On perturbations of generalized Landau-Lifshitz dynamics

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Abstract

We consider deterministic and stochastic perturbations of dynamical systems with conservation laws in $\mathbb{R}^3$. The Landau-Lifshitz equation for the magnetization dynamics in ferromagnetics is a special case of our system. The averaging principle is a natural tool in such problems. But bifurcations in the set of invariant measures lead to essential modification in classical averaging. The limiting slow motion in this case, in general, is a stochastic process even if pure deterministic perturbations of a deterministic system are considered. The stochasticity is a result of instabilities in the non-perturbed system as well as of existence of ergodic sets of a positive measure. We effectively describe the limiting slow motion.

Keywords: Magnetization dynamics, Landau-Lifshitz equation, averaging principle, stochasticity in deterministic systems.

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1 Introduction

The analytical study of magnetization dynamics governed by the Landau-Lifshitz equation (see [18]) has been the focus of considerable research for many years. In normalized form this equation reads as (see [5], equations (2.51) and (2.53)):

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}), \quad \mathbf{m}(r,0) = \mathbf{m}_0(r) \in \mathbb{R}^3, \quad |\mathbf{m}_0(r)| = 1.$$  \(1.1\)

Here $\mathbf{h}_{\text{eff}}$ is an effective field. The three-dimensional vector $\mathbf{m}(r,t)$ is the magnetization of the material at a fixed point $r \in \mathbb{R}^3$ at time $t$; The term $\alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}})$ is the Landau-Lifshitz damping term, $0 < \alpha << 1$. One can check that (1.1) preserves a first integral $F(\mathbf{m}) = \frac{1}{2}|\mathbf{m}|^2$. Therefore for fixed $r$, the system (1.1) describes a motion on the sphere in $\mathbb{R}^3$.

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One can introduce an energy density function $G$ such that $\nabla G = -h_{\text{eff}}$. Then equation (1.1) can be written as follows:

$$\frac{d\mathbf{m}}{dt} = \mathbf{m} \times \nabla G + \alpha \mathbf{m} \times (\mathbf{m} \times \nabla G), \quad \mathbf{m}(0) = \mathbf{m}_0 \in \mathbb{R}^3, \quad |\mathbf{m}_0| = 1.$$  

(1.2)

We assume that $G$ is a smooth generic function. Considered on the unit sphere $S^2$ in $\mathbb{R}^3$, such a function may have three types of critical points: maxima, minima and saddle points. Without the damping $\alpha \mathbf{m} \times (\mathbf{m} \times \nabla G)$ the energy density $G$ is preserved. One easily checks that $\nabla G \cdot (\alpha \mathbf{m} \times (\mathbf{m} \times \nabla G)) = -\alpha |\mathbf{m} \times \nabla G|^2$ so that the damping term is a kind of ”friction” for the system (1.2), just like the classical friction in Hamiltonian systems (compare with [4]).

If $0 < \alpha << 1$, the dynamics of (1.2) has two distinct time scales: the fast time scale of the precessional dynamics and the relatively slow time scale of relaxational dynamics caused by the small damping term $\alpha \mathbf{m} \times (\mathbf{m} \times \nabla G)$. Therefore it is natural to use the averaging principle to describe the long-time evolution of energy density $G$. However the classical averaging principle here should be modified: existence of saddle points of $G(m)$ on the sphere $\{|m| = 1\}$ leads to stochastic, in a certain sense, behavior of the slow motion even in the case of purely deterministic damping term (compare with [4]). Moreover, in Section 5, we consider a more general class of equations, where level set components of first integrals, which are compact two-dimensional surfaces may have topological structure different from a sphere. If genus of such a surface is positive, the non-perturbed system can have positive area ergodic sets. Existence of such sets lead to an ”additional stochasticity”. Description of the stochastic process which characterizes the long-time evolution of the energy is one of the main goals of this paper.

Random perturbation caused by thermal fluctuations become increasingly pronounced in nano-scale devices. To take this into account one can include in the right-hand side of (1.2) a small stochastic term. This stochastic term, in general, introduces one more time scale in the system. Interplay between the influence of small damping and even smaller stochastic term leads to certain changes in the metastability of the system. Description of the metastable distributions is another goal of this paper. There are some other asymptotic regimes of the Landau-Lifshitz dynamics which we mention briefly and we will consider them in more details elsewhere.

2 Sketch of the paper

In this section we give an informal sketch of the results.

In the next two sections we consider perturbations of the following equation...
\[ \dot{X}_t = \nabla F(X_t) \times \nabla G(X_t), \quad X_0 = x_0 \in \mathbb{R}^3, \quad (2.1) \]

which could be regarded as a generalized Landau-Lifshitz equation.

Here \( G(x) \) and \( F(x), \ x \in \mathbb{R}^3 \), are smooth enough generic functions (this means that each of these functions has a finite number of critical points which are assumed to be non-degenerate), \( \lim |x| \to \infty F(x) = \infty \). The initial point \( x_0 = x_0(z) \) is chosen in such a way that \( F(x_0(z)) = z \). As before we call \( G(x) \) energy (to be precise, \( G(x) \) in (2.2) is the energy density but for brevity we call it energy).

It is easy to see that \( F(x) \) and \( G(x) \) are first integrals of system (2.1). For instance,

\[ \frac{dF(X_t)}{dt} = \nabla F(X_t) \cdot (\nabla F(X_t) \times \nabla G(X_t)) = 0. \]

Note also that the Lebesgue measure in \( \mathbb{R}^3 \) (the volume) is invariant for system (2.1):

\[ \text{div}(\nabla F(x) \times \nabla G(x)) = \nabla G(x) \cdot (\nabla \times \nabla F(x) - \nabla F(x) \cdot (\nabla \times \nabla G(x)) = 0. \]

This implies, in particular, that \( \frac{1}{|\nabla F(x)|} \) is the density of an invariant measure of system (2.1) considered on the surface \( \tilde{S}(z) = \{ x \in \mathbb{R}^3 : F(x) = z \} \) with respect to the area on \( \tilde{S}(z) \). Notice that the surface \( \tilde{S}(z) \) may have several connected components. For brevity in the next two sections, and in the rest of this section (except the last four paragraph), we assume that the level surface \( \tilde{S}(z) = \{ x \in \mathbb{R}^3 : F(x) = z \} \) has only one connected component and this component is homeomorphic to \( S^2 \). In Sections 5 and 6 we will drop this assumption and consider more general situations.

As we already mentioned, the damping term in (1.2) preserves the first integral \( \frac{1}{2} |m|^2 \), so that we consider, first, perturbations of (2.1) preserving \( F(x) \). The perturbed equation can be written in the form

\[ \dot{X}_t^\varepsilon = \nabla F(X_t^\varepsilon) \times \nabla G(X_t^\varepsilon) + \varepsilon \nabla F(X_t^\varepsilon) \times \tilde{b}(X_t^\varepsilon), \quad X_0^\varepsilon = x_0 \in \mathbb{R}^3. \quad (2.2) \]

Here \( \tilde{b}(\bullet) \) is a smooth vector field in \( \mathbb{R}^3 \). In the next two sections we assume for brevity that the perturbation \( \varepsilon \nabla F \times \tilde{b} \) is of “friction” type:

\[ \nabla G(x) \cdot (\nabla F(x) \times \tilde{b}(x)) < 0, \quad x \in \tilde{S}(z) \subset \mathbb{R}^3. \quad (2.3) \]

Note that any vector field \( \nabla F(x) \times \tilde{b}(x) \) can be written in the form \( \nabla F(x) \times (\nabla F(x) \times b(x)) \) for some vector field \( b(x) \in \mathbb{R}^3 \). Indeed, without loss of generality one can assume that \( \tilde{b}(x) \perp \nabla F \). Each such vector \( \tilde{b}(x) \) can be represented as \( \nabla F(x) \times b(x) \). So that the perturbed equation can be written as
\[ \hat{X}_t^\varepsilon = \nabla F(\hat{X}_t^\varepsilon) \times \nabla G(\hat{X}_t^\varepsilon) + \varepsilon \nabla F(\hat{X}_t^\varepsilon) \times (\nabla F(\hat{X}_t^\varepsilon) \times \mathbf{b}(\hat{X}_t^\varepsilon)) , \hat{X}_0^\varepsilon = x_0 \in \mathbb{R}^3 . \quad (2.4) \]

Furthermore, using the identity \( \mathbf{A} \cdot (\mathbf{B} \times (\mathbf{C} \times \mathbf{D})) = (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) \) we can check that
\[ \nabla G \cdot (\nabla F \times (\nabla F \times \mathbf{b})) = - (\nabla F \times \mathbf{b}) \cdot (\nabla F \times \nabla G) . \quad (2.5) \]

Therefore the "friction-like" condition (2.3) becomes
\[ (\nabla F \times \mathbf{b}) \cdot (\nabla F \times \nabla G) > 0 . \quad (2.6) \]

The equation (1.2) corresponds to the case that \( \mathbf{b}(\hat{X}_t^\varepsilon) = \nabla G(\hat{X}_t^\varepsilon) \) and \( F(\hat{X}_t^\varepsilon) = \frac{1}{2} |\hat{X}_t^\varepsilon|^2 \). One easily checks that system (2.4) preserves \( F \) so that \( \hat{X}_t^\varepsilon \) is moving on a certain level surface \( \{ F = z \} \).

We make some geometric assumptions that are used in Sections 3 and 4. Suppose that the set \( S(z) = \{ x \in \mathbb{R}^3 : G(x) \leq G(x_0(z)) + 1 \} \cap \{ x \in \mathbb{R}^3 : F(x) = z \} \) is a 2-dimensional Riemannian manifold which is \( C^\infty \)-diffeomorphic to \( R = \{ (a,b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1 \} \). Let the \( C^\infty \) diffeomorphism be \( f : S(z) \to R \). To be specific, we denote \( f(x_1,x_2,x_3) = (f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3)) \) for \( (x_1,x_2,x_3) \in S(z) \). We assume that the diffeomorphism \( f \) is non-singular for \( (x_1,x_2,x_3) \in S(z) \). We denote by \( d(\cdot,\cdot) \) the metric on \( S(z) \) induced by standard Euclidean metric in \( \mathbb{R}^3 \). Let our function \( G \) on \( S(z) \) have only one saddle point and two minima, and these critical points are non-degenerate. Assume that the level surfaces \( \{ G = g \} \) are transversal to the level surface \( \{ F = z \} \): \( \nabla F(x) \) and \( \nabla G(x) \) are not parallel. We denote by \( C(g,z) \) the set \( \{ G(x) = g \} \cap \{ F(x) = z \} \). Without loss of generality we can assume that \( C(0,0) = \{ G = 0 \} \cap \{ F = z \} \) is the \( \infty \)-shaped curve (homoclinic trajectory) on \( \{ F = z \} \) corresponding to the saddle point of \( G \). Let the saddle point of \( G \) on \( \{ F = z \} \) be \( O_2(z) \) and the two minima be \( O_1(z) \) and \( O_3(z) \). Suppose that as \( z \) varies, the curves \( O_1(z) \), \( O_2(z) \) and \( O_3(z) \) are transversal to \( \{ F = z \} \) (see Fig.1). Notice that when \( g > 0 \), \( C(g,z) \) has only one connected component which we call \( C_2(g,z) \). When \( g < 0 \), \( C(g,z) \) has two connected components \( C_1(g,z) \) and \( C_3(g,z) \) bounding domains on \( S(z) \) containing \( O_1(z) \) and \( O_3(z) \) respectively. Let \( C_1(0,0) \) and \( C_3(0,0) \) be the parts of the homoclinic trajectory \( C(0,z) \) bounding domains containing \( O_1(z) \) and \( O_3(z) \) respectively. Let \( C_2(0,0) = C(0,0) \). Let \( D_i(g,z) \) \( (i = 1,2,3) \) be the region bounded by \( C_i(g,z) \).

In the next two sections when we speak about a stochastic process or a motion on the surface \( S(z) \), for example \( X_t^\varepsilon \), \( X_t^{c,\delta} \) etc. , we are assuming that they are stopped once they hit \( \partial S(z) \).

To study equation (2.3), we make a time change \( t \mapsto \frac{t}{\varepsilon} \). Let \( X_t^\varepsilon = \hat{X}_{t/\varepsilon}^\varepsilon \). We get from (2.2) that
\[ \dot{X}_t^\varepsilon = \frac{1}{\varepsilon} \nabla F(X_t^\varepsilon) \times \nabla G(X_t^\varepsilon) + \nabla F(X_t^\varepsilon) \times (\nabla F(X_t^\varepsilon) \times b(X_t^\varepsilon)), X_0^\varepsilon = x_0 \in \mathbb{R}^3, 0 < \varepsilon << 1. \] (2.7)

Therefore the fast motion is defined by the vector field \( \frac{1}{\varepsilon} \nabla F \times \nabla G \), and the slow motion is due to \( \nabla F \times (\nabla F \times b) \) (we will sometimes ignore the arguments since they could be directly understood from the context). In order to study the limiting behavior of the process \( X_t^\varepsilon \), we introduce a graph \( \Gamma \) (compare with [14, Chapter 8]). The graph \( \Gamma \) is constructed in the following way. Let us identify the points of each connected component of the level sets of \( G \) on \( S(z) \). Let the identification mapping be \( \mathcal{Y} \). The set obtained after such an identification, equipped with the natural topology, is a graph \( \Gamma \) with an interior vertex \( O_2(z) \) corresponding to the saddle point \( O_2(z) \) on \( S(z) \) and related homoclinic curve (in the following we will use the same symbol for either the critical point of \( G \) on \( S(z) \) or the corresponding vertex on \( \Gamma \)), and two exterior vertices \( O_1(z) \) and \( O_3(z) \) corresponding to the stable equilibriums \( O_1(z) \) and \( O_2(z) \) on \( S(z) \), together with another exterior vertex \( P \) corresponding to \( \partial S(z) \) (notice that by our definition \( S(z) = \{ x \in \mathbb{R}^3 : G(x) \leq G(x_0(z)) + 1 \} \cap \{ x \in \mathbb{R}^3 : F(x) = z \} \) so that \( \partial S(z) \) is a level curve of \( G \) on \( \{ F = z \} \)). The edges of the graph are defined as follows: edge \( I_2 \) corresponds to trajectories on \( S(z) \) lying outside \( C(0, z) \); edges \( I_1 \) and \( I_3 \) correspond to those trajectories on \( S(z) \) belonging to the wells containing \( O_1(z) \) and \( O_3(z) \), respectively. A point \( \mathcal{Y}(x) = y \in \Gamma \) can be characterized by two coordinates \( (g, k) \) where \( g = G(x) \) is the value of function \( G \) at \( x \in \mathcal{Y}^{-1}(y) \subset S(z) \), and \( k = k(x) \) is the number of the edge of the graph \( \Gamma \) to which \( y = \mathcal{Y}(x) \) belongs. Notice that \( k \) is not chosen in a unique way since for \( y = O_2(z) \) the value of \( k \) can be either 1, 2 or 3. The distance \( \rho(y_1, y_2) \) between two points \( y_1 = (G(x_1), k) \) and \( y_2 = (G(x_2), k) \) is simply \( \rho(y_1, y_2) = |G(x_1) - G(x_2)|. \)
For \( y_1, y_2 \in \Gamma \) belonging to different edges of the graph it is defined as \( \rho(y_1, y_2) = \rho(y_1, O_2(z)) + \rho(O_2(z), y_2) \).

The slow component of \( X^\varepsilon_t \) is the projection of \( X^\varepsilon_t \) on \( \Gamma \): \( Y^\varepsilon_t = \mathcal{Q}(X^\varepsilon_t) \). Using the classical averaging principle one can describe the limiting motion of \( Y^\varepsilon_t \) as \( \varepsilon \downarrow 0 \) inside the edges. But it turns out that the trajectory \( Y^\varepsilon_t \), when hitting the interior vertex \( O_2(z) \) on \( \Gamma \), is very sensitive to \( \varepsilon \). This means that \( Y^\varepsilon_t = \mathcal{Q}(X^\varepsilon_t), G(X^\varepsilon_0) > G(O_2(z)) \), hits \( O_2(z) \) in a finite time \( t_{\varepsilon 0}^\varepsilon \) such that \( \lim_{\varepsilon \downarrow 0} t_{\varepsilon 0}^\varepsilon = t_0 \) exists and finite, and after that alternately as \( \varepsilon \downarrow 0 \) goes to \( I_1 \) or \( I_3 \). The limit of \( Y^\varepsilon_t \) as \( \varepsilon \downarrow 0 \) for \( t > t_0 \) does not exist (compare with \([4]\)). In order to describe the limiting behavior, we have to regularize the problem. To this end one can add a small stochastic perturbation of order \( \delta \) either to the initial condition or to the equation. Let \( X^\varepsilon_{t,\delta} \) be the result of addition of such a perturbation. Then, under certain mild assumptions, the slow component \( \mathcal{Q}(X^\varepsilon_{t,\delta}) \) of \( X^\varepsilon_{t,\delta} \) converges weakly in the space of continuous trajectories on any finite time interval \([0, T]\) to a stochastic process \( Y_t \) on the graph \( \Gamma \) as first \( \varepsilon \downarrow 0 \) and then \( \delta \downarrow 0 \). Since small random perturbations, as a rule, are available in the system, exactly this weak limit characterizes the behavior of \( \tilde{X}^\varepsilon_{t,\varepsilon} \) as \( 0 < \varepsilon << 1 \). We will introduce different types of regularization and prove that all these regularizations lead to the same limiting stochastic process \( Y_t \) on \( \Gamma \), which we calculate.

The proofs, in Section 3 and, partly, in Section 4, are similar to the case of perturbations of Hamiltonian systems ([14, Chapter 8], [4]), and we pay most of the attention to the arguments which are not presented in these works. For instance, in the case of regularization by a random perturbation of the initial point, bounds for the hitting time of the homoclinic trajectory are considered in details.

So far we considered just deterministic perturbations preserving the first integral \( F \). Stochastic perturbations were used just for regularization of the problem. One can consider also white-noise-type perturbations preserving \( F \) of the same or of a larger order than deterministic perturbations. Then, in an appropriate time scale, the limiting slow motion converges to a diffusion process on a graph (Section 4). In general, deterministic and stochastic perturbations have different order, so that, after time rescaling \( t \rightarrow \frac{t}{\varepsilon} \), the perturbed equation has the form

\[
X^\varepsilon_{t,\delta} = \frac{1}{\varepsilon} \nabla F(X^\varepsilon_{t,\delta}) \times \nabla G(X^\varepsilon_{t,\delta}) + \nabla F(X^\varepsilon_{t,\delta}) \times \tilde{b}(X^\varepsilon_{t,\delta}) + \delta \sigma(X^\varepsilon_{t,\delta}) \circ \tilde{W}_t, \quad X^\varepsilon_{0,\delta} = x_0. \tag{2.8}
\]

Here \( \tilde{W}_t \) is the standard Gaussian white noise, \( \sigma(x) \) is a smooth matrix-function such that \( \sigma^T \nabla F \equiv 0 \). If we denote by \( a(x) = \sigma(x)\sigma^T(x) \) the diffusion matrix, the condition \( \sigma^T(x) \nabla F(x) \equiv 0 \) is equivalent to the assumption that \( a(x) \nabla F(x) \equiv 0 \). The stochastic term in (2.8) is understood in the Stratonovich sense, then \( F(X^\varepsilon_{t,\delta}) \equiv F(x_0) \) with probability 1. We assume that the matrix \( a \) is non-degenerate on \( \{ F = z \} \). (We will specify the non-degeneracy in Section 4.)
The process \( X_t^{\varepsilon,\delta} \) defined by (2.8) lives on the surface \( \{ x \in \mathbb{R}^3 : F(x) = z \} \) and has a slow and a fast component as \( \varepsilon << 1 \) and \( \delta > 0 \) fixed. The slow component is again the projection \( \mathcal{P}(X_t^{\varepsilon,\delta}) \) of \( X_t^{\varepsilon,\delta} \) on the graph \( \Gamma \). We consider the case \( 0 < \varepsilon << \delta << 1 \) and assume that the deterministic perturbation is friction-like.

If \( \varepsilon > 0 \) is small enough and \( \delta = 0 \), the system \( X_t^{\varepsilon,0} \), \( X_0^{\varepsilon,0} = x \in S(z) \), has three critical points \( O_1'(z) \), \( O_2'(z) \), \( O_3'(z) \) of the same type as the corresponding points \( O_i(z) \). The distance between corresponding points tends to zero together with \( \varepsilon \). If \( 0 < \delta << 1 \), \( X_t^{\varepsilon,\delta} \), \( X_0^{\varepsilon,\delta} = x \), at a time \( t = T_\delta(\lambda) \), \( \lim_{\delta \downarrow 0} \delta^2 \ln T_\delta(\lambda) = \lambda > 0 \), is situated in a small neighborhood of the metastable state \( M^\varepsilon(x,\lambda) \); \( M^\varepsilon(x,\lambda) \) is one of the stable equilibriums of \( X_t^{\varepsilon,0} \). The function \( M^\varepsilon(x,\lambda) \) is defined by the action functional for the family \( X_t^{\varepsilon,\delta} \) as \( \delta \downarrow 0 \) (see [8], [10], [13], [14], [21]).

But if \( \varepsilon \) tends to zero, the situation is different: \( X_t^{\varepsilon,\delta}, X_0^{\varepsilon,\delta} = x_0 \), converges to a random variable distributed between \( O_1(z) \) and \( O_2(z) \) as \( 0 < \varepsilon << \delta << 1 \). The set of possible distributions between the minima is finite and is independent of the stochastic part of perturbations. But which of these distributions is realized at a time \( T_\delta(\lambda) \) depends on \( \lambda \) and \( x_0 = X_0^{\varepsilon,\delta} \), as well as on stochastic perturbations. We describe these metastable distributions in Section 4.

Perturbations of a more general equation than (2.1) are considered in Section 5. The non-perturbed motion in this case, in general, has just one smooth first integral and the averaging procedure essentially depends on the topological structure of the connected components of level sets of the existing first integral. Each connected component is two dimensional orientable compact manifold. The topology of such a manifold is determined by its genus. We show that if the genus is greater than zero (for instance, when the component is a 2-torus \( \mathbb{T}^2 \)), the limiting slow motion spends an exponentially distributed random time at some vertices.

Perturbations of system (2.1) may have different origin and they may have different order. In the last Section 6, we briefly consider such a situation.

Perturbations of (2.1) breaking both first integrals \( F(x) \) and \( G(x) \) can be considered: (after time change)

\[
X_t^\varepsilon = \frac{1}{\varepsilon} \nabla F(X_t^\varepsilon) \times \nabla G(X_t^\varepsilon) + B(X_t^\varepsilon), \quad X_0^\varepsilon = x_0(z) \in \mathbb{R}^3, \quad 0 < \varepsilon << 1.
\]  

(2.9)

Here \( B(\cdot) \) is a general smooth vector field on \( \mathbb{R}^3 \). Then the perturbed motion is not restricted to the level surface \( \{ F = z \} \). In this case the slow component of the perturbed motion lives on an "open book" \( \sqcap \) homeomorphic to the set of connected components of the level sets \( C(z_1, z_2) = \{ x \in \mathbb{R}^3 : F(x) = z_1, G(x) = z_2 \} \), \( (z_1, z_2) \in \mathbb{R}^2 \) (compare with [16]). The slow component of the motion is equal to \( \mathcal{P}(X_t^\varepsilon) = Y_t^\varepsilon \), where \( \mathcal{P} : \mathbb{R}^3 \to \sqcap \) is
the identification mapping. After an appropriate regularization, \( Y_\varepsilon \) approaches as \( \varepsilon \downarrow 0 \) a stochastic process \( Y_t \). We will consider this question in more details elsewhere.

3 Regularization by perturbation of the initial condition

We study in this section the regularization of system (2.7) by a stochastic perturbation of the initial condition.

Let \( U_\delta(x) = \{ y \in S(z) : d(x, y) < \delta \} \).

Consider the equation:

\[
\dot{X}_t^{\varepsilon,\delta} = \frac{1}{\varepsilon} \nabla F(X_t^{\varepsilon,\delta}) \times \nabla G(X_t^{\varepsilon,\delta}) + \nabla F(X_t^{\varepsilon,\delta}) \times \nabla F(X_t^{\varepsilon,\delta}) \times \nabla F(X_t^{\varepsilon,\delta}) \times b(X_t^{\varepsilon,\delta}) \ , X_0^{\varepsilon,\delta} = x_0(z, \delta) \in \mathbb{R}^3 .
\]

(3.1)

Here \( 0 < \delta \ll 1 \) is a small parameter. The initial position \( x_0(z, \delta) = X_0^{\varepsilon,\delta} \) is a random variable distributed uniformly in \( U_\delta(x_0(z)) \subset \{ F = z \} \). We are choosing \( \delta \) small enough so that \( U_\delta(x_0(z)) \subset S(z) \).

Our goal is to prove the following

**Theorem 3.1.** Let \( X_t^{\varepsilon,\delta} \) be the solution of equation (3.1), and \( Y_t^{\varepsilon,\delta} = Y_t(X_t^{\varepsilon,\delta}) \) be the slow component of \( X_t^{\varepsilon,\delta} \). Then, for each \( T > 0 \), \( Y_t^{\varepsilon,\delta} \) converges weakly in the space of continuous functions \( f : [0, T] \rightarrow \Gamma \) to a stochastic process \( \overline{Y}_t(x_0(z)) \) as, first, \( \varepsilon \downarrow 0 \) and then \( \delta \downarrow 0 \).

We will define the process \( \overline{Y}_t(x_0(z)) \) later in this section.

Let us start with the perturbed, but not regularized system (2.7). The motion of \( X_t^\varepsilon \) is on the surface \( S(z) \). The change of \( G(X_t^\varepsilon) \) is governed by the equation

\[
\frac{dG(X_t^\varepsilon)}{dt} = \nabla G \cdot \left( \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times b) \right)
= \nabla G \cdot (\nabla F \times (\nabla F \times b))
= -(\nabla F \times b) \cdot (\nabla F \times \nabla G) .
\]

The function \( G \) is a first integral of the unperturbed system (2.1) and the damping term \( \nabla F \times (\nabla F \times b) \) of (2.7) plays the role of "friction" which makes the value of \( G \) smaller and smaller.

The stable, but not asymptotically stable equilibriums \( O_1(z) \) and \( O_3(z) \) of (2.1) become asymptotically stable equilibriums \( O_1'(z) \) and \( O_3'(z) \) for the perturbed system (2.7). The saddle point \( O_2(z) \) becomes the saddle point \( O_2'(z) \). The distances between \( O_1(z) \) \( (O_2(z), O_3(z)) \) and \( O_1'(z) \) \( (O_2'(z), O_3'(z)) \) are less than \( A\varepsilon \) for a constant \( A > 0 \). When \( \varepsilon \) is small enough, the pieces of the curves formed by \( O_1'(z) \), \( O_2'(z) \) and \( O_3'(z) \) (as
Fig. 2: White and grey ribbons

For $z$ varies) are transversal to $\{F = z\}$. Separatrices of the saddle point $O'_{2}(z)$ are shown in Fig.2. They, roughly speaking, divide the part of the surface $S(z)$ outside the $\infty$-shaped curve $C(0, z)$ in ribbons: the gray ribbon enters the neighborhood of $O'_{1}(z)$, and the white ribbon enters the neighborhood of $O'_{3}(z)$. The width of each ribbon is of order $\varepsilon$ as $\varepsilon \downarrow 0$.

The trajectory $X_{\varepsilon}t$ has a fast component, which is close to the non-perturbed motion (2.1) (with the speed of order $\frac{1}{\varepsilon}$), and the slow component, which is the projection $Y_{\varepsilon}t = \Pi(X_{\varepsilon}t)$ of $X_{\varepsilon}t$ on the graph $\Gamma$ corresponding to $G(x)$. Within each edge of the graph, say edge $I_{i}$, $i = 1, 2, 3$, standard averaging principle works. Let $G_{i}^{\varepsilon} = G(X_{i}^{\varepsilon})$. We have, by the standard averaging principle (cf. [1], Ch.10),

$$
\lim_{\varepsilon \downarrow 0} \sup_{0 \leq t \leq T < \infty} |G_{i}^{\varepsilon} - G_{i}| = 0.
$$
The function $G_t$ satisfies $G_0 = G(x_0(z))$ and
\[
\frac{dG_t}{dt} = B^{(i)}(G_t), \text{ where }
\]
\[
B^{(i)}(g) = \frac{1}{T_i(g)} \oint_{C_i(g,z)} \frac{\nabla \cdot (\nabla F \times (\nabla F \times b))}{|\nabla F \times \nabla G|} \cdot dm
\]
(3.2)

Here $T_i(g) = \oint_{C_i(g,z)} \frac{dl}{|\nabla F \times \nabla G|}$ is the period of rotation for the unperturbed system (2.1) along the curve $C_i(g,z)$. The vector $v = \frac{\nabla F \times \nabla G}{|\nabla F \times \nabla G|}$ is the unit velocity vector for the unperturbed system (2.1); $\mathbf{n} = \mathbf{n}(x) = \frac{\nabla F(x)}{|\nabla F(x)|}$ is normal to the level surface $\{F = z\}$. The area element on $\{F = z\}$ is denoted by $dm$. We used the Stokes formula in the last step.

Fix a point $x_0(z)$ on the level surface $\{F = z\}$ outside the $\infty$-shaped curve $C(0,z)$. To be specific, let $x_0(z)$ belong to the white ribbon. Let $\gamma_s(z)$ be the curve on $\{F = z\}$ containing $x_0(z)$ and orthogonal to the perturbed trajectories (2.7). Let $a(z), b(z), c(z)$ be the intersection points of $\gamma_s(z)$ with separatrices neighboring to $x_0(z)$. To be specific, let $x_0(z)$ lie between $b(z)$ and $c(z)$ (see Fig.3, where a part of the flow is shown). By our transversality condition, we can take $\lambda > 0$ small enough and a curve $\xi(\tilde{z})$, $\tilde{z} \in [z - 2\lambda, z + 2\lambda]$ which lies on the surface $\{G = G(x_0(z))\}$ and is transversal to the level surface $\{F = z\}$, containing the point $x_0(z)$ ($\xi(z) = x_0(z)$). Let $x_0(\tilde{z}) = \xi(\tilde{z})$.

Consider the curve $\gamma_s(\tilde{z})$ on $\{F = \tilde{z}\}$ containing $x_0(\tilde{z})$ and orthogonal to the trajectories of (2.7). We also consider corresponding neighboring points $a(\tilde{z}), b(\tilde{z}), c(\tilde{z})$ defined for $x_0(\tilde{z})$ in the same way as we did for $x_0(z)$. For fixed $\varepsilon > 0$, we choose $\lambda$ small enough such that as $\tilde{z}$ varies in $[z - 2\lambda, z + 2\lambda]$, the curves $a(\tilde{z}), b(\tilde{z})$ and $c(\tilde{z})$ are transversal to $\{F = z\}$. The part of $\gamma_s(\tilde{z})$ between $a(\tilde{z}) (b(\tilde{z}))$ and $b(\tilde{z}) (c(\tilde{z}))$ belongs to the grey (white) ribbon for the trajectories of (2.7) on $\{F = \tilde{z}\}$. Now we consider the curvilinear rectangle $\square_1$ with vertices $a(z + \lambda), a(z - \lambda), b(z - \lambda), b(z + \lambda)$ constructed in the following way: $\square_1$ consists of the parts of the curves of $\gamma_s(\tilde{z})$ from $a(\tilde{z})$ to $b(\tilde{z})$ as $\tilde{z}$ varies in $[z - \lambda, z + \lambda]$. We construct another curvilinear rectangle $\square_2$ with vertices $b(z + \lambda), b(z - \lambda), c(z - \lambda), c(z + \lambda)$ in exactly the same way as $\square_1$, but consisting of curves $\gamma_s(\tilde{z})$ from $b(\tilde{z})$ to $c(\tilde{z})$ as $\tilde{z}$ varies in $[z - \lambda, z + \lambda]$.

Let vector $\vec{\nu}$ be the unit vector outward normal to these two curvilinear rectangles...
\[ \int_{\Box_k} \left( \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times b) \right) \cdot \vec{v} \, dm = \int_{E_k} \text{div}(\nabla F \times (\nabla F \times b)) \, dV \]

\[ = -\int_{z-\lambda}^{z+\lambda} d\tilde{z} \int_{S_k(\tilde{z})} \nabla \times (\nabla F \times b) \cdot \frac{\nabla F}{|\nabla F|} \, dm \]

\[ = -2\lambda \int_{S_k(\tilde{z})} \nabla \times (\nabla F \times b) \cdot \frac{\nabla F}{|\nabla F|} \, dm + o(\lambda). \tag{3.3} \]

Let \( L(a(z), b(z)) \) and \( L(b(z), c(z)) \) be, respectively, the arc length of \( \gamma_s(z) \) between \( a(z) \) and \( b(z) \), and between \( b(z) \) and \( c(z) \). The flux of the vector field \( \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times b) \)
through $\square_k$ ($k = 1, 2$) is equal to 
\[- \int_{\partial_1} \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times \mathbf{b}) \, dm.\]

Let $\text{Area}(\bullet)$ denote the area of some domain. Let $|J(\tilde{z}, \gamma_s(\tilde{z}))| \neq 0$ be the Jacobian factor between the area element on $\square_1 \cup \square_2$ and $d\tilde{z}d\gamma_s(\tilde{z})$. We have

\[
\text{Area}(\square_1 \cup \square_2) = \int_{\tilde{z} - \lambda}^{\tilde{z} + \lambda} d\tilde{z} \int_{a(\tilde{z})}^{c(\tilde{z})} |J(\tilde{z}, \gamma_s(\tilde{z}))|d\gamma_s(\tilde{z})
\]
\[= 2\lambda \int_{a(z)}^{c(z)} |J(z, \gamma_s(z))|d\gamma_s(z) + (I)
\]
\[= 2\lambda|J(z, b(z))|L(a(z), c(z)) + 2\lambda(II) + (I).
\]

Here

\[(I) = \int_{\tilde{z} - \lambda}^{\tilde{z} + \lambda} d\tilde{z} \left( \int_{a(\tilde{z})}^{c(\tilde{z})} |J(\tilde{z}, \gamma_s(\tilde{z}))|d\gamma_s(\tilde{z}) - \int_{a(z)}^{c(z)} |J(z, \gamma_s(z))|d\gamma_s(z) \right),
\]

and

\[(II) = \int_{a(z)}^{c(z)} (J(z, \gamma_s(z)) - J(z, b(z)))d\gamma_s(z).
\]

Note that $|(I)| \leq C_1\lambda^2$ since the function $I(\tilde{z}) = \int_{a(\tilde{z})}^{c(\tilde{z})} |J(\tilde{z}, \gamma_s(\tilde{z}))|d\gamma_s(\tilde{z})$ satisfies $|I(\tilde{z}_1) - I(\tilde{z}_2)| \leq C_2|\tilde{z}_1 - \tilde{z}_2|$. We also have $|(II)| \leq C_3L(a(z), c(z))^2$ since $|J(z, \gamma_s(z)) - J(z, b(z))| \leq C_4|\gamma_s(z) - b(z)| \leq C_5L(a(z), c(z))$. Combining these estimates with (3.3) and the fact that for some constants $C_6, C_7 > 0$,

\[
\frac{C_6}{\varepsilon} \leq \frac{1}{\text{Area}(\square_1 \cup \square_2) \int_{\square_1 \cup \square_2} \left| \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times \mathbf{b}) \right| dm \leq \frac{C_7}{\varepsilon},
\]

we see that as $\varepsilon \downarrow 0$, the asymptotic widths of the grey and white ribbons (i.e. $L(a(z), b(z))$ and $L(b(z), c(z)))$ are of order $O(\varepsilon)$. The next lemma gives the asymptotic ratio of the widths:

**Lemma 3.1.** Let $x_0(z)$ and the points $a(z), b(z), c(z)$ be defined as above. Then

\[
\lim_{\varepsilon \downarrow 0} \frac{L(a(z), b(z))}{L(b(z), c(z))} = \frac{\int_{D_1(0,z)} \nabla \times (\nabla F \times \mathbf{b}) \cdot d\mathbf{m}}{\int_{D_3(0,z)} \nabla \times (\nabla F \times \mathbf{b}) \cdot d\mathbf{m}}.
\]

Here the domains $D_1(0,z)$ and $D_3(0,z)$ are the regions bounded by $C_1(0,z)$ and $C_3(0,z)$. 

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The proof of the lemma is similar to the proof of Lemma 3.4 in [4] but based on (3.3), rather than on the divergency theorem in $\mathbb{R}^2$, as in [4]. We provide the details in the Appendix. \(\square\)

In the following we will fix an initial point \(x\) (not necessarily \(x_0(z)\)) on \(S(z)\). We put \(\tilde{x} = f(x) \in f(S(z)) = R = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}\). Let us consider the trajectory \(X^\ell_t(x)\) of (2.7) starting from point \(x\). Let \(\tilde{X}^\ell_t(\tilde{x}) = f(X^\ell_t(x))\). Our goal now is to estimate the time of "one rotation" of \(X^\ell_t(x)\) around either \(O^1_2(z)\) or \(O^2_2(z)\) or around both of them.

Note that (in two dimensional case), a neighborhood \(U\) of a saddle point of \(G\) on \(S(z)\) exists such that the system can be reduced to a linear one in \(\tilde{U} \subset \mathbb{R}^2\) by a non-singular diffeomorphism of the class \(C^{1,\alpha}\), \(\alpha > 0\). This comes from the corresponding result in \(\mathbb{R}^2\) ([17, Theorem 7.1]) and the fact that our surface \(S(z)\) is \(C^\infty\)-diffeomorphic to \(R = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}\).

In our case, the system depends on a parameter \(\varepsilon\), but one can check that neighborhood \(U\) and \(\alpha > 0\) can be chosen the same for all small enough \(\varepsilon\), and the \(C^{1,\alpha}\)-norm of the functions defining the diffeomorphism are bounded uniformly in \(\varepsilon\).

For the reason above, it is sufficient to consider the corresponding flow \(\tilde{X}^\ell_t(\tilde{x})\) on \(R\). Such a flow has the same structure consisting of grey and white ribbons on \(R\). For notational convenience we will use the same symbols for objects related to such a flow, corresponding to our original \(X^\ell_t(x)\). For example, we will write \(\tilde{X}^\ell_t(\tilde{x})\) simply as \(X^\ell_t(x)\), and the set \(f(U_\delta(x))\) as \(U_\delta(x)\), etc. The reader could easily understand which specific flow we are referring to from the context.

The system on \(R\) can be linearized in a neighborhood of \(O^2_2(z)\), as described above.

First, note that if \(x\) is situated outside a fixed (independent of \(\varepsilon\)) neighborhood of the \(\infty\)-shaped curve \(C(0, z)\), the trajectory \(X^\ell_t(x)\) comes back to corresponding curve \(\gamma \ni x\), orthogonal to the perturbed trajectory, at least, if \(\varepsilon > 0\) is small enough. The time of such a rotation \(t_\varepsilon(x) < \varepsilon A(x)\) (recall that we made time change \(t \to \frac{t}{\varepsilon}\)); \(A(x)\) here is independent of \(\varepsilon\) and bounded uniformly in each compact set disjoint with \(C(0, z)\).

If \(x\) is close to \(C(0, z)\), then \(X^\ell_t(x)\) comes to a \(\delta\)-neighborhood \(U_\delta(O^2_2(z))\) of \(O^2_2(z)\) in a time less than \(\varepsilon A_\delta\), \(A_\delta < \infty\). But the time spent by the trajectory inside the neighborhood \(U_\delta(O^2_2(z))\) of \(O^2_2(z)\) can be large for small \(\varepsilon\); in particular, the separatrices entering \(O^\delta_2(z)\) never leave \(U_\delta(O^2_2(z))\). So we should consider trajectories started at distance \(\delta\) from \(O^2_2(z)\) in more detail.

Let \(\delta > 0\) be so small that \(U_{2\delta}(O^2_2(z))\), for \(\varepsilon\) small enough, belongs to the neighborhood \(U\) of \(O^2_2(z)\) where our perturbed system can be linearized. The saddle point \(O^2_2(z)\) under this transformation goes to the origin \(O\), the separatrices of \(O^2_2(z)\) go to the axis \(\hat{x}\) and \(\hat{y}\), the trajectories \(X^\ell_t\) go to the trajectories of the linear system (Fig.4).
Fig. 4: Linearized system

One can explicitly calculate the time $\theta(\hat{h}, \hat{\delta})$ which the linear system trajectory needs to go from a point $(\hat{h}, \hat{y}_0)$ to $(\hat{\delta}, \hat{y}_1)$ (Fig.4):

$$\theta(\hat{h}, \hat{\delta}) = \text{const} \cdot \left| \ln \frac{\hat{h}}{\delta} \right|. \quad (3.5)$$

Let a perturbed trajectory enters $U_\delta(O'_2(z))$ at a point $x \in \partial U_\delta(O'_2(z))$, $G(x) > 0$, and exits $U_\delta(O'_2(z))$ at a point $y \in \partial U_\delta(O'_2(z))$. We can assume that $x$ and $y$ are close enough to the pieces of the separatrices which go to the axises $\hat{x}$, $\hat{y}$ after the linearization so that the curves $\gamma$ and $\gamma'$ orthogonal to perturbed trajectories and containing $x$ and $y$ respectively cross these pieces of separatrices (these pieces are shown in Fig.5 as bold lines and denoted by numbers 1,2,3,4) at points $a$ and $a'$ (Fig.5). Let the distance between $x$ and the closest last piece of the separatrix entering $O'_2(z)$ be equal to $h$ (here and below we are using the distance defined by minimal geodesics since we are working in a sufficiently small neighborhood). Consider the closest to $x$ separatrix crossing $\gamma$ at a point $b$ such that $G(b) > G(x)$. Let $l$ be the distance between $y$ and this separatrix.

If at least one whole ribbon intersects the curve $\gamma$ between $x$ and the piece of the separatrix entering $O'_2(z)$ (and containing point $a$), the trajectory $X'_\varepsilon(x)$ makes a complete rotation around both $O'_1(z)$ and $O'_3(z)$ and crosses $\gamma$ at a point $x' \in \gamma$ (case 1). The time spent by this trajectory outside $U_\delta(O'_2(z))$ is bounded from above by $A_1\varepsilon$. Since the perturbed system can be linearized in $U_{2\delta}(O'_2(z))$ by a $C^{1,\alpha}$-diffeomorphism, equality (3.5) implies that the transition from $x$ to $y$ takes time less than $A_2\varepsilon|\ln h|$: $A_1$ and $A_2$, in particular, depend on $\delta$, but are independent of $\varepsilon$.

The trajectory $X'_\varepsilon(x)$ comes to $\partial U_\delta(O'_2(z))$ again at the point $z$ (Fig.5). It follows from the divergence theorem that the distance from $z$ to the last piece of the separatrix
entering \( O'_2(z) \) (and containing the point \( v \) in Fig.5), in the case when \( X^\varepsilon_t(x) \) comes back to \( x' \in \gamma \), is bounded from below and from above by \( A_3h \) and \( A_4h \) respectively. Therefore the transition from \( z \) to \( z' \) also takes time less than \( A_5\varepsilon|\ln h| \).

Consider now the case when between the initial point \( y \in \partial U_\delta(O'_2(z)) \) and the last piece of the separatrix entering \( O'_2(z) \) there is no whole ribbon (Fig.6). Transition between \( y \) and \( y' \), because of the same reasons as above, takes time less than \( A_6\varepsilon|\ln h| \), where \( h \) is distance between \( y \) and the last piece of separatrix entering \( O'_2(z) \). But complete rotation of the trajectory \( X^\varepsilon_t(y) \) includes also the transition from \( z \) to \( y'' \). It is easy to check using divergence theorem that, the distance from \( z \) to the separatrix entering \( O'_2(z) \) is bounded from below and from above by \( A_7l \) and \( A_8l \) respectively, where \( l \) is the distance between \( y \) and the separatrix crossing \( \gamma \) at a point \( b \) such that \( H(b) > H(y) \) (Fig.6). Therefore, the transition time between \( z \) and \( y'' \) is less than \( A_9\varepsilon|\ln l| \), and the whole rotation time for \( X^\varepsilon_t(y) \) is less than \( A_{10}\varepsilon(|\ln h| + |\ln l|) \) for \( \varepsilon > 0 \) small enough.

Denote by \( t_\varepsilon(x) \) the time of complete rotation for the trajectory \( X^\varepsilon_t(x) \). Suppose \( x \) is not a critical point of \( G \). We have

\[
t_\varepsilon(x) = \min\{t > 0 : X^\varepsilon_t(x) \text{ crosses twice one of the curves } \gamma \text{ or } \gamma' \}.
\]

Summarizing the above bounds and taking into account that outside \( U_\delta(O'_1(z)) \cup U_\delta(O'_2(z)) \cup U_\delta(O'_3(z)) \) the trajectory \( X^\varepsilon_t(x) \) moves with the speed of order \( \varepsilon^{-1} \), we get,
Lemma 3.2. Let $X^\varepsilon_t(x)$ enter $U_\delta(O'_2(z))$ at a point $y = y(x) \in \partial U_\delta(O'_2(z))$, and let $h = h(x)$ be the distance between $y(x)$ and the last piece of a separatrix entering $O'_2(z)$. Let $\gamma$ be the curve orthogonal to perturbed trajectories and containing $y(x)$.

If in one complete rotation, $X^\varepsilon_t(y(x))$ come back to $\gamma$, then

$$t^\varepsilon(x) \leq A_{11}\varepsilon|\ln h(x)|. \quad (3.6)$$

If $X^\varepsilon_t(y(x))$ does not come back to $\gamma$, and $l(x)$ is the distance from $y(x)$ to the closest separatrix, which crosses $\gamma$ at a point $b$, such that $G(b) > G(y(x))$, then for $\varepsilon > 0$ small enough,

$$t^\varepsilon(x) < A_{12}\varepsilon(|\ln h(x)| + |\ln l(x)|). \quad (3.7)$$

Now we come back to our original system (2.7) on $S(z)$. Let $\alpha$ be a small positive number. Denote by $E_\alpha = E_\alpha(\varepsilon)$ the set of points $x \in S(z)$ such that the distance between $x$ and the closest separatrix is greater than $\varepsilon\alpha$ (since $\varepsilon$ is small we can work with minimal geodesics). Let $E^g_\alpha$ be the intersection of $E_\alpha$ with the gray ribbon; $E^w_\alpha$ be the intersection with the white ribbon.

Denote by $\Lambda^\varepsilon_t(x,\beta)$ the time when $X^\varepsilon_t(x)$ reaches $C(\beta,z)$:

$$\Lambda^\varepsilon_t(x,\beta) = \inf \{t > 0 : G(X^\varepsilon_t(x)) = \beta\};$$

if $G(x) > 0$ and $|\beta|$ is small, $\Lambda^\varepsilon_t(x,\beta) < \infty$ for all small $\varepsilon > 0$. 

Fig. 6: Case 2
Lemma 3.3. Let $G(x_0(z)) > 0$ and let $\mu > 0$ be so small that $G(x) > 0$ for $x \in U_{2\mu}(x_0(z))$. There exist $\alpha_0$, $\beta_0 > 0$ and $A_{13}$ such that for each $x \in U_{\mu}(x_0(z)) \cap E_{\alpha}$, $\alpha \in (0, \alpha_0)$, $\beta \in (0, \beta_0)$,

$$\Lambda_\varepsilon(x, -\beta) - \Lambda_\varepsilon(x, \beta) < A_{13}\beta |\ln \beta|$$

(3.8)

for $\varepsilon < \varepsilon_0$. Here $A_{13}$, in particular, depends on $\alpha$ and $\beta$ but is independent of $\varepsilon$; $\varepsilon_0 > 0$ depends on $\alpha$ and $\beta$.

The proof of this lemma is based on Lemma 3.2 and the fact that each rotation decreases the value of $G$ on an amount of order $O(\varepsilon)$. Therefore the total time is less than $A_{14} \sum_{k=1}^{\varepsilon} |\ln(\kappa z)| \sim \int_0^\beta |\ln z|dz \leq A_{13}\beta |\ln \beta|$ for $\varepsilon > 0$ small enough. □

Proof of Theorem 3.1. Equation (3.2) can be considered for each of three edges of the graph $\Gamma$ corresponding to $G(x)$ on $S(z)$: for $i = 1, 2, 3$, we have

$$\dot{g}^{(i)}_t = \frac{1}{T_i(g^{(i)}_t)} B^{(i)}(g^{(i)}_t, z),$$

$$T_i(g) = \oint_{C_i(g, z)} \frac{dl}{\nabla F \times \nabla G},$$

$$B^{(i)}(g, z) = -\int_{D_i(g, z)} \nabla \times (\nabla F \times b) \cdot ndm.$$  

Equation (3.10) for $i = 2$ can be solved for each initial condition $g^{(2)}_0 = g > 0$, $g < \max\{G(w) : w \in \partial S(z)\}$. Such a solution is unique, and $g^{(2)}_t$ reaches 0 in a finite time $\tau_0(g, z)$. If $i = 1, 3$, equation (3.10) with initial condition $g^{(i)}_0 = g < 0$ has a unique solution; if $g^{(i)}_0 = 0$, equation (3.10) has a unique solution $\hat{g}^{(i)}_t$ if we additionally assume that $\hat{g}^{(i)}_t < 0$ for $t > 0$.

Define two continuous functions $\hat{g}^{1}_{t}(g)$ and $\hat{g}^{3}_{t}(g)$, $t \geq 0$, as follows: $\hat{g}^{1}_{0} = \hat{g}^{3}_{0} = g > 0$,

$$\hat{g}^{1}_{t}(g) = \begin{cases} g^{(2)}_t, g^{(2)}_0 = g, 0 \leq t \leq \tau_0(g, z), \\ \hat{g}^{(1)}_{t-\tau_0(g, z)}, \tau_0(g, z) \leq t < \infty; \end{cases}$$

$$\hat{g}^{3}_{t}(g) = \begin{cases} g^{(2)}_t, g^{(2)}_0 = g, 0 \leq t \leq \tau_0(g, z), \\ \hat{g}^{(3)}_{t-\tau_0(g, z)}, \tau_0(g, z) \leq t < \infty; \end{cases}$$

Let us cut out $\alpha\varepsilon$-neighborhoods of the separatrices ($\mu$-neighborhood of a point $x_0$, $G(x_0) > 0$, is shown in Fig.7); recall that $E_\alpha$ is the exterior of the $\varepsilon\alpha$-neighborhood of the separatrices, $E^g_\alpha$ is the intersection of $E_\alpha$ with the gray ribbon, $E^w_\alpha$ is the intersection of $E_\alpha$ with the white ribbon. In particular, $E^g_0$ ($E^w_0$) is whole gray (white).
The classical averaging principle together with Lemma 3.3 imply that for each 
\( x \in U_\delta(x_0) \cap \mathcal{L}_x^\delta, \ G(x) = g > 0, \) for any \( \lambda,T > 0, \) and any small enough \( \alpha,\delta > 0, \) 
there exists \( \varepsilon_0 > 0 \) such that 
\[
\max_{0 \leq t \leq T} |H(X_\varepsilon^\delta_t(x)) - \hat{g}_1^\delta(g)| < \lambda \tag{3.11}
\]
for \( 0 < \varepsilon < \varepsilon_0. \)

Similarly, for each \( x \in U_\delta(x_0) \cap \mathcal{L}_x^w, \ G(x) = g > 0, \)
\[
\max_{0 \leq t \leq T} |H(X_\varepsilon^\delta_t(x)) - \hat{g}_3^\delta(g)| < \lambda \tag{3.12}
\]
for \( 0 < \varepsilon < \varepsilon_0. \)

Let \( G(x) > 0 \) for \( x \in U_\delta(x_0(z)) \) so that \( \mathcal{Q}(U_\delta(x_0(z))) \subset I_2 \subset \Gamma. \) Define a stochastic 
process \( Y_\ell^\delta(x_0(z)), \ t \geq 0, \) on \( \Gamma \) as follows: (recall that the pair \( (k,g), \) where \( k \) is the number of an edge, \( k \in \{1,2,3\}, \) and \( g \) is the value of \( G(x) \) on \( \mathcal{Q}^{-1}(y), \ y \in \Gamma, \) form a 
global coordinate system on \( \Gamma \))
\[
Y_\ell^\delta(x_0(z)) = (2, \hat{g}_2^\delta(G(x_0(z,\delta)))) \text{ for } 0 \leq t \leq \tau_0(x_0(z,\delta)).
\]
(Recall that \( \tau_0(x_0(z,\delta)) \) is the first time when the process \( g_1^{(2)}, g_0^{(2)} = G(x_0(z,\delta)) > 0 \) 
in (3.10) reaches 0.)

At the time \( \tau_0(x_0(z,\delta)) \) the process \( Y_\ell^\delta(x_0(z,\delta)) \) reaches \( O_2(z) \) and without any
delay goes to $I_1$ or $I_3$ with probabilities

$$p_1 = \frac{\int \int_{D_1(0,z)} \nabla \times (\nabla F \times b) \cdot ndm}{\int \int_{D_1(0,z)} \nabla \times (\nabla F \times b) \cdot ndm + \int \int_{D_3(0,z)} \nabla \times (\nabla F \times b) \cdot ndm},$$

(3.13)

$$p_3 = \frac{\int \int_{D_3(0,z)} \nabla \times (\nabla F \times b) \cdot ndm}{\int \int_{D_1(0,z)} \nabla \times (\nabla F \times b) \cdot ndm + \int \int_{D_3(0,z)} \nabla \times (\nabla F \times b) \cdot ndm},$$

(3.14)

respectively; $Y_t^\delta(x_0(z)) = (1, g_{t \to \tau_0(x_0(z), \delta)}(x_0(z, \delta))$ for $\tau_0(x_0(z, \delta)) \leq t < \infty$ if $Y_t^\delta(x_0(z))$ enters $I_1$ at time $\tau_0(x_0(z, \delta))$, and $Y_t^\delta(x_0(z)) = (3, g_{t \to \tau_0(x_0(z), \delta)}(x_0(z, \delta))$ for $\tau_0(x_0(z, \delta)) \leq t < \infty$ if $Y_t^\delta(x_0(z, \delta))$ enters $I_3$ at time $\tau_0(x_0(z, \mu))$.

One can consider a process $Y_t(x_0(z)) = Y_t^0(x_0(z))$ on $\Gamma$: $Y_t(x_0(z))$ is deterministic inside the edges and governed by equations (3.10); its stochasticity concentrated at the vertex $O_2(z)$: after reaching $O_2(z)$, $Y_t(x_0(z))$ immediately goes to $I_1$ or to $I_3$ with probabilities $p_1$ or $p_3$ defined by equalities (3.13) and (3.14).

Denote by Area$(D)$, $D \subset S(z)$, the area of a domain $D$. Since the point $x_0(z, \delta)$ is distributed uniformly in $U_\delta(x_0(z))$,

$$\left| \mathbb{P}\{X_t^{\varepsilon, \delta} \text{ enters } D_1(0,z) \} - \frac{\text{Area}(E_0^\varepsilon \cap U_\delta(x_0(z)))}{\text{Area}(U_\delta(x_0(z)))} \right| \to 0,$$

as $\varepsilon \downarrow 0$. According to Lemma 3.2,

$$\lim_{\varepsilon \downarrow 0} \frac{\text{Area}(E_0^\varepsilon \cap U_\mu(x_0(z)))}{\text{Area}(U_\mu(x_0(z)))} = p_1, \quad \lim_{\varepsilon \downarrow 0} \frac{\text{Area}(E_0^\varepsilon \cap U_\delta(x_0(z)))}{\text{Area}(U_\delta(x_0(z)))} = p_3,$$

(3.16)

where $p_1$ and $p_3$ are defined in (3.13) and (3.14).

Taking into account that $\text{Area}(E_0^\varepsilon \cap U_\delta(x_0(z))) \rightarrow \text{Area}(E_0^\varepsilon \cap U_\delta(x_0(z)))$ and $\text{Area}(E_0^\varepsilon \cap U_\delta(x_0(z))) \rightarrow \text{Area}(E_0^\varepsilon \cap U_\delta(x_0(z)))$ as $\alpha \downarrow 0$, we derive from (3.10)–(3.16) that, for each $T > 0$, the slow component $\mathcal{E}(X_t^{\varepsilon, \delta})$ of $X_t^{\varepsilon, \delta}$ converges weakly in the space of continuous functions on $[0, T]$ with values in $\Gamma$ to the process $Y_t(x_0(z))$.

It is easy to see that $Y_t^\delta(x_0(z))$ converges weakly to $Y_t(x_0(z))$ as $\delta \downarrow 0$.

This gives the proof of Theorem 3.1. \qed
4 Regularization by stochastic perturbation of the dynamics

Let now the perturbation have deterministic and stochastic parts:

\[ X_t^{\varepsilon, \delta} = \frac{1}{\varepsilon}(\nabla F \times \nabla G)(X_t^{\varepsilon, \delta}) + \nabla F \times (\nabla F \times b)(X_t^{\varepsilon, \delta}) + \delta \sigma(X_t^{\varepsilon, \delta}) \circ \dot{W}_t, \tag{4.1} \]

where the stochastic term is understood in the Stratonovich sense. Here vector \( c(s, x) \in \mathbb{R}^d \) has \( i \)-th component

\[ \sum_{j,k=1}^d \frac{\partial \sigma_{ij}}{\partial x_k} \sigma_{kj}, \quad 1 \leq i \leq d. \]

(One can directly check that equality \( \sigma^T \nabla F = 0 \) implies \( \nabla F \cdot \sigma \circ \dot{W}_t = 0 \).) In particular, if \( b(x) \equiv 0 \), we have pure stochastic perturbations. Therefore \( F \) is a first integral for system (4.1), i.e., the process \( X_t^{\varepsilon, \delta} \) never leaves the surface \( \{ F = z \} \). We also assume that \( \mathbf{e} \cdot (a(x) \mathbf{e}) \geq a|\mathbf{e}|^2 \) for a constant \( a > 0 \) and every \( \mathbf{e} \in \mathbb{R}^3 \) such that \( \mathbf{e} \cdot \nabla F(x) = 0 \). This means that the process \( X_t^{\varepsilon, \delta} \) is non-degenerate if considered on the manifold \( S(z) \subset \{ x \in \mathbb{R}^3 : F(x) = z \} \). Recall that we stop our process \( X_t^{\varepsilon, \delta} \) once it hits \( \partial S(z) \). The resulting process is still called \( X_t^{\varepsilon, \delta} \).

We will make use of the following simple Lemma (see, for instance, [20, page 36, formula (3.3.6)]):

**Lemma 4.1.** Let \( b(s, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \) be Lipschitz and bounded in \( s \) and \( x \). Let \( \sigma(s, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) be bounded, Lipschitz in \( s \), and differentiable in \( x \). Let \( \sigma_{ij} \) is the \( (i, j)-\)th element of matrix \( \sigma \). Consider the diffusion process

\[ X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s) \circ dW_s \]

in \( \mathbb{R}^d \), where the stochastic term is understood in Stratonovich sense. Then we have

\[ X_t = x + \int_0^t b(s, X_s)ds + \frac{1}{2} \int_0^t c(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \]

where the stochastic term is understood in the Itô sense. Here vector \( c(s, x) \in \mathbb{R}^d \) has \( i \)-th component

\[ \sum_{j,k=1}^d \frac{\partial \sigma_{ij}}{\partial x_k} \sigma_{kj}, \quad 1 \leq i \leq d. \]
Using this Lemma, we easily write equation (4.1) in the Itô sense:

\[
X^\varepsilon,\delta_t = \frac{1}{\varepsilon}(\nabla F \times \nabla G)(X^\varepsilon,\delta_t) + \nabla F \times (\nabla F \times b)(X^\varepsilon,\delta_t) + \delta \sigma(X^\varepsilon,\delta_t)\,\mathrm{d}W_t + \frac{\delta^2}{2} \Sigma(X^\varepsilon,\delta_t). \tag{4.2}
\]

Here \(\Sigma\) is a vector in \(\mathbb{R}^3\) with the \(i\)-th component \(\Sigma_i = \sum_{j,k=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_k} \sigma_{kj}\) for \(i = 1, 2, 3\).

The generator \(L\) of the process \(X^\varepsilon,\delta\) is written as

\[
Lu(x) = \frac{\delta^2}{2} \sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \left(\frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times b) + \frac{\delta^2}{2} \Sigma\right) \cdot \nabla u(x).
\]

Using Itô’s formula we see that

\[
G(X^\varepsilon,\delta) - G(x_0(z)) = \delta \int_0^t (\nabla G)^T (X^\varepsilon,\delta_s) \sigma(X^\varepsilon,\delta_s) \,\mathrm{d}W_s + \int_0^t \left(\nabla G \cdot (\nabla F \times (\nabla F \times b)) + \frac{\delta^2}{2} \nabla G \cdot \Sigma + \frac{\delta^2}{2} \sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 G}{\partial x_i \partial x_j}\right)(X^\varepsilon,\delta_s) \,\mathrm{d}s. \tag{4.4}
\]

Now we are in a position to use the standard averaging principle (see, for example, [14, Chapter 8]), to check that within edge \(I_i (i = 1, 2, 3)\) of the graph \(\Gamma\), as \(\varepsilon \downarrow 0\) and \(\delta\) is fixed, the process \(G(X^\varepsilon,\delta)\) converges weakly to the process \(G^\delta\) governed by the operator

\[
T_i = \frac{1}{T_i(g)} \left(A_i(g, z) + \frac{\delta^2}{2} A_{1,i}(g, z) + \frac{\delta^2}{2} A_{2,i}(g, z)\right) \frac{dl}{g} + \frac{\delta^2}{2} \frac{1}{T_i(g)} B_i(g, z) \frac{d^2}{dg^2}. \tag{4.5}
\]

The coefficients are

\[
\begin{align*}
A_i(g, z) &= \oint_{C_i(g, z)} \nabla G \cdot (\nabla F \times (\nabla F \times b)) \frac{dl}{|\nabla F \times \nabla G|}, \\
A_{1,i}(g, z) &= \oint_{C_i(g, z)} \nabla G \cdot \Sigma \frac{dl}{|\nabla F \times \nabla G|}, \\
A_{2,i}(g, z) &= \oint_{C_i(g, z)} \sum_{j,k=1}^{3} a_{ij} \frac{\partial^2 G}{\partial x_i \partial x_j} \frac{dl}{|\nabla F \times \nabla G|}, \\
B_i(g, z) &= \oint_{C_i(g, z)} |(\nabla G)^T \sigma|^2 \frac{dl}{|\nabla F \times \nabla G|}. \tag{4.6}
\end{align*}
\]

Here \(T_i(g)\) is the period of rotation of the unperturbed system (2.1) on \(C_i(g, z)\):

\[
T_i(g) = \oint_{C_i(g, z)} \frac{dl}{|\nabla F \times \nabla G|}, \text{ where } dl \text{ is the length element on } C_i(g, z).
\]

We define a process \(Y^\delta\) on \(\Gamma\) as follows: \(Y^\delta\) is a Markov process on \(\Gamma\), stopped once it hits exterior vertex \(P\) (recall that we stop our process \(X^\varepsilon,\delta\) once it hits \(S(\varepsilon)\)); also recall
that by our definition $S(z) = \{ x \in \mathbb{R}^3 : G(x) \leq G(x_0(z)) + 1 \} \cap \{ x \in \mathbb{R}^3 : F(x) = z \}$ so that $\partial S(z)$ is a trajectory, corresponding to vertex $P$ on $\Gamma$ and governed by a generator $A$. The operator $A$ is defined as follows. The domain of definition for the generator $A$ consists of functions $f(g,k)$ on $\Gamma$ which are twice continuously differentiable in the variable $g$ within the interior part of each edge $I_i$; inside $I_i$, $Af(g,i) = \mathbf{L}_i f(g,i)$, and finite limits $\lim_{y \to O_1(z)} Af(y)$ (which are taken as the value of $Af$ at vertex $O_1(z)$) and finite one sided limits $\lim_{y \to G(O_1(z))} \frac{\partial f}{\partial g}(g,i)$, $\lim_{y \to G(P)} \frac{\partial f}{\partial g}(g,i)$ exist. We set $Af(y) = 0$ (taken as the value of $Af$ at point $P$, this means that the process $Y^\delta_t$ is stopped at the point $P$). For the interior vertex $O_2(z)$, $f$ satisfies the gluing condition:

$$\sum_{i=1}^3 (\pm) \beta_{2,i} \lim_{g \to G(O_2(z))} \frac{\partial f}{\partial g}(g,i) = 0 , \quad (4.7)$$

where $+$ sign is for the limit taking within edge $I_2$ and $-$ sign is for the limit taking within edge $I_1$ and $I_3$. The coefficients $\beta_{2,i}$ are defined by

$$\beta_{2,i} = \oint_{C_s(0,z)} |(\nabla G)^T \sigma|^2 \frac{dl}{|\nabla F \times \nabla G|} . \quad (4.8)$$

Exterior vertex $O_1(z)$ and $O_2(z)$ are inaccessible. Such a process $Y^\delta_t$ on $\Gamma$ exists and is unique ([14, Chapter 8]).

**Theorem 4.1.** As $\varepsilon \downarrow 0$ and $\delta$ is fixed, the process $\mathfrak{F}(X^{\varepsilon,\delta}_t)$ converges weakly in the space of continuous functions $f : [0,T] \to \Gamma$, $0 < T < \infty$, to the process $Y^\delta_t$.

The proof of this Theorem is based on the fact that we can carry the dynamics of (3.1) on $S(z)$ to a corresponding one on $R \subset \mathbb{R}^2$ by the $C^\infty$-diffeomorphism $f : S(z) \to \mathbb{R}^2$. We denote $f(x_1,x_2,x_3) = (f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3))$, $(x_1,x_2,x_3) \in S(z)$. Let $Z^{\varepsilon,\delta}_t = f(X^{\varepsilon,\delta}_t)$ be the image of the diffusion process on $\mathbb{R}^2$. Using the Itô formula for Stratonovich integrals, we have

$$\begin{align*}
    dZ^{\varepsilon,\delta}_t &= df(X^{\varepsilon,\delta}_t) \\
    &= (Df)(f^{-1}(Z^{\varepsilon,\delta}_t)) dX^{\varepsilon,\delta}_t \\
    &= \left( \frac{1}{\varepsilon^2} \beta(Z^{\varepsilon,\delta}_t) + \beta_1(Z^{\varepsilon,\delta}_t) \right) dt + \delta(Df)(f^{-1}(Z^{\varepsilon,\delta}_t)) \sigma(f^{-1}(Z^{\varepsilon,\delta}_t)) \circ dW_t \\
    &= \left( \frac{1}{\varepsilon^2} \beta(Z^{\varepsilon,\delta}_t) + \beta_1(Z^{\varepsilon,\delta}_t) \right) dt + \delta\tilde{\sigma}(Z^{\varepsilon,\delta}_t) \circ d\tilde{W}_t
\end{align*} \quad (4.9)$$

so that $Z^{\varepsilon,\delta}_t = (f_1(X^{\varepsilon,\delta}_t), f_2(X^{\varepsilon,\delta}_t))$ is a diffusion process on $\mathbb{R}^2$, stopped once it hits $\partial R$.

Here the matrix $Df$ is the differential of $f$: $Df = \left( \frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq 2,1 \leq j \leq 3}$. The vector fields $\beta(Z) = (Df)(\nabla F \times \nabla G)(f^{-1}(Z))$ and $\beta_1(Z) = (Df)(\nabla F \times (\nabla F \times b))(f^{-1}(Z))$. 

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The $2 \times 2$ matrix $\tilde{\sigma}$ is defined in the following way: $\tilde{\sigma}(Z) \circ d\tilde{W}_t = (Df)(f^{-1}(Z))\sigma(f^{-1}(Z)) \circ dW_t$, where $W_t$ is the standard 3-dimensional Wiener process and $\tilde{W}_t$ is the standard 2-dimensional Wiener process. The integral curves of the vector field $\tilde{\beta}$ has one saddle point $f(O_2(z))$ and two stable equilibriums $f(O_1(z))$ and $f(O_3(z))$.

We define $G(Z) = G(f^{-1}(Z))$ for $Z \in \mathbb{R}^2$. The function $G$ serves as the first integral for the vector field $\vec{\beta}$: $\nabla G \cdot \vec{\beta} = 0$. Furthermore, it is easy to check that $\tilde{\beta}(Z) = \kappa(Z)\nabla G(Z)$ with $\kappa \neq 0$, so that our system just by a non-singular time change differs from a Hamiltonian system with one degree of freedom. Therefore one can use the same arguments as in the case of 2-dimensional Hamiltonian systems (see, [14, Chapter 8], [13], [11]) to calculate the limiting behavior the process $Z^\varepsilon_\delta$ as $\varepsilon \downarrow 0$. (In the calculation of the gluing conditions, the problem caused by additional drift term $\vec{\beta}_1$ and another drift term related to the Stratonovich integral can be resolved using the absolute continuous transformation; detailed estimates see [11] and Appendix.2.) The coefficients of the gluing condition at the interior vertex are given as follows:

$$\beta_{2,i} = \oint_{f(C_i(0,z))} \left| (\nabla G)^T \tilde{\sigma} \right|^2 dl_z \left| \beta \right|,$$

where $dl_z$ is the length element on $f(C_i(0,z))$. Note that they coincide with (4.8), since equality

$$(\nabla G)^T \tilde{\sigma} \circ d\tilde{W}_t = (Df)^T(Df)\sigma \circ dW_t = (\nabla G)^T \sigma \circ dW_t$$

implies $\left| (\nabla G)^T \tilde{\sigma} \right|^2 = \left| (\nabla G)^T \sigma \right|^2$, and $\frac{dl_z}{\left| \beta \right|} = \frac{dl}{\left| \nabla F \times \nabla G \right|}$. □

The next step is to consider the limit as $\delta \downarrow 0$ of the process $Y^\delta_t$. This follows the same line of argument as in [4, Section 2]. In particular, one can do a similar calculation as in Lemma 2.2 of [4]. The additional small drift term depending on $\delta$ (caused by the Stratonovich integral) in (4.5) will disappear as $\delta \downarrow 0$. (We briefly indicate how to calculate this in Appendix.3.) We therefore have a limiting process $Y_t$ on $\Gamma$ defined as follows: $Y_t = (g_t^{(i)}, k_t)$ is a deterministic motion inside each edge of $\Gamma$ with $g_t^{(i)}$ satisfying the differential equation (3.10) and the branching probability for $Y_t$ at vertex $O_2(z)$ is given by (3.13) and (3.14). The process $Y_t$ spends time zero at the vertex $O_2(z)$. These arguments imply

**Theorem 4.2.** As $\delta \downarrow 0$, the process $Y^\delta_t$ converges weakly in the space of continuous functions $f : [0,T] \to \Gamma$, $0 < T < \infty$, to the process $Y_t$. 

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Theorem 4.1 and 4.2 imply that the slow component \( \mathbb{P}(X_t^{\varepsilon, \delta}) \) of the process \( X_t^{\varepsilon, \delta} \) converges weakly to the process \( Y_t \) on the graph \( \Gamma \). Note that \( Y_t \) is independent of the diffusion matrix \( a(x) = \sigma(x)\sigma^T(x) \) and is the same process which we had using regularization by stochastic perturbation of the initial point.

Consider process \( X_t^{\varepsilon, \delta} \) defined by (4.1). Under the assumption that the deterministic perturbation in (4.1) is friction-like, for \( \varepsilon > 0 \) small enough and fixed, the equilibrium \( O_1^t(z) \) and \( O_3^t(z) \) are asymptotically stable for the dynamical system \( X_t^{\varepsilon, 0} \) on \( \{ F(x) = z \} \).

The process \( X_t^{\varepsilon, \delta} \) is close to \( X_t^{\varepsilon, 0} \) on any fixed time interval if \( \delta \) is small enough. But on time intervals of order \( \exp \left\{ \frac{\lambda}{\delta^2} \right\} \) for \( \lambda > 0 \), \( X_t^{\varepsilon, \delta} \) may perform transitions between the neighborhoods of \( O_1^t(z) \) and \( O_3^t(z) \) due to the large deviations from \( X_t^{\varepsilon, 0} \). In a generic case, for \( x \in \{ F(x) = z \} \) and \( \lambda > 0 \), there exists just one stable equilibrium \( M^\varepsilon(x, \lambda) \) (in the case of two stable equilibriums, \( M^\varepsilon(x, \lambda) = O_1^t(z) \) or \( M^\varepsilon(x, \lambda) = O_3^t(z) \) such that with probability close to 1 as \( \delta \downarrow 0 \), \( X_t^{\varepsilon, \delta} \) is situated in a small neighborhood of \( M^\varepsilon(x, \lambda) \), if \( X_0^{\varepsilon, \delta} = x \), \( \lim_{\delta \downarrow 0} \delta^2 \ln T^\delta(\lambda) = \lambda \). The state \( M^\varepsilon(x, \lambda) \) is called metastable state for a given initial point \( x \) and time scale \( \lambda > 0 \) (see [8], [10] where the procedure for calculating \( M^\varepsilon(x, \lambda) \) is described).

But it turns out that the function \( M^\varepsilon(x, \lambda) \) is very sensitive to \( \varepsilon \) as \( \varepsilon \downarrow 0 \): For \( \lambda \) not very large, \( M^\varepsilon(x, \lambda) \) alternatively is equal to \( O_1^t(z) \) or to \( O_3^t(z) \) as \( \varepsilon \downarrow 0 \). Moreover, for small \( \varepsilon \), \( M^\varepsilon(x, \lambda) \) is sensitive to changes of the initial point \( x \) as well. Therefore, if \( \varepsilon \ll 1 \), the notion of metastability should be modified (compare with [3], [9]): For given \( x \) and \( \lambda \), one should consider the set of metastable distributions between the stable equilibriums. In general, there exists a finite number of distributions on the set of stable equilibriums which serve as limiting distributions of \( X_t^{\varepsilon, \delta} \) as first \( \varepsilon \downarrow 0 \) and then \( \delta \downarrow 0 \).

The set of metastable distributions is independent of the stochastic terms in (4.1) and defined just by the deterministic system and deterministic perturbations. But which of those distributions serves as limiting distribution of \( X_t^{\varepsilon, \delta} \), \( X_0^{\varepsilon, \delta} = x \), is defined by the stochastic term in (4.1).

In our case, when we have just two stable equilibriums \( O_1(z) \) and \( O_3(z) \), three distributions can serve as metastable distribution: first, the distribution concentrated at \( O_1(z) \), second, the distribution concentrated at \( O_3(z) \), and third, the distribution between \( O_1(z) \) and \( O_3(z) \) with \( P\{ O_1(z) \} = p_1 \), \( P\{ O_3(z) \} = p_3 \) where \( p_1 \) and \( p_3 \) defined by (3.13), (3.14).

**Theorem 4.3.** Let \( \lambda_1 = -\int_{G(O_1(z))} A_1(g, z)dg < \int_{G(O_3(z))} A_3(g, z)dg = \lambda_3 \), where \( A_i(g, z) \) and \( B_i(g, z) \) are defined by (4.6). Let \( \lim_{\delta \downarrow 0} \delta^2 \ln T^\delta(\lambda) = \lambda > 0 \). Then for each small enough \( h > 0 \),
\[
\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P_x \{|X^{\varepsilon, \delta}_{T^\delta(\lambda)} - O_1(z)| < h\} = 1 \text{ if } \mathfrak{y}(x) \in I_1 \text{ and } \lambda < \lambda_1,
\]

\[
\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P_x \{|X^{\varepsilon, \delta}_{T^\delta(\lambda)} - O_3(z)| < h\} = 1 \text{ if } \mathfrak{y}(x) \in I_2 \text{ and } \lambda > 0 \text{ or if } \lambda > \lambda_3 \text{ for any } x \in \{F(x) = z\},
\]

\[
\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P_x \{|X^{\varepsilon, \delta}_{T^\delta(\lambda)} - O_i(z)| < h\} = p_i, \ i \in \{1, 3\}, \text{ if } \mathfrak{y}(x) \in I_3 \text{ and } \lambda < \lambda_1.
\]

The probabilities \(p_1\) and \(p_3\) are defined by (3.13)-(3.14).

The proof follows from Theorem 4.1 and the fact that the transition time from \(O_1(z)\) to \(O_3(z)\) (from \(O_3(z)\) to \(O_1(z)\)) for the process \(Y^\delta_t\) on \(\Gamma\) is logarithmically equivalent as \(\delta \downarrow 0\) to \(\exp \left\{ \frac{\lambda_1}{\delta^2} \right\} \exp \left\{ \frac{\lambda_3}{\delta^2} \right\}\) (Theorem 4.4.2 in [14]). \(\square\)

**Remark:** We assumed in Sections 3 and 4 that the function \(G(x)\) has in \(S(z)\) just one saddle point and two minima. We also assumed that the deterministic perturbations are friction-like. Then each minimum point become asymptotically stable for the perturbed system. It is not difficult to check that if \(G(x)\) has on the set \(\tilde{S}(z) = \{F = z\}\) (we assumed it has only one connected component) more than two minima points and several saddle points but just one local maximum, and the deterministic perturbations are friction-like, then the system can be regularized by an addition of stochastic perturbations of the initial point or of the dynamics. Corresponding graph in this case has several interior vertices corresponding to the saddle points of \(G(x)\) and exterior vertices corresponding to the extremums.

Inside each edge, the limiting slow motion is governed by corresponding equation (3.2). The exterior vertices are inaccessible in finite time. The limiting slow motion spends time zero at interior vertices, and the branching at each interior vertex occurs exactly as in the case of a unique saddle point. The branching at each interior point is independent of the previous behavior of the limiting slow motion.

But situation is a bit different if \(G(x)\) has on \(\tilde{S}(z)\) more than one maxima or if the perturbations are not friction-like. In this case, in general, it is impossible to regularize the problem by a random perturbation of the initial point: the limit of \(\mathfrak{y}(X^{\varepsilon, \delta}_t)\) as \(\varepsilon \downarrow 0\) may not exist (compare with [4]). The regularization by stochastic perturbations of the equation, as we did in Section 4, is possible under mild additional assumptions. One should keep in mind that, if the deterministic perturbation is not friction-like, the stochastic branching occurs just at those interior vertices where there are two ”exit” edges and one ”entrance” edge (this means that the limiting slow motion along an edge attached to the vertex is, respectively, directed from or to the vertex).
Note that, since we assume that $\lim_{|x| \to \infty} F(x) = \infty$, at least one local maximum of $G(x)$ is available on each connected component of every level set of $F(x)$.

5 Positive genus level set components

Consider a slightly more general equation

$$\tilde{X}_t = \nabla F(\tilde{X}_t) \times d(\tilde{X}_t), \quad \tilde{X}_0 = x(z),$$

(5.1)

where the initial point $x(z)$ belongs to one of the connected components $M = M(z)$ of the level set $\{x \in \mathbb{R}^3 : F(x) = z\}$. As before, we assume that $F(x)$ is smooth enough, $\lim_{|x| \to \infty} F(x) = \infty$, and $\nabla F(x) \neq 0$ for $x \in M$, so that $M$ is a compact connected orientable two-dimensional surface in $\mathbb{R}^3$.

The vector field $d(x)$, $x \in \mathbb{R}^3$, is assumed to be smooth and the vector field $\nabla F(x) \times d(x)$ has, at most, a finite number of rest points on $M$. Moreover, assume that

$$\nabla \times d(x) = 0 \text{ for } x \in M.$$ 

Note that in the case of equation (2.1), $d(x) = \nabla G(x)$, and the last assumption is satisfied.

We will make use of the following

**Lemma 5.1.** The measure on $M(z)$ with the density with respect to the surface area proportional to $1/|\nabla F(x)|$ is invariant for the flow (5.1) on $M(z)$.

**Proof.** Let us consider an auxiliary system

$$\tilde{\tilde{X}}_t = \frac{\nabla F(\tilde{X}_t)}{|\nabla F(\tilde{X}_t)|} \times d(\tilde{X}_t), \quad \tilde{\tilde{X}}_0 = x(z),$$

which is a time change of system (5.1). Take any closed non self-intersecting curve $\gamma$ on $M$ bounding a region $D(\gamma)$ on $M$. Let the unit vector field $e_1$ be outward normal to $\gamma$, but tangent to $M$. Let $e_3 = \frac{\nabla F}{|\nabla F|}$. Let the unit vector field $e_2$ be tangent to $\gamma$ and $M$: $e_2 = e_3 \times e_1$. We have

$$\oint_\gamma \left( \frac{\nabla F}{|\nabla F|} \times d \right) \cdot e_1 dl$$

$$= - \oint_\gamma (d \times e_3) \cdot e_1 dl$$

$$= - \oint_\gamma d \cdot (e_3 \times e_1) dl = - \oint_\gamma d \cdot e_2 dl = - \iint_{D(\gamma)} \nabla \times d dm = 0.$$
trajectories is similar to the case of area-preserving systems on a two-dimensional torus. The general structure of an area-preserving flow on a torus is described in [2] (Also see [19, Theorem 3.1.7]. Here not exactly the area is preserved, but a measure with strictly positive and bounded density. Then the structure of the trajectories of the dynamical system (5.1) in each domain $U_k$ for $k = 1, \ldots, n$, bounded by the separatrices of the flow, such that the trajectories of the dynamical system (5.1) in each $U_k$ behaves as in a part of the plane: they are either periodic or tend to a point where the vector field is equal to zero. Outside of the domains $U_k$ the trajectories form one ergodic class. Let this ergodic class be $E = \mathbb{T}^2 \setminus \bigcup_{k=1}^n \overline{U}_k$ (here and below $\overline{U}_k$ is the closure of $U_k$). Within each $U_k$ the system (5.1) behaves like a standard Hamiltonian system with a Hamiltonian $H_k$. For brevity let us assume that each $U_k$ contains only one maxima or minima of $H_k$ and no saddles (the case when there is a saddle can be resolved using the results of previous sections). Let us denote the maxima or minima of $H_k$ in $U_k$ by $M_k$. Let $A_k$ be the saddles of (5.1) on $\mathbb{T}^2$: $A_k$ is situated on the boundary of $U_k$. Let us introduce a family of functions $h_k(x) = H_k(x) - H_k(A_k)$ when $x \in U_k$ and $h_k(x) = 0$ when $x \in \mathbb{T}^2 \setminus \overline{U}_k$, $k = 1, \ldots, n$. Let the set $\{x \in \overline{U}_k : h_k(x) = h\}$ be $\gamma_k(h_k)$. We notice that $\gamma_k(0)$ is the separatrix bounding $U_k$ and containing $A_k$.

Identify all points of the ergodic class $E$ as well as the points belonging to each level set of each function $H_k(x), x \in \overline{U}_k$. Let $\mathcal{Y}$ be the identification mapping. Then $\mathcal{Y}(M)$, in the natural topology, is homeomorphic to a graph $G$. This graph is a tree, and $\mathcal{Y}$ maps the entire ergodic class $E$ to the root of the graph which is denoted by $O$. Let $\gamma_k(h) = \{x \in \overline{U}_k : h_k(x) = h\}$. Define a metric $\rho(y_1, y_2)$ on $G$ as follows: If $y_1 = \mathcal{Y}(\gamma_k(h_1)), y_2 = \mathcal{Y}(\gamma_l(h_2))$, put $\rho(y_1, y_2) = |h_1 - h_2|$ for $k = l$, and $\rho(y_1, y_2) = \rho(y_1, O) + \rho(O, y_2)$ if $k \neq l$. In this way the region $\overline{U}_k$ will be mapped into a segment $I_k$ of the form either $[0, h_k(M_k)]$ (if $M_k$ is a maximum) or $[h_k(M_k), 0]$ (if $M_k$ is a minimum).
Every point \( y \) as \( Y \) of the edge containing \( y \) \( A \) governed by a generator \( A \) is the period of one rotation along \( \gamma \).

We also recall that we have the non-degeneracy conditions of \( a \) to our system (5.1). After the time change \( t \rightarrow \frac{t}{\varepsilon} \), our perturbed system has the form

\[
\dot{X}_i^{\varepsilon, \delta} = \frac{1}{\varepsilon} \nabla F(X_i^{\varepsilon, \delta}) \times d(X_i^{\varepsilon, \delta}) + \nabla F(X_i^{\varepsilon, \delta}) \times p(X_i^{\varepsilon, \delta}) + \delta \sigma(X_i^{\varepsilon, \delta}) \dot{W}_i, \quad X_0^{\varepsilon, \delta} = x_0(z). \tag{5.2}
\]

Here \( p(\bullet) \) is a smooth vector field in \( \mathbb{R}^3 \) and \( \sigma \) is the same matrix defined in Section 4. We remind the reader that \( \sigma^T \nabla F = 0 \) and \( a = (a_{ij}) = \sigma \sigma^T \) is the diffusion matrix. We also recall that we have the non-degeneracy conditions of \( a \) on \( M: \mathbf{e} \cdot (a(x) \mathbf{e}) \geq \underline{a} ||\mathbf{e}||^2 \) for some \( \underline{a} > 0 \) and all \( \mathbf{e} \) such that \( \mathbf{e} \cdot \nabla F = 0 \). The process \( X_i^{\varepsilon, \delta} \) lives on the surface \( M \).

Let us define a strong Markov process \( Y_i^{\delta} \) on \( G \) as the diffusion process on \( G \) governed by a generator \( A \) such that, at each interior point \( (k, h_k) \) of an edge \( I_k \), \( Af(k, h_k) = T_k f(k, h_k) \), where

\[
T_k f(k, h_k) = \frac{1}{T_k(h_k)} \left( a_k(h_k) + \frac{\delta^2}{2} a_{1,k}(h_k) + \frac{\delta^2}{2} a_{2,k}(h_k) \right) \frac{\partial f}{\partial h_k} + \frac{\delta^2}{2} \frac{1}{T_k(h_k)} b_k(h_k) \frac{\partial^2 f}{\partial h_k^2}, \tag{5.3}
\]

with

\[
\begin{align*}
a_k(h_k) &= \oint_{\gamma_k(h_k)} \nabla H_k \cdot (\nabla F \times p) \frac{dl}{|\nabla F \times d|}, \\
a_{1,k}(h_k) &= \oint_{\gamma_k(h_k)} \nabla H_k \cdot \Sigma \frac{dl}{|\nabla F \times d|}, \\
a_{2,k}(h_k) &= \oint_{\gamma_k(h_k)} \sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 H_k}{\partial x_i \partial x_j} \frac{dl}{|\nabla F \times d|}, \\
b_k(h_k) &= \oint_{\gamma_k(h_k)} |(\nabla H_k)^T \sigma|^2 \frac{dl}{|\nabla F \times d|},
\end{align*}
\tag{5.4}
\]

and

\[
T_k(h_k) = \oint_{\gamma_k(h_k)} \frac{dl}{|\nabla F \times d|}
\]

is the period of one rotation along \( \gamma_k(h_k) \). Here the vector \( \Sigma \) is the same vector as in Section 4.

The domain \( D(A) \) of \( A \) consists of those functions \( f \) that are continuous on \( G \) and have the following properties.

- Function \( f \) is twice continuously differentiable in the interior of each of the edges.
We have the one sided limits \( \lim_{h_k \to 0} \mathcal{T}_k f(k, h_k) \) and \( \lim_{h_k \to h_k(M_k)} \mathcal{T}_k f(k, h_k) \) at the endpoints of each of the edges. The values of the limit \( q = \lim_{h_k \to 0} \mathcal{T}_k f(k, h_k) \) are the same for all the edges.

The following gluing condition is satisfied at \( O \):

\[
\sum_{k=1}^{n} (\pm) \beta_k \lim_{h_k \to 0} \frac{\partial f}{\partial h_k}(k, h_k) = q ,
\]

with sign + if \( A_k \) is a local minimum of \( H_k \) restricted on \( U_k \) and sign − otherwise. Here

\[
\beta_k = \frac{1}{\lambda(E)} \int_{\gamma_k(0)} |(\nabla H_k)^T \sigma|^2 \frac{dl}{|\nabla F \times d|}
\]

with

\[
\lambda(E) = \int_E \frac{dm}{|\nabla F|} .
\]

(Here \( dm \) is the area element on \( M \).) These conditions define the process \( Y_t^{\delta} \) on \( G \) in a unique way.

We have the following

**Theorem 5.1.** The process \( Y_t^{\varepsilon, \delta} = \mathcal{W}(X_t^{\varepsilon, \delta}) \) converges weakly in the space of continuous trajectories \([0, T] \to G\) as \( \varepsilon \downarrow 0 \) to \( Y_t^{\delta} \).

The proof of this theorem is an application of Theorem 1 of [6]. To be precise, in formula (5) of [6], we set \( \kappa = \delta^2 \), \( v(X_t^{\varepsilon, \delta}) = \nabla F(X_t^{\varepsilon, \delta}) \times d(X_t^{\varepsilon, \delta}) \), \( \beta(X_t^{\varepsilon, \delta}) = \nabla F(X_t^{\varepsilon, \delta}) \times p(X_t^{\varepsilon, \delta}) \), \( u(X_t^{\varepsilon, \delta}) = \frac{\bar{c}(X_t^{\varepsilon, \delta})}{2} \) (a term which comes from the Stratonovich integral), \( \sigma(X_t^{\varepsilon, \delta}) = \sigma(X_t^{\varepsilon, \delta}) \). Furthermore, we can write down the generator \( L \) of \( X_t^{\varepsilon, \delta} \) in self-adjoint form

\[
Lu = \frac{\delta^2}{2} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{3} a_{ij} \frac{\partial u}{\partial x_j} \right) + \left( \frac{1}{\varepsilon} \nabla F \times d + \nabla F \times p - \frac{\delta^2}{2} \Pi \right) \cdot \nabla u .
\]

Here \( \Pi \) is a 3-vector with the \( i \)-th component \( \Pi_i = \sum_{j=1}^{3} \frac{\partial \sigma_{kj}}{\partial x_k} \sigma_{ij} \). Notice that since \( \sigma^T \nabla F = 0 \), we have \( \nabla F \cdot \Pi = \sum_{j,k=1}^{3} \frac{\partial \sigma_{kj}}{\partial x_k} \sum_{i=1}^{3} \sigma_{ij} \frac{\partial F}{\partial x_i} = 0 \). Also notice that since we have checked the fact that \( F(X_t^{\varepsilon, \delta}) \) is a constant of motion, Itô’s formula imply \( LF(x) = 0 \). Therefore, we have \( LF(x) = 0 \) where

\[
\mathcal{L}u = \frac{\delta^2}{2} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{3} a_{ij} \frac{\partial u}{\partial x_j} \right) .
\]
From here we see that the auxiliary process $X^{ε,δ}_t$, $X^{ε,δ}_0 = x_0(z)$ corresponding to the operator $L$ lives on the surface $M = M(z)$. Since $L$ is self-adjoint in $R^3$, the (degenerate) process $X^{ε,δ}_t$ has an invariant measure proportional to $R^3$ Lebesgue measure. This implies, that the process $X^{ε,δ}_t$, viewed as a non-degenerate diffusion process on $M$, has a unique invariant measure with density proportional to $1/|∇F(x)|$. Since we have checked that the deterministic flow (5.1) on $M$ also has an invariant measure with density proportional to $1/|∇F(x)|$, we see that the auxiliary process $X^{ε,δ}_t$, $X^{ε,δ}_t = x_0(z)$ governed by the operator

$$ Lu = \frac{δ^2}{2} \sum \frac{∂}{∂x_i} \left( \sum_{j=1}^{3} a_{ij} \frac{∂u}{∂x_j} \right) + \left( \frac{1}{ε} ∇F \times d \right) \cdot ∇u $$

is a non-degenerate diffusion process on $M$ with a unique invariant measure which has a density proportional to $1/|∇F(x)|$. This fact, together with the standard method of absolutely continuous change of measure (see [11] and compare with Appendix.2), allow us to calculate the gluing condition (5.5).

Since the small random perturbation term $δσ \circ \dot{W}_t$ in (5.2) is only introduced as a regularization, we must study the limit of $Y_t^δ$ as $δ \downarrow 0$. It follows from the same argument as in Section 3 of [6] that the limiting process $Y_t$ should be described as follows. Let

$$ \overline{ψ}_k = 2 \oint_{γ_k(0)} ∇H_k \cdot (∇F \times p) \frac{dl}{|∇F \times d|} \neq 0. $$

Let $s_k$, $1 ≤ k ≤ n$, take values 0 and 1. We set $s_k = 1$ if $\overline{ψ}_k > 0$ and $M_k$ is a local maximum of $H_k$ as well as if $\overline{ψ}_k < 0$ and $M_k$ is a local minimum of $H_k$. Otherwise we set $s_k = 0$. Let

$$ r_k = \frac{s_k |\overline{ψ}_k|}{2λ(E)}, 1 ≤ k ≤ n. $$

Then we can describe $Y_t$ as follows.

• The process $Y_t$ is a strong Markov process with continuous trajectories.
• If $Y_0 = O$, where $O$ is the root of $G$, then the process spends a random time $τ$ in $O$. There is a random variable $ξ$ that is independent of $τ$, taking values in the set $\{1, ..., n\}$, such that $Y_t \in I_ξ$ for $t > τ$. If $s_k = 0$ for all $k$, $1 ≤ k ≤ n$ then $τ = ∞$. If $s_k = 1$ for some $k$ then $τ$ is distributed as an exponential random variable with expectation $\sum_{k=1}^{n} r_k$. If $s_k = 1$ for some $k$ then

$$ P(Y_t \in I_k, t > τ) = \frac{r_k}{\sum_{k=1}^{n} r_k}. $$

If $Y_0 \in \text{Int}I_k$ then
\[
\frac{dY_t}{dt} = \overline{B}_k(Y_t)
\]
for $t < \sigma$ where $\sigma = \inf(t : Y_t = 0)$ and $\overline{B}_k(h_k) = \overline{\psi}_k(h_k) / 2T_k(h_k)$.

**Theorem 5.2.** As $\delta \downarrow 0$, the process $Y_\delta^t$ converges weakly in the space of continuous trajectories $[0, T] \to G$, to the process $Y_t$.

The proof is an application of Theorem 2 in [6]. (See the explanation in the proof of Theorem 5.1.)

In the more general situation when the surface $M$ has higher genus, the situation is similar (compare with [7]). In particular, corresponding graph may be not a tree; it can have more than one special vertices where the limiting Markov process spends random time with exponential distribution; transitions between those special vertices are possible.

6 Multiscale perturbations

Equation (2.1) has two first integrals $F(x)$ and $G(x)$. These integrals may have different nature and their perturbations may have different order. Consider the case when the perturbed system has the form

\[
\dot{X}_{t}^{\varepsilon, \kappa} = \nabla F(X_{t}^{\varepsilon, \kappa}) \times \nabla G(X_{t}^{\varepsilon, \kappa}) + \sqrt{\varepsilon} \sigma_1(X_{t}^{\varepsilon, \delta}) \ast \dot{W}_1^t + \sqrt{\varepsilon} \sigma_2(X_{t}^{\varepsilon, \delta}) \ast \dot{W}_2^t,
\]

$\varepsilon, \kappa > 0$, $X_{0}^{\varepsilon, \kappa} = x \in M \subset \{y \in \mathbb{R}^3 : F(y) = z\}$,

where $M$ is a connected component of the level set $\{F(x) = z\}$; $\sigma_1(x)$ and $\sigma_2(x)$ are $3 \times 3$-matrices; $\dot{W}_1^t$ and $\dot{W}_2^t$ are independent white noises in $\mathbb{R}^3$. Put $a_1(x) = \sigma_1(x)\sigma_1^T(x)$, $a_2(x) = \sigma_2(x)\sigma_2^T(x)$. Sign "\ast" in the stochastic terms means that the stochastic integrals are defined in such a way, that the generator of the process $X_{t}^{\varepsilon, \kappa}$ is as follows

\[
L^{\varepsilon, \kappa}u(x) = (\nabla F(x) \times \nabla G(x)) \cdot \nabla u(x) + \frac{\kappa}{2} \text{div}(a_1(x)\nabla u(x)) + \frac{\varepsilon}{2} \text{div}(a_2(x)\nabla u(x)).
\]

We assume that $a_1(x)\nabla F(x) = 0$ and $e \cdot (a_1(x)e) \geq a_1|e|^2$ for each $e$ such that $e \cdot \nabla F(x) = 0$, $a_1$ is a positive constant. The matrix $a_2(x)$ is assumed to be non-degenerate. The assumptions concerning $a_1(x)$ imply that the process $X_{t}^{0, \kappa}$ moves on the surface $M$: $P\{X_{t}^{0, \kappa} \in M\} = 1$. This follows directly from the Itô formula (we refer the reader to the proof of Theorem 5.1 in Section 5, where we did a similar calculation).
Moreover, the process $X_t^{0,\kappa}$ on $M$ is non-degenerate. This implies that, for any $\kappa > 0$, the process $X_t^{0,\kappa}$ has on the compact manifold $M$ (we assume that $\lim_{|x| \to \infty} F(x) = \infty$) a unique invariant measure. On the other hand, the drift in (6.2) is divergence-free and the main part is formally self-adjoint. Therefore the Lebesgue measure is invariant for the process $X_t^{\varepsilon,\kappa}$, and in particular for $X_t^{0,\kappa}$, in $\mathbb{R}^3$. This implies that

$$C = \left( \int_M \frac{dm}{|\nabla F(x)|} \right)^{-1},$$

where $dm$ is the surface area on $M$, is the density of the unique invariant measure of $X_t^{0,\kappa}$ on $M$ for each $\kappa > 0$.

Assume that $0 < \varepsilon << \kappa < 1$. This means that we have relatively large perturbations of the first integral $G(x)$ and much smaller perturbations of $F(x)$. On the time intervals of order $\frac{1}{\varepsilon}$, one can omit the term $\sqrt{\varepsilon} \sigma_2(x) \ast W_t^2$ in (6.1): the first integral $F(X_t^{\varepsilon,\kappa})$ does not change on such intervals as $0 \leq \varepsilon << \kappa << 1$, and the evolution of $G(X_t^{\varepsilon,\kappa})$ asymptotically coincides with the evolution of $G(X_t^{0,\kappa})$ and can be described using the results of Section 4.

But on time intervals of order $\frac{1}{\varepsilon} > \frac{1}{\kappa}$, the situation is different. Consider process $\hat{X}_t^{\varepsilon,\kappa} = \frac{X_t^{\varepsilon,\kappa}}{\varepsilon}$. The process $\hat{X}_t^{\varepsilon,\kappa}$ is governed by the generator $\frac{1}{\varepsilon} L_t^{\varepsilon,\kappa} = \hat{L}_t^{\varepsilon,\kappa}$. It has a fast and a slow components as $\varepsilon \downarrow 0$. The fast component of the process $\hat{X}_t^{\varepsilon,\kappa}$ can be approximated by the process $\hat{\hat{X}}_t^{\varepsilon,\kappa}$ corresponding to the generator

$$\hat{\hat{L}}_t^{\varepsilon,\kappa} u(x) = \frac{1}{\varepsilon} (\nabla F(x) \times \nabla G(x)) \cdot \nabla u + \frac{\kappa}{2\varepsilon} \text{div}(a_1(x) \nabla u).$$

The process $\hat{\hat{X}}_t^{\varepsilon,\kappa}$ lives on the surface $M$ and, up to a simple time change $t \to \frac{t}{\varepsilon}$, coincides with $X_t^{0,\kappa}$. In particular, it has the same invariant density $C|\nabla F(x)|^{-1}$.

To describe the slow component of $\hat{X}_t^{\varepsilon,\kappa}$, one should introduce a graph. Identify points of each connected component of every level set of the function $F(x)$. Let $\mathcal{G}$ be the identification mapping. Then the set $\mathcal{G}(\mathbb{R}^3)$ is homeomorphic to a graph provided with the natural topology which we denote by $\Gamma$.

Note that all connected components of level sets not containing critical points of $F(x)$ are two-dimensional compact (we assume that $\lim_{|x| \to \infty} F(x) = \infty$ manifolds). Each local maximum or minimum of $F(x)$ corresponds to an exterior vertex belonging just to one edge. The saddle points correspond to the interior vertices. Unlike in the case of generic functions of two variables, not every interior vertex belongs to three edges: If $O$ is a saddle point of $F(x)$, the surface $\{ y \in \mathbb{R}^3 : F(y) = F(O) \}$ divides each small neighborhood of $O$ in three parts. But two of these parts, in the case of functions of three variables can come together far from $O$ (compare with [12]). One can introduce a global coordinate system on $\Gamma$: Number the edges of $\Gamma$. Then each point $y \in \Gamma$ can be identified by two numbers $k$ and $z$, where $k$ is the number of an edge containing $y$ and $z = F(\mathcal{G}^{-1}(y))$.  

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The slow component of $X^\varepsilon,\kappa$ is the (not Markovian, in general) process $\mathcal{Y}(\hat{X}^\varepsilon,\kappa) = Y^\varepsilon,\kappa$ on $\Gamma$.

Define a diffusion process $Y_t$ on $\Gamma$ which inside each edge $I_k \subset \Gamma$ is governed by an ordinary differential operator $\mathcal{T}_k = \frac{1}{2T_k(z)} \frac{d}{dz} (\pi_k(z) \frac{d}{dz})$, where

$$T_k(z) = \int_{\mathcal{Y}^{-1}(k,z)} \frac{d\nu}{|\nabla F(x)|} , \quad \pi_k(z) = \int_{G(k,z)} \text{div}(a_2 \nabla F(x)) dx ,$$

where $G(k, z) \subset \mathbb{R}^3$ is the domain bounded by the surface $\mathcal{Y}^{-1}(k, z)$; $a_2(x) = \sigma_2(x) \sigma^T_2(x)$, $dm$ is the area element on $\mathcal{Y}^{-1}(k, z)$.

The operators $\mathcal{T}_k$ define the process $Y_t$ inside the edges. To define the behavior of $Y_t$ at the vertices, we describe the domain $D_A$ of the generator of $Y_t$ (see Ch.8 in [14]).

We say that a continuous on $\Gamma$ and smooth inside the edges function $f \in D_A$ if and only if the following holds.

- The function defined inside the edges by the formula $\mathcal{T}_k f(k, z)$ can be extended to a continuous on the whole graph function.
- If edges $I_{i_1}, I_{i_2}, I_{i_3}$ are attached to an interior vertex $O$, then

$$\sum_{k=1}^3 (\pm \pi_{i_k}(O) D_k f(O) = 0 ,$$

where $\pi_{i_k}(O) = \lim_{z \to F(O)} \pi_{i_k}(z)$ ($\pi_{i_k}(z)$ is defined by (6.3)), and

$$D_k f(O) = \lim_{z \to F(O)} \frac{f(k, z) - f(k, F(O))}{z - F(O)}$$

(compare with [12]). The sign convention in the gluing condition is as follows: Let $\mathcal{Y}^{-1}(I_{i_1})$ belong to the set $\{ x \in \mathbb{R}^3 : F(x) \geq F(O) \}$, and $\mathcal{Y}^{-1}(I_{i_2}), \mathcal{Y}^{-1}(I_{i_3}) \subset \{ x \in \mathbb{R}^3 : F(x) \leq F(O) \}$. Then sign $+$ should be taken in front of $\pi_{i_1}(O)$ and sign $-$ in front of $\pi_{i_2}(O)$ and $\pi_{i_3}(O)$.

- If just two edges $I_{i_1}$ and $I_{i_2}$ are attached to an interior vertex $O$, then $D_{i_1} f(O) = D_{i_2} f(O)$.

For functions $f(k, z)$ with these properties, $A f(k, z) = T_k f(k, z)$. These conditions define the Markov process $Y_t$ on $\Gamma$ in a unique way. Exterior vertices are inaccessible for $Y_t$.

**Theorem 6.1.** The process $Y^\varepsilon,\kappa = \mathcal{Y}(\hat{X}^\varepsilon,\kappa)$ converges weakly in the space of continuous functions $[0, T] \to \Gamma$ for each finite $T > 0$ as $\varepsilon \downarrow 0$ to the (independent of $\kappa$ and $\sigma_1(x)$) process $Y_t$ defined above.
The proof of this statement follows from Theorem 2.1 of [12]. We omit the details. Using the absolute continuity arguments which we mentioned earlier one can consider more general perturbations in (6.1).

Appendix

1. We provide here the proof of Lemma 3.1. By a similar calculation as we did before stating Lemma 3.1 we have

\[
\left| \frac{L(a(z), b(z))}{L(b(z), c(z))} - \frac{\text{Area}(\Box_1)}{\text{Area}(\Box_2)} \right| < \delta_1(\lambda) + \delta_2(\varepsilon) .
\]

(A.1.1)

(Here and below we use symbol \( \delta_k(\mu) \) to denote a positive quantity which goes to zero as the parameter \( \mu \downarrow 0 \).)

We can also check, by mean value theorem, that

\[
\left| \frac{\text{Area}(\Box_1)}{\text{Area}(\Box_2)} - \frac{\int_{\Box_1} \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times \mathbf{b}) \, dm}{\int_{\Box_2} \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times \mathbf{b}) \, dm} \right| < \delta_3(\lambda \varepsilon) .
\]

(A.1.2)

By (3.3), it is easy to check that

\[
\left| \frac{\int_{\Box_1} \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times \mathbf{b}) \, dm}{\int_{\Box_2} \frac{1}{\varepsilon} \nabla F \times \nabla G + \nabla F \times (\nabla F \times \mathbf{b}) \, dm} - \frac{\int_{S_1(z)} \nabla \times (\nabla F \times \mathbf{b}) \cdot \mathbf{n} \, dm}{\int_{S_2(z)} \nabla \times (\nabla F \times \mathbf{b}) \cdot \mathbf{n} \, dm} \right| < \delta_4(\lambda) .
\]

(A.1.3)

By using the averaging principle, it is possible to show that the ratio \( \frac{\text{Area}(\Box_1)}{\text{Area}(\Box_2)} \) is asymptotically preserved along the flow of (2.7) (compare with [4]). Therefore we can take \( \Box_1 \) and \( \Box_2 \) as close to the separatrices hitting and exiting \( O'_2(z) \) as we wish. This fact, together with the estimates (A.1.1)-(A.1.3), imply our Lemma 3.1, by letting first \( \lambda \downarrow 0 \) and then \( \varepsilon \downarrow 0 \).

2. We explain here the missing details in the proof of Theorem 4.1. As we have explained in that proof, our process \( Z_t^{\varepsilon, \delta} \) satisfies the equation

\[
\dot{Z}_t^{\varepsilon, \delta} = \frac{1}{\varepsilon} \kappa(Z_t^{\varepsilon, \delta}) \nabla \mathcal{G}(Z_t^{\varepsilon, \delta}) + \beta_1(Z_t^{\varepsilon, \delta}) + \frac{\delta^2}{2} \tilde{c}(Z_t^{\varepsilon, \delta}) + \delta \tilde{\sigma}(Z_t^{\varepsilon, \delta}) \tilde{W}_t, \quad Z_0^{\varepsilon, \delta} = z_0 = (f_1(x_0(z)), f_2(x_0(z))) .
\]

(A.2.1)
Here the term $\frac{\delta^2}{2}\tilde{c}(Z_{i,t}^{\varepsilon,\delta})$ comes from the Stratonovich integral in (4.9).

As before, we can identify the connected components of the level sets of the Hamiltonian $G$ to obtain a graph $\Gamma$. Let $\Psi$ be the identification mapping. Let us use the same symbols to denote vertices and edges as those we use for the graph corresponding to $X_{i,t}^{\varepsilon,\delta}$ (see Section 2).

System (A.2.1), by a non-singular time change, can be reduced to a perturbed Hamiltonian system with Hamiltonian $G$. The form of the operators governing the limiting diffusion inside the edges is obtained by standard averaging. To get the gluing conditions, we first consider an auxiliary process

$$\hat{Z}_{i,t}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\kappa(\hat{Z}_{i,t}^{\varepsilon,\delta})\nabla G(\hat{Z}_{i,t}^{\varepsilon,\delta}) + \delta\tilde{\sigma}(\hat{Z}_{i,t}^{\varepsilon,\delta})\tilde{W}_t, \quad \hat{Z}_{0}^{\varepsilon,\delta} = z_0. \quad \text{(A.2.2)}$$

Such a process, by a non-singular time change, is equivalent to a perturbed Hamiltonian system which has Lebesgue measure as its invariant measure. Using this fact, via a standard proof of [14, Chapter 8, Section 6], we conclude that the gluing condition for the weak limit of $\Psi(\hat{Z}_{i,t}^{\varepsilon,\delta})$ as $\varepsilon \downarrow 0$ at vertex $O_2(z)$ is given by the coefficients

$$\beta_{2,i} = \int_{f(C(0,z))} |(\nabla G)^T\tilde{\sigma}|^2 d\tilde{z},$$

for $i = 1, 2, 3$. Here $\beta = \kappa\nabla G$.

The measure $\hat{\mu}^{\varepsilon,\delta}$ corresponding to $\hat{Z}_{i,t}^{\varepsilon,\delta}$ ($0 \leq t \leq T$) is related to the measure $\mu^{\varepsilon,\delta}$ corresponding to $Z_{i,t}^{\varepsilon,\delta}$ ($0 \leq t \leq T$) via the Girsanov formula

$$\frac{d\hat{\mu}^{\varepsilon,\delta}}{d\mu^{\varepsilon,\delta}} = I_{0T}^{\varepsilon,\delta} = \exp \left\{ \frac{1}{\delta} \int_0^T \sigma^{-1}(\hat{Z}_{i,t}^{\varepsilon,\delta})[\beta_1(\hat{Z}_{i,t}^{\varepsilon,\delta}) + \frac{\delta}{2}\tilde{c}(\hat{Z}_{i,t}^{\varepsilon,\delta})] \cdot d\tilde{W}_t - \frac{1}{2\delta^2} \int_0^T \left| \sigma^{-1}(\hat{Z}_{i,t}^{\varepsilon,\delta})[\beta_1(\hat{Z}_{i,t}^{\varepsilon,\delta}) + \frac{\delta}{2}\tilde{c}(\hat{Z}_{i,t}^{\varepsilon,\delta})] \right|^2 dt \right\}. $$

**Lemma A.2.1.** There exist constants $A_1 > 0$, $T_0 > 0$ such that $E_{z_0}(I_{0T}^{\varepsilon,\delta} - 1)^2 \leq A_1 T$ for all $T < T_0$.

To prove this lemma, we first apply Itô’s formula to $(I_{0T}^{\varepsilon,\delta} - 1)^2$ and taking expected value. After that we use the fact that

$$E_{z_0}\exp \left\{ \frac{2}{\delta} \int_0^T \sigma^{-1}(\hat{Z}_{i,t}^{\varepsilon,\delta})[\beta_1(\hat{Z}_{i,t}^{\varepsilon,\delta}) + \frac{\delta^2}{2}\tilde{c}(\hat{Z}_{i,t}^{\varepsilon,\delta})] \cdot d\tilde{W}_t - \frac{2}{\delta^2} \int_0^T \left| \sigma^{-1}(\hat{Z}_{i,t}^{\varepsilon,\delta})[\beta_1(\hat{Z}_{i,t}^{\varepsilon,\delta}) + \frac{\delta^2}{2}\tilde{c}(\hat{Z}_{i,t}^{\varepsilon,\delta})] \right|^2 dt \right\} = 1.$$
and the Cauchy-Schwarz inequality. The proof is essentially the same as that of Lemma 2.3 in [11].

For small $\lambda > 0$ we let

$$D_2(\lambda) = \{ x \in \mathbb{R}^2 : G(x) \in [-\lambda, \lambda] \}.$$  

For $i = 1, 3$, we let

$$D_i(\lambda) = \{ x \in \mathbb{R}^2 : G(f(O_i(z))) \leq G(x) \leq G(f(O_i(z))) + \lambda, x \text{ is in the well } D_i(0, z) \text{ containing } O_i(z) \}.$$  

For $k = 1, 2, 3$ we let

$$\tau_{\varepsilon, \delta}^{x, \lambda}(\lambda) = \inf\{ t > 0, Z_t^{\varepsilon, \lambda} \notin D_k(\lambda) \}.$$  

We have

**Lemma A.2.2.** For any positive $\mu > 0$ and $\kappa > 0$ there exists $\lambda_0 > 0$ such that for $0 \leq \lambda < \lambda_0$ for sufficiently small $\varepsilon$ and all $x \in D_2(\lambda)$

$$\mathbb{E}_{z_0} \int_0^{\tau_{\varepsilon, \delta}^{x, \lambda}(\lambda)} \exp(-\mu t) dt < \kappa \lambda,$$

and for all $x \in D_i(\lambda)$ ($i = 1, 3$) we have

$$\mathbb{E}_{z_0} \int_0^{\tau_{\varepsilon, \delta}^{x, \lambda}(\lambda)} \exp(-\mu t) dt < \kappa .$$

The proof of this Lemma is based on corresponding estimates for the process $\hat{Z}_t^{\varepsilon, \delta}$ and Lemma A.2.1. It is essentially the same as that of Lemma 2.4 in [11].

**Lemma A.2.3.** Let $q_i = \frac{\beta_{2,i}}{3}$ where $i = 1, 2, 3$. We have, for any $\kappa > 0$ there exist $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ there exist $\lambda' > 0$ such that for sufficiently small $\varepsilon$ we have

$$\left| \mathbb{P}_{z_0} \{ Z_t^{\varepsilon, \delta} \in C_i((-1)^i \lambda, z) \} - q_i \right| < \kappa$$

for all $x \in D_2(\lambda') \cup \partial D_2(\lambda')$.

The proof of this Lemma is also the same as that of Lemma 2.5 in [11].
The slow component $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ of the process $\hat{Z}_{t}^{\varepsilon,\delta}$ converges weakly as $\varepsilon \downarrow 0$ to a diffusion process $\hat{Y}_{t}^{\delta}$ on $\Gamma$. The process $\hat{Y}_{t}^{\delta}$ is defined by a family of differential operators, one on each edge of $\Gamma$, and by gluing conditions at the vertices. The operators and gluing conditions were calculated in Chapter 8 of [14]. The convergence of $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ to $\hat{Y}_{t}^{\delta}$ was also proved in [14].

To find the weak limit of the slow component $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ of $Z_{t}^{\varepsilon,\delta}$ as $\varepsilon \downarrow 0$, note that the family $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ is weakly compact as $\varepsilon \downarrow 0$. Inside each edge, the limit is a diffusion process with the generator defined by the standard averaging principle. The limiting process $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ and $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ inside an edge, in general, are different. But as it follows from Lemmas A.2.1-A.2.3, the gluing conditions are the same. This implies that the family $\mathcal{Y}(\hat{Z}_{t}^{\varepsilon,\delta})$ converges weakly as $\varepsilon \downarrow 0$ and identifies the limiting process as the process $Y_{t}^{\delta}$ in Theorem 4.1.

3. We indicate here how to calculate the branching probabilities as claimed in Theorem 4.2. Let $Y_{t}^{\delta}$ be the diffusion process on graph $\Gamma$ described in Theorem 4.1. Let $E_{h}(u) = \{ v \in \Gamma : \rho(u, v) < h \}$ for $u \in \Gamma$,

$$\tau_{h}^{\delta} = \min \{ t : Y_{t}^{\delta} \notin E_{h}(u) \} .$$

Let $p_{1}$ and $p_{3}$ be defined as in (3.13) and (3.14). We have

**Lemma A.3.1.** We have, for a small enough $h$,

$$\lim_{\delta \downarrow 0} P_{O_{2}(z)}(Y_{\tau_{h}^{\delta}}^{\delta} \in I_{3}) = 0 ,$$

$$\lim_{\delta \downarrow 0} P_{O_{2}(z)}(Y_{\tau_{h}^{\delta}}^{\delta} \in I_{i}) = p_{i} \text{ for } i = 1, 3 .$$

To prove this Lemma, we let $u = (g, i) \in E_{h}(O_{2}(z))$. We set $v_{j}^{\delta}(u) = v_{j}^{\delta}(g, i) = P_{(g, i)}(Y_{\tau_{h}^{\delta}}^{\delta} \in I_{j})$. The function $v_{j}^{\delta}(g, i)$ is the unique continuous solution of the following problem

$$\begin{align*}
L_{i}v_{j}^{\delta}(g, i) &= 0 , \quad (g, i) \in E_{h}(O_{2}(z)) \setminus \{ O_{2}(z) \} , \quad i = 1, 2, 3 , \\
v_{j}^{\delta}(g, i)\big|_{(g, i)\in \partial E_{h}(O_{2}(z)) \cap I_{i}} &= 0 \text{ for } i \neq j , \\
v_{3}^{\delta}(g, j)\big|_{(g, j)\in \partial E_{h}(O_{2}(z)) \cap I_{j}} &= 1 , \\
\sum_{k=1}^{3}(\pm)\beta_{2,k} \lim_{g \to \partial O_{2}(z)} \frac{\partial v_{j}^{\delta}}{\partial g}(g, k) &= 0 .
\end{align*}$$
Here $T_i$ are defined in (4.5) and $\beta_{2,i}$ are defined in (4.7) and (4.8), with " + " sign for $k = 2$ and " − " sign for $k = 1, 3$. One can solve this problem explicitly and derive the statement of Lemma A.3.1 similarly to Lemma 2.2 of [4].

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