A Gross–Kohnen–Zagier Type Theorem for Higher-Codimensional Heegner Cycles

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Introduction

Heegner points, which are certain CM points on modular curves, have been the subject of extensive research in the last few decades. In particular, the seminal paper [GZ] of Gross and Zagier relates their heights to derivatives of certain $L$-series, a result which established the first step towards the Birch-Swinnerton-Dyer conjectures. Gross, Kohnen, and Zagier investigated the properties of these points further in [GKZ] and proved that certain combinations of these points, namely the Heegner divisors, correspond to coefficients of a modular form of weight $\frac{3}{2}$ when considered in components of the Jacobian variety of the modular curve. In another direction, Borcherds extended in [B1] the theta correspondence (investigated earlier by Howe and many others) to weakly holomorphic modular forms. He then used these results in [B2] to give an alternative proof of the modularity theorem of [GKZ], a proof which generalizes to many other cases: CM points on Shimura curves (still in dimension 1), the Hirzebruch–Zagier divisors defined in [HZ] on Hilbert modular surfaces (in dimension 2), Humbert surfaces on Siegel threefolds, etc.

Just as modular curves can be interpreted as moduli spaces for elliptic curves (with some level structure), Shimura curves are the moduli spaces for Abelian surfaces with quaternionic multiplication. These moduli interpretations lead to universal families (Kuga–Sato varieties) as well as to variations of Hodge structures over these curves. The classical moduli interpretation of the modular curves leads to the variation of Hodge structures $V_1$ of Shimura, and Besser showed in [Be] that $Sym^2V_1$ can be seen as a part of variations of Hodge structures on Shimura curves. The cohomology groups of the universal families involve many parts which arise as cohomology groups of the underlying curves with coefficients in the local systems underlying $V_1$ and its symmetric powers.

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The Abelian surfaces with quaternionic multiplication contain many generic cycles. The CM points on Shimura (and modular) curves represent Abelian surfaces which contain, apart from these generic cycles, additional cycles, the CM cycles. In this work we give a general definition for these cycles (which is not based only on their normalized fundamental cohomology classes), and show that they form all the Abelian subvarieties of such Abelian surfaces, up to the generic cycles in the case of the split quaternion algebra. Our object of interest is these CM cycles, considered as vertical cycles inside the universal family. More generally, we consider the case of symmetric powers of the cycles inside higher-dimensional Kuga–Sato type varieties. The \(m\)th symmetric power of a CM cycle is considered as a vertical cycle of codimension \(m + 1\) inside the \((2m + 1)\)-dimensional Kuga–Sato variety of type \(W_{2m}\).

A rough version of our main result can now be stated. The Heegner cycles, of codimension \(m + 1\) in \(W_{2m}\), are certain combinations of these CM cycles, which are supported over the classical Heegner divisors on modular (and Shimura) curves. For these cycles we have

**Theorem.** The images of the \((m + 1)\)-codimensional Heegner cycles correspond, in an appropriate quotient group, to the coefficients of a modular form of weight \(\frac{3}{2} + m\).

Our method of proof follows \[B2\] closely, but it requires some additional technical preliminaries. Borcherds constructed in \[B1\] a more general theta correspondence, using theta functions which are associated to homogeneous polynomials. We use the singular theta lift corresponding to a lattice \(L\) of signature \((2, b_-)\) and the polynomial \(P_{m, m, 0}(Z, \lambda) = (\lambda, Z, \lambda)^m_{(2, b_-)}\). The resulting functions are not meromorphic on the complex manifold \(G(L_R)\), but are eigenfunctions of eigenvalue \(-2mb_-\) with respect to the corresponding weight \(m\) Laplacian on \(G(L_R)\), with known singularities. However, in the case \(b_- = 1\) of Shimura and modular curves, the images of these theta lifts under the weight raising operators are meromorphic (with known poles), hence could be related to algebraic objects. In fact, this result establishes a new singular Shimura-type correspondence, as stated in the following

**Theorem.** Let \(m\) be an even positive integer and let \(f = \sum_n c_n q^n\) be a weakly holomorphic modular form of weight \(\frac{1}{2} - m\) which lies in the Kohnen plus-space. Then the function

\[
\frac{r^m}{2} \delta_{2m} \Phi_{L, m, 0}(v, F) = \sum_r \left( \sum_{d|r} d^{m+1} c_d^2 \right) r^m q^r
\]

is a modular form of weight \(2m + 2\) for \(SL_2(\mathbb{Z})\), which has poles of order \(m + 1\) at some CM points on \(\mathcal{H}\).

This paper is divided into 4 sections. In Section 1 we present the Siegel theta functions which we use, and derive a differential equation which these theta functions satisfy. This differential equation is later used in order to prove
that our theta lift is an eigenfunction of the corresponding Laplacian, an obser-
vation which plays a key role in our arguments. Section 2 introduces Borcherds’
singular theta lift, and then generalizes some results of Section 4 of [Bru]
about the “self-adjointness” property of the weight $k$ Laplacian for Borcherds’
regularized theta lift. We then turn to analyze the properties of the particular theta
lift which we use for our main result, first in the general case and then also for
the $\delta_{2n}$-image in the case $b_- = 1$, which yields the singular Shimura-type corre-
spondence stated above. The arithmetic part is presented in Section 3, where we
discuss universal families over Shimura curves and the corresponding variations
of Hodge structures. In the later parts of this section we define CM cycles as
actual cycles in Abelian surfaces with quaternionic multiplication, and obtain a
classification of the Abelian subvarieties of Abelian surfaces with quaternionic
multiplication. Finally, Section 4 defines the relations which we consider be-
tween the CM (or Heegner) cycles in question, and proves the main result of
this paper. We conclude with some remarks suggesting possible connections
between our results and existing theorems and conjectures.

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1 Relations Between Actions of Differential Op-
erators on Theta Functions

In this Section we present Siegel theta functions with polynomials as in [B1],
as well as the description of the Grassmannian of a vector space of signature
$(2, b_-)$ as in, among others, [B1] and [Bru]. We give an explicit formula for the
weight $m$ Laplacian operator on functions on the Grassmannian in this case,
and conclude by proving a relation between the actions of the two Laplacians
on the theta functions associated to a polynomial of specific type. This relation
is a crucial step in showing that the theta lifts considered in the next Section
are eigenfunctions of the Laplacian on the Grassmannian (with the appropriate
weight), a property which is central in the following applications.

1.1 The Basic Summand in a Theta Function and First
Properties

Let $V$ be a real vector space with a non-degenerate bilinear form of signature
$(b_+, b_-)$, in which the pairing of $\lambda$ and $\mu$ in $V$ is denoted $(\lambda, \mu)$ and $(\lambda, \lambda)$ is
shortened to $\lambda^2$ (the norm of $\lambda$). The quadratic form $q(\lambda)$ of [Bru] corresponds
to $\frac{\lambda^2}{2}$. The Grassmannian of $V$ is defined by

$$G(V) = \{ V = \langle v_+ \rangle \oplus \langle v_- \rangle | v_+ \gg 0, v_- \ll 0, v_+ \perp v_- \},$$

i.e., the set of decompositions of $V$ as an orthogonal direct sum of a $(b_+\text{-dimensional})$ positive definite space $v_+$ and a $(b_-\text{-dimensional})$ negative definite space $v_-$. Then, for every $\lambda \in V$ we write $\lambda_{v_\pm}$ for the projection of $\lambda$ onto $v_\pm$, or equivalently the $v_\pm$-part of $\lambda$. We let $\Delta_{v_\pm}$ be the Laplacian on the definite space $v_\pm$, i.e., $\Delta_{v_\pm} = \pm \sum_{h=1}^{b_\pm} \frac{\partial^2}{\partial \lambda_h^2}$ if $\lambda \in v_\pm$ is $\pm \sum_h \lambda_h v_h$ where the $v_h$ form an orthonormal basis for $v_\pm$ (hence $\lambda_0 = (\lambda, v_0)$). The Laplacian $\Delta_V$ of $V$, which is invariant under the action of the orthogonal group $O(V)$ of $V$, is $\Delta_{v_+} + \Delta_{v_-}$. However, we shall also use the Laplacian corresponding to $v \in G(V)$, denoted $\Delta_v$ and defined as $\Delta_{v_+} - \Delta_{v_-}$ (i.e., the Laplacian $V$ would have with the majorant corresponding to $v$). We denote the upper half plane $\{ \tau \in \mathbb{C} | \Re \tau > 0 \}$ by $\mathcal{H}$, and wherever an element $\tau \in \mathcal{H}$ is involved, $x$ and $y$ denote its real and imaginary parts respectively. For $\tau \in \mathcal{H}$, $v \in G(V)$, a polynomial $p$ on $V$, and $\lambda \in V$, define the function

$$F(\tau, v, p, \lambda) = e^{-\Delta_{v} / 2 \pi y}(\lambda) e\left(\frac{\lambda_0^2}{2} + \frac{\lambda_0^2}{2} \right),$$

where $e(z)$ is the useful shorthand for $e^{2 \pi i z}$ for any complex number $z$. The polynomial $p$ involved with these functions will usually be homogenous of degree $(m_+, m_-)$ with respect to $v$, which means that for every element $\lambda$ of $V$, which decomposes as $\lambda_{v_+} + \lambda_{v_-}$ with respect to $v$, and real numbers $\alpha_+$ and $\alpha_-$, we have $p(\alpha_+ \lambda_{v_+} + \alpha_- \lambda_{v_-}) = \alpha_+^{m_+} \alpha_-^{m_-} p(\lambda)$.

An even lattice $L$ in $V$ is a (free) discrete $\mathbb{Z}$-submodule of $V$ of maximal rank $b_+ + b_-$ such that $\lambda^2$ is an even integer for any $\lambda \in L$. Hence $(\lambda, \mu) \in \mathbb{Z}$ for any $\lambda$ and $\mu$ in $L$. The dual lattice $L^*$ is the set of elements $\lambda \in L$ such that $(\lambda, \mu) \in \mathbb{Z}$ for all $\mu \in L$, and $L^*$ contains $L$ as a subgroup of finite index. Then $\mathbb{C}[L^*/L]$ is a finite-dimensional vector space, with basis elements $(e_\gamma)_{\gamma \in L^*/L}$. Define the Siegel theta function of $L$, with respect to the element $v \in G(V)$ and a polynomial $p$ which is homogenous of degree $(m_+, m_-)$ with respect to $v$, to be the $\mathbb{C}[L^*/L]$-valued function

$$\Theta_L(\tau, v, p) = \sum_{\gamma \in L^*/L} \theta_\gamma(\tau, v, p)e_\gamma, \quad \theta_\gamma(\tau, v, p) = \sum_{\lambda \in L + \gamma} F(\tau, v, p, \lambda).$$

Recall that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\tau \in \mathcal{H}$ we have the automorphy factor $j(M, \tau) = c\tau + d$, associated to the action of $SL_2(\mathbb{R})$ on $\mathcal{H}$ in which $M$ takes $\tau \in \mathcal{H}$ to $M\tau = \frac{a\tau + b}{c\tau + d}$. The metaplectic group $Mp_2(\mathbb{R})$ of $SL_2(\mathbb{R})$ is a central double cover of $SL_2(\mathbb{R})$, a realization of which is the group of pairs of an element $M \in SL_2(\mathbb{R})$ and a square root of the function $\tau \mapsto j(M, \tau)$. Explicitly, an element of $Mp_2(\mathbb{R})$ is a pair $(M, \varphi)$ with $M \in SL_2(\mathbb{R})$ and $\varphi : \mathcal{H} \to \mathbb{C}$ such that $\varphi(\tau)^2 = j(M, \tau)$, with the product of two such elements $(M, \varphi)$ and $(N, \psi)$ being $(MN, (\varphi \circ N) \cdot \psi)$. We denote the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$
by $Mp_2(\mathbb{Z})$. It is generated by $T = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$, and $S = \left( \begin{array}{cc} -1 & \sqrt{\tau} \\ 1 & 1 \end{array} \right)$, with the relations $S^2 = (ST)^3 = Z$, $Z^4 = I$. The element $Z = (-I, i)$ generates the center of $Mp_2(\mathbb{R})$ and $Mp_2(\mathbb{Z})$, and the kernel of their projections onto $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$ is generated by the order 2 element $Z^2 = (I, -1)$. We denote by $\rho_V$ the Weil representation of $Mp_2(\mathbb{R})$ on the space $S(V)$ of Schwartz functions on $V$. Explicitly, for an element $M \in Mp_2(\mathbb{R})$ (with the usual $SL_2$ entries) the operator $\rho_V(M)$ takes the Schwartz function $\Phi$ on $V$ to the function

$$\lambda \mapsto \delta^{b_- - b_+} |a|^{(b_+ + b_-)/2} \Phi(a \lambda) e\left( \frac{ab \lambda^2}{2} \right)$$

if $c = 0$ and $\sqrt{\det(M, \tau)} = \delta \sqrt{|d|}$, and to the function sending $\lambda \in V$ to

$$\left( \text{sgn}(\mathbb{R} \sqrt{\det(M, \tau)}) \right) \zeta_8^{\text{sgn}(c)} \delta^{b_- - b_+} |c|^{-(b_+ + b_-)/2} \int_V \Phi(\mu) e\left( \frac{d\mu^2}{2c} - \frac{(\lambda, \mu)}{c} + \frac{a\lambda^2}{2c} \right) d\mu$$

if $c \neq 0$. Here $\zeta_8 = e\left( \frac{\pi}{4} \right)$ is the basic 8th root of unity, so that $\zeta_8^{b_- - b_+}$ is the complex conjugate of the Weil index of the self-dual locally compact group $V$ corresponding to its identification with its dual via the bilinear form and the character $t \mapsto e(t)$ of $\mathbb{R}$. Furthermore, we denote $\rho_L$ the Weil representation of $Mp_2(\mathbb{Z})$ on the finite-dimensional vector space $\mathbb{C}[L^* / L]$. This representation is defined by the action of the generators as

$$\rho_L(T)(e_\gamma) = e(\gamma^2/2)e_\gamma,$$

$$\rho_L(S)(e_\gamma) = \frac{\zeta_8^{b_- - b_+}}{\sqrt{\Delta_L}} \sum_{\delta \in L^*/L} e(-\gamma, \delta)e_\delta,$$

where $\Delta_L = |L^*/L|$ (see, for example, Section 4 of [B1] or Section 2 of [B2]). The root of unity $\zeta_8^{b_- - b_+}$ is also the complex conjugate of the Weil index of the finite group $L^*/L$ through its identification with its dual. The equality of the Weil indices of $V$ and $L^*/L$ follows either from direct evaluation or from the general argument of Cartier’s extension of Weil’s seminal work [W]. This representation factors through a double cover of $SL_2(\mathbb{Z}/NZ)$ for some integer $N$ called the level of $L$, or through $SL_2(\mathbb{Z}/NZ)$ itself if the signature of $L$ is even. This property has long been established, and a direct derivation has recently been obtained as Theorem 3.2 (or Proposition 8.2) of [Zc]. Formulæ for the action of a general element of $Mp_2(\mathbb{Z})$ via the Weil representation $\rho_L$ are given in [Schc] for even signature and in [Sitc] for $\mathbb{Z}$ or [Zc] for the general case.

The stabilizer of $i \in \mathcal{H}$ in the action of $Mp_2(\mathbb{R})$ on $\mathcal{H}$ is classically denoted $K$ (not to be confused with the lattice $K$ below) and consists of the elements $k_\theta$ with $SL_2(\mathbb{R})$-images $\left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right)$. The element $k_\theta$ satisfies $j(k_\theta, i) = e^{i\theta}$, and the metaplectic square root is chosen to attain $e^{i\theta/2}$ at $i$. Hence for an element of $Mp_2(\mathbb{R})$ the angle $\theta$ is considered as an element of $\mathbb{R}/4\pi \mathbb{Z}$ (with the projection into $SL_2(\mathbb{R})$ being the map taking $\theta$ to its image in $\mathbb{R}/2\pi \mathbb{Z}$). $K$ is a maximal compact subgroup of $Mp_2(\mathbb{R})$. Some properties of the functions $F$ and $\Theta_L$, which will turn out useful in this paper, are summarized in the following.
Theorem 1.1. (i) For any $M \in M_{p_2}(\mathbb{R})$ we have the equality

$$
\rho_V(M)F(\tau, v, p, \lambda) = j(M, \tau)^{-\frac{b_+}{2} - m} j(M, \tau)^{-\frac{b_-}{2} + m} F(M\tau, v, p, \lambda)
$$

of functions of $\lambda \in V$. (ii) The action of $\rho_V(k_0)$ multiplies $F(i, v, p, \lambda)$ by $(e^{it})^{\frac{b_+}{2} + m - \frac{b_-}{2} - m}$. (iii) If $M \in M_{p_2}(\mathbb{Z})$ then the equality

$$
\rho_L(M)\Theta_L(\tau, v, p) = j(M, \tau)^{-\frac{b_+}{2} - m} j(M, \tau)^{-\frac{b_-}{2} + m} \Theta_L(M\tau, v, p)
$$

holds in $\mathbb{C}[L^*/L]$, i.e., $\Theta_L$ is modular of weight $(\frac{b_+}{2} + m, \frac{b_-}{2} - m)$ and representation $\rho_L$ with respect to $M_{p_2}(\mathbb{Z})$ (as defined below).

Proof. Part (i) is essentially Lemma 1.2 of [Sn] (in fact, only the case $b_+ = 2$ and certain harmonic polynomials of homogeneity degree $(m_+, 0)$ are considered there, but using the results of Section 3 of [B1] the proof extends to the general case). Part (ii) is the special case in which we take $M = k_0 \in K$ and $\tau = i$ in part (i). Part (iii) is the case $\alpha = \beta = 0$ in Theorem 4.1 of [B1].

For $\tau \in H$ we define $g_\tau$ to be the matrix $\frac{1}{\sqrt{\tau}}(y \, 1)$ with the metaplectic matrix $\sqrt{j(g_\tau, \phi)}$ being the constant $+\sqrt{\tau}$. Then the first assertion of Theorem 1.1 implies $\rho_V(g_\tau)F(i, v, p, \lambda) = y^{\frac{b_+}{2} + m + \frac{b_-}{2} - m} F(\tau, v, p, \lambda)$. On the other hand, after extending the Weil representation $\rho_V$ to the Lie algebra $\mathfrak{sl}(2)(\mathbb{R})$ of $M_{p_2}(\mathbb{R})$ and then to its universal enveloping algebra, Lemma 1.4 of [Sn] and the second assertion of Theorem 1.1 show, with $m = 2k$ and $k = \frac{b_+}{2} + m_+ - \frac{b_-}{2} - m_-$, that

$$
\rho_V(C)y^{\frac{b_+}{2} + \frac{b_-}{2} + m} F(\tau, v, p, \lambda) = 4\Delta_{\frac{b_+}{2} + \frac{b_-}{2} + m} y^{\frac{b_+}{2} + \frac{b_-}{2} + m} F(\tau, v, p, \lambda).
$$

(1)

Here the $\Delta_{\tau,s}$ (in the variable $\tau$) is the operator

$$
y^2(\partial_x^2 + \partial_y^2) - iy(r - s)\partial_x + y(r + s)\partial_y = 4y^2\partial_\tau - 2iy\partial_\tau + 2is\partial_\tau
$$

on functions on $H$ for any $r$ and $s$, and $C = 2EF + 2FE + H^2$ is the Casimir element of the universal enveloping algebra of $\mathfrak{sl}(2)(\mathbb{R})$. The weigt $$(r, s)$$ Laplacian on $H$ is defined to be $\Delta_{\tau,s} = \Delta_{\tau,s} + (r - 1)s$, and the equality $\Delta_{\tau,s}y^2 = y^2\Delta_{\tau+t,s+t}$ holds for any $t$. The right hand side of equation (1) can therefore be written as

$$
y^{\frac{b_+}{2} + \frac{b_-}{2} + m} F(\tau, v, p, \lambda) \Delta_k = \Delta_{k,0} \text{ being the weight } k \text{ Laplacian.}
$$

Moreover, by Lemma 1.5 of [Sn] we know that the action of $\rho_V(C)$ on functions $V$ coincides with that of the Laplacian $\Delta_{SO_{b_+, b_-}}$ of $O(V)$ (i.e., the action of the Casimir element of the universal enveloping algebra of $\mathfrak{so}_{b_+, b_-}$) plus the constant $\frac{n(n - 1)}{2}$ (for $n = \dim V = b_+ + b_-$). This implies

$$
\Delta_{SO_{b_+, b_-}} y^{\frac{b_+}{2} + \frac{b_-}{2} + m} F(\tau, v, p, \lambda) = 4\Delta_{k,0} y^{\frac{b_+}{2} + \frac{b_-}{2} + m} F(\tau, v, p, \lambda) +
$$

$$
\left[(m_+ - m_-)(m_+ - m_- + b_+ - b_- - 2) - (b_+ - 2)b_-\right] y^{\frac{b_+}{2} + \frac{b_-}{2} + m} F(\tau, v, p, \lambda),
$$

(2)
write vectors of \( V \) vector stands for \( \eta \) its norm is \( \eta \)

Choosing some \( \zeta \) vector space \( K \) \( z \) in the variable of the Grassmannian
\( v \) on the \( \text{SO} \) actions of \( \text{SO} \)

since in this case the action of \( \text{SO} \)
defines a holomorphic section \( z \) where the constant is the evaluation of \( Z \)
also satisfy \( ( \) \( ) \) this dependence affects strongly the form which \( \Delta \)

we now present a particular case in which the actions can be related.

1.2 Grassmannians with Complex Structures and Differential Operators

We take \( b_+ = 2 \), so that the Grassmannian admits a structure of a complex manifold, as we now describe (see also Section 13 of [Bru] or Section 3.2 of [Bru]). Let \( (V_C)_0 \) be the algebraic subvariety \( \{ Z_V \in V_C | Z_V^2 = 0 \} \) of \( V_C \), and let \( P \) denote the (analytically open) subset of \( (V_C)_0 \) defined by such \( Z \) which also satisfy \( (Z_V, Z_V) > 0 \). Every \( Z_V \in P \) determines the 2-dimensional positive definite subspace \( v_+ \) of \( V \) which is spanned by the real and imaginary parts of \( Z_V \). Moreover, choosing an order on this basis of \( v_+ \) determines an orientation on \( v_+ \), and the elements of the basis are orthogonal to one another and have the same norm. Conversely, given a 2-dimensional positive definite subspace \( v_+ \) of \( V \) and an oriented orthogonal basis \( (X_V, Y_V) \) of \( V \) in which \( X_V \) and \( Y_V \) have the same norm, the vector \( Z_V = X_V + iY_V \) is an element of \( P \) which maps to \( v_+ \). Now, the conditions defining \( (V_C)_0 \) and \( P \) are stable under multiplication from \( \mathbb{C}^* \), hence define an algebraic subvariety \( \mathbb{P}(V_C)_0 \) of \( \mathbb{P}(V_C) \) and an analytically open subset of \( \mathbb{P}(V_C)_0 \) respectively. Since two elements of \( P \) map to the same element of \( G(V) \) if and only if they are scalar multiples of one another, the (smooth) map \( P \to G(V) \) factors through the image of \( P \) in \( \mathbb{P}(V_C)_0 \) and gives a diffeomorphism between \( G(V) \) and the latter manifold. The complex structure on \( G(V) \) is defined by pulling back the complex structure on the latter manifold via this diffeomorphism.

Another form of the Grassmannian is obtained by the following construction
(which is interesting on its own right). Assume that \( V \) is indefinite, and choose some non-zero vector \( z \in V \) with \( z^2 = 0 \). One then defines the non-degenerate vector space \( K_\mathbb{R} = z^\perp / \mathbb{R}z \) of signature \( (b_+ - 1, b_- - 1) \) (for arbitrary \( b_+ \)). Choosing some \( \zeta \in V \) with \( (z, \zeta) = 1 \), we can identify \( K_\mathbb{R} \) with \( \{ z, \zeta \}^\perp \) and write vectors of \( V \) as triplets \( (\eta, a, b) \), with \( \eta \in K_\mathbb{R} \) and \( a \) and \( b \) in \( \mathbb{R} \). This vector stands for \( \eta + a\zeta + bz \) (under the identification of \( K_\mathbb{R} \) with \( \{ z, \zeta \}^\perp \)), and its norm is \( \eta^2 + a^2\zeta^2 + 2ab \). Returning to the case where \( b_+ = 2 \), the choice of \( z \) defines a holomorphic section \( G(V) \to P \) by the condition that the pairing of the image of the section with \( z \) is 1. Every norm 0 vector \( Z_V \in V_C \) which satisfies \( (Z_V, z) = 1 \) must be of the form \( Z_V = (Z, 1, \frac{z^2 - \zeta^2}{2}) \) for some vector
\[ Z = X + iY \in K_\mathbb{C}, \text{ and the condition } (Z_V, \overline{Z_V}) > 0 \text{ corresponds to } Y^2 > 0. \]

Since \( K_\mathbb{R} \) is Lorentzian, the set of positive norm vectors in \( K_\mathbb{R} \) consists of two connected components (called cones), and the orientation we chose on the space \( v_+ \) for each element of \( G(V) \) implies that if \( Z_V \) is in the image of the section \( G(V) \to P \) then \( Y = \Im Z \) lies in one of these cones. We call this cone the positive cone, and denote it by \( C \). The vector \( Z \) is determined as the \( K_\mathbb{C} \)-image of \( Z_V - \zeta \), and by considering \( Z_V \) as a function of \( Z \) we obtain a biholomorphic diffeomorphism between \( G(V) \) (or, more precisely, the image of the section into \( P \)) and the tube domain \( K_\mathbb{R} + iC \). In the following we interchange \( v \in G(V) \) and \( Z \in K_\mathbb{R} + iC \) freely, where in expressions where both \( v \) and \( Z \) appear they are to be understood to correspond to one another via these maps.

In some contexts it will be important to emphasize the dependence of the vector \( Z_V \) on \( Z \in K_\mathbb{R} + iC \) or on \( v \in G(V) \). In these cases we shall write \( Z_{v,V} \) or \( Z_{v,V} \) instead of \( Z_V \). The vectors \( X_V \) and \( Y_V \) in \( V \) (or \( X_{Z,V} \) and \( Y_{Z,V} \), or \( X_v, V \) and \( Y_v, V \)) stand for the real and imaginary parts of \( Z_V \) (or \( Z_{Z,V} \), or \( Z_{v,V} \)), namely \( (X, 1, \frac{\nu^2 - \zeta^2}{2}) \) and \( (Y, 0, -(X, Y)) \) respectively. They are orthogonal to one another and have the common norm \( Y^2 \). We remark that replacing \( C \) by the negative cone \( -C \) takes the complex structure on \( G(V) \) to the complex conjugate structure and inverts the orientations on the spaces \( v_+ \).

The polynomials we consider are of the form \( P_{r,s,t}(Z, \lambda) = (\lambda, Z_{\nu,V})^r (\lambda, \overline{Z_{\nu,V}})^s, \) for non-negative integers \( r, s, \) and \( t \). In particular \( P_{r,s,t} \) is homogenous of degree \( (r + t, 0) \) with respect to \( v \). For such \( p \) the constant in Equation (2) reduces to \( m_+ (m_+ - b_-) \), where \( m_+ = r + t \) is the homogeneity degree of \( P_{r,s,t}(\lambda, Z) \) in the \( \lambda \) variable.

We recall that the action of the orthogonal group \( O^+(V) \cong O_{2,+b}^+ \) (where the + refers to those elements of the general orthogonal group which preserve the orientation on the positive definite part) admits a factor of automorphy on \( G(V) \cong K_\mathbb{R} + iC \). This factor of automorphy is denoted \( j(\sigma, Z) \) for \( \sigma \in O^+(V) \) and \( Z \in K_\mathbb{R} + iC \), and is defined by the equation \( \sigma(Z_{\nu,V}) = j(\sigma, Z) Z_{\sigma, V} \), or equivalently \( j(\sigma, Z) = \sigma(Z_{\nu,V}), z \) for \( v \in G(V) \) corresponding to \( Z \in K_\mathbb{R} + iC \). Since \( (Z_{\nu,V}, \overline{Z_{\nu,V}}) = 2Y^2 \) it follows (see, for example, Lemma 3.20 of [Bru]), that

\[ (3(\sigma Z))^2 = \frac{Y^2}{|j(\sigma, Z)|^2}. \]

This implies

\[ P_{r,s,t}(\sigma Z, \lambda) = j(\sigma, Z)^{r-s} j(\sigma, Z)^{s-t} P_{r,s,t}(Z, \sigma^{-1} \lambda) \]

for all \( Z \in K_\mathbb{R} + iC, \lambda \in V, \) and \( \sigma \in O^+(V) \). Since \( \Delta_v P_{r,s,t} = \Delta_{\nu_v} P_{r,s,t} \) is just \( 4rt P_{r-1, s-1, t-1} \) and this operation preserves the differences \( s-r \) and \( s-t \), Equation (3) continues to hold if we replace \( P_{r,s,t} \) by \( e^{-\Delta_v / 8\pi y} P_{r,s,t} \). The fact that \( e^{\left( \frac{r \lambda^2}{2} + \frac{s \lambda_{-1}}{2} \right)} \) attains the same values on \( (\sigma Z, \lambda) \) and on \( (Z, \sigma^{-1} \lambda) \) implies that Equation (3) is valid also for \( F_{r,s,t}(\tau, Z, \lambda) = F(\tau, v, P_{r,s,t}, \lambda) \). Assume now that \( L \) is an even lattice in \( V \). Then the components \( \theta_{\lambda}(\tau, v, p) \) are invariant under replacing \( \lambda \) by \( \lambda^{-1} \) for \( \lambda \in \Gamma = \ker (\text{Aut}(L) \to \text{Aut}(L^*/L)) \) (this group \( \Gamma \) is called the discriminant kernel of \( \text{Aut}(L) \)). Therefore Equation (3) for \( F_{r,s,t} \).
implies that the corresponding theta function \( \Theta_{L,r,s,t}(\tau, v) \) satisfies the equation
\[
\Theta_{L,r,s,t}(\tau, \sigma v) = j(\sigma, Z)^{s-r} j(\sigma, Z)^{s-t} \Theta_{L,r,s,t}(\tau, v).
\] (4)

We recall that a function \( \Phi \) on \( G(V) \) is automorphic of weight \( (k, l) \) with respect to some discrete subgroup \( \Gamma \) of \( O^+(L) \) if
\[
\Phi(\sigma v) = j(\sigma, Z)^k j(\sigma, Z)^l \Phi(v).
\]
(a standard argument shows that this is equivalent to the definition given in [B1], in which automorphy was defined using homogeneity and \( \Gamma \)-invariance on \( P \)). Therefore for fixed \( \tau \in H \), \( \Theta_{L,r,s,t}(\tau, v) \) is automorphic of weight \( (s-r, s-t) \) with respect to \( \Gamma \) as a function on \( G(V) \sim K_R + iC \).

We recall that the operator \( \Delta_{SO_{b+,b-}^+} \) is characterized (up to multiplicative and additive constants) by the fact that it is a second order differential operator which commutes with the action of \( SO_{b+,b-}^+ \) on the variable \( \lambda \). Equation (3) (for \( F_{r,s,t} \)) indicates that the operator on \( Z \) corresponding to \( \Delta_{SO_{b+,b-}^+} \) must be an operator (of order 2) which commutes with the action of the slash operators \( [\sigma]_{s-r,s-t} \) on the \( Z \) variable, which are defined (as usual) by
\[
\Phi[\sigma]_{k,l}(Z) = \Phi(\sigma Z) j(\sigma, Z)^{-k} j(\sigma, Z)^{-l}.
\]
As with the standard theory of Laplacians and Casimir operators in simple Lie groups, there should be only one such second order differential operator, up to multiplicative and additive constants. Let \( \Delta^G \) be the operator which in an orthonormal basis for \( K_R \) takes the form
\[
8 \sum_{g,h} y_g y_h \partial_g \overline{\partial}_h - 4Y^2 \left( \partial_1 \overline{\partial}_1 - \sum_{k>1} \partial_k \overline{\partial}_k \right)
\]
(this is \( 8 \) times the operator \( \Delta_1 \) of [Na] or \( 8\Omega \) in the notation of [Bru]), and let \( D^* = \sum y_h \partial_h \) and \( D = \sum y_h \overline{\partial}_h \). The differential operator we are looking for is given in the following

**Lemma 1.2.** the combination
\[
\Delta^G \kappa_{k,l} = \Delta^G - 4ikD^* + 4ilD^*
\]
commutes with the slash operators \( [\sigma]_{k,l} \) for every \( \sigma \in O^+(V) \).

**Proof.** [Na] has shown that \( O^+(V) \) is generated by the elements \( p_\xi \) for \( \xi \in K_R \), \( k_{a,A} \) for \( a \in \mathbb{R}^* \) and \( A \in O^{sgn(a)}(K_R) \), and \( w \). In the \( K_R + \mathbb{R} \zeta + \mathbb{R} z \) notation, under the assumption \( \xi^2 = 0 \) (which is made in [Na] and can always be satisfied by replacing \( \zeta \) by \( \zeta - \frac{\xi^2}{2} z \)), these elements take the form
\[
p_\xi = \begin{pmatrix} 1 & 0 & -\xi^* \\ \xi & 1 & -\zeta \\ 0 & 0 & 1 \end{pmatrix}, \quad k_{a,A} = \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & a \end{pmatrix}, \quad w = \begin{pmatrix} \bar{w} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},
\]

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where $\xi^* : \mathbb{K}_{\mathbb{R}} \to \mathbb{R}$ is defined by pairing with $\xi$ and $\bar{w}$ is the reflection with respect to the hyperplane perpendicular to a pre-fixed positive norm vector in $\mathbb{K}_{\mathbb{R}}$. The sign condition on $A$ ensures that $k_{a,A} \in O^+(V)$, and $w \in SO^+(V)$. Hence it suffices to verify the commutativity of $\Delta_{k,l}$ with $[\sigma]_{k,l}$ for $\sigma$ being one of the elements $p_\xi$, $k_{a,A}$, or $w$.

For $\sigma = p_\xi$ and for $\sigma = k_{a,A}$ the automorphy factor $j(\sigma, Z)$ is a constant function of $Z$ (1 for $p_\xi$, $\frac{1}{\bar{a}}$ for $k_{a,A}$), so the assertion follows easily from the fact that the three operators $\Delta^G$, $D^*$, and $\bar{D}^G$ are invariant under such $\sigma$. For $\sigma = w$, the action of the part $-4ik\bar{D}^G + 4iD^* \Delta_{k,l}$ on $F[\sigma]_{k,l}$ yields

$$-4ik[\bar{D}^G(F \circ w)j^{-l} + (F \circ w)D^*j^{-l}]j^{-k} + 4il[D^*(F \circ w)j^{-k} + (F \circ w)D^*j^{-k}]j^{-l}.$$}

On the other hand, $\Delta^G$ is the sum of two operators, hence apart from the expression $\Delta^G(F \circ w)j^{-k}j^{-l}$, the combination $\Delta^G(F[\sigma]_{k,l})$ involves

$$8\bar{D}^G(F \circ w)D^*j^{-k}j^{-l} + 8D^*(F \circ w)j^{-k}\bar{D}^Gj^{-l} + 8(F \circ w)D^*j^{-k}\bar{D}^Gj^{-l}$$

from the action of the operator $\sum_{g,h} y_j y_h \partial_g \partial_h$ and similar three expressions from the action of the other operator. The automorphy factor $j(w, Z)$ is $\frac{Z^2}{2}$, and evaluating $D^*j^{-k}$, $\bar{D}^Gj^{-l}$, and the other derivatives of $j^k$ and $\overline{j^{-l}}$ shows that $\Delta^G_{k,l}(F[\sigma]_{k,l})$ equals

$$\Delta^G(F \circ w)j^{-k}j^{-l} - 2ik\overline{D}^G(F \circ w)j^{-k-1}j^{-l} + 2ilZ^2D^*(F \circ w)j^{-k}j^{-l-1} +$$

$$+ 4kY^2D(F \circ w)(wZ)j^{-k-1}j^{-l-1} + 4ilY^2D(F \circ w)j^{-k}j^{-l-1}$$

(and the coefficients of $F(wZ)j^{-k-1}j^{-l-1}$ cancel out). Now, the Theorem of [Na] shows that $\Delta^G(F \circ w)(Z) = (\Delta^G F)(wZ)$, and the formulae concerning $D^*$ and $\bar{D}^G$ in [Na] translate to $D^*(F \circ w)(Z) = \overline{W^2}(DF)(wZ) - 2i(3\overline{W})^2$, $(DF)(wZ)$, and $D^*(F \circ w)(Z) = \overline{W^2}(DF)(wZ) + 2iW^2$, $(DF)(wZ)$, with $W = w(Z)$ (as the expressions denoted $\delta$ and $d$ in [Na] are $\frac{Z^2}{2}$ and $\frac{Z^2}{4}$ respectively). One also verifies that $D(F \circ w)(Z) = -(DF)(wZ)$ and $\overline{D}(F \circ w)(Z) = -(\overline{D}F)(wZ)$, while $W^2 = \frac{Z^2}{2}$, $\overline{W}^2 = \frac{Z^2}{2}$, and $(3\overline{W})^2 = \frac{3Z^2}{2}$. Therefore, $\Delta^G_{k,l}(F[\sigma]_{k,l})(Z)$ equals

$$(\Delta^G F)(wZ)j^{-k}j^{-l} - 4ik\overline{D}^G(F \circ w)(wZ)j^{-k}j^{-l} + 4ilD^*F(wZ)j^{-k}j^{-l},$$

which agrees with the value $(\Delta^G_{k,l} F)[\sigma]_{k,l}(Z)$. This proves the lemma. \hfill \Box

The weight $(k,l)$ Laplacian on $G(V)$ is the operator $\Delta^G_{k,l} = \Delta^G_{k,l} - 2(2k - b_-)l$. Lemma 3.20 of [Bru] shows that multiplication by $(Y^2)^t$ takes an automorphic form of weight $(k + t, l + t)$ to an automorphic form of weight $(k, l)$, and one verifies that the operators $\Delta^G_{k,l}$ satisfy the relation $\Delta^G_{k,l}(Y^2)^t = (Y^2)^t \Delta^G_{k+l,t+t}$.  

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We remark that the results of [Na] are stated for $b_-$ (or $q$ in the notation of [Na]) being at least 3. However, the proof holds equally well for $q = 2$, and the same applies for our Lemma 1.2. For $b_- = 1$ the Grassmannian $K_R + iC$ is $\mathcal{H}$, the operator $\Delta^G$ (or $8\Delta_1$ in the notation of [Na]) is the usual Laplacian $4y^2\partial\overline{\partial}$ on $\mathcal{H}$, and $\Delta^G_{k,l}$ is $\Delta_{2k,2l}$. The factor 2 if the weights comes, as mentioned in [B1], from the fact that $SL_2(\mathbb{R})$ is a double cover of $SO^+_{2,1}$. Therefore, Lemma 1.2 holds for any value of $b_-.

1.3 A Differential Equation for $\Theta_{L,r,s,t}$

The following generalization of Proposition 4.5 of [Bru] will turn out very important for our purposes:

**Proposition 1.3.** Let $L$ be an even lattice in the space $V$ of signature $(2,b_-)$, and let $k = 1 + \frac{b_-}{2} + r + t$. Then the theta function $\Theta_{L,r,s,t}$ satisfies the differential equation

$$4\Delta_{k,r} y^k \Theta_{L,r,s,t}(\tau, Z) = [\Delta^G_{s-r-s-l,Z} + 2r(b_- - 2t)] y^k \Theta_{L,r,s,t}(\tau, Z).$$

**Proof.** We follow the proof of Proposition 4.5 of [Bru], with the necessary adjustments. It suffices to prove that the basic summand $F_{r,s,t}(\tau, Z, \lambda)$ satisfies this differential equation for any $\lambda$. Let $\rho_{k,l}$ be the representation of $O^+(V)$ on $C^\infty(G(V))$ using the weight $(k,l)$ slash operators, namely $\rho_{k,l}(\sigma) \Phi = \Phi[\sigma^{-1}]_{k,l}$, and extend it to universal enveloping algebra of $\mathfrak{g}(V)$. Lemma 1.2 shows that the action of the Casimir operator of $O^+(V)$ via $\rho_{k,l}$ must be the same as that of $\alpha \Delta^G_{k,l} + \beta$ for some constants $\alpha$ and $\beta$. Equation (3) and the paragraph following it imply the equality

$$F_{r,s,t}(\tau, Z, \sigma^{-1}\lambda) = \rho_{s-r-s-l}(\sigma) F_{r,s,t}(\tau, Z, \lambda)$$

for every $\tau \in \mathcal{H}$, $Z \in K_R + iC$, $\lambda \in V$, and $\sigma \in O^+(V)$. Since the Casimir operator of $O^+(V)$ acts on functions of $\lambda$ as $\Delta_{SO^+_{2,b_-}}$, we obtain

$$\Delta_{SO^+_{2,b_-}} F_{r,s,t}(\tau, Z, \lambda) = \pm(\alpha \Delta^G_{s-r-s-l} + \beta) F_{r,s,t}(\tau, Z, \lambda).$$

Equation (2) now yields

$$4\Delta_{k,r} y^k F_{r,s,t}(\tau, Z, \lambda) = (\alpha \Delta^G_{s-r-s-l} + \beta) y^k F_{r,s,t}(\tau, Z, \lambda)$$

for some constants $\tilde{\alpha}$ and $\tilde{\beta}$ which are independent of $\lambda$. Choose a basis for $K_R$ in which the first two basis elements span a hyperbolic plane and the rest are orthonormal (or with common norm $-2$, as in [Bru] and [Na]), and take $\lambda$ to be the second basis element. Then evaluating $e^{-\Delta_{s-r-s-l}/8\pi y} F_{r,s,t}(\tau, Z, \lambda)$ shows that

$$F_{r,s,t}(\tau, Z, \lambda) = \sum_{j=0}^{\min(r,t)} \frac{j!}{(2\pi)^j} \left(\frac{\tau}{j}ight) \left(\frac{r\tau}{j}ight) \left(\frac{s\tau}{j}ight) e^{-2\pi y \frac{2r+s}{2j}}.$$
with $z_1$ being the first coordinate of $Z$ in this basis. A straightforward computation shows that $\alpha = 1$ and $\beta = 2r(b_+ - 2t)$. This proves the proposition. \qed

The validity of Equation (3) extends to the case in which $P_{r,s,t}$ is multiplied by a power of $\lambda_{v_+}^2$. Since it holds also after applying the differential operator $e^{-\Delta_v/8\pi y}$, Equation (3) extends to this case as well. Multiplying by $\lambda_{v_+}^2$ just adds 1 to each index of $P_{r,s,t}$, hence it gives only functions which have already been considered above. In any case, it follows that Proposition 1.3 holds also for the theta function associated to the polynomial $P_1$ to each index of $L\theta$ and applying this relation for $k = 1 + \frac{b_+}{2} + r + t - 2h$ and the constant $\beta$ being $2(r-h)(b_+ - 2(t-h))$. However, we shall not need this generalization in this paper.

We remark that since we are concerned with explicit functions and operators, Equations (1) and (2) and Proposition 1.3 can be obtained by direct evaluation of the corresponding derivatives. In addition, we recall the weight raising operator $R_k = 2i \frac{\partial}{\partial \tau} + \frac{k}{y}$ and the weight lowering operator $L = -2iy^2 \frac{\partial}{\partial \tau}$ (note the sign difference relative to [Bru]!). The action of the operators $R_k$ and $L$ on the function $y^{k-m-F}(\tau, v, p, \lambda)$, for arbitrary signature and homogeneous polynomial $p$, takes it to

$$-2\pi y^{k-m-F}(\tau, v, \lambda_{v_+}^2 p, \lambda) - \frac{1}{8\pi} y^{k-m-2F}(\tau, v, \Delta_{v_-} p, \lambda)$$

(5a)

and

$$2\pi y^{k-m-F}(\tau, v, \lambda_{v_+}^2 p, \lambda) + \frac{1}{8\pi} y^{k-m-F}(\tau, v, \Delta_{v_-} p, \lambda)$$

(5b)

respectively. Thus similar formulae describe the action of these operators on arbitrary theta functions with polynomials. One verifies that the action of $\Delta_{SO_{k,b_-,b_+}^*}(\frac{b_+}{2} - m_-)$ on $y^{k-m-F}(\tau, v, p, \lambda)$ takes it to $y^{k-m-q(\lambda)} \lambda_{v_-}^2(\frac{\lambda_{v_+}^2}{2} + \frac{\lambda_{v_-}^2}{2})$ with $q$ being the image of $e^{-\Delta_v/8\pi y}$ under the operator

$$\Delta_{SO_{k,b_-,b_+}^*} + 8\pi y\lambda_{v_-}^2 I_{v_+} - 8\pi y\lambda_{v_+}^2 I_{v_-} + 4\pi yb_+\lambda_{v_-}^2 - 4\pi yb_-\lambda_{v_+}^2 - 16\pi^2 y^2\lambda_{v_+}^2\lambda_{v_-}^2,$$

(6)

where for a space $U$ with basis $u_j$, the operator $I_U$ stands for $\sum_{j} u_j \frac{\partial}{\partial u_j}$ (and is independent of the basis chosen). The Laplacian $\Delta_k$ can be written as $R_{k-2} \cdot L$, and applying this relation for $k = \frac{b_+}{2} + m_+ - \frac{b_+}{2} - m_-$ shows that $\Delta_k$ takes $y^{k-m-F}(\tau, v, p, \lambda)$ to the function described in Equation (6) up to the constant multiple of $y^{k-m-F}(\tau, v, p, \lambda)$ appearing in Equation (2). As for the case $b_+ = 2$ and the polynomial $P_{r,s,t}$, one can check directly that the operator from Equation (6) takes $P_{r,s,t}$ to

$$-4\lambda_{v_+}^{2} a_{r,t} P_{r-1,s-1,t-1} - 4\pi y b_- P_{r+1,s+1,t+1} + 2tb_- P_{r,s,t},$$

(7)
where $a_{r,t}$ stands for

$$(r - 2\pi y \lambda_{v_+}^2)(t - 2\pi y \lambda_{v_+}^2) - 2\pi y \lambda_{v_+}^2 = rt - (r + t + 1)2\pi y \lambda_{v_+}^2 + (2\pi y \lambda_{v_+}^2)^2.$$  

A direct evaluation of the action of the operator $\Delta^G_{s-r,s-t} + (r-t)^2 + b_- (r-t)$ on $P_{r,s,t}(Z,\lambda)e^{\frac{\lambda_{v_+}^2}{2} + \frac{\lambda_{v_+}^2}{2}}$ gives the expression from Equation (7) multiplied by $e^{\frac{\lambda_{v_+}^2}{2} + \frac{\lambda_{v_+}^2}{2}}$. Therefore, the fact that $\Delta^G_{s-r,s-t} + (r-t)^2 + b_- (r-t)$ and $\Delta_{SO_{b_+,b_-,\lambda}}$ give the same result also on $F_{r,s,t}$ follows from the observation that $\Delta_{v_+}^j P_{r,s,t}$ is some constant multiple of $P_{r-j,s-j,t-j}$ for any $j \leq \min\{r,t\}$ and the difference $(r - j) - (t - j)$ equals $r - t$. This argument suggests an alternative proof of Equations (1) and (2) and of Proposition 1.3 (as well as of Proposition 4.5 of [Bru]), a proof which is independent of Theorem 1.1, the results of [Sh], and theorems about actions of Casimir operators and Laplacians in general.

2 Theta Lifts of Almost Weakly Holomorphic Forms

In this Section we evaluate the theta lifts of certain almost weakly holomorphic modular forms, which will give the main tool for the arithmetic application later (see Definitions 4.1 and 1.2 below). Many parts of this Section are very technical, and skipping most of it except the statements of Theorem 2.7 for general $b_-$ and Theorem 2.8 for $b_- = 1$ may suffice on the first reading.

2.1 Modular Forms

Let $\Gamma$ be a Fuchsian subgroup of $Mp_2(\mathbb{R})$ (i.e., the volume of $\Gamma \backslash \mathcal{H}$ with respect to the invariant measure $d\tau/\tau$ is finite), and let $\rho$ be a representation of $\Gamma$ on some finite-dimensional complex vector space $V_{\rho}$. We recall that a modular form of weight $(k,l)$ and representation $\rho$ with respect to $\Gamma$, for $k$ and $l$ being half-integers, is a real-analytic function $f : \mathcal{H} \to V_{\rho}$ which satisfies the functional equation

$$f(M \tau) = j(M,\tau)^{k} j(M,\tau)^{l} \rho(M) f(\tau)$$

for every $\tau \in \mathcal{H}$ and $M \in \Gamma$. The half-integral powers of $j(M,\tau)$ and its complex conjugate are defined by the metaplectic choice of the square root. Multiplying a modular form of weight $(k + t, l + t)$ by $\gamma^t$ gives a modular form of weight $(k,l)$, and by a modular function we refer to a scalar-valued modular form of weight 0 and trivial representation, i.e., a $\Gamma$-invariant function on $\mathcal{H}$.

A function $f : \mathcal{H} \to V$, for $V$ a complex vector space, is called almost holomorphic if it can be written as a sum $\sum_{k=0}^{k_{max}} \frac{f_k(\tau)}{\lambda_{v_+}^k}$, for some integer $k_{max}$, with the $f_k$ holomorphic functions $\mathcal{H} \to V$. This condition is stable under the action of $Mp_2(\mathbb{R})$. In case $\Gamma$ has a cusp, we call a modular form of weight $k$ and representation $\rho$ with respect to $\Gamma$ weakly holomorphic if it is holomorphic
on $\mathcal{H}$ and may have a pole in every cusp. We call such a function *almost weakly holomorphic* when the requirement of holomorphicity on $\mathcal{H}$ is replaced by almost holomorphicity. Recall that the weight raising operator $R_k$ takes modular forms of weight $k$ to modular forms of weight $k+2$, and the weight lowering operators $L$ takes such forms to modular forms of weight $k-2$. Moreover, both operators preserve almost holomorphicity and almost weakly holomorphicity, with the latter operator annihilating (almost) holomorphic functions.

A modular form $f$ of any weight $(k,l)$ and representation $\rho_L$ with respect to $Mp_2(\mathbb{Z})$ has a Fourier expansion

$$f(\tau) = \sum_{\gamma \in L^* / L} \sum_{n \in \mathbb{Q}} c_{\gamma,n}(y) q^n e_{\gamma}$$

with the standard notation $q = e(\tau)$, and where $c_{\gamma,n}$ are smooth functions of $y$ such that $c_{\gamma,n}$ vanishes unless $n \in \mathbb{Z} + \frac{\gamma^2}{4}$. For almost weakly holomorphic $f$ the functions $c_{\gamma,n}$ are polynomials of bounded degree in $\frac{1}{y}$ and vanish unless $n \gg -\infty$, and for weakly holomorphic $f$ modular form these polynomials are constants.

### 2.2 Theta Lifts

Let $L$ be an even lattice of signature $(b_+,b_-)$, and take for every $v \in G(L_\mathbb{R})$ a polynomial $p_v$ on $L_\mathbb{R}$ which is homogeneous of degree $(m_+,m_-)$ with respect to $v$. Following [B1] (but using a different presentation of the polynomial part) and others, we define the theta lift of a (not necessarily holomorphic) modular form $F$ of weight $\frac{b_+}{2} + m_+ - \frac{b_-}{2} - m_-$ and representation $\rho_L$ as follows. The dual of the Weil representation $\rho_L$ is identified with its complex conjugate $\rho_L^*$ by declaring the canonical basis $\{e_\gamma\}_{\gamma \in L^* / L}$ orthonormal, so that for modular forms $f$ and $g$ of weights $(k,l)$ and $(r,s)$ and representation $\rho_L$ the (Hermitian) pairing $\langle f,g \rangle_{\rho_L}$ is a scalar valued modular form of weight $(k+s,l+r)$ with respect to $Mp_2(\mathbb{Z})$. Specifying $f = F$ and $g = \Theta_L$ and multiplying by $y^{\frac{b_+}{2} + m_+}$ gives us a modular function of $\tau$. If $F$ decreases exponentially towards the cusp, then we define the *theta lift of $F$* as the integral

$$\Phi_L(v,F,p_v) = \int_{X(1)} y^{\frac{b_+}{2} + m_+} (F(\tau), \Theta_L(\tau,v,p_v))_{\rho_L} \frac{dxdy}{y^2}, \quad (8)$$

where $X(1)$ is the level one modular curve $X(SL_2(\mathbb{Z}))$. The integral in Equation (8) converges, and we refer to it as the theta lift of $F$ at $v \in G(L_\mathbb{R})$ with respect to the polynomial $p_v$. We consider it as a function on $G(L_\mathbb{R})$, which depends on the polynomial $p$ (where $p$ stands for the function $v \mapsto p_v$). In this case $\Phi_L(v,F,p_v)$ is a smooth function of $v$ if $p_v$ depends smoothly on $v$.

If $F$ grows exponentially toward the cusp, then the integral in Equation (8) diverges, but in many cases it gives a meaningful value when regularized as follows. Let

$$D = \{\tau \in \mathcal{H} | |\Re \tau| \leq 1/2, |\tau| \geq 1\}$$
be the classical fundamental domain for $SL_2(\mathbb{Z})$ (or $Mp_2(\mathbb{Z})$), and define, for every $w > 1$, $D_w = \{ \tau \in D | y \leq w \}$. Since $D_w$ is a compact domain, the integral over it converges, and the same assertion holds when the integrand is multiplied by $y^{-s}$ for any complex $s$. Under certain conditions on $F$ the limit as $w \to \infty$ defines a holomorphic function of $s$ for some right half-plane, which can be extended to a meromorphic function for all $s \in \mathbb{C}$. The regularized integral is then defined to be the constant term of the Laurent expansion of this function at $s = 0$.

As stated in Section 6 of [B1], the equality

$$\Phi_L(\sigma v, F, p_v \circ \sigma^{-1}) = \Phi_L(v, f, p_v)$$

holds for every $\sigma$ in the discriminant kernel $\Gamma$ of $Aut(L)$. Indeed, this follows from the observation that if $p_v$ is homogenous of degree $(m_+, m_-)$ with respect to $v$ then $p_v \circ \sigma^{-1}$ is homogenous of degree $(m_+, m_-)$ with respect to $\sigma v$ for every $\sigma \in O(L_\mathbb{R})$, and the equality $\Delta_{\sigma v}(p \circ \sigma^{-1}) = (\Delta_v p) \circ \sigma^{-1}$ holds. This is the "automorphy property" of $\Phi_L$ in our notation, using $p_v$ for each $v$ rather than treating $v$ as an isometry $L_\mathbb{R} \to \mathbb{R}^{b_+, b_-}$ and $p$ as a polynomial on the latter space. For $p = 1$ this gives the $\Gamma$-invariance of $\Phi_L$. For a general polynomial $p_v$ the automorphic behavior of $\Phi_L$ depends on the relation between $p_v \circ \sigma^{-1}$ and $p_{\sigma v}$, i.e., on the behavior of $v \mapsto p_v$ under $\Gamma$. In general we cannot say anything about this relation, but for the case $b_+ = 2$ and the polynomials $P_{r,s,t}$ defined above (or even for $P_{r,s,t}(\lambda_{\mathbb{C}}^2)$), Equation (4) implies

$$\Phi_{L,r,s,t}(\sigma v, F) = j(\sigma, Z)^{s-t} \overline{j(\sigma, Z)}^{-t} \Phi_{L,r,s,t}(v, F)$$

for any $Z \in K_\mathbb{R} + iC$ and $\sigma \in \Gamma^+ = \Gamma \cap O^+(L_\mathbb{R})$ (the complex conjugation on $\Theta_L$ in $\Phi_L$ interchanges the powers of $j$ and $\overline{j}$), with $\Phi_{L,r,s,t}(v, F)$ denoting $\Phi_L(v, F, P_{r,s,t})$.

If $F$ vanishes at the cusp then the theta lift $\Phi_L$ is a smooth function of $v$. On the other hand, if $F$ grows exponentially at the cusp this is no longer true, even when $p_v$ is a smooth function of $v$. We restrict attention to the case of $F$ being almost weakly holomorphic, i.e., is of the form

$$F(\tau) = \sum_{\gamma \in L^*/L} \sum_{k=0}^{k_{\text{max}}} \sum_{n > -\infty} c_{\gamma, n, k} q^n y^{\gamma}$$

for some integer $k_{\text{max}}$, where $c_{\gamma, n, k}$ are constants. Since $c_{\gamma, n, k} \neq 0$ only for $n \in \frac{\gamma}{2} + \mathbb{Z}$, the polar part of $F$ is a finite sum. Then Theorem 6.2 of [B1] shows that, in our notation, the singularities of $\Phi_L$ are along the real-$b_+$-codimensional sub-Grassmannians of the form

$$\lambda^+ = \{ v \in G(L_\mathbb{R}) | \lambda \in v_- \} \subseteq G(L_\mathbb{R})$$

where $\lambda$ is an element of $L$ (or $L^*$) of negative norm. Along such $\lambda^+$ the singularity (in the sense of not being smooth rather than being just discontinuous)
takes the form
\[
\sum_{\alpha \in \mathbb{L}^*} \sum_{j,k} c_{\alpha,\lambda} \varphi_{x,\lambda} \Delta_{\lambda}^j(x) \left( \frac{\Gamma(\beta)}{(-2\pi \alpha^2 \lambda^2)^{\beta}} \beta \not\in -\mathbb{N} \right) \left( \frac{\ln(\alpha^2 \lambda^2)}{(-2\pi \alpha^2 \lambda^2)^{\beta}(-\beta)!} \right) \beta \in -\mathbb{N}
\]

where \( \beta \) stands for \( \frac{b_1}{2} + m_+ - j - k - 1 \) (and \( \mathbb{N} \) includes 0). The sum over \( \alpha \) is essentially finite since there are only a finite number of non-zero coefficients \( c_{\gamma,n,k} \) with \( n < 0 \). The sum over \( j \) (which includes only those terms for which \( \Delta_{\lambda}^j(x) \) does not vanish) and the sum over \( 0 \leq k \leq k_{\text{max}} \) are also finite.

At this point we need to sort out some inaccuracies in the proofs of \([B1]\) which are relevant for our discussion. In particular, the argument based on Lemma 14.1 of this reference (specifically Corollary 6.3 and Corollary 14.2 there) can be replaced by the following

**Lemma 2.1.** Let \( C \) be a positive integer, and let \( p \) be a polynomial of degree smaller than \( C \). Then \( \sum_{j=0}^{C} (-1)^j \left( \begin{array}{c} C \\ j \end{array} \right) p(j) = 0 \).

**Proof.** Since we can write \( p(x) = \sum_{r=0}^{\deg p} a_r \binom{x}{r} \) where \( \binom{x}{r} \) is the polynomial \( \prod_{i=0}^{r-1} (x-i)/r! \) (of degree \( r \)), it suffices to prove the claim for \( p(x) = \binom{x}{r} \) with \( 0 \leq r < C \). In this case only terms with \( j \geq r \) contribute, and the contribution is given by \( \binom{C}{r} \binom{j}{r} \binom{C-r}{j-r} \). Hence we find that
\[
\sum_{j=r}^{C} (-1)^j \binom{C}{j} \binom{j}{r} (-1)^r \binom{C-r}{i} = 0,
\]
as \( r < C \).

Indeed, both Corollary 6.3 and Corollary 14.2 of \([B1]\) involve sums over binomial coefficients of the form \( \binom{A+D-2j}{A-2j} = \binom{A+D-2j}{D} \) for \( A \) and \( D \) non-negative integers with \( D < C \) which are independent of \( j \). This expression is a polynomial function of \( j \) of degree \( D < C \), hence our Lemma applies for both cases.

Somewhat more disturbing are the assertions in Corollary 6.3 that state that certain expressions are polynomials in an oriented norm 1 vector \( v_1 \) in a Lorentzian space. Indeed, in our notation (with \( v \mapsto p_v \)), the “wall crossing formula” from Corollary 6.3 states that the difference between the values of the theta lift on two adjacent Weyl chambers \( W \) and \( W' \) with separating “wall” \( \lambda^+ \) for \( (\lambda, W') > 0 \) is
\[
\sum_{x,\lambda \in \mathbb{L}^*} \sum_{j,k} 4c_{x,\lambda} \varphi_{x,\lambda} \Delta_{\lambda}(x) \left( \frac{\Gamma(m_+ - j - k - \frac{4}{2})}{(-8\pi)^{3/4}} \right) \left( \sqrt{2\pi x(x, v_1)} \right)^{1/2} \frac{1}{(-8\pi)^{3/4}j!} \binom{m_+ - j - k - \frac{4}{2}}{j!} \left( \Gamma(m_+ - j - k - \frac{4}{2}) \right) \left( \frac{\Gamma(\beta)}{(-2\pi \alpha^2 \lambda^2)^{\beta}(-\beta)!} \right)
\]
with only \( x > 0 \) and with \( v_1 \) being an oriented norm 1 vector spanning \( v_+ \), and the proof there shows that only positive powers of \( (\lambda, v_1) \) appear. However, this
is not a polynomial in $v_1$ because of the unknown dependence of $p_v$ on $v$ (or on $v_1$). The fact that in [B1] one does not work with $v \in G(M)$ but with $v$ an isometry from $L_R$ to $\mathbb{R}^{k_+ \cdot b_-}$ does not overcome this problem, since then the expression in Equation (11) depends on $v$ as an isometry and not only on its image in $G(L_R)$, hence cannot be described as a function of $v_1$ alone. Only the smoothness of this difference as a function on all of $G(L_R)$ survives (at least in our conventions with $p_v$). On the other hand, when $p_v(\lambda) = (\lambda, v_1)^{m^+}$ of homogeneity degree $(m_+, 0)$ (and no multiplying constant), the expression in Equation (11) is indeed the restriction of a polynomial on $L_R$ of the asserted degree to the set of vectors of the form $v_1$.

In order to discuss the properties of the theta lift in the Lorentzian case, we consider again the decomposition of $p$ appearing just above Lemma 5.1 of [B1]. In our notation, for any $v \in G(L_R)$, a polynomial $p_v$ on $L_R$ which is homogenous of degree $(m_+, m_-)$ with respect to $v$, and natural numbers $h_+$ and $h_-$, there is a polynomial $p_{v,h_+,h_-}$ on $K_R$ which is homogenous of degree $(m_+-h_+, m_- - h_-)$ with respect to the decomposition $w$ of $K_R$ into the images $w_{\pm}$ of $z_{v_+}^\pm \subseteq v_\pm$ in $K_R$ such that

$$p_v(\lambda) = \sum_{h_+ \cdot h_-} (\lambda, z_{v_+})^{h_+} (\lambda, z_{v_-})^{h_-} p_{v,h_+,h_-} \left( \left( \frac{(\lambda, z) z_{v_+}}{z_{v_+}^2} \right) / \mathbb{R} \right)$$

for all $\lambda \in L_R$. We leave $v$ (rather than $w$) in the notation since different elements $v \in G(L_R)$ may map, with different $p_v$, to the same $w \in G(K_R)$. The map $\lambda \mapsto (\lambda - (\lambda, z) z_{v_+} / z_{v_+}^2) / \mathbb{R}$ corresponds to the map denoted $w$ in [B1]. A careful verification of Sections 5 and 7 of [B1] shows that the reduction formula in Theorem 7.1 of [B1] holds in our notation with each $p_{v,h_+,h_-}$ replaced by $p_{v,h_+,h_-}$. Now, Theorem 10.2 of [B1] implies that the theta lift $\Phi_L$ is a smooth function inside any Weyl chamber. However, the assertion that it is a polynomial does not necessarily hold, because of the same problem discussed in the previous paragraph. Only in the special case where $p_v(\lambda) = (\lambda, v_1)^{m^+}$ we have $h_- = 0$ and $p_{v,m_+,0} = 1$, so that the assertion of Theorem 10.3 of [B1] holds as stated for this case. This is related to the fact that $p_v \circ \sigma^{-1} = p_{sv}$ for this $p_v$, since $\Gamma$-invariance is used in the proof of that theorem.

These remarks show that one must be careful when investigating properties of theta lifts of modular forms using theta functions with polynomials. However, in the applications appearing in [B1] one considers only the case $p = 1$ (in Sections 11, 13, and 15 there), or some multiple of $\eta \mapsto (\eta, v_1)^{m^+}$ (in Section 14). Hence the results of these sections hold as stated.

### 2.3 Differential Properties of Some Theta Lifts

For the proof of the main result of this paper, Theorem 13 below, we shall need to use Serre duality against principal parts of weakly holomorphic modular forms of weight $1 - b_- - m$ in order to establish that a certain series is a modular form of weight $1 + b_+ + m$. On the other hand, the polynomial $P_{r,s,t}$ is of homogeneity
degree \((r + t, 0)\), so that for \(r + t = m\) we have to lift modular forms of weight \(1 - \frac{b}{2} - m\). Therefore the modular forms which we shall lift are the images of weakly holomorphic modular forms of weight \(1 - \frac{b}{2} - m\) under \(m\) applications of the corresponding weight raising operator. The normalized weight raising operator for weight \(l\) is

\[
\delta_l = -\frac{R_l}{4\pi} = \frac{\partial_l}{2\pi i} - \frac{l}{4\pi iy},
\]

where \(\partial_l\) denotes \(\frac{\partial}{\partial_l}\) from now on. Note that \(\frac{\partial_l}{2\pi iy}q^m = nq^m\), so that this normalization of the weight raising operator has a simple action on Fourier expansions of modular forms. A well-known formula for the composition of these operators, which is easily proved by induction, states that

\[
\delta_l^m = \delta_{l+2m-2} \circ \ldots \circ \delta_l = \sum_k \binom{m}{k} \prod_{s=m-k}^{m-1} \left( \frac{l+s}{4\pi y} \right) \frac{\partial_l}{(2\pi i)^m-k}.
\]

(see, for example, Equation (56) in \(\text{Za}\)). Moreover, \(\delta_l\) sends eigenfunctions of the weight \(l\) Laplacian \(\Delta_l\) and eigenvalue \(\lambda\) (i.e., which are annihilated by \(\Delta_l + \lambda\)—note that we are using the convention where the Laplacian has a positive definite principal symbol hence eigenvalues must be negated) to eigenfunctions of \(\Delta_{l+2}\) with eigenvalue \(\lambda + l\), so that the \(\delta_l^m\)-images of the former functions have the eigenvalue \(\lambda + m(l - m - 1)\). Thus, if \(f = \sum_{\gamma,n,c} c_{\gamma,n}q^n e_\gamma\) is a weakly holomorphic (hence harmonic, with \(\lambda = 0\)) modular form of weight \(1 - \frac{b}{2} - m\) and representation \(\rho_L\) for some even lattice \(L\) of signature \((2, b_-)\) then

\[
F = \delta_l^m \frac{1}{y^{b_-+m}} f = \sum_{\gamma,n,k} c_{\gamma,n,k} q^n y^k c_\gamma, \quad c_{\gamma,n,k} = \binom{m}{k} n^{-m-k} \prod_{r=0}^{k-1} \left( r + \frac{b_-}{2} \right), \quad c_{\gamma,n} \left( \frac{4\pi}{\tau} \right)^k
\]

(12)

is an almost weakly holomorphic modular form of weight \(1 - \frac{b}{2} - m\) and representation \(\rho_L\) which is an eigenfunction of \(\Delta_{1 - \frac{b}{2} - m}\) with eigenvalue \(-\frac{mb_-}{2}\).

We now consider the function \(\Phi_{L,m,m,0}(v, F)\), which by Equation (11) is an automorphic form on \(O^+(L_R)\) which has weight \(m\) with respect to \(\Gamma^+\). This is the theta lift used in Theorem 14.3 of \(\text{[B1]}\) (in our notation), but the modular form \(F\) which we lift is not weakly holomorphic. One of the crucial properties of this function is its behavior under the operator \(\Delta^G_m = \Delta^G_{m,0}\). This property is proved using Proposition 1.3 after one transfers the action of the Laplacian \(\Delta_k\) from \(F\) to \(\Theta_L\), as in Section 4.1 of \(\text{[Bru]}\). Since the regularization in \(\text{[Bru]}\) is different from ours, we give the proofs of all assertions, with an emphasis on the differences relative to \(\text{[Bru]}\).

**Lemma 2.2.** Let \(f\) and \(g\) be modular forms of weights \(k\) and \(k+2\) respectively, and representation \(\rho_L\). The equality

\[
\int_{D_w} (g^{k+2}(R_k f, g) + y^k \langle f, L g \rangle) y^{-s} d\mu = s \int_{D_w} y^{k+1} \langle f, g \rangle y^{-s} d\mu - \int_{\mathbb{R}/\mathbb{Z} + ivw} w^k \langle f, g \rangle w^{-s} dx
\]
holds for any \( w > 1 \) and \( s \in \mathbb{C} \).

**Proof.** For the proof, see Lemma 4.2 [Bru] (with some sign differences). The first term on the right hand side arises from the difference between \( y^{-s} d\omega \) and \( d(y^{-s}\omega) \), for \( \omega \) being the \( SL_2(\mathbb{Z}) \)-invariant 1-form \( y^k(f, g) d\tau \) appearing in [Bru].

Note that while \( y^{-s} \omega \) is not \( Mp_2(\mathbb{Z}) \)-invariant in general, the fact that \( y^{-s} \) is \( T \)-invariant and its restriction to the curve \( |\tau| = 1 \) is also \( S \)-invariant (since \( \Im(S\tau) \) allows us to apply the argument also in this case).

Using Lemma 2.2 twice we obtain the following analog of Lemma 4.3 of [Bru]:

**Lemma 2.3.** Let \( f \) and \( g \) be two modular forms of weight \( k \) and representation \( \rho_L \), and let \( w > 1 \) and \( s \in \mathbb{C} \) be given. The difference

\[
\left( \int_{D_w} y^k(\Delta_k f, g) y^{-s} d\mu - \int_{D_w} y^k(f, \Delta_k g) y^{-s} d\mu \right)
\]

equals

\[
s \int_{D_w} y^{k-1}((Lf, g) - (f, Lg)) y^{-s} d\mu - \int_{\mathbb{R}/\mathbb{Z} + iw} w^{k-2}((Lf, g) - (f, Lg)) w^{-s} dx.
\]

We are interested in the case where \( g(\tau) = y^{\frac{b+}{2}+m-} \Theta_L(\tau, v, p_v) \) (so that \( k = \frac{b+}{2} + m_+ - \frac{b-}{2} - m_- \)). The line integrals at the limit \( w \to \infty \) are dealt with in the following

**Lemma 2.4.** Let \( L \) be an even lattice of signature \( (b_+, b_-) \), let \( v \) be an element of \( G(L_R) \) which does not belong to any \( \lambda^\perp \) for \( \lambda \in L^* \), and let \( p \) be a polynomial on \( L_R \) which is homogenous of degree \( (m_+, m_-) \) with respect to \( v \). Let \( F \) be a modular form of weight \( \frac{b+}{2} + m_+ - \frac{b-}{2} - m_- \) and representation \( \rho_L \), and write \( F(\tau) \) as

\[
\sum_{\gamma \in L^*/L} \sum_{n \gg -\infty} c_{\gamma, n}(y) q^n e_\gamma
\]

with \( c_{\gamma, n} \) smooth for every \( \gamma \in L^*/L \) and \( n \in \mathbb{Z} + \frac{b^2}{4} \). Assume that \( c_{\gamma, n} \) has sub-exponential growth (i.e., \( c_{\gamma, n}(y) = o(e^{y}) \) as \( y \to \infty \) for every \( \varepsilon > 0 \) for every \( \gamma \) and \( n \), and in the case where \( m_+ + m_- \) is even and the constant \( \Delta_j(p) \) for \( j = \frac{m_+ + m_-}{4} \) does not vanish, assume that \( c_{0, 0}(y) = o(y^T) \) as \( y \to \infty \) for some \( T \)). Then

\[
\lim_{w \to \infty} \int_{\mathbb{R}/\mathbb{Z}} w^{\frac{b+}{2} + m_+ - 2} \langle F(x + iw), \Theta_L(x + iw, v, p_v) \rangle_{\rho_L} dx = 0
\]

for large enough \( \Re s \).
Proof. For fixed \( w \), the integral equals some power of \( w \) times the constant term of the Fourier expansion of \((F, \Theta_L)\) at \( y = w \). Hence we are considering the limit of the expression

\[
w^\frac{b_+ + m_+ - 2 - s}{2} \sum_{\lambda \in \mathbb{L}^*} e^{-\Delta_\lambda/8\pi w} (\mathcal{J})(\lambda) c_{\lambda, \lambda^2}^L (w) e^{-2\pi w \lambda^2} \tag{13}
\]
as \( w \to \infty \). The assumption that \( v \not\in \lambda^\perp \) for any \( \lambda \in \mathbb{L}^* \) implies that \( \lambda^2 \not\in \mathcal{M}_{v_+} \) for any non-zero \( \lambda \) in \( \mathbb{L}^* \), so that \( e^{-2\pi w \lambda^2} \) eliminates the sub-exponential growth of \( c_{\lambda, \lambda^2}^L (w) \) (hence of \( c_{\lambda, \lambda^2}^L (w)w^{-j} \) for any \( j \) coming from \( e^{-\Delta_\lambda/8\pi y} (\mathcal{J})(\lambda) \)). To evaluate the remaining term with \( \lambda = 0 \), we see that \( e^{-\Delta_\lambda/8\pi y} (\mathcal{J})(\lambda = 0) \) may have non-zero contribution only if \( m_+ + m_- \) is even, and then the only contribution comes from the constant \( \Delta_0 (p) \) for \( j = \frac{m_+ + m_-}{2} \). If this constant does not vanish, we have to consider some constant multiple of \( c_{0,0}(w)w^{\frac{b_+ + m_+ - 2 - j - s}{2}} \), which tends to 0 for \( Re s >> 0 \) because of the polynomial growth of \( c_{0,0} \). This proves the lemma.

Using Lemmas 2.3 and 2.4 we generalize Lemma 4.4 of [Bru] as follows.

**Lemma 2.5.** Let \( L, v, p, \) and \( F \) be as in Lemma 2.2. Assume that the regularized theta lift of \( F \) is well-defined (i.e., we obtain a function on \( s \) for some right half-plane and can meromorphically continue it to \( \mathbb{C} \)). If \( e^{-\Delta_\lambda/8\pi y} (q)(\lambda = 0) \) vanishes for either \( q = p, q = \Delta_\nu p, \) or \( q = \lambda^2_{v_-} p \), then the theta lift of \( \Delta_k F (\tau) \), with the weight \( k \) being \( \frac{b_+}{2} + m_+ - \frac{b_+}{2} + m_- \), gives the same result as the regularized integral

\[
\int_{X(1)} \langle F(\tau), \Delta_k y^{\frac{b_+}{2} + m_+} \Theta_L (\tau, v, p) \rangle \frac{dxdy}{y^2}. \tag{13}
\]

**Proof.** Fix \( w > 1 \) and \( s \in \mathbb{C} \). By Lemma 2.3 the difference between the corresponding integrals on \( D_w \) is the sum of a line integral at \( y = w \) and a certain integral over \( D_w \) multiplied by \( s \). The weight raising operator \( L \) preserves the properties of \( F \) needed for Lemma 2.4 and by Equation \( \text{(13)} \) the image of \( y^{\frac{b_+}{2} + m_+} \Theta_L (\tau, v, p) \) under \( L \) is the sum of two theta functions with polynomials. Thus, for large enough \( Re s \), Lemma 2.4 implies that the line integral vanishes as \( w \to \infty \). It remains to consider the constant term in the Laurent expansion at \( s = 0 \) of the limit \( w \to \infty \) of the other integral. Because of the factor \( s \), changing the integral by a finite number does not affect this value. We thus replace the domain \( D_w \) by the rectangle \( x \in \mathbb{R}/ \mathbb{Z} \) and \( 1 \leq y \leq w \). By Equation \( \text{(13)} \) we have to consider a linear combination of integrals of the form

\[
s \int_1^w \int_{\mathbb{R}/ \mathbb{Z}} y^{\alpha - s} \langle G(\tau), \Theta_L (\tau, v, q_\alpha) \rangle dxdy
\]
for \( (G, q) \) being \( (L \nu, p), (F, \Delta_\nu p), \) and \( (F, \lambda^2_{v_-} p) \), each with the corresponding power \( \alpha \). Expanding \( \langle G, \Theta_L \rangle \) and integrating over \( x \) we obtain \( s \) times the
integral of a function of the form appearing in Equation (13). Our assumption on \( v \) implies that the integral over \([1, \infty)\) of every term with \( \lambda \neq 0 \) is finite for every \( s \), so that the factor \( s \) eliminates it at \( s = 0 \). This shows that the difference between the theta lift and the integral asserted in the lemma is the constant term at \( s = 0 \) of a linear combination of expressions of the form

\[
s\Delta_v(\mathcal{F}) \int_1^\infty y^{s-j} c_{0,0}(y) dy
\]

for the appropriate \( j \). Since we assume that the coefficients \( \Delta_v(\mathcal{F}) \) in Equation (14) vanish for the three possible polynomials \( q \), the proof of the lemma is complete.

In general the expression in Equation (14) does not vanish. This happens, for example, if \( p = 1 \), a case in which the difference between the theta lift of \( \Delta_k F \) and the integral in Lemma 2.5 is a constant (i.e., independent of \( v \)). If \( F \) is almost weakly holomorphic then for large enough \( \Re s \) Equation (14) becomes a linear combination of expressions of the form \( \frac{\beta}{\beta-m} \), so that at \( s = 0 \) the constant in question is the coefficient of the term in which \( \beta = 0 \).

The conditions required by Lemma 2.4 and Lemma 2.5 are satisfied by a large variety of modular forms. All the almost weakly holomorphic modular forms clearly satisfy these conditions, and the same holds for various eigenforms of \( \Delta_k \).

For example, the behavior of the Whittaker functions discussed in Section 1.3 of [Bru] implies that the Fourier coefficients of the functions \( F_\beta,m(\tau, s) \), evaluated explicitly in Theorem 1.9 there, satisfy these conditions for every \( s \) (not to be confused with our parameter \( s \!\!]\)). To see this, note that in this case the Fourier coefficients are \( c_{\beta,m}(\gamma, n; y, s) \) while we consider coefficients of \( q^n e^{2\pi ny} \).

Returning to the theta lift \( \Phi_{L,m,m,0}(v, F) \) for \( F = \delta^m \), we obtain

**Corollary 2.6.** For \( m > 0 \), or for \( m = 0 \) (with \( F = f \)) under the condition \( c_{0,0} = 0 \), the action of the operator \( \Delta^G_m \) multiplies \( \Phi_{L,m,m,0}(v, F) \) by \( 2mb_- \).

**Proof.** The operator \( \Delta^G_m = \Delta^G_{0,m} \) acts on the conjugated theta function \( \overline{\Theta_{L,m,m,0}} \) which shows up in the definition of \( \Phi_{L,m,m,0} \). Hence its action is the same as the action of the conjugated operator \( \Delta^G_{0,m} = \Delta^G_{0,m} + 2mb_- \) on the theta function \( \Theta_{L,m,m,0} \) itself (after conjugating). But with \( r = s = m \) and \( t = 0 \), the combination \( \Delta^G_{0,m} + 2mb_- \) coincides with the operator appearing on the right hand side of Proposition 1.3. It follows that \( \Delta^G_m \Phi_{L,m,m,0}(v, F) \) is 4 times the integral from Lemma 2.5. Since for \( m > 0 \) the polynomial \( e^{-\Delta_v/8\pi y} \) vanishes at \( \lambda = 0 \) and for \( m = 0 \) we assume \( c_{0,0} = 0 \), the expression in Equation (14) vanishes in both cases. Hence Lemma 2.5 implies that \( \Delta^G_m \Phi_{L,m,m,0}(v, F) \) coincides with \( 4\Phi_{L,m,m,0}(v, \Delta_k F) \). As \( \Delta_k \) multiplies \( F \) by \( \frac{mb_-}{2} \), the corollary follows.\[\square\]
2.4 The Theta Lift of $F = \delta^m_{1 - \frac{b}{2} - m} f$

In order to evaluate $\frac{im}{2} \Phi_{L,m,m,0}(v, F)$ using Theorem 7.1 of [BH], we need to evaluate $p_{v,h_+,h_-}$ for $p_v = (-i)^m \frac{P_{m,0}}{2}$. Only elements with $h_- = 0$ appear (since $m_- = 0$), and as $z_{v} = \frac{X_v}{Y}$, the binomial decomposition

$$
\frac{(-i)^m}{2} \cdot \frac{(\lambda, X_v, V + iY_v, V)}{(Y^2)^m} = \sum_{h_+} \frac{m}{h_+} \left( \frac{(\lambda, X_v, V)}{Y^2} \right)^{h_+} \cdot \left( \frac{(\lambda, Y_v, V)}{Y^2} \right)^{m-h_+}
$$

implies

$$
p_{v,h_+,0}(\eta) = \frac{(-i)^{h_+}}{2} \frac{m}{h_+} \frac{\eta}{(Y^2)^{m-h_+}}.
$$

This polynomial is homogenous of degree $(m - h_+, 0)$ with respect to the element $w \in G(K)$ in which $w_+$ is spanned by $Y$ (or by the normalized generator $\frac{Y}{\sqrt{Y}}$). This illustrates the reason why we cannot use the notation $p_{w,h_+,0}$ of [BH] for a general polynomial: This notation implies dependence only on $w$, i.e., on $\frac{1}{\sqrt{Y}}$, while Equation (15) displays dependence on $Y$ itself (hence on $v$). On the other hand, we can write $p_{v,0,0}(\eta)$ as $\frac{1}{\sqrt{Y}}(\eta, w_1)$ with $w_1 = \frac{1}{\sqrt{Y}}$ (as in Theorem 14.3 of [BH]), so that the term with $\Phi_K$ in Theorem 7.1 of [BH] can be evaluated using the polynomial $\bar{p}(\eta) = (\eta, w_1)$. Since for this polynomial Theorem 10.3 of [BH] remains valid, we deduce that $\Phi_K(v, F, p_{v,0,0})$ is a polynomial in $\frac{1}{\sqrt{Y}}$ divided by $|Y|^m$.

We can now state the properties of the theta lift $\Phi_{L,m,m,0}(v, F)$.

**Theorem 2.7.** For $F = \delta^m_{1 - \frac{b}{2} - m} f$ and $f$ a weakly holomorphic modular form of weight $1 - \frac{b}{2} - m$, the theta lift $\frac{im}{2} \Phi_{L,m,m,0}(v, F)$ is a function of $Z \in K_{R+iC}$ whose singularity along $\lambda^{-1}$ for negative norm $\lambda$ is given by

$$
\frac{1}{2} \sum_{\alpha \in L} c_{\alpha \lambda, \frac{b}{2} + \frac{b}{2}} \frac{(i\alpha)^m}{(2\pi)^m} \prod_{r=0}^{m-1} \left( r + \frac{b}{2} \right) \cdot \left( \frac{(\lambda, Z_v, V)^m}{2m(Y^2)^m} \right) \cdot \ln \left( \frac{|(\lambda, Z_v, V)|^2}{Y^2} \right) + 
$$

$$
\sum_{k=0}^{m-1} \frac{m!}{k!} \frac{\lambda^2}{2} \cdot \frac{\lambda^{k-m}}{(\lambda, Z_v, V)^{k-(Y^2)^k}} \cdot \frac{1}{m-k},
$$

and which is annihilated by $\Delta_{\lambda}^m - 2mb_-$ outside its singularities. Its Fourier expansion at the primitive norm 0 vector $z$ of $L$ (if it exists) decomposes, in a Weyl chamber $W$ containing $z$ in its closure, as

$$
\frac{\sqrt{Z}}{\pi^{m-1}|Y|^{m-1}} \cdot \sum_{k=0}^{m} \sum_{\rho \in L} \frac{A_{k,C,\rho}}{(Y^2)^k} \cdot (\rho, W) > 0 \times \left\{ e((\rho, Z)) \right\} \cdot \left\{ e((\rho, W)) \right\} \cdot (\rho, W) < 0.
$$

Here $\varphi(Y)$ is a polynomial of degree not exceeding $m + 1$ in $\frac{Y}{\sqrt{Y}}$, plus some constant divided by $|Y|^m$. The coefficients $A_{k,C,\rho}$ are constants which belong
to the field generated by the Fourier coefficients of \( f \) over some finite cyclotomic extension of \( \mathbb{Q} \).

Proof. The fact that \( \Delta_m \Phi_{L,m,m,0} = 2mb_+ \Phi_{L,m,m,0} \) is the content of Corollary 2.6. The singularities can be read off Equation (10), with \( b_+ = 2, m_+ = m, k \leq m \), and \( j = 0 \) since \( p_v \) is harmonic. The condition \( \beta \in -\mathbb{N} \) holds only for \( k = m \), and when we substitute the coefficients \( c_{\gamma,n,k} \) from Equation (12) and the formula for \( \lambda_k^2 \) and note that \( \ln(\alpha^2) \) multiplies a smooth function, we obtain the asserted singularity near \( \lambda_k^\perp \). The rest of the proof follows the proof of Theorem 14.3 of [B1]. We assume that \( L \) contains a primitive norm 0 vector (otherwise the assertion about Fourier expansions is vacuous), and that \( v \) (or \( w \)) is not on any wall between two Weyl chambers. Since the polynomial \( p_v \) is of homogeneity degree \( (m,0) \), we can take \( h_- = h = 0 \) in Theorem 7.1 of [B1], and find that \( \frac{1}{h} \Phi_{L,m,m,0}(v,F) \) is the sum of \( \sqrt{2} \Phi_{K}(w,F_{K},p_v,0,0) \), a term coming from the element \( 0 \in K^* \), and the expression

\[
\sqrt{2} |Y| \sum_{h_+,j,k,n} \eta^{h_i} \sum_{0 \neq \eta \in K^*} \Delta_w(\eta) \sum_{(\gamma)} \frac{e(n((\eta,X) + (\gamma,\zeta)))}{(2\pi n)_{\eta,Y}} \times \left( \frac{Y^2}{2\pi n} \right)^{m-h_+ - j - k - \frac{1}{2}} K^{n-h_+ - j - k - \frac{1}{2}} (2\pi n)_{\eta,Y}) .
\]

Here \( \gamma \) is the \( L^*/L \)-image of an element of \( L^* \) whose restriction to \( z^\perp \subseteq L \) is the pull-back of \( \eta : K \to \mathbb{Z} \) under the projection \( z^\perp \to K \). The derivation employs Lemma 7.2 of [B1], together with the fact that \( b_+ = 2, z_{v+}^2 = \frac{1}{(2\pi)^r}, \mu = X, \) and \( \eta_{w+} = \frac{(\eta,Y)^2}{(2\pi)^2} \).

Next, we recall that \( \Delta_w \) differentiates twice with respect to the pairing with \( \frac{Y}{|Y|} \), hence applying it to the complex conjugate of \( p_v, \eta_{w+} \) from Equation (13) we find that

\[
\Delta_w(p_v, \eta_{w+}^2) = \frac{m!}{2} h_+! \frac{(\eta,Y)^{m-h_+ - 2j}}{(Y^2)^{m-h_+ - j}} .
\]

Furthermore, we quote from the proof of Theorem 14.3 of [B1] the formula

\[
K_{\nu+\frac{1}{4}}(t) = \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{\nu=0}^{t} \frac{\nu!}{(\nu-r)!} \frac{1}{(2\pi)^r} .
\]

for the \( K \)-Bessel function of half-integral index (here \( \nu \in \mathbb{N} \), including 0). Since inverting the index leaves the \( K \)-Bessel function invariant, the same expansion holds for \( K_{-\nu+\frac{1}{4}}(t) \). Substituting, and collecting the total powers of \( 2, \pi, n, (\eta,Y), |(\eta,Y)| \), and \( Y^2 \) we find that this expression reduces to

\[
\sum_{h_+,j,k,n} \frac{m! \sum_{r=0}^{\nu} \frac{(\nu+r)!}{(\nu-r)!} \frac{1}{(2\pi)^r}}{h_+! \frac{(\eta,Y)^{m-h_+ - 2j}}{(Y^2)^{m-h_+ - j}}} \times \sum_{h_+,j,k,n} \frac{m! \sum_{r=0}^{\nu} \frac{(\nu+r)!}{(\nu-r)!} \frac{1}{(2\pi)^r}}{h_+! \frac{(\eta,Y)^{m-h_+ - 2j}}{(Y^2)^{m-h_+ - j}}} \times \sum_{h_+,j,k,n} \frac{m! \sum_{r=0}^{\nu} \frac{(\nu+r)!}{(\nu-r)!} \frac{1}{(2\pi)^r}}{h_+! \frac{(\eta,Y)^{m-h_+ - 2j}}{(Y^2)^{m-h_+ - j}}}.
\]
\[ \times |(\eta, Y)|^{k-j-r} \text{sgn}(\eta, Y)^{m-h+}(Y^2)^{-k}e(n[\eta, X] + i[(\eta, Y)] + (\gamma, \zeta)) \] \quad (17)

where \( \nu \) stands for \( m - h_+ - j - k - 1 \) when this integer is non-negative and for \( j + k + h_+ - m \) otherwise. We observe that \( j \) and \( r \) appear in exponents only through their sum \( C = j + r \), and we wish to show that only terms with \( C \leq k \) appear. Considering the terms in which \( \nu = j + k + h_+ - m \), we recall that \( 2j \) is bounded by \( m - h_+ \). Hence we have the inequality \( j + k + m - \leq -j \), which implies \( r \leq \nu \leq j - \) hence \( C = j + r \leq k \). For the other terms we fix \( C \) and write \( r = C - j \), so that the only part depending on \( j \) is

\[
\frac{1}{C!} \sum_j (-1)^j \binom{C}{j} \frac{(m-h_+ + C-k-1-2j)!}{(m-h_+ - 2j)!}.
\]

For \( C \geq k + 1 \) this sum equals

\[
\frac{(C-k-1)!}{C!} \sum_j (-1)^j \binom{C}{j} \frac{(m-h_+ + C-k-1-2j)}{C-k-1},
\]

hence vanishes by Lemma 2.1 since the rightmost binomial coefficient is a polynomial of degree \( C-k-1 < C \) in \( j \). Thus, only terms with \( C \leq k \) survive, which implies that \(|(\eta, Y)|^{k-C}\) does not lead to a “polar” singularity along \( \eta \) for negative norm vectors \( \eta \).

Now, by Equation (12), \( c_{\gamma, e}^{(2)} \eta, k \) is some rational multiple of \( c_{\gamma, e}^{(2)} \eta, k \). Thus, given \( k, C, n, \eta, \) and \( \gamma \) (with \( C \leq k \) and the usual relation between \( \gamma \) and \( \eta \)), the corresponding terms in Equation (17) take the form

\[
ak_{k,C,e} \frac{n^{m-k-C-1}(\eta^2)^{m-k}}{\pi^{k+C}} c_{\gamma, e}^{(2)} \frac{(\eta, Y)^{k-C}}{(Y^2)^k} e(n[\eta, X] + i[(\eta, Y)] + (\gamma, \zeta)),
\]

where \( \varepsilon = \text{sgn}(\eta, Y) \) and the coefficients \( a_{k,C,e} \) are rational numbers. We note that (B1) shows that \( a_{0,0,0,e} \) equals \( 2^m \) for \( \varepsilon = +1 \) and 0 for \( \varepsilon = -1 \). Now, the power of \( n \) lies in \( \mathbb{Q} \), and \( e((\gamma, \zeta)) \) gives roots of unity whose order is bounded by the denominators appearing in \((\zeta, \gamma)\) for \( \gamma \in L^* \) (which can be restricted to \( \gamma \) in the lattice denoted \( L^*_0 \) in (B1)). The remaining part of the exponent is \( e(n(\eta, Z)) \) if \((\eta, Y) > 0\) and \( e(n(\eta, Z)) \) if \((\eta, Y) < 0\). Since the dependence on \( n \) and \( \eta \) can be written as \( \frac{n^m}{(\eta^2)^k} \) times \( e((\rho, Z)) \) or \( e((\rho, Z)) \) is

\[
A_{k,C,e} = a_{k,C,e} (\rho^2)^{m-k} \sum_{n>0} \frac{c_{\gamma, e}^{(2)}}{n^{m+1}} e(n(\gamma, \zeta)).
\]

Equation (18) shows that the coefficients \( A_{k,C,e} \) have the asserted properties. Indeed, the dependence on \( X \) (i.e., on \( \mu \)) in Theorem 7.1 of (B1) comes only from \( e((n, \mu)) \), so that the terms with \( \Phi_K \) or with \( n = 0 \) all go into \( \varphi(Y) \).

It remains to show that \( \varphi(Y) \) has the asserted properties. The term involving \( \Phi_K \) was discussed above (note the factor \( \frac{1}{|Y|^{1/2}} \) that shows up in \( p_{v,0,0} \) but also
the coefficient $\sqrt{2}/Y$ multiplying $\Phi_K$ in Theorem 7.1 of [B1]). The reason
for the coefficient $\frac{\sqrt{2}}{Y}$ is explained below. The polynomial from Theorem 10.3
of [B1] has degree at most $m+1$ since $m_+ = k_{\max} = m$ and $m_- = 0$. We
turn to the term with $\eta = 0$ and see that it gives a constant divided by $(Y^2)^m$. Given $h_+$
and $j$, the expression $\Delta_{\ell}(\gamma_{v,h_+})/\eta(\eta)$ does not vanish for $\eta = 0$ only
if $2j = m - h_+$, so we carry out the sum only over $j$ and write $h_+ = m - 2j$. Using Equation (16) (with $h_+ = m - 2j$), Lemma 7.3 of [B1], and the fact that
the corresponding $\gamma \in L^* / L$ are the images of $\frac{\delta}{\eta}$ (with pairing $\frac{\delta}{\eta}$ with $\zeta$) for
$\delta \in \mathbb{Z}/N\mathbb{Z}$ (where $N$ is defined by $(L, z) = N\mathbb{Z}$) we obtain the expression

$$\sqrt{2}/Y \sum_{j,k,n} \frac{n^{m-2j}}{(2j)!^{m-2j}(-8\pi)^j} \cdot \frac{j^{m-2j}}{(m-2j)!^{(Y_2)^j}} \sum_{\delta \in \mathbb{Z}/N\mathbb{Z}} e\left(\frac{\delta n}{N}\right) \times$$

$$\times c_{\eta,0,k} \left(\pi n^2 Y^2 \right)^{j-k-s} \left(\frac{2}{Y^2}\right)^{k-s} \Gamma\left(s + \frac{1}{2} + k - j\right).$$

We need the constant term of the Laurent expansion of this expression at $s = 0$. Collecting powers gives

$$\sum_{k,\delta} c_{\eta,0,k} \frac{2^{k-m}}{(Y^2)^{k,m+2}} \sum_j \frac{(-1)^j m!}{(2j)!^{(m-2j)!}} \Gamma\left(s + \frac{1}{2} + k - j\right) \left(\frac{2}{\pi Y^2}\right)^{j} \times$$

$$\times \sum_n e\left(\frac{\delta n}{N}\right) n^{m-2k-1-2s},$$

(19)
in which the sum over $n$ equals $N^{m-2k-1-2s} \sum_{\epsilon} e\left(\frac{\delta \epsilon}{N}\right) \zeta(1 + 2k - m + 2s, \frac{\epsilon}{N})$ and involves values of the Hurwitz zeta function. Now, Equation (12) implies
that $c_{\eta,0,k}$ vanishes unless $k = m$ where it equals $\prod_{r=0}^{k-1} (r + \frac{b}{m}) \cdot \frac{\delta_0}{\mathfrak{m}}$. Since
the Hurwitz zeta function is holomorphic at $m+1$ we may substitute $s = 0$ here, and since $\Gamma\left(\frac{1}{2} + k - j\right)$ is a rational multiple of $\sqrt{\pi}$ we see that the expression from Equation (19) equals

$$\frac{\tilde{a}}{(Y^2)^m \pi 2^m} \sum_{\delta \in \mathbb{Z}/N\mathbb{Z}} c_{\eta,0,0} \sum_{\epsilon} e\left(\frac{\delta \epsilon}{N}\right) \zeta\left(m + 1, \frac{\epsilon}{N}\right)$$

where $\tilde{a}$ is a rational number. Since this is a constant times $\frac{1}{(Y^2)^m}$, adding it to the term with $\Phi_K$ shows that $\varphi(Y)$ has the desired form. \hfill \square

A remark about the evaluation of the constant term $P_2$ in Theorem 14.3 of [B1] is in order here. In the case of weakly holomorphic $F$ (considered in Theorem 14.3 of [B1]) only $k = 0$ appears, so that the sum over $n$ in Equation (19) gives us indeed $N^{m-1} \sum_{\epsilon} e\left(\frac{\delta \epsilon}{N}\right) \cdot -B_m\left(\frac{\delta}{N}\right)/m$ (after evaluating the Hurwitz zeta function). The symmetry and duplication formulae for the gamma function show that $\Gamma\left(\frac{1}{2} - j\right) = \frac{\pi^{(j+1)}}{\sqrt{\pi 2^{-j} \Gamma(2j+1) \sin \pi (\frac{1}{2} - j)}}$, which for integral $j$ is just
Therefore the sum over \( j \) is simply \( \sum_{j}(2j)!(m - 2j)! = 2^{m-1} \). Hence the true value of \( P_2 \) is half the value given in the proof of Theorem 14.3 of [B1], though the constant appearing in the assertion of that theorem is correct. This means that one should be careful when evaluating \( P_2 \) as the limit of the expression for \( \eta \neq 0 \)—indeed, doing so with \((\eta, W) < 0 \) yields the value 0. Interestingly, taking the limit \( \eta \to 0 \) of the average of the expressions for \( \eta \) and for \(-\eta \) gives the correct answer, and is independent of signs of products with Weyl chambers. In any case, the proof of Theorem 2.7 shows how to evaluate this term directly.

A more detailed analysis of the polynomial part of \( \varphi(K) \) shows that the coefficients of this polynomial involve the lattice \( K \), some cyclotomic numbers, and the Fourier coefficients of the modular form \( f \). Explicitly, the contribution of \( \varphi(K) \) is the limit of \( \frac{-4y^4}{(2j)!} \) in the assertion of Theorem 2.7. Hence for “algebraic” \( f \) these coefficients are algebraic. On the other hand, evaluating the constant coming from Equation (19) for \( \eta = 0 \) is very hard. For example, for \( N = 1 \) with non-zero \( c_{0,0} \) we have constant terms only for even \( m \), hence we need the values of the Riemann zeta function at odd positive integers, whose properties (not to mention a finite formula) are not yet known.

If \( c_{0,0} = 0 \), our results also hold for the case \( m = 0 \). Indeed, in this case the assertion about the eigenvalue 0 of \( \frac{m}{2}\Phi_{L,m,m,0}(v, F) \) follows from Corollary 2.6, and even though the Hurwitz zeta functions are no longer holomorphic at 1, the combination \( \sum_{N=1}^{\infty} e^{i(N)}(1 + 2s, \frac{N}{2}) \) is holomorphic at \( s = 0 \) for \( \delta \neq 0 \). On the other hand, with a non-vanishing value for \( c_{0,0} \) Lemma 2.5 implies that \( \Delta_0^G \frac{m}{2}\Phi_{L,0,0,0}(v, F) \) is a constant, and the Laurent expansion of this combination at \( s = 0 \) gives a constant minus \( c_{0,0} \ln |Y| \). Indeed, in this case one obtains the usual theta lift with \( p = 1 \) from Theorem 13.3 of [B1] (up to a factor of 2), for which \( \ln |\Psi|^2 \) is harmonic for meromorphic \( \Psi \) and Equation (4.3) of [Bru] gives that applying \( \Delta_0^G = 8\Omega \) to \( \ln |Y| \) gives a constant. For a related result, see Theorem 4.7 of [Bru]. Finally, we mention that Theorem 2.7, as well as Theorems 4.6 and 4.7 of [Bru], provide answers to Problem 16.6 of [B1] in some (interesting) special cases.

We remark that \( \frac{m}{2}\Phi_{L,m,m,0}(v, F) \) can be obtained by lifting the weakly holomorphic modular form \( f \) itself, using the theta function with a different polynomial. Explicitly, \( \frac{m}{2}\Phi_{L,m,m,0}(v, F) = (\frac{-4y^4}{(2j)!})^{m}\Phi_L(v, f, p_v) \) with the polynomial \( p_v(\lambda) = P_{m,m,0}(\lambda, Z)(\lambda_Z^2)^m \). For \( m = 0 \) both sides give the same expression, and for \( m > 0 \) the equality follows from Equation (55) (recall that \( P_{m,m,0} \) is \( \Delta_{v,+}\)-harmonic), Lemma 2.2, Lemma 2.4, and an argument similar to the proof of Lemma 2.5. In some cases we need to consider the automorphic form as the difference of restrictions of automorphic forms on larger Grassmannians, of signature \((2, b_+ + 24)\), as in Section 8 of [B1]. For this purpose it is more appropriate to consider our function as \( \frac{-4y^4}{(2j)!}\Phi_L(v, f, p_v) \), since the functions on the larger Grassmannians arise from lifts of the modular form \( \frac{y^4}{2} \) where \( \Delta \) is the classical holomorphic cusp form of weight 12 (not to be confused with the various Laplacians, whose notation also involves the symbol \( \Delta \)). Theorem 2.7 implies
that \( \frac{i^m}{2} \Phi_{L,m,m,0}(v,F) \) is obtained in this case from restrictions of automorphic forms of weight \( m \) on the larger Grassmannians which are eigenfunctions with eigenvalue \(-2m(b_++24)\) of the corresponding Laplacians. Certain properties of these automorphic forms can be read off their Fourier expansions, from which one deduces properties of \( \frac{i^m}{2} \Phi_{L,m,m,0}(v,F) \). We note that in order to obtain such description for \( \frac{i^m}{2} \Phi_{L,m,m,0}(v,F) \) we must consider it as \( \frac{(-i)^m}{2m+2} \Phi_L(v,f,p_v) \), since \( \frac{1}{2} \) is weakly holomorphic if \( f \) is, while the fact that \( F \) is an eigenfunction of the Laplacian \( \Delta_{1-\frac{v}{2}+m} \) does not imply a similar property for \( \frac{1}{2} \).

### 2.5 The Case \( b_- = 1 \)

In this paper we will be considering mainly the case \( b_- = 1 \). We shall consider our lattices as embedded in the fixed space \( V = M_2(\mathbb{R})_0 \) of traceless \( 2 \times 2 \) real matrices, which is of signature \((2,1)\) with the pairing taking traceless matrices \( A \) and \( B \) to \( Tr(AB) \) (hence \( A^2 = -2 \det A \) in this pairing). We recall that by choosing \( z \) to be the norm 0 vector \( z = \left( \begin{smallmatrix} 1/\beta \end{smallmatrix} \right) \) for some non-zero \( \beta \) and \( \zeta \) to be \( \left( \beta \right) \) (or \( \left( -h/2 \beta \right) \) if we want \( \zeta^2 \) to be some non-zero number \( h \)), so that \( K_\mathbb{R} \cong \{ z, \zeta \}^\perp \) is the space of traceless diagonal matrices. We identify \( x \in \mathbb{R} \) with the element \( \left( \begin{smallmatrix} \beta x & 0 \\ 0 & -\beta x \end{smallmatrix} \right) \), so that \( K_\mathbb{R} \cong \mathbb{R} \) with the norm of \( x \in \mathbb{R} \) being \( 2\beta^2 x^2 \), \( C \) with the positive reals, and \( G(V) \cong K_\mathbb{R} + iC \) with \( \mathcal{H} \). For each \( \tau \in K_\mathbb{R} + iC \cong \mathcal{H} \), the vector \( Z_{\nu,V} \) is \( \beta M_\tau \) with \( M_\tau = \left( \begin{smallmatrix} \tau & -|\tau|^2 \\ \bar{\tau} & \tau \end{smallmatrix} \right) \), and the negative definite space \( v_- \) is spanned by \( J_\tau = \frac{1}{\sqrt{2}} \left( \begin{smallmatrix} \sqrt{\tau} & -\sqrt{\tau} \\ \sqrt{\tau} & \sqrt{\tau} \end{smallmatrix} \right) \). The group \( SL_2(\mathbb{R}) \) maps onto \( SO^+(V) \) according to the action on \( V \) by conjugation, and the relation \( \gamma M_\tau \gamma^{-1} = j(\gamma, \tau)M_{\gamma \tau} \) shows the “doubling of the weights” from automorphic forms on \( G(V) \) to modular forms on \( \mathcal{H} \). We remark that in Example 5.1 of [B2] an equivalent model is used, in which the vector space consists of symmetric \( 2 \times 2 \) matrices, the norm of \( A \) is \(-2N \det A \), and the action of \( \gamma \in SL_2(\mathbb{R}) \) takes a (symmetric) matrix \( M \) to \( g M g^t \). The isomorphism between these two models is given by right multiplication by \( S \) and by the scalar \( \sqrt{N} \). Thus, in order for the vector \( z \) to be a primitive norm 0 vector in the lattice \( L \) considered there we must take \( \beta = \sqrt{N} \).

Consider now the properties of the function \( \frac{i^m}{2} \Phi_{L,m,m,0}(v,F) \) in the case \( b_- = 1 \). We consider the group \( \Gamma_{0+}^\perp \), which is the intersection of \( \Gamma^+ \) with \( SO^+(L_\mathbb{R}) \), as embedded as a Fuchsian group into \( PSL_2(\mathbb{R}) \) (or its inverse image in \( SL_2(\mathbb{R}) \) or \( Mp_2(\mathbb{R}) \)). Theorem 2.7 shows that \( \frac{i^m}{2} \Phi_{L,m,m,0}(v,F) \) is a modular form of weight \( 2m \) on \( \mathcal{H} \) with respect to \( \Gamma_{0+}^\perp \). The fact that it has eigenvalue \(-2m \) with respect to \( \Delta_{0m}^\perp = \Delta_{2m} \) implies that applying the weight raising operator \( \delta_{2m} \) to it yields a meromorphic modular form of weight \( 2m + 2 \). We recall that a negative norm vector in this case is a multiple of \( J_\tau \) for some \( \sigma = s + it \in \mathcal{H} \), and for such a vector the sub-Grassmannian \( \lambda^\perp \) is the point \( \sigma \in \mathcal{H} \) (these are the CM points in this case—see Proposition 3.1 below and its relation to the Grassmannian). By a slight abuse of notation, we use the variable \( \tau = x + iy \) for this \( \mathcal{H} \) as well, though it is the variable \( v \) of \( \Phi \) (and \( \Theta \)). Since the \( \tau \) variable of \( \Theta \)
does not play a role in the following; this notation should lead to no confusion. The description of \( \frac{m}{2} \Phi_{L,m,m,0}(v, F) \) is given in the following

**Theorem 2.8.** The function \( \frac{m}{2} \delta_{2m} \Phi_{L,m,m,0}(v, F) \) is a meromorphic modular form of weight \( 2m + 2 \) with respect to \( \Gamma_0^+ \), whose poles are at points \( \sigma \in \mathcal{H} \) for which a real multiple of \( J_\sigma \) lies in (the isomorphic copy of) \( L^* \). The principal part at such \( \sigma \) is

\[
\frac{i}{(4\pi)^{m+1}} \sum_{\alpha_J \in L} c_{\alpha_J,-\alpha} \frac{(2m)!}{(2\pi)^m} \frac{\alpha^m}{\beta^m} \frac{m!(2t)^{m+1}}{(\tau - \sigma)^{m+1}(\tau - \sigma)^{m+1}}.
\]

In case \( \Gamma_0^+ \) has cusps, the Fourier expansion at such a cusp is

\[
\sum_{r \geq 0} \frac{r^m}{\beta^{2m}} \sum_{d|r} \frac{d^{m+1}}{\gamma|_{\sigma} = \frac{1}{2\pi i}} \sum_{\alpha_J \in L^*} c_{\alpha_J,-\alpha} e\left( \frac{r}{d}(\gamma, \zeta) \right) q^r
\]

(plus some constant if \( m = 0 \)), so that the (positive) Fourier coefficients lie in the field generated by the Fourier coefficients of the weight \( \frac{1}{2} - m \) weakly holomorphic modular form \( f \) over some finite cyclotomic extension of \( \mathbb{Q} \).

**Proof.** We begin by analyzing the singularities of \( \frac{m}{2} \delta_{2m} \Phi_{L,m,m,0}(v, F) \), which are the \( \delta_{2m} \)-images of the singularities of \( \frac{m}{2} \Phi_{L,m,m,0}(v, F) \) given in Theorem 2.7. The constant \( \prod_{\tau \in \mathbb{Q} \setminus \mathbb{Z}} (\tau + \mathbb{Z}) \) is \( (2k)! \), for \( \lambda = J_\sigma \) we have \( \lambda^2 = -2 \), \( Y^2 \) equals \( 2\beta^2 y^2 \), and the pairing of \( Z_{v,V} = j_{L_\tau} \) and \( \mathbb{F}_{v,V} = \beta M_\tau \) with \( \lambda = J_\sigma \) gives \(-\frac{\pi}{4}(\tau - \sigma)(\tau - \sigma)\) and its complex conjugate respectively. Hence the singularity of \( \frac{m}{2} \Phi_{L,m,m,0}(v, F) \) at \( \sigma \) is

\[
\frac{1}{2} \sum_{\alpha_J \in L^*} c_{\alpha_J,-\alpha} \frac{(ia)^m}{(2\pi)^m} \frac{(2m)!}{(4\pi)^m} \frac{(-\sigma + \tau)(\tau - \sigma)^m}{\beta^m} \ln \frac{||\sigma - \tau||^2 ||\tau - \sigma||^2}{2y^2} + \frac{m!}{2k! \gamma_k \gamma_{m-k}} \frac{\lambda^2}{\beta^2} \quad \text{for} \quad k \leq m.
\]

which can be written as

\[
\frac{m!}{2} \sum_{\alpha_J \in L^*} c_{\alpha_J,-\alpha} \frac{(ia)^m}{(2\pi)^m} \frac{(2k)!}{(k!)^2} \frac{(-1)^k}{(4\pi y^2)^k} \frac{\gamma_k}{\beta^m} \frac{\lambda^2}{\beta^m} 
\]

with \( g_k(\tau) \) being \( (m-k)(\tau)^{m-k} (\tau - \sigma)^{m-k} \) for \( 0 \leq k \leq m \) and \( -\ln \frac{||\sigma - \tau||^2 ||\tau - \sigma||^2}{2y^2} \) for \( k = m \). When we apply the fact that \( (\tau - \sigma)^k (\tau - \sigma)^k \) is an anti- holomorphic function of \( \sigma \) and \( \delta_{2m} \frac{\gamma_k(\tau)}{\gamma_{2k}} = \frac{\delta_{2m-2k} \gamma_k(\tau)}{\gamma_{2k}} \) implies that we can simply apply \( \delta_{2m-2k} \) to \( g_k(\tau) \) for each \( k \). Thus, the singularity of \( \frac{m}{2} \delta_{2m} \Phi_{L,m,m,0}(v, F) \) at \( \sigma \) is

\[
\frac{m!}{4\pi} \sum_{\alpha_J \in L^*} c_{\alpha_J,-\alpha} \frac{(ia)^m}{(2\pi)^m} \frac{(2k)!}{(k!)^2} \frac{(-1)^k}{(4\pi y^2)^k} \frac{\gamma_k}{\beta^m} \frac{\lambda^2}{\beta^m} \times
\]
The sum of the combinatorial expressions reduces to
\[
\frac{(\tau - \sigma) + (\tau - \sigma) + 2(\tau - \sigma)(\tau - \sigma)}{(\tau - \sigma)^{m+1-k}(\tau - \sigma)^{m+1-k}} :=
\]
Indeed, \(\delta_{2m-2k}g_k(\tau)\) gives the last factor multiplied by \(\frac{1}{2}\) for both \(k < m\) and \(k = m\). We proceed to show that this agrees with the asserted singularity.

For ease of presentation, we omit the coefficient in front of the sum over \(k\) in what follows, and put it back at the last step. We begin by expanding the powers of \(\tau - \sigma = (\tau - \sigma) - 2iy\) and \(\tau - \sigma = (\tau - \sigma) - 2iy\) and writing \(\frac{(-1)^k}{(4y)^k}\) as \(\frac{1}{(2t)^k(-2iy)^k}\), which gives

\[
\sum_{k=0}^{m} \frac{(2k)!}{(2t)^{2k}} \sum_{a,b} \frac{1}{a!(k-a)!b!(k-b)!(-2iy)^{a+b}} \left[ \frac{1}{(\tau - \sigma)^{m+1-k-a}(\tau - \sigma)^{m-k-b} + \frac{2}{2iy}} \right].
\]

Now fix \(c\) and \(l\), and collect the terms involving \(\frac{1}{(\tau - \sigma)^{m+1-l+1}(-2iy)^c}\). We thus take \(a = l - 1\) and \(b = c - l + 1\) in the first summand within the square brackets, \(a = l - 1\) and \(b = c - l + 1\) in the second summand, and \(a = l - 1\) and \(b = c - l\) in the third summand, giving

\[
\sum_{k,c,l} \frac{(2k)!}{(2t)^{2k}} \sum_{a,b} \frac{1}{a!(k-a)!b!(k-b)!(-2iy)^{a+b}} \left[ \frac{1}{l!(k-l)!(c-l)!(k+l-c)! + \frac{2}{l!(k-l)!(c-l)!(k+l-c)!}} \right].
\]

The sum of the combinatorial expressions reduces to \(\frac{1}{l!(k+1)!(c+1)!(k+1-l)!(c+1-l)!}\), so that we can write this as

\[
\sum_{c} \frac{(c+1)(-2iy)^{-c}}{(\tau - \sigma)^{m+1-l+1}(\tau - \sigma)^{m+1-c}} \sum_{k,l} \frac{(2k)!}{(2t)^{2k}l!(k+1-l)!(c-l+1)!(k+l-c)!}.
\]

The next step is to expand \((\tau - \sigma)^{k+1-l}\) using \(\tau - \sigma = \tau - \sigma + 2it\) and the Binomial Theorem. Writing \((2t)^{2k}\) as \((-1)^k(2it)^{2k}\), we obtain for every \(c\)

\[
\sum_{k,l,h} \frac{(-1)^k(2k)!}{(2it)^{k+l+h}l!(c-l+1)!(k+l-c)!}.
\]

The index change \(k = s - l - h\) yields

\[
\sum_{c,s,t} \frac{(-1)^s(c+1)(-2iy)^{-c}(2it)^{-s}(\tau - \sigma)^{m+1-s}(\tau - \sigma)^{m+1-c}} \sum_{l,h} \frac{(-1)^{s+h}(2s-2l-2h)!}{l!(c-l+1)!(s-h-c)!h!(s+1-2l-2h)!}.
\]
and then for every c and s write \( r = h + l \) and obtain that the corresponding term is

\[
\sum_r \frac{(-1)^r (2s - 2r)! (s - r + 1)}{(s + 1 - 2r)! (c + 1)! (s - c)!} \sum_l \left( \frac{c + 1}{l} \right) \binom{s - c}{r - l}.
\]

A well-known combinatorial identity shows that the sum over \( l \) is simply \( \binom{s + 1}{r} \), and writing \( (s + 1 - r) \binom{s + 1}{r} \) as \( (s + 1) \binom{s}{r} \) and \( \frac{c + 1}{(c + 1)! (s - c)!} \) as \( \frac{1}{s!} \binom{s}{r} \), the expression reduces to

\[
\sum_{c,s} \frac{(-1)^s (s + 1) (-2iy)^{-c} (2it)^{1-s}}{(\tau - \sigma)^{m + 1 - s} (\tau - \sigma)^{m + 1 - s}} \binom{s}{c} \sum_r (-1)^r \binom{s}{r} \frac{(2s - 2r)!}{(s + 1 - 2r)!}.
\]

As \( \frac{(2s - 2r)!}{(s + 1 - 2r)!} = \prod_{s = 2}^{2s} (i - 2r) \) is a polynomial of degree \( s - 1 < s \) for all \( s > 0 \), Lemma 2.1 shows that the sum over \( r \) vanishes unless \( s = 0 \) (where this sum equals 1). Since \( s = 0 \) implies \( c = 0 \) because of the factor \( \binom{s}{r} \), only the corresponding term \( \frac{1}{(\tau - \sigma)^{m + 1} (\tau - \sigma)^{m + 1}} \) survives. In particular, we indeed obtain the singularity of a meromorphic function. Multiplying by the factor we obtained above now gives the asserted singularity at \( \sigma \).

Assume that \( \Gamma_0^+ \) has a cusp, corresponding to a primitive norm 0 vector \( z \in \mathcal{L} \) which is normalized as above. The description of \( \frac{i^m}{\pi} \Phi_{\mathcal{L},m,0}(v,F) \) in Theorem 2.7 implies that in this case its expansion near the cusp looks like the sum of an almost holomorphic function of depth \( \leq 2m \), the complex conjugate of such a function, and an expression of the form \( \frac{D}{\rho} + \frac{D}{\rho^*} \) for \( B \) and \( D \) constants. Note that the condition on the sign of \( (\rho, W) \) implies that this expansion is almost holomorphic (or its conjugate) rather than just almost weakly holomorphic. Since the image of \( \frac{i^m}{\pi} \Phi_{\mathcal{L},m,0}(v,F) \) under \( \delta_{2m} \) must be meromorphic, it follows that the conjugate almost holomorphic part must be annihilated and only the part divided by \( y^0 \) of the almost holomorphic part remains. A direct evaluation of \( \delta_{2m} \sum_r \psi_r(\tau) \psi_r(\bar{\tau}) \) with the functions \( \psi_r \) being holomorphic (or meromorphic) shows that the part divided by \( y^0 \) is \( \psi_r(\tau) \psi_r(\bar{\tau}) \). Moreover, as \( \delta_{2m} \frac{1}{y^0} = 0 \) and \( \delta_{2m} \frac{1}{y^m} = 0 \), the holomorphicity implies \( B = 0 \) (unless \( m = 0 \)).

We normalize \( z \) and \( \beta \) such that \( K^* \) is identified with \( \mathbb{Z} \subseteq \mathbb{R} \), and the condition \( (\rho, W) > 0 \) means that \( \rho \) is mapped to \( \mathbb{R}^+ \) with \( r > 0 \) by this isomorphism. Under this normalization, the coefficient \( A_{k,C,\rho} \) with \( (\rho, W) > 0 \) in the expansion of \( \frac{i^m}{\pi} \Phi_{\mathcal{L},m,0}(v,F) \) multiplies \( \frac{\tau^k - C}{(2\pi)^3 (\pi B)!} e(\tau r) \). Thus we need to consider only the terms with \( k = C = 0 \), i.e., \( \psi_0(\tau) = \sum_{r>0} A_{0,0,0} \frac{q^r}{(2\pi)^3 (\pi B)!} \). Differentiating gives \( \frac{\psi_0(\tau)}{2\pi z} = \sum_{r>0} r A_{0,0,0} \frac{q^r}{2\pi z} \eta \), and recall that \( a_{0,0,0} = 2^m \) the norm of the element \( \frac{\tau}{2\pi} \in K^* \). Using the expression for the coefficients \( A_{k,C,\rho} \) given in Equation (18), the substitution \( d = \frac{\tau}{2\pi} \) yields the asserted Fourier expansion for \( \frac{i^m}{\pi} \delta_{2m} \Phi_{\mathcal{L},m,0}(v,F) \).

Note that the constant coming from \( \eta = 0 \) in Theorem 2.7 which is very hard to evaluate, is the coefficient denoted \( D \) in the proof of Theorem 2.8.
Since it multiplies $\frac{1}{m}$ and $\delta_{2m} \frac{1}{m} = 0$, it has no effect on the properties of $\frac{1}{m} \delta_{2m} \Phi_{L,m,m,0}(v,F)$. We also note that the singularity in Theorem 2.8 is invariant under the weight $2m + 2$ slash operators, hence it gives the singular part of a modular form of this weight. Indeed, the set of $\alpha$ satisfying $\alpha J_\sigma \in L^*$ and the coefficients $c_{\alpha J_\sigma, -\alpha^2}$ for such $\alpha$ are the same for $\sigma$ and for $\gamma \sigma$ with $\gamma \in \Gamma_0^*$, and the formula for $\gamma \tau - \gamma \sigma$ and $\gamma \tau - \gamma \sigma$ shows that applying the slash operator $[\gamma]_{2m+2}$ to the singularity at $\gamma \sigma$ gives the singularity at $\sigma$.

As an example for the map $f \mapsto \frac{1}{m} \delta_{2m} \Phi_{L,m,m,0}(v,F)$ we take $L$ to be the lattice spanned by $z = \left( \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}} \right)$, $\zeta = \left( \frac{\sqrt{N}}{N}, \frac{-\sqrt{N}}{N} \right)$, and the vector $\left( \frac{\sqrt{N}}{N}, \frac{-\sqrt{N}}{N} \right)$, where the latter spans $K$. In this case $\beta = \sqrt{N}$, the generator of $K$ has norm $2N$, $L^*/L = K^*/K$ is cyclic of order $2N$, $\rho_L$ equals $\rho_K$, and the group $\Gamma_0^*$ is $\Gamma_0(N)$ acting by conjugation. Since $(z,L) = \mathbb{Z}$ and $\zeta \in L$, the exponent in Theorem 2.8 is trivial and the sum over $\gamma$ involves only the class $d \in \mathbb{Z}/2N\mathbb{Z}$. Hence Theorem 2.8 yields a singular Shimura-type correspondence, as follows.

**Corollary 2.9.** Let $m$ and $N$ be positive integers, let $K$ be the 1-dimensional lattice generated by an element of norm $2N$, and let $f = \sum_{\gamma \in \mathbb{Z}/2N\mathbb{Z}} \sum_{\alpha \in \mathbb{Q}} c_{\gamma,n} q^{\alpha} e_{\gamma}$ be a weakly holomorphic modular form of weight $\frac{1}{2} + m$ and representation $\rho_K$. The $q$-series

$$\sum_{r>0} \frac{r^m}{N^m} \left( \sum_{d|r} q^{m+1} c_{d,c_{d,\frac{1}{d}}^2} \right) q^r$$

defines a meromorphic modular form of weight $2m + 2$ with respect to $\Gamma_0(N)$, which has poles of order $m + 1$ at some elements of $\mathcal{H}$ which are solutions of quadratic equations over $\mathbb{Z}$.

Note that the weight of $f$ implies that the Fourier coefficients $c_{\gamma,n}$ are $(-1)^m$-symmetric, namely the equality $c_{-\gamma,n} = (-1)^m c_{\gamma,n}$ holds for any $\gamma \in L^*/L$ and $n \in \mathbb{Q}$. In particular, if $L^*/L$ has exponent 2 (or 1) then there are no modular forms of weight $1 - \frac{b}{2} \tau \overline{\tau} m$ and representation $\rho_L$ if $m$ is odd. For this reason, the case $N = 1$ in Corollary 2.9 gives non-trivial results only for even $m$. In this case the space of modular forms of weight $\frac{1}{2} - m$ is isomorphic to the space of weakly holomorphic modular forms of the same weight with respect to $\Gamma_0(4)$ which lie in the corresponding Kohnen Plus-space (see Chapter 5 of [EZ] or Proposition 1 of [K]—both results extend to the weakly holomorphic case using powers of the cusp form $\Delta$). Under this isomorphism, the assertion of Corollary 2.9 becomes the statement given in the corresponding Theorem in the introduction.

Theorem 2.8 improves over Theorem 2.7 since it takes the weakly holomorphic modular form $f$ to a meromorphic function on $G(L_\mathbb{R})$. It would be interesting to obtain similar results for higher values of $b$. Indeed, for $b_\gamma = 2$ the Grassmannian $G(L_\mathbb{R})$ is isomorphic to $\mathcal{H}^2$ and $\frac{1}{m} \delta_{2m} \Phi_{L,m,m,0}$ is a Hilbert modular form of weight $(m, m)$ which is an eigenfunction of weight $-4m$ with respect to $\Delta_m^G = 2\Delta_{m,\tau} + 2\Delta_{m,\sigma}$ (where $\tau$ and $\sigma$ are the coordinates of $\mathcal{H}^2$). Applying the combined weight raising operator $\delta_{m,\tau} \delta_{m,\sigma}$ gives a Hilbert modular form of weight $(m + 2, m + 2)$ on $\mathcal{H}^2$ which is harmonic. Unlike the case $b_\gamma = 1$, in
which the eigenvalue was sufficient to establish meromorphicity after the operation of $\delta_{2m}$, here one needs to work harder in order to see whether the result is meromorphic. If this were indeed the case, then one would be able to obtain a Gross–Kohnen–Zagier type theorem for higher-codimensional Heegner cycles in universal families over Hilbert modular surfaces along the lines of the proof of the main result of this paper. It would also be interesting to find “weight raising operators” for Siegel modular forms of degree 2, and to see whether a similar argument can work for the case $b_\infty = 3$ as well. We leave these questions for future research.

3 CM Cycles in Universal Families over Modular and Shimura Curves

In this Section we present the arithmetic objects to which our main result, Theorem 4.3 below, applies. We discuss universal families of Abelian surfaces over Shimura (and modular) curves, the variations of Hodge structures arising from them, and the CM points on the underlying curves, mainly following [Be] (though more explicitly). We then generalize the classical “graphs of CM isogenies” and define CM cycles as actual cycles inside the corresponding Abelian surfaces in the general case, rather than the fundamental classes considered in [Be] as CM cycles. Finally, we show that the cycles we define (together with the generic ones in the case of the split quaternion algebra) yield all the Abelian subvarieties of an Abelian surface with quaternionic multiplication.

3.1 Variations of Hodge Structures over Shimura Curves

We introduce, following Section 5 of [Be] and Section 3 of [Bry], the Shimura variation of Hodge structure, denoted $V_1$, over the upper half-plane $H$. Working over $\mathbb{R}$, the underlying local system is trivial with fiber $\mathbb{R}^2$ (column vectors). With the natural action of the group $SL_2(\mathbb{R})$ (and more generally $GL_2^+(\mathbb{R})$) on $\mathbb{R}^2$, this local system is $SL_2(\mathbb{R})$-equivariant. As a variation of Hodge structure, tensoring the fiber over $\tau \in H$ with $\mathbb{C}$, the vector $(\tau_1)$ is of Hodge weight $(1, 0)$ and the complex conjugate vector $(\bar{\tau}_1)$ has Hodge weight $(0, 1)$. $V_1$ arises naturally in two different settings. The interpretation of $H$ as the moduli space for (oriented) complex structure on $\mathbb{R}^2$ yields $V_1$ as follows. Every complex structure on the vector space $\mathbb{R}^2$ is (up to a sign) $J_\tau$ for some $\tau \in H$, and the Hodge structure on $\mathbb{C}^2 = \mathbb{R}^2 \otimes \mathbb{C}$ corresponding to $J_\tau$ is the Hodge structure on the fiber of $V_1$ over $\tau$. Another construction which yields $V_1$ uses the universal elliptic curve $\pi: \mathcal{E} \to H$, in which the fiber over $\tau \in H$ is the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. Then $V_1$ is $R^1\pi_*\mathbb{R}$ (in the analytic category), since the holomorphic 1-form $dz$, of Hodge weight $(1, 0)$, has periods $\tau$ and $1$ with respect to the chosen basis for $H_1(E_\tau, \mathbb{Z})$ (and $\bar{\pi}$ has the complex conjugate periods). The polarization thus obtained is explicitly given by the bilinear pairing which takes two vectors to minus the determinant of the matrix they form (as columns). These two constructions are related: By considering the real manifold $E_\tau$ as $(\mathbb{R}^2)^t/(\mathbb{Z}^2)^t$
(row vectors), the complex structure on \( E_\tau \) is given by \( J_\tau \) acting from the right. Identifying an element of the holomorphic tangent space of \( E_\tau \) with its first coordinate multiplied by \( iy \) yields the representation of \( E_\tau \) as \( \mathbb{C} / (\mathbb{Z} \tau \oplus \mathbb{Z}) \).

For any natural \( m \), we define \( V_m \) to be the \( m \)th symmetric power of \( V_1 \). We use multiplicative notation for symmetric powers: For \( m \) vectors \( v_i, 1 \leq i \leq m \) in a space \( V \), the image of the element \( v_1 \otimes \ldots \otimes v_m \) of \( V^\otimes m \) under the projection to \( \text{Sym}^m V \) is denoted \( v_1 \ldots v_m = \prod_{i=1}^{m} v_i \), with powers to indicate repetitions. Hence in the Hodge decomposition of the fiber of \( \text{Sym}^m V \) in a space \( M \), the vector \( (\tau)^m \) is of type \((1,0)\) \((\tau)^m \) of type \((0,1)\) \((\tau)^m \) is of type \((m_+,m_-)\) \((\tau)^m \) is of type \((m_+,m_-)\) and it spans the space of vectors of this type in this fiber.

We make frequent use of the simple and classical equation stating that the equality

\[
M \begin{pmatrix} \tau \\ 1 \end{pmatrix} = j(M,\tau) \begin{pmatrix} M \tau \\ 1 \end{pmatrix},
\]

holds for every \( \tau \in \mathcal{H} \) and every \( M \in \text{SL}_2(\mathbb{R}) \) (and, more generally, in \( \text{GL}^+_2(\mathbb{R}) \)). Equation (20) shows that \( V_1 \) is \( \text{GL}^+_2(\mathbb{R}) \)-equivariant, hence the same holds for its symmetric powers \( V_m \). Thus, these variations of Hodge structures descend to variations of Hodge structures on quotients of \( \mathcal{H} \) by Fuchsian groups. Moreover, letting the larger group \( \text{GL}_2(\mathbb{R}) \) act on \( \mathcal{H} \cup \overline{\mathcal{H}} = \mathbb{C} \setminus \mathbb{R} \), Equation (20) still holds.

Furthermore, it will turn out useful for the construction of generic cycles below to extend Equation (20) to the case where \( M \) has a vanishing determinant, provided that \( j(M,\tau) \neq 0 \). Such a matrix \( M \) takes every element \( \tau \in \mathbb{C} \) (except for the value for which \( j(M,\tau) = 0 \)) to a fixed element of \( \mathbb{R} \), independently of the value of \( \tau \). Equation (20) also shows that the matrix \( M = dI + cJ_\tau \) multiplies \( (\tau)_1 \) by \( j(M,\tau) = ci + d \), so that multiplying \( (\tau)_1 \) by any complex number can be carried out via the action of elements of \( \text{GL}^+_2(\mathbb{R}) \).

Another formula which will turn out to be very useful states that

\[
\delta J_\tau \delta^{-1} = J_{\delta \tau},
\]

for \( \tau \in \mathcal{H} \) and \( \delta \in \text{GL}^+_2(\mathbb{R}) \) (and more generally, for \( \tau \in \mathcal{H} \cup \overline{\mathcal{H}} \) and \( \delta \in \text{GL}_2(\mathbb{R}) \), where we have the obvious relation \( J_{\tau} = -J_\tau \)).

The geometric construction of \( V_1 \) is defined over \( \mathbb{Z} \), hence \( V_m \) is defined over \( \mathbb{Q} \). However, following [Be] (though more explicitly) we show that over \( \mathbb{R} \), \( V_2 \) arises as a component of many different geometric variations of Hodge structures, which arise from various non-trivial quaternion algebras and give rise to different rational structures on \( V_2 \). We remark at this point that as a representation of \( \text{SL}_2(\mathbb{R}) \), \( V_2 \) is equivalent to the action on \( M_2(\mathbb{R})_0 \) by conjugation described above. In fact, by extending the action on \( M_2(\mathbb{R})_0 \) to \( \text{GL}_2^+(\mathbb{R}) \) using the formula \( g : u \mapsto g \cdot u \cdot \text{adj} g \) (this is conjugation for \( g \in \text{SL}_2(\mathbb{R}) \)) we have an isomorphism between \( V_2 \) and \( M_2(\mathbb{R})_0 \) as representations of \( \text{GL}_2^+(\mathbb{R}) \). An explicit isomorphism takes \( (\begin{smallmatrix} 1 & i \\ 0 & 0 \end{smallmatrix})^2 \) to \( (\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}) \), \( (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^2 \) to \( (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \), and \( (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})^2 \) to \( (\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}) \). Under this isomorphism, the type \((2,0)\) vector \( (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})^2 \) is identified with the matrix \( M_\tau \), the type \((1,1)\) vector \( (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})^* \) is sent to \( y J_\tau \), and the type \((0,2)\) vector \( (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})^2 \) is sent
to $M_\tau = \overline{M_\tau}$. The representation $V_2$ is also isomorphic to the representation on symmetric matrices considered in Example 5.1 of [B2], and the isomorphism takes $\left(\tau \right)^2$ to the norm 0 vector considered in that example. However, for our purposes the first choice is more convenient.

We now present the Abelian surfaces with QM and their cohomology, following Section 3 of [Bc]. Let $B$ be a rational quaternion algebra, i.e., a 4-dimensional central simple algebra over $\mathbb{Q}$, with (reduced) trace $Tr : B \to \mathbb{Q}$ and (reduced) norm $N : B^* \to \mathbb{Q}^*$. The main involution of $B$ will be denoted $x \mapsto \overline{x}$. For background on (rational) quaternion algebras we refer the reader to [Vi], for example. Assume that $B$ splits over $\mathbb{R}$, and fix an embedding $i : B \to M_2(\mathbb{R})$, so that $B_\mathbb{R} \cong M_2(\mathbb{R})$ ($i$ is unique up to conjugation in $M_2(\mathbb{R})$ by the Skolem-Noether theorem). We identify $B$ with its image under $i$, hence omit $i$ from the notation from now on.

Let $\mathcal{M}$ be an order (a subring which is a full integral lattice) in $B$. We consider Abelian surfaces with quaternionic multiplication (QM) from $\mathcal{M}$. As a real manifold, $A = V/\Lambda$ where $V$ is the 4-dimensional real vector space $T_0(A)$ and $\Lambda = H_1(A,\mathbb{Z})$ is a full integral lattice in $V$. Since $\mathcal{M}$ acts as endomorphisms of $A$, the lattice $\Lambda$ is an $\mathcal{M}$-module, and $V = \Lambda \otimes \mathbb{R}$ is isomorphic to the space $\mathcal{M} \otimes \mathbb{R} = M_2(\mathbb{R})$. The submodule $\Lambda$ of $V$ is isomorphic to an ideal in $B$, hence we fix an ideal $I$ in $B$ such that $L(I) = \{ x \in B | xI \subseteq I \}$ is equal to $\mathcal{M}$, and we consider Abelian surfaces with QM from $\mathcal{M}$ whose underlying real manifold is $I \setminus M_2(\mathbb{R})$.

The action of $\mathcal{M} \subseteq \text{End}(A)$ on $A$ is by multiplication from the left. For these endomorphisms to be endomorphisms of $A$ as an Abelian surface, they must commute with the complex structure on $A$. Hence the complex structure must be given by multiplication from the right by a matrix $J$ in $M_2(\mathbb{R})$ whose square is $-I$. Such a matrix is $\pm J_\tau$ for $\tau \in \mathcal{H}$, and we define $A_\tau$ to be the Abelian surface thus obtained from the complex structure $J_\tau$. Identifying elements of the holomorphic tangent space of $A_\tau$ with their first column multiplied by $iy$, the manifold $A_\tau$ takes the form $\mathbb{C}^2/I(\tau)$. Every complex Abelian surface with QM from $\mathcal{M}$ is isomorphic to such $A_\tau$ with some ideal $I$, and multiplying the ideal by an element of $B^*$ from the right gives rise to an isomorphic Abelian surface. Hence for every order $\mathcal{M}$ in $B$ the moduli space for Abelian surfaces with QM from $\mathcal{M}$ decomposes according to classes of ideals $I$ in $B$ such that $L(I) = \mathcal{M}$ modulo right multiplication from $B^*$. These classes are orbits in the action of $B^*$ on the set of ideals in $B$ by right multiplication, and the order $L(I)$ is well-defined on such orbits. When $\mathcal{M}$ is an Eichler order then every ideal of $M$ is principal. Hence there is just one class (or orbit), and one can take, without loss of generality, $I = \mathcal{M}$. Otherwise, this statement may not be true, but in any case there exist only finitely many such classes (see, for example, [Vi]).

Let $B_0$ be the space of traceless elements of $B$. [Bc] introduces two maps from $B_0$ into $H^2(A,\mathbb{Q})$, which we now present. Since $H_1(A,\mathbb{Z}) = I$, we consider $H^2(A,\mathbb{Q})$ as the space of alternating rational-valued bilinear maps on $I$ (or on
B). Define, for every \( b \in B_0 \),

\[
i(b) : (x, y) \mapsto Tr(bx\overline{y}), \quad \breve{i}(b) : (x, y) \mapsto Tr(b\overline{xy})
\]

(with \( x \) and \( y \) in \( I \), or in \( B \)). The fact that \( b \in B_0 \) implies that \( i(b) \) and \( \breve{i}(b) \) are indeed alternating. Tensoring with \( \mathbb{R} \) allows these maps to act on pairs of elements of \( B_{\mathbb{R}} = M_2(\mathbb{R}) \) (and be defined for \( b \in M_2(\mathbb{R})_0 \), and tensoring with \( \mathbb{C} \) gives the Hodge decomposition. Theorem 3.10 of \([B\mathcal{E}]\) states that \( H^2(A, \mathbb{Q}) \) is \( i(B_0) \oplus \breve{i}(B_0) \) with the direct sum being orthogonal with respect to the cup product, and that the cup product takes \( i(b) \) and \( \breve{i}(c) \) to \( disc(I)Tr(b\overline{c}) \) and pairs \( i(b) \) and \( \breve{i}(c) \) to \(-disc(I)Tr(b\overline{c})\). Hence \( i(B_0) \) is of signature \((1, 2)\) and \( \breve{i}(B_0) \) is of signature \((2, 1)\). The action of an element \( a \in \mathcal{M} \), as an endomorphism of \( A \), on these elements of \( H^2(A, \mathbb{Q}) \), is given by

\[
a : i(b) \mapsto i(ab\overline{a}), \quad a : \breve{i}(b) \mapsto N(a)\breve{i}(b).
\]

We recall that the complex structure acts from the right, and that a complex-valued alternating bilinear map \( \varphi \) on \( B_{\mathbb{R}} \) is of type \((2, 0)\) if \( \varphi(x, yJ) = i\varphi(x, y) \), of type \((0, 2)\) if \( \varphi(x, yJ) = -i\varphi(x, y) \), and of type \((1, 1)\) if \( \varphi(xJ, yJ) = \varphi(x, y) \) (see, for example, Lemma 3.9 of \([B\mathcal{E}]\)). This shows that in the Hodge decomposition of \( H^2(A, \mathbb{C}) \), for \( A = A_\tau \) for some \( \tau \in \mathcal{H} \), \( i(M_\tau) \) spans \( H^{2,0}(A) \), its complex conjugate \( \breve{i}(M_\tau) \) spans \( H^{0,2}(A) \), and \( H^{1,1}(A) = i((B_0)_C) \oplus \breve{i}(C.J_\tau) \). In particular, \( i((B_0)_C) \) is isomorphic, as Hodge structures, to the fiber over \( \tau \) of the variation of Hodge structures \( V_2 \).

Let \( \Lambda \) be the lattice of elements \( b \in B_0 \) such that \( Tr(bx\overline{y}) \in \mathbb{Z} \) for every \( x \) and \( y \) in \( I \), and let \( \breve{\Lambda} \) be the lattice of elements \( b \in B_0 \) such that \( Tr(b\overline{xy}) \in \mathbb{Z} \) for every \( x \) and \( y \) in \( I \). Then, by the Lefschetz Theorem on \((1, 1)\) classes, \( \Lambda \) embeds via \( i \) as a rank 3 subgroup of the Néron-Severi group \( NS(A) \) of algebraic cycles in \( A \). On the other hand, the lattice \( \breve{\Lambda} \) will be used later for the theta lifts. In particular, the polarization on \( A \) is \( i(v) \) for some \( v \in \Lambda \), which satisfies \( v^2 = -\frac{1}{\text{disc}(I)} \) so that the cup product of the polarization with itself will be 2. Therefore \( v = \pm \frac{1}{\sqrt{\text{disc}(I)}} J_\sigma \) for some \( \sigma \in \mathcal{H} \), and the positivity condition \(-Tr(vxJ\overline{x}) > 0\) for every non-zero \( x \in B \) implies that the appropriate choice of sign for \( v \) is + (i.e., the polarization has “the same sign” as the complex structure). The fact that an endomorphism \( a \in \mathcal{M} \cap B^* \) of \( A_\tau \) takes \( i(v) \) to \( i(ava^{-1}) = N(a)i(ava^{-1}) \) shows that the endomorphisms of \( A_\tau \) relate all the possible polarizations on \( A_\tau \) one to another, which implies that no further characterization of the polarization is possible.

Let \( \pi : A \rightarrow \mathcal{H} \) be the universal Abelian surface with QM from the order \( \mathcal{M} \) (say with ideal \( I \)), i.e., such that the fiber over \( \tau \in \mathcal{H} \) is the Abelian surface \( A_\tau \) (see, for example, Section 4 of \([B\mathcal{E}]\)). \( \mathcal{A} \) is the quotient of \( \mathcal{H} \times \mathbb{C}^2 \) by the action of \( I \) in which \( \alpha \in I \) takes \( (\tau, (\omega, z)) \) to \( (\tau, (\omega, z + \alpha(z))) \). The previous two paragraphs show that the variation of Hodge structures \( R^2\pi_*\mathcal{C} \) splits as the direct sum of a constant 3-dimensional variation of Hodge structures of type \((1, 1)\) and a variation of Hodge structures isomorphic to \( V_2 \) (but with the polarization multiplied by \( disc(I) \)). For non-trivial \( B \), the rational structure
here does not come from any rational structure on $V_1$. We remark that for non-trivial $B$ the variation of Hodge structures $R^1\tau_*\mathbb{C}$ is isomorphic to $V_1 \oplus V_1$, but again the rational structure does not come from that on one copy of $V_1$. Note that when comparing the interpretations of $\mathcal{H}$ as the base space for $V_2$ and as the Grassmannian $G(L_B)$ considered above, the Hodge types $(2,0)$ and $(0,2)$ correspond to the space $v_+$ (as $Z_{v,v}$ is a multiple of $M_\tau$) and the Hodge type $(1,1)$ is spanned by $J_\tau \in v_-$. We recall that in the case of $B = M_2(Q)$ and $\mathcal{M} = I = M_2(\mathbb{Z})$, the Abelian surface $A$ is the product $E \times E$ of an elliptic curve $E$ with itself. In this case Section 3 of [Bry] decomposes $H^2(A)$ as

$$H^2(E \times E) = (H^2(E) \otimes H^0(E)) \oplus (H^1(E) \otimes H^1(E)) \oplus (H^0(E) \otimes H^2(E))$$ (22)

(by the Künneth formula), and the middle term $H^1(E) \otimes H^1(E)$ decomposes further as $\text{Sym}^2 H^1(E) \oplus \wedge^2 H^1(E)$. The first and last terms in Equation (22) as well as $\wedge^2 H^1(E)$ are of rank 1 and type $(1,1)$. Identifying $H^1(E,\mathbb{C})^*$ with $H^1(E,\mathbb{C})(1)$ via the polarization, the middle term in Equation (22) is identified with $\text{End}(E)$, where $\wedge^2 H^1(E)$ maps to multiples of $\text{Id}_E$ and $\text{Sym}^2 H^1(E)$ corresponds to traceless endomorphisms. The relation to cycles in $E \times E$ is as follows. The oriented generators of the first and last terms in Equation (22) correspond to the axes $\{0\} \times E$ and $E \times \{0\}$ respectively, and and the oriented generator of $\wedge^2 H^1(E)$ corresponds to normalized diagonal $\Delta_E = \Delta_E - \{0\} \times E - E \times \{0\}$. The first and last terms of Equation (22) generate a hyperbolic plane which is perpendicular to the middle term of that equation with respect to the cup product. The spaces $\text{Sym}^2 H^1(E)$ and $\wedge^2 H^1(E)$ are also orthogonal with respect to the cup product. The universal family $A$ is $E \times H E$ (with projection $\pi_2$ onto $\mathcal{H}$), with fiber $A_{\tau} = E_\tau \times E_\tau$ over $\tau$. The corresponding variation of Hodge structures $R^2\pi_2_*\mathbb{C}$ decomposes, according to Equation (22) and the further decomposition of the middle term there, as

$$R^2\pi_2_*\mathbb{C} = (R^2\pi_*\mathbb{C} \otimes R^0\pi_*\mathbb{C}) \oplus (R^0\pi_*\mathbb{C} \otimes R^2\pi_*\mathbb{C}) \oplus \wedge^2 V_1 \oplus V_2$$ (23)

(recall that $R^1\pi_*\mathbb{C} = V_1$). The first three terms in Equation (23) are 1-dimensional and of type $(1,1)$, and correspond to the 1-codimensional cycles $\bigcup_{\tau \in \mathcal{H}} \{0\} \times E_\tau |_{\tau}$, $\bigcup_{\tau \in \mathcal{H}} E_\tau \times \{0\} |_{\tau}$, and $\bigcup_{\tau \in \mathcal{H}} \Delta_{E_{\tau}} |_{\tau}$, in $E \times H E$, respectively. The action of congruence subgroups of $SL_2(\mathbb{Z})$ on these variations of Hodge structures (and cycles) is trivial, so that (under mild restrictions on the congruence subgroup chosen) they descend to (horizontal) 1-codimensional cycles on the universal families over the (open) modular curves, i.e., in the open Kuga-Sato varieties $W_2$. The fourth term in Equation (23) is just $V_2$ with the rational structure coming from its description as $\text{Sym}^2 V_1$ with the rational structure on $V_1$ presented above. In this case the rational structure on $R^1\pi_2_*\mathbb{C} = V_1 \oplus V_1$ also arises from the rational structures on both terms $V_1$. Equation (20) shows that for $\tau \in \mathcal{H}$ and $M \in R(I) = \{x \in B | x \subseteq I\}$ with positive norm, scalar multiplication by $j(M,\tau)$ on $C^2$ takes the lattice $I(M,\tau)$ to

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hence defines a map from $A_v$ to $A_r$. This map is an isogeny of degree $N(M)$, hence it is an isomorphism if and only if $M$ is in the group $R(I)_1^\ast$ of norm 1 elements of $R(I)$. Conversely, any map from $A_v$ to $A_r$ which respects the QM is of this type. Hence there exists a non-trivial map from $A_v$ to $A_r$ if and only if $\sigma = M\tau$ with $M \in R(I)$ with positive norm, and in particular $A_v \cong A_r$ if and only if $\sigma = M\tau$ with $M \in R(I)_1^\ast$. Hence the Shimura curve $X(\Gamma) = \Gamma \backslash \mathcal{H}$ with $\Gamma = R(I)_1^\ast$ is the moduli space for Abelian surfaces with QM from $\mathcal{M}$ and with $H_1(A) \cong I$ as $\mathcal{M}$-modules. For non-trivial $B$ the quotient $X(\Gamma)$ is already compact, so we denote the quotient $X(\Gamma)$ rather than $Y(\Gamma)$. The condition of the positive norm is required in order to work with $\mathcal{H}$ rather than with $\mathcal{H} \cup \overline{\mathcal{H}}$. The map $\Gamma \times \mathcal{A} \to \mathcal{A}$ defined by $(M, v + I(\mathcal{I}_1)) \mapsto \frac{\tau}{\gamma(M, \tau)} + I(M\tau)|_{M\tau}$ gives a well-defined action of $\Gamma$ on $\mathcal{A}$ above $\mathcal{H}$, hence the quotient is (at least when $\Gamma$ has no elements with fixed points in $\mathcal{H}$) a universal Abelian surface over $X(\Gamma)$, which is a Kuga-Sato type variety similar to $W_2$. Topologically, $\mathcal{A}$ is just $\mathcal{H} \times B_2/I$, and we observe that under the isomorphism between $B_2/I$ and $A_r$ described above, this action of $\Gamma$ on $\mathcal{A}$ corresponds to the action $M(\tau, v) = (M\tau, vM^{-1})$ on this topologically trivial bundle. Note that this universal family is, as a complex manifold, a quotient of $\mathcal{H} \times \mathbb{C}^2$ by the action of the semi-direct product in which $\Gamma$ acts on $I$ (see also Section 4 of [Be]). This observation may lead to a theory of Jacobi-like forms on $\mathcal{H} \times \mathbb{C}^2$, with the group acting on $\mathcal{H}$ being $\Gamma$ rather than a usual congruence subgroup of $SL_2(\mathbb{Z})$.

Replacing the $\mathcal{M}$-module $I$ by $Ix$ for some $x \in B^*$ moves $\Gamma$ to $x\Gamma x^{-1}$ and gives an isomorphic Shimura curve (and an isomorphic universal Abelian surface). Thus, for every orbit of ideals $I$ in $B$ with $L(I) = \mathcal{M}$ modulo the right action of $B^*$ there is a Shimura curve corresponding to this class, well-defined up to isomorphism. Hence for every order $\mathcal{M}$ the moduli space of Abelian surfaces with QM from $\mathcal{M}$ is a finite disjoint union of these Shimura curves. In particular, for $\mathcal{M}$ an Eichler order this moduli space is one Shimura curve, for example the one obtained by taking $I = \mathcal{M}$. A related result is Part 6 of Theorem 3.10 of [Be], which shows that the isomorphism of $I$ defined by right multiplication by $t \in R(I)^\ast$ takes the cohomology class $i(b)$ to $N(t)i(b)$. This implies that these classes are invariant under the isomorphisms coming from $\Gamma$, hence give well-defined cohomology classes in the fiber of the universal family over the Shimura curve. The cohomology class $i(b)$ is taken to $i(tb^{-1})$ (which for $t \in \Gamma$ is $i(tb\Gamma^{-1})$). Take now $B = M_2(\mathbb{Q})$ and $\mathcal{M}$ (and $I$) to be the Eichler order of level $N$ consisting of matrices in $M_2(\mathbb{Z})$ with lower left entry divisible by $N$. In this case the group $\Gamma$ is $\Gamma_0(N)$, the Shimura curve is the modular curve $X_0(N)$ (or rather the open modular curve $Y_0(N)$), and the Abelian surface $A_r$ for $\tau \in \mathcal{H}$ is $E_\tau \times E_{N\tau}$ (with the description given above for elliptic curves). The classical isogeny between $E_\tau$ and $E_{N\tau}$, which is multiplication by $N$ in the classical complex-analytic presentation, is related to the element $(\begin{smallmatrix} 0 & 0 \\ N & 0 \end{smallmatrix}) \in \mathcal{M}$. This returns us to the interpretation of the open modular curve $Y_0(N)$ as the moduli space of cyclic isogenies of degree $N$ between elliptic curves.
3.2 De-Rham Cohomology Classes

We later define CM cycles in Abelian surfaces with QM as actual cycles (rather than cohomology classes as in Section 5 of [Be]). Evaluating their fundamental classes is most easily carried out in the de-Rham setting, since it has the advantage that the calculation is independent of the choice of quaternion algebra B with which we work. Along the way we show how working in this setting simplifies the proofs of several assertions in [Be]. We recall that for any Abelian variety (or complex torus) A the space $H_1(A, \mathbb{R})$ is naturally identified with the tangent space $V = T_0(A)$, and the de-Rham cohomology groups of A consist of algebraic differential forms on V. Since in our case of Abelian surfaces with QM the space V is $M_2(\mathbb{R})$ (regardless of the quaternion algebra B), a single calculation suffices for all the quaternion algebras. Taking the tensor product with $\mathbb{C}$ gives the decomposition to Hodge types, which is based on the choice of $\tau \in \mathcal{H}$ (but again independent of B).

Let us first demonstrate this point of view for the case of elliptic curves. We write $E_\tau$ (with $\tau \in \mathcal{H}$) as $(\mathbb{R}^2)^t/((\mathbb{Z}^2)^t$ as above, hence $V = (\mathbb{R}^2)^t$. A basis for $H^1(E_\tau, \mathbb{R})$ (which is a $\mathbb{Z}$-basis for $H^1(E_\tau, \mathbb{Z})$ in this case) consists of the differential forms $a$ and $b$ taking each vector in V to its first and second entry respectively. Since with the complex structure we have identified $(\mathbb{Z}^2)^t \subseteq (\mathbb{R}^2)^t$ with $\mathbb{Z}\tau \oplus \mathbb{Z} \subseteq \mathbb{C}$, the differential form $dz$ on $\mathbb{C}$ is $a\tau + b \in H^1(E_\tau, \mathbb{C})$, and $d\tau$ is the complex conjugate $a\bar{\tau} + b$. These forms span $H^{1,0}(E_\tau)$ and $H^{0,1}(E_\tau)$ respectively. For the second cohomology group, $b \wedge a$ is an oriented generator of $H^2(E_\tau, \mathbb{Z})$, and the integral of $b \wedge a$ over $E_\tau$ is 1 by definition. This is in correspondence with the fact that the integral of $dz \wedge d\tau$ over $E_\tau$ is $-2iVol(E_\tau)$, while $dz \wedge d\tau = -2iyb \wedge a$ and the volume of $E_\tau$ is $y$.

Let now $A = A_\tau = \mathbb{C}^2/I(\tau)$, with $I$ an ideal in a rational quaternion algebra B as above, be an Abelian surface with QM from the order $\mathcal{M} = L(I)$. Then $V = M_2(\mathbb{R})$, and writing matrices in $M_2(\mathbb{R})$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define a, b, c, and d to be the 1-forms associated with this notation. These forms give a real basis for $H^1(A, \mathbb{R})$, but they are not in $H^1(A, \mathbb{Q})$ unless $B = M_2(\mathbb{Q})$. A direct evaluation of the maps $i$ and $\iota$ on $(B_0)_{\mathbb{R}} = M_2(\mathbb{R})$ gives

$$
i \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = d \wedge c, \quad \iota \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a \wedge d - b \wedge c, \quad \iota \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = b \wedge a,$$

$$\iota \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = c \wedge a, \quad \iota \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = d \wedge a + c \wedge b, \quad \iota \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = d \wedge b.$$

Note again that these classes are in $H^2(A, \mathbb{R})$, but not in $H^2(A, \mathbb{Q})$ in general. We have thus established that $H^2(A, \mathbb{R}) = i(M_2(\mathbb{R})_0) \oplus \iota(M_2(\mathbb{R})_0)$, whence part 3 of Theorem 3.10 of [Be] also follows.

Tensoring with $\mathbb{C}$ we obtain Hodge types, and as in the case of elliptic curves we find that

$$dz_1 = a\tau + b, \quad dz_2 = c\tau + d, \quad d\tau_1 = a\bar{\tau} + b, \quad d\tau_2 = c\bar{\tau} + d.$$
A corollary of these calculations is that \( dz_2 \wedge dz_1 = \iota(M_\tau) \) (this agrees with Proposition 5.12 of [Be]) and \( dz_2 \wedge dz_1 = \iota(M_\tau) \), which again gives their Hodge types. The Hodge types of the other elements of \( H^2(A, \mathbb{C}) \) can also be obtained in this way: First observe that the equalities \( d \wedge c = \frac{dz_2 \wedge dz_1}{2iy} + b \wedge a = \frac{dz_1 \wedge dz_2}{2iy} \), and \( a \wedge d - b \wedge c = \frac{dz_2 \wedge dz_1 + dz_1 \wedge dz_2}{2iy} \) provide an alternative proof to the assertion that the image of \( \iota \) is contained in \( \text{H}^{1,1}(A_\tau) \) for any \( \tau \). The last combination, \( \frac{dz_2 \wedge dz_1 - dz_1 \wedge dz_2}{2iy} \), equals \( i(yJ_\tau) \) and is the \((1,1)\) part of \( \iota(M_2(\mathbb{R})) \). Hence we obtain the asserted Hodge decomposition of \( H^2(A, \mathbb{C}) \) from another viewpoint.

The forms corresponding to algebraic cycles are identified using Poincaré duality. Recall that the fundamental (cohomology) class of a \((1\text{-codimensional})\) cycle \( C \) on \( A_\tau \) is the cohomology class represented by a 2-form \( \omega \) which satisfies the formula

\[
\int_C \eta = \int_{A_\tau} \omega \wedge \eta = \int_{A_\tau} \eta \wedge \omega, \quad \eta \in H^2(A_\tau, \mathbb{C}).
\]

The form \( a \wedge b \wedge c \wedge d = b \wedge a \wedge d \wedge c = d \wedge c \wedge b \wedge a \) is oriented, and we need its integral over \( A_\tau \) in order to obtain the right constant for Poincaré duality. For \( I = M_2(\mathbb{Z}) \) the integral is 1. We observe that \( M_2(\mathbb{Z}) \) is self-dual with respect to the symmetric pairing \((x, y) \mapsto \text{Tr}(x\overline{y})\) on \( M_2(\mathbb{R}) \) (with \( \overline{y} = \text{adj}y \)). Let \( I \) be an arbitrary ideal in an arbitrary quaternion algebra \( B \), and let \( T \in M_4(\mathbb{R}) \) be the matrix representing a basis of \( I \) over \( \mathbb{Z} \) by \( a, b, c, \) and \( d \). In this case the integral equals \(| \det T | \). Since the dual basis for \( I^* = \{ x \in M_2(\mathbb{R}) | (x, I) \subseteq \mathbb{Z} \} \) is represented by \((T^*)^{-1} \), we find that \( \text{disc}(I)^2 \) equals \( \det T / \det(T^*)^{-1} = (\det T)^2 \). Therefore the integral over \( B_2/I \) is \( \text{disc}(I) \) for any ideal \( I \). It follows that if the wedge product of two 2-forms gives \( N \) times \( a \wedge b \wedge c \wedge d \) then their cup product equals \( N \cdot \text{disc}(I) \). This observation can be used to prove parts 1 and 2 of Theorem 3.10 of [Be]: Indeed, the wedge product of \( \iota(b) \) and \( \iota(c) \) vanishes, and one can verify directly on the \( \mathbb{R} \)-basis given above for \((B_0)_\mathbb{R} = M_2(\mathbb{R})_0 \) that the wedge Replace products of \( \iota(b) \) with \( \iota(c) \) and of \( \iota(b) \) with \( \iota(c) \) give \( \text{Tr}(\overline{c}) = -\text{Tr}(bc) \) and \( -\text{Tr}(\overline{b}) = +\text{Tr}(bc) \) respectively. As another application of the above observation we establish that the 2-form representing the fundamental class of the cycle \( C \) is

\[
\frac{1}{\text{disc}(I)} \left[ \left( \int_C d \wedge c \right) b \wedge a + \left( \int_C b \wedge a \right) d \wedge c - \left( \int_C d \wedge b \right) c \wedge a + \right. \\
- \left. \left( \int_C c \wedge a \right) d \wedge b + \left( \int_C c \wedge b \right) d \wedge a + \left( \int_C d \wedge a \right) c \wedge b \right]. \tag{24}
\]

We begin by recovering the known results for the generic cycles in the split \( B \) case using this method. Take \( B = M_2(\mathbb{Q}) \) and \( I = \mathcal{M} \) is the classical Eichler order of level \( N \) (and discriminant \( N \)), so that \( A_\tau = E_\tau \times E_{N_\tau} \) for all \( \tau \in \mathcal{H} \). We define the (generic) isogeny \( \phi_M : E_\tau \rightarrow E_{N_\tau} \) to be the isogeny which complex-analytically is multiplication by an integer \( M \in \mathbb{N} \), and the isogeny \( \psi_L : E_{N_\tau} \rightarrow E_\tau \) which complex-analytically is multiplication by an integer \( L \).
(these are not isogenies for \( M = 0 \) or \( L = 0 \), but nonetheless give rise to generic cycles). The cycle corresponding to the graph of \( \varphi_M \) is defined by the relation \( dz_2 = Mdz_1 \) (hence also \( \mathcal{O}z_2 = M\mathcal{O}z_1 \)), which translates to \( c = Ma \) and \( d = Mb \).

The integral of the 2-form \( b \wedge a \) over this cycle coincides with the integral over \( E_\tau \), which equals 1. Using the relations \( c = Ma \) and \( d = Mb \) on this cycle we can evaluate the other integrals appearing in Equation (22), and find that the fundamental class of this cycle is

\[
\frac{M^2}{N}b \wedge a + \frac{M}{N}(a \wedge d - b \wedge c) + \frac{d \wedge c}{N} = \iota\left( \frac{M/N - 1/N}{M^2/N - M/N} \right)
\]

(this element is indeed in \( H^2(A_\tau, \mathbb{Z}) \), since \( \frac{d_\varphi}{N} \in H^1(A_\tau, \mathbb{Z}) \)). On the graph of \( \psi_L \) we have \( a = Lc \) and \( b = Ld \), and the integral of \( d \wedge c \) over \( E_{\psi_L} \) (and over the cycle) is \( N \) (since \( H^1(E_{\psi_L}, \mathbb{Z}) \) is spanned by \( d \) and \( \frac{d_\varphi}{N} \)). Therefore Equation (24) shows that the fundamental class of the graph of \( \psi_L \) is

\[
b \wedge a + L(a \wedge d - b \wedge c) + L^2d \wedge c = \iota\left( \frac{L}{1} - \frac{L^2}{L} \right).
\]

In particular, the axis \( E_\tau \times \{0\} \) is the graph of \( \varphi_0 \) whose fundamental class is \( \frac{d_\varphi}{N} = \iota(0_{0,0}^{1,0}, \iota(0_{0,0}^{1,0}) \), and the axis \( \{0\} \times E_{N\tau} \) is the graph of \( \psi_0 \) with fundamental class \( b \wedge a = \iota(1_{0,0}^{1,0}) \). The third “basis element” of \( \Lambda \), i.e., \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) with \( \iota \)-image \( a \wedge d - b \wedge c \), appears (with some multiplying coefficients) in the true isogenies, and for each such isogeny the coefficients of the first basis elements, i.e., \( d \wedge c \), and \( b \wedge a \), are the intersection numbers of the graph of the isogeny in question with the axes. The basis element itself, with coefficient 1, is obtained in the fundamental classes of either the classical isogeny \( \varphi_N \) or the dual isogeny \( \psi_1 \), after normalization by subtracting the appropriate multiples of the axes. In particular, for \( N = 1 \) we have \( \varphi_1 = \psi_1 = Id_{E_\tau} \), its graph is the diagonal \( \Delta_{E_\tau} \), and \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the fundamental class of the cycle \( \Delta_{E_\tau} \) mentioned above. Hence this basis of \( \Lambda \) corresponds to the decomposition of the generic part of \( NS(A_\tau) \) according to Equation (22) and the further decomposition of the middle term there. Note that for nonzero \( L \), \( \psi_L \) is the isogeny dual to \( \varphi_{NL} \), and dual isogenies give the same normalized cycles.

In order to motivate the definition of CM cycles in Abelian surfaces with QM from orders in general quaternion algebras, we note that for any \( M \in \mathbb{Z} \) the graph of the isogeny \( \varphi_M \) is the image in \( A_\tau = E_\tau \times E_{N\tau} \) of the line \( \mathbb{C} \mathcal{M}(1) \), and for \( L \in \mathbb{Z} \) the graph of \( \psi_L \) is the image in \( A_\tau = E_\tau \times E_{N_L} \) of the line \( \mathbb{C} \mathcal{M}(1) \), via the natural projection \( \mathbb{C}^2 \to A_\tau = \mathbb{C}^2/\mathcal{M}(1) \). In particular, the axes are the images of the lines \( \mathbb{C} (1) \) and \( \mathbb{C} (0) \) in \( A_\tau \).

### 3.3 CM Points and CM Isogenies

Let \( E \) be an elliptic curve, and assume that \( E \) has complex multiplication (CM) from an order \( \mathcal{O} \) of (negative) discriminant \( -D \) in the imaginary quadratic extension \( \mathbb{K} = \mathbb{Q}(\sqrt{-D}) \) of \( \mathbb{Q} \). This means that “multiplication by \( \sqrt{-D} \) is
an endomorphism of \( E \), and one defines the (classical) CM cycle to be the (normalized) graph of this endomorphism inside \( E \times E \). More generally, assume that \( E_\tau \) and \( E_{N\tau} \) have CM from this order \( O \) in \( \mathbb{K} \), which occurs if and only if \( \tau \) is in \( \mathbb{K} \) and is of the form \( \tau = \sqrt{-D}/g \) for \( g > 0 \) and \( h \) integers such that \( N|g \) and \( D+h^2 \) is integral. Such points \( \tau \) are the CM points (for the case of split \( B \)), since in the moduli interpretation of modular curves they represent (isogenies between) elliptic curves with complex multiplication. Over our element \( \tau \) of \( \mathcal{H} \), the (normalized) graph of the CM-isogeny “multiplication by \( N\sqrt{-D} \)” from \( E_\tau \) to \( E_{N\tau} \), and the (normalized) graph of the isogeny “multiplication by \( \sqrt{-D} \)” from \( E_{N\tau} \) to \( E_\tau \) give CM cycles in \( A_\tau = E_\tau \times E_{N\tau} \). These isogenies are not dual to one another, but one to minus the other. Their graphs (before normalization) are the images in \( A_\tau \) of the lines \( \mathbb{C}(N\sqrt{-D}) \) and \( \mathbb{C}(\sqrt{-D}) \) respectively. We can always multiply \( D \) by an integral square and obtain a different isogeny with a different graph, hence a different CM cycle in \( A_\tau \). It is important to note that a line in \( \mathbb{C}^2 \) gives an Abelian subvariety of \( A_\tau \) if and only if the lattice \( I(\tau) \) (i.e., the kernel of the projection \( \mathbb{C}^2 \to A_\tau \)) intersects the line in question in a full lattice in the line.

Let us now find the fundamental class of the graph of the isogeny “multiplication by \( \sqrt{-D} \)” from \( E_{N\tau} \) to \( E_\tau \). Along this graph we have \( d\tau_1 = \sqrt{-D}d\tau_2 \) and \( d\tau_2 = -\sqrt{-D}d\tau_1 \), which translate to \((a \ b) = (c \ d)\sqrt{-D}J_\tau \) (as row vectors of length 2). The integral of \( d \wedge c \) over the cycle is \( N \) (like with \( \psi_L \)). Taken together, these results reduce Equation (24) to

\[
b \wedge a + Dd \wedge c + \frac{\sqrt{-D}}{g} \left[ |\tau|^2 c \wedge a + x(d \wedge a + c \wedge b) + d \wedge b \right],
\]

or equivalently \( b \wedge a + Dd \wedge c + i(\sqrt{-D}J_\tau) \). The generic part consists again of multiples of the axes \( E_\tau \times \{0\} \) and \( \{0\} \times E_{N\tau} \) coming from the intersection numbers, but we observe that it equals \( i(\sqrt{-D}J_{\tau_0}) \) for \( \tau_0 \) being the element \( \sqrt{-D} \in \mathcal{H} \). Further note that the Abelian surface corresponding to this point \( \tau_0 \) has CM from the same field \( \mathbb{K} \) as our \( A_\tau \). Thus, the fundamental class of this CM cycle is \( i(\sqrt{-D}J_{\tau_0}) + i(\sqrt{-D}J_{\tau}) \), and subtracting the generic part leaves the normalized, transversal class \( i(\sqrt{-D}J_\tau) \). As we show below, this observation is important, since this way of looking at the CM cycles extends to the general \( B \) case. In a similar manner we evaluate the fundamental class of the graph of the other CM-isogeny, “multiplication by \( N\sqrt{-D} \)” from \( E_\tau \) to \( E_{N\tau} \), and obtain \( d\psi_L + NDb \wedge a - i(\sqrt{-D}J_\tau) \). The minus sign in front of \( i(\sqrt{-D}J_\tau) \) here is related both to the fact that this isogeny is dual to minus the previous one, and to the fact that the corresponding \( \tau_0 \) is \( \frac{1}{N\sqrt{-D}} \in \mathcal{H} \). These assertions will show up more clearly in the general case.

For the case \( B = M_2(\mathbb{Q}) \), the CM points for the variation of Hodge structures on \( \mathcal{H} \) which is the \( V_2 \)-part of \( R^2\pi_{2,*}\mathbb{C} \) can be characterized in several ways: As points lying in some imaginary quadratic field, as the points \( \tau \in \mathcal{H} \) such that \( E_\tau \) has CM, or equivalently as points \( \tau \) such that some real multiple of the complex structure \( J_\tau \) has integral (or rational) entries. Indeed, for \( \tau \in \mathcal{H} \) of the form
We have that \( \sqrt{D} \mathbf{I} = \left( \begin{array}{cc} h & \alpha h \\ -g & h \end{array} \right) \) is in \( M_2(\mathbb{Z}) \), and conversely if \( \alpha J_\tau \) is in \( M_2(\mathbb{Z}) \) for some \( \alpha \), then \( x \) and \( y^2 \) are rational hence \( \tau = x + iy \) lies in an imaginary quadratic field. This observation is independent of the order we choose inside \( M_2(\mathbb{Q}) \). It is the third characterization which carries over to non-trivial quaternion algebras. The corresponding CM cycles are the normalized cycles just described, but lying inside \( \mathbf{A} \) as 2-codimensional (vertical) cycles. We have seen that inside an Abelian surface \( A_\tau \) with CM there are several (actually, an infinite number of) such CM cycles; In fact, multiples of \( \sqrt{-D} \) do not provide all the possible CM cycles, as we can take also graphs of isogenies of the form “multiplication by \( a + b\sqrt{-D} \) with non-zero \( a \). Hence a specific cycle depends on a (rather arbitrary) choice, though we shall soon see (with an arbitrary quaternion algebra) that the particular choice has a weak effect on the normalized fundamental class.

Let \( B \) now be arbitrary, with \( i, M, I, A_\tau \) for \( \tau \in \mathcal{H} \), and \( \pi : A \to \mathcal{H} \) as before. We let elements of positive norm in \( B \) (which are hence in \( GL_2^+(\mathbb{R}) \) via \( i \)) act on \( \mathcal{H} \), and we have the variation of Hodge structures \( R^2_i A_\tau \mathbb{C} \) and the part isomorphic to \( V_2 \), in which \( i(J_\tau) \) spans the (1,1) part of the fiber over \( \tau \). For every \( \tau \in \mathcal{H} \), \( NS(A_\tau) \) contains the generic part \( i(\Lambda) \) (hence is of rank at least 3), but it cannot be of rank exceeding \( \dim_{\mathbb{C}} H^{1,1}(A_\tau) = 4 \). There is a difference between the split and non-split \( B \) cases, because for non-split \( B \) the generic cycles cannot be written as the image of a line in the universal cover. Indeed, such a line gives an Abelian subvariety of \( A_\tau \), in which case \( A_\tau \) cannot be simple, contradicting the fact that for a generic \( \tau \), \( End(A_\tau) \) is an order in a division algebra. For more on this, see Theorem \cite{KK} below. On the other hand, if \( A_\tau \) also has CM, then the CM cycles themselves are still obtained this way, as we show below.

First we characterize the CM points. They depend on \( B \) and on the isomorphism \( i \). We prove the following extension of Lemma 7.2 of \cite{Be}:

**Proposition 3.1.** For \( \tau \in \mathcal{H} \), the following conditions are equivalent: (i) There exists some \( b \in B \setminus \mathbb{Q} \) of positive norm such that \( b = i(b) \in GL_2^+(\mathbb{R}) \) fixes \( \tau \). (ii) Some (real) multiple of \( J_\tau \) lies in \( B = i(B) \). (iii) The rank of \( NS(A_\tau) \) is 4 (rather than the generic 3). (iv) There is a non-trivial endomorphism of \( A_\tau \) commuting with the action of \( M \).

**Proof.** The equivalence of (i) and (ii) follows from the fact that the stabilizer of \( \tau \) in \( GL_2^+(\mathbb{R}) \) consists of the matrices of the form \( dI + cJ_\tau \) with \( c \) and \( d \) real (and not both 0). (ii) implies (iii) by the Lefschetz Theorem on (1,1) classes, since \( i(J_\tau) \) has Hodge type (1,1). (iii) implies (ii) since \( H^{1,1} \cap i(B) \) consists of multiples of \( i(J_\tau) \). In order to see that (iv) is equivalent to (ii), observe that an endomorphism of \( A_\tau \) commuting with the action of \( M \) must be given by multiplication from the right by a matrix \( b \), which must lie in \( R(I) \) and commute with the complex structure \( J_\tau \). The fact that the centralizer of \( J_\tau \) consists of matrices of the from \( dI + cJ_\tau \) implies the equivalence of (ii) and (iv). This proves the proposition. \( \square \)
The points \( \tau \in \mathcal{H} \) satisfying the equivalent conditions of Proposition 3.1 are called CM points (with respect to \( B \) and \( i \)), or just CM points when the specification of \( B \) and \( i \) is clear.

The equivalence of (iv) and (ii) in Proposition 3.1 shows that if \( A_\tau \) has CM then \( \text{End}_M(A_\tau) \) is an order in an imaginary quadratic extension of \( \mathbb{Q} \). Indeed, it is generated by the element \( b = cJ_\tau \) of \( B \), which satisfies \( b^2 = -N(b) \in \mathbb{Q} \). It turns out useful to fix the imaginary quadratic field \( K \), and concentrate on the Abelian surfaces with CM from some order in this field \( K \). If \( A_\tau \) has no CM then \( \text{End}_M(A_\tau) = \mathbb{Z} \), and since every Abelian surface with QM is isomorphic (over \( \mathbb{C} \)) to some \( A_\tau \), the same holds for any such surface.

Fix an imaginary quadratic field \( K \), which splits \( B \). The field \( K \) can be embedded into \( \mathbb{C} \) in two ways, and into \( B \) (thus into \( M_2(\mathbb{R}) \) via \( i \)) in many ways. In order to investigate the CM cycles in Abelian surfaces with QM having CM from an order in \( K \) we need these embeddings to relate to one another in a way we now describe. We fix an embedding of \( K \) into \( \mathbb{C} \) (without notation), and treat \( K \) as a subfield of \( \mathbb{C} \). Thus \( \sqrt{-D} \in K \) for positive rational \( D \) means the corresponding element in \( K \cap \mathcal{H} \). Now, in any embedding of \( K \) into \( B \subseteq M_2(\mathbb{R}) \), the element \( \sqrt{-D} \in K \) is taken to a multiple of \( J_\tau \) for some \( \tau \in \mathcal{H} \) (this \( \tau \) is the fixed point of \( K^* \subseteq GL_2^+(\mathbb{R}) \) acting on \( \mathcal{H} \)). The multiplier is \( \pm \sqrt{D} \) and complex conjugation on \( K \) changes the sign. The normalization that we choose implies that this multiplier is positive. Since in this case \( \sqrt{D} J_\tau \), as the non-trivial endomorphism from Proposition 3.1 (iv), acts on \( H^{1,0}(A_\tau) \) as multiplication by \( \sqrt{-D} \in K \cap \mathcal{H} \) (Equation (20) again), this normalization is equivalent to the one given in Section 7 of [Be]. We use this normalization for all embeddings in what follows.

In the split case, all the (classical) CM points lying in \( K \cap \mathcal{H} \) for a fixed imaginary quadratic field \( K \) are taken to one another by the action of \( \text{GL}_2^+(\mathbb{Q}) \), or equivalently of \( M_2(\mathbb{Z}) \cap \text{GL}_2^+(\mathbb{R}) \). We now show that the same holds in the general case, an observation which will be used for the definition and analysis of CM cycles. Let \( \tau_0 \in \mathcal{H} \) be the fixed point of such a (normalized) embedding of some imaginary quadratic field \( K \) (of discriminant \( -D \)) into \( B \). Thus \( A_{\tau_0} \) has CM from an order in \( K \), and \( \sqrt{D} J_{\tau_0} \in B \). We arrive at the following

**Lemma 3.2.** For a point \( \tau \in \mathcal{H} \), the following are equivalent: (i) The Abelian surface \( A_\tau \) (with QM from \( M \)) has an additional structure of CM from an order in the field \( K \). (ii) \( \tau_0 = \gamma \tau \) for some \( \gamma \in B^* \) with positive norm.

**Proof.** We have seen that \( \sqrt{D} J_{\tau_0} \in B \), and condition (ii) is equivalent to the assertion that \( \sqrt{D} J_{\tau_0} \in B \). Hence (ii) implies (i) by Equation (21). Conversely, the Skolem-Noether Theorem shows that the equality \( \gamma(\sqrt{D} J_{\tau})\gamma^{-1} = \sqrt{D} J_{\tau_0} \) holds for some \( \gamma \in B^* \) (as both elements define embeddings of \( K \) into \( B \)). Equation (21) and the fact that the complex structure \( J_\tau \) determines \( \tau \) imply \( \tau_0 = \gamma \tau \), and \( \det \gamma > 0 \) follows since both \( \tau_0 \) and \( \tau \) are in \( \mathcal{H} \). Hence (i) implies (ii) and the proof is complete.

Two remarks are in order here. First, the element \( \gamma \) in condition (ii) in Lemma 3.2 can be taken in any order or ideal in \( B \) of our choice. Second, we
can consider also the fixed point \( \tau_0 \in \overline{\mathcal{H}} \) of an embedding of \( \mathbb{K} \) into \( B \), and then Lemma 3.2 holds with \( \gamma \) on negative norm (and with the corresponding embedding \( \mathbb{K} \) into \( B \) not being normalized). Indeed, we shall consider below CM cycles constructed from \( \tau_0 \) both from \( \mathcal{H} \) and from \( \overline{\mathcal{H}} \).

### 3.4 CM Cycles and their Fundamental Classes

Let \( \tau_0 \) and \( \mathbb{K} \) be as in Lemma 3.2 (or in its extended version with \( \tau_0 \in \overline{\mathcal{H}} \)), and let \( A_\tau \) be as before and such that \( A_\tau \) has CM from an order in the same field \( \mathbb{K} \).

We define the CM cycle corresponding to \( \tau_0 \) in \( A_\tau \) to be the image in \( A_\tau \) of the line \( \mathbb{C}(\tau_0) \subseteq \mathbb{C}^2 \). We first need to show that this is indeed an Abelian subvariety of \( A_\tau \), i.e., that \( I(\tau_0) \cap \mathbb{C}(\tau_0) \) is a full lattice in that line. From Lemma 3.2 we have an element \( \gamma \in B \) such that \( \gamma \tau = \tau_0 \), and we can assume that \( \gamma \in I \) and is primitive. Then Equation (20) shows that the image of the vector \( \gamma(\tau_0) \) is indeed an Abelian subvariety of \( A_\tau \) (21). This Abelian subvariety is an elliptic curve, and it is expedient to find its isomorphism type. In order to do so we take integers \( g > 0 \) and \( h \) such that \( \gamma(\tau_0) \) and \( \gamma(J_g, -h)(\tau_0) \) generate the lattice \( I(\tau_0) \cap \mathbb{C}(\tau_0) \). This elliptic curve is isomorphic to \( E_\tau \) for \( \tilde{\tau} = \sqrt{-D}/g \in \mathcal{H} \), so that it has CM by \( \sqrt{-D} \). This description must be independent of the choice the primitive element \( \gamma \in I \) taking \( \tau \) to \( \tau_0 \), and indeed replacing \( \gamma \) by a different element \( \delta \) with the same properties and taking the appropriate \( g \) and \( h \) gives the elliptic curve corresponding to an \( SL_2(\mathbb{Z}) \)-image of \( \tilde{\tau} \), which is an elliptic curve isomorphic to \( E_\tau \).

Observe that the equality \( dz_1 = \tau_0 dz_2 \) holds along the CM cycle corresponding to \( \tau_0 \). Hence in the trivial \( B \) case the graphs of the CM isogenies “multiplication by \( \sqrt{-D} \)” from \( E_{N,\tau} \) to \( E_\tau \) and “multiplication by \( N \sqrt{-D} \)” from \( E_\tau \) to \( E_{N,\tau} \) are indeed obtained by taking \( \tau_0 = \sqrt{-D} \in \mathcal{H} \) and \( \tau_0 = \frac{1}{N\sqrt{-D}} \in \overline{\mathcal{H}} \) respectively. Moreover, the generic cycles in the split case can be obtained in a similar manner, since in this description the graph of the generic isogeny \( \psi_L \) corresponds to \( \tau_0 = L \in \mathbb{R} \), and the graph of the isogeny \( \varphi_M \) corresponds to \( \tau_0 = \frac{1}{M} \in \mathbb{P}^1(\mathbb{R}) \). Extending the validity of Equation (20) to some matrices \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of determinant 0 allows us to show that the images of these lines in \( A_\tau \) are cycles, if we define \( M \tau \) for \( \tau \in \mathcal{H} \) (or in \( \mathcal{H} \cup \overline{\mathcal{H}} \)) by the usual formula \( \frac{a \tau + b}{c \tau + d} \). This number is real (or more generally, an element of \( \mathbb{P}^1(\mathbb{R}) \)), and is in fact independent of \( \tau \).

We now calculate the fundamental class of the CM cycle corresponding to \( \tau_0 \) (both in \( \mathcal{H} \) in \( \overline{\mathcal{H}} \)) in \( A_\tau \). We consider the cycle as described in the previous paragraph, and note that as a subset of \( A_\tau \), this cycle is \( \mathbb{C}(\tau_0)/j(\gamma, \tau)(Z(\tau + Z))(\tau) \) with \( \gamma \) and \( \tilde{\tau} \) as defined above (recall the coefficient \( j(\gamma, \tau) \) in Equation (20)).
The isomorphism of this cycle with $E_{\tilde{\tau}}$ is given by the second coordinate divided by $J(\gamma, \tau)$. This observation will soon turn out useful. Along this cycle the relations $dz_1 = \pi_0 dz_2$ and $d\tilde{\tau}_1 = \pi_{10} d\tilde{\tau}_2$ hold, and writing $\tau_0 = x_0 + iy_0$, we see that in $A_+$ these relations are equivalent to $(a \ b) = (c \ d)(x_0 I + y_0 J_\tau)$. As in the split case, we evaluate the integral of $d \wedge c$ over this cycle, and obtain the values of all the other integrals from these relations. Equation (24) then gives the fundamental class of this cycle. When $B$ is split, evaluating this integral is easy. In the general case, this task requires some caution, as we now show.

Recalling that $d \wedge c = \frac{dz_2 \wedge d\tilde{\tau}_2}{2iy_0}$, we evaluate the integral of $dz_2 \wedge d\tilde{\tau}_2$. The isomorphism between our cycle and $E_{\tilde{\tau}}$ implies that the coordinate $z_2$ is $J(\gamma, \tau)$ times the natural coordinate $dz$ on $E_{\tilde{\tau}}$, hence $d\tilde{\tau}_2 = J(\gamma, \tau) dz$. A straightforward evaluation now gives

$$
\int_{(\mathbb{C}(\gamma) \rightarrow A_+)} dz_2 \wedge d\tilde{\tau}_2 = |J(\gamma, \tau)|^2 \int_{E_{\tilde{\tau}}} dz \wedge d\tilde{\tau} = -2i|J(\gamma, \tau)|^2 |\tilde{\tau}| = -2iy_0 \frac{\det \gamma \cdot \Im \tilde{\tau}}{y_0},
$$

where in the last step we have used the equality $y_0 = \frac{\det \gamma y_0}{J(\gamma,\tau)^2}$ (recall that $\tau_0 = \gamma \tau$, and det $\gamma$ is not necessarily 1). The relations between $a$, $b$, $c$, and $d$ and Equation (24) now give the fundamental class of the CM cycle under consideration as

$$
\frac{\det \gamma \cdot \Im \tilde{\tau}}{\text{disc}(I)} \left[ \frac{b \wedge a}{y_0} + \frac{|\tau|^2}{y_0} d \wedge c + \frac{d \wedge b}{y_0}  + \left( \frac{x}{y} - \frac{x_0}{y_0} \right) d \wedge a + \left( \frac{x}{y} + \frac{x_0}{y_0} \right) c \wedge b \right],
$$

which reduces to $\frac{\det \gamma \cdot \Im \tilde{\tau}}{\text{disc}(I)}$ for the product $\det \gamma \cdot \Im \tilde{\tau}$, hence on the choice of $\gamma$. However, choosing an alternative element $\delta$ and the appropriate $g$ and $h$ for the other lattice generator (directed appropriately) gives a matrix $\alpha \in SL_2(\mathbb{Z})$ such that the CM cycle in question now looks like $E_{\alpha \tilde{\tau}}$. Then one verifies that $\delta$ is obtained from $\gamma$ by multiplication by a matrix of determinant $|J(\gamma, \tau)|^2$, so that $\det \delta \cdot \Im \alpha \tilde{\tau} = \det \gamma \cdot \Im \tilde{\tau}$ and our expression for the fundamental class is intrinsic.

The coefficient $\frac{\det \gamma \cdot \Im \tilde{\tau}}{\text{disc}(I)}$ belongs to $\mathbb{Q} \cdot \sqrt{D}$, so by writing it as $\beta \sqrt{D}$ for some $\beta \in \mathbb{Q}$ the fundamental class becomes $\iota(\beta \sqrt{D} J_\tau) + \iota(\beta \sqrt{D} J_\tau)$ (with images of elements of $B$). The sign of $\beta$ equals the sign of $\det \gamma$, which is positive for $\tau_0 \in \mathcal{H}$ and negative if $\tau_0 \in \overline{\mathcal{H}}$. In the latter case we write $J_\tau = -J_{\overline{\tau}}$ (for $\overline{\tau_0} \in \mathcal{H}$) and replace $\beta$ by $|\beta| = -\beta$, so that the fundamental class takes the form $\iota(\beta \sqrt{D} J_{\overline{\tau}}) - \iota(\beta \sqrt{D} J_\tau)$ with $\beta \in \mathbb{Q}$ and positive. The actual value of $\beta$ is probably the minimal rational number such that either the sum or both terms give elements of $H^2(A_+, \mathbb{Z})$, but this point requires further investigation. The normalized class (or the fundamental class of the normalized cycle) is only the $\iota$ part, in which the dependence on the choice of the CM cycle itself appears only via the coefficient $\beta$ (and the sign in the second presentation). This case corresponds to $r = 1$ (without symmetric powers) of Definition 7.4 of [Be] for the CM cycles. The advantage of our presentation is that it shows directly why these classes come from the actual (CM) cycles inside $A_+$. The sign of the normalized cycle contains the information about the “direction” of the line in
C^2, whether its slope is in H or in \overline{H}. The formulae presented above for the fundamental classes of CM isogenies in the split B case are now seen to be the special cases \( \tau_0 = \sqrt{-D} \) and \( \tau_0 = \frac{1}{\sqrt{-D}} \) of this more general construction.

We remark that the generic cycles in the split B case can be obtained by applying a special case of the above argument, but some extra care is required. The graphs of the generic isogenies are obtained by the same construction, but with the line having a real (or infinite) slope. Observe again that cycles with slopes in H correspond to a positive coefficient in the \( i \) part while cycles with slopes in \( \overline{H} \) have a negative coefficient in that part. The observation that the fundamental classes of the generic cycles we considered have no \( i \) part is related, in this point of view, to the fact that \( \mathbb{R} \) lies between \( H \) and \( \overline{H} \). One can check that with real \( \tau_0 \) we can proceed as above, but leaving the expression \( |j(\gamma, \tau)|^2 \) in Equation (24) unchanged (since now both \( y_0 \) and \( \gamma \) vanish). One obtains the same explicit formula, but with the global coefficient \( \frac{|j(\gamma, \tau)|^2}{y\cdot \text{disc}(I)} \) and with every element in the parentheses multiplied by \( y_0 \). In this expression (valid for real as well as non-real \( \tau_0 \)) we now substitute \( y_0 = 0 \), which eliminates the terms which are multiplied by \( y_0 \) (i.e., the \( i \) part), leaving \( |j(\gamma, \tau)|^2 \frac{\gamma}{y\cdot \text{disc}(I)} I(\gamma, \tau) \). Recall that \( \tau_0 = x_0 \) in this case since \( \tau_0 \) is real. The coefficient is independent of the choice of \( \gamma \) as discussed above, and the argument of \( \tau \) here is the limit of \( y_0 J_\tau_0 \) for real \( \tau_0 \). This covers the case of a finite real slope. On the other hand, dividing the argument by \( |\tau_0|^2 \) and multiplying the coefficient by the same factor gives the argument \( \frac{y_0}{|\tau_0|^2} J_\tau_0 \) of \( \tau \) and a coefficient of \( \frac{|\tau_0|^2}{y\cdot \text{disc}(I)} |j(\gamma, \tau)|^2 \). Since both these parts have finite, non-zero limits as \( \frac{y_0}{|\tau_0|^2} \to 0 \), while the resulting argument of \( \tau \) has a vanishing limit, this covers the case of \( \tau_0 \to \infty \) (as well as any non-zero real value). The cases \( \tau_0 = L \) and \( \tau_0 = \frac{1}{M} \) reproduce the previous results for the graphs of the generic isogenies \( \psi_L \) and \( \varphi_M \), by using the matrices \( \gamma = \begin{pmatrix} 0 & L \\ 1 & 0 \end{pmatrix} \) (with \( j(\gamma, \tau) = 1 \) and \( \tau = N\tau \)) for the former and \( \gamma = \begin{pmatrix} 0 & 1 \\ M & 0 \end{pmatrix} \) (with \( j(\gamma, \tau) = M \) and \( \tau = \tau \)) for the latter with \( M \neq 0 \) (and for \( M = 0 \) take the limit). More generally, we obtain such a cycle for every \( \tau_0 \in \mathbb{P}^1(\mathbb{Q}) \) in this case, and every such cycle can be considered as the graph of a correspondence between \( E_\tau \) and \( E_{N\tau} \).

This method of calculating fundamental classes of CM cycles is equivalent to the more direct approach of finding intersection numbers of the cycles themselves. This is so, since the wedge product of differential forms corresponds to the intersection pairing of cycles. Working with the intersection numbers, though appearing more “algebraic”, probably requires choosing \( i \) to be a specific embedding into \( M_2(\mathbb{R}) \) and probing deeper into the details of the structure of the specific quaternion algebra \( B \) in question. As already mentioned, our method avoids this complexity, and has the advantage of working uniformly over all the quaternion algebras.

We have seen that if the Abelian surface \( A_\tau \) with QM has CM from an order in \( \mathbb{K} \) then there are many different CM cycles in \( A_\tau \), one for each choice of \( \tau_0 \in \mathbb{K} \setminus \mathbb{Q} \). Lemma 5.2 implies that for two such cycles, corresponding to \( \tau_0 \) and to another element \( \sigma_0 \in \mathbb{K} \setminus \mathbb{Q} \), there is a generic endomorphism of \( A_\tau \),
given by $\gamma \in \mathcal{M}$, which takes one CM cycle to the other. Hence there is no “canonical” choice of a CM cycle in $A_\tau$. On the level of cohomology, part 5 of Theorem 3.10 of \cite{Be} and Equation (21) show that the action of such $\gamma$ on the fundamental class $i(\beta \sqrt{DJ_0}) + i(\beta \sqrt{DJ_\tau})$ of the CM cycle corresponding to $\tau_0$ (with $\beta$ of any sign) gives $N(\gamma)[i(\beta \sqrt{DJ_0}) + i(\beta \sqrt{DJ_\tau})]$. The restriction of such an endomorphism to the CM cycles themselves is an isogeny of elliptic curves, and the degree of the isogeny depends on the relation between $N(\gamma)$, the rational number $\beta$ corresponding to $\tau_0$, and the one corresponding to $\sigma_0$. We remark that the same assertions extend to the action of endomorphisms of non-zero norm on the generic cycles in the split case (as already mentioned above in the context of polarizations), and, properly interpreted, also to endomorphisms of vanishing norm. We further observe that over $X(\Gamma)$, $A_\tau$ is identified with $A_{M\tau}$ for $M \in \Gamma = R(1)^+_\Gamma$ using the isomorphism which is given by multiplication by the scalar $j(M, \tau)$. This isomorphism preserves the line $\mathbb{C}^2(\tau_0)$ in $\mathbb{C}^2$, hence identifies the CM cycles corresponding to $\tau_0$ in $A_\tau$ and in $A_{M\tau}$ with each other. Therefore the CM cycle corresponding to $\tau_0$ is well-defined in the fiber of $A$ over a CM point in $X(\Gamma)$. At the level of cohomology, part 6 of Theorem 3.10 of \cite{Be} and Equation (21) show that the action of the isomorphism $M \in \Gamma$ takes the class $i(\beta \sqrt{DJ_0}) + i(\beta \sqrt{DJ_\tau})$ in $H^2(A_{M\tau}, \mathbb{Z})$ to $i(\beta \sqrt{DJ_0}) + i(\beta \sqrt{DJ_\tau})$ in $H^2(A_\tau, \mathbb{Z})$ (recall that elements of $\Gamma$ have norm 1). This is the way we consider the fundamental class of the CM cycle in the corresponding fiber over $X(\Gamma)$. This result also holds for the generic cycles in the split $B$ case.

Section 7 of \cite{Be} states that an Abelian surface with QM having also CM structure has models defined over number fields. \cite{Be} uses some abstract tools (such as polarizations, the Poincaré divisor, etc.) in order to show that the CM cycles are algebraic and are defined over the same field as the Abelian surface and its endomorphisms (both the QM and the CM). However, using our construction this result is immediate, and requires no abstract tools: The scalar multiples of $\mathbb{C}^2(\tau_0)$ are precisely those elements of $\mathbb{C}^2$ on which the QM endomorphism $\sqrt{-D}$ from $\mathcal{M} \subseteq B$ acts like the CM endomorphism of scalar multiplication by $\sqrt{-D} \in \mathcal{H}$ from $\mathcal{O} \subseteq \mathbb{K}$. Therefore it is related to the kernel of the endomorphism $\overline{\sqrt{DJ_0}} - \sqrt{-D} \in \mathcal{M} \otimes \mathcal{O} = \text{End}(A_\tau)$. In fact, it is the connected component of 0 of this kernel. If $A_\tau$ and its endomorphisms are defined over a number field $\mathbb{F}$ then so is this kernel, and we claim that the connected component is defined over $\mathbb{F}$ as well. Indeed, any automorphism of $\mathbb{C}/\mathbb{F}$ gives an isomorphism of Abelian varieties from $A_\tau$ to itself and preserves the kernel of the endomorphism in question, but it also takes connected components to connected components and 0 to 0 hence leaves the connected component of 0 invariant (this argument was shown to me by Philipp Habegger). This description also explains why these cycles are defined in the universal Abelian surface over the Shimura curve, and not only in the universal family over $\mathcal{H}$. As $\tau_0$ is arbitrary, these assertions hold for every CM cycle. In the split $B$ case, the graphs of isogenies are algebraic by definition, and a general generic cycle is algebraic as the graph of an algebraic correspondence between elliptic curves. Alternatively, the generic cycles are also obtained as kernels of generic
endomorphisms, hence they are algebraic by the same argument used for the CM cycles.

An Abelian surface \( A \) with QM having CM from an order \( \mathcal{O} \subseteq \mathbb{K} \) must be isogenous to the self-product of an elliptic curve with CM from \( \mathbb{K} \) (see, for example, statement 7.7 of \[\text{[Be]}\]). This observation also follows directly from our construction of the CM cycles. Indeed, each CM cycle is an elliptic curve with CM from \( \mathbb{K} \), and the map from the product of two different CM cycles to \( A \) is an isogeny. As any two elliptic curves with CM from the same field are always isogenous, the claim follows. Moreover, this construction shows that if all these endomorphisms are defined over a number field then so is this isogeny.

### 3.5 Abelian Subvarieties of an Abelian Surface with QM

We have now two kinds of (1-dimensional) Abelian subvarieties of an Abelian surface \( A \) with QM: The split \( B \) case gives rise to generic cycles (graphs of generic isogenies or correspondences), while the CM case (with general \( B \)) introduces CM cycles. We claim that this classification exhausts all the possibilities for Abelian subvarieties of \( A \).

**Theorem 3.3.** (i) If the Abelian surface \( A_\tau \) with QM has also CM then the lines with finite non-real slopes in \( \mathbb{C}^2 \) which give Abelian subvarieties of \( A_\tau \) are exactly the lines spanned by \( (\gamma^0) \) with \( \gamma \in \mathcal{H} \cup \mathcal{H} \) for which there is some \( \gamma \in B \) (non-zero norm) such that \( \gamma \cdot \tau = \gamma \). (ii) If \( A_\tau \) has no CM then no lines of non-real slopes give Abelian subvarieties of \( A_\tau \). (iii) If \( B = M_2(\mathbb{Q}) \) (with the usual i) a line of slope from \( \mathbb{P}^1(\mathbb{R}) \) gives an Abelian subvariety of \( A \) if and only if the slope is in \( \mathbb{P}^1(\mathbb{Q}) \) (i.e., rational or infinite). (iv) If \( B \) is non-trivial then no lines of real (or infinite) slope give Abelian subvarieties of \( A \).

**Proof.** We need to find all slopes \( \gamma \) such that there exist two elements \( \gamma \) and \( \delta \) in \( I \) (or equivalently in \( B \)), which are linearly independent over \( \mathbb{Q} \) (hence also over \( \mathbb{R} \)), and which take \( (1, 0) \) into the line \( \mathbb{C}(\tau) \). Equivalently, by Equation (20) and its extensions we need two such elements of \( B \) which take \( \gamma \cdot \tau \) for any \( \gamma \) taking \( (1, 0) \) to \( \mathbb{C}(\tau) \), the element \( \gamma \cdot \sqrt{D}J_\tau \) does the same and is linearly independent of \( \gamma \). For (ii) we observe that if \( \gamma \) and \( \delta \) take \( \tau \) to \( \tau_0 \in \mathcal{H} \cup \mathcal{H} \) and are linearly independent, then \( \gamma \in B^* \) and \( \gamma^{-1} \delta \in B \setminus \mathbb{Q} \) stabilizes \( \tau_0 \). Hence in this case \( A_\tau \) has CM by property (i) of Proposition 3.1, which proves (ii). Turning our attention to real (infinite) slopes, since \( \gamma_0 = \gamma \tau \) with non-zero \( \gamma \) and \( \gamma_0 \in \mathbb{P}^1(\mathbb{R}) \) imply \( N(\gamma) = 0 \), the fact that a non-trivial \( B \) does not contain such elements implies (iv). For part (iii) we observe that a slope \( \gamma_0 = \frac{L}{M} \in \mathbb{P}^1(\mathbb{Q}) \) with \( L \) and \( M \) (coprime) integers \( \infty = \frac{1}{0} \) gives an Abelian subvariety of \( A \), since the elements \((0 \ 1) \) and \((L \ 0) \) of \( B \) take \( (1, 0) \) to \( (L) \) and \( \gamma \). On the other hand, a matrix \( \gamma \) taking \( \tau \in \mathcal{H} \) to \( \gamma \) must be of the form \( \frac{c \delta}{d} \) with \( c \) and \( d \) not both 0, and such \( \gamma \) can be in \( B = M_2(\mathbb{Q}) \) only if \( \gamma \in \mathbb{Q} \). This proves (iii), and completes the proof of the theorem. \( \square \)
We can rephrase Theorem 3.3 as follows. An Abelian surface with QM from an order in a non-trivial quaternion algebra $B$ which has no CM has no Abelian subvarieties (indeed, it is simple since its endomorphism ring is an order in a division algebra). In an Abelian surface with QM from an order in a non-trivial quaternion algebra $B$ but having CM, the only Abelian subvarieties are the CM cycles. In an Abelian surface with QM from an order in $M_2(Q)$ with no CM, the Abelian subvarieties are the images of lines with slopes from $P^1(Q)$, and are all generic. In an Abelian surface with QM from an order in $M_2(Q)$ and CM from an order in an imaginary quadratic field $K$, the Abelian subvarieties are the images of lines with slopes from $P^1(K)$ (since then $\tau$ is in $K$ and its $M_2(Q)$-images give exactly all the elements of $K \cup \infty$). Those lines with slopes from $P^1(Q)$ are generic, and those lines with slopes from $K \setminus Q$ are CM cycles. This completes the characterization of the Abelian subvarieties in an Abelian surface with QM, and all of them are are algebraic subvarieties which are defined over the field of definition of $A_\tau$ and its ring of endomorphisms.

Theorem 3.3 is related to the well-known result that any Abelian subvariety $B$ of an Abelian variety $A$ admits a “complementary” subvariety $C$ of $A$ such that $A$ is isogenous to $B \times C$ (see, for example, the proof of Proposition 10.1 of [M]). Therefore any Abelian subvariety of $A$ is the connected component of $0$ of the kernel of an endomorphism of $A$ which vanishes on $B$ and embeds the complement $C$ (up to isogeny) back into $A$, and the rank of this endomorphism (as a matrix, for example) is $\dim(A) - \dim(B)$. The subvarieties of $A_\tau$ are thus related to endomorphisms of rank 1 of $A_\tau$, and they are in one-to-one correspondence with the equivalence classes of such endomorphisms under the equivalence relation identifying endomorphisms whose kernel have the same connected component of $0$. In the split $B$ case the generic cycle corresponding to $\tau_0 = \frac{\lambda}{\mu} \in P^1(Q)$ is the connected component of $0$ of the kernel of the rank $1$ generic endomorphism $\left( \lambda M - \lambda L \right)$ for any $\lambda$ and $\mu$ in $\mathbb{Z}$ (which are not both 0). If $A_\tau$ also has CM from an order $O$ in an imaginary quadratic field $K$, then the CM cycle corresponding to $\tau_0 \in K \setminus Q$ is the connected component of $0$ of the kernel of any rank $1$ endomorphism of the form $\left( \lambda M - \lambda L \right)$ with $\lambda$ and $\mu$ in $O$ (again not both 0) such that $\lambda \tau_0$ and $\mu \tau_0$ are also in $O$. For non-trivial $B$, if $A_\tau$ has no CM, then $\text{End}(A_\tau) = \mathcal{M}$ contains no elements of rank 1, and indeed $A_\tau$ contains no Abelian subvarieties in this case. If $A_\tau$ does have CM from $K = \mathbb{Q}(\sqrt{-D})$ then one shows that every rank $1$ endomorphism of $A_\tau$ is of the form $\delta(\sqrt{D}\gamma \tau_0 - \sqrt{-D})$, with a uniquely determined $\tau_0 \in H \cup \overline{H}$ which satisfies $\tau_0 = \gamma \tau$ for some $\gamma \in B$, and with some $\delta \in B$ such that this endomorphism does not vanish. The connected component of $0$ in the kernel of such endomorphism is the CM cycle corresponding to $\tau_0$. In general we may consider the Abelian subvarieties of $A_\tau$ as the $K$-rational points of a $1$-dimensional Brauer-Severi variety over $K = Z(\text{End}(A) \otimes \mathbb{Q})$ (this is $\mathbb{Q}$ in case of no CM and the CM field $K$ if $A_\tau$ has CM), and the Brauer-Severi variety corresponds to $\text{End}(A) \otimes \mathbb{Q}$ as a simple $K$-algebra. Thus, our Theorem 3.3 agrees with the expected result, but it also provides an explicit, set-theoretic description of the Abelian subvarieties in this case of Abelian surfaces with QM.
We call these cycles \((\text{cycles in } A)\) class of such a cycle in \(H\) algebraic as products of algebraic subvarieties. The fundamental cohomology \(H^2\) Hodge structures isomorphic to \((R\text{ of } H\text{ to } V)\) \(\text{Sym}\) namely \(R\) within the variation of Hodge structures \(\text{Sym}\), \(\text{Sym}\) arising from lines with slopes from \(\delta\text{ of } \mathbb{P}^1(\mathbb{Q})\) \(\subseteq \mathbb{P}^1(\mathbb{R})\) if \(\tau \in \delta(K \cap H)\) then the lines with slopes in \(\delta(K \setminus \mathbb{Q})\) yield CM cycles, and in total the lines giving cycles in \(A_\tau\) have slopes from \(\delta(\mathbb{P}^1(K))\).

We now consider symmetric powers of our CM cycles. Let \(A^m\) be the fibered product of \(A\) with itself \(m\) times over \(H\), and let \(\pi_m : A^m \rightarrow H\) be the projection. The \(m\)th power of a CM cycle \(C \subseteq A_\tau\) is the \(m\)-cycle in \(A^m_\tau\) defined by

\[
C^m = \{(z_1, \ldots, z_m) \in A^m \mid z_j \in C, 1 \leq j \leq m\} \subseteq A^m_\tau.
\]

We call these cycles \((m\text{-codimensional})\) CM cycles in \(A^m_\tau\). They are vertical cycles in \(A^m\) which lie over the same CM points in \(H\). Moreover, they are algebraic as products of algebraic subvarieties. The fundamental cohomology class of such a cycle in \(H^2(A^m_\tau)\) lies in the Künneth component \(H^2(A_\tau)\otimes^m\) of \(H^2(A^m_\tau)\), and more precisely in the \(S_m\)-invariant part of this component, namely \(\text{Sym}^m H^2(A_\tau)\). Turning to variations of Hodge structures, we find that within the variation of Hodge structures \(R^2\pi_m^*\mathbb{C}\) over \(H\) lies a variation of Hodge structures isomorphic to \((R^2\pi_*\mathbb{C})\otimes^m\), which contains a part isomorphic to \(V_2^\otimes^m\), and we shall be interested in its symmetric part \(\text{Sym}^m V_2\). We treat the normalized fundamental classes of the \(m\)-dimensional CM cycles as elements of \(\text{Sym}^m V_2\). There exists a map, defined over \(\mathbb{Q}\), from \(\text{Sym}^m V_2\) to \(\text{Sym}^{m-2} V_2\), whose kernel is a variation of Hodge structures isomorphic to \(V_{2m}\). The projection onto \(V_{2m}\) is the map denoted \(P\) in Section 5 [Be], and the CM cycles defined there are the images of the fundamental classes of our \(m\)-dimensional CM under \(P\). We note that related modularity results, with the local system \(V_{2m}\) (but using other means like the Shintani and Kudla–Millson lifts), are presented, for example, in [FM1] and [FM2]. However, our result applies for the full local system \(\text{Sym}^m V_2\). We also note that in the split \(B\) case with \(\tau_0 = \sqrt{-D}\) (a CM isogeny), these \(m\)-dimensional cycles resemble those cycles defined in page 123 of [Zh] (for \(m = k - 1\)). However, [Zh] considers the action of \(S_{2m}\) while here we only have the action of \(S_m\). Taking the images of the fundamental classes in the quotient by the action of the full group \(S_{2m}\) (rather than of \(S_m\), which acts trivially on these cycles) is equivalent to the action of the projector \(P\). The fact that the Grassmannian \(G(L_\mathbb{R})\) has a moduli interpretation also for other values of \(b\) suggests that we can apply the same considerations (including the symmetric powers) to the corresponding universal families. It would be interesting to investigate if there are maps corresponding to \(P\) for this setting.
4 Relations between CM Cycles

We now combine the two interpretations of $H$ that we have, i.e., as a Grassmannian of signature $(2, 1)$ and as the base space of universal families (or variations of Hodge structures). We take the lattice $L$ to be $\tilde{\Lambda}$ from the $\tilde{\iota}$ part of the integral cohomology $H^2(A, \mathbb{Z})$ with $A = \mathbb{C}^2/i(I) \mathbb{Z}$ for $I \subseteq B$ and $i$ as above. As the cup product of the $\tilde{\iota}$-images of two elements $b$ and $c$ in this lattice is $\text{disc}(I) \cdot Tr(bc)$, the isometry taking $L_\mathbb{R}$ into $M_2(\mathbb{R})_0$ is not $i$ but $i$ divided by $\sqrt{\text{disc}(I)}$. As $J_v$ spans $v_{-}$ over $v \in G(L_\mathbb{R}) \cong H$, condition $(ii)$ of Proposition 3.1 can be translated as some element of $L$ (or $L^*$, or $L_\mathbb{Q} = B_0$) being in the negative definite space $v_{-}$ corresponding to $v$. Therefore the CM points characterized in Proposition 3.1 are precisely the points $v = \lambda^\perp$ for some $\lambda \in L$ with negative norm. Borcherds used this observation in [B2] to show that the meromorphic automorphic forms from Theorem 13.3 of [B1] have divisors based on CM points, and the relations between the CM (or Heegner) cycles stated in [GKZ] follow. For our application we shall use the fact that the automorphic forms constructed in our Theorem 2.7, and the meromorphic automorphic forms from Theorem 2.8, have singularities (poles) at these points as well. The method follows [B2] rather closely, with some specific adjustments to suit our purposes.

In view of the fact that higher-dimensional Grassmannians (at least for the cases $b_+ = 2$ and $b_+ = 3$) also admit interpretations as base spaces for universal families or variations of Hodge structures, we phrase and prove our result in the full generality of the main theorem of [B2]. For this purpose we use strong principality (see Definition 4.1 below), even though for $b_+ = 1$ (the case of modular and Shimura curves considered here), Definition 4.2 for principality is more natural. As we remarked after Theorem 2.8 it is possible that images of our function $\Phi_{L,m,m,0}^m$ under certain operators become meromorphic automorphic forms on $G(L_\mathbb{R})$ also for other values of $b_+$. If this is indeed the case, then one can replace Definition 4.1 by a definition of principality suitable for applications for these cases.

4.1 $m$-Divisors, Heegner $m$-Divisors, and their Classes

We now define the objects for which we prove our result. Let $L$ be an even lattice of signature $(2, b_+)$, and let $G(L_\mathbb{R})$ be its positive Grassmannian. An $m$-divisor on $G(L_\mathbb{R})$ is a locally finite sum of elements in $\text{Sym}^m L_\mathbb{C} \otimes \text{Div}(G(L_\mathbb{R}))$, namely of expressions of the sort

$$\left( \prod_{i=1}^m \lambda_i \right) \otimes Y$$

where $\lambda_i \in L_\mathbb{C}$ (and $\prod_{i=1}^m \lambda_i$ is the corresponding element of $\text{Sym}^m L_\mathbb{C}$) and $Y$ is an irreducible divisor on $G(L_\mathbb{R})$. The group $O(L_\mathbb{R})$ acts on both $G(L_\mathbb{R})$ and
$\text{Sym}^m L_{\mathbb{C}}$, and for a discrete subgroup $\Gamma$ of $O(L_{\mathbb{R}})$, we define an $m$-divisor on the quotient $X(\Gamma) = \Gamma \backslash G(L_{\mathbb{R}})$ to be just a $\Gamma$-invariant $m$-divisor on $G(L_{\mathbb{R}})$. An $m$-divisor on $G(L_{\mathbb{R}})$ (or on $X(\Gamma)$) is of totally negative type if it satisfies the following condition: For every term of the form appearing in Equation (26), each $\lambda_i$ lies in (the complexification of) the space $v_-$ corresponding to every point $v \in Y$. This condition is well-defined, and is equivalent to every term being of the form $a_{\lambda} \lambda^m \otimes \lambda^\perp$ for some negative norm $\lambda \in M_{\mathbb{R}}$, with $a_{\lambda} \in \mathbb{C}$.

The heart of the matter here is to define the “correct” equivalence relation among these $m$-divisors. In order to do so, we first need to introduce a certain property of automorphic forms on $G(L_{\mathbb{R}})$. Let $\Phi$ be an automorphic form of weight $m$ on $G(L_{\mathbb{R}}) \cong K_{\mathbb{R}} + i \mathbb{C}$ which is an eigenform of $\Delta$ with eigenvalue $-2mb_-$, and assume first that the lattice $L$ contains a primitive norm zero vector $z$. The expansion of $\Phi$ around $z$ in a Weyl chamber $W$ containing $z$ in its closure is of the form

$$\tilde{\varphi}(Y) + \sum_{k=0}^{m} \sum_{C=0}^{k} \sum_{\rho \in K^*} \frac{4\pi^{k+C}}{(Y^2)^k} \left\{ \begin{array}{ll} e((\rho, Z)) & (\rho, W) > 0 \\ e((\rho, \overline{Z})) & (\rho, W) < 0. \end{array} \right.$$  

(as in Theorem 2.7). We call $\Phi$ algebraic if the coefficients $A_{k,C,\rho}$ are algebraic over $\mathbb{Q}$. If $L$ contains no norm zero vectors, then as in Section 8 of [B1], we can embed $L$ in two different lattices $M$ and $N$ of signature $(2, b_- + 24)$, which gives embeddings of $G(L_{\mathbb{R}})$ into the larger Grassmannians $G(M_{\mathbb{R}})$ and $G(N_{\mathbb{R}})$. Assume that we can present $\Phi$ as the difference of weight $m$ automorphic forms on $G(M_{\mathbb{R}})$ and $G(N_{\mathbb{R}})$, which are eigenforms with eigenvalue $-2mb_-$ of the corresponding Laplacians, minus their singularities. We call $\Phi$ algebraic if it is obtained in this way from algebraic automorphic forms on the larger Grassmannians. We note again that one has to apply the argument carefully in this case—see the last remark following Theorem 2.7 above. In the case $b_-=1$, we call a meromorphic automorphic (modular) form of weight $2m+2$ with respect to a subgroup $\Gamma$ of $SL_2(\mathbb{R})$ having $\infty$ as a cusp algebraic if its expansion around $\infty$ involves Fourier coefficients which are algebraic over $\mathbb{Q}$. This definition agrees with the classical $q$-expansion principle, and such modular forms are meromorphic sections of the line bundle $\Omega_{X(\mathbb{C})}^{\otimes (m+1)}$ which are defined over the algebraic closure of $\mathbb{Q}$. If $\Gamma$ has no cusps then we use the algebraicity of meromorphic sections of $\Omega_{X(\mathbb{C})}^{\otimes (m+1)}$ in order to characterize the algebraic meromorphic modular forms. Given an algebraic automorphic form $\Phi$ of orthogonal weight $m$ (hence modular weight $2m$) and eigenvalue $-2m$ on $G(L_{\mathbb{R}}) \cong \mathcal{H}$, whose algebraicity is characterized using embeddings into larger lattices as described above, it is likely that the meromorphic modular form $\delta_{2m} \Phi$ is an algebraic modular form of weight $2m+2$.

Let $\Gamma$ be a discrete subgroup of $O^+(L_{\mathbb{R}})$ (which we take in our applications to be the part $\Gamma^+_0$ of the discriminant kernel of $SO^+(L)$). The general definition is

**Definition 4.1.** The $m$-divisor of totally negative type on $X(\Gamma)$, specified as $\sum_{\lambda \in L_{\mathbb{R}}, \lambda^2 < 0} a_{\lambda} \lambda^m \otimes \lambda^\perp$ (a locally finite, $\Gamma$-invariant sum) is called strongly
principal if there exists an algebraic automorphic form of weight \( m \) on \( G(\mathcal{L}) \) whose singularities lie on the divisors \( \lambda \), and along each such \( \lambda \) the singularity is

\[
\frac{a_{\lambda}}{2} \left( \frac{i}{2\pi} \right)^m \prod_{r=0}^{m-1} \left( r + \frac{b_-}{2} \right) \cdot \frac{(\lambda, Z_{v,V})^m}{2^m(Y^2)^m} \cdot - \ln \frac{|(\lambda, Z_{v,V})|^2}{Y^2} + \\
+ \sum_{k=0}^{m-1} \frac{m! \left( \frac{\lambda^2}{2} \right)^{m-k}}{2^k k!} \prod_{r=0}^{k-1} \left( r + \frac{b_-}{2} \right) \cdot \frac{(\lambda, Z_{v,V})^k}{(\lambda, Z_{v,V})^{m-k}(Y^2)^k} \cdot \frac{1}{m-k}.
\]

If \( b_- = 1 \) we adopt the alternative definition

**Definition 4.2.** We call the \( m \)-divisor of totally negative type on \( X(\Gamma) \) defined by \( \sum_{\lambda \in \mathcal{L}^L, \lambda^2 < 0} a_{\lambda} \lambda^m \otimes \lambda^\perp \) (again a locally finite, \( \Gamma \)-invariant sum) principal if there exists an algebraic modular form of weight \( 2m + 2 \) on \( G(\mathcal{L}) \cong \mathcal{H} \) whose poles are at the points \( \lambda \), and at each point \( \lambda = \sigma \) the singularity is

\[
\frac{i}{(4\pi)^{m+1}} \sum_{\alpha} \frac{m!(2i)^{m+1}}{(\tau - \sigma)^{m+1}}
\]

We remark that while the (principal) divisor of a rational function depends also on the zeros of the function, our (strongly) principal \( m \)-divisors involve only the singularities of the function. Our definitions agree with the classical definition, since for the case \( m = 0 \) considered in [B2], Definition 4.1 is based on the singularities of the theta lift \( \Phi \), which is roughly the logarithm of the absolute value of the Borcherds product \( \Psi \). Moreover, in the case \( b_- = 1 \) Definition 4.2 considers the singularities of \( \delta_0 \Phi = \frac{a}{2\pi \mathcal{G}} \), which is (under some technical assumptions) the logarithmic derivative \( \Psi' / \Psi \) of \( \Psi \) (see also the remarks at the end of this Section).

The concepts of principal and strongly principal \( m \)-divisors in Definitions 4.1 and 4.2 are well-defined in the following sense. In a term of the form \( a_{\lambda} \lambda^m \otimes \lambda^\perp \) we can substitute \( \lambda = x\mu \) for some real \( x \neq 0 \), making this term equal \( a_{\mu} \mu^m \otimes \mu^\perp \) with \( a_{\mu} = x^m a_{\lambda} \). Now, \( (\frac{\lambda^2}{2})^{-m-k} \cdot \frac{(\lambda, Z_{v,V})^k}{(\lambda, Z_{v,V})^{-m-k}} \) is the same as \( x^m (\frac{\mu^2}{2})^{-m-k} \cdot \frac{(\mu, Z_{v,V})^k}{(\mu, Z_{v,V})^{-m-k}} \) for every \( 0 \leq k \leq m \) in this case (and the difference in the logarithm gives a smooth function of \( Z \)). Moreover, if \( b_- = 1 \) then \( \lambda = \alpha J_\sigma \) for some \( \alpha \), hence \( \mu = \frac{\alpha}{2} J_\sigma \) and we write \( \alpha^m = x^m (\frac{\alpha^2}{2})^m \). It follows from the relation between \( a_{\lambda} \) and \( a_{\mu} \) that both Definitions 4.1 and 4.2 are independent of this presentation, hence are well-defined. We further remark that the algebraicity constraint corresponds to the requirement that the weakly holomorphic modular form appearing in Theorem 13.3 of [B1] must have integral Fourier coefficients with negative indices.

Let \( y_n^{(m)} \) be the \( m \)-divisor \( \sum_{\lambda \in \mathcal{L}^L, \lambda^2 = -2n} \lambda^m \otimes \lambda^\perp \) for any \( \gamma \in M^*/M \) and positive \( n \in \mathbb{Q} \). This \( m \)-divisor is non-trivial on \( X(\Gamma) \) only if \( n \equiv \gamma^2/2 \mod \mathbb{Z} \). Moreover, if \( \gamma \) is of order 2 in \( L^*/L \) and \( m \) is odd then \( y_n^{(m)} = 0 \), since the
contributions of \( \lambda \) and \(-\lambda \) cancel. Hence if \( L^*/L \) has exponent 2 (or 1) then there are non-zero \( m \)-divisors \( y^{(m)}_{n,\gamma} \) only for even \( m \). Let \( \text{Heeg}^{(m)}(X(\Gamma)) \) be the free Abelian group of \( m \)-divisors which is generated by the \( m \)-divisors \( y^{(m)}_{n,\gamma} \). An element of \( \text{Heeg}^{(m)}(X(\Gamma)) \) is called a Heegner \( m \)-divisor. The Abelian group \( \text{PrinHeeg}^{(m)}_{st}(X(\Gamma)) \) is the subspace of \( \text{Heeg}^{(m)}(X(\Gamma)) \) consisting of those Heegner \( m \)-divisors which are strongly principal, and if \( b_- = 1 \) we denote the group of principal Heegner \( m \)-divisors by \( \text{PrinHeeg}^{(m)}(X(\Gamma)) \). The spaces \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \) and \( \text{Heeg}^{(m)}Cl(X(\Gamma)) \), whose elements are called (strong) Heegner \( m \)-divisor classes on \( X(\Gamma) \), are the corresponding quotient groups. The image of \( y^{(m)}_{n,\gamma} \) in \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \), and for \( b_- = 1 \) also in \( \text{Heeg}^{(m)}Cl(X(\Gamma)) \), will be denoted \( y^{(m)}_{n,\gamma} \) as well.

We can now state and prove the main result of this paper.

**Theorem 4.3.** The formal power series \( \sum_{n,\gamma} y^{(m)}_{n,\gamma} q^n e^{\gamma} \) is a modular form of weight \( 1 + b_-/2 + m \) and representation \( \rho^*_L \) with respect to \( Mp_2(\mathbb{Z}) \), with coefficients in the finite-dimensional space \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \otimes \mathbb{Q} \).

**Proof.** The proof follows [B2]. Consider the space of weight \( k \) modular forms with representation \( \rho^*_L \) (as a finite-dimensional subspace of the space of power series) and the space of singular parts of weakly holomorphic modular forms of weight \( 2 - k \) and representation \( \rho_L \) (which is a subspace of finite codimension of the space of singular parts). By Theorem 3.1 of [B2], these two spaces are the full perpendicular spaces of one another for any half-integral \( k \). Let \( f \) be a weakly holomorphic modular form of weight \( 1 - b_-/2 - m \) and representation \( \rho_L \) with respect to \( Mp_2(\mathbb{Z}) \), and expand \( f(\tau) \) as \( \sum_{\gamma \in M^*/M} \sum_{n \in \mathbb{Z}} c_{\gamma, n} q^n e^{\gamma} \). Then by Theorem 2.7 we find that the singularities of the theta lift \( \Phi_{L,m,m,0}(v, F) \) of \( F = \delta^m \sum_{1 - b_-/2 - m} f \) are along the divisors \( \lambda^\perp \) with \( \lambda \in L^* \), where every \( \lambda \in L^* \) contributes a singularity as in Definition 4.1 with \( a_{\lambda} = c_{\lambda, \lambda^\perp} \) along \( \lambda^\perp \). If we assume that \( f \) has algebraic Fourier coefficients, then it follows from the description of the coefficients \( A_{k,C,\rho} \) that the automorphic form \( \sum_{v,F} \Phi_{L,m,m,0}(v, F) \) is algebraic. Assuming that the Fourier coefficients \( c_{n,\gamma} \) with \( n < 0 \) are integral, this implies that the \( m \)-divisor \( \sum_{n > 0, \gamma} c_{-n,\gamma} y^{(m)}_{n,\gamma} \) is strongly principal, hence it vanishes in \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \). Let \( \xi^{(m)} \) be the map which takes a “singular part” \( \sum_{n < 0, \gamma} a_{\gamma,n} q^n e^{\gamma} \) to the element \( \sum_{n > 0, \gamma} a_{\gamma,-n} y^{(m)}_{n,\gamma} \) in the class group \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \). The map \( \xi^{(m)} \) is surjective, and since the space of weakly holomorphic modular forms of any weight and representation \( \rho_L \) has a basis consisting of modular forms with integral Fourier coefficients (see [McG]), \( \xi^{(m)} \) factors through the quotient of the space of singular parts by singular parts of weakly holomorphic modular forms. By Lemmas 4.2, 4.3, and 4.4 of [B2] this quotient space is finite-dimensional. Hence \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \) is finite-dimensional as well. We can thus tensor the space of singular parts with \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \), and extend the Serre duality pairing to take an element from this tensor product and a power series to an element of \( \text{Heeg}^{(m)}_{st}Cl(X(\Gamma)) \). This
forms with rational Fourier coefficients, hence
\[ \sum \text{modular form with coefficients in } \text{Heeg} \]
rem. in the case \( b \)
Whether or not the notions of principality and strong principality are equivalent
pairing is algebraic over \( Q \)
generating series of the Heegner
does not show up, so that our expression is in fact a “cusp form”.
form the appropriate principal part corresponds to a weakly holomorphic modular
representation \( \rho \)
L is (hence also \( \beta \)) in \([K]\) (here \( E_r \) is the function introduced in \([CG]\), and \( H_{2r} \) is normalized
to attain 1 at \( \infty \) and to have integer coefficients), let \( E_r \) be the classical (nor-
extended pairing continues to be non-degenerate.
Now, for an algebraic modular form \( f \) as above, the vanishing expression
\[ \sum_{n>0,\gamma} c_{-n,\gamma} y_{n,\gamma}^{(m)} \] is obtained as the Serre duality pairing of the formal power series \( \sum_{n,\gamma} y_{n,\gamma}^{(m)} q^n e_\gamma \) with the singular part of \( f \). Hence this power series is perpendicular to all the singular parts of weakly holomorphic meromorphic modular form of weight \( 1 - b_-/2 - m \) and representation \( \rho_L \) with respect to \( Mp_2(\mathbb{Z}) \). By
Lemma 4.3 of \([B2]\) and the result of \([McG]\), the formal power series in question lies in the space of (holomorphic) modular forms of weight \( 1 + b_-/2 + m \) and representation \( \rho_L^* \) tensored with \( \text{Heeg}^{(m)}_{st}(X_\Gamma) \otimes \mathbb{C} \). Finally, the Serre duality pairing is algebraic over \( \mathbb{Q} \) and we use spaces which are spanned by modular forms with rational Fourier coefficients, hence \( \sum_{n,\gamma} y_{n,\gamma}^{(m)} q^n e_\gamma \) is an “algebraic” modular form with coefficients in \( \text{Heeg}^{(m)}(X_\Gamma) \otimes \mathbb{Q} \). This proves the theorem.

In the case \( b_- = 1 \) the power series of Theorem 4.3 is a modular form with coefficients in \( \text{Heeg}^{(m)}(X(\Gamma)) \otimes \mathbb{Q} \) (with no subscript \( st \)). One way to prove this assertion is to replace Definition 4.1 by Definition 4.2 in the proof of Theorem 4.3. Another way is to observe that every strongly principal Heegner \( m \)-divisor is principal. This is so, since if \( \Phi \) is an automorphic form (which shows that some \( m \)-divisor is strongly principal) then \( \delta_{2m} \Phi \) reveals the principality of this \( m \)-divisor. Hence \( \text{Heeg}^{(m)}(X(\Gamma)) \otimes \mathbb{Q} \) is a quotient group of \( \text{Heeg}^{(m)}_{st}(X(\Gamma)) \otimes \mathbb{Q} \), and the assertion follows directly from Theorem 4.3. Whether or not the notions of principality and strong principality are equivalent in the case \( b_- = 1 \) remains an interesting question.

We also remark that unlike the result of \([B2]\), here the “constant term” \( y_{0,0} \) does not show up, so that our expression is in fact a “cusp form”.

Let us give an example of an automorphic form yielding the principality of a Heegner \( m \)-divisor, which also illustrates Corollary 2.9 above. Take \( N = 1 \) (hence also \( \beta = 1 \)) in the example preceding Corollary 2.9, namely the lattice \( L \) is \( M_2(\mathbb{Z})_0 \subseteq M_2(\mathbb{R})_0 \), and let \( m \) be even. Theorem 4.3 implies that the generating series of the Heegner \( m \)-divisors is a cusp form of weight \( \frac{3}{2} + m \) and representation \( \rho_L^* \).

Equivalently, it lies in the Kohnen Plus-space of cusp forms of weight \( \frac{3}{2} + m \) on \( \Gamma_0(4) \) (see \([EZ]\) or \([K]\)), which is isomorphic to the direct sum of cusp forms of weight \( m - 2 \) of \( SL_2(\mathbb{Z}) \) and cusp forms of weight \( m - 4 \) of \( SL_2(\mathbb{Z}) \) by Proposition 1 of \([K]\). Hence this series is non-zero only for \( m \geq 8 \).

Take \( m = 2 \), and we present the function yielding the principality \( y^{(2)}_{4,1} \). The weight \( -\frac{3}{2} \) weakly holomorphic modular form \( f \) with representation \( \rho_L \) having the appropriate principal part corresponds to a weakly holomorphic modular form \( g \) in the Kohnen Plus-space of weight \( -\frac{3}{2} \) which is of the form \( q^{-3} + O(1) \). Define
\[ \theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad H_{\frac{3}{2}}(\tau) = 120 \sum_{n} H(2, n) q^n \]
as in \([K]\) (here \( H(2, n) \) is the function introduced in \([CG]\), and \( H_{\frac{3}{2}} \) is normalized to attain 1 at \( \infty \) and to have integer coefficients), let \( E_r \) be the classical (nor-
malized) Eisenstein series of even weight $r \geq 4$ on $SL_2(\mathbb{Z})$, and let $\Delta$ be the classical cusp form of weight 12. The function
\[
g(\tau) = \frac{E_{10}(4\tau)\theta(\tau) - E_8(4\tau)H_2(\tau)}{\Delta(4\tau)} = q^{-3} - 56 + 384q - 15024q^4 + 39933q^5 - 523584q^8 + 1129856q^9 + O(q^{12})
\]
has the desired principal part in the Kohnen Plus-space, and let $f$ be the corresponding weakly holomorphic modular form with representation $\rho_L$. The lift $\frac{l^2}{T^2}\delta_4\Phi_{L,2,0}(v,F)$ of $F = \delta^2_T f$ is given by the Fourier expansion
\[
384q - 479232q^2 + 274558464q^3 - 118219210752q^4 + 43867326009600q^5 + O(q^6)
\]
around $\infty$, and it is a modular form of weight 6 with respect to $SL_2(\mathbb{Z})$ which has a pole of the form $\frac{18\sqrt{3}/(4\pi)^3}{(\tau - \sigma)(\tau - \bar{\sigma})^2}$ at $\sigma$ the 3rd root of unity in $\mathcal{H}$ (recall that the Fourier coefficient is 1 and $\alpha = 2t = \sqrt{3}$). These properties determine $\frac{l^2}{T^2}\delta_4\Phi_{L,2,0}(v,F)$ as $\frac{384E_6\Delta}{T^4}$, a fact which can be verified directly by explicit evaluation of the Fourier coefficients.

### 4.2 Harmonic Weak Maass Forms

We now explain why the condition of algebraicity of the automorphic form is required in the definition of principal $m$-divisors (both in Definition 411 and in Definition 412). One of the main properties of the automorphic form $\frac{l^m}{T^m}\Phi_{L,m,m,0}(v,F)$ in Theorem 2.7 is its eigenvalue under the action of the Laplacian $\Delta^G_m$, which follows from the fact that $F = \delta^m_{1-b_{-m}} f$ has a specific eigenvalue under the weight $1 - \frac{b_{-}}{T} + m$ Laplacian on $\mathcal{H}$. This, in turn, is due to the observation that the initial modular form $f$ is weakly holomorphic hence harmonic. Thus, by taking $f$ to be any harmonic weak Maass form, the theta lift $\frac{l^m}{T^m}\Phi_{L,m,m,0}(v,F)$ for $F = \delta^m_{1-b_{-m}} f$ has the same eigenvalue (hence in the case $b_{-} = 1$ its image under $\delta_{2m}$ is again meromorphic). We now present briefly the properties of $\frac{l^m}{T^m}\Phi_{L,m,m,0}$ in this case, and explain why a “careless” definition of the principal $m$-divisors leads to trivial results.

Section 3 of [BF] shows that every harmonic weak Maass form $f$ of weight $l$ and representation $\rho_L$ with respect to $Mp_2(\mathbb{Z})$ decomposes as the sum of a holomorphic part and a non-holomorphic part. The holomorphic part has an ordinary Fourier expansion around the cusp $\infty$ (with coefficients $c_{\gamma,n}$), and the non-holomorphic part consists of the “constant terms” $a_{\gamma,n}y^{1-l}$ for isotropic elements $\gamma \in L^*/L$, and the other terms are of the form $a_nq^n\Gamma(1-l,-4\pi ny)$ where the “incomplete Gamma function” $\Gamma(1-l,-4\pi ny)$ is defined as $\int_{4\pi ny}^{\infty} e^{-t}t^{-l}dt$. Consider now the function $F = \delta^m_T f$. The part of $F$ coming from the holomorphic part of $f$ looks like the expression in Equation (12), and the image of the constant term $a_{\gamma,n}y^{1-l}$ is some multiple of $y^{1-l-m}$ (which is $y^{b_{-l}}$ for $l = 1 - \frac{b_{-}}{T} - m$). The other part is the sum of two expressions: One expression
Indeed, for such a representation $\rho_L$ with respect to $Mp_2(\mathbb{Z})$, is $\delta^m f$ for some harmonic Maass form $f$. Moreover, for $l < 1 - m$ every modular form $F$ of weight $l + 2m$ and representation $\rho_L$ is not holomorphic at the cusp. Then, the singularity arises only from the holomorphic part of $F$. It remains to consider the lift of $F = \delta^m_{1-\frac{b}{m}} f$ in the case where $\xi_{1-\frac{b}{m}} f$ is holomorphic at the cusp. Then, the singularity arises only from the holomorphic part of $F$, which also yields a Fourier expansion as in Theorem 2.7 around a norm zero vector $z \in \mathbb{L}$ (if it exists). The non-holomorphic “constant terms” $a_{\gamma,0} b^{\lambda-1}$ (with $\gamma$ isotropic) of $f$ changes the form of the function $\varphi(Y)$. The contribution of the other non-holomorphic parts of $f$ to the integral in Theorem 7.1 of [B1] can be evaluated as values of Bessel $K$-functions at $2\pi n|Y| \cdot |\eta_{w-}|$, and $|\eta_{w-}| = \sqrt{(\eta_1 Y)^2 - 2 Y^2 \eta^2} / |Y| > 0$ (recall that only elements with $\eta^2 \leq 0$ contribute, by the condition on $\xi_{1-\frac{b}{m}} f$). It follows that there is no contribution to the coefficients of expressions of the form $e((\rho, Z))$ or $e((\rho, \overline{Z}))$. Moreover, the fact that only nonzero $\rho \in K^*$ with $\rho^2 \leq 0$ contribute to the “generalized Fourier expansion” implies that in the case $b_- = 1$ only the holomorphic part of $f$ is visible in the theta lift (since $K$ is positive definite). It is also worth noting that for $b_- = 2$, with $(\rho, Z) = \rho_1 \sigma + \rho_2 \tau$, the contribution of the non-holomorphic part of $f$ gives rise to expressions in which the exponents are $e(\rho_1 \sigma + \rho_2 \tau)$ or $e(\rho_1 \overline{\sigma} + \rho_2 \tau)$ (according to the sign of $\rho$). These expressions may be related to the class map defined in Section 5 of [Bru], since the operator $\partial \overline{\partial}$ applied in that reference is related to $\partial, \partial \overline{\sigma} + \partial_{\sigma} \overline{\sigma}$.  

\[ - \sum_{\gamma, n \neq 0} a_{n, \gamma} \sum_{k=0}^{m-1} \frac{n^m e(n \tau)}{(-4\pi n y)^{k+1}} \sum_{h=0}^{m-1} \prod_{l=0}^{k} \frac{(m-1-k+h)!}{h!} (m-1-k)! \]
Bearing this in mind, one sees why the algebraicity condition in Definitions 4.1 and 4.2 is necessary, especially for the case $b_− = 1$: Proposition 3.11 of [BF] shows that any principal part at $∞$ can be obtained as the principal part of a harmonic weak Maass form of every weight. Hence using general automorphic forms with singularities yields all possible relations, so that the (strong) Heegner $m$-divisor class group reduces to 0. For $b_− > 1$ one may distinguish the theta lifts arising from weakly holomorphic modular forms by the fact that their Fourier expansions involve only terms in which the exponents are $e\left(\rho, Z\right)$ or $e\left(\rho, \sqrt{\eta} Y\right)$, while the other theta lifts include also terms of the form $e\left(\rho, X + \frac{\sqrt{(\eta, Y)^2 - \eta^2 Y^2}}{Y}\right)$. However, for $b_− = 1$, where the contribution of the non-holomorphic parts of harmonic weak Maass forms is not visible in the lift, this argument does not hold. This is the stage where we use algebraicity: The (algebraic) weakly holomorphic modular forms give rise to algebraic theta lifts, while for harmonic weak Maass forms we pose the following

**Conjecture 4.4.** Let $f$ be a harmonic weak Maass form of weight $\frac{1}{2} - m$ (with $m > 0$) and representation $\rho_L$, and assume that $\xi_{\frac{1}{2} - m} f$ is holomorphic also at $∞$. Assume further that the principal part of $f$ is algebraic (i.e., involves only Fourier coefficients which are algebraic over $\mathbb{Q}$), or even integral. Then if $\xi_{\frac{1}{2} - m} f \neq 0$ (i.e., $f$ is not weakly holomorphic) then there is some Fourier coefficient of the holomorphic part of $f$ which is transcendental over $\mathbb{Q}$.

Conjecture 4.4 is supported by Corollary 1.4 of [BO], for example, as well as by the conjecture preceding it in that reference. Indeed, in the weight $\frac{1}{2}$ considered there one has certain theta series of weight $\frac{3}{2}$ which map by the Shimura correspondence to Eisenstein series of weight 2. Harmonic weak Maass forms which map to these forms under the operator $\xi_{\frac{1}{2}}$ are known to have algebraic Fourier coefficients. Since in higher half-integral weights the Shimura correspondence takes cusp forms to cusp forms, we expect in Conjecture 4.4 only the weakly holomorphic modular forms to have algebraic coefficients. In any case, it will be interesting to find, under Conjecture 4.4 whether every relation between the Heegner $m$-divisors in $H_{ee}^{(m)} Cl\left(X(\Gamma)\right)$ (or in $H_{ee}^{(m)} Cl\left(X(\Gamma)\right)$ for $b_− = 1$) arises from a weakly holomorphic modular form as we described.

### 4.3 Relations to Other Works and Possible Interpretations

A result related to the case $b_− = 1$ for modular curves is given in [Zh], and states that the images of $m$-codimensional CM cycles in the Kuga–Sato variety $W_{2m}$ over modular curves in a certain vector space are related to a Hecke submodule of a certain vector-valued Hecke module of modular forms of weight $2m + 2$ (see Theorem 0.3.1 of that reference, with $k = m + 1$). Theorem 0.2.1 of [Zh] points to the direction that these cycles are coefficients of a modular form of weight $2m + 2$ in this vector space. Applying the Shimura correspondence to the modular form of weight $\frac{3}{2} + m$ obtained from our Theorem 4.3 gives a (vector-valued) modular form of weight $2m + 2$, which may be related to the modular form of [Zh].
Furthermore, \( H \) defines, for any \( m \), a map on the Heegner divisors on \( X_0(N) \) into some elliptic curve, and conjectures that the images of these divisors under this map correspond to the coefficients of a modular form of weight \( 2m + 2 \). In some cases this map coincides with the Abel-Jacobi map on the CM cycles into a certain sub-torus of the intermediate Jacobian of the Kuga–Sato variety \( W_{2m} \), and \( H \) supplies numerical evidence that this is true in some other cases as well.

Let us indicate how our results may be related to those mentioned in the previous paragraph. This is also related to the question, which algebro-geometric object is obtained by division by our principal \( m \)-divisors. In the case \( m = 0 \) considered by Borcherds, the weight 1 automorphic form \( \delta_0 \frac{1}{2} \Phi_{L,0,0,0} \), which is meromorphic only if \( c_{0,0} = 0 \), is roughly the logarithmic derivative of the meromorphic automorphic function \( \Psi \) (the Borcherds product). Hence it is (under some normalization) a meromorphic modular form of weight 2 with only simple poles in CM points, and the residues in these poles are integral. Hence it corresponds to a differential of the first kind in the description of [Sch] or Section 3 of [BO], and its algebraicity corresponds to the fact that its residue divisor vanishes in the Jacobian of the curve \( X(\Gamma) \) (being the divisor of the rational function \( \Psi \))—see Theorem 1 of [Sch] or Theorem 3.2 of [BO]. Returning to the pairing with Hodge type \((0,0)\), since only \( X \) vanishes in the Jacobian of the curve \( W_{2m} \), and its algebraicity corresponds to the fact that its residue divisor vanishes in the Jacobian of the curve \( X(\Gamma) \) (being the divisor of the rational function \( \Psi \))—see Theorem 1 of [Sch] or Theorem 3.2 of [BO]. Returning to the pairing with Hodge type \((0,0)\), since only \( X \) vanishes in the Jacobian of the curve \( W_{2m} \), and its algebraicity corresponds to the fact that its residue divisor vanishes in the Jacobian of the curve \( X(\Gamma) \) (being the divisor of the rational function \( \Psi \))—see Theorem 1 of [Sch] or Theorem 3.2 of [BO].

For every \((2m + 1)\)-dimensional cycle \( Z \) in \( W_{2m} \), we consider the \( H^1(X, Z) \) part of the cohomology class in \( H^{2m+1}(W_{2m}) \) which is Poincaré dual to \( Z \). This part is of the form \( g(\tau) \frac{\tau}{\eta} \frac{2m}{2} d\tau + \overline{g(\tau)} \frac{\tau}{\eta} \frac{2m}{2} d\overline{\tau} \) for some cusp form \( g \). The integral of \( \frac{\tau}{\eta} \frac{2m}{2} \Phi_{L,m,m,0} \) along this cycle gives the integral over \( X(\Gamma) \) of the form \((2iy)^{2m} \delta_{2m} \frac{\tau}{\eta} \frac{2m}{2} \Phi_{L,m,m,0} \overline{g(\tau)} d\tau d\overline{\tau} \) (up to a certain contribution from the poles, which we shall soon consider), since only the pairing with Hodge type \((0,2m+1)\) has to be accounted for. This form is exact, being \( d\left( \frac{\tau}{\eta} \frac{2m}{2} \Phi_{L,m,m,0} \overline{g(\tau)} d\tau \right) \). Thus the integral vanishes, while the contribution from the poles also vanishes since \( \overline{g(\tau)} \) is smooth and the integral over a circle of radius \( \varepsilon \) around each pole \( \sigma \) gives an expression which vanishes as \( \varepsilon \to 0 \). This evaluation process is well-defined up to the location of the poles of \( \frac{\tau}{\eta} \frac{2m}{2} \Phi_{L,m,m,0} \) with respect to the choice of the cycle. However, jumping over a pole \( \sigma \) changes the value by a totally imaginary multiple of the residue of
\(\delta_{2m} \frac{\Phi_{L,m,m,0}}{\tau} (\tau - \sigma)^m (\tau - \overline{\tau})^m\), which is assumed to be integral. Therefore if we could establish some generalization of Theorem 1 of [Sch] assuring us that the algebraicity of this differential form with local coefficients implies the vanishing of a certain expression inside some generalized Jacobian (up to torsion), then we would know that the images of our Heegner \(m\)-divisors in \(Heeg(m)\text{Cl}(X(\Gamma))\) are the same as their images in this Jacobian.

The map \(\alpha\) of [H] seems also related to the \((H^{2m+1,0})^*\) part of the intermediate Jacobian of the Kuga–Sato variety \(W_{2m}\): For any CM point \(\sigma\), consider the CM cycle corresponding to \(\sigma\) in \(A_{\sigma}\), subtract some generic cycles in \(A_{\sigma}\) in order to obtain a cycle with normalized fundamental class, and let \(z^m_{\sigma}\) be the \(m\)th symmetric power of this cycle. We bound \(z^m_{\sigma}\) by the following cycle. Consider the \(2m + 1\)-dimensional cycle \([\sigma, \infty) \times z^m_{\sigma}\) in the notation of Section 8 of [Ba] (in the modular case this is possible), and take the closure of this cycle plus some cycle bounding the fiber over \(\infty\). The latter part is the counterpart of the combination of the cycles \(\Delta_k\) from [Ba], and the entire cycle resembles, in the case \(m = 1\), the cycle considered in [Sc]. Then the integral of some form of type \((2m + 1, 0)\), which looks like \(g(\tau)(dz_1 \wedge dz_2)^m d\tau\), over this cycle, decomposes as in the proof of Theorem 8.5 of [Be]. The integral from \(\sigma\) to \(\infty\) is of the function \(g(\tau)(\tau - \sigma)^m (\tau - \overline{\tau})^m\) (up to some constant), and since \(\sigma\) is a (modular) CM point on \(\mathcal{H}\) this gives (under the correct normalization) the integral considered in [H]. The integral over the cycle in the fiber over \(\infty\) probably gives some period of \(g\), and when we focus on one newform \(g\) (of weight \(2m + 2\)) this reduces to the image in the lattice in \(\mathbb{C}\) appearing in [H]. It will be very interesting to investigate what relations can be established between the existence of the form \(\delta_{2m} \frac{\Phi_{L,m,m,0}}{\tau}\) and the values of these integrals.

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