Harmonic mode-locking and sub-round-trip time nonlinear dynamics of electro-optically controlled solid state laser

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Abstract. Harmonic mode-locking in a solid state laser due to optoelectronic control is studied numerically on the basis of two methods. The first one is detailed numeric simulation taking into account laser radiation fine time structure. It is shown that optimally chosen feedback delay leads to self-started mode-locking with generation of desired number of pulses in the laser cavity. The second method is based on discrete maps for short laser pulse energy. Both methods show that the application of combination of positive and negative feedback loops allows to reduce the period of regular nonlinear dynamics down to a fraction of a laser cavity round trip time.

1. Introduction

Despite the fact that the “laser era” lasts for more than 50 years, the problem of regular and chaotic temporal structure formation remains the matter of topical interest. Evidence of this is a great number of articles, reviews, and book publications, [1–4] to name a few. Soon after the creation of the first solid state laser, negative feedback was applied for spike elimination in a pulsed free-running laser [5]. Later it was shown that feedback control allows not only to stabilize laser output, but to realize the regimes that are based on nonlinear dynamics, which is inherent in this control method due to the presence of time delay [1,2]. In the present time, it is clear that the delayed feedback control makes output stabilization possible only in a limited gain range [2,6]. Maximum gain can be determined from the system of equations that describe the laser generation process. Further increase of gain above this value results initially in regular nonlinear dynamics (in the form of envelope modulation), and then in chaotic laser output. Problems of time delayed control arise frequently in the analysis of self-oscillating systems of other types. Generation of regular bursts with period in the range from several round trip times \( T_r \) up to several hundred \( T_r \) via nonlinear dynamics manifestation (using nonlinear dynamics methods) [7–9] was not only of theoretical relevance but also of practical interest. It is combination of positive and negative feedback loops that opens up new control possibilities. Such control allows to choose new time scale of regular dynamics: it was shown that combination of feedback loops can realize the time scale in the range of \((1–100)/T_r\) [8,10,11]. Feedback erasure [12] allows to overcome the inertial action of feedback and to reduce the scale of dynamics. The control which combines two
feedback loops and external modulation of losses allowed to look at the problem of the period
doubling bifurcation in solid state lasers in a new fashion [10, 11]. Matter of current interest is
the question of further diminishing of the time scale. Is it possible to observe nonlinear dynamics
on the time scale less than $T_r$? Is the combination of inertial feedback loops sufficient for this
purpose?
In this paper we consider the prospects for pushing the regular and chaotic nonlinear
laser dynamics into the time scale below $T_r$ based on the idea of combining harmonic mode
locking [13, 14] with dual optoelectronic feedback [9]. Recent experiments on harmonic laser
mode locking by means of optoelectronic feedback [15] play in favor of this perspective.

2. Harmonic mode locking in solid state laser with optoelectronic control
Feedback control gives the unique possibility to order laser radiation fine time structure. The
responding idea of ordering radiation fine time structure was stated in [16]. The authors
concluded that, by choosing the delay $T_d$ of sufficiently fast negative feedback, it is possible to
obtain either the regime of mode-locking with generation of one pulse per axial interval or the
regime of smoothing of radiation fine time structure.

The importance of feedback delay $T_d$ is clearly demonstrated by the following consideration.

Figure 1. Delay-free-feedback controlled laser: a) $RC$ circuit of the optoelectronic control
scheme: $i(t) —$ photodiode current, $R —$ discharge resistor, $C —$ modulator capacity; b) the
simplified scheme of the laser with the electro-optical modulator when delay is set to minimal
technical value $\Delta$, $M1, M2 —$ cavity mirrors.

Figure 2. Temporal dependence of laser intensity $I(t)$, photocurrent $i(t)$, Pockels cell control
voltage $U(t)$, and the cell transmission $P(t)$. $T_d$ is the feedback delay, i.e. the time interval
between the laser pulse arrival in the cell and the beginning of photocurrent response caused by
the same pulse in the feedback control circuit. a — the case of a single pulse per axial interval,
b — three pulses per axial interval. Time is normalized to $T_r$. 

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We assume the sensitivity of the negative feedback to be large enough to exclude the active medium saturation. An example of a laser controlled by the negative feedback loop based on an intracavity electro-optical modulator (Pockels cell) is presented schematically in Fig. 1. The Pockels cell is controlled by the photocurrent $i(t)$ generated in a fast photodiode by the light deflected to the feedback loop by a beam splitter. In the simplest case, electric circuit includes the Pockels cell crystal capacity $C$, the resistor $R$, and the photodiode as a current source. The time constant of this control circuit is $\tau = RC$ which characterizes the feedback inertiality. The Pockels cell transmission is

$$P(U) = \cos^2 \left( \frac{\pi}{2} \cdot \frac{U + U_0}{U_{\lambda/4}} \right),$$

where $U_0$ is the static bias voltage, $U$ is the voltage generated by laser pulse, $U_{\lambda/4}$ is the quarter-wave voltage (i.e. needed to turn off the cell). If the control is designed as negative feedback ($0 < U + U_0 < U_{\lambda/4}$), then with the increase of the laser intensity $I$ voltage $U$ increases and $P$ decreases.

The choice of the feedback delay for the mode-locking regime is illustrated in Fig. 2. Suppose that a single and short enough laser pulse has been formed and circulates in a laser cavity, and the cavity round trip time is $T_r$. The sequence of ultrashort light pulses would generate photocurrent pulses with period $T_r$ (we assume here that the duration of a laser pulse $\tau_l$ and the response time of the photodiode $\tau_{ph}$ are constrained by the inequality $\tau_l < \tau_{ph} \ll T_r$). In its turn, the sequence of photocurrent pulses, when fed to the $RC$ circuit, forms on the capacitor $C$ a sawtooth voltage with a short front (determined by the photocurrent pulse duration) and a long tail (determined by $\tau_l$). The sawtooth voltage has maximum and minimum values $U_{max}$ and $U_{min}$, that are related by the feedback memory

$$U_{min} = U_{max} e^{-T_r/\tau_l}. \quad (2)$$

The regime of mode-locking occurs in a number of regions of feedback delay time $T_d$. The first region corresponds to the minimal technical delay of the control action $\Delta$ (Fig. 2a), see [15] for details. In this case, photodiode current generated by the control pulse, i.e. a part of laser pulse deflected by the beam splitter, charges cell capacitor in time delay $\Delta$ after the reflection of the laser pulse by mirror M1 and its passage through Pockels cell. The rest of the same laser pulse amplified in active medium returns to the cell at the moment of its nearly maximum discharge (nearly maximum transmission). Therefore the transmission of the cell in the moment of the laser pulse passage is determined by this pulse energy and by the energy of preceding pulses taking into account the discharge of capacity in the control circuit. In the case of mode-locking with one pulse per axial interval, $[T_r - 2T_r]$ is a suitable range of the discharge time $\tau$ for self-sustained short pulse generation [17, 18]. The increase of feedback delay by one or more $T_r$ gives the possibility to tune the delay so that each pulse passes the cell exactly at its maximum transmission. This allows one to obtain the shortest laser pulse duration.

It is possible to determine the delay required to generate $k \geq 2$ pulses per $T_r$. An example with three pulses is presented in Fig. 2b. Consider the case of mode-locking with generation of a train with pulse separation interval $T_r/k$. We consider the case when the pulse is controlled by its part reflected into feedback loop. It passes through the modulator at the maximum discharge of its capacity. This is achieved after time interval $T_r/k$. Thus the delay of feedback should be equal to $(1 - 1/k)T_r \quad (15)$. The increase of the delay by $mT_r$, $m = 0, 1, 2, ...,$ conserves the conditions for $k$ pulses generation per $T_r$. In general, ordered fine time structures of a certain kind arise periodically with the increase of delay exactly by $T_r$. It should be noted that for short pulse generation the optimal value of discharge time $\tau$ decreases with the increase of $k$. The maximum number of short pulses per $T_r$ is limited by the photocurrent response time of the photodiode used in the feedback loop.
Finally, to find the condition for smoothing fine time structure, one can use the approach similar to that in the case of a single pulse generation. It is clear that if the feedback delay is chosen in such a way that a speculative pulse passes through the cell in the moment of minimum transmission, then feedback control results in the suppression of the pulse and thus in the smoothing of laser intensity. The best smoothing is realized when time delay is smaller than $T_r$ approximately by $2\tau_{ph}$.

3. Numerical model of the feedback control by negative and positive loops

Let us trace the field evolution in a laser cavity one cavity-round-trip forward taking into account the action of intracavity elements in the laser with combination of two feedback loops (see Fig. 3). The dual-feedback control can be organized on the basis of two-sectional electro-optic modulator, e.g. in [7] we used temperature-compensated lithium tantalate modulator. Assume that at a laser round-trip with number $K$, $K = 0, 1, 2, ... K_{max}$, the laser field amplitude is expressed as $A_K(t)$, $t \in [K \cdot T_r, (K + 1) \cdot T_r]$. Below each function is calculated on the same interval. The number of round-trip is incremented after the passage of intracavity polarizer. The alteration of $A_K(t)$ in the laser cavity is attributed to several factors described in Subsections 3.1–3.2.

3.1. Amplification in active medium

To describe the laser radiation passage through active medium at round-trip with number $K$ we follow the approach of [19] which is valid for large gain values. The relation between incoming radiation amplitude $A_K^{in}(t)$ and outcoming radiation amplitude $A_K^{out}(t)$ for two-pass case is:

$$A_K^{out}(t) = \frac{\exp\left(\frac{3}{2}gK\right)}{2\sqrt{\pi}r_0\sqrt{gK}} \int_{-\infty}^{\infty} A_K^{in}(t') \exp \left(\frac{t - t'}{2\tau_a} - \frac{(t - t')^2}{4\tau_a^2gK}\right) dt'. \quad (3)$$

Here $\tau_a = \frac{1}{\Delta \nu_\Gamma}$ is the lifetime of upper laser level, $\Delta \nu_\Gamma$ is active medium linewidth, the form of Lorentz line $\Gamma(\nu_0 - \nu)$ is

$$\Gamma(\nu_0 - \nu) = 2\tau_a \frac{1}{1 + (\nu_0 - \nu)^2 \Delta \nu_\Gamma^2}. \quad (4)$$

At each round-trip, the spontaneous emission noise $N(t)$ is added to the laser radiation field. In calculation it is convenient to set the field amplitude $N(t)$ of spontaneous emission of an active medium in time representation as a sum of $n_{max}$ harmonics with arbitrary phases $\phi_n$. We take $n_{max} = 2n_g$, with the number of involved harmonics $n_g$ determined by the gain line width [20]: $n_g = \Delta \nu_t / \Delta \nu_r = 2L/(c\tau_a)$, where $\nu_r$ is the interharmonics spectral interval, $c$ is the speed of light, $L$ is the cavity length.
\[ N_K(t) = N_s \sum_{n=0}^{n_{\text{max}}} \exp(-n^2/n_0^2) \exp \left( i\phi_n + i \frac{2\pi nt}{T_r} \right), \quad t \in [K \cdot T_r, (K + 1) \cdot T_r], \quad (5) \]

where \( N_s \) is spontaneous emission noise amplitude, the first factor in the sum is mode amplitude according to the distribution of inside the gain linewidth, while the second factor corresponds to phase evolution of each longitudinal mode. Phase \( \phi_n \) is randomly chosen from the interval \([0, 2\pi]\).

### 3.2. Losses and shaping in electrooptic modulator

Let the beam splitter and the output coupler reflect the fractions \( B \) and \( R \) of incident intensity correspondingly. Total passive losses equal \( L = 1 - (1 - B)^2R \). Linear decrease of amplitude \( A_K(t) \) due to passive losses is described by a factor \( P_L = \sqrt{1 - L} = (1 - B) \cdot \sqrt{R} \).

The laser radiation is temporarily shaped in the electro-optic modulator. Dependence of the laser radiation on feedback control voltage \( U(t) \) and \( U_2(t) \) is given by

\[ P(t) = P(U_1(t), U_2(t)) = \cos^2 \left( \frac{\pi U_0 + U_1(t) + U_2(t)}{2U_{\Lambda/4}} \right). \quad (6) \]

Control voltages \( U_1(t) \) and \( U_2(t) \) at the capacitors \( C_1 \) and \( C_2 \) are described by the following equations:

\[ \frac{dU_1(t)}{dt} + \frac{U_1(t)}{\tau_1} = \frac{i\alpha_1(t)}{C_1}, \quad \frac{dU_2(t)}{dt} + \frac{U_2(t)}{\tau_2} = \frac{i\alpha_2(t)}{C_2}, \quad (7) \]

where \( \tau_1 = R_1C_1, \tau_2 = R_2C_2 \) are the time constants of discharge circuits. Photocurrent \( i_{\alpha 1,2} \) in each control circuit is calculated as the convolution of the laser intensity with the photocurrent response function taking into account feedback loop sensitivities and optical delays:

\[ i_{\alpha 1,2} = k_{1,2} \int_0^{+\infty} I(t-t'-T_r-T_{d,1,2})(1-P(t-t'-T_r-T_{d,1,2}))P\alpha_{1,2}(t')dt', \quad (8) \]

where \( P\alpha_{1,2}(t) \) are time response functions of photodiodes in feedback circuits. The simulation provides either the triangular or the Gaussian and rectangular photocurrent response function of a photodiode. In each case the response time of photodiode \( \tau_{\alpha ph} \) is full width at half maximum of the response function. Parameters \( k_1 \) and \( k_2 \) correspond to feedback sensitivity, \( T_{d,1} \) and \( T_{d,2} \) are optical delays in feedback circuits. Relative sensitivity of feedback circuits is \( \alpha = k_2/k_1 \). Laser radiation intensity is

\[ I_K(t) = A_K(t)A_K^*(t), \quad (9) \]

where \( A_K^* \) is complex conjugate of \( A_K \).

### 3.3. Final form for amplitude transformation

By collecting all terms that describe energy alteration from Subsections 3.1–3.2 we write final form of pass-to-pass amplitude \( A \) transformation:

\[ A_{K+1}(t) = P(t)P_L \frac{\exp(\gamma K)}{2\sqrt{\pi}\tau_{\alpha} \sqrt{gK}} \int_{-\infty}^{\infty} A_K(t') \exp \left( \frac{t-t'-\gamma a \tau \phi}{2\gamma a \sqrt{gK}} \right)^2 dt' + N_K(t). \quad (10) \]

To investigate the dynamics we calculate total energy of laser radiation at each round trip \( E_K \):

\[ E_K(t) = \int_{KT_r}^{(K+1)T_r} I(t)dt. \quad (11) \]
For convenience, we operate with gain $G_K$ in percent of generation threshold value which is connected to $g_K$ via the formula:

$$G_K = e^{g_K}.$$  \hfill (12)

As an output result we obtain arrays of total radiation energy $\{E\}$, gain $\{G\}$, intensity of a maximum spike $\{I_m\}$, its location on the round-trip $\{T_m\}$ and duration $\{\tau_p\}$. For each $K$ we obtain fine-time-structure variables, namely $I(t)$, $U_{1,2}(t)$, $i_{1,2}(t)$, $P(t)$ for $t \in [K \cdot T_r, (K+1) \cdot T_r]$ with resolution of 0.5-1 ps. Calculation of each parameter set takes from several minutes up to 24 hours with Intel PC 2.4 GHZ, 3.25 GB RAM.

The approach allows us to investigate the laser generation dynamics from self start of mode-locking regime and remains valid for pump power levels sufficient for nonlinear dynamics manifestation.

4. Nonlinear dynamics of electro-optically controlled solid state laser
Let us discuss the graphical representation of the nonlinear laser dynamics in detail. In papers written previously by other authors, the regimes with characteristic time much longer than $T_r$ were considered [1, 2]. Our goal is to investigate the possibility of scale reduction by use of a powerful concept of feedback loops combination.

4.1. Single feedback controlled dynamics
The fastest way to test chosen combination of feedback loops is to calculate and plot $E(t)$ (total energy at each round-trip time interval) dynamics diagram. It gives the information analogous to well-known bifurcation diagrams that show dependence of steady-state amplitudes versus control parameter values. The $E(t)$ diagram is calculated with slow variation of gain in time (see Fig. 4). We fix maximum gain and find the feedback parameters that require fine tuning: relative sensitivity and delays. Then we calculate $E(t)$ while $G$ is linearly increased from round-trip to round-trip. The gain rise is slow enough (several hundred percents per 5000–50000 $T_r$) for the plot to show most important features of the dynamical systems. Different regimes can be distinguished in $E(t)$ dynamics diagrams:

![Figure 4](image.png)

**Figure 4.** Dynamics of single feedback controlled laser in the case of $k = 32$ short pulses per $T_r = 10$ ns. **a:** total laser radiation energy $E(t)$ at each $T_r$ versus time $t$. Gain $G$ is linearly increased with time and shown at the upper $x$-axis in units of the threshold gain for reference; **b-g:** normalized laser radiation intensity fine time structure $I(t)$ versus time $t$. Time $t$ is in units of $T_r$. 

• when the gain is low enough, the laser operates in regular mode-locking regime and there is one branch in the diagram,
• when gain reaches bifurcation point, periodic bursting appears as several branches when period is a multiple of $T_r$,
• if bursting period is not a multiple of $T_r$, dense point mapping is observed (gray area),
• when the gain is high, it is typical to observe chaotic regimes but gray area at the diagram cannot be distinguished from the previous case. Insets with fine time structure provide detailed information on laser dynamics.

If the nonlinear dynamics is regular, and if period is a fraction of $T_r$, we can distinguish parallel branches at the $E(t)$ diagram (see, e.g., Fig. 4d). The most interesting regimes are then investigated with constant gain: insets I–IV show steady-state behavior at constant gain values. Each inset corresponds to the behavior type shown at the diagram with arrow, and represents laser radiation intensity versus time $t$ (in round-trips $T_r$) in steady state. The dynamics studies allow to determine the parameters necessary to generate laser radiation with given properties. In simulation the following parameters of laser were used: $T_r = 10$ ns, $P_L = 50\%$, active-medium line width 120 GHz (corresponds to Nd:YAG), and photocurrent response time for both feedback loops $\tau_{ph} = 500$ ps.

It looks reasonable to seek the nonlinear dynamics regimes at a time scale less than $T_r$ when laser is mode-locked with generation of large number of pulses on $T_r$. Maximum number is determined by the value of $\tau_{ph}$ and can be much larger than $T_r/\tau_{ph}$. Example of the case with $\tau = 0.057T_r = 500$ ps $= \tau_{ph}$. $T_d = 0.98T_r$, if the laser is mode-locked by a single negative inertial feedback loop is presented in Fig. 4. The number of short laser pulses per $T_r$ was 32, when loss modulation due to feedback control was only 5%. Maximum gain for stabilization, reaches 620% of threshold value. Despite the dense fill-in of the envelope, the period of regular nonlinear dynamics is not less than $2T_r$.

To simplify the control scenario (neighboring pulses do not interact) in search of short-period dynamics and exclude the simultaneous charging of modulator capacity by two photocurrent pulses, we consider the case when triangular photocurrent pulses do not intersect at a time line. Since the photocurrent response has full width of 1 ns at zero level, the number of short pulses per $T_r$ $k = 7$ short pulses per $T_r = 10$ ns.

![Figure 5](image-url)

**Figure 5.** Dynamics of single feedback controlled laser in the case of $k = 7$ short pulses per $T_r = 10$ ns. a: total laser radiation energy $E(t)$ at each $T_r$ versus time $t$. Gain $G$ is linearly increased with time and shown at the upper $x$-axis in units of the threshold gain for reference; b–g: normalized laser radiation intensity fine time structure $I(t)$ versus time $t$. Time $t$ is in units of $T_r$. 
laser pulses \( k = 7 \) (Fig. 6) is a good choice with the photocurrent gap of few hundred ps if \( T_r = 10 \) ns. It is worth to note that at higher gain values the short pulse development takes less time. All regimes below are characterized by wide stability regions which we consider to be an important advantage of proposed systems. The scenario of nonlinear dynamics development for \( k = 7 \) presented in Fig. 5 is similar to that in the case of \( k = 32 \). Our investigation shows that in a single feedback controlled laser the dynamics with period less than \( T_r \) is not observed but chaotic regimes have high frequency: the characteristic time is interpulse interval \( T_r/k \). In the following Section we show how the dynamics changes if we add the second loop in feedback control system.

4.2. Dual feedback controlled dynamics

In this Subsection we investigate the possibility to observe dynamics with periods less than \( T_r \) under dual feedback control. In our previous works, see e.g. [12], we used the method of memory erasure to switch off the action of negative inertial feedback after one \( T_r \). This allowed to realize the dynamics of logistic map at a time scale of \( T_r \) when the relative feedback delay was set to \( T_r \). In that study when relative delay was 2\( T_r \) we obtained pulsations with period 3\( T_r \) and 4\( T_r \).

Here we can generalize the memory erasure technique for the use in the case of arbitrary choice of feedback delays provided that the delay of negative feedback is shorter. If the relative sensitivity of positive feedback is chosen according to the formula

\[
\alpha = e^{-\delta T_d/\tau},
\]

where \( \delta T_d \) is the relative delay of positive feedback, \( \tau = \tau_1 = \tau_2 \), then the action of negative feedback is switched off after a time \( \delta T_d \). Note that \( \alpha < 1 \) has a certain physical meaning: only negative feedback can play the stabilization role, therefore its contribution should be larger.

We continue the consideration of 7th harmonics mode locking. If the relative feedback delay is set to an interpulse interval, it results in dynamics of 7 independent laser pulses (Fig. 7). Each pulse is controlled by itself on the previous round-trip. Thus the observed dynamics is close to the dynamics of independent logistic maps. We are able to trace the period doubling cascade, but the period 3 is not observable if we switch on the noise. In this case periods less than \( T_r \) are not observed but chaotic dynamics has a scale of \( T_r/7 \).

![Figure 6](image)

Figure 6. Development of harmonic mode-locking regime under single feedback control, \( k = 7 \) short pulses per \( T_r = 10 \) ns, \( G = 3 \). Laser radiation fine time structure: intensity \( I(t) \) over time \( t \) on the time scale of one round-trip time.
Memory erasure technique allows to operate with feedback depth and to organize the control scenario when each pulse is controlled by a given number of pulses by increasing $\delta T_d$. If the depth is not greater than two interpulse intervals, each laser pulse is controlled by itself and its neighbor on the previous round-trip. To accomplish this we used the following parameters: negative feedback delay time $0.887T_r$ (which corresponds to the generation of 7 pulses per $T_r$), positive feedback delay $1.162T_r$ (relative delay $\delta T_d \sim 2T_r/7$), $\tau = 0.143T_r \sim T_r/7$ for both feedback loops, relative sensitivity of positive feedback $\alpha$ was set to 0.135 which corresponds to memory erasure. The behavior is presented in Fig. 9. We obtained high frequency train of short laser pulse microgroups with period $3T_r/7$.

**Figure 7.** Dynamics of dual feedback controlled laser in the case of $k = 7$ short pulses per $T_r = 10$ ns. a: total laser radiation energy $E(t)$ at each $T_r$ versus time $t$. Gain $G$ is linearly increased with time and shown at the upper $x$-axis in units of the threshold gain for reference; b–g: normalized laser radiation intensity fine time structure $I(t)$ versus time $t$. Time $t$ is in units of $T_r$.

**Figure 8.** Dynamics of dual feedback controlled laser in the case of $k = 10$ short pulses per $T_r = 10$ ns at $T_{d1} = 0.908T_r$, $T_{d2} = 0.113T_r$, $\alpha = 0.135$, $\tau_1 = \tau_2 = T_r/9$. a: total laser radiation energy $E(t)$ at each $T_r$ versus time $t$. Gain $G$ is linearly increased with time and shown at the upper $x$-axis in units of the threshold gain for reference; b–g: normalized laser radiation intensity fine time structure $I(t)$ versus time $t$. Time $t$ is in units of $T_r$. Three-stage structure with a stage width less than $T_r$ is labelled with e.
More exotic regular nonlinear dynamics was obtained when positive feedback, in contrast to erasure regime, acts earlier than negative one. The dynamics in the form of three-stage structure was observed in the gain range of 3.78–3.83 (Fig. 8, III). This was demonstrated in a mode-locking regime with generation of $k=10$ pulses per $T_r$, when $\delta T_d = -0.785 T_r$.

The set of parameter such as harmonics number, feedback combination, relative sensitivity of loops, their inertiality form a multidimensional space which is not easy to explore due to slow procedure of numeric simulation. For better coverage of the variety of dynamical regimes it is worth to apply a simpler approach based on discrete maps.

5. Discrete maps

The discrete map approach was introduced in [21] for a picosecond laser with single non-inertial negative feedback. In the case of small changes of pulse duration the energy of $(n+1)$th pulse in the cavity was described by a recurrence relation

$$E_{n+1} = R\alpha_n^2 \beta_n T_n E_n,$$

where $R$ is the loss coefficient, $\alpha_n$ is the coefficient of amplification by active medium, $\beta_n$ is the transmission coefficient of saturable absorber (dependent on pulse energy $E_n$), $T_n$ is the transmission coefficient of electro-optic modulator. In the simplest case of delay-free negative feedback

$$T_n = T_0 - \mu E_n,$$  \hspace{1cm} (15)

where $T_0$ is the initial transmission of the modulator, $\mu$ is the feedback coefficient. By substitution of $x_n = \mu E_n / T_0$, authors of [21] obtained the map

$$x_{n+1} = r_n x_n (1 - x_n),$$  \hspace{1cm} (16)

where $r_n = R\alpha_n^2 \beta_n T_0$. If the variation of active medium amplification coefficient is not taken into account, than the obtained relation is reduced to the well-known logistic map.

Later the method of discrete maps was applied for the case of combination of feedback loops in our works [9,17]. Such an approach opened up the following possibilities:

1) To allow the laser output being non-dependent of pump power and total intracavity losses variation for both continuous and pulsed lasers [17]. This is obtained due to a) the giant stability zone which is intrinsic in short-delay control and b) short steady-state build-up time which is provided by the round-trip-time delayed positive feedback.

2) In burst mode, $T_r$-shorter (compared to negative) positive feedback delayed control [7] allows the period increase up to several hundred round-trip time by variation of relative feedback sensitivity. Hence, the period pushed the burst repetition frequency to sub-MHz range.

3) In a dual-feedback-controlled mode-locked solid state laser energy of output pulse demonstrates the behavior typical for the logistic map: period doubling cascade, periodicity windows, intermittency, and deterministic chaos on the time scale of the cavity round trip time [12]. To achieve this, it is necessary that inertial delay-free negative feedback is combined with $T_r$-delayed positive feedback, which sensitivity is chosen to erase the residual action of inertial negative feedback. In this case the control parameter is pump power. This approach makes the laser to be a spectacular device for nonlinear dynamics demonstration.

4) In a dual-feedback-controlled mode-locked solid state laser burst mode with minimal period $3T_r$ can be realized. To achieve this, it is necessary that inertial delay-free negative feedback is combined with $2T_r$-delayed positive feedback, which sensitivity is chosen to erase the residual action of inertial negative feedback [11]. In this case the burst repetition frequency is equal to several tens of MHz.
Matter of current interest is to extend the map-based approach to the case of harmonically mode-locked laser. For the case of one inertial feedback and a single laser pulse in the cavity (see, e.g. [9]), we obtained a map known also as logistic map with memory

\[ x_{n+1} = rx_n \left( 1 - \sum_{m=0}^{\infty} x_{n-m} \gamma^m \right), \]  

(17)

where \( \gamma = e^{-T_r/\tau} \). For harmonic mode-locking with \( k > 1 \) we can either write down a system of \( k \) maps or compactify the system of maps into a single map with discrete time count \( T_r/k \) using

**Figure 9.** Dynamics of dual feedback controlled laser in the case of \( k = 7 \) short pulses per \( T_r = 10 \) ns and memory erasure with relative delay of \( 2T_r/7 \). a: total laser radiation energy \( E(t) \) at each \( T_r \) versus time \( t \). Gain \( G \) is linearly increased with time and shown at the upper \( x \)-axis in units of the threshold gain for reference; b–g: normalized laser radiation intensity fine time structure \( I(t) \) versus time \( t \). Time \( t \) is in units of \( T_r \). Dynamics with period \( 3T_r/7 \) is labelled with f.

**Figure 10.** Dynamics of map (21). The representation is analogous to that of Fig. 9. Inset with period \( 3T_r/7 \) is labelled with f. \( x(n) \) is the normalized energy of a short laser pulse, \( n \) is the discrete time count in units of \( T_r/7 \), \( k = 7 \).
the cyclicality of equations. Thus the map is

\[ x_{n+1} = r x_{n-(k-1)} \left( 1 - \sum_{i=0}^{M} x_{n-(k-1)-k \cdot M-i \cdot \gamma^i} \right), \quad n = 0, 1, 2..., \] (18)

where \( n \) is the number of laser pulse, \( x_n \) is the pulse energy, \( r \) is the total gain, \( 0 \leq \gamma < 1 \) describes feedback inertiality. In this case all laser pulses in the cavity interact with each other due to feedback inertiality. This case is the discrete representation of laser dynamics shown in Figs. 5 and 4 in Subsection 4.1.

Non-inertial feedback case is the system of \( k \) independent logistic maps

\[ x_{n+1} = r x_{n-(k-1)} \left( 1 - x_{n-(k-1)} \right), \quad n = 0, 1, 2..., \] (19)

that describe the situation when laser pulses do not interact (analogue of the case in Fig. 7 in Subsection 4.2). The combination of negative and delayed positive feedback loops gives

\[ x_{n+1} = r x_{n-(k-1)} \left( 1 - \sum_{i=0}^{M} x_{n-(k-1)-k \cdot M-i \cdot \gamma^i} + \alpha \sum_{i=p}^{\infty} x_{n-(k-1)-k \cdot M-i \cdot \gamma^{i-p}} \right), \quad n = 0, 1, 2..., \] (20)

where \( p \) is the delay and \( \alpha \) is the relative feedback sensitivity of positive feedback loop. By choosing the value of \( \alpha \) it is possible to cancel the members of the first sum and thus to switch off the feedback action.

Bursting with period \( 3T_r/7 \) (Fig. 10) was observed in map

\[ x_{n+1} = r x_{n-6} (1 - x_{n-6} - 0.367x_{n-7}). \] (21)

The map describes the case when a pulse interacts only with its neighbour in the laser cavity. This example illustrates that the introduction of limited interpulse interaction drastically changes the generation regime in the laser system resulting in regular collective dynamics of output pulses. The results of map simulation agree with the numeric simulation taking into account the evolution of fine time structure of laser radiation (compare Figs. 9 and 10).

To summarize in this Section, discrete maps are a simple and fast instrument to describe the features of laser dynamics not only over the stability region border but far in the nonlinear dynamics region, even predicting the appearance of regular subroundtrip dynamics in a laser. On top of that, due to their simplicity discrete maps are a promising opportunity to build a bridge from a laser to other dynamical systems.

6. Conclusion

This study shows a high potential of combining the dual feedback control with self-starting harmonic mode-locking. The proposed approach to laser control gives new possibility for ordering of output time structure and may be useful for a wide community of scientists and engineers that work in the areas of nonlinear dynamics, solid state lasers and optoelectronics.

We have shown that in the case of self-started harmonic mode-locking a laser controlled by optoelectronic feedback loop is able to generate \( k > T_r/\tau_{ph} \), short pulses per cavity round-trip time. Such control allowed to go into the collective dynamics of laser pulses coupled by inertial feedback action. The proposed approach is distinguished by high output radiation stability in the gain range of several hundred percent over the threshold. Single inertial optoelectronic feedback control allows one to initiate regular and chaotic nonlinear laser dynamics at a time scale of less than a laser cavity round-trip time \( T_r \).
Regular sequence of pulses with sub-round-trip period is realized when positive feedback delay is added to erase the negative feedback action but the interaction remains at least between two pulses of $k$. If the pulse interaction is switched off, we are able to trace period doubling cascade of $k$ independent logistic maps and chaotic dynamics has a scale of $T_r/k$. Such set of $k$ self-starting chaotic generators in a single laser cavity is a promising object for high-frequency chaotic encoding. The detailed numerical simulation proved that all presented regimes can be realized in a solid state laser with two electro-optic feedback loops.

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