Twisted Quantum Lax Equations

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Abstract

We give the construction of twisted quantum Lax equations associated with quantum groups. We solve these equations using factorization properties of the corresponding quantum groups. Our construction generalizes in many respects the AKS construction for Lie groups and the construction of M. A. Semenov Tian-Shansky for the Lie-Poisson case.

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1 Introduction

In this paper we discuss the quantum integrable systems related to quasitriangular Hopf algebras in a way that generalizes the classical theory based on the construction of Adler \cite{Adler}, Kostant \cite{Kostant}, Symes \cite{Symes} for the Lie groups and its subsequent generalization to the Lie-Poisson case due to Semenov-Tian-Shansky \cite{SemenovTianShansky}. The classical theory which we briefly summarize in this section gives the construction and solution of integrable systems possessing a (twisted) Lax pair and $r$-matrix formulation. The rich structure of these integrable systems appears naturally as a consequence of the factorization properties of the groups under consideration. Within this approach the fundamental methods (inverse scattering method \cite{InverseScatteringMethod}, algebro-geometric methods of solution) and the fundamental notions of the soliton theory, such as $\tau$-function \cite{tauFunction} and Baker-Akhiezer function \cite{BakerAkhiezer}, found their unifying and natural group-theoretical explanation. This paper deals with the quantum case. The theory of integrable models in quantum mechanics and quantum field theory made remarkable progress with the quantum version of the inverse scattering method, which goes back to the seminal Bethe ansatz for the solution of the Heisenberg spin chain. We refer the reader for a review of related topics to the books \cite{IntroductionToQuantumGroups}, \cite{QuantumIntegrableSystems} and to the papers \cite{QuantumFactorizationTheorem}, \cite{QuantumLieAlgebras}. This development suggested the introduction of quantum groups \cite{QuantumGroups}, \cite{QuantumAlgebras} as algebraic objects that play in the quantum case a role analogous to that of Lie groups in the classical theory. However, we were still missing (with the exception of the quantum integrable systems with discrete time evolution \cite{DiscreteTimeEvolution}) a quantum analogue of the factorization theorem for the solution of the Heisenberg equations of motion for quantum integrable systems. We have also to mention the remarkable paper \cite{QuantumFactorization} in this context.

The quantum systems we consider are quantum counterparts of those described by the classical factorization theorem. In a recent paper by M. Schlieker and one of the coauthors \cite{Schlieker} the quantum version of the theory in the case without twisting was formulated. The main result of the present article, the quantum factorization theorem (formulated in the Section 4), as well as the remaining discussion extends all constructions in the presence of twisting, which seems to be nontrivial.
1.1 Classical integrable systems

Here we briefly review the construction of integrable systems and their solution by factorization which is due to M. Semenov-Tian-Shansky [27] and which generalizes to the case of the Poisson Lie groups the construction of Adler, Kostant and Symes [1], [19], [31]. Let $G$ be a quasitriangular Poisson Lie group, which is for simplicity assumed to be a matrix group. Let $g$ be the corresponding Lie algebra and $r \in g \otimes g$ the classical $r$-matrix, a solution to the classical Yang-Baxter equation. In the following we will use a notation that does not distinguish between the universal element and its matrix representative. We will denote by $G_r$ and $g_r$ the dual Poisson Lie group and its Lie algebra respectively. The pairing between $g$ and $g_r$ is denoted by $\langle \cdot, \cdot \rangle$. The Poisson structure on $G$ is given by the Sklyanin bracket

$$\{ g_1, g_2 \} = [r, g \otimes g].$$

(1)

Here we used the standard tensor notation: $g_1 = g \otimes 1$, $g_2 = 1 \otimes g$, with $g \in G$ being a group element (matrix) and 1 the unit matrix. The commutator on the right-hand side is the usual matrix commutator in $g \otimes g$. With the universal $r$-matrix $r$ we can associate two mappings $r_{\pm}: g_r \to g_r$

$$r_{\pm}(X) = \langle X \otimes id, r \rangle \equiv X_{\pm}, \quad r_{\mp}(X) = -\langle id \otimes X, r \rangle \equiv X_{\mp}. \quad (2)$$

These mappings are algebra homomorphism. Let $g_{\pm} = \text{Im}(r_{\pm})$ be the corresponding subalgebras in $g$. Consider the combined mapping $i_r = r_{\mp} \oplus r_{\pm}: g_r \to g \oplus g$. Let us assume that the mapping $r_{\pm} - r_{\mp}: g_r \to g$ is an isomorphism of linear spaces. In that case $g$ is called factorizable and any $X \in g$ has a unique decomposition

$$X = X_+ - X_-,$$

(3)

with $(X_-, X_+) \in \text{Im}(i_r)$. The map $i_r$ gives rise to a Lie group embedding $I_r: G_r = G_- \times G_+ \to G \times G$. Followed by the group inversion in the first factor and a subsequent group multiplication of factors it defines a local homeomorphism on the Poisson Lie groups $G_r$ and $G$. This means that in the neighborhood of the group identity any group element $g \in G$ admits a unique decomposition

$$g = g_+^{-1} g_+,$$

(4)
with \((g_-, g_+) \in \text{Im}(I_r)\).

The group manifold equipped with Sklyanin bracket (\(\mathfrak{g}\)) plays the role of the phase space for a classical dynamical system governed by a Hamiltonian constructed in the following way. Let \(\phi\) by an automorphism of \(\mathfrak{g}\) which preserves the classical \(r\)-matrix

\[(\phi \otimes \phi) r = r\] (5)

and which defines an automorphism of \(G\) denoted by the same symbol. The Hamiltonian \(h\) is taken as any smooth function on \(G\) invariant with respect to the twisted conjugation. This means that

\[h(g) = h(g_1^\phi g(g_1)^{-1}), \quad g^\phi \equiv \phi(g)\] (6)

holds for any two group elements \(g, g_1 \in G\). The functions satisfying (\(\mathfrak{g}\)) are in involution with respect to the Sklyanin bracket, so they play the role of integrals of motion for the dynamical system on \(G\) just described above. We shall denote this involutive subset of \(C^\infty(G)\) as \(I^\phi(G)\).

For any smooth function \(f\) on \(G\) let us introduce \(D_f(g) \in \mathfrak{g}\), by the following equality

\[\langle D_f(g), X \rangle = (d/dt)_{t=0} f(g e^{tX}).\] (7)

We shall also use the symbol \(\nabla_f(g)\) for the corresponding element of \(\mathfrak{g}\)

\[\nabla_f(g) = (D_f(g))_+ - (D_f(g))_-\] (8)

and we shall refer to it as to the gradient of \(f\). If \(h_1, h_2 \in I^\phi(G)\) then the corresponding gradients commute.

Now we can formulate the main classical theorem:

**Theorem 1.1 (main classical theorem)**  (i) Functions \(h \in I^\phi(G)\) are in involution with respect to the Sklyanin bracket on \(G\). (ii) The equations of motion defined by Hamiltonians \(h \in I^\phi(G)\) are of the Lax form

\[dL/dt = \phi(M^\pm)L - LM^\pm,\] (9)

with \(M^\pm = (D_h)(L)_\pm\) and \(L \in G\). (iii) Let \(g^\pm(t)\) be the solutions to the factorization problem (\(\mathfrak{g}\)) with the left hand side given by

\[g(t) = \exp(t \nabla_h(L(0))).\] (10)

The integral curves of equation (\(\mathfrak{g}\)) are given by

\[L(t) = \phi(g^\pm(t))L(0)g^\pm(t)^{-1}.\] (11)
We shall not to prove this theorem. Interested reader can consult [27], [24] for the proof. We just mention that there is an easy direct proof and a more conceptual one, which reflects the all rich structure of the theory of Poisson Lie groups and the geometry related to the theory of integrable systems. The strategy of the second proof is to show that the Lax equations are obtained by a change of variables from the simplest $G_- \times G_+$-invariant Hamiltonian systems on the so-called classical Heisenberg double (the Poisson Lie generalization of the of the cotangent bundle $T^*(G)$). This construction allows also, using the Poisson Lie variant of the symplectic reduction, to give one more description of the symplectic leaves of $G$ which are known to be the orbits of the dressing action of $G_r$ on $G$ [27].

It is often useful to consider the Lax equations (9) corresponding to different Hamiltonians from $I^\phi(G)$ simultaneously, using different time parameters corresponding to the different Hamiltonians. Usually there is given a hierarchy (a complete set) of functionally independent Hamiltonians $h_\alpha$, with $\alpha$ running over some label set $I$. The corresponding time parameters are $\{t_\alpha\}_{\alpha \in I} \equiv t$ and we index by $\alpha$ also the corresponding gradients and matrices $M^\pm$ entering the Lax equations. Then we obtain the following equations for $L(t) \in G$, $M^\pm_\alpha(t) \in \mathfrak{g}$ and $M_\alpha \equiv \nabla h_\alpha \in \mathfrak{g}$

\[
\frac{\partial L}{\partial t_\alpha} = \phi(M^\pm_\alpha)L - LM^\pm_\alpha, \tag{12}
\]

\[
\frac{\partial M_\alpha}{\partial t_\beta} = [M^\pm_\beta, M_\alpha] \tag{13}
\]

and

\[
\frac{\partial M^\pm_\alpha}{\partial t_\beta} - \frac{\partial M^\pm_\beta}{\partial t_\alpha} = [M^\pm_\beta, M^\pm_\alpha]. \tag{14}
\]

Here the integral curves $L(t)$ corresponding to the commuting dynamical flows on $G$ are given as above by the twisted conjugation like in (11) with the factors of

\[
g(t) = \exp(\sum_\alpha t_\alpha M_\alpha(0)). \tag{15}
\]

In the case of a concrete dynamical system, we pick up a proper symplectic leaf on $G$. Usually it is taken in a way that $I^\phi(G)$ contains enough first integrals to ensure that the system is completely integrable (in a proper sense).
The group element $g(t)$ can then be brought by a similarity transformation to the form
\[ g(t) = \varphi(0)\exp(\sum t_\alpha X_\alpha)\varphi(0)^{-1}, \tag{16} \]
where $X_\alpha$ are generators of some abelian subalgebra of $G$, so that the group element $g(t)$ describes an embedding of one of the commutative subgroups of $G$ into $G$ itself.

Let us now consider the quantization of the Lie-Poisson structure. The reader is referred to the existing monographs on quantum groups (e.g. [4]) for the necessary information on quantum groups.

### 1.2 Heisenberg equations of motion

As quantized phase space we take the non-commutative Hopf algebra $F$ of functions on a quantum group, dual via the pairing $\langle ., . \rangle : U \otimes F \to k$ to a quasitriangular Hopf algebra $U$. The quantum analog of Sklyanin’s bracket is then [9]
\[ RT_1 T_2 = T_2 T_1 R. \tag{17} \]

The $R$-matrix can be expanded as $R = 1 + hr + \mathcal{O}(h^2)$, where $h$ is a deformation parameter that gives the correspondence to classical mechanics:
\[ \frac{[f, g]}{h} = \{f, g\} \mod h, \]
such that for instance $0 = RT_1 T_2 - T_2 T_1 R = [T_1, T_2] \mod h$ and
\[ 0 = \frac{RT_1 T_2 - T_2 T_1 R}{h} = \{T_1, T_2\} - [T_1 T_2, r] \mod h, \]
i.e. $\{T_1, T_2\} = [T_1 T_2, r]$ is the classical limit of (17) as desired. (Note that (locally) all integrable systems have such a classical $r$-matrix [3]).

In these and the following expressions $T$ may either be interpreted simply as a Matrix $T \in M_n(F)$ or, much more general, as the canonical element of $U \otimes F$ : Let $\{e_i\}$ and $\{f^i\}$ be dual linear bases of $U$ and $F$ respectively. It is very convenient to work with the canonical element in $U \otimes F$ (also called the universal $T$-matrix [3], for an elementary overview see [5]),
\[ T = \sum_i e_i \otimes f^i \in U \otimes F, \tag{18} \]
because equations expressed in terms of it will be reminiscent of the familiar expressions for matrix representation—but we still keep full Hopf algebraic generality. For the same reason we will often write “$T_1$” in place of “$T_{12}$” when we use a notation that suppresses direct reference to the second tensor space of $T$; multiplication in $F$ is understood in that case. Example: $R_{12}T_1T_2 = T_2T_1R_{12}$ is short for $R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12}$.

The set of cocommutative elements of $F$ form a commutative subalgebra $I \subset F$. If we choose a Hamiltonian from this set, it will commute with all other cocommutative elements, which will consequently be constants of motion. This observation can be generalized to twisted cocommutative elements: Let $\phi$ be an automorphism of $U$ that preserves the universal $R$-matrix,

$$ (\phi \otimes \phi)(R) = R. \quad (19) $$

($\phi$ is the quantum analog of an automorphism of a Poisson-Lie Group.) Let $\phi^*$ be the pullback of $\phi$ to $F$, i.e. $\langle x, \phi^*(f) \rangle = \langle \phi(x), f \rangle$. The Hamiltonian $h \in F$ shall be a twisted cocommutative function.\(^1\) i.e.

$$ \Delta' h = (\phi^* \otimes \text{id})(\Delta h), \quad (20) $$

where $\Delta' = \tau \circ \Delta$ is the opposite comultiplication. The set of twisted cocommutative functions also form a commutative subalgebra $I^\phi \subset F$.

In the following we shall present the more general twisted case. The untwisted formulation can be obtained by omitting “$\phi$” in all expressions. The dynamics of our system is governed by the Heisenberg equations of motion

$$ i \dot{f} = [h, f], \quad \forall f \in F. \quad (21) $$

These can equivalently be written in terms of the universal $T$ as

$$ i \dot{T} \equiv i \sum e_i \otimes f^i = [\text{id} \otimes h, T] = \langle T_{13}T_{23} - T_{23}T_{13}, h \otimes \text{id}^2 \rangle, $$

or short

$$ i \dot{T}_2 = \langle [T_1, T_2], h \otimes \text{id} \rangle. \quad (22) $$

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1Quantum traces are twisted cocommutative functions with a non-trivial $\phi$ given by the square of the antipode, while ordinary traces are simply cocommutative.
Our strategy to solve this equation will be to embed the quantized phase-space $F$ into a bigger algebra (the Heisenberg double, $D_H \approx F \otimes U$ as a vector space), where the equations take on a particularly simple form: The image under this embedding of a Hamiltonian $h \in I^\phi$ is a casimir in $U$ which leads to trivial time evolution in $U \subset D_H$ and simple (linear) evolution in $D_H$. Projecting the solution back to $F$ we will find that Heisenberg’s equations can be written in (twisted) Lax form and our original problem is solved by factorization just like in the classical case.

2 Heisenberg double with twist

The Heisenberg Double of $F$ shall refer to the the semi-direct product algebra $D_H = F \rtimes U$ \cite{[33] 1-2 25}. It is also known as the quantum algebra of differential operators \cite{[33] 25} or the quantum cotangent bundle on $F$; see e.g. \cite{[10]} for the corresponding classical Heisenberg double. $D_H$ is isomorphic to $F \otimes U$ as a vector space; it inherits the product structures of $F$ and $U$; mixed products are obtained from the left action of $U$ on $F$. All relations can be conveniently summarized in terms of the canonical element $T$ of $F \otimes U$ \cite{[5]}:

$$T_{23}T_{12} = T_{12}T_{13}T_{23}.$$  \hspace{1cm} Heisenberg Double \hspace{1cm} (23)

This equation gives commutation relations for elements $x \in U$ with elements $f \in F$ that equip $F \otimes U$ with an algebra structure: $x \cdot f = \langle \text{id} \otimes \Delta x, \Delta f \otimes \text{id} \rangle \in F \otimes U$. In the setting of interest to us, $F$ is a co-quasitriangular Hopf algebra who’s structure \cite{[17]} is determined by a universal $R$-matrix. Following Drinfel’d’s construction \cite{[6]} we shall assume that $U$ is itself the quantum double of a Hopf algebra $U_\pm$; the universal $R$ arises then as the canonical element in $U_- \otimes U_+$, where $U_- = U_+^{\text{top} \Delta}$. The Yang-Baxter equation

$$R_{23}R_{12}R_{13} = R_{12}R_{13}R_{23}$$  \hspace{1cm} YBE - Quantum Double \hspace{1cm} (24)

plays the same role for the quantum double $U = U_+ \rtimes U_-$ as \cite{[23]} plays for the Heisenberg Double. The spaces $U_+$ and $U_-$ are images of the two mappings $R^\pm : F \to U_{\pm}$ associated with the universal $R \in U_- \otimes U_+$:

$$R^+(f) = \langle R_{21}, \text{id} \otimes f \rangle, \quad R^-(f) = \langle R_{12}^{-1}, \text{id} \otimes f \rangle.$$  \hspace{1cm} (25)
The twisted Heisenberg double was introduced in [28]. The most convenient description for our purposes of the dual Hopf algebra $U$ with twist is in terms of the universal invertible twisted-invariant 2-tensor $Y \in U \otimes U$:

$$Y_1 R^\phi_{12} Y_2 R_{21} = R_{12} Y_2 R^\phi_{21} Y_1.$$  \hfill (26)

Here as well as in the following the superscript $\phi$ denotes the application of the automorphism $\phi$ to the first tensor space: $R^\phi \equiv (\phi \otimes \text{id})(R)$. Twisted invariance means:

$$Y_{12} T^{\phi}_{12} = T_{21} Y_{12}.$$  \hfill (27)

The mixed relations

$$Y_1 T_2 = T_2 R_{21} Y_1 R^\phi_{12}$$  \hfill (28)

complete the description of the Heisenberg double. (In the case of a trivial twist $\phi = \text{id}$ we may chose $Y = R_{21} R_{12}$; equation (26) is then a consequence of the Yang-Baxter equation (24).)

Crucial for the construction that we are going to present is that $T$ factorizes in $U \otimes F$ as [4]

$$T = \Lambda Z, \quad \Lambda \in U_- \otimes F_-, \quad Z \in U_+ \otimes F_+, \quad F_\pm = (U_\pm)^\ast.$$  \hfill (29)

We have $U = U_- \otimes U_+$ as a linear space and coalgebra and $F = F_- \otimes F_+$ as a linear space and algebra:

$$(\Delta \otimes \text{id})(\Lambda) = \Lambda_1 \Lambda_2, \quad (\Delta \otimes \text{id})(Z) = Z_1 Z_2, \quad \Lambda_1 Z_2 = Z_2 \Lambda_1.$$  \hfill (30)

The universal elements $Z$ and $\Lambda$ of $U_\pm \otimes F_\pm$ define projections $F \to F_\pm$ and $U \to U_\pm$ that can be used to extract the $F_\pm$ parts of any element of $F$ and the $U_\pm$ parts of $(-+)$-ordered expressions in $U$. Let us denote by $(L^+)^{-1} \in U \otimes U_+$ and $L^- \in U \otimes U_-$ the corresponding images of $Y^{-1}$, such that $Y = L^+(L^-)^{-1}$. Using the maps based on $Z$ and $\Lambda$ we can derive a host of relations between $Y$, $Z$, $\Lambda$ and $L^\pm$:

**Proposition 2.1** The Heisenberg Double is defined by relations (17), (26) and (28). The following relations are consequences of these and (23):

$$Z_1 Y_1 Z_2 = Z_2 Z_1 Y_1 R^\phi_{12}.$$  \hfill (31)
\begin{align}
R_{12}Z_1Z_2 &= Z_2Z_1R_{12} 	ag{32} \\
R_{12}\Lambda_1\Lambda_2 &= \Lambda_2\Lambda_1R_{12} 	ag{33} \\
Z_1L_1^+Z_2 &= Z_2Z_1L_1^+ 	ag{34} \\
Z_1L_1^+\Lambda_2 &= \Lambda_2R_{21}Z_1L_1^+ 	ag{35} \\
L_1^-Z_2 &= Z_2R_{12}^\phi^{-1}L_1^- 	ag{36} \\
L_1^-\Lambda_2 &= \Lambda_2L_1^- 	ag{37} \\
R_{12}L_2^\pm L_1^\pm &= L_1^\pm L_2^\pm R_{12} 	ag{38} \\
R_{12}^\phi L_2^\pm L_1^\mp &= L_1^-L_2^\pm R_{12} 	ag{39} \\
Z_{23}\Lambda_{12} &= \Lambda_{12}Z_{23} 	ag{40} \\
Z_{23}Z_{12} &= Z_{12}Z_{13}Z_{23}. 	ag{41}
\end{align}

**Remark:** Similar relations involving $R$, $L^\pm$ and $T$ in the presence of twisting were proposed in \[28\]. Further relations involving $Y$ and $T$ can be easily obtained from the ones given above. The apparent asymmetry between relations involving $Z$ versus those involving $\Lambda$ is due to our choice of factorizing $T$ as $\Lambda Z$. We could have also based our analysis on $T = SV$ with $S \in U_+ \otimes F_+$ and $V \in U_- \otimes F_-; \text{ this would restore the } +/\text{-symmetry.}$

**Proof:** We shall only proof relation (31) that we are going to use extensively in the next sections. (The other relations follow similarly; see also [17] and the discussion in [16].)

The second tensor space of relation (28) is not $(U_-U_+)$-ordered so we have to resort to a trick: with the help of (17) we can derive a new relation

\[
T_1Y_1T_2 = R_{21}T_2^{-1}T_1R_{12}^\phi
\]

whose second tensor space is $(U_-U_+)$-ordered as is easily verified. Projecting to $U_+$ we obtain $T_1Y_1Z_2 = Z_2T_1Y_1R_{12}^\phi$ which can be simplified using $T_1 = \Lambda_1Z_1$ and $\Lambda_1Z_2 = Z_2\Lambda_1$ to yield (31). □
A remark on quantum traces and twisting

We have argued that the cocommutative elements of $F$ are natural candidates for Hamiltonians. Classically cocommutativity is equivalent to ad-invariance, so it would also be natural to look for Hamiltonian functions in the quantum case that are invariant under the quantum adjoint coaction:

$$\Delta^\text{Ad}(h) \equiv h_{(2)} \otimes S(h_{(1)})h_{(3)} = h \otimes 1,$$

(42)

It turns out that both these and the cocommutative Hamiltonians are treated on equal footing in the twisted formulation: Requirement (42) is equivalent to

$$\Delta h = (\text{id} \otimes S^2)(\Delta'h),$$

(43)

i.e. corresponds to a twisted cocommutative function with the pullback of the twist $\phi^*$ given by the square of the antipode. The twist $\phi = S^2$ is here generated via conjugation by an element $u \in U$:

$$S^2(x) = u x u^{-1}, \quad \forall x \in U.$$  

(44)

It seems interesting to study the general case of a twist $\phi$ that is given via conjugation by some element $\varphi$, i.e.

$$\phi(x) = \varphi x \varphi^{-1}, \quad (\varphi \otimes \varphi)R = R(\varphi \otimes \varphi).$$

(45)

If $\varphi \in U$ then $\phi^*(f) = \langle \varphi, f_{(1)} \rangle f_{(2)} \langle \varphi^{-1}, f_{(3)} \rangle$ for all $f \in F$. (Here we see by the way that $\phi^*(h) = h$ holds both for cocommutative and twisted cocommutative $h$.) Due to (45), $f \mapsto \langle \text{id} \otimes \varphi^{-1}, \Delta f \rangle$ defines an algebra isomorphism of $F$, that maps $I \subset F$ to $I^\phi \subset F$, i.e. cocommutative elements to twisted cocommutative elements.

It is easily verified that all expressions containing $\phi$, e.g. (26), (31), etc. continue to hold if we omit $\phi$ and replace $Y$ by $Y \cdot (\varphi \otimes 1)$. Examples:

$$R_{21}Y_1\varphi_1 R_{12}Y_2\varphi_2 = Y_2\varphi_2 R_{21}Y_1\varphi_1 R_{12},$$

(46)

$$Z_1Y_1\varphi_1 Z_2 = Z_2Z_1Y_1\varphi_1 R_{12},$$

(47)

This gives a nice mnemonic for where to put the $\phi$’s—even when an element $\varphi$ does not exist in $U$: First we write expressions without $\phi$, then we formally

\footnote{This holds for instance for quantum traces.}
replace all \((L^-)^{-1}\)s by \((L^-)^{-1} \cdot (\varphi \otimes 1)\) (and consequently \(Y\) by \(Y \cdot (\varphi \otimes 1)\)), finally we remove all \(\varphi\)'s from the expression with the help of relation (45).

**Remark:** \(Y = L^+(L^-)^{-1}\) but \(Y \neq R_{21} R_{12}\) in the twisted case. In case we know an element \(\varphi\) that satisfies (45), we can realize \(L^\pm\) in terms of the universal \(R\) for instance as \(L^+ = R_{21}\) and \(L^- = \varphi R_{12}^{-1}\). There is however some remaining ambiguity in this choice.

### 2.1 Embedding the operator algebra into the double

Here we will show how to embed \(F\) into \(D_H\) in such a way that any (twisted) cocommutative element of \(F\) is mapped to a casimir operator of \(U \subset D_H\).

**Proposition 2.2** The following element of \(U \otimes D_H\)

\[
\tilde{T} = \phi(Z)Y^{-1}Z^{-1}, \tag{48}
\]

where \(\phi(Z) \equiv (\phi \otimes \text{id})(Z)\), satisfies

\[
R_{12} \tilde{T}_1 \tilde{T}_2 = \tilde{T}_1 \tilde{T}_2 R_{12} \tag{49}
\]

and thus defines an embedding of \(F \hookrightarrow D_H : f \mapsto \langle \tilde{T}, f \otimes \text{id} \rangle\), that is an algebra homomorphism. (The picture of \(F\) in \(D_H\) by this embedding will be denoted \(\tilde{F}\).)

**Proof:** Start with (26) in form

\[
Y_1^{-1}R_{21}^{\phi} Y_2^{-1} = R_{21}^{-1} Y_2^{-1} R_{12}^{\phi} Y_1^{-1} R_{12},
\]

multiply by \(Z_2^{-1}Z_1^{-1}\) from the right and use (32) to obtain

\[
Y_1^{-1}Z_1^{-1}Y_2^{-1}Z_2^{-1} = R_{21}^{-1} Y_2^{-1} R_{12}^{\phi} Y_1^{-1} Z_1^{-1} Z_2^{-1} R_{12}.
\]

Applying equation (31) to the underlined part gives

\[
Y_1^{-1}Z_1^{-1}Y_2^{-1}Z_2^{-1} = R_{21}^{-1} Y_2^{-1} Z_2^{-1} Y_1^{-1} Z_1^{-1} R_{12}\tag{50}
\]

and as a corollary: \(R_{12}Z_2Y_2Z_1Y_1 = Z_1Y_1Z_2Y_2R_{21}\). Now use equation (31) twice: once in the form \(Z_1Y_1\phi(Z_2) = \phi(Z_2)Z_1Y_1R_{12}\), which follows from \((\phi \otimes \phi)(R) = R\), to replace \(Y_1^{-1}Z_1^{-1}\) on the LHS of (50) and once to replace \(R_{21}^{-1} Y_2^{-1} Z_2^{-1}\) on the RHS of (31). Multiplying the resulting expression by \(\phi(Z_i)\) from the left and using (32) in the form \(R_{12}\phi(Z_1)\phi(Z_2) = R_{12}\phi(Z_1)\phi(Z_2)\) gives our result (49). □
Proposition 2.3 The image $\tilde{h}$ of the Hamiltonian $h$ under the embedding $F \to \hat{F}$ is a casimir in $U \subset D_H$. We can find the following explicit expression:

$$\tilde{h} = \langle T, h \otimes \text{id} \rangle = \langle u_1^{-1}Y_1^{-1}, h \otimes \text{id} \rangle,$$

(51)

where $u_1^{-1} = (S^2 \otimes \text{id})(R)_{121}$, and satisfies $u_1^{-1}x = S^2(x)u_1^{-1}, \forall x \in U$.

Proof: We have to proof two things: 1) $\tilde{h}$ commutes with all elements of $U$ and 2) $\tilde{h}$ is an element of $U$ with the given expression.

Ad 1): Here is a nice direct calculation that shows that $\tilde{h}$ commutes with $Y_1^{-1}$ and hence (in the factorizable case) with all of $U$.

Start with the twisted reflection equation (23) in the form

$$Y_1^{-1}R_{21}^{-1}Y_2^{-1} = R_{21}^{-1}Y_2^{-1}R_{12}^{-1}Y_1^{-1}R_{12},$$

apply (31) with subscripts 1 and 2 exchanged to the underlined part, rearrange and multiply by $\phi(Z_1)$ from the left to obtain:

$$\phi(Z_1)Y_1^{-1}Z_1^{-1}Y_2^{-1} = \phi(Z_1)R_{21}^{-1}Y_2^{-1}R_{12}^{-1}Y_1^{-1}R_{12}Z_2^{-1}Z_1^{-1}Z_2.$$

Now we can use (31) twice, first in the form $\phi(Z_1)R_{21}^{-1}Y_2^{-1} = Y_2^{-1}Z_2^{-1}\phi(Z_1)Z_2$ and then in the form $Z_2R_{12}^{-1}Y_1^{-1} = Y_1^{-1}Z_1^{-1}Z_2Z_1$, to remove two $R$'s from the RHS. The resulting expression, simplified with the help of (32), is

$$\phi(Z_1)Y_1^{-1}Z_1^{-1}Y_2^{-1} = Y_2^{-1}Z_2^{-1}\phi(Z_1)Y_1^{-1}Z_1^{-1}R_{12}Z_2.$$

Contracting with $h$ in the first tensor space and using $h_{(1)} \otimes \ldots \otimes h_{(4)} = h_{(2)} \otimes \ldots \otimes h_{(4)} \otimes \phi^*h_{(1)}$, which follows from the twisted cocommutativity of $h$, we can move $R_{12}$ three places to the left:

$$\langle \phi(Z_1)Y_1^{-1}Z_1^{-1}Y_2^{-1}, h \otimes \text{id} \rangle = \langle Y_2^{-1}Z_2^{-1}R_{12}^{-1}\phi(Z_1)Y_1^{-1}Z_1^{-1}Z_2, h \otimes \text{id} \rangle.$$

Applying (31) once more to the underlined part and simplifying the resulting expression with the help of (32) in the form $R_{12}^{-1}\phi(Z_1)Z_2 = Z_2\phi(Z_1)R_{12}$ we finally obtain

$$\langle \phi(Z_1)Y_1^{-1}Z_1^{-1}Y_2^{-1}, h \otimes \text{id} \rangle = \langle Y_2^{-1}\phi(Z_1)Y_1^{-1}Z_1^{-1}, h \otimes \text{id} \rangle, \text{ i.e.}$$

$$[1 \otimes \tilde{h}, Y_1^{-1}] = 0.$$

Here and in the following we will use the following convenient notation that was brought to our attention by C. Chryssomalakos: The second subscripts denote the order of multiplication in a given tensor space. Consider for example $R = \sum_i \alpha_i \otimes \beta_i$, then $(S^2 \otimes \text{id})(R)_{121}$ equals $\sum_i \beta^i S^2(\alpha_i)$ and lives in tensor space 1.
Ad 2): Now we will derive the explicit expression for \( \tilde{h} \). (Using that expression it is also possible to prove that \( \tilde{h} \) is a casimir in \( U \).) We start with equation (31), written as

\[
Z_2 R_{12}^{-1} Y_1^{-1} Z_1^{-1} = Y_1^{-1} Z_1^{-1} Z_2,
\]

and move the \( R \) to the RHS with the help of its opposite inverse \( \bar{R} \), which satisfies \( \bar{R} \phi = (S^2 \otimes \text{id})(R \phi) \), where

\[
\bar{R} \phi 
\]

satisfies \( \bar{R} \phi \phi^{-1} = 1 \otimes 1 \). We find

\[
Z_2 Y_1^{-1} Z_1^{-1} = \bar{R} \phi \phi^{-1} Y_1^{-1} Z_1^{-1} Z_2.
\]

Let us now multiply tensor spaces 1 and 2 so that the two \( Z \)'s on the RHS cancel

\[
Z_1 Y_1^{-1} Z_1^{-1} = \bar{R} \phi \phi^{-1} Y_1^{-1} Z_1^{-1} Z_2 \]

If we now contract this expression with \( h \) in the first tensor space, we can use the twisted cocommutativity of \( h \) in the form

\[
h(1) \otimes h(2) \otimes h(3) = h(2) \otimes h(3) \otimes \phi^* h(1)
\]

to change the order of multiplication in the first tensor space on both sides of the equation:

\[
\langle \phi(Z_1 Y_1^{-1} Z_1^{-1}, h \otimes \text{id}) = \langle \bar{R}_1 \phi \phi^{-1} Y_1^{-1}, h \otimes \text{id} \rangle, \]

i.e. \( \langle \bar{T}, h \otimes \text{id} \rangle = \langle u^{-1} Y_1^{-1}, h \otimes \text{id} \rangle \). This is precisely the expression (51) that we wanted to prove.

We would like to briefly sketch how to prove that \( \tilde{h} \) is a casimir starting from (51): \( h' = \langle u^{-1} \otimes \text{id}, \Delta(h) \rangle \) is an element of \( F \) which is coinvariant with respect to the twisted adjoint action. (This follows from twisted cocommutativity of \( h \) and the fact that \( u^{-1} \) generates \( S^2 \).) \( Y \) on the other hand is a twisted invariant 2-tensor in \( U \otimes U \). Being the contraction of \( Y \) by \( h' \), \( \tilde{h} \) is itself an ad-invariant element of \( U \) and hence a casimir operator:

\[
T_2 \langle Y_1^{-1}, h' \otimes \text{id} \rangle = \langle T_1^{-1} Y_1^{-1} T_1, h' \otimes \text{id} \rangle T_2 = \langle Y_1^{-1}, h' \otimes \text{id} \rangle T_2.
\]

\[
\square
\]

2.2 Dynamics in the double

Now that we have found the image of the Hamiltonian under the embedding of the quantized phase space \( F \) into the Heisenberg double we can study Heisenberg’s equations of motion in the double. These are

\[
i \dot{\mathcal{O}} = [\tilde{h}, \mathcal{O}], \quad \forall \mathcal{O} \in D_H.
\]

Time evolution in the \( U \)-part of \( D_H \) is trivial (because \( \tilde{h} \) is central in \( U \))

\[
i \dot{x} = [\tilde{h}, x] = 0, \quad \forall x \in U \subset D_H.
\]

In the \( F \)-part we find simple linear equations

\[
i \dot{T} = [1 \otimes \tilde{h}, T] = T(\Delta \tilde{h} - 1 \otimes \tilde{h}) =: T \xi
\]
that are solved by exponentiation because $\xi$ is an element of $U \otimes U \subset U \otimes D_H$
and hence time-independent, see (54),
\[ T(t) = T(0)e^{-it\xi}. \] (56)
Here are some alternative useful expressions for $\xi = \Delta \tilde{h} - 1 \otimes \tilde{h}$: Equation
(55) slightly rewritten gives
\[ T_2\xi_2 = \langle \tilde{T}_1T_2 - T_2\tilde{T}_1, h \otimes \text{id} \rangle. \] (57)
Starting from (28) one can derive
\[ \xi = \langle u_1^{-1}(R_{12}^\phi - 1)R_{21}^{-1} - Y_1^{-1}, h \otimes \text{id} \rangle \] (58)
and
\[ \xi = \langle (S^{-1} \circ \phi \otimes \text{id})(R_{12}R_{21}) - 1 \rangle u_1^{-1}Y_1^{-1}, h \otimes \text{id} \rangle. \] (59)
We have thus far been able to give the explicit time evolution in the Heisenberg double. In section [3] we will come closer to the solution to the original problem—Heisenberg’s equations of motion—via explicit expressions for the evolution of $\tilde{T}(t)$.

3 Quantum Lax equation

We will now derive an explicit expression for the time evolution of $\tilde{T}$. Using the time-independence of $Y^{-1} \in U \otimes U \subset U \otimes D_H$ we find
\[ \tilde{T}(t) = \phi(Z(t))Y^{-1}Z^{-1}(t) = \phi(Z(t)Z^{-1}(0))\tilde{T}(0)Z(0)Z^{-1}(t). \] (60)
If we had started with an alternative $\tilde{T}$ expressed in terms of $\Lambda$ and $Y$ we would have found an expression involving $\Lambda$ instead of $Z$. Such considerations lead to the following proposition:

**Proposition 3.1** Let $\tilde{g}_+(t) = Z(t)Z(0)^{-1}$, $\tilde{g}_-(t) = \Lambda^{-1}(t)\Lambda(0)$ and $\tilde{M}^\pm(t) = i\tilde{g}_\pm(t)\tilde{g}_\pm^{-1}(t)$. The time-evolution of $\tilde{T}$ is given via conjugation by
\[ \tilde{g}_\pm(t) = \exp(-it(1 \otimes \tilde{h}))\exp(it(1 \otimes \tilde{h} - \tilde{M}_\pm(0))) : \] (61)
\[ \tilde{T}(t) = \phi(\tilde{g}_+(t))\tilde{T}(0)\tilde{g}_+(t)^{-1} = \phi(\tilde{g}_-(t))\tilde{T}(0)\tilde{g}_-(t)^{-1} \] (62)
and Heisenberg’s equation of motion can be written in Lax form
\[ i \frac{d}{dt} \tilde{T} = \phi(\tilde{M}^+ \tilde{T} - \tilde{T}\tilde{M}^+) = \phi(\tilde{M}^-)\tilde{T} - \tilde{T}\tilde{M}^-. \] (63)
Proof: The definition of $\tilde{M}^{\pm}(t)$ can be used to express $\bar{g}_\pm(t)$ in terms of $\tilde{M}^{\pm}(t)$. From (53) we have
\[ i \frac{d}{dt} \bar{g}_\pm(t) = \tilde{M}_\pm(t) \bar{g}_\pm(t) = e^{-it(1 \otimes \hbar)} \tilde{M}_\pm(0) e^{it(1 \otimes \hbar)} \bar{g}_\pm(t); \]

this can be integrated with the initial condition $\bar{g}_\pm(0) = 1$ to give equation (61). If we differentiate (62), we find equation (63). What is left to prove is equation (62). $\tilde{T}(t) = \phi(\bar{g}_+(t)) \tilde{T}(0) \bar{g}_+(t)^{-1}$ is simply (61) expressed in terms of $\bar{g}_+(t)$. The time evolution in $D_H$ is an algebra homomorphism and so we can decompose $T(t) = \Lambda(t) Z(t)$ with $\Lambda(t) \in U_- \otimes F_-(t)$ and $Z(t) \in U_+ \otimes F_+(t)$. It is now easy to see that $\phi(\bar{g}_+(t)) \tilde{T}(0) \bar{g}_+(t)^{-1} = \phi(\bar{g}_-(t)) \tilde{T}(0) \bar{g}_-(t)^{-1}$ is equivalent to $\phi(T(t)) Y^{-1} T^{-1}(t) = \phi(T(0)) Y^{-1} T^{-1}(0)$, i.e. we need to show that $\phi(T) Y^{-1} T^{-1}$ is time-independent: From (53) we get
\[ i \frac{d}{dt} \left( \phi(T) Y^{-1} T^{-1} \right) = \phi(T) \left( \phi(\xi) Y^{-1} - Y^{-1} \xi \right) T^{-1} = 0. \]

(That this is zero can be seen from the explicit expression $\xi = \Delta \hbar - 1 \otimes \hbar$: $(\phi \otimes \text{id})(\Delta \hbar) Y^{-1} - Y^{-1} \Delta \hbar = 0$ because $Y^{-1}$ is a twisted invariant 2-tensor and $[1 \otimes \hbar, Y^{-1}] = 0$ because $\hbar$ is a casimir operator.) \(\square\)

We will now proceed to derive explicit expressions for $\tilde{M}^{\pm}$ in terms of $\hbar$. We will not use the expressions for $\xi$ but rather work directly with $Z$ and $\Lambda$. First we prove the following lemma:

**Lemma 3.2** The following two relations hold in $U \otimes U \otimes D_H$:
\[ \Lambda_2^{-1} \tilde{T}_1 \Lambda_2 = \tilde{T}_1 R_{21}^{-1}, \] (64)
\[ Z_2 \tilde{T}_1 Z_2^{-1} = R_{12}^{\phi} \tilde{T}_1. \] (65)

**Proof:** We need to use
\[ Z_1 Y_1 \Lambda_2 = \Lambda_2 R_{21} Z_1 Y_1, \] (66)
which follows with $T = \Lambda Z$ from (28) and (31). We have
\[ \Lambda_2^{-1} \tilde{T}_1 \Lambda_2 = \Lambda_2^{-1} \phi(Z_1) Y_1^{-1} Z_1^{-1} \Lambda_2 = \Lambda_2^{-1} \phi(Z_1) \Lambda_2 Y_1^{-1} Z_1^{-1} R_{21}^{-1} = \tilde{T}_1 R_{21}^{-1} \] which proves (64). Similarly:
\[ Z_2 \tilde{T}_1 Z_2^{-1} = Z_2 \phi(Z_1) Y_1^{-1} Z_1^{-1} Z_2^{-1} = Z_2 \phi(Z_1) R_{12}^{\phi} Z_2^{-1} Y_1^{-1} Z_1^{-1} = R_{12}^{\phi} \tilde{T}_1, \] which proves (65). \(\square\)
Proposition 3.3 It holds that
\[
\begin{align*}
\tilde{M}^+(t) &= 1 \otimes \tilde{h} - \langle \tilde{T}_1(t) R_{12}, h \otimes \text{id} \rangle \in U_+ \otimes \tilde{F}, \\
\tilde{M}^-(t) &= 1 \otimes \tilde{h} - \langle \tilde{T}_1(t) R_{21}^{-1}, h \otimes \text{id} \rangle \in U_- \otimes \tilde{F}.
\end{align*}
\] (67) (68)

Proof: \[
\tilde{M}^+(t) = i\tilde{Z}(t) Z(t)^{-1} = \frac{d}{dt} \left( e^{-it(1 \otimes \tilde{h})} Z(0) e^{it(1 \otimes \tilde{h})} \right) Z(t)^{-1}
\]
\[
eq e^{-it(1 \otimes \tilde{h})} \left( 1 \otimes \tilde{h} - Z(0)(1 \otimes \tilde{h}) Z(0)^{-1} \right) e^{it(1 \otimes \tilde{h})}
\]
\[
= 1 \otimes \tilde{h} - e^{-it(1 \otimes \tilde{h})} ((Z_2 \tilde{T}_1 Z_2^{-1})(0), h \otimes \text{id} ) e^{it(1 \otimes \tilde{h})} = 1 \otimes \tilde{h} - \langle R_1^\phi \tilde{T}_1(t), h \otimes \text{id} \rangle
\]
\[
= 1 \otimes \tilde{h} - \langle \tilde{T}_1(t) R_{12}, h \otimes \text{id} \rangle,
\]
where we have used (63) and \((\phi^* \otimes \text{id})(\Delta h) = \Delta' h\).
The proof of (68) is based on (64) but otherwise completely analogous. □

Just like \(T\) factorizes as \(T(t) = \Lambda(t) Z(t)\), we shall think of \(\tilde{g}_\pm \in U_\pm \otimes D_H\) as factors of an element of \(U \otimes D_H\):
\[
\tilde{g}(t) = \tilde{g}_-^{-1}(t) \tilde{g}_+(t).
\] (69)

Proposition 3.4 \(\tilde{g}(t) \equiv \tilde{g}_-^{-1}(t) \tilde{g}_+(t)\) and its factors \(g_-(t)\) and \(g_+(t)\) are in fact elements of \(U \otimes \tilde{F} \subset U \otimes D_H\) as is apparent from the following expression:
\[
\tilde{g}(t) = Z(0) \exp(-it\xi) Z^{-1}(0) = \exp(-it\tilde{M}(0)),
\] (70)
where \(\tilde{M} \equiv \tilde{M}_+ - \tilde{M}_- = \langle \tilde{T}_1(R_{21}^{-1} - R_{12}), h \otimes \text{id} \rangle \in U \otimes \tilde{F}\).

Proof: \[
\tilde{g}(t) = \tilde{g}_-^{-1}(t) \tilde{g}_+(t) = \Lambda^{-1}(0) T(t) Z^{-1}(0) = Z(0) \exp(-it\xi) Z^{-1}(0).
\]
From equation (57) and \(T = \Lambda Z\): \(Z \xi Z^{-1} = \Lambda_2^{-1}(\tilde{T}_1 T_2 - T_2 \tilde{T}_1, h \otimes \text{id}) Z_2^{-1} = (\Lambda_2^{-1} \tilde{T}_1 \Lambda_2 - Z_2 \tilde{T}_1 Z_2^{-1}, h \otimes \text{id}) = (\tilde{T}_1 R_{21}^{-1} - R_1^\phi \tilde{T}_1, h \otimes \text{id}) = \tilde{M}_+ - \tilde{M}_-\) and hence \(Z \exp(-it\xi) Z^{-1} = \exp(-it(\tilde{M}_+ - \tilde{M}_-)) = \exp(-it\tilde{M})\). □

So far we have learned a great deal about the equations of motion in the Heisenberg double and their solution. We are now ready to go back to our original problem, i.e. the formulation of the equations of motion in the quantized phase space \(F\) in terms of quantum Lax equations and their solution by factorization, thus generalizing what has become known as the "Main Theorem" to the realm of quantum mechanics. Let us mention that the Lax equations presented in this section formalize and generalize the concrete examples known for particular integrable models [13], [29], [20], [21], [34], [30].
4 Solution by factorization

Using the fact that the embedding via $\tilde{T}$ of $F$ into $D_H$ is an algebra homomorphism we can drop all the $\tilde{\cdot}$’s in the previous section, thereby projecting the solution of the time evolution of $\tilde{F}$ back to $F$. The result can be summarized in a quantum mechanical analog of theorem [1.1]

Theorem 4.1 (main quantum theorem)

(i) The set of twisted cocommutative functions $I^\phi$ is a commutative subalgebra of $F$.

(ii) The equations of motion defined by Hamiltonians $h \in I^\phi$ are of the Lax form

\[ i \frac{dT}{dt} = \phi(M^\pm)T - TM^\pm, \tag{71} \]

with $M^\pm = \langle T_1(1 - R^\pm)21, h \otimes \text{id} \rangle \in U_\pm \otimes F$, $R^+ \equiv R_{21}$, $R^- \equiv R_{12}^{-1}$ and $T \in U \otimes F$.

(iii) Let $g^\pm(t) \in U_\pm \otimes F$ be the solutions to the factorization problem

\[ g^{-1}_-(t)g^+_+(t) = \exp(-itM(0)) \in U \otimes F, \tag{72} \]

where $M(0) = M^+ - M^-$, then

\[ T(t) = \phi(g^\pm(t))T(0)g^\pm(t)^{-1} \tag{73} \]

solves the Lax equation (71); $g^\pm(t)$ are given by

\[ g^\pm(t) = \exp(-it(1 \otimes h)) \exp(it(1 \otimes h - M^\pm(0))) \exp(it(1 \otimes h - M^\pm(0))) \tag{74} \]

and are the solutions to the differential equation

\[ i \frac{dg^\pm(t)}{dt} = M^\pm(t)g^\pm(t), \quad g^\pm(0) = 1. \tag{75} \]

This theorem follows from the geometric construction given in the previous sections, but we shall also present a direct proof:
Proof:
(i) Let $f, g \in I^\phi \subset F$, then $fg \in I^\phi$. Using (17), (20) and (19) we can show that $f$ and $g$ commute:

$$fg = \langle T_1T_2, f \otimes g \rangle = \langle R_{12}^{-1}T_2T_1R_{12}, f \otimes g \rangle = \langle R_{13}^{-1}T_3T_1R_{24}, \Delta f \otimes \Delta g \rangle = \langle R_{13}^{-1}T_3T_1 (\phi \otimes \phi)(R_{24}), \Delta' f \otimes \Delta' g \rangle = \langle R_{13}^{-1}T_3T_1R_{24}, \Delta' f \otimes \Delta' g \rangle = \langle R_{12}R_{12}^{-1}T_2T_1, f \otimes g \rangle = gf.$$ 

(ii) From $R_{21}^\pm T_1T_2 = T_2T_1R_{21}^\pm$, (21) and (19) it follows that

$$\langle T_1 (id \otimes \phi)(R_{21}^\pm)T_2, h \otimes id \rangle = \langle T_2T_1R_{21}^\pm, h \otimes id \rangle$$

and as a consequence all terms that contain $R_{21}^\pm$ cancel on the RHS of equation (71); we are left with

$$\phi(M^\pm)T - TM^\pm = \langle T_1T_2 - T_2T_1, h \otimes id \rangle = [h, T] = idT/dt.$$ 

(iii) The Lax equation (71) follows immediately from (73) and (73). The rest can be proven in three steps:

a) Let $m_{\pm} = 1 \otimes h - M_{\pm}$: $g_{\pm}(t) = e^{-it(1 \otimes h)}e^{itm_{\pm}}$ are elements of $U_\pm \otimes F$ and are the solutions to (73) as can be checked by differentiation:

$$i \frac{d}{dt} g_{\pm}(t) = e^{-it(1 \otimes h)}(1 \otimes h - m_{\pm})e^{itm_{\pm}} = M_{\pm}(t)e^{-it(1 \otimes h)}e^{itm_{\pm}} = M_{\pm}(t)g_{\pm}(t),$$

and $g_{\pm}(0) = 1$

b) $m_+ = \langle T_1R_{12}, h \otimes id \rangle$ and $m_- = \langle T_1R_{21}, h \otimes id \rangle$ commute: Using (17), (24), and (20) we find

$$m_+m_- = \langle T_1R_{13}T_2R_{32}^{-1}, h \otimes h \otimes id \rangle = \langle R_{12}^{-1}T_2T_1R_{12}R_{13}^{-1}R_{32}^{-1}, h \otimes h \otimes id \rangle = \langle R_{12}^{-1}T_2R_{32}^{-1}T_1R_{13}R_{12}, h \otimes h \otimes id \rangle = \langle T_2R_{32}^{-1}T_1R_{13}R_{12}R_{12}^{-1}, h \otimes h \otimes id \rangle = m_-m_+$$

From a) and b) follows

c) $g_{-1}(t)g_+(t) = e^{-itm_{-}e^{itm_{+}}} = e^{itm_{+}-m_{-}} = e^{-itm(0)}$, i.e. the $g_{\pm}(t)$ of (73) solve the factorization problem (72). \( \Box \)

Remark: If we replace $R_{\pm}$ in the definition of $M_{\pm}$ in the previous theorem by $(L_{\pm}^{-1})$, then the $\phi$'s in (71) and (73) will not appear explicitly anymore.

Dressing transformations

We have found two (identical) solutions for the time-evolution in $F$:

$$f(t) = \langle T(t), f(0) \otimes id \rangle, \quad f(0) \in F$$

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with $T(t)$ given in (73). Let us verify that

$$\phi(g_+(t))T(0)g_+(t)^{-1} = \phi(g_-(t))T(0)g_-(t)^{-1}.$$  

Let $g(t) = g_-(t)^{-1}g_+(t) = \exp(-itM(0))$; we have to show that

$$\phi(g(t)) = T(0)g(t)T(0)^{-1}$$

which is implied by:

$$TMT^{-1} = \langle T_2T_1(R_{21}^{-1} - R_{12})T_2^{-1}, h \otimes \text{id} \rangle = \langle (R_{21}^{-1} - R_{12})T_1, h \otimes \text{id} \rangle = \langle T_1(\text{id} \otimes \phi)(R_{21}^{-1} - R_{12}), h \otimes \text{id} \rangle = \phi(M).$$

With (74) we can re-express (73) as

$$T(t) = \left(T(0)g(t)T(0)^{-1}\right)_{-1} T(0) \left(g(t)\right)^{-1}_{-1}$$

and thus find that the time-evolution in $F$ has the form of a dressing transformation. More precisely we can identify elements of $F$ with elements of $U \otimes F$ (via the factorization map):

$$e^{ith} \mapsto e^{itm_{\pm}} = R_{\pm} \cdot e^{it(1 \otimes h)} \cdot (R_{\pm})^{-1} \mapsto g = e^{-itm_{-}}e^{itm_{+}}$$

and hence have a map

$$F \ni e^{ith} : U \otimes F \to U \otimes F : T(0) \mapsto T(t).$$

Let us choose the same $\pm$-conventions for $T = \Lambda Z$ and $Y^{-1} = L^{-1}(L^+)^{-1}$ as we did for $g = g_+g_+$, i.e. $(T)_- = \Lambda^{-1}$, $(T)_+ = Z$, etc. We can then write the embedding of section 2.1 in a way that parallels the classical theory:

$$\phi(TL^+)_- \cdot (\phi(T)L^-) \cdot (TL^+)_+^{-1} = \phi((\Lambda^{-1})^{-1}ZL^+)_- \cdot (\phi(T)L^-) \cdot ((\Lambda^{-1})^{-1}ZL^+)_+ = \phi(\Lambda^{-1}) \cdot (\phi(T)L^-) \cdot (L^+)^{-1}Z^{-1} = \phi(Z)Y^{-1}Z^{-1}.$$  

(“$\pm$” refers to the first tensor space.)

**Remark 1:** Note that the multiplication “$\cdot$” is taken in $U \otimes D_H$ rather than $U \otimes (F \otimes U)$; this and the form in which the dressing transformations appear in this article are somewhat non-standard.
Remark 2: The formal factorization problem in the case of $U$ being factorizable [22] remains the same as in the untwisted case. See Appendix 2 in reference [17].

Remark 3: Also in the quantum case we can consider Lax equations corresponding to different twisted cocommutative Hamiltonians simultaneously. The equations (12), (13), (14) and (15) are still valid.

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