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THE DISCRETE LAPLACIAN OF A 2-SIMPLICIAL COMPLEX

YASSIN CHEBBI

Abstract. In this paper, we introduce the notion of oriented faces especially triangles in a connected oriented locally finite graph. This framework then permits to define the Laplace operator on this structure of the 2-simplicial complex. We develop the notion of $\chi$-completeness for the graphs, based on the cut-off functions. Moreover, we study essential self-adjointness of the discrete Laplacian from the $\chi$-completeness geometric hypothesis.

Résumé. Dans cet article, nous introduisons la notion de faces orientées et plus particulièrement de triangles dans un graphe connexe orienté localement fini. Ce cadre permet alors de définir l’opérateur de Laplace sur cette structure d’un 2-complexe simplicial. Nous développons la notion de $\chi$-complétude pour les graphes, basée sur les fonctions de coupure. De plus, nous étudions le caractère essentiellement auto-adjoint du Laplacien discret à partir de l’hypothèse géométrique $\chi$-complétude.

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1. Introduction

The impact of the geometry on the essential self-adjointness of the Laplacians is studied in many areas of mathematics on Riemannian manifolds; see ([C], [EL], [G], [M1]) and also on one-dimensional simplicial complexes; see ([AT], [CTT], [FLW], [HKMW], [M2], [T]). Indeed, Laplacians on Riemannian manifolds and simplicial complexes share a lot of common elements. Despite of this, various geometric notions such as distance and completeness in the Riemannian framework have no immediate analogue in the discrete setting. Combinatorial Laplacians were originally studied on graphs, beginning with Kirchhoff and his...
study of electrical networks [K]. Simplicial complexes can be viewed as generalizations of graphs, as from any graph, we can form its clique complex, a 2-simplicial complex whose faces correspond to the cliques of the graph. In this article, we take a connected oriented locally finite graph and we introduce the oriented faces especially triangles in such a way that every face is a triangle, so we can regard it as a two-dimensional simplicial complex. This work presents a more general framework for the Laplacians defined in terms of the combinatorial structure of a simplicial complex. The main result of this work gives a geometric hypothesis to ensure essential self-adjointness for the discrete Laplacian. We develop the \( \chi \)-completeness hypothesis for triangulations. This hypothesis on locally finite graphs covers many situations that have been already studied in [AT]. The authors prove that the \( \chi \)-completeness is satisfied by graphs which are complete for some intrinsic metric, as defined in [FLW] and [HKMW].

The paper is structured as follows: In the second section, we will first present the basic concepts about graphs or rather one-dimensional simplicial complexes. Next, we introduce the notion of oriented faces more particularly triangles where all the faces are triangles. This special structure of 2-simplicial complex is called triangulation. Without loss of generality, we can assume that every triangle is a face for simplicity sake. So this permits to define the Gauß-Bonnet operator \( T = d + \delta \) acting on triplets of functions, 1-forms and 2-forms. After that, we define the discrete Laplacian by \( L := T^2 \) which admits a decomposition according to the degree

\[
L := L_0 \oplus L_1 \oplus L_2.
\]

In the third and fourth sections, we study the closability of the operators which are used in the following sections. Next, we get started with refer to [AT] for the notion \( \chi \)-completeness of the graphs and we develop this geometric hypothesis for the triangulations in Definition 4.2. Moreover, we have developed it through optimal example of the "triangular tree" to produce a concrete way to prove a triangulation which is not \( \chi \)-complete, based on the offspring function, we refer here to [BGJ] for this notion.

In the fifth section, we address the main results concerning essential self-adjointness for \( T \) and \( L \). In the case of complete manifolds, there is a result of Chernoff; see [C], and we also have for the discrete setting; see [AT], which conclude that the Dirac operator is essentially self-adjoint. As a result, they prove essential self-adjointness of the Laplace-Beltrami operator. So, we take this idea to make the relationship between \( T \) and \( L \) about the essential self-adjointness, when the triangulation is \( \chi \)-complete.

In the final section, we present a particular example of a triangulation where we study the \( \chi \)-completeness hypothesis. Moreover, we show that \( L_1 \) and \( L_2 \) is not necessarily essentially self-adjoint on the simple case.

We can extend the results in this paper to more general 2-simplicial complex, where the oriented faces are not necessarily triangles. Particularly we can give a more general expression of the operator \( d^1 \). More precisely, one can take 2-simplicial complexes with the number of edges of an oriented face bounded. Indeed this hypothesis is important to give a meaning of the inequality in Definition 4.2.

2. Preliminaries

2.1. The basic concepts. A graph \( K \) is a pair \( (V, \mathcal{E}) \), where \( V \) is the countable set of vertices and \( \mathcal{E} \) the set of oriented edges, considered as a subset of \( V \times V \). When two vertices \( x \) and \( y \) are connected by an edge \( e \), we say they are neighbors. We denote \( x \sim y \) and \( e = [x, y] \in \mathcal{E} \). We assume that \( \mathcal{E} \) is symmetric, ie. \( [x, y] \in \mathcal{E} \Rightarrow [y, x] \in \mathcal{E} \). An oriented graph \( K \) is given by a partition of \( \mathcal{E} : \)

\[ \mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+ \]

\[ (x, y) \in \mathcal{E}^- \Leftrightarrow (y, x) \in \mathcal{E}^+ \]

In this case for \( e = (x, y) \in \mathcal{E}^- \), we define the origin \( e^- = x \), the termination \( e^+ = y \) and the opposite edge \( -e = (y, x) \in \mathcal{E}^+ \). Let \( e : V \to (0, \infty) \) the weight on the vertices. We also have \( r : \mathcal{E} \to (0, \infty) \) the weight on the oriented edges with

\[ \forall e \in \mathcal{E}, \ r(-e) = r(e). \]
A path between two vertices \(x, y \in \mathcal{V}\) is a finite set of oriented edges \(e_1, ..., e_n, n \geq 1\) such that
\[e_1^− = x, e_n^+ = y\] and, if \(n \geq 2\), \(\forall j, 1 \leq j \leq n - 1 \implies e_j^+ = e_{j+1}^−\).

The path is called a cycle or closed when the origin and the end are identical, i.e. \(e_1^− = e_n^+\), with \(n \geq 3\).

If no cycles appear more than once in a path, the path is called a simple path. The graph \(\mathcal{K}\) is connected if any two vertices \(x, y\) can be connected by a path with \(e_1^− = x\) and \(e_n^+ = y\).

We say that the graph \(\mathcal{K}\) is locally finite if each vertex belongs to a finite number of edges. The graph \(\mathcal{K}\) is without loops if there is not the type of edges \((x, x)\), i.e.
\[
\forall x \in \mathcal{V} \implies (x, x) \notin \mathcal{E}.
\]

2.1.1. The set of neighbors of \(x \in \mathcal{V}\) is denoted by
\[
\mathcal{V}(x) := \{y \in \mathcal{V} : y \sim x\}.
\]

2.1.2. The degree of \(x \in \mathcal{V}\) is by definition \(\text{deg}(x)\), the number of neighbors of \(x\).

2.1.3. The combinatorial distance \(d_{\text{comb}}\) on \(\mathcal{K}\) is
\[
d_{\text{comb}}(x, y) = \min\{n, \{e_i\}_{1 \leq i \leq n} \subseteq \mathcal{E} \text{ a path between the two vertices } x, y\}\.
\]

2.1.4. Let \(B\) be a finite subset of \(\mathcal{V}\). We define the edge boundary \(\partial_\mathcal{E}B\) of \(B\) by
\[
\partial_\mathcal{E}B := \{e \in \mathcal{E} \text{ such that } \{e^−, e^+\} \cap B \neq \emptyset \text{ and } \{e^−, e^+\} \cap \mathcal{B}^c \neq \emptyset\}.
\]

In the sequel, we assume that
\(\mathcal{K}\) is without loops, connected, locally finite and oriented.

**Definition 2.1.** An oriented face of \(\mathcal{K}\) is a surface limited by a simple closed path, considered as an element of \(\mathcal{E}^n\) with \(n \geq 3\), i.e \(\varpi\) an oriented face \(\implies \exists n \geq 3, \varpi = (e_1, e_2, ..., e_n) \in \mathcal{E}^n\) such that \(\{e_i\}_{1 \leq i \leq n} \subseteq \mathcal{E}\) is a simple closed path.

Let \(\mathcal{F}\) be the set of all oriented faces of \(\mathcal{K}\), we consider the pair \((\mathcal{K}, \mathcal{F})\) as a 2-simplicial complex, we denote it by \(\mathcal{T}\). We can denote also \(\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mathcal{F})\).

**Remark 2.2.** Care should be taken not to confuse the simple cycles and the oriented faces. Indeed, one can have simple cycles that are not oriented faces.

For a face \(\varpi = (e_1, e_2, ..., e_n) \in \mathcal{F}\), we have
\[
\varpi = (e_i, ..., e_n, e_1, ..., e_{i-1}) \in \mathcal{F}, \forall 3 \leq i \leq n - 1.
\]

We can denote also
\[
\varpi = (e_2, e_3, ..., e_n) = .... = (e_n, e_1, e_2, ..., e_{n-1}) \in \mathcal{F}.
\]

Because \(\mathcal{K}\) is an oriented graph, we demand
\[
(e_1, e_2, ..., e_n) \in \mathcal{F} \Rightarrow (-e_n, -e_{n-1}, ..., -e_2, -e_1) \in \mathcal{F}.
\]

Given \(\varpi = (e_1, e_2, ..., e_n) \in \mathcal{F}\), the opposite face of \(\varpi\) is denoted by
\[
-\varpi = (-e_n, -e_{n-1}, ..., -e_2, -e_1) \in \mathcal{F}.
\]

Let \(B\) be a finite subset of \(\mathcal{V}\). We define the face boundary \(\partial_\mathcal{F}B\) of \(B\) by
\[
\partial_\mathcal{F}B := \{\sigma = (e_1, e_2, ..., e_n) \in \mathcal{F}, \exists i \text{ such that } e_i \in \partial_\mathcal{E}B, n \geq 3\}.
\]

**Definition 2.3.** (Triangulation) A triangulation is a 2-simplicial complex such that all the faces are triangles.
Remark 2.4. In the definition of a triangulation we demand that faces are triangles. In the sequel, we assume also that each triangular cycle is an oriented face for simplicity reasons. Indeed all the results of this work can be extended easily to any triangulation.

In the sequel we will represent the oriented faces by their vertices
\[ ω = (e_1, e_2, e_3) = [e_1^−, e_2^−, e_3^−] ∈ F. \]

For a face \( ω = [x, y, z] \) ∈ \( F \). Let us set
\[ ω = [x, y, z] = [y, z, x] = [z, x, y] ∈ F \Rightarrow −ω = [y, x, z] = [x, z, y] = [z, y, x] ∈ F. \]

To define weighted triangulations we need weights, let us give \( s : F → (0, ∞) \) the weight on oriented faces such that for all \( ω ∈ F, s(−ω) = s(ω) \). The weighted triangulation \((T, c, r, s)\) is given by the triangulation \( T = (V, E, F) \). We say that \( T \) is simple if the weights of the vertices, the edges and faces equals 1. For an edge \( e ∈ E \), we also denote the oriented face \([e^−, e^+, x]\) by \([e, x]\), with \( x ∈ V(e^−) \cap V(e^+) \).

The set of vertices belonging to the edge \( e ∈ E \) is given by
\[ F_e := \{x ∈ V, (e, x) ∈ F\} = V(e^−) \cap V(e^+). \]

2.2. Functions spaces. We denote the set of 0-cochains or functions on \( V \) by:
\[ C(V) = \{f : V → C\} \]
and the set of functions of finite support by \( C_∞(V) \).

Similarly, we denote the set of 1-cochains or 1-forms on \( E \) by:
\[ C(E) = \{φ : E → C, φ(−e) = −φ(e)\} \]
and the set of 1-forms of finite support by \( C_∞(E) \).

Moreover, we denote the set of 2-cochains or 2-forms on \( F \) by:
\[ C(F) = \{φ : F → C, φ(−ω) = −φ(ω)\} \]
and the set of 2-forms of finite support by \( C_∞(F) \).

Let us define the Hilbert spaces \( l^2(V) \), \( l^2(E) \) and \( l^2(F) \) as the sets of cochains with finite norm, we have

(a) \[ l^2(V) := \{f ∈ C(V); \sum_{x ∈ V} c(x)|f(x)|^2 < ∞\}, \]
with the inner product
\[ ⟨f, g⟩_{l^2(V)} := \sum_{x ∈ V} c(x)f(x)g(x). \]

(b) \[ l^2(E) := \{φ ∈ C(E); \sum_{e ∈ E} r(e)|φ(e)|^2 < ∞\}, \]
with the inner product
\[ ⟨φ, ψ⟩_{l^2(E)} := \frac{1}{2} \sum_{e ∈ E} r(e)φ(e)ψ(e). \]

(c) \[ l^2(F) := \{φ ∈ C(F); \sum_{ω ∈ F} s(ω)|φ(ω)|^2 < ∞\}, \]
with the inner product
\[ ⟨φ_1, φ_2⟩_{l^2(F)} = \frac{1}{6} \sum_{[x, y, z] ∈ F} s(x, y, z)φ_1(x, y, z)φ_2(x, y, z). \]
The direct sum of the spaces $l^2(V), l^2(\mathcal{E})$ and $l^2(F)$ can be considered as a new Hilbert space denoted by $\mathcal{H}$, that is
$$\mathcal{H} = l^2(V) \oplus l^2(\mathcal{E}) \oplus l^2(F),$$
with the norm
$$\forall F = (f, \varphi, \phi) \in \mathcal{H}, \quad \|F\|_\mathcal{H}^2 = \|f\|_{l^2(V)}^2 + \|\varphi\|_{l^2(\mathcal{E})}^2 + \|\phi\|_{l^2(F)}^2.$$ 

2.3. Operators. We give in this part the expressions of the operators introduced on graphs which are already well known and we also give other operators acting on triangulations.

2.3.1. The difference operator. By analogy to electric networks of voltage differences across edges leading to currents [LP], we define the difference operator $d^0 : C_c(\mathcal{V}) \rightarrow C_c(\mathcal{E})$ by
$$\forall f \in C_c(\mathcal{V}), \quad d^0(f)(e) = f(e^+) - f(e^-).$$

2.3.2. The co-boundary operator. It is the formal adjoint of $d^0$, denoted $\delta^0 : C_c(\mathcal{E}) \rightarrow C_c(\mathcal{V})$, (see [AT]) acts as
$$\forall \varphi \in C_c(\mathcal{E}), \quad \delta^0(\varphi)(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e) \varphi(e).$$

2.3.3. The exterior derivative. It is the operator $d^1 : C_c(\mathcal{E}) \rightarrow C_c(\mathcal{F})$, given by
$$\forall \psi \in C_c(\mathcal{E}), \quad d^1(\psi)(x, y, z) = \psi(x, y) + \psi(y, z) + \psi(z, x).$$

2.3.4. The co-exterior derivative. It is the formal adjoint of $d^1$, denoted $\delta^1 : C_c(\mathcal{F}) \rightarrow C_c(\mathcal{E})$, which satisfies
$$\langle d^1 \psi, \phi \rangle_{l^2(\mathcal{F})} = \langle \psi, \delta^1 \phi \rangle_{l^2(\mathcal{E})}, \forall (\psi, \phi) \in C_c(\mathcal{E}) \times C_c(\mathcal{F}). \quad (2.1)$$

Lemma 2.5. The formal adjoint $\delta^1 : C_c(\mathcal{F}) \rightarrow C_c(\mathcal{E})$, is given by
$$\delta^1(\phi)(e) = \frac{1}{r(e)} \sum_{x \in F_e} s(e, x) \phi(e, x).$$

Proof:

Let $(\psi, \phi) \in C_c(\mathcal{E}) \times C_c(\mathcal{F})$. The equation (2.1) gives
$$\langle d^1 \psi, \phi \rangle_{l^2(\mathcal{F})} = \frac{1}{6} \sum_{[x, y, z] \in \mathcal{F}} s(x, y, z) d^1(\psi)(x, y, z) \overline{\phi}(x, y, z)$$
$$= \frac{1}{2} \sum_{[x, y, z] \in \mathcal{F}} s(x, y, z) \psi(x, y) \overline{\phi}(x, y, z)$$
$$= \langle \psi, \delta^1 \phi \rangle_{l^2(\mathcal{E})}.$$ 

To justify it note that the expression of $d^1$ contributing to the first sum is divided into three similar parts. So it remains to show only
$$\sum_{[x, y, z] \in \mathcal{F}} s(x, y, z) \psi(x, y) \overline{\phi}(x, y, z) = \sum_{e \in \mathcal{E}} \psi(e) \sum_{x \in F_e} s(e, x) \overline{\phi}(e, x)$$
$$= \sum_{e \in \mathcal{E}} r(e) \psi(e) \left( \frac{1}{r(e)} \sum_{x \in F_e} s(e, x) \phi(e, x) \right)$$
\[\square\]
2.3.5. *Gauß-Bonnet operator on* \( T \). By analogy to Riemannian geometry, we use the decomposition of the operators in [EL] to define the Gauß-Bonnet operator. Let us begin by defining the operator
\[
d : C_c(V) \oplus C_c(E) \oplus C_c(F) \circlearrowright
\]
by
\[
\forall (f, \varphi, \phi) \in C_c(V) \oplus C_c(E) \oplus C_c(F), \quad d(f, \varphi, \phi) = (0, d^0 f, d^1 \varphi),
\]
and \( \delta \) the formal adjoint of \( d \). Thus it satisfies
\[
\langle d(f_1, \varphi_1, \phi_1), (f_2, \varphi_2, \phi_2) \rangle_H = \langle (f_1, \varphi_1, \phi_1), \delta(f_2, \varphi_2, \phi_2) \rangle_H,
\]
for all \( (f_1, \varphi_1, \phi_1), (f_2, \varphi_2, \phi_2) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \).

**Lemma 2.6.** Let \( T = (\mathcal{K}, \mathcal{F}) \) be a triangulation. Then
\[
\delta : C_c(V) \oplus C_c(E) \oplus C_c(F) \circlearrowright
\]
is given by
\[
\delta(f, \varphi, \phi) = (\delta^0 \varphi, \delta^1 \phi, 0), \quad \forall (f, \varphi, \phi) \in C_c(V) \oplus C_c(E) \oplus C_c(F).
\]

**Proof:**

Let \( (f_1, \varphi_1, \phi_1), (f_2, \varphi_2, \phi_2) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \). Using the equation (2.2)
\[
\langle d(f_1, \varphi_1, \phi_1), (f_2, \varphi_2, \phi_2) \rangle_H = \langle (0, d^0 f_1, d^1 \varphi_1), (f_2, \varphi_2, \phi_2) \rangle_H
\]
\[
= \langle d^0 f_1, \varphi_2 \rangle_{\mathcal{H}(E)} + \langle d^1 \varphi_1, \phi_2 \rangle_{\mathcal{H}(F)}
\]
\[
= \langle f_1, \delta^0 \varphi_2 \rangle_{\mathcal{H}(V)} + \langle \varphi_1, \delta^1 \phi_2 \rangle_{\mathcal{H}(E)}
\]
\[
= \langle (f_1, \varphi_1, \phi_1), (\delta^0 \varphi_2, \delta^1 \phi_2, 0) \rangle_H.
\]
\[\square\]

**Definition 2.7.** Let \( T = (\mathcal{K}, \mathcal{F}) \) be a triangulation, the Gauß-Bonnet operator defined as
\[
T := d + \delta : C_c(V) \oplus C_c(E) \oplus C_c(F) \circlearrowright
\]
is given by
\[
T(f, \varphi, \phi) = (\delta^0 \varphi, d^0 f + \delta^1 \phi, d^1 \varphi)
\]
for all \( (f, \varphi, \phi) \in C_c(V) \oplus C_c(E) \oplus C_c(F) \). Moreover, the matrix representation of \( T \) is given by
\[
T \equiv \begin{pmatrix}
0 & \delta^0 & 0 \\
\delta^0 & 0 & \delta^1 \\
0 & \delta^1 & 0
\end{pmatrix}
\]

**Lemma 2.8.** If \( T = (\mathcal{K}, \mathcal{F}) \) is a triangulation then \( d^1 d^0 = \delta^0 \delta^1 = 0 \).

**Proof:**

Let \( f \in C_c(V) \), we have that
\[
d^1(d^0 f)(x, y, z) = d^0 f(x, y) + d^0 f(y, z) + d^0 f(z, x)
\]
\[
= (f(y) - f(x)) + (f(z) - f(y)) + (f(x) - f(z)) = 0.
\]

Since \( d^1 d^0 = 0 \) and the operator \( \delta^0 \delta^1 \) is the formal adjoint of \( d^1 d^0 \). Then \( \delta^0 \delta^1 = 0 \).
\[\square\]
\[ \tilde{f}(x, y, z) := \frac{1}{3} \left( \tilde{f}(x, y) + \tilde{f}(y, z) + \tilde{f}(z, x) \right) = \frac{1}{3} (f(x) + f(y) + f(z)). \]

The exterior product of two 1-forms defined as \( \wedge \) is given by:

\[ (\psi \wedge \varphi)(x, y, z) = [\psi(x, y) + \psi(y, z)] \varphi(x, y) + [\psi(x, y) + \psi(x, z)] \varphi(y, z) + [\psi(y, z) + \psi(y, x)] \varphi(z, x). \]

It satisfies \( \psi \wedge \varphi = - (\varphi \wedge \psi) = - \varphi \wedge \psi \), for all \( \varphi, \psi \in C(\mathcal{E}) \).

**Lemma 2.9.** (Derivation properties) Let \( (f, \varphi, \phi) \in C_c(\mathcal{V}) \times C_c(\mathcal{E}) \times C_c(\mathcal{F}). \) Then

\[ d^1(\tilde{f}\varphi)(x, y, z) = \tilde{f}(x, y, z)d^1(\varphi)(x, y, z) + \frac{1}{6} (d^0(f) \wedge \varphi)(x, y, z). \tag{2.3} \]

\[ \delta^1(\tilde{f}\varphi)(e) = \tilde{f}(e)\delta^1(\varphi)(e) + \frac{1}{6r(e)} \sum_{x \in \mathcal{F}_e} s(e, x) [d^0(f)(e^-, x) + d^0(f)(e^+, x)] \phi(e, x). \tag{2.4} \]

**Proof:**

1. Let \( (f, \varphi, \phi) \in C_c(\mathcal{V}) \times C_c(\mathcal{E}) \), we have

\[ d^1(\tilde{f}\varphi)(x, y, z) = \tilde{f}(x, y)\varphi(x, y) + \tilde{f}(y, z)\varphi(y, z) + \tilde{f}(z, x)\varphi(z, x) \]
\[ = \left[ \tilde{f}(x, y) + \tilde{f}(y, z) + \tilde{f}(z, x) \right] \left[ \varphi(x, y) + \varphi(y, z) + \varphi(z, x) \right] \]
\[ - \left( \tilde{f}(y, z) + \tilde{f}(z, x) \right) \varphi(x, y) - \left( \tilde{f}(z, x) + \tilde{f}(x, y) \right) \varphi(y, z) \]
\[ - \left( \tilde{f}(x, y) + \tilde{f}(y, z) \right) \varphi(z, x) \]
\[ = \tilde{f}(x, y, z)d^1(\varphi)(x, y, z) \]
\[ + \left( \frac{2}{3} \tilde{f}(x, y) - \frac{1}{3} \tilde{f}(y, z) + \tilde{f}(z, x) \right) \varphi(x, y) \]
\[ + \left( \frac{2}{3} \tilde{f}(y, z) - \frac{1}{3} \tilde{f}(z, x) + \tilde{f}(x, y) \right) \varphi(y, z) \]
\[ + \left( \frac{2}{3} \tilde{f}(z, x) - \frac{1}{3} \tilde{f}(x, y) + \tilde{f}(y, z) \right) \varphi(z, x). \]

On the other hand, we have

\[ \left( \frac{2}{3} \tilde{f}(x, y) - \frac{1}{3} \tilde{f}(y, z) + \tilde{f}(z, x) \right) \varphi(x, y, z) = \frac{1}{6} (d^0(f)(z, x) + d^0(f)(y, z)). \]

Similarly, we get

\[ \left( \frac{2}{3} \tilde{f}(y, z) - \frac{1}{3} \tilde{f}(z, x) + \tilde{f}(x, y) \right) \varphi(y, z) = \frac{1}{6} (d^0(f)(x, y) + d^0(f)(x, z)) \varphi(y, z). \]

and

\[ \left( \frac{2}{3} \tilde{f}(z, x) - \frac{1}{3} \tilde{f}(x, y) + \tilde{f}(y, z) \right) \varphi(z, x) = \frac{1}{6} (d^0(f)(y, z) + d^0(f)(y, x)) \varphi(z, x). \]
Hence, we have
\[\delta^1(\tilde{f}\phi)(e) = \frac{1}{r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \tilde{f}(e, x) \phi(e, x)\]
\[= \frac{1}{3r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \left(\tilde{f}(e) + \tilde{f}(e^+, x) + \tilde{f}(x, e^-)\right) \phi(e, x)\]
\[= \frac{1}{3} \tilde{f}(e) \delta^1(\phi)(e) + \frac{1}{3r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \tilde{f}(e^+, x) \phi(e, x)\]
\[+ \frac{1}{3r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \tilde{f}(x, e^-) \phi(e, x).\]
\[= \tilde{f}(e) \delta^1(\phi)(e)\]
\[+ \frac{1}{3r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \left(\tilde{f}(e^+, x) - \tilde{f}(e)\right) \phi(e, x)\]
\[+ \frac{1}{3r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \left(\tilde{f}(x, e^-) - \tilde{f}(e)\right) \phi(e, x)\]
\[= \tilde{f}(e) \delta^1(\phi)(e)\]
\[+ \frac{1}{6r(e)} \sum_{x \in \mathcal{F}_n} s(e, x) \left[d^0(\tilde{f})(e^-, x) + d^0(\tilde{f})(e^+, x)\right] \phi(e, x).\]

\[\square\]

2.3.6. **Laplacian.** Through the Gauß-Bonnet operator $T$, we can define the discrete Laplacian on $T$. So, Lemma 2.8 induces the following definition

**Definition 2.10.** Let $T = (\mathcal{K}, \mathcal{F})$ be a triangulation, the Laplacian on $T$ defined as

\[L := T^2 : C_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \oplus C_c(\mathcal{F})\]

is given by

\[L(f, \varphi, \phi) = (\delta^0 d^0 f, (d^0 \delta^0 + \delta^1 d^1) \varphi, d^1 \delta^1 \phi).\]

for all $(f, \varphi, \phi) \in C_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F})$.

**Remark 2.11.** We can write

\[L := L_0 \oplus L_1 \oplus L_2,\]

where $L_0$ is the discrete Laplacian acting on functions given by

\[L_0(f)(x) := \delta^0 d^0 (f)(x) = \frac{1}{c(x)} \sum_{e \in \mathcal{E}^+ x} r(e) d^0 (f)(e),\]
with \( f \in C_c(\mathcal{V}) \), and where \( L_1 \) is the discrete Laplacian acting on 1-forms given by

\[
L_1(\varphi)(x, y) := (d^0 \delta^0 + \delta^1 d^1)(\varphi)(x, y)
\]

\[
= \frac{1}{c(y)} \sum_{e, e^+ = y} r(e) \varphi(e) - \frac{1}{c(x)} \sum_{e, e^- = x} r(e) \varphi(e) + \frac{1}{r(x, y)} \sum_{z \in \mathcal{F}_{[x, y]}} s(x, y, z) d^1(\varphi)(x, y, z),
\]

with \( \varphi \in C_c(\mathcal{E}) \), and where also \( L_2 \) is the discrete Laplacian acting on 2-forms given by

\[
L_2(\phi)(x, y, z) := d^3 \delta^1(\phi)(x, y, z)
\]

\[
= \frac{1}{r(x, y)} \sum_{u \in \mathcal{F}_{[x, y]}} s(x, y, u) \phi(x, y, u)
\]

\[
+ \frac{1}{r(y, z)} \sum_{u \in \mathcal{F}_{[y, z]}} s(y, z, u) \phi(y, z, u)
\]

\[
+ \frac{1}{r(z, x)} \sum_{u \in \mathcal{F}_{[z, x]}} s(z, x, u) \phi(z, x, u),
\]

with \( \phi \in C_c(\mathcal{F}) \).

**Remark 2.12.** The operator \( L_1 \) is called the full Laplacian and defined as \( L_1 = L_1^- + L_1^+ \), where \( L_1^- = d^0 \delta^0 \) (resp. \( L_1^+ = \delta^1 d^1 \)) is called the lower Laplacian (resp. the upper Laplacian). In both articles [AT] and [BGJ], the authors denote \( \Delta_0 = \delta^0 d^0 \) and \( \Delta_1 = d^0 \delta^0 \). In this work, we have \( L_0 = \Delta_0 \) and \( L_1^- = \Delta_1 \).

### 3. Closability

On a connected locally finite graph, the operators \( d^0 \) and \( \delta^0 \) are closable (see [AT]). The next Lemma proves the closability of the operators \( d^1 \) and \( \delta^1 \) on a triangulation.

**Lemma 3.1.** Let \( \mathcal{T} = (\mathcal{K}, \mathcal{F}) \) be a weighted triangulation. Then the operators \( d^1 \) and \( \delta^1 \) are closable.

**Proof:**

- Let \( (\varphi_n)_{n \in \mathbb{N}} \) be a sequence from \( C_c(\mathcal{E}) \) and \( \phi \in L^2(\mathcal{F}) \) such that

\[
\lim_{n \to \infty} (\|\varphi_n\|_{L^2(\mathcal{F})} + \|d^1 \varphi_n - \phi\|_{L^2(\mathcal{F})}) = 0,
\]

then for each edge \( e \), \( \varphi_n(e) \) converges to 0 and for each face \( \varpi \), \( d^1(\varphi_n)(\varpi) \) converges to \( \phi(\varpi) \). But by the expression of \( d^1 \) and local finiteness of \( \mathcal{T} \), for each face \( \varpi \), \( d^1(\varphi_n)(\varpi) \) converges to 0. Thus we have that \( \phi = 0 \).

- The same can be done for \( \delta^1 \): Let \( (\phi_n)_{n \in \mathbb{N}} \) be a sequence from \( C_c(\mathcal{F}) \) and \( \varphi \in L^2(\mathcal{E}) \) such that

\[
\lim_{n \to \infty} (\|\phi_n\|_{L^2(\mathcal{F})} + \|\delta^1 \phi_n - \varphi\|_{L^2(\mathcal{E})}) = 0,
\]

then for each face \( \sigma \), \( \phi_n(\sigma) \) converges to 0 and for each edge \( e \), \( \delta^1(\phi_n)(e) \) converges to \( \varphi(e) \). But by the expression of \( \delta^1 \) and local finiteness of \( \mathcal{T} \), for each edge \( e \), \( \delta^1(\phi_n)(e) \) converges to 0. Thus we have that \( \varphi = 0 \).

The smallest extension is the closure (see [S],[RSv1]), denoted \( \overline{\delta^0} := \delta^0_{\min} \) (resp. \( \overline{d^0} := \delta^0_{\min}, \overline{d^1} := d^1_{\min} \)), \( \overline{\mathcal{T}} := \delta^1_{\min}, \overline{T} := T_{\min}, \overline{\mathcal{T}} := L_{\min} \) has the domain

\[
\text{Dom}(\overline{d^0}_{\min}) = \left\{ f \in L^2(\mathcal{V}) \mid \exists (f_n)_{n \in \mathbb{N}}, f_n \in C_c(\mathcal{V}), \lim_{n \to \infty} \|f_n - f\|_{L^2(\mathcal{V})} = 0, \lim_{n \to \infty} d^0(f_n) \text{ exists in } L^2(\mathcal{E}) \right\},
\]
for such an \( f \), one puts
\[
\delta_{\min}^0(f) = \lim_{n \to \infty} \delta^0(f_n).
\]

We notice that \( \delta_{\min}^0(f) \) is independent of the sequence \( (f_n)_{n \in \mathbb{N}} \), because \( \delta^0 \) is closable.

The largest is \( \delta_{\max}^0 = (\delta^0)^* \), the adjoint operator of \( \delta_{\min}^0 \), (resp. \( \delta_{\max}^1 = (\delta^1)^* \), the adjoint operator of \( \delta_{\min}^1 \).

We also note \( \delta_{\max}^1 = (\delta^1)^* \), the adjoint operator of \( \delta_{\min}^1 \), (resp \( \delta_{\max}^0 = (\delta^0)^* \), the adjoint operator of \( \delta_{\min}^0 \).

**Proposition 3.2.** Let \( T = (\mathcal{K}, \mathcal{F}) \) be a weighted triangulation. Then
\[
\text{Dom}(T_{\min}) \subseteq \text{Dom}(\delta_{\min}^0) \oplus (\text{Dom}(\delta_{\min}^0) \cap \text{Dom}(\delta_{\min}^1)) \oplus \text{Dom}(\delta_{\min}^1).
\]

**Proof:**

Let \( F = (f, \varphi, \phi) \in \text{Dom}(T_{\min}) \), so there exists a sequence \( (F_n)_n = ((f_n, \varphi_n, \phi_n))_{n \in \mathbb{N}} \subseteq \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \) such that \( \lim_{n \to \infty} F_n = F \) in \( \mathcal{H} \) and \( (TF_n)_{n \in \mathbb{N}} \) converges in \( \mathcal{H} \). Let us denote by \( l_0 = (f_0, \varphi_0, \phi_0) \) this limit. Therefore
\[
\|TF_n - l_0\|^2_{\mathcal{H}} = \|\delta^0 \varphi_n - f_0\|^2_{\mathcal{V}} + \|f_n - \varphi_0\|^2_{\mathcal{E}} + \|\delta^1 \phi_n - \delta^0 \phi_0\|^2_{\mathcal{F}}.
\]

Hence \( \delta^0 \varphi_n \to f_0 \) and \( \delta^1 \phi_n \to \phi_0 \) respectively in \( l^2(\mathcal{V}) \) and in \( l^2(\mathcal{E}) \). So, by definition, \( \varphi \in \text{Dom}(\delta_{\min}^0) \cap \text{Dom}(\delta_{\min}^1) \), \( f_0 = \delta_{\min} \varphi \) and \( \phi_0 = \delta_{\min} \phi \). Moreover, we combine the parallelogram identity with Lemma 2.8 to obtain the following result
\[
\|(\delta^0 + \delta^1)(f_n, \phi_n)\|^2_{\mathcal{F}} = \|\delta^0(f_n)\|^2_{\mathcal{V}} + \|\delta^1(\phi_n)\|^2_{\mathcal{E}}, \quad \forall n \in \mathbb{N}, \forall (f, \phi) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{F}).
\]

Since \( ((\delta^0 + \delta^1)(f_n, \phi_n))_n \) converges in \( l^2(\mathcal{E}) \), then by completeness of \( l^2(\mathcal{E}) \) \( (\delta^0(f_n))_n \) and \( (\delta^1(\phi_n))_n \) are convergent in \( l^2(\mathcal{E}) \). Thus, we conclude that \( f \in \text{Dom}(\delta_{\min}^0) \) and \( \phi \in \text{Dom}(\delta_{\min}^1) \).

**Proposition 3.3.** Let \( T = (\mathcal{K}, \mathcal{F}) \) be a weighted triangulation. Then
\[
\text{Dom}(L_{\min}) \subseteq \text{Dom}(\delta_{\min}^0 \delta_{\min}^0) \oplus (\text{Dom}(\delta_{\min}^0 \delta_{\min}^0) \cap \text{Dom}(\delta_{\min}^0 \delta_{\min}^1)) \oplus \text{Dom}(\delta_{\min}^0 \delta_{\min}^1).
\]

**Proof:**

i) We will show that \( (L_0)_{\min} \subseteq \delta_{\min}^0 \delta_{\min}^0 \). First, we note that
\[
\text{Dom}(\delta_{\min}^0 \delta_{\min}^0) = \{ f \in \text{Dom}(\delta_{\min}^0), \delta_{\min}^0 f \in \text{Dom}(\delta_{\min}^0) \}.
\]

Let \( f \in \text{Dom}((L_0)_{\min}) \), so there exists a sequence \( (f_n)_n \subseteq \mathcal{C}_c(\mathcal{V}) \) such that
\[
f_n \to f \text{ in } l^2(\mathcal{V}), \quad \delta^0 \delta^0 f_n \to (\delta^0 \delta^0)_{\min} f \text{ in } l^2(\mathcal{V}).
\]

So, \( (L_0 f_n)_n \) is a Cauchy sequence. Moreover, we have
\[
\forall n, m \in \mathbb{N}, \quad \|\delta^0 f_n - \delta^0 f_m\|^2_{l^2(\mathcal{E})} = \langle \delta^0(f_n - f_m), \delta^0(f_n - f_m) \rangle_{l^2(\mathcal{E})}
\]
\[
= \langle \delta^0 \delta^0(f_n - f_m), f_n - f_m \rangle_{l^2(\mathcal{V})}
\]
\[
= \langle L_0(f_n - f_m), f_n - f_m \rangle_{l^2(\mathcal{V})}
\]
\[
\leq \|L_0(f_n - f_m)\|_{l^2(\mathcal{V})} \|f_n - f_m\|_{l^2(\mathcal{V})}.
\]

Thus \( (\delta^0 f_n)_n \) is a Cauchy sequence because \( (L_0 f_n)_n \) is a Cauchy sequence and \( (f_n)_n \) is convergent. So, it is convergent in \( l^2(\mathcal{E}) \). By closability, we conclude that \( f \in \text{Dom}(\delta_{\min}^0 \delta_{\min}^0) \).
ii) First, for all $\varphi, \psi \in C_c(\mathcal{E})$ we have
\[
\langle L_1 \varphi, \psi \rangle_{L^2(\mathcal{E})} = \langle (L_1 + L_1^+) \varphi, \psi \rangle_{L^2(\mathcal{E})} = \langle \delta^0 \varphi, \delta^0 \psi \rangle_{L^2(\mathcal{V})} + \langle d^1 \varphi, d^1 \psi \rangle_{L^2(\mathcal{F})}. \tag{3.1}
\]
Using the same method as in i) with (3.1) we obtain that $(L_1^-)_{\min} \subseteq d_{\min}^0$ and $(L_1^+)_{\min} \subseteq d_{\min}^1$. It remains to show that we have
\[
(L_1^-)_{\min} \subseteq (L_1^{\pm})_{\min} + (L_1^+)_{\min}
\]
Let $\varphi \in \text{Dom}((L_1^-)_{\min})$, so there exists a sequence $(\varphi_n)_{n} \subseteq C_c(\mathcal{E})$ such that $\varphi = \lim_{n \to \infty} \varphi_n$ in $L^2(\mathcal{E})$ and $(L_1^\chi \varphi_n)_{n} \subseteq L^2(\mathcal{E})$. Then, by the parallelogram identity with Lemma 2.8 we obtain
\[
\|(L_1^- + L_1^+)\varphi_n\|_{L^2(\mathcal{E})}^2 = \|L_1^- \varphi_n\|_{L^2(\mathcal{E})}^2 + \|L_1^+ \varphi_n\|_{L^2(\mathcal{E})}^2, \quad \forall n \in \mathbb{N}.
\]
Then $(L_1^- (\varphi_n))_{n}$ and $(L_1^+ (\varphi_n))_{n}$ are convergent in $L^2(\mathcal{E})$. Moreover, by the closability of $L_1^-$ and $L_1^+$, we conclude that $\varphi \in \text{Dom}((L_1^-)_{\min}) \cap \text{Dom}((L_1^+)_{\min})$.

iii) Since, for any $\phi, \Theta \in C_c(\mathcal{F})$, we have
\[
\langle L_2 \phi, \Theta \rangle_{L^2(\mathcal{F})} = \langle d^1 \delta^1 \phi, \delta^1 \Theta \rangle_{L^2(\mathcal{F})} = \langle \delta^1 \phi, \delta^1 \Theta \rangle_{L^2(\mathcal{F})}, \tag{3.2}
\]
Using the same method as in i) with (3.2) we obtain that $(L_2)_{\min} \subseteq d_{\min}^1$.

\[\square\]

4. Geometric hypothesis

4.1. $\chi$-completeness. In this subsection, we give the geometric hypothesis for the triangulation $\mathcal{T}$. First we recall the definition of $\chi$-completeness given in [AT] for the case of graphs. A graph $\mathcal{K} = (\mathcal{V}, \mathcal{E})$ is $\chi$-complete if there exists an increasing sequence of finite sets $(\mathcal{B}_n)_{n \in \mathbb{N}}$ such that $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ and there exist related functions $\chi_n$ satisfying the following three conditions:

i) $\chi_n \in C_c(\mathcal{V})$, $0 \leq \chi_n \leq 1$.

ii) $x \in \mathcal{B}_n \Rightarrow \chi_n(x) = 1$.

iii) $\exists C > 0$ such that $\forall n \in \mathbb{N}$, $x \in \mathcal{V}$
\[
\frac{1}{c(x)} \sum_{e \in \mathcal{E}, e^\pm = x} r(e)|d^0\chi_n(e)|^2 \leq C.
\]

Remark 4.1. The $\chi$-completeness is related to the notion of intrinsic metric for weighted graphs. This geometric hypothesis covers many situations that have been already studied. Particularly in [AT], the authors prove that it is satisfied by locally finite graphs which are complete for some intrinsic pseudo metric, as defined in [FLW] and [HKMW].

Definition 4.2. A triangulation $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ is $\chi$-complete, if

(C1) $\mathcal{K}$ is $\chi$-complete.

(C2) $\exists M > 0$, $\forall n \in \mathbb{N}$, $e \in \mathcal{E}$, such that
\[
\frac{1}{r(e)} \sum_{x \in \mathcal{E}} s(e,x)|d^0\chi_n(e^-, x) + d^0\chi_n(e^+, x)|^2 \leq M.
\]

For this type of 2-simplicial complexes one has

\[
\forall p \in \mathbb{N}, \exists p_n, n \geq p_n \Rightarrow \forall e \in \mathcal{E}, \text{ such that } e^+ \in B_{p_n}, d^0\chi_n(e) = 0. \tag{4.1}
\]
\[
\mathcal{E} = \bigcup_{n \in \mathbb{N}} \mathcal{E}_n \text{ if } \mathcal{E}_n := \{e \in \mathcal{E}, e^+ \in \mathcal{B}_n \text{ or } e^- \in \mathcal{B}_n\}. \tag{4.2}
\]
\[
\forall q \in \mathbb{N}, \exists q_n, n \geq q_n \Rightarrow \forall (e, x) \in \mathcal{F}, \text{ such that } e^-, e^+ \in \mathcal{B}_n, d^0\chi_n(e^-, x) = 0. \tag{4.3}
\]
\[ \mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \text{ if } \mathcal{F}_n := \{ [x, y, z] \in \mathcal{F}, x \in B_n \text{ or } y \in B_n \text{ or } z \in B_n \}. \tag{4.4} \]

\[ \forall f \in l^2(\mathcal{V}), \|f\|^2_{l^2(\mathcal{V})} = \lim_{n \to \infty} \langle \chi_n f, f \rangle_{l^2(\mathcal{V})}. \tag{4.5} \]

\[ \forall \varphi \in l^2(\mathcal{E}), \|\varphi\|^2_{l^2(\mathcal{E})} = \lim_{n \to \infty} \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \chi_n(e^+) \|\varphi(e)\|^2. \tag{4.6} \]

\[ \forall \phi \in l^2(\mathcal{F}), \|\phi\|^2_{l^2(\mathcal{F})} = \lim_{n \to \infty} \frac{1}{6} \sum_{e \in \mathcal{F}} \chi_n(e) \left( \sum_{x \in \mathcal{F}_e} s(e, x) \|\phi(e, x)\|^2 \right). \tag{4.7} \]

\[ \lim_{n \to \infty} \sum_{e \in \mathcal{F}^\ast(n)} r(e) \|\varphi(e)\|^2 = 0, \tag{4.8} \]

where

\[ \mathcal{E}^\ast(n) := \{ e \in \mathcal{E}, \exists x \in \mathcal{F}_e \text{ such that } (e^\pm, x) \in \text{supp}(d^0 \chi_n) \} \]

\[ \lim_{n \to \infty} \sum_{e \in \mathcal{E}} \sum_{x \in \mathcal{F}^\ast(n)} s(e, x) \|\phi(e, x)\|^2 = 0, \tag{4.9} \]

where

\[ \forall e \in \mathcal{E}, \mathcal{F}_e^\ast(n) := \{ x \in \mathcal{F}_e, (e^\pm, x) \in \text{supp}(d^0 \chi_n) \}. \]

**Proposition 4.3.** Let \( \mathcal{T} \) be a simple triangulation of bounded degree, i.e. \( \exists \lambda > 0, \forall x \in \mathcal{V}, \deg(x) \leq \lambda. \)

Then \( \mathcal{T} \) is a \( \chi \)-complete triangulation.

**Proof:**

Let us consider \( \mathcal{T} \) an infinite triangulation. Given \( o \in \mathcal{V}, \) let \( B_n \) be a ball of radius \( n \in \mathbb{N} \) centered by the vertex \( o: \)

\[ B_n = \{ x \in \mathcal{V}, \ d_{\text{comb}}(o, x) \leq n \}. \]

We set the cut-off function \( \chi_n \in C_c(\mathcal{V}) \) as follow:

\[ \chi_n(x) := \left( \frac{2n - d_{\text{comb}}(o, x)}{n} \vee 0 \right) \wedge 1, \forall n \in \mathbb{N}^*. \]

- If \( x \in B_n \Rightarrow \chi_n(x) = 1 \) and \( x \in B^n_{2n} \Rightarrow \chi_n(x) = 0. \)

- For \( e \in \mathcal{E}, \) we have that

\[ |d^0 \chi_n(e)| \leq \frac{1}{n} \left| d_{\text{comb}}(o, e^+) - d_{\text{comb}}(o, e^-) \right| = \frac{1}{n}. \]

Hence

\[ \forall x \in \mathcal{V}, \sum_{e \in \mathcal{E}, e^\pm = x} |d^0 \chi_n(e)|^2 \leq \frac{\lambda}{n^2} \]

and

\[ \forall e \in \mathcal{E}, \sum_{x \in \mathcal{F}_e} |d^0 \chi_n(e^+, x) + d^0 \chi_n(e^-, x)|^2 \leq \frac{2\lambda}{n^2}. \]

\[ \square \]

**Example 4.4.** (A \( \chi \)-complete triangulation)

We consider \( \mathcal{T} \) a 6-regular simple triangulation, i.e. \( \deg(x) = 6, \forall x \in \mathcal{V}. \) Then, by Proposition 4.3 we have that \( \mathcal{T} \) is a \( \chi \)-complete triangulation.
Proposition 4.5. Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a $\chi$-complete triangulation. Then
\[
\text{Dom } (L_0)_{\text{min}} = \text{Dom} (\delta_{\text{min}}^0 d_{\text{min}}^0).
\]

Proof:

In Proposition 3.3, we have already $(L_0)_{\text{min}} \subseteq \delta_{\text{min}}^0 d_{\text{min}}^0$. Indeed, we will show that $\delta_{\text{min}}^0 d_{\text{min}}^0 \subseteq (L_0)_{\text{min}}$. Let $f \in \text{Dom}(\delta_{\text{min}}^0 d_{\text{min}}^0)$, by the $\chi$-completeness of $\mathcal{T}$, we now consider a sequence $(\chi_n f)_n \subseteq C_c(\mathcal{V})$. It remains to show that:

\[
\lim_{n \to \infty} \| f - \chi_n f \|_{L^2(\mathcal{V})} + \| L_0 (f - \chi_n f) \|_{L^2(\mathcal{V})} = 0. \tag{4.10}
\]

For the first term of (4.10), since $f \in L^2(\mathcal{V})$ we have
\[
\| f - \chi_n f \|^2_{L^2(\mathcal{V})} \leq \sum_{x \in B_n^c} c(x) |f(x)|^2 \to 0, \text{ when } n \to \infty.
\]

For the second term of (4.10), we need a derivation formula of $d^0$, see [M2]. Let $e \in \mathcal{E}$, for each $(f, g) \in C_c(\mathcal{V}) \times C_c(\mathcal{V})$ we have
\[
d^0(fg)(e) = f(e^+)d^0(g)(e) + d^0(f)(e)g(e^-). \tag{4.11}
\]

By the definition of $L_0$, we have
\[
\| L_0 (f - \chi_n f) \|^2_{L^2(\mathcal{V})} = \sum_{x \in \mathcal{V}} \frac{1}{c(x)} \left| \sum_{e, e^+ = x} r(e) d^0((1 - \chi_n)f)(e) \right|^2.
\]
Using the derivation formula (4.11), we get
\[
\|L_0(f - \chi_n f)\|^2_{L^2(V)} \leq 2 \sum_{x \in V} \frac{1}{c(x)} \left( \sum_{e, e^+ = x} r(e)(1 - \chi_n)(e^+)d^0(f)(e) \right)^2 \\
+ 2 \sum_{x \in V} \frac{1}{c(x)} \left( \sum_{e, e^+ = x} r(e)f(e^-)d^0(\chi_n)(e) \right)^2 \\
= 2 \left( \|(1 - \chi_n)L_0(f)\|^2_{L^2(V)} + \sum_{x \in V} \frac{1}{c(x)} \left( \sum_{e, e^+ = x} r(e)f(e^-)d^0(\chi_n)(e) \right)^2 \right).
\]

Since \(L_0(f) \in L^2(V)\), we have
\[
\lim_{n \to \infty} \|(1 - \chi_n)L_0(f)\|_{L^2(V)} = 0.
\]

On the other hand, by the hypothesis iii) of \(\chi\)-completeness and the Cauchy-Schwarz inequality, we get
\[
\sum_{x \in V} \frac{1}{c(x)} \left| \sum_{e, e^+ = x} r(e)f(e^-)d^0(\chi_n)(e) \right|^2 \leq \sum_{x \in V} \frac{1}{c(x)} \left( \sum_{e, e^+ = x} r(e)|d^0(\chi_n)(e)|^2 \right) \\
\sum_{e \in \text{supp}(d^0(\chi_n)), e^+ = x} r(e)|f(e^-)|^2 \\
\sum_{e \in \text{supp}(d^0(\chi_n)), e^+ = x} r(e)|f(e^-)|^2 \\
\leq C \sum_{e \in \text{supp}(d^0(\chi_n))} r(e)|f(e^-)|^2.
\]

The properties (4.1) and (4.2) permit to conclude that this term tends to 0 when \(\infty\).

\[\square\]

4.2. The case of a not \(\chi\)-complete triangulation. In [BGJ], the authors use the offspring function on the trees to give a counter example of a graph which is not \(\chi\)-complete. The same thing for the triangulations is not always \(\chi\)-complete. To prove it, we will study the triangular tree in Definition 4.6.

Let \(\mathcal{T}\) a weighted triangulation, one can take any point \(o \in V\). Given \(n \in \mathbb{N}\), we denote the spheres by
\[
S_n := \{x \in V, \ d_{\text{comb}}(o, x) = n\}.
\]

**Definition 4.6.** A triangular tree \(\mathcal{T} = (V, \mathcal{F})\) with the origin vertex \(o\) is a triangulation where \(V = \bigcup_{n \in \mathbb{N}} S_n\), such that
\[
\forall x \in S_n \setminus \{o\}, \ V(x) \cap S_{n-1} = \{\overline{y}\}.
\]
\[
\forall x \in S_n, \ y \in V(x) \cap S_{n+1} \iff \overline{y} = x.
\]
\[
(x, y) \in \mathcal{E} \cap (S_n \setminus \{o\})^2 \Rightarrow \overline{x} = \overline{y}.
\]

where \(\overline{x}\) the unique vertex in \(S_{n-1}\), which is related with \(x \in S_n \setminus \{o\}\).

Let \(\mathcal{T}\) be a simple triangular tree. The offspring of the \(n\)-th generation (see [BGJ]) is given by
\[
\text{off}(n) = \frac{\#S_{n+1}}{\#S_n}.
\]
Figure 2. A Triangular Tree

Proposition 4.7. Let $T$ be a simple triangular tree with the origin vertex $o$. Assume that

$$\sup_{n \in \mathbb{N}} \sup_{x \in S_n} \frac{\# (\mathcal{V}(x) \cap S_{n+1})}{\text{off}(n)} < \infty.$$  

Then

$$T \text{ is } \chi\text{-complete } \iff \sum_{n \geq 1} \frac{1}{\sqrt{\text{off}(n)}} = \infty.$$  

Proof:

$\Rightarrow$ In a proof by contradiction, we start by assuming that $T$ is $\chi$-complete and the series converges. So, there exists a sequence $(\chi_n)_n$ included in $\mathcal{C}(\mathcal{V})$, satisfying

$$\exists C > 0, \forall n \in \mathbb{N}, \sum_{y \sim x} |\chi_n(x) - \chi_n(y)|^2 \leq C, \ x \in \mathcal{V}.$$  

Given $n, m \in \mathbb{N}$ and $x_m \in S_m$. By the local finiteness of the triangulation, we find $x_{m+1} \in \mathcal{V}(x_m) \cap S_{m+1}$, such that

$$|\chi_n(x_m) - \chi_n(x_{m+1})| = \min_{y \in \mathcal{V}(x_m) \cap S_{m+1}} |\chi_n(x_m) - \chi_n(y)|.$$  

But,

$$\sum_{y \in \mathcal{V}(x_m) \cap S_{m+1}} |\chi_n(x_m) - \chi_n(y)|^2 \leq C.$$  

Hence

$$|\chi_n(x_m) - \chi_n(x_{m+1})| \leq \frac{\sqrt{C}}{\sqrt{\text{off}(m)}}.$$  

Moreover, by convergence of the series, there is $N \in \mathbb{N}$ such that

$$\sum_{k \geq N} \frac{1}{\sqrt{\text{off}(k)}} < \frac{1}{2\sqrt{C}}.$$
Corollary 4.8. Let $\mathcal{T}$ be a simple triangular tree, endowed with an origin such that

$$\text{off}(n) = \#(\mathcal{V}(x) \cap \mathcal{S}_{n+1}), \text{ for all } x \in \mathcal{S}_n,$$

then $\mathcal{T}$ is $\chi$-complete if and only if

$$\sum_{n \geq 1} \frac{1}{\sqrt{\text{off}(n)}} = \infty.$$
Example 4.9. Set $\alpha > 0$. Let $T$ be a simple triangular tree, endowed with an origin such that
\[ \text{off}(n) = \#(V(x) \cap S_{n+1}) = \lfloor n^\alpha \rfloor + 1, \text{ for all } x \in S_n, \]
then $T$ is $\chi$-complete if only if $\alpha \leq 2$.

5. Essential self-adjointness

In [AT], the authors use the $\chi$-completeness hypothesis on a graph to ensure essential self-adjointness for the Gauss-Bonnet operator and the Laplacian. In this section, with the same idea we will prove the main result, when the triangulation is $\chi$-complete. Let us begin from

Proposition 5.1. Let $T = (K, F)$ be a $\chi$-complete triangulation then the operator $d^1 + \delta^1$ is essentially self-adjoint on $\mathcal{C}_c(E) \oplus \mathcal{C}_c(F)$.

Proof:

It suffices to show that $d^1_{\min} = d^1_{\max}$ and $\delta^1_{\min} = \delta^1_{\max}$. Indeed, $d^1 + \delta^1$ is a direct sum and if $F = (\varphi, \phi) \in \text{Dom}((d^1 + \delta^1)_{\max})$ then $\varphi \in \text{Dom}(d^1_{\max})$ and $\phi \in \text{Dom}(\delta^1_{\max})$. By hypothesis, we have $\varphi \in \text{Dom}(d^1_{\min})$ and $\phi \in \text{Dom}(\delta^1_{\min})$, thus $F \in \text{Dom}((d^1 + \delta^1)_{\min})$.

1) Let $\varphi \in \text{Dom}(d^1_{\max})$, we will show that
\[ \|\varphi - \overline{\chi_n} \varphi\|_{\mathcal{C}(E)} + \|d^1 (\varphi - \overline{\chi_n} \varphi)\|_{\mathcal{C}(E)} \rightarrow 0 \text{ when } n \rightarrow \infty. \]

By the properties (4.1) and (4.2), we know that
\[ \forall p \in \mathbb{N}, \exists n_p, \forall n \geq n_p, \|\varphi - \overline{\chi_n} \varphi\|_{\mathcal{C}(E)} \leq \sum_{e \in E^p} r(e) |\varphi(e)|^2 \]
so $\lim_{n \rightarrow \infty} \|\varphi - \overline{\chi_n} \varphi\| = 0$.

From the derivation formula (2.3) in Lemma 2.9, we have
\[ d^1 (\varphi - \overline{\chi_n} \varphi)(e, x) = d^1 \left( \left( 1 - \overline{\chi_n} \right) \varphi \right)(e, x) \]
\[ = \left( 1 - \overline{\chi_n} \right)(e, x) d^1 \varphi(e, x) \]
\[ + \frac{1}{6} \left( d^0(1 - \chi_n)(x, e^-) + d^0(1 - \chi_n)(x, e^+) \right) \varphi(e) \]
\[ + \frac{1}{6} \left( d^0(1 - \chi_n)(e) + d^0(1 - \chi_n)(e^-, x) \right) \varphi(e^+, x) \]
\[ + \frac{1}{6} \left( d^0(1 - \chi_n)(e^+, x) + d^0(1 - \chi_n)(-e) \right) \varphi(x, e^-) \]
\[ = \left( 1 - \overline{\chi_n} \right)(e, x) d^1 \varphi(e, x) \]
\[ + \frac{1}{6} \left( d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x) \right) \varphi(e) \]
\[ + \frac{1}{6} \left( d^0 \chi_n(-e) + d^0 \chi_n(x, e^-) \right) \varphi(e^+, x) \]
\[ + \frac{1}{6} \left( d^0 \chi_n(x, e^+) + d^0 \chi_n(e) \right) \varphi(x, e^-). \]

Since $d^1 \varphi \in l^2(F)$, one has
\[ \lim_{n \rightarrow \infty} \left\| \left( 1 - \overline{\chi_n} \right) d^1 \varphi \right\|_{l^2(E)} = 0. \]
On the other hand,
\[
\sum_{(e,x) \in E} s(e, x)|\varphi(e)|^2|d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x)|^2 = \sum_{e \in E} |\varphi(e)|^2 \sum_{x \in F_e} s(e, x)|d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x)|^2 \leq M \sum_{e \in E^* (n)} r(e)|\varphi(e)|^2.
\]

The property (4.8) allows to conclude that this term tends to 0 as \( n \to \infty \). Applying the same process to the other terms, we have
\[
\sum_{(e,x) \in E} s(e, x)|\varphi(e^+ , x)|^2|d^0 \chi_n(-e) + d^0 \chi_n(x, e^-)|^2 = \sum_{(e^+, x) \in E} |\varphi(e^+ , x)|^2 \sum_{y \in F_{(e^+, x)}} s(e^+, x, y) |d^0 \chi_n(e^+, y) + d^0 \chi_n(x, y)|^2 \leq M \sum_{(e^+, x) \in E^* (n)} r(e^+, x)|\varphi(e^+, x)|^2
\]
and
\[
\sum_{(e,x) \in E} s(e, x)|\varphi(x, e^-)|^2|d^0 \chi_n(x, e^+) + d^0 \chi_n(e^-)|^2 = \sum_{(x,e^-) \in E} |\varphi(x, e^-)|^2 \sum_{y \in F_{(x,e^-)}} s(x, e^-, y) |d^0 \chi_n(x, y) + d^0 \chi_n(e^-, y)|^2 \leq M \sum_{(x,e^-) \in E^* (n)} r(x, e^-)|\varphi(x, e^-)|^2.
\]

2) Let \( \phi \in \text{Dom}(\delta^1_{n \to x}) \), we will show that
\[
\|\phi - \tilde{\chi}_n \phi\|_{\ell^2(E)} + \|\delta^1(\phi - \tilde{\chi}_n \phi)\|_{\ell^2(E)} \to 0 \text{ when } n \to \infty.
\]
By the properties (4.3) and (4.4), we know that
\[
\|\phi - \tilde{\chi}_n \phi\|_{\ell^2(E)}^2 = \frac{1}{6} \sum_{(x,y,z) \in E} s(x, y, z)|1 - \tilde{\chi}_n(x, y, z)|^2|\phi(x, y, z)|^2 \leq \sum_{(x,y,z) \in F_{E}^q} s(x, y, z)|\phi(x, y, z)|^2 \to 0, \text{ when } n \to \infty.
\]
By the derivation formula (2.4) in Lemma 2.9, we have
\[
\delta^1(\phi - \tilde{\chi}_n \phi)(e) = \delta^1 \left( (1 - \chi_n) \phi \right)(e)
\]
\[
= (1 - \tilde{\chi}_n)(e) \delta^1(\phi)(e) + \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)d^0(1 - \chi_n)(e^-, x)\phi(e, x) + \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)d^0(1 - \chi_n)(e^+, x)\phi(e, x)
\]
\[
= (1 - \tilde{\chi}_n)(e) \delta^1(\phi)(e) + \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)d^0(\chi_n)(x, e^-)\phi(e, x)
\]
\[
+ \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)d^0(\chi_n)(x, e^+)\phi(e, x) + \frac{1}{6r(e)} \sum_{x \in F_e} s(e, x)d^0(\chi_n)(x, e^+)\phi(e, x).
\]
We know that
\[ \lim_{n \to \infty} \| (1 - \bar{\chi}_n) \delta^1(\phi) \| = 0 \]
because \( \delta^1 \phi \in L^2(\mathcal{E}) \). For the second and third terms, we use the inequality of Definition 4.2 and the Cauchy-Schwarz inequality. Fix \( e \in \mathcal{E} \), then
\[
\left| \sum_{x \in \mathcal{F}_{x}} s(e, x) \left( d^0(\chi_n)(x, e^-) + d^0(\chi_n)(x, e^+) \right) \phi(e, x) \right|^2 \leq \sum_{x \in \mathcal{F}_{x}} s(e, x) \left| d^0(\chi_n)(x, e^-) + d^0(\chi_n)(x, e^+) \right|^2 \\
\times \sum_{x \in \mathcal{F}_{\mathcal{F}}(n)} s(e, x) |\phi(e, x)|^2 \\
\leq M r(e) \sum_{x \in \mathcal{F}_{\mathcal{F}}(n)} s(e, x) |\phi(e, x)|^2.
\]
Therefore,
\[
\sum_{e \in \mathcal{E}} r(e) \left| \frac{1}{r(e)} \sum_{x \in \mathcal{F}_{x}} s(e, x) \left( d^0(\chi_n)(x, e^-) + d^0(\chi_n)(x, e^+) \right) \phi(e, x) \right|^2 \leq M \sum_{e \in \mathcal{E}} \sum_{x \in \mathcal{F}_{\mathcal{F}}(n)} s(e, x) |\phi(e, x)|^2.
\]

By property (4.9), this terms tends to 0.

\[ \Box \]

**Corollary 5.2.** Let \( T = (\mathcal{K}, \mathcal{F}) \) be a \( \chi \)-complete triangulation then the operator \( L^+_1 \oplus L_2 \) is essentially self-adjoint on \( C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \).

**Proof:**

First we have that \( L^+_1 \oplus L_2 = (d^1 + \delta^1)^2 \) and \( L^+_1 \oplus L_2 (C_c(\mathcal{E}) \oplus C_c(\mathcal{F})) \subseteq C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \).

As Proposition 13 in [AT] we prove that \( d^1 + \delta^1 \) is essentially self-adjoint if and only if \( L^+_1 \oplus L_2 \) is essentially self-adjoint.

\[ \Box \]

**Theorem 5.3.** Let \( T = (\mathcal{K}, \mathcal{F}) \) be a \( \chi \)-complete triangulation then the operator \( T \) is essentially self-adjoint on \( C_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \).

**Proof:**

1st Step: We will show that
\[ \text{Dom}(T_{\text{min}}) = \text{Dom}(d^0_{\text{min}}) \oplus \left( \text{Dom}(\delta^0_{\text{min}}) \cap \text{Dom}(\delta^1_{\text{min}}) \right) \oplus \text{Dom}(\delta^1_{\text{min}}). \]

Let \( F = (f, \varphi, \phi) \in \text{Dom}(d^0_{\text{min}}) \oplus \left( \text{Dom}(\delta^0_{\text{min}}) \cap \text{Dom}(\delta^1_{\text{min}}) \right) \oplus \text{Dom}(\delta^1_{\text{min}}) \). Then there exist \( (f_n) \subseteq C_c(\mathcal{V}) \) and \( (\phi_n) \subseteq C_c(\mathcal{F}) \) such that:
- \( f_n \to f \) in \( L^2(\mathcal{V}) \) and \( d^0 f_n \to d^0 f \) in \( L^2(\mathcal{E}) \).
- \( \phi_n \to \phi \) in \( L^2(\mathcal{F}) \) and \( \delta^1 \phi_n \to \delta^1 \phi \) in \( L^2(\mathcal{E}) \)

On the other hand, let \( \varphi \in \text{Dom}(\delta^0_{\text{min}}) \cap \text{Dom}(\delta^1_{\text{min}}) \). By the \( \chi \)-completeness of \( T \), we now consider the sequence \( (\bar{\chi}_n \varphi) \). It remains to show that
\[ \| \varphi - \bar{\chi}_n \varphi \|_{L^2(\mathcal{F})} + \| \delta^0 \varphi - \bar{\chi}_n \varphi \|_{L^2(\mathcal{V})} + \| (\varphi - \bar{\chi}_n \varphi) \|_{L^2(\mathcal{F})} \to 0, \text{ when } n \to \infty. \]

The first and the third terms have already been shown in Proposition 5.1. For the following we need a derivation formula of \( \delta^0 \) taken in [M2]. Let \( x \in \mathcal{V} \), for each \( (f, \varphi) \in C_c(\mathcal{V}) \times C_c(\mathcal{E}) \) we have
\[
\delta^0(f \varphi)(x) = f(x) \delta^0(\varphi)(x) - \frac{1}{2c(x)} \sum_{e, e^+ = x} r(e) d^0(f)(e) \varphi(e).
\]
Therefore, by derivation formula (5.1), we get
\[
\delta^0(\varphi - \tilde{\chi}_n\varphi)(x) = (1 - \chi_n)(x)\delta^0(\varphi)(x) + \frac{1}{2c(x)} \sum_{e,e^+ = x} r(e)d^0\chi_n(e)\varphi(e).
\]

As a consequence, because $\delta^0\varphi \in l^2(\mathcal{V})$, we have
\[
\lim_{n \to \infty} \|(1 - \chi_n)\delta^0\varphi\|_{l^2(\mathcal{V})} = 0.
\]

For the second term, we combine the property iii) of $\chi$-completeness for a graph with the Cauchy-Schwarz inequality to obtain for all $x \in \mathcal{V},$
\[
| \sum_{e,e^+ = x} r(e)\delta^0\chi_n(e)\varphi(e) |^2 \leq \sum_{e,e^+ = x} r(e)|\delta^0\chi_n(e)|^2 \sum_{e \in \text{supp}(\delta^0\chi_n), e^+ = x} r(e)|\varphi(e)|^2
\]
\[
\leq Cc(x) \sum_{e \in \text{supp}(\delta^0\chi_n), e^+ = x} r(e)|\varphi(e)|^2.
\]

So,
\[
\sum_{x \in \mathcal{V}} \frac{1}{c(x)} | \sum_{e,e^+ = x} r(e)\delta^0\chi_n(e)\varphi(e) |^2 \leq \sum_{x \in \mathcal{V}} C \sum_{e \in \text{supp}(\delta^0\chi_n), e^+ = x} r(e)|\varphi(e)|^2
\]
\[
\leq C \sum_{e \in \text{supp}(\delta^0\chi_n)} r(e)|\varphi(e)|^2 \to 0, \text{ when } n \to \infty,
\]
by the properties (4.1) and (4.2).

Hence
\[
F_n \to F \text{ in } \mathcal{H}, \quad TF_n \to T_{\text{min}}F \text{ in } \mathcal{H},
\]
where $F_n = (f_n, \tilde{\chi}_n\varphi, \phi_n)$ and $T_{\text{min}}F(f, \varphi, \phi) = (\delta^0_{\text{min}}\varphi, \delta^0_{\text{min}}f + \delta^1_{\text{min}}\phi, d^1_{\text{min}}\varphi)$. Then $F \in \text{Dom}(T_{\text{min}})$.

2nd Step: To show that $T$ is essentially self-adjoint, we will prove that $T_{\text{max}} = T_{\text{min}}$. By the first step, Theorem 1 in [AT] and Proposition 5.1 it remains to show that:
\[
\text{Dom}(T_{\text{max}}) \subseteq \text{Dom}(d^0_{\text{max}}) \oplus (\text{Dom}(\delta^0_{\text{max}}) \cap \text{Dom}(d^1_{\text{max}})) \oplus \text{Dom}(d^1_{\text{max}}).
\]

Let $F = (f, \varphi, \phi) \in \text{Dom}(T_{\text{max}})$ then $TF \in \mathcal{H}$. This implies that $\delta^0\varphi \in l^2(\mathcal{V})$, $\delta^0f + \delta^1\phi \in l^2(\mathcal{E})$ and $d^1\varphi \in l^2(F)$. As consequence, by the definition of $\delta^0_{\text{max}}$ and $d^1_{\text{max}}$ we have $\varphi \in \text{Dom}(\delta^0_{\text{max}}) \cap \text{Dom}(d^1_{\text{max}})$. Moreover, by $\chi$-completeness of $\mathcal{T}$, there exists a sequence of cut-off functions $(\chi_n)_n \subseteq C_c(\mathcal{V})$. Then, the parallelogram identity with Lemma 2.8 we get
\[
||d^0(\chi_n f) + \delta^1(\tilde{\chi}_n \phi)||_{l^2(\mathcal{E})}^2 = ||d^0\chi_n f||_{l^2(\mathcal{E})}^2 + ||\delta^1\tilde{\chi}_n \phi||_{l^2(\mathcal{E})}^2.
\]

Now, it remains to prove that $d^0(\chi_n f) + \delta^1(\tilde{\chi}_n \phi)$ converges in $l^2(\mathcal{E})$. Indeed, we need some formulas taken in Lemma 2.9 and [M2] to give that:
\[
d^0(\chi_n f) = \tilde{\chi}_n d^0(f) + \tilde{f}d^0(\chi_n).
\]
\[
\delta^1(\tilde{\chi}_n \phi)(e) = \tilde{\chi}_n(e)\delta^1(\phi)(e) + \frac{1}{6r(e)} \sum_{z \in \mathcal{F}_n} s(e, x) \left[ d^0(\chi_n)(e^-, x) + d^0(\chi_n)(e^+, x) \right] \phi(e, x).
\]
Therefore, we have
\[ \|d^0(f - \chi_n f) + \delta^1(\phi - \chi_n \bar{\phi})\|_\mathcal{I}(\mathcal{E})^2 = \|(1 - \chi_n)(d^0 f + \delta^1 \phi) + f d^0 \chi_n + \mathcal{I}_n\|_\mathcal{I}(\mathcal{E})^2 \leq 3 \left( \|(1 - \chi_n)(d^0 f + \delta^1 \phi)\|_\mathcal{I}(\mathcal{E})^2 + \|f d^0 \chi_n\|_\mathcal{I}(\mathcal{E})^2 + \|\mathcal{I}_n\|_\mathcal{I}(\mathcal{E})^2 \right) \]
Because \(d^0 f + \delta^1 \phi \in \mathcal{I}(\mathcal{E})\), we have
\[ \lim_{n \to \infty} \|(1 - \chi_n)(d^0 f + \delta^1 \phi)\|_\mathcal{I}(\mathcal{E})^2 = 0. \]
By Proposition 5.1 we have
\[ \lim_{n \to \infty} \|\mathcal{I}_n\|_\mathcal{I}(\mathcal{E})^2 = 0. \]
Moreover, by the hypothesis iii) of \(\chi\)-completeness we have
\[ \|f d^0 \chi_n\|_\mathcal{I}(\mathcal{E})^2 = \frac{1}{2} \sum_{\epsilon \in \mathcal{E}} r(\epsilon) |\bar{f}(\epsilon) d^0(\chi_n)(\epsilon)|^2 \leq \sum_{\epsilon \in \mathcal{E}} |f(\epsilon) d^0(\chi_n)(\epsilon)|^2 = \sum_{\epsilon \in \mathcal{E}} |f(\epsilon) d^0(\chi_n)(\epsilon)| \sum_{\epsilon, \epsilon^+ = \epsilon} |r(\epsilon) d^0(\chi_n)(\epsilon)| \leq C \sum_{\epsilon \in \mathcal{V}_n} c(\epsilon) |f(\epsilon)|^2 \]
where \(\mathcal{V}_n := \{ x \in \mathcal{V}, \exists \epsilon \in \text{supp}(d^0 \chi_n) \text{ such that } \epsilon^+ = x \} \). This term tends to 0 by the property (4.2).

\[ \square \]

**Theorem 5.4.** Let \( \mathcal{T} = (\mathcal{K}, \mathcal{F}) \) be a \(\chi\)-complete triangulation. Then \( T \) is essentially self-adjoint on \( C_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \) if and only if \( L \) is essentially self-adjoint on \( C_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \).

**Proof:**

Since
\[ T(\mathcal{C}_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F})) \subseteq \mathcal{C}_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F}), \]
using the same technique in the proof of Proposition 13 in [AT], the result holds.

\[ \square \]

**Corollary 5.5.** Let \( \mathcal{T} = (\mathcal{K}, \mathcal{F}) \) be a \(\chi\)-complete triangulation then \( L \) is essentially self-adjoint on \( C_c(\mathcal{V}) \oplus C_c(\mathcal{E}) \oplus C_c(\mathcal{F}) \).

6. **Examples**

6.1. **A triangulation with 1-dimensional decomposition.** We now strengthen the previous example and follow ideas of [BG] and [BGJ].

**Definition 6.1.** (1-dimensional decomposition) A 1-dimensional decomposition of the graph \( \mathcal{K} = (\mathcal{V}, \mathcal{E}) \) is a family of finite sets \( (\mathcal{S}_n)_{n \in \mathbb{N}} \) which forms a partition of \( \mathcal{V} \), that is \( \mathcal{V} = \sqcup_{n \in \mathbb{N}} \mathcal{S}_n \), such that for all \( x \in \mathcal{S}_n, y \in \mathcal{S}_m \),
\[ (x, y) \in \mathcal{E} \Rightarrow |n - m| \leq 1. \]
Given such a 1-dimensional decomposition, we write \( B_n := \cup_{i=0}^n S_i \). We set,

\[
\begin{align*}
\deg_{S_n}^+(x) &:= \frac{1}{c(x)} \sum_{y \in V(x) \cap S_{n+1}} r(x, y) \quad \text{for all } x \in S_n, \\
\deg_{S_n}^0(x) &:= \frac{1}{c(x)} \sum_{y \in V(x) \cap S_n} r(x, y) \quad \text{for all } x \in S_n, \\
\deg_{S_n \times S_{n+1}}(e) &:= \frac{1}{r(e)} \sum_{x \in F_e \cap (S_n \cup S_{n+1})} s(e, x) \quad \text{for all } e \in S_n \times S_{n+1}, \\
\deg_{S_n^2}^0(e) &:= \frac{1}{r(e)} \sum_{x \in F_e \cap S_n} s(e, x) \quad \text{for all } e \in S_n^2, \\
\deg_{S_n^2}^+(e) &:= \frac{1}{r(e)} \sum_{x \in F_e \cap S_{n+1}} s(e, x) \quad \text{for all } e \in S_n^2.
\end{align*}
\]

We denote

\[
\eta_n^+ := \sup_{x \in S_n} \deg_{S_n}^+(x), \quad \beta_n := \sup_{e \in S_n \times S_{n+1}} \deg_{S_n \times S_{n+1}}(e), \quad \gamma_n^+ := \sup_{e \in S_n^2} \deg_{S_n^2}^+(e).
\]

**Theorem 6.2.** Let \( T = (K, F) \) be a triangulation and \( (S_n)_{n \in \mathbb{N}} \) a 1-dimensional decomposition of the graph \( K \). Assume that

\[
\sum_{n \in \mathbb{N}} \frac{1}{\sqrt{\xi(n, n+1)}} = \infty,
\]

with \( \xi(n, n+1) = \eta_n^+ + \eta_{n+1}^- + \beta_n + \gamma_n^+ + \gamma_{n+1}^- \). Then \( T \) is \( \chi \)-complete and in particular, \( L \) is essentially self-adjoint on \( C_c(V) \oplus C_c(E) \oplus C_c(F) \).

**Proof:**

We set

\[
\chi_n(x) = \begin{cases} 
1 & \text{if } d_{\text{comb}}(o, x) \leq n, \\
\max \left( 0, 1 - \sum_{k=n}^{d_{\text{comb}}(o, x)-1} \frac{1}{\sqrt{\xi(k, k+1)}} \right) & \text{if } d_{\text{comb}}(o, x) > n.
\end{cases}
\]
Since the series diverges, \( \chi_n \) is with finite support. Note that \( \chi_n \) is constant on \( S_n \). If \( x \in S_m \) with \( m > n \), we have

\[
\frac{1}{c(x)} \sum_{y \in V(x) \cap S_{n+1}} r(x, y)|\chi_n(x) - \chi_n(y)|^2 \leq \frac{\deg_{S_m}^+(x)}{\xi(m, m + 1)} \leq 1.
\]

\[
\frac{1}{c(x)} \sum_{y \in V(x) \cap S_m} r(x, y)|\chi_n(x) - \chi_n(y)|^2 = 0.
\]

\[
\frac{1}{c(x)} \sum_{y \in V(x) \cap S_{m-1}} r(x, y)|\chi_n(x) - \chi_n(y)|^2 \leq \frac{\deg_{S_m}^-(x)}{\xi(m - 1, m)} \leq 1.
\]

On the other hand,
- If \( e \in S_m \times S_{m+1} \), we get

\[
\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap (S_m \cup S_{m+1})} s(e, x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 \leq \frac{\deg_{S_m \times S_{m+1}}^+(x, y)}{\xi(m, m + 1)} \leq 1.
\]

- If \( e \in S_m^2 \), we get

\[
\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap S_m} s(e, x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 = 0
\]

\[
\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap S_{m-1}} s(e, x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 \leq \frac{\deg_{S_m}^-(e)}{\xi(m, m + 1)} \leq 1.
\]

Then \( T \) is \( \chi \)-complete and in particular, \( L \) is essentially self-adjoint by Corollary 5.5.

\[\square\]

6.2. **A triangular tree.** Let \( T \) be a triangular tree, endowed with an origin. Due to its structure, one can take

\[
(**) \quad \begin{cases}
\deg_{S_n}^- (x) := \frac{1}{c(x)} r(x, e) & \text{for all } x \in S_n, \\
\deg_{S_n \times S_{n+1}}^+(e) := \frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap S_{n+1}} s(e, x) & \text{for all } e \in S_n \times S_{n+1}, \\
\deg_{S_n^2}^- (e) := \frac{1}{r(e)} s(e, e) & \text{for all } e \in S_n^2,
\end{cases}
\]

where \( e \) is a unique vertex in \( S_{n-1} \cap \mathcal{F}_e \).

**Proposition 6.3.** Let \( T \) be a triangular tree with its origin \( o \). Assume that

\[
\sum_{n \in \mathbb{N}} \frac{1}{\sqrt{\xi(n, n + 1)}} = \infty,
\]

with \( \xi(n, n + 1) = \eta_n^+ + \eta_{n+1}^- + \beta_n + \gamma_{n+1}^- \). Then \( T \) is \( \chi \)-complete and in particular, \( L \) is essentially self-adjoint on \( C_c(V) \oplus C_c(E) \oplus C_c(F) \).

**Proof:**

Use the same method as Theorem 6.2 with (**)
6.3. Essential self-adjointness on the simple case. In [W] and [D], they prove that $L_0$ is essentially self-adjoint on $C_c(V)$ when the graph is simple. But the self-adjointness property does not always hold with other operators in the simple case. We recall the operator $L_1^{-1}$ is not necessarily essentially self-adjoint on simple tree, see [BGJ]. Moreover, we refer to [GS] for the adjacency matrix $A_K = \text{deg} - L_0$ where $\text{deg}$ denotes the operator of multiplication with the functions which shows that the deficiency indices of $A_K$ are infinite. In this framework, it is important to notice that $L_1$ and $L_2$ are not necessarily essentially self-adjoint on a simple triangulation.

**Proposition 6.4.** Let $\mathcal{T}$ be a simple triangular tree. Assume that

$$\text{off}(n) = \#(V(x) \cap S_{n+1}), \ x \in S_n$$

$$n \mapsto \frac{\text{off}^2(n)}{\text{off}(n + 1)} \in l^1(\mathbb{N}).$$

Then, $L_1$ is not essentially self-adjoint on $C_c(\mathcal{E})$ and the deficiency indices are infinite.

**Proof:**

We construct $\varphi \in l^2(\mathcal{E}) \setminus \{0\}$, such that $\varphi \in \text{Ker}(L_1^* + i)$ and such that $\varphi$ is equal to constant $C_n$ on $S_n \times S_{n+1}$. It takes the value 0 on $S_n^2$. Given the fact that $(x,y) \in S_n^2$, we get

$$C_n \left(\#(V(x) \cap S_{n+1}) - \#(V(y) \cap S_{n+1})\right) = 0.$$ 

It holds because of the condition (6.1). Now, we set $(x,y) \in S_n \times S_{n+1}$ and with the condition (6.1), we have

$$(\text{off}(n) + 1 + i)C_n - \text{off}(n + 1)C_{n+1} - C_{n-1} = 0.$$ 

We can then apply Theorem 5.1 in [BGJ] to obtain the conclusion.

We will see now also that $L_2$ is not necessarily essentially self-adjoint on simple triangulation.

**Proposition 6.5.** Let $\mathcal{T}$ be a simple triangulation with 1-dimensional decomposition as shown in Figure 4. Assume that

$$n \mapsto \frac{\#S_{2n}}{\#S_{2(n+1)}} \in l^1(\mathbb{N}).$$

Then, $L_2$ is not essentially self-adjoint on $C_c(\mathcal{F})$.

**Proof:**

[Diagram of a particular triangulation with 1-dimensional decomposition]
We consider the faces in Figure 4 as follows:

\[ \varpi \in \mathcal{F} \iff \text{there exists } n \in \mathbb{N} \cup \{0\} \text{ such that } \varpi = (e_{2n+1}, x) \in (S_{2n+1}^2 \times S_{2n}) \cup (S_{2n+1}^2 \times S_{2n+2}) \cdot \]

Set \( \phi \in l^2(\mathcal{F}) \setminus \{0\} \) such that \( \phi \in \text{Ker}(L_2^* + i) \). For \( n \in \mathbb{N} \), it is given by

\[
\phi(e_{2n+1}, x) := \begin{cases}
C_{2n+2} & \text{for all } x \in S_{2n+2}, \\
C_{2n} & \text{for all } x \in S_{2n}.
\end{cases}
\]

Let \( x \in S_{2n+2} \), we have

\[
(L_2^* + i)(\phi)(e_{2n+1}, x) = \sum_{u \in \mathcal{F}_{e_{2n+1}}} \phi(e_{2n+1}, u) + \sum_{u \in \mathcal{F}_{(e_{2n+1}^* +)} \phi(e_{2n+1}, u) + \sum_{u \in \mathcal{F}_{(x, e_{2n+1}^* +)}} \phi(x, e_{2n+1}, u)
\]

+ \( i\phi(e_{2n+1}, x) = 0. \)

Hence, we get

\[
(\#S_{2n+2} + 2 + i)C_{2n+2} + (\#S_{2n})C_{2n} = 0. \tag{6.3}
\]

By the equation (6.3), we get

\[
\|\phi_{|S_{2n+1}^2 \times S_{2n+2}}\|^2_{l^2(\mathcal{F})} = \frac{1}{6} \sum_{[x, y, z] \in S_{2n+1}^2 \times S_{2n+2}} |\phi(x, y, z)|^2
\]

= \frac{1}{6} (\#C_{2n+2})^2 (\#S_{2n+2})^2

= \frac{1}{6} (\#C_{2n})^2 (\#S_{2n+2})^2

= \frac{(\#S_{2n})(\#S_{2n+2})}{|\#S_{2n+2} + 2 + i|^2} \|\phi_{|S_{2n+1}^2 \times S_{2n}}\|^2_{l^2(\mathcal{F})}
\]

Since \( \lim_{n \to \infty} \frac{\#S_{2n}}{\#S_{2n+1}} = 0 \), we get by induction

\[
C := \sup_{n \in \mathbb{N}} \|\phi_{|S_{2n+1}^2 \times S_{2n+2}}\|^2_{l^2(\mathcal{F})} < \infty.
\]

Thus

\[
\|\phi_{|S_{2n+1}^2 \times S_{2n+2}}\|^2_{l^2(\mathcal{F})} \leq C \frac{(\#S_{2n})(\#S_{2n+2})}{|\#S_{2n+2} + 2 + i|^2}.
\]

From (6.2), we conclude that \( \phi \in l^2(\mathcal{F}) \). By mimicking the proof of Theorem X.36 of [RSv2] one shows that \( \dim\text{Ker}(L_2^* + i) \geq 1 \) and thus \( L_2^* \) is not essentially self-adjoint on \( C_c(\mathcal{F}) \).

\[ \square \]

**Remark 6.6.** By one of Propositions 6.5 and 6.4, we conclude that \( L \) is not necessarily essentially self-adjoint on a simple triangulation.

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