Highly-collimated, magnetically-dominated jets around rotating black holes

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Abstract

In this paper, we propose a general method for perturbative solutions to Blandford-Znajek mechanism. Instead of solving the nonlinear Grad-Shafranov equation directly, we introduce an alternative way to determine relevant physical quantities based on the horizon boundary condition and the convergence requirement. Both the angular velocity and the toroidal magnetic fields are self-consistently specified according to our method. As an example, stationary axisymmetric and force-free jet models around rotating black holes are self-consistently constructed according to the method we proposed. This jet solution distinguishes itself from prior known analytic solutions in that it is highly collimated and asymptotically approaches a magnetic cylinder. This jet solution is helically twisted, since toroidal magnetic field is generated when the black hole spin is taken into account. For a given magnetic flux threading the black hole, the jet power and energy extraction rate of the collimated jet are compared with previous solutions. We find that our new solution agrees better with current state-of-the-art numerical simulation results. Some interesting properties of the collimated jet and effects of field line rotation on the jet stability are also briefly discussed.
I. INTRODUCTION

Many high-energy astrophysical objects, such as active galactic nuclei (AGNs) and gamma-ray bursts (GRBs) as well as ultra-strongly magnetized neutron stars (magnetars) involve relativistic magnetically-dominated plasma. Under such circumstances, magnetic fields play crucial roles in the dynamics of these astrophysical scenarios, which can drive powerful winds/jets from these astrophysical objects. It is widely accepted that, in these objects, the magnetic energy density conspicuously exceeds the thermal and rest mass energy density of particles. The force-free electrodynamics behave well in such extreme magnetically dominated scenarios as the less important terms, such as the inertia and pressure, are entirely ignored. In the force free electrodynamics the Lorentz force $F_{\mu\nu}J^\nu$ disappears. Based on the force-free electrodynamics, Blandford & Znajek (1977) studied an axisymmetric steady-state plasma surrounding a spinning black hole and proposed that the rotation energy of a Kerr black hole could be extracted via the action of force free electromagnetic fields, in the form of Poynting flux via magnetic field lines penetrating the central black hole. This BZ mechanism is proved to be one of the most powerful energy releasing processes in our universe and it is one of promising candidates as the central engine of AGNs and GRBs.

The configuration of the ordered magnetic field around the black hole has been discussed both in analytical and numerical studies. A self-consistent description of the highly magnetized plasma around strongly curved space-time of rotating black holes involves the nonlinear Grad-Shafranov (GS) equation. First examples of the force-free field configurations were constructed in [1], henceforth BZ77, in which perturbation techniques were applied to get self-consistent field configurations. Some following efforts, in which the angular velocity of magnetic field is prescribed rather than self-consistently determined from the GS equation, were made to model central regions of stationary axisymmetric magnetosphere of black holes [2][3]. Due to the extreme nonlinearity of the GS equation, there has been almost no further development in the analytic solution to the BZ mechanism. General relativistic magnetohydrodynamics (GRMHD) and magnetodynamics (GRMD) simulations, however, provide us an opportunity to look into the nature of the BZ mechanism. GRMHD simulations of black hole accretion system show that the perturbative split monopole solution is consistent with the numerical results in the low-density polar regions [4][5][6]. GRMD simulations suggest that the split monopole solution is accurate and stable for slow rotating
black holes ($a \ll 1$), where $a$ is the specific momentum of the Kerr black hole, and is also a rather good approximation for even fast rotating black hole ($a = 0.9$) \cite{7,8,9,10}. The question about the magnetic field configuration in the vicinity of a black hole still remains an open issue. Recent numerical investigation \cite{11} shows that the standard split monopole jet power is about 60% larger than the GRMHD simulation results, which indicates that the split monopole jet model may not account for the simulation results properly. In this paper, we propose a highly collimated jet model, which can reduce the jet power to be more consistent with recent simulations.

There are some concerns about stabilities of jet launched by the BZ mechanism \cite{12,13,14}, but many numerical simulations imply that the jet is stable. Especially, three-dimensional GRMHD simulations have been performed to investigate the stability of relativistic jets and no instability was discovered \cite{15,16}. The possible origin for the discrepancy is that analytical work, which applies the Kruskal-Shafranov (KS) criteria to the highly magnetized magnetosphere, does not consider the stabilizing effects present in simulations, including field rotation, gradual shear, a surrounding sheath, sideways expansion and non-linear saturation \cite{15}. Including the field rotation, the split monopole solution was analytically proved to be stable against screw unstable modes satisfying the KS criteria \cite{17}. In this paper, we also study stabilities of the new collimated jet solution.

This paper is organized as follows: basic equations governing stationary axisymmetric force-free fields around Kerr black holes are introduced in section 2. In section 3 we describe a perturbative approach to obtain self-consistent highly collimated jet solutions. Physical properties, such as energy extraction rate, stability of the jet solution, are discussed and compared with previous known solutions in section 4. Discussions are given in section 5.

II. STATIONARY AXISYMMETRIC FORCE-FREE FIELDS AROUND KERR BLACK HOLES

We adopt the Kerr-Schild coordinate (horizon penetrating, \cite{4}), in which the line element is

$$ds^2 = -\left(1 - \frac{2r}{\Sigma}\right)dt^2 + \left(\frac{4r}{\Sigma}\right)drdt + \left(1 + \frac{2r}{\Sigma}\right)dr^2 + \Sigma d\theta^2 - \frac{4ar \sin^2 \theta}{\Sigma} d\phi dt$$
\[-2a \left( 1 + \frac{2r}{\Sigma} \right) \sin^2 \theta d\phi dr + \sin^2 \theta \left[ \Delta + \frac{2r(r^2 + a^2)}{\Sigma} \right] d\phi^2, \tag{1}\]

where \(\Sigma = r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2r + a^2,\) and \(\sqrt{-g} = \Sigma \sin \theta\).

Since the magnetosphere around the black hole is magnetically dominated, we adopt the force-free approximation, which ensures that the electromagnetic field dominates over matter

\[T^{\mu\nu} = T^{\mu\nu}_{\text{matter}} + T^{\mu\nu}_{\text{EM}} \approx T^{\mu\nu}_{\text{EM}}. \]

The energy-momentum tensor of the electromagnetic field is

\[T^{\mu\nu}_{\text{EM}} = F^{\mu\tau} F^{\nu}_{\tau} - \frac{1}{4} \delta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \]

where the Faraday tensor is defined as

\[F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}. \]

It is easy to prove that the energy-momentum conservation of electromagnetic field is equivalent to the force-free condition (18),

\[T^{\mu\nu}_{\phi\sigma} = F^{\mu\nu} J_{\nu} = 0. \tag{2}\]

The force-free condition implies the vanishing of the electric field in the local rest-frame of the current, and thus \(F^\mu_\nu F^\nu_\mu = 0, \) where \(F^\mu_\nu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) is the dual of the Faraday tensor. It is straightforward to prove that \(A_{t,\phi} A_{t,r} = A_{t,\theta} A_{t,\phi}, \) which indicates that \(A_t\) is a function of \(A_\phi. \) We can define the angular velocity of the magnetic field \(\Omega(r, \theta)\) as follows,

\[-\Omega \equiv \frac{dA_t}{dA_\phi} = \frac{A_{t,\theta}}{A_{\phi,\theta}} = \frac{A_{t,r}}{A_{\phi,r}}, \tag{3}\]

which is an unspecified function and will be determined self-consistently in Section III. For simplicity, we consider an stationary and axisymmetric model, which implies that \(F^t_\phi = 0\) and the non-vanishing components of the antisymmetric Faraday tensor \(F^\mu_\nu\) are as follows:

\[F^r_\phi = -F^\phi_r = A^\phi_r, \quad F^\theta_r = -F^r_\theta = A^\phi_\theta, \quad F^\phi_r = -F^r_\phi = A^\theta_r, \tag{4}\]

\[F^{t\theta} = -F^{\theta t} = \Omega A^\phi_r, \quad F^{t\phi} = -F^{\phi t} = \Omega A^\phi_\theta, \tag{5}\]

\[F^{r\theta} = -F^{\theta r} = \sqrt{-g} B^\phi. \tag{6}\]

The above five non-zero components of \(F^\mu_\nu\) can be specified in terms of three free functions \(\Omega(r, \theta), A_\phi(r, \theta), B^\phi(r, \theta). \) According to the definition of the energy-momentum tensor, we can further have that \(T^\theta_\theta = -\Omega T^\phi_\phi\) and \(T^r_t = -\Omega T^\phi_\phi. \) With these two relations, the energy conservation and angular momentum conservation equations \(T^\mu_{t,\mu} = 0\) and \(T^\mu_{\phi,\mu} = 0\) can be cast as \(\Omega_r A_{\phi,\theta} = \Omega_\theta A_{\phi,r}\) and \((\sqrt{-g} F^\theta_r)_r A^\phi_\theta = (\sqrt{-g} F^\phi_r)_\theta A^\phi_r. \) It is obvious that \(\Omega\) and \(\sqrt{-g} F^\phi_r\) are functions of \(A_\phi, \) i.e., \(\Omega \equiv \Omega(A_\phi)\) and \(\sqrt{-g} F^\phi_r \equiv H(A_\phi),\) where \(\Omega\) and \(H\) are as-yet unspecified functions. Substitute Equations (4), (5), (6), and the relation \(F^\theta_r = H/\sqrt{-g}\)
into the equation $F^{\theta r} = g^{\theta \mu} g^{r \nu} F_{\mu \nu}$, we can readily arrive at

$$B^\phi = -\frac{H\Sigma + (2\Omega r - a) \sin \theta A_{\phi, \theta}}{\Delta \Sigma \sin^2 \theta},$$

(7)

which relates the toroidal magnetic field $B^\phi$ to the functions $A_{\phi}(r, \theta)$, $\Omega(A_{\phi})$, and $H(A_{\phi})$. To determine the unknown functions of $\Omega(A_{\phi})$ and $H(A_{\phi})$, the remaining momentum conservation equations in the $r$ and $\theta$ direction $T^\mu_{r, \mu} = 0$ and $T^\mu_{\theta, \mu} = 0$ have to be considered in greater details. The two conservation equations in the $r$ and $\theta$ directions are equivalent and read

$$-\Omega[\sqrt{-g} F^{\theta r}]_{,r} + (\sqrt{-g} F^{r \theta})_{,\theta} + F^r_{\theta \theta} H'(A_{\phi}) + [(\sqrt{-g} F^{\phi r})_{,r} + (\sqrt{-g} F^{\phi \theta})_{,\theta}] = 0,$$

(8)

where the prime denotes derivative with respect to $A_{\phi}$. Note that above equation is equivalent to Equation (3.14) in ([1]), which is also widely called Grad-Shafranov equation. The three functions $A_{\phi}(r, \theta)$, $\Omega(A_{\phi})$, and $H(A_{\phi})$ are related by the nonlinear equation (8).

### III. COLLIMATED JET SOLUTIONS – A PERTURBATIVE APPROACH

Since the Grad-Shafranov equation is highly nonlinear, our strategy is to find its solution in the simplest case for non-rotating black holes, i.e., $a = 0$ and then to perturb the simplest solution by allowing the black hole’s spin, $a$, to increase slowly. Namely, the corresponding solution can be expressed, up to $O(a^2)$, as

$$A_{\phi} = A_{\phi}^{(0)} + a^2 A_{\phi}^{(2)} + O(a^4),$$

(9)

$$\Omega = a \Omega^{(1)} + O(a^3),$$

(10)

$$B^\phi = a B_{\phi}^{(1)} + O(a^3).$$

(11)

Keep in mind that $\Omega$ and $\sqrt{-g} F^{\theta r}$ are both functions of $A_{\phi}$, and they should be in the form of

$$\Omega = \Omega(A_{\phi}) = a \omega(A_{\phi}), \quad \sqrt{-g} F^{\theta r} = H(A_{\phi}) = ah(A_{\phi}).$$

(12)

We now consider the zeroth-order solution $A_{\phi}^{(0)}$ with $a = 0$. The simplest force-free field around non-rotating black holes is actually the potential field in the Schwarzschild space-time (19). In this case $\Omega(r, \theta) = 0$ and $B^\phi(r, \theta) = 0$. The non-vanishing components of the Faraday tensor $F_{\mu \nu}$ are $F_{r \phi}, F_{\phi r}, F_{\theta \phi}, F_{\phi \theta}$. It is easy to know that Equation (2) holds
automatically for the $\mu = t, \phi$ components, and the $\mu = r, \theta$ component equations give the identical result as follows:

$$\mathcal{L} A^{(0)}_{\phi} \equiv \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial r} \left( 1 - \frac{2}{r} \right) \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right\} A^{(0)}_{\phi} = 0 .$$  \hspace{1cm} (13)

There exists a zeroth-order collimated, uniform magnetic field solution

$$A^{(0)}_{\phi} = r^2 \sin^2 \theta .$$  \hspace{1cm} (14)

The field line of this solution is of a highly collimated cylindrical shape. If the black hole is spinning, toroidal magnetic fields will be generated, and the magnetic cylinder will be twisted and turns into a helically twisted structure. Note that different zeroth-order solutions, i.e., the monopole solution and the paraboloidal solution, were adopted in ([1]). The explicit dependence of zeroth-order solutions on the coordinate $r$ and $\theta$ are displayed in Table 1.

According to Equation (7), we find

$$B^{\phi(1)} = \frac{2 \cos \theta (1 - 2 r \omega (A_{\phi})) - h(A_{\phi})/ \sin^2 \theta}{r^2 - 2r} .$$  \hspace{1cm} (15)

If we require $B^{\phi}$ to be well-behaved on the horizon (Znajek horizon condition [20]), then $r = 2$ must be a root to the equation, $2 \cos \theta (1 - 2 r \omega) - h(A_{\phi})/ \sin^2 \theta = 0$, namely, $h(4 \sin^2 \theta) = 2 \cos \theta \sin^2 \theta \left[ 1 - 4 \omega (4 \sin^2 \theta) \right]$. This equation can be written in a more compact form as

$$h(x) = \sqrt{4 - x} \left( \frac{1}{4} - \omega(x) \right) x ,$$

where $x = 4 \sin^2 \theta$. Since $h = h(A_{\phi})$ and this equation can be written as

$$h(A_{\phi}) = \sqrt{4 - A_{\phi}} \left( \frac{1}{4} - \omega(A_{\phi}) \right) A_{\phi} .$$  \hspace{1cm} (16)

Note that this equation only applies to the upper hemisphere $\theta < \pi/2$. For the lower hemisphere, an extra minus sign is required.

Before diving into solving the GS equation, it is helpful to analyze the behavior of energy flux first. The energy flux is defined as ([21][20])

$$T^{r}_{t} = F^{r\theta} F_{t\theta} = -\frac{1}{\sqrt{-g}} H(A_{\phi}) \Omega(A_{\phi}) A_{\phi, \theta} , \quad T^{\theta}_{t} = F^{\theta r} F_{t r} = \frac{1}{\sqrt{-g}} H(A_{\phi}) \Omega(A_{\phi}) A_{\phi, r} .$$  \hspace{1cm} (17)

On the cylinder surface, $r \sin \theta = 2$, it is easy to know that $T^{r}_{t} \equiv 0$ and $T^{\theta}_{t} \equiv 0$. So $r \sin \theta = 2$ serves as a boundary that no energy flux penetrates. In the outer region, $r \sin \theta > 2$, we can simply choose

$$H(A_{\phi}) = 0 , \quad \Omega(A_{\phi}) = 0 \quad (r \sin \theta > 2) ,$$  \hspace{1cm} (18)
which make sure that $T^r_t = T^\theta_t \equiv 0$ in the outer region. In the following we will see that our choice naturally makes the global solution continuous across the interface at $r \sin \theta = 2$. With some tedious manipulations, Equation (8) can be reduced, accurate to order $O(a^2)$, to

$$\mathcal{L}A^{(2)}_\phi = S(r, \theta) = \begin{cases} 
S_{\text{in}}(r, \theta) & \text{if } r \sin \theta < 2 \\
S_{\text{out}}(r, \theta) & \text{if } r \sin \theta > 2
\end{cases},$$

(19)

where the source term in the inner region ($r \sin \theta < 2$) is

$$S_{\text{in}}(r, \theta) = 8 \sin \theta \cos^2 \theta/r^3 - 6 \sin \theta \cos^2 \theta/r^2 + (1 - 2 \omega r)(\sin^2 \theta B^{(1)}_\phi)_r \theta$$

$$+ 4 \sin \theta \omega^2 r^2 + 4 \sin \theta(3 \cos^2 \theta - 1) \omega^2 r + r^2 \sin \theta B^{(1)}_\phi h'$$

$$- 8 \omega' r \sin^5 \theta + 4 r^3 \sin^3 \theta \omega \theta' \left( r + 2 \cos^2 \theta \right),$$

(20)

and the source term in the outer region ($r \sin \theta > 2$) is

$$S_{\text{out}}(r, \theta) = \frac{8 \sin \theta \cos^2 \theta}{r^3} - \frac{2 \sin \theta}{r^2} + \frac{4 \sin \theta}{r^2(2 - r)}(3 \cos^2 \theta - 1).$$

(21)

Note that the prime in the above equations represents the derivative with respect to $A_\phi$. The Znajek horizon condition imposes the equation (16) between $H$ and $\Omega$. To specify the collimated jet, we still need to know the behavior of angular velocity of the magnetic field, $\Omega(A_\phi)$. Usually we need to solve the above inhomogeneous Grad-Shafranov equation before we get the angular velocity of the magnetic field. Fortunately, we find that the convergence condition can be applied to get the further details about this solution. According to BZ77, the sufficient and necessary condition for the existence of convergent solution of $A^{(2)}_\phi$ is that the integral $\int_2^\infty dr \int_0^\pi d\theta |S(r, \theta)|/r$ converges (convergence condition). It is easy to prove that the contribution from the outer region

$$\int_2^\infty dr \int_{-\delta}^{\pi-\delta} d\theta |S(r, \theta)|/r = \int_2^\infty dr \int_{\delta}^{\pi-\delta} d\theta |S_{\text{out}}(r, \theta)|/r$$

(22)

converges, where $\delta = \arcsin(2/r)$. So we only need to require the contribution from inner region

$$\int_2^\infty dr \int_0^\delta d\theta |S(r, \theta)|/r = \int_2^\infty dr \int_0^\delta d\theta |S_{\text{in}}(r, \theta)|/r$$

(23)

to be convergent. Assuming $\omega, \omega' \sim O(1)$, the following three source terms in Equation (20)

$$4 \omega^2 r^2 \sin \theta \sim 4 \omega \omega' r^4 \sin^3 \theta \sim r^2 \sin \theta B^{(1)}_\phi h'$$

(24)
are of the same order $O(r)$ and will lead to logarithm divergence of the integral above, where we have used the fact that $0 \leq \theta \leq \delta \sim O(1/r)$ for the inner region. Note that except the above three terms, the contribution from all other terms in Equation (20) is convergent and are not listed here. So the convergence condition requires

$$4\omega^2 r^2 \sin \theta + 4\omega \omega' r^4 \sin^3 \theta + r^2 \sin \theta B^{(1)} h' = 0 . \quad (25)$$

Accurate to $O(r)$, the above equation can be written equivalently as

$$4\omega^2 A_\phi + 4\omega \omega' A^2_\phi - \frac{(h^2)'}{2} = 2(\omega^2 A^2_\phi)' - \frac{(h^2)'}{2} = 0 , \quad (26)$$

where we have used the result of Eq.(15). Obviously the above equation can be integrated as

$$\text{const} = 4\omega^2 A^2_\phi - h^2 = 4\omega^2 A^2_\phi - (4 - A_\phi) \left( \frac{1}{4} - \omega \right)^2 A^2_\phi . \quad (27)$$

Note that $A_\phi = 0$ at the polar axis, hence the integration constant vanishes. This equation constitutes a quadratic equation for the unknown function $\omega = \omega(A_\phi)$, which can be solved explicitly as,

$$\omega = \frac{\sqrt{4 - A_\phi}}{4 \left( 2 + \sqrt{4 - A_\phi} \right)} \quad (r \sin \theta < 2) , \quad (28)$$

which is consistent with the result of [22] who obtained the same solution using a different approach. Using Eq.(10), we get

$$h = \frac{\sqrt{4 - A_\phi} A_\phi}{2 \left( 2 + \sqrt{4 - A_\phi} \right)} \quad (r \sin \theta < 2) . \quad (29)$$

Surely we may explicitly write $A^{(2)}_\phi$ as sum of infinite series (11). But we do not plan to do that, because we have obtained all quantities that are of physical interests even not knowing the details about $A^{(2)}_\phi$, which only provides information about the distortion of magnetic field lines in $r - \theta$ plane.

IV. PHYSICAL PROPERTIES OF COLLIMATED, MAGNETICALLY DOMINATED JETS

We have obtained the explicit analytical expressions of $\omega = \omega(A_\phi)$ and $h = h(A_\phi)$. Some interesting physical properties of this collimated jet solution can be further explored, such as
the energy extraction rate, the energy extraction efficiency, the stability of the jet solution, the comparison with solutions of BZ77 and numerical simulations.

The energy extraction rate is defined as $\dot{E} = -2\pi \int_0^\pi \sqrt{-gT^r_r} d\theta = 2\pi \int_0^\pi H(A_\phi)\Omega(A_\phi) dA_\phi$. Direct integration leads to

$$\dot{E}/2 = 2\pi \int_0^4 H\Omega dA_\phi = 8\pi a^2 \left(2 - 2\ln 2 - \frac{7}{12}\right) \approx 0.24\pi a^2,$$

where the factor 2 on the left hand side is to include contributions from both hemispheres.

The comparison between previous analytic solutions and the jet solution is listed in Table 1. Note that the paraboloidal solution is not as complete as the other two, in which $\Omega$ and $H$ are only available on horizon, so in the table 'NA' means generally 'Not Available'. In the table, the energy extraction efficiency, $\bar{\epsilon}$, is defined as

$$\bar{\epsilon} = \frac{\int H\Omega dA_\phi}{\int (H\Omega/4\omega) dA_\phi}.$$

All the zeroth-order solutions are normalized to keep the amount of magnetic flux crossing the horizon identical. It is clear that the collimated jet power is reduced by a factor about 30% compared to the split monopole jet model, which is more consistent with recent simulation results ([11]). If the electromagnetic torque is further counteracted by the accretion material, the jet power would be further reduced. In addition, for the collimated jet solution, it is easy to know the angle-averaged angular velocity on the horizon $\bar{\Omega} = (1 - \ln 2)\Omega_H \approx 0.31\Omega_H$ which is also consistent with the $t - \theta - \phi$ averaged angular velocity $\bar{\Omega} \approx 0.3\Omega_H$ (see Fig. 8 of [11]).

In Fig.1, we show the variation total electric current $H(A_\phi)$ on the horizon, which matches numerical simulation results, such as the bottom panel of Fig.5 of [23]. Komissarov [23] noticed a sharp transition between the rotating jet column of magnetic field lines penetrating the black hole horizon and the non-rotating field lines which are not attached to the black hole, and they interpreted the sharp transition as a discontinuity smeared by numerical viscosity. Our analytic solution clearly shows that the transition is only a sharp turn (see Fig.1) instead of a discontinuity, which is also confirmed by recent simulations of higher resolution ([24]).

The stability of jets is also an issue of vital astrophysical importance. Time dependent GRMD simulations of black hole magnetospheres have demonstrated the stability of split monopole solution ([7]) and the collimated jet solution ([23]). Recent calculations show that
TABLE I. Comparison between our collimated jet solution and prior solutions

| solution          | $A_\phi^{(0)}$ | $\Omega$ | $H(A_\phi)$ | $\hat{E}$ | $\hat{\epsilon}$ |
|-------------------|----------------|----------|-------------|-----------|-------------------|
| Split Monopole    | $-4 \cos \theta$ | $a/8$   | $4\Omega \sin^2 \theta$ | $0.67 \pi a^2$ | $0.50$ |
| Paraboloidal      | $\frac{1}{\ln 2} [r(1 - \cos \theta) + 2(1 + \cos \theta)(1 - \ln(1 + \cos \theta))]$ | NA     | NA          | $0.55 \pi a^2$ | $0.38$ |
| Collimated Jet    | $r^2 \sin^2 \theta$ | $a\omega$ | $2\Omega A_\phi$ | $0.48 \pi a^2$ | $0.36$ |

FIG. 1. The comparison of three solutions: the variation of total poloidal current $H$ (we fix $r = 2$ for curves of BZ77 solutions).

Mode growth rates are far lower than those predicted by the Kruskal-Shafranov stability criterion, suggesting that it may not be appropriate for jet stability analysis \(\text{(25)}\). Thus we adopt the criterion proposed by \(\text{(17)}\), since it explicitly accounts for the effects of field line rotation. This criterion states that the magnetosphere will possibly be unstable only when

\[
\left| \frac{\hat{B}_\phi}{\hat{B}_z} \right| > \Omega r \sin \theta . \tag{32}
\]

where $\hat{B}_\phi$ and $\hat{B}_z$ are local toroidal and poloidal field in the zero-angular-momentum observers’ frame. To be specific, accurate to $O(a)$, the toroidal and poloidal magnetic field are

\[
|\hat{B}_\phi| = \frac{r - 2}{r^2} \sqrt{g} B^\phi \approx \frac{H}{r \sin \theta} = 2\Omega r \sin \theta , \quad |\hat{B}_z| = 2 , \quad \tag{33}
\]
respectively. The ratio of the two is approximately,

\[ \frac{\hat{B}_\phi}{\hat{B}_z} = \Omega r \sin \theta . \]  \hspace{1cm} (34)

It is clear that the instability threshold is not satisfied and the collimated jet solution is a stable solution.

V. SUMMARY AND DISCUSSION

We present a general method for perturbative solutions of Blandford-Znajek mechanism. Assuming stationary axisymmetric and force-free magnetospheres, the energy-momentum equations can be divided into 2 constraint equations \( \Omega \equiv \Omega(A_\phi) \) and \( \sqrt{-gF^{\theta r}} \equiv H(A_\phi) \) and GS equation (Eq.8), where GS equation is a second order differential equation which requires two boundary conditions. We propose the horizon regularity condition and the convergence condition as the two boundary conditions. With the boundary conditions, all physical quantities such as the angular velocity of magnetic fields \( \Omega \), the total current \( H \) and energy extraction rate \( \dot{E} \) are self-consistently specified.

As an example, we construct a highly-collimated and magnetically-dominated jet solution in the vicinity of spinning black holes. The nonlinear GS equation (8) is investigated analytically to get axisymmetric steady-state force-free jet solutions. This equation is solved by a perturbation technique. In this paper we choose a uniform and collimated zeroth-order solution, \( A^{(0)}_\phi = r^2 \sin^2 \theta \), in the Schwarzschild spacetime. The higher order solution, \( A^{(2)}_\phi \), which accounts for the effect of black hole spin, can be obtained based on the zeroth-order solution around nonrotating black holes. It is straightforward yet tedious to get the solution of inhomogeneous GS equation (19) for \( A^{(2)}_\phi \). Nevertheless, \( A^{(2)}_\phi \) can not provide further details about the physical properties of the jet, such as the angular velocity of the field lines, \( \Omega \), and the poloidal current, \( H \), we take an alternative way to get these important physical quantities. According to the Znajek horizon boundary condition, we know that the angular velocity of the magnetic field, \( \Omega \), and poloidal current, \( H \), are closely related by Equation (16). The condition for the existence of convergent solution to the inhomogeneous GS equation (19) imposes another constraint between \( \Omega \) and \( H \), i.e., Equation (27). With these two equations, \( \Omega \) and \( H \) could be explicitly determined. It is clear that even we do
not get the explicit expression of the second-order solution $A_{\phi}^{(2)}$, we still self-consistently

determine all the valuable physical variables.

Based on the known the angular velocity and toroidal magnetic field of the collimated
jet, we further explore the physical properties of the jet solution, such as the jet power,
energy extraction rate. It is found that, given the magnetic field flux threading the black
hole horizon, the power of the highly collimated jet is about 30% lower than the standard
split monopole jet power and our solution is more consistent with recent GRMHD simulation
results.

The jet field lines are helically twisted by the black hole’s spin. It has been shown the
KS criterion for screw instability may not be valid for rotating jet stability analysis and
we adopt a new criterion proposed by [17]. We further study how the effects of field line
rotation influence the jet stability. We find the rotation tends to stabilize the jet.

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[1] R. D. Blandford and R. L. Znajek, MNRAS 179, 433 (1977).
[2] D. A. MacDonald, MNRAS 211, 313 (1984).
[3] V. S. Beskin, Y. N. Istomin, and V. I. Parev, SVA 36, 642 (1992).
[4] J. C. McKinney and C. F. Gammie, Astrophys. J. 611, 977 (2004), astro-ph/0404512.
[5] J. C. McKinney, ApJL 630, L5 (2005), astro-ph/0506367.
[6] J. C. McKinney, MNRAS 368, 1561 (2006), astro-ph/0603045.
[7] S. S. Komissarov, MNRAS 326, L41 (2001).
[8] S. S. Komissarov, MNRAS 336, 759 (2002), astro-ph/0202447.
[9] S. S. Komissarov, MNRAS 350, 1431 (2004), astro-ph/0402430.
[10] S. S. Komissarov, MNRAS 350, 427 (2004).
[11] R. F. Penna, R. Narayan, and A. Sadowski, MNRAS 436, 3741 (2013).
[12] L.-X. Li, *ApJL* 531, L111 (2000), astro-ph/0001420.

[13] M. Lyutikov, *New Journal of Physics* 8, 119 (2006), astro-ph/0512342.

[14] D. Giannios and H. C. Spruit, *A&A* 450, 887 (2006), astro-ph/0601172.

[15] J. C. McKinney and R. D. Blandford, *MNRAS* 394, L126 (2009), arXiv:0812.1060.

[16] J. C. McKinney, A. Tchekhovskoy, and R. D. Blandford, *Science* 339, 49 (2013), arXiv:1211.3651 [astro-ph.CO].

[17] A. Tomimatsu, T. Matsuoka, and M. Takahashi, *Phys. Rev. D* 64, 123003 (2001), astro-ph/0108511.

[18] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (1980).

[19] P. Ghosh, *MNRAS* 315, 89 (2000), astro-ph/9907427.

[20] R. L. Znajek, *MNRAS* 179, 457 (1977).

[21] T. Damour, in *Seventh Texas Symposium on Relativistic Astrophysics*, Annals of the New York Academy of Sciences, Vol. 262, edited by P. G. Bergman, E. J. Fenyves, and L. Motz (1975) pp. 113–122.

[22] V. Beskin, *MHD Flows in Compact Astrophysical Objects: Accretion, Winds and Jets*, 1st ed. (2009).

[23] S. S. Komissarov, *MNRAS* 359, 801 (2005), astro-ph/0501599.

[24] D. Alic, P. Moesta, L. Rezzolla, O. Zanotti, and J. L. Jaramillo, *Astrophys. J.* 754, 36 (2012), arXiv:1204.2226 [gr-qc].

[25] R. Narayan, J. Li, and A. Tchekhovskoy, *Astrophys. J.* 697, 1681 (2009), arXiv:0901.4775 [astro-ph.HE].