Umkehr Maps Patched via the Compactified Cleavage Operad

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1 Introduction

String topology is concerned with homotopical structures on $M^{S^1}$, the free loop space of $M$. Higher dimensional string topology is similarly concerned with homotopical structures on the mapping spaces $M^N$, where $N$ is a suitable manifold. We wish to give a spectral description of higher-dimensional spherical string topology. That is, this paper revolves around the following

**Theorem 1.1** For $M$ a compact orientable manifold, there are maps in the category of spectra

$$
\left((\text{Cleav}_{S^n} \rtimes \text{SO}(n + 1))(-; k)\right) \times \left(M^{S^n}\right)^k \rightarrow \left(M^{S^n}\right) \wedge S^{\dim(M) - (k-1)}
$$

such that taking homology, we get an action of $H_*(\text{Cleav}_{S^n} \rtimes \text{SO}(n + 1))$ on $H_*(M^{S^n})$ that generalizes string topology, where $H_*(-) := H_*(-; \dim(M))$.

The coloured operad $\text{Cleav}_{S^n}$ and its semidirect product $\text{Cleav}_{S^n} \rtimes \text{SO}(n + 1)$ are given in [Bar11, Ch. 3+6]. The crucial part of our argument for proving the theorem above will be to provide an action of $\text{Cleav}_{S^n}$, from here the extended action of the semidirect product slits into place.

One naïve approach would be to – pointwise, for any $[T, P] \in \text{Cleav}_{S^n}(-; k)$ – provide umkehr maps $\varphi^1: \left(M^{S^n}\right)^k \rightarrow M_{[T, P]}^{S^n}$ residing in the category of spectra for the correspondence $M^{S^n} \longrightarrow M_{[T, P]}^{S^n} \xrightarrow{\varphi^1} \left(M^{S^n}\right)^k$ described in [Bar11, Ch. 4]. There are several methods available in the literature for obtaining umkehr maps for a fixed $[T, P]$; as a homotopical application of [Kle01] we give one such method in section 3.

Being able to construct umkehr maps pointwise only hints at the potential for the map 1. It does not automatically give these umkehr maps parametrized by $\text{Cleav}_{S^n}(-; k)$ as a topological entity. The bulk of this paper is aimed at patching umkehr maps obtained locally – closely related to the pointwise version – into a global umkehr map.
The duality theory that goes into umkehr maps is interlinked with that of Poincaré duality; since we are going for a global umkehr map Poincaré duality comes in play not only for the manifold $M$ – but also for the spaces involved in the operad $\text{Cleav}_{S^n}$.

In [Bar11], we show that $\text{Cleav}_{S^n}(\cdot; k)$ as a space is weakly equivalent to ordered euclidean $(n + 1)$-dimensional configuration spaces on $k$ points. Poincaré duality does not work well for non-compact spaces such as these, and therefore we are lead to consider the larger punctured cleavage operad $\text{Cleav}^\text{p}_n$, which essentially allows for the cleaving hyperplanes, defining $\text{Cleav}_{S^n}$ to be tangential to $S^n$.

Having cleaving hyperplanes become tangential to $S^n$ means that we allow a point in $\text{Cleav}^\text{p}_n(\cdot; k)$ to be a set consisting of cleaving hyperplanes, along with some points – or punctures – decorated with elements of the commutative operad. The arity of the commutative operad counts the amount of tangential hyperplanes present at the point. The topology of $\text{Cleav}^\text{p}_n(\cdot; k)$ is rigged such that the punctures can transfer between being points and hyperplanes cleaving $S^n$. This is illustrated in picture II where we have shown four steps in a path of $\text{Cleav}_{S^n}(S^n; \tau)$.

![Diagram](image)

Figure 1: Starting from (A) the central cleaving hyperplane moves upwards, transferring cleaving hyperplanes to punctures along the way. From (C) punctures slide away, giving a lower arity of the associated commutative operads.
As indicated in picture 1, punctures can freely slide past cleaving hyperplanes and other punctures. This simplifies the homotopical type of the operad compared to $\text{Cleav}_{S^n}$, among other things making it compact, and we have the following:

**Theorem 1.2** There is a homotopy equivalence

$$\overrightarrow{\text{Cleav}}_{S^n}(-;k) \simeq (S^n)^{k-1}.$$ 

While the explicit homotopy is slightly more involved, the above homotopical identification bears resemblance to what one obtains by extending ordered $(n+1)$-dimensional euclidean configuration spaces to allow all but one special point to agree with each other; in this case a homotopy to $(S^n)^{k-1}$ is obtained by translating all point so that the special point is at the origin of $\mathbb{R}^{n+1}$, and subsequently bringing all other points but the special point in unit distance to the origin of $\mathbb{R}^{n+1}$.

We do indeed construct an action of $\overrightarrow{\text{Cleav}}_{S^n}$ similar to Theorem 1.1. We do this by decomposing $\overrightarrow{\text{Cleav}}_{S^n}(-;k)$ into smaller spaces that naturally act on a suitably downshifted spectral version of $M^{S^n}$. A homotopy colimit patches the action together to a global action.

What we hereby obtain is however not a generalization of string topology; rather, it is an extension of the intersection product of the underlying manifold in $M^{S^n}$. Extended in the sense that the topology of $\text{Cleav}_{S^n}$ provides a higher bracket; taking the limit we however get associated to the mapping space $M^{S^n}$, homotopy equivalent to $M$, an $E_{\infty}$-action whose homology recovers the intersection product.

The Chas-Sullivan product is not obtained by just restricting the above action to $\text{Cleav}_{S^n}$. In the last section of this paper, we show how the action of the operad $\overrightarrow{\text{Cleav}}_{S^n}$ can be modified produce an action of $\text{Cleav}_{S^n}$ that does recover string topology; this modification basically disallows parametrised mappings to tend towards zero length by seeing the timber of $\text{Cleav}_{S^n}(-;k)$ as segments of a sphere that parametrizes the entire $k$-ary action. The direct and more trivial action hence provides a homotopical link to the original string topology inspiration from intersection theory [CS99].

The operad $\overrightarrow{\text{Cleav}}_{S^n}$ can hence be seen as an intermediate operad that tells a tale about the fragility of string topology. From birth, string topology has been born with defensive mechanisms that disallow parametrized loops to tend towards zero length. Our tour around giving a spectral version of string topology hence shows that these defensive mechanisms are essential to provide the full structure of string topology.

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2 The Compactified Cleavage Operad

By the definition of the Cleavage Operad, [Bar11, Ch. 3.1], a point of the \( k \)-ary operations of the coloured operad \( \text{Cleav}_{S^n} \), denoted \([T, P] \in \text{Cleav}_{S^n}(-; k)\) is determined by a certain set of hyperplanes \( \{P_1, \ldots, P_{k-1}\} \) decorated on the binary tree \( T \), subject to an equivalence relation called chop-equivalence. As mentioned in the introduction, we shall extend \( \text{Cleav}_{S^n} \) through a larger coloured operad \( \overrightarrow{\text{Cleav}}_{S^n} \) with marked points mimicking the cleaving hyperplanes \( P \) vanishing towards tangent planes of \( S^n \).

**Definition 2.1** Given \( U \in \text{Timber}_{S^n} \), we let the \( U \)-punctured operad with \( k \)-ary term be given by the space

\[
\mathcal{Pun}_U(k) := \left( \coprod_{k=k_1 + \cdots + k_j} U^j \times \text{Comm}(k_1) \times \cdots \times \text{Comm}(k_j) \right) / \sim
\]

Here, the disjoint union is taken over all partitions of \( k \in \mathbb{N} \), meaning that \( j \) vary and \( k_i > 0 \) for all \( i \in \{1, \ldots, j\} \). The commutative operad \( \text{Comm}(k_i) \) has operadic output labelled by the number \( 1, \ldots, k_i \).

The equivalence relation \( \sim \) is given by whenever \((u_1, \ldots, u_j)\) has \( u_p = u_q \) where \( p < q \) we equavalate this point with the point \((u_1, \ldots, \hat{u}_q, \ldots, u_j)\) in the space indexed by the partition \( k = k_1 + \cdots + k_{p-1} + (k_p + k_q) + k_{p+1} + \cdots + k_q + \cdots + k_j \), where \( \hat{\cdot} \) is to be interpreted as omitting the symbol the hat is over from the formulae.

In order to obtain an operadic structure on \( \mathcal{Pun}_U \), we relabel the outputs of \( \text{Comm}(k_i) \) for each \( i \in \{1, \ldots, j\} \) monotonely by the number \((\sum_{m=1}^{i-1} k_m) + 1, \ldots, \left(\sum_{m=1}^{i-1} k_m\right) + k_i\), and let \( \circ_m \)-composition be induced from the \( \text{Comm}(k_m) \)-constituent.

As in picture [1] of the introduction, a point in \( \mathcal{Pun}_U(k) \) is an ordered configuration of \( j \) points in \( U \) decorated with \( j \) corollas, with the corolla associated to \( x_i \) having \( k_i \) leaves, and representing an element of the commutative operad \( \text{Comm}(k_i) \). We call a point labelled by \( \text{Comm}(k) \) a \( k \)-clustering of punctures.

To define a coloured operad, where the punctures mimick tangential hyperplanes, we first of all need to define the space of colours that the operad lives over. From [Bar11, Ch 3.2], we have a space of timber, denoted \( \text{Timber}_{S^n} \). In this space, a point is given by a certain open subset of \( S^n \). We extend the space to \( \overrightarrow{\text{Timber}}_{S^n} \). As a set it will be given by \( \text{Timber}_{S^n} \cup S^n \), and we endow it with a topology from \( \text{Timber}_{S^n} \) by taking limits of sequences \( \{U\} \in \text{Timber}_{S^n} \) that converge towards \( s \in S^n \). Hereby, points of \( \overrightarrow{\text{Timber}}_{S^n} \) will be given by certain open submanifolds of \( S^n \), called timber, as well as points \( s \in S^n \) called punctures.

**Construction 2.2** To defined the Compactified Cleavage Operad, let \( \Sigma_k \) be the permutation group of the letters \( \{1, \ldots, k\} \). To \( U \in \overrightarrow{\text{Timber}}_{S^n} \), let \( \overrightarrow{\text{Cleav}}_{S^n}(U; i) = \emptyset \) if \( U \) is a puncture.

We let

\[
\overrightarrow{\text{Cleav}}_{S^n}(U; k) := \left( \coprod_{i=1}^{k} \overrightarrow{\text{Cleav}}_{S^n}(U; i) \times \mathcal{Pun}_U(k - i) \right) \times_{\Sigma_k} \Sigma_k
\]
Where $\Sigma_k$ acts on the right on itself by multiplication, and on the left by permuting outgoing colours of the two operads for every $i \in \{1, \ldots, k\}$.

Using $\mathrm{Timber}_{S^n}$, we put a topology on $\mathrm{Cleav}_{S^n}(U; k)$ that connects the disjoint unions of (2). Given a sequence $\{(T, \mathcal{P}), x_{i+1}, \ldots, x_k\}$ in the space $\mathrm{Cleav}_{S^n}(U; i) \times \mathcal{Pun}_U(k - i)$ such that if we take a sequence associated to the $j$'th output of $\{(T, \mathcal{P})\}$ as a sequence in $\mathrm{Timber}_{S^n}$, this sequence converges towards the puncture $y \in S^n \subset \mathrm{Timber}_{S^n}$.

As a limit point to the sequence $\{(T, \mathcal{P})\}$, we assign the element of $\mathrm{Cleav}_{S^n}(U; i - 1) \times \mathcal{Pun}_U(k - (i - 1))$, indexed by $i - 1$ in the disjoint union of (2), given by the following: Delete the hyperplane $P'$ that gives rise to the $j$'th output of $\{(T, \mathcal{P})\}$ to obtain the cleaving element $[T', \mathcal{P}]$. Note that the definition of $\mathrm{Cleav}_{S^n}$ ensures that the hyperplane $P'$ is uniquely defined from a certain step in the sequence, where it is sufficiently close to the tangenthyperplane of $S^n$ at $y$.

The limit point will thus be given by the limit of $\sigma_{j,i+1} \cdot \{(T', \mathcal{P}), y, x_1, \ldots, x_k\}$, where $\sigma_{j,i+1} \in \Sigma_k$ acts on every element in the sequence, and is the transposition that interchanges $j$ with $i + 1$, and hence lets $y$ be the output labelled by the output of $\mathrm{Cleav}_{S^n}(U; i)$ we are forgetting when passing to $\mathrm{Cleav}_{S^n}(U; i - 1)$.

Remark 2.3 An immediate consequence of the construction of the coloured operad $\overrightarrow{\mathrm{Cleav}}_{S^n}$ is that there is an inclusion of coloured operads $\overrightarrow{\mathrm{Cleav}}_{S^n} \hookrightarrow \overrightarrow{\mathrm{Cleav}}_{S^n}$, induced by the inclusion of $\overrightarrow{\mathrm{Cleav}}_{S^n}(U; k)$ into the $i = k$ part of (2).

To an element $\chi \in \overrightarrow{\mathrm{Cleav}}_{S^n}(U; k)$, let $[T, \mathcal{P}]_\chi \in \overrightarrow{\mathrm{Cleav}}_{S^n}(U; i)$ denote the element in the first factor of (2) indexed by $i$, describing cleaving hyperplanes, and $P_\chi \in \mathcal{Pun}_U(k - i)$ the factor of the second, describing punctures. In other words, we let $(p_{i_1}, \ldots, p_{i_k})_\chi$ denote the configuration of $k - i$ punctures in $U$, counted with clustering multiplicity in the sense that points who agree in $U$ are both counted.

Construction 2.4 To $\chi \in \overrightarrow{\mathrm{Cleav}}_{S^n}(-; k)$ we shall form a diagram akin to the action diagram for $\overrightarrow{\mathrm{Cleav}}_{S^n}$ acting upon $M^{S^n}$ given in [Bar11, 4.2]. That is, we consider the following pullback diagram:

$$
\begin{array}{ccc}
M^{S^n} & \to & (M^{S^n})^k \\
\downarrow & & \downarrow N(\sigma_\chi) \\
M^{\sigma_0(\beta_\chi)} & \to & B \\
\end{array}
$$

Whenever $\chi \in \overrightarrow{\mathrm{Cleav}}_{S^n}(-; k)$, that is $\chi$ exhibits no punctures, the diagram will be precisely the diagram given in [Bar11, 4.2], with $B = M^{1_{\mathrm{disc}} \cup N_i}$.

To account for the punctures, we impose the following modifications to extend the action for a general element $\chi \in \overrightarrow{\mathrm{Cleav}}_{S^n}(-; k)$:

- We let the blueprint of $\chi$ be given as the following subset of $D^{n+1}$:

$$
\beta_\chi := \beta_{[T, \mathcal{P}]_\chi} \cup \{p_1\} \cup \cdots \cup \{p_m\}
$$

where $\beta_{[T, \mathcal{P}]_\chi}$ is described in [Bar11, 4.1], and $p_1, \ldots, p_m$ the punctures of $\chi$.  

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To properly handle the punctures, if \( i \in \{1, \ldots, k\} \) labels a puncture \( p_j \) of \( \chi \), we in \( B \) replace the factor given by \( M^{C_N} \) with \( M^{S^n}_{/ \sim_{p_j}} \), where \( \sim_{p_j} \) is the equivalence relation that identifies \( f \) with \( g \) if they both satisfies \( f(-p_j) = g(-p_j) \). This hence gives a homeomorphism \( M^{S^n}_{/ \sim_{p_j}} \cong M \). We are considering this quotient by \( \sim_{p_j} \) rather than \( M \) directly, for extending this pointwise action to an actual action of \( \text{Cleav}_{S^n} \).

The pullback space \( M^{S^n}_{\chi} \) will no longer be identified as a subspace of \( M^{S^n} \), since whenever there is a puncture \( p_i \), this amounts to a space of maps from \( S^n \) that are constant along the blueprint, as well as an extra sphere that have been joined as a wedge at the point \( p_i \). While we don’t have a canonical inclusion map from this space into \( M^{S^n} \), we nonetheless have a map to \( M^{S^n} \) given by forgetting all extra the spheres that have been wedged on. This map will be used to describe the action of \( \text{Cleav}_{S^n} \).

### 3 Umkehr Maps Along Manifolds

Recall that a surjective map \( p: A \to B \) is called a quasifibration if for every \( b \in B \), the canonical map from \( p^{-1}(b) \) to the homotopy fiber at \( b \) is a weak equivalence.

Let \( F \to X \overset{p}{\to} Y \) denote a quasifibration \( p \), with \( F \) the fiber. Extending the quasifibration via the homotopy fiber \( F_p := \{(x, f) \in X \times Y \mid f(0) = *, f(1) = p(x)\} \), leads to the usual fibration sequence \( \Omega Y \to F_p \to X \to Y \).

Note by the above that the monoid \( \Omega Y \) acts on \( F_p \) by concatenation of based loops, and the associated Borel construction \( E\Omega Y \times_{\Omega Y} F_p \) fits into a morphism of fibration sequences

\[
\begin{array}{ccc}
F_p & \to & E\Omega Y \times_{\Omega Y} F_p \\
\downarrow & & \downarrow \quad \text{pr} \\
F & \to & X \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
\]

Assuming that \( Y \) is path connected, the two outmost morphisms are the standard homotopy equivalences, and the middle morphism is the projection onto \( X \) from \( F_p \). We shall use the notation \( F_{h\Omega Y} := E\Omega Y \times_{\Omega Y} F_p \). The long exact sequence of homotopy groups now tells us that we have a weak equivalence:

**Lemma 3.1** \( F_{h\Omega Y} \cong X \)

The way we have introduced quasifibrations, they are not necessarily preserved by taking pullbacks, consider therefore the pullback-diagram, where we assume that the pullback, \( f^* \) of the quasifibration \( f \) is a quasifibration.

\[
\begin{array}{ccc}
f^*(X) & \overset{\varphi^*}{\to} & X \\
\downarrow & & \downarrow \\
A & \overset{\varphi}{\to} & B
\end{array}
\]
This has the effect that we can identify the map $\varphi^*$ as a map $\varphi^*: F_{h\Omega A} \to F_{h\Omega B}$.

In [Kle01], the dualizing spectrum of a group $G$ is the $G$-equivariant function spectrum $\text{Map}_G(EG_+, G)$, with source and target suspension spectra. From [Kle01, Th. D] there are norm maps $\eta_B: F_{h\Omega B} \wedge D_{\Omega B} \to F_{h\Omega B}$ and $\eta_A: F_{h\Omega A} \wedge D_{\Omega A} \to F_{h\Omega A}$.

Here, $X^{hG}$ denotes the homotopy fixed-point spectrum $\text{Map}(EG_+, X)^G$, that is the fixed point of the associated mapping space, where $X$ is a $G$-spectrum.

**Definition 3.2** Taking the induced map of $\varphi: A \to B$ produces an umkehr map $\varphi^! = \text{Map}(E(\varphi)_+, X): F_{h\Omega B} \to F_{h\Omega A}$.

This map is what leads to the following:

**Lemma 3.3** If $A$ and $B$ in [H] are Poincaré duality spaces, and $f$ is a quasifibration, then the umkehr map $\varphi^!$ is up to homotopy specified as a map in spectra $\varphi^{h!}: X \wedge D_{\Omega B} \to f^*(X) \wedge D_{\Omega A}$.

**Proof.** Since $A$ and $B$ are Poincaré duality spaces, [Kle01, Th. D] gives us that the norm-maps involved in the following string of morphisms are homotopy-equivalences of spectra:

$$
\begin{align*}
X \wedge D_{\Omega B} &\longrightarrow F_{h\Omega B} \wedge D_{\Omega B} \quad \eta_B \quad F_{h\Omega B} \\
\downarrow &\quad \downarrow \\
f^*(X) \wedge D_{\Omega A} &\longrightarrow F_{h(\Omega A)} \wedge D_{\Omega A} \quad \eta_A \quad F_{h\Omega A}
\end{align*}
$$

The two extremal morphisms are homotopy equivalences by 3.1.

From [Kle01, Th. A], we furthermore have that $D_{\Omega A}$ is a sphere spectrum of dimension $-\dim(A)$. The proof of 3.3 relies on the norm map, which in [Kle01, Ch. 3] is constructed as a morphism that only lives in the homotopy category of topological spaces. When we use the notation $\varphi^{h!}: X \wedge D_{\Omega B} \to f^*(X) \wedge D_{\Omega A}$, we shall similarly mean a morphism residing in the homotopy category of topological spaces.

Consider a diagram of the form

$$
\begin{array}{ccc}
X & \xrightarrow{f} & B \\
\downarrow & & \downarrow \varphi \\
X' & \xrightarrow{f'} & B' \\
\end{array}
$$

with the circular morphisms being retractions, $\rho_E \circ \iota_E = 1_E$, $\rho_B \circ \iota_B = 1_B$ and $\rho_A \circ \iota_A = 1_A$ — and where these retractive properties are commuting with the morphisms $f$ and $\varphi$.

Assume that $f$ and $f'$ are quasifibrations, with fibers $F$ and $F'$.

The morphisms $\iota_X$ and $\iota_{f^*(X)}$ induced from the diagram [6] gives a diagram in the homotopy category of topological spaces.
E \wedge D_{\Omega B} \xrightarrow{\varphi^h} f^*(E) \wedge D_{\Omega A} \tag{7}

E' \wedge D_{\Omega B} \xrightarrow{f^*(E')} \wedge D_{\Omega A}

of maps, with the top map being the umkehr map of The diagram also gives extension diagrams of loop-spaces

\Omega A \xrightarrow{\iota_A} \Omega A' \xrightarrow{\Omega(A'/A)}

\Omega B \xrightarrow{\iota_B} \Omega B' \xrightarrow{\Omega(B'/B)}

By [Kle01, Th. B+C], there is an equivalence of spectra $D_{\Omega A'} \simeq D_{\Omega A} \wedge D_{\Omega A'/A}$ and $D_{\Omega B'} \simeq D_{\Omega B} \wedge D_{\Omega B'/B}$. Taking the morphism into the first factor hence produces morphisms of spectra that extends (7) by composition to a morphism

E \wedge D_{\Omega B} \xrightarrow{\varphi^h} f^*(E) \wedge D_{\Omega A} \tag{8}

E' \wedge D_{\Omega B'} \xrightarrow{f^*(E')} \wedge D_{\Omega A'}

By (8), there are retraction maps $\hat{\rho}_E$ and $\hat{\rho}_{f^*(E)}$ fitting into the above diagram.

**Lemma 3.4** Assuming that $A, A', B, B'$ are Poincaré duality spaces, in (6) we have the identity

$\varphi^h = \hat{\rho}_E \circ \varphi^h \circ \iota_E$

**Proof.** By [Kle01, Th. A] we have an identification of $D_{\Omega C}$ as a sphere spectrum of dimension $- \dim(C)$ if $C$ is Poincaré duality space. The progression from (7) to (8) is hence a desuspension map, so $\varphi^h$ fits into the lower morphism of (7) and the identity follows.

4 Patching Local Actions

We now go into detail with the promised spectral action of $\overline{\text{Cleav}}_{S^n}(-; k)$ on $M^{S^n}$. We shall do this by decomposing $\overline{\text{Cleav}}_{S^n}$ into smaller spaces that describe actions on $M^{S^n}$. For these smaller spaces, we apply the methods of section 3 to obtain umkehr maps for the constituents. Via an explicit homotopy colimit, we then show that these patch together to a global umkehr map, for the correspondance diagrams given in 2.4.
4.1 Patching Actions In Families

To $U \in \text{Ob}(\text{Cleav}_{S^n}(U; k))$, we cover $\text{Cleav}_{S^n}(U; k)$ by a set of closed subspaces. To $\chi \in \text{Cleav}_{S^n}(U; k)$, let $|\pi_0(\beta_\chi)|$ denote the amount of components of the blueprint.

**Definition 4.1** Let the $m$-evasion space $A_m \subseteq \text{Cleav}_{S^n}(U; k)$ be the subspace given by the elements $\chi \in \text{Cleav}_{S^n}(U; k)$ such that $\chi \in A_m$ has $|\pi_0(\beta_\chi)| = m + 1$

Hereby, $A_0$ will consist of punctured cleavages where all the hyperplanes and punctures involved intersect non-trivially inside $D^{n+1}$. On the other hand $A_{k-2}$ consist of cleaving data where all the involved hyperplanes and punctures give disjoint subsets of $D^{n+1}$.

Let $\overline{A_m}$ denote the componentwise closure of the $m$-evasion space inside $\text{Cleav}_{S^n}(U; k)$. Componentwise, in the sense that if two components $C_1, C_2 \subseteq A_m$ have $\overline{C_1} \cap \overline{C_2} \neq \emptyset$, we let $\overline{C_1}$ and $\overline{C_2}$ be disjoint spaces in $\overline{A_m}$. To describe these components, we introduce the following notation:

For $j \in \{0, \ldots, k-2\}$, partition the set $\{1, \ldots, k-1\}$ into $j + 1$ disjoint, nonempty sets $I_0, \ldots, I_j$ with $\bigcup_{i=1}^j I_i = \{1, \ldots, k-1\}$. For each of these partitions, label the hyperplanes and punctures defining the cleaving configurations by elements of $I_0, \ldots, I_j$. We let $\overline{A}_j^{I_0, \ldots, I_j}$ denote the component of $\overline{A}_j$ where the hyperplanes and punctures labelled by $I_i$ cluster together; that is, precisely the hyperplanes and punctures determined by $I_i$ form a connected blueprint. For a cleaving configuration in $\overline{A}_j^{I_0, \ldots, I_j}$, call the hyperplanes and punctures labelled by $I_i$ a cluster.

This covering defines a category, $\text{Pat}(\{\overline{A}_m\})$ with objects given as intersections of components of elements of $\{A_m\}$ and morphisms given by inclusions of subspaces.

For $\overline{A}_j^{I_0, \ldots, I_j}$ and $\overline{A}_l^{I_1, \ldots, I_l}$, the intersection $\overline{A}_j^{I_0, \ldots, I_j} \cap \overline{A}_l^{I_1, \ldots, I_l}$ of these two subspaces will be equipped with a splice inclusion to the object of $\text{Pat}(\{\overline{A}_m\})$ indexed by the partition spliced from $I_0, \ldots, I_j$ and $I_0', \ldots, I_l'$ where any two $I_p$ and $I_p'$ have been joined together if there is an element of $\{1, \ldots, k-1\}$ that is contained in both $I_p$ and $I_p'$.

For $A \in \text{Pat}(\{\overline{A}_m\})$, call this inclusion $\text{spl}(A)$; we emphasize that for $A = \overline{A}_m$ we let $\text{spl}(A) = A$.

**Lemma 4.2** For any $A \in \text{Ob}(\text{Pat}(\{A_i\}))$, let $\# \text{spl}(A)$ denote the amount of clusters in $\text{spl}(A)$. There is a homotopy equivalence from $A$ to a product of spheres:

$$A \simeq (S^n)^{\# \text{spl}(A)}$$

To prove this, we shall supply an explicit homotopy equivalence. The following type of cleaving configurations will play an essential role:

An element $\chi \in \text{Cleav}_{S^n}(\cdot; m)$ is called an embedded $m$-cactus if $\beta_T \cdot \chi$ is connected and each hyperplane intersects nontrivially with at least one other hyperplane or puncture of $[T, \mathcal{P}]_\chi$ in such a fashion that these intersections are a single point. Without specification of $m$ we shall simply call such clusters of hyperplanes for embedded cacti.
Proof. Applying the homotopy given in [Bar11, 3.17], we can allow ourselves to assume that \( A \) is a subset of \( \text{Cleav}_{S^n}(S^n; k) \). This accounts for the first step in the homotopy we shall supply. The homotopy will have a target where all hyperplanes have been pushed towards punctures, so the resulting product of spheres will be a space of all possible configurations of punctures on \( S^n \) the given space of \( \text{Ob}(\text{Pat} (\{A_i\})) \) exhibits.

To \( a \in A \in \text{Ob}(\text{Pat} (\{A_i\})) \), consider the timber \( U_k \) of \( a \) at the output labelled by \( k \). Note that the only hyperplanes that do not have a well-defined normal-vector pointing away from \( U_k \) are the ones that do intersect \( U_k \) as subsets of \( \mathbb{R}^{n+1} \), but are dominated by other hyperplanes in the cleaving configuration making them not cleave \( U_k \).

The image of \( A \) under the splice inclusion \( \text{spl}(A) \) specifies a partition, stating which hyperplanes and punctures are bound to cluster together as cleaving configurations of \( A \). As we shall see, each of these clusters contribute to precisely one factor in \( (S^n)^\# \text{spl}(A) \).

To describe the homotopy, we shall first assume that all these clusters of hyperplanes and punctures are forming embedded cacti. For each of the clusters that define \( U_k \) as timber by having one of its hyperplanes participate in the boundary of \( U_k \), fix one hyperplane \( P \) that defines part of the boundary of \( U_k \).

Since we have assumed that they sit as embedded cacti, there will for all hyperplanes in these clusters always be a well-defined normal-vector pointing away from \( U_k \).

Take any of the other hyperplanes \( P' \) in the embedded cactus, different from \( P \). Since \( P' \) are sitting in embedded cacti, there is a well-defined point that connects \( P' \) to \( P \), potentially along other hyperplanes in the embedded cactus. Rotating each hyperplane around this point with normal-vector given by the normal-vector \( \nu_{P'} \) is hence well-defined and we run this rotation until the hyperplane becomes a puncture at the point of rotation. All punctures and hyperplanes connected to \( P' \) part of the embedded cactus, we rotate along with \( P' \) - in the sense that for \( P'' \) with a point of rotation at \( P' \), the point of rotation will move for \( P'' \) as \( P' \) rotates. Note furthermore that in this process, \( P' \) might start intersecting other hyperplanes not participating in the same embedded cactus as \( P' \), while doing this we shall consider \( P' \) as dominating these hyperplanes. These hyperplanes of other clusters hence turn into punctures as \( P' \) sweep past. Naturally, these punctures will not follow \( P' \) in the course of the homotopy.

Eventually in the rotation, \( P' \) will be tangential to \( S^n \), and hence turn into a puncture at the point \( P' \) has been rotating about.

The end-result will hence be \( P \) sitting as the single hyperplane in the cluster of hyperplanes and punctures, with all punctures situated on \( P \cap S^n \). We finally translate \( P \) in the direction of its normal-vector, transporting all the punctures from the cluster on \( P \) along with it, until \( P \) itself is a tangent-vector to \( S^n \) and hence becomes a cluster of punctures.

In this final translational process, \( P \) might meet other clusters of hyperplanes. To avoid cases where \( P \) is perpendicular to one of the hyperplanes in the other cluster, we first apply the above procedure to this cluster for the timber directly cleaved by \( P \). Since there are only finitely many clusters of hyperplanes, this iteration of the process terminates in finite time.

To deal with clusters of hyperplanes that are not sitting as embedded cacti, the defect from being an embedded cacti will be given by some hyperplanes that are dominated
by other hyperplanes in the cluster. We can simply allow ourselves to ignore these dominated hyperplanes and apply the procedure to the portion of the cluster that is an embedded cacti. The above procedure will eventually either turn them into a puncture by a hyperplane sweeping past them - or the above procedure will eventually have the hyperplanes participate in the cluster by having them connected by a single point - in which case they participate in an embedded cacti and start rotating by the above procedure.

This hence gives the desired homotopy equivalence where all clusters of hyperplanes specified by $\text{spl}(A)$ give rise to one factor of $S^n$, specified by puncture the clusters of hyperplanes eventually collect into.

Although not directly contained in the lemma above, we note that the proof for the case of $A = A_0$ in the proof above transfers easily to the case of $\text{Cleav}_{S^n}(-; k)$. this computes the homotopy type of $\text{Cleav}_{S^n}$, and hence proves 1.2 given in the introduction.

**Definition 4.3** To $\chi \in \overline{A_m}$, there are associated timber $N^\chi_1, \ldots, N^\chi_k$. We shall generally be working with the closure of the complement of the associated timber $\bigsqcup_{i=1}^k \overline{\bigcup N^\chi_i}$. Whenever $i \in \{1, \ldots, k\}$ does not label a puncture, this give rise to a factor consisting of a disjoint union of wedges of disks and points, as described in [Bar11, 3.8].

A point $\chi \in \overline{A_m} \backslash A_m$ will have $|\pi_0(\beta_\chi)| < m + 1$. Hence, for some $j \in \{1, \ldots, k\}$, the closure inside $S^n$ of the complement of the timber of $\chi$ will have $\bigcup N^\chi_j$ consisting of less components compared to the case of $\chi' \in A_m$. The lesser amount of components of $\bigcup N^\chi_j$ is signified by a wedge $\bigvee_i C_i$ of disks that are disjoint components $\bigsqcup_i C_i$ when $\chi$ moves to $A_m$.

We shall invoke the rule that whenever $\bigvee_i C_i$ occurs as as above in $\overline{A_m}$, the symbol $\bigcup N^\chi_j$ will have $\bigvee_i C_i$ replaced by $\bigsqcup_i C_i$. Hence justifying the name evasion space for $\overline{A_m}$, since when considering its action on mapping spaces, the timber will always ‘evoke collision’ inside the evasion space.

We shall define a functor $F$ that goes from the category $\text{Pat}\left(\{\overline{A_m}\}\right)$ to the subcategory of the diagram-category of topology spaces, given by pullbacks, i.e. diagrams of the form

$$
\begin{array}{ccc}
F(A)_{1,1} & \longrightarrow & F(A)_{1,2} \\
\downarrow & & \downarrow \\
F(A)_{2,1} & \longrightarrow & F(A)_{2,2}
\end{array}
$$

Call the category of these diagrams $\text{Pull}$.

Our first step in defining the umkehr map is an intermediate functor $G$ that will be enough for providing an umkehr each of the the evasion spaces $\overline{A_m}$.

**Construction 4.4** Define $G$ from the category $\{\overline{A_m}\}$ consisting only of the evasion spaces $\overline{A_m}$, with no morphisms between them into $\text{Pull}$ by letting $G(\overline{A_m})$ be written as
We first define $G(A_m)_{2,2}$, and the morphism $\hat{\text{res}}$. As a set, we let

$$M_{\hat{\mathcal{A}}_m}^{\mathcal{N}_i} := \bigsqcup_{\chi \in A_m} \mathcal{C}N_i$$

where $\mathcal{C}N_i$ is the closure inside $S^n$ of the $i$'th output of the element $\chi \in A_m$ as in [1,3].

We specify a topology on this space as the quotient space under the restriction maps from $(M^{S^n})^{k+m} \times A_m$. Each component of $\mathcal{C}N_i$ is included into one of the $k+m$ copies of $S^n$ involved in the domain. Since we have taken componentwise closure of $\mathcal{A}_m$, and evade collision in the sense of [1,3] the map is well defined as the $\mathcal{C}N_i$ exercising multiple components will be constant throughout each component of $\mathcal{A}_m$.

Whenever $i \in \{1, \ldots, k\}$ label a puncture of $\chi$, we have that the closure of the complement of the timber will have $\mathcal{C}N_i = S^n$. Since we shall want to have $\varphi_{\mathcal{A}_m}$ be a morphism of Poincaré duality spaces in order to supply the umkehr map, we shall whenever $i$ labels the puncture $p_i$ apply the equivalence relation $f \sim g \in M^{S^n}$ of the second bullet in [2,3].

Hereby, collapsing $M^{S^n}$ to a single copy of $\mathcal{A}_m$. The base space of the pullback diagram will be given by the quotient space

$$G(A_m)_{2,2} := M_{\hat{\mathcal{A}}_m}^{\mathcal{N}_i}/\sim.$$ 

Where $\sim$ is applied at each puncture of $\chi \in A_m$.

The map $\hat{\text{res}}$ is given as the restriction map $\text{res} : (M^{S^n})^{k+m} \times A_m \to M_{\hat{\mathcal{A}}_m}^{\mathcal{N}_i}$, composed with the quotient map to $G(A_m)_{2,2}$.

The map $\varphi_{\mathcal{A}_m}$ can be defined, pointwise in $A_m$ as given in [Bar11, 4.2] – the constant maps along the components that share points with the component of the blueprint.

For a disk of $\mathcal{C}N_i \neq S^n$, we can define a point specifying the centre – for instance by maximizing the time with which any geodesic flow will reach the boundary of the disk. For a sequence in $A_m$ with $N_i$ converging to the puncture $\{p_i\}$, the centres of the complementary converging disks will tend towards $\{-p_i\}$, in which case we have $M^{S^n}/\sim \cong M$. We can hence define a map $\text{ev}_c$, by evaluating at these centres in a tuple $(f_1, \ldots, f_{k+m})_\chi \in M_{\hat{\mathcal{A}}_m}^{\mathcal{N}_i}$, and $-p_i$ where $N_i = \{p_i\}$. We in turn have a retraction

$$G(A_m)_{2,2} \xrightarrow{\text{ev}_c} M^{k+m} \times A_m.$$

Where $\text{cst}$ maps $(m_1, \ldots, m_{k+m})$ to the maps that are constantly $m_1, \ldots, m_{k+m}$.
Since we are mapping from a family of contractible disks, the above specifies a deformation retraction.

**Lemma 4.5** \(G(\overline{A}_m)_{2,2}\) deformation retracts onto \(M^{k+m} \times \overline{A}_m\).

The following lemma tells us that the extension of \(\text{Cleav}_{S^n}\) to \(\overline{\text{Cleav}}_{S^n}\), morally introducing some 'mild singularities', makes the fibrations involved in the action diagrams turn from fibrations into quasifibrations.

**Lemma 4.6** For any evasion space \(\overline{A}_m\), the maps \(\hat{\text{res}}\) and \(\tilde{\text{res}}\) are quasifibrations.

**Proof.** We shall apply the Dold-Thom criterion, \([\text{DT58}, 2.15]\), to show that the maps are quasifibrations.

To do this, for \(\chi \in \overline{A}_m\), denote by \(\chi, \ldots, \chi_{k+m}\) the \(k+m\) disks or points that are determined in 4.3 as the complements of timber of \(\chi\), or by the punctures of \(\chi\).

To apply the criterion, we shall first filter the targets \(M^{k+m}_{\overline{A}_m} \chi\) and \(M^{m-1} \times \overline{A}_m\) by closed subsets \(\emptyset = X_0 \subseteq \cdots \subseteq X_{k+m+1}\), and \(\emptyset = \overline{X}_0 \subseteq \cdots \subseteq \overline{X}_{k+m+1}\).

Let
\[
Y_i = \{\chi \in \overline{A}_m \mid \chi_i, \ldots, \chi_{k+m} \text{ is a puncture}\}
\]

In turn, this gives the filtrations by letting \(X_i = M^{k+m}_{Y_i} \chi\) and \(\overline{X}_i = M^{m-1} \times Y_i\).

To employ the Dold-Thom criterion, we first of all need that we have fibrations when considering the restrictions \(\hat{\text{res}}|_{X_i \setminus X_{i-1}}\) and \(\tilde{\text{res}}|_{\overline{X}_i \setminus \overline{X}_{i-1}}\) determined by letting the domain of \(\hat{\text{res}}\) be restricted to the preimage of \(X_i \setminus X_{i-1}\).

Notice that since \(\hat{\text{res}}|_{\overline{X}_i \setminus \overline{X}_{i-1}}\) is the pullback along \(\hat{\text{res}}|_{X_i \setminus X_{i-1}}\), this fibration condition only needs to be checked for \(\hat{\text{res}}|_{X_i \setminus X_{i-1}}\). The condition however follows since considering the map for each factor of \(\{M^{S^n}\}^{k+m}\) these will for all \(\chi \in X_i \setminus X_{i-1}\) either be given by disks of continuously varying sizes for the first \(i\) components, and evaluation maps at single points for the remaining factors. Since each factor is a fibration, we can specify lifts factorwise to show that we have a fibration.

The final condition in the Dold-Thom criterion is that we should have open neighborhoods \(U_i \supset X_i\) and \(\hat{U}_i \supset \overline{X}_i\), such that there are deformation retractions

- \(r_i : U_i \to X_i\)
- \(\hat{r}_i : \hat{U}_i \to \overline{X}_i\)
- \(\rho_i : \hat{\text{res}}^{-1}(U_i) \to \hat{\text{res}}^{-1}(X_i)\)
- \(\hat{\rho}_i : \tilde{\text{res}}^{-1}(\hat{U}_i) \to \tilde{\text{res}}^{-1}(\overline{X}_i)\).

These should commute with the map we are trying to show is a quasifibration, that is - restricted accordingly - \(r_i \circ \hat{\text{res}} = \hat{\text{res}} \circ \rho_i\) and \(\hat{r}_i \circ \tilde{\text{res}} = \tilde{\text{res}} \circ \hat{\rho}_i\).

There are \(i\) designated punctures involved in \(Y_i\), and we can define \(\hat{U}_i\) and \(\tilde{U}_i\) from \(Y_i\) by to \(y \in Y_i\) letting \(0 < \varepsilon_y\) be less than the smallest distance between any two of the \(i\)
punctures of $y$; and allow the punctures of $Y_i$ to move into hyperplanes that are within $\varepsilon_y$ of the actual puncture. $U_i$ and $\hat{U}_i$ are given by allowing all points in the second factor along this extension of $Y_i$.

Since we have chosen $\varepsilon_y$ sufficiently small, the punctures that turn into hyperplanes will never intersect with each other, so there is a well-defined homotopy moving the hyperplanes towards the puncture of $y$. Under the quotient $\sim$ of \ref{4.4} we have identified functions that don’t agree at the opposite of the puncture; and this ensures that the puncture is involved in all the disks of the second factor, providing a well-defined collapse onto this point. This defines $r_i$ and $\tilde{r}_i$ as deformation retractions.

To define $\rho_i$, we can again simply extend the deformation of $U_i$ onto $Y_i$ again.

The deformation retraction $\tilde{\rho}_i$ is defined by letting the disk on the side opposing the hyperplane moving away from the puncture parametrise the extra wedge of a sphere arising in puncture at the target $Y_i$. Using these parametrisations while moving the hyperplane away from the puncture shows that it is a deformation retraction.

Since the deformation retractions all have been constructed from the open neighborhood of $Y_i$, it is easy to see that they commute with the maps $\tilde{\res}$ and $\hat{\res}$.

\[ \square \]

By \ref{4.5} and \ref{4.2} we have a pullback-diagram with the lower portion of the spaces being manifolds. Since we have vertical quasifibrations, we can in light of \ref{3.3} produce an umkehr map, for each U-diagram indexed by $A_m$ independently. However, we shall need one more step in extending our umkehr-diagrams in order to patch the local umkehr maps into a global umkehr map.

We shall therefore replace the functor $G$ by a functor $F$: $\Pat (\{A_m\}) \to \text{Pull}$.

**Construction 4.7** Fixing $\chi \in \overline{A_m}$, as seen in \ref{4.4} the set of complements of associated timber $\bigsqcup_{i=1}^{k} \overline{N_i}$ consist of $k + m$ contractible components. The functor $F$ will be dependent on choosing any $k - 2 - m$ of these components $D_1, \ldots, D_{k-2-m}$, that hereby gives a restriction map in the same way as in \ref{4.4}

\[ \text{res}: (M^{S_m})^{2(k-1)} \times \overline{A_m} \to M_{\overline{A_m}}^{\bigsqcup_{i=1}^{k-2-m} D_i}, \]

where the target is topologised analogously to \ref{4.4}.

Again, when $\chi$ has a puncture $p_i$ at any $D_i$ or $\overline{N_i}$, we apply the same quotient under the equivalence relation $\sim$. We thus let

\[ F(\overline{A_m})_{2,2} := M_{\overline{A_m}}^{\bigsqcup_{i=1}^{k-2-m} D_i} / \sim. \]

Using this, we consider the diagram, with the square involving $G(\overline{A_m})_{1,1}$ and $G(\overline{A_m})_{2,2}$ being the diagram \ref{3}:
The maps labelled $\Delta$ are all a suitable iteration of diagonal maps, making the diagram commute. The map $\eta$ is specified by noting that for $\chi \in \overline{A_m}$ we have $|\pi_0(\beta_\chi)| = m + 1$. The difference between $k-1$ and $m+1$ is $k-2-m$, and having chosen the components $D_1, \ldots , D_{k-2-m}$, we specify $\eta$ as the map that is given by $\varphi_{\overline{A_m}}$ along $m+1$ of the factors of $M^{k-1}$, and for each $j \in k-2-m$, we choose one of the other factors in $M^{k-1}$ doubled by the diagonal map, and let $\eta$ be constant along $D_j$.

We let $F: \text{Pat}(\{\overline{A_m}\}) \to \text{Pull}$ be given as the pullback square involving $F(\overline{A_m})_{1,1}$.

Let $F(\overline{A_m})_0$ be given by noting that the extra choices involved in this construction will in the pullback space $F(\overline{A_m})_{1,1}$ give rise to $k$ mappings from $S^n \to M$ that fit together as in 2.4 as well $k-2$ mappings given by the choices of components involved in $F$ and $G$. The map $\text{pr}$ is the projection map that projects these extra $k-2$ mappings away. To any $\chi \in A_m$, the associated space $F(\overline{A_m})_0$ restricted to $\chi$ will be given by $M^S_{\chi}$ of 2.4.

The explicit definition of $F$ is dependent on the choices made above, we allow these choices to not be part of the notation since we shall see that 3.4 will make these choices cancel out in taking a pullback along the top-maps in the diagram of 4.7.

Note that – up to the underlying choice of disks – the target of $F$ does not depend on whether we choose a point of intersecting $\overline{A_m}$ or $\overline{A_j}$ so it is also defined for the intersections $\overline{A_m} \cap \overline{A_j}$, meaning that $F$ can be extended to a functor $F: \text{Pat}(\{\{A_m\}\}) \to \text{Pull}$. This is the functor for which we need to show that the umkehr maps for $\overline{A_m}$ defined by 3.4 patch together to a global umkehr map of the entire operad $\text{Cleav}_{S^n}$:

**Theorem 4.8** The diagrams (11) exhibit an action of $\text{Cleav}_{S^n}$ in the category of spectra. This action is provided through maps for any $k \in \mathbb{N}$:

$$\Gamma_k: (M^{S^n})^k \times \text{Cleav}_{S^n}(-;k) \to M^{S^n} \wedge S^{\dim(M)(k-1)}.$$
Proof. To $A \in \text{Ob}(\text{Pat} \{\{A_i\}\})$, we take a choice of functor $F$. Lemma 4.2 tells us that $A$ is a Poincaré duality space, and therefore, using 4.3 the spaces $F(A)_{2,2}$ and $F(A)_{2,1}$ of 4.7 are Poincaré duality spaces as well.

Since 4.4 tells us that the vertical maps are quasifibrations, for each individual $A$ we get by 3.3 that there is an umkehr map

$$\varphi_A^h : F(A)_{1,2} \wedge D_{\Omega F(A)_{2,2}} \to F(A)_{1,1} \wedge D_{\Omega F(A)_{2,1}}$$

By [Kle01, Th. A], the dualizing spectra will be given by suitably desuspended sphere spectra. The domain of $\varphi_A^h$ will be a sphere desuspended by $\dim p \cdot M q \cdot p \cdot k - 1 q - \dim p \cdot A q$ and the target desuspended by $\dim p \cdot M q \cdot p \cdot k - 1 q - \dim p \cdot A q$, where $\dim p \cdot A q$ denotes the dimension of the product of spheres given by the deformation retraction of 4.2.

Smashing $(M^{S^n})^k \times A$ and $M^{S^n}_A$ with the same dualizing spectra as above, and in that order, we get that composition with the induced map in spectra with the maps $\Delta$ and $pr$ above provide an action

$$A \times (M^{S^n})^k \to F(A)_0 \wedge S^{\dim p \cdot M q \cdot p \cdot (k-1)}$$

where we have suspended the map suitably to have no desuspensions on the domain of the action.

Let $A' \to A$ be a morphism in $\text{Pat} \{\{A_i\}\}$, these are inclusions. Restricting the action of $A$ to $A'$ will by 3.4 yield the same umkehr map, as producing the action for $A'$ directly.

Therefore, taking the colimit of the associated umkehr maps provide a morphism in spectra, and using 3.3 we get the desired morphism

$$\Gamma_k : (M^{S^n})^k \times \overrightarrow{\text{Cleav}_{S^n}} \to \text{hocolim}_{A \in \text{Pat} \{\{A_i\}\}} F(A)_0 \wedge S^{\dim p \cdot M q \cdot p \cdot (k-1)} \quad (12)$$

Further composing this map with the canonical map onto the actual colimit, we can identify the target as in 2.4 where we see that the colimit is equivalent to the space of maps $S^n \to M$ constant along the blueprint of $\chi \in \overrightarrow{\text{Cleav}_{S^n}}(-; k)$, with extra functions $f_i$ wedged on at $-p_i$ of $f_i$ whenever $i$ labels a puncture of $\chi$. Forgetting these extra $f_i$ in the sense of 2.3 defines a map to $M^{S^n}$ which in turn provides the action of $\overrightarrow{\text{Cleav}_{S^n}}$.

The following tells us that the constructed action as such does not give string topology operations.

**Proposition 4.9** The operations $\rho_{p,q} : H_p(M^{S^n}) \otimes H_q(M^{S^n}) \otimes H_0(\overrightarrow{\text{Cleav}_{S^n}}(-; 2)) \to H_{p+q-d}(M^{S^n})$ recovers the intersection product through the morphism $H_*(M) \to H_*(M^{S^n})$ induced by constant maps. All other operations are zero.

**Proof.** Restricting to either endpoints of $[-1,1]$ in $\overrightarrow{\text{Cleav}_{S^n}}(-; 2) \cong S^n \times [-1,1]$, denoted $\overrightarrow{\text{Cleav}_{S^n}}^\pm$ , gives the following diagram

\[\text{Diagram}\]
where the upwards and downwards arrows are inclusions, and the vertical arrows are the restrictions of a factorisation of the map \( \Gamma_2 \) from [4.8]. In particular, for a fixed \( s \in S^n \), the space \( M^S_s \) is given by the space of maps from two copies of \( S^n \) wedged at \( s \) and \( -s \) respectively. The maps \( \gamma_1 \) and \( \gamma_{-1} \) each forgets a distinctive one of these copies of \( S^n \).

This means that for a fixed direction of the normal vector of \( \overrightarrow{\text{Cleav}}_{S^n}(-; 2) \), i.e. fixing \( s \) for \( (s, t) \in S^n \times [-1, 1] \cong \overrightarrow{\text{Cleav}}_{S^n}(-; 2) \), corestricting the image of \( \Gamma_2(f, g) \in M^S_n \wedge S^{\dim(M)} \), the definition of \( \gamma_1 \) and \( \gamma_{-1} \) tells us that this corestriction in the upper and lower lines in the above diagram is generated by the \( f \) and \( g \) that are constant maps. That is, only homology classes obtained by families of constant maps survives in the target of the action.

To see that this does indeed recover the intersection product, note that to \( A, B \subseteq M \) submanifolds of \( M \), the sets of maps \( \{ \text{cst}_a : S^n \to \{ a \} \subset M \}_{a \in A} \) and \( \{ \text{cst}_b : S^n \to \{ b \} \subset M \}_{b \in B} \) will in the pullback amount to the subspace \( A \cap B = \{ \text{cst}_x \}_{x \in A \cap B} \subset M^S_n \).

The map \( \{ \text{cst}_x \}_{x \in A \cap B} \to M^S_n \times M^S_n \) is the diagonal map; the cohomology hence represents the cup product of the classes represented by \( A \) and \( B \), and by the norm map isomorphism of [Kle01, Th. D] taking the dualizing spectrum along with homology amounts to Poincaré duality – which yields the intersection product of the classes associated to \( A \) and \( B \).

\[ \square \]

One upshot of [4.9] is that since the equitorial embeddings \( S^1 \hookrightarrow S^2 \hookrightarrow \cdots \) of unit spheres provide morphisms of coloured operads \( \overrightarrow{\text{Cleav}}_{S^1} \to \overrightarrow{\text{Cleav}}_{S^2} \to \cdots \) by at each morphism extending the cleaving hyperplanes parallel to the extra coordinate axis in \( \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \).

This means that there is a sequence of operads whose homotopy type is determined by [4.2] The homology of these operads provide higher brackets - and taking the limit of this string of actions these brackets vanish and provide the following:

**Corollary 4.10** The \( E_\infty \)-operad \( \overrightarrow{\text{Cleav}}_{S^\infty} \) acts on \( M^{S^\infty}_n \) in the category of spectra with the same suspension as in [4.8] such that the homology of this action provides the intersection product on \( \mathbb{H}_s(M) \).
5 Recovering String Topology

We now go in detail with how the action of \( \text{Cleav}_{S^n} \) gives rise to an action of \( \text{Cleav}_{S^n} \) generalizing string topology, and hence does not contain the triviality issues that are present in [LS]. The basic change we infer is an introduction of 'outer' parametrizing maps that specifies how each timber is mapped onto \( M \) from a portion of \( S^n \). The technical difference between just restricting the action of \( \text{Cleav}_{S^n} \) is that we factor the action map via an extra parametrizing map. This hence gives an additional factorization after the usage of the umkehr map, and while the homotopy type of the first intermediate space between \( M_{\text{S}^n} (\cdot; k) \) and \( M_{\text{S}^n} \wedge S^{\dim(M)-(k-1)} \) will be the same as restricting from \( \text{Cleav}_{S^n} \), the additional factorization will at one hand disallow the triviality arising in 4.8 in the sense that we no longer can have paramatrized maps to \( S^n \) that tends to zero length. On the other hand, this additional parametrization is also part of string topology as defined through the cactus operad; meaning that we can show that it also generalizes string topology.

Note that we can use the deformation retraction (10) used to prove 4.5 to build the following diagram involving two pullback-diagrams:

\[
\begin{array}{ccc}
C(A_m) & \xrightarrow{\text{cst}_*} & (M_{\text{S}^n})_{k+m} \times A_m \\
\downarrow \text{(ev)} & & \downarrow \text{res} \\
G(A_m)_{1,1} & \xrightarrow{\text{res}} & M_{\text{dim}(\beta(f,T))} \times A_m \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
G(A_m)_{2,2} & \xrightarrow{\text{cst}} & M_{k+m} \times A_m
\end{array}
\]

where \( C(A_m) \) is the pullback space of the outer maps \( \text{ev}_c \) and \( \text{ev}_c \circ \varphi_{A_m} \). The retractive maps \( (\text{ev})_* \) and \( \text{cst}_* \) and are obtained from the universal properties accompanying pullback diagram.

Note that \( C(A_m) \) can be identified as a map out of a wedge of spheres, that sit together in \( m+1 \)-wedge point in a cacti-like configuration – see the definition of the cactus operad in [Vor05] or [CV06, 2.2]. Consider points \( \chi \in A_m \) that satisfy \( \chi \in \text{Cleav}_{S^n} (\cdot; k) \), so that timber of \( \chi \) will never be a puncture. For these \( \chi \), the map \( \text{cst}_* \) produces from the map \( f_\chi : S^N \to M \) that is constant along the blueprint of \( \chi \) a map to the element of \( C(A_m) \). This specifies the map from the wedge of spheres to \( M \) where each sphere, labelled by \( i \) is parametrized by \( f_\chi \) restricted to the \( i \)th \( S^n \)-timber of \( \chi \).

This hence says that as long \( A_m \ni \chi \in \text{Cleav}_{S^n} (\cdot; k) \), we can factor the map induced by inclusions \( G(A_m)_{\text{Cleav}_{S^n} (\cdot; k)} \to M_{\text{S}^n} \) through the space \( C(A_m) \). Note that this factorization is no longer well-defined if we attempt to extend it to the map \( G(A_m) \to M_{\text{S}^n} \).
specifying the action of $\overrightarrow{\text{Cleav}}_{S^n}$ given in (13).

From (12) we see that simply restricting $\overrightarrow{\text{Cleav}}_{S^n}$ to $\text{Cleav}_{S^n}$ does not give string topology, note however that in the proof of (8) we can restrict the maps $\varphi_A^{h_1}$ to $A \cap \text{Cleav}_{S^n} (-; k)$. For these restrictions, we hence have both of the pullback-diagrams given in the diagram (13). In (12), these restricted maps patch together to a map into the homotopy colimit over $\text{Pat}(\{A_m\})$. To $A \in \text{Pat}(\{A_m\})$, let $C(A \cap \text{Cleav}_{S^n} (-; k))_0$ be as $F(A \cap \text{Cleav}_{S^n} (-; k))_0$ in the diagram (11) the space that forgets the superfluous $m$ mappings from $S^n \to M$. Since we in the proof of (11) are taking a homotopy colimit, we can use the deformation retraction involving $\text{cst}$ and $(\text{ev}_c)_*$ to obtain a map

$$\theta: \left( M^{S^n} \right)^k \times \text{Cleav}_{S^n} (-; k) \to \text{hocolim}_{A \in \text{Pat}(\{A_m\})} C(A \cap \text{Cleav}_{S^n})_0 \wedge S^{\dim(M) \cdot (k-1)}.$$

We let $\gamma$ denote the map to the canonical map to the colimit, and let

$$\rho: \text{colim}_{A \in \text{Pat}(\{A_m\})} C(A \cap \text{Cleav}_{S^n} (-; k))_0 \to \text{colim}_{A \in \text{Pat}(\{A_m\})} F(A \cap \text{Cleav}_{S^n} (-; k))_0$$

be the map induced by the morphisms $(\text{ev}_c)_*$ of (13) - this map specifies the promised parametrization of the cactus-like mappings of spheres into $M$.

Finally, $\iota: \text{colim}_{A \in \text{Pat}(\{A_m\})} F(A \cap \text{Cleav}_{S^n} (-; k))_0 \wedge S^{\dim(M) \cdot (k-1)} \to M^{S^n} \wedge S^{\dim(M) \cdot (k-1)}$ denotes the map induced by the pointwise inclusion in $M^{S^n}$ of maps that are constant along the blueprint.

**Theorem 5.1** The map of spectra

$$\iota \circ \rho \circ \gamma \circ \theta: \left( M^{S^n} \right)^k \times \text{Cleav}_{S^n} (-; k) \to M^{S^n} \wedge S^{\dim(M) \cdot (k-1)}$$

generalizes string topology, in the sense that letting $n = 1$, the homology of the above map provides the string topology operations on $\mathbb{H}_*(M^{S^n})$.

**Proof.** Note that in [CJM02, Ch. 3], the cacti operad is seen as acting on $M^{S^1}$ by providing umkehr maps for the left map in the correspondance

$$M^{S^1} \xrightarrow{\iota} M^c_{S^1} \xrightarrow{c} \left( M^{S^1} \right)^k.$$  \hfill (14)

Where $c$ is a cactus, described by a configuration of $k$ copies of $S^1$, denoted $c_1 \cup \cdots \cup c_k \subset \mathbb{R}^2$ sitting in a tree-like configuration, together with a map $\pi: S^1 \to c_1 \cup \cdots \cup c_k$. The rightmost map in (13) is precisely described by the composite $\iota \circ \rho$, since the map $(\text{ev}_c)_*$ that is used in the definition of $\rho$ parametrizes the $k$ circles in the exact same way that $\pi$ is given. \hfill $\square$

Note that we have an action of $\text{SO}(n+1)$ on $\text{Cleav}_{S^n}$, in the sense of [Bar11, Ch. 6]. Similarly, $\text{SO}(n+1)$ acts on $M^{S^n}$ by rotating the domain.

As is shown in [Bar11, Ch. 6], the above action on $\text{Cleav}_{S^n}$ translates into a semidirect product $\text{Cleav}_{S^n} \rtimes \text{SO}(n+1)$ with $\text{Ob}(\text{Cleav}_{S^n} \rtimes \text{SO}(n+1)) = \text{Ob}(\text{Cleav}_{S^n})$ and $(\text{Cleav}_{S^n} \rtimes \text{SO}(n+1))(; k) = \text{Cleav}_{S^n}(-; k) \times (\text{SO}(n+1))^k$.

We can extend the action of (5) to an action of $\text{Cleav}_{S^n} \rtimes \text{SO}(n+1)$ by letting the $k$ copies of $\text{SO}(n+1)$ act on the $k$ mapping spaces $M^{S^n}$ in the domain of the action map.
Proposition 5.2 The above action extending from $\text{Cleav}_{S^n}$ to $\text{Cleav}_{S^n} \rtimes \text{SO}(n+1)$ is well-defined.

Proof. We only need to check that the action map given in 5.1 is equivariant with respect to these two actions; in the sense that acting upon the target of the action map precisely corresponds to acting on acting on $\text{Cleav}_{S^n}(-; k)$ by rotating the hyperplanes forming the cleavage, as well as rotating the target of the $k$ mappings $S^n \to M$. That the $k$ mappings are forced to be acted upon by $\text{SO}(n+1)$ follows since the inner pullback-diagram of (13) is $\text{SO}(n+1)$-equivariant, meaning that the inner pullback automatically satisfies this equivariance. Further, the homotopy equivalence $(ev_c)_*$ is equivariant. This can be seen since the centre-points of the complements of timber moves along with the rotation. Hereby the homotopical replacement of the umkehr map to also satisfy the desired equivariance.

The above two propositions leads us to the following corollary, giving the generalization of the Chas-Sullivan loop product. Recall from [SW03, Th. 5.4] and [Bar11, Ch. 6] that in uneven degrees actions over $\text{Cleav}_{S^n} \rtimes \text{SO}(n+1)$ are higher version of Batalin-Vilkovisky algebra, whereas in even degrees they are a trivial extension over $E_n$-algebras by $\text{SO}(n + 1)$. This gives content to the following

Corollary 5.3 $\mathbb{H}_*(M^{S^n})$ is an algebra over $\mathbb{H}_*(f\text{Disk}_{n+1})$, such that for $n = 1$, this gives the Chas-Sullivan product [CS99].
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