Investigating and improving student understanding of the probability distributions for measuring physical observables in quantum mechanics

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Abstract
A solid grasp of the probability distributions for measuring physical observables is central to connecting the quantum formalism to measurements. However, students often struggle with the probability distributions of measurement outcomes for an observable and have difficulty expressing this concept in different representations. Here we first describe the difficulties that upper-level undergraduate and PhD students have with the probability distributions for measuring physical observables in quantum mechanics. We then discuss how student difficulties found in written surveys and individual interviews were used as a guide in the development of a quantum interactive learning tutorial (QuILT) to help students develop a good grasp of the probability distributions of measurement outcomes for physical observables. The QuILT strives to help students become proficient in expressing the probability distributions for the measurement of physical observables in Dirac notation and in the position representation and be able to convert from Dirac notation to position representation and vice versa. We describe the development and evaluation of the QuILT and findings about the effectiveness of the QuILT from in-class evaluations.

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1. Introduction

Learning quantum mechanics is challenging partly due to the abstract and non-intuitive nature of the subject matter. In the most common interpretation of quantum mechanics, measurement outcomes are probabilistic and measurement collapses the state of the system to an eigenstate of the operator corresponding to the observable measured. The issues related to the probability distributions of measuring different physical observables such as energy or position for a system in a given quantum state are central in connecting the theoretical quantum formalism to measurements conducted in laboratories. Therefore, it is important for upper-level undergraduate and PhD students to develop proficiency with the probability distributions of measurement outcomes when different physical observables are measured in a given quantum state of the system. Indeed, several prior studies have found that many advanced students struggle with the foundational issues in quantum mechanics including measurement (e.g., see [1–12]). Studies have focused on diverse pedagogical approaches for helping students learn quantum mechanics better (e.g., see [13–26]), and visualization tools such as QuVIS and QuTIP have been developed to help students develop intuition about quantum mechanical phenomena [23, 24].

Furthermore, students must grapple with various representations when learning about probability distributions for measuring different physical observables. For example, concepts involving probability distributions for measuring observables can be expressed in Dirac notation and in the position representation without using Dirac notation. The ability to express a concept in different representations is a hallmark of expertize. Physics experts use multiple representations of concepts and have the ability to construct, interpret, and transform between different representations of knowledge. However, students often struggle to develop a robust functional understanding of these concepts and it is often challenging for students to express probability distributions for measurement outcomes in different forms and translate between different representations.

Dirac notation is an elegant, compact notation that is used commonly in undergraduate and PhD level quantum courses. Some curricula, such as the spins-first approach to quantum mechanics, introduce Dirac notation early on in a course instead of beginning with a ‘particle in a box’ approach [25]. It is possible that research-based learning tools that focus on the probability distributions for measuring physical observables in the context of Dirac notation can aid students in developing a robust understanding of these topics and help them translate their expressions for the probability distribution for the measurement of an observable from Dirac notation to position representation and vice versa. In order to help students learn these important concepts and procedures and be able to connect their conceptual and procedural knowledge, we first conducted research on students’ conceptual difficulties with probability distributions for the measurement of observables in a given quantum state of the system in Dirac notation and in position representation. Then, we used the research on students’ difficulties as a guide to develop a quantum interactive learning tutorial (QuILT) to help students become proficient in expressing the probability distributions for the measurement outcomes of observables in Dirac notation and position representation and be able to translate from Dirac notation to position representation and vice versa. The QuILT does not involve visualization tools but it does help students synthesize different notations when learning about probability distributions for the measurement outcomes of observables. It uses a guided
inquiry-based approach to learning and was developed using an iterative approach to development and assessment.

Below, we start with a brief background on quantum probability distributions for observables and how to express them in Dirac notation and position representation (which summarizes the concepts and procedures students should learn). We then describe the methodology for the investigation of students’ difficulties and categorize the difficulties found. Next, we discuss the development and assessment of the QuILT including data from upper-level undergraduate and PhD students suggesting that the QuILT was effective in improving students’ understanding of probability distributions of measurement outcomes.

2. Background

We first discuss the requisite knowledge we want students to have, e.g., related to the probability distribution for measuring position or energy, including the probability density amplitude and probability density for position measurement as shown in Table 1. For example, if a particle in a generic quantum state $|\Psi\rangle$ is confined in one spatial dimension and $|x\rangle$ is an eigenstate of the position operator $\hat{x}$ with eigenvalue $x$, the probability density amplitude for measuring position is the wavefunction in position representation $\Psi(x)$ (or $\langle x | \Psi \rangle$ when written in Dirac notation). The probability density for measuring the position of a particle in a given quantum state is $|\Psi(x)|^2$ (or $|\langle x | \Psi \rangle|^2$ when written in Dirac notation). The probability for measuring the particle between $x$ and $x + dx$ is the probability density multiplied by an infinitesimal length $dx$, i.e., $|\Psi(x)|^2 dx$ (or $|\langle x | \Psi \rangle|^2 dx$ written in Dirac notation). Furthermore, if a particle in a generic quantum state $|\Psi\rangle$ is confined to one-dimension and the stationary state wavefunctions are $\psi_n(x)$ (or $\langle x | n \rangle$ when written in Dirac notation) in which $|n\rangle$ is the nth stationary state in Dirac notation without choosing a representation) and the corresponding non-degenerate energy eigenvalues are $E_n$ (in which $n = 1, 2, 3 \ldots \infty$), the probability amplitude for measuring energy $E_n$ and collapsing the state to $|n\rangle$ in the position representation is $\int_{-\infty}^{+\infty} \psi_n^*(x) \Psi(x) dx$ (or equivalently, $\langle n | \Psi \rangle$ written in Dirac notation). Correspondingly, the probability for obtaining energy $E_n$ and collapsing the state to $|n\rangle$ in the position representation is $\int_{-\infty}^{+\infty} \psi_n^*(x) \Psi(x) dx$ (or $|\langle n | \Psi \rangle|^2$ in Dirac notation).

| Table 1. Summary of probability density amplitude, probability density, and probability distribution for measuring position or energy in Dirac notation and position representation. |
|--------------------------------------|--------------------------------------|
| Probability density amplitude for measuring position | $\langle x | \Psi \rangle$ |
| Probability density for measuring the position | $|\langle x | \Psi \rangle|^2$ |
| Probability for measuring the particle between $x$ and $x + dx$ | $|\Psi(x)|^2 dx$ |
| Probability amplitude for measuring energy $E_n$ | $\int_{-\infty}^{+\infty} \psi_n^*(x) \Psi(x) dx$ |
| Probability for measuring energy $E_n$ | $|\langle n | \Psi \rangle|^2$ |

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Table 2. Questions involving probability distributions for measuring energy and position (correct answers to multiple-choice questions are bolded) and the number of upper-level undergraduate (UG) students and PhD students (G) who were asked each question (N).

| Question | N  |
|----------|----|
| **Question Q1** A particle interacts with a one-dimensional infinite square well of width \( a \) (\( V(x) = 0 \) for \( 0 \leq x \leq a \) and \( V(x) = +\infty \) otherwise). The stationary state wavefunctions are \( \psi_n(x) = \frac{1}{\sqrt{a}} \sin(n\pi x/a) \) and the allowed energies are \( E_n = \frac{E_n^0}{2m} \) where \( n = 1, 2, 3 \ldots \infty \). The wave function at time \( t = 0 \) is \( \Psi(x, 0) = Ax(a - x) \) for \( 0 \leq x \leq a \), where \( A \) is a suitable normalization constant. Choose all of the following statements that are correct at time \( t = 0 \):

1. If you measure the position of the particle at time \( t = 0 \), the probability density for measuring \( x \) is \( |Ax(a - x)|^2 \).

2. If you measure the energy of the system at time \( t = 0 \), the probability of obtaining \( E_1 \) is \( \int_0^a |\psi_1^*(x)Ax(a - x)|^2 \, dx \).

3. If you measure the position of the particle at time \( t = 0 \), the probability of obtaining a value between \( x \) and \( x + dx \) is \( \int_x^{x+dx} |\Psi(x, 0)|^2 \, dx \).

A. 1 only B. 3 only C. 1 and 2 only D. 1 and 3 only E. All of the above

**Question Q2.** A particle interacts with a one-dimensional infinite square well of width \( a \) (\( V(x) = 0 \) for \( 0 \leq x \leq a \) and \( V(x) = +\infty \) otherwise). The stationary state wavefunctions are \( \psi_n(x) = \frac{1}{\sqrt{a}} \sin(n\pi x/a) \) and the allowed energies are \( E_n = \frac{E_n^0}{2m} \) where \( n = 1, 2, 3 \ldots \infty \). The wave function at time \( t = 0 \) is \( \Psi(x, 0) = (\psi_1(x) + \psi_2(x))/\sqrt{2} \). Choose all of the following statements that are correct at time \( t = 0 \):

1. If you measure the position of the particle at time \( t = 0 \), the probability density for measuring \( x \) is \( |(\psi_1(x) + \psi_2(x))/\sqrt{2}|^2 \).

2. If you measure the energy of the system at time \( t = 0 \), the probability of obtaining \( E_1 \) is \( \int_0^a |\psi_1^*(x)(\psi_1(x) + \psi_2(x))/\sqrt{2}|^2 \, dx \).

3. If you measure the position of the particle at time \( t = 0 \), the probability of obtaining a value between \( x \) and \( x + dx \) is \( \int_x^{x+dx} |\Psi(x, 0)|^2 \, dx \).

A. 1 only B. 3 only C. 1 and 2 only D. 1 and 3 only E. All of the above

**Question Q3.** For a spinless particle confined in one spatial dimension, the state of the quantum system at time \( t = 0 \) is denoted by \( |\Psi\rangle \). \( |x\rangle \) and \( |p\rangle \) are the eigenstates of position and momentum operators. Choose all of the following statements that are correct.
Table 2. (Continued.)

| Question | \( N \) |
|----------|--------|
| 1. \( |\Psi\rangle = \int \langle p|\psi(p)dp \) |  |
| 2. \( |\Psi\rangle = \int \psi(x)|x\rangle dx \) |  |
| 3. If you measure the position of the particle in the state \(|\Psi\rangle\), the probability of finding the particle between \(x\) and \(x + dx\) is \(|\langle x|\psi\rangle|^2 dx\). |  |

A. 1 only  
B. 1 and 2 only  
C. 1 and 3 only  
D. 2 and 3 only  
E. all of the above

**Question Q4.** Suppose \(|\Psi\rangle\) is a generic state and the energy eigenstates \(|n\rangle\) are such that \(H|n\rangle = E_n|n\rangle\), where \(n = 1, 2, 3 \ldots \infty\). Choose all of the following statements that are correct.

1. \( |\Psi\rangle = \sum_n \langle n|\Psi\rangle |n\rangle \)
2. \( e^{-i\hat{H}/\hbar}|\Psi\rangle = \sum_n e^{-iE_n\hbar} \langle n|\Psi\rangle |n\rangle \)
3. If you measure the energy of the system in the state \(|\Psi\rangle\), the probability of obtaining \(E_n\) and collapsing the state to \(|n\rangle\) is \(|\langle n|\Psi\rangle|^2\).

A. all of the above  
B. 1 and 2 only  
C. 1 and 3 only  
D. 2 and 3 only  
E. 3 only

**Question Q5.** An operator \(\hat{Q}\) corresponding to a physical observable \(Q\) has a continuous non-degenerate spectrum of eigenvalues. The states \(|q\rangle\) are eigenstates of \(\hat{Q}\) with eigenvalues \(q\). At time \(t = 0\), the state of the system is \(|\Psi\rangle\). Choose all of the following statements that are correct.

1. A measurement of the observable \(Q\) must return one of the eigenvalues of the operator \(\hat{Q}\).
2. If you measure \(Q\) at time \(t = 0\), the probability of obtaining an outcome between \(q\) and \(q + dq\) is \(|\langle q|\Psi\rangle|^2 dq\).
3. If you measure \(Q\) at time \(t = 0\), the probability of obtaining an outcome between \(q\) and \(q + dq\) is \(\int_{-\infty}^{\infty} \psi_q^* (x) \psi(x) dx \int_{-\infty}^{\psi_q(x)} dq\) in which \(\psi_q(x)\) and \(\psi(x)\) are the wavefunctions corresponding to states \(|q\rangle\) and \(|\Psi\rangle\) respectively.

A. 1 only  
B. 1 and 2 only  
C. 1 and 3 only  
D. 2 and 3 only  
E. all of the above

**Question Q6.** Write an expression for the probability that a generic state \(|\Psi\rangle\) will collapse into an eigenstate \(|\psi_i\rangle\) of \(\hat{Q}\) upon measurement of the observable \(Q\), given that \(\hat{Q}|\psi_i\rangle = \lambda_i|\psi_i\rangle\), where \(i = 1, 2, 3 \ldots N\).

**Question Q7.** Write an expression for the probability of measuring observable \(Q\) in the interval between \(q\) and \(q + dq\) in the state \(|\Psi\rangle\), given...
Similar to the basics of probability distributions for the measurement outcomes for position and energy summarized in table 1, students should be able to represent the probability distributions for the measurement outcomes of any physical observable in an analogous manner, whether the eigenvalue spectrum of its corresponding operator is discrete or continuous. Since learning about the probability distributions for the measurement outcomes of different physical observables in a given quantum state is critical for being able to connect the abstract quantum mechanics formalism with measurement, students should develop procedural proficiency in determining the probability distributions of measurement outcomes as well as a conceptual understanding of these issues. Therefore, we investigated the conceptual and procedural difficulties (which were often intertwined) that students in upper-level undergraduate and PhD level courses have with the probability distributions for measuring observables via responses to free-response and multiple-choice questions and by conducting individual interviews with some student volunteers. Below, we describe the methodology for the investigation of student difficulties and the development of a QuILT. We also discuss how the research on student difficulties informed the development of the QuILT, which strives to help students learn these concepts better. We also describe findings from the in-class evaluations that suggest that the QuILT was effective in helping students learn these concepts.

### 3. Methodology for the investigation of student difficulties

Both before and during the preliminary development of the QuILT, we investigated the conceptual difficulties students have with issues related to the probability distributions for measuring different physical observables in quantum mechanics in order to effectively address them. Student difficulties were investigated by administering written free-response and multiple-choice questions to students after traditional lecture-based instruction in relevant concepts. Questions were administered to upper-level undergraduate and PhD level courses have with the probability distributions for measuring observables via responses to free-response and multiple-choice questions and by conducting individual interviews with some student volunteers. Below, we describe the methodology for the investigation of student difficulties and the development of a QuILT. We also discuss how the research on student difficulties informed the development of the QuILT, which strives to help students learn these concepts better. We also describe findings from the in-class evaluations that suggest that the QuILT was effective in helping students learn these concepts.

#### Table 2. (Continued.)

| Question | $N$ |
|----------|-----|
| that $|\psi\rangle$ is a generic state of a quantum system and the states $\{|q\rangle\}$ are eigenstates of $\hat{Q}$ with continuous eigenvalues $q$. |     |
Questions Q3 and Q4 involve concepts relevant for the probability distributions for measuring position, momentum, or energy in Dirac notation. Question Q5 involves the probability distribution for measuring a generic physical observable \( Q \) in both the position representation and Dirac notation. The free-response question Q6 is posed in Dirac notation and evaluates student understanding of the probability distribution for measuring an observable \( Q \) with a discrete eigenvalue spectrum. The free-response question Q7 is posed in Dirac notation and involves the probability distribution for measuring an observable \( Q \) with a continuous eigenvalue spectrum.

We also conducted individual interviews with 23 students. The individual interviews employed a think-aloud protocol to better understand the rationale for students’ written responses. During these semi-structured interviews, students were asked to ‘think aloud’ while answering questions. They first read the questions and reasoned about them without interruptions except that they were prompted to think aloud if they were quiet for a long time. After students had finished answering a particular question to the best of their ability, we asked them to further clarify and elaborate issues that they had not clearly addressed on their own. Below, we describe some of the common difficulties found in written responses and interviews and also point out cases in which students were proficient.

4. Student difficulties

**Difficulty identifying expressions for the probability distribution of measuring the position of a particle in a narrow range between \( x \) and \( x + dx \).** Questions Q1 and Q2 are analogous because the initial quantum states \( \Psi(x, 0) \) in both questions are not energy eigenstates and the methods for finding the probability density for measuring position are similar. In question Q1, the probability density for measuring the position \( x \) is \( |\Psi(x, 0)|^2 = |Ax(a - x)|^2 \) and the probability for finding the particle between the positions \( x \) and \( x + dx \) is \( |\Psi(x, 0)|^2 dx = |Ax(a - x)|^2 dx \). On question Q2, the probability density for measuring \( x \) is \( |\Psi(x, 0)|^2 = \left| \left( \psi_1(x) + \psi_2(x) \right) / \sqrt{2} \right|^2 \) and the probability for measuring the particle between \( x \) and \( x + dx \) is \( |\Psi(x, 0)|^2 dx = \left| \left( \psi_1(x) + \psi_2(x) \right) / \sqrt{2} \right|^2 dx \). Table 3 shows that, on Questions Q1 and Q2, approximately 80% of the students recognized that the probability density for measuring \( x \) is \( |Ax(a - x)|^2 \) (or \( \left| \left( \psi_1(x) + \psi_2(x) \right) / \sqrt{2} \right|^2 \)) and selected statement 1 as correct (they chose answer option A, C, D, or E). However, since statement 1 is in four out of five answer choices, there was a \( 4/5 = 80\% \) chance of guessing correctly so student performance on this concept is at the level of random guessing. Table 3 shows that, on

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Question} & \% & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \\
\hline
\text{Q1} & & 16\% & 12\% & 17\% & 29\% & 26\% \\
\text{Q2} & & 19\% & 14\% & 14\% & 29\% & 24\% \\
\text{Q3} & & 10\% & 10\% & 27\% & 22\% & 31\% \\
\text{Q4} & & 51\% & 10\% & 17\% & 8\% & 13\% \\
\text{Q5} & & 14\% & 10\% & 26\% & 8\% & 28\% \\
\text{Q6} & & 23\% & & & & \\
\text{Q7} & & 10\% & & & & \\
\hline
\end{array}
\]
questions Q1 and Q2, 67% of the students incorrectly claimed that the probability of finding the particle between \(x\) and \(x + dx\) is \(\int_{x}^{x+dx} x|\Psi(x, 0)|^2dx\) and selected statement 3 as correct (chose answer option B, D, or E). Interviews suggest that some students who thought that statement 3 in questions Q1 and Q2 was correct confused the probability of measuring position of the particle between \(x\) and \(x + dx\) with the expectation value of position (although the integral in statement (3) is not from \(x = 0\) to \(x = a\), necessary for the expectation value). Similar difficulties have been found in previous research [6]. For example, in a previous study, a student who wrote that the probability for measuring the particle between \(x\) and \(x + dx\) is \(\int_{x}^{x+dx} x|\Psi|^2dx\) stated that \(|\Psi|^2\) gives the probability of the wavefunction being at a given position and if you multiply it by \(x\) you get the probability of measuring the position \(x'\) [6]. This student, and others, had difficulty distinguishing between the related concepts of probability and expectation value.

**Difficulty identifying expressions for the probability of measuring position of a particle between \(x\) and \(x + dx\) in Dirac notation.** In contrast to questions Q1 and Q2, Question Q3 focuses on concepts involving the probability distribution of measurement outcomes for position measurement (or momentum measurement) in Dirac notation. For example, Question Q3 asks students to evaluate the correctness of statement (3): ‘if you measure the position of the particle in the state \(|\Psi\rangle\), the probability of finding the particle between \(x\) and \(x + dx\) is \(|\langle x|\Psi\rangle|^2dx\)’. Table 3 shows that 80% of the students correctly recognized that the statement is true and selected an answer that included statement (3) (i.e., options C, D, or E). In addition, table 3 shows that on Q3, 63% of the students correctly recognized that the expansion of a generic state \(|\Psi\rangle\) in terms of the eigenstates of position \(|x\rangle\) is \(|\Psi\rangle = \int \Psi(x)|x\rangle dx\) and selected an answer that included statement (2) (i.e., options B, D or E). However, since statement 2 is in three out of the five answer choices, there was a 3/5 = 60% chance of guessing correctly and student performance on the expansion of a generic state \(|\Psi\rangle\) in terms of the eigenstates of position is almost at the level of random guessing. Having the ability to express \(|\Psi\rangle\) as \(|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x)|x\rangle dx\) can be helpful when answering questions related to the probability distribution of measurement outcomes of position measurements since \(|\Psi(x)\rangle\) is the probability density amplitude for measuring position. Similarly, having the ability to express \(|\Psi\rangle\) as \(|\Psi\rangle = \int \langle p|\Psi\rangle |p\rangle dp\) as in statement (1) on Q3 can be helpful when answering questions about the probability distribution of measurement outcomes for momentum since \(\langle p|\Psi\rangle\) is the probability density amplitude for measuring momentum \(p\).

**Difficulty identifying the probability of measuring energy \(E_1\) in position representation.** Using the notations in the problems students were given, the probability of measuring energy \(E_1\) given an initial wave function \(\Psi(x, 0)\) of the system in the position representation is \(\int_{-\infty}^{\infty} \psi_{n}^*(x)\Psi(x, 0)dx\). In question Q1, the probability for measuring energy \(E_1\) is \(\int_{0}^{a} \psi_{1}^*(x)(\psi_{1}(x) - \Delta x) dx\) and in question Q2, the probability for measuring energy \(E_1\) is \(\int_{0}^{a} \psi_{2}^*(x)((\psi_{1}(x) + \psi_{2}(x))\sqrt{2}) dx\). Table 3 shows that, on questions Q1 and Q2, only approximately 40% of students selected statement 2 as the correct expression for the probability of measuring energy \(E_1\) and chose answer option C or E. On question Q2, the initial wave function is \(\Psi(x, 0) = (\psi_{1}(x) + \psi_{2}(x))/\sqrt{2}\). An expert would immediately recognize that the probability of measuring \(E_1\) is \(1/2\) and can be obtained using \(\int_{0}^{a} \psi_{1}^*(x)((\psi_{1}(x) + \psi_{2}(x))/\sqrt{2}) dx\) due to the orthogonality of energy eigenstates.
Interviews suggest that even students who recognized that the probability of measuring energy $E_i$ is $\frac{1}{2}$ for this wave function, which is an equal superposition of ground and first excited states, did not recognize that the integral in statement 2 in question Q2 gives the component of the quantum state along the ground state and is related to the energy measurement amplitude. Previous research has shown that when students were asked to identify the correct expression for the probability of measuring energy $E_n$ for the wavefunction $\Psi(x,0) = A x (a-x)$ for a particle in a one-dimensional infinite square well of width $a$, approximately half of the students selected an incorrect answer option that included the Hamiltonian operator $\hat{H}$, i.e., $\int_0^a \psi^*_n(x) \hat{H} \psi_n(x,0) dx$ [6, 7]. Interviews suggest that the fact that few students were able to recognize the correct expression for the probability of measuring energy $E_i$ in questions Q1 and Q2 may partly be due to the fact that they thought that the expression for probability of measuring energy $E_i$ should include the Hamiltonian operator $\hat{H}$.

**Difficulty identifying expressions for the probability of measuring energy in Dirac notation.** In contrast to questions Q1 and Q2, Question Q4 involves identifying the probability distribution for measuring a particular value of energy in Dirac notation. Table 3 shows that, on question Q4, 89% of the students recognized that if the energy of the system in the state $|\Psi\rangle$ is measured, the probability of obtaining $E_n$ and collapsing the state to $|n\rangle$ is $|\langle n|\Psi\rangle|^2$ and selected an answer that included statement (3) (i.e., options A, C, D, or E). However, since statement (3) is in four out of the five answer choices, there was a $4/5 = 80\%$ chance of guessing correctly and student performance on identifying the probability of obtaining $E_n$ is only somewhat higher than the level of random guessing. In addition, 69% of the students correctly recognized that the expansion of a generic state $|\Psi\rangle$ in terms of the energy eigenstates $|n\rangle$ is $|\Psi\rangle = \sum_n \langle n|\Psi\rangle |n\rangle$ and selected an answer that included statement (1) (i.e., options A, B, or D). However, since statement (1) is in three out of the five answer choices, there was a $3/5 = 60\%$ chance of guessing correctly and student performance on the expansion of a generic state $|\Psi\rangle$ in terms of the energy eigenstates is only somewhat higher than the level of random guessing.

**Difficulty with identifying expressions for the probability of measuring a generic observable in Dirac notation and position representation in the same question.** Question Q5 involves statements about the probability of measuring a generic observable $Q$ in both Dirac notation and position representation. Students were told that a generic operator $\hat{Q}$ corresponding to a physical observable $Q$ has a continuous, non-degenerate spectrum of eigenvalues and the states $|q\rangle$ are eigenstates of $\hat{Q}$ with eigenvalues $q$. In Dirac notation, the probability of obtaining an outcome between $q$ and $q + dq$ is $|\langle q|\Psi\rangle|^2 dq$. In position representation, the probability of obtaining an outcome between $q$ and $q + dq$ is $\int_{-\infty}^{\infty} \psi^*_q(x) \Psi(x) dx dq$ in which $\psi_q(x)$ and $\Psi(x)$ are the wavefunctions in position representation corresponding to states $|q\rangle$ and $|\Psi\rangle$, respectively. Table 3 shows that, on question Q5, 58% of the students correctly recognized that the probability of obtaining an outcome between $q$ and $q + dq$ in Dirac notation is $|\langle q|\Psi\rangle|^2 dq$ and selected answer option B, D, or E (which is at the level of random guessing). In addition, 60% of the students recognized that the probability of obtaining an outcome between $q$ and $q + dq$ in the position representation is $\int_{-\infty}^{\infty} \psi_q^*(x) \Psi(x) dx dq$ and selected answer option C, D, or E (which is at the level of random guessing). Furthermore, only 32% of the students selected an answer choice that included both expressions $|\langle q|\Psi\rangle|^2 dq$ and $\int_{-\infty}^{\infty} \psi_q^*(x) \Psi(x) dx dq$ (i.e., answer options D or E). Many students did not
realize that the expressions \(|\langle q|\Psi\rangle|^2dq\) and \(\int_{-\infty}^{\infty} \psi^n_\phi(x)\Psi(x)dx \int^2 dq\) both denote the probability of obtaining an outcome between \(q\) and \(q + dq\).

**Difficulty in expressing the probability distribution for measurement outcomes of a generic observable \(Q\) (with corresponding operator \(\hat{Q}\)).** Both questions Q6 and Q7 in table 2 focus on probability distributions for measurement outcomes. Table 3 shows that on the free-response question Q6, only 23% of the undergraduate students were able to write a correct expression for the probability of obtaining an eigenvalue \(\lambda\) of an observable \(Q\) with discrete eigenvalues, i.e., \(|\langle \psi|\lambda\rangle|^2\) on question Q6. Furthermore, on the free response question Q7 that evaluates student understanding of the probability distribution for measuring an observable \(Q\) whose corresponding operator has continuous eigenvalues, table 3 shows that only 10% of undergraduates and 45% of PhD students wrote a correct expression for the probability of obtaining an outcome between \(q\) and \(q + dq\), i.e., \(|\langle q|\Psi\rangle|^2dq\). In both the discrete and continuous cases, the common difficulty involved students writing an expression for probability that explicitly involved the operator \(\hat{Q}\). On question Q6, 23% of the students wrote an expression that involved the operator \(\hat{Q}\) and on question Q7, 32% of the undergraduate students and 32% of the PhD students wrote an expression that involved the operator \(\hat{Q}\). Students who wrote the operator \(\hat{Q}\) in their expressions for probability usually either (1) wrote an expression that resembled an expectation value of the operator \(\hat{Q}\), or (2) wrote an expression that involved \(\hat{Q}\) acting on the generic state |\(\Psi\)\(\rangle\) (but did not resemble an expectation value).

On questions Q6 and Q7, students who wrote an expression for the probability that resembled an expectation value usually wrote that the probability is \(\langle \psi|\hat{Q}|\psi\rangle = \langle \psi|\lambda|\psi\rangle\). Some students also incorrectly wrote \(\langle \Psi|\hat{Q}|\Psi\rangle\). In response to question Q7 that involved the continuous case, some students wrote that the probability for measuring the observable \(Q\) in the interval between \(q\) and \(q + dq\) is \(|\langle \psi|\hat{Q}|\psi\rangle|^2dq\). In think-aloud interviews, many students had difficulty distinguishing between the probability of measuring a particular value of an observable in a given state and the expectation value. Previous research has shown that when students were asked to find an expression for the probability of measuring \(E_n\) in the general state |\(\psi_n\)\(\rangle\), students often wrote \(\langle \psi_n|\hat{H}|\psi_n\rangle\) or \(\langle \psi|\hat{H}|\psi\rangle\) as the probability of measuring \(E_n\). When these students were explicitly asked to compare their expressions for the probability of measuring a particular value of energy and the expectation value of energy, some students appeared concerned. They recognized that the probability and expectation value were different, but they still struggled to distinguish between these concepts. They could not write an expression for the probability of measuring \(E_n\) either using the Dirac notation or in the position representation [6, 7]. Difficulties of this type indicate that students often struggle to differentiate between the concepts of probability and expectation value which can lead to attempts to apply concepts in inappropriate situations.

In response to question Q6 or Q7, some students wrote an expression that involved \(\hat{Q}\) acting on the generic state |\(\Psi\)\(\rangle\) that did not resemble an expectation value. For example, some students wrote that the probability of measuring the observable \(Q\) in the interval between \(q\) and \(q + dq\) was, e.g., \(\hat{Q}|\Psi\rangle, \hat{Q}|\Psi\rangledq, \int_{q}^{q+dq}\langle \hat{Q}|\Psi\rangle dq\), or \(\int_{q}^{q+dq}|\hat{Q}|\Psi\rangle|^2 dq\). Previous research has shown that students often think that the operator should appear in the expression for calculating probability of measuring a particular eigenvalue of the operator [6, 7]. Some interviewed students who incorrectly claimed that the operator should appear in their expression for calculating probability made conceptual arguments claiming that, in quantum mechanics, an operator acting on a generic state must yield an eigenvalue of the corresponding operator. These students often stated that an operator acting on a state corresponds to
the measurement of the corresponding observable and measurement collapses the state and gives one of the eigenvalues as the measured value, i.e., \( \hat{O} |\psi\rangle = q \). In another study, students claimed that the position operator \( \hat{x} \) acting on a position eigenfunction \( \delta(x - x') \) corresponds to a measurement of position and must yield a position eigenvalue \( x' \) according to the postulates of quantum mechanics \([10]\). A similar difficulty has been found in the context of the Hamiltonian operator \( \hat{H} \) acting on a generic state \( |\psi\rangle \) or an energy eigenstate \( |\psi_n\rangle \)—many students incorrectly claim that \( \hat{H} |\psi\rangle = E_n \) or \( \hat{H} |\psi_n\rangle = E_n \) \([6, 7]\) because \( \hat{H} \) acting on its eigenstate corresponds to the measurement of energy. The students who incorrectly claimed that \( \hat{H} |\psi_n\rangle = E_n |\psi_n\rangle \), similar to students who incorrectly wrote \( \delta(x - x') = x' \delta(x - x') \) and did not realize that \( \delta(x - x') = x' \delta(x - x') \).

In summary, written responses and individual interviews suggest that students have many common difficulties identifying and expressing the probability distributions for measuring physical observables after traditional instruction in relevant concepts. Many students had great difficulty in identifying an expression for probability in Dirac notation and the position representation. Within the same multiple-choice question (question Q5 in table 2) that asked students to identify expressions for the probability for measurement of a generic observable \( Q \) in Dirac notation and position representation, less than half of the students recognized that the expressions for probability, i.e., \( \langle q |\psi\rangle^2 dq \) and \( \int_{-\infty}^{\infty} \psi_n^*(x) \psi(x) dx \int_{-\infty}^{\infty} dq \), are equivalent. Furthermore, students had difficulty writing an expression for probability in Dirac notation in a free-response format. Many of the difficulties stem e.g., from an inability to distinguish between related concepts such as the expectation value and the probability distribution of measurement outcomes, and an incorrect conception that the expressions for the probability distributions of measuring physical observables should involve the corresponding quantum mechanical operators.

5. QuILT development

5.1. Development and validation of the QuILT

The difficulties described in the previous section indicate that students struggled with the probability distributions for measuring observables both in the position representation and Dirac notation after traditional instruction in relevant concepts. Therefore, we developed a QuILT that takes into account the common difficulties found, makes use of Dirac notation, and strives to help students develop a better grasp of these concepts. The research-validated QuILT is inspired by a model of student learning centered on Vygotsky’s notion of the ‘zone of proximal development’ (ZPD). The ZPD refers to the zone defined by the difference between what a student can do on his/her own and what a student can do with the help of an instructor who is familiar with his/her prior knowledge and skills \([27]\). Providing scaffolding support is at the heart of this ZPD model and can be used to stretch students’ learning beyond their current knowledge using carefully crafted learning tools that provide scaffolding support. Furthermore, a cognitive task analysis of the underlying concepts from an expert’s perspective \([28]\) was also used as a guide to develop the QuILT. The cognitive task analysis involves a careful analysis of the underlying concepts in the order in which those concepts should be invoked and applied in each situation to accomplish a task (i.e., answer the quantum physics questions in our case). The QuILT actively engages students in the learning process using a guided inquiry-based approach in which various concepts build on each other. It strives to provide appropriate scaffolding support to students in order to help them remain
in the ZPD. The QuILT can be used in upper-level undergraduate and graduate-level quantum mechanics courses after students have had instruction in relevant topics.

The development of the QuILT went through a cyclic, iterative process which included the following stages before the in-class implementation.

1. Development of the preliminary version based on a cognitive task analysis of the underlying knowledge and research on student difficulties with relevant concepts.
2. Implementation and evaluation of the QuILT by administering it individually to students and obtaining feedback from faculty members who are experts in these topics.
3. Determining its impact on student learning and assessing what difficulties were not adequately addressed by the QuILT based upon the feedback obtained.
4. Refinements and modifications based on the feedback from the implementation and evaluation.

In addition to written free-response and multiple-choice questions administered to students in various classes, we conducted individual interviews with 23 student volunteers. The interviews used a think-aloud protocol to better understand the rationale for their responses throughout the development of various versions of the QuILT and the development of the corresponding pretest and posttest given to students before and after they engaged in learning via the QuILT. After each individual interview with a particular version of the QuILT (along with the administration of the pretest and posttest), modifications were made based upon the feedback obtained from the interviewed students. For example, if students got stuck at a particular point and could not make progress from one question to the next with the scaffolding already provided to them, suitable modifications were made to the QuILT. Thus, the administration of the QuILT to several PhD students and upper-level undergraduate students individually was used to ensure that the guided inquiry-based approach was effective and the questions were unambiguously interpreted. The QuILT was also iterated several times with three faculty members and two graduate students who conduct physics education research to ensure that the content and wording of the questions were appropriate. Modifications were made based upon their feedback. The QuILT strives to provide enough scaffolding to allow students to build a robust knowledge structure while keeping them engaged in the learning process. When we found that the QuILT was working well in individual administration and the posttest performance was significantly improved compared to the pretest performance, it was administered to upper-level undergraduates and PhD students in various classes.

5.2. Structure of the QuILT

The QuILT includes a pretest to be administered right after traditional instruction on the relevant concepts but before students engage with the QuILT and a posttest to be administered after students work on the QuILT. The pretest is not returned to students but the posttest is returned to them after grading. The questions on the pretest and posttest are in free-response format. The free-response format requires that students generate answers based upon a robust understanding of the topics as opposed to memorization. The QuILT begins with a ‘warm-up’ that builds on students’ prior knowledge about a vector in a physical three-dimensional vector space they are familiar with from introductory physics and helps them make connections between a force vector \( \mathbf{F} \) in a physical three-dimensional vector space and a quantum state vector \( |\psi\rangle \) in an abstract vector space. Then, students learn about the basics of Dirac notation including scalar products before learning about the expansion of a state using a complete set of eigenstates of an operator corresponding to different observables, probability distributions for measurement of observables and expectation values of observables in a given quantum
state, projection operators, and completeness relations. The last section of the QuILT focuses on connecting Dirac notation with position and momentum representations. In the QuILT, students first learn about probability distributions for the measurement of observables and other relevant concepts in the context of a generic operator $\hat{Q}$ corresponding to a physical observable $Q$. Then, students are guided to generalize concepts they learned about in the generic context to the concrete contexts of position, momentum and energy measurements.

The QuILT strives to help students remain in the ZPD by explicitly bringing out common conceptual difficulties and then providing appropriate scaffolding to help students develop a coherent understanding. Throughout the QuILT, students select an answer based on their understanding up to that point and then are given opportunities to check the answer via follow up questions and discussions with a peer. If a student’s answer is inconsistent with the correct answer, further scaffolding is provided throughout the QuILT to ensure that students remain in the ZPD. The QuILT is best used in class to give students an opportunity to work together in small groups and discuss their thoughts with peers, which provides peer learning support. However, students can be asked to work on the parts they could not finish in class at home as homework or students can be asked to work on the entire QuILT as a self-paced learning tool so long as the pretest and posttest are administered in class. Below, we give some typical examples of how some of the common difficulties found via research are incorporated as resources and how student learning is scaffolded via the QuILT.

5.3. Addressing student difficulties with the probability distributions of measurement outcomes of observables via the QuILT

Addressing the difficulty with recognizing the probability of measuring position in a narrow range between $x$ and $x + dx$ in the position representation. As noted earlier, many students had difficulty recognizing the probability distribution for measuring position and sometimes incorrectly claimed that the probability of measuring position in a narrow range between $x$ and $x + dx$ was $\int_{x}^{x+dx} x |\Psi(x, 0)|^2 dx$. In addition, some students had difficulty identifying that the probability density for measuring position is $|\Psi(x)|^2$ and the expansion of a generic state $|\Psi\rangle$ in terms of a complete set of position eigenstates $|x\rangle$ is $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x)|x\rangle dx$. Having the ability to write $|\Psi\rangle$ as $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x)|x\rangle dx$ can be beneficial when answering questions related to the probability distribution for position measurements so long as students learn to make sense of the relevance of this expansion and what the expansion coefficients $\Psi(x)$ mean. In particular, students should learn that in order to find the probability distribution for the measurement outcomes of an observable in a state $|\Psi\rangle$ they should expand the state in terms of a complete set of orthonormal eigenstates of the operator corresponding to the observable measured. Then, the expansion coefficients will be related to the probability distribution for the measurement of the observable. The following questions are part of a guided inquiry-based sequence in the QuILT that strives to help students learn about the probability density and probability in the context of position measurement and build on students’ prior knowledge and difficulties found during the investigation of difficulties.

Consider the following conversation between three students:

- **Student A**: How does the expansion of $|\Psi\rangle$ in terms of a complete set of position eigenstates, $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x)|x\rangle dx$, help you in questions involving the measurement of the position of the particle?
Student B: $|\Psi(x_0)|^2 dx = |\langle x_0 | \Psi \rangle|^2 dx$ gives us the probability of finding the particle in a narrow range between $x_0$ and $x_0 + dx$ when we measure the position of the particle.

Student C: But I thought that the probability of finding the particle in a narrow range between $x_0$ and $x_0 + dx$ is \( \int_{x_0}^{x_0 + dx} x |\Psi(x)|^2 dx \).

Student A: So is the probability of finding the particle in a narrow range between $x_0$ and $x_0 + dx$ represented mathematically as \( \int_{x_0}^{x_0 + dx} x |\Psi(x)|^2 dx = x_0 |\Psi(x_0)|^2 dx \)?

Student B: No. It is just $|\Psi(x_0)|^2 dx$ not $x_0 |\Psi(x_0)|^2 dx$. $|\Psi(x_0)|^2$ is the probability density at position $x_0$. We multiply $|\Psi(x_0)|^2$ by a width $dx$ to obtain the probability of finding the particle in a narrow range between $x_0$ and $x_0 + dx$.

Do you agree with Student B’s explanation? Explain your reasoning.

Write the probability of finding the particle with a momentum between $p_0$ and $p_0 + dp$ when we measure the momentum of the particle in state $|\Psi\rangle$.

While working on these questions, students can discuss with peers and articulate their reasons for agreeing with Student A or Student B, which can aid learning. After this question, further scaffolding support is provided to ensure that students are in the ZPD. They are asked to check whether their responses are consistent with follow up questions, reconcile possible differences between their initial responses and the correct concepts, and build a coherent understanding of the concepts.

Addressing the difficulty with the probability for measuring energy $E_i$ in the position representation. In questions Q1 and Q2 in table 2, students had difficulty identifying a correct expression for the probability of measuring energy in the position representation \( \left( \text{i.e., } \left| \int_0^a \psi^* l(x) (a - x) dx \right|^2 \right) \) or \( \left| \int_0^a \psi^* l(x) (\psi_1(x) + \psi_2(x)) \sqrt{2} \right|^2 \) as shown in table 3. The QuILT first strives to help students learn about how the probability of measuring a particular observable for the quantum system in a generic state $|\Psi\rangle$ is related to the expansion coefficients of the state vector $|\Psi\rangle$ in an appropriate basis consisting of the eigenstates of the corresponding operators, e.g., if a generic state vector $|\Psi\rangle$ is expanded in terms of energy eigenstates $|\Psi\rangle = \sum_n |n\rangle \langle n| \Psi\rangle$, the probability for measuring energy $E_n$ is the absolute square of the expansion coefficient, i.e., $|\langle n| \Psi\rangle|^2$. After the students learn to make sense of why such an expansion is useful, the QuILT helps students develop facility in expanding a generic state vector $|\Psi\rangle$ in a chosen basis and internalize that representing a generic state vector $|\Psi\rangle$ in a suitable basis can help them determine the probability distributions for the measurement outcomes of observables. For example, they learn that one can expand a generic state vector $|\Psi\rangle$ in terms of a complete set of the position eigenstates as $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x) |x\rangle dx$ where $\Psi(x) = \langle x| \Psi\rangle$ is the probability density amplitude for the position measurement. Similarly, using the expansion $|\Psi\rangle = \sum_n \langle n| \Psi\rangle |n\rangle$, the probability of measuring a particular energy $E_n$ when the quantum system is in state $|\Psi\rangle$ would be $|\langle n| \Psi\rangle|$. Then, students are guided to learn how to insert an identify operator $I$ written in terms of a complete set of position eigenstates $|x\rangle$, i.e., $I = \int_{-\infty}^{\infty} |x\rangle \langle x| dx$, in order to write the probability of a particular energy measurement $|\langle n| \Psi\rangle|^2$ in the position representation (i.e., $|\langle n| \Psi\rangle|^2 = \int_{-\infty}^{\infty} \langle n| \Psi(x) \rangle \langle x| dx \right|^2 = \int_{-\infty}^{\infty} \psi_n^*(x) \Psi(x) \sqrt{2} \right|^2$ in which $\psi_n (x) = \langle x| n\rangle$ is the nth energy eigenstate in the position representation).

Addressing the difficulty with discerning the underlying similarity between different cases including those in which the eigenvalue spectrum of the operator corresponding to the
observable is continuous versus discrete. The following two questions in the QuILT first help students become proficient with the concept of the probability distribution of measurement outcomes of an observable whose corresponding operator has a discrete eigenvalue spectrum by having students reflect upon the coefficients in the expansion of a generic state $|\Psi\rangle$ in the context of a generic operator $\hat{Q}$ with orthonormal eigenstates $|q_n\rangle$, $n = 1, 2, 3 \ldots N$ and discrete eigenvalues $q_n$.

1. Earlier you learned that any vector in the Hilbert space can be expanded as a linear superposition of a complete set of eigenstates of a Hermitian operator $\hat{Q}$. For an operator $\hat{Q}$ with eigenstates $|q_n\rangle$, $n = 1, 2, 3 \ldots N$ (which form an orthonormal basis for an $N$ dimensional vector space) and discrete eigenvalues $q_n$, choose all of the following statements that are correct about the coefficients $C_n$ in the expansion $|\Psi\rangle = \sum_n C_n |q_n\rangle$.

(I) To find $C_n$ we take the scalar product of both sides of the equation with an eigenstate $|q_n\rangle$. Then, $\langle q_n | \Psi \rangle = \sum_n C_n \langle q_n | q_n \rangle = \sum_n C_n \delta_{nn} = C_n$.

(II) To find $C_n$ we take the scalar product of both sides of the equation with an eigenstate $|q_n\rangle$. Then, $\langle q_n | \Psi \rangle = \sum_n C_n \langle q_n | q_n \rangle = \sum_n C_n$.

(III) The coefficient $C_n$ for a particular eigenstate $|q_n\rangle$ in the expansion $|\Psi\rangle = \sum_n C_n |q_n\rangle$ is related to the probability of measuring the corresponding eigenvalue $q_n$ when a measurement of observable $\hat{Q}$ is made in the state $|\Psi\rangle$.

(a) (I) and (II) only
(b) (I) and (III) only
(c) (I) and (II) only
(d) All of the above.

2. For an arbitrary physical observable $\hat{Q}$ with corresponding operator with a complete set of eigenstates $|q_n\rangle$ with discrete eigenvalues $q_n$ where $n = 1, 2, \ldots N$, write the probability of measuring eigenvalue $q_n$ as a result of a measurement of $\hat{Q}$ performed when the system is in the state $|\Psi\rangle$.

After this question, further scaffolding is provided to help students check their work and reconcile possible differences between their initial responses and the correct concepts.

The expansion of a state in terms of a complete set of eigenstates of the operator corresponding to the observable measured is helpful for finding the probability distributions of measurement outcomes regardless of whether the probability distribution is for an observable whose corresponding operator has a discrete or continuous eigenvalue spectrum. Therefore, the following two questions in the QuILT help students reflect upon how to generalize the expansion of a generic state vector $|\Psi\rangle$ as a linear superposition of a complete set of orthonormal eigenstates with discrete eigenvalues to that of the continuous case of the position eigenstates $|x\rangle$ chosen as the basis for expansion.

1. Consider a generic state vector $|\Psi\rangle = \sum_{n=1}^{\infty} a_n |q_n\rangle$, where $a_n = \langle q_n | \Psi \rangle$. Then, the state vector $|\Psi\rangle$ can be represented as a column vector like this: $|\Psi\rangle \doteq \begin{pmatrix} \langle q_1 | \Psi \rangle \\ \langle q_2 | \Psi \rangle \\ \vdots \\ \langle q_n | \Psi \rangle \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, where the $\doteq$ sign means that this is a representation of $|\Psi\rangle$ in the chosen basis, e.g., $\{ |q_n\rangle, n = 1, 2, \ldots \infty \}$. Consider the following conversation between two students about the situation where basis vectors are chosen to be position eigenstates $|x\rangle$ or momentum eigenstates $|p\rangle$, each of which have a continuous eigenvalue spectrum.
Student 1: We cannot write the state vector $|\Psi\rangle$ as a column vector if position eigenstates $|x\rangle$ are chosen as the basis vectors. State vector $|\Psi\rangle$ written as a linear superposition of position eigenstates $|x\rangle$ is $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x)|x\rangle dx$, where $\Psi(x) = \langle x|\Psi\rangle$. But since this expansion of $|\Psi\rangle$ involves an integral instead of a summation, we cannot write $|\Psi\rangle$ as a column vector with respect to the basis vectors $|x\rangle$.

Student 2: I disagree. Even though the expansion of $|\Psi\rangle$ is an integral instead of a sum, when we choose the position eigenstates $|x\rangle$ as the basis vectors, we can still envision $|\Psi\rangle$ as a column vector with respect to the basis vectors $|x\rangle$. Like this:

$$|\Psi\rangle = \begin{pmatrix} \langle x_1| \Psi \rangle \\ \langle x_2| \Psi \rangle \\ \vdots \\ \langle x_n| \Psi \rangle \end{pmatrix} = \begin{pmatrix} \Psi(x_1) \\ \Psi(x_2) \\ \vdots \\ \Psi(x_n) \end{pmatrix} = \Psi(x).$$

Student 1: But why do the $x$'s have indices? Don’t position eigenstates $|x\rangle$ have a continuous eigenvalue spectrum $x$, not a discrete eigenvalue spectrum?

Student 2: Yes, you are correct. Actually, you should think of $x_1 = \Delta x, x_2 = 2\Delta x \ldots$ etc, and take the limit as $\Delta x \to 0$. I was simply making an analogy with the discrete eigenvalue spectrum case. However, the best way to write $|\Psi\rangle$ when position eigenstates $|x\rangle$ are chosen as the basis vectors is as $\Psi(x)$, which is also called the position space wavefunction. $\Psi(x)$ is a column vector with position eigenvalues $x$ as a continuous index.

Do you agree with Student 2? Explain your reasoning.

2. Write the generic state vector $|\Psi\rangle$ as a column vector in the momentum representation when momentum eigenstates $|p\rangle$ are chosen as basis vectors.

Following these questions and peer discussion, an inquiry-based approach builds on student difficulties and strives to help students develop a coherent understanding of the probability distributions for measurement outcomes of observables whose corresponding operators have a discrete or continuous eigenvalue spectrum. The goal is to help students conceptually understand what the probability distribution of a measurement outcome means and how to compute it in a given situation.

Addressing the difficulty with identifying expressions for the probability distribution of measurement outcomes of a generic observable in Dirac notation and position representation in the same question. On question Q5, many students did not realize that the expressions $\int |q|^2 dq$ and $\int_{-\infty}^{\infty} \psi^*_q(x)\Psi(x)dx \int dq$ both denote the probability of obtaining an outcome between $q$ and $q + dq$ when an observable $Q$ is measured in state $|\Psi\rangle$. The following question is part of a guided-inquiry-based sequence in the QuILT that strives to help students learn to make sense of and become proficient in writing an expression for the probability density for measuring position in a narrow range between $x$ and $x + dx$ first in Dirac notation as $\langle x|\Psi\rangle^2$ and then without using Dirac notation (using the familiar Born interpretation) as follows.

Write each of these expressions (probability density amplitude, probability density, and probability of measuring position in a narrow range between $x$ and $x + dx$) in Dirac notation.

- $\Psi(x) =$
- $|\Psi(x)|^2 =$
- $|\Psi(x)|^2 dx =$
First, students learn how to connect the probability of measuring position in a narrow range between \( x \) and \( x + dx \) in Dirac notation with the expression in position representation \( |\psi(x)|^2 dx \) in the familiar context of Born’s interpretation of the wave function. Then, the QuILT uses a guided inquiry-based approach that strives to help students learn to generalize the results to write the probability of measuring observable \( Q \) between \( q \) and \( q + dq \) as a result of a measurement performed when the system is in the state \( |\Psi\rangle \), i.e., \( |\langle q|\Psi\rangle|^2 dq \). In particular, students are asked to use what they learned in the context of position measurement and generalize what they learned about the probability of measuring position (in Dirac notation and in position representation without using Dirac notation) to an arbitrary observable \( Q \) whose corresponding quantum operator has continuous eigenvalues \( q \) with the following type of questions.

For an arbitrary physical observable \( Q \) whose corresponding operator has eigenstates \( |q\rangle \) and continuous eigenvalues \( q \), write the probability of measuring observable \( Q \) between \( q \) and \( q + dq \) as a result of a measurement performed when the system is in the state \( |\Psi\rangle \).

Students also use a guided inquiry-based approach to learn how the expansion of a generic state vector \( |\Psi\rangle \) in terms of position eigenstates \( |x\rangle \) as \( |\Psi\rangle = \int_{-\infty}^{\infty} \psi(x)|x\rangle dx \) can be useful when determining the probability of measuring observable \( Q \) between \( q \) and \( q + dq \) in the position representation, i.e., \( |\langle q|\Psi\rangle|^2 dq = \int_{-\infty}^{\infty} \langle q|x\rangle \psi(x) dx \int \psi^*(x) \psi(x) dx \) (in which \( \psi_q(x) \) and \( \psi(x) \) are the wavefunctions in the position representation corresponding to states \( |q\rangle \) and \( |\Psi\rangle \), respectively). Moreover, we also included questions in the QuILT to help students develop facility with using the identity operator \( I \) written in terms of position eigenstates \( |x\rangle \), i.e., \( I = \int_{-\infty}^{\infty} |x\rangle \langle x| dx \), to decompose a state vector into its components along each of the basis vectors \( |x\rangle \) in order to write the states in position representation. Additional questions in the QuILT strive to scaffold student learning such that students are in the ZPD by providing opportunities for students to check their responses and reconcile possible differences between their responses and the correct understanding.

Helping students differentiate between the probability distribution for measuring a generic observable \( Q \) and the expectation value of \( Q \). As noted earlier, in response to questions Q6 and Q7 in table 2, students sometimes confused the probability for measuring a particular value of an observable \( Q \) with the expectation value of \( Q \). Other students wrote expressions for the probability of measuring a particular value of the observable \( Q \) that included the operator \( \hat{Q} \) corresponding to the observable \( Q \). One guided inquiry-based sequence in the QuILT strives to help students differentiate between the related concepts of probability of measuring a particular value of an observable and its expectation value. For example, the following question helps students distinguish between the related concepts of probability and expectation value.

Consider the following statement from Student A:

- **Student A:** \( |\langle q_n|\Psi\rangle|^2 \) is the probability of measuring \( q_n \) when you measure observable \( Q \) in the state \( |\Psi\rangle \). The expectation value is the average value of a large number of measurements performed on identically prepared systems. Since we know the probability of measuring each eigenvalue \( q_n \) of the operator \( \hat{Q} \), the expectation value is \( \langle \Psi|\hat{Q}|\Psi\rangle = \sum_{n=1}^{N} q_n |\langle q_n|\Psi\rangle|^2 \).

Do you agree with Student A’s statement? Explain your reasoning.
After answering this question, students use the expansion $|\psi\rangle = \sum_{n=1}^{N} a_n |q_n\rangle$ to derive an expression for the expectation value $\langle \hat{Q} | \psi \rangle = \sum_{n=1}^{N} q_n \langle \psi | q_n \rangle$. Then, the students are asked to verify whether their answer agrees with Student A’s statement. If their answer does not agree with Student A’s statement, they are asked to go back and check their work with a partner to obtain the equation for the expectation value of observable $\hat{Q}$ in terms of its complete set of eigenstates $\{|q_n\rangle, n = 1, 2, \ldots N\}$ and eigenvalues $q_n$, i.e., $\langle \psi | \hat{Q} | \psi \rangle = \sum_{n=1}^{N} q_n |\langle q_n | \psi \rangle|^2$.

After working on the QuILT, students are expected to be able to make sense of the probability distributions of measurement outcomes conceptually and be able to express the probability distributions for the measurement of different observables (both for the cases when the corresponding operators have a discrete or continuous eigenvalue spectrum) in Dirac notation as well as position representation (without the use of Dirac notation). The QuILT strives to help students develop a coherent understanding of probability distribution for the measurement of an arbitrary observable so that they can apply it in diverse situations (e.g., measurement of position in the interval between $x$ and $x + dx$ or measurement of energy for a system in a given state).

### 6. Evaluation of the QuILT

After the QuILT appeared to be effective in individual administration to students during interviews, it was administered to upper-level undergraduate and PhD students. Undergraduate students ($N = 62$) in three upper-level undergraduate quantum mechanics courses first had traditional instruction in relevant concepts. Then, students were given a pretest on these topics in class. All students had sufficient time to work through the pretest. Then, students worked through the QuILT in class and were given one week to work through the rest of the QuILT as homework. The pretest and QuILT counted as a small portion of their homework grade for the course. The pretest was not returned to students. Undergraduate students were then given a posttest in class (all students had sufficient time to take the posttest). The posttests were graded for correctness as a quiz for the quantum mechanics course. In addition, the upper-level undergraduate students were aware that topics discussed in the tutorial could also appear in future exams since the tutorial was part of the course material.

The QuILT was also administered to PhD students ($N = 66$) who were simultaneously enrolled in the first semester of a PhD-level core quantum mechanics course and a course for training teaching assistants in two consecutive years. In the teaching assistant training class, the PhD students learned about instructional strategies for teaching introductory physics courses (e.g., tutorial-based approaches to learning). They first worked on the pretest (all students had sufficient time to take the pretest). The PhD students worked through the QuILT in the teaching assistant training class to learn about the effectiveness of the tutorial approach to teaching and learning. They were given one week to work through the rest of the QuILT as homework. Then, a posttest was administered to the PhD students in class (all students had sufficient time to take the posttest). The PhD students were given credit for completing the pretest, QuILT, and posttest, but they were not given credit for correctness. The PhD students’ scores on the posttest did not contribute to the final grade for the teaching assistant training class (which was a Pass/Fail course).

To evaluate the effectiveness of the QuILT in helping students develop a coherent understanding of probability, we compared the scores on questions Q1–Q5 (shown in table 2).
Table 4. Distribution of student responses on questions Q1–Q5 for students who did not work through the QuILT (non-QuILT group) and students who worked through the QuILT (QuILT group) and the p-values for each question for the comparison of the means of the QuILT and non-QuILT groups (which shows that the differences in the means of the QuILT and non-QuILT groups are statistically significant for all questions).

| Question | Non-QuILT group | QuILT group | p-value |
|----------|-----------------|-------------|---------|
| Q1       | N = 158 | A (16%) B (12%) C (17%) D (29%) E (26%) | N = 89 | A (6%) B (12%) C (45%) D (15%) E (22%) | p < 0.001 |
| Q2       | N = 158 | A (19%) B (14%) C (14%) D (29%) E (24%) | N = 89 | A(8%) B (9%) C (40%) D (16%) E (26%) | p < 0.001 |
| Q3       | N = 184 | A (10%) B (10%) C (27%) D (22%) E (31%) | N = 124 | A (0%) B (6%) C (14%) D (9%) E (71%) | p < 0.001 |
| Q4       | N = 184 | A (51%) B (10%) C (17%) D (8%) E (13%) | N = 124 | A (90%) B (5%) C (3%) D (1%) E (1%) | p < 0.001 |
| Q5       | N = 184 | A (14%) B (26%) C (28%) D (4%) E (28%) | N = 124 | A (2%) B (33%) C (12%) D (2%) E (50%) | p < 0.001 |

of students who worked on the QuILT versus students who did not work on the QuILT. Table 4 shows the distribution of students’ average responses on questions Q1–Q5 for the students who did not work through the QuILT (non-QuILT group) and for those who did (QuILT group). Questions Q1–Q5 were administered to undergraduate students at five universities in the US who did not work through the QuILT but had at least one semester of upper-level undergraduate quantum mechanics. Questions Q1–Q5 were also administered to undergraduate and PhD students who worked through the QuILT (at least one month after the students had worked through the QuILT) and can be considered to test how much the students retained what they had learned after working through the QuILT at least one month later. Furthermore, the questions Q1–Q5 are not exact questions from the QuILT and require students to transfer their learning to a slightly different context. The performance of the undergraduate and PhD students was not significantly different on questions Q1–Q5 so we do not differentiate between the two groups in table 4.

Questions Q1 and Q2. Identifying expressions for the probability for measuring position of a particle in a narrow range between x and x + dx in position representation. As noted earlier, in questions Q1 and Q2, 67% of the students who had not worked through the QuILT incorrectly claimed that the probability of measuring the particle between x and x + dx is is \( \int_x^{x+dx} x |\Psi(x, 0)|^2 dx \) and selected statement 3 (chose answer option B, D, or E). Students who had worked through the QuILT were less likely to select statement 3 as correct—table 4 shows that, on questions Q1 and Q2, approximately 50% of the students who worked through the QuILT incorrectly claimed that the probability of measuring the particle between x and x + dx is \( \int_x^{x+dx} x |\Psi(x, 0)|^2 dx \).
Question Q3. Identifying expressions for the probability of measuring position in Dirac notation. As noted earlier, students who had not worked through the QuILT performed well on question Q3 that asks students to evaluate the correctness of statement (3): ‘if you measure the position of the particle in the state $|\Psi\rangle$, the probability of finding the particle between $x$ and $x + dx$ is $|\langle x |\Psi\rangle|^2 \, dx$. Table 4 shows that 80% recognized that the statement is true and selected an answer that included statement (3) (i.e., option C, D, or E). Furthermore, 63% of the students who had not worked through the tutorial were able to identify the expansion of a generic state $|\Psi\rangle$ as $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x) |x\rangle \, dx$ and selected option B, D, or E. Still, students who had worked through the QuILT performed better than those who did not on question Q3—table 4 shows that 94% of them correctly recognized that the probability of finding the particle between $x$ and $x + dx$ is $|\langle x |\Psi\rangle|^2 \, dx$ and selected an answer that included statement (3) (i.e., option C, D, or E). 86% of the students who worked through the QuILT were able to identify that the expansion of a generic state $|\Psi\rangle$ as $|\Psi\rangle = \int_{-\infty}^{\infty} \Psi(x) |x\rangle \, dx$.

Questions Q1 and Q2. Identifying an expression for probability of measuring energy $E_i$ in position representation. As noted earlier, on questions Q1 and Q2, approximately 40% of students who did not work through the QuILT were able to recognize the correct expression for the probability of measuring energy $E_i$ and selected statement 2 as correct (chose answer option C or E). Students who had worked through the tutorial performed better—table 4 shows that, on questions Q1 and Q2, almost 70% of the students who had worked through the QuILT were able to recognize the correct expression for the probability of measuring energy $E_i$ in the position representation and selected statement 2 as correct (chose answer option C or E).

Question Q4. Identifying expression for the probability of measuring energy $E_i$ in Dirac notation. Students who had not worked through the QuILT performed well on question Q4 that asked them to evaluate the correctness of statement (3): if you measure the energy of the system in the state $|\Psi\rangle$, the probability of obtaining $E_i$ and collapsing the state to $|n\rangle$ is $|\langle n |\Psi\rangle|^2$. Table 4 shows that 89% of the students recognized that the statement is true and selected an answer that included statement (3) (i.e., options A, C, D, or E). Furthermore, 69% of the students who had not worked through the QuILT were able to identify that the expansion of a generic state $|\Psi\rangle$ in terms of the energy eigenstates $|n\rangle$ is $|\Psi\rangle = \sum_n \langle n |\Psi\rangle |n\rangle$ and selected option A, B, or D. Still, students who had worked through the QuILT performed better than those who did not on question Q4—table 4 shows that 95% of them correctly recognized that the probability of obtaining $E_i$ and collapsing the state to $|n\rangle$ is $|\langle n |\Psi\rangle|^2$ and 96% of them were able to identify that the expansion of a generic state $|\Psi\rangle$ in terms of the energy eigenstates $|n\rangle$ is $|\Psi\rangle = \sum_n \langle n |\Psi\rangle |n\rangle$.

Question Q5. Identifying expressions for probability in Dirac notation and position representation in the same question. Students who had not worked through the QuILT had difficulty with identifying expressions for the probability for obtaining an outcome between $q$ and $q + dq$ written in Dirac notation and position representation (without the use of Dirac notation) on question Q5. Only 32% of the students recognized that both expressions $|\langle q |\Psi\rangle|^2 dq$ and $\int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) dx \, dq$ denote the probability for obtaining an outcome between $q$ and $q + dq$ and selected answer option D or E. Table 4 shows that students who worked through the QuILT performed better on question Q5 and 52% of them recognized that both expressions $|\langle q |\Psi\rangle|^2 dq$ and $\int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) dx \, dq$ denote the probability for obtaining an outcome between $q$ and $q + dq$.
Question Q7. Generating an expression for the probability distribution for measuring a generic observable $Q$ with a continuous eigenvalue spectrum. We also evaluated the effectiveness of the QuILT in improving students’ understanding of the probability distributions of measurement outcomes by giving question Q7 (shown in table 2) on the QuILT pretest and posttest. Students were given full credit if they wrote $(\langle q | \Psi \rangle)^2$, $\Psi^*(q)\Psi(q)$ or $(\Psi^*(q)\Psi(q))^2$, or $\langle q | \Psi \rangle^2$, $\langle \Psi(q) | \Psi(q) \rangle^2$. Table 5 shows the percentages of students who correctly answered question Q7 related to probability distributions of measurement outcomes on the pretest and posttest. Table 6 shows the pretest and posttest average scores on question Q7. The number of students on the posttest does not match the pretest because students’ scores on the posttest were not counted if they did not work through the entire tutorial. Average normalized gain $[29]$ is commonly used to determine how much the students learned and takes into account their initial scores on the pretest. It is defined as $\gamma = (\% (S_f) - \% (S_i)) / (100 - \% (S_i))$, in which $S_f$ and $S_i$ are the final (post) and initial (pre) class averages, respectively [29]. The average normalized gain on question Q7 for undergraduate students was 0.67 and for PhD students was 0.59. We also calculate the effect size denoted by $d$ in the form of Cohen’s $d$ $d = (\mu_1 - \mu_2) / \sigma_{\text{pooled}}$, where $\mu_1$ and $\mu_2$ are the averages of the two groups being compared and $\sigma_{\text{pooled}} = \sqrt{(\sigma_1^2 + \sigma_2^2) / 2}$, where $\sigma_1$ and $\sigma_2$ are the standard deviations of the two groups [30]. The effect size on question Q7 is 1.5 for undergraduate students (which is considered a large effect size) and 0.69 for PhD students (which is a medium effect size).

On the pretest, on question Q7, approximately 30% of the undergraduate and PhD students wrote an expression which involved the operator $\hat{Q}$. On the posttest when question Q7 was given, students performed better and fewer students wrote an expression that involved the operator $\hat{Q}$. In particular, on the posttest on Q7, 19% of the undergraduates and 9% of the PhD students wrote an expression that involved the operator $\hat{Q}$.

Overall, students who had worked through the QuILT performed better than those who did not work on the QuILT on multiple-choice questions about probability distributions for measurement outcomes expressed in Dirac notation and position representation administered one month after they had worked through the QuILT. Students who had worked through the

| Question Q7 | Pretest (% correct) | Posttest (% correct) |
|-------------|---------------------|----------------------|
|             | UG (N = 62)         | G (N = 69)           |
|             | 11%                 | 45%                  |
|             | UG (N = 58)         | G (N = 66)           |
|             | 66%                 | 74%                  |

| Table 5. Percentages of students who correctly answered the pretest and posttest question Q7 that involved generating the probability distributions for measurement outcomes for an observable whose corresponding operator has a continuous eigenvalue spectrum. |

| Question Q7 | Pretest (average score) | Posttest (average score) |
|-------------|-------------------------|--------------------------|
|             | UG (N = 62)             | G (N = 69)               |
|             | 15%                     | 51%                      |
|             | UG (N = 58)             | G (N = 66)               |
|             | 72%*                   | 80%*                     |

*The difference between the means on the pretest and posttest question was significantly different ($p < 0.001$).
QuILT also performed better on a free-response posttest question (see Q7 in table 2 and scores on Q7 in table 5) on the probability distributions of measurement outcomes for an observable $Q$ whose corresponding operator $\hat{Q}$ has a continuous eigenvalue spectrum.

7. Summary and discussion

One foundational issue in quantum mechanics is that the measurement outcomes are probabilistic. However, there have been relatively few investigations of student difficulties with this issue in the context of Dirac notation and position and momentum representations. Therefore, we explored the difficulties students have with this concept. Students exhibited many common difficulties with the probability distributions for measurement outcomes of physical observables and have difficulty translating expressions for probability distributions from Dirac notation to position representation and vice versa. We developed a QuILT that strives to help students learn these concepts better. Although these concepts are very challenging, the evaluations of the QuILT are encouraging. Moreover, in an end of semester survey in one of the undergraduate quantum mechanics courses in which this QuILT was incorporated, many students reported that they felt the QuILT was very helpful in helping them learn these concepts.

While this study focused on student understanding of probability distribution of measurement outcomes, future investigations will focus on other equally challenging topics in quantum mechanics and strategies to help students learn those concepts better. For example, students also have difficulties with topics such as possible outcomes of measurements, expectation values, and time development of quantum states and time dependence of expectation values [7]. Furthermore, in the study described here, we found that students had difficulties both with Dirac notation and position and momentum representations in the context of probability distributions for measurement outcomes. However, since Dirac notation is a compact and elegant notation, one hypothesis is that learning the formalism of quantum mechanics using Dirac notation first can be a stepping stone for learning these concepts in position and momentum representations. Future investigations will focus on whether learning quantum mechanics topics in the context of Dirac notation first can indeed aid students in developing a better understanding of quantum mechanics (including situations in which position and momentum representations are used).

We note that, in our investigation of student difficulties, students were asked questions that used both Dirac notation and wave function in the position representation in the same question. It is possible that the percentages of students answering these questions correctly were impacted by the mix of notations used in the same question. Future work will focus on how students respond to questions that use only one notation, i.e., only Dirac notation or only wave functions in the position representation.

Furthermore, in our study, we used the state $|\Psi\rangle$ to represent a generic quantum state and the wave function in position representation was expressed as $\langle x | \Psi \rangle = \Psi(x)$. The QuILT strives to help students make the distinction between the ket state $|\Psi\rangle$ and the wave function $\Psi(x)$ and reason about the fact that $\Psi(x)$ is a representation of the state $|\Psi\rangle$ in position representation (and that $|\Psi\rangle = \Psi(x)$). It is possible that some students still have difficulty differentiating between the generic state $|\Psi\rangle$ and the wave function in position representation $\Psi(x)$ because of the similarity in notation. Future studies will investigate whether representing a generic quantum state as $|\alpha\rangle$ (instead of $|\Psi\rangle$) and using the notation $\langle \chi | \alpha \rangle = \Psi_\chi(\chi)$ in which $\Psi_\chi(\chi)$ denotes the wave function in position representation for the ket state $|\chi\rangle$ can help
students better differentiate between the generic ket state $|\alpha \rangle$ and the wave function $\Psi_\alpha(x)$ (for example, see [31]).

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