subsequence, which we denote by the same labeling, we have $u_{n_k} \to t' \in [0, T']$, and by the triangle inequality,

$$\max\{|\xi_{n_k}(u_{n_k}) - \xi_{n_k}(t')|, |\xi_{n_k}(v_{n_k}) - \xi_{n_k}(t')|\} \geq \varepsilon/2$$

for all $k$. Thus, setting $t_{n_k}$ as either $u_{n_k}$ or $v_{n_k}$ yields (52).

We note that $\tau_{n_k, \pm}^{-1} \underset{u}{\to} \tau_{0, \pm}^{-1}$ on $[0, T']$, as follows from choosing an appropriate compact of $(-a, a) = (-a, 0) \cup [0, a)$ and using Lemmas 2.5 and 2.6 on each half interval. Thus, for $x' := \tau_{0, -}^{-1}(t')$ and $y' := \tau_{0, +}^{-1}(t')$, the points $x_{n_k}, y_{n_k}$ and $x'_{n_k}, y'_{n_k}$ welded together by $\xi_{n_k}$ at times $t_{n_k}$ and $t'$, respectively, satisfy

$$\max\{|x_{n_k} - x'|, |x'_{n_k} - x'|, |y_{n_k} - y'|, |y'_{n_k} - y'|\} \to 0$$

as $k \to \infty$. In particular, $x_{n_k} + y_{n_k} \to 0$ and $x'_{n_k} + y'_{n_k} \to 0$. Since by (52), we have either $|\xi_{n_k}(t_{n_k})| \geq \varepsilon_1/2$ or $|\xi_{n_k}(t')| \geq \varepsilon_1/2$, by Lemma 3.11, there is some fixed $\delta > 0$ such that

$$\min\{\tau_{n_k}(x_{n_k}), \tau_{n_k}(x'_{n_k})\} < \tau_0(x') - \delta$$

for all large $k$. Since this contradicts $\tau_{n_k} \underset{u}{\to} \tau_0$ on a sufficiently large compact of $(-a, a)$, we conclude that the sequence $\{\xi_n\}$ is, indeed, equicontinuous on $[0, T']$.

Take any subsequential limit $\xi_{n_k} \to \tilde{\xi}$ on $[0, T']$. We wish to show that the hitting time function $\bar{\tau}$ for $\tilde{\xi}$ is the same as $\tau_0$ and then apply part (i) of the theorem. (Unfortunately, we cannot jump to use Theorem 3.4 because, a priori, we do not know that $\tilde{\xi} \in S$. ) Fix $0 < y_0$ and $0 < \varepsilon < y_0/2$. Then, by (27),

$$\tau_{n_k}(y_0 - \varepsilon) \leq \bar{\tau}(y_0) \leq \tau_{n_k}(y_0 + \varepsilon)$$

for all sufficiently-large $k$, and so, in the limit, we find

$$\frac{(y_0 - \varepsilon)^2}{4} \leq \bar{\tau}(y_0) \leq \frac{(y_0 + \varepsilon)^2}{4}.$$
Sending $\epsilon \to 0$, we see $\bar{\tau}(y_0) = y_0^2/4$, and similarly that $\bar{\tau}(-y_0) = y_0^2/4$. By assumption on $\tau_n$ and part (i), we conclude $\bar{\tau} = 0$ on $[0, T']$. Hence, all subsequential limits are the same, showing $\xi \to 0$ uniformly on $[0, T']$.

3.4.1 Positive evidence for the nonzero case

In considering whether hitting time convergence implies driver convergence, we ran some numerical experiments to gain intuition. We thought that we may have generated a method to produce a counterexample, but the simulations actually produced convergent driving functions, thus yielding some positive evidence for a generalization of Theorem 3.13(ii). We share the construction and this evidence here.

Let $T_0$ be the collection of hitting-time functions $\tau(\cdot; \xi)$ generated by $\xi \in S_0([0, T])$, and choose $\tau(\cdot; \xi) \in T_0$ corresponding to $\xi \neq 0$. Say $\tau : [-a, b] \to \mathbb{R}$ with $\tau(a) = \tau(b) = T$. At stage $n$, consider the pairs

$$(x_j, y_j) = (\tau^{-1}(jT/n; \xi), \tau^{-1}(jT/n; \xi)),$$

that $\xi$ welds together at times $jT/n$. We construct a driver $\xi_n \in S_0$ that also welds $x_j$ to $y_j$ in time $jT/n$ but with large oscillations, potentially hindering the desired uniform convergence.

Indeed, start $\xi_n$ at zero and have it move linearly with extremely large speed until it reaches $(x_1(\epsilon; \xi_n) + y_1(\epsilon; \xi_n))/2$ at some time $\epsilon$. The true welding time for $(x_1, y_1)$ is

$$\frac{T}{n} \leq \frac{(y_1 - x_1)^2}{16}$$

by (45), where the maximum is attained by the driver that is constantly $(y_1 + x_1)/2$. So, if we set $\xi_n$ to be $(x_1(\epsilon; \xi_n) + y_1(\epsilon; \xi_n))/2$ for $\epsilon \leq t \leq \frac{T}{n} - \epsilon$, it does not weld these points, as $\epsilon$ is very small and $\xi$ is not constant in general. At time $\frac{T}{n} - \epsilon$, we then use a large oscillation in $\xi_n$ to weld both points together in time $\epsilon$ (see below in the proof of Lemma 4.9 for a description on how to do this). Thus, $\xi_n$ welds the pair $(x_1, y_1)$ in exactly time $t_1 = T/n$.

We next have $\xi_n$ move in time $\epsilon$ to the average $(x_2(t_1 + \epsilon; \xi_n) + y_2(t_1 + \epsilon; \xi_n))/2$ of the next pair. Intuitively, since $\xi_1$ stayed “far away” from $x_1$ and $y_1$ until the large $\epsilon$-oscillation at the end, $x_2$ and $y_2$ have not moved as far as they normally would have under $\xi$ in $[0, T/n]$, and thus, we expect

$$\frac{T}{n} < \frac{(x_2(t_1; \xi_n) + y_2(t_1; \xi_n))^2}{16}.$$ 

So, we have $\xi_n$ wait constantly at the average $(x_2(t_1 + \epsilon; \xi_n) + y_2(t_1 + \epsilon; \xi_n))/2$ on $\frac{T}{n} + \epsilon \leq t \leq \frac{2T}{n} - \epsilon$, and then, use another large oscillation to weld the images of $x_2$ and $y_2$ together in time $\epsilon$. Thus, $\xi_n$ also welds $(x_2, y_2)$ in exactly the same amount of time as $\xi$.

We continue in this way to weld all the $(x_j, y_j)$ at times identical to $\tau(\cdot; \xi)$, which by the monotonicity of $\tau_n$ implies $\tau(\cdot; \xi_n) \overset{u}{\to} \tau(\cdot; \xi)$.

We wondered if in the $\epsilon \to 0$ limit, the oscillations of $\xi_n$ at the end of each interval $[jT/n, (j+1)T/n]$ would grow so large that $||\xi_n - \xi||_{\infty[0,T]}$ would be bounded below. In
Figure 3: Numerical simulations of drivers $\xi_n$ as constructed in §3.4.1, in the $\epsilon \to 0$ limit (where the drivers move instantly and hence yield vertical lines). Here, the true driver $\xi$, pictured in blue, generates a line with angle $\pi/3$ to the positive reals on $0 \leq t \leq 2$, followed by a vertical line segment on $2 \leq t \leq 3$. With a coarse approximation, the constructed driver $\xi_{n_1}$ in the upper figure struggles to stay close, but we see in the lower figure that the $\xi_n$ still converge as the mesh becomes finer. Convergence at the corner is apparently only logarithmically fast in the mesh size, however (we quadruple the points in each iteration). Given the difficulty at the nonsmooth point of $\xi$, it is unclear whether the $\xi_n$ would converge for a rough fractal driver, such as $\xi$ for the Von Koch snowflake [24, Figure 2].

Given the difficulty in the simulations, this was not the case, however, suggesting that the convergence of hitting times is fairly robust. We show a characteristic simulation in Figure 3, and submit this as positive evidence for a generalization of Theorem 3.13(ii).

We comment that our constructed driver $\xi_n$ is not the worst-case scenario, as it is not necessarily the one that minimizes the shrinking of the last interval $y_n - x_n$ on $0 \leq t \leq \frac{(n-1)T}{n}$ among all the drivers that weld the pairs $(x_j, y_j)$ at time $jT/n$ for $1 \leq j \leq n - 1$. If such an extremal driver could be approximated and simulated, one could then follow it with the above construction to weld $(x_n, y_n)$ on $\frac{n-1}{nT} \leq t \leq T$ and produce the largest possible oscillation in $\xi_n$ for welding these
last points. The intuition is that the pair \((x_n, y_n)\) is “far away” when the driver is welding the other points, and so does not move very much; if we minimize its movement, we then have the opportunity for an extremely large fluctuation in \(\xi_n\) at the very end. It would be interesting to see if the oscillation would remain macroscopic in the \(n \to \infty\) limit.

4 CONTINUITY PROPERTIES OF DRIVERS TO WELDINGS AND HITTING TIMES TO WELDINGS

In this chapter, we use drivers \(\xi\) belonging to \(S_0\), which is natural to have the welding map \(\varphi\) exchange intervals around the origin. We also restrict to times \(T < \infty\), as we will need domain compactness for uniform continuity.

4.1 Continuity of drivers to weldings

**Theorem 4.1.** Let \(0 < T < \infty\) and let \(\xi, \xi_n \in S_0([0, T])\). Let \(\varphi : [-a, 0] \to [0, b]\) and \(\varphi_n : [-a_n, 0] \to [0, b_n]\) be the conformal weldings corresponding to \(\xi\) and \(\xi_n\), respectively, where \(\tau(a) = \tau(b) = \tau_n(-a_n) = \tau_n(b_n) = T\). If \(\xi_n \overset{u}{\to} \xi\) on \([0, T]\), then \(a_n \to a\), \(b_n \to b\), and \(\varphi_n \overset{u}{\to} \varphi\) on \([-c, 0]\) for any \([-c, 0] \subset (-a, 0]\).

**Remark 4.2.** As in Theorem 3.4, note that we also have

\[
\max\{|a_n - a|, |b_n - b|\} \leq \|\xi_n - \xi\|_{\infty[0, T]} \tag{53}
\]

by Theorem 3.8.

**Proof.** Since \(a_n \to a\) and \(b_n \to b\) by (53), \(\varphi_n \in C_0([-c, 0])\) for all large \(n\). Noting \(\varphi = \tau_+^{-1} \circ \tau_-\) and writing \(\tau_{\pm}\) and \(\tau_{n, \pm}\) for the hitting-time functions generated by \(\xi\) and \(\xi_n\), respectively, we see for \(-c \leq x \leq 0\) and sufficiently large \(n\) that

\[
|\varphi_n(x) - \varphi(x)|
\]

\[
= |\tau_{n, +}^{-1}(\tau_{n, -}(x)) - \tau_{+}^{-1}(\tau_{-}(x))|
\]

\[
\leq |\tau_{n, +}^{-1}(\tau_{n, -}(x)) - \tau_{+}^{-1}(\tau_{-}(x))| + |\tau_{+}^{-1}(\tau_{n, -}(x)) - \tau_{+}^{-1}(\tau_{-}(x))|\n\]

\[
\leq \|\xi_n - \xi\|_{\infty[0, T]} + \omega\left(\|\tau_{n, -} - \tau_{-}\|_{\infty[-c, 0]; \tau_{+}^{-1}}\right) \tag{54}
\]

by Theorem 3.8, where \(\omega(\cdot; \tau_{+}^{-1})\) is the modulus of continuity of \(\tau_{+}^{-1}\) on \([0, T]\), which exists by Lemma 3.1. Since \(\|\tau_{n, -} - \tau_{-}\|_{\infty[-c, 0]} \to 0\) by Theorem 3.4, we have \(\|\varphi_n - \varphi\|_{\infty[-c, 0]} \to 0\). □

Even though \(\xi \to \varphi\) is continuous in the sense of Theorem 4.1, the next lemma says that there is again no universal modulus of continuity.

**Lemma 4.3.** There exist \(\xi_n, \tilde{\xi}_n \in S_0([0, T])\) and \(\varepsilon > 0\) such that

\[
\|\xi_n - \tilde{\xi}_n\|_{\infty[0, T]} \to 0,
\]

and

\[
\|\varphi_n - \varphi\|_{\infty[-c, 0]} \to 0.
\]
but where for all \( n \),

\[
|\varphi_n(x_n) - \varphi_n(x_n)| \geq \varepsilon \tag{55}
\]

for some \( x_n \) welded by every \( \xi_n \) and \( \bar{\xi}_n \), where \( \varphi_n \) and \( \bar{\varphi}_n \) are the weldings corresponding to \( \xi_n \) and \( \bar{\xi}_n \), respectively.

**Proof.** We use the same drivers \( \xi(\cdot, \delta) \) and \( \bar{\xi}(\cdot, \delta) \) as in the proof Lemma 3.6, including the modification in the last paragraph so that both are in \( S_0 \). We show that

\[
|\varphi^{-1}(y_0) - \bar{\varphi}^{-1}(y_0)| \geq \varepsilon,
\]

with \( y_0 = y_{0,\delta} = 2\sqrt{2}\delta \) the same point selected in the last paragraph of that proof. (To obtain (55) instead of a statement about the inverse weldings, reflect the drivers across the origin and take \( x_{0,\delta} := -y_{0,\delta} \).) Consider the situation at time \( t = \delta^2 + \delta \), when \( \xi \) and \( \bar{\xi} \) arrive at \(-1\) and \(-1 - \delta\), respectively, for the first time. We have that \( y(\delta^2 + \delta; \xi) = 2\delta - 1 \), and thus, the point \( x_0 \) that will weld to it under \( \xi \) is, at that moment, at

\[
x(\delta^2 + \delta; \xi) = -2\delta - 1. \tag{56}
\]

By (33), the position of \( y_0 \) under \( \bar{\xi} \) satisfies

\[
y(\delta^2 + \delta; \bar{\xi}) - \bar{\xi}(\delta^2 + \delta) \geq \frac{1}{3} + 3\delta,
\]

and thus, the point \( \bar{x}_0 \) that welds to \( y_0 \) under \( \bar{\xi} \) satisfies

\[
\bar{x}(\delta^2 + \delta; \bar{\xi}) \leq -1 - \delta - \left(\frac{1}{3} + 3\delta\right) = -\frac{4}{3} - 4\delta,
\]

and so, by (56), we see

\[
x(\delta^2 + \delta; \xi) - \bar{x}(\delta^2 + \delta; \bar{\xi}) \geq \frac{1}{3} + 2\delta. \tag{57}
\]

As \( \delta \to 0^+ \), simple estimates with the Loewner equation show \( x(\delta^2 + \delta; \xi) - x_0 \to 0 \) and \( \bar{x}(\delta^2 + \delta; \bar{\xi}) - \bar{x}_0 \to 0 \), and thus, by (57),

\[
x_0 - \bar{x}_0 = \varphi^{-1}(y_0) - \bar{\varphi}^{-1}(y_0) \geq \frac{1}{4}
\]

for all small \( \delta \).

As in Lemma 3.7, we can easily conclude from above results that \( (x; \xi) \mapsto \varphi(x; \xi) \) is pointwise jointly continuous in \( x \) and \( \xi \). This generalizes [3, Thm. 1.2(b)], as our statement covers all drivers \( \xi \) generating simple curves, not just the a.s. Brownian motion case.
Lemma 4.4. Let $0 < T < \infty$ and $\xi \in S_0([0, T])$ a driver with welding $\varphi : [-a, 0] \rightarrow [0, b]$. If $x \in (-a, 0]$ and $\epsilon > 0$, there exists $\delta = \delta(\epsilon, x, \xi)$ such that whenever $\bar{x} \leq 0$ and $\bar{\xi} \in S_0([0, T])$ satisfy
\[
\max\{|\bar{x} - x|, \|\bar{\xi} - \xi\|_{\infty[0, T]}\} < \delta,
\]
then
\[
|\varphi(\bar{x}; \bar{\xi}) - \varphi(x; \xi)| < \epsilon.
\]

Proof. Write $\bar{\varphi}$ and $\varphi$ for $\varphi(\cdot; \bar{\xi})$ and $\varphi(\cdot; \xi)$, respectively, and similarly for their drivers and hitting times. For $\frac{x-a}{2} \leq \bar{x} \leq 0$, we have that $\bar{\varphi}(\bar{x})$ is defined whenever $\|\bar{\xi} - \xi\|_{\infty[0, T]}$ is sufficiently small by Theorem 3.8, and for such $\bar{x}$, the triangle inequality yields
\[
|\bar{\varphi}(\bar{x}) - \varphi(x)| \leq \|\bar{\varphi} - \varphi\|_{\infty[(x-a)/2, 0]} + \omega(|\bar{x} - x|; \varphi)
\]
by (54), where $\omega(\cdot; \varphi)$ is the modulus of continuity of $\varphi$ on $[-a, 0]$, and $\omega(\cdot; \tau_-^{-1})$ that for $\tau_-^{-1}$ on $[0, T]$. As $\bar{x}$, $\tau_-^{-1}$, and $\varphi$ are fixed, and the map $\bar{x} \mapsto \tau$ is continuous by Theorem 3.4, we may choose $\delta$ small enough such that each of these three terms is less than $\epsilon/3$. \hfill $\square$

4.2 Continuity of hitting times to weldings

The continuity of $\tau \mapsto \varphi$, as expressed in the following lemma, is an immediate consequence of writing $\varphi = \tau_+^{-1} \circ \tau_-$ via Lemma 3.1 and then using Lemmas 2.5 and 2.6; we leave the details to the interested reader.

Lemma 4.5. Let $0 < T < \infty$ and $\xi, \xi_n \in S_0([0, T])$ be drivers with $\tau : [-a, b] \rightarrow [0, T]$ and $\tau_n : [-a_n, b_n] \rightarrow [0, T]$ their respective hitting times and $\varphi : [-a, 0] \rightarrow [0, b]$ and $\varphi_n : [-a_n, 0] \rightarrow [0, b]$ their respective conformal weldings. If $a_n \rightarrow a$, $b_n \rightarrow b$ and $\tau_n \rightarrow \tau$ on any $[c, d] \subset (-a, b)$, then $\varphi_n \rightarrow \varphi$ on $[-d, 0]$ for any $[-d, 0] \subset (-a, 0]$.

We can use the linear drivers $\xi(t) = ct$ to show that the continuity in the preceding lemma is not uniform, and, in fact, is fairly egregiously nonuniform.

Lemma 4.6. For $n \in \mathbb{N}$, there exist drivers $\xi_n, \xi_n^\ast \in S_0([0, T_n])$ whose hitting times $\tau_n : [-a_n, b_n] \rightarrow [0, T_n], \bar{\tau}_n : [-\bar{a}_n, \bar{b}_n] \rightarrow [0, T_n]$ satisfy $\bar{a}_n < a_n$, $\bar{a}_n \rightarrow a_n$, $\bar{b}_n = b_n$ and
\[
\|\tau_n - \bar{\tau}_n\|_{\infty[-a_n, b_n]} \rightarrow 0
\]
as $n \rightarrow \infty$, but where the corresponding conformal weldings $\varphi_n, \bar{\varphi}_n$ satisfy
\[
\|\varphi_n - \bar{\varphi}_n\|_{\infty[-a, 0]} \rightarrow \infty.
\]
An informal explanation of this phenomenon is as follows: the drivers we will construct are moving toward the “boundary” of $S_0$, and the modulus of continuity of $\tau \mapsto \varphi$ blows up as one moves toward the “boundary.”

**Proof.** For $n \in \mathbb{N}$, we begin by considering the linear drivers

$$
\xi_n(t) := nt \quad \text{and} \quad \tilde{\eta}_n(t) := \left( n - \frac{1}{n} \right)t,
$$

where we will modify $\tilde{\eta}_n$ to create the desired $\tilde{\xi}_n$. Note that the point $-a_n := -\frac{2}{n} + \frac{1}{n!}$, which has finite hitting time under $\xi_n$ by Lemma 2.4, and thus also under $\tilde{\eta}_n \leq \xi_n$. We consider the associated interval $[0, T_n]$, where

$$
T_n := \tau \left( -\frac{2}{n} + \frac{1}{n!} \right) := \tau \left( -\frac{2}{n} + \frac{1}{n!}; \xi_n \right).
$$

By (18) and Stirling’s formula,

$$
T_n = \frac{-2}{n^2} + \frac{1}{n \cdot n!} + 2 \log \left( \frac{2(n-1)!}{n^2} \right) = O \left( \frac{\log(n)}{n^2} \right) \to 0 \quad (59)
$$
as $n \to \infty$. Since $\varphi_n(-a_n)$ is the point $0 < b_n$ that satisfies $\tau_n(b_n) = \tau_n(-a_n)$, we see by (59) that

$$
b_n > \xi_n(T_n) = O(\log(n)), \quad n \to \infty,
$$

and thus, $b_n = \varphi_n(-a_n) \to \infty$.

On the other hand, however,

$$
\tilde{T}_n := \tau(-a_n; \tilde{\eta}_n) = \frac{-2}{n^2 - 1} + \frac{1}{(n^2 - 1)(n-1)!} + \frac{2n^2}{(n^2 - 1)^2} \log \left( \frac{n^2}{1 + \frac{n^2 - 1}{2(n-1)!}} \right)
$$

$$
= O \left( \frac{\log(n)}{n^2} \right) \quad n \to \infty,
$$

and thus, $\tilde{\eta}_n(\tilde{T}_n) = O(\log(n)/n) \to 0$. Since $\tilde{T}_n \to 0$ as well, it easily follows from the Loewner equation that $\varphi(-a_n; \tilde{\eta}_n) = o(1)$ as $n \to \infty$, and we conclude

$$
|\varphi_n(-a_n) - \varphi(-a_n; \tilde{\eta}_n)| = O(\log(n)). \quad (60)
$$

This is the desired divergence on the weldings; we proceed to build from $\tilde{\eta}_n$ a driver $\tilde{\xi}_n$ that is defined on all of $[0, T_n]$ and for which (60) still holds.

Indeed, set $\tilde{\xi}_n$ to be the piece-wise linear driver that is identical to $\tilde{\eta}_n$ on $[0, \tilde{T}_n]$, and is followed by the linear driver that reaches the image of $b_n$ at time $T_n > \tilde{T}_n$. That is, $b_n = \tau_{n,+}^{-1}(T_n) = \tau_{n,+}^{-1}(\tilde{T}_n; \tilde{\xi}_n)$ arrives at some point $b_n(\tilde{T}_n; \tilde{\eta}_n) > \tilde{\eta}_n(\tilde{T}_n)$ at time $T_n$, under the upward flow generated by $\tilde{\eta}_n$, and we follow $\tilde{\eta}_n$ by the linear driver $\tilde{\xi}_n$ that moves extremely fast toward the image of $b_n(\tilde{T}_n; \tilde{\eta}_n)$, reaching it in exactly $T_n - \tilde{T}_n$ units of time. We call the entire driver $\tilde{\xi}_n$, and write

$$
-\tilde{a}_n := \tau_{n,+}^{-1}(b_n(T_n; \tilde{\xi}_n); \tilde{\xi}_n).
$$
By construction, the hitting times \( \tilde{\tau}_n \) of \( \tilde{\xi}_n \) are finite on \([-\tilde{a}_n, b_n]\), where

\[
\lim_{t \to \infty} \tau^{-1}_n(t; \tilde{\eta}_n) = -\frac{2}{n - 1/n} < -\tilde{a}_n < -\tilde{a}_n = \frac{2}{n} + \frac{1}{n!},
\]

and so \( \tilde{a}_n \to a_n \). Here, the first inequality is because \((n - 1/n)t \leq \tilde{\xi}_n(t)\) (at least for large \(n\)). Note furthermore that (58) trivially holds, as

\[
\|\tau_n - \tilde{\tau}_n\|_{[a_n, b_n]} \leq \|\tau_n\|_{[a_n, b_n]} + \|\tilde{\tau}_n\|_{[a_n, b_n]} \leq 2T_n \to 0.
\]

Since \( \tilde{\xi}_n \) welds identically to \( \tilde{\eta}_n \) on \([0, \tilde{T}_n]\), (60) yields

\[
\|\varphi_n - \tilde{\varphi}_n\|_{[-a_n, 0]} \geq |\varphi_n(-a_n) - \tilde{\varphi}_n(-a_n)| = O(\log(n)),
\]

where \( \tilde{\varphi} \) is the welding associated to \( \tilde{\xi}_n \). \( \square \)

**Remark 4.7.** We expect that similar reasoning could yield \( \xi_n, \tilde{\xi}_n \in S_0([0, T_n]) \) such that \( \|\xi_n - \tilde{\xi}_n\|_{\infty} \to 0 \) but where \( \|\varphi_n - \tilde{\varphi}_n\|_{\infty} \to \infty \).

### 4.3 Weldings to hitting times and weldings to drivers are not continuous

We first consider when the maps \( \varphi \mapsto \tau \) and \( \varphi \mapsto \xi \) are well defined. As in the case of Theorem 3.13 (ii), standard results show that this includes when the welding arises from a conformally removable curve (recall Definition 3.12).

**Lemma 4.8.** The maps \( \varphi \mapsto \tau \) and \( \varphi \mapsto \xi \) are both well defined on the collection of \( \varphi \) arising from curves \( \gamma \) that are conformally removable in \( \mathbb{H} \).

The potential issue, of course, is that the same welding may correspond to two different curves. This cannot happen given the assumption of conformal removability.

**Proof.** The proof is identical to that of Theorem 3.13 (ii): the welding uniquely determines the curve on the collection of curves that are conformally removable in \( \mathbb{H} \). Since the curve determines the driver and the hitting times, the result follows. \( \square \)

The next lemma shows that uniform welding convergence is rather weak, in the sense of implying neither hitting time nor driving function convergence.

**Lemma 4.9.** Let \( \varphi(x) = -x \) on \([-1, 0]\) be the welding for the vertical line segment \([0, i]\), with corresponding driver \( 0 \) and hitting time function \( \tau_0 \). There exist \( \varphi_n \) corresponding to simple curves \( \gamma_n \) with drivers \( \xi_n \) and hitting times \( \tau_n \) such that \( \|\varphi_n - \varphi\|_{[0, 1]} \to 0 \) but where \( \xi_n \not\to 0 \) and \( \tau_n \not\to \tau_0 \).

**Remark 4.10.** This should be contrasted with Theorem 3.13(ii), which showed that if \( \tau_n \to \tau_0 \), then \( \xi_n \to 0 \).
Proof. We build $\gamma_n$ by a piece-wise linear driving function $\xi_n$ that welds each $k/n$ to $-k/n$, $k = 1, \ldots, n$, under its upward Loewner flow. The intuition is that $\xi_n$ will capture the initial points extremely fast through rapid oscillations, and hence, the latter points will not have time to move very far, affording $\xi_n$ the opportunity to travel far from 0 to capture them. By construction, the welding $\varphi_n$ generated by $\xi_n$ will satisfy $\varphi_n(-k/n) = k/n$ for all $k \in \{1, \ldots, n\}$, and since the weldings are monotone, this yields $\varphi_n \longrightarrow \varphi$ on $[-1, 0]$.

We first note that we can weld $1/n$ to $-1/n$ in an arbitrarily small amount of time. Write $x_1 = -1/n$ and $y_1 = 1/n$, with $x_1(t)$ and $y_1(t)$ their images after time $t$ in the upward Loewner flow generated by $\bar{\xi}_1$, which we now construct. Starting from $\bar{\xi}_1(0) = 0$, move $\bar{\xi}_1$ linearly to $y_1(\epsilon_1) - \epsilon_1$ in time $\epsilon_1$, for some small $\epsilon_1 > 0$. Then $y_1(\epsilon_1) = -2/\epsilon_1$, and have $\bar{\xi}_1$ rush back toward $x_1(t)$ at precisely this same speed, until the time $t_1$ when it is exactly half-way between $x_1(t_1)$ and $y_1(t_1)$. Then, we freeze $\bar{\xi}_1$ at this point $\bar{\xi}_1(t_1)$ and let $x_1$ and $y_1$ flow together until they weld at time $T_1$.

We have that $\bar{\xi}_1$ is piece-wise linear thus an element of $S_0([0, T_1])$, and we note that $T_1$ is small: when moving back toward $x_1$, the distance $\bar{\xi}_1$ travels is less than $2/n$, and so, the time required is less than $\epsilon_1/n$. Once $\bar{\xi}_1$ stops, the image of $y_1$ is still $\epsilon_1$ away, and so, takes $\epsilon_1^2/4$ units of time to reach $\bar{\xi}_1(t_1)$. Thus,

$$T_1 < \epsilon_1 + \epsilon_1/n + \epsilon_1^2/4.$$ 

After flowing up with such a driver to weld the pair $(x_1, y_1)$, we can repeat the idea to capture subsequent points with $\bar{\xi}_j$'s, and can thus generate a $\xi_n$ that welds the first $n - 1$ pairs in $T_{n-1} < 1/n^2$ time. The last mesh point remaining on the right is the image $y_n(T_{n-1})$ of $y_n(0) = 1$, and since for all $0 \leq s \leq T_{n-1}$, we have the very coarse estimate

$$y_n(s) - \xi(s) \geq y_n(s) - y_n-1(s) \geq y_n(0) - y_n-1(0) = \frac{1}{n}$$

by (22), we see $y_n$ has moved toward $\xi_n$ no more than $2n(1/n^2) = 2/n$ units, showing

$$\sup |\xi_n| > 1 - \frac{3}{n}$$

when $\xi_n$ welds the last two points fast enough.

We conclude $\varphi_n \longrightarrow \varphi$ but $\xi_n \nrightarrow 0$. Furthermore, we may build $\xi_n$ to weld all the points in time $2/n^2$, showing the hitting times also do not converge to $\tau_0$. \hfill $\Box$

Figure 4 gives a numerical approximation for the curves $y_n$ generated by the $\xi_n$, which collapse to the real interval $[-1, 1]$ as $n \rightarrow \infty$.

4.4 | An application: Convergence of a zipper-like algorithm for welding using minimal-energy curves

Let $y([0, T])$ be a finite simple curve in $\mathbb{H} \cup \{y(0)\}$ with associated upward driver $\xi$ and conformal welding $\varphi : [-a, 0] \to [0, b]$. Recall that $y$ has finite Loewner energy if the Dirichlet energy of $\xi$ is finite, that is, if $\xi$ is absolutely continuous and

$$I_L(y) = I_L(\xi) := \frac{1}{2} \int_0^T \xi(t)^2 \, dt < \infty.$$ (61)
FIGURE 4  Numerical simulations of curves $\gamma_n$ (left) and the drivers $\xi_n$ (right) that show that the map $\varphi \mapsto \xi$ is not continuous, and hence, the converse of Theorem 4.1 does not hold. Here, the weldings $\varphi_n$ of the $\gamma_n$ converge uniformly on $[-1, 0]$ to $\varphi(x) = -x$, but the drivers stay far in supremum distance from the zero driver. (This illustration simplifies both the $\gamma_n$ and $\xi_n$ by using linear interpolation between numerically-accurate data points.)

The Loewner energy was introduced (without the name) in [10] and subsequently saw rapid development in [31, 43–46], to give an incomplete list. In short, it has fascinating connections to a diverse array of fields: probability theory, complex analysis, hyperbolic geometry, geometric measure theory, and Teichmüller theory (see, e.g., [5, 6, 40]). See [47] for a helpful overview.

We use Loewner energy minimizers in this section to address a conformal welding approximation question. Given a partition $P = \{(x_j, y_j)\}_{j=1}^N$ of $[-a, b]$,

$$-a = x_N < x_{N-1} < \cdots < x_1 < x_0 = 0 = y_0 < y_1 < \cdots < y_N = b$$  \hspace{1cm} (62)

with $\varphi(x_j) = y_j$ for each $j$, an interesting question is when a curve $\tilde{\gamma}$ that welds each pair $(x_j, y_j)$ together is close to $\gamma$. We have seen in §4.3 that this is not always the case. Indeed, given that both $\varphi \mapsto \xi$ and $\xi \mapsto \gamma$ are not continuous, we expect $\varphi \mapsto \gamma$ to exhibit a number of pathologies.

The question of closeness of curves given closeness of weldings is related to the two zipper algorithms of Don Marshall in complex analysis [26].

\begin{itemize}
    \item The first of these, the domain zipper algorithm, seeks to approximate a conformal map $f$ from $\mathbb{D}$ to a domain $\Omega$ via a map $f_n$ taking $\mathbb{D}$ to a domain $\Omega_n$ whose boundary agrees with $\partial \Omega$ on a given mesh/discretization $Q \subset \partial \Omega$. The map $f_n$ is built from composing $\#Q$ conformal maps, where each subsequent map, in effect, draws a boundary
\end{itemize}

\footnote{All “zippers” in this section are distinct from Sheffield’s quantum zipper in probability theory.}
segment between the next two points in \( \Omega \). The question for the domain zipper is: when \( \Omega \) is very fine, and one “draws” a reasonable arc between successive points in \( \Omega \), is \( \partial \Omega_n \) uniformly close to \( \partial \Omega \)?

The second algorithm, Marshall’s welding zipper algorithm, seeks to reconstruct \( \gamma \) through composing \( N \) conformal maps that “zip up” the partition (62) discretizing \( \varphi \) one pair at a time, producing a curve \( \gamma_n \) whose welding \( \varphi_n \) by construction agrees with \( \varphi \) at each \( x_j \). The question for the welding zipper is: what conformal maps can you use for each zip to guarantee that \( \gamma_n \) is close to \( \gamma \)?

Both versions of the zipper, it turns out, work remarkably well in practice and have become something of industry standards for numerically computing conformal maps. Proofs of convergence have been elusive, however. For the domain zipper, the only convergence result is for when one draws hyperbolic geodesic segments between subsequent boundary points [26], and the welding zipper remains an open problem (though see [28] for a partial result and further discussion).

In this subsection, we give as a corollary of Theorem 4.1 a positive convergence result to an algorithm similar to Marshall’s welding zipper algorithm. We construct curves \( \gamma_n \) whose weldings match \( \varphi \) on \( P_n \), but we create each \( \gamma_n \) “all at once” through minimizing Loewner energy among all such curves, instead of building the approximating curve through \( N \) compositions. That is, whereas Marshall’s zipper welds the first two points \( x_1, y_1 \) with some map \( F_1 \), sending both \( x_1 \) and \( y_1 \) to the base of a first curve segment, and then welds the images \( F_1(x_2), F_1(y_2) \) of the next two points under \( F_1 \) with some \( F_2 \), sending them to the base of a second curve segment, and so on, we start with curves that already weld all pairs in \( P \) and minimize energy among them. This makes the problem more tractable, and we only need existing results once Theorem 3.4 gives the existence of energy minimizers.

Let us write \( |P| := \max \{ x_{j-1} - x_j, y_j - y_{j-1} \} \) for the norm of the partition and say \( \varphi \) welds \( P \) if \( \varphi(x_j) = y_j \) for all \( 1 \leq j \leq N \). We will also say that a curve \( \gamma \) welds \( P \) if its conformal welding \( \varphi \) welds \( P \).

**Theorem 4.11.** Let \( \gamma : [0, T] \to \mathbb{H} \cup \{ x \} \) be a finite curve of finite Loewner energy, with upward driver \( \xi \in S_0([0, T]) \) and corresponding welding \( \varphi : [-a, 0] \to [0, b] \).

(i) For any partition \( P \) of \([-a, b]\) as in (62), there exists a curve \( \gamma_P \) with driver \( \xi_P \in S_0 \) that minimizes the Loewner energy among all curves welding \( P \).

(ii) If \( \{P_n\} \) is any sequence partitions with \( |P_n| \to 0 \), and \( \{\gamma_n\} \) a sequence of corresponding Loewner-energy minimizers from (i), then in the half-plane capacity parametrizations of the curves,

\[
\lim_{n \to \infty} \|\gamma_n - \gamma\|_{[0, T']} = 0
\]

for any \([0, T'] \subset [0, T]\). Furthermore, the Loewner energies of the entire curves satisfy

\[
\lim_{n \to \infty} I_L(\gamma_n) = I_L(\gamma).
\]

If the partitions are furthermore nested, \( P_n \subset P_{n+1} \) for all \( n \), the sequence \( \{I_L(\gamma_n)\} \) is nondecreasing.

---

1 We do not propose concrete means to actually compute the minimizers \( \gamma_n \), and so, we are admittedly using the term “algorithm” rather loosely. Our point is to allude to the welding zipper algorithm, our source of inspiration.
We note that the reason we must restrict to \([0, T'] \subset [0, T)\) in (63) is that, \textit{a priori}, the total time \(T_n\) of \(\gamma_n\) could be less than \(T\) for all \(n\).‡ The proof will show that if we rescale all the \(\gamma_n\)'s to have the “correct” time \(T\) via setting \(\tilde{\gamma}_n := \sqrt{\frac{T}{T_n}} \gamma_n\), then

\[
\|\tilde{\gamma}_n - \gamma\|_{\infty[0,T]} \to 0. \tag{65}
\]

In fact, our strategy to prove (63) will be to first show (65).

Note also that we normalize so that \(\xi(0) = 0\), corresponding to the conformal welding exchanging intervals on either side of the origin. Considering \(\gamma\) as generated by \(\xi\) on \([0, T]\), we thus have \(\gamma(0) = x = \xi(T)\), which is not necessarily zero.

We precede the proof by recalling two useful known results.

**Proposition 4.12** (Lemma 2.3 of [42], Lemma 6.1 of [17]). Let \(h_t, \tilde{h}_t\) be the upward flow maps satisfying (1) for drivers \(\xi, \tilde{\xi} \in C([0, T])\). Then, for \(0 \leq t \leq T\), \(z, \tilde{z} \in \mathbb{H}\) with \(\min\{|\text{Im}(z)|, |\text{Im}(\tilde{z})|\} =: Y\),

\[
|h_t(z) - \tilde{h}_t(\tilde{z})| \leq \sqrt{1 + \frac{4T}{Y} (|z - \tilde{z}| + \|\xi - \tilde{\xi}\|_{\infty[0,T]})}.
\]

**Proposition 4.13** (Proposition 2.1(i), (iii) of [10]). Let \(\gamma : [0, T] \to \mathbb{H}\) be a simple curve with Loewner energy \(I_L(\gamma) \leq M < \infty\) and downward driver \(\lambda\).

- \textit{The downward flow maps} \(g_t\) \textit{satisfy}

\[
|(g_t^{-1})'(\lambda(t) + iy)| \leq e^{\frac{1}{2}M}.
\]

\textit{for all} \(0 < y\) \textit{and} \(0 \leq t \leq T\).

- \textit{The half-plane capacity parametrization} \(\gamma\) \textit{is Hölder-1/2} with Hölder seminorm \(|\gamma|_{1/2} \leq Ce^{CM}\) for some \(C > 0\).†

**Proof of Theorem 4.11.** (i) In [29, Lemma 4.1], the second author proved the existence of Loewner-energy minimizers for a single pair \((x_1, y_1)\), also using Theorem 4.1. We extend the idea for the existence of \(\gamma_P\), repeating some details for the convenience of the reader.

We begin by observing that the set

\[
D(P, C) := \{\xi \in S_0 : I_L(\xi) \leq C, \gamma^\xi \text{welds} P\}
\]

is nonempty for sufficiently large \(C\). This is because one can repeatedly map up with conformal maps to the complement of circular arc segments orthogonal to \(\mathbb{R}\), welding two points at a time to the base of the arc, to obtain a simple curve, driven by some \(\xi\), and with welding \(\phi\) that welds \(P\). Each circular arc segment has finite energy, and so, \(I_L(\xi) < \infty\) because there are finitely many

‡ Recall the similar technicality due to the changing domains of the \(\tau_n\) discussed in Example 3.5.

† We note that this second result also follows, although without the explicit bound on \(|\gamma|_{1/2}\), from [23, proof of Lemma 4.1] and the fact that finite-energy curves have locally-small Hölder seminorm.
pairs. Next, take a sequence \( \xi_n \in D_0 : = D(P, I_L(\tilde{\xi})) \) such that

\[
\lim_{n \to \infty} I_L(\xi_n) = \inf \{ I_L(\xi) : \xi \in D_0 \}.
\]

We claim that we may suppose all the \( \xi_n \) are defined on a universal interval \([0, T_U] \) of capacity time. Indeed, since the diameter of any curve \( \tilde{\gamma} \) welding \(-a\) to \(b\) is comparable to \(b + a\) \([18, \text{top of p.74}]\), and

\[
\text{hc}(\tilde{\gamma}) \leq \text{hc} \left( \text{diam}(\tilde{\gamma}) B_1(\tilde{\gamma}(0)) \cap H \right) 
\leq C^2(a + b)^2 \text{hc} \left( B_1(0) \cap H \right) \leq C^2(a + b)^2
\]

by scaling and monotonicity of \(\text{hc}(\cdot)\) and (7), the times \(T_n\) for \(\xi_n\) to weld \(-a\) to \(b\) are all bounded. Furthermore, we can extend any \(\xi_n\) past \(T_n\) by the constant function \(\xi_n(T_n)\) without adding energy, and thus, some such \(T_U\) exists.

Since \(\{I_L(\xi_n)\}\) is bounded, by (61) and Hölder’s inequality, \(\{\xi_n\}\) is bounded and equicontinuous on \([0, T_U]\) and thus precompact. If \(\xi_{n_k} \to \eta\) is any subsequential uniform limit, by the lower semicontinuity of the energy in this topology \([45, \S2.2]\),

\[
I_L(\eta) \leq \liminf_{k \to \infty} I_L(\xi_{n_k}) = \inf \{ I_L(\xi) : \xi \in D_0 \},
\]

and so, \(\eta\) is a minimizer so long as it belongs to \(D_0\). That is, we must show \(\eta \in S_0\) and that the curve \(\gamma^\eta\) it generates welds \(P\). The first property is immediate since \(I_L(\eta) < \infty\); in fact, \(\gamma^\eta\) is a \(K\)-quasiarc for some \(K = K(I_L(\eta))\) \([45, \text{Prop. 2.1}]\). We claim that Theorem 4.1 shows \(\gamma^\eta\) welds \(P\). Indeed, by extending \(\eta\) past \(T_U\) to \([0, T'_U]\) with the constant value \(\eta(T_U)\) on \([T_U, T'_U]\), if necessary, we may assume that \([x_N, y_N]\) is in the interior of the interval welded by \(\eta\). Similarly, extending all the \(\xi_{n_k}\) on \([T_U, T'_U]\) to be constantly their terminal value \(\xi_{n_k}(T_U)\), we still have uniform convergence on \([0, T'_U]\), and Theorem 4.1 then yields

\[
y_j = \varphi_{n_k}(x_j) \to \varphi^\eta(x_j)
\]

for each \(j\), where \(\varphi_{n_k}\) and \(\varphi^\eta\) are the weldings associated to \(\xi_{n_k}\) and \(\eta\), respectively. Thus, \(\eta \in D_0\), and so, energy minimizers \(\gamma_p\) exist among all curves welding \(P\).

(ii) Now suppose that we have a sequence of partitions \(P_n = \{(x_j, y_j)\}_j \) with \(|P_n| \to 0\), and let \(\gamma_n\) be a minimizer for welding \(P_n\) with upward driver \(\xi_n \in S_0([0, T_n])\) that welds \(-a\) to \(b\) at time \(T_n\). By extending the drivers identically by their ending values, we may again assume that there is a single interval \([0, T_U]\) on which all the \(\xi_n\) are defined, and furthermore, that we have \(\varepsilon > 0\) such that each \(\xi_n\) welds an interval including

\[
[-a - \varepsilon, b + \varepsilon]
\]

by time \(T_U\).

† This iterative construction is an example of the welding zipper algorithm. The energy of a single orthogonal circular arc segment is explicitly computable; see [28, Lemma 5.7], for instance.
We show $\xi_n \to \xi$ uniformly on $[0, T_U]$ (we have also extended $\xi$ by constantly $\xi(T)$, if needed). First note that the weldings $\varphi_n$ for $\xi_n$ converge uniformly to the welding $\varphi$ for $\xi$, which is clear from monotonicity and the agreement on $P_n$ with $|P_n| \to 0$. As above, uniformly bounded energy implies that $\{\xi_n\}$ is precompact, and taking any uniform limit $\xi_{n_k} \to \eta$ on $[0, T_U]$, we have

$$I_L(\eta) \leq \liminf_{k \to \infty} I_L(\xi_{n_k}) \leq I_L(\xi)$$

by lower semicontinuity and minimization, and so, $\eta$ has finite energy and generates a simple curve $\gamma^n$, as noted above. In particular, we find

$$\max\{\tau(-a - \epsilon/2; \eta), \tau(b + \epsilon/2; \eta)\} < T_U$$

by (66) and Theorem 3.4, and so, by Theorem 4.1, $\eta$ welds identically to $\xi$ on $[-a, b]$. By injectivity of the welding-to-curve map in the category of quasi-arcs, the curves $\eta$ and $\xi$ generate by welding $[-a, 0]$ to $[0, b]$ are the same, up to postcomposition by affine map $z \mapsto cz + d$ for some $c, d \in \mathbb{R}$. However, since both $\gamma^n$ and $\gamma$ are normalized by the Loewner flow (recall (4)), we have $c = 1, d = 0$, and consequently, that $\eta \equiv \xi$ on $[0, T_U]$.

All subsequential limits of $\{\xi_n\}$ are therefore $\xi$, and we conclude $\xi_n \overset{u}{\to} \xi$ on $[0, T_U]$. In particular, convergence of hitting times yields

$$T_n = \tau(-a; \xi_n) \to \tau(-a; \xi) = T,$$  \hspace{1cm} (67)

and hence, by uniformity of convergence,

$$\xi_n(T_n) \to \xi(T).$$  \hspace{1cm} (68)

It remains to show that the curves $\gamma_n$ generated by $\xi_n$ on $[0, T_n]$ converge to $\gamma$ on any $[0, T'] \subset [0, T)$. (Recall that since the inverse Loewner transform $\eta \mapsto \gamma^n$ is not continuous from $S_0([0, T_U])$ to $C([0, T_U])$ [18, Ex. 4.49], this is not immediate.) We start by rescaling the minimizers to all the have the same capacity time $T$. Recall that the curve $\gamma_n$ generated by the nonextended driver $\xi_n$ welds $P_n$ in time $T_n$. We thus set $\alpha_n := \sqrt{T/T_n}$ and $\tilde{\gamma}_n := \alpha_n \gamma_n$, and we claim that the $\tilde{\gamma}_n$ converge uniformly in their half-plane capacity parametrizations to $\gamma$ on $[0, T]$. Note that the $\tilde{\gamma}_n$ have upward driving functions $\tilde{\xi}_n(\cdot) = \alpha_n \xi(\cdot/\alpha_n^2)$ that satisfy

$$|\tilde{\xi}_n(t) - \xi(t)| \leq \sqrt{\frac{T}{T_n}} \left|\xi_n\left(t \frac{T_n}{T}\right) - \xi \left(t \frac{T}{T_n}\right)\right| + \sqrt{\frac{T}{T_n}} \left|\xi \left(t \frac{T}{T_n}\right) - \xi(t)\right|,$$

where we are using the extension of $\xi$ to $[0, T_U]$ as needed. For large $n$, this is small by (67), the uniform continuity of $\xi$ on $[0, T_U]$, and the convergence $\xi_n \overset{u}{\to} \xi$ on $[0, T_U]$, showing

$$\|\tilde{\xi}_n - \xi\|_{\infty[0,T]} \to 0.$$

We can use this, along with boundedness of the Loewner energies of $\gamma_n, \gamma$, to show that $|\tilde{\gamma}_n(t) - \gamma(t)|$ is uniformly small. The downward driving functions of $\tilde{\gamma}_n$ and $\gamma$ are $\tilde{\lambda}_n(t) = \tilde{\xi}_n(T - t)$ and $\lambda(t) = \xi(T - t)$, respectively, and we have, for $0 \leq t \leq T$,  

$$|\tilde{\gamma}_n(t) - \gamma(t)| = |\tilde{\xi}_n^{-1}(t, \tilde{\lambda}_n(t)) - g^{-1}(t, \lambda(t))|, $$
where $\tilde{g}_n(t, z)$ and $g(t, z)$ are the downward flow maps satisfying (8) for $\tilde{\gamma}_n$ and $\gamma$, respectively. For fixed $t$, by Lemma 2.1 with $T$ replaced by $t$, we have that $\tilde{g}_n^{-1}(t, \cdot) = \tilde{h}_n(t, \cdot)$, where $\tilde{h}_n$ is the upward flow map satisfying (1) generated by the driver $s \mapsto \tilde{\xi}_n(s)$ on $T - t \leq s \leq T$, while similarly $g^{-1}(t, \cdot) = h(t, \cdot)$, with the latter generated by upward driver $s \mapsto \xi(s)$ on $T - t \leq s \leq T$. We thus see, for fixed $0 < \gamma$,

$$\left|\tilde{\gamma}_n(t) - \gamma(t)\right| = \left|\tilde{h}_n(t, \tilde{\lambda}_n(t)) - h(t, \lambda(t))\right|$$

$$\leq \left|\tilde{h}_n(t, \tilde{\lambda}_n(t)) - \tilde{h}_n(t, \tilde{\lambda}_n(t) + iy)\right|$$

$$+ \left|\tilde{h}_n(t, \tilde{\lambda}_n(t) + iy) - h(t, \lambda(t) + iy)\right|$$

$$+ \left|h(t, \lambda(t) + iy) - h(t, \lambda(t))\right|,$$

where we are estimating similarly to [12, Lemma 3.4], for example. Proposition 4.13 yields

$$\left|\tilde{h}_n(t, \tilde{\lambda}_n(t)) - \tilde{h}_n(t, \tilde{\lambda}_n(t) + iy)\right| = \left|\tilde{g}_n^{-1}(t, \tilde{\lambda}_n(t)) - \tilde{g}_n^{-1}(t, \tilde{\lambda}_n(t) + iy)\right|$$

$$\leq \int_0^y \left|\tilde{g}_n^{-1}(\tilde{\lambda}(t) + iv)\right| dv \leq ye^{\frac{1}{2}I_L(\gamma)},$$

since $I_L(\gamma_n) < I_L(\gamma)$ by construction. The third term in our previous estimate having the same bound, we have

$$\left|\tilde{\gamma}_n(t) - \gamma(t)\right| \leq ye^{\frac{1}{2}I_L(\gamma)} + \left|\tilde{h}_n(t, \tilde{\lambda}_n(t) + iy) - h(t, \lambda(t) + iy)\right|$$

$$\leq ye^{\frac{1}{2}I_L(\gamma)} + 2\sqrt{1 + \frac{4T}{y}} \left\|\tilde{\xi}_n - \xi\right\|_{\infty[0, T]}$$

by Proposition 4.12. Choosing small $\gamma$ and then large $n$ shows that

$$\left\|\tilde{\gamma}_n - \gamma\right\|_{\infty[0, T]} \to 0,$$

(69)

as claimed.

Note that the convergence (63) of the unscaled curves $\gamma_n$ is an immediate consequence. Indeed, the $\gamma_n$ are defined on $[0, T'] \subset [0, T)$ for large $n$ by (67), and for $0 \leq t \leq T'$, we observe

$$\left|\gamma_n(t) - \gamma(t)\right| \leq \left|\gamma_n(t) - \sqrt{\frac{T}{T_n}}\gamma_n\left(\frac{T_n}{T}t\right)\right| + \left\|\tilde{\gamma}_n - \gamma\right\|_{\infty[0, T]}. $$

(70)

Using the uniform Hölder continuity of the $\gamma_n$ from Proposition 4.13 (which, in particular, implies that the $\gamma_n$ are uniformly bounded by some $C'$), we estimate

$$\left|\gamma_n(t) - \sqrt{\frac{T}{T_n}}\gamma_n\left(\frac{T_n}{T}t\right)\right| \leq \left|\gamma_n(t) - \gamma_n\left(\frac{T_n}{T}t\right)\right| + \left|\gamma_n\left(\frac{T_n}{T}t\right)\right| \left|1 - \sqrt{\frac{T}{T_n}}\right|$$

$$\leq Ce^{C_I_L(\gamma)}\sqrt{T - T_n} + C' \left|1 - \sqrt{\frac{T}{T_n}}\right|,$$
which is small for large $n$ by (67). Combined with (70) and (69), we conclude that $\|\gamma_n - \gamma\|_{\infty[0,T']} \to 0$, as claimed.

We lastly turn to the energy limit (64). By minimization,

$$I_L(\gamma_n) = \frac{1}{2} \int_0^{T_n} \xi_n^2(t) dt \leq \frac{1}{2} \int_0^{T} \xi^2(t) dt = I_L(\gamma)$$

for all $n$, and so, $\limsup_{n \to \infty} I_L(\gamma_n) \leq I_L(\gamma)$. Write $I_{L,t}(\cdot)$ for the Loewner energy on an interval $[0,t]$. Since our extended drivers satisfy $\xi_n \overset{u}{\to} \xi$ on $[0,T_U]$, by lower semicontinuity and minimization,

$$I_{L,T}(\gamma) = I_{L,T_U}(\xi) = \liminf_{n \to \infty} I_{L,T_U}(\xi_n) = \liminf_{n \to \infty} I_{L,T_n}(\xi_n) \leq I_{L,T}(\gamma),$$

where we recall that extending the drivers did not add any energy. Thus, $\limsup I_L(\gamma_n) \leq \liminf I_L(\gamma_n)$ and (64) follows. If the partitions are nested, $\gamma_{n+1}$ also welds $P_n$, showing $I_L(\gamma_n) \leq I_L(\gamma_{n+1})$ by minimization. \hfill \Box

## 5 | PROBLEMS

We close with several problems that appear natural from our study of drivers, weldings, and hitting times.

**Problem 5.1.** Suppose that weldings $\varphi_n, \varphi$ correspond to drivers $\xi_n, \xi \in S_0$. If all the weldings share a fixed modulus of continuity $\omega$, does $\varphi_n \overset{u}{\to} \varphi$ imply $\xi_n \overset{u}{\to} \xi$?

**Problem 5.2.** Is $\tau \mapsto \xi$ pointwise continuous at all hitting times arising from curves that are conformally removable in $\mathbb{H}$?

**Problem 5.3.** The contrast between Theorem 3.13(ii) and Lemma 4.9 shows that hitting times $\tau$ are “better behaved” than conformal weldings $\varphi$. Is the map $\tau \mapsto \xi$ well defined (and continuous) on all of $S_0$?

**Problem 5.4.** Suppose $\xi, \tilde{\xi} \in S_0$ have hitting times $\tau$ and $\tilde{\tau}$, respectively. If $\tau_- = \tilde{\tau}_-$, does $\tau_+ = \tilde{\tau}_+?$

**Problem 5.5.** Jordan curves $\gamma \subset \hat{\mathbb{C}}$ have conformal weldings $\varphi : S^1 \to S^1$ and drivers $\xi$ defined on all of $\mathbb{R}$ (see [46, §6] for an explanation of the latter). Do our results extend to this setting?

**APPENDIX: INTEGRAL FORMULAS RELATING DRIVERS, HITTING TIMES AND WELDINGS**

**Lemma A.1.** Let $\xi$ be an upward driver generating a simple curve, and let $x_0$ be a point with $\tau(x_0; \xi) < \infty$. The hitting time $\tau := \tau(x_0; \xi)$ satisfies

$$\tau = \frac{1}{4} (x_0^2 - x(\tau)^2) - \int_0^{\tau} \frac{\xi(t)}{x(t) - \xi(t)} dt. \quad (A.1)$$
Furthermore, if $y_0 = \varphi(x_0)$ is the image of $x_0$ under the conformal welding $\varphi$ generated by $\xi$, then

$$\varphi(x_0)^2 - x_0^2 = 4 \int_0^\tau \frac{\xi(t)(y(t) - x(t))}{(\xi(t) - x(t))(y(t) - \xi(t))} dt. \quad (A.2)$$

**Proof.** If $x_0 = \xi(0)$, the first formula is obvious. If not, we observe that $\partial_t (x(t)^2) = -4x(t)/(x(t) - \xi(t))$, yielding

$$x(\tau - \epsilon)^2 - x_0^2 = -4 \int_0^{\tau - \epsilon} \frac{x(t)}{x(t) - \xi(t)} dt.$$

Since $t \mapsto x(t)$ is continuous up to $t = \tau$, sending $\epsilon \to 0$ gives

$$x(\tau)^2 - x_0^2 = -4 \int_0^\tau \frac{x(t)}{x(t) - \xi(t)} dt = -4\tau - 4 \int_0^\tau \frac{\xi(t)}{x(t) - \xi(t)} dt,$$

which yields (A.1).

Second, since $x(\tau) = y(\tau) = \xi(\tau)$, using (A.1) for both $x_0$ and $y_0$ and equating the right-hand sides yields

$$x_0^2 - 4 \int_0^\tau \frac{\xi(t)}{x(t) - \xi(t)} dt = y_0^2 - 4 \int_0^\tau \frac{\xi(t)}{y(t) - \xi(t)} dt,$$

which is equivalent to (A.2). \qed

**ACKNOWLEDGEMENTS**

The authors are thankful to Yizheng Yuan for pointing our attention to [39, Lemma 5.1] and suggesting how it could yield welding convergence, and for looking at a draft of this paper. We also thank Don Marshall and Steffen Rohde for comments on a very early draft, and we are grateful to have learned the trick (25) from Steffen Rohde (perhaps, it goes back to Oded Schramm), and to have seen it applied in a similar manner, albeit rougher, to what we do in Theorem 3.8. We are also deeply grateful to the anonymous referee for meticulously combing through the manuscript and providing us with very helpful feedback.

This research was partially conducted while the authors were at Mathematical Sciences Research Institute during the spring 2022 semester and is thus partially supported by the US National Science Foundation under Grant No. DMS-1928930.

**DATA AVAILABILITY STATEMENT**

No datasets were used in this research.

**JOURNAL INFORMATION**

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.
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