THE $P_2^1$ MARGOLIS HOMOLOGY OF CONNECTIVE
TOPOLOGICAL MODULAR FORMS

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Abstract. The element $P_2^1$ of the mod 2 Steenrod algebra $\mathcal{A}$ has the property
$(P_2^1)^2 = 0$. This property allows one to view $P_2^1$ as a differential on $H_*(X, F_2)$
for any spectrum $X$. Homology with respect to this differential, $\mathcal{M}(X, P_2^1)$,
is called the $P_2^1$ Margolis homology of $X$. In this paper we give a complete
calculation of the $P_2^1$ Margolis homology of the 2-local spectrum of topological
modular forms $tmf$ and identify its $F_2$ basis via an iterated algorithm. We
apply the same techniques to calculate $P_2^1$ Margolis homology for any smash
power of $tmf$.

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Convention. Throughout this paper we work in the stable homotopy category of
spectra localized at the prime 2.

1. Introduction

The connective $E_\infty$ ring spectrum of topological modular forms $tmf$ has played a
vital role in computational aspects of chromatic homotopy theory over the last
two decades [Goe10], [DFHH14]. It is essential for detecting information about
the chromatic height 2, and it has the rare quality of having rich Hurewicz image.
There is a $K(2)$-local equivalence [HM14]

$$L_{K(2)}tmf \simeq E_2^{hG_{18}}$$

where $E_2$ is the second Morava $E$-theory at $p = 2$ and $G_{18}$ is the maximal finite
subgroup of the Morava stabilizer group $G_2$. The spectrum $E_2^{hG_{18}}$ can be used to
build the $K(2)$-local sphere spectrum (see [BG18]). The homotopy groups of $tmf$
approximate both the stable homotopy groups of spheres and the ring of integral
modular forms. In many senses, $tmf$ is the chromatic height 2 analogue of connective
real $K$-theory $ko$. Further, the homotopy groups of $tmf$ are completely known
[Bau08].

Let us now recall the definition of the element $P_2^1 \in \mathcal{A}$. Milnor described the mod
2 dual Steenrod algebra $\mathcal{A}_*$ as the graded polynomial algebra [Mil58, App. 1]

$$\mathcal{A}_* \cong F_2[\xi_1, \xi_2, \xi_3, \ldots]$$
where $|\xi| = 2^j - 1$. The Steenrod algebra $\mathcal{A}$ has an $\mathbb{F}_2$-basis dual to the monomial basis of $\mathcal{A}$. The elements of the $\mathbb{F}_2$-basis of $\mathcal{A}$ which are dual to $\xi^{2^j}$ are denoted by $P^+_i$, and the elements $P^-_i$ are denoted by $Q_i$, for $i < 0$. When $s < t$, the elements $P^+_s$ are exterior power generators, i.e. $(P^+_1)^2 = 0$. Thus, any left $\mathcal{A}$-module $K$ can be regarded as a complex with differential given by the left multiplication by $P^+_1$ (for $s < t$). This leads to the following definition.

**Definition 1.1 ([Mar83]).** Let $K$ be any left $\mathcal{A}$-module and $0 \leq s < t$. Let

$$L^{P^+_s} : K \to K$$

denote the left action by $P^+_s$. The left $P^+_s$ Margolis homology group of $K$, $\mathcal{M}^L(K, P^+_t)$, is defined as

$$\mathcal{M}^L(K, P^+_t) := \frac{\text{Ker} \ L^{P^+_s} : K \to K}{\text{Im} \ L^{P^+_s} : K \to K}.$$ 

For a right $\mathcal{A}$-module $K$, one can similarly define the right $P^+_t$ Margolis homology group of $K$ as

$$\mathcal{M}^R(K, P^+_t) := \frac{\text{Ker} \ R^{P^+_s} : K \to K}{\text{Im} \ R^{P^+_s} : K \to K}$$

where $R^{P^+_s}$ is the right action by $P^+_t$ on $K$.

**Notation 1.2.** For a spectrum $X$, $\mathcal{M}(X, P^+_t)$ will denote $\mathcal{M}^L(H^*(X), P^+_t)$ or equivalently $\mathcal{M}^R(H_*(X), P^+_t)$.

Computations of Margolis homology underly many essential computations in homotopy theory. For example, Adams work on $BP(1)$ cooperations [Ada74] relies on the computations of $\mathcal{M}(BP(1), Q_i)$ for $i = 0, 1$. Calculations like $\mathcal{M}(bo, Q_i)$ for $i = 0, 1$ are essential ingredients in the work of Mahowald on bo-resolutions [Mah81]. More recently, Culver described $BP(2)$ resolutions [Cul] by understanding $\mathcal{M}(BP(2), Q_i)$ for $i = 0, 1, 2$. Computation of $\mathcal{M}(tmf^\Lambda, Q_2)$ is an essential ingredient in [BBB^+b].

The element $Q_i$ is primitive for all $i \in \mathbb{N}$. In other words, the comultiplication map $\Delta$ on $\mathcal{A}$ sends $Q_i$ to

$$(1.3) \quad \Delta(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i.$$ 

Consequently, $Q_i$ acts on $H_*(X)$ as a derivation, namely it follows the Leibniz rule

$$Q_i(xy) = Q_i(x) \cdot y + x \cdot Q_i(y),$$

whenever $X$ is a ring spectrum. The Leibniz rule implies the Künneth isomorphism [Mar83, Proposition 17, pg 343]

$$\mathcal{M}(X \otimes Y, Q_i) \cong \mathcal{M}(X, Q_i) \otimes \mathcal{M}(Y, Q_i)$$

and hence, $\mathcal{M}(X, Q_i)$ is an $\mathbb{F}_2$ algebra whenever $X$ is a ring spectrum. As a result, computation of $Q_i$ Margolis homology and its description is often fairly straightforward.

On the other hand, for $s > 0$, $P^+_t$ is not a primitive element of $\mathcal{A}$. In particular,

$$\Delta(P^+_s) = P^+_s \otimes 1 + Q_1 \otimes Q_1 + 1 \otimes P^+_1.$$
and its action on $H_*(X)$ for a ring spectrum $X$, does not follow the Leibniz rule. Instead, we have

$$P_2^1(xy) = P_2^1(x)y + Q_1(x)Q_1(y) + xP_2^1(y).$$

(1.4)

As a result, the product of two $P_2^1$ cycles may not necessarily be a $P_2^1$ cycle, hence $\mathcal{M}(X, P_2^1)$ may not admit any multiplicative structure even if $X$ is a ring spectrum. This is the main reason why the $P_2^1$ Margolis homology calculations are significantly more complicated.

Let us now consider the spectrum $\text{tmf}$. It is well-known ([HM14], [Mat16]) that

$$H_*(\text{tmf}; F_2) \cong F_2[\zeta^8, \zeta^4, \zeta^2, \zeta, 1, \zeta_3, \zeta_5, \ldots] \subset A_*$$

is a subalgebra of $A_*$. Here the elements $\zeta_i$ are the images of $\xi_i$ under the antipode of the Hopf algebra $A_*$ (see Section 2). The right action of $Q_i$ is given by the formula (see [Cul, §2] for details)

$$Q_i(\zeta_n) = \zeta_n^{2^i-1}.$$  

Then, since the $Q_i$ are derivations, it can be easily seen that

$$\mathcal{M}(\text{tmf}, Q_0) = F_2[\zeta^8, \zeta^4]$$

(1.5)

$$\mathcal{M}(\text{tmf}, Q_1) = \frac{F_2[\zeta^8, \zeta^4, \zeta^2, \zeta, 1, \zeta_3, \zeta_5, \ldots]}{\langle \zeta^3, \zeta_4, \ldots \rangle}$$

(1.6)

$$\mathcal{M}(\text{tmf}, Q_2) = \frac{F_2[\zeta^8, \zeta^4, \zeta^2, \zeta, 1, \zeta_3, \zeta_5, \ldots]}{\langle \zeta^2, \zeta^8, \zeta^4, \ldots \rangle}.$$  

(1.7)

In this paper, we give a complete calculation of $\mathcal{M}(\text{tmf}^\wedge r, P_2^1)$ for arbitrary $r \geq 1$. In fact, the calculation for $r > 1$ follows from the case $r = 1$, because after forgetting the internal grading one can construct a non-canonical isomorphism (see Section 4)

$$\mathcal{M}(\text{tmf}^\wedge r, P_2^1) \cong \mathcal{M}(\text{tmf}, P_2^1).$$

For the case $r = 1$, we give an iterated algorithm (see Definition 3.16) that constructs an $F_2$-basis of $\mathcal{M}(\text{tmf}, P_2^1)$. We give a complete description of $\mathcal{M}(\text{tmf}, P_2^1)$ in Theorem 3.18 which is the main result of this paper. Although $\mathcal{M}(\text{tmf}, P_2^1)$ is not an algebra, we notice that $\mathcal{M}(\text{tmf}, P_2^1)$ is a module over an infinitely generated exterior algebra $S$ (see Lemma 3.1 for a description of $S$). Theorem 3.18 also describes $\mathcal{M}(\text{tmf}, P_2^1)$ as an $S$-module.

The key tool we use is the **length spectral sequence** (2.14), which we define in Section 2. The length spectral sequence admits a $d_0$ differential and a $d_2$ differential and collapses at the $E_3$ page. The Leibniz rule does hold for the $d_0$, but not for $d_2$. In order to work around this issue, we notice that the $E_2$ page admits an action of $S$ (i.e. $d_2$ are $S$ linear) and we use it to simplify the computation of $E_\infty = E_3$.

We also notice that almost identical calculations lead to a complete description of $\mathcal{M}((BZ/2^\infty)_+, P_2^1)$. The methods developed in this paper can be considered as a blueprint for computations of $P_1$ Margolis homology of a variety of other $A$-modules.

Our calculations of $\mathcal{M}(\text{tmf}^\wedge r, P_2^1)$ have many applications, as the spectrum $\text{tmf}^\wedge r$ has a wide range of applications, particularly in chromatic homotopy theory. First
note that the cohomology of \( \text{tmf} \), as a module over the Steenrod algebra \( A \), is isomorphic to (see [HM14], [Mat16])

\[
H^*(\text{tmf}; \mathbb{F}_2) \cong A/\langle A(2) \rangle
\]

(1.8)

where \( A(2) \) is the subalgebra of \( A \) generated by \( Sq^1 \), \( Sq^2 \) and \( Sq^4 \). This, and a change of rings isomorphism, imply that the \( E_2 \) page of the Adams spectral sequence converging to \( \text{tmf}_+ X \) (for a spectrum \( X \)) is

\[
E_2^{s,t} := \text{Ext}^{s,t}_A(H^*(X), \mathbb{F}_2).
\]

(1.9)

One can detect infinite families in the \( E_2 \) page via the map

\[ q : \text{Ext}^{s,t}_A(H^*(X), \mathbb{F}_2) \to \text{Ext}^{s,t}_{A(2)}(H^*(X), \mathbb{F}_2). \]

The codomain of \( q \) can be understood by calculating \( M(X, P^1_2) \). Note that

\[
\text{Ext}^{s,t}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{2,1}],
\]

where \( h_{2,1} \) is the first example of a finite complex at \( p = 2 \) anyway. Recently, Bhattacharya and Egger introduced a family of finite spectra \( Z \) [BEa], and \( \pi_* L_{K(2)} Z \) has been computed [BBB\textsuperscript{+}b, BEb], the first example of a finite complex at \( p = 2 \) whose \( K(2) \)-local homotopy groups are completely determined. The finite complex \( Z \) can be constructed from the sphere spectrum, by a succession of cofiber sequences of self-maps (see [BEa]), the last one of which is

\[
\Sigma^5 A_1 \wedge C \nu \xrightarrow{w} A_1 \wedge C \nu \to Z.
\]

In a quest to leverage the knowledge of \( \pi_* L_{K(2)} Z \) to \( \pi_* L_{K(2)} S^0 \), one must first attempt to compute the \( K(2) \)-local homotopy groups of \( A_1 \wedge C \nu \). Very briefly, our strategy is to use the \( v_2 \)-local \( \text{tmf} \)-based Adams spectral sequence

\[
E_1^{s,t} = v_2^{-1} \pi_*(\text{tmf} \wedge \text{tmf}^{\wedge r} \wedge A_1 \wedge C \nu) \Rightarrow \pi_{t-s}(L_{K(2)} A_1 \wedge C \nu)
\]

and compare it with that of \( Z \). One can identify the \( E_1 \)-page of the above spectral sequence using the classical Adams spectral sequence

\[
E_2^{s,t} = \text{Ext}^{s,t}_A(H^*(\text{tmf} \wedge \text{tmf}^{\wedge r} \wedge A_1 \wedge C \nu), \mathbb{F}_2) \Rightarrow \pi_{t-s}(\text{tmf} \wedge \text{tmf}^{\wedge r} \wedge A_1 \wedge C \nu).
\]

(1.10)

Because of (1.8) and the fact that \( H^*(A_1 \wedge C \nu) \cong A(2)/\Lambda(Q_2, P^1_2) \), and the change of rings isomorphism, the \( E_2 \)-page of the spectral sequence (1.10) has the form

\[
\text{Ext}^{s,t}_{A(Q_2, P^1_2)}(H^*(\text{tmf}^{\wedge r}), \mathbb{F}_2)
\]

Hence, computation of \( M(\text{tmf}^{\wedge r}, P^1_2) \) is essential for understanding the \( E_2 \)-page of (1.10).
Motivation II - \textit{tmf} resolution of the sphere spectrum. The connective spectrum $bo$ is not a flat ring spectrum, hence the $E_2$ page of the $bo$-based Adams spectral sequence does not have a straightforward expression like the classical Adams spectral sequence. However, Lellmann and Mahowald [LM87] were able to calculate the $d_1$ differentials (also see [BBB+a]) and gave a description of the “$v_1$-periodic part” of the $E_2$-page. They identified the free Eilenberg–MacLane summand of $bo^\wedge r$. To identify this free summand one needs to identify the $\mathcal{A}(1)$ free summand of

$$H^*(bo^\wedge r) \cong \mathcal{A//A(1)^{\otimes r}}.$$  

This can be done by calculating $\mathcal{M}(bo^\wedge r, Q_0)$ and $\mathcal{M}(bo^\wedge r, Q_1)$ and using the following theorem due to Margolis.

**Theorem 1.11** ([Mar83, Chapter 19, Theorem 6]). An $\mathcal{A}(n)$-module $K$ is free if and only if $\mathcal{M}(K, P_1^0) = 0$ whenever $s + t \leq n + 1$ with $s < t$.

To emulate the strategy of Lellmann and Mahowald to understand the $tmf$-based Adams spectral sequence for $S^0$ one needs to first identify the $\mathcal{A}(2)$-free part of

$$H^*(tmf^\wedge r) \cong (\mathcal{A//A(2)})^{\otimes r}.$$  

Potentially, this can be identified using the knowledge of $\mathcal{M}(tmf^\wedge r, Q_i)$ for $i = 0, 1, 2$ and $\mathcal{M}(tmf^\wedge r, P_2^1)$, along with Theorem 1.11.

Motivation III - Infinite loop space of $tmf$. There are $\mathcal{A}$-modules $J(k)$, called Brown–Gitler modules [BG73], which assemble into a doubly graded $\mathcal{A}$-algebra, denoted here by $J(*)^\ast$. Moreover, there is an $\mathcal{A}$-module isomorphism $J(*)^\ast \cong \mathbb{F}_2[x_1, x_2, \ldots]$ where $x_i \in J(2^i)^\ast$ and the left $\mathcal{A}$ action on $J(*)^\ast$ is [Sch94]

$$S^q(x_i) = x_i + x_{i-1}^2.$$  

In fact, $J(k)^\ast$ can be thought of as inheriting this action by virtue of being a subobject of $\mathcal{A}$. Because of this, minor modifications to methods of this paper apply to the calculation of $\mathcal{M}(J(k), P_2^1)$. By [KM13] there is a spectral sequence, obtained by studying Goodwillie towers, relating the knowledge of $H_*(tmf; \mathbb{F}_2)$ to that of $H_*(\Omega^\infty tmf; \mathbb{F}_2)$ (also see [HM16] which provides a spectral sequence relating the cohomology of $tmf$ to the cohomology of its infinite loop-space $H^*(\Omega^\infty tmf; \mathbb{F}_2)$). Roughly speaking, this relies on computing certain derived functors, usually labeled $\Omega^\ast_{\mathcal{A}}$, in the category of unstable modules over $\mathcal{A}$. It turns out that there is an isomorphism (see [Goe86] or [HK00])

$$\Omega^\ast_{\mathcal{A}} \Sigma^{-1}(\mathcal{A//A(2)}) \cong \text{Ext}_{\mathcal{A}(2)}^{s,t}(\mathbb{F}_2, J(*)^\ast),$$

so that these computations require an understanding of the $J(k)$ as modules over $\mathcal{A}(2)$, the hardest part of which is understanding how $P_2$ acts.

Organization of the paper. In Section 2, we recall some facts about the Steenrod algebra and its dual. We introduce the spectral sequence (2.14), which computes the $P_2^1$ Margolis homology of $tmf$, and discuss the $d_0$ differentials in it.

In Section 3, we compute the $E_3 = E_\infty$ page of the spectral sequence (2.14). We do that by introducing building blocks $M_J$ and computing $\mathcal{M}(M_J, P_2^1)$. Then we establish the relationship between $\mathcal{M}(tmf, P_2^1)$ and $\mathcal{M}(M_J, P_2^1)$ in Theorem 3.18.
In Section 4, we show how to apply the same methods to calculate $P^1_2$ Margolis homology for $tmf^{r,r}$ and $(B\mathbb{Z}/2^{<k})_+$. Theorem 3.18 essentially gives complete answer in these cases.

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2. Action of $P^1_2$ and the length spectral sequence

The dual Steenrod algebra $A_*$ is the subalgebra of $\pi_*(HPF_2 \wedge HPF_2)$ which Milnor [Mil58] showed to be a polynomial algebra

$$A_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$$

where $|\xi_i| = 2^i - 1$. Milnor defined $S\mathcal{q}(r_1, r_2, \ldots) \in A$ as the dual of $\xi_1^r \xi_2^{r_2} \cdots$ and showed that they form an $\mathbb{F}_2$ basis of the Steenrod algebra $A$, known as the Milnor basis. The $P^t_i$ elements are defined as

$$P^t_i = S\mathcal{q}(r_1, \ldots), \text{ where } r_i = \begin{cases} 0, & i \neq t \\ 2^s, & i = t. \end{cases}$$

The action of an element $a \in A$ on an $A$-algebra follows the product rule given by the Cartan formula, i.e.

$$a(x \cdot y) = \Sigma_i a'_i(x) \cdot a''_i(y),$$

where $\Delta(a) = \Sigma_i a'_i \otimes a''_i$ is the comultiplication in the Hopf algebra $A$.

Remark 2.1. We would like to note that standard commonly used notation for the generators of the dual Steenrod algebra at $p = 2$ differs from the notation in the original paper [Mil58], and we are grateful to John Rognes for explaining this to us. In [Mil58, Appendix 1], Milnor denotes the polynomial generators of the dual Steenod algebra at $p = 2$ by $\zeta_i$, so that $A_* \cong \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$ and defines $S\mathcal{q}(r_1, r_2, \ldots)$ as dual to the element $\zeta_1^r \zeta_2^{r_2} \cdots$. It has since become standard in the literature [MT68, Ada74, Mar83] to use a different notation and to denote the polynomial generators which were denoted by $\zeta_i$ in [Mil58, Appendix 1] by $\xi_i$, in order to match the notation for the odd primary Steenrod algebra. Hence in current standard notation $S\mathcal{q}(r_1, r_2, \ldots)$ is dual to $\xi_1^r \xi_2^{r_2} \cdots$. The symbol $\xi_i$ is now usually used to denote the image of $\xi_i$ under the antipode of the Hopf algebra $\chi : A_* \to A_*$, induced by the ‘flip map’ on $HF_2 \wedge HF_2$. The elements $\xi_i = \chi(\xi_i)$ can be computed recursively using the formula

$$\sum_{i+j=k} \xi_i^{2^j} \chi(\xi_j) = 0,$$

together with the assumption that $\xi_0 = 1$ and $\xi_i = 0$ when $i < 0$.

The homology of $tmf$ is the subalgebra of $A_*$ ([HM14], [Mat16, Theorem 5.13])

$$A := H_*(tmf; \mathbb{F}_2) \cong (A//A(2))^* = \mathbb{F}_2[\xi_1^4, \xi_2^4, \xi_3^2, \xi_4, \xi_5, \ldots].$$

Thus the action of $A$ on $\mathcal{X}$ is simply the restriction of the action of $A$ on $A_*$. 

The right action of $A$ on $A_*$ is determined by the action of the total squaring operation $Sq = 1 + \sum_{i>0} Sq^i$ [Pea14, Lemma 3.6]

\[(\xi_i) Sq = \xi_i + \xi_{i-1}^2 + \xi_{i-2}^4 + \cdots + \xi_1^{2^{i-1}} + 1\]

which is a ring homomorphism.

**Remark 2.4** (Action of the total squaring operation). There are multiple ways to define the action of $A$ on $A_*$. While we will be using the action defined by (2.3), we would like to collect other commonly used actions here. By [Mah81], the right and left actions of $Sq$ on $\xi_i$ are given by the formulas

\[Sq(\xi_i) = \xi_i + \xi_{i-1}^2\]
\[(\xi_i) Sq = \xi_i + \xi_{i-1},\]

while the left action on $\xi_i$ is

\[Sq(\xi_i) = \xi_i + \xi_{i-1} + \cdots + \xi_1 + 1.\]

From these formulas we can derive

\[Q_{i-1}(\xi_n) = \xi_{n-i}^2\]
\[(\xi_n) Q_{i-1} = \xi_{n-i}^2;\]

the second equation can also be found in [Cul].

**Important Notation 2.5.** Since we only work with the right action of $Sq$ in this paper, we will write $a(x)$ to denote the right action of $a \in A$ on $x \in H_* tmf$ for the rest of the paper. Thus, from now on

\[a(x) := (x)a.\]

We now focus on the action of $P^2_2 = Sq(0, 2) = Sq^2, Sq^4 + Sq^4, Sq^2$ on $\mathcal{I}$. From (2.3), one can easily see that $Sq^{2i}$ acts trivially on $\xi_n$, when $i > 0$ and $n \neq 1$. It follows immediately that

\[P^1_2(\xi_i) = 0.\]

Beware! This *does not* mean that $P^1_2(\xi_2\xi_j) = 0$, as the Leibniz rule does not hold.

Since $\Delta(P^1_2) = P^2_2 \otimes 1 + Q_1 \otimes Q_1 + 1 \otimes P^1_2$, we obtain the product formula

\[(2.6) \quad P^1_2(xy) = P^1_2(x)y + Q_1(x) Q_1(y) + x P^1_2(y).\]

Using $Q_1(\xi_i) = \xi_i^4$, we get

\[(2.7) \quad P^1_2(\xi_i) = \xi_i^4 \quad P^1_2(\xi_i^2) = \xi_i^8.\]

Formulas become more complicated for triple products, e.g.

\[P^1_2(\xi_i\xi_j\xi_k) = \xi_i^4 \xi_j^4 \xi_k^4 + \xi_i^4 \xi_j^4 \xi_{k-2}^4 + \xi_i^4 \xi_j^4 \xi_{k-2}^4,\]

and in general we have the following result.

**Lemma 2.8.** The action of $P^1_2$ on $\mathcal{I}$ is given by the formula

\[P^1_2(\xi_1 \cdots \xi_n) = \sum_{1 \leq j < k \leq n} \frac{\xi_j \cdots \xi_k}{\xi_j \xi_k} Q_1(\xi_j) Q_1(\xi_k) \]
\[= \sum_{1 \leq j < k \leq n} \xi_i^4 \cdots \xi_{j-2}^4 \xi_{j-1}^4 \xi_{j+1}^4 \cdots \xi_{k-2}^4 \xi_{k+1}^4 \cdots \xi_n^4,\]
where indices are allowed to repeat.

**Proof.** Follows from an inductive argument on \( n \), using (2.6) and the facts that \( P_2^i(\zeta_i) = 0 \) and \( Q_1(\zeta_i) = \zeta_{i-2} \).

The technique developed in this paper begins with the following observation. Consider the subalgebra

\[
\mathcal{E} := \mathbb{F}_2[\zeta_1^4, \zeta_2^4, \zeta_3^4, \zeta_4^4, \ldots] \subset \mathfrak{T} = \mathbb{F}_2[\zeta_1^4, \zeta_2^4, \zeta_3^4, \zeta_4, \zeta_5, \ldots]
\]

which we will call the even subalgebra of \( \mathfrak{T} \), as every element in \( \mathcal{E} \) has even grading. Since \( |Q_1| = 3 \) and every element in \( \mathcal{E} \) has even grading, \( Q_1 \) must act trivially on \( \mathcal{E} \).

Thus, \( P_2^i \) restricted to \( \mathcal{E} \) follows the Leibniz rule, hence \( \mathcal{M}(\mathcal{E} \otimes_r, P_2^s) \) is an algebra. Using (2.7) and the Künneth isomorphism for a derivation, we can easily deduce the following result.

**Lemma 2.9.** The \( P_2^4 \) Margolis homology of \( \mathcal{E} \) is given by

\[
\mathcal{M}(\mathcal{E}, P_2^4) \cong \Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \ldots).
\]

Moreover

\[
\mathcal{M}(\mathcal{E} \otimes_r, P_2^4) \cong \mathcal{M}(\mathcal{E}, P_2^4) \otimes_r \cong (\Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \ldots)) \otimes_r.
\]

Now consider the quotient \( K := \mathfrak{T} \parallel \mathcal{E} \cong \mathbb{F}_2 \otimes_{\mathcal{E}} \mathfrak{T} \). We have an isomorphism \( K \cong \Lambda(\zeta_4, \zeta_5, \ldots) \), and the induced action of \( Q_1 \) and \( P_2^4 \) on \( K \) is trivial. For a set \( A \), we let \( \mathbb{F}_2(A) \) denote the \( \mathbb{F}_2 \)-vector space which has the generating set \( A \). The algebra \( K \) admits a natural increasing filtration

\[
G^p(K) := \mathbb{F}_2(\zeta_1, \ldots, \zeta_k | k \leq p).
\]

induced by the length of the monomials. We call it the *length filtration*.

This length filtration on \( K \) induces an increasing filtration \( \{G^p(\mathfrak{T})\}_{p \geq 0} \) on \( \mathfrak{T} \), where \( G^p(\mathfrak{T}) \) is the pullback of \( G^p(K) \) (in vector spaces) along the quotient map \( \mathfrak{T} \to K \).

\[
\begin{array}{ccc}
G^p(\mathfrak{T}) & \longrightarrow & \mathfrak{T} \\
\downarrow & & \downarrow \\
G^p(K) & \longrightarrow & K.
\end{array}
\]

**Definition 2.10.** Let \( I \) be a finite tuple of natural numbers, and for \( I = \{i_1, \ldots, i_n\} \) let \( \zeta^I \) denote the monomial \( \zeta_1^{i_1} \cdots \zeta_n^{i_n} \). Then the *length* \( L \) of \( \zeta^I \) is defined by

\[
L(\zeta^I) = \sum_{j=1}^{\ell(I)} (i_j \text{ mod } 2).
\]

In other words, \( L(\zeta^I) \) counts the number of odd exponents in \( \zeta^I \). Then \( G^p(\mathfrak{T}) \) is the span of monomials \( \zeta^I \) of length less than or equal to \( p \)

\[
G^p(\mathfrak{T}) \cong \mathbb{F}_2(\zeta^I | L(\zeta^I) \leq p).
\]

The length function \( L \) measures “how far” a given monomial in \( \mathfrak{T} \) is from the even subalgebra \( \mathcal{E} \). Since there is an \( \mathbb{F}_2 \)-vector space isomorphism

\[
\mathfrak{T} \cong \mathcal{E} \otimes \mathfrak{K} // \mathcal{E} = \mathcal{E} \otimes K
\]

any monomial \( m \in \mathfrak{T} \) can be uniquely written as \( e \cdot k \) where \( e \in \mathcal{E} \) and \( k \in K \).
Commutativity follows from the fact that

\[ P \circ Q = Q \circ P \quad \text{and} \quad \text{for any monomials } \zeta_1, \ldots, \zeta_n, \text{ where indices do not repeat.} \]

The action of \( Q_1 \) is given by the formula

\[ Q_1(\zeta_1, \ldots, \zeta_n) = \sum_{k=1}^{n} \zeta_1 \cdots \zeta_{k-1} \zeta_{k-1} - 1 \zeta_{k+1} \cdots \zeta_n \]

where we allow repetition of indices. Since \( Q_1 \) acts trivially on \( \mathcal{E} \), it follows that

\[ Q_1(e \cdot k) = e \cdot Q_1(k). \]

From the formula above we see that \( Q_1(k) \neq 0 \) and \( L(Q_1(k)) = L(k) - 1 \). Hence,

\[ L(Q_1(m)) = L(e \cdot Q_1(k)) = L(Q_1(k)) - 1 = L(e \cdot k) - 1 = L(m) - 1. \]

Next, note that

\[ P_2^1(m) = P_2^1(e) \cdot k + Q_1(e) \cdot Q_1(k) + e \cdot P_2^1(k) = P_2^1(e) \cdot k + e \cdot P_2^1(k) \]

From Lemma 2.8, we see that \( L(P_2^1(k)) = L(P_2^1(k)) - 2 \) assuming \( P_2^1(k) \neq 0 \). Now set \( m_L = P_2^1(e) \cdot k \) and \( m_{L-2} = e \cdot P_2^1(k) \).

**Lemma 2.13.** The Hopf algebra \( \Lambda(Q_1, P_2^1) \) is commutative and cocommutative.

**Proof.** Commutativity follows from the fact that \( P_2^1 \) and \( Q_1 \) commute [AM71, Lemma 1.3(2)] (in the notation of [AM71], \( P_2^1 = P_2(2) \) and \( Q_1 = P_2(1) \)).

Co-cocommutativity follows from the fact that the diagram

\[ \Lambda(Q_1, P_2^1) \xrightarrow{\Delta} \Lambda(Q_1, P_2^1) \otimes \Lambda(Q_1, P_2^1) \]

\[ \Delta \downarrow \quad \text{flip} \]

\[ \Lambda(Q_1, P_2^1) \otimes \Lambda(Q_1, P_2^1) \]

commutes, because of (1.3) and (1.4). \( \square \)
If $M$ is a $\Lambda(Q_1, P^1_2)$-module then let $C^*_M$ denote the periodic chain complex

\[ \ldots \xrightarrow{p_1^i} M \xrightarrow{p_1} M \xrightarrow{p_1^i} \ldots \]

Its homology groups are isomorphic in each degree, i.e.

\[ H_i(C^*_M) \cong H_j(C^*_M) \]

for all $i, j \in \mathbb{Z}$. We use $\mathcal{M}(M, P^1_2)$ to denote this common homology group. When $M = \Xi$, the filtration $G^\bullet(\Xi)$ induces a filtration on $C^*_M$. By Lemma 2.12, $P^1_2$ respects the length filtration. This means we have a short exact sequence of chain complexes

\[
0 \rightarrow \bigoplus_{p \in \mathbb{Z}} G^{p-1}(C^*_\Xi) \rightarrow \bigoplus_{p \in \mathbb{Z}} G^p(C^*_\Xi) \rightarrow \bigoplus_{p \in \mathbb{Z}} G^p(C^*_\Xi) \rightarrow 0.
\]

Upon taking the homology, this short exact sequence of chain complexes produces an exact couple, resulting in a spectral sequence

\[
E^1_1 := H^q \left( \frac{G^p(C^*_\Xi)}{G^{p-1}(C^*_\Xi)} \right) \Rightarrow H^q(C^*_\Xi).
\]

We rewrite this spectral sequence as

\[
E^p_1 := \mathcal{M} \left( \frac{G^p(\Xi)}{G^{p-1}(\Xi)}, P^1_2 \right) \Rightarrow \mathcal{M}(tmf, P^1_2).
\]

and we call it the length spectral sequence.

The $E_1$ page of (2.14) is easy to calculate. Note that the associated graded

\[
\bigoplus_{p \geq 0} \frac{G^p(\Xi)}{G^{p-1}(\Xi)} \cong \mathcal{E} \otimes \mathcal{K}
\]

as an $\mathbb{F}_2$-algebra. The action of $\Lambda(Q_1, P^1_2)$ on $\mathcal{E} \otimes \mathcal{K}$ is defined using the Cartan formula as in the definition below.

**Definition 2.15** ([Mar83], p.186). Let $\Gamma$ be any Hopf algebra. For two $\Gamma$-modules $M$ and $N$, the underlying $\mathbb{F}_2$ vector space of $M \otimes N$ is simply $M \otimes_{\mathbb{F}_2} N$, and $\Gamma$ acts via the diagonal map, i.e.

\[
a(m \otimes n) = \sum_i a_i(m) \otimes a'_i(n),
\]

where $a \in \Gamma$ and $\Delta(a) = \sum_i a_i \otimes a'_i$, where $\Delta$ is the coproduct of the Hopf algebra.

Now we describe the action of $P^1_2$ on a monomial $m \in \bigoplus_{p \geq 1} \frac{G^p(\Xi)}{G^{p-1}(\Xi)}$. Write $m = e \otimes k$ for some $e \in \mathcal{E}$ and $k \in \mathcal{K}$. By Definition 2.15

\[
P^1_2(m) = P^1_2(e \otimes k) = P^1_2(e) \otimes k.
\]

Since the length filtration $G^\bullet(\Xi)$ is multiplicative, i.e.

\[
G^p(\Xi) \cdot G^{p'}(\Xi) \subset G^{p+p'}(\Xi),
\]

and $P^1_2$ restricted to $\mathcal{E}$ follows the Leibniz rule, the $E_1$ page of (2.14) is an $\mathbb{F}_2$-algebra and isomorphic to

\[
E^*_1 \cong \mathcal{M}(\mathcal{E} \otimes \mathcal{K}, P^1_2) \cong \mathcal{M}(\mathcal{E}, P^1_2) \otimes \mathcal{K} \cong \Lambda(\zeta^4_2, \zeta^4_3, \ldots) \otimes \Lambda(\zeta_4, \zeta_5, \ldots).
\]

In order to avoid confusion regarding the multiplicative structure of $E^*_1$, it is convenient to rename the generators.
Notation 2.16. We set \( x_i := \zeta_{i+3} \) and \( t_i := \zeta_{i+1}^4 \). Further, for finite subsets \( I = \{ i_1, \ldots, i_n \} \subset \mathbb{N} \) and \( J = \{ j_1, \ldots, j_m \} \subset \mathbb{N} \), we let \( t_I \) and \( x_I \) denote the monomials \( t_{i_1} \cdots t_{i_n} \) and \( x_{j_1} \cdots x_{j_m} \) respectively. We use \( t_I x_J \) to denote the tensor product \( t_I \otimes x_J \).

Lemma 2.12 and Lemma 2.13 imply that we have a commutative diagram of chain complexes

\[
\begin{array}{ccccccc}
0 & \rightarrow & \bigoplus_p G^p(C^n_{\Sigma}) & \rightarrow & \bigoplus_p G^{p+1}(C^n_{\Sigma}) & \rightarrow & \bigoplus_p G^{p+2}(C^n_{\Sigma}) & \rightarrow & 0 \\
0 & \rightarrow & \bigoplus_p G^{p-1}(C^n_{\Sigma}) & \rightarrow & \bigoplus_p G^p(C^n_{\Sigma}) & \rightarrow & \bigoplus_p G^{p+1}(C^n_{\Sigma}) & \rightarrow & 0.
\end{array}
\]

Consequently there is an action of \( Q_1 \) on each page of (2.14), which shifts the length filtration by \(-1\). In particular, we note \( Q_1(x_i) = t_i \) and in general

\[
(2.17) \quad Q_1(t_I x_J) = \sum_{j \in J} t_j t_I x_{J - \{ j \}}.
\]

Let \( m \in \Sigma \) be any monomial, \( m_L \) and \( m_{L-2} \) be as in Lemma 2.12, and let \([m]\) denote the equivalence class in the \( E_1 \) page of (2.14) represented by \( m \). Lemma 2.12 implies that the \( d_1 \) differential of (2.14) is trivial,

\[
d_2([m]) = [m_{L-2}]
\]

for the class of the monomial \( m \in \Sigma \) in the \( E_1 \) page, and the spectral sequence (2.14) collapses at the \( E_3 \) page. If we write \( m \in \Sigma \) as \( m = e \cdot k \), where \( e \in \mathcal{E} \) and \( k \in \mathcal{K} \), then

\[
d_2([m]) = [e \cdot P_2^1(k)] = [e] \cdot [P_2^1(k)].
\]

This means that the \( d_2 \) differential of (2.14) is \( \mathcal{M}(\mathcal{E}, P_2^1) \)-linear. It follows from the formula of Lemma 2.8 that

\[
(2.18) \quad d_2(t_I x_J) = \sum_{K \in J/2} t_K t_I x_{J - K},
\]

where \( J/2 \) is the set of subsets of \( J \) which contain two elements.

The formula for the \( d_2 \) differentials is intimately related to the action of \( Q_1 \) on the \( E_2 \) page of (2.14). The \( \Lambda(Q_1) \)-module structure on \( E_2^* \) (see (2.17)) can extend it to the \( \Lambda(P_2^1) \)-module structure using the algebra structure of \( E_2^* \) and the product formula (2.6)), along with the initial condition

\[
(2.19) \quad P_2^1(x_i) = P_2^1(t_i) = 0.
\]

The action of \( P_2^1 \) that results from this procedure is

\[
(2.20) \quad P_2^1(t_I x_J) = \sum_{K \in J/2} t_K t_I x_{J - K}
\]

on the monomial basis, which can be extended to all of \( E_2^* \) using \( \mathbb{F}_2 \)-linearity. Notice that the action we obtain through this process coincides with the formula for the \( d_2 \) differentials (2.18).
3. The reduced length

For convenience, we denote the $E_2$-page of (2.14) by

$$R = \Lambda(t_i : i \geq 1) \otimes \Lambda(x_i : i \geq 1),$$

which is an $F_2$-algebra, as well as a $\Lambda(Q_1, P_2^1)$-module, where the actions of $Q_1$ and $P_2^1$ are given by (2.17) and (2.20) respectively. In this section we analyze the $\Lambda(Q_1, P_2^1)$-module structure of $R$, which leads us to a description of

$$E_{\infty}^0 \cong \ldots \cong E_3^0 \cong H(E_2^*, d_2) \cong M(R, P_2^1).$$

The main idea here is to notice that the action of $P_2^1$ is linear with respect to the subalgebra

$$S := \Lambda(t_ix_i | i \in \mathbb{N}_+) \subset R,$$

which implies that $M(R, P_2^1)$ admits an $S$-module structure.

**Lemma 3.1.** The subalgebra $S \subset R$ is a trivial $\Lambda(Q_1, P_2^1)$-submodule which splits off as a $\Lambda(Q_1, P_2^1)$-module.

**Proof.** For any element $t_I x_I \in S$, it is clear from (2.17) and (2.18) that

$$Q_1(t_I x_I) = 0 = P_2^1(t_I x_I).$$

Thus $S$ is a trivial submodule.

Now observe from (2.17) and (2.18) that none of the monomials $t_I x_I \in S$ is a summand of $Q_1(t_I x_{I'})$ or $P_2^1(t_I x_{I'})$ for any choice of $I'$ and $J'$. Hence, $S$ is a split summand. □

**Corollary 3.2.** Every element of $S$ is a nonzero cycle in the $M(R, P_2^1)$.

**Lemma 3.3.** The action of $P_2^1$ on $R$ is $S$-linear.

**Proof.** It is enough to show that

$$P_2^1(t_i x_i, t_I x_J) = (t_i x_i) \cdot P_2^1(t_I x_J).$$

If $i \in I$, then $t_i t_I = 0$. Hence both the LHS and the RHS of (3.4) are zero.

If $i \in J$, then $x_i x_J = 0$, hence LHS of (3.4) is zero. On the other hand,

$$RHS = t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_J x_{-K}$$

$$= \sum_{i \in K \in J[2]} t_i t_K t_I x_i x_J x_{-K} + t_i \cdot \left( \sum_{i \notin K \in J[2]} t_i t_I x_i x_{J - K} \right) = 0,$$

as $t_i t_K = 0$ when $i \in K$ and $x_i x_{J - K} = 0$ when $i \notin K$. 

Now consider the case when $i \notin I \cup J$. Let $I' = I \cup \{i\}$ and $J' = J \cup \{i\}$. Then,

\[ P_2^1(t_i x_i \cdot t_1 x_j) = \sum_{K \in J'[2]} t_{K} t_{I'} x_{J' - K} \]

\[ = \sum_{i \notin K \in J'[2]} t_{K} t_{I'} x_{J' - K} + \sum_{i \in K \in J'[2]} t_{K} t_{I'} x_{J' - K} \]

\[ = \sum_{i \notin K \in J'[2]} t_{K} t_{I'} x_{J' - K} \]

\[ = t_i x_i \cdot \sum_{K \in J[2]} t_{K} t_{I} x_{J - K} \]

\[ = t_i x_i \cdot P_2^1(t_1 x_j). \]

**Remark 3.5.** While the $E_2$ page of (2.14) admits an $\mathbb{F}_2$-algebra structure, the $E_3$ page does not admit any multiplicative structure. This is because the $d_2$ differentials do not follow Leibniz rule and the product of $d_2$ cycles may not be a cycle. For example, $x_i$ for all $i \in \mathbb{N}_+$, is a $d_2$-cycle, whereas $x_i x_j$ for $i \neq j$ supports a differential $d_2(x_i x_j) = t_i t_j$ by (2.18). Even if $\alpha, \beta$ and $\alpha \cdot \beta$ are $P_2^1$ cycles it is unclear that the pairing $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ is well-defined in the $E_3$ page.

**Corollary 3.6.** $\mathcal{M}(\mathcal{R}, P_2^1)$ is a module over the ring $\mathcal{S}$.

**Proof.** By Lemma 3.3, there exists a pairing $\mu : S \otimes R \to R$ such that the diagram

\[ \begin{array}{ccc}
S \otimes R & \xrightarrow{\mu} & R \\
1 \otimes P_2^1 & \downarrow & \downarrow P_2^1 \\
S \otimes R & \xrightarrow{\mu} & R.
\end{array} \]

commutes. It follows that $\mathcal{M}(\mathcal{R}, P_2^1)$ is an $\mathcal{S}$ module. \hfill \Box

As a result, we only need to understand the action of $P_2^1$ on the generators of $\mathcal{R}$ when viewed as an $\mathcal{S}$-module. In order to approach this problem we introduce the notion of reduced length.

**Definition 3.7.** For any monomial $t_1 x_j \in \mathcal{R}$ the reduced length $\ell$ is

\[ \ell(t_1 x_j) = |J - I| = |J \cap I^c| = |J| - |J \cap I|, \]

where $I^c$ denotes the complement of $I$.

Note that the length of $t_1 x_j \in \mathcal{R}$ is given by the formula $L(t_1 x_j) = |J|$; in other words, it is counting the number of factors of $x_j$. Whereas, $\ell(t_1 x_j)$ counts only those factors $x_j$ in $x_j$ for which $t_j$ is not a factor of $t_j$. For example,

\[ \ell(x_1) = \ell(t_1 x_1 x_2) = \ell(t_1 t_2 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 x_4) = 1 \]

\[ \ell(x_1 x_2) = \ell(t_1 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 x_5 x_6) = 2. \]

**Remark 3.8.** The reduced length function $\ell$ measures “how far” a given monomial in $\mathcal{R}$ is from the subalgebra $\mathcal{S}$.
For each \( i \in \mathbb{N}_+ \), let \( M_i := \Lambda(Q_1) \{ x_i \} \subset \mathcal{R} \) denote the \( \Lambda(Q_1, P_1^2) \)-submodule isomorphic to \( \Lambda(Q_1) \) and generated by \( x_i \). For an indexing set \( K \subset \mathbb{N}_+ \), let
\[
M_K := \bigotimes_{j \in K} M_j \subset \mathcal{R}
\]
with the convention that \( M_\emptyset := \mathbb{F}_2 \). If the indexing set is \( [n] = \{1, \ldots, n\} \subset \mathbb{N}_+ \), then we write \( M_{[n]} \) to denote \( M_{\{1, \ldots, n\}} \).

In Figure 1, Figure 2 and Figure 3 we present \( M_i \), \( M_{\{1,2\}} \) and \( M_{\{1,2,3\}} \) respectively as a \( \Lambda(Q_1, P_1^2) \)-module. In these figures the blue curved lines depict the action of \( Q_1 \) and red boxed lines depict the action of \( P_1^2 \).

\[ x_i \]
\[ (t_i) \]

**Figure 1:** \( M_i \) as a module over \( \Lambda(Q_1, P_1^2) \)

Note that the set \( W := \{ t_{I \cap J} \in \mathcal{R} | I \cap J = \emptyset \} \) forms a generating set for \( \mathcal{R} \) as an \( \mathcal{S} \)-module as any monomial \( t_{I \cap J} \in \mathcal{R} \) can be uniquely written as a product of an element of \( W \) and a monomial in \( \mathcal{S} \):
\[
t_{I \cap J} = t_{I \cap J} x_{I \cap J} \cdot t_{I - (I \cap J)} x_{J - (I \cap J)}.
\]
For any finite subset \( K \subset \mathbb{N}_+ \),
\[
W_K := \{ t_{I \cap J} | I \cup J = K, I \cap J = \emptyset \} \subset W
\]
forms an \( \mathbb{F}_2 \)-basis for \( M_K \), i.e. \( \mathbb{F}_2(W_K) = M_K \). Since
\[
W = \bigsqcup_{K \subset \mathbb{N}_+} W_K
\]
and \( \mathbb{F}_2(W_K) = M_K \) is closed under the action of \( Q_1 \) and \( P_1^2 \) (these actions preserve the total indexing set \( K \), by (2.17) and (2.18)), we learn that
\[
\mathcal{R} / \mathcal{S} \cong \mathbb{F}_2 \otimes_\mathcal{S} \mathcal{R} \cong \mathcal{R} \otimes_\mathcal{S} \mathbb{F}_2 \cong \bigoplus_K M_K
\]
is an isomorphism of \( \Lambda(Q_1, P_1^2) \)-modules. Consequently, we have the following lemma.

**Lemma 3.9.** Let \( \mathcal{S}_K \subset \mathcal{S} \) denote the subalgebra \( \Lambda(t_{I \cap J} | I \subset \mathbb{N}_+ - K) \). There is a \( \Lambda(Q_1, P_1^2) \)-module isomorphism
\[
\bigoplus_{K \subset \mathbb{N}_+} \mathcal{S}_K \otimes M_K \cong \mathcal{R}.
\]

**Proof.** Consider the \( \mathbb{F}_2 \)-vector space isomorphism
\[
\iota : \mathcal{R} \to \bigoplus_{K \subset \mathbb{N}_+} \mathcal{S}_K \otimes M_K
\]
Figure 2: \( M[2] \) as a module over \( \Lambda(Q_1, \mathbb{P}_2) \), where \( [2] = \{1, 2\} \)

\[
\begin{align*}
\begin{array}{c}
t_1 x_2 \\
\downarrow \\
t_1 t_2 \\
\end{array} & \quad \begin{array}{c}
x_1 x_2 \\
\downarrow \\
t_2 x_1 + t_1 x_2 \\
\end{array} \\
\end{align*}
\]

Figure 3: \( M[3] \) as a module over \( \Lambda(Q_1, \mathbb{P}_2) \), where \( [3] = \{1, 2, 3\} \)

\[
\begin{align*}
\begin{array}{c}
t_1 t_2 x_3 + t_1 t_3 x_2 + t_2 t_3 x_2 \\
\downarrow \\
t_1 t_2 t_3 \\
\end{array} & \quad \begin{array}{c}
x_1 x_2 x_3 \\
\downarrow \\
t_3 x_1 x_2 + t_2 x_1 x_3 + t_1 x_2 x_3 \\
\end{array} \\
\end{align*}
\]

which sends

\[
t_I x_J \mapsto t_{I \cap J} x_{I \cap J} \otimes t_{I-(I \cap J)} x_{J-(I \cap J)} \in S_K \otimes M_K,
\]

where \( K = I \cup J - I \cap J \). The map \( \iota^{-1} \) sends

\[
t_B x_B \otimes t_{K-A} x_A \mapsto t_{B \cup (K-A)} \cdot x_{B \cup A},
\]

where \( A \subset K \). This map is also a \( \Lambda(Q_1, \mathbb{P}_2) \)-module isomorphism as \( S_K \subset S \) is a trivial \( \Lambda(Q_1, \mathbb{P}_2) \)-module by Lemma 3.1.

Hence we can reduce the problem of computing \( \mathcal{M}(\mathcal{R}, \mathbb{P}_2) \) to computing \( \mathcal{M}(M_K, \mathbb{P}_2) \) for various finite subsets \( K \) of \( \mathbb{N}_+ \). Thus, we first need to understand the structure of \( M_K \) as a \( \Lambda(Q_1, \mathbb{P}_2) \)-module.

Remark 3.10. Let \( [n] \) denote the indexing set \( \{1, ..., n\} \subset \mathbb{N}_+ \). If \( |K| = n \), then there exists the unique order preserving bijection

\[
\iota : [n] \rightarrow K
\]

and it induces an isomorphism \( \iota : M_{[n]} \cong M_K \). Thus it is enough to understand \( \Lambda(Q_1, \mathbb{P}_2) \)-module structure of \( M_{[n]} \) for all \( n \in \mathbb{N}_+ \).

As depicted in Figure 3, \( M[3] \) splits as a \( \Lambda(Q_1, \mathbb{P}_2) \)-module

\[
M[3] \cong \Lambda(Q_1, \mathbb{P}_2) \{x_1 x_2 x_3\} \\
\oplus \Lambda(Q_1) \{t_3 x_1 x_2 + t_1 x_2 x_3\} \\
\oplus \Lambda(Q_1) \{t_2 x_1 x_3 + t_3 x_1 x_2\}
\]

as summand of a free \( \Lambda(Q_1, \mathbb{P}_2) \)-module and two copies of \( \Lambda(Q_1) \).
Remark 3.12. The splitting of (3.11) is a consequence of Lemma 2.13. Since \( \Lambda(Q_1, P_1^2) \) is cocommutative, for any \( \Lambda(Q_1, P_1^2) \)-module \( M \) and \( \sigma \in F_2[\Sigma_n] \), the induced map

\[
\sigma : M^\otimes n \to M^\otimes n
\]

is a map of \( \Lambda(Q_1, P_1^2) \)-modules. Note that in the group ring \( F_2[\Sigma_3] \), the identity element can be written as a sum of idempotent elements

\[
1 = e + f_1 + f_2.
\]

For example, one can choose \( e = 1 + (1 2 3) + (1 2 3) \), \( f_1 = 1 + (1 2) + (1 3) + (1 3 2) \) and \( f_2 = 1 + (1 2) + (1 3) + (1 2 3) \). Then we have

\[
M^{\otimes 3} \cong e(M^{\otimes 3}) \oplus f_1(M^{\otimes 3}) \oplus f_2(M^{\otimes 3}).
\]

When \( M \cong \Lambda(Q_1) \), we get the decomposition of \( (3.11) \).

The splitting of (3.11), along with the following fact about finite dimensional Hopf algebras, is the key to understanding the structure of \( M_K \).

Theorem 3.13 ([NZ89]). If \( H \) is a finite dimensional connected Hopf algebra over a field \( F \), then for any \( H \)-module \( M \), \( H \otimes M \) is a free \( H \)-module.

Let us denote by \( A \) the \( \Lambda(Q_1, P_1^2) \)-module isomorphic to \( \Lambda(Q_1) \) and let \( B := A \otimes A \).

Then using (3.11) and Theorem 3.13, we notice that

\[
M_3 \cong B \otimes A \cong \{\text{Free}\} \oplus A^{\otimes 2}
\]

\[
M_4 \cong \{\text{Free}\} \oplus B^{\otimes 2}
\]

\[
M_5 \cong \{\text{Free}\} \oplus A^{\otimes 4}
\]

where \( \{\text{Free}\} \) denotes a free \( \Lambda(Q_1, P_1^2) \) module. This iterative process can be continued as described in Lemma 3.14 below. We use \( A\{y\} \), resp. \( B\{y\} \), to specify that \( y \) generates \( A \), resp. \( B \), as a \( \Lambda(Q_1, P_1^2) \) module. For example, \( M_i \cong A\{x_i\} \).

Lemma 3.14. As a \( \Lambda(Q_1, P_1^2) \)-module,

\[
M_{2r+1} \cong \{\text{Free}\} \oplus (\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}),
\]

where \( \ell(h_{2r+1,i}) = r + 1 \).

As a \( \Lambda(Q_1, P_1^2) \)-module,

\[
M_{2r} \cong \{\text{Free}\} \oplus \bigoplus_{i=1}^{2^{r-1}} B\{h_{2r,i}\},
\]

where \( \ell(h_{2r,i}) = r + 1 \).

Proof. Our proof is by induction on \( r \). From Figure 1, Figure 2 and Figure 3, the claim is true for \( k = 1, 2, 3 \). Note that

\[
h_{1,1} = x_1 \quad h_{2,1} = x_1 x_2 \quad h_{3,1} = (t_3 x_1 + x_3 t_1) x_2 \quad h_{3,2} = (t_2 x_3 + t_3 x_2) x_1
\]

Now assume that the result is true for \( 2r - 1 \), i.e.

\[
M_{2r-1} \cong \{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} A\{h_{2r-1,i}\}
\]
where $\ell(h_{2r-1,i}) = r$ and $\{\text{Free}\}$ is a free $\Lambda(Q_1, P_2)$-module. It follows that
\[ M_{[2r]} \cong M_{[2r-1]} \otimes M_{2r} \cong (\{\text{Free}\} \otimes A\{x_{2r}\}) \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} B\{h_{2r-1,i}, x_{2r}\}. \]

By Theorem 3.13, the first summand is, again, a free module. Set
\[ h_{2r,i} = h_{2r-1,i} \cdot x_{2r} \]
and notice $\ell(h_{2r-1,i}x_{2r}) = \ell(h_{2r-1,i}) + \ell(x_{2r}) = r + 1$.

To complete the inductive argument, observe
\[ M_{[2r+1]} \cong M_{[2r-1]} \otimes B\{x_{2r}x_{2r+1}\} \]
\[ \cong (\{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} A\{h_{2r-1,i}\}) \otimes B\{x_{2r}x_{2r+1}\} \]
\[ \cong \{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} (A\{h_{2r-1,2i-1}\} \oplus A\{h_{2r+1,2i}\}), \]
where one can define the generators $h_{2r-1,j}$ from Figure 3 by replacing $x_1, x_2, x_3$ with $h_{2r-1,i}, x_2r$, and $x_{2r+1}$ respectively. More specifically, one can define
\[ h_{2r+1,2i-1} = Q_1(h_{2r-1,i} \cdot x_{2r+1}) \cdot x_{2r}, \]
\[ h_{2r+1,2i} = h_{2r-1,i} \cdot Q_1(x_{2r}x_{2r+1}). \]

It is easy to check that $\ell(h_{2r+1,j}) = r + 1$. \hfill \Box

Following the proof of Lemma 3.14, we can provide an explicit basis of $M(M_K, P_2)$. By Remark 3.10 it suffices to provide a basis for $M(M_{[n]}, P_2)$ for all $n \geq 1$. We do so inductively (see Definition 3.16), however we must treat the odd and the even case separately, essentially because of Lemma 3.14. Since $A$ is a trivial $\Lambda(P_2)$-module, $M(A, P_2) \cong A$, and we get
\[ M(M_{[2r+1]}, P_2) \cong M\left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}, P_2\right) \cong \bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}. \]

Thus the collection
\[ \{h_{2r+1,i} : 1 \leq i \leq 2^r\} \cup \{Q_1(h_{2r+1,i}) : 1 \leq i \leq 2^r\} \]
is a $\mathbb{F}_2$-basis of $M(M_{[2r+1]}, P_2)$. When $n$ is even, say $n = 2r$, then
\[ M(M_{[2r]}, P_2) \cong \bigoplus_{i=1}^{2^{r-1}} M(B\{h_{2r,i}\}, P_2). \]

Now note that, if $B\{x \otimes y\} = A\{x\} \otimes A\{y\}$ (where $x$ and $y$ are generators), then
\[ \{Q_1(x) \otimes y, x \otimes Q_1(y)\} \]
is a $\mathbb{F}_2$-basis of $M(B\{x \otimes y\}, P_2)$. Using the fact that
\[ h_{2r,i} = h_{2r-1,i} \cdot x_{2r}, \]
we get Corollary 3.15 and Definition 3.16 thereafter.
Corollary 3.15. Let $\mathcal{M}(M_K, P_2) = \{ x \in \mathcal{M}(M_K, P_2) | \ell(x) = l \}$.

If $|K| = 2r + 1$, then

$$\dim \mathcal{M}(M_K, P_2)_l = \begin{cases} 2^r, & \text{if } l = r, r + 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $|K| = 2r$, then

$$\dim \mathcal{M}(M_K, P_2)_l = \begin{cases} 2^r, & \text{if } l = r \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Lemma 3.14 implies

$$M_{[2r+1]} \cong \{ \text{Free} \} \oplus \bigoplus_{1 \leq i \leq 2^r} A\{ h_{2r+1, i} \}$$

where $\ell(h_{2r+1, i}) = r + 1$. By Lemma 2.12 we have $\ell(Q_1(h_{2r+1, i})) = r$. Thus $\{ h_{2r+1, i} \}$ is the basis for $\mathcal{M}(M_{[2r+1]}, P_2)^{r+1}$ and $\{ Q_1(h_{2r+1, i}) \}$ is the basis for $\mathcal{M}(M_{[2r+1]}, P_2)^r$. Applying Remark 3.10 we deduce the statement about dimension for any $M_K$ with $|K| = 2r + 1$.

For the even case we have from Lemma 3.14

$$M_{[2r]} \cong \{ \text{Free} \} \oplus \bigoplus_{1 \leq i \leq 2^{r+1}} B\{ h_{2r, i} \}$$

where $\ell(h_{2r, i}) = r + 1$. Then for each $i$, $\mathcal{M}(B\{ h_{2r, i} \}, P_2) = \mathcal{M}(B\{ h_{2r, i} \}, P_2)_r$ is an $F_2$ vector space of dimension 2 generated by $\{ h_{2r-1, i}, t_{2r}, Q_1(h_{2r-1, i}) \cdot x_{2r} \}$. \hfill \Box

Definition 3.16. We define the basis $B_{[n], l}$ of $\mathcal{M}(M_{[n]}, P_2)_l$ for $0 \leq l \leq n$ inductively starting with $B_{[1], 0} = \{ t_1 \}$ and $B_{[1], 1} = \{ x_1 \}$. Suppose we have defined

$$B_{[2r-1], l} := \begin{cases} \{ h_{2r-1, 1}, \ldots, h_{2r-1, r-1} \} & \text{if } l = r \\ \{ Q_1(h_{2r-1, 1}), \ldots, Q_1(h_{2r-1, r-1}) \} & \text{if } l = r - 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Then define:

$$B_{[2r], r} := \{ h_{2r-1, 1}, t_{2r}, \ldots, h_{2r-2, r-2} \cdot t_{2r} \} \cup \{ Q_1(h_{2r-1, 1}) \cdot x_{2r}, \ldots, Q_1(h_{2r-1, r-2}) \cdot x_{2r} \}$$

and set $B_{[2r], l} := \emptyset$ if $l \neq r$.

Now define $h_{2r+1, 2r} = Q_1(h_{2r-1, 1}) \cdot (x_{2r+1} \cdot x_{2r})$ and $h_{2r+1, 2r+1} = h_{2r-1, 1} \cdot Q_1(x_{2r} \cdot x_{2r+1})$ and set

$$B_{[2r+1], l} := \begin{cases} \{ h_{2r+1, 1}, \ldots, h_{2r+1, r-1} \} & \text{if } l = r + 1 \\ \{ Q_1(h_{2r+1, 1}), \ldots, Q_1(h_{2r+1, r-1}) \} & \text{if } l = r \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $B_{[n]}$ denote the union $\bigcup_l B_{[n], l}$. Let $B_K$ denote the image of the $B_{[n]}$ under the isomorphism $\iota : M_{[n]} \rightarrow M_K$ of Remark 3.10.

Example 3.17 (Examples of $B_{[n]}$). We explicitly identify $B_{[n]}$ using Definition 3.16 for $n \leq 4$, and for $n = 1, 2, 3$ we can compare to Figures 1, 2 and 3, to see that $B_{[n]}$ is indeed the basis for $\mathcal{M}(M_{[n]}, P_2)$.

- $B_{[1]} = \{ t_1, x_1 \}$,
- $B_{[2]} = \{ t_1 x_2, t_2 x_1 \}$,
THE $P_1^2$ MARGOLIS HOMOLOGY OF $tmf$

\[
B_{[3]} = \{ t_1 t_2 x_3 + t_1 t_3 x_2, t_1 t_2 x_3 + t_2 t_3 x_1 \} \cup \{ t_3 x_1 x_2 + t_2 x_1 x_3, t_3 x_1 x_2 + t_1 x_2 x_3 \},
\]
and,

\[
B_{[4]} = \{ t_1 t_2 x_3 + t_1 t_3 x_2 x_4, t_1 t_2 x_3 + t_2 t_3 x_1 x_4, t_3 t_4 x_1 x_2 + t_2 t_4 x_1 x_3, t_3 t_4 x_1 x_2 + t_1 x_2 x_3 \}.
\]

Let $P_K := \mathbb{F}_2(B_K) \subset M_K$. By construction, $P_K$ is a trivial split $\Lambda(P_1^1)$-submodule of $M_K$. The splitting of $P_K$ follows from the fact that its complement is free and the fact that $\Lambda(P_1^1)$, like any other finite dimensional connected Hopf algebra, is self-injective \cite[Ch. 12, §2]{Mar83}.

**Theorem 3.18.** Let $K$ be a finite subset of $\mathbb{N}_+$. Let

\[
SB_K := \{ t_I x_J \cdot b | I \cap K = \emptyset \text{ and } b \in B_K \} \subset \mathbb{R}.
\]

Then

\[
B := \bigcup_{K \subset \text{finite} \mathbb{N}_+} SB_K
\]
forms a basis of the $\mathbb{F}_2$-vector space $\mathcal{M}(tmf, P_1^1)$ and

\[
\mathcal{M}(tmf, P_1^1) \cong \bigoplus_{K \subset \text{finite} \mathbb{N}_+} \mathbb{F}_2(SB_K) \cong \bigoplus_{K \subset \text{finite} \mathbb{N}_+} S_K \otimes \mathcal{M}(M_K, P_1^1).
\]

is an isomorphism of $\mathbb{F}_2$-vector spaces.

**Proof.** By Lemma 3.9, we have a $\Lambda(Q_1, P_1^1)$ module isomorphism

\[
\mathcal{R} \cong \bigoplus_{K \subset \text{finite} \mathbb{N}_+} S_K \otimes M_K.
\]

Therefore, the linearity of the action of $P_1^1$ (see Corollary 3.6) with respect to elements in $S$ gives us

\[
\mathcal{M}(tmf, P_1^1) \cong \mathcal{M}(\mathcal{R}, P_1^1) \cong \mathcal{M}(\bigoplus_{K \subset \text{finite} \mathbb{N}_+} S_K \otimes M_K, P_1^1) \cong \bigoplus_{K \subset \text{finite} \mathbb{N}_+} S_K \otimes \mathcal{M}(M_K, P_1^1) \cong \bigoplus_{K \subset \text{finite} \mathbb{N}_+} \mathcal{S}_K \otimes M_K \cong \bigoplus_{K \subset \text{finite} \mathbb{N}_+} \mathbb{F}_2(SB_K). \quad \Box
\]

**Remark 3.19.** Let $e$ denote the exchange map $e : \mathcal{R} \to \mathcal{R}$ which sends

\[
e : t_I x_J \mapsto t_J x_I.
\]

It seems to be the case that $[m] \in \mathcal{M}(tmf, P_1^1)$ if and only if $[e(m)] \in \mathcal{M}(tmf, P_1^1)$. The source of such symmetry is unclear to the authors, although it might be related to Spanier–Whitehead duality.
Finally, we would like to say a word about the module structure of $\mathcal{M}(tmf, P_2)$ over $S$. Note that the collection of elements

$$B_S := \{ t_1 x_1 | I_{\text{finite}} \}$$

forms an $\mathbb{F}_2$-basis of $S$. The $S$-module structure on $\mathcal{M}(tmf, P_2)$ is extended from a pairing at the level of bases

$$B_S \otimes SB_K \xrightarrow{\mu} SB_K$$

$$s \otimes (s' \cdot b) \mapsto \begin{cases} (s \cdot s') \cdot b, & \text{if } I \cap K = \emptyset \\ 0, & \text{if } I \cap K \neq \emptyset. \end{cases}$$

**Remark 3.20.** Recall that $H_*(tmf)$ was described in terms of $\zeta_i$. We can convert an element of the Margolis homology expressed in terms of $t_i$ and $x_i$ back to an expression involving $\zeta_i$ using the identifications of Notation 2.16. For example,

$$t_4 t_9 x_2 x_6 + t_2 t_9 x_4 x_6$$

can be identified with the class represented by element $\zeta_5^5 \zeta_9^4 + \zeta_4^5 \zeta_7 \zeta_9 \in \mathfrak{T}$.

## 4. $P_2^1$ Margolis homology of $tmf^{\wedge r}$ and $B(\mathbb{Z}/2^{\times n})_+$

### 4.1. $P_2^1$ Margolis homology of $tmf^{\wedge r}$.

Note that

$$H_*(tmf^{\wedge r}) \cong H_* (tmf)^{\otimes r} \cong \mathfrak{T}^{\otimes r}.$$

We first extend the notion of length to $\mathfrak{T}^{\otimes r}$ for a monomial $\zeta^{i_1} \cdots | \zeta^{i_r}$. For $\zeta^{i_1} \in \mathfrak{T}^{\otimes r}$, we define

$$L(\zeta^{i_1} | \cdots | \zeta^{i_r}) = L(\zeta^{i_1}) + \cdots + L(\zeta^{i_r}).$$

We define the even subalgebra $E_r$ of $\mathfrak{T}^{\otimes r}$ as the span of those monomials in $\mathfrak{T}^{\otimes r}$ whose lengths are zero. Observe that,

$$E_r \cong \mathcal{E}^{\otimes r}.$$

Notion of length leads to an increasing filtration on $\mathfrak{T}^{\otimes r}$, call it the length filtration, by setting

$$G^p(\mathfrak{T}^{\otimes r}) = \{(\zeta^{i_1} | \cdots | \zeta^{i_r}) | L(\zeta^{i_1} | \cdots | \zeta^{i_r}) \leq p\}.$$

Let $K_r = K^{\otimes r}$, where $K$ is as defined in Section 2. Just like in the case $r = 1$, we get a length spectral sequence and its $E_1$ page is

$$E_1^r \cong \mathcal{M}(E_r, P_2^1) \otimes K_r \Rightarrow \mathcal{M}(tmf^{\wedge r}, P_2^1).$$

Since action of $P_2$ follows the Leibniz rule when restricted to $E$, we get

$$\mathcal{M}(E_r, P_2^1) \cong \mathcal{M}(E, P_2^1)^{\otimes r}.$$

**Notation 4.2.** For shorthand, we denote $x_{i,j} = (1 \cdots | \zeta_{i+3}^{j-1} | \zeta_{r-j}^{i} | \cdots | 1)$ and $t_{i,j} = (1 \cdots | \zeta_{r-j}^{i} | \zeta_{i+j}^{j-1} | \cdots | 1)$. With this notation we have

$$Q_1(x_{i,j}) = t_{i,j}.$$
Using Notation 4.2, we see that the $E_1$ page of the length spectral sequence (4.1), as an algebra, is isomorphic to

$$\mathcal{R}_r := \Lambda(t_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leq j \leq r) \otimes \Lambda(x_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leq j \leq r).$$

It is easy to see that the map induced by the reindexing map

$$\iota : (i, j) \mapsto r(i - 1) + j,$$

produces a (non-canonical) isomorphism of algebras between $\mathcal{R}_r$ (the $E_2$ page of (4.1)) and $\mathcal{R}$ (the $E_2$ page of (2.14)), after forgetting the internal grading. This is also an isomorphism of $\Lambda(Q_1, P_2^1)$-modules. Thus we have an isomorphism

$$\iota_* : \mathcal{M}(tmf, P_2^1) \xrightarrow{\cong} \mathcal{M}(tmf^{\wedge r}, P_2^1)$$

induced by the $\iota$. Therefore, Theorem 3.18 essentially gives a complete calculation of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$.

**Example 4.3.** For example, let us assume $r = 3$. Then the element $t_2t_4x_6x_9 + t_2t_6x_4x_9 \in \mathcal{M}(tmf, P_2^1)$ (see Example 3.17) corresponds to the element

$$t_{1,2}t_{2,1}x_{2,3}x_{3,3} + t_{1,2}t_{2,3}x_{2,1}x_{3,3} \in \mathcal{M}(tmf^{\wedge 3}, P_2^1)$$

under the bijection obtained from the above reindexing. When expressed in terms of $\zeta_i$s (see Notation 4.2), the same element can be expressed as

$$\zeta_4^4|\zeta_5^2|\zeta_6^2 + 1 + \zeta_5^4|\zeta_4^2|\zeta_5^2|\zeta_6^2|.$$  

**Remark 4.4 (P_2 Margolis homology of Brown–Gitler spectra).** It is well-known that

$$H_*(tmf) \cong \bigoplus_{i \geq 0} H_*(\Sigma^{8i}bo_i)$$

where $bo_i$ are certain Brown–Gitler spectra associated with $bo$. In [Mah81] Mahowald defined a multiplicative weight function, which is given by $w(\zeta_i) = 2^i - 1$. $H_*(\Sigma^{8i}bo_i)$ is the summand of $H_*(tmf)$ which consists of elements of Mahowald weight exactly equal to $8i$. We assign Mahowald weight of $t_{i,j}$ and $x_{i,j}$ as

$$w(t_{i,j}) = w(x_{i,j}) = 2^{i+1}.$$  

It follows that the Margolis homology $\mathcal{M}(bo_{q_1} \wedge \cdots \wedge bo_{q_n}, P_2^1)$ is a summand of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$. It consists of all polynomials of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$ expressed in terms of $x_{i,j}$ and $t_{i,j}$ such that $w(x_{i,j}) = w(t_{i,j}) = 4q_j$.

**Remark 4.5.** While it is true that $\mathcal{R}_r \cong \mathcal{R}^{\otimes r}$, as an $\mathbb{F}_2$-algebra as well as an $\Lambda(Q_1, P_2^1)$-module, it is not useful for the purposes of calculating $\mathcal{M}(\mathcal{R}_r, P_2^1)$. This is because $P_2^1$ does not obey Leibniz rule and

$$\mathcal{M}(\mathcal{R}_r, P_2^1) \neq \mathcal{M}(\mathcal{R}, P_2^1)^{\otimes r}.$$  

However we overcome this difficulty by producing an $\Lambda(Q_1, P_2^1)$-module isomorphism $\iota_*$ at the expense of forgetting the internal grading.
4.2. \(P_2^1\) Margolis homology of \((BZ/2^{x_k})_+\). The space \(BZ/2\) is also known as \(\mathbb{RP}^\infty\), the real infinite-dimensional projective space. It is well-known that

\[H^*((BZ/2)_+, \mathbb{F}_2) \cong \mathbb{F}_2[x]\]

and therefore

\[H^*((BZ/2^{x_k})_+, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_k].\]

It can be seen that \(P_2^1(x_i) = 0\) and \(Q_1(x_i) = x_i^4\). We again define the length function on the monomials in the usual way

\[L(x_1^{i_1} \ldots x_k^{i_k}) = (i_1 \text{ mod } 2) + \cdots + (i_k \text{ mod } 2).\]

The even complex \(\mathcal{E}\), which is the span of elements of length zero, is isomorphic to

\[\mathcal{E} = \mathbb{F}_2[x_1^2, \ldots, x_k^2].\]

It can be seen that \(P_2^1(x_1^2) = x_1^8\). Now observe that \(Q_1\) acts trivially on \(\mathcal{E}\), hence \(P_2^1\) acts as a derivation and, therefore,

\[\mathcal{M}(\mathcal{E}, P_2^1) \cong \Lambda(x_1^4, \ldots, x_k^4)\]

Now the length function gives us an increasing length filtration

\[G^p(\mathbb{F}_2[x_1, \ldots, x_k]) = \mathbb{F}_2[x_1^{i_1} \ldots x_k^{i_k} : L(x_1^{i_1} \ldots x_k^{i_k}) \leq p].\]

This results in a length spectral sequence which only has \(d_0\) and \(d_2\) differentials. If we denote \(x_i^i\) by \(t_i\) for convenience, we can see that the length spectral sequence

\[E_2^\bullet = \Lambda(t_1, \ldots, t_k) \otimes \Lambda(x_1, \ldots, x_k) \Rightarrow \mathcal{M}((BZ/2^{x_k})_+, P_2^1)\]

is a sub spectral sequence of \((2.14)\) and is, in fact, isomorphic to it when \(k = \infty\). Thus, when \(k\) is finite, we can recover a complete description of \(\mathcal{M}((BZ/2^{x_k})_+, P_2^1)\) from Theorem 3.18. More precisely, we obtain

\[\mathcal{M}((BZ/2^{x_k})_+, P_2^1) \cong \bigoplus_{K \subset [k]} S_K \otimes \mathcal{M}(M_K, P_2^1),\]

where \(S_K = \Lambda(t_ix_i \mid i \in [k] - K)\). \(\mathcal{M}((BZ/2^{x_k})_+, P_2^1)\) is a module over \(S_{[k]}\).

Example 4.6. \(\mathcal{M}(\mathbb{RP}^\infty, P_2^1) \cong \mathbb{F}_2(x_1, t_1, t_1x_1)\), where the internal degrees of \(x_1\) and \(t_1\) are 1 and 4 respectively and \(S_{[1]} = \Lambda(t_1x_1)\). Similarly,

\[\mathcal{M}(\mathbb{RP}^\infty \times \mathbb{RP}^\infty, P_2^1) \cong \mathbb{F}_2(x_1, x_2, t_1, t_2, t_1x_1, t_2x_2, t_1x_2, t_2x_1, t_1x_1x_2, t_2x_2x_1, t_1t_2x_2, t_1t_2x_1, t_1t_2x_1, t_1t_2x_2)\]

where the internal degrees of \(x_i\) and \(t_i\) are 1 and 4 respectively. Here \(S_{[2]} = \Lambda(t_1x_1, t_2x_2)\). If we denote

\[H^*((\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_+, \mathbb{F}_2) \cong \mathbb{F}_2[y, z]\]

where \(|y| = |z| = 1\), then one may choose \(x_1 = [x], x_2 = [y], t_1 = [x^4]\) and \(t_2 = [y^4]\).
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