Long-Range Nematic Order in Two-Dimensional Active Matter

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Working in two space dimensions, we show that the orientational order emerging from self-propelled polar particles aligning nematically is quasi-long-ranged beyond $\ell_s$, the scale associated to induced velocity reversals, which is typically extremely large and often cannot even be measured. Below $\ell_s$, nematic order is long-range. We construct and study a hydrodynamic theory for this de facto phase and show that its structure and symmetries differ from conventional descriptions of active nematics. We check numerically our theoretical predictions, in particular the presence of $\pi$-symmetric propagative sound modes, and provide estimates of all scaling exponents governing long-range space-time correlations.

Studies of active matter continue to flourish, exploring more and more complex situations in an increasingly quantitative manner [1]. Evidence accumulates showing that active matter exhibits collective properties impossible in thermal equilibrium or even in driven systems [2]. In spite of all this progress, important fundamental questions remain open. A long-standing such issue is whether true long-range nematic order can emerge in two space dimensions (2D).

Whereas it is now well known, notably thanks to the seminal work by Toner and Tu, that long-range polar order can arise in 2D active systems [3][11], the debate has remained opened for active nematics: On the one hand, theoretical results conclude that nematic order can at best be quasi-long-range [12], as in equilibrium, albeit with important differences [13–19]. On the other hand numerical and experimental results obtained on self-propelled particles without spontaneous velocity reversals yielded convincing data demonstrating true long-range nematic order over a large range of scales [20][21].

In this Letter, we study 2D dry dilute active nematics —the framework in which the question of the asymptotic nature of nematic order was mostly discussed— using numerical simulations and theory. We show that the homogeneous ordered phase of a Vicsek-style model of polar self-propelled particles aligning nematically actually displays true long-range nematic order only up to $\ell_s$, the scale associated to typical time between velocity reversals $\ell_s$, which is typically extremely large and often cannot even be measured. Below $\ell_s$, nematic order is long-range. We derive a hydrodynamic theory for this regime and show that it possesses a structure and symmetries different from those of standard active nematics. Our analysis of this field theory predicts $\pi$-symmetric sound modes and the scaling form of space-time fluctuations. Finally, numerical results confirm the theory and allow us to estimate all scaling exponents.

We use the Vicsek-style model of polar particles with nematic alignment first introduced in [20]. Particles $i=1, \ldots, N$ evolve at discrete timesteps with constant speed $v_0$ in square domains of linear size $L$ with periodic boundary conditions, interacting with neighbors within unit distance. Their positions $\mathbf{r}_i$ and unit-length orientations $\mathbf{e}_i = \mathbf{e}(\theta_i)$ obey:

$$\mathbf{r}_i^{t+1} = \mathbf{r}_i^t + v_0 \mathbf{e}_i^{t+1}, \quad (1a)$$
$$\mathbf{e}_i^{t+1} = (\mathcal{R}_\eta \circ \vartheta) (\langle \mathbf{e}_i^t \cdot \mathbf{e}_j^t \rangle_{j \sim i}, (1b)$$

where $\vartheta$ normalizes vectors ($\vartheta(\mathbf{u}) = \mathbf{u}/|\mathbf{u}|$), and $\mathcal{R}_\eta$ rotates them by a random angle drawn from a uniform distribution in $[-\pi \eta, \pi \eta]$, independently for every particle at every timestep. The two main parameters are the global density $\rho = N/L^2$ and the noise strength $\eta$. The phase diagram in the $(\rho, \eta)$ plane is typical of Vicsek-style models [11]. All results presented below were obtained with $v_0 = 0.5$ and $\rho = 2$.

We focus on the homogeneous nematic liquid that exists for $\eta \lesssim 0.21$, where the global nematic order parameter $S = |\langle \mathbf{e}^{2\vartheta(\mathbf{u})} \rangle_k|_1$ takes $O(1)$ values. In this state, particles can be split into two ‘polar’ subpopulations according to which of the two opposite directions defined by the nematic order their orientation is closest. The nematic interaction in Eq. (1b) aligns particles belonging to the same population and anti-aligns particles belonging to opposite populations, so that particles mostly stay in the same population. Nevertheless, under the action of interactions and noise, they can eventually turn enough that they join the other population. It was shown in [20] that the distance traveled between such reversals is distributed exponentially with a characteristic length $\ell_s$ independent of system size. In Fig. 1(a), we show that $\ell_s$ grows very fast when the noise strength $\eta$ decreases. A good fit of our data is that $\ell_s \sim \eta^{-8}$.

In [20], the global nematic order parameter $S$ was found to decrease slower than a power of $L$ and consistent with an algebraic decay to a finite asymptotic value $(S(L) - S(\infty) \sim L^{-\tau})$. These results led to conclude to
true long-range nematic order, but they were obtained on a range of system sizes barely encompassing \( \ell_c \). Here, choosing a noise strength such that \( \ell_c \) is not too large, we find that for \( L > \ell_c \), \( S \) decays like a small power of \( L \), in departure from the \( L < \ell_c \) behavior (Fig. 1(b)). Asymptotically, nematic order is only quasi-long-range, in agreement with standard theories [18].

Nevertheless, in most of the homogeneous nematic phase, \( \ell_c \) is so large that only the \( L < \ell_c \) regime is accessible and it is thus important to study it per se. Working in this regime, we confirm that nematic order is fully long-range; moreover, the scaling of the local slope \( \sigma(L) \) showing a plateau for \( L \gg \ell_c \) (Fig. 1(c)) allows to identify an internal crossover scale \( \ell_c \) separating two regimes with different values of \( \sigma \) (Fig. 1(c)).

We now present a theory of the long-range-ordered nematics present on scales much smaller than \( \ell_c \). Full details of calculations are given in [22]. Our approach is not a perturbative version of active nematics: We directly consider two populations, \( R \) and \( L \), of polar active particles with speed \( v_0 \) aligning their velocity with neighbors if those belong to the same population, and anti-align it otherwise. This is not equivalent to usual nematic alignment: two particles of the same population will align even if their relative angle is obtuse, and they will anti-align if they belong to different populations, irrespective of their angle. We further assume that the populations exchange members randomly at rate \( 1/\tau_r \approx \ell_c/v_0 \). We first write Boltzmann equations ruling the evolution of the one-body probability density functions \( f_L(r, \theta, t) \) and \( f_R(r, \theta, t) \):

\[
\partial_t f_L + v(\theta) \cdot \nabla f_L = \frac{1}{\tau_r} (f_R - f_L) + I_{st}[f_L] + I_{co}[f_L, f_R],
\]

and the equation governing \( f_R \) is given by swapping the \( L \) and \( R \) subscripts. In these, \( v(\theta) = v_0 \epsilon(\theta) \) is the velocity of particles with orientation \( \theta \), whereas the integrals \( I_{st} \) and \( I_{co} \), given in [22], describe the effects of angular self-diffusion and collisions.

Introducing the more convenient \( f = f_R + f_L \) and \( g = f_R - f_L \), expanding \( f \) and \( g \) in Fourier series of \( \theta \) (e.g., \( f(r, \theta, t) = \sum_{\ell} f_\ell(r, t)e^{-i\ell \theta} \)), the Boltzmann equations are de-dimensionalized and transformed into a hierarchy of partial differential equations for the \( f_\ell \) and \( g_\ell \) fields. As shown in [22], a linear stability analysis of the disordered solution \( \rho \equiv f_0 = \bar{\rho} \) (the total density), \( f_{k>0} = g_k = 0 \) reveals that it is unstable to \( g_1 \) perturbations at large density and/or weak noise. The field \( g_1 \) is thus responsible for the onset of orientational order. Note that \( g_1 \) measures polar order within each population, i.e. is a proxy for global nematic order. The equations for \( g_1 \) and \( g_0 \) read

\[
\partial_t \rho = -\text{Re}[\nabla^* f_1],
\]

\[
\partial_t g_0 = -2\tau_r^{-1} \rho - \text{Re}[\nabla^* g_1],
\]

where \( \nabla \equiv \partial_x + i\partial_y \) denotes the complex gradient.

Following the Boltzmann-Ginzburg-Landau approach [11, 23, 25], one can build step by step a scaling ansatz using a small parameter \( \varepsilon \) marking the magnitude of order near onset \((|g_1| \sim \varepsilon)\). As detailed in [22], this leads to: \( |g_{k>1}| \sim \varepsilon^k \); \( |f_{k>1}| \sim \varepsilon^k \), and \( \partial_t \rho \sim \nabla \sim \varepsilon^k \). In addition, considering Eqs. (3a,3b), one completes the scaling ansatz by \( |g_0| \sim \varepsilon \), [\( |\delta g_1| \sim |f_1| \) \( \sim \varepsilon^2 \). Truncating and closing the Boltzmann hierarchy at order \( \varepsilon^2 \) yields hydrodynamic equations for \( f_1 \) and \( g_1 \):

\[
\partial_t f_1 = -\frac{1}{2} \nabla \rho + (\mu[|\rho| - \zeta |g_1|^2] f_1 + D_1 \nabla f_1 + \kappa_1[|\rho| - \chi_{|g_1|}^2] g_1 + D_2 g_1 \nabla g_1 + \kappa_2 |\nabla^* g_1|^2), \quad (3c)
\]

\[
\partial_t g_1 = -\frac{1}{2} \nabla g_0 + (\nu[|\rho| - \Gamma |\rho| |g_1|^2] g_1 + \Omega \nabla g_1 + \lambda_1 |\nabla^* g_1|^2 + \lambda_2 |\nabla g_1|^2 \nabla g_1 + \lambda_3 |\nabla^* g_1|^2 + \lambda 4 |\nabla^* (g_1 f_1) + \lambda_5 f_1 |\nabla^* g_1 | (3d)
\]

where all coefficients depend on the particle-level parameters \( \bar{\rho}, \eta, \) and \( \tau_r \). (see [22] for their explicit expressions), and local dependencies on \( \rho \) and \( g_0 \) are indicated.

Eqs. (3), are structurally different from hydrodynamic theories written for active nematics. The \( 2\pi \)-symmetry of the interaction between our polar particles makes the pairs of equations for \((\rho, f_1)\) and \((g_0, g_1)\) resemble two coupled Toner-Tu (TT) systems. Both \( \rho \) and \( g_0 \) are affected by the corresponding order fields \( f_1 \) and \( g_1 \), which are not \( \pi \)-symmetric. Discarding the couplings to \( \rho \) and \( f_1 \), Eqs. (5b) and (5d) are almost like the TT equations in the limit \( \tau_r \to \infty \). They however miss terms \( \sim g_0 g_1 \) and \( \sim g_1 \nabla g_1 \) that are forbidden by the \( R \leftrightarrow L \) symmetry of the problem, which imposes the equations to be invariant under \( g \leftrightarrow -g \).

Eqs. (5), even if formally derived at the onset of order, reflect the symmetries of the deeply ordered phase. We now focus on fluctuations in that phase, i.e. around the homogeneous ordered solution \( \rho = \bar{\rho}, g_0 = f_1 = 0, g_1 = \hat{g} \equiv \sqrt{\nu[\bar{\rho}] / \Gamma[\bar{\rho}]} \) that exists when \( \nu[\bar{\rho}] > 0 \) [27]. Linearizing Eqs. (5) around this solution, separating parallel
(\|) and transverse (\perp) components, we obtain a system of 6 equations governing small perturbations \( \delta \rho \), \( \delta g_0 \), \( \delta f_\parallel \), \( \delta f_\perp \), \( \delta g_\parallel \), and \( \delta g_\perp \). We find that \( \delta \rho \) and \( \delta g_\perp \) are hydrodynamic modes, while \( \delta f_\parallel \), \( \delta f_\perp \), and \( \delta g_\parallel \) decay rapidly. Since \( \tau_r \) can take arbitrary large values, we also consider \( \delta g_0 \) as hydrodynamic. Enslaving the fast modes, we obtain the following linear system:

\[
\begin{align*}
\partial_t \delta \rho &= (D_\parallel \delta^2 \rho_\parallel + D_\perp \delta^2 \rho_\perp) \delta \rho - \lambda_0 \partial_i \delta \rho_0 + D_\rho \partial^2 \rho_\perp \delta g_\parallel + \eta_1 \partial^2 \rho \delta g_0, \\
\partial_t \delta g_0 &= (D_\parallel \delta^2 \rho_\parallel + D_\perp \delta^2 \rho_\perp - 2 \tau_r^{-1}) \delta g_0 - \kappa_0 \partial_i \delta \rho - \nu_0 \partial_i \delta g_0 + \eta_2 \partial^2 \rho_\parallel \delta \rho, \\
\partial_t \delta g_\perp &= (D_\parallel \delta^2 \rho_\parallel + D_\perp \delta^2 \rho_\perp + \gamma \partial^2 \rho_\perp \delta \rho - \omega_0 \partial_\perp \delta g_0 + \xi_0 \partial^2 \rho \delta g_0),
\end{align*}
\]

where we split the complex gradient into \( \nabla = \partial_\parallel + i \partial_\perp \), and all the (bare) coefficients are given in \cite{23,28}.

We first note that in the small \( \tau_r \) limit, such that particles reverse their orientation many times on the scale at which we observe fluctuations, \( \delta g_0 \) is non-hydrodynamic (cf. Eq. (4b)). Eqs. (4) then reduce to those of an homogeneous active nematic (with \( \delta g_\perp \) playing the role of the transverse fluctuations of nematic order, see \cite{18,22}).

In the \( \tau_r \to \infty \) limit of main interest here, on the other hand, we neglect the term \( 2 \tau_r^{-1} \delta g_0 \) in Eq. (4b). To compute space and time correlation functions of the three hydrodynamic fields \( \delta \rho \), \( \delta g_0 \), and \( \delta g_\perp \), we equip Eqs. (4) with additive, uncorrelated, zero-mean noise terms. For Eq. (4a), governing density fluctuations, this noise is conserved and we write it \( \partial_\parallel h_\parallel + \partial_\perp h_\perp \). Writing the (stochastic) Eqs. (4) in Fourier space, taking the long wavelength, low frequency limit \( q, \omega \to 0 \), rather tedious but standard calculations detailed in \cite{22} lead to:

\[
\begin{align}
\langle |\delta \rho(\omega,q)|^2 \rangle &\sim \omega_q^{-1} \left[ (q^2 \Delta_\rho)^2 + q^2 \Delta_{\rho,\perp}(\omega^2 - v_0 \alpha_0 q^2_\perp)^2 + \Delta_0 q^2_\parallel \omega^2 + \Delta_\parallel (v_0 \lambda_0) q^2_\perp \right], \\
\langle |\delta g_0(\omega,q)|^2 \rangle &\sim \omega_q^{-1} \left[ (q^2 \Delta_\rho)^2 + q^2 \Delta_{\rho,\perp} \kappa_0 q^4 \omega^2 + 2 \Delta_0 q^2_\parallel \omega^2 + \Delta_\parallel (\omega^2 - \kappa_0 \lambda_0 q^2_\perp) \right], \\
\langle |\delta g_\perp(\omega,q)|^2 \rangle &\sim \omega_q^{-1} \left[ (q^2 \Delta_\rho)^2 + q^2 \Delta_{\rho,\perp} (\alpha_0 \kappa_0) q^2_\parallel q^2_\perp + \Delta_0 q^2_\parallel \omega^2 + \Delta_\parallel (\omega^2 - \kappa_0 \lambda_0 q^2_\perp) \right],
\end{align}
\]

where \( \Delta_\rho \) and \( \Delta_{\rho,\perp} \) are the amplitudes of the conserved \( \rho \) noise, \( \Delta_0 \) and \( \Delta_\parallel \) those of the \( g_0 \) and \( g_\perp \) noises, and

\[
\mathcal{D}(\omega,q) \equiv |\omega - i \varepsilon_d(q)|^2 \times |\omega - c(\theta_q)q + i \varepsilon_p(q)|^2 \times |\omega + c(\theta_q)q + i \varepsilon_p(q)|^2.
\]

As shown in \cite{22}, where their explicit forms are given, \( \varepsilon_{d,p}(q) \sim q^2 \), whereas the anisotropic speed is

\[
c(\theta_q) = \sqrt{\kappa_0 \lambda_0 \cos^2(\theta_q) + v_0 \alpha_0 \sin^2(\theta_q)},
\]

where \( \theta_q \) denotes the angle between \( q \) and the mean order. Eqs. (6) are fundamentally different from their counterparts in both active nematics and the TT class: at most orientations \( \theta_q \) correlations have a diffusive peak and two symmetric propagative peaks at \( \omega = \pm c(\theta_q)q \).

Equal-time correlation functions are easily obtained by integrating Eqs. (6) over \( \omega \). They all diverge as \( q^2 \) for most \( \theta_q \), which means that nematic order is only quasi-long-range at this linear level, a situation similar to that of polar order in TT theory. To resolve this marginal situation, one needs to study nonlinear hydrodynamics. We first repeat the calculations leading to Eqs. (6) keeping the leading order nonlinearities (in fields and gradients). The structure of our theory shares similarities with the polar case. We thus limit ourselves to terms of order 3 in fields and gradients \cite{6}. After lengthy but straightforward manipulations (detailed in \cite{22}), we obtain:

\[
\begin{align}
\partial_t \delta \rho &= \mathcal{L}_\rho + j_1 \partial_i (\delta g_0 \delta \rho) + j_2 \partial_\perp (\delta g_0 \delta g_\perp), \\
\partial_t \delta g_0 &= \mathcal{L}_g_0 + c_1 \partial_i \delta \rho^2 + c_2 \partial_\perp \delta \rho^2 + c_3 \partial_\parallel \delta g_\parallel^2, \\
\partial_t \delta g_\perp &= \mathcal{L}_g_\perp + w_1 \delta g_\parallel \partial_\parallel \delta g_\perp + w_2 \delta g_\parallel \delta g_\perp + w_3 \delta g_\perp \partial_\perp \delta g_\perp + w_4 \delta g_\parallel \delta g_\perp \delta \rho,
\end{align}
\]

where \( \mathcal{L} \) is the linear part (Eqs. (4a)). Introducing the scaling exponents via \( x_\perp \to b x_\perp, x_\parallel \to b^5 x_\parallel, t \to b^4 t, \delta g_\perp \to b^3 \delta g_\perp, \delta g_0 \to b^3 \delta g_0, \delta \rho \to b^5 \delta \rho \) and imposing a fixed point condition on Eqs. (8) considered valid in any dimension \( d \) yields the following values of the exponents

\[
z = 2, \xi = 1, \chi = \chi_0 = \chi_0 = 1 - \frac{d}{2} \text{ (linear level)}.
\]

We thus have isotropic (\( \xi = 1 \)) diffusive (\( z = 2 \)) scaling with quasi-long-range order in \( d = 2 \) (\( \chi = 0 \)) at the linear level, as for both active nematics and TT theory.

At the linear fixed point, 9 of the 10 nonlinear terms in Eqs. (8) scale like \( b^{\ell(d_\ell)/2} \), i.e. are relevant in \( d \leq d_c = 4 \) (the exception is \( \omega_5 \)). This means that the linear theory breaks down in \( d \leq 4 \), and that we should in principle embark on a complete renormalization group analysis to
obtain exponent values. We leave this challenging task for future studies. Instead we rely on general considerations and formal similarities with TT theory to make predictions that we test numerically.

Replacing the eigenfrequencies $\omega = c(\theta_q)q$ into (4a) and (4b) (expressed in Fourier space), we find at leading order $c(\theta_q)q\delta \rho \sim q_0\delta \rho_0$ and $c(\theta_q)q\delta \eta \sim q_0\delta \eta + q_1\delta \eta_\perp$. Therefore, taking $\theta_q = \pi/2$ we get that $|\delta \rho_0| \sim |\delta \eta_\parallel|$, such that $\chi_0 = \chi$, while for any orientation of $q$ not purely longitudinal or transverse we have $|\delta \rho| \sim |\delta \rho_0|$, which implies $\chi_\rho = \chi_0$. It is thus likely that the equality $\chi = \chi_0 = \chi_\rho$ holds even at the nonlinear level.

Given that the structure of Eqs. (3) is similar to that found in TT theory, we follow [6] and conjecture that the scaling of correlation functions in the nonlinear theory is obtained using renormalized noise coefficients $\Delta = q_1^{-1} f_\Delta(q_1/q_0^c)$ and renormalized dampings $\tilde{\varepsilon} = q_1^2 f_\varepsilon(q_1/q_0^c)$ (where we have defined $\zeta = d - 1 + 2\chi + \xi$, with functions $f_\Delta$ and $f_\varepsilon$ expected to be universal and to satisfy $f_{\Delta,\varepsilon}(x) \to x^\zeta/\zeta$). The black dashed line is a fit by the theoretical predictions of Eqs. (4). (b) Angular dependence of the measured sound speed $c(\theta_q)$: Inset: polar plot showing the $\pi$-symmetry of $c(\theta_q)$.

We now come back to our Vicsek-style model at noise strength $\eta = 0.1$ and show data for the order correlations confirming the structure of the above theory and providing estimates of the scaling exponents. Additional results for the densities $\rho$ and $\rho_0$ will be published elsewhere [29]. We actually measure the transverse nematic order $\delta Q_\perp$, which, when aligned along the horizontal direction and assuming small angular deviations, is a good proxy of $\delta g_\perp$ ($\delta Q_\perp \sim \cos(\theta)\sin(\theta) \sim \theta \sim \rho^{-1}\delta g_\perp$).

The frequency spectra do have the qualitative structure predicted by Eqs. (3): two symmetric propagative peaks and a central diffusive one (Fig. 2a). As expected, peak locations, at a fixed angle $\theta_q$ are proportional to $q$, allowing the easy measurement of the sound speed $c(\theta_q)$, which we find in perfect quantitative agreement with Eq. (1) (Fig. 2b)). Peak widths provide estimates of $z$ and $z/\xi$ in the $\parallel$ and $\perp$ directions, as in TT theory. As shown in Fig. 2c), we find a crossover at the same scale $\xi$ as observed in Fig. 1c). For scales below $\ell_c$, we find $z \sim 1.75$ and $z/\xi \sim 1.4$, while we are only able to estimate $z/\xi \sim 1.1$ in the asymptotic regime (we use primes to denote exponent values measured below $\ell_c$). The equal-time order correlation function shown in Fig. 2c) in the $\parallel$ and $\perp$ directions, also exhibits a crossover at $\ell_c$. From the pre-crossover scaling we estimate $\zeta' \sim 1.75$ and $\zeta'/\xi' \sim 1.4$, while we find $\zeta/\xi \sim 1.1$ in the $q < 2\pi/\ell_c$ regime.

We thus have two sets of scaling exponents: for scales below $\ell_c$, the above estimates lead to $\nu' = \zeta' \sim 1.4$, $\xi' \sim 1.25$, and $2\chi' \sim -0.5$. Note that this yields $-2\chi'/\xi' \sim 0.4$, in agreement with our estimate of $\omega' \sim 0.45$ in Fig. 1c). For scales beyond $\ell_c$, we have $z = \zeta$, but cannot estimate $\xi$ from correlation functions. Using $\omega \sim 0.8$ (Fig. 1c), yields $\xi \sim 1.1$ and $2\chi \sim -0.9$, and finally $z = \zeta \sim 1.2$. A few remarks are in order: (i) both below and above $\ell_c$, $z = \zeta = 1 + 2\chi + \xi$, a hyperscaling relation also verified by polar flocks that implies that the dominant noises are additive and their amplitude is not renormalized; (ii) in our nematic phase the anisotropy exponent $\zeta \sim 1.1 > 1$, at odds with 2D polar flocks for which $\zeta \sim 0.95 < 1$ [8], but in both cases we cannot exclude that scaling is asymptotically isotropic.

To summarize, the orientational order emerging from self-propelled polar particles aligning nematically is always quasi-long-range asymptotically, but this regime is only observed beyond $\ell_c$, the scale associated to induced velocity reversals, which can easily take very large values and often cannot even be measured. Below $\ell_c$, nematic order is fully long-range. Constructing a hydrodynamic theory from microscopic grounds, we showed that this de facto phase has a structure and symmetries distinct from both conventional descriptions of active nematics and Toner and Tu theory. Consequently, systems in the
corresponding class exhibit features never reported so far, such as long-range nematic order and the presence $\pi$-symmetric propagative sound modes.

Finally, we believe our findings can be observed experimentally, as long as the rate of velocity reversals, be they induced or spontaneous, is small. After all, nematic alignment resulting from inelastic collisions between elongated objects is quite generic. Confined bacteria and motility assays are promising systems in this regard.

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