Bank regulation under fire sale externalities

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Abstract

We examine the optimal design of and interaction between capital and liquidity regulations. Banks, not internalizing fire sale externalities, overinvest in risky assets and underinvest in liquid assets in the competitive equilibrium. Capital requirements can alleviate the inefficiency, but banks respond by decreasing their liquidity ratios. When capital requirements are the only available tool, the regulator tightens them to offset banks’ lower liquidity ratios, leading to fewer risky assets and less liquidity compared with the second best. Macroprudential liquidity requirements that complement capital regulations implement the second best, improve financial stability, and allow for more investment in risky assets.

Keywords: Bank capital regulation, liquidity regulation, fire sale externalities, Basel III

JEL Codes: G20, G21, G28.

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1 Introduction

The recent financial crisis led to a redesign of bank regulations. Prior to the crisis, capital requirements were the dominant tool of bank regulators around the world. Liquidity requirements for internationally active banks were always part of the discussion in the Basel Committee for Banking Supervision, but several factors delayed their introduction until recently. One main factor was the argument that capital and liquidity requirements are substitutes. It was believed that capital requirements would also address liquidity risk by creating incentives for banks to hold assets with lower risk weights, which should have better liquidity characteristics.\(^1\)

The crisis, however, revealed that even well-capitalized banks can experience a deterioration of their capital ratios due in part to illiquid positions (Brunnermeier, 2009). Without the unprecedented liquidity and asset price supports of leading central banks, liquidity problems faced simultaneously by several financial institutions could have resulted in a dramatic collapse of the financial system. The experience brought liquidity into the spotlight and provided the supervisory momentum to introduce harmonized liquidity regulations.\(^2\) As a result, a third generation of bank regulation principles, popularly known as Basel III, strengthens the previous Basel capital adequacy accords by adding macroprudential aspects and liquidity requirements such as the liquidity coverage ratio (LCR) and the net stable funding ratio.

Several countries, including the United States and the countries in the European Union, have already adopted Basel III liquidity requirements together with the enhanced capital requirements. However, guidance from the theoretical literature on the regulation of liquidity and the interaction between liquidity and capital regulations is quite limited, as also emphasized by Tirole (2011) and Bouwman (2012). This paper is one of the first attempts to fill this gap in the literature, and it makes two main contributions. First, we show that banks’ choices of capital and liquidity ratios in an unregulated competitive equilibrium are inefficient under fire sale externalities. Both ratios have distinct effects on the extent of fire sale risk and, hence, on the externalities that banks impose on each other. Therefore, we argue that implementing the second-best allocations in a decentralized economy requires regulating banks on both channels. In a more general setup, optimal regulation should target all independent choices that have a direct effect on the externality. Such regulation would also align the remaining unregulated choices with their efficient levels. In our model, both the liquidity and capital ratios interact directly

\(^1\)See Goodhart (2011) and Bonner and Hilbers (2015) for a review.
\(^2\)See Rochet (2008), Bouwman (2012), Stein (2013), Allen (2014), Claessens (2014), Tarullo (2014) and Bonner and Hilbers (2015) for recent discussions on the regulation of bank liquidity.
with the fire sale externality in an intuitive way, and thus are subjects of optimal bank regulation.

Second, the paper contributes to the literature by analyzing the interaction between capital and liquidity regulations in addressing this inefficiency. In particular, we uncover novel results on the effects of a capital-regulation-only regime on banks’ risk-taking and financial stability. We show that banks respond to tightening capital requirements by decreasing their liquidity buffers, a result consistent with the empirical evidence from several developed countries after the introduction of Basel I in 1988 and Basel II in 2004 (Bonner and Hilbers, 2015). Studying capital regulation alone is important because it represents the pre-Basel III era and thus is informative for understanding the development of systemic risk in that period.

We consider a three-period model in which a continuum of banks have access to two types of assets. Banks have to decide in the initial period how many risky and liquid assets to carry in their portfolio. We allow for a flexible balance sheet size so that banks can increase both their risky and liquid assets at the same time. The risky asset has a constant return but requires, with a known probability, additional investment in the future before collecting returns. This additional investment cost creates a liquidity need, which is proportional to the amount of risky assets on a bank’s balance sheet. The liquid asset provides zero net return; however, it can be used to cover the additional investment cost. If liquidity carried from the initial period is insufficient to offset the shock, banks’ only option is to sell some of their risky assets to firms in the traditional sector. This sell-off of risky assets takes the form of fire sales because firms in the traditional sector are less productive in managing the risky asset and their demand for risky assets is downward-sloping: The marginal product of each additional asset is lower under their management. Thus, traditional firms offer a lower price when banks try to sell a higher quantity of risky assets.

Atomistic banks do not take into account the effect of their initial portfolio choices on the fire sale price. If banks hold more risky assets, then they need more liquidity to cover the additional investment cost. As a result, there are more fire sales and a lower fire sale price. Similarly, smaller liquidity buffers lead to greater fire sales and a lower fire sale price. Given this setup, we compare the unregulated competitive equilibrium in which banks freely choose their capital and liquidity ratios to the allocations of a constrained planner. Though subject to the same contracting constraints, the constrained planner internalizes the effect of initial allocations on the fire sale price, whereas banks do not, leading them to overinvest in the risky asset (lower capital ratios) and underinvest in liquid assets. We investigate how the constrained efficient (second-best) allocations can
be implemented using quantity-based capital and liquidity regulations.

Our results indicate that the constrained efficient allocations can be achieved with joint implementation of capital and liquidity regulations (complete regulation). The regulation required is macroprudential because it targets the aggregate capital and liquidity ratios. Banks hold liquid assets for precautionary reasons because they can use these resources to protect against liquidity shocks. Liquidity has a social benefit as well: Higher aggregate liquidity leads to less-severe fire sales. However, banks fail to internalize this benefit of aggregate liquidity, which results in inefficiently low liquidity ratios when there is no regulation. Similarly, banks neglect the social aspect of capital ratios and end up choosing inefficiently low capital ratios in the competitive equilibrium.

We then use this model to study a regulatory framework with capital requirements alone, similar to the pre-Basel III episode, which we call partial regulation. In this setup, banks respond to the introduction of capital regulations by decreasing their liquidity ratios further below the already inefficiently low levels in the competitive equilibrium. If there is no regulation, banks choose a composition of risky and safe assets in their portfolio that reflects their privately optimal level of risk-taking. When the level of risky investment is limited by capital regulations, banks reduce the liquidity of their portfolio in order to get closer to their privately optimal level of fire sale risk. Thus, in a sense, banks’ responses constitute a counterforce to regulation. *Capital regulation improves financial stability by limiting aggregate risky investment, which in turn weakens banks’ incentives to hold liquidity because the marginal benefit of liquidity decreases with financial stability.* Given this counterforce, the regulator applies capital regulation stringently to offset banks’ lower liquidity ratios, reducing socially profitable risky investment. As a result, bank capital ratios under partial regulation are higher and risky investment is lower compared with the second-best allocation.

The aforementioned findings have important policy implications. The lack of complementary liquidity requirements leads to lower levels of bank liquidity and risky investments as well as more severe financial crises compared with the second best. We also show that the welfare and financial stability benefits of a liquidity requirement that supplements capital regulation are quantitatively substantial. Our results indicate that the pre-Basel III regulatory framework, with its focus on capital requirements, was ineffective in addressing systemic instability caused by fire sales and that Basel III liquidity regulations are a step in the right direction.

We continue after this section with a review of the related literature. After presenting our benchmark results in Sections 3 and 4, we explore quantitative implications of our model in Section 5. In Section 6, we consider an extension. In our benchmark model,
banks hold liquidity for precautionary reasons. In the extended model, we introduce a strategic motive to hoard liquidity, whereby banks consider the probability of surviving the crisis episodes and purchase the assets of failed banks at fire sale prices. We show that if such a motive is strong compared to the precautionary motive, banks may hold too much liquidity in the competitive equilibrium. In Section 7, we discuss assumptions and provide a summary of further extensions. Finally, Section 8 concludes. The online appendix contains the proofs, closed-form solutions of the model, and additional extensions.

2 Literature review

Even though capital regulations have been studied extensively on their own, we are aware of only a few papers that investigate the jointly optimal determination of capital and liquidity regulations. Kashyap, Tsomocos, and Vardoulakis (2014) investigate the effectiveness of several regulations in the presence of run risk and credit risk. Their paper does not consider fire sale externalities, and optimal regulation does not necessarily involve capital or liquidity regulations. In Walther (2016), the socially optimal outcome is to have no fire sales, whereas in our paper partial fire sales are optimal. Furthermore, unlike us, Walther does not study the implications of regulating only capital or liquidity on banks’ investment decisions and financial stability.

Some studies have pointed out the inefficiency of banks’ liquidity choices in laissez-faire equilibrium under market incompleteness, informational frictions, or externalities. In Bhattacharya and Gale (1987), Farhi, Golosov, and Tsyvinski (2009), and Calomiris, Heider, and Hoerova (2013) liquidity in the competitive equilibrium is suboptimally low compared with the second best, whereas in Allen and Gale (2004) and Arseneau, Rappoport, and Vardoulakis (2015), liquidity can be too low or too high compared with the second best. Therefore, these papers provide a rationale for the regulation of liquidity. Cifuentes, Ferrucci, and Shin (2005) and Perotti and Suarez (2011) have also argued for liquidity regulations to address systemic externalities.

Our paper is also related to the literature that features financial amplification and asset fire sales, which includes the seminal contributions of Fisher (1933), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Krishnamurthy (2003, 2010), and Brunnermeier and Pedersen (2009). Fire sales in our model are similar to those in Lorenzoni (2008), Korinek (2011), and Kara (2016). These papers show that under pecuniary externalities arising

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3 The authors consider the following regulations: deposit insurance, loan-to-value limits, dividend taxes, and capital and liquidity ratio requirements.
from asset fire sales, there exists overinvestment in risky assets. Relatedly, in He and Kondor (2016) there is overinvestment in risky assets in boom periods and underinvestment during recessions under pecuniary externalities. However, unlike our paper, none of these papers consider jointly optimal determination of risky investment levels and liquidity.

In our model fire sales are socially costly because assets are transferred from more productive to less productive agents, as suggested originally by Shleifer and Vishny (1992). If fire sales constitute only a transfer between equally productive agents, and hence do not imply a social deadweight loss, then limiting the risky investment or liquidity might be unnecessary or even harmful. Such fire sales lead to excessive liquidity holding in Acharya, Shin, and Yorulmazer (2011), too little debt and underinvestment in risky assets in Gale and Gottardi (2015), and underused deposits (versus equity) and overinvestment in risky assets in Gale and Yorulmazer (2017). In Stein (2012), banks, not internalizing the fire sale externalities, rely too much on short-term debt, a cheap form of financing, which in turn supports socially excessive lending.

Pecuniary externalities are categorized into two types by Davila and Korinek (2017): distributive externalities that are due to marginal rates of substitution of different agents not being equalized and collateral externalities that arise from market price affecting the value of collateral. In our case, banks are financially constrained and market incompleteness impedes the equalization of the marginal rate of substitutions. The resulting distributive externalities lead to overinvestment in risky assets and underinvestment in liquid assets.

In our framework, pecuniary externalities are the only source of inefficiency.\textsuperscript{4} The Pareto suboptimality due to pecuniary externalities is well known in the literature.\textsuperscript{5} Greenwald and Stiglitz (1986), for instance, show that pecuniary externalities by themselves are not a source of inefficiency but can lead to welfare losses when markets are incomplete or when there is imperfect information. If the markets were complete, there would be no reason for fire sales and the first-best world would be established, with no role for regulation.

\textsuperscript{4}We do not model agency or information problems that the literature has traditionally used to justify capital or other bank regulations.

\textsuperscript{5}The Pareto suboptimality of competitive markets when the markets are incomplete goes back at least to the work of Borch (1962). The idea was further developed in the seminal papers of Hart (1975), Stiglitz (1982), and Geanakoplos and Polemarchakis (1986), among others.
3 Model

The model consists of three periods, $t = 0, 1, 2$, along with a continuum of consumers and of banks, each with a unit mass. Bankers and consumers are risk neutral, and bankers consume only in period 2.

There are two types of goods, a consumption and an investment good (the liquid and illiquid assets). Consumers are endowed with $\omega$ units of consumption goods in each period. In addition to providing deposits for banks, each consumer owns a firm in the traditional sector, which we discuss in Section 3.1. Banks have two technologies: a storage technology and a technology that converts consumption goods into investment goods one-to-one at $t = 0$. Investment goods that are managed by a bank until the last period yield $R > 1$ consumption goods per unit, and they fully depreciate after the return is collected at $t = 2$. However, investment goods are risky, as they are subject to a restructuring shock at $t = 1$, which we discuss in detail below. Risky assets can be thought as mortgage-backed securities or a portfolio of loans to firms.

Banks choose at $t = 0$ how many risky assets to hold, denoted by $n_i$, and how many liquid (safe) assets, denoted by $b_i$, to put aside for each unit of risky assets. The total amount of liquid assets held by each bank is then $n_i b_i$, and $b_i$ can be interpreted as a liquidity ratio. Therefore, the total asset size of a bank is $n_i + n_i b_i = (1 + b_i) n_i$. On the liability side, each bank is endowed with $e$ units of equity. The fixed-equity assumption captures the fact that it is difficult to raise equity in the short term (see, for example, Almazan, 2002; Repullo, 2005; Dell’Ariccia and Marquez, 2006). Hence, each bank raises $l_i = (1 + b_i) n_i - e$ units of deposits at $t = 0$. We assume that each bank is a local monopsony in the deposit market so that consumers earn zero net expected interest from their deposits. This non-contingent debt is the only allowed contract between banks and consumers at the initial period, and, therefore, the asset markets are incomplete.

Because we are interested in studying the interactions between capital and liquidity, we endow banks with two independent choice variables: the amount of safe and risky assets. As a result, the bank size is not fixed in the model. To have a well-defined problem when the bank size is flexible, we introduce a nonpecuniary cost of operating a bank, captured by $\Phi((1 + b_i) n_i)$. Furthermore, similar to the ones imposed by Van den

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6We assume that the initial endowment of consumers is sufficiently large and that it is not a binding constraint in equilibrium.

7To simplify the exposition, we abstract from modeling the relationship between banks and firms. Instead, we assume that banks directly invest in risky projects. This assumption is equivalent to assuming that there are no contracting frictions between banks and firms, as more broadly discussed by Stein (2012).

8Deposits are simple long-term debt contracts that are to be repaid at $t=2$ and cannot be withdrawn early. Moreover, the composition of the liability side does not play any role in our model. All our results hold if banks are fully equity financed.
Heuvel (2008), Acharya (2003, 2009), and Davila and Korinek (2017), we assume that the operational cost is increasing in the size of the balance sheet, \( \Phi'(\cdot) > 0 \), and it is convex, \( \Phi''(\cdot) > 0 \), to ensure that there is an interior solution to the banks’ problem. We discuss the effect of this cost function on our results in detail in Section A.5 in the online appendix.

Investment and deposit collection decisions are made at time \( t = 0 \). The only uncertainty is about the risky asset and is resolved at the beginning of \( t = 1 \): The economy lands in good times with probability \( 1 - q \) and in bad times with probability \( q \). In good times, no bank is hit with restructuring shocks, and therefore no further action is taken. However, in bad times, the risky assets are distressed and have to be restructured, as in Holmstrom and Tirole (1998) and Lorenzoni (2008). If the restructuring cost—\( c \) per risky asset—is not paid, the risky investment is scrapped.

A bank can use its liquid assets, \( n_i^b \), to carry out the restructuring at \( t = 1 \). If the liquid assets are not sufficient, the bank needs external finance. However, we assume that banks cannot borrow the required resources from the household sector. Banks’ inability to raise further external financing at date 1 can be explained, for example, by a combination of debt-overhang and limited-commitment problems. The only way for banks to raise the necessary funds is to sell some of their risky assets to firms in the traditional sector. The sequence of events is illustrated in Figure 1.

### 3.1 Crisis and fire sales

Agents’ decisions at time \( t = 0 \) depend on their expectations regarding the events at time \( t = 1 \). Thus, applying a backward induction, we first analyze the equilibrium at the interim period. Note that if the good state is realized at \( t = 1 \), banks take no further action and obtain a total return of \( \pi_i^{Good} = Rn_i + b_i n_i \) at the final period, \( t = 2 \). We start with the

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9In Section A.1 in the online appendix we describe a general setting in which banks can pledge only a fraction of their returns in the final period to the lenders. We then derive the parameter region that gives rise to this basic setup in which the pledgeability constraint does not bind in the initial period but it does in the bad state of the interim period because of debt overhang.
problem of traditional firms in bad times, then we analyze the problem of banks.

### 3.1.1 Traditional sector

Firms in the traditional sector can buy investment goods from banks. They produce $F(y)$ units of consumption goods using $y$ units of investment goods purchased from banks. Let $P$ denote the market price of the investment good at $t = 1$ in the bad state.\(^\text{10}\) Each firm in the traditional sector takes the market price as given and chooses the amount of investment goods to buy, $y$, in order to maximize net returns from investment, $F(y) - Py$. The first-order condition of this problem, $F'(y) = P$, determines the traditional firms’ demand function for the investment good: $Q^d(P) ≡ F'(P)^{-1} = y$.

**Assumption 1 (Efficiency).** $F'(y) > 0$ and $F''(y) < 0$ for all $y ≥ 0$, with $R ≥ F'(0) > \nu ≡ qR(1 + c)/(R − 1 + q)$.

Under the Efficiency assumption, firms’ production technology is concave and thus yields a downward-sloping demand function for investment goods (see Figure 2). Firms are also less productive than banks at each level of investment goods employed due to $F'(0) ≤ R$.\(^\text{11}\) As a result, banks have to accept a price lower than the fundamental value, $R$, to sell any assets to them and accept even lower prices to sell more assets.\(^\text{12}\) In addition, we assume that $F'(0)$ is not too small to ensure that a limited fire sale does not decrease the price dramatically below $R$.

**Assumption 2 (Elasticity).** $\epsilon^d = \frac{\partial Q^d(P)}{\partial P} = \frac{F'(y)}{yF''(y)} < -1$ for all $y ≥ 0$.

Rewriting the assumption as $F'(y) + yF''(y) > 0$, it implies that banks’ proceeds from selling assets, $Py = F'(y)y$, are strictly increasing in the amount of assets sold, $y$.\(^\text{13}\)

**Assumption 3 (Regularity).** $F'(y)F'''(y) - 2F''(y)^2 ≤ 0$ for all $y ≥ 0$.

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\(^\text{10}\)The price of the investment good at $t = 0$ will be 1 as long as there is positive investment, and the price at $t = 2$ will be 0 because the investment good fully depreciates at this point.

\(^\text{11}\)The origins of this idea can be found in Williamson (1988) and Shleifer and Vishny (1992), who claim that some assets are industry-specific and, hence, less productive when managed by outsiders. Thus, transfer of assets from banks to outsiders via fire sales creates a deadweight cost.

\(^\text{12}\)A decreasing returns to scale technology for outsiders, as in the works of Kiyotaki and Moore (1997), Lorenzoni (2008), and Korinek (2011), arises if the industry-specific assets are heterogeneous. The traditional sector would initially purchase assets that are easy to manage, but as they continue to purchase more assets, they would need to buy those that require increasingly sophisticated management and operation skills.

\(^\text{13}\)Without this assumption, different levels of asset sales would raise the same level of funds, leading to multiple equilibria. This assumption is also imposed by Lorenzoni (2008), Korinek (2011), and Kara (2016) to rule out multiple equilibria under fire sales.
The *Regularity* assumption holds for log-concave demand functions implied by \( F(\cdot) \), yet it is weaker than log-concavity. We use this assumption to guarantee that the objective functions of banks and the planner are concave and yield interior solutions.\(^{14}\)

**Assumption 4 (Technology).** \( 1 + qc < R < 1/(1 - q) \).

The first inequality states that the net expected return on the risky asset is positive. The second inequality, \( R < 1/(1 - q) \), implies that scrapping investment in the bad state yields negative expected profits, and, thus, it is never optimal.

### 3.1.2 Banks’ problem in the bad state

Consider the problem of bank \( i \) when bad times are realized at \( t = 1 \). The bank has an investment level, \( n_i \), and liquid assets of \( b_i n_i \) chosen at the initial period. If \( b_i \geq c \), the bank has enough liquid resources to restructure all of the assets. However, if \( b_i < c \), then the bank does not have enough liquidity to cover the restructuring cost entirely and, thus, decides what fraction of these assets to sell \((1 - \gamma_i)\). The bank chooses \( \gamma_i \) to maximize total returns from that point on,

\[
\max_{0 \leq \gamma \leq 1} R\gamma_i n_i + P(1 - \gamma_i)n_i + b_i n_i - cn_i, \tag{1}
\]

subject to the budget constraint \( P(1 - \gamma_i)n_i + b_i n_i - cn_i \geq 0 \). Banks want to choose the highest possible \( \gamma_i \) because they receive \( R \) by keeping assets on the balance sheet, whereas by selling them they get \( P \leq R \).\(^{15}\) Therefore, banks sell just enough assets to cover their liquidity shortage and the budget constraint binds, which implies \( \gamma_i = 1 - (c - b_i)/P \). As a result, the fraction of investment goods sold by each bank is \( 1 - \gamma_i = \frac{c - b_i}{P} \). The supply of investment goods by each bank, \( i \), is then equal to

\[
Q_s^i(P, n_i, b_i) = (1 - \gamma_i)n_i = \frac{c - b_i}{P}n_i \tag{2}
\]

for \( c \leq P \leq R \). This supply curve is downward-sloping and convex, which is standard in the fire sale literature (see Figure 2, left panel). A negative slope implies that if there is a decrease in the price of assets, banks have to sell more assets in order to generate the resources needed for restructuring. We can substitute the optimal value of \( \gamma_i \) into (1) and write the maximized expected returns of banks in the bad state as \( \pi_i^{Bad} = R\gamma_i n_i = \ldots \)

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\(^{14}\)Please see Kara (2016) for a discussion on this assumption. Two examples that satisfy assumptions 1–3 are \( F(y) = R \ln(1 + y) \) and \( F(y) = \sqrt{y + (1/2R)^2} \).

\(^{15}\)In equilibrium the price must satisfy \( R \geq P > c \) and banks never scrap investment goods. We provide a proof of these claims in the proof of Proposition 1 in the online appendix.
Figure 2: Equilibrium in the fire sale market and comparative statics

\[ R(1 - \frac{c-b_i}{P})n_i \] for a given \( n_i \) and \( b_i \).\(^{16}\)

3.1.3 Asset market equilibrium at date 1

We consider a symmetric equilibrium where \( n_i = n \) and \( b_i = b \) for all banks. Therefore, the aggregate risky investment level is given by \( n \) and the liquidity ratio is given by \( b \). The equilibrium price of investment goods in the bad state, \( P \), is determined by the market clearing condition \( Q^d(P) - Q^s(P; n, b) = 0 \). This equilibrium is illustrated in the left panel of Figure 2. Note that the equilibrium price of the risky asset and the amount of fire sales at \( t = 1 \) are functions of the initial aggregate investment in the risky asset and the aggregate liquidity ratio. Therefore, we denote the fire sale price in terms of state variables as \( P(n, b) \).

Lemma 1. The fire sale price of a risky asset, \( P(n, b) \), is decreasing in \( n \) and increasing in \( b \). The fraction of risky assets sold, \( 1 - \gamma(n, b) \), is increasing in \( n \) and decreasing in \( b \).

When banks enter the interim period with larger holdings of risky assets, they have to sell more at each price. This shift in the supply, as shown in the right panel of Figure

}\(^{16}\)Depending on the parameters, the expected returns from the retained assets \( (R\gamma_i n_i) \) may not be sufficient to cover the promised return on deposits, in which case the bank becomes technically insolvent. We assume that insolvent banks are required to continue to manage assets until the final period and hand over the proceeds to consumers. In such a situation, banks have to pay a positive interest rate on deposits to satisfy consumers’ participation constraint, as explored explicitly in Section A.1 in the online appendix. In this setup, whether banks become insolvent after conducting fire sales has no effect on our results.
lowers the equilibrium price. A lower initial liquidity ratio also shifts the aggregate supply by increasing the liquidity shortage in the bad state, \((c - b)n\).

### 3.2 Competitive equilibrium

At the initial period, each bank \(i\) chooses the amount of risky asset \(n_i\) and the liquidity ratio \(b_i\) to maximize its expected profits,

\[
\Pi_i(n_i, b_i) = (R + b_i - qc)n_i - D(n_i(1 + b_i)) - I(b_i < c)q(R - P)Q_i^s(P, n_i, b_i),
\]

subject to its budget constraint, \(e + l_{i0} \geq n_i + b_in_i\), at \(t = 0\). Let \(\Gamma(n_i, b_i) \equiv (R + b_i - qc)n_i - D(n_i(1 + b_i))\) represent the basic profits that would be obtained if there were no fire sales. \(D(n_i(1 + b_i)) = n_i(1 + b_i) + \Phi(n_i(1 + b_i))\) is the sum of the initial cost of funds and the operational costs. Note that because consumers earn zero net expected return on their lending to banks, the cost of funds to a bank is \(e + l_{i0} = n_i(1 + b_i)\). The last term in (3) is the expected cost of fire sales: If liquidity hoarded at \(t = 0\) is not sufficient to cover the shock in the bad state at \(t = 1\)—that is \(b_i < c\)—then the bank sells \(Q_i^s(P, n_i, b_i)\) units of assets and loses \(R - P \geq 0\) on each unit sold.

Whether fire sales take place in equilibrium depends on the initial liquidity ratios. If banks fully insure themselves against the fire sale risk—that is, if they choose \(b_i \geq c\) at \(t = 0\)—then fire sales are avoided. However, the following lemma shows that in the competitive equilibrium, banks choose less than full insurance.

**Lemma 2.** Under the Efficiency and Technology assumptions, banks always take fire sale risk in equilibrium; that is, \(b_i < c\) for all banks.

Even though both the amount \((c)\) and frequency \((q)\) of the liquidity shock are exogenous, whether and to what extent a fire sale takes place are endogenously determined. In Lemma 2 we show that perfect insurance is never optimal. The intuition is as follows. The expected marginal return on liquid assets exceeds unity as long as there are fire sales. Perfect insurance guarantees that no fire sale takes place and, as a result, the expected marginal return on liquid assets is equal to 1, which is dominated by the return on risky assets. But then no bank would hoard liquidity, suggesting that we cannot have an equilibrium where \(b_i \geq c\). In other words, banks would not hoard any liquidity if there was no fire sale risk. Lemma 2 allows us to focus on the imperfect insurance case; that is, \(b_i < c\). We can write banks’ profit function under this result as

\[
\Pi_i(n_i, b_i) = \Gamma(n_i, b_i) - q(R - P)Q_i^s(P, n_i, b_i).
\]
The unique symmetric equilibrium in which \( n_i = n^c \) and \( b_i = b^c \) for all banks is determined by the first-order conditions of banks’ and traditional firms’ problems and market clearing at date 1:

\[
\frac{\partial \Gamma}{\partial x_i} - q(R - P)\frac{\partial Q^p_i}{\partial x_i} = 0, \quad \forall x_i \in \{n_i, b_i\},
\]

\( F'(y) = P \)

\( y = Q^*(P, n, b) \).

We first show that the competitive equilibrium price, \( P \), is independent of the functional form of the traditional sector’s demand and the operational cost of banks.

**Proposition 1.** Under the Efficiency, Elasticity, Regularity, and Technology assumptions, the competitive equilibrium price of assets is given by

\[
P^c = \frac{qR(1 + c)}{R - 1 + q}.
\]

The equilibrium price, \( P^c \), is increasing in the probability of the liquidity shock, \( q \), and the size of the shock, \( c \), but decreasing in the return on the risky assets, \( R \).

Proposition 1 shows that the price of assets in the bad state is positively related to the expected size of the liquidity shock, \( qc \). When banks expect to incur a smaller additional cost for the investment, or when they face this cost with a lower probability, they increase risky investment and decrease liquidity buffers, as we show in the next proposition. As a result, when a crisis hits, there are more fire sales and a lower price for risky assets in equilibrium.

### 3.2.1 A closed-form solution for the competitive equilibrium

Because we are interested in comparing equilibrium values of \( n \) and \( b \) to the constrained efficient allocations, we need functional form assumptions.\(^{17}\) For analytical convenience, suppose that the operational cost of a bank is given by \( \Phi(x) = \phi x^2 \). On the demand side, suppose that the traditional sector’s return function is given by \( F(y) = R \ln(1 + y) \). We will refer to these two functional form assumptions as the “log-quadratic assumptions” and clarify whenever a result is obtained under these assumptions. However, in Section A.5 in the online appendix we show that our results hold under assumptions 1–4, and “log-quadratic assumptions” are used only for closed-form solutions. Proposition 2 presents the comparative statics for the competitive equilibrium.

\(^{17}\)This necessity is discussed in detail in Section A.5 in the online appendix.
Proposition 2. Under the log-quadratic functional form assumptions, the comparative statics for the competitive equilibrium risky investment level, \( n^c \), and liquidity ratio, \( b^c \), are as follows:

1. The risky investment level \( (n^c) \) is increasing in the return on the risky asset \( (R) \) and decreasing in the size of the liquidity shock \( (c) \), the probability of the bad state \( (q) \), and the marginal cost parameter \( (\phi) \).

2. The liquidity ratio \( (b^c) \) is increasing in the return on the risky asset \( (R) \), the size of the liquidity shock \( (c) \), and the probability of the bad state \( (q) \), and it is decreasing in the marginal cost parameter \( (\phi) \).

Proposition 2 shows that \( b^c \) and \( n^c \) increase (decrease) simultaneously as a response to an increase (decrease) in \( R \), which is possible thanks to the flexible bank balance sheet size. Proposition 2 implies that banks act less prudently by increasing risky investment and reducing liquidity when they expect financial shocks to be less frequent (lower \( q \)) or less severe (lower \( c \)). This in turn leads to more severe disruption to financial markets through lower asset prices and more fire sales if shocks do materialize, as shown by Proposition 1.

3.3 Constrained planner’s problem

A constrained planner is subject to the same market constraints as the private agents. In particular, the planner takes the borrowing constraints of banks in the bad state as given. However, unlike banks, the constrained planner takes into account the effect of initial portfolio allocations on the price of assets in the bad state. The constrained planner maximizes the expected profits of banks subject to a constraint that, after the transfers, consumers are at least as well off as they are in the competitive equilibrium.

The planner makes these compensatory transfers between banks and consumers to ensure that reallocation of resources leads to a Pareto improvement. We assume that transfers happen only in good times and in the final period—that is, when the pledge-ability constraint of banks does not bind. We denote these transfers by \( T_2 \). Crucially, the planner cannot use transfers to circumvent the financial constraints of bankers at date 1.

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18This result is reminiscent of the financial instability hypothesis of Minsky (1992, p. 8), who suggests that “over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system.”

19The generic inefficiency results in Geanakoplos and Polemarchakis (1986) and Greenwald and Stiglitz (1986) require a rank condition to hold, which ensures that there are as many independent goods to be taxed or subsidized as the number of agents that need to receive or provide compensation. In more recent papers, such as Lorenzoni (2008) and Davila and Korinek (2017), the constrained planner is endowed with lump-sum transfers to compensate the agents who lose from reduced fire sales, and these transfers function inherently in the same way that the rank condition allows compensation in more general models.
in the bad state. Hence, the planner solves the following optimization problem:

\[
\max_{n, b, y} \Gamma(n, b) - I(b < c)q(R - P)Q^s(P, n, b) - (1 - q)T_2, \quad (9)
\]

subject to

\[
y = Q^s(P, n, b), \quad \text{and } F'(y) = P;
\]

\[(1 - q)T_2 + 3\omega + q[F(y) - Py] \geq U_i^c. \quad (10)
\]

The last constraint states that consumers’ utility must be at least as much as \(U_i^c\), their expected utility in the competitive equilibrium. The term \(q[F(y) - Py]\) represents consumers’ expected profits from fire sales. After the planner has determined allocations and transfers at \(t = 0\), private agents follow their optimal strategies. The next lemma addresses the question of whether the constrained planner would avoid fire sales completely by setting \(b \geq c\).

Lemma 3. Under the risk neutrality, Efficiency, and Technology assumptions, it is optimal for the constrained planner to take fire sale risk; that is, the constrained optimal liquidity ratio satisfies \(b^* < c\).

The intuition of Lemma 2 also applies here: Holding liquidity is optimal if and only if fire sale risk exists. Full insurance is socially excessive because it ensures that there is no fire sale risk, and thus the marginal benefit of liquidity is zero, whereas its opportunity cost is always positive. Both the private and social risk-return trade-off leads to taking some fire sale risk, although at differing degrees. Lemma 3 allows us to focus on the \(b < c\) case when analyzing the constrained planner’s problem. We can simplify the optimality conditions for planner’s problem to

\[
\frac{\partial \Gamma}{\partial x} - q(R - P) \frac{\partial Q^s}{\partial x} - q(R - P) \frac{\partial Q^s}{\partial P} \frac{\partial P}{\partial x} = 0, \quad \forall x \in \{n, b\}. \quad (11)
\]

We denote the constrained efficient allocations by \(n^s, b^s\) and the associated price of assets in the bad state by \(P^s\), where the superscript “s” stands for the second best.

These first-order conditions are similar to the first-order conditions of the banks’ problem, shown in (5), except that each condition contains an additional term: \(-q(R - P) \frac{\partial Q^s}{\partial P} \frac{\partial P}{\partial x}\) for \(x \in \{n, b\}\). This wedge arises because, unlike the individual banks, the constrained planner takes into account how initial choices affect the price of assets, \(P\), and, hence, the amount of assets sold to the traditional sector, \(Q^s\). In other words, the planner internal-

\[\text{To keep the exposition of the model simple, we model the transfers in the good state at } t=2. \text{ The transfer could be made ex-ante at date 0 as well. What is important here is not the timing of transfers but the fact that the planner cannot use transfers to sidestep the contracting problems between private parties in the bad state when banks are constrained. In Section A.4 in the online appendix we extensively discuss when the transfers are needed and how these transfers relate to real world examples.} \]
izes the fact that larger risky investments or lower liquidity ratios lead to a lower asset price and more fire sales in the bad state. We can show that the competitive equilibrium is constrained inefficient under some general conditions, and we compare the competitive equilibrium level of risky assets and liquidity ratios with the constrained efficient allocations.

**Proposition 3.** Under the risk neutrality, **Efficiency, Elasticity, and Technology** assumptions, the competitive equilibrium is constrained inefficient. Furthermore, under the log-quadratic functional form assumptions, competitive equilibrium allocations compare to the constrained efficient allocations as follows:

1. **Risky investment levels:** $n^c > n^s$
2. **Liquidity ratios:** $b^c < b^s$

Proposition 3 shows that in the competitive equilibrium, unregulated banks overinvest in the risky asset, $n^c > n^s$, as in Lorenzoni (2008) and Korinek (2011). We built upon these models by adding a liquidity choice, which provides banks with an option to insure against the fire sale risk. Nevertheless, Proposition 3 shows that banks inefficiently insure against liquidity shocks by holding too low liquidity, $b^c < b^s$. Hence, this result suggests that both capital and liquidity choice margins of banks are distorted under fire sale externalities.

### 3.4 Implementing the constrained efficient allocations: Complete regulation

The constrained efficient allocations $(n^s, b^s)$ can be implemented using simple quantity regulations—in particular, by imposing a minimum liquidity ratio as a fraction of risky assets ($b_i \geq b^s$) and an upper limit on risky investment ($n_i \leq n^s$). The latter corresponds to a minimum risk-weighted capital ratio—that is, $e/n_i \geq e/n^s$—because of the fixed inside equity of banks. For analytical convenience, we use the upper bound on risky investment formulation for capital regulation in the rest of the paper.\(^{21}\)

The quantity-based rules can be mapped to the capital and liquidity regulations in the Basel III accord. First, the risk-weighted capital ratio, $e/n_i$, corresponds to the Basel definition, as it gives liquid assets, $n_i b_i$, a zero risk weight while giving risky assets, $n_i$, a weight of 1 in the denominator. In reality, banks carry several risky assets on their balance sheet for which Basel Accords require different risk weights. However, introducing assets with different risk profiles to our setup would complicate the analysis without adding further insight.

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\(^{21}\)The constrained efficient allocations can also be implemented using Pigouvian taxation instead of quantity-based rules. In this case, introducing two linear Pigouvian taxes, one for risky investment and one for the liquidity ratio, will be sufficient.
Second, our liquidity regulation is similar to the LCR requirement proposed in Basel III. The LCR requires banks to hold high-quality liquid assets against the outflows expected in the next 30 days under a stress scenario. In our setup, the expected cash outflow in the bad state is the liquidity need, $c$, per each risky asset. Therefore, the liquidity requirement can be written as $b_i n_i/cn_i \geq b_s n_s/cn_s$. It is true that the LCR focuses on liquidity shocks on the liability side, whereas we consider liquidity shocks on the asset side. However, this modeling choice is not essential to our result; all we need is a liquidity requirement in some states of the world that cannot be fully met by raising external financing. If we instead model liquidity shock as a proportion of deposits, we would then need capital regulation to limit the size of deposits and liquidity requirement to increase the high-quality liquid assets.

4 Partial regulation: Regulating only capital ratios

The liquidity requirement was missing in the pre-Basel III era. In order to understand the implications for both the banks and regulators, in this section we consider an economy in which the capital ratios of banks are regulated but there is no requirement on their liquidity ratios. This setup also allows us to study the interaction of banks’ capital and liquidity ratios and answer the following questions: What happens to banks’ liquidity when their capital ratios are regulated? Do banks manage their liquidity in an efficient way, or does capital regulation distort their choice of liquidity? Moreover, the lack of liquidity regulation impacts the stringency of optimal capital regulation. This setup allows us to compare the optimal capital ratios with and without supporting liquidity regulation.

We consider the problem of a planner who is endowed with only one tool. In particular, the planner moves first and chooses the level of risky investment, $n$, at $t=0$ but allows banks to freely choose their liquidity ratio, $b_i$. In this sequential setup, the planner anticipates how banks will set their liquidity ratios for a given regulatory limit on risky investment and incorporates banks’ responses when selecting the optimal risky investment level. As in the previous section, the planner is subject to the same contracting constraints as the private agents but internalizes the fire sale externalities. Because banks choose inefficiently high levels of risky investment in the competitive equilibrium, the planner wants to limit them with regulation.\textsuperscript{22} Therefore, an upper bound set by the planner on risky investment is going to be binding for banks and hence will implement the risky investment choice of the planner. We call this case a “partially regulated economy” and compare it to the competitive equilibrium and second-best allocations. We start by studying the banks’

\textsuperscript{22}We prove this claim formally in the next section.
problem. For a given regulatory upper bound on investment level, \( n \), banks set \( n_i = n \) and choose the liquidity ratio, \( b_i \), to maximize their expected profits:

\[
\max_{b_i} \Pi_i(b_i; n) = \max_{b_i} (R + b_i - qc)n - D(n_i(1 + b)) - q(R - P)Q^s(P, n, b_i).
\]  

(12)

From the first-order condition with respect to \( b_i \), we can obtain the banks’ reaction function to the regulatory investment level as follows:

\[
b_i(n) = D^{-1}(1 - q + qR/P) - 1.
\]  

(13)

The planner takes this reaction function into account while choosing the risky investment level. The resulting constrained optimization problem can be formalized as follows:

\[
\max_{n, y} \Gamma(n, b(n)) - q(R - P)Q^s(P, n, b(n)) - (1 - q)T_2,
\]

subject to \( y = Q^s(P, n, b(n)) \), and \( F'(y) = P \),

\[
\frac{d\Pi_i(b_i; n)}{db_i} = 0,
\]

\[(1 - q)T_2 + 3\omega + q[F(y) - Py] \geq U^c_i.
\]

We can simplify the condition for the planner’s choice \( n \) to:

\[
\frac{\partial \Gamma}{\partial n} + \frac{\partial \Gamma}{\partial b} b'(n) - q(R - P) \left( \frac{\partial Q^s}{\partial n} + \frac{\partial Q^s}{\partial b} b'(n) \right) - q(R - P) \frac{\partial Q^s}{\partial P} \frac{dP}{dn} = 0.
\]  

(14)

We denote the optimal risky investment level that solves the first-order condition (14) by \( n^p \), the associated optimal liquidity choice of banks under partial regulation by \( b^p \), and the price of assets in the bad state by \( P^p \), where the superscript “\( p \)” stands for partial regulation. Changing \( n \) has a direct and an indirect effect on the fire sale price. The direct effect is due to the amount of fire sales being proportional to initial investment in the risky asset, while the indirect effect stems from banks changing their liquidity ratios in response to a change in \( n \). In the following proposition we characterize banks’ response to a tightening of capital regulations.

**Proposition 4.** Let the operational cost of a bank be given by \( \Phi(x) = \phi x^2 \). Then, banks decrease their liquidity ratio as the regulator tightens capital requirements; that is, \( b'(n) \geq 0 \) for any concave technology function for the traditional sector, \( F(\cdot) \), that satisfies the Elasticity and Regularity assumptions along with either

(i) \( F'(0) = R \), or (ii) \( F'(0) \leq R \) and \( R < \frac{F'(F'+yF'')}{F'+2yF''} \) for all \( y \geq 0 \).

In Proposition 4, the regulator attempts to correct banks’ excessive risk-taking by re-
quiring a higher risk-weighted capital ratio. However, because this regulation prevents banks from reaching their privately optimal level of risk, they react by reducing their liquidity ratios. In other words, banks undermine the capital regulation by shifting risk from the regulated channel to the unregulated channel through less-liquid portfolios. It would not be surprising to observe banks holding fewer liquid assets after being asked to decrease their risky asset holdings. However, what is stated in Proposition 4 goes further: Banks also decrease their liquidity ratios—that is, banks hoard less liquidity per unit of risky asset.

The mechanics of this risk shifting occurs as follows. When the regulator raises the capital requirement, all banks invest less in the risky asset \((\text{n})\), and the fire sale price rises. As the regulation limits the aggregate risky investment in equilibrium, banks correctly anticipate this increase in the fire sale price. However, a higher price reduces the marginal benefit of liquidity per risky asset, \((\text{R/P} - 1)\), and hence, the banks' response is to decrease their liquidity ratios.\(^{23}\)

An analogy from automobile safety regulations provides further intuition. Peltzman (1975) and Crandall and Graham (1984) show that whether regulations such as safety belts and airbags reduce the fatality rate depends on the response of drivers to the increased protection. They provide empirical evidence that drivers do indeed increase their driving intensity as a response to safety regulations, resulting in a less-than-expected reduction in fatality rates. Similarly, in our setup, capital regulations intend to make the financial system safer, but individual banks respond by taking on more risk in the liquidity channel.

The result in Proposition 4 is also consistent with empirical evidence from the United States and several European countries during the implementation of Basel capital regulations. Bonner and Hilbers (2015) show that banks' capital ratios increased significantly between the Basel I proposal in 1988 and its final implementation in late 1992, whereas liquidity ratios declined over the same period. Similar negative correlations between capital and liquidity ratios were observed in some countries following the introduction of Basel II in 2004. These negative correlations are not observed over longer horizons when capital is not regulated tightly. Therefore, the authors conclude that tightening capital regulation is correlated with declining liquidity buffers due to banks shifting risk from one channel to another, similar to what we show in Proposition 4.

\(^{23}\)Note that a planner endowed with only capital regulation tools would be willing to tolerate some reduction in liquidity when all banks decrease their risky investment level because there is a substitution between capital and liquidity ratios from the planner’s perspective. A higher capital ratio, by increasing the fire sale price of assets and hence making the system safer, reduces the dependence on liquidity. But the banks’ reduction in liquidity in response to the introduction of a capital requirement goes further because they do not internalize that lower liquidity ratios lead to a lower fire sale price. If banks had acknowledged the price decrease, they would have reduced liquidity less.
Because Proposition 4 is a key result of our paper and plays a crucial role in understanding the results in the next section, we further clarify the mechanism behind this result. The proof of Proposition 4 relies on banks’ profit function exhibiting increasing differences in $b_i$ and $n$, which is the case if the cross partial derivative $\left( \frac{\partial^2 \Pi(n,b_i)}{\partial b \partial n} \right)$ is positive. We obtain the cross partial derivative of banks’ expected profit as

$$\frac{\partial^2 \Pi(n,b_i)}{\partial b \partial n} = \left\{ (1 - q) + qR \left( \frac{1}{P} - \frac{n}{P^2} \frac{\partial P}{\partial n} \right) - 1 \right\} - \Phi''(n(1 + b_i)).$$ (15)

The terms in braces can be simplified as $q \left( \frac{R}{P} - 1 \right) - qR \frac{n}{P^2} \frac{\partial P}{\partial n}$ and are positive with minimal requirements: $P$ is less than $R$ and $\frac{\partial P}{\partial n}$ is negative, as shown in Lemma 1. In other words, the fire sale price should be less than the fundamental value and decrease with the amount of risky investment. Thus, the only condition necessary on the cost function is that it should not be too steep, if it is convex. For instance, a fixed balance sheet size assumption, i.e. $\Phi''(n(1 + b_i)) = \infty$, would mechanically overturn the result. The proof would also be complete when the nonpecuniary cost is assumed away or when any linear ($\Phi''(x) = 0$) or concave ($\Phi''(x) < 0$) cost function is assumed. Thus, the current functional form ($\Phi''(x) > 0$) in fact makes $b'(n) > 0$ only harder to obtain. This relationship is driven purely by how the risky investment size and liquidity ratio interact through the fire sale market: A lower investment increases the fire sale price, which then translates into a lower marginal benefit of liquidity for banks. The conditions in the proposition are thus stricter than needed and assumed only for consistency with the closed-form solutions used in the remaining propositions. The requirements on the cost function and the generality of our other results are further discussed in Section A.5 in the online appendix.

4.1 Complete versus partial regulation: Do we need liquidity requirements?

In this section we investigate whether capital regulation alone can restore the second-best allocations. For this reason, we compare the equilibrium outcomes in three different settings: a decentralized equilibrium without any regulation, a partially regulated economy in which there is only capital regulation, and a complete regulation (second best) case that has both capital and liquidity regulations. Proposition 5 summarizes the results.

**Proposition 5.** Under the log-quadratic functional form assumptions, risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium, partial regulation equilibrium, and second best compare as follows:

1. Risky investment levels: $n^c > n^s > n^p$
2. Liquidity ratios: $b^s > b^c > b^p$
3. Financial stability measures

(a) Price of assets in the bad state: $P^s > P^p > P^c$

(b) Fraction of assets sold: $1 - \gamma^c > 1 - \gamma^p > 1 - \gamma^s$

(c) Total fire sales: $(1 - \gamma^c)n^c > (1 - \gamma^p)n^p > (1 - \gamma^s)n^s$

Proposition 5 highlights how, among the three regimes, the partial regulation is the harshest in terms of limiting the risky investment while being the least liquid regime at the same time. First, we show that the investment in risky assets under partial regulation is not only lower than the competitive equilibrium level—that is, $n^p < n^c$—but also lower compared to the second best: $n^p < n^s$. That is, if capital regulation is introduced in isolation, it must subject banks to a more stringent requirement than the constrained efficient level. Second, under partial regulation banks choose to become even less liquid than they were in competitive equilibrium ($b^p < b^c$). Note that in Proposition 3 we established that competitive equilibrium is characterized with too little liquidity. As a result, partial regulation liquidity ratios are lower compared with the second-best level as well.

Partial regulation features lower investment in both liquid and risky assets compared with complete regulation. These two results are intimately related, and they are both driven by how banks respond to a capital requirement, as shown in Proposition 4. When capital regulation limits the risky investment, banks choose less-liquid portfolios, which partially offsets the positive impact of the reduction in risky investment. The precautionary behavior of the regulator is then to implement the capital regulation in a more restrictive way, which increases the fire sale price but leads to a lower level of risky investment compared with the second best. In other words, endowed with only one tool and anticipating that banks would undermine the regulation, the regulator excessively uses this single tool. In contrast, in the second best, the planner can control both ratios and, hence, raises the total welfare by using a more balanced combination of liquid and risky assets. Higher liquidity ratios under complete regulation allow banks to hold more risky assets without increasing the fire sale risk.

In order to see the interaction between the capital and liquidity requirements, consider the following scenario: A country transitions from partial regulation to complete regulation by imposing new liquidity rules in addition to existing capital rules. This transition can be compared to moving from the Basel I/II regulatory approach to the Basel III regulatory approach. Assuming that capital regulation had been set optimally during the pre-Basel III period, capital requirements can be relaxed after the introduction of liquidity requirements. Therefore, our results would predict that more long-term profitable risky investments can be financed via the banking system after the implementation of liquidity requirements.

21
Given that capital regulation is costly, as it limits risky investment, what are the associated financial stability benefits? How effective is capital regulation in addressing financial instability when applied in isolation, without accompanying liquidity requirements? To answer these questions, we can compare the measures of financial instability across the two regulatory regimes. More fire sales and a lower price of the risky asset in the bad state are associated with greater financial instability. Proposition 5 shows that the introduction of capital regulation in isolation increases the fire sale price compared to the competitive equilibrium price. However, the price is still below the constrained optimal price level. The message is the same when we compare both the fraction and the total amount of risky assets that must be sold under the two regulatory regimes, as shown in items 3-b and 3-c in Proposition 5. In general, minimum capital requirements may serve several other purposes, such as countering moral hazard problems generated by the existence of limited liability and deposit insurance, that we do not analyze in this paper. However, what we show here is that, under fire sale externalities, capital regulations are not effective in achieving the second-best allocations unless they are combined with liquidity requirements. Furthermore, in Section 5 we show that the quantitative benefits of additional liquidity regulation are also substantial.

Our results indicate that neither capital nor liquidity ratios alone are perfect predictors of potential instability: A better-capitalized banking system may end up conducting larger fire sales. Under partial regulation, for instance, although the capital ratios are higher than under complete regulation, more fire sales take place when the shock hits. Similarly, a more-liquid banking system may experience greater financial instability: Banks are more liquid in the unregulated competitive equilibrium compared with partial regulation, but they end up conducting more fire sales and obtain a lower price for risky assets in the former. We end this section by comparing bank sizes across three different regimes.

**Proposition 6.** Under the log-quadratic functional form assumptions, bank balance sheet sizes across different regimes compare as follows: \( n(1 + b) = n^* (1 + b^*) > n^p (1 + b^p) \)

Proposition 6 shows that bank size in the competitive equilibrium is equal to the socially optimal size. However, bank size is smaller under partial regulation, as there are both less risky and less liquid assets in this regime compared with the constrained optimum. Proposition 6 also shows that the optimal simple leverage ratio, \( \frac{b^*}{n^*(1+b^*)} \), is the same under the second best and unregulated competitive equilibrium. Therefore, in the current setup, a leverage regulation in the traditional sense that puts a lower limit on this ratio and that is applied in isolation would be ineffective.\(^24\) However, an unorthodox lever-

\(^{24}\)Nevertheless, leverage ratio regulation might be an important method of addressing other market failures, such as
age regulation that puts an upper bound—rather than a lower bound—on the leverage ratio would be sufficient to replicate the constrained social optimum when combined with a capital regulation.\textsuperscript{25}

4.2 Can regulating only liquidity be the solution?

In our model, fire sales are triggered by a restructuring shock in the bad state. Banks are solvent as long as they can cover this liquidity requirement, because the return on the risky asset ($R$) is greater than the cost of restructuring ($c$). Therefore, one may wonder if the second-best allocations can be implemented using liquidity regulation alone—that is, without using capital requirements at all. The short answer is no. First, note that, in Lemma 3, we show that it is not optimal to avoid fire sales completely by forcing banks to perfectly insure against the liquidity shock by setting $b = c$. Second, regulating only liquidity means that banks are free to choose their capital ratios. The question then becomes whether banks choose the optimal capital ratio when the minimum liquidity requirement is set optimally.

**Proposition 7.** Under the Efficiency, Elasticity, Regularity, and Technology assumptions, banks do not choose the constrained optimal risky investment level, $n^s$, if the regulator sets the minimum liquidity ratio at the constrained optimal level, $b^s$; that is, $n_i(b^s) \neq n^s$.

In fact, in the proof of the proposition we show that banks choose higher than the second-best level of risky investment; that is, $n_i(b^s) > n^s$, or, equivalently, banks choose lower capital ratios compared with the second best. Therefore, the second-best allocations cannot be implemented by regulating liquidity alone. Banks can take on the fire sale risk through both liquidity and capital channels. As a result, implementing the second best requires restraining banks on both channels. Otherwise, banks use the unregulated channel to take more risk, undermining the regulator’s intent to eliminate the inefficiency.\textsuperscript{26}

\textsuperscript{25}This result is obtained simply because, for a fixed level of equity capital ($e$), the unorthodox leverage requirement imposes a lower limit on the bank size—that is, $n_i(1 + b_i) \geq n^s(1 + b^s)$ for all banks. If this rule is combined with a capital regulation ($n_i \leq n^s$), it implies a minimum liquidity regulation—that is, $b_i \geq b^s$.

\textsuperscript{26}In reality, there are many other choices that banks make. Then, should optimal regulation target all of these choices? The insight that comes from our model is that it depends. A bank’s choice should be regulated only if it affects the fire sale price directly. Regulating these choices will make sure that the remaining unregulated choices are aligned with their optimal levels as well.
5 Quantitative impact of additional liquidity regulation

In this section we explore the quantitative benefits of a liquidity requirement that supplements capital regulation within the scope of our benchmark model.\footnote{A few recent studies have focused on the quantitative effects of liquidity requirements as a standalone regulation or when combined with capital regulation. In De Nicoló, Gamba, and Lucchetta (2012) and Covas and Driscoll (2014), liquidity regulations only reduce lending and welfare, whereas, in some other studies, liquidity requirements increase welfare, such as by lowering the likelihood of systemic distress without reducing consumption growth in Adrian and Boyarchenko (2013) and by increasing the credit quality in Boissay and Collard (2016). These quantitative studies impose regulatory constraints and study their implications, whereas in our paper optimal regulatory constraints emerge endogenously to correct for specific market failures. Van den Heuvel (2018) quantifies the welfare costs of capital and liquidity requirements in a neoclassical growth model.} We show that the additional welfare and financial stability benefits of liquidity requirements are substantial. We establish this result first by providing a numerical example based on a reasonable set of parameters and second by reporting the distribution of quantitative benefits and associated summary statistics based on a wide range of parameters.

In our illustrative numerical example, we set our model period to be two years so that the total model length is four years. We set the expected return on the risky investment to be $R = 1.25$, which means that the annual return is 5.7 percent ($1.057^4 = 1.25$). We let the probability of the bad state be $q = 0.25$ so that a crisis is expected to occur every 16 years ($4/0.25$). We choose the magnitude of the liquidity shock to be $c = 0.1$, which means that once a crisis hits, banks have to invest an additional 10 percent to keep the risky asset productive. Lastly, we choose the marginal operating cost parameter $\phi = 0.01$, a small number, and set the initial equity of banks to $e = 1$, without loss of generality.

Table 1: Numerical example

|                                | Competitive equilibrium | Partial regulation | Constrained efficient alloc. |
|--------------------------------|-------------------------|--------------------|-----------------------------|
| Price of risky asset (P)       | 0.688                   | 0.695              | 0.815                       |
| Total profits                  | 1.0922                  | 1.0928             | 1.1018                      |
| Bank profits                   | 1.046                   | 1.048              | 1.077                       |
| Traditional sector’s profits   | 0.0462                  | 0.0447             | 0.0249                      |
| Risky investment (n)           | 9.81                    | 9.58               | 9.69                        |
| Liquidity ratio (b)            | 0.043                   | 0.042              | 0.055                       |
| Bank size (n(1+b))             | 10.227                  | 9.988              | 10.227                      |
| Fraction of assets sold        | 0.08341                 | 0.08337            | 0.05504                     |
| Total amount of fire sales     | 0.82                    | 0.80               | 0.53                        |

Table 1 collects the equilibrium values of interest from three different economies: competitive equilibrium, partial regulation, and the constrained planner’s solution. When we move from competitive equilibrium to partial equilibrium by introducing capital regula-
tions alone, we observe that the price of risky assets and the total profits barely increase (1.06 percent and 0.05 percent, respectively). Having additional liquidity requirements provides a much bigger impact: The price increases further, by 17.3 percent, while total profits also increase by an additional 0.82 percent.

This numerical example shows that capital regulations are not very effective when they are used without a complementary liquidity regulation. The impact of additional liquidity regulation on the price (total profits) is 16 times (15 times) higher compared with the impact of capital regulation alone. Significantly, this large figure is not dependent on the particular parameters we choose. We obtain substantial quantitative benefits for complementary liquidity regulation when we make similar comparisons in a large parameter space. In particular, we calculate the equilibrium for a Cartesian product of the parameters—that is, when \((R, c, q) \in [1.05, 2] \times [0.05, 0.95] \times [0.05, 0.95]\), while setting \(\phi = 0.01\). For each parameter combination, we solve the equilibrium for all three economies and compare the quantitative benefits of the transitions from competitive equilibrium to partial regulation and from partial regulation to complete regulation. In this exercise we compare the changes in prices, which generally mimic the changes in total profits.

**Figure 3:** Quantitative benefits of additional liquidity requirements

(a) Distributions of \(\frac{P_{p} - P_{c}}{P_{c}}\) and \(\frac{P_{s} - P_{p}}{P_{p}}\)

(b) Distribution of \(\frac{P_{s}}{P_{p}}\)

Figure 3a shows the distribution of the improvement in prices in percentages from the competitive equilibrium to partial regulation (small dark region in the bottom left) and from partial regulation to complete regulation (light color). This exercise indicates
that, on average, capital regulation alone increases the price less than 1 percent, and the maximum impact is 3.3 percent. An additional liquidity ratio regulation increases the price 7.6 percent on average and up to a maximum of 45 percent for some parameters. As is evident in Figure 3a, no combination of parameters produces a benefit from regulating only capital ratios that is close to the benefits of complete regulation.

Figure 3b better captures the relative improvement in the price. Figure 3b plots the distribution of the ratio of the respective price improvements in the Cartesian set of parameters. The figure indicates that the increase in price due to additional liquidity requirements is, on average, 24 times higher than that using capital regulations alone. There is a parameter space in which the additional impact of a liquidity requirement can be very small; however, these are exactly the same parameters for which the impact of capital regulation is also small. Thus, the ratio of the impact of complete regulation over the impact of only capital regulation is never small: The minimum is 2.8, and the maximum is 45. Furthermore, extending the upper end of the parameter range of $R$ shifts this distribution to the right, making the impact of additional liquidity regulation strictly larger.28

The numerical example also confirms that, without transfers, more regulation makes consumers worse off. As can be seen in Table 1, profits in the traditional sector decrease somewhat when we introduce capital regulations in isolation and decrease significantly further when we add liquidity requirements. However, bank profits always increase with regulation and increase in absolute value more than the traditional sector’s profits decrease. As a result, total profits increase with regulation.

6 Would banks overhoard liquidity?

In our benchmark model, banks hold liquidity for precautionary reasons: to meet a potential liquidity need and, hence, to reduce their exposure to fire sale risk. In doing so, however, banks do not internalize pecuniary externalities and end up holding “too little liquidity” in equilibrium. This result is in contrast to a recent line of literature that focuses on a strategic motive for holding liquidity and shows that banks may hold “too much liquidity” in equilibrium (see, for example, Acharya, Shin, and Yorulmazer, 2011).29

A strategic motive emerges if in a crisis state some banks fail and the remaining banks have excess liquidity to purchase failing banks’ assets at fire sale prices. A social plan-

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28Using a smaller range ([0.05, 0.5]) for $q$ and $c$ reduces to mean of the ratio only to 19, which is still large.

29We should note, however, that our discussion is limited to fire sale related channels. A broader literature shows that banks may hold inefficiently high or low levels of liquidity due to various other mechanisms (see, for example, Bhattacharya and Gale, 1987; Allen and Gale, 2004; Farhi, Golosov, and Tsyvinski, 2009). Also, here we focus on ex-ante liquidity choices only. There is also a recent line of literature that focuses on “liquidity hoarding” as the crisis unfolds in stages (see, for example, Diamond and Rajan, 2011; Gale and Yorulmazer, 2013).
ner would not hold liquidity for strategic reasons, as it does not create any social value. Assets are only transferred from failing banks to equally efficient surviving banks. Thus, banks’ liquidity holdings are excessive from the planner’s perspective.

The precautionary and strategic motives are studied in isolation in the literature. In this section we extend our benchmark model by combining these different channels in one model. To model distinct precautionary and strategic motives, we make two changes to the benchmark model. First, we allow shocks to be idiosyncratic, so that banks that are not hit by the shock can purchase the risky assets from the shock-receiving banks. As before, the good state arises with probability $1 - q$ and a crisis state with probability $q$. However, in the crisis state, each bank gets a shock with probability $\Lambda$. Second, we allow the shock to be one of the two types: bad or ugly. Conditional on the crisis state, the shock is the bad type with probability $\lambda_b$ and the ugly type with $1 - \lambda_b$ probability. The bad shock is what we have in the benchmark model, where shock-receiving banks have to pay $cn_i$ to save their risky assets. They can use their liquidity holdings or sell their risky assets at fire sale prices to cover the liquidity shortage. Hence, banks have a precautionary reason to hoard liquidity for the bad state. The ugly shock causes banks to default and all of their risky assets to be sold, similar to Acharya, Shin, and Yorulmazer (2011), Gale and Gottardi (2015), and Gale and Yorulmazer (2017).³⁰ Importantly, in the ugly state, defaulting banks’ liquidity holdings do not help them in any way, so liquidity has no precautionary benefit. Thus, we isolate the strategic motive—the opportunity to purchase the assets of failing banks at a deep discount—as the sole motive for holding liquidity for the ugly state.

From a social point of view, whether banks hold inadequate or excessive liquidity depends on the relative likelihood of the bad and ugly states, captured by $\lambda_b$. In Figure 4 we plot banks’ and the planner’s optimal choices of liquidity ratio for $\lambda_b \in (0, 1)$.³¹ We see that for low values of $\lambda_b$ banks’ liquidity ratio is higher than that of the planner’s because, in this region, the ugly state is more likely and the strategic motive dominates. For larger values of $\lambda_b$, the precautionary motive dominates, and the liquidity ratio in the competitive equilibrium falls below the constrained efficient level, as in our benchmark model.

We should note two properties regarding the strategic channel. First, unlike the precautionary channel, it does not necessarily rely on pecuniary externalities: fire sales would

³⁰To keep the exposition simple, we assume that default happens due to an exogenous reason, but it is possible to internalize default by, for example, assuming that depositors demand early withdrawal in the ugly state.

³¹We obtain this graph by numerically solving for the competitive equilibrium and planner’s problem under log-quadratic assumptions for the following parameters: $R = 1.5; q = 0.2; \Lambda = 0.5; c = 0.2; \phi = 0.01$ for each value of $\lambda_b$ between $[0.01, 0.99]$ with 0.01 increments. In Section A.6 in the online appendix we describe the banks’ and planner’s problems in this extended setup.
still constitute a transfer between banks even if the fire sale price is constant, which eliminates any pecuniary externality. Second, for the strategic channel to operate, more liquid banks should not be less likely to default. If we believe that more liquid banks are less likely to default in a given state, then the liquidity holdings become precautionary in addition to being strategic, and the implications of a strategic channel do not apply. We circumvent this issue by writing a model where these channels appear in different states of the world, while Acharya, Shin, and Yorulmazer (2011) avoid modeling the precautionary channel by assuming that bank defaults are always independent of their liquidity holdings.

7 Discussion of assumptions and extensions

We devote this section to a summary of some extensions provided in the online appendix. Here we also provide a general a discussion of our assumptions, welfare criteria, and transfers used in the constrained planner’s problem. We provide more in-depth discussion of these issues in the online appendix.

In the benchmark model banks are not allowed to raise funds in any form once the shock hits. However, this strict assumption is for brevity and for keeping the model tractable. With this simplification our goal is to capture banks’ inability to attract further funds—even if these funds are available in the economy—because banks may be facing information asymmetry or debt overhang problems. In Section A.1 of the online appendix, we explicitly model how limited commitment and debt-overhang problems can prevent banks from raising external finance at the interim date. An ex-post liquidity provision policy can also help banks in limiting fire sales. In Section A.2 we introduce
a lending facility (similar to discount window lending) by the central bank from which banks can borrow funds when they need liquidity in the interim period, at a penalty interest rate. In such a setup, we show that banks would still hoard liquidity ex-ante and conduct fire sales ex-post. This availability of ex-post liquidity would alter banks ex-ante decisions and decrease their incentives to hold liquidity. However, as long as banks hold some liquidity to shield their risky assets from fire sales, their liquidity will be lower than the planner’s, because when banks decide how much liquidity to hold ex-ante they will fail to internalize the social benefit of liquidity (higher fire sale price). Thus, although these external funding sources can mitigate fire sales, banks’ liquidity (and risky asset) choices will remain inefficient from a social perspective.

Bank regulations might fall short of providing the expected financial stability if fire sale externalities extend to a broader financial system. To discuss this, in Section A.3 we introduce a new group of financial institutions identical to banks in our model, but not regulated. We show that regulations on capital and liquidity make the financial system more stable by increasing the fire sale price, which in turn creates incentives for unregulated institutions to invest more in risky assets and to decrease their liquidity buffer. This result suggests that if we regulate only some institutions, unregulated institutions that engage in similar investment behavior will free ride on the improved stability and, hence, undermine the existing regulations. Therefore, similar to Farhi, Golosov, and Tsyvinski (2009), we argue that efficient regulations should have a wide scope and apply to all relevant financial institutions.

The constrained efficiency concept we use in this paper relies on the planner making compensating transfers from banks to households. In Section A.4 we clarify issues related to transfers and welfare criteria. We discuss questions such as: What form do these transfers take in the real world? When would transfers not be required? Briefly, decreasing fire sales using capital and liquidity regulations hurts firms in the traditional sector, the fire buyers. This distributional aspect is a general feature of models that involve a social deadweight cost for fire sales, in the spirit of Shleifer and Vishny (1992). If the planner does not care about the welfare of fire buyers (as in the private interest theory of bank regulation), then fire buyers do not need to be compensated. If, instead, one takes a public interest approach to bank regulation, then they need to be compensated directly or indirectly (through transfers or by subsidizing their consumption). However, banks’ profits always increase with regulation and increase in absolute value more than the traditional sector’s profits decrease. As a result, total profits increase with regulation. Thus, it is possible to implement capital and liquidity regulations in a Pareto-improving way by taxing fire buyers.

See Kroszner and Strahan (2001) for a discussion of alternative approaches to regulation.
banks and transferring resources to consumers.

Finally, in Section A.5 we clarify the generality of our results and the role of functional form assumptions. We can show numerically that all propositions hold under assumptions 1 to 4. However, to prove these propositions analytically we need closed-form solutions, which we obtain under log-quadratic functional forms. Closed-form solutions are needed because when banks have two or more independent choice variables it is generally not possible to compare equilibrium allocations between the competitive equilibrium and the planner’s solutions by analyzing only the first-order conditions. In Section A.5, we first establish this necessity formally. Second, we discuss how our results are robust to using other functional forms. Third, we discuss why we need an operational cost assumption, how important this assumption is for our results, and how our model can accommodate non-convex cost functions.

8 Conclusion

In this paper, we investigate the optimal design of bank regulation and the interaction between capital and liquidity requirements. Our model is characterized by fire sale externalities, because atomistic banks do not take into account the effect of their initial portfolio choices on fire sale prices. In an unregulated competitive equilibrium, banks overinvest in the risky asset and underinvest in the liquid asset compared with a constrained planner’s allocations. We investigate whether the constrained efficient allocations can be implemented using quantity-based capital and liquidity regulations, as in the Basel Accords.

Our results indicate that the pre-Basel III regulatory framework, with its reliance on capital requirements alone, was ineffective in addressing systemic instability caused by fire sales. Capital requirements can lead to less severe fire sales by forcing banks to reduce risky assets. However, we show that banks respond to stricter capital requirements by decreasing their liquidity ratios. Anticipating this response, the regulator preemptively sets capital ratios at high levels. Ultimately, this interplay between banks and the regulator leads to lower levels of risky assets and liquidity compared with the second-best allocations. Liquidity requirements that complement capital regulations, as in Basel III, can implement the second best, improve financial stability, and allow for a higher level of investment in risky assets.

It is important to highlight that our results cannot be interpreted as indicating that the actual capital regulation requirements were too high in a particular country (such as the United States) in the pre-crisis period, which corresponds to the pre-Basel III framework, and that current capital requirements should be relaxed. Our results only say that if cap-
ital regulations were set optimally in the absence of liquidity regulation, they would be set at higher levels compared with the second-best.

Beyond bank regulation, our results imply that capital ratios are not a perfect predictor of the stability of the banking system or any individual bank under a potential distress scenario. Without sufficient liquidity buffers, banks’ capital can easily erode with fire sale losses. Under fire sale externalities, a well-capitalized banking system may experience greater losses than a less-capitalized banking system with strong liquidity buffers. Thus, capital ratios alone cannot measure the soundness of individual banks or of a banking system.

The Basel III liquidity ratio LCR currently applies to only large banks in the United States. In contrast, our results suggest that liquidity regulations should also apply to small banks because in our model all banks are as small by definition. Whether liquidity regulations should be applied differently to large and small banks and whether they should be applied differently to well-capitalized and poorly capitalized banks is beyond the scope of our paper. We leave these interesting theoretical and policy questions to future research.
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Bank regulation under fire sale externalities

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Online Appendix

A  Extensions and further discussions

A.1  Debt overhang and limited commitment

In our benchmark model we assume that banks cannot raise external funding when they receive the liquidity shock. In this section we discuss how a debt overhang problem can prevent banks from raising external financing at the interim date. We assume that there is limited liability for banks and banks can only pledge a fraction $\alpha \in \{\alpha_H, \alpha_L\}$ of their revenues in the final period to lenders. The fraction that can be pledged is higher in good times than bad times, that is, $\alpha_H > \alpha_L$. $1 - \alpha$ can be interpreted as a haircut for banks’ assets. Haircuts tend to rise during times of financial distress, as documented by Shin (2008) and Gorton and Metrick (2012). We set $\alpha_H = 1$ without loss of generality and assume that $\alpha_L < 1$.

Banks’ problem at $t = 0$ in the decentralized equilibrium is:

$$\max_{n_i, b_i, \gamma_i, l_{i0}, l_{i1}, r_i} (1 - q)[(R + b_i)n_i - r_i l_{i0}] + q \max\{R\gamma_i n_i - r_i l_{i0}, 0\} - e - \Phi((1 + b_i)n_i), \quad (16)$$

subject to

$$l_{i0} \leq (1 - q)(R + b_i)n_i + \alpha_L q R\gamma_i n_i \quad \text{Collateral constraint at } t=0 \quad (17)$$

$$l_{i1b} \leq \max\{\alpha_L \gamma_i R n_i - r_i l_{i0}, 0\} \quad \text{Collateral constraint at } t=1, \text{ bad state} \quad (18)$$

$$(1 - q)r_i l_{i0} + q \min\{r_i l_{i0}, \alpha_L R\gamma_i n_i\} \geq l_{i0} \quad \text{PC of lenders at } t=0 \quad (19)$$

$$e + l_{i0} \geq n_i + b_i n_i \quad \text{BC of banks at } t=0 \quad (20)$$

$$l_{i1b} + P(1 - \gamma_i)n_i + b_i n_i - c n_i \geq 0 \quad \text{BC of banks in the bad state at } t=1 \quad (21)$$

where $0 \leq \gamma_i \leq 1$ is the fraction of assets that a bank chooses to carry forward after receiving the liquidity shock in the bad state at $t = 1$.

The collateral constraint of banks at $t = 0$, depicted in (17), puts an upper bound on the leverage banks can take. Banks can borrow from consumers by pledging their future cash flows and the right side of the constraint (17) represents the expected pledgeable
cash flow. In case of a good state at $t = 1$, no action is required: banks do not need any additional financing and they will wait until $t = 2$ to collect project returns. In case of a bad shock, however, banks prefer to borrow more if they have debt capacity. The collateral constraint depicted in (18) determines if banks have any debt capacity left given their existing debt. If their existing debt, $r_i l_{i0}$, exceeds their debt capacity $\alpha_L \gamma_i Rn_i$, the right side of constraint equals zero, which implies that $l_{i1b} = 0$. In other words, if a bank enters the interim period with high amount of debt, it cannot raise additional funds due to a debt-overhang problem. Note that once the bad state is realized, banks’ debt capacity is lower compared with $t = 0$. The participation constraint of consumers at $t = 0$ is shown in (19) which states that expected payoff from lending to banks should be at least as much as the amount lent $l_{i0}$. Banks’ budget constraints at $t = 0$ and $t = 1$ are provided in (20) and (21). In a bad state at $t = 1$ a bank has three potential sources of liquidity to cover restructuring cost of risky assets: liquidity hoarded from the initial period, $b_i n_i$, amount raised by fire sales, $P(1 - \gamma_i)n_i$, and additional borrowing $l_{i1b}$.

We would like to focus on an equilibrium where the collateral constraint does not bind at the initial period, but it does in the bad state at $t = 1$ due to a debt-overhang problem. Banks raise $l_{i0} = (1 + b_i)n_i - e$ amount of long-term debt from lenders at $t = 0$ and promise them a non-contingent payment of $r_i l_{i0}$ at $t = 2$, where $r_i \geq 1$ is the associated gross interest rate. A debt-overhang problem arises when the pledgeable return of a bank is, at most, enough to honor the existing debt even if the fire sale can be completely avoided—that is, if $r_i l_{i0} \geq \alpha_L Rn_i$. In such a case, the bank cannot raise any additional funds in the bad state, that is, $l_{i1b} = 0$. Therefore, we call $r_i l_{i0} \geq \alpha_L Rn_i$ as the debt-overhang condition, and under this condition the collateral constraint at $t = 1$, given by (18), binds. Using $l_{i0} = (1 + b_i)n_i - e$ we can rewrite the collateral constraint at $t = 0$, given by (17), and the debt-overhang condition at $t = 1$ as follows:

\[
(1 + b_i)n_i - e \leq (1 - q)(R + b_i)n_i + \alpha_L q \gamma_i n_i \quad \text{Collateral constraint at } t=0 \tag{22}
\]

\[
r_i(1 + b_i)n_i - r_i e \geq \alpha_L Rn_i \quad \text{Debt-overhang condition at } t=1 \tag{23}
\]

We are interested in a parameter region in which conditions (22) and (23) are both satisfied. It is clear that when $\alpha_L$ or $e$ is high, the first collateral constraint (22) is more likely to be satisfied whereas the debt-overhang condition (23) is less likely to be so. Thus, there is an intermediate range of $\alpha_L$ and $e$ such that while the first constraint is satisfied, the debt overhang problem arises in equilibrium at $t = 1$. With high equity, $e$, banks need less outside funds at $t = 0$ thus the first constraint is less likely to bind. When $\alpha_L$ is high enough, the debt capacity of banks are high and accordingly the collateral constraint at $t = 0$ would not bind. Banks enter $t = 1$ with a given debt of $r_i l_{i0}$. Therefore
if $\alpha_L$ is low, or if $r_i l_{i0}$ is high, the debt capacity at $t = 1$ can be already saturated with the existing debt. Because lower $e$ implies higher initial debt, $l_{i0}$, it also leads to debt-overhang problem at $t = 1$. Thus $\alpha_L$ and $e$ jointly determine whether a debt-overhang problem will arise at $t = 1$, and we focus on the parameter region where it does arise. Note that the composition of the liability side has no effect on the fire sale externality, and hence, on the results in our setup. Therefore, instead of the level of equity $e$ we can look at the simple leverage ratio $k_i = e/(n_i + b_in_i)$. Using the closed-form solutions in the appendix we can determine the range for leverage ratio, $k_i$, and fraction of pledgeable revenue in the bad state $\alpha_L$ that leads to debt-overhang problem. As we can see by the shaded region in Figure 5, debt-overhang problem arises for a reasonably large parameter space without violating the collateral constraint at $t = 0$.

When the parameters $e$ and $\alpha_L$ lie in the shaded region in Figure 5 and the debt-overhang arises, the bank will be in default in the bad state.\footnote{If a bank is in default after fire sales, we assume that it is required by law to manage the remaining assets until final period and deliver returns to consumers.} Hence it has to pay initial lenders a positive interest rate, $r_i > 1$, in good times to compensate. For the PC condition of consumers to be satisfied, $r_i$ has to be such that

$$r_i \geq \frac{l_{i0} - \alpha_L q R \gamma_i n_i}{(1 - q) l_{i0}} \equiv r^*_i. \quad (24)$$
To obtain this constraint we rearrange the PC condition (19) and note that \( \min\{\alpha L R \gamma_i n_i, r_i l_{i0}\} = \alpha L R \gamma_i n_i \). In order to maximize profits, banks set \( r_i = r_i^* \). Note that, profit of banks in the bad state in this case is \( R \gamma_i n_i - \alpha L R \gamma_i n_i = (1 - \alpha L) R \gamma_i n_i \). Now, we can substitute the optimal \( r_i^* \) back into the banks’ objective function (16) and simplify to obtain:

\[
\max_{n_i, b_i, \gamma_i, l_{i0}} (1 - q)(R + b_i)n_i + qR \gamma_i n_i - (1 + b_i)n_i - \Phi((1 + b_i)n_i),
\]

subject to

\[
\begin{align*}
 e + l_{i0} & \geq n_i + b_i n_i & \text{BC of banks at } t=0 \\
 P(1 - \gamma_i)n_i + b_i n_i - c n_i & \geq 0 & \text{BC of banks in the bad state at } t=1
\end{align*}
\]

(25)

(26)

Thus, in the parameter space in which banks are not collateral constrained at \( t=0 \) but a debt-overhang problem arises at \( t = 1 \), banks’ optimization problem that we obtain is identical to the one presented in our benchmark model.

### A.2 Ex-post liquidity provision by the central bank

Our goal in this paper is to study the optimal design of ex-ante regulations and the interaction between them. Optimal design of ex-post policies and the interactions between ex-ante and ex-post policies have also received considerable attention in the post-crisis period.\(^{34}\) Studying these important but challenging topics in a full-fledged model is beyond the scope of this paper. Nevertheless, to explore under what conditions our results would be robust to an ex-post intervention by the regulatory authority—in particular, a liquidity provision to banks—we provide an extension of our benchmark model in this section. The scope of this extension is limited to analyzing the response of banks, both ex-ante and ex-post, to the introduction of a liquidity provision by the central bank.

In the benchmark model banks are not allowed to raise funds in any form once the shock hits. However, this strict assumption is for brevity and for keeping the model tractable. With this simplification our goal is to capture banks’ inability to attract further funds—even if these funds are available in the economy—because banks may be facing information asymmetry or debt overhang problems, as we model in Section A.1 in this online appendix. In this section, we relax this assumption and allow banks to have additional liquidity access from the central bank in the interim period after a bad shock is realized. This ex-post liquidity access alters banks’ ex-ante decisions, yet we show that our main results remain robust under this extension.

\(^{34}\)See, for example, Acharya and Yorulmazer (2008); Bebchuk and Goldstein (2011); Farhi and Tirole (2012); Benigno et al. (2013); Philippon and Schnabl (2013); Jeanne and Korinek (2013).
Our goal is to show that limited ex-post policy interventions do not eliminate fire sales and pecuniary externalities, and hence, the associated inefficiencies with ex-ante choices in the competitive equilibrium remain. Therefore, we argue that even though a regulator can improve upon the competitive equilibrium using ex-post policy interventions, he still needs to use both capital and liquidity requirements to implement the constrained efficient allocations, as long as these ex-post interventions are costly or limited in scope. Note that, as emphasized by Benigno et al. (2012) and Philippon and Schnabl (2013), if ex-post interventions were free and unlimited, it would be possible to undo externalities and implement the first best. However, such policies are not realistic because ex-post interventions generally involve social deadweight costs of one form or another, such as distortionary taxes or support for inefficient banks.\footnote{For instance, Acharya and Yorulmazer (2008), Farhi and Tirole (2012), Philippon and Schnabl (2013), and Jeanne and Korinek (2013) study ex-post interventions that are financed by distortionary taxes, which effectively limit the scope and effectiveness of these interventions.}

Unlike recent literature on ex-post interventions, we do not model such costs explicitly as our focus is not on the optimal design of ex-post interventions. We simply assume that the intervention is limited.

In particular, we introduce a lending facility (similar to discount window lending) by the central bank from which banks can borrow funds when they need liquidity in the interim period, potentially at a penalty interest rate.\footnote{We abstract from banks’ concern about a stigma attached to discount window borrowing.} We assume that the central bank announces this policy at $t = 0$ and can commit to it. Therefore, banks take the ex-post policy into account when they make choices at $t = 0$. According to the policy, the central bank lends banks an amount that is proportional to banks’ risky asset $\tau_{cb,i}$ at a gross interest rate $r_{cb} \geq 1$. Therefore, banks have to pay back $r_{cb}\tau_{cb,i}$ at $t = 2$. If the central bank was not constrained in its interventions, in principle it could just satisfy every bank’s liquidity needs. But then banks would not have any incentive to hold liquidity ex-ante. We do not allow this case by assuming that liquidity available from the central bank is lower than the liquidity shock per risky asset, that is, $\tau_{cb} < c$. We consider this assumption as a shortcut because such a liquidity provision is not likely to be costless, and if it were to be set optimally $\tau_{cb}$ would be less than the shock $c$.\footnote{We could imagine a scenario in which the marginal cost of liquidity provision is increasing with $\tau_{cb}$, and an interior solution is obtained where $\tau_{cb} < c$.} The effect of this ex-post policy intervention is then similar to making the restructuring shock smaller, that is, we can define the new shock as $\tilde{c} = c - \tau_{cb}$. As the restructuring shock is dampened but not eliminated banks will continue to hoard liquidity ex-ante to insure against it.

Banks also need to make a decision at $t = 1$ in the bad state: to borrow from the central bank, conduct fire sales, or both. To see whether banks would still conduct fire sales when they can borrow from the lending facility, we need to compare the costs of each of the two
options. Selling a unit of the risky asset brings \( P \) but results in foregoing \( R \), so the net cost of conducting fire sales is \( \frac{R-P}{P} \). Borrowing from the central bank has a net cost of \( \frac{r_{cb}-1}{1} \). We assume that the policy is designed so that \( \frac{R}{P} > r_{cb} \): The fire sale is more costly at the equilibrium price of assets. Otherwise, banks would not use a facility that is costlier than fire sales and the introduction of such a facility would not have any effect on the banking system. When \( \frac{R}{P} > r_{cb} \), banks borrow as much as possible from the central bank and then fire sell assets to cover the remaining liquidity need.

Having established that banks still hoard liquidity ex-ante and conduct fire sales ex-post, we can now analyze how the introduction of this liquidity facility affects banks’ choices at \( t = 0 \) and \( t = 1 \). We solve the equilibrium of the new model by backwards induction. This new facility alters banks’ problem in the bad state of the interim period as follows. Banks choose what fraction of assets to sell, \( 1 - \tilde{\gamma}_i \), to maximize total returns at \( t = 2 \):

\[
\max_{0 \leq \tilde{\gamma}_i \leq 1} R\tilde{\gamma}_in_i + P(1 - \tilde{\gamma}_i)n_i + \tau_{cb}n_i + b_in_i - cn_i - r_{cb}\tau_{cb},
\]

subject to the budget constraint

\[
P(1 - \tilde{\gamma}_i)n_i + \tau_{cb}n_i + b_in_i - cn_i \geq 0.
\]

The new term, \( \tau_{cb}n_i \), represents the relaxation of the constraint thanks to the liquidity facility of the central bank. Profit maximization requires banks to keep as many risky asset as possible in the interim period because \( R \geq P \). Thus, banks sell the minimum required amount of risky assets to cover the liquidity shortage, \((c - \tau_{cb} - b_i)n_i\). As a result, the budget constraint binds and yields \( \tilde{\gamma}_i = 1 - \frac{c-\tau_{cb}-b_i}{P} \).\(^{38}\) We can then simplify banks’ profit conditional on a bad state as:

\[
\Pi_{i|\text{Bad}} = R\tilde{\gamma}_in_i - r_{cb}\tau_{cb}n_i = R \left( 1 - \frac{\tilde{c} - b_i}{P} \right) n_i - r_{cb}\tau_{cb}n_i,
\]

where we use \( \tilde{c} = c - \tau_{cb} \). The first part, \( R\tilde{\gamma}_in_i \), represents returns on the remaining assets at the bank after fire sales and has the same functional form as in the benchmark model. The second part, \( r_{cb}\tau_{cb}n_i \), is the cost of utilizing the central bank’s lending facility. Therefore,

\(^{38}\)When \( \tau_{cb} = 0 \), we get \( \tilde{\gamma}_i = \gamma_i = 1 - \frac{c-b_i}{P} \), which is the fraction of assets banks keep when there is no ex-post policy intervention.
expected profit of banks at \( t = 0 \) can be written as:

\[
\Pi_i = (1 - q)(R + b_i)n_i + qR \left( 1 - \frac{\bar{c} - b_i}{P} \right) n_i - D(n_i + n_i b_i) - qr_{cb} n_i \tau_{cb},
\]

\[
= (1 - q)(Rn_i + B_i) + qR \left( 1 - \frac{c - b_i}{P} \right) n_i - D(n_i + n_i b_i) + qn_i \tau_{cb} \left( \frac{R}{P} - r_{cb} \right).
\]

The last term summarizes the impact of the new lending facility on banks’ problem. The first-order conditions of the banks’ problem with respect to \( n_i \) and \( b_i \) respectively are:

\[
(1 - q)(R + b_i) + qR \left( 1 - \frac{\bar{c} - b_i}{P} \right) = D' \left( n_i (1 + b_i) \right)(1 + b_i) + qr_{cb} \tau_{cb},
\]

\[
(1 - q)n_i + qR \frac{1}{P} n_i = D' \left( n_i (1 + b_i) \right) n_i.
\]

Comparing the first-order conditions above with the corresponding conditions in the benchmark model, given by (B.4) and (B.5) in Section B in the online appendix, reveals that the introduced policy has a direct impact on banks’ choice of risky assets. One the one hand, because the liquidity support provided by the central bank dampens the effect of the restructuring shock \((\bar{c} = c - \tau_{cb})\), banks’ marginal benefit from holding risky asset increases. On the other hand, the policy imposes an additional marginal cost for risky assets, captured by the term \( qr_{cb} \tau_{cb} \), due to the penalty rate on the borrowed funds. The policy affects the choice of liquidity ratio only indirectly, through its effect on the price of risky assets at \( t=1 \), to which liquidity ratio choice responds to significantly. Therefore, banks will change their risky asset and liquidity holdings as a response to the lending facility terms. Combining banks’ first order conditions yields the price in competitive equilibrium as:

\[
P_{cb} = \frac{qR (1 + c - \tau_{cb})}{R - 1 + q(1 - \tau_{cb} r_{cb})},
\]

In the next lemma we describe how the parameters of the liquidity facility affect the price of risky assets at date 1.

\textbf{Lemma 4.} While a loose ex-post policy decreases the fire sale price, a tight policy can increase it: \( \frac{\partial P_{cb}}{\partial r_{cb}} > 0 \) and \( \frac{\partial P_{cb}}{\partial \tau_{cb}} < 0 \).

When ex-post policy becomes looser, either by lowering the borrowing rate for banks \( r_{cb} \) or by increasing borrowing limit from the central bank for each unit of risky assets—that is, a higher \( \tau_{cb} \)—fire sale price decreases. Thus, such a policy has the potential to be detrimental to financial stability as it can increase the amount of assets sold at fire sales.

\footnote{Note that when \( \tau_{cb} = 0 \) we get the price in the benchmark competitive equilibrium: \( P^c = \frac{qR (1 + c)}{R - 1 + q} \).}
and decrease the price. The next lemma establishes that for policy to increase the fire sale price and decrease the amount of fire sales, the cost of borrowing from the central bank lending facility should be above a threshold value.

**Lemma 5.** For a given $\tau_{cb}$, it is possible to improve the fire sale price, $P_{cb} > P^c$, with a sufficiently high $r_{cb}$.

Lemma 5 establishes that the central bank can improve upon the unregulated competitive equilibrium by using this ex-post policy alone, that is, without using capital and liquidity regulations. Note that this lemma does not show that ex-post policy can implement the constrained efficient allocations. It only claims that banks would conduct less fire sales and fire sale price would improve. Furthermore, existence of this ex-post facility does not change the interaction dynamics between capital and liquidity in our framework. We can show that under this extension, banks would still reduce their liquidity when a capital regulation is implemented in isolation. Therefore, the regulator would still need to use both capital and liquidity regulations to implement the constrained efficient allocations.

In summary, our analysis suggests that as long as banks hold some liquidity to shield their risky assets from fire sales, their liquidity will be lower than the planner’s. Although we do not allow in our benchmark model, in reality banks can raise funds from the central bank or even from private sources when a liquidity need emerges. Provided that these external liquidity sources are costly, they will not eliminate banks’ liquidity hoarding motive against fire sale risk. When banks decide how much liquidity to hold ex-ante they will correctly calculate the costs of such holdings, but they will fail to internalize the social benefit of liquidity in mitigating fire sales. Thus, although these external funding sources affect the level of liquidity banks hold, banks’ liquidity (and risky asset) choices will remain inefficient from a social perspective.

**A.3 Existence of unregulated financial institutions**

Central banks and regulatory institutions around the world mainly focus on regulating banks to improve financial stability. However, actions of nonbank financial institutions affect the stability of the system as well. Yet, some financial institutions are partially or totally exempt from bank regulations. We analyze how unregulated financial institutions react to bank regulation as well as what their reactions imply for financial stability. For that purpose, we introduce a new group of financial institutions that are identical to banks but not regulated.

We denote the choices of regulated institutions with $(\tilde{n}, \tilde{b})$ and those of unregulated
institutions with \((n, b)\). As before, \(n\) and \(\tilde{n}\) are the amount of risky investment while \(b\) and \(\tilde{b}\) denote the liquid asset per unit risky investment. Liquidity needs of regulated and unregulated institutions in bad times at \(t = 1\) respectively are: \((c - \tilde{b})\tilde{n}\) and \((c - b)n\). The market clearing condition in the fire sale market is \((c - \tilde{b})\tilde{n} + (c - b)n = Py\). Thus, the fire sale price is a function of \(\tilde{n}, \tilde{b}, n, b\). Below we analyze the response of unregulated institutions to bank regulation. In particular, we study the risky asset choice of an unregulated institution and see how it changes as the regulator limits the total risky investment \(\tilde{n}\), in the regulated segment. An unregulated institution chooses \(n_i, b_i\) to maximize its expected profits, given by:

\[
\Pi_i(n_i, b_i) = \Gamma(n_i, b_i) - q(R - P)\frac{c - b_i}{P}Q_s^i(P, n_i, b_i),
\]

where \(Q_s^i(P, n_i, b_i) = (c - b_i)n_i/P\). Here, each atomistic institution takes the fire sale price \(P(\tilde{n}, \tilde{b}, n, b)\) as given and we treat \(\tilde{n}, \tilde{b}\) as parameters of the model because unregulated institutions take them as given. The regulator effectively determines the aggregate amount of \(\tilde{n}\) using capital regulations and aggregate amount of \(\tilde{b}\) using liquidity requirements. Therefore, the first-order condition of the unregulated institution with respect to \(n_i\) is

\[
\frac{\partial \Pi(n_i, b_i)}{\partial n_i} = \frac{\partial \Gamma}{\partial n_i} - q(R - P)\frac{c - b_i}{P} = 0.
\]

To see how optimal \(n_i\) changes with \(\tilde{n}\), we need to evaluate the sign of the cross-partial derivative of the profit function:

\[
\frac{\partial^2 \Pi(n_i, b_i)}{\partial \tilde{n} \partial n_i} = qR\frac{c - b_i}{P^2} \frac{\partial P}{\partial \tilde{n}} < 0.
\]

Using the monotone comparative statics techniques outlined by Vives (2001), the negative sign of the cross-partial derivative (by Lemma 1 and 2) indicates that \(n'(\tilde{n}) < 0\), that is, as regulation tightens risky investment level of banks, unregulated institutions respond by increasing their risky investment. Therefore, \(n_i\) and \(\tilde{n}\) are imperfect substitutes from the unregulated institution’s point of view. Similarly, we can show that \(b'(\tilde{b}) < 0\), that is, as the regulation require banks to increase their liquidity ratios, unregulated institutions respond by decreasing their liquidity ratios. Thus, unregulated institutions free ride on the liquidity of regulated institutions. In similar ways, we can also show that \(n'(\tilde{b}) > 0\), and \(b'(\tilde{n}) > 0\).

Regulations on \(\tilde{n}\) and \(\tilde{b}\) make the financial system more stable by increasing the fire sale price, which in turn create incentives for the unregulated institutions to invest more in risky assets and decrease their liquidity buffer. The behavior of unregulated institutions creates a counterforce to the regulation. To further explain the intuition behind these
results, we can consider an analogy from automotive safety regulations in the spirit of Peltzman (1975): Cars and motorcycles usually share the same roads. If we introduce speed restrictions on cars but not on motorcycles, roads will initially become safer, but this will create incentives for motorcycle riders to increase their driving intensity, creating a counterforce to the regulation.

The effect analyzed in this section is similar to the one examined in international policy coordination literature such as Acharya (2003), Dell’Ariccia and Marquez (2006), and Kara (2016). These papers show that bank regulations across countries are strategic substitutes, and hence, countries have an incentive to free ride and relax regulations when others tighten. Similarly, we show above that bank capital and liquidity regulations in a given country have a public good property under fire sale externalities. If we regulate only some institutions, unregulated institutions that engage in similar investment behavior will free ride on the improved stability brought by disciplined institutions. Therefore, as argued by Farhi et al. (2009), efficient regulations should have a wide scope and apply to all relevant financial institutions.

A.4 On welfare and transfers

The constrained efficiency concept we use in this paper relies on the planner making compensating transfers from banks to households. The constrained planner’s solution implies a higher price of assets in the interim state, which benefits banks while reducing the profits of traditional firms. The increase in banks’ profits is larger than the loss in the traditional sector. Thus, banks would be better-off even if they were to compensate the owners of the traditional sector for their losses. Hence, with compensating transfers, the constrained planner’s solution provides a Pareto improvement over competitive equilibrium allocations. As our paper is aspiring to be relevant for policy makers, we would like to address some theoretical and practical questions about the welfare criterion and transfers. In particular, the following questions are of interest: Why transfers are needed? Who does the regulator/planner have in mind, or represent? What form do these transfers take in the real world? Who gets them? Some of these questions transcend the specific research questions of the current paper, and we discuss answers to those based on the existing literature.

First of all, transfers are only necessary when a public interest approach to bank regulation is taken. Both in practice and in theory, the rule-setters might aim to maximize the social welfare or the well being of private interest groups.\footnote{See Kroszner and Strahan (2001) for a discussion of alternative approaches to regulation.} Regarding bank capital regulations, several papers in the literature have relied on public interest rationales such as
spillover effects for general macroeconomic performance from a more stable financial system to justify these regulations (see Kroszner and Strahan, 2001, and references therein). Nevertheless, in the context of inefficient fire sales as in our model, it is possible that the regulation—following a private interest theory of bank regulation (e.g., Olson et al., 1965; Stigler, 1971; Peltzman, 1976; Peltzman et al., 1989; Becker, 1983)—is designed to maximize the profits of the banking industry alone. Such an approach does not require transfers as regulators are not concerned about the well being of the groups who lose due to the regulation.\textsuperscript{41} If we take a private interest approach to regulation in our model, and the planner maximizes the profits of the banking sector (both in the cases of partial and complete regulation), all of our results remain unchanged as we have shown in an earlier version of our paper.

Although straightforward in theory, the transfers associated with regulations are subtle in real life and often take the form of concessions given to the consumers (or any other losing group) when industry regulation harms them. When governments plan to enact new regulations with distributional consequences, there is often heated horse trading that takes place in legislative branches as different representatives or parties represent different constituencies that are differently affected by the new regulations (Peltzman, 1984, 1985; Mian, Sufi, and Trebbi, 2010). As a result of this haggling the final bills often contain provisions to satisfy varied constituencies.\textsuperscript{42}

A common device used to share rents generated by regulation that primarily benefits banks is directed credit allocation (Kroszner, 2000; Kroszner and Strahan, 2001). In the United States, one example of such distribution is the Credit Reinvestment Act (CRA), which was signed into law in 1977 to encourage banks and saving institutions to serve to the needs of all communities, including low and middle income groups. On several occasions when Congress or its agencies implemented (de)regulations that empowered the banking sector, they also made amendments to the CRA to fortress the enforcement or broaden the reach of its mandate to serve all communities. For example, the Gramm-Leach-Bliley Act of 1999, while benefiting banks by repealing the Glass-Steagall Act, also included amendments to the CRA stating that no institution would be allowed to open new lines of business without receiving satisfactory compliance under CRA guidelines. The compromise was a result of heated debates between White House and Senator Gramm.\textsuperscript{43} On signing the Gramm-Leach-Bliley Act of 1999, President Clinton shows that

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\textsuperscript{41}In practice there are reasons why regulator would not care about the traditional sector, for example, purchaser of assets at fire sales might be foreign investors or the regulators might care about the welfare of only the institutions that regulations apply.

\textsuperscript{42}The empirical evidence in Kroszner and Strahan (2001) from the voting on the 1991 Federal Deposit Insurance Corporation Improvement Act (FDICIA) in the U.S. Congress provides an excellent example.

\textsuperscript{43}https://archive.nytimes.com/www.nytimes.com/library/financial/102399banks-congress.html.
policy makers are aware of positive and negative impacts of policy changes on different communities and try to balance out by saying: “as we expand the powers of banks, we will expand the reach of the [Community Reinvestment] Act”. Similarly, further proposals to enhance the CRA were made in the aftermath of the global financial crisis, a period in which bank capital and liquidity regulations expanded vastly, especially on the largest banks. These examples suggest that transfers in practice are provided via other legislative changes bundled with regulatory legislation. This bundling often happens through the direct and explicit recognition by interested parties and legislators about who gains and who loses from proposed regulation. Importantly, these transfers do not overcome contracting and enforcement issues between private parties. They merely compensate groups who lose from changes in regulations. We conclude that the transfers in our model should be thought as an approximation of these compensations in practice.

A.5 The generality of the results

In this section our aim is to clarify the generality of our results and the role of functional form assumptions. We can show numerically that all propositions hold under assumptions 1 to 4. However, to prove these propositions analytically, as we do in Section B of the online appendix, we need closed-form solutions, which we obtain under log-quadratic functional forms. Closed-form solutions are needed because when banks have two or more independent choice variables, generally, it is not possible to compare equilibrium allocations between competitive equilibrium and planner’s solutions by only analyzing the first-order conditions. Below, we establish this necessity formally and show that it is not specific to our model. For instance, this necessity is not generated by the nonpecuniary cost function per se, but is due to having two independent choice variables in the model.

We start by comparing the first-order conditions with respect to the risky asset in the competitive equilibrium and in the planner’s solution, given respectively by:

\[
\frac{\partial \Gamma(n_i, b_i)}{\partial n_i} - q(R - P) \frac{\partial Q_i(P, n_i, b_i)}{\partial n_i} = 0,
\]

\[
\frac{\partial \Gamma(n, b)}{\partial n} - q(R - P) \frac{\partial Q(P, n, b)}{\partial n} + qQ(P, n, b) \frac{\partial P}{\partial n} \left\{1 - (R - P) \frac{\partial Q}{\partial P}\right\} = 0,
\]

which we can represent as follows for brevity: \(h(n^e, b^e) = 0\) for the competitive equilib-
rium, and \( h_n(n^s, b^s) + \kappa_n(n^s, b^s) = 0 \) for the planner. Although they look similar, the planner’s first order condition has an additional term \( \kappa_n(n^s, b^s) = qQ(\cdot) \frac{\partial P}{\partial n} \left\{ 1 - (R - P) \frac{\partial Q}{\partial P} \right\} \) because the planner internalizes the effect on the equilibrium price. Given that \( \frac{\partial Q}{\partial P} < 0 \) and \( \frac{\partial P}{\partial n} < 0 \), this additional term is negative. If we had only \( n_i \) as an independent choice variable, for example, by assuming a fixed balance sheet such that \( n_i + n_i b_i = A \) where \( A \) is a constant, then the two first-order conditions above would be sufficient to compare \( n^c \) to \( n^s \). In particular, with a concave objective function, \( \kappa_n(n^s, b^s) < 0 \) would imply that \( h_n(n^s) > h_n(n^c) \) which, in turn, would lead to \( n^s < n^c \). Hence, when there is only one choice variable it is possible to compare values satisfying the two equilibrium conditions without requiring any functional form assumptions. However, when there are more than one choice variables (and thus first-order conditions), comparing the first-order conditions do not usually reveal much about the ordering of variables from different economies.

In our case with two choice variables, the negative sign of the extra term in the planner’s economy still implies \( h_n(n^s, b^s) > h_n(n^c, b^c) \). However, the existence of another choice variable in \( h_n(\cdot) \) prevents a comparison between \( n^s \) and \( n^c \), unless we know that \( b^c = b^s \). The same reasoning also applies to the other choice variable \( b \), which we do not repeat. Thus, given that we have two independent choice variables we cannot compare the equilibrium values from the competitive economy with the planner’s choices. This conclusion is not specific to our model but applies generally to comparing equilibrium values of two endogenous variables from any two different economies.\(^{45}\) That is why we use functional form assumptions to get closed-form solutions for \( n \) and \( b \) under all three economies.

Using numerical solutions we confirm that the results in propositions 2, 3, 5, and 6 are robust to using several other functional forms.\(^{46}\) In particular, different return functions for the traditional sector that satisfy our basic assumptions, such as \( F(y) = y(R - \alpha y) \) and \( F(y) = \sqrt{y + (1/2R)^2} \), and several convex cost functions, including \( \Phi(x) = \phi x^{\alpha} \) where \( \alpha > 1 \), \( \Phi(x) = \phi_0 \exp^{\phi_1 x} \), and \( \Phi(x) = \phi_0 x \text{Log}(\phi_1 x) \) work as well.\(^{47}\) The only requirement, as we discuss below, is that the convex cost function should not be too steep.

We should note that the result in Proposition 4, \( b'(n) > 0 \), does not rely on a specific cost function. As emphasized in the discussions that follows the proposition, any convex or concave cost function would work as long as it allows a relatively flexible balance

---

\(^{45}\)We can identify the specific cases for which the comparison of equilibrium values is possible without solving for equilibrium. Denoting the first order condition for \( b \) similarly as \( h_b \), and terms due to pecuniary externality as \( \kappa_b \), these specific cases (sufficiency conditions) requires certain sign restrictions on \( \kappa_n, \kappa_b, h_{nb}, h_{nn} \) and \( h_{bb} \). However, these sufficiency conditions are not met in our model. Details are available upon request from the authors.

\(^{46}\)Propositions 1, 4, and 7 do not require any functional form assumptions.

\(^{47}\)These results are available upon request from the authors.
Thus, we rule out only very steep convex cost functions that practically imply a fixed-size balance sheet. For example, suppose that the operational costs take the following general convex form: $\Phi(x) = \phi x^\alpha$ where $\alpha > 1$. There is an upper bound on $\alpha$ above which the cost function gets too steep and adjusting the balance sheet size becomes very expensive, making balance sheet size effectively inflexible. We can characterize this upper bound, denoted by $\bar{\alpha}$, in terms of model parameters and show that $\bar{\alpha} > 2$ in the whole parameter space. Thus, for $1 < \alpha \leq 2$ the result holds without additional assumption on parameters. In other words, for convex functions that are steeper than the quadratic form, the result holds in the parameter region that satisfies $\alpha < \bar{\alpha}$. This also implies that $b'(n) > 0$ result holds for power functions steeper than the quadratic one, but the parameter set for which this result holds will get smaller as the power $\alpha$ increases.

Next, we discuss why we need an operational cost assumption and how important this assumption is for our results. In our model, the net interest rate on bank deposits is zero. Without an additional cost (such as an operational cost) banks can borrow more from depositors and park these funds as liquid assets (cash) in their portfolios. In that way they could freely insure against the fire sale risk. We believe that such a scenario is not realistic. First, banks do face costs to attract deposits. Second, an unlimited amount of funds from depositors at zero cost would cancel out the opportunity cost of holding liquid assets, namely the cost of bygone profits from other investments. A constant balance sheet size would emphasize this opportunity cost mechanism, as it does in many papers in the literature. However, with a fixed balance sheet size, the choice between risky assets and liquid assets boils down to a portfolio allocation problem. A setup with a single choice variable does not allow the type of interactions we study. Thus, by employing a flexible balance sheet size, we avoid two extreme assumptions: namely, that banks have an unlimited amount of funds at their disposal, and that bank balance sheet size is inflexible.

Finally, we consider if the cost function have to be convex. The necessity of a convex cost function depends on our assumption of banks’ technology for risky assets. The reason our cost function is convex, similar to the ones imposed by Van den Heuvel (2008) and Acharya (2003, 2009), and Davila and Korinek (2017), is because it ensures the existence of an equilibrium given that banks are endowed with constant returns to scale (CRTS) technology. With CRTS technology we need a convex cost function for bank profit function to be concave in the size of the balance sheet. CRTS technology simplifies the analysis of fire sales tremendously and allows us obtain analytical solutions, which are crucial for some of our results as we discuss above. Otherwise, a convex cost assumption would not be necessary if banks’ technology showed decreasing returns to scale. In that
case, a concave cost function would also yield an interior solution.

Furthermore, whether bank size matters for financial stability and the inefficiencies banks create is also discussed in the context of the recent financial crisis, as well as how bank regulation might affect bank size. To speak to these discussions, a flexible balance sheet size is important because it also allows us to study the optimal size of banks’ balance sheets. Our result in Proposition 6 emphasizes that the composition of a bank’s balance sheet matters more than its size, and that regulation does not necessarily imply a reduction in balance sheet size.

A.6 Precautionary and strategic channels for holding liquidity

This appendix complements Section 6 Would Banks Overhoard Liquidity by describing how the precautionary and strategic channels for holding liquidity show up in the banks’ and planner’s problems. We start with the banks’ profit maximization problem at $t = 0$:

$$E[\Pi] = (1 - q)E[\Pi_{\text{good}}] + qE[\Pi_{\text{crisis}}] - n_i(1 + b_i) - \Phi(n_i(1 + b_i)).$$

With probability $1 - q$, the good state prevails and there is no shock. Expected revenue in the good state is given by $E[\Pi_{\text{good}}] = RN_i + b_i n_i$. The main modification happens in the crisis state, which arises with probability $q$. $\lambda_b$ is the probability of a bad shock, given the crisis state, while $1 - \lambda_b$ is the probability of an ugly shock. The expected revenue conditional on the crisis state is

$$E[\Pi_{\text{crisis}}] = \lambda_b \left\{ \Lambda R \left( 1 - \frac{c - b_i}{P_b} \right)n_i + (1 - \Lambda)(RN_i + b_i n_i \frac{R}{P_b}) \right\} + (1 - \lambda_b)(1 - \Lambda)b_i n_i \frac{R}{P_u}.$$

The terms in the curly brackets represent a bank’s expected revenue conditional on the crisis being a bad one: With probability $\Lambda$ the bank incurs the restructuring cost and keeps a $1 - \frac{c - b_i}{P_b}$ fraction of its risky assets after fire sales. This state corresponds to the bad state of our benchmark model, which gives banks a precautionary motive. The second term represents the bank’s expected profit if it does not receive the shock in the bad state, which happens with probability $1 - \Lambda$. In this case, the bank’s risky assets are intact and will produce a total of $RN_i$ in the final period. The bank also purchases $\frac{b_i n_i}{P_b}$ units of additional risky assets at fire sale prices from shock-receiving banks and earns a return of $R$ on purchased assets.

With probability $1 - \lambda_b$, the crisis is an ugly shock: The terms outside the curly brackets represent the bank’s profit in this state. In the ugly state, the bank receives a shock and defaults with probability $\Lambda$; thus its profit is zero due to limited liability. With probability
$1 - \Lambda$ the bank does not receive the shock and hence does not default. Surviving banks purchase risky assets at a price of $P_u$ from defaulting banks. Note that when $\Lambda = 1$ the idiosyncratic shock turns to an aggregate one, and when $\lambda_b = 1$ the ugly state disappears and only the bad state remains. When both of these parameters are equal to 1, this model is identical to our benchmark model.\(^{48}\)

Note that because the shock is idiosyncratic, exactly a $\Lambda$ fraction of banks receive the shock by the law of large numbers. Hence, the market clearing condition in the bad state is as follows: Shock-receiving banks need to raise $(cn_i - bn_i)$, which they cover by selling risky assets. $1 - \Lambda$ percent of banks have excess liquidity $b_i n_i$, and they purchase risky assets from the shock-receiving banks together with firms in the traditional sector. Traditional firms buy $y_b$ units, spending a total of $P_b y_b$. Thus, the market clearing condition that defines the price $P_b$ is: $\Lambda(cn - bn) = (1 - \Lambda)bn + P_b y_b$.

In the ugly state, $\Lambda$ fraction of banks are liquidated; $\Lambda n$ risky assets are sold at fire sale prices. The total liquidity available from surviving banks and traditional firms to purchase assets is $(1 - \Lambda)bn + P_u y_u$. Hence, the market clearing in the ugly state requires $P_u \Lambda n = (1 - \Lambda)bn + P_u y_u$.

In the ugly state, a wedge between the planner’s and the competitive banks’ liquidity choices arises due to the transfer of risky assets through fire sales. Each surviving bank purchases $\frac{b_i n_i}{P_u}$ units of risky assets from defaulting banks at fire sale prices. Hence, in total $(1 - \Lambda)\frac{bn}{P_u}$ risky assets are transferred from defaulted banks to surviving ones. A planner does not care about these transfers and maximizes the expected output of the banking sector, given by \(^{49}\)

$$E[W] = (1 - q)E[W_{\text{good}}] + qE[W_{\text{crisis}}] - n_i(1 + b_i) - \Phi(n_i(1 + b_i)),$$

where $E[W_{\text{good}}] = Rn + bn$, and

$$E[W_{\text{crisis}}] = \lambda_b [Rn + bn - y_b (R - P_b) - cn] + (1 - \lambda_b) [Rn + bn - y_u (R - P_u)].$$

As we see in the welfare function, the planner only cares about the amount of risky assets sold to traditional firms in the crisis states, denoted by $y_b$ and $y_u$.\(^{50}\) The asset sales among banks are pure transfer and do not show up in the planner’s objective function.

\(^{48}\)When only $\lambda_b$ is equal to 1, this model corresponds to the benchmark model in which we replace aggregate shock with idiosyncratic shocks. We study this case in an earlier version of the paper and show that it has the same implications as the benchmark model. In particular, banks always hoard too little liquidity, indicating that this result is driven by a precautionary motive regardless of whether the shock is aggregate or idiosyncratic.

\(^{49}\)To simplify the exposition, we assume that the planner maximizes the profits of the banking sector only. Our results are robust to solving a Pareto problem where, as in the benchmark model, the planner uses transfers to compensate the consumers.

\(^{50}\)If there were no outsiders in the ugly state, $y_u$ would be zero because all the risky assets are sold among banks.
B Proofs omitted in the main text

Lemma 1. The fire sale price of a risky asset, \( P(n, b) \), is decreasing in \( n \) and increasing in \( b \). The fraction of risky assets sold, \( 1 - \gamma(n, b) \), is increasing in \( n \) and decreasing in \( b \).

Proof. Part 1: \( P(n, b) \), is decreasing in \( n \) and increasing in \( b \).

The asset market clearing condition in the bad state at \( t = 1 \) is

\[
Q_s(P) = c - b P_n = Q^d(P),
\]

which can be written as

\[
(c - b)n = PQ^d(P). \tag{B.1}
\]

First, take the partial derivative of both sides of this last equation with respect to \( n \):

\[
c - b = \frac{\partial P}{\partial n} Q^d(P) + P \frac{\partial Q^d(P)}{\partial P} \frac{\partial P}{\partial n} = \frac{\partial P}{\partial n} \left\{ Q^d(P) + P \frac{\partial Q^d(P)}{\partial P} \right\} = \frac{\partial P}{\partial n} Q^d(P) \left\{ 1 + \epsilon^d \right\},
\]

where \( \epsilon^d = \frac{\partial Q^d(P)}{\partial P} \frac{P}{Q^d} \) is the price elasticity of the traditional sector’s demand function. Rearranging the last equation gives

\[
\frac{\partial P}{\partial n} = \frac{c - b}{Q^d(P)(1 + \epsilon^d)} < 0
\]

since \( 1 + \epsilon^d < 0 \) by the Elasticity assumption, and \( c - b > 0 \) by assumption here because we are examining the case with fire sales. We later show in Lemma 2 and 3 that \( c - b > 0 \) indeed holds in equilibrium.

Now take the partial derivative of both sides of (B.1) with respect to \( b \):

\[
-n = \frac{\partial P}{\partial b} Q^d(P) + P \frac{\partial Q^d(P)}{\partial P} \frac{\partial P}{\partial b} = \frac{\partial P}{\partial b} \left\{ Q^d(P) + P \frac{\partial Q^d(P)}{\partial P} \right\} = \frac{\partial P}{\partial b} Q^d(P) \left\{ 1 + \epsilon^d \right\}.
\]

Rearranging the last equation yields

\[
\frac{\partial P}{\partial b} = -\frac{n}{Q^d(P)(1 + \epsilon^d)} > 0.
\]

because \( 1 + \epsilon^d < 0 \) holds by Elasticity assumption.

Part 2: \( 1 - \gamma(n, b) \), is increasing in \( n \) and decreasing in \( b \).

In Section 3.1.2 we derived the fraction of risky assets sold by banks in equilibrium as

\[
1 - \gamma(n, b) = \frac{c - b}{P(n, b)}. \tag{17}
\]

Note that

\[
\frac{\partial (1 - \gamma)}{\partial n} = \frac{\partial (1 - \gamma)}{\partial P} \frac{\partial P}{\partial n} = -\frac{c}{P^2} \frac{\partial P}{\partial n} > 0,
\]
because by Lemma 1 we have that $\frac{\partial P}{\partial n} < 0$. Similarly, we can obtain

$$\frac{\partial (1 - \gamma)}{\partial b} = -\frac{1}{P} + \frac{\partial (1 - \gamma)}{\partial P} \frac{\partial P}{\partial b} = -\frac{1}{P} - \frac{c}{P^2} \frac{\partial P}{\partial b} < 0,$$

because by Lemma 1 we have that $\frac{\partial P}{\partial b} > 0$.

**Lemma 2.** Under the Efficiency and Technology assumptions, banks always take fire sale risk in equilibrium; that is, $b_i < c$ for all banks.

**Proof.** It is straightforward to show that banks never carry excess liquidity in equilibrium, that is, $b_i > c$. This is because when $b_i > c$ the liquid assets in excess of the shock, $(b_i - c)n$, have no use even in the bad state; the expected return on liquid assets is equal to one and dominated by the expected return on the risky asset, $R - cq$, by the Technology assumption.

To prove $b_i < c$, we start with the full insurance case, that is $b_i = c$, and move $\varepsilon$ amount of investment from liquid asset to risky asset, and show that this reallocation is profitable. Deviating bank get exposed to fire sale risk as a result of this reallocation.

First, we rewrite expected profit in terms of the total amount of liquid assets at the bank, defined as $B_i \equiv b_i n_i$, rather than the liquidity ratio, $b$:

$$\Pi_i = (1 - q)(R + b_i) n_i + qR \left(1 - \frac{c - b_i}{P}\right) n_i - D(n_i + n_i b_i),$$

$$= (1 - q)(R n_i + B_i) + qR \left(n_i - \frac{cn_i - B_i}{P}\right) - D(n_i + B_i),$$

In case of perfect insurance the size of liquidity hoarded at the initial period is equal to the size of the liquidity need in the bad state, that is, $B_i = cn_i$. Expected profit in the full insurance case boils down to $\Pi_{fi} = R n_i + (1 - q) B_i - D(n_i + B_i)$. Now, moving some amount of funds in the initial period from liquid assets to the risky investment, yields $B_{i,new} = B_i - \varepsilon$ and $n_{i,new} = n_i + \varepsilon$. Let us denote the fire sale price after the reallocation by $P_\varepsilon$. Expected profit is as follows after the reallocation of funds

$$\Pi_{i,new} = \Pi_{i,fi} + \varepsilon(R - 1 + q) - qR \frac{(1 + c)\varepsilon}{P_\varepsilon},$$

where $P_\varepsilon = F'(0)$. The following equation provides the condition for the deviating bank to profit from this reallocation of funds away from full insurance:

$$\varepsilon(R - 1 + q) - qR \frac{(1 + c)\varepsilon}{P_\varepsilon} \geq 0. \quad (B.2)$$

Using Efficiency assumption, $P_\varepsilon = F'(0) > \nu$ implies that this condition is satisfied and deviation is profitable. 

\[ \square \]
**Proposition 1.** Under the Efficiency, Elasticity, Regularity, and Technology assumptions, the competitive equilibrium price of assets is given by

\[ P^c = \frac{qR(1 + c)}{R - 1 + q}. \]  

(B.3)

The equilibrium price, \( P^c \), is increasing in the probability of the liquidity shock, \( q \), and the size of the shock, \( c \), but decreasing in the return on the risky assets, \( R \).

**Proof.** First of all, we show that the equilibrium price must satisfy \( R \geq P > c \). By the Efficiency assumption, the equilibrium price of assets must satisfy \( P \leq F'(0) \leq R \), otherwise the traditional sector would not purchase any assets. In equilibrium, we must also have \( P > c \), otherwise in the bad state banks would scrap assets rather than selling them; that is, there would be no fire sale. However, if there is no fire sale, then there is an incentive for each bank to deviate and to sell some assets to outsiders. The deviating bank would receive a price close to \( F'(0) \), which is greater than the cost of restructuring, \( c \) by the Efficiency and Technology assumptions. Note that \( F'(0) > \nu \equiv \frac{qR(1 + c)}{(R - 1 + q)} \) together with \( R < 1/(1 - q) \) implies that \( F'(0) > c \). Having \( P > c \) together with the Technology assumption implies that investment goods are never scrapped in equilibrium.

Next, we obtain the closed-form solution for \( P \). The first-order conditions of the banks’ problem (3) with respect to \( n_i \) and \( b_i \) respectively are:

\[
(1 - q)(R + b_i) + qR \gamma_i = D'(n_i(1 + b_i))(1 + b_i),
\]

(B.4)

\[
(1 - q)n_i + qR \frac{1}{P} n_i = D'(n_i(1 + b_i))n_i,
\]

(B.5)

where \( \gamma_i = 1 - (c - b_i)/P \) as obtained in the previous section. Combining the two equations yields

\[
(1 - q)R + (1 - q)b_i + qR \left( \frac{b_i - c}{P} \right) = (1 - q) + (1 - q)b_i + \frac{qR}{P} + \frac{qR}{P} b_i.
\]

In this last equation, the terms that involve the liquidity ratio, \( b_i \), on both sides cancel out each other, and hence we can solve for \( P^c \), the competitive equilibrium price of assets. It is straightforward to obtain the signs of the derivatives of \( P^c \) with respect to model parameters, \( R, c, q \).

**Proposition 2.** Under the log-quadratic functional form assumptions, the comparative statics for the competitive equilibrium risky investment level, \( n^c \), and liquidity ratio, \( b^c \), are as follows:

1. The risky investment level \( (n^c) \) is increasing in the return on the risky asset \( (R) \) and decreasing in the size of the liquidity shock \( (c) \), the probability of the bad state \( (q) \), and the marginal cost parameter \( (\phi) \).

2. The liquidity ratio \( (b^c) \) is increasing in the return on the risky asset \( (R) \), the size of the liquidity shock \( (c) \), and the probability of the bad state \( (q) \), and it is decreasing in the marginal cost parameter \( (\phi) \).
Proof. The derivatives below use the following closed-form solution for the competitive equilibrium risky investment level and liquidity ratio as obtained in Section C.1:

\[ n^c = \frac{[R - 1 - qc][R - 1 + q + 2\phi R(1 + c)]}{(R - 1 + q)(1 + c)^22\phi}, \quad b^c = \frac{cq - \frac{2\phi R}{\tau + 1}}{q + \frac{2\phi R}{\tau + 1}}. \]

In most derivatives below, we use the Technology assumption \((R - 1 - qc > 0)\) to obtain the sign. The derivatives for the risky investment level and their signs can be obtained as follows after some algebraic manipulation:

\[ \frac{\partial n^c}{\partial R} = \frac{(R - 1 + q)^2 + 2\phi(1 + c)[(R + q - 1)^2 + (1 - q)q(1 + c)]}{(R - 1 + q)^2(1 + c)^22\phi} > 0. \]
\[ \frac{\partial n^c}{\partial c} = \frac{-2[2(R - 1) + q(1 - c) + 2\phi R(1 + c)]}{2\phi(1 + c)^3} < 0. \]
\[ \frac{\partial n^c}{\partial q} = \frac{2\phi(1 + c)^3}{(R - 1 + q)^2(1 + c)^22\phi} < 0. \]
\[ \frac{\partial n^c}{\partial \phi} = \frac{-c(R - 1 + q)^2 - 2\phi R(1 + c)(R - 1)(1 + c)}{2(R - 1 + q)(1 + c)^22\phi} < 0. \]

Similarly, the derivatives for the liquidity ratio and their signs can be obtained as follows:

\[ \frac{\partial b^c}{\partial R} = \frac{\frac{2\phi(1-q)}{(\tau+1)^2}}{\frac{2\phi R}{\tau+1} + q} > 0. \]
\[ \frac{\partial b^c}{\partial c} = \frac{q^2}{\frac{2\phi R}{\tau+1} + q} > 0. \]
\[ \frac{\partial b^c}{\partial q} = \frac{\frac{2\phi R}{\tau+1}q(1+c)}{(\tau+1)^2} > 0. \]
\[ \frac{\partial b^c}{\partial \phi} = \frac{-\frac{2R}{\tau+1}q(1+c)}{\frac{2\phi R}{\tau+1} + q} < 0. \]

\[ \square \]

Lemma 3. Under the risk neutrality, Efficiency, and Technology assumptions, it is optimal for the constrained planner to take fire sale risk; that is, the constrained optimal liquidity ratio satisfies \(b^s < c\).

Proof. In principle, it is possible to completely insure against the fire sale risk. Under full insurance, similar to some interpretation of narrow banking (Freixas and Rochet, 2008, Chapter 7.2.2), the banks are able to cover the liquidity need even in the worst scenario by using their liquid holdings. However, we show that full insurance is not optimal and the constrained social planner takes some fire sale risk, by setting the aggregate liquidity ratio less than the liquidity need in the bad state, that is, by setting \(b < c\).

To show this, we start with the full insurance case, that is \(b = c\), and move \(\varepsilon\) amount of investment from liquid asset to risky asset, and show that this reallocation provides a Pareto improvement. Banks get exposed to fire sale risk as a result of this reallocation. As we show in Lemma 2 this reallocation does not hurt banks as long fire sale price is not dramatically different from the fundamental value of risky assets, i.e. \(R \geq F'(0) > \nu\) as
given by the Efficiency assumption. Consumer are better off with this reallocation because the expected profit of traditional sector they own is now positive, \( q[F(y_e) - y_eP_e] > 0 \) while it was zero under the full insurance case.

**Proposition 3.** Under the risk neutrality, Efficiency, Elasticity, and Technology assumptions, the competitive equilibrium is constrained inefficient. Furthermore, under the log-quadratic functional form assumptions, competitive equilibrium allocations compare to the constrained efficient allocations as follows:

1. Risky investment levels: \( n^c > n^s \)
2. Liquidity ratios: \( b^c < b^s \)

**Proof.** To prove that the competitive equilibrium is constrained inefficient we compare the equations that define the competitive equilibrium and constrained planner’s allocations, and show that the additional terms in the constrained planner’s problem are strictly different than zero. The first-order conditions of the competitive equilibrium are defined in Section 3.2, shown in equation (5), and the ones for constrained planner’s case is derived in Section 3.3, shown in equations (11) and (C.11).

\[
\frac{\partial \Gamma}{\partial x_i} - q(R - P) \frac{\partial Q_i^s}{\partial x_i} = 0, \quad \forall x_i \in \{n_i, b_i\} \tag{B.6}
\]

\[
\frac{\partial \Gamma}{\partial x} - q(R - P) \frac{\partial Q^s}{\partial x} - q(R - P) \frac{\partial Q^s}{\partial P} \frac{\partial P}{\partial x} = 0, \quad \forall x \in \{n, b\} \tag{B.7}
\]

where \( Q_i^s(P, n_i, b_i) = c - b_iP_n^i \).

Using (2), we obtain that \( \frac{\partial Q^s}{\partial P} = -(c - b)n/P^2 < 0 \), that is, the supply of assets is downward-sloping for banks. Therefore, the extra term in constrained planner’s problem is negative for risky investment because in Lemma 1 we have shown that \( \frac{\partial P}{\partial n} < 0 \). This term captures the extra units of fire sales by other banks, caused by each banks’ additional investment in the risky asset. Similarly, when comparing the first-order conditions with respect to the liquidity ratio, the extra term in constrained planner’s problem is positive because we have shown in Lemma 1 that \( \frac{\partial P}{\partial b} > 0 \). This term captures the public good property of liquidity: The liquid asset held by banks not only insures them against the fire sale risk but also constitutes a positive externality on other banks via greater fire sale prices.

We defer the proof of the second part of this proposition to Lemmas 7 and 8, which are under the proof of Proposition 5 below.

**Proposition 4.** Let the operational cost of a bank be given by \( \Phi(x) = \phi x^2 \). Then, banks decrease their liquidity ratio as the regulator tightens capital requirements; that is, \( b'(n) \geq 0 \) for any concave technology function for the traditional sector, \( F(\cdot) \), that satisfies the Elasticity and Regularity assumptions along with either

(i) \( F'(0) = R \), or (ii) \( F'(0) \leq R \) and \( R < \frac{F'(F'^2 + yF''^2)}{F'^2 + 2yF''} \) for all \( y \geq 0 \).
Proof. We are studying the partial regulation case, in which banks are free to choose their liquidity ratio $b_i$, but the regulator limits their choice of $n_i$. Therefore, we can write banks’ expected profit function at $t = 0$ as follows
\[
\max_{b_i} \Pi_i(b_i; n) = (1 - q)\{R + b_i\}n + qR\gamma n - D(n(1 + b_i)). \tag{B.8}
\]
Here, we can treat $n$ like a parameter of the model because banks take it as given. The regulator, in a sense, determines the aggregate amount of $n$. Therefore, the first-order condition of the banks’ problem is
\[
\frac{\partial \Pi(b_i; n)}{\partial b_i} = (1 - q)n + qRn\frac{\partial \gamma}{\partial b_i} - n - 2\phi n^2(1 + b_i) = 0
\]
\[
= (1 - q)n + qR\frac{1}{P} - n - 2\phi n^2(1 + b_i) = 0,
\]
which can be simplified as
\[
q \left(\frac{R}{P} - 1\right) = 2\phi n(1 + b_i). \tag{B.9}
\]
Note that we can obtain the derivative of the equilibrium price with respect to the regulatory parameter, $n$, as follows:
\[
\frac{\partial P}{\partial n} = \frac{F_3(c - b_i)}{F_1 + yF_2}, \tag{B.10}
\]
where $F_k \equiv \frac{d^kF(y)}{dy^k}$ for $k = 1, 2$, and $y$ shows the quantity of assets sold to the traditional sector in fire sales.

Banks’ profit function exhibits increasing differences in $b_i$ and $n$ if its cross derivative is positive. Increasing differences means that $b’(n) > 0$, that is, the optimal choice of $b_i$ in banks’ problem is increasing with the regulatory parameter, $n$. We can obtain the cross derivative of banks’ expected profit as
\[
\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} = (1 - q) + qR\left(\frac{1}{P} - n \frac{\partial P}{P^2 \partial n}\right) - 1 - 4\phi n(1 + b_i).
\]
Substituting for $\phi n(1 + b_i)$ from the banks’ first-order condition (B.9) and using the ex-
pression for $\partial P/\partial n$, given by (B.10), and we can simplify the cross derivative as follows

\[
\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} = (1 - q) + qR \left( \frac{1}{P} - \frac{n F_2(c - b_i)}{P^2 F_1 + y F_2} \right) - 1 - 2q \left( \frac{R}{P} - 1 \right),
\]

\[
= -q + qR \left( \frac{1}{P} - \frac{n(c - b_i)}{P y} \frac{1}{P F_1 + y F_2} \right) - \frac{2qR}{P} + 2q,
\]

\[
= q + qR \left( \frac{1}{P} - \frac{n(c - b_i)}{P y} \frac{1}{P F_1 + y F_2} \right) - \frac{2qR}{P},
\]

where in the second line we manipulated the second term within the parentheses by multiplying and dividing by $y$. Now, use of the equality of $y = n(c - b_i)/P$ in equilibrium and finally substitute $P = F_1$ to get:

\[
\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} = q + qR \left( \frac{1}{P} - \frac{1}{P} \frac{y F_2}{F_1 + y F_2} \right) - \frac{2qR}{P} = q - qR \left( \frac{1}{P} + \frac{1}{P} \frac{y F_2}{F_1 + y F_2} \right) = q \left\{ 1 - R \frac{[F_1 + 2y F_2]}{F_1(F_1 + y F_2)} \right\}.
\]

Increasing differences hold if

\[
\frac{\partial^2 \Pi(b_i; n)}{\partial b_i \partial n} > 0 \Leftrightarrow R < \frac{F_1(F_1 + y F_2)}{F_1 + 2y F_2} \equiv \kappa. \quad (B.11)
\]

Therefore, if we assume that the traditional sector’s technology $F$ satisfies (B.11), we are done. If we do not make this assumption, we can instead assume that $F_1(0) = R$ and show that (B.11) holds for all $y > 0$. Note that when $y$ is equal to zero $\kappa$ is equal to $F_1$ by definition, and we have that $F_1(0) = R$ by assumption. Therefore, in order to show that $\kappa > R$ for all $y > 0$, all we need to show is that $\kappa$ is increasing in $y$. Below we show that the derivative of $\kappa$ with respect to $y$ is indeed positive:

\[
\frac{d\kappa}{dy} = \frac{F_2(F_1 + y F_2) + F_1(F_2 + F_2 + y F_3)][F_1 + 2y F_2]}{(F_1 + 2y F_2)^2},
\]

\[
= \frac{3F_1 F_2 + 2y F_2 + F_1 F_3 y)[F_1 + y F_2 + y F_2] - [F_1(F_1 + y F_2)][F_2 + 2F_2 + 2y F_3]}{(F_1 + 2y F_2)^2}.
\]

(B.12)

Because the denominator of the derivative is positive we focus on the numerator to obtain
the sign of the derivative. The numerator of (B.12) can be simplified as follows:

\[
\frac{d\kappa}{dy} \times (F_1 + 2yF_2)^2 = y(F_2^2 - F_1F_3)(F_1 + yF_2) + yF_2[3F_1F_2 + yF_2 + 2F_1F_3]
\]

\[
= y(F_2^2 - F_1F_3)F_1 + yF_2[yF_2^2 - yF_1F_3 + 3F_1F_2 + yF_2 + 2F_1F_3],
\]

\[
= y(F_2^2 - F_1F_3)F_1 + yF_2[3F_1F_2 + 2yF_2^2].
\]

Divide both sides with \(y\) to simplify further:

\[
\frac{d\kappa}{dy} \times \frac{(F_1 + 2yF_2)^2}{y} = \frac{F_1F_2^2 - F_1F_3 + 3F_1F_2^2 + 2yF_2^3}{y}
\]

\[
= 4F_1F_2 - F_1F_3 + 2yF_2^3 = 2F_1F_2 - F_1F_3 + 2F_1F_2 + 2yF_2^3 = F_1(2F_2^2 - F_1F_3) + 2F_2^2(F_1 + yF_2) > 0.
\]

\(2F_2^2 - F_1F_3\) is positive due to the **Regularity** assumption, and \(F_1 + yF_2\) is positive due to the **Elasticity** assumption.

**Proposition 5.** Under the log-quadratic functional form assumptions, risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium, partial regulation equilibrium, and second best compare as follows:

1. Risky investment levels: \(n^c > n^s > n^p\)
2. Liquidity ratios: \(b^s > b^c > b^p\)
3. Financial stability measures
   - (a) Price of assets in the bad state: \(P^s > P^p > P^c\)
   - (b) Fraction of assets sold: \(1 - \gamma^c > 1 - \gamma^p > 1 - \gamma^s\)
   - (c) Total fire sales: \((1 - \gamma^c)n^c > (1 - \gamma^p)n^p > (1 - \gamma^s)n^s\)

**Proof.** Proof of this proposition is established through a series of lemmas below.

**Lemma 6.** \(P^s > P^p > P^c\)

**Proof.** **Part 1:** \(P^p > P^c\). First, note that we obtain the competitive equilibrium price in Proposition 1 as

\[
P^c = \frac{qR(1 + c)}{R - 1 + q} = \frac{\beta}{R\sigma},
\]

using the definitions of \(\sigma, \beta\) from (C.28) and (C.32). Now, take the cubic equation given by (C.33) which defines the price in partial regulation and divide it by \(R\sigma\) to obtain:

\[
R \left\{ 2\phi \sigma P^3 + (\sigma qR - 2\phi \beta)P - q\beta \right\} + 2\phi \beta P^2 - 2\phi(1 + c)P^3 = 0
\]

\[
R \left[ 2\phi \frac{P^3}{R} + \left( q - \frac{2\phi \beta}{\sigma R} \right) P - \frac{q\beta}{\sigma R} \right] + 2\phi \beta \frac{P^2}{R} - 2\phi(1 + c)\frac{P^3}{R} = 0
\]
Note that $(1 + c)/\sigma = P^c$, and substitute this into the equation above and manipulate:

$$R \left[ \frac{2\phi}{R} P^3 + (q - 2\phi P^c) P - qP^c \right] + 2\phi P^c P^2 - \frac{2\phi}{R} P^c P^3 = 0$$

$$R \left( \frac{2\phi}{R} P^2 + q \right) P - \left( 2\phi R P + qR - 2\phi P^2 + \frac{2\phi}{R} P^3 \right) P^c = 0$$

From this last equivalence we can obtain the price ratios in these two cases as:

$$\frac{P^c}{P} = \frac{2\phi P^2 + qR}{2\phi P^3 - 2\phi P^2 + 2\phi RP + qR} = \frac{2\phi RP^2 + qR^2}{2\phi P^3 - 2\phi RP^2 + 2\phi R^2 P + qR^2}, \quad \text{(B.13)}$$

where $P = P^p$. In order to show that $P^c < P^c$, we need to show that the numerator of this ratio is less then its denominator, that is

$$2\phi RP^2 + qR^2 < 2\phi P^3 - 2\phi RP^2 + 2\phi R^2 P + qR^2$$

$$0 < (R - P)^2$$

The last inequality holds because we must have $P^p < R$ in equilibrium. $P^p < R$ holds in equilibrium for the following reason: Assumption Efficiency states that $P^p \leq R$, yet the equality cannot arise in equilibrium as $P^p = R$ implies $P^c = R$ as well due to (B.13). However, given the solution for $P^c$ in Proposition 1, $P^c < R$ holds due to the Technology assumption, $R - cq - 1 > 0$. Thus, we must have $P^p < R$.

**Part 2: $P^s > P^p$.** First, note that

$$R - 1 - qc = R - 1 + q - q - qc = R - 1 + q - q(1 + c) = qR\sigma - q(1 + c) = q(\sigma R - 1 - c),$$

where $\sigma, \beta$ are defined by (C.28) and (C.32). Using this equivalence we can write the polynomial equation that defines $P^s$, equation (C.17), as

$$(R - 1 - qc)P^s^2 + q\beta P^s - qR\beta = 0$$

$$q(\sigma R - 1 - c)P^s^2 + q\beta P^s - qR\beta = 0$$

$$\sigma R - 1 - c \frac{R^2}{P^s} + \frac{\beta}{R} P^s = \beta$$

Now substitute $\beta$ using the last equation above into the cubic equation that gives $P^p$,
equation (C.33):

\[
2\phi(\sigma R - 1 - c)P^3 + 2\phi\beta P^2 + R(\sigma q R - 2\phi \beta)P - q R\beta = 0
\]

\[
2\phi(\sigma R - 1 - c)P^3 + 2\phi\beta P^2 + R\left(\sigma q R - 2\phi \frac{\sigma R - 1 - c}{P} - 2\phi \frac{\beta}{P^2}\right)P - q R\beta = 0
\]

\[
2\phi(\sigma R - 1 - c)P^3 + 2\phi\beta P^2 + \sigma q R^2 P - 2\phi(\sigma R - 1 - c)P^2 P - 2\phi\beta P P - q R\beta = 0
\]

\[
(2\phi\sigma R - 1 - c)P(P^2 - P^2) + 2\phi\beta P(P - P) + q R(\sigma R P - \beta) = 0 \quad \text{(B.14)}
\]

where \( P \) solves this equation is \( P_p \). Note that the first two terms in (B.14) must have the same sign, and their sum will have the opposite sign of the last term, \( q R(\sigma R P^p - \beta) \). Therefore, in order to show that \( P^p - P^s < 0 \), we need to show that \( q R(\sigma R P^p - \beta) > 0 \). We can write this last terms as

\[
q R(\sigma R P^p - \beta) = q R^2 \sigma P^p - q(1 + c)R^2 > 0 \iff \sigma P^p - 1 - c > 0.
\]

Note that by Part 1, we know that \( P^c < P^p \). Hence, if \( \sigma P^c - 1 - c \geq 0 \) then we must necessarily have \( \sigma P^p - 1 - c > 0 \). Using the closed-form solution of the competitive equilibrium, given by (8), we can show that:

\[
\sigma P^c - 1 - c = \frac{R - 1 + q}{q R} \frac{q R(1 + c)}{R - 1 + q} - 1 - c = 0
\]

Therefore, we must have \( \sigma P^p - 1 - c > 0 \), which implies that \( P^s > P^p \) in order for equation (B.14) to hold.

\[\square\]

**Lemma 7.** \( b^s > b^c > b^p \)

**Proof.** Part 1: \( b^s > b^c \). Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (C.4) and (C.18) as

\[
b^c = \frac{c q (\tau^c + 1) - 2\phi R}{q (\tau^c + 1) + 2\phi R}, \quad b^s = \frac{c q (\tau^s + 1)^2 - 2\phi R}{q (\tau^s + 1)^2 + 2\phi R}.
\]

Comparing the liquidity ratios under competitive equilibrium \( (b^c) \) and under the constrained planner’s solution \( (b^s) \), we see that they have the same following functional form:

\[
f(x) = \frac{c q x - 2\phi R}{q x + 2\phi R} \quad \text{(B.15)}
\]

The only difference is \( x = \tau^c + 1 \) in the competitive case versus \( x = (\tau^s + 1)^2 \) in the constrained planner’s problem. First, note that

\[
f'(x) = \frac{c q (q x + 2\phi R) - (c q x - 2\phi R) q}{(q x + 2\phi R)^2} = \frac{2\phi R q (1 + c)}{(q x + 2\phi R)^2} > 0 \quad \text{(B.16)}
\]

Therefore, in order to show that \( b^s > b^c \), all we need to show is that \( (\tau^s + 1)^2 > \tau^c + 1, \)
which can be written equivalently as:

\[
\frac{R^2}{P^s} > \frac{R}{P^c} \iff P^s < RP^c.
\]

Now, substitute \(P^s\) from the solution to the constrained planner’s problem, given by (C.17) and the competitive equilibrium price, \(P^c\), from (8) to write this inequality as:

\[
q\beta(R - P^s) < \frac{qR(1 + c)}{R - 1 + q}, \quad R - P^s < \frac{R - 1 - qc}{R - 1 + q}, \quad R \left(1 - \frac{R - 1 - qc}{R - 1 + q}\right) < P^s, \quad \frac{Rq(1 + c)}{R - 1 + q} = P^c < P^s.
\]

The last inequality holds by Lemma 6. Therefore, \((\tau^s + 1)^2 > \tau^c + 1\), which implies that \(b^s > b^c\).

**Part 2:** \(b^c > b^p\). Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (C.4) and (C.21) as:

\[
b^c = \frac{cq(\tau^c + 1) - 2\phi R}{q(\tau^c + 1) + 2\phi R}, \quad b^p = \frac{cq(\tau^p + 1) - 2\phi R}{q(\tau^p + 1) + 2\phi R}.
\]

Comparing the liquidity ratios under competitive equilibrium \((b^c)\) and under the partial regulation case \((b^p)\), we see that they have the same functional form, \(f(x)\), given above by (B.15). The only difference is \(x = \tau^c + 1\) in the competitive case versus \(x = \tau^p + 1\) in the partial case. We have shown above, by (B.16), that \(f'(x) > 0\). Therefore, in order to show that \(b^c > b^p\), all we need to show is that \(\tau^c > \tau^p\). Note that because \(\tau^p = R/P^p - 1\) and \(\tau^c = R/P^c - 1\), and \(P^p > P^c\) by Lemma 6, we have that \(\tau^c > \tau^p\). This completes the proof. 

**Lemma 8.** \(n^c > n^s > n^p\)

**Proof.** **Part 1:** \(n^c > n^s\). Using the closed-form solution for the competitive equilibrium, (C.5), and for the constrained planner’s problem, (C.19), the difference in risky investment
levels across these two cases can be written as

\[
    n^c - n^s = \frac{\tau^c}{\tau^c + 1} \frac{q(\tau^c + 1) + 2\phi R}{2\phi(1 + c)} - \frac{\tau^s}{\tau^s + 1} \frac{q(\tau^s + 1)^2 + 2\phi R}{2\phi(1 + c)}
\]

\[
= \left\{ \frac{1}{2\phi(1 + c)} \left[ \frac{\tau^c}{\tau^c + 1} q(\tau^c + 1) + 2\phi R \left( \tau^c + 1 \right) - \tau^s \left( \tau^s + 1 \right) q(\tau^s + 1)^2 + 2\phi R \left( \tau^s + 1 \right) \right] \right\}
\]

\[
= \left\{ \frac{1}{2\phi(1 + c)} \left[ \left( \tau^c - \tau^s(\tau^s + 1) + \frac{2\phi R}{\tau^c + 1} \left( \tau^c + 1 \right) \right) - \frac{2\phi R}{\tau^s + 1} \right]\right\}
\]

First, note that \( \tau^c = R/P - 1 > \tau^s = R/P^s - 1 \) by Lemma 6, and this implies that \( 2\phi R \left( \tau^c + 1 \right) - 2\phi R \left( \tau^s + 1 \right) \) is positive. Therefore, \( n^c - n^s \) is positive if \( q \tau^c - q \tau^s(\tau^s + 1) \geq 0 \). Next, we show that this inequality indeed holds. From (C.16) we have \( R - 1 - qc = \frac{qR(R-P^s)(1+c)}{P^c} \), which implies that:

\[
    \tau^c = \frac{R - 1 - qc}{q(1 + c)} = \frac{R(R - P^s)}{P^s} = \frac{R}{P^s} \left( \frac{R}{P^s} - 1 \right) = \tau^s(\tau^s + 1),
\]

where we use that \( \tau^c = R/P^c - 1 \) and \( P^c = \frac{qR(1+c)}{R-1+q} \), as given by 8.

**Part 2: \( n^s > n^p \).** For this part, we use the fact that \( P^s > P^p \) as shown by Lemma 6. Using the closed-form solution for \( n^s \) from (C.19) and \( n^p \) from (C.22), we can write the difference in risky investment levels across these two cases as:

\[
n^s - n^p = \frac{\tau^s}{\tau^s + 1} \frac{q(\tau^s + 1)^2 + 2\phi R}{2\phi(1 + c)} - \frac{\tau^p}{\tau^p + 1} \frac{q(\tau^p + 1) + 2\phi R}{2\phi(1 + c)},
\]

\[
= \left\{ \frac{1}{2\phi(1 + c)} \left[ \frac{\tau^s}{\tau^s + 1} q(\tau^s + 1)^2 + 2\phi R \left( \tau^s + 1 \right) - \tau^p \left( \tau^p + 1 \right) q(\tau^p + 1) + 2\phi R \right] \right\},
\]

\[
= \left\{ \frac{1}{2\phi(1 + c)(\tau^p + 1)(\tau^s + 1)} \right\}
\]

where

\[
\Theta \equiv q(\tau^s + 1)(\tau^p + 1)\left[\tau^s(\tau^s + 1) - \tau^p \right] + 2\phi R \left[ \tau^s(\tau^p + 1) - \tau^p(\tau^s + 1) \right]
\]

\[
= q(\tau^s + 1)(\tau^p + 1)\left[\tau^c - \tau^p \right] + 2\phi R \left[ \tau^s - \tau^p \right],
\]

where we use the equivalence, \( \tau^c = \tau^s(\tau^s + 1) \), obtained in Part 1 above. Since the denominator of (B.17) is positive, in order to prove that \( n^s - n^p > 0 \), it suffices to show that \( \Theta > 0 \). In order to show that this inequality holds, first, we write \( 2\phi R \) that shows up \( \Theta \) in terms of \( \tau^c \)'s. For that, we start from the cubic equation that gives the partial equilibrium price as obtained by (C.33):
This inequality holds because

This inequality can be simplified further as follows:

\[
0 = \frac{2\phi}{q} (R - 1 - qc) P^2 + 2\phi R (1 + c) P^2 + R^2 (\sigma q - 2\phi (1 + c)) P - (1 + c) q R^2,
\]

\[
0 = \frac{2\phi}{q} (1 + c) R P^3 + 2\phi R (1 + c) P^2 + R (R - 1 + q - 2\phi R (1 + c)) P - (1 + c) q R^2,
\]

\[
0 = 2\phi (1 + c) R P^3 + 2\phi R (1 + c) P^2 + R \left( \frac{q R (1 + c)}{P c} - 2\phi R (1 + c) \right) P - (1 + c) q R^2,
\]

\[
0 = 2\phi (1 + c) R P^3 + 2\phi R (1 + c) P^2 + \frac{q R^2 (1 + c)}{P c} P - 2\phi R^2 (1 + c) P - (1 + c) q R^2,
\]

\[
0 = 2\phi (1 + c) \frac{R^3}{(\tau + 1)^3} + 2\phi (1 + c) \frac{R^2}{(\tau + 1)^2} \frac{q R^2 (1 + c)}{P c} \frac{R}{\tau + 1} - 2\phi R^2 (1 + c) \frac{R}{\tau + 1} - (1 + c) q R^2,
\]

\[
0 = 2\phi (1 + c) R^3 \frac{R^3}{(\tau + 1)^3} + 2\phi (1 + c) \frac{R^3}{(\tau + 1)^2} + q R^2 (1 + c) \frac{\tau^c + 1}{R} \frac{R}{\tau + 1} - 2\phi (1 + c) \frac{R^3}{\tau + 1} - (1 + c) q R^2,
\]

\[
0 = 2\phi (1 + c) \frac{R^3}{(\tau + 1)^3} \left[ \tau^c + \tau + 1 - (\tau + 1)^2 \right] + q R^2 (1 + c) \left[ \frac{\tau^c + 1}{\tau + 1} - 1 \right],
\]

\[
0 = \frac{2\phi R}{(\tau + 1)^2} [\tau^c - (\tau + 1)] - q(\tau - \tau^c),
\]

in which \( P \) stands for \( P^p \) and \( \tau \) for \( \tau^p \). In the first line we use definition of \( \sigma \), given by (C.28), to write \( \sigma R - 1 - c = (R - 1 - qc)/q \), while using \( \tau^c = R/P^c - 1 = (R - 1 - qc)/(q(1 + c)) \) in the second line. In the third line we replaced \( R - 1 + q \) with \( \frac{q R (1 + c)}{P c} \) using equation (8) for price in competitive equilibrium and later we use \( P = R/(\tau + 1) \) to replace \( P \). From the last equation above we can obtain:

\[
2\phi R = \frac{q(\tau^p + 1)^2 (\tau^p - \tau^c)}{\tau^c - \tau^p (\tau^p + 1)} = \frac{q(\tau^p + 1)^2 (\tau^p - \tau^c)}{\tau^s (\tau^s + 1) - \tau^p (\tau^p + 1)},
\]

where we use the equivalence, \( \tau = \tau^s (\tau^s + 1) \), again. Now we plug this expression for \( 2\phi R \) back into (B.18) and show below that \( \Theta > 0 \) holds:

\[
q(\tau^s + 1)(\tau^p + 1)[\tau^c - \tau^p] > 2\phi R [\tau^p - \tau^s] = \frac{q(\tau^p + 1)^2 (\tau^p - \tau^c)}{\tau^s (\tau^s + 1) - \tau^p (\tau^p + 1)}[\tau^p - \tau^s]
\]

\[
\tau^s + 1 > \frac{(\tau^p + 1)(-1)(\tau^p - \tau^s)}{\tau^s (\tau^s + 1) - \tau^p (\tau^p + 1)}
\]

\[
\tau^s + 1 > \frac{(\tau^p + 1)(\tau^p - \tau^s)}{\tau^p (\tau^p + 1) - \tau^s (\tau^s + 1)}
\]

This inequality can be simplified further as follows:

\[
(\tau^s + 1) \tau^p (\tau^p + 1) - \tau^s (\tau^s + 1)^2 > (\tau^p + 1)(\tau^p - \tau^s)
\]

\[
(\tau^p + 1)[\tau^p (\tau^s + 1) - (\tau^p - \tau^s)] > \tau^s (\tau^s + 1)^2
\]

\[
(\tau^p + 1)\tau^s (\tau^p + 1) > \tau^s (\tau^s + 1)^2
\]

\[
(\tau^p + 1)^2 > (\tau^s + 1)^2.
\]

This inequality holds because \( P^s > P^p \), as shown by Lemma 6, which implies that \( \tau^p > \tau^s \),
using the definitions $\tau_p = \frac{R}{P^p} - 1$ and $\tau_s = \frac{R}{P^s} - 1$.

**Lemma 9.** $1 - \gamma^c > 1 - \gamma^p > 1 - \gamma^s$

**Proof.**

$$1 - \gamma = \frac{c - b}{P}$$
together with $b^s > b^p$ and $P^s > P^p \implies 1 - \gamma^p > 1 - \gamma^s$

To obtain $(1 - \gamma^c) > (1 - \gamma^p)$, we can equivalently show that $\frac{c - b^c}{P^c} > 1$.

Using equations (C.4) and (C.21) for $b^c$ and $b^p$ respectively, $b^c = \frac{2\phi R(1+c)}{2\phi R + q(\tau^c + 1)} \implies c - b^p = \frac{2\phi R(1+c)}{2\phi R + q(\tau^p + 1)}$. Writing $\tau^c$ and $\tau^p$ in terms of $P^p$ and $P^c$ we get the following,

$$\frac{c - b^p}{P^p} = \frac{c - b^c}{P^p} = \frac{2\phi P^c + q (P^c)}{2 \phi P^c + q P^p} > 1.$$

The last inequality holds because $P^p > P^c$ by Lemma 6.

**Lemma 10.** $(1 - \gamma^c)n^c > (1 - \gamma^p)n^p > (1 - \gamma^s)n^s$

**Proof.** Given that the demand function for risky assets in the interim period is downward sloping (continuous and differentiable as well), the prices disclose the amount of fire sales. Hence, we can use the results in Lemma 6 to prove this lemma:

$$(1 - \gamma)n = \frac{R}{P} - 1 \text{ and } P^s > P^p \implies (1 - \gamma^s)n^s < (1 - \gamma^p)n^p < (1 - \gamma^c)n^c.$$  

**Proposition 6.** Under the log-quadratic functional form assumptions, bank balance sheet sizes across different regimes compare as follows:

$$n^c(1 + b^c) = n^s(1 + b^s) > n^p(1 + b^p).$$

**Proof.** Using the closed-form solutions in Sections C.1 and C.2, we can write the bank size under the competitive equilibrium and constrained planner’s problem as follows:

$$n^c(1 + b^c) = \frac{\tau^c}{\tau^c + 1} \frac{2\phi R + q(\tau^c + 1)}{2\phi (1 + c)} = \frac{\tau^c}{\tau^c + 1} \frac{q(\tau^c + 1)}{2\phi} = \frac{q \tau^c}{2\phi}.$$

$$n^s(1 + b^s) = \frac{\tau^s}{\tau^s + 1} \frac{2\phi R + q(\tau^s + 1)^2}{2\phi (1 + c)} = \frac{\tau^s}{\tau^s + 1} \frac{q \tau^s (\tau^s + 1)}{2\phi}.$$
Part I of Lemma 8 we show that \( \tau^c = \tau^s(\tau^s + 1) \). Thus, comparing the equations above we conclude \( n^c(1 + b^c) = n^s(1 + b^s) \).

Lastly, \( b^* > b^c > b^p \), as shown in Lemma 7, and \( n^c > n^s > n^p \), as shown in Lemma 8, together imply that \( n^c(1 + b^c) > n^p(1 + b^p) \), that is, the bank balance sheet size is the smallest under partial regulation.

**Proposition 7.** Under the Efficiency, Elasticity, Regularity, and Technology assumptions, banks do not choose the constrained optimal risky investment level, \( n^s \), if the regulator sets the minimum liquidity ratio at the constrained optimal level, \( b^s \); that is, \( n_i(b^s) \neq n^s \).

**Proof.** We first study bank behavior under liquidity regulation alone. In this case, the regulator chooses the optimal liquidity ratio, \( b \), at \( t = 0 \) to maximize the net expected social welfare but allows banks to freely choose their risky investment level, \( n_i \). Consider the problem of a bank first: For a given regulatory liquidity ratio, \( b \), a bank chooses the level of risky investment, \( n_i \), to maximize its expected profits:

\[
\max_{n_i} \Pi_i(n_i; b) = \max_{n_i} (R + b - qc)n_i - D(n_i(1 + b)) - q(R - P)Q_i^s(P, n_i, b) \quad (B.19)
\]

The first-order condition of the banks’ problem (B.19) with respect to \( n_i \) is

\[
R + b - qc - D'(n_i(1 + b))(1 + b) - q(R - P)\frac{\partial Q_i^s}{\partial n_i} = 0, \quad (B.20)
\]

where \( Q_i^s(P, n_i, b) = (1 - \gamma)n_i - \frac{c - bP}{P}n_i \).

We then compare this first-order condition with the corresponding one of the constrained planner’s problem, given by (C.12). These two first-order conditions are written explicitly below for comparison:

\[
\Psi \equiv (1 - q)(R + b) + qR\left\{ \gamma + \frac{\partial \gamma}{\partial n} \right\} + q \left\{ F'((1 - \gamma)n) \left( 1 - \gamma - \frac{\partial \gamma}{\partial n} \right) - c + b \right\} - D'(\cdot)(1 + b) = 0.
\]

\[
\Upsilon \equiv (1 - q)(R + b) + qR\gamma - D'(\cdot)(1 + b) = 0.
\]

The constrained planner’s first-order condition, \( \Psi \), includes extra terms because the planner internalizes the effects of fire sale externalities. These extra terms are:

\[
Z = qR\frac{\partial \gamma}{\partial n} + q \left\{ F'((1 - \gamma)n) \left( 1 - \gamma - \frac{\partial \gamma}{\partial n} \right) - c + b \right\}
\]
Hence, we can write $\Psi = \Upsilon + Z$. We first show that the sum of these extra terms is negative:

$$Z = qR \frac{\partial \gamma}{\partial n} n + q \left\{ F'((1 - \gamma)n) \left( 1 - \gamma - \frac{\partial \gamma}{\partial n} n \right) - c + b \right\}$$

$$= qR \frac{\partial \gamma}{\partial n} n + q \left\{ P \left( \frac{c - b}{P} - \frac{\partial \gamma}{\partial n} n \right) - c + b \right\}$$

$$= qR \frac{\partial \gamma}{\partial n} n + q \left\{ c - b - \frac{\partial \gamma}{\partial n} nP - c + b \right\}$$

$$= qR \frac{\partial \gamma}{\partial n} n - qP \frac{\partial \gamma}{\partial n} = q \frac{\partial \gamma}{\partial n} (R - P) < 0,$$

where we use that in equilibrium $F'((1 - \gamma^s)n^s) = P^s$. The sign of $Z$ is negative because $R > P^s$ by the Efficiency assumption, and $\partial \gamma/\partial n < 0$ by Lemma 1.

$Z < 0$ implies that banks’ first-order condition, $\Upsilon$, evaluated at the constrained efficient allocations, $n^s, b^s$ is positive, that is $\Upsilon(n^s, b^s) > 0$. On the contrary, we have $\Upsilon(n(b^s), b^s) = 0$ by definition of optimality. Furthermore, we can show that $\Upsilon$ is decreasing in $n$ for a given $b$, that is:

$$\frac{\partial \Upsilon}{\partial n} = qR \frac{\partial \gamma}{\partial n} - D''(\cdot)(1 + b)^2 < 0,$$

because $D''(\cdot) > 0$ by assumption and $\partial \gamma/\partial n < 0$ by Lemma 1. Therefore, we must have $n(b^s) > n^s$.

\[ \square \]

\textbf{Lemma 5.} If $r_{cb}$ is high enough, it is possible to improve upon the competitive equilibrium, by inducing a higher price for risky assets, that is, $P_{cb} > P_c$.

\textit{Proof.}

$$\frac{qR(1 + c - \tau_{cb})}{R - 1 + q(1 - \tau_{cb} r_{cb})} > \frac{qR(1 + c)}{R - 1 + q}$$

$$(R - 1 + q)(1 + c) - (R - 1 + q)r_{cb} > (R - 1 + q)(1 + c) - q\tau_{cb} r_{cb} (1 + c)$$

$$- (R - 1 + q)r_{cb} > - q\tau_{cb} r_{cb} (1 + c)$$

$$R - 1 + q < r_{cb}(1 + c)$$

$$\frac{R - 1 + q}{(1 + c)} < r_{cb}$$

\[ \square \]
C  Closed-form solutions

C.1  A closed-form solution for the competitive equilibrium

Suppose that the operational cost of a bank is given by $\Phi(x) = \phi x^2$ and let the traditional sector’s technology function be given by $F = R \ln(1 + y)$. Firms in traditional sector choose how much assets, $y$, to buy from banks in the bad state at $t = 1$ to maximize their profits, $F(y) - Py$, where $P$ is the price of assets. The first-order condition of this problem yields (inverse) demand function of firms in traditional sector for risky assets:

$$P = F'(y) = \frac{R}{1 + y} \quad \text{and hence} \quad y = F'^{-1}(P) = \frac{R}{P} - 1 \equiv Q^d(P). \quad (C.1)$$

We solve for the competitive equilibrium price, $P$, in the main text, as shown by (8).\(^{51}\) Now, use this solution in the demand side function and define the total amount of assets purchased by the traditional sector, $\tau$, in terms of the exogenous variables as follows:

$$y = \frac{R}{P} - 1 = \frac{R - 1 + q}{q(1 + c)} - 1 \equiv \tau^c. \quad (C.2)$$

We obtain the total supply of asset by banks as $(1 - \gamma)n$ by (2) in Section 3.1.2. Hence, the market clearing condition, $(1 - \gamma)n = \tau$, yields:

$$(c - b)n = P\tau \implies n = \frac{P\tau}{c - b}. \quad (C.3)$$

This equation gives the investment level, $n$, as a function of the liquidity ratio, $b$. We can solve for the latter from the first-order conditions of banks’ problem in the decentralized case, given by (B.4-B.5), as derived in the proof of Proposition 1 below. Using $\frac{R}{P} = \tau + 1$ from (C.2) and the functional form of the operational cost, $\Phi'(n(1 + b)) = 2\phi n(1 + b)$, in the first-order condition with respect to $b$, given by (B.5) yields:

$$1 - q + q(\tau + 1) = 1 + 2\phi n(1 + b),$$

$$1 + q\tau = 1 + 2\phi \frac{P\tau}{c - b}(1 + b),$$

where in the second line we use $n = P\tau/(c - b)$ from (C.3). Substituting $R/(\tau + 1)$ for $P$ from (C.2) yields

$$c - b = 2\phi \frac{R(1 + b)}{q \tau + 1}.\quad (C.4)$$

---

\(^{51}\)There is a unique equilibrium in the fire sale market under the Elasticity assumption. Gai et al. (2008) provide an example in which this assumption is not satisfied and thus multiple equilibria exist. The ex-ante beliefs of agents determine the choice of equilibrium, and the authors show that the irrespective of the beliefs, the competitive equilibrium is constrained inefficient and leads to overinvestment.
Finally, rearrange to obtain the liquidity ratio in the competitive equilibrium as

$$b^c = \frac{cq(\tau^c + 1) - 2\phi R}{q(\tau^c + 1) + 2\phi R}. \tag{C.4}$$

To obtain the risky investment level in the competitive equilibrium substitute this expression for $b$ in (C.3):

$$n^c = \frac{\tau^c q(\tau^c + 1) + 2\phi R}{2\phi(1 + c)} \tag{C.5}$$

### C.2 A closed-form solution for the constrained planner’s problem

Lemma 3 allows us to focus on the $b < c$ case when analyzing the constrained planner’s problem which simplifies as follows:

$$\max_{n,b,y} \Gamma(n,b) - q\{(R - P)Q^s(P,n,b)\} - (1 - q)T_2,$$

subject to $y = Q^s(P,n,b)$,

$$F'(y) = P,$$

$$(1 - q)T_2 + 3\omega + q[F(y) - Py] \geq U_{i,CE}, \tag{C.7}$$

where $Q^s = \frac{c-b}{P} n$. The transfers must be such that consumers receive in expectation what they lose from the change in the amount of fire sales and price of assets:

$$(1 - q)T_2 = q[F(y^c) - P^c y^c - F(y) + Py]. \tag{C.8}$$

In other words, the constraint on consumers’ expected utility binds. Substituting this value for transfers back into the planner’s problem helps us to get rid of the constraint on consumers’ expected utility:

$$\max_{n,b,y} \Gamma(n,b) - q\{(R - P)Q^s(P,n,b)\} + q[F(y) - Py] - q[F(y^c) - P^c y^c],$$

subject to $y = Q^s(P,n,b)$,

$$F'(y) = P.$$

Corresponding first-order conditions with respect to $x \in \{n,b\}$ are, respectively,

$$\frac{\partial \Gamma}{\partial x} - q\left[(R - P)\left(\frac{\partial Q^s}{\partial x} + \frac{\partial Q^s}{\partial P} \frac{\partial P}{\partial x}\right) - Q^s \frac{\partial P}{\partial x} - (F'(y) - P)\left(\frac{\partial y}{\partial x} + \frac{\partial y}{\partial P} \frac{\partial P}{\partial x}\right) + y \frac{\partial P}{\partial x}\right] = 0. \tag{C.10}$$

Using an envelope argument, $F'(y) = P$, and the market clearing condition $y = Q^s$, we can simplify the optimality condition for welfare to:

$$\frac{\partial \Gamma}{\partial x} - q(R - P) \frac{\partial Q^s}{\partial x} - q(R - P) \frac{\partial Q^s}{\partial P} \frac{\partial P}{\partial x} = 0, \ \forall x \in \{n,b\}. \tag{C.11}$$

Using the equilibrium condition $y = Q^s = (1 - \gamma)n = \frac{(c-b)n}{P}$ and $\Gamma(n_i, b_i) \equiv (R + b_i -
\( q_c n_i - D(n_i(1 + b_i)) \) we can write the the first-order conditions of the planner’s problem with respect to \( n \) and \( b \) are respectively:

\[
(1 - q)(R + b) + qR\left\{ \gamma + \frac{\partial \gamma}{\partial n} n \right\} + q \left\{ F'((1 - \gamma)n) \left( 1 - \gamma - \frac{\partial \gamma}{\partial n} n \right) - c + b \right\} = D'(n(1 + b))(1 + b), \quad (C.12)
\]

\[
(1 - q)n + qR\frac{\partial \gamma}{\partial b} n + q \left\{ F'((1 - \gamma)n) \left( -\frac{\partial \gamma}{\partial b} \right) n + n \right\} = D'(n(1 + b))n, \quad (C.13)
\]

where \( \gamma = 1 + \frac{b - c}{P} \) from banks’ problem in the bad state, as obtained in Section 3.1.2.

Combining the two first-order conditions to obtain:

\[
(1 - q)(R + b) + qR\left\{ \gamma + \frac{\partial \gamma}{\partial n} n \right\} + q \left\{ F'((1 - \gamma)n) \left( 1 - \gamma - \frac{\partial \gamma}{\partial n} n \right) - c + b \right\} = \\
\left[ (1 - q) + qR\frac{\partial \gamma}{\partial b} + qF'((1 - \gamma)n) \left( -\frac{\partial \gamma}{\partial b} \right) + q \right] (1 + b). \quad (C.14)
\]

From this point on we use the log-quadratic functional form assumptions in order to get closed form solutions to the planner’s problem. First, note that using the functional form for traditional sector’s demand, given by (C.1), in the market clearing condition yields the price of assets in the bad state as a function of initial portfolio allocations:

\[
E(P, n, b) = Q^d(P) - Q^s(P, n, b) = 0 \implies \frac{R - P}{P} = \frac{c - b}{P} n \implies P = R - (c - b)n. \quad (C.15)
\]

Substituting \( \frac{\partial \gamma}{\partial n} = -\frac{(c-b)^2}{P^2} \) and \( \frac{\partial \gamma}{\partial b} = \frac{R}{P^2} \), and later \( P = R - (c - b)n \) into (C.14) and noting that \( F'((1 - \gamma)n) = P \) yields:

\[
(1 - q)(R + b) + qR\left\{ 1 - \frac{c - b}{P} - \frac{(c-b)^2}{P^2} n \right\} + q \left\{ P \left( \frac{c - b}{P} + \frac{(c-b)^2}{P^2} n \right) - c + b \right\} = \\
\left[ (1 - q) + qR\frac{R}{P^2} - qP\frac{R}{P^2} + q \right] (1 + b),
\]

or equivalently:

\[
(1 - q)(R - 1) + (1 - q)(1 + b) + qR - qR\left\{ \frac{(c-b)P + (c-b)^2}{P^2} n \right\} + q \left\{ P\left( c-b\right)P + \frac{(c-b)^2}{P^2} n \right\} - c + b \right\} = \\
(1 - q)(1 + b) + q\frac{R}{P^2}(R - P)(1 + b) + q(1 + b).
\]

Note that \( (c-b)P + (c-b)^2n = (c-b)[R - (c-b)n] + (c-b)^2n = R(c-b) \). Substitute this equivalance into the equation above and simplify:

\[
R - 1 + q - qR\frac{R(c-b)}{P^2} + qP\frac{R(c-b)}{P^2} - qc + qb = q\frac{R}{P^2}(R - P)(1 + b) + q + qb.
\]
\[ R - 1 - qc = \frac{qR}{P^2} \{(R - P)(1 + b) + R(c - b) - P(c - b)\} \]
\[ R - 1 - qc = \frac{qR}{P^2} \{(R - [R - (c - b)n](1 + b) + R(c - b) - [R - \gamma(c - b)n](c - b)\} \]
\[ R - 1 - qc = \frac{qR}{P^2} \{(R - R + (c - b)n(1 + b) + R(c - b) - R(c - b) + (c - b)^2n\} \]
\[ R - 1 - qc = \frac{qR}{P^2} \{(c - b)n(1 + b) + (c - b)^2n\} \]

Further simplification yields:

\[ R - 1 - qc = \frac{qR(c - b)n(1 + c)}{P^2} \]
\[ R - 1 - qc = \frac{qR(R - P)(1 + c)}{P^2}, \tag{C.16} \]

where we substitute \( P = R - (c - b)n \) in the second line using the market clearing condition (C.15), and \((c - b)n = R - P\) using the same condition again in the last line above. From (C.16) we obtain the following quadratic equation in \( P \):

\[ (R - 1 - qc)P^2 + qR(1 + c)P - qR^2(1 + c) = 0, \tag{C.17} \]

which we can solve for the price of assets under constrained planner’s solution, \( P^s \):

\[ P^s = \frac{-qR(1 + c) + \sqrt{q^2R^2(1 + c)^2 + 4(R - 1 - qc)qR^2(1 + c)}}{2(R - 1 - qc)}. \]

We can define \( \tau^s \equiv R/P^s - 1 \) similar to (C.2) to represent the total amount of assets sold under fire sales to the traditional sector in terms of the model parameters, and write risky investment as a function of the liquidity ratio as \( n^s = P^s \tau^s/(c - b^s) \) using the market clearing condition, similar to (C.15).

We use these equations to solve for the constrained efficient portfolio allocations \( n^s, b^s \). For that start from the first-order condition with respect to \( b \) given above by (C.13):

\[ 1 - q + qR \frac{\partial \gamma}{\partial b} + q \left\{ F'(1 - n) \left\{ \frac{\partial \gamma}{\partial b} \right\} + 1 \right\} = D'(n(1 + b)), \]
\[ 1 - q + qR \frac{R}{P^2} + q \left\{ -P \frac{R}{P^2} + 1 \right\} = 1 + 2\phi n(1 + b), \]
\[ q \frac{R^2}{P^2} - q \frac{R}{P} = 2\phi n(1 + b). \]
Writing all endogenous variables in terms of $\tau^s$ and simplifying yields
\[
q(\tau^s + 1)^2 - q(\tau^s + 1) = 2\phi \frac{P_{\tau^s}}{c - b^s}(1 + b^s),
\]
\[
q(\tau^s + 1)(\tau^s + 1 - 1) = 2\phi \frac{R}{\tau^s + 1 c - b^s}(1 + b^s),
\]
\[
q(\tau^s + 1)^2 \tau^s (c - b^s) = 2\phi R \tau^s (1 + b^s),
\]
\[
q(\tau^s + 1)^2 c - 2\phi R = b^s \{2\phi R + q(\tau^s + 1)^2\},
\]
where we use $R/P^s = \tau^s + 1$ and $n^s = P^s \tau^s/(c - b^s)$. For future reference, using the second from the last number, we can obtain the liquidity shortage per risky asset in the constrained planner’s solution as
\[
c - b^s = \frac{2\phi R (1 + b^s)}{q(\tau^s + 1)^2}.
\]
We can obtain the closed-form solution for the constrained efficient liquidity ratio, $b^*$, by rearranging the last equation above, as
\[
b^* = \frac{cq(\tau^s + 1)^2 - 2\phi R}{q(\tau^s + 1)^2 + 2\phi R}.
\]
Finally, we can obtain the closed-form solution for the risky investment level by substituting $b^*$ into $n^s = P^s \tau^s/(c - b)$ and using $P^s = R/(\tau^s + 1)$
\[
n^s = \frac{P_{\tau^s}}{c - b^s} = \frac{R_{\tau^s} q(\tau^s + 1)^2 + 2\phi R}{\tau^s + 1 2\phi R(1 + c)} = \frac{\tau^s}{\tau^s + 1} \frac{q(\tau^s + 1)^2 + 2\phi R}{2\phi(1 + c)}.
\]

C.3 A closed-form solution for the partial regulation case

In the partial regulation case, we consider the problem of a planner who chooses the level of risky investment, $n_i$, at $t = 0$ in a Pareto efficient way but allows banks to freely choose their liquidity ratio, $b_i$. We first analyze banks’ problem in this setting and then turn to the planner’s problem. When the planner’s optimal investment level is introduced as a regulatory upper bound on investment level, $n_i$, banks set $n_i = n$ and choose the liquidity ratio, $b_i$, to maximize their expected profits:
\[
\max_{b_i} \Pi_i(b_i; n) = (1 - q)\{R + b_i\}n + qR\gamma_2n - D(n(1 + b_i)).
\]
The first-order condition of the banks’ problem (C.3) with respect to $b_i$ is
\[
1 - q + qR \frac{1}{P} = D'(n(1 + b_i)).
\]
We use the log-quadratic functional form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section C.1 and constrained planner’s problem in Section C.2. We can also define $\tau^* = R/P^* - 1$ similar to (C.2) to represent the total amount of assets sold under fire sales to the traditional sector in terms of
the model parameters, and write risky investment as a function of the liquidity ratio as \( n^* = P^* \tau^*/(c-b) \) using the market clearing condition, similar to (C.15). Now, use the functional-form for the operational cost in banks’ first-order condition and manipulate

\[
1 - q + \frac{q R}{P} = 1 + 2\phi n(1+b),
\]

\[
q \left( \frac{R}{P} - 1 \right) = 2\phi \frac{P \tau}{c-b}(1+b),
\]

\[
n \tau = 2\phi \frac{R}{\tau + 1} \frac{\tau}{c-b}(1+b),
\]

where we first use \( n = \frac{P \tau}{c-b} \) and then substitute \( P = \frac{R}{\tau+1} \). From the last equation we can obtain an expression for the liquidity ratio in this case in terms of \( \tau \) as follows

\[
b^p = \frac{q c (\tau^p + 1) - 2\phi R}{q(\tau^p + 1) + 2\phi R}.
\]

Using \( n = \frac{P \tau}{c-b} \) and \( P = \frac{R}{\tau+1} \) once more, we can obtain a similar expression for the risky investment level in this case in terms of \( \tau^p \) as follows:

\[
n^p = \frac{\tau^p}{\tau^p + 1} \frac{q(\tau^p + 1) + 2\phi R}{2\phi(1+c)}.
\]

All that remains now is to obtain a closed-form solution for \( \tau^p = R/P^p - 1 \), and substitute that in (C.21) and (C.22) to obtain closed-form solutions for \( n^p \) and \( b^p \). To obtain a closed-form solution for \( P^p \) we analyze the regulator’s problem. The regulator takes into account that for any given \( n \), the banks optimally choose their liquidity ratio \( b(n) \), as shown by the response function (13).

The planner takes this reaction function into account while choosing the risky investment level to maximize the expected bank profits subject to the constraint that consumers’ utility after transfers is at least as high as in the competitive equilibrium:

\[
\max_{n,y} \Gamma(n, b(n)) - q\{ (R-P)Q^s(P, n, b(n)) - (1-q)T_2 \},
\]

subject to \( y = Q^s(P, n, b(n)) \),

\[
F'(y) = P,
\]

\[
\frac{d\Pi_i(b_i; n)}{db_i} = 0
\]

\[
(1-q)T_2 + 3\omega + q[F(y) - Py] \geq U_i^c.
\]

The transfers must be such that consumers receive in expectation what they lose from the change in the amount of fire sales and price of assets:

\[
(1-q)T_2 = q[F(y^*) - P^*y^* - F(y) + Py]
\]

(C.23)

In other words, the constraint on consumers’ expected utility binds. Substituting this value for transfers back into the planner’s problem helps us to get rid of the constraint on
consumers’ expected utility:

$$\max_{n,y} \Gamma(n, b(n)) - q \{(R - P)Q^s(P, n, b(n))\} + q[F(y) - Py] - q[F(y^c) - P^c y^c], \quad (C.24)$$

subject to $y = Q^s(P, n, b(n))$,

$F'(y) = P$,

$$\frac{d\Pi_i(b_i; n)}{db_i} = 0$$

The optimal risky investment level in this case is determined by the following first-order condition of the planner’s problem with respect to $n$:

$$\frac{\partial \Gamma}{\partial n} + \frac{\partial \Gamma}{\partial b} b'(n) - q \left[ (R - P) \left( \frac{\partial Q^s}{\partial n} + \frac{\partial Q^s}{\partial b} b'(n) + \frac{\partial Q^s}{\partial P} \frac{dP}{dn} \right) - Q^s \frac{dP}{dn} \right]$$

$$+ q \left[ (F'(y) - P) \left( \frac{\partial y}{\partial n} + \frac{\partial y}{\partial b} b'(n) + \frac{\partial y}{\partial P} \frac{dP}{dn} \right) + y \frac{dP}{dn} \right] = 0,$$

where

$$Q^s = \frac{c - b}{P} n$$

and

$$\frac{dP}{dn} = \frac{\partial P}{\partial n} + \frac{\partial P}{\partial b} b'(n).$$

Using an envelope argument, $F'(y) = P$, and the market clearing condition $y = Q^s$, we can simplify the optimality condition for welfare to:

$$\frac{\partial \Gamma}{\partial n} + \frac{\partial \Gamma}{\partial b} b'(n) - q (R - P) \left( \frac{\partial Q^s}{\partial n} + \frac{\partial Q^s}{\partial b} b'(n) \right) - q (R - P) \frac{\partial Q^s}{\partial P} \frac{dP}{dn} = 0. \quad (C.25)$$

Using the equilibrium condition $y = Q^s = (1 - \gamma)n \frac{(c-b)n}{P}$ and $\Gamma(n, b_i) \equiv (R + b_i - qc)n - D(n(1 + b_i))$ we can write the first-order condition as

$$(1 - q) \{R + b(n) + nb'(n)\} + q R \left\{ \gamma + n \frac{d\gamma}{dn} \right\} + q \left[ F'(\cdot) \left( 1 - \gamma - \frac{d\gamma}{dn} \right) - c + b(n) + nb'(n) \right] = D'(n(1 + b)) \{1 + b(n) + nb'(n)\}$$

(C.26)

We use the log-quadratic functional form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section C.1 and constrained planner’s problem in Section C.2. First, note that substituting for $P$ using (C.15) into $\gamma$ we get

$$\gamma = 1 + \frac{b(n) - c}{P} = 1 + \frac{b(n) - c}{R + (b(n) - c)n},$$

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Using this equivalence, we can obtain the total derivative of $\gamma$ with respect to $n$ as:

$$\frac{d\gamma}{dn} = \frac{\partial\gamma}{\partial b} b'(n) + \frac{\partial\gamma}{\partial n}$$

$$= \frac{P - (b(n) - c)(b(n) - c)}{p^2} - \frac{(b(n) - c)^2}{p^2}$$

$$= \frac{b'(n)}{p} - \frac{nb(n)(b(n) - c)}{p^2} - \frac{(b(n) - c)^2}{p^2}.$$  \quad (C.27)

Replacing $d\gamma/\ dn$ in the first-order condition (C.26) with (C.27) and rearranging yields

$$(1 - q)\{R + b(n)\} + qR \left(1 + \frac{b(n) - c}{p} - \frac{n(b(n) - c)^2}{p^2}\right) + nb'(n) \left\{1 - q + \frac{qR}{p} - D(\cdot) - \frac{qR(b(n) - c)n}{P^2}\right\}$$

$$+ q \left[\left(-\frac{b(n) - c}{p} - \frac{b'(n)n}{p} + \frac{n^2b'(n)(b(n) - c)}{p^2} + \frac{n(b(n) - c)^2}{p^2}\right)\right] P + (b(n) - c) + nb'(n)$$

$$- D(\cdot)\{1 + b(n)\} = 0,$$

where we replace $F'(1 - \gamma)n = P$ using the market clearing condition (C.15) in the second line. We have that $1 - q + qR/P - D(\cdot) = 0$ from the banks’ first-order condition (C.20). Hence, the first-order condition above can further be simplified as follows:

$$R - 1 + q = \frac{qR^2(1 + c)}{p^2} - \frac{qRn(b(n) - c)(1 + b(n))}{p^2} - \frac{qRb'(n)n^2(b(n) - c)}{p^2}$$

$$+ \frac{qn(b(n) - c)^2P}{p^2} + \frac{q(b'(n)n)(b(n) - c)P}{p^2} = 0.$$

Divide the last equation by $qR$ to obtain

$$\frac{R - 1 + q}{qR} = \frac{R(1 + c)}{p^2} - \frac{n(b(n) - c)(1 + b(n))}{p^2} - \frac{b'(n)n^2(b(n) - c)}{p^2}$$

$$+ \frac{n(b(n) - c)^2P}{RP^2} + \frac{b'(n)n)(b(n) - c)P}{RP^2} = 0.$$

Let us define

$$\sigma \equiv \frac{R - 1 + q}{qR}.$$ \quad (C.28)

Using this definition, we can write this first-order condition as

$$\frac{1}{p^2} \left\{\sigma P^2 - R(1 + c) - n(b(n) - c)(1 + b(n)) - b'(n)(b(n) - c)n^2\right\}$$

$$+ \frac{1}{p^2} \left\{\frac{(b - c)^2nP}{R} + \frac{b'(n)n^2(b - c)P}{R}\right\} = 0.$$  \quad (C.29)

We focus on the terms inside the braces because in equilibrium price must be strictly positive. Using this term, we would like to write endogenous variables $n$ and $b$ in terms of the parameters of the model and $P$, and then, use these expression in the first-order conditions of the banks’ problem (C.20) to obtain a closed-form solution for $P$. For that,
first, below we obtain $1 + b(n), n(b(n) - c)$ and $b'(n)$ in terms of the parameters of the model and $P$ starting from the banks’ first-order condition (C.20):

$$
(1 - q) + q \frac{R}{P} = 1 + 2\phi n(1 + b),
$$
(C.30)

$$q(R - P) = P2\phi n(1 + b),
$$
$$q(R - P) = \lbrack R + (b - c)n\rbrack2\phi n(1 + b),
$$
$$-q(b - c)n = 2\phi n(1 + b)R + 2\phi n(1 + b)(b - c)n,
$$
$$-(b - c)[q + 2\phi n(1 + b)] = 2\phi(1 + b)R,
$$

where we substitute for $P = R + (b - c)n$ using (C.15). Now, take the derivative of both sides with respect to $n$, and collect terms that involve $b'(n)$:

$$
-b'(n)[q + 2\phi n(1 + b)] - 2\phi(b - c)[1 + b + nb'(n)] = 2\phi Rb'(n),
$$
$$-b'(n)[q + 2\phi n(1 + b)] - 2\phi(b - c)(1 + b) - 2\phi(b - c)nb'(n) = 2\phi Rb'(n),
$$
$$-b'(n)[2\phi R + 2\phi n(b - c) + q + 2\phi n(1 + b)] = 2\phi(b - c)(1 + b),
$$
$$-b'(n)[2\phi R + q + 2\phi n(2b + 1 - c)] = 2\phi(b - c)(1 + b).
$$

From the last equation we obtain:

$$b'(n) = \frac{-2\phi(b - c)(1 + b)}{2\phi R + q + 2\phi n(2b + 1 - c)}.
$$
(C.31)

We further simplify $b'(n)$ in order to eliminate $b$ from this expression. In order to do this simplification, note that first, from the market clearing condition at $t = 1, P = R + (b - c)n$, as derived in (C.15), we can obtain that

$$b - c = -\frac{R - P}{n}.
$$

Second, from the banks’ first-order condition, given by (C.30), we can obtain that

$$1 + b = \frac{q}{2\phi n}\left(\frac{R}{P} - 1\right).
$$

Use these values for $1 + b$ and $b - c$ into (C.31) to write $b'(n)$ as a function of $n, P$ and the parameters of the model as follows

$$
b'(n) = \frac{-2\phi(-1)\frac{R - P}{n} - \frac{q}{2\phi n}\frac{R}{P} - 1}{2\phi R + q - 2\phi(R - P) + 2\phi\frac{q}{2\phi}\left(\frac{R}{P} - 1\right)}.
$$

$$= \frac{q}{\frac{n^{2}}{2}P(R - P)^{2}}\left[\frac{2\phi R P + q P - 2\phi P(R - P) + q(R - P)}{P(R - P)^{2}}\right]
$$

$$= \frac{q}{n^{2}}\left[\frac{2\phi R P + q P - 2\phi R P + 2\phi P^{2} + q R - q P}{n^{2}}\right],
$$

$$= \frac{q}{n^{2}}\left[\frac{2\phi P^{2} + q R}{n^{2}}\right].
$$
Eventually, use the expressions obtained for $1 + b(n)$, $n(b(n) - c)$ and $b' (n)$ above to rewrite the term inside the braces in (C.29) as:

$$\begin{align}
\sigma P^2 - R(1 + c) + (R - P) \frac{q(R - P)}{2 \phi P n} + \frac{q(R - P)^2}{n^2[2 \phi P^2 + qR]} \frac{R - P}{n} n^2 + \frac{P(R - P)^2}{n R} - \frac{q(R - P)^2}{n^2[2 \phi P^2 + qR]} \frac{R - P}{n R} P n^2 & = 0 \\
\sigma P^2 - R(1 + c) + \frac{q(R - P)^2}{n} \left[ \frac{1}{2 \phi P} + \frac{R - P}{2 \phi P^2 + qR} \right] + \frac{(R - P)^2 P 2 \phi P^2 + qP}{n R} & = 0
\end{align}$$

Note that the last equation takes the form of $A + B/n + C/n = 0$ where $A, B, C$ group relevant terms. Therefore, we can obtain $n$ in the form of $n = -B/A - C/A$, that is, from the last equation we can obtain $n$ in terms of $P$ and the parameters of the model:

$$n = \frac{q(R - P)^2 \left[ \frac{1}{2 \phi P} + \frac{R - P}{2 \phi P^2 + qR} \right]}{R(1 + c) - \sigma P^2} + \frac{(R - P)^2 (2 \phi P^2 + qP) P}{(2 \phi P^2 + qR) R} \equiv \psi_1(P) + \psi_2(P).$$

We can similarly obtain an expression for $b$ in terms of $P$ and the parameters of the model using the equilibrium price function $P = R + (b - c)n$, which implies that

$$b = \frac{P - R}{n} + c = \frac{P - R + cn}{n} = \frac{P - R + c[\psi_1(P) + \psi_2(P)]}{\psi_1(P) + \psi_2(P)}.$$

Now, substitute these expressions for $n$ and $b$ back into the banks’ first-order condition (C.30) in order to obtain a fixed-point equation that involves only $P$ as an endogenous variable, from which we can solve for the equilibrium price $P$:

$$2 \phi n(1 + b) = -q + \frac{q R}{P},$$

$$2 \phi \left[ \psi_1(P) + \psi_2(P) \right] \left[ \frac{P - R + c[\psi_1(P) + \psi_2(P)]}{\psi_1(P) + \psi_2(P)} + 1 \right] + q = \frac{q R}{P},$$

$$2 \phi P \left( P - R + (1 + c) [\psi_1(P) + \psi_2(P)] \right) + q P = q R,$$

$$-2 \phi P (R - P) + 2 \phi P (1 + c) [\psi_1(P) + \psi_2(P)] = q (R - P),$$

$$2 \phi (1 + c) P [\psi_1(P) + \psi_2(P)] = (2 \phi P + q)(R - P)$$

$$2 \phi (1 + c) P (R - P)^2 \left\{ q \left[ \frac{1}{2 \phi P} + \frac{R - P}{2 \phi P^2 + qR} \right] + \frac{P (2 \phi P^2 + qP)}{R (2 \phi P^2 + qR)} \right\} = [R(1 + c) - \sigma P^2](R - P)(2 \phi P + q),$$

$$2 \phi (1 + c) P (R - P) \left\{ q \left[ \frac{R (2 \phi PR + qR)}{2 \phi P^2 + qR} \right] + \frac{2 \phi P^2 (2 \phi P^2 + qP)}{2 \phi P (2 \phi P^2 + qR)} \right\} = [R(1 + c) - \sigma P^2](2 \phi P + q)$$

Now, we sum the terms in side the braces on the left-hand side and multiply both sides with the
common denominator of the left-hand side after summation and simplify further to get:

\[
2\phi(1 + c)P(R - P)(qR(2\phi PR + qR) + 2\phi P^2(2\phi P^2 + qP)) = [R(1 + c) - \sigma P^2](2\phi P + q)2\phi PR(2\phi P^2 + qR)
\]

\[
(1 + c)(R - P)(qR^2(2\phi P + q) + 2\phi P^3(2\phi P + q)) = [R(1 + c) - \sigma P^2](2\phi P + q)R(2\phi P^2 + qR)
\]

\[
(1 + c)(R - P)(2\phi P + q)(qR^2 + 2\phi P^3) = [R(1 + c) - \sigma P^2](2\phi P + q)R(2\phi P^2 + qR)
\]

\[
(1 + c)(R - P)(qR^2 + 2\phi P^3) = [R(1 + c) - \sigma P^2]R(2\phi P^2 + qR)
\]

\[
(1 + c)R(2\phi P^2 - (1 + c)qR^2 - (1 + c)2\phi P^3) = R^2(1 + c)(2\phi P^2 + qR) - \sigma P^2 R(2\phi P^2 + qR)
\]

We can rearrange this last equation to obtain a cubic equation in terms of the partial equilibrium price:

\[
2\phi(\sigma R - 1 - c)P^3 + 2\phi R(1 + c)P^2 + R^2(\sigma q - 2\phi(1 + c))P - (1 + c)qR^2 = 0.
\]

Define

\[
\beta \equiv R(1 + c).
\]

Replacing \(\beta\) for \(R(1 + c)\) we can also write the cubic equation for the partial regulation price as follows:

\[
2\phi(\sigma R - 1 - c)P\beta^3 + 2\phi\beta P\beta^2 + R(\sigma q R - 2\phi\beta)P\beta - qR\beta = 0
\]

It is easy to show that this cubic equation has only one real root and two complex conjugate roots. The only real root can easily be obtained using Vieta’s substitution for cubic equations.