An adaptive central-upwind scheme on quadtree grids for variable density shallow water equations

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Abstract
Minimizing computational cost is one of the major challenges in the modeling and numerical analysis of hydrodynamics, and one of the ways to achieve this is by the use of quadtree grids. In this article, we present an adaptive scheme on quadtree grids for variable density shallow water equations. A scheme for the coupled system is developed based on the work of [M.A. Ghazizadeh, A. Mohammadian, and A. Kurganov, Computers & Fluids, 208 (2020)]. The scheme is capable of exactly preserving “lake-at-rest” steady states. A continuous piecewise bi-linear interpolation of the bottom topography function is used to achieve higher-order in space in order to preserve the positivity of water depth for the point values of each computational cell. Necessary conditions are checked to be able to preserve the positivity of water depth and density, and to ensure the achievement of a stable numerical scheme. At each timestep, local gradients are examined to find new seeding points to locally refine/coarsen the computational grid.

KEYWORDS
central-upwind scheme, positivity-preserving scheme, quadtree grids, shallow water equations, variable density, well-balanced scheme

1 | INTRODUCTION
Quadtree grids are two-dimensional (2-D) semistructured Cartesian grids that can be very accommodating for various problems in the field of computational hydrodynamics. One of the advantages of quadtree grids over structured and unstructured grids is grid coarsening/refining. The accuracy is increased/maintained while the grid refines/coarsens wherever it is needed and thus, the computational cost is reduced. There are a number of studies on how to generate quadtree grids; see, for example, References 1-8.

The main goal of this paper is to develop an adaptive well-balanced positivity-preserving central-upwind scheme on quadtree grids for the 2-D Ripa system.\textsuperscript{9-11} The Ripa model which is popular in ocean modeling, and in fact is the variable density shallow water equations (SWEs), can be written in terms of conservative variables of \( w \) (water surface), \( hu \) and \( hv \) (the unit discharges), and \( h\rho \) in 2-D:

\[
\begin{align*}
\frac{w_t}{w} + (hu)_x + (hv)_y &= 0, \\
(hu)_t + (hu^2 + \frac{g}{2\rho}h^2\rho)_x + (huv)_y &= -\frac{g}{\rho}h\rho B_x, \\
(hv)_t + (huv)_x + (hv^2 + \frac{g}{2\rho}h^2\rho)_y &= -\frac{g}{\rho}h\rho B_y, \\
(h\rho)_t + (h\rho u)_x + (h\rho v)_y &= 0.
\end{align*}
\]

(1)
where \( t \) is time, \( g \) is the gravitational constant, \( x \) and \( y \) are the directions in the 2-D Cartesian coordinate system, \( u(x, y, t) \) and \( v(x, y, t) \) are the water velocities in the \( x \)- and \( y \)-directions, respectively, \( B(x, y) \) is the bottom topography, \( h(x, y, t) = w(x, y, t) - B(x, y) \) is the water depth, \( \rho \) is the density, and \( \rho_0 \) is the reference density.

The Ripa system (1) admits “lake-at-rest” steady state solutions, which is called the well-balanced property,

\[
\rho \equiv \text{Const}, \quad w \equiv \text{Const}, \quad u = v \equiv 0, \quad (2)
\]

and

\[
B \equiv \text{Const}, \quad p := \frac{g}{2\rho_0} h^2 \rho \equiv \text{Const}, \quad u = v \equiv 0. \quad (3)
\]

which can be obtained from (1).\(^{12} \) The proposed quadtree scheme is capable of exactly reserving “lake-at-rest” steady state in (2). Another important attribute of the following method is its ability to preserve the non-negativity of \( h \) and \( \rho \), which is called the positivity-preserving property (see Reference 13 for a comprehensive review on these subjects).

A number of numerical schemes on quadtree grids for the SWEs have been introduced in recent years. For example, an adaptive well-balanced positivity-preserving central-upwind high-order scheme on quadtree grids was proposed in Reference 3. In addition, a well-balanced scheme on quadtree-cut-cell grids was proposed in Reference 14. This scheme is based on the hydrostatic reconstruction from Reference 15. An adaptive second-order Roe-type scheme was proposed in Reference 16. An adaptive well-balanced Godunov-type scheme for the shallow water for the wet-dry over complex topography was introduced in Reference 17 and an adaptive quadtree Roe-type scheme for 2-D two-layer SWEs was presented in Reference 18. Furthermore, a residual distribution (RD) discretization method in Reference 19 and a discontinuous Galerkin (DG) scheme in Reference 20 for the SWE over quadtree grids were introduced recently.

A finite volume shock capturing scheme for the SWE with suspended sediment transport in Reference 21 and the variable water-sediment mixture density\(^{22} \) are some of the recent works similar to the variable density system that we would like to mention. Besides the aforementioned numerical methods, several well-balanced positivity-preserving central-upwind schemes for the shallow water equations have been proposed in the past years; see, for example, References 15, 23-36, yet, to our knowledge, none of these methods has been extended to the coupled variable density SWEs over quadtree grids.

In Reference 37 the coupled variable density SWEs were studied with a Godunov-type HLLC approximate Riemann solver. There are other studies that have been conducted on variable density SWEs and variable horizontal temperature SWEs (which have mathematically similar properties) with different numerical schemes; see, for example, References 12, 38-40.

In this article, we propose a central-upwind quadtree scheme which is based on the one from Reference 3 that can be implement on problems with discontinuous bottom topography. Central-upwind schemes are finite-volume methods that are Godunov-type Riemann-problem-solver-free.\(^{41-44} \) Central-upwind schemes have been referred to as “black-box” solvers for general multidimensional systems of hyperbolic systems of conservation laws, and have been extended to shallow water models.\(^{13} \) The proposed scheme is the first well-balanced positivity-preserving central-upwind scheme for the variable density SWEs over quadtree grids that is capable of solving problems with discontinuous bottom topography. This method is simple, efficient, and robust (we emphasize on References 3, 33).

The article is organized as follows. In Section 2, we briefly describe the quadtree grid generation terminology. In Section 3, we construct a central-upwind quadtree scheme for the variable density SWEs with the mentioned features and test it on four different numerical examples that are presented in Section 4.

### 2 | QUADTREE GRIDS

In this section, we denote the terminology for how to generate quadtree grids (see References 2-4, 45):

**Seeding points:** A set of points that helps to locally refine/coarsen the computational grid when needed.

**Level of refinement:** Level of the quadtree, in which the size of the smallest cell is inversely proportional to the maximum level of \( m \).

**Regularized quadtree grid:** In a regularized grid, no cell can have both an adjacent neighboring cell and a diagonally neighboring cell with a refinement level difference greater than one (Figure 1). The proposed scheme is based on regularized quadtree grids to prevent complicated formulations and improve stability.
3 Adaptive Semidiscrete Central-Upwind Scheme

We write system (1) in the following vector form:

\[ U_t + F(U, B)_x + G(U, B)_y = S(U, B), \]

(4)

where

\[ U := (w, hu, hv, h\rho)^\top, \]

and the fluxes and source term are:

\[ F(U, B) = \left( hu, \frac{(hu)^2}{w-B}, \frac{g}{2\rho} (w-B)^2, \frac{(hu)(hv)}{w-B}, h\rho \right)^\top, \]

(5)

\[ G(U, B) = \left( hv, \frac{(hu)(hv)}{w-B}, \frac{(hv)^2}{w-B}, \frac{g}{2\rho} (w-B)^2, hv\rho \right)^\top, \]

(6)

\[ S(U, B) = \left( 0, -\frac{g}{\rho} (w-B)B_x, -\frac{g}{\rho} (w-B)B_y, 0 \right)^\top. \]

(7)

In the following, an adaptive well-balanced semidiscrete central-upwind scheme for (4) is presented. The proposed scheme will be designed according to the algorithm in Reference 3:

3.1 Finite-volume semidiscretization over quadtree grids

Let us represent each cell \( C_{j,k} \) of size \( \Delta x_{j,k} \times \Delta y_{j,k} \) centered at \((x_{j,k}, y_{j,k})\) as a finite volume quadtree cell in the proposed scheme. The approximate averages of the cell read as:

\[ \bar{U}_{j,k}(t) \approx \frac{1}{\Delta x_{j,k} \Delta y_{j,k}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{y_{k-\frac{1}{2}}}^{y_{k+\frac{1}{2}}} U(x, y, t) \, dy \, dx, \]

(8)

where \( x_{j \pm \frac{1}{2}} := x_{j,k} \pm \Delta x_{j,k}/2 \) and \( y_{k \pm \frac{1}{2}} := y_{j,k} \pm \Delta y_{j,k}/2 \). Let us consider the neighboring cells of \( C_{j,k} \). In the regularized quadtree grid, there exists eight possible neighboring cell configurations in the \( x \)-direction and eight possible configurations in the \( y \)-direction.\(^3\) For the sake of brevity, we only present the proposed scheme for the configuration in Figure 2. We note that the implementation of the rest of configurations is similar to what is presented in what follows and all configurations need to be considered in order for the scheme to hold the properties demonstrated in this study. The
left-neighboring cells of $C_{j,k}$ are denoted by $I$ and $II$ which are centered at $(x_{j,k} - 3\Delta x_{j,k}/4, y_{j,k} \pm \Delta y_{j,k}/4)$ with a size of $\Delta x_{j,k}/2 \times \Delta y_{j,k}/2$.

The evolution of time-dependant cell averages $\overline{U}_{j,k}$, which are obtained after the semidiscretization of the system (4)–(7), are computed by solving the following system of Ordinary Differential Equations (ODE):

$$
\frac{d}{dt} \overline{U}_{j,k} = \frac{H^x_{j+\frac{1}{2},k} - \frac{H^x_{j+\frac{3}{2},k} + H^x_{j+\frac{1}{2},k+\frac{1}{2}}}{2}}{\Delta x_{j,k}} - \frac{H^y_{j,k+\frac{1}{2}} - \frac{H^y_{j,k+\frac{3}{2}}}{2}}{\Delta y_{j,k}} + \overline{S}_{j,k}.
$$

(9)

In (9), $H^x_{j+\frac{1}{2},k}$, $H^x_{j+\frac{3}{2},k}$, $H^y_{j,k+\frac{1}{2}}$, $H^y_{j,k+\frac{3}{2}}$ and $\overline{S}_{j,k}$ are the numerical fluxes, and $\overline{S}_{j,k}$ is a cell average of the source term:

$$
\overline{S}_{j,k} \approx \frac{1}{\Delta x_{j,k} \Delta y_{j,k}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{y_{k-\frac{1}{2}}}^{y_{k+\frac{1}{2}}} S(U, B) \, dy \, dx.
$$

(10)

For the sake of brevity, we have omitted all of the time-dependent indexed quantities in (9)–(10).

### 3.2 Piecewise bilinear reconstruction of $B$

Cells of different sizes $\Delta x \times \Delta y$, $\Delta x_1 \times \Delta y_1, \ldots, \Delta x_n \times \Delta y_n$ exist in the quadtree grid. The set of cells of the corresponding size is indicated by $G^{(r)}$, that is, $G^{(r)} = \{ C_{j,k} : |C_{j,k}| = \Delta x_r \times \Delta y_r \}$. We exactly follow the steps that were introduced in Reference 3 to reconstruct the bottom topography $\tilde{B}(x,y)$.

**Step 1.** Set $\ell := 1$.

**Step 2.** Reconstruct bilinear pieces $\tilde{B}$ for all $(j,k)$ such that $C_{j,k} \in G^{(r)}$ and

$$
\tilde{B}(x,y) = B_{j+\frac{1}{2},k+\frac{1}{2}} + (B_{j+\frac{1}{2},k+\frac{1}{2}} - B_{j-\frac{1}{2},k+\frac{1}{2}}) \frac{x - x_{j+\frac{1}{2},k+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2},k+\frac{1}{2}}} + (B_{j-\frac{1}{2},k+\frac{1}{2}} - B_{j-\frac{1}{2},k-\frac{1}{2}}) \frac{y - y_{j+\frac{1}{2},k+\frac{1}{2}}}{\Delta y_{j+\frac{1}{2},k+\frac{1}{2}}} \quad \forall (x,y) \in C_{j,k}.
$$

**Step 3.** Set $\ell := \ell + 1$.

**Step 4.** If $\ell \leq m$, then go to Step 2.

The cell average of $\tilde{B}$ over the cell $C_{j,k}$ is equal to its value at the center, that is

$$
B_{j,k} := \tilde{B}(x_j, y_k) = \frac{1}{\Delta x_{j,k} \Delta y_{j,k}} \int_{C_{j,k}} \tilde{B}(x,y) \, dx \, dy = \frac{B_{j+\frac{1}{2},k} + B_{j-\frac{1}{2},k} + B_{j,k+\frac{1}{2}} + B_{j,k-\frac{1}{2}}}{4}.
$$

(11)
where

$$B_{j \pm \frac{1}{2}, k} := \tilde{B}(x_{j \pm \frac{1}{2}}, y_{k}) = \frac{1}{2} \left( B_{j \pm \frac{1}{2}, k + \frac{1}{2}} + B_{j \pm \frac{1}{2}, k - \frac{1}{2}} \right),$$

and

$$B_{j, k \pm \frac{1}{2}} := \tilde{B}(x_{j}, y_{k \pm 1}) = \frac{1}{2} \left( B_{j + \frac{1}{2}, k \pm \frac{1}{2}} + B_{j - \frac{1}{2}, k \pm \frac{1}{2}} \right).$$

### 3.3 Piecewise linear reconstructions

In this section, we construct a spatial second-order scheme, which employs a piecewise polynomial interpolation \( \tilde{U} \), where

$$\tilde{U}(x,y) = (U_{x})_{j,k}[x-x_{j}] + (U_{y})_{j,k}[y-y_{k}], \quad (x,y) \in C_{j,k}. \quad (14)$$

With such a reconstruction, it is possible to build a well-balanced scheme; however, the density equation adds non-physical oscillations in the solution due to violating conservation of mass. Thus, instead of reconstructing conservative variables in \( U \), we introduce \( \mathcal{U} := (w, hu, hv, \rho)^{T} \) and then obtain the point values of \( \mathcal{U} \) for the cell \( C_{j,k} \) in Figure 2 which results in

$$\mathcal{U}_{j+\frac{1}{2},k}^{+} = \overline{U}_{j+\frac{1}{2},k} + \frac{\Delta x_{j+1,k}}{2} (\overline{U}_{x})_{j+1,k}, \quad \mathcal{U}_{j+\frac{1}{2},k}^{-} = \overline{U}_{j+\frac{1}{2},k} + \frac{\Delta x_{j,k}}{2} (\overline{U}_{y})_{j,k},$$

$$\mathcal{U}_{j-\frac{1}{2},k}^{+} = \overline{U}_{j-\frac{1}{2},k} + \frac{\Delta x_{j,k}}{2} (\overline{U}_{x})_{j-1,k}, \quad \mathcal{U}_{j-\frac{1}{2},k}^{-} = \overline{U}_{j-\frac{1}{2},k} + \frac{\Delta x_{j+1,k}}{2} (\overline{U}_{y})_{j+1,k},$$

where \( \overline{U} \) denote the cell averages of \( \mathcal{U} \). Note that in (15) the density variable \( \rho_{j,k} \) is computed as

$$\rho_{j,k} := \frac{(h \rho)_{j,k}}{h_{j,k}}, \quad \tilde{h}_{j,k} := \overline{w}_{j,k} - B_{j,k},$$

and the point values of \( h \) and \( h \rho \) read as

$$h_{j-\frac{1}{2},k \pm \frac{1}{2}}^{+} = w_{j-\frac{1}{2},k \pm \frac{1}{2}}^{+} - B_{j-\frac{1}{2},k \pm \frac{1}{2}}, \quad h_{j+\frac{1}{2},k}^{-} = w_{j+\frac{1}{2},k}^{-} - B_{j+\frac{1}{2},k},$$

$$h_{j,k-\frac{1}{2}}^{+} = w_{j,k-\frac{1}{2}}^{+} - B_{j,k-\frac{1}{2}}, \quad \text{and} \quad h_{j,k+\frac{1}{2}}^{-} = w_{j,k+\frac{1}{2}}^{-} - B_{j,k+\frac{1}{2}},$$

and

$$(h \rho)^{+}_{j-\frac{1}{2},k \pm \frac{1}{2}} = h_{j-\frac{1}{2},k \pm \frac{1}{2}}^{+} \cdot \rho^{+}_{j-\frac{1}{2},k \pm \frac{1}{2}}, \quad (h \rho)^{-}_{j+\frac{1}{2},k} = h_{j+\frac{1}{2},k}^{-} \cdot \rho^{-}_{j+\frac{1}{2},k},$$

$$(h \rho)^{+}_{j-\frac{1}{2},k-\frac{1}{2}} = h_{j-\frac{1}{2},k-\frac{1}{2}}^{+} \cdot \rho^{+}_{j-\frac{1}{2},k-\frac{1}{2}}, \quad \text{and} \quad (h \rho)^{-}_{j,k+\frac{1}{2}} = h_{j,k+\frac{1}{2}}^{-} \cdot \rho^{-}_{j,k+\frac{1}{2}}.$$ \( \text{We compute the slopes } (\mathcal{U}_{x}) \text{ and } (\mathcal{U}_{y}) \text{ by using the minmod limiter in order to minimize oscillations:} \)

$$\left( \mathcal{U}_{x} \right)_{j,k} = \text{minmod} \left( \frac{\overline{w}_{j,k} - \overline{w}_{j+1,k}}{3\Delta x_{j,k}/4}, \frac{\overline{w}_{j,k} - \overline{w}_{j+1,k+1}}{3\Delta x_{j,k}/4}, \frac{\overline{w}_{j+1,k} - \overline{w}_{j,k}}{\Delta x_{j,k}} \right),$$

$$\left( \mathcal{U}_{y} \right)_{j,k} = \text{minmod} \left( \frac{\overline{w}_{j,k} - \overline{w}_{j,k-1}}{\Delta y_{j,k}}, \frac{\overline{w}_{j,k+1} - \overline{w}_{j,k}}{\Delta y_{j,k}} \right).$$ \( \quad (17) \)
where the minmod function is defined by
\[ \min\{z_1, z_2, \ldots\} := \begin{cases} \min_j \{z_j\}, & \text{if } z_j > 0 \quad \forall j, \\ \max_j \{z_j\}, & \text{if } z_j < 0 \quad \forall j, \\ 0, & \text{otherwise}. \end{cases} \]

### 3.3.1 Positivity-preserving correction of \( w \)

Using the reconstruction (17) cannot guarantee the non-negativity of
\[ \tilde{h}(x, y) := \tilde{w}(x, y) - \tilde{B}(x, y). \]

The point values of \( \tilde{h} \) should remain nonnegative and \( \tilde{w} \) would need correction if at least one of them becomes negative. We denote the four corner point values of \( \tilde{w} \) over the cell \( C_{j,k} \) by
\[ w_{j,k}^{\text{NE}} := \tilde{w}(x_{j+\frac{1}{2}, k+\frac{1}{2}} - 0, y_{k+\frac{1}{2}} - 0), \quad w_{j,k}^{\text{SE}} := \tilde{w}(x_{j+\frac{1}{2}, k+\frac{1}{2}} - 0, y_{k+\frac{1}{2}} + 0), \]
\[ w_{j,k}^{\text{NW}} := \tilde{w}(x_{j-\frac{1}{2}} + 0, y_{k+\frac{1}{2}} - 0), \quad w_{j,k}^{\text{SW}} := \tilde{w}(x_{j-\frac{1}{2}} + 0, y_{k+\frac{1}{2}} + 0). \]

If,
\[ w_{j,k}^{\text{NE}} \geq B_{j+\frac{1}{2}, k+\frac{1}{2}}, \quad w_{j,k}^{\text{SE}} \geq B_{j+\frac{1}{2}, k+\frac{1}{2}}, \quad w_{j,k}^{\text{NW}} \geq B_{j-\frac{1}{2}, k+\frac{1}{2}}, \quad w_{j,k}^{\text{SW}} \geq B_{j-\frac{1}{2}, k+\frac{1}{2}}, \]
then we set
\[ \tilde{w}(x, y) = w_{j,k}^{\text{SW}} + \left( w_{j,k}^{\text{NE}} - w_{j,k}^{\text{SW}} \right) \frac{x-x_{j+\frac{1}{2}}}{\Delta x} + \left( w_{j,k}^{\text{NW}} - w_{j,k}^{\text{SW}} \right) \frac{y-y_{k+\frac{1}{2}}}{\Delta y} \]
\[ + \left( w_{j,k}^{\text{NE}} - w_{j,k}^{\text{SW}} - w_{j,k}^{\text{NW}} + w_{j,k}^{\text{SW}} \right) \frac{x-x_{j+\frac{1}{2}}}{\Delta x} \frac{y-y_{k+\frac{1}{2}}}{\Delta y}, \quad (x, y) \in C_{j,k}. \]

If at least one of the inequalities in (18) is not satisfied, correction for the point values of \( w \) at the vertices of \( C_{j,k} \) should be implemented. The point values of \( \tilde{w} \) at the cell \( C_{j,k} \) in Figure 2 are
\[ w_{j-\frac{1}{2}, k+\frac{1}{2}} = \tilde{w}\left(x_{j-\frac{1}{2}} + 0, y_{k+\frac{1}{2}}\right), \quad w_{j+\frac{1}{2}, k} = \tilde{w}\left(x_{j+\frac{1}{2}} - 0, y_{k}\right), \]
\[ w_{j+\frac{1}{2}, k+\frac{1}{2}} = \tilde{w}\left(x_{j+\frac{1}{2}} + 0, y_{k+\frac{1}{2}} + 0\right), \quad w_{j-\frac{1}{2}, k} = \tilde{w}\left(x_{j} + y_{k+\frac{1}{2}} - 0\right), \]
and therefore, the corresponding corrected values of \( h_{j-\frac{1}{2}, k+\frac{1}{2}}, h_{j+\frac{1}{2}, k}, h_{j+\frac{1}{2}, k+\frac{1}{2}}, h_{j-\frac{1}{2}, k+\frac{1}{2}} \) are non-negative.

### 3.3.2 Desingularization

In Section 3.3.1, we showed the required correction on \( w \). Employing the minmod limiter (17) guarantees the positivity of the point values of \( \rho \). To prevent very small or even zero values of cell averages \( \tilde{h} \), and point values of \( h \) of cell \( C_{j,k} \), the corresponding point values of \( u, v \), and \( \rho \) are computed as follows:
\[ u := \sqrt{\frac{2}{h}} (hu) \frac{h}{\sqrt{h^2 + \max\{h^2, \varepsilon\}}}, \quad v := \sqrt{\frac{2}{h}} (hv) \frac{h}{\sqrt{h^2 + \max\{h^2, \varepsilon\}}}, \quad \rho := \sqrt{\frac{2}{h}} (h\rho) \frac{h}{\sqrt{h^2 + \max\{h^2, \varepsilon\}}}, \]
where we choose \( \varepsilon = \max\{\min_{j,k}\{(\Delta x_{j,k})^4\}, \min_{j,k}\{(\Delta y_{j,k})^4\}\} \). The conservative variables recalculation is done by setting:
\[ (hu) := h \cdot u, \quad (hv) := h \cdot v, \quad (h\rho) := h \cdot \rho. \]

Note that all of the indices have been omitted in the above equations.
3.4 | Local speeds

The one-sided local speeds of propagation, denoted at the corresponding cell interfaces by \( a_{α,β}^+ \) and \( b_{γ,δ}^+ \), can be computed by using the largest and smallest eigenvalues of the Jacobian matrices \( \frac{∂F}{∂U} \) and \( \frac{∂G}{∂U} \)

\[
\begin{align*}
    a_{α,β}^+ &= \max \left\{ u_{α,β}^+ + \sqrt{\frac{\xi}{ρ_{α,β}} h_{α,β}^+ ρ_{α,β}^+}, \ u_{α,β}^- + \sqrt{\frac{\xi}{ρ_{α,β}} h_{α,β}^- ρ_{α,β}^-} \right\}, \\
    a_{α,β}^- &= \min \left\{ u_{α,β}^+ - \sqrt{\frac{\xi}{ρ_{α,β}} h_{α,β}^+ ρ_{α,β}^+}, \ u_{α,β}^- - \sqrt{\frac{\xi}{ρ_{α,β}} h_{α,β}^- ρ_{α,β}^-} \right\}, \\
    b_{γ,δ}^+ &= \max \left\{ v_{γ,δ}^+ + \sqrt{\frac{\xi}{ρ_{γ,δ}^+ h_{γ,δ}^+}}, \ v_{γ,δ}^- + \sqrt{\frac{\xi}{ρ_{γ,δ}^- h_{γ,δ}^-}} \right\}, \\
    b_{γ,δ}^- &= \min \left\{ v_{γ,δ}^+ - \sqrt{\frac{\xi}{ρ_{γ,δ}^+ h_{γ,δ}^+}}, \ v_{γ,δ}^- - \sqrt{\frac{\xi}{ρ_{γ,δ}^- h_{γ,δ}^-}} \right\}.
\end{align*}
\]

(20)

where \((α, β) ∈ \{(j − \frac{1}{2}, k − \frac{1}{2}), (j + \frac{1}{2}, k + \frac{1}{2})\}\) and \((γ, δ) ∈ \{(j, k − \frac{1}{2}), (j, k + \frac{1}{2})\}\) in Figure 2.

3.5 | Central-upwind numerical fluxes

We use the central-upwind fluxes from Reference 33:

\[
\begin{align*}
    H_{α,β}^c &= \frac{a_{α,β}^+ - a_{α,β}^-}{a_{α,β}^+ - a_{α,β}^-} \left[ U_{α,β}^+ - U_{α,β}^- \right], \\
    H_{γ,δ}^c &= \frac{b_{γ,δ}^+ - b_{γ,δ}^-}{b_{γ,δ}^+ - b_{γ,δ}^-} \left[ U_{γ,δ}^+ - U_{γ,δ}^- \right].
\end{align*}
\]

(21)

3.6 | Well-balanced discretization of the source term

When the discretized cell average of the source term, \( \overline{S}_{j,k} = \left(0, \overline{S}_{j,k}^{(2)}, \overline{S}_{j,k}^{(3)}\right)^T \), exactly balances the numerical fluxes in Equation (9) at the “lake-at-rest” steady state (2), the numerical scheme is well-balanced. This means that the right-hand side (RHS) of (9) vanishes as long as \( \overline{U}_{j,k} = (\overline{w}, 0, 0, \overline{ρ})^T \) for all \((j, k)\), where \( \overline{w} \) and \( \overline{ρ} \) are constants.

Notice that at the “lake-at-rest” state, all of the reconstructed point values are \( w^± = \overline{w}, u^± = v^± = 0 \) and \( ρ^± = \overline{ρ} \), and thus, \( a_{α,β}^± = a_{α,β}^± \), \( \forall (α, β) \), \( b_{γ,δ}^± = b_{γ,δ}^± \), \( \forall (γ, δ) \), and the numerical fluxes (21) reduce to:

\[
\begin{align*}
    H_{α,β}^c &= \left(0, 0, \frac{g}{2\overline{ρ}} \overline{ρ} \left(\overline{w} - B_{α,β}\right)^2, 0, 0\right)^T, \\
    H_{γ,δ}^c &= \left(0, 0, \frac{g}{2\overline{ρ}} \overline{ρ} \left(\overline{w} - B_{γ,δ}\right)^2, 0, 0\right)^T,
\end{align*}
\]

and the flux terms on the RHS of (9) then become

\[
\begin{align*}
    H_{α,β}^c &= \left(0, 0, \frac{g}{2\overline{ρ}} \overline{ρ} \left(\overline{w} - B_{α,β}\right)^2, 0, 0\right)^T, \\
    H_{γ,δ}^c &= \left(0, 0, \frac{g}{2\overline{ρ}} \overline{ρ} \left(\overline{w} - B_{γ,δ}\right)^2, 0, 0\right)^T.
\end{align*}
\]

(22)

By applying Barrow’s theorem, we notice that:

\[
\begin{align*}
    -\frac{\xi}{\overline{ρ}} (w - B) B_x &= \frac{\xi}{2\overline{ρ}} \left[ ρ (w - B)^2 \right]_x - \frac{\xi}{\overline{ρ}} (w - B) w_x, \\
    -\frac{\xi}{\overline{ρ}} (w - B) B_y &= \frac{\xi}{2\overline{ρ}} \left[ ρ (w - B)^2 \right]_y - \frac{\xi}{\overline{ρ}} (w - B) w_y.
\end{align*}
\]
We now rewrite the cell averages of the second and third components of the integral in (10)

\[
\frac{g}{2\rho_0} \int_{\Delta x_{j,k}}^{\Delta x_{j+1,k}} \left( \rho (w - B)^2 \right)_{x = x_{j,k}} \, dy - \frac{g}{\rho_0} \int_{\Delta x_{j,k}}^{\Delta x_{j+1,k}} \int_{y_{k-1/2}}^{y_{k+1/2}} \rho (w - B) w_x \, dy 
\]

and

\[
\frac{g}{2\rho_0} \int_{y_{k-1/2}}^{y_{k+1/2}} \left( \rho (w - B)^2 \right)_{y = y_{k-1/2}} \, dx - \frac{g}{\rho_0} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho (w - B) w_y \, dx
\]

We then approximate the integrals in (23) and (24), which results in the following quadrature for the second and third components of the source term:

\[
\overline{S}_{j,k}^{(2)} \approx \frac{g}{2\rho_0,\Delta x_{j,k}} \left[ \rho_{j+1/2}^+ \left( \frac{w_{j+1/2}^+ - B_{j+1/2}}{2} \right) - \rho_{j-1/2}^- \left( \frac{w_{j-1/2}^- - B_{j-1/2}}{2} \right) \right]
\]

and

\[
\overline{S}_{j,k}^{(3)} \approx \frac{g}{2\rho_0,\Delta y_{j,k}} \left[ \rho_{j,k+1/2}^+ \left( \frac{w_{j,k+1/2}^+ - B_{j,k+1/2}}{2} \right) - \rho_{j,k-1/2}^- \left( \frac{w_{j,k-1/2}^- - B_{j,k-1/2}}{2} \right) \right]
\]

The scheme now preserves the solution of “lake-at-rest” where, \((w_x)_{j,k} = (w_y)_{j,k} \equiv 0\), \(\forall (j,k)\) and thus (22) and (25) express that the RHS of (9) vanishes and therefore, the scheme is well-balanced. We finally state that, we did not consider the second lake-at-rest condition in (3). Further studies can be done on this condition for system (1).

### 3.7 | Positivity-preserving property

In this section, we extend the positivity-preserving proof from Reference 3 to implement on the coupled variable density system (see also References 13 and 33). We use a forward Euler method to integrate Equation (9) in time, which results in

\[
\overline{W}_{j,k}^{n+1} = \overline{W}_{j,k}^n - \lambda_{j,k}^n \left( H_{x,k}^{n+1} - H_{x,k}^n \right)
\]

and

\[
\overline{(h\rho)}_{j,k}^{n+1} = \overline{(h\rho)}_{j,k}^n - \lambda_{j,k}^n \left( H_{x,k}^{n+1} - H_{x,k}^n \right)
\]

where \(\overline{W}_{j,k} := \overline{W}_{j,k}(t^n)\), \(\overline{W}_{j,k}^{n+1} := \overline{W}_{j,k}(t^{n+1})\), \(\overline{(h\rho)}_{j,k} := \overline{(h\rho)}_{j,k}(t^n)\), and \(\overline{(h\rho)}_{j,k}^{n+1} := \overline{(h\rho)}_{j,k}(t^{n+1})\) with \(t^{n+1} = t^n + \Delta t^n\), \(\lambda_{j,k}^{n+1} := \Delta t^n / \Delta x_{j,k}\), \(\mu_{j,k}^n := \Delta t^n / \Delta y_{j,k}\), and the numerical fluxes on the RHS are evaluated at time level \(t = t^n\) using (21):

\[
H_{a,b}^{n+1} = \frac{\alpha_{a,b}^+ (w^+ - w_-)}{\alpha_{a,b}^+ - \alpha_{a,b}^-} \left[ w^+_{a,b} - w^-_{a,b} \right]
\]

and

\[
H_{a,b}^{n+1} = \frac{\alpha_{a,b}^+ (w^+ - w_-)}{\alpha_{a,b}^+ - \alpha_{a,b}^-} \left[ (h\rho)^+_{a,b} - (h\rho)^-_{a,b} \right]
\]

with

\[
\alpha_{a,b}^+ = \frac{\alpha_{a,b}^+}{\alpha_{a,b}^+ - \alpha_{a,b}^-} \sum_{i=1}^{4} \left( \frac{\alpha_{a,b}^+ - \alpha_{a,b}^-}{\alpha_{a,b}^+} \right) \left[ (w^+ - w^-) \right]
\]

and

\[
\alpha_{a,b}^+ = \frac{\alpha_{a,b}^+}{\alpha_{a,b}^+ - \alpha_{a,b}^-} \sum_{i=1}^{4} \left( \frac{\alpha_{a,b}^+ - \alpha_{a,b}^-}{\alpha_{a,b}^+} \right) \left[ (h\rho)^+_{a,b} - (h\rho)^-_{a,b} \right]
\]
where \((h\rho)_{n,\beta}^+ = h_{n,\beta}^+ \cdot \rho_{n,\beta}^+\) and \((h\rho)_{r,\delta}^+ = h_{r,\delta}^+ \cdot \rho_{r,\delta}^+\), and \((\alpha, \beta) \in \left\{ (j - \frac{1}{2}, k - \frac{1}{4}), (j - \frac{1}{2}, k + \frac{1}{4}), (j + \frac{1}{2}, k) \right\}\) and \((\gamma, \delta) \in \left\{ (j, k - \frac{1}{2}), (j, k + \frac{1}{2}) \right\}\) in Figure 2.

If \(\bar{h}_{j,k}^n \geq 0\) for all \((j, k)\), then the point values of the computed \(h\) are nonnegative.\(^{33}\) Moreover, using the bilinear piecewise reconstruction (11)–(13) for the bottom topography and the similar relationships for the reconstructed point values of \(w\), we have

\[
\bar{h}_{j,k}^n = \frac{1}{4} \left( \frac{h_{j+\frac{1}{2},k-\frac{1}{2}}^+ + h_{j+\frac{1}{2},k+\frac{1}{2}}^+}{2} + h_{j+\frac{1}{2},k}^- + h_{j+\frac{1}{2},k}^+ + h_{j+\frac{1}{2},k}^- \right) \tag{30}
\]

for the grid configuration in Figure 2.

We now subtract \(B_{j,k}^n\) from both sides of (26) and use (28) and (30) to rewrite (26) as follows:

\[
\begin{aligned}
-\bar{h}_{j,k}^{n+1} &= -\lambda^n_{j, k} a_{j+\frac{1}{2},k}^- \cdot \frac{a_{j+\frac{1}{2},k}^+ - u_{j+\frac{1}{2},k}^-}{a_{j+\frac{1}{2},k}^+ - a_{j+\frac{1}{2},k}^-} \cdot h_{j+\frac{1}{2},k}^- + \left[ \frac{1}{4} - \lambda^n_{j,k} a_{j+\frac{1}{2},k}^+ \cdot \frac{u_{j+\frac{1}{2},k}^- - a_{j+\frac{1}{2},k}^+}{a_{j+\frac{1}{2},k}^+ - a_{j+\frac{1}{2},k}^-} \right] h_{j+\frac{1}{2},k}^-
+& \frac{\lambda^n_{j,k} a_{j-\frac{1}{2},k+\frac{1}{2}}^+}{2} \cdot \frac{u_{j-\frac{1}{2},k+\frac{1}{2}}^- - a_{j-\frac{1}{2},k+\frac{1}{2}}^-}{a_{j-\frac{1}{2},k+\frac{1}{2}}^+ - a_{j-\frac{1}{2},k+\frac{1}{2}}^-} \cdot h_{j-\frac{1}{2},k+\frac{1}{2}}^-
+ & \frac{1}{2} \left[ \frac{1}{4} - \lambda^n_{j,k} a_{j-\frac{1}{2},k+\frac{1}{2}}^- \cdot \frac{a_{j+\frac{1}{2},k+\frac{1}{2}}^+ - u_{j+\frac{1}{2},k+\frac{1}{2}}^-}{a_{j+\frac{1}{2},k+\frac{1}{2}}^- - a_{j+\frac{1}{2},k+\frac{1}{2}}^-} \right] h_{j+\frac{1}{2},k+\frac{1}{2}}^-
+& \frac{1}{2} \left[ \frac{1}{4} - \lambda^n_{j,k} a_{j-\frac{1}{2},k+\frac{1}{2}}^- \cdot \frac{a_{j-\frac{1}{2},k+\frac{1}{2}}^+ - u_{j-\frac{1}{2},k+\frac{1}{2}}^-}{a_{j-\frac{1}{2},k+\frac{1}{2}}^- - a_{j-\frac{1}{2},k+\frac{1}{2}}^-} \right] h_{j-\frac{1}{2},k+\frac{1}{2}}^-
- \lambda^n_{j,k} b_{j,k+\frac{1}{2}}^+ \cdot \frac{h_{j+\frac{1}{2},k+\frac{1}{2}}^+ - v_{j,k+\frac{1}{2}}^-}{b_{j,k+\frac{1}{2}}^+ - b_{j,k+\frac{1}{2}}^-} \cdot h_{j+\frac{1}{2},k+\frac{1}{2}}^- + \left[ \frac{1}{4} - \mu^n_{j,k} b_{j,k+\frac{1}{2}}^+ \cdot \frac{v_{j,k+\frac{1}{2}}^- - b_{j,k+\frac{1}{2}}^-}{b_{j,k+\frac{1}{2}}^+ - b_{j,k+\frac{1}{2}}^-} \right] h_{j+\frac{1}{2},k+\frac{1}{2}}^-
+ & \mu^n_{j,k} b_{j,k-\frac{1}{2}}^+ \cdot \frac{h_{j-\frac{1}{2},k-\frac{1}{2}}^- - v_{j,k-\frac{1}{2}}^-}{b_{j,k-\frac{1}{2}}^+ - b_{j,k-\frac{1}{2}}^-} \cdot h_{j-\frac{1}{2},k-\frac{1}{2}}^- + \left[ \frac{1}{4} + \mu^n_{j,k} b_{j,k-\frac{1}{2}}^- \cdot \frac{v_{j,k-\frac{1}{2}}^- - b_{j,k-\frac{1}{2}}^-}{b_{j,k-\frac{1}{2}}^- - b_{j,k-\frac{1}{2}}^-} \right] h_{j+\frac{1}{2},k+\frac{1}{2}}^-
\right] \tag{31}
\end{aligned}
\]

This shows that the cell averages of \(h\) are linear combinations of the reconstructed nonnegative point values of \(h\). Thus, \(\bar{h}_{j,k}^{n+1} \geq 0\) as all of the terms in this linear combination are nonnegative.\(^{3,33}\)

One can obtain a similar proof for positivity of \((h\rho)_{j,k}^{n+1}\) by using the following statements,\(^{12}\)

\[
\bar{h}_{j,k} = \frac{h_{j+\frac{1}{2},k}^- + h_{j+\frac{1}{2},k}^+ + h_{j+\frac{1}{2},k+\frac{1}{2}}^+}{2}, \quad \bar{\rho}_{j,k} = \frac{\rho_{j+\frac{1}{2},k}^- + \rho_{j+\frac{1}{2},k}^+ + \rho_{j+\frac{1}{2},k+\frac{1}{2}}^+}{2}, \tag{32}
\]

and thereby, utilizing (16), one obtains:

\[
\bar{(h\rho)}_{j,k} = \frac{1}{4} \left[ h_{j+\frac{1}{2},k}^- \rho_{j+\frac{1}{2},k}^- + \frac{h_{j+\frac{1}{2},k-\frac{1}{2}}^+ \rho_{j+\frac{1}{2},k-\frac{1}{2}}^+ + h_{j+\frac{1}{2},k+\frac{1}{2}}^+ \rho_{j+\frac{1}{2},k+\frac{1}{2}}^+}{2} \right] \]
\[
\frac{(h\rho)^n}{(h\rho)_{j,k}} = \frac{1}{4} \left[ \frac{h_{j+\frac{1}{2},k}^+ \rho_{j+\frac{1}{2},k+\frac{1}{2}}^- + h_{j+\frac{1}{2},k}^- \rho_{j+\frac{1}{2},k+\frac{1}{2}}^+ + h_{j-\frac{1}{2},k-\frac{1}{2}}^+ \rho_{j-\frac{1}{2},k+\frac{1}{2}}^- + h_{j-\frac{1}{2},k+\frac{1}{2}}^- \rho_{j-\frac{1}{2},k+\frac{1}{2}}^+}{2} \right] \\
+ \frac{1}{4} \left[ \frac{h_{j-\frac{1}{2},k-\frac{1}{2}}^+ \rho_{j-\frac{1}{2},k+\frac{1}{2}}^- + h_{j-\frac{1}{2},k+\frac{1}{2}}^+ \rho_{j-\frac{1}{2},k-\frac{1}{2}}^-}{4} \right].
\]  

(33)

Similarly, it can be shown from desingularization that

\[
\frac{(h\rho)^n}{(h\rho)_{j,k}} = \frac{1}{4} \left[ \frac{h_{j,k+\frac{1}{2}}^- \rho_{j,k+\frac{1}{2}}^- + h_{j,k-\frac{1}{2}}^+ \rho_{j,k-\frac{1}{2}}^+}{2} \right] + \frac{1}{4} \left[ \frac{h_{j+\frac{1}{2},k}^- \rho_{j+\frac{1}{2},k}^- + h_{j+\frac{1}{2},k}^+ \rho_{j+\frac{1}{2},k}^+}{2} \right].
\]  

(34)

Finally, from (33) and (34) we have

\[
\frac{(h\rho)^n}{(h\rho)_{j,k}} = \frac{1}{8} \left[ \frac{h_{j+\frac{1}{2},k}^- \rho_{j+\frac{1}{2},k}^- + h_{j+\frac{1}{2},k+\frac{1}{2}}^+ \rho_{j+\frac{1}{2},k+\frac{1}{2}}^- + h_{j-\frac{1}{2},k-\frac{1}{2}}^+ \rho_{j-\frac{1}{2},k-\frac{1}{2}}^-}{2} \right] \\
+ \frac{1}{8} \left[ \frac{h_{j-\frac{1}{2},k-\frac{1}{2}}^- \rho_{j-\frac{1}{2},k-\frac{1}{2}}^- + h_{j-\frac{1}{2},k+\frac{1}{2}}^+ \rho_{j-\frac{1}{2},k+\frac{1}{2}}^- + h_{j+\frac{1}{2},k+\frac{1}{2}}^- \rho_{j+\frac{1}{2},k+\frac{1}{2}}^-}{2} \right] \\
+ \frac{1}{8} \left[ \frac{h_{j+\frac{1}{2},k}^- \rho_{j+\frac{1}{2},k}^- + h_{j+\frac{1}{2},k}^+ \rho_{j+\frac{1}{2},k}^+}{4} \right].
\]  

(35)

We rewrite (27) as follows:

\[
\frac{(h\rho)^{n+1}}{(h\rho)_{j,k}} = -\lambda_j^a a_{j+\frac{1}{2},k}^- \frac{a_{j+\frac{1}{2},k}^+ - u_{j+\frac{1}{2},k}^+}{4} + \frac{1}{8} \left[ 1 - \lambda_j^a a_{j+\frac{1}{2},k}^+ \frac{u_{j+\frac{1}{2},k}^- - a_{j+\frac{1}{2},k}^-}{4} \right] h_{j+\frac{1}{2},k}^- \rho_{j+\frac{1}{2},k}^- \\
+ \frac{\lambda_j^a a_{j+\frac{1}{2},k}^+}{2} \frac{u_{j+\frac{1}{2},k-\frac{1}{2}}^- - a_{j+\frac{1}{2},k-\frac{1}{2}}^-}{4} h_{j+\frac{1}{2},k-\frac{1}{2}}^- \rho_{j+\frac{1}{2},k-\frac{1}{2}}^- \\
+ \frac{\lambda_j^a a_{j-\frac{1}{2},k}^+}{2} \frac{u_{j-\frac{1}{2},k+\frac{1}{2}}^- - a_{j-\frac{1}{2},k+\frac{1}{2}}^-}{4} h_{j-\frac{1}{2},k+\frac{1}{2}}^- \rho_{j-\frac{1}{2},k+\frac{1}{2}}^- \\
+ \frac{1}{2} \left[ \frac{1}{8} \left[-\lambda_j^a a_{j-\frac{1}{2},k}^- \frac{a_{j-\frac{1}{2},k}^+ - u_{j-\frac{1}{2},k}^+}{4} \right] h_{j-\frac{1}{2},k}^- \rho_{j-\frac{1}{2},k}^- \\
+ \frac{\lambda_j^a a_{j-\frac{1}{2},k}^-}{2} \frac{u_{j-\frac{1}{2},k-\frac{1}{2}}^- - a_{j-\frac{1}{2},k-\frac{1}{2}}^-}{4} h_{j-\frac{1}{2},k-\frac{1}{2}}^- \rho_{j-\frac{1}{2},k-\frac{1}{2}}^- \\
+ \frac{\lambda_j^a a_{j-\frac{1}{2},k}^-}{2} \frac{u_{j-\frac{1}{2},k+\frac{1}{2}}^- - a_{j-\frac{1}{2},k+\frac{1}{2}}^-}{4} h_{j-\frac{1}{2},k+\frac{1}{2}}^- \rho_{j-\frac{1}{2},k+\frac{1}{2}}^- \\
+ \frac{1}{8} \left[ 1 - \lambda_j^a a_{j-\frac{1}{2},k}^- \frac{a_{j-\frac{1}{2},k}^+ - u_{j-\frac{1}{2},k}^+}{4} \right] h_{j-\frac{1}{2},k}^- \rho_{j-\frac{1}{2},k}^- \right].
\]

Quadtree grid adaptivity

(17) guarantees positivity of the point values of \( \rho \). When the grid locally refines or coarsens, at the end of the evolution step, the solution in constants that depend on the problem at hand (e.g., the maximum level of the quadtree, the Froude number, and the 

\[
\begin{align*}
+ \mu_{j,k}^n b_{j,k-\frac{1}{2}}^{-} - b_{j,k-\frac{1}{2}}^{-} & \cdot h_{j-\frac{1}{2},k}^{-} \rho_{j-\frac{1}{2},k}^{-} \\
+ \mu_{j,k}^n b_{j,k-\frac{1}{2}}^{-} - b_{j,k-\frac{1}{2}}^{-} & \cdot h_{j-\frac{1}{2},k}^{-} \rho_{j-\frac{1}{2},k}^{-} \\
+ \mu_{j,k}^n b_{j,k-\frac{1}{2}}^{-} - b_{j,k-\frac{1}{2}}^{-} & \cdot h_{j-\frac{1}{2},k}^{-} \rho_{j-\frac{1}{2},k}^{-} \\
+ \mu_{j,k}^n b_{j,k-\frac{1}{2}}^{-} - b_{j,k-\frac{1}{2}}^{-} & \cdot h_{j-\frac{1}{2},k}^{-} \rho_{j-\frac{1}{2},k}^{-} \\
\end{align*}
\]

(36)

Finally, in all of the numerical experiments, we have used the three-stage third-order strong stability preserving (SSP) Runge-Kutta solver (see, for example, References 3,46,47).

3.8 Quadtree grid adaptivity

At the new time level \( t = n+1 \), the quadtree grid locally refines or coarsens for the next timestep. We first need to compute the slopes \( \{(w_x)_{j,k}^{n+1}\} \) and \( \{(w_y)_{j,k}^{n+1}\} \), and \( \{(\rho_x)_{j,k}^{n+1}\} \) and \( \{(\rho_y)_{j,k}^{n+1}\} \) on the old grid (which is denoted by \( C_{j,k}^{\text{old}} \)) according to Section 3.3 and then select the centers of those cells \( C_{j,k}^{\text{old}} \), at which (see Reference 3):

\[
(w_x)_{j,k}^{n+1} \geq C_{w, \text{seed}} \quad \text{or} \quad (w_y)_{j,k}^{n+1} \geq C_{w, \text{seed}},
\]

and

\[
(\rho_x)_{j,k}^{n+1} \geq C_{\rho, \text{seed}} \quad \text{or} \quad (\rho_y)_{j,k}^{n+1} \geq C_{\rho, \text{seed}}.
\]

We denote the required seeding points to generate the new grid by \( C_{j,k}^{\text{new}} \). In (37) and (38), \( C_{w,\text{seed}} \) and \( C_{\rho, \text{seed}} \) are constants that depend on the problem at hand (e.g., the maximum level of the quadtree, the Froude number, and the bottom topography function). When the grid locally refines or coarsens, at the end of the evolution step, the solution in terms of the computed cell averages \( \{ \overline{U}_{j,k}^{n+1} \} \) over the grid \( C_{j,k}^{\text{old}} \), should be projected onto the new grid \( C_{j,k}^{\text{new}} \). This should be done in a conservative manner as follows:

Case 1: If \( C_{j,k}^{\text{new}} = C_{j,k}^{\text{old}} \) for some \((j',k')\), that is, if the cell \( C_{j,k}^{\text{old}} \) does not need to be refined/coarsened, then

\[
\begin{align*}
\left( \overline{U}_{j,k}^{n+1} \right)_{\text{new}} &= \left( \overline{U}_{j',k'}^{n+1} \right)_{\text{old}}.
\end{align*}
\]
Numerical Examples

In this section, we investigate the performance of the proposed scheme in four numerical examples. Note that as the scheme is based on the one in Reference 3, we therefore, do not provide a grid independency analysis. In all of the examples, we take $g = 1$ and $\rho_o = 997$.

Example 1  (Circular dam break with constant density). We demonstrate the ability of the proposed scheme to generate adaptive grids at each timestep and maintain symmetry in this example. A circular water column collapses on a horizontal flat bottom topography $40 \times 40$ dimensions plane where:

$$w(x, y, 0) = \begin{cases} 2, & (x - 20)^2 + (y - 20)^2 < 2.5^2, \\ 1, & \text{otherwise}, \end{cases}$$

$$u(x, y, 0) = v(x, y, 0) \equiv 0, \quad \rho(x, y, 0) \equiv \rho_o.$$

Furthermore, we take $m = 9$ refinement levels of the quadtree grid and set $C_{w, \text{seed}} = 5 \times 10^{-4}$ in (37). The solution runs until the final time $t = 4$. Water surface contours and the respective quadtree grids are illustrated in Figure 3. The quadtree grid starts with 2,134 cells and ends with 35,200 cells at $t = 4$. The results in Figure 3 show that the solution follows the same evolution in comparison with the ones obtained in References 48 and 37.

Example 2  (Dam break with density discontinuity over a hump). This example is based on the benchmark in References 49 and 37. We show the capability of the central-upwind quadtree scheme to maintain the well-balanced property, symmetry, and to generate adaptive grids. We consider the computational domain to be $[0, 2] \times [0, 1]$ with the following initial conditions:

$$\rho(x, y, 0) = \begin{cases} 997, & x < 1, \\ 1200, & x \geq 1, \end{cases} \quad u(x, y, 0) = v(x, y, 0) \equiv 0, \quad w(x, y, 0) \equiv 1,$$

and the given bottom topography

$$B(x, y) = 0.8\rho_o^{-5(x-0.9)^2-50(y-0.5)^2}.$$

We set $C_{w, \text{seed}} = 10^{-2}$ and $C_{\rho, \text{seed}} = 10$ in (37) and (38), and $m = 8$. A solid wall boundary condition is used at the top and bottom boundaries. For the sake of simplicity, we set the left and the right boundaries to Dirichlet boundary conditions. We run the solution up to the final time $t = 0.8$ with the nonwell-balanced and well-balanced schemes. For the nonwell-balanced scheme, the source approximations read as:

$$\frac{\partial \rho}{\partial t} = -\frac{\rho \Delta x_{jk}}{\rho_o} \left[ \frac{B_{j-k+1/2} + B_{j-k-1/2}}{2} - \frac{B_{j-k+1/2} + B_{j-k-1/2}}{2} \right].$$
FIGURE 3  Example 1: initial and computed water surface $w(x, y, t)$ (left column) and corresponding quadtree grids (right column) for $t = 0, 1, 2, 3,$ and $4$ (from top to bottom) [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 4  Example 2: computed water surface $w(x, y, t)$ (left column) and corresponding quadtree grids (right column) for $t = 0.1, 0.2, 0.4, 0.6,$ and $0.8$ (from top to bottom) obtained using the nonwell-balanced scheme [Colour figure can be viewed at wileyonlinelibrary.com]
Example 2: computed water surface $w(x, y, t)$ (left column) and corresponding quadtree grids (right column) for $t = 0.1, 0.2, 0.4, 0.6,$ and $0.8$ (from top to bottom) obtained using the well-balanced scheme. [Colour figure can be viewed at wileyonlinelibrary.com]
Example 3: computed water surface $w(x, y, t)$ (left column) and corresponding quadtree grids (right column) for $t = 0.6, 0.9, 1.2, 1.5, \text{ and } 1.8$ (from top to bottom) obtained using the nonwell-balanced scheme [Colour figure can be viewed at wileyonlinelibrary.com]

\[ S_{jk}^{(3)} = \frac{g(h\rho)_{jk}}{\rho_0 \Delta x_{jk}} \left[ \frac{B_{j+\frac{1}{2},k+\frac{1}{2}} + B_{j+\frac{1}{2},k-\frac{1}{2}}}{2} - \frac{B_{j-\frac{1}{2},k+\frac{1}{2}} + B_{j-\frac{1}{2},k-\frac{1}{2}}}{2} \right]. \]

Water surface contours and the respective quadtree grids of the solution of the nonwell-balanced scheme are demonstrated in Figure 4. The quadtree grid starts with 1,184 cells and ends with 27,749 cells at $t = 0.8$. The well-balanced solution is presented in Figure 5 where the quadtree has a minimum of 1,184 and a maximum of 18,497 cells. Utilizing
the well-balanced scheme reduces the number of cells in the quadtree grid; thereby, the computational cost is reduced as there are fewer cells in the quadtree grid at each timestep, and unphysical oscillations are eliminated.

**Example 3** (Small perturbations of a stationary steady-state solution). This numerical example tests the capability of the proposed scheme to capture small perturbations of a steady state solution.\(^3\text{28,32,50-52}\) We choose a computational domain \([-2, 2] \times [0, 1]\) to prevent complicated boundary conditions. The initial conditions are

\[
w(x, y, 0) = \begin{cases} 
1.01, & 0.05 < x < 0.15, \\
1, & \text{otherwise},
\end{cases} \quad u(x, y, 0) = v(x, y, 0) \equiv 0,
\]
and the given bottom topography function in Example 2.

We set boundary conditions similar to Example 2 for this test. $m = 9$ is taken, and we set $C_{w,seed} = 10^{-2}$ and $C_{\rho,seed} = 10$ in (37) and (38). The nonwell-balanced solution is computed until the final time $t = 1.8$ and plot the snapshots of $w$ (left) and the quadtree grids (right) at times $t = 0.6, 0.9, 1.2, 1.5,$ and $1.8$ in the domain of $[0, 2] \times [0, 1]$ in Figure 6. The
quadtree grid starts with 2,530 cells and reaches a maximum number of 13,804 cells during the time evolution. The well-balanced solution is illustrated in Figure 7, respectively. In this solution, the number of cells reaches the maximum of 10,222. Figure 7 demonstrates that the proposed scheme captures small perturbations of the “lake-at-rest” steady state. Furthermore, the total execution time (without considering the grid generation) for the WB solution over a quadtree grid is 513 s and for the same solution over a Cartesian grid is 1,816 s respectively.

**Example 4** (Sudden contraction with variable density inflow). The last example is a modification of the example in References 3 and 53. The purpose of this example is to demonstrate the positivity-preserving property of the proposed scheme.

We consider an open channel with a sudden contraction. The geometry of the channel is established on its contraction, where

\[
y_b(x) = \begin{cases} 
  0.5, & x \leq 1, \\
  0.4, & \text{otherwise}.
\end{cases}
\]

The computational domain is \([0, 3] \times [0.5 - y_b(x), 0.5 + y_b(x)]\). Solid wall boundary conditions are imposed at all of the boundaries except for part of the left inflow boundary, with \(u(0, y_i, t) \equiv 2\) and \(\rho(0, y_i, t) \equiv 1007\), where \(y_i \in [0.4, 0.6]\). In addition, we set the right boundary to a zero-order extrapolation. The following initial conditions are prescribed:

\[
w(x, y, 0) \equiv 1, \quad u(x, y, 0) = v(x, y, 0) \equiv 0, \quad \rho(x, y, 0) \equiv \rho_s.
\]

In this example, we take \(m = 8\) refinement levels of the quadtree grid and set \(C_{w, \text{seed}} = 2\) and \(C_{\rho, \text{seed}} = 20\) in (37) and (38). We compute the solution with the bottom topography given in Example 2, where the water depth at the top of the humps is quite shallow, which makes it a good example to test the positivity-preserving property.

We compute the solution until the final time \(t = 1.9\) and plot the evolution of \(w\) and \(\rho\) at times \(t = 0.4, 0.8, 1.2, 1.6,\) and 1.9 in Figure 8. The quadtree grid in this solution starts with a minimum of 298 and reaches a maximum of 9,928 cells. As one can see, the proposed central-upwind quadtree scheme preserves the positivity of the computed water depth and density.

**ACKNOWLEDGMENTS**

The work of the authors was supported by NSERC grant 210717. The authors warmly thank Carlos Parés from the University of Málaga for providing resources on the well-balanced property of the shallow water equations. The authors also thank Yangyang Cao and Philippe LeFloch from the Laboratoire Jacques-Louis Lions of Sorbonne Université for their useful discussions on the well-balanced property on conservation laws.

**DATA AVAILABILITY STATEMENT**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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**How to cite this article:** Ghazizadeh MA, Mohammadian A. An adaptive central-upwind scheme on quadtree grids for variable density shallow water equations. *Int J Numer Meth Fluids.* 2022;94(5):461-481. doi: 10.1002/fld.5062