B-spline-like bases for $C^2$ cubics on the Powell-Sabin 12-split

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Abstract. For spaces of constant, linear, and quadratic splines of maximal smoothness on the Powell-Sabin 12-split of a triangle, the so-called S-bases were recently introduced. These are simplex spline bases with B-spline-like properties on the 12-split of a single triangle, which are tied together across triangles in a Bézier-like manner.

In this paper we give a formal definition of an S-basis in terms of certain basic properties. We proceed to investigate the existence of S-bases for the aforementioned spaces and additionally the cubic case, resulting in an exhaustive list. From their nature as simplex splines, we derive simple differentiation and recurrence formulas to other S-bases. We establish a Marsden identity that gives rise to various quasi-interpolants and domain points forming an intuitive control net, in terms of which conditions for $C^0$-, $C^1$-, and $C^2$-smoothness are derived.

1. Introduction

1.1. Motivation

Piecewise polynomials, or splines, defined over triangulations have applications in many branches of science, ranging from scattered data fitting to finding numerical solutions to partial differential equations. See [1,12] for comprehensive monographs.

In applications like geometric modelling [4] and solving PDEs by isogeometric methods [5] one often desires a low degree spline with higher smoothness. For a general triangulation, it was shown in Theorem 1(ii) of [24] that the minimal degrees of a triangular $C^1$ and $C^2$ element are 5 and 9, respectively. To obtain smooth splines of lower degree one can split each triangle in the triangulation into several subtriangles. Three such splits are the Clough-Tocher split $\Delta$ (CT), the Powell-Sabin 6-split $\Delta$ (PS6) and 12-split $\Delta$ (PS12) of a triangle $\Delta$, with 3, 6, and 12 subtriangles, respectively. On these splits global $C^1$-smoothness can be obtained with degree 3 for CT, degree 2 and 3 for PS6 and degree 2 for PS12 [9,19]. $C^2$-smoothness is achieved for PS6 and PS12 using degree 5; see [11,21,22] on a general (planar) triangulation.

To compute with splines we need a basis for the spline space. In the univariate case B-splines have many advantages. They lead to banded matrices with good stability properties for low degrees and can be computed efficiently using stable recurrence relations. We would like similar bases for splines on triangulations. In [3] a basis, called the S-basis, was introduced for $C^1$ quadratics on the PS12-split. The S-basis consists of simplex splines [16,20] and has all the usual properties of univariate B-splines, including a recurrence relation down to piecewise linear polynomials and a Marsden identity. Global
C¹-smoothness is achieved by connecting neighboring triangles using classical Bézier techniques. This basis has been applied for swift assembly of the stiffness matrices in the finite element method [23]. For a quintic B-spline-like basis on the PS12-split see [14, 15]. This basis has C³ supersmoothness on each macro triangle and has global C²-smoothness. Moreover, in addition to giving C¹- and C²-smooth spaces on any triangulation, these spaces on the PS12-split are suitable for multiresolution analysis [6, 8, 14, 18]. A similar B-spline-like simplex basis has been constructed on the CT-split [13], while for the PS6-split a B-spline basis has been developed for the C¹-smooth quadratics and cubics and C²-smooth quintics [7, 9, 22]. The latter bases have many of the nice B-spline properties, but have to be computed by conversion to Bernstein bases on each subtriangle.

In this paper we systematically enumerate the simplex splines and determine the possible S-bases for the spaces of C⁰−¹ splines of degree d on the PS12-split for d = 0, 1, 2, 3. In the cubic case and for a general triangulation, we argue that these cannot be extended to globally smooth bases. Instead, we envision applications for local constructions, such as hybrid meshes and extra-ordinary points, which are important issues in isogeometric analysis.

1.2. Main result

For d = 0, 1, 2, 3, we consider the space Sₗ(d) of C⁰−¹ smooth splines of degree d on the Powell-Sabin 12-split of a triangle (see definition below). We consider bases s of Sₗ(d) satisfying the following properties:

P₁ s is invariant under the dihedral symmetry group G of the equilateral triangle (cf. §2.1.8).

P₂ s reduces to a B-spline basis on the boundary (cf. §2.1.9).

P₃ s forms a positive partition of unity and satisfies a Marsden identity, for which the dual polynomials only have real linear factors (cf. §4).

P₄ s has all its domain points inside Δ, with precisely d + 2 domain points on each edge of Δ (cf. Figure 3).

P₅ s admits a stable recurrence relation (cf. §3).

P₆ s admits a differentiation formula (cf. §3.6).

P₇ s comes equipped with quasi-interpolants (cf. §4.3).

P₈ s can be smoothly tied together across adjoining triangles using Bézier-type conditions (cf. §5).

P₁ makes sure that basic properties of s are left invariant under affine transformations. P₂–P₄ allow to establish an Bézier-like control net. Together with the differentiation formula (P₆), this makes it possible to establish Bézier-type conditions (P₈) for smooth joins to neighbouring triangles.

In addition some of the bases s satisfy:

P₉ s has local linear independence.

We call any basis for Sₗ(d) satisfying P₁–P₈ an S-basis. This space has dimension nₛ as in (1.10) and simplex spline bases

\[ sₗ^d = [S₁,d, \ldots, Sₙₛ,d], \ d = 0, 1, 2, 3, \quad \tilde{sₗ}^d = [\tilde{S}_₁,d, \ldots, \tilde{S}_ₙₖ,d], \ d = 2, 3, \quad (1.1) \]

listed in Table 4. Generalizing a similar result for the bases s₀, s₁, s₂ in [3], the main result of the paper is the following.
Theorem 1.1. The sets \( s = s^T, \tilde{s}^T_4 \) as in (1.1) are the only simplex spline bases for \( S_d(\mathbb{A}) \) satisfying P1–P4. Moreover, these bases also satisfy P5–P8.

The structure of the paper is as follows. In the next section we recall some basic facts from splines on triangulations, focusing on simplex splines on the 12-split. Theorem 1.1, as well as the local linear independence property (P9), are known [3] to hold for the constant, linear, and quadratic bases \( s^0, s^1, s^2 \). For the remaining bases \( \tilde{s}^2, s^3, \tilde{s}^3 \), Theorem 1.1 is established property-by-property in individual sections in the paper. Supplementary computational results are presented in a Jupyter notebook [17].

1.3. Basic tools
We recall some basic tools used throughout the paper.

1.3.1. Conventions
We use small Greek letters (e.g. \( \alpha, \beta \)) for scalar values, small boldface letters (e.g. \( s \)) to denote vectors, capital boldface letters (e.g. \( R, T, U \)) for matrices. Scalar-valued univariate functions are denoted by small letters, scalar-valued multivariate functions are denoted by capital letters (e.g. \( S, M, Q \)), while vector-valued multivariate are, like matrices, denoted by capital boldface letters. Calligraphic fonts (e.g. \( \mathcal{K} \)) are reserved for (multi)sets, expressed as

\[
\mathcal{K} = \{k_1, \ldots, k_1, \ldots, k_s, \ldots, k_s\} = \{k_1^{\mu_1} \cdots k_s^{\mu_s}\},
\]

with \( \mu_i \) the multiplicity of \( k_i \). The size \( |\mathcal{K}| \) of \( \mathcal{K} \) is its number of elements counting multiplicities, i.e., \( |\mathcal{K}| = \mu_1 + \cdots + \mu_s \). Generalizing the notation for closed and half-open intervals, we write \([\mathcal{K}]\) for the convex hull of \( \mathcal{K} \). Whenever \( \mathcal{K} \) consists of vertices of \( \mathbb{A} \), we write \(|\mathcal{K}|\) for the half-open convex hull of \( \mathcal{K} \) obtained as union of the half-open subtriangles shown in Figure 1 (right).

Blackboard bold (e.g. \( \mathbb{P}_d, \mathbb{S}_d \)) is used to denote function spaces. In particular, identifying matrices with linear maps, the symbol \( \mathbb{R}^{m,n} \) denotes the space of \( m \times n \) real matrices. We identify \( \mathbb{R}^m \) with \( \mathbb{R}^{m,1} \) (column vectors), and denote the standard basis vectors in \( \mathbb{R}^m \) by \( e_1, \ldots, e_m \).
For an \( m \times n \) matrix \( A \) and \( i = [i_1, \ldots, i_r]^T, j = [j_1, \ldots, j_s]^T \) with \( 1 \leq i_1 < \cdots < i_r \leq m, 1 \leq j_1 \leq \cdots \leq j_s \leq n \), then \( A(i, j) \) is the \( r \times s \) matrix whose \((k, \ell)\) element is \( a_{i_k,j_\ell} \). In particular, \( e(i) \) denotes the vector whose \( j \)th element is \( c_{ij} \).

The support of a function \( F \), denoted by \( \text{supp}(F) \), is the closure of the set of values in the domain of \( F \) at which \( F \) is nonzero. Empty products are assumed to be 1.

### 1.3.2. The Powell-Sabin 12-split

Consider the triangle \( \triangle \) with vertices \( p_1, p_2, p_3 \in \mathbb{R}^2 \) and midpoints

\[
p_4 := \frac{p_1 + p_2}{2}, \quad p_5 := \frac{p_2 + p_3}{2}, \quad p_6 := \frac{p_3 + p_1}{2}.
\]

(1.2)

Taking the complete graph on these six points, one obtains additional points

\[
p_7 := \frac{p_4 + p_6}{2}, \quad p_8 := \frac{p_4 + p_5}{2}, \quad p_9 := \frac{p_5 + p_6}{2}, \quad p_{10} := \frac{p_1 + p_2 + p_3}{3}.
\]

(1.3)

and subtriangles \( \triangle_1, \ldots, \triangle_{12} \) as in Figure 1. The resulting split is called the **Powell-Sabin 12-split** \( \mathcal{A} \) of \( \triangle \).

### 1.3.3. Barycentric and directional coordinates

The **barycentric coordinates** \( \beta = (\beta_1, \beta_2, \beta_3) \) of a point \( x \in \mathbb{R}^2 \) with respect to the triangle \( \triangle = [p_1, p_2, p_3] \) is the unique solution to

\[
x = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3, \quad 1 = \beta_1 + \beta_2 + \beta_3.
\]

(1.4)

Similarly, we write \( \beta^{i,j,k} = (\beta_1^{i,j,k}, \beta_2^{i,j,k}, \beta_3^{i,j,k}) \) for the barycentric coordinates of \( x \) with respect to the triangle \( [p_i, p_j, p_k] \subset \triangle \). To save space in the recursion and differentiation matrices, we use the short-hands

\[
\gamma_j := 2\beta_j - 1, \quad \beta_{i,j} = \beta_i - \beta_j, \quad \sigma_{i,j} = \beta_i + \beta_j, \quad \text{for } i, j = 1, 2, 3.
\]

(1.5)

Note that

\[
\gamma_j \geq 0 \text{ at } \Delta_i, \quad i \neq 2j - 1, 2j.
\]

(1.6)

For any \( u = [u_1, u_2]^T \in \mathbb{R}^2 \), consider the corresponding directional derivative \( D_u := u \cdot \nabla = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} \). The unique solution \( \alpha := [\alpha_1, \alpha_2, \alpha_3]^T \) of

\[
u = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3, \quad 0 = \alpha_1 + \alpha_2 + \alpha_3.
\]

(1.7)

is called the **directional coordinates** of \( u \). If \( u = q^1 - q^2 \), with \( q^1, q^2 \in \mathbb{R}^2 \), then \( \alpha_j := \beta_j^1 - \beta_j^2 \), \( j = 1, 2, 3 \), where \( \beta^i := [\beta_1^i, \beta_2^i, \beta_3^i]^T \) is the vector of barycentric coordinates of \( q^i, i = 1, 2 \).

### 1.3.4. Function spaces

Let \( \mathbb{P}_d(\mathbb{R}^2) \) denote the space of bivariate polynomials of total degree at most \( d \) and with real coefficients, which has dimension \( \nu_d := (d + 1)(d + 2)/2 \). On a triangle \( \triangle \) with barycentric coordinates \( \beta_1, \beta_2, \beta_3 \), a convenient basis of \( \mathbb{P}_d \) is formed by the **Bernstein polynomials**

\[
B_{i_1,i_2,i_3}^d := \frac{d!}{i_1!i_2!i_3!} \beta_1^{i_1} \beta_2^{i_2} \beta_3^{i_3}, \quad i_1 + i_2 + i_3 = d.
\]

(1.8)

Analogously, on the 12-split \( \mathcal{A} \) of a triangle \( \triangle \), we consider the spaces

\[
S_d(\mathcal{A}) := \{ f \in C^{d-1}(\triangle) : f|_{\triangle_k} \in \mathbb{P}_d, \text{ for } k = 1, \ldots, 12 \}, \quad d = 0, 1, 2, \ldots
\]

(1.9)
The dimension \( n_d \) of this space is [15, Theorem 3]

\[
(n_0, n_1, n_2, n_3, \ldots) = (12, 10, 12, 16, \ldots), \quad n_d = \frac{1}{2}d^2 + \frac{3}{2}d + 7, \quad d \geq 2. \tag{1.10}
\]

For \( d = 0, 1, 2 \), we equip these spaces with the S-bases \( s_d^J = [S_{j,d}]_{j=1}^{n_d} \) presented in [3].

Each piecewise polynomial on \( \Delta \) can be represented as an element of the \( \mathbb{P}_d \)-module \( \mathbb{P}_{12}^d \), i.e., as a vector with components the polynomial pieces on the faces \( \triangle_k \) of \( \Delta \).

2. Simplex splines

In this section we recall the definition and some basic properties of simplex splines, and determine a list of all simplex splines in \( S_d(\Delta) \) for \( d = 0, 1, 2, 3 \).

2.1. Definition and properties

First we provide the definition of simplex spline convenient for our purposes, and recall properties necessary for the remainder of the paper.

2.1.1. Geometric construction

Let \( k_1, \ldots, k_{d+3} \in \mathbb{R}^2 \) be a sequence of points in the plane, called knots, defining a multiset \( K \).

Let \( \sigma = [k_1, \ldots, k_{d+3}] \subset \mathbb{R}^{d+2} \) be a simplex whose projection \( P : \mathbb{R}^{d+2} \rightarrow \mathbb{R}^2 \) onto the first two coordinates satisfies \( P(k_i) = k_i \), for \( i = 1, \ldots, d + 3 \). For any integer \( k \geq 1 \), let \( \text{vol}_k \) denote the \( k \)-dimensional volume. We define the integral normalized simplex spline

\[
M[K] : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad M[K](x) := \frac{\text{vol}_d(\sigma \cap P^{-1}(x))}{\text{vol}_{d+2}(\sigma)}.
\]

We will restrict ourselves to simplex splines on the 12-split \( \Delta \) of a triangle \( \Delta \), in which case \( K = \{p_1^{\mu_1} \cdots p_{10}^{\mu_{10}}\} \). While \( M[K] \) is the simplex spline most commonly encountered in the literature, our discussion is simpler in terms of the (area normalized) simplex spline, defined as

\[
Q[K] := \text{vol}_2(\Delta) \cdot \left(\frac{|K| - 1}{2}\right)^{-1} M[K].
\]

Whenever \( \mu_7 = \mu_8 = \mu_9 = 0 \), we use the graphical notation

\[
\Delta := Q[p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}p_5^{\mu_5}p_6^{\mu_6}p_7^{\mu_7}p_8^{\mu_8}p_9^{\mu_9}p_{10}^{\mu_{10}}].
\]
2.1.2. Piecewise polynomial structure

$Q[K]$ is a piecewise polynomial on the convex hull $|K|$ of $K$, with knot lines formed by the complete graph of $K$; see Figure 2.

2.1.3. Degree

Each polynomial piece of $Q[K]$ has total degree bounded as

$$\text{deg } Q[K] \leq d := |K| - 3.$$  \hfill (2.1)

2.1.4. Smoothness

The smoothness across a knot line can be controlled locally. More precisely, for any $x \in \triangle$, let $\mu$ be the maximum number of knots of $K$ (counting multiplicities), at least two of which are distinct, along any affine line containing $x$. Then $Q[K]$ will have continuous derivatives up to order $d + 1 - \mu$ at $x$, which we will express with the notation

$$Q[K] \in C^{d+1-\mu} \text{ at } x.$$  \hfill (2.2)

For example, if $Q[K]$ is a $C^{d-1}$-smooth simplex spline of degree $d$, then any line segment in $\triangle$ can contain at most two distinct knots.

2.1.5. Recursion

For $d \geq 1$, the simplex spline can be expressed in terms of simplex splines of lower degree,

$$Q[K](x) = \sum_{j=1}^{d+3} \beta_j Q[K \setminus k_j](x), \quad \sum_j \beta_j = 1, \quad \sum_j \beta_j k_j = x \in \mathbb{R}^2.$$  \hfill (2.3)

For simplex splines with knot multiset $K = \{p_1^{\mu_1} \cdots p_{10}^{\mu_{10}}\} \subset \mathbb{R}^2$ composed of vertices of $\triangle$, we can therefore equivalently define $Q[K]$ recursively by

$$Q[K](x) := \begin{cases} 
0 & \text{if } \text{area}(|K|) = 0, \\
1_{|K|}(x) \frac{\text{area}(\triangle)}{\text{area}(|K|)} & \text{if } \text{area}(|K|) \neq 0 \text{ and } |K| = 3, \\
\sum_{j=1}^{10} \beta_j Q[K \setminus k_j](x) & \text{if } \text{area}(|K|) \neq 0 \text{ and } |K| > 3,
\end{cases}$$  \hfill (2.4)

with $x = \beta_1 p_1 + \cdots + \beta_{10} p_{10}$, $\beta_1 + \cdots + \beta_{10} = 1$, and $\beta_i = 0$ whenever $\mu_i = 0$. For instance, with $\beta_1, \beta_2, \beta_3$ the barycentric coordinates of $\triangle$,

$$= \beta_1 \cdot 1_\triangle + \beta_2 \cdot 0 + \beta_3 \cdot 0 = \beta_1 \cdot 1_\triangle.$$

2.1.6. Differentiation

When it is defined, the directional derivative of the simplex spline of degree $d$ can be expressed in terms of simplex splines of lower degree,

$$D_u Q[K] = d \sum_{j=1}^{d+3} \alpha_j Q[K \setminus k_j], \quad \sum_j \alpha_j = 0, \quad \sum_j \alpha_j k_j = u \in \mathbb{R}^2.$$  \hfill (2.5)

For instance, with $\alpha_1, \alpha_2, \alpha_3$ directional coordinates of $u$ with respect to the triangle $\triangle$,

$$\frac{1}{3} D_u = \alpha_1 + \alpha_2 + \alpha_3.$$
2.1.7. Knot insertion

The simplex spline admits the knot insertion formula
\[
Q[\mathcal{K}] = \sum_{j=1}^{d+3} c_j Q[(\mathcal{K} \sqcup y) \setminus k_j], \quad \sum_{j} c_j = 1, \quad \sum_{j} c_j k_j = y \in \mathbb{R}^2. \tag{2.6}
\]

For instance, repeatedly applying knot insertion at the midpoints \( p_k = c_i p_i + c_j p_j, \) \( c_i = c_j = \frac{1}{2}, \) at the cost of the end points \((p_i, p_j) \in \{p_1, p_2, p_3\},\)
\[
= \frac{1}{2} \bigtriangleup + \frac{1}{2} \bigtriangleup = \frac{1}{4} \bigtriangleup + \frac{1}{4} \bigtriangleup + \frac{1}{4} \bigtriangleup + \frac{1}{4} \bigtriangleup \tag{2.7}
\]

2.1.8. Symmetries

The dihedral group \( \mathcal{G} \) of the equilateral triangle consists of the identity, two rotations and three reflections, i.e.,

The affine bijection sending \( p_k \) to \((\cos 2\pi k/3, \sin 2\pi k/3),\) for \( k = 1, 2, 3,\) maps \( \bigtriangleup \) to the 12-split of an equilateral triangle. Through this correspondence, the dihedral group permutes the vertices \( p_1, \ldots, p_{10} \) of \( \bigtriangleup. \) Every such permutation \( \sigma \) induces a bijection \( Q[p_1^{\mu_1} \cdots p_{10}^{\mu_{10}}] \mapsto Q[\sigma(p_1)^{\mu_1} \cdots \sigma(p_{10})^{\mu_{10}}] \) on the set of all simplex splines on \( \bigtriangleup. \) For any set \( s \) of simplex splines, we write
\[
[s]_{\mathcal{G}} := \{Q[\sigma(\mathcal{K})] : Q[\mathcal{K}] \in s, \ \sigma \in \mathcal{G}\}
\]
for the \( \mathcal{G} \)-equivalence class of \( s, \) i.e., the set of simplex splines related to \( s \) by a symmetry in \( \mathcal{G}. \) In particular, the bases in \([1,1]\) shown in Table \( \square \) take the compact form

\[
\begin{align*}
s_0 &= \left[\frac{1}{8} \bigtriangleup, \frac{1}{24} \bigtriangleup\right]_{\mathcal{G}}, \\
s_1 &= \left[\frac{1}{4} \bigtriangleup, \frac{1}{3} \bigtriangleup, \frac{1}{6} \bigtriangleup, \frac{1}{4} \bigtriangleup\right]_{\mathcal{G}}, \\
s_2 &= \left[\frac{1}{4} \bigtriangleup, \frac{1}{2} \bigtriangleup, \frac{3}{4} \bigtriangleup\right]_{\mathcal{G}}, \\
\tilde{s}_2 &= \left[\frac{1}{4} \bigtriangleup, \frac{1}{2} \bigtriangleup, \frac{3}{4} \bigtriangleup\right]_{\mathcal{G}}, \\
s_3 &= \left[\frac{1}{4} \bigtriangleup, \frac{1}{2} \bigtriangleup, \frac{3}{4} \bigtriangleup\right]_{\mathcal{G}}, \\
\tilde{s}_3 &= \left[\frac{1}{4} \bigtriangleup, \frac{1}{2} \bigtriangleup, \frac{3}{4} \bigtriangleup\right]_{\mathcal{G}}.
\end{align*}
\]

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We say that \( s \) is \( \mathcal{G} \)-invariant whenever \( [s]_\mathcal{G} = s \) (property P1). One sees immediately that this is the case for the bases in (1.1).

2.1.9. Restriction to an edge

Let \( e = [p_i, p_k] \) be an edge of \( \Delta \) with midpoint \( p_j \) and let \( \varphi_{ik}(t) := (1 - t)p_i + tp_k \). By induction on \(|\mathcal{K}|\),

\[
Q[\mathcal{K}] \circ \varphi_{ik}(t) = \begin{cases} 0 & \text{if } \mu_i + \mu_j + \mu_k < |\mathcal{K}| - 1, \\ \frac{\text{area}(\Delta)}{\text{area}(\mathcal{K})} B(t) & \text{if } \mu_i + \mu_j + \mu_k = |\mathcal{K}| - 1, \end{cases}
\]

(2.8)

where \( B \) is the univariate B-spline with knot multiset \( \{0^\mu_0 0^\mu_1 1^\mu_k\} \).

We say that \( Q[\mathcal{K}] \) reduces to a B-spline on the boundary when \( B \) is one of the consecutive univariate B-splines \( B_1^d, \ldots, B_{d+2}^d \) on the open knot multiset \( \{0^{d+1} 0^1 1^d\} \), i.e.,

\[
B_1^d := B[0^{d+1} 0^1], \quad B_2^d := B[0^d 0^1 1], \quad \ldots, \quad B_{d+2}^d := B[0^1 1^d].
\]

(2.9)

Similarly a basis \( \{S_1, \ldots, S_{nd}\} \) of \( S_d(\Delta) \) reduces to a B-spline basis on the boundary (property P2) when, for \( 1 \leq i < k \leq 3 \), as multisets,

\[
\{ S_1 \circ \varphi_{ik}, \ldots, S_{nd} \circ \varphi_{ik} \} = \{(B_1^d)^1 \cdots (B_{d+2}^d)^1 0^{r-d-2}\}.
\]

One sees that this is the case for the bases in (1.1).

2.2. Enumeration on the 12-split

Any simplex spline \( Q[\mathcal{K}] \) on \( \Delta \) is specified by a multiset \( \mathcal{K} = \{p_1^{\mu_1} \cdots p_{10}^{\mu_{10}}\} \). Let us see how various properties of \( Q[\mathcal{K}] \) translate into conditions on the knot multiplicities.

Certain segments, like \([p_1, p_3]\), do not appear as edges in the 12-split, meaning that \( C^\infty \)-smoothness is required across these segments. Hence, by (2.2),

\[
\mu_1\mu_8 = \mu_1\mu_9 = \mu_8\mu_9 = 0, \quad \mu_2\mu_7 = \mu_2\mu_9 = \mu_7\mu_9 = 0, \quad \mu_3\mu_7 = \mu_3\mu_8 = \mu_7\mu_8 = 0.
\]

(2.10)

If \( Q[\mathcal{K}] \) has degree \( d \), then, by (2.1),

\[
\mu_1 + \mu_2 + \cdots + \mu_{10} = d + 3.
\]

(2.11)

To achieve \( C^r \)-smoothness across the knotlines in \( \Delta \), necessarily

\[
\begin{align*}
\mu_1 + \mu_5 + \mu_7 + \mu_{10} &\leq d + 1 - r, & \mu_4 + \mu_6 + \mu_7 &\leq d + 1 - r, \\
\mu_2 + \mu_6 + \mu_8 + \mu_{10} &\leq d + 1 - r, & \mu_4 + \mu_5 + \mu_8 &\leq d + 1 - r, \\
\mu_3 + \mu_4 + \mu_9 + \mu_{10} &\leq d + 1 - r, & \mu_5 + \mu_6 + \mu_9 &\leq d + 1 - r,
\end{align*}
\]

(2.12)

whenever two of the multiplicities are nonzero.

**Lemma 2.1.** Suppose \( Q[p_1^{\mu_1} \cdots p_{10}^{\mu_{10}}] \) is a \( C^r \)-smooth simplex spline on \( \Delta \) of degree \( d \). If \( d \leq 2r + 1 \), then

\[
\mu_7 = \mu_8 = \mu_9 = 0.
\]

(2.13)

If \( d \leq \lceil \frac{3}{2}r \rceil \), then

\[
\mu_{10} = 0,
\]

(2.14)

\[
\mu_1 + \mu_5, \mu_2 + \mu_6, \mu_3 + \mu_4, \mu_4 + \mu_6, \mu_4 + \mu_5, \mu_5 + \mu_6 \leq d + 1 - r,
\]

(2.15)

(whenever both multiplicities are nonzero),

\[
\mu_4 + \mu_5 + \mu_6 \leq \frac{3}{2}(d + 1 - r),
\]

(2.16)
\[ \mu_1 + \mu_2 + \mu_3 \geq \frac{1}{2}(3r + 3 - d), \quad (2.17) \]

**Proof.** Suppose \( \mu_7 \geq 1 \). Then, by (2.10), \( \mu_2 = \mu_3 = \mu_8 = \mu_9 = 0 \). Adding the first row in (2.12) and subtracting (2.11), yields \( \mu_7 \leq d - 1 - 2r \). This is a contradiction whenever \( d \leq 2r + 1 \), establishing the first statement of the theorem.

Next assume \( d \leq \lceil \frac{3}{2}r \rceil \). Adding the first column in (2.12) and using (2.11), yields \( d + 3 + 2\mu_{10} \leq 3d + 3 - 3r \). Solving for \( \mu_{10} \), we obtain the second statement of the theorem.

The third statement follows immediately from the first two and (2.12). Moreover, adding the inequalities in the second column in (2.12), dividing by two, and using (2.11), one obtains the fourth statement. Finally, together with (2.11), we obtain the fifth statement. \( \blacksquare \)

Next we determine the \( C^{d-1} \)-smooth simplex splines of degree \( d \) on \( \triangle \) for \( d = 0, 1, 2, 3 \). Selecting those that reduce to either zero or a B-spline on the boundary, we arrive Table 1.

### 2.2.1. The case \( d = 0 \) and \( r = -1 \)

The \( C^{-1} \)-smooth constant simplex splines \( Q[K] \) on \( \triangle \) have \( |K| = 3 \), corresponding to triples of knots not lying on a line.

### 2.2.2. The case \( d = 1 \) and \( r = 0 \)

The \( C^0 \)-smooth linear simplex splines \( Q[K] \) on \( \triangle \) have knot multiplicities satisfying \( \mu_7 = \mu_8 = \mu_9 = 0 \) by Lemma 2.1 and therefore \( \mu_1 + \cdots + \mu_6 + \mu_{10} = 4 \) by (2.11). By (2.12), \( \mu_{10} \leq 1 \). If \( \mu_{10} = 1 \), then (2.12) implies \( \mu_1, \mu_2, \ldots, \mu_6 \leq 1 \). Up to symmetry, and systematically distinguishing cases by the number of corner knots, we obtain the simplex splines

If \( \mu_{10} = 0 \), then, again distinguishing cases by the number of corner knots, we obtain the simplex splines

### 2.2.3. The case \( d = 2 \) and \( r = 1 \)

The \( C^1 \)-smooth quadratic simplex splines \( Q[K] \) on \( \triangle \) have knot multiplicities satisfying \( \mu_7 = \mu_8 = \mu_9 = \mu_{10} = 0 \) by Lemma 2.1 and

\[ \mu_1 + \cdots + \mu_6 = 5, \quad \mu_1 + \mu_2 + \mu_3 \geq 2, \quad \mu_4 + \mu_5 + \mu_6 \leq 3. \quad (2.18) \]

Distinguishing cases by the number of corner knots, yields

### 2.2.4. The case \( d = 3 \) and \( r = 2 \)

The \( C^2 \)-smooth cubic simplex splines \( Q[K] \) on \( \triangle \) have, by Lemma 2.1, knot multiplicities satisfying \( \mu_7 = \mu_8 = \mu_9 = \mu_{10} = 0 \)

\[ \mu_1 + \mu_2 + \mu_3 \geq 3, \quad \mu_4 + \mu_5 + \mu_6 \leq 3. \quad (2.19) \]
Let \( e = [p_i, p_k] \) be any edge of \( \Delta \) with midpoint \( p_j \). If \( \mu_i + \mu_j + \mu_k < 5 \), then \( Q[K]|_e = 0 \) by (2.8).

In the remaining case \( \mu_i + \mu_j + \mu_k = 5 \) we demand that \( Q[K] \) reduces to a B-spline on the boundary, yielding the conditions

\[
\begin{align*}
\text{not}(\mu_1 + \mu_4 + \mu_2 = 5 \text{ and } \mu_4 \geq 2), \\
\text{not}(\mu_1 + \mu_4 + \mu_2 = 5 \text{ and } \mu_1 \geq 1 \text{ and } \mu_2 \geq 1 \text{ and } \mu_4 \neq 1), \\
\text{not}(\mu_2 + \mu_5 + \mu_3 = 5 \text{ and } \mu_5 \geq 2), \\
\text{not}(\mu_2 + \mu_5 + \mu_3 = 5 \text{ and } \mu_2 \geq 1 \text{ and } \mu_3 \geq 1 \text{ and } \mu_5 \neq 1), \\
\text{not}(\mu_1 + \mu_6 + \mu_3 = 5 \text{ and } \mu_6 \geq 2), \\
\text{not}(\mu_1 + \mu_6 + \mu_3 = 5 \text{ and } \mu_1 \geq 1 \text{ and } \mu_3 \geq 1 \text{ and } \mu_6 \neq 1). 
\end{align*}
\]

(2.20)

**Theorem 2.2.** With one representative for each \( G \)-equivalence class, Table 1 presents an exhaustive list of the \( C^2 \) cubic simplex splines on \( \Delta \) that reduce to either zero or a B-spline on the boundary.

**Proof.** By (2.10), it suffices to consider the following cases according to the support \([K]\) of \( Q[K]\), up to a symmetry of \( G \).

**Case 0, no corner included, \([K] = [p_4, p_5, p_6]\):** By (2.11), \( \mu_4 + \mu_5 + \mu_6 = 6 \), contradicting (2.15). Hence this case does not happen.

**Case 1a, 1 corner included, \([K] = [p_1, p_4, p_6]\):** For a positive support \( \mu_1, \mu_4, \mu_6 \geq 1 \), and since \( \mu_4 + \mu_6 \leq 2 \) by (2.15), we obtain .

**Case 1b, 1 corner included, \([K] = [p_1, p_4, p_5, p_6]\):** By (2.11) and (2.19) one has \( \mu_1 = 6 - \mu_4 - \mu_5 - \mu_6 \geq 3 \), contradicting \( \mu_1 + \mu_5 \leq 2 \) from (2.15). Hence this case does not occur.

**Case 2a, 2 corners included, \([K] = [p_1, p_2, p_6]\):** For a positive support, \( \mu_1, \mu_2, \mu_6 \geq 1 \). Since \( \mu_2 + \mu_6 \leq 2 \) by (2.15), it follows \( \mu_2 = \mu_6 = 1 \). Moreover, \( \mu_4 = 1 \) by (2.20), and we obtain .

**Case 2b, 2 corners included, \([K] = [p_1, p_2, p_5, p_6]\):** Since \( \mu_1 + \mu_5, \mu_2 + \mu_6 \leq 2 \) by (2.15), implying \( \mu_1 = \mu_2 = \mu_5 = \mu_6 = 1 \). Then \( \mu_4 = 2 \) by (2.11), contradicting (2.20). Hence this case does not occur.

**Case 3, 3 corners included, \([K] = [p_1, p_2, p_3]\):** We distinguish cases for \( (\mu_4, \mu_5, \mu_6) \), with \( \mu_4 \geq \mu_5 \geq \mu_6 \).

By (2.15), \( \mu_4, \mu_5, \mu_6 \leq 1 \).
Figure 3. Domain meshes (solid) with numbering of the domain points (circles) and remaining dual point averages (hexagons), used in the quasi-interpolant \((4.12)\), for the bases (A) \(s_2\), (B) \(\tilde{s}_2\), (C) \(s_3\), (D) \(\tilde{s}_3\) on the Powell-Sabin 12-split (dotted).

(0,0,0) One has \(\mu_1, \mu_2, \mu_3 = 2\) by \((2.20)\), and we obtain \(\bigcirc\).

(1,0,0) One has \(\mu_3 + \mu_4 \leq 2\) by \((2.15)\) implying \(\mu_3 = \mu_4 = 1\), yielding \(\bigcirc\) and \(\bigcirc\).

(1,0,1) One has \(\mu_3 + \mu_4, \mu_2 + \mu_6 \leq 2\) by \((2.15)\), implying \(\mu_2 = \mu_3 = \mu_4 = \mu_6 = 1\). It follows from \((2.11)\) that \(\mu_1 = 2\), yielding \(\bigcirc\).

(1,1,1) From \((2.11)\) one immediately obtains \(\bigcirc\).

3. S-bases on the 12-split

For \(d = 0, 1, 2, 3\), consider the S-bases \(s_d = [S_{i,n_d}]\), listed in Table 4. In this section, we relate these bases through a matrix recurrence relation (property P5), generalizing Theorem 2.3 and Corollary 2.4 in \([3]\) for \(d \leq 2\).

Theorem 3.1. We have
\[
s_d^T = s_{d-1}^T R_d, \quad d = 1, 2, 3,
\]
where \(R_1 \in \mathbb{P}_{11}^{10,10}\) is given by \((3.4)\), \(R_2 \in \mathbb{P}_{1}^{10,12}\) by \((3.5)\), and \(R_3 \in \mathbb{P}_{1}^{12,16}\) by \((3.7)\). Moreover, \(R_d(i,j)S_{i,d-1}(x) \geq 0\) for all \(i,j\) and \(x \in \triangle\).

Corollary 3.2. Suppose \(x \in \triangle_k\) for some \(1 \leq k \leq 12\). Then
\[
s_d^T = e_k^T R_1 \cdots R_d, \quad d = 0, 1, 2, 3.
\]

In the remainder of the section we build up this recurrence relation, starting from degree 0. We will make use of the short-hands \((1.5)\) involving the barycentric coordinates \(\beta_1, \beta_2, \beta_3\) of \(x\) with respect to the triangle \(\triangle\).
3.1. Constant S-basis

Since $S_0(\Delta)$ has dimension $n_0 = 12$, it is easy to see that there is a unique basis $s_0 = [S_{1,0}, \ldots, S_{12,0}]$ forming a partition of unity. Explicitly,

$$S_{j,0}(x) = 1_{\Delta_j}(x) := \begin{cases} 1, & x \in \Delta_j, \\ 0, & \text{otherwise}, \end{cases} \quad j = 1, \ldots, 12, \quad (3.3)$$

where the $\Delta_j$ are the half-open subtriangles in Figure 1(right), with disjoint union $\Delta_1 \sqcup \cdots \sqcup \Delta_{12} = \Delta$. This implies that Corollary 3.2 follows immediately from Theorem 3.1.

3.2. Linear S-basis

The basis $s_1 = [S_{1,1}, \ldots, S_{10,1}]$ of $S_1(\Delta)$ is the nodal basis dual to the point evaluations at the vertices of $\Delta$, i.e., $S_{j,1}(p_i) = \delta_{i,j}$, $i, j = 1, \ldots, 10$. Represented as elements of $\mathbb{P}_1^{12}$, the basis functions $S_{1,1}, \ldots, S_{10,1}$ are precomputed and assembled as the columns of the matrix

$$R_1 = \begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 2\beta_{3,2} & 4\beta_2 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 2\beta_{2,3} & 0 & 0 & 4\beta_3 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 2\beta_{3,1} & 0 & 0 & 0 & 4\beta_3 & 0 & 0 \\ 0 & \gamma_2 & 0 & 2\beta_{2,1} & 0 & 0 & 0 & 4\beta_1 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 2\beta_{3,2} & 0 & 0 & 0 & 4\beta_1 & 0 \\ 0 & 0 & \gamma_3 & 0 & 2\beta_{3,1} & 0 & 0 & 0 & 4\beta_3 & 0 \\ 0 & 0 & 0 & \gamma_3 & 0 & 2\beta_{2,1} & 0 & 0 & 0 & 4\beta_2 \\ 0 & 0 & 0 & 0 & \gamma_3 & 0 & 2\beta_{2,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_3 & 0 & 2\beta_{2,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3 & 0 & 2\beta_{2,1} & 0 \end{bmatrix} \in \mathbb{P}_1^{12,10}. \quad (3.4)$$

The element $R_1(i,j)$ in row $i$ and column $j$ gives the value of $S_{j,1}(x)$ in subtriangle $\Delta_i$, which can be seen to be positive in $\Delta_i$. For instance, for the last column this follows from (1.6).

3.3. Quadratic S-basis

Next we consider the quadratic S-basis $s_2^T = [S_{1,2}, \ldots, S_{12,2}]$. This basis is precomputed using the recurrence relation (2.4). With appropriate choices of the coefficients in this relation, the result of the precomputation is the matrix

$$R_2 = \begin{bmatrix} \gamma_1 & 2\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\beta_3 \\ 0 & 0 & 0 & 2\beta_1 & \gamma_2 & 2\beta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\beta_2 & \gamma_3 & 2\beta_1 & 0 & 0 \\ 0 & \beta_{3,1} & 3\beta_3 & \beta_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{P}_1^{10,12}. \quad (3.5)$$

The element in row $i$ and column $j$ of the matrix product $R_1(x)R_2(x)$ gives the value of $S_{j,2}(x)$ in triangle $\Delta_i$. This computation only involves nonnegative combinations of nonnegative quantities. Thus the computation of the $S_{j,2}$ is fast and stable.
**Remark 3.3** (Alternative quadratic S-basis). The basis \( s_2 \) is the unique quadratic simplex spline basis with local linear independence, as changing out any of its elements with another spline in the second row of Table 4 will cause the outer subtriangles \( \Delta_1, \Delta_2, \ldots, \Delta_6 \) to become overloaded.

Consider the basis \( s_2 \) as in Table 4 which only differs from \( s_2 \) in the entries 3,7,11, satisfying the relation

\[
\begin{bmatrix}
\frac{3}{4} \\
\frac{3}{4} \\
\frac{3}{4}
\end{bmatrix} \quad = \quad T_2^T \begin{bmatrix}
\frac{3}{4} \\
\frac{3}{4} \\
\frac{3}{4}
\end{bmatrix}, \quad T_2^T = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}, \quad (3.6)
\]

which follows from knot insertion \((2.6)\) at the midpoints in terms of the endpoints. Hence \( s_2^T = s_2^T T_2 \), where \( T_2 \in \mathbb{R}^{12 \times 12} \) is obtained from the identity matrix by replacing its principal \((3,7,11)\)-submatrix by \( T_2 ' \). Hence \((3.1), (3.2)\) hold, for \( d = 2 \), with \( s_2 \) replaced by \( s_2 \) and \( R_2 \) replaced by \( R_2 := R_2 T_2 \).

### 3.4. Cubic S-basis

Finally we consider the cubic S-basis \( s_3 = [S_{1,3}, \ldots, S_{16,3}]^T \). This basis is precomputed using the recurrence relation \((2.4)\). With appropriate choices of the coefficients in this relation, the result of the precomputation is the matrix

\[
R_3 = \begin{bmatrix}
\gamma_1 & 2\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\beta_3 & 0 & 0 & 0 & 0 \\
0 & \beta_1,3 & \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sigma_1}{3} & 0 & 0 & 0 & \frac{\beta_1}{3} & 0 & 0 & 0 & 0 & \frac{2\beta_1}{3} & \frac{2\beta_1}{3} & 0 & \frac{2\beta_1}{3} \\
0 & 0 & 0 & \beta_1 & \beta_2,3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_2 & 2\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & \beta_2,3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} & \beta_3,1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\beta_2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \in \mathbb{P}_1^{12\times16}.
\]

**Proof.** [Proof of Theorem 3.1] It remains to show the statement in the cubic case. Using the \( G \)-invariance of the basis, it suffices to show the recursion relations for the columns \( j = 1, 2, 3, 13, 16 \) of \( R_3 (\beta) \). We find

\[
S_{1,3} := \frac{1}{4} \begin{bmatrix}
\frac{2\beta_1}{3} \\
\frac{1}{4} \beta_1,4,6 \\
\frac{1}{4} \beta_1,4,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6 \\
\frac{1}{2} \beta_2,1,2,6
\end{bmatrix}
\]

which follows from knot insertion \((2.6)\) at the midpoints in terms of the endpoints. Hence \( s_2^T = s_2^T T_2 \), where \( T_2 \in \mathbb{R}^{12 \times 12} \) is obtained from the identity matrix by replacing its principal \((3,7,11)\)-submatrix by \( T_2 ' \). Hence \((3.1), (3.2)\) hold, for \( d = 2 \), with \( s_2 \) replaced by \( s_2 \) and \( R_2 \) replaced by \( R_2 := R_2 T_2 \).
\[ S_{13,3} := \beta_1 \begin{pmatrix} 2.3 \\ 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 1.6 \\ 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 2.6 \\ 1 \end{pmatrix} + \beta_1 \left( \begin{pmatrix} 1.2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1.3 \\ 1 \end{pmatrix} \right) + \beta_2 \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 2.5 \\ 1 \end{pmatrix} \]
\[ = \beta_1 \left( \frac{1}{2} S_{1,2} + \frac{1}{2} S_{3,2} \right) + 2\beta_2 S_{1,2} + 2\beta_3 S_{2,2} = 2\beta_3 S_{2,2} + \frac{2}{3} \beta_1 S_{3,2} + \frac{2}{3} \beta_1 S_{1,2} + 2\beta_2 S_{1,2}, \]
\[ S_{16,3} := \frac{1}{4} \begin{pmatrix} 2.3 \\ 1 \end{pmatrix}, \quad \frac{1}{4} \begin{pmatrix} 1.6 \\ 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 2.6 \\ 1 \end{pmatrix} = \frac{1}{4} \beta_1 S_{7,2} + \frac{1}{4} \beta_2 S_{11,2} + \frac{1}{4} \beta_3 S_{3,2}. \]

Clearly all coefficients in the recurrence relations for \( S_{3,3}, S_{13,3}, S_{16,3} \) are nonnegative on \( \Delta \). The same holds for \( S_{1,3} \) on the triangle \([p_1, p_4, p_6]\) and for \( S_{2,3} \) on the triangle \([p_1, p_2, p_6]\). The remaining columns of \( R_3 \) can be found similarly, or using \( G \)-invariance.

**Remark 3.4** (No local linear independence). Since \( S_3(\Delta) \) contains all 10 cubic polynomials on \( \Delta \), the basis \( s_3 \) has local linear independence (property P9) if exactly 10 cubic \( S \)-splines in \( s_3 \) overlap each triangle \( \Delta_k \). Now the support of \( S_{1,3}, j = 3,7,11,13,14,15,16 \), a total of 7 functions, contains all the triangles. While the inner triangles \( \Delta_i, i = 7,8,9,10,11,12 \), contain the support of 10 cubic \( S \)-splines, the border triangles \( \Delta_i, i = 1,2,3,4,5,6 \), contain the support of 11 cubic \( S \)-splines. Hence the basis \( s_3 \) does not have local linear independence.

**Remark 3.5** (Alternative cubic \( S \)-basis). Consider the alternative basis \( \tilde{s}_3 \), which only differs from \( s_3 \) in the entries 13,14,15,16. From (3.7) it follows that
\[ \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} = T_3^T \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} \]
\[ , \quad T_3^T = \begin{bmatrix} 3/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1 \end{bmatrix}, \]

and therefore \( \tilde{s}_3^T = s_3^T T_3 \), where \( T_3 \in \mathbb{R}^{16,16} \) is obtained from the identity matrix by replacing its principal \((13,14,15,16)\)-submatrix by \( T_3^T \). Hence (3.1), (3.2) hold, for \( d = 3 \), with \( s_3 \) replaced by \( \tilde{s}_3 \) and \( R_3 \) replaced by \( \tilde{R}_3 := R_3 T_3 \) (but keeping \( s_2 \) and \( R_2 \) the same).

### 3.5. Fast evaluation

Since the support of most splines in the bases \( s_d \) only cover part of \( \Delta \), the evaluation procedure (3.2) of \( s_d \) (and similarly for its derivatives) for points on a given triangle can be efficiently implemented using multiplication of submatrices. For this purpose we define the index sets
\[ G_d^k := \{ j : \Delta_k \subseteq \text{supp}(S_{j,d}) \}, \quad k = 1, \ldots, 12, \quad d = 0, 1, 2, 3, \]
\[ H_d^k := \{ j : R_d(k, j) \neq 0 \}, \quad k = 1, \ldots, n_{d-1}, \quad d = 1, 2, 3. \]

Here \( G_d^k \) encodes the splines in \( s_d^T \) that are nonzero over \( \Delta_k \), and \( H_d^k \) encodes the splines in \( s_d^T \) that appear in the recurrence relation for \( S_{k,d} \). In particular \( G_d^0 = \{ k \} \), and the remaining sets are listed explicitly in Table 2. We use the symbols \( g_d^k \) and \( h_d^k \) for the vectors consisting of the elements in \( G_d^k \) and \( H_d^k \), respectively, arranged in increasing order.
For $d = 0, 1, 2$, it is easily verified that each $G_d^k$ contains $\nu_d = (d+1)(d+2)/2 = \dim \mathbb{P}_d(\mathbb{R}^2)$ elements. Hence $g_d^k = [i_1, \ldots, i_{\nu_d}]^T$ with $i_1 \leq \cdots \leq i_{\nu_d}$ (cf. Table 4). Also note that

$$g_d^k = H_1^k \quad \text{and} \quad g_d^k = H_2^k \cup H_3^k, \quad [i_1, i_2, i_3] = g_d^k, \quad k = 1, \ldots, 12. \quad (3.10)$$

For $d = 0, 1, 2, 3$, let $S_{j,d}$ be the polynomial representing $S_{j,d}$ on $\bigtriangleup_k$, and let

$$s_d^k = [S_{1,d}, S_{2,d}]^T, \quad s_d^k = s_d^k(g_d^k),$$

which represents the (ordered) vector whose elements form the set $S_{j,d} := \{S_{j,d} : j \in G_d^k\}$. Next, for $1 \leq k \leq 12$, define submatrices

$$R_1^k := R_1(k, g_1^k) \in \mathbb{R}^{1,3}, \quad R_2^k := R_2(g_1^k, g_2^k) \in \mathbb{R}^{3,6}, \quad R_3^k := R_3(g_2^k, g_3^k) \in \mathbb{R}^{6,12}$$

where $g_d^k$ is defined in (3.9), $\eta_1 = \cdots = \eta_6 = 11$ and $\eta_7 = \cdots = \eta_{12} = 10$.

Example 3.6. Since $g_1^k = [1, 6, 7]$ and $g_1^k = [1, 2, 3, 10, 11, 12]$ as in Table 2

$$R_1^k(x)R_2^k(x) = \begin{bmatrix} \gamma_1 & 2\beta_2 & 0 & 0 & 0 & 0 & 2\beta_3 \\ 0 & 0 & 0 & \beta_{3,2} & 3\beta_2 & \beta_{1,2} \\ 0 & \frac{3\beta_2}{2} & \frac{3\beta_2}{2} & 0 & \frac{3\beta_2}{2} & \frac{7\beta_2}{2} \end{bmatrix}.$$

We are now ready to state the polynomial version of Corollary 3.2

**Corollary 3.7.** For $d = 0, 1, 2, 3$, $k = 1, \ldots, 12$, coefficient vector $c \in \mathbb{R}^{n_d}$ with subvector $c(g_d^k)$, $F_d = s_d^T c \in S_d(\bigtriangleup)$, and $F_d = F_d|\bigtriangleup \in \mathbb{P}_d(\mathbb{R}^2)$,

$$s_d^k = R_1^k \cdots R_4^k, \quad F_d = R_1^k \cdots R_4^k c(g_d^k). \quad (3.11)$$

**Proof.** Clearly $s_0^k = 1$ and $F_0^k = c(k)$, showing the result for $d = 0$. By Corollary 3.2

$$s_d^k = R_1(k, j), \quad j = 1, \ldots, 10,$$
and (3.11) follows for \( d = 1 \). Now
\[
S^k_{j,2} = e_k^T R_1 R_2(:, j) = \sum_{i=1}^{10} R_1(k, i) R_2(i, j)
\]
\[
= \sum_{i \in G_1^k} R_1(k, i) R_2(i, j) = R_1(k, g_1^k) R_2(g_1^k, j), \quad j = 1, \ldots, 12,
\]
\[
S^k_{j,3} = e_k^T R_1 R_2 R_3(:, j) = \sum_{m=1}^{12} \sum_{l=1}^{10} R_1(k, l) R_2(l, m) R_3(m, j)
\]
\[
= R_1(k, g_1^k) R_2(g_1^k, g_2^k) R_3(g_2^k, j), \quad j = 1, \ldots, 16.
\]
Hence (3.11) follows for \( d = 2, 3 \) as well.

Remark 3.8. For the alternative bases \( \tilde{s}_2 \) (resp. \( \tilde{s}_3 \)) the set \( H_2^k \) (resp. \( H_3^k \)) needs to be recomputed from the modified recursion matrices \( \tilde{R}_2 \) (resp. \( \tilde{R}_3 \)). The splines in \( \tilde{s}_3 \) and \( s_3 \) have identical support, so that they can be evaluated by slicing their recursion matrices using the same index vectors. In the quadratic case, \( S_{3,2}, S_{7,2}, S_{11,2} \) have full support, as opposed to the trapezoidal support of \( S_{3,2}, S_{7,2}, S_{11,2} \). Hence \( G_2^1, G_2^2 \) (resp. \( G_3^2, G_2^2 \), reps. \( G_2^3, G_2^3 \)) need to be augmented by \{7\} (resp. \{11\}, reps. \{3\}).

3.6. Derivatives

Analogous to the evaluation procedure for splines expressed in an S-basis, this section presents a formula and evaluation procedure for their (higher-order) directional derivatives (property P6). This is achieved by applying the Leibniz rule to (3.2), and making use of special properties of the recursion matrices \( R_i \), made precise in the following two lemmas. As before, we consider barycentric coordinates \( \beta \) and directional coordinates \( \alpha \) with respect to a triangle \( \Delta = [p_1, p_2, p_3] \).

Lemma 3.9. Let \( m \geq 1 \) and \( f \in C^m(\mathcal{U}) \), where \( \mathcal{U} \subset \mathbb{R}^2 \) is a region and \( x \in \mathcal{U} \) with barycentric coordinates \( \beta = (\beta_1, \beta_2, \beta_3) \). For \( i = 1, \ldots, m \), consider vectors \( u_i \in \mathbb{R}^2 \) with directional coordinates \( \alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}) \). Then
\[
D_{u_1} \cdots D_{u_m} f = \sum_{i_1=1}^{3} \cdots \sum_{i_m=1}^{3} \alpha_{i_1}^1 \cdots \alpha_{i_m}^m \frac{\partial^m f}{\partial \beta_{i_1} \cdots \partial \beta_{i_m}}.
\]
Moreover, with \( \alpha_1, \alpha_2 \) directional coordinates of \( e_1, e_2 \),
\[
\frac{\partial^m f}{\partial x_1^{i_1} \partial x_2^{i_2}} = \sum_{|i|=m} \sum_{i_1+i_2=i} \left( \begin{array}{c} |i_1| \\ i_1 \end{array} \right) \left( \begin{array}{c} |i_2| \\ i_2 \end{array} \right) \alpha_{i_1}^1 \alpha_{i_2}^2 \frac{\partial^{|i|} f}{\partial \beta^{|i|}} f,
\]
where we used standard multi-index notation.

Proof. With \( x = x(\beta_1, \beta_2, \beta_3) = \beta_1(p_1 + \beta_2(x)p_2 + \beta_3(x)p_3) \), we can consider \( f(x) = f(x(\beta_1, \beta_2, \beta_3)) \) as a function of \( \beta_1, \beta_2, \beta_3 \). For any \( t \in \mathbb{R} \) and \( j = 1, \ldots, m \), the barycentric coordinates of \( x + tu_j \) are \( \beta + t\alpha_j \), implying
\[
D_{u_j} f(x) = \frac{d}{dt} f((\beta_1 + t\alpha_{j,1})p_1 + (\beta_2 + t\alpha_{j,2})p_2 + (\beta_3 + t\alpha_{j,3})p_3) \big|_{t=0}
\]
\[
= \alpha_{j,1} \frac{\partial f}{\partial \beta_1} + \alpha_{j,2} \frac{\partial f}{\partial \beta_2} + \alpha_{j,3} \frac{\partial f}{\partial \beta_3}.
\]
Hence the action of \( D_u \) on \( f \) is that of the differential polynomial
\[
\alpha_1^j \frac{\partial}{\partial \beta_1} + \alpha_2^j \frac{\partial}{\partial \beta_2} + \alpha_3^j \frac{\partial}{\partial \beta_3}, \quad j = 1, \ldots, m. \tag{3.15}
\]
Since these differential polynomials commute, we can apply polynomial arithmetic to compute their product, and thus arrive at (3.12).

Next consider the standard basis vectors \( e_1 \) and \( e_2 \), with corresponding directional coordinates \( \alpha_1 = (\alpha_1^1, \alpha_1^2, \alpha_1^3) \) and \( \alpha_2 = (\alpha_2^1, \alpha_2^2, \alpha_2^3) \), and let \( \{u_1, \ldots, u_m\} = \{e_1^{m_1} e_2^{m_2}\} \) as multisets. Then, taking the product of (3.15) in this case and applying the multinomial theorem twice,
\[
\frac{\partial^n f}{\partial x_1^{m_1} \partial x_2^{m_2}} = \left( \sum_{|i_1|=m_1} \left( \frac{|i_1|}{i_1} \right) \alpha_1^{i_1} \frac{\partial |i_1|}{\partial \beta_1} \right) \left( \sum_{|i_2|=m_2} \left( \frac{|i_2|}{i_2} \right) \alpha_2^{i_2} \frac{\partial |i_2|}{\partial \beta_2} \right) f,
\]
from which (3.13) follows. \( \blacksquare \)

**Lemma 3.10.** For any \( x, y, u \in \mathbb{R}^2 \) and \( i = 1, 2 \),
\[
R_i(x) R_{i+1}(y) = R_i(y) R_{i+1}(x), \quad (D_u R_i) R_{i+1}(x) = R_i(x) (D_u R_{i+1}). \tag{3.16}
\]

**Proof.** Fix \( x, y \in \mathbb{R}^2 \) with barycentric coordinates \( \beta^x \) and \( \beta^y \), respectively. Equation (3.16) will follow from
\[
R_i(\beta^x) R_{i+1}(\beta^y) = R_i(\beta^y) R_{i+1}(\beta^x), \quad i = 1, 2. \tag{3.18}
\]
For \( i = 1 \) this was proved in [3]. For \( i = 2 \) it was checked symbolically in the Jupyter notebook. Taking the derivative with respect to \( x \) on both sides of (3.16) and setting \( y = x \) we obtain (3.17). \( \blacksquare \)

Note that, for fixed \( u \), the matrices \( D_u R_i \) and \( D_u R_{i+1} \) are constant. In fact, with \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) the directional coordinates of \( u \), Lemma 3.9 implies
\[
U_{d,u} := D_u R_d(\beta) = \alpha_1 \frac{\partial R_d(\beta)}{\partial \beta_1} + \alpha_2 \frac{\partial R_d(\beta)}{\partial \beta_2} + \alpha_3 \frac{\partial R_d(\beta)}{\partial \beta_3}, \quad d = 1, 2, 3. \tag{3.19}
\]
From the definition (1.5) of \( \gamma_j := 2 \beta_j - 1, \beta_{i,j} := \beta_i - \beta_j, \) and \( \sigma_{i,j} := \beta_i + \beta_j, \) it follows that
\[
\frac{\partial \gamma_j}{\partial \beta_k} = 2 \delta_{k,j}, \quad \frac{\partial \beta_{i,j}}{\partial \beta_k} = \delta_{k,i} - \delta_{k,j}, \quad \frac{\partial \sigma_{i,j}}{\partial \beta_k} = \delta_{k,i} + \delta_{k,j}.
\]
Hence one obtains the matrix \( U_{d,u} \) from \( R_d \) by replacing
\[
\beta_j \mapsto \alpha_j, \quad \beta_{i,j} \mapsto \alpha_{i,j} := \alpha_i - \alpha_j, \quad \gamma_j \mapsto 2 \alpha_j, \quad \sigma_{i,j} \mapsto \tau_{i,j} := \alpha_i + \alpha_j.
\]
Analogous to the recursive evaluation (3.2) of the value of \( s_d \), there exist recursive formulas for its directional derivatives.

**Theorem 3.11.** For any point \( x \in \mathbb{R}^2 \) and direction vectors \( u, v, w \in \mathbb{R}^2 \),
\[
D_u(R_1(x) R_2(x) R_3(x)) = 3 R_1(x) R_2(x) U_{3,u}, \tag{3.20}
\]
\[
D_v D_u(R_1(x) R_2(x) R_3(x)) = 6 R_1(x) U_{2,v} U_{3,u}, \tag{3.21}
\]
\[
D_w D_v D_u(R_1(x) R_2(x) R_3(x)) = 6 U_{1,w} U_{2,v} U_{3,u}. \tag{3.22}
\]
where in (3.22) we assume that \( x \) is not on a knot line of \( \mathbf{A} \).
\[\begin{tabular}{|c|c|c|c|c|c|}
\hline
\(i\) & \(\sigma_i\) & \(S_{\sigma_i,3}\) & \(S_{\sigma_i,3}|_e\) & \(\frac{1}{3}D(e_{\alpha_1,\alpha_3,\alpha_3})S_{\sigma_i,3}|_e\) & \(\frac{1}{3}D^2(e_{\alpha_1,\alpha_3,\alpha_3})S_{\sigma_i,3}|_e\) \\
\hline
1 & 1 & \(\frac{1}{4}\) & \(B_1^3\) & \(2\alpha_1B_1^2\) & \(4\alpha_1^2B_1^1\) \\
2 & 2 & \(\frac{1}{2}\) & \(B_2^3\) & \(2\alpha_2B_2^2 + \alpha_13B_2^2\) & \(2\alpha_2(4\alpha_1 + \alpha_2)B_2^1 + \alpha_1^2B_2^1\) \\
3 & 3 & \(\frac{1}{4}\) & \(B_3^3\) & \(\alpha_2B_3^2 + \alpha_1B_3^2\) & \(2\alpha_2^2B_3^1 + 2\alpha_1\alpha_2B_3^2 + \alpha_1^2B_3^1\) \\
4 & 4 & \(\frac{1}{2}\) & \(B_4^3\) & \(\alpha_23B_4^2 + 2\alpha_1B_4^2\) & \(\alpha_2^3B_4^1 + 2\alpha_1(\alpha_1 + \alpha_2)B_3^1\) \\
5 & 5 & \(\frac{1}{4}\) & \(B_5^3\) & \(2\alpha_2B_5^2\) & \(4\alpha_2^2B_3^1\) \\
6 & 12 & \(\frac{1}{2}\) & 0 & \(2\alpha_3B_6^2\) & \(2\alpha_3(4\alpha_1 + \alpha_3)B_1^1\) \\
7 & 13 & \(\frac{1}{2}\) & 0 & \(2\alpha_3B_7^2\) & \(8\alpha_2\alpha_3B_1^1 + 2\alpha_3(3\alpha_1 + \alpha_2)B_2^1\) \\
8 & 14 & \(\frac{1}{2}\) & 0 & \(2\alpha_3B_8^2\) & \(2\alpha_3(3\alpha_2 + \alpha_1)B_2^1 + 8\alpha_1\alpha_3B_1^1\) \\
9 & 6 & \(\frac{1}{2}\) & 0 & \(2\alpha_3B_9^2\) & \(2\alpha_3(4\alpha_2 + \alpha_3)B_1^1\) \\
10 & 11 & \(\frac{1}{4}\) & 0 & 0 & \(2\alpha_3^2B_1^1 + 2\alpha_3^2B_2^1\) \\
11 & 16 & \(\frac{1}{4}\) & 0 & 0 & \(\alpha_3^2B_1^2\) \\
12 & 7 & \(\frac{1}{4}\) & 0 & 0 & \(\alpha_3^2B_3^2 + 2\alpha_3^2B_3^1\) \\
13 & 15 & \(\frac{1}{4}\) & 0 & 0 & 0 \\
14 & 10 & \(\frac{1}{2}\) & 0 & 0 & 0 \\
15 & 8 & \(\frac{1}{2}\) & 0 & 0 & 0 \\
16 & 9 & \(\frac{1}{4}\) & 0 & 0 & 0 \\
\hline
\end{tabular}\]

Table 3. With \(\sigma_i\) the reordering \((5.2)\), restrictions of \(S_{\sigma_i,3}\) and its directional derivatives to \(e = [p_1,p_2]\) are expressed as linear combinations of the univariate B-splines \(B_1^d,\ldots,B_{d+2}^d\).

**Proof.** By the product rule

\[D_u(R_1R_2R_3) = (D_uR_1)R_2R_3 + R_1(D_uR_2)R_3 + R_1R_2(D_uR_3).\]

Using \((3.17)\) repeatedly we obtain \((3.20)\). Differentiating \((3.20)\) using the product rule, applying \((3.17)\) and that \(D_uU_{3,u} = 0\), we obtain

\[D_uD_u(R_1R_2R_3) = 3((D_uR_1)R_2U_{3,u} + R_1(D_uR_2)U_{3,u})\]

\[= 6R_1(x)(D_uR_2)U_{3,u},\]

and \((3.21)\) follows. The proof of \((3.22)\) is similar. \(\blacksquare\)

Splines in \(S_3(\Delta)\), and their directional derivatives of order \(k\), restrict to univariate \(C^{2-k}\)-smooth splines of degree \(3-k\) on each boundary edge, with a single knot at the midpoint. Hence they can, after
a reparametrization \(2.8\), be expressed as linear combinations of the univariate B-splines \(B^d_1, \ldots, B^d_{d+2}\) on the open knot multiset \(\{0^{d+1} 0.5^1 1^{d+1}\}\); see Table 3. Here the directional derivatives \(D_u\) are written in terms of the directional coordinates \(\alpha_1, \alpha_2, \alpha_3\) of \(u\) with respect to the triangle \(\Delta = [p_1, p_2, p_3]\).

**Example 3.12.** The directional coordinates of \(u\) with respect to the triangles \(\Delta = [p_1, p_2, p_3]\) and \(\Delta' = [p_1, p_4, p_6]\) are
\[
\begin{align*}
\alpha_1 & = 2\alpha_1 p_1 + 2\alpha_2 p_4 + 2\alpha_3 p_6, \\
\alpha_1 + \alpha_2 + \alpha_3 & = 0.
\end{align*}
\]
Repeatedly applying the differentiation formula \(2.5\) with respect to \(\Delta'\),
\[
\begin{align*}
\frac{1}{3} D_u \begin{array}{c}
\end{array} & = 2\alpha_1, \\
\frac{1}{3} \cdot 2 D_u^2 \begin{array}{c}
\end{array} & = 4\alpha_1^2.
\end{align*}
\]
When applying \(2.8\), the weight of \(S_{1,3}\) cancels the ratio \(\frac{\text{area}(\Delta)}{\text{area}(\Delta')}\), yielding the first row in Table 3.

4. Marsden identity and ensuing properties

In this section we derive and apply Marsden identities for the bases in \(1.1\), establishing property P3. These identities imply polynomial reproduction, i.e., \(P_d(\Delta) \subset S_d(\mathbf{A})\), yield the construction of quasi-interpolants, imply stability of the bases in the \(L_\infty\) norm, and yield a bound for the distance between spline values and corresponding control points.

4.1. Derivation of the Marsden identity

To the vertices \(p_j\) of \(\mathbf{A}\) we associate linear polynomials
\[
c_j = c_j(y) := 1 - p_j^T y \in \mathbb{P}_1, \quad j = 1, \ldots, 10,
\]
which satisfy, by \(1.2\) and \(1.3\),
\[
c_4 = \frac{c_1 + c_2}{2}, \quad c_5 = \frac{c_2 + c_3}{2}, \quad c_6 = \frac{c_3 + c_4}{2}, \quad c_{10} = \frac{c_1 + c_2 + c_3}{3}.
\]
Taking the products of these shown in Table 4 yields the dual polynomials
\[
\Psi_{j,d}(y) := \prod_{k=1}^{d} (1 - p_{j,d,k}^T y) \in \mathbb{P}_5, \quad j = 1, \ldots, n_d, \quad d = 0, 1, 2, 3,
\]
with \(n_d\) the dimension in \(1.10\), and referring to the \(p_{j,d,k}\) as the dual points of degree \(d\). For each basis, the dual polynomials are assembled in a vector
\[
\psi_d := [\Psi_{1,d}, \ldots, \Psi_{n_d,d}]^T, \quad d = 0, 1, 2, 3.
\]
Corresponding to the dual polynomials \(\Psi_{j,d}\) (or basis functions \(S_{j,d}\)), we define the domain points
\[
\xi_{j,d} := \frac{p_{j,d,1} + \cdots + p_{j,d,d}}{d}, \quad j = 1, \ldots, n_d,
\]
as the averages of the corresponding dual points. Figure 3 shows each set of domain points, symmetrically connected in a domain mesh. Note that each set of domain points satisfies property P4.

**Theorem 4.1.** For \(x, y \in \mathbb{R}^2\) we have
\[
R_d(x) \psi_d(y) = (1 - x^T y) \psi_{d-1}(y), \quad d = 1, 2, 3,
\]
where \(R_d(x)\) is given by \(3.4\), \(3.5\), \(3.7\).
Proof. This holds for \( d = 1, 2 \) by Theorem 3.4 in [3]. Consider \( d = 3 \). Let \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \). Let \((\beta_1, \beta_2, \beta_3)\) be the barycentric coordinates of \( \mathbf{x} \) with respect to \( \Delta \). From (1.4), it follows
\[
\beta_1 c_1(\mathbf{y}) + \beta_2 c_2(\mathbf{y}) + \beta_3 c_3(\mathbf{y}) = 1 - \mathbf{x}^T \mathbf{y}.
\]
Thus, it is enough to show that
\[
(R_3 \psi_3)_i = (\beta_1 c_1 + \beta_2 c_2 + \beta_3 c_3) \psi_{1,2}, \quad i = 1, \ldots, 12.
\]
We verify this statement for \( i = 1, 2, 3 \), by taking the product of the \( i \)th row of \( R_3 \) as in (3.7) with \( \psi_3 \) as in Table 4, which gives
\[
(R_3 \psi_3)_1 = \gamma_1 \psi_{1,3} + 2 \beta_2 \psi_{2,3} + 3 \beta_3 \psi_{12,3} = (2 \beta_1 - 1) c_1^3 + 2 \beta_2 c_1^2 c_4 + 2 \beta_3 c_1 c_6
\]
Thus, it is enough to show that
\[
(R_3 \psi_3)_2 = \beta_1 \psi_{1,3} + \beta_2 \psi_{2,3} + 2 \beta_3 \psi_{12,3} = (\beta_1 - 3)c_1^3 + 2 \beta_2 c_1 c_6 + 2 \beta_3 c_1 c_6
\]
We verify this statement for \( i = 1, 2, 3 \), by taking the product of the \( i \)th row of \( R_3 \) as in (3.7) with \( \psi_3 \) as in Table 4, which gives
\[
(R_3 \psi_3)_3 = \beta_1 \psi_{1,3} + \beta_2 \psi_{2,3} + \beta_3 \psi_{11,3} + 2 \beta_1 \psi_{13,3} + 2 \beta_2 \psi_{14,3} + 2 \beta_3 \psi_{16,3}
\]
The remaining components are found similarly, or using \( G \)-invariance.

From Theorem 4.1 we immediately obtain the following Marsden identity, generalizing Theorem 3.1 in [3].

Corollary 4.2 (Marsden identity). With \( S_{j,d} \) and \( \Psi_{j,d} \) as in Table 4,
\[
(1 - \mathbf{x}^T \mathbf{y})^d = \sum_{j=1}^{n_d} S_{j,d}(\mathbf{x}) \Psi_{j,d}(\mathbf{y}) = s_d(\mathbf{x})^T \psi_d(\mathbf{y}), \quad d = 0, 1, 2, 3.
\]

As was shown in [15, Theorem 5], the Marsden identity can be brought into the following barycentric form, which is independent of the vertices of the triangle.

Corollary 4.3 (Barycentric Marsden identity). Let \( \beta_j = \beta_j(\mathbf{x}) \), \( j = 1, 2, 3 \), be the barycentric coordinates of \( \mathbf{x} \in \mathbb{R}^2 \) with respect to \( \Delta = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] \). Then (4.7) is equivalent to
\[
(\beta_1 c_1 + \beta_2 c_2 + \beta_3 c_3)^d = \sum_{j=1}^{n_d} S_{j,d}(\beta_1 \mathbf{p}_1 + \beta_2 \mathbf{p}_2 + \beta_3 \mathbf{p}_3) \Psi_j(c_1, c_2, c_3),
\]
where \( \mathbf{x} \in \Delta \), \( c_1, c_2, c_3 \in \mathbb{R} \), and, for \( j = 1, \ldots, n_d \),
\[
\Psi_j(c_1, c_2, c_3) = \prod_{k=1}^d (\beta_1 (\mathbf{p}_{j, d, k}) c_1 + \beta_2 (\mathbf{p}_{j, d, k}) c_2 + \beta_3 (\mathbf{p}_{j, d, k}) c_3).
\]

Example 4.4. The barycentric Marsden identity for the cubic \( S \)-basis \( s_3 \) is
\[
(c_1 \beta_1 + c_2 \beta_2 + c_3 \beta_3)^3 = \left( \begin{array}{ccc}
\frac{1}{4} c_1 & c_2 & c_3 \\c_1 & c_2 & c_3
\end{array} \right)^3
\]

Where \( \left( \begin{array}{ccc}
\frac{1}{4} c_1 & c_2 & c_3 \\c_1 & c_2 & c_3
\end{array} \right) \) is the 3rd section of the table.
Remark 4.5. Substituting (3.6) and (3.8), we also obtain Marsden identities for the alternative bases $\tilde{S}_d = [\tilde{S}_{j,d}]$, for $d = 2, 3$, with corresponding dual polynomials $\tilde{\psi}_d = [\tilde{\psi}_{j,d}]$, shown in Table 1.

4.2. Polynomial reproduction

The barycentric Marsden identity can directly be applied to express Bernstein polynomials on $\Delta$ in terms of the S-basis. In particular, applying the multinomial theorem to the left hand side of (4.8), one notices that the Bernstein polynomial $B_{i_1,i_2,i_3}^d$ appears as the coefficient of $c_1^{i_1}c_2^{i_2}c_3^{i_3}$. Hence, defining the “coefficient of” operator $\partialc{i_1,i_2,i_3}$

\[ [c_1^{i_1}c_2^{i_2}c_3^{i_3}]F := \frac{1}{i_1!i_2!i_3!} \partialc{i_1} \partialc{i_2} \partialc{i_3} F(0,0,0) \]

for any formal power series $F(c_1, c_2, c_3)$ and nonnegative integers with sum $i_1 + i_2 + i_3 = d$,

\[ B_{i_1,i_2,i_3}^d = \sum_{i=1}^{n_d} S_{j,d}(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3)(c_1^{i_1}c_2^{i_2}c_3^{i_3}) \psi_j(c_1, c_2, c_3). \]

Thus one immediately sees from the monomials in the dual polynomials which simplex splines appear in the above linear combination. For instance, substituting the dual polynomials from Table 4 and the short-hands (4.2), one obtains

\[ B_{300}^3 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}, \quad B_{310}^3 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}, \quad B_{311}^3 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}, \]

consistent with the result achieved by repeatedly applying knot insertion, see for instance (2.7).

4.3. Quasi-interpolation

Based on a standard construction, the Marsden identity gives rise to quasi-interpolants in terms of the de Boor-Fix functionals. In this section, we present a different quasi-interpolant that solely involves point evaluations at averages of dual points (property P7).

Let $(\beta_1, \beta_2, \beta_3)$ be the barycentric coordinates with respect to the triangle $\Delta$. As explained in [15 §6.1], the Bernstein polynomial $B_{i_1,i_2,i_3}^d$ can be expressed in terms of the simplex spline basis by replacing each dual polynomial in (4.7), after substituting (4.2), by its coefficient of $c_1^{i_1}c_2^{i_2}c_3^{i_3}$.

**Theorem 4.6.** For $d = 1, 2, 3$ and each basis in (1.1) with dual points $p_{j,d,k}$, consider the map

\[ Q_d : C^0(\Delta) \rightarrow \mathbb{S}_d(\Delta), \quad Q_d(F) = \sum_{j=1}^{n_d} l_{j,d}(F) S_{j,d}. \]
### Table 4. Basis functions $S_{j,d}, \tilde{S}_{j,d}$, domain points $\xi_{j,d}, \tilde{\xi}_{j,d}$, and corresponding dual polynomials $\Psi_{j,d}, \tilde{\Psi}_{j,d}$, factored into linear polynomials $c_j$ as in (4.1).
where the functionals \( l_{j,d} : C^0(\Delta) \longrightarrow \mathbb{R} \) are given by

\[
l_{j,d}(F) := \sum_{m=1}^{d} \frac{m^d}{d!} (-1)^{d-m} \sum_{1 \leq k_1 < \cdots < k_m \leq d} F \left( \frac{p_{j,d,k_1} + \cdots + p_{j,d,k_m}}{m} \right).
\]

Then \( Q_d \) is a quasi-interpolant reproducing polynomials up to degree \( d \).

**Proof.** For \( d = 1, 2 \) the statement is shown in \([3]\) \S 6.1. We give an explicit proof for the remaining case \( d = 3 \), analogous to the proof provided in \([13]\). In that case \([l_{1,3}(f), \ldots, l_{16,3}(f)] = \left[ \frac{1}{6}, \frac{1}{6}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, 9 \right] \) \( f(M) \), applied component-wise, where

\[
M = \begin{bmatrix}
  p_{1,3,1} & \cdots & p_{16,3,1} \\
p_{1,3,2} & \cdots & p_{16,3,2} \\
p_{1,3,3} & \cdots & p_{16,3,3} \\
\frac{p_{1,3,1} + p_{1,3,2}}{2} & \cdots & p_{16,3,1} + p_{16,3,2} \\
\frac{p_{1,3,2} + p_{1,3,3}}{2} & \cdots & p_{16,3,2} + p_{16,3,3} \\
\frac{p_{1,3,1} + p_{1,3,2} + p_{1,3,3}}{3} & \cdots & p_{16,3,1} + p_{16,3,2} + p_{16,3,3}
\end{bmatrix}
\]

with \( l_j = \xi_{j,3} \) for \( j = 1, \ldots, 16 \) the domain points, and with the quarterpoints

\[
\begin{align*}
l_{17} &= \frac{p_1 + p_4}{2}, & l_{18} &= \frac{p_2 + p_4}{2}, & l_{19} &= \frac{p_2 + p_5}{2}, \\
l_{20} &= \frac{p_3 + p_5}{2}, & l_{21} &= \frac{p_3 + p_6}{2}, & l_{22} &= \frac{p_1 + p_6}{2}, \\
l_{23} &= \frac{p_4 + p_6}{2}, & l_{24} &= \frac{p_4 + p_5}{2}, & l_{25} &= \frac{p_5 + p_6}{2},
\end{align*}
\]

as in Figure 3c. To prove that \( Q_3 \) reproduces polynomials up to degree 3, i.e., \( Q_3(B^3_{ijk}) = B^3_{ijk} \) whenever \( i + j + k = 3 \), it suffices to show this for \( B^3_{300}, B^3_{210}, B^3_{111} \) using the symmetries. Evaluating these polynomials at the dual point averages yields Table 5.

For instance, the Bernstein polynomial \( B^3_{111} \) is only nonzero for \( l_{13}, l_{14}, l_{15}, l_{16} \) and \( l_{23}, l_{24}, l_{25} \). Hence only the entries in the last 4 columns and last 2 rows in \( f(M) \) can be nonzero, yielding the coefficients

\[
\begin{bmatrix}
  l_{13,3} & \cdots & l_{16,3} \\
\end{bmatrix} B^3_{111} = \begin{bmatrix}
  -\frac{4}{3} & 9 \\
\end{bmatrix} B^3_{111} \begin{bmatrix}
  l_{23} & l_{24} & l_{25} & l_{17} \\
l_{13} & l_{14} & l_{15} & l_{16}
\end{bmatrix}
= \begin{bmatrix}
  -\frac{4}{3} & 9 \\
\end{bmatrix} \begin{bmatrix}
  \frac{3}{15} & \frac{3}{15} & \frac{3}{15} & 0 \\
  \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & 1 & 1
\end{bmatrix},
\]

consistent with (4.11). Similarly one establishes reproduction of \( B^3_{300} \) and \( B^3_{210} \); additional details are shown in the Jupyter notebook. \( \blacksquare \)
The quasi-interpolant $Q_d$ is bounded independently of the geometry of $\Delta$, since, using that $s_d$ forms a partition of unity,

$$\|Q_d(F)\|_{L_\infty(\Delta)} \leq \max_j |l_{j,d}(F)| \leq C_d \|F\|_{L_\infty(\Delta)}, \quad C_d = \sum_{m=1}^d \frac{m!}{d!} \left(\frac{d}{m}\right).$$

In particular, $[C_1, C_2, C_3] = [1, 3, 9]$. Therefore, by a standard argument, $Q_d$ is a quasi-interpolant that approximates locally with order 4 smooth functions whose first four derivatives are in $L_\infty(\Delta)$.

**Remark 4.7.** In [3] Lemma 6.1 it was shown, for $d = 1, 2$, that the functionals $l_{j,d}$ form the dual basis to $s_d$, i.e., $l_{j,d}(S_{1,d}) = \delta_{ij}$. This is equivalent to the statement that $Q_d$ reproduces all splines in $S_d(\Delta)$.

However, for $d = 3$ this is not the case. For instance, repeatedly using the recurrence relation (2.4) with respect to the triangle $\Delta' := [p_1, p_4, p_6]$,

$$S_{1,3} = \frac{1}{4} \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right] = \frac{1}{4} \beta_1^{1,4,6} \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right] = \ldots = \frac{1}{4} (\beta_1^{1,4,6})^3 \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right] = (\beta_1^{1,4,6})^3 \mathbf{1}_{\Delta'}.$$

As immediately seen from the support of $S_{1,3}$ and Figure 3c, $S_{1,3}(l_i) \neq 0$ for $i \neq 1, 2, 12, 13, 17, 22$. Hence $l_{16,3}(S_{1,3}) = \frac{1}{6} S_{1,3}(l_1) = \frac{1}{6} \neq 0$.

**Remark 4.8.** Although $Q_d$ involves $n_d$ dual functionals each involving a sum with $\sum_{m=1}^d \binom{n_d}{m}$ terms, many of the dual point averages (and hence the point evaluations) coincide. In particular the quasi-interpolant for the linear basis $s_1$ involves 10 point evaluations at the domain points. The quadratic bases $s_2, \tilde{s}_2$ (respectively cubic bases $s_3, \tilde{s}_3$) involve 16 (respectively 25) point evaluations, whose carriers are shown in Figure 3.

### 4.4. $L_\infty$ stability and distance to the control points

Each basis $s$ in (1.1) is stable in the $L_\infty$ norm with a condition number bounded independent of the geometry of $\Delta$.

**Theorem 4.9.** Let $F = s^Tc$ with $s$ as in (1.1). There is a constant $\kappa > 0$ independent of the geometry of $\Delta$, such that

$$\kappa^{-1} \|e\|_\infty \leq \|F\|_{L_\infty(\Delta)} \leq \|e\|_\infty. \tag{4.13}$$
Proof. For \( s = s_0, s_1, s_2 \) this was shown in [3 Theorem 6.2], with the best possible constants. It remains to show this for \( s = [S_1, \ldots, S_n] = \tilde{s}_2, \tilde{s}_3, \tilde{s}_3 \). Let \( \lambda = [\lambda_1, \ldots, \lambda_{n_\ell}]^T \) be the point evaluations at the corresponding domain points \( \xi = [\xi_1, \ldots, \xi_{n_\ell}] \).

Applying these functionals to \( F = s^T c \) yields a system \( f := \lambda F = M c \), with collocation matrix \( M := \lambda s^T = [S_j(\xi_i)]_{i,j=1}^{n_\ell} \). A computation (see the Jupyter notebook) shows that each collocation matrix is nonsingular, and its elements are rational numbers independent of the geometry of \( M \) and leaves \( p \) and \( \hat{M} \) collocation matrices \( \xi_s \).

Smooth surface joins 5.

Proof. For \( s \) let \( \xi, \hat{\xi}, \xi_1, \xi_2, \xi_3, \xi_3 \) be triangles sharing the edge \( e \) under the affine map \( A : \Delta \rightarrow \Delta \) that maps \( \tilde{p}_3 \) to \( p_3 \) and leaves \( p_1, p_2 \) invariant, i.e., \( \hat{s}_3 = s_3 \circ A \). In this section we derive conditions for smooth joins of

\[
F(x) := \sum_{i=1}^{16} c_i S_{\sigma_3}(x), \quad x \in \Delta, \quad \hat{F}(x) := \sum_{i=1}^{16} \hat{c}_i \hat{S}_{\sigma_3}(x), \quad x \in \hat{\Delta},
\]

where we used the reordering \( i \mapsto \sigma \) (cf. Figure 4) defined by

\[
\sigma = [\sigma_i]_{i=1}^{16} = [1, 2, 3, 4, 5, 12, 13, 14, 6, 11, 16, 7, 15, 10, 8, 9].
\]

Remark 5.1. The reordering \( [5, 2] \) is chosen such that the splines \( S_{\sigma_3} \) have an increasing number of knots outside of \( e \). In particular, there is 1 such knot for \( S_{\sigma_3}, \ldots, S_{\sigma_3} \), 2 knots for \( S_{\sigma_3}, \ldots, S_{\sigma_3} \), 3 knots for \( S_{\sigma_3}, S_{\sigma_3}, S_{\sigma_3} \), and more than 3 knots for \( S_{\sigma_3}, S_{\sigma_3}, S_{\sigma_3} \). By (2.5), this implies that after this reordering only the first 5 (resp. 5 + 4, resp. 5 + 4 + 3) splines in \( s_3^2 \) are involved in the \( C^0 \) (resp. \( C^1 \), resp. \( C^2 \)) conditions, as only these (resp. their derivatives, resp. their 2nd order derivatives) are not identically zero on \( e \).

Imposing a smooth join of \( F \) and \( \hat{F} \) along \( e \) translates into Bézier-like linear relations among the ordinates \( c_i \) and \( \hat{c}_i \) (property P8).
Theorem 5.2. Let $\beta_1, \beta_2, \beta_3$ be the barycentric coordinates of $\tilde{p}_3$ with respect to the triangle $\triangle$. Then $F$ and $\tilde{F}$ meet with $C^0$-smoothness if and only if
\[
\hat{c}_1 = c_1, \quad \hat{c}_2 = c_2, \quad \hat{c}_3 = c_3, \quad \hat{c}_4 = c_4, \quad \hat{c}_5 = c_5;
\]
$C^1$-smoothness if and only if in addition
\[
\hat{c}_6 = \beta_1 c_1 + \beta_2 c_2 + \beta_3 c_6, \quad \hat{c}_7 = \beta_1 c_2 + \beta_2 \left(\frac{c_2 + c_3}{2}\right) + \beta_3 c_7,
\]
\[
\hat{c}_8 = \beta_1 c_4 + \beta_1 \left(\frac{c_3 + c_4}{2}\right) + \beta_3 c_8;
\]
$C^2$-smoothness if and only if in addition
\[
\begin{align*}
\hat{c}_{10} &= 2\beta_1 \beta_2 \left(\frac{3c_2 - c_1}{2}\right) + 2\beta_1 \beta_3 \left(\frac{3c_6 - c_1}{2}\right) + 2\beta_2 \beta_3 \left(\frac{4c_7 - c_2 - c_6}{2}\right) + \beta_1^2 c_1 + \beta_2^2 c_3 + \beta_3^2 c_{10}, \\
\hat{c}_{12} &= 2\beta_1 \beta_2 \left(\frac{3c_4 - c_5}{2}\right) + 2\beta_2 \beta_3 \left(\frac{3c_9 - c_5}{2}\right) + 2\beta_1 \beta_3 \left(\frac{4c_8 - c_4 - c_9}{2}\right) + \beta_1^2 c_3 + \beta_2^2 c_5 + \beta_3^2 c_{12}, \\
\hat{c}_{11} &= +2\beta_1 \beta_2 \left(\frac{c_1 - 2c_2 + 4c_3 - 2c_4 + c_5}{2}\right) + \beta_3^2 c_{11} \\
&\quad + 2\beta_1 \beta_3 \left(\frac{c_1 - 2c_2 + c_3 - 3c_6 + 6c_7 - 2c_8 + c_9}{2}\right) + \beta_2^2 (2c_4 - c_5) \\
&\quad + 2\beta_2 \beta_3 \left(\frac{c_3 - 2c_4 + c_5 - 3c_9 + 6c_8 - 2c_7 + c_6}{2}\right) + \beta_1^2 (2c_2 - c_1).
\end{align*}
\]

Proof. By the barycentric nature of the statement, we can change coordinates by the linear affine map that sends $p_1 \mapsto (0,0), p_2 \mapsto (1,0), \text{ and } p_3 \mapsto (0,1)$. In these coordinates, $\tilde{p}_3 = \beta_1 (0,0) + \beta_2 (1,0) + \beta_3 (0,1) = (\beta_2, \beta_3)$. Let $u := \tilde{p}_3 - p_1 = (\beta_2, \beta_3)$. For $r = 0,1,2$, the splines $F$ and $\tilde{F}$ meet with $C^r$-smoothness along $e$ if and only if $D_k^u F(\cdot,0) = D_k^u \tilde{F}(\cdot,0)$ for $k = 0,\ldots,r$. Substituting (5.1) this is equivalent to
\[
\sum_{i=1}^{16} c_i D_k^u S_{\sigma_i,3}(\cdot,0) = \sum_{i=1}^{16} \hat{c}_i D_k^u \hat{S}_{\sigma_i,3}(\cdot,0), \quad k = 0,\ldots,r,
\]
(5.3)
which using Table 3 reduces to a sparse system
\[
\sum_{j=1}^{5-k} r_{kj} \phi_j^{3-k} = 0, \quad k = 0, \ldots, r,
\]
where \( r_{kj} \) is a linear combination of the \( c_i, \hat{c}_i \) with \( i = 1, \ldots, n_{k+1} \), with \( n_1 = 5, n_2 = 5 + 4, \) and \( n_3 = 5 + 4 + 3 \). This system holds identically if and only if \( r_{kj} = 0 \) for \( j = 1, \ldots, 5 - k \) and \( k = 0, \ldots, r \). Let \( n_0 = 0 \). For \( k = 0, 1, 2, 3 \) one solves for \( \hat{c}_{n_k+1}, \ldots, \hat{c}_{n_{k+1}} \), each time eliminating the ordinates \( \hat{c}_i \) that were previously obtained, resulting in the smoothness relations of the Theorem; see the Jupyter notebook for details.

**Remark 5.3.** Since we forced our S-bases to restrict to B-spline bases on the boundary, the presented local bases can trivially be extended to global bases with \( C^0 \)-smoothness across the macrotriangles.

**Remark 5.4.** To obtain global \( C^2 \)-smoothness for \( d = 3 \), maximal sharing of the degrees of freedom is obtained by specifying values, first-order and second-order derivatives at the vertices of the coarse triangulation. This would amount to 18 degrees of freedom on a single macro triangle, exceeding the 16 available degrees of freedom. In particular, the degrees of freedom along the edges will be overdetermined. Hence global \( C^2 \)-smoothness cannot be obtained in this way.

**Remark 5.5.** For \( d = 3 \), it is not clear how to extend these local bases to bases with \( C^1 \)-smoothness on triangulations. One strategy is to attempt to construct a dual basis that determines the value of the spline, as well as the value of its cross-boundary derivative, at the edges of the macro triangles, thus forcing \( C^0 \) and \( C^1 \)-smoothness across these edges. On a single edge, this would require fixing 5 degrees of freedom for obtaining \( C^0 \)-smoothness and an additional 4 degrees of freedom for attaining \( C^1 \)-smoothness. Again maximal sharing is obtained by locating degrees of freedom (i.e., values and first-order derivatives) at the vertices of the macro triangles. For each edge, one would additionally need 1 degree of freedom for \( C^0 \) and 2 degrees of freedom for \( C^1 \), exceeding the remaining 7 of the available 16 degrees of freedom. Hence global \( C^1 \)-smoothness on a general triangulation cannot be obtained in this way. Whether this is possible on specific triangulations, such as cells, is an open problem.

**Bibliography**

[1] Philippe G. Ciarlet, *The finite element method for elliptic problems*, Society for Industrial and Applied Mathematics, 2002.

[2] Ray W. Clough and J. L. Tocher, *Finite element stiffness matrices for analysis of plate bending*, Proceedings of the conference on matrix methods in structural mechanics, 1965, pp. 515–546.

[3] Elaine Cohen, Tom Lyche, and Richard F. Riesenfeld, *A B-spline-like basis for the Powell-Sabin 12-split based on simplex splines*, Mathematics of Computation 82 (2013), no. 283, 1667–1707.

[4] Elaine Cohen, Richard F. Riesenfeld, and Gershon Elber, *Geometric modeling with splines: an introduction*, AK Peters/CRC Press, 2001.

[5] J. Austin Cottrell, Thomas J.R. Hughes, and Yuri Bazilevs, *Isogeometric analysis: toward integration of CAD and FEA*, John Wiley & Sons, 2009.

[6] Oleg Davydov and Wee Ping Yeo, *Refinable C2 piecewise quintic polynomials on Powell-Sabin-12 triangulations*, Journal of Computational and Applied Mathematics 240 (2013), 62–73.

[7] Paul Dierckx, *On calculating normalized Powell-Sabin B-splines*, Computer Aided Geometric Design 15 (1997), no. 1, 61–78.

[8] Nira Dyn and Tom Lyche, *A Hermite subdivision scheme for the evaluation of the Powell-Sabin 12-split element*, Approximation theory IX 2 (1998), 33–38.

[9] Jan Grošelj and Hendrik Speleers, *Construction and analysis of cubic Powell–Sabin B-splines*, Computer Aided Geometric Design 57 (2017), 1–22.
Donald Knuth, *Bracket notation for the “coefficient of” operator*, A Classical Mind: Essays in Honour of C. A. R. Hoare, 1994.

Ming-Jun Lai and Larry L. Schumaker, *Macro-elements and stable local bases for splines on Powell-Sabin triangulations*, Math. Comp. 72 (2003), no. 241, 335–354.

Tom Lyche and Jean-Louis Merrien, *Simplex Splines on the Clough-Tocher Element*, Computer Aided Geometric Design 65 (October 2018), 76–92.

Tom Lyche and Georg Muntingh, *A Hermite interpolatory subdivision scheme for $C^2$-quintics on the Powell-Sabin 12-split*, Comput. Aided Geom. Design 31 (2014), no. 7-8, 464–474.

Georg Muntingh, *Notebook: B-spline-like bases for $C^2$ cubics on the Powell-Sabin 12-split*, 2019. [https://github.com/georgmuntingh/SSplines/blob/master/examples/C2-cubic.ipynb](https://github.com/georgmuntingh/SSplines/blob/master/examples/C2-cubic.ipynb).

Peter Oswald, *Hierarchical conforming finite element methods for the biharmonic equation*, SIAM J. Numer. Anal. 29 (1992), no. 6, 1610–1625.

Michael J. D. Powell and Malcolm A. Sabin, *Piecewise quadratic approximations on triangles*, ACM Trans. Math. Software 3 (1977), no. 4, 316–325.

Hartmut Prautzsch, Wolfgang Boehm, and Marco Paluszny, *Bézier and B-spline techniques* (2002), xiv+304.

Larry L. Schumaker and Tatyana Sorokina, *Smooth macro-elements on Powell-Sabin 12 splits*, Math. Comp. 75 (2006), no. 254, 711–726 (electronic).

Hendrik Speleers, *A normalized basis for quintic Powell-Sabin splines*, Comput. Aided Geom. Design 27 (2010), no. 6, 438–457.

Ivar Haugaløkken Stangeby, *Simplex splines on the Powell-Sabin 12-split: components of the finite element method*, Master’s thesis, 2018.

Alexander Ženišek, *A general theorem on triangular finite $C^{(m)}$-elements*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 8 (1974), no. R-2, 119–127 (English, with Loose French summary).