PREVALENCE OF GENERIC LAVER DIAMOND

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(Communicated by Mirna Džamonja)

In memory of Richard Laver, 1942-2012

Abstract. Viale (2012) introduced the notion of Generic Laver Diamond at $\kappa$—which we denote $\dot{\diamond}^{\text{Lav}}(\kappa)$—asserting the existence of a single function from $\kappa \to H_\kappa$ that behaves much like a supercompact Laver function, except with generic elementary embeddings rather than internal embeddings. Viale proved that the Proper Forcing Axiom (PFA) implies $\dot{\diamond}^{\text{Lav}}(\omega_2)$. We strengthen his theorem by weakening the hypothesis to a statement strictly weaker than PFA. We also show that the principle $\dot{\diamond}^{\text{Lav}}(\kappa)$ provides a uniform, simple construction of 2-cardinal diamonds, and prove that $\dot{\diamond}^{\text{Lav}}(\kappa)$ is quite prevalent in models of set theory; in particular:

1. $L$ satisfies $\dot{\diamond}^{\text{Lav}}(\kappa)$ whenever $\kappa$ is a successor cardinal, or when the appropriate version of Chang’s Conjecture fails.

2. For any successor cardinal $\kappa$, there is a $\kappa$-directed closed class forcing—namely, the forcing from Friedman-Holy (2011)—that forces $\dot{\diamond}^{\text{Lav}}(\kappa)$.

Prediction principles have been central topics in set theory ever since Jensen introduced the $\dot{\diamond}$ principle in the 1960s. Not only does $\dot{\diamond}$ hold in canonical inner models such as $L$, it also is frequently introduced by forcing; for example, adding a Cohen subset of $\kappa$ introduces a $\dot{\diamond}_\kappa(\kappa)$ sequence. The $\dot{\diamond}$ principle is frequently used to prove one direction of an independence result. Two-cardinal variations of $\dot{\diamond}$—i.e. versions of $\dot{\diamond}$ which guess subsets of some fixed $\wp(\kappa,\lambda)$, rather than just guessing subsets of $\kappa$—were introduced by Jech [8]. Donder-Matet [5] (with a correction by Shioya [17]) proved that mild cardinal arithmetic assumptions are enough to guarantee such 2-cardinal versions of $\dot{\diamond}$; for example, $\kappa^{<\kappa} = \kappa$ implies that $\dot{\diamond}(\kappa,\lambda)$ holds for all $\lambda \geq \kappa^+$.

Laver [13] proved that if $\kappa$ is a supercompact cardinal, then there is a function $F : \kappa \to H_\kappa$ which essentially behaves like a universal $\dot{\diamond}_\kappa(\kappa)$ sequence with respect to supercompactness measures (rather than merely with respect to the club filter, as is the case with usual $\dot{\diamond}_\kappa(\kappa)$ and its 2-cardinal variants). For this reason it is commonly called a Supercompact Laver Diamond/Function, and notably appears in the consistency proofs of the Proper Forcing Axiom, Martin’s Maximum, and an indestructibly supercompact cardinal.

Received by the editors April 5, 2013 and, in revised form, April 18, 2014.
2010 Mathematics Subject Classification. Primary 03E57, 03E55, 03E35, 03E05.
Part of this work was done while the author participated in the Thematic Program on Forcing and its Applications at the Fields Institute, which was partially supported from NSF grant DMS-1162052.

1E.g. independence of Suslin’s Hypothesis, Whitehead’s Problem, existence of non-inner automorphisms of the Calkin algebra, Naimark’s Problem, and many others.

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The concept of a Laver function has been generalized in two distinct directions; unfortunately both are referred to as “Generalized Laver Diamond” in the literature:

1. Generalizations of (supercompact) Laver Diamond to some other large cardinal notions (e.g. strong cardinals) have been developed (e.g. [11], [2], [6], and [7]).
2. Viale [19] generalized the notion of Laver Diamond so that it makes sense at successor cardinals.

To distinguish Viale’s version from the large cardinal versions mentioned above, we will refer to Viale’s version as “Generic” Laver Diamond at \( \kappa \), denoted by \( \diamondsuit_{Lav}(\kappa) \). This is a function from \( \kappa \to H_\kappa \) which behaves somewhat like a Laver function, except with generic rather than internal ultrapowers (it can also be defined without reference to forcing; see Section 1). In particular, \( \diamondsuit_{Lav}(\kappa) \) can hold when \( \kappa \) is a successor cardinal, and \( \diamondsuit_{Lav}(\kappa) \) provides a particularly elegant, uniform way to produce 2-cardinal diamond sequences; see Section 3 for such a construction.

Viale’s main result was that the Proper Forcing Axiom (PFA) implies \( \diamondsuit_{Lav}(\omega_2) \). We strengthen this theorem by weakening the hypothesis to a statement strictly weaker than PFA; moreover our proof is completely elementary and does not make use of the coding of \( H_{\omega_2} \) which appeared in Viale’s proof.

**Theorem 1.** If \( \varepsilon = \omega_2 \) and the class \( \text{GIC}_{\omega_1} \) of \( \omega_1 \)-guessing, internally club sets is stationary, then \( \diamondsuit_{Lav}(\omega_2) \) holds.

More generally: whenever \( \Gamma \subset \varphi_\kappa(V) \) is a \( \Pi_1(V) \) definable stationary class in some parameter from \( H_\kappa \) such that:
- \( \Gamma \) projects downward; and
- \( \Gamma \) satisfies the Isomorphism Property,
then there is a \( \diamondsuit_{Lav}(\Gamma) \) function.

Our proof of Theorem 1 gives an explicit construction of the \( \diamondsuit_{Lav}(\Gamma) \) function; namely, the function defined recursively in Claims 33.1 and 33.2.

We also prove—as is the case with the weaker \( \diamondsuit_\kappa \) and its 2-cardinal variants—that the principle \( \diamondsuit_{Lav}(\kappa) \) is quite prevalent in models of set theory:

**Theorem 2.** If Stationary Condensation holds at \( \kappa \) and \( \kappa \) is a successor cardinal, then \( \diamondsuit_{Lav}(\kappa) \) holds. More generally, if the appropriate version of weak Chang’s Conjecture fails at \( \kappa \), then \( \diamondsuit_{Lav}(\kappa) \) holds.

Combining this theorem with the results of Friedman-Holy [10] yields:

**Corollary 3.** If \( \kappa \) is a successor cardinal, then there is a \( \kappa \)-directed closed class forcing extension that models \( \diamondsuit_{Lav}(\kappa) \).

We also prove:

**Theorem 4.** \( L \) satisfies \( \diamondsuit^+_\text{Lav}(\kappa) \) whenever \( \kappa \) is a successor cardinal, or whenever the appropriate version of weak Chang’s Conjecture fails at \( \kappa \). By Theorem 14 \( \diamondsuit_{Lav}(\kappa) \) also holds for such \( \kappa \).

\(^2\)The hypothesis that \( \varepsilon = \omega_2 \) and \( \text{GIC}_{\omega_1} \) is a stationary class is a consequence of PFA (by Viale-Weiß [20] and Todorcevic [18]), but is strictly weaker than PFA; see Remark 33. In particular, it does not imply the existence of the Caicedo-Velickovic coding.
The paper is structured as follows: Section 1 provides the definitions and basic facts about $\Diamond_{\text{Lav}}(\kappa)$ and its variants; Section 2 proves that $\Diamond_{\text{Lav}}(\kappa)$ is equivalent to $\Diamond_{\text{Lav}}(\kappa)$; Section 3 proves Theorems 2 and 4; Section 5 proves Theorem 1; and Section 6 concludes with a question.

We will use the following notation throughout the paper:

**Definition 5.** Suppose $X$ is a set and $(X, \in \upharpoonright (X \times X))$ is extensional. Then $H_X$ denotes the transitive collapse of $X$ and $\sigma_X : H_X \rightarrow X$ denotes the inverse of the Mostowski collapsing map of $X$.

We also will use the following convention:

**Convention 6.** If $p_1, \ldots, p_n$ are each elements of some $H_\theta$, then each $p_i$ in the structure $(H_\theta, \in, p_1, \ldots, p_n)$ will be understood to be the natural interpretation of a constant symbol $\check{p}_i$.

The author would like to thank the anonymous referee for many helpful suggestions, especially regarding the structure of the proof of Theorem 14.

1. $\Diamond_{\text{Lav}}(\Gamma)$, $\Diamond_{\text{Lav}}^+(\Gamma)$, and $\Diamond_{\text{Lav}}^-(\Gamma)$

We recall the definition of Generic Laver Diamond from Viale [19]: a function $F : \kappa \rightarrow H_\kappa$ is a Generic Laver function iff for every set $b$ and for every sufficiently large regular $\theta$, there are stationarily many $M \in \wp_\kappa(H_\theta)$ such that $b \in M$ and the Mostowski collapse of $M$ sends $b$ to $F(M \cap \kappa)$. The role of the class $\Gamma$ in the following definition is to allow for refinements; similarly to the way that $\Diamond(S)$ refines $\Diamond_\kappa$ (where $S$ is a stationary subset of $\kappa$).

**Definition 7.** Suppose $\kappa$ is a regular uncountable cardinal and $F : \kappa \rightarrow V$ is a function. If $b$ is a set and $\theta$ is a regular uncountable cardinal, define $G_{b, \theta}^\Gamma$ to be the set of those $M$ such that:

(a) $b \in M \prec (H_\theta, \in)$,
(b) $M \cap \kappa \in \kappa$,
(c) $\sigma_{\upharpoonright M}^{-1}(b) = F(M \cap \kappa)$ (recall that $\sigma_M : H_M \rightarrow M$ denotes the inverse of the Mostowski collapse of $M$).

For any class $\Gamma$, we say that $F$ is a $\Diamond_{\text{Lav}}(\Gamma)$ function iff for every set $b$ and every sufficiently large regular $\theta$, the following set is stationary:

$$\Gamma \cap G_{b, \theta}^\Gamma.$$  

We say that $\Diamond_{\text{Lav}}(\Gamma)$ holds iff there is a $\Diamond_{\text{Lav}}(\Gamma)$ function.

We also define variants of Generic Laver Diamond which are analogous to what Kunen [12] calls $\Diamond^-$ and $\Diamond^+$:

**Definition 8.** Suppose $\kappa$ is a regular uncountable cardinal and $F : \kappa \rightarrow V$ is a function. For each set $b$ and regular uncountable $\theta$, the set $G_{b, \theta}^\Gamma$ is defined the same way that $G_{b, \theta}^\Gamma$ was defined in Definition 7 except requirement (c) is replaced with the following requirement:

(c) $\sigma_{\upharpoonright M}^{-1}(b) \in F(M \cap \kappa)$.

For any class $\Gamma$, we say that $F$ is a $\Diamond_{\text{Lav}}^-(\Gamma)$ (resp. $\Diamond_{\text{Lav}}^+(\Gamma)$) function iff

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3E.g. if $X \prec (H_\theta, \in)$ for some $\theta$.
4Viale’s formulation was actually more similar to the characterization in Lemma 12 below.
We say:

Lemma 12. A function $\tilde{\varphi} (\Gamma)$ (resp. $\tilde{\varphi} (\Gamma)$) holds if there exists a $\tilde{\varphi} (\Gamma)$ (resp. $\tilde{\varphi} (\Gamma)$) function.

Remark 9. In the special case where $\Gamma$ is the natural class $\varphi (\kappa)(V)$, we will just write $\tilde{\varphi}(\kappa)$ to mean $\tilde{\varphi}(\Gamma)$, and similarly for $\tilde{\varphi}(\kappa)$ and $\tilde{\varphi}(\kappa)$.

There are obvious local variations of Definitions 7 and 8, but we will not make use of such local variations in this paper.

Remark 10. If $\Gamma \subseteq \varphi (\kappa)(V)$ and $F$ is a $\tilde{\varphi}(\Gamma)$ function, then so is

$$F \upharpoonright \{ \alpha < \kappa \mid F(\alpha) \in H_{\kappa} \}.$$  

Proof. Let $b$ be any set. Let $\theta$ be a sufficiently large regular cardinal so that $\Gamma \cap G_{\theta}^{b,\theta}$ is stationary. Since $\Gamma \subseteq \varphi (\kappa)(V)$ then $H_M$ is an element of $H_{\kappa}$ for every $M \in \Gamma \cap G_{\theta}^{b,\theta}$; so $F(M \cap \kappa) = \sigma_M^{-1}(b) \in H_M \in H_{\kappa}$ for any such $M$.

Remark 11. If $\Gamma \subseteq \varphi (\kappa)(V)$ and $\tilde{\varphi}(\Gamma)$ holds, then so does $\tilde{\varphi}(\Gamma')$. This is analogous to the trivial fact that $\tilde{\varphi}(S)$ implies $\tilde{\varphi}(S')$ whenever $S \subseteq S'$ are stationary subsets of $\kappa$.

Definition 7 can also be rephrased in terms of generic elementary embeddings, a characterization which more closely resembles the definition of supercompact Laver functions. This is the version which appeared in Viale [19] (albeit in the presence of Woodin cardinals). While the following lemma is technically a second-order scheme, it is also possible to obtain a first-order definition involving generic embeddings.

Lemma 12. A function $F : \kappa \rightarrow V$ is a $\tilde{\varphi}(\Gamma)$ function if and only if for every $b$ and every sufficiently large regular $\theta$, there is a generic elementary embedding $j : V \rightarrow N$ with critical point $\kappa$ such that:

1. $H_{\theta}^V$ is an element of the (transitivized) wellfounded part of $N$;
2. $j[H_{\theta}^\Gamma] \in N$;
3. $j(F)(\kappa) = b$; and
4. $N \models j[H_{\theta}^\Gamma] \in j(\Gamma)$.

Proof. Assume $F : \kappa \rightarrow H_{\kappa}^\Gamma$ is a $\tilde{\varphi}(\Gamma)$ function. Fix some $b$ and let $\theta$ be sufficiently large so that $S := \Gamma \cap G_{\theta}^{b,\theta}$ is stationary (see Definition 7 for the meaning of $G_{\theta}^{b,\theta}$). Let $I$ be the restriction of the non-stationary ideal to the stationary set $S$. Let $U$ be $(V, \varphi(S)/I)$-generic and $j : V \rightarrow \mathcal{U} N_{\mathcal{U}}$ the generic ultrapower embedding; here $N_{\mathcal{U}}$ may not be wellfounded, but standard applications of Los' Theorem imply that $H_{\theta}^V$ is an element of its (transitivized) wellfounded part and moreover:

- $j[H_{\theta}^\Gamma] = [\text{id}]_\mathcal{U} \in j(\Gamma)$ (since $S \subseteq \Gamma$);
- $j(F)(\kappa) = b$ (since $S \subseteq G_{\theta}^{b,\theta}$).

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5Namely, one could weaken $\tilde{\varphi}(\Gamma)$ by restricting attention to only those $b$ up to some fixed cardinality.

6Namely, in the statement of Lemma 12 if one replaces “there is a generic elementary embedding $j : V \rightarrow N^\kappa$” with “there is a normal ideal $I$ such that $\vert \mathcal{U} \cap \subseteq \kappa$ the ultrapower embedding $j : H_{\mathcal{U}}^\kappa \rightarrow \mathcal{U} \text{ult} (H_{\mathcal{U}}^\kappa, \mathcal{U} \cdot \mathcal{U})$ has critical point $\kappa$ and has the following properties . . . ”, then the resulting statement is first order, and the proof is similar to the proof given here.
Conversely, suppose $F$ is a function from $\kappa \to H_\kappa$ and for every $b$ and sufficiently large $\theta$ there is a generic embedding $j = j_{b,H_\theta} : V \to N$ satisfying the requirements listed in the statement of the lemma. Let $\A = (H_\theta, \in, \{b\}, \ldots)$ be arbitrary and let $M' := j[H_\theta^V]$. Then $\sigma_{M'} = j | H_\theta^V$ and $N$ models the following facts about $M'$:

- $M' \prec j(\A)$,
- $j(F)(M' \cap j(\kappa)) = b = \sigma_{M'}^{-1}(j(b))$,
- $M' \in j(\Gamma)$.

By elementarity of $j$, $V$ believes there is an $M$ such that $M \prec \A$, $F(M \cap \kappa) = \sigma_M^{-1}(b)$, and $M \in \Gamma$. \hfill $\Box$

We will use the following definition; many natural classes (e.g. classes of internally approachable structures, etc.) have this property:

**Definition 13.** A class $\Gamma$ projects downward iff whenever $X \in \Gamma$ and $\theta \leq \sup(X \cap \text{ORD})$ is a regular uncountable cardinal, then $X \cap H_\theta \in \Gamma$.

2. **Equivalence of $\Diamond_{\text{Lav}}(\Gamma)$ with $\Diamond^-_{\text{Lav}}(\Gamma)$**

Theorem 14 below—which is a generalization of Kunen’s proof that $\Diamond$ is equivalent to $\Diamond^-$ (cf. Theorem 7.14 of [12])—is crucial to the proof of Theorem 2.

**Theorem 14.** Let $\Gamma$ be a downward-projecting class (in the sense of Definition 13) and $\kappa$ a regular uncountable cardinal, and suppose that $M \cap \kappa \in \kappa$ for every $M \in \Gamma$. Then

$$\Diamond^-_{\text{Lav}}(\Gamma) \iff \Diamond_{\text{Lav}}(\Gamma).$$

**Proof.** The $\Leftarrow$ direction is trivial, since if $F : \kappa \to H_\kappa$ is a $\Diamond_{\text{Lav}}(\Gamma)$ function, then easily $\alpha \mapsto \{ F(\alpha) \}$ is a $\Diamond^-_{\text{Lav}}(\Gamma)$ function. Now we prove the $\Rightarrow$ direction. Suppose $F : \kappa \to V$ witnesses $\Diamond^-_{\text{Lav}}(\Gamma)$. Let $\{z_{\alpha,i} \mid i < \alpha\}$ be an enumeration of $F(\alpha)$ for each $\alpha < \kappa$. For each $i < \kappa$ define a function $F_i : \kappa \to H_\kappa$ as follows: if $z_{\alpha,i}$ is a function with $i$ in its domain, then let $F_i(\alpha) := z_{\alpha,i}(i)$; otherwise $F_i(\alpha) := \emptyset$. We finish the proof by showing:

**Claim 14.1.** There is some $i < \kappa$ such that $F_i$ is a $\Diamond_{\text{Lav}}(\Gamma)$ function.

**Proof.** Suppose not; so for every $i < \kappa$ there is some $b_i$ and some algebra $\A_i = (H_{b_i}, \in, b_i)$ such that:

$$\forall M \in \Gamma \ M < \A_i \implies \sigma_M^{-1}(b_i) \not\in F_i(M \cap \kappa).$$

Let $B := \langle b_i \mid i < \kappa \rangle$ and fix some regular $\Omega$ such that $B$ and $\langle \A_i, b_i \mid i < \kappa \rangle$ are elements of $H_\Omega$. Let $T$ denote the set of $M \in \Gamma$ such that $M \prec \B := (H_\Omega, \in, B, \A, \bar{b})$ and $\sigma_M^{-1}(B) \in F(M \cap \kappa)$; $T$ is stationary because $F$ is a $\Diamond^-_{\text{Lav}}(\Gamma)$ function. Now $\sigma_M^{-1}(B) \in F(M \cap \kappa) = \{ z_{M \cap \kappa, i} \mid i < M \cap \kappa \}$, so there is some $i_M < M \cap \kappa$ such that $\sigma_M^{-1}(B) = z_{M \cap \kappa, i_M}$. Since $i_M \in M$ for all $M \in T$ then by Fodor’s Lemma there is a stationary $T' \subseteq T$ and some fixed $\hat{i}$ such that $i_M = \hat{i}$ for all $M \in T'$. Fix an $M \in T'$. First observe that

$$\sigma_M^{-1}(B) = \sigma_M^{-1}(\langle b_i \mid i < \kappa \rangle) = \langle \sigma_M^{-1}(b_i) \mid i \in M \cap \kappa \rangle$$

so in particular $\sigma_M^{-1}(B)$ is a function with $\hat{i}$ in its domain. Now

$$\sigma_M^{-1}(B) = z_{M \cap \kappa, i_M} = z_{M \cap \kappa, \hat{i}}$$
and so in particular
\[(4) \quad \sigma^{-1}_M(B)(\hat{i}) = z_{M \cap \kappa, i}(\hat{i}).\]

So (1) and (2) imply:
\[(5) \quad \sigma^{-1}_M(b_i) = z_{M \cap \kappa, i}(\hat{i}).\]

Now by the definition of \(F_i\):
\[(6) \quad F_i(M \cap \kappa) = z_{M \cap \kappa, i}(\hat{i}).\]

Finally (3) and (5) imply:
\[(7) \quad F_i(M \cap \kappa) = \sigma^{-1}_M(b_i).\]

On the other hand, since \(M \prec \mathfrak{B}\) and \(\hat{i} \in M\) then \(M_\hat{i} := M \cap H_{\theta_i} \prec \mathfrak{B}_i\); moreover \(M_\hat{i} \in \Gamma\) by our assumption that \(\Gamma\) projects downward. So (1) implies:
\[(8) \quad F_i(M_\hat{i} \cap \kappa) \neq \sigma^{-1}_{M_\hat{i}}(b_i).\]

But \(M_\hat{i} \cap \kappa = M \cap \kappa\) and \(\sigma^{-1}_M(b_i) = \sigma^{-1}_{M_\hat{i}}(b_i)\), so (8) implies:
\[(9) \quad F_i(M \cap \kappa) \neq \sigma^{-1}_M(b_i)\]

which contradicts (7) and completes the proof of the claim. \(\square\)

3. A simple proof of 2-cardinal Diamond

Jech [8] introduced a 2-cardinal Diamond principle, which is a guessing principle for subsets of \(\varphi_\kappa(\lambda)\). Precisely, \(\lozenge(\kappa, \lambda)\) asserts the existence of a function \(\langle A_z \mid z \in \varphi_\kappa(\lambda) \rangle\) such that for every \(A \subseteq \lambda\), the following set is stationary in \(\varphi_\kappa(\lambda)\):

\[\{z \in \varphi_\kappa(\lambda) \mid A \cap z = A_z\}.\]

Viale’s principle \(\lozenge_{Lav}(\kappa)\) easily implies that \(\kappa^{<\kappa} = \kappa\), which in turn—using a theorem of Donder-Matet [5] and a correction by Shioya [17]—implies that \(\lozenge(\kappa, \lambda)\) holds for all \(\lambda > \kappa\).\(^7\) However, there is an especially simple, direct proof of the implication

\[\lozenge_{Lav}(\kappa) \implies \forall \lambda > \kappa \lozenge(\kappa, \lambda)\]

which does not require going through the theorem of Donder-Matet and Shioya (though of course their proof is much more general, as it assumes only that \(2^{<\kappa} < \lambda\)). Suppose \(F : \kappa \to H_\kappa\) witnesses \(\lozenge_{Lav}(\kappa)\), and let \(\lambda \geq \kappa\) be any cardinal. For each (extensional) \(M \in P_\kappa(\lambda)\), recall that \(\sigma_M\) denotes the inverse of the Mostowski collapsing map of \(M\), and set

\[A_M := \sigma_M[F(M \cap \kappa)].\]

Then \(\langle A_M \mid M \in P_\kappa(\lambda) \rangle\) is a \(\Diamond_{\kappa, \lambda}\) sequence: let \(A \subseteq \lambda\). By Laverness of \(F\), there is a stationary set \(S'\) of \(M' \prec (H_{2\lambda})^+, \in, \{A\}\) such that \(\sigma^{-1}_{M'}(A) = F(M' \cap \kappa)\). Then for any \(M' \in S'\), setting \(M := M' \cap \lambda\) we have:

\[A \cap M = A \cap (M' \cap H_\lambda) = \sigma_{M'}(F(M' \cap \kappa)) \cap (M' \cap H_\lambda) = \sigma_{M'}[F(M' \cap \kappa)] = \sigma_M[F(M \cap \kappa)] = A_M.\]

\(^7\)Donder-Matet [5], with the correction by Shioya [17], proved that \(\lozenge(\kappa, \lambda)\) holds whenever \(2^{<\kappa} < \lambda\).
Thus, setting
\[ S := \{ M' \cap \lambda \mid M' \in S' \} \]
we have that \( A \cap M = A_M \) for every \( M \in S \).

4. \( \diamondsuit_{\text{Lav}}(\kappa) \), Condensation, and weak Chang’s Conjecture

We will describe a natural attempt to define a \( \diamondsuit_{\text{Lav}}(\kappa) \) function, which often works in the presence of Condensation and/or the appropriate failure of Chang’s Conjecture. For regular uncountable cardinals \( \kappa < \theta \), the Chang’s Conjecture \( (\theta, \kappa) \rightarrow (\kappa, \kappa) \) means that for every first order structure \( \mathfrak{A} \) on \( \theta \) (in a countable language) there is an \( M \prec \mathfrak{A} \) such that \( |M| = \kappa \) and \( |M \cap \kappa| < \kappa \); this is equivalent to saying that
\[ \{ M \subset \theta \mid |M| = \kappa \text{ and } |M \cap \kappa| < \kappa \} \]
is a (weakly) stationary set. Weak Chang’s Conjecture holds at \( \kappa, \theta \)—abbreviated \( \text{wCC}(\kappa, \theta) \)—iff for every first-order structure \( \mathfrak{A} = (\theta, \in, \ldots) \) in a countable language, there are stationarily many \( \alpha < \kappa \) such that
\[ \sup \{ \text{otp}(X \cap \theta) \mid X \prec \mathfrak{A} \text{ and } X \cap \kappa = \alpha \} \geq \alpha^+. \]

It is easy to see that decreasing the parameter \( \theta \) in \( \text{wCC}(\kappa, \theta) \) increases the strength; i.e. \( \text{wCC}(\kappa, \theta) \implies \text{wCC}(\kappa, \theta') \) whenever \( \theta \leq \theta' \). Thus the strongest is when \( \theta = \kappa^+ \); the principle \( \text{wCC}(\kappa, \kappa^+) \) is the well-known “weak Chang’s Conjecture at \( \kappa \)”, denoted \( \text{wCC}(\kappa) \) in Definition 1.6 of Donder-Levinski [4].

Remark 15. Under \( V = L \), a cardinal \( \kappa \) is ineffable iff \( \text{wCC}(\kappa) \) holds. See Corollary 1.13 of Donder-Levinski [4].

Remark 16. Standard techniques (e.g. as in Foreman-Magidor [2]) enable reformulation of, say, \( \text{wCC}(\kappa, \theta) \) which only refers to a single structure. For example, fixing a wellorder \( \Delta \) of \( H(2^\theta)^+ \), the principle \( \text{wCC}(\kappa, \theta) \) is equivalent to saying there are stationarily many \( \alpha < \kappa \) such that
\[ \sup \{ \text{otp}(X \cap \theta) \mid X \prec (H(2^\theta)^+, \in, \Delta) \text{ and } X \cap \kappa = \alpha \} \geq \alpha^+. \]

For a regular uncountable \( \kappa \) and a class \( \Gamma \subset \wp_\kappa(V) \), we say that \( \Gamma \) is \( \text{ORD stationary} \) (resp. \( \Gamma \) is \( \text{ORD club} \)) if \( \Gamma \cap \wp_\kappa(H_\theta) \) is stationary (resp. contains a club) for all sufficiently large \( \theta \). Note that \( \diamondsuit_{\text{Lav}}(\Gamma) \) trivially implies that \( \Gamma \) is \( \text{ORD stationary} \).

Definition 17. Let \( \Gamma \subseteq \wp_\kappa(V) \). We say that \( \text{wCC}(\Gamma) \) holds iff
(a) \( \Gamma \) is \( \text{ORD stationary} \); and
(b) there are stationarily many \( \alpha < \kappa \) such that
\[ \sup \{ \text{otp}(X \cap \text{ORD}) \mid X \in \Gamma \land X \prec_{\Sigma_1} (V, \in) \land X \cap \kappa = \alpha \} \geq \alpha^+. \]
The principle \( \text{wCC}^*(\Gamma) \) is defined similarly, except (11) is replaced by:
\[ \bigcup \{ H_X \mid X \in \Gamma \land X \prec_{\Sigma_1} (V, \in) \land X \cap \kappa = \alpha \} \bigg| \geq \alpha^+. \]

Observation 18. If \( \Gamma \subseteq \wp_\kappa(V) \), the principle \( \text{wCC}(\Gamma) \) implies the principle \( \text{wCC}^*(\Gamma) \), since
\[ \{ \text{otp}(X \cap \text{ORD}) \mid X \in \Gamma \text{ and } X \cap \kappa = \alpha \} = \{ \text{height}(H_{X \cap \text{ORD}}) \mid X \in \Gamma \text{ and } X \cap \kappa = \alpha \}. \]
**Definition 19.** Suppose \( \kappa \) is regular and uncountable and that \( \Gamma \subseteq \{ X \mid X \cap \kappa \in \kappa \} \).

For each \( \alpha < \kappa \) define:
\[
A^\Gamma_\alpha := \bigcup \{ H_M \mid M \in \Gamma \text{ and } M \cap \kappa = \alpha \}.
\]

Define the map \( F^\Gamma \) with domain \( \kappa \) by \( \alpha \mapsto A^\Gamma_\alpha \).

**Lemma 20.** Suppose \( \Gamma \) is a \( < \text{ORD} \)-stationary \( \text{(resp. club)} \) subclass of \( \varphi_\kappa(V) \), and let \( F^\Gamma \) be as in Definition 19. Then:

1. \( F^\Gamma \) satisfies Requirement \( \text{(A)} \) in Definition 18 of a \( \Diamond^{\text{Lav}}_\Gamma(\Gamma) \) function.
2. There is some \( \theta \) such that \( F^\Gamma = F^\Gamma \cap \varphi_\kappa(H_\theta) \).
3. Suppose \( \text{wCC}^\ast(\Gamma) \) fails. Then \( F^\Gamma \) also satisfies requirement \( \text{(B)} \) of Definition 18 and thus \( F^\Gamma \) is a \( \Diamond^{\text{Lav}}_\Gamma(\Gamma) \) \( \text{(resp. } \Diamond^{\text{Lav}}_\Gamma(\Gamma) \text{)} \) function.

**Proof.** Consider any set \( b \) and any structure \( \mathfrak{A} = (H_\theta, \in, \{ b \}, \ldots) \) in a countable language. Let \( S := \{ M \in \Gamma \mid M \prec \mathfrak{A} \} \). Then for every \( M \in S \)
\[
\sigma_M^{-1}(b) \in H_M \subseteq A^\Gamma_{M \cap \kappa} = F^\Gamma(M \cap \kappa).
\]

Part 2 just follows from the class Pigeonhole Principle and the definition of \( A^\Gamma_\alpha \) in (13).

Finally, suppose \( \text{wCC}^\ast(\Gamma) \) fails. Then \( |F^\Gamma(\alpha)| < \alpha^+ \) for almost every \( \alpha < \kappa \).

Combined with item 1 of the current lemma, this implies that \( F^\Gamma \) is a \( \Diamond^{\text{Lav}}_\Gamma(\Gamma) \) function (in the case that \( \Gamma \) was \( < \text{ORD} \)-stationary) or a \( \Diamond^{\text{Lav}}_\Gamma(\Gamma) \) function (in the case that \( \Gamma \) was \( < \text{ORD} \)-club).

**Corollary 21.** Suppose \( \kappa \) is regular uncountable and \( \Gamma \) is a \( < \text{ORD} \)-stationary \( \text{(resp. } \text{club}) \) subclass of \( \varphi_\kappa(V) \) such that \( \text{wCC}^\ast(\Gamma) \) fails. Then \( \Diamond^{\text{Lav}}_\Gamma(\Gamma) \) \( \text{(resp. } \Diamond^{\text{Lav}}_\Gamma(\Gamma) \text{)} \) holds.

**Proof.** This follows immediately from Lemma 20 and Theorem 14.

The following remark shows that the converse of Corollary 21 is false (however, Question 34 asks whether a natural variation of the converse must hold).

**Remark 22.** The converse of Corollary 21 is false; i.e. it is possible for \( \Diamond^{\text{Lav}}_\Gamma(\Gamma) \) and \( \text{wCC}(\Gamma) \) to simultaneously hold. Suppose \( \kappa \) is a supercompact cardinal, \( \Gamma = \varphi_\kappa(V) \), and \( F : \kappa \to V_\kappa \) is a (classical) supercompact Laver function. Then \( \text{wCC}(\Gamma) \) holds.

It is also possible to obtain a counterexample to the converse of Corollary 27 where \( \kappa \) is a successor cardinal. Suppose \( \kappa \) is supercompact and \( F : \kappa \to V_\kappa \) is a (classical) Laver function; suppose also that \( \kappa \) is almost huge. Let \( \mathbb{P} \) be any \( \kappa \)-cc forcing which is a subset of \( V_\kappa \) and turns \( \kappa \) into a successor cardinal, and let \( G \) be \( (V, \mathbb{P}) \)-generic. It is easy to see that the function \( \tilde{F} : \kappa \to H_\kappa^{V[G]} \) defined by
\[
\alpha \mapsto \left( F(\alpha) \right)_{G \cap V_\kappa}
\]
is a \( \Diamond^{\text{Lav}}(\kappa) \) function in \( V[G] \). But the almost hugeness of \( \kappa \) in \( V \) implies that, in \( V[G] \), the class \( \varphi_\kappa(V[G]) \) has the wCC property.

---

8Note \( S \) is stationary or contains a club, depending on whether we assume \( \Gamma \) is \( < \text{ORD} \) stationary or \( < \text{ORD} \) club.

9In fact \( \tilde{F} \) is a \( \Diamond^{\text{Lav}}(\Gamma) \) function for any class \( \Gamma \) which has the property that for every \( \theta \) there is a (possibly illfounded) generic embedding \( i : V[G] \to W \) such that \( i[H_\theta^{V[G]}] \in i(\Gamma) \).
Abstract Condensation principles have been extensively studied, for example, by Law [14], Woodin [21], and Friedman-Holy [10]. Friedman and Holy considered several versions of Condensation, and proved that Stationary Condensation (even stronger versions called Local Club Condensation) are consistent with $\kappa$ being a very large cardinal. This is to be contrasted with the severe restraints that Club Condensation (as in [21]) place on the large cardinal properties of $\kappa$.

**Definition 23.** Suppose $\kappa$ is a regular uncountable cardinal. Stationary (resp. Club) Condensation holds at $\kappa$ iff there exists an $\in$-increasing and $\subseteq$-continuous sequence $\langle M_\eta \mid \eta < \kappa \rangle$ of transitive sets such that, letting

$$\Gamma_{\vec{M}} := \{ X \in \wp_\kappa(V) \mid (\exists \eta < \kappa)(H_X = M_\eta) \},$$

then for every regular $\theta \geq \kappa$, the set $\Gamma_{\vec{M}} \cap \wp_\kappa(H_\theta)$ is stationary (resp. Club).

They also proved:

**Theorem 24** (Friedman-Holy [10]). Suppose $\kappa$ is regular. Then there is a $\kappa$-directed closed class forcing extension which satisfies ZFC and Stationary Condensation at $\kappa$ (and all larger cardinals).

The forcing first forces GCH above $\kappa$ and then performs a reverse Easton iteration of adding Cohen subsets of cardinal successors above $\kappa$.

**Remark 25.** All of the results of this section actually hold using a weaker, non-linear form of condensation. In particular, for the Stationary Condensation theorems about a successor cardinal $\kappa$, all we need is some stationary $\Gamma \subseteq \wp_\kappa(V)$ such that for most $\alpha < \kappa$, the map

$$s^\Gamma_\alpha : \{ X \in \Gamma \mid X \cap \kappa = \alpha \} \to H_\kappa$$

defined by

$$X \mapsto H_X$$

is at most $|\alpha|$-to-one.

**Lemma 26.** Suppose $\vec{M} = \langle M_\eta \mid \eta < \kappa \rangle$ is an $\in$-increasing, $\subseteq$-continuous sequence of (transitive) sets. Let $T \subseteq \Gamma_{\vec{M}}$, where $\Gamma_{\vec{M}}$ is defined as in (14). Then:

$$wCC(T) \iff wCC^*(T).$$

**Proof.** The $\Leftarrow$ direction of (15) is obvious, by Observation 18. For the $\Rightarrow$ direction we show the contrapositive: assume $wCC^*(T)$ holds, and consider a typical $\alpha < \kappa$ which witnesses this fact. Since $\vec{M}$ is $\in$-increasing then the map $\eta \mapsto \text{height}(M_\eta)$ is one-to-one; it follows that for any pair $X, Y \in T$:

$$\text{otp}(X \cap \theta) = \text{otp}(Y \cap \theta) \implies H_X = H_Y.$$

Thus since the set

$$\{ H_X \mid X \in T \text{ and } X \cap \kappa = \alpha \}$$

has cardinality $\geq \alpha^+$, then so does the set

$$\{ \text{height}(H_X) \mid X \in T \text{ and } X \cap \kappa = \alpha \}$$

which means that $\alpha$ witnesses that $wCC(T)$ holds. $\square$
Corollary 27. Assume Stationary Condensation (resp. Club Condensation) holds at \( \kappa \), as witnessed by some \( \vec{M} = (M_\eta \mid \eta < \kappa) \). Let \( \Gamma \subseteq \wp_\kappa(V) \) be the class of structures that condense to \( \vec{M} \). Then:

\[ \neg wCC(\Gamma) \implies \diamond_{Lav}(\Gamma) \text{ (resp. } \diamond_{Lav}^+(\Gamma)) \]

Proof. This follows directly from Corollary 21 and Lemma 26.

Lemma 28. If \( \kappa \) is a successor cardinal and Stationary Condensation holds at \( \kappa \) as witnessed by some \( \vec{M} = (M_\eta \mid \eta < \kappa) \), then \( wCC^+(\vec{M}) \) fails.

Proof. Say \( \kappa = \mu^+ \). Let \( \alpha \in (\mu, \mu^+) \); then there is some \( \eta_\alpha < \kappa \) such that \( M_{\eta_\alpha} \models \alpha \notin \text{CARD} \).

If \( X \not\prec_\Sigma_1 (V, \in) \) and \( \alpha = X \cap \kappa \), then \( H_X \models \alpha \in \text{CARD} \). It follows that

\[ R_\alpha := \{ H_X \mid X \in \Gamma_{\vec{M}} \text{ and } X \not\prec_\Sigma_1 (V, \in) \text{ and } X \cap \kappa = \alpha \} \subset M_{\eta_\alpha} \]

and so

\[ |R_\alpha| \leq |M_{\eta_\alpha}| < \kappa = \alpha^+. \]

Corollary 29. \( L \) satisfies \( \diamond_{Lav}^+(\kappa) \) whenever \( \kappa \) is a successor cardinal.

Proof. This follows from Lemma 28 and Corollary 27; note that \( L \) satisfies Club Condensation.

5. Strengthening and Simplification of Main Theorem from Viale

The key ingredient of Viale’s [19] proof of that \( PFA \) implies \( \diamond_{Lav}(\omega_2) \) is his “Isomorphism Theorem” about a particular subclass of \( \wp_{\omega_2}(V) \) which was shown by Viale-Weiβ [20] to be stationary in \( \wp_{\omega_2}(H_\theta) \) for all \( \theta \) (under the assumption of \( PFA \)). This subclass of \( \wp_{\omega_2}(V) \) is called the class of \( \omega_1 \)-internally club, \( \omega_1 \)-guessing models, and denoted \( \text{GIC}_{\omega_1} \). The stationarity of this class is responsible for much of the consistency strength and many of the consequences of \( PFA \), and is widely conjectured to be equiconsistent with \( PFA \) [14]. We will not need to define \( \text{GIC}_{\omega_1} \), but only use a few of its key properties.

For transitive \( ZF^- \) models \( H \) and \( H' \), we say that \( H \) is a hereditary initial segment of \( H' \) iff either \( H = H' \) or \( H = (H_\lambda)^{H'} \) for some \( \lambda \in \text{CARD}^{H'} \).

Definition 30. A class \( \Gamma \) has the \( \kappa \)-Isomorphism Property iff whenever \( X, X' \) are elements of \( \Gamma \) and \( X \cap \kappa = X' \cap \kappa \), then one of \( H_X, H_{X'} \) is a hereditary initial segment of the other.

The class \( \text{GIC}_{\omega_1} \) easily projects downward (in the sense of Definition 13) [12] and Viale proved [13].

Theorem 31 (Viale). Assume \( H \) and \( H' \) are transitive \( ZF^- \) models such that \( H \cap \text{GIC}_{\omega_1} = H' \cap \text{GIC}_{\omega_1} \), and \( H, H' \) are both in \( \text{GIC}_{\omega_1} \). Then one of \( H, H' \) is a hereditary initial segment of the other.

\[ \text{To see this: let } f : \mu \to \alpha \text{ be surjective. Let } X \not\prec (H_\kappa, \in, \{ f \}) \text{ with } \alpha \subset X; \text{ then } X = H_X = M_\eta \text{ for some } \eta, \text{ and sees that } \alpha \text{ is not a cardinal.} \]

\[ \text{Even without the “internally club” part.} \]

\[ \text{See Lemma 10 of [3].} \]

\[ \text{A simplified and elementary proof of the Isomorphism Theorem for } \text{GIC}_{\omega_1} \text{ can be found in Section 2.3 of Cox-Viale [3].} \]
Corollary 32 (Viale). Suppose $c = \omega_2$ and $p : \omega_2 \leftrightarrow H_{\omega_1}$ is a bijection. Then

$$\Gamma := \bigcup_{\theta \geq \omega_2} \{ X \in \text{GIC}_{\omega_1} \mid X < (H_{\theta}, \in, p) \}$$

satisfies the $\omega_2$-Isomorphism Property.

Proof. This follows immediately from Theorem 31 since $H_X \cap H_{\omega_1} = p[X \cap \omega_2] = p[X' \cap \omega_2] = H_X' \cap H_{\omega_1}$. \hfill $\square$

Note also that the class $\Gamma$ from Corollary 32 is $\Pi_1(V)$ definable from the parameter $p \in H_{\omega_3}$. This parameter $p$ itself—i.e. this wellorder of $H_{\omega_2}$ in ordertype $\omega_2$—is not assumed to be definable in any way for our proof below of Theorem 1. Viale’s original construction of a $\diamondsuit_{\text{Lav}}(\omega_2)$ function made use of a definable wellorder of $H_{\omega_2}$ that exists under PFA, as proved by Caicedo-Veličković [1]. We show that this coding mechanism turns out to be unnecessary, and we also provide a simplified, direct construction of the Laver function.

Remark 33. The conjunction of $c = \omega_2$ and stationarity of GIC$_{\omega_1}$ is strictly weaker than PFA. It follows from PFA by Todorcević [18] and Viale-Weiß [20]; but it does not imply PFA as shown by two different constructions:

1. The author has shown that the model obtained by forcing with Neeman’s [15] pure side condition poset using models of 2 types below a supercompact cardinal produces a model of $c = \omega_2$ plus stationarity of GIC$_{\omega_1}$ which is not a model of PFA.

2. Menachem Magidor has shown that adding a Cohen real over an arbitrary model of PFA preserves the stationarity of GIC$_{\omega_1}$; and by Shelah [16] this forcing extension does not even model Martin’s Axiom.

5.1. Proof of Theorem 1. Suppose $\Gamma$ is a stationary subclass of $\varphi_\kappa(V)$ satisfying the assumptions of Theorem 1 i.e. it projects downward, it satisfies the $\kappa$-Isomorphism Property, and it is $\Pi_1(V)$ definable from some parameter $p$, where $p \in H_{\kappa^+}$. Given a function $F : \kappa \rightarrow H_\kappa$, let us say that a set $b$ is a witness to non-$\Gamma$-Laverness of $F$ iff there is some algebra $\mathcal{A}_b = (H_{\text{trcl}(b)^{+}_\kappa}, \in, \{p, b\}, \ldots)$ such that $\sigma^M_M(b) \neq F(M \cap \kappa)$ for every $M \in \Gamma \cap \varphi_\kappa(H_\theta)$ such that $M < \mathcal{A}_b$. We say that a regular cardinal $\theta$ is the least cardinal witnessing non-$\Gamma$-Laverness of $F$ iff $\theta$ is the least regular cardinal such that there is a $b \in H_\theta$ witnessing the non-$\Gamma$-Laverness of $F$.

For any $\alpha < \kappa$ and any partial $g : \alpha \rightarrow H_\kappa$, let $W^\Gamma_g$ be the set of transitive $ZF^-$ models $W$ such that:

- there is an elementary $\sigma : W \rightarrow \Sigma_1 V$ with $\alpha = \text{crit}(\sigma)$ and $\sigma(\alpha) = \kappa$;
- $\text{range}(\sigma) \in \Gamma$ and $p \in \text{range}(\sigma)$;
- $W$ has a largest cardinal $\theta_W$;
- $g \in W$ and $W$ believes that $\theta_W$ is the least regular cardinal witnessing non-$\Gamma^W$-Laverness of $g$, where $\Gamma^W$ is the subset of $W$ defined over $W$ using the definition of $\Gamma$ and the parameter $\sigma^{-1}(p)$.

14 Each construction also shows that the conjunction of $c = \omega_2$ with stationarity of GIC$_{\omega_1}$ does not imply the existence of the Caicedo-Veličković coding.
Claim 33.1. For any $\alpha < \kappa$ and any $g : \alpha \to H_\kappa$, the set $\mathcal{W}_g^{\Gamma}$ has at most one element.

Proof of Claim 33.1 Suppose $W$, $W'$ were two distinct elements of $\mathcal{W}_g^{\Gamma}$; let $\sigma : W \to \Sigma_1$, $V$ and $\sigma' : W' \to \Sigma_1$, $V$ be the maps required by the definition of $\mathcal{W}_g^{\Gamma}$, and set $M := \text{range}(\sigma)$ and $M' := \text{range}(\sigma')$. Note that $p \in M \cap M' \cap H_{\kappa^+}$, and it follows easily (by coding $p$ as a subset of $\kappa$ in an absolute manner and using the assumption that $M \cap \kappa = M' \cap \kappa = \alpha$) that $p_W := \sigma^{-1}(p)$ is equal to $p_{W'} := \sigma'^{-1}(p)$. By the assumption that $\Gamma$ has the $\kappa$-Isomorphism Property, one of $W$, $W'$ is a hereditary initial segment of the other; WLOG assume $W$ is a strict hereditary initial segment of $W'$. Note then that $\theta_W$ (the largest cardinal of $W$) is strictly smaller than $\theta_{W'}$ (the largest cardinal of $W'$). Note also, since $p_W = p_{W'}$, $\Gamma$ is $\Pi_1$ definable in $p$, and $W$ is a hereditary initial segment of (in particular a $\Sigma_1$ elementary substructure of) $W'$, then

$$\Gamma^W = \Gamma^{W'} \cap W.$$ 

Let $\bar{H} := (H_{\theta_W})^W = (H_{\theta_W})^{W'}$. Then

$$X := (\bar{\varphi}_\alpha(\bar{H}))^W \cap \Gamma^W.$$ 

Let $b \in \bar{H}$ witness (from the point of view of $W$) that $g$ is not a $\Diamond_{\text{Lav}}(\Gamma^W)$ function. Then there is some algebra $A_b \in W$ on $H$ such that:

$$W \models (\forall M \in X) \left(M < A_b \implies \sigma^{-1}_M(b) \neq g(M \cap \alpha)\right).$$

This is a $\Sigma_1$ statement in the parameters $X$, $A_b$, and $g$, and is thus upward absolute to $W'$; but since $X$ is also equal to $(\bar{\varphi}_\alpha(\bar{H}))^{W'} \cap \Gamma^{W'}$, this contradicts the minimality of $\theta_{W'}$. \hfill $\square$

Now for any $\alpha < \kappa$ and any $g : \alpha \to H_\kappa$, let $W_g$ denote the unique element of $\mathcal{W}_g^{\Gamma}$ given by Claim 33.1 (if it exists) and let $\theta_g$ be the largest cardinal of $W_g$. Let $b_g$ be any witness in $(H_{\theta_g})^{W_g}$ to the non-Laverness of $g$ w.r.t. $\Gamma^{W_g}$.

Claim 33.2. The function $F : \kappa \to H_\kappa$ defined recursively by $\alpha \mapsto b_g|\alpha$ (if this exists; 0 otherwise) is a $\Diamond_{\text{Lav}}(\Gamma)$ function.

Proof of Claim 33.2. This is where we use the assumption that $\Gamma$ projects downward, along with the assumption that $\Gamma$ is stationary at every $\bar{\varphi}_\kappa(H_\Omega)$. Suppose for a contradiction that $F$ is not a $\Diamond_{\text{Lav}}(\Gamma)$ function; let $\theta$ be the least regular cardinal witnessing non-$\Gamma$-Laverness of $F$. Let $\Omega := \theta^+$ and, using stationarity of $\Gamma$, pick an $M' \prec (H_\Omega, \in, \{F, p\})$ such that $M' \in \Gamma$. Let $\alpha := M' \cap \kappa$, $\sigma_{M'} : H_{M'} \to M'$ be the inverse of the Mostowski collapse of $M'$, and $\bar{\Gamma} := \sigma^{-1}_{M'}(\Gamma \cap \bar{\varphi}_\kappa(H_\theta))$. Then $\sigma^{-1}_{M'}(F) = F \upharpoonright \alpha$ and $H_{M'}$ is the unique element of $\mathcal{W}_{F|\alpha}^{\Gamma}$; so by the recursive definition of the function $F$, we know that $F(\alpha)$ is some element of $\bar{\Gamma}$ witnessing the non-$\bar{\Gamma}$-Laverness of $F \upharpoonright \alpha$ from the point of view of $H_{M'}$. Set $\bar{b} := F(\alpha) \in H_{M'}$ and let $\bar{A} = (\bar{H}, \in, \{\bar{b}, \bar{p}, \ldots\}) \in H_{M'}$ be an algebra corresponding to the witness $\bar{b}$. Let $b := \sigma_{M'}(\bar{b})$ and $A := \sigma_{M'}(\bar{A})$; by elementarity of $\sigma_{M'}$:

\begin{equation}
H_\Omega \models (\forall M) \left(M < A \land M \in \Gamma \implies \sigma^{-1}_M(b) \neq F(M \cap \kappa)\right).
\end{equation}

Set $M := M' \cap H_\theta$; by the downward projection assumption on the class $\Gamma$, we know that $M \in \Gamma$. 

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Furthermore, since $M' \prec (H_\Omega, \in, \{A\})$ then
\[ M = M' \cap H_\theta \prec A. \]

Finally:
\[ \sigma_M^{-1}(b) = \sigma_{M'}^{-1}(b) = \bar{b} = F(\alpha) = F(M' \cap \kappa) = F(M \cap \kappa). \]
These properties of $M$ contradict (16), and complete the proof of the claim. \square

This completes the proof of Theorem 1.

6. Concluding remarks

Recall that Corollary 27 said that if $\Gamma \subseteq \wp(\kappa)(V)$ and the appropriate version of Chang’s Conjecture fails for $\Gamma$, then $\Diamond_{\text{Lav}}(\Gamma)$ holds; and Remark 22 demonstrated that the converse was not literally true. However, it’s still natural to wonder if $\Diamond_{\text{Lav}}(\Gamma)$ must always be essentially due to some failure of Chang’s Conjecture:

**Question 34.** Suppose $\kappa$ is a successor cardinal, $\Gamma \subset \wp(\kappa)(V)$, and $F : \kappa \to H_\kappa$ is a $\Diamond_{\text{Lav}}(\Gamma)$ function. Must there be some (definable) $\Gamma' \subseteq \Gamma$ such that:

1. $F$ is still a $\Diamond_{\text{Lav}}(\Gamma')$ function; and
2. $\text{wCC}(\Gamma')$ fails?

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