SHARP INEQUALITIES FOR THE PSI FUNCTION AND HARMONIC NUMBERS

FENG QI AND BAI-NI GUO

Abstract. In this paper, two sharp inequalities for bounding the psi function \( \psi \) and the harmonic numbers \( H_n \) are established respectively, some results in [I. Muqattash and M. Yahdi, Infinite family of approximations of the Digamma function, Math. Comput. Modelling 43 (2006), 1329–1336.] are improved, and some remarks are given.

1. Introduction

It is well-known that the classical Euler’s gamma function is defined by

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad (1) \]

for \( x > 0 \) and the derivative of its logarithm is called the psi or digamma function and denoted by \( \psi(x) \) for \( x > 0 \).

In [6], an infinite family of approximations for the psi function \( \psi(x) \) on \((0, \infty)\), denoted as \( \{I_a, a \in [0, 1]\} \), where

\[ I_a(x) = \ln(x + a) - \frac{1}{x}, \quad (2) \]

was constructed. Among other things, Corollary 2.3 and Theorem 3.2 in [6] may be recited as follows:

1. For all \( x \in (0, \infty) \),

\[ \ln(x + 1) - \frac{1}{x} \geq \psi(x) \geq \ln x - \frac{1}{x}, \quad (3) \]

2. For every \( x \in (0, \infty) \), there exists an \( a \in [0, 1] \) such that \( \psi(x) = I_a(x) \).

The inequality (3) may be rearranged as

\[ 1 \geq \exp\left( \psi(x) + \frac{1}{x} \right) - x \triangleq Q(x) \geq 0 \quad (4) \]

for \( x \in (0, \infty) \). Since \( \Gamma(x + 1) = x\Gamma(x) \) for \( x > 0 \), taking the logarithm of this recurrent formula and differentiating yields

\[ \psi(x + 1) = \psi(x) + \frac{1}{x} \quad (5) \]

As a result, the function \( Q(x) \) defined in (4) may be rearranged as

\[ Q(x) = e^{\psi(x+1)} - x \quad (6) \]
for $x \in (0, \infty)$. The graph of $Q(x)$ in the interval $(0, 9)$, plotted by the famous software Mathematica 5.2, and the limits

$$\lim_{x \to 0^+} Q(x) = e^{-\gamma} \quad \text{and} \quad \lim_{x \to \infty} Q(x) = \frac{1}{2},$$

(7)
calculated also by Mathematica 5.2, see Figure 1, show that the function $Q(x)$

![Figure 1. The graph on (0, 9) and two limits as $x \to 0^+$ and $\infty$ of $Q(x)$](image)

is possibly decreasing on $(0, \infty)$ and $\frac{1}{2} < Q(x) < e^{-\gamma} = 0.56 \cdots$ for $x \in (0, \infty)$, where $\gamma = 0.577 \cdots$ stands for the Euler-Mascheroni’s constant. These analyses imply that conclusions obtained in [6] may possibly be refined and restated more accurately.

Our main results are included in the following theorems.

**Theorem 1.** For all $x \in (0, \infty)$,

$$\ln \left( x + \frac{1}{2} \right) - \frac{1}{x} < \psi(x) < \ln(x + e^{-\gamma}) - \frac{1}{x}.$$  

(8)

The constants $\frac{1}{2}$ and $e^{-\gamma} = 0.56 \cdots$ in (8) are the best possible.

**Theorem 2.** The function $Q(x)$ defined in (6) is strictly decreasing and strictly convex on $(-1, \infty)$.

**Theorem 3.** For every $x \in (0, \infty)$, there exists a unique number $a \in \left( e^{-\gamma}, \frac{1}{2} \right)$ such that $\psi(x) = I_a(x)$. Conversely, for every $a \in \left( e^{-\gamma}, \frac{1}{2} \right)$, there exists a unique number $x \in (0, \infty)$ such that $\psi(x) = I_a(x)$.

In [6, Definition 3.5], the so-called error of the approximation $\psi(x) \approx I_a(x)$ was defined by

$$E_a(x) = \psi(x) - I_a(x) = \psi(x) - \ln(x + a) - \frac{1}{x}$$

(9)

for $x > 0$ and $a \in [0, 1]$. In [6, Theorem 3.7], it was proved that the errors $E_a(x)$ for $x \in [2, \infty)$ and $a \in [0, 1]$ are uniformly bounded between $-\ln \frac{a}{2}$ and $\ln \frac{1}{2}$. This can be restated as follows.

**Theorem 4.** Let $c = 1.4616321 \cdots$ is the only positive root of the psi function $\psi(x)$ on $(0, \infty)$. For $x \in (c, \infty)$ and $a \in \left( e^{-\gamma}, \frac{1}{2} \right)$, the following conclusions are valid:

1. The errors $E_a(x)$ are uniformly bounded between $-\ln \frac{x + e^{-\gamma}}{x + 1/2}$ and $\ln \frac{x + e^{-\gamma}}{x + 1/2}$;
2. $\psi(x) = \ln(x + a) - \frac{1}{x} + O\left( \ln \frac{x + e^{-\gamma}}{x + 1/2} \right)$. 


It is well-known that the $n$-th harmonic numbers are defined for $n \in \mathbb{N}$ by
\[ H_n = \sum_{k=1}^{n} \frac{1}{k} \] (10)
and that $H_n$ can be expressed in terms of the psi function $\psi(x)$ by
\[ H_n = \psi(n + 1) + \gamma. \] (11)

By virtue of the decreasing monotonicity of the function $Q(x)$ and the formula (11), the following new bounds for $H_n$ are derived as follows.

**Theorem 5.** For $n \in \mathbb{N}$,
\[ \ln \left( n + \frac{1}{2} \right) + \gamma < H_n(n) \leq \ln(n + e^{-\gamma} - 1) + \gamma. \] (12)

In the final section, some remarks about above conclusions are given.

2. Proofs of theorems

**Proof of Theorem 1.** Since $\psi(1) = -\gamma$, the first limit in (7) is valid clearly.

In [10], it was derived that
\[ \frac{1}{2x} - \frac{1}{12x^2} < \psi(x + 1) - \ln x < \frac{1}{2x} \] (13)
for $x > 0$. Then
\[ xe^{1/(2x)-1/(12x^2)} - x < Q(x) < xe^{1/(2x)} - x. \]

It is easy to check that both bounds for $Q(x)$ above tend to $\frac{1}{2}$ as $x \to \infty$. The second limit in (7) follows.

In [4, Lemma 1.1] and [5, Lemma 1.1], the inequality
\[ \psi'(x)e^{\psi'(x)} < 1 \] (14)
for $x > 0$ was obtained. This means that
\[ \frac{Q(x)}{[Q(x)]'} = \psi'(x + 1)e^{\psi'(x + 1)} - 1 < 0 \] (15)
for $x > -1$. Consequently, the function $Q(x)$ is strictly decreasing on $(-1, \infty)$.

Combining the decreasing monotonicity with two limits in (7) of $Q(x)$ leads obviously to Theorem 1.

**Proof of Theorem 2.** The decreasing monotonicity has been proved in the proof of Theorem 1.

Differentiating (15) once again gives
\[ [Q(x)]'' = \left\{ \psi''(x + 1) + [\psi'(x + 1)]^2 \right\} e^{\psi'(x + 1)}. \]

In [2, p. 208], [4, Lemma 1.1], [5, Lemma 1.1] and [7, 8], the function
\[ [\psi'(x)]^2 + \psi''(x) \]
for $x \in (0, \infty)$ was verified to be positive. Hence, the function $[Q(x)]'' > 0$ and $Q(x)$ is strictly convex on $(-1, \infty)$.

**Proof of Theorem 3.** Since the function $Q(x)$ is strictly decreasing from $(0, \infty)$ onto $(e^{-\gamma}, 1/2)$, that is, the mapping $Q : (0, \infty) \mapsto (e^{-\gamma}, 1/2)$ is bijective, then the proof of Theorem 3 is easily completed.
Proof of Theorem 4. By the inequality (8), it is easy to see that
\[ 0 \leq |\psi(x) - I_a(x)| \leq |I_{1/2} - I_{e^{-\gamma}}| \]
for \( x \in (c, \infty) \) and \( a \in (e^{-\gamma}, \frac{1}{2}) \), which is equivalent to
\[ 0 \leq |E_a(x)| \leq \ln \frac{x + e^{-\gamma}}{x + 1/2} \leq \ln \frac{e + e^{-\gamma}}{c + 1/2}. \]
The proof of Theorem 4 is complete. \( \Box \)

Proof of Theorem 5. Since \( Q(x) \) is strictly decreasing on \((0, \infty)\), it follows that
\[ \lim_{x \to \infty} Q(x) < Q(x) \leq Q(1) \]
for \( x \in [1, \infty) \), which is equivalent to
\[ \frac{1}{2} < e^{\psi(x+1)} - x \leq e^{\psi(2)} - 1, \]
\[ x + \frac{1}{2} < e^{\psi(x+1)} \leq x + e^{1-\gamma} - 1, \]
\[ \ln \left( x + \frac{1}{2} \right) < \psi(x + 1) \leq \ln (x + e^{1-\gamma} - 1). \]
Taking \( x = n \in \mathbb{N} \) and using the formula (11) in the above inequality leads to the inequality (12). The proof of Theorem 5 is complete. \( \Box \)

3. Remarks

Remark 1. Note that the inequality (8) in Theorem 1 refines the double inequality in Corollary 2.3 of [6].

Remark 2. The conclusions in Theorem 3 clarify the uncertain claims between lines 6–7 in [6, p. 1332].

Remark 3. In [10], the following sharp bounds for \( H_n \) were established: For \( n \in \mathbb{N} \),
\[ \ln n + \gamma + \frac{1}{2n + 1/(1-\gamma) - 2} \leq H(n) < \ln n + \gamma + \frac{1}{2n + 1/3}. \] (17)
The constants \( \frac{1}{1-\gamma} - 2 \) and \( \frac{1}{3} \) are the best possible.

In [3, pp. 386–387] and [9], alternative sharp bounds for \( H_n \) were presented: For \( n \in \mathbb{N} \),
\[ 1 + \ln(\sqrt{e} - 1) - \ln(e^{1/(n+1)} - 1) \leq H_n < \gamma - \ln(e^{1/(n+1)} - 1). \] (18)
The constants \( 1 + \ln(\sqrt{e} - 1) \) and \( \gamma \) in (18) are the best possible.

There have been a lot of literature devoted to bounding harmonic numbers \( H_n \). For more information on \( H_n \), please refer to [3, 9, 10] and related references therein.

Now we compare analytically the bounds among (12), (17) and (18). For this purpose, let
\[ f(x) = \ln \left( x + \frac{1}{2} \right) + \gamma - \left[ 1 + \ln(\sqrt{e} - 1) - \ln(e^{1/(x+1)} - 1) \right] \]
and
\[ g(x) = \ln(x + e^{1-\gamma} - 1) + \ln(e^{1/(x+1)} - 1). \]
for \( x \geq 1 \). The function \( f(x) \) may be rearranged as
\[ f(x) = \ln \left( \left( x + \frac{1}{2} \right) (e^{1/(x+1)} - 1) \right) + \left[ \gamma - 1 - \ln(\sqrt{e} - 1) \right] \triangleq \ln \frac{f_1(x)}{2} + A, \]
where

\[ f_1'(x) = \frac{(2x^2 + 2x + 1)[e^{1/(x+1)} - 2(x+1)^2/(2x^2 + 2x + 1)]}{(x+1)^2} \]

\[ \triangleq \frac{(2x^2 + 2x + 1)f_3(1/(x+1))}{(x+1)^2}, \]

\[ f_2(u) = e^u - \frac{2}{1 + (1-u)^2} = \frac{1 + (1-u)^2}{1 + (1-u)^2} \triangleq f_3(u) \]

and \( f_3'(u) = u^2e^u > 0 \) for \( u \in (0, \frac{1}{2}) \). Since \( f_3(u) \) is increasing on \( (0, \frac{1}{2}) \) and \( \lim_{u \to 0^+} f_3(u) = 0 \), it follows that \( f_3(u) > 0 \) and \( f_2(u) > 0 \) on \( (0, \frac{1}{4}) \). This leads to \( f_1'(x) > 0 \) and \( f_1(x) \) being strictly increasing for \( x \geq 1 \). Therefore, the function \( f(x) \) is increasing for \( x \geq 1 \), with

\[ f(1) = \ln\left(\left(\sqrt{e} - 1\right)/2\right) + \gamma - 1 = -0.01731922699030 \cdots \]

and

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left[ \frac{x+1/2}{x+1} \cdot \frac{e^{1/(x+1)} - 1}{1/(x+1)} \right] + \gamma - 1 - \ln(\sqrt{e} - 1) \]

\[ = \gamma - 1 - \ln(\sqrt{e} - 1) \]

\[ = 0.00996779446872 \cdots. \]

As a result, the left-hand side inequalities (12) and (18) are not included each other. Numerical calculation gives

\[ f(2) = -0.001061745178 \cdots \quad \text{and} \quad f(3) = 0.004039213518 \cdots, \]

thus, as \( n \geq 3 \), the lower bound in (12) is better than the one in (18).

Numerical computation gives

\[ g(2) = -0.00060205073286 \cdots \quad \text{and} \quad g(3) = 0.00153070402207 \cdots. \]

It is easy to see that

\[ \lim_{x \to \infty} g(x) = \ln\left[ \frac{x + e^{-1} - 1}{x + 1} \cdot \frac{e^{1/(x+1)} - 1}{1/(x+1)} \right] = 0. \]

Combining with the graph of \( g(x) \) on (1, 29), see Figure 2, it can be possibly claimed that, as \( n \geq 3 \), the upper bound in (12) is not better than the one in (18).

Similarly, it may be showed that the upper bound in (12) is not better than the one in (17) as \( n \geq 2 \), that the lower bound in (12) is not better than the one in (17) at all, and that the inequality (18) is not better than the one (17) at all. Thus,
it may be said that the inequality (17) is the best among inequalities (12), (17) and (18).

Remark 4. In [7, 8], the positivity of the function (16) was generalized to a much general result including, as a particular case, its being completely monotonic on $(0, \infty)$.

Remark 5. It is conjectured that the function $Q(x)$ defined in (6) is completely monotonic on $(-1, \infty)$, that is, $(-1)^k[Q(x)]^{(k)} \geq 0$ on $(-1, \infty)$ for $k \geq 0$.

Acknowledgements. This manuscript was completed during the first author’s visit to the RGMIA, Victoria University, Australia between March 2008 and February 2009. The first author express his sincere appreciations on local colleagues at the RGMIA for their invitation and hospitality throughout this period.

References

[1] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 9th printing, Washington, 1970.

[2] H. Alzer, Sharp inequalities for the digamma and polygamma functions, Forum Math. 16 (2004), 181–221.

[3] H. Alzer, Sharp inequalities for the harmonic numbers, Expo. Math. 24 (2006), no. 4, 385–388.

[4] N. Batir, Some new inequalities for gamma and polygamma functions, J. Inequal. Pure Appl. Math. 6 (2005), no. 4, Art. 103; Available online at http://jipam.vu.edu.au/article.php?sid=577.

[5] N. Batir, Some new inequalities for gamma and polygamma functions, RGMIA Res. Rep. Coll. 7 (2004), no. 3, Art. 1; Available online at http://www.staff.vu.edu.au/rpgmia/v7n3.asp.

[6] I. Muqattash and M. Yahdi, Infinite family of approximations of the Digamma function, Math. Comput. Modelling 43 (2006), 1329–1336.

[7] F. Qi, A completely monotonic function involving divided differences of psi and polygamma functions and an application, RGMIA Res. Rep. Coll. 9 (2006), no. 4, Art. 8; Available online at http://www.staff.vu.edu.au/rpgmia/v9n4.asp.

[8] F. Qi, The best bounds in Kershaw’s inequality and two completely monotonic functions, RGMIA Res. Rep. Coll. 9 (2006), no. 4, Art. 2; Available online at http://www.staff.vu.edu.au/rpgmia/v9n4.asp.

[9] F. Qi and B.-N. Guo, A short proof of monotonicity of a function involving the psi and exponential functions, Available online at http://arxiv.org/abs/0902.2519v1.

[10] F. Qi, R.-Q. Cui, Ch.-P. Chen, and B.-N. Guo, Some completely monotonic functions involving polygamma functions and an application, J. Math. Anal. Appl. 310 (2005), no. 1, 303–308.