PARABOLIC AND ELLIPTIC SYSTEMS IN DIVERGENCE FORM WITH VARIABLY PARTIALLY BMO COEFFICIENTS

HONGJIE DONG AND DOYOON KIM

Abstract. We establish the solvability of second order divergence type parabolic systems in Sobolev spaces. The leading coefficients are assumed to be only measurable in one spatial direction on each small parabolic cylinder with the spatial direction allowed to depend on the cylinder. In the other orthogonal directions and the time variable the coefficients have locally small mean oscillations. We also obtain the corresponding $W^{1,p}$-solvability of second order elliptic systems in divergence form. Our results are new even for scalar equations and the proofs simplify the methods used previously in [12].

1. Introduction

This paper concerns the unique solvability of divergence type parabolic and elliptic systems in Sobolev spaces when the leading coefficients are in the class of variably partially BMO (bounded mean oscillation) functions. The parabolic system we consider has the form

$$P u - \lambda u = \text{div} g + f,$$

where $\lambda \geq 0$ is a constant, $g = (g_1, g_2, \ldots, g_d)$, and

$$P u = -u_t + D_\alpha(A^{\alpha\beta}D_\beta u) + D_\alpha(B^\alpha u) + \hat{B}^\alpha D_\alpha u + Cu.$$

The coefficients $A^{\alpha\beta}$, $B^\alpha$, $\hat{B}^\alpha$, $C$ are $m \times m$ matrices, which are bounded and measurable, and the leading coefficients $A^{\alpha\beta}$ are uniformly elliptic. Note that $u = (u^1, \ldots, u^m)^{tr}$, $g_\alpha = (g_{\alpha}^1, \ldots, g_{\alpha}^m)^{tr}$, $f = (f^1, \ldots, f^m)^{tr}$ are (column) vector-valued functions defined on $(S,T) \times \mathbb{R}^d = \{(t,x) : t \in (S,T), x = (x_1, \ldots, x_d) \in \mathbb{R}^d\}$, where $-\infty \leq S < T \leq \infty$. For given $g$ and $f \in L_p$, $1 < p < \infty$, we seek a unique solution $u$ in the parabolic Sobolev space $H^1_p$ (for a definition of $H^1_p$, see Section 2).

We also consider the following elliptic system

$$L u - \lambda u = \text{div} g + f,$$

where

$$L u = D_\alpha(A^{\alpha\beta}D_\beta u) + D_\alpha(B^\alpha u) + \hat{B}^\alpha D_\alpha u + Cu.$$

In this case $A^{\alpha\beta}$, $B^\alpha$, $\hat{B}^\alpha$, $C$, $g$, and $f$ are independent of $t$ and satisfy the same conditions as in the parabolic case. Naturally, the solution space is $W^{1,p}_p$.

1991 Mathematics Subject Classification. 35K15, 35R05.

Key words and phrases. Second-order systems, bounded mean oscillation, variably partially BMO coefficients, Sobolev spaces.

H. Dong was partially supported by a start-up funding from the Division of Applied Mathematics of Brown University, NSF grant number DMS-0635607 from IAS, and NSF grant number DMS-0800129.
There are many papers concerning elliptic and parabolic equations/systems in Sobolev spaces with VMO (vanishing mean oscillation) or BMO type coefficients. Chiarenza, Frasca, and Longo first proved the interior estimate for non-divergence form elliptic equations with VMO coefficients. Then the solvability of elliptic and parabolic equations in Sobolev space were presented in [7] and [5]. These are the earliest papers about non-divergence type equations with VMO coefficients. For divergence type equations with VMO coefficients, results of similar type were obtained in [1]. Later, Byun and Wang studied in their papers (see, e.g., [3] and references therein) divergence type equations/systems with BMO coefficients in non-smooth domains. Krylov also treated BMO coefficients in [22, 23], where he gave a unified approach to investigating the $L_p$-solvability of both divergence and non-divergence form parabolic and elliptic equations with coefficients BMO in the spatial variables (and only measurable in $t$ in the parabolic case). For other related results, we also refer the reader to [25, 26, 21, 15, 2, 8], and references therein.

To explain the class of coefficients in this paper, we first mention partially BMO coefficients, which are characterized as having no regularity assumptions with respect to one (fixed) variable and having locally small mean oscillations with respect to the other variables. This class of coefficients was first introduced in [19], where the $W^{2,p}_p$-solvability of elliptic equations in non-divergence form were obtained by adapting some ideas in [22]. Since then, non-divergence type equations/systems with partially VMO/BMO coefficients have been considered in [20, 16, 17, 13, 14]. Partially BMO coefficients are quite general so that they include VMO coefficients as in [6, 7, 5] as well as BMO coefficients as in [3, 4]. As to divergence type equations, the authors of this paper proved in [12] the $W^{1,p}_p$-solvability of elliptic equations with partially BMO coefficients. Then parabolic equations as well as systems in divergence form were treated in [10, 13].

In this paper, we deal with the class of variably partially BMO coefficients, which is a generalization of partially BMO coefficients. This class of coefficients was first introduced by Krylov in [24] for elliptic equations in non-divergence form in the whole space and $p \in (2, \infty)$. Variably partially BMO coefficients are measurable in one spatial direction and have small mean oscillation in the other directions via a diffeomorphism on each small cylinder (or ball in the elliptic case). Diffeomorphisms may be chosen differently for each cylinder, so the direction in which coefficients are only measurable (have no regularity assumption) may vary from one cylinder to another. In other words, there is no global fixed direction, with respect to which the coefficients are only measurable. It is easily seen that the class of partially BMO coefficients is a special case of variably partially BMO coefficients with the identity diffeomorphism. Later, non-divergence type parabolic equations with similar type of coefficients and any $p \in (1, \infty)$ were dealt with in [13].

Having variably partially BMO coefficients in hand we establish in this paper the corresponding results of [24] and [13] for divergence type equations, which generalize the results of [12]. Moreover, in contrast to [24, 13, 12] we deal with systems as well, so our results extend all results in [12] to the system case. The main steps are $L_2$-oscillation estimates of the derivatives of solutions, and then applying a generalized Fefferman-Stein theorem proved by Krylov in [24]. However, there are

*In fact, the authors of [13] considered partially VMO coefficients, which are only measurable in one fixed variable and have vanishing mean oscillations in the other variables. However, in the same spirit as in [24, 21] the proofs there also work for equations with partially BMO coefficients.*
additional difficulties due to the divergence structure of the equations/systems and the fact that coefficients are (locally) only measurable in one direction.

To overcome the difficulty due to the divergence structure, in [12] we used a scaling argument. Roughly speaking, we considered a rescaled function \( u(\mu^{-1}x_1, x') \) instead of \( u(x_1, x') \) to get a priori estimates of \( Du \), where \( \mu \) is a large constant and \( (x_1, x') \in \mathbb{R}^d \). The coefficients considered in this paper, as noted above, have no specific fixed direction to which we can apply the scaling argument. This prompted us to develop a new method, the key step of which is to estimate the mean oscillations of \( \sum_{j=1}^{d} a^{ij}D_ju \) and \( D_iu, i = 2, \cdots, d \), instead of the full gradient of \( u \), if the given equation is

\[
D_i(a^{ij}D_ju) = \text{div} g.
\]

Applying our new method to the equations in [12], it not only removes the necessity of the scaling procedure, but also simplifies the proofs there. Moreover, the method allows us to treat systems, whereas in [12, 10] we were only able to deal with scalar equations due to the fact that a certain change of variables had to be used.

Comparing to elliptic equations considered in [12], another obstacle in the parabolic case is in the estimate of \( \|u_t\|_{L^2} \). In contrast to the case of non-divergence form equations, the estimate of \( \|u_t\|_{L^2} \) does not follows directly from those of spatial derivatives of \( u \). To circumvent this obstacle, in Lemma 3.2 we use an iteration argument combined with suitably chosen weights. For equations with symmetric coefficient matrices, a simpler proof can be found in [14].

Unlike [12], where equations are considered in the whole space, a half space and a bounded domain, here we only concentrate on equations in the whole space for the simplicity of the presentation. For a discussion about different approaches for equations with VMO, BMO, or partially BMO coefficients, see [12] and references therein.

The paper is organized as follows. We introduce some notation and present the main results in Section 2. In Section 3 we prove some preliminary results, which are necessary in the proofs of the main results. We prove in Section 4 Theorem 2.2. The other main results are easily derived from Theorem 2.2. Finally in Section 5 we make some remarks on elliptic systems with some less regularity assumptions on diffeomorphisms.

2. Notation and main results

2.1. Notation and function spaces. We begin this section by introducing some notation, which will be used throughout the paper. Let \( m, d \geq 1 \) be integers. A typical point in \( \mathbb{R}^d \) is denoted by \( x = (x_1, \cdots, x_d) = (x_1, x') \). We set

\[
D_\alpha u = u_{x_\alpha}, \quad D_{\alpha\beta} u = u_{x_\alpha x_\beta}, \quad D_t u = u_t.
\]

By \( Du \) and \( D^2u \) we mean the gradient and the Hessian matrix of \( u \). On many occasions we need to take these objects relative to only part of variables. In such cases, we use the following notation:

\[
D_{x'} u = u_{x'}, \quad D_{x_1, x'} u = u_{x_1, x'}, \quad D_{xx'} u = u_{xx'},
\]

where, for example, \( D_{xx'} u \) means one of \( u_{x_\alpha x_\beta}, \alpha = 1, \cdots, d, \beta = 2, \cdots, d \), or the whole collection of them.
Throughout the paper, we always assume that \( 1 < p < \infty \) unless explicitly specified otherwise. By \( N(d, m, p, \cdots) \) we mean that \( N \) is a constant depending only on the prescribed quantities \( d, m, p, \ldots \).

For a function \( f(t, x) \) in \( \mathbb{R}^{d+1} \), we set \( (f)_D \) to be the average of \( f \) over an open set \( D \) in \( \mathbb{R}^{d+1} \), i.e.,

\[
(f)_D = \frac{1}{|D|} \int_D f(t, x) \, dx \, dt = \int_D f(t, x) \, dx \, dt,
\]

where \( |D| \) is the \( d + 1 \)-dimensional Lebesgue measure of \( D \).

For \( -\infty \leq S < T \leq \infty \), we denote

\[
\mathcal{H}_p^1((S, T) \times \mathbb{R}^d) = (1 - \Delta)^{1/2} L_{p/2}^1((S, T) \times \mathbb{R}^d),
\]

\[
\mathcal{H}_p^{-1}((S, T) \times \mathbb{R}^d) = (1 - \Delta)^{1/2} L_p((S, T) \times \mathbb{R}^d).
\]

For any \( T \in (-\infty, \infty] \), we use

\[
\mathbb{R}_T = (-\infty, T), \quad \mathbb{R}_T^{d+1} = \mathbb{R}_T \times \mathbb{R}^d
\]

to abbreviate, for example, \( L_p((-\infty, T) \times \mathbb{R}^d) = L_p(\mathbb{R}_T^{d+1}) \). When \( T = \infty \), we frequently use the abbreviations \( L_p = L_p(\mathbb{R}^{d+1}), \mathcal{H}_p^1 = \mathcal{H}_p^1(\mathbb{R}^{d+1}), \) etc.

Set

\[
B_r(x') = \{ y \in \mathbb{R}^{d-1} : |x' - y'| < r \}, \quad B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \},
\]

\[
Q_r(t, x) = (t - r^2, t) \times B_r(x'), \quad Q_r(t, x) = (t - r^2, t) \times B_r(x),
\]

and

\[
B_r' = B_r'(0), \quad B_r = B_r(0), \quad Q_r' = Q_r'(0), \quad Q_r = Q_r(0, 0).
\]

As above, \(|B_r'|, |B_r|, |Q_r'|, \) and \(|Q_r|\) mean the volume of \( B_r', B_r, Q_r', \) and \( Q_r \) respectively.

For a function \( g \) defined on \( \mathbb{R}^{d+1} \), we denote its (parabolic) maximal and sharp function, respectively, by

\[
\mathcal{M}_g(t, x) = \sup_{Q \in \mathcal{Q}, (t, x) \in Q} \int_Q |g(s, y)| \, dy \, ds,
\]

\[
g^#(t, x) = \sup_{Q \in \mathcal{Q}, (t, x) \in Q} \int_Q |g(s, y) - (g)_Q| \, dy \, ds,
\]

where \( \mathcal{Q} \) is the collection of all cylinders in \( \mathbb{R}^{d+1} \), i.e.

\[
\mathcal{Q} = \{ Q_r(t, x) : (t, x) \in \mathbb{R}^{d+1}, r \in (0, \infty) \}.
\]

### 2.2. Main results.

We state our assumptions on the coefficients precisely. We assume that all the coefficients are bounded and measurable, and \( A^{\alpha \beta} \) are uniformly elliptic, i.e. there exist \( \delta \in (0, 1) \) and \( K \geq 1 \) such that for any vectors \( \xi_\alpha = (\xi^i_\alpha) \in \mathbb{R}^m, \alpha = 1, \cdots, d \) and any \( (t, x) \in \mathbb{R}^{d+1} \) we have

\[
\delta \sum_{\alpha=1}^d \sum_{i=1}^m |\xi^i_\alpha|^2 \leq \sum_{\alpha, \beta=1}^d \sum_{i,j=1}^m A^{\alpha \beta}_{ij}(t, x) \xi^i_\alpha \xi^j_\beta \leq \delta^{-1}
\]

\[
|A^{\alpha \beta}(t, x)| \leq \delta^{-1}, \quad |B^\alpha(t, x)| \leq K, \quad |B^{\alpha}(t, x)| \leq K, \quad |C(t, x)| \leq K,
\]

where \( \alpha, \beta = 1, 2, \cdots, d \).

Denote by \( \mathcal{A} \) the set of \( m d \times m d \) matrix-valued measurable functions \( \tilde{A} = (\tilde{A}^{\alpha \beta}(y_1)) \) of one spatial variable such that (2.1) holds with \( \tilde{A} \) in place of \( A \).
Let $\Psi$ be the set of $C^{1,1}$ diffeomorphisms $\psi: \mathbb{R}^d \to \mathbb{R}^d$ such that the mappings $\psi$ and $\phi = \psi^{-1}$ satisfy
\[
|D\psi| + |D^2\psi| \leq \delta^{-1}, \quad |D\phi| + |D^2\phi| \leq \delta^{-1}.
\]  
(2.2)

**Assumption 2.1** $(\gamma)$. There exists a positive constant $R_0 \in (0, 1]$ such that, for any parabolic cylinder $Q$ of radius less than $R_0$, one can find an $\tilde{A} \in \mathcal{A}$ and a $\psi = (\psi_1, \ldots, \psi_d) \in \Psi$ such that
\[
\int_Q |A(t, x) - \tilde{A}(\psi_1(x))| \, dx \, dt \leq \gamma |Q|.
\]  
(2.3)

Now we state the main results of this paper. Our first theorem is about the solvability of (1.1) in $\mathbb{R}^{d+1}_T$.

**Theorem 2.2.** Let $p \in (1, \infty)$ and $T \in (-\infty, \infty]$ and $u \in \mathcal{H}^1_p(\mathbb{R}^{d+1}_T)$. Then there exist constants $\gamma = \gamma(d, m, p, \delta) > 0$, and $\lambda_0 \geq 0$ and $N > 0$, depending only on $d$, $m$, $p$, $R_0$, $\delta$ and $K$, such that under Assumption 2.1 $(\gamma)$ the following assertions hold.

(i) For any $u \in \mathcal{H}^1_p(\mathbb{R}^{d+1}_T)$, we have
\[
\lambda \|u\|_{L^p(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda} \|Du\|_{L^p(\mathbb{R}^{d+1}_T)} + \|u\|_{\mathcal{H}^{-1}_p(\mathbb{R}^{d+1}_T)} \leq N(\sqrt{\lambda} + 1)\|P - \lambda u\|_{\mathcal{H}^{-1}_p(\mathbb{R}^{d+1}_T)}
\]  
(2.4)

for all $\lambda > \lambda_0$.

(ii) For any $\lambda > \lambda_0$ and $f, g \in L^p(\mathbb{R}^{d+1}_T)$, there exists a unique $u \in \mathcal{H}^1_p(\mathbb{R}^{d+1}_T)$ solving
\[
P u - \lambda u = \text{div} \, g + f
\]

in $\mathbb{R}^{d+1}_T$. Moreover, $u$ satisfies the estimate
\[
\lambda \|u\|_{L^p(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda} \|Du\|_{L^p(\mathbb{R}^{d+1}_T)} + \|u\|_{\mathcal{H}^{-1}_p(\mathbb{R}^{d+1}_T)} \leq N\sqrt{\lambda} \|g\|_{L^p(\mathbb{R}^{d+1}_T)} + N\|f\|_{L^p(\mathbb{R}^{d+1}_T)}.
\]

The next result is regarding the initial value problem of (1.1). For $-\infty < S < T \leq \infty$ we define $\mathcal{H}^1_{q,p}((S, T) \times \Omega)$ to be the subspace of $\mathcal{H}^1_{q,p}((S, T) \times \Omega)$ consisting of functions satisfying $u|_{t \geq S} \in \mathcal{H}^1_{q,p}((-\infty, T) \times \Omega)$.

**Theorem 2.3.** Let $p \in (1, \infty)$, $T \in (0, \infty]$. Then there exists a constant $\gamma > 0$ depending only on $d$, $m$, $p$ and $\delta$, such that under Assumption 2.1 $(\gamma)$, for any $f, g \in L^p((0, T) \times \mathbb{R}^d)$, there exists a unique $u \in \mathcal{H}^1_p((0, T) \times \mathbb{R}^d)$ satisfying
\[
P u = \text{div} \, g + f
\]

in $(0, T) \times \mathbb{R}^d$. Moreover, there is a constant $N$ depending only on $d$, $m$, $p$, $T$, $R_0$, $\delta$ and $K$ such that
\[
\|u\|_{\mathcal{H}^1_p((0, T) \times \mathbb{R}^d)} \leq N \left( \|f\|_{L^p((0, T) \times \mathbb{R}^d)} + \|g\|_{L^p((0, T) \times \mathbb{R}^d)} \right).
\]

Indeed, by considering $v := e^{-(\lambda_0 + 1)t} \cdot u$ instead of $u$ the operator $P$ becomes $P - (\lambda_0 + 1)I$. Now we extend $f$ and $g$ to be zero for $t < 0$ and solve the system for $v$ in $\mathbb{R}^{d+1}_T$ using Theorem 2.2. By the uniqueness, we have $v = 0$ when $t \leq 0$. Thus $u = e^{(\lambda_0 + 1)t} \cdot v$ solves the original initial value problem and the estimate follows as well.

As a consequence of Theorem 2.3, we obtain the $W^{1, 1}_p$-solvability of elliptic systems (1.2) with variably partially BMO coefficients with locally small BMO semi-norms.
Assumption 2.4 ($\gamma$). There exists a positive constant $R_0 \in (0, 1]$ such that, for any ball $B$ of radius less than $R_0$, one can find an $\tilde{A} \in \mathcal{A}$ and a $\psi = (\psi_1, \cdots, \psi_d) \in \Psi$ such that

$$\int_B |A(x) - \tilde{A}(\psi_1(x))| \, dx \leq \gamma |B|.$$ 

Theorem 2.5. Let $p \in (1, \infty)$. Then there exist constants $\gamma = \gamma(d, m, p, \delta) > 0$, and $\lambda_0 \geq 0$, $N > 0$ depending only on $d$, $m$, $p$, $K$, $\delta$ and $R_0$ such that under Assumption 2.4 ($\gamma$) the following assertions hold.

(i) For any $u \in W^1_p(\mathbb{R}^d)$, we have

$$\lambda \|u\|_{L^p(\mathbb{R}^d)} + \sqrt{\lambda} \|Du\|_{L^p(\mathbb{R}^d)} \leq N(\sqrt{\lambda} + 1) \|\mathcal{L}u - \lambda u\|_{W^{-1,p}(\mathbb{R}^d)}$$

for all $\lambda \geq \lambda_0$.

(ii) For any $\lambda > \lambda_0$ and $f, g \in L^p(\mathbb{R}^d)$, there exists a unique $u \in W^1_p(\mathbb{R}^d)$ solving (1.2) in $\mathbb{R}^d$. Moreover, $u$ satisfies the estimate

$$\lambda \|u\|_{L^p(\mathbb{R}^d)} + \sqrt{\lambda} \|Du\|_{L^p(\mathbb{R}^d)} \leq N\sqrt{\lambda} \|g\|_{L^p(\mathbb{R}^d)} + N \|f\|_{L^p(\mathbb{R}^d)}.$$ 

Theorem 2.3 is deduced from Theorem 2.4 by using the idea that solutions to elliptic systems can be viewed as steady state solutions to parabolic systems. We omit the details and refer the reader to the proof of Theorem 2.6 [22]. In Section 3, we shall give an outline of the proof of Theorem 2.5 under a weaker regularity assumption on $\psi$ and $\phi$.

Remark 2.6. The $\mathcal{H}^{1}_{p}$-solvability results in this paper admit the extension to the mixed norm spaces $\mathcal{H}^{1}_{q,p}$ by following the idea in [22]; see also, for instance, [18] and [10]. Here we do not pursue this, and leave it to interested readers.

Remark 2.7. As an application of Theorem 2.2 and 2.5, one can obtain the solvability of parabolic and elliptic systems on a half space or a bounded Lipschitz domain with a small Lipschitz constant, with either the homogeneous Dirichlet boundary condition or the conormal derivative boundary condition. For systems on a half space, near the boundary we require $A^{\alpha\beta}$ to be measurable in the normal direction and have locally small mean oscillation in the other directions. For systems on a Lipschitz domain, near the boundary we require $A^{\alpha\beta}$ to have locally small mean oscillation in the spatial directions (and measurable in $t$ in the parabolic case; cf. Theorem 5.1 [13]). In both cases, $A^{\alpha\beta}$ is also assumed to satisfy Assumption 2.4 (or Assumption 2.4 in the elliptic case) in the interior of the domain. We omit the detail and refer interested readers to the discussions in [12].

3. Estimates of mean oscillations

In this section we assume $B = \bar{B} = 0$ and $C = 0$. The main objective of this section is to estimate the $L^2$-oscillations of solutions to $\mathcal{P}u = \text{div} \, g$, which is the key ingredient in the proofs of our main results. We start with the well-known $\mathcal{H}^{1}_{2}$-solvability of (1.1) with measurable coefficients.

Lemma 3.1.

(i) Let $T \in (-\infty, \infty]$ and $\lambda \geq 0$. Assume $u \in \mathcal{H}^{1}_{2}(\mathbb{R}^{d+1}_T)$ and $\mathcal{P}u - \lambda u = \text{div} \, g$, where $f, g \in L^2(\mathbb{R}^{d+1}_T)$. Then just under the uniform ellipticity condition (with no regularity assumption on $A^{\alpha\beta}$), there exists a constant $N = N(d, m, \delta)$ such that

$$\sqrt{\lambda} \|Du\|_{L^2(\mathbb{R}^{d+1}_T)} + \lambda \|u\|_{L^2(\mathbb{R}^{d+1}_T)} \leq N\sqrt{\lambda} \|g\|_{L^2(\mathbb{R}^{d+1}_T)} + N \|f\|_{L^2(\mathbb{R}^{d+1}_T)}.$$
If $\lambda = 0$ and $f = 0$, we have
\[ ||Du||_{L^2(\mathbb{R}^d_{T})} \leq N ||g||_{L^2(\mathbb{R}^d_{T})}. \]

(ii) For $\lambda > 0$ and any $f, g \in L^2(\mathbb{R}^d_{T})$, there exists a unique $u \in \mathcal{H}^1(\mathbb{R}^d_{T})$ solving $Pu - \lambda u = \nabla g + f$ in $\mathbb{R}^d_{T}$.

(iii) Let $T \in (0, \infty)$. For any $f, g \in L^2((0,T) \times \mathbb{R}^d)$, there exists a unique $u \in \mathcal{H}^1((0,T) \times \mathbb{R}^d)$ solving $Pu = \nabla g + f$ in $(0,T) \times \mathbb{R}^d$. Moreover,
\[ ||u||_{\mathcal{H}^1((0,T) \times \mathbb{R}^d)} \leq N ||g||_{L^2((0,T) \times \mathbb{R}^d)} + N ||f||_{L^2((0,T) \times \mathbb{R}^d)}, \]
where $N = N(d,m,\delta,T) > 0$. If $f = 0$, we have
\[ ||Du||_{L^2((0,T) \times \mathbb{R}^d)} \leq N ||g||_{L^2((0,T) \times \mathbb{R}^d)}, \]
where $N = N(d,m,\delta) > 0$.

We recall the following Caccioppoli-type inequality for parabolic systems in divergence form.

**Lemma 3.2.** Let $0 < r < R < \infty$. Assume $u \in \mathcal{H}^1_{2,loc}$ and $Pu = \nabla g + f$ in $Q_R$, where $f, g \in L^\infty(Q_R)$. Then there exists a constant $N = N(d,m,\delta)$ such that
\[ ||Du||_{L^2(Q_r)} \leq N (||g||_{L^2(Q_R)} + (R-r)||f||_{L^2(Q_R)} + (R-r)^{-1}||u||_{L^2(Q_R)}). \]

**Proof.** We provide a sketchy proof for the sake of completeness. Take a $\zeta \in C_0^\infty$ such that
\[ \zeta = \begin{cases} 1 & \text{on } Q_r, \\ 0 & \text{on } \mathbb{R}^d \setminus (-R^2, R^2) \times B_R \end{cases} \]
and
\[ |D\zeta| \leq N(R-r)^{-1}, \quad |\zeta| \leq N(R-r)^{-2}. \]
After multiplying both sides of the system by $\zeta^2 u$ and integrating on $Q_R$, we get
\[ \int_{Q_R} u \cdot u \, dx dt + \int_{Q_R} D_\alpha(u \zeta^2) \cdot (A^{\alpha\beta} D_\beta u) \, dx dt = -\int_{Q_R} (\nabla g + f) \cdot u \zeta^2 \, dx dt. \]
Integrating by parts and using Young’s inequality yield
\[ \int_{Q_R} D_\alpha(u) \cdot (A^{\alpha\beta} D_\beta u) \zeta^2 \, dx dt \leq N \int_{Q_R} ((R-r)^{-2}||u||^2 + (R-r)^2||f||^2 + ||g||^2) \, dx dt + \frac{\delta}{2} \int_{Q_R} |Du|^2 \zeta^2 \, dx dt, \]
where $N = N(d,m,\delta) > 0$. To finish the proof, it suffices to use (2.1) and absorb the last term on the right-hand side of (2.1) to the left-hand side. \qed

On account of the above lemma, we make a frequent use of the following argument. Let $A^{\alpha\beta} = A^{\alpha\beta}(x_1)$, that is, they are functions of $x_1 \in \mathbb{R}$ only. Also let $u \in C_0^\infty$ and assume $Pu = 0$ in $Q_R$. Then since $D_\alpha u$, $\alpha = 2, \cdots, d$, satisfies $P(D_\alpha u) = 0$ in $Q_R$, by the above lemma, it follows that
\[ ||DD_\alpha u||_{L^2(Q_{r_1})} \leq N ||D_\alpha u||_{L^2(Q_{r_2})} \leq N ||u||_{L^2(Q_R)}, \]
where $r_1 < r_2 < R$ and $N$ depends only on $d$, $m$, $\delta$, and radii $r_1$, $r_2$, $R$. If we further consider $D_\alpha D_\beta u$, $\alpha, \beta = 2, \cdots, d$, which satisfies $P(D_\alpha D_\beta u) = 0$ in $Q_R$, again by the above lemma
\[ ||DD_\alpha D_\beta u||_{L^2(Q_{r_0})} \leq N(d,m,\delta,r_0,r_1) ||D_\alpha D_\beta u||_{L^2(Q_{r_1})}, \]
where \( r_0 < r_1 \). By combining the above two inequalities we obtain
\[
\|DD_{\alpha\beta}u\|_{L_2(Q_{r_0})} \leq N\|Du\|_{L_2(Q_R)},
\]
where we may also have \( \|u\|_{L_2(Q_R)} \) instead of \( \|Du\|_{L_2(Q_R)} \) on the right-hand side. By repeating the same reasoning, we have
\[
\|DD^k_{\alpha\beta}u\|_{L_2(Q_{r_0})} \leq N\|Du\|_{L_2(Q_R)},
\]
where \( k \) is a positive integer, \( r < R \) and \( N = N(d, m, \delta, r, R, k) \). Considering the derivatives of \( u \) in time as well, in general we have the following lemma. With an additional assumption that the coefficient matrices are symmetric, a similar result was proved in [10].

**Lemma 3.3.** Let \( 0 < r < R < \infty \) and \( A^{\alpha\beta} = A^{\alpha\beta}(x_1) \), \( \alpha, \beta = 1, 2, \cdots, d \). Assume \( u \in C_{loc}^\infty \) satisfies
\[
\mathcal{P}u = 0.
\]
in \( Q_R \). Then we have
\[
\|D_1D^j_{ij}u\|_{L_2(Q_{r_0})} + \|D_1D_{ij}D_1u\|_{L_2(Q_{r_0})} \leq N\|Du\|_{L_2(Q_R)},
\]
where \( i, j \) are nonnegative integers satisfying \( i + j \geq 1 \) and \( N = N(d, m, R, r, i, j) \).

**Proof.** Note that \( \mathcal{P}(D_1D^j_{ij}u) = 0 \) in \( Q_R \). Thus, thanks to the argument shown before this lemma, it suffices to prove
\[
\|u_i\|_{L_2(Q_{r_0})} \leq N(d, m, \delta, R, r)\|Du\|_{L_2(Q_R)}.
\]
Set
\[
r_0 = r, \quad r_n = r + \sum_{k=1}^n \frac{R-r}{2^k}, \quad n = 1, 2, \cdots,
\]
\[
s_n = \frac{r_n + r_{n+1}}{2}, \quad Q^{(n)} = Q_n, \quad \tilde{Q}^{(n)} = Q_s, \quad n = 0, 1, 2, \cdots.
\]
We choose \( \zeta_n(t, x) \in C_{0}^{\infty} \) such that
\[
\zeta_n = \begin{cases} 1 & \text{on } Q_n \quad \text{on } \mathbb{R}^{d+1} \setminus (-s_n^2, s_n^2) \times B_{s_n} \end{cases}
\]
and
\[
|D\zeta_n| \leq N \frac{2^n}{R - r}.
\]
Also set
\[
A_n = \|u_i\|_{L_2(Q^{(n)})}, \quad B = \|Du\|_{L_2(Q_R)}.
\]
After multiplying both sides of (3.2) from the left by \( \zeta_n^2(u_1, \cdots, u_m) \) and integrating on \( Q_R \), we get
\[
\sum_{i=1}^d \int_{Q_R} (u_1^i\zeta_n)^2 \, dx \, dt + \sum_{\alpha, \beta=1}^d \sum_{i,j=1}^m \int_{Q_R} D_\alpha(u_1^i\zeta_n^2)A_{ij}^\alpha D_\beta u_j^i \, dx \, dt = 0.
\]
From this and the Young’s inequality, we have
\[
\sum_{i=1}^d \int_{Q_R} (u_1^i\zeta_n)^2 \, dx \, dt
\]
\[
= - \sum_{\alpha, \beta=1}^d \sum_{i,j=1}^m \int_{Q_R} \left( D_\alpha u_1^i A_{ij}^\alpha D_\beta u_j^i \zeta_n^2 + 2u_1^i A_{ij}^\alpha D_\beta u_j^i \zeta_n D_\alpha \zeta_n \right) \, dx \, dt
\]
We multiply both sides of (3.5) by $3^{-n}$ and sum over $n$ to obtain
\[ \sum_{n=0}^{\infty} 3^{-n}A_n \leq \sum_{n=0}^{\infty} 3^{-n-1}A_{n+1} + N \sum_{n=0}^{\infty} (2/3)^n B. \]
Therefore, \[ A_0 \leq N \sum_{n=0}^{\infty} (2/3)^n B. \]

The lemma is proved.

Owing to the structure of the divergence form systems, the same type of inequality as in the above lemma holds true if $u$ in the left-hand side is replaced by $U$, the definition of which is
\[ U := \sum_{\beta=1}^{d} A^{1\beta} D_{\beta} u, \quad \text{i.e.,} \quad U^i = \sum_{\beta=1}^{d} \sum_{j=1}^{m} A_{ij}^{1\beta} D_{\beta} u^j, \quad i = 1, \ldots, m. \]

Lemma 3.4. Let $A^{\alpha\beta} = A^{\alpha\beta}(x_1)$. Assume $u \in C_{\text{loc}}^\infty$ satisfies (3.2) in $Q_4$. Then, for nonnegative integers $i, j$
\[ \|D_1^i D_{x}^j U\|_{L^2(Q_2)} + \|D_1^i D_{x} D_{1} U\|_{L^2(Q_2)} \leq N \|D_1 U\|_{L^2(Q_4)}, \quad (3.6) \]
where $N = N(d, m, \delta, i, j) > 0$.

Proof. As before, to prove (3.6) it is enough to show
\[ \|U\|_{L^2(Q_2)} + \|D_1 U\|_{L^2(Q_2)} \leq N \|D_1 U\|_{L^2(Q_4)}. \quad (3.7) \]
Indeed, if this holds true, due to the fact that $D_1 D_{x}^j u$ also satisfies (3.2) we have
\[ \|D_1^i D_{x}^j U\|_{L^2(Q_2)} + \|D_1^i D_{x} D_{1} U\|_{L^2(Q_2)} \leq N \|D_1^i D_{x}^j U\|_{L^2(Q_4)}. \]
Then by Lemma 3.3 we bound the right-hand side of the above inequality by a constant times $\|D_1 U\|_{L^2(Q_4)}$, so we arrive at the inequality (3.6).

To prove (3.7), we observe that in $Q_4$,
\[ D_1 U = u_t - \sum_{\alpha=2, \beta=1}^{d} D_{\alpha} (A^{\alpha\beta} D_{\beta} u) = u_t - \sum_{\alpha=2, \beta=1}^{d} A^{\alpha\beta} D_{\alpha\beta} u, \]
where the last equality is due to the independency of $A^{\alpha\beta}$ in $x' \in \mathbb{R}^{d-1}$. Therefore,
\[ \|D_1 U\|_{L^2(Q_2)} \leq N \|u_t\|_{L^2(Q_2)} + N \|D_{x} u\|_{L^2(Q_2)}. \]
By Lemma 3.3, the right-hand side of the above inequality is bounded by a constant times \( \|Du\|_{L^2(Q_2)} \). Thus
\[
\|D_1U\|_{L^2(Q_2)} \leq N\|Du\|_{L^2(Q_2)}.
\]
It is clear that
\[
\|U\|_{L^2(Q_2)} \leq N\|Du\|_{L^2(Q_2)}.
\]
Therefore, the inequality (3.7) and thus Lemma 3.4 are proved.

As usual, for \( \mu \in (0, 1) \) and a function \( w \) defined on \( D \subset \mathbb{R}^{d+1} \), we denote
\[
[w]_{C^{\mu}(D)} = \sup_{(t,x), (s,y) \in D; (t,x) \neq (s,y)} \frac{|w(t,x) - w(s,y)|}{|t-s|^{\mu/2} + |x-y|^{\mu}}.
\]

Lemma 3.5. Let \( A^{\alpha\beta} = A^{\alpha\beta}(x) \). Assume that \( u \in C^{\infty}_{loc} \) satisfies (3.2) in \( Q_4 \). Then we have
\[
[U]_{C^{1/2}(Q_1)} \leq N\|Du\|_{L^2(Q_4)},
\]
\[
[D_{x^i}u]_{C^{1/2}(Q_1)} \leq N\|Du\|_{L^2(Q_4)},
\]
where \( N = N(d,m,\delta) > 0 \).

Proof. We first prove (3.8). By the triangle inequality, we have
\[
\sup_{(t,x), (s,y) \in Q_1; (t,x) \neq (s,y)} \frac{|U(t,x) - U(s,y)|}{|t-s|^{1/4} + |x-y|^{1/2}} \leq \sup_{x_1,y_1 \in (-1,1), x_1 \neq y_1} \frac{|U(t,x_1,x') - U(t,y_1,y')|}{|x_1 - y_1|^{1/2}}
\]
\[
+ \sup_{y_1 \in (0,1)} \frac{|U(t,y_1,x') - U(s,y_1,y')|}{|t-s|^{1/4} + |x'_1 - y'_1|^{1/2}} := I_1 + I_2.
\]

Hence the inequality (3.8) follows if we prove \( I_i \leq \|Du\|_{L^2(Q_4)}, i = 1,2 \).

Estimate of \( I_1 \): By the Sobolev embedding theorem \( U(t,x,x') \), as a function of \( x_1 \in (-1,1), \) satisfies
\[
\sup_{x_1,y_1 \in (-1,1), x_1 \neq y_1} \frac{|U(t,x_1,x') - U(t,y_1,x')|}{|x_1 - y_1|^{1/2}} \leq N\|U(t,\cdot,x')\|_{W^{1/2}_2(-1,1)}.
\]
On the other hand, there exists a positive integer \( k \) such that \( U(t,x_1,x') \) and \( D_1U(t,x_1,x') \), as functions of \( x_1 \in Q_1', \) satisfy
\[
\sup_{(t,x') \in Q_1'} (|U(t,x_1,x')| + |D_1U(t,x_1,x')|)
\]
\[
\leq \|U(t,\cdot,x')\|_{W^{1/2}_2} + \|D_1U(t,\cdot,x')\|_{W^{1/2}_2(Q_1')}.
\]
This implies that, for all \( (t,x') \in Q_1', \)
\[
\int_{-1}^{1} |U(t,x_1,x')|^2 \, dx_1 + \int_{-1}^{1} |D_1U(t,x_1,x')|^2 \, dx_1
\]
\[
\leq N \sum_{i=1,j_1 + j_2 \leq k} \|D^i_1 D^{j_1} D^{j_2}_D U\|_{L^2(Q_2)}^2.
\]
This combined with (3.10) shows that
\[
I_1 \leq N \sum_{i \leq i_{j_1 + j_2 k}} \|D^i_1 D^{j_1} D^{j_2}_D U\|_{L^2(Q_2)} \leq N\|Du\|_{L^2(Q_4)},
\]
where the last inequality is due to Lemma 3.4.

**Estimate of \( I_2 \):** Again using the Sobolev embedding theorem, we find a positive integer \( k \) such that \( U(t, y_1, x') \), as a function of \((t, x') \in Q_1'\), satisfies

\[
\sup_{(s, y, y') \in Q_1' \atop (t, x') \neq (s, y')} |U(t, y_1, x') - U(s, y_1, y')| \leq N\|U(\cdot, y_1, \cdot)\|_{W^{1/4}_2(Q_1')}.
\]  

(3.11)

For each \( i, j \) such that \( i + j \leq k \), \( D^i_x D^j_x U(t, y_1, x') \), as a function of \( y_1 \in (-1, 1) \), satisfies

\[
\sup_{y_1 \in (-1, 1)} \|D^i_x D^j_x U(t, y_1, x')\| \leq N\|D^i_x D^j_x U(t, \cdot, x')\|_{L^2(-1, 1)} + N\|D^i_x D^j_x D^1_x U(\cdot, \cdot, x')\|_{L^2(-1, 1)}.
\]

This together with (3.11) gives

\[
I_2 \leq N \sum_{i+j+k \leq k} \|D^i_x D^j_x D^1_x U\|_{L^2(Q_2)} \leq N\|Du\|_{L^2(Q_d)},
\]

where the last inequality follows from Lemma 3.4. Hence the inequality (3.8) is proved. The proof of (3.9) is done by repeating the same reasoning as above with the help of Lemma 3.3.

By using a scaling argument, we have the following corollary.

**Corollary 3.6.** Let \( r \in (0, \infty) \), \( \kappa \in [4, \infty) \), and \( A^{\alpha\beta} = A^{\alpha\beta}(x_1) \). Assume \( u \in C^\infty_{\text{loc}} \) satisfies (3.2) in \( Q_{\kappa r} \). Then we have

\[
(|U - (U)_{Q_r}|)_{Q_r} \leq N\kappa^{-1/2}(\|Du\|_{Q_{\kappa r}}^2)^{1/2}, \quad (3.12)
\]

\[
(|D^\alpha u - (D^\alpha u)_{Q_r}|)_{Q_r} \leq N\kappa^{-1/2}(\|Du\|_{Q_{\kappa r}}^2)^{1/2}, \quad (3.13)
\]

where \( N = N(d, m, \delta) > 0 \).

**Proof.** We prove only (3.12). The inequality (3.13) is proved similarly. By a scaling argument, i.e., by considering \( u(r^2 t, r x) \) and \( A^{\alpha\beta}(r x_1) \), it suffices to prove (3.12) when \( r = 1 \). In this case, to use again the same type of scaling argument we define

\[
\hat{u}(t, x) = u((\kappa/4)^2 t, (\kappa/4) x), \quad A^{\alpha\beta}(x_1) = A^{\alpha\beta}(\kappa/4)(x_1).
\]

Since \( Pu = 0 \) in \( Q_\kappa \), we have

\[
-\hat{u}_t + D_\alpha(\hat{A}^{\alpha\beta} D_\beta \hat{u}) = 0
\]

in \( Q_4 \). Then by Lemma 3.3 applied to \( \hat{U} = \sum_{\beta=1}^d \hat{A}^{1\beta} D_\beta \hat{u} \), we have

\[
[\hat{U}]_{C^{1/2}(Q_1)} \leq N\|D\hat{u}\|_{L^2(Q_4)}.
\]

Note that

\[
(|U - (U)_{Q_1}|)_{Q_1} \leq [U]_{C^{1/2}(Q_1)} \leq [U]_{C^{1/2}(Q_{\kappa r})} \leq \kappa^{-3/2}[\hat{U}]_{C^{1/2}(Q_1)} \leq N\kappa^{-3/2}\|D\hat{u}\|_{L^2(Q_4)} = N\kappa^{-1/2}(\|Du\|_{Q_{\kappa r}}^2)^{1/2}.
\]

The corollary is proved. \( \Box \)

The following is the main result of this section.
Proposition 3.7. Let $\kappa \in [8, \infty)$, $r \in (0, \infty)$, $A^{\alpha \beta} = A^{\alpha \beta}(x_1)$, and $g \in L_{2,loc}$. Assume that $u \in \mathcal{H}_{2,loc}^1$ satisfies

$$\mathcal{P}u = \text{div} \ g$$

in $Q_{kr}$. Then we have

$$\left( |U - (U)_{Q_r}|_{Q_r} + |D_x u - (D_x u)_{Q_r}|_{Q_r} \right) \leq N\kappa -1/2 |Du|^{1/2}_{Q_{kr}} + N\kappa^{(d+2)/2} |g|^{1/2}_{Q_{kr}},$$

(3.14)

where $N$ depends only on $d, m, \kappa$ and $\delta$.

Proof. By performing the standard mollifications, we may assume $u, g$ and $A^{\alpha \beta}$ are smooth. Take $\zeta \in C_0^\infty$ such that

$$\zeta = 1 \text{ on } Q_{kr/2}, \quad \zeta = 0 \text{ outside } (- (kr)^2, (kr)^2) \times B_{kr}.$$  

By Lemma 3.1 (iii), there exists a unique $w \in \mathcal{H}_{2,loc}^1((- (kr)^2, 0) \times \mathbb{R}^d)$ satisfies

$$\mathcal{P}w = \text{div}(\zeta g).$$

Moreover,

$$\|Du\|_{L_2(Q_{kr})} \leq \|\zeta g\|_{L_2((- (kr)^2, 0) \times \mathbb{R}^d)} \leq \|g\|_{L_2(Q_{kr})}.$$  

(3.15)

Let $v = u - w$ so that $\mathcal{P}v = 0$ in $Q_{kr/2}$. Note that by the classical result $w$ is in fact infinitely differentiable in $Q_{kr}$ because the coefficients of the operator as well as $\zeta g$ are smooth. Hence $v$ is also infinitely differentiable in $Q_{kr}$.

Denote $V = A^{1/2}D_\beta v$. By Corollary 3.4 we have

$$\left( |V - (V)_{Q_r}|_{Q_r} + |D_x v - (D_x v)_{Q_r}|_{Q_r} \right) \leq N\kappa -1/2 |Dv|^{1/2}_{Q_{kr}}.$$  

(3.16)

Now we observe that

$$\left( |U - (U)_{Q_r}|_{Q_r} + |D_x u - (D_x u)_{Q_r}|_{Q_r} \right) \leq 2 \left( |U - (U)_{Q_r}|_{Q_r} + 2 \left( |D_x u - (D_x u)_{Q_r}|_{Q_r} \right) \right).$$

Therefore, upon using (3.15), (3.16) and the triangle’s inequality, we bound the left-hand side of (3.14) by

$$2 \left( |V - (V)_{Q_r}|_{Q_r} + 2 \left( |D_x v - (D_x v)_{Q_r}|_{Q_r} \right) \right) \leq N\kappa -1/2 |Dv|^{1/2}_{Q_{kr}} + N\kappa^{(d+2)/2} |g|^{1/2}_{Q_{kr}}$$

$$\leq N\kappa -1/2 |Du|^{1/2}_{Q_{kr}} + N\kappa^{(d+2)/2} |g|^{1/2}_{Q_{kr}}.$$  

The proposition is proved. \hfill \Box

We will also make use of a generalization of the Fefferman–Stein Theorem proved recently in [24]. Let $C_n = \{C_n(i_0, i_1, \ldots, i_d), i_0, \ldots, i_d \in \mathbb{Z}\}, n \in \mathbb{Z}$ be the filtration of partitions given by parabolic dyadic cubes, where

$$C_n(i_0, i_1, \ldots, i_d) = [i_0 2^{-2n}, (i_0 + 1) 2^{-2n}) \times [i_1 2^{-n}, (i_1 + 1) 2^{-n}) \times \cdots \times [i_d 2^{-n}, (i_d + 1) 2^{-n}).$$
Theorem 3.8 (Theorem 2.7 [24]). Let \( p \in (0, 1) \), \( F, G, H \in L_1 \). Assume \( G \geq |F| \), \( H \geq 0 \) and for any \( n \in \mathbb{Z} \) and \( C \in \mathbb{C}_n \) there exists a measurable function \( F^C \) defined on \( C \) such that \( |F| \leq F^C \leq G \) on \( C \) and

\[
\int_C |F^C - (F^C)_C| \, dx \, dt \leq \int_C H \, dx \, dt.
\]

Then we have

\[
\|F\|_{L_p}^p \leq N\|H\|_{L_p}\|G\|_{L_p}^{p-1},
\]

provided that \( H,G \in L_p \).

4. Proof of Theorem 2.2

In this section we complete the proof of Theorem 2.2.

Lemma 4.1. Let \( \kappa \geq 8 \), \( r > 0 \), \( \tilde{A} \in \mathcal{A} \), \( \psi \in \Psi \), \( u \in C_{loc}^\infty \) and \( g \in L_{2,loc} \). Assume

\[
-(u_t(x,t) + D_k \left( \tilde{A}^{kl}(y) D_l u(t,x) \right)) = \text{div} g,
\]

where \( y = \psi(x), \phi = \psi^{-1} \), and

\[
\tilde{A}^{kl}(y) = \sum_{\alpha,\beta=1}^d D_{y\alpha}\phi_k(y)\tilde{A}^{\alpha\beta}(y_1)D_{y\beta}\phi_l(y), \quad k,l = 1, \ldots, d.
\]

Then there exist constants \( \nu = \nu(d,\delta) \geq 1 \) and \( N = N(d,m,\delta) > 0 \) such that

\[
\left(\|Ju - (Ju)_{Q_r,1}\|_{Q_r} \right) + \sum_{\beta=2}^d \left(\|Ju_\beta - (Ju)_{Q_r,1}\|_{Q_r}\right) \leq N\kappa^{(d+2)/2} \left(\|g\|^2 + \|u_t\|^2\right)^{1/2}_{Q_{vr}} + N\kappa^{-1/2} \left(\|Du_t\|^2\right)^{1/2}_{Q_{vr}},
\]

where

\[
u_\beta(t,x) = (D_{y\beta}g)(t,\psi(x)), \quad \psi(t,y) = u(t,\phi(y)), \quad J(y) = \det(\partial \phi / \partial y), \quad U(t,x) = \sum_{\beta=1}^d \tilde{A}^{1\beta}(\psi_1(x))u_\beta(t,x).
\]

Proof. Without loss of generality, we assume \( \psi(0) = 0 \). From the integral formulation of (4.1), we see that \( v \) satisfies

\[
-(Jv)_t + D_{y\alpha} \left( \tilde{A}^{\alpha\beta}(y_1)JD_{y\beta}v \right) = \text{div} \tilde{g},
\]

where

\[
\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_d), \quad \tilde{g}_\alpha = J \sum_{\beta=1}^d g_\beta(t,\phi(y))(D_{\beta}y_\alpha)(\phi(y)).
\]

So \( Jv \) satisfies

\[
-(Jv)_t + D_{y\alpha} \left( \tilde{A}^{\alpha\beta}(y_1)D_{y\beta}(Jv) \right) = \text{div} \tilde{g},
\]

where

\[
\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_d), \quad \tilde{g}_\alpha = J \sum_{\beta=1}^d g_\beta(t,\phi(y))(D_{\beta}y_\alpha)(\phi(y)) + \sum_{\beta=1}^d \tilde{A}^{\alpha\beta}(y_1)u(t,y)D_{y\beta}J.
\]
Then by Proposition 3.7,

\[
(|V - (V)_{Q_r}|)_{Q_r} + (|D_{y'}(Jv) - (D_{y'}(Jv))_{Q_r}|)_{Q_r}
\leq N\kappa^{-1/2}(|D(Jv)|^2)^{1/2} + N\kappa^{(d+2)/2}(\|g\|^2)^{1/2},
\]

(4.4)

where

\[
V(t, y) = \sum_{\beta=1}^{d}\hat{A}^{1\beta}(y_1)D_{y_\beta}(Jv).
\]

Now we observe that

\[
(|JU - (JU)_{Q_r}|)_{Q_r} + \sum_{\beta=2}^{d}(|Ju_\beta - (Ju_\beta)_{Q_r}|)_{Q_r}
\leq 2(|JU - (V)_{Q_r}|)_{Q_r} + 2\sum_{\beta=2}^{d}(|Ju_\beta - (D_{y_\beta}(Jv))_{Q_r}|)_{Q_r} := 2(I_1 + I_2),
\]

where \( \nu > 1 \) is a constant obtained in the following observation. There exist constants \( \nu \) as well as \( N \) depending only on \( d \) and \( \delta \) such that, for a nonnegative measurable function \( f(t, x) \),

\[
\int_{Q_r} f(t, x) \, dx \, dt \leq N \int_{Q_{\nu r}} f(t, \phi(y)) \, dy \, dt,
\]

\[
\int_{Q_r} f(t, \phi(y)) \, dy \, dt \leq N \int_{Q_{\nu r}} f(t, x) \, dx \, dt.
\]

(4.5)

Thus

\[
I_1 = \int_{Q_r} |J(\psi(x))| \sum_{\beta=1}^{d}\hat{A}^{1\beta}(\psi_1(x))(D_{y_\beta}v)(t, \psi(x)) - (V)_{Q_{\nu r}} \, dx \, dt
\]

\[
\leq N \int_{Q_{\nu r}} |J(y)| \sum_{\beta=1}^{d}\hat{A}^{1\beta}(y_1)(D_{y_\beta}v)(t, y) - (V)_{Q_{\nu r}} \, dy \, dt
\]

\[
\leq N (|V - (V)_{Q_{\nu r}}|)_{Q_{\nu r}} + N(|v|)_{Q_{\nu r}}.
\]

Similarly,

\[
I_2 = \int_{Q_r} |J(\psi(x))(D_{y_\beta}v)(t, \psi(x)) - (D_{y_\beta}(Jv))_{Q_{\nu r}}| \, dx \, dt
\]

\[
\leq N \int_{Q_{\nu r}} |J(y)(D_{y_\beta}v)(t, y) - (D_{y_\beta}(Jv))_{Q_{\nu r}}| \, dy \, dt
\]

\[
\leq N (|D_{y'}(Jv) - (D_{y'}(Jv))_{Q_{\nu r}}|)_{Q_{\nu r}} + N (|v|)_{Q_{\nu r}}.
\]

Using the above two sets of inequalities for \( I_1 \) and \( I_2 \) as well as using (4.4) with \( \nu r \) in place of \( r \) and (4.5), we obtain the desired inequality in the lemma with \( \nu^2 \) in place of \( \nu \). The proof is completed upon simply replacing \( \nu^2 \) by another constant \( \nu \).

With the aid of Lemma 4.1, we estimate the mean oscillations of \( JU \) and \( Ju_\beta \) for general operators.
Proposition 4.2. Let $\gamma > 0$ and $\tau, \sigma \in (1, \infty)$ satisfy $1/\tau + 1/\sigma = 1$. Let $\nu = \nu(d, \delta) \geq 1$ be the constant in Lemma 4.1, $B^\alpha = \bar{B}^\alpha = C = 0$, and $g \in L^2_{2,loc}$. Assume that $u \in C^\infty_0$ vanishes outside $Q_R$ for some $R \in (0, R_0]$ and satisfies $P \partial_t = \operatorname{div} g$. Then under Assumption \(2.1(\gamma)\), for each $r \in (0, \infty)$, $\kappa \geq 8$, and $(t_0, x_0) \in \mathbb{R}^{d+1}$, there exist a diffeomorphism $\psi \in \Psi$, coefficients $\hat{A}^{ij}, \beta = 1, \cdots, d$ (independent of $u$), and a positive constant $N = N(d, m, \delta, \tau)$ such that

$$
\left| \left( (JU - (JU)_{Qr(t_0, x_0)} ) \right)_{Qr(t_0, x_0)} + \sum_{\beta=2}^{d} \left| (J \partial_t^\beta - (J \partial_t^\beta)_{Qr(t_0, x_0)} ) \right|_{Qr(t_0, x_0)} \right|
\leq N \kappa^{(d+2)/2} \left( |g|^2 + |u|^2 \right)^{1/2}_{Qr(t_0, x_0)} + N \kappa^{(d+2)/2} \left( \left| D \partial_t u \right|^2 \right)^{1/(2\tau)}_{Qr(t_0, x_0)} + N \kappa^{(d+2)/2} R + \kappa^{-1/2} \left( \left| D \partial_t u \right|^2 \right)^{1/2}_{Qr(t_0, x_0)} ,
$$

(4.6)

where $\partial_t$, $J$ and $U$ are defined as in \((4.2)\) and \((4.3)\).

Proof. We fix a $\kappa \geq 8$ and $r \in (0, \infty)$. Choose $Q$ to be $Q_{\kappa r}(t_0, x_0)$ if $\nu r < R$ and $Q_{\kappa r}$ if $\nu r \geq R$. Let $(t^*, x^*)$ be the center of $Q$ and $y^* = \psi(x^*)$. By Assumption \((2.1)(\gamma)\), we can find $\psi \in \Psi$ and $\bar{A} = \bar{A}(r) \in \mathcal{A}$ satisfying \((2.3)\). We set

$$
\hat{A}^{\alpha \beta}(y) = \sum_{l=1}^{d} \sum_{l=1}^{d} D_k \partial_t \phi_l(y) \hat{A}^{k l}(y), \quad \alpha, \beta = 1, \cdots, d,
$$

and

$$
\hat{A}^{k l}(y) = \sum_{l=1}^{d} D_{y^\alpha} \phi_l(y) \hat{A}^{\alpha \beta}(y_0) D_{y^\beta} \phi_l(y), \quad k, l = 1, \cdots, d,
$$

where $y = \psi(x)$. The ellipticity constants of $\hat{A}$ and $\hat{\bar{A}}$ may not be $\delta$, but they depend only on $\delta$. Note that

$$
- \partial_t u + D_k (\hat{A}^{k l} D_l u) = \operatorname{div} g + D_k \left( \hat{A}^{k l} D_l u - \hat{\bar{A}}^{k l} D_l u \right).
$$

Thus by Lemma 4.1 with a shift of the coordinates,

$$
\left| \left( (JU - (JU)_{Qr(t_0, x_0)} ) \right)_{Qr(t_0, x_0)} + \sum_{\beta=2}^{d} \left| (J \partial_t^\beta - (J \partial_t^\beta)_{Qr(t_0, x_0)} ) \right|_{Qr(t_0, x_0)} \right|
\leq N \kappa^{(d+2)/2} \left( |g|^2 + |u|^2 \right)^{1/2}_{Qr(t_0, x_0)} + N \kappa^{-1/2} \left( \left| D \partial_t u \right|^2 \right)^{1/2}_{Qr(t_0, x_0)} ,
$$

(4.7)

where $\nu = \nu(d, \delta) \geq 1$, $N = N(d, m, \delta) > 0$, and

$$
\int_{Q_{\kappa r}(t_0, x_0)} |g|^2 dx \ dt := \int_{Q_{\kappa r}(t_0, x_0)} |\hat{\bar{A}}^{k l} D_l u - \hat{A}^{k l} D_l u|^2 dx \ dt 
\leq 2 \int_{Q_{\kappa r}(t_0, x_0)} |\hat{\bar{A}}^{k l} D_l u - \bar{A}^{k l} D_l u|^2 dx \ dt + 2 \int_{Q_{\kappa r}(t_0, x_0)} |\bar{A}^{k l} D_l u - \bar{A}^{k l} D_l u|^2 dx \ dt 
= 2( I_1 + I_2).
$$

Note that

$$
\hat{A}^{k l}(y) = \sum_{\alpha, \beta=1}^{d} D_{y^\alpha} \phi_l \hat{A}^{\alpha \beta}(y_0) D_{y^\beta} \phi_l(y^*).
$$

Thus by \((2.2)\)

$$
I_1 = \int_{Q_{\kappa r}(t_0, x_0) \cap Q_R} \left| \left( D_{y^\alpha} \phi_l D_{y^\beta} \phi_l \hat{A}^{\alpha \beta}(y) \right) (\psi(x)) - \left( D_{y^\alpha} \phi_l D_{y^\beta} \phi_l \hat{A}^{\alpha \beta}(y^*) \right) \right| D \partial_t u|^2 dx \ dt
$$

and

$$
I_2 = \int_{Q_{\kappa r}(t_0, x_0) \cap Q_R} \left| \left( D_{y^\alpha} \phi_l D_{y^\beta} \phi_l \hat{A}^{\alpha \beta}(y) \right) (\psi(x)) - \left( D_{y^\alpha} \phi_l D_{y^\beta} \phi_l \hat{A}^{\alpha \beta}(y^*) \right) \right| D \partial_t u|^2 dx \ dt.
$$
\[ \leq N\|(D_{y^*}\phi_k D_{y^*}\phi_l})(\psi(\cdot)) - (D_{y^*}\phi_k D_{y^*}\phi_l)(y^*)\|_{L^\infty(Q_R)}^2 \int_{Q_{\nu,r}(t_0,x_0)} |Du|^2 \, dx \, dt \]
\[ \leq NR^2 \int_{Q_{\nu,r}(t_0,x_0)} |Du|^2 \, dx \, dt. \] (4.8)

On the other hand, by the Hölder’s inequality, we have
\[ I_2 \leq NI_2^{1/\tau} I_2^{1/\tau}, \] (4.9)

where
\[ I_{21} = \sum R \int_{Q_{\nu,r}(t_0,x_0) \cap Q_R} |\tilde{A}^{kl}(\psi_1(x)) - A^{kl}(t,x)|^{2\sigma} \, dx \, dt, \]
\[ I_{22} = \int_{Q_{\nu,r}(t_0,x_0)} |Du|^{2\tau} \, dx. \]

Due to Assumption \((2.2) (\gamma)\),
\[ I_{21} \leq N\gamma |Q| \leq N(\nu\kappa R)^{d+2}\gamma, \]

This together with \((4.7) - (4.9)\) yields \((4.6)\). The proposition is proved. \(\square\)

The next corollary follows immediately from Proposition 4.2 by using the triangle inequality.

**Corollary 4.3.** Let \(\gamma > 0, \kappa \geq 8\) and \(\tau, \sigma \in (1, \infty)\) satisfy \(1/\tau + 1/\sigma = 1\). Suppose that \(B^\alpha = \tilde{B}^\alpha = C = 0\), \(\bm{g} \in L^2_{2,loc}\), and \(\bm{u} \in C^\infty_\nu\) vanishes outside \(Q_R\) for some \(R \in (0, R_0]\) satisfying \(\mathcal{P}\bm{u} = \text{div} \bm{g}\). Under assumption \((2.2)(\gamma)\), for each \(n \in \mathbb{Z} \) and \(C \in \mathbb{C}_n\), there exist a diffeomorphism \(\psi \in \Psi\), coefficients \(\tilde{A}^{1\beta}, \beta = 1, \cdots, d\) (independent of \(\bm{u}\)), and a constant \(N = N(d, m, \delta, \tau)\) such that
\[ (|J\mathcal{U} - (J\mathcal{U})_C|_C) + \sum_{\beta=2}^d (|J\bm{u}_\beta - (J\bm{u}_\beta)_C|_C) \leq N(H)_C, \] (4.10)

where \(\bm{u}_\beta\), \(J\) and \(\mathcal{U}\) are defined as in \((4.2)\) and \((4.3)\), and
\[ H = \gamma^{d+2} (M(\|\bm{g}\|^2 + \|\bm{u}\|^2_2))^{1/2} + \gamma^{d+2} (M(\|Du\|^{2\tau}_2))^{1/2} + (\gamma^{d+2} R + \gamma^{-\frac{1}{2}})(M(\|Du\|^2))^{1/2}. \]

**Proposition 4.4.** Let \(p \in (2, \infty)\). Assume \(B^\alpha = \tilde{B}^\alpha = C = 0\). Then there exist positive constants \(\gamma, N\) and \(R \in (0, 1]\) depending only on \(d, m, p, \) and \(\delta\) such that under Assumption \((2.2)(\gamma)\), for any \(\bm{u} \in C^\infty_\nu\) vanishing outside \(Q_{RR_0}\) and \(\bm{g} \in L^p\), we have
\[ \|Du\|_{L^p} \leq N\|\bm{g}\|_{L^p} + N\|\bm{u}\|_{L^p}, \] (4.11)

provided that \(\mathcal{P}\bm{u} = \text{div} \bm{g}\).

**Proof.** Let \(\gamma > 0, \kappa \geq 8\) and \(R \in (0, 1]\) be constants to be specified later. Let \(\tau = (p + 2)/4 > 1\) such that \(p > 2\tau\). We take \(n \in \mathbb{Z}\), \(C \in \mathbb{C}_n\) and let \(\psi \in \Psi\) be the diffeomorphism from Corollary 4.3 corresponding to the chosen \(n\) and \(C\). We also obtain corresponding \(\bm{u}_\beta\), \(J\) and \(\mathcal{U}\) as in \((4.2)\) and \((4.3)\).

It is easily seen that
\[ |Du| \leq N_2(d, \delta) \sum_{\beta=2}^d |J\bm{u}_\beta| + N_2(d, \delta)|J\mathcal{U}| \leq N_3(d, \delta)|Du|, \]
We set
\[ F = |Du|, \quad F^C = N_2 \sum_{\beta=2}^d |u_\beta| + N_2 |JU|, \quad G = N_3 |Du|. \]

By the triangle inequality and (4.10),
\[ \left(||F^C - (F^C)_C||_C \right) \leq N(H)_C, \]
where \( H \) is defined in Corollary 4.3. Now by Theorem 3.8, we get
\[ \|Du\|_{L^p} = \|F\|_{L^p} \leq N\|H\|_{L^p} \leq N(\epsilon)\|H\|_{L^p} + \epsilon\|G\|_{L^p}. \]

Upon taking a small \( \epsilon > 0 \), it holds that
\[ \|Du\|_{L^p} \leq N\|H\|_{L^p}. \quad (4.12) \]

We use the definition of \( H \) and the Hardy-Littlewood maximal function theorem (recall \( p > 2 \tau > 2 \)) to deduce from (4.12)
\[ \|Du\|_{L^p} \leq N\kappa^{d/2} (\|g\|_{L^p} + \|u\|_{L^p}) + N(\kappa^{d/2} \gamma^3 + \kappa^{d+2} R R_0 + \kappa^{-1})\|Du\|_{L^p}. \quad (4.13) \]

By choosing \( \kappa \) sufficiently large, then \( \gamma \) and \( R \) sufficiently small in (4.13) such that
\[ N(\kappa^{d+2} \gamma^3 + \kappa^{d+2} R R_0 + \kappa^{-1}) \leq 1/2, \]
we come to (4.11). The proposition is proved. \( \Box \)

Proof of Theorem 2.2. Thanks for the duality argument, it suffices to prove the case \( p > 2 \). For \( T = \infty \), the theorem follows from Proposition 4.4 by using a partition of unity and an idea by S. Agmon; see, for instance, the proof of Theorem 1.4 [24]. For general \( T \in (-\infty, \infty] \), we use the fact that \( u = w \) for \( t < T \), where \( w \in \mathcal{H}_p \) solves
\[ \mathcal{P}w - \lambda w = \chi_{t<T}(\mathcal{P}u - \lambda u). \]

This finishes the proof of the theorem. \( \Box \)

5. A REMARK ABOUT ELLIPTIC SYSTEMS

For elliptic systems, the condition on diffeomorphisms \( \psi \) and \( \phi \) can be relaxed. Indeed, we only require \( \psi \) and \( \phi \) to be in \( C^{0,1} \) and \( D\psi \) has locally small mean oscillations. More precisely, we impose the following assumption on \( \psi \) and \( A \), which is weaker than the one in Section 2.

Let \( \Psi \) be the set of \( C^{0,1} \) diffeomorphisms \( \psi : \mathbb{R}^d \to \mathbb{R}^d \) such that the mappings \( \psi \) and \( \phi = \psi^{-1} \) satisfy
\[ |D\psi| \leq \delta^{-1}, \quad |D\phi| \leq \delta^{-1}. \]

Assumption 5.1 (\( \gamma \)). There exists a positive constant \( R_0 \in (0,1] \) such that, for any ball \( B \) of radius less than \( R_0 \), one can find an \( \hat{A} \in \mathcal{A} \) and a \( \psi = (\psi_1, \cdots, \psi_d) \in \Psi \) such that
\[ \sum_{k,l} \int_B |\hat{A}^{kl}(\psi_1(x)) - A^{\alpha\beta} D_\alpha \psi_k D_\beta \psi_l J(x)| \, dx \leq \gamma |B|. \quad (5.1) \]
Lemma 5.2. Let $\kappa \geq 8$, $r > 0$, $\hat{A} \in \mathcal{A}$, $\psi \in \Psi$, $u \in C_{0}^{\infty}(\mathbb{R}^{d})$ and $g \in L_{2,\text{loc}}(\mathbb{R}^{d})$.

Assume
\[
D_{\alpha} \left( \hat{A}^{k}(y)D_{y_{k}}\phi_{\alpha}(y)D_{y_{\beta}}\phi_{\beta}(y)J^{-1}D_{\beta}u(x) \right) = \text{div } g,
\]
where $y = \psi(x)$ and $\phi = \psi^{-1}$. Then there exist constants $\nu = \nu(d, \delta) \geq 1$ and $N = N(d, m, \delta)$ such that
\[
(U - (U)_{B_{r}})_{B_{r}} + \sum_{\beta = 2}^{d} (u_{\beta} - (u_{\beta})_{B_{r}})_{B_{r}} \leq N \kappa^{(d+2)/2} \left( \|g\|_{B_{\infty,r}(x_{0})}^{1/2} + \kappa^{-1/2} \left( \|Du\|_{B_{\infty,r}(x_{0})}^{2} \right) \right),
\]
where $u_{\beta}(x) = (D_{y_{\beta}}\nu)(\psi(x))$, $v(y) = u(\phi(y))$, \begin{equation} J(x) = \det(\partial \psi/\partial x)^{-1}, \quad U(x) = \hat{A}^{1\beta}(\psi_{1}(x))u_{\beta}(t, x). \end{equation}

Proof. The proof is similar to that of Lemma 4.1. From the integral formulation, it is easy to see that $v$ satisfies
\[
D_{y_{\alpha}}(\hat{A}^{\alpha\beta}(y^{1})D_{y_{\beta}}v) = D_{y_{\alpha}}(JD_{\beta}\psi_{\alpha}g_{\beta}).
\]
The lemma then follows from Proposition 3.7.

Proposition 5.3. Let $\gamma > 0$ and $\tau, \sigma \in (1, \infty)$ satisfy $1/\tau + 1/\sigma = 1$. Let $\nu = \nu(d, \delta) > 1$ be the constant in Lemma 4.1. $B^{0} = \hat{B}^{0} = \hat{C} = 0$, and $g \in L_{2,\text{loc}}(\mathbb{R}^{d})$.

Assume that $u \in C_{0}^{\infty}(\mathbb{R}^{d})$ vanishes outside $B_{R}$ for some $R \in (0, R_{0}]$ and satisfies $Lu = \text{div } g$. Then under Assumption 5.1 ($\gamma$), for each $r \in (0, \infty)$, $\gamma \geq 8$, and $x_{0} \in \mathbb{R}^{d}$, there exist a diffeomorphism $\psi \in \Psi$, coefficients $\hat{A}^{1\beta}$, $\beta = 1, \cdots, d$ (independent of $u$), and a positive constant $N = N(d, m, \delta, \tau)$ such that
\[
(U - (U)_{B_{r}(x_{0})})_{B_{r}(x_{0})} + \sum_{\beta = 2}^{d} (u_{\beta} - (u_{\beta})_{B_{r}(x_{0})})_{B_{r}(x_{0})} \leq N \kappa^{(d+2)/2} \left( \|g\|_{B_{\infty,r}(x_{0})}^{1/2} + \kappa^{-1/2} \left( \|Du\|_{B_{\infty,r}(x_{0})}^{2} \right) \right),
\]
where $u_{\beta}$, $J$ and $U$ are defined as in (5.2) and (5.3).

Proof. We fix a $\kappa \geq 8$, and $r \in (0, \infty)$. Choose $B$ to be $B_{\nu\kappa r}(t_{0}, x_{0})$ if $\nu\kappa r < R$ and $B_{R}$ if $\nu\kappa r \geq R$. By Assumption 5.1 ($\gamma$), we can find $\psi \in \Psi$ and $\hat{A} = \hat{A}(s) \in \mathcal{A}$ satisfying (5.1). By Lemma 5.2 with a shift of the coordinates,
\[
(U - (U)_{B_{r}(x_{0})})_{B_{r}(x_{0})} + \sum_{\beta = 2}^{d} (u_{\beta} - (u_{\beta})_{B_{r}(x_{0})})_{B_{r}(x_{0})} \leq N \kappa^{(d+2)/2} \left( \|g + g\|_{B_{\infty,r}(x_{0})}^{1/2} \right) + N \kappa^{-1/2} \left( \|Du\|_{B_{\infty,r}(x_{0})}^{2} \right),
\]
where $N = N(d, m, \delta) > 0$ and
\[
\hat{g}_{\alpha} = \left( \hat{A}^{k}(y)D_{y_{k}}\phi_{\alpha}(y)D_{y_{\beta}}\phi_{\beta}(y)J^{-1} - A^{\alpha\beta}(x) \right) D_{\beta}u(x).
\]
By the definition of \( \hat{A} \),
\[
\int_{B_{\nu r}(x_0)} |\mathbf{g}_\alpha|^2 \, dx \leq N \int_{B_{\nu r}(x_0) \cap B_R} \left| \left( \hat{A}^{kl}(\psi_1) - A^{\alpha\beta} D_\alpha \psi_k D_\beta \psi_l J \right) \right|^2 |Du|^2 \, dx \\
\leq NI_1^{1/\sigma} I_2^{1/\tau}, \tag{5.6}
\]
where
\[
I_1 = \sum_{k,l} \int_{B_{\nu r}(x_0) \cap B_R} \left| \hat{A}^{kl}(\psi_1) - A^{\alpha\beta} D_\alpha \psi_k D_\beta \psi_l J \right|^{2\sigma} \, dx,
\]
\[
I_2 = \int_{B_{\nu r}(x_0)} |Du|^{2\sigma} \, dx.
\]

Due to Assumption 5.1 (\( \gamma \)),
\[
I_1 \leq \sum_{k,l} \int_B \left| \hat{A}^{kl}(\psi_1) - A^{\alpha\beta} D_\alpha \psi_k D_\beta \psi_l J \right|^{2\sigma} \, dx \leq N \gamma |B| \leq N (\nu K)^d \gamma.
\]
This together with (5.5) and (5.6) yields (5.4). The proposition is proved. \( \Box \)

Following the arguments in the previous section, we obtain the result of Theorem 2.5 under Assumption 5.1. We omit the details.

### References

[1] P. Auscher, M. Qafsaoui, Observations on \( W^{1,p} \) estimates for divergence elliptic equations with VMO coefficients, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.*, 5 (2002), 487–509.
[2] E. Acerbi and G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.* 136 (2007), no. 2, 285–320.
[3] S. Byun, L. Wang, Parabolic equations in Reifenberg domains, *Arch. Ration. Mech. Anal.*, 176 (2005), 271–301.
[4] S. Byun, L. Wang, Gradient estimates for elliptic systems in non-smooth domains, *Math. Ann.*, 341 (2008), 629–650.
[5] M. Bramanti, M. Cerutti, \( W^{1,2}_p \) solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, *Comm. Partial Differential Equations* 18 (1993), no. 9–10, 1735–1763.
[6] F. Chiarenza, M. Frasca, P. Longo, Interior \( W^{2,p} \) estimates for nondivergence elliptic equations with discontinuous coefficients, *Ricerche Mat.* 40 (1991), 149–168.
[7] , \( W^{2,p}_\ast \)-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.* 336 (1993), no. 2, 841–853.
[8] R. Denk, M. Hieber, J. Prüss, Optimal \( L^p-L^q \)-estimates for parabolic boundary value problems with inhomogeneous data, *Math. Z.* 257 (2007), no. 1, 193–224.
[9] G. Di Fazio, \( L^p \) estimates for divergence form elliptic equations with discontinuous coefficients. (Italian summary) *Boll. Un. Mat. Ital. A* (7) 10 (1996), no. 2, 409–420.
[10] H. Dong, Solvability of parabolic equations in divergence form with partially BMO coefficients, preprint (2008). (http://www.dam.brown.edu/people/hdong/partiallyVMO4.pdf).
[11] , Parabolic equations with variably partially VMO coefficients, preprint (2008), arXiv:0811.4124 [math.AP].
[12] H. Dong, D. Kim, Elliptic equations in divergence form with partially BMO coefficients, *Arch. Ration. Mech. Anal.*, to appear (2009), arXiv:0810.4716 [math.AP].
[13] , \( L^p \) solvability of divergence type parabolic and elliptic systems with partially BMO coefficients, preprint (2008).
[14] H. Dong, N. V. Krylov, Second-order elliptic and parabolic equations with \( B(\mathbb{R}^2, VMO) \) coefficients, *Trans. Amer. Math. Soc.*, to appear (2009), arXiv:0810.2739 [math.AP].
[15] R. Haller-Dintelmann, H. Heck, M. Hieber, \( L^p-L^q \)-estimates for parabolic systems in non-divergence form with VMO coefficients, *J. London Math. Soc. (2)* 74 (2006), no. 3, 717–736.
[16] D. Kim, Parabolic equations with measurable coefficients. II. (English summary) *J. Math. Anal. Appl.* 334 (2007), no. 1, 534–548.
[17] ______, Elliptic and parabolic equations with measurable coefficients in $L_p$-spaces with mixed norms, *Methods Appl. Anal.*, 15 (2008), no. 4, 437–468.

[18] ______, Parabolic equations with partially BMO coefficients and boundary value problems in Sobolev spaces with mixed norms, *Potential Anal.*, to appear.

[19] D. Kim, N. V. Krylov, Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others, *SIAM J. Math. Anal.* 39 (2007), no. 2, 489–506.

[20] ______, Parabolic equations with measurable coefficients, *Potential Anal.* 26 (2007), no. 4, 345–361.

[21] J. Kinnunen, J. Lewis, Higher integrability for parabolic systems of $p$-Laplacian type, *Duke Math. J.* 102 (2000), no. 2, 253–271.

[22] N. V. Krylov, Parabolic and elliptic equations with VMO coefficients, *Comm. Partial Differential Equations* 32 (2007), no. 3, 453–475.

[23] ______, Parabolic equations with VMO coefficients in spaces with mixed norms, *J. Funct. Anal.* 250 (2007), no. 2, 521–558.

[24] ______, Second-order elliptic equations with variably partially VMO coefficients, *J. Funct. Anal.* 257 (2009), no. 6, 1695–1712.

[25] D. Palagachev, Quasilinear elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.* 347 (1995), no. 7, 2481–2493.

[26] D. Palagachev, L. Softova, A priori estimates and precise regularity for parabolic systems with discontinuous data, *Discrete Contin. Dyn. Syst.*, 13 (2005), no. 3, 721–742.

(H. Dong) Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

E-mail address: Hongjie_Dong@brown.edu

(D. Kim) Department of Mathematics, University of Southern California, 3620 South Vermont Avenue, KAP 108, Los Angeles, CA 90089, USA

E-mail address: doyoonki@usc.edu