Geometric U-folds in four dimensions

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Abstract
We describe a general construction of geometric U-folds compatible with a non-trivial extension of the global formulation of four-dimensional extended supergravity on a differentiable spin manifold. The topology of geometric U-folds depends on certain flat fiber bundles which encode how supergravity fields are globally glued together. We show that smooth non-trivial U-folds of this type can exist only in theories where both the scalar and space-time manifolds have non-trivial fundamental group and in addition the scalar map of the solution is homotopically non-trivial. Consistency with string theory requires smooth geometric U-folds to be glued using subgroups of the effective discrete U-duality group, implying that the fundamental group of the scalar manifold of such solutions must be a subgroup of the latter. We construct simple examples of geometric U-folds in a generalization of the axion–dilaton model of $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet, whose scalar manifold is a generally non-compact Riemann surface of genus at least two endowed with its uniformizing metric. We also discuss the relation between geometric U-folds and a moduli space of flat connections defined on the scalar manifold, which involves certain character varieties not studied in the literature.

Keywords: supergravity, U-folds, differential geometry

1. Introduction and main results
U-folds are consistent backgrounds that incorporate in a non-trivial manner the natural symmetries of a supergravity or string theory [1–7]. In this note, a U-fold means a supergravity background which can be obtained by gluing local solutions using U-dualities, aside from local diffeomorphisms and gauge transformations. Such solutions can be promoted to string theory backgrounds only when all U-dualities involved belong to the discrete subgroup allowed by charge quantization.
Many particular constructions of U-folds have been considered in the literature (see, for example, [8–18]), where it was often suggested that some of them do not admit any geometric description. However, no fully general and precise mathematical definition of U-folds has yet been given. Due to this fact, it is unclear which U-folds may admit equivalent (though possibly non-standard) descriptions through ordinary objects of differential geometry, namely objects obtained via constructions involving manifolds and smooth maps satisfying various conditions—the latter of which include fiber bundles. It is thus possible that many backgrounds currently postulated to be ‘non-geometric’ may in fact admit descriptions within the framework of global differential geometry—albeit such a description may be ‘non-standard’.

García-Fernández–Shahbazi [19, 20] considered a particular construction of non-standard geometric solutions based on a large class of non-simply-connected complex manifolds with properties that are quite different from those of traditional supersymmetric backgrounds. It was argued that those solutions, though constructed geometrically in terms of manifolds and bundles, can be interpreted as U-folds and hence appear to be ‘non-geometric’ when viewed from the perspective of patching traditional local solutions using an open cover, in the sense that the gluing of the restrictions of the global solution to the open sets of a cover involves non-trivial U-duality transformations. This shows how it is possible to construct large classes of apparently non-geometric backgrounds using ordinary manifolds and bundles, provided that globally the geometric objects involved are topologically non-trivial.

In this note we propose a general geometric construction of a class of U-folds (called below geometric U-folds), which can be considered in any four-dimensional extended supergravity theory. The general global formulation of such theories on a space-time manifold \( M \) requires one to specify a certain flat symplectic vector bundle \( \mathcal{S} \) defined over the target manifold \( \mathcal{M} \) of the kinetic sigma model of scalar fields. The structure group of \( \mathcal{S} \) is the U-duality group \( G_{\sigma} \) of the theory acting in an appropriate symplectic representation \( \rho \) which encodes electromagnetic dualities. More precisely, \( \mathcal{S} \) is the vector bundle associated through \( \rho \) to a flat principal \( G \)-bundle \( Q \) defined over \( \mathcal{M} \), where \( Q \) must be specified when defining the theory. Given a classical solution \( \text{Sol} \) of the equations of motion with underlying space-time \( M \), the pull-back of \( \mathcal{S} \) through the sigma model map \( \Phi : M \rightarrow \mathcal{M} \), which encodes the scalar fields of \( \text{Sol} \), gives a flat vector bundle \( \mathcal{S}_\Phi \) defined over \( M \). When restricted to the open sets of a trivializing cover, the transition functions of \( \mathcal{S}_\Phi \) encode U-duality transformations. The crucial observation is that the flat bundle \( \mathcal{S} \) can be topologically non-trivial, hence \( \mathcal{S}_\Phi \) can also be non-trivial provided that \( M \) and \( \mathcal{M} \) are not simply-connected and that the homotopy class of \( \Phi \) differs from that of the constant map. In this case, at least one of the transition functions of \( \mathcal{S}_\Phi \) must be non-trivial, so \( \text{Sol} \) can be interpreted as a non-trivial U-fold. As remarked in reference [7], a non-trivial space-time fundamental group is a standard underlying assumption in the construction of non-trivial U-folds. In this note, we are able to show in precise mathematical terms what is the general relation between the fundamental group of the space-time manifold and the global topology of a given U-fold solution. In fact, we obtain that such relation involves in a crucial way the fundamental group of the scalar manifold and the homotopy class of the scalar map in a very specific way. To the best knowledge of the authors, these are some of the first generic results on the global topology of U-folds available in the literature.

Under the same assumptions on \( M, \mathcal{M} \) and \( \Phi \), the U-fold interpretation applies even for fluxless solutions (solutions for which all electromagnetic field strengths are identically zero), since in that case \( \text{Sol} \) can be viewed as a multivalued solution of the standard supergravity theory (the theory constructed using the universal cover \( \mathcal{M}_0 \) of \( \mathcal{M} \)), which is glued from local solutions of the latter using U-duality transformations. Thus a global geometric solution can appear to be ‘non-geometric’ when interpreted by patching its restrictions to the open sets of a cover, the reason being, as in [19, 20], that the global solution involves topologically
non-trivial geometric objects. As familiar from the cosmic string literature, non-simply-connected spacetimes can often be mimicked by considering source-full solutions defined on simply-connected spacetimes containing localized codimension two sources. In such set-ups, the regular part of the full solution is a source-free solution defined on the complement of all localized sources, a complement which is incomplete and need not be simply connected. In this paper, we consider source-free solutions (which may be restrictions of source-full solutions to the complement of all localized sources).

The standard formulation of four-dimensional supergravity theories involves scalar manifolds $M_0$ which are simply-connected; for $\mathcal{N} \geq 3$, these manifolds are symmetric spaces of non-compact type, which are in fact contractible. As a consequence, the flat bundles $Q_0$ and $S_0$ of the standard formulation are always trivial. However, the local computations leading to the construction of supergravity theories only fix the scalar manifold $\mathcal{M}$ up to local isometries, i.e. they only determine its universal cover—this cover is the simply-connected manifold $M_0$ used in the standard formulation. This observation implies that one can consider generalized supergravity models in which $\mathcal{M}$ is a smooth quotient of $M_0$ through the action of a discrete subgroup $\Gamma$ of the effective\textsuperscript{3} U-duality group $G_{\text{eff}}$, in which case $\pi_1(\mathcal{M}) \simeq \Gamma \subset G_{\text{eff}}$. It is such generalized models that admit geometric U-fold solutions. We will see that such U-folds are glued using U-duality transformations belonging to $\Gamma$, so they can be lifted to string theory U-folds only when $\pi_1(\mathcal{M})$ is a subgroup of the discrete U-duality group $G_0(\mathbb{Z}) \subset G_0$ which survives \cite{21} in string theory.

We point out a close relation between geometric U-folds and certain moduli spaces of flat connections defined on the scalar manifold. The latter lead to character varieties that, to our best knowledge, have not been systematically studied in the literature. Finally, we illustrate our construction with two examples. The first is the ‘generalized axion–dilaton model’, namely $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet with $M$ given by a (generally non-compact) Riemann surface of genus $g \geq 2$ endowed with its uniformizing metric. The second is $\mathcal{N} = 8$ supergravity with scalar manifold given by a double coset $\Gamma \backslash \mathcal{E}_7(7)/\langle SU(8)/\mathbb{Z}_2 \rangle$, where $\Gamma$ is a discrete subgroup of $\mathcal{E}_7(7)$. For the generalized axion–dilaton model, we construct explicit geometric U-folds which are similar to those of \cite{22}, being sourced by cosmic strings.

The construction outlined in this note leads to various questions for further research. First of all, we lack the global mathematical formulation of four-dimensional supergravity on topologically non-trivial four-manifolds. Obtaining such formulation, aside from a mathematically interesting problem, is a mandatory step in order to understand the global structure of supergravity solutions, and in particular supergravity U-folds. Work in this direction is already in progress \cite{23} and in fact the project has evolved into a long term research program focused on understanding the global mathematical structure of supergravity. Aside from studying the global structure of particular supergravity U-fold solutions, it would be interesting to properly define and study the moduli space of geometric U-folds in order to understand the space of inequivalent U-folds that can be obtained as solutions of a particular supergravity theory. This problem would require studying the relevant character varieties of discrete subgroups of U-duality groups. It would also be interesting to consider similar constructions in $\mathcal{N} = 1$ supergravity and to systematically analyze source-full U-fold solutions. Finally, it would be interesting to construct further explicit examples of geometric U-folds and to study their properties and physics consequences.

\textsuperscript{3}In $\mathcal{N} = 2$ theories, $G_{\text{eff}}$ is a discrete quotient of $G_0$ which acts effectively on the scalar manifold, while for $\mathcal{N} \geq 3$ theories we have $G_{\text{eff}} = G_0$. 

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Table 1. The simply-connected scalar manifolds \(M_0\) of standard four-dimensional supergravity theories, where \(n_v\) denotes the number of vector multiplets. For \(\mathcal{N} \geq 3\), these scalar manifolds are diffeomorphic with \(\mathbb{R}^{\dim M_0}\) and hence contractible. In the table, \(\text{SO}(6, n_v)\) denotes the connected component of the identity in the group \(\text{SO}(6, n_v)\) (which has two connected components). The abbreviation PSK means a (simply-connected) projective special Kähler manifold.

| \(\mathcal{N}\) | \(M_0\) | \(\dim_{\mathbb{R}} M_0\) | Indecomposable |
|---|---|---|---|
| 2 | PSK | \(2n_v\) | Not necessarily |
| 3 | \(\text{SU}(1, n_v)\) | \(6n_v\) | Yes |
| 4 | \(\text{SU}(2, 2) \times \text{U}(1)\) \(\times \text{SO}(5, n_v)\) | \(2 + 6n_v\) | No |
| 5 | \(\text{SU}(1, 5)\) \(\times \text{SO}(7, n_v)\) | 10 | Yes |
| 6 | \(\text{SU}(1, 7) \times \text{SO}(10, n_v)\) \(\times \text{U}(6)\) | 30 | Yes |
| 8 | \(\text{SO}(16)\) \(\times \text{U}(8)\) \(\times \text{SO}(8)\) | 70 | Yes |

2. The global formulation of extended four-dimensional supergravity theories

In this section we outline the global formulation of un-gauged \(\mathcal{N} \geq 2\) supergravity\(^4\), following references [24, 25], describing all non-scalar fields of the theory as global sections of appropriate fiber bundles.

Consider a four-dimensional, oriented, Lorentzian spin manifold \((M, g)\), whose bundle of complex chiral spinors we denote by \(S\). The standard four-dimensional supergravity theories are constructed using certain simply-connected Riemannian scalar manifolds \((M_0, G_0)\) which are summarized in table 1. The U-duality group \(G_0\) of these theories is summarized in table 2. The Lagrangian contains a total number \(n\) of \(U(1)\) gauge fields, whose transformation under U-dualities is determined by a certain group morphism \(\rho : G_0 \to \text{Sp}(2n, \mathbb{R})\). We have \(n = n_v + m\), where \(n_v\) is the number of vector multiplets (which can be non-zero only for \(\mathcal{N} \in \{2, 3, 4\}\) while \(m\) is the number of \(U(1)\) gauge fields in the gravity multiplet. Namely:

1. When \(\mathcal{N} = 2\), \(M_0\) is a simply-connected projective special Kähler (PSK) manifold \([26–28]\) and \(G_0\) is a discrete cover of its group of special isometries\(^5\) \(\text{Iso}_0(M_0)\). The dimension of \(M_0\) (as a real manifold) equals \(2n_v\) and we have \(n = n_v + 1\), the supplementary \(U(1)\) gauge field being the graviphoton.

2. When \(\mathcal{N} \geq 3\), \(M_0\) is a certain simply-connected globally symmetric space of non-compact type, which is de Rham irreducible except for \(\mathcal{N} = 4\) and \(n_v \geq 1\), in which case it has two simply-connected and irreducible factors of non-compact type. By the Hadamard–Cartan theorem, it follows that \(M_0\) is diffeomorphic with \(\mathbb{R}^{\dim M_0}\). Moreover, the U-duality group \(G_0\) is the connected component \(\text{Iso}_0(M_0, G_0)\) of the identity in the isometry group \(\text{Iso}(M_0, G_0)\) (see table 2). The U-duality group acts transitively on \(M_0\) and we have \(M_0 = G_0/H_0\) where \(H_0 \subset G_0\) is the isotropy group of this action. For \(\mathcal{N} \in \{3, 4\}\), the pure supergravity theory is coupled to \(n_v\) vector multiplets and \(M_0\) is uniquely determined by \(\mathcal{N}\) and \(n_v\). For \(\mathcal{N} \geq 5\), the theory does not admit coupling to vector multiplets, thus \(n_v = 0\), \(n = m = 0\) and \(M_0\) is uniquely determined by \(\mathcal{N}\).

For \(\mathcal{N} = 2\) theories, the U-duality group \(G_0\) is a cover of \(\text{Iso}_0(M_0)\) and hence the action of \(G_0\) on \(M\) induced through the covering map \(G_0 \to \text{Iso}_0(M)\) may be non-effective (we will

\(^4\)For \(\mathcal{N} = 2\), we consider only the theory coupled to vector multiplets.

\(^5\)Those isometries of \(M_0\) which preserve the complex structure as well as the flat symplectic connection (sometimes called ‘duality symmetries’ or ‘duality invariances’ [29]).
see an example of this in section 5). Define the effective U-duality group to be the group \( G_{\text{eff}} = \text{Iso}_0(\mathcal{M}) \) of special isometries, which acts effectively on \( \mathcal{M}_0 \). For \( \mathcal{N} \geq 3 \) theories, we set \( G_{\text{eff}} = G_0 \).

In this paper, we will work with the more general choice of scalar manifold:

\[ \mathcal{M} = \Gamma \backslash \mathcal{M}_0, \tag{2.1} \]

where \( \Gamma \subset G_{\text{eff}} \) is a discrete subgroup of the effective U-duality group such that \( \Gamma \backslash \mathcal{M}_0 \) is smooth. We endow \( \mathcal{M} \) with the metric \( \mathcal{G} \) induced from \( \mathcal{G}_0 \) and let \( \pi : \mathcal{M}_0 \rightarrow \mathcal{M} \) denote the canonical projection. Then \( (\mathcal{M}_0, \mathcal{G}_0) \) is the Riemannian universal cover of \( \mathcal{M} \) and \( \Gamma = \text{Aut}(\pi) \cong \pi_1(\mathcal{M}) \) is the deck group of this cover. Moreover:

1. When \( \mathcal{N} = 2 \) and \( \Gamma \) is non-trivial, the manifold \( \mathcal{M} \) is projective special Kähler.
2. When \( \mathcal{N} \geq 3 \) and \( \Gamma \) is non-trivial, the manifold \( \mathcal{M} \) is a locally symmetric space.

Let \( \text{Iso}(\mathcal{M}, \mathcal{G}) \) denote the isometry group of \( (\mathcal{M}, \mathcal{G}) \) and \( K_\Gamma \) be the largest subgroup of \( \text{Iso}(\mathcal{M}_0, \mathcal{G}_0) \) which contains \( \Gamma \) as its normal subgroup. Then there exists a short exact sequence \([30]\):

\[ 1 \longrightarrow \Gamma \longrightarrow K_\Gamma \longrightarrow \text{Iso}(\mathcal{M}, \mathcal{G}) \longrightarrow 1, \]

which can be used to determine the isometry group of \( (\mathcal{M}, \mathcal{G}) \). The bosonic Lagrangian of the theory based on the scalar manifold (2.1) is globally determined (up to a discrete ambiguity described below) by:

- A principal \( H_0 \)-bundle \( P \) over \( \mathcal{M} \).
- A flat principal \( G_0 \)-bundle \( Q \) over \( \mathcal{M} \).

Namely:

1. When \( \mathcal{N} = 2 \), we have \( H_0 = \text{U}(1) \) and \( P \) is the canonical circle bundle of \( \mathcal{M} \) (the circle bundle of that holomorphic line bundle whose first Chern class equals the Kähler class).
2. When \( \mathcal{N} \geq 3 \), \( H_0 \) is the isotropy group of the symmetric space \( \mathcal{M}_0 \), while \( P \) be the principal \( H_0 \)-bundle \( \Gamma \backslash \mathcal{M}_0 \rightarrow \mathcal{M} \).

When \( \Gamma = 1 \), the corresponding bundles (which are topologically trivial) are denoted by \( P_0 \) and \( Q_0 \) and are the bundles used in the standard theory. Notice that \( Q_0 \) is trivial as a flat principal bundle since \( \mathcal{M}_0 \) is simply-connected, so \( Q_0 \) is determined by \( \mathcal{M}_0 \) and \( G_0 \) up to isomorphism of flat principal bundles. In the general theory (when \( \Gamma \) is non-trivial), the flat connection of \( Q \) defines the holonomy representation:

\[ \Delta : \Gamma \cong \pi_1(\mathcal{M}, y) \rightarrow G_0, \]
where $y \in \mathcal{M}$ is an arbitrary point. The universal cover $\mathcal{M}_0$ can be viewed as a principal bundle $C$ over $\mathcal{M}$ with discrete structure group given by $\Gamma$. Then $Q$ is isomorphic with the principal $G_\rho$-bundle $C \times_{\Delta} G_\rho$ associated to $C$ through $\Delta$. Consider the flat vector bundle $S = Q \times_{\rho \Delta} \mathbb{R}^{2n}$ of rank $2n$ over $\mathcal{M}$, which is associated to $Q$ through the representation $\rho$. Then $S$ is isomorphic with the vector bundle $C \times_{\rho \Delta} \mathbb{R}^{2n}$ associated to $C$ through the representation:

$$\rho \circ \Delta : \Gamma \to \text{Sp}(2n, \mathbb{R}).$$

The bosonic fields appearing in the Lagrangian are:

- The Lorentzian metric $g$ of $M$.
- A smooth map $\Phi : M \to \mathcal{M}$. Using this map we can pull back $P$, $Q$ and $S$ to the following bundles defined over $M$:

  $$P_\Phi \overset{\text{def}}{=} \Phi^*(P), \quad Q_\Phi \overset{\text{def}}{=} \Phi^*(Q), \quad S_\Phi \overset{\text{def}}{=} \Phi^*(S).$$

- An $S_\Phi$-valued closed 2-form $F \in \Omega^2(M, S_\Phi)$, which describes the electric and magnetic field strengths of the U(1) gauge fields.

Notice that $Q_\Phi$ is a flat principal $G_\rho$-bundle defined over $M$, whose holonomy representation is given by:

$$\Delta \circ \Phi_* : \pi_1(M, x) \to G_\rho,$$

where $\Phi_* : \pi_1(M, x) \to \pi_1(M, y)$ is the homotopy push-forward through $\Phi$ and we took $y = \Phi(x)$ for some $x \in M$. Similarly, $S_\Phi \overset{\text{def}}{=} Q_\Phi \times_{\rho \Delta} \mathbb{R}^{2n}$ is a flat symplectic vector bundle defined over $M$, whose holonomy representation is given by $\rho \circ \Delta \circ \Phi_*$. The fermionic field content is determined by two vector bundles:

$$E_G = P_\Phi \times_{\theta_G} V_G, \quad E_f = P_\Phi \times_{\theta_f} V_f$$

associated to $P_\Phi$ through complex representations of $H_0$, namely the gravitino representation $\theta_G : H_0 \to \text{Aut}(V_G)$ and the fermionic matter representation $\theta_f : H_0 \to \text{Aut}(V_f)$, whose precise choice depends on the theory$^6$ Here $V_G$ and $V_f$ are complex vector spaces of appropriate dimensions. The fermionic fields in the Lagrangian are:

- The gravitino field, which is a one-form $\Psi \in \Omega^1(M, S \otimes E_G)$ valued in the vector bundle $S \otimes E_G$, where $S$ is the complex spinor bundle of $M$.
- A spinor $\chi \in \Omega^0(M, S \otimes E_f)$, which is a smooth section of $E_f$.

The reader may check that the objects introduced above reproduce the standard local (index) formulation of the field content of the theory, including the appropriate local description of the symmetries.

### 3. Geometric U-folds

Let $\text{Sol}$ be a finite ordered set of fields satisfying the equations of motion of the supergravity theory defined on $M$. Even though $\text{Sol}$ is defined geometrically$^7$, it can in certain cases be interpreted as a U-fold when understood by gluing local solutions defined on the sets of an

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$^6$ For certain supergravities, the spinor fields have ‘Kähler weight’ 1/2, so the representations $\theta_G$ and $\theta_f$ involve taking the square root of a U(1) sub-bundle $R$ of $P_\Phi$ which corresponds to R-symmetry. In these cases, fermions should strictly speaking be understood as sections of vector bundles associated to the principal bundle obtained from $P_\Phi$ upon replacing $R$ with a square root $R^{1/2}$, which exists only when the first Chern class $c_1(R)$ is even and whose choice introduces a discrete ambiguity in the global construction of the theory; see appendix.

$^7$ It is manifestly described using manifolds and maps of such, including sections of fiber bundles.
open cover of \( M \). Let \( \mathcal{U} \overset{\text{def}}{=} \{ U_a \}_{a \in I} \) be an open cover of \( M \) which is a trivializing cover for both \( Q_\Phi \) and \( P_\Phi \). Restricting \( \text{Sol} \) to \( U_\alpha \) gives a family \( \{ \text{Sol}_a \}_{a \in I} \), where:

\[
\text{Sol}_a \overset{\text{def}}{=} \text{Sol}|_{U_a}, \quad a \in I
\]

is a solution of the theory defined on \( U_a \). We are interested in how this family glues to yield the global solution \( \text{Sol} \). For intersecting open sets \( U_a \) and \( U_b \) of the cover, we have two possibilities:

1. The local solutions \( \text{Sol}_a \) and \( \text{Sol}_b \) are glued through transformations which do not involve a non-trivial U-duality.
2. The local solutions \( \text{Sol}_a \) and \( \text{Sol}_b \) are glued through transformations involving a non-trivial U-duality.

If \( Q_\Phi \) is topologically trivial, then we can arrange that the first case occurs for all pairs of intersecting open sets; in this case, we may in fact find a trivializing cover consisting of the single open set \( U = M \). In this situation, we say that \( \text{Sol} \) is trivial as a U-fold. If \( Q_\Phi \) is topologically non-trivial then the second possibility arises for at least one pair of intersecting open sets of any trivializing open cover. In this case, we say that \( \text{Sol} \) is a non-trivial geometric U-fold. Indeed, all fields of the theory, except for the metric and the scalar fields encoded by \( \Phi \), are either tensor fields defined on \( M \) or global sections of \( \mathcal{S}_\Phi, E_\rho \) or \( E_\mathcal{F} \). For example, the field-strength \( F \in \Omega^2(M, \mathcal{S}_\Phi) \) is a closed two-form taking values in \( \mathcal{S}_\Phi \). If \( Q_\Phi \) is topologically non-trivial then the open cover \( \{ U_a \}_{a \in I} \) contains at least two intersecting sets \( U_a, U_b \) with a non-trivial transition function:

\[
g_{ab} : U_a \cap U_b \to G_0
\]

for \( Q_\Phi \). Denote by \( F_a = F|_{U_a} \in \Omega^2(U_a, \mathcal{S}_\Phi|_{U_a}) \) the field-strength of the local solution in \( U_a \) and by \( F_b = F|_{U_b} \in \Omega^2(U_b, \mathcal{S}_\Phi|_{U_b}) \) the field-strength of the local solution in \( U_b \). Then:

\[
F_a = (\rho \circ g_{ab}) F_b.
\]

When \( Q_\Phi \) is non-trivial, we are thus forced to glue \( F_a \) to \( F_b \) using a non-trivial U-duality transformation for every intersecting pair for which \( g_{ab} \) is not identically 1 on \( U_a \cap U_b \). Hence:

- A smooth global solution \( \text{Sol} \) of extended supergravity for which \( Q_\Phi \) is topologically non-trivial and \( F \) is not identically zero is a non-trivial U-fold of the theory based on \( (\mathcal{M}, \mathcal{G}) \).

Recall that \( \Phi \) induces a map \( \Phi_\ast : \pi_1(M, x) \to \pi_1(\mathcal{M}, y) \), where \( x \in M \) is such that \( \Phi(x) = y \). Even when \( \mathcal{S}_\Phi \) is trivial or the gauge fields vanish, solutions for which the group \( \Phi_\ast(\pi_1(M, x)) \) is non-trivial can be viewed as U-folds of the standard theory based on the scalar manifold \( (\mathcal{M}_0, \mathcal{G}_0) \). To see this, let \( \mathcal{M}_0 \) denote the universal cover of \( M \) and \( p : \mathcal{M}_0 \to M \) be the canonical projection. For any choice of \( x \in M \) and of points \( x_0 \in \mathcal{M}_0 \) and \( y_0 \in \mathcal{M}_0 \) such that \( p(x_0) = x \) and \( \pi(y_0) = \Phi(x) = y \), the map \( \Phi \) lifts to a uniquely-determined map \( \Phi_0 : \mathcal{M}_0 \to \mathcal{M}_0 \) such that \( \Phi_0(x_0) = y_0 \) and such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{M}_0 & \xrightarrow{\Phi_0} & \mathcal{M}_0 \\
p \downarrow & & \downarrow \pi \\
M & \xrightarrow{\Phi} & \mathcal{M}
\end{array}
\]

Similarly, all remaining constituent fields of the solution \( \text{Sol} \) lift to fields defined on \( \mathcal{M}_0 \) which together with \( \Phi_0 \) form a solution \( \text{Sol}_0 \) (defined on \( \mathcal{M}_0 \)) of the standard supergravity
theory (which is constructed using $(\mathcal{M}_0, G_0)$). The original solution $\text{Sol}$ of the theory based on $(\mathcal{M}, G)$ can be identified with $\text{Sol}_0$, viewed as a \textit{multivalued} solution of the standard theory defined on $M$, with monodromies controlled by $U$-duality transformations belonging to $\Gamma \subset G_{\text{eff}}$. Indeed, let us view the universal cover $\pi : \mathcal{M}_0 \to \mathcal{M}$ as a principal $\Gamma$-bundle $C$ defined over $\mathcal{M}$. This pulls back through $\Phi$ to a principal $\Gamma$-bundle $C_\Phi \underset{\text{def}}{=} \Phi^*(C)$ defined over $M$, which in turn pulls back through $\rho$ to a principal $\Gamma$-bundle $C_\Phi^\rho \underset{\text{def}}{=} \rho^*(C_\Phi)$ defined over $M_0$.

The map $\Phi_0$ can now be viewed as a section $\hat{\Phi}$ of $C_\Phi$, i.e. as a multivalued global section\(^8\) of $C_\Phi$. This multivalued section is one-valued (i.e., it descends to an ordinary global section of $C_\Phi$) only when $\Phi_0$ factors through $p$, i.e. when $\Phi$ lifts to a map from $M$ to $M_0$. In turn, this happens iff $\Phi_*(\pi_1(M, x)) = 1$. When $\Phi_*(\pi_1(M, x)) \neq 1$, the multivalued section $\hat{\Phi}$ has monodromy valued in the structure group $\Gamma$ of $C_\Phi$, which is a subgroup of the effective $U$-duality group $G_{\text{eff}}$. Thus:

- A smooth global solution $\text{Sol}$ of the supergravity theory based on the scalar manifold $(\mathcal{M}, G)$ which has the property that $\Phi_*(\pi_1(M, x)) \neq 1$ is a non-trivial geometric U-fold.

The condition $\Phi_*(\pi_1(M, x)) \neq 1$ requires that both $M$ and $\mathcal{M}$ have non-trivial first homotopy group and that $\Phi$ be a homotopically non-trivial map. Similarly, the flat principal bundle $Q_\Phi$ (and hence also the flat vector bundle $S_\Phi$) is trivial unless the holonomy representation $\Delta \circ \Phi : \pi_1(M) \to G_0$ is non-trivial, which requires $\Phi_*(\pi_1(M)) \neq 1$ and that the group morphism $\Delta$ be non-trivial (i.e. that $Q$ be a non-trivial flat principal bundle over $\mathcal{M}$). In particular:

- Every smooth global solution $\text{Sol}$ of a standard extended supergravity theory (with simply-connected scalar manifold $\mathcal{M}_0$) is trivial as a U-fold, as is any solution of the generalized theory based on $\mathcal{M}$ whose underlying space-time is simply-connected or whose scalar field configuration $\Phi$ is homotopically trivial.

4. \textbf{Pre-classifying geometric U-folds}

Let:

\[ \mathcal{M}_{G_\Phi}(\mathcal{M}) = \text{Hom}(\pi_1(\mathcal{M}), G_0) / G_0 \]
\[ \mathcal{M}_{G_\Phi}(M) = \text{Hom}(\pi_1(M), G_0) / G_0 \]

denote the moduli spaces of flat principal $G_0$-bundles over $\mathcal{M}$ and $M$ respectively, where the quotient is through the adjoint action of $G_0$. A smooth map $\Phi : M \to \mathcal{M}$ is called \textit{admissible} if there exist fields defined on $M$ which, together with $\Phi$, form a solution $\text{Sol} = (\Phi, \ldots)$ of the theory defined on $M$. Let $C_{\text{ad}}^\infty(M, \mathcal{M})$ denote the space of admissible maps from $M$ to $\mathcal{M}$. The homotopy space $[M, \mathcal{M}]_{\text{ad}}$ of admissible maps is the space of connected components of $C_{\text{ad}}^\infty(M, \mathcal{M})$ with respect to the natural topology and can be obtained upon dividing through the homotopy equivalence relation $\sim$:

\[ [M, \mathcal{M}]_{\text{ad}} \underset{\text{def}}{=} \pi_0(C_{\text{ad}}^\infty(M, \mathcal{M})) = C_{\text{ad}}^\infty(M, \mathcal{M}) / \sim. \]

Since the isomorphism class of $Q_\Phi \underset{\text{def}}{=} \Phi^*(Q)$ as a flat bundle depends only on the homotopy class of $\Phi$ and on the isomorphism class of $Q$ as a flat bundle, the pull-back operation induces a well-defined map:

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\(^8\)By definition, a multivalued global section of a fiber bundle $F \to M$ is an ordinary global section of the bundle $\rho^*(F) \to M_0$. 

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whose image we denote by $\mathcal{M}^\text{ad}_0(M)$ and call the pre-moduli space of geometric U-folds. The moduli space $\mathfrak{M}(M)$ of geometric U-folds on $M$ (if properly defined) should map to $\mathcal{M}^\text{ad}_0(M)$ with fiber given by those geometric U-folds which have isomorphic bundles $Q_f$. Effectively describing $\mathfrak{M}(M)$ and $\mathcal{M}^\text{ad}_0(M)$ appears to be a formidable problem, given the complicated nature of the equations of motion of the theory. Notice that (4.1) is controlled by the homotopy push-forward:

$$[M, \mathcal{M}]_\text{ad} \times \mathcal{M}^\text{ad}_0(M) \ni ([\Phi], [Q]) \mapsto [\Phi^*(Q)] \in \mathcal{M}^\text{ad}_0(M),$$

(4.1)

5. $\mathcal{N} = 2$ supergravity coupled to one vector multiplet

The standard axion–dilaton model is $\mathcal{N} = 2$ supergravity coupled to one vector multiplet with simply-connected scalar manifold given by the (open) upper-half plane:\(^9\)

$$\mathcal{M}_0 = \mathbb{H} = \text{PSL}(2, \mathbb{R})/U(1),$$

equipped with the rescaled Poincaré metric $G_0$ of constant Gaussian curvature $-2$ (scalar curvature $-4$). The latter has the squared line element:

$$ds^2_0 = \frac{1}{2(\text{Im } \tau)} d\tau d\bar{\tau} = 2(G_0)_{\tau\bar{\tau}} d\tau d\bar{\tau}$$

and is the unique PSL$(2, \mathbb{R})$-invariant metric of the given scalar curvature. The connected component of the isometry group is $\text{Iso}_0(\mathcal{M}_0, G_0) = \text{PSL}(2, \mathbb{R})$, which is also the group of orientation-preserving isometries. The manifold $\mathcal{M}_0$ is projective special Kähler with the following global prepotential defined on the conical special Kähler domain $\mathcal{D} = \{(X^0, X^1) \in \mathbb{C}^2| \text{Re}(X^0 X^1) > 0\}$:

$$F_0(X^0, X^1) = -i X^0 \bar{X}^1 = -(X^0)^2 F_0(\tau),$$

where $\tau \equiv i \frac{X^1}{X^0}$ and $F_0(\tau) = \tau$. We have $(G_0)_{\tau\bar{\tau}} = \frac{\partial K_0}{\partial \tau \partial \bar{\tau}} = \frac{1}{2(\text{Im } \tau)}$, where $K_0$ is the (global) Kähler potential in the gauge $X^0 = \frac{1}{2}$.

$$K_0(\tau) = -\ln(\text{Im } \tau);$$

notice that we are using the Riemannian (positive-definite) metric on $\mathcal{M}_0$. The effective U-duality group is the group of special isometries $G_{\text{eff}} = \text{Iso}_0(\mathcal{M}_0) = \text{PSL}(2, \mathbb{R})$, while the U-duality group is its double cover $G_0 = \text{SL}(2, \mathbb{R})$. The canonical circle bundle is the trivial $U(1)$-bundle $P_0 = \mathcal{M}_0 \times U(1)$. We have $n_c = 1$ and $n_v = 2$, while the symplectic representation $\rho : G_0 \to \text{Sp}(4, \mathbb{R})$ is given (up to equivalence of representations) by:

$$\text{SL}(2, \mathbb{R}) \ni A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \rho(A) = \begin{bmatrix} a\Theta_2 & b\Theta_2 \\ c\Theta_2 & d\Theta_2 \end{bmatrix},$$

\(^9\)Since $U(1)/Z_2$ is isomorphic with $U(1)$ through the isogeny $z \to z^2$, while $\text{SL}(2, \mathbb{R}) \simeq SU(1, 1)$, we also have the coset presentations $\mathbb{H} \simeq \text{SL}(2, \mathbb{R})/U(1) \simeq SU(1, 1)/U(1)$, which are not minimal since the group appearing in the numerator is a double cover of $\text{Iso}_0(\mathcal{M}_0, G_0) = \text{PSL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{R})$ acts non-effectively on $\mathbb{H}$.\(^{10}\)Every Riemann surface of genus $g \geq 2$ is projective special Kähler when endowed with its uniformizing metric [31].
where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Theta_2 \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This determines the trivial rank four flat symplectic vector bundle $S_0 = M_0 \times \mathbb{R}^4$.

Let $\Gamma \subset G_{\text{eff}} = \text{PSL}(2, \mathbb{R})$ be a Fuchsian group without elliptic elements. The generalized axion–dilaton model determined by $\Gamma$ is $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet with smooth scalar manifold:

$$\mathcal{M} = \Gamma \backslash \mathbb{H},$$

endowed with the constant negative curvature metric $\mathcal{G}$ induced by $\mathcal{G}_0$. Thus $\mathcal{M}$ is a (possibly non-compact) smooth Riemann surface of genus $g \geq 2$ while $\mathcal{G}$ is its (rescaled) uniformizing metric (the unique metric on $\mathcal{M}$ which has constant Gaussian curvature equal to $-2$). By the uniformization theorem, any smooth Riemann surface of genus $g \geq 2$ endowed with its uniformizing metric can be presented as in (5.1). Notice that $\mathcal{M}$ has finite volume when $\Gamma$ is co-finite and that it is compact when $\Gamma$ is co-compact. When endowed with the complex structure $J$ induced from $\mathcal{H}$, the Hermitian manifold $(\mathcal{M}, J, \mathcal{G})$ is projective special Kähler.

Any group morphism $\Delta : \Gamma \to G_0 = \text{SL}(2, \mathbb{R})$ determines a flat principal $\text{SL}(2, \mathbb{R})$-bundle $Q = C \times_\Delta \text{SL}(2, \mathbb{R})$ and a rank four flat symplectic vector bundle $S = C \times_{\rho \circ \Delta} \mathbb{R}^4$ with monodromy representation $\rho = \Delta$. The moduli space of flat $\text{SL}(2, \mathbb{R})$-bundles on $\mathcal{M}$ is the well-studied character variety:

$$\mathcal{M}_{\text{SL}(2, \mathbb{R})}(\mathcal{M}) = \text{Hom}(\Gamma, \text{SL}(2, \mathbb{R}))/\text{SL}(2, \mathbb{R}),$$

which is closely related to the Teichmüller space of $\mathcal{M}$. This model admits non-trivial geometric U-folds, which can be promoted to string theory backgrounds only when $\Gamma$ is a subgroup of $\text{PSL}(2, \mathbb{Z})$. Below, we construct examples of such U-folds.

5.1. Example: fluxless axion–dilaton U-folds

Consider the generalized axion–dilaton model defined by a Fuchsian group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ without elliptic elements. Focusing on solutions for which the two gauge field strengths vanish identically, we can truncate the bosonic part of the action to:

$$S[g, \tau] = \int_M \left\{ *R + \frac{d\tau \wedge *d\tau}{2(\text{Im}\tau)^2} \right\}.$$  (5.2)

Take the space-time manifold to be of the form:

$$M = \mathbb{R}^2 \times \Sigma,$$

where $\Sigma$ is an oriented connected surface without boundary which admits a (possibly incomplete) flat Riemannian metric $g_2$ and take $g$ to be a flat Lorentzian metric of the form $g = g_2 \times 2$. Further, assume that $\tau$ does not depend on the coordinates of $\mathbb{R}^2$, so that it can be viewed as a smooth map $\tau : \Sigma \to \mathcal{M}$. Then the Einstein equation is satisfied while the equation of motion for $\tau$ reduces to:

$$\partial \bar{\partial} \tau + \frac{\partial \tau \bar{\partial} \tau}{\tau \bar{\tau}} = 0,$$  (5.3)

where $\partial$ and $\bar{\partial}$ are the Dolbeault differentials defined by the complex structure of $\Sigma$ corresponding to the conformal class of $g_2$. A particular class of solutions of (5.3) is given by maps $\tau$ which satisfy $\partial \tau = 0$ and hence are holomorphic on $\Sigma$. As explained in section 3, these can be viewed as multivalued holomorphic maps $\bar{\tau}$ from $\Sigma$ to $\mathbb{H}$ whose monodromy representation
takes values in $\Gamma$ and hence involves U-duality transformations. Let $\tau_0 : \Sigma_0 \to \mathbb{H}$ denote the lift of $\tau$ at a point $x_0 \in M_0$, where $\Sigma_0$ is the universal cover of $\Sigma$. The universal cover of $M$ is $M_0 = \mathbb{R}^2 \times \Sigma_0$. Distinguish the cases:

1. $(\Sigma, g_2)$ is complete. Then $(\Sigma, g_2)$ is an oriented Euclidean space-form and hence must be the Euclidean plane (conformally, the complex plane $\mathbb{C}$), the flat infinite cylinder (conformally, the complex punctured plane $\mathbb{C} \setminus \{0\}$), or a flat torus (conformally, an elliptic curve). Since $\mathcal{M}$ has genus at least two, the Picard theorem for Riemann surfaces forces $\tau$ to be constant, so such solutions are trivial as U-folds.

2. $(\Sigma, g_2)$ is incomplete. Then $\Sigma$ can be any open domain of the Euclidean plane. This leads to non-trivial geometric U-folds provided that $\Sigma$ has non-trivial fundamental group. A physically interesting example is $\Sigma = \mathbb{C} \setminus A$, where $A = \{p_1, \ldots, p_k\}$ is a non-empty finite set of points of the complex plane. In this case, the Riemannian universal cover $\Sigma_0$ is conformally equivalent with the complex plane or with the Poincaré disk (depending on whether $k = 0, 1$ or $k \geq 2$) and $\hat{\tau}$ can have non-trivial $\Gamma$-valued monodromies around each of the points $p_j$, giving a four-dimensional solution similar to the cosmic string of [22]. The Hodge dual $H_0 \equiv *_{M_0} d\tau_0 \in \Omega^4(M_0)$ satisfies $dH_0 = 0$ since $\tau_0$ is holomorphic. Thus $H_0 = dR_0$ for a globally-defined two-form $R_0 \in \Omega^2(M_0)$. The standard supergravity theory with scalar manifold $(M_0, \mathcal{G}_0)$ admits cosmic strings which couple to this potential. Then the solution $\tau$ can be interpreted as being sourced by strings with worldvolume $\mathbb{R}^2 \times \{p_j\}$, which are responsible for the monodromies of $\hat{\tau}$.

The supergravity backgrounds constructed above can be lifted to string theory only when $\Gamma$ is a subgroup of $\text{PSL}(2, \mathbb{Z})$.

Unlike our solutions, the backgrounds of [22] arise in ten-dimensional IIB supergravity. Also notice that the construction of loc. cit. uses the Fuchsian group $\text{PSL}(2, \mathbb{Z})$, which contains elliptic elements. As a consequence, the quotient $\text{PSL}(2, \mathbb{Z}) \setminus \mathbb{H}$ (endowed with the constant negative curvature metric induced by $\mathcal{G}_0$) is a projective special Kähler orbifold (topologically, a once-punctured sphere with two conical orbifold points of orders $2$ and $3$) which coincides with the moduli space of elliptic curves endowed with its Weil–Petersson metric. This orbifold is in principle not an admissible scalar manifold in supergravity, so the construction of [22] makes physics sense only in non-perturbative IIB string theory, where it gives an F-theory background. By contrast, the construction above works classically in four-dimensional $\mathcal{N} = 2$ supergravity (where it produces solutions containing cosmic string sources), since it involves smooth target manifolds for the scalar field $\tau$.

6. $\mathcal{N} = 8$ supergravity

The scalar manifold of standard $\mathcal{N} = 8$ supergravity is given by:

$$\mathcal{M}_0 = \text{E}_{7(7)}/(\text{SU}(8)/\mathbb{Z}_2).$$

In this case, $H_0 = \text{SU}(8)/\mathbb{Z}_2$ is the maximal compact sub-group of $\text{E}_{7(7)}$ and $P_0$ is the principal $H_0$-bundle given by the canonical projection $\text{E}_{7(7)} \to \mathcal{M}_0$. The U-duality group is $G_0 = \text{E}_{7(7)}$ and $\rho$ is the 56-dimensional representation of $G_0$. The U-duality group has the following polar decomposition [32]:

$$\text{E}_{7(7)} \simeq H_0 \times \mathbb{R}^{70}.$$
Since $\pi_1(H_0) = \mathbb{Z}_2$, this gives $\pi_1(E_{7(7)}) = \mathbb{Z}_2$ and shows that $\mathcal{M}_0$ is diffeomorphic with $\mathbb{R}^{70}$, hence contractible. Thus every fiber bundle on $\mathcal{M}_0$ is topologically trivial and the standard theory does not admit non-trivial geometric U-folds.

Let $\Gamma$ be a discrete subgroup of $E_{7(7)}$. Then any smooth Clifford–Klein form:

$$\mathcal{M} = \Gamma \backslash \mathcal{M}_0 = \Gamma \backslash E_{7(7)}/(SU(8)/\mathbb{Z}_2)$$

is admissible as a scalar manifold of $\mathcal{N} = 8$ supergravity. At least when $\mathcal{M}$ is non-compact, we expect such generalized models to admit geometric U-fold solutions, which could be promoted to string theory backgrounds when $\Gamma$ is a subgroup of $E_{7(7)}(\mathbb{Z})$. The pre-moduli space of such geometric U-folds is controlled by the character variety:

$$\mathcal{M}_{E_{7(7)}}(\mathcal{M}) = \text{Hom}(\Gamma, E_{7(7)}/E_{7(7)}),$$

which, to our best knowledge, was not systematically studied in the supergravity literature.

7. Final remarks

In this note, we have considered supergravity U-folds from the point of view of the generic ungauged four-dimensional (Lorentzian) effective theory that could potentially correspond to the effective theory describing string theory compactified on certain locally geometric U-fold compactification backgrounds. This is in contrast to the scenario usually explored and studied in the literature, which focuses on approaching U/T-fold compactification backgrounds from the ten-dimensional string/supergravity point of view as well as the local structure of the corresponding effective gauged supergravity, see for example [33–35] and references therein.

Therefore, it would be useful to understand in detail the relation between the set up we consider in this note and the higher-dimensional constructions already present in the literature, see for example [2, 4, 8, 36–39]. In this note we do not explicitly address the way in which U-fold compactification backgrounds are constructed from the ten-dimensional point of view. Instead, we focus on the global structure of the generic bosonic sector of four-dimensional supergravity, which should correspond to the effective theory describing some (certainly not all) string compactifications on certain locally geometric U-fold backgrounds. In practical terms, the way we do this is by imposing global consistency with the global U-duality transformations which act on the fields of the local theory. In other words, we glue local standard supergravities in four dimensions by using U-duality transformations. These are global symmetries of the local equations of motion and hence can be used to define a consistent globally defined theory whose solutions are local solutions of standard supergravity glued by U dualities.

It is not immediately clear how to relate in detail the construction proposed in this note to the explicit way in which some U/T-fold compactification backgrounds are constructed. What we can say is that ten-dimensional U dualities (such as the ones used to construct T/U-fold compactification backgrounds) correspond to isometries of the scalar manifold of the four-dimensional theory, which act in a symplectic representation on the gauge fields and their duals. The isometry group acts in a very non-linear way on the scalar manifold of the theory, which corresponds to the moduli space of the compactification, and hence contains information about the extra dimensions and the way the theory has been obtained from ten dimensions through compactification on a U-fold compactification background. In order to find a direct relationship between the way a U-fold compactification background is constructed and the twisting (as described in our manuscript) of the corresponding four-dimensional effective theory, one should actually perform the compactification and find the global structure of the
corresponding effective theory. To the best of our knowledge, this has not been studied in the literature. This topic deserves further attention and we plan to address it in the future.

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Appendix. Square root ambiguity of spinor fields

As mentioned above, for certain supergravities the representations $\theta_G$ and $\theta_f$ used in the construction of $E_G$ and $E_f$ involve taking the square root of the fundamental representation of a $U(1)$-sub-bundle of $P_\Phi$, a procedure which generally is obstructed and non-unique. For example, the spinors of $\mathcal{N} = 2$ supergravity are properly-speaking valued in complex vector bundles associated to a square root $P_{\Phi}^{1/2}$ of the $U(1)$-bundle $P_\Phi$. Thus $c_1(P_\Phi) = \Phi^*c_1(P)$ must be even. The square roots of $P_\Phi$ have first Chern classes lying in the preimage of $c_1(P_\Phi)$ through the endomorphism of $H^2(M, \mathbb{Z})$ given by multiplication with 2. This phenomenon does not seem to have been studied systematically in the supergravity literature.

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