\textbf{\textit{L}^p\textit{-ESTIMATES FOR THE VARIATION FOR SINGULAR INTEGRALS ON UNIFORMLY RECTIFIABLE SETS}}

ALBERT MAS AND XAVIER TOLSA

Abstract. The $L^p$ ($1 < p < \infty$) and weak-$L^1$ estimates for the variation for Calderón-Zygmund operators with smooth odd kernel on uniformly rectifiable measures are proven. The $L^2$ boundedness and the corona decomposition method are two key ingredients of the proof.

1. Introduction

This article is devoted to obtain $L^p$ ($1 < p < \infty$) and weak-$L^1$ estimates for the variation for Calderón-Zygmund operators with smooth odd kernel with respect to uniformly rectifiable measures. As a matter of fact, we prove that if the $L^2$ estimates hold then the $L^p$ and weak-$L^1$ estimates follow; the results in [16] deal with the $L^2$ case.

Regarding the Calderón-Zygmund operators, given $1 \leq n < d$ integers, in this article we consider kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ such that $K(-x) = -K(x)$ for all $x \neq 0$ ($K$ is odd) and

\begin{equation}
|K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x_i}K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x_i}\partial_{x_j}K(x)| \leq \frac{C}{|x|^{n+2}}
\end{equation}

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\}$ and all $1 \leq i, j \leq d$, where and $C > 0$ is some constant. Given a Radon measure $\mu$ in $\mathbb{R}^d$, $f \in L^1(\mu)$ and $x \in \mathbb{R}^d$, we set

\[ T^\mu f(x) = T_\epsilon(f\mu)(x) := \int_{|x-y| > \epsilon} K(x-y)f(y) \, d\mu(y), \]

and we denote $T^\mu f(x) = \sup_{\epsilon > 0} |T^\mu f(x)|$, $\mathcal{T} = \{T_\epsilon\}_{\epsilon > 0}$ and $\mathcal{T}^\mu = \{T^\mu_\epsilon\}_{\epsilon > 0}$. Given $\rho > 2$ and $f \in L^1_{\text{loc}}(\mu)$, the $\rho$-variation operator acting on $\mathcal{T}^\mu f = \{T^\mu_\epsilon f\}_{\epsilon > 0}$ is defined as

\[ (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f(x) := \sup_{\epsilon_m} \left( \sum_{m \in \mathbb{Z}} |T^\mu_{\epsilon_m} f(x) - T^\mu_{\epsilon_{m+1}} f(x)|^\rho \right)^{1/\rho} \]

where the pointwise supremum is taken over all the non-increasing sequences of positive numbers $\{\epsilon_m\}_{m \in \mathbb{Z}}$.

Concerning the notion of uniform rectifiability, recall that a Radon measure $\mu$ in $\mathbb{R}^d$ is called $n$-rectifiable if there exists a countable family of $n$-dimensional $C^1$ submanifolds $\{M_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^d$ such that $\mu(E \setminus \bigcup_{i \in \mathbb{N}} M_i) = 0$ and $\mu \ll \mathcal{H}^n$, where $\mathcal{H}^n$ stands for the $n$-dimensional Hausdorff measure. Moreover, $\mu$ is said to be $n$-dimensional Ahlfors-David.
regular, or simply AD regular, if there exists some constant $C > 0$ such that
\[ C^{-1}r^n \leq \mu(B(x, r)) \leq Cr^n \]
for all $x \in \text{supp}\mu$ and $0 < r \leq \text{diam}(\text{supp}\mu)$. Finally, one says that $\mu$ is uniformly $n$-rectifiable if there exist $\theta, M > 0$ so that, for each $x \in \text{supp}\mu$ and $0 < r \leq \text{diam}(\text{supp}\mu)$, there is a Lipschitz mapping $g$ from the $n$-dimensional ball $B^n(0, r) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ such that $\text{Lip}(g) \leq M$ and
\[ \mu(B(x, r) \cap g(B^n(0, r))) \geq \theta r^n, \]
where $\text{Lip}(g)$ stands for the Lipschitz constant of $g$. In particular, uniform rectifiability implies rectifiability. A set $E \subset \mathbb{R}^d$ is called $n$-rectifiable (or uniformly $n$-rectifiable) if $\mathcal{H}^n|E$ is $n$-rectifiable (or uniformly $n$-rectifiable, respectively).

We are ready now to state our main result. In the statement $M(\mathbb{R}^d)$ stands for the Banach space of finite real Radon measures in $\mathbb{R}^d$ equipped with the total variation norm.

**Theorem 1.1.** If $\mu$ is a uniformly $n$-rectifiable measure in $\mathbb{R}^d$ and $\rho > 2$, then
\[ \mathcal{V}_\rho \circ T^\mu : L^p(\mu) \to L^p(\mu) \quad (1 < p < \infty) \quad \text{and} \quad \mathcal{V}_\rho \circ T : M(\mathbb{R}^d) \to L^{1,\infty}(\mu) \]
are bounded operators. In particular, $\mathcal{V}_\rho \circ T^\mu : L^1(\mu) \to L^{1,\infty}(\mu)$ is bounded.

The variation operator has been examined in different contexts during the last years, being probability, ergodic theory, and harmonic analysis three areas where variational inequalities turned out to be a powerful tool to prove new results or to enhance already known ones (see for example [1, 7, 8, 9, 10, 12, 17], and the references therein). Inspired by the results on variational inequalities for Calderón-Zygmund operators in $\mathbb{R}^n$ like [2, 3], in [15] we began our study of such type of inequalities when one replaces the underlying space $\mathbb{R}^n$ and its associated Lebesgue measure by some reasonable measure in $\mathbb{R}^d$, being the Hausdorff measure on a Lipschitz graph a first natural candidate. In this regard, Theorem 1.1 should be considered as a natural generalisation of variational inequalities for Calderón-Zygmund operators in $\mathbb{R}^n$ from a geometric measure-theoretic point of view.

A big motivation to prove Theorem 1.1 is its connection to the so called David-Semmes problem regarding the Riesz transform and rectifiability. Given a Radon measure $\mu$ in $\mathbb{R}^d$, one defines the $n$-dimensional Riesz transform of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where
\[ R^\mu_\epsilon f(x) = \int_{|x-y| > \epsilon} \frac{x-y}{|x-y|^{n+1}} f(y) \, d\mu(y), \quad x \in \mathbb{R}^d. \]
Note that the kernel of the Riesz transform is the vector $(x^1, \ldots, x^d)/|x|^{n+1}$ (so, in this case kernel $K$ in (1) is vectorial). We also use the notation $R^\mu f(x) := \{R^\mu_\epsilon f(x)\}_{\epsilon > 0}$ and, as usual, we define the maximal operator $R^\mu f(x) = \sup_{\epsilon > 0} |R^\mu_\epsilon f(x)|$.

G. David and S. Semmes asked more than twenty years ago the following question, which is still open (see, for example, [15, Chapter 7]):

**Question 1.2.** Is it true that an $n$-dimensional AD regular measure $\mu$ is uniformly $n$-rectifiable if and only if $R^\mu_\epsilon$ is bounded in $L^2(\mu)$?

By [8], the “only if” implication of this question above is already known to hold. Also in [5], G. David and S. Semmes gave a positive answer to the other implication if one replaces the $L^2$ boundedness of $R^\mu_\epsilon$ by the $L^2$ boundedness of $T^\mu_\epsilon$ for a wide class of odd kernels $K$. In the case $n = 1$ the “if” implication was proved in [13] using the notion of curvature of measures. Later on, the same implication was answered affirmatively for $n = d - 1$ in the work [11] by combining quasiorthogonality arguments with some variational estimates which
use the maximum principle derived from the fact that the Riesz kernel is (a multiple) of the gradient of the fundamental solution of the Laplacian in $\mathbb{R}^d$ when $n = d - 1$. Question 1.2] is still open for the general case $1 < n < d - 1$. However, thanks to Theorem 1.1 and 16 Theorem 2.3] we get the following corollary, which characterizes uniform rectifiability in terms of variational inequalities for the Riesz transform and more general Calderón-Zygmund operators.

**Corollary 1.3.** Let $\mu$ be an $n$-dimensional $AD$ regular Radon measure in $\mathbb{R}^d$. Then, the following are equivalent:

(a) $\mu$ is uniformly $n$-rectifiable,
(b) for any odd kernel $K$ as in (1) and any $\rho > 2$, $V_\rho \circ T^\mu$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, and from $L^1(\mu)$ into $L^{1,\infty}(\mu)$,
(c) for some $\rho > 0$, $V_\rho \circ R^\mu$ is bounded in $L^2(\mu)$.

Comparing Corollary 1.3 to Question 1.2 note that the corollary asserts that if we replace the $L^2(\mu)$ boundedness of $R^\mu_\ast$ by the stronger assumption that $V_\rho \circ R^\mu$ is bounded in $L^2(\mu)$, then $\mu$ must be uniformly rectifiable. On the other hand, the corollary claims that the variation for singular integral operators with any odd kernel satisfying (1), in particular for the $n$-dimensional Riesz transforms, is bounded in $L^p(\mu)$ for all $1 < p < \infty$ and it is of weak-type $(1, 1)$, which is a stronger conclusion than the one derived from an affirmative answer to Question 1.2.

The proof of (c) $\implies$ (a) in Corollary 1.3 is not as hard as the converse implications. Essentially, a combination of the arguments in [19] with the fact that, in a sense, $V_\rho \circ R^\mu$ controls $R^\mu_\ast$ does the job (see [16]). Theorem 1.1 is used to prove that (a) $\implies$ (b) in Corollary 1.3 inasmuch as the corresponding result in [10] was only proved for $p = 2$. Theorem 1.1 allows us to get it in full generality, completing the whole picture on variation for singular integrals and uniform rectifiability. As far as we know, neither the $L^p$ estimates with $1 < p < \infty$ nor the weak-$L^1$ estimate for $V_\rho \circ T^\mu$ on uniform rectifiable measures $\mu$ were known, except for the case $p = 2$ treated in [10] and the case where $1 < p < \infty$ but $supp \mu$ is a Lipschitz graph with slope strictly smaller than 1, solved in [14]. Let us stress that from the latter result one can not easily deduce the $L^p$ estimates on uniformly rectifiable measures (as in the standard situation in Calderón-Zygmund theory), basically because the good-$\lambda$ method does not work properly for $V_\rho \circ T$. To avoid this obstacle, our method relies on the corona decomposition technique combined with some ideas from the Lipschitz case in [14] and from [2] and [12] to deal with variational inequalities, as well as the $L^2$ result from [16].

2. Preliminaries and auxiliary results

2.1. Notation and terminology. As usual, in the paper the letter ‘$C$’ (or ‘$c$’) stands for some constant which may change its value at different occurrences, and which quite often only depends on $n$ and $d$. Given two families of constants $A(t)$ and $B(t)$, where $t$ stands for all the explicit or implicit parameters involving $A(t)$ and $B(t)$, the notation $A(t) \lesssim B(t)$ ($A(t) \gtrsim B(t)$) means that there is some fixed constant $C$ such that $A(t) \leq CB(t)$ ($A(t) \geq CB(t)$) for all $t$, with $C$ as above. Also, $A(t) \approx B(t)$ is equivalent to $A(t) \lesssim B(t) \lesssim A(t)$.

Throughout all the paper we assume that $1 \leq n \leq d$ are integers and that $\mu$ is an $n$-dimensional AD-regular measure in $\mathbb{R}^d$. Given a bounded Borel set $A \subset \mathbb{R}^d$ and $f \in L^1_{loc}(\mu)$, we write the mean of $f$ on $A$ with respect to $\mu$ as follows:

$$m_A f := \frac{1}{\mu(A)} \int_A f \, d\mu.$$
We consider the centered maximal Hardy-Littlewood operator:

$$\mathcal{M} f(x) = \sup_{r > 0} m_{B(x,r)} |f|.$$  

This is known to be bounded in $L^p(\mu)$, for $1 < p \leq \infty$, and from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. For $1 \leq q < \infty$, we also set

$$\mathcal{M}_q f := \mathcal{M}(|f|^q)^{1/q}.$$  

This is bounded in $L^p(\mu)$, for $q < p \leq \infty$, and from $L^p(\mu)$ to $L^{1,\infty}(\mu)$.  

Given $0 \leq a < b$, consider the closed annulus

$$A(x,a,b) := \overline{B(x,b)} \setminus B(x,a).$$  

Given $k \in \mathbb{Z}$, set

$$I_k := [2^{-k-1}, 2^{-k}).$$  

One defines the short and long variation operators $\mathcal{V}_p^S \circ T^\mu$ and $\mathcal{V}_p^L \circ T^\mu$, respectively, by

$$\langle \mathcal{V}_p^S \circ T^\mu \rangle f(x) := \sup_{\{\epsilon_m\}} \left( \sum_{k \in \mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_k} |T^\mu_{\epsilon_m} f(x) - T^\mu_{\epsilon_{m+1}} f(x)|^p \right)^{1/p},$$

$$\langle \mathcal{V}_p^L \circ T^\mu \rangle f(x) := \sup_{\{\epsilon_m\}} \left( \sum_{m \in \mathbb{Z}; \epsilon_m \in I_j, \epsilon_{m+1} \in I_k} \text{for some } j < k \right) |T^\mu_{\epsilon_m} f(x) - T^\mu_{\epsilon_{m+1}} f(x)|^p \right)^{1/p},$$

where, in both cases, the pointwise supremum is taken over all the non-increasing sequences of positive numbers $\{\epsilon_m\}_{m \in \mathbb{Z}}$. For convenience of notation, given $0 < \epsilon = \delta$ set $T_{\delta,\epsilon} := T_{\delta} - T_{\epsilon}$.

Let $\varphi_R : [0, +\infty) \to [0, +\infty)$ be a non-decreasing $C^2$ function with $\chi_{[4, \infty)} \leq \varphi_R \leq \chi_{[1/4, \infty)}$ and set $\varphi_\epsilon(x) = \varphi_R(|x|^2/\epsilon^2)$. Given a finite Borel measure $\mu$ in $\mathbb{R}^d$, we define

$$T_{\varphi, \mu}(x) := \int \varphi_\epsilon(x-y) K(x-y) d\mu(y) \quad \text{for } x \in \mathbb{R}^d.$$  

Finally, write $\mathcal{T}_\varphi := \{T_{\varphi, \epsilon}\}_{\epsilon > 0}$. Compare the operator in (2) to

$$T_\epsilon \mu(x) = \int \chi_\epsilon(x-y) K(x-y) d\nu(y),$$

where $\chi_\epsilon() := \chi_{(1,\infty)}(|\cdot|/\epsilon)$, and the family $\mathcal{T}_\varphi$ to $\mathcal{T}$.

2.2. Dyadic lattices. For the study of the uniformly rectifiable measures we will use the “dyadic cubes” built by G. David in [4, Appendix 1] (see also [6, Chapter 3 of Part I]). These dyadic cubes are not true cubes, but they play this role with respect to a given $n$-dimensional AD regular Radon measure $\mu$, in a sense.

Let us explain which are the precise results and properties of this lattice of dyadic cubes. Given an $n$-dimensional AD regular Radon measure $\mu$ in $\mathbb{R}^d$ (for simplicity, we may assume diam(supp$\mu$) = $\infty$), for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_j^\mu$ of Borel subsets of supp$\mu$ (the dyadic cubes of the $j$-th generation) such that:

(a) each $\mathcal{D}_j^\mu$ is a partition of supp$\mu$, i.e. supp$\mu = \bigcup_{Q \in \mathcal{D}_j^\mu} Q$ and $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{D}_j^\mu$ and $Q \neq Q'$;

(b) if $Q \in \mathcal{D}_j^\mu$ and $Q' \in \mathcal{D}_k^\mu$ with $k \leq j$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;

(c) for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j^\mu$, we have $2^{-j} \lesssim \text{diam}(Q) \lesssim 2^{-j}$ and $\mu(Q) \approx 2^{-jn}$;
(d) there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}^\mu_j$, and $0 < \tau < 1$,

$$
\mu\{x \in Q : \text{dist}(x, \supp \mu \setminus Q) \leq 2^{-j}\} \quad (3)
$$

$$
\mu\{x \in \supp \mu \setminus Q : \text{dist}(x, Q) \leq 2^{-j}\} \leq C\tau^{1/C}2^{-jn}.
$$

This property is usually called the small boundaries condition. From (3), it follows that there is a point $z_Q \in Q$ (the center of $Q$) such that $\text{dist}(z_Q, \supp \mu \setminus Q) \gtrsim 2^{-j}$ (see [6, Lemma 3.5 of Part I]).

We set $\mathcal{D}^\mu := \bigcup_{j \in \mathbb{Z}} \mathcal{D}^\mu_j$.

Given a cube $Q \in \mathcal{D}^\mu_j$, we say that its side length is $2^{-j}$, and we denote it by $\ell(Q)$. Notice that $\text{diam}(Q) \leq \ell(Q)$. For $\lambda > 1$, we also write

$$
\lambda Q = \{x \in \supp \mu : \text{dist}(x, Q) \leq (\lambda - 1)\ell(Q)\}.
$$

We denote

$$
B_Q := B(z_Q, c_1\ell(Q)),
$$

where $c_1 \geq 1$ is some fixed constant which will be chosen below, depending on other parameters.

Given $R \in \mathcal{D}_j$ for some $j \in \mathbb{Z}$, let $P(R)$ denote the cube in $\mathcal{D}^\mu_{j+1}$ which contains $R$ (the parent of $R$), and set

$$
\text{Ch}(R) := \{Q \in \mathcal{D}^\mu_{j+1} : Q \subset R\},
$$

$$
V(R) := \{Q \in \mathcal{D}^\mu_j : \text{dist}(Q, R) \leq \ell(Q)\}.
$$

(Ch($R$) are the children of $R$, and $V(R)$ stands for the vicinity of $R$).

Notice that $P(R)$ is a cube from $\mathcal{D}^\mu$ but Ch($R$) and V($R$) are collections of cubes from $\mathcal{D}^\mu$. It is not hard to show that the number of cubes in Ch($R$) and V($R$) is bounded by some constant depending only on $n$ and the AD regularity constant of $\mu$.

For a cube $Q \in \mathcal{D}^\mu_j$, we write

$$
I_Q := I_j = [\ell(Q)/2, \ell(Q))
$$

2.3. The corona decomposition. Given an $n$-dimensional AD regular Radon measure $\mu$ on $\mathbb{R}^d$ consider the dyadic lattice $\mathcal{D}^\mu$ introduced in Subsection 2.2. Following [6] Definitions 3.13 and 3.19 of Part I, one says that $\mu$ admits a corona decomposition if, for each $\eta > 0$ and $\theta > 0$, one can find a triple $(\mathcal{B}, \mathcal{G}, \text{Trs})$, where $\mathcal{B}$ and $\mathcal{G}$ are two subsets of $\mathcal{D}^\mu$ (the “bad cubes” and the “good cubes”) and Trs is a family of subsets $S \subset \mathcal{G}$ (that we will call trees), which satisfy the following conditions:

(a) $\mathcal{D}^\mu = \mathcal{B} \cup \mathcal{G}$ and $\mathcal{B} \cap \mathcal{G} = \emptyset$.

(b) $\mathcal{B}$ satisfies a Carleson packing condition, i.e., $\sum_{Q \in \mathcal{B}, Q \subset R} \mu(Q) \lesssim \mu(R)$ for all $R \in \mathcal{D}^\mu$.

(c) $\mathcal{G} = \bigcup_{S \in \text{Trs}} S$, i.e., any $Q \in \mathcal{G}$ belongs to only one $S \in \text{Trs}$.

(d) Each $S \in \text{Trs}$ is coherent. This means that each $S \in \text{Trs}$ has a unique maximal element $Q_S$ which contains all other elements of $S$ as subsets, that $Q' \in S$ as soon as $Q' \in \mathcal{D}^\mu$ satisfies $Q \subset Q' \subset Q_S$ for some $Q \in S$, and that if $Q \in S$ then either all of the children of $Q$ lie in $S$ or none of them do (if $Q \in \mathcal{D}^\mu_j$, the children of $Q$ is defined as the collection of cubes $Q' \in \mathcal{D}^\mu_{j+1}$ such that $Q' \subset Q$).

(e) The maximal cubes $Q_S$, for $S \in \text{Trs}$, satisfy a Carleson packing condition. That is, $\sum_{S \in \text{Trs}, Q_S \subset R} \mu(Q_S) \lesssim \mu(R)$ for all $R \in \mathcal{D}^\mu$.

(f) For each $S \in \text{Trs}$, there exists an $n$-dimensional Lipschitz graph $\Gamma_S$ with constant smaller than $\eta$ such that $\text{dist}(x, \Gamma_S) \leq \theta \text{diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$ (one
can replace “$x \in 2Q$” by “$x \in c_2 Q$” for any constant $c_2 \geq 2$ given in advance, by [6 Lemma 3.31 of Part I]).

It is shown in [5] (see also [6]) that if $\mu$ is uniformly rectifiable then it admits a corona decomposition for all parameters $k > 2$ and $\eta, \theta > 0$. Conversely, the existence of a corona decomposition for a single set of parameters $k > 2$ and $\eta, \theta > 0$ implies that $\mu$ is uniformly rectifiable.

We set

$$\text{Top}_\mathcal{G} = \{Q_S : S \in \text{Trs}\} \quad \text{and} \quad \text{Top} = \text{Top}_\mathcal{G} \cup \mathcal{B}.$$ 

If $\mu$ is uniformly rectifiable, then, by the properties (b) and (e) above, for all $R \in \mathcal{D}^\mu$ we have

$$\sum_{Q \in \text{Top}_\mathcal{G} \cup \mathcal{B} \subset R} \mu(Q) \lesssim \mu(R).$$

If $R \in S$ for some $S \in \text{Trs}$, we denote by $\text{Tree}(R)$ the set of cubes $Q \in S$ such that $Q \subset R$ (the tree of $R$). Otherwise, that is, if $R \in \mathcal{B}$, we set $\text{Tree}(R) := \{R\}$. Finally, $\text{Stp}(R)$ stands for the set of cubes $Q \in \mathcal{B} \cup (\mathcal{G} \setminus \text{Tree}(R))$ such that $Q \subset R$ and $P(Q) \in \text{Tree}(R)$ (the stopping cubes relative to $R$), so actually $Q \subset R$. Notice that if $R \in \mathcal{B}$, then we have $\text{Stp}(R) = \text{Ch}(R)$.

2.4. Auxiliary results.

**Lemma 2.1** (Calderón-Zygmund decomposition). Let $\mu$ be a compactly supported uniformly $n$-rectifiable measure in $\mathbb{R}^d$. For every $\nu \in M(\mathbb{R}^d)$ with compact support and every $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$, the following hold:

(a) There exists a finite or countable collection of almost disjoint cubes $\{Q_j\}_j$, that is

$$\sum_j \chi_{Q_j} \leq C,$$

and a function $f \in L^1(\mu)$ such that

$$\nu(Q_j) > 2^{-d-1}\lambda \mu(2Q_j),$$

$$\nu(\eta Q_j) \leq 2^{-d-1}\lambda \mu(2\eta Q_j) \quad \text{for } \eta > 2,$$

$$\nu = f\mu \text{ in } \mathbb{R}^d \setminus \Omega \text{ with } |f| \leq \lambda \mu \text{-a.e., where } \Omega = \bigcup_j Q_j.$$

(b) For each $j$, let $R_j := 6Q_j$ and denote $w_j := \chi_{Q_j} (\sum_k \chi_{Q_k})^{-1}$. Then, there exists a family of functions $\{b_j\}_j$ with $\text{supp} b_j \subset R_j$ and with constant sign satisfying

$$\int b_j d\mu = \int w_j d\nu,$$

$$\|b_j\|_{L^\infty(\mu)} \mu(R_j) \leq C\nu(Q_j), \text{ and}$$

$$\sum_j |b_j| \leq C_0 \lambda, \quad \text{where } C_0 \text{ is some absolute constant}.$$

Let us remark that the cubes in the preceding lemma are “true cubes”, i.e. they do not belong to $\mathcal{D}^\mu$.

Notice that from [5] it follows that, for every $j$, $\eta Q_j \cap \text{supp} \mu \neq \emptyset$ for $\eta > 4$, which implies that

$$\mu(\eta Q_j) \approx \ell(\eta Q_j)^n \quad \text{for } \eta > 5,$$

and, in particular, $\mu(R_j) \approx \ell(R_j)^n \approx \ell(Q_j)^n$.

We will need the following version of the dyadic Carleson embedding theorem.
**Theorem 2.2** (Dyadic Carleson embedding theorem). Let \( \mu \) be a Radon measure on \( \mathbb{R}^d \). Let \( \mathcal{D} \) be some dyadic lattice from \( \mathbb{R}^d \) and let \( \{ a_Q \}_{Q \in \mathcal{D}} \) be a family of non-negative numbers. Suppose that for every cube \( R \in \mathcal{D} \) we have

\[
\sum_{Q \in \mathcal{D}: Q \subseteq R} a_Q \leq c_3 \mu(R).
\]

Then every family of non-negative numbers \( \{ \gamma_Q \}_{Q \in \mathcal{D}} \) satisfies

\[
\sum_{Q \in \mathcal{D}} \gamma_Q a_Q \leq c_3 \int \sup_{Q \ni x} \gamma_Q \, d\mu(x).
\]

Also, for \( p \in (1, \infty) \), if \( f \in L^p(\mu) \),

\[
\sum_{Q \in \mathcal{D}} |m_Q f|^p a_Q \leq c c_3^3 \| f \|_{L^p(\mu)},
\]

where \( m_Q f = \int_Q f \, d\mu/\mu(Q) \) and \( c \) is an absolute constant.

In the preceding theorem, the lattice \( \mathcal{D} \) can be, for example, either the usual dyadic lattice of \( \mathbb{R}^d \) or, in the case when \( \mu \) is AD-regular, the lattice of cubes associated with \( \mu \). For the proof of this classical result, see [20, Theorem 5.8], for example.

We say that \( C \subset \mathcal{D} \) is a Carleson family of cubes if

\[
\sum_{Q \in \mathcal{D}: Q \subset R} \mu(Q) \leq c_3 \mu(R) \quad \text{for all } R \in \mathcal{D}.
\]

By (14), it follows that for such a family \( C \) and any \( f \in L^p(\mu) \),

\[
\sum_{Q \in C} |m_Q f|^p \mu(Q) \leq c c_3^3 \| f \|_{L^p(\mu)}^p.
\]

**Lemma 2.3.** Let \( \nu \in M(\mathbb{R}^d) \) be a positive measure with compact support and \( \lambda > 2^{d+1} \| \nu \|/\| \mu \| \) and consider cubes \( \{ Q_j \}_j \) and \( \{ R_j \}_j \) as in Lemma 2.1. Denote

\[
\nu_b := \sum_j (w_j \nu - b_j \mu),
\]

where the \( b_j \)'s satisfy (8), (9) and (10), and \( w_j := \chi_{Q_j} (\sum_k \chi_{Q_k})^{-1} \). Let \( \mathcal{C} \subset \mathcal{D}^\mu \) be a Carleson family of cubes. For each \( S \in \mathcal{C} \) consider the ball \( B_S \) given by (1), so it is centered on \( S, S \subset B_S \) and \( r(B_S) \lesssim \ell(S) \). Suppose that each \( B_S \), with \( S \in \mathcal{C} \), contains some cube \( R_j \). Then, for every \( p \in (1, \infty) \),

\[
\sum_{S \in \mathcal{C}} \left( \frac{|\nu_b|(B_S)}{\ell(S)^n} \right)^p \ell(S)^n \lesssim \lambda^{p-1} \| \nu \|
\]

and

\[
\sum_{S \in \mathcal{C}} \left( \frac{\nu(B_S)}{\ell(S)^n} \right)^p \ell(S)^n \lesssim \lambda^{p-1} \| \nu \|
\]

with the implicit constants depending on \( p \) and the constant involved in the Carleson character of the family \( \mathcal{C} \).
Proof. First we will show (15). Taking into account that \( \mu(S) \approx \ell(S)^n \) for every \( S \in C \), by (13) in Theorem 2.2 one gets
\[
\sum_{S \in C} \left( \frac{|\nu_b|(B_S)}{\ell(S)^n} \right)^p \ell(S)^n \leq c \int \left( \sup_{S \ni x} \frac{|\nu_b|(B_S)}{\ell(S)^n} \right)^p d\mu(x).
\]
Write
\[
\tilde{\nu}_b = \sum_j w_j \nu \quad \text{and} \quad \tilde{g} = \sum_j b_j,
\]
so that, for every \( S \in C \),
\[
|\nu_b|(B_S) \leq \tilde{\nu}_b(B_S) + \int_{B_S} \tilde{g} \, d\mu.
\]
Note that the measure \( \tilde{\nu}_b \) and the functions \( b_j, g \) are positive because \( \nu \) is assumed to be a positive measure. By (17) then we have
\[
\sum_{S \in C} \left( \frac{|\nu_b|(B_S)}{\ell(S)^n} \right)^p \ell(S)^n \lesssim \int \left( \sup_{S \ni x} \tilde{\nu}_b(B_S) \right)^p d\mu(x) + \int \left( \sup_{S \ni x} m_{B_S} \tilde{g} \right)^p d\mu(x),
\]
where \( m_{B_S} \tilde{g} = \int_{B_S} \tilde{g} \, d\mu / \mu(B_S) \) and we have taken into account that \( \mu(B_S) \approx \ell(S)^n \).

To deal with the last integral on the right hand side of (18) we use the non-centered maximal Hardy-Littlewood operator defined by
\[
\widetilde{M} f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu,
\]
where the supremum is take over all the balls which contain \( x \) and whose center lies on \( \text{supp} \mu \). Recalling that \( \widetilde{M} \) is bounded in \( L^p(\mu) \), and using that \( \| \tilde{g} \|_{L^1(\mu)} \leq c \lambda \) (by (10)) and \( \| \tilde{g} \|_{L^1(\mu)} \leq c \| \nu \| \) (by (13)), we obtain
\[
\int \left( \sup_{S \ni x} m_{B_S} \tilde{g} \right)^p d\mu(x) \leq c \int (\widetilde{M} \tilde{g})^p d\mu \leq c \int \tilde{g}^p d\mu \leq c \lambda^{p-1} \int \tilde{g} \, d\mu \leq c \lambda^{p-1} \| \nu \|.
\]

Now we turn our attention to the first integral on the right hand side of (18). We write
\[
\int \left( \sup_{S \ni x} \tilde{\nu}_b(B_S) \right)^p d\mu(x) = \int_{\cup_{j} 2Q_j} \ldots + \int_{\mathbb{R}^n \setminus \cup_{j} 2Q_j} \ldots =: I_1 + I_2.
\]
To estimate \( I_1 \), we claim that
\[
\frac{\tilde{\nu}_b(B_S)}{\ell(S)^n} \lesssim \lambda.
\]
This follows from the fact that \( B_S \) contains some cube \( R_j \), which in turn implies that, for some \( \eta \geq 6 \), \( B_S \) is contained in some cube \( \eta Q_j \) with \( \ell(\eta Q_j) \approx \ell(S) \), and then
\[
\frac{\tilde{\nu}_b(B_S)}{\ell(S)^n} \lesssim \frac{\nu(\eta Q_j)}{\ell(\eta Q_j)^n},
\]
which together with (11) and (6) yields the claim above. Then, using also (13) and the fact the cubes \( \{Q_j\}_j \) have finite overlap, we deduce that
\[
I_1 \lesssim \lambda^p \sum_j \mu(2Q_j) \lesssim \lambda^p \sum_j \frac{\nu(Q_j)}{\lambda} \lesssim \lambda^{p-1} \| \nu \|.
\]
Finally we deal with the integral $I_2$. Consider $x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j$ and $S$ such that $x \in S \in \mathcal{C}$ (which, in particular, tells us that $S \setminus \bigcup_j 2Q_j \neq \emptyset$). Notice that

$$\tilde{\nu}_b(B_S) \leq \sum_{i : Q_i \cap B_S \neq \emptyset} \nu(Q_i).$$

From the conditions $Q_i \cap B_S \neq \emptyset$ and $S \setminus \bigcup_j 2Q_j \neq \emptyset$, we infer that $r(B_S) \geq \frac{1}{2} \ell(Q_i)$. So we deduce that $Q_i \subset c_4 B_S$, for some constant $c_4 \gtrsim 1$. Hence,

$$\tilde{\nu}_b(B_S) \leq \sum_{i : Q_i \subset c_4 B_S} \nu(Q_i) \leq \sum_{i : Q_i \subset c_4 B_S} \int b_i \, d\mu,$$

where we used (8) for the last estimate. Observe now that if $Q_i \subset c_4 B_S$, then $R_i \subset c_5 B_S$, for some absolute constant $c_5 \geq c_4$. So recalling that $\tilde{g} = \sum_j b_j$, we obtain

$$\tilde{\nu}_b(B_S) \lesssim \int_{c_5 B_S} \tilde{g} \, d\mu,$$

Therefore,

$$\frac{\tilde{\nu}_b(B_S)}{\ell(S)^n} \lesssim \frac{1}{\mu(B_S)} \int_{c_5 B_S} \tilde{g} \, d\mu \lesssim \tilde{M}\tilde{g}(x)$$

for every $x \in S$. So arguing as in (19) we deduce that

$$I_2 \lesssim \int (\tilde{M}\tilde{g}(x))^p \, d\mu(x) \lesssim \lambda^{p-1} \|\nu\|.$$ 

Together with the estimate we obtained for $I_1$, this yields

$$(20) \quad \int \left( \sup_{S \ni x} \frac{\tilde{\nu}_b(B_S)}{\ell(S)^n} \right)^p \, d\mu(x) \lesssim \lambda^{p-1} \|\nu\|,$$

and so using (19) we get (15).

The inequality (16) follows from the fact that for each $S \in \mathcal{C}$, $\nu(B_S) \leq c \tilde{\nu}_b(B_S)$, applying (13) in Theorem 2.2 and (20). \hfill \square

Let $\mu$ be a uniformly $n$-rectifiable measure in $\mathbb{R}^d$. Consider the splitting $\mathcal{D}^\mu = \mathcal{B} \cup (\bigcup_{T \in \text{Trs}} T)$ given by the corona decomposition of $\mu$. For a fixed constant $A \geq 1$, we denote by $\partial T$ the family of cubes $Q \in T$ for which there exists some $P \in \mathcal{D}^\mu \setminus T$ such that

$$\frac{1}{2} \ell(P) \leq \ell(Q) \leq 2\ell(P) \quad \text{and} \quad \text{dist}(P, Q) \leq A \ell(Q).$$

We call $\partial T$ the boundary of $T$. If $T = \text{Tree}(R)$, with $R \in \text{Top}_G$, we also write $\partial \text{Tree}(R) := \partial T$. We set

$$\partial \text{Trs} := \bigcup_{T \in \text{Trs}} \partial T.$$

Notice that $\partial \text{Trs} \subset T$.

The following lemma has been proved in [3, (3.28) in page 60].

**Lemma 2.4.** Let $\mu$ be a uniformly $n$-rectifiable measure in $\mathbb{R}^d$. The family $\partial \text{Trs}$ is a Carleson family.

We will also need the following auxiliary result.
**Lemma 2.5** (Anuli estimates). Assume that the constants $\eta$ and $\theta$ in property (f) of the corona decomposition (see Section 2.3) are small enough. Let $Q \in D$, $x \in Q$ and $\epsilon \in [(\ell(Q)/2, \ell(Q)]$. Let $k \in \mathbb{Z}$ and $C > 0$ be such that $C2^{-k} \leq \ell(Q)/2$. Given $R \in V(Q)$, denote

$$\Lambda_k := \left\{ P \in \text{Tree}(R) \cup \text{Stp}(R) : \ell(P) = 2^{-k}, P \subset A(x, \epsilon - C2^{-k}, \epsilon + C2^{-k}) \right\}.$$  

Then

$$\mu \left( \bigcup_{P \in \Lambda_k} P \right) \lesssim 2^{-k} \ell(R)^{n-1},$$

where the implicit constant in the last inequality above only depends on $n$, $d$, $\mu$ and $C$.

For the proof, see [16, Lemma 5.9]. In fact, in this reference the anuli estimates are proved only for $R \in G$. However, for $R \in B$, the inequality (21) is trivial.

3. $V_\rho \circ T : M(R^d) \to L^{1, \infty}(\mu)$ is a bounded operator

In this section we will prove the following result.

**Theorem 3.1.** If $\mu$ is a uniformly $n$-rectifiable measure in $\mathbb{R}^d$ and $\rho > 2$, then

(i) $V_\rho^S \circ T : M(R^d) \to L^{1, \infty}(\mu)$ is bounded,

(ii) $V_\rho^C \circ T : M(R^d) \to L^{1, \infty}(\mu)$ is bounded.

In particular, $V_\rho \circ T$ is a bounded operator from $M(R^d)$ to $L^{1, \infty}(\mu)$ for all $\rho > 2$.

Notice that by the triangle inequality we can easily split the variation operator into the short and long variations, that is, $(V_\rho \circ T^\mu)f \leq (V_\rho^S \circ T^\mu)f + (V_\rho^C \circ T^\mu)f$. Therefore, that $V_\rho \circ T$ is a bounded operator from $M(R^d)$ to $L^{1, \infty}(\mu)$ for all $\rho > 2$ follows from (i) and (ii) above, whose proofs are given below.

We will use the next result, which is contained in [16, Theorem 1.3 and Corollary 4.2].

**Theorem 3.2.** If $\mu$ is a uniformly $n$-rectifiable measure in $\mathbb{R}^d$ and $\rho > 2$, then

(i) $V_\rho \circ T^\nu : L^2(\mu) \to L^2(\mu)$ is bounded,

(ii) $V_\rho \circ T^\nu : M(R^d) \to L^{1, \infty}(\mu)$ is bounded.

**Proof of Theorem 3.1.** We will deal with the long variation $V_\rho^C \circ T$ by comparing it with the smoothened version $V_\rho \circ T_\rho$, using Theorem 3.2(ii), estimating the error terms by the short variation $V_\rho^S \circ T$, and applying Theorem 3.1(i). More precisely, the triangle inequality yields

$$|T_\epsilon \nu(x) - T_\delta \nu(x)| \leq |T_{\rho_1} \nu(x) - T_{\rho_3} \nu(x)| + |T_{\rho_2} \nu(x) - T_{\rho_4} \nu(x)| + |T_{\delta} \nu(x) - T_{\rho_1} \nu(x)|$$

for any $0 < \delta \leq \epsilon$. Therefore,

$$\left( (V_\rho^C \circ T) \nu(x) \right)^p \leq \left( (V_\rho \circ T_\rho) \nu(x) \right)^p + \sum_{\{\epsilon_m\} \in \mathbb{Z}, \epsilon_m \in I_j, \epsilon_m+1 \in I_k} \left| T_{\epsilon_m} \nu(x) - T_{\epsilon_{m+1}} \nu(x) \right|^p,$$

$$+ \sup_{\{\epsilon_m\} : \epsilon_m \in I_k, \epsilon_m+1 \in I_j \text{ for some } j < k} \left| T_{\epsilon_m} \nu(x) - T_{\epsilon_{m+1}} \nu(x) \right|^p,$$

$$\lesssim \left( (V_\rho \circ T_\rho) \nu(x) \right)^p + \sup_{\{\epsilon_m\} : \epsilon_m \in I_m, \epsilon_m \in \mathbb{Z}} \sum_{\epsilon_m \in I_m} \left| T_{\epsilon_m} \nu(x) - T_{\epsilon_{m+1}} \nu(x) \right|^p.$$
Let us estimate the second term on the right hand side of (22). Since \( \chi_{[4,\infty]} \leq \varphi_R \leq \chi_{[1/4,\infty)} \) by definition, we have

\[
\chi_{[1,\infty)}(t) - \varphi_R(t) = \int_{1/4}^4 \varphi'_R(s) (\chi_{[1,\infty)} - \chi_{[s,\infty)}) (t) \, ds
\]

for all \( t \geq 0 \). This means that \( \chi_{[1,\infty)} - \varphi_R \) is a convex combination of the functions \( \chi_{[1,\infty)} - \chi_{[s,\infty)} \) for \( 1/4 \leq s \leq 4 \). Then, Fubini’s theorem gives

\[
T_t \nu(x) - T_{\varphi_R} \nu(x) = \int \left( \chi_{[1,\infty)}(|x-y|^2/\varepsilon^2) - \varphi_R(|x-y|^2/\varepsilon^2) \right) K(x-y) \, d\nu(y)
\]

(23)

\[
= \int_{1/4}^4 \varphi'_R(s) \left( (\chi_{[s,\infty)} - \chi_{[\varepsilon\sqrt{\pi},\infty)}) (|x-y|) \right) K(x-y) \, d\nu(y)
\]

(24)

Finally, applying (25) to (22) yields

\[
\sup_{\{\varepsilon_m \}} \int_{1/4}^4 \varphi'_R(s) \left( \sum_{m \in \mathbb{Z}} |T_{\varepsilon_m} \nu(x) - T_{\varphi_R} \nu(x)|^\rho \right)^{1/\rho} \, ds
\]

(25)

\[
\leq \int_{1/4}^4 \varphi'_R(s) (\mathcal{V}^\mathcal{S}_\rho \circ T) \nu(x) \, ds \lesssim (\mathcal{V}^\mathcal{S}_\rho \circ T) \nu(x).
\]

and Theorem 3.1(ii) follows by Theorems 3.2(ii) and 3.1(i). \( \square \)

Proof of Theorem 3.1(i). We have to prove that there exists some constant \( C > 0 \) such that

(26)

\[
\mu \left( \{ x \in \mathbb{R}^d : (\mathcal{V}^\mathcal{S}_\rho \circ T) \nu(x) > \lambda \} \right) \leq \frac{C}{\lambda} ||\nu||
\]

for all \( \nu \in M(\mathbb{R}^d) \) and all \( \lambda > 0 \). The proof of (26) combines the Calderón-Zygmund decomposition developed in Lemma 2.4, the corona decomposition of \( \mu \) described in Subsection 2.3, and other standard techniques for proving variational inequalities. We will start following the lines of the proof of [14, Theorem 1.4], until the application of the corona decomposition. Since \( \mathcal{V}^\mathcal{S}_\rho \circ T \) is sublinear, we can assume without loss of generality that \( \nu \) is a positive measure. Let us first check that we can also assume both \( \mu \) and \( \nu \) to be compactly supported. Given \( \nu \in M(\mathbb{R}^d) \) and \( N \in \mathbb{N} \), set

\[
\mu_N := \chi_{B(0,N)} \mu \quad \text{and} \quad \nu_N := \chi_{B(0,N)} \nu.
\]
and suppose that there exists some constant $C > 0$ such that
\begin{equation}
\mu_{N_0}\left(\{x \in \mathbb{R}^d : (V^S_\rho \circ \mathcal{T})\nu_N(x) > \lambda\}\right) \leq \frac{C}{N} \|\nu_N\|\tag{27}
\end{equation}
for all $\lambda > 0$, all $\nu \in M(\mathbb{R}^d)$ and all $N_0, N \in \mathbb{N}$. It is not hard to show that
\[ |(V^S_\rho \circ \mathcal{T})\nu(x) - (V^S_\rho \circ \mathcal{T})\nu_N(x)| \leq \frac{C}{N - N_0} \nu(\mathbb{R}^d \setminus B(0, N)) \]
for all $x \in \operatorname{supp}(\mu_{N_0})$ and all $N$ big enough, where $C > 0$ does not depend on $N_0$. In particular, if $N \to \infty$ then $(V^S_\rho \circ \mathcal{T})\nu_N(x) \to (V^S_\rho \circ \mathcal{T})\nu(x)$ for all $x \in \operatorname{supp}(\mu_{N_0})$. Since (27) holds for $\nu_N$ by assumption, we deduce that it also holds for $\nu$. Now, by letting $N_0 \to \infty$ and using monotone convergence, (27) with $\nu_N$ replaced by $\nu$ yields (26), as desired. In conclusion, for proving the theorem, we only have to verify (26) when
\[ \lambda \geq L \]
conclusion, for proving the theorem, we only have to verify (26) when $\nu$ is compact supported. Moreover, since (26) obviously holds for $\lambda \leq 2^{d+1} \|\nu\|/\|\mu\|$, we can also restrict ourselves to the case $\lambda > 2^{d+1} \|\nu\|/\|\mu\|$.

Let $\{Q_j\}_j$ be the almost disjoint family of cubes of Lemma 2.1 and set $\Omega := \bigcup_j Q_j$ and $R_j := 6Q_j$. Then we can write $\nu = g\mu + \nu_b$, with
\[ g\mu := \chi_{\mathbb{R}^d \setminus \Omega} + \sum_j b_j \mu \quad \text{and} \quad \nu_b := \sum_j \nu_b^j := \sum_j (w_j \nu - b_j \mu), \]
where the $b_j$’s satisfy (8), (9) and (10), and $w_j := \chi_{Q_j} (\sum_k \chi_{Q_k})^{-1}$.

Since $V^S_\rho \circ \mathcal{T}$ is sublinear,
\begin{equation}
\mu\left(\{x \in \mathbb{R}^d : (V^S_\rho \circ \mathcal{T})\nu(x) > \lambda\}\right) \leq \mu\left(\{x \in \mathbb{R}^d : (V^S_\rho \circ \mathcal{T})\nu_b(x) > \lambda/2\}\right) + \mu\left(\{x \in \mathbb{R}^d : (V^S_\rho \circ \mathcal{T})\nu_b(x) > \lambda/2\}\right).\tag{28}
\end{equation}
We obviously have $V^S_\rho \circ \mathcal{T} \mu \leq V_\rho \circ \mathcal{T} \mu$, so Theorem 3.2(i) yields that $V^S_\rho \circ \mathcal{T} \mu$ is bounded in $L^2(\mu)$. Note that $|g| \leq C\lambda$ by (7) and (10). Hence, using (9),
\begin{align}
\mu\left(\{x \in \mathbb{R}^d : (V^S_\rho \circ \mathcal{T})g(x) > \lambda/2\}\right) &\leq \frac{1}{\lambda^2} \int |(V^S_\rho \circ \mathcal{T})g|^2 \, d\mu \leq \frac{1}{\lambda^2} \int |g|^2 \, d\mu \\
&\leq \frac{1}{\lambda} \int |g| \, d\mu \leq \frac{1}{\lambda} \left( \nu(\mathbb{R}^d \setminus \Omega) + \sum_j \int_{R_j} |b_j| \, d\mu \right) \\
&\leq \frac{1}{\lambda} \left( \nu(\mathbb{R}^d \setminus \Omega) + \sum_j \nu(Q_j) \right) \lesssim \|\nu\|/\lambda. \tag{29}
\end{align}

Set $\hat{\Omega} := \bigcup_j 2Q_j$. By (5), we have $\mu(\hat{\Omega}) \leq \sum_j \mu(2Q_j) \lesssim \lambda^{-1} \sum_j \nu(Q_j) \lesssim \lambda^{-1} \|\nu\|$. We are going to prove that
\begin{equation}
\mu\left(\{x \in \mathbb{R}^d \setminus \hat{\Omega} : (V^S_\rho \circ \mathcal{T})\nu_b(x) > \lambda/2\}\right) \lesssim \|\nu\|/\lambda. \tag{30}
\end{equation}
Then (26) follows directly from (28), (29), (30) and the estimate $\mu(\hat{\Omega}) \lesssim \lambda^{-1} \|\nu\|$ above-mentioned, finishing the proof of Theorem 3.1(i).

To prove (30), given $x \in \mathbb{R}^d \setminus \hat{\Omega}$ we first write
\begin{equation}
(V^S_\rho \circ \mathcal{T})\nu_b(x) \leq (V^S_\rho \circ \mathcal{T})\left( \sum_j \chi_{2R_j}(x) \nu_b^j \right)(x) + (V^S_\rho \circ \mathcal{T})\left( \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \nu_b^j \right)(x). \tag{31}
\end{equation}
Notice that $\chi_{2R_j}(x)$ and $\chi_{\mathbb{R}^d \setminus 2R_j}(x)$ are evaluated at the fixed point $x$ on the right hand side.
The first term on the right hand side of (31) is easily handled using the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho^S \circ T^\mu$ and standard estimates. More precisely, since $\mathcal{V}_\rho^S \circ T$ is sublinear,

\begin{equation}
(\mathcal{V}_\rho^S \circ T) \left( \sum_j \chi_{2R_j}(x) v_j^\rho \right)(x)
\end{equation}

\begin{equation}
\leq \sum_j \chi_{2R_j}(x)(\mathcal{V}_\rho^S \circ T^\mu)b_j(x) + \sum_j \chi_{2R_j}(x)(\mathcal{V}_\rho^S \circ T^\nu)w_j(x)
\end{equation}

because $v_j^\rho = w_j \nu - b_j \mu$. On one hand, using Theorem 32(i), that $\mu(2R_j) \lesssim \mu(R_j)$ (because $\frac{1}{2} R_j \cap \text{supp} \mu \neq \emptyset$) and (9), we get

\begin{equation}
\int_{2R_j} (\mathcal{V}_\rho^S \circ T^\mu)b_j \, d\mu \leq \left( \int_{2R_j} |(\mathcal{V}_\rho^S \circ T^\mu)b_j|^2 \, d\mu \right)^{1/2} \mu(2R_j)^{1/2}
\lesssim \|b_j\|_{L^2(\mu)} \mu(2R_j)^{1/2} \lesssim \|b_j\|_{L^\infty(\mu)} \mu(R_j) \lesssim \nu(Q_j).
\end{equation}

On the other hand, if $x \in 2R_j \setminus 2Q_j$ then dist$(x, Q_j) \approx \ell(Q_j)$. Therefore, given $k \in \mathbb{Z}$,

\begin{equation}
B(x, 2^{-k}) \cap Q_j = \emptyset \iff \text{dist}(x, Q_j) \geq 2^{-k} \iff \ell(Q_j) \geq 2^{-k}.
\end{equation}

Since the $\ell^p$-norm is not bigger than the $\ell^1$-norm for $p \geq 1$, and since supp$w_j \subset Q_j$ and $|w_j| \leq 1$, from (34) we get

\begin{equation}
(\mathcal{V}_\rho^S \circ T^\nu)w_j(x) \leq \sup \sum_{\{e_m\} \in \mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_k} |T_{e_m, \epsilon_m+1} w_j(x)|
\lesssim \nu(Q_j) \sum_{k \in \mathbb{Z}: B(x, 2^{-k}) \cap Q_j \neq \emptyset} 2^{kn} \lesssim \nu(Q_j) \ell(Q_j)^{-n},
\end{equation}

and therefore, using again that $\mu(2R_j) \lesssim \mu(R_j) \approx \ell(R_j)^n \approx \ell(Q_j)^n$, we obtain

\begin{equation}
\int_{2R_j \setminus 2Q_j} (\mathcal{V}_\rho^S \circ T^\nu)w_j \, d\mu \lesssim \nu(Q_j) \ell(Q_j)^{-n} \mu(2R_j) \lesssim \nu(Q_j).
\end{equation}

Finally, applying (35) and (36) to (32), we conclude that

\begin{equation}
\int_{\mathbb{R}^d \setminus \hat{\Omega}} (\mathcal{V}_\rho^S \circ T) \left( \sum_j \chi_{2R_j}(x) v_j^\rho \right)(x) \, d\mu(x)
\end{equation}

\begin{equation}
\leq \sum_j \int_{2R_j} (\mathcal{V}_\rho^S \circ T^\mu)b_j \, d\mu + \sum_j \int_{2R_j \setminus 2Q_j} (\mathcal{V}_\rho^S \circ T^\nu)w_j \, d\mu \lesssim \sum_j \nu(Q_j) \lesssim \|\nu\|.
\end{equation}

Thanks to (31, 36) and Chebyshev's inequality, to prove (30) it is enough to verify that

\begin{equation}
\mu \left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : (\mathcal{V}_\rho^S \circ T) \left( \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) v_j^\rho \right)(x) > \lambda/4 \right\} \right) \lesssim \frac{\|\nu\|}{\lambda}.
\end{equation}

Our task now is to prove (37). Given $x \in \text{supp} \mu$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a non-increasing sequence of positive numbers (which depends on $x$, i.e. $\epsilon_m \equiv \epsilon_m(x)$) such that

\begin{equation}
(\mathcal{V}_\rho^S \circ T) \left( \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) v_j^\rho \right)(x) \leq 2 \sum_{k \in \mathbb{Z}, \epsilon_m, \epsilon_{m+1} \in I_k} \left| \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) T_{\epsilon_m, \epsilon_{m+1}} v_j^\rho(x) \right|^p \frac{1}{\rho}.
\end{equation}

Typically, the problem of the existence of such a sequence can be avoided by defining an auxiliary operator $\mathcal{V}_\rho^S \circ T$ along the same lines of $\mathcal{V}_\rho^S \circ T$ and requiring the supremum to be taken over a finite set of indices $I$ (thus the supremum is a maximum in this case). One
then proves the desired estimate for $\mathcal{V}_\rho^S \circ T$ with bounds independent of $I$ and deduces the general result by taking the supremum over all finite sets $I$ and using monotone convergence. For the sake of shortness, we omit the details.

Define the interior and boundary sum, respectively, by

$$S_i(x) := \left( \sum_{k \in Z} \sum_{\epsilon_m, \epsilon_{m+1} \in I_k} \left| \sum_{j : R_j \subseteq A(x, \epsilon_{m+1}, \epsilon_m)} \chi_{\mathbb{R}^d \setminus 2R_j}(x) T_{\epsilon_m, \epsilon_{m+1}} \nu_b^j(x) \right|^p \right)^{1/p},$$

$$S_b(x) := \left( \sum_{k \in Z} \sum_{\epsilon_m, \epsilon_{m+1} \in I_k} \left| \sum_{j : R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} \chi_{\mathbb{R}^d \setminus 2R_j}(x) T_{\epsilon_m, \epsilon_{m+1}} \nu_b^j(x) \right|^p \right)^{1/p}.$$

If $R_j \cap A(x, \epsilon_{m+1}, \epsilon_m) = \emptyset$ then $T_{\epsilon_m, \epsilon_{m+1}} \nu_b^j(x) = 0$, thus

$$(\mathcal{V}_\rho^S \circ T) \left( \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \nu_b^j(x) \right)(x) \leq 2(S_i + S_b)$$

by (38) and the triangle inequality, and so

$$\mu \left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : (\mathcal{V}_\rho^S \circ T) \left( \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \nu_b^j(x) \right)(x) > \lambda/4 \right\} \right) \leq \mu \left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : S_i(x) > \lambda/16 \right\} \right) + \mu \left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : S_b(x) > \lambda/16 \right\} \right).$$

To estimate $\mu \left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : S_i(x) > \lambda/16 \right\} \right)$ we use the fact that the $\ell^p$-norm is not bigger than the $\ell^1$-norm for $\rho \geq 1$, and that $\supp(\nu_b^j) \subset R_j$:

$$S_i(x) \leq \sum_{m \in Z} \left| \sum_{j : R_j \subseteq A(x, \epsilon_{m+1}, \epsilon_m)} \chi_{\mathbb{R}^d \setminus 2R_j}(x) T_{\epsilon_m, \epsilon_{m+1}} \nu_b^j(x) \right| \leq \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \sum_{m \in Z : A(x, \epsilon_{m+1}, \epsilon_m) \supseteq R_j} |T_{\epsilon_m, \epsilon_{m+1}} \nu_b^j(x)| \leq \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) |T \nu_b^j(x)|,$$

Recall that $\nu_b^j(R_j) = 0$ and $\|\nu_b^j\| \lesssim \nu(Q_j)$ by (9). Thus, if $z_j$ denotes the center of $R_j$, we have

$$\int_{\mathbb{R}^d \setminus 2R_j} |T \nu_b^j| \, d\mu \leq \int_{\mathbb{R}^d \setminus 2R_j} \int_{R_j} |K(x - y) - K(x - z_j)| \, d|\nu_b^j|(y) \, d\mu(x) \lesssim \int_{\mathbb{R}^d \setminus 2R_j} \int_{R_j} \frac{|y - z_j|}{|x - z_j|^{n+1}} \, d|\nu_b^j|(y) \, d\mu(x) \lesssim \|\nu_b^j\| \int_{\mathbb{R}^d \setminus 2R_j} \frac{|x - z_j|^{n+1}}{|x - z_j|} \, d\mu(x) \lesssim \|\nu_b^j\| \lesssim \nu(Q_j).$$

Finally, from Chebyshev’s inequality, (40) and (41) we conclude that

$$\mu \left( \left\{ x \in \mathbb{R}^d \setminus \hat{\Omega} : S_i(x) > \lambda/16 \right\} \right) \leq \frac{16}{\lambda} \sum_j \int_{\mathbb{R}^d \setminus 2R_j} |T \nu_b^j| \, d\mu \lesssim \frac{1}{\lambda} \sum_j \nu(Q_j) \lesssim \frac{\|\nu\|}{\lambda}.$$
The proof of this estimate is much more involved than the previous ones and requires the use of the corona decomposition of $\mu$, that is, we need to introduce the splitting $D^\mu = B \cup (\bigcup_{S \in \text{Trs } S})$. We denote

$$T_{j,m}(x) := \chi_{\mathbb{R}^d \setminus 2R_j}(x)T_{m,\epsilon m+1}v_j(x).$$

Recall that $P \in D_k$, we write $I_P = [2^{-j-1}, 2^{-j})$. Since $\rho > 2$, the $\ell^\rho$-norm is not bigger than the $\ell^2$-norm, and we get

$$\int_{\mathbb{R}^d \setminus \Omega} S_b^2 \mu \leq \sum_{P \in B} \int_{P \setminus \Omega} \sum_{\epsilon m, \epsilon m+1 \in I_P} \sum_{j : R_j \cap \partial A(x, \epsilon m+1, \epsilon m) \neq \emptyset} T_{j,m}(x) \mu(dx)$$

(44)

$$+ \sum_{S \in \text{Trs }} \sum_{P \in S} \int_{P \setminus \Omega} \sum_{\epsilon m, \epsilon m+1 \in I_P} \sum_{j : R_j \cap \partial A(x, \epsilon m+1, \epsilon m) \neq \emptyset} T_{j,m}(x) \mu(dx).$$

Observe that

$$|T_{j,m}(x)| \lesssim \ell(P)^{-n} \chi_{\mathbb{R}^d \setminus 2R_j}(x)|v_j^2|(A(x, \epsilon m+1, \epsilon m))$$

(45)

for all $\epsilon m, \epsilon m+1 \in I_P$. If in addition $x \in P \setminus 2R_j$ and $R_j \cap A(x, \epsilon m+1, \epsilon m) \neq \emptyset$, taking into account that $\epsilon m \approx \epsilon m+1 \approx \ell(P) \gtrsim \text{dist}(x, R_j) \gtrsim \ell(R_j)$, we deduce that

$$R_j \subset B_P,$$

assuming the constant $c_1$ in (44) big enough.

Concerning the first term on the right hand side of (44), from (45) and using that $\|v_j^2\| \lesssim \nu(Q_j)$, that the $Q_j$'s have bounded overlap and that $Q_j \subset B_P$ for all $j$ such that $R_j \subset B_P$, we get

$$\sum_{P \in B} \int_{P \setminus \Omega} \sum_{\epsilon m, \epsilon m+1 \in I_P} \sum_{j : R_j \cap \partial A(x, \epsilon m+1, \epsilon m) \neq \emptyset} T_{j,m}(x) \mu(dx)$$

(47)

$$\lesssim \sum_{P \in B} \int_{P} \left( \sum_{\epsilon m, \epsilon m+1 \in I_P} \sum_{j : R_j \subset B_P} \ell(P)^{-n} |v_j^2|(A(x, \epsilon m+1, \epsilon m)) \right)^2 \mu(dx)$$

$$\lesssim \sum_{P \in B} \int_{P} \left( \sum_{j : R_j \subset B_P} \frac{\|v_j^2\|^2}{\ell(P)^n} \right) \mu(dx) \lesssim \sum_{P \in B : \exists R \subset B_P} \left( \frac{\nu(B_P)}{\ell(P)^n} \right)^2 \ell(P)^n \lesssim \lambda \|\nu\|,$$

where we also used Lemma 2.3 in the last inequality, because $B$ is a Carleson family.

From now on, all our efforts are devoted to estimate the second term on the right hand side of (44).

Claim 3.3. Given $Q \in \text{Top}_G$, $P \in \text{Tree}(Q)$ and $R_j \subset B_P$, at least one of the following holds:

(i) There exists $R \in \text{Tree}(Q)$ such that $R \subset B_P$, $R_j \subset B_R$ and $\ell(R_j) \in I_R$.

(ii) There exists $R \in \partial \text{Tree}(Q)$ such that $R \subset B_P$ and $R_j \subset B_R$.

We postpone the proof of the preceding statement till the end of the proof of the theorem. Thanks to this claim, given $Q \in \text{Top}_G$ and $P \in \text{Tree}(Q)$, we can split

$$\{j : R_j \subset B_P\} \subset J_1 \cup J_2,$$

where

$$J_1 := \{j : R_j \subset B_P, \exists R \in \text{Tree}(Q) \text{ such that } R \subset B_P, R_j \subset B_R, \ell(R_j) \in I_R\},$$

$$J_2 := \{j : R_j \subset B_P, \exists R \in \partial \text{Tree}(Q) \text{ such that } R \subset B_P, R_j \subset B_R\}.$$
Recall that if \( x \in P \setminus 2R_j \), \( \epsilon_m, \epsilon_{m+1} \in I_P \) and \( R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset \) then \( R_j \subset B_P \). Thus, we can decompose the second term on the right hand side of (48) using \( J_1 \) and \( J_2 \) as follows

\[
\sum_{S \in \text{Trs}} \sum_{P \in S} \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \left| \sum_{j; R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} T_{j, m}(x) \right|^2 \, d\mu(x)
\]

(48)

\[
\lesssim \sum_{Q \in \text{Top}_Q} \sum_{P \in \text{Tree}(Q)} \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \left| \sum_{j \in J_1; R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} T_{j, m}(x) \right|^2 \, d\mu(x)
\]

\[
+ \sum_{Q \in \text{Top}_Q} \sum_{P \in \text{Tree}(Q)} \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \left| \sum_{j \in J_2; R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} T_{j, m}(x) \right|^2 \, d\mu(x).
\]

Despite that the arguments to estimate both terms on the right hand side of (48) are similar, we will deal with them separately, due to its different nature with respect to the structure of the corona decomposition.

**Claim 3.4.** Let \( Q, P, x, \epsilon_m \) and \( \epsilon_{m+1} \) be as on the right hand side of (48). We have

\[
\left| \sum_{j \in J_1; R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} |v^j_b| (A(x, \epsilon_{m+1}, \epsilon_m)) \right|^2
\]

(49)

\[
\lesssim \lambda \ell(P)^n \sum_{k; 2^{-k} \leq \ell(P)} \left( \frac{2^k - 1}{\ell(P)} \right)^{1/2} \sum_{j \in J_1; \ell(R_j) \in I_k} |v^j_b| (A(x, \epsilon_{m+1}, \epsilon_m)).
\]

Given \( j \in J_2 \), denote by \( R(j) \in \partial \text{Tree}(Q) \) some cube such that \( R(j) \subset B_P \) and \( R_j \subset B_{R(j)} \). We have

\[
\left| \sum_{j \in J_2; R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} |v^j_b| (A(x, \epsilon_{m+1}, \epsilon_m)) \right|^2
\]

(50)

\[
\lesssim \lambda^{1/2} \ell(P)^{n/2} \nu(B_P)^{1/2} \sum_{R \in \partial \text{Tree}(Q); R \subset B_P} \sum_{j \in J_2; R(j) = R} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} |v^j_b| (B_R \cap A(x, \epsilon_{m+1}, \epsilon_m)).
\]

Again we postpone the proof of the preceding claim till the end of the proof of the theorem.
For the case \( j \in J_1 \) in \((18)\), using \((15), (19)\) and \((16)\) we get

\[
\sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \sum_{j \in J_1: R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} T_{j,m}(x) \, d\mu(x)
\]
\[
\leq \lambda \sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \ell(P)^{-n} \times \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \sum_{k: 2^{-k} \leq \ell(P)} \left( \frac{2^{-k}}{\ell(P)} \right)^{1/2} \sum_{j \in J_1: \ell(R_j) \in I_k} |\nu_j^i(j)(A(x, \epsilon_{m+1}, \epsilon_m))| \, d\mu(x)
\]
\[
\leq \lambda \sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \ell(P)^{-n} \times \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \sum_{k: 2^{-k} \leq \ell(P)} \left( \frac{2^{-k}}{\ell(P)} \right)^{1/2} \sum_{j \in J_1: \ell(R_j) \in I_k} |\nu_j^i| \, d\mu(x)
\]
\[
\leq \lambda \sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \ell(P)^{-n} \times \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \sum_{k: 2^{-k} \leq \ell(P)} \left( \frac{2^{-k}}{\ell(P)} \right)^{1/2} \sum_{j \in J_1: \ell(R_j) \in I_k} \nu_j^i \nu_j \, d\mu(x)
\]

In the third inequality we used that \( j \in J_1 \) implies that \( R_j \subset B_P \).

Concerning the case \( j \in J_2 \) in \((18)\), by \((15)\) and \((16)\) we see that

\[
\sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \sum_{j \in J_2: R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset} T_{j,m}(x) \, d\mu(x)
\]
\[
\leq \lambda^{1/2} \sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \ell(P)^{-n} \left( \frac{\nu(B_P)}{\ell(P)^n} \right)^{1/2} \times \int_{P \setminus \Omega} \sum_{\epsilon_m, \epsilon_{m+1} \in I_P} \sum_{R \subset B_P} \sum_{R \subset \text{Tree}(Q): R \cap \partial \Omega = \emptyset} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} |\nu_j^i(B_R \cap A(x, \epsilon_{m+1}, \epsilon_m))| \, d\mu(x)
\]
\[
\leq \lambda^{1/2} \sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \left( \frac{\nu(B_P)}{\ell(P)^n} \right)^{1/2} \sum_{R \subset B_P} \sum_{R \subset \text{Tree}(Q): R \cap \partial \Omega = \emptyset} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} |\nu_j^i| \, d\mu(x)
\]
\[
\leq \lambda^{1/2} \sum_{Q \in \topg} \sum_{P \in \text{Tree}(Q)} \sum_{R \subset B_P} \sum_{R \subset \text{Tree}(Q): R \cap \partial \Omega = \emptyset} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} \left( \frac{\nu(B_P)}{\ell(P)^n} \right)^{1/2} \left( \frac{\nu(B_R)}{\ell(R)^n} \right) \ell(R)^n,
\]

where we also used in the last inequality above that \(|\nu_j^i| \leq \nu(Q_j)\) and that the \( Q_j \)'s have bounded overlap, and we assumed the constant \( c_1 \) in \((4)\) big enough. Since \( a^{1/2} \leq a^{3/2} + b^{3/2} \)
for all $a,b \geq 0$, we obtain

$$
\sum_{Q \in \mathcal{T} \cap P \in \text{Tree}(Q)} \sum_{P \in \text{Tree}(Q)} \int_{P} d\mu(x) \left| \sum_{j \in J : R_{j} \cap \partial A(x_{j}, \epsilon_{m} \ell_{j}) \neq \emptyset} T_{j,m}(x) \right|^2 d\mu(x)
$$

\[ \lesssim \lambda^{1/2} \sum_{Q \in \mathcal{T} \cap P \in \text{Tree}(Q)} \sum_{P \in \text{Tree}(Q)} \sum_{R \subseteq R_{P} \subseteq B_{P}} \left( \frac{\nu(B_{P})}{\ell(P)} \right)^{3/2} + \left( \frac{\nu(B_{R})}{\ell(R)} \right)^{3/2} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} \ell(R)^{n}, \]

where we have set $a_{P} := \sum_{R \subseteq \mathcal{T} \cap P \in \text{Tree}(Q)} (\ell(R)/\ell(P))^{1/4} \ell(R)^{n}$ whenever $P \in \text{Tree}(Q)$ for some $Q \in \mathcal{T} \cap P \in \text{Tree}(Q)$ (otherwise, we set $a_{P} = 0$). Since $\partial \text{Trs}$ is a Carleson family, we see that the $a_{P}$’s satisfy a Carleson packing condition because, for a given $T \in \mathcal{D}^{n},$

$$
\sum_{P \subseteq T} a_{P} \leq \sum_{P \subseteq T} \sum_{Q \in \mathcal{T} \cap P \in \text{Tree}(Q)} \sum_{R \subseteq R_{P} \subseteq B_{P}} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} \ell(R)^{n}
$$

\[ = \sum_{P \subseteq T} \sum_{R \in \partial \text{Trs}} \sum_{R \subseteq B_{P} \subseteq B_{T}} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} \ell(R)^{n}
\]

\[ = \sum_{R \subseteq \partial \text{Trs}} \ell(R)^{n} \sum_{P \subseteq T} \left( \frac{\ell(R)}{\ell(P)} \right)^{1/4} \lesssim \sum_{R \subseteq \partial \text{Trs}} \ell(R)^{n} \lesssim \ell(T)^{n}. \]

Therefore,

$$
\sum_{Q \in \mathcal{T} \cap P \in \text{Tree}(Q)} \int_{P} d\mu(x) \left| \sum_{j \in J : R_{j} \cap \partial A(x_{j}, \epsilon_{m} \ell_{j}) \neq \emptyset} T_{j,m}(x) \right|^2 d\mu(x)
$$

\[ \lesssim \lambda^{1/2} \sum_{P \subseteq \mathcal{D}^{n}} \left( \frac{\nu(B_{P})}{\ell(P)} \right)^{3/2} \left( a_{P} + \ell(P)^{n} \chi_{\partial \text{Trs}}(P) \right) \lesssim \lambda \|\nu\|,
\]

because the coefficients $a_{P} + \ell(P)^{n} \chi_{\partial \text{Trs}}(P)$ satisfy a Carleson packing condition and thus we can use Lemma 2.3.

Finally, (3.3) follows from (1.1), (4.1), (5.1), (6.1) and (7.2), so Theorem 3.1(i) is proved except for the claims.

Proof of Claim 3.3. Let $Q \in \mathcal{T} \cap P \in \text{Tree}(Q)$ and $R_{j} \subseteq B_{P}$. For the purpose of the claim, we can assume that $\ell(Q) \geq \ell(R_{j})$. Without loss of generality, we can also assume that $\ell(P) \geq \ell(R_{j})$ (recall that $R_{j} \subseteq B_{P}$, so $\ell(P) \geq \ell(R_{j})$). Otherwise, we replace $P$ by a suitable ancestor from Tree$(Q)$ with side length comparable to $\ell(R_{j})$.

Let $R \subseteq \text{Tree}(Q)$ be a cube with minimal side length such that $R_{j} \subseteq B_{R}$ and $\ell(R) \geq \ell(R_{j})$, that is, $\ell(R) \leq \ell(S)$ for all $S \subseteq \text{Tree}(Q)$ with $R_{j} \subseteq B_{S}$ and $\ell(S) \geq \ell(R_{j})$. In particular, notice that $P$ may coincide with $R$, and in any case $\ell(R) \leq \ell(P)$. If $\ell(R_{j}) \in I_{R}$, that is $\ell(R) \geq \ell(R_{j}) \geq \ell(R)/2$, then $R$ fulfills (i) and we are done. On the contrary, assume that $\ell(R)/2 > \ell(R_{j})$. Since $R_{j} \subseteq B_{R}$, there exists $R' \subseteq \mathcal{D}^{n}$ such that $\ell(R') = \ell(R)$, $\text{dist}(R', R) \leq \ell(R)$ and $R' \cap R_{j} \cap \text{supp}(\mu) \neq \emptyset$. Therefore, there exists a son $R''$ of $R'$ such that $R'' \cap R_{j} \cap \text{supp}(\mu) \neq \emptyset$, so $R_{j} \subseteq B_{R''}$. By the minimality of $R$, we must have $R'' \notin \text{Tree}(Q)$, thus $R \in \partial \text{Tree}(Q)$ and (ii) is fulfilled. \qed
\textbf{Proof of Claim 3.4.} Let us first prove (49). If \( j \in J_1 \) then \( R_j \subset B_P \) and, in particular, \( \ell(R_j) \lesssim \ell(P) \). Thus, by Cauchy-Schwarz inequality,

\[
\left| \sum_{j \in J_1: R_j \cap \partial A(x, \epsilon_m, \epsilon_m) \neq \emptyset} \left| \nu_j^1(A(x, \epsilon_m, \epsilon_m)) \right|^2 \right| = \left| \sum_{k: 2^{-k} \lesssim \ell(P)} \left( \frac{2^{-k}}{\ell(P)} \right)^{1/4} \left( \frac{\ell(P)}{2^{-k}} \right)^{1/4} \sum_{j \in J_1: \ell(R_j) \in I_k} \left| \nu_j^1(A(x, \epsilon_m, \epsilon_m)) \right|^2 \right| \\
\leq \left| \sum_{k: 2^{-k} \lesssim \ell(P)} \left( \frac{\ell(P)}{2^{-k}} \right)^{1/2} \sum_{j \in J_1: \ell(R_j) \in I_k} \left| \nu_j^1(A(x, \epsilon_m, \epsilon_m)) \right|^2 \right|.
\]

(53)

Using that \( |\nu_j^1(A(x, \epsilon_m+1, \epsilon_m))| \lesssim \nu(Q_j) \) and that the \( Q_j \)'s have bounded overlap, assuming that \( c_1 \) in (48) is big enough, from the definition of \( J_1 \) we see that

\[
\sum_{j \in J_1: \ell(R_j) \in I_k} \left| \nu_j^1(A(x, \epsilon_m+1, \epsilon_m)) \right| \lesssim \sum_{R \in \text{Tree}(Q): \ell(R) \in I_k} \nu(B_R)
\]

(54)

where we also used (42) and that \( \mu(B_R) \lesssim \mu(R) \) in the second inequality and the estimates on annuli from Lemma 2.3 in the third one. Then, (49) follows from (53) and (54).

Let us turn our attention to (50) now. Recall that, given \( j \in J_2, R(j) \in \partial \text{Tree}(Q) \) denotes some cube such that \( R(j) \subset B_P \) and \( R_j \subset B_{R(j)} \). Similarly to (53), by Hölder’s inequality we get

\[
\left| \sum_{j \in J_2: R_j \cap \partial A(x, \epsilon_m+1, \epsilon_m) \neq \emptyset} \left| \nu_j^1(A(x, \epsilon_m+1, \epsilon_m)) \right|^{3/2} \right| \\
\leq \left| \sum_{R \in \partial \text{Tree}(Q): R \subset B_P} \sum_{j \in J_2: R(j) = R} \left| \nu_j^1(B_R \cap A(x, \epsilon_m+1, \epsilon_m)) \right|^{3/2} \right| \\
\leq \left| \sum_{k: 2^{-k} \lesssim \ell(P)} \left( \frac{\ell(P)}{2^{-k}} \right)^{1/4} \sum_{R \in \partial \text{Tree}(Q): R \subset B_P, \ell(R) = 2^{-k}} \sum_{j \in J_2: R(j) = R} \left| \nu_j^1(B_R \cap A(x, \epsilon_m+1, \epsilon_m)) \right|^{3/2} \right|.
\]

(55)
Using that \( \|\nu_b^j\| \lesssim \nu(Q_j) \), that the \( Q_j \)'s have bounded overlap, that \( \nu(B_R) \lesssim \lambda \mu(B_R) \) for the cubes \( R \) in the last sum above and Lemma 2.3 we deduce that

\[
\sum_{R \in \partial \text{Tree}(Q_\ell)} \sum_{j \in J_\ell: R(j) = R} |\nu_b^j|(B_R \cap A(x, \epsilon_{m+1}, \epsilon_m)) \lesssim \sum_{R \in \partial \text{Tree}(Q)} \sum_{j \in J_\ell: R(j) = R} \nu(Q_j) \lesssim \sum_{R \in \partial \text{Tree}(Q)} \nu(B_R) \lesssim \lambda 2^{-k} l(P)^{n-1},
\]

which combined with (55) yields

\[
\sum_{R \in \partial \text{Tree}(Q_\ell)} \sum_{j \in J_\ell: R(j) = R} |\nu_b^j|(A(x, \epsilon_{m+1}, \epsilon_m))^{3/2} \lesssim \lambda^{1/2} l(P)^{n/2} \sum_{k: 2^{-k} \lesssim l(P)} \left( \frac{2^{-k}}{l(P)} \right)^{1/4} \sum_{R \in \partial \text{Tree}(Q_\ell)} \sum_{j \in J_\ell: R(j) = R} |\nu_b^j|(B_R \cap A(x, \epsilon_{m+1}, \epsilon_m))^{1/4} \lesssim \lambda^{1/2} l(P)^{n/2} \sum_{R \in \partial \text{Tree}(Q)} \sum_{j \in J_\ell: R(j) = R} \left( \frac{l(R)}{l(P)} \right)^{1/4} |\nu_b^j|(B_R \cap A(x, \epsilon_{m+1}, \epsilon_m)).
\]

Finally, (50) is a consequence of (55) and the trivial estimate

\[
\sum_{j \in J_\ell: R(j) = R} |\nu_b^j|(A(x, \epsilon_{m+1}, \epsilon_m)) \lesssim \nu(B_P),
\]

which holds if \( c_1 \) in (3) is big enough because \( \|\nu_b^j\| \lesssim \nu(Q_j) \) and the \( Q_j \)'s have bounded overlap.

4. \( \mathcal{V}_\rho \circ \mathcal{T}^\mu : L^p(\mu) \to L^p(\mu) \) is a bounded operator for \( 1 < p < \infty \)

The boundedness of \( \mathcal{V}_\rho \circ \mathcal{T}^\mu \) in \( L^p(\mu) \) for \( 1 < p < 2 \) follows by interpolation, taking into account that it is bounded in \( L^2(\mu) \) and from \( L^1(\mu) \) to \( L^{1,\infty}(\mu) \), by Theorem 3.2 and Theorem 3.1. So it only remains to prove the boundedness in \( L^p(\mu) \) for \( 2 < p < \infty \). This task is carried out in the next theorem.

Theorem 4.1. If \( \mu \) is a uniformly \( n \)-rectifiable measure in \( \mathbb{R}^d \), then \( \mathcal{V}_\rho \circ \mathcal{T}^\mu \) is a bounded operator in \( L^p(\mu) \) for all \( \rho > 2 \) and all \( 2 < p < \infty \).

Proof. We are going to prove that if \( \mu \) is a uniformly \( n \)-rectifiable measure then \( \mathcal{M}^\mu_{\mathcal{D}^\rho} \circ \mathcal{V}_\rho \circ \mathcal{T}^\mu \) is a bounded operator in \( L^p(\mu) \) for all \( 2 < p < \infty \), where \( \mathcal{M}^\mu_{\mathcal{D}^\rho} \) denotes the dyadic sharp maximal function, that is,

\[
\mathcal{M}^\mu_{\mathcal{D}^\rho} f(x) = \sup_{D \in \mathcal{D}^\rho: x \in D} m_D |f - m_D f|.
\]

The theorem will then follow by standard estimates.

Fix \( f \in L^p(\mu) \) and \( x_0 \in \text{supp} \mu \). Then,

\[
(\mathcal{M}^\mu_{\mathcal{D}^\rho} \circ \mathcal{V}_\rho \circ \mathcal{T}^\mu) f(x_0) = \sup_{D \in \mathcal{D}^\rho: x_0 \in D} m_D |(\mathcal{V}_\rho \circ \mathcal{T}^\mu) f - m_D ((\mathcal{V}_\rho \circ \mathcal{T}^\mu) f)|. 
\]
Given $D \in \mathcal{D}^\mu$ such that $x_0 \in D$, we decompose $f = f_1 + f_2$ with $f_1 := f \chi_{3D}$ and $f_2 := f - f_1$. Since $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is sublinear and positive, $|\mathcal{V}_\rho \circ \mathcal{T}^\mu f - (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2| \leq (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_1$ and so $|(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f - c| \leq (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_1 + |(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2 - c|$ for all $c \in \mathbb{R}$. If we take $c = (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2(z_D)$, where $z_D$ denotes the center of $D$ (we may assume that $c < \infty$), then

$$m_D(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f - m_D((\mathcal{V}_\rho \circ \mathcal{T}^\mu)f)$$

(58)

$$\leq 2m_D((\mathcal{V}_\rho \circ \mathcal{T}^\mu)f - (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2(z_D))$$

$$\lesssim m_D((\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_1 + m_D((\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2 - (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2(z_D))$$

$$=: I_1 + I_2.$$

A good estimate for $I_1$ can be easily derived using Cauchy-Schwarz’s inequality, Theorem 3.2(i) and that $\mu$ is n-AD regular. More precisely,

$$I_1 \lesssim \left( \frac{1}{\mu(D)} \int_D |(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_1|^2 d\mu \right)^{1/2} \lesssim \left( \frac{1}{\mu(D)} \int_{3D} |f|^2 d\mu \right)^{1/2} \lesssim \mathcal{M}_2 f(x_0).$$

The estimate of $I_2$ is much more involved. Given $x \in D$, since $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is sublinear and positive, it is not hard to check that

$$|(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2(x) - (\mathcal{V}_\rho \circ \mathcal{T}^\mu)f_2(z_D)|$$

(60)

$$\leq \sup_{\{\epsilon_m\} \in Z} \left( \sum_{m \in \mathbb{Z}} |\mathcal{T}_{\epsilon_m,\epsilon_{m+1}} f_2(x) - \mathcal{T}_{\epsilon_m,\epsilon_{m+1}} f_2(z_D)|^\rho \right)^{1/\rho},$$

where the supremum is taken over all non-increasing sequences $\{\epsilon_m\}_{m \in \mathbb{Z}}$ of positive numbers $\epsilon_m$. In order to estimate the right hand side of (60), take one of such sequences $\{\epsilon_m\}_{m \in \mathbb{Z}}$ and note that, by the triangle inequality,

$$|\mathcal{T}_{\epsilon_m,\epsilon_{m+1}} f_2(x) - \mathcal{T}_{\epsilon_m,\epsilon_{m+1}} f_2(z_D)|$$

(61)

$$\leq \int \chi_{[\epsilon_{m+1}, \epsilon_m]}(|x - y|) |K(x - y) - K(z_D - y)| |f_2(y)| d\mu(y)$$

$$+ \int \left| \chi_{[\epsilon_{m+1}, \epsilon_m]}(|x - y|) - \chi_{[\epsilon_{m+1}, \epsilon_m]}(|z_D - y|) \right| |K(z_D - y)| |f_2(y)| d\mu(y)$$

$$=: a_m + b_m.$$

Since $x$ and $z_D$ belong to $D$ and $f_2$ vanishes in $3D$, we can assume that $\epsilon_{m+1} > \ell(D)$ in the definition of $a_m$ and $b_m$ for all $m \in \mathbb{Z}$.

Let us first look at the sum relative to the $a_m$’s for $m \in \mathbb{Z}$. Using that $\rho > 1$, the regularity of the kernel $K$, that $f_2$ vanishes in $3D$, and that $\mu$ is n-AD regular, for each $x \in D$ we have

$$\left( \sum_{m \in \mathbb{Z}} a_m^\rho \right)^{1/\rho} \leq \sum_{m \in \mathbb{Z}} \int_{\epsilon_{m+1} < |x - y| \leq \epsilon_m} |K(x - y) - K(z_D - y)||f_2(y)| d\mu(y)$$

(62)

$$\lesssim \ell(D) \sum_{m \in \mathbb{Z}} \int_{\epsilon_{m+1} < |x - y| \leq \epsilon_m} \frac{|f_2(y)|}{|y - z_D|^{n+1}} d\mu(y)$$

$$\lesssim \ell(D) \int_{\mathbb{R}^n \setminus 3D} \frac{|f(y)|}{|y - z_D|^{n+1}} d\mu(y) \lesssim \mathcal{M} f(x_0) \lesssim \mathcal{M}_2 f(x_0),$$

where we also used Cauchy-Schwarz’s inequality in the last estimate above.
The sum relative to the \( b_m \)'s for \( m \in \mathbb{Z} \) requires a more delicate analysis. We split \( \mathbb{Z} = J_1 \cup J_2 \), where

\[
J_1 := \{ m \in \mathbb{Z} : \epsilon_m - \epsilon_{m+1} > \ell(D) \}, \\
J_2 := \{ m \in \mathbb{Z} : \epsilon_m - \epsilon_{m+1} \leq \ell(D) \}.
\]

To shorten notation, we also set

\[
A_m^1(z_D) := A(z_D, \epsilon_m - \ell(D), \epsilon_m + \ell(D)) \quad \text{and} \quad A_m^2(x) := A(x, \epsilon_{m+1}, \epsilon_m).
\]

Since we are assuming \( \epsilon_{m+1} > \ell(D) \) for all \( m \in \mathbb{Z} \), both \( A_m^1(z_D) \) and \( A_m^1(z_D) \) are well defined for all \( m \in J_1 \). Moreover, since \( |x - z_D| \leq \ell(D) \) for all \( x \in D \), we easily get

\[
|\chi_{(\epsilon_{m+1}, \epsilon_m)}(x - i) - \chi_{(\epsilon_{m+1}, \epsilon_m)}(z_D - i)| \leq \chi_{(\epsilon_{m+1}, \epsilon_m)}(z_D) + \chi_{(\epsilon_{m+1}, \epsilon_m)}(z_D) \quad \text{for all } m \in J_1,
\]

\[
|\chi_{(\epsilon_{m+1}, \epsilon_m)}(x - i) - \chi_{(\epsilon_{m+1}, \epsilon_m)}(z_D - i)| \leq \chi_{(\epsilon_{m+1}, \epsilon_m)}(z_D) + \chi_{(\epsilon_{m+1}, \epsilon_m)}(z_D) \quad \text{for all } m \in J_2.
\]

We are going to split the sum associated with the \( b_m \)'s in terms of \( J_1 \) and \( J_2 \), using in each case the corresponding estimate from (63).

Concerning the sum over \( J_1 \), since \( \rho > 2 \), (63) yields

\[
\left( \sum_{m \in J_1} b_m^\rho \right)^{1/\rho} \lesssim \left( \sum_{m \in J_1} \left( \int_{A_m^1(z_D)} |K(z_D - y)||f_2(y)| \, d\mu(y) \right)^2 \right)^{1/2}
\]

\[
+ \left( \sum_{m \in J_1} \left( \int_{A_m^1(z_D)} |K(z_D - y)||f_2(y)| \, d\mu(y) \right)^2 \right)^{1/2}
\]

\[
=: S_1 + S_2.
\]

The arguments for estimating \( S_1 \) and \( S_2 \) are almost the same, so we will only give the details for \( S_1 \). Since \( f_2 \) vanishes in \( 3D \),

\[
S_1^2 = \sum_{k \in \mathbb{Z}} \sum_{m \in J_1; \epsilon_m \in I_k} \left( \int_{A_m^1(z_D)} |K(z_D - y)||f_2(y)| \, d\mu(y) \right)^2 \lesssim \sum_{Q \supseteq D} \sum_{\epsilon_m \in I_Q} \frac{|(f_2)(A_m^1(z_D))|^2}{\ell(Q)^{2n}}
\]

Our task now is to bound \( |(f_2)(A_m^1(z_D))|^2 \). This is done by splitting the annulus \( A_m^1(z_D) \), whose width equals \( 2\ell(D) \), into disjoint cubes \( P \in \mathcal{D} \) such that \( \ell(P) = \ell(D) \) and grouping them properly in terms of the corona decomposition. More precisely, for \( Q \supseteq D \) and \( \epsilon_m \in I_Q \), we have \( A_m^1(z_D) \cap \text{supp}(\mu) \subseteq \bigcup_{R \in V(Q)} R \). Recall also that the number of cubes in \( V(Q) \) is bounded independently of \( Q \). Therefore,

\[
|\{(f_2)(A_m^1(z_D))|^2 \lesssim \sum_{R \in V(Q)} \left| \sum_{P \in \text{Tree}(R)} (f_2)(A_m^1(z_D) \cap P) \right|^2
\]

\[
+ \sum_{R \in V(Q)} \left| \sum_{P \in \text{Stp}(R)} (f_2)(A_m^1(z_D) \cap P) \right|^2.
\]

(66)
The first term on the right hand side of \((66)\) can be easily estimated using Cauchy-Schwarz’s inequality, that the \(P\)'s such that \(\ell(P) = \ell(D)\) are disjoint and Lemma 2.5. That is, 

\[
\left| \sum_{P \in \text{Tree}(R): \ell(P) = \ell(D)} (|f_2| \mu)(A^1_m(z_D) \cap P) \right|^2 = \left| \int \left( \sum_{P \in \text{Tree}(R): \ell(P) = \ell(D)} \chi_{A^1_m(z_D) \cap P} \right) |f_2| d\mu \right|^2 
\]

\[
\leq \left( \sum_{P \in \text{Tree}(R): \ell(P) = \ell(D)} \mu(A^1_m(z_D) \cap P) \right) \left( \sum_{P \in \text{Tree}(R): \ell(P) = \ell(D)} (|f_2|^2 \mu)(A^1_m(z_D) \cap P) \right) 
\]

\[
\lesssim \ell(D) \ell(R)^{n-1} \sum_{P \in \text{Tree}(R): \ell(P) = \ell(D)} (|f_2|^2 \mu)(A^1_m(z_D) \cap P). 
\]

The second term on the right hand side of \((65)\) is estimated similarly but, since the cubes in \(\text{Stp}(R)\) may have different side length, we need to introduce an auxiliary splitting of the sum in terms of the side length. This extra splitting, combined with an application of Cauchy-Schwarz inequality yields 

\[
\left| \sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} (|f_2| \mu)(A^1_m(z_D) \cap P) \right|^2 = \left| \sum_{j \geq 0} 2^{j/4} \sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} (|f_2| \mu)(A^1_m(z_D) \cap P) \right|^2 
\]

\[
\leq \sum_{j \geq 0} 2^{j/2} \left( \sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} \mu(A^1_m(z_D) \cap P) \right) \left( \sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} (|f_2|^2 \mu)(A^1_m(z_D) \cap P) \right), 
\]

where we also used in the last inequality above that the \(P\)'s which belong to \(\text{Stp}(R)\) are disjoint and Cauchy-Schwarz’s inequality. Since the width of the annulus \(A^1_m(z_D)\) equals \(2\ell(D)\), if \(P \in \text{Stp}(R)\) is such that \(\ell(P) = 2^{-j} \ell(R) \geq \ell(D)\) and \(A^1_m(z_D) \cap P \neq \emptyset\) then 

\[
P \subset A(z_D, \epsilon_m - C2^{-j} \ell(R), \epsilon_m + C2^{-j} \ell(R)) 
\]

for some \(C > 0\) depending only on \(n, d\) and \(\mu\). Hence, Lemma 2.5 gives 

\[
\sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} \mu(A^1_m(z_D) \cap P) \lesssim 2^{-j} \ell(R)^n, 
\]

which plugged into \((65)\) yields 

\[
\left| \sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} (|f_2| \mu)(A^1_m(z_D) \cap P) \right|^2 \lesssim \sum_{j \geq 0} 2^{-j/2} \ell(R)^n \sum_{P \in \text{Stp}(R): \ell(P) \geq \ell(D)} (|f_2|^2 \mu)(A^1_m(z_D) \cap P) 
\]

\[
\leq \sum_{P \in \text{Stp}(R)} \left( \frac{\ell(P)}{\ell(R)} \right)^{1/2} \ell(R)^n (|f_2|^2 \mu)(A^1_m(z_D) \cap P). 
\]
Applying (67) and (68) to (66), we see that

$$
|\sum_{Q \in V(Q)} \sum_{P \in \text{Tree}(R)} \frac{\ell(D)}{\ell(R)} \ell(R)^n |f|_2\mu (A^l_m(x_D)) \Big| \leq \sum_{Q \in D^\mu} \sum_{P \in \text{Tree}(R)} \frac{\ell(D)}{\ell(R)} \ell(R)^n |f|_2\mu (A^l_m(x_D)) \big( A^l_m(x_D) \cap P \big)
$$

(70)

\[ + \sum_{Q \in V(Q)} \sum_{P \in \text{Tree}(R)} \frac{\ell(P)}{\ell(R)} \ell(R)^n |f|_2\mu (A^l_m(x_D)) \big( A^l_m(x_D) \cap P \big). \]

Now that we have estimated \(|\sum_{Q \in V(Q)} \sum_{P \in \text{Tree}(R)} \frac{\ell(D)}{\ell(R)} \ell(R)^n |f|_2\mu (A^l_m(x_D)) \big( A^l_m(x_D) \cap P \big)|^2\), we can derive a bound for \(S_1^2\). Since \(\ell(Q) = \ell(R)\) for all \(R \in V(Q)\), (65) and (70) imply that

$$
S_1^2 \lesssim \sum_{Q \in D^\mu} \sum_{m \in J_1} \sum_{\ell \in m \in I_Q} \sum_{Q \in D^\mu} \frac{\ell(D)}{\ell(R)} \ell(R)^n \big( |f|_2\mu (A^l_m(x_D)) \big) \big( A^l_m(x_D) \cap P \big)
$$

(71)

\[ + \sum_{Q \in V(Q)} \sum_{P \in \text{Tree}(R)} \frac{\ell(P)}{\ell(R)} \ell(R)^n \big( |f|_2\mu (A^l_m(x_D)) \big) \big( A^l_m(x_D) \cap P \big). \]

Note that the \(A^l_m(x_D)\)'s have bounded overlap for \(m \in J_1\) because \(\{\epsilon_m\}_{m \in \mathbb{Z}}\) is a non-increasing sequence. Therefore, from (71) we deduce that

$$
S_1^2 \lesssim \sum_{Q \in D^\mu} \sum_{m \in J_1} \sum_{\ell \in m \in I_Q} \sum_{Q \in D^\mu} \frac{\ell(D)}{\ell(R)} \ell(R)^n \big( |f|_2\mu (P) \big)
$$

(72)

\[ + \sum_{Q \in V(Q)} \sum_{P \in \text{Tree}(R)} \frac{\ell(P)}{\ell(R)} \ell(R)^n \big( |f|_2\mu (P) \big). \]

For the first term on the right hand side of (72), using that the \(P\)'s in \(D^\mu\) such that \(\ell(P) = \ell(D)\) are disjoint, that \(\mu\) is \(n\)-AD regular and that \(x_0 \in D\), we have

$$
\sum_{Q \in D^\mu} \sum_{P \in \text{Tree}(R)} \frac{\ell(D)}{\ell(R)} \ell(R)^n \big( |f|_2\mu (P) \big) \leq \sum_{Q \in D^\mu} \sum_{P \in V(Q)} \frac{\ell(D)}{\ell(R)} \ell(R)^n \big( |f|_2\mu (P) \big)
$$

(73)

\[ \lesssim \sum_{Q \in D^\mu} \sum_{P \in V(Q)} \frac{\ell(D)}{\ell(Q)} \mathcal{M}_2 f(x)^2 \]

\[ \lesssim \mathcal{M}_2 f(x)^2. \]

In order to estimate the second term on the right hand side of (72), note that \(R \in V(Q)\) if and only if \(Q \in V(R)\) and that if \(D \subset Q\) and \(R \in V(Q)\) then \(D \subset 3R\), thus by changing the order of summation and using that the number of cubes in \(V(R)\) is bounded independently
of $R$ and that $D_\mu = \bigcup_{S \in \text{Top}} \text{Tree}(S)$ we see that
\[
\sum_{Q \in D_\mu} \sum_{R \in \text{V}(Q)} \sum_{P \in \text{Stp}(R)} \left( \frac{\ell(P)}{\ell(R)} \right)^{1/2} \ell(R)^{-n} \left( |f_2|^2 \mu \right)(P)
\leq \sum_{R \in D_\mu} \sum_{Q \in \text{V}(R)} \sum_{P \in \text{Stp}(R)} \left( \frac{\ell(P)}{\ell(R)} \right)^{1/2} \ell(R)^{-n} \left( |f_2|^2 \mu \right)(P)
\leq \sum_{S \in \text{Top} \cap \text{Tree}(S) : P \in \text{Stp}(R)} \sum_{3R \supset D \cup P} \left( \frac{\ell(P)}{\ell(R)} \right)^{1/2} \ell(R)^{-n} \left( |f_2|^2 \mu \right)(P)
\leq \sum_{S \in \text{Top} \cap \text{Tree}(S)} \sum_{P \in \text{Stp}(S)} \ell(P)^{1/2} \left( |f_2|^2 \mu \right)(P) \sum_{R \in \text{Tree}(S) : 3R \supset D \cup P} \ell(R)^{-n-1/2},
\]
where we also used in the last inequality above that, for $S \in \text{Top}$, if $P \in \text{Stp}(R)$ for some $R \in \text{Tree}(S)$ then $P \in \text{Stp}(S)$ and $P \subset R$. Moreover, denoting $$D(P, D) := \ell(P) + \text{dist}(P, D) + \ell(D),$$ we have
\[
\sum_{R \in \text{Tree}(S) : 3R \supset D \cup P} \ell(R)^{-n-1/2} \lesssim \sum_{j \in \mathbb{Z}} \sum_{R \in \text{Tree}(S) : 3R \supset D \cup P} (2^j D(P, D))^{-n-1/2}
\lesssim D(P, D)^{-n-1/2},
\]
because the number of cubes $R \in D_\mu$ such that $3R \supset D \cup P$ and $2^j D(P, D) < \ell(R) \leq 2^{j+1} D(P, D)$ is bounded independently of $j \in \mathbb{Z}$, and the statements “$3R \supset D \cup P$” and “$2^j D(P, D) < \ell(R) \leq 2^{j+1} D(P, D)$” are compatible each other only if $j \geq j_0$ for some $j_0 \in \mathbb{Z}$ which only depends on $d, n$ and $\mu$. Plugging (75) into (74), we get
\[
\sum_{Q \in D_\mu} \sum_{R \in \text{V}(Q)} \sum_{P \in \text{Stp}(R)} \left( \frac{\ell(P)}{\ell(R)} \right)^{1/2} \ell(R)^{-n} \left( |f_2|^2 \mu \right)(P)
\lesssim \sum_{S \in \text{Top} \cap \text{Tree}(S)} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} \left( |f_2|^2 \mu \right)(P) \ell(P)^n.
\]
Finally, by (73), (76), and (72), we conclude that
\[
S_2^2 \lesssim M_2 f(x_0)^2 + \sum_{S \in \text{Top} \cap \text{Tree}(S)} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} m_P \left( |f|^2 \right).
\]
As we pointed out before, the same estimate holds for $S_2^2$, because the only properties that we used from the annuli $A_{j_0}(z_D)$’s are that they have bounded overlap for $m \in J_1$, that their width is comparable to $\ell(D)$, that they are centered in some point lying in $D \subset Q$ and that they have diameter comparable to $\ell(Q)$. Of course, these properties are also shared by the annuli $A_{m+1}(z_D)$’s. Actually, for estimating $S_2$, one can argue exactly as in the case of $S_1$ but replacing $\{m \in J_1 : \epsilon_m \in I_Q\}$ by $\{m \in J_1 : \epsilon_{m+1} \in I_Q\}$ in the involved arguments.
Therefore, by (77), the analogous estimate for \( S_2 \), and (64), we see that
\[
(78) \quad \left( \sum_{m \in J_1} b_m^\rho \right)^{1/\rho} \lesssim M_2 f(x_0) + \left( \sum_{S \in \text{Top} \atop P \in \text{Stp}(S)} \sum_{m} \left( \frac{\ell(P)}{D(P,D)} \right)^{n+1/2} m_P \left( |f|^2 \right) \right)^{1/2}.
\]

We now deal with the sum relative to the \( b_m \)'s for \( m \in J_2 \). The estimates are essentially as in the case of \( m \in J_1 \), but we include the sketch of the arguments for the reader's convenience. Since \( \rho > 2 \), (63) yields
\[
(79) \quad \left( \sum_{m \in J_2} b_m^\rho \right)^{1/\rho} \lesssim \left( \sum_{m \in J_2} \left( \int_{A_m^2(z_D)} |K(z_D - y)||f(y)| d\mu(y) \right)^2 \right)^{1/2} + \left( \sum_{m \in J_2} \left( \int_{A_m^2(y)} |K(z_D - y)||f(y)| d\mu(y) \right)^2 \right)^{1/2} =: S_3 + S_4.
\]

The arguments to estimate \( S_3 \) and \( S_4 \) are almost the same, so we will only give the details for \( S_3 \). Since \( f_2 \) vanishes in 3D,
\[
(80) \quad S_3^3 = \sum_{k \in \mathbb{Z}} \sum_{m \in J_2} \sum_{\epsilon_m \in I_k} \left( \int_{A_m^2(z_D)} |K(z_D - y)||f_2(y)| d\mu(y) \right)^2 \lesssim \sum_{Q \in \mathcal{D}^u} \sum_{\epsilon_m \in I_Q} \sum_{P \in \text{Stp}(Q)} \left( |(f_2|\mu)(A_m^2(z_D))| \frac{\ell(Q)^2}{\ell(Q)^2} \right)^2.
\]

Once again, our task now is to estimate \( \left( |(f_2|\mu)(A_m^2(z_D))| \right)^2 \). As before, this is done by splitting the annulus \( A_m^2(z_D) \), whose width is \( \epsilon_m - \epsilon_{m+1} \), into disjoint cubes \( P \in \mathcal{D}^u \) such that \( \epsilon_m - \epsilon_{m+1} \in I_P \) and grouping them properly in terms of the corona decomposition. Arguing as in (63), we now have
\[
(81) \quad \left( |(f_2|\mu)(A_m^2(z_D))| \right)^2 \lesssim \sum_{R \in \mathcal{V}(Q)} \sum_{\epsilon_m - \epsilon_{m+1} \in I_P} \left( |(f_2|\mu)(A_m^2(z_D) \cap P)| \right)^2 + \sum_{R \in \mathcal{V}(Q)} \sum_{\epsilon_m - \epsilon_{m+1} \in I_P} \left( |(f_2|\mu)(A_m^2(z_D) \cap P)| \right)^2.
\]

The first term on the right hand side of (81) can be easily estimated using Cauchy-Schwarz’s inequality, that the \( P \)'s in Tree(\( R \)) such that \( \epsilon_m - \epsilon_{m+1} \in I_P \) are disjoint and Lemma 2.5. Similarly to what we did in (67), we now obtain
\[
(82) \quad \left( \sum_{P \in \text{Tree}(R): \epsilon_m - \epsilon_{m+1} \in I_P} |(f_2|\mu)(A_m^2(z_D) \cap P)| \right)^2 \lesssim (\epsilon_m - \epsilon_{m+1}) \ell(R) \sum_{P \in \text{Tree}(R): \epsilon_m - \epsilon_{m+1} \in I_P} |(f_2|^2|\mu)(A_m^2(z_D) \cap P)| \leq \ell(D) \ell(R) \sum_{P \in \text{Tree}(R): \epsilon_m - \epsilon_{m+1} \in I_P} \left( |(f_2|^2|\mu)(A_m^2(z_D) \cap R) \right),
\]

where we also used in the last inequality above that \( \epsilon_m - \epsilon_{m+1} \leq \ell(D) \), because we are assuming \( m \in J_2 \). As before, the second term on the right hand side of (81) is estimated similarly to (82) but introducing an auxiliary splitting of the sum in terms of the side length.
of the cubes. By applying the Cauchy-Schwarz inequality, we can proceed exactly as in (68) and (69), but replacing \( \ell(D) \) by \( \epsilon_m - \epsilon_{m+1} \), and then we deduce that

\[
\sum_{P \in \text{Stp}(R) : \ell(P) \geq \epsilon_m - \epsilon_{m+1}} \left( |f_2| \mu(A^1_m(z_D) \cap P) \right)^2 \leq \sum_{P \in \text{Stp}(R)} \left( \frac{\ell(P)}{\ell(R)} \right)^{1/2} \ell(R)^n \left( \frac{\ell(D)}{\ell(R)} \right)^n \left( \frac{|f_2| \mu(A^1_m(z_D) \cap P)}{|f_2| \mu} \right)^2.
\]

(83)

Combining (80) and (81) with (82) and (83), and using that \( \ell(R) = \ell(Q) \) for all \( R \in V(Q) \), and that the annuli \( A^2_m(z_D) \)'s have bounded overlap for \( m \in \mathbb{Z} \) because \( \{\epsilon_m\}_{m \in \mathbb{Z}} \) is a non-decreasing sequence, we conclude that

\[
S^2_3 \lesssim \sum_{Q \in D'} \sum_{R \in V(Q)} \frac{\ell(D)}{\ell(R)} \ell(R)^{-n} \left( \frac{|f_2| \mu}{|f| \mu} \right)(R).
\]

(84)

Plugging (83) and (84) into (81) finally yields

\[
S^2_3 \lesssim \mathcal{M}_2 f(x_0)^2 + \sum_{S \in \text{Top}} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} m_P \left( |f|^2 \right).
\]

(85)

Similarly to what we said below (77), the same estimate that we have for \( S_3 \) also holds for \( S_4 \). Therefore, applying (85) (and the same estimate for \( S_4 \)) to (79), we see that

\[
\left( \sum_{m \in I_2} b_m^2 \right)^{1/\rho} \lesssim \mathcal{M}_2 f(x_0) + \left( \sum_{S \in \text{Top}} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} m_P \left( |f|^2 \right) \right)^{1/2}.
\]

(86)

To complete the proof of the theorem it only remains to put all the estimates together and to use standard arguments. From (92), (78) and (80), we see that

\[
\left( \sum_{m \in \mathbb{Z}} (a_m + b_m)^2 \right)^{1/\rho} \lesssim \mathcal{M}_2 f(x_0) + \left( \sum_{S \in \text{Top}} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} m_P \left( |f|^2 \right) \right)^{1/2},
\]

which, by (80) and (81), implies that

\[
I_2 = \frac{1}{\mu(D)} \int_D \left| (V_{\rho} \circ T^\mu) f_2 - (V_{\rho} \circ T^\mu) f_2(z_D) \right| d\mu \lesssim \mathcal{M}_2 f(x_0) + \left( \sum_{S \in \text{Top}} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} m_P \left( |f|^2 \right) \right)^{1/2}.
\]

(87)
Finally, combining (57) and (58) with (59) and (87), and using that $\bigcup_{S \in \text{Top}} \text{Stp}(S) \subset \text{Top}$, we conclude that

\[
(M_{D_\mu}^p \circ \mathcal{V}_p \circ \mathcal{T}_\mu)f(x_0) \\
\lesssim M_2 f(x_0) + \sup_{D \in D_\mu: x_0 \in D} \left( \sum_{S \in \text{Top}} \sum_{P \in \text{Stp}(S)} \left( \frac{\ell(P)}{D(P, D)} \right)^{n+1/2} m_P (|f|^2) \right)^{1/2} \\
\lesssim M_2 f(x_0) + \left( \sum_{P \in \text{Top}} \left( \frac{\ell(P)}{D(P, x_0)} \right)^{n+1/2} m_P (|f|^2) \right)^{1/2} \\
=: M_2 f(x_0) + \mathcal{E}_{1/2} f(x_0),
\]

for all $x_0 \in \text{supp}(\mu)$, where we denoted

\[
D(P, x_0) := \ell(P) + \text{dist}(P, x_0).
\]

In Lemma 4.2 below we prove that $\mathcal{E}_{1/2}$ is a bounded operator in $L^p(\mu)$ for all $2 < p < \infty$. Assuming this for the moment, by (55) and the $L^p(\mu)$-boundedness of $M_2$, we see that $M_{D_\mu}^p \circ \mathcal{V}_p \circ \mathcal{T}_\mu$ is also bounded in $L^p(\mu)$ for all $2 < p < \infty$. Then we obtain

\[
\| (\mathcal{V}_p \circ \mathcal{T}_\mu) f \|_{L^p(\mu)} \lesssim \| (M \circ \mathcal{V}_p \circ \mathcal{T}_\mu)^2 f \|_{L^p(\mu)} \lesssim \| (M^2 \circ \mathcal{V}_p \circ \mathcal{T}_\mu) f \|_{L^p(\mu)} \lesssim \| f \|_{L^p(\mu)}
\]

for all $2 < p < \infty$, and the theorem is proved. \hfill \square

**Lemma 4.2.** Given $\delta > 0$, set

\[
\mathcal{E}_\delta f(x) := \left( \sum_{P \in \text{Top}} \left( \frac{\ell(P)}{D(P, x)} \right)^{n+\delta} m_P (|f|^2) \right)^{1/2}
\]

for $f \in L^p(\mu)$ and $x \in \mathbb{R}^d$, where $D(P, x)$ is defined in (59). Then $\mathcal{E}_\delta$ is a bounded operator in $L^p(\mu)$ for all $2 < p < \infty$.

**Proof.** The proof follows by duality and Carleson’s embedding theorem. Since $2 < p < \infty$, if $q$ is such that $2/p + 1/q = 1$ then $1 < q < \infty$, thus

\[
\| \mathcal{E}_\delta f \|_{L^p(\mu)} = \| (\mathcal{E}_\delta f)^2 \|_{L^{p/2}(\mu)}^{1/2} = \sup_{\|g\|_{L^q(\mu)} \leq 1} \left| \int (\mathcal{E}_\delta f)^2 g \, d\mu \right|^{1/2}.
\]

Note that

\[
\left| \int (\mathcal{E}_\delta f)^2 g \, d\mu \right| \leq \sum_{P \in \text{Top}} m_P (|f|^2) \int \left( \frac{\ell(P)}{D(P, x)} \right)^{n+\delta} |g(x)| \, d\mu(x).
\]

Integrating over dyadic annuli and using that $\mu$ is $n$-AD regular, it is easy to check that

\[
\frac{1}{\mu(P)} \int \left( \frac{\ell(P)}{D(P, x)} \right)^{n+\delta} |g(x)| \, d\mu(x) \lesssim M g(y) \quad \text{for all } y \in P
\]
(here it is crucial that $\delta > 0$). Thus, by \((91), (92)\), Hölder’s inequality and Carleson’s embedding theorem (recall that $p/2$ and $q$ belong to $(1, \infty)$),

\begin{equation}
\left| \int (\mathcal{E}_\delta f)^2 g \, d\mu \right| \lesssim \sum_{P \in \text{Top}} m_P \left( \left( \int |f|^2 \right)^{p/2} \mu(P) \right)^{2/p} \left( \sum_{P \in \text{Top}} (m_P(Mg))^{q/2} \mu(P) \right)^{1/q} \lesssim \|f\|_{L^p(\mu)}^2 \|Mg\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.
\end{equation}

From \((90)\) and \((93)\) we conclude that $\|\mathcal{E}_\delta f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$, as wished. \hfill \Box

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Albert Mas. Departament de Matemàtica Aplicada I, ETSEIB, Universitat Politècnica de Catalunya, Avda. Diagonal 647, 08028 Barcelona (Spain)
E-mail address: amasblesa@gmail.com

Xavier Tolsa. Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, Catalonia
E-mail address: xtolsa@mat.uab.cat