Spatially inhomogeneous structures in the solution of Fisher-Kolmogorov equation with delay

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Abstract. We consider the problem of density wave propagation in a logistic equation with delay and diffusion (Fisher–Kolmogorov equation with delay). A Ginzburg–Landau equation was constructed in order to study the qualitative behavior of the solution near the equilibrium state. The numerical analysis of wave propagation shows that for a sufficiently small delay this equation has a solution similar to the solution of a classical Fisher–Kolmogorov equation. The delay increasing leads to existence of the oscillatory component in spatial distribution of solutions. A further increase of delay leads to destruction of the traveling wave. That is expressed in the fact that undamped spatio-temporal fluctuations exist in a neighborhood of the initial perturbation. These fluctuations are close to the solution of the corresponding boundary value problem with periodic boundary conditions. Finally, when the delay is sufficiently large we observe intensive spatio-temporal fluctuations in the whole area of wave propagation.

1. Introduction

A. N. Kolmogorov, I. G. Petrovskii and N. S. Piskunov proposed the logistic equation with diffusion \[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u[1 - u]. \] (1) We will use abbreviation KPP for this equation.

Here \( u(t, x) \) is a density distribution of the individuals number which have a dominant gene, \( t \geq 0 \) is a time variable and \( x \in (-\infty, \infty) \) is a spatial variable. We consider the population density wave propagation problem from some nonzero initial conditions. Article by R. A. Fisher \cite{2}, which devoted to the same problem, was published almost simultaneously. Therefore the equation (1) is sometimes called the Fisher equation or the Fisher-Kolmogorov equation.

The equation (1) is used in a wide range of applications related to the space distribution of different waves — from the concentration waves to the population density waves. Among the large number of publications on the topic, we note a books \cite{3, 4, 5}, which contains problem summarizing results and we note the extensive bibliography of \cite{5}.

For any of classical boundary conditions (Neumman boundary conditions: \( \frac{\partial u}{\partial x} \big|_{x=0} = \frac{\partial u}{\partial x} \big|_{x=b} = 0 \); periodic boundary conditions \( u(t, x+T) \equiv u(t, x) \) and some other) the equation (1) has only one attractor — a homogeneous equilibrium state \( u_0 \equiv 1 \). All other equilibrium states are unstable.
Let’s consider the generalization of the KPP equation with delay \( h > 0 \)

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u[1 - u(t - h, x)].
\] (2)

Under certain assumptions this equation with diffusion and delay can be derived from the equations of population density dynamics which are studied in a series of articles by Britton N., Kuang Y., Gourley S., So J., Wu J. [6, 7, 8] (also refer the bibliography of [9]). This equation is one of the simplest problems of “reaction-diffusion” type and it was seen in various formulations [10, 11, 12, 13, 14, 15, 16, 17].

For the equation (2) we investigate the behavior of solutions with the initial conditions of a sufficiently small neighborhood (in \( C[-\infty, \infty] \times C[-h, 0] \)) of the equilibrium state \( u_0 \equiv 1 \).

It is shown that the behavior of solutions (2) in the neighborhood of \( u_0 \) is determined by a special non-linear parabolic equations of Ginzburg-Landau in the case of parameter \( h \) is close to \( \pi/2 \). Then we construct the equation of wave propagation for (2) and prove some statements about their dynamic properties. At the end of the article we presented the results of numerical modeling of the density wave in the equation (2) in the case of an unbounded \( x \) region.

2. Problem statement

The logistic equation with delay

\[
\dot{u} = u[1 - u(t - h)]
\] (3)

has been studied in [18, 19, 20, 21, 22, 23, 24, 25, 26]. Solution of (3) monotonically tends to 1 for \( h < e^{-1} \) and oscillatory tends to 1 for \( e^{-1} < h < \pi/2 \). In the article [18] it was shown that all positive solutions (3) tend to 1 for \( h \leq 37/24 \) and \( t \to +\infty \). The equilibrium state \( u_0 \) is asymptotically stable for \( h \leq \pi/2 \). Apparently ([21]) \( u_0 \) is globally stable for \( h \leq \pi/2 \). For \( h > \pi/2 \) we have slowly oscillating stable periodic solution \( u_*(t) \) (the distance between a peaks is greater than \( h \)). In the case of \( h - \pi/2 \ll 1 \) and \( h \gg 1 \) asymptotic behavior of solution is given in [26].

Situation for the equation (2) with classical boundary conditions is more complicated. Suppose, for example, we have a periodic boundary conditions

\[ u(t, x + T) \equiv u(t, x). \] (4)

Of course, the boundary value problem (2), (4), as well as the equation (3), have spatially homogeneous periodic solution \( u_*(t) \) at \( h > \pi/2 \). If parameter \( h \) is close to \( \pi/2 \) and \( h \gg 1 \) then this solution is stable. If \( h \) and \( T \) are sufficiently large then it is possible that the solution \( u_*(t) \) loses its stability and complex spatially inhomogeneous structures appear [24, 25, 26, 27, 28].

Lets us consider the case of delay \( h \) proximity to \( \pi/2 \) in detail and suppose the following relation holds:

\[ h = \pi/2 + \varepsilon h_1 \quad \text{and} \quad 0 < \varepsilon \ll 1. \] (5)

If we additionally suppose that the parameter \( T \) of (4) satisfies

\[ T \gg 1, \] (6)

then the dynamic of boundary value problem (2), (4) becomes more complicated. In the main the behavior of solutions in a small neighborhood of the equilibrium state \( u_0 \equiv 1 \) is determined by nonlocal behavior of solutions of the normalized complex equation (Ginzburg-Landau equation)

\[
\frac{\partial \xi}{\partial \tau} = \sigma \delta \frac{\partial^2 \xi}{\partial y^2} + h_1 \delta \xi + d|\xi|^2 \xi,
\] (7)
\[ \xi(\tau, y + 1) \equiv \xi(\tau, y). \]  
(8)

Here \( \tau = \varepsilon t, y = T^{-1}x \) — new time and spatial variables, 
\( \sigma = T^{-2}\varepsilon^{-1} \) is of order of 1, 
\[ \delta = (4 - 2\pi i)(4 + \pi^2)^{-1}, \quad d = -2(3\pi - 2 + i(\pi + 6))(5(4 + \pi^2))^{-1}. \]  
(9)

Solution of (2), (4) and solutions of (7), (8) are related by the formula  
\[ u(t, x) = 1 + \varepsilon^{1/2}[\xi(\tau, y) \exp(it) + \bar{\xi}(\tau, y) \exp(-it)] + O(\varepsilon). \]  
(10)

The dynamic properties (7) – (8) depend essentially on the parameter \( \sigma \). For example, for sufficiently small \( \sigma \) all simple periodic solution of the form  
\[ \rho_m \exp(2\pi i m \tau) \quad (m = 0, \pm 1, \pm 2, \ldots) \]  
(11)

are unstable [29].

In the articles [27, 30] for problem (7) with periodic boundary conditions (8) and Neumann boundary conditions a numerical experiment was performed. As a result of numerical experiment we can say that if we decrease the parameter \( \sigma \) then oscillations modes of solution became disordered and have more complex structure of the spatial variable. It should be noted that the approval of the correspondence between the solutions of quasinormal form (7), (8) and initial boundary value problem can be substantiated only in the case when problem (7), (8) has hyperbolic attractor. Therefore, in an article [31] for the equation (2) with Neumann boundary conditions and the boundary conditions (8) authors make the transition to a special difference analog and then performed a numerical analysis of the discrete system.

3. The normalized equation construction

Consider the behavior of equation solution near the equilibrium state \( u_0 \equiv 1 \)

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \left[ 1 - u \left( t - \frac{\pi}{2} - \varepsilon, x \right) \right]. \]  
(12)

The characteristic (wave) equation for the linearized on \( u_0 \) equation (12) have a form  
\[ \lambda = -k^2 - \exp \left( - \left( \frac{\pi}{2} + \varepsilon \right) \lambda \right), \quad k \in (-\infty, \infty). \]  
(13)

With sufficiently small \( k^2 \) equation (13) has a root \( \lambda(k, \varepsilon) \). This root has sufficiently small real part. Let us state this conclusion more precisely. Let’s \( k = \varepsilon^{1/2} m \).

Then equation (13) has a root \( \lambda_m(\varepsilon) = \lambda(\varepsilon^{1/2} m, \varepsilon) \) and the following asymptotic equality holds for \( \varepsilon \to 0 \)

\[ \lambda_m(\varepsilon) = i + 2\varepsilon(1 - m^2)(2 - i\pi)(4 + \pi^2)^{-1} + O(\varepsilon). \]  
(14)

Thus for each \( |m| < 1 \) a real part of \( \lambda_m(\varepsilon) \) is close to zero and imaginary part is close to 1.

Let us consider the formal series

\[ u(t, x) = 1 + \varepsilon^{1/2}[\xi(\tau, y) \exp(it) + \bar{\xi}(\tau, y) \exp(-it)] + \varepsilon u_2(\tau, y, t) + \varepsilon^{3/2} u_3(\tau, y, t) + \ldots, \]  
(15)

and \( \tau = \varepsilon t, y = \varepsilon^{1/2} x, \xi(\tau, y) \) — unknown ”amplitude” to be defined, and function \( u_j(\tau, y, t) \) has \( 2\pi \)-periodic dependence of \( t \). Substitute (15) in (12) and equate the coefficients of corresponding powers of \( \sqrt{\varepsilon} \). At the second step we find \( u_2(\tau, y, t) \). At the third step we collect the coefficients for \( \varepsilon^{3/2} \) and obtain the equation with respect to \( u_3 \). From the condition of solvability of this equation
in a specified class of functions we got the Ginzburg-Landau equation (as in the preceding section) to find \( \xi(\tau, y) : \)

\[
\frac{\partial \xi}{\partial \tau} = \delta \frac{\partial^2 \xi}{\partial y^2} + \delta \xi + d|\xi|^2 \xi,
\]

(16)

here \( \delta \) and \( d \) are determined by formulas (9). Note that \( \text{Re} \delta > 0 \) and \( \text{Re} d < 0 \).

The main result of this section is the following:

**Theorem 1.** Let the equation (16) have a bounded solution \( \xi_0(\tau, y) \) for \( \tau \to +\infty \) and \( y \to \pm \infty \). Then equation (12) has asymptotic solution \( u(t, x, \varepsilon) \) with residual \( O(\varepsilon^{3/2}) \):

\[
u(t, x, \varepsilon) = 1 + \varepsilon^{1/2}[\xi_0(\tau, y) \exp(it) + \bar{\xi}_0(\tau, y) \exp(-it)] + 0.2\varepsilon[(2 - i)\xi^2(\tau, y) \exp(2it) + (2 + i)\bar{\xi}^2(\tau, y) \exp(-2it)] + O(\varepsilon^{3/2}).
\]

Equation (16) has one-parameter family of the simplest periodic solutions

\[
u_m(\tau, y) = \rho_m \exp(i\omega_m \tau),
\]

(17)

where \( \rho_m = (m^2 - 1)\text{Re} \delta(\text{Re} d)^{-1}, \) \( |m| < 1, \) \( \omega_m = \left[\text{Im} \delta - \text{Im} \text{Re} \delta (\text{Re} d)^{-1}\right](1 - m^2) - \text{Im} \delta. \)

Since we have \( 1 + \text{Im} \alpha (\text{Re} \alpha)^{-1} \cdot \text{Im} \text{Re} (\text{Re} d)^{-1} = -(\pi^2 + 4)/(6\pi - 4) < 0 \), then from [29] we conclude that all solutions of (17) are unstable. Taking into account this result, in the next section we consider some properties of the wave solutions of (2).

### 4. Some properties of the wave equation

Consider a substitution in the equation (2) in the form of a traveling wave \( u(t, x) = w(2t \pm x) \) and introduce a new time \( s = 2t \pm x \), then for \( w(s) \) we have the following second order equations with delay

\[
w'' - 2w' + w[1 - w(s - 2h)] = 0,
\]

(18)

here \( \cdot \) is a derivative with respect to \( s \). The properties of stability of the zero solution of (18) do not depend on \( h \). This solutions is unstable. It corresponds to a multiple root equal to 1. The stability properties of equilibrium state 1 are determined by the location of the roots of the characteristic quasipolinom

\[
P(\lambda) \equiv \lambda^2 - 2\lambda - \exp(-2h\lambda).
\]

(19)

We prove the following lemmas.

**Lemma 1.** Quasipolinom \( P(\lambda) \) has three real roots at \( 0 < h < h^*_1 \). One root is positive and two roots are negative. \( P(\lambda) \) has only one positive real root at \( h > h^*_1 \). Here \( h_* = (\lambda_*-1)/(\lambda_*^2-2\lambda_*) \approx 0.5608, \) and \( \lambda_* \) is a root of transcendental equation \( \lambda^2 - 2\lambda - \exp[(2 - 2/\lambda)(\lambda/2 - 2)] = 0. \)

**Lemma 2.** All roots of quasipolinom \( P(\lambda) \) lie in the left half-plane for \( 0 < h < h^*_2, \) except for one real positive root. Here \( h^*_2 = (\sqrt{5} - 2)^{-1/2} \cdot \text{arccos}(\sqrt{5} + 2)/2 \approx 1.8617, \) The pair \( \lambda = \pm i\omega_0 \) of pure imaginary roots goes to the imaginary axis at \( h = h^*_2 \) and \( \omega_0 = (\sqrt{5} - 2)^{-1/2} \approx 0.4859. \)

In a neighborhood of solution \( w(s) = 1 \) we find the asymptotic of the cycle, branching from this solution at \( h = h^*_2 + \mu, \) \( 0 < \mu < 1 \). In this case, the following statement holds.

**Lemma 3.** There exists \( \mu_0 > 0 \) such that for all \( 0 < \mu < \mu_0 \) equation (18) has dichotomous cycle which one-dimensional unstable manifold and following asymptotic

\[
w(s, \mu) = 1 + 2\sqrt{-\mu \omega_0/d_0} \cos((\omega_0 + \mu(\psi_0 - c_0 \varphi_0/d_0))s + \gamma) + O(\mu),
\]

(20)

here \( \varphi_0 + i\psi_0 = 2\omega_0^2(-1 + i\omega_0)(P'(i\omega_0))^{-1}, \) \( d_0 + ic_0 = \left(2\omega_0^2(1 - \omega_0^2 - 2i\omega_0) + b((\omega_0^2 + 2i\omega_0)^2 - (\omega_0^2 + 2i\omega_0)^{-1})\right)(P'(i\omega_0))^{-1}, \) \( b = (\omega_0^2 + 2i\omega_0)(4\omega_0^2 + 4i\omega_0 + (\omega_0^2 + 2i\omega_0)^2)\) and \( \gamma \) is an arbitrary constant, which determines the phase shift along the cycle. Approximate values of the coefficients are \( \varphi_0 + i\psi_0 \approx 0.1368 - 0.2066i \) and \( d_0 + ic_0 \approx -0.0443 - 0.0366i. \)
5. Description of the numerical experiment
Described in the above sections analytical results have allowed us to make a numerical experiment. In this experiment we distinguish the qualitative features of the structure of the concentration wave of (2) from spatially localized initial disturbance. We considered (2) on the interval [a, b] and |a − b| was a sufficiently large. The boundary values were zero. Initial condition at −h ≤ t ≤ 0 were localized near central part of the space variable pulse of 0.1 amplitude.

Below we present the results of numerical experiments obtained for different values of delay (0, 1.6, 1.8 and 2) that are the most revealing to study of the structure of the wave front. The figures 1–3 show an image of solutions density distribution of the KPP equation with delay and some cuts in different planes, allowing to clarify the process of emergence and propagation of the oscillations on the border of the wave front and on the central part of it.

6. Main results
In order to clarify some of the features of the qualitative behavior of solutions of KPP with delay we built Ginzburg-Landau equation that describes the dynamic properties of the equations near equilibrium state u_0 ≡ 1.

We studied the wave propagation equation (18) and found the critical parameter of delay for which changes in the structure of the spatial distribution of the solution appear.

Numerical analysis, based on analytical results, allowed us to distinguish the following intervals of delay: 1) the behavior of solutions of problem with delay close to the behavior of the problem without delay; 2) in the center part of the wave solutions area appears a region with complex spatial distribution; 3) in the spatial distribution of solutions there is preserved a region where the solution tends to 1; 4) the whole area of wave propagation is filled with the
Figure 3. The solution of (2) at \( h = 2 \) a) density distribution \( u(t,x) \) in a grayscale; b) cross-section \( t = 200 \); c) fragment at \( t \in [200 \times 220] \) and \( x \in [300 \times 600] \); intensive oscillations in space and time variables.

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