The Microstructure of Stochastic Volatility Models with Self-Exciting Jump Dynamics∗

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Abstract

We provide a general probabilistic framework within which we establish scaling limits for a class of continuous-time stochastic volatility models with self-exciting jump dynamics. In the scaling limit, the joint dynamics of asset returns and volatility is driven by independent Gaussian white noises and two independent Poisson random measures that capture the arrival of exogenous shocks and the arrival of self-excited shocks, respectively. Various well-studied stochastic volatility models with and without self-exciting price/volatility co-jumps are obtained as special cases under different scaling regimes. We analyze the impact of external shocks on the market dynamics, especially their impact on jump cascades and show in a mathematically rigorous manner that many small external shocks may trigger endogenous jump cascades in asset returns and stock price volatility.

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1 Introduction

Affine stochastic volatility models have been extensively investigated in the mathematical finance and financial economics literature in the last decades. In the classical Heston [31] model the variance (squared volatility) process follows square-root mean-reverting Cox-Ingersoll-Ross [11] process. The Heston model introduces a dynamics for the underlying asset that can take into account the asymmetry and excess kurtosis that are typically observed in financial assets returns and provides analytically tractable option pricing formulas. However, it is unable to capture large volatility movements. To account for large volatility movements, the model has been extended to jump-diffusion models by numerous authors. Bates [7] adds a jump component in the asset price process. Barndorff-Nielsen and Shephard [6] consider volatility processes of Ornstein-Uhlenbeck type driven by Lévy processes. Affine models allowing for jumps in prices and volatilities are considered in Bakshi et al. [5], Bates [7], Duffie et al. [15], Pan [52] and Sepp [54], among many others. Empirical evidence for the presence of (negatively correlated) co-jumps in returns and volatility is given in, e.g. Eraker [19], Eraker et al. [20], and Jacod and Todorov [38].

In a standard jump model with arrival rates calibrated to historical data, jumps are inherently rare. Even more unlikely are patterns of multiple jumps in close succession over hours or days. Large moves, however, tend to appear in clusters. For example, as reported in Aıt-Sahalia et al. [1] “from mid-September to mid-November 2008, the US stock market jumped by more than 5% on 16 separate days. Intraday fluctuations were even more pronounced: during the same two months, the range of intraday returns exceeded 10% during 14 days.” Jump clusters can also be observed in the volatility. Figure 1 displays the evolution of the Chicago Board of Exchange VIX index and indicates up movements of more than 2.5% in daily closing values for the above mentioned period; on 26 out of 49 days, the index jump up by 2.5% or more. Bates [8] argues that the dramatic decline in futures prices on Monday, October 19, 1987, from the previous Friday’s closing value “was the result of an estimated 34 jumps” in volatility.

Jump clusters over time have been discussed in the financial econometrics literature by many authors including Aıt-Sahalia et al. [1], Fulop et al. [24], Lee and Mykland [48], Maheu and McCurdy [50], and Yu [55]. Among the most relevant papers for our work are the ones by Andersen et al. [2] and Bates [8]. They consider continuous-time models of self-exciting price/volatility co-jumps in intraday stock returns and volatility. Every small intraday

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jump substantially increases the probability of more intradaily cojumps in volatility and returns, and these multiple price jumps can accumulate into the major outliers in daily returns. Bates [8] finds “that multifactor models with both exogenous and self-exciting but short-lived volatility spikes” substantially improve model fits both in-sample and out-of-sample. He also shows that such models provide more accurate predictions of implied volatility. A similar conclusion on implied volatilities was reached in the recent work by Jiao et al. [41].

In order to account for self-exiting jump dynamics one needs to leave the widely applied class of Lévy jump processes. Lévy processes have independent increments and hence do not allow for any type of serial dependence. Hawkes processes are capable of displaying mutually exciting jumps. Originally introduced by Hawkes [28, 29] to model the occurrence seismic events, Hawkes processes have received considerable attention in the financial mathematics and economics literature as a powerful tool to model financial time series in recent years; we refer to Bacry et al. [4] and references therein for reviews on Hawkes processes and their applications to science and finance. On the more mathematical side, a series of functional limit theorem and large deviation principles for Hawkes processes and marked Hawkes processes has recently been established by, e.g. Bacry et al. [3], Gao and Zhu [25, 26], and Karabash and Zhu [43]. Horst and Xu [35] introduced Hawkes random measures in order to study limit theorems for limit order book models with self-exciting cross-dependent order flow. In [34] they established functional limit theorems for marked Hawkes point measures with homogeneous immigration under a light-tailed condition on the arrival dependencies of different events. Under a light-tailed condition on the arrival dependencies of different events, Jaisson and Rosenbaum [39] proved that the rescaled intensity process of a Hawkes process converges weakly to a Feller diffusion and that the rescaled point process converges weakly to the integrated diffusion. Under a heavy-tailed condition they proved that the rescaled point process converges weakly to the integral of a rough fractional diffusion; see [40]. Their result provides a microscopic foundation for the rough Heston model; see [16, 17].

Motivated by the recent empirical works on stochastic volatility models with self-exciting jumps, we provide a unified microscopic foundation for stochastic volatility models with price/volatility co-jumps based on Hawkes processes. Many of the existing jump diffusion stochastic volatility models including the classical Heston model [31], the Heston model with jumps [7, 15, 52], the OU-type volatility model [6], the multi-factor model with self-exciting volatility spikes [8] and the alpha Heston model [41] are obtained as scaling limits under different scaling regimes. As such, our work contributes to the rich literature on scaling limits for financial market models\(^1\) as well as to the

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\(^1\)Much of the earlier work including [22, 23, 32] focussed on the temporary occurrence and bubbles, due to imitation and contagion effects rather than volatility. More recently, the focus seems to have shifted to order books models and volatility.

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Figure 1: CBOE VIX index based on daily closing values from Sep. 12 to Nov. 20, 2008. Orange dots indicate a jump of 2.5% or more from previous day closing value.
growing literature on Hawkes processes by establishing novel scaling limits for Hawkes systems.

Figure 2: CBOE VIX index based on daily closing values from 1999-2019.

Our analysis uses the link between Hawkes processes and continuous-state branching processes (CB-processes). The extinction and the first jump behavior of CB-processes has been extensively investigated in the probability literature; see [27, 30] for details. To the best of our knowledge ours is the first paper to provide a comprehensive analysis of the jump distribution of CB-processes. In particular, we provide an asymptotic analysis of the distribution of the last jump time. Last jump times are of particular interest to us as they allow is lengths of jump cascades triggered by exogenous shocks. Most importantly, we prove the existence of a critical value at which the lengths of jump cascades follow a Pareto distribution. This critical case introduces a regime in which the financial system may be quite fragile in the sense that a single exogenous shocks may result in large number of co-jumps.

Branching processes are a particular class of affine processes. Affine processes have been widely used in the financial mathematics literature. Duffie et al. [14] define an affine process as a time-homogeneous Markov process, whose characteristic function is the exponential of an affine function of the state vector under a regularity assumption. They showed that this type of process unifies the concepts of continuous-state branching processes with immigration (CBI-processes) and Ornstein-Uhlenbeck type processes. They also provide a rigorous mathematical approach to affine processes, including the characterization of affine processes in terms of admissible parameters (comparable to the characteristic triplet of a Lévy process). Dawson and Li [12] provide a construction of affine process as the unique strong solution to a system of stochastic differential equations with non-Lipschitz coefficients and Poisson-type integrals over some random sets. Keller-Ressel [45] considers the long-term behavior of affine stochastic volatility models, including an expression for the invariant distribution and provides explicit expressions for the time at which a moment of given order becomes infinite. We repeatedly draw on the results in [12, 14, 45].

We consider a microstructure model of a financial market with two types of market orders that we refer to as exogenous orders and induced orders, respectively. We think of exogenous orders as exogenous shocks; they arrive at an exogenous Poisson dynamics. Exogenous orders generate a random environment for the arrival of induced orders. Induced orders arrive according to a marked Hawkes process with exponential kernel. The marks represent the magnitudes by which the orders change prices as well as their impact on the arrival rate of future induced order flow. In particular, jumps in prices and/or volatility may trigger cascades of co-jumps and hence volatility clusters. Jump cascades are particularly likely to occur after large exogenous shocks. Figure 2 shows the evolution of the CBOE VIX index from 1990 to 2019; the time series clearly displays the occasional occurrence of exogenous shocks. Apart from the already mentioned clustering of jumps during the height of the global financial crisis there seem to be further jump clusters, for instance after the 1998 Russian and the 2011 Eurozone debt crisis.
Our first main contribution is to analyze the genealogical decomposition (2.7) of the benchmark model, and to analyze the impact duration of large external shocks on induced order flow. To this end, we provide a decomposition of the benchmark model as the sum of a sequence of independent self-enclosed sub-models that describe the impact of exogenous shocks on induced order flow. We then provide four regimes for the long-run impact of an exogenous shock to the market dynamics. The impact of an exogenous shock will last forever with positive probability in the supercritical case and vanish at an exponential rate in the subcritical case. In the critical case the impact duration of external shocks is heavy-tailed despite the exponential decay of individual events on future dynamics.

Our second main contribution is a scaling limit for the benchmark model when the frequency of order arrivals tends to infinity and the impact of an individual order on the market dynamics tend to zero. Depending on the choice of scaling parameters, various well-known jump-diffusion stochastic volatility models are obtained in the scaling limit. Different to the arguments in [39] on the convergence of nearly unstable Hawkes processes, we give sufficient conditions on the model parameters that guarantee the existence of a non-degenerate scaling limit based on the link between Hawkes processes and branching particle systems. Loosely speaking, we require the convergence of the sequence of branching mechanisms. From this, we conclude that the sequence of generators converges to a limiting generator when restricted to exponential functions. Since we do not require any moment conditions and because the linear span of exponential functions is not dense in the domain of the limit generator, methods and techniques based in the convergence of generators as given in [21] can not be applied to establish the convergence of the rescaled market models. Instead, we use general convergence results for infinite dimensional stochastic integrals established in Kurtz and Protter [47]. Their methods have previously been applied to prove diffusion approximations for limit order book models in [33]. We prove (Theorem 3.12, Corollary 3.13) that the rescaled sequence of market models converges in distribution to the unique solution of a stochastic differential equation driven by two independent Gaussian white noise processes, a Poisson random measure that describes the arrivals of large exogenous shocks and an independent Poisson random measure that describes the dynamics of endogenously induced self-excited jumps.

Our third main contribution is to analyze the genealogical decomposition of the limiting jump-diffusion volatility model (Theorem 4.1). We provide an economically intuitive decomposition in terms of three sub-models. The first sub-model is self-enclosed and captures the impact of all events prior to time 0 on future order flow. The second sub-model describes the cumulative impact of the exogenous shocks of positive magnitude on the market dynamics; this sub-model can be further decomposed into a sequence of self-enclosed sub-models that capture the impact of individual shocks. The third sub-model is the most interesting one. This self-enclosed sub-model describes the impact of exogenous shocks “of insignificant magnitude” on the market dynamics. In the scaling limit large exogenous shocks translate into jumps while vanishingly small exogenous shocks translate in a well defined sense into a non-trivial mean-reversion level of the stochastic volatility process that keeps the volatility bounded away from zero at all times. Due to the dependence of the jump arrivals on the volatility process, this shows in a mathematically rigorous manner how many small exogenous events may trigger endogenous jump cascades in the limit. We furthermore show that the third sub-model can be further decomposed into sub-models where the volatility evolves according to an excursion process selected by a Poisson random measure (Theorem 4.2). Our decomposition is different from that in [41] where the volatility process is decomposed into a truncated variance process plus a variance process that captures all jumps larger than some threshold. Unlike ours, their sub-models are not self-enclosed and do not classify jumps by their origin.

We do not only classify jumps by their origin but also establish a full characterization of the distribution of jumps of different magnitudes and different origins in the scaling limit. We give an explicit expression for the joint distribution of the number and magnitudes of jumps induced by an exogenous shocks of given size and the time of the last induced jump in terms of the unique continuous solution to a Riccati equation with singular initial condition. To the best of our knowledge, last jump times have not been analyzed in the branching literature before. We prove that exogenously (by exogenous shocks) triggered jumps tend to cluster much more than endogenously (by the volatility) triggered ones. Figure 3 illustrates this effect. It shows the evolution of a sample path of our model with (blue) and without (orange) external shocks. There are two external shocks; the times of the shocks originating from the external shocks are indicated by red dots. Light blue dots times of indicate endogenously triggered jumps. For our choice of parameters they are more evenly distributed across time.

The remainder of this paper is organized as follows. Our benchmark financial market model is introduced in Section 2. The scaling result is established in Section 3. In this section we also show how various well studied stochastic volatility models can be obtained as scaling limits. The genealogical structure of the market dynamics of

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\footnote{We emphasize that we provide an SDE representation for CB-processes with jump as the scaling limits of continuous-time branching particle systems. By contrast, most scaling limits analyzed the literature so far require discrete-time prelimit models. Many key results used to establish scaling limits such as Corollary 8.9 in [21, p.233] do only work for discrete-time prelimit models.}
the benchmark model and the scaling limit is analyzed in Section 4. This section establishes a detailed description of the full jump dynamics of our CB-processes.

2 Hawkes market model

In this section, we introduce a benchmark stochastic volatility model for which we drive a scaling limit in a later section. There are two types of buy/sell orders in our model that we refer to as exogenous orders and induced orders, respectively. Exogenous orders are orders that arrive at an exogenous Poisson dynamics (“exogenous shocks”). They generate a random environment for the arrival of induced orders. Induced orders will arrive at much higher frequencies than exogenous orders and follow a self-exciting dynamics.

In what follows all random variables are defined on a common probability space \((\Omega, F, P)\) endowed with filtration \(\{F_t : t \geq 0\}\) that satisfies the usual hypotheses.

2.1 The benchmark model

The arrivals of exogenous/induced market buy/sell orders are recorded by \((F_t)\)-random point processes \(\{N_{t}^{e,b,s} : t \geq 0\}\) with respective arrival times \(\{r_{k}^{e/b/s} : k = 1, 2, \cdots\}\). We denote by \(\{J_{k}^{e/b/s} : k = 1, 2, \cdots\}\) the sequences of price changes (in ticks) resulting from exogenous/induced market buy/sell orders. For any time \(t \geq 0\), the (logarithmic) price \(P_t\) is given by

\[
P_t = P_0 + \sum_{k=1}^{N_{t}^{e,b}} \delta \cdot J_{k}^{e,b} - \sum_{k=1}^{N_{t}^{e,s}} \delta \cdot J_{k}^{e,s} + \sum_{k=1}^{N_{t}^{i,b}} \delta \cdot J_{k}^{i,b} - \sum_{k=1}^{N_{t}^{i,s}} \delta \cdot J_{k}^{i,s},
\]

where \(P_0\) is the price at time 0 and \(\delta\) is the tick size on the logarithmic scale, i.e. the minimum log-price movement.

We now formulate three assumptions on the order flow dynamics that greatly simplify the subsequent analysis but that do not change our main results on the occurrence of jumps and jump-cascades. First, we assume that price increments are independent conditionally on the state of the market and past events.

Assumption 2.1 Price changes are described by independent sequences \(\{J_{k}^{e/b/s} : k = 1, 2, \cdots\}\) of i.i.d. \(\mathbb{Z}_+\)-valued
random variables.

Second, we assume that price increments are uncorrelated.

**Assumption 2.2** For any \( l, l' \in \{ e, i \} \) and \( j, j' \in \{ b, s \} \),
\[
\mathbb{E}[dN^l_{t} \cdot dN^{l'}_{t} | \mathcal{F}_t^-] = 0, \quad \text{if } l \neq l' \text{ or } j \neq j'.
\]

Third, we assume that exogenous buy/sell orders arrive according to independent Poisson processes.

**Assumption 2.3** \( \{N^{e,b}_t : t \geq 0\} \) and \( \{N^{e,s}_t : t \geq 0\} \) are two independent Poisson processes with rates \( p^e_b \) and \( p^e_s \) respectively.

Induced orders arrive according to Hawkes processes with \( \beta \)-exponential kernel for some \( \beta > 0 \). We assume that both past exogenous and past induced orders increase the arrival intensity of induced orders.

**Assumption 2.4** The processes \( \{N^{i,b}_t : t \geq 0\} \) and \( \{N^{i,s}_t : t \geq 0\} \) are marked Hawkes processes with intensities \( p^i_b V_t \cdot dt \) and \( p^i_s V_t \cdot dt \) respectively, where \( p^i_b + p^i_s = 1 \), and where the intensity process \( \{V_t : t \geq 0\} \) is given by
\[
V_t = \mu_t + \sum_{j \in \{b,s\}} \sum_{k=1}^{N^{i,j}_t} X^{i,j}_k e^{-\beta(t-t^{i,j}_k)} + \sum_{j \in \{b,s\}} \sum_{k=1}^{N^{i,j}_t} X^{i,j}_k e^{-\beta(t-t^{i,j}_k)}, \quad t \geq 0.
\]

Here, \( \{\mu_t : t \geq 0\} \) is an \( \mathcal{F}_0 \)-measurable functional-valued random variable that represents the impact of all the orders that arrived prior to time 0 on the arrival rate of future orders, and \( \{X^{e,s/b,s}_k : k = 1, 2, \cdots\} \) are four mutually independent sequences of nonnegative i.i.d. random variables that represent the impact of each order arrival on the intensity process.

**Remark 2.5** We emphasize that the kernel appearing in the definition of the Hawkes process is of exponential type. Although the results in [34] suggest that most of our convergence results hold for more general kernels, the genealogy analysis presented in Section 2.2 and Section 4 requires the Markov property of the market models. This is why we restrict our analysis to exponential Hawkes processes.

We now provide a stochastic integral representation of the price process in terms of marked Hawkes point measures as introduced in Horst and Xu [34]. To this end, we associate with each order a mark from the mark space \( \mathbb{U} = \mathbb{R} \times \mathbb{R}_+ \). A mark comprises the amount by which the order changes the price (in ticks) along with its

\[
\nu_{e/s}(du) := p^{e/b}_s \cdot \mathbb{P} \left\{ (J^{e/b}_{s}, X^{e/b}_{s}) \in (du_1, du_2) \right\} + p^{e/s}_b \cdot \mathbb{P} \left\{ (-J^{s/b}_{s}, X^{s/b}_{s}) \in (du_1, du_2) \right\}.
\]

Let \( \{\tau^{e/s}_k : k = 1, 2, \cdots\} \) be the arrival times of exogenous/induced (buy and sell) orders. In view of Assumption 2.3, we can associate with the sequence \( \{(\tau^e_k, \xi^e_k) : k = 1, 2, \cdots\} \) a Poisson point measure
\[
N_e(dt, du) := \sum_{k=1}^{\infty} 1_{\tau^e_k \in dt, \xi^e_k \in du}
\]
on \( [0, \infty) \times \mathbb{U} \) with intensity \( dt \nu_e(du) \). Likewise, associated with the sequence \( \{(\tau^i_k, \xi^i_k) : k = 1, 2, \cdots\} \) is an \( (\mathcal{F}_t) \)-random point measure
\[
N_i(dt, du) := \sum_{k=1}^{\infty} 1_{\tau^i_k \in dt, \xi^i_k \in du}
\]
The sub-models are given by a sequence of models \( \{ \text{self-enclosed price dynamics} \} \). To this end, we first establish a representation of the market dynamics in terms of independent processes. Having introduced our Hawkes market model, we are now going to analyze the impact of exogenous shocks on the market. Under the above assumptions, and using the fact that the integral representation can be further simplified if we assume that the impact of all the orders that arrive prior to time 0 on the arrival rate of future events decreases exponentially.

**Assumption 2.6** The process \( \{ \mu_t : t \geq 0 \} \) satisfies \( \mu_t := V_0 \cdot e^{-\beta t} \) for any \( t \geq 0 \).

Under the above assumptions, and using the fact that \( e^{-\beta t} = 1 - \int_0^t \beta e^{-\beta s} ds \), the model (2.7)-(2.8) can be rewritten into

\[
\begin{align*}
P_t &= P_0 + \int_0^t \int_U \delta \cdot u_1 N_1(ds, du) + \int_0^t \int_U \delta \cdot u_1 N_1(ds, du, dx), \\
V_t &= V_0 - \int_0^t \beta V_s ds + \int_0^t \int_U u_2 N_1(ds, du) + \int_0^t \int_U V_s \cdot u_2 e^{-\beta(t-s)} N_1(ds, du, dx).
\end{align*}
\]

By Theorem 6.2 in [12], there exists a unique \( U \)-valued strong solution \( \{ (P_t, V_t) : t \geq 0 \} \) to (2.9)-(2.10). We call this solution the Hawkes market model with parameter \( (\delta, \beta; \nu_\theta) \). The solution is a strong Markov process whose infinitesimal generator \( A_\delta \) acts on any function \( f \in C^2(U) \) according to

\[
A_\delta f(p, v) = -v \cdot \beta \frac{\partial}{\partial v} f(p, v) + v \cdot \int_U \left[ f(p + \delta \cdot u_1, v + u_2) - f(p, v) \right] \nu_\theta(du).
\]

**2.2 The genealogy of market dynamics**

Having introduced our Hawkes market model, we are now going to analyze the impact of exogenous shocks on the price dynamics. To this end, we first establish a representation of the market dynamics in terms of independent and identically distributed self-enclosed sub-models. We call a market model self-enclosed if \( \nu_\theta(U) = 0 \); in this case, we denote the model by \( \{ (P_{0,t}, V_{0,t}) : t \geq 0 \} \). In view of (2.9)-(2.10) it satisfies the following dynamics

\[
\begin{align*}
P_{0,t} &= P_0 + \int_0^t \int_U \int_U \delta \cdot u_1 N_1(ds, du, dx), \\
V_{0,t} &= V_0 - \int_0^t \beta V_s ds + \int_0^t \int_U \int_U \delta \cdot u_1 N_1(ds, du, dx).
\end{align*}
\]

From the cluster representation of the Hawkes process, we can decompose the Hawkes market model into a sum of self-enclosed models \( \{ (P_{k,t}, V_{k,t}) : t \geq 0 \} \). In terms of the sequence of arrival times and marks \( \{ (\tau_k, \zeta_k) : k = 1, 2, \ldots \} \), the sub-models are given by a sequence of models \( \{ (P_{k,t}, V_{k,t}) : k = 1, 2, \ldots \} \) that satisfy \( (P_{k,t}, V_{k,t}) = (0, 0) \) for \( t < \tau_k \) while for \( t \geq \tau_k \),

\[
\begin{align*}
P_{k,t} &= \xi_{k,P} + \int_{\tau_k}^t \int_U \int_U \delta \cdot u_1 N_1(ds, du, dx), \\
V_{k,t} &= \xi_{k,V} - \int_{\tau_k}^t \beta V_s ds + \int_{\tau_k}^t \int_U \int_U \delta \cdot u_1 N_1(ds, du, dx).
\end{align*}
\]
**Theorem 2.7** The Hawkes market model \( \{(P_t, V_t) : t \geq 0\} \) defined by \((2.9)-(2.10)\) admits the decomposition

\[
\{(P_t, V_t) : t \geq 0\} \overset{\text{a.s.}}{=} \left\{ \sum_{k=0}^{\infty} (P_{k,t}, V_{k,t}) : t \geq 0 \right\}.
\]

**Proof.** It suffices to prove that the infinite sum is well defined and equals \( \{(P_t, V_t) : t \geq 0\} \). For any \( K \geq 1 \), let \( (P^K_t, V^K_t) := \sum_{k=0}^{K} (P_{k,t}, V_{k,t}) \), which solves

\[
P^K_t = P_0 + \sum_{k=1}^{K} \xi_{k,P} e^{t \tau_k^P} + \int_{\tau_k^P}^{t} \int_{0}^{\tau_k^P} \delta \cdot u_1 N_1(ds, du, dx),
\]

\[
V^K_t = V_0 + \sum_{k=1}^{K} \xi_{k,V} e^{t \tau_k^V} - \int_{\tau_k^V}^{t} \beta V^K_s ds + \int_{\tau_k^V}^{t} \sum_{k=1}^{K} \int_{0}^{\tau_k^V} u_2 N_2(ds, du, dx).
\]

For any \( T \geq 0 \), the set \( \{k \in \mathbb{N} : \tau_k^P \leq T\} \) is a.s. finite. Hence \( \{(P^K_t, V^K_t) : t \in [0,T]\} \overset{\text{a.s.}}{=} \{(P_t, V_t) : t \in [0,T]\} \) as \( K \to \infty \).

The sub-models \( \{(P_{k,t}, V_{k,t}) : t \geq 0\}_{k \geq 0} \) are self-enclosed, mutually independent and identically distributed. That is, for any \( j, k \geq 1 \),

\[
\{(P_{j,t+\tau^P_k}, V_{j,t+\tau^V_k}) : t \geq 0\} \overset{\text{d}}{=} \{(P_{k,t+\tau^P_k}, V_{k,t+\tau^V_k}) : t \geq 0\}.
\]

Conditioned on \( (\xi^P_1, \xi^V_1) = (P_0, V_0) \), the model \( \{(P_{1,t+\tau^P_k}, V_{1,t+\tau^V_k}) : t \geq 0\} \) equals \( \{(P_{0,t}, V_{0,t}) : t \geq 0\} \) in law. As a result, the impact of exogenous orders on the market dynamics can be analyzed by analyzing the model \( \{(P_{0,t}, V_{0,t}) : t \geq 0\} \). The duration of the impact of an exogenous order on the price dynamics can be described in terms of the random variable

\[
\mathcal{T}_0 := \sup \{t \in [0,\infty) : |P_{0,t} - P_{0,t-}| > 0\}.
\]

Its distribution along with the distribution of the jumps in prices and volatility will be analyzed within a broader context in Section 4.1 below. In that section we also prove that the long-run impact of an exogenous shock of magnitude \( (P_0, V_0) \) on the market dynamics depends on the decay in the long run of the impact of induced orders on the volatility as described by the quantity

\[
\tilde{\beta} := \beta - \int_{U} u_2 \nu_1(du).
\]

(2.16)

The critical case \( \tilde{\beta} = 0 \) is particularly relevant economically. In this case, the impact of exogenous orders on induced ones is slowly decaying. In the scaling limit it corresponds to a loss of mean-reversion of the volatility process. Specifically, we have the following result, which is a special case of Corollary 4.11 below.

**Corollary 2.8** If \( \int_U |u_2| \nu_1(du) < \infty \), then \( P\{\mathcal{T}_0 = \infty\} > 0 \) if and only if \( \tilde{\beta} < 0 \). More precisely, there are the four regimes for the long-run impact of exogenous shocks of magnitude \( (P_0, V_0) \) on induced order flow.

1. If \( \tilde{\beta} < 0 \), then as \( t \to \infty \),

\[
P\{\mathcal{T}_0 \geq t\} \to C_0 \in (0,1).
\]

2. If \( \tilde{\beta} > 0 \), then for any \( t > 0 \),

\[
P\{\mathcal{T}_0 \geq t\} \leq \frac{V_0}{\beta} \cdot e^{-\tilde{\beta} \cdot t}.
\]

3. If \( \tilde{\beta} = 0 \) and \( \nu_1(|u_2|^2) := \frac{1}{2} \int_U |u_2|^2 \nu_1(du) < \infty \), then as \( t \to \infty \),

\[
P\{\mathcal{T}_0 \geq t\} \sim \frac{V_0}{\nu_1(|u_2|^2)} \cdot t^{-1}.
\]

4. If \( \tilde{\beta} = 0 \) and \( \nu_1(\mathbb{Z} \times [x,\infty)) \sim C(1+x)^{-1-\alpha} \) as \( x \to \infty \) for some \( \alpha \in (0,1) \), then as \( t \to \infty \),

\[
P\{\mathcal{T}_0 \geq t\} \sim C \cdot V_0 \cdot t^{-1/\alpha}.
\]
2.3 Examples

We close this section with three specific examples. Within each example we clarify when co-jumps in prices and volatilities are negatively correlated. We revisit all three examples when analyzing scaling limits.

We say that an \((\mathbb{Z}_+ \times \mathbb{R}_+)-valued\) random variable \(\xi = (\xi_1, \xi_2)\) has \textit{bivariate exponential distribution} \(\text{BVE}(\lambda)\) with parameter \(\lambda := (\lambda_1, \lambda_2, \lambda_{12}) \in \mathbb{R}^3_+\) if for any \((k, x) \in \mathbb{Z}_+ \times \mathbb{R}_+\),

\[
P\{\xi_1 \geq k, \xi_2 \geq x\} = \exp\{-\lambda_1(k - 1) - \lambda_2 x - \lambda_{12}((k - 1) \lor x)\}.
\]

The first moment \(M_1^\alpha(\lambda) := (M_{1,k}^\alpha(\lambda))_{k=1,2}\) and the second moment \(M_2^\alpha(\lambda) := (M_{2,jk}^\alpha(\lambda))_{j,k=1,2}\) are given by

\[
M_{1,1}^\alpha(\lambda) := E[\xi_1] = \frac{1}{1 - e^{-\lambda_1 - \lambda_1}}, \quad M_{1,2}^\alpha(\lambda) := E[\xi_2] = \frac{1}{\lambda_2 + \lambda_{12}}
\]

and

\[
M_{2,11}^\alpha(\lambda) := E[\|\xi_1\|^2] = \frac{e^{-\lambda_1 - \lambda_1} + (2e^{-\lambda_1 - \lambda_1} - 1)^2}{(1 - e^{-\lambda_1 - \lambda_1})^2}, \quad M_{2,22}^\alpha(\lambda) := E[\|\xi_2\|^2] = \frac{2}{\lambda_2 + \lambda_{12}},
\]

\[
M_{2,12}^\alpha(\lambda) := E[\xi_1\xi_2] = \frac{1/\lambda_2 - 1/(\lambda_2 + \lambda_{12})}{1 - e^{-(\lambda_1 + \lambda_1 + \lambda_{12})}} + \frac{1/\lambda_2 - 1/(\lambda_2 + \lambda_{12})}{1 - e^{-(\lambda_1 + \lambda_1 + \lambda_{12})}} \
\]

This implies that

\[
C_{\xi}(\lambda) := \text{Cov}(\xi_1, \xi_2) = \frac{1/\lambda_2 - 1/(\lambda_2 + \lambda_{12})}{1 - e^{-(\lambda_1 + \lambda_1 + \lambda_{12})}} + \frac{1/\lambda_2 - 1/(\lambda_2 + \lambda_{12})}{1 - e^{-(\lambda_1 + \lambda_1 + \lambda_{12})}} \geq 0
\]

with equality if \(\lambda_{12} = 0\), which holds if and only if \(\xi_1\) and \(\xi_2\) are independent.

**Example 2.9** (Exponential market model) For \(j \in \{e, 1\}\) and some constants \(\lambda_j^\alpha, \lambda_j^b \in \mathbb{R}_+^3\), let

\[
(J^{j:b}, X^{j:b}) \overset{d}{=} \text{BVE}(\lambda_j^\alpha) \quad \text{and} \quad (J^{j:z}, X^{j:z}) \overset{d}{=} \text{BVE}(\lambda_z^\alpha).
\]

In this case, we call the market model (2.9)-(2.10) exponential market model with parameter \((\delta, \beta, p_{e/1}^{b/e_1}, \lambda_{b/e_1}^{e})\). From (2.19), the jumps in prices and volatilities are negatively correlated if \(p_j^e C_{\xi}(\lambda_j^b) < p_j^z C_{\xi}(\lambda_j^z)\) for \(j \in \{e, 1\}\).

We say that an \((\mathbb{Z}_+ \times \mathbb{R}_+)-valued\) random variable \(\xi = (\xi_1, \xi_2)\) has a \textit{bivariate Pareto distribution} \(P(\alpha, \theta)\) with parameters \(\alpha > 0\) and \(\theta = (\theta_1, \theta_2) \in (0, \infty)^2\) if for any \((k, x) \in \mathbb{Z}_+ \times \mathbb{R}_+\),

\[
P\{\xi_1 = k, \xi_2 \in dx\} = \left[\left(1 + \frac{k - 1}{\theta_1} + \frac{x}{\theta_2}\right)^{-\alpha} - \left(1 + \frac{k}{\theta_1} + \frac{x}{\theta_2}\right)^{-\alpha}\right] \frac{\alpha}{\theta_2} dx.
\]

The probability law of \(\xi = (\xi_1, \xi_2)\) is multivariate regularly varying with index \(\alpha\) and \(E[\|\xi\|^{\kappa}] < \infty\), for any \(\kappa < \alpha\). When \(\alpha > 2\), the first moment \(M_1^\alpha(\alpha, \theta) := (M_{1,k}^\alpha(\alpha, \theta))_{k=1,2}\) and the second moment \(M_2^\alpha(\alpha, \theta) := (M_{2,jk}^\alpha(\alpha, \theta))_{j,k=1,2}\) have the following representation:

\[
M_{1,1}^\alpha(\alpha, \theta) := E[\xi_1] = \sum_{k=0}^{\infty} \left(1 + \frac{k}{\theta_1}\right)^{-\alpha} \quad \text{and} \quad M_{1,2}^\alpha(\alpha, \theta) := E[\xi_2] = \frac{\theta_2}{\alpha - 1}
\]

and

\[
M_{2,11}^\alpha(\alpha, \theta) := E[\|\xi_1\|^2] = \sum_{k=0}^{\infty} \left(2k + 1\right) \left(1 + \frac{k}{\theta_1}\right)^{-\alpha} \quad \text{and} \quad M_{2,22}^\alpha(\alpha, \theta) := E[\|\xi_2\|^2] = \frac{2\theta_2^2}{(\alpha - 1)(\alpha - 2)},
\]

\[
M_{2,12}^\alpha(\alpha, \theta) := E[\xi_1\xi_2] = \frac{\theta_2}{\alpha - 1} \sum_{k=0}^{\infty} \left(1 + \frac{k}{\theta_1}\right)^{-\alpha + 1}.
\]

This implies that

\[
C_{\xi}(\alpha, \theta) := \text{Cov}(\xi_1, \xi_2) = \frac{\theta_2}{\alpha - 1} \sum_{k=0}^{\infty} \frac{k}{\theta_1} \left(1 + \frac{k}{\theta_1}\right)^{-\alpha} > 0.
\]
Example 2.10 (Pareto market model) For $j \in \{e, i\}$ and some constants $\alpha_j > 0$, $\theta^b_j, \theta^e_j \in (0, \infty)^2$, let
\[
(J_j^b, X_j^b) \overset{d}{=} \mathcal{P}(\alpha_j, \theta^b_j) \quad \text{and} \quad (J_j^e, X_j^e) \overset{d}{=} \mathcal{P}(\alpha_j, \theta^e_j).
\]
In this case, we call the market model (2.9)-(2.10) Pareto market model with parameter $(\delta, \beta, \theta^b_j, \alpha_j, \theta^e_j)$. From (2.23), when $\alpha_j > 2$, the jumps in prices and volatilities are negatively correlated if $p^e_j \mathcal{C}_p(\alpha_j, \theta^e_j) < p^b_j \mathcal{C}_p(\alpha_j, \theta^b_j)$ for all $j \in \{e, i\}$.

We say that a $(\mathbb{Z}_+ \times \mathbb{R}_+)$-valued random variable $\xi = (\xi_1, \xi_2)$ has an exponential-Pareto mixing distribution with parameter $(\lambda, \alpha, \theta)$ if for any $(k, x) \in \mathbb{Z}_+ \times \mathbb{R}_+$, and some $q \in (0, 1)$
\[
\mathbb{P}(\xi_1 \geq k, \xi_2 \geq x) = q \left( 1 + \frac{k - 1}{\theta_1} + \frac{x}{\theta_2} \right)^{-\alpha} (1 - q) \exp \{-\lambda_1(k - 1) - \lambda_2 x - \lambda_2((k - 1) \lor x)\}.
\]

Example 2.11 (Exponential-Pareto mixing market model) If the mark of each event has an exponential-Pareto mixing distribution, then we call the market model (2.9)-(2.10) exponential-Pareto mixing market model with parameter $(\delta, \beta, \theta^b_j, \theta^e_j, \alpha_j, \theta^e_j)$ and selecting mechanism $g_{e/i}$.

### 3 The scaling limit

In this section, we consider the weak convergence of a sequence of rescaled market models. Our scaling limit provides a microscopic foundation for an array of stochastic volatility models including the classic Heston model, the Heston model with jumps, the jump-diffusion stochastic volatility model in [15] and the alpha Heston model in [41].

#### 3.1 Assumptions and asymptotic results

For each $n \in \mathbb{N}$, we consider a market model $\{(P_n(t), V_n(t)) : t \geq 0\}$ of the form (2.9)-(2.10) with initial state $(P_0(n), V_0(n))$ and parameter $(1/n, \beta(n); \nu^{(n)}(t))$. Without loss of generality, we assume that all models are defined on the common filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We are interested in the weak convergence of the rescaled market models $\{(P_n(t), V_n(t)) : t \geq 0\}$ defined by
\[
\left( \begin{array}{c}
P_n(t) \\ V_n(t)
\end{array} \right) = \left( \begin{array}{c}
P_n(0) \\ V_n(0)
\end{array} \right) - \int_0^t \left( \begin{array}{c}
0 \\ \gamma_n \beta(n)
\end{array} \right) V_n(s) ds + \int_0^t \int_U \frac{u}{n} N^{(n)}_e(\gamma_n \cdot ds, du) \\
+ \int_0^t \int_U \int_0^{V_n(s)} \frac{u}{n} N^{(n)}_1(\gamma_n \cdot ds, du, n \cdot dx)
\right)
\]
where $\{\gamma_n\}_{n \geq 1}$ is a sequence of positive numbers that converges to infinity as $n \to \infty$. The market models are driven by the Poisson random measures $N^{(n)}_e$ and $N^{(n)}_1$ whose respective compensators are given by
\[
\tilde{N}^{(n)}_e(\gamma_n \cdot ds, du) := \gamma_n \cdot ds \nu^{(n)}_e(du) \quad \text{and} \quad \tilde{N}^{(n)}_1(\gamma_n \cdot s, du, n \cdot dx) := n \gamma_n \cdot ds \nu^{(n)}_1(du) dx.
\]
In particular, the arrival rate of exogenous orders in the $n$-th market model is $\gamma_n$, and the arrival rate of induced orders is $n \gamma_n$.

This justifies our interpretation of induced orders as originating from high-frequency trading.

**Remark 3.1** The dynamics of external order flow could be scaled differently to generate diffusive behavior in the price process we cannot expect to obtain diffusive behavior in the volatility process. The impact of external orders on the intensity process is given by
\[
\int_0^t \int_U u_2 N^{(n)}_e(\gamma_n \cdot ds, du),
\]
which is a compound Poisson process with positive jumps and hence is a subordinator. Thus the scaling limit will still be a subordinator. The Lévy-Itô decomposition of subordinator then tells us that external events will never generate diffusive behavior in the volatility.

---

3 Since order sizes are scaled by a factor $\frac{1}{n}$, this suggests that exogenous orders will not generate a diffusive behavior in the limit; they only generate a drift and/or jumps. Induced orders, on the other hand, may well generate diffusive behavior.
The third term on the right side of equation (3.1) is a two-dimensional, compound Poisson process. Its weak convergence has been extensively studied in the literature; see [37, Chapter VII] or [42, Corollary 15.20]. The following condition is necessary for its weak convergence.

**Assumption 3.2** There exists a constant \( \gamma \in [0, \infty) \) such that \( \gamma_n \sim \gamma \cdot n \) as \( n \to \infty \).

We now give sufficient conditions on the parameters \( \beta^{(n)} \) and \( \nu^{(n)}_\epsilon \) that guarantee the existence of a non-degenerate scaling limit. Our arguments are based on the link between the benchmark model and general branching particle systems. By (2.11) the infinitesimal generator \( A^{(n)} \) of \( \{(P^{(n)}(t), V^{(n)}(t)) : t \geq 0\} \) acts on functions \( f \in C^2(U) \) according to,

\[
A^{(n)} f(p, v) = -v \cdot \gamma_n \beta^{(n)} \frac{\partial}{\partial v} f(p, v) + v \cdot n \gamma_n \int_U \left[ f((p, v) + u/n) - f(p, v) \right] \nu^{(n)}_1(du) + \gamma_n \int_U \left[ f((p, v) + u/n) - f(p, v) \right] \nu^{(n)}_\epsilon(du). \tag{3.2}
\]

By Corollary 8.9 in [21, p.232], the sequence of \( \{(P^{(n)}(t), V^{(n)}(t)) : t \geq 0\} \) is weakly convergent if there exits an infinitesimal generator \( A \) with domain \( \mathcal{D}(A) \) such that for any \( f \in \mathcal{D}(A) \),

\[
A f(p, v) \to A f(p, v), \quad \text{as } n \to \infty. \tag{3.3}
\]

In what follows, we identify a candidate limit generator by analyzing the limit of the sequence \( A^{(n)} f \) for the special case where \( f \) is the exponential function. To this end, let \( U_+ := i \mathbb{R} \times \mathbb{C}_- \) with \( i \mathbb{R} := \{i \cdot x : x \in \mathbb{R}\} \) and \( \mathbb{C}_- := \{x + i \cdot y : (x, y) \in \mathbb{R}_+\times\mathbb{R}\} \), where \( i := \sqrt{-1} \). For any \( z = (z_1, z_2) \in U_+ \),

\[
A^{(n)} \exp\{z_1 p + z_2 v\} = \exp\{z_1 p + z_2 v\} \cdot \left[ \gamma_n \int_U u_1 \nu^{(n)}_1(du) \cdot v z_2 + \gamma_n \int_U (u_2 - \beta^{(n)}) \nu^{(n)}_1(du) \cdot v z_2 + \gamma_n \int_U (e^{\frac{1}{n}(z_1 u)} - 1) \nu^{(n)}_\epsilon(du) \right]. \tag{3.4}
\]

Convergence as \( n \to \infty \) holds if all the four terms in the above sum are convergent. The first two terms converge if and only if the following condition holds.

**Assumption 3.3** Assume that \( \int_U \|u\| \nu^{(n)}_1(du) < \infty \) for any \( n \geq 1 \) and that there exists a constant \( b := (b_1, b_2) \in \mathbb{R}^2 \) such that

\[
-\gamma_n \int_U u_1 \nu^{(n)}_1(du) \to b_1 \quad \text{and} \quad \gamma_n \int_U (\beta^{(n)} - u_2) \nu^{(n)}(du) \to b_2.
\]

We split the last two terms in (3.4) into four parts according to the direction of price movements. Specifically, let \( U_\pm = \mathbb{R}_\pm \times \mathbb{R}_+ \), and for \( j \in \{+, -\} \) and \( z \in U_j \) let

\[
G_j^{(n)}(z) := n \gamma_n \int_U \left( e^{\frac{1}{n}(z, u)} - 1 + \frac{z, u}{n} \right) \nu^{(n)}_1(du) \quad \text{and} \quad H_j^{(n)}(z) := \gamma_n \int_U \left( e^{-\frac{1}{n}(z, u)} - 1 \right) \nu^{(n)}_\epsilon(du).
\]

The next condition guarantees that the remaining terms on the right hand side of equation (3.4) converge.

**Condition 3.4** There exists a constant \( C > 0 \) such that \( \sup_{n \geq 1} n \gamma_n \int_{U_+} \|u\|^2 \wedge \|u\| \nu^{(n)}_1(du) \leq C \). Moreover, \( G^{(n)}_\pm(\cdot) \) and \( H^{(n)}_\pm(\cdot) \) converge to continuous functions \( G_\pm(\cdot) \) and \( H_\pm(\cdot) \) respectively, as \( n \to \infty \).

The following proposition provides an exact representation of the limit functions \( G_\pm(\cdot) \) and \( H_\pm(\cdot) \).

**Proposition 3.5** Under Condition 3.4, the limit functions \( G_\pm(\cdot) \) and \( H_\pm(\cdot) \) have the following representations: for any \( z = (z_1, z_2) \in U_\pm \),

\[
G_\pm(z) = \langle z, \sigma^\pm z \rangle + \int_{U_\pm} (e^{\langle z, u \rangle} - 1 + \langle z, u \rangle) \hat{\nu}_1(du) \quad \text{and} \quad H_\pm(z) = \langle z, a^\pm \rangle + \int_{U_\pm} (e^{\langle -z, u \rangle} - 1) \hat{\nu}_\epsilon(du),
\]

where \( a^\pm \in U_\pm \), \( \sigma^\pm := (\sigma^\pm)^j_{l=1,2} \) is a symmetric, nonnegative definite matrix, and \( \langle \|u\| \wedge \|u\|^2 \rangle \hat{\nu}_1(du) \) and \( \langle \|u\| \wedge 1 \rangle \hat{\nu}_\epsilon(du) \) are finite measures on \( U \setminus \{0\} \).

\(^4\)We emphasize that \( \gamma = 0 \) is allowed. In Section 3 we consider the case \( \gamma_n = n \) as well as the case \( \gamma_n = n^\alpha \) for some \( \alpha \in (0, 1) \).
Proof. We prove the result for the function $G_+ \cdot$. All other results can be established in the same way. From Theorem 8.1 in [53], the representation of $G_+ \cdot$ in terms of $\sigma^+ \cdot$ and $\nu_1(\cdot) \cdot$ is unique if it exists. For any $n \geq 1$, let

$$
\theta^{(n)}_1 := n\gamma_n \int_{\mathbb{U}_+} (\|u\|^2 \wedge 1) \zeta_{\theta}^{(n)}(n \cdot u) \quad \text{and} \quad P_1^{(n)}(\cdot) := n\gamma_n \frac{\|u\|^2 \wedge 1}{\theta^{(n)}_1} \cdot 1_{\{u \in \mathbb{U}_+\} \cdot \zeta_{\theta}^{(n)}(n \cdot u)},
$$

which is a probability law on the compact space $\mathbb{U}_+ := \mathbb{U}_+ \cup \{\infty\}$. Thus, we can always find a subsequence that converges weakly to some limit probability law $P_1$ on $\mathbb{U}_+$. We notice that $P_1$ may have a point mass in 0. Among other things, we need to prove that $P_1\{+\infty\} = 0$.

Let $C := \{r \geq 0 : P_1\{\|u\| = r\} = 0\}$; its complement is at most countable. For any $r \in C$, we put $B_r := \{u \in \mathbb{U} : \|u\| \leq r\}$ and decompose the function $G_+^{(n)}$ as

$$
G_+^{(n)}(z) = \theta^{(n)}_1 \left( \langle \hat{\beta}^{(n)}, z \rangle + \sum_{j,l=1}^{2} \sigma^{(n)}_{j,l}(r) z_j z_l + J^{(n)}(z, r) + E^{(n)}(z, r) \right),
$$

where

$$
\hat{\beta}^{(n)} := \int_{\mathbb{U}_+} (u - u \wedge 1) \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1},
$$

$$
J^{(n)}(z, r) := \int_{\mathbb{U}_+ \setminus B_r} [e^{-\langle z, u \rangle} - 1 + \langle z, u \wedge 1 \rangle] \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1},
$$

$$
\sigma^{(n)}_{j,l}(r) := \frac{1}{2} \int_{B_r} (u_j \wedge 1)(u_l \wedge 1) \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1} \in [-1, 1], \quad j, l = 1, 2,
$$

$$
E^{(n)}(z, r) := \int_{B_r} [e^{-\langle z, u \rangle} - 1 + \langle z, u \wedge 1 \rangle - \frac{1}{2} \sum_{j,l=1}^{2} (u_j \wedge 1)(u_l \wedge 1) z_j z_l] \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1}.
$$

The weak convergence of $\{P_1^{(n)}\}_{n \geq 1}$ along with the fact that $P_1(\partial B_r) = 0 \ (r \in C)$ implies that

$$
\lim_{n \to \infty} \sigma^{(n)}_{j,l}(r) = \frac{1}{2} \int_{B_r} (u_j \wedge 1)(u_l \wedge 1) \frac{P_1(du)}{\|u\|^2 \wedge 1} =: \sigma_{j,l}(r),
$$

$$
\lim_{n \to \infty} E_n(z, r) = \int_{B_r} [e^{-\langle z, u \rangle} - 1 + \langle z, u \wedge 1 \rangle - \frac{1}{2} \sum_{j,l=1}^{2} (u_j \wedge 1)(u_l \wedge 1) z_j z_l] \frac{P_1(du)}{\|u\|^2 \wedge 1} =: E(z, r).
$$

By Condition 3.4, the sequences $\{\theta^{(n)}_1, n \in \mathbb{N}\}$ and $\{\hat{\beta}^{(n)}, n \in \mathbb{N}\}$ are bounded. Hence, we may w.l.o.g. assume that

$$
(P_1^{(n)}, \theta^{(n)}_1, \hat{\beta}^{(n)}) \to (P_1, \theta_1, \hat{\beta})
$$

as $n \to \infty$. If $\theta_1 = 0$, the convergence of $G_+^{(n)}$ implies $G_+ = 0$, due to the moment condition on the measures $\nu^{(n)}_1$. Hence, we may assume that $\theta_1 > 0$. Thus, as $n \to \infty$

$$
\int_{\mathbb{U}_+ \setminus B_r} \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1} \to c_0(r) \quad \text{and} \quad \int_{\mathbb{U}_+ \setminus B_r} (u \wedge 1) \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1} \to c(r)
$$

for some non-negative numbers $c_0(r)$ and $c(r)$. As a result, for any $z \in \mathbb{R}_+^2$ (noting that $u \geq 0$) it follows from (3.6) that

$$
\lim_{n \to \infty} \int_{\mathbb{U}_+ \setminus B_r} e^{-\langle z, u \rangle} \frac{P_1^{(n)}(du)}{\|u\|^2 \wedge 1} = \int_{\mathbb{U}_+ \setminus B_r} e^{-\langle z, u \rangle} \frac{P_1(du)}{\|u\|^2 \wedge 1} = \frac{G_+(z)}{\theta_1} - \langle \hat{\beta}, z \rangle - \sum_{j,l=1}^{2} \hat{\sigma}_{j,l}(r) z_j z_l - E(z, r) + c_0(r) - \langle c(r), z \rangle.
$$
Since \( G_+() \) is continuous by assumption and \( \mathcal{E}(\cdot, r) \) is continuous by the dominated convergence theorem, the function \( \int_{U_n} e^{-(u \cdot \theta)} P_n(du) \) is continuous and hence \( P_1 \) is a probability law on \( U_+ \), i.e. \( P_1\{\infty\} = 0 \). In particular, 
\[
\lim_{n \to \infty} \frac{\hat{\beta}^{(n)}}{\|u\|^2 + 1} = \beta = \int_{U_+} (u - u \wedge 1) \frac{P_1(du)}{\|u\|^2 + 1}.
\]

Moreover, \( \lim_{r \to 0^+} \lim_{n \to \infty} \sigma_{j,l}^{+(n)}(r) = \hat{\sigma}_{j,l}^+ \) and \( \lim_{r \to 0^+} \lim_{n \to \infty} \mathcal{H}^{(n)}(z, r) = \int_{U_+ \setminus \{0\}} \left[ e^{-(z, u)} - 1 + (z, u) \right] P_1(du) \). Taking these back into (3.6), we have 
\[
G_+(z) = \sum_{j,l=1}^2 \sigma_{j,l}^+ z j z l + \int_{U_+ \setminus \{0\}} \left[ e^{-(z, u)} - 1 + (z, u) \right] \nu_4(du),
\]
where \( \sigma_{j,l}^+ := \theta_j \hat{\sigma}_{j,l}^+ \) and \( \nu_4(du) := \theta_1 \frac{P_1(du)}{\|u\|^2 + 1} \) on \( U_+ \setminus \{0\} \). \( \square \)

Let \( a := (a_1, a_2) = a^+ + a^- \), and \( \sigma := (\sigma_{j,l}) = \sigma^+ + \sigma^- \), and for any \( z = (z_1, z_2) \in U_+ \), let 
\[
G(z) := -\langle b, z \rangle + G_+(z) - G_-(z) = -\langle b, z \rangle + \langle z, \sigma z \rangle + \int_U \left( e^{\langle z, u \rangle} - 1 - (z, u) \right) \nu_4(du),
\]
\[
H(z) := H_+(z) + H_-(z) = \langle a, z \rangle + \int_U \left( e^{\langle z, u \rangle} - 1 \right) \nu_4(du).
\]

Under Conditions 3.3 and 3.4 we thus have that 
\[
\mathcal{A}^{(n)} \exp\{z_1 p + z_2 v\} \to \exp\{z_1 p + z_2 v\} \cdot [H(z) + v \cdot G(z)], \quad (p, v) \in U, \quad z \in U_+
\]
as \( n \to \infty \). It will turn out that the infinitesimal generator \( \mathcal{A} \) of the limit process does indeed act on \( f \in C^2(U) \) according to 
\[
\mathcal{A} f(p, v) = \langle (a - bv), \nabla f(p, v) \rangle + v \cdot \langle \nabla, \sigma \nabla f(p, v) \rangle + \int_U [f((p, v) + u) - f(p, v)] \nu_4(du) + v \cdot \int_U [f((p, v) + u) - f(p, v) - \langle u, \nabla f(p, v) \rangle] \nu_4(du).
\]

**Remark 3.6** As pointed out in the proof of the previous proposition the measure \( P_1 \) may has a point-mass in 0. Likewise, the corresponding measure \( P_\nu \) arising in the analysis of the functions \( H_l \) may have a point mass in 0 as well. If there are no point measures in zero, then \( \sigma^\pm \), respectively \( a^\pm \) are zero. Loosely speaking, the quantities \( \sigma \) and \( a \) account for the arrival of infinitely many induced, respectively exogenous orders of “insignificant magnitude”. This turns out to be very important for the analysis of the jump dynamics of the scaling limit.

Before we prove the weak convergence of the rescaled market models, we provide an alternative link between the parameters \((1/n, \beta^{(n)}; \nu_4^{(n)})\) and \((a, b, \sigma, \nu_4/1)\) that clarifies their interpretation as the drift, diffusion and jump measure of the limiting model.

**Proposition 3.7** Condition 3.4 holds if and only if \( \sup_{n \geq 1} n \gamma_n \int_{U_+} \|u\|^2 \wedge \|u\|^{\nu_4^{(n)}(du)} < \infty \) and there exist parameters \((a^\pm, \sigma^\pm, \nu_4/1)\) such that the following holds.

(a) As \( n \to \infty \),
\[
\gamma_n \int_{U_+} \left( \frac{u}{n} \wedge 1 \right) \nu_4^{(n)}(du) \to a^\pm + \int_{U_+} (u \wedge 1) \nu_4(du)
\]
and
\[
n \gamma_n \int_{U_+} \left( \frac{u}{n} \wedge 1 \right)^T \left( \frac{u}{n} \wedge 1 \right) \nu_4^{(n)}(du) \to 2 \sigma^\pm + \int_{U_+} (u \wedge 1)^T (u \wedge 1) \nu_4(du);
\]

(b) For any \( f_1, f_2 \in C_b(\mathbb{R}^2) \) that satisfy \( f_k(u) = O(\|u\|^k) \) as \( \|u\| \to 0 \) for \( k = 1, 2 \), if \( n \to \infty \),
\[
\gamma_n \int_U f_1 \left( \frac{u}{n} \right) \nu_4^{(n)}(du) \to \int_U f_1(u) \nu_4(du) \quad \text{and} \quad n \gamma_n \int_U f_2 \left( \frac{u}{n} \right) \nu_4^{(n)}(du) \to \int_U f_2(u) \nu_4(du).
\]
Proof. A direct computation shows that conditions (a) and (b) imply the convergence of the terms (3.8)-(3.10) to continuous functions, which immediately implies the convergence of $G_{+}^{(n)}$ to $G_{+}$. Using the same arguments as in the proof of the previous lemma, the converse follows from direct computation using the convergence along a subsequence of $\{(P_{1}^{(n)}, \theta_{1}^{(n)})\}_{n \geq 1}$. \hfill $\square$

3.2 Weak convergence

The weak convergence of Markov processes has been extensively analyzed in the probability literature. We encounter two problems when proving the weak convergence of our market models via the convergence of generators. First, although we identified a candidate limit generator $\mathcal{A}$ using exponential functions, since the linear span of the set $\{e^{(z)} : z \in \mathbb{U}_{+}\}$ is not dense in $\mathcal{G}(\mathcal{A})$ this is not enough to deduce the convergence of the generators. Second we do not impose any moment conditions on the processes $(P^{(n)}, V^{(n)})$, which prevents us from applying standard convergence arguments.

Remark 3.8 Let us further clarify why we cannot expect to prove the desired weak convergence by simply appealing to existing convergence results as established in, e.g. the textbook by Ethier and Kurtz [21]. One approach to establish weak convergence of Markov processes uses [21, Corollary 8.7]. Without moment estimates, it would be challenging to verify the compact containment condition. Moreover, one would have to check the convergence of the generators $\mathcal{A}^{(n)}$ for all functions from the domain of the limit generator $\mathcal{A}$. A second approach uses [21, Theorem 9.4]. However, Condition 4.3 in [21] is not enough to provide moment estimates on the prelimit processes. Indeed, the measure $\nu_{\kappa}$ in Proposition 3.5 does not have any finite moment, i.e. for any $\kappa > 0$

$$\int_{\Omega} \|u\| \wedge \|u\|^\kappa \nu_{\kappa}(du) = \infty$$

and hence $\sup_{n \geq 1} E[|P^{(n)}(t)|^\kappa + |V^{(n)}(t)|^\kappa] = \infty$. As a result, (9.18) in [21, Theorem 9.4] does not hold and test functions from the set $C_{0}^{\infty}$ will not satisfy (9.17) in [21]. A third approach uses [21, Corollary 8.9]. However, this corollary can only be applied to rescaled Markov chains, and hence does not apply to our setting.

We prove the weak convergence of the rescaled market models using the general convergence results for infinite dimensional stochastic integrals established by Kurtz and Protter [47]. To this end, we consider the separable Banach space $L^{1,2}(\mathbb{R}_{+}) := L^{1}(\mathbb{R}_{+}) \cap L^{2}(\mathbb{R}_{+})$, endowed with the norm $\|\cdot\|_{L^{1,2}} := \|\cdot\|_{L^{1}} \vee \|\cdot\|_{L^{2}}$ and the Haar basis $\{\varphi_{j}^{k} : j \geq -1, k \geq 0\}$. The Haar basis is defined by $\varphi_{1}^{0}(x) = 1_{(0,1)}(x)$ and

$$\varphi_{j}^{k}(x) = \begin{cases} 2^{j/2}, & x \in [k \cdot 2^{-j}, (k+1/2) \cdot 2^{-j}) \\ -2^{j/2}, & x \in [(k+1/2) \cdot 2^{-j}, (k+1) \cdot 2^{-j}) \\ 0, & \text{else}, \end{cases}$$

Let the separable Banach space $\mathbb{H} := \mathbb{R}^{2} \times L^{1,2}(\mathbb{R}_{+})$ be endowed with norm $\|\cdot\|_{\mathbb{H}}$ defined by

$$\|Z\|_{\mathbb{H}} := |Z_{1}| + |Z_{2}| + \|Z_{3}\|_{L^{1,2}}, \quad \text{for any} \quad Z := (Z_{1}, Z_{2}, Z_{3}) \in \mathbb{H}.$$ 

Following Kurtz and Protter [47], we say that an $(\mathcal{F}_{t})$-adapted process $\{Y(t) := (Y_{1}(t), Y_{2}(t), Y_{3}(t)) : t \geq 0\}$ is a $\mathbb{H}^{\#}$-semimartingale if $\{Y_{1}(t) : t \geq 0\}$ and $\{Y_{2}(t) : t \geq 0\}$ are two $\mathbb{R}$-semimartingales and $\{Y_{3}(t) : t \geq 0\}$ is an $L^{1,2}(\mathbb{R}_{+})^{\#}$-semimartingale random measure on $\mathbb{R}_{+}$, i.e.

1. for any $f \in L^{1,2}(\mathbb{R}_{+})$, $\{Y_{3}(f, t) := \int_{\mathbb{R}_{+}} f(x)Y_{3}(t, dx) : t \geq 0\}$ is a cádlág, $(\mathcal{F}_{t})$ semimartingale with $Y_{3}(f, 0) = 0$;
2. for any $f_{1}, \cdots, f_{m} \in L^{1,2}(\mathbb{R}_{+})$ and $w_{1}, \cdots, w_{m} \in \mathbb{R}$, $Y_{3}(\sum_{i=1}^{m} w_{i} f_{i}, t) = \sum_{i=1}^{m} w_{i} Y_{3}(f_{i}, t)$ a.s. for any $t \geq 0$.

Let us recall the definition of stochastic integrals w.r.t. the $\mathbb{H}^{\#}$-semimartingale $Y$. To this end, let $\mathcal{S}_{\mathbb{H}}^{0}$ be the collection of $\mathbb{H}$-valued, simple processes of the form

$$X(t) := (X_{1}(t), X_{2}(t), X_{3}(t)) = \left( \sum_{i=1}^{m} \xi_{1,i} \cdot 1_{[\tau_{i}, \tau_{i+1})}(t), \sum_{i=1}^{m} \xi_{2,i} \cdot 1_{[\tau_{i}, \tau_{i+1})}(t), \sum_{i=1}^{m} \xi_{3,i,j} \cdot 1_{[\tau_{i}, \tau_{i+1})}(t) \cdot \varphi_{i}^{j} \right), \quad (3.15)$$
where $0 = t_1 < t_2 < \cdots$, and $\{\xi_{1,k}\}, \{\xi_{2,k}\}, \{\xi_{2,k,i,j}\}$ are sequences of bounded, $\mathbb{R}$-valued, $(\mathcal{F}_t)$-adapted random variables, all but finitely many of which being zero. For any $X \in \mathcal{S}_{\mathbb{H}}^0$, the stochastic integral w.r.t. $Y$ is defined as

$$\int_0^t X(s-) \cdot Y(ds) = \sum_{k=1}^{\infty} \xi_{1,k}[Y_1(t_{k+1} \land t) - Y_1(t_k \land t)] + \sum_{k=1}^{\infty} \xi_{2,k}[Y_2(t_{k+1} \land t) - Y_2(t_k \land t)]$$

$$+ \sum_{k=1}^{\infty} \sum_{i,j=0}^{\infty} \xi_{3,k,i,j} \cdot [Y_3(t_{k+1} \land t, \varphi_i^j) - Y_3(t_k \land t, \varphi_i^j)].$$

Let $\mathbb{H}$ be the completion of the linear space $\mathcal{S}_{\mathbb{H}}^0$ with respect to the norm $\| \cdot \|_{\mathbb{H}}$, i.e. for any $\{X(t) : t \geq 0\} \in \mathbb{H}$, there exists a sequence of simple processes $\{X_n(t) : t \geq 0\} \in \mathcal{S}_{\mathbb{H}}^0$ such that for any $T \geq 0$,

$$\lim_{n \to \infty} \int_0^T \|X(t) - X_n(t)\|_{\mathbb{H}} dt = 0.$$

The stochastic integral for $X(\cdot) \in \mathbb{H}$ is then defined as

$$\int_0^t X(s-) \cdot Y(ds) := \lim_{n \to \infty} \int_0^t X_n(s-) \cdot Y(ds).$$

We say that a $(\mathcal{F}_t)$-adapted process $\{Y(t) : t \geq 0\}$ is a standard $\mathbb{H}$-semimartingale if $\int_0^t X(s-) \cdot Y(ds) \in \mathcal{D}([0, \infty); \mathbb{R}^2)$ for any $X \in \mathcal{S}_{\mathbb{H}}^0$ and

$$\mathcal{H}_t := \left\{ \left\| \int_0^t X(s-) \cdot Y(ds) \right\| : X \in \mathcal{S}_{\mathbb{H}}^0, \sup_{s \leq t} \|X(s)\|_{\mathbb{H}} \leq 1 \right\}$$

is stochastically bounded for each $t \geq 0$.

We are now ready to provide an alternative representation of the market model (3.1). For any $t \geq 0$ and $n \geq 1$, let us define $Y^{(n)}(t) := (Y_1^{(n)}(t), Y_2^{(n)}(t), Y_3^{(n)}(t))$ by

$$Y_1^{(n)}(t) := \gamma_n \left( \int_{\mathcal{U}} u Y_1^{(n)}(du) - \left( \begin{array}{c} 0 \\ \beta_n \end{array} \right) \right), \quad Y_2^{(n)}(t) := \int_0^t \int_{\mathcal{U}} \frac{u}{n} N_1^{(n)}(\gamma_n \cdot ds, du)$$

and

$$Y_3^{(n)}(t) := \int_0^t \int_{\mathcal{U}} \frac{u}{n} \tilde{N}_1^{(n)}(\gamma_n \cdot ds, du, n \cdot dx).$$

The process $\{Y^{(n)}(t) : t \geq 0\}$ is a real-valued process, $\{Y_2^{(n)}(t) : t \geq 0\}$ is a compound Poisson process, and $\{Y_3^{(n)}(t) : t \geq 0\}$ is an $L^{1,2}(\mathbb{R}_+)^3$-martingale random measure on $\mathbb{R}_+$. We can rewrite the market model (3.1) as

$$\begin{pmatrix} P^{(n)}(t) \\ V^{(n)}(t) \end{pmatrix} = \begin{pmatrix} P^{(n)}(0) \\ V^{(n)}(0) \end{pmatrix} + \int_0^t F(V^{(n)}(s-)) \cdot Y^{(n)}(ds),$$

where the function $F : \mathbb{R}_+ \to (\mathbb{R}_+^2 \times L^{1,2}(\mathbb{R}_+))^2$ is defined by

$$F(v) := \begin{pmatrix} v & 1 & 1_{\{v < v_0\}} \\ v & 1 & 1_{\{v < v_0\}} \end{pmatrix}.$$
Lemma 3.9 Under Condition 3.3 and 3.4, we have \( \{ (Y_1^{(n)}(t), Y_2^{(n)}(t)) : t \geq 0 \}_{n \geq 1} \) converges weakly to \( \{ (Y_1(t), Y_2(t)) : t \geq 0 \} \) in \( D([0, \infty); \mathbb{R}^2) \) as \( n \to \infty \), where

\[
Y_1(t) = -bt \quad \text{and} \quad Y_2(t) = at + \int_0^t \int_U uN_1(ds, du),
\]

and \( N_1(ds, du) \) is a Poisson random measure on \( (0, \infty) \times U \) with intensity \( ds\check{v}_d(du) \).

We now turn to the convergence of the sequence \( \{ Y_3^{(n)}(t) : t \geq 0 \}_{n \geq 1} \). Following Kurtz and Protter [47] we say that this sequence converges weakly to \( \{ Y_3(t) : t \geq 0 \} \) as \( n \to \infty \) and write \( Y_3^{(n)} \to Y_3 \) if

\[
\left( Y_3^{(n)}(f_1, \cdot), \ldots, Y_3^{(n)}(f_m, \cdot) \right) \to \left( Y_3(f_1, \cdot), \ldots, Y_3(f_m, \cdot) \right)
\]

weakly in \( D([0, \infty), \mathbb{R}^{2 \times m}) \) for any \( f_1, \ldots, f_m \in L^{1,2}(\mathbb{R}_+) \) and any \( m \in \mathbb{N} \). The following lemma establishes the weak convergence. It uses the fact that \( 2\sigma \) is symmetric and nonnegative-definite so that the square root \( \sqrt{2\sigma} \) is well defined.

Lemma 3.10 Under Condition 3.3 and 3.4, we have that \( Y_3^{(n)} \to Y_3 \) where \( \{ Y_3(t) : t \geq 0 \} \) is an \( L^{1,2}(\mathbb{R}_+) \)-martingale with the following representation:

\[
Y_3(t) = \int_0^t \sqrt{2\sigma} \left( \frac{W_1(ds, dx)}{W_2(ds, dx)} \right) + \int_0^t \int_U u\check{N}_0(ds, du, dx).
\]

Here, \( W_1(dt, dx) \) and \( W_2(dt, dx) \) are orthogonal Gaussian white noises on \( (0, \infty)^2 \) with intensities \( dsdx \), \( N_0(ds, du, dx) \) is a Poisson random measure on \( (0, \infty) \times U \times \mathbb{R}_+ \) with intensity \( ds\check{v}_d(du)dx \), and

\[
\check{N}_0(ds, du, dx) := N_0(ds, du, dx) - ds\check{v}_d(du)dx.
\]

Proof. We prove the weak convergence of \( \{ Y_3^{(n)}(f, t) : t \geq 0 \}_{n \geq 1} \) for any \( f \in L^{1,2}(\mathbb{R}_+) \); the general case be proved in the same way. From Kurtz’s criterion [21, p. 137] tightness of this sequence follows from the following estimate: for any \( t \in [0, 1] \) and \( \epsilon > 0 \),

\[
E_{\mathcal{F}_t} \left[ \| Y_3^{(n)}(f, t + \epsilon) - Y_3^{(n)}(f, t) \| \right] \leq 2(\epsilon + \sqrt{\epsilon}). \tag{3.18}
\]

In order to verify this inequality, we first apply Jensen’s inequality to get

\[
E_{\mathcal{F}_t} \left[ \| Y_3^{(n)}(f, t + \epsilon) - Y_3^{(n)}(f, t) \| \right] = E \left[ \| \int_t^{t+\epsilon} \int_0^\infty f(x) \frac{u}{n} \check{N}_1^{(n)}(\gamma_n \cdot ds, du, n \cdot dx) \| \right] \\
\leq E \left[ \int_t^{t+\epsilon} \int_{\|u\| \leq 1} \int_0^\infty f(x) \frac{u}{n} \check{N}_1^{(n)}(\gamma_n \cdot ds, du, n \cdot dx) \|^{2} \right]^{1/2} \\
+ E \left[ \int_t^{t+\epsilon} \int_{\|u\| > 1} \int_0^\infty f(x) \frac{u}{n} \check{N}_1^{(n)}(\gamma_n \cdot ds, du, n \cdot dx) \| \right]. \tag{3.19}
\]

Applying the Burkholder-Davis-Gundy inequality to the first expectation on the right side of the last inequality, this term can be bounded by

\[
C E \left[ \int_t^{t+\epsilon} \int_{\|u\| \leq 1} \int_0^\infty f(x)^2 \frac{\|u\|^2}{n^2} \check{N}_1^{(n)}(\gamma_n \cdot ds, du, n \cdot dx) \right] \leq C \cdot \epsilon \cdot \| f \|_{L^2} \| \gamma_n \| \int_{\|u\| \leq 1} \| u \|^{2} \check{v}_d^{(n)}(n \cdot du).
\]

Moreover, the second term on the right side of the last inequality in (3.19) can be bounded by

\[
2\epsilon \cdot \| f \|_{L^1} \gamma_n \int_{\|u\| > 1} \| u \| \check{v}_d^{(n)}(n \cdot du).
\]

Taking these two upper estimates back into (3.19) and using Condition 3.4 yields (3.18). We may hence assume that \( \{ Y_3^{(n)}(f, t) : t \geq 0 \}_{n \geq 1} \) converges weakly in \( D([0, \infty); \mathbb{R}^3) \) along a subsequence. By Skorokhod’s representation theorem, we may actually assume almost sure convergence and need to prove that
\[ Y_3^{(n)}(f, t) \rightarrow \int_0^t \int_0^\infty f(x)\sqrt{2\sigma} \left( \frac{W_1(ds, dx)}{W_2(ds, dx)} \right) + \int_0^t \int_\mathbb{U} \int_0^\infty f(x)u\tilde{N}_0(ds, du, dx) =: Y_3(f, t). \]

The next step, then, consists in proving that the limit is a local martingale; subsequently we apply a general representation result for local martingales to conclude. In order to prove the local martingale property of the \( \phi \in C_b(U) \) according to

\[ A_f^{(n)}(z) = n\gamma_n \int_0^\infty dx \int_{U^\prime} [\phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle] \nu_4^{(n)}(n \cdot du) \]

\[ = n\gamma_n \int_0^\infty dx \int_{U^\prime} [\phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle] \nu_4^{(n)}(n \cdot du) \]

\[ + n\gamma_n \int_0^\infty dx \int_{U^\prime} [\phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle] \nu_4^{(n)}(n \cdot du). \] (3.20)

We can rewrite the two terms on the right side of the last equality as

\[ n\gamma_n \int_0^\infty dx \int_{U^\prime} [\phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle] \nu_4^{(n)}(n \cdot du) \]

\[ = \frac{1}{2} \sum_{j,l=1}^2 \frac{\partial^2 \phi(z)}{\partial z_j \partial z_l} n\gamma_n \int_0^\infty |f(x)|^2 dx \int_{U^\prime} (u_j \wedge 1)(u_l \wedge 1) \nu_4^{(n)}(n \cdot du) \]

\[ + n\gamma_n \int_0^\infty dx \int_{U^\prime} [\phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle] \]

\[ - \frac{1}{2} \sum_{j,l=1}^2 \frac{\partial^2 \phi(z)}{\partial z_j \partial z_l} |f(x)|^2 (u_j \wedge 1)(u_l \wedge 1) \nu_4^{(n)}(n \cdot du). \]

From Proposition 3.7, as \( n \to \infty \) the first term on the right side of the equality above converges to

\[ \int_0^\infty dx \int_{U^\prime} [\phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle] \sigma_{j,l}^+ \frac{1}{2} \int_{U^\prime} (u_j \wedge 1)(u_l \wedge 1) \nu_4^{(n)}(du). \]

and the second term converges to

\[ \int_0^\infty dx \int_{U^\prime} \left[ \phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle - \frac{1}{2} \sum_{j,l=1}^2 \frac{\partial^2 \phi(z)}{\partial z_j \partial z_l} |f(x)|^2 (u_j \wedge 1)(u_l \wedge 1) \right] \nu_4^{(n)}(du). \]

Hence, the generator \( A_f^{(n)} \) converges to the linear operator \( A_f \) defined by

\[ A_f \phi(x) := \sum_{j,l=1}^2 \sigma_{j,l}^+ \frac{\partial^2 \phi(z)}{\partial z_j \partial z_l} \int_0^\infty |f(x)|^2 dx + \int_0^\infty dx \int_{U^\prime} \left[ \phi(z + f(x)u) - \phi(z) - \langle \nabla \phi(z), f(x)u \rangle \right] \nu_4^{(n)}(du). \] (3.21)

Applying Itô’s formula to the function \( e^{i(z,Y_3^{(n)}(f,s))} \), we see that

\[ \mathcal{M}_4^{(n)}(f) := e^{i(z,Y_3^{(n)}(f,t))} - \int_0^t A_f^{(n)} e^{i(z,Y_3^{(n)}(f,s))} ds \]

is an \( (\mathcal{F}_t) \)-local martingale. Since

\[ \sup_{s \in [0,t]} |A_f^{(n)} e^{i(z,Y_3^{(n)}(f,s))}| = \sup_{s \in [0,t]} |e^{i(z,Y_3^{(n)}(f,s)))} | \cdot n\gamma_n \int_0^\infty dx \int_{U^\prime} |e^{i(z,f(x)u)} - 1 - i(z, f(x)u)| \nu_4^{(n)}(n \cdot du) \]

\[ \leq n\gamma_n \int_0^\infty dx \int_{U^\prime} |(z, f(x)u)|^2 \wedge |(z, f(x)u)| \nu_4^{(n)}(n \cdot du) \]

\[ \leq \int_0^\infty \|f(x)z\|^2 \vee \|f(x)z\|dx \cdot n\gamma_n \int_U \|u\|^2 \wedge \|u\| \nu_4^{(n)}(n \cdot du), \] (3.22)
and because the last term is uniformly bounded in $n$, due to Condition 3.4, the process $\{M^{(n)}_t(f) : t \geq 0\}$ is in fact a true martingale.

By the almost sure convergence in the Skorohod topology we know that $Y_3^{(n)}(f, t) \to Y_3(f, t)$ a.s. for all but countably many $t > 0$. Thus, it follows from (3.21) and the dominated convergence theorem that
\[
\int_0^t A_f^{(n)} e^{i(z, Y_3^{(n)}(f, s))} ds \to \int_0^t A_f e^{i(z, Y_3(f, s))} ds
\]
almost surely in the space $C([0, \infty), \mathbb{C})$. This implies that for any $z \in \mathbb{R}^2$,
\[
M_t^{(n)}(f) \to M_t(f) := e^{i(z, Y_3(f, t))} - \int_0^t A_f e^{i(z, Y_3(f, s))} ds
\]
almost surely in the Skorohod space $D([0, \infty), \mathbb{C})$ and hence that $M_t^{(n)}(f) \to M_t(f)$ almost surely for almost all $t > 0$. In view of (3.22) we also have convergence in $L^1$ for almost all $t > 0$. Since $\{M_t(f) : t \geq 0\}$ is right-continuous, the martingale property of $\{M_t^{(n)}(f) : t \geq 0\}$ carries over to $\{M_t(f) : t \geq 0\}$. By [37, Theorem 2.42] this is equivalent to the local martingale property of $\{Y_3(f, t) : t \geq 0\}$.

We now prove the desired representation. The local martingale $\{Y_3(f, t) : t \geq 0\}$ admits the the canonical representation
\[
Y_3(f, t) = M_t(f) + \int_0^t u \tilde{N}_f^d(ds, du'),
\]
where $N_f^d(ds, du')$ is an integer-valued random measure on $[0, \infty) \times \mathbb{U}$ with compensator $ds \nu_f^d(du')$ and
\[
\nu_f^d(du') = \int_0^\infty dx \int_{\mathbb{U}} 1_{\{f(x)u \in du'\}} \hat{\nu}_x(du),
\]
and $\{M_t(f) : t \geq 0\}$ is a continuous, $\mathbb{R}^2$-valued local martingale with quadratic covariation process
\[
\langle M_j^f(f), M_l^g(f) \rangle_t = t \cdot 2\sigma_{j,l} \int_0^\infty |f(x)|^2 dx, \quad j, l = 1, 2.
\]

Similarly, for any $f, g \in L^{1,2}(\mathbb{R}_+)$, we have
\[
\langle M_j^f(f), M_l^g(g) \rangle_t = t \cdot 2\sigma_{j,l} \int_0^\infty f(x)g(x) dx, \quad j, l = 1, 2.
\]

By [18, Theorem III-7], there exist two orthogonal Gaussian white noise $W_1(ds, dx)$ and $W_2(ds, dx)$ on $(0, \infty)^2$ with density $ds dx$ such that for any $f \in L^{1,2}(\mathbb{R}_+)$,
\[
M_j^f(f) = \int_0^t \int_{\mathbb{R}_+} f(x) \sqrt{2\sigma_{1,j}} W_1(ds, dx) + \int_0^t \int_{\mathbb{R}_+} f(x) \sqrt{2\sigma_{2,j}} W_2(ds, dx), \quad j = 1, 2.
\]

By [36, Theorem 7.4], there exists a Poisson random measure $N_0(ds, du, dx)$ on $(0, \infty) \times \mathbb{U} \times \mathbb{R}_+$ with intensity $ds \nu_1(du) dx$ such that
\[
\int_0^t \int_{\mathbb{U}} u \tilde{N}_f^d(ds, du) = \int_0^t \int_{\mathbb{U}} \int_0^\infty f(x) \cdot u \tilde{N}_0(ds, du, dx).
\]

Taking these two representations back into (3.23) yields the desired result. \qed

### 3.2.2 Uniform tightness and weak convergence of market models

The next steps towards the proof of the weak convergence of the market models is to prove that $\{Y^{(n)}(t) : t \geq 0\}_{n \geq 1}$ is a sequence of uniformly tight standard $\mathbb{H}^\#$-semimartingales. That is, the sequence $\{\mathcal{H}^{(n)}_t\}_{n \geq 1}$ is uniformly stochastically bounded for any $t \geq 0$, where $\mathcal{H}^{(n)}_t$ is defined as in (3.16) with $Y$ replaced by $Y^{(n)}$. 

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Lemma 3.11 Under Conditions 3.3 and 3.4, \( \{Y^{(n)}(t) : t \geq 0\}_{n \geq 1} \) is a sequence of uniformly tight \( H^F \)-semimartingales.

Proof. We just prove that \( \{\mathcal{H}^{(n)}_1\}_{n \geq 1} \) is uniformly stochastically bounded; the general case can be proved similarly. For any simple \( H \)-valued processes \( X \),

\[
\int_0^1 X(s-) \cdot Y^{(n)}(ds) = \sum_{k=1}^3 \int_0^1 X_k(s-) \cdot Y^{(n)}_k(ds), \tag{3.24}
\]

where

\[
\int_0^1 X_1(s-) \cdot Y_1^{(n)}(ds) := \gamma_n \left( \int U_1^{(n)}(du) - \left( \frac{0}{\beta_n} \right) \right) \sum_{k=1}^1 \xi_1,k \mathbf{1}_{\{\xi_{1,k}\}}(s) ds,
\]

\[
\int_0^1 X_2(s-) \cdot Y_2^{(n)}(ds) := \int_0^1 \sum_{k=1}^1 \xi_2,k \mathbf{1}_{\{\xi_{2,k}\}}(s) \int \mathbf{1}_{\{\xi_{2,k}\}}(\gamma_n \cdot ds, du),
\]

\[
\int_0^1 X_3(s-) \cdot Y_3^{(n)}(ds) := \int_0^1 \int_0^\infty \sum_{k=1}^1 \xi_3,k,i,j \mathbf{1}_{\{\xi_{3,k,i,j}\}}(s) \varphi_i^j(x) \int \mathbf{1}_{\{\xi_{3,k,i,j}\}}(\gamma_n \cdot ds, du, n \cdot dx).
\]

Since \( \sup_{s \in [0,1]} \|X(s)\|_H \leq 1 \) a.s., we have \( |\xi_{1,k}|, |\xi_{2,k}|, |\xi_{3,k,i,j}| \leq 2 \) a.s. for any \( k \geq 1, i \geq -1 \) and \( j \geq 0 \). By Condition 3.3 there exists a constant \( C > 0 \) such that for any \( n \geq 1 \),

\[
\gamma_n \left\| \int U_1^{(n)}(du) - \left( \frac{0}{\beta_n} \right) \right\| \leq C
\]

and from the Markov inequality,

\[
P\left\{ \left\| \int_0^1 X_1(s-) \cdot Y_1^{(n)}(ds) \right\| \geq K \right\} \leq \frac{C}{K} \int_0^1 \mathbf{E}[\|\xi_{1,k}\|^{(n)} \mathbf{1}_{\{\xi_{1,k}\}}(s) ds] \leq \frac{2C}{K}. \tag{3.25}
\]

For the second term in (3.24), we have

\[
\left\| \int_0^1 X_2(s-) \cdot Y_2^{(n)}(ds) \right\| \leq 2 \int_0^1 \int U \mathbf{1}_{\{\xi_{1,k}\}}(\gamma_n \cdot ds, du).
\]

For any \( M > 0 \), we have

\[
P\left\{ \int_0^1 \int U \mathbf{1}_{\{\xi_{1,k}\}}(\gamma_n \cdot ds, du) \geq K \right\} \leq P\left\{ \int_0^1 \int U \left( \frac{\|u\|}{n} \wedge M \right) \mathbf{1}_{\{\xi_{1,k}\}}(\gamma_n \cdot ds, du) \geq K \right\}
\]

\[
+ P\left\{ \int_0^1 \int U \left( \frac{\|u\|}{n} \wedge M \right) \mathbf{1}_{\{\|u\| \geq nM\}} \mathbf{1}_{\{\xi_{1,k}\}}(\gamma_n \cdot ds, du) \geq K \right\}. \tag{3.26}
\]

Applying Jensen’s inequality to the first term on the right side of this inequality, it can be bounded by

\[
\frac{1}{K} \mathbf{E} \left[ \int_0^1 \int U \left( \frac{\|u\|}{n} \wedge M \right) \mathbf{1}_{\{\xi_{1,k}\}}(\gamma_n \cdot ds, du) \right] \leq \frac{1}{K} \gamma_n \int U \left( \frac{\|u\|}{n} \wedge M \right) \nu^{(n)}(du).
\]

Moreover, the second term on the right side of (3.26) can be bounded by

\[
P\left\{ \int_0^1 \mathbf{1}_{\{\|u\| \geq nM\}} \mathbf{1}_{\{\xi_{1,k}\}}(\gamma_n \cdot ds, du) \geq K \right\} \leq 1 - \exp\{-\gamma_n \nu^{(n)}(\|u\| \geq nM)\} \leq C \cdot \gamma_n \nu^{(n)}(\|u\| \geq nM).
\]

By Proposition 3.7(b), there exist constants \( C > 0 \) independent of \( M \) and \( C_M > 0 \) such that for any \( n \geq 1 \),

\[
\gamma_n \int U \left( \frac{\|u\|}{n} \wedge M \right) \nu^{(n)}(du) \leq C_M \gamma_n \nu^{(n)}(\|u\| \geq nM) \quad \text{and} \quad \gamma_n \nu^{(n)}(\|u\| \geq nM, \infty) \leq C \cdot \nu^{(n)}(\|u\| \geq nM).
\]

Altogether, this yields

\[
P\left\{ \left\| \int_0^1 X_2(s-) \cdot Y_2^{(n)}(ds) \right\| \geq K \right\} \leq \frac{C_M}{K} + C \nu^{(n)}(\|u\| \geq M). \tag{3.27}
\]
which goes to 0 when first sending \( K \) and then \( M \) to \( \infty \). We now consider the third term on the right side of (3.24). Applying Jensen’s inequality again, we have

\[
P\left\{ \left\| \int_0^1 X_3(s-) \cdot Y_3^{(n)}(ds) \right\| \geq K \right\} \leq P\left\{ \left\| M_{\leq 1} \right\| \geq K/2 \right\} + P\left\{ \left\| M_{> 1} \right\| \geq K/2 \right\}
\]

\[
\leq \frac{1}{K^2} E \left[ \left\| M_{\leq 1} \right\|^2 \right] + \frac{2}{K} E \left[ \left\| M_{> 1} \right\| \right],
\]

where

\[
M_{\leq 1} := \int_0^1 \int_0^\infty \int_{\|u\| \leq n} \xi_{3,k,i,j}1_{(t_k,t_{k+1})}(s)\varphi_{i}^{x}(s)\frac{u}{n}\tilde{N}_i^{(n)}(\gamma_n ds, du, n \cdot dx),
\]

\[
M_{> 1} := \int_0^1 \int_0^\infty \int_{\|u\| > n} \xi_{3,k,i,j}1_{(t_k,t_{k+1})}(s)\varphi_{i}^{x}(s)\frac{u}{n}\tilde{N}_i^{(n)}(\gamma_n ds, du, n \cdot dx).
\]

Applying the Burkholder-Davis-Gundy inequality to the first term on the right side of the last inequality in (3.28), we have

\[
E[\left\| M_{\leq 1} \right\|^2] \leq \int_0^1 \sum \xi_{3,k,i,j}1_{(t_k,t_{k+1})}(s)ds \int_0^\infty |\varphi_{i}^{x}(s)|^2 ds n\gamma_n \int_{\|u\| \leq 1} \frac{\|u\|^2}{n^2} \nu_1^{(n)}(du)
\]

\[
\leq \|X_3\|_{L^2} \cdot n\gamma_n \int_{\|u\| \leq 1} \frac{\|u\|^2}{n^2} \nu_1^{(n)}(du) \leq C n\gamma_n \int_{\|u\| \leq 1} \frac{\|u\|^2}{n^2} \nu_1^{(n)}(du).
\]

Moreover, we also have

\[
E[\left\| M_{> 1} \right\|] \leq 2 \int_0^1 \sum \xi_{3,k,i,j}1_{(t_k,t_{k+1})}(s)ds \int_0^\infty |\varphi_{i}^{x}(s)| ds n\gamma_n \int_{\|u\| > 1} \frac{\|u\|^2}{n^2} \nu_1^{(n)}(du)
\]

\[
= 2 \|X_3\|_{L^1} \cdot n\gamma_n \int_{\|u\| > 1} \frac{\|u\|^2}{n^2} \nu_1^{(n)}(du) \leq C \cdot n\gamma_n \int_{\|u\| > 1} \frac{\|u\|^2}{n^2} \nu_1^{(n)}(du).
\]

Along with the assumption that \( \sup_{n \geq 1} n\gamma_n \int_{\|u\| > 1} \|u\|^2 \cdot \|u\| \nu_1^{(n)}(du) < \infty \), we have

\[
P\left\{ \left\| \int_0^1 X_3(s-) \cdot Y_3^{(n)}(ds) \right\| \geq K \right\} \leq \frac{C}{K} + \frac{C}{K^2}.
\]

We are now ready to prove the weak convergence of the rescaled Hawkes market models.

**Theorem 3.12** Under Condition 3.3 and 3.4, if \((P^{(n)}(0), V^{(n)}(0)) \rightarrow (P(0), V(0))\) in distribution, the rescaled processes \(\{(P^{(n)}(t), V^{(n)}(t)) : t \geq 0\}\) converge weakly to \(\{(P(t), V(t)) : t \geq 0\}\) in \(\mathcal{D}([0, \infty); \mathbb{U})\) as \(n \rightarrow \infty\), where \(\{(P(t), V(t)) : t \geq 0\}\) is the unique strong solution to the following stochastic system:

\[
\left( \begin{array}{c} P(t) \\ V(t) \end{array} \right) = \left( \begin{array}{c} P(0) \\ V(0) \end{array} \right) + \int_0^t (a - bV(s)) ds + \int_0^t \int_0^V \sqrt{2\sigma} \left( W_1(ds, dx) \right) + \int_0^t \int_U uN_1(ds, du) + \int_0^t \int_{U} \left( V(s-) - V(s) \right) u\tilde{N}_0(ds, du, dx).
\]

**Proof.** By [13, Theorem 2.5] the stochastic system (3.30) admits a unique strong solution \((P, V)\). In order to establish the convergence we recall that

\[
\left( \begin{array}{c} P^{(n)}(t) \\ V^{(n)}(t) \end{array} \right) = \left( \begin{array}{c} P^{(n)}(0) \\ V^{(n)}(0) \end{array} \right) + \int_0^t F(V^{(n)}(s) - V^{(n)}(s)) ds.
\]

The function \(F\) is unbounded and does not satisfy the requirements of [47, Theorem 7.5]. To overcome this problem, we use a standard localization argument; see [46] for details. Let

\[
\eta^c(\omega) = \inf_{t \geq 0} \{ \omega(t) : \omega(t- \geq c \} \quad \text{and} \quad F^c(\omega, s-) := 1_{[0, \eta^c]}(s)F(\omega(s-))
\]
for $\omega \in \mathcal{D}([0, \infty) \times \mathbb{R})$ and $c > 0$, and consider the equation

$$
\left( \begin{array}{c}
P^{(n),c}(t) \\ V^{(n),c}(t)
\end{array} \right) = \left( \begin{array}{c} P^{(n)}(0) \\ V^{(n)}(0)
\end{array} \right) + \int_0^t F^c(V^{(n),c}, \sigma) \cdot Y^{(n)}(ds).
$$

This system satisfies the assumptions of [47, Theorem 7.5]. Thus, as $n \to \infty$,

$$(P^{(n),c}, V^{(n),c}) \to (P^c, V^c)$$

weakly in $\mathcal{D}([0, \infty))$ and hence $\eta^c(V^{(n),c}) \to \eta^c(V^c)$ weakly in $\mathcal{D}([0, \infty) \times \mathbb{R})$, where $(P^c, V^c)$ is the unique strong solution to (3.30), restricted to $[0, \eta^c(V^c))$. Uniqueness of strong solutions to (3.30) also yields $(P^c, V^c) \to (P, V)$ a.s. as $c \to \infty$. This shows that $(P^{(n)}, V^{(n)}) \to (P, V)$ weakly in $\mathcal{D}([0, \infty), \mathbb{U})$. \qed

The process $(\{(P(t), V(t)) : t \geq 0\})$ is a strong Markov process with infinitesimal generator $\mathcal{A}$ defined by (3.14). Standard arguments (see [36, Theorem 7.1' and 7.4]) show that the limit process $(\{(P(t), V(t)) : t \geq 0\})$ also can be represented as a stochastic integral driven by two Brownian motions.

**Corollary 3.13** Let $\{B_1(t) : t \geq 0\}$ and $\{B_2(t) : t \geq 0\}$ be two Brownian motions with correlation coefficient $\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11} \sigma_{22}}}$. The unique strong solution to the stochastic system $^5$

$$
\left( \begin{array}{c}
P(t) \\ V(t)
\end{array} \right) = \left( \begin{array}{c} P(0) \\ V(0)
\end{array} \right) + \int_0^t (a - bV(s))ds + \int_0^t \sqrt{\sigma(V(s))} dB_1(s) \\
+ \int_0^t \int_0^t uN_1(ds, du) + \int_0^t \int_0^t u\tilde{N}_0(ds, du, dx)
\right),
$$

(3.31)
is a realization for the limiting market system. The solution is an affine process with characteristic function

$$
E\left[ \exp\{z_1P(t) + z_2V(t)\} \right] = \exp\left\{ z_1P(0) + \psi_G(z)V(0) + \int_0^t H(z_1, \psi_G(z))ds \right\}, \quad z \in \mathbb{U}_+,
$$

(3.32)

where $\{\psi_G(z) : t \geq 0\}$ is the unique solution to the following Riccati equation

$$
\psi_G(z) = z_2 + \int_0^t G(z_1, \psi_G(z))ds.
$$

(3.33)

### 3.3 Examples

We now provide scaling limits for each of the examples considered in Section 2.2. We show that the rescaled market models converge to Heston-type stochastic volatility models with or without jumps, depending on the arrival frequency of large orders.

**Example 3.14** (Heston volatility model) Let $\{(P_t^{(n)}, \psi_t^{(n)}) : t \geq 0\}$ be an exponential market as defined in Example 2.9 with parameters $(1/n, \beta(n), \nu_{(n)}^{b/s}, \lambda_{(n)}^{b/s}, \lambda_{(n)}^{b/s})$ satisfying

$$
\lambda_{(n)}^{b/s} \to \lambda^{b/s}, \quad n\left[\nu_{(n)}^{sM_{1,1}}(\lambda_{(n)}^{b/s}) - \nu_{(n)}^{bM_{1,1}}(\lambda_{(n)}^{b/s})\right] \to 0, \quad n\left[\beta(n) - \nu_{(n)}^{bM_{1,2}}(\lambda^{b/s}) - \nu_{(n)}^{bM_{3,2}}(\lambda^{b/s})\right] \to b_2 > 0
$$

(3.34)
as $n \to \infty$. Let $\gamma = n$. Then Condition 3.3 holds. The conditions in Proposition 3.7 hold as well with limiting parameters

$$
\nu_{\gamma}(U) = \nu_{\gamma}(U) = 0, \quad \sigma^{\gamma} = \frac{\sigma_2}{2} \cdot M_{1,1}(\lambda^{b/s}), \quad \sigma^{-\gamma} = -\frac{\sigma_2}{2} \cdot M_{1,1}(\lambda^{b/s}), \quad \sigma^{\gamma} = \frac{\sigma_2}{2} \cdot M_{1,1}(\lambda^{b/s}),
$$

(3.34)

Since all order size distributions are light-tailed, there are no jumps in the scaling limit, and the limit model (3.31) reduces to the standard Heston volatility model

$$
P(t) = P(0) + a_1 t + \int_0^t \sqrt{2\sigma_1 V(s)} dB_1(s),
$$

$$
V(t) = V(0) + \int_0^t b_2 \left[ \frac{\sigma_2}{b_2} - V(s) \right] ds + \int_0^t \sqrt{2\sigma_2 V(s)} dB_2(s).
$$

$^5$The existence and uniqueness of strong solution follows from, e.g. [12, Theorem 6.2].
Next, we consider an exponential-Pareto mixing market model as defined in Example 2.11. In this case, we obtain the jump-diffusion Heston volatility model as analyzed in [15, 52] among many others in the scaling limit.

**Example 3.15 (Pareto-jump-diffusion volatility model)** Let \( \{(P_t^{(n)}(t), V_t^{(n)}(t)) : t \geq 0\} \) be an exponential-Pareto mixing market model as defined in Example 2.11 with parameters \((1/n, \beta(n), p_{b/s}, \alpha_b/\alpha_s, \alpha_b/\alpha_s, \theta_{b/s}(n)) \) and selecting mechanism \( \theta_{s_i}^{(n)} \) satisfying

\[
\alpha_b > 0, \quad \alpha_s = 1, \quad q_1^{(n)} = \frac{1}{n^2}, \quad q_s^{(n)} = \frac{1}{n}, \quad \theta_{s_i}^{b/s(n)} = n \cdot \theta_{s_i}^{b/s},
\]

and (3.34) as \( n \to \infty \). Let \( \gamma_n = n \). Then, Condition 3.3 as well as the conditions in Proposition 3.7 hold with limit parameters \( a^\pm, \sigma^\pm \) as defined in Example 3.14, with \( \hat{\nu}_e(U) = 0 \), and

\[
\hat{\nu}_e(du) = \sum_{j \in \{b, s\}} 1_{U_j}(u) \cdot p_d^j \cdot \frac{\alpha_b(\alpha_s + 1)}{\theta_{t,1}^j \theta_{s,1}^j} \left( 1 + \frac{|u_1|}{\theta_{t,1}^j} + \frac{u_2}{\theta_{s,1}^j} \right)^{-a_\pm - 2} du_1 du_2,
\]

where \( U_b = U_s \) and \( U_s = U_- \). In this case, the limit stochastic system (3.31) reduces to the following jump-diffusion Heston volatility model:

\[
\begin{align*}
P(t) &= P(0) + a_1 t + \int_0^t \sqrt{2\sigma_1 V(s)} dB_1(s) + \sum_{k=1}^{N_t} \xi_{k,p}^e, \\
V(t) &= V(0) + \int_0^t b_2 \left[ \frac{u_2}{p_2} - V(s) \right] ds + \int_0^t \sqrt{2\sigma_2 V(s)} dB_2(s) + \sum_{k=1}^{N_t} \xi_{k,v}^e,
\end{align*}
\]

where \( \{N_t : t \geq 0\} \) is a Poisson process with rate 1 and \( \{\xi_{k,p}^e, \xi_{k,v}^e : k = 1, 2, \cdots\} \) is a sequence of i.i.d. \( U \)-valued random variables with probability law \( \hat{\nu}_e(du) \).

In the previous example co-jumps in prices and volatility emerged in the scaling limit as a result of occasional large exogenous shocks. The key assumption was that \( \gamma_n = n \). The following example considers the scaling limit for \( \gamma_n = n^\alpha \) for some \( \alpha \in (0, 1) \). Two types of jumps emerge in our limit: jumps originating from large exogenous orders as well as self-exciting child-jumps. For the special case where the induced orders do not contribute to the intensity of order arrivals, the child-jumps drop out of the model. In this case, the limiting volatility process reduces to a non-Gaussian process of Ornstein-Uhlenbeck type as analyzed in Barndorff-Nielsen and Shephard [6].

**Example 3.16 (Stable-volatility model without diffusion)** Let \( \{(P_t^{(n)}(t), V_t^{(n)}(t)) : t \geq 0\} \) be a Pareto market model as defined in Example 2.10 with parameters \((1/n, \beta(n), p_{b/s}, \alpha_b/\alpha_s, \alpha_b/\alpha_s, \theta_{b/s}(n)) \) satisfying that \( \alpha_b = \alpha_s = 1 \in (0, 1) \). Moreover, as \( n \to \infty \) we have \( p_{b/s}^{(n)} \to p_{b/s} \) with \( p_b^2 + p_s^2 = 1 \) and

\[
n^{\alpha_b} [p_1^{(n)} M_{1,1}^{b/s}(\alpha_1, \theta_1) - p_1^{(n)} M_{1,1}^{b/s}(\alpha_1, \theta_1)] \to b_1, \quad n^{\alpha_s} [\beta(n) - p_1^{(n)} M_{1,2}^{b/s}(\alpha_1, \theta_1) - p_1^{(n)} M_{1,2}^{b/s}(\alpha_1, \theta_1)] \to b_2. \quad (3.35)
\]

Let \( \gamma_n = n^\alpha \). Then Condition 3.3 as well as the conditions in Proposition 3.7 hold with the following limit parameters: \( a^+ = a^- = 0, \quad \sigma^+ = \sigma^- = 0 \), and

\[
\hat{\nu}_e(du) = \sum_{j \in \{b, s\}} 1_{U_j}(u) \cdot p_d^j \cdot \frac{\alpha_b(\alpha_s + 1)}{\theta_{t,1}^j \theta_{s,1}^j} \left( 1 + \frac{|u_1|}{\theta_{t,1}^j} + \frac{u_2}{\theta_{s,1}^j} \right)^{-a_\pm - 2} du_1 du_2, \quad i \in \{e, i\}.
\]

In this case, the limit stochastic system (3.31) reduces to the following pure-jump model:

\[
\begin{pmatrix}
P(t) \\
V(t)
\end{pmatrix} = \begin{pmatrix}
P(0) \\
V(0)
\end{pmatrix} - \int_0^t b V(s) ds + \int_0^t \int_U u N_1(\omega, ds, du) + \int_0^t \int_U \int_0^V(u s) u \tilde{N}_0(\omega, ds, du, dx). \quad (3.36)
\]

The dynamics (3.36) can be represented in more convenient way. Let \( Z_{\alpha}(t) \) denote the third term on the right side of (3.36). By Theorem 14.3(ii) in [53], \( \{Z_{\alpha}(t) : t \geq 0\} \) is a \( U \)-valued \( \alpha \)-stable processes. As for the last term, let us introduce a random measure \( N_{\alpha}(\omega, ds, du) \) on \([0, \infty) \times \Omega \) as follows: for any \( t \geq 0 \) and \( U \subset \Omega \),

\[
N_{\alpha}(((0, t], U)) = \int_0^t \int_U \int_0^V(u s) 1_{\{u \in \alpha \sqrt{V(s-)} \omega \cap U\}} N_0(\omega, ds, du, dx).
\]

\[\text{Constant time-changes as allowed in Barndorff-Nielsen and Shephard [6] can easily be incorporated into our model.}\]
Standard computations show that its predictable compensator has the following representation:

\[
\tilde{N}_{\alpha_1}(\{0, t\}, U) = \sum_{j \in \{b, s\}} P_j \int_0^t ds \int_{U_j} V(s-1) \mathbf{1}_{\{s \in \sqrt{V(s-1)} U\}} \frac{\alpha_1 (\alpha_1 + 1)}{\theta_1^2, \theta_2^2} \left( \frac{|u_1|}{\theta_1} + \frac{u_2}{\theta_2^2} \right)^{-\alpha_1 - 1} du_1 du_2
\]

\[
= \sum_{j \in \{b, s\}} P_j \int_0^t ds \int_{U_j} \mathbf{1}_{\{s \in \sqrt{V(s) U}\}} \frac{\alpha_1 (\alpha_1 + 1)}{\theta_1^2, \theta_2^2} \left( \frac{|u_1|}{\theta_1} + \frac{u_2}{\theta_2^2} \right)^{-\alpha_1 - 1} du_1 du_2.
\]

Thus, \(N_{\alpha_1}(ds, du)\) is a Poisson random measure on \(U\) with intensity \(d\tilde{v}_i(du)\) and the Lévy processes \(\{Z_{\alpha_1}(t) : t \geq 0\}\) defined by

\[
Z_{\alpha_1}(t) := \int_0^t \int_U u \tilde{N}_{\alpha_1}(ds, du)
\]

is a compensated, \(U\)-valued \(\alpha_1\)-stable process. We can now rewrite (3.36) into

\[
\left( \begin{array}{c}
P(t) \\ V(t)
\end{array} \right) = \left( \begin{array}{c}
P(0) \\ V(0)
\end{array} \right) + Z_{\alpha_1}(t) - \int_0^t bV(s) ds + \int_0^t \sqrt{V(s-)} dZ_{\alpha_1}(s).
\]

So far, we obtained jump-diffusion models with exogenous jump dynamics as well as pure jump models with endogenous jump dynamics as scaling limits. The next example combines both dynamics into a single model. We call this model generalized \(\alpha\)-stable Heston volatility model. We assume that induced orders arrive at a rate \(n^2\) as in Example 3.15. However, we now assume that large orders arrive with much higher probabilities. In Example 3.15 large exogenous and induced orders arrive at rate one per unit time. Now we assume they arrive at rate \(n^2\) per unit time; despite the increased rate, their proportion among all orders will still be negligible in the limit. Several existing models are obtained as special cases. If the arrival intensity of induced orders depends on exogenous orders only, the model reduces to the stochastic volatility model studied in Barndorff-Nielsen and Shephard [6]. If, in addition, there are no jumps in the volatility, the model reduces to the one analyzed in Bates [7]. The special case with no child-jumps and no exogenous jumps in prices corresponds to the model studied in Nicolato et al. [1]. The special case without exogenous jumps corresponds to the alpha Heston model that has recently been studied in Jiao et al. [41]. The multi-factor model of Bates [8] with both exogenous and self-excited shocks is also contained as a special case.

Example 3.17 (Generalized \(\alpha\) Heston volatility model) Let \(\{(P^{(n)}_t, V^{(n)}_t) : t \geq 0\}\) be an exponential-Pareto mixing market model as defined in Example 2.11 with parameters \((1/n, \beta^{(n)}_e, \beta^{b/\text{m}}_e, \beta^{b/\text{s}}_e, \alpha^{b/\text{s}}_e, \alpha^{b/\text{m}}_e, \alpha^{b/\text{s}}_e, \alpha^{b/\text{m}}_e, \theta_1^{b/\text{m}}_e, \theta_1^{b/\text{s}}_e)\) and selecting mechanism \(d_{\alpha_i^e}\) satisfying that \(\alpha^e \in (0, 1), \alpha^i \in (1, 2), q^{(n)}_i = n^{\alpha^i - 2}, q^e = n^{\alpha^e - 1}\) and (3.34) holds as \(n \to \infty\). Let \(\gamma_n = n^2\), we see that Condition 3.3 and conditions in Proposition 3.7 hold with limit parameters \(a^+, \sigma^\pm\) defined in Example 3.14 and \(b, \tilde{v}_i(du)\) defined in Example 3.16. In this case, the limit stochastic system (3.31) reduces to

\[
\left( \begin{array}{c}
P(t) \\ V(t)
\end{array} \right) = \left( \begin{array}{c}
P(0) \\ V(0)
\end{array} \right) + \int_0^t (a - bV(s)) ds + \int_0^t \sqrt{V(s)} d\left( \frac{\sqrt{2\sigma_{11}} B_1(s)}{\sqrt{2\sigma_{22}} B_2(s)} \right) + Z_{\alpha_1}(t) + \int_0^t \sqrt{V(s)} dZ_{\alpha_1}(s).
\]

4 The genealogy of the limiting market dynamics

In this section, we analyze the impact of exogenous shocks on the jump dynamics of the limiting model. Exogenous shocks can be split in two groups: shocks of “significant magnitude” and shocks of “insignificant magnitude”. Shocks of “significant magnitude” are captured by the Poisson random measure \(N_1\). As argued in Section 3 (see Remark 3.6), shocks of “insignificant magnitude” are captured by the drift vector \(a \in \mathbb{R} \times \mathbb{R}_+\). Both types of shocks can trigger jump cascades. Cascades triggered by shocks of significant magnitude are exogenous while cascades triggered by shocks of insignificant magnitude are endogenous in nature.

Rewriting the characteristic function (3.32) as

\[
E \left[ e^{z_1 F(t) + z_2 V(t)} \right] = \exp \left( z_1 P(0) + \psi^{\alpha_1}_2(z) V(0) \right) \cdot \exp \left\{ \int_0^t ds \int_U \left[ e^{z_1 u_1 + \psi^{\alpha_1}_2(z) u_2 - 1} \tilde{v}_e(du) \right] \right\}
\]
\[ x = \exp \left\{ z_1 \cdot a_1 t + a_2 \int_0^t \psi_s^G(z) \, ds \right\} \tag{4.1} \]

the limit model (3.30) can be decomposed into into a hierachy of self-enclosed sub-models in terms of the strong Markov processes induced by each of the exponential functions on the right side of the above equation.

(i) From the semigroup property of \( \{ \psi_t^G(z) : t \geq 0 \} \) we conclude that the first term on the right side of (4.1) induces a Markov semigroup \( (Q_{0,t})_{t \geq 0} \) on \( U \), via

\[ \int_0^\infty e^{(z,u)} Q_{0,t}(u',du) = \exp \left\{ z_1 u'_1 + \psi_t^G(z) u'_2 \right\}. \tag{4.2} \]

Applying the Kolmogorov consistency theorem and the relationship between the solution to (3.30) and its characteristic function (3.32), we see that the Markov process \( \{ (P_0(t),V_0(t)) : t \geq 0 \} \) that solves

\[ \begin{pmatrix} P_0(t) \\ V_0(t) \end{pmatrix} = \begin{pmatrix} P(0) \\ V(0) \end{pmatrix} - \int_0^t bV_0(s)ds + \int_0^t \int_0^{V_0(s)} \sqrt{2\sigma} \begin{pmatrix} W_1(ds,dx) \\ W_2(ds,dx) \end{pmatrix} \]

+ \int_0^t \int_\mathbb{U} \int_0^{V_0(s-)} u\tilde{N}_0(ds,du,dx) \tag{4.3} \]

is a realization of the transition semigroup \( (Q_{0,t})_{t \geq 0} \). This self-enclosed market model captures the impact of all events prior to time 0 on future order flow. We emphasize that the model is independent of the drift and that the volatility mean-reverts to the level zero if \( b_2 > 0 \).

(ii) Let \( \{ P_{e,t}() : t \geq 0 \} \) be the family of probability laws induced by second term on the right side of (4.1). The family of kernels \( \{ Q_{e,t}() \}_{t \geq 0} \) on \( U \) defined by \( Q_{e,t}(u',du) := P_{e,t} \ast Q_{0,t}(u',du) \) forms a Markov semigroup. The Markov process \( \{ (P_{e,t}(t),V_{e,t}(t)) : t \geq 0 \} \) that solves

\[ \begin{pmatrix} P_{e}(t) \\ V_{e}(t) \end{pmatrix} = \int_0^t uN_1(ds,du) + \int_0^t \int_0^{V_0(s)+V_1(s)} \sqrt{2\sigma} \begin{pmatrix} W_1(ds,dx) \\ W_2(ds,dx) \end{pmatrix} \]

- \int_0^t bV_{e}(s)ds + \int_0^t \int_0^{V_0(s-)+V_1(s-)} u\tilde{N}_0(ds,du,dx) \tag{4.4} \]

is a realization of the transition semigroup \( (Q_{e,t})_{t \geq 0} \). We can interpret this model as describing the cumulative impact of the exogenous orders on the market dynamics. Again, this model is independent of the drift and the volatility mean-reverts to the level zero if \( b_2 > 0 \).

(iii) Let \( \{ P_{a,t}() : t \geq 0 \} \) be the family of probability laws induced by last term on the right side of (4.1), and \( \{ Q_{a,t}() \}_{t \geq 0} \) be the Markov semigroup on \( U \) given by \( Q_{a,t}(u',du) := P_{a,t} \ast Q_{0,t}(u',du) \). The Markov process \( \{ (P_{a,t}(t),V_{a,t}(t)) : t \geq 0 \} \) that solves

\[ \begin{pmatrix} P_{a}(t) \\ V_{a}(t) \end{pmatrix} = \int_0^t (a-bV_a(s))ds + \int_0^t \int_0^{V_0(s)+V_1(s)} \sqrt{2\sigma} \begin{pmatrix} W_1(ds,dx) \\ W_2(ds,dx) \end{pmatrix} \]

+ \int_0^t \int_\mathbb{U} \int_0^{V_0(s-)+V_1(s-)} u\tilde{N}_0(ds,du,dx) \tag{4.5} \]

is a realization of the transition semigroup \( (Q_{a,t})_{t \geq 0} \). Unlike the two other sub-models this sub-model depends on the drift \( a \). We can interpret this model as describing the impact of the drift of the volatility process on the market dynamics.

By the spatial-orthogonality of the Gaussian white noises \( W_1(ds,du), W_2(ds,du) \) and the Poisson random measure \( N_0(ds,du,dx) \), the processes defined by (4.3)-(4.5) are mutually independent. Their sum equals the process (3.30).

**Theorem 4.1** The following decomposition of the solution \( \{(P(t),V(t)) : t \geq 0\} \) to (3.30) holds:

\[ \{(P(t),V(t)) : t \geq 0\} \overset{a.s.}{=} \left\{ \left( \frac{P_0(t)+P_e(t)+P_a(t)}{V_0(t)+V_e(t)+V_a(t)} \right) : t \geq 0 \right\}. \]
The sub-model \(\{(P_\varepsilon(t), V_\varepsilon(t)) : t \geq 0\}\) can be decomposed into a sum of independent and identically distributed sub-models of the form (i). To this end, we denote by \(Q_{p,v}\) the distribution of the process (4.3) with initial state \((P(0), V(0)) = (p, v)\). Then,

\[
(P_\varepsilon(t), V_\varepsilon(t)) := \int_0^t \int \mathbb{D}([0, \infty), \mathbb{U}) \omega(t - s)N_\varepsilon(ds, d(p, v), d\omega)
\]

where \(N_\varepsilon(dt, d\omega)\) is a Poisson random measure on \((0, \infty) \times \mathbb{U} \times \mathbb{D}([0, \infty), \mathbb{U})\) with intensity \(ds\nu_\varepsilon(d(p, v))Q_{p,v}(d\omega)\).

The sub-model \(\{(P_a(t), V_a(t)) : t \geq 0\}\) can also be decomposed into self-enclosed sub-models where the volatility process evolves as an excursion process selected by a Poisson random measure. In order to make this more precise, we put \(\tau_0(\omega) := \inf\{t > 0 : \omega_2(t) = 0\}\) for any \(\omega(\cdot) := (\omega_1(\cdot), \omega_2(\cdot)) \in \mathbb{D}(\mathbb{R}_+, \mathbb{U})\), denote by \(\mathbb{D}_0(\mathbb{R}_+, \mathbb{U})\) the subspace of \(\mathbb{D}(\mathbb{R}_+, \mathbb{U})\) defined by

\[
\mathbb{D}_0(\mathbb{R}_+, \mathbb{U}) := \{\omega \in \mathbb{D}(\mathbb{R}_+, \mathbb{U}) : \omega(0) = (0, 0) \text{ and } \omega_2(t) = 0 \text{ for any } t \geq \tau_0(\omega)\},
\]

endowed with the filtration \(\mathcal{F}_t := \sigma(\omega(s) : s \in [0, t])\), for \(t \geq 0\), and by \((Q_{0,t}^a)_{t \geq 0}\) the restriction of the semigroup \((Q_{0,t})_{t \geq 0}\) on \(\mathbb{R} \times (0, \infty)\). The following result is proved in the appendix.

**Theorem 4.2** Let \(\sigma_{22} > 0\). There exists an \(\sigma\)-finite measure \(Q(d\omega)\)\(^7\) on \(\mathbb{D}([0, \infty), \mathbb{U})\) with support \(\mathbb{D}_0([0, \infty), \mathbb{U})\), and a Poisson random measure \(N_a(ds, d\omega)\) on \((0, \infty) \times \mathbb{D}_0([0, \infty), \mathbb{U})\) with intensity \(a_2d\omega Q(d\omega)\) such that the stochastic system \(\{(\hat{P}_a(t), \hat{V}_a(t)) : t \geq 0\}\) defined by

\[
(\hat{P}_a(t), \hat{V}_a(t)) := \int_0^t (a_1, 0)ds + \int_0^t \int_{\mathbb{D}_0([0, \infty), \mathbb{U})} \omega(t - s)N_a(ds, d\omega) \tag{4.6}
\]

is Markov with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) with \(\mathcal{F}_t := \sigma(N_a((0, s], U) : s \in [0, t], U \in \mathcal{G}_{t-})\) and has the same finite dimensional distributions as \(\{(P_a(t), V_a(t)) : t \geq 0\}\).

**Remark 4.3** Let us compare our decomposition result with that of Jiao et al. [41]. The variance process studied in their paper is a special case of (4.5). By considering the jumps larger than some threshold \(\hat{y} > 0\) as immigrations of a CB-process, they provided a decomposition for the variance processes as a sum of a truncated variance process \(V^{(y)}\) with jump threshold \(y\) and a sub-model of the form (4.4) with \(N_1(ds, du)\) replaced by a Poisson random measure with intensity \(V_s^{(y)}ds1_{u_2 \geq \hat{y}}\tilde{\nu}_1(du)\)\(^8\). Their decomposition offers a way to refine the sub-model (4.5), which is different to our cluster representation in Theorem 4.2. However, their truncated variance process \(V^{(y)}\) is not self-enclosed. Moreover, we believe that our decomposition results is economically more intuitive as it is based on the different origins of the jumps.

The preceding analysis shows that the sub-models \(\{(P_0(t), V_0(t)) : t \geq 0\}\) and \(\{(P_\varepsilon(t), V_\varepsilon(t)) : t \geq 0\}\) form the building blocks of the limiting market model. In what follows we analyse these two models in greater detail. Since each external shock is associated with a sub-model of the form (i) while the sub-model (iii) describes the entire impact of all shocks of insignificant magnitude we expect the jump dynamics to be different in nature. Specifically, we expect the former model to display short-lived jump clusters while we expect the latter one to display a uniform jump dynamics in the longer run.

### 4.1 The sub-model \(\{(P_0(t), V_0(t)) : t \geq 0\}\)

In this section we study the sub-model \(\{(P_0(t), V_0(t)) : t \geq 0\}\). By [44, Theorem 1.1], arguments given in [27], the volatility process \(\{V_0(t) : t \geq 0\}\) forms a continuous-state branching process. In the following subsection we recall some well-known properties of continuous-state branching process; we refer to the textbook of Li [49] for a comprehensive analysis of continuous-state branching processes.

---

\(^7\)We construct such a measure \(Q(d\omega)\) in (A.7).

\(^8\)We emphasize that the assumption of a Poisson arrival of exogenous shocks was made for mathematical convenience and to better distinguish the effects of Poisson and Hawkes arrivals. An extension to Hawkes arrivals is not difficult.
4.1.1 Continuous-state branching process

The Lamperti transform between continuous-state branching processes and spectrally positive Lévy processes shows that the volatility process \( V_0 \) tends to either 0 or \( \infty \) with probability one, i.e. \( \mathbb{P}\{\lim_{t\to\infty} V_0(t) \in \{0, \infty\}\} = 1 \) and \( \mathbb{P}\{\lim_{t\to\infty} V_0(t) \in 0\} = 1 \) if and only if \( b_2 \geq 0 \). Let

\[
\mathcal{T}_0 := \inf\{t \geq 0 : V_0(t) = 0\}
\]

be its first hitting time of 0. Since 0 is a trap for this process, \( V_0(t) = 0 \) for all \( t \geq \mathcal{T}_0 \) almost surely and hence \( \mathcal{T}_0 \) is usually referred to as the extinction time. Grey [27] provides a necessary and sufficient condition for the continuous-state branching process \( V_0 \) to extinct in finite time with positive probability in terms of the function

\[
G_0(x) := G(0, -x) := b_2 x + \sigma_{22} x^2 + \int_{\mathbb{U}} (e^{-u_2 x} - 1 + u_2 x) \hat{\nu}_1(du), \quad x \geq 0.
\]

Lemma 4.4 (Grey (1974)) For any \( t > 0 \), \( \mathbb{P}\{\mathcal{T}_0 \leq t\} > 0 \) if and only if there exists a constant \( \vartheta > 0 \) such that

\[
G_0(\vartheta) > 0 \quad \text{and} \quad \int_0^\infty \frac{1}{G_0(x)} \, dx < \infty.
\]

In this case, \( \mathbb{P}\{\mathcal{T}_0 \leq t\} = \exp\{-V(0) \cdot \vartheta t\} \), where \( \{\vartheta_t : t \geq 0\} \) is the minimal solution to the Riccati equation

\[
\frac{d}{dt} \vartheta_t = -G_0(\vartheta_t)
\]

with singular initial condition \( \vartheta_0 = \infty \). Moreover, \( \mathbb{P}\{\mathcal{T}_0 < \infty\} = \exp\{-V(0) \cdot \vartheta_\infty\} \) with \( \vartheta_\infty \) being the largest root of \( G_0(x) = 0 \), and \( \vartheta_\infty = 0 \) if and only if \( b_2 = G_0(0) \geq 0 \).

Remark 4.5 i) The integral condition (4.8) holds if, for instance, \( \sigma_{22} > 0 \). In the absence of jumps, this condition is also necessary.

ii) The case \( \sigma_{22} \) corresponds to the discrete benchmark model analyzed in Section 2. In this case,

\[
G_0(x) = \hat{\beta}_x + \int_{\mathbb{U}} (e^{-u_2 x} - 1) \hat{\nu}_1(du).
\]

He and Li [30, Theorem 3.1] provide an explicit representation of the first time a continuous-state branching process jumps with the jump size belonging to a given set. Specifically, within our setting let

\[
\tau_{A_0} := \inf\{t > 0 : V_0(t) - V_0(t-1) \in A_0\}
\]

where \( A_0 \) of \( \mathbb{R}_+ \) is a Borel subset with \( \hat{\nu}_1(\mathbb{R} \times A_0) < \infty \). Then,

\[
\mathbb{P}\{\tau_{A_0} \geq t\} = \exp \{ -V_0 \cdot u_t^{A_0}(\hat{\nu}_1(\mathbb{R} \times A_0)) \},
\]

where \( u_t^{A_0}(\lambda) \) is the unique nonnegative solution of

\[
\frac{d}{dt} u_t^{A_0}(\lambda) = \lambda - G_0(u_t^{A_0}(\lambda)), \quad u_0^{A_0}(\lambda) = 0.
\]

In what follows we provide a comprehensive analysis of the entire jump dynamics of the market model \( \{(P_0(t), V_0(t)) : t \geq 0\} \) with a particular emphasis on last jump times.

4.1.2 Jump distribution

Throughout this section we fix a set \( A \subset \mathbb{U} \) that satisfies \( \hat{\nu}_1(A) < \infty \) and denote by

\[
\mathcal{T}_A := \sup\{t \geq 0 : (\Delta P_0(t), \Delta V_0(t)) \in A\}
\]

the arrival time of the last jump of the price/volatility process whose magnitude belongs to the set \( A \). Moreover, for any \( T \in [0, \infty] \) and any nonnegative bounded function function \( g \) on \( \mathbb{U} \), let

\[
\mathcal{J}_A^g(T) := \int_0^T \int_A \int_0^{V_0(s-)} g(u) N_0(ds, du, dx)
\]
where \( \psi \) denotes the \( g \)-functional of jumps with magnitudes in \( A \) during the time interval \([0, T]\). The case \( g(u) \equiv 1 \) corresponds to the number of jumps with magnitudes in \( A \) during the time interval \([0, T]\), i.e.

\[
J_A(T) := J_A^1(T) = \#\{t \in [0, T] : (\Delta P_0(t), \Delta V_0(t)) \in A\}.
\]

Since the process \( \{V_0(t) : t \geq 0\} \) is conservative, i.e. \( \mathbb{P}\{\sup_{s \in [0, T]} V_0(s) < \infty\} = 1 \) we see that \( T_A < \infty \) if and only if \( J_A(\infty) < \infty \). Moreover, by definition,

\[
\{ T_A \leq r, \mathcal{F}_0 \leq t \} = \left\{ \int_r^t \int_A \int_0^{V_0(s)-} N_0(ds, du, dx) = 0, V_0(t) = 0 \right\}
\]

almost surely, and so

\[
\mathbb{P}\{ T_A \leq r, \mathcal{F}_0 \leq t \} = \mathbb{E} \left[ \exp \left\{ -\dot{\nu}_4(A) \cdot \int_r^t V_0(s) ds \right\} 1_{\{ T_A \leq r, V_0(t) = 0 \}} \right].
\]

The indicator function is inconvenient when computing the expected value. To bypass this problem we use a result from He and Li [30]. Under the assumptions of Lemma 4.4, the event \( \{ T_A \leq r \} \) has strictly positive probability for any \( r > 0 \). Conditioned on this event, the process \( \{V_0(t) : t \geq r\} \) almost surely equals the process \( \{V_0^A(t) : t \geq r\} \) defined by

\[
V_0^A(t) = V_0(r) - \int_r^t \left( b + \int_A u_2 \hat{\nu}_4(du) \right) V_0^A(s) ds + \int_r^t \sqrt{2 \sigma_{22} V_0^A(s)} dB_2(s)
+ \int_r^t \int_A \int_0^{V_0^A(s)-} u_2 N_0(ds, du, dx).
\]

By Theorem 2.2 in [13], \( V_0^A(t) \leq V_0(t) \) for all \( t \geq r \). In particular, under the conditions of Lemma 4.4, \( \mathbb{P}\{ V_0^A(t) = 0 \} > 0 \) for all \( t > r \). Using the fact that 0 is a trap for the volatility process, straightforward modifications of arguments given in the proof of [30, Theorem 3.1] show that for any \( t > r \),

\[
\{ T_A \leq r, \mathcal{F}_0 \leq t \} = \left\{ \int_r^t \int_A \int_0^{V_0^A(s)-} N_0(ds, du, dx) = 0, V_0^A(t) = 0 \right\}.
\]

Using the dominated convergence theorem, and the fact that \( \{ (V_0^A(t), \int_0^t V_0^A(w) dw) : t \geq r \} \) is an affine process,

\[
\mathbb{P}\{ T_A \leq r, \mathcal{F}_0 \leq t \} = \lim_{\lambda_0 \to \infty} \mathbb{E}_{\mathcal{F}_r}\left[ \exp \left\{ -\lambda_0 \cdot V_0^A(t) \right\} \int_r^t \int_A \int_0^{V_0^A(s)-} N_0(ds, du, dx) = 0 \right]
= \lim_{\lambda_0 \to \infty} \mathbb{E}_{\mathcal{F}_r}\left[ \exp\left\{ -\dot{\psi}_4(A) \int_r^t V_0^A(s) ds - \lambda_0 \cdot V_0^A(t) \right\} \right]
= \lim_{\lambda_0 \to \infty} \exp\left\{ -\dot{\psi}_4^A(\lambda_0) V_0^A(r) \right\},
\]

where \( \dot{\psi}_4^A(\lambda_0) : [0, \infty) \to \mathbb{R}_+ \) is the unique positive solution to the Riccati differential equation

\[
\psi^A_4(\lambda_0) = \lambda_0 + \int_0^t \left[ \hat{\nu}_4(A) - G_0^A(\dot{\psi}_4^A(\lambda_0)) \right] du,
\]

and the function \( G_0^A : \mathbb{R}_+ \to \mathbb{R} \) is defined by

\[
G_0^A(x) := G_0(x) - \int_A \left( e^{-xu_2} - 1 \right) \hat{\nu}_4(du);
\]

see Theorem 2.7 in Duffie et al. [14] for details. The following theorem shows that the distribution of the random variable \( (\mathcal{F}_0, T_A, J_A(T)) \) can be expressed in terms of the unique continuous solution to the Riccati equation (4.10) with singular initial condition, and the unique non-negative continuous solution to the Riccati differential equation

\[
\frac{d}{ds} \phi^A_s(x, g) = -G_0(\phi^A_s(x, g)) - \int_A \left[ e^{-g(u)} - 1 \right] \exp\left\{ -\phi^A_s(x, g) \cdot u_2 \right\} \hat{\nu}_4(du)
\]

with initial condition \( \phi^A_0(x, g) = x \) for \( x \geq 0 \).
Then Corollary 4.7 it follows from sup

From this and the uniqueness of the solution, we conclude that the ODE satisfies a comparison principle. In particular, \( \psi_t^A(\lambda_0) \) is increasing in \( \lambda_0 \) for any \( t \geq 0 \), and hence the limit \( \bar{\psi}_t^A := \lim_{\lambda_0 \to \infty} \psi_t^A(\lambda_0) \) exists in \([0, \infty)\). In order to prove that \( \bar{\psi}_t^A < \infty \) for any \( t > 0 \), we first conclude from (4.9) that

\[
\exp \left\{ - \bar{\psi}_{t-r}^A V_0(r) \right\} = \lim_{\lambda_0 \to \infty} E_{\mathfrak{E}_t} \left[ \exp \left\{ - \dot{\varphi}_t^A(\lambda_0) \int_r^t V_0^A(s) ds - \lambda_0 \cdot V_0^A(t) \right\} \right] = E \left[ \exp \left\{ - \dot{\varphi}_t^A \int_r^t V_0^A(s) ds ; V_0^A(t) = 0 \right\} \right].
\]

Since \( \{V_0^A(t) : t \geq r\} \) is conservative, so is \( \{V_0^A(t) : t \geq r\} \) and hence \( \int_r^t V_0^A(s) ds < \infty \). Thus, the expectation on the right side of the last equality is positive since \( \text{P} \{V_0^A(t) = 0\} > 0 \) for all \( t > r \). This shows that \( \bar{\psi}_{t-r}^A < \infty \) for any \( t > r \). Stochastic continuity of \( \{V_0^A(t) : t \geq r\} \) yields continuity of \( \bar{\psi}_t^A \) on \((0, \infty)\).

We now show that \( \{\bar{\psi}_t^A : t > 0\} \) solves (4.12) with singular initial condition. Using the semigroup property \( \psi_{t+s}^A(\lambda_0) = \psi_t^A(\psi_s^A(\lambda_0)) \) for any \( t, s > 0 \) and letting \( \lambda_0 \to \infty \) shows that \( \bar{\psi}_{t+s}^A = \psi_t^A(\psi_s^A) \), from which we deduce that \( t \mapsto \bar{\psi}_t^A \) is a continuous solution to (4.12). Moreover, for any sequence \( \{s_n : n \geq 1\} \) that satisfies \( s_n \to 0^+ \), it follows from \( \sup_{n \geq 1} \bar{\psi}_{s_n}^A \geq \sup_{n \geq 1} \psi_{s_n}^A(\lambda_0) = \lambda_0 \) that \( \lim_{t \to 0^+} \bar{\psi}_t^A = \infty \). Finally, the constructed solution is also the minimal solution in the class of continuous functions. Indeed, if \( \psi(\cdot) \) is another continuous solution to (4.12) with \( \psi(0^+) = \infty \), then the semigroup property yields \( \psi(t) = \lim_{s \to 0^+} \psi(t+s) = \lim_{s \to 0^+} \psi_t^A(\psi(s)) \geq \psi_t^A(\lambda_0) \) for any \( t > 0 \) and all \( \lambda_0 \geq 0 \), from which the assertion follows.

We finally prove (4.13). For any \( t > r \), it follows from (4.9) that

\[
E \left[ \exp \left\{ - \int_{\mathcal{T}} \mathcal{J}_A^T \right\} ; \mathcal{T}_A \leq t, \mathcal{T}_t \leq t \right] = E \left[ \exp \left\{ - \int_{\mathcal{T}} \mathcal{J}_A^T \times E_{\mathfrak{S}, \mathcal{T}_A} \left[ \exp \left\{ - \bar{\psi}_{t-r}^A V_0(r) \right\} \right] \right\} \right] = E \left[ \exp \left\{ - \int_{\mathcal{T} \land T} V_0^A(s) g(u) N_0(ds, du, dx) + \psi_{t-r}^G(0, -\bar{\psi}_{t-r}^A) \cdot V_0(r \land T) \right\} \right].
\]

Following the standard arguments given in, e.g. proof of [14, Theorem 2.12], we have

\[
\exp \left\{ - \int_{\mathcal{T} \land T} V_0^A(s) g(u) N_0(ds, du, dx) + \psi_{t-r}^G(0, -\bar{\psi}_{t-r}^A) \cdot V_0(r \land T) \right\} = \exp \left\{ - \phi_t^A \left( - \psi_{t-r}^G, (0, -\bar{\psi}_{t-r}^A), \lambda \right) \cdot V(0) \right\} + \text{Martingale}.
\]

Taking expectations on both sides of above equation, yields the desired result.

\[\square\]

**Corollary 4.7** For \( \lambda \geq 0 \), let \( \varphi_t^A(x, \lambda) \) be the unique non-negative continuous solution to (4.11) with \( g(u) \equiv \lambda \). Then

\[
E \left[ \exp \left\{ - \lambda \mathcal{J}_A(T) \right\} ; \mathcal{T}_A \leq r, \mathcal{T}_0 \leq t \right] = \exp \left\{ - \phi_t^A \left( - \psi_{t-r}^G, (0, -\bar{\psi}_{t-r}^A), \lambda \right) \cdot V(0) \right\}.
\]

From (4.8), we see that \( G_0^A(\infty) = \infty \). Hence, the right inverse \( G_0^{A^{-1}}(y) := \inf \left\{ x \geq 0 : G_0^A(x) > y \right\} \) of \( G_0^A \) is well defined. This allows us to obtain the distribution of the random variable \((\mathcal{J}_A(T), \mathcal{T}_A)\).

**Corollary 4.8** Suppose that (4.8) holds for some \( \vartheta > 0 \). As \( t \to \infty \), the function \( t \mapsto \bar{\psi}_t^A \) decreases to \( \bar{\psi}_\infty^A := G_0^{A^{-1}}(\dot{\varphi}_1(A)) \). Moreover, for any \( r \geq 0 \) and \( \lambda \geq 0 \),

\[
E \left[ \exp \{ -\vartheta \mathcal{J}_A(T) \} ; \mathcal{T}_A \leq r \right] = \exp \left\{ - \phi_t^A \left( - \psi_{t-r}^G, (0, -\bar{\psi}_{t-r}^A), \lambda \right) \cdot V(0) \right\}.
\]
Proof. From (4.14), for any \( \lambda \geq 0 \), we see that \( \phi^G_{r,T}(0, -\psi^G_{r,T}(0, -\bar{\psi}_t^A), \lambda) \) is decreasing in \( t \). Moreover, by continuity and the uniqueness of the solution to (4.11), the equation satisfies a comparison principle, and so the mapping \( x \mapsto \phi^G_t(x, 0) \) is increasing. As a result, the mapping \( t \mapsto -\psi^G_{r,T}(0, -\bar{\psi}_t^A) \) is decreasing. Moreover, from (3.33), 
\[
-\psi^G_{t}(0, -x) = x - \int_t^0 G_0(\psi^G_s(0, -x))ds,
\]
which is increasing in \( x \). The last two results imply that the mapping \( t \mapsto \bar{\psi}_t^A \) is decreasing. In particular, the limit \( \bar{\psi}_\infty^A := \lim_{t \to \infty} \bar{\psi}_t^A \) exists. Since \( \bar{\psi}_{t+s}^A = \phi^A_t(\bar{\psi}_s^A) \), as \( s \to \infty \) we have \( \bar{\psi}_\infty^A = \phi^A_t(\bar{\psi}_\infty^A) \). Taking this into (4.10) we conclude that \( \int_0^T \bar{\nu}_t(A) - G^A_0(\bar{\psi}_\infty^A)ds \equiv 0 \) and hence
\[
\bar{\psi}_\infty^A = G^A_0, \quad (\bar{\nu}_t(A)).
\]

The following corollary shows that the quantity \( b_2 \) determines whether or not the impact duration of an external shock is almost surely finite.\(^9\)

**Corollary 4.9** We have \( P\{T_A < \infty\} = P\{J_A(\infty) < \infty\} = \exp\{-V(0) \cdot \bar{\nu}_\infty\} \), where \( \bar{\nu}_\infty \) is the largest root of \( G_0(x) = 0 \). Moreover, \( P\{T_A < \infty\} = P\{J_A(\infty) < \infty\} = 1 \) if and only if \( b_2 = G_0'(0) \geq 0 \).

**Proof.** By equation (3.32) and Corollary 4.8, \( P\{T_A \leq r\} = \exp\{-\phi^A_t(\bar{\psi}_\infty^A, 0) \cdot V(0)\} \). This implies that \( \phi^A_t(\bar{\psi}_\infty^A, 0) \) decreases to some limit \( \phi^A_\infty(\bar{\psi}_\infty^A, 0) \) as \( r \to \infty \). From (4.11), we see that semigroup property holds for \( \phi^A_t(x, 0) \) for any \( x \geq 0 \) and hence \( G_0(\phi^A_\infty(x, 0)) = 0 \). We now show that \( \phi^A_\infty(x, 0) \equiv \bar{\nu}_\infty \). From (4.7), we see that \( G_0(x) = 0 \) and that \( G_0 \) is strictly convex. Hence, \( G_0(x) > 0 \) for any \( x > 0 \) if and only if \( b_2 = G_0'(0) \geq 0 \). In this case, \( \bar{\nu}_\infty = 0 \) is the only root of \( G_0(x) = 0 \) and \( P\{T_A < \infty\} = P\{J_A(\infty) < \infty\} = 1 \). If \( b_2 = G_0'(0) < 0 \), since \( G_0(x) \to \infty \) as \( x \to \infty \), there is only one positive root \( \bar{\nu}_\infty \) of \( G_0(x) = 0 \) and \( G_0(y) < 0 \) for \( y \in (0, \bar{\nu}_\infty) \). It suffices to prove \( \phi^A_\infty(x, 0) > 0 \), which follows directly from the fact that \( \phi^A_\infty(x, 0) > 0 \) continuously decreases to \( \phi^A_\infty(x, 0) > 0 \) as \( t \to \infty \).

Taking expectation on both sides of (4.3), we have \( E[V_0(t)] = V(0) \exp\{-b_2t\} \) for any \( t \geq 0 \). From the definition of \( J_A(T) \), we have
\[
E[J_A(T)] = E\left[ \int_0^T \int_A^{V_0(s)-} N_0(ds, du, dx) \right] = \hat{\nu}_t(A) \int_0^T E[V_0(s)]ds = \hat{\nu}_t(A) \cdot \frac{V(0)}{b_2} \cdot (1 - e^{-b_2T}).
\]

In particular, the expected number of jumps \( E[J_A(T)] \) converges as \( T \to \infty \) if and only if \( b_2 > 0 \). In this case the number of induced jumps of a given magnitude following an initially shock is finite and the expected number of shocks is proportional to the shock size. Specifically, we have the following corollary.

**Corollary 4.10** We have \( E[J_A(\infty)] < \infty \) if and only if \( b_2 > 0 \). In this case, \( E[J_A(\infty)] = \hat{\nu}_t(A) \cdot V(0)/b_2 \).

The following corollary is the analogue of Corollary 2.8. It provides four regimes for the impact duration of exogenous shocks on the market dynamics. If the volatility strictly mean-reverts to 0, then the impact decreases exponentially. In the critical case \( b_2 = 0 \), the impact decays only slowly.

**Corollary 4.11** We have the following four regimes:

1. If \( b_2 < 0 \), we have as \( t \to \infty \),
\[
P\{T_A > t\} \sim P\{T_0 > t\} \to 1 - \exp\{-V(0) \cdot \bar{\nu}_\infty\};
\]

2. If \( b_2 > 0 \), there exists a constant \( C > 0 \) such that for any \( t \geq 1 \),
\[
P\{T_A > t\} \leq P\{T_0 > t\} \leq C \cdot V(0) \cdot e^{-b_2t};
\]

3. If \( b_2 = 0 \) and \( \hat{\nu}_t(\{|u_2|^2\}) := \frac{1}{2} \int_0^T |u_2|^2 \hat{\nu}_t(du) < \infty \), we have \( t \to \infty \),
\[
P\{T_A > t\} \sim P\{T_0 > t\} \sim \frac{V(0)}{\sigma^2 + \hat{\nu}_t(\{|u_2|^2\}) \cdot t^{-1};
\]

\(^9\)In the discrete benchmark model \( b_2 = \beta \).
which implies that applying the nonlinear Gr"{o}nwall's inequality, we have

Similarly, we also have for any $\epsilon \in (0, \sigma_{22} + \nu_{4}(|u_{2}|)^{2})$,

This shows (3) as $\epsilon$ is arbitrary. For (4), from [10, Theorem 8.1.6], we have as $\lambda \to 0+$,

As before, we also have as $t \to \infty$,

We now prove the last two regimes for $P\{T_{A} > t\}$. From Corollary 4.8 and 4.9, we have as $t \to \infty$,

From (4.11), we also have

From this and the argument before, we also can prove that the last two regimes for $P\{T_{A} > t\}$. 

\[ \begin{array}{ll}
\end{array} \]
4.2 The sub-model \{((P_{a}(t), V_{a}(t)) : t \geq 0)\}

We now study the distribution of induced jumps in the system \{((P_{a}(t), V_{a}(t)) : t \geq 0)\} assuming that \(a_2 > 0\). The analysis is much simpler than the preceding one because now the volatility process is recurrent. For any \(T > 0\), let

\[ J^a_A(T) := \#\{t \in [0, T] : (\Delta P_{a}(t), \Delta V_{a}(t)) \in A\} = \int_{0}^{T} \int_{A} \int_{0}^{V_{a}(s^{-})} N_0(ds, du, dx). \]

Taking expectation on both sides of above equation, we see that the expected number of jumps is given by

\[ \mathbf{E}[J^a_A(T)] = \hat{\nu}_a(A) \int_{0}^{T} \mathbf{E}[V_{a}(s)] ds. \]

Taking expectation on both sides of (4.5), yields

\[ \mathbf{E}[V_{a}(t)] = \int_{0}^{t} (a_2 - b_2 \mathbf{E}[V_{a}(s)]) ds = \frac{a_2}{b_2} (1 - e^{-b_2 t}) \]

and so

\[ \mathbf{E}[J^a_A(T)] = \hat{\nu}_a(A) \cdot \frac{a_2}{b_2} \cdot [T - (1 - e^{-b_2 T})/b_2]. \]

**Lemma 4.12** For any \(T \in [0, \infty)\), we have

\[ \mathbf{E}\{\exp\{-\lambda J^a_A(T)\}\} = \exp\left\{-\int_{0}^{T} a_2 \psi_{a}(\lambda) ds\right\}, \]

where \(s \mapsto \psi_{a}(\lambda)\) is the unique solution to the following Riccati equation:

\[ \psi_{a}(\lambda) = \hat{\nu}_a(A) \cdot (1 - e^{-\lambda}) \cdot t - \int_{0}^{t} G_0(\psi_{a}(\lambda)) ds. \]

**Proof.** From the exponential formula of Poisson random measure; see [9, p.8], we have

\[ \mathbf{E}\{\exp\{-\lambda J^a_A(T)\}\} = \mathbf{E}\left\{\exp\left\{-\hat{\nu}_a(A) \cdot (1 - e^{-\lambda}) \cdot \int_{0}^{T} V_{a}(s) ds\right\}\right\}. \]

Since \(\{(\int_{0}^{t} V_{a}(s) ds, V_{a}(t)) : t \geq 0\}\) is affine, the result follows from [14, Theorem 2.7]. \(\square\)

**Remark 4.13** Let us compare exogenously and endogenously triggered jumps assuming that the volatility process strictly mean-reverts and that the arrival rate of exogenous shocks \(\lambda_a := \nu_{a}(U)\) is finite. The expected number of exogenously triggered jumps with magnitude in a set \(A\) triggered by a shock of size \(V(0)\) equals

\[ \mathbf{E}[J_A(T)] = \hat{\nu}_a(A) \cdot \frac{V(0)}{b_2} \cdot (1 - e^{-b_2 T}). \]

Loosely speaking, jumps triggered by an exogenous shock of unit size can be viewed as arriving at a constant rate \(\lambda_a \hat{\nu}_a(A)\) for an exponentially distributed (with parameter \(b_2\)) amount of time. Moreover, as

\[ \mathbf{E}[J_A(T)] = \hat{\nu}_a(A)V(0)T + O(T^2) \text{ for } T \to 0 \]

the expected number of shocks is linear over small time periods. By contrast, the expected number of endogenously triggered jumps satisfies

\[ \mathbf{E}[J^a_A(T)] = O(T^2) \text{ for } T \to 0 \quad \text{and} \quad \mathbf{E}[J^a_A(T)] = O(T) \text{ for } T \to \infty. \]

This shows that exogenously triggered jumps are on average more likely to occur shortly after the exogenous event and hence to cluster. By contrast, endogenously triggered jumps tend to arrive at constant rates in the longer run. Figure 3 illustrates this phenomena. The reason is that \(V_{a}(0) = 0\), which makes jumps close to the initial time very unlikely. In view of Theorem 4.2 the sub-model \{((P_{a}(t), V_{a}(t)) : t \geq 0)\} decomposes into self-enclosed sub-models where the volatility process evolves as an excursion process selected by a Poisson random measure. Whenever a new excursion “arrives” the probability of jumps increases and then decreases again when the excursion process starts reversing back to zero.
A  A cluster representation for \( \{(P_a(t), V_a(t)) : t \geq 0\} \)

This appendix proves Theorem 4.2. The proof uses arguments given in Li [49] where the result is established for the volatility process. We assume throughout that \( \sigma_{22} > 0 \) and start with the following simple but useful lemma.

Lemma A.1 For any two finite measures \( (u_2 \land 1)\nu_1(du) \) and \( (u_2 \land 1)\nu_2(du) \) on \( \mathbb{U} \setminus \{0\} \), we have \( \nu_1(du) = \nu_2(du) \) if and only if for any \( z = (z_1, z_2) \in \mathbb{U}_* \),

\[
\int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})\nu_1(du) = \int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})\nu_2(du).
\] (A.1)

Proof. We first extend \( \nu_1(du) \) and \( \nu_2(du) \) to \( \mathbb{U} \) with \( \nu_1(\{0\}) = \nu_2(\{0\}) = 0 \). From (A.1) with \( z \) replaced by \( z + (0,1) \), we have

\[
\int_{\mathbb{U}} (e^{(z,u)} - u_2 - e^{z_1u_1})\nu_1(du) = \int_{\mathbb{U}} (e^{(z,u)} - u_2 - e^{z_1u_1})\nu_2(du).
\] (A.2)

Taking the difference between (A.1) and (A.2), we have

\[
\int_{\mathbb{U}} e^{(z,u)}(1 - e^{-u_2})\nu_1(du) = \int_{\mathbb{U}} e^{(z,u)}(1 - e^{-u_2})\nu_2(du).
\]

By assumption \( (1 - e^{-u_2})\nu_1(du) \) and \( (1 - e^{-u_2})\nu_2(du) \) are finite measure on \( \mathbb{U} \). By the one-to-one correspondence between measures and their characteristic function, \( (1 - e^{-u_2})\nu_1(du) = (1 - e^{-u_2})\nu_2(du) \) and \( \nu_1(du) = \nu_2(du) \). \( \square \)

For any \( t > 0 \) and \( u' \in \mathbb{U} \), the probability measure \( Q_{0,t}(u', du) \) introduced in (4.2) is infinitely divisible, i.e., for any \( n \geq 1 \),

\[
\int_{\mathbb{U}} e^{(z,u)\eta_{0,t}(u', du)} = \left( \exp \left\{ z_1 \frac{u'_1}{n} + \psi_t^G(z) \frac{u'_2}{n} \right\} \right)^n = \int_{\mathbb{U}} e^{(z,u)\eta_{0,t}^{(sn)}(u'/n, du)}. \] (A.3)

Using the Lévy-Khintchine formula for infinite divisible distributions (see [9, Theorem 1]), the representation (3.33) for the characteristic exponent \( \psi_t^G(z) : z \in \mathbb{U}_* \) and Theorem 3.13 in [49] along with the assumption that \( \sigma_{22} > 0 \), we obtain that

\[
\psi_t^G(z) = b_1(t)z_1 + \sigma_{11}(t)|z_1|^2 + \int_{\mathbb{U} \setminus \{0\}} (e^{(z,u)} - 1 - z_1u_1)\eta(du), \] (A.4)

where \( b_1(t) \in \mathbb{R}, \sigma_{11}(t) \geq 0 \) and \( (|u_1| \lor |u_1|^2 + |u_2| \lor 1)\eta(du) \) is a finite measure on \( \mathbb{U} \setminus \{0\} \).

Lemma A.2 The family of \( \sigma \)-finite measures \( \{\eta_t(du) : t \geq 0\} \) is an entrance law for \( (Q_{0,t})_{t \geq 0} \), i.e.

\[
\eta_{t+s}(du) = \eta_t Q_{0,s}(du), \ \ \ s, t \geq 0.
\]

Proof. In view of Lemma A.1, it suffices to prove that for any \( s, t \geq 0 \)

\[
\int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})\eta_{t+s}(du) = \int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})\eta_t Q_{0,s}(du). \] (A.5)

From (A.4),

\[
\int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})\eta_t(du) = \int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})\eta_t(du) = \psi_t^G(z_1, 0) - \psi_t^G(z_1, z_2). \] (A.6)

Moreover, from the definition of \( Q_{0,s}(u', du) \) and (4.2), we have

\[
\int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})Q_{0,s}(u', du) = \int_{\mathbb{U}} (e^{(z,u)} - e^{z_1u_1})Q_{0,s}(u', du)
= \exp \left\{ z_1u'_1 + \psi_t^G(z)u'_2 \right\} - \exp \left\{ z_1u'_1 + \psi_t^G(z_1, 0)u'_2 \right\}.
\]
Taking this into the term on the right side of (A.5), we obtain from (A.6) that
\[
\int_U (e^{(z,u)} - e^{z_{1,u_1}}) \eta u_2 Q_{0,t} (du) = \int_U \eta (du') \int_U (e^{(z,u)} - e^{z_{1,u_1}}) Q_{0,t} (du', du) = \int_U [e^{z_{1,u_1} + \psi^G (z) u_2'} - e^{z_{1,u_1} + \psi^G (z_{1,0}) u_2}] \eta (du') = \psi^G (z, \psi^G (z_{1,0})) - \psi^G (z, \psi^G (z)).
\]

From the semigroup property of \((\psi_t^G)_{t \geq 0}\), we also have \(\psi_t^G (z_1, \psi^G (z_{1,0})) = \psi^G (z_1, \psi^G (z_{1,0})) = \psi^G (z_1, \psi^G (z)).\) Along with (A.6) this yields the desired result. \(\square\)

In view of the preceding lemma, we can define an \(\sigma\)-finite measure \(Q (d\omega)\) on \(D([0, \infty), \mathbb{U})\) as follows: for any \(0 < t_1 < t_2 < \cdots < t_n\) and \(u^{(1)}, \ldots, u^{(n)} \in \mathbb{R} \times (0, \infty),\)
\[
Q (\omega (t_1) \in du^{(1)}, \omega (t_2) \in du^{(2)}, \cdots, \omega (t_n) \in du^{(n)}) := \eta_{t_1} (du^{(1)}) Q_{0, t_2 - t_1} (u^{(1)}, du^{(2)}) \cdots Q_{0, t_n - t_{n-1}} (u^{(n-1)}, du^{(n)}).
\]

The following lemma provides the analogue of equation (3.17) in [49].

**Lemma A.3** For any \(t > 0\), we have \(\frac{1}{u_2} Q_{0,t} ((0, u_2'), du) \to \eta_t (du)\) as \(u_2 \to 0+\).

**Proof.** From (A.6) we have for any \(z := (z_1, z_2) \in \mathbb{U}_*\),
\[
\lim_{u_2 \to 0+} \int_U (e^{(z,u)} - e^{z_{1,u_1}}) \frac{1}{u_2} Q_{0,t} ((0, u_2'), du) = \lim_{u_2 \to 0+} \frac{1}{u_2} (e^{u_2 \psi^G (z)} - e^{u_2 \psi^G (z_{1,0})}) = \psi^G (z) - \psi^G (z_{1,0}) = \int_U (e^{(z,u)} - e^{z_{1,u_1}}) \eta_t (du).
\]
The same arguments as in the proof of Lemma A.1 yield
\[
\lim_{u_2 \to 0+} \int_U e^{(z,u)} (1 - e^{-u_2}) \frac{1}{u_2} Q_{0,t} ((0, u_2'), du) = \int_U e^{(z,u)} (1 - e^{-u_2}) \eta_t (du),
\]
and hence the desired result. \(\square\)

By Lemma A.3, we have formally that
\[
Q (\omega (t_1) \in du^{(1)}, \omega (t_2) \in du^{(2)}, \cdots, \omega (t_n) \in du^{(n)}) = \lim_{u_2 \to 0+} \frac{1}{u_2} Q_{0,t_1} (u^{(1)}, du^{(2)}) \cdots Q_{0, t_2 - t_1} (u^{(1)}, du^{(2)}) \cdots Q_{0, t_n - t_{n-1}} (u^{(n-1)}, du^{(n)}),
\]
which shows that \(Q (d\omega)\) is supported on \(D_0([0, \infty), \mathbb{U})\). We refer to the proof of Theorem 6.1 in [49] for further details.

**Proof of Theorem 4.2.** It suffices to prove that \((\hat{P}_a (t), \hat{V}_a (t))\) is a Markov process with transition semigroup \((Q_{a,t})_{t \geq 0}\) on \(\mathbb{U}\). For this, it is enough to prove that for any \(0 \leq t_1 \leq r \leq t_2\), any \(z = (z_1, z_2) \in \mathbb{U}_*\) and every \(\mathcal{E}_{t_1}\)-measurable \(\mathbb{C}\)-valued random variable \(X_{t_1}\),
\[
\mathbb{E} [X_{t_1} \cdot e^{z_{1,t} a} V_a (z)] = \mathbb{E} [X_{t_1} \cdot \exp \left\{ z_1 P_a (r) + \psi^G_{t_2-r} (z) V_a (r) + \int_0^{t_2 - r} [a_1 z_1 + a_2 \psi^G (z)] ds \right\}]. \quad (A.8)
\]

From the definition of \((\mathcal{E}_t)_{t \geq 0}\), we just need to prove this statement with
\[
X_{t_1} = \exp \left\{ \int_0^{t_1} \int_{D_0 ([0, \infty), \mathbb{U})} h (\omega) N_{a} (ds, d\omega) \right\},
\]
where \(h (\omega) : D([0, \infty), \mathbb{U}) \to \mathbb{C}_-\). From this and (4.6),
\[
\mathbb{E} \left\{ \exp \left\{ \int_0^{t_1} \int_{D_0 ([0, \infty), \mathbb{U})} h (\omega) N_{a} (ds, d\omega) + z_1 P_a (t_2) + z_2 V_a (t_2) \right\} \right\}
\]

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By the exponential formula for Poisson random measures, the last term equals

\[
\exp \left\{ a_1 t_2 z_1 + \int_0^{t_2} \int_{D_0([0,\infty],U)} \left[ h(\omega) \mathbf{1}_{\{s \leq t_1\}} + \langle z, \omega(t_2 - s) \rangle \right] N_a(ds,d\omega) \right\}.
\]

Moreover, from (A.7) and Lemma A.3 we have

\[
\int_{D_0([0,\infty],U)} \left[ \exp\{h(\omega)\mathbf{1}_{\{s \leq t_1\}} + \langle z, \omega(t_2 - s) \rangle\} - 1\right] Q(d\omega)
\]

which equals to

\[
E\left[ \exp \left\{ \int_0^{t_1} \int_{D_0([0,\infty],U)} h(\omega)N_a(ds,d\omega) + z_1 P_a(r) + \psi^G_{t_2-r}(z) V_a(r) \right\} \right].
\]

Taking these back into (A.9), we have

\[
E\left[ \exp \left\{ \int_0^{t_1} \int_{D([0,\infty),U)} h(\omega)N_a(ds,d\omega) + z_1 P_a(t_2) + z_2 V_a(t_2) \right\} \right]
\]

Here we have got the desired result (A.8).
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