CLASSIFICATION OF $\mathbb{C}P^2$-MULTIPlicative HIRZEBRUCH GENERA

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Abstract. The short article \cite{1} states results on $\mathbb{C}P^2$-multiplicative Hirzebruch genera. The aim of the following text is to give a proof of Theorem 3 from \cite{1}. This proof uses only the technique of functional differential equations.

1. Preliminaries

Let $R$ be a commutative torsion-free ring with unity and no zero divisors, and let $L_f : \Omega_U \to R$ be the Hirzebruch genus determined by the series $f(x) = x + \sum_{k=1}^{\infty} f_k \frac{x^{k+1}}{(k+1)!}$, where $f_k \in R$.

A Hirzebruch genus $L_f : \Omega_U \to R$ is called $\mathbb{C}P^2$-multiplicative, if we have $L_f[M] = L_f[\mathbb{C}P^2]L_f[B]$ for any bundle of stably complex manifolds $M \to B$ with fiber $\mathbb{C}P^2$ and structure group $G$ such that $U^*(BG)$ is torsion-free. From the localization theorem for the universal toric genus (see \cite{2}) for the standard action of the torus $T^3$ on the complex projective plane $\mathbb{C}P(2)$, theorem holds:

**Theorem 1.** A genus $L_f$ is $\mathbb{C}P^2$-multiplicative if and only if $f(x)$ solves the functional equation

$$
egin{align*}
\frac{1}{f(t_1 - t_2)f(t_1 - t_3)} + \frac{1}{f(t_2 - t_1)f(t_2 - t_3)} + \frac{1}{f(t_3 - t_1)f(t_3 - t_2)} = C, \quad C \in R.
\end{align*}
$$

In \cite{3} it was shown with the help of equation (1) that for bundles of oriented manifolds the universal $\mathbb{C}P^2$-multiplicative genus is determined by the signature of the manifold. We have $C = L_f[\mathbb{C}P(2)] = \frac{3f_1^2 - f_2}{2}$.

2. Theorem

In \cite{1} the following theorem is proposed. Its proof is given in the next section.

**Theorem 2.** Let $L_f$ be a $\mathbb{C}P^2$-multiplicative genus.

If $L_f[\mathbb{C}P(2)] \neq 0$, then $L_f$ is the two-parametric Todd genus, and

$$
f(x) = \frac{e^{\alpha x} - e^{\beta x}}{e^{\alpha x} - \beta e^{\beta x}}, \quad f_1 = - (\alpha + \beta), \quad f_2 = 2\alpha\beta + f_1^2, \quad f_3 = 4f_1f_2 - 3f_1^3.
$$

If $L_f[\mathbb{C}P(2)] = 0$, then it is a two-parametric case of general elliptic genus in the terminology of \cite{4}, and

$$
f(x) = -\frac{2\psi(x) + a^2}{\psi'(x) - a\psi(x) + b - \frac{a^2}{4}}.
$$

Here $\psi$ and $\psi'$ are Weierstrass functions of the elliptic curve with parameters $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{27}(8b^2 - 12a^3b + 3a^6)$, and discriminant $\Delta = -b^3(3b - a^3)$. The parameters $a$ and $b$ are related to the coefficients of the series $f(x)$ by $f_1 = -a$, $f_2 = 3a^2$, $f_3 = 12b - 9a^3$.

**Proposition 3.** The genus determined by $f(x)$ as in \cite{3} was first introduced in \cite{5}. As we need a name for it we propose to name it Buchstaber–Netay genus.
3. Proof

The proof of the theorem follows as a compilation of theorem \[1\] with the following three lemmas, each given with its own proof.

For convenience set \(q(x) = \frac{1}{f(x)}\). Denote \(x = t_1 - t_2, y = t_2 - t_3\). Equation \[1\] takes the form

\[
 q(x)q(x + y) + q(-x)q(y) + q(-x - y)q(-y) = C.
\]

Lemma 4. The function \(q(x) = \frac{1}{f(x)}\), where

\[
f(x) = \frac{e^{\alpha x} - e^{\beta x}}{e^{\alpha x} - e^{\beta x}}
\]
satisfies the functional equation \[4\] for \(C = \alpha^2 + \alpha \beta + \beta^2\).

Proof. The proof is a straightforward substitution, namely, equation \[4\] takes the form

\[
\frac{(ae^{\alpha x} - be^{\beta x})(ae^{\alpha(x+y)} - be^{\beta(x+y)})}{(e^{\alpha x} - e^{\beta x})(e^{\alpha(x+y)} - e^{\beta(x+y)})} + \frac{(ae^{-\alpha x} - be^{-\beta x})(ae^{\alpha y} - be^{\beta y})}{(e^{-\alpha x} - e^{-\beta x})(e^{\alpha y} - e^{\beta y})} + \frac{(ae^{-\alpha(x+y)} - be^{-\beta(x+y)})(ae^{-\alpha y} - be^{-\beta y})}{(e^{-\alpha(x+y)} - e^{-\beta(x+y)})(e^{-\alpha y} - e^{-\beta y})} = C,
\]

which after multiplication of the nominator and denominator by a relevant factor becomes

\[
\frac{(ae^{\alpha x} - be^{\beta x})(ae^{\alpha(x+y)} - be^{\beta(x+y)})}{(e^{\alpha x} - e^{\beta x})(e^{\alpha(x+y)} - e^{\beta(x+y)})} + \frac{(\beta e^{\alpha x} - \alpha e^{\beta x})(ae^{\alpha y} - be^{\beta y})}{(e^{\alpha x} - e^{\beta x})(e^{\alpha y} - e^{\beta y})} + \frac{(\beta e^{-\alpha(x+y)} - \alpha e^{-\beta(x+y)})(be^{\alpha y} - \alpha e^{\beta y})}{(e^{-\alpha(x+y)} - e^{-\beta(x+y)})(e^{-\alpha y} - e^{-\beta y})} = C.
\]

Bringing to a common factor one gets

\[
\begin{align*}
 (ae^{\alpha x} - be^{\beta x})(e^{\alpha y} - e^{\beta y})(ae^{\alpha(x+y)} - be^{\beta(x+y)}) &+ (\beta e^{\alpha x} - \alpha e^{\beta x})(ae^{\alpha y} - be^{\beta y})(e^{\alpha(x+y)} - e^{\beta(x+y)}) + \\
 + (e^{\alpha x} - e^{\beta x})(\beta e^{\alpha y} - \alpha e^{\beta y})(ae^{\alpha(x+y)} - be^{\beta(x+y)}) &+ (\beta e^{-\alpha(x+y)} - \alpha e^{-\beta(x+y)})(be^{\alpha y} - \alpha e^{\beta y})(e^{\alpha(x+y)} - e^{\beta(x+y)}) = C(e^{\alpha x} - e^{\beta x})(e^{\alpha y} - e^{\beta y})(e^{\alpha(x+y)} - e^{\beta(x+y)}).
\end{align*}
\]

Now this expression is available for term-by-term check, like at \(e^{2\alpha(x+y)}\) we have

\[
\alpha^2 + \alpha \beta + \beta^2 = C
\]

and the same for all other coefficients. \(\square\)

Lemma 5. The function \(q(x) = \frac{1}{f(x)}\), where

\[
f(x) = -\frac{2\varphi(x) + \frac{a^2}{2}}{\varphi'(x) - a\varphi(x) + b - \frac{a^2}{4}}
\]

with parameters \(g_2 = \frac{1}{4}(8b - 3a^3)a\) and \(g_3 = \frac{1}{4}(8b^2 - 12a^3b + 3a^6)\) of the Weierstrass \(\varphi\)-function satisfies the functional equation \[4\] for \(C = 0\).

Proof. We have

\[
q(x) = \frac{a}{2} - \frac{b}{2\varphi(x) + \frac{a^2}{2}} - \frac{\varphi'(x)}{2\varphi(x) + \frac{a^2}{2}}.
\]

For \(C = 0\) equation \[4\] after the substitution of \(q(x)\) takes the form (here we take into account that \(\varphi\) is an even function and \(\varphi'\) is odd)

\[
\begin{align*}
 \left(\frac{a}{2} - \frac{b}{2\varphi(x) + \frac{a^2}{2}} - \frac{\varphi'(x)}{2\varphi(x) + \frac{a^2}{2}}\right) &\left(\frac{a}{2} - \frac{b}{2\varphi(x + y) + \frac{a^2}{2}} - \frac{\varphi'(x + y)}{2\varphi(x + y) + \frac{a^2}{2}}\right) + \\
 + \left(\frac{a}{2} - \frac{b}{2\varphi(x + y) + \frac{a^2}{2}} + \frac{\varphi'(x)}{2\varphi(x) + \frac{a^2}{2}}\right) &\left(\frac{a}{2} - \frac{b}{2\varphi(y) + \frac{a^2}{2}} - \frac{\varphi'(y)}{2\varphi(y) + \frac{a^2}{2}}\right) + \\
 + \left(\frac{a}{2} - \frac{b}{2\varphi(x + y) + \frac{a^2}{2}} + \frac{\varphi'(x + y)}{2\varphi(x + y) + \frac{a^2}{2}}\right) &\left(\frac{a}{2} - \frac{b}{2\varphi(y) + \frac{a^2}{2}} + \frac{\varphi'(y)}{2\varphi(y) + \frac{a^2}{2}}\right) = 0.
\end{align*}
\]
After bringing this expression to a common denominator we obtain that it is required to prove the relation

\[
\left( \varphi(y) + \frac{a^2}{4} \right) \left( \varphi'(x) + b - a \left( \varphi(x) + \frac{a^2}{4} \right) \right) \left( \varphi'(x + y) + b - a \left( \varphi(x + y) + \frac{a^2}{4} \right) \right) - \\
- \left( \varphi(x + y) + \frac{a^2}{4} \right) \left( \varphi'(x) - b + a \left( \varphi(x) + \frac{a^2}{4} \right) \right) \left( \varphi'(y) + b - a \left( \varphi(y) + \frac{a^2}{4} \right) \right) + \\
+ \left( \varphi(x) + \frac{a^2}{4} \right) \left( \varphi'(x + y) - b + a \left( \varphi(x + y) + \frac{a^2}{4} \right) \right) \left( \varphi'(y) - b + a \left( \varphi(y) + \frac{a^2}{4} \right) \right) = 0.
\]

Consider the left part as a function of \( x \) where \( y \) is a parameter. It is a two-periodic function, it might have poles only in points comparable to \( x = 0 \) and \( x = -y \). Consider this function for \( x = 0 \). We obtain 0 at \( \frac{1}{x^7} \), while at \( \frac{1}{x^8} \) we obtain

\[
32(3a^4 - 8ab - 4g_2)\varphi(y) + 12a^6 - 64a^3b + 16a^2g_2 - 192g_3 + 64b^2,
\]

which gives 0 after substituting \( g_2 \) and \( g_3 \). At \( \frac{1}{x^7} \) we get

\[
16(3a^4 - 8ab - 4g_2)\varphi'(y)
\]

which gives 0 again. Therefore the left part has no poles in points comparable to \( x = 0 \). As the equation is invariant under substitutions \((x \rightarrow y, y \rightarrow x, a \rightarrow -a, b \rightarrow -b)\) and \( x \rightarrow y, y \rightarrow -x - y, a \rightarrow a, b \rightarrow b \), thus it has no poles comparable to \( x = -y \). Therefore the left part of the expression, being a meromorphic function without poles, must be constant. Calculation of the free term at \( x = 0 \) shows that this expression is equal to zero. □

**Lemma 6.** The functional equation (4) does not have solutions other then stated in Lemmas 4 and 5.

**Proof.** The series decomposition of (4) in \( y \) taking into account the initial conditions gives at \( y^k \) for \( k = 0 \) the equation

\[
q(x)^2 - f_1q(-x) + q'(-x) = C,
\]

for \( k = 1 \) the derivative of (5) and for \( k = 2 \) the equation

\[
6q(x)q''(x) - Kq(-x) + (3f_1^2 - 2f_2)q'(-x) - 3f_1q''(-x) + 2q'''(-x) = 0, \quad K = 3f_1^3 - 4f_1f_2 + f_3.
\]

Decomposing the equations (5) and (6) and taking into account initial conditions, we obtain from (5) at \( x^0 \) the relation \( 2C = 3f_1^2 - f_2 \). Further from \( x^k \) in (5) and \( x^{k-2} \) in (6) for \( k = 2, 3, 4, 5 \) we get coinciding relations

\[
\begin{align*}
f_4 &= 15f_1^4 - 25f_1^2f_2 + 7f_1f_3 + 4f_2^2, \\
f_5 &= 15f_1^2f_2 - 15f_2^3 + 10f_1f_3 + 5f_2f_3, \\
f_6 &= 315f_1^6 - 945f_1^4f_2 + 345f_1^3f_3 + 660f_1^2f_2^2 - 93f_1^2f_4 - 290f_1f_2f_3 - 140f_2^3 - 60f_3^3 + 18f_1f_5 + 32f_2f_4 + 20f_3^2, \\
f_7 &= 210f_1^8 - 210f_1^4f_2 - 420f_1^3f_2^2 + 105f_1^2f_3 + 420f_1^2f_2^2f_3 + 140f_1f_2f_3^2 + 35f_2f_5 + 112f_1f_3f_4 + 70f_2f_3^2 - \\
&\quad - 70f_2f_5 + 8f_1f_6 + 14f_2f_4 + 21f_3f_4.
\end{align*}
\]

At \( x^0 \) in (5) and at \( x^4 \) in (6) we get different relations:

\[
\begin{align*}
f_8 &= 8505f_1^8 - 36855f_1^6f_2 + 14805f_1^4f_3 + 48300f_1^2f_2^2 - 4599f_1^4f_4 - 29820f_1^2f_2f_3 - 19320f_1^2f_3^2 + 1134f_1^4f_5 + \\
&\quad + 6552f_1^2f_2^2 + 4095f_1^2f_3^2 + 10500f_1f_2^3 + 1120f_2^4 - 222f_2f_6 - 980f_1f_2f_5 - 1470f_1f_3f_4 - 924f_2f_4 - \\
&\quad - 115f_3f_5^2 + 33f_1f_7 + 80f_2f_6 + 140f_3f_5 + 84f_1f_6; \\
f_9 &= 53865f_1^9 - 233415f_1^7f_2 + 94500f_1^5f_3 + 304920f_1^3f_2^2 - 29862f_1^5f_4 - 188370f_1^3f_2f_3 - 121380f_1^2f_2^2 + \\
&\quad + 7497f_1^2f_5 + 41706f_1^2f_2f_3 + 25515f_1^2f_3^2 + 65730f_1f_2^3f_3 + 7000f_1^2f_6 - 1476f_2f_6 - 6300f_1f_2f_5 - \\
&\quad - 9072f_1f_3f_4 - 5796f_2f_4 - 7140f_2f_5 + 216f_1f_7 + 516f_2f_6 + 861f_3f_5 + 504f_2^2; \\
\end{align*}
\]

Comparing this expressions for \( f_8 \) taking into account the expressions for \( f_4, f_5, f_6, f_7 \) and \( C \) we obtain the relation

\[
CK^2 = 0.
\]

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Therefore we obtain either $C = 0$, which gives solution (3), or $K = 0$, which gives solution (2).

From (5) and initial conditions it follows that for given $f_1, f_2, f_3$ all $f_k$ for $k \geq 4$ are uniquely defined, thus there are no other solutions.

References

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