Fluctuations around Periodic BPS-Density Waves in the Calogero Model

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Abstract: The collective field formulation of the Calogero model supports periodic density waves. An important set of such density waves is a two-parameter family of BPS solutions of the equations of motion of the collective field theory. One of these parameters is essentially the average particle density, which determines the period, while the other parameter determines the amplitude. These BPS solutions are sometimes referred to as “small amplitude waves” since they undulate around their mean density, but never vanish. We present complete analysis of quadratic fluctuations around these BPS solutions. The corresponding fluctuation hamiltonian (i.e., the stability operator) is diagonalized in terms of bosonic creation and annihilation operators which correspond to the complete orthogonal set of Bloch-Floquet eigenstates of a related periodic Schrödinger hamiltonian, which we derive explicitly. Remarkably, the fluctuation spectrum is independent of the parameter which determines the density wave’s amplitude. As a consequence, the sum over zero-point energies of the field-theoretic fluctuation hamiltonian, and its ensuing normal-ordering and regularization, are the same as in the case of fluctuations around constant density background, namely, the ground state. Thus, quadratic fluctuations do not shift the energy density tied with the BPS-density waves studied here, compared to its ground state value. Finally, we also make some brief remarks concerning fluctuations around non-BPS density waves.

Keywords: Calogero Model, Collective-Field Theory, BPS, Solitons, Fluctuations, Floquet-Bloch.
1. Introduction

In the limit of infinite number of particles and finite density, the Calogero-Sutherland models \[1, 2\] may be reformulated in terms of hydrodynamic variables, namely, the particle density and current. The Hamiltonian of this collective field reformulation is given by \[3\]

\[
H_{\text{coll}} = \frac{1}{2m} \int dx \partial_x \pi(x) \rho(x) \partial_x \pi(x) + \frac{1}{2m} \int dx \rho(x) \left( \frac{\lambda - 1}{2} \frac{\partial x}{\rho} + \lambda \int \frac{dy \rho(y)}{x - y} \right)^2 + H_{\text{sing}},
\]

where \(H_{\text{sing}}\) denotes a singular contribution \[4, 5\]

\[
H_{\text{sing}} = -\frac{\lambda}{2m} \int dx \rho(x) \frac{\partial_x P}{x - y} \bigg|_{y=x} - \frac{\lambda - 1}{4m} \int dx \partial_x^2 \delta(x-y) |_{y=x},
\]
and $P$ is the principal part symbol.

Here,

$$\rho(x) = \sum_{i=1}^{N} \delta(x - x_i) \quad (1.3)$$

is the collective - or density - field, and

$$\pi(x) = -i \frac{\delta}{\delta \rho(x)} \quad (1.4)$$

is its canonically conjugate momentum. It follows from (1.3) that the collective field is a positive operator

$$\rho(x) \geq 0 , \quad (1.5)$$

and that it obeys the normalization condition

$$\int_{-\infty}^{\infty} dx \rho(x) = N . \quad (1.6)$$

The latter constraint is implemented by adding to (1.1) a term $\mu \left( \int_{-\infty}^{\infty} dx \rho(x) - N \right)$, where $\mu$ is a Lagrange multiplier (the chemical potential).

The first term in (1.2) is linear in $\rho(x)$. Therefore, its singular coefficient

$$-\frac{\lambda^2 m}{2} \frac{\partial}{\partial x} \bigg|_{y=x}$$

amounts to a shift of the chemical potential $\mu$ by an infinite constant. The last term in (1.2) is, of course, a field independent constant - an infinite shift of energy.

For the sake of being self-contained, we briefly explain in the Appendix how to derive the collective Hamiltonian (1.1) and its singular part (1.2) from the microscopic Calogero Hamiltonian.

It is worth mentioning at this point that the Calogero model enjoys a strong-weak-coupling duality symmetry [6, 7]. At the level of the collective Hamiltonian (1.1), these duality transformations read

$$\tilde{\lambda} = \frac{1}{\lambda} , \quad \tilde{m} = -\frac{m}{\lambda} , \quad \tilde{\mu} = -\frac{\mu}{\lambda} ; \quad \tilde{\rho}(x) = -\lambda \rho(x) , \quad \text{and} \quad \tilde{\pi}(x) = -\frac{\pi(x)}{\lambda} \quad (1.7)$$

and it is straightforward to see that these transformations leave (1.1) (including the chemical potential term) invariant. The minus signs which occur in (1.7) are all important: We interpret all negative masses and densities as those pertaining to holes, or antiparticles. Thus, the duality transformations (1.7) exchange particles and antiparticles. With this interpretation we always have

$$\frac{\rho(x)}{m} > 0 , \quad (1.8)$$
and thus the first two terms in (1.1) are manifestly positive. (For more details see e.g. Section 3 of [8], and references therein.)

One of the important and interesting features of this collective-field formulation of the Calogero model is that it bears soliton solutions, which are known explicitly [4, 10, 11, 12]. They arise (to leading order in the $\frac{1}{N}$ expansion) as solutions of the classical equations of motion resulting from (1.1) [13].

Recently there has been considerable renewed interest in solitons in the Calogero-Sutherland models. In [14] it was shown that the collective-field theory of Calogero model is equivalent to a quantum version of an integrable Benjamin-Ono equation [15]. In this way, it can be seen that the semiclassical one-soliton solutions studied in [4, 10, 11, 12] correspond to a single pole in a certain pole ansatz.

The authors of [16, 17] have studied a specific duality-based generalization of the hermitian matrix model which is equivalent to a two-family Calogero model [18, 19, 20, 21]. The multi-vortex solutions of the coupled BPS equations were interpreted as giant gravitons [22]. In [23] we have shown that the coupled BPS equations collapse effectively into a single-family BPS equation at special loci in parameter space of the two family model. A similar observation was also made concerning the more general coupled non-BPS variational equations. All this was the consequence of the invariance of the collective-field Hamiltonian of the two-family Calogero model under an Abelian group of strong-weak-coupling dualities [8], analogous to (1.7). In [24] and [25] it was shown that a large class of solitons in the two-family Calogero model can be obtained by reducing it to the effectively one-family Calogero model.

These results on the collective-field solitons in various variants of the two-family Calogero model motivated us to revisit the collective-field theory of the original single-family Calogero model in [23], where we studied, among other things, the periodic soliton crystal solutions originally discovered in [10]. In this paper we focus on a particular type of such soliton crystals - the periodic BPS density waves of the collective field hamiltonian, and study their quantum stability. We present complete analysis of quadratic fluctuations around these BPS solutions. The corresponding fluctuation hamiltonian (i.e., the stability operator) is diagonalized in terms of bosonic creation and annihilation operators which correspond to the complete orthogonal set of Bloch-Floquet eigenstates of a related periodic Schrödinger hamiltonian, which we derive explicitly. The resulting fluctuation spectrum is positive, and therefore these density waves are stable.

This paper is organized as follows: In Section 2 we review the solution of the static BPS equation associated with the collective Hamiltonian (1.4). This is done by converting it into a Riccati equation which can then solve explicitly. The solution is a static periodic soliton - the finite amplitude wave solution of (10).

In Section 3 we study quadratic fluctuations around these BPS density waves and discuss their quantum-mechanical stability as was described in the paragraph
next to previous. The complete orthogonal set of Floquet-Bloch eigenstates of the related periodic Schrödinger hamiltonian, in terms of which we diagonalize the fluctuation hamiltonian, are derived explicitly in Section 4. Remarkably, the fluctuation spectrum is independent of the parameter which determines the density wave’s amplitude. As a consequence, the sum over zero-point energies of the field-theoretic fluctuation hamiltonian, and its ensuing normal-ordering and regularization, are the same as in the case of fluctuations around constant density background, namely, the ground state. We close by making some brief remarks concerning fluctuations around non-BPS density waves in Section 5. Finally, some technical details of the collective field formalism are relegated to the Appendix.

2. Review of periodic BPS solutions of the collective field equation

The Hamiltonian (1.1) is essentially the sum of two positive terms. Its zero-energy classical solutions are zero-momentum, and therefore time independent configurations of the collective field (1.3), which are also solutions of the BPS equation

\[ B[\rho] \equiv \frac{\lambda - 1}{2} \frac{\partial_x \rho}{\rho} + \lambda \int dy \frac{\rho(y)}{x - y} = 0. \]  

(2.1)

It is easy to check that the duality transformation (1.7) maps a solution \( \rho(x) \) of (2.1) with coupling \( \lambda \) onto another solution \( \tilde{\rho}(x) = -\lambda \rho(x) \) of that equation with coupling \( \tilde{\lambda} = \frac{1}{\lambda} \). As we shall see below in Eq. (2.21), all solutions of (2.1) are of definite sign, and never vanish along the real axis. Thus, such a positive solution of (2.1) is mapped by (1.7) onto a negative solution, and vice-versa.

The BPS equation (2.1) may be written alternatively as

\[ (\lambda - 1) \partial_x \rho = 2\pi \lambda \rho \rho^H, \]  

(2.2)

where \( \rho^H \) is the Hilbert-transform

\[ \rho^H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{\rho(y)}{y - x} \]  

(2.3)

of \( \rho \). Note that for \( \lambda = 1 \), where the CM describes non-interacting fermions, the only solution of (2.2) is \( \rho = \rho_0 = \text{const} \). This is also the case at the bosonic point \( \lambda = 0 \). Henceforth, we shall assume \( \lambda \neq 0, 1 \). Space independent constant configurations \( \rho = \rho_0 \) are obviously solutions of (2.1) also for \( \lambda \neq 0, 1 \). However, for such values of \( \lambda \), (2.1) bears also the periodic space-dependent density wave solutions, whose derivation we review in this section.

The proper way to solve this nonlinear integro-differential equation is to consider it together with its Hilbert-transform\[14, 17, 23\]

\[ (\lambda - 1) \partial_x \rho^H = \pi \lambda ((\rho^H)^2 - \rho^2 + \rho_0^2), \]  

(2.4)
where on the RHS we used the identity†

\[ 2(\rho \rho^H)^H = (\rho^H)^2 - \rho^2 + \rho_0^2 \tag{2.5} \]

(and the fact that \( \partial_x \rho^H = (\partial_x \rho)^H \) on the LHS). Here \( \rho_0 \) is a real parameter such that

\[ \int_{-\infty}^{\infty} dx (\rho(x) - \rho_0) = 0 . \tag{2.6} \]

It arises from the fact that we seek a solution of \( \rho(x) \) which need not necessarily decay at spatial infinity. Note that (2.4) is even in \( \rho_0 \). By definition, the sign of \( \rho_0 \) coincides with that of \( \rho(x) \), the solution of (2.4). A positive solution \( \rho(x) \geq 0 \) corresponds to a BPS configuration of particles, and a negative one, to a configuration of antiparticles, as was mentioned following (1.7).

We proceed as follows. Given the density \( \rho(x) \), consider the resolvent

\[ \Phi(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{\rho(y)}{y - z} \tag{2.7} \]

associated with it, in which \( z \) is a complex variable.

The resolvent \( \Phi(z) \) is evidently analytic in the complex plane, save for a cut along the support of \( \rho(x) \) on the real axis. From the identity

\[ \frac{1}{x \mp i0} = \frac{P}{x} \pm i\pi \delta(x) , \tag{2.8} \]

we obtain

\[ \Phi_{\pm}(x) \equiv \Phi(x \pm i0) = \rho^H(x) \pm i\rho(x) . \tag{2.9} \]

Thus, if \( \Phi(z) \) is known, \( \rho(x) \) can be determined from the discontinuity of \( \Phi(z) \) across the real axis.

An important property of \( \Phi(z) \), which follows directly from the definition (2.7), is

\[ \Im \Phi(z) = \Im \frac{\rho(y) \, dy}{\pi \, |z - y|^2} . \tag{2.10} \]

Thus, if \( \rho(x) \) does not flip its sign throughout its support, we have

\[ \text{sign} \left( \Im \Phi(z) \right) = \text{sign} \left( \Im z \right) \text{sign} \left( \rho(x) \right) . \tag{2.11} \]

We shall use this property to impose certain further conditions on the solution of (2.13) below.

† For a compendium of useful identities involving Hilbert-transforms see Appendix A of [23] and also Appendix A of the second paper cited in [14].
It follows from (2.9) that (2.2) and (2.4) are, respectively, the imaginary and real parts of the Riccati equation

\[(\lambda - 1) \partial_x \Phi_\pm (x) = \pi \lambda (\Phi_\pm^2 (x) + \rho_0^2)\]  

(2.12)

obeyed by both complex functions \(\Phi_\pm (x)\). Let \(\Phi_\pm (z)\) be the analytic continuations of \(\Phi_\pm (x)\) into the \(z\)-upper and lower half planes, respectively. These functions are evidently the two solutions of

\[(\lambda - 1) \partial_z \Phi(z) = \pi \lambda (\Phi(z)^2 + \rho_0^2),\]  

(2.13)

subjected to the boundary conditions \(\Phi^*_+ (x+i0) = \Phi_- (x-i0)\) and sign \((\Im \Phi_+ (x + i0)) = \text{sign} (\rho(x)) = \text{sign} \rho_0\), from (2.9). The resolvent (2.7) is then obtained by patching together \(\Phi_+ (z)\) in the upper half-pane and \(\Phi_- (z)\) in the lower half-plane.

The standard way to solve (2.13) is to write it as

\[
\left( \frac{1}{\Phi(z) - i\rho_0} - \frac{1}{\Phi(z) + i\rho_0} \right) \partial_z \Phi(z) = ik, 
\]

(2.14)

where

\[k = \frac{2\pi \lambda \rho_0}{\lambda - 1},\]  

(2.15)

is a real parameter.

Straightforward integration of (2.14) then yields the solutions

\[
\Phi_\pm (z) = i\rho_0 \frac{1 + e^{i(kz-u_\pm)}}{1 - e^{i(kz-u_\pm)}}, 
\]

(2.16)

where \(u_\pm\) are integration constants. The boundary condition \(\Phi^*_+ (x+i0) = \Phi_- (x-i0)\) then tells us that \(u_- = -u^*_+\). Clearly, \(\Im u_+\) can be absorbed by a shift in \(x\). Therefore, with no loss of generality we set \(\Im u_+ = 0\). The second boundary condition sign \((\Im \Phi_+ (x + i0)) = \text{sign} \rho_0\) then tells us that \(u \equiv \Re u_+ > 0\). Thus, \(\Phi_\pm (z)\) are completely determined and we obtain (2.7) as

\[
\Phi(z) = i\rho_0 \frac{1 + e^{i(kz-u \text{sign} (\Im z))}}{1 - e^{i(kz-u \text{sign} (\Im z))}}, 
\]  

(2.17)

As can be seen in (2.21) below, the density \(\rho(x)\) associated with (2.17) is indeed of definite sign, namely, \(\text{sign} \rho_0\).

The asymptotic behavior of (2.17) is such that

\[
\Phi(\pm i\infty) = \pm i\rho_0 \text{ sign } k. 
\]

(2.18)

This must be consistent with (2.11), which implies (together with the fact that \(\text{sign} (\rho(x)) = \text{sign} \rho_0\)) that \(k\) must be positive. In other words, as can be seen from (2.15), positive BPS density configurations \((\rho_0 > 0)\) exist only for \(\lambda > 1\), and negative
BPS densities \((\rho_0 < 0)\) arise only for \(0 < \lambda < 1\). The duality symmetry (1.7), which interchanges the domains \(0 < \lambda < 1\) and \(\lambda > 1\), maps these two types of BPS configurations onto each other. Positivity of \(k\), the condition (1.8) and the fact that \(\text{sign } \rho(x) = \text{sign } \rho_0\) imply that we always have

\[
\text{sign } \rho_0 = \text{sign } m = \text{sign } (\lambda - 1).
\tag{2.19}
\]

Now that we have determined \(\Phi(z)\), let us extract from it the BPS density \(\rho(x)\) and its Hilbert transform \(\rho^H(x)\). From (2.17) we find that

\[
\Phi_+(x) = \Phi(x + i0) = \rho_0 \frac{-\sin kx + i \sinh u}{\cosh u - \cos kx},
\tag{2.20}
\]

from which we immediately read-off the solution of the BPS-equation (2.1) as

\[
\rho(x) = \rho_0 \frac{\sinh u}{\cosh u - \cos kx},
\]

\[
\rho^H(x) = -\rho_0 \frac{\sin kx}{\cosh u - \cos kx},
\tag{2.21}
\]

where both \(k > 0\) and \(u > 0\), and the sign of \(\rho(x)\) coincides with that of \(\rho_0\). That \(\rho^H\) in (2.21) is indeed the Hilbert-transform of \(\rho\) can be verified by explicit calculation.

The static BPS-soliton, given by \(\rho(x)\) in (2.21), is nothing but the finite-amplitude solution of (10). It comprises a two-parameter family of spatially periodic solutions, all of which have zero energy density, by construction. The period is

\[
T = \frac{2\pi}{k} = \frac{\lambda - 1}{\lambda \rho_0}.
\tag{2.22}
\]

It can be checked by explicit calculation\textsuperscript{\ref{footnote:integral}} that

\[
\frac{1}{T} \int_{\text{period}} \rho(x) \, dx = \rho_0,
\tag{2.23}
\]

and therefore that \(\int_{-\infty}^{\infty} (\rho(x) - \rho_0) \, dx = 0\), as required by definition of \(\rho_0\). Thus, the parameter \(\rho_0\) determines both the period of the solution \(\rho(x)\), as well as its period-average, and the other (positive) parameter \(u\) determines the amplitude of oscillations between its extremal values

\[
\rho_{\text{min}} = \rho_0 \tanh \frac{u}{2} \quad \text{and} \quad \rho_{\text{max}} = \rho_0 \coth \frac{u}{2}.
\tag{2.24}
\]

\textsuperscript{\ref{footnote:integral}}The best way to do this computation is to change variables to \(t = e^{ikx}\) and transform the integral into a contour integral around the unit circle.
Note also from (2.23), that the number of particles per period is

\[ T\rho_0 = \frac{\lambda - 1}{\lambda}. \]  

(2.25)

A few limiting cases of (2.21) are worth mentioning. Thus, if we let \( u \to 0 \), we obtain a comb of Dirac \( \delta \)–functions

\[ \rho(x) = \frac{\lambda - 1}{\lambda} \sum_{n \in \mathbb{Z}} \delta(x - nT). \]  

(2.26)

If, in addition to \( u \to 0 \), we also let \( k \) tend to zero (or equivalently, let the period \( T \) diverge), such that \( b = \frac{\pi}{k} \) remains finite, we obtain the BPS soliton solution [10, 11]

\[ \rho(x) = \frac{\lambda - 1}{\lambda} \frac{b}{\pi b^2 + x^2}. \]  

(2.27)

In fact, the original construction of the periodic soliton (2.21) in [10] was done by juxtaposing infinite solitons (2.27) in a periodic array.

Note that the relation (2.25) is preserved in both limiting cases discussed above, since the RHS of (2.25) depends neither on \( u \) nor on \( k \).

Finally, by letting \( u \to \infty \) in (2.21), we obtain the uniform solution \( \rho = \rho_0 \) of (2.1), independently of \( k \). As can be seen from (2.13), \( k \) blows up as \( \lambda \to 1 \), namely, at the point of non-interacting fermions. Consequently, in this limit, the BPS density wave oscillates wildly between its extrema (2.24). This is clearly a pathological situation, unless these extrema coincide, which happens only when \( u \to \infty \), namely, the uniform configuration \( \rho = \rho_0 \). The latter is the only solution of (2.1) at \( \lambda = 1 \).

Thus, if we seek match smoothly our rapidly oscillating solutions in the vicinity of \( \lambda = 1 \) with the constant solution \( \rho = \rho_0 \) precisely at \( \lambda = 1 \), we have to insist that the amplitude of the density wave should vanish as \( \lambda \to 1 \), i.e., let \( u \to \infty \). More quantitatively, according to (2.24), a typical estimate of a derivative of the density wave (2.21) is \( k(\rho_{\text{max}} - \rho_{\text{min}}) = \frac{2k\rho_0}{\sinh u} \), which for large \( k \) and \( u \) behaves asymptotically like \( ke^{-u} \sim \frac{e^{-u}}{\lambda - 1} \). Thus, we must have \( e^u \gg \frac{1}{\lambda - 1} \) in (2.21), in order for the density waves to cross-over smoothly to the uniform solution at \( \lambda = 1 \). None of these complications arise at \( \lambda = 0 \), corresponding to non-interacting bosons. In this limit \( k \to 0 \) independently of \( u \), and (2.21) tends to a uniform solution trivially.

### 3. Fluctuations

The collective-field formalism provides a systematic framework for the \( \frac{1}{N} \) expansion [4, 5, 26]. By expanding the collective Hamiltonian (1.1) around the static BPS solution in (2.21), which we shall henceforth denote by \( \rho_s(x) \), we can go beyond the leading order and obtain the spectrum of low-lying excitations above \( \rho_s(x) \). Here we
shall concentrate on the next-to-leading terms, embodied in the quadratic fluctuations around \( \rho_s(x) \).

To this end we write

\[
\rho(x) = \rho_s(x) + \eta(x),
\]

(3.1)

where \( \eta(x) \) is a small density fluctuation around the wave solution \( \rho_s(x) \), being typically of order \( 1/N \). Similarly, we shift \( \mu = \mu_s + \delta \mu \), where \( \mu_s = 0 \) is the value of the chemical potential corresponding to \( \rho_s(x) \). Due to (1.6), which is already satisfied by \( \rho_s(x) \), we clearly must have

\[
\int dx \eta = 0.
\]

This latter constraint is enforced by the shifted chemical potential \( \delta \mu \), as can be seen in (3.3) below.

A convenient intermediate step is to expand the BPS combination \( B[\rho] \) around \( \rho_s(x) \). We obtain

\[
B[\rho_s + \eta] = \left[ \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho_s} \right) - \lambda \pi \eta^H \right] - \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho_s} \right)^2 + \ldots ,
\]

(3.2)

where the ellipsis stands for terms cubic in \( \eta \) and higher, and where we used the BPS equation \( B[\rho_s(x)] = 0, (2.1) \). Then, substituting (3.2) in (1.1), and expanding \( H_{\text{coll}} \) to second order in \( \eta(x), \pi(x) \) and \( \delta \mu \), we obtain the quadratic fluctuation Hamiltonian \( H_{\text{coll}}^{(2), \text{BPS}} \) around \( \rho_s(x) \) as

\[
H_{\text{coll}}^{(2), \text{BPS}} = \frac{1}{2m} \int dx \rho_s(x) \left( \partial_x \pi(x) \right)^2 + \frac{1}{2m} \int dx \rho_s(x) \left[ \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho_s} \right) - \lambda \pi \eta^H \right]^2
\]

\[
- \delta \mu \int dx \eta + H_{\text{sing}}.
\]

(3.3)

Here we used the constraint (1.6) and the BPS equation (2.1) to eliminate all terms linear in the shifted quantities \( \eta \) and \( \delta \mu \).

Let us now evaluate \( H_{\text{sing}} \) in the background of \( \rho_s(x) \). This will be also useful for later reference. The singular term multiplying \( \rho(x) \) in the first term in (1.2) is a constant, \( C \). We can evaluate this constant as a divergent integral in momentum space. To this end, note that according to (3.13) below, \( C \) is proportional to the singular matrix element \( \langle x|p|y \rangle \). Thus

\[
C = -\frac{\lambda}{2m} \partial_x \frac{P}{x-y} \bigg|_{y=x} = -\frac{\lambda \pi}{2m} \langle x|p|y \rangle = -\frac{(\lambda - 1)k}{4m \rho_0} \int \frac{dq}{2\pi} |q|,
\]

(3.4)

where we have also used (2.15). Since \( C \) is constant, according to (2.6) we may replace \( \rho_s(x) \) under the integral in (1.2) by \( \rho_0 \). Therefore, \( H_{\text{sing}} \) in this background is the same as in the familiar constant background \( \rho(x) = \rho_0 \). Combining \( \int dx C \rho_0 \),

\[\text{This integral should, of course, be cut-off at some large momentum. The point is that the same regularization should be used for both the uniform and density wave backgrounds.}\]
the first term in (1.2), with the momentum space representation of the second term there, we obtain our desired result as

\[ H_{\text{sing}} = \left( \int dx \right) \int dq \frac{\lambda - 1}{4m} \left( q^2 - k|q| \right), \]  

(3.5)

We shall make use of this explicit form later.

The hamiltonian (3.3) is essentially the sum of squares of local hermitian operators, each of which is multiplied by the positive function \( \frac{\rho_s(x)}{m} \). Thus, the expectation value of (3.3) with respect to any (normalizable) wave functional \( \Psi[\eta] \) is positive, for all values of \( \lambda \). The fluctuation spectrum about \( \rho_s(x) \) is therefore positive, and the BPS density waves \( \rho_s(x) \) are stable. This, of course, comes at no surprise, since the energy density tied with \( \rho_s(x) \) is strictly zero - the lowest possible value for (1.1).

This positivity of (3.3), i.e., its quadratic structure, clearly calls for the introduction of the operator

\[ A(x) = \partial_x \pi(x) + i \left[ \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho_s} \right) - \lambda \pi H \right] \]  

(3.6)
along with its hermitian adjoint

\[ A^\dagger(x) = \partial_x \pi(x) - i \left[ \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho_s} \right) - \lambda \pi H \right]. \]  

(3.7)

It follows from the canonical commutation relations

\[ [\eta(x), \pi(y)] = i \delta(x - y), \quad [\eta(x), \eta(y)] = [\pi(x), \pi(y)] = 0 \]  

(3.8)

that \( A \) and \( A^\dagger \) satisfy the commutation relation

\[ [A(x), A^\dagger(y)] = (1 - \lambda) \partial_x \partial_y \left( \frac{\delta(x - y)}{\rho_s(x)} \right) + 2\lambda \partial_x \frac{P}{x - y}, \]  

(3.9)

with all other commutators vanishing.

In terms of the operators (3.6) and (3.7), we may write the quadratic hamiltonian (3.3) as

\[ H_{\text{coll}}^{(2),\text{BPS}} = \frac{1}{2m} \int dx \rho_s(x) A^\dagger(x) A(x) + \frac{1}{4m} \int dx \rho_s(x) [A(x), A^\dagger(x)] + H_{\text{sing}} - \delta \mu \int dx \eta. \]  

(3.10)

As can be clearly seen by comparing (3.9) and (1.2), the commutator term in (3.10) would cancel the singular term \( H_{\text{sing}} \) if \( \rho_s \) were constant, in accordance with (3.5).
We shall see below (see (3.21)) that this commutator term, in fact, exactly cancels the expression in (3.5) also in the background $\rho_s(x)$. Thus, we obtain
\[
H_{\text{coll}}^{(2),BPS} = \frac{1}{2m} \int dx \rho_s(x) A^\dagger(x) A(x). \tag{3.11}
\]

As we shall see below, this hamiltonian requires one last zero-point energy subtraction to render it finite.

### 3.1 Diagonalization of $H_{\text{coll}}^{(2),BPS}$ in terms of bosonic creation and annihilation operators

Our goal is to diagonalize the manifestly positive hamiltonian (3.11). This could be achieved by decomposing $\sqrt{\rho_s(x)/2m} A(x)$ and $\sqrt{\rho_s(x)/2m} A^\dagger(x)$ into orthonormal modes. To this end, let us first observe that we may rewrite (3.9) as
\[
\sqrt{\rho_s(x)/2m} \left[ A(x), A^\dagger(y) \right] \sqrt{\rho_s(y)/2m} = 1_{\text{Fock}} \cdot \langle x|H|y \rangle, \tag{3.12}
\]
where $H$ is the single-particle hamiltonian
\[
H = (1 - \lambda) \sqrt{\rho_s(x)/2m} \left[ p \frac{1}{\rho_s(x)} p - \frac{k}{\rho_0} |p| \right] \sqrt{\rho_s(x)/2m}, \tag{3.13}
\]
in which $x$ and $p$ are canonically conjugate position and momentum operators. The first term in (3.13), quadratic in $p$, can be read-off from (3.9) in a straightforward manner. In fact, it has the standard non-relativistic hamiltonian form, since it is proportional to
\[
\left( \frac{1}{\sqrt{\rho_s}} p \sqrt{\rho_s} \right) \dagger \left( \frac{1}{\sqrt{\rho_s}} p \sqrt{\rho_s} \right) = p^2 + \left( \frac{1}{2} \partial_x \log \rho_s(x) \right)^2 - \frac{1}{2} \partial_x^2 \log \rho_s(x)
= p^2 + \frac{k^2}{4} \left[ \left( \frac{\rho_s(x)}{\rho_0} \right)^2 - 1 \right], \tag{3.14}
\]
where in the last step we used the BPS equation (2.1). The second term in (3.13) arises because the matrix-element
\[
\langle x||p||y \rangle = \frac{1}{\pi} \partial_x \frac{P}{x - y}, \tag{3.15}
\]
which follows from the identity\(^\S\)

\[
|p| \psi(x) = -\partial_x (\psi^H(x)). \tag{3.16}
\]

\(^\ddagger\)The term $\delta \mu \int dx \eta$ merely constrains the zero-momentum Fourier mode of $\eta(x)$ to vanish, and we shall henceforth not write it explicitly.

\(^\S\)This identity can be easily established by applying the operator $|p|$ to the Fourier integral representation of $\psi(x)$, and using the formula $(e^{ipx})^H = i e^{ipx} \text{sign} p$. 

---
In addition, in deriving the second term in (3.13), one has to invoke equations (2.1) and (2.15).

The operator $H$ is therefore a hermitian periodic Schrödinger operator, with period $T = \frac{2\pi}{k}$ (Eq.(2.22)). Consequently, it has a complete set of orthogonal Floquet-Bloch eigenstates $\varphi_q(x)$ and corresponding energy eigenvalues $\omega(q)$, where $q$ is quasi-momentum. Note that $H$ is not positive definite. Hence $\omega(q)$ may become negative over some range of $q$.

As we shall show in Section 4 below, the spectrum of $H$ has no gaps! This peculiar feature is obviously the result of the unconventional $|p|$ term in $H$. Since there are no gaps in the spectrum, we shall take the quasi-momentum $q$ to range from $-\infty$ to $+\infty$ (i.e., work in the extended-Brillouine zone scheme).

We shall now prove that the complete orthogonal set of Floquet-Bloch eigenstates $\varphi_q(x)$ can be used to diagonalize the fluctuation hamiltonian (3.11) over an appropriate Fock space. This can be done exclusively on basis of the general properties of the periodic Schrödinger hamiltonian $H$, without any reference to the explicit form of the eigenstates $\varphi_q(x)$. We shall therefore defer derivation of the explicit form of the eigenstates $\varphi_q(x)$ and their corresponding energy eigenvalues $\omega(q)$ to Section 4.

Let us now list a few standard facts about the spectrum of $H$. By definition, the Floquet-Bloch eigenstates are quasi-periodic, and satisfy

$$\varphi_q(x + T) = e^{i q T} \varphi_q(x).$$

The completeness relation $\int_{-\infty}^{\infty} dq |\omega(q)\rangle \langle \omega(q)| = \mathbb{1}$, obeyed by the Floquet-Bloch eigenstates, may be written as

$$\int_{-\infty}^{\infty} dq \varphi_q(x) \varphi^*_q(y) = \delta(x - y).$$

Here, as usual, $\langle x | \omega(q) \rangle = \varphi_q(x)$. Orthogonality is expressed as

$$\langle \omega(q) | \omega(q') \rangle = \int_{-\infty}^{\infty} dx \varphi^*_q(x) \varphi_{q'}(x) = \delta(q - q').$$

Finally, using the spectral decomposition $H = \int_{-\infty}^{\infty} dq \omega(q) |\omega(q)\rangle \langle \omega(q)|$ we obtain the matrix element of $H$ in the position basis as

$$\langle x | H | y \rangle = \int_{-\infty}^{\infty} dq \omega(q) \varphi_q(x) \varphi^*_q(y).$$
Let us revert back to the discussion leading from (3.10) to (3.11), namely cancellation of (3.5) by the commutator term in (3.10). By combining (3.12), (3.19) and (3.20) we can express that commutator term as

\[
\frac{1}{4m} \int dx \rho_s(x) \left[ A(x), A^\dagger(x) \right] = \frac{1}{2} \int dx \langle x | H | x \rangle = \left( \int dx \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \frac{\omega(q)}{2}. \tag{3.21}
\]

In other words, this commutator term is nothing but the zero-point energy of a non-interacting bosonic field theory with eigenmodes given by the eigenvalues of \( H \) in (3.13), \( \omega(q) \). We compute this dispersion relation in the next section. It is given by (4.26). Substituting (4.26) in (3.21) we see that it exactly cancels \( H_{\text{sing}} \) in (3.5), as promised.

Since we seek a diagonalization of the fluctuation hamiltonian (3.11) over an appropriate Fock space, we shall associate with each Floquet-Bloch eigenstate \( \varphi_q(x) \) a pair of bosonic creation and annihilation operators \( a(q), a^\dagger(q) \) which obey the standard bosonic commutator algebra

\[
\left[ a(q), a(q') \right] = \delta(q - q'), \\
\left[ a(q), a(q') \right] = \left[ a^\dagger(q), a^\dagger(q') \right] = 0. \tag{3.22}
\]

It is then a matter of straightforward calculation to show that the algebra (3.9) is realized by the normal-mode expansions

\[
\sqrt{\frac{\rho_s(x)}{2m}} A(x) = \int_{-\infty}^{\infty} dq |\omega(q)|^{1/2} \varphi_q(x) \left[ \theta(\omega(q)) a(q) + \theta(-\omega(q)) a^\dagger(q) \right], \\
\sqrt{\frac{\rho_s(x)}{2m}} A^\dagger(x) = \int_{-\infty}^{\infty} dq |\omega(q)|^{1/2} \varphi_q^*(x) \left[ \theta(\omega(q)) a^\dagger(q) + \theta(-\omega(q)) a(q) \right], \tag{3.23}
\]

which take into account the fact that \( \omega(q) \) flips its sign. Indeed, it is easy to see that (3.23) are consistent with \( [A(x), A(y)] = 0 \). In addition, fulfillment of (3.12), namely,

\[
\left[ \sqrt{\frac{\rho_s(x)}{2m}} A(x), \sqrt{\frac{\rho_s(y)}{2m}} A^\dagger(y) \right] = \langle x | H | y \rangle, \tag{3.24}
\]

follows from (3.23), (3.22), (3.20), (3.19) and (3.13). Thus, (3.23) is a legitimate realization of the operators \( A(x), A^\dagger(x) \). Finally, upon substituting the normal mode expansions (3.23) in (3.11) we obtain, using the orthogonality relation (3.19), the
desired diagonal form of $H_{coll}^{(2), BPS}$ as\footnote{Here we used $\theta^2(\omega) = \theta(\omega)$, as well as $\theta(\omega)\theta(-\omega) = 0$. The latter is responsible for the absence of $aa$ and $a^\dagger a^\dagger$ terms on the RHS of (3.23).}

$$H_{coll}^{(2), BPS} = \int_{-\infty}^{\infty} dq |\omega(q)| \left[ \theta (\omega(q)) a^\dagger(q)a(q) + \theta (-\omega(q)) a(q)a^\dagger(q) \right]. \quad (3.25)$$

This manifestly positive operator is not normal-ordered, due to the expansions (3.23), which contain both creation and annihilation operators. We should perhaps mention that a similar situation arises in the two dimensional anyonic model studied in [27]. The divergent expectation value of (3.25) in the Fock vacuum is given by the sum over zero-point energies of negative $\omega(q)$ modes, namely,

$$\langle 0 | H_{coll}^{(2), BPS} | 0 \rangle = \left( \int dx \right) \int_{-\infty}^{\infty} dq \frac{dq}{2\pi} |\omega(q)| \theta (-\omega(q)) = - \left( \int dx \right) \int_{|q|>k} dq \frac{\omega(q)}{2\pi}, \quad (3.26)$$

where in the last step we used (3.3) and (4.26).

Note that the Floquet-Bloch eigenvalue $\omega(q)$ given in (1.24) coincides with the dispersion relation of quasi-particles in the presence of uniform condensate $\rho_0$, namely, the ground state. Thus, according to (3.26), quantum corrections to the energy density tied with the BPS-density wave studied here, due to quadratic fluctuations of the collective field, coincide with the analogous corrections around the uniform condensate background. In other words, as one starts from the uniform solution of (2.1), namely, (2.21) taken at infinite $u$, and then reduces $u$ continuously to some positive finite value, there are no (next-to-leading order) quantum corrections to the energy density tied with the BPS-density wave relative to the ground state. (The classical values of energies in both cases, are of course null.) Moreover, since the single BPS soliton, or lump, is obtained as the limiting case (2.27) of the BPS-density wave, its mass it not corrected by quadratic fluctuations either.

Finally, using the commutator algebra (3.22) to normal-order (3.25), and subtracting the divergent contribution (3.26), we obtain the desired diagonalized fluctuation hamiltonian simply as

$$: \left( H_{coll}^{(2), BPS} \right) : = \int_{-\infty}^{\infty} dq |\omega(q)| a^\dagger(q)a(q). \quad (3.27)$$

It is manifestly positive definite, as required. We need not worry about positivity of $\omega(q)$ since only its absolute value enters (3.27).
4. The exact Floquet-Bloch eigenfunctions and energy band

Now that we have diagonalized $H_{coll}^{(2)}$, it remains to determine the Floquet-Bloch eigenstates $\varphi_q(x)$ and corresponding eigenvalues $\omega(q)$ explicitly, and establish their orthogonality and completeness.

The eigenvalue equation $H\varphi_q(x) = \omega(q)\varphi_q(x)$ may be written explicitly as

$$ (1 - \lambda)\sqrt{\frac{\rho_s}{2m}} p \frac{1}{p} \left( \sqrt{\frac{\rho_s}{2m}} \phi \right) + 2\pi\lambda \frac{\rho_s}{2m} |p| \left( \sqrt{\frac{\rho_s}{2m}} \phi \right) = \omega \phi, $$

where we have suppressed any $q$-dependence for brevity. Note also the slight change in notation of eigenstates from $\varphi_q$ to $\phi_q$. (This is done in order to avoid possible confusion in the discussion below, and we shall return to the original notation toward the end of this section.) This equation clearly calls for defining a new unknown function

$$ \psi = \sqrt{\frac{\rho_s}{2m}} \phi $$

which has the same quasi periodicity (3.17) as $\phi$. In terms of $\psi$ we have

$$ \left( \frac{1 - \lambda}{2m} \right) \rho_s p \frac{1}{p} \psi + 2\pi\lambda \left( \frac{\rho_s}{2m} \right) |p| \psi = \omega \psi. $$

Recalling (3.16) and the BPS equation (2.1), we may write (4.3) more explicitly as

$$ (\lambda - 1)\partial_x^2 \psi - 2\pi\lambda \left[ \rho_s^H \partial_x \psi + \rho_s \partial_x \psi^H \right] = 2m\omega \psi. $$

4.1 Solving Schrödinger’s Equation

Equation (4.4) is very suggestive of taking its Hilbert transform, using the identity\(^\dagger\)

$$ (fg^H + f^Hg)^H = f^Hg^H - fg + f_0g_0, $$

where $f_0$ and $g_0$ are subtraction constants of their corresponding functions, analogous to $\rho_0$ in (2.6). Since $\partial_x$ and the Hilbert transform commute, we find the Hilbert transform of (4.4) as

$$ (\lambda - 1)\partial_x^2 \psi^H - 2\pi\lambda \left[ \rho_s^H \partial_x \psi^H - \rho_s \partial_x \psi + \rho_0 \psi_0' \right] = 2m\omega \psi^H. $$

Life is made easier by the fact that $\psi_0'$, the subtraction constant in (4.6), needed to render the integral of $\psi'(x)$ over the real axis, is null. This should be expected, due to Cauchy’s theorem, under the assumption that $\psi(x)$ is analytic in at least one of

\(^\dagger\)For derivation of (4.5) for real functions see Appendix A of [23]. For a derivation of (4.3) for complex valued functions analytic in one half-plane, where they also decay at infinity (e.g., exponentials $e^{ipx}$), see Appendix A in the second paper in [14]. If $f$ and $g$ are of the latter type, they are eigenfunctions of the Hilbert transform, with eigenvalue +$i$ for analyticity in the UHP, and $-i$ for the LHP. For such functions $f_0 = g_0 = 0$ due to Cauchy’s theorem.
the half-planes where it also decays at infinity. We can also prove that \( \psi_0' = 0 \) more directly. Since both \( \psi(x) \) and \( \psi'(x) \) are quasi-periodic and obey similar relations like (3.17), it follows that
\[
\int_{-\infty}^{\infty} \psi'(x) \, dx = \left( \int_0^T \psi'(x) \, dx \right) \sum_{n=-\infty}^{\infty} e^{i n q T} = 2\pi (\psi(T) - \psi(0)) \delta_P(qT)
\]
\[
= 2\pi \psi(0) (e^{i q T} - 1) \delta_P(qT),
\]
where \( \delta_P(x) \) is the periodic delta-function of period \( 2\pi \). The last expression obviously vanishes, since \( \psi(0) \) is finite by assumption.\(^\dagger\) Thus, \( \psi_0' = 0 \) as promised and it can be dropped from (4.6). Reversing the argument, this independent proof that \( \psi_0' = 0 \) implies that if \( \psi'(x) \) is analytic in one of the complex half-planes, it must decay to zero at infinity there.

The natural thing to do now is to combine (4.4) and (4.6), in a manner similar to (2.12). Thus, adding and subtracting \( i \) times (4.4) from (4.6), we obtain the pair of equations
\[
(\lambda - 1) \partial_x^2 F_\pm(x) - 2\pi \lambda \Phi_\pm \partial_x F_\pm = 2m\omega F_\pm,
\]
where
\[
F_\pm(x) = \psi^H(x) \pm i \psi(x).
\]

Recall that the two functions \( \Phi_\pm(x) \), defined in (2.9), are the boundary values
\[
\Phi_\pm(x) = \Phi(x \pm i0)
\]
of the meromorphic function \( \Phi(z) \) in (2.17), as one approaches the real axis. Thus, \( \Phi_+(x) \) is analytic in the \( x-UHP \), and \( \Phi_-(x) \) is analytic in the \( x-LHP \). These are the coefficient functions in (4.8). It thus follows from the theory of differential equations that \( F_+(x) \) should be analytic in the \( x-UHP \), and that \( F_-(x) \) should be analytic in the \( x-LHP \).

The two equations (4.8) can be united into a single equation
\[
(\lambda - 1) F''(z) - 2\pi \lambda \Phi(z) F'(z) = 2m\omega F(z),
\]
or more compactly,
\[
F''(z) - k \frac{\rho_0(\Phi(z) F'(z) = \frac{2m\omega}{\lambda - 1} F(z),
\]
for a single meromorphic function \( F(z) \), such that
\[
F_\pm(x) = F(x \pm i0).
\]
\(^\dagger\)Otherwise, if \( \psi(0) \) diverges, we can pick any other period of \( \psi(x) \) at which endpoints \( \psi \) is finite.
Our first step in solving (4.11) is to remove the first-derivative term. Thus, following standard methods, we substitute

$$F(z) = \xi(z)f(z)$$

(4.13)

where $f(z)$ is a new unknown function, and $\xi(z)$ is determined by demanding that upon substituting (4.13) in (4.11), the coefficient of $f'(z)$ will be null. Thus, we find

$$\frac{\xi'}{\xi} = \frac{k}{2\rho_0} \Phi.$$  

(4.14)

Using (2.17) we may write this equation as

$$\frac{\xi'}{\xi} = \frac{ik}{2} + i\kappa \frac{e^{ikz-u} \text{sign}(\Im z)}{1 - e^{ikz-u} \text{sign}(\Im z)}.$$  

(4.15)

The solution in each of the complex half-planes is immediate and is independent of the solution in the other half-plane. Thus,

$$\xi = iC_+ \theta(\Im z) \frac{e^{ikz}}{1 - e^{ikz-u}} - iC_- \theta(-\Im z) \frac{e^{ikz}}{1 - e^{ikz+u}},$$

(4.16)

with integration constants $C_\pm$. In each half-plane, $\xi(z)$ is of course analytic, with discontinuity along the real axis. Granted more information on the analytic properties of our solution $\xi(z)$ and $F(z)$ we shall be able to determine these integration constants.

We also need an expression for $\xi''/\xi$ which appears in the equation for $f(z)$. Taking the derivative of (4.14) and using (2.13) we find

$$\frac{\xi''}{\xi} = \frac{k}{2\rho_0} \left( \Phi' + \frac{k}{2\rho_0} \Phi^2 \right) = \left( \frac{k}{2\rho_0} \right)^2 \left( 2\Phi^2 + \rho_0^2 \right).$$

(4.17)

With (4.17) at our disposal, we finally obtain the very simple equation

$$f''(z) + \left( \frac{k}{2} \right)^2 f(z) = \frac{2m\omega}{\lambda-1} f(z)$$

(4.18)

for $f(z)$. We readily find the solution as

$$f(z) = e^{ipz}$$

(4.19)

with $p$ a real momentum-like parameter.\footnote{More precisely, the two independent solutions of (4.18) are $e^{\pm ipz}$. Since they are related by flipping the sign of $p$, it is enough to consider only (4.19), since we allow $p$ to range over positive and negative values.} The resulting dispersion relation is, of course,

$$\omega(p) = \frac{\lambda - 1}{2m} \left[ \left( \frac{k}{2} \right)^2 - p^2 \right].$$

(4.20)
Finally, combining (4.19), (4.16) and (4.13) we obtain the solution of (4.11) as
\[ F(z) = iC_+(q) \theta(3z) \frac{e^{iqz}}{1 - e^{ikz - u}} - iC_-(q) \theta(-3z) \frac{e^{iqz}}{1 - e^{ikz + u}}, \] (4.21)
where
\[ q = \frac{k}{2} + p \] (4.22)
is the quasi-momentum, as indeed, \( F(z + T) = e^{iqT}F(z) \), in accordance with (3.17).

In terms of \( q \), the dispersion relation (4.20) may be written as
\[ \omega(q) = \frac{\lambda - 1}{2m} q(k - q). \] (4.23)

Note that \( \omega(q) \) is independent of the amplitude of \( \rho_s(x) \), which is governed by the parameter \( u \). It depends on \( \rho_s \) only through \( k \), or equivalently, through \( \rho_0 \).

We are not done yet, since we have to impose the condition that \( F(\pm x + iT) = e^{iqT}F(\pm x) \) be consistent with (4.9) and also with the fact that \( \psi'_0 = 0 \) in (4.6), meaning that \( \psi(x) \) has to decay at infinity in the half-plane where it is analytic. Bearing in mind the last requirement, we obtain from (4.9) that
\[ \psi(x) = \frac{F(x + i0) + F(x - i0)}{2i} = \frac{C_+(q) \theta(q)}{2} \frac{e^{iqx}}{1 - e^{ikx - u}} + \frac{C_-(q) \theta(k - q)}{2} \frac{e^{iqx}}{1 - e^{ikx + u}}, \]
\[ \psi^H(x) = \frac{F(x + i0) - F(x - i0)}{2i} = \frac{iC_+(q) \theta(q)}{2} \frac{e^{iqx}}{1 - e^{ikx - u}} - \frac{iC_-(q) \theta(k - q)}{2} \frac{e^{iqx}}{1 - e^{ikx + u}}. \] (4.24)

It would be perhaps useful to spend a few words to clarify these equations. Let us concentrate on \( \psi(x) \). The first term in \( \psi(x) \), analytic in the \( x-UHP \), decays there at infinity only for \( q > 0 \), which is guaranteed by the \( \theta(q) \) prefactor. Similarly, the second therm in \( \psi(x) \), analytic in the \( x-LHP \), decays there at infinity only for \( q < k \), which is guaranteed by the other prefactor \( \theta(k - q) \). With this assignment of domains of quasi-momentum, the first term in \( \psi(x) \) is an eigenvector of the Hilbert transform with eigenvalue \( i \), and the second term is an eigenvector with eigenvalue \( -i \), and therefore, \( \psi^H(x) \) in the second line of (4.24), is indeed the Hilbert transform of \( \psi(x) \).

The domains of quasi-momentum in (4.24) appear asymmetric. In order to render them more symmetric, we shift \( q \) in the second terms in both lines of (4.24) by one unit of the reciprocal lattice, \( q = q' + k \), where, of course, \( q' < 0 \). After a little algebra (including absorbing a factor \( -e^{-u} \) in \( C_- \) and an appropriate redefinition thereof), and reinstating \( q \) for \( q' \), we can write \( \psi(x) \) more symmetrically as
\[ \psi(x) = \frac{C_+(q) \theta(q)}{2} \frac{e^{iqx}}{1 - e^{ikx - u}} + \frac{C_-(q) \theta(-q)}{2} \frac{e^{iqx}}{1 - e^{-ikx - u}}. \] (4.25)
With this new assignment of quasi-momenta, the dispersion relation \( (4.23) \) may be written more symmetrically as
\[
\omega(q) = \frac{\lambda - 1}{2m}(k|q| - q^2),
\]
(4.26)
where \( q > 0 \) corresponds to the first term in \( (4.23) \), and \( q < 0 \) to the second term. This function is made of two parabolas soldered together at \( q = 0 \). It vanishes at \( q = 0, \pm k \), with a cusp at \( q = 0 \). In addition, since \( \frac{\lambda - 1}{2m} > 0 \), it has two degenerate maxima at \( q = \pm \frac{k}{2} \), with maximal value \( \omega_{\text{max}} = \frac{\lambda - 1}{4m}k^2 \). Therefore, each value of \( \omega(q) \) between \( \omega = 0 \) and \( \omega_{\text{max}} \) is four-fold degenerate, and occurs at \( 0 < q < k \) as well as at \( \pm (k - q) \) and \( -q \). All other possible values of \( \omega(\pm q) \), with \( |q| > k \), are just doubly-degenerate.

We readily identify \( \omega(q) \) as the dispersion relation of quasi-particles in the presence of uniform condensate \( \rho_0 \) \([5, 23]\), as we have already mentioned following \( (3.26) \). In particular, it is independent of the parameter \( u \), which governs the amplitude of oscillations of the BPS-soliton.

It is interesting to study the behavior of \( (4.26) \) in the limit \( \lambda \to 1 \). In this limit, according to \( (2.17) \), \( k \) diverges such that \( (\lambda - 1)k \to 2\pi \rho_0 \), and therefore \( \omega(q) \to \frac{\pi \rho_0}{m} |q| \). This is, of course, the dispersion relation of sound waves propagating in the ground state of a one-dimensional Fermi gas of uniform density \( \rho_0 \) and Fermi momentum \( k_F = \pi \rho_0 \). This uniform Fermi gas configuration is the only solution of \( (2.1) \) at \( \lambda = 1 \). Thus, the spectrum \( (4.26) \) of fluctuations around the BPS-soliton \( (2.21) \) behaves smoothly as \( \lambda \) goes through \( \lambda = 1 \). This supports our assertion, made at the end of Section 2, that these BPS-solitons should also be taken to be smooth around \( \lambda = 1 \), which is achieved by setting \( e^u \gg \frac{1}{\lambda - 1} \) as \( \lambda \to 1 \).

### 4.2 The complete orthogonal set of Floquet-Bloch eigenfunctions

Equipped with the explicit solution \( (4.25) \) (and the definition \( (4.3) \)), we can read-off the Floquet-Bloch wave-functions as
\[
\phi_q(x) = \sqrt{\frac{1 - e^{-2u}}{2\pi}} \frac{\rho_0}{\rho_s(x)} \frac{e^{iqx}}{1 - e^{ikx - u}}, \quad q > 0
\]
\[
\tilde{\phi}_q(x) = \sqrt{\frac{1 - e^{-2u}}{2\pi}} \frac{\rho_0}{\rho_s(x)} \frac{e^{iqx}}{1 - e^{-ikx - u}}, \quad q < 0,
\]
(4.27)
such that
\[
\tilde{\phi}_q = \phi^*_{-q}.
\]
(4.28)
Here, the normalization factors $C_\pm$ in (4.25) are chosen such that $\phi_q$ and $\tilde{\phi}_q$ be normalized to a delta-function, namely,

$$
\int_{-\infty}^{\infty} dx \, \phi_q(x) \phi_p^*(x) = \delta(q - p) \quad q, p > 0
$$

$$
\int_{-\infty}^{\infty} dx \, \tilde{\phi}_q(x) \tilde{\phi}_p^*(x) = \delta(q - p) \quad q, p < 0 , \tag{4.29}
$$

which can be easily established by using the factorization

$$
\frac{1}{\rho_s(x)} = \frac{e^u}{2\rho_0 \sinh u} (1 - e^{ikx-u})(1 - e^{-ikx-u}) . \tag{4.30}
$$

Our work is not done yet, since $\phi_q$ and $\tilde{\phi}_q$ do not form orthogonal sets. To see this choose some $p > 0$ and $q < 0$ and consider

$$
\phi_p(x) \tilde{\phi}_q^*(x) = \frac{e^{-i(p+|q|)}}{2\pi} \frac{1 - e^{-ikx-u}}{1 - e^{ikx-u}} , \tag{4.31}
$$

where we used (4.30) once more. The denominator can be expanded into a geometric series which we integrate term by term. Thus, we obtain

$$
\int_{-\infty}^{\infty} dx \, \phi_p(x) \tilde{\phi}_q^*(x) = \sum_{n=0}^{\infty} e^{-nu} \left[ \delta(p + |q| + nk) - e^{-u} \delta((n-1)k + p + |q|) \right]
$$

$$
= -e^{-u} \delta(p + |q| - k) , \tag{4.32}
$$

where in the last step we used the fact that $k, p$ and $|q|$ are all positive. In this region of parameters, the solutions of $p + |q| - k = 0$ are any $0 < p < k$ and $q = p - k$ (and therefore $-k < q < 0$). Thus, $\phi_p(x)$, with $0 < p < k$, and $\tilde{\phi}_{p-k}(x)$ are not orthogonal, and we need to rotate them into mutually orthogonal combinations. This rotation will mix the two states, which is allowed physically, since both states are degenerate in energy according to (4.26), and moreover, have the same quasi-momentum mod-$k$, i.e., both states acquire phase $e^{ipT}$ upon spatial shift by one period $T$. Obtaining these orthogonal combinations is a straightforward task, which essentially amounts to diagonalizing Pauli’s matrix $\sigma_x$. The desired orthogonal combinations are found as

$$
\phi_q^{(+)}(x) = \frac{\phi_q(x) + \tilde{\phi}_{q-k}(x)}{\sqrt{2(1 - e^{-u})}} = \sqrt{\frac{\rho_s(x)}{4 \pi \rho_0 (1 + e^{-u})}} \left( e^{iqx} + e^{i(q-k)x} \right)
$$

$$
\phi_q^{(-)}(x) = \frac{\phi_q(x) - \tilde{\phi}_{q-k}(x)}{\sqrt{2(1 + e^{-u})}} = \sqrt{\frac{\rho_s(x)}{4 \pi \rho_0 (1 - e^{-u})}} \left( e^{iqx} - e^{i(q-k)x} \right) , \quad 0 < q < k . \tag{4.33}
$$
It is easy to check, using (4.32), that they are normalized to a delta-function

\[ \int_{-\infty}^{\infty} dx \phi_q^+(x)\phi_p^+(x) = \int_{-\infty}^{\infty} dx \phi_q^-(x)\phi_p^-(x) = \delta(q - p) \quad 0 < q, p < k. \quad (4.34) \]

By construction, they are also orthogonal to all the other eigenfunctions, with quasi-momentum larger than \( k \) or smaller than \(-k\). In order to make our notations more symmetric with respect to assignment of quasi-momenta, let us rename in (4.33)

\[ \tilde{\phi}_q^-(x) = -\phi_{k+q}^-(x) = \frac{-\phi_q(x) - \phi_{q+k}(x)}{\sqrt{2(1 + e^{-u})}} = \sqrt{\frac{\rho_s(x)}{4\pi \rho_0(1 - e^{-u})}} (e^{iqx} - e^{i(q+k)x}) , \quad -k < q < 0, \]

which of course, has no effect on the orthogonality relations. This concludes our derivation of the normalized orthogonal set of orthogonal Floquet-Bloch functions. For convenience, let us summarize them in the following list:

\[
\varphi_q(x) = \begin{cases} 
\phi_q(x) = \sqrt{\frac{1 - e^{-2u}}{2\pi}} \frac{\rho_0}{\rho_s(x) 1 - e^{ikx - u}} e^{iqx}, & q > k \\
\phi_q^+(x) = \sqrt{\frac{\rho_s(x)}{4\pi \rho_0(1 + e^{-u})}} \frac{e^{iqx} + e^{i(q-k)x}}{2(1 + e^{-u})}, & 0 < q < k \\
\tilde{\phi}_q^-(x) = \frac{-\phi_q(x) - \phi_{q+k}(x)}{\sqrt{2(1 + e^{-u})}} = \sqrt{\frac{\rho_s(x)}{4\pi \rho_0(1 - e^{-u})}} (e^{iqx} - e^{i(q+k)x}) , \quad -k < q < 0 \\
\tilde{\phi}_q^+(x) = \sqrt{\frac{1 - e^{-2u}}{2\pi}} \frac{\rho_0}{\rho_s(x) 1 - e^{-ikx - u}} e^{iqx}, & q < -k,
\end{cases}
\]

for which

\[ \int_{-\infty}^{\infty} dx \varphi_q(x)\varphi_p^+(x) = \delta(q - p) \quad (4.37) \]

for all \( q, p \in \mathbb{R} \), in accordance with (3.19).

### 4.2.1 The completeness relation

We shall now verify that the functions in (4.36) comprise a complete set. Thus, consider the LHS of (3.18), namely,

\[ \Gamma(x, y) = \int_{-\infty}^{\infty} dq \varphi_q^+(x)\varphi_q(y), \quad (4.38) \]
and split it into contributions of the four sets of functions in (4.36)

\[\Gamma_1(x, y) = \int_{-\infty}^{\infty} dq \, \phi_q^*(x) \phi_q(y)\]
\[\Gamma_2(x, y) = \int_{0}^{k} dq \, \phi_q^{(+)*}(x) \phi_q^{(+)}(y)\]
\[\Gamma_3(x, y) = \int_{-k}^{0} dq \, \tilde{\phi}_q^{-*}(x) \tilde{\phi}_q^{-}(y)\]
\[\Gamma_4(x, y) = \int_{-\infty}^{-k} dq \, \tilde{\phi}_q^{*}(x) \tilde{\phi}_q(y) .\] (4.39)

From the first and fourth lines in (4.36) we obtain, by using the identities

\[\int_{-\infty}^{\infty} e^{iqx} dq = \frac{i e^{ikx}}{x + i \epsilon} = \frac{i e^{ikx} P_x}{x} + \pi \delta(x)\]
\[\int_{-\infty}^{k} e^{iqx} dq = \frac{-i e^{ikx}}{x - i \epsilon} = -\frac{i e^{ikx} P_x}{x} + \pi \delta(x)\] (4.40)

that

\[\Gamma_1(x, y) = \frac{1 - e^{-2u}}{2\pi} \frac{|\rho_0|}{\sqrt{\rho_s(x)\rho_s(y)}} \frac{1}{(1-e^{-ikx-u})(1-e^{iky-u})} \frac{i e^{ik(y-x)}}{y - x + i \epsilon}\]
\[\Gamma_4(x, y) = \Gamma_1^*(x, y),\] (4.41)

where, the last line follows from (4.28). After some work we obtain

\[\Gamma_1(x, y) + \Gamma_4(x, y) = \delta(x-y) - \frac{\sqrt{\rho_s(x)\rho_s(y)}}{\pi |\rho_0| (1-e^{-2u})} \frac{\sin \frac{k(x-y)}{2}}{\frac{x-y}{2} - e^{-u} \cos \frac{k(x+y)}{2}} \left[ \cos \frac{k(x-y)}{2} - e^{-u} \cos \frac{k(x+y)}{2} \right],\] (4.42)

where we used the last equalities in (4.41). In particular, the last cumbersome term in (4.42) arises from the principal parts in (4.41), which combine in such a way that the singular principal part \(\frac{P_x}{x-y}\) is multiplied by a function which has a simple zero at \(x = y\), which allows us to drop the \(P\) symbol. \(\Gamma_2(x, y)\) and \(\Gamma_3(x, y)\) are evidently non-singular kernels. From the second and third lines in (4.36), we can show, after some straightforward but tedious calculation that \(\Gamma_2(x, y) + \Gamma_3(x, y)\) exactly cancels the second, regular term in (4.42). Thus,

\[\Gamma(x, y) = \delta(x-y),\] (4.43)

proving completeness of the set of Floquet-Bloch functions (4.36).
4.3 Comments on uniqueness of the eigenstates

Recall from the discussion following (4.26) that when \(-k < q < k\), \(\omega(q)\) is four-fold degenerate. Thus, e.g., for \(0 < q < k\), also quasi-momenta \(k - q\), \(-q\) and \(q - k\) all have a common value \(\omega(q)\) (which ranges between \(\omega = 0\) and \(\omega_{\text{max}}\) as \(q\) varies in this domain). To this common eigenvalue correspond the four orthogonal eigenstates \(\phi_q^+(x), \tilde{\phi}_{q-k}^-(x), \tilde{\phi}_{q-k}^-(x)\) and \(\phi_{q-k}^-(x)\). This quartet of states splits into two pairs of states with common quasi periodicity, namely, \(\phi_q^+(x)\) and \(\tilde{\phi}_{q-k}^-(x)\), which acquire phase \(e^{iqT}\) under a shift of \(x\) by one period \(T\) (as can be seen from (4.36), in accordance with (3.17)), and \(\tilde{\phi}_{-q}^-(x)\) and \(\phi_{k+q}^+(x)\), which acquire phase \(e^{-iqT}\). We can rotate each pair of these states by a \(2 \times 2\) unitary matrix, which may even possibly be \(q\)-dependent, while leaving orthogonality (4.37) and completeness (3.18) in tact. The rotated basis is as good as the original one given in (4.36). Thus, in this range of quasi-momenta, namely, \(-k < q < k\), there is extra local energy dependent \(SU(2) \times SU(2)\) symmetry. It would be interesting to investigate the origins of this symmetry further.

In the remaining range of quasi-momenta, \(|q| > k\), the energy spectrum is doubly degenerate, however, the corresponding two quasi-momenta \(\pm q\) are not generically separated by an integer multiple of \(k\), and these states cannot be mixed, except for a discrete set of values \(q_n = \pm \frac{n}{2} k\), with \(n\) a positive integer, where there is an extra \(SU(2)\) symmetry.

Finally, let us briefly comment on the zero-energy solutions. Consider approaching \(\omega = 0\) from within the domain of four-fold degeneracy. Let us pair the quartet of degenerate states according to their quasi-periodicities, as was discussed just above, namely, \((\phi_q^+(x), \tilde{\phi}_{q-k}^-(x))\) on one hand, and \((\tilde{\phi}_{-q}^-(x), \phi_{k+q}^+(x))\) on the other. These four limiting states are of course strictly periodic, and we can mix them by a unitary transformation. Thus, the energy dependent \(SU(2) \times SU(2)\) is enhanced at \(\omega = 0\) to the much larger symmetry \(SU(4)\). However, let us return to the paired states of the lower \(SU(2) \times SU(2)\) symmetry. As can be seen from the explicit expressions in (4.36), we can take linear combinations of the members of each pair which are proportional to \(\sqrt{\frac{\omega}{\rho_0}}\). For such combinations, according to (4.2), we have \(\psi(x) \propto \rho_s(x)\). It is straightforward to check directly that \(\psi(x) = \rho_s(x)\) is indeed a solution of (4.3) when \(\omega = 0\). Indeed, upon substituting these \(\psi\) and \(\omega\) in (4.4) we obtain the derivative of the BPS equation (2.1) multiplied by \(2\rho_s\). One physical origin of this zero-mode has to do with the translational collective coordinate of the BPS density wave - i.e., it can be shifted arbitrarily in space\(^4\). However, due to the enhanced \(SU(4)\) symmetry, this cannot be the sole origin of zero eigenvalues in the spectrum of \(H\), which requires further study.

\(^4\)Recall the imaginary parts \(\Im u_+ = \Im u_-\) which we absorbed as a shift of \(x\) following (2.16).
5. Conclusion and discussion

In this paper we have completely diagonalized the Hamiltonian of quadratic collective field fluctuations in the background of BPS-density waves, which appear as static solutions of the equations of motion of the collective field formulation of the Calogero model. The fluctuation spectrum is positive, demonstrating linear stability of the BPS-density waves. Remarkably, the fluctuation spectrum \( \omega(q) \) around these BPS-density waves coincides with that of fluctuations around uniform condensates.

The only difference between fluctuations around these two background types is the explicit form of the orthogonal complete set of Floquet-Bloch mode functions \( \phi_q(x) \). We computed these functions explicitly for the BPS-density wave background, by diagonalizing explicitly a related periodic Schrödinger operator, with a non-standard term, containing the absolute value \(|p|\) of the momentum operator. Contrary to standard periodic Schrödinger operators, familiar from solid state physics, there are no gaps in the spectrum of the Schrödinger operator studied in this paper.

We close this paper by making some brief comments concerning fluctuations around non-BPS density wave solutions of the collective field equations of motion. These variational equations, for static configurations \( \rho(x) \), boil down to

\[
B[\rho] = \frac{\lambda - 1}{\rho} \partial_x (\rho B[\rho]) + 2\pi \lambda \left( \rho B[\rho] \right)^H - 2m\mu = 0, \tag{5.1}
\]

where the BPS combination \( B[\rho] \) was defined in (2.1). Note that the BPS-density wave configuration \( \rho_s(x) \), given in (2.21), is a solution of (5.1) corresponding to vanishing chemical potential \( \mu = 0 \). In [23] we have found solutions of (5.1), which basically amount to subtracting from the BPS-density wave (2.21) either its minimum or maximum value, thus obtaining extremal density waves which do vanish periodically, as opposed to the BPS-density wave. These solutions, which we named vortex crystals (corresponding to subtraction of the minimum, hence positive configurations) and anti-vortex crystals (subtraction of the maximum, hence negative configurations), coincide with the large amplitude waves of [12] in the static limit. We have discussed these vortex and anti-vortex crystals in [23] in detail, and computed the energy densities tied with them, which are all positive, hence above the zero-energy density tied with the BPS-density waves. Therefore the question of their linear stability poses an interesting problem, which is still open.

This stability problem is more difficult to analyze than the stability of BPS-density waves, which we worked out here. To appreciate this difficulty, let us derive the quadratic fluctuation Hamiltonian in the background a non-BPS static solution \( \rho \) of (5.1). A convenient intermediate step is to expand

\[
B[\rho + \eta] = B[\rho] + \left[ \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho} \right) - \lambda \pi \eta^H \right] - \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho} \right)^2 + \ldots, \tag{5.2}
\]
where the ellipsis stands for terms cubic in $\eta$ and higher. Then, substituting (5.2) in (1.1), and expanding $H_{\text{coll}}$ to second order in $\eta(x), \pi(x)$ and $\delta \mu$, we obtain the quadratic piece $H_{\text{coll}}^{(2)}$ as

$$H_{\text{coll}}^{(2)} = V_{\text{coll}}[\rho] - \delta \mu \int dx \eta + H_{\text{sing}}$$

$$+ \frac{1}{2m} \int dx \rho(x) \left( \partial_x \pi(x) \right)^2 + \frac{1}{2m} \int dx \rho(x) \left[ \frac{\lambda - 1}{2} \partial_x \left( \frac{\eta}{\rho} \right) - \lambda \pi \eta H \right]^2$$

$$- \frac{\pi \lambda}{m} \int dx B[\rho] \eta \eta H ,$$

(5.3)

where we used the constraint (1.6) and the variational equation (5.1) to eliminate all terms linear in the shifted quantities $\eta$ and $\delta \mu$. Here $V_{\text{coll}}[\rho]$ is the energy tied in the extremal configuration $\rho(x)$ (which was null in the BPS case).

Now we can see the crux of the problem: $\rho(x)$ is not a BPS configuration, and thus $B[\rho] \neq 0$. This therefore generates a new term in (5.3), which did not arise in the BPS case, namely, the last term in (5.3), $- (\pi \lambda / m) \int dx B[\rho] \eta \eta H$. This term prevents us from carrying out the factorization of the quadratic pieces in (5.3) in a manner analogous to (3.11), which proved so useful to the complete and explicit diagonalization of the quadratic fluctuation hamiltonian around BPS-density waves. A new idea is clearly needed to solve the non-BPS fluctuation spectrum.

A. The collective field formulation of the Calogero model: derivation of $H_{\text{coll}}$ and its singular part $H_{\text{sing}}$

For the sake of being self-contained, we briefly review in this appendix the derivation of the collective field hamiltonian $H_{\text{coll}}$ in (1.1), and its singular part $H_{\text{sing}}$ in (1.2). Standard references on the collective field formalism are [13, 28]. Here we shall follow [29], as well as [3, 5], which focus specifically on the Calogero model.

The Calogero model, whose quantum hamiltonian is given by

$$H = - \frac{1}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \lambda (\lambda - 1) \sum_{i \neq j} \frac{1}{2m} \frac{1}{(x_i - x_j)^2}$$

(A.1)

describes $N$ identical particles of mass $m$ living in one dimension. These particles are subjected to inverse-square pair interactions with dimensionless coupling $\lambda$. \footnote{Note that we did not include in (A.1) a confining potential. This is not really a problem, as we can always add a very shallow confining potential to regulate the problem (in the case of purely repulsive interactions), or else, consider the particles confined to a very large circle (i.e., consider (A.1) as the large radius limit of the Calogero-Sutherland model [3]). We shall henceforth tacitly assume that the system is thus properly regularized at large distances.}
many-body wave functions are of the general form \( \Psi(x_1, \ldots, x_N) = \Delta^\lambda S(x_1, \ldots, x_N) \),
where \( S(x_1, \ldots, x_N) \) is a function totally symmetric under any permutation of the
particles, and \( \Delta = \prod_{i<j} (x_i - x_j) \) is the Vandermonde determinant. The so-called
Jastrow-factor \( \Delta^\lambda \) arises due to the singular pair-interaction, which requires the wave
function to vanish when any two particles coincide. The precise power-like vanishing
is dictated by the requirement that the hamiltonian be self-adjoint.

We see that for a generic value of \( \lambda \), under the interchange of any two particles,
the wave function suffers a phase change of \( e^{i\pi\lambda} \). These particles are therefore anyons.
The singular pair-wise interactions vanish, of course, at \( \lambda = 0 \), which corresponds
to non-interacting bosons, and also at \( \lambda = 1 \), which corresponds to non-interacting
fermions.

Powers of moments of the collective field \( \rho(x) \) in (1.3) are clearly the building
blocks of all functions which are totally symmetric in the particle coordinates. Thus,
the symmetric factor in the many-body wave-function, \( S(x_1, \ldots, x_N) \), assuming it
has a well-behaved large-\( N \) limit, should become a well-behaved functional of \( \rho(x) \).
It is precisely these symmetric wave-functions \( S(x_1, \ldots, x_N) \) on which the collective
field operators, as well as the collective hamiltonian \( H_{\text{coll}} \) in (1.1) act. Thus, in order
to transform (A.1) into a form amenable to collective field reformulation, we have to
strip-off the Jastrow factors from the many-body wave function. This we achieve by
performing on (A.1) the similarity transformation

\[
H \rightarrow \tilde{H} = \Delta^{-\lambda} H \Delta^\lambda, \tag{A.2}
\]

It is straightforward to check that \[3\]

\[
\frac{1}{\Delta^\lambda} \left( \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} - \lambda(\lambda - 1) \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \right) \Delta^\lambda = \frac{1}{\Delta^{2\lambda}} \sum_{i=1}^N \frac{\partial}{\partial x_i} \Delta^{2\lambda} \frac{\partial}{\partial x_i} \equiv \nabla^2_s. \tag{A.3}
\]

Thus,

\[
\tilde{H} = -\frac{1}{2m} \nabla^2_s. \tag{A.4}
\]

We can naturally interpret \( \nabla^2_s \) as part of a laplacian in some set of curvilinear coordinates \( q^a \),
which in addition to the \( x_i \), also contain additional coordinates orthogonal
to them. \( \nabla^2_s \) is therefore the projection of the larger laplacian onto the subspace
which depends exclusively on the original coordinates \( x_i \). In particular, it is invariant
under coordinate transformations which involve only the \( x_i \).

The space parametrized by the coordinates \( q^a \) is endowed with a metric \( g_{ab} \). We
need not concern ourselves with the details of this metric (and of the additional
coordinates, orthogonal to the \( x_i \)) except for the following two facts: First, we must
clearly have\[4\]

\[
\sqrt{g} = \Delta^{2\lambda}. \tag{A.5}
\]

\[4\] The volume element must be positive. Thus, we should really interpret \( \Delta^{2\lambda} = (\Delta^2)^\lambda = |\Delta|^{2\lambda} \).
Second, in the subspace of the original coordinates obviously $ds^2 = \sum_{i=1}^{N} dx_i^2$, and the corresponding block of $g_{ab}$ is therefore the unit matrix.

At the particular values $\lambda = \frac{1}{2}, 1$ and 2, these curvilinear coordinates are those of the symmetric spaces corresponding to real-symmetric, complex-hermitian and quaternionic-self-dual matrices, respectively, and $\nabla_s^2$ is the projection of the laplacian into the singlet sector of the corresponding matrix space.

By construction, $\tilde{H}$ acts on the symmetric functions $S(x_i)$, and inner-products and matrix elements are computed with the integration measure

$$d\mu = \Delta^{2\lambda} d^N x,$$

in respect to which $\tilde{H}$ is hermitian.

The collective field formalism amounts to performing a point canonical transformation from the $N$ position operators $x_i$ and their conjugate momenta $p_i = -i \frac{\partial}{\partial x_i}$, to a new set of coordinates, namely, the collective field operators $\rho(x)$ and their conjugate momenta $\Pi(x) = -i \frac{\delta}{\delta \rho(x)}$, and then expressing $\tilde{H}$ in terms of $\rho(x)$ and $\Pi(x)$.

A.1 Point canonical transformations

More precisely, we should look upon this as a special case of point canonical coordinate transformations

$$q^a \rightarrow Q^a = Q^a(q)$$

of the larger space, parametrized by the entire collection of coordinates $q^a$, in which only the subspace parametrized by the $x_i$ is transformed into the new set of coordinates $\rho(x)$, while the subspace orthogonal to the $x_i$ remains unchanged.

The metric in the new coordinates is given, in the usual manner, by

$$\Omega_{ab}(Q) = g_{mn}(q(Q)) \frac{\partial q^m}{\partial Q^a} \frac{\partial q^n}{\partial Q^b}$$

and its inverse is given by

$$\Omega^{ab}(Q) = g^{mn}(q(Q)) \frac{\partial Q^a}{\partial q^m} \frac{\partial Q^b}{\partial q^n}.$$ 

In these equations $q^a(Q)$ are, of course, the inverse coordinate transformations. It follows from (A.8) that

$$\Omega(Q) = \det \Omega_{ab} = g J^2$$

where

$$J = \det \left( \frac{\partial q}{\partial Q} \right)$$

is the jacobian of the transformation. Thus, we have

$$\sqrt{\Omega} = \sqrt{g} J$$

(A.12)
rendering the volume element invariant
\[
\sqrt{\Omega} d\mathbf{Q} = \sqrt{\gamma} d\mathbf{q}.
\] (A.13)

Our next step is to transform the hamiltonian \( \tilde{H} \) in (A.4) to the new coordinates. Since \( \nabla^2_s \) in (A.3) is that part of the invariant laplacian in our space, which in particular, remains invariant under transformations that change only the \( x_i \) while keeping the remaining orthogonal subspace unchanged, we simply have
\[
\tilde{H}_Q = -\frac{1}{2m} \nabla^2_Q s,
\] (A.14)
where
\[
\nabla^2_Q = \frac{1}{\sqrt{\Omega}} \frac{\partial}{\partial Q^a} \left( \Omega^{ab} \sqrt{\Omega} \frac{\partial}{\partial Q^b} \right),
\] (A.15)
and the subscript \( s \) in \( \nabla^2_Q s \) means a projection on the subspace originally parametrized by the \( x_i \). By construction, of course, \( \nabla^2_Q s = \nabla^2_s \).

This new expression for \( \tilde{H} = \tilde{H}_Q \) is symmetric with respect to the measure \( \sqrt{\omega} d\mathbf{Q} \). Life would be much easier if we could rid ourselves of this potentially complicated measure, and map \( \tilde{H} \) onto an effective hamiltonian \( H_{eff} \) which is symmetric with respect to the flat measure \( d\mathbf{Q} \). This we can achieve by performing the similarity transformation
\[
H_{eff} = \Omega^{\frac{1}{4}} \tilde{H}_Q \Omega^{-\frac{1}{4}}.
\] (A.16)

Let us now massage \( H_{eff} \) into a more transparent form. In order to avoid cluttering of our formulas, we shall henceforth not display the subscript \( s \) explicitly. Thus, \textit{all the formulas below should be properly projected onto the subspace originally parametrized by the \( x_i \)}. It is a matter of straightforward calculation to show that
\[
\Omega^{\frac{1}{4}} \nabla^2_Q \Omega^{-\frac{1}{4}} = \Omega^{-\frac{1}{4}} \frac{\partial}{\partial Q^a} \left( \Omega^{ab} \sqrt{\Omega} \frac{\partial}{\partial Q^b} \right) \Omega^{-\frac{1}{4}}
\]
\[
= \left( \Omega^{-\frac{1}{4}} \frac{\partial}{\partial Q^a} \Omega^{\frac{1}{4}} \right) \Omega^{ab} \left( \Omega^{\frac{1}{4}} \frac{\partial}{\partial Q^b} \Omega^{-\frac{1}{4}} \right)
\]
\[
= \frac{\partial}{\partial Q^a} \Omega^{ab} \frac{\partial}{\partial Q^b} - \left( \frac{\partial}{\partial Q^a} \left( \Omega^{ab} C_b \right) - C_a \Omega^{ab} C_b \right)
\] (A.17)
where we have defined
\[
C_a = \frac{1}{4} (\log \Omega)_a
\] (A.18)
and where \((\cdot)_a\) indicates a derivative with respect to \( Q^a \). The operator \( \Omega^{\frac{1}{4}} \nabla^2_Q \Omega^{-\frac{1}{4}} \) is manifestly symmetric with respect to the flat measure \( d\mathbf{Q} \), as is evident in each of the lines in (A.17). Thus, \( H_{eff} \) is indeed the desired hamiltonian we set out to find, which, following (A.17), we may write explicitly as
\[
H_{eff} = \frac{1}{2m} P_a \Omega^{ab} P_b + \frac{\hbar^2}{2m} C_a \Omega^{ab} C_b + \frac{\hbar^2}{2m} \left( \Omega^{ab} C_b \right)_a
\] (A.19)
where we introduced the momentum operators

\[ P_a = -i\hbar \frac{\partial}{\partial Q^a}, \quad (A.20) \]

and displayed \( \hbar \) dependence explicitly. The terms in \([A.13]\) quadratic in \( \hbar \) may be thought of as a generalization of the centrifugal barrier which arises in the radial hamiltonian in \( D \) dimensions\(^3\). Evidently, these terms are purely a quantum mechanical effect.

It is easy to see that

\[ C_a = \frac{1}{2} \Gamma^b_{ba} \quad (A.21) \]

where \( \Gamma^b_{ba} \) is the second Christoffel symbol (i.e., the connection) of \( \Omega_{ab} \). However, sometimes a direct computation of the \( C_a \) from their definition \([A.18]\), or from the identity \([A.21]\), may be too difficult to carry in practice. Thus, in order to bypass these potential difficulties, we shall now derive an identity satisfied by the \( C_a \), from which we could compute them with somewhat less effort.

To this end we argue as follows: The invariant laplacian acting on a function which is a scalar under coordinate transformation produces yet another scalar function. Thus,

\[ \nabla_q^2 \psi(q) = \nabla^2 Q \psi(q(Q)). \quad (A.22) \]

In particular, the coordinate functions themselves are scalars (their differentials are one-forms). Let us define the quantities

\[ \omega^a = -\hbar \nabla^2 Q^a = -\hbar \nabla^2 Q^a. \quad (A.23) \]

It follows from the definitions \([A.23]\) and \([A.18]\) that

\[ \omega^a = -\hbar \nabla^2 Q^a = -\frac{\hbar}{\sqrt{\Omega}} \frac{\partial}{\partial Q^b} \left( \Omega^{ab} \sqrt{\Omega} \right) = -\frac{\hbar}{2} \Omega^{ab} (\log \Omega), -\hbar \Omega^{ab}, b \]

or

\[ \omega^a + 2 \hbar \Omega^{ab} C_b + \hbar \Omega^{ab}, b = 0, \quad (A.24) \]

which is the desired identity to determine the \( C_a \). To simplify the computation, we are free to choose in \([A.23]\) the coordinates \( q^a \) in which the computation of \( \omega^a = -\hbar \nabla^2 Q^a \) is as simple as possible. In our case, these are just the original coordinates.

\(^3\)The radial part of the \( D \)-dimensional laplacian \( \nabla^2_r = r^{-(D-1)} \partial_r (r^{D-1} \partial_r) \), defined with respect to the measure \( r^{D-1}dr \), may be transformed by a similarity transformation into \( \tilde{nabla}^2_r = \frac{\partial^2}{\partial r^2} - \frac{(D-1)(2-D)}{r^2} = \partial^2_r - C_r \Omega^{rr} C_r \), which is defined with respect to the flat measure \( dr \).
A.2 Application of the point canonical transformations to the collective field problem

The transformation from the \( x_i \)'s to the density field \( \rho(x) \) makes sense only in the limit \( N \to \infty \), since on one side there are \( N \) position operators \( x_i \), and a continuum of density operators \( \rho(x) \) on the other side. The continuum of these density operators are not all independent. For example, they are subjected to the constraints (1.3) and (1.3). Thus, in order to facilitate this transformation, one has to regularize the continuum theory. This is most conveniently achieved in momentum space. As our independent collective variables we choose the first \( N \) Fourier modes

\[
\rho_k = \int dx e^{-ikx} \rho(x) = \sum_{i=1}^{N} e^{-ikx_i} \tag{A.25}
\]
cut-off at some \( k_{\text{max}} \), where \( k \) is properly discretized, e.g., by putting our system in a large box of size \( L \), and imposing appropriate boundary conditions. The details of this discretization are not important for our discussion of the large-\( N \) limit. Assume next that the particles condense and that the mean particle density in the condensate is of the order of some value \( \rho \). Thus, the microscopic inter-particle distance will be of the order \( l \sim \frac{1}{\rho} \). Consequently, the maximal Fourier components should be of the order \( k_{\text{max}} = \frac{1}{l} \sim \rho \). Thus, the high density limit makes \( k_{\text{max}} \to \infty \) (and letting \( L \to \infty \) in the end, makes \( k \) continuous). The collective field reformulation of the model is therefore valid in the high density regime, where the system behaves like a continuous medium.

For large but finite spatial box size \( L \), momenta are discrete. We then take the first lowest \( N \) Fourier modes \( \rho_k \) as our new coordinates \( Q^a \).\footnote{As was stressed above, this transformation affects only the subspace originally parametrized by the \( x_i \). The coordinates of the orthogonal complement subspace remain unchanged.} Since the metric \( g_{ab} \) in the original coordinates is given by \( ds^2 = \sum_{i=1}^{N} (dx_i)^2 + \ldots \), i.e., its block in the subspace of interest is simply the unit matrix, we obtain from (A.8) that

\[
\Omega^{\rho_k \rho_{k'}} = \Omega(k, k'; [\rho]) = \sum_{i=1}^{N} \frac{\partial \rho_k}{x_i} \frac{\partial \rho_{k'}}{x_i} = -kk' \rho_{k+k'} \tag{A.26}
\]
in obvious notations, where momentum modes play the role of the new coordinate indices.

The effective hamiltonian (A.19) contains also the quantities \( C_a \), which we will determine from the identity (A.24). Thus, we have to compute \( \omega^{\rho k} = \omega(k; [\rho]) \). From the definition (A.23) we obtain\footnote{From this point on we set \( \hbar = 1 \) again.}

\[
\omega(k; [\phi]) = -\nabla_{\phi}^{2} \phi_k , \tag{A.27}
\]
where $\nabla_s^2$ was defined in (A.3). Thus, we obtain (see Eqs.(3.11)-(3.13) in [3])

$$\omega(k; [\phi]) = k^2 \sum_i e^{ikx_i} - 2ik\lambda \sum_i e^{ikx_i} \sum_{j,j\neq i} \frac{1}{x_i - x_j}.$$  
(A.28)

Adding and subtracting a $\sum_i$ term in the second term in (A.28) leads us to the final expression

$$\omega(k; [\phi]) = (1 - \lambda)k^2 \rho_k + \lambda k^2 \int_0^1 d\alpha \rho_{k\alpha} \rho_{k(1-\alpha)}.$$  
(A.29)

We should now substitute (A.26) and (A.29) in (A.24). It is easy to see that the last term there vanishes:

$$\Omega_{ab,b} = -\sum_{k'} kk' \delta \rho_{k+k'} = -\sum_{k'} kk' \delta_k = 0.$$  
(A.30)

Thus, we obtain from (A.24) that

$$\omega[k; [\rho]) + 2 \sum_{k'} \Omega(k, -k'; [\rho]) C(k'; [\rho]) = 0,$$  
(A.31)

which we can use to determine $C[k; [\rho])$. In the combined limits of large density and infinite spatial box, the $k$-sums tend to Fourier integrals. Thus, in the limit,

$$\omega[k; [\rho]) + 2 \int \frac{dk'}{2\pi} \Omega(k, -k'; [\rho]) C(k'; [\rho]) = 0,$$  
(A.32)

which we shall now transform to $x$-space. To this end we need (see Eq.(6) in [4])

$$\omega(x; [\rho]) = \int \frac{dk}{2\pi} e^{ikx} \omega(k; [\rho]) = (\lambda - 1)\partial_x^2 \rho(x) + 2\lambda \partial_x \int \frac{\rho(x)\rho(y)}{x-y} dy,$$  
(A.33)

as well as

$$\Omega(x, y; [\rho]) = \int \frac{dk dk'}{(2\pi)^2} e^{ikx+ik'y} \Omega(k, k'; [\rho]) = \partial_x \partial_y (\rho(x)\delta(x-y)).$$  
(A.34)

Using (A.33) and (A.34), we Fourier transform (A.32) to $x$-space and obtain

$$\omega(x; [\rho]) - 2\partial_x (\rho(x)\partial_x C(x; [\rho])) =$$

$$2\partial_x \left[ \frac{\lambda - 1}{2} \partial_x \rho(x) + \lambda \int \frac{\rho(x)\rho(y)}{x-y} dy - \rho(x)\partial_x C(x; [\rho]) \right] = 0,$$  
(A.35)

which we can readily solve for $\partial_x C$, and obtain

$$\partial_x C(x; [\rho]) = \frac{\lambda - 1}{2} \partial_x \rho(x) + \lambda \int \frac{\rho(x)\rho(y)}{x-y} dy.$$  
(A.36)
We now have all the ingredients required for computing the effective hamiltonian (A.19). Thus, substituting (A.34), (A.36) and \( P_a := \Pi(x) = -i \frac{\partial}{\partial \rho(x)} \) in the first two terms in (A.19) we readily obtain the first two terms in (1.1). The singular piece \( H_{\text{sing}} \) in (1.2) arises from the last, divergence term in (A.19). From (A.24) and from (A.30) we thus have
\[
\frac{1}{2m} \left( \Omega^{ab} C_b \right)_a = -\frac{1}{4m} \omega^a := -\frac{1}{4m} \int dx \frac{\delta \omega(x; [\rho])}{\delta \rho(x)},
\]
from which \( H_{\text{sing}} \) in (1.2) follows.

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