Einstein’s Equations and a Cosmology with Finite Matter

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Abstract

We discuss various space-time metrics which are compatible with Einstein’s equations and a previously suggested cosmology with a finite total mass [1]. In this alternative cosmology the matter density was postulated to be a spatial delta function at the time of the big bang thereafter diffusing outward with constant total mass. This proposal explores a departure from standard assumptions that the big bang occurred everywhere at once or was just one of an infinite number of previous and later transitions.

keywords: Cosmology, Solutions of Einstein’s equations, Finite Mass, String Landscape, String cosmology, Multiverse, Supersymmetry, preferred frame

1 Introduction

It is currently widely believed that the universe is, on large scales, isotropic and homogeneous although there are many nagging discrepancies from this standard cosmological model including many unexplained asymmetries and correlations [3]. In this paper we address four aspects of the standard cosmological model deserving of discussion.

1. In the standard cosmological model and in any homogeneous cosmology with a non-zero probability per unit space-time volume to produce an individual of any species,
there results an infinite replication of each possible human, quasi-human, and monster individual. We referred to this property as infinite cloning.

2. The standard cosmological model is vague with regard to initial conditions. Although well-founded studies [1] have deduced that matter must have originated at a finite time in the past, many authors continue to seek a cosmology extending into the infinite past.

3. In the standard cosmological model, the matter density is homogeneous on large scales. If matter is homogeneous on large scales there should be no larger structures. Surprisingly, however, clusters of galaxies have been found with dimensions as great as any probed scale.

4. The standard cosmological model is often described in the inflationary era as an infinite de-Sitter space with a large cosmological constant. The current era is then described as another infinite de-Sitter space with a much smaller cosmological constant. Relativity demands that the transition cannot occur everywhere at once but should have an expanding bubble topology which is not evident in the standard model.

The first of these four issues may be primarily philosophical in nature as long as the infinite cloning is at causally disconnected patches of space-time. Nevertheless it is, perhaps, interesting to ask whether there is a viable cosmological model without this cloning property. This was the motivation of ref [1]. In addition, this infinite replication of everything is also at the heart of the measure problem which is acknowledged to be a serious puzzle in standard cosmology.

It has been suggested that the question of an origin of time is also somewhat philosophical since, even if time has an origin, it may be at such a great time in the past as to be for all practical purposes infinitely remote. To this one could answer that a model that addresses the initial condition problem is potentially better than one that avoids the question. In quantum physics the initial state can be “prepared” in any quasi-stationary state after which the evolution of the system follows from the equations of motion and quantum jumps to the ground state or other quasi-stationary states are allowed. An initial state in a slow roll between two quasi-stationary states would be hard to reconcile with quantum theory.

Thirdly, even if the homogeneity of the matter distribution on large scales becomes consistent with observation, one should ask whether there are viable models where inhomogeneity sets in at larger scales such as the model we propose. Faced with an apparent symmetry of matter, it is a time-honored tradition in physics to construct a parameter-dependent model that agrees with the apparent symmetry only in some limit thus replacing a theoretical question with the experimental question of observational constraints.

Much effort has been expended to resolve or soften the big bang singularity while little attention has been given to dealing with the possibility of an actual singularity. Taking the time of the big bang at \( t = 0 \), the proposal of ref. [1] is that the matter density \( \rho_m(r, t) \)
satisfies:
\[
\lim_{t \to 0} \rho_m(\vec{r}, t) = \begin{cases} 
\infty & r = 0 \\
0 & r > 0
\end{cases}
\] (1.1)

with, at all positive times,
\[
\int d^3r \rho_m(\vec{r}, t) = M .
\] (1.2)

These equations define \( \rho_m(\vec{r}, t) \) as proportional to a spatial delta function at time \( t = 0 \):
\[
\lim_{t \to 0} \rho_m(\vec{r}, t) = M \delta^3(\vec{r}) .
\] (1.3)

A particular example of a matter density satisfying these equations is
\[
\rho_m(r, t) = \frac{M}{(R \sqrt{\pi} a(t))^3} e^{-r^2/(Ra(t))^2} .
\] (1.4)

where the scale factor \( a(t) \) is a function of time that vanishes at \( t = 0 \). The model becomes consistent with standard cosmological model homogeneity at distances such that \( r/a(t) \) is much smaller than the parameter \( R \). Thus \( R \) plays the role of a scale of inhomogeneity. A Gaussian density is the ground state of the three dimensional harmonic oscillator so the model could have the effect of matter confined by a quadratic potential. A possible future extension of the current treatment might be to take \( \rho_m \) to be a temperature dependent superposition of excited states which could bring in the angular variations suggested in ref. [1]. The number density of equivalent nucleons is \( 1/m_N \) of this matter density (including dark matter) and the number density of photons is proportional to the number density of nucleons.

As we shall show, the model predicts a flat space-time at very small (and very large) \( r/R \). In regions of flat space, the equation of continuity for \( \rho_m \)
\[
\nabla \cdot (\rho_m \vec{v}(r)) + \frac{\partial \rho_m}{\partial t} = 0
\] (1.5)
implies, independent of \( R \), Hubble’s law:
\[
\vec{v}(r) = \frac{\dot{a}}{a} r .
\] (1.6)
The deceleration parameter is
\[
q = -\frac{\ddot{a}a}{\dot{a}^2} .
\] (1.7)

In the matter dominated regime of a Friedmann-Robertson-Walker model, the scale factor should vary as \( t^p \) with \( p = 2/3 \). In the vacuum energy dominated regime of that model, the
scale factor should vary as $e^{H_v t}$ with $H_v$ being the vacuum Hubble parameter. In the current model we might take

$$a(t) = \frac{(1 - \Omega_m)(e^{H_v t} - 1) + \Omega_m (H_m t)^p}{(1 - \Omega_m)(e^{H_v t_0} - 1) + \Omega_m (H_m t_0)^p}$$

(1.8)

where $H_v$ and $H_m$ are parameters with dimensions of inverse time and $\Omega_m$ gives the matter fraction. The scale factor is normalized to unity at the current time $t_0$. With these choices, matter is born in the big bang with an outward velocity and inward acceleration. After some time, $t_1$, the deceleration parameter becomes negative. From a phenomenological point of view, at some time, $t_2$, $a(t)$ should make a transition to a lower vacuum energy and, perhaps, a different value of $p$.

The main question of this paper is whether the assumptions of eqs. 1.1 are consistent with Einstein’s equations or whether they would require modifications of Einstein’s theory.

According to Einstein’s theory, there is a proportionality between the energy-momentum tensor, $T_{\mu\nu}$ and the Einstein tensor, $G_{\mu\nu}$, which is itself a function of the metric tensor $g$.

$$G_{\mu\nu}(g) = -8\pi G_N T_{\mu\nu}$$

(1.9)

By a choice of units we suppress the factor of $8\pi G_N$. The energy density is then

$$\rho(\vec{r}, t) = T^4_{44} = -4G^4$$

(1.10)

In general, the equations have a solution for any postulated energy-momentum tensor. Rather than attack this problem head-on with a supercomputer, we prefer to examine simple choices for a metric which lead analytically to a matter density satisfying eqs. 1.1, 1.2. We use standard packages in Mathematica and Maple to perform this calculation.

2 A modified continuous Schwarzschild-deSitter metric

We consider a metric of the form

$$ds^2 = -g^{-1}_{44}(r, t)dr^2 + r^2d\Omega^2 + g_{44}(r, t)dt^2$$

(2.11)
With this metric the non-zero components of the Einstein tensor are

\[ G_4^4 = -(1/r^2)(1 + g_{44}(r, t) + r g'_{44}(r, t)) \]
\[ G_r^r = G_4^4 \]
\[ G_\theta^\theta = \dot{g}_{44}(r, t)/r \]
\[ G^r_\phi = \ddot{g}_{44}(r, t)/r g_{44}(r, t)^2 \]
\[ G^\phi_r = 0 \]

(2.12)

Here, prime refers to a derivative with respect to the radial coordinate and dot refers to a derivative with respect to time.

The proposed metric and its resulting Einstein tensor share some properties with the Vaidya metrics \[ \text{2} \] which are defined as

\[ ds^2_{\text{Vaidya}} = -(1 - 2h(u)/r)(du)^2 + r^2d\Omega - 2dudr \]

(2.13)

where \( u + 2h \ln \left( \frac{r}{2h} - 1 \right) = t - r \).

As in our eq. 2.12 the Vaidya metrics result in off-diagonal Einstein Tensor elements (mass currents) and lead to an equality of the \( G_4^4 \) and \( G_r^r \), a property of “null dusts”. The Vaidya metrics were invented to model a black hole accreting or radiating. They represent examples of inhomogeneous, time dependent matter densities which, presumably however, cannot be made to satisfy our eqs. 1.1, 1.2.

The Schwarzschild solution for a black hole of mass \( m \) corresponds to

\[ g_{44}(r, t) = -1 + 2m/r \]

(2.14)

According to Birkhoff’s theorem this time-independent metric is the unique zero vacuum energy solution of Einstein’s equations in the region external to a mass \( m \). If the mass is point like, the solution corresponds to a vanishing energy-momentum tensor at positive \( r \) but to a delta function singularity comprising mass \( m \) at the origin.

The Schwarzschild-deSitter solution generalizes this to a space with vacuum energy density \( \Lambda \):

\[ g_{44}(r, t) = -1 + 2m/r + r^2\Lambda/3 \]

(2.15)

It corresponds again to vanishing density and pressure of matter in the external region but to a constant vacuum energy density and pressure. Both of these metrics define a light-trapping region where \( g_{44}(r, t) \) changes sign. They require either restricting the region...
of applicability or attaching a complementary patch of space time. We will see a similar phenomenon in the metrics we study.

In this section we further generalize the Schwarzschild-deSitter metric by replacing the point-like mass $m$ with a function of space and time.

$$g_{44}(r, t) = -1 + h(r, t) + r^2 \Lambda/3 \ . \quad (2.16)$$

If $h(r, t)$ extends continuously over all space there is no contradiction with Birkhoff’s theorem. We aim to satisfy Einstein’s equations with a matter density similar to that of eq.1.4. To this end we might consider

$$h(r, t) = c_0 \frac{r^2}{(Ra(t))^3} e^{-(r/(Ra(t)))^2} \ . \quad (2.17)$$

According to which the energy density from eq.2.12 is

$$\rho(r, t) = -G_4^4 = \Lambda + \frac{c_0}{(Ra(t))^3} (3 - 2(r/(Ra(t)))^2) e^{-(r/(Ra(t)))^2} \ . \quad (2.18)$$

In this case the matter density proportional to $c_0$ becomes negative at large $r$ and integrates to zero.

In the following section we study the possibility of avoiding this by multiplying $h(r, t)$ by an infinite series in $(r/(Ra(t)))^2$. In section 4 we consider restricting $h(r, t)$ to the region of positive matter density in eq.2.18. The matter density would then describe an expanding bubble.

### 3 A positive definite density with infinite range

In this section we avoid the appearance of negative energy densities by writing the infinite series

$$h(r, t) = \frac{r^2}{(Ra(t))^3} e^{-(r/(Ra(t)))^2} \sum_{n=0}^{\infty} c_n \left( \frac{r}{Ra(t)} \right)^{2n} \ . \quad (3.19)$$

Then,

$$\dot{g}_{44}(r, t) = \frac{\dot{a}(t)}{a(t)} \frac{r^2}{(Ra(t))^3} e^{-(r/(Ra(t)))^2} \sum_{n=0}^{\infty} c_n \left( \frac{r}{Ra(t)} \right)^{2n} (-2n + 3 + 2(\frac{r}{Ra(t)})^2) \ . \quad (3.20)$$

We can choose

$$c_n = c_0 \frac{\Gamma(5/2)}{\Gamma(5/2 + n)} \quad (3.21)$$
so that

\[ \frac{1 + g_{44}(r, t) + r g'_{44}(r, t)}{r^2} = \Lambda + \frac{3c_0}{(R_a(t))^3} e^{-r/(R_a(t))^2} \]  

(3.22)

and

\[ \dot{g}_{44}(r, t) = -3c_0 \frac{\dot{a}(t)}{a(t)} \frac{r^2}{(R_a(t))^3} e^{-r/(R_a(t))^2} \]  

(3.23)

One can see from this that the off-diagonal terms in the Einstein tensor vanish rapidly at large times and at small and asymptotically large radius.

Ignoring the off-diagonal components, the density and (negative) pressure are

\[ \rho(r, t) = \Lambda + \frac{3c_0}{(R_a(t))^3} e^{-r/(R_a(t))^2} \]  

(3.24)

The Christoffel symbols, $\Gamma^\mu_{\alpha\beta}$ vanish at $r = 0$ for $\mu$ being either time or radial direction which implies that the metric is spatially flat for small $r$ at fixed positive $t$. One could also see this from the vanishing of $h(r, t)$ at $r = 0$. This justifies the neglect of the curvature in eq. (1.5) near the origin.

The matter density then agrees with eq. (1.4) if we choose

\[ c_0 = \frac{M}{3\pi^{3/2}} \]  

(3.25)

with $M$ being the total mass of matter in the universe.

To insure that the energy density reduces to the FRW model in the absence of matter ($c_0 = 0$), it would be natural to take

\[ \Lambda = 3H_v^2 \]  

(3.26)

with the dark energy scale factor

\[ a_v(t) = e^{H_v t} \]  

(3.27)

### 4 A finite mass model with bubble topology

In this section we seek a metric that corresponds to matter contained within an expanding bubble. In this case the metric beyond the bubble boundary is constrained by Birkhoff’s theorem to be as in eq. (2.15).

We retain the form of eq. (2.16) for $g_{44}(r, t)$. However, in order to incorporate a growing bubble of small vacuum energy within a background of high vacuum energy we allow the vacuum energy $\Lambda$ to be discontinuous with the following form:
\[
\Lambda(r, t) = \Lambda_1 \theta(t_2 - t) + \theta(t - t_2) (\Lambda_1 \theta(r - t + t_2) + \Lambda_2 \theta(t - t_2 - r)) \quad .
\] (4.28)

Similarly we would take at positive \( t \):

\[
a(t) = a_1(t) \theta(t_2 - t) + \theta(t - t_2) (a_1(t) \theta(r - t + t_2) + a_2(t) \theta(t - t_2 - r)) \quad .
\] (4.29)

and

\[
h(r, t) = c_0 \frac{r^2}{(Ra(t))^3} e^{-r/(Ra(t))^2} \theta(\sqrt{3/2Ra(t)} - r) + \theta(r - \sqrt{3/2Ra(t)}) 2M/r \quad .
\] (4.30)

The sharp transitions between phases in eqs. 4.28 and 4.29 should be interpreted as the thin
wall limit of continuous phase transitions described by a hyperbolic tangent function.

\[
\theta(x) = \frac{1}{2} \lim_{b \to 0^+} (1 + \tanh(x/b)) \quad .
\] (4.31)

To be consistent one should define \( \theta(0) = 1/2 \). These equations describe a universe beginning
at \( t = 0 \) and undergoing a transition from \( \Lambda_1 \) to a possibly much smaller \( \Lambda_2 \) at time \( t = t_2 \).
Cosmological data on the length of the inflationary era constrains \( t_2 \). Since, in the thin wall
limit, \( \Lambda \) is time independent between jumps, the \( \theta \) functions modify the previously deduced
Einstein tensor only by delta functions at the phase transition jumps and a contribution to
the surface tension of the bubble which we ignore. \( M \) is the (finite and constant) total mass
inside the bubble. Outside the bubble the metric reduces to the Schwarzschild-deSitter form.

Ignoring the delta function contributions at the discontinuities the mass density and
pressure are as before except that the vacuum energy and scale factor have the theta function
form of eqs. 4.28 and 4.29. Inside the bubble, the density is

\[
\rho(r, t) = \Lambda(t) + \frac{c_0}{(Ra(t))^3} e^{-r/(Ra(t))^2} (3 - 2(r/(a(t)R))^2) \theta(\sqrt{3/2Ra(t)} - r) \quad .
\] (4.32)

Integrating numerically, the total mass contained in the bubble is

\[
M \approx 5.15c_0 \quad .
\] (4.33)

As a crude estimate of \( c_0 \) we could note that, if the local matter density were valid up the
Hubble length and then dropped to zero, the total mass defining \( c_0 \) would be

\[
M \approx \frac{4}{3} \pi L_H^3 \cdot (1 GeV/c^2/m^3) \approx 10^{79} GeV/c^2 \quad .
\] (4.34)

If at some future time there is a transition to zero \( H_v \) with \( p = 1 \), the boundary of the matter
bubble thereafter expands outward with constant speed which can be taken to be the speed
of light.
5 Conclusions

In this article we have addressed four conceptual issues in the standard cosmology. With respect to the question of infinite cloning, we noted previously that this property of the standard model is eliminated if there exists no more than a finite mass. Similarly, the measure problem of the standard model is avoided since there are no infinite occurrences of any event at any given time. Of course the standard Friedmann-Robertson-Walker (FRW) metric with a negative curvature also has a finite total mass and thus could be an alternate resolution of the problem. However the WMAP and Planck results put stringent limits on departures from zero curvature in an FRW analysis. In the FRW model with spatial curvature, the matter density is constant up to the limiting radius whereas the current model predicts a matter inhomogeneity. The isotropy of the cosmic background radiation is often taken to imply matter homogeneity to within one part in $10^5$ but, in fact, observations merely require either that the scale of inhomogeneity, $R$ in the above equations, is above $10^5$ relative to the Hubble length or that the Earth is close to the center of the CBR distribution. In ref.[1] we took the first alternative as the default assumption but here we would also like to explore the other possibility. This latter possibility may lead to the only way to incorporate a correlation between asymmetries in the CBR and the plane of the solar system [3] if the corresponding data survives.

For any finite $R$, the number of identical human clones is finite but, depending on the size of $R$ this number could still be large. To estimate the degree of human individuality we can make the following analysis. We can estimate that there are, have been, or will be about $10^{11}$ humans per solar system per Hubble time, there are about $10^{11}$ solar systems per galaxy, and about $10^{11}$ galaxies per Hubble volume. If $R$ is less than $10^6$ Hubble lengths, there are or will be about $10^{51}$ humans per Hubble time. The number of base pair loci on the human genome is about $6 \cdot 10^9$ at each of which there are one of four possible ordered base pairs. Thus there are $4^{6 \cdot 10^9}$ possible different human body types. Since $10^{51}$ is negligible compared to this number, the probability of random cloning of two identical humans is infinitesimal in the current model although it is unity in the standard model. [4]

It might seem that the present proposal would involve severe fine tuning and be contrary to the “Copernican Principle” that the earth does not stand at a privileged position in the universe. In fact, however, the origin of the Earth’s present rest frame is [1] 5.1 Mpc from the origin of the rest frame of the Cosmic Background Radiation as determined by the dipole asymmetry in the CBR. This is about 0.1% of the Hubble length, a minor amount of fine tuning compared to other cosmological coincidences. In fact, Copernicus did not say there was no center of the universe (solar system) but merely that the earth was displaced from this center by an amount coincidentally similar to 0.1%. On the other hand, the standard cosmology assumes that every observer is slightly displaced from its own center of a spherical shell of background radiation as part of a homogeneous universe.

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Observationally, it is known that galaxy cluster counts are vastly lower than predicted from the Planck results. The observed density of very distant galaxies ($z \approx 7$) in the Hubble Ultra Deep Field is only a tenth of the local density. This could be due to an underestimate of systematic errors since distant galaxies of low luminosity are undercounted. Numerically, the matter density of eq. 4.32 falls to one tenth of its $r = 0$ value at $r = r_0 = 1.032Ra(t)$. At present $a(t) = 1$. The relation to redshift is

$$r = L_H \frac{(z + 1)^2 - 1}{(z + 1)^2 + 1}$$

with Hubble length $L_H = ca(t)/\dot{a}(t)$. $z = 7$ corresponds therefore to a distance about 3% less than the Hubble length and thus a value of $R$ about 6% less than the Hubble length could be suggested. Given the uncertainty in systematic errors, one might want to consider this a lower limit.

In the standard model, the universe is expected to be homogeneous at scales above some minimum. A fractal analysis suggests this minimum is about 370 Mpc, about that of the Sloan Great Wall and roughly a tenth of the Hubble length. Any observed structures of greater dimension than this would represent a significant challenge to the standard cosmological model. Typical clusters of galaxies are 2 to 3 Mpc across with large quasar groups (LQG) extending typically up to 200 Mpc across. However, recently a huge quasar group with longest dimension of 1240 Mpc has been discovered more than three times the expected limit and uncomfortably close to the size of the visible universe. This calls into question the basic standard assumption of large scale matter homogeneity based on the isotropy of the CBR to within one part in $10^5$. We feel that this is another reason to consider inhomogeneous models such as ours although, clearly, our model is not intended to deal with high resolution details such as galactic clusters.

With respect to the question of initial conditions, we note that if the matter distribution takes the form of eq. 4.32 with a squared scale factor $a(t)^2$ behaving as a non-integer power of $t$ near the big bang as in eq. 1.8, the matter density at negative values of $t$ is complex and everywhere infinite as the time of the big bang is approached from below. Thus, in this model, time is undefined before the big bang; i.e. time is positive definite. We suggest that the universe with at least two states of differing vacuum energy is born at the big bang in a state of high vacuum energy with a matter density that is a spatial delta function of position. Unlike the standard picture in which surviving matter is created out of dark energy at the end of inflation, in our model matter and dark energy are born together in the big bang. For a finite time the matter density dominates over the vacuum energy followed by an inflationary era in which the vacuum energy dominates until a quantum transition to a much lower vacuum energy occurs. We have discussed elsewhere the chance that there will be a still later transition to a possibly supersymmetric state of zero vacuum energy and constant scale factor in order to avoid the total energy within some radius exceeding the Schwarzschild energy.

Since the initial state of a physical system is not determined by the equations of motion
but must merely be one of the states of the system, one cannot seek an explanation for an initial state big bang within physics theory. On the other hand the precise form of $a(t)$ might have an explanation within physics and could be different from the simple form we have studied here.

In this article we have established a consistency between Einstein’s equations with a given metric and our previously studied inhomogeneous model with finite matter. The model [1] also suggested possible new approaches to other long-standing questions in physics which are not further discussed here.

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