ARENS REGULARITY OF CERTAIN WEIGHTED SEMIGROUP ALGEBRA AND COUNTABILITY

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Abstract. It is known that every countable semigroup admits a weight $\omega$ for which the semigroup algebra $\ell_1(S, \omega)$ is Arens regular and no uncountable group admits such a weight; see [4]. In this paper, among other things, we show that for a large class of semigroups, the Arens regularity of the weighted semigroup algebra $\ell_1(S, \omega)$ implies the countability of $S$.

1. Introduction and Preliminaries

Arens [2] introduced two multiplications on the second dual $A^{**}$ of a Banach algebra $A$ turning it into Banach algebra. If these multiplications are coincide then $A$ is said to be Arens regular. The Arens regularity of the semigroup algebra $\ell_1(S)$ has been investigated in [7]. The Arens regularity of the weighted semigroup algebra $\ell_1(S, \omega)$ has been studied in [4] and [3]. In [3] Baker and Rejali obtained some nice criterions for Arens regularity of $\ell_1(S, \omega)$. Recent developments on the Arens regularity of $\ell_1(S, \omega)$ can be found in [5]. For the algebraic theory of semigroups our general reference is [6].

In this paper we first show that the Arens regularity of a weighted semigroup algebra is stable under certain homomorphisms of semigroups (Lemma 2.2). Then we study those conditions under which the Arens regularity of $\ell_1(S, \omega)$ necessitates the countability of $S$. The most famous example for such a semigroup is actually a group, as Craw and Young have proved in their nice paper [4]. As the main aim of the paper we shall show that for a wide variety of semigroups the Arens regularity of $\ell_1(S, \omega)$ implies that $S$ is countable; (see Theorems 3.4 and 3.5).

2. Arens Regularity of $\ell_1(S, \omega)$ and Some Hereditary Properties

Let $S$ be a semigroup and $\omega : S \to (0, \infty)$ be a weight on $S$, i.e. $\omega(st) \leq \omega(s)\omega(t)$ for all $s, t \in S$, and let $\Omega : S \times S \to (0, 1]$ be defined by $\Omega(s, t) = \frac{\omega(st)}{\omega(s)\omega(t)}$, for $s, t \in S$. Following

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we call \( \Omega \) to be 0–cluster if for each pair of sequences \((x_n), (y_m)\) of distinct elements of \(S\), \(\lim_n \lim_m \Omega(x_n, y_m) = 0 = \lim_m \lim_n \Omega(x_n, y_m) \) whenever both iterated limits exist.

We define,

\[
\ell_\infty(S, \omega) := \{ f : S \to \mathbb{C} : \|f\|_{\omega, \infty} = \sup \{ \frac{|f(s)|}{\omega(s)} : s \in S \} < \infty \}
\]

\[
\ell_1(S, \omega) := \{ g : S \to \mathbb{C} : \|g\|_{\omega, 1} = \sum_{s \in S} |g(s)| \omega(s) < \infty \}.
\]

For ease of reference we quote the following criterion from [3] which will be frequently used in the sequel.

**Theorem 2.1.** [3, Theorems 3.2, 3.3] For a weighted semigroup algebra \(\ell_1(S, \omega)\), the following statements are equivalent.

(i) \(\ell_1(S, \omega)\) is regular.

(ii) The map \((x, y) \mapsto \chi_A(xy)\Omega(x, y)\) is cluster on \(S \times S\) for each \(A \subseteq S\).

(iii) For each pair of sequences \((x_n), (y_m)\) of distinct points of \(S\) there exist subsequences \((x'_n), (y'_m)\) of \((x_n), (y_m)\) respectively such that either

(a) \(\lim_n \lim_m \Omega(x'_n, y'_m) = 0 = \lim_m \lim_n \Omega(x'_n, y'_m)\), or

(b) the matrix \((x'_n y'_m)\) is of type C.

In particular, if \(\Omega\) is 0–cluster then \(\ell_1(S, \omega)\) is regular.

Let \(\psi : S \to T\) be a homomorphism of semigroups. If \(\omega\) is a weight on \(T\) then trivially \(\omega'(s) := \omega(\psi(s))\) defines a weight on \(T\).

If \(\psi : S \to T\) is an epimorphism and \(\omega\) is a bounded below (that is, \(\inf \omega(S) > 0\)) weight on \(S\) then a direct verification reveals that

\[
\omega^\ast(t) := \inf \omega(\psi^{-1}(t)), \quad (t \in T),
\]

defines a weight on \(T\). We commence with the next elementary result concerning to the stability of regularity under the semigroup homomorphism.

**Lemma 2.2.** Let \(\psi : S \to T\) be a homomorphism of semigroups.

(i) If \(\psi\) is onto and \(\omega\) is bounded below weight on \(S\) then the regularity of \(\ell_1(S, \omega)\) necessities the regularity of \(\ell_1(T, \omega^\ast)\). Furthermore if \(\Omega\) is 0-cluster, then \(\Omega^\ast\) is 0-cluster.

(ii) For a weight \(\omega\) on \(T\) if \(\ell_1(S, \omega^\ast)\) is regular, then \(\ell_1(T, \omega)\) is regular.
Proof. (i) Since $\omega$ is bounded below, we can assume that, $\inf \omega(S) \geq \varepsilon > 0$, for some $\varepsilon < 1$. Hence $\overrightarrow{\omega} \geq \varepsilon$. Let $(x_n), (y_m)$ be sequences of distinct elements in $T$. Then there are sequences of distinct elements $(s_n), (t_m)$ in $S$ such that

$$
\begin{cases}
\overrightarrow{\omega}(x_n) > \omega(s_n)(1 - \varepsilon) \quad &\text{and} \quad \psi(s_n) = x_n, \\
\overrightarrow{\omega}(y_m) > \omega(t_m)(1 - \varepsilon) \quad &\text{and} \quad \psi(t_m) = y_m.
\end{cases}
$$

It follows that $\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m) > \omega(s_n)\omega(t_m)(1 - \varepsilon)^2$ and so from $\overrightarrow{\omega}(x_n y_m) \leq \omega(s_n t_m)$ we get

$$
\frac{\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m)}{\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m)} \leq \frac{1}{(1 - \varepsilon)^2} \frac{\omega(s_n t_m)}{\omega(s_n)\omega(t_m)}, \quad \text{or equivalently,}
$$

$$
\overrightarrow{\Omega}(x_n, y_m) \leq \frac{1}{(1 - \varepsilon)^2} \Omega(s_n, t_m), \quad (n, m \in \mathbb{N}). \quad (2.1)
$$

Applying the inequality (2.1), an standard argument based on Theorem 2.1 shows that if $\ell_1(S, \omega)$ is regular then $\ell_1(T, \overrightarrow{\omega})$ is regular. \qed

Corollary 2.3. Let $\psi : S \to T$ be a homomorphism of semigroups. If $\ell_1(S)$ is Arens regular then $\ell_1(T, \omega)$ is Arens regular, for every weight function $\omega$ on $T$.

Proof. Let $\ell^1(S)$ be Arens regular and let $\omega$ be a weight on $T$. Then $\ell^1(S, \overrightarrow{\omega})$ is Arens regular by [3, Corollary 3.4]. Lemma 2.2 implies that $\ell_1(T, \omega)$ is Arens regular. \qed

3. Arens Regularity of $\ell_1(S, \omega)$ and Countability of $S$

We commence with the next result of Craw and Young with a slightly simpler proof.

Corollary 3.1. (See [4, Corollary 1]) Let $S$ be a countable semigroup. Then there exists a bounded below weight $\omega$ on $S$ such that $\Omega$ is 0-cluster. In particular, $\ell_1(S, \omega)$ is Arens regular.

Proof. Let $F$ be the free semigroup generated by the countable semigroup $S = \{a_k : k \in \mathbb{N}\}$. For every element $x \in F$ (with the unique presentation $x = a_{k_1}a_{k_2} \cdots a_{k_r}$) set $\omega_1(x) = 1 + k_1 + k_2 + \cdots + k_r$. A direct verification shows that $\omega_1$ is a weight on $F$ with $1 \leq \omega_1$, and that $\Omega_1$ is 0-cluster. Let $\psi : F \to S$ be the canonical epimorphism. Set $\omega := \overrightarrow{\omega_1}$. By Lemma 2.2, $\omega$ is our desired weight on $S$. \qed

In the sequel the following elementary lemma will be frequently used.

Lemma 3.2. A nonempty set $X$ is countable if and only if there exists a function $f : X \to (0, \infty)$ such that the sequence $(f(x_n))$ is unbounded for every sequence $(x_n)$ with distinct elements in $X$. 
Proof. If $X = \{x_n : n \in \mathbb{N}\}$ is countable the $f(x_n) = n$ is the desired function. For the converse, suppose that such a function $f : X \to (0, \infty)$ exists. Since $X = \cup_{n \in \mathbb{N}}\{x \in X : f(x) \leq n\}$ and each of the sets $\{x \in X : f(x) \leq n\}$ is countable, so $X$ is countable. □

**Theorem 3.3.** If $\ell^1(S)$ is not Arens regular and $S$ admits a bounded below weight for which $\Omega$ is 0-cluster, then $S$ is countable.

Proof. Let $\omega$ be a bounded below weight for which $\Omega$ is 0-cluster. Let $\epsilon > 0$ is so that $\omega \geq \epsilon$. Let $S$ be uncountable. By Lemma 3.2 there is a sequence $(s_n)$ of distinct elements in $S$ and $n_0 \in \mathbb{N}$ such that $\omega(s_n) \leq n_0$ for all $n \in \mathbb{N}$. As $\ell_1(S)$ is not Arens regular, there exist subsequences $(s_{n_k})$, $(s_{m_l})$ of $(s_n)$ such that $\{s_{n_k}s_{m_l} : k < l\} \cap \{s_{n_k}s_{m_l} : k > l\} = \emptyset$. We thus get

$$\Omega(s_{n_k}, s_{m_l}) = \frac{\omega(s_{n_k}s_{m_l})}{\omega(s_{n_k})\omega(s_{m_l})} \geq \frac{\epsilon}{n_0^2}, \quad (k, l \in \mathbb{N}),$$

contradicts the 0-clusterlity of $\Omega$. □

Abtehi et al. [1] have shown that for a wide variety of semigroups (including Brandt semigroups, weakly cancellative semigroups, (0-)simple inverse semigroups and inverse semigroups with finite set of idempotents) the Arens regularity of the semigroup algebra $\ell^1(S)$ necessities the finiteness of $S$ (see [1, Corollary 3.2, Proposition 3.4 and Theorem 3.6]). Applying these together with Theorem 3.3 we arrive to the next result.

Note that as it has been reminded in Theorem 2.1, if $\Omega$ is 0-cluster then $\ell_1(S, \omega)$ is regular and the converse is also true in the case where $S$ is weakly cancellative; (see [3, Corollary 3.8]).

**Theorem 3.4.** If $S$ admits a bounded below weight for which $\Omega$ is 0-cluster then $S$ is countable in either of the following cases.

1. $S$ is a Brandt semigroup.
2. $S$ is weakly cancellative.
3. $S$ is a simple (resp. 0-simple) inverse semigroup.
4. $S$ is an inverse semigroup with finitely many idempotents.

In the next result we shall show that the same result holds when $S$ is a completely simple semigroup.

**Theorem 3.5.** If $S$ admits a bounded below weight for which $\Omega$ is 0-cluster then $S$ is countable in the case where $S$ is completely simple [resp. 0-simple].
Proof. Suppose that \( \omega \) is a bounded below weight on \( S \) such that \( \Omega \) is 0–cluster. Let 
\( S \) be completely 0–simple, then as it has been explained in \([6]\), \( S \) has the presentation 
\( S \cong M^0(G, I, \Lambda; P) = (I \times G \times \Lambda) \cup \{0\} \), equipped with the multiplication 
\[
(i, a, \lambda)(j, b, \mu) = \begin{cases} 
(i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\
0 & \text{if } p_{\lambda j} = 0,
\end{cases}
\]
\((i, a, \lambda)0 = 0(i, a, \lambda) = 0\).

Fix \( i_0 \in I, \lambda_0 \in \Lambda \) and define \( f : I \rightarrow (0, \infty) \) by 
\[
f(i) = \begin{cases} \omega(i, p_{\lambda_0 i_0}^{-1}, \lambda_0) & \text{if } p_{\lambda_0 i_0} \neq 0 \\
\omega(i, 1, \lambda_0) & \text{if } p_{\lambda_0 i_0} = 0.
\end{cases}
\]

Let \((i_n)\) be a sequence of distinct elements in \( I \) and set 
\[
x_n = \begin{cases} (i_n, p_{\lambda_0 i_n}^{-1}, \lambda_0) & \text{if } p_{\lambda_0 i_n} \neq 0 \\
(i_n, 1, \lambda_0) & \text{if } p_{\lambda_0 i_n} = 0.
\end{cases}
\]

It is readily verified that if \( p_{\lambda_0 i_n} \neq 0 \) then \( x_n x_m = x_n \), for all \( m \in \mathbb{N} \); indeed 
\[
x_n x_m = (i_n, p_{\lambda_0 i_n}^{-1}, \lambda_0)(i_m, p_{\lambda_0 i_m}^{-1}, \lambda_0) = (i_n, p_{\lambda_0 i_n}^{-1}p_{\lambda_0 i_m}^{-1}, \lambda_0) = (i_n p_{\lambda_0 i_n}^{-1}, \lambda_0) = x_n.
\]

And if \( p_{\lambda_0 i_n} = 0 \) then \( x_n x_m = 0 \), for all \( m \in \mathbb{N} \).

Hence \( \frac{1}{f(i_m)} = \frac{1}{\omega(x_m)} = \frac{\omega(x_n x_m)}{\omega(x_n) \omega(x_m)} = \Omega(x_n, x_m) \) in the case where \( p_{\lambda_0 i_n} \neq 0 \) and 
\[
(\frac{\omega(0)}{f(i_m)})^2 = (\frac{\omega(0)}{\omega(x_m)})^2 = (\frac{\omega(0)}{\omega(x_n) \omega(x_m)}) \text{ whenever } p_{\lambda_0 i_n} = 0. \]

These observations together with the 0–clusterity of \( \Omega \) imply that \( (f(i_m)) \) is unbounded. Hence \( I \) is countable, by Lemma 3.2. Similarly \( \Lambda \) is countable. We are going to show that \( G \) is also countable. To this end, let \( \omega_0(g) = \omega(i_0, gp_{\lambda_0 i_0}^{-1}, \lambda_0) \) \((g \in G)\). Then \( \omega_0 \) is a weight on \( G \) such that \( \Omega_0 \) is 0–cluster and so \( G \) is countable, by Theorem 3.4. Therefore \( S \) is countable as claimed. Proof for the case that \( S \) completely simple semigroup is similar. \( \square \)

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