Geometry and Temperature Chaos in Mixed Spherical Spin Glasses at Low Temperature: The Perturbative Regime

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Abstract

We study the Gibbs measure of mixed spherical $p$-spin glass models at low temperature, in (part of) the 1-RSB regime, including, in particular, models close to pure in an appropriate sense. We show that the Gibbs measure concentrates on spherical bands around deep critical points of the (extended) Hamiltonian restricted to the sphere of radius $\sqrt{N} \rho_*$, where $\rho_*$ is the rightmost point in the support of the overlap distribution. We also show that the relevant critical points are pairwise orthogonal for two different low temperatures. This allows us to explain why temperature chaos occurs for those models, in contrast to the pure spherical models. © 2019 Wiley Periodicals, Inc.

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1 Introduction

We study here the Gibbs measure of the mixed spherical $p$-spin glass model at low temperature, in part of the 1-RSB regime. The model, which is a variant of that introduced in the seminal paper [51], is defined as follows. Let $S^{N−1}(\sqrt{N})$ denote the (euclidean) sphere of radius $\sqrt{N}$ in dimension $N$. Let $J^{(p)}_{i_1,...,i_p}$ denote i.i.d. real standard Gaussian random variables, and let $\{\gamma_p\}_{p\geq2}$ be a sequence of nonnegative deterministic constants. The Hamiltonian is defined as

$$H_N(\sigma) = H_{N,v}(\sigma) := \sum_{p=2}^{\infty} \frac{\gamma_p}{N^{(p−1)/2}} \sum_{i_1,...,i_p=1}^{N} J^{(p)}_{i_1,...,i_p} \sigma_{i_1}...\sigma_{i_p},$$

with ground state

$$\text{GS}_N = \min_{\sigma \in S^{N−1}(\sqrt{N})} H_N(\sigma);$$

the associated Gibbs measure is the random probability measure on $S^{N−1}(\sqrt{N})$ given by

$$\frac{dG_{N,\beta}}{d\sigma}(\sigma) := \frac{1}{Z_{N,\beta}} e^{-\beta H_N(\sigma)};$$

here $Z_{N,\beta}$ is a normalization constant and $d\sigma$ denotes the normalized Haar measure on $S^{N−1}(\sqrt{N})$.

Let

$$v(x) = \sum_{p=2}^{\infty} \gamma_p^2 x^p.$$  

We refer to the model as pure if $v(x)$ is a monomial, and as mixed otherwise. Throughout the paper, we assume that $\gamma_p$ decays exponentially, so that $v(\cdot)$ is defined on an open interval that includes $[0, 1]$. Following [5], we normalize $v$ by setting $v(1) = 1$. 

For many models of spin glasses including the spherical models, properties of the Gibbs measure in terms of their overlaps (i.e., the distribution of the distance between two or more points sampled independently from the Gibbs measure), which serve here as an order parameter, are available through a version of the Parisi formula; see [31, 43, 56, 57], and through the ultrametricity properties of $G_{\beta,N}$, see [40]. We refer to [41, 58] for comprehensive introductions to the mathematical theory of spin glasses.

Our goal in this paper is different: we aim at developing a geometric description of the Gibbs measure at low temperature. For the pure model, this was achieved by one of us in [53], where it was shown that at low temperature $\beta \gg 1$, $G_{\beta,N}$ concentrates in thin bands (or rings) centered at the locations of the deepest local minima of $H_N$.

Our results apply to spherical mixed models that satisfy a certain decoupling condition (Condition M, defined below) related to critical points. As we shall see, Condition M dictates 1-RSB at very low temperature. A particularly important class of models that satisfy our conditions are perturbations of pure $p$-spin spherical models; see Section 6 below. We show that the geometric description of the support of the Gibbs measure in low temperature, developed in [53] for the pure $p$-spin model, needs to be modified in the mixed case. The Gibbs measure in the mixed case is supported on thin bands that are centered at critical points not of the Hamiltonian (1.1) but rather of its extension to the sphere $S^{N-1}(\rho_*, \sqrt{N})$, for appropriate $\rho_* = \rho_*(\beta) < 1$; see Theorem 1.2. (These centers are close to critical points of the Hamiltonian (1.1) with low, but not minimal, energy.) As a by-product, we are able to show that states of asymptotic positive mass are pure and nearly orthogonal (see Theorem 1.3), and explain why those models exhibit chaos in temperature while the pure models do not; see Theorem 1.4 below and Section 12.

Our geometric description of the Gibbs measure at low temperature is closely related to the “state following method” in the physics literature. The structure of the Gibbs measure at low temperature for some mixed spherical models has indeed been described in terms very similar to ours in [10], an important and early work in the huge physics literature. In [10], the “states” are defined as minima of the so-called TAP free energy, and “followed” as a function of temperature, using the so-called Franz-Parisi potential. The mixed models studied by [10] must satisfy a simple condition on the mixture: the function $(v''(\rho))^{-3/2} v(\rho)$ must be decreasing. This condition, which is true for example in the case $v(x) = (x^3 + x^4)/2$, is different from our Condition M given below. The “state following method” is fully presented in the more recent work [59]. We refer to sections IV.D and IV.E, and in particular to figures 4 and 6 in [59]. This work in fact deals with different models of spin glasses (mainly the Ising p-spin model rather than the mixed

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1 The value of the global minimum can be inferred from Parisi’s formula. It was evaluated via a study of the limiting expected complexity in [6] (pure) and [5] (mixed), and complemented (for the pure p-spin spherical model) by a study of second moments in [52].
spherical one). Our results establish here rigorously what [59] calls “level crossing,” which is identified there as the origin of temperature chaos, as was proposed before in [34, 36, 48].

To add to this very short review of the most relevant works in the huge physics literature, we point to a different result on the “state following method” for mixtures of spherical spin glasses, which is beyond what we can understand at this point. In [3], a breaking of the BRST-supersymmetry is proposed for small perturbations of a spherical p-spin glass. The mathematical literature relevant for this question of “state following,” “level crossing,” and even simply for the TAP free energy is much sparser. In [19, 58] (high-) and recently [7] (low-temperature) the TAP equations were rigorously mathematically established, and in [16] a recursive scheme was constructed for the solution of the TAP equations. In [23] a deep link between the TAP free energy extrema and “pure states” is given for generic mixtures of Ising p-spin spin glass models, and the TAP representation for the free energy is derived. In [11] the representation for the free energy is derived for the 2-spin spherical model, and in [54] it is established for general spherical models, by analysis of the free energy landscape associated to bands as in the current work.

1.1 Main Results

We turn to a detailed description of our results. We introduce the function

\[ G(v) = \log \frac{v''(1)}{v'(1)} - \frac{(v''(1) + v'(1))(v''(1)v(1) + v'(1)^2 - v'(1)v(1))}{v''(1)v'(1)^2} \]

(1.5) (See (3.14) for an interpretation of \( G(v) \) in terms of a first-moment calculation involving the number of critical points of \( H_N \).) Following [5], we call the model pure-like, critical, or full according to whether \( G(v) > 0 \), \( = 0 \), or \( < 0 \), respectively. In what follows, we deal exclusively with pure-like models.

Next, consider the limiting expected complexity at level \( u \) and radial derivative \( x \), defined as

\[ \Theta_{v,1}(u, x) = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \text{critical points } \sigma \in S^{N-1}(\sqrt{N}) \text{ with} \right. \right. \right. \]

\[ \left. \left. \left. \left| \frac{1}{N} H_N(\sigma) - u \right| \leq \delta \text{ and } \left| \frac{1}{\sqrt{N}} \frac{d}{dR} H_N(\sigma) - x \right| \leq \delta \right\} \right\}; \]

(1.6)

where \( \frac{d}{dR} H_N(\sigma) := \frac{1}{\sqrt{N}} \frac{d}{d\rho} H_N(\rho \sigma)|_{\rho=1} \) is the radial derivative of \( H_N(\sigma) \) at \( \sigma \in S^{N-1}(\sqrt{N}) \). Recapitulating one of the main results in [5], one has that the limit in (1.6) exists and is explicit; see Theorem 3.1 below.

Let

\[ -E_0 := -E_0(v) = \min \left\{ E : \sup_{x \in \mathbb{R}} \Theta_{v,1}(E, x) = 0 \right\}. \]

(1.7)
The level $-E_0 N$ is the threshold beyond which the number of critical points decays exponentially in expectation; see $[5]$.

In particular, by Markov’s inequality, for large $N$,

\[(1.8) \quad \frac{1}{N} \min_{\sigma \in S^{N-1}(\sqrt{N})} H_N(\sigma) \geq -E_0 - o(1) \quad \text{with high probability.}\]

As we show below in Lemma 5.1, there exists a unique maximizer

\[(1.9) \quad -x_0 := -x_0(\nu) = \arg \max_{x \in \mathbb{R}} \Theta_{\nu,1}(-E_0, x). \]

Define the overlap between $\sigma, \sigma' \in \mathbb{R}^N$ as

\[(1.10) \quad R(\sigma, \sigma') := \|\sigma\| \|\sigma'\|. \]

where \((\cdot, \cdot)\) denotes the standard inner product in $\mathbb{R}^N$. We next introduce a function $\Psi_{\nu,1,1}(r, u, x)$ that will play a crucial role in second-moment computations, and which we refer to as the pair complexity at level $u$, radial derivative $x$, and overlap $r$:

\[
\Psi_{\nu,1,1}(r, u, x) = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \text{pairs of critical points } \sigma, \sigma' \text{ with } |R(\sigma, \sigma') - r| \leq \delta, \right. \right.

\left. \left. \frac{1}{N} H_N(\sigma) - u \right| \leq \delta, \quad \frac{1}{N} H_N(\sigma') - u \right| \leq \delta, \quad \left| \frac{1}{\sqrt{N}} \frac{d}{dR} H_N(\sigma) - x \right| \leq \delta, \quad \right. \right.

\left. \left. \left| \frac{1}{\sqrt{N}} \frac{d}{dR} H_N(\sigma') - x \right| \leq \delta \right\} \right). \]

An explicit expression for the function $\Psi_{\nu,1,1}$ appears in (3.10). Finally, set, for $r \in (-1, 1)$,

\[(1.11) \quad \Psi_0^\nu(r) := \Psi_{\nu,1,1}(r, -E_0, -x_0). \]

and for $r = \pm 1$, let $\Psi_0^\nu(\pm 1)$ be the corresponding $r \to \pm 1$ limit. The function $\Psi_0^\nu(\cdot)$, which is determined by $\nu$, is a continuous function from $[-1, 1]$ to $\mathbb{R} \cup \{-\infty\}$, and determines the pair complexity at level $-E_0$ and radial derivative $x$ as a function of the overlap.

All our results will be under the following assumption.

**CONDITION M.** Assume that $\nu$ is mixed and pure-like, that $\frac{d^2}{d\nu^2} \Psi_0^\nu(0) < 0$, and that the maximum of $\Psi_0^\nu(r)$ on the interval $[-1, 1]$ is obtained uniquely at $r = 0$.

Condition M implies that the pair complexity at the relevant levels $(-E_0, -x_0)$ is maximal at zero overlap. We will see below (see Theorem 1.3) that Condition M implies in particular that the model belongs to the so-called 1-1RSB class. We will

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2 The definition of $E_0$ given in this paper coincides with the definition in $[5]$ for pure-like or critical models, but not for full models. See sections 4 and 5 of $[5]$.

3 The $r \to 1$ limit always exists and is finite; see the proof of Lemma 3.5 below. The $r \to -1$ limit is finite if and only if $\nu$ is even. Otherwise, it is $-\infty$. 
also see that Condition M is an open condition, and that small perturbations of pure models satisfy Condition M; see Propositions 6.2 and 6.3.

We are ready to state our first result. Recall the ground state $GS_N$; see (1.2).

**THEOREM 1.1.** Assume Condition M. Then,

$$
\lim_{N \to \infty} \frac{GS_N}{N} = -E_0 \quad \text{a.s.}
$$

Thus, expected complexity determines the ground state. Our proof of Theorem 1.1 relies on a two-moment analysis presented in Sections 3, 4, and 5, and avoids the use of Parisi’s formula.

We next turn to the description of the support of the Gibbs measure $G_{\beta, N}$ for large $\beta$. Let $F_1, \ldots, F_{N-1}$ be a piecewise smooth frame field on $S^{N-1}(\sqrt{N})$, and extend it to $x \in \mathbb{R}^N \setminus \{0\}$ by setting $F_i(x) = F_i(\sqrt{N}x/\|x\|)$ (under the usual identification of tangent spaces with affine subspaces of $\mathbb{R}^N$). Denote

$$
\nabla_{sp} H_N(\sigma) := \{F_i H_N(\sigma)\}_{i \leq N-1},
$$

$$
\nabla_{sp}^2 H_N(\sigma) := \{F_i F_j H_N(\sigma)\}_{i,j \leq N-1}.
$$

We shall call the points $\sigma \in S^{N-1}(\rho \sqrt{N})$ with $\nabla_{sp} H_N(\sigma) = 0$, $\rho$-critical points. For any set $B \subseteq \mathbb{R}$ let

$$
\mathcal{C}_{N, \rho}(B) := \{\sigma \in S^{N-1}(\rho \sqrt{N}) : \nabla_{sp} H_N(\sigma) = 0, H_N(\sigma) \in B\}
$$

denote the set of $\rho$-critical points with values in $B$. As we are about to see, the support of the Gibbs measure at inverse temperature $\beta \gg 1$ is asymptotically contained in thin bands around $\rho$-critical points at level $-NE$ for particular values of $\rho_\star = \rho_\star(\beta)$ and $E_\star = E_\star(\beta)$ defined in (8.6) and (12.1), respectively. $(-E_\star$ is the normalized ground state of the Hamiltonian (1.1) on $S^{N-1}(\rho_\star \sqrt{N})$.) To define what we mean by bands, set

$$
\text{Band}(\sigma_0, \epsilon) := \{\sigma \in S^{N-1}(\sqrt{N}) : \|R(\sigma, \sigma_0) - \sigma_0 \| / \sqrt{N} \leq \epsilon\}.
$$

Given a sequence $\epsilon_N > 0$, set $B = (-E_\star - \epsilon_N, -E_\star + \epsilon_N)$ and $\mathcal{C}_\star = \mathcal{C}_\star(\beta) := \mathcal{C}_{N, \rho_\star}(NB)$. As the next theorem shows, the union (over $\mathcal{C}_\star$) of these bands asymptotically supports $G_{N, \beta}$.

**THEOREM 1.2 (Support of the Gibbs measure).** Assume that $\nu$ satisfies Condition M. Then there exist positive $\epsilon_N \to 0$ such that for large enough $\beta$ the following hold:

1. subexponential number of critical points:

$$
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\{|\mathcal{C}_\star|\} = 0;
$$

2. asymptotic support:

$$
\lim_{N \to \infty} \mathbb{E}\left\{G_{N, \beta}\left(\bigcup_{\sigma_0 \in \mathcal{C}_\star} \text{Band}(\sigma_0, \epsilon_N)\right)\right\} = 1.
$$
We also obtain a detailed description of the states associated with the Gibbs measure $G_{N,\beta}$.

**Theorem 1.3.** Assume that $v$ satisfies Condition M. Let $\epsilon_N \to 0$ as in Theorem 1.2 and $\beta$ large enough. Let $\sigma$ and $\sigma'$ be independent samples from the Gibbs measure $G_{N,\beta}$. Then, for any $\delta > 0$, the following holds:

1. **States are pure:**
   \[
   \lim_{N \to \infty} \mathbb{P} \left\{ \sigma, \sigma' \in \text{Band}(\sigma_0, \epsilon_N) \text{ for some } \sigma_0 \in \mathcal{C}_*, \right. \\
   \left. \text{and } |R(\sigma, \sigma') - \rho_*^2| \geq \delta \right\} = 0, \\
   \lim_{N \to \infty} \mathbb{P} \left\{ \sigma \in \text{Band}(\sigma_0, \epsilon_N), \sigma' \in \text{Band}(-\sigma_0, \epsilon_N) \text{ for some } \sigma_0 \in \mathcal{C}_*, \right. \\
   \left. \text{and } |R(\sigma, \sigma') + \rho_*^2| \geq \delta \right\} = 0.
   \]

2. **Orthogonality of states:**
   \[
   \lim_{N \to \infty} \mathbb{P} \left\{ \sigma \in \text{Band}(\sigma_0, \epsilon_N), \sigma' \in \text{Band}(\sigma_0', \epsilon_N) \text{ for some } \sigma_0 \neq \pm \sigma_0' \in \mathcal{C}_*, \right. \\
   \left. \text{and } |R(\sigma, \sigma')| \geq \delta \right\} = 0.
   \]

Theorem 1.3 implies that at low temperature, the model is in the 1-RSB phase. Its free energy can be computed from Parisi’s formula as a minimization over a manageable space of measures. We obtain an alternative description of the free energy as a simple maximization over a subinterval of $[0, 1]$; see Remark 12.1 below.

We next turn to the question of temperature chaos. The physics literature on temperature chaos is too large (and controversial) to be fully described here. We refer to the review [46] for a wider discussion and references, as well as [34]. Temperature chaos has been studied extensively back in the foundational works [9, 18, 29, 30, 35, 39] by using scaling arguments and real space renormalization group analysis. Temperature chaos’ existence has been argued as well as its absence, for various spin glass models. For instance, [4, 34, 47, 49] argue in favor, and [15, 37, 45] argue against it. In any case, this question of temperature chaos is seen as important, as temperature chaos has been proposed as a mechanism for memory and rejuvenation experiments (see [32, 50]). A more recent and fascinating line of work on temperature chaos “as a rare event” relies on a very rich large-deviation analysis, first proposed in [44] and pursued in [27, 28].

The mathematical literature on temperature chaos is obviously much easier to summarize (see [20–22, 42, 53]). After early results in [20–21], temperature chaos for generic even $p$-spin models with Ising spins is proven in [42]. [53] proves that temperature chaos does not happen for the pure $p$-spin spherical model at low

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4 Note that the second limit in (1.17) is only relevant in case $v$ is an even polynomial, since otherwise $\mathcal{C}_* \cap -\mathcal{C}_* = \emptyset$ for $\epsilon_N$ small.

5 Theorem 1.3 per se actually does not preclude the possibility that the model is in the replica symmetric phase, with the Gibbs measure not giving mass to any region of vanishing normalized volume. However, the last possibility is easily ruled out by a use of Parisi’s formula.
enough temperature. Finally, [22] proves temperature chaos for spherical models either when the model satisfies a certain decoupling condition, or when the model is an infinitesimal perturbation of a pure $p$-spin model (similarly to the models covered in Proposition 6.2 below), at all temperatures and without the restriction to the 1RSB phase, but without the full geometric description we present here.

As mentioned above, one of the main results in [53] is that pure spherical models do not exhibit temperature chaos: for $\beta > 0$ large enough, the supports of $G_{\beta, N}$ and $G_{\beta', N}$, while essentially disjoint, are close to each other, and independent samples from these two measures have significant overlap with uniformly positive probability. On the other hand, in view of the above-mentioned results of [22, 42], generically one expects to find temperature chaos in spin glasses. We can confirm that indeed, temperature chaos exists in the mixed models we consider.

**Theorem 1.4 (Chaos in low temperature).** Assume that $v$ satisfies Condition M and let $\beta \neq \beta'$ be large enough. Let $\epsilon_N, \epsilon'_N \to 0$ and $\mathcal{C}, \mathcal{C}'$ be the corresponding widths and sets of critical points as in Theorem 1.2. If $\mathbf{\sigma}$ and $\mathbf{\sigma}'$ are independent samples from $G_{N, \beta}$ and $G_{N, \beta'}$, respectively, then for any $\delta > 0$,

$$
\lim_{N \to \infty} \mathbb{P} \left\{ \mathbf{\sigma} \in \text{Band}(\mathbf{\sigma}_0, \epsilon_N), \mathbf{\sigma}' \in \text{Band}(\mathbf{\sigma}'_0, \epsilon'_N) \right. \text{for some } \mathbf{\sigma}_0 \in \mathcal{C}, \mathbf{\sigma}'_0 \in \mathcal{C}'', \text{ and } |R(\mathbf{\sigma}, \mathbf{\sigma}')| > \delta \left\} = 0.
$$

**2 Outline of Proofs**

At low temperature the Gibbs measure concentrates on regions where the Hamiltonian is very low. For some models, those regions are believed to have fairly simple topology, sometimes referred to as “deep, separated valleys” in the physics literature, at the bottom of which one finds a local minimum of the landscape. A natural approach to analyze the Gibbs measure is therefore to study the distribution of critical points, investigate the local structure of the Hamiltonian $H_N(\mathbf{\sigma})$ around them, and use those to analyze the Gibbs weights of various regions around the deep critical points. Below we outline how we use this approach in our setting, and compare with [53] where a similar method was used for pure models.

**2.1 Critical Points**

Before directly considering Gibbs weights of different regions of the sphere, we need to investigate the distribution of $\rho$-critical points of the Hamiltonian, whose analysis will be based on moment computations. The first-moment calculation was carried out in [5] for the original sphere $S^{N-1}(\sqrt{N})$, i.e., for 1-critical points. In Theorem 3.1 we generalize the latter to general $\rho$. To establish the concentration of the number of $\rho$-critical points at low enough energies, we carry out the corresponding second-moment computation. See Theorem 3.2 and Corollary 3.6 which generalize to mixed models the second-moment computation of [52] for the pure case at logarithmic scale.

While our second-moment calculation is valid for general mixed models, its matching to the first moment squared is not guaranteed. In fact, for deep levels,
matching at exponential scale is equivalent to $\Psi^0_\nu(r)$ being maximized at $r = 0$, an assumption we make in Condition M.

Another consequence of the second-moment calculation is that deep $\rho$-critical points are approximately orthogonal; see Corollary 3.7 Moreover, for different values $\rho_1, \rho_2$ close to 1, the corollary implies pairwise approximate orthogonality of the collection of deep $\rho_1$-critical points and $\rho_2$-critical points. This will be crucial to understanding chaos in temperature.

2.2 Deep Sub-level Sets, Critical Points, and Bands

Given a lower bound on the free energy, for an appropriate energy level the Gibbs mass of the corresponding super-level is negligible. Exploiting this, we will be able to restrict our attention to the temperature dependent sub-level set

$$\{\sigma \in S^{N-1}(\sqrt{N}) : H_N(\sigma) \leq -N(E_0(\nu) - \tau(\beta))\},$$

with $\tau(\beta) \to 0$ as $\beta \to \infty$, which asymptotically carries all the mass; see Corollary 11.2.

The geometry of the set in (2.1) and its relation to $\rho$-critical points will play an important role in our analysis. We shall see (Proposition 9.1) that for large $\beta$, the sub-level set (2.1) splits into exponentially many connected components, each of which contains exactly one local minimum of $H_N(\sigma)$ on the sphere $S^{N-1}(\sqrt{N})$. By increasing $\beta$, we can make those $1$-critical points be as close to orthogonal as we wish (Corollary 3.7) and the diameter of the components as small as we wish (Proposition 9.1).

Only a small number of these many connected components will significantly contribute to the partition function. To characterize which do and identify the relevant region inside those components responsible for such contribution, we will “scan” them using bands as in (1.14). More precisely, defining for any $\sigma_0$ in $B^N(\sqrt{N})$, the $N$-dimensional ball of radius $\sqrt{N}$,

$$S(\sigma_0) := \{\sigma \in S^{N-1}(\sqrt{N}) : R(\sigma, \sigma_0) = \|\sigma_0\|/\sqrt{N}\},$$

we will see that for some constant $c$, for each $1$-critical point $\sigma_*$ in the sub-level set (2.1), there is a differentiable path $\sigma_\rho, \rho \in [1 - c \tau(\beta), 1]$, such that $\sigma_1 = \sigma_*$, each $\sigma_\rho \in S^{N-1}(\rho \sqrt{N})$ is a $\rho$-critical point, and the union $\bigcup_\rho S(\sigma_\rho)$ of sections covers the corresponding connected component of (2.1) (see Proposition 9.1 and Lemma 11.3).

Recall the normalization constant $Z_{N,\beta}$ (see (1.3)), set $F_{N,\beta} = \frac{1}{N} \log Z_{N,\beta}$, and define the free energy $F_\beta$ as

$$F_\beta = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} F_{N,\beta}.$$
(The existence of the limit in (2.3) is well-known (see, e.g., [56]) and follows also from our analysis.) For any point $\sigma_0 \in \mathbb{B}^N(\sqrt{N})$, we define the corresponding weight

\begin{equation}
Z_{N,\beta}(\sigma_0) := (1 - \|\sigma_0\|^2 / N)^{N} \int_{S(\sigma_0)} e^{-\beta H_N(\sigma)} d\sigma,
\end{equation}

where the integration over $S(\sigma_0)$ is with respect to the uniform probability measure on $S(\sigma_0)$, and where the factor before the integral accounts for the volume of a thin band centered at $\sigma$, and logarithmically scales like

$$\text{Vol}_{N-2}(S(\sigma_0)) / \text{Vol}_{N-1}(S^{N-1}(\sqrt{N})).$$

Using the information above on the sub-level set in (2.1), to prove part 2 of Theorem 1.2, we will need to prove a lower bound and upper bounds on certain free energies. The lower bound is on the free energy related to the collection of sections around $\rho_\ast$-critical with energy $-E_\ast$. That is, we will need to show that

\begin{equation}
\lim_{N \to \infty} \mathbb{P} \left\{ \frac{1}{N} \log \sum_{\sigma_0 \in \mathbb{R}_N \rho_\ast(\{E_0 - \epsilon, -E_\ast + \epsilon\})} Z_{N,\beta}(\sigma_0) \geq F_\beta - \delta \right\} = 1
\end{equation}

for any positive $\epsilon$ and $\delta$. The upper bounds we shall need are of the form

\begin{equation}
\lim_{N \to \infty} \mathbb{P} \left\{ \frac{1}{N} \log \sum_{\sigma_0 \in \mathbb{R}_N \rho(\{E - \epsilon, E + \epsilon\})} Z_{N,\beta}(\sigma_0) \leq F_\beta - \delta \right\} = 1,
\end{equation}

and we will need to prove them for any

$$\rho \in [1 - c \tau(\beta), 1] \quad \text{and} \quad E \in [-E_0(v), -E_0(v) - c \tau(\beta)]$$

such that $(E, \rho) \neq (-E_\ast, \rho_\ast)$ (and $\epsilon$ and $\delta$ that are allowed to depend on $(E, \rho)$).

### 2.3 Conditional Models on Bands

To obtain bounds of the form (2.5) and (2.6), we shall compute certain related expectations. By an application of the Kac-Rice formula, those will be expressed through various probabilities or expectations involving the restriction of $H_N(\sigma)$ to sections $S(\sigma_0)$ for points $\sigma_0 \in \mathbb{B}^N(\sqrt{N})$ conditional on

\begin{equation}
H_N(\sigma_0) = NE, \quad \nabla_{0p} H_N(\sigma_0) = 0.
\end{equation}

This restriction can be mapped from $S(\sigma_0)$ to the “standard” sphere of the same dimension, $S^{N-2}(\sqrt{N-1})$, and the random field thus obtained should be thought of as a random spherical Hamiltonian. In Section 7 we will extend the decomposition obtained in [53, sec. 3] for pure models to general mixed models, and show that this spherical Hamiltonian is in fact a mixed $p$-spin model, with mixture coefficients depending on $\rho = \|\sigma_0\| / \sqrt{N}$. The key to analyzing the model under the conditioning of (2.7) is the observation that each of the different $p$-spin interactions can be written in terms of the euclidean derivatives of order $p$ of $H_N(\sigma)$ at $\sigma_0$. In particular, the conditioning only affects the lowest two interactions, the 0- and 1-spins. Here, by 0-spin we mean a model $H_{N-1}(\sigma) = J_0$ that is constant on
and is determined by a single Gaussian variable \( J_0 \sim N(0, N) \), and by 1-spin, a model of the form \( H_{N-1}(\sigma) = \sum \sigma_i J_i \) with \( J_i \sim N(0, 1) \). We will see that the conditioning amounts to determining the constant value of the 0-spin to be \( NE \), and removing the 1-spin interaction term. The corresponding conditional model is, therefore, a mixture of \( p \)-spins with \( p \geq 2 \), “shifted” by a factor of \( NE \). In particular, we remark that for \( \|\sigma_0\| = \rho \sqrt{N} \), this model is replica symmetric for large \( \beta \).

2.4 A Comparison with the Pure Case

In Section 2.2 we explained how the connected components of the sub-level set (2.1) can be covered by “moving” sections \( S(\sigma_\rho) \). A simpler approach would be to use concentric sections centered at the 1-critical point \( \sigma_1 \) and avoid altogether constructing the path \( \sigma_\rho \) and investigating it. In fact, this is exactly what was done for the pure case in [53].

However, the pure case is very special: since its Hamiltonian is a homogeneous polynomial, \( \sigma_\rho = \rho \sigma_1 \) is a (degenerate) path of \( \rho \)-critical points, and (2.7) with \( \sigma_0 \in S^{N-1}(\sqrt{N}) \) dictates the same for any point \( \rho \sigma_0 \) on the same “fiber,” with \( E \) scaled to \( \rho^p E \). In particular, in [53] the derivation of bounds of the form (2.6) always (i.e., independently of \( \rho \)) involved conditioning as in (2.7) with \( \sigma_0 = \sigma_1 \) being the corresponding 1-critical point.

The problem with applying the same approach in the mixed case is that if we work with \( \sigma_\rho = \rho \sigma_1 \), and thus do not impose the condition that \( \nabla_{sp} H_N(\sigma_\rho) = 0 \), the conditional models we have to deal with involve a nonzero 1-spin component, leading to a more complicated analysis of the corresponding weights. In particular, since we need to analyze the weights of exponentially many critical points, we must understand their large-deviation probabilities, which for nonzero 1-spin, have speed \( N \) matching the complexity, i.e., the number of critical points at a given energy.

By working with \( \rho \)-critical \( \sigma_\rho \), we manage to avoid the difficult analysis of weights, and obtain a replica symmetric description for the restriction of \( H_N(\sigma) \) to relevant bands.

2.5 Upper Bounds on the Free Energies of the Conditional Models

We now return to the bounds (2.6). For any \( \rho \in (0, 1) \) and \( E < 0 \), we will show that with \( B = B(E, \epsilon) = [E - \epsilon, E + \epsilon] \),

\[
\frac{1}{N} \log \mathbb{E} \left\{ \sum_{\sigma_0 \in \mathbb{C}_{N,\rho}(NB)} Z_{N,\beta}(\sigma_0) \right\} \leq \sup_{x \in \mathbb{R}} \Theta_{\nu,\rho}(E, x) + \Lambda_{Z,\beta}(E, \rho) + O(\epsilon),
\]

(2.8)

where \( \Lambda_{Z,\beta}(E, \rho) \) is \( \frac{1}{N} \log \) the expectation of a single weight \( Z_{N,\beta}(\sigma_0) \), conditional on (2.7) (see Corollary 8.1 and Lemma 11.5). By Markov’s inequality,
asymptotically we have that (2.8) holds without the expectation, with high probability. This bound, however, will be useful only for large enough $\rho$ in $[1 - c \tau(\beta), 1]$ — the range we used to cover the sub-level set of (2.1).

More precisely, we will define a critical value $\rho_c = \rho_c(\beta)$ (see (8.8)) such that for $\rho \geq \rho_c$ the conditional model of Section 2.3 is replica symmetric and typically the corresponding free energy matches the simple bound we get from expectations, and for $\rho < \rho_c$ that free energy is typically smaller at exponential scale.

For the latter range, we will use the fact that for large $\beta$ the conditional model is dominated, in an appropriate sense, by its 2-spin component (see Section 2.3). In Section 11 (see Lemmas 11.6 and 11.7), we will prove that for $\rho \in [1 - c \tau(\beta), \rho_c]$ and large $\beta$, with high probability,

$$\frac{1}{N} \log \sum_{\sigma_0 \in \mathcal{E}_{N,\rho}(NB)} Z_{N,\beta}(\sigma_0)$$

$\leq \sup_{x \in \mathbb{R}} \Theta \times, \rho(E, x) + \Lambda_{F, \beta}^2(E, \rho) + O(\epsilon) + K(\rho, E, 0) + T_{\rho}.$

The function $\Lambda_{F, \beta}^2(E, \rho)$ is the asymptotic (normalized) free energy $N^{-1} \log \sum_{\sigma_0 \in \mathcal{E}_{N,\rho}(NB)} Z_{N,\beta}^2(\sigma_0)$ corresponding to the 2-spin component only, conditional on (2.7). The term $K(\rho, E, 0)$, defined in Lemma 11.6, accounts for atypically large weights $Z_{N,\beta}(\sigma_0)^{2-\rho}$ which may occur for some of the exponentially many points in $\mathcal{E}_{N,\rho}(NB)$ (for $E$ not too negative). $T_{\beta}$, introduced in Lemma 11.7, bounds the error resulting from using only the 2-spin part in our computation, which for large $\beta$ and hence $\rho$ close to 1 becomes negligible compared to the other terms.

By an abuse of notation, let $-E_*(\rho)$ denote the limiting ground state of $H_N(\sigma)$ restricted to $\mathbb{S}^{N-1}(\sqrt{N})$. Note that the complexity $\Theta \times, \rho(E, x)$ does not scale with $\beta$. Therefore, from the definition (5.3) of $\Lambda_{Z, \beta}(E, \rho)$, one has that for any $\rho \in [\rho_c, 1]$, for large $\beta$ the right-hand side of (2.8) is maximized over $(-E_*(\rho), \infty)$ with $E = -E_*(\rho)$ (where, of course, lower $E$ are not relevant as there are typically no $\rho$-critical points with such energy). For the range $\rho \in [1 - c \tau(\beta), \rho_c]$ further analysis based on the concentration of the free energy will be needed, but our conclusion will be the same — the right-hand side of (2.9) is also maximized with $E = -E_*(\rho)$.

For the ground state $E = -E_*(\rho)$, the complexity term $\sup_{x \in \mathbb{R}} \Theta \times, \rho(E, x)$ vanishes (see Remarks 3.4 and 5.12). Moreover, if we take $\epsilon$ to be small, so that at exponential scale the number of points in $\mathcal{E}_{N,\rho}(NB)$ is small, the large-deviation term $K(\rho, E, 0)$ becomes negligible. Combining the above, to prove (2.6), roughly

---

6 Atypical in the sense that $Z_{N,\beta}^2(\sigma_0)$ is much larger than its expectation conditional on (2.7), with $E = H_N(\sigma_0)/N$. 
what we will have to show is that for \( \rho \in [1 - c \tau (\beta), 1] \setminus \{ \rho_\ast \} \),

\[
\Lambda_\beta (\rho_\ast) = F_\beta > \Lambda_\beta (\rho) := \begin{cases} 
\Lambda_{Z, \beta} (-E_\ast (\rho), \rho), & \rho \in [\rho_c, 1], \\
\Lambda_{F, \beta}^2 (-E_\ast (\rho), \rho), & \rho \in [1 - c \tau (\beta), \rho_c],
\end{cases}
\]

for large \( \beta \). In fact, we will only need to prove the inequality, but our proof will go through showing that \( \Lambda_\beta (\rho_\ast) = F_\beta \).

### 2.6 The Lower Bound on the Free Energy

In order to lower bound the limiting free energy \( F_\beta \), we shall consider the collections of bands corresponding to \( \rho_\ast \) and \(-E_\ast (\rho) = -E_\ast (\rho_\ast) \). From the moment matching of the number of \( \rho_\ast \)-critical points in \( C \) (see Corollary 3.6), we have that with probability that decays slower than exponentially in \( N \), \( \mathcal{C}_\ast \) is nonempty.\(^7\)

To prove the lower bound we will show in Section 10 that for any \( \delta > 0 \), there are no points \( \sigma_0 \) in \( \mathcal{C}_\ast \) for which

\[
Z_{N, \beta} (\sigma_0) < \Lambda_{Z, \beta} (-E_\ast (\rho_\ast), \rho_\ast) - \delta.
\]

Consequently, with probability that decays slower than exponentially in \( N \),

\[
(2.11) \quad F_\beta > \Lambda_{Z, \beta} (-E_\ast (\rho_\ast), \rho_\ast) - \delta.
\]

From the concentration of the free energy around its mean, we then conclude that \( (2.11) \) occurs with probability that tends to 1 as \( N \to \infty \).

### 3 Critical Points: Main Results and Notation

A crucial step in the analysis of the Gibbs measure \( G_{N, \beta} \) is the study of the critical points of \( H_N (\cdot) \) on \( S^{N-1}(\rho \sqrt{N}) \). We carry out this analysis by applying the second-moment method to

\[
(3.1) \quad \text{Crt}_{N, \rho} (B, D) := |\mathcal{C}_{N, \rho} (NB, \sqrt{N}D)|,
\]

where

\[
\mathcal{C}_{N, \rho} (B, D) := \left\{ \sigma \in S^{N-1}(\rho \sqrt{N}) : \nabla_{sp} H_N (\sigma) = 0, \ H_N (\sigma) \in B, \ \frac{d}{dR} H_N (\sigma) \in D \right\}
\]

denotes the set of \( \rho \)-critical points with values in \( B \) and “normal” derivative

\[
\frac{d}{dR} H_N (\sigma) := \left. \frac{d}{d\rho} \right|_{\rho = |\sigma|} H_N (\rho \sigma / \| \sigma \|) = \frac{1}{\| \sigma \|} \left. \frac{d}{d\rho} H_N (\rho \sigma) \right|_{\rho = 1}
\]

in \( D \).

The logarithmic asymptotics of \( \mathbb{E} \text{Crt}_{N, 1} (B, \mathbb{R}) \) were calculated by Auffinger and Ben Arous \([5]\) (see also \([6]\) for the pure case). We shall need the next theorem,\(^7\)

---

\(^7\)This is what follows from matching at exponential scale. Had we established matching at scale \( O(1) \), as in the pure case \([52]\), the same probability would have gone to 1.
which is a straightforward extension of their computation, accounting for general
$D$ and $\rho$.

Let $\mu^*$ denote the semicircle probability measure, the density of which with
respect to Lebesgue measure is

$$
d\mu^* = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{|x| \leq 2}. 
$$

(3.2)

Define the functions (see, e.g., [26, prop. II.1.2])

$$
\Theta_{\nu, \rho}(u, x) \triangleq \int_{\mathbb{R}} \log |\lambda - x| d\mu^*(\lambda)
$$

(3.3)

$$
= \begin{cases} 
\frac{x^2}{4} - \frac{1}{2} \left[ \frac{\sqrt{x^2 - 4}}{x^2 - 4} - \log \left( \frac{\sqrt{x^2}}{2} + \frac{|x|}{2} \right) \right] & \text{if } |x| > 2, \\
\frac{x^2}{4} - \frac{1}{2} \left[ \frac{\sqrt{x^2 - 4}}{x^2 - 4} - \log \left( \frac{\sqrt{x^2}}{2} + \frac{|x|}{2} \right) \right] & \text{if } 0 \leq |x| \leq 2 
\end{cases}
$$

where

$$
\Sigma_{\rho} := \begin{pmatrix}
\nu(\rho^2) & \rho^2 \nu'(\rho^2) \\
\rho^2 \nu'(\rho^2) & \rho^4 \nu''(\rho^2) + \rho^2 \nu'(\rho^2)
\end{pmatrix}
$$

is the covariance matrix of the vector $(H_N(\sigma) / \sqrt{N}, \frac{d}{d\rho} H_N(\sigma))$, and is invertible
by Lemma [3.8] whose statement and proof are given at the end of this section. The
next theorem, whose proof appears in Section 4, is an evaluation of the exponential
rate of growth of the expectation of (3.1).

**Theorem 3.1 (First moment).** For any intervals $B$ and $D$, with $\Theta_{\nu, \rho}(u, x)$ as
defined in (3.3),

$$
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Cr}_{t N, \rho}(B, D) = \sup_{u \in B, x \in D} \Theta_{\nu, \rho}(u, x).
$$

(3.6)

We shall also need an asymptotic upper bound on the corresponding second
moment. For any subsets $I \subset [-1, 1]$, $B_i, D_i \subset \mathbb{R}$, define the “contribution” of
pairs of points with overlap in $I$ to $\prod_{i=1,2} \mathbb{E}_{N, \rho_i}(NB_i, \sqrt{N} D_i)$ by

$$
\left[ \text{Cr}_{t N, \rho_1, \rho_2}(B_1, B_2, D_1, D_2, I) \right]_2 
$$

$$
:= \# \left\{ (\sigma, \sigma') \in \prod_{i=1,2} \mathbb{E}_{N, \rho_i}(NB_i, \sqrt{N} D_i) : R(\sigma, \sigma') \in I \right\}.
$$

(3.7)
Define the function
\[
\begin{align*}
\Psi_{\nu, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2) \\
:= 1 + \frac{1}{2} \log \left( \frac{1 - r^2}{v'(\rho_1^2)v'(\rho_2^2)} - \frac{\rho_1^2\rho_2^2 v''(\rho_1^2)v''(\rho_2^2)}{v'(\rho_1^2)v'(\rho_2^2)} \right) \\
&- \frac{1}{2} (u_1, u_2, x_1, x_2)^T \Sigma_{U, X}^{-1}(r, \rho_1, \rho_2)(u_1, u_2, x_1, x_2) + \Omega \left( \frac{x_1}{\rho_1 \sqrt{v''(\rho_1^2)}} \right) + \Omega \left( \frac{x_2}{\rho_2 \sqrt{v''(\rho_2^2)}} \right),
\end{align*}
\] (3.8)

where \(\Sigma_{U, X}(r, \rho_1, \rho_2)\), computed explicitly in (A.4), is the covariance matrix of the vector
\[
\begin{align*}
\left( H_N(\sigma)/\sqrt{N}, H_N(\sigma')/\sqrt{N}, \frac{d}{dR} H_N(\sigma), \frac{d}{dR} H_N(\sigma') \right)
\end{align*}
\] (3.9)

with \((\sigma, \sigma') \in \mathbb{S}^{N-1}(\rho_1 \sqrt{N}) \times \mathbb{S}^{N-1}(\rho_2 \sqrt{N})\) with overlap \(r\), conditioned on \(\nabla_{sp} H_N(\sigma), \nabla_{sp} H_N(\sigma')\); see Lemma 4.2. Lemma 3.8 implies that this covariance matrix is invertible. With a slight abuse of notation, we write
\[
\Psi_{\nu, \rho_1, \rho_2}(r, u, x) = \Psi_{\nu, \rho_1, \rho_2}(r, u, u, x, x).
\]

We note for later use that if we substitute \(r = 0\), by simple algebra,
\[
\Psi_{\nu, \rho_1, \rho_2}(0, u_1, u_2, x_1, x_2) = \Theta_{\nu, \rho_1}(u_1, x_1) + \Theta_{\nu, \rho_2}(u_2, x_2).
\]

The following theorem and lemma are extensions of theorem 5 of [52], which concerns the pure case with \(\rho_i = 1\). The proofs are given in Section 4.

**Theorem 3.2 (Second moment).** For any intervals \(I \subset (-1, 1)\), \(B_i, D_i \subset \mathbb{R}\), with \(\Psi_{\nu, \rho_1, \rho_2}\) as defined in (3.8),
\[
\begin{align*}
\limsup_{N \to \infty} N \log \mathbb{E}[\text{Crt}_{N, \rho_1, \rho_2}(B_1, B_2, D_1, D_2, I)]_2 \\
\leq \sup_{r \in I, u_j \in B_j, x_j \in D_j} \Psi_{\nu, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2).
\end{align*}
\] (3.12)

The next lemma shows that the bound in Theorem 3.2 is tight in the region of interest. Besides its intrinsic reassuring value, this fact will be used explicitly in the proof of Lemma 5.7 below.

**Lemma 3.3.** If \(D_i \subset (-\infty, -2 - \sqrt{v''(\rho_i^2)\rho_i - \tau})\) for some \(\tau > 0\), then (3.12) holds with equality.

We next turn to the study of consequences of Condition M. In addition to \(E_0\), which was introduced in (1.7), the energy level
\[
\begin{align*}
E_\infty := E_\infty(\nu) = \frac{v''(1)v(1) + v'(1)^2 - v'(1)v(1)}{v'(1)\sqrt{v''(1)}}
\end{align*}
\] (3.13)
was defined in \([5, 6]\), as a threshold energy related to the spectrum of the Hessian matrix \(r_{2sp}H_N(\sigma)\) at critical points. As we will see, it plays a role in several large-deviation problems and concentration of statistics, and for pure-like models it is related to the function \(G(v)\) defined in \([1, 5]\) (see \([5, \text{eq} \ (1.22), \ (4.1)]\) through the relation

\[(3.14) \quad G(v) = \sup_{x \in \mathbb{R}} \Theta_{v,1}(-E_\infty, x).\]

For \(\rho \in (0, 1]\), set \(v_\rho(x) = \sum_{\nu} \gamma^2_{\nu} \rho^{2\nu} x^\nu\) and define \(E_0(\rho) := E_0(v, \rho) = E_0(v_\rho)\) and similarly define \(E_\infty(\rho)\). For pure-like \(v,\) define \(x_0(\rho) := x_0(v, \rho) = \frac{1}{\rho} x_0(v_\rho)\). The next remark summarizes scaling relations associated to these quantities.

**Remark 3.4.** Fix the disorder coefficients \(J_{i_1, \ldots, i_p}\) in \([1, 1]\), and let \(H_N(\sigma)\) be the Hamiltonian corresponding to the mixture \(v_\rho(x)\). Then \(H_N(\sigma) = H(\sigma)\) and \(\frac{d}{d\rho} H_N(u_\rho) = \frac{d}{d\rho} H_N(u)\). Therefore, \(\Theta_{v, \rho}(u, x) = \Theta_{v, \rho}(u, \rho x)\) and similarly to \((1.7)\) and \((1.9)\), we have

\[(3.15) \quad -E_0(\rho) = \min \{E : \sup_{x \in \mathbb{R}} \Theta_{v, \rho}(E, x) = 0\},\]

\[(3.16) \quad -x_0(\rho) = \arg \max_{x \in \mathbb{R}} \Theta_{v, \rho}(-E_0(\rho), x).\]

The following lemma implies the matching of the second and first moment squared of \((3.1)\) at exponential scale as \(N \to \infty\) for small \(E\) and \(D\) around \(-E_0(\rho)\) and \(-x_0(\rho)\). The proof is contained in Section 5.

**Lemma 3.5.** For any \(v\) satisfying Condition M, there exists some \(\delta > 0\) such that if \(|\rho - 1| < \delta, B_i \subset -E_0(\rho_i) + (-\delta, \delta)\), and \(D_i \subset -x_0(\rho_i) + (-\delta, \delta)\), then for any \(\epsilon > 0,
\[
\sup_{|r| \in \epsilon, 1, u_i \in B_i, x_i \in D_i} \psi_{v, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2)
\leq \sum_{i = 1, 2} \sup_{u_i \in B_i, x_i \in D_i} \Theta_{v, \rho_2}(u_i, x_i),
\]

whenever both summands on the right-hand side of \((3.17)\) are nonnegative.

As a consequence of Lemma 3.5 and Theorem 3.1 we have the matching of the moments at exponential scale.

**Corollary 3.6 (Matching of moments).** With \(v\) and \(\delta\) as in Lemma 3.5 assume \(|\rho - 1| < \delta, B \subset -E_0(\rho) + (-\delta, \delta)\), and \(D \subset -x_0(\rho) + (-\delta, \delta)\). Then

\[(3.18) \quad \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}(\text{Crt}_{N, \rho}(B, D)^2) = 2 \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N, \rho}(B, D),\]

as soon as the expectation on the right-hand side tends to \(\infty\) as \(N \to \infty\). The equality in \((3.18)\) continues to hold if we let \(B = B_N = u + (-\delta_N, \delta_N)\) and
Another interesting consequence of Lemma 3.5 is the next corollary, whose proof appears in Section 5.

**Corollary 3.7 (Orthogonality of deep critical points).** Assume Condition M. Then there exists $\delta_0 > 0$ so that for any $0 < \delta < \delta_0$, $\rho_1, \rho_2 \in (1 - \delta, 1 + \delta)$, and $\epsilon > 0$, there exist constants $\eta = \eta(\epsilon), c = c(\epsilon) > 0$ so that, with $B_i(\eta) := (-E_0(\rho_i) - \eta, -E_0(\rho_i) + \eta)$.

\[
\Pr\left\{ \exists \sigma_i \in \mathcal{C}_N, R(\sigma_1, \sigma_2) \noleq \pm \sigma_2 : |R(\sigma_1, \sigma_2)| \geq \epsilon \right\} < e^{-cN}.
\]

Moreover, for any $\eta_N = o(1)$, setting $B_{N,i} := (-\infty, -E_0(\rho_i) + \eta_N)$, there exists a sequence $\epsilon_N = o(1)$ such that

\[
\lim_{N \to \infty} \Pr\left\{ \exists \sigma_i \in \mathcal{C}_N, \sigma_1 \neq \pm \sigma_2 : |R(\sigma_1, \sigma_2)| \geq \epsilon_N \right\} = 0.
\]

We finish this section with the statement and proof of the following lemma concerning the invertibility of the matrices $\Sigma_{U,X}(r, \rho_1, \rho_2)$ and $\Sigma_\rho$.

**Lemma 3.8.** If $v$ is not a monomial, then for any $r \in (-1, 1)$, $\Sigma_{U,X}(r, \rho_1, \rho_2)$ and $\Sigma_\rho$ are invertible (and therefore strictly positive definite) for any $\rho_1, \rho_2, \rho \in (0, 1]$.

**Proof.** Recall that $\Sigma_{U,X}(r, \rho_1, \rho_2)$ is the covariance matrix of the vector (3.9) conditional on the gradients at the two corresponding points. Suppose $H^v_N(\sigma) = H^v_{U,1}(\sigma) + H^v_{U,2}(\sigma)$ is the Hamiltonian corresponding to $v = v_1 + v_2$, where the $H^v_{U,i}(\sigma)$ are independent. Using a similar notation for $\Sigma_{U,X}(r, \rho_1, \rho_2)$, we have that if $\Sigma^v_{U,X}(r, \rho_1, \rho_2) + \Sigma^v_{U,X}(r, \rho_1, \rho_2)$ is invertible, then so is $\Sigma^v_{U,X}(r, \rho_1, \rho_2)$ (since the former corresponds to the distribution obtained by conditioning on each of the gradients corresponding to $H^v_{U,i}(\sigma)$, $i = 1, 2$, and the latter corresponds to conditioning on the sum of those gradients being 0). Thus, since $\Sigma^v_{U,X}(r, \rho_1, \rho_2)$ is positive semidefinite, to prove invertibility for a general non-pure mixture, it is enough to prove that $\Sigma^v_{U,X}(r, \rho_1, \rho_2) + \Sigma^v_{U,X}(r, \rho_1, \rho_2)$ is invertible for any $v_1(x) = \gamma x^p$ and $v_2(x) = \gamma' x^{p'}$ with $p \neq p'$ and $\gamma, \gamma' > 0$.

From the formula [A, 4] for $\Sigma_{U,X}(r, \rho_1, \rho_2)$, we have that if $v_i(x) = x^{\rho_i}$ and $a(r, \rho, p) := \rho^{2(p-1)} \left[ 1 - \frac{p r^{2(p-1)}(1 - r^2)}{1 - (r p - (p - 1) r p^2(1 - r^2))^2} \right]$, $b(r, p) := r^p (\rho_1 \rho_2)^{p-1}$

\[
\begin{aligned}
1 - p r^{p-2}(1 - r^2) & \frac{r^p - (p - 1) r p^2(1 - r^2)}{1 - (r p - (p - 1) r p^2(1 - r^2))^2} \\
\end{aligned}
\]


then
\[ \Sigma_{U,X}^{v_i}(r, \rho_1, \rho_2) = \begin{pmatrix} \rho_1^2 a(r, \rho_1, \rho_2) & \rho_1 \rho_2 b(r, \rho_i) & \rho_i \rho_1 a(r, \rho_1, \rho_i) & \rho_i \rho_1 b(r, \rho_i) \\ \rho_1 \rho_2 b(r, \rho_i) & \rho_2^2 a(r, \rho_2, \rho_i) & \rho_i \rho_2 b(r, \rho_i) & \rho_i \rho_2 a(r, \rho_2, \rho_i) \\ \rho_i \rho_1 a(r, \rho_1, \rho_i) & \rho_i \rho_2 b(r, \rho_i) & \rho_i^2 a(r, \rho_1, \rho_i) & \rho_i^2 b(r, \rho_i) \\ \rho_i \rho_1 b(r, \rho_i) & \rho_i \rho_2 b(r, \rho_i) & \rho_i^2 b(r, \rho_i) & \rho^2 a(r, \rho_2, \rho_i) \end{pmatrix}. \]

Therefore, if \((U_1, U_2) \sim N(0, \Sigma_{U_i}^{v_i}(r, \rho_1, \rho_2))\), where \(\Sigma_{U_i}^{v_i}(r, \rho_1, \rho_2)\) is the upper left \(2 \times 2\) submatrix of \(\Sigma_{U,X}^{v_i}(r, \rho_1, \rho_2)\), then
\[ \left( U_1, U_2, \frac{\bar{p}_i}{\rho_1} U_1, \frac{\bar{p}_i}{\rho_2} U_2 \right) \sim N(0, \Sigma_{U,X}^{v_i}(r, \rho_1, \rho_2)). \]
Since \(\Sigma_{U_i}^{v_i}(r, \rho_1, \rho_2)\) is invertible whenever \(|r| \neq 1\), we have that
\[ (x_1, x_2, y_1, y_2) \Sigma_{U,X}^{v_i}(r, \rho_1, \rho_2) = 0 \]
if and only if \(x_1 + \frac{\bar{p}_i}{\rho_1} y_1 = 0 \) and \(x_2 + \frac{\bar{p}_i}{\rho_2} y_2 = 0\). Using the positive definiteness of \(\Sigma_{U,X}^{v_i}\) we deduce that \((x_1, x_2, y_1, y_2) \Sigma_{U,X}^{v_i}(r, \rho_1, \rho_2) = 0\) iff \((x_1, x_2, y_1, y_2) = 0\).
This proves the invertibility of \(\Sigma_{U,X}^{v_i}(r, \rho_1, \rho_2)\) for general mixtures.
Noting that \(\Sigma_{\rho_1}\) is the \(2 \times 2\) submatrix obtained from \(\Sigma_{U,X}(0, \rho_1, \rho_2)\) by deleting the second and fourth rows and columns, we conclude the invertibility of \(\Sigma_{\rho_1}\).

Strict positive definiteness follows from invertibility, since both matrices are covariance matrices.

\[ \square \]

4 Computation of Moments: Proofs of Theorems 3.1, 3.2 and Lemma 3.3

This section is devoted to moment computations, yielding the proofs of the results in the title. The proofs rely on tedious computations of certain covariance matrices, which are contained in Appendix A.

4.1 Proof of Theorem 3.1 (First moment)

By an application of the Kac-Rice formula \([1,\text{ theorem 12.1.1}]\), using the stationarity of \((H_N(\sigma), \frac{d}{d \rho} H_N(\sigma))\) on \(S^{N-1}(\sqrt{N} \rho)\), we obtain
\[ \mathbb{E} \text{Crt}_{N, \rho}(B, D) = \rho^{-1} \omega_N \varphi \nabla_{\varphi} H_N(\sigma)(0) \times \mathbb{E} \left\{ \det(\nabla_{\varphi}^2 H_N(\sigma)) \right\} \frac{d}{d \rho} H_N(\sigma) \in \sqrt{N} D \bigg| \nabla_{\varphi} H_N(\sigma) = 0 \right\}, \]
where \(\sigma \in S^{N-1}(\sqrt{N} \rho)\) is arbitrary, \(\varphi \nabla_{\varphi} H_N(\sigma)(0)\) is the density of \(\nabla_{\varphi} H_N(\sigma)\) at 0, and \(\omega_N = 2\pi^{N/2}/\Gamma(N/2)\) is the surface area of the \((N - 1)\)-dimensional
unit sphere. By a covariance computation contained in Lemma A.1 of Appendix A (applied with $r = 1$), the three variables

$$
\left( H_N(\sigma), \frac{d}{dR} H_N(\sigma), \frac{\nabla^2_{sp} H_N(\sigma)}{\sqrt{N}} + \frac{d}{\sqrt{N}\rho} dR H_N(\sigma), \frac{\nabla_{sp} H_N(\sigma)}{\sqrt{N}} \right)
$$

are independent, $\nabla_{sp} H_N(\sigma) \sim N(0, \nu'(\rho^2)I)$,

(4.1) $$
\left( \frac{1}{\sqrt{N}} H_N(\sigma), \frac{d}{dR} H_N(\sigma) \right) \sim N(0, \Sigma_{\rho})
$$

with $\Sigma_{\rho}$ as defined in (3.5), and

$$
G = \sqrt{\frac{N}{(N-1)\nu'(\rho^2)}} \left( \frac{\nabla^2_{sp} H_N(\sigma)}{\sqrt{N}} + \frac{d}{\sqrt{N}\rho} dR H_N(\sigma)I \right)
$$

is a (normalized) GOE matrix, that is, a real, symmetric $N - 1 \times N - 1$ matrix such that all elements are centered Gaussian variables that, up to symmetry, are independent with variance given by

$$
\mathbb{E}\{G^2_{ij}\} = \begin{cases} 
1/(N-1), & i \neq j, \\
2/(N-1), & i = j.
\end{cases}
$$

Combining the above, after some algebra, we have that as $N \to \infty$,

$$
\mathbb{E} \text{Crt}_{N,\rho}(B, D) = \frac{N}{2} + o(N) \left( \frac{\rho^2 \nu'(\rho^2)}{\nu'(\rho^2)} \right) \left( \frac{N}{2} \right) \times 
\int \limits_{\sqrt{N}(B \times D)} dudx \exp \left\{ -\frac{1}{2}(u, x)\Sigma_{\rho}^{-1}(u, x)^{\top} \right\}
\right\}
\times \mathbb{E} \left\| \log \left( \frac{\det(G - \frac{x}{\rho \sqrt{(N-1)\nu'(\rho^2)}} I)}{\sqrt{N}} \right) \right\|.
$$

(4.2)

The determinant in (4.2) can be written as $\exp(\sum \log |\lambda_i|)$, where $\lambda_i$ are the corresponding eigenvalues. An upper bound on the right-hand side of (4.2), which gives the inequality \leq in (3.6), is obtained by combining Varadhan's integral lemma [25, theorem 4.3.1, exercise 4.3.11] and the large-deviation principle satisfied by the empirical measure of eigenvalues of GOE matrices [14, theorem 2.1.1] (together with a truncation argument based on the upper bound for top eigenvalue [12, lemma 6.3] of GOE matrices). We will discuss a similar argument in the much more complicated case of bounding the expectation of $[\text{Crt}_{N,\rho_1,\rho_2}(B_1, B_2, D_1, D_2, I)]_2$ in the proof of Theorem 3.2. Therefore, we refrain from going into the details here.
To obtain the reverse inequality \( \geq \) in (3.6), it is enough to show that for any \( t \in \mathbb{R} \),

\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \int_{t-\epsilon}^{t+\epsilon} \mathbb{E} \{ |\det(G - uI)| \} \, du = \Omega(t),
\]

where \( \Omega \) is as in (3.3). A direct proof of (4.3) can be obtained using the exponential in \( N^2 \) convergence of the empirical measure of eigenvalues of \( G \) to the semicircle law, together with the overcrowding estimates in [38, theorem 1.12] and a truncation of the logarithm. Instead of providing the details, we use the following observation. For the pure case \( v_p(x) = x^p \), it was shown in [6, (3.21)] that

\[
\mathbb{E} \text{Cr}t_{N,1}(B, \mathbb{R}) = e^{\frac{N}{2} + o(N)} (p - 1) \frac{N}{2} \int_{\sqrt{N}B} e^{-\frac{u^2}{2}} \mathbb{E} \left\{ \left| \det \left( G - u \sqrt{\frac{p}{(N-1)(p-1)}} \right) \right| \right\} \, du.
\]

On the other hand, it is proved in [6, theorem 2.8] that

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Cr}t_{N,1}(B, \mathbb{R}) = \sup_{u \in B} \left\{ \frac{1}{2} \log(p - 1) - \frac{u^2}{2} + \Omega \left( u \sqrt{\frac{p}{p-1}} \right) \right\}.
\]

By considering the intervals \( B = (t - \epsilon, t + \epsilon) \), the above implies (4.3) and completes the proof. \( \square \)

### 4.2 Proof of Theorem 3.2 (Second moment)

Throughout the proof we fix the intervals \( B_i, D_i \subset \mathbb{R} \), and \( I \subset (-1, 1) \) and the numbers \( \rho_1, \rho_2 \in (0, 1] \). The proof follows closely that of [52, theorem 5] (see section 5.4 there) and requires, in particular, variants of the auxiliary lemmas 11–16 of [52]. An application of the Kac-Rice formula [1, theorem 12.1.1] and isotropy yield the integral formula

\[
\mathbb{E} \left\{ \text{Cr}t_{N,\rho_1,\rho_2}(B_1, B_2, D_1, D_2, I) \right\} = \omega_{N\rho_1}^{N-1} (N-1)^{N-1} \left( \rho_1^2 \rho_2^2 \right)^{\frac{N-1}{2}} \times \int_{I\mathbb{R}} dr \cdot \left( 1 - r^2 \right)^{\frac{N-1}{2}} \varphi \nu_{\rho_1} H_N(\rho_1 \mathfrak{a}), \nu_{\rho_2} H_N(\rho_2 \sigma(r)) (0,0) \times
\]
Their computationally heavy proof is postponed to Appendix A.

The following three lemmas, generalizing [52, lemmas 12 and 13] to the mixed case, are concerned with the joint law of the random variables appearing in (4.4). Their computationally heavy proof is postponed to Appendix A.

\begin{equation}
\sigma (r) := \sqrt{N} (0, \ldots, 0, \sqrt{1-r^2}, r),
\end{equation}

\( \varphi_{\nabla sp H_N (\rho_1 \hat{n})} \), \( \nabla sp H_N (\rho_2 \sigma (r)) \) is the joint Gaussian density of \( \nabla sp H_N (\rho_1 \hat{n}) \) and \( \nabla sp H_N (\rho_2 \sigma (r)) \). This has been worked out in [52, lemma 11] for pure spherical models and \( \rho_1 = \rho_2 = 1 \), \( B_1 = B_2 \), and \( D_1 = D_2 = \mathbb{R} \), but the proof in the mixed case is similar.

The following three lemmas, generalizing [52, lemmas 12 and 13] to the mixed case, are concerned with the joint law of the random variables appearing in (4.4). Their computationally heavy proof is postponed to Appendix A.

**Lemma 4.1 (Density of gradients).** For any \( r \in (-1, 1) \) and \( \rho_1, \rho_2 \in (0, 1] \), there exists a choice of the orthonormal frame field \( F = (F_i)_{i=1}^{N-1} \) such that the density of \( (\nabla sp H_N (\rho_1 \hat{n}), \nabla sp H_N (\rho_2 \sigma (r))) \) at \( (0, 0) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \) is given by

\begin{equation}
(2\pi)^{-(N-1)} \left[ \nu' (\rho_1^2) \nu' (\rho_2^2) - (\nu' (\rho_1 \rho_2 r))^2 \right]^{N-2 \over 2} \times \left[ \nu' (\rho_1^2) \nu' (\rho_2^2) - (r \nu' (\rho_1 \rho_2 r) - \rho_1 \rho_2 \nu'' (\rho_1 \rho_2 r) (1-r^2))^2 \right]^{-1 \over 2}.
\end{equation}

**Lemma 4.2 (Conditional law of Hamiltonians and normal derivatives).** With notation as in Lemma 4.1 conditional on

\begin{equation}
(\nabla sp H_N (\rho_1 \hat{n}), \nabla sp H_N (\rho_2 \sigma (r))) = (0, 0),
\end{equation}

the vector \( V(r) := V(r, \rho_1, \rho_2) \) defined by

\begin{align}
V(r) &= \left( \frac{1}{\sqrt{N}} H_N (\rho_1 \hat{n}), \frac{1}{\sqrt{N}} H_N (\rho_2 \sigma (r)), \right.
onumber \\
&\quad \left. \frac{d}{dR} H_N (\rho_1 \hat{n}) \right) \frac{d}{dR} H_N (\rho_2 \sigma (r))
\end{align}

is a centered Gaussian vector with covariance matrix \( \Sigma U, \chi (r, \rho_1, \rho_2) \) (see (A.4)).
LEMMA 4.3 (Conditional law of Hessians). With notation as in Lemma 4.1 conditional on (4.7), the joint distribution of the matrices

\[
\begin{align*}
&\left(\sqrt{\frac{N}{(N-1)\nu''(\rho_1^2)}} \left( \nabla^2_{sp} H_N(\rho_1 \hat{n}) + \frac{1}{\sqrt{N\rho_1}} \frac{d}{dR} H_N(\rho_1 \hat{n}) I \right) \right) \times \\
&\quad \left(\sqrt{\frac{N}{(N-1)\nu''(\rho_2^2)}} \left( \nabla^2_{sp} H_N(\rho_2 \sigma(r)) + \frac{1}{\sqrt{N\rho_2}} \frac{d}{dR} H_N(\rho_2 \sigma(r)) I \right) \right)
\end{align*}
\]

is the same as that of

\[
(A^{(1)}_{N-1}(r, \rho_1, \rho_2), A^{(2)}_{N-1}(r, \rho_1, \rho_2)),
\]

where

\[
A^{(i)}_{N-1}(r, \rho_1, \rho_2) = \hat{M}^{(i)}_{N-1}(r, \rho_1, \rho_2) + \sqrt{\frac{N}{(N-1)\nu''(\rho_i^2)}} m_i(r, \rho_1, \rho_2) e_{N-1, N-1},
\]

\[
m_i(r, \rho_1, \rho_2) = \frac{1}{\sqrt{N}} (1 - r^2) V(r, \rho_1, \rho_2) \Sigma_{U, X}(r, \rho_1, \rho_2) \xi_i(r, \rho_1, \rho_2).
\]

e_{N-1, N-1} is the \( N-1 \times N-1 \) matrix whose \( N-1, N-1 \) entry is equal to 1 and all other entries are 0. \( V(r, \rho_1, \rho_2) \) is given by (4.8). \( \Sigma_{U, X}(r, \rho_1, \rho_2) \) and \( \xi_i(r, \rho_1, \rho_2) \) are given by (A.4) and (A.3), and \( \hat{M}^{(i)}_{N-1}(r, \rho_1, \rho_2) \) are \( N-1 \times N-1 \) Gaussian random matrices independent of (4.8) with block structure

\[
\begin{align*}
\hat{M}^{(i)}_{N-1}(r, \rho_1, \rho_2) &= \begin{pmatrix} G^{(i)}_{N-2}(r) & Z^{(i)}(r) \\ (Z^{(i)}(r))^T & Q^{(i)}(r) \end{pmatrix},
\end{align*}
\]

satisfying the following:

1. \( (\hat{G}^{(1)}_{N-2}(r), \hat{G}^{(2)}_{N-2}(r)), (Z^{(1)}(r), Z^{(2)}(r)), \) and \( (Q^{(1)}(r), Q^{(2)}(r)) \) are independent.

2. \( \hat{G}^{(i)}(r) = \hat{G}^{(i)}_{N-2}(r) \) are \( N-2 \times N-2 \) random matrices such that \( \sqrt{\frac{N-1}{N-2}} \hat{G}^{(i)}(r) \) is a GOE matrix and, in distribution,

\[
\begin{pmatrix} \hat{G}^{(1)}(r) \\ \hat{G}^{(2)}(r) \end{pmatrix} = \begin{pmatrix} (1 - \phi(r))^{1/2} \hat{G}^{(1)} + \text{sgn}(\nu''(\rho_1 \rho_2 r))\phi(r)^{1/2} \hat{G} \\ (1 - \phi(r))^{1/2} \hat{G}^{(2)} + \phi(r)^{1/2} \hat{G} \end{pmatrix},
\]

where

\[
\phi(r) := \frac{|\nu''(\rho_1 \rho_2 r)|}{\sqrt{\nu''(\rho_1^2)\nu''(\rho_2^2)}}
\]
and where $G = G_{N-2}$, $G^{(1)} = G_{N-2}^{(1)}$, and $G^{(2)} = G_{N-2}^{(2)}$ are independent of each other and have the same law as $\hat{G}^{(i)}(r)$, that is, scaled GOE.

3. $Z^{(i)}(r) = (Z^{(i)}_j(r))_{j=1}^{N-2}$ are Gaussian vectors such that $(Z^{(1)}_j(r), Z^{(2)}_j(r))$ are independent for different $j$ and, with $\Sigma_Z(r)$ given by (A.2),

$$(Z^{(1)}_j(r), Z^{(2)}_j(r)) \sim N \left(0, \frac{1}{(N-1)} \Sigma_Z(r, \rho_1, \rho_2) \right).$$

4. With $\Sigma_Q(r)$ given by (A.2),

$$(Q^{(1)}(r), Q^{(2)}(r)) \sim N \left(0, \frac{1}{(N-1)} \Sigma_Q(r, \rho_1, \rho_2) \right).$$

By Lemma 4.2, conditional on (4.7), the vector (4.8) has the same distribution as

$$(4.11) \quad (U_1(r), U_2(r), X_1(r), X_2(r)) \sim N \left(0, \Sigma_{U, X}(r, \rho_1, \rho_2) \right).$$

>From (4.4), Lemmas 4.1 and 4.3 and some calculus,

$$(4.12) \quad \mathbb{E} \left[ \text{Crt}_{N, \rho_1, \rho_2}(B_1, B_2, D_1, D_2, I) \right]_2 = C_N \int_1^r d \tau \cdot (D(r))^{N-3} F(r) \mathbb{E} \left\{ \prod_{i=1,2} \left| \text{det} \left( M^{(i)}_{N-1}(r) \right) \right|^{\frac{1}{2}} \right\},$$

where

$$E_i = \{ U_i(r) \in \sqrt{N} B_i, X_i(r) \in \sqrt{N} D_i \}.$$

$$M^{(i)}_{N-1}(r) = A^{(i)}_{N-1}(r) - \sqrt{\frac{1}{(N-1) r''(\rho_i^2)}} \frac{1}{\rho_i} X_i(r) I,$$

$A^{(i)}_{N-1}(r) := A^{(i)}_{N-1}(r, \rho_1, \rho_2)$ are defined by (4.9) and assumed to be independent of (4.11), and

$$(4.13) \quad C_N = \omega_N \omega_{N-1} \left( \frac{1}{2 \pi (N-1) \rho_1 \rho_2} \frac{r''(\rho_1^2) r''(\rho_2^2)}{r'(\rho_1^2) r'(\rho_2^2)} \right)^{N-1},$$

$$D(r) = (1 - r^2)^{\frac{1}{2}} \left( 1 - \frac{(v'(\rho_1 \rho_2 r)^2}{r'(\rho_1^2) r'(\rho_2^2)} \right)^{-\frac{1}{2}},$$

$$F(r) = \left( 1 - \frac{(v'(\rho_1 \rho_2 r)^2}{r'(\rho_1^2) r'(\rho_2^2)} \right)^{-\frac{1}{2}} \times \left( 1 - \frac{r v'(\rho_1 \rho_2 r) - \rho_1 \rho_2 v'(\rho_1 \rho_2 r)(1 - r^2)}{\sqrt{r'(\rho_1^2) r'(\rho_2^2)}} \right)^{-\frac{1}{2}}.$$
Next, we relate the determinant of $M_{N-1}(r)$ to that of its $N-2 \times N-2$ upper left submatrix, which we denote by $G_{N-2}(r)$. With $\hat{G}_{N-2}(r)$ as in (4.10) we have

$$G_{N-2}(r) = \hat{G}_{N-2}(r) - \sqrt{\frac{1}{(N-1)p_i^2} \frac{1}{X_i(r)}} I.$$  

Set

$$W_i(r) = W_{i,N}(r) := \left( 2 \sum_{j=1}^{N-2} \left( (M_{N-1}(r))_{j,N-1} \right)^2 + \left( (M_{N-1}(r))_{N-1,N-1} \right)^2 \right)^{\frac{1}{2}}.$$

For any $\kappa > \epsilon > 0$ define

$$h_\epsilon(x) = \max\{\epsilon, x\}$$

and

$$h_\kappa^\epsilon(x) = \begin{cases} \epsilon & \text{if } x < \epsilon, \\ x & \text{if } x \in [\epsilon, \kappa], \\ \kappa & \text{if } x > \kappa, \end{cases}$$

so that $h_\epsilon(x)h_\kappa^\epsilon(x) = h_\epsilon(x)$. Lastly, define

$$\log^\kappa_\epsilon(x) = \log(h_\kappa^\epsilon(x)).$$

For a general real symmetric matrix $C$, let $\lambda_j(C)$ denote the eigenvalues of $C$.

By exactly the same proof as for lemma 14 of [52] (which does not involve probabilistic arguments) we have that for any $\epsilon > 0, r \in (-1, 1)$, almost surely,

$$|\det(M_{N-1}(r))| \leq \frac{W_i(r)(W_i(r) + \epsilon)}{\epsilon} \prod_{j=1}^{N-2} h_\epsilon(\lambda_j(G_{N-2}(r))).$$

With $\kappa > \epsilon$ and $2 \leq m \in \mathbb{N}$ arbitrary, set $t = t(m) := m/(m-1)$, and

$$\mathcal{E}_{\epsilon,\kappa}^{(1)}(r) = \mathbb{E}\left\{ \prod_{i=1,2} \prod_{j=1}^{N-2} h_\kappa^\epsilon(\lambda_j(G_{N-2}(r))) \right\} \prod_{i=1,2} E_i,$$

$$\mathcal{E}_{\epsilon,\kappa}^{(2)}(r) = \mathbb{E}\left\{ \prod_{i=1,2} \prod_{j=1}^{N-2} h_\kappa^\infty(\lambda_j(G_{N-2}(r))) \right\}^{2m},$$

$$\mathcal{E}_{\epsilon,\kappa}^{(3)}(r) = \mathbb{E}\left\{ \left( \frac{W_1(r)(W_1(r) + \epsilon)}{\epsilon} \right)^{4m} \right\} \mathbb{E}\left\{ \left( \frac{W_2(r)(W_2(r) + \epsilon)}{\epsilon} \right)^{4m} \right\}. $$
Then, from (4.12), (4.19), and Hölder’s inequality we have that
\[
\mathbb{E}\left\{ \prod_{i=1,2} \left| \det(M^{(i)}_{N-1}(r)) \right|^{\frac{1}{\|E_i\}} \right\} \leq \left( \xi_{\epsilon,\kappa}^{(1)}(r) \right)^{1/2} \left( \xi_{\epsilon,\kappa}^{(2)}(r) \right)^{1/4m} \left( \xi_{\epsilon,\kappa}^{(3)}(r) \right)^{1/4m} \]
and
\[
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E}\left\{ \left[ \text{Crt}_{N,\rho_1,\rho_2}(B_1, B_2, D_1, D_2, I) \right]_2 \right\} \right) \leq \limsup_{N \to \infty} \frac{1}{N} \log(C_N) + \limsup_{N \to \infty} \frac{1}{mN} \log \left( \int_I (F(r))^m (\xi_{\epsilon,\kappa}^{(2)}(r))^{1/2} (\xi_{\epsilon,\kappa}^{(3)}(r))^{1/4} \, dr \right) + \limsup_{N \to \infty} \frac{1}{tN} \log \left( \int_I (D(r))^{(N-3)} (\xi_{\epsilon,\kappa}^{(1)}(r)) \, dr \right) =: \Delta_1 + \Delta_II + \Delta_{III}. \tag{4.21}
\]
We note that
\[
\Delta_1 = 1 + \log \left( \rho_1 \rho_2 \frac{v''(\rho_1^2)}{v''(\rho_2^2)} \frac{1}{\rho_1} \frac{1}{\rho_2} \right).
\]
To complete the proof of Theorem 3.2 we will show that \( \Delta_{III} \leq 0 \) if \( \kappa \) is large enough, and that
\[
\lim_{\epsilon \to 0} \lim_{\kappa \to \infty} \lim_{m \to \infty} \Delta_{III} = \sup_{r \in I, u_i \in B_i, x_i \in D_i} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) - \Delta_1. \tag{4.22}
\]
By a similar proof to that of lemma 16 of [52] (essentially all that is needed is to replace \( \tilde{U}_i(r) \) by
\[
\sqrt{\frac{1}{(N-1)v''(\rho_i^2)} \frac{1}{\rho_i} X_i(r)}
\]
everywhere in the proof), using the large-deviation principle satisfied by the empirical measure of eigenvalues [14, theorem 2.1.1] and the upper bound for top eigenvalue [12, lemma 6.3] of GOE matrices, we have the following two inequalities:

1. For all \( t, \epsilon > 0 \) and \( \kappa > \max\{\epsilon, 1\} \) there exists a constant \( c = c(\epsilon, \kappa) > 0 \) such that for any intervals \( B_i \subset \mathbb{R} \) and large enough \( N \), uniformly in \( r \in (-1, 1) \),
\[
\mathbb{E}\left\{ \prod_{i=1,2} \prod_{j=1}^{N-2} (h_i^\epsilon(\lambda_j(G^{(i)}_{N-2}(r))))^t \cdot \|E_i\| \right\} \leq \exp\{-cN^2\} + \exp\{2t\epsilon N\} \mathbb{E}\left\{ \|E_1\| \|E_2\| \right\}
\]
\[
\leq \exp\left\{ \sum_{i=1,2} tN \int \log \epsilon \left( \frac{1}{(N-1)v''(\rho_i^2)} \frac{1}{\rho_i} X_i(r) \right) \, d\mu^*(\lambda) \right\}, \tag{4.23}
\]

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where \( \mu^* \) is the semicircle law, given by (3.2).

(2) For any \( m \geq 1 \), there exists some large \( \kappa > 0 \) so that, uniformly in \( r \in (-1, 1) \) and \( N \),

\[
\mathbb{E}\left\{ \prod_{i=1,2}^{N-2} \prod_{j=1}^{N-2} \left( h_\kappa(\lambda_J(G^{(i)}_{N-2}(r))) \right)^{2m} \right\} \leq 2.
\]

The equality in (4.22) follows from (4.23) and Varadhan’s integral lemma [25, theorem 4.3.1, exercise 4.3.11]. (See the proof of [52, theorem 5] for details.)

It remains to consider \( \Delta_\Pi \). By (4.24), for fixed \( m \) and large enough \( r \),

\[
\Delta_\Pi \leq \lim_{N \to \infty} \sup_{r} \frac{1}{mN} \log \left( \int_I |(\mathcal{F}(r))^m(e_{e,K}^{(3)}(r)) |^{1/4} d r \right).
\]

We control the right-hand side of (4.25) differently according to whether \( r_1 = r_2 \) or not.

Assume first that \( r_1 = r_2 = r \). In that case, we will show that the integrand in (4.25) is bounded uniformly in \( r \in (-1, 1) \) by some (possibly \( m \)-dependent) constant independent of \( N \). We have to be particularly careful with the limit as \( |r| \to 1 \) since \( (\mathcal{F}(r))^m \) can explode as \( |r| \to 1 \). Note that

\[
\lim_{r \to 1} (v'(\rho^2 r))^2 = (v'(\rho^2))^2
\]

if and only if \( v \) is either an even or odd polynomial. Thus, only in this case \( \lim_{r \to 1} \mathcal{F}(r) = \infty \). Therefore, to complete the proof it is sufficient to show that for \( r_1 = r_2 = r \),

\[
\lim_{r \to 1} \sup_{r \neq 1} (\mathcal{F}(r))^m (e_{e,K}^{(3)}(r)) |^{1/4} < \infty,
\]

and in the case that \( |v(r)| = |v(-r)| \), the same holds for the \( r \to -1 \) limit.

By the same proof as in lemma 15 of [52], if \( X_{N-1} = \sum_{i=1}^{N-1} X_i^2, X_i \sim N(0, 1) \) i.i.d., is a Chi-squared variable of \( N - 1 \) degrees of freedom, then

\[
\mathbb{E}\left\{ (W_i(r))^2 \right\} \leq (2V(r))^m \mathbb{E}\left\{ X_{N-1}^m \right\} = (2V(r))^m (N - 1)(N + 1) \cdots (N - 3 + 2m),
\]

where \( V(r) \) is the maximum of the variance of the elements \( (M^{(i)}_{N-1}(r))_1, N-1 \) and of \( (M^{(i)}_{N-1}(r))_N, N-1 \). By (A.7), the variance of the former element is equal to \( (N - 1)^{-1} \Sigma_{Z,11}(r, \rho, \rho) \) (see (A.2)). By Lemmas 4.2 and 4.3, the variance of the latter is equal to the conditional variance of the \( N - 1, N - 1 \) entry of

\[
\sqrt{\frac{N}{(N - 1)v''(\rho_2^2)\nabla^2 H_N(\rho_2 \vartheta (r))}}
\]
conditioned on \( \nabla_{\text{sp}} H_N(\hat{\rho}) \) and \( \nabla_{\text{sp}} H_N(\rho(\sigma(r))) \). By the same calculation by which we arrive at (A.7), this conditional variance is equal to

\[
(N - 1)^{-1} \left[ \rho^4 \psi''(\rho^2 r)(1 - r^2)^2 - 6 \rho^2 \psi''(\rho^2 r) r(1 - r^2) + \psi''(\rho^2 r)(3r^2 - 4(1 - r^2)) + r \rho^{-2} \psi'(\rho^2 r) - a_2(r, \rho, \hat{\rho})(1 - r^2) \right.
\]

\[
	imes (-\rho^3 \psi''(\rho^2 r)(1 - r^2) + 3r \rho \psi''(\rho^2 r) + \rho^{-1} \psi'(\rho^2 r))^2 \].
\]

By some calculus, one can verify that each of those variances when multiplied by \((N - 1)^{-1}\) converges as \( r \to 1 \) to a constant. On the other hand, as \( r \to 1 \), \( F(r)(1 - r) \) converges to a positive constant. Similarly, when \( j(r) = j \), the same convergences hold \( r \to 1 \) with \( 1 - r \) replaced by \( 1 + r \). Combined with (4.27), this implies (4.26) and the corresponding \( r \to 1 \) limit, when needed, and completes the proof for the case \( \rho_1 = \rho_2 \).

Assume next that \( \rho_1 \neq \rho_2 \). In that case, the argument is simpler, since \( F(r) \) is uniformly bounded in \( r \), while \( W_i(r)^2 \) has the law of a sum of \( N - 1 \) squares of Gaussian variables, conditioned on the vanishing of the spherical gradient; compare with (4.27). Before the conditioning, these variables are independent and each of them has variance uniformly bounded by a multiple of \( 1/(N - 1) \), and the variance of the sum after the conditioning is not larger than the variance before conditioning. Therefore, \( \epsilon_{\psi}(3) \leq C(m) \), and thus \( \Delta \ll 0 \) if \( \rho_1 \neq \rho_2 \). This completes the proof in the case \( \rho_1 \neq \rho_2 \), and thus the proof of Theorem 4.2 is complete.

### 4.3 Proof of Lemma 3.3 (Matching of first and second moments)

The lemma will follow from (4.12) if we can show that for \( x_i \in D_i \), \( r_0 \in (-1, 1) \), and \( \epsilon_N \to 0 \) slowly enough, say \( \epsilon_N = N^{-1} \), uniformly in \( r \in (r_0 - \epsilon_N, r_0 + \epsilon_N) \),

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left\{ \neg \bigg \| \det \left( M_{N-1}^{(E)}(r) \right) \bigg \| E' \right\}
\]

\[
\geq \Omega \left( \frac{x_1}{\rho_1 \sqrt{\psi''(\rho_1^2)}} \right) + \Omega \left( \frac{x_2}{\rho_2 \sqrt{\psi''(\rho_2^2)}} \right),
\]

where

\[
E' = \left\{ \forall i = 1, 2 : U_i(r) \in \sqrt{N}(u_i - \epsilon_N, u_i + \epsilon_N), \right. \]

\[
X_i(r) \in \sqrt{N}(x_i - \epsilon_N, x_i + \epsilon_N) \bigg \}.
\]

since by (4.11),

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left\{ E_i' \right\} \geq -\frac{1}{2} (u_1, u_2, x_1, x_2) \Sigma_{U, X}^{-1}(r, \rho_1, \rho_2)(u_1, u_2, x_1, x_2)^T.
\]
By (4.13), (4.9), and (4.10), the matrix $M_{N-1}(r)$ is of the form

$$M_{N-1}(r) = \left( \frac{\hat{G}_{N-2}(r)}{0} \right) - \frac{1}{(N-1)\psi'(\rho_i^2)} \frac{1}{\rho_i} X_i(r) I + T^{(i)},$$

where $\hat{G}_{N-2}(r)$ is a GOE matrix and $T^{(i)}$ is a matrix whose only nonzero elements are in the last column and row.

Let $\tilde{M}_{N-2}(r)$ be the upper left $N-2 \times N-2$ submatrix of $M_{N-1}(r)$, and let $\delta > 0$ be arbitrary. From our assumption on $D_i$ and the convergence of the top eigenvalue of Wigner matrices (see [2, theorem 2.1.22]), the eigenvalues $\lambda_j^{(i)}$ of $\tilde{M}_{N-2}(r)$ are smaller than some $-\tau' < 0$, independent of $N$, with probability approaching 1 as $N \to \infty$. Therefore, by Wigner’s theorem (see [2, theorem 2.1.1]), again with probability approaching 1 as $N \to \infty$,

$$\frac{1}{N} \log |\det \tilde{M}_{N-2}(r)| = F^{(i)} \geq \Omega \left( \frac{x_i}{\rho_i \sqrt{\psi'(\rho_i^2)}} \right) - \delta,$$

where $F^{(i)} = \frac{1}{N} \sum_{j=1}^{N-2} \log |\lambda_j^{(i)}|$. Finally, since the variance of $X_i(r)$ is bounded from below by some $c = c(\rho_0) > 0$ uniformly in $r \in (\rho_0 - \epsilon_N, \rho_0 + \epsilon_N)$,

$$\frac{1}{N} \log \left( \frac{\det M_{N-1}(r)}{\det \tilde{M}_{N-2}(r)} \right) = \frac{1}{N} \log \left( Z - V^T (\tilde{M}_{N-2}(r))^{-1} V \right) \in (-\delta, \delta),$$

again with probability approaching 1 as $N \to \infty$, where $V$ is the vector composed of the first $N-2$ elements of the last column of $M_{N-1}(r)$ and $Z$ is the $N-1$, $N-1$ element of $M_{N-1}(r)$. This completes the proof. \qed

5 Matching of Moments and Orthogonality: Proofs of Theorem 1.1, Lemma 3.5, and Corollaries 3.6 and 3.7

In this section we use the second-moment computations of Section 4 to show that the ground state is determined by a first-moment computation. We also show that deep $\rho$-critical points are nearly orthogonal. We begin stating and proving several consequences of Condition M. This is then followed by the proofs of the statements in the title of the section.

5.1 Consequences of Condition M

For pure-like models, $-x_0(\rho)$ was defined as the maximizer of the complexity $\Theta_{\psi,\rho}(-E_0(\rho), x)$. As the next lemma shows, this definition makes sense.
LEMMA 5.1. If \( \nu_\rho \) is pure-like, then there exists a unique \( x_0(\rho) \) such that
\[
-x_0(\rho) = \arg \max_{x \in \mathbb{R}} \Theta_{\nu_\rho}(E_0(\rho), x).
\]
and it satisfies \(- \frac{x_0(\rho)}{\rho \sqrt{\nu_\rho(\rho^2)}} < -2.\)

Next we state several auxiliary lemmas regarding \( E_0(\rho) \) and \( x_0(\rho) \), which will be needed later.

LEMMA 5.2. For \( \rho \) such that \( \nu_\rho \) is pure-like, \((E, x) \mapsto \Theta_{\nu_\rho}(E, x)\) is strictly concave on \( \{(E, x) : x < -2\rho \sqrt{\nu_\rho(\rho^2)}\} \). Moreover, for any \( E \) and \( x < -2\rho \sqrt{\nu_\rho(\rho^2)} \), if \( \Theta_{\nu_\rho}(E, x) = 0 \) then \( \frac{d}{dE}\Theta_{\nu_\rho}(E, x) \neq 0.\)

Note that the conclusion of Lemma 5.2 holds if \( E = -E_0(\rho) \) and \( x = -x_0(\rho) \).

A useful corollary of Lemma 5.2 is the following.

LEMMA 5.3. For any \( \rho \) such that \( \nu_\rho \) is pure-like, \( E_0(\rho) \) and \( x_0(\rho) \) are smooth functions of \( \rho \).

The operator \( \| v \| = \sum_{p=-2}^{\infty} \gamma_p \rho^p \) defines a norm on the space of (possibly infinite) polynomials \( v(x) = \sum_{p=-2}^{\infty} \gamma_p x^p \) such that \( \| v \| < \infty \). We will denote \( v^{(\delta)} \to v^{(0)} \) whenever \( \| v^{(\delta)} - v^{(0)} \| \to 0 \) as \( \delta \to 0.\)

Remark 5.4. If \( \| \bar{V} - v \| < \delta \), then
\[
\sup_{r \in [-1, 1]} \left| \frac{d^i}{dr^i} \bar{V}(r) - \frac{d^i}{dr^i} v(r) \right| < \delta \quad \text{for} \quad i = 0, \ldots, 4. \]

We will need the following stability results with respect to the norm \( \| \cdot \| \).

LEMMA 5.5. If \( v^{(\delta)} \to v^{(0)} \) for some pure-like or pure mixture \( v^{(0)} \), then
\[
E_0(v^{(\delta)}) \to E_0(v^{(0)}) \quad \text{and} \quad x_0(v^{(\delta)}) \to x_0(v^{(0)}) \quad \text{as} \quad \delta \to 0. \]

In the next lemma we implicitly assume that \( \rho \) and \( \rho_1 \) are positive numbers for which \( \Psi_{\nu, \rho, \rho} \) and \( \Psi_{\nu, \rho_1, \rho_2} \) are well-defined.

LEMMA 5.6. Assume \( v \) is non-pure. Then the following hold:

1. \( \Psi_{\nu, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2) \) and its first and second derivatives in \( r \) are continuous functions of \( \rho_1, \rho_2 \), \( r \in (-1, 1), u_i \in \mathbb{R}, x_i \in \mathbb{R} \), and \( v \) (w.r.t. the norm \( \| \cdot \| \)).

2. If \( v \) satisfies Condition M then, for any \( \tau > 0 \), for small enough \( \delta = \delta(\tau) > 0 \) and any \( \epsilon > 0 \), the following holds: If for some mixture \( \bar{V} \), \( \| \bar{V} - v \|, |1 - \rho_1|, |u_i + E_0(1)|, \) and \( |x_i + x_0(1)| \) are all smaller than \( \delta \), then
\[
\sup_{\epsilon \leq |r| \leq 1 - \tau} \Psi_{\bar{V}, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2) < \Psi_{\bar{V}, \rho_1, \rho_2}(0, u_1, u_2, x_1, x_2). \]

\( ^8 \)That is, \( \rho \) and \( \rho_1 \) are such that expressions like \( v(\rho^2) \) or its derivatives, which appear in the definitions of \( \Psi_{\nu, \rho, \rho} \) and \( \Psi_{\nu, \rho_1, \rho_2} \), are finite. From our assumption that \( \lim_{p \to 1} \log \gamma_p < 0 \), the range that \( \rho \) and \( \rho_1 \) are allowed to be in contains an open interval containing \((0, 1] \).
(3) With \( E = -E_0(v) \) and \( x = -x_0(v) \),

\[
\limsup_{r \rightarrow 1} \Psi_{\tau, \rho, \rho}(r, u_1, u_2, x_1, x_2) - (v, 1, 1, E, x, x) \leq \limsup_{r \rightarrow 1} \Psi_{v, 1, 1}(r, E, E, x, x) = \Psi^0_v(1).
\]

The same also holds with the \( r \rightarrow 1 \) limit replaced by \( r \rightarrow -1 \) and \( \Psi^0_v(1) \) by \( \Psi^0_v(-1) \).

The rest of the subsection is devoted to proofs.

**Proof of Lemma 5.5.** We begin with an auxiliary computation. Let \( \Sigma = \Sigma_1 \) (see (3.5)) and denote its elements by \( \Sigma_{ij} \). By Lemma 3.8, \( \Sigma \) is invertible. Note that, by a direct computation or from the standard formula for a conditional Gaussian distribution [1, pp. 10–11],

\[
(5.1) \quad (u, x) \Sigma^{-1}(u, x)^T = u^2 \Sigma^{-1}_{11} + (x - \Sigma_{12} \Sigma^{-1}_{11}) u^2 (\Sigma_{22} - \Sigma_{22} \Sigma^{-1}_{11})^{-1}.
\]

By the definition (3.4) of \( \Theta_{v, \rho, \rho} \), (5.1), and substitution of the values of \( \Sigma_{ij} \),

\[
(5.2) \quad \Theta_{v, 1}(-E, x) = \frac{1}{2} + \frac{1}{2} \log \left( \frac{v''(1)}{v'(1)} \right) - \frac{E^2}{2} - \frac{(x + v'(1) E)^2}{2(v''(1) + v'(1) - v'(1)^2)} + \Omega \left( \frac{x}{\sqrt{v''(1)}} \right).
\]

Turning to the proof of the lemma, assume first that the limiting polynomial \( v^{(0)} \) is not a pure mixture. Since \( \Omega(x) \) from (3.3) is a Lipschitz function and \( \Sigma_1 \) is positive definite, for small \( \delta, E_0(v^{(\delta)}) \) and \( x_0(v^{(\delta)}) \) belong to some compact set \([-T, T]^2\), and the same for \( v^{(0)} \). On \([-T, T]^2\), \( \Theta_{v, \rho, \rho}(E, x) \) converges uniformly to \( \Theta_{v, \rho, \rho}(E, x) \) as \( \delta \rightarrow 0 \). Since \( -E_0(v^{(\delta)}) < -E_{\infty}(v^{(0)}) \) and \( -x_0(v^{(\delta)}) \) are unique,\(^9\) this proves the lemma in the current case.

Next, assume that \( v^{(0)}(x) = \psi_\rho(x) = x^p \) is pure. We may and will assume that any of the \( v^{(\delta)} \) is not a pure mixture. Setting \( \alpha^2 = v''(1) + v'(1) - v'(1)^2 \) and denoting by \( C > 0 \) the Lipschitz constant of \( \Omega(x) \) for any \( v \), we obtain from (5.2) that

\[
(5.3) \quad \Theta_v(-E, x - v'(1) E) - \Theta_v(-E, -v'(1) E) \leq -\frac{x^2}{2\alpha^2} + \frac{C}{\sqrt{v''(1)}},
\]

and the left-hand side of (5.3) is negative whenever \( |x| > 2C \alpha^2 / \sqrt{v''(1)} \). Moreover,

\[
(5.4) \quad \max_{x \in \mathbb{R}} \Theta_v(-E, x - v'(1) E) - \Theta_v(-E, -v'(1) E) \leq C^2 \alpha^2 / v''(1).
\]

---

\(^9\) Uniqueness of \( E_0(v^{(0)}) \) follows from [5, prop. 1, theorem 1.4]; uniqueness of \( x_0 \) follows from Lemma 5.1 below.
with the maximum above being obtained with some $|x| \leq 2C\alpha^2_\nu / \sqrt{v''(1)}$. Since $\alpha^2_{\nu,0} \to 0$, this proves that $x_0(\nu(\delta)) \to x_0(\nu(0))$.

Since $\Theta_{\nu,\delta}(-E, -v'(1))$ converges to $\Theta_p(-E)$ uniformly on compacts, the convergence $E_0(\nu(\delta)) \to E_0(\nu(0))$ can be deduced by first restricting to a compact range $E \in [-T, T]$, as we did for the mixed case. □

**Proof of Lemma 5.1.** By remark 5.4 it is enough to prove the lemma assuming that $p = 1$ and $v(1) = 1$, which we will. Since $\Omega$ from (3.3) is a symmetric function satisfying $d \Theta_{\nu,1}(E_0, x) = d \mu(x) = 0$ if and only if $x = 0$, we deduce from (5.2) that
\[
\sup_{x < -v'(1)E_0} \Theta_{\nu,1}(-E_0, x) > \sup_{x \geq -v'(1)E_0} \Theta_{\nu,1}(-E_0, x).
\]
For any $-2\sqrt{v''(1)} < x < -v'(1)E_0$, since $\Omega(x) = x^2/4 - 1/2$ for $|x| \leq 2$,
\[
\frac{d}{dx} \Theta_{\nu,1}(-E_0, x) = x \left( \frac{1}{2v''(1)} - \frac{1}{v''(1) + v'(1) - v'(1)^2} \right) - \frac{v'(1)E_0}{v''(1) + v'(1) - v'(1)^2} < - \frac{1}{\sqrt{v''(1)}} + \frac{2\sqrt{v''(1)}}{v''(1) + v'(1) - v'(1)^2} - \frac{v'(1)E_0}{v''(1) + v'(1) - v'(1)^2} \leq 0,
\]
where the first inequality follows since the expression in parentheses is negative, as can be checked using that $v(1) = 1$, and the second inequality follows by calculus since for pure-like or critical models $E_0 \geq E_\infty$ from (5.14). Therefore,
\[
\sup_{x \leq -2\sqrt{v''(1)} \atop x \geq -2\sqrt{v''(1)}} \Theta_{\nu,1}(-E_0, x) \geq \sup_{x < -v'(1)E_0} \Theta_{\nu,1}(-E_0, x),
\]
with strict inequality if the supremum of the left-hand side is obtained at some $x < -2\sqrt{v''(1)}$. We conclude that
\[
\sup_{x \leq -2\sqrt{v''(1)} \atop x < -\sqrt{2}} \Theta_{\nu,1}(-E_0, x) = \sup_{x \leq -2\sqrt{v''(1)} \atop x < -\sqrt{2}} \Theta_{\nu,1}(-E_0, x \sqrt{2v''(1)}) = \sup_{x < -\sqrt{2}} \frac{1}{2} \log \left( \frac{v''(1)}{v'(1)} \right) - \frac{u^2}{2} + \frac{x^2}{2} - \frac{(\sqrt{2}v''(1)x + v'(1)E_0)^2}{2(v''(1) + v'(1) - v'(1)^2)} - I_1(|x|),
\]
where
\[
I_1(x) = \frac{1}{2} \left( x \sqrt{x^2 - 2} + \log 2 - 2 \log(x + \sqrt{x^2 - 2}) \right) \quad x \geq 2,
\]
is the rate function for the top eigenvalue of a GOE matrix (with a different normalization than we use); see [5, eq. (2.9)]. In the proof of [3, theorem 1.1] it is shown that by replacing $-E_0$ with some $u < -E_\infty$ in (5.5), the supremum in...
is obtained at a unique point \( x_* < -\sqrt{2} \). This completes the proof since, for pure-like models, \(-E_0 < -E_\infty\) from (3.14).

**Proof of Lemma 5.2** On \( D := \{(E, x) : x < -2\rho\sqrt{v''(\rho^2)}\} \), the term involving \( \Omega \) in the definition of \( \Theta_{v, \rho}(E, x) \) is strictly concave. Since \( \Sigma_\rho \) is positive-definite (see Lemma 3.8), \((E, x) \mapsto \Theta_{v, \rho}(E, x)\) is strictly concave on \( D \) as well.

To see the second part of the lemma, assume for a proof by contradiction that both \( \Theta_{v, \rho}(E, x) = 0 \) and \( \frac{d}{dE} \Theta_{v, \rho}(E, x) = 0 \) for some \((E, x) \in D\). Then, by the concavity of \( \Theta_{v, \rho} \) on \( D \), it follows that the maximum of \( \Theta_{v, \rho}(E, x) \) over \( D \) is equal to 0. Observe that in the proof of Lemma 5.1 only information used on \( E_0 \) was that \( E_0 < E_1 \). Therefore, following the same proof we have that for any \( E \in \left( -E_0(\rho), -E_\infty(\rho) \right) \), there exists a unique \( x \) such that

\[
(5.6) \quad \Theta_{v, \rho}(E, x) = \max_{x \in \mathbb{R}} \Theta_{v, \rho}(E, x),
\]

and for that \( x, (E, x) \in D \). Also, since \( v_\rho \) is pure-like, by (3.14) and Remark 3.4, the right-hand side of (5.6) with \( E = -E_\infty(\rho) \) is strictly positive. By continuity, we also have some \( E \in \left( -E_0(\rho), -E_\infty(\rho) \right) \) for which the right-hand side of (5.6) is strictly positive, which implies that the maximum of \( \Theta_{v, \rho}(E, x) \) over \( D \) is strictly positive. Since we arrive at a contradiction, the proof is completed.

**Proof of Lemma 5.3** The proof is an application of the implicit function theorem. Let \( \bar{\rho} \) be a positive number such that \( v_\bar{\rho} \) is pure-like. Define the function

\[
F(\rho, (E, x)) = \begin{pmatrix} \Theta_{v, \rho}(E, x) \\ \frac{d}{dx} \Theta_{v, \rho}(E, x) \end{pmatrix}.
\]

By Remark 3.4, \(-x_0(\bar{\rho})\) is the maximum point of \( x \mapsto \Theta_{v, \bar{\rho}}(-E_0(\bar{\rho}), x) \), and thus \( F(\bar{\rho}, g(\bar{\rho})) = 0 \). By Lemma 5.1, \(-x_0(\bar{\rho}) \neq -2\bar{\rho}/\sqrt{v''(\bar{\rho})} \). From the definition of \( \Theta_{v, \bar{\rho}}(E, x) \), one can verify that \( F \) is a smooth function of \( \rho, E, \) and \( x \) on a neighborhood of \((\bar{\rho}, -E_0(\bar{\rho}), -x_0(\bar{\rho}))\). Therefore, by the implicit function theorem, if we show that the Jacobian

\[
\begin{pmatrix}
\frac{d}{dE} \Theta_{v, \rho}(E, x) & \frac{d}{dx} \Theta_{v, \rho}(E, x) \\
\frac{d}{dE} \frac{d}{dx} \Theta_{v, \rho}(E, x) & \frac{d}{dx} \frac{d}{dx} \Theta_{v, \rho}(E, x)
\end{pmatrix}
\]

is invertible at \((E, x) = (-E_0(\bar{\rho}), -x_0(\bar{\rho}))\), then there exists a smooth function \( g(\rho) = (E(\rho), x(\rho)) \) in the neighborhood of \( \bar{\rho} \) such that \( F(\rho, (E(\rho), x(\rho))) = 0 \). Hence, \((E(\rho), x(\rho)) = (-E_0(\rho), -x_0(\rho)) \) (since by Lemmas 5.2 and 5.1 there is a unique \(-E < 0\) such that \( \sup_x \Theta_{v, \rho}(-E, x) = 0 \)).

What remains is to prove the invertibility of the matrix above. Since

\[
\frac{d}{dx} \Theta_{v, \rho}(-E_0(\rho), -x_0(\rho)) = 0,
\]

the proof is completed by invoking Lemma 5.2.
PROOF OF LEMMA 5.6 Using Remark 5.4, point 1 follows directly from the formula (3.8) for \(\Psi_{r_0,1,\rho_2}\) and from the definitions (3.3) and (A.4) of \(\Omega(x)\) and \(\Sigma_{U,u}(r,\rho_1,\rho_2)\).

Turning to the proof of point 2, for any \(\xi > 0\), define
\[
\eta(\xi) = \Psi_v^0(0) - \sup_{|r| \leq \xi} \Psi_v^0(r) > 0,
\]
where the inequality is due to Condition M. From continuity, for small enough \(\xi\) and \(\delta = \delta(\tau, \xi), \) uniformly in \(\rho_1, u_1, x_1, \) and \(v\) as in point 2
\[
\sup_{|r| \leq \xi} d^2 \frac{d^2}{dr^2} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) < 0,
\]
\[
\sup_{|r| \leq 1 - \tau} \left| \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) - \Psi_v^0(r) \right| < \frac{\eta(\xi)}{2}.
\]
Since the derivative in \(r\) at \(r = 0\) of any of the entries of \(\Sigma_{U,u}(r,\rho_1,\rho_2)\) (see (A.4)) is 0,
\[
\left. \frac{d}{dr} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) \right|_{r=0} = 0.
\]
The above implies point 2 for the \(r \to 1\) limit. Since \(\Omega(x)\) is continuous, it is enough to prove that
\[
\liminf_{(u, r, \rho, u_1, u_2, x_1, x_2) \to (v, 1, 1, -E_0, -E_0, -x_0, -x_0)} (u_1, u_2, x_1, x_2) \Sigma_{U,u}^{-1}(u_1, u_2, x_1, x_2)^T
\]
\[
\geq \liminf_{r \to 1} (r, -E_0, -E_0, -x_0, -x_0) \Sigma_{U,u}^{-1}(r, 1, 1, r, -E_0, -E_0, -x_0, -x_0)^T
\]
(where the dependence of \(\Sigma_{U,u}\) in \(v\) is expressed by its addition to the subscript).

In Appendix A we define (see (A.2) and (A.4))
\[
\Sigma_{U,u}(r, \rho, \rho) = \begin{pmatrix}
\Sigma_U(r, \rho, \rho) & \Sigma_b(r, \rho, \rho) \\
\Sigma_{b}(r, \rho, \rho) & \Sigma_X(r, \rho, \rho)
\end{pmatrix}.
\]
By setting
\[
\Sigma_{U,u}(1, \rho, \rho) = \begin{pmatrix}
z_1(\rho) & z_2(\rho) \\
z_2(\rho) & z_3(\rho)
\end{pmatrix},
\]
where \(z_{1,2,3}(\rho)\) are the \(2 \times 2\) matrix whose entries are all \(1\) and
\[
z_1(\rho) = \bar{v}(\rho^2) - \rho^2 \frac{\bar{v}'(\rho^2)^2}{\bar{v}'(\rho^2) + 3 \rho^2 \bar{v}''(\rho^2)},
\]
\[
z_2(\rho) = \rho \bar{v}'(\rho^2) - \rho \bar{v}'(\rho^2) \frac{\rho^2 \bar{v}''(\rho^2) + \bar{v}'(\rho^2)^2}{\bar{v}'(\rho^2) + 3 \rho^2 \bar{v}''(\rho^2)}.
\]
we continuously extend the elements of (5.8) for \(r = 1\).
For any $r \in (-1, 1]$, each of the blocks of (5.8) is a $2 \times 2$ matrix of the form
\[
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}.
\]
Therefore, the covariance matrix $\Sigma_{U, X}(r, \rho, \rho)$ has two orthogonal unit eigenvectors, say $v_i(r, \rho)$, $i = 1, 2$, of the form $(u, u, x, x)$ and two orthogonal unit eigenvectors of the form $(u, -u, x, -x)$, say $v_i(r, \rho)$, $i = 3, 4$. By [33, pp. 106–108], from the continuity of the elements of $\Sigma_{U, X}(r, \rho, \rho)$, we can choose the eigenvectors $v_i(r, \rho)$ and the corresponding eigenvalues $\lambda_i(r, \rho)$ so that they are continuous in $r$ and $x$ at the point $(r, \rho, \nu) = (1, 1, \nu)$ (though we do not necessarily have continuity on a neighborhood of this point).

For a general vector $w \in \mathbb{R}^4$,
\[
w^T \Sigma_{U, X}^{-1}(r, \rho, \rho) w \geq \sum_{i=1, 2} \frac{1}{\lambda_i(r, \rho)} |w, v_i(r, \rho)|^2,
\]
and for $w_0 = (-E_0, -E_0, -x_0, -x_0)$ we have an equality, since it is orthogonal to $v_i(r, \rho)$, $i = 3, 4$. This implies (5.7), and therefore point 3 for the $r \to 1$ limit.

Finally, we prove point 3 for the $r \to -1$ limit. If $\nu(x)$ is an even or an odd function, a similar argument can be applied using eigenvalues and eigenvectors. Lastly, if $\nu(x)$ is neither even nor odd, then the logarithmic term in the definition of $\Psi_{1,1}(r, u_1, u_2, x_1, x_2)$ goes to $-\infty$ as $r \to -1$, and so do both the limits in point 3.

5.2 Proof of Lemma 3.5 (Large overlaps are negligible for second moments)

Recall that at $r = 0$, (3.11) and that $E_0(\rho)$ and $x_0(\rho)$ are continuous in $\rho$ by Lemma 5.5. Thus, to prove Lemma 3.5 we need to show that for small enough $\delta > 0$, if $|1 - \rho_i| < \delta$ and
\[
B_i \times D_i \subset A(\delta) = \{(u, x) : |u + E_0(\nu)|, |x + x_0(\nu)| < \delta\},
\]
then, with $I(\epsilon) = \{x : |x| \in [\epsilon, 1]\}$,
\[
\sup_{r \in I(\epsilon), u_i \in B_i, x_i \in D_i} \Psi_{v, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2) < \sup_{u_i \in B_i, x_i \in D_i} \Psi_{v, \rho_1, \rho_2}(0, u_1, u_2, x_1, x_2).
\]
We proceed by treating the cases $\rho_1 = \rho_2$ and $\rho_1 \neq \rho_2$ separately.

The Case $\rho_1 = \rho_2 = \rho$

>From the continuity stated in point 1 of Lemma 5.6,
\[
\lim_{(\rho, u_j, x_j) \to (1, -E_0(\nu), -x_0(\nu))} \Psi_{v, \rho, \rho}(0, u_1, u_2, x_1, x_2) = \Psi_{v, \rho}^0(0) > \max\{\Psi_{v, \rho}^0(1), \Psi_{v, \rho}^0(-1)\}.
\]
Thus, from point 3 of Lemma 5.6 for small enough \( \delta \) it is enough to prove (5.9) with \( I(\epsilon) \) replaced by \( I(\epsilon, \tau) = [-1 + \tau, -\epsilon] \cup [\epsilon, 1 - \tau] \) for some small \( \tau \). Moreover, from the continuity of \( \Psi_{v,\rho,\rho} \) in all its variables,

\[
(5.10) \quad \sup_{r \in I(\epsilon, \tau), u_i \in B_i, x_i \in D_i} \Psi_{v,\rho,\rho}(r, u_1, u_2, x_1, x_2) = \Psi_{v,\rho,\rho}(r', u'_1, u'_2, x'_1, x'_2)
\]

for some \( r' \in I(\epsilon, \tau) \) and \( u'_i \) and \( x'_i \) in the closure of \( B_i \) and \( D_i \), respectively. (Recall that the latter sets are bounded by assumption.) Therefore, the proof of Lemma 3.5 in the case \( \rho_1 = \rho_2 \) follows from point 2 of Lemma 5.6.

**The Case \( \rho_1 \neq \rho_2 \)**

The main ingredient in the proof of the current case is the following lemma, the proof of which is deferred to the end of the subsection.

**Lemma 5.7.** For small enough \( \tau, \delta > 0 \), for any \( \rho_1 \neq \rho_2 \) such that \( |1 - \rho_i| < \delta \) and any \( (u_i, x_i) \in A(\delta) \),

\[
(5.11) \quad \sup_{1-\tau \leq |r| < 1} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) \leq \min_{i=1,2} \Theta_{v,\rho_i}(u_i, x_i).
\]

Note that for \( \rho_1 \neq \rho_2 \), as \( |r| \to 1 \) the logarithmic term in the definition of \( \Psi_{v,\rho_1,\rho_2} \) goes to \(-\infty\). Since the quadratic term involving \( \Sigma_{U,X}^{-1}(r, \rho_1, \rho_2) \) in (3.8) is nonpositive, we conclude that, for fixed \( \rho_1 \neq \rho_2 \),

\[
(5.12) \quad \lim_{|r| \to 1} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) = -\infty,
\]

and the convergence is uniform in \( u_1, u_2, x_1, x_2 \) in compact sets (but not in \( \rho_1, \rho_2 \)). Combining (5.12) with point 1 of Lemma 5.6 we have that

\[
\sup_{r \in I(\epsilon), u_i \in B_i, x_i \in D_i} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) = \Psi_{v,\rho_1,\rho_2}(r', u'_1, u'_2, x'_1, x'_2)
\]

for some \( r \in I(\epsilon) \) and \( u_i \) and \( x_i \) in the closure of \( B_i \) and \( D_i \), respectively, which are assumed here to be bounded sets. Thus, in light of point 2 of Lemma 5.6 to prove (5.9) it is enough to show that for small enough \( \delta > 0 \), if \( |1 - \rho_i| < \delta \) and \((u_i, x_i) \in A(\delta)\), then

\[
(5.13) \quad \sup_{1-\tau \leq |r| < 1} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) < \Psi_{v,\rho_1,\rho_2}(0, u_1, u_2, x_1, x_2)
\]

(\( = \Theta_{v,\rho_1}(u_1, x_1) + \Theta_{v,\rho_2}(u_2, x_2) \)),

where the equality follows from (5.11) and \( \tau \) is a fixed number that can be assumed to be small. In fact, in light of our assumption that the summands in (3.17) are nonnegative, it will be enough to prove (5.13) only for \( \rho_i, u_i, \) and \( x_i \) such that \( \Theta_{v,\rho_i}(u_i, x_i) \geq 0 \). In this case, if one of \( \Theta_{v,\rho_i}(u_i, x_i) \) is strictly positive, then (5.13) follows from (5.11). Hence, to complete the proof of Lemma 3.5 it remains to prove Lemma 5.7 and the following one.
Lemma 5.8. For small enough $\tau, \delta > 0$, if $|1 - \rho_1| < \delta$, $(u_i, x_i) \in A(\delta)$, and $\Theta_{v,\rho_1}(u_1, x_1) = \Theta_{v,\rho_2}(u_2, x_2) = 0$, then

$$\sup_{1 - \tau \leq |r| < 1} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2) < 0.$$  

Lemma 5.8 is a direct consequence of Lemma 5.7 and the following two short lemmas applied with $g_i = \Theta_{v,\rho_i}$, $g = \Psi_{v,\rho_1,\rho_2}$, and $B = (-1, 1) \setminus (-1 + \tau, 1 - \tau)$.

Lemma 5.9. Let $B \subset \mathbb{R}$, let $g_i(t)$, $i = 1, 2$, be real functions defined on open sets $T_i \subset \mathbb{R}^k$, and let $g(s, t_1, t_2)$ be a real function defined on $B \times T_1 \times T_2$. Suppose that for any $t_1, t_2$

$$\sup_{s \in B} g(s, t_1, t_2) \leq \min_{i=1,2} g_i(t_i).$$

If for some $t_1^*, t_2^*$ and $s^* \in B$,

1. $g(s^*, t_1^*, t_2^*) = \sup_{s \in B} g(s, t_1^*, t_2^*)$,
2. $g_1(t_1^*) = g_2(t_2^*)$,
3. $(\nabla g_1)(t_1^*) \neq 0$ (in particular, the gradient exists), and
4. the gradient of $g$ in the coordinate $t_1$ only, $(\nabla_{t_1} g)(s^*, t_1^*, t_2^*)$, exists, then

$$\sup_{s \in B} g(s, t_1^*, t_2^*) < g_1(t_1^*) = g_2(t_2^*).$$

Proof. Assume for contradiction that $g(s^*, t_1^*, t_2^*) = \min_{i=1,2} g_i(t_i^*)$. Let $v \in \mathbb{R}^k$ be a vector for which $(\nabla g_1(t_1^*), v) < 0$. From (5.14) we also must have that $(\nabla_{t_1} g)(s^*, t_1^*, t_2^*)v < 0$. However, then, for small enough $\epsilon > 0$, we have that $g(s^*, t_1^* - \epsilon v, t_2^*) > g(s^*, t_1^*, t_2^*) = \min\{g_1(t_1^* - \epsilon v), g_2(t_2^*)\}$, in contradiction to (5.14).

Lemma 5.10. Assume that $\rho_1 \neq \rho_2$. Then the supremum in the left-hand side of (5.11) is obtained at some point $r^* \in (-1, 1) \setminus (-1 + \tau, 1 - \tau)$, and the gradient in $(u_1, x_1)$ only of $\Psi_{v,\rho_1,\rho_2}$ at $(r^*, u_1, u_2, x_1, x_2)$ exists. Further, for small enough $\delta$, if $|1 - \rho| < \delta$, $(u, x) \in A(\delta)$, and $\Theta_{v,\rho}(u, x) = 0$, then $\nabla \Theta_{v,\rho}(u, x) \neq 0$.

Proof. The existence of $r^*$ as in the lemma follows from point 1 of Lemma 5.6 and 5.12. The fact that $\nabla \Theta_{v,\rho}(u, x) \neq 0$ follows from Lemmas 5.1 and 5.2. 

It thus remains to prove Lemma 5.7. For the proof, we need the following deterministic inequality.

Lemma 5.11. For any $r \in (\cos \pi/8, 1)$, if we set $r_0 := \cos(4\cos^{-1}(r)) \in (0, 1),$ we have that, deterministically,

$$[\text{Crt}_{N,\rho_1,\rho_2}(B_1, B_2, D_1, D_2, I(r))]_2 \leq \sum_{i=1,2} [\text{Crt}_{N,\rho_i,\rho_i}(B_i, B_i, D_i, D_i, I(r_0))]_2 + 2 \text{Crt}_{N,\rho_1}(B_1, D_1),$$

where $\text{Crt}_{N,\rho_i}$ is the cross-ratio of the line $\rho_i$ with respect to the points $B_i, B_i, D_i, D_i$. 

\[ M \]
where \( I(r) = (-1, 1) \setminus [-r, r] \). In particular, under the conditions of Lemma 3.3,

\[
\sup_{|s| \in [1-r, 1)} \Psi_{v, u_1, u_2, x_1, x_2}(s, u_1, u_2, x_1, x_2) \leq \max \left\{ \min_{i=1,2} \Theta_{v, u_i}(u_i, x_i), \sup_{|s| \in [1-r, 1)} \Psi_{v, u_1, u_2, x_1, x_2}(s, u_1, u_2, x_1, x_2) \right\}.
\]

(5.15)

**Proof.** In the current proof, for a point \( \sigma \in S^{N-1}(1) \) denote by \( S^\pm_\sigma(\delta) \) the set of points in \( S^{N-1}(1) \) that are different from both \( \sigma \) and \( -\sigma \) and that have minimal distance from \( \sigma \) or \( -\sigma \) less than \( \delta \), under the usual metric on the sphere. For any point \( \sigma \in \mathbb{R}^N \setminus \{0\} \) define the cone

\[
B^*_\sigma(\delta) := \{ c\sigma' : c \in \mathbb{R}, \sigma' \in S^*_\pm(\sigma/\|\sigma\|, \delta) \}.
\]

The overlap \( r \) defines the distance

\[
\epsilon := \cos^{-1}(r) \in [0, \pi]
\]

on the sphere \( S^{N-1}(1) \). Assuming that \( r \in (\cos \pi/8, 1) \), we define \( r_0 \) as the overlap that corresponds to four times that distance, \( r_0 := \cos(4\cos^{-1}(r)) \in (0, 1) \).

Note that

\[
\left[ \text{Crt}_{N,1,2}(B_1, B_2, D_1, D_2, I(r)) \right]_2 = \sum_{\sigma \in \mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1)} |\mathcal{C}_{N,2}(NB_2, \sqrt{N}D_2) \cap B^*_\pm(\sigma, \epsilon)|.
\]

(5.16)

Denote by \( A_0 \) the set of points \( \sigma \in \mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1) \) for which

\[
|\mathcal{C}_{N,2}(NB_2, \sqrt{N}D_2) \cap B^*_\pm(\sigma, \epsilon)| > |\mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1) \cap B^*_\pm(\sigma, 4\epsilon)| + 2.
\]

(5.17)

Denoting \( a_c = |\mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1) \setminus A_0| \), we have that

\[
\sum_{\sigma \in \mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1) \setminus A_0} |\mathcal{C}_{N,2}(NB_2, \sqrt{N}D_2) \cap B^*_\pm(\sigma, \epsilon)| \\
\leq 2a_c + \sum_{\sigma \in \mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1) \setminus A_0} |\mathcal{C}_{N,1}(NB_1, \sqrt{N}D_1) \cap B^*_\pm(\sigma, 4\epsilon)| \\
\leq 2a_c + [\text{Crt}_{N,1,2}(B_1, B_2, D_1, D_2, I(r_0))]_2.
\]

To complete the proof we will show that there exists an injective mapping

\[ X : A_0 \to \mathcal{C}_{N,2}(NB_2, \sqrt{N}D_2). \]
such that for any \( \sigma \in A_0 \),

\[
\lvert C_{N,\rho_2}(NB_2, \sqrt{N} D_2) \cap B^*_\pm(\sigma, \epsilon) \rvert \\
\leq \lvert C_{N,\rho_2}(NB_2, \sqrt{N} D_2) \cap B^*_\pm(\mathcal{X}(\sigma), 4\epsilon) \rvert + 2.
\]

(5.18)

This will imply that the sum in (5.16) over \( A_0 \) only is bounded from above by

\[
\left[ \text{Crt}_{N,\rho_2,\rho_2}(B_2, B_2, D_2, D_2, I(r_0)) \right]_2 + 2|A_0|.
\]

Our definition is inductive, starting with an arbitrary point \( \sigma_0 \in A_0 \). From the definition of \( A_0 \), the number of points in \( A_0 \cap B^*_\pm(\sigma, 2\epsilon) \) is smaller than

\[
\lvert C_{N,\rho_1}(NB_1, \sqrt{N} D_1) \cap B^*_\pm(\sigma, 2\epsilon) \rvert
\]

and smaller than the number of points in

\[
\lvert C_{N,\rho_2}(NB_2, \sqrt{N} D_2) \cap B^*_\pm(\sigma_0, \epsilon) \rvert.
\]

(5.19)

so we can define \( \mathcal{X} \) injectively on \( A_0 \) and the proof of the first statement

in Lemma 5.11 is completed. The second statement follows from letting \( B_i \) and \( D_i \) shrink to a point as \( N \to \infty \) and applying Theorem 3.1, point 1 of Lemma 5.6, and (5.12). \( \square \)

We have completed all preparatory steps and can proceed to the proof of Lemma 5.7.

**Proof of Lemma 5.7.** By Lemma 5.1, \( -x_0/\sqrt{v'(1)} < -2 \). Therefore for small enough \( \delta \) if \( \rho_i \) and \( D_i \) are as in Lemma 3.5 then \( D_i \) are as in Lemma 3.3.
Hence, from the second statement of Lemma 5.11, we obtain that

\[
\sup_{|r| \in [1 - \tau, 1)} \Psi_{v, \rho_1, \rho_2}(r, u_1, u_2, x_1, x_2)
\]

\[
\leq \max \left\{ \min_{i=1,2} \Theta_{v, \rho_i}(u_i, x_i), \sup_{|r| \in [1 - \tau_0, 1)} \Psi_{v, \rho_1, \rho_1}(r, u_1, u_1, x_1, x_1), \sup_{|r| \in [1 - \tau_0, 1)} \Psi_{v, \rho_2, \rho_2}(r, u_2, u_2, x_2, x_2) \right\}
\]

(5.20)

for \( \rho_i \in (1 - \delta, 1 + \delta) \) and \((u_i, x_i) \in A(\delta)\), where \( l - \tau_0 \) is related to \( l - \tau \) as \( \tau_0 \) is related to \( r \) in Lemma 5.11. By Condition M, \( \Psi_{v}^{\theta}(\pm 1) < \Psi_{v}^{\theta}(0) = 0 \). Hence, from points 1 and 3 of Lemma 5.6 uniformly in \( \rho_i \in (1 - \delta, 1 + \delta) \) and \((u_i, x_i) \in A(\delta)\),

\[
\sup_{|r| \in [1 - \tau_0, 1)} \Psi_{v, \rho_1, \rho_1}(r, u_i, u_i, x_i, x_i) \leq -c
\]

for some constant \( c > 0 \), assuming \( \delta \) and \( \tau \), and therefore \( \tau_0 \), are small enough. Since \( \Theta_{v, \rho}(u, x) \) is continuous in \( \rho, u, \) and \( x \), and \( \Theta_{v,1}(-E_0, -x_0) = 0 \), the maximum in (5.20) is equal to the first term for small \( \delta \). Namely, we have proved (5.11).

5.3 Proof of Corollary 3.6 (Matching of moments)

Note that

\[
\mathbb{E} \left( \text{Crt}_{N, \rho}(B, D) \right)^2 \leq 2 \mathbb{E} \text{Crt}_{N, \rho}(B, D) + \mathbb{E} \text{Crt}_{N, \rho, \rho}(B, B, D, D, (-1, 1))
\]

so that, since we assume that \( \mathbb{E} \text{Crt}_{N, \rho}(B, D) \to \infty \),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left( \text{Crt}_{N, \rho}(B, D) \right)^2 \leq \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N, \rho, \rho}(B, B, D, D, (-1, 1)) \leq 2 \sup_{u \in B, x \in D} \Theta_{v, \rho}(u, x),
\]

where the second inequality follows from Theorem 3.2, Lemma 3.5, and (3.11). The fact that

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left( \text{Crt}_{N, \rho}(B, D) \right)^2 \geq 2 \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}(B, D)
\]

\[
= 2 \sup_{u \in B, x \in D} \Theta_{v, \rho}(u, x)
\]

follows from Theorem 3.1. The \( N \)-dependent case follows from a standard diagonalization argument.
5.4 Proof of Theorem 1.1 (Ground state computation)

>From Theorem 3.1 and the definition of $-E_0$ (combined with Markov’s inequality and the Borel-Cantelli lemma),

\[
\lim_{N \to \infty} \frac{1}{N} \text{GS}_N \geq -E_0 \quad \text{almost surely.}
\]

>From Corollary 3.6, for some $\delta_N = o(1)$, with $B = B_N = -E_0 + (-\delta_N, \delta_N)$ and $D = D_N = -x_0 + (-\delta_N, \delta_N)$,

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}(\text{Crt}_{N,1}(B, D))^2 = 2 \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N,1}(B, D).
\]

By appealing to the Cauchy-Schwarz inequality, this implies that

\[
\mathbb{P} \left( \frac{1}{N} \text{GS}_N \leq -E_0 + \frac{\delta_N}{N} \right) \geq \mathbb{P}(\text{Crt}_{N,1}(B, D) > 0) \geq \frac{(\mathbb{E}(\text{Crt}_{N,1}(B, D))^2}{\mathbb{E}(\text{Crt}_{N,1}(B, D))^2}
\]

does not decay exponentially in $N$.

Using the Borell-TIS inequality [17, 24] (see also [1, theorem 2.1.1]), which implies that the $\text{GS}_N/N$ has exponential in $N$ tails, this is in fact sufficient to conclude the matching upper bound to (5.21). For the full argument, see appendix IV of [52], where this is carried out in the pure setting. □

Remark 5.12. By Proposition 6.3 below, for $\rho$ close enough to 1, the mixture $\nu_\rho(x) = \sum_{p=2}^{\infty} \gamma_p^2 \rho^{2p} x^{p}$ satisfies Condition M. Thus, by Theorem 1.1

\[
\lim_{N \to \infty} \frac{1}{N} \min_{\sigma \in S^{N-1}(\sqrt{N})} H_N(\rho \sigma) = -E_0(\rho) \quad \text{a.s.}
\]

5.5 Proof of Corollary 3.7 (Orthogonality of deep critical points)

We begin with two preparatory lemmas.

**Lemma 5.13.** For any compact set $K \subseteq (0, \infty)$, there exists some large $T > 0$ such that $\Theta_{v,\rho}(E, x) < 0$ uniformly in $\rho \in K$ and

\[
(E, x) \in \{(E, x) : \max(|E|, |x|) > T\}.
\]

**Proof.** Since the matrix $\Sigma_\rho$ (appearing in the definition (3.4) of $\Theta_{v,\rho}(E, x)$) is positive definite and its elements are continuous in $\rho$, the eigenvalues of $\Sigma_\rho$ are bounded from below by some positive constant uniformly in $\rho \in K$. The lemma therefore follows from the definition of $\Theta_{v,\rho}(E, x)$ and the fact that $\Omega$ is Lipschitz continuous. □

**Lemma 5.14.** Assume that $v$ is pure-like. Then there exist $\delta > 0$ so that for any $\rho \in 1 + (-\delta, \delta)$ and any $\epsilon > 0$, there exists $U_c = c(\epsilon) > 0$ so that for small enough $\epsilon > 0$ and large enough $N$, setting $B(\epsilon) := -E_0(\rho) + (-\epsilon, \epsilon)$ and $D(\epsilon') := -x_0(\rho) + (-\epsilon', \epsilon')$,

\[
\mathbb{P} \left\{ \text{Crt}_{N,1}(B(\epsilon), \mathbb{R} \setminus D(\epsilon')) > 0 \right\} \leq e^{-c N},
\]

where $B(\epsilon) := -E_0(\rho) + (-\epsilon, \epsilon)$ and $D(\epsilon') := -x_0(\rho) + (-\epsilon', \epsilon')$. 

From Theorem 3.1 and Lemma 5.13, to prove (5.23) it will be enough to show that for some \( c > 0 \) and large enough \( T \),
\[
(E, x) \in B(e) \times [-T, T] \setminus D(e^\ell) \implies \Theta_v,\rho(-E, -x) < -2c.
\]
Assume \( \delta \) is small enough so that \( \nu_\rho \) is pure-like, and therefore from Lemma 5.1, \( \Theta_v,\rho(-E_0(\rho), x) < 0 \) for any \( x \neq -x_0(\rho) \). Lemma 5.14 therefore follows from the continuity of \( \Theta_v,\rho(u, x) \) in \( u \) and \( x \).

We can now provide the proof of Corollary 3.7.

**Proof of Corollary 3.7.** From Lemma 5.14, to conclude the proof it will be enough to show that for any \( \epsilon > 0 \), if \( \eta, c > 0 \) are small enough, then
\[
|E_\rho(\sigma_1, \sigma_2)| \geq \epsilon < e^{-cN},
\]
where we define \( B_i(\eta) = -E_0(\rho_i) + (-\eta, \eta) \) and \( D_i(\eta) = -x_0(\rho_i) + (-\eta, \eta) \).

By Theorem 3.2, the corresponding number of pairs of points \((\sigma_1, \sigma_2)\),
\[
K(\epsilon, \eta) := \left[ \text{Crt}_{N, \rho_1, \rho_2}(I(\epsilon), B_1(\eta), B_2(\eta), D_1(\eta), D_2(\eta)) \right],
\]
where \( I(\epsilon) = (-1, 1) \setminus (-\epsilon, \epsilon) \) satisfies
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} K(\epsilon, \eta) \leq \sup_{r \in I(\epsilon), u_i \in B_i(\eta), x_i \in D_i(\eta)} \Psi_{v,\rho_1,\rho_2}(r, u_1, u_2, x_1, x_2).
\]
From Lemma 3.5, the fact that \( \Theta_v,\rho_i(u_i, x_i) \) is continuous in \( u_i \) and \( x_i \) and equal to 0 when \( u_i = -E_0(\rho_i) \) and \( x_i = -x_0(\rho_i) \), we have that if \( |1 - \rho_i| \) and \( \eta \) are small enough, then the right-hand side of (5.25) is negative. By Markov’s inequality, this proves (5.24). The bound of (3.20) follows by a standard diagonalization argument.

### 6 Stability of Condition M and Further Consequences

In this short section, we provide several further consequences of Condition M. The proofs utilize some of the results in Section 5.

**Lemma 6.1.** Assume Condition M. Then there exists \( \delta > 0 \) so that, for any \( \rho \in (1 - \delta, 1) \), \( \frac{d}{d\rho} E_0(\rho) = x_0(\rho) \).

The next two propositions concern the stability of Condition M under perturbations of \( \nu \).

**Proposition 6.2.** For any \( \rho \geq 3 \) there exists a \( \delta > 0 \) such that if \( \|v(x) - x^\rho\| < \delta \), then \( v \) also satisfies Condition M.

**Proposition 6.3.** If \( v \) is a non-pure mixture satisfying Condition M, then for some \( \delta > 0 \), any \( \overline{v} \) for which \( \|\overline{v} - v\| < \delta \) also satisfies Condition M.
The rest of the section is devoted to the proofs.

**Proof of Proposition 6.3.** Since, in the definition of \( \Psi_{v, \rho_1, \rho_2} \), all the expressions, including the elements of the matrix \( \Sigma_{U,X} \), involve only polynomials in \( r \) that do not have a linear term, we have that \( \frac{d}{d r} \Psi_{v, \rho_1, \rho_2}(0, u_1, u_2, x_1, x_2) = 0 \) for any \( v, \rho_1, u_i, \) and \( x_i \). Therefore Proposition 6.3 is a direct consequence of Lemmas 5.5 and 5.6. \( \square \)

**Proof of Lemma 6.1.** By Remark 5.12, 
\[
\lim_{N \to \infty} \frac{1}{N} \min_{\sigma \in \mathbb{S}^{N-1} (\sqrt{N})} H_N (\rho \sigma) = -E_0(\rho) \quad \text{almost surely}
\]
for any \( \rho \in (1 - \delta, 1] \) if \( \delta > 0 \) is small enough.

Combined with (5.23), this implies that, with probability tending to 1, if \( \sigma_\rho \) is a global minimum point of \( \mathbb{S}^{N-1} (\rho \sqrt{N}) \Rightarrow \sigma \mapsto H_N (\sigma) \), then
\[
\left| \frac{1}{N} H_N (\sigma_\rho) + E_0(\rho) \right|, \quad \left| \frac{1}{\sqrt{N}} \frac{d}{d R} H_N (\sigma_\rho) + x_0(\rho) \right| < \epsilon_N,
\]
for some sequence \( \epsilon_N = o(1) \).

From the uniform bound on the Lipschitz constant of the Hessian from Corollary C.2 (with \( k = 2 \)), with probability tending to 1,
\[
\frac{1}{N} \left| H_N \left( \frac{1}{\rho}, \sigma_\rho \right) - \left( H_N (\sigma_\rho) + (1 - \rho) \frac{1}{\sqrt{N}} \frac{d}{d R} H_N (\sigma_\rho) \right) \right| \leq \tilde{C}_2 (1 - \rho)^2,
\]
\[
\frac{1}{N} \left| H_N (\rho \sigma_1) - \left( H_N (\sigma_1) - (1 - \rho) \frac{1}{\sqrt{N}} \frac{d}{d R} H_N (\sigma_1) \right) \right| \leq \tilde{C}_2 (1 - \rho)^2,
\]
for some constant \( \tilde{C}_2 > 0 \). Therefore,
\[
-E_0(1) \leq -E_0(\rho) - x_0(\rho)(1 - \rho) + \tilde{C}_2 (1 - \rho)^2,
\]
\[
-E_0(\rho) \leq -E_0(1) + x_0(1)(1 - \rho) + \tilde{C}_2 (1 - \rho)^2,
\]
and
\[
x_0(\rho) - \tilde{C}_2 (1 - \rho) \leq \frac{E_0(1) - E_0(\rho)}{1 - \rho} \leq x_0(1) + \tilde{C}_2 (1 - \rho).
\]
Since \( \rho \mapsto x_0(\rho) \) is continuous by Lemma 5.5 the proof is completed. \( \square \)

**Proof of Proposition 6.2.** Throughout the proof we shall use the notation \( \Psi_v (r, u, x) := \Psi_{v,1,1}(r, u, u, x, x) \).

We will also always assume that \( \rho, \rho_1, \) and \( \rho_2 \) are equal to 1 and omit them from notation, writing, for example, \( \Sigma_{N}(B, \mathbb{R}) \) or \( \Sigma_{U}(r) \) for \( \Sigma_{N,1}(B, \mathbb{R}) \) and \( \Sigma_{U}(r, 1, 1) \).
For the pure case \( v_p(x) = x^p \), similarly to (3.6), it was proved in [6, theorem 2.8] that for any intervals \( B \subset (-\infty, 0), \)

\[
\lim_{N \to \infty} \frac{1}{N} \log(\mathbb{E} \text{Crt}_N(B, \mathbb{R})) = \sup_{u \in B} \Theta_p(u),
\]

where

\[
\Theta_p(u) := \frac{1}{2} + \frac{1}{2} \log(p - 1) - \frac{u^2}{2} + \Omega\left(\sqrt{\frac{p}{p - 1}} u\right).
\]

Define \( E_\infty(v_p) = E_\infty(p) = 2\sqrt{(p - 1)/p} \) and \( E_0(v_p) = E_0(p) \) as the unique number \( E \in (E_\infty(p), \infty) \) such that \( \Theta_p(E) = 0 \), and set \( -x_0(p) = -v_p'(1)E_0(p) \).

The above definitions can be appropriately extended to unnormalized pure models \( v(x) = x^p \).

Similarly to (3.12), it was proved in [52, theorem 5] that for the pure case \( v_p(x) = x^p \), for any intervals \( B \subset (-\infty, 0), I \subset (-1, 1), \)

\[
\limsup_{N \to \infty} \frac{1}{N} \log(\mathbb{E}[\text{Crt}_N(B, B, \mathbb{R}, I)]) \leq \sup_{r \in I, u \in B} \Psi_p(r, u_1, u_2),
\]

where, with \( \Sigma_U(r, 1, 1) \) as defined for the mixed case (A.2),

\[
\Psi_p(r, u_1, u_2) := 1 + \frac{1}{2} \log\left((p - 1)^2 \frac{1 - r^2}{1 - r^2 p - 2}\right) - \frac{1}{2}(u_1, u_2) \Sigma_U^{-1}(r, 1, 1) \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) + \Omega\left(\sqrt{\frac{p}{p - 1}} u_1\right) + \Omega\left(\sqrt{\frac{p}{p - 1}} u_2\right).
\]

By Lemma 7 of [52], for any \( \epsilon > 0 \),

\[
(6.1) \quad \Psi_p(0, -E_0(p), -E_0(p)) > \sup_{|r| \leq (\epsilon, 1)} \Psi_p(r, -E_0(p), -E_0(p)).
\]

From Lemma 4.2 for mixed \( H_N(\sigma) \), the conditional mean and covariance of \( (d/dR) H_N(\tilde{\sigma}) \) given \( (d/dR) H_N(\sigma(r)) \) given

\[
(6.2) \quad H_N(\tilde{\sigma}) = H_N(\sigma(r)) = u, \quad \nabla sp H_N(\tilde{\sigma}) = \nabla sp H_N(\sigma(r)) = 0,
\]

are equal, respectively, to

\[
m(r, u) = (\Sigma_{b, 11}(r), \Sigma_{b, 21}(r)) \Sigma_U^{-1}(r, u, u)^T,
\]

\[
\Sigma_{\tilde{X}}(r) = \Sigma_X(r) - \Sigma_{b}^{-1}(r) \Sigma_U^{-1}(r) \Sigma_{b}(r),
\]

where invertibility follows from Lemma 3.8.

Denoting \( \tilde{X} = x - m(r, u) \), for any non-pure \( v \),

\[
\Psi_v(r, u, x) = 1 + \frac{1}{2} \log\left((1 - r^2)\frac{v''(1)^2}{v'(1)^2 - (v'(r))^2}\right) + 2\Omega\left(\frac{x}{\sqrt{v'(1)}}\right)
\]

\[-\frac{1}{2}(u, u) \Sigma_U^{-1}(r, u, u)^T - \frac{1}{2}(\tilde{X}, \tilde{X}) \Sigma_{\tilde{X}}^{-1}(r, \tilde{X}, \tilde{X})^T.
\]
We note that $\Sigma_{U,11}(r) = \Sigma_{U,22}(r)$ and $\Sigma_{U,12}(r) = \Sigma_{U,21}(r)$, and therefore $\Sigma_{U,11}(r) \pm \Sigma_{U,12}(r)$ are the eigenvalues of $\Sigma_U(r)$ that correspond to the eigenvectors $(1, \pm 1)$. The same holds for $\Sigma_X(r)$ and $\Sigma_{\tilde{X}}(r)$. Thus,

$$m(r, u) = \frac{\Sigma_{b,11}(r) + \Sigma_{b,21}(r)}{\Sigma_{U,11}(r) + \Sigma_{U,12}(r)},$$

and we have that

$$\Psi_{\nu}(r, u, x) = 1 + \frac{1}{2} \log \left( \frac{(1 - r^2)^2}{v'(1)^2 - (v'(r))^2} \right) + 2\Omega \left( \frac{\chi}{\sqrt{\nu'(1)}} \right) - \frac{u^2}{\Sigma_{U,11}(r) + \Sigma_{U,12}(r)} - \frac{\bar{x}^2}{\Sigma_{\tilde{X},11}(r) + \Sigma_{\tilde{X},12}(r)}.$$  

(6.4)

Set, for any mixture $\nu$,

$$\tilde{\Psi}_{\nu}(r, u) = 1 + \frac{1}{2} \log \left( \frac{(1 - r^2)^2}{v'(1)^2 - (v'(r))^2} \right) - \frac{u^2}{\Sigma_{U,11}(r) + \Sigma_{U,12}(r)}.$$  

We note that

$$\tilde{\Psi}_{\nu_p}(r, u) = \Psi_{\nu}(r, u, u) - 2\Omega \left( \frac{\sqrt{p}}{p - 1} u \right).$$  

(6.5)

**Lemma 6.4.** Assume $\nu$ is non-pure. Then, for any $p \geq 3$, the following holds:

1. $\tilde{\Psi}_{\nu}(r, u)$ and its first and second derivatives in $r$ are continuous functions of $r \in (-1, 1), u \in \mathbb{R}$, and $\nu$ in a small neighborhood of $\nu_p$ (w.r.t. the norm $\| \cdot \|$).
2. We have that

$$\limsup_{(v, r, u) \to (v_p, 1, -E_0(p))} \tilde{\Psi}_{\nu}(r, u) \leq \limsup_{r \to 1} \tilde{\Psi}_{\nu_p}(r, -E_0(p)),$$

and the same also holds with the $r \to 1$ limits replaced by $r \to -1$.

**Proof.** The lemma follows from (6.5) and from the definition of the function $\tilde{\Psi}_{\nu}(r, u)$, since as $\nu \to v_p$ the corresponding derivatives in $r$, up to order 4, converge by Remark 5.4, and since $\Omega$ is smooth in the neighborhood of $E_0(p) > E_\infty(p) = 2\sqrt{(p - 1)/p}$.

Continuing with the proof of Proposition 6.2, we use the notation

$$Q(r) = Q_\nu(r) := \frac{(-\lambda_0(\nu) - m(r, -E_0(\nu)))^2}{\Sigma_{\tilde{X},11}(r) + \Sigma_{\tilde{X},12}(r)}.$$
for the quadratic term as in (6.4) with \( (u, x) = (-E_0(v), -x_0(v)) \). Recall that using (5.4) we had that \( |\nabla x_0(v) + v'(1)E_0(v)| \leq 2C\alpha_v^2 / \sqrt{v''(1)} \), for an appropriate constant \( C \). For \( r = 0 \),

\[
m(0, -E_0(v)) = -v'(1)E_0(v) \quad \text{and} \quad \Sigma_{\bar{X},11}(r) + \Sigma_{\bar{X},12}(r) = \alpha_v^2,
\]

and therefore

\[
|Q(0)| \leq 4C^2\alpha_v^2 / v''(1).
\]

By straightforward algebra, (assuming \( p \geq 3 \))

\[
\frac{d^2}{dr^2} \tilde{\Psi}_{v_p}(0, -E_0(p)) = -1 \quad \forall v, E : \frac{d}{dr} \tilde{\Psi}_v(0, E) = 0,
\]

and, by (6.1) and (6.5),

\[
\tilde{\Psi}_{v_p}(0, -E_0(p)) > \sup_{|r| \leq \epsilon} \tilde{\Psi}_{v_p}(r, -E_0(p)).
\]

Therefore, from Lemmas 5.5 and 6.4, if \( \|v - v_p\| \) is small enough, then for some \( \epsilon > 0 \) and any \( r \) with \( |r| \leq \epsilon \),

\[
\tilde{\Psi}_v(r, -E_0(v)) \leq \tilde{\Psi}_v(0, -E_0(v)) - r^2 / 4.
\]

Since \( Q(r) \leq 0 \) and (6.6), from the above, if \( \|v - v_p\| \) is small enough, then for any \( r \in (-1, 1) \) with \( |r| > 4C\alpha_v / \sqrt{v''(1)} \),

\[
\Psi_v(r, -E_0(v), -x_0(v)) < \Psi_v(0, -E_0(v), -x_0(v)).
\]

Since for any \( v, E, \) and \( x, \frac{d}{dr} \Psi_v(0, E, x) = 0 \), in light of (6.9), the proof of Proposition 6.2 will be complete if we prove that

\[
|r| \leq 4C\alpha_v / \sqrt{v''(1)} \implies |Q(r) - Q(0)| \leq r^2 / 8,
\]

assuming \( \|v - v_p\| \) is small enough. This follows from Taylor-expanding each of the terms in the definition of \( Q(r) \). More precisely, for any \( r \in (-\eta, \eta) \) we have that \( a_2(r) \) and \( a_4(r) \) are bounded in absolute value by some constant \( c' > 0 \) and

\[
\Sigma_{U,11}(r) - v(1), \quad \Sigma_{U,12}(r), \quad \Sigma_{b,11}(r) - v'(1), \quad \Sigma_{b,12}(r), \quad \Sigma_{\bar{X},11}(r) - v''(1) - v'(1), \quad \text{and} \quad \Sigma_{\bar{X},12}(r)
\]

are bounded in absolute value by \( cr^2 \), where \( c \) can be taken to be as small as we wish provided that \( \eta \) and \( v''(1) \) are small enough. For arbitrary \( C' > 0 \), assuming that \( \|v - v_p\| \) is sufficiently small, we therefore have that for any \( r \in [-C'\alpha_v, C'\alpha_v] \),

\[
|\Sigma_{\bar{X},11}(r) + \Sigma_{\bar{X},12}(r) - \alpha_v^2| \leq c_2' r^2,
\]

\[
|m(r, -E_0(v)) - v'(1)E_0(v)| \leq c_2' r^2,
\]

10We remind the reader that we are omitting \( \rho = 1 \) from our notation, so that for example, \( a_2(r) \) stands for \( a_2(r, 1, 1) \), etc.
from which (6.10) follows, since $\alpha_2 \to 0$ as $\nu \to \nu_p$. This completes the proof of Proposition 6.2. $\square$

## 7 Conditional Models on Sections

In this section, we show that conditionally on the value of $H_N$ at a point and its first-order derivatives there, one obtains an effective mixed-model on the $N - 2$ dimensional sections of the sphere determined by a fixed overlap with that point, that is, on appropriate “bands”. The main result are Lemmas 7.1 and 7.3.

For indices $i_1, \ldots, i_p$, denote by $\gamma_{\{i_1, \ldots, i_p\}}^p$ the sum of all $\gamma_{\{i_0, \ldots, i_p\}}^p$ such that $\{i_0, \ldots, i_p\} = \{i_1, \ldots, i_p\}$ as multisets. For $x \in \mathbb{R}^N$ with $\|x\| \leq \sqrt{N}$, we may write

\[(7.1)\]

\[
H_N(x) = \sum_{p=2}^\infty \frac{\gamma_p}{N^{(p-1)/2}} \sum_{i_1, \ldots, i_p \leq N} \sum_{i_1 \leq \cdots \leq i_p} x_{i_1} \cdots x_{i_p}.
\]

We are interested in the structure of the Hamiltonian $H_N(\sigma)$ on the section

\[(7.2)\]

\[
S(\rho \hat{n}) = \{\sigma \in S^{N-1}(\sqrt{N}) : R(\sigma, \hat{n}) = \rho\}
\]

formed by points with fixed overlap relative to the point $\hat{n} = (0, \ldots, 0, \sqrt{N})$; see (2.2). We therefore introduce, for $\sigma \in S^{N-1}(\sqrt{N})$,

\[(7.3)\]

\[
\tilde{\sigma} = \sqrt{\frac{N - 1}{N}} \frac{(\sigma_1, \ldots, \sigma_{N-1})}{\sqrt{1 - \rho^2(\sigma)}} \in S^{N-2}(\sqrt{N - 1}),
\]

\[
(\sigma) = \frac{\sigma_N}{\sqrt{N}} = R(\sigma, \hat{n}).
\]

Thinking of $\tilde{\sigma}$ as coordinates in an $(N - 1)$-dimensional sphere, we group the terms in (7.1) corresponding to $k$-spin interactions,

\[(7.4)\]

\[
\tilde{H}_N^{\hat{n}, k}(\sigma) = \sum_{p=k}^\infty \frac{\gamma_p}{N^{(p-1)/2}} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq N - 1} \sum_{\sigma_{i_1}, \ldots, \sigma_{i_k}} \gamma_{\{i_1, \ldots, i_k\}}^{\{i_1, \ldots, i_k\}} \sigma_{i_1} \cdots \sigma_{i_k} \frac{p-k}{N - 1} H_{N-1,k}(\tilde{\sigma}),
\]

where

\[
H_{N-1,k}(\tilde{\sigma}) := (N - 1)^{-\frac{k+1}{2}} \binom{p}{k}^{-1/2} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq N - 1} \sum_{\sigma_{i_1}, \ldots, \sigma_{i_k}} \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_k}.
\]
Since for different \((p, k)\) the model \(H_{N-1,k}(\bar{\sigma})\) are measurable w.r.t. to disjoint sets of the coefficients \((\bar{J}_{i_1,\ldots,i_p}^{(p)})\), we have that

\[
H_N(\sigma) = \sum_{k=0}^{\infty} H_{N}^{k}(\sigma) \quad \text{and} \quad \bar{H}_{N}^{k}(\sigma) \quad \text{are independent.}
\]

Also note that, since

\[
\text{Var}(J_{i_1,\ldots,i_k}) = \left(\frac{p}{k}\right) \text{Var}(J_{i_1,\ldots,i_k}) \quad \text{for each} \quad k \geq 0,
\]

\(H_{N-1,k}(\bar{\sigma})\) is a pure \(k\)-spin models on \(\mathbb{S}^{N-2}(\sqrt{N-1})\) (where for \(k = 0\), the “0-spin” model \(H_{N-1,0}(\bar{\sigma})\) \(\equiv (N-1)^{1/2} \bar{J}_{N,N}^{(p)} \equiv (N-1)^{1/2} \bar{J}_{N,N}^{(p)} \) is a random variable that is constant as function of \(\bar{\sigma}\)).

Hence, setting

\[
(7.6) \quad \alpha_k(p) := (1 - \rho^2)^{k/2} \left(\sum_{p-k}^{\infty} \gamma_p \left(\frac{p}{k}\right) \rho^{2(p-k)}\right)^{1/2}
\]

and letting \(H_{N-1,k}(\bar{\sigma})\) denote a pure \(k\)-spin model, we have that

\[
(7.7) \quad \bar{H}_{N}^{k}(\sigma) = \sqrt{\frac{N}{N-1} \alpha_k(\rho(\sigma))} H_{N-1,k}(\bar{\sigma}).
\]

Thus, on \(S(\rho_{\hat{n}})\), we can view the representation \(7.4\) as a Taylor series of \(H_N(\sigma)\) around \(\rho_{\hat{n}}\). Namely, for any \(\sigma \in S(\rho_{\hat{n}})\),

\[
(7.8) \quad \frac{d}{dx_{i_1}} \cdots \frac{d}{dx_{i_k}} H_N(\rho_{\hat{n}})
\]

\[
= \sum_{p-k}^{\infty} \gamma_p \rho^{p-k} \prod_{j=1}^{N-1} \frac{1}{1} \left|\{i : i = j\}\right|! (N-1)\bar{J}_{i_1,\ldots,i_k,N,N}^{(p)}(p)_{i_1,\ldots,i_k,N,N}.
\]

We would like to relate \(\nabla_{\sigma} H_N(\sigma)\) and \(\nabla^2_{\sigma} H_N(\sigma)\), defined using the frame field \(F_i\) as in \(1.13\), to the euclidean derivatives \(7.8\). Therefore, in this section we will
assume that \( F_i \) is chosen so that (see [53, footnote 7])

\[
\nabla_{sp} H_N(\rho \hat{n}) = \left( \frac{d}{dx_i} \right)_{x=0} H_N \left( (x_1, \ldots, x_{N-1}, \rho \sqrt{N - \|x\|^2}) \right)_{i \leq N-1},
\]

(7.10) \( \nabla_{sp}^2 H_N(\rho \hat{n}) \)

\[
\nabla_{sp} H_N(\rho \hat{n})
= \left( \frac{d}{dx_i} H_N(\rho \hat{n}) \right)_{i \leq N-1} = \left( \sum_{p=2}^{\infty} \left[ \sum_{j=1}^{N-1} \frac{\gamma_p \rho^{p-1}}{\rho^p} \tilde{T}_i,j,N,\ldots,N \right]_{i \leq N-1},
\]

(7.11) \( \nabla_{sp}^2 H_N(\rho \hat{n}) \)

\[
\nabla_{sp} H_N(\rho \hat{n})
= \left( \frac{d}{dx_i} \frac{d}{dx_j} H_N(\rho \hat{n}) \right)_{i,j \leq N-1} - \frac{1}{\rho \sqrt{N}} \frac{d}{dR} H_N(\rho \hat{n}) \mathbf{I},
\]

where the \( N-1 \times N-1 \) matrix \( G(\rho \hat{n}) = G_{N-1}(\rho \hat{n}) \) defined by

\[
G(\rho \hat{n}) := \left( \frac{d}{dx_i} \frac{d}{dx_j} H_N(\rho \hat{n}) \right)_{i,j \leq N-1}
\]

(7.12) has the same law as \( \sqrt{\frac{N-1}{N}} \mathbf{V}(\rho^2) \mathbf{M} \), where \( \mathbf{M} = \mathbf{M}_{N-1} \) is a GOE matrix.

For random variables that are continuous functions of the Gaussian disorder \( (J_{i_1, \ldots, i_p})_{p \geq 2} \), we will denote by \( \mathbb{P}_{u,v,A}^{\rho} \{ \cdot \} \) the conditional probability given

\[
H_N(\rho \hat{n}) = u, \quad \frac{d}{dR} H_N(\rho \hat{n}) = v, \quad \nabla_{sp} H_N(\rho \hat{n}) = 0, \quad \text{and} \quad G_{N-1}(\rho \hat{n}) = \mathbf{A},
\]

interpreted in the usual way by restricting \( (J_{i_1, \ldots, i_p})_{p \geq 2} \) to the appropriate affine subspace with the restriction of the density of \( (J_{i_1, \ldots, i_p})_{p \geq 2} \), normalized. Similarly, \( \mathbb{P}_{u,v}^{\rho} \{ \cdot \} \) (respectively, \( \mathbb{P}_{u,A}^{\rho} \{ \cdot \} \)) will denote the conditional probability given only the first three equalities (respectively, all equalities but the second). The corresponding expectations will be denoted with \( \mathbb{P} \) replaced by \( \mathbb{E} \).

Let \( \theta_\rho : S^{N-2}(\sqrt{N-1}) \to S(\rho \hat{n}) \) denote the (left) inverse of \( \sigma \mapsto \sigma \) (see (7.3)), given by

\[
\theta_\rho((\sigma_1, \ldots, \sigma_{N-1})) = \sqrt{\frac{N}{N-1}} \left( 1 - \rho^2 \right)(\sigma_1, \ldots, \sigma_{N-1}, 0) + \rho \hat{n}.
\]
For any function $h : S^{N-1}(\sqrt{N}) \to \mathbb{R}$, define $h|_{\rho} : S^{N-2}(\sqrt{N} - 1) \to \mathbb{R}$ by
\begin{equation}
    h|_{\rho}(\sigma) = h \circ \theta_{\rho}(\sigma).
\end{equation}

**Lemma 7.1.** Under $\mathbb{P}_{\rho}^\rho\{ \cdot \}$ we have that
\begin{equation}
    H_N|_{\rho}(\sigma) = u + \sum_{k=2}^{\infty} \alpha_k(\rho) \sqrt{\frac{N}{N-1}} H_{N-1,k}(\sigma),
\end{equation}
where $H_{N-1,k}(\sigma)$ are independent pure $k$-spin models of dimension $N-1$. Under $\mathbb{P}_{\rho,u,v,A}\{ \cdot \}$,
\begin{equation}
    H_N|_{\rho}(\sigma) = u + \frac{N}{2(N-1)} (1 - \rho^2) \sigma^T A \sigma + \sum_{k=-3}^{\infty} \alpha_k(\rho) \sqrt{\frac{N}{N-1}} H_{N-1,k}(\sigma).
\end{equation}

Also, if $H_N(\sigma)$ is replaced by $\sum_{k=0}^{l} \bar{H}^k_N(\sigma)$ above, then the conditional law is given by the same formula but with summation up to $l$ instead of $\infty$.

An important aspect of Lemma 7.1 is that the expressions in (7.15) and (7.16) do not contain linear terms (i.e., with $k = 1$).

**Proof.** By (7.5), we can write $H_N|_{\rho} = \sum_{k=0}^{\infty} \bar{H}^k_N|_{\rho}(\sigma)$. By (7.9), if $H_N(\rho \hat{n}) = u$, then $\bar{H}^0_N|_{\rho}(\sigma) = u$. By (7.9) and (7.11), if $\nabla_{\mathbb{P}} H_N(\rho \hat{n}) = 0$, then $\bar{H}^1_N|_{\rho}(\sigma) = 0$. By (7.9) and (7.12), if $G_{N-1}(\rho \hat{n}) = A$, then $\bar{H}^2_N|_{\rho}(\sigma) = \frac{N}{2(N-1)} (1 - \rho^2) \sigma^T A \sigma$. Further, from (7.11),
\begin{equation}
    \bar{H}^k_N|_{\rho}(\sigma) = \sqrt{\frac{N}{N-1}} \alpha_k(\rho) H_{N-1,k}(\sigma), \quad k \geq 1.
\end{equation}

Note that $\frac{d}{d \rho} H_N(\rho \hat{n})$ and $H_N(\rho \hat{n})$ are measurable w.r.t. the disorder coefficients $\bar{F}^{(p)}_{N-1,N}$. Similarly, by (7.11), $\nabla_{\mathbb{P}} H_N(\rho \hat{n})$ is measurable w.r.t. the coefficients of the form $\bar{F}^{(p)}_{i_1,N-1,N}$. By (7.12), $G_{N-1}(\rho \hat{n})$ is measurable w.r.t. the coefficients of the form $\bar{F}^{(p)}_{i_1,i_2,N-1,N}$. Lastly, for any $k \geq 3$, by (7.4), $\{ \bar{H}^k_N|_{\rho}(\sigma) \}_{\sigma}$ is measurable w.r.t. the coefficients of the form $\bar{F}^{(p)}_{i_1,\ldots,i_k,N-1,N}$.

The lemma follows by combining these facts, using that the disorder coefficients $\bar{F}^{(p)}_{i_1,\ldots,i_k,N-1,N}$ are independent for different values of $(k,p)$.

The random fields $\bar{H}^k_N(\sigma)$ can be developed around a general point $\sigma_0 \in S^{N-1}(\sqrt{N})$ instead of $\hat{n}$. One way to do so is by using (7.9) and rotating, in an appropriate sense, our coordinate system to be aligned with the direction corresponding to $\sigma_0$. More intrinsically, we can use the connection to Taylor expansions. That is, for any $\sigma \in S^{N-1}(\sqrt{N})$ such that $R(\sigma, \sigma_0) = \rho$, we define $\bar{H}^{\rho,0,k}_N(\sigma)$ as the
\( k \)-degree term in the Taylor series (in \( \mathbb{R}^N \)) of \( H_N(x) \) around \( H_N(\rho \sigma_0) \) evaluated at \( \sigma \). Denote by
\[
\nabla^k E H_N(x) = \left( \frac{d}{dx_{i_1}} \cdots \frac{d}{dx_{i_k}} H_N(x) \right)_{i_1, \ldots, i_k}
\]
the tensor of (euclidean) derivatives of order \( k \) of the Hamiltonian \( H_N(x) \), and for any tensor \( T = (t_{i_1, \ldots, i_k})_{i_1, \ldots, i_k \leq N} \) define
\[
(7.17) \quad \| T \|_\infty := \frac{1}{\sqrt{N}} \sup_{|y^{(j)}|_{L^1}} \left| \sum_{i_1, \ldots, i_p = 1}^N t_{i_1, \ldots, i_k} y_{i_1}^{(1)} \cdots y_{i_p}^{(p)} \right|.
\]
where \( y^{(i)} = (y_1^{(i)}, \ldots, y_N^{(i)}) \), and for a linear subspace \( V \subset \mathbb{R}^N \) define \( \| T \|_{V} \) similarly to (7.17) with the additional restriction that \( y_i^{(j)} \in V \). For use later, we record the following direct corollary of the definitions.

**Corollary 7.2.** Let \( \sigma_0 \in \mathbb{S}^{N-1}(\sqrt{N}) \) and \( \rho \in (0, 1) \), and denote by \( V_0 \) the tangent space to \( \mathbb{S}^{N-1}(\sqrt{N}) \) at \( \sigma_0 \). If \( \| \nabla^k E H_N(\rho \sigma_0) \|_{V_0} < C N^{-k/2} \), then for all \( \sigma \in \mathbb{S}^{N-1}(\sqrt{N}) \) with \( R(\sigma, \sigma_0) = \rho \),
\[
|\bar{H}_N^{\sigma_0, k}(\sigma)| \leq NC(1 - \rho^2)^{k/2} / k!.
\]

We conclude this section with a bound on the error of a finite-degree approximation of the expansion.

**Lemma 7.3.** For any mixture \( \nu \), there exist \( C, c > 0 \) such that
\[
(7.18) \quad \mathbb{P} \left\{ \exists k \geq 0, \sigma_0, \sigma \in \mathbb{S}^{N-1}(\sqrt{N}) : 1 - \rho(\sigma)^2 < c, \right. \left. \left| H_N(\sigma) - \sum_{i=0}^k \bar{H}_N^{\sigma_0, i}(\sigma) \right| \geq NC \left( \frac{1 - \rho(\sigma)^2}{c} \right)^{(k+1)/2} \right\} \leq e^{-cN},
\]
where we abbreviate \( \rho(\sigma) := R(\sigma, \sigma_0) \).

**Proof.** Since \( H_N(\sigma) = \sum_{i=0}^\infty \bar{H}_N^{\sigma_0, i}(\sigma) \), we need to derive an appropriate bound for \( |\sum_{i=k+1}^{\infty} \bar{H}_N^{\sigma_0, i}(\sigma)| \), for any \( k \geq 0 \). From Corollary 7.2 and the concentration inequality in (C.8) of Corollary C.2, with \( K \) the universal constant of Lemma C.1, we have the following: On an event whose complement has exponentially small-in-\( N \) probability, uniformly over all \( \sigma_0, \sigma \in \mathbb{S}^{N-1}(\sqrt{N}) \) and \( i \geq 1 \), writing \( \xi = \xi(\sigma) = 1 - \rho(\sigma)^2 \),
\[
|\bar{H}_N^{\sigma_0, i}(\sigma)| \leq 2NK \sum_{p \geq i} \gamma_p p^{1/2} \left( \frac{p}{i} \right)^{i/2},
\]
and
\[
\left| \sum_{i-k+1}^{\infty} \tilde{H}_N^{0,i}(\sigma) \right| \leq 2NK \sum_{i-k+1}^{\infty} \sum_{p-i}^{\infty} \gamma_p p^{1/2} \left( \frac{p}{i} \right)^{i/2} \left( \frac{\xi}{c} \right)^{i/2}
\]
\[
\leq 2NK \sum_{i-k+1}^{\infty} \sum_{p-i}^{\infty} \gamma_p p^{1/2} \left( \frac{p}{i} \right)^{i/2},
\]
(7.19)
where in the last inequality we used that $\xi/c \leq 1$.

We next claim that for small enough $c > 0$,
\[
\sum_{i-1}^{\infty} \sum_{p-i}^{\infty} \gamma_p p^{1/2} \left( \frac{p}{i} \right)^{i/2} < \infty.
\]
(7.20)
Indeed, interchanging the order of summation, the left-hand side of (7.20) equals
\[
\sum_{p-1}^{\infty} \gamma_p p^{1/2} \left( \sum_{i=1}^{\infty} \left( \frac{p}{i} \right)^{i/2} \right) < \infty,
\]
where in the last inequality we used that $\sum_{p=1}^{\infty} \gamma_p^2 \beta p < \infty$ for some $\beta > 1$ by assumption. Combining (7.19) with (7.20) completes the proof of the lemma.

\section{The Logarithmic Weight Functions $\Lambda_{Z,\beta}$ and $\Lambda_{F,\beta}$}

In Section 7, we studied the structure of the Hamiltonian on the section $S(\sigma_0)$ of codimension 1 around an arbitrary point $\sigma_0 \in \mathbb{B}^N(\sqrt{N})$; see (2.2). Those results can be used to compute the weight of thin bands conditional on the center being a $\rho$-critical point $\sigma_\rho$ such that $H_N(\sigma_\rho) \approx NE$. To discuss these, we introduce notation.

For any measurable set $B \subset \mathbb{S}^{N-1}(\sqrt{N})$, set
\[
Z_{N,\beta}(B) = \int_B e^{-\beta H_N(\sigma)} d\sigma.
\]
(8.1)
Using the uniform Lipschitz bounds of Corollary C.2 for the gradient, one has that with high probability, for any $\rho \in (0, 1)$ and $\delta > 0$,
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{P} \left\{ \sup_{\sigma_0 \in \mathbb{S}^{N-1}(\rho \sqrt{N})} \left| \frac{1}{N} \log \frac{Z_{N,\beta}(\text{Band}(\sigma_0, \epsilon))}{Z_{N,\beta}(\sigma_0)} \right| > \delta \right\} = 0,
\]
where $Z_{N,\beta}(\sigma_0)$ is defined in (2.4). Therefore, we focus on analyzing weights of the form $Z_{N,\beta}(\sigma_0)$. We begin with the following consequence of Lemma 7.1.

\textbf{Corollary 8.1.} For any $\rho \in (0, 1)$,
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\rho \hat{\nu}} \{ Z_{N,\beta}(\rho \hat{\nu}) \} = \Lambda_{Z,\beta}(E, \rho).
\]
(8.2)
where, with $\alpha_k(\rho)$ as defined in (7.6),

$$
\Lambda_{Z,\beta}(E, \rho) := -\beta E + \frac{1}{2} \log(1 - \rho^2) + \frac{1}{2} \beta^2 \sum_{k=2}^{\infty} \alpha_k^2(\rho),
$$
(8.3)

$$
= -\beta E + \frac{1}{2} \log(1 - \rho^2) + \frac{1}{2} \beta^2 (\nu(1) - \alpha_0^2(\rho) - \alpha_1^2(\rho)).
$$

**Proof.** The logarithmic term in (8.3) is equal to the limit of $\frac{1}{N} \log$ of the ratio of volumes in (2.4). By Lemma 7.1,

$$
\frac{1}{N} \log \mathbb{P}_{\rho}^N \{ H_N(\sigma) \} = E \quad \text{and} \quad \frac{1}{N} \log \text{Var}_{\rho}^N \{ H_N(\sigma) \} = \sum_{k=2}^{\infty} \alpha_k^2(\rho),
$$

for any $\sigma \in S(\rho \hat{\pi})$, where $\text{Var}_{\rho}^N$ denotes the variance under $\mathbb{P}_{\rho}^N$. Using the expression for the moment-generating function of Gaussian variables, $\mathbb{E}e^{\beta x} = e^{\beta \mu + \frac{1}{2} \sigma^2}$ for $x \sim N(\mu, \sigma^2)$, completes the proof. □

We shall see that for large $\beta$, the $\rho$-critical points that contribute most to the partition function are the deepest ones, that is, points with $H_N(\sigma_\rho) = -N(E_0(\rho) + o(1))$. We are thus particularly interested in the function $\rho \mapsto \Lambda_{Z,\beta}(-E_0(\rho), \rho)$, which by an abuse of notation we will denote by $\Lambda_{Z,\beta}(\rho)$.

For pure-like $\nu$ and $\rho < 1$ close enough to 1, using Lemmas 6.1 and 5.5,

$$
\frac{d}{d \rho} \Lambda_{Z,\beta}(\rho) = \beta x_0(\rho) - \frac{\rho}{1 - \rho^2} - \beta^2 (1 - \rho^2) \sum_{p=2}^{\infty} \gamma_p^2 p(p-1) \rho^{2p-3},
$$

$$
\frac{d^2}{d \rho^2} \Lambda_{Z,\beta}(\rho) = \beta \frac{d}{d \rho} x_0(\rho) - \frac{1 + \rho^2}{(1 - \rho^2)^2} + 2 \beta^2 \sum_{p=2}^{\infty} \gamma_p^2 p(p-1) \rho^{2p-2}
$$

$$
- \beta^2 (1 - \rho^2) \sum_{p=2}^{\infty} \gamma_p^2 p(p-1)(2p-3) \rho^{2p-4}.
$$

Hence, for any $0 < T = T(\beta) = o(\sqrt{\beta})$, uniformly on $(0, T]$, as $\beta \to \infty$,

$$
(8.4) \quad \frac{1}{\beta} \frac{d}{d \rho} \Lambda_{Z,\beta}(1 - \frac{t}{\beta}) \longrightarrow x_0(1) - \frac{1}{2t} - 2\nu''(1) t,
$$

$$
(8.5) \quad \frac{1}{\beta^2} \frac{d^2}{d \rho^2} \Lambda_{Z,\beta}(1 - \frac{t}{\beta}) \longrightarrow -\frac{1}{2t^2} + 2\nu''(1).
$$

We conclude that for pure-like $\nu$ and large $\beta$, $\Lambda_{Z,\beta}(\rho)$ has exactly two critical points in $[1 - T/\beta, 1)$,

$$
(8.6) \quad \rho_* := \rho_*(\beta) = 1 - \frac{t}{\beta} + o\left( \frac{1}{\beta} \right) \quad \text{and} \quad \rho_{**} := \rho_{**}(\beta) = 1 - \frac{t_*}{\beta} + o\left( \frac{1}{\beta} \right).
$$
where
\[ t_\pm = \frac{x_0(1) \pm \sqrt{x_0^2(1) - 4v''(1)}}{4v''(1)} \]
are the roots of (8.4). Note that by Lemma 5.1, the discriminant above is positive.
Also, from (8.5) and the fact that (8.4) is 0 at the critical points, the corresponding
second derivatives normalized by \( \beta^2 \) converge to
\[ -\frac{x_0(1)}{t_\pm} + 4v''(1) = \pm \frac{4v''(1) \sqrt{x_0^2(1) - 4v''(1)}}{x_0(1) \sqrt{x_0^2(1) - 4v''(1)}}. \]
Therefore, for large \( \beta \), \( \rho_*(\beta) \) is a local maximum and \( \rho_{**} \) is a local minimum.

The quantity \( \Lambda_{\beta(\rho)} \) always gives an upper bound for the (conditional) free energy defined as
\begin{equation}
F_{E,v}^\rho := \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{NE,v} \log \{Z_{N,\beta(\rho \hat{n})}\},
\end{equation}
by appealing to Markov’s inequality (compare with (8.2)). However, to derive a
matching lower bound we will need the 2-spin model corresponding to the expansion
of Section 7, namely, \( \overline{H}^{\rho_{(2)}}_{N,\beta} \) \( \sigma \), to have an effective high temperature.
From known facts concerning the spherical 2-spin; see, e.g., [56, theorem 1.1, proposition 2.2], the (first) transition in \( \rho \) from high to low temperature occurs at
\begin{equation}
\rho_c := \rho_c(\beta) = \max \left\{ \rho \in (0, 1) : \alpha_2(\rho) = \frac{1}{\beta \sqrt{2}} \right\},
\end{equation}
where
\[ \alpha_2(\rho) = (1 - \rho^2) \left( \sum_{p=2}^{\infty} \frac{\gamma_p^2(p \beta)}{p^2} \rho^{2p-4} \right)^{1/2}. \]
We note for later use that, since
\[ \alpha_2(\rho) = (1 - \rho) \sqrt{2v''(1)} + O((1 - \rho)^2) \quad \text{as} \ \rho \to 1, \]
(8.9) \[ \rho_c(\beta) = 1 - \frac{t_c}{\beta} + O\left(\frac{1}{\beta^2}\right) \quad \text{as} \ \beta \to \infty \quad \text{where} \ t_c = \frac{1}{2 \sqrt{v''(1)}}. \]
We can write
\[ t_\pm = \frac{1}{2 \sqrt{v''(1)}} (z \pm \sqrt{z^2 - 1}) \quad \text{and} \quad z = \frac{x_0(1)}{2 \sqrt{v''(1)}} > 1. \]
Since \( z - \sqrt{z^2 - 1} \) decreases in \( z > 1 \) and is equal to 1 for \( z = 1 \), by (8.6) and
(8.9), for large \( \beta \),
\begin{equation}
t_- < t_c < t_+ \quad \text{and} \quad \rho_{**} < \rho_c < \rho_*.
\end{equation}
Also note for later use that, from (8.3), we have a constant gap in the limit:

$$\lim_{\beta \to \infty} (\Lambda_{Z, \beta}(\rho_\star) - \Lambda_{Z, \beta}(\rho_c))$$

(8.11)

$$= (t_c - t_0) x_0(1) + \frac{1}{2} \log(t_0/t_c) + v''(1)(t_c^2 - t_0^2) > 0,$$

where the strict inequality follows by writing the limit as the integral, over \((\rho_c, \rho_\star)\), of (8.4) times \(\beta\), and noting that (8.4) is positive in this range.

As mentioned above, for \(\rho < \rho_c\), \(\Lambda_{Z, \beta}(E, \rho) \neq \mathbb{P}_{E, v}^{\rho}\); see (8.7). For the asymptotics we shall consider, what will be relevant is the free energy corresponding to the 2-and-below spins on \(S(\sigma_0)\), defined similarly to (2.4) by

$$Z_{N, \beta}^2(\sigma_0) := (1 - \|\sigma_0\|/\sqrt{N}) \frac{N}{2} \int_{S(\sigma_0)} e^{-\beta \sum_{i=0} \mathbb{P}_{E, v}^{0,i}(\sigma)} d\sigma.$$

The (logarithmic) error between \(Z_{N, \beta}^2(\sigma_0)\) and \(Z_{N, \beta}(\sigma_0)\) will be controlled using Lemma 7.3; see (8.16) below. Our immediate task is to provide, in the next lemma, an expression for the free energy of the former.

**Lemma 8.2.** For any \(\rho \in (0, 1)\) such that \(\beta \alpha_2(\rho) \geq 1/\sqrt{2}\) (in particular, for \(\rho \in (\rho_c - \delta, \rho_c)\) with small fixed \(\delta\), independent of \(\beta\)),

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{NE, v}^{\rho} (\log Z_{N, \beta}^2(\rho_\star)) = \Lambda_{F, \beta}^2(E, \rho),$$

where

$$\Lambda_{F, \beta}^2(E, \rho) := -\beta E + \frac{1}{2} \log(1 - \rho^2) + \sqrt{2} \beta \alpha_2(\rho)$$

(8.13)

$$- \frac{1}{2} \log(\beta \alpha_2(\rho)) - \frac{3}{4} - \frac{1}{4} \log 2$$

$$= -\beta E + \sqrt{2} \beta \alpha_2(\rho) - \frac{1}{4} \log(\beta^2 v''(\rho^2)) - \frac{3}{4}.$$ 

**Proof.** The term \(1/2 \log(1 - \rho^2)\) comes from the volumes ratio as in Corollary 8.1. With \(\sigma_0 = \rho_\star\) the integral in (8.12) is determined by \(\sum_{j=0}^2 \mathbb{H}_N^{\hat{j}} |_{\rho}(\sigma)\). By the argument used to prove Lemma 7.1 under \(\mathbb{P}_{NE, v}^{\rho}\),

$$\sum_{j=0}^2 \mathbb{H}_N^{\hat{j}} |_{\rho}(\sigma) \overset{d}{=} NE + \alpha_2(\rho) \sqrt{\frac{N}{N-1}} H_{N-1,2}(\sigma).$$

(8.14)

where \(H_{N-1,2}(\sigma)\) is the pure 2-spin model on \(\mathbb{S}^{N-2}(\sqrt{N-1})\). Our assumption on \(\rho\) is equivalent to requiring the effective inverse temperature \(\beta \alpha_2(\rho)\) being in the high-temperature phase of the 2-spin model and the proof is completed by the expression for its free energy from [56, theorem 1.1, proposition 2.2]. \(\square\)
We note that for any $0 < T = T(\beta) = o(\sqrt{\beta})$, uniformly on $(0, T]$, as $\beta \to \infty$,

\begin{equation}
\frac{d}{d \rho} \Lambda_{F, \beta}^{2-}(1 - \frac{t}{\beta}) \to x_0(1) - 2 \sqrt{v''(1)} > 0,
\end{equation}

where we denote $\Lambda_{F, \beta}^{2-}(\rho) := \Lambda_{F, \beta}^{2-}(-E_0(\rho), \rho)$. Also,

\[ \Lambda_{F, \beta}^{2-}(\rho_c) = \beta E_0(\rho_c) + \frac{1}{2} \log(1 - \rho_c^2) + \frac{1}{2} \beta^2 \alpha_2^2(\rho_c). \]

and therefore

\begin{equation}
\left| \Lambda_{F, \beta}^{2-}(\rho_c) - \Lambda_{Z, \beta}(\rho_c) \right| = O\left( \frac{1}{\beta} \right).
\end{equation}

9 The Structure of Deep Level Sets

In this section we study the structure of the sub-level set

\[ A_t := \{ \sigma \in S^{N-1}(\sqrt{N}) : H_N(\sigma) \leq - (E_0 - t)N \} \]

for small $t$, and relate it to deep critical points. The main result we prove is the following:

**Proposition 9.1.** Assume that $v$ satisfies Condition M. For large enough $c_{LS} = c_{LS}(v)$ and small enough $\delta_{LS} = \delta_{LS}(v)$ we have the following. For any $t < \delta_{LS}/c_{LS}$ and any $\eta > 0$, with probability tending to 1 as $N \to \infty$:

1. Each connected component $A$ of the sub-level set $A_t$ contains exactly one 1-critical point $\sigma_1$, which is in particular a local minimum of $H_N(\sigma)$ on $S^{N-1}(\sqrt{N})$.

2. For each 1-critical point $\sigma_1 \in A_t$, there exists a differentiable path $G : [1 - \delta_{LS}, 1] \to S^{N-1}(1)$ such that $\sqrt{N}G(1) = \sigma_1$ and for any $\rho \in [1 - \delta_{LS}, 1]$, $\sigma_\rho := \sqrt{N}\rho G(\rho)$ is a $\rho$-critical point. Moreover, the speed $\| \frac{d}{d \rho} G(\rho) \|$ of $G(\rho)$ under the standard Riemannian metric on $S^{N-1}(1)$ is bounded by $c_{LS} > 0$.

3. For each 1-critical point $\sigma_1 \in A_t$, along the path $\sigma_\rho$ defined above,

\begin{equation}
- E_0(\rho) - \eta < \frac{1}{N} H_N(\sigma_\rho) < - E_0(1) + m x_0(1)(1 - \rho),
\end{equation}

where $m = m(\delta_{LS})$ is a constant that is determined by $\delta_{LS}$ and satisfies $\lim_{\delta_{LS} \to 0} m(\delta_{LS}) = 1$.

4. For each 1-critical point $\sigma_1 \in A_t$, the spherical cap

\[ \text{Cap}(t) := \{ \sigma \in S^{N-1}(\sqrt{N}) : R(\sigma, \sigma_1 - c_{LS}t) \geq 1 - c_{LS}t \} \]

contains the connected component of $\sigma_1$ in $A_t$.

The proof occupies the rest of this section. We first prove part 2, then parts 3 and 4, and finally part 1.
9.1 Proof of Proposition 9.1 Part 2

The construction of the path \( G \) will be based on an application of the implicit function theorem. Define \( G : (0, 1 + \tau) \times U \to \mathbb{R}^{N-1} \), where \( U \subset \mathbb{R}^{N-1} \) is a small neighborhood of the origin, by

\[
G(\rho, x) = \left(F_i H_N(\rho T_{n}(x))\right)_{i \leq N-1} \\
T_{n}(x) = \left(x_1, \ldots, x_{N-1}, \sqrt{N - \|x\|^2}\right),
\]

where \( x = (x_1, \ldots, x_{N-1}) \), and we recall that \( F_i, i \leq N - 1 \), is a piecewise smooth frame field that we defined before (1.13), and that we will assume to satisfy (7.10) (in particular, the frame from Lemma A.1). We choose this definition so that if \( G(\rho, x) = 0 \), then \( \rho T_{n}(x) \) is a \( \rho \)-critical point. Denote

\[
J_x G(\rho, x) = \left(\frac{d}{dx_j} F_i H_N(\rho T_{n}(x))\right)_{i, j \leq N-1},
\]

\[
\frac{d}{d\rho} G(\rho, x) = \left(\frac{d}{d\rho} F_i H_N(\rho T_{n}(x))\right)_{i \leq N-1},
\]

and note that at \( x = 0 \),

\[
J_x G(\rho, 0) = (\rho F_i F_j H_N(\rho \hat{n}))_{i, j \leq N-1} = \rho \nabla^{2}_{sp} H_N(\rho \hat{n}),
\]

\[
\frac{d}{d\rho} G(\rho, 0) = \frac{1}{\rho} \sum \gamma_p (p - 1) \rho^{p-1} \nabla_{sp} H_{N, \rho}(\hat{n}),
\]

where for the second equality we used the fact that

\[
\nabla_{sp} H_N(\rho \sigma) = \sum \gamma_p \rho^{p-1} \nabla_{sp} H_{N, \rho}(\sigma).
\]

By the implicit function theorem, if \( \rho_0 \hat{n} \) is \( \rho_0 \)-critical, that is, \( G(\rho_0, 0) = 0 \), and \( J_x G(\rho_0, 0) \) is invertible, then on a small neighborhood of \( \rho_0 \) there exists a unique \( g_{\rho_0 \hat{n}}(\rho) = (g^{(i)}_{\rho_0 \hat{n}}(\rho))_{i \leq N-1} \in \mathbb{R}^{N-1} \) such that \( G(\rho, g_{\rho_0 \hat{n}}(\rho)) = 0 \) and

\[
\frac{d}{d\rho} g_{\rho_0 \hat{n}}(\rho_0) := \left(\frac{d}{d\rho} g^{(i)}_{\rho_0 \hat{n}}(\rho)\right)_{i \leq N-1} = -[J_x G(\rho_0, 0)]^{-1} \frac{d}{d\rho} G(\rho_0, 0).
\]

In this case, defining the path \( G_{\rho_0 \hat{n}}(\rho) = T_{n}(g_{\rho_0 \hat{n}}(\rho))/\|T_{n}(g_{\rho_0 \hat{n}}(\rho))\| \in S^{N-1}(1) \), we have that the speed of \( G_{\rho_0 \hat{n}}(\rho) \) at \( \rho_0 \), relative to the standard Riemannian metric, is given by \( \|\frac{d}{d\rho} g_{\rho_0 \hat{n}}(\rho_0)\|/(\rho_0 \sqrt{N}) \).

Of course, instead of \( \rho \hat{n} \), the same argument can be applied to a general point \( \rho \sigma \in S^{N-1}(\rho \sqrt{N}) \), and therefore, if \( \rho \sigma \) is a \( \rho \)-critical point such that

\[
\frac{1}{\rho} \sum \gamma_p (p - 1) \rho^{p-1} \nabla_{sp} H_{N, \rho}(\sigma) \leq \sqrt{N} c',
\]

\[
\frac{1}{\rho} \left\| \nabla_{sp} H_{N, \rho}(\sigma) \right\|_{op} < c,
\]
where by (9.6) we mean in particular that the inverse exists, then there exists a path \( \mathcal{G}_\theta (\rho') \) of \( \rho' \)-critical points, defined on a neighborhood of \( \rho \), whose speed at \( \rho \) is bounded by \( cc' \).

To complete the proof, we need to prove that for some \( c, c' \), and \( \delta_{LS} \), with probability tending to 1, for every 1-critical point \( \sigma_1 \) with \( H_N (\sigma_1) \leq -(E_0 - t)N \) if \( \rho \in [1 - \delta_{LS}, 1] \) and \( \sigma \in S^{N-1}(\sqrt{N}) \) has geodesic distance to \( \sigma_1 \) smaller than \( \sqrt{N}\delta_{LS}cc' \), then (9.5) and (9.6) hold.\footnote{We note that when we prove the bounds (9.5) and (9.6) we may allow the frame field \( F_i \) to depend on \( \sigma \).}

Proving (9.5) is easier. If we define \( \tau_p = \gamma_p (\rho - 1) \) and let \( \tilde{H}_N (\sigma) \) be the corresponding mixed model, then by (9.4) the left-hand side of (9.5) is exactly the norm of \( \frac{1}{\rho} \nabla^2_{\mathbb{S}_p} \tilde{H}_N (\rho \sigma) \). Therefore for large enough \( c' \), the concentration inequality (C.8) of Corollary C.2 implies (9.5) with high probability uniformly over all \( \sigma \in S^{N-1}(\sqrt{N}) \) and \( \rho \in [1 - \delta_{LS}, 1] \).

To prove (9.6), we first relate the spherical Hessian \( \nabla^2_E H_N (x) \) to the euclidean Hessian matrix \( \nabla^2_E H_N (x) = \{ \frac{d}{dx} \frac{d}{dx} H_N (x) \}_{i,j \leq N} \), where \( x \in \mathbb{R}^N \), \( \| x \| \leq \sqrt{N} \).

Assuming \( \| x \| = \rho \sqrt{N} \), let \( \mathcal{T}_x = \mathcal{T}_x S^{N-1}(\rho \sqrt{N}) \) be the tangent space to the sphere at \( x \), viewed as a linear subspace of \( \mathbb{R}^N \) using the usual identification. Let \( \mathbf{A}(x) \subset \mathbb{R}^{N-1 \times N} \) be some matrix whose rows form an orthonormal basis of \( \mathcal{T}_x \). For an appropriate choice of the frame \( F_i \) on a neighborhood of \( x \); see (7.11),

\[
\nabla^2_E H_N (x) = \mathbf{A}(x) \nabla^2_E H_N (x) (\mathbf{A}(x))^T - \frac{1}{\| x \|} \frac{d}{dR} H_N (x) \mathbf{I}.
\]

Therefore, \( \| [\nabla^2_E H_N (x)]^{-1} \|_{op} = f_N (x) \) where

\[
(9.7) \quad f_N (x) := \left( \min_{v \in \mathcal{T}_x \cap \| v \|=1} \left| \left[ \nabla^2_E H_N (x), v v^T \right] - \frac{1}{\| x \|} \frac{d}{dR} H_N (x) \right| \right)^{-1},
\]

and \( f_N (x) = \infty \) whenever \( \nabla^2_E H_N (x) \) is not invertible.

In light of the the concentration inequality (C.8) of Corollary C.2, Part 2 of Proposition 9.1 will follow if we prove the following two lemmas; in both lemmas, \( f_N \) is as in (9.7).

**Lemma 9.2.** For small enough \( t > 0 \) and large enough \( c > 0 \),

\[
(9.8) \quad \lim_{N \to \infty} \mathbb{P} \left\{ \exists \sigma_1 \in \mathcal{C}_{N,1} \left( (\infty, -(E_0 - t)N) \right) : f_N (\sigma_1) \geq c/2 \right\} = 0.
\]

**Lemma 9.3.** For any \( C, c > 0 \), for small enough \( \delta > 0 \), if \( x, x' \in \mathbb{R}^N \) are points such that

\[
(1) \quad \frac{1}{\sqrt{N}} \| x \| \in \left( \frac{1}{2}, 1 \right],
\]

\[
(2) \quad \frac{1}{\sqrt{N}} \| \nabla_E H_N (x) \|, \| \nabla_E H_N (x) \|_{op} \leq C,
\]

...
(3) \( \frac{1}{N} \| \nabla E H_N(x) - \nabla E H_N(x') \| \leq \| \nabla^2 E H_N(x) - \nabla^2 E H_N(x') \|_{op} \leq \frac{1}{\sqrt{N}} \| x - x' \|_2 \)

are all smaller than \( \delta \),

then,

\( f_N(x) < c/2 \implies f_N(x') < c \).

PROOF OF LEMMA 9.2. By Lemma 5.14 to prove (9.8) for small \( t \) it will be sufficient to show that for small \( \epsilon \),

\[ \lim_{N \to \infty} P \left\{ \exists \sigma_1 \in \mathbb{C}_{N,1}(NB(\epsilon), \sqrt{N}D(\epsilon)) : f_N(\sigma_1) \geq c/2 \right\} = 0, \]

where \( B(\epsilon) = -E_0 + (-\epsilon, \epsilon) \) and \( D(\epsilon) = -x_0 + (-\epsilon, \epsilon) \).

We show below that

\[ c := \limsup_{N \to \infty} \sup_{u \in B(\epsilon), x \in D(\epsilon)} \frac{1}{N} \log \left( \mathbb{P}^{1}_{N u, \sqrt{N} x} \{ f_N(\hat{\mathbf{n}}) \geq c/2 \} \right) < 0. \]

Since \( \Theta_{\nu,1}(u, x) \) is continuous and \( \Theta_{\nu,1}(-E_0(1), -x_0(1)) = 0 \), an application of Lemma 5.1 with \( \varphi(u, x) = -c/2 \) and \( g_N = f_N \) completes the proof, reducing \( \epsilon \) if needed.

We thus turn to the proof of (9.9). From (7.12) and the sentence following it, the conditional probability in (9.9) is equal to

\[ P \{ \exists i \leq N - 1 : |\lambda_i(\hat{\mathbf{n}}) - x| \leq 2/c \}, \]

where \( \lambda_i(\hat{\mathbf{n}}) \) are the eigenvalues of the matrix \( G(\hat{\mathbf{n}}) \), which has the same law as

\[ \sqrt{N - 1/N} \nu''(1) \mathbf{M} \]

with \( \mathbf{M} \) being a GOE matrix. By Lemma 5.1, \( -x_0 / \sqrt{\nu''(1)} < -2 \), and for small \( \epsilon \) and large \( c \), (9.10) is exponentially small in \( N \), by the bound on the top eigenvalue of [12, lemma 6.3]. This implies (9.9) and completes the proof of the lemma.

PROOF OF LEMMA 9.3. Throughout the proof, we write \( \delta \) for various positive quantities satisfying \( \lim_{\delta \to 0} \delta = 0 \).

Since \( \frac{d}{dR} H_N(x) = \langle \nabla E H_N(x), \frac{x}{\| x \|} \rangle \), from our assumptions,

\[ \left\| \frac{1}{\| x \|} \frac{d}{dR} H_N(x) - \frac{1}{\| x' \|} \frac{d}{dR} H_N(x') \right\| \leq \epsilon_1(\delta), \]

for some \( \epsilon_1(\delta) \).

Since, by assumption, \( \| x \| / \sqrt{N} > \frac{1}{2} \) and \( \| x - x' \| / \sqrt{N} < \delta \), there exists \( \epsilon_2(\delta) \) such that for any \( v' \in T_{x'} \) with \( \| v' \| = 1 \) there exists \( v \in T_x \) with \( \| v \| = 1 \) such that \( \| v - v' \| \leq \epsilon_2(\delta) \), and vice versa. Write

\[ \left| \langle \nabla^2 E H_N(x), vv^\top \rangle - \langle \nabla^2 E H_N(x'), vv^\top \rangle \right| \]

\[ \leq \left| \langle \nabla^2 E H_N(x) - \nabla^2 E H_N(x'), vv^\top \rangle \rangle \right| + \left| \langle \nabla^2 E H_N(x), vv^\top - vv^\top \rangle \rangle \right|, \]

(9.11)
where \( \langle A, B \rangle = \text{Tr}(A^T B) = \sum_i \langle Ae_i, Be_i \rangle \), with \( e_i \) being an orthonormal basis, denotes the Hilbert-Schmidt inner product. The first summand in (9.11) is bounded by \( \delta \), by assumption (3). The second summand is bounded by

\[
\| \nabla^2_{\infty} H_N(x) \|_{op} \| vv^T - vv'^T \|_{op} \cdot \text{Rank}(vv^T - vv'^T) \leq 2C \| vv^T - vv'^T \|_{op}.
\]

Since \( \| v - v' \| \leq \epsilon_2(\delta) \), (9.12) is bounded from above by some \( \epsilon_3(\delta) \).

Combining the above we have that \( |(f_N(x'))^{-1} - (f_N(x))^{-1}| \leq \epsilon_4(\delta) \) for some \( \epsilon_4(\delta) \). In particular, for small \( \delta \), \( f_N(x') < c \).

Having completed the proof of Lemmas 9.2 and 9.3, the proofs of Proposition 9.1 Part 2 is complete.

9.2 Proof of Proposition 9.1 Parts 3 and 4

For small \( \epsilon \), if \( \sigma, \sigma' \in \mathbb{S}^{N-1}(\sqrt{N}) \) are points with geodesic distance less than \( \sqrt{N} \epsilon \), then

\[
R(\sigma, \sigma') \geq \sqrt{1 - \epsilon^2} = 1 - \epsilon^2/2 + O(\epsilon^4).
\]

Let \( c_{LS} \) be the constant from the (already proved) Part 2 of Proposition 9.1. Then, assuming \( t \) is small, with high probability, any 1-critical point \( \sigma_1 \) satisfies

\[
d(\sigma_1, \frac{\sqrt{N} \sigma_1 - c_{LS} t}{\| \sigma_1 - c_{LS} t \|}) \leq \sqrt{N} \epsilon^2_{LS} t
\]

and therefore

\[
R(\sigma_1, \sigma_1 - c_{LS} t) \geq 1 - (\epsilon_{LS}^2 t)^2 \geq 1 - c_{LS} t,
\]

that is, \( \sigma_1 \in \text{Cap}(t) \).

Let \( \partial \text{Cap}(t) = \{ \sigma \in \mathbb{S}^{N-1}(\sqrt{N}) : R(\sigma, \sigma_1 - c_{LS} t) = 1 - c_{LS} t \} \) denote the boundary of \( \text{Cap}(t) \). Let \( \sigma_1 \in A_t \cap \text{Cap}(t) \) be some 1-critical point and assume that \( \partial \text{Cap}(t) \cap A_t = \emptyset \), that is

\[
(9.13) \min_{\sigma \in \partial \text{Cap}(t)} H_N(\sigma) = -(E_0 - t) N.
\]

Let \( \sigma_0 \) be some point in the connected component \( K \) of \( A_t \) containing \( \sigma_1 \), and let \( Q : [0, 1] \to \mathbb{S}^{N-1}(\sqrt{N}) \) be a continuous path, wholly contained in the connected component \( K \), that connects \( \sigma_0 = Q(0) \) to \( \sigma_1 = Q(1) \). Note that \( \sigma_0 \) must be contained in \( \text{Cap}(t) \), since otherwise we would have had some \( s' \) such that \( Q(s') \in \partial \text{Cap}(t) \cap K \). In other words, if \( \sigma_1 \in \text{Cap}(t) \) and (9.13), then \( K \subset \text{Cap}(t) \).

Therefore, to prove Part 2 of Proposition 9.1 we need to show that with high probability, for any 1-critical point \( \sigma_1 \in A_t \), the corresponding \( (1 - c_{LS} t) \)-critical point \( \sigma_{1 - c_{LS} t} \) satisfies (9.13).

In the rest of the proof, \( \epsilon > 0 \) will be a constant that can be taken to be as small as we wish, provided \( t \) and \( \delta_{LS} \) are small enough. Also, whenever we fix a 1-critical point \( \sigma_1 \in A_t \), \( \sigma_\rho \) will denote the corresponding \( \rho \)-critical point defined by Part 2 of Proposition 9.1 (we always restrict to the event that those points exist for all
\( \sigma_1 \in A_t \). We shall prove that, with probability tending to 1, for any 1-critical point \( \sigma_1 \in A_t \) and \( \rho \in [1 - \delta_{LS}, 1) 

(a) With \( T_x \) as defined in the proof of Part 2

\[
\min_{v \in T_{\sigma_\rho}; |v| = 1} \left| \langle \nabla^2 E H_N(\sigma_\rho), vv^\top \rangle \right| < 2 \sqrt{\nu}(1) + \epsilon.
\]

(b) For some sequence \( \eta_N = o(1), H_N(\sigma_\rho) > -N \epsilon_0(\rho) - N \eta_N \).

(c) For some constant \( c_2 > 0 \),

\[
|H_N(\sigma_\rho) - (H_N(\sigma_1) + N(1 - \rho) \epsilon_0)| < N \epsilon(1 - \rho) + Nc_2(1 - \rho)^2.
\]

By an argument similar to the proof of Lemma 9.2, namely, replacing the condition \( f_N(\sigma_1) \geq c/2 \) by (9.14) with \( \rho = 1 \) and using the bound

\[
\limsup_{N \to \infty} \max_{x \in D(\epsilon)} \frac{1}{N} \log \mathbb{P}_{x,N} \left\{ \min_{v \in T_x; |v| = 1} \left| \langle \nabla^2 E H_N(\hat{\mathbf{n}}), vv^\top \rangle \right| < 2 \sqrt{\nu}(1) + \frac{\epsilon}{2} \right\} < 0,
\]

which follows from (7.12), we conclude that with probability tending to 1 as \( N \to \infty \), for each of the points \( \sigma_1 \in A_t \), (9.14) holds with \( \rho = 1 \) instead of \( \epsilon \). Combined with Corollary C.2 and the fact that, by Part 2 of Proposition 9.1,

\[
|\sigma_1 - \sigma_\rho|_2^2 \leq (|\sigma_1 - \rho \sigma_1| + \|\rho \sigma_1 - \sigma_\rho\|_2)^2 \leq N(1 - \rho)^2(1 + c_{LS})^2,
\]

this implies point (a) above.

Point (b) follows from Theorem 3.1 and Remark 3.4, by Markov’s inequality.

By Lemma 5.14, for all \( \sigma_1 \in A_t \) with small \( \epsilon \), with probability tending to 1 as \( N \to \infty \),

\[
\frac{d}{dR} H_N(\sigma_1) + \sqrt{N} \epsilon_0(1) \leq \sqrt{N} \epsilon.
\]

Let \( \mathcal{E} = \mathcal{E}_N \) be the intersection of the event that Part 2 of Proposition 9.1 holds (with some \( c_{LS} \)) and the fact that, by Part 2 of Proposition 9.1,

\[
|\sigma_1 - \sigma_\rho|_2^2 \leq (|\sigma_1 - \rho \sigma_1| + \|\rho \sigma_1 - \sigma_\rho\|_2)^2 \leq N(1 - \rho)^2(1 + c_{LS})^2,
\]

this implies point (c) above.

On the event \( \mathcal{E}_N \), for any 1-critical point \( \sigma_1 \in A_t \) and any \( \rho \in [1 - \delta_{LS}, 1] \), by a Taylor approximation,

\[
H_N(\sigma_\rho) - \left[ H_N(\sigma_1) - (1 - \rho) \sqrt{N} \frac{d}{dR} H_N(\sigma_1) \right] \leq \tilde{C}_2 |\sigma_1 - \sigma_\rho|_2^2 \leq N(1 - \rho)^2 \tilde{C}_2(1 + c_{LS})^2,
\]

where we used (9.16) and the fact that \( \langle \nabla E H_N(\sigma_1), v \rangle = 0 \) for any direction \( v \) tangent to \( S^{N-1}(\sqrt{N}) \) at \( \sigma_1 \). Combined with (9.17), this proves point (c) above.
Points (a) and (b) above imply that (9.2) holds for any finite number of values of \( \rho \) simultaneously, with probability tending to 1 as \( N \to \infty \). To complete the proof of Part 3 of Proposition 9.1, the same needs to be proved for a whole range of values of \( \rho \) simultaneously. The latter follows from the bound on the speed of \( \sigma_\rho \) from Part 2 of Proposition 9.1, the fact that by (C.8), \( H_N : \mathbb{R}^N (\sqrt{N}) \to \mathbb{R} \) is a Lipschitz function, and the fact that \(-E_0(\rho)\) is continuous in a neighborhood of 1 by Lemma 5.3.

Finally, we turn to the proof of (9.13). Recall the definition of \( \overline{H}_N^{\sigma_{0,k}}(\sigma) \) from Section 7 and that \( H_N(\sigma) = \sum_{k=0}^{\infty} \overline{H}_N^{\sigma_{0,k}}(\sigma) \). Set \( \overline{\sigma}_\rho = \sigma_\rho / \rho \) and let \( \sigma \in \mathbb{S}^{N-1}(\sqrt{N}) \) be some point such that \( R(\sigma, \overline{\sigma}_\rho) = \rho \). Assuming points (b) and (c) above,

\[
\overline{H}_N^{\sigma_{0,0}}(\sigma) / N = H_N(\sigma_\rho) / N \\
> -E_0 + (1 - \rho)x_0 - \epsilon(1 - \rho) - c_2(1 - \rho)^2 - \eta_N.
\]

Since \( \sigma_\rho \) is \( \rho \)-critical, \( \overline{H}_N^{\sigma_{0,1}}(\sigma) = 0 \). Using point (a) above, by Corollary 7.2 we have

\[
\left| \overline{H}_N^{\sigma_{0,2}}(\sigma) \right| \leq N(1 - \rho^2)(\sqrt{\nu''(1)} + \epsilon / 2).
\]

From Lemma 7.3 with probability tending to 1, for some constants \( c, C > 0 \), \( \rho \) close to 1, and all points \( \overline{\sigma}_\rho \),

\[
\sum_{k=3}^{\infty} \overline{H}_N^{\sigma_{0,k}}(\sigma) \leq N C \left( \frac{1 - \rho^2}{c} \right)^{3/2}.
\]

Combining the above, for sufficiently small \( \delta \), and for any \( \rho \in [1 - \delta_{LS}, 1) \) and large enough \( N \),

\[
H_N(\sigma) = \sum_{k=0}^{\infty} \overline{H}_N^{\sigma_{0,k}}(\sigma) > -NE_0 + N(1 - \rho)(x_0 - 2 \sqrt{\nu''(1)}) - \epsilon',
\]

where \( \epsilon' > 0 \) is a constant that can be taken to be as small as we wish assuming \( \delta_{LS} \) and \( \tau \) are small enough. Finally, since by Lemma 5.1, \( x_0 > 2 \sqrt{\nu''(1)} \) for some \( \tau > 0 \),

\[
H_N(\sigma) > -E_0 N + (1 - \rho) \tau N.
\]

For \( \rho = 1 - c_{LS} \tau \), assuming that \( c_{LS} > 1 / \tau \), we have that \( H_N(\sigma) > -E_0 N + \tau N \). This yields (9.13) and completes the proof.

9.3 Proof of Proposition 9.1 Part 1

The fact that a.s. each component \( A \) contains a local minimum point \( \sigma_1 \) of \( H_N(\sigma) \) on \( \mathbb{S}^{N-1}(\sqrt{N}) \) is a direct consequence of the fact that

\[
H_N : \mathbb{S}^{N-1}(\sqrt{N}) \to \mathbb{R}
\]
is a Morse function a.s., as can be verified using [1] theorem 11.3.1. To prove that \( \sigma_1 \) is the only \( l \)-critical point in \( A \), assuming Parts 2 and 4 and that \( \delta_{LS}/c_{LS} \) is
small enough, it is sufficient to show that for small fixed \( c > 0 \), if \( t \) is small enough, then there are no two critical points \( \sigma, \sigma' \) with \( H_N(\sigma) \leq -(E_0 - t)N \) such that \( R(\sigma, \sigma') > 1 - c \), with probability tending to 1 as \( N \to \infty \). This is a consequence of Corollary 3.7 (which, in fact, states that the critical points are either antipodal or close to orthogonal for small \( t \)). □

10 The Lower Bound on the Free Energy

This section is devoted to proving the following lower bound on the free energy.

Recall the variables \( Z_{N,\beta} \), \( \Lambda_{Z,\beta} \), and \( Z_{N,\beta}(\sigma_0) \) (see (1.3), (8.3), and (2.4)), and the variables \( \rho_* \) and \( E_0(\rho_*) \) (see (8.6) and (3.15)).

**Proposition 10.1.** Assuming Condition M, for large enough \( \beta \) and any \( \epsilon > 0 \), there exist constants \( c, C > 0 \) (depending on \( \epsilon \)) such that

\[
P\left\{ \frac{1}{N} \log Z_{N,\beta} \leq \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \epsilon \right\} \leq C e^{-Nc}.
\]

The rest of the section is devoted to the proof of Proposition 10.1. Consider the following three statements:

1. For some \( \epsilon_N = o(1) \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}\left\{ \frac{1}{N} \log Z_{N,\beta} > \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \epsilon_N \right\} = 0.
\]

2. Let \( \delta_N = o(1) \) be arbitrary and set \( B_N = -E_0(\rho_*) + (-\delta_N, \delta_N) \) and \( D_N = -x_0(\rho_*) + (-\delta_N, \delta_N) \). Then, for some \( \epsilon_N = o(1) \), as \( N \to \infty \),

\[
\mathbb{P}\{ LC_{\text{rt}}_{N,\beta} = o(1) \} \cdot \mathbb{P}\{ \text{Crt}_{N,\rho_*}(B_N, D_N) \}.
\]

where, with \( Z_{N,\beta}(\sigma_0) \) defined by (2.4),

\[
LC_{\text{rt}}_{N,\beta} = \# \{ \sigma_0 \in \mathcal{C}_{N,\rho_*}(NB_N, \sqrt{N}D_N) : \frac{1}{N} \log Z_{N,\beta}(\sigma_0) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \epsilon_N \}
\]

denotes the number of “light” points.

3. Fix a sequence \( \delta_N = o(1) \) in the definition of \( B_N, D_N \). Then, if \( \epsilon_N = \epsilon_N(\delta_N) = o(1) \) (decays slowly enough, then as \( N \to \infty \), uniformly in \( (u, x) \in NB_N \times \sqrt{N}D_N \),

\[
\mathbb{P}_{u,x}\left\{ \frac{1}{N} \log Z_{N,\beta}(\rho_* \hat{n}) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \epsilon_N \right\} = o(1).
\]

We will prove (3) and the implications (3) \( \implies \) (2) \( \implies \) (1) \( \implies \) Proposition 10.1.
(1) \implies \textbf{Proposition 10.1}

By Corollary C.4

\begin{equation}
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}\left\{ \frac{1}{N} \log Z_{N, \beta} - \frac{1}{N} \mathbb{E} \log Z_{N, \beta} \geq N t \right\} \\
\leq -t^2/2\beta^2 v(1).
\end{equation}

Therefore, (10.2) implies that

\begin{equation}
\liminf_{N \to \infty} \frac{1}{N} \log Z_{N, \beta} \geq \Lambda_{Z, \beta}(-E_0(\rho_*), \rho_*).
\end{equation}

Another application of Corollary C.4, together with (10.6), yields (10.1).

(2) \implies (1)

Let

\begin{equation}
\text{HCrt}_{N, \beta} = \#\left\{ \sigma_0 \in \mathcal{C}_{N, \rho_*}(NB_N, \sqrt{N} D_N) : \frac{1}{N} \log Z_{N, \beta}(\sigma_0) \geq \Lambda_{Z, \beta}(-E_0(\rho_*), \rho_*) - \epsilon_N \right\}
\end{equation}

denote the number of “heavy” critical points. We show below that one can choose \( \delta_N, \epsilon_N \to_N \infty 0 \) so that

\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}\{\text{HCrt}_{N, \beta} \geq 1\} = 0.
\end{equation}

By (C.8) of Corollary C.2, there exist constants \( c, C > 0 \) and an event \( A_N \) with \( \mathbb{P}(A_N) \leq e^{-c N} \) such that \( \sup_{\sigma \in S_N - 1(\sqrt{N})} \| \nabla H_N(\sigma) \|_\infty \leq C \) on \( A_N \), and therefore, on that event, if \( \text{HCrt}_{N, \beta} \geq 1 \) then for some \( t_N \to_N \infty 0 \) with \( \sigma_0 \) being an arbitrary point as in (10.7),

\[
\frac{1}{N} \log Z_{N, \beta}(\sigma_0) \geq \Lambda_{Z, \beta}(-E_0(\rho_*), \rho_*) - \epsilon_N - C t_N.
\]

Using (10.8), this gives (1) with \( \epsilon_N \) in the latter replaced by \( \epsilon_N + C t_N \).

It remains to prove (10.8). Choose \( \delta_N \) so that \( \mathbb{E} \text{Crt}_{N, \rho_*}(B_N, D_N) \to_N \infty \infty \), which is possible by Theorem 3.1. By (10.3), this implies that \( \mathbb{E} \text{HCrt}_{N, \beta} \to_N \infty \infty \infty \). By the Cauchy-Schwarz inequality,

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}\{\text{HCrt}_{N, \beta} \geq 1\} \geq \liminf_{N \to \infty} \frac{1}{N} \log \frac{\mathbb{E}(\text{HCrt}_{N, \beta})^2}{\mathbb{E}[(\text{HCrt}_{N, \beta})^2]}.
\]

The conclusion (10.8) now follows from Corollary 3.6, since obviously

\[
\mathbb{E}[(\text{HCrt}_{N, \beta})^2] \leq \mathbb{E}[(\text{Crt}_{N, \rho_*}(B_N, D_N))^2].
\]
In the proof of Theorem 3.1, we expressed $E_{C_{rtN},p}$ (with $\rho = \rho_*, B = B_N, D = D_N$) by (4.2) (with $d = d_N$, $\rho = \rho_*$, $B = B_N$, $D = D_N$). The same argument gives a similar expression for $E_{LC_{rtN}}$, but with the expectation in (4.2) replaced by

$$E_{\rho_*,N}^{p,\rho_*} \left[ \log Z_{N,\beta}(\rho_*) \right]$$

where $t(x) = x/\rho_* \sqrt{(N-1)\omega^2(\rho_*)}$ and $G$ is a GOE matrix given by

$$G = \sqrt{\frac{N}{(N-1)\omega^2(\rho_*)}} \left( \nabla_N^2 H_N(\rho_*) + \frac{1}{\sqrt{N}\rho_*} \frac{d}{dR} H_N(\rho_*) \right).$$

Thus, (2) follows if we show that the expectation of (10.9) is $o(1)$ times the same expectation without the indicator, uniformly over the domain of integration in (4.2), i.e., over $(u, x) \in \sqrt{N} B_N \times \sqrt{N} D_N$. By Lemma 5.1, $t(x) > 2 + \delta$ for some $\delta > 0$, uniformly in $x \in \sqrt{N} D$ (for large $\beta$, so that $x_0(\rho_*)$ is close to $x_0(1)$). Therefore, by [52, cors. 22, 23], as $N \to \infty$,

$$\mathbb{E} |\det G - t(x) I|^2 \leq C (\mathbb{E} |\det G - t(x) I|)^2,$$

for some $C > 0$, uniformly in $x \in \sqrt{N} D$. Hence, by the Cauchy-Schwarz inequality, (3) $\implies$ (2).

**Proof of (3)**

We need the following two lemmas. Recall the notation $Z_{2-N,\beta}(\rho)$; see (8.12).

**Lemma 10.2.** For large enough $\beta$ and $\epsilon_N = o(1)$ decaying slowly enough, uniformly in $(u, x) \in NB_N \times \sqrt{N} D_N$,

$$\mathbb{P}_{\rho_*,N}^{\rho_*} \left[ \frac{1}{N} \log Z_{2-N,\beta}^{2-N}(\rho_*) \right] = o(1).$$

**Lemma 10.3.** For large enough $\beta$ and $\epsilon_N = o(1)$ decaying slowly enough, uniformly in $(u, x) \in NB_N \times \sqrt{N} D_N$,

$$\mathbb{P}_{\rho_*,N}^{\rho_*} \left[ \frac{1}{N} \log Z_{N,\beta}(\rho_*) - \frac{1}{N} \log Z_{N,\beta}^{2-N}(\rho_*) \right] = o(1).$$
PROOF OF LEMMA 10.2 With $H_{N-1,2}(\sigma)$ denoting the pure 2-spin mode as in (8.12), set

$$
\tilde{Z}_{N,\beta} := \int_{S^{N-2}(\sqrt{N}/N-1)} \exp \left\{ -\beta \alpha_2(\rho_*) \sqrt{\frac{N}{N-1}} H_{N-1,2}(\sigma) \right\} d\sigma,
$$

(10.12)

$$
\tilde{F}_{N,\beta} := \frac{1}{N} \log \tilde{Z}_{N,\beta},
$$

where the integration is with respect to the uniform Hausdorff measure on the sphere $S^{N-2}(\sqrt{N}/N-1)$. We will show that

$$
\mathbb{P} \left\{ \tilde{F}_{N,\beta} < \frac{1}{2} \beta^2 \alpha_2^2(\rho_*) + \beta \delta_N - \frac{\epsilon_N}{2} \right\} = o(1).
$$

(10.13)

By Lemma 7.1 the probability in (10.10) is independent of $x$ and depends on $u$ through the uniform “shift” in the conditional law, and (10.13) implies (10.10) since the limit of $\frac{1}{N} \log$ of the ratio of volumes from the definition (8.12) equals $\frac{1}{2} \log(1 - \rho^2_*)$.

Let $\tilde{F}_{N,\beta}$ be defined similarly to $\tilde{F}_{N,\beta}$, only without the $\sqrt{N/(N-1)}$ term in (10.12), and note that it is enough to prove (10.13) with $\tilde{F}_{N,\beta}$ instead of $\tilde{F}_{N,\beta}$ (by increasing $\epsilon_N$ if needed). Baik and Lee [8, theorem 1.2] proved that the free energy of the pure 2-spin converges in distribution as $N \to \infty$. In particular, their result shows that $N(\tilde{F}_{N,\beta} - \frac{1}{2} \beta^2 \alpha_2^2(\rho_*))$ converges to a Gaussian variable. This implies (10.13) provided that $N(\epsilon_N - 2\beta \delta_N) \to \infty$.

□

PROOF OF LEMMA 10.3. The proof is built upon the argument used in [53, sec. 6.4]. First, note that by Lemma 7.1 the probability in (10.11) does not depend on $u$ and $x$. In fact, the difference of free energies in (10.11) is equal in distribution to

$$
\frac{1}{N} \log \mathbb{E} X_N = \frac{1}{2N} \beta^2 \text{Var} \left( \sum_{k=3}^{\infty} H_{N-1,k}(\rho_*) \right) = \frac{1}{2} \beta^2 \sum_{k=3}^{\infty} \alpha_k^2(\rho_*).
$$

We will define a sequence of events $\mathcal{E}_N$ measurable w.r.t. $(H_{N-1,2}(\sigma))_\sigma$ such that $\lim_{N \to \infty} \mathbb{P} \{ \mathcal{E}_N \} \to 1$ and

$$
\lim_{N \to \infty} \frac{\mathbb{E} \{ X_N^2 \mid \mathcal{E}_N \}}{(\mathbb{E} X_N^2)} \leq 1.
$$

(10.14)
Since \( \mathbb{E}\{X_N \mid (H_N - 1, 2(\sigma))_\sigma\} = \mathbb{E}X_N \), also \( \mathbb{E}\{X_N \mid \mathcal{E}_N\} = \mathbb{E}X_N \). Thus, from Chebyshev’s inequality (10.14) will imply (10.11), even with \( \epsilon_N \) of any order larger than \( 1/N \).

Denote

\[ T_N(I) := \{(\sigma_1, \sigma_2) \in (\mathbb{S}^{N-1} (\sqrt{N}))^2 : R(\sigma_1, \sigma_2) \in I\} \]

Using the co-area formula with the mapping

\[ (\sigma_1, \sigma_2) \mapsto R(\sigma_1, \sigma_2) \quad \forall \sigma \in \mathbb{S}^{N-1} (\sqrt{N}) \]

we have that, for measurable \( I \subset [-1, 1] \),

\[
\mathbb{E}W(\bar{H}_N, I) := \mathbb{E} \int_{T_N(I)} \exp\{-\beta \bar{H}_N(\sigma_1) - \beta \bar{H}_N(\sigma_2)\} d\sigma_1 d\sigma_2
\]

(10.15)

\[
= \int_1^{\omega_N} \frac{1}{\omega_N} (1 - \xi^2)^{N-3} \exp\{\beta^2 \theta_N(\xi)\} d\xi,
\]

where \( \bar{H}_N(\sigma) \) is a general mixed model and, with \( \sigma \) and \( \sigma_{\xi} \) being two points with \( R(\sigma, \sigma_{\xi}) = \xi \),

\[ \theta_N(\xi) := \text{Var}(\bar{H}_N(\sigma)) + \text{Cov}(\bar{H}_N(\sigma), \bar{H}_N(\sigma_{\xi})). \]

Thus, whenever \( \theta_N(\xi)/N \rightarrow \theta(\xi) \) uniformly in \( \xi \in [-1, 1] \),

\[
\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}W(\bar{H}_N, I) = \sup_{\xi \in I} w(\bar{H}_N, \xi) := \sup_{\xi \in I} \frac{1}{2} \log(1 - \xi^2) + \beta^2 \theta(\xi).
\]

Now consider \( \bar{H}_{N-1}^{(1)}(\sigma) = \alpha_2(\rho_\ast) \sqrt{N/(N-1)} \bar{H}_{N-1, 2}(\sigma) \). The derivative

\[
\frac{d}{d\xi} w(\bar{H}_{N-1}^{(1)}, \xi) = -\frac{\xi}{1 - \xi^2} + 2\beta^2 \alpha_2^2(\rho_\ast) \xi
\]

is negative for any \( \xi \in (0, 1] \) and positive for any \( \xi \in [-1, 0) \), if \( 2\beta^2 \alpha_2^2(\rho_\ast) < 1 \).

From (8.6),

\[
\lim_{\beta \rightarrow \infty} 2\beta^2 \alpha_2^2(\rho_\ast) - 1 = 4\tau_2^2 v'(1) - 1 < 0,
\]

where the inequality follows since \( 4\tau_2^2 v'(1) - 1 \) has the same sign as the normalized limiting second derivative of (8.5) at the local maximum \( \rho_\ast \). We conclude that for large enough \( \beta \) and any \( \tau > 0 \),

\[
\frac{1}{N} \log \mathbb{E}W(\bar{H}_{N-1}^{(1)}, (-\tau, \tau)) = \beta^2 \alpha_2^2(\rho_\ast) \geq \frac{1}{N} \log \mathbb{E}W(\bar{H}_{N-1}^{(1)}, [-1, 1] \setminus (-\tau, \tau)).
\]

In particular, for some \( \tau_N = o(1) \),

\[
\lim_{N \rightarrow \infty} \frac{\mathbb{E}W(\bar{H}_{N-1}^{(1)}, [-1, 1] \setminus (-\tau_N, \tau_N))}{\exp\{N\beta^2 \alpha_2^2(\rho_\ast)\}} = 0.
\]
Setting \( I_N = \{-\tau_N, \tau_N\} \setminus (-N^{-\alpha}, N^{-\alpha}) \) with some \( a \in (\frac{1}{3}, \frac{1}{2}) \), using the fact \( \sqrt{N}\omega_N/\omega_{N-1} \rightarrow \sqrt{2\pi} \) and the change of variables \( \sqrt{N} \xi \mapsto \xi' \), we obtain

\[
\lim_{N \to \infty} \frac{\mathbb{E} W(\bar{H}_{N-1}^{(1)}, I_N)}{N} = \lim_{N \to \infty} \int_{\sqrt{N}I_N} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2+\beta^2\alpha_2^2(\rho_\star)\xi^2+o(\xi^2)} d\xi.
\]

For \( \rho_\star > \rho_c \), \( \beta^2\alpha_2^2(\rho_\star) < \frac{1}{2} \) and therefore the limit of \( \text{(10.16)} \) is equal to 0. By Markov’s inequality, we conclude that with probability tending to 1 as \( N \to \infty \),

\[
W(\bar{H}_{N-1}^{(1)}, [-1, 1] \setminus (-N^{-\alpha}, N^{-\alpha})) < \eta_N \exp\{N\beta^2\alpha_2^2(\rho_\star)\},
\]

for some \( \eta_N = o(1) \).

We are now ready to define the events \( \mathcal{E}_N \). As in the proof of Lemma 10.2 [8, theorem 1.2] implies that with probability tending to 1 as \( N \to \infty \),

\[
Z_{N,\beta}^2 = W(\bar{H}_{N-1}^{(1)}, [-1, 1]) > \eta_N^{1/2} \exp\{N\beta^2\alpha_2^2(\rho_\star)\}.
\]

Define \( \mathcal{E}_N \) as the intersection of this event and \( \text{(10.17)} \), so that on \( \mathcal{E}_N \) we also have that

\[
W(\bar{H}_{N-1}^{(1)}, (-N^{-\alpha}, N^{-\alpha})) > (\eta_N^{1/2} - \eta_N) \exp\{N\beta^2\alpha_2^2(\rho_\star)\}.
\]

Next consider \( \bar{H}_{N-1}^{(2)}(\sigma) = \sum_{k=2}^\infty \alpha_k(\rho_\star) \sqrt{N(N-1)} H_{N-1,k}(\sigma) \). In this case, as \( \beta \to \infty \),

\[
\beta^2 \vartheta_{N-1}(\xi) = \beta^2 \sum_{k=2}^\infty \alpha_k^2(\rho_\star) N(1 + \xi^k)
= \beta^2 \sum_{k=2}^\infty \alpha_k^2(\rho_\star) N + \beta^2\alpha_2^2(\rho_\star) N \xi^2 + \xi^3 O\left(\frac{1}{\beta}\right).
\]

Using this and a similar argument to the one used above for the Hamiltonian corresponding to \( k = 2 \) only, we obtain that for the current Hamiltonian

\[
\mathbb{E} W(\bar{H}_{N-1}^{(2)}, [-1, 1] \setminus (-N^{-\alpha}, N^{-\alpha})) < \eta_N \exp\left\{ N\beta^2 \sum_{k=2}^\infty \alpha_k^2(\rho_\star) \right\},
\]

where we may need to increase \( \eta_N = o(1) \). Therefore, for large \( N \),

\[
\mathbb{E} \left[ \frac{W(\bar{H}_{N-1}^{(2)}, [-1, 1] \setminus (-N^{-\alpha}, N^{-\alpha}))}{Z_{N,\beta}^2(\mathbb{E} X_N)^2} \mid \mathcal{E}_N \right] < \eta_N^{1/2} (1 + o(1)).
\]
We conclude that

$$
\mathbb{E} \{ \chi_N^2 \mid \mathcal{E}_N \} = \mathbb{E} \left[ \frac{W(\widetilde{H}_{N-1}^{(2)}, [-1, 1])}{W(\widetilde{H}_{N-1}^{(1)}, [-1, 1])} \bigg| \mathcal{E}_N \right] 
$$

$$
= \mathbb{E} \left[ \frac{W(\widetilde{H}_{N-1}^{(2)}, (-N^{-\alpha}, N^{-\alpha}))}{W(\widetilde{H}_{N-1}^{(1)}, (-N^{-\alpha}, N^{-\alpha}))} \bigg| \mathcal{E}_N \right] (1 + O(1)).
$$

By conditioning on $\widetilde{H}_{N-1}^{(1)}(\sigma)$, similar to the above

$$
\lim_{N \to \infty} \mathbb{E} \frac{W(\widetilde{H}_{N-1}^{(2)}, (-N^{-\alpha}, N^{-\alpha}))}{W(\widetilde{H}_{N-1}^{(1)}, (-N^{-\alpha}, N^{-\alpha}))}(\mathbb{E}X_N)^2
$$

$$
= \lim_{N \to \infty} \int_{N^{1/2-\alpha}}^{N^{1/2-\alpha}} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2+\beta^2 \sum_{k=1}^{\infty} \sigma_k(\rho_*) \xi^k N^{-k/2+1+o(\xi^2)} d\xi} = 1.
$$

This proves (10.14) and completes the proof.

PROOF OF POINT (3). Using the formula (8.3) for $\Lambda_{\mathcal{Z}, \beta}(E, \rho)$, point (3) follows from Lemmas 10.2 and 10.3.

11 Upper Bounds on the Free Energy

We prove in this section an upper bound that is complementary to the lower bound of Proposition 10.1. Both bounds will play a crucial role in the proof of the first part of Theorem 1.2. For any measurable $D \subset S^{N-1}(\sqrt{N})$, we define $Z_{N, \beta}(D) = \int_D e^{-\beta H_N(\sigma)} d\sigma$.

PROPOSITION 11.1. Assume Condition M. For large enough $\beta$ and any $\epsilon > 0$, for small enough $\delta > 0$,

$$
\frac{1}{N} \log Z_{N, \beta} \left( S^{N-1}(\sqrt{N}) \setminus \bigcup_{\sigma_0 \in \mathcal{E}_{N, \rho_+}(NB)} \text{Band}(\sigma_0, \epsilon) \right)
$$

$$
< \Lambda_{\mathcal{Z}, \beta} (-E_0(\rho_*) - \epsilon, -E_0(\rho_*) + \epsilon) - \delta
$$

with probability tending to 1 as $N \to \infty$, where $B = (-E_0(\rho_*) - \epsilon, -E_0(\rho_*) + \epsilon)$.

The rest of the section is devoted to the proof of Proposition 11.1. In Section 11.1 using the results on the structure of sub-level sets from Section 9, we show how Proposition 11.1 can be deduced from bounds on weights of sections $Z_{N, \beta}(\sigma_\rho)$, defined in (2.4), centered at $\rho$-critical points of a given depth. In Section 11.2 we prove two general upper bounds for the latter, which are relevant for different ranges of $\rho$. Finally, in Section 11.3 we use those to conclude Proposition 11.1.
11.1 A Reduction to Bounds on Weights $Z_{N,\beta}(\sigma_\rho)$ of $\rho$-Critical Points

We begin with the observation that, since the integration is w.r.t. the probability Haar measure on the sphere,

$$
\frac{1}{N} \log \int_{S^{N-1}(\sqrt{N}) \setminus A_\tau} e^{-\beta H_N(\sigma)} d\sigma \leq \beta (E_0 - \tau),
$$

where $A_\tau$ is the sub-level of $-(E_0 - \tau) N$ (see (9.1)). The following is a direct consequence.

**Corollary 11.2.** It is enough to prove Proposition 11.1 with $S^{N-1}(\sqrt{N})$ in (11.1) replaced by the (random) subset $A_\tau$ from (9.1), with

$$
\tau := \tau(\beta, \delta) = E_0(v) - \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*)/\beta + \delta.
$$

>From the asymptotics of $\rho_* $ in (8.6), the definition of $\Lambda_{Z,\beta}$, see (8.3), and the continuity of $E_0(\rho)$ proved in Lemma 6.1, $\tau \to 0$ as $\beta \to \infty$ and $\delta \to 0$. Hence, with high probability, the sub-level set $A_\tau$ is covered by the caps in (9.3) from Proposition 9.1. >From this it is not difficult to move to a cover by sections of the form $S(\sigma)$; see (2.2).

**Lemma 11.3.** Assume that $v$ satisfies Condition M. Then, for small enough $t$, with the notation of Proposition 9.1 for each connected component $A$ of $A_\tau$,

$$
A \subset \bigcup_{\rho \in [1-c_{LS}t, 1]} S(\sigma_\rho),
$$

with probability tending to 1 as $N \to \infty$.

**Proof.** Let $\sigma \in A$ and define the continuous function $d_\sigma(\rho) = R(\sigma, \sigma_\rho) - \rho$, for $\rho \in [1 - c_{LS}t, 1]$. If $\sigma \neq \sigma_1$ and $\sigma \notin S(\sigma_{1-c_{LS}t})$, then $d_\sigma(\rho) < 0$ and $d_\sigma(1-c_{LS}t) > 0$. By the mean value theorem, there exists some $\rho \in [1-c_{LS}t, 1]$ such that $d_\sigma(\rho) = 0$, which exactly means that $\sigma \in S(\sigma_\rho)$.

Finally, we translate the bound we need to bounds treating each pair $(\rho, E)$ separately, in an appropriate sense. Denote

$$
W_{\tau, \eta} := \{(\rho, E) : \rho \in [1 - c_{LS} \tau, 1],
E \in [-E_0(\rho) - \eta, -E_0(1) + 2\gamma_0(1)c_{LS} \tau]\}
$$

and

$$
B_{\tau}(\rho, E, \epsilon) := (\rho - \epsilon, \rho + \epsilon) \times (E - \epsilon, E + \epsilon).
$$

**Lemma 11.4.** Assume that $v$ satisfies Condition M. Assume that for large enough $\beta$ and any small $\epsilon > 0$, there exist $\eta, \delta$, and $v$ (depending on $\epsilon, v, \rho$, and $\beta$), so that with $\tau$ given by (11.2), for any

$$
(\rho, E) \in W_{\tau, \eta} \setminus B_{\tau}(\rho_*, -E_0(\rho_*), \epsilon),
$$
we have that, with probability tending to 1 as \( N \to \infty \),

\[
(11.6) \quad \frac{1}{N} \log \sum_{\sigma \in \mathcal{C}_{N,\rho}(\{E-v, E+v\})} Z_{N,\beta}(\sigma) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta.
\]

Then, Proposition [11.1] holds true.

PROOF. Throughout the proof we implicitly restrict to the event that all the statements of Proposition [9.1], Lemma [11.3], and Corollary [C.2] hold. (The probability of this event converges to 1, as \( N \to \infty \).) All the statements below should be interpreted as “occurring with probability tending to 1,” and we will refrain from repeatedly writing so. Note that since we assume that \( \beta \) is large, \( \tau \) given by (11.2) can be made arbitrarily small.

By an abuse of notation, denote by \( \mathcal{C}_{N,\rho}(NE) \) the set of \( \rho \)-critical points \( \sigma_0 \) such that \( H_N(\sigma_0) = NE \). From Proposition [9.1], for any \( \rho \in [1-c_{LS}, 1] \) and connected component \( A \) of \( A_\tau \), there is a corresponding \( \rho \)-critical point \( \sigma_\rho \). Denote the subset of those points for which \( H_N(\sigma_\rho) = NE \) by \( \mathcal{C}_{N,\rho}^\tau(NE) \). For any \( W \subset \mathbb{R}^2 \) denote, by an abuse of notation,

\[
S(W) := \bigcup_{(\rho, E) \in W} \bigcup_{\sigma \in \mathcal{C}_{N,\rho}(NE)} S(\sigma),
\]

\[
\text{Band}(W, \epsilon) := \bigcup_{(\rho, E) \in W} \bigcup_{\sigma \in \mathcal{C}_{N,\rho}(NE)} \text{Band}(\sigma, \epsilon),
\]

and define \( S_\tau(W) \) and \( \text{Band}_\tau(W, \epsilon) \) similarly, with \( \mathcal{C}_{N,\rho}(NE) \) being replaced by \( \mathcal{C}_{N,\rho}^\tau(NE) \).

Let \( A \) be some connected component of \( A_\tau \), and let \( \sigma_\rho \) be the corresponding path of \( \rho \)-critical points. From Part [3] of Proposition [9.1], since \( \tau \) is small, for arbitrary \( \eta \) and large enough \( N \),

\[
\frac{1}{N} H_N(\sigma_\rho) \in [-E_0(\rho) - \eta, -E_0(1) + 2\chi_0(1)c_{LS} \tau]
\]

for all \( \rho \in [1-c_{LS}, 1] \). Combining this with Lemma [11.3] we obtain that

\[
A_\tau \subset S_\tau(W_\tau, \eta).
\]

Next, we construct a cover using bands corresponding to a finite number of values of \( \rho \). For two points \( \sigma, \sigma' \in \mathbb{B}^N(\sqrt{N}) \), if \( (\|\sigma\| - \|\sigma'\|)/\sqrt{N} \) and the distance between \( \sigma'/\|\sigma'\| \) and \( \sigma/\|\sigma\| \) (w.r.t. the standard metric on the sphere) are both in \((-v/2, v/2)\), then

\[
S(\sigma) \subset \text{Band}(\frac{\|\sigma'\|}{\|\sigma\|}, \sigma, \frac{v}{2}) \subset \text{Band}(\sigma', v).
\]

With \( A \) and \( \sigma_\rho \) as above, fix some \( \rho \in [1-c_{LS}, 1] \). From the Lipschitz bound of (C.8) and Point [2] of Proposition [9.1] we have the following. For any given
$v > 0$, for small enough $v/2 > v'> 0$ (independent of $\rho$ and $E$), if $|\rho - \rho'| < v'$ and $\left| \frac{1}{N} H_N(\sigma_{\rho'}) - E \right| < v'$, then, with notation as in Proposition 9.1,
\[ |G(\rho') - G(\rho)| < \frac{v}{2} \quad \text{and} \quad \left| \frac{1}{N} H_N(\sigma_{\rho}) - E \right| < v, \]
and thus
\[ S(\sigma_{\rho'}) \subset \text{Band}(\sigma_{\rho}, v). \]

Denoting
\[ B_{\epsilon}(\rho, E, v) = \{\rho\} \times (E - v, E + v), \]
we therefore have that
\[
(11.7) \quad S_\epsilon \left( B_{\epsilon}(\rho, E, v') \right) \subset \text{Band}_\epsilon \left( B_{\epsilon}(\rho, E, v), v \right).
\]

Applying the above with $(\rho, E) = (\rho_*, -E_0(\rho_*))$ and $v = \epsilon$, we have that for small enough $\epsilon' > 0$,
\[
\bigcup_{\sigma_\epsilon \in \mathcal{E} \cap \sigma_{\epsilon_0}(N\mathbb{B})} \text{Band}(\sigma_\epsilon, \epsilon) = \text{Band}(B_{\epsilon}(\rho_*, -E_0(\rho_*), \epsilon), \epsilon) \supset S_\epsilon \left( B_{\epsilon}(\rho_*, -E_0(\rho_*), \epsilon') \right).
\]
From Corollary [11.2] and Lemma [11.3], we conclude that in order to prove Proposition [11.2], it is enough to show that for any arbitrarily small $\epsilon$, there exists some $\delta$ such that
\[
\frac{1}{N} \log Z_{N,\beta} \left( S_\epsilon \left( W_{\epsilon, \eta} \setminus B_{\epsilon}(\rho_*, -E_0(\rho_*), \epsilon) \right) \right) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta
\]
with probability tending to 1 as $N \to \infty$.

Now, fix $\epsilon > 0$ and let $v$ and $\eta$ (see [11.4]) be some small numbers. Let $v'$ be the value corresponding to $v$ by the relation above. Choose some cover for the region $W_{\epsilon, \eta} \setminus B_{\epsilon}(\rho_*, -E_0(\rho_*), \epsilon)$ by a finite number of boxes $B_{\epsilon}(\rho, E, v')$, with each of the centers $(\rho, E)$ belonging to $W_{\epsilon, \eta} \setminus B_{\epsilon}(\rho_*, -E_0(\rho_*), \epsilon)$.

From [11.7],
\[
\frac{1}{N} \log Z_{N,\beta} \left( S_\epsilon \left( B_{\epsilon}(\rho, E, v') \right) \right) \leq \frac{1}{N} \log Z_{N,\beta} \left( \text{Band}_\epsilon \left( B_{\epsilon}(\rho, E, v), v \right) \right).
\]
Hence, since we are dealing with a finite number of boxes, for (11.8) to hold, it is enough to establish that for each of the boxes $B_{\epsilon}(\rho, E, v')$,
\[
\frac{1}{N} \log Z_{N,\beta} \left( \text{Band}_\epsilon \left( B_{\epsilon}(\rho, E, v), v \right) \right) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta
\]
with probability going to 1. From the Lipschitz bound of [C.8], it is thus enough to show with such probability that
\[
\frac{1}{N} \log Z_{N,\beta} \left( S_\epsilon \left( B_{\epsilon}(\rho, E, v) \right) \right) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta,
\]
where we may need to decrease $\delta$. This completes the proof of Lemma [11.4] \(\square\)
11.2 General Bounds on Weights $Z_{N,\beta}(\sigma_0)$ at a Given Depth

This section is devoted to the proof of the following three lemmas, bounding from above the contribution to the free energy coming from $\rho$-critical points. We recall that $Z_{N,\beta}(\sigma_0)$ and $Z_{N,\beta}^2(\sigma_0)$ below are as defined in (2.4) and (8.12).

**Lemma 11.5.** For any $\delta > 0$ there exists a constant $c = c(\nu) > 0$ such that for any $\rho \in (\delta, 1)$, $E \in (-2E_0(\rho), 0)$, and $\epsilon > 0$, setting $B = (E - \epsilon, E + \epsilon)$,

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \sum_{\sigma_0 \in \mathcal{G}_{N,\rho}(NB)} Z_{N,\beta}(\sigma_0)
\leq \sup_{x \in \mathbb{R}} \Theta_{\nu,\rho}(E, x) + \Lambda_{\beta}(E, \rho) + (\beta + c)\epsilon,
\]

where $\Theta_{\nu,\rho}(E, x)$ and $\Lambda_{\beta}(E, \rho)$ are given by (3.4) and (8.3).

**Lemma 11.6.** For any $\delta > 0$ there exists a constant $c = c(\nu) > 0$ such that for any $\rho \in (\delta, 1)$ with $\beta \alpha_2(\rho) \geq 1/\sqrt{2}$, $E \in (-2E_0(\rho), 0)$, and $\epsilon > 0$, setting $B = (E - \epsilon, E + \epsilon)$,

\[
\lim_{N \to \infty} \mathbb{P} \left\{ \frac{1}{N} \log \mathbb{E} \sum_{\sigma_0 \in \mathcal{G}_{N,\rho}(NB)} Z_{N,\beta}^2(\sigma_0) \right\}
\geq \theta + K_{\beta}(E, \rho, \epsilon) + \Lambda_{\beta/\rho}(E, \rho) + \beta\epsilon = 0,
\]

where $\theta = \sup_{x \in \mathbb{R}} \Theta_{\nu,\rho}(E, x)$, $K_{\beta}(E, \rho, \epsilon) = \beta \alpha_2(\rho) \sqrt{2\theta + c\epsilon}$, and $\Lambda_{\beta/\rho}(E, \rho)$ is defined by (8.13).

**Lemma 11.7.** There exist constants $C, c > 0$ such that for any $\rho \in (0, 1)$, and with $T_{\rho} = C(1 - \rho^{2})^{3/2}$,

\[
\mathbb{P} \left\{ \exists \sigma_0 \in \mathcal{S}_{N,\rho}(\sqrt{N}) : \frac{1}{N} \log Z_{N,\beta}(\sigma_0) - \log Z_{N,\beta}^2(\sigma_0) > T_{\rho} \right\} \leq e^{-cN}.
\]

**Proof of Lemma 11.5.** From Corollary 8.1, for any $\rho > 0$ and $v \in \mathbb{R}$,

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\gamma,\rho} \{ Z_{N,\beta}(\rho \mathbf{n}) \} = \Lambda_{\beta}(E, \rho).
\]

By Lemma 7.1 replacing $E$ by $E + \delta$ in the conditional expectation above amounts to shifting the conditional law of the random field $H_{N}|_{\rho}(\sigma)$, uniformly in $\sigma$, by $N\delta$. Moreover, by the same corollary the conditional law $H_{N}|_{\rho}(\sigma)$ is independent of $\nu$. Thus,

\[
\lim_{N \to \infty} \sup_{E \in \mathbb{R}, \nu \in \mathbb{R}} \frac{1}{N} \log \mathbb{E}_{\gamma,\rho} \{ Z_{N,\beta}(\rho \mathbf{n}) \} = \Lambda_{\beta}(E, \rho) + \beta\epsilon.
\]
Hence, by the Kac-Rice formula contained in Lemma B.2,
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \sum_{\sigma_0 \in \mathcal{C}_{N,\rho}(NB)} Z_{N,\beta}(\sigma_0) \\
\leq \sup_{E' \in B, x \in \mathbb{R}} \Theta_{v,\rho}(E', x) + \Lambda Z_{N,\beta}(E, \rho) + \beta \varepsilon,
\]
from which (11.9) follows by the fact that \(\sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x)\) is Lipschitz in \(E \in (-2E_0(\rho), 0)\), uniformly over \(\rho \in (\delta, 1)\), and therefore for some \(c > 0\).

(11.12) \[
\sup_{E' \in B} \sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E', x) < \sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x) + c\varepsilon. \square
\]

**Proof of Lemma 11.6.** The conditional variance of \(\sum_{i=0}^{2} \hat{H}_N^{\hat{n}_i} |_{\rho(\sigma)}\) under \(\mathbb{P}_{NE,\nu}^\rho\) is equal to \(N\alpha_2^2(\rho)\) (see (8.14)). Thus, setting
\[
\Delta_N(\sigma_0) = \left| \frac{1}{N} \log Z_{N,\beta}(\sigma_0) - \frac{1}{N} \mathbb{E}_{NE,\nu}^\rho \log Z_{N,\beta}(\rho \hat{n}) \right|,
\]
by Corollary C.4 for any \(v \in \mathbb{R}\) and \(t > 0\),
\[
\mathbb{P}_{NE,\nu}^\rho\{\Delta_N(\rho \hat{n}) > t\} \leq 3 \exp\left\{-(N-1)\gamma^2/2\beta^2\alpha_2^2(\rho)\right\}.
\]

Lemma B.1 therefore implies that
\[
\limsup_{N \to \infty} \frac{1}{N} \log(\mathbb{E}\{\sigma_0 \in \mathcal{C}_{N,\rho}(NB) : \Delta_N(\sigma_0) > t\}) \leq \sup_{E' \in B, x \in \mathbb{R}} \Theta_{v,\rho}(E', x) - t^2/2\beta^2\alpha_2^2(\rho).
\]

By (11.12), for
\[
t \geq \beta \alpha_2(\rho) \sqrt{2(\theta + c\varepsilon)},
\]
where \(\theta = \sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x)\), the left-hand side of (11.13) is negative.

Thus, with probability tending to 1 as \(N \to \infty\), for all the points \(\sigma_0 \in \mathcal{C}_{N,\rho}(NB)\) we have that \(\Delta_N(\sigma_0) < t\). From Theorem 3.1 the number of points in \(\mathcal{C}_{N,\rho}(NB)\) is bounded by \(\theta + c\varepsilon\), with probability tending to 1. The proof of the lemma therefore follows from the fact that the complement of the event in (11.10) is contained in the intersection of those two events, and since, by Lemma 8.2, similarly to (11.11),
\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{P}_{NE',\nu}^\rho\{\log Z_{N,\beta}(\rho \hat{n})\} = \Lambda Z_{N,\beta}(E, \rho) + \beta \varepsilon. \square
\]

**Proof of Lemma 11.7.** Recall that by definition (2.4) and (8.12),
\[
\frac{Z_{N,\beta}(\sigma_0)}{Z_{N,\beta}^2(\sigma_0)} = \frac{\int_{S(\sigma_0)} \exp(-\beta H_N(\sigma)) d\sigma}{\int_{S(\sigma_0)} \exp(-\beta \sum_{i=0}^{2} \hat{H}_N^{\sigma_0,i}(\sigma)) d\sigma}.
\]
From Lemma 7.3 applied with $k = 2$, for some constants $C, c > 0$, 

$$
\mathbb{P}\left\{ \exists \sigma_0 \in \mathbb{S}^{N-1}(\rho \sqrt{N}) : 
\frac{1}{N} \sup_{\sigma \in \mathbb{S}^{N-1}(\sigma_0)} \left| H_N(\sigma) - \sum_{i=0}^{k} \hat{H}_N^{\sigma_0,i}(\sigma) \right| > C(1 - \rho^2)^{3/2} \right\} \leq e^{-cN}.
$$

Lemma 11.7 directly follows from those two facts. □

### 11.3 Proof of Proposition 11.1

Let $\beta > 0$ be some large number, and $\epsilon > 0$ be some small number. We will show that there exist $\eta, \delta$, and $\nu$ satisfying the bound (11.6) as in Lemma 11.4.

Suppose that $(\rho, E) \in W_{\epsilon, \eta} \setminus B(\rho_*, -E_0(\rho_*), \epsilon)$ (see (11.4) and (11.2)) and further assume that $\rho \geq \rho^{**}$ (see (8.8)). From Lemma 11.3 with probability tending to 1 as $N \to \infty$,

$$
\frac{1}{N} \log \sum_{\sigma \in \mathbb{S}^{N-1}(\epsilon \sqrt{N})} Z_{N,B}(\sigma) < \sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x) + \Lambda_{Z,\beta}(E, \rho) + (\beta + c)\nu
$$

for some constant $c$. Since we can choose $\nu$ as small as we wish, (11.6) follows if we show that for $(\rho, E)$ as above,

$$
(11.15) \quad \sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x) + \Lambda_{Z,\beta}(E, \rho) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta/2.
$$

>From continuity of the left-hand side in $(\rho, E)$ (uniformly on compacts), if we prove (11.15) for any $(\rho, E) \in W_{\epsilon, 0} \setminus B(\rho_*, -E_0(\rho_*), \epsilon)$ with $\rho \geq \rho^{**}$, i.e., with $\eta = 0$, then the same will follow for some small $\eta > 0$.

Note that the dependence of $\Lambda_{Z,\beta}(E, \rho)$ in $E$ is through the term $\beta E$ in its definition (8.3). $\Theta_{v,\rho}(E, x)$ does not depend on $\beta$, and $\sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x)$ is a Lipschitz function of $E$ in a compact set uniformly over $\rho$ in a compact subset of $(0, 1]$. Thus, for large enough $\beta$, any constant $C > 0$, and fixed $\rho \in [1 - \rho^{**}, 1]$,

$$
\forall E \in (-E_0(\rho), -E_0(\rho) + C):
$$

$$
\sup_{x \in \mathbb{R}} \Theta_{v,\rho}(E, x) + \Lambda_{Z,\beta}(E, \rho) < \sup_{x \in \mathbb{R}} \Theta_{v,\rho}(-E_0(\rho), x) + \Lambda_{Z,\beta}(-E_0(\rho), \rho) = \Lambda_{Z,\beta}(-E_0(\rho), \rho).
$$

As we saw in Section 8, the maximum of $\rho \mapsto \Lambda_{Z,\beta}(-E_0(\rho), \rho)$ over $[\rho^{**}, 1]$ is obtained at $\rho_*$. From continuity, this completes the proof in the ease where $\rho \geq \rho^{**}$.
Now assume that \((\rho, E) \in W_{\tau, \eta} \setminus B_{\varepsilon}(\rho_*, -E_0(\rho_*), \varepsilon)\) and \(\rho \in [1 - c_{LS}, \rho_*]\).

From Lemma 11.6, with probability tending to 1 as \(N \to \infty\),
\[
\frac{1}{N} \log \sum_{\sigma \in \mathcal{G}_N} Z_{N, \beta}(\sigma) \leq \theta_{\rho, E} + \beta \alpha_2(\rho) \sqrt{2\theta_{\rho, E}} + \Lambda_{E, \beta}(E, \rho) + \beta \nu,
\]
for some constant \(c\), where \(\theta_{\rho, E} = \sup_{x \in \mathbb{R}} \Theta_{\nu, \rho}(E, x)\). As before, by assuming that \(\nu\) is small enough, we absorb the \(\beta \nu\) and \(c \nu\) terms into \(\delta/2\), so that we need to show that
\[
\theta_{\rho, E} + \beta \alpha_2(\rho) \sqrt{2\theta_{\rho, E}} + \Lambda_{E, \beta}(E, \rho) \leq \Lambda_{Z, \beta}(E_0(\rho_*), \rho) - \delta/2.
\]

To prove (11.16), we will develop the \(\beta \to \infty\) asymptotics of the terms above. Below \(c\) and \(C\) will be constants that are assumed to be sufficiently small or large, respectively, whenever needed. We also allow them to change from line to line. Assume henceforth, that \(\delta\) and \(\eta\), which are allowed to depend on \(\beta\), are both smaller than \(c \log \beta / \beta\).

First, we note that with \(t_-\) as in (8.6), from (8.3) and (11.2), as \(\beta \to \infty\),
\[
\beta \alpha_2(\rho_*) = \sqrt{2v'(1)t_-} + O \left( \frac{1}{\beta} \right),
\]
\[
\Lambda_{Z, \beta}(E_0(\rho_*), \rho_*) = \beta E_0(\rho_*) + \frac{1}{2} \log \left( \frac{2t_-}{\beta} \right) + \tau_2 v''(1) + O \left( \frac{1}{\beta} \right),
\]
(11.17)
\[
\tau(\beta, \delta) = E_0(1) - E_0(\rho_*) + \frac{\log \beta}{2\beta} + \delta + O \left( \frac{1}{\beta} \right) \leq \frac{\log \beta}{\beta},
\]
where the inequality follows since, by Lemma 5.1, \(E_0(\rho)\) is differentiable at \(\rho = 1\) and \(\rho_* = 1 - O(1/\beta)\).

For any compact \(K\), for large enough \(T > 0\),
\[
\sup_{x \in \mathbb{R}} \Theta_{\nu, \rho}(E, x) = \sup_{|x| \leq T} \Theta_{\nu, \rho}(E, x)
\]
uniformly over \(E \in K\) and \(\rho\) close enough to 1 (see Lemma 5.13). Using this and the fact that \(\theta_{1, -E_0(1)} = 0\), one can verify that for some \(C > 0\),
\[
\theta_{\rho, E} \leq |1 - \rho|C + |E + E_0(1)|C
\]
for \((\rho, E)\) in a small neighborhood of \((1, -E_0(1))\). Thus, for \((\rho, E) \in W_{\tau, \eta}\) and \(\beta\) large enough, we have from (11.17) and (8.13) that \(\beta \theta_{\rho, E} + \beta \alpha_2(\rho)\), and
\[
\left| \Lambda_{E, \beta}(E, \rho) - \Lambda_{E, \beta}(E_0(\rho), \rho) \right|
\]
are all smaller than \(C \log \beta\).
>From the above, (11.16) will follow if we show that for any $\rho \in [1-c_{LS} \tau, \rho_*]$
\[\frac{C \log \beta}{\beta} + C \frac{(\log \beta)^{3/2}}{\beta^{1/2}} + \Lambda_{F,\beta}^2(-E_0(\rho), \rho) \leq \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta/2,\]
or, since we assume $\beta$ is large,
\[\Lambda_{F,\beta}^2(-E_0(\rho), \rho) \leq \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta/4.\]

>From (8.15), reparametrizing
\[\tilde{\Lambda}_{F,\beta}(t) = \Lambda_{F,\beta}(-E_0(\rho), \rho)|_{\rho=1-\frac{1}{\beta t}},\]
we deduce that, uniformly in $t \in (0, c_{LS} \log \beta] \supset (0, c_{LS} \tau \beta]$, as $\beta \to \infty$,
\[\tilde{\Lambda}_{F,\beta}(t) = \kappa' - tk(1 + o(1)).\]
where $\kappa = x_0(1) - 2\sqrt{v''(1)} > 0$ and $\kappa' = \lim_{s \to 0} \tilde{\Lambda}_{F,\beta}(s)$.

Therefore, (11.18) follows from (8.11) and (8.16). This concludes the proof of (11.6) for small enough $\eta, \delta, v$, and thus also the proof of Proposition 11.1. □

12 Proofs of the Main Results: Theorems 1.2, 1.3, and 1.4

Recall the definition of $\rho_*$; see (8.6). The energy $E_*$ that was used in the statements of the proofs is defined as the limiting normalized ground state (see Remark 5.12).

\[E_* := E_*(\beta) = E_0(\rho_*).\]
Throughout the proofs we will use the notation $B(\epsilon) = -E_0(\rho_*) + (-\epsilon, \epsilon)$ and $D(\epsilon) = -x_0(\rho_*) + (-\epsilon, \epsilon)$.

12.1 Proof of Theorem 1.2 (Support of the Gibbs measure)

>From continuity,
\[\lim_{\epsilon \to 0} \sup_{E \in B(\epsilon), x \in D(\epsilon)} \Theta_{\epsilon,\rho_*}(E, x) = \Theta_{\epsilon,\rho_*}(-E_0(\rho_*), -x_0(\rho_*)) = 0.\]

Thus, by Theorem 3.1 and Lemma 5.14 (1.15) holds for any choice of $\epsilon_N = o(1)$.

From the lower bound of Proposition 10.1 and the upper bound of Proposition 11.1 we have that for any $\epsilon > 0$ and small enough $\delta > 0$, with probability tending to 1 as $N \to \infty$,
\[\frac{1}{N} \log Z_{N,\beta} > \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta/2\]
and
\[\frac{1}{N} \log Z_{N,\beta} \left(\mathbb{S}^{N-1}(\sqrt{N}) \setminus \bigcup_{\sigma_0 \in \mathcal{E}_N(\rho)} \text{Band}(\sigma_0, \epsilon)\right) < \Lambda_{Z,\beta}(-E_0(\rho_*), \rho_*) - \delta.\]
This proves the statement of Part 2 with \( \epsilon > 0 \) instead of \( \epsilon_N = o(1) \). By a standard diagonalization argument, we obtain the same with \( \epsilon_N = o(1) \), assuming the rate of decay is slow enough. \( \square \)

Remark 12.1. Using (12.2), (12.3), (8.6), and (8.8), we obtain that

\[
F_\beta = \Lambda_{Z, \beta} (-E_0(\rho_*), \rho_*) = \sup_{\rho \in [\rho_*, 1]} \Lambda_{Z, \beta} (-E_0(\rho), \rho).
\]

12.2 Proof of Theorem 1.3, Part 1 (States are pure)

Let \( \epsilon_N \) be the sequence defined in Theorem 1.2. First note that by Lemma 5.14, instead of \( C \), it will be enough to prove the theorem with points only from (12.4)

\[
C := \epsilon_N(\rho_*), \sqrt{N} D(\epsilon_N)).
\]

where \( \epsilon_N \) may have to be increased. By a union bound, the first limit of Part 1 will follow if we show that

\[
\lim_{N \to \infty} \mathbb{E} \sum_{\sigma_0 \in \epsilon_+^+} G_{N, \beta} \times G_{N, \beta} \{ \sigma, \sigma' \in \text{Band}(\sigma_0, \epsilon_N), \ |R(\sigma, \sigma') - \rho_*^2| > \delta \} = 0,
\]

where the above notation means that \( \sigma, \sigma' \) are sampled independently from the Gibbs measure \( G_{N, \beta} \).

For any \( \delta > 0 \) and \( \sigma_0 \in \mathbb{B}^N(\sqrt{N}) \), define

\[
\mathbb{Z}_{\beta, 0, \sigma_0} = \int_{(\text{Band}(\sigma_0, \epsilon_N))^2} \mathbb{P}(\{ |R(\sigma, \sigma') - \rho_*^2| > \delta \}) e^{-\beta(\mathcal{H}_N(\sigma) + \mathcal{H}_N(\sigma'))} d\sigma \ d\sigma',
\]

where the integration is w.r.t. the product measure of the probability Hausdorff measure on the sphere with itself. Note that the probability under the product Gibbs measure in (12.5) is equal to \( \mathbb{Z}_{\beta, 0, \sigma_0} / (Z_{N, \beta})^2 \).

Denote \( \Lambda_{\beta} := \Lambda_{Z, \beta} (-E_0(\rho_*), \rho_*) \). Assume that for some \( C \) independent of \( N \), \( \mathcal{H}_N(\sigma) \) is \( \sqrt{N} C \)-Lipschitz continuous on \( \mathbb{B}^N(\sqrt{N}) \). Then for \( \eta > 0 \), since \( \epsilon_N \to 0 \), for large \( N \),

\[
\frac{1}{N} \log Z_{\beta, 0, \sigma_0} < 2\Lambda_{\beta} - \eta, \quad \text{then} \quad \frac{1}{N} \log \mathbb{Z}_{\beta, 0, \sigma_0} < 2\Lambda_{\beta} - \eta/2,
\]

where we define

\[
Z_{\beta, 0, \sigma_0} := \left(1 - \rho_*^2\right)^N \int_{(S(\sigma_0))^2} \mathbb{P}(\{ |R(\sigma, \sigma') + \rho_*^2| > \delta \}) e^{-\beta(\mathcal{H}_N(\sigma) + \mathcal{H}_N(\sigma'))} d\sigma \ d\sigma',
\]

with the integration being w.r.t. the product of the probability Hausdorff measure on \( S(\sigma_0) \) with itself, and where the term \( (1 - \rho_*^2)^N \) accounts for the volume of band (at exponential level for large \( N \)).
In light of the lower bound on the free energy (12.2), the implication (12.6), and the Lipschitz bound of (C.8), to prove (12.5) it will be enough to show that for any $\delta > 0$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{C}_+^*} Z_{N, \beta, \delta}^{\otimes 2} (\sigma_0) \right\} < 2 \Lambda \beta.$$ (12.7)

Recall that $\Theta_{v, \rho_\ast} (E_0 (\rho_\ast), - x_0 (\rho_\ast)) = 0$, and note that $\Theta_{v, \rho_\ast} (E, x)$ is continuous. Thus, similarly to Lemma B.2 to prove (12.7) it is sufficient to show that

$$\limsup_{N \to \infty} \sup_{E \in \mathcal{B}(\varepsilon_N), \ n \in D(\varepsilon_N)} \frac{1}{N} \log \mathbb{E} \rho_* \left\{ Z_{N, \beta, \delta}^{\otimes 2} (\rho_* \hat{n}) \right\} < 2 \Lambda \beta.$$ (12.8)

>From Lemma [7.1] and (10.15), the left-hand side of (12.8) is equal to

$$\log (1 - \rho_*^2) + 2 \beta E_0 (\rho_\ast) + \sup_{\| \xi \| \leq \frac{\delta}{1 - \rho_*^2}} \left\{ \frac{1}{2} \log (1 - \xi^2) + \beta^2 \sum_{k=2}^{\infty} \alpha_k^2 (\rho_\ast)(1 + \xi^k) \right\},$$ (12.9)

where we used the fact that for points in $S(\rho_\ast \hat{n}), |R(\sigma, \sigma')| > \delta$ if and only if

$$|R(\sigma - \rho_\ast \hat{n}, \sigma' - \rho_\ast \hat{n})| > \frac{\delta}{1 - \rho_*^2}.$$ (12.9)

The right-hand side of (12.8) is equal to the expression in (12.9) with $\xi = 0$ (and no supremum). The inequality (12.8) therefore follows from a similar analysis to that following (10.15). This proves the first of the two limits in Part 1 of Theorem 1.3.

If $\psi$ is neither an odd nor even polynomial, then almost surely $H_N (\sigma)$ has no antipodal critical points on $\mathbb{S}^{-1} (\rho_\ast \sqrt{N})$. (This can be verified by applying the Kac-Rice formula [10] theorem 12.1.1 to compute the expected number of pairs $(\sigma_1, \sigma_2) \in (\mathbb{S}^{-1} (\rho_\ast \sqrt{N}))^2$ of critical points with overlap $|R(\sigma_1, \sigma_2) + 1| < \epsilon$, and taking $\epsilon \to 0$.) If $\psi$ is odd, then for any $\sigma_0 \in \mathcal{C}_+^*, - \sigma_0$ is also a critical point, but since $H_N (- \sigma_0) = - H_N (\sigma_0), - \sigma_0 \notin \mathcal{C}_+^*$. Lastly, if $\psi$ is even, then for any $\sigma_0 \in \mathcal{C}_+^*, H_N (- \sigma_0) = H_N (\sigma_0)$ and $- \sigma_0 \in \mathcal{C}_+^*$.

Hence, we only need to prove the second limit of Part 1 in the case where $\psi$ is even. This case, however, follows directly from the symmetry $H_N (\sigma) = H_N (- \sigma)$. $\Box$

12.3 Proof of Theorem 1.3, Part 2 (Orthogonality of states)

The key element in the current proof is combining Part 1 of Theorem 1.3 with the following corollary, which is a direct conclusion of lemma 11 in [53] and the argument used in the proof of theorem 3 in [53], both of which rely on basic linear algebra and do not involve probabilistic arguments.
COROLLARY 12.2. For some function \( \zeta(\epsilon) > 0 \) satisfying \( \lim_{\epsilon \to 0} \zeta(\epsilon) = 0 \) we have the following for every \( N \). For \( i = 1, 2 \), let \( \epsilon > 0, \rho_i \in (0, 1) \), and \( \sigma_0^i \in S_N^{-1}(\rho_i \sqrt{N}) \). If \( M_i \) is a measure supported on \( \text{Band} (\sigma_0^i, \epsilon) \) such that
\[
M_i 	imes M_i \left\{ |R(\sigma, \sigma') - \rho_1^2| > \epsilon \right\} < \epsilon,
\]
then
\[
M_1 	imes M_2 \left\{ |R(\sigma, \sigma') - \rho_1 \rho_2 R(\sigma_0^1, \sigma_0^2)| > \zeta(\epsilon) \right\} < \zeta(\epsilon).
\]

Denote by \( \mathcal{C}_*^\delta \) the set of points \( \sigma_0 \in \mathcal{C}_* \) for which
\[
\frac{1}{N} \log Z_{N, \beta} (\text{Band} (\sigma_0, \epsilon_N)) > \Lambda_{N, \beta} (-E_0(\rho_*), \rho_*) - \delta.
\]
Since by (1.15) the number of points in \( \mathcal{C}_* \) is sub-exponential, from the lower bound on the free energy in Proposition 10.1, if \( \delta_N = o(1) \) decays slow enough, then
\[
\lim_{N \to \infty} \mathbb{E} G_{N, \beta} \left( \bigcup_{\sigma_0 \in \mathcal{C}_*^\delta_N} \text{Band} (\sigma_0, \epsilon_N) \right) = 1.
\]

From Corollary 3.7, for large \( \beta \) (and therefore \( \rho_* \) close to 1), the bands corresponding to different points in \( \mathcal{C}_* \) are disjoint, with probability tending to 1 as \( N \to \infty \). Therefore, to prove (1.18) it will be enough to show that
\[
\lim_{N \to \infty} \mathbb{P} \left\{ \forall \sigma_0 \neq \pm \sigma_0' \in \mathcal{C}_*^\delta_N : G_{N, \beta}^{\sigma_0} \times G_{N, \beta}^{\sigma_0'} \left\{ |R(\sigma, \sigma')| > \delta \right\} < \zeta_N \right\} = 1
\]
for some \( \zeta_N = o(1) \), where we denote by \( G_{N, \beta}^{\sigma_0} \) the conditional Gibbs measure given \( \text{Band} (\sigma_0, \epsilon_N) \).

Note that from (12.7), (12.6), and Corollary 3.7 if \( \epsilon_N' = o(1) \) decays sufficiently slow, then with probability tending to 1, uniformly in \( \sigma_0 \in \mathcal{C}_*^\delta_N \),
\[
G_{N, \beta}^{\sigma_0} \times G_{N, \beta}^{\sigma_0'} \left\{ |R(\sigma, \sigma') - \rho_1^2| > \epsilon_N' \right\} < \epsilon_N',
\]
and uniformly in \( \sigma_0, \sigma_0' \in \mathcal{C}_*^\delta_N \) with \( \sigma_0 \neq \pm \sigma_0', |R(\sigma_0, \sigma_0')| < \epsilon_N' \). Combined with Corollary 12.2, this implies (12.10) and completes the proof. \( \square \)

12.4 Proof of Theorem 1.4

Let \( \beta \neq \beta' \) be two different inverse temperatures and define \( \rho_* = \rho_* (\beta) \neq \rho_* (\beta') \) by (8.6). Note that almost surely there are no pairs of critical points \( (\sigma_1, \sigma_2) \in S_N^{-1}(\rho_* \sqrt{N}) \times S_N^{-1}(\rho_*' \sqrt{N}) \) such that \( \sigma_1 = \pm \sigma_2 \). This can be verified, e.g., by applying the Kac-Rice formula [12.1.1] to compute the expected number of pairs \( (\sigma_1, \sigma_2) \in S_N^{-1}(\rho_* \sqrt{N}) \times S_N^{-1}(\rho_*' \sqrt{N}) \) of critical points with overlap \( |R(\sigma_1, \sigma_2)| \pm | \) < \( \epsilon \), and taking \( \epsilon \to 0 \). (We emphasize, however, that for pure models, for any \( \rho_* \)-critical point \( \sigma_1 \), the point \( \sigma_2 = \sigma_1 \).)
\( \rho_0 \) is a \( \rho_0 \)-critical point deterministically, in which case the Kac-Rice formula cannot be applied due to the degeneracy of the covariance matrix.

From the argument that precedes (12.10) (which is based on Corollary 3.7 valid also for \( \rho_1 \neq \rho_2 \)), to complete the proof of Theorem 1.4 it will be enough to show that for some \( \zeta_N = o(1) \),

\[
\lim_{N \to \infty} \mathbb{P} \left\{ \forall \sigma_0 \in \mathcal{C}_N, \sigma'_0 \in \mathcal{C}_N, \sigma_0 \neq \pm \sigma'_0 : \right. \\
G_{N,\beta}^{\sigma_0} \times G_{N,\beta'}^{\sigma'_0} \left\{ |R(\sigma, \sigma')| > \delta \right\} < \zeta_N \right\} = 1,
\]

where \( \sigma_0 \in \mathcal{C}_N \) and \( G_{N,\beta}^{\sigma_0} \) are as in the proof of Part 2 of Theorem 1.3 and \( \sigma_0 \in \mathcal{C}_N \) and \( G_{N,\beta'}^{\sigma'_0} \) are defined similarly. Similarly to (12.11), we have that uniformly in \( \sigma_0 \in \mathcal{C}_N \),

\[
G_{N,\beta}^{\sigma'_0} \times G_{N,\beta'}^{\sigma'_0} \left\{ \left| R(\sigma, \sigma') - \rho_0^2 \right| > \epsilon_N \right\} < \epsilon_N,
\]

and uniformly in \( \sigma_0 \in \mathcal{C}_N \), \( \sigma'_0 \in \mathcal{C}_N \) with \( \sigma_0 \neq \pm \sigma'_0 \), \( |R(\sigma_0, \sigma'_0)| < \epsilon_N \) with probability tending to 1, for \( \epsilon_N \) decaying slowly enough. Combining the above with Corollary 12.2 the proof is completed.

\[ \square \]

Appendix A Covariances

In this appendix we prove Lemmas 4.1, 4.2 and 4.3. We begin with a study of the joint covariance of

\[ H_N(\sigma), \quad \nabla_{sp} H_N(\sigma), \quad \nabla^2_{sp} H_N(\sigma), \quad \frac{d}{dR} H_N(\rho \sigma), \]

at two points of the form

\[ \rho \sigma = \rho_1 \hat{n} \quad \text{and} \quad \rho \sigma = \rho_2 \sigma(r) = \rho_2 \sqrt{N} (0, \ldots, 0, \sqrt{1 - r^2}, r). \]

With the usual notation

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases} \]

in the lemma below, we denote

\[ \delta_{i-j} = \delta_{ij}, \quad \delta_{i-j-k} = \delta_{ij} \delta_{jk}, \quad \delta_{i-j\neq k} = \delta_{ij} (1 - \delta_{jk}), \quad \text{etc.} \]

\textbf{Lemma A.1.} For any \( r \in [-1, 1] \) and \( \rho_1, \rho_2 \in (0, 1] \) there exists a frame field \( F = (F_i) \) (orthonormal when restricted to any sphere centered at the origin) satisfying

\[
F_i H_N(\rho_1 \hat{n}) = \frac{d}{dx_i} \Bigg|_{x=0} H_N \left( x_1, \ldots, x_{N-1}, \rho_1 \sqrt{N - \|x\|^2} \right),
\]

\[
(A.1) \quad F_i F_j H_N(\rho_1 \hat{n}) = \frac{d}{dx_i} \frac{d}{dx_j} \Bigg|_{x=0} H_N \left( x_1, \ldots, x_{N-1}, \rho_1 \sqrt{N - \|x\|^2} \right),
\]

\[ \text{for } r \in [-1, 1]. \]
such that

\[
\frac{1}{N} \mathbb{E} \left\{ H_N(\rho_1 \hat{n}) H_N(\rho_2 \sigma(r)) \right\} = \nu(\rho_1 \rho_2 r).
\]

\[
\frac{1}{\sqrt{N}} \mathbb{E} \left\{ H_N(\rho_1 \hat{n}) \frac{d}{dR} H_N(\rho_2 \sigma(r)) \right\} = \frac{1}{\sqrt{N}} \mathbb{E} \left\{ \frac{d}{dR} H_N(\rho_2 \hat{n}) H_N(\rho_1 \sigma(r)) \right\} = \rho_1 r \nu'(\rho_1 \rho_2 r).
\]

\[
\frac{1}{\sqrt{N}} \mathbb{E} \left\{ H_N(\rho_1 \hat{n}) F_I H_N(\rho_2 \sigma(r)) \right\}
= -\frac{1}{\sqrt{N}} \mathbb{E} \left\{ F_I H_N(\rho_2 \hat{n}) H_N(\rho_1 \sigma(r)) \right\}
= -\rho_1 r \nu'(\rho_1 \rho_2 r)(1 - r^2)^{1/2} \delta_{l-N-1}.
\]

\[
\mathbb{E} \left\{ H_N(\rho_1 \hat{n}) F_k F_I H_N(\rho_2 \sigma(r)) \right\}
= \mathbb{E} \left\{ F_k F_I H_N(\rho_2 \hat{n}) H_N(\rho_1 \sigma(r)) \right\}
= \rho_1^2 r \nu'(\rho_1 \rho_2 r)(1 - r^2) \delta_{l-k-N-1} - \frac{\rho_1}{\rho_2} r \nu'(\rho_1 \rho_2 r) \delta_{l-N-1}.
\]

\[
\mathbb{E} \left\{ \frac{d}{dR} H_N(\rho_1 \hat{n}) \frac{d}{dR} H_N(\rho_2 \sigma(r)) \right\} = \rho_1 \rho_2 r^2 \nu''(\rho_1 \rho_2 r) + r \nu'(\rho_1 \rho_2 r).
\]

\[
\mathbb{E} \left\{ \frac{d}{dR} H_N(\rho_1 \hat{n}) F_I H_N(\rho_2 \sigma(r)) \right\}
= -\mathbb{E} \left\{ F_I H_N(\rho_2 \hat{n}) \frac{d}{dR} H_N(\rho_1 \sigma(r)) \right\}
= -\left( \rho_1 \rho_2 r \nu''(\rho_1 \rho_2 r) + \nu'(\rho_1 \rho_2 r) (1 - r^2)^{1/2} \delta_{l-N-1} \right).
\]

\[
\sqrt{N} \mathbb{E} \left\{ \frac{d}{dR} H_N(\rho_1 \hat{n}) F_k F_I H_N(\rho_2 \sigma(r)) \right\}
= \sqrt{N} \mathbb{E} \left\{ F_k F_I H_N(\rho_2 \hat{n}) \frac{d}{dR} H_N(\rho_1 \sigma(r)) \right\}
= (\rho_1^2 \rho_2 r \nu''(\rho_1 \rho_2 r) + 2 \rho_1 \nu''(\rho_1 \rho_2 r) (1 - r^2) \delta_{l-k-N-1} - \left( \rho_1 \rho_2 r \nu''(\rho_1 \rho_2 r) + \frac{r}{\rho_2} \nu'(\rho_1 \rho_2 r) \right) \delta_{l-N-1}.
\]

\[
\mathbb{E} \left\{ F_j H_N(\rho_1 \hat{n}) F_I H_N(\rho_2 \sigma(r)) \right\}
= \nu'(\rho_1 \rho_2 r) \delta_{l-j-N-1}
+ (r \nu'(\rho_1 \rho_2 r) - \rho_1 \rho_2 \nu''(\rho_1 \rho_2 r) (1 - r^2) \delta_{l-j-N-1}.
\]
\[
\sqrt{N} \mathbb{E} \left\{ F_j H_N(\rho_1 \hat{\mathbf{n}}) F_k F_l H_N(\rho_2 \sigma(r)) \right\} \\
= -\sqrt{N} \mathbb{E} \left\{ F_k F_l H_N(\rho_2 \hat{\mathbf{n}}) F_j H_N(\rho_1 \sigma(r)) \right\} \\
= \rho_1^2 \rho_2 v''(\rho_1 \rho_2 r) (1 - r^2)^{3/2} \delta_{j-k-l-N-1} \\
- \rho_1 v'(\rho_1 \rho_2 r) (1 - r^2)^{1/2} \\
\cdot \left[ \delta_{j-k-N-1} \delta_{l-N-1} + \delta_{j-l-N-1} \delta_{k-N-1} + 2r \delta_{j-k-l-N-1} \right] \\
- (r \rho_1 v''(\rho_1 \rho_2 r) + \rho_2^{-1} v'(\rho_1 \rho_2 r))(1 - r^2)^{1/2} \delta_{k-l} \delta_{j-N-1}.
\]

\[
N \mathbb{E} \left\{ F_i F_j H_N(\rho_1 \hat{\mathbf{n}}) F_k F_l H_N(\rho_2 \sigma(r)) \right\} \\
= \rho_1^2 \rho_2 v''''(\rho_1 \rho_2 r) (1 - r^2)^2 \delta_{j-k-l-N-1} \\
- \rho_1 \rho_2 v'''(\rho_1 \rho_2 r) (1 - r^2) \left[ 6r \delta_{i-j-k-l-N-1} + r \delta_{i-j-N-1} \delta_{k-l-N-1} \\
+ r \delta_{i-j-N-1} \delta_{k-l-N-1} + \delta_{k-i-N-1} \delta_{j-l-N-1} + \delta_{i-k-N-1} \delta_{j-l-N-1} \right] \\
+ \delta_{j-k-N-1} \delta_{l-N-1} + \delta_{j-l-N-1} \delta_{k-N-1} \\
+ \delta_{j-i-N-1} \delta_{k-l-N-1} + 2r \delta_{j-k-l-N-1} \\
+ (\delta_{j-i-N-1} + r \delta_{j-l-N-1}) (\delta_{k-i-N-1} + r \delta_{k-l-N-1}) \\
+ (\delta_{j-l-N-1} + r \delta_{j-i-N-1}) (\delta_{j-k-N-1} + r \delta_{j-k-N-1}) \\
+ \rho_1^{-1} \rho_2^{-1} v'(\rho_1 \rho_2 r) \delta_{i-j-k-l}.
\]

**Proof.** The lemma follows by straightforward algebra from the relations

\[
H_N(\rho \sigma) = \sum \gamma_p \rho^p H_{N,p} \sigma, \\
v_p H_N(\rho \sigma) = \sum \gamma_p \rho^p H_{N,p} \sigma, \\
v^2_p H_N(\rho \sigma) = \sum \gamma_p \rho^p H_{N,p} \sigma, \\
\frac{d}{dR} H_N(\rho \sigma) = \frac{1}{\sqrt{N}} \sum \gamma_p \rho^p H_{N,p} \sigma,
\]

and the covariance computations of \[52\] lemma 30], which dealt with the pure case. \(\square\)

For any \(r \in (-1, 1)\) and \(\rho_1, \rho_2 \in (0, 1]\), define

\[
a_1(r, \rho_1, \rho_2) = \frac{v'(\rho_2^2)}{v'(\rho_1^2) v'(\rho_2^2) - (v'(\rho_1 \rho_2 r))^2}, \\
a_2(r, \rho_1, \rho_2) = \frac{v'(\rho_2^2)}{v'(\rho_1^2) v'(\rho_2^2) - (v'(\rho_1 \rho_2 r))^2}, \\
a_3(r, \rho_1, \rho_2) = \frac{-v'(\rho_1 \rho_2 r)}{v'(\rho_1^2) v'(\rho_2^2) - (v'(\rho_1 \rho_2 r))^2}, \\
a_4(r, \rho_1, \rho_2) = \frac{-v'(\rho_1 \rho_2 r)}{v'(\rho_1^2) v'(\rho_2^2) - (v'(\rho_1 \rho_2 r))^2}.
\]
\[ u_1(r, \rho_1, \rho_2) = \rho_1 \rho_2 r v''(\rho_1 \rho_2 r) + v'(\rho_1 \rho_2 r), \]
\[ u_2(r, \rho_1, \rho_2) = -\rho_1^2 \rho_2 v''(\rho_1 \rho_2 r)(1 - r^2) + 2\rho_2 v'(\rho_1 \rho_2 r), \]
\[ u_3(r, \rho_1, \rho_2) = \rho_1^2 \rho_2 r v'''(\rho_1 \rho_2 r) + 2\rho_1 v''(\rho_1 \rho_2 r). \]

For any of \( T = U, X, b, Z, Q \), define the matrix

\[ \Sigma_T(r, \rho_1, \rho_2) = \begin{pmatrix} \Sigma_{T,11}(r, \rho_1, \rho_2) & \Sigma_{T,12}(r, \rho_1, \rho_2) \\ \Sigma_{T,12}(r, \rho_1, \rho_2) & \Sigma_{T,22}(r, \rho_1, \rho_2) \end{pmatrix} \]

by the following:

(A.2) \[ \Sigma_{U,11}(r, \rho_1, \rho_2) = \Sigma_{U,22}(r, \rho_2, \rho_1) \]
\[ = v'(\rho_1^2) - \rho_1^2 a_2(r, \rho_2, \rho_1) \left( v'(\rho_1 \rho_2 r) \right)^2 (1 - r^2). \]
\[ \Sigma_{U,12}(r, \rho_1, \rho_2) = \Sigma_{U,21}(r, \rho_1, \rho_2) \]
\[ = v(\rho_1 \rho_2 r) + \rho_1 \rho_2 a_4(r, \rho_1, \rho_2) \left( v'(\rho_1 \rho_2 r) \right)^2 (1 - r^2). \]
\[ \Sigma_{X,11}(r, \rho_1, \rho_2) = \Sigma_{X,22}(r, \rho_2, \rho_1) \]
\[ = \rho_1^2 v''(\rho_1^2) + v'(\rho_1^2) - (v_1(r, \rho_1, \rho_2))^2 (1 - r^2) a_2(r, \rho_2, \rho_1). \]
\[ \Sigma_{X,12}(r, \rho_2, \rho_1) = \Sigma_{X,21}(r, \rho_1, \rho_2) \]
\[ = \rho_1 \rho_2 r^2 v''(\rho_1 \rho_2 r) + v'(\rho_1 \rho_2 r) \]
\[ + (v_1(r, \rho_1, \rho_2))^2 (1 - r^2) a_4(r, \rho_1, \rho_2). \]
\[ \Sigma_{b,11}(r, \rho_1, \rho_2) = \Sigma_{b,22}(r, \rho_2, \rho_1) \]
\[ = \rho_1 v'(\rho_1^2) - \rho_1 (1 - r^2) v'(\rho_1 \rho_2 r) v_1(r, \rho_1, \rho_2) a_2(r, \rho_2, \rho_1). \]
\[ \Sigma_{b,12}(r, \rho_1, \rho_2) = \Sigma_{b,21}(r, \rho_1, \rho_2) \]
\[ = \rho_1 r v'(\rho_1 \rho_2 r) + \rho_1 (1 - r^2) v'(\rho_1 \rho_2 r) v_1(r, \rho_1, \rho_2) a_4(r, \rho_1, \rho_2). \]
\[ \Sigma_{Z,11}(r, \rho_1, \rho_2) = \Sigma_{Z,22}(r, \rho_2, \rho_1) \]
\[ = 1 - \frac{1}{v''(\rho_1^2)} \rho_2^2 (1 - r^2) \left( v''(\rho_1 \rho_2 r) \right)^2 a_1(r, \rho_2, \rho_1). \]
\[ \Sigma_{Z,12}(r, \rho_1, \rho_2) = \Sigma_{Z,21}(r, \rho_1, \rho_2) \]
\[ = - \frac{1}{\sqrt{v''(\rho_1^2) v'''(\rho_1^2)}} \left[ \rho_1 \rho_2 (1 - r^2) v'''(\rho_1 \rho_2 r) + r v''(\rho_1 \rho_2 r) \right] \]
\[ + \rho_1 \rho_2 (1 - r^2) \left( v''(\rho_1 \rho_2 r) \right)^2 a_3(r, \rho_1, \rho_2). \]
\[
\Sigma_{Q,11}(r, \rho_1, \rho_2) = 2 - \frac{(1 - r^2)}{v''(\rho_1^2)} a_2(r, \rho_2, \rho_1)(v_2(r, \rho_1, \rho_2))^2
\]

\[
- (1 - r^2)^2 (\xi_1(r, \rho_1, \rho_2))^\top \Sigma_{U,X}^{-1}(r, \rho_1, \rho_2) \xi_1(r, \rho_1, \rho_2).
\]

\[
\Sigma_{Q,22}(r, \rho_1, \rho_2) = 2 - \frac{(1 - r^2)}{v''(\rho_2^2)} a_2(r, \rho_1, \rho_2)(v_2(r, \rho_2, \rho_1))^2
\]

\[
- (1 - r^2)^2 (\xi_2(r, \rho_1, \rho_2))^\top \Sigma_{U,X}^{-1}(r, \rho_1, \rho_2) \xi_2(r, \rho_1, \rho_2).
\]

\[
\Sigma_{Q,12}(r, \rho_1, \rho_2) = \Sigma_{Q,21}(r, \rho_1, \rho_2) = \frac{1}{\sqrt{v''(\rho_1^2)v''(\rho_2^2)}}
\]

\[
\times \left[ \rho_1 \rho_2 v'''(\rho_1 \rho_2 r)(1 - r^2)^2 - 2 \rho_1 \rho_2 (1 - r^2) r v'''(\rho_1 \rho_2 r)
\right.
\]

\[
+ 2r^2 (1 - r^2) v''(\rho_1 \rho_2 r) + v_2(r, \rho_1, \rho_2)v_2(r, \rho_2, \rho_1)a_4(r, \rho_1, \rho_2)
\]

\[
- (1 - r^2)^2 (\xi_1(r, \rho_1, \rho_2))^\top \Sigma_{U,X}^{-1}(r, \rho_1, \rho_2) \xi_2(r, \rho_1, \rho_2).
\]

where

\[
\xi_1(r, \rho_1, \rho_2) = \frac{1}{\sqrt{v''(\rho_1^2)}}
\]

\[
\times \begin{pmatrix}
\rho_1 v'(\rho_1 \rho_2 r) v_2(r, \rho_1, \rho_2) a_2(r, \rho_2, \rho_1) \\
\rho_2 v'(\rho_1 \rho_2 r) - \rho_2 v'(\rho_1 \rho_2 r) v_2(r, \rho_1, \rho_2) a_4(r, \rho_1, \rho_2) \\
v_3(r, \rho_2, \rho_1) - v_1(r, \rho_1, \rho_2) v_2(r, \rho_1, \rho_2) a_2(r, \rho_2, \rho_1)
\end{pmatrix},
\]

(A.3)

\[
\xi_2(r, \rho_1, \rho_2) = \frac{1}{\sqrt{v''(\rho_2^2)}}
\]

\[
\times \begin{pmatrix}
\rho_1^2 v''(\rho_1 \rho_2 r) - \rho_1 v'(\rho_1 \rho_2 r) v_2(r, \rho_2, \rho_1) a_4(r, \rho_1, \rho_2) \\
\rho_2^2 v''(\rho_1 \rho_2 r) v_2(r, \rho_2, \rho_1) a_2(r, \rho_1, \rho_2) \\
v_3(r, \rho_1, \rho_2) - v_1(r, \rho_1, \rho_2) v_2(r, \rho_2, \rho_1) a_4(r, \rho_1, \rho_2)
\end{pmatrix},
\]

and

\[
\Sigma_{U,X}(r, \rho_1, \rho_2) = \begin{pmatrix}
\Sigma_U(r, \rho_1, \rho_2) & \Sigma_B(r, \rho_1, \rho_2) \\
\Sigma_B^\top(r, \rho_1, \rho_2) & \Sigma_X(r, \rho_1, \rho_2)
\end{pmatrix}.
\]

(A.4)
Remark A.2. For \( r \in (-1, 1) \) and \( \rho_1, \rho_2 \in (0, 1] \), we note that
\[
\nu'(\rho_1^2)\nu'(\rho_2^2) = \sum_{p \geq m} \frac{\gamma_p^2 \gamma_m}{1 + \delta_{m-p}} (\rho_1^{2(p-1)} \rho_2^{2(m-1)} + \rho_1^{2(m-1)} \rho_2^{2(p-1)}),
\]
\[
\nu'(\rho_1 \rho_2 r)^2 = \sum_{p \geq m} \frac{\gamma_p^2 \gamma_m}{1 + \delta_{m-p}} -2(\rho_1 \rho_2 r)^{p-1+m-1},
\]
and thus by comparing summand by summand, \( \nu'(\rho_1^2)\nu'(\rho_2^2) > \nu'(\rho_1 \rho_2 r)^2 \), and that
\[
r\nu'(\rho_1 \rho_2 r) - \rho_1 \rho_2 \nu''(\rho_1 \rho_2 r)(1 - r^2)
\]
(A.5)
\[
= \sum_{p, m} \gamma_p^2 \gamma_m (pr - (p-1)r^{p-2})(m r^{m} - (m-1)r^{m-2}),
\]
which, since \( |pr - (p-1)r^{p-2}| < 1 \) for any \( p \geq 2 \), similarly implies that (A.5) is strictly smaller, in absolute value, than \( \nu'(\rho_1^2)\nu'(\rho_2^2) \). Therefore, the denominators in the definitions of \( a_i(r, \rho_1, \rho_2) \) are positive for any \( r \in (-1, 1) \).

Proof of Lemmas 4.1, 4.2, and 4.3. The proof closely follows that of [52, lemmas 12 and 13]. Fix \( r \in (-1, 1) \) and \( \rho_1, \rho_2 \in (0, 1] \). Assume all vectors in the proof are column vectors and denote the concatenation of any two vectors \( v_1, v_2 \) by \((v_1; v_2)\). The covariance matrix of the vector \((\nabla_{sp} H_N (\rho_1 \hat{n}); \nabla_{sp} H_N (\rho_2 \sigma (r)))\) can be extracted from Lemma [A.1]. By computation, one has that (4.6) holds and that the inverse of the latter covariance matrix is the block matrix
\[
A_0(r) = \begin{pmatrix}
    a_1(r, \rho_1, \rho_2) I_{N-1} & 0 & a_3(r, \rho_1, \rho_2) I_{N-1} & 0 \\
    0 & a_2(r, \rho_1, \rho_2) & 0 & a_4(r, \rho_1, \rho_2) \\
    a_3(r, \rho_1, \rho_2) I_{N-1} & 0 & a_1(r, \rho_2, \rho_1) I_{N-1} & 0 \\
    0 & a_4(r, \rho_1, \rho_2) & 0 & a_2(r, \rho_2, \rho_1)
\end{pmatrix},
\]
where \( I_{N-1} \) is the \( N - 1 \times N - 1 \) identity matrix.

For any random vector \( V \) let \( \mathbb{E} V \) denote the corresponding vector of expectations. From Lemma [A.1] denoting by \( e_i \) the \( 1 \times (2N - 2) \) vector with the \( i^{th} \) entry equal to 1 and all others equal to 0, we obtain that
\[
\frac{1}{\sqrt{N}} \mathbb{E} \left\{ H_N (\rho_1 \hat{n}) \cdot \left( \nabla_{sp} H_N (\rho_1 \hat{n}); \nabla_{sp} H_N (\rho_2 \sigma (r)) \right) \right\} = -\rho_1 \nu'(\rho_1 \rho_2 r)(1 - r^2)^{1/2} e_{2N-2},
\]
\[
\frac{1}{\sqrt{N}} \mathbb{E} \left\{ H_N (\rho_2 \sigma (r)) \cdot \left( \nabla_{sp} H_N (\rho_1 \hat{n}); \nabla_{sp} H_N (\rho_2 \sigma (r)) \right) \right\} = \rho_2 \nu'(\rho_1 \rho_2 r)(1 - r^2)^{1/2} e_{N-1},
\]
\[
\mathbb{E} \left\{ \frac{d}{dR} H_N (\rho_1 \hat{n}) \cdot \left( \nabla_{sp} H_N (\rho_1 \hat{n}); \nabla_{sp} H_N (\rho_2 \sigma (r)) \right) \right\} = - (\rho_1 \rho_2 r) \nu''(\rho_1 \rho_2 r) + \nu'(\rho_1 \rho_2 r)(1 - r^2)^{1/2} e_{2N-2},
\]
\[
\mathbb{E} \left\{ \frac{d}{dR} H_N(\rho_2 \sigma(r)) \cdot (\nabla_{sp} H_N(\rho_1 \hat{n}); \nabla_{sp} H_N(\rho_2 \sigma(r))) \right\} \\
= (\rho_1 \rho_2 r \nu'(\rho_1 \rho_2 r) + \nu'(\rho_1 \rho_2 r))(1 - r^2)^{1/2} e_{N-1}.
\]

and
\[
\sqrt{N} \mathbb{E} \left\{ E_{i} E_{j} H_N(\rho_1 \hat{n}) \cdot (\nabla_{sp} H_N(\rho_1 \hat{n}); \nabla_{sp} H_N(\rho_2 \sigma(r))) \right\} \\
= \begin{cases} 
0, & \lfloor i, j, N - 1 \rfloor \leq 3, \\
(\rho_2 \nu'(\rho_1 \rho_2 r) + \rho_1^{-1} \nu'(\rho_1 \rho_2 r))(1 - r^2)^{1/2} e_{2N-2}, & i = j \neq N - 1, \\
\rho_2 \nu'(\rho_1 \rho_2 r)(1 - r^2)^{1/2} e_{N-1+i}, & i \neq j = N - 1, \\
\rho_1 \rho_2 \nu'(\rho_1 \rho_2 r)(1 - r^2)^{1/2} e_{N-1+j}, & j \neq i = N - 1, \\
[-\rho_1 \rho_2^2 \nu''(\rho_1 \rho_2 r)(1 - r^2) + 3 \rho_2 \nu''(\rho_1 \rho_2 r) \\
+ \rho_1^{-1} \nu'(\rho_1 \rho_2 r)](1 - r^2)^{1/2} e_{2N-2}, & i = j = N - 1.
\end{cases}
\]

Denoting by Cov\{X, Y\} the covariance of two random variables X and Y conditional on \nabla_{sp} H_N(\rho_1 \hat{n}) and \nabla_{sp} H_N(\rho_2 \sigma(r)) (and the covariance with no conditioning by Cov\{X, Y\}), we have (cf. \cite{1} pp. 10–11)
\[
\text{Cov}\{X, Y\} = \text{Cov}\{X, Y\} - (\mathbb{E}\{X \cdot V_Y\})^T \mathbf{A}_0(r) \mathbb{E}\{Y \cdot V_Y\},
\]
where
\[
V_Y = (\nabla_{sp} H_N(\rho_1 \hat{n}); \nabla_{sp} H_N(\rho_2 \sigma(r))).
\]

Using this formula and the above by straightforward algebra, one finds that conditional on \nabla_{sp} H_N(\rho_1 \hat{n}) = \nabla_{sp} H_N(\rho_2 \sigma(r)) = 0, the random vector (4.8) is a centered Gaussian vector with covariance matrix \Sigma_{U,X}(r, \rho_1, \rho_2), completing the proof of Lemma \cite{4,2}. Moreover, defining the random matrices
\[
\mathbf{K}^{(1)} = \frac{1}{\sqrt{\nu'(\rho_1^2)}} \left( \nabla_{sp}^2 H_N(\rho_1 \hat{n}) + \frac{1}{\sqrt{N} \rho_1} \frac{d}{dR} H_N(\rho_1 \hat{n}) \mathbf{I} \right),
\]
\[
\mathbf{K}^{(2)} = \frac{1}{\sqrt{\nu'(\rho_2^2)}} \left( \nabla_{sp}^2 H_N(\rho_2 \sigma(r)) + \frac{1}{\sqrt{N} \rho_2} \frac{d}{dR} H_N(\rho_2 \sigma(r)) \mathbf{I} \right).
\]
we have that for \( l = 1, 2 \) the vectors

\[
\begin{align*}
&\text{Cov}\left\{ K_{ij}^{(l)}, H_N(\rho_1 \hat{n}) \right\} \\
&\text{Cov}\left\{ K_{ij}^{(l)}, H_N(\rho_2 \sigma(r)) \right\} \\
&\text{Cov}\left\{ K_{ij}^{(l)}, \sqrt{N} \frac{d}{dR} H_N(\rho_1 \hat{n}) \right\} \\
&\text{Cov}\left\{ K_{ij}^{(l)}, \sqrt{N} \frac{d}{dR} H_N(\rho_2 \sigma(r)) \right\}
\end{align*}
\]

(A.6)

are equal to \( \delta_{i-j-N-1}(1 - r^2) \xi_j(r, \rho_1, \rho_2) \), respectively, and that

\[
N \text{Cov}\left\{ K_{ij}^{(1)}, K_{kl}^{(1)} \right\} = \begin{cases} 
2 - \delta_{i=N-1} \frac{1}{\sqrt{\rho_1^2}} (1 - r^2) \\
\times a_2(r, \rho_2, \rho_1)(u_2(r, \rho_1, \rho_2))^2, & i = j = k = l, \\
1, & N - 1 \notin \{i, j\} = \{k, l\}, i \neq j, \\
\Sigma_{Z,11}(r, \rho_1, \rho_2), & N - 1 \in \{i, j\} = \{k, l\}, i \neq j, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
N \text{Cov}\left\{ K_{ij}^{(2)}, K_{kl}^{(2)} \right\} = \begin{cases} 
2 - \delta_{i=N-1} \frac{1}{\sqrt{\rho_2^2}} (1 - r^2) \\
\times a_2(r, \rho_1, \rho_2)(u_2(r, \rho_2, \rho_1))^2, & i = j = k = l, \\
1, & N - 1 \notin \{i, j\} = \{k, l\}, i \neq j, \\
\Sigma_{Z,22}(r, \rho_1, \rho_2), & N - 1 \in \{i, j\} = \{k, l\}, i \neq j, \\
0, & \text{otherwise,}
\end{cases}
\]

(A.7)

\[
N \text{Cov}\left\{ K_{ij}^{(1)}, K_{ij}^{(2)} \right\} = \begin{cases} 
0, & i \neq j, \\
2 \frac{\sqrt{\rho_1 \rho_2}}{\sqrt{\rho_1^2} \sqrt{\rho_2^2}}, & i = j \neq N - 1,
\end{cases}
\]

\[
= \frac{1}{\sqrt{\rho_1^2} \sqrt{\rho_2^2}} \rho_1 \rho_2 u^\prime(\rho_1 \rho_2 r)(1 - r^2)^2 \\
- 2 \rho_1 \rho_2 u^\prime(\rho_1 \rho_2 r)(1 - r^2) r + 2 r^2 u(\rho_1 \rho_2 r) \\
+ u_2(r, \rho_1, \rho_2) u_2(r, \rho_2, \rho_1) a_4(r, \rho_1, \rho_2)(1 - r^2)^2, & i = j = N - 1,
\]

\[
N \text{Cov}\left\{ K_{ij}^{(1)}, K_{ij}^{(2)} \right\} = \begin{cases} 
\frac{\sqrt{\rho_1 \rho_2}}{\sqrt{\rho_1^2} \sqrt{\rho_2^2}}, & |i, j, N - 1| = 3, \\
\Sigma_{Z,12}(r, \rho_1, \rho_2), & |i, j, N - 1| = 2, i \neq j,
\end{cases}
\]

\[
N \text{Cov}\left\{ K_{ij}^{(1)}, K_{kl}^{(2)} \right\} = 0 \quad \text{if } \exists m: \sum_{i \in \{i, j, k, l\}} 1 |t = m| = 1.
\]

Let \( \text{Cov}_{H_N, \frac{d}{dR}} \{ X, Y \} \) and \( \mathbb{E}_{H_N, \frac{d}{dR}} \{ X \} \) denote the covariance of two random variables \( X \) and \( Y \) and the expectation of \( X \), respectively, conditional on

\[
\nabla_{\sigma} H_N(\sigma) = 0, \quad H_N(\sigma), \quad \frac{d}{dR} H_N(\sigma) \quad \sigma \in \{ \rho_1 \hat{n}, \rho_2 \sigma(r) \}.
\]

(A.8)
Since the covariances (A.6) are nonzero only if \( i = j = N - 1 \), unless \( i = j = k = l = N - 1 \), for any \( \kappa, \kappa' \in \{1, 2\} \),
\[
\mathbb{E}_{H_N, \frac{d}{dR}} \nabla \{K^{(\kappa)}_{ij}\} = 0, \\
\text{Cov}_{H_N, \frac{d}{dR}} \nabla \{K^{(\kappa)}_{ij}, K^{(\kappa')}_{kl}\} = \text{Cov} \{K^{(\kappa)}_{ij}, K^{(\kappa')}_{kl}\}.
\]
Lastly, for \( \kappa, \kappa' \in \{1, 2\} \),
\[
N \text{Cov}_{H_N, \frac{d}{dR}} \nabla \{K^{(\kappa)}_{N-1,N-1}, K^{(\kappa')}_{N-1,N-1}\} = \Sigma_{\kappa\kappa'}(r, \rho_1, \rho_2)
\]
\[
= N \text{Cov} \{K^{(\kappa)}_{N-1,N-1}, K^{(\kappa')}_{N-1,N-1}\}
- (1 - r^2)^2 \left( \xi_{\kappa}(r, \rho_1, \rho_2) \right) \left( \right)^{-1} \Sigma_{U,X}^{-1}(r, \rho_1, \rho_2) \xi_{\kappa}(r, \rho_1, \rho_2),
\]
\[
\mathbb{E}_{H_N, \frac{d}{dR}} \nabla \{K^{(\kappa)}_{N-1,N-1}\}
= (1 - r^2) \left( \frac{1}{\sqrt{N}} V(r, \rho_1, \rho_2) \Sigma_{U,X}^{-1}(r, \rho_1, \rho_2) \xi_{\kappa}(r, \rho_1, \rho_2),
\]
where \( V = V(r, \rho_1, \rho_2) \) is given by
\[
V = \left( \frac{1}{\sqrt{N}} H_N(\rho_1 \hat{u}), \frac{1}{\sqrt{N}} H_N(\rho_2 \sigma(r)), \frac{d}{dR} H_N(\rho_1 \hat{u}), \frac{d}{dR} H_N(\rho_2 \sigma(r)) \right). \quad \square
\]

Appendix B  The Kac-Rice formula

Our analysis uses the two auxiliary lemmas below, based on a variant of the Kac-Rice formula \cite[theorem 12.1.1]{1}. In the notation of \cite[theorem 12.1.1]{1}, we are interested in situations where, with some random function \( g_N(\sigma) \),
\[
M = S^{N-1}(\rho \sqrt{N}), \\
f(\sigma) = \nabla_{sp} H_N(\sigma), \\
u = 0 \in \mathbb{R}^{N-1}, \\
h(\sigma) = \left( H_N(\sigma), \frac{d}{dR} H_N(\sigma), g_N(\sigma) \right).
\]

**Lemma B.1.** Let \( B \), \( D \), and \( L \) be some intervals and let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a continuous function. Assume that \( H_N(\sigma) \) is a mixed model such that
\[
(B.1) \quad \lim_{N \to \infty} \sup_{u \in B, x \in D} \left\{ \frac{1}{N} \log \left( \mathbb{P}_{\rho N, \sqrt{N}}^\rho \left\{ g_N(\hat{u}) \in NL \right\} \right) - \varphi(u, x) \right\} \leq 0.
\]
Then
\[
(B.2) \quad \lim_{N \to \infty} \frac{1}{N} \log(\mathbb{E}[g_{N,\rho}(NB, \sqrt{N}D) : g_N(\sigma_0) \in NL])
\]
\[
\leq \sup_{u \in B, x \in D} \{ \Theta_{\varphi, \rho}(u, x) + \varphi(u, x) \}.
\]

**Lemma B.2.** Let \( B \) and \( D \) be some intervals and \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a continuous function. Suppose that \( H_N(\sigma) \) is a mixed model such that, with \( Z_{N,\rho}(\sigma_0) \) defined
by (2.4),

\[
\limsup_{N \to \infty} \sup_{u \in B, x \in D} \left\{ \frac{1}{N} \log \left( \mathbb{E}^p_{N u, \sqrt{N} x} \left\{ Z_{N, \beta(\rho \hat{n})} \right\} \right) - \varphi(u, x) \right\} \leq 0.
\]

Then

\[
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \sum_{\sigma \in \mathbb{E}_{\mathfrak{N}, p}(NB, \sqrt{N} D)} Z_{N, \beta(\sigma_0)} \right\} \right)
\leq \sup_{u \in B, x \in D} \left\{ \Theta_{v, \rho}(u) + \varphi(u, x) \right\}.
\]

Lemmas B.1 and B.2 follow from a similar argument for the proof of [53, lemmas 14, 16], which dealt with the pure case with one exception. After applying the Kac-Rice formula, instead of integrating only over the variable \( H_N \), one has to integrate over both \( H_N \) and \( dR \), exactly as we have done in the proof of Theorem 3.1.

### Appendix C Lipschitz Estimates

The following lemma is based on the proof of [13, lemma 2.2].

**Lemma C.1 (Lipschitz continuity of derivatives).** For any \( p \geq 2 \), the pure \( p \)-spin model \( H_{N, p}(x) \) satisfies

\[
\mathbb{P} \left\{ \exists 1 \leq k \leq p - 1, x, x' \in B_N : \right.
\]

\[
\begin{align*}
\frac{1}{k!} \left\| \nabla^k_E H_{N, p}(x) - \nabla^k_E H_{N, p}(x') \right\|_\infty \\
\geq 2 K p^{3/2} \binom{p - 1}{k} \| x - x' \| \frac{N^{k/2}}{N^{k/2}} \leq e^{-K^2 p N/2}
\end{align*}
\]

and

\[
\mathbb{P} \left\{ \exists 1 \leq k \leq p : \sup_{x \in B_N} \frac{1}{k!} \| \nabla^k_E H_{N, p}(x) \|_\infty \right. \\
\begin{align*}
\geq 2 K p^{1/2} \binom{p}{k} N^{-(k-1)/2} \left\| x \right\| \leq e^{-K^2 p N/2},
\end{align*}
\]

where \( B_N = \{ x \in \mathbb{R}^N : \| x \| \leq \sqrt{N} \} \), \( K > 0 \) is a universal constant. For \( k = p \), \( \nabla^k_E H_{N, p}(x) \) is constant in \( x \in B_N \), and for \( k > p \), \( \nabla^k_E H_{N, p}(x) = 0 \), almost surely.
PROOF. Throughout the proof we will use \( \{i_1, \ldots, i_p\} \) to denote multisets of indices. For any such multiset \( A \), define \( J_A^{(p)} \) as the average of all \( J_{i_1, \ldots, i_p}^{(p)} \) with \( \{i_1, \ldots, i_p\} = A \). We can write the pure \( p \)-spin Hamiltonian and its derivatives as

\[
H_{N,p}(\mathbf{x}) = N^{-(p-1)/2} \sum_{i_1, \ldots, i_p=1}^{N} J_{i_1, \ldots, i_p}^{(p)} x_{i_1} \cdots x_{i_p},
\]

\[
(C.3) \quad \frac{d}{dx_{j_1}} \cdots \frac{d}{dx_{j_k}} H_{N,p}(\mathbf{x}) = N^{-(p-1)/2} \frac{p!}{(p-k)!} \sum_{i_1, \ldots, i_{p-k}=1}^{N} J_{i_1, \ldots, i_{p-k}, j_1, \ldots, j_k}^{(p)} x_{i_1} \cdots x_{i_{p-k}},
\]

where \( k \leq p \), as otherwise the derivatives are 0. For any points \( \mathbf{x}, \mathbf{x'} \in \mathcal{B}_N \) and \( y^{(1)}, \ldots, y^{(k)}, k - 1 \leq p \), with \( \|y^{(j)}\| \),

\[
\frac{1}{\sqrt{N}} \sum_{j_1, \ldots, j_k=1}^{N} \left( \frac{d}{dx_{j_1}} \cdots \frac{d}{dx_{j_k}} H_{N,p}(\mathbf{x}) - \frac{d}{dx_{j_1}} \cdots \frac{d}{dx_{j_k}} H_{N,p}(\mathbf{x'}) \right) 
\times y_{j_1}^{(1)} \cdots y_{j_k}^{(k)}
\]

\[
(C.4) = N^{-p/2} \frac{p!}{(p-k)!} \times \sum_{j_1, \ldots, j_k=1}^{N} \sum_{i_1, \ldots, i_{p-k}=1}^{N} J_{i_1, \ldots, i_{p-k}, j_1, \ldots, j_k}^{(p)} 
\times (x_{i_1} \cdots x_{i_{p-k}} - x'_{i_1} \cdots x'_{i_{p-k}}) y_{j_1}^{(1)} \cdots y_{j_k}^{(k)}
\]

\[
= \sum_{l=1}^{p-k} N^{-p/2} \frac{p!}{(p-k)!} \times \sum_{j_1, \ldots, j_k=1}^{N} \sum_{i_1, \ldots, i_{p-k}=1}^{N} J_{i_1, \ldots, i_{p-k}, j_1, \ldots, j_k}^{(p)} 
\times x_{i_1} \cdots x_{i_{l-1}} (x_{i_l} - x'_{i_l}) x'_{i_{l+1}} \cdots x'_{i_{p-k}} y_{j_1}^{(1)} \cdots y_{j_k}^{(k)}
\]

\[
\leq N^{-(p-1)/2} \frac{p!}{(p-k-1)!} \|J_{N,p}\|_{\infty} (\max\{\|x\|, \|x'\|\})^{p-k-1} \|x - x'\|,
\]
where the second equality follows by writing \( x_{i_1} \cdots x_{i_{p-1}} - x_{i_1}^{(p)} \cdots x_{i_{p-1}}^{(p)} \) as a telescopic sum and
\[
J_{N,p} \coloneqq \left( J_{i_1, \ldots, i_p}^{(p)} \right)_{i_1, \ldots, i_p \leq N}.
\]
Note that for \( k = p \), the derivative (C.3) is independent of \( x \), and the difference of derivatives (C.4) is 0.

By [60, theorem 2.1], if \( T \) is a symmetric tensor, then the supremum in \((7.17)\) is obtained with \( y^{(1)} = \cdots = y^{(p)} \). Hence, since \( J_{N,p} \) is symmetric by definition,
\[
\|J_{N,p}\|_{\infty} = \frac{1}{N} \sup_{|y| = \sqrt{N}} |H_{N,p}(y)|.
\]

Using Dudley’s entropy bound \([1, \text{theorem 1.3.3}]\) it is standard to show that

\[
\mathbb{E} \|J_{N,p}\|_{\infty} \leq K \sqrt{p},
\]

where \( K > 0 \) is a universal constant. This is a consequence, for example, of lemma 19 of \([55]\) and a bound on the canonical metric of \( H_{N,p}(\sigma) \), which can be proved similarly to lemma 20 of \([55]\). The event in (C.1) is contained in \( \{\|J_{N,p}\|_{\infty} \geq 2K \sqrt{p}\} \). Thus, by the Borell-TIS inequality \([17, 24]\) (see also \([1, \text{theorem 2.1.1}]\)), its probability is bounded from above by

\[
\mathbb{P} \{\|J_{N,p}\|_{\infty} \geq 2K \sqrt{p}\} \leq e^{-K^2 pN/2},
\]

as required.

To prove (C.2), note that similarly to (C.4), for \( k \leq p \),
\[
\frac{1}{\sqrt{N}} \sum_{j_1, \ldots, j_k = 1}^{N} \frac{d}{dx_{j_1}} \cdots \frac{d}{dx_{j_k}} H_{N,p}(x) y^{(1)}_{j_1} \cdots y^{(k)}_{j_k}
\]
\[
= N^{-p/2} \frac{p!}{(p-k)!} \sum_{j_1, \ldots, j_k = 1}^{N} \sum_{i_1, \ldots, i_{p-k-1} = 1}^{N} J_{i_1, \ldots, i_{p-k}, j_1, \ldots, j_k}^{(p)}
\]
\[
\times x_{i_1} \cdots x_{i_{p-k}} y^{(1)}_{j_1} \cdots y^{(k)}_{j_k}
\]
\[
\leq N^{-(p-1)/2} \frac{p!}{(p-k)!} \|J_{N,p}\|_{\infty} \|x\|^{p-k}.
\]

\[\begin{aligned}
\text{There, a bound is computed for the canonical (pseudo) metric of a modified Hamiltonian } \\
\tilde{H}_{N,p}(T(x)), \text{ where } \\
T(x_1, \ldots, x_N) = \sqrt{N} \left( x_1, \ldots, x_{N-1}, \sqrt{1 - \sum x_i^2} \right). \\
\text{Here a similar, but simpler, bound is needed for the metric } d(x, y) = \left( \mathbb{E} \left( H_{N,p}(T(x)) - H_{N,p}(T(y)) \right)^2 \right)^{1/2}. \text{ Following the general argument of lemma 20 of } \ [55] \text{ and using the estimate the Taylor expansion between (7.26) and (7.27) of the same paper, it is straightforward to show that } d^2(x, y) \leq NpC |x - y| \text{ for some constant } C \text{ independent of } p.
\end{aligned}\]
and proceed as before. \[\square\]

**Corollary C.2.** For any mixed model \(H_N(x)\), assuming that \(\lim p^{-1} \log \gamma_p < 0\), there exists a constant \(c > 0\) such that, with \(C_k = 2K \sum_{p \geq k+1} \gamma_p p^{3/2} (p^{-1})\) and \(\tilde{C}_k = 2K \sum_{p \geq k} \gamma_p p^{1/2} (p^{-k})\) where \(K\) is as in Lemma C.1,

\[
\mathbb{P} \left\{ \exists k \geq 1, x, x' \in B_N : \frac{1}{k!} \| \nabla^{k} \mathcal{H}_N(x) - \nabla^{k} \mathcal{H}_N(x') \|_\infty \geq C_k \frac{\| x - x' \|}{N^{k/2}} \right\} \leq e^{-cN},
\]

and

\[
\mathbb{P} \left\{ \exists k \geq 1 : \frac{1}{k!} \sup_{x \in B_N} \| \nabla^{k} \mathcal{H}_N(x) \|_\infty \geq N^{-(k-1)/2} \tilde{C}_k \right\} \leq e^{-cN}.
\]

**Proof.** The bound of (C.7) (resp., (C.8)) follows by a union bound from (C.1) (resp., (C.2)), since

\[
\| \nabla^{k} \mathcal{H}_N(x) \|_\infty \leq \sum_{p \geq 2} \gamma_p \| \nabla^{k} \mathcal{H}, p(x) \|_\infty
\]

and, for large \(N\), \(\sum_{p \geq 2} e^{-K^2 p N/2} \leq e^{-K^2 N/2} =: e^{-cN}. \square\)

**Lemma C.3.** For any finite mixture \(v(x) = \sum_{p=0}^{p_0} \gamma_p x^p\) \(F_{N, \beta} = \frac{1}{N} \log Z_{N, \beta}\) is a Lipschitz function of the Gaussian disorder coefficients \(J_{i_1, \ldots, i_p}\) with Lipschitz constant \(\beta \sqrt{v(1)/N}\).

**Proof.** Write

\[
H_N(J, \sigma) = \sum_{p=0}^{p_0} \gamma_p N^{-\frac{p-1}{2}} \sum_{i_1, \ldots, i_p} J_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p},
\]

where \(J = (J_{j_1, \ldots, j_p})\) is the array of all the disorder coefficients. For any \(i_1, \ldots, i_k\), set

\[
D_{i_1, \ldots, i_k} := \frac{d}{d x_{i_1, \ldots, i_p}} \log \left( \int_{S^{N-1}((\sqrt{N})} \exp\{-\beta H_N(J, \sigma)\} d\sigma \right)
\]

\[
= - \beta \gamma_p N^{-\frac{p-1}{2}} \frac{\int_{S^{N-1}((\sqrt{N})} \sigma_{i_1} \cdots \sigma_{i_p} \exp\{-\beta H_N(J, \sigma)\} d\sigma}{\int_{S^{N-1}((\sqrt{N})} \exp\{-\beta H_N(J, \sigma)\} d\sigma}.
\]

The ratio of integrals in the last equation can be viewed as an expectation under the Gibbs measure. Denote expectation by this measure by \(\langle \cdot \rangle\), so that the ratio is
simply $|\sigma_1 \cdots \sigma_p\rangle$. We then have

\begin{equation}
(C.9) \sum_{p=2}^{p_0} \sum_{i_1,\ldots,i_p=1}^{N} (D_{i_1,\ldots,i_p})^2 = \sum_{p=2}^{p_0} \beta^2 \gamma_p^2 N^{-(p-1)} \sum_{i_1,\ldots,i_p=1}^{N} |\sigma_1 \cdots \sigma_p\rangle^2 \leq \sum_{p=2}^{p_0} \beta^2 \gamma_p^2 N^{-(p-1)} \sum_{i_1,\ldots,i_p=1}^{N} \langle(\sigma_1 \cdots \sigma_p)^2\rangle.
\end{equation}

Note that

$$\sum_{i_1,\ldots,i_p=1}^{N} \langle(\sigma_1 \cdots \sigma_p)^2\rangle = \left\|\sigma\right\|_2^{2p} = N^p.$$ 

Therefore, $\frac{1}{N} \log Z_{N,\beta}$ has Lipschitz constant bounded from above by

$$\frac{1}{N} \left( \sum_{p=2}^{\infty} \beta^2 \gamma_p^2 N^{1/2} \right)^{1/2} = \beta \sqrt{\frac{\nu(1)}{N}}.$$ 

\[\square\]

**Corollary C.4.** For any mixture $v(x) = \sum_{p=2}^{\infty} \gamma_p^2 x^p$ where $\lim p^{-1} \log \gamma_p < 0$ and any $t > 0$, the free energy $F_{N,\beta} = \frac{1}{N} \log Z_{N,\beta}$ satisfies

\begin{equation}
(C.10) \mathbb{P}\{ |F_{N,\beta} - \mathbb{E}F_{N,\beta}| > t \} \leq 3 \exp\left\{-Nt^2/2\beta^2 \nu(1)\right\}.
\end{equation}

**Proof.** For finite mixtures, i.e., such that $\gamma_p = 0$ for all $p \geq p_0$, the corollary follows from Lemma C.3 and standard concentration results (see, e.g., [2, lemma 2.3.3]) with prefactor 2 instead of 3. For the infinite case we will truncate the mixture.

Let $F_{N,\beta}^{(p)}$ be the partition function corresponding to the truncated Hamiltonian

$$H_{N,\beta}^{(p)}(\sigma) = \sum_{k=2}^p \gamma_k N^{-\frac{k-1}{2}} \sum_{i_1,\ldots,i_k=1}^{N} J_{i_1,\ldots,i_k} \sigma_{i_1} \cdots \sigma_{i_k},$$

which we assume to be defined by using the same disorder variables $J_{i_1,\ldots,i_k}$ as $H_N(\sigma)$, defined by the corresponding infinite sum. For any $p \geq 2$ and $\epsilon > 0$, the left-hand side of (C.10) is bounded from above by

\begin{equation}
(C.11) \mathbb{P}\{ |F_{N,\beta}^{(p)} - \mathbb{E}F_{N,\beta}^{(p)}| > (t - 2\epsilon) \} + \mathbb{P}\{ |F_{N,\beta} - F_{N,\beta}^{(p)}| > \epsilon \} + \mathbb{P}\{ |\mathbb{E}F_{N,\beta}^{(p)} - \mathbb{E}F_{N,\beta}| > \epsilon \}.
\end{equation}

For any $\delta$, for large enough $p$, from (C.5),

$$\mathbb{E} \sup_{\sigma} |H_{N,\beta}^{(p)}(\sigma) - H_{N,\beta}(\sigma)| < N \delta.$$
Hence, for fixed $\epsilon$, for large enough $p$, the last two summands above are smaller than the right-hand side of (C.10), with prefactor 1. Thus, from the finite case we have that

$$\mathbb{P} \{|F_{N,\beta} - \mathbb{E} F_{N,\beta}| > t\} \leq 3 \exp\{-N(t - 2\epsilon)^2 / 2\beta^2 v(1)\}.$$ 

By taking $\epsilon \to 0$, we obtain (C.10).

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