KISSING NUMBER IN SPHERICAL SPACE

MARIA DOSTERT AND ALEXANDER KOLPAKOV

ABSTRACT. This paper investigates the behaviour of the kissing number $\kappa(n,r)$ of congruent radius $r > 0$ spheres in $S^n$, for $n \geq 2$. Such a quantity depends on the radius $r$, and we plot the approximate graph of $\kappa(n,r)$ with relatively high accuracy by using new upper and lower bounds that are produced via semidefinite programming and by using spherical codes, respectively.

Key words: spherical geometry, kissing number, semidefinite programming.
2010 AMS Classification: Primary: 05B40. Secondary: 52C17, 51M09.

1. INTRODUCTION

Let $X = \mathbb{E}^n$, $\mathbb{H}^n$ or $S^n$, be the Euclidean, hyperbolic or spherical space, respectively. This means that $\mathbb{E}^n$ is the $n$-dimensional Euclidean space with the usual metric, while $\mathbb{H}^n$ is the $n$-dimensional hyperbolic (or Lobachevski) space with metric of constant sectional curvature $-1$, and $S^n = \{v \in \mathbb{R}^{n+1} | v \cdot v = 1\}$ is the unit sphere in $\mathbb{R}^{n+1}$ with the induced metric of constant sectional curvature $+1$.

Let $d(x,y)$ be the geodesic distance between two points $x,y \in X$. A sphere in $X$ of radius $r > 0$ centred at $p \in X$ is defined as $S(p,r) = \{x \in X | d(p,x) = r\}$. A configuration $S_0, S_1, \ldots, S_k$ of $(k+1)$ congruent radius $r > 0$ spheres in $X$ is called a kissing configuration if all $S_i, 1 \leq i \leq k$ are tangent to $S_0$ and have non-intersecting interiors. Then maximal possible such $k$ is called the kissing number $\kappa(n,r)$, that depend a priori on the dimension $n \geq 1$ of the space, and the radius $r$.

It is easy to see that $k(1,r) = 2$ for $X = \mathbb{E}^1$ and $\mathbb{H}^1$ (the latter being isometric to $\mathbb{E}^1$). However, for $X = S^1$ we have that $k(1,r) = 2$, if $r \leq \frac{\pi}{2}$, $k(1,r) = 1$, if $\frac{\pi}{2} < r \leq \pi$, and $0$, if $\pi < r \leq 2\pi$.

Moreover, in $X = \mathbb{E}^n$ the kissing number does not depend on $r$ (because of rescaling), and thus $\kappa(n) = \kappa(n,1)$. In contrast, if $X = \mathbb{H}^n$ or $S^n$, with $n \geq 2$, the dependence of $\kappa(n,r)$ on the radius $r$ is essential. The case of $X = \mathbb{H}^n$ was studied in [2] for $n = 2$, and in [3] for arbitrary dimensions $n \geq 2$. The case of $X = S^n$, $n \geq 2$, is the subject of the present paper.

ACKNOWLEDGEMENTS

A. K. was supported by the Swiss National Science Foundation, project no. PP00P2-170560. This collaboration resulted from several visits and a series of mini-courses given by M. D. at the University of Neuchâtel: the authors are grateful for its hospitality and support. They also feel obliged to Fernando M. de Oliveira (TU Delft) and Oleg Musin (University of Texas Rio Grande Valley) for fruitful discussions.

2. GEOMETRIC CONSIDERATIONS

In this section we produce some upper and lower bounds for the kissing number $\kappa(n,r)$, so that we can analyse its general behaviour, and some limiting values when the radius $r$ approaches $0$ or $\frac{\pi}{2}$. In this respect, we use classical geometric approach. We refer the reader to [10] §2.1 for the necessary basics of spherical geometry.

2.1. UPPER BOUND. First we prove the following upper bound for $\kappa(n,r)$, with $0 \leq r \leq \frac{\pi}{2}$.

Theorem 2.1. For any integer $n \geq 2$ and a non-negative number $r \leq \frac{\pi}{2}$, we have that

$$\kappa(n,r) \leq U(n,r) = \frac{2B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\sec^2 r, \frac{n-1}{2}, \frac{1}{2}\right)},$$
where \( B(x; y, z) = \int_0^x t^{y-1}(1 - t)^{z-1} dt \) is the incomplete beta-function, and \( B(y, z) = B(1; y, z) \), for all \( x \in [0, 1] \), and \( y, z > 0 \).

**Proof.** Let \( S_0 \) be a radius \( r \) sphere in \( \mathbb{S}^n \) with centre \( O \), and let \( S_i, i = 1, \ldots, k \), be its neighbours in a kissing configuration that are also radius \( r \) spheres with centres \( O_i \), \( i = 1, \ldots, k \), respectively. Consider a configuration of two tangent spheres: \( S_0 \) and \( S_i \) for some \( i, 1 \leq i \leq k \). Let \( OT_i \) be the geodesic ray emanating from \( O \) that is tangent to \( S_i \) at point \( T_i \). Let also \( L_i \) be the point of tangency between \( S_0 \) and \( S_i \), while \( N_i \) is the intersection point of \( S_0 \) with \( OT_i \).

Then in the \( OO_i T_i \) spherical plane we have a right spherical triangle with vertices exactly \( O, O_i \) and \( T_i \). Let \( \theta \) be the angle at \( O \). Then by a version of the spherical Pythagorean theorem [10, §2.5, Exercise 2(3)], we obtain

\[
\sin |O_i T_i| = \sin |OO_i| \sin \theta,
\]

which implies, once we substitute the lengths \( |O_i T_i| = r \) and \( |OO_i| = 2r \),

\[
\sin \theta = \frac{\sec r}{2}.
\]

The condition \( r < \frac{\pi}{4n} \) ensures that the triangle \( OO_i T_i \) does not degenerate into a spherical geodesic “lune” of angle \( \theta = \frac{\pi}{2} \).

If we project \( S_i \) onto \( S_0 \) along the geodesic rays emanating from the centre \( O \) of \( S_0 \), we obtain a “cap” \( C_i \) on \( S_0 \), and all such caps resulting from a kissing configuration have non-intersecting interiors. However, we shall consider a purely Euclidean picture instead that can be obtained as follows.

Since \( S_0 \) is a section of \( \mathbb{S}^n \) by a hyperplane \( P_0 \), let us consider the orthogonal projection \( p \) of \( \mathbb{R}^{n+1} \) onto \( P_0 \). Then the centre \( O \) of \( S_0 \) is projected down to a point \( O^* = p(O) \) in \( P_0 \), while we have \( L_i^* = p(L_i) = L_i, N_i^* = p(N_i) = N_i \). Thus, the cap \( C_i \) project to a cap \( C_i^* = p(C_i) = C_i \) on the sphere \( S_0^* = p(S_0) = S_0 \) in the plane \( P_0 \). The cap \( C_i^* \) has angular radius \( \theta \) as measured on the surface of \( S_0^* \), for all \( 1 \leq i \leq k \).

Thus we obtain that

\[
\text{Area } S_0^* \geq \sum_{i=1}^{k} \text{Area } C_i^* = \kappa(n, r) \cdot \text{Area } C_i^*,
\]

where

\[
\text{Area } C_i^* = \frac{1}{2} \cdot \text{Area } S_0^* \cdot \frac{B \left( \sin^2 \theta, \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \frac{n-1}{2}, \frac{1}{2} \right)}.
\]

We remark that the above formula is valid only for spherical caps of angular radius \( \theta \leq \frac{\pi}{2} \). Otherwise, the resulting area will be that of the complementary region to \( C_i^* \) in \( S_0^* \). As our condition \( r \leq \frac{\pi}{4} \) implies \( \theta \leq \frac{\pi}{2} \), the theorem follows. \( \square \)

**2.2. Lower bound.** The following lemma is a simple relation between packing and covering of the \( n \)-dimensional unit sphere \( \mathbb{S}^n \) by closed metric balls. A packing of \( \mathbb{S}^n \) by closed metric balls of angular radius \( r > 0 \) with non-intersecting interiors is called maximal if it cannot be enlarged by adding more such balls without overlapping their interiors.

**Lemma 2.2.** Let \( \mathbb{S}^n \) be packed by closed metric balls \( B_i, i = 1, 2, \ldots \), of equal (angular) radius \( r \), and let such packing be maximal. Then \( \mathbb{S}^n \) is covered by closed metric balls \( B'_i, i = 1, 2, \ldots, \) concentric to \( B_i \), of radius \( 2r \).

**Proof.** Subject to the conditions of the lemma, if there is a point \( p \in \mathbb{S}^n \) not covered by any of the \( B'_i \)'s, then a metric ball of radius \( r \) centred at \( p \) can be added to the initial packing. The latter contradicts its maximality. \( \square \)

Now a lower bound on \( \kappa(n, r) \) can be obtained.
Theorem 2.3. For any integer \( n \geq 2 \) and a non-negative number \( r \), we have that
\[
\kappa(n, r) \geq L(n, r) = \begin{cases} 
\frac{2B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\sec^2 r - \frac{\pi}{4}, \frac{n-1}{2}, \frac{1}{2}\right)}, & \text{if } 0 \leq r \leq \frac{\pi}{4}, \\
\frac{2B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{2B\left(\frac{n+1}{2}, \frac{1}{2}\right) - B\left(\sec^2 r - \frac{\pi}{4}, \frac{n+1}{2}, \frac{1}{2}\right)}, & \text{if } \frac{\pi}{4} \leq r \leq \frac{\pi}{3}, 
\end{cases}
\]
where \( B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1}dt \) is the incomplete beta-function, and \( B(y, z) = B(1; y, z) \), for all \( x \in [0, 1] \), and \( y, z > 0 \).

Proof. Let us observe that the packing of \( S_0^* \) by the spherical caps \( C_i^*, i = 1, 2, \ldots, k \), from Theorem 2.1 is maximal, if \( k = \kappa(n, r) \). Since rescaling does not change angular distances, we may assume that \( S_0^* \) has unit radius.

Let \( C_i' \) be a spherical cap concentric to \( C_i^* \) of angular radius \( \theta \). By Lemma 2.2, \( C_i' \)'s cover \( S_0^* \).

Then,
\[
\kappa(n, r) \cdot \text{Area } C_i' = \sum_{i=1}^k \text{Area } C_i' \geq \text{Area } S_0^*,
\]
where
\[
\text{Area } C_i' = \frac{1}{2} \cdot \text{Area } S_0^* \begin{cases} 
\frac{B\left(\sin^2(2\theta), \frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\frac{n+1}{2}, \frac{1}{2}\right)}, & \text{if } 0 \leq \theta \leq \frac{\pi}{4}, \\
2 \frac{B\left(\sin^2(2\theta), \frac{n+1}{2}, \frac{1}{2}\right)}{B\left(\frac{n+1}{2}, \frac{1}{2}\right)}, & \text{if } \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3},
\end{cases}
\]
depending on whether \( \theta \leq \frac{\pi}{4} \) (then the first value realises the cap area) or \( \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \) (then the second value realises the cap area, while the first one gives the complementary region area), and with \( \theta \) satisfying \( \sin \theta = \frac{\cos \theta}{b} \), as before.

By using the formula \( \sin(2\theta) = 2\sin \theta \cos \theta \), the theorem follows after a straightforward computation. \( \square \)

The above bound comes from an argument completely analogous to that by Wyner [11]. According to the recent results by Jenssen, Joos, and Perkins, it can be improved by a linear factor in \( n \) provided \( 0 \leq r \leq \frac{\pi}{4} \), c.f. [6] Theorem 2.

2.3. Limiting values of kissing numbers. By putting \( r = 0 \) in the above formulas, we obtain that \( \sin \theta = \frac{1}{2} \), or \( \theta = \frac{\pi}{6} \), which produces the usual (and rather imprecise) estimates for the Euclidean kissing number \( \kappa(n) \).

Another limiting case is \( \kappa(n, \pi/3) = 2 \). First of all, we obtain from Theorems 2.1, 2.3 that \( 1 \leq \kappa(n, \pi/3) \leq 2 \). Now, let us consider the points \( a = (0, -1, 0, \ldots, 0) \), \( b = (\sqrt{3}/2, 1/2, 0, \ldots, 0) \) and \( c = (-\sqrt{3}/2, 1/2, 0, \ldots, 0) \) in \( S^n \). Notice that \( a, b \) and \( c \) are placed at mutually equal distances of \( 2\pi/3 \). Thus, the spheres \( S_a, S_b \) and \( S_c \) of radii \( \pi/3 \), centred at the respective points, are mutually tangent. Each of them has two congruent neighbours and thus \( \kappa(n, \pi/3) \geq 2 \).

It is also clear from the upper bound in Theorem 2.1 that \( \kappa(n, r) = 1 \) for \( \frac{\pi}{3} < r \leq \frac{\pi}{2} \) and \( n \geq 1 \). As \( r = \frac{\pi}{2} \), the sphere of radius \( r \) fills a hemisphere of \( S^n \), and thus \( \kappa(n, r) = 0 \) for \( \frac{\pi}{2} < r \leq \pi \) for \( n \geq 1 \).

Below, by using semidefinite programming (SDP) and by constructing concrete configurations of kissing spheres, respectively, we produce good enough upper and lower bounds to approximately plot \( \kappa(n, r) \) as a step-function for dimensions \( n = 2, 3, 4 \) and radii \( 0 \leq r \leq \pi \). Moreover, the obtained bounds often provide exact values of kissing numbers for various values of radii.

3. Lower bounds via spherical codes

A set of points \( C \subset S^{n-1} \) with \( x \cdot y \leq \cos \theta \) for all distinct \( x, y \in C \) is called a spherical code with minimal angular distance \( \theta \). Then the kissing number \( \kappa(n, r) \) of radius \( r \) spheres in \( S^n \), \( n \geq 2 \), is equal to the cardinality of a maximal spherical code \( C \subset S^{n-1} \) with \( x \cdot y \leq 1 - \frac{1}{1 + \cos(2r)} \) for \( x, y \in C \).
Lemma 3.1.
\[
\kappa(n, r) = \max \left\{ |C| : C \in S^{n-1} \text{ and } x \cdot y \leq 1 - \frac{1}{1 + \cos(2r)} \text{ for all distinct } x, y \in C \right\}.
\]

Proof. Consider a spherical triangle with side length $2r$, and let $\theta$ be one of its inner angles. By applying the spherical law of cosines [10, Theorem 2.5.3], we obtain
\[
\cos \theta = \frac{\cos(2r) - \cos^2(2r)}{\sin^2(2r)} = \frac{\cos(2r)(1 - \cos(2r))}{1 - \cos^2(2r)} = \frac{\cos(2r)}{1 + \cos(2r)} = 1 - \frac{1}{1 + \cos(2r)}.
\]

Let $x_i \in \mathbb{R}^n$, for $i = 1, \ldots, k$, be the approximate numeric coordinates of the code elements in a spherical code $C \subset S^{n-1}$. Here we use [5] as a source of putatively optimal spherical codes on $S^{n-1}$, for $n = 2, 3, 4$. We define $\tilde{x}_i \in \mathbb{Q}^n$ to be a rational approximation of $x_i$ (in many cases, $x_i$ is a real number with 16 digit precision, and thus is already approximated by a rational). After normalizing $\tilde{x}_i$ to norm 1, we obtain that there exist $a_i \in \mathbb{Q}, b_i \in \mathbb{Q}^n$ such that $\tilde{x}_i = \sqrt{a_i} \cdot b_i$.

Using interval arithmetic in SageMath [9], we compute the maximal inner product of $\tilde{x}_i$ and $\tilde{x}_j$ for $i, j \in \{1, \ldots, k\}, i \neq j$. Let $r \in \mathbb{R}$ be such that the maximal inner product is at most $1 - \frac{1}{1 + \cos(2r)}$. Since $\tilde{x}_i \in S^{n-1}$ for all $i \in \{1, \ldots, k\}$, this exact spherical code (having exact values for its elements) defines a feasible kissing configuration of $k$ radius $r$ spheres in $S^n$. Note that while turning the approximate solution into an exact kissing configuration we might have to vary $r$ slightly. The SageMath code converting the approximate codes from [5] to their rationalised forms is included in the ancillary files.

4. Upper bounds via semidefinite optimization

By using semidefinite optimisation techniques, we can graph the upper bound for the function $k(n, r)$. In [1], Bachoc and Vallentin developed a semidefinite program (SDP) for computing upper bounds for the cardinality of a spherical code with a given minimal angular distance. Due to Lemma 3.1, we can adapt the SDP from [1] in order to obtain upper bounds for $\kappa(n, r)$.

The new SDP is given in the theorem below, where we use the following notation. First of all, $J$ denotes the “all 1’s” matrix. Then, for $n \geq 3$, let $P_k^n(u)$ denote the Jacobi polynomial of degree $k$ and parameters $((n - 3)/2, (n - 3)/2)$, normalized by $P_k^n(1) = 1$. If $n = 2$, then $P_k^n(u)$ denotes the Chebyshev polynomial of the first kind of degree $k$. For a fixed integer $d > 0$, we define $Y_k^n$ to be a $(d - k + 1) \times (d - k + 1)$ matrix whose entries are polynomials on the variables $u, v, t$ defined by
\[
(Y_k^n)_{i,j}(u, v, t) = P_{i+2k}^n(u)P_{j+2k}^n(v)Q_{i}^{n-1}(u, v, t),
\]
for $0 \leq i, j \leq d - k$, where
\[
Q_{k}^{n-1}(u, v, t) = ((1 - u^2)(1 - v^2))^{k/2}P_{k}^{n-1}\left(\frac{t - uv}{\sqrt{(1 - u^2)(1 - v^2)}}\right).
\]

The symmetric group on three elements $S_3$ acts on a triple $(u, v, t)$ by permuting its components. This induces the action
\[
\sigma p(u, v, t) = p(\sigma^{-1}(u, v, t))
\]
on $\mathbb{R}[u, v, t]$, where $\sigma \in S_3$. By taking the group average of $Y_k^n$, we obtain the matrix
\[
S_k^n(u, v, t) = \frac{1}{6} \sum_{\sigma \in S_3} \sigma Y_k^n(u, v, t),
\]
whose entries are invariant under the action of $S_3$.

For any two square matrices $A, B \in \mathbb{R}^{n \times n}$, let $\langle A, B \rangle = \text{tr}(B^T A)$ be the trace inner product.
Theorem 4.1. Any feasible solution of the following optimization program gives an upper bound on $\kappa(n,r)$:

$$\min \ 1 + \sum_{k=1}^{d} a_k + b_{11} + \langle J, F_0 \rangle,$$

$$a_k \geq 0 \ \text{for} \ k = 1, \ldots, d,$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \succeq 0,$$

$$F_k \in \mathbb{R}^{(d-k+1)\times(d-k+1)} \ \text{and} \ F_k \succeq 0 \ \text{for} \ k = 0, \ldots, d,$$

$$(i) \sum_{k=1}^{d} a_k S^n_k(u, u, 1, F_k) \leq -1 \ \text{for} \ (u, u, 1) \in \Delta_0,$$

$$(ii) \ b_{22} + \sum_{k=0}^{d} \langle S^n_k(u, v, t) F_k \rangle \leq 0 \ \text{for} \ (u, v, t) \in \Delta,$$

where

$$\Delta = \left\{ (u, v, t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq 1 - \frac{1}{1 + \cos(2r)} \ \text{and} \ 1 + 2uvt - u^2 - v^2 - t^2 \geq 0 \right\}$$

and

$$\Delta_0 = \left\{ (u, u, 1) : -1 \leq u \leq 1 - \frac{1}{1 + \cos(2r)} \right\}.$$

The above SDP is similar to the SDP for the upper bounds on the kissing number in hyperbolic space [3]. Therefore, we use the same techniques to verify the numerical results of the SDP solver rigorously. Like in [3], the verification script is a slightly modification of the verification program by Machado and Oliveira [7]. The verification script runs in SageMath 6.6 [9] and is available together with other ancillary files.

5. Lower and upper bounds

From Lemma 3.1 we immediately deduce that $\kappa(n,r)$ is a decreasing function in $0 \leq r \leq \frac{\pi}{3}$, for any $n \geq 1$, since $\frac{1}{1 + \cos(2r)}$ is increasing with $r$. Indeed, the set of possible codes $C$ in the definition of $\kappa(n,r)$ from Lemma 3.1 becomes smaller as $r$ increases. Thus $\kappa(n,r)$ is a decreasing step-function of $r$ for any fixed dimension $n \geq 1$.

Having the upper bounds from SDP and lower bounds from concrete configurations for any given value of $0 \leq r \leq \frac{\pi}{3}$ that are sufficiently close to each other provides us with an approximate shape of $\kappa(n,r)$. Also, we observe that $\kappa(n+1,r) \geq \kappa(n,r)$, for $n \geq 1$, and $0 \leq r \leq \frac{\pi}{3}$, since $S^{n-1} \subset S^{n}$, and the above argument applies in Lemma 3.1 again.

For dimension 2, 3, and 4 we consider the spherical codes for $0 \leq r \leq \pi/3$ determined by the approach of Section 3. For each radius $r$ of these exact spherical codes, we compute the lower bound given by Theorem 2.3, the upper bound given by Theorem 2.1 as well as the upper bound by solving the SDP in Theorem 4.1. Below we give for each of these dimensions a plot of the computed lower and upper bounds as well as a table with the results corresponding to $r$. Note that the SDP upper bounds given in the tables are all rigorously verified. Whereas the SDP upper bounds in the plots are numerical results from the SDP solver. For solving the SDP with high-precision arithmetic we use the SDPA-GMP [4] solver.

For certain values of $r$, we obtain the lower and upper bounds for $\kappa(2,r)$ given in Table 1. In order to compute the SDP upper bounds from Table 1 we use $d = 6$ in Theorem 4.1. For $r > 0.9$ often numerical issues occur in the verification process. Therefore we can not verify the SDP result for
Table 1. Bounds for the kissing number in $S^2$

| $r$     | theoretical lower bound | lower bound by construction | SDP upper bound | theoretical upper bound |
|---------|--------------------------|-----------------------------|-----------------|-------------------------|
| 0       | 3                        | 6                           | 6.00001         | 6                       |
| 0.001   | 3                        | 5                           | 5.99999994      | 6                       |
| 0.552   | 2.50281                  | 5                           | 5.00073         | 5.00562                 |
| 0.554   | 2.49924                  | 4                           | 4.997299        | 4.99848                 |
| 0.785   | 2.00101                  | 4                           | 4.00006         | 4.00203                 |
| 0.79    | 1.98824                  | 3                           | 3.93569         | 3.97648                 |
| 0.91    | 1.64972                  | 3                           | 3.2046          | 3.2994                  |
| 0.951   | 1.51505                  | 3                           | 3.2046          | 3.03009                 |
| 0.956   | 1.4976                   | 2                           | 3.2046          | 2.9952                  |
| 1.0     | 1.32881                  | 2                           | 2.0029          | 2.65761                 |
| $\pi/3$ | 1                        | 2                           | 2.0029          | 2                       |

$r \in \{0.951, 0.956, \pi/3\}$. In these cases we use the verified SDP upper bound for the next smaller $r$. These upper bounds are written in italic in Table 1.

Figure 1. Kissing number in $S^2$.

In Table 1 one can see that the obtained kissing configurations are optimal, since their difference to the best upper bounds are less than 1. Since $\kappa(n, r)$ is a decreasing function, we also give the optimal kissing number for certain intervals of $r$. E.g. $\kappa(2, r) = 5$ for all $r \in [0.001, 0.552]$. We compute lower and upper bounds for further values of $r$ and give a plot of the results in Figure 1. Note that the cardinality of the computed kissing configurations always coincide with the value of the SDP step function obtained by rounding down the SDP upper bound.

In Table 2 we give the computed lower and upper bounds for $\kappa(3, r)$ for certain values of $r$. In the computation of the SDP upper bounds we use $d = 6$. Like in the computation of the SDP upper bounds for $\kappa(2, r)$, there often occur numerical issues in the verification of the SDP upper bounds for $r > 0.9$. For some values of $r$ we can still make the verification work by sacrificing a bit of the obtained result. Therefore, the best SDP upper bound which we can verify for $\kappa(3, 0.9117)$ is larger than the best verified SDP upper bound for $\kappa(3, 0.9)$. Since an upper bound for $\kappa(n, r)$ is also an upper bound for $\kappa(n, r')$ where $r' \leq r$, we use for $r = 0.9117$ the SDP upper bound of $\kappa(3, 0.9)$. For some values of
we could not obtain a rigorous SDP upper bounds due to numerical issues. In theses case we use the SDP upper bound of the next smaller $r$. In Table 2, these values are written in italic.

| $r$          | theoretical lower bound | theoretical upper bound | SDP step function | SDP upper bound |
|--------------|-------------------------|-------------------------|-------------------|-----------------|
| 0            | 4                       | 12                      | 12.718793         | 14.9282         |
| 0.3141592653585 | 3.61803                | 12                      | 12.00001          | 13.3915         |
| 0.3141592659   | 3.61803                | 10                      | 11.99999999993    | 13.3914         |
| 0.412223419962516 | 3.35793                | 10                      | 10.298991         | 12.3436         |
| 0.438         | 3.28046                | 9                       | 9.989757          | 12.0312         |
| 0.52359877553501 | 3                       | 9                       | 9.045288          | 10.899          |
| 0.529         | 2.98123                | 8                       | 8.989455          | 10.8231         |
| 0.604714601326872 | 2.70711               | 8                       | 8.003919          | 9.71366         |
| 0.604925      | 2.70632                | 7                       | 7.999957          | 9.71074         |
| 0.65075453732723 | 2.53209               | 7                       | 7.070565          | 9.00341         |
| 0.66          | 2.49635                | 6                       | 6.903651          | 8.85815         |
| 0.785398163397449 | 2                      | 6                       | 6.0000018         | 6.82843         |
| 0.7853982657   | 2                      | 4                       | 5.9999997         | 6.82843         |
| 0.911738290968488 | 1.5                  | 4                       | 4.028392          | 4.73205         |
| 0.945         | 1.37238                | 3                       | 4.028392          | 4.17451         |
| 0.95531661686152 | 1.3333                | 3                       | 4.028392          | 4               |
| 1.0           | 1.16771                | 2                       | 2.0029            | 3.22047         |
| $\pi/3$       | 1                      | 2                       | 2.0029            | 2               |

In Table 2 one can see that except for $r = 0.95531661686152$, the kissing configurations are optimal. Whereas for $r = 0.95531661686152$, it is not clear whether the kissing number is 3 or 4. Furthermore, due to our computations we assume that there is no radius $r$ such that there exists a maximal kissing configuration in $S^3$ with cardinality $5 + 1 = 6$ or $11 + 1 = 12$.

![Figure 2. Kissing number in $S^3$.](image-url)
In Figure 2 we give a plot of the computed lower and upper bounds for \( \kappa(3, r) \) for further values of \( r \). One can see that for most of the values of \( r \) the cardinality of the computed kissing configuration coincides with the value of the SDP step function.

**Table 3. Bounds for the kissing number in \( \mathbb{S}^4 \)**

| \( r \)     | theoretical lower bound | lower bound by construction | SDP upper bound | theoretical upper bound |
|------------|--------------------------|----------------------------|-----------------|------------------------|
| 0          | 5.11506                  | 24 c.f. \[8\]             | 24.43544        | 34.6807                |
| 0.064960281031 | 5.0847                 | 22                         | 24.25999        | 34.4481                |
| 0.135      | 4.98499                  | 21                         | 23.698995       | 33.6845                |
| 0.2348312007464 | 4.72978                | 21                         | 22.343847       | 31.7315                |
| 0.315      | 4.43922                  | 20                         | 20.975086       | 29.5112                |
| 0.3478604258810 | 4.30116                | 20                         | 20.418654       | 28.4574                |
| 0.3743605576995 | 4.18278                | 18                         | 20.039183       | 27.5544                |
| 0.393      | 4.09608                  | 17                         | 19.493801       | 26.8935                |
| 0.3966966954949 | 4.07857                | 17                         | 19.336889       | 26.7601                |
| 0.439      | 3.87137                  | 16                         | 17.528082       | 25.182                 |
| 0.44269036900123 | 3.85274              | 16                         | 17.387671       | 25.0403                |
| 0.49       | 3.60742                  | 15                         | 15.92363        | 23.1747                |
| 0.49969620570817 | 3.55583               | 15                         | 15.650850       | 22.7827                |
| 0.53       | 3.3923                   | 14                         | 14.877753       | 21.5411                |
| 0.54100885503509 | 3.33217               | 14                         | 14.632380       | 21.0849                |
| 0.55183    | 3.27277                  | 13                         | 14.402314       | 20.6343                |
| 0.55558271937072 | 3.2521                | 13                         | 14.313536       | 20.4776                |
| 0.595      | 3.03363                  | 12                         | 12.970691       | 18.8222                |
| 0.61547970865277 | 2.91955               | 12                         | 12.302214       | 17.9586                |
| 0.6299     | 2.8392                   | 11                         | 11.902489       | 17.3507                |
| 0.63337378793619 | 2.81986               | 11                         | 11.780786       | 17.2044                |
| 0.653      | 2.71075                  | 10                         | 10.990971       | 16.3794                |
| 0.68471920300192 | 2.53556              | 10                         | 10.000004       | 15.0555                |
| 0.6847193    | 2.53556                  | 9                          | 9.99999994      | 15.0555                |
| 0.68811601660265 | 2.51692               | 9                          | 9.8530813       | 14.9147                |
| 0.71       | 2.3976                   | 8                          | 8.9684325       | 14.0131                |
| 0.785398163397449 | 2                   | 8                          | 8.0000293       | 11.0078                |
| 0.78539828   | 2                       | 5                          | 7.9999982       | 11.0078                |
| 0.88607712356268 | 1.52096              | 5                          | 5.008075        | 7.3467                 |
| 0.9        | 1.46106                  | 4                          | 4.5958861       | 6.8789                 |
| 0.91173828638360 | 1.4121               | 4                          | 4.5958861       | 6.4921                 |
| 0.9206     | 1.37591                  | 3                          | 4.5958861       | 6.2043                 |
| 0.95531661577188 | 1.2431               | 3                          | 4.5958861       | 5.1151                 |
| \( \pi/3 \) | 1                       | 2                          | 2.00000029      | 2                     |

In Table 3 we give lower and upper bounds for \( \kappa(4, r) \) for certain values of \( r \). In order to obtain the given SDP upper bounds we use \( d = 8 \) for \( 0 \leq r \leq 0.5999 \). For \( d > 0.5999 \) we use \( d = 6 \) for solving the SDP. Similary to the SDP computations in dimension 2 and 3, often there occur numerical issues if \( r > 0.9 \). Therefore, we do not obtain any verified SDP upper bounds for \( r \in \{ 0.9206, 0.95531661577188, \pi/3 \} \). As in dimension 2 and 3, we use the rigorous SDP upper bounds for smaller radius \( r \). In Table 3 these bounds are written in italic. For \( \kappa(4, \pi/3) \) we use the rigorous SDP upper bound with \( r = 1.0 \).
Due to our results in Table 3, some of the considered kissing configurations are optimal, though in a visibly lesser proportion as compared to dimensions 2 and 3. In Figure 3, we give a plot of lower and upper bounds on $\kappa(4,r)$ for further values of $r$.

REFERENCES

[1] C. Bachoc and F. Vallentin, New upper bounds for kissing numbers from semidefinite programming, Journal Amer. Math. Soc. 21 (2008), 909–924.
[2] L. Bowen, Circle packing in the hyperbolic plane, Math. Physics Electronic J. 6 (2000), 1–10.
[3] M. Dostert and A. Kolpakov, Kissing number in hyperbolic space, arXiv:1907.00255
[4] K. Fujisawa, M. Fukuda, K. Kobayashi, M. Kojima, K. Nakata, M. Nakata, and M. Yamashita, SDPA (Semidefinite Programming Algorithm) User’s Manual – Version 7.0.5, Research Report B-448, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo, 2008, http://sdpa.sourceforge.net
[5] R. H. Hardin, W. D. Smith, N. J. A. Sloane, et al. Spherical Codes, http://neilsloane.com/packings/
[6] M. Jenssens, F. Joos, and W. Perkins, On kissing numbers and spherical codes in high dimensions, arXiv:1803.02702
[7] F. C. Machado and F. M. de Oliveira, Improving the semidefinite programming bound for the kissing number by exploiting polynomial symmetry, arXiv:1609.05167
[8] O. R. Musin, The kissing number in four dimensions, Ann. Math. 168 (2008), 1–32.
[9] SAGEMath, Sage Mathematics Software (Version 6.6), The Sage Development Team, 2012, http://www.sagemath.org
[10] J. G. Ratcliffe, Foundations of hyperbolic manifolds, Graduate Texts in Mathematics 149. New–York: Springer, 2013.
[11] A. D. Wyner,Capabilities of bounded discrepancy decoding, Bell Syst. Tech. J. 44 (1965), 1061–1122.

EPFL, SB TN, Station 8, 1015 Lausanne, Suisse / Switzerland
E-mail address: maria.dostert@epfl.ch

Institut de mathématiques, Rue Emile–Argand 11, 2000 Neuchâtel, Suisse / Switzerland
E-mail address: kolpakov.alexander@gmail.com