Some new results for Hasimoto surfaces
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Abstract – Let $\sigma = \sigma(s,t)$ be the position vector of a curve $\Gamma$ moving on surface $M$ in $E^3$ such that $\sigma = \sigma(s,t)$ is a unit speed curve for all $t$. If the surface $M$ is a Hasimoto surface, then, the position vector $\sigma$ satisfy the following condition

$$\sigma_t = \sigma \wedge \sigma_{ss}$$

also called as smoke ring equation or vortex filament [1]. In that work, we investigate the geometric properties according to Bishop frame of Hasimoto surfaces in Euclidean 3-space. Also, we give some characterization of parameter curves given according to Bishop frame of Hasimoto surfaces.

Keywords – Hasimoto Surface, Euclidean Space, vortex filament, Bishop frame, smoke ring equation.

Introduction

In [8], Da Rios invoked what is now known as the localized induction approximation to derive a pair of coupled nonlinear equations guiding the time evolution of the torsion and curvature of vortex filament also called smoke ring equations. Additively In 1972, Hasimoto [2] demonstrated that the Da Rios equations may be associated to generate the celebrated nonlinear Schrodinger (NLS) equation of soliton theory and also in this work, he considered a proximity to the selfinduced motion of a thin isolated vortex filament moving without extending in an incompressible fluid. Finally he obtain that if the position vector of vortex filament is $\sigma = \sigma(s,t)$, then the formula

$$\sigma_t = \sigma \wedge \sigma_{ss}$$

is hold. In [8], the Da Rios equations and their composition, the NLS equation, are derived in a purely geometric manner via a binormal motion of an inextensible curve. In [3], authors discussed on the Hasimoto surface in $E^3$, where they invastiged its geometric properties and also gave some characterizations of parametric curves of this surface.

In that work, we move the study of Hasimoto surfaces started in [3] into the Minkowski space. First, we investigate the geometric properties according to Bishop frame of Hasimoto surfaces in Euclidean 3-space. Finally, we give some characterization of parameter curves given according to Bishop frame of Hasimoto surfaces.

Preliminiaries

Let $E^3$ denote the three-dimensional Euclidean space, that is, the real vector space $R^3$ endowed with the Riemann metric

$$\langle \cdot, \cdot \rangle = (d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2$$

where $(\xi_0, \xi_1, \xi_2)$ is rectangular coordinate system of $E^3$. Let $u$ an arbitrary vector in $E^3$. So, the norm of $u$ is given by $||u|| = \sqrt{\langle u, u \rangle}$, [7].

Let $\mathbb{D}$ be a simply-connected domain in $E^3(t; s)$ and $\sigma: \mathbb{D} \rightarrow E^3$ an immersion in $E^3$. If $\sigma = \sigma(s,t)$ is a parametrization of surface $M$ in $E^3$, then the unit normal vector field $N$ on $M$ is given by
where $s = \partial\sigma / \partial s$, and $\sigma_t = \partial\sigma / \partial t$, $\times$ stands for the Euclidean cross product of $E^3$ [7].

The metric $\langle, \rangle$ on each tangent plane of $M$ is determined by the first fundamental form

$$I = \langle d\sigma, d\sigma \rangle = E ds^2 + 2Fdsdt + Gdt^2$$

with differentiable coefficients

$$E = \langle s_s, s_s \rangle, F = \langle s_s, s_t \rangle, G = \langle s_t, s_t \rangle$$

Since we have,

$$det I = EG - F^2$$

The shape operator of the immersion is indicated by the second fundamental form

$$II = -(dN, d\sigma) = es^2 + 2fdsdt + gdt^2$$

with differentiable coefficients

$$e = \langle s_{st}, N \rangle, f = \langle s_{st}, N \rangle, g = \langle s_{tt}, N \rangle.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve $\Gamma$ has vanishing second derivative. One can state parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [5]. The tangent vector and any convenient arbitrary basis for remainder of the frame are used [5,6]. The Bishop frame is expressed as:

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix}_t = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ z \end{bmatrix},$$

(1)

where the set of $\{t, y, z\}$ is called as Bishop trihedra and the functions $k_1$ and $k_2$ are the Bishop curvatures (see for details in [4]).

**Main results**

In this section, as we mentioned before, we move the study of Hasimoto surfaces started in [3] into the Minkowski space. So, we would like to give our main aim as following:

**Main theorem:** Let $\sigma = \sigma(s,t)$ be the position vector of a curve $\Gamma$ moving on surface $M$ in Euclidean 3-space such that $\sigma = \sigma(s,t)$ is a unit speed curve for all $t$. Then the derivatives of followings are satisfied:
where \( \{t,y,z\} \) is the Bishop frame field and the functions \( k_1 \) and \( k_2 \) are the Bishop curvature functions of the curve \( \Gamma \) for all \( t \).

**Proof.** We would like to obtain time derivatives of the Bishop frame \( \{t,y,z\} \) which is given the form

\[
\begin{bmatrix}
  t \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
  0 & -(k_2)_t & (k_1)_t \\
  (k_2)_t & 0 & -k_1^2 + k_2^2/2 \\
-(k_1)_t & k_1^2 + k_2^2/2 & 0
\end{bmatrix}
\begin{bmatrix}
  t \\
y \\
z
\end{bmatrix}
\tag{2}
\]

On the other hand from imposition of condition \( \sigma_{st} = \sigma_{ts} \), we find the following equalities

\[
\alpha_s = (k_1)_s - k_2 y, \\
\beta_s = (k_2)_s + k_1 y, \\
\gamma_s = -k_1 \beta - k_2 \alpha.
\]

Now, we assume that the velocity of the curve is of the form

\[
\sigma_t = \lambda t + \mu y + \theta z.
\]

One can choose the correspondence for the surface \( M \) as \( \{\lambda, \mu, \theta\} \rightarrow \{0, -k_2, k_1\} \). Thus, the velocity vector is given by

\[
\sigma_t = \sigma_x \times \sigma_{ss} = -k_2 y + k_1 z
\]

which is solution of smoke ring equation. Hence, we can rewrite (4)2,3 under the correspondence as

\[
\alpha = -(k_2)_s, \quad \beta = (k_1)_s.
\]

Substituting the last equations into (4)1 gives

\[
\gamma = -\frac{k_1^2 + k_2^2}{2}.
\]

Thus, the proof of main theorem is completed.

**Some Characterization of Parameter Curves of Hasimoto surfaces**

In this section, we would like to give new characterizations of parameter curves of Hasimoto surfaces in Euclidean 3-spaces.

**Theorem.** Assume \( \sigma = \sigma(s,t) \) is a Hasimoto surface in \( \mathbb{E}^3 \). Then the followings are satisfied:

i. \( s \)-parameter curves of the surface \( \sigma = \sigma(s,t) \) are geodesics,

ii. \( t \)-parameter curves of the surface \( \sigma = \sigma(s,t) \) are geodesics if and only if

\[
k_1(k_1)_t + k_2(k_2)_t = 0
\]
where, \( k_1 \) and \( k_2 \) are Bishop curvature functions of the curve for all \( t \).

**Theorem.** Assume \( \sigma = \sigma(s,t) \) is a Hasimoto surface in \( \mathbb{E}^3 \). Then the followings are satisfied:

i. \( s \)-parameter curves of the surface \( \sigma = \sigma(s,t) \) are asymptotics if and only if \( \kappa = 0 \),

ii. \( t \)-parameter curves of the surface \( \sigma = \sigma(s,t) \) are asymptotics if and only if

\[
2(k_2(k_1)_t + k_1(k_2)_t) = (k_1^2 + k_2^2)^2
\]

where, \( k_1 \) and \( k_2 \) are Bishop curvature functions of the curve for all \( t \).

**Corollary.** If \( s \)-parameter curves of a Hasimoto surface \( \sigma = \sigma(s,t) \) in \( \mathbb{E}^3 \) are asymptotics, then the \( t \)-parameter curves are also asymptotics.

**Corollary.** The parameter curves of a Hasimoto surface \( \sigma = \sigma(s,t) \) in \( \mathbb{E}^3 \) are lines of curvature if and only if

\[
k_1(k_1)_s + k_2(k_2)_s = 0.
\]

**Conclusion**

In this paper we studied the Hasimoto surfaces in Euclidean 3-spaces. Also we obtained the time derivatives of Bishop trihedra \( \{t, y, z\} \) of the curve moving on Hasimoto surfaces. After, we obtained some characterizations of parameter curves of Hasimoto surfaces in \( \mathbb{E}^3 \).

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