A model of color confinement

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A simple model is presented that describes the free energy \( W(J) \) of QCD coupled to an external current that is a single plane wave, \( J(x) = H \cos(k \cdot x) \). The model satisfies a bound obtained previously on \( W(J) \) that comes from the Gribov horizon. If one uses this model to fit recent lattice data — which give for the gluon propagator \( D(k) \) a non-zero value, \( D(0) \neq 0 \), at \( k = 0 \) — the data favor a non-analyticity in \( W(J) \).

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I. INTRODUCTION

Recent numerical studies on large lattices of the gluon propagator \( D(k) \) in Landau gauge in 3 and 4 Euclidean dimensions, reviewed recently in \[1\], yield finite values for \( D(0) \neq 0 \) \[2 - 7\], in apparent disagreement with the theoretical expectation that \( D(0) = 0 \), originally obtained by Gribov \[8\], and argued in \[9\]. Upon reviewing the argument \[9\] which leads to \( D(0) = 0 \), one hypothesis stands out which should perhaps be dropped in view of the apparent disagreement. This is the hypothesis that the free energy \( W(J) \) in the presence of sources \( J \) is analytic in \( J \). This is an important point because a non-analyticity in the free energy is characteristic of a change of phase.

The free energy \( W(J) \) enters the picture because it is the generating function of the connected gluon correlators. In particular the gluon propagator \( D_{x,y}(J) \) is a second derivative of \( W(J) \) at \( J = 0 \),

\[
D_{x,y} = \frac{\partial^2 W(J)}{\partial J_x \partial J_y} \bigg|_{J=0}.
\]

(1)

Here a condensed index notation is used, where \( J_x \) represents \( J^b_{\mu}(x) \), \( \mu \) is a Lorentz index, and \( b \) is a color index, and we write

\[
(J, A) = \sum_x J_x A_x = \int d^D x J^b_{\mu}(x) A^b_{\mu}(x).
\]

(2)

The free energy \( W(J) \) in the presence of sources \( J \) is given by

\[
\exp W(J) = \langle \exp(J, A) \rangle = \int_{\Omega} dA \rho(A) \exp(J, A),
\]

(3)

where \( \rho(A) \) is a positive, normalized probability distribution. The integral over \( A \) is effected in Landau gauge \( \partial_{\mu} A_{\mu} = 0 \). The domain of integration is restricted to the Gribov region \( \Omega \), a region in \( A \)-space where the Faddeev-Popov operator is non-negative, \( M(A) \equiv -\partial_{\mu} D_{\mu}(A) \geq 0 \).

The model will be defined for the special case where the source is given by

\[
J^b_{\mu}(x) = H^b_{\mu} \cos(k \cdot x),
\]

(4)

so the free energy

\[
\exp W_k(H) = \langle \exp \left[ \sum_x H^b_{\mu} \cos(k \cdot x) A^b_{\mu}(x) \right] \rangle,
\]

(5)

depends only on the parameters \( k_{\mu} \) and \( H^b_{\mu} \). Because \( A_{\mu}(x) \) is transverse, only the transverse part of \( H \) is operative, and we impose

\[
k_{\mu} H^b_{\mu} = 0.
\]

(6)

By analogy with spin models, \( H^b_{\mu} \) may be interpreted as the strength of an external magnetic field, with a color index \( b \), which is modulated by a plane wave \( \cos(k \cdot x) \).

A rigorous bound for \( W_k(H) \) on a finite lattice is given in \[8\]. One can easily show that in the limit of large lattice volume \( V \), and in the continuum limit, this implies the Lorentz-invariant continuum bound in \( D \) Euclidean dimensions,

\[
w_k(H) \equiv \frac{W_k(H)}{V} \leq (2Dk^2)^{1/2} |H|,
\]

(7)

where \( |H|^2 = \sum_{\mu,b} (H^b_{\mu})^2 \) is the color- and Lorentz-invariant norm of \( H^b_{\mu} \). In our notation the vector potential is given by \( A(x) = g_A \text{pert}(x) \), and has engineering dimension in mass units \( [A(x)] = 1 \) in all Euclidean dimension \( D \), while \( [H] = D - 1 \).

This bound yields in the zero-momentum limit

\[
w_0(H) = \lim_{k \to 0} w_k(H) = 0.
\]

(8)

As discussed in \[8\], this states that the system does not respond to a constant external color-magnetic field \textit{no matter how strong}. In this precise sense, the color degree of freedom is absent, and color is confined.
II. THE MODEL

The model of color confinement is defined by the expression for the free energy

\[ W_k(H) = c|k|V\{[H^2 + f^2(k)]^{1/2} - f(k)\}, \quad (9) \]

where \( c \) is a dimensionless constant, and \( f(k) \geq 0 \) is an as yet undetermined function. For \( c = (2D)^{1/2} \) this expression for \( W_k(H) \) satisfies the bound \( (7) \), as one sees from \((H^2 + f^2)^{1/2} \leq |H| + f\). The bound \( (7) \) is saturated for small \( k \) if

\[ f(0) = 0. \quad (10) \]

For example we shall later take

\[ f(k) = m^{D-2}|k|, \quad (11) \]

where \( m \) is a mass.

One can easily verify that expression \( (9) \) also satisfies \( W_k(0) = 0 \), as required for normalized probability \( \rho(A) \), and that the gluon propagator, defined below, is a positive matrix, as required.

By definition the “classical field” is the expectation-value

\[ a^b_{\mu}(k, H) = \langle a^b_{\mu}(k) \rangle_{H} \quad (12) \]

of the field

\[ a^b_{\mu}(k) \equiv \int \frac{d^Dx}{V} \cos(k \cdot x) A^b_{\mu}(x) \quad (13) \]

in the presence of the source \( H \), and at momentum \( k \). It is expressed in terms of \( W_k(H) \), by

\[ a^b_{\mu}(k, H) \equiv \frac{\partial W_k(H)}{\partial H^b_{\mu}} = c|k|V \frac{H^b_{\mu}}{[H^2 + f^2(k)]^{1/2}}, \quad (14) \]

and the gluon propagator by

\[ D^{bc}_{\mu\nu}(k, H) \equiv \frac{\partial^2 W_k(H)}{\partial H^b_{\mu} \partial H^c_{\nu}} = c|k|P^{bc}_{\mu\nu}(k) \delta^{bc}[H^2 + f^2(k)] + H^b_{\mu} H^c_{\nu} \frac{1}{[H^2 + f^2(k)]^{3/2}}. \quad (15) \]

Here \( P^{bc}_{\mu\nu}(k) = \delta_{\mu\nu} - k_{\mu} k_{\nu}/k^2 \) is the transverse projector which appears because \( H^b_{\mu} \) is identically transverse \( k_{\mu} H^b_{\mu} = 0 \), and \( \partial H^b_{\mu}/\partial H^c_{\nu} = P^{bc}_{\mu\nu}(k) \delta^{bc} \).

The free energy \( W_k(H) \) vanishes at \( k = 0 \) for all \( H \),

\[ W_0(H) = 0. \quad (16) \]

If \( W_k(H) \) were analytic in \( H \) in the limit \( k \to 0 \), this would imply that all derivatives of \( W_0(H) \) vanish and with it the gluon propagator \( (15) \) at \( k = 0 \), \( D(0) = 0 \). However this disagrees with recent lattice data \[1] \ which give a finite result, \( D(0) \neq 0 \), in Euclidean dimensions 3 and 4. However the second (and higher) derivative \( W''_k(H) \) is non-analytic in \( H \) in the limit \( k \to 0 \) when \( f(0) = 0 \), and the gluon propagator \( (15) \)

\[ D^{bc}_{\mu\nu}(k) = D^{bc}_{\mu\nu}(k,0) = \frac{c|k|}{f(k)} P^{bc}_{\mu\nu} \quad (17) \]

does not necessarily vanish for \( k \to 0 \). Indeed with \( f(k) = m^{D-2}|k| \), we have

\[ D(0) = c/m^{D-2}, \quad (18) \]

which is finite. Thus our model, with \( f(k) = m^{D-2}|k| \) satisfies the bound \( (7) \) and accords with lattice data for \( D = 3 \) and \( D = 4 \). In contradistinction to the previous treatment \[9\], the hypothesis that \( W_k(H) \) is analytic in \( H \) for \( k \to 0 \) is relaxed. Moreover, for \( f(k) = m^{D-2+\alpha}|k|^{1-\alpha} \), we get

\[ D(k) = \frac{ck^\alpha}{m^{D-2+\alpha}}, \quad (19) \]

which fits the data in dimension \( D = 2 \) for \( \alpha = 1/5 \) \[10\].

If one fits the lattice data with this model, the data favor the value \( f(0) = 0 \) in \( D = 2, 3 \), and 4 dimensions. With the value \( f(0) = 0 \), the model gives asymptotically at low \( k \)

\[ W_k(H) \approx W^{as}_k(H) = c|k||H|V. \quad (20) \]

This is linear in \( |H| \), whereas normally one expects the free energy \( W_k(H) \) to be a power series in \( H \), with leading term of order \( H^2 \).

III. MODEL QUANTUM EFFECTIVE ACTION

As in statistical mechanics we define, for each momentum \( k \), the analog of the bulk magnetization

\[ M^b_{\mu}(k, H) = \frac{\partial W_{k}(H)}{\partial H^b_{\mu}}, \quad (21) \]

and make the Legendre transformation from \( W_k(H) \) to

\[ \Gamma_k(M_k) = M_k H - W_k(H). \quad (22) \]

In fact the “magnetization” \( M_k(H) \), coincides in our gauge theory with the classical gluon field given in \[14\],

\[ a^b_{\mu}(H) = M_k(H), \quad (23) \]

and the Legendre transformation \( (22) \) yields the quantum effective action of the gauge theory

\[ \Gamma_k(a_k) = a^b_{\mu} H - W_k(H), \quad (24) \]

More precisely this is the quantum effective action with all variables set to 0 except \( a^b_{\mu}(k) \) for fixed \( k \). To find \( \Gamma_k(a_k) \) we note that

\[ \Gamma_k(H) = \frac{c|k||H^2}{[H^2 + f^2(k)]^{1/2}} - W_k(H) \]

\[ = c|k||f(k)|V \left\{ 1 - \frac{f(k)}{[H^2 + f^2(k)]^{1/2}} \right\}. \quad (25) \]
and from squaring (14) we find

$$\frac{f}{[H^2 + f^2(k)]^{1/2}} = \left[1 - \left(\frac{a_k}{c|k|V}\right)^2\right]^{1/2}, \quad (26)$$

which gives for the quantum effective action with all variables except $a_k$ set to 0

$$\Gamma_k(a_k) = c|k|f(k)V\left\{1 - \left[1 - \left(\frac{|a_k|}{c|k|V}\right)^2\right]^{1/2}\right\}. \quad (27)$$

It is singular at

$$|a(k)| = c|k|V. \quad (28)$$

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