EIGHTY NEW INVARIANTS
IN THE ELLIPTIC BILLIARD

DAN REZNIK, RONALDO GARCIA, AND JAIR KOILLER

Abstract. We introduce several-dozen experimentally-found invariants of
Poncelet $N$-periodics in the confocal ellipse pair (Elliptic Billiard). Recall
this family is fully defined by two integrals of motion (linear and angular
momentum), so any “new” invariants are dependent upon them. Nevertheless,
proving them may require sophisticated methods. We reference some two-dozen
proofs already contributed. We hope this article will motivate contributions
for those still lacking proof.

Keywords: elliptic billiard, invariant, optimization, experimental.

MSC 51N20, 51M04, 65-05

1. Introduction

The Elliptic Billiard (EB) is a special case of Poncelet’s Porism [13], where the
conic pair are two confocal ellipses; it therefore admits a 1d family of $N$-periodic
trajectories [34, 18, 12] which at every vertex are bisected by the normals of the
outer ellipse in the pair (hence the term “billiard”); see Figure 1.

The EB is an integrable system (in fact it is conjectured as the only integrable
planar billiard [19]). Integrability implies invariant perimeter $L$; a second classic

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invariant is Joachimsthal’s constant \( J \), which is simply a statement that all trajectory segments are tangent to the confocal caustic [34, 28].

Continuing our work on properties of N-periodics in the EB [26, 16], here we introduce dozens of new invariants detected experimentally. These involve distances, areas, angles and centers of mass of N-periodics and several derived polygons defined below. Some invariants depend on the parity of \( N \), while others on positional constraints.

Note that since the N-periodics in the EB are fully defined by \( L, J \), any “new” invariants listed here or elsewhere must be ultimately dependent upon said quantities. Nevertheless, proving a specific functional dependence may require sophisticated techniques. Several proofs have already been contributed and are referenced below. We hope to motivate more contributions and/or new discoveries.

Admittedly, the number of possible invariants is infinite as one may select any functional combination of \( L, J \). Our selection criterion can be loosely defined as any quantity constant over the N-periodic family and/or derived objects, which is an elementary function of lengths, angles, areas, etc.

This article is organized as follows: preliminary definitions are given in Section 2. Invariants are introduced in Section 3, in several clusters, involving: (i) lengths, areas, and angles of N-periodics and associated polygons; (ii) pedal polygons to N-periodics and (iii) their outer polygons; (iv) antipedal polygons (defined below); (v) area-ratios related to the Steiner curvature centroid [33]; (vi) pairs of pedal polygons; (vii) area-ratios of evolute polygons [8]; (viii) focus-inversive objects and (ix) pairs of focus-inversive objects.

Details about our experimental toolbox are covered in Section 4. All symbols used in this article are listed on Table 12 in Appendix A.

We encourage the reader to watch the videos included in Section 5 which provide more insight into invariant phenomena.

Related Work. In a companion article [15] we derive explicit expressions for some of the invariants listed herein for certain “low” \( N \), e.g., 3–6. Methods for obtaining N-periodic trajectories based on Cayley’s condition are surveyed in [14, 10]. A few explicit expressions for the caustic parameter (for \( N = 3, 4, 6, 8 \)) appear in [23].

2. Preliminaries

Let the EB have center \( O \), semi-axes \( a > b > 0 \), and foci \( f_1, f_2 \) at \([\pm \sqrt{a^2 - b^2}, 0]\). Let \( a', b' \) denote the major, minor semi-axes of the confocal caustic, whose values are given by a method due to Cayley [12], though we obtain them numerically, see Section 4.

As mentioned above, the perimeter \( L \) is invariant for a given N-periodic family, as is Joachimsthal’s constant \( J = \langle Ax, v \rangle \), where \( x \) is a bounce point (called \( P_i \) above), \( v \) is the unit velocity vector \((P_i - P_{i-1})/||v||\), \( \langle . , . \rangle \) stands for dot product, and [34]:

\[
A = \text{diag}[1/a^2, 1/b^2]
\]

Hellmuth Stachel derived [31] an elegant expression for Joachimsthal’s constant \( J \) in terms of the axes of the EB and its caustic:

\[
J = \frac{\sqrt{a^2 - a'^2}}{ab}
\]
Note: holding a constant, for each $N$, $a''$ and therefore $J$ assume a distinct value.

Let a polygon have vertices $W_i, i = 1, ..., N$. In this paper all polygon areas are signed, i.e., obtained from a sum of cross-products [22]:

$$S = \frac{1}{2} \sum_{i=1}^{N} W_i \times W_{i+1}$$

Let $W_i = (x_i, y_i)$, then $W_i \times W_{i+1} = (x_i y_{i+1} - x_{i+1} y_i)$.

The area centroid $\overline{W}$ of a polygon is given by [22]:

$$\overline{W} = \frac{1}{6S} \sum_{i=1}^{N} (W_i \times W_{i+1})(W_i + W_{i+1})$$

The curvature $\kappa$ of the ellipse at point $(x, y)$ at distance $d_1, d_2$ to the foci is given by [35, Ellipse]:

$$\kappa = \frac{1}{a^2b^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^{-3/2} = ab(d_1d_2)^{-3/2} = (\kappa_a d_1d_2)^{-3/2}$$

Where $\kappa_a = (ab)^{-2/3}$ is the constant affine curvature of the ellipse [21].

Given a polygon with vertices $W_i$ and angles $\theta_i$, its Steiner Centroid of Curvature $K$ is given by [33, p. 22]:

$$K = \frac{\sum_{i=1}^{N} \rho_i R_i}{\sum \rho_i}, \text{ with } \rho_i = \sin(2\theta_i)$$

3. Invariants

In this section we present the invariants found so far in several tables. Each invariant is given an identifier $k_n$ where the first digit of $n$ refers to a cluster of invariants; see Table 1.
On the invariant tables below, column “invariant” provides an expression for the conserved quantity; column “value” provides a closed-form expression for the invariant (when available) in terms of the fundamental constants, or a ‘?’ when not available (note that the invariant may already have been proved but no closed-form expression has yet been found); column “which N” specifies whether the invariant only holds for certain N (even, odd, etc.); column “date” specifies the month and year (mm/yy) when the invariant was first experimentally detected. Column “proven” references available proofs if already communicated and/or published, else it displays a ‘?’.

3.1. Basic Invariants. Invariants involving angles and areas of N-periodics and its tangential and internal polygons are shown on Table 2. There $\theta_i, A$ (resp. $\theta_i', A'$) are angles, area of an N-periodic (resp. outer polygon to the N-periodic). $A''$ is the area of the internal polygon (where orbit touches caustic), see Figure 1. All sums/products go from $i = 1$ to $N$. $k_1, k_2, k_3$ originally studied in [26]. $l_i$ and $r_i$ denote $|P''_i - P_i|$ and $|P_{i+1} - P''_i|$, respectively and $d_{ij} = |P_i - f_j|$. $\kappa_i$ denotes the curvature of the EB at $P_i$ (3). $\alpha_{j,i}$ denotes the angle $P_if_jP_{i+1}$.

3.2. Pedal Polygons. Tables 3 and 4 describe invariants found for the pedal polygons of N-periodics and the outer polygon, see Figure 2.

3.3. Pedals with respect to N-periodic. Let $Q_i$ be the feet of perpendiculars dropped from a point $M$ onto the sides of the N-periodic. Let $A_m$ denote the area of the polygon formed by the $Q_i$, Figure 2. Let $\phi_i$ denote the angle between two consecutive perpendiculars $Q_i - M$ and $Q_{i+1} - M$. Table 3 lists invariants so far observed for these quantities.
Table 2. Distance, area, and angle invariants displayed by the N-periodic, its outer and/or inner polygon. \( k_{i}, i = 116, 117, 118 \) were discovered by Hellmuth Stachel. \( k_{119} \) was co-discovered with Pedro Roitman [27] and is equivalent to \( k_{902} \). \( k_{120} \) was suggested by A. Akopyan.

![Figure 2](image-url)
3.4. Pedals with respect to the Outer Polygon. Let \( Q_i' \) be the feet of perpendiculars dropped from a point \( M \) onto the outer polygon. Let \( \phi_i' \) denote the angle between two consecutive perpendiculars \( Q_i' - M \) and \( Q_{i+1}' - M \). Let \( A_{m}' \) denote the area of the polygon formed by the \( Q_i' \).

In the spirit of [29] we also analyze centers of mass: \( C_0' = \sum Q_i' / N \) is the vertex centroid, and the area centroid \( C_2' \) of the polygon defined by the \( Q_i' \) (2). Table 4 lists invariants so far observed for these quantities.

| code          | invariant          | value | which N | M    | date | proven |
|---------------|--------------------|-------|---------|------|------|--------|
| k201          | \( |Q_1' - O| \)       | \( a'' \) | all     | \( f_1, f_2 \) | 4/20  | [4]    |
| k202.a        | \( \prod |Q_1' - M| \) (\( b'' \))^N     | even | \( f_1, f_2 \) | 4/20 | [9]   |
| k202.b        | \( \prod |Q_1' - M| \) (\( a'' b'' \))^N/2 | \( \equiv 0 \pmod{4} \) | O     | 4/20 | [9]   |
| k203.a        | \( A_{A_m}' \)      |       | \( \equiv 0 \pmod{4} \) | all  | 4/20 | ?      |
| k203.b        | \( A_{A_m}' \)      |       | \( \not\equiv 2 \pmod{4} \) | O    | 4/20 | ?      |
| k204          | \( A/A_{m}' \)      |       | \( \equiv 2 \pmod{4} \) | all  | 4/20 | ?      |
| k205          | \( \sum \cos \phi_i' \) |       | all     | all  | 4/20 | [4]    |

Table 3. Invariants of pedal polygon with respect to \( N \)-Periodic sides. \( \dagger \) \( k_{201} \) means the locus of the vertices of a pedal with respect to a focus is a circle.

3.5. Antipedal Polygons. The antipedal polygons to the \( N \)-periodic and the outer polygon are shown in Figure 3. The antipedal polygon \( Q_i^* \) of \( P_i \) with respect to \( M \) is defined by the intersections of rays shot from every \( P_i \) along \( (P_i - M)^\perp \).

Let \( A_{m} \) denote the area of the \( Q_i^* \) polygon and \( C_{0}^*, C_{2}^* \) its vertex- and signed \(^2\) area-centroids. \( C_{0}^*, C_{2}^* \) refer to centers of antipedals of the outer polygon. Table 5 lists invariants found so far for these polygons.

| code          | invariant          | value | which N | M    | date | proven |
|---------------|--------------------|-------|---------|------|------|--------|
| \( \dagger k_{301} \) | \( |Q_1^* - O| \)       | \( a \) | all     | \( f_1, f_2 \) | 4/20  | [4]    |
| k302          | \( \sum |Q_1^* - M|^2 \) |       | all     | all  | 4/20 | [9]   |
| k303.a        | \( A'/A_{m}' \)     |       | \( \equiv 2 \pmod{4} \) | all  | 4/20 | ?      |
| k303.b        | \( A'/A_{m}' \)     |       | \( \not\equiv 0 \pmod{4} \) | O    | 4/20 | ?      |
| k304          | \( A'/A_{m}' \)     |       | \( \equiv 0 \pmod{4} \) | all  | 4/20 | ?      |
| k305          | \( \prod \cos \phi_i' \) |       | all     | all  | 4/20 | [1]    |
| k306          | \( C_0^* \)         |       | all     | all  | 4/20 | [9]   |
| k307          | \( C_2^* \)         |       | even    | all  | 4/20 | ?      |

Table 4. Invariants of pedal polygon with respect to the sides of the outer polygon. \( \dagger \) \( k_{301} \) means the locus of the outer pedal with respect to a focus is a circle.

3.6. Pedals of Steiner Curvature Centroids. Referring to Figure 4, let \( P, P', P'' \) denote as before the \( N \)-periodic, outer, and inner polygons, \( A, A', A'' \) their areas, and \( K, K', K'' \) their Steiner centroids of curvature (4). Let \( P_k, P'_k, P''_k \) denote the pedal polygons of \( P, P', P'' \) with respect to \( K, K', K'' \), and \( A_k, A'_k, A''_k \) their areas.

\(^2\)Antipedals can be self-intersecting.
Figure 3. Left (resp. right): Antipedal polygons for $N = 5$ from a point $m$ with respect to the $N$-periodic (resp. its outer polygon). Vertex and area centroids $C_0^*, C_2^*$ are also shown.

| code   | invariant | value | which $N$ | $M$   | date | proven |
|--------|-----------|-------|-----------|-------|------|--------|
| $k_{401}$ | $A'_m A^*_m$ | ?     | $\equiv 2 \pmod{4}$ | all   | 4/20 | ?      |
| $k_{402}$ | $A'/A^*_m$ | ?     | $\equiv 0 \pmod{4}$ | all   | 4/20 | ?      |
| $k_{403,a}$ | $A_m A'_m$ | ?     | odd       | O     | 4/20 | ?      |
| $k_{403,b}$ | $A_m A^*_m$ | ?     | $\equiv 0 \pmod{4}$ | $f_1, f_2$ | 4/20 | ?      |
| $k_{404}$ | $A^*_m/A_m$ | ?     | $\equiv 2 \pmod{4}$ | $f_1, f_2$ | 4/20 | ?      |
| $k_{405}$ | $C_0^*$ | ?     | even      | $O, f_1, f_2$ | 4/20 | ?      |
| $k_{406,a}$ | $C_0', C_2'$ | O     | even      | O     | 4/20 | ?      |
| $k_{406,b}$ | $C_0', C_2'$ | ?     | 4         | $f_1, f_2$ | 4/20 | ?      |
| $k_{407}$ | $C_0'$ | ?     | even      | $f_1, f_2$ | 4/20 | ?      |

Table 5. Invariants of antipedal polygons.

When $N$ even, the curvature centroids are stationary at the origin, so invariants described before involving $A, A_m$ (and primed quantities) for $M = O$ apply. For odd $N$, the Curvature Centroids move along individual ellipses concentric with the EB. Invariants are observed appear on Table 6.

Combining the above with $k_{103}$ and $k_{106}$ one obtains as corollaries the fact that $A_k/A'_k$, $A_k/A''_k$, and $A'_k/A''_k$ are invariant for odd $N$.

3.7. Pairs of Focal Pedals and Antipeals. Let $\bar{Q}_{1,i}$ and $\bar{Q}_{2,i}$ be the vertices of the pedal polygon with respect to $f_1$ and $f_2$. Define $q_{1,i} = |\bar{Q}_{1,i} - f_1|$ and
Figure 4. An N-periodic $P$ is shown along with its outer $P'$ and inner $P''$ polygons. Also shown are their Steiner centroids of curvature $K, K', K''$ and the the pedal polygons $P_k, P'_k, P''_k$ with respect to said centroids.

| code | invariant | value | which N | date | proven |
|------|-----------|-------|---------|------|--------|
| $k_{501}$ | $A/A_k$ | ? | odd | 7/20 | ? |
| $k_{502}$ | $A'/A'_k$ | ? | odd | 7/20 | ? |
| $k_{503}$ | $A''/A''_k$ | ? | odd | 7/20 | ? |

Table 6. Invariants of pedal polygons of $N$-periodic, outer, and inner polygons, with respect to their Steiner Curvature Centroids.

$Q_{2,1} = |Q_{2,1} - f_2|$. Likewise, let $Q_{1,1}^*$ and $Q_{2,1}^*$ be the vertices of the antipedal polygon with respect to $f_1$ and $f_2$. Define $q_{1,1} = |Q_{1,1} - f_1|$ and $q_{2,1} = |Q_{2,1} - f_2|$.

Let $A_1$ (resp. $A_2$) denote the area of pedal polygon to $N$-periodics wrt $f_1$ (resp. $f_2$) onto the $N$-periodic, and similarly $A'_1, A'_2$ for the outer polygon focus-pedal. Table 7 list invariants so far detected involving pairs of these quantities.

Note $k_{604,a}, k_{604,b}$ can be proven via a symmetry argument, namely, area pair are equal since opposite vertices of an even $N$-periodic are reflections about the origin, as will be the pedal polygons from either focus.

3.8. Evolute Polygons. After [8], let the evolute\(^3\) polygon $R_{ev}$ of a generic polygon $R$ have vertices at the intersections of successive pairs of perpendicular bisectors to the sides of $R$; see Figure 5. So $P_{ev}, P'_{ev}, P''_{ev}$ denote the evolute polygons of $P, P', P''$, respectively, and $A_{ev}, A'_{ev}, A''_{ev}$ their areas. Trivially, at $N = 3$ the latter vanish since perpendicular bisectors concur. At $N = 4$, $P'$ is a rectangle, so $A'_{ev} = 0$. Area invariants observed for $N > 4$ appear on Table 8.

Combining the above with $k_{103}$ and $k_{106}$ one obtains as corollaries the fact that $A_{ev}/A'_{ev}, A_{ev}/A''_{ev}$ and $A'_{ev}/A''_{ev}$ are invariant for all $N > 4$.

\(^3\)The evolute of a smooth curve is the envelope of the normals [35, Evolute]. The perpendicular bisector is its discrete version.
3.9. Inversive Objects. Referring to Figure 6, let \( P_{j,i}^{-1} \) denote the inversion of \( P_i, i = 1, \ldots, N \) with respect to a unit-radius circle centered on focus \( f_j, j = 1, 2 \), and \( d_{j,i} = |P_i - f_j| \). Let \( P_j^\dagger \) denote the polygon with vertices at \( P_{j,i}^{-1} \). Let \( L_j^\dagger \) denote its perimeter, \( A_j^\dagger \) its area, and \( \theta_j^\dagger \) its \( i \)th internal angle. Identical but primed symbols

\[ \text{Table 7. Invariants between pairs of pedal polygons defined with respect to the foci.} \]

| code  | invariant | value | which N | date | proven |
|-------|-----------|-------|---------|------|--------|
| \( k_{601} \) | \( \sum q_{1,i} \sum q_{2,i} \) | ? | odd | 4/20 | ? |
| \( k_{602} \) | \( \prod q_{1,i} \prod q_{2,i} \) | ? | all | 4/20 | ? |
| \( k_{603} \) | \( \sum q_{1,i}^2 \sum q_{2,i}^2 \) | 1 | all | 5/20 | ? |
| \( k_{604,a} \) | \( \bar{A}_1 \bar{A}_2 \) | ? | odd | 4/20 | ? |
| \( k_{604,b} \) | \( \bar{A}_1 / \bar{A}_2 \) | 1 | even | 4/20 | symmetry |
| \( k_{605,a} \) | \( \bar{A}_1^2 / \bar{A}_2^2 \) | ? | odd | 4/20 | ? |
| \( k_{606} \) | \( \sum q_{1,i} / \sum q_{2,i} \) | ? | all | 4/20 | ? |
| \( k_{607} \) | \( \bar{A}_1^2 / \bar{A}_2^2 \) | 1 | \( \equiv 0 \pmod{4} \) | 10/20 | ? |
| \( k_{608} \) | \( \bar{A}_1^2 / \bar{A}_2^2 \) | 1 | even | 10/20 | ? |
| \( k_{609} \) | \( \bar{A}_1^2 / \bar{A}_2^2 \) | 1 | even | 10/20 | ? |
| \( k_{610} \) | \( \bar{A}_1^2 / \bar{A}_2^2 \) | 1 | even | 10/20 | ? |

\[ \text{Table 8. Area-ratio invariants displayed by the evolute polygons of } N \text{-periodic, outer, and inner polygons.} \]

| code  | invariant | value | which N | date | proven |
|-------|-----------|-------|---------|------|--------|
| \( k_{701} \) | \( A/A_{ev} \) | ? | > 4 | 7/20 | ? |
| \( k_{702} \) | \( A'/A'_{ev} \) | ? | > 4 | 7/20 | ? |
| \( k_{703} \) | \( A''/A''_{ev} \) | ? | > 4 | 7/20 | ? |
Figure 6. The vertices $P^{-1}_{j,i}$ of the inversive polygon $P_1$ are obtained by inverting N-periodic vertices with respect to a unit-radius circle (not shown) centered on the left focus. The inversive focal spokes connect said focus to the vertices of $P_1$.

refer to the inversion of the outer polygon (vertices $P'_i$) with respect to $f_j$; see Figure 7.

Let $P'^\oplus_i$ (resp. $P'^\otimes_i$) denote the inversion of outer (resp. N-periodic) vertices with respect to the billiard (resp. caustic) ellipse. Recall the inversion of a point wrt to an ellipse is the midpoint of the chord joining the tangents from said point to said ellipse [17]. So the polygon $P'^\oplus_i$ (resp. $P'^\otimes_i$) has vertices at the side midpoints of N-periodic (resp. inner polygon). Let their areas be denoted $A'^\oplus_i$ and $A'^\otimes_i$, respectively.

Referring to Figure 8, let $A_{j,\text{pol}}$ and $A_{j,\text{dual}}$ denote the areas of the polar and dual polygons with respect to $f_j$, respectively. Let $\psi_{j,i}$ (resp. $w_i$) refers to the polar’s ith angle (resp. dual’s ith sidelength). Let $A_{j,\text{ant}}$ denote the area of the antipedal polygon wrt $f_j$, i.e., $A^\ast_M$ for $M = f_j$. Recall that the dual is the inverse of the pedal [7], therefore the inversive and antipedal polygons are also inverses of each other. Let $f'_1, f'_2$ be the foci of the elliptic locus of the outer polygons’ vertices (non-confocal though concentric and axis aligned with the billiard pair). Let $P'^\dagger_j$ denote the inversion of the $P'_i$ wrt $f'_j$ and $A'^\dagger_j$ denote the of the polygon defined by the $P'^\dagger_j$.

Table 9 lists invariants of inversive objects defined with respect to a chosen focus (e.g. $f_1$), whereas Table 10 presents invariants involving a pair of inversive objects, defined with respect of both foci.

4. Experimental Method

A numeric/visualization toolbox was developed in Wolfram Mathematica [36] to accurately calculate and display N-periodics while reporting their areas, angles, etc., and those of some derived objects (pedal and inversive polygons, etc.); see Figure 9.
Figure 7. The pair of inversive orbit (resp. outer) polygons is obtained by inverting the N-periodic (resp. outer polygon) with respect to a circle centered on each focus.

Table 9. Invariants of inversive objects over the N-periodic family. **$k_{803}$ Co-discovered with Pedro Roitman [27]. †$k_{817}$–$k_{818}$ The inverted vertices $P_i^\oplus$ (resp. $P_i^{\ominus}$) are at the midpoints of the inner (resp. N-periodic orbit) segments. †$k_{814}$ Discovered by A. Akopyan [5].
Figure 8. Polygons derived from N-periodics: (i) inversive: vertices are inversions of the \( P_i \) wrt to a unit circle centered on a focus, e.g., \( f_1 \); (ii) polar: antipedal of the inversive wrt to \( f_1 \). The locus of its vertices is a circle non-concentric with an also circular caustic. Let \( O_p \) denote the latter’s center. (iii) dual: inversion of the polar with respect to \( O_p \). This produces a Poncelet family inscribed in a circle centered on \( O_p \) and circumscribed about an ellipse with one focus on \( O_p \).

Table 10. Invariants of pairs of inversive objects defined with respect to the foci \( f_1, f_2 \). As observed by A. Akopyan, \(*k_{902}\) is in fact equivalent to \( k_{119} \), see (3).
Since all trajectories in the Billiard family are tangent to the same confocal caustic, a crucial calculation is to obtain one caustic semiaxis, e.g., $a''$ for a given choice of $a, b, \text{and } N$. We achieve this for all $N$ via non-linear least-squares minimization [20] of the bisection error. Namely:

- Initialize $N$ vertices $P_i$ evenly across the ellipse (pick $t_i, i = 1, \ldots, N$ for each), and let $P_1 = (a, 0)$.
- Let $b_i$ be the unit bisector of the $N$-gon sides incident at $P_i$. Let $n_i$ denote the ellipse normal at the $P_i$. The $P_i$ will be a legitimate closed billiard trajectory if all bisectors are perfectly aligned with the local normals, i.e., if $P_i^*$ can be found which make the following error vanish:

$$E = \sum_{i=1}^{N} (n_i^T b_i)^2$$

- Obtain the unique confocal ellipse tangent to $[a, 0]P_2^*$.

Notice only $N/2$ vertices for $N$ odd (resp. $N/4$ for $N$ even) need to be optimized if one exploits the symmetries of odd (resp. even) vertex positions when $P_1 = (a, 0)$.

In terms of identifying invariants, we look for quantities which over hundreds of configurations of a given $N$-family are statistically constant, maintained over a range of Billiard aspect ratios.

5. Video List

Videos of some of the above phenomena have been placed on a Youtube playlist [25] and are listed individually on Table 11.
| Id | Title                                                                 | N  | Link                                      |
|----|----------------------------------------------------------------------|----|-------------------------------------------|
| 01 | Area Invariants of Pedal and Antipedal Polygons                      | 3  | LN623VjeefFQ                              |
| 02 | Exploring invariants of N-Periodics and pedal polygons               | 3-12 | 2yXb0V7qf7k                             |
| 03 | Centroid Stationarity of Pedal Polygons                              | even | j_GD_g8aIbg                            |
| 04 | Equal sum of distances from foci to vertices of Antipedal Polygon   | 3-6 | 6F7Y3UKJzdkg                           |
| 05 | Conyclic feet of focal pedals and product of sums of lengths for odd N | 5,6 | OT-xAdb0p8o                            |
| 06 | Invariant altitudes of N-Periodics and outer polygons I             | 3,4 | MvZhWbI6iB8                             |
| 07 | Invariant altitudes of N-Periodics and outer polygons II            | 5,6 | ZMHlWxeKrM                              |
| 08 | Sum of focal squared altitudes to outer polygon                     | 3-8 | VUtBRzmb0YU                             |
| 09 | Sum of square altitudes from arbitrary point to outer polygon       | 5  | RImeROZNgj8                              |
| 10 | Area products of focal pedal polygons                               | 5  | sw8pJFMW00w                              |
| 11 | Area ratios of Pedal Polygons                                       | 5,6 | 6F7Y3UKJzdkg                           |
| 12 | Invariant Area Ratios to Minimum-Area Steiner Pedal Polygons        | 5  | f0JwRlu7iaY                              |
| 13 | N-Periodic Inversive Invariants                                    | 5  | wkstGKq5jo0                              |
| 14 | N-Periodic Inversive Objects                                        | 5  | bFeshskizls                             |
| 15 | Odd N-Periodics in the Elliptic Billiard: Invariant Area Product of Focus-Inversive Polygons | 5  | bTkbdEPNUOY                             |
| 16 | Invariants of Inversive, Polar, and Dual Polygons derived from Billiard N-Periodics | 5  | qyAHOw32NXy                             |
| 17 | Centers of N-Periodic Inversive Arcs: Bicentric Poncelet Family w/ Invariants | 5  | mXkK_4RYnU                              |
| 18 | \( N = 6, a/b = 2 \) antipedal polygon has zero signed area         | 5  | f0AES-CzjNI                              |

Table 11. Youtube list of videos about invariants of N-Periodics, the last column provides the link.

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### Appendix A. Table of Symbols

| symbol         | meaning                                                                                                                                 |
|----------------|-----------------------------------------------------------------------------------------------------------------------------------------|
| $O, f_1, f_2, N$ | center and foci of billiard                                                                                                             |
| $(a, b), (a'', b'')$ | billiard (resp. caustic) major, minor semi-axes                                                                                      |
| $N, L, J$      | trajectory sides, inv. perimeter and Joachimsthal’s constant                                                                           |
| $P_i, P'_i, P''_i$ | N-periodic, outer, inner polygon vertices                                                                                           |
| $d_{j,i}, l_i, r_i$ | distances $|P_i - f_j|, |P''_i|, |P_{i+1} - P''_i|$                                                                         |
| $\theta_i, \theta'_i, \alpha_{j,i}$ | N-periodic, outer poly, and $P_i f_j P_{i+1}$ angles                                                                                 |
| $A, A', A''$ | N-periodic, outer, inner areas                                                                                                       |
| $M$            | a point in the plane of the billiard                                                                                                 |
| $Q_i, Q'_i, Q''_i$ | vertices of the pedal polygon of $N$-periodic, outer, inner polygon wrt $M$                                                          |
| $Q'_i, Q''_i$ | vertices of the antipedal polygon of the $N$-periodic, outer, inner polygon wrt $M$                                                   |
| $\phi_i, \phi'_i, \phi''_i$ | ith angle of pedal polygon of $N$-periodic, outer, inner polygons wrt $M$                                                               |
| $A_{M}, A'_{M}, A''_{M}$ | areas of $Q_i, Q'_i, Q''_i$ polygons                                                                                               |
| $C_0, C'_0, C''_0$ | vertex centroids of the $Q_i, Q'_i, Q''_i$ polygons                                                                                   |
| $C_2, C'_2, C''_2$ | area centroids of the $Q_i, Q'_i, Q''_i$ polygons                                                                                     |
| $Q_{j,i}, Q'_{j,i}$ | vertices of $N$-periodic pedal, antipedal polygon wrt. $f_j$                                                                       |
| $g_{j,i}, g'_{j,i}$ | $[Q_{j,i} - f_j]$ and $[Q'_{j,i} - f_j]$                                                                                            |
| $A_{j}, A'_{j}$ | areas of $Q_{j,i}$ and $Q'_{j,i}$ polygons                                                                                            |
| $K, K', K''$ | Steiner centroids of curvature of $P, P', P''$                                                                                         |
| $P_k, P'_k, P''_k$ | Pedal Polygons of $P, P', P''$ wrt. $K, K', K''$                                                                                      |
| $A_k, A'_k, A''_k$ | Areas of $P_k, P'_k, P''_k$                                                                                                           |
| $P_{ev}, P'_{ev}, P''_{ev}$ | Evolute Polygons of $P, P', P''$                                                                                            |
| $A_{ev}, A'_{ev}, A''_{ev}$ | Areas of $P_{ev}, P'_{ev}, P''_{ev}$                                                                                              |
| $P^{-1}_{j,i}$ | inversion of $P_i$ wrt. to unit-radius circle centered on $f_j$                                                                       |
| $P^+_j, L^+_j, A^+_j$ | polygon defined by $P^{-1}_{j,i}$, its perimeter, and area                                                                          |
| $\theta^+_j$ | internal angle of $P^+_j$ at $P^{-1}_{j,i}$                                                                                           |
| $f^+_j, f''_j$ | foci of the elliptic locus of the $P'_j$                                                                                              |
| $P''_{j}, A''_{j,i}$ | inversion of $P'_j$ wrt to $f'_j$, area of $P''_j$ poly                                                                               |
| $P^{\oplus}, P^{\ominus}$ | inversion of $P_i$ (resp. $P'_i$) wrt caustic (resp. billiard)                                                                      |
| $A^{\oplus}, A^{\ominus}$ | areas of $P^{\oplus}, P^{\ominus}$                                                                                                 |
| $\psi_{j,i}, \psi_{j,i}$ | ith angle of polar (side length of dual) polygon wrt $f_j$                                                                               |
| $A_{j, pol}, A_{j, dual}$ | area of polar, dual polygon wrt $f_j$                                                                                               |
| $A_{j, ped}, A_{j, ant}$ | area of pedal, antipedal polygon wrt $f_j$                                                                                           |

Table 12. Symbols used in the invariants. Note $i = 1, ..., N$ and $j = 1, 2$. Note any single (resp. double) primed quantities apply to the outer (resp. inner) polygons.
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Data Science Consulting, Rio de Janeiro, Brazil
Email address: dreznik@gmail.com

Math & Statistics Institute, Federal University of Goiás, Goiânia, Brazil
Email address: ragarcia@ufg.br

Federal University of Rio de Janeiro, Rio de Janeiro, Brazil
Email address: jairkoiller@gmail.com